A discrete resampling technique to correct for Doppler effect in continuous gravitational wave search

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Abstract. The detection of continuous gravitational waves has to deal with the Doppler effect induced by the Earth motion with respect to the source. This frequency shift must be taken into account to recover the signal energy as a monochromatic peak with a high signal-to-noise ratio. The correction to be applied to the antenna output depends on the source sky direction, on the source spin and also on the spin-down rate. Since they are, in general, unknown parameters a large computational effort is necessary to correct for any possible value. A correction technique independent of the source frequency is discussed here. The method consists to anticipate or delay the antenna clock by removing or doubling one of its digital signal sample, in order to maintain the clock of the moving observer well locked to the rest one. The method, which requires just a little computational effort, appears to be very effective for “semi-targeted” searches, where the source direction is known but the emission frequency is not.

1. Introduction
Rotating neutron stars are expected to emit permanently gravitational waves mainly at the double of their rotational frequency (from fractions of Hz up to several hundreds of Hz). Long observation times are necessary in order to distinguish the peak at the emission frequency in the antenna spectral floor, since the signal-to-noise ratio scales as the square root of the measurement period [1]. The small spectral peak is smeared, and thus buried in the noise, by the phase modulation induced by the Doppler effect due to the antenna motion, and by the phase shift induced by the slowing down of the source rotational frequency \textit{spin-down}. This last effect, whose values range between $10^{-17}$ and a few $10^{-10}$ Hz/s, is induced by the energy losses due to emissions of electromagnetic and gravitational waves and can be measured by the electromagnetic observations of the galaxy Pulsars. The correction for recovering a monochromatic peak at the frequency of the source depends on the spin-down rate $\dot{\nu}_0$, on the source sky direction, and on the source emission frequency $\nu_0$. Here, we propose a correction technique that, once fixed the source direction and the spin-down rate, is valid at the same time for all the possible emission frequencies in the detection band. The technique consists in a resampling of the antenna output signal, aimed at synchronizing with a large accuracy the detector time with the source one, so to put the antenna “at rest” with respect to the source. The method, as other techniques [1], is based on a discrete resampling. In particular, in our case, we accelerate (or slow down) the antenna proper time by removing (or doubling) in a timely manner single samples of the detector digitized signal in order to keep the synchronization with...
the source clock. The applied correction, and then the possibility to recover a signal peak in the spectrum of the antenna output, is valid for all the possible emission frequencies. If the source direction is known, but the emission frequency \( \nu_0 \) and the spin-down rate \( \dot{\nu}_0 \) are unknown (semi-targeted search), one can perform several times the resampling, assuming different possible values of the spin-down rates (more precisely of the fractional spin-down, \( \dot{\nu}_0/\nu_0 \)), and then search for a peak coming out from the spectral floor in the resampled data streams. In the usual techniques, once fixed the sky direction, the correction must be applied for any possible value of the couple of parameters \( \dot{\nu}_0 \) and \( \nu_0 \), with a larger computational effort.

Let us consider a monocromatic signal with amplitude \( A_0 \) and frequency \( \nu_0 \). A Fourier analysis performed on a 1 year-long data bench by an observer at rest with respect to the source exhibits a peak in the spectrum with amplitude \( A_0 \) and width \( \delta \nu = \frac{1}{\text{Year}} = 3.17 \times 10^{-8} \text{ Hz} \). Let us considered the simple case of a detector covering uniformly in one year a circular orbit centered on the Sun (the orientation of the detector is assumed to be constant, that means to neglect any modulation effect. The detected signal will be:

\[
A(t) = A_0 \cos(\omega_0 t + \varphi(t)) \quad \text{where} \quad \varphi(t) + \epsilon \sin(\omega_y t) \quad (1)
\]

While \( \omega_0 = 2\pi \nu_0 \), \( \omega_y = 2\pi \nu_y \) and \( \nu_y \) denotes the frequency corresponding to 1 year orbital period. \( \epsilon \) is the amplitude of the phase variation due to the Doppler effect in one year (modulation index). This phase modulation induces a spread of the spectral line whose half bandwidth, for a source located on the ecliptic plane, is given by:

\[
\Delta \nu = \nu_0 \frac{v}{c} \approx 10^{-4} \nu_0 \quad (2)
\]

where \( v \) is the orbital speed (around 30 km/s for the Earth), while \( c \) is the light speed. Since the energy detected by the antenna (and thus the power spectrum integral) is not modified by the Doppler effect, the square of the signal amplitude will be reduced by a factor of the order of \( \frac{A_0^2}{A_0^2} \). For a wave at a frequency \( \nu_0 = 100 \text{ Hz} \), this means:

\[
\frac{A_0^2}{A_0^2} = \frac{\delta \nu}{2 \Delta \nu} \approx \frac{3.17 \times 10^{-8}}{2 \times 10^{-4} \times 10^2} = 1.58 \times 10^{-6} \quad (3)
\]

and thus a ratio of the signal amplitudes around \( 1.25 \times 10^{-3} \), with a reduction of signal to noise ratio around 58 dB.

