On the 2nd Order Corrections to the Hard Pomeron and the Running Coupling

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Abstract

It is shown that solutions to the 2nd order BFKL eigenvalue equation exist for arbitrary large real values of the complex angular momentum $j$. This corresponds to a cut in the complex $j$ plane along the whole real axis, and it makes the use of the complex angular momentum variable for the calculation of the high-energy behavior somewhat questionable. The eigenfunctions contain non-perturbative pieces which behave as $\exp(-1/\alpha_s b)$ and have no counterpart in the leading-log BFKL equation. The high-energy behavior of the 2nd order BFKL Green function as found by other authors, is reproduced by excluding these non-perturbative pieces of the eigenfunctions.
1 Introduction

Recently corrections of the 2nd order in the coupling constant to the hard pomeron (BFKL) equation have been calculated [1]. For the intercept they are accompanied by a large negative coefficient [1, 2], which makes its prediction reliable only for extremely low coupling constants and translates into extremely large relevant mass scales. Although this conclusion has been questioned in [3], different calculations [4, 5, 6] show features of this approach which clearly limit its applicability.

In [4] the corrections to the BFKL kernel have been summed to obtain the corrected high-energy behavior of the corresponding amplitudes. This summation resulted in a multiple integral over intermediate rapidities, which was performed in the saddle point approximation and using the asymptotic form of the 1s order (leading-log) BFKL Green function. Even after these simplifications this leads to a highly nontrivial multiple integral, the calculation of which required much skill and ingenuity. A similar result for the high behavior has been found also in [5].

In this note we draw attention to the fact that there exists a simple way of exactly solving the BFKL eigenvalue equation with the kernel being known in the two first orders. The solution turns out to exist for any real value of the complex angular momentum \( j \), from \(-\infty\) to \(+\infty\). It means that the singularities of the BFKL amplitude occupy the whole real axis in the \( j \) plane, so that there is no rightmost singularity and no intercept (the first hint of a “non-Regge” behavior was noted in [4] upon the observation of a non-usual form of the \( s \)-dependence). All solutions to the eigenvalue equation turn out to have non-perturbative features. They separate into “normal” pieces, which can be related to the lowest order solutions and behave similarly both in \( q \) and the energy, and “abnormal” contributions, which are non-oscillatory and proportional to \( \exp(-1/\alpha_s b) \). If we neglect the abnormal pieces, we obtain an asymptotic behavior at high \( s \), which coincides with [4].

Our method easily generalizes to the case when the coupling constant is summed to all orders to run with the momentum scale. In the lowest order in this running coupling the equation still can be written, in spite of the evident difficulties at small momenta. Solutions of this equation also exist for any real \( j \). This indicates that the physical equation should have a kernel which is different from the lowest order one, not only in the running of the coupling but also in its functional dependence, to give a meaningful result in terms of a Mellin transform. When this paper was being completed, a preprint
by E. M. Levin appeared [3], in which a very similar method has been applied in order to solve the 2nd order BFKL equation, with the running coupling constant summed to all orders and being modified in a specific manner in the confinement region. The contents of this preprint partly overlap with our Section 4. However the central point of our paper - the existence of a cut in the $j$ plane along the whole real axis - is not discussed in [3].

2 Basic equations

We start by presenting basic formulas necessary for our derivation. They all are either standard or can be taken from Refs. [1, 4]. We restrict ourselves to the forward case and to the azimuthally symmetric wave function $\psi(q)$, which is known to dominate in the high-energy limit. The pomeron equation will be written in the form of a Schrödinger equation

$$H\psi(q) = E\psi(q),$$

(1)

where the “energy” is related to the complex angular momentum by $E = 1 - j (= -\omega$ in the usual notation [1]). The “Hamiltonian” $H$ has been calculated up to terms of the second order in the coupling constant $\alpha_s(\mu^2) \equiv \alpha_s$:

$$H = H^{(1)} + H^{(2)}.\tag{2}$$

Proper functions of the lowest order (BFKL) Hamiltonian $H^{(1)}$ are well-known:

$$H\psi_\nu(q) = \epsilon^{(1)}_{\nu}\psi_\nu(q),\tag{3}$$

where

$$\psi_\nu(q) = \frac{1}{\pi q\sqrt{2}} q^{2\nu},\tag{4}$$

the unperturbed energies are (in units of $\alpha_s N_c/\pi$)

$$\epsilon^{(1)}_{\nu} = \psi\left(\frac{1}{2} + i\nu\right) + \psi\left(\frac{1}{2} - i\nu\right) - 2\psi(1)\tag{5}$$

($\epsilon^{(1)}_{\nu} = -\chi(\gamma), \gamma = 1/2 + i\nu$, in the usual notation [1]). In this equation $\psi(z) = d\ln \Gamma(z)/dz$, and $\nu$ runs from $-\infty$ to $+\infty$. The wave functions (4) are correctly normalized:

$$\langle \psi_\nu|\psi_{\nu'}\rangle \equiv \int d^2q\psi^*_\nu(q)\psi_{\nu'}(q) = \delta(\nu - \nu').\tag{6}$$
The second order part of the Hamiltonian can be conveniently written in terms of its action on the proper functions of the first order Hamiltonian $\psi_\nu$ [4]. Namely

$$H \psi_\nu(q) = [\kappa_{sc}(\nu) + \kappa_r(\nu, q)] \psi_\nu(q).$$  

(7)

The first term in the bracket is scale invariant:

$$\kappa_{sc}(\nu) = -\frac{\alpha_s N_c}{4\pi} c(\nu) \epsilon^{(1)}_\nu,$$

(8)

where the function $c(\nu)$ can be found in [4]. The second term provides for the running of the coupling:

$$\kappa_r(\nu, q) = [-2\alpha_s b \ln(q/\mu)] \epsilon^{(1)}_\nu,$$

(9)

where $b = (11N_c - 2N_f)/(12\pi)$. We put $\mu = 1$ for simplicity.

