A Mixed Finite Element Method for Cavitation Computation in Incompressible Nonlinear Elasticity

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Abstract

A mixed finite element method, which couples a dual-parametric biquadratic finite element approximation for the deformation and a piecewise affine approximation for the pressure like Lagrange multiplier (DP-Q2-P1), is developed and analyzed for the numerical computation of cavitation problem for incompressible nonlinear elastic materials, and a damped Newton method is applied to solve the resulted discrete problem. The method is proved to be stable, locking free and convergent on properly constructed meshes. The accuracy and efficiency of the method are illustrated by numerical experiments and results.

Key words: DP-Q2-P1 mixed finite element, cavitation computation, incompressible elasticity, locking-free, convergent

1 Introduction

Cavitation is a widely observed phenomenon in ductile nonlinear elastic materials, such as rubbers and polymers, under sufficiently large tension, and it is considered to be one of the most important mechanisms that lead to the failure of the materials. In two well known mathematical models, namely the defect model [1–3] and the perfect model [4], the cavitation phenomenon is characterized as material instability and a bifurcation.

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phenomenon. Both models have been intensively studied \[5, 6\] and the close relationship between their solutions has also been revealed to certain extent analytically and numerically \[7–10\]. Based on these two models, there are numerous work in the analysis and computation of the cavitation phenomenon, devoted by researchers in various fields of material sciences, mechanics, mathematics, etc. (see \[1\]- \[23\] and the references therein). In the numerical computation of cavitation problems, the defect model is so far dominantly used (see \[10\], \[13]-[23\] among many others), because it avoids the so called Lavrentiev phenomenon \[24\]-[26\] intrinsically embedded in the perfect model for the cavitation problem, or vaguely speaking in this particular case, avoids the difficulty of making holes out of nowhere in the deformed configuration.

Let \(0 \in \Omega = B_1(0) = \{x \in \mathbb{R}^n : |x - 0| < 1\}\), where \(n = 2, 3\) is the spatial dimension, and let \(B_\rho(0) = \{x \in \mathbb{R}^n : |x - 0| < \rho\} \subset \Omega\). Let \(\Omega_\rho = \Omega \setminus B_\rho(0)\) be the domain occupied by an elastic body in its reference configuration, where \(B_\rho(0)\) denotes the pre-existing defects of radius \(\rho\) centered at \(0\). Then, in its simplest form of the defect model for incompressible elastic materials, the cavitation problem can be expressed as to find a deformation \(u\) to minimize the total energy

\[
E(u) = \int_{\Omega_\rho} W(\nabla u(x)) \, dx - \int_{\partial\Omega} t \cdot u \, ds,
\]

in the set of admissible deformation functions

\[
\mathcal{A}_I = \{u \in W^{1,s}(\Omega_\rho; \mathbb{R}^n) \text{ is } 1\text{-to-1 a.e.} : \int_{\Omega_\rho} u \, dx = 0, \det \nabla u = 1 \text{ a.e.}\}, \tag{1.2}
\]

where \(W: M_{+}^{n \times n} \to \mathbb{R}^+\) is the stored energy density function of the material with \(M_{+}^{n \times n}\) being the set of \(n \times n\) matrices with eigenvalues all positive, \(t\) is the prescribed traction, and \(1 < s < n\) is a given Sobolev index.

To relax the rather restrictive condition \(\det \nabla u = 1\), a.e. appeared in \(\mathcal{A}_I\), a mixed formulation of the following form (see \[27, 28\]) is usually used in computation:

\[
(u, p) = \arg \sup_{p \in L^2(\Omega_\rho)} \inf_{u \in \mathcal{A}} E(u, p), \tag{1.3}
\]

where \(p \in L^2(\Omega_\rho)\) is a pressure like Lagrangian multiplier introduced to relax the constraint of incompressibility, and where the Lagrangian functional \(E(u, p)\) and the set of admissible deformations \(\mathcal{A}\) are defined as

\[
E(u, p) = \int_{\Omega_\rho} (W(\nabla u) - p(\det \nabla u - 1)) \, dx - \int_{\partial\Omega} t \cdot u \, ds, \tag{1.4}
\]
\[ A = \{ u \in W^{1,\infty}(\Omega_{\rho}; \mathbb{R}^n) \text{ is 1-to-1 a.e. : } \int_{\Omega_{\rho}} u \, dx = 0 \} \]  \hspace{1cm} (1.5)

In the numerical computation of the cavitation problems with the defect model, the main difficulty comes from the extremely large anisotropic deformation near the cavity surface in the form of increasingly severe compression in the radial direction and correspondingly large stretches in circumferential one, especially when the radius \( \rho \) of the initial defect is very small, which causes serious trouble for discrete functions to satisfy the orientation preservation condition, \( i.e. \det \nabla u_h > 0 \), and at the same time to have reasonably accurate cavity approximations one would normally expect \([16, 17]\).

The situation becomes even tougher for incompressible materials, where the orientation preservation condition is replaced by a much more restrictive condition \( \det \nabla u_h = 1 \) or \( \det \nabla u_h \approx 1 \) for the mixed formulation. In addition, for incompressible elasticity, it is well known that, even in the case of small deformation and linear elasticity, the notorious locking phenomenon can happen and ultimately leads to the failure of some finite element approximations \([27–30]\).

Recently, the orientation preservation conditions and error analysis for some numerically successful finite element methods in the cavitation computation for compressible nonlinear elastic materials \([10, 16]\) have been developed, and the results led to efficient meshing strategies which helped greatly in improving the numerical performance of the methods \([20, 22, 23]\).

In the present paper, based on the mixed formulation of \((1.3)\), a mixed finite element method (DP-Q2-P1), which couples a dual-parametric biquadratic finite element approximation for the deformation \( u \) and a piecewise affine approximation for the pressure like Lagrange multiplier \( p \), is developed and analyzed for the numerical computation of cavitation problem for 2-D incompressible nonlinear elastic materials, and a damped Newton method is applied to solve the resulted discrete problem. The method is shown to be stable, locking free and convergent on properly constructed meshes. The performance of the method is illustrated by numerical experiments and results.

