ON THE GRAVITATIONAL FIELD OF A POINT MASS IN EINSTEIN UNIVERSE BACKGROUND

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Abstract

Some properties of an exact solution due to Vaidya, describing the gravitational field produced by a point particle in the background of the static Einstein universe are examined. The maximal analytic extension and the nature of the singularities of the model are discussed. By using the Euclidean approach, some quantum aspects are analysed and the thermodynamics of this spacetime is also discussed.
1 Introduction

The standard Schwarzschild solution is described in flat background. Because a black hole is a cosmological object, it is worthwhile to examine the effect of the cosmological background on the black hole properties. We recall that by considering a non-zero cosmological constant we obtain the Schwarzschild-(anti)de Sitter solution with rather different properties. Also, we would like to take into account the fact that a black hole may also be surrounded by a local mass distribution.

A model taking into account the deviation from flatness on a large scale was proposed a long time ago by Einstein and Strauss [1]. In this model, the vacuum Schwarzschild field matches across a spherical comoving boundary to a pressure-free Friedmann-Robertson-Walker (FRW) universe. This would provide for example a description of the effect of the cosmic expansion on the gravitational field of the Sun.

Another way to deal with the embedding of massive objects in a FRW universe is to solve Einstein’s field equations exactly or approximately, in such a way that the resulting solution can be interpreted as an embedding of some massive object in the considered background. In 1933, McVittie found solutions of Einstein’s field equations for a perfect fluid energy-tensor, representing a Schwarzschild field embedded in the FRW spacetime [3] (see also ref. [4,5,6,7] for an up-to-date discussion of the Mc-Vittie solutions’ properties).

A rather different approach has been considered by Vaidya, who derived a perfect fluid solution, interpreted as the Kerr metric in the cosmological background of the Einstein space [5,6].

When specializing the Vaidya solution for a vanishing angular momentum, we obtain the simpler case of a Schwarzschild metric in the Einstein space background.

It is the purpose of this paper to study some properties of this spacetime, in an attempt to address the fundamental question of how the cosmological background will affect the black hole properties.

The paper is structured as follows: in section 2 we outline the basic features of the Vaidya solution, while in section 3 we analyse the global properties of this spacetime and the issue of singularities of the model. Based on these results, some thermodynamic properties are discussed in section 4. We conclude with section 5 where the results are compiled.

2 Einstein equations and matter content

In the particular case of zero angular momentum, the metric proposed by Vaidya reduces to the form

$$ds^2 = \frac{dt^2}{1 - \frac{2m}{R_0} \cot \frac{r}{R_0}} + R_0^2 \sin^2 \left( \frac{r}{R_0} \right) (d\theta^2 + \sin^2 \theta d\phi^2) - (1 - \frac{2m}{R_0} \cot \frac{r}{R_0}) dt^2. \quad (1)$$
This metric is also a particular case of the Wahlquist solution \[8\]. \(R_0\) and \(m\) are two positive constants; when setting \(m\) equal to zero, the metric (1) reduces to the metric of the Einstein universe; for \(R_0 \to \infty\) the vacuum Schwarzschild solution is recovered. The parameter \(R_0\) can be considered as a measure of the cosmological influence on the black hole properties. Therefore it is natural to interpret \(m\) and \(R_0\) as representing the Schwarzschild mass and the radius of the universe, respectively.

It is also interesting that the only metric of the form
\[
\text{(2)}
\]
satisfying the condition
\[
R^r_r = R^\theta_\theta = R^\phi_\phi
\]
(with \(R^b_a\) the Ricci tensor) is \[1\] (such an universe will be isotropic around one observer, but not spatially homogeneous). When solving the Einstein equations, we find, in addition to the matter content of the Einstein universe, a perfect fluid with \(\rho + 3p = 0\), violating in some regions the energy conditions. As opposed to McVittie’s solution, this matter content is not spatially homogeneous, since
\[
\begin{align*}
8\pi\rho &= \frac{6m}{R_0^2} \cot\left(\frac{r}{R_0}\right), \\
8\pi p &= -\frac{2m}{R_0^2} \cot\left(\frac{r}{R_0}\right).
\end{align*}
\]
This new contribution to the total energy-momentum tensor, associated with the energy source at the origin of the radial coordinate, will spoil the homogeneous nature of the Einstein universe. We can think about the above relations as describing the perturbation of the local matter content of the Einstein cosmological background induced by a point mass. It is also tempting to suppose a quantum origin of this stress-energy tensor.

