SUMMATION FORMULAE INVOLVING MULTIPLE HARMONIC NUMBERS

Dongwei Guo and Wenchang Chu

By means of the generating function approach, we derive several summation formulae involving multiple harmonic numbers $H_{n,\ell}(\sigma)$, as well as other combinatorial numbers named after Bernoulli, Euler, Bell, Genocchi and Stirling.

1. INTRODUCTION AND MOTIVATION

The harmonic numbers and the alternating ones are well-known that are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \geq 1;$$

$$\mathcal{H}_0 = 0 \quad \text{and} \quad \mathcal{H}_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \text{for} \quad n \geq 1;$$

as well as their generating functions

$$\sum_{n=0}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x} \quad \text{and} \quad \sum_{n=0}^{\infty} \mathcal{H}_n x^n = \frac{-\ln(1+x)}{1-x}.$$

For their wide applications in combinatorics, number theory and computer science (for example, the analysis of algorithms), properties and identities that

*Corresponding author. Wenchang Chu
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Dongwei Guo and Wenchang Chu

Involuntary methods have been explored by various methods (cf. [1, 3, 7]). Further generalizations can be found in the papers [4, 5, 8, 9, 10]. By introducing a variable \( \sigma \), we can unify the both numbers by

\[
H_0(\sigma) = 0 \quad \text{and} \quad H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \quad \text{for} \quad n \geq 1
\]

with the generating function

\[
\sum_{n=0}^{\infty} H_n(\sigma) x^n = -\ln(1 - x\sigma) \frac{1}{1 - x}.
\]

In this paper, we shall examine the following harmonic–like numbers

\[
H_{n,0}(\sigma) \equiv 1 \quad \text{and} \quad H_{n,\ell}(\sigma) = \sum_{\substack{k_1, k_2, \ldots, k_\ell \geq 1 \\ k_1 + k_2 + \cdots + k_\ell \leq n}} \frac{\sigma^{k_1 + \cdots + k_\ell}}{k_1 k_2 \cdots k_\ell} \quad \text{for} \quad n \geq 1.
\]

When \( \ell = 1 \) and \( \sigma = \pm 1 \), they will reduce to \( H_n \) and \( H_n \), respectively.

Classifying according to the sum \( m = k_1 + k_2 + \cdots + k_\ell \), we have

\[
H_{n,\ell}(\sigma) = \sum_{m=1}^{n} G_{m,\ell}(\sigma) \quad \text{where} \quad G_{m,\ell}(\sigma) := \sum_{\substack{k_1, k_2, \ldots, k_\ell \geq 1 \\ k_1 + k_2 + \cdots + k_\ell = m}} \frac{\sigma^m}{k_1 k_2 \cdots k_\ell}.
\]

Since the generating function of \( G_{m,\ell}(\sigma) \) is equal to

\[
\sum_{m=0}^{\infty} G_{m,\ell}(\sigma) x^m = \ln^\ell \frac{1}{1 - x\sigma},
\]

the generating function for the sequence \( H_{n,\ell}(\sigma) \) of their partial sums results in

\[
\sum_{n=0}^{\infty} H_{n,\ell}(\sigma) x^n = \left\{-\ln(1 - x\sigma)\right\}^\ell \frac{1}{1 - x}.
\]

Let \( [x^n]f(x) \) stand for the coefficient of \( x^n \) in the formal power series \( f(x) \). Then we have the following expression

\[
H_{n,\ell}(\sigma) = [x^n] \left\{-\ln(1 - x\sigma)\right\}^\ell \frac{1}{1 - x} = \sum_{m=1}^{n} \frac{\ell!}{m!} \left[ m \atop \ell \right] \sigma^m,
\]

where the signless Stirling numbers of the first kind is given by the generating function

\[
\sum_{m=0}^{\infty} \frac{\ell!}{m!} \left[ m \atop \ell \right] x^m = \ln^\ell \frac{1}{1 - x}.
\]
When \( \sigma = 1 \), Cheon and El-Mikkawy \[2\] found not only the generating function
\[
\sum_{n=0}^{\infty} H_{n,\ell}(1)x^n = \frac{\{-\ln(1-x)\}^\ell}{1-x},
\]
but also the following explicit expression
\[
H_{n,\ell}(1) = \left[\frac{n + 1}{\ell + 1}\right]^{\ell!} n!.
\]
Unfortunately, for \( H_{n,\ell}(-1) \), there does not exist such an elegant expression.

