Pseudographs and the Lax–Oleinik semi-group: a geometric and dynamical interpretation

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Abstract

Let \( H : T^* M \to \mathbb{R} \) be a Tonelli Hamiltonian defined on the cotangent bundle of a compact and connected manifold and let \( u : M \to \mathbb{R} \) be a semi-concave function. If \( E(u) \) is the set of all the super-differentials of \( u \) and \((\varphi_t)\) the Hamiltonian flow of \( H \), we prove that for \( t > 0 \) small enough, \( \varphi_t(E(u)) \) is an exact Lagrangian Lipschitz graph.

This provides a geometric interpretation/explanation of a regularization tool that was introduced by Bernard (2007 Ann. Sci. École Norm. Sup. 40 445–52) to prove the existence of \( C^{1,1} \) subsolutions.

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1. Introduction

In the recent developments of the so-called ‘weak K.A.M. theory’, the notion of ‘pseudograph’ did appear recently in a paper of Bernard (see [2]) to prove some results concerning Arnold’s and Mather’s diffusion. Let us explain quickly how this notion appeared.

We consider the Hamilton–Jacobi equation: \( H : T^* M \to \mathbb{R} \) for a Hamiltonian function \( H : T^* M \to \mathbb{R} \) defined on a cotangent bundle that is \( C^2 \), superlinear and convex in the fibre. In the 1980s, Lions and Crandall introduced the notion of viscosity solution for this equation (see [6]). In the case \( M = \mathbb{T}^n \), Lions, Papanicolaou and Varadhan proved the existence of a viscosity solution. Then, in [7], Fathi proved the existence of a viscosity solution (that he called a weak K.A.M. solution) for any manifold. Such a weak K.A.M. solution is semi-concave and hence locally Lipschitz (see, for example, [8]). A semi-concave function \( u : M \to \mathbb{R} \)

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is Lipschitz and hence differentiable on a set \( E \subset M \) with full (Lebesgue) measure, and the graph \( \mathcal{G}(u) = \{(q, du(q)); q \in E\} \) of the derivatives of any semi-concave function is what we call a pseudograph.

When \( u \) is \( C^2 \), the pseudograph \( \mathcal{G}(u) \) is in fact a graph above the whole manifold \( M \) and is a Lagrangian graph.

That is why a very natural question is:

**Questions.** Are the pseudographs Lagrangian manifolds in general? And, as a pseudograph is not a smooth manifold, in which sense?

As we will explain more precisely in this introduction, the answer is yes in the following sense:

we can add to the pseudograph \( \mathcal{G}(u) \) of any semi-concave function \( u \) some ‘vertical parts’ in such a way that we obtain a Lipschitz Lagrangian manifold \( \mathcal{E}(u) \). Moreover, we will see that the images of \( \mathcal{E}(u) \) by any small enough negative time of the flow of any Tonelli Hamiltonian is a Lipschitz Lagrangian graph.

To illustrate this, let us draw three pictures for the simple pendulum. In the first picture, we draw a pseudograph, made with some parts of the separatrices of the pendulum; in the second one, we draw the corresponding enlarged pseudograph, i.e. we add some vertical parts to obtain a Lipschitz Lagrangian submanifold. Finally, we draw the image of this Lagrangian submanifold by the flow of the pendulum for a small negative time.

\[
\begin{align*}
\mathcal{G}(u) & \quad \mathcal{E}(u) & \quad \epsilon(\mathcal{E}(u))
\end{align*}
\]

Let us note that in the other sense, Chaperon proved in [5] that every Lagrangian submanifold of \( T^*M \) that is Hamiltonianly isotopic to the zero section can be ‘cut’ in such a way that we obtain the graph of the differential of a Lipschitz function defined on \( M \). Let us describe Chaperon’s method: he associates a generating function to any such submanifold and then selects a critical value \( \Phi(q) \) of this generating function above every point \( q \in M \). The function \( \Phi : M \to \mathbb{R} \) is then Lipschitz and above an open set \( U \subset M \) of full Lebesgue measure, the function \( \Phi \) is differentiable and the point \( (q, d\Phi(q)) \) belongs to the Lagrangian submanifolds. The ‘cut graph’ is then the graph of \( d(\Phi|_U) \).

