Minimum Fill-In: Inapproximability and Almost Tight Lower Bounds*

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Abstract

Given an $n \times n$ sparse symmetric matrix with $m$ nonzero entries, performing Gaussian elimination may turn some zeroes into nonzero values. To maintain the matrix sparse, we would like to minimize the number $k$ of these changes, hence called the minimum fill-in problem. Agrawal, Klein, and Ravi [FOCS 1990; Graph Theory & Sparse Matrix Comp. 1993] developed the first approximation algorithm for the problem, based on early work on heuristics by George [SIAM J. Numer. Anal. 10(1973)] and by Lipton, Rose, and Tarjan [SIAM J. Numer. Anal. 16(1979)]. The objective function they used is $m + k$, the number of nonzero elements in the matrix after elimination. An approximation algorithm using $k$ as the objective function was presented by Natanzon, Shamir, and Sharan [STOC 1998; SIAM J. Comput. 30(2000)]. These two versions are incomparable to each other in terms of approximation.

Parameterized algorithms for the problem was first studied by Kaplan, Shamir, and Tarjan [FOCS 1994; SIAM J. Comput. 28(1999)]. Fomin and Villanger [SODA 2012; SIAM J. Comput. 42(2013)] recently gave an algorithm running in time $2^{O(\sqrt{k \log k})} + n^{O(1)}$.

Hardness results of this problem are surprisingly scarce, and the few known ones either are weak or have to use nonstandard complexity conjectures. The only inapproximability result by Wu et al. [IJCAI’15; J. Artif. Intell. Res. 49(2014)] applies to only the objective function $m + k$, and is grounded on the Small Set Expansion Conjecture. The only nontrivial parameterized lower bounds, by Bliznets et al. [SODA 2016], include a very weak one based on the Exponential Time Hypothesis (ETH), and a strong one based on hardness of subexponential-time approximation of the minimum bisection problem on regular graphs. For both versions of the problem, we exclude the existence of polynomial time approximation schemes, assuming $P \neq NP$, and the existence of $2^{O(n^{1-\delta})}$-time approximation schemes for any positive $\delta$, assuming ETH. It also implies a $2^{O(k^{1/2-\delta})} \cdot n^{O(1)}$ parameterized lower bound. Behind these results is a new reduction from vertex cover, which might be of its own interest: All previous reductions for similar problems are from some kind of graph layout problems.

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1 Introduction

The minimum fill-in problem arises from the application of Gaussian elimination to a sparse matrix, where we want to minimize the number of zero entries that are turned into nonzero values, called the fill-in, during the elimination process. As usual, here we ignore the accidental transformation from a nonzero value to zero. Natanzon et al. [29] gave a graph-theoretic interpretation of the minimum fill-in problem on symmetric matrices. A graph $G$ can be easily extracted from an $n \times n$ symmetric matrix $M$ as follows: introduce $n$ vertices, each for a row, and an edge between the $i$th vertex and the $j$th vertex if and only if $i \neq j$ and $M_{ij} \neq 0$. The minimum fill-in of $M$ is exactly the minimum number of edges we need to add to $G$ to make it chordal, i.e., free of induced cycles of length at least four.

This correlation turns out to be crucial in the sense that almost all known algorithmic and hardness results to this problem use the graph formulation. One early heuristic approach is to choose a vertex with the minimum degree and add edges to make its neighborhood a clique [16], behind which is the observation that every chordal graph has a vertex whose neighborhood form a clique. Using another fact that every minimal separator of a chordal graph is a clique, George [15, 16] proposed the nested dissection heuristic, which recursively finds a balanced separator and add edges to make it a clique. Its performance relies thus on how good the separators we can find; e.g., combined with the Lipton-Tarjan planar separator theorem [26], it immediately leads to approximation algorithms for planar and bounded-genus graphs [25, 17]. Using the approximation algorithm of Leighton and Rao [24] for finding balanced separators, Agrawal et al. [2] developed the first algorithm with a nontrivial performance guarantee for the minimum fill-in problem on general graphs.

The minimum fill-in problem, minimizing the number of added edges (the number of zeroes turned into nonzero values), can also be formulated as minimizing the number of edges in the resulting chordal supergraph (the number of nonzero elements in the matrix after elimination). When discussing approximation algorithms, we need to specify the objective functions: They may behave quite different from the point of view of approximation. For convenience, we use the name minimum fill-in when the objective function is the size of the fill-in, and chordal completion otherwise.

