Aeroelastic stability of cylindrical shells interacting with internal annular fluid flow

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Abstract. This paper is devoted to the analysis of the dynamic behavior of cylindrical shells, containing an internal annular layer of ideal fluid and subject to the external supersonic gas flow. The aerodynamic pressure is calculated based on the quasi-static aerodynamic theory. The behavior of the compressible fluid is described in terms of the perturbation velocity potential. A mathematical formulation of the problem is developed based on the classical theory of shells and virtual displacement principle. A solution of the problem involves computation of complex eigenvalues of the coupled system of equations. The paper presents the results of numerical experiments, which were performed to estimate the influence of the fluid flow velocity on the value of the static pressure in the unperturbed gas flow for shells, interacting with fluid layers of different thicknesses. The numerical simulation shows that a reduction of the fluid layer thickness and increase of the fluid velocity produce a stabilizing effect by virtue of increasing the threshold of aerodynamic stability. However, an essential reduction of the layer thickness can lead, depending on the preset combinations of boundary conditions, to a considerable growth of the stability threshold or to the onset of instability.

1. Introduction
Thin-walled plate and shell structures interacting with fluids, as parts of many industrial applications are used in various technical fields. For a rather long time they have been the focus of numerous theoretical studies [1, 2]. The characteristic feature of these works is that the interaction between the elastic body and liquid or gaseous medium takes place only on the side of one structure surface - outer or inner surface. An exception is coaxial shells, in which the inner shell interacts both with the internal and annular flows. On the one hand, depending on the prescribed boundary conditions and flow velocity a combined action of both flows might be of additive nature, providing a strong destabilizing effect on the examined system and, on the other hand, might enlarge the stability boundaries [3–6]. However, in recent years there have been a number of publications, in which the aeroelastic stability is studied taking into account the influence of a quiescent liquid with a free surface located inside the shell or at its outer surface [7–10]. The panel flutter of isotropic and heated functionally-graded cylindrical shells completely filled with a quiescent or flowing ideal liquid has been analyzed in our recent papers [11, 12]. The objective of this study is to investigate the aeroelastic characteristics of cylindrical shells interacting with the internal annular fluid flow.
Figure 1. Cylindrical shell, containing an internal annular fluid flow and subject to the external supersonic gas flow.

2. Statement of the problem

Let us consider an elastic cylindrical shell of length $L$ (figure 2), thickness $h$ and radius $b$. Inside, it interacts with a layer of ideal compressible liquid of radius $a$, and outside it is subject to a supersonic gas flow with velocity $U_\infty$. Here we will concern ourselves with the influence of the velocity $U$ of the fluid flowing in the annular channel of different heights on the boundary of aeroelastic stability of the shell, at the edges of which we prescribe different combinations of boundary conditions.

The components of the strain vector $\varepsilon = \{\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \kappa_1, \kappa_2, 2\kappa_{12}\}^T$ in the curvilinear coordinate system $(s, \theta, z)$ are determined in terms of the classical theory of shells, which is based on the Kirchhoff–Love hypotheses, and are written as [13]

$$
\varepsilon_1 = \frac{\partial u}{\partial s}, \quad \varepsilon_2 = \frac{1}{b} \left( \frac{\partial v}{\partial \theta} + w \right), \quad \varepsilon_{12} = \frac{1}{b} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial s},
$$

$$
\kappa_1 = -\frac{\partial^2 w}{\partial s^2}, \quad \kappa_2 = \frac{1}{b^2} \left( \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right), \quad \kappa_{12} = \frac{1}{b} \left( \frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial s \partial \theta} \right)
$$

(1)

Here $u$, $v$ and $w$ are the meridional, circumferential, and normal components of the displacement vector of the shell.

The physical equations relating the vector of forces and moments $\mathbf{T}$ to the vector of strains $\varepsilon$, can be written in the matrix form as

$$
\mathbf{T} = \{T_{11}, T_{22}, T_{12}, M_{11}, M_{22}, M_{12}\}^T = \mathbf{D}\varepsilon
$$

(2)

The non-zero elements of the stiffness matrix $\mathbf{D}$ for isotropic material are generally determined in terms of the elasticity modulus $E$, Poisson’s ratio $\nu$ and shear modulus [13].

