Lattice study of the Boer-Mulders transverse momentum distribution in the pion

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Fundamental TMD correlator

\[ \Phi^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S \vert q(0) \Gamma U[0, \ldots, b] q(b) \vert P, S \rangle \]

\[ \Phi^{[\Gamma]}(x, k_T, P, S, \ldots) \equiv \int \frac{d^2 b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi) P^+} \exp \left( i x (b \cdot P) - i b_T \cdot k_T \right) \frac{\Phi^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots)}{S(b^2, \ldots)} \bigg|_{b^+ = 0} \]

- “Soft factor” \( S \) required to subtract divergences of Wilson line \( \mathcal{U} \)
- \( S \) is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel
Gauge link structure motivated by SIDIS

Gauge link structure:

In matrix element \( \Phi^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \)

\[
\frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \ldots, b] q(b) | P, S \rangle
\]

Staple-shaped gauge link \( U[0, \eta v, \eta v + b, b] \)

incorporates SIDIS final state effects

\[ l + H(P) \longrightarrow l' + h(P_h) + X \]
Gauge link structure motivated by SIDIS

Staple-shaped links incorporate SIDIS final state effects:

• Gauge link roughly follows direction of ejected quark, (close to) light cone
• Effective resummed description of gluon exchanges between ejected quark and remainder of nucleon in evolving final state
• Beyond tree level: Rapidity divergences suggest taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes $v$ space-like. Parametrize in terms of Collins-Soper parameter

$$\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$$

Light-like staple for $\hat{\zeta} \to \infty$. Perturbative evolution equations for large $\hat{\zeta}$.

• In this approach, have “modified universality”, $f^{T\text{-odd}}$, SIDIS = $-f^{T\text{-odd}}$, DY (initial state interactions in DY case). SIDIS: $\eta v \cdot P \to \infty$, DY: $\eta v \cdot P \to -\infty$. 
Fundamental TMD correlator

\[ \tilde{\Phi}_\text{unsubtr.}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \eta v, \eta v + b, b] q(b) | P, S \rangle \]

\[ \Phi^{[\Gamma]}(x, k_T, P, S, \ldots) \equiv \int \frac{d^2b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi)^+} \exp(i x (b \cdot P) - i b_T \cdot k_T) \left. \frac{\tilde{\Phi}_\text{unsubtr}^{[\Gamma]}(b, P, S, \ldots)}{S(b^2, \ldots)} \right|_{b^+ = 0} \]

- “Soft factor” \( S \) required to subtract divergences of Wilson line \( U \)
- \( S \) is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel
Decomposition of $\Phi$ into TMDs

All leading twist structures:

$$\Phi[\gamma^+] = f_1 - \left[ \epsilon_{ij} k_i S_j \left( \frac{f}{m_H} \right) \right]_{\text{odd}}$$

$$\Phi[\gamma^+ \gamma^5] = \Lambda g_1 + \frac{k_T \cdot S_T}{m_H} g_{1T}$$

$$\Phi[i\sigma^i + \gamma^5] = S_i h_1 + \left( \frac{2k_i k_j - k_T^2 \delta_{ij}}{2m_H^2} \right) S_j h_{1T} + \frac{\Lambda k_i}{m_H} h_{1L} + \left[ \epsilon_{ij} k_j \frac{h_1}{m_H} \right]_{\text{odd}}$$
TMD Classification

All leading twist structures:

| q → | H | U | L | T |
|-----|---|---|---|---|
| H   |   | f₁ |   | h₁⊥ |
| U   |   | g₁ |   | h₁⊥ |
| L   |   |   | g₁ | h₁⊥ |
| T   |   | f₁T | g₁T | h₁ | h₁⊥ |

↑
Sivers (T-odd)

← Boer-Mulders (T-odd)
Decomposition of $\Phi$ into amplitudes

$$\Phi_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma U[0, \eta v, \eta v + b, b] q(b) | P, S \rangle$$

Decompose in terms of invariant amplitudes; at leading twist,

$$\frac{1}{2P^+} \Phi^{[\gamma^+]}_{\text{unsubtr.}} = A_{2B} + i m_H \epsilon_{ij} b_i S_j \bar{A}_{12B}$$

$$\frac{1}{2P^+} \Phi^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} = -\Lambda A_{6B} + i [(b \cdot P) \Lambda - m_H (b_T \cdot S_T)] \bar{A}_{7B}$$

$$\frac{1}{2P^+} \Phi^{[i\sigma^i+\gamma^5]}_{\text{unsubtr.}} = i m_H \epsilon_{ij} b_j \bar{A}_{4B} - S_i \bar{A}_{9B}$$

$$- i m_H \Lambda b_i \bar{A}_{10B} + m_H [(b \cdot P) \Lambda - m_H (b_T \cdot S_T)] b_i \bar{A}_{11B}$$
Fourier-transformed TMDs

\[ \tilde{f}(x, b_T^2, \ldots) \equiv \int d^2k_T \exp(ib_T \cdot k_T) f(x, k_T^2, \ldots) \]

\[ \tilde{f}^{(n)}(x, b_T^2, \ldots) \equiv n! \left( -\frac{2}{m_H^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \ldots) \]

