AdS. Klein-Gordon equation

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Abstract

I propose a generalization of the Klein-Gordon equation in the framework of AdS space-time and exhibit a four parameter family of solutions among which there is a two parameter family of time-dependent bound states.

Introduction

In 1973 E. Alvarez and I, [1], suggested that the so-called expansion of the Universe could be due to a decreasing of the so called ”speed of light constant $c$“, quantified by the very simple formula:

$$\frac{\dot{c}}{c} = -H$$

(1)

$H$ being the so called ”Hubble constant". This corresponds to a decreasing of $c$ by $10^{-8} m/s$ in an interval of time greater than a century, not directly observable, but it gives a meaning to start with establishing a relationship between two quantities that both depend on time, escaping thus to the apparently solid argument that only dimensionless fundamental constants could depend on time.

I have personally kept developing this point of view on several occasions [4], [5], this paper being my last effort in this direction, while others points of view, [6], [7], [9] have also been developed and some of them severely criticized in [10].

Space-time model

Using polar coordinates, let us consider the Robertson-Walker space-time model of the Universe:

$$ds^2 = -dt^2 + \frac{1}{c^2} \left( \frac{dr^2}{1 - br^2} + r^2 d\Omega^2 \right)$$

(2)

where $b$ is the curvature of space and $c = c(t)$ is a time dependent function such that $c_0 = c(0)$ is the speed of light at the present epoch. Using $c(t)$ as a description of the evolution of the Universe is formally strictly equivalent to using the scale factor $a(t) = c_0/c(t)$ except that in this case it looks
queer to require that a dimensionless quantity as \( a(t) \) is equal to 1 at the present epoch, while \( c(t) \) having dimensions of velocity, we can always assume that \( c_0 = 1 \) using an appropriate system of units.

**D’Alembertian**

Let us consider the D’Alembertian operator corresponding to the space-time model above acting on a function \( \psi(t, r, \theta, \phi) \). A straightforward calculation yields:

\[
\Delta_4 \Psi = -\frac{\partial^2 \Psi}{\partial t^2} + 3\frac{\partial \ln c}{\partial t} \frac{\partial \Psi}{\partial t} + c^2(1 - b r^2) \frac{\partial^2 \Psi}{\partial r^2} + \frac{2 c^2}{r} \left( 1 - \frac{3}{2} b r^2 \right) \frac{\partial \Psi}{\partial r} + \frac{c^2}{r^2} \left( \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right).
\]  

(3)

**Variables separation**

Let us assume now that \( \psi \) is the following product of three functions:

\[
\Psi = B(t)f(r)Y(\theta, \phi) \tag{4}
\]

Assuming that \( Y \) is a spherical harmonic, so that:

\[
LY \equiv \frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial^2 Y}{\partial \phi^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial Y}{\partial \theta} = -l(l + 1)Y, \tag{5}
\]

also that \( f \) is a solution of:

\[
Lf \equiv (1 - b r^2) \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \left( 1 - \frac{3}{2} b r^2 \right) \frac{\partial f}{\partial r} - \frac{l(l + 1)}{r^2} f = -k_1^2 f \tag{6}
\]

where \( k_1 \) is a constant. And also that \( B \) is a solution of:

\[
LB \equiv -\frac{\partial^2 B}{\partial t^2} + 3\frac{\partial \ln c}{\partial t} \frac{\partial B}{\partial t} = k_0^2 c^2 B \tag{7}
\]

where \( k_0 \) is another constant, by direct substitution into (3) we get:

\[
\Delta_4 \Psi = (k_0^2 - k_1^2)c^2 \Psi \tag{8}
\]

I chose the signs of the second members of (6) and (7) so that:
\[ \Psi = e^{i(k_0ct \pm k_1r)}Y(\theta, \varphi) \] (9)

when \( \Lambda \to 0 \), and \( b \to 0 \).