A mathematical formulation of the problem is based on the virtual displacements principle, which in the matrix form is written as

$$
\int_{S_s} \delta \varepsilon^T \mathbf{T} dS + \int_{S_s} \delta \mathbf{d}^T \rho_0 \ddot{d} dS - \int_{S_s} \delta \mathbf{d}^T \mathbf{P} dS = 0
$$

(3)

Here $\mathbf{d}$ and $\mathbf{P} = \{0 0 p_a + p_f\}$ are the vectors of the generalized displacements and surface loads; $p_a$ and $p_f$ are the aerodynamic and hydrodynamic pressures acting on the external and
internal surfaces of the shell; \( \rho_0 = \int_h \rho_s \, dz \); \( \rho_s \) is the specific density of the material of the shell; \( S_s \) is the surface that bounds the body of the shell.

The aerodynamic pressure exerted by the gas flow on the elastic surface is calculated in the framework of quasi-static aerodynamic theory using the approximate formula \[14\]

\[
p_a = - \left( q \frac{\partial w}{\partial s} + q_1 \frac{\partial w}{\partial t} - q_2 w \right)
\]

where \( q = \rho_\infty U_\infty^2 / \beta = \kappa p_\infty M_\infty^2 / \beta \), \( q_1 = q(M_\infty^2 - 2) / (U_\infty \beta^2) \), \( q_2 = 1/2 q R \beta \).

Here, \( M_\infty = U_\infty / c_\infty \) is the Mach number in a gas medium; \( \rho_\infty, p_\infty \) and \( c_\infty \) are the density, static pressure and sound speed in the unperturbed gas flow; \( q \) is the modified dynamic pressure; \( \kappa \) is the adiabatic index; \( \beta = (M_\infty^2 - 1)^{1/2} \).

The hydrodynamic pressure exerted by the liquid, occupying the space \( V_f \), on the wetted surface of the shell \( S_s = S_f \cap S_s \) is calculated using the linearized Bernoulli equation \[15\]

\[
p_f = -\rho_f \left( \frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} \right)
\]

Here \( \rho_f \) is the fluid density, \( S_f \) is the bounding surface of the fluid volume, \( \phi \) is the perturbation velocity potential, which in the cylindrical coordinates \( (r, \theta, x) \) is described by the wave equation

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{1}{c^2} \left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right]^2 \phi
\]

where \( c \) is the speed of sound in the fluid.

On the shell-fluid interface we impose the impermeability condition

\[
\frac{\partial \phi}{\partial n} = \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial s}
\]

where \( n \) is the normal to the boundary. It is assumed that the internal surface of the annular channel comes into contact with the rigid cylinder, which corresponds to the boundary condition \( \partial \phi / \partial n |_{r=a} = 0 \).

The perturbation velocity potential at the shell inlet and outlet obeys the following boundary conditions

\[
\phi |_{x=0} = 0, \quad \phi / \partial x |_{x=L} = 0
\]

For numerical implementation of the problem, the equation of perturbation velocity potential \(6\) and boundary conditions \(7-8\) are transformed using the Bubnov–Galerkin method \[16\].

### 3. Numerical implementation

For numerical implementation we used a semi-analytical variant of the finite element method (FEM) \[17\], in which the solution is represented as a Fourier series with respect to the circumferential coordinate \( \theta \)

\[
(u, w, \phi) = \sum_{j=0}^{\infty} (u_j, w_j, \phi_j) \cos j \theta, \quad v = \sum_{j=0}^{\infty} v_j \sin j \theta
\]

where \( j \) is the number of harmonic.

Expressing the unknown variables in \(9\) in terms of their node values, we obtain the known matrix relations.
\[ U = \{u, v, w\}^T = N d^e, \quad \phi = F f^e, \quad \varepsilon = B d^e \]

Here \( N \) and \( F \) are the matrices of the shape functions for the shell finite element and the perturbation velocity potential, \( d^e \) and \( f^e \) are the vectors of the nodal values, \( B \) is the matrix of the relation between the strains and nodal displacements.