$\kappa_{sc}$ is independent of $q$. So it simply shifts the energy

$$\epsilon^{(1)}_\nu \rightarrow \epsilon_\nu = \epsilon^{(1)}_\nu \left(1 - \frac{\alpha_s N_c}{4\pi} c(\nu)\right).$$

(10)

As a result the lowest energy level at $\nu = 0$ acquires a factor

$$1 - \frac{\alpha_s N_c}{4\pi} c(0) \simeq 1 - \frac{\alpha_s N_c}{\pi} 6.562$$

for $N_c = 3$ and $N_f = 2$, which is so disturbing from the point of practical application of this formalism to present momentum scales.

The second term $\kappa_r$ depends on $q$ and evidently changes the equation itself. In [4] the influence of this factor was investigated by summing all orders in its action. We are going to propose a different approach.

3 The $\nu$-representation

Our idea is quite trivial (and not new, see [5]). Since the action of the second order Hamiltonian on the proper functions $\psi_\nu$ is known, we are going to pass to the representation which these functions provide. To this aim we present any solution $\Psi(q)$ of the total Hamiltonian $H$ as a superposition of $\psi_\nu$,

$$\Psi(q) = \int d\nu f(\nu) \psi_\nu(q).$$

(11)

Mathematically it is nothing but a Fourier transformation of $q \Psi(q)$ with respect to $\ln q$. Using [3] we have

$$f(\nu) = \int d^2 q \psi_\nu^\ast(q) \Psi(q).$$

(12)
In the $\nu$ representation the Schrödinger equation (1) reads
\[
\int d\nu' H(\nu, \nu') f(\nu') = E f(\nu),
\] (13)
where, of course,
\[
H(\nu, \nu') = \langle \psi_\nu | H | \psi_{\nu'} \rangle = \int d^2q \psi^*_\nu(q) H \psi_{\nu'}(q).
\] (14)

Using (3) we obtain
\[
H^{(1)}(\nu, \nu') = \epsilon^{(1)}(\nu) \delta(\nu - \nu').
\] (15)
The scale invariant part in $H^{(2)}$ gives similarly
\[
H^{(2)}_{sc}(\nu, \nu') = -\frac{\alpha s N_c}{4\pi} c(\nu) \epsilon^{(1)}(\nu) \delta(\nu - \nu').
\] (16)
The running coupling part involves $\ln q$, which leads to a derivative of the $\delta$-function,
\[
H^{(2)}_r(\nu, \nu') = -i\alpha s b \epsilon^{(1)}(\nu') \delta'(\nu - \nu').
\] (17)

We note that in all second order terms we can substitute $\epsilon^{(1)}(\nu)$ by $\epsilon(\nu)$ defined by (10) in the adopted approximation.

Thus the Schrödinger equation in the $\nu$ representation turns out to be a differential equation of the first order in $\nu$:
\[
(-i\alpha s b) \frac{d}{d\nu} [\epsilon(\nu) f(\nu)] + \epsilon(\nu) f(\nu) = E f(\nu).
\] (18)

This equation is solved trivially. The solution can be taken as
\[
f_E(\nu) = \frac{C}{\epsilon(\nu)} \exp \left( \frac{iE}{\alpha s b} \int_0^\nu \frac{d\nu'}{\epsilon(\nu')} - \frac{i\nu}{\alpha s b} \right),
\] (19)
where $C$ is a constant. Note that the 1st order $\epsilon^{(1)}(\nu)$ has zeros at $\nu = \nu_0 \simeq \pm 0.6375$. So, in this approximation, one has to specify the rule to circumvent the pole singularity in the integral, say, change $\epsilon(\nu) \rightarrow \epsilon(\nu) - i0$. At the 2nd order $\epsilon(\nu)$ does not vanish for real $\nu$, so that this change is not necessary. However in both cases $\epsilon(\nu)$ contains an imaginary part. Correspondingly, for the conjugate equation one has to take $\tau(\nu) = \epsilon^*(\nu)$. Then it will have similar solutions $\overline{f}_E$ as (19) without the prefactor $1/\epsilon(\nu)$ and with an opposite sign of $\text{Im} \epsilon(\nu)$. Imaginary parts of $\epsilon(\nu)$ and $\tau(\nu)$ will produce certain real factors in $f_E$ and $\overline{f}_E$, of a non-perturbative nature (with $1/\alpha s$ in the exponent), the product of which gives unity. Because of that in a product $f_E \overline{f}_E$, which is of practical importance, these
products always cancel. Due to this circumstance, in future we shall neglect the real factors generated, for real \( \nu \), by the imaginary part of \( \epsilon_\nu \).