The Doppler effect can be corrected for a given sky direction by resampling the detected signal with an average frequency equal to the source one, but phase modulated so to compensate for phase modulation due to observer motion. The goal is to recover a large fraction of the signal energy in the original FFT bin, so to have a signal to noise ratio similar to the one measured by the rest observer. The accuracy we need in the Doppler compensation can be computed by developing the phase-modulated signal in Bessel functions. It is easy to demonstrate that the term at the source frequency \( (J_0) \) keeps the 60% of the signal energy, if the residual phase modulation amplitude, after the Doppler correction, is less than one radian (\( \epsilon < 1 \text{ rad} \)). The signal to noise ratio reduction, expressed in amplitude, will be small, less than 6 dB. This is taken as our specification, even if the entire argument can be easily scaled to ask for a better accuracy in signal to noise ratio recovering, without remarkable impact on the computational effort.

**2. Correction for the Doppler effect**

The method discussed here aims at synchronizing the moving observer clock to the rest one with an accuracy better than a sampling interval \( \Delta t = 1/\nu_s \), where \( \nu_s \) is the sampling frequency of the antenna digitized output signal (typically between 10 and 20 kHz). Let us consider for the
purpose a “virtual” (not physical) sinusoidal wave with a frequency equal to the sampling one, coming from the direction of the source. Its equiphasic surfaces are planes perpendicular to the wave vector $\overrightarrow{k_s}$, traveling at the speed of light. In the rest frame, the travelling equiphasic plane equation (for a given phase $\phi$) is given by:

$$\overrightarrow{k_s} \cdot \overrightarrow{r} - \omega_s t = \phi$$

(4)

where $\omega_s = 2\pi \nu_s$ and $|\overrightarrow{k_s}| = \omega_s/c$.

Let us consider the family of planes whose phase $\phi$ is an integer multiple of $2\pi$. Looking at two successive planes in Eq.(4), it straightforward to understand that these planes travel parallel each other, separated by a time $\Delta t$ (that means by a distance $c\Delta t$), corresponding to a phase $\omega_s \Delta t$. The rest observer that monitors (in a stroboscopic way) at the sampling frequency $\nu_s$, (i.e. each $\Delta t$ seconds) the positions of this family of planes, sees at any sample time the $j_{th}$ plane takes the place that the $(j+1)_{th}$ one had at the previous sampling time, and to be replaced by the $(j-1)_{th}$ one. In other words, the equiphasic planes of these virtual signal at the sampling frequency, appear to shift each other, but the entire family will have at any sample the same positions. We will use these fixed set of planes as a reference grid in the rest frame, where the motion of the moving observer (i.e. of the antenna) is described by a trajectory $\overrightarrow{r}(t)$. Without any loss of generality, one can assume that at time $t = 0$ (start of the analysis) the two clocks (moving and rest one) are synchronized to $\phi = 0$ in the origin of the rest frame. The origin can be chosen coincident with the time-zero position of the antenna trajectory. At a time $t$, the phase measured by the moving observer is $\overrightarrow{k_s} \cdot \overrightarrow{r}(t) - \omega_s t$ (Eq.(6)). This must be compared with the one detected by the observer at rest in the origin, $\omega_s t$. The dephasing ($\overrightarrow{k_s} \cdot \overrightarrow{r}(t)$) is thus ruled by the moving observer position with respect to the start grid plane where synchronization occurred.