Discretization of the shell domain was carried out with the use of high-precision finite element in the form of truncated cone and approximation of the meridional and circumferential components of the displacement vector by a cubic polynomial and the normal component — by the 7th degree polynomial [18]. Discretization of the fluid domain was performed with the use of triangular finite element with linear approximation of the perturbation velocity potential [17].

Using the standard FEM procedures and taking into account equations (2,4,5,9,10), we obtain from the transformed wave equation for the perturbation velocity potential and virtual displacement principle (3) a coupled system of two equations, which

\[
M\{\ddot{d} \quad \dot{\phi}\}^T + C\{\dot{d} \quad \dot{\phi}\}^T + (K + A)\{d \quad \phi\}^T = 0
\]

where

\[
M = \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix}, \quad C = \begin{bmatrix} C_s & C_{sf} \\ C_{fs} & C_f \end{bmatrix}, \quad K = \begin{bmatrix} K_s & 0 \\ 0 & K_f \end{bmatrix}, \quad A = \begin{bmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{bmatrix}
\]

\[
K_s = \sum_{m_s} \int_{S_s} B^T D B dS, \quad M_s = \sum_{m_s} \int_{S_s} N^T \rho_0 \kappa dS, \quad M_f = \sum_{m_f} \int_{V_f} \frac{1}{c^2} F^T F dV
\]

\[
K_f = \sum_{m_f} \int_{V_f} \left( F_{,x} F_{,x} + \frac{1}{2} F_{,x} \phi \phi + F_{,x} F_{,x} \right) dV, \quad C_{fs} = -\sum_{m_s} \int_{S_s} F^T \tilde{N} dS
\]

\[
C_{sf} = \rho_f \sum_{m_s} \int_{S_s} \tilde{N}^T F dS, \quad A_{sf} = \rho_f \sum_{m_s} \int_{S_s} U \tilde{N}^T F_{,x} dS
\]

\[
A_f = -\sum_{m_f} \int_{V_f} M^2 F_{,x} F_{,x} dV, \quad C_f = \sum_{m_f} \int_{V_f} \frac{2U}{c^2} F_{,x} F_{,x} dV, \quad C_s = q_1 \sum_{m_s} \int_{S_s} \tilde{N}^T \tilde{N} dS
\]

\[
A_{fs} = -\sum_{m_s} \int_{S_s} U F^T \tilde{N}_{,x} dS, \quad A_s = \sum_{m_s} \int_{S_s} (q \tilde{N}^T \tilde{N}_{,x} + q_2 \tilde{N}^T \tilde{N}) dS
\]

Here, \( m_f \) and \( m_s \) is the number of finite elements used to decompose the fluid and shell domains, \( \tilde{N} \) is the shape function matrix for a normal component of the displacement vector. Representing the disturbed motion of the shell and the fluid as \( (d, \phi) = (q, f) \exp(i\lambda t) \), we finally arrive at

\[
(-\lambda^2 M + i\lambda C + K + A)\{q \quad f\}^T = 0
\]

where \( q \) and \( f \) are some functions of coordinates, \( i = \sqrt{-1} \) and \( \lambda = \lambda_1 + i\lambda_2 \) is the characteristic index. The eigenvalue problem was solved using the iterative algorithm, which is based on the Muller method [19].

4. The results of computations

For numerical simulation we chose a cylindrical shell \( (h = 3 \times 10^{-4} \text{ m}) \) made of aluminum (\( E = 70 \text{ GPa}, \nu = 0.3, \rho = 2707 \text{ kg/m}^3 \)) with the following variants of kinematic boundary conditions prescribed at the ends: simply supported \((v = w = 0, S)\), clamped \((u = v = w = \partial w/\partial s = 0, C)\) and free-end support \((F)\) conditions. The gas and fluid flows have the following characteristics \( \rho_f = 1000 \text{ kg/m}^3, c = 1500 \text{ m/s}, \kappa = 1.4, M_\infty = 3, T_\infty = 48.89^\circ\text{C} \) is the drag temperature in
Figure 2. The non-dimensional parameter of the dynamic pressure $\gamma$ versus the number of harmonics $j$ at different values of the annular channel width $k$ (a) and annular flow velocity $\Upsilon$ (b) for clamped shells.