In some cases, Otto and Viterbo proved in [10] that this Lipschitz function is a semi-concave one, and hence the ‘cut graph’ is a pseudograph. Let us also note that we proved in [1] that any invariant Lagrangian manifold that is Hamiltonianly isotopic to the zero section and invariant by a Tonelli Hamiltonian is the graph of a smooth function. Hence if the pseudograph of a weak KAM solution is obtained by cutting an invariant Lagrangian submanifold that is Hamiltonianly isotopic to the zero section, then this pseudograph has to be a true smooth submanifold.

Once we have proved that the pseudographs are some ‘Lagrangian manifolds’ (in some sense that we will be explained soon), we know that their images by the Hamiltonian flows are Lagrangian too because a Hamiltonian flow is symplectic. But in general the image of a pseudograph by a Hamiltonian flow is not a pseudograph (it may happen that it is not a
graph). To stay in the class of the graphs, let us consider the two Lax–Oleinik semi-groups $T_t$, $\tilde{T}_t : C^0(M, \mathbb{R}) \to C^0(M, \mathbb{R})$ associated with the considered Tonelli Hamiltonian (they will be precisely defined). Let us recall some well-known results concerning the relationships between the action of these two semi-groups and the action of the Hamiltonian flow on the pseudographs (see [2, 8]). We denote the associated Hamiltonian flow by $(\phi_t)$.

1. for every $t > 0$, all the functions of $T_t(C^0(M, \mathbb{R}))$ (respectively, $\tilde{T}_t(C^0(M, \mathbb{R}))$) are semi-concave (respectively, semi-convex);
2. if $u : M \to \mathbb{R}$ is a semi-concave function, then for all $t > 0$, we have $\bar{G}(\tilde{T}_t u) \subset \phi_t(G(u))$; the action of the negative Lax–Oleinik semi-group on the derivative of a semi-concave function is what follows: we take the image $\phi_t(G(u))$ of the graph of $u$ by the positive flow and we remove some part of this set;
3. similarly, if $u$ is semi-convex, we have for all $t > 0$: $\bar{G}(T_t u) \subset \varphi_{-t}(G(u))$;
4. if we just assume that $u$ is continuous, then for all $t > 0$ the set $\varphi_{-t}(G(T_t u))$ is a subset of the set of the sub-derivatives of $u$ and $\varphi_t(G(T_t u))$ is a subset of the set of the super-derivative of $u$. Hence the positive Lax–Oleinik semi-group maps any continuous function on a function $T_t u$ such that $G(T_t u)$ is a part of the image by the negative flow of what we will call the enlarged pseudograph (i.e. the set of all the super-derivatives of $u$).

Hence there is a deep link between the action of the Lax–Oleinik semi-group on the graphs of super/sub-derivatives and the action of the Hamiltonian flow. Our purpose is to give a precise statement concerning the action of the positive Lax–Oleinik semi-group ($\tilde{T}_t$) on the semi-concave functions and to prove simultaneously that the enlarged pseudographs of the semi-concave functions are some Lipschitz Lagrangian submanifolds.

Before explaining our result, let us introduce precisely some notions. At first, we recall what is a semi-concave function and we define the enlarged pseudographs.

**Definition.**

1. Let $U$ be an open subset of $\mathbb{R}^d$, $K \geq 0$ be a constant and $u : U \to \mathbb{R}$ be a function. We say that $u$ is $K$-semi-concave if for every $x \in U$, there exists a linear form $p_x$ defined on $\mathbb{R}^d$ such that:

$$\forall y \in U, u(y) \leq u(x) + p_x(y - x) + K \|y - x\|^2;$$

(where $\|\cdot\|$ is the usual Euclidean norm). Then we say that $p_x$ is a $K$-super-differential of $u$ at $x$.

2. Let $M$ be a compact and connected manifold with a finite atlas $A = \{(U_i, \Phi_i : U_i \to \mathbb{R}^d); 1 \leq i \leq N\}$ and $u : M \to \mathbb{R}$ be a function; we say that $u$ is $K$-semi-concave if for every $i \in \{1, \ldots, N\}$, the function $u \circ \Phi_i^{-1} : \Phi_i(U_i) \to \mathbb{R}$ is $K$-semi-concave. Then, a $K$-superdifferential of $u$ is a $p_x \circ D\Phi_i(x)$ where $p_x$ is a $K$-super-differential of $u \circ \Phi_i^{-1}$ at $\Phi_i(x)$.

