Multi-distributed entanglement in finitely correlated chains

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Abstract. – The entanglement-sharing properties of an infinite spin-chain are studied when the state of the chain is a pure, translation-invariant state with a matrix-product structure [1]. We study the entanglement properties of such states by means of their finitely correlated structure [2]. These states are recursively constructed by means of an auxiliary density matrix $\rho$ on a matrix algebra $\mathcal{B}$ and a completely positive map $E : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}$, where $\mathcal{A}$ is the spin $2 \times 2$ matrix algebra. General structural results for the infinite chain are therefore obtained by explicit calculations in (finite) matrix algebras. In particular, we study not only the entanglement shared by nearest-neighbours, but also, differently from previous works [3], the entanglement shared between connected regions of the spin-chain. This range of possible applications is illustrated and the maximal concurrence $C = \frac{1}{\sqrt{2}}$ [4] for the entanglement of connected regions can actually be reached.

The recent developments in quantum information and computation theory are witnessing an increasing interest in the entanglement properties of multi-qubit systems like quantum spin-chains, correlated electrons and interacting bosons. These systems had so far been studied in condensed matter physics especially in relations to their critical behaviour in phase-transitions [5]. With its emergence as a precious resource for quantum computation and communication tasks, entanglement is now investigated in condensed matter physics, too; on one hand to detect, extract and manipulate it, on the other hand to understand its role in solid state phenomena [6–13].

The entanglement properties of the class of states of quantum spin chains known as Matrix-Product States (MPS) [1], have been proved powerful tools in density matrix renormalization groups techniques [14] and in universal quantum computation to show the equivalence of teleportation-based and one-way quantum computation [15]. They can be also achieved by sequential generation [16]. Actually, these states were originally introduced in [2] as Finitely Correlated States (FCS) and their properties were there studied in great detail; in particular, it was showed that FCS can be implemented as ground states of Hamiltonians with short range interaction (see also [15]). This motivates the study of entanglement in FCS both from the point of view of the large variety of scaling laws provided and from the point of view of the possible physical applications.
Two main different approaches have emerged: one dealing with the scaling behaviour of entanglement-sharing between various subsets of localized spins [17, 18, 20, 21], the other one studying how much entanglement can be localized on two distant spins by measuring the others [19, 22]. In this Letter we choose the first approach and study the entanglement of FCS by means of their recursive structure [2], that is by studying the specific completely positive maps between finite dimensional algebras and finite dimensional density matrices on which it is based. This provides a rich phenomenology of total translation invariant states over infinite spin chains whose entanglement properties are nevertheless determined by, and can consequently be studied in a finite-dimensional setting. Thus, we are able to compare nearest neighbours with non-nearest neighbours entanglement or, more generally, to analytically study the entanglement between different subsets of lattice points. In particular, we shall give a general necessary condition for one spin being entangled with a subset of others and prove that in some cases this condition is also sufficient. Finally, we show that entanglement sharing can achieve its maximum in the sense of [4].

**Construction of translation invariant FCS:** We will denote by \( \mathcal{A}_Z \) an infinite spin-chain, the spins at sites \( i \in Z \) being described by the algebra \( (A)_i = M_2 \) of \( 2 \times 2 \) complex matrices. The infinite algebra \( \mathcal{A}_Z \) arises as a suitable limit of the local tensor-product algebras \( A_{[-n,n]} := \otimes_{j=-n}^n (A)_j \). Any state \( \omega \) over \( \mathcal{A}_Z \) is specified by density matrices \( \rho_{[1,n]} \) defining the action of \( \omega \) as an expectation over local operators \( A_{[1,n]} \in \mathcal{A}_{[1,n]} \):

\[
\omega(A_{[1,n]}) = \text{Tr}_{[1,n]}(\rho_{[1,n]} A_{[1,n]}) .
\]

(1)

The \( \rho_{[1,n]} \)'s must satisfy the compatibility conditions

\[
\text{Tr}_{n+1}(\rho_{[1,n+1]} A_{[1,n]} \otimes 1_{n+1}) = \text{Tr}(\rho_{[1,n]} A_{[1,n]}) ,
\]

(2)

whereas, translation-invariance requires

\[
\text{Tr}_{n+1}(\rho_{[1,n+1]} 1_1 \otimes A_{[2,n+1]}) = \text{Tr}(\rho_{[1,n]} A_{[1,n]}) .
\]

(3)

