INDUCTIVE LIMITS FOR SYSTEMS OF TOEPLITZ ALGEBRAS

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Abstract. This article deals with inductive systems of Toeplitz algebras over arbitrary directed sets. For such a system the family of its connecting injective $\ast$-homomorphisms is defined by a set of natural numbers satisfying a factorization property. The motivation for the study of those inductive systems comes from our previous work on the inductive sequences of Toeplitz algebras defined by sequences of numbers and the limit automorphisms for the inductive limits of such sequences. We show that there exists an isomorphism in the category of unital $C^\ast$-algebras and unital $\ast$-homomorphisms between the inductive limit of an inductive system of Toeplitz algebras over a directed set defined by a set of natural numbers and a reduced semigroup $C^\ast$-algebra for a semigroup in the group of all rational numbers. The inductive systems of Toeplitz algebras over arbitrary partially ordered sets defined by sets of natural numbers are also studied.

1. Introduction

The main part of motivation for the present article comes from our work on inductive sequences of Toeplitz algebras and limit automorphisms of $C^\ast$-algebras generated by isometric representations for semigroups of rational numbers (see [1]). We transfer some results from [1] concerning inductive sequences of Toeplitz algebras defined by sequences of numbers to the case of inductive systems over arbitrary directed sets defined by sets of natural numbers satisfying a factorization property. In turn, the results in [1] are closely related to those in [2] [3] [4] [5] [6] [7] which are devoted to mappings of compact topological groups.

A part of motivation for studying inductive systems of $C^\ast$-algebras comes from algebraic quantum field theory [8, 9, 10, 11]. The general framework of algebraic quantum field theory is given by a covariant functor. Usually that functor acts from a category associated to a partially ordered set into a category describing the algebraic structure of observables. The standard assumption in quantum physics is that the second category consists of unital $C^\ast$-algebras and unital $\ast$-homomorphisms between $C^\ast$-algebras. Thus one has an inductive system of $C^\ast$-algebras.

A simple example of an inductive system $\mathcal{F} = (K, \{A_a\}, \{\sigma_{ba}\})$ of $C^\ast$-algebras over a directed set $(K, \leq)$ is that in which $\{A_a \mid a \in K\}$ is a net of $C^\ast$-subalgebras of a given $C^\ast$-algebra $A$. By this, one means that each $A_a$ is a $C^\ast$-subalgebra containing the unit $I_A$ of the algebra $A$, $A_a \subset A_b$ and $\sigma_{ba} : A_a \to A_b$ is the inclusion mapping whenever $a, b \in K$ and $a \leq b$. Given such a net $\mathcal{F}$, the norm

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closure of the union of all $A_a$ is itself a $C^*$-subalgebra of $A$ which is a simple example of an inductive limit in the category of $C^*$-algebras.

The basic tool of the algebraic approach to quantum fields over a spacetime is a net of $C^*$-algebras over a set defined as a suitable set of regions of the spacetime ordered under inclusion [8, 9, 10]. In [12, 13, 14] nets consisting of $C^*$-algebras of quantum observables for the case of curved spacetimes are studied. The paper [15] deals with a net that is constructed by means of the semigroup $C^*$-algebra generated by the path semigroup for a partially ordered. In [16] the authors consider a net of $C^*$-algebras associated to a net over a partially ordered set consisting of Hilbert spaces.

In this article we study inductive limits for systems of Toeplitz algebras over arbitrary directed sets. Here, by the Toeplitz algebra we mean the reduced semigroup $C^*$-algebra for the additive semigroup of non-negative integers. We recall that the reduced semigroup $C^*$-algebra can be constructed for an arbitrary left cancellative semigroup. The study of such semigroup $C^*$-algebras goes back to L. A. Coburn [18, 19], R. G. Douglas [20], G. J. Murphy [21, 22] and is continued at the present time (see, for example, [23, 24] and references there in).

To define a family of connecting injective $*$-homomorphisms for an inductive system of Toeplitz algebras over a directed set we make use of a set of natural numbers satisfying factorization equalities (see Section 3, Definition 3.1) and Coburn’s Theorem [17, Theorem 3.5.18].