One can see that \( f_E \) and \( \mathcal{F}_E \) form an orthonormal system: Indeed

\[
\langle \mathcal{F}_E | f_{E'} \rangle = |C|^2 \int \frac{d\nu}{\epsilon_\nu} \exp \left( \frac{i(E' - E)z(\nu)}{\alpha_s b} \right),
\]

where we have defined

\[
z(\nu) = \int_0^\nu \frac{d\nu'}{\epsilon_{\nu'}}
\]

and used the fact that in \( \mathcal{F}_E \) appears \( z(\nu) = z^*(\nu) \). Evidently

\[
\langle \mathcal{F}_E | f_{E'} \rangle = |C|^2 \int dz \exp \left( \frac{i(E' - E)z}{\alpha_s b} \right) = 2\pi \alpha_s b |C|^2 \delta(E - E'),
\]

and choosing \( |C| = (2\pi \alpha_s b)^{-1/2} \) we obtain a correct normalization.

The most remarkable property of the solution is that it exists for any real value of the energy \( E \). In the next Section we will show that there exists a normalizable eigenfunction \( \Psi_E(q) \) for any \(-\infty < E < +\infty\). Some doubts may arise in view of the pole singularity at \( \nu = \nu_0 \) generated by the denominator \( \epsilon^{(1)}(\nu) \). However, inspection shows that at this point the exponential provides a rapidly oscillating factor which ensures the convergence of integrals involving \( f(\nu) \). So the spectrum of the Hamiltonian, with the second order correction included, extends from \(-\infty\) to \(+\infty\). Such a dramatic change in the spectrum is due to a highly singular character of the “interaction term” proportional to \( a \ln q \). From the mathematical point of view it is an operator unbounded from below. With the rest of the Hamiltonian bounded from below, this shifts the spectrum to arbitrary large negative values of \( E \). In the \( \ln q \) space it is a linear potential which cannot be considered as small perturbation irrespective of the magnitude of the coupling constant which it accompanies. The solution (19) is accordingly non-perturbative, the coupling constant appearing in the denominator of the exponent. There seems to be no simple one-to-one correspondence between the perturbative (i.e. 1st order) and non-perturbative (i.e. 2nd order) wave functions. A detailed discussion of the connection between perturbative and non-perturbative solutions will be given in the next Section.
4 Transition to the $q$ space. The high-energy limit

In the $q$ space the found eigenfunctions are given by Eq. (11). We obtain

$$\Psi_E(q) = \frac{C}{\pi q \sqrt{2}} \int \frac{d\nu}{\epsilon_\nu} \exp \left( i\nu \ln q^2 - i\nu \frac{1}{\alpha_s b} + \frac{iEz(\nu)}{\alpha_s b} \right),$$

(23)

with $z(\nu)$ defined by (21). Since $\alpha_s$ is supposed to be small we can study the integral in $\nu$ by the saddle point method. The same method will also provide us with the asymptotics of (23) at very small and very large $q$. The saddle point is determined by an equation

$$Ez' - 1 + \alpha_s b \ln q^2 = 0,$$

which determines $\nu$ for a given eigenvalue $E$. It follows

$$E = \epsilon_\nu(1 - \alpha_s b \ln q^2).$$

(24)

Putting this into (23) one gets a crude asymptotic estimate

$$\Psi_E(q) \sim \frac{1}{q\epsilon_\nu} \exp \left\{ -i \left( \frac{1}{\alpha_s b} - \ln q^2 \right) [\nu - \epsilon_\nu z(\nu)] \right\},$$

(25)

where $\nu$ should be determined from (24). Eq. (24) is nothing but the usual expression for the pomeron energy $E$ as a function of $\nu$ with a running coupling constant to the 2nd order $\alpha_s(q^2) = \alpha_s(1 - \alpha_s b \ln q^2)$. However now we have to consider it as an equation for $\nu$ at a given $E$.

Let us discuss the behavior in $q$ of the solutions, for a fixed value of $E$ (see Fig. 1). In all the arguments presented below, $\alpha_s$ is small enough (smaller than $\sim 0.05$, in order to avoid the complications found in [3]) for the minimal value of $\epsilon_\nu$, $\epsilon_{min}$, to be\footnote{$\epsilon_{min} < 0$; if we neglect the second order correction in $\epsilon_\nu$, $\epsilon_{min} = -(\alpha_s N_c/\pi) 4 \ln 2$.} at $\nu = 0$.

First we take $E > 0$. For $\ln q^2 < 1/(\alpha_s b)$ (or $\alpha_s(q^2) > 0$), (24) has a pair of solutions for real $\nu$ (Fig. 2), starting from $\pm \nu_0$ at $\ln q^2 = -\infty$ and going to $\nu = \pm \infty$ when $\ln q^2$ approaches $1/(\alpha_s b)$. According to (23), for such $q$ the eigenfunctions $\Psi_E(q)$ will essentially have a similar behavior in $q$ as the unperturbed eigenfunctions $\psi_\nu(q)$, with the correspondence between $E$ and $\nu$ established by means of Eq. (24): they are plane waves in the variable $\ln q^2$, apart from the common factor $1/q$ and a certain distortion due to the third term in the exponent of (23) (clearly visible in the form (23)). Thus we find an one-to-one correspondence between the 1st order perturbative
and the 2nd order non-perturbative eigenfunctions for this part of the \( \ln q^2 \) space; we call these oscillatory pieces of the solutions “normal”. With \( \alpha_s b \) small this behavior remains valid up to very large values of \( q \) limited by the restriction \( \ln q^2 < 1/(\alpha_s b) \).