During the study of a combined action of aerodynamic and hydrodynamic loads, we operate on the assumption that the fluid velocity $U$ is fixed and we search for the static pressure in the unperturbed flow $p_\infty$, at which the system loses its stability.

Figure 4 shows the plots of non-dimensional parameter of the dynamic pressure $\gamma$ as a function of the harmonic number $j$, obtained for rigidly clamped shells at different values of flow velocity $\Upsilon$, and channel width $k$. For clarity, we also presents in the figure the results of computations made for a similar shell completely filled with the fluid. With increasing fluid velocity ($\Upsilon \neq 0$), the monotonic curve of the non-dimensional aerodynamic parameter changes to the non-monotonic curve. Such behavior is explained by the fact that at different harmonics the flowing fluid may have different effects. In this case, the critical number of the harmonic, i.e., the number of the harmonic with the lowest value of parameter $\gamma$, passes into the region of lower values. The influence of the thickness of the annular flow on the size of the instability region for different harmonics is shown in figure 4(a). Here, to represent the non-dimensional size of the fluid layer, we use notation proposed in [1]. From the data displayed in the figures, it follows that the width of the annular channel has only a qualitative impact on the instability boundaries, which is most pronounced in the case of very narrow annular channel.

At a certain height of the annular channel and prescribed boundary conditions the fluid flow exerts a stabilizing effect, which is demonstrated in figure 4(a), depicting the stability boundaries at three values of parameter $k$. It is evident from the figure, that actually, the critical dynamic pressure in the narrowest channel experiences an exponential growth. This can be attributed to the growth of the hydrodynamic pressure acting on the elastic surface, which strengthens it and prevents thereby the loss of stability.
The non-monotonic behavior of curves in figure 4(a) suggests that with increase of the fluid velocity the critical harmonic number changes. In figure 4(b), this effect is illustrated in greater detail. It is seen that for some harmonics an increase in the velocity of the fluid flow can lead either to an abrupt growth or gradual reduction of the critical parameter \( \gamma \), violating thereby the smoothness of stability threshold variation.

A combined action of the gas and fluid flows leads to a change in the instability type. In the case of a quiescent fluid the loss of stability occurs in the form of the coupled-mode flutter. Whereas an increase in the velocity of annular flow leads to a change in the type of instability to the single-mode flutter. Note that this form of instability remains invariable at any combinations of boundary conditions.

The dependence of the stability boundary on the thickness of the annular fluid layer is analyzed in detail in figure 4. The data shown in figure 4(a) indicate that a common feature of all harmonics, beginning with a certain value of \( b/a \), is a variation of the parameter \( \gamma \), which in the case of a thin fluid layer occurs in a jump-wise manner. On the other hand, from the results given in figure 4(b) we can conclude that in the specified parameter range for the examined combination of boundary conditions take place a rise of the stability threshold, which becomes more pronounced with increasingly growing velocity of the fluid.

The analysis of the behavior patterns found for other combinations of boundary conditions has shown that at a certain thickness of the annular fluid layer, the internal fluid flow has a stabilizing effect on the system, which is found to be more pronounced in cantilever shells (figure 4(a)). However, for simply supported shells there is a threshold value of the fluid layer thickness, beyond which even an inessential value of the hydrodynamic pressure parameter leads to the loss of stability (figure 4(b)). It should be noted that according to [11] for thin and long shells completely filled with liquid and subject to similar boundary conditions, a combined action of the aerodynamic and hydrodynamic loads is of the additive nature and leads to an abrupt decrease of the stability threshold.