The main result, Theorem 4.1, states that the inductive limit for an inductive system of Toeplitz algebras over a directed set defined by a set of natural numbers is isomorphic in the category of unital $C^*$-algebras and their unital $*$-homomorphisms to a reduced semigroup $C^*$-algebra for a semigroup in the group of all rational numbers.

The article is organized as follows. It consists of Introduction, Preliminaries and three more sections containing the results. Section 3 deals with the auxiliary statements that are used for proving the main result. In Section 4 is devoted to the proof of Theorem 4.1. Section 5 contains the theorem on inductive systems of Toeplitz algebras over arbitrary partially ordered sets defined by sets of natural numbers satisfying factorization equalities.

The results in this article were announced without proofs in [25] and are closely related to those in [26].

2. Preliminaries

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of all natural numbers, the additive group of all integers, the additive group of all rational numbers and the field of all complex numbers respectively. For a sequence of numbers $M = (m_1, m_2, \ldots)$, where $m_s \in \mathbb{N}$, $s \in \mathbb{N}$, we shall consider the subgroup $\mathbb{Q}_M$ of the group $\mathbb{Q}$ defined as follows:

$$\mathbb{Q}_M := \left\{ \frac{m}{m_1 m_2 \ldots m_s} \mid m \in \mathbb{Z}, s \in \mathbb{N} \right\}.$$  

It is known (see, for example, [22, Proposition 1], [28, Lemma 1]) that the group $\mathbb{Q}_M$ is the inductive (direct) limit in the category of groups and their homomorphisms
for the following inductive (direct) sequence

\[ \mathbb{Z} \xrightarrow{\tau_1} \mathbb{Z} \xrightarrow{\tau_2} \mathbb{Z} \xrightarrow{\tau_3} \ldots, \]

where the connecting homomorphisms \( \tau_s \) are given by \( \tau_s(m) = m_s'm, \quad m \in \mathbb{Z} \), \( s \in \mathbb{N} \). By a directed set we mean an upward directed partially ordered set. We recall the definition of reduced semigroup \( C^* \)-algebras for semigroups in \( \mathbb{Q} \). To do this we assume that \( \Gamma \) is an arbitrary subgroup in the group \( \mathbb{Q} \). The positive cone in the ordered group \( \Gamma \) is denoted by the symbol

\[ \Gamma^+ := \Gamma \cap [0, +\infty) . \]

As usual, the symbol \( l^2(\Gamma^+) \) stands for the Hilbert space of all square summable complex-valued functions on the additive semigroup \( \Gamma^+ \):

\[ l^2(\Gamma^+) := \{ f : \Gamma^+ \rightarrow \mathbb{C} : \sum_{\gamma \in \Gamma^+} |f(\gamma)|^2 < +\infty \} . \]

Recall that the inner product in the space \( l^2(\Gamma^+) \) is given by the formula \( <f, g> := \sum_{\gamma \in \Gamma^+} f(\gamma) \overline{g(\gamma)} \). The canonical orthonormal basis in the Hilbert space \( l^2(\Gamma^+) \) is denoted by \( \{ e_g \mid g \in \Gamma^+ \} \). That is, for arbitrary elements \( g, h \in \Gamma^+ \), we set \( e_g(h) = \delta_{g,h} \), where

\[ \delta_{g,h} := \begin{cases} 1, & \text{if } g = h; \\ 0, & \text{if } g \neq h. \end{cases} \]

Let us consider the \( C^* \)-algebra of all bounded linear operators \( B(l^2(\Gamma^+)) \) in the Hilbert space \( l^2(\Gamma^+) \). For every element \( g \in \Gamma^+ \), we define the isometry \( V_g \in B(l^2(\Gamma^+)) \) by \( V_ge_h := e_{g+h} \), where \( h \) is an element of the semigroup \( \Gamma^+ \).