What happens if \( q \) becomes larger, so as \( 1/(\alpha_s b) < \ln q^2 < (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \)? One can see that the picture changes and Eq. (24) will now give imaginary saddle points (Fig. 2), starting from \( \nu = \pm i/2 \) for \( \ln q^2 = 1/(\alpha_s b) \) and approaching \( \nu = 0 \) as \( \ln q^2 \) comes near \( (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \), so that the wave function becomes damped in \( q \) by some power factors (as for the case \( E < 0 \), Eq. (27) but with an opposite sign in the exponent). This piece of the eigenfunction which is governed by imaginary \( \nu \) values will be called “abnormal”.

Now we take \( \ln q^2 > (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \). Eq. (24) has now solutions for real \( \nu \), starting from \( \nu = 0 \) and going to \( \pm \nu_0 \) for \( \ln q^2 \to -\infty \). It is easy to see the limiting asymptotics when \( \ln q^2 \to +\infty \). Taking into account the denominator \( \epsilon_\nu \) in (23) one then finds an oscillating behavior

\[
\Psi_E(q) \sim \frac{1}{q} \exp \left( i\nu_0 \ln q^2 - i \frac{E}{c\alpha_s b} \ln \ln q^2 \right),
\]

where \( c = \epsilon_{\nu_0} \). This behavior is valid both for very small and very large values of \( q \) (and also for any sign of \( E \), see below).

We consider now \( E < 0 \) fixed in (24). Again we find three pieces in the solutions, corresponding to different values of \( \ln q^2 \). For \( \ln q^2 < (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \), (24) will have solutions for real \( \nu \), starting from \( \pm \nu_0 \) at \( \ln q^2 = -\infty \) and going to \( \pm \infty \) for \( \ln q^2 \to (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \). This piece is equivalent to the last piece discussed in the case \( E > 0 \).

For \( (1 - E/\epsilon_{\text{min}})/(\alpha_s b) < \ln q^2 < 1/(\alpha_s b) \), the corresponding values of the saddle point \( \nu \) will have a non-zero imaginary part. If we neglect the second order correction to \( \epsilon_\nu \), Eq. (24) will give a pair of pure imaginary points \( \pm i|\nu| \), with \( |\nu| \) going from 0 at \( \ln q^2 = (1 - E/\epsilon_{\text{min}})/(\alpha_s b) \) to 1/2 for \( \ln q^2 = 1/(\alpha_s b) \). To get rid of the singularities along the real axis it is convenient to pass to the integration variable \( z \) in (23). Then, for \( \ln q^2 \) close to \( 1/(\alpha_s b) \), in the complex \( z \) plane the integrand will have singularities at points where \( 1/\epsilon_\nu \) vanishes, that is, at poles of \( \epsilon_\nu \). They occur at pure imaginary \( \nu = \pm i/2, \pm 3i/2, \ldots \). Function \( z \) at these points will also take pure imaginary values \( \pm i|z^{(k)}|, k = 1, 2, \ldots \), with \( |z^{(1)}| < |z^{(2)}| < \ldots \). In the strip \( |\text{Im } z| < |z^{(1)}| \) the integrand

\footnote{Actually a sum of contributions from the two points \( \nu_0 \), which differ only in sign if we neglect the 2nd order correction in \( \epsilon_\nu \), should be taken.}
will be analytic, so that the integration contour in $z$ can be freely shifted up and down parallel to the real axis. The solutions of Eq. (24) for $E < 0$ stay inside this strip, tending to its boundaries as $E \to -\infty$ and correspondingly $\nu$ tends to $\pm i/2$. So one can always shift the integration contour to pass through the saddle point. Note that if one takes into account the 2nd order correction to $\epsilon_{\nu}$ then the first order poles at $\nu = \pm i/2$ are changed to third order poles at the same point. As a consequence, the saddle points will acquire a real part and tend to $\pm i/2$ at certain angles when $E \to -\infty$, which will however not influence the final result. If we take $\alpha_s$ very small, with $q$ in this range, the two saddle points approach their limiting values $\pm i/2$. The product $\epsilon_{\nu} z(\nu)$ will be also pure imaginary with the same sign and its modulus greater than $|\nu|$, since $|\epsilon_{\nu}|$ grows towards $\nu = \pm i/2$. Then from (25) it follows that we should shift the integration contour down to pass through the saddle point with a negative imaginary part. As a result we shall obtain

$$\Psi_E(q) \sim \frac{1}{q \epsilon_{\nu}} \exp \left\{ - \left( \frac{1}{\alpha_s b} - \ln q^2 \right) |\nu - \epsilon_{\nu} z(\nu)| \right\},$$

(27)

which shows that at finite $q$ the abnormal pieces are damped by the non-perturbative damping factor $\exp(-\text{const}/\alpha_s)$. This piece will correspond to the second piece discussed for $E > 0$ (but, as mentioned above, with the opposite sign in the exponent: for $E < 0 \ln q^2$ approaches $1/(\alpha_s b)$ from below and for $E > 0$ from above, see Fig. 1).

If we are now interested in the region $\ln q^2 > 1/(\alpha_s b)$, then the saddle point starts from $\pm \infty$ at $\ln q^2 = 1/(\alpha_s b)$ and tends, for $\ln q^2 \to +\infty$, to the point $\pm \nu_0$ at which $\epsilon_{\nu} = 0$. As a result one obtains the same oscillating asymptotic behavior as for the first and third pieces in $E > 0$.