The rest of the paper is organized as follows. In § 2, we discuss the solution properties of the Newton method for the equilibrium equations derived from the mixed formulation of the cavitation problem. § 3 is devoted to the construction of the DP-Q2-P1 mixed finite element method and its stability analysis. Convergence analysis of the finite element solutions is given in § 4. Numerical experiments and results are presented in § 5 to show the accuracy and efficiency of the method. Some concluding remarks are given in § 6.
2 The Newton method for the mixed formulation

Consider the mixed formulation of a 2-dimensional ($n = 2$) cavitation problem given by (1.3)-(1.5) for incompressible nonlinear elastic materials. Since $\det \nabla \mathbf{u} = 1$, without loss of generality (see [5]), we consider the energy density of the form

$$W(\nabla \mathbf{u}) = \mu |\nabla \mathbf{u}|^s + d(\det \nabla \mathbf{u}), \quad 1 < s < 2,$$

(2.1)

where $\mu > 0$ is a material constant, $|\cdot|$ denotes the Frobenius norm of a matrix, and $d(\xi) = \kappa (\xi - 1)^2 + d_1(\xi)$, which appears in $E(\mathbf{u})$ only as a constant but plays a crucial role in the convergence analysis of the numerical solutions (see § 4), with $\kappa > 0$ and $d_1 : \mathbb{R}_+ \to \mathbb{R}_+$ being a strictly convex function satisfying

$$d_1(\xi) \to +\infty \text{ as } \xi \to 0, \text{ and } \frac{d_1(\xi)}{\xi} \to +\infty \text{ as } \xi \to +\infty.$$  

(2.2)

2.1 The equilibrium equation and its Newton method

Under certain hypotheses (see (H1)-(H3)), the weak form of the Euler-Lagrange equation, i.e. the equilibrium equation, of the mixed formulation (1.3), can be expressed as

$$\left\{ \begin{array}{l}
\int_{\Omega_\rho} \left( \frac{\partial W(\nabla \mathbf{u})}{\partial \nabla \mathbf{v}} - p (\text{cof} \nabla \mathbf{u} : \nabla \mathbf{v}) \right) \, d\mathbf{x} = \int_{\partial \Omega_\rho} \mathbf{t} \cdot \mathbf{v} \, ds, \quad \forall \mathbf{v} \in \mathcal{X}, \\
\int_{\Omega_\rho} q (\det \nabla \mathbf{u} - 1) \, d\mathbf{x} = 0, \quad \forall q \in \mathcal{M},
\end{array} \right.$$  

(2.3)

where, and throughout the paper, $\text{cof} \nabla \mathbf{u}$ denotes the cofactor matrix of $\nabla \mathbf{u}$,

$$\mathcal{X} = \left\{ \mathbf{v} \in H^1(\Omega_\rho; \mathbb{R}^2) : \int_{\Omega_\rho} \mathbf{v} \, d\mathbf{x} = 0 \right\}, \quad \mathcal{M} = L^2(\Omega_\rho),$$

(2.4)

are the test function spaces for the deformation and pressure respectively.

The Newton method to solve the equation (2.3) is formally given as: For a given initial deformation $(\mathbf{u}_0, p_0) \in (\mathcal{A} \cap W^{1,\infty}(\Omega_\rho; \mathbb{R}^2)) \times L^2(\Omega_\rho)$, and $k = 0, 1, \cdots$, find $(\mathbf{u}_{k+1}, p_{k+1}) \in (\mathcal{A} \cap W^{1,\infty}(\Omega_\rho; \mathbb{R}^2)) \times L^2(\Omega_\rho)$ such that

$$\delta E(\mathbf{u}_k, p_k)[\mathbf{v}, q] + \delta^2 E(\mathbf{u}_k, p_k)[\mathbf{u}_{k+1} - \mathbf{u}_k, \mathbf{v}; p_{k+1} - p_k, q] = 0, \quad \forall (\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}. \quad (2.5)$$

Denote $\underline{\mathbf{u}} := \mathbf{u}_k$, $\underline{p} := p_k$ and $\mathbf{w} := \mathbf{u}_{k+1} - \mathbf{u}_k$, $\underline{p} := p_{k+1} - p_k$, and let

$$a(\mathbf{w}, \mathbf{v}; \underline{\mathbf{u}}, \underline{p}) := \int_{\Omega_\rho} \left( \mu s (s - 2)|\nabla \underline{\mathbf{u}}|^{s-4}(\nabla \underline{\mathbf{u}} : \nabla \mathbf{w})(\nabla \underline{\mathbf{u}} : \nabla \mathbf{v}) \\
+ \mu s |\nabla \underline{\mathbf{u}}|^{s-2}(\nabla \mathbf{w} : \nabla \mathbf{v}) + d''(\det \nabla \underline{\mathbf{u}})(\text{cof} \nabla \underline{\mathbf{u}} : \nabla \mathbf{w})(\text{cof} \nabla \underline{\mathbf{u}} : \nabla \mathbf{v}) \\
+ (d''(\det \nabla \underline{\mathbf{u}}) - \underline{p}) \text{cof} \nabla \mathbf{w} : \nabla \mathbf{v} \right) \, d\mathbf{x}, \quad (2.6)$$

$$b(\mathbf{v}, q; \underline{\mathbf{u}}) := \int_{\Omega_\rho} q \text{cof} \nabla \underline{\mathbf{u}} : \nabla \mathbf{v} \, d\mathbf{x}, \quad (2.7)$$
\begin{align*}
  f(v; u, p) &:= \int_{\partial \Omega} t \cdot v \, ds - \int_{\Omega_p} (\mu_s |\nabla u|^{s-2} \nabla u + (d' (\det \nabla u) - p) \operatorname{cof} \nabla u) : \nabla v \, dx, \quad (2.8) \\
  g(q; u) &:= -\int_{\Omega_p} q (\det \nabla u - 1) \, dx. \quad (2.9)
\end{align*}

Then, in an iteration of the Newton method, one solves the following problem:

\begin{align*}
  \begin{cases}
    \text{Find } (w, p) \in X \times M, \text{ such that } \\
    a(w, v; u, p) + b(v, p; u) = f(v; u, p), \quad \forall v \in X, \\
    b(w, q; u) = g(q; u), \quad \forall q \in M.
  \end{cases} \quad (2.10)
\end{align*}

To simplify the notation, \(a(w, v; u), b(w, q; u), f(v; u), g(q; u)\) will be denoted as \(a(w, v), b(w, q), f(v), g(q)\) whenever \((u, p)\) is not directly involved in the calculation.