We denote by \( C^*_r(\Gamma^+) \) the \( C^* \)-subalgebra in the \( C^* \)-algebra \( B(l^2(\Gamma^+)) \) generated by the set of isometries \( \{ V_g \mid g \in \Gamma^+ \} \). The algebra \( C^*_r(\Gamma^+) \) is called the reduced semigroup \( C^* \)-algebra of the semigroup \( \Gamma^+ \), or the Toeplitz algebra generated by \( \Gamma^+ \). As was mentioned in Introduction, in the similar way a semigroup \( C^* \)-algebra can be defined for an arbitrary left cancellative semigroup (see, for example, \[24\] Section 2). In the case when \( \Gamma \) is the group \( \mathbb{Z} \), we also denote the semigroup \( C^* \)-algebra \( C^*_r(\mathbb{Z}^+) \) by \( \mathcal{T} \) and use the symbols \( T \) and \( T^n \) instead of \( V_1 \) and \( V_n \), respectively, where \( n \in \mathbb{Z}^+ \).

The following statement is an immediate consequence of Coburn’s theorem \[17\] Theorem 3.5.18.

**Lemma 2.1.** For every number \( n \in \mathbb{N} \), there exists a unique unital \( * \)-homomorphism of \( C^* \)-algebras \( \varphi : \mathcal{T} \rightarrow \mathcal{T} \) such that \( \varphi(T) = T^n \). Moreover, \( \varphi \) is isometric.

In the sequel, we abbreviate those self-homomorphisms of the Toeplitz algebra as follows:

\[ \varphi : \mathcal{T} \rightarrow \mathcal{T} : T \mapsto T^n. \]

We note that a straightforward proof of Lemma 2.1 is also contained in \[29\] Proposition 3. For necessary results in the theory of \( C^* \)-algebras we refer the reader, for example, to books \[17\] and \[30\] Ch. 4, § 7.

Further, we recall the definition and some facts concerning the inductive limits for inductive systems of \( C^* \)-algebras (see, for example, \[31\] Section 11.4). Necessary facts from the theory of categories and functors are contained, for example, in \[30\] Ch. 0, § 2 and \[32\].
In what follows, up to Section 5 we shall consider a directed set \((K, \leq)\). Note that in Section 5 we will deal with an arbitrary partially ordered set \((K, \leq)\) that is not necessarily directed.

The category associated to the set \((K, \leq)\) is denoted by the same letter \(K\). We recall that the objects of this category are the elements of the set \(K\), and for any pair \(a, b \in K\) the set of morphisms \(\text{Mor}_K(a, b)\) from \(a\) to \(b\) is defined as follows:

\[
\text{Mor}_K(a, b) = \begin{cases} 
\{(a, b)\}, & \text{if } a \leq b; \\
\emptyset, & \text{otherwise.} 
\end{cases}
\]

Let us take a covariant functor \(F\) from the category \(K\) into the category of unital \(C^*\)-algebras and their unital \(*\)-homomorphisms. Such a functor is called an inductive system in the category of \(C^*\)-algebras over the set \((K, \leq)\). It may be given by a collection \((K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})\) satisfying the properties from the definition of a functor. We shall write \(F = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})\). Here, \(\{\mathfrak{A}_a \mid a \in K\}\) is a family of unital \(C^*\)-algebras. We also suppose that all morphisms \(\sigma_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b\), where \(a \leq b\), are embeddings of \(C^*\)-algebras, i.e., unital injective \(*\)-homomorphisms.

Recall that the diagram

\[
\begin{array}{ccc}
\mathfrak{A}_a & \xrightarrow{\sigma_{ca}} & \mathfrak{A}_c \\
\sigma_{ba} \downarrow & & \downarrow \sigma_{cb} \\
\mathfrak{A}_b & & \\
\end{array}
\]

is commutative for all elements \(a, b, c \in K\) satisfying the conditions \(a \leq b\) and \(b \leq c\), that is, the following equation holds:

\[(2.2) \quad \sigma_{ca} = \sigma_{cb} \circ \sigma_{ba} .\]

Furthermore, for each element \(a \in K\) the morphism \(\sigma_{aa} : \mathfrak{A}_a \to \mathfrak{A}_a\) is the identity mapping. In the case of a countable set \(K\) the system \(F\) is called an inductive sequence of \(C^*\)-algebras.