In limit \( |b_T| \to 0 \), recover \( k_T \)-moments:

\[ \tilde{f}^{(n)}(x, 0, \ldots) \equiv \int d^2k_T \left( \frac{k_T^2}{2m_H^2} \right)^n f(x, k_T^2, \ldots) \equiv f^{(n)}(x) \]

ill-defined for large \( k_T \), so will not attempt to extrapolate to \( b_T = 0 \), but give results at finite \( |b_T| \).

Also, we can only access limited range of \( b \cdot P \), so cannot Fourier-transform to obtain \( x \)-dependence. For now, consider only first \( x \)-moments (accessible at \( b \cdot P = 0 \)):

\[ f^{[1]}(k_T^2, \ldots) \equiv \int_{-1}^{1} dx \ f(x, k_T^2, \ldots) \]
Relation between Fourier-transformed TMDs and invariant amplitudes $\tilde{A}_i$

Invariant amplitudes directly give selected $x$-integrated TMDs in Fourier ($b_T$) space (showing just the ones relevant for Sivers, Boer-Mulders shifts), up to soft factors:

\[
\tilde{f}_1^{[1](0)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\tilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b_T^2, \ldots)
\]

\[
\tilde{f}_{1T}^{[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = -2\tilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b_T^2, \ldots)
\]

\[
\tilde{h}_1^{[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\tilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\tilde{S}(b_T^2, \ldots)
\]
Generalized shifts

Form ratios in which soft factors, (Γ-independent) multiplicative renormalization factors cancel

Boer-Mulders shift:

$$\langle k_y \rangle_{UT} \equiv m_H \frac{\tilde{h}^{1\|1\}[1]}{f_1^{1\|1}[0]} = \frac{\int dx \int d^2 k_T k_y \Phi[\gamma^+ + s^j i \sigma^j + \gamma^5](x, k_T, P, \ldots)}{\int dx \int d^2 k_T \Phi[\gamma^+ + s^j i \sigma^j + \gamma^5](x, k_T, P, \ldots)} \bigg|_{s_T=(1,0)}$$

Average transverse momentum of quarks polarized in the orthogonal transverse ("T") direction in an unpolarized ("U") hadron; normalized to the number of valence quarks. "Dipole moment" in $b_T^2=0$ limit, "shift".

Issue: $k_T$-moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at nonzero $b_T^2$,

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}^{1\|1\}[1]}{f_1^{1\|1}[0]}(b_T^2, \ldots)$$

(remember singular $b_T \to 0$ limit corresponds to taking $k_T$-moment). "Generalized shift".
Generalized shifts from amplitudes

Now, can also express this in terms of invariant amplitudes:

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}_1^{[1]}(1)(b_T^2, \ldots)}{f_1^{[1]}(0)(b_T^2, \ldots)} = m_H \frac{\tilde{A}_{1B}(b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\tilde{A}_{2B}(b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Analogously, Sivers shift (in a polarized hadron):

$$\langle k_y \rangle_{TU}(b_T^2, \ldots) = -m_H \frac{\tilde{A}_{12B}(b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\tilde{A}_{2B}(b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$
Lattice setup

- Evaluate directly \( \tilde{\Phi}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2}\langle P, S| \bar{q}(0) \Gamma \mathcal{U}[0, \eta v, \eta v + b, b] q(b) |P, S\rangle \)

- Euclidean time: Place entire operator at one time slice, i.e., \( b, \eta v \) purely spatial

- Since generic \( b, v \) space-like, no obstacle to boosting system to such a frame!

- Parametrization of correlator in terms of \( \tilde{A}_i \) invariants permits direct translation of results back to original frame

- Form desired ratios of \( \tilde{A}_i \) invariants

- Extrapolate \( \eta \to \infty, \hat{\zeta} \to \infty \) numerically. Pion: largest \( \hat{\zeta} = 2.03 \)

- Use MILC 2+1-flavor gauge ensemble with \( a \approx 0.12 \text{ fm}, m_{\pi} = 518 \text{ MeV} ; 20^3 \times 64 \)

- Use variety of \( P, b, \eta v \); note \( b \perp P, b \perp v \) (lowest \( x \)-moment, kinematical choices/constraints)
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $|b_T|$

\[
m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \quad \text{(GeV)}
\]