Excluding the region $\ln q^2 > 1/(\alpha_s b)$, which for small values of $\alpha_s$ covers only exceptionally large values of $q$, we can summarize the situation saying that normal solutions (pieces) have an oscillating behavior and abnormal ones are power damped both for small and large values of $q$ and also contain a non-perturbative damping factor $\exp(-\text{const}/\alpha_s)$. Abnormal solutions have no correspondence with the perturbative ones. Since there exist normalizable solutions for arbitrary large negative values of energy, that is, for arbitrary large positive values of the angular momentum $j$, they will lead to contributions to the amplitude which grow infinitely fast at high energies.

In fact even the passage to the energy representation results impossible due to their existence. The Green function of the total Hamiltonian as a function of the angular
momentum $j$ is given by the spectral representation

$$G(j, q_1, q) = \int dE \frac{\Psi_E(q_1) \overline{\Psi}_E(q)}{j - 1 + E},$$  \hspace{1cm} (28)

where the integration runs over all the spectrum. The Green function as a function of rapidity $Y = \ln s$ is obtained by integrating (28) with a weight $\exp [Y(j - 1)]$,

$$G(Y, q_1, q) = \int dE \exp(-YE) \Psi_E(q_1) \overline{\Psi}_E(q).$$  \hspace{1cm} (29)

Evidently this integral is ill-defined for $Y > 0$ if the integration goes from arbitrary large negative values of $E$, as in our case.

However, throwing out non-perturbative contributions which behave $\propto \exp(-1/\alpha_s)$, one obtains an apparently reasonable asymptotics, which coincides with the one found in [4]. Let us cutoff the integral over $E$ in (29) by some negative lower limit $E_0 < \epsilon_{\text{min}}$, where $\epsilon_{\text{min}}$ is the discussed minimal value of energy $E$ for which Eq. (24) gives real solutions for $\nu$:

$$G(Y, q_1, q) = \int_{E_0}^{\infty} dE \exp(-YE) \Psi_E(q_1) \overline{\Psi}_E(q).$$  \hspace{1cm} (30)

Putting our solutions we get then

$$G(Y, q_1, q) = \frac{|C|^2}{2\pi^2 q q_1} \int_{E_0}^{\infty} dE \exp(-YE) \int \frac{d\nu d\nu_1}{\epsilon_{\nu_1}} \exp[i\nu_1 \ln q_1^2 - i\nu \ln q^2 - i\beta(\nu_1 - \nu) + i\beta E(z_1 - z)],$$  \hspace{1cm} (31)

where we denoted $\beta = 1/(\alpha_s b)$ for brevity. We integrate over $E$ to obtain

$$G(Y, q_1, q) = \frac{|C|^2}{2\pi^2 q q_1} \exp(-YE_0) \int \frac{d\nu d\nu_1}{\epsilon_{\nu_1}} \frac{1}{[Y - i\beta(z_1 - z)]^{-1}} \exp[i\nu_1 ln q_1^2 - i\nu ln q^2 - i\beta(\nu_1 - \nu) + i\beta E_0(z_1 - z)].$$  \hspace{1cm} (32)

Now we pass to the variable $z_1$. The integrand in (32) has an explicit pole in it, and also singularities along the imaginary axis due to the singularities of $\nu_1(z_1)$. The latter are situated at finite distance from the real axis on both its sides. The explicit pole, on the contrary, is quite close to the real axis, due to smallness of $\alpha_s$. It lies slightly below the real axis in the $z_1$ plane. With $E_0 < 0$ we can close the contour around the singularities in the lower $z_1$ semiplane. The contribution from the cut along the negative semiaxis will contain a damping factor $\exp(-1/\alpha_s)$. We shall neglect it, which is certainly true in the limit $\alpha_s \to 0$ and amounts to throwing out all anomalous solutions. Then we are left with only the residue at

$$Y - i\beta(z_1 - z) = 0.$$  \hspace{1cm} (33)
In this approximation the dependence on $E_0$ disappears and we get an integral over $\nu$:

$$G(Y, q_1, q) = \frac{|C|^2}{\pi qq_1 \beta} \int d\nu \exp[i\nu_1 \ln q_1^2 - i\nu \ln q^2 - i\beta(\nu_1 - \nu)],$$

(34)

in which $\nu_1$ should be determined as a function of $\nu$ from Eq. (33). Solution of this equation can be accomplished by perturbation theory, recalling that $\beta = 1/(\alpha_s b)$ and is large. Then we obtain in the first three orders

$$\Delta \nu = \nu_1 - \nu = -\frac{iy}{z'} + \frac{y^2 z''}{2(z')^3} - \frac{iy^3}{6(z')^5}[z' z''' - 3(z'')^2],$$

(35)

where we denoted $y = \alpha_s b Y$. To further simplify we take into account that the saddle point $\nu$ in the integration in (34) is small, of order $1/Y$. This allows to use an approximation

$$\epsilon_\nu = \epsilon_0 + i\delta \nu + \pi \nu^2,$$

(36)

with

$$\epsilon_0 = -\frac{N_c \alpha_s}{\pi} 4 \ln 2 \left[ 1 - \frac{N_c \alpha_s}{4\pi} \left( 25.8387 + 0.1869 \frac{N_f}{N_c} + 3.8442 \frac{N_f}{N_c^2} \right) \right],$$

$$\delta = \left( \frac{N_c \alpha_s}{\pi} \right)^2 \left( 15.4262 - 2.8048 \frac{N_f}{N_c} \right),$$

$$\pi = a - \left( \frac{N_c \alpha_s}{\pi} \right)^2 \left( 322.188 - 3.10189 \frac{N_f}{N_c} + 21.6732 \frac{N_f}{N_c^2} \right),$$