### 2.2 Solvability of the Newtonian iteration problem

Let \(a(\cdot, \cdot) : X \times X \to \mathbb{R}, b(\cdot, \cdot) : X \times M \to \mathbb{R}\) be the bilinear forms defined by (2.6) and (2.7) for given \((u, p)\). Let the operators \(B : X \to M'\) and \(B' : M \to X'\) be defined by \(Bv(q) := b(v, q)\) and \(B'q(v) := b(v, q)\) respectively, and denote their kernels as

\[ \text{Ker}B := \{ v \in X : b(v, q) = 0, \forall q \in M \}, \quad \text{Ker}B' := \{ q \in M : b(v, q) = 0, \forall v \in X \}. \]

Noticing that \(a(\cdot, \cdot)\) is symmetric, by the well known theorem given by Brezzi & Fortin (see Theorem 1.1 on page 42 in [27]), one has that the Newtonian iteration problem (2.10) has a unique solution \((u, p) \in X \times (M/\text{Ker}B')\) which continuously depends on \(f \in X'\) and \(g \in \text{Im}B\), if the bilinear forms satisfy the following conditions:

\begin{itemize}
  \item[(i)] The symmetric bilinear form \(a(\cdot, \cdot)\) is continuous and invertible on \(\text{Ker}B\), i.e. there exists a constant \(\alpha > 0\) such that
    \[ \inf_{w \in \text{Ker}B} \sup_{v \in \text{Ker}B} \frac{a(w, v)}{\|w\|_X \|v\|_X} \geq \alpha. \quad (2.11) \]
  \item[(ii)] The bilinear form \(b(\cdot, \cdot)\) is continuous and satisfies the inf-sup condition, i.e. there exists a constant \(\beta > 0\) such that
    \[ \sup_{v \in X} \frac{b(v, q)}{\|v\|_X} \geq \beta \|q\|_{M/\text{Ker}B'}. \quad (2.12) \]
\end{itemize}

We restrict ourselves to the cavity solution \(\bar{u}\) which is a stable absolute minimizer of \(E(u)\) in \(A_I\). The failure of invertibility condition (2.11) implies that \(\bar{u}\) is a bifurcation point (see Chapter 6 on pages 379-384 in [31]), which can not be a stable cavitation solution. To proceed with the Newtonian iterations with (2.10), we assume...
(H1) There exists a solution \((\tilde{u}, \tilde{p}) \in (A \cap W^{1,\infty}(\Omega_\rho; \mathbb{R}^2)) \times L^\infty(\Omega_\rho)\) to the equilibrium equation \((2.3)\), where \(\tilde{u}\) is an absolute energy minimizer of \(E(u)\) in \(A_I\).

(H2) Given \((f, g) \in \mathcal{X}' \times \text{Im} B\), there exists a unique solution \((w, p) \in \mathcal{X} \times \mathcal{M}\) to \((2.10)\).

(H3) For given \((u, p) \in (A \cap W^{1,\infty}(\Omega_\rho; \mathbb{R}^2)) \times L^2(\Omega_\rho)\), the solution \((w, p) \in \mathcal{X} \times \mathcal{M}\) to \((2.10)\), if exists, satisfies \((u + w, p + p) \in (A \cap W^{1,\infty}(\Omega_\rho; \mathbb{R}^2)) \times L^2(\Omega_\rho)\).

In fact, in the radially symmetric case one has \(|\tilde{u}|_{1,\infty} = O(\rho^{-1})\) with the bound achieved on the cavity surface (see [3]), and in general (H1) holds for the defect model when \(\partial \Omega_\rho\) as well as the boundary data given are sufficiently smooth. (H2) equivalent to that \(b(v, q)\) satisfies the inf-sup condition \((2.12)\) and \(a(w, v)\) satisfies the invertibility condition \((2.11)\) (see [32]). The regularity hypothesis (H3), for the Newtonian iteration problem \((2.10)\), is expected to hold at least in a \(W^{1,\infty}\)-neighborhood of the stable cavity solution \(\tilde{u}\) which is assumed to be an absolute energy minimizer of \(E(u)\) in \(A_I\).

Remark 1. If \(\nabla u\) in each iteration of the Newton method is positive definite, which can somehow be numerically verified, and satisfies certain regularity condition, then \(b(v, q)\) can be proved to satisfy the inf-sup condition \((2.12)\) (see [33]).

3 The mixed finite element method and its stability

In this section, we will establish the discrete version of the equilibrium equation \((2.3)\) and the Newtonian iteration problem \((2.10)\), using a dual-parametric biquadratic finite element for the deformation and a linear finite element for the pressure (DP-Q2-P1 element) which is proved to be stable and locking free for the cavitation problem.

3.1 The DP-Q2-P1 mixed finite element

Let \((\hat{T}, \hat{P}, \hat{\Sigma})\) be the standard biquadratic-linear mixed rectangular element,

\[
\begin{aligned}
\hat{T} &= [-1, 1] \times [-1, 1] \text{ is the standard reference quadrilateral element,} \\
\hat{P} &= \{Q_2(\hat{T}), P_1(\hat{T})\}, \\
\hat{\Sigma} &= \{\hat{u}(\hat{a}_i), 0 \leq i \leq 8; \ \hat{p}(\hat{b}_0), \partial x_1 \hat{p}(\hat{b}_0), \partial x_2 \hat{p}(\hat{b}_0)\},
\end{aligned}
\]

where \(\{\hat{a}_i\}_{i=0}^8\) are the vertices of \(\hat{T}\), \(\{\hat{a}_i\}_{i=4}^7\) represent the nodes on the middle points of the corresponding edges of \(\hat{T}\), and \(\hat{a}_8 = \hat{b}_0 = (0, 0)\), as shown in Figure 1.
For a given set of four points $a_i = (R_i \cos \theta_i, R_i \sin \theta_i)$, $i = 0, 1, 2, 3$, satisfying $R_0 = R_3 < R_1 = R_2$, $\theta_0 = \theta_1 < \theta_2 = \theta_3$ as shown in Figure 2, define $F_T : \hat{T} \to \mathbb{R}^2$ by

$$
\begin{align*}
  R &= R_0 + \frac{x_1 + 1}{2}(R_1 - R_0), \\
  \theta &= \theta_0 + \frac{x_2 + 1}{2}(\theta_3 - \theta_0), \\
  x_1 &= R \cos \theta, \\
  x_2 &= R \sin \theta.
\end{align*}
$$

Obviously, $F_T$ is a injection, thus $T = F_T(\hat{T})$ defines an element. We define the dual-parametric biquadratic-linear (DP-Q2-P1) mixed finite element $(T, P, \Sigma)$ as follows:

$$
\begin{align*}
  T &= F_T(\hat{T}) \text{ being a circular ring sector element,} \\
  P_T &= \{ (\mathbf{u}, p) : T \to \mathbb{R}^2 \times \mathbb{R} | \mathbf{u} = \hat{\mathbf{u}} \circ F_T^{-1}, \hat{\mathbf{u}} \in Q_2, \\
  &\quad p = \hat{p} \circ F_T^{-1}, \hat{p} \in P_1 \}, \\
  \Sigma_T &= \{ \mathbf{u}(a_i), a_i = F_T(\hat{a}_i), 0 \leq i \leq 8; p(b_0), \partial_R p(b_0), R^{-1} \partial_\theta p(b_0), b_0 = F_T(\hat{b}_0) \},
\end{align*}
$$

and denoted as $\bar{Q}_2 \times \bar{P}_1 = F_T(Q_2) \times F_T(P_1)$. Accordingly, the discrete test and trial function spaces for the admissible finite element deformation and pressure are given as