The inductive limit of the system \(F = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})\) over a directed set \(K\) is a pair \((\mathfrak{A}, \{\sigma^K_a\})\), where \(\mathfrak{A}\) is a \(C^*\)-algebra and \(\{\sigma^K_a : \mathfrak{A}_a \to \mathfrak{A} \mid a \in K\}\) is a family of unital injective \(*\)-homomorphisms such that the following two properties are fulfilled [31, Proposition 11.4.1]:

1) For every pair of elements \(a, b \in K\) satisfying the condition \(a \leq b\) the diagram

\[
\begin{array}{ccc}
\mathfrak{A}_a & \xrightarrow{\sigma^K_a} & \mathfrak{A}_b \\
\sigma^K_{ba} \downarrow & & \downarrow \sigma^K_{ab} \\
\mathfrak{A} & & \\
\end{array}
\]

is commutative, that is, the equality for mappings

\[(2.3) \quad \sigma^K_a = \sigma^K_b \circ \sigma_{ba} \]

holds. Moreover, we have the following equality:

\[(2.4) \quad \mathfrak{A} = \bigcup_{a \in K} \sigma^K_a(\mathfrak{A}_a),\]

where the bar means the closure of the set with respect to the norm topology in the \(C^*\)-algebra \(\mathfrak{A}\).
2) If \( B \) is a \( C^* \)-algebra, \( \psi_a : A_a \rightarrow B \) is an injective \(*\)-homomorphism for each \( a \in K \), and conditions analogous to those in (2.3) and (2.4) are satisfied, then there exists a unique \(*\)-isomorphism \( \theta \) from \( A \) onto \( B \) such that the diagram

\[
\begin{array}{ccc}
A_{a} & \xrightarrow{\psi_{a}} & B \\
\sigma_{a}^{K} & \downarrow \theta & \\
A & \xrightarrow{\psi_{a}} & B \\
\end{array}
\]

is commutative for every \( a \in K \), that is, the equality \( \psi_{a} = \theta \circ \sigma_{a}^{K} \) holds.

The \( C^* \)-algebra \( A \) itself is often called the inductive limit. It is denoted as follows: \( \lim_{\rightarrow} F := A \). As is well known, the inductive limit can always be constructed for an inductive system in the category of \( C^* \)-algebras and their \(*\)-homomorphisms. For the details we refer the reader to Proposition 11.4.1 in [31].

3. Auxiliary results

The inductive sequences of Toeplitz algebras defined by arbitrary sequences of prime numbers are the objects for studying in [1]. Now, by analogy with the definition of such a sequence (see [1, Definition 1]), we are going to give the definition of the inductive system of Toeplitz algebras over a partially ordered set defined by a set of natural numbers possessing an additional property.

Let \(( K, \leq )\) be a directed set. In what follows, we consider a set of natural numbers

\[(3.1) \quad N = \{n_{ba} \in \mathbb{N} \mid a, b \in K, a \leq b\} \]

such that the factorization equalities

\[(3.2) \quad n_{ca} = n_{cb} \cdot n_{ba}\]

hold for all elements \( a, b, c \in K \) satisfying the conditions \( a \leq b \) and \( b \leq c \). It follows immediately from (3.2) that the equality \( n_{aa} = 1 \) holds for every \( a \in K \).

Further, using Lemma 2.1 for every number \( n_{ba} \in N \) we define the isometric \(*\)-homomorphism by the formula

\[(3.3) \quad \sigma_{ba} : T \rightarrow T : T \mapsto T^{n_{ba}}.\]

It is clear that the equalities (2.2) are valid for all elements \( a, b, c \in K \) whenever the conditions \( a \leq b \) and \( b \leq c \) hold, and for each \( a \in K \) the \(*\)-homomorphism \( \sigma_{aa} : A_a \rightarrow A_a \) is the identity mapping. Thus we can give the definition of an inductive system of Toeplitz algebras (compare with [1, Definition 1]) over a directed set defined by a set of natural numbers satisfying factorization equalities.

**Definition 3.1.** Let \(( K, \leq )\) be a directed set and \( N \) be a set of natural numbers (3.1) satisfying (3.2). An inductive system of \( C^* \)-algebras

\[(3.4) \quad F = (K, \{T_a\}, \{\sigma_{ba}\}),\]

where \( T_a = T \) for all \( a \in K \) and the connecting \(*\)-homomorphisms \( \sigma_{ba} \) are given by (3.3), is called the inductive system of Toeplitz algebras over \( K \) defined by \( N \).