$a = \frac{N_c \alpha_s}{\pi} 14\zeta(3)$

($\zeta(z)$ being Riemann’s zeta function), and to restrict ourselves to terms up to the second order in $\nu$ in the exponent in (34). The exponent then becomes a polynomial

$$P(\nu) = p_0 - ip_1 \nu - p_2 \nu^2,$$

(37)

where from (35) and (36) we find (up to terms $\propto \alpha_s^{n+2} Y^n$)

$$p_0 = -Y \epsilon_\nu (1 - \alpha_s b \ln q_1^2) + \frac{1}{3} (\alpha_s b \epsilon_0)^2 a Y^3 - \frac{1}{2} \delta \alpha_s b \epsilon_0 Y^2,$$

$$p_1 = \ln \frac{q^2}{q_1^2} - \alpha_s b a \epsilon_0 Y^2 + \delta Y,$$

$$p_2 = a Y.$$

(38)

Integration over $\nu$ gives the desired asymptotics at large $Y$:

$$G(Y, q_1, q) = \frac{|C|^2}{\pi qq_1 \beta} \sqrt{\pi} \frac{p_2}{4p_2} \exp \left( p_0 - \frac{p_1^2}{4p_2} \right).$$

(39)

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3The values of $\alpha_s$ considered here are those small enough to keep $\pi > 0$, i.e. $\alpha_s$ smaller than $\sim 0.05$. The case $\pi < 0$ has been examined in [3, 5] and leads to an oscillatory behavior of the Green function.
Putting expressions (38) into (39), retaining only terms up to $Y^3$ in the exponent and neglecting terms of order $(\delta/a)$ we obtain

$$G(Y, q_1, q) = \frac{|C|^2}{\pi q q_1 \beta} \sqrt{\frac{\pi}{aY}} \exp \left( -Y \epsilon_0 \left[ 1 - \frac{\alpha_s b}{2} (\ln q^2 + \ln q_1^2) \right] - \frac{\ln^2(q^2/q_1^2)}{4aY} + \frac{1}{12} (\alpha_s b \epsilon_0)^2 aY^3 \right).$$

This expression coincides with the asymptotics found in [4], if we take into account that our $a = 4D$ in [4].

5 Running coupling in all orders

An amusing application of the described formalism is a possibility to write an infrared stable equation for the case when the interaction is taken only to the first order in the coupling, but the coupling is taken running in all orders. In other words

$$H \psi \nu = \epsilon^{(1)} \nu(q) \psi \nu(q),$$

where

$$\epsilon^{(1)} \nu(q) = \frac{N_c}{\pi b \ln \frac{x}{\Lambda} \epsilon^{(1)}}$$

and $\epsilon^{(1)} \nu$ is defined by (5). In the following we put $\Lambda = 1$.

Of course, we understand that (12) is quite unsatisfactory from the physical point of view: it implies that the coupling constant is prolonged as a pure imaginary quantity into the confinement region $q < 1$. So the following serves only as an illustration of the power of the employed technique and also as a warning against a simple-minded use of the saddle point approximations. Note that in [4], where, as mentioned in the Introduction, a similar problem is solved by a similar technique, a different choice was made:

$$\ln q^2 \to r(q),$$

where $r(q)$ was to be determined from a transcendental equation

$$r = \ln q^2 + \frac{1}{2} \ln(br).$$

However for $q^2 < \sqrt{2e/b}$ this equation leads to complex values of $r$ and consequently for $\alpha_s(q^2) = 1/(br)$, which seems to be even worse than our choice.
In the \( \nu \) representation we obtain the Hamiltonian as

\[
H(\nu, \nu') = \frac{N_c}{2\pi b} \epsilon_{\nu'}^{(1)} \text{sign}(\nu' - \nu). \tag{43}
\]

The eigenvalue equation becomes

\[
\frac{N_c}{2\pi b} i \int d\nu' \epsilon_{\nu'}^{(1)} \text{sign}(\nu' - \nu) f(\nu') = E f(\nu). \tag{44}
\]

Differentiating with respect to \( \nu \) we obtain

\[
-\frac{N_c}{\pi b} i \epsilon_{\nu'}^{(1)} f(\nu) = E f'(\nu), \tag{45}
\]

with a solution

\[
f(\nu) = \frac{C}{E} \exp \left( -i \frac{N_c}{\pi b E} \int_0^\nu d\nu' \epsilon_{\nu'}^{(1)} \right). \tag{46}
\]

This seems to be a valid solution for any real \( E \), positive or negative. The conjugate equation is

\[
-\frac{N_c}{\pi b} i \bar{f}(\nu) = E \left( \bar{f}(\nu) / \epsilon_{\nu'}^{(1)} \right)', \tag{47}
\]

with solutions similar to (46) but with an extra factor \( \epsilon_{\nu'}^{(1)} \). They also exists for any \( E \).