$$
\begin{align*}
  \mathcal{X}_h &= \mathcal{A}_h := \{ \mathbf{u}_h \in C(\bar{\Omega}_p; \mathbb{R}^2) : \mathbf{u}_h|_T \in F_T(Q_2), \\ 
  &\quad \int_{\Omega_p} \mathbf{u}_h \, d\mathbf{x} = 0 \}, \quad (3.1) \\
  \mathcal{M}_h &= \mathcal{P}_h := \{ p_h \in L^2(\bar{\Omega}_p) : p_h|_T \in F_T(P_1) \}. \quad (3.2)
\end{align*}
$$
A typical mesh $T_h$ consisting of well defined circular ring sector elements on $\Omega_\rho = B_1(0) \setminus B_\rho(0)$ is shown in Figure 3 where we have $N = 8$ evenly spaced elements in each of the 3 circular ring layers. A typical circular ring sector element $T$ in a prescribed circular ring with inner radius $\rho_T$ and thickness $\tau_T$ is shown in Figure 4.

Noticing that $F_T$ is dual-parametric with $|F_T| = (\frac{\tau_T^2}{4} + \frac{R^2\pi}{N})^2$ and $\det F_T = \frac{R \pi \tau_T}{2N}$, it is not difficult for us to show, by the standard scaling argument, that

$$|\hat{v}|_{\gamma,2,T} \cong h_{\gamma,T}^{-1}|v|_{\gamma,2,T}, \quad \gamma = 0, 1, \quad \forall T \in T_h, \quad \text{and} \quad \forall v \in H^1(T; \mathbb{R}^2), \quad (3.3)$$

which will be used in our stability and convergence analysis in § 4, remains valid if the triangulation $T_h$ satisfies further

$$(H4) \quad \text{The triangulation } T_h \text{ is regular, i.e. } |\widehat{a_0a_3}| \cong |a_0a_1|, \quad h_T \cong h, \quad \forall T \in T_h.$$  

Here and throughout the paper, $X \cong Y$, or equivalently $Y \lesssim X \lesssim Y$, means that $C^{-1}Y \leq X \leq CY$ holds for a generic constant $C \geq 1$ independent of $\rho, T$ and $h$.

Recall $X_h := A_h \subset H^1(\Omega_\rho; \mathbb{R}^2)$, $M_h := P_h \subset L^2(\Omega_\rho)$. In the DP-Q2-P1 mixed finite element method, the equilibrium equation (2.3) is discretized into the following form

$$\begin{align*}
\int_{\Omega_\rho} \frac{\partial W(\nabla u_h)}{\nabla u_h} : \nabla v_h - p_h \cof \nabla u_h : \nabla v_h \, dx &= \int_{\partial \Omega} t \cdot v_h \, ds, \quad \forall v_h \in X_h, \\
\int_{\Omega_\rho} q_h (\det \nabla u_h - 1) \, dx &= 0, \quad \forall q_h \in M_h,
\end{align*} \quad (3.4)$$

and, in an iteration of the damped Newton method to solve this nonlinear discrete problem, one solves the following discrete linear problem

$$\begin{align*}
\text{Find } (w_h; p_h) \in X_h \times M_h, \text{ such that } \\
a(w_h; v_h; u_h; p_h) + b(v_h; p_h; u_h) &= f(v_h; u_h; p_h), \quad \forall v_h \in X_h, \\
b(w_h; q_h; u_h) &= g(q_h; u_h), \quad \forall q_h \in M_h,
\end{align*} \quad (3.5)$$

to obtain a direction $(w_h; p_h)$, and conduct an incomplete linear search so that the new guess $(u_h, p_h) + \alpha(w_h, p_h)$ is orientation preserving and satisfies (H5) given below.

### 3.2 Stability of the DP-Q2-P1 mixed finite element method

The key for the DP-Q2-P1 mixed finite element method to be stable and locking free for the problem (3.5) is that the discrete inf-sup condition (or LBB condition)

$$\sup_{v_h \in X_h} \frac{b(v_h; q_h; u_h)}{\|v_h\|_{1,2,\Omega_\rho}} \geq \beta \|q_h\|_{0,2,\Omega_\rho}, \quad \forall q_h \in M_h \quad (3.6)$$

8
holds for a constant $\beta$ independent of the mesh size $h$, which can be established by means of the famous Fortin criterion (see Proposition 2.8 on page 58 in [27]) and a general two steps construction frame as given in Lemma 2 (see Proposition 2.9 on page 59 in [27]), under the regularity hypothesis (H4) for the mesh $T_h$, and the the regularity hypothesis (H5) given below for $u_h$. In fact, (H5) generally holds for a proper discrete cavity deformation $u_h$, which is not necessarily the cavity solution of the problem.

(H5) $\rho_T \lesssim \lambda_1(\nabla u_h) \leq \lambda_2(\nabla u_h) \lesssim 1/\rho_T$, and $0 < c \leq \det \nabla u_h \leq C$, $\forall x \in T \in T_h$,

where $\lambda_1(\nabla u_h) \leq \lambda_2(\nabla u_h)$ are the eigenvalues of $\nabla u_h$, and $0 < c < 1 < C$ are constants independent of $\rho$ and $h$.

**Lemma 1.** (see Fortin Criterion [27]) The LBB condition (3.6) holds with a constant $\beta$ independent of $h$ if and only if there exists an operator $\Pi_h \in \mathcal{L}(X, X_h)$ satisfying:

$$\begin{cases}
    b(v - \Pi_h v, q_h; u_h) = 0, & \forall q_h \in M_h, \forall v \in X, \\
    \|\Pi_h v\|_{1,2,\Omega_o} \leq c\|v\|_{1,2,\Omega_o}, & \forall v \in X
\end{cases}$$

with a constant $c > 0$ independent of $h$.

**Lemma 2.** Let $\Pi_1 \in \mathcal{L}(X, X_h)$ and $\Pi_2 \in \mathcal{L}(X, X_h)$ be such that

$$\begin{cases}
    \|\Pi_1 v\|_{1,2,\Omega_o} \leq c_1\|v\|_{1,2,\Omega_o}, & \forall v \in X, \\
    \|\Pi_2(I - \Pi_1)v\|_{1,2,\Omega_o} \leq c_2\|v\|_{1,2,\Omega_o}, & \forall v \in X, \\
    b(v - \Pi_2 v, q_h; u_h) = 0, & \forall v \in X, \forall q_h \in M_h.
\end{cases}$$

Set $\Pi_h v = \Pi_1 v + \Pi_2(v - \Pi_1 v)$, then $\Pi_h \in \mathcal{L}(X, X_h)$ satisfies (3.7).