To obtain the main result of the article we shall prove in this section several auxiliary assertions. For formulations of these assertions we introduce additional notation.

Firstly, for a directed set \( K \) and its arbitrary element \( a \in K \) the cofinal subset \( K^a \) of \( K \) is defined as follows: \( K^a := \{b \in K \mid a \leq b\} \).
Secondly, for a subset $S$ in the set $K^a$ we shall deal with the set of natural numbers

$$N(S) := \{n_{ba} \in N \mid b \in S\}.$$ 

Throughout this section $(K, \{T_b\}, \{\sigma_{cb}\})$ is an inductive system of Toeplitz algebras over $K$ defined by set (3.1) satisfying (3.2). Moreover, for an element $a \in K$ we consider the inductive system

$$(K^a, \{T_b\}, \{\sigma_{cb}\})$$

of Toeplitz algebras over $K^a$ defined by the set

$$(n_{cb} \in N \mid b, c \in K^a).$$

Using property 2) of the inductive limit for the system (3.5) (see Preliminaries), it is straightforward to prove the following statement.

**Lemma 3.2.** For every element $a \in K$ there exists an isomorphism between the inductive limits

$$\lim_{\rightarrow} (K^a, \{T_b\}, \{\sigma_{cb}\}) \simeq \lim_{\rightarrow} (K, \{T_b\}, \{\sigma_{cb}\})$$

in the category of unital $C^*$-algebras and unital $\ast$-homomorphisms.

**Lemma 3.3.** The following two assertions are valid:

1) if for elements $b, c \in K^a$ the equality for numbers

$$n_{ba} = n_{ca}$$

holds then we have the equality for the images of Toeplitz algebras

$$\sigma_b^{K^a}(T_b) = \sigma_c^{K^a}(T_c):$$

2) if for some $k \in \mathbb{N}$ and $b, c \in K^a$ the equality for numbers

$$n_{ca} = k \cdot n_{ba}$$

holds then we have the inclusion for the images of Toeplitz algebras

$$\sigma_b^{K^a}(T_b) \subset \sigma_c^{K^a}(T_c).$$

**Proof.** 1) Assume that for some elements $b, c \in K^a$ equality (3.7) holds. Since $K$ is a directed set there exists an element $d \in K^a$ such that $b \leq d$ and $c \leq d$. Then factorization equalities (3.2) yield the following equalities for the corresponding natural numbers:

$$n_{db} \cdot n_{ba} = n_{da} = n_{dc} \cdot n_{ca}.$$ 

Therefore, using condition (3.7), one gets immediately the equality $n_{db} = n_{dc}$, which implies the following equality for the images of Toeplitz algebras:

$$\sigma_{db}(T_b) = \sigma_{dc}(T_c).$$

By (2.3) and (3.12), we obtain desired relation (3.8):

$$\sigma_b^{K^a}(T_b) = \sigma_b^{K^a}(\sigma_{db}(T_b)) = \sigma_d^{K^a}(\sigma_{dc}(T_c)) = \sigma_c^{K^a}(T_c).$$

2) Again, we choose an element $d \in K^a$ such that the conditions $b \leq d$ and $c \leq d$ are satisfied. Then, by (3.11) and (3.9), we get the equality $n_{db} \cdot n_{ba} = n_{dc} \cdot k \cdot n_{ba}$. Consequently, we have the equality $n_{db} = n_{dc} \cdot k$ that together with (3.3) imply the following inclusion for the images of Toeplitz algebras:

$$\sigma_{db}(T_b) \subset \sigma_{dc}(T_c).$$
Thus, using equality (2.3) and inclusion (3.13), we obtain required inclusion (3.10):
\[ \sigma^K_b(T_b) = \sigma^K_d(\sigma_d(T_b)) \subset \sigma^K_d(\sigma_d(T_c)) = \sigma^K_c(T_c). \]

\[ \square \]

Lemma 3.4. There exists a totally ordered countable subset \( \Lambda^a \) in the set \( K^a \) satisfying the following property. For every element \( b \in K^a \) there is an element \( c \in \Lambda^a \) and a number \( k \in \mathbb{N} \) such that the equality
\[ n_{ca} = k \cdot n_{ba} \]
holds, where \( n_{ca} \in N(\Lambda^a) \), \( n_{ba} \in N(K^a) \).