Requiring \( \langle f_E \mid f_{E'} \rangle = \delta(E - E') \), we get \( |C| = (2\pi^2 b/N_c)^{-1/2} \). Solutions (46) and its conjugate can be readily expressed in terms of the \( \Gamma \)-function. Indeed

\[
z(\nu) = \int_0^\nu d\nu' \epsilon_{\nu'}^{(1)} = -i \ln \frac{\Gamma(1/2 + i\nu)}{\Gamma(1/2 - i\nu)} - 2\nu \psi(1). \tag{48}
\]

In the momentum representation the eigenfunctions have the form

\[
\Psi_E(q) = \frac{C}{\pi E q^{1/2}} \int d\nu \exp \left( i\nu \ln q^2 - i \frac{N_c}{2\pi b E} z \right). \tag{49}
\]

To find the asymptotics at large \( |\ln q^2| \) we employ the saddle point approximation. The saddle point is determined by a simple equation

\[
\ln q^2 = \frac{N_c}{\pi b E} \epsilon_{\nu'}^{(1)}
\]

or

\[
E = \frac{N_c}{\pi b \ln q^2} \epsilon_{\nu'}^{(1)}, \tag{50}
\]

where we used the definition (12). This equation is again nothing but the relation between the energy and \( \nu \), expected from (11). It has to be considered as an equation for the saddle point \( \nu \) for a given \( E \). We will now analyze, as in the previous Section, the behavior of the solutions in momentum space, for fixed \( E \) (having in mind that the saddle point will only give a reliable solution for very large \( |\ln q^2| \)).
Let us take $E > 0$. Then, for $\ln q^2 < N_c \epsilon^{(1)}_{\text{min}}/(\pi bE) = -4N_c \ln 2/(\pi bE)$, (50) has solutions for imaginary $\nu$, going from $\nu = \pm i/2$ at $\ln q^2 = -\infty$ to $\nu = 0$ at $\ln q^2 = N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$. As a result, for (49) we obtain a falling asymptotics (see below the case $E < 0$); this would correspond, in the language of the previous Section, to an abnormal piece.

Now, if $\ln q^2 > N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$, (50) has solutions for real $\nu$, going from $\nu = 0$ at $\ln q^2 = N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$ to $\nu = \pm \infty$ for $\ln q^2 = +\infty$. This will result in an oscillating asymptotics for $\Psi_E(q)$ (the normal piece), apart from the dimensionful factor $1/q$.

We consider now the case $E < 0$. For $\ln q^2 < N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$, the saddle point equation (50) will give real solutions, going from $\nu = \pm \infty$ for $\ln q^2 = -\infty$ to $\nu = 0$ at $\ln q^2 = N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$. The solution (49) will be, as for the second piece in the case $E > 0$, a normal oscillating piece.

For $\ln q^2 > N_c \epsilon^{(1)}_{\text{min}}/(\pi bE)$, (50) will have a pair of conjugate purely imaginary solutions for $\nu$. At large $\ln q^2$ they approach points $\pm i/2$. In the vicinity of, say, $\nu = i/2$ we have

$$\epsilon^{(1)}_\nu \simeq -(1/2 + i\nu)^{-1},$$

so that from (50) we find

$$\frac{1}{2} + i\nu = -\frac{N_c}{\pi bE \ln q^2}.$$  

Function $z$ has a logarithmic singularity at this point so that it will be approximately given by

$$z \simeq i \ln \left| \frac{1}{2} + i\nu \right| = i \ln \frac{N_c}{\pi b|E| \ln q^2},$$  

where we have taken into account that in the considered region $E < 0$. It follows that at this saddle point the exponent in (49) becomes

$$i\nu \ln q^2 - i\frac{N_c}{\pi bE} z \simeq -\ln q^2 \left[ \frac{1}{2} - \frac{N_c}{\pi b|E| \ln q^2} \left( 1 - \ln \frac{N_c}{\pi b|E| \ln q^2} \right) \right].$$

At the complex conjugate saddle point, in the vicinity of $\nu = -(1/2)i$, the exponent will have an opposite sign. In the complex $\nu$ plane the integrand in (49) has singularities along the imaginary axis for $|\text{Im } \nu| > 1/2$. In the strip $|\text{Im } \nu| < 1/2$ it is analytic, so that the integration contour can be freely shifted above or below the real axis in this interval. Depending on the sign of

$$\frac{1}{2} - \frac{N_c}{\pi b|E| \ln q^2} \left( 1 - \ln \frac{N_c}{\pi b|E| \ln q^2} \right),$$
one can always shift the contour to pass over one of the two conjugate saddle points so as to have a negative coefficient before $\ln q^2$. As a result we obtain a falling asymptotic (abnormal) behavior in this $\ln q^2$ range (corresponding to the first piece in the case $E > 0$):

$$\Psi_E(q) \propto \frac{1}{q} q^{-\left|1 - \frac{2N_c}{\pi b|E| \ln q^2}\right| \left(1 - \ln \frac{N_c}{\pi b|E| \ln q^2}\right)}.$$  \hspace{1cm} (53)

Summarizing, both for $E > 0$ or $E < 0$ we find that the solution has two pieces, normal and abnormal, depending on the $\ln q^2$ range we are studying. The eigenfunction (49) is normalizable for any real value of $E$. Restricting ourselves to the case $\ln q^2 > N_c\epsilon_{\text{min}}/(\pi bE)$ (corresponding to $\ln q^2 > 0$ for large negative $E$), we find a normal oscillating solution for $E > 0$ and an abnormal falling solution for $E < 0$.