It is well known that the Einstein universe solution is unstable (see e.g. \[12\]). Also, in the median region \(r \sim R_0\pi/2\), the local properties of the Vaidya model are well described by the Einstein universe. However, a local perturbation of the cosmic background in this region will propagate towards antipodes. Therefore, the radius \(R_0\) would continue to change in the same direction once the model started to expand or contract. One may expect that the new time-dependent configuration will represent the static case of a general solution briefly noted in \[6\] and interpreted as the Kerr field embedded in a FRW universe.

This raises the interesting question if this time-dependent solution has a central singularity and event horizon, and if so, how they are affected by the cosmic expansion.

These issues are currently being studied and will be discussed elsewhere.

We note also the form of the line-element \[1\] in Schwarzschild coordinates
\[
\text{(3)}
\]
\[-(1 - \frac{2m}{\varpi}) \sqrt{1 - \left(\frac{r}{R_0}\right)^2} dt^2, \quad (5)\]

where
\[\varpi = R_0 \sin \frac{r}{R_0}. \quad (6)\]

This helps us to compare with the results obtained in [9], where the gravitational field produced by a point mass in the background of the static Einstein universe is studied with many similar conclusions. For small \(m/R_0\), the line element (5) reproduces the approximate solution derived in [9].

### 3 Maximal analytic extension

A metric of the form (2) has singularities where \(e^{2A}\) vanishes or becomes infinite. However, some of these singularities can be pseudosingularities, caused by an inappropriate coordinate system. Following the usual techniques of analytic extension across pseudosingularities [12], we obtain Kruskal-like form of the metric (4):

\[ds^2 = \frac{32m^3 \exp\left(-\frac{r}{2m}\right)}{1 + \left(\frac{2m}{R_0}\right)^2} R_0 \sin\left(\frac{r}{R_0}\right) (dz^2 - dT^2) + R_0^2 \sin^2\left(\frac{r(z, T)}{R_0}\right) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (7)\]

which is related to the original form by

\[z^2 - T^2 = \frac{R_0}{2m} \exp\left(\frac{r}{2m}\right) \sin \frac{r}{R_0} \left(1 - \frac{2m}{R_0} \cot \frac{r}{R_0}\right), \quad (8)\]

\[\frac{T}{z} = \tanh\left(\frac{t}{4m} \left[1 + \left(\frac{2m}{R_0}\right)^2\right]\right) \quad r > r_H, \quad (9)\]

\[\frac{z}{T} = \tanh\left(\frac{t}{4m} \left[1 + \left(\frac{2m}{R_0}\right)^2\right]\right) \quad r < r_H, \quad (10)\]

where \(r_H = R_0 \arctan\left(\frac{2m}{R_0}\right)\).

No singularities occur here, except the genuine singularities at \(r = 0\) and \(r = \pi R_0\), which cannot be removed by any coordinate transformations.

On the manifold defined by coordinates \((z, \theta, \phi, T)\) with metric (8), we define four regions (fig.1): the region I with \(z > |T|\) is isometric to the region of the metric (4) for which \(r > r_H\). There is also a region I', defined by \(z < -|T|\), which turns out to be again isometric with the same region of the metric (4). This can be regarded as another universe on the other side of the throat \(r = r_H\).

The regions II and II' are isometric with the region \(r < r_H\) of the metric (4). The surface \(r = r_H\) presents all the characteristics of an event horizon. For a freely falling observer an infinite time \(t\) is required to traverse the finite distance \(L_0\) between an exterior point and a point on the horizon, but that destination is reached in a finite proper time. A photon would require an infinitely long time
to cover the finite stretch $L_0$. Once a particle has fallen inside $r = r_H$ it cannot avoid the singularity. In Schwarzschild coordinates, the horizon is located at $r = \frac{2m}{\sqrt{1 + (\frac{2m}{R_0})^2}}$; the effect of the curvature of the universe on the radius of the event horizon is to reduce it. We note also that both $\rho$ and $p$ defined by (4) are well defined at $r = r_H$.