As mentioned above, the goal is to lock the antenna clock to the rest one so to have a time difference not larger than the sampling interval ($\Delta t$). When the moving observer crosses one of the two planes nearest to the origin (the first neighbor toward the wave vector or the one on the opposite side), a dephasing $\omega_s \Delta t$ has been cumulated. We must compensate for this by slowing down (or accelerating, depending on the dephase sign - see below) the moving observer proper time, by a time $\Delta t$. This compensation is performed in an easy way, just repeating (or deleting) one of the digitized signal sample. In particular, if the observer motion versus is opposite to the wave one (i.e. $\overrightarrow{v} \cdot \hat{n} < 0$) a negative dephasing with respect to the rest clock occurs. This means that the moving clock is anticipating the rest one (since detects in advance the running equiphasic plane). The antenna clock has thus to be slowed by repeating a sample of its output signal stream. Vice versa, when $\overrightarrow{v} \cdot \hat{n} > 0$, a delay occurs, and we compensate by removing a sample. The correction just described must be performed each time the moving observer crosses one of the grid planes. In this way the two clocks are always synchronized with an accuracy less or equal of a sampling interval $\Delta t$.

The entire argument has been developed considering a virtual wave, coming from the source direction, having a non-physical emission frequency, equal to the digital signal sampling rate (several kHz). For a generic wave with frequency $\nu_0$, the achieved time synchronization between the two clocks, corresponds to a phase locking accuracy of $\omega_0 \Delta t$, i.e. $2\pi \nu_0 / \nu_s$. To meet the required 1 rad phase accuracy it thus necessary to operate with a sampling frequency at least $2\pi$ times larger than the source one. In gravitational wave antennas, where use of sampling frequencies between 10 and 20 kHz is made, the technique can thus be applied up to several kHz, enough for continuous gravitational waves by semi-targeted analysis, that usually stop around 200 Hz. This leaves a margin for the achievement of a better recovering accuracy and/or a preliminary sampling frequency reduction of the original data stream. In the usual cases the moving observer velocity does not change too much during the crossing between two grid planes. The time to cross two successive equiphasic planes ($t_{crossing}$) is well approximated by
their distance \((c\Delta t)\), divided by the amplitude of the antenna velocity \((\vec{v})\) projection along the wave vector \((|\vec{v} \cdot \hat{n}|)\):
\[
t_{\text{crossing}} = 1/(|\vec{\beta} \cdot \hat{n}| \nu_s)
\]  
\[ (5) \]

The minimum possible value for \(t_{\text{crossing}} = 1/(\beta \nu_s)\), occurs when the velocity direction is parallel to the wave vector. This means that the time between the two successive planes (and thus the correction), independently of the sampling frequency, cannot occur before than \(1/\beta\) samples (in the case of Earth, where \(\beta \simeq 10^{-4}\), one each about ten thousands). Vice versa, when the velocity of the orbit is almost perpendicular to the wave vector, long times between two successive crossings take place.

\section{Spin-down correction}

The spin-down effect can be described by the Taylor expansion of the rotational frequency of the Neutron star as a function of time:
\[
\nu(t) = \nu_0 + \dot{\nu}_0 t + \frac{1}{2} \ddot{\nu}_0 t^2 + ...
\]  
\[ (6) \]
where \(\dot{\nu}_0, \ddot{\nu}_0, \ldots\) are named spin-down parameters. The cumulated phase of the signal, as measured by the observed at rest with respect to the source, is obtained by integrating \(\nu(t)\) over time:
\[
\varphi(t) = 2\pi \int_0^t \nu(\tau) d\tau = 2\pi \left( \nu_0 t + \frac{1}{2} \dot{\nu}_0 t^2 + \frac{1}{6} \ddot{\nu}_0 t^3 + ... \right).
\]  
\[ (7) \]
Taking the first order of the expansion, the signal emitted by the Pulsar is
\[
S(t) = h_0 \sin \left( \omega_0 t + \frac{1}{2} \dot{\omega}_0 t^2 \right)
\]  
\[ (8) \]
where \(h_0\) is the gravitational wave amplitude, \(\omega_0 = 2\pi \nu_0\) and \(\dot{\omega}_0 = 2\pi \dot{\nu}_0\), with typical values of \(|\dot{\nu}_0|\), ranging between a few \(10^{-17}\) and a few \(10^{-10}\) Hz/s (see [2]). Since the spin-down induces a decrease of the frequency in time, it is clear that in all cases \(\dot{\nu}_0 < 0\).