Let $G$ be a graph on $n$ vertices and $m$ edges, and let $\phi(G)$ denote the size of minimum fill-ins of $G$. The algorithm of Agrawal et al. [2, 1] always produces a chordal supergraph of at most $O((m + \phi(G))^{0.75}\sqrt{m} \log^{3.5} n)$ edges, thereby having a ratio $O(\sqrt{m} \log^{3.5} n) = O(\sqrt{n} \log^{3.5} n)$ for the chordal completion problem. The first (and only) approximation algorithm for minimum fill-in was reported by Natanzon et al. [29], which has a ratio $8\phi(G)$, i.e., it always finds a fill-in of size at most $8\phi^2(G)$. We remark that these two results are incomparable in general. They also provided algorithms with better approximation ratios on graphs of degrees at most $d$, $O(\sqrt{d} \log^4 n)$ for chordal completion [2] and $O(d^{2.5} \log^3(n\delta))$ [29] for minimum fill-in.

Thus far there are no constant-ratio approximation algorithms known for either of them. Even more embarrassing might be the progress on hardness results. We could not even exclude polynomial time approximation schemes for the minimum fill-in problem. The only known inapproximability result for chordal completion was given by Wu et al. [36], who excluded constant-ratio approximation on the assumption of the Small Set Expansion Conjecture, which is related to the Unique Games Conjecture but less established [31]. We give the first inapproximability result for both problems on the assumption $P \neq NP$.

**Theorem 1.1.** If either of the minimum fill-in and the chordal completion problems has a polynomial time approximation scheme, then $P = NP$.

We actually show a stronger result, which however needs a stronger complexity assumption, namely the Exponential Time Hypothesis, which states that the satisfiability problem with at most 3 variables per clause (3SAT) cannot be solved in $2^{o(m + n)}$ time, where $m$ and $n$ denote the number of clauses and variables in the Boolean formula [18, 19]. The Exponential Time Hypothesis (ETH) is the standard working hypothesis of fine-grained complexity, which aims to understand the exact time complexity of problems and to prove lower bounds.

**Theorem 1.2.** Assuming ETH, there is some positive $c$ such that no algorithm can find a $(1 + c)$ approximation for the minimum fill-in problem or the chordal completion problem in time $2^{O(n^{1-\delta})}$, for any positive constant $\delta$.

This makes also a significant contribution to an important problem in parameterized computation, i.e., the parameterized lower bound of minimum fill-in. Recall that given a graph $G$, and an integer parameter $k$, the

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1For the reader unfamiliar with this line of research, we refer to [27] and [10, Chapter 14] for reference.
parameterized minimum fill-in problem asks whether there is a fill-in of size at most \(k\). (Note that it does not make much sense to use \(m + k\) as the parameter.) Kaplan et al. [20] designed the first parameterized algorithm for the minimum fill-in problem, which runs in time \(O(16^k k^6 + k^2 n^m)\), and proposed an \(O(k^3)\)-vertex kernel. Natanzon et al. [29] managed to improve it to \(O(k^2)\), which played a crucial role in their approximation algorithm mentioned above. Fomin and Villanger [12] developed a parameterized algorithm running in time \(2^{O(\sqrt{k}\log k)} + O(k^2 n^m)\), thereby placing this problem in the class of very few problems that admit subexponential-time parameterized algorithms on general graphs.

Note that the problem is trivial when \(k > n^2/2\); otherwise \(k^{1/2} = O(n)\). Thus, Theorem 1.2 immediately implies an almost tight lower bound on parameterized algorithms for this problem.

**Theorem 1.3.** Assuming ETH, there is no algorithm that can solve the minimum fill-in problem in time \(2^{O(k^{1/2-k})} \cdot n^{O(1)}\), for any positive constant \(\delta\).