Both $r = 0$ and $r = \pi R_0$ are curvature singularities with corresponding infinities of the functions $\rho$, $p$. However the $r = 0$ singularity is less dangerous being located inside an event horizon. The situation with an antipodal singularity at $r = \pi R_0$ is rather different. Since there is no horizon, this is a naked singularity, contradicting the cosmic censorship conjecture (however, it is hazardous to claim that the Vaidya metric (1) describes the final state of a collapsing matter distribution). This singularity appears to be repulsive: no timelike geodesic hits them, though a radial null geodesic can. Then we can interpret the metric (1) as describing a black hole with an antipodal naked singularity in the cosmological background of Einstein universe.

In fact, given the high symmetry and the simple matter content of the solution we have to expect the existence of an antipodal singularity. However, we suspect that this is not a generic property. We can hope that by considering a more general matter content this unpleasant feature can be avoided by the introducing an additional event horizon.

## 4 Thermodynamic properties

The presence of a naked singularity makes difficult the formulation of a quantum field theory on the background of metric (1). However, the standard arguments of the Euclidean approach for deriving the Hawking radiation seems to be valid in this case, too. We assume the possibility of extending path integral formulation of gravitational thermodynamics to the situation under consideration. This is a very strong assumption, given the existence of the naked singularity; we recall that one of the reasons to deal with the Euclideanization procedure was to avoid the space-time singularities [1].

Proceed by making in (8) the formal substitution $\xi = iT$ to yield

$$ds^2 = \frac{32m^3}{1 + (\frac{2m}{R_0})^2} \left(dz^2 + d\xi^2\right) + R_0^2 \sin^2\left(\frac{r}{R_0}\right) \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

On the section on which $z, \xi$ are real, $r$ will be real and great or equal to $r_H$ (the singularity at $r = \pi R_0$ is still present). Define the imaginary time $\tau$ by $\tau = -it$. It follows from eq. (8) that $\tau$ is periodic with period

$$\beta = \frac{8\pi m}{1 + (\frac{2m}{R_0})^2}.$$  \begin{equation} \tag{12} \end{equation}

The periodicity in the imaginary (Euclidean) time is usually interpreted as evidence of a thermal bath of temperature $T = \frac{\hbar}{\beta}$, so that the Hawking temperature
of a Schwarzschild black hole in the Einstein universe background is identified as

$$T_H = \frac{1}{8\pi m} \left[ 1 + \left( \frac{2m}{R_0} \right)^2 \right]. \quad (13)$$

The same result can be obtained by direct calculation of the surface gravity $k$ ($T_H = \frac{k}{2\pi}$). The Hawking temperature of the system appears to be increased relative to that of Schwarzschild black hole of equal horizon area.

From the formula (12) one can see that $\beta$ has a maximum value $2\pi R_0$ for $m = R_0/2$ and therefore $T_H$ has a minimum value of $T_0 = (2\pi R_0)^{-1}$ when $r_H = \pi R_0/4$. For $m > \frac{R_0}{2}$, the temperature $T_H$ increases with the mass; for a large enough $m$ we have $T_H \approx \frac{m^2}{2\pi R_0^2}$.

According to "Tolman relation", the local temperature $T(r)$ measured by a moving, accelerated detector is related to the Hawking temperature $T_H$ measured by a detector in the asymptotic region by

$$T(r) = (-g_{rr})^{-\frac{1}{2}} T_H = \frac{1}{8\pi m} \left( 1 + \left( \frac{2m}{R_0} \right)^2 \right) \left( 1 - \frac{2m}{R_0} \cot \frac{r}{R_0} \right)^{-\frac{1}{2}}. \quad (14)$$

We can consider this solution as describing a "dirty" black hole [13] since the interaction with a classical matter field is present; however, the metric (1) is not asymptotically flat and the general formalism for analysing the thermodynamic properties of a dirty black hole cannot be applied [13, 14].

Accordingly to Gibbons and Hawking [11], thermodynamic functions including the entropy can be computed directly from the saddle point approximation to the gravitational partition function (namely the generating functional analytically continued to the Euclidean spacetime).