Proof. First of all, in the set of natural numbers \( N(K^a) \) we consider the subset
\[ \{n_{ba} \in N \mid s \in \mathbb{N}\} \]
which is uniquely determined by the following three properties:

- \( b_1 = a; \)
- the inequality \( n_{b_1} < n_{b_{s+1}} \) is valid for every \( s \in \mathbb{N}; \)
- for every number \( n_{ba} \in N(K^a) \) there exists a natural number \( n_{ba} \) in set (3.15) such that the equality \( n_{ba} = n_{ba} \) holds.

In other words, we throw out repeating numbers from the set \( N(K^a) \). Moreover, the elements of set (3.15) constitute an increasing sequence of natural numbers indexed by \( s \). To construct the desired set \( \Lambda^a \) we use set (3.15) and proceed as follows.

As the first element of the set \( \Lambda^a \) denoted by \( c_1 \) we choose the element \( a \). We note that the condition \( b_1 = a \leq c_1 = a \) is satisfied. Then equality (3.14) holds with \( c = c_1 = a, b = b_1 = a \) and \( k = n_{c_1b_1} = n_{ba} = 1. \)

As the second element of the set \( \Lambda^a \) we take any element \( c_2 \in K^a \) satisfying the conditions \( c_1 \leq c_2 \) and \( b_1 \leq c_2 \). Such an element exists because \( K^a \) is a directed set. In this case we have equality (3.14) with \( c = c_2, b = b_2 \) and \( k = n_{c_2b_2}. \) Continuing to argue in this way, we shall construct a countable totally ordered subset
\[ \Lambda^a := \{c_s \in K^a \mid s \in \mathbb{N}\} \]
in the set \( K^a \) possessing the required property. It is worth noting that the condition \( c_s \leq c_{s+1} \) holds for every \( s \in \mathbb{N}. \)

In the next lemma we consider the inductive sequence of Toeplitz algebras
\[ (\Lambda^a, \{\mathcal{T}_c\}, \{\sigma_{c tc}\}) \]
over the set \( \Lambda^a \) defined by the subset \( \{n_{ca} \in N \mid b, c \in \Lambda^a\} \) in the set \( N. \) The proof of the following statement is similar to that of Proposition 1 in [1].

Lemma 3.5. There exists a subgroup \( Q_M \) in the group \( \mathbb{Q} \) such that the reduced semigroup \( C^*\)-algebra of the semigroup \( Q^+_M \) is isomorphic to the inductive limit of the inductive sequence (3.17):
\[ C^*_r(Q^+_M) \simeq \text{lim} \ (\Lambda^a, \{\mathcal{T}_c\}, \{\sigma_{c tc}\}). \]

Remark. In [21] Theorem 1 it is shown that the functor sending a partially directed set to the corresponding Toeplitz algebra is continuous. Consider the inductive sequence of groups (2.1) with the connecting homomorphisms \( \tau_s \) given by \( \tau_s(m) = n_{c_{s+1}c}m, m \in \mathbb{Z}, s \in \mathbb{N}. \) Applying the above-mentioned functor to
this inductive sequence of groups and making use of Theorem 1 in [21], we obtain another proof of Lemma 3.5.

4. The main result

In this section we prove the following statement concerning the inductive limits for inductive systems of Toeplitz algebras over directed sets defined by sets of natural numbers (3.1) satisfying factorization equalities (3.2). To do this we shall use the results from the previous section.

**Theorem 4.1.** Let $\mathcal{F}$ be an inductive system of Toeplitz algebras over a directed set $K$ defined by a set of natural numbers $N$ satisfying factorization equalities. Then there exists a subgroup $Q_M$ in the group of all rational numbers such that the reduced semigroup $\mathcal{C}^*$-algebra of the semigroup $Q_M^+$ is isomorphic to the inductive limit of the inductive system $\mathcal{F}$:

$$C^*_r(Q_M^+) \simeq \lim_{\longrightarrow} \mathcal{F}.$$  

**Proof.** As in the previous section we use the notation $\mathcal{F} = (K, \{T_a\}, \{\sigma_{ba}\})$.

Let us fix an element $a \in K$. Then we take a totally ordered countable subset $\Lambda^a$ in the set $K^a$ which is constructed in the proof of Lemma 3.4 (see (3.16)).