Let us put Theorem 1.3 into context. Again, compare to the algorithmic progress, the hardness results of parameterized algorithms lay far behind. The only nontrivial lower bounds were given by Bliznets et al. [6] very recently. To relate their results, however, we need to start from 1970s. The complexity of the minimum fill-in problem was among the open problems of Garey and Johnson [13], and settled by Yannakakis [37] with a simple reduction from the optimal linear arrangement problem. His reduction, however, is not very much helpful for deriving inapproximability and other hardness results we want for the problem. For example, there is no inapproximability result for the optimal linear arrangement problem on the assumption \(P \neq NP\); to exclude polynomial time approximation schemes for it, Ambühl [4] had to use the assumption that NP-complete problems cannot be solved in randomized subexponential time.

On the other hand, to derive lower bounds on exact or parameterized algorithms, we need to prevent the graph or the parameter of the reduced instance from increasing too much with the reduction. Therefore, if we want to reuse the reduction of Yannakakis, we need to trace the whole sequence of reductions from 3SAT, which, if we spell out, has five steps, namely, max-2SAT, maximum cut, optimal linear arrangement, and chain completion, before eventually minimum fill-in. As said, the last two reductions are by Yannakakis [37]; while the first three reductions are due to Garey et al. [14]. For the prospect of deriving tight lower bounds on the minimum fill-in problem from Yannakakis’ reduction, the main obstacles lie in step 3, from maximum cut to optimal linear arrangement, and step 4, from optimal linear arrangement to chain completion. In [14], the original version of the reduction for step 3 blows up an \(n\)-vertex graph to an \(n^4\)-vertex graph. The selection of 4 in the exponent turns out to be for the convenience of presentation, and any constant larger than 3 would suffice. Step 4 then blows up the graph size by another quadratic factor. Therefore, as already mentioned in [12, 6] (without explanation), assuming ETH, one can only derive lower bounds of \(2^{O(\sqrt{n})}\) and \(2^{O(\sqrt{n}/\log^c n)}\) from these reductions in their original form [14, 37].

Bliznets et al. [6] sedulously retraced the tortuous (and torturous) five-step reduction, and managed to decrease the size of the reduced instance by a stronger complexity conjecture and heavier constructions. First, they managed to improve step 3 such that it produces a graph of almost linear size, which allows them to derive improved but still weaker lower bounds of \(2^{O(\sqrt{n}/\log^c n)}\) and \(2^{O(\sqrt{n}/\log^c n)} \cdot n^{O(1)}\) for some constant \(c\). Then, to avoid the quadratic explosion in step 4, they had to introduce a new conjecture on the subexponential-time approximation hardness of the minimum bisection problem on \(d\)-regular graphs.

Instead of further improving these reductions or finding alternatives, we start from scratch and devise a completely new reduction, which turns out to be surprisingly simple. Our reduction is from the vertex cover problem, whose NP-hardness is derived directly from SAT [21]. The following theorem summarizes a simple version of our reduction, which serves an alternative, and far simpler, proof for the NP-hardness of the minimum fill-in problem.

**Theorem 1.4.** Given an \(n\)-vertex graph \(G\), we can construct in polynomial time another graph \(H\) on \(n^3 + n\) vertices such that \(G\) has a vertex cover of size \(c\) if and only if \(H\) has a fill-in of size at most \((c + 1)n^2 - 1\).

The rest of the paper is organized as follows. Section 2 presents the reduction summarized by Theorem 1.4. Section 3 improves it to a linear reduction that proves Theorem 1.1–1.3: It uses instead a graph of bounded degree, and a gap version of vertex cover. Section 4 contrasts our reduction to that of Yannakakis, and proposes some possible remedy to make our reduction work for the related interval completion problem.
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no difference between minimizing |E_+|, i.e., the number of added edges, and |E(G)| + |E_+|, i.e., the number of

edges in the resulting graph. They behave, however, quite different with respect to approximation; consider, for

example, a dense graph (|E(G)| = O(n^2)) . In approximation algorithms for minimum fill-in, the ratio is defined
to be |E_+|/φ(G), while for chordal completion it is (|E(G)| + |E_+|)/(|E(G)| + φ(G)).

We give here the reduction announced in Theorem 1.4.

Reduction 1. Let G be a graph on n vertices. For each vertex v of G, introduce a set of n^2 new vertices, and add
edges to make them adjacent to all other vertices of G but v itself. Let U denote the set of n^3 new vertices. Add all
possible edges to connect U into a clique.