The class of translation-invariant FCS over \( \mathcal{A}_Z \) is defined by a triple \( (B, \rho, \mathcal{E}) \) where \( B \) is a \( b \times b \) matrix algebra, \( \rho \in B \) a density matrix and \( \mathcal{E} : A \otimes B \rightarrow B \) a completely positive unital map, which in Kraus-Stinespring form reads

\[
\mathcal{E}(A \otimes B) = \sum_j V_j (A \otimes B)V_j^\dagger , \quad V_j : \mathbb{C}^2 \otimes \mathbb{C}^b \rightarrow \mathbb{C}^b ,
\]

(4)

with \( A \in A \) and \( B \in B \). Unitality means that identities are preserved: \( \mathcal{E}(1_A \otimes 1_B) = 1_B \).

Let \( \mathcal{E}^{(1)}(A) := \mathcal{E}(A \otimes 1_B) \); this defines a completely positive map from \( A \) into \( B \). Analogously, the recursive compositions \( \mathcal{E}^{(n)} := \mathcal{E} \circ (\text{id}_A \otimes \mathcal{E}^{(n-1)}) \) are completely positive maps from \( A_{[1,n]} \) into \( B \). Setting

\[
\text{Tr}(\rho_{[1,n]} A_{[1,n]}) := \text{Tr}_B(\rho \mathcal{E}^{(n)}(A_{[1,n]})) ,
\]

(5)

the r.h.s. recursively defines local density matrices \( \rho_{[1,n]} \) over \( A_{[1,n]} \) and a total state \( \omega \) on \( \mathcal{A}_Z \) [2]. Further, \( \text{Tr}_B(\rho \mathcal{E}(A \otimes B)) = \text{Tr}_B(\rho \mathcal{B}) , \forall B \in B \) yields translation-invariance [2].

Concretely, we choose \( B = M_2 \) and \( \mathcal{E} \) in \( \mathcal{H} \) with just one Kraus operator \( V : \mathbb{C}^2 \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^2 \). This is such that

\[
V|\phi_j \otimes \psi\rangle = u_i |\psi\rangle , \quad V^\dagger |\psi\rangle = \sum_i |\phi_i \rangle \otimes v_i |\psi\rangle
\]

and

\[
\mathcal{E}
\]

(6)
where \(|\phi_{1,2}\rangle \in \mathbb{C}^2\) are orthonormal and \(v_{1,2}\) are \(2 \times 2\) matrices satisfying

\[
v_1 v_1^\dagger + v_2 v_2^\dagger = 1_B \quad \text{(unitality)}, \quad \sum_{j=1}^2 v_j^\dagger \rho v_j = \rho \quad \text{(translation invariance)}.
\] (7)

If there exists a unique \(\rho\) fulfilling the previous condition, the resulting translation–invariant FCS are pure states over \(A_2\) [2], namely they cannot be decomposed as mixtures of other states. These pure states can be interpreted as ground states for appropriate constructed Hamiltonians of finite range, but higher correlations [2].

Consider \(\mathbb{B}\) with \(n = 2\) and \(A_{[1,2]} = A_1 \otimes A_2\), then

\[
\Tr\left(\rho_{[1,2]} A_1 \otimes A_2\right) := \Tr_{\mathbb{B}}\left(\rho \mathbb{E}\left(A_1 \otimes \mathbb{E}\left(A_2 \otimes 1_B\right)\right)\right).\] (8)

Using the properties of the trace-operation, the action of \(\mathbb{E}\) becomes the action of its dual map \(F\) onto the state \(\rho \in \mathbb{B}: \Tr_{\mathbb{B}}(\rho \mathbb{E}(A \otimes B)) = \Tr_{A \otimes B}(F(\rho) A \otimes B)\). This provides a state \(\rho_{A \otimes B} := F(\rho) = V^\dagger \rho V\) on \(A \otimes B\):

\[
\rho_{A \otimes B} = \sum_{s,t=1}^2 |\phi_s\rangle \langle \phi_t|^\dagger \otimes v_s^\dagger \rho v_t = \begin{pmatrix} v_1^\dagger \rho v_1 & v_1^\dagger \rho v_2 \\ v_2^\dagger \rho v_1 & v_2^\dagger \rho v_2 \end{pmatrix}.\] (9)