We claim that there exists an isomorphism between the inductive limits of the inductive system $\mathcal{F}$ and inductive sequence (3.17) of Toeplitz algebras, that is,

$$\lim_{\longrightarrow} \mathcal{F} \simeq \lim_{\longrightarrow} (\Lambda^a, \{T_c\}, \{\sigma_{c,cs}\})$$

in the category of unital $\mathcal{C}^*$-algebras and unital $*$-homomorphisms.

Indeed, we consider the inductive system (3.5) of Toeplitz algebras over the set $K^a$ defined by the subset (3.6) in the set $N$. By (2.4) we have the following equality for the inductive limit of this system:

$$\lim_{\longrightarrow} (K^a, \{T_b\}, \{\sigma_{cb}\}) = \bigcup_{b \in K^a} \sigma_b^a(T_b).$$

By Lemma 3.2, to show the existence of isomorphism (4.2) it is enough to prove that one has the isomorphism between the inductive limits of systems (3.17) and (3.5) in the category of unital $\mathcal{C}^*$-algebras and their unital $*$-homomorphisms:

$$\lim_{\longrightarrow} (\Lambda^a, \{T_c\}, \{\sigma_{c,cs}\}) \simeq \lim_{\longrightarrow} (K^a, \{T_b\}, \{\sigma_{cb}\}).$$

To construct an isomorphism (4.4) we shall use the universal property and property 2) of the inductive limit for system (3.17) (see Preliminaries). To this end, for elements $c_s, c_t \in \Lambda^a$ satisfying the condition $c_s \leq c_t$ we consider the following
commutative diagram (firstly without the dashed arrow):

\[ \text{(4.5)} \]

\[
\begin{array}{ccc}
\mathcal{T}_{c_s} & \xrightarrow{\sigma_{ctc_s}} & \mathcal{T}_{c_t} \\
\sigma^{K_a}_{c_t} & \parallel & \sigma^{K_a}_{c_t} \\
\text{lim} \langle \Lambda^a, \{T_{c_s}\}, \{\sigma_{ctc_s}\} \rangle & \xrightarrow{\theta} & \text{lim} \langle K^a, \{T_b\}, \{\sigma_{cb}\} \rangle \\
\end{array}
\]

The universal property of the inductive limits for systems of C*-algebras and the injectivity of *-homomorphisms in diagram (4.5) yield the injective *-homomorphism

\[ \theta : \text{lim} \langle \Lambda^a, \{T_{c_s}\}, \{\sigma_{ctc_s}\} \rangle \rightarrow \text{lim} \langle K^a, \{T_b\}, \{\sigma_{cb}\} \rangle \]

such that diagram (4.5) complemented by \( \theta \) is also commutative. To show that the homomorphism \( \theta \) is surjective it is sufficient (see property 2) in Preliminaries) to prove the equality

\[ \text{(4.6)} \]

\[ \text{lim} \langle K^a, \{T_b\}, \{\sigma_{cb}\} \rangle = \bigcup_{s=1}^{+\infty} \sigma^{K_a}_{c_s}(T_{c_s}). \]

Since we have the inclusion of sets \( \Lambda^a \subset K^a \) and representation (4.3), the space on the left-hand side of (4.6) contains the space on its right-hand side.

To prove the reverse inclusion for the spaces in (4.6) we use (4.3) again. For this aim let us fix an arbitrary element \( b \in K^a \). We state that the image set \( \sigma^{K_a}_{c_b}(T_{b}) \) is contained in the set \( \sigma^{K_a}_{c_s}(T_{c_s}) \) for an element \( c_s \in \Lambda^a \).