An example of Reduction 1 is illustrated in Figure 1. Let H be the graph obtained from G by Reduction 1. It has n + n \cdot n^2 = O(n^3) vertices. Apart from the edges from E(G), the graph H contains n^3(n^3 - 1)/2 edges among U, and n \cdot (n^3 - n^2) edges between V(G) and U, hence E(H) = O(n^6). For any given vertex cover C of G, the set V(G) \ C is an independent set of G; since we do not add edges in V(G) during the reduction, V(G) \ C is also an independent set of H. Therefore, if we add all edges among C \ U to make it a clique, then we end with a
split graph. Recall that a split graph is chordal; we have thus constructed a fill-in of H from a vertex cover of G.

We now consider the other direction, i.e., how to extract a vertex cover of G from a fill-in of H. We say that a
vertex v of G is full with respect to a fill-in E_+ if E_+ contains all the missing edges between v and U, and we
simply say it is full when the fill-in E_+ is clear from the context. The following simple observation is crucial for
the whole paper.

Figure 1: Illustration for Reduction 1. The original graph G, with black vertices and black edges, is inside the
dashed ellipse. Only two sets of the new vertices (blue squares) are shown in the right of the figure; they are
 corresponding to u, v \in V(G) respectively. The n^2 vertices for u are connected to v by blue edges, and they are
nonadjacent to u, indicated by red dashed lines; likewise for the n^2 new vertices for v. Edges between them and
other vertices of G are omitted for clarity. The whole graph is H.
Proposition 2.1. Let $H$ be the graph obtained from $G$ by Reduction 1. For any fill-in $E_+$ of $H$, the set $C$ of vertices that are full with respect to $E_+$ is a vertex cover of $G$.

Proof. Let $\hat{H} = H + E_+$ and let $uv$ be any edge of $G$. We argue that at least one of $u$ and $v$ is full with respect to $E_+$. Suppose otherwise, then we can find $u', v' \in U$ such that $uu'$ and $vv'$ are non-edges of $\hat{H}$; see Figure 1. By the construction, $u'$ and $v'$ are distinct and $uu', u'v', v'u' \in E(H) \subseteq E(\hat{H})$, and then $uu'v'$ is a $C_4$ of $\hat{H}$, contradicting that $\hat{H}$ is chordal. Therefore, between the two vertices of each edge of $G$, at least one of them is full with respect to $E_+$, and the proposition follows. \hfill ∎

We are now ready to present our main result of this section, which is an easy consequence of the aforementioned two-way constructions.

Lemma 2.2. Let $G$ be an $n$-vertex graph, and let $H$ be the graph obtained from $G$ by Reduction 1. Then $\tau(G)n^2 \leq \varphi(H) < (\tau(G) + 1)n^2$.

Proof. The lower bound follows directly from Proposition 2.1. For the upper bound, we construct a fill-in for $H$ as follows. Let $C$ be any minimum vertex cover of $G$. We add all edges among $C \cup U$ to make $G$ a split graph. These include $|C|n^2$ edges between $C$ and $U$ and $(\binom{|C|}{2} - |E(G[C])|)$ non-edges in $G[C]$. Thus,

$\varphi(H) \leq |C|n^2 + \left(\binom{|C|}{2} - |E(G[C])|\right) \leq |C|n^2 + \left(\binom{|C|}{2}\right) = \tau(G)n^2 + \left(\frac{\tau(G)}{2}\right) < \tau(G)n^2 + \left(\frac{n}{2}\right) < (\tau(G) + 1)n^2$.

This concludes the proof. \hfill ∎

To see why Lemma 2.2 implies Theorem 1.4, note that if $\tau(G) \leq c$, then $\varphi(H) \leq (\tau(G) + 1)n^2 - 1 \leq (c + 1)n^2 - 1$; otherwise, $\varphi(H) \geq \tau(G)n^2 > (c + 1)n^2 > (c + 1)n^2 - 1$.

3 The hardness results

A graph is $d$-degree-bounded if every vertex has degree at most $d$. Any $d$-degree-bounded graph can be trivially colored with $d + 1$ colors. According to Brooks’ theorem [8], if $d \geq 3$ and the graph does not contain a clique on $d + 1$ vertices, then it can be colored by $d$ colors, and such a coloring can be found in linear time. Our second reduction starts from such a colored $d$-degree-bounded graph. We introduce a set of vertices for each color class instead of each vertex, and the size of each set is $bn$ for some constant $b$ to be specified later.