The r.h.s. of \(\mathbb{B}\) reads \(\Tr_{A \otimes B}(\rho_{A \otimes B} A_1 \otimes \left(\mathbb{E}(A_2 \otimes 1_B)\right))\), by turning \(\text{id}_A \otimes \mathbb{E}\) into its dual, nearest-neighbours states arise as \(\rho_{12} := \rho_{[1,2]} = \Tr_{\mathbb{B}}(\text{id}_A \otimes F(\rho_{A \otimes B}))\) and read

\[
\rho_{12} = \sum_{i,j,l,m=1}^2 |\phi_i\rangle \langle \phi_j| \otimes \begin{pmatrix} R_{11|1} & R_{11|2} \\ R_{22|1} & R_{22|2} \end{pmatrix} = \begin{pmatrix} R_{1111} & R_{1112} & R_{1121} & R_{1122} \\ R_{2211} & R_{2212} & R_{2221} & R_{2222} \end{pmatrix},\] (10)

where \(R_{ijlm} = \Tr(v_i^\dagger v_j^\dagger \rho v_l v_m)\), while general local density matrices are given by

\[
\rho_{[1,n]} = \sum_{s,t} |\phi_s\rangle \langle \phi_t| \Tr(v_s^\dagger \rho v_t),\] (11)

where \(|\phi_s\rangle = |\phi_{s_1} \otimes \phi_{s_2} \otimes \cdots \phi_{s_n}\rangle, v_t := v_{t_1} \cdots v_{t_n}\).

We shall now relate the entanglement of \(\rho_{A \otimes B}\) to that of the spin at site 1 with those at sites in \([p,n]\), \(p > 1\). This means investigating the entanglement of the restricted state \(\omega(A_1) \otimes A_{[p,n]}\). By the previous construction, the restriction amounts to the expectations

\[
\omega\left(A_1 \otimes 1_{[2,p-1]} \otimes A_p \otimes \cdots A_n\right) = \Tr_{A \otimes B}(\rho_{A \otimes B} A_1 \otimes G(A_p \otimes \cdots A_n))\], (12)

where \(1_{[2,p-1]} = (12) \otimes \cdots (1p-1)\) and \(G\) is a completely positive map from \(A_{[p,n]}\) into \(\mathbb{B}\). Notice that while \(\rho_{12}\) is a nearest-neighbours state, \(\rho_{A \otimes B}\) encodes the entanglement of the spin at site 1 with any subset of spins \(A_{[p,n]}\), \(p > 1\), after being embedded into \(\mathcal{B} = \mathcal{A} = \mathcal{M}_2\) by \(G\). The advantage of the abstract structure presented above emerges in that the
total state over the chain is determined by the triple \((\mathcal{B}, E, \rho)\), so that properties like the maximal entanglement shared by nearest neighbours in a translation invariant chain, and, more generally, the entanglement between different subsets of lattice points, can be studied by means of that triple only.

**Distribution of the entanglement along the chain:** As a measure of the entanglement of a state \(\nu_{12}\) over the tensor product algebra \(\mathcal{N}_1 \otimes \mathcal{N}_2\), we shall use the entanglement of formation [25]

\[
E_{\mathcal{N}_1 \otimes \mathcal{N}_2}(\nu_{12}) = \inf_{\nu_{12} = \sum_j \lambda_j \nu_{12}^j} \left\{ \sum_j \lambda_j S\left(\nu_{12}^j | \mathcal{N}_1\right) \right\},
\]

where \(\nu_{12} = \sum_j \lambda_j \nu_{12}^j\) is a convex decomposition into pure states and \(S\left(\nu_{12}^j | \mathcal{N}_1\right)\) the von Neumann entropy of their restrictions to the subalgebra \(\mathcal{N}_1\), namely of the state obtained by tracing over the second factor, the density matrix representing the expectation functional \(\nu_{12}\).

It turns out that

\[
E_{(\mathcal{A}_1 \otimes \mathcal{A})_p}(\omega) \leq E_{(\mathcal{A}_1 \otimes \mathcal{A}[p,n])}(\omega) \leq E_{\mathcal{A} \otimes \mathcal{B}}(\rho_{\mathcal{A} \otimes \mathcal{B}}).
\]