**Step 1.** The construction of $\Pi_1 \in \mathcal{L}(X, X_h)$. Let $(X_h, M_h)$ be given by

$$\begin{align}
    X_h &= \{v_h \in X : v_h|_T \in \hat{Q}_1 \oplus \text{span}\{p_1, p_2, p_3, p_4\}\}, \\
    \mathcal{M}_h &= \{q_h \in M_h : q_h|_T \in \hat{P}_0\},
\end{align}$$

where $\hat{Q}_1 = F_T(Q_1)$, $\hat{P}_0 = F_T(P_0)$, and $\{p_i\}_{i=1}^4$ are the edge bubble functions with respect to the edges $\{e_i\}_{i=1}^4$ of $T$: for $i = 1$, let $x = (x_1, x_2)$ and $\hat{x} = F_T^{-1}(x) = (\hat{x}_1, \hat{x}_2)$, then

$$p_1(x) = (\hat{q}_1 \circ F_T^{-1}(\hat{x}))n_1(F_T(-1, \hat{x}_2)),$$

where $\hat{q}_1 = (1 - \hat{x}_1^2)(1 - \hat{x}_2)$ and $n_1$ is the unit out normal of the edge $e_1$, obviously $p_1(x) = 0, \forall x \in \partial T \setminus e_1$, formulae for $\{p_i\}_{i=2}^4$ are similar. In particular, we notice that $\{p_i\}_{i=1}^4$ have zero tangential components on the edges of $T = F_T(T)$. 

Firstly, let \( \tilde{\Pi}_1 : \mathcal{X} \to \{ \mathbf{v}_h \in C(\Omega_\rho; \mathbb{R}^2) : \mathbf{v}_h|_T \in \tilde{Q}_1, \forall T \in \mathcal{T}_h \} \) be the Clément interpolation operator, since (H4) is satisfied, it follows from the standard scaling argument (see for example Corollary 2.1 on page 106 in [26]) that
\[
\sum_{T \in \mathcal{T}_h} h_T^{2\gamma-2} |\mathbf{v} - \tilde{\Pi}_1 \mathbf{v}|_{\gamma,2,T}^2 \lesssim |\mathbf{v}|_{1,2,\Omega_\rho}^2, \quad \gamma = 0, 1. \quad (3.10)
\]
Define \( \bar{\Pi}_1 \mathbf{v} = \tilde{\Pi}_1 \mathbf{v} - \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} \tilde{\Pi}_1 \mathbf{v} \, dx \), then \( \bar{\Pi}_1 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h) \). Since, \( \int_{\Omega_\rho} \mathbf{v} \, dx = 0 \), it follows from the Hölder inequality, \( (3.10) \) and \( h_T \equiv h \) (see (H4)) that
\[
\left| \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} \bar{\Pi}_1 \mathbf{v} \, dx \right| \leq \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} |\bar{\Pi}_1 \mathbf{v} - \mathbf{v}| \, dx \lesssim \|\bar{\Pi}_1 \mathbf{v} - \mathbf{v}\|_{0,2,\Omega_\rho} \lesssim h |\mathbf{v}|_{1,2,\Omega_\rho}.
\]
Consequently, by \( (3.10) \) and \( h_T \equiv h \), we have
\[
|\mathbf{v} - \tilde{\Pi}_1 \mathbf{v}|_{\gamma,2,\Omega_\rho} \lesssim h^{1-\gamma} |\mathbf{v}|_{1,2,\Omega_\rho}, \quad \gamma = 0, 1. \quad (3.11)
\]
Next, let \( \tilde{\Pi}_2 : \mathcal{X} \to \{ \mathbf{v}_h \in C(\Omega_\rho; \mathbb{R}^2) : \mathbf{v}_h|_T \in \text{span}\{ \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \} \} \) be defined by
\[
\begin{aligned}
\tilde{\Pi}_2 \mathbf{v}|_T &\in \text{span}\{ \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \}, \\
\int_{e_i} (\text{cof} \, \nabla \mathbf{u}_h^T \tilde{\Pi}_2 \mathbf{v}) \cdot \mathbf{n}_i \, ds &= \int_{e_i} (\text{cof} \, \nabla \mathbf{u}_h^T \mathbf{v}) \cdot \mathbf{n}_i \, ds, \quad \forall e_i = F^{-1}_T(\hat{e}_i), \quad i = 1, 2, 3, 4.
\end{aligned} \quad (3.12)
\]
Define \( \Pi_2 \mathbf{v} = \tilde{\Pi}_2 \mathbf{v} - \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} \tilde{\Pi}_2 \mathbf{v} \, dx \), then we have \( \nabla \Pi_2 \mathbf{v} = \nabla \tilde{\Pi}_2 \mathbf{v} \), for all \( \mathbf{v} \in H^1(\Omega_\rho; \mathbb{R}^2) \), \( \tilde{\Pi}_2 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h) \), and in particular, as \( \text{div}(\text{cof} \, \nabla \mathbf{u}_h|_T) = 0 \), we have
\[
\int_T \text{cof} \, \nabla \mathbf{u}_h : \nabla (\Pi_2 \mathbf{v} - \mathbf{v}) \, dx = \int_{\partial T} (\text{cof} \, \nabla \mathbf{u}_h (\tilde{\Pi}_2 \mathbf{v} - \mathbf{v})) \cdot \mathbf{n} \, ds = 0. \quad (3.13)
\]
Now, define \( \Pi_1 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h) \) by setting \( \Pi_1 \mathbf{v} \triangleq \bar{\Pi}_1 \mathbf{v} = \tilde{\Pi}_1 \mathbf{v} + \tilde{\Pi}_2 (\mathbf{v} - \bar{\Pi}_1 \mathbf{v}), \forall \mathbf{v} \in \mathcal{X} \).