- up-quarks
- $\hat{\zeta} = 1.01$
- $|b| = 0.12 \text{ fm}$
- $m_\pi = 518 \text{ MeV}$
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $|b_T|$
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $|b_T|$

$m_\pi \bar{A}_{4B}/\bar{A}_{2B}$ (GeV)

- up-quarks
  - $\hat{\zeta} = 1.01$
  - $|b| = 0.36$ fm
  - $m_\pi = 518$ MeV

$\eta|v|$ (lattice units)
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $|b_T|$

$m_\pi \tilde{A}_{4B}/\tilde{A}_{2B}$ (GeV)

- up-quarks
- $\hat{\zeta} = 1.01$
- $|b| = 0.48$ fm
- $m_\pi = 518$ MeV

$\eta|v|$ (lattice units)
Results: Boer-Mulders shift

Dependence of SIDIS limit on $|b_T|$
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$

\[
m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \quad (\text{GeV})
\]

Graph showing dependence of $m_\pi \tilde{A}_{4B}/\tilde{A}_{2B}$ on $\eta |v|$ (lattice units) for up-quarks, $\hat{\zeta} = 0$, $|b| = 0.36 \text{ fm}$, $m_\pi = 518 \text{ MeV}$.
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$

$$m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \quad (\text{GeV})$$

up-quarks
$\hat{\zeta} = 1.01$
$|b| = 0.36 \text{ fm}$
$m_\pi = 518 \text{ MeV}$

$\eta|v|$ (lattice units)
Results: Boer-Mulders shift

Dependence on staple extent; sequence of panels at different $\hat{\eta}$

\[ m_\pi \bar{A}_{4B} / \bar{A}_{2B} \] (GeV)

- up-quarks
  - $\hat{\eta} = 2.03$
  - $|b| = 0.36$ fm
- $m_\pi = 518$ MeV

$\eta |v|$ (lattice units)
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$

\[ m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \]

\[ (GeV) \]

\[ |b_T| = 0.36 \text{ fm} \]

\[ m_\pi = 518 \text{ MeV} \]

\[ P \sim (1, 0, 0) \]

\[ P \sim (1, 1, 0) \]

Contribution $\tilde{A}_4$ only
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$

$$m_\pi \tilde{A}_{4B} / \tilde{A}_{2B} \ (GeV)$$

up-quarks $|b_T| = 0.34 \text{ fm}$

$P \sim (1, 0, 0)$

$P \sim (1, 1, 0)$

$P \sim (1, 1, 1)$

$m_\pi = 518 \text{ MeV}$

Contribution $A_4$ only
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$

$$m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \quad (GeV)$$

| $|b_T| = 0.27 \text{ fm}$ | $m_\pi = 518 \text{ MeV}$ | $P \sim (1, 0, 0)$ | Contribution $\tilde{A}_4$ only |
|---------------------|---------------------|-------------------|-----------------------|
| up-quarks           |                     |                   |                       |
Results: Boer-Mulders shift

Boer–Mulders Shift

\[ \frac{m_N \tilde{b}_4}{\tilde{f}_1} \]

- **total**
- **contrib. from \( \tilde{a}_4 \)**

up – quarks,
\( \hat{\zeta} = 0.39 \),
\( |b| = 0.36 \text{ fm} \)

connected only

\[ m_\pi \frac{\tilde{A}_{4B}}{\tilde{A}_{2B}} \]

- **up-quarks**
  \( \hat{\zeta} = 1.01 \)
  \( |b| = 0.36 \text{ fm} \)
  \( m_\pi = 518 \text{ MeV} \)

connected only

\[ m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \]

**Nucleon**

**Pion**

staple extent \( \eta|v| \) (lattice units)

\[ m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \]

\[ m_\pi \tilde{A}_{4B}/\tilde{A}_{2B} \]
Results: Boer-Mulders shift

Nucleon; flavor separated
Conclusions

• Study of T-odd Boer-Mulders observable in the pion using staple-shaped gauge link structures.

• To avoid soft factors, multiplicative renormalization constants, constructed appropriate ratio of Fourier-transformed TMDs (“shift”).

• $v$ taken off light cone: Dependence on Collins-Soper parameter $\hat{\zeta}$. In addition to $\eta v \to \infty$, need to also consider $\hat{\zeta} \to \infty$.

• $\eta v \to \infty$ seems under good control; plateaux reached at moderate values.

• Significant progress concerning the $\hat{\zeta} \to \infty$ limit compared with earlier study in nucleon. Tentative statements concerning light cone limit possible. Perspective: Develop high momentum interpolating operators.

• Quantitative correspondence between $u$-quark Boer-Mulders ratios in proton, $\pi^+$ meson.