**Figure 3.** The non-dimensional parameter of the dynamic pressure \( \gamma \) versus the non-dimensional fluid velocity \( \Upsilon \) for clamped shells at different values of the annular channel width \( k \) (a), and different harmonics \( j \) (b).
5. Conclusion
We have considered the results of numerical simulation made to investigate the dynamic behavior of circular cylindrical shells of revolution, containing an annular layer of flowing, ideal, compressible fluid and subject to the external supersonic gas flow. The influence of the velocity of the annular fluid flow on the boundaries of aeroelastic stability has been analyzed for shells under different boundary conditions. It has been found that an increase of the fluid velocity, irrespective of the prescribed combination of the boundary conditions, leads to a change in the type of stability loss — a coupled mode flutter changes to a single-mode flutter. The results of computation allowed us to conclude that there exists a critical thickness of the annular fluid layer, at which the annular fluid flow in shells under any of the examined combinations of
the boundary conditions produces a stabilizing effect and increases the aerodynamic stability threshold. For simply supported shells a decrease in the thickness of the annular fluid layer to some critical minimum causes a reverse reaction, namely, a destabilizing effect.

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References
[1] Paidoussis M P 2004 Fluid-Structure Interactions: Slender Structures and Axial Flow vol 2 (London: Elsevier Academic Press) p 1040
[2] Algazin S D and Kijko I A 2015 Aeroelastic Vibrations and Stability of Plates and Shells (Studies in Mathematical Physics vol 25) (Berlin: Walter de Gruyter GmbH) p 231
[3] Paidoussis M P, Chan S P and Misra A K 1984 Dynamics and stability of coaxial cylindrical shells containing flowing fluid J. Sound Vib. 97 201-35
[4] Paidoussis M P, Nguyen V B and Misra A K 1991 A theoretical study of the stability of cantilevered coaxial cylindrical shells conveying fluid J. Fluids Struct. 5 127-64
[5] Bochkarev S A and Matveenko V P 2010 The dynamic behaviour of elastic coaxial cylindrical shells conveying fluid J. Appl. Math. Mech. 74 467-74
[6] Bochkarev S A, Lekomtsev S V and Matveenko V P 2014 Parametric investigation of the stability of coaxial cylindrical shells containing flowing fluid Eur. J. Mech. A Solids 47 174-81
[7] Farhat C, Chiu E K-Y, Amsallem D, Schotté J-S and Ohayon R 2013 Modeling of fuel sloshing and its physical effects on flutter AIAA J. 51 2252-65
[8] Sabri F and Lakis A A 2010 Hybrid finite element method applied to supersonic flutter of an empty or partially liquid-filled truncated conical shell J. Sound Vib. 329 302-16
[9] Sabri F and Lakis A A 2011 Effects of sloshing on flutter prediction of liquid-filled circular cylindrical shell J. Aircr. 48 1829-39
[10] Noorian M, Haddadpour H and Firouz-Abadi R 2015 Investigation of panel flutter under the effect of liquid sloshing J. Aerosp. Eng. 28 04014059
[11] Bochkarev S A and Lekomtsev S V 2015 An aeroelastic stability of the circular cylindrical shells containing a flowing fluid Vestn. Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki 19 750-67 (in Russian)
[12] Bochkarev S A, Lekomtsev S V and Matveenko V P 2017 Aeroelastic stability of heated functionally graded cylindrical shells containing fluid Mech. Adv. Mater. Struct. 24 DOI: 10.1080/15376494.2016.1232457
[13] Biderman V L 1977 Mechanics of Thin-walled Structures (Moscow: Mashinostroenie) p 488 (in Russian)
[14] Bismarck-Nasr M N 1992 Finite element analysis of aeroelasticity of plates and shells Appl. Mech. Rev. 45 461-82
[15] Vol’mir A S 1979 Shells in Fluid and Gas Flow. Problems of Hydroelasticity (Moscow: Nauka) p 320 (in Russian)
[16] Bochkarev S A and Matveenko V P 2008 Numerical study of the influence of boundary conditions on the dynamic behavior of a cylindrical shell conveying a fluid Mech. Solids 43 477-86
[17] Zienkiewicz O C 1971 The Finite Element Method in Engineering Science (London: McGraw-Hill) p 521
[18] Shivakumar K N and Krishna Murty A V 1978 A high precision ring element for vibrations of laminated shells J. Sound Vib. 58 311-8
[19] Matveenko V P, Sevodin M A and Sevodina N V 2014 Applications of Muller’s method and the argument principle to eigenvalue problems in solid mechanics Comput. Continuum Mech. 7 331-6 (in Russian)