Really, by Lemma 3.4 there exists an element \( c_s \in \Lambda^a \) and a number \( k \in \mathbb{N} \) such that the factorization equality (3.14) is valid for natural numbers \( n_{c_s,a} \in N(\Lambda^a) \), \( n_{ba} \in N(K^a) \). By assertion 2) in Lemma 3.3 for the image sets \( \sigma^{K_a}_{c_b}(T_{b}) \) and \( \sigma^{K_a}_{c_s}(T_{c_s}) \) we obtain inclusion (3.10) with \( c_s \) instead of \( c \). It follows from (4.3) that the inclusion for C*-algebras

\[ \text{lim} \langle K^a, \{T_b\}, \{\sigma_{cb}\} \rangle \subset \bigcup_{s=1}^{+\infty} \sigma^{K_a}_{c_s}(T_{c_s}) \]

holds. Hence, equality (4.6) is proved. Therefore, the *-homomorphism \( \theta \) between the inductive limits is an isomorphism of C*-algebras. Thus, isomorphism (4.4) exists.

Furthermore, we obtain the existence of isomorphism (4.2), as claimed. Finally, by Lemma 3.5 there exists a subgroup \( \mathbb{Q}_M \) in the group \( \mathbb{Q} \) for which we have isomorphism (4.18). Thus, using isomorphism (4.2), we obtain isomorphism (4.1), as required. \( \square \)
5. **Inductive systems of Toeplitz algebras over arbitrary partially ordered sets**

Throughout this section a pair \((K, \leq)\) denotes a partially ordered set that is not necessarily directed. By analogy with Definition 3.1 one can define the inductive system of Toeplitz algebras over \((K, \leq)\) defined by set (3.1) satisfying factorization equalities (3.2). Below we shall consider such an inductive system \(\mathcal{F}\) and use notation (3.4).

Taking the family of all directed subsets of the set \((K, \leq)\) and making use of Zorn’s lemma, one can easily prove the following statement.

**Lemma 5.1.** Let \((K, \leq)\) be a partially ordered set. Then the following equality holds:

\[
K = \bigcup_{i \in I} K_i,
\]

where \(\{K_i \mid i \in I\}\) is the family of all maximal directed subsets of the set \((K, \leq)\).

Now, for a given inductive system of Toeplitz algebras (5.1) over the partially ordered set \((K, \leq)\) defined by set (3.1) satisfying (3.2) we consider representation (5.2)

\[
\mathcal{F}_i = (K_i, \{a_i\}, \{\sigma_{ba}\})
\]

over the directed set \((K_i, \leq)\) defined by the set of natural numbers \(\{n_{ba} \in N \mid a, b \in K_i\}\)

and its inductive limit \(\lim_{\rightarrow} \mathcal{F}_i\).

We consider the direct product of \(\mathbb{C}^*\)-algebras \(\prod_{i \in I} \lim_{\rightarrow} \mathcal{F}_i\). That is, the \(\mathbb{C}^*\)-algebra

\[
\prod_{i \in I} \lim_{\rightarrow} \mathcal{F}_i := \left\{(f_i) \mid \|f_i\| = \sup_i \|f_i\| < +\infty \right\}
\]

relative to the pointwise operations and the supremum norm.

To formulate the result of this section it is convenient for us to give the definition.

**Definition 5.2.** The inductive system \(\mathcal{F}_i\) defined by (5.2) is called the inductive subsystem of \(\mathcal{F}\) over the set \(K_i\).

The following statement is an immediate consequence of Theorem 4.1.

**Theorem 5.3.** Let \(K\) be a partially ordered set and let \(\{K_i \subset K \mid i \in I\}\) be a family of all maximal directed subsets in the set \(K\). Let \(\mathcal{F}\) be an inductive system of Toeplitz algebras over \(K\) defined by a set of natural numbers \(N\) satisfying factorization equalities. Let \(\mathcal{F}_i\), where \(i \in I\), denote the inductive subsystem of \(\mathcal{F}\) over the set \(K_i\). Then there exists a family \(\{Q_{M_i} \subset \mathbb{Q} \mid i \in I\}\) consisting of subgroups in the group of all rational numbers \(\mathbb{Q}\) and an isomorphism between the direct products of \(\mathbb{C}^*\)-algebras

\[
\prod_{i \in I} \lim_{\rightarrow} \mathcal{F}_i \simeq \prod_{i \in I} C^*_\mathbb{r}(Q_{M_i}^+)
\]

in the category of unital \(\mathbb{C}^*\)-algebras and their unital \(*\)-homomorphisms.

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