The Euclidean gravitational part of the action has the general form [15]

$$I_E = \frac{1}{16\pi} \int_M (R - 2\Lambda) \sqrt{g} d^4x + \frac{1}{8\pi} \int_{\partial M} K \sqrt{h} d^3x. \quad (15)$$

In this case, due to the global structure, it is not necessary a so called "reference action" substraction (this substraction was needed for a Schwarzschild black hole in order to get a finite result [11]). In the semiclassical approximation, the dominant contribution to the path integral will come from the neighborhood of saddle points of the action, that is, of classical solution; the zeroth order contribution to $\log Z$ will be $-I_E$.

The integral (15) is evaluated on the Euclidean section (with $r \geq r_H$) containing the naked singularity at $r = \pi R_0$; however it takes a finite value

$$I_E = \frac{\beta m}{2} \left[ 1 - \left( \frac{2m}{R_0} \right)^2 \right] - \frac{R_0}{2m} \arctan \left( \frac{2m}{R_0} \right) + \beta \pi R_0 \frac{R_0}{4}. \quad (16)$$

All thermodynamic properties can be deduced from the partition function; for example the intrinsic entropy has the value

$$S = \frac{\beta^2}{16\pi} \left( 4 - \frac{1 + \left( \frac{2m}{R_0} \right)^2 \left[ 3 + 2 \left( \frac{2m}{R_0} \right)^2 \right]}{1 - \left( \frac{2m}{R_0} \right)^2} \right), \quad (17)$$

6
revealing a complex dependence on the parameters $m, R_0$. We are facing two well-known limits, namely, the Schwarzschild limit (a vanishing cosmological constant, $R_0 \to \infty$) with $S = 4\pi m^2$ and the Einstein limit ($m \to 0$), with $S = 0$ as expected. A curious result is obtained for a large enough $m$ ($\frac{m}{R_0} \gg 1$); in this limit the asymptotic behavior of the entropy is $S \approx 8\pi m^2$.

However, in the "extremal" case $r_H = \frac{\pi R_0}{4}$ (i.e. $R_0 = 2m$) we obtain a diverging value for the entropy. This divergence is to be associated with the peculiar global structure of the solution and the existence of the naked singularity.

For $\frac{m}{R_0} \ll 1$ we obtain a set of corrections to the (asymptotically flat) Schwarzschild black hole thermodynamic quantities. In this case, a curious result is the rather benign character of the antipodal naked singularity.

5 Further discussions

The exact solution proposed by Vaidya in [6] is usually regarded as a possible approach to the study of black holes in a non-flat background.

We have discussed in this paper some properties of this solution. After considering the maximal analytic extension of this model, an interpretation has been proposed as describing a black hole with an antipodal naked singularity in the cosmological background of Einstein universe. Assuming the possibility of extending path integral formulation of gravitational thermodynamics to this case, the expression of the Hawking temperature, the Euclidean action and the intrinsic entropy have been derived. These relations reveal a complex dependence on the two parameters of the model $m, R_0$.

The existence of a naked singularity in the line element (1) is certainly the most unpleasant feature of the Vaidya model.

For $\frac{m}{R_0} \ll 1$ a possibly way to deal with the naked singularity is to suppose that the line element (1) is valid for $r < r_0$ only (for a suitable $r_0 > r_H$) and to match it to a different metric which goes over into the Einstein line-element for large enough $r$. In this way the Vaydia model becomes an isolated island in a Einstein universe background, and the problems associated with the case $R_0 = 2m$ can be avoided too. A possible choice for $r_0$ is $r_0 = \pi R_0/2$, where the line element (1) becomes the line element of the Einstein universe. Unfortunately, a surface density distribution of matter seems to be always necessary.

A further extension of this paper could be the inclusion of a nonzero angular momentum and considering the immersion in an FRW universe along the line suggested by Vaidya in [6]. Some features of the simplest case discussed in this paper (for example the existence of a naked singularity) are still present in the general case given the strong correlation between the particle and the universe.

The Vaidya model represents an oversimplified picture and can not be considered a live candidate for describing a physical situation, but can be a source of insight into the possibilities allowed by relativity theory.

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Figure 1: Penrose diagram for the maximally extended Vaidya metric [1]