Reduction 2. Let $G$ be an $n$-vertex $d$-degree-bounded graph that does not contain a clique on $d + 1$ vertices. We find a proper coloring of $G$ with $d$ colors. For each color, introduce a set of $bn$ new vertices, and add edges to make them adjacent to all vertices of $G$ that receive a different color. Let $U$ denote all the $bdn$ new vertices; add all possible edges to connect $U$ into a clique.

An example of Reduction 2 is illustrated in Figure 2. The produced graph $H$ has $(bd + 1)n = O(n)$ vertices. For any vertex cover $C$ of $G$, adding all edges among $C \cup U$ to $H$ makes it a split graph. Thus,

$\varphi(H) \leq \tau(G) \cdot bn + \left(\frac{\tau(G)}{2}\right) < bn\tau(G) + \frac{1}{2}\tau^2(G)$.

Although we use a constant number of sets, the following facts from Reduction 1 remain true: (1) each vertex of $G$ is nonadjacent to one set of new vertices; and (2) for each edge of $G$, its two vertices are nonadjacent to two different sets of new vertices. The second fact is ensured by the proper coloring. We define the full vertices in exactly the same way as before: a vertex $v$ of $G$ is full with respect to a fill-in $E_+$ if $E_+$ contains all the missing edges between $v$ and $U$. It is easy to verify that Proposition 2.1 remains true for the new reduction: Actually, a word-by-word copy of the proof works.

Proposition 3.1. Let $G$ be an $n$-vertex $d$-degree-bounded graph that does not contain a clique on $d + 1$ vertices, and let $H$ be the graph obtained from $G$ by Reduction 2. For any fill-in $E_+$ of $H$, the set of vertices that are full with respect to $E_+$ is a vertex cover of $G$. 


Before proving the main theorems of this paper, let us recall some simple facts on minimum vertex covers of a d-degree-bounded graph $G$ with $d \geq 3$. Trivially, $\tau(G) < |V(G)|$, while an easy degree counting tells us that $\tau(G) \geq |V(G)|/(d+1)$. If $G$ does contain a clique on $d+1$ vertices, this clique is necessarily a component of $G$. For the vertex cover problem, such a clique component would not concern us: We take $d$ vertices from it, which is optimal. For approximation, if we can get approximation ratio $\alpha$ for the rest of the graph with all such components removed, then we can have a ratio for the original graph no worse than $\alpha$. In other words, any approximation lower bounds of the vertex cover problem on $d$-degree-bounded graphs hold for $d$-degree-bounded graphs with no $(d+1)$-cliques.

**Lemma 3.2.** Let $\epsilon$ be a positive constant and let $d \geq 3$ be an integer. If there is an $f(N, \epsilon)$-time approximation algorithm with ratio $1 + \epsilon/3$ for the minimum fill-in problem on $N$-vertex graphs, then there is an $O(f(cN, \epsilon) + N^2)$-time approximation algorithm with ratio $1 + \epsilon$ for the minimum vertex cover problem on $N$-vertex $d$-degree-bounded graphs, where $c$ is a constant depending on only $d$ and $\epsilon$.

**Proof.** Let $\alpha = 1 + \epsilon/3$. We use the $\alpha$-approximation algorithm for the minimum fill-in problem to construct a $(1 + \epsilon)$-approximation algorithm for the vertex cover problem on $d$-degree-bounded graphs as follows. Since it is easy to find a 2-approximation for the vertex cover problem, it suffices to consider $\epsilon < 1$.

Let $G$ be a $d$-degree-bounded graph on $n$ vertices; we may assume without loss of generality that $G$ contains no clique on $d+1$ vertices. We apply Reduction 2 to $G$ with $b := \lceil \epsilon^{-1} \rceil$, and let $H$ be the obtained graph. Then we use the $\alpha$-approximation algorithm to find a fill-in $E_+$ of $H$, with $|E_+| \leq \alpha \cdot \phi(H)$. Our algorithm for vertex cover simply returns the set $C$ of full vertices: By Proposition 3.1, it is a vertex cover of $G$.