3. A function is semi-concave if it is $K$-semi-concave for a certain $K$; while the quantitative notion of ‘$K$-semi-concave function’ depends on the considered atlas of $M$ that we choose, the notion of ‘semi-concave function’ is independent of this atlas. The notion of super-differential too does not depend on the atlas.

4. if $u : M \to \mathbb{R}$ is semi-concave, its enlarged pseudograph is the set $E(u)$ of all the super-differentials of $u$:

$$E(u) = \{(x, p_x); p_x \text{ is a superdifferential of } u \text{ at } x\}.$$
The enlarged pseudograph $\mathcal{E}(u)$ of a semi-concave function $u$ contains its pseudograph $\mathcal{G}(u)$; in general, $\mathcal{E}(u)$ is no longer a graph and $\mathcal{E}(u)$ is compact (it is clearly closed and Bernard proved in [2] that it is bounded).

**Remark.** In fact, even if it does not appear in the notation, the definition of $\mathcal{E}(u)$ depends on the choice of the constant $K$ of semi-concavity that we choose, and in the proofs we will fix such a constant $K$. But *a posteriori*, because of theorem 1, we see that $\mathcal{E}(u)$ is independent of this constant.

For a survey of the principal properties of the semi-concave functions, the reader may have a look at the appendix of [2] and [8].

Let us now explain which kind of submanifolds will interest us:

**Definition.** Let $M$ be a $d$-dimensional compact and connected manifold.

1. a non-empty subset $N$ of $T^*M$ is a $d$-dimensional Lipschitz submanifold of $T^*M$ if for every $x \in N$, there exists a (smooth) chart $(U, \Phi)$ of $T^*M$ at $x$ such that $\Phi(N \cap U)$ is the graph of a Lipschitz map $\ell : V \to \mathbb{R}^2$ defined on a open subset $V$ of $\mathbb{R}^d$. Of course, this notion is invariant by $C^1$-diffeomorphism.
2. a Lipschitz graph is $\{(s(x); x \in M)\}$ where $s : M \to T^*M$ a Lipschitz section. Of course, a Lipschitz graph is a $d$-dimensional Lipschitz submanifold of $T^*M$.
3. a $d$-dimensional Lipschitz submanifold $N$ of $T^*M$ is exact Lagrangian if it is exact Lagrangian in the sense of distributions, that is if for every $\gamma : [a, b] \to N$ closed Lipschitz arc drawn on $N$, we have $0 = \int_\gamma \lambda$ (where $\lambda$ designates the Liouville 1-form of $T^*M$). This notion is invariant under $C^1$ exact symplectic diffeomorphisms.

Then the Lipschitz graph of a Lipschitz section $s : M \to T^*M$ is exact Lagrangian if and only if there exists a $C^{1,1}$ function $u : M \to \mathbb{R}$ (that is a $C^1$ function whose derivative is Lipschitz) such that: $s = du$.

The result that we obtain is:

**Theorem 1.** Let $M$ be a compact and connected manifold, let $u : M \to \mathbb{R}$ be a semi-concave function and let $\mathcal{E}(u)$ be its enlarged pseudograph. Let $(\phi_t)$ be a Tonelli Hamiltonian flow of $T^*M$. Then there exists $\varepsilon > 0$ such that for all $t \in ]0, \varepsilon]$, we have: $\mathcal{G}(d\phi_t u) = \phi_{-t}(\mathcal{E}(u))$ is a Lipschitz graph above the whole manifold.

We immediately deduce:

**Corollary 2.** The enlarged pseudograph of any semi-concave function of $M$ is a Lipschitz exact Lagrangian submanifold of $T^*M$.

Let us recall that in [3], using [9], Bernard proved the following regularization result (that he used to prove the existence of $C^{1,1}$ subsolutions): for each semi-concave function $u : M \to \mathbb{R}$, for every $t > 0$ small enough, the function $\tilde{T}_t u$ is $C^{1,1}$.