By making use of Riordan arrays, Cheon and El-Mikkawy \[2\] proved four summation formulae about the multiple harmonic numbers \( H_{n,\ell}(1) \). Examining carefully the structure of their sums, we find that they fit into the following general scheme about formal power series.

Observe that the numbers \( H_{n,\ell}(\sigma) \) form a Riordan array generated by the formal power series pair
\[
\left(\frac{1}{1-x}, \ln \frac{1}{1-x\sigma}\right).
\]
If \( \Lambda(x) \) is another formal power series
\[
\Lambda(x) := \sum_{\ell=0}^{\infty} \lambda\ell x^\ell
\]
then we can evaluate the sum
\[
\sum_{\ell=0}^{\infty} \lambda\ell H_{n,\ell}(\sigma) = \sum_{\ell=0}^{\infty} \lambda\ell \left[\frac{\{-\ln(1-x\sigma)\}^\ell}{1-x}\right] = \left[x^n\right] \left[\frac{\Lambda(-\ln(1-x\sigma))}{1-x}\right].
\]

According to this scheme, we shall establish seven classes of summation formulae about \( H_{n,\ell}(\sigma) \), including the aforementioned four identities of Cheon and El-Mikkawy \[2, Theorem 3.2\]. Then the paper will end in Section 3 with a comment about a composite sum involving the derangement numbers.

Throughout the paper, the following well–known classical numbers (cf. Comtet \[6, §1.14\]) will be used without being recalled:

- Bernoulli numbers \( B_n \) with the generating function
  \[
  \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.
  \]

- Euler numbers \( E_n \) with the generating function
  \[
  \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k = \frac{2e^x}{e^{2x} + 1}.
  \]
• Genocchi numbers $G_n$ with the generating function
\[
\sum_{k=1}^{\infty} \frac{G_k}{k!} x^k = \frac{2x}{e^x + 1}.
\]
• Bell numbers $B_n$ with the generating function
\[
\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = e^{e^x - 1}.
\]
• The derangement numbers $D_n$ with the generating function
\[
\sum_{k=0}^{\infty} \frac{D_k}{k!} x^k = \frac{e^{-x}}{1 - x}.
\]
• The Stirling number of the second kind with the generating function
\[
\sum_{n\geq k} \frac{x^n}{n!} S(n,k) = \frac{(e^x - 1)^k}{k!}.
\]

In order to reduce lengthy expressions, we shall make use of the following notations. As usual, the logical function is defined by $\chi(true) = 1$ and $\chi(false) = 0$. For a real number $x$, the smallest integer $\geq x$ and the greatest integer $\leq x$ will be denoted $\lceil x \rceil$ and $\lfloor x \rfloor$, respectively. When $m$ is a natural number, $i \equiv m j$ stands for that “$i$ is congruent to $j$ modulo $m$”.

2. SEVEN CLASSES OF SUMMATION FORMULAE

In this section, we prove seven summation theorems, where all the formulae are valid for $n \geq 1$ because there are always the same initial value “1” for the results corresponding to the trivial case $n = 0$. To our knowledge, all the formulae displayed in this section are new except for those being explicitly given by references.

We start with the following summation theorem.

**Theorem 1.** For the sum defined by
\[
A_n(\tau, \sigma) := \sum_{\ell=0}^{n} \frac{\tau^\ell}{\ell!} H_n,\ell(\sigma)
\]
the following identity holds
\[
A_n(\tau, \sigma) = \sum_{k=0}^{n} \binom{\tau + k - 1}{k} \sigma^k.
\]
Two particular cases are highlighted below, where the former integrates the first two formulae given by Cheon and El-Mikkawy \[2, \text{Theorem 3.2: i & ii}]:

\[ A_n(\pm 1, 1) = \begin{cases} n + 1, & " + "; \\ 0, & " - "; \end{cases} \]

\[ A_n(\pm 1, -1) = \begin{cases} 1 - \chi(n \equiv 2 \, \text{mod} \, 1), & " + "; \\ 2, & " - "; \end{cases} \]

**Proof.** Consider the exponential function \( \Lambda(x) = e^{\tau x} \) in (1). Then we have immediately

\[
\sum_{\ell=0}^{n} \frac{\tau^\ell}{\ell!} H_n,\ell(\sigma) = [x^n] \frac{(1-x\sigma)^{-\tau}}{1-x} = \sum_{k=0}^{n} \left( \frac{\tau + k - 1}{k} \right) \sigma^k.
\]

The next theorem is a variant of Theorem 1.