We consider first the approximation ratio of this algorithm. Since $E_+$ contains all the missing edges between
C and U, it follows \(|E_+| \geq |C| \cdot bn\). On the other hand, \(|E_+| \leq \alpha \cdot \varphi(H)\). Combining them with (1), we have
\[
|C| \leq \frac{|E_+|}{bn} \leq \frac{\alpha \cdot \varphi(H)}{bn} < \frac{\alpha \cdot bn \tau(G) + 0.5 \tau^2(G)}{bn} = \alpha \tau(G) \left(1 + \frac{\tau(G)}{2bn}\right).
\]
By \(b = \lfloor \epsilon^{-1} \rfloor \geq \epsilon^{-1}\) and the fact \(\tau(G) < n\), we can conclude
\[
|C| \leq \alpha \left(1 + \frac{\tau(G)}{2bn}\right) \leq \alpha \left(1 + \frac{\epsilon}{2}\right) = \left(1 + \frac{\epsilon}{3}\right) \left(1 + \frac{\epsilon}{2}\right),
\]
which is smaller than \(1 + \epsilon\) for any \(\epsilon < 1\).

We now calculate the running time of this algorithm. Note that \(|V(H)| = bd + n\). Let \(c = (\epsilon^{-1} + 1)d + 1 > bd + 1\). The construction takes \(O((cN)^2) = O(n^2)\) time; the approximation algorithm for \(H\) takes \(f(cn, \epsilon)\) time; and it takes another \(O(n^2)\) time to find and return the vertex cover \(C\). Thus, the total running time is \(O(f(cn, \epsilon) + n^2)\). This concludes the proof.

\begin{lemma}
Let \(c\) be a positive constant and let \(d \geq 3\) be an integer. If there is an \(f(N, \epsilon)\)-time approximation algorithm with ratio \(1 + \epsilon^2/(10d^2)\) for the chordal completion problem on \(N\)-vertex graphs, then there is an \(O(f(cN, \epsilon) + N^2)\)-time approximation algorithm with ratio \(1 + \epsilon\) for the minimum vertex cover problem on \(N\)-vertex \(d\)-degree-bounded graphs, where \(c\) is a constant depending on only \(d\) and \(\epsilon\).
\end{lemma}

\begin{proof}
Let \(\alpha = 1 + \epsilon^2/(10d^2)\). We use the \(\alpha\)-approximation algorithm for the minimum chordal completion problem to construct a \((1 + \epsilon)\)-approximation algorithm for the vertex cover problem on \(d\)-degree-bounded graphs as follows. Since it is easy to find a \(2\)-approximation for the vertex cover problem, it suffices to consider \(\epsilon < 1\).

Let \(G\) be a \(d\)-degree-bounded graph on \(n\) vertices; we may assume without loss of generality that \(G\) contains no clique on \(d + 1\) vertices. We apply Reduction 2 to \(G\) with \(b := \lfloor \epsilon^{-1} \rfloor\), and let \(H\) be the obtained graph. Then we use the \(\alpha\)-approximation algorithm to find a chordal completion of \(H\): let \(\tilde{H}\) be the obtained chordal supergraph. Recall that the minimum number of edges a chordal supergraph of \(H\) can have is \(|E(H)| + \varphi(H)\), thus \(|E(\tilde{H})| \leq \alpha(|E(H)| + \varphi(H))\). Denote by \(E_+\) the fill-in produced by this algorithm, i.e., \(E_+ = E(\tilde{H}) \setminus E(H)\). Our algorithm for vertex cover simply returns the set \(C\) of full vertices: By Proposition 3.1, it is a vertex cover of \(G\).