**Step 2.** The construction of \( \Pi_2 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h) \). Denote the bi-quadratic bubble function space on \( \hat{T} \) by \( \hat{\mathcal{B}} = \{ \mathbf{b}(\hat{x}) = (b_1(1 - \hat{x}_1^2)(1 - \hat{x}_2^2), b_2(1 - \hat{x}_1^2)(1 - \hat{x}_2^2)) \} \). Define
\[
\mathcal{B}_h = \{ \mathbf{b} \in C(\hat{\Omega}_\rho; \mathbb{R}^2) : \mathbf{b}|_T = \hat{\mathbf{b}} \circ F_{T}^{-1}, \hat{\mathbf{b}} \in \hat{\mathcal{B}} \}.
\] (3.14)
Notice that \( \mathcal{X}_h = \tilde{\mathcal{X}}_h + \mathcal{B}_h \). Define \( \Pi_2 : \{ \mathbf{v} \in \mathcal{X} : \int_T \text{cof} \, \nabla \mathbf{u}_h : \nabla \mathbf{v} \, dx = 0 \} \to \mathcal{B}_h \) as the unique solution of
\[
\int_T \text{cof} \, \nabla \mathbf{u}_h : \nabla (\Pi_2 \mathbf{v} - \mathbf{v}) \, q_h \, dx = 0, \quad \forall q_h \in \hat{P}_1(T) \setminus \hat{P}_0(T), \forall T \in \mathcal{T}_h. \quad (3.15)
\]
Since \( \Pi_2 \mathbf{v} \) is a bubble function on \( T \in \mathcal{T}_h \) and \( \text{div}(\text{cof} \, \nabla \mathbf{u}_h|_T) = 0 \), we have
\[
\int_T \text{cof} \, \nabla \mathbf{u}_h : \nabla \Pi_2 \mathbf{v} \, dx = \int_{\partial T} (\text{cof} \, \nabla \mathbf{u}_h) \mathbf{n} \cdot \Pi_2 \mathbf{v} \, ds - \int_T \text{div}(\text{cof} \, \nabla \mathbf{u}_h) \cdot \Pi_2 \mathbf{v} \, dx = 0. \quad (3.16)
\]
Lemma 3. Let $\mathcal{T}_h$ and $u_h$ satisfy hypothesis (H4) and (H5) respectively. Then, $\Pi_1 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h)$, as defined in step 1, satisfies

\[
\frac{\|\Pi_1 v\|_{1, 2, \Omega}}{\rho_T} \lesssim \frac{1}{\rho_T^2} \|v\|_{1, 2, \Omega}, \quad \forall v \in \mathcal{X},
\]

\[
\int_T \operatorname{cof} \nabla u_h : \nabla (\Pi_1 v - v) \, dx = 0, \quad \forall v \in \mathcal{X}, \forall T \in \mathcal{T}_h.
\]  

(3.17)

Proof. Since $\bar{\Pi}_2 \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}_h)$ is defined by (3.12), thus, by solving the linear system, $\bar{\Pi}_2 v$ can be explicitly expressed as $\bar{\Pi}_2 v = \sum_{i=1}^{4} \alpha_i(v) p_i$, where

\[
\alpha_i = \left[ \int_{e_i} (\operatorname{cof} \nabla u_h) \cdot n_i \, ds \right] / \left[ \int_{e_i} (\operatorname{cof} \nabla u_h) \cdot n_i \, ds \right], \quad i = 1, 2, 3, 4.
\]  

(3.18)

Noticing that (H5) also implies $\lambda_2(\operatorname{cof} \nabla u_h) \lesssim \rho_T^{-1}$, by the trace theorem, we have

\[
\left| \int_{e_i} (\operatorname{cof} \nabla u_h) \cdot n_i \, ds \right| \lesssim \lambda_2(\operatorname{cof} \nabla u_h) \int_{e_i} |\nabla u_h| \, ds \lesssim \frac{h_T}{\rho_T} \int_{e_i} |\nabla u_h| \, ds \lesssim \frac{h_T}{\rho_T} \|\nabla u_h\|_{1, 2, \tilde{T}}.
\]  

(3.19)

and similarly, since $\lambda_1(\operatorname{cof} \nabla u_h) \gtrsim \rho_T$, we have

\[
\left| \int_{e_i} (\operatorname{cof} \nabla u_h) \cdot p_i \, ds \right| = \left| \int_{e_i} p_i \cdot (\operatorname{cof} \nabla u_h) \, ds \right| \gtrsim \frac{\rho_T}{h_T} \bar{n}_i \, d\bar{s}.
\]  

(3.20)

Therefore, (3.18)-(3.20) yields that

\[
|\alpha_i| \lesssim \frac{1}{\rho_T^2} \|\nabla u_h\|_{1, 2, \tilde{T}}, \quad i = 1, 2, 3, 4.
\]  

(3.21)

Hence, by the standard scaling argument, we obtain

\[
|\bar{\Pi}_2 v|_{1, 2, T}^2 = \left| \sum_{i=1}^{4} \alpha_i p_i \right|_{1, 2, T}^2 \lesssim \frac{1}{\rho_T^4} (h_T^{-2} \|v\|_{0, 2, T}^2 + |v|_{1, 2, T}^2).
\]  

(3.22)

Since $\nabla \bar{\Pi}_2 v = \nabla \bar{\Pi}_2 v$, it follows from (3.11) and (3.22) that

\[
|\Pi_1 v|_{1, 2, \Omega}^2 \lesssim |\bar{\Pi}_1 v|_{1, 2, \Omega}^2 + \sum_{T} |\bar{\Pi}_2 (v - \bar{\Pi}_1 v)|_{1, 2, T}^2 \lesssim |\bar{\Pi}_1 v|_{1, 2, \Omega}^2 + \frac{1}{\rho_T^4} (h_T^{-2} \|v - \bar{\Pi}_1 v\|_{0, 2, T}^2 + |v - \bar{\Pi}_1 v|_{1, 2, T}^2) \lesssim \frac{1}{\rho_T^2} |v|_{1, 2, \Omega}^2.
\]

Recall $\int_{\Omega_T} v \, dx = 0, \forall v \in \mathcal{X}$, this and the Poincaré inequality lead to the inequality in (3.17). On the other hand, by (3.13), we have, for all $v \in \mathcal{X}$,

\[
\int_T \operatorname{cof} \nabla u_h : \nabla (\Pi_1 v - v) \, dx = \int_T \operatorname{cof} \nabla u_h : \nabla (\bar{\Pi}_2 (v - \bar{\Pi}_1 v) - (v - \bar{\Pi}_1 v)) \, dx = 0.
\]

This completes the proof of the lemma. \qed
Theorem 1. Let \( b(v,q;\mathbf{u}) \) be given by (2.7). Let \( \mathcal{T}_h \) and \( \mathbf{u}_h \) satisfy hypothesis (H4) and (H5) respectively. Then, there exists a constant \( \beta > 0 \) independent of \( h \) such that \( b(v_h,q_h;\mathbf{u}_h) \) satisfies the LBB condition (3.6).

**Proof.** Set \( \Pi_h = \Pi_1 + \Pi_2(I - \Pi_1) \). Then, by Lemma 1, Lemma 2 (see (3.8)), and (3.13)-(3.17), what remains for us to show is that \( \|\Pi_2(I - \Pi_1)v\|_{1,2,\Omega} \leq c_2\|v\|_{1,2,\Omega}, \forall v \in \mathcal{X} \).