The proof of this fact follows from a slight generalization of an argument in [24]. Let \(\mathcal{N}_3\) be another algebra, \(\nu_{13}\) a state over \(\mathcal{N}_1 \otimes \mathcal{N}_3\) and \(\Gamma : \mathcal{N}_2 \rightarrow \mathcal{N}_3\) a unital \((\Gamma (1_{\mathcal{N}_2}) = 1_{\mathcal{N}_3})\) completely positive map. Then, \(\nu_{12} = \nu_{13} \circ (\text{id}_{\mathcal{N}_1} \otimes \Gamma)\) is a state over \(\mathcal{N}_1 \otimes \mathcal{N}_2\), where \(\circ\) denotes the composition of the expectation functional \(\nu_{13}\) with the completely positive unital map \(\text{id}_{\mathcal{N}_1} \otimes \Gamma\). As the decompositions of \(\nu_{12}\) induced by those of \(\nu_{13}\) are not all possible ones, it follows that \(E_{\mathcal{N}_1 \otimes \mathcal{N}_2}(\nu_{12})\) is bounded from above by

\[
\inf_{\nu_{13} = \sum_j \lambda_j \nu_{13}^j} \left\{ \sum_j \lambda_j S\left(\nu_{13}^j \circ (\text{id}_{\mathcal{N}_1} \otimes \Gamma) | \mathcal{N}_1\right) \right\}.
\]

From the unitality of \(\Gamma\), the states \(\nu_{13}^j \circ (\text{id}_{\mathcal{N}_1} \otimes \Gamma)\) restricted to \(\mathcal{N}_1\), that is evaluating mean values of operators of the form \(\mathcal{N}_1 \otimes 1_{\mathcal{N}_2}\), coincide with the restrictions of \(\nu_{13}^j\) themselves. Thus, \(E_{\mathcal{N}_1 \otimes \mathcal{N}_2}(\nu_{12}) \leq E_{\mathcal{N}_1 \otimes \mathcal{N}_2}(\nu_{13})\). The first inequality in the hierarchy follows by taking \(\mathcal{N}_1 = (\mathcal{A}_1), \mathcal{N}_2 = (\mathcal{A}_p), \mathcal{N}_3 = \mathcal{A}[p,n]\) and as \(\Gamma\) the natural embedding of \((\mathcal{A}_1)\) into \(\mathcal{A}[p,n]\). The second one, by choosing \(\mathcal{N}_3 = (\mathcal{A}_1), \mathcal{N}_2 = \mathcal{A}[p,n], \mathcal{N}_3 = \mathcal{B}\) and \(\Gamma = \mathcal{G} : \mathcal{A}[p,n] \rightarrow \mathcal{B}\); in fact, \(\omega (\mathcal{A}_1 \otimes \mathcal{A}[p,n]) = \rho_{\mathcal{A} \otimes \mathcal{B}} \circ \text{id}_{\mathcal{A} \otimes \mathcal{G}}\), see also Eq. (12). Thus, the general necessary condition follows that, for \((\mathcal{A}_1)\) to be entangled with \(\mathcal{A}[p,n], p > 1\), the state \(\rho_{\mathcal{A} \otimes \mathcal{B}}\) must be entangled over \(\mathcal{A} \otimes \mathcal{B}\). Translation-invariance makes this necessary condition independent of rigid shifts of the two algebras; also, though the next examples go in the right direction, unfortunately, there is so far no general argument for its sufficiency namely that entanglement between \(\mathcal{A}\) and \(\mathcal{B}\) should imply entanglement between \(\mathcal{A}\) and some \(\mathcal{A}[p,n]\).

**Examples:**Because \(\mathcal{A} = \mathcal{B} = \mathcal{M}_2\), we study the concurrence [26] of which the entanglement of formation is a monotonically increasing function. The concurrence of \(\rho\) is given by \(C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}\), where \(\lambda_j\) are the square roots of the eigenvalues in decreasing order of the matrix \(\rho \bar{\rho}\) where \(\bar{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)\) and \(\rho^*\) denotes complex conjugation in the standard basis. In terms of concurrence, inequality reads

\[
C\left(\omega | (\mathcal{A}_{1} \otimes \mathcal{A}_p)\right) \leq C\left(\omega | (\mathcal{A}_{1} \otimes \mathcal{A}[p,n])\right) \leq C\left(\rho_{\mathcal{A} \otimes \mathcal{B}}\right).
\]