As shown in the following, a downsampling of the data will be done in order to perform an FFT of a few months long data series with an acceptable number of points. In this frame an effective way to correct also for the spin-down (and synchronize the clocks) is to compute at each downsampling time the value of time \(t'\) at which the phase detected by the moving observed \(\varphi'(t')\) (left term of next equation) is equal to the one of a perfect monochromatic signal, at the same (generic) emission frequency \(\omega_0\). This means to impose:
\[
\omega_0 \left( \frac{\vec{r}(t') \cdot \hat{n}}{c} + t' \right) + \frac{1}{2} \dot{\omega}_0 \left( \frac{\vec{r}(t') \cdot \hat{n}}{c} + t' \right)^2 = \omega_0 k t_0 \quad k = 1, \ldots T/t_0
\]  
\[ (9) \]
with the usual notations, and where \(T\) is the whole observation time. The Doppler effect in the left term is evident. \(t' - k t_0\) represents the time difference between the antenna clock and the pure monochromatic signal. Solving Eq.(9), this time difference \(\delta t\) can be obtained:
\[
\delta t = t' - k t_0 = - \frac{\vec{r}(t') \cdot \hat{n}}{c} + \frac{\varepsilon (k t_0)^2}{1 - \varepsilon k t_0 + \sqrt{1 - 2\varepsilon k t_0}}
\]  
\[ (10) \]
where \(\varepsilon = \dot{\nu}_0/\nu_0\) is the fractional spin-down rate. \(\delta t\) is the delay/anticipation to be compensated, in order to equalize the two phases at the given downsampling time. Obviously, the time shift can be expressed in terms of the number of samples to be shifted in downsampling extraction,
The resampling method: the downsampled stream is generated by taking from the high-frequency stream the sample shifted with respect to the one selected without correction (see dotted arrows), that means exactly at each downsampling time (i.e. 100 s in this case). The number of samples to be skipped at each downsampling time in order to maintain the synchronization is computed from Eq.10. A backward shift is used to induce a delay in the antenna clock, while a forward shift gives an anticipation with respect to the rest signal. The correction is performed with the shifted downsampled depicted in Fig.1, including simultaneously both the spin-down and the Doppler effects. \( \delta t \) has to be computed only a few times over the observation time (each downsampling time), and the applied correction is valid for all the sources in the \( \hat{n} \) direction, with the same value of the fractional spin-down rate \( \varepsilon \), independently of the emission frequency. Typical downsampled frequencies range from the case of the targeted searches, where the emission frequency is known (a thousandths of Hz band and less is enough, corresponding to hundreds of s downsampled time), to semitargeted case, where a several Hz band (fraction of s downsampled time) is required. Slightly different methods to compute sampling shifts (Eq.10) in the two cases are used in these two scenarios [3].

4. Validation of the pipeline
Several tests of the resampling method have been performed by simulating signals with different frequency \( \nu_0 \) and spin-down rates \( \dot{\nu}_0 \), considering several source sky directions, and signal sampling frequencies \( \nu_s \). In order to test the robustness of the method to trajectory changes, different parameters for the antenna orbit have been used, including the real Virgo antenna trajectory during its first scientific (VSR1, May-Oct 2007). The amplitude modulation due to the variation of the response of the antenna, induced by the different orientations during the Earth rotation [4], has not been included in the analysis. As shown in [5], this can be done at the end of the pipeline, on the final FFT, after the Doppler and spin-down correction. Both the phase locking accuracy and the peak recovering have been tested by simulating only the signal affected by Doppler and spin down, with no noise injected in the data stream. Spin-down rates ranging from \( \dot{\nu}_0 = -10^{-12} \) Hz/s up to non-physical values (\( \dot{\nu}_0 = -10^{-4} \) Hz/s) have been taken into account. The different tests will be discussed in a better detail in a forthcoming paper [3].
In Fig.2, the comparison of the phase of the modulated signal, coming from a fixed direction of the sky, with the rest one is performed before and after the correction.

![Figure 2](image.jpg)

**Figure 2.** Phase locking accuracy test using a circular orbit around the Sun (with a radius of $1.5 \cdot 10^8$ m and a uniform orbital velocity of $\beta = 10^{-4}$) with $\nu_0 = 20$ Hz, $\dot{\nu}_0 = -10^{-11}$ Hz/s and $\nu_s = 1000$ Hz. The plot on the left shows, as a function of time, the phase difference between the rest signal and the modulated one (red curve), and the phase difference (close to zero) between rest and corrected signals (black curve). The large Doppler modulation of the phase induced by the circular orbit and the very small drift due to spin-down can be appreciated in the red curve. The residual dephasing after the correction, visible in the $y-axis$ zoom of the black curve (right plot), never exceeds the expected value, $\pm \pi \nu_0 / \nu_s = \pm 0.063$ rad.