**Theorem 2.** For the sum defined by

\[
\mathbb{B}_n(\tau, \sigma) := \sum_{\ell=1}^{n} \frac{\tau^\ell}{(\ell-1)!} H_n,\ell(\sigma)
\]

the following identity holds

\[
\mathbb{B}_n(\tau, \sigma) = \tau \sum_{k=0}^{n} \left( \frac{\tau + k - 1}{k} \right) \sigma^k H_{n-k}(\sigma).
\]

We record two special cases, where the first one corresponding to "-" is due to Cheon and El-Mikkawy \[2, \text{Theorem 3.2: iii}]:

\[ \mathbb{B}_n(\pm 1, 1) = \begin{cases} (n + 1)H_n - n, & " + "; \\ \frac{-1}{n}, & " - "; \end{cases} \]

\[ \mathbb{B}_n(\pm 1, -1) = \begin{cases} \frac{1}{2} \{ H_n + (-1)^n H_n \}, & " + "; \\ -H_n - H_{n-1}, & " - "; \end{cases} \]

**Proof.** Analogously let \( \Lambda(x) = \tau x e^{\tau x} \) in (1). Then we can evaluate

\[
\sum_{\ell=1}^{n} \frac{\tau^\ell}{(\ell-1)!} H_n,\ell(\sigma) = [x^n] \frac{\tau}{(1-x\sigma)^{\tau}} \times \frac{\ln(1-x\sigma)}{x-1}
\]

\[
= \tau \sum_{k=0}^{n} \left( \frac{\tau + k - 1}{k} \right) \sigma^k H_{n-k}(\sigma).
\]

\]
Theorem 3. For the sum defined by

\[ \mathcal{C}_n(\tau, \sigma) := \sum_{\ell=0}^{n} B_{\ell} \frac{\tau^{\ell}}{\ell!} H_{n, \ell}(\sigma) \]

the following identities hold

\[ \mathcal{C}_n(1, \sigma) = \frac{H_{n+1}(\sigma)}{\sigma} - H_n(\sigma), \]
\[ \mathcal{C}_n(-1, \sigma) = \frac{H_{n+1}(\sigma)}{\sigma}. \]

In particular for \( \sigma = \pm 1 \), we deduce further, where the former with the minus sign “−” is equivalent to Cheon and El-Mikkawy [2, Theorem 3.2: iv]:

\[ \mathcal{C}_n(\pm 1, 1) = \begin{cases} 
1, & \text{“+”;} \\
\frac{1}{n+1}, & \text{“-”;} 
\end{cases} \]
\[ \mathcal{C}_n(\pm 1, -1) = \begin{cases} 
-H_n - H_{n+1}, & \text{“+”;} \\
-H_{n+1}, & \text{“-”}. 
\end{cases} \]

Proof. Choose \( \Lambda(x) = \frac{\tau x}{e^{\tau x} - 1} \), the generating function of Bernoulli numbers. We get from (1)

\[ \sum_{\ell=0}^{n} B_{\ell} \frac{\tau^{\ell}}{\ell!} H_{n, \ell}(\sigma) = [x^n] \frac{\tau}{(1 - x\sigma)^{-\tau} - 1} \times \frac{\ln(1 - x\sigma)}{x - 1}. \]

For \( \tau = 1 \), we can determine the coefficient

\[ [x^n] \frac{1 - x\sigma}{x\sigma} \times \frac{\ln(1 - x\sigma)}{x - 1} = \frac{H_{n+1}(\sigma)}{\sigma} - H_n(\sigma). \]