It is not very hard to deduce theorem 1 and its corollary from this result. The novelty in our paper is:

- the connection with the Lagrangian submanifolds;
- related to this connection, the proofs, that give a dynamical interpretation of the action of Lax–Oleinik semi-group on the enlarged pseudographs.
2. Proof of theorem 1

We assume that $M$ is a compact and connected manifold with a finite atlas $\mathcal{A}$ and $u : M \to \mathbb{R}$ is a $K$-semi-concave function. We consider any Tonelli Hamiltonian function $H : T^*M \to \mathbb{R}$ and denote by $(\varphi_t)_{t \in \mathbb{R}}$ its Hamiltonian flow.

2.1. Proof that $\varphi_t(\mathcal{E}(u))$ is a graph for $t \in [-\varepsilon, 0[$

Given $\varepsilon \in [0, 1]$ small enough, we want to know if it is possible that for a $t \in [-\varepsilon, 0[$, $\varphi_t(\mathcal{E}(u))$ is not a graph above a certain part of $M$.

To prove that, we will need some inequalities given in the following lemmata.

**Lemma 3.** We assume that $(q_0, p_0)$, $(q_1, p_1) \in \mathcal{E}(u)$ are in the same chart of the atlas. Then

$$(p_1 - p_0)(q_1 - q_0) \leq 2K\|q_1 - q_0\|^2.$$

**Proof.** We also know that $p_j$ is a $K$-super-differential of $u$ at $q_j$, for $j = 0, 1$. Hence

$$u(q_1) - u(q_0) \leq p_0(q_1 - q_0) + K\|q_1 - q_0\|^2;$$

$$u(q_0) - u(q_1) \leq p_1(q_0 - q_1) + K\|q_1 - q_0\|^2.$$

By adding up these two inequalities, we deduce the lemma.

Before stating the following lemma, let us give a definition.

**Definition.** A subset $K$ of $T^*M$ is convex in the fibre if for every $q \in M$, the intersection $K \cap T^*_qM$ is a convex subset of the linear space $T^*_qM$.

**Lemma 4.** Let $K$ be a compact subset of $T^*M$ that is convex in the fibre and let $c, C$ be two constants such that

$$\forall \tau \in [-1, 1], \forall x \in K, \forall v \in \mathbb{R}^d, c\|v\|^2 \leq H_{p,p}(\varphi_{\tau}(x))(v, v) \leq C\|v\|^2.$$

Then there exists $\varepsilon > 0$ such that, for every $t \in [0, \varepsilon]$ and every $(q, p), (q, p + \Delta p) \in K$, if we use the notations: $(q_0, p_0) = \varphi_t(q, p)$ and $(q_1, p_1) = \varphi_t(q, p + \Delta p)$, we have

$$(p_1 - p_0)(q_1 - q_0) \geq \frac{c}{2}t\|\Delta p\|^2 \quad \text{and} \quad \|q_1 - q_0\| \leq 2Ct\|\Delta p\|.$$

**Proof.** Because $K$ is compact, if we choose $\varepsilon > 0$ small enough, then $q_0$, $q_1$ and $\pi \circ \varphi_t(q_0, p_0) \in [-\varepsilon, 0]$ are in the same chart of the atlas $\mathcal{A}$.

Then from now we work in the coordinates given by such a chart, i.e. in $\mathbb{R}^d$ and $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ and we write $(q_1, p_1) = \varphi_t(q, p + \Delta p)$ with $t \in [0, \varepsilon]$.

As $K$ is compact and $t \in [0, 1]$, there exists a constant $R > 0$ such that, necessarily: $\|\Delta p\| \leq R$ (for the usual Euclidian norm in $\mathbb{R}^d$).

We compute (let us note that for every $s \in [0, 1]$, we have $(q, p + s\Delta p) \in K$ because $K$ is convex in the fibres)

$$\varphi_t(q, p + \Delta p) - \varphi_t(q, p) = \int_0^1 D\varphi_t(q, p + s\Delta p)(0, \Delta p)\, ds.$$