We consider first the approximation ratio of this algorithm. Apart from edges of \(G\), the constructed graph \(H\) contains \(\binom{|U|}{2}\) edges among \(U\) and \(n \cdot (|U| - bn)\) edges between \(V(G)\) and \(U\). Thus,
\[
|E(H)| = |E(G)| + \frac{bdn}{2} + n(bdn - bn)
\]
\[
\leq \frac{dn}{2} + \frac{b^2d^2n^2 - bdn}{2} + b(d - 1)n^2
\]
\[
= \frac{dn - bdn}{2} + \frac{b^2d^2n^2 + 2b(d - 1)n^2}{2}
\]
\[
< b^2d^2n^2,
\]
where the last inequality follows from that \(b \geq \epsilon^{-1} \geq 1\) and \(d \geq 3\). Since \(E_+\) contains all the missing edges between \(C\) and \(U\), it follows \(|E_+| \geq |C| \cdot bn\). Combining (1), we have
\[
|C| \leq \frac{|E_+|}{bn} = \frac{|E(\tilde{H})| - |E(H)|}{bn}
\]
\[
\leq \frac{\alpha(|E(H)| + \varphi(H)) - |E(H)|}{bn}
\]
\[
= \frac{[\alpha - 1]|E(H)| + \alpha \cdot \varphi(H)}{bn}
\]
\[
< \frac{(\alpha - 1)bd^2n^2 + \alpha(bn \tau(G) + \tau^2(G)/2)}{bn}
\]
\[
= (\alpha - 1)bd^2n + \alpha \tau(G) + \frac{\alpha \tau^2(G)}{2bn}
\]
\[
< (\alpha - 1)bd^2n + \alpha \tau(G) + \frac{\alpha \tau(G)}{2b},
\]
where the last inequality follows from that $\tau(G) < n$. And then

\[
\frac{|C|}{\tau(G)} < \frac{(\alpha - 1)bd^2n}{\tau(G)} + \alpha + \frac{\alpha}{2b} < 2(\alpha - 1)bd^3 + \frac{\alpha}{2b} \leq 2(\alpha - 1)bd^3 + \frac{\alpha e}{2} \leq 2(\alpha - 1)(1 + \frac{e}{2}) + 1 + \frac{e}{2} = (\alpha - 1)(2bd^3 + 1 + \frac{e}{2}) + 1 + \frac{e}{2} < (\alpha - 1)(4d^3 + 1 + \frac{e}{2}) + 1 + \frac{e}{2} < (\alpha - 1)\left(\frac{5d^3}{e} + 1 + \frac{e}{2}\right) \quad (b < \frac{2}{e})
\]

Together with the hardness results of the vertex cover problem on degree-bounded graphs, it is now quite straightforward to derive the main results announced in Section 1.

\textbf{Proof of Theorem 1.1.} Papadimitriou and Yannakakis [30] and Alimonti and Kann [3] showed that vertex cover is APX-hard on $d$-degree-bounded graphs for all $d \geq 3$. If there is a polynomial time approximation scheme for the minimum fill-in problem or the chordal completion problem, then according to Lemma 3.2 and Lemma 3.3 respectively, we can use it to derive a polynomial time approximation scheme for the vertex cover problem on 3-degree-bounded graphs, which implies $P = NP$. □

The following theorem has been essentially observed and used by Marx [28, Lemma 2.5].\footnote{His proof, which is omitted in the conference version, uses the reduction with bounded-degree expander graphs of Papadimitriou and Yannakakis [30] and the almost-linear size PCP of Dinur [11] (personal communication).} A simpler proof was given by Bonnet et al. [7, Proposition 3, Theorem 9].

\textbf{Lemma 3.4 ([28, 7])}. Assuming ETH, there exist an integer $d$ and a positive constant $\epsilon$ such that there exists no $(1 + \epsilon)$-approximation algorithm for vertex cover on $d$-degree-bounded graphs in time $2^{O(n^{1-\delta})}$ for any positive constant $\delta$.

\textbf{Proof of Theorem 1.2.} Suppose that there is a $2^{O(n^{1-\delta})}$-time algorithm for some positive constant $\delta < 1$ that can approximate the minimum fill-in problem within ratio $1 + \epsilon$ for any positive $\epsilon$. Then for any integer $d \geq 3$, we can use Lemma 3.2 to derive a $2^{O(n^{1-\delta})}$-time $(1 + 3\epsilon)$-approximation algorithm for the vertex cover problem on $d$-degree-bounded graphs. Since $\epsilon$ can be made arbitrarily small, together with Lemma 3.4, the algorithm refutes ETH. A similar argument works for the chordal completion problem. □