As a first finitely correlated structure, let us choose \(v_1 = \begin{pmatrix} c & s \\ 0 & 0 \end{pmatrix}\), \(v_2 = \begin{pmatrix} 0 & s \\ s & c \end{pmatrix}\), with \(c = \cos \varphi\) and \(s = \sin \varphi\). Then the conditions are satisfied by \(\rho = 1/2 \begin{pmatrix} 1 & 2cs \\ 2cs & 1 \end{pmatrix}\) and,
using (11), the state of three adjacent qubits explicitly reads

\[ \rho_{[1,3]} = \frac{1}{2} \sum_{i,j=1}^{2} \Omega_{ij} \otimes |\phi_i\rangle \otimes |\phi_j\rangle, \]  
where

\[ \Omega_{11} = \begin{pmatrix} c^2 & 2c^2s^2 \\ 2c^2s^2 & s^2 \end{pmatrix}, \quad \Omega_{22} = \begin{pmatrix} s^2 & 2c^2s^2 \\ 2c^2s^2 & c^2 \end{pmatrix}, \quad \Omega_{12} = \Omega_{21}^\dagger = \begin{pmatrix} cs & 2c^3s \\ 2c^3s & cs \end{pmatrix}, \]  

whereas the state \( \rho_{AB} \) amounts to

\[ \rho_{AB} = \frac{1}{2} \begin{pmatrix} c^2 & cs & 2c^2s^2 & 2c^3s \\ cs & s^2 & 2c^2s^2 & 2c^3s \\ 2c^2s^2 & 2c^2s^2 & s^2 & cs \\ 2c^3s & 2s^2c^2 & cs & c^2 \end{pmatrix}. \]  

From (17), tracing over site 2 and 3, one gets the one-site qubit-states

\[ \rho_1 = \frac{1}{2} \begin{pmatrix} 1 & 4c^2s^2 \\ 4c^2s^2 & 1 \end{pmatrix}, \]  
while tracing over site 3 we get the following two-sites states,

\[ \rho_{12} = \frac{1}{2} \begin{pmatrix} c^2 & 2c^2s^2 & 2c^2s^2 & 4c^4s^2 \\ 2c^2s^2 & s^2 & 4c^2s^4 & 4c^2s^4 \\ 2c^2s^2 & 4c^2s^4 & s^2 & 2c^2s^2 \\ 4c^4s^2 & 2s^2c^2 & 2c^2s^2 & c^2 \end{pmatrix}. \]  
and tracing over site 2 gives

\[ \rho_{13} = \frac{1}{4} \Omega_{11} \otimes \Omega_{11} + \frac{1}{4} \Omega_{22} \otimes \Omega_{22}. \]  
The latter state is clearly separable, there is no entanglement between site 1 and site 3, i.e. no next nearest neighbour entanglement. While partially transposing \( \rho_{12} \), i.e. \( 4c^2s^4 \leftrightarrow 4c^4s^2 \) the positivity is surely lost for \( 0 \leq \varphi < \pi/4 \), thus we have entanglement between nearest neighbours. The same happens to the state of one site with the rest \( \rho_{AB} \), namely, if \( s^2 - 4c^6 < 0 \), that is for \( 0 \leq \varphi < \pi/4 \), this state is surely not positive under partial transposition.

A finer picture can be obtained by looking at the concurrences:

\[ C_{AB} := C(\rho_{AB}) = |\sin 2\varphi \cos(2\varphi)| \]  
vanishes at \( \varphi = \pi/4, 5\pi/4 \), while \( C_{12} := C(\rho_{12}) \), though explicitly computable, is not as easily readable. Their behaviours as functions of \( \alpha = \sin(2\varphi) \) are reported in Figure 1(a) which shows firstly that, in agreement with (16), \( C_{AB} \geq C_{12} \) and secondly that \( \rho_{AB} \) is entangled whenever \( \rho_{12} \) is entangled, namely for all \( \sin(2\varphi) \neq 0, \pm 1 \).

Another finitely correlated state is given by choosing

\[ v_1 = \begin{pmatrix} ac & as \\ -s & c \end{pmatrix}, \quad v_2 = \sqrt{1-a^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 0 \leq a \leq 1, \]  

**Fig. 1** – (a) \( C_{AB} \), green; \( C_{12} \), red; abscissa \( \alpha = \sin(2\varphi) \). (b) \( C_{AB} \), green; \( C_{12} \), red; abscissa \( \varphi \).
with \( s = \sin \varphi \) and \( c = \cos \varphi \). Conditions (11) give

\[
\begin{align*}
\rho &= \begin{pmatrix} x & y \\ y & 1-x \end{pmatrix}, \\
x &= \frac{(1-a)s^2 + 2s^2}{(a-1)c^2 + 2s^2}, \\
y &= \frac{(a-1)x}{(a-1)c^2 + 2s^2}.
\end{align*}
\] (22)