**Solution of the radial equation**

Maple16 gives right away two independent solutions of the radial equation (6)

\[ f_1 = \frac{1}{\sqrt{r}}\text{LegendreP}\left( \frac{-1}{2} \frac{\sqrt{b} - 2\sqrt{b + k_1^2}}{\sqrt{b}}, l + \frac{1}{2}, \sqrt{1 - br^2} \right) \] (10)

\[ f_2 = \frac{1}{\sqrt{r}}\text{LegendreQ}\left( \frac{-1}{2} \frac{\sqrt{b} - 2\sqrt{b + k_1^2}}{\sqrt{b}}, l + \frac{1}{2}, \sqrt{1 - br^2} \right) \] (11)

**Bound states, \( l=0 \) or \( l=-1, b < 0 \)**

Let us assume now that \( b \neq 0 \). In this case the two independent solutions of (6) are:

\[ f_{\pm} = \frac{1}{r} \left( b r + \sqrt{b(b r^2 - 1)} \right)^{\alpha}, \quad \alpha = \pm \sqrt{1 + \frac{k_1^2}{b}} \] (12)

and their behavior near the origin is:

\[ f_{\pm} = e^{\alpha \ln(-b)} + O(r). \] (13)

For \( b > 0 \) the solution is not regular near the origin and therefore from now on I shall assume that \( b < 0 \). The behavior of the solution above when \( r \to \infty \) is:

\[ f_{\pm} = \left( \frac{1}{2} \frac{1}{r^2} + O\left( \frac{1}{r^3} \right) \right) \frac{1}{r^\alpha}, \] (14)

so that the space integral

\[ |f|^2 = 4\pi \int_0^\infty \frac{f^2 r^2 dr}{\sqrt{1 - br^2}} \] (15)

is finite if \( \alpha > 0 \), i.e., if \( f = f^+ \) and \( k_1^2 < |b| \). Any other solution has an infinite norm.

**Time dependence**

To discuss the equation LB, (7), I shall assume that \( c \) is the function of \( t \) describing the Anti de Sitter model (AdS) of the Universe. It has
therefore maximal space-time symmetry with negative space curvature, $b < 0$, and positive cosmological constant $\Lambda > 0$. In particular when $c$ is a decreasing function of time it satisfies the differential equation:

$$\dot{c} = -c\sqrt{\lambda^2 - bc^2} \quad \text{where} \quad \Lambda = 3\lambda^2$$  \hfill (16)

that integrated yields:

$$c = \frac{\lambda}{p} \csc \left( \lambda t + \text{arccsch} \left( \frac{pc_0}{\lambda} \right) \right), \quad p = \sqrt{-b}$$  \hfill (17)

Two other useful relations can be derived from (16), namely:

$$\dot{c}^2 = \lambda^2 c^2 - bc^4$$  \hfill (18)

and:

$$\ddot{c} = \lambda^2 c - 2bc^3$$  \hfill (19)

that follows from the preceding one after derivation and simplification.

Since $c$ is a monotonous decreasing function of $t$, it is possible to consider $B$ as a function of $c$. So that $B(t) = B(c(t))$. Using (18) and (19) leads then to the consideration of the differential equation:

$$LB \equiv -c^2(\lambda^2 - bc)\frac{\partial^2 B}{\partial c^2} + c(2\lambda^2 - bc^2)\frac{\partial B}{\partial c} - k_0c^2B.$$  \hfill (20)

$c = 0$ is a regular singular value and therefore the solutions of this equation admit formal series solutions:

$$B = c^s(1 + a_1c + \cdots)$$  \hfill (21)

$s$ being a solution of the indices equation:

$$-s^2 + 3s = 0$$  \hfill (22)

so that $s = 0$ or $s = 3$.

Maple16 gives the general solution of (20) as a linear combination with constant coefficients of the two particular solutions.

$$B_1 = c^{3/2}\text{LegendreP}\left(-\frac{1}{2} + \sqrt{1 + \frac{k_0^2}{b^2}}\frac{3}{2}, \frac{1 - \frac{bc^2}{\lambda^2}}{\sqrt{\pi}}\right)$$  \hfill (23)

$$B_2 = c^{3/2}\text{LegendreQ}\left(-\frac{1}{2} + \sqrt{1 + \frac{k_0^2}{b^2}}\frac{3}{2}, \frac{1 - \frac{bc^2}{\lambda^2}}{\sqrt{\pi}}\right)$$  \hfill (24)