4 Concluding remarks

We have shown the first inapproximability result and almost tight lower bound for parameterized algorithms and exact algorithms for the minimum fill-in problem. We also get the first inapproximability result for the chordal completion problem under the assumption $P \neq NP$. All these results are consequences of our new reduction from the vertex cover problem to the minimum fill-in problem.
It is easy to verify that our reductions work for the completion problem to any graph class that forbids \( C_4 \)
and contains all split graphs, e.g., split graphs, \( C_4 \)-free graphs, and even-hole-free graphs (i.e., \( \{C_{2\ell} : \ell \geq 2\} \)-free graphs).\(^3\) On the one hand, Proposition 2.1 applies to all completion to graph classes that forbid \( C_4 \). On the other hand, the solution obtained by connecting \( C \cup U \) into a clique is a solution for any completion problem to a superclass of the class of split graphs. In other words, our reductions work for an objective graph class as long as the set \( \mathcal{F} \) of its forbidden induced subgraphs satisfies the following conditions: (1) \( C_4 \in \mathcal{F} \); and (2) any other \( F \in \mathcal{F} \) contains \( 2K_2 \) or \( C_5 \).

Yannakakis’ reduction is far more powerful in this sense. It directly applies to interval graphs, unit interval
graphs, and strongly chordal graph, while a slight modification works for trivially perfect graphs and threshold
graphs (see, e.g., Bliznets et al. [6] for details). Interval graphs are the intersection graphs of intervals on the
real line. The completion problem to interval graphs is another graph completion problem that finds important
application in sparse matrix computation. Tarjan [34] showed that it is equivalent to the profile minimization
problem. Many algorithms have been developed in literature, e.g., [32, 35]. Since all previous hardness results
use Yannakakis’ reduction or some variations, the status of hardness results on these problems are the same.

We propose here a possible remedy to make our reductions work for interval graphs. It needs a stronger
claim on coloring subcubic (3-degree-bounded) graphs containing no \( K_4 \)—namely, we want to color it in a
way that there is a maximum independent set \( X \) receiving only two colors. If we use such a colored graph in
Reduction 2, then to get the split graph, we can also add all edges between the independent set and \( V(G) \setminus X \),
which is minimum vertex cover of \( G \). The resulting graph would then be an interval graph. However, whether this
approach works relies on two questions that we have no answer: (1) is it true that every cubic graph containing
no \( K_4 \) admits a 3-coloring that uses at most two colors for some maximum independent set of this graph? and
(2) if the answer to the first question is yes, can such a coloring be found in polynomial time? Note that, however,
this cannot be generalized to \( d \)-degree-bounded graphs, because they would imply a 2-approximation algorithm
for the maximum independent set problem on \( d \)-degree-bounded graphs: by taking the vertices from a largest
color class.

The proof of Yannakakis is very influential for another reason. He built the correlation between linear
arrangement and minimum fill-in and related problems, including treewidth (recall that the treewidth of a graph
\( G \) can be defined as one less than the size of a maximum clique of a chordal supergraph of \( G \) with the smallest
clique number), interval completion, and pathwidth. Indeed, Yannakakis defined the chain graphs to capture
the ordering property in a graph (he has used chain graphs in early work [38] without giving a name). The
linear arrangement problem is a special case of the general family of graph layout problems, which ask for
an ordering of the vertices of a graph to minimize/maximize some measures. Similar idea had actually been
used by Kashiwabara and Fujisawa [22, 23]. All hitherto known reductions on related problems, including
[5, 6, 36], followed this idea, and used graph layout problems as the source problems. An important benefit of
this approach is that they can be (usually in an effortless way) applied to related problems on (proper) interval
graphs. However, these reductions usually explode the graphs too much, and their approximation hardness has
not been settled.

Another open problem is whether our reduction can be adapted to the treewidth problem. Similar as all
previous reductions, the graphs produced by our reductions are very dense. Therefore, none of them applies
to sparse graphs and planar graphs. Indeed, the complexity of treewidth on planar graphs is rather a famous
open problem. As a final remark, a planar graph has treewidth \( \Theta(\sqrt{n}) \) and a fill-in of size \( o(n^2) \) \([9]\),
while a the treewidth of a cubic graph can be \( \Theta(n) \) and its minimum fill-in can be \( \Theta(n^2) \) \([2]\).

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\(^3\)We omit the definitions of these graph classes that are nonessential for our results and the following discussion. The reader unfamiliar
with these notations is referred to http://www.graphclasses.org/ for details.
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