Instead, when \( \tau = -1 \), the coefficient becomes

\[ [x^n] \frac{1}{x\sigma} \times \frac{\ln(1 - x\sigma)}{x - 1} = \frac{H_{n+1}(\sigma)}{\sigma}. \]

Theorem 4. For the sum defined by

\[ \mathbb{D}_n(\tau, \sigma) := \sum_{\ell=0}^{n} S(\ell, m) \frac{\tau^{\ell}}{\ell!} H_{n, \ell}(\sigma) \]

the following identities hold

\[ \mathbb{D}_n(1, \sigma) = \frac{\sigma^m}{m!} \sum_{k=m}^{n} \binom{k-1}{m-1} \sigma^k, \]
\[ \mathbb{D}_n(-1, \sigma) = \frac{(-\sigma)^m}{m!} \chi(m \leq n). \]
When \( \sigma = \pm 1 \), they can be restated as follows:

\[
\mathbb{D}_n(\pm 1, 1) = \begin{cases} 
\frac{1}{m!} \binom{n}{m}, & \text{"+";} \\
\frac{(-1)^m}{m!} \chi(m \leq n), & \text{"-"}; 
\end{cases}
\]

\[
\mathbb{D}_n(\pm 1, -1) = \begin{cases} 
\left(\frac{-1}{m!}\sum_{k=m}^{n} \binom{k-1}{m-1} (-1)^k \right), & \text{"+";} \\
\frac{\chi(m \leq n)}{m!}, & \text{"-"}. 
\end{cases}
\]

**Proof.** Let \( \Lambda(x) = \left(\frac{e^{x}-1}{m!}\right)^m \) be the exponential generating function of Stirling numbers of the second kind. We have from (1)

\[
\sum_{\ell=0}^{n} S(\ell, m) \frac{x^\ell}{\ell!} H_{n,\ell}(\sigma) = \left[x^n\right] \frac{(1-x\sigma)^{-m} - 1}{m!(1-x)}.
\]

For \( \tau = 1 \), we can determine the coefficient

\[
\left[x^n\right] \frac{(x\sigma)^m (1-x\sigma)^{-m}}{m!(1-x)} = \frac{\sigma^m}{m!} \sum_{k=m}^{n} \binom{k-1}{m-1} \sigma^k.
\]

Instead, when \( \tau = -1 \), the coefficient becomes

\[
\left[x^n\right] \frac{(-x\sigma)^m}{m!(1-x)} = \frac{(-\sigma)^m}{m!} \chi(m \leq n). \quad \Box
\]

**Theorem 5.** For the sum defined by

\[
\mathbb{E}_n(\tau, \sigma) := \sum_{\ell=0}^{n} B_{\ell} \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)
\]

the following identities hold

\[
\mathbb{E}_n(1, \sigma) = 1 + \sum_{k=1}^{n} \sum_{i=1}^{k} \binom{k-1}{i-1} \frac{\sigma^i}{i!},
\]

\[
\mathbb{E}_n(-1, \sigma) = \sum_{k=0}^{n} \frac{(-\sigma)^k}{k!}.
\]
For $\sigma = \pm 1$, these formulae can be further simplified into

$$
\mathbb{E}_n(\pm 1, 1) = \begin{cases} 
1 + \sum_{k=1}^{n} \frac{1}{k!} \binom{n}{k}, & \text{"+";} \\
\frac{D_n}{n!}, & \text{"-";} 
\end{cases}
$$

$$
\mathbb{E}_n(\pm 1, -1) = \begin{cases} 
1 + \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^k}{i!} \binom{k-1}{i-1}, & \text{"+";} \\
\sum_{k=0}^{n} \frac{1}{k!}, & \text{"-"}. 
\end{cases}
$$

**Proof.** Consider $\Lambda(x) = \exp\left(e^{x\tau} - 1\right)$, the generating function of Bell numbers. We can express the sum in (1) as

$$
\sum_{\ell=0}^{n} B_\ell \frac{x^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n] \exp\{(1 - x\sigma)^{-\tau} - 1\}.
$$

For $\tau = 1$, we can determine the coefficient

$$
[x^n] \exp\left(\frac{x\sigma}{1-x\sigma}\right) = \sum_{k=0}^{n} [x^k] \exp\left(\frac{x\sigma}{1-x\sigma}\right).
$$