In all the tested cases the signal phase has been correctly recovered, with a residual dephasing with respect to the rest signal within the expected limit ($\pm \pi \nu_0 / \nu_s$). One of the end-to-end tests performed on the semi-targeted pipeline application is illustrated in Fig.3 and its caption. We tested the accuracy in the peak reconstruction applying the technique to signals with different frequency, sampling frequency and spin-down rates. As shown in 4, the amplitude reconstruction accuracy results to depend, as expected, only on the ratio between the signal frequency and the sampling frequency (and not independently on the signal frequency, and on the spin-down rate). One can also observe that, once the signal frequency is one fifth of the sampling frequency, an acceptable level of reconstruction (amplitude losses well below 10 per cent) is achieved. This means that to investigate signals below 200 Hz (as usual in semi-targeted searches), 1 kHz sampling rate is enough.

The full pipeline for semi-targeted searches works as follows. The high-frequency signal (4 kHz or 20 kHz) is pass-banded around a given central frequency with a large band (a few Hz). The chosen bandwidth depends only on the maximum dimension of the FFT one wants to perform, and not by our method that is valid at the same time for all the emission frequencies (small enough with respect to the sampling frequency - see Fig.4). Indeed, after the correction taking place on the pass-banded stream, still at high frequency, one can heterodyne the signal, and thus makes a down-sampling of the signal at a frequency around the double of the bandwidth (again at a few Hz). The correction is valid for all the frequencies of the small band and thus, even considering a long integration time (months), one can perform the FFT on the full set of data, since, after the downsampling, we have a reasonable number of points. For instance, considering 120 days of integration time, and investigating a band whose width is 2 Hz (downsampling frequency of 5 Hz), after the correction (for a given spin-down rate), one can make an FFT on around 50 millions of points (small enough for effective FFT algorithms).
Figure 3. Linear spectral amplitude of the signal before (gray) and after (black) correction. A 30 days-long data stream, sampled at 4 kHz and made by random noise, with an amplitude similar to the Virgo one (in the concerning band), is filtered by a Chebyshev pass-band ($\nu_{\text{stop}} = 2.5$ Hz and central frequency $\nu_c = 19.0$ Hz). A gravitational wave signal, coming from the center of the galaxy, as it would be detected by the moving Virgo antenna (with the same orbital motion of VSR1), is injected in the data stream, before the pass-band filtering at $\nu = 20.025$ Hz, with $\dot{\nu}/\nu \simeq -5 \times 10^{-13}$ s$^{-1}$, and amplitude $h_0 = 10^{-19}$. The smearing of the spectral energy induced by Doppler effect is visible in the gray curve, such as the recovering of the signal peak at 1.025 Hz in the black one. The losses with respect to the original signal are, after the reconstruction, below 1%.

In order to avoid ultra-long FFT, it is more effective to perform a broad-band semi-targeted search, splitting the analysis on different smaller bands, and repeating the method several times for different “spectral slices”. The computational costs for each steps of the pipeline have been measured on a standard computer. The cost of the resampling has a negligible impact on the entire pipeline ($10^{-5}$ of the length of the period investigated), with respect to the data reading (I/O) ($4 \times 10^{-4}$ of the period, for data input at 4 kHz). Details on the optimization of the pipeline for searches on many directions of the sky will be published in [3].

Conclusions

The proposed resampling method to compensate for Doppler and spin-down effect has been successfully tested. Good performance in terms of accuracy in spectral peak recovering, absence of artifacts in the noise, and fast execution time has been achieved. Since the Doppler and the spin-down correction, for a given fractional spin-down rate and direction, is valid simultaneously for all the investigated frequencies, the method results to be very effective in
Figure 4. Peak amplitude losses as a function of the source-sampling frequency ratio \((\nu_0/\nu_s)\). The plot is repeated for three different sampling frequencies (400 Hz, 1 kHz, 4 kHz). The losses, as expected, depend only on the frequency ratio, and not directly on the source frequency. The estimated errors are smaller than the plot resolution.

semitargeted searches, where the emission frequency is unknown. An analysis based on this method, concerning the data of the two recent Virgo scientific runs is in progress by using the GRID environment.

Acknowledgments
The authors acknowledge the support of the Rome 1 Virgo group, in particular S.Frasca, C.Palomba, F.Antonucci and P.Astone, for their smart suggestions. Thanks are also due the Ligo Scientific Community-Virgo joint group aimed to Continuous Wave data analysis, for the discussion of the method and the many suggestions received.
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