Since \( \Pi_2v \) is a bubble function on \( T \in \mathcal{T}_h \) and \( \text{div}(\text{cof} \nabla \mathbf{u}_h|_T) = 0 \), by the integral by parts and the change of integral variables, equation (3.15) can be rewritten as

\[
\int_T \Pi_2v \cdot (\text{cof} \nabla_{\hat{x}} \mathbf{u}_h \nabla_{\hat{x}} \hat{q}_h) \, d\hat{x} = \int_T \hat{q}_h \text{cof} \nabla_{\hat{x}} \mathbf{u}_h : \nabla_{\hat{x}} \hat{v} \, d\hat{x}, \quad \forall \hat{q}_h \in P_1(\hat{T}) \setminus P_0(\hat{T}). \tag{3.23}
\]

where the gradient operator \( \nabla_{\hat{x}} := (\partial_{x_1}, \partial_{x_2}) \). By solving the linear system, we can write \( \Pi_2v(\hat{x}) \) explicitly as

\[
\alpha = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) = \left( \int_T \hat{b} \text{cof} \nabla_{\hat{x}} \mathbf{u}_h \, d\hat{x} \right)^{-1} \left( \int_T \text{cof} \nabla_{\hat{x}} \mathbf{u}_h : \nabla_{\hat{x}} \hat{v} \, d\hat{x} \right), \tag{3.24}
\]

where \( \hat{b}(\hat{x}) = (1 - \hat{x}_1^2)(1 - \hat{x}_2^2) \). By the Hölder inequality and hypothesis (H5),

\[
\left| \int_T \text{cof} \nabla_{\hat{x}} \mathbf{u}_h : \nabla_{\hat{x}} \hat{v} \, d\hat{x} \right| \lesssim \frac{h_T}{\rho_T} |\hat{v}|_{1,2,\hat{T}} \lesssim \frac{h_T}{\rho_T} |\hat{v}|_{1,2,T}, \quad i = 1, 2. \tag{3.25}
\]

On the other hand, noticing that \( \hat{u}_h(\hat{x}) \in Q_2(\hat{T}) \), by direct calculations (similar to that in the proof of Theorem 3.1 in [20]), we have, by (H5),

\[
\det \left( \int_T \hat{b} \text{cof} \nabla_{\hat{x}} \mathbf{u}_h(\hat{x}) \, d\hat{x} \right) \cong \det \nabla \mathbf{u}_h(\alpha_8) h_T^2 \cong h_T^2. \tag{3.26}
\]

Thus, again by hypothesis (H5),

\[
\left| \left( \int_T \hat{b} \text{cof} \nabla_{\hat{x}} \mathbf{u}_h \, d\hat{x} \right)^{-1} \right| \cong h_T^{-2} \left| \int_T \hat{b} \nabla_{\hat{x}} \mathbf{u}_h \, d\hat{x} \right| \lesssim h_T^{-2} \|\nabla_{\hat{x}} \mathbf{u}_h\|_{0,2,\hat{T}} \lesssim \frac{1}{\rho_T h_T}. \tag{3.27}
\]

As a consequence of (3.24)-(3.27) and the standard scaling argument, we are led to

\[
\|\Pi_2v\|_{1,2,T} \cong \|\hat{\Pi}_2v\|_{1,2,\hat{T}} \cong |\alpha| \lesssim \frac{1}{\rho_T} |\hat{v}|_{1,2,\hat{T}} \cong \frac{1}{\rho_T} |v|_{1,2,T}. \tag{3.28}
\]

Finally, by (3.15), (3.17), (3.28) and the Poincaré inequality, we obtain

\[
\|\Pi_2(I - \Pi_1)v\|_{1,2,\Omega} \lesssim \frac{1}{\rho^4} \|(I - \Pi_1)v\|_{1,2,\Omega} \lesssim \frac{1}{\rho^4} \|v\|_{1,2,\Omega}, \quad \forall v \in \mathcal{X}, \tag{3.29}
\]

and complete the proof of the theorem. \( \square \)
Remark 2. Notice that, if \( \mathbf{u}_h \) is a proper cavity deformation (not necessarily anywhere close to the solution of the problem), then the eigenvectors of \( \nabla \mathbf{u}_h \) corresponding to \( \lambda_1(\nabla \mathbf{u}_h) \) and \( \lambda_2(\nabla \mathbf{u}_h) \) are approximately collinear with \( \mathbf{n}_1, \mathbf{n}_3 \) and \( \mathbf{n}_2, \mathbf{n}_4 \) respectively. Therefore, (3.19) and (3.20) can be zoomed to the same eigenvalue \( \lambda_j(\nabla \mathbf{u}_h), j = 1 \) or 2, thus the inequality in (3.17) can be improved to \( \| \Pi_1 \mathbf{v} \|_{1,2,\Omega_\rho} \lesssim \| \mathbf{v} \|_{1,2,\Omega_\rho} \). As a consequence, the coefficient in (3.29) reduces to \( 1/\rho^2 \).

4 Convergence of DP-Q2-P1 cavitation solutions

The convergence analysis of numerical cavitation solutions generally consists of two parts: (1) to show that \( E(\mathbf{u}_h) \rightarrow E(\mathbf{u}) = \inf_{\mathbf{v} \in A} E(\mathbf{v}) \); (2) to apply some prominent theorems in the theory of cavitation problems to show that subsequences, which converge to an analytical cavitation solution \( \mathbf{u} \), can be extracted from \( \{\mathbf{u}_h\}_{h>0} \).

To show \( E(\mathbf{u}_h) \rightarrow E(\mathbf{u}) \), we need the interpolation error estimates given in [20], where the cavitation solution \( \mathbf{u} \) is naturally assumed to be in a set of the form

\[
\mathcal{U}(\Upsilon) = \left\{ \mathbf{u} \in C^4(\Omega_\rho; \mathbb{R}^2) : \mathbf{u}(\mathbf{x}) = (r \cos \phi, r \sin \phi), \text{ where } r = r(R, \theta), \phi = \phi(R, \theta), \right. \\
\left. \text{with } \left| \frac{\partial^{i+j} \mathbf{u}}{\partial R^i \partial \theta^j} \right| \leq \Upsilon, \left| \frac{\partial^{i+j} \phi}{\partial R^i \partial \theta^j} \right| \leq \Upsilon, \forall i, j \geq 0, i + j \leq 4 \right\},
\]

here \( \Upsilon \) is a constant independent of the initial defect size \( \rho \). Let \( \bar{\Pi}_h^2 \) be the interpolation operator defined on \( C(\overline{\Omega}_\rho; \mathbb{R}^2) \) such that \( \bar{\Pi}_h^2 \mathbf{u}|_{T} \in F_T(Q_2), \forall T \in \mathcal{T}_h \). Define a linear operator \( \Pi_h^2 : A \cap C(\overline{\Omega}_\rho; \mathbb{R}^2) \rightarrow A_h \) (see (3.1)) by \( \Pi_h^2 \mathbf{u} \triangleq \bar{\Pi}_h^2 \mathbf{u} - \frac{1}{|\Omega_{\rho}|} \int_{\Omega_{\rho}} \bar{\Pi}_h^2 \mathbf{u} \, d\mathbf{x} \).