Instead, when $\tau = -1$, the coefficient becomes

$$
[x^n] \exp\left(-x\sigma\right) = \sum_{k=0}^{n} \frac{(-\sigma)^k}{k!}.
$$

\[\Box\]

**Theorem 6.** For the sum defined by

$$
\mathbb{F}_n(\tau, \sigma) := \sum_{\ell=0}^{n} G_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)
$$

the following identities hold

$$
\mathbb{F}_n(1, \sigma) = H_n(\sigma) - \sum_{k=1}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma),
$$

$$
\mathbb{F}_n(-1, \sigma) = -\sum_{k=0}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma).
$$

Observe that

$$
\sum_{k=1}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma) = \sum_{k=1}^{n-1} \frac{\sigma^k}{2^k} \sum_{i=1}^{n-k} \frac{\sigma^i}{i} = \sum_{i=1}^{n-1} \frac{\sigma^i}{i} \sum_{k=1}^{n-1} \frac{\sigma^k}{2^k} = \sum_{i=1}^{n-1} \frac{\sigma^i}{i} \left(\frac{\sigma}{2} - \left(\frac{\sigma}{2}\right)^{n-i+1}\right)\frac{1 - \frac{\sigma}{2}}{1 - \frac{\sigma}{2}}.
$$
When \( \sigma = \pm 1 \), the formulae in Theorem 6 can be stated explicitly as follows:

\[
F_n(\pm 1, 1) = \begin{cases} 
\sum_{k=1}^{n} \frac{2^k}{2^n k}, & " + "; \\
-2H_n + \sum_{k=1}^{n} \frac{2^k}{2^n k}, & " - "; 
\end{cases}
\]

\[
F_n(\pm 1, -1) = \begin{cases} 
\frac{4}{3}H_n - \frac{1}{3} \sum_{k=1}^{n} \frac{2^k}{2^n k}, & " + "; \\
-\frac{2}{3}H_n - \frac{1}{3} \sum_{k=1}^{n} \frac{2^k}{2^n k}, & " - ".
\end{cases}
\]

**Proof.** Let \( \Lambda(x) = \frac{2x\tau}{1 + e^{2x\tau}} \) be the generating function of Genocchi numbers. We have from (1)

\[
\sum_{\ell=0}^{n} \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n] \frac{2\tau}{1 + (1-x\sigma)^{-\tau}} \times \frac{\ln(1-x\sigma)}{x-1}.
\]

For \( \tau = 1 \), we can determine the coefficient

\[
[x^n] \frac{2(1-x\sigma)}{2-x\sigma} \times \frac{\ln(1-x\sigma)}{x-1} = H_n(\sigma) - \sum_{k=1}^{n} \frac{\sigma^k}{2k} H_{n-k}(\sigma).
\]

Instead, when \( \tau = -1 \), the coefficient becomes

\[
[x^n] \frac{-2}{2-x\sigma} \times \frac{\ln(1-x\sigma)}{x-1} = -\sum_{k=0}^{n} \frac{\sigma^k}{2k} H_{n-k}(\sigma). \quad \Box
\]

**Theorem 7.** For the sum defined by

\[
G_n(\sigma) := \sum_{\ell=0}^{n} E_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)
\]

the following identities hold

\[
G_n(1) = \chi(n \not\equiv 3) \frac{(-1)^{\lfloor \frac{n}{4} \rfloor}}{2^{\lfloor \frac{n}{2} \rfloor}};
\]

\[
G_n(-1) = \frac{4}{5} + \frac{(-1)^{\lfloor \frac{n}{4} \rfloor}}{5 \cdot 2^{\lfloor \frac{n}{2} \rfloor}} \times \begin{cases} 
1, & n \equiv 0; \\
1, & n \equiv 1; \\
-3, & n \equiv 2; \\
2, & n \equiv 3.
\end{cases}
\]
Proof. Finally specify \( \Lambda(x) = \frac{2e^x}{1+e^{-2x}} \) by the generating function of Euler numbers in (1), where another variable \( \tau \) is suppressed because this function is even. Then the corresponding sum can be reformulated as

\[
\sum_{\ell=0}^{n} \frac{E_{\ell}}{\ell!} H_{n,\ell}(\sigma) = [x^n] \frac{2(1-x\sigma)}{(1-x)(1+(1-x\sigma)^2)}.
\]