But since (20) is real and $B_1$ and $B_2$ are complex we have in fact four real solutions of (20). The first two terms of the power series expansions of $\text{Re}(B_1)$ and $\text{Im}(B_2)$ are:

$$\text{Im}(B_2) = \frac{\pi}{2}\text{Re}(B_1) = -\frac{\sqrt{\pi}}{8} \frac{2^{3/4}}{(-\frac{1}{2})^{3/4}\lambda\sqrt{\pi}} (2\Lambda + 3k_0^2c^2)$$  \hfill (25)

This proves that they belong to the index $s = 0$ and that they are proportional with a factor $(1/2)\pi$. Extending the series a few more terms it is easy to prove that $\text{Im}(B_1) = 0$ and that $\text{Re}(B_2)$ belongs to the index $s = 3$. This distinguishes this latter function as the only one that goes to zero when $c$ goes to zero.
The function $B_2$ and its complex conjugate $\bar{B}_2$ can therefore be considered as the fundamental complex solution of (20).

I have thus proved that there exists a system of modes:

$$\psi = B_2(t, k_0) f^+(r, k_1) Y_1^m(\theta, \varphi)$$

(26)
depending on four parameters $(k_0, k_1, l, m)$ that are solutions of a generalized Klein-Gordon:

$$\Delta_4 \psi = (k_0^2 - k_1^2)c^2 \psi$$

(27)

Noteworthy is the fact that with $l = 0$ or $l = -1$ and $k_1^2 < |b|$ the corresponding $f^+$ time-independent factor norm is finite and therefore $\psi$ in this case describes a time-dependent bound state.

A concomitant consequence to assuming that $c$ is a function of time is that it might be necessary or plausible to consider also the time dependence of some of the other so called "fundamental constants", [7], [4]. In this latter arXiv paper I found plausible to accept that Newtons gravitational constant $G$ and the fine structure constant $\alpha$ should be kept constants. And that on the contrary the elementary charge $e$, the electric permittivity $\epsilon$, the magnetic permeability $\mu$, the mass of the elementary particles $m$ and the Planck's constant $h$ should vary as follows:

$$\epsilon = \epsilon_0 \frac{c_0}{c}, \quad \mu = \mu_0 \frac{c_0}{c}, \quad e = e_0 \frac{c}{c_0}, \quad h = h_0 \frac{c^2}{c_0}, \quad e = e_0 \frac{c}{c_0}, \quad m = m_0 \frac{c}{c_0}.$$

(28)

If this is the case then we have that:

$$\frac{m^2 c^2}{h^2} = \frac{m_0^2 c_0^2}{h_0^2}$$

(29)

and (27) can equivalently be written:

$$\Delta_4 \psi = \frac{m^2 c^4}{h^2} \psi, \quad \text{with} \quad \frac{m^2 c^2}{h^2} = k_0^2 - k_1^2.$$  

(30)

Figure 1 is the graph of $c$ corresponding to $b = -0.45$, and $\Lambda = 1.65$ (units as in [4]).

Figures 2 are the graphs of the real and imaginary parts of $B_2(c(t))$ with $k_0 = 0$.

References

[1] E. Alvarez and L. Bel Astro. Phys. J. 179, pp.391-393 (1973)
[2] S. Weinberg, Gravitation... Chap. 13, John Wiley & Sons (1972)
[3] W. Rindler, Essential Relativity Chap. 9, Springer-Verlag (1977)
[4] Ll. Bel arXiv:0309031v1; arXiv:0306091v2;
[5] Ll. Bel [arXiv:1003.5675v2] [physics.gen-ph];
[6] A. Albrecht and J. Magueijo, Phys. Rev. D59, 043416, [astro-ph/9811018]
[7] J. Magueijo, J. Barrow and H. Sandvik, Phys. Lett. B549 pp. 284-289, (2002)
[8] J. W. Moffat [arXiv:gr-qc/9211020v2; arXiv:1404.5567v3] [astro-ph.CO]
[9] J. W. Moffat
[10] G. F. Ellis and J.-P. Uzan, Amer. J. Phys 73, 240, (2005); gr-qc/0305099