Lemma 4. Let \( \Omega_\rho = B_1(0) \setminus B_\rho(0) \) with a pre-existing defect of radius \( \rho \), and let \( \mathbf{u}(\mathbf{x}) \in A \cap \mathcal{U}(\Upsilon) \). Let \( \mathcal{T}_h \) be a circular ring layered mesh (see Fig. 3 and Fig. 4) satisfying that, for a given constant \( \alpha \in (0, 1) \), \( \tau_T \lesssim \min\{\sqrt{\rho_T}, \rho_T^{(1-\alpha)/4} h\}, \forall T \in \mathcal{T}_h \), and \( N^{-1} \lesssim \rho^{(1-\alpha)/4} h \). Then, the following error estimates hold:

\[
|\nabla \Pi_h^2 \mathbf{u}(\mathbf{x}) - \nabla \mathbf{u}(\mathbf{x})| \lesssim \tau_T^2 + |\mathbf{x}|^{-1} N^{-2}, \quad \forall \mathbf{x} \in T, \quad (4.1)
\]
\[
|\det \nabla \Pi_h^2 \mathbf{u}(\mathbf{x}) - \det \nabla \mathbf{u}(\mathbf{x})| \lesssim |\mathbf{x}|^{-1} (\tau_T^2 + N^{-2}), \quad \forall \mathbf{x} \in T, \quad (4.2)
\]
\[
\left| \int_{\Omega_{\rho}} d(\det \nabla \Pi_h^2 \mathbf{u}) \, d\mathbf{x} - \int_{\Omega_{\rho}} d(\det \nabla \mathbf{u}) \, d\mathbf{x} \right| \lesssim h^3. \quad (4.3)
\]
\[
\| \det \nabla \Pi_h^2 \mathbf{u} - \det \nabla \mathbf{u} \|_{0,2,\Omega_\rho} \lesssim h^2, \quad (4.4)
\]
\[
|\Pi_h^2 \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x})| \lesssim h^3, \quad \forall \mathbf{x} \in \overline{\Omega}_{\rho}. \quad (4.5)
\]
Proof. Since $\nabla \Pi_h^2 u = \nabla \tilde{\Pi}_h^2 u$, (4.1)-(4.3) follow directly from (20) (see Theorem 3.1, Theorem 4.3 and the proof of Theorem 5.2 there for details).

Since $N^{-1} \lesssim \rho_T^{(1-\alpha)/4} h$ and $\tau_T \lesssim \rho_T^{(1-\alpha)/4} h$, (4.4) follows from (4.2) and
\[
\left( \int_{\rho}^{1} \frac{1}{R^2} (\tau^4 + N^{-4} + \tau^2 N^{-2}) \cdot R \, dR \right)^{1/2} \leq \left( \int_{\rho}^{1} \frac{1}{R} \cdot R \, dR \right)^{1/2} \lesssim \frac{h^2}{\sqrt{1-\alpha}}.
\]

By Theorem 4.1 in (20), we have $\| \tilde{\Pi}_h^2 u(x) - u(x) \| \lesssim \tau_T^3 + N^{-3}, \forall x \in T$, this leads to $\| \frac{1}{|T|} \int_{T} \tilde{\Pi}_h^2 u \, dx \| \lesssim h^3$, and consequently (4.5) holds.

Lemma 5. Let $(\tilde{u}, \tilde{p}) \in (A \cap \mathcal{U}(\mathcal{Y})) \times H^{\gamma+1}(\Omega)$, $\gamma = 0$ or $1$, be a solution to problem (2.3) with $\tilde{u}$ being an absolute minimizer of $E(\cdot)$ in $\mathcal{A}_T$. Let $\mathcal{T}_h$ be circular ring layered meshes satisfying (H4) and that, for a given constant $\alpha \in (0,1)$, $\tau_T \lesssim \min \{ \sqrt{\rho_T}, \rho_T^{(1-\alpha)/4} h \}$, $\forall T \in \mathcal{T}_h$. Let $(u_h, p_h) \in \mathcal{X}_h \times \mathcal{M}_h$ be absolute energy minimizing finite element solutions to problem (3.4) with $\| p_h \|_{0,2,\Omega} \lesssim h^{-\beta}$ for a constant $\beta \in [0,2)$. Then
\[
\begin{align*}
- h^{\gamma+1} & \lesssim E(u_h, p_h) - E(\tilde{u}, \tilde{p}) \lesssim h^{2-\beta}, \quad (4.6) \\
\| u_h \|_{1,s,\Omega} & \lesssim 1, \quad \| \det \nabla u_h \|_{0,2,\Omega} \lesssim 1. \quad (4.7)
\end{align*}
\]

Proof. Firstly, by assumption, $E(u_h, p_h) \leq E(\Pi_h^2 \tilde{u}, p_h)$. While
\[
E(\Pi_h^2 u, p_h) = E(\tilde{u}, \tilde{p}) + \int_{\Omega} \mu(\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}) \cdot \nabla \tilde{u} \, dx - \int_{\Omega} p_h (\nabla \Pi_h^2 \tilde{u} - 1) \, dx - \int_{\partial \Omega} t \cdot (\Pi_h^2 \tilde{u} - \tilde{u}) \, ds = E(\tilde{u}, \tilde{p}) + I_1 + I_2 + I_3 + I_4,
\]

since $\det \nabla \tilde{u} = 1$, a.e. in $\Omega$. By means of Taylor expansion, we have
\[
\begin{align*}
\int_{\Omega} |\nabla \Pi_h^2 \tilde{u}|^s \, dx & \leq \int_{\Omega} |\nabla \tilde{u}|^s \left( 1 + \frac{|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|}{|\nabla \tilde{u}|} \right)^s \, dx \\
& = \int_{\Omega} |\nabla \tilde{u}|^s \left( 1 + s \left( 1 + \eta \frac{|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|}{|\nabla \tilde{u}|} \right)^{s-1} \frac{|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|}{|\nabla \tilde{u}|} \right) \, dx, \quad \eta \in (0,1).
\end{align*}
\]

By (4.1) and $|\nabla \tilde{u}(x)| \approx \frac{1}{|x|}$ (see (3, 20)), we have $|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|/|\nabla \tilde{u}| \lesssim h^2$, hence
\[
|I_1| \leq \mu s \left( 1 + \eta \frac{|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|}{|\nabla \tilde{u}|} \right)^{s-1} \frac{|\nabla \Pi_h^2 \tilde{u} - \nabla \tilde{u}|}{|\nabla \tilde{u}|} \|_{0,\infty,\Omega} \int_{\Omega} |\nabla \tilde{u}|^s \, dx \lesssim h^2.
\]

By