When \( \sigma = 1 \), by making use of partial fractions

\[
\frac{2}{1+(1-x)^2} = \frac{1}{1-i-x} - \frac{i}{1+i-x} - \frac{1}{1-x},
\]

we can extract the coefficient of \( x^n \)

\[
[x^n] \frac{2}{1+(1-x)^2} = \frac{i}{(1+i)^{n+1}} - \frac{i}{(1-i)^{n+1}} = 2^{n+1} \left\{ i e^{-\frac{n+1}{4} \pi i} - i e^{\frac{n+1}{4} \pi i} \right\}.
\]

Then the first formula for \( G_n(1) \) follows from the simplification

\[
\left\{ i e^{-\frac{n+1}{4} \pi i} - i e^{\frac{n+1}{4} \pi i} \right\} = (-1)^{\frac{n}{4}} \times \begin{cases} \sqrt{2}, & n \equiv 0 \mod 4; \\ 2, & n \equiv 1 \mod 4; \\ \sqrt{2}, & n \equiv 2 \mod 4; \\ 0, & n \equiv 3 \mod 4. \end{cases}
\]

Alternatively for \( \sigma = -1 \), we can decompose the rational fraction

\[
\frac{2(1+x)}{(1-x)(1+(1+x)^2)} = \frac{4}{5(1-x)} + \frac{2-i}{5(1+x+1)} + \frac{2+i}{5(1+x-1)}
\]

and determine the coefficient of \( x^n \)

\[
[x^n] \frac{2(1+x)}{(1-x)(1+(1+x)^2)} = \frac{4}{5} + \frac{(-1)^n(2-i)}{5(1+i)^{n+1}} + \frac{(-1)^n(2+i)}{5(1-i)^{n+1}}
\]

\[
= \frac{4}{5} + \frac{(-1)^n}{5 \cdot 2^{n+2}} + \left\{ (2+i)e^{\frac{n+1}{4} \pi i} + (2-i)e^{-\frac{n+1}{4} \pi i} \right\}.
\]

Then the second formula for \( G_n(-1) \) follows from the simplification

\[
\left\{ (2+i)e^{\frac{n+1}{4} \pi i} + (2-i)e^{-\frac{n+1}{4} \pi i} \right\} = (-1)^{\frac{n}{2}} \times \begin{cases} \sqrt{2}, & n \equiv 0 \mod 4; \\ 2, & n \equiv 1 \mod 4; \\ 3\sqrt{2}, & n \equiv 2 \mod 4; \\ 4, & n \equiv 3 \mod 4. \end{cases}
\]
3. CONCLUDING COMMENTS

There should be a number of further choices for the formal power series $\Lambda(x)$ in (1), that may lead us to more summation formulae. For instance, Specifying in (1) by the generating function $\Lambda(x) = \frac{e^{-x}}{1-x}$ of the derangement numbers, we have the sum

$$\sum_{\ell=0}^{n} \frac{\tau^{\ell}}{\ell!} H_{n}(\ell) = [x^n] \frac{(1-x\sigma)^{\tau}}{(1-x)(1 + \tau \ln(1-x\sigma))}.$$ 

When $\tau = \sigma = 1$, we can evaluate the sum

$$\sum_{\ell=0}^{n} \frac{D_{\ell}}{\ell!} H_{n}(\ell)(1) = [x^n] \frac{1}{1 + \ln(1-x)} = \sum_{k=0}^{n} \frac{k!}{n!} \left[ \frac{n}{k} \right].$$

However, for other values of $\tau$ and $\sigma$, the corresponding sums don’t admit such nice expressions.

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Dongwei Guo
School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou (Henan), P. R. China
E-mail: guo.dongwei2018@outlook.com

Wenchang Chu
Department of Mathematics and Physics
University of Salento (P. O. Box 193)
73100 Lecce, Italy
E-mail: chu.wenchang@unisalento.it