Using the linearized Hamilton equations, we obtain that the quantity in the integral is equal to

$$t(H_{\pi, p}(\varphi_t(q, p + s\Delta p)\Delta p + \|\Delta p\|\varepsilon_1(s, t, \Delta p)), \Delta p + \|\Delta p\|\varepsilon_2(s, t, \Delta p))$$

where the functions $\varepsilon_j$ tend uniformly to 0 when $t$ tends to 0. We deduce

$$(p_1 - p_0)(q_1 - q_0) = t\int_0^1 [H_{\pi, p}(\varphi_t((q, p + s\Delta p)(\Delta p, \Delta p) + t\|\Delta p\|^2\varepsilon(s, t, \Delta p))]\, ds$$




where the function $\eta$ tends uniformly to 0 when $t$ tends to 0. Hence if $\varepsilon$ has been chosen small enough, we have

$$(p_1 - p_0)(q_1 - q_0) \geq \frac{c}{2} t \|\Delta p\|^2 \quad \text{and} \quad \|q_1 - q_0\| \leq 2Ct \|\Delta p\|.$$ \hfill \Box$$

If $x = (q, p) \in T^*M$, we denote by $\mathcal{V}(x)$ its vertical: $\mathcal{V}(x) = T_y^*M = \{y \in T^*M; \pi(y) = \pi(x) = q\}$ where $\pi : T^*M \rightarrow M$ designates the usual projection.

Then we want to know if it is possible for a $t \in [0, \varepsilon]$ and a $x \in \mathcal{E}(u)$ that $\mathcal{V}(\varphi_-(x)) \cap \varphi_-(\mathcal{E}(u))$ contains at least two points. It means that there exists two different points $(q_0, p_0), (q_1, p_1) \in \mathcal{E}(u)$ such that $(q_1, p_1) \in \varphi_t(\mathcal{V}(\varphi_-(q_0, p_0)))$. We use the notation: $(q, p) = \varphi_-(q_0, p_0)$ and $(q, p + \Delta p) = \varphi_-(q_1, p_1)$.

As $\mathcal{E}(u)$ is compact subset of $T^*M$ that is compact in the fibre, we can use lemma 4 to choose $\varepsilon > 0$. Then we have

$$(p_1 - p_0)(q_1 - q_0) \geq \frac{c}{2} t \|\Delta p\|^2 \quad \text{and} \quad \|q_1 - q_0\| \leq 2Ct \|\Delta p\|.$$ Now lemma 3 tells us that $(p_1 - p_0)(q_1 - q_0) \leq 2K \|q_1 - q_0\|^2$. Then

$$(p_1 - p_0)(q_1 - q_0) \leq 2K \|q_1 - q_0\|^2 \leq 8C^2Kt^2 \|\Delta p\|^2.$$ Finally, we have proved that there exist two strictly positive constants $c$ and $C$ such that

$$\frac{c}{2} t \|\Delta p\|^2 \leq (p_1 - p_0)(q_1 - q_0) \leq 8C^2Kt^2 \|\Delta p\|^2.$$ It is obviously impossible for $t > 0$ small enough and $\Delta p \neq 0$.

2.2. **Proof that** $\pi \circ \varphi_t(\mathcal{E}(u)) = M$

We want to prove that for $t \in [-\varepsilon, 0]$, the graph $\varphi_t(\mathcal{E}(u))$ covers the whole $M$.

We have recalled in point 4 of the introduction that for all $t > 0$, we have $\mathcal{G}(\mathcal{T}_t, u) \subset \varphi_-(\mathcal{E}(u))$; this applies directly that $\pi \circ \varphi_t(\mathcal{E}(u)) = M$.

2.3. **Proof that** $\varphi_t(\mathcal{E}(u))$ is a Lipschitz graph

We have proved that for $t \in [0, \varepsilon_0]$, $\varphi_-(\mathcal{E}(u))$ is a graph above $M$. Because this graph is compact ($\mathcal{E}(u)$ is compact), it is the graph of a continuous section $s_t : M \rightarrow T^*M$.

We have to prove that $s_t$ is Lipschitz. We may eventually change $\varepsilon_0$ in such a way that $K < \frac{1}{C_\varepsilon}$.