With this choice \( (v_2)^2 = 0 \) and all terms in (10) with two adjacent \( v_2 \) or \( v_1^\dagger \) vanish; therefore, independently of \( a, s \) in (22), nearest-neighbours states are of the form

\[
\rho_{12} = \begin{pmatrix}
1 - 2\gamma & \alpha & \alpha & 0 \\
\alpha & \gamma & \beta & 0 \\
\alpha & \beta & \gamma & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\] (23)

and have concurrence \( C_{12} = 2|\beta| \). The entanglement of formation is maximal when

\[
C_{12} = 2\text{Tr}(v_2^\dagger v_1^\dagger \rho v_2 v_1) = \frac{2(1 - a^2)s^2c^2}{(1-a)^2c^2 + 2s^2},
\] (24)

is maximal, that is when \( a = \sqrt{2} - 1, \cos(2\phi_1) = \sqrt{2} - 1 \) and \( \beta = \frac{1}{\sqrt{2}}(\sqrt{2} - 1) \), whence \( C_{12} = \sqrt{2} - 1 = 0.41. \) On the other hand, the concurrence of \( \rho_{A\otimes B} \) amounts to

\[
C_{A\otimes B} = \frac{2\sqrt{1 - a^2}s^2c^2}{(1-a)^2c^2 + 2s^2}.
\] (25)

In Figure 1(b), \( a \) is set \( \sqrt{2} - 1 \) and (24–25) are plotted against \( \varphi \in [0, 2\pi] \): again agreement is shown with the general monotonicity of concurrence expressed by (16). Namely, nearest-neighbours are entangled if the state of one site with the rest \( \rho_{A\otimes B} \) is entangled, but we do not know if entanglement of one site with rest always implies nearest neighbours entanglement. We believe this to be peculiar of our choice of \( B \): since it is two dimensional both \( \rho_{12} \) and \( \rho_{A\otimes B} \) have rank 2 and separable density matrices of rank 2 have 0 measure.

When three qubits share equal entanglement, then the maximum entanglement of one qubit with each of the two other qubits is the sum of the squares of the two concurrences which is less than or equal to one [4]. Thus nearest neighbour concurrence cannot exceed \( 1/\sqrt{2} \), but it is an open problem whether this upper bound can be achieved by an entangled chain. In [3] an entangled chain is considered in a translational-invariant state with nearest-neighbours states as in (23) but with \( \alpha = 0 \) and an upper bound for shared entanglement corresponding to a greater concurrence \( C_{12} = 0.434467. \)

The translation invariant state over the system \( A_{[-\infty,-1]} \otimes A_{0} \otimes A_{[1,\infty]} \) can be defined with \( \bar{G} \otimes 1 \otimes G \) mapping the system into \( B \otimes A \otimes B \). Now we have a effectively three qubit state sharing equal amount of entanglement and due to the considerations in [4] the upper limit for the concurrence is given by \( 1/\sqrt{2} \). In the following example, we show how the maximum can indeed be reached by a finitely correlated structure. We set \( v_1 \) as in (21), \( v_2 = \sqrt{1-a^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \end{pmatrix} \) and

\[
\rho = \frac{1}{1 + a^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \end{pmatrix}, \quad \text{then} \quad \rho_{A\otimes B} = \begin{pmatrix}
\frac{a^2}{\sqrt{1-a^2}} & 0 & ac & 0 \\
0 & \frac{a^2}{\sqrt{1-a^2}} & as & 0 \\
ac & as & \sqrt{1-a^2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\] (26)
whence $C(\rho_{A_0 \otimes B}) = \frac{2a\sqrt{1-a^2}}{1+a^2} \sin \varphi$ which attains its maximum $1/\sqrt{2}$ at $\varphi = \pi/2$ and $a = 1/\sqrt{3}$. This state differs from the one in Eq. (23). Therefore optimizing the entanglement of $A_0 \otimes A_{[1,\infty]}$ reduces the entanglement of $A_0 \otimes A_1$.

**Conclusions:** We studied the entanglement properties of FCS over infinite quantum spin chains by means of their recursive structure and illustrated some possible behaviors by examples with $2 \times 2$ matrices. Investigation of higher dimensional contexts as the AKLT model [23] and comparison with different approaches [17] is in progress.

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