With the spectrum of the Hamiltonian (11) extending from $-\infty$ to $\infty$, we again have grave problems in passing from the angular momentum to rapidities. From the start the integral (29), which determines the Green function as a function of rapidity, is badly divergent at large negative $E$. It is amusing that nevertheless one obtains a seemingly reasonable asymptotics at high $Y$ and momenta if one forgets about the initial divergence and makes some crude approximations. Indeed, with the eigenfunctions given by (49), we have

$$G(Y, q_1, q) = \frac{|C|^2}{2\pi^2 q q_1} \int \frac{dE}{E^2} \exp(-YE) \int d\nu d\nu_1 \epsilon^{(1)}_{\nu} \exp \left( i\nu \ln q_1^2 - i\nu \ln q^2 - i\frac{N_c}{\pi bE}(z_1 - z) \right),$$  \hspace{1cm} (54)

where $z_1 = z(\nu_1)$. The integral (54) does not exist. However let us forget this and calculate its asymptotics for large $\ln q^2$ and $\ln q_1^2$. Then small values of $\nu$ and $\nu_1$ and consequently of $z$ and $z_1$ evidently give the dominant contribution. At small $\nu$

$$z = \epsilon_0^{(1)} \nu,$$

so that we obtain

$$G(Y, q_1, q) = \frac{2|C|^2\epsilon_0^{(1)}}{q q_1} \int \frac{dE}{E^2} \exp(-YE) \delta \left( \ln q^2 - \frac{N_c\epsilon_0^{(1)}}{\pi bE} \right) \delta \left( \ln q_1^2 - \frac{N_c\epsilon_0^{(1)}}{\pi bE} \right).$$

This integral exists, since the $\delta$-functions ensure that the integrand is zero nearly for all $E$ and for large negative $E$ in particular. We obtain in this manner

$$G(Y, q_1, q) = \frac{|C|^2\pi b}{N_c} \frac{\delta(q - q_1)}{q} \exp \left( -Y \frac{N_c\epsilon^{(1)}}{\pi b \ln q^2} \right).$$  \hspace{1cm} (55)
The behavior in $Y$ is just what one would expect from naive expectations of having a running coupling in the intercept.

Of course, this exercise has very little meaning. It only shows that using poorly controlled approximations one can arrive at seemingly sensible results even though the exact expression has no meaning at all. In our case it shows that the gluon interaction should have a different functional dependence to cure its bad qualities, which arise when one simply changes a fixed coupling by a running one. In mathematical terms the interaction then results not bounded from below, which shifts the spectrum down to $-\infty$.

6 Conclusions

Different aspects of the 2nd order corrections to the BFKL pomeron [1, 2] have been studied recently [3, 4, 5, 6] using different techniques. Some of the resulting features, as the existence of a “non-Regge” term $\propto \exp (Y^3)$ [4, 5] or of oscillatory solutions [3, 6] for $\alpha_s$ greater than $\sim 0.05$, together with the diffusion into the infrared region [4, 7], impose severe limits on the possible applicability of these results to the experimental situation.

In this note we find the solutions to the 2nd order BFKL Hamiltonian and analyze the resulting spectrum in the energy plane ($E = 1 - j = -\omega$). It turns out not to be bounded from below, and we show that for any $-\infty < E < +\infty$ there exists a normalizable eigenfunction $\Psi_E(q)$. This solution exhibits non-perturbative features, with the coupling constant appearing in the denominator of the exponent, Eq. (23). Both for the cases $E > 0$ and $E < 0$, solutions to the eigenvalue equation have three different pieces in momentum space, depending on the relation of the value of $\ln q^2$ with $\alpha_s$ and $E$. The different pieces can be classified as being “normal”, with an oscillating behavior as in the 1st order case, and “abnormal”, which are non-oscillatory in $q$ and have no equivalence to the 1st order solutions. For very small $\alpha_s$, the case $E > 0$ is almost entirely normal, whereas for $E < 0$ the eigenfunction is almost everywhere abnormal. From our study we conclude that, in the presence of the 2nd order corrections, the unbounded spectrum of the kernel makes the use of the complex angular momentum formalism to study the high-energy behavior somewhat doubtful.

A calculation of the high energy behavior of the 2nd order BFKL Green function has been presented in [4]. In this paper, Mellin transforms were applied only to the
leading-log BFKL Green function, and the problem of the unbounded spectrum of the
second order corrections was avoided. So the iteration techniques used in [4] seem to
offer a possibility to compute the high energy behavior of the 2nd order BFKL Green
function bypassing the difficulties of the unbounded spectrum found in our paper. We
have shown that with a modification of the Mellin transform the Green function can be
defined by simply excluding the abnormal pieces of the eigenfunctions; this definition
leads to a high energy behavior which agrees with [4].

In [5] also the high-energy behavior of the 2nd order BFKL Green function is
investigated. The result is consistent with Ref. [4]. Although the methods used in [5]
are similar to parts of our study, no particular attention has been paid to the energy
spectrum of the 2nd order BFKL kernel which constitutes the main purpose of our
paper.

So far we have concentrated on the spectrum of the 2nd order BFKL kernel, as
it was presented in [1]. In the context of the leading-order BFKL equation [8, 9] it has already been emphasized that the physical spectrum of the Pomeron depends
upon the (nonperturbative) infrared behaviour, i.e. upon the (unknown) extrapolation
of the perturbative BFKL kernel into the small-\(q\) region. We have not yet adressed
this question, but we feel that the arguments which in [8, 9] have been given for the
leading-order BFKL equation should now be reconsidered in presence of the 2nd order
corrections which lead to this dramatic change of the energy spectrum.

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Figure 1: Behavior in $q$ of the solutions (23) for fixed $E > 0$ and $E < 0$. Real $\nu$ correspond to an oscillatory (normal) piece, while imaginary $\nu$ correspond to an exponentially damped (abnormal) one.
Figure 2: Location of zeros \((\nu = \pm \nu_0)\) and poles \((\nu = \pm i/2)\) of \(\epsilon_{\nu}\) in the complex \(\nu\) plane.