We will use the so-called Bouligand’s paratingent cone (see [4]):

**Definition.** Let $E$ be a subset of $T^*M$. The paratangent cone to $E$ at $(q, p) \in E$ is defined (in chart but it does not depend on the chart) as the subset of $T_{(q, p)}(T^*M)$ whose elements are the limits of the sequences:

$$(\frac{1}{t_n}(q_n - q'), \frac{1}{t_n}(p_n - p'))_{n \in \mathbb{N}} \text{with } t_n \in \mathbb{R}^+_*, q_n, q', p_n, p' \in E \text{ and } \lim q_n = \lim q' = q, \lim p_n = \lim p' = p. \text{ It is denoted by } C_{(q, p)}E.$$ If $(q, p), (q', p') \in \mathcal{E}(u)$ are in a same chart, we have proved in lemma 3 that $(p' - p)(q' - q) \leq 2K \|q' - q\|^2$. We deduce that for all $(\delta q, \delta p) \in C_{(q, p)}\mathcal{E}(u)$, we have $\delta p. \delta q \leq 2K \|\delta q\|^2$.

Moreover, we deduce easily from lemma 4 that if $R > 0$, there exists $\varepsilon > 0$ such that for every $(q, p)$, $(q, p + \Delta p) \in T^*M$ that satisfy $\|p\| \leq R$ and $\|p + \Delta p\| \leq R$, we have if we use the notations $\varphi_t(q, p) = (q_0, p_0)$ and $\varphi_t(q, p + \Delta p) = (q_1, p_1)$ for a $t \in [0, \varepsilon]$:

$$(p_1 - p_0)(q_1 - q_0) \geq \frac{c}{2} t \|\Delta p\|^2 \quad \text{and} \quad \|q_1 - q_0\| \leq 2Ct \|\Delta p\|.$$
Looking at what happens when $\Delta p$ tends to 0, we deduce that for every $(q, p) \in \varphi_-(\mathcal{E}(u))$, for every $\delta p_0 \in T_{(q, p)}(T^*_q M)$, if we use the notation $D\varphi_t(q, p)(0, \delta p_0) = (\delta q, \delta p)$, then we have

$$\delta p \cdot \delta q \geq \frac{C}{2} \|\delta p_0\|^2 \quad \text{and} \quad \|\delta q\| \leq 2Ct \|\delta p_0\|.$$

As a consequence: $\|\delta q\|^2 \leq 8\frac{C}{t}\|\delta p\|\cdot \|\delta q\|$.

Finally, we have proved for $(q, p) \in \mathcal{E}(u)$ that

- for all $(\delta q, \delta p) \in C_{(q, p)} \mathcal{E}(u)$, we have $\delta p \cdot \delta q \leq 2K \|\delta q\|^2$;
- for all $(\delta q, \delta p) \in T^*_{(q, p)} M$ that is in the image by $D\varphi_t$ of the vertical $V(\varphi_-(q, p)) = \ker D\pi(\varphi_-(q, p))$, we have $\|\delta q\|^2 \leq 8\frac{C}{t}\|\delta p\|\cdot \|\delta q\|$.

If we choose $\varepsilon < \frac{K}{8C}$, we obtain that $D\varphi_t V(\varphi_-(q, p)) \cap C_{(q, p)} \mathcal{E}(u) = \{0\}$, and then that $V(\varphi_-(q, p)) \cap D\varphi_t (C_{(q, p)} \mathcal{E}(u)) = \{0\}$.

Finally, we have proved that the paratingent cone to $\varphi_-(\mathcal{E}(u))$, which is the graph of $s_t$, contains no vertical line. Let us deduce that $s_t$ is Lipschitz. We assume that there are two sequences of points $(q_n, p_n), (q'_n, p'_n)$ of $\varphi_-(\mathcal{E}(u))$ such that $\lim_{n \to \infty} \frac{p'_n - p_n}{q'_n - q_n} = +\infty$. Using a subsequence, because $\varphi_-(\mathcal{E}(u))$ is compact, we may assume that the two sequences converge. Then necessarily $(q_n)$ and $(q'_n)$ have the same limit (because the previous limit is $+\infty$ and $\|p'_n - p_n\|$ is bounded). Hence by continuity of $s_t$, $(p_n)$ and $(p'_n)$ too have the same limit. But if we write $t_n = \|p_n - p'_n\|$ and if we use a subsequence in such a way that $(\frac{p'_n - p_n}{q'_n - q_n})$ converges to a $u$, we obtain that $\lim_{n \to \infty} \frac{1}{t_n} (q'_n - q_n, p'_n - p_n) = (0, u)$ is in the paratingent cone to $\varphi_-(\mathcal{E}(u))$ at $(q, p)$, it contradicts the fact that this paratingent cone contains no vertical line. Hence $s_t$ is Lipschitz.

2.4. Proof that $\varphi_-(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph

We have to prove that there exists a $C^1$ function (hence it will be $C^{1,1}$) $u_t : M \to \mathbb{R}$ such that $s_t = du_t$. It is enough to prove that for any closed Lipschitz arc $\gamma : [a, b] \to M$, then $\int_a^b s_t(\gamma(t))\dot{\gamma}(t) \, dt = 0$. Let us define a closed loop of $T^* M$ by $\forall t \in [a, b], \eta(t) = (\eta_1(t), \eta_2(t)) = (\eta_1(t), \eta_2(\gamma(t)))$. The arc $\gamma$ being Lipschitz and $s_t$ being Lipschitz, the arc $\eta$ is Lipschitz too. Hence we can define $\int_{\eta} \lambda$ where $\lambda$ is the Liouville 1-form. The flow being exact symplectic, we have $\int_{\eta} \lambda = \int_a^b s_t(\gamma(t))\dot{\gamma}(t) \, dt$. We are reduced to compute $\int_{\eta} \lambda = \int_a^b \eta_2(t) \dot{\eta}_1(t) \, dt$.

Let us recall that $\eta$ is a closed Lipschitz arc drawn on $\mathcal{E}(u)$; then $\eta_2(t)$ is a $K$-superdifferential of $u$ at $\eta_1(t)$ and

$$u(\eta_1(t) + \delta \tau) - u(\eta_1(t)) \leq u(\eta_2(t)(\eta_1(t) + \delta \tau) - \eta_1(t)) + K \|\eta_1(t) + \delta \tau - \eta_1(t)\|^2.$$

Moreover, $u, \eta_1$ and $\eta_2$ are Lipschitz, then (Lebesgue) almost everywhere derivable. If $\tau$ is a point where $u \circ \eta_1$ and $\eta_1$ are derivable, we obtain by dividing by $\delta \tau$ (positive or negative) and taking the limit when $\delta \tau$ tends to 0:

$$\frac{d}{dt}(u \circ \eta_1)(\tau) = \eta_2(t) \dot{\eta}_1(t) \quad \text{and by integration}
\int_a^b \eta_2(t) \dot{\eta}_1(t) \, dt = \int_a^b \frac{d}{dt}(u \circ \eta_1)(\tau) \, dt = u(\eta_1(b)) - u(\eta_1(a)) = 0.$$
2.5. Proof that $\mathcal{G}(d\tilde{T}_t u) = \varphi_{-t}(E(u))$

Even if we will not use it, let us recall the definition of the two Lax–Oleink semi-groups for completeness.

**Definition.** We recall the definition of the semi-groups $(T_t)_{t>0}$ and $(\tilde{T}_t)_{t>0}$:

$$T_t u(q) = \min_{q' \in M} (u(q') + A_t(q', q)) \quad \text{and} \quad \tilde{T}_t u(q) = \max_{q' \in M} (u(q') - A_t(q, q'))$$

where $A_t(q, q') = \min_{\gamma} \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds$.

We have proved that $\varphi_{-t}(E(u))$ is an exact Lagrangian Lipschitz graph for every $t \in [-\epsilon, 0]$; we write $\varphi_{-t}(E(u)) = \mathcal{G}(u_t)$ with $u_t$ that is $C^{1,1}$. Moreover, we have noted in the introduction that $\mathcal{G}(\tilde{T}_t u) \subset \varphi_{-t}(E(u)) = \mathcal{G}(u_t)$; as $\tilde{T}_t u$ is Lipschitz, we deduce that for Lebesgue almost every $q \in M$, we have $d u_t = d \tilde{T}_t u$. The derivatives of the two Lipschitz functions $u_t$ and $\tilde{T}_t u$ are almost everywhere equal, then $\tilde{T}_t u - u_t$ is a constant function and then $\tilde{T}_t u$ is $C^{1,1}$ and we have the equality: $\mathcal{G}(d\tilde{T}_t u) = \varphi_{-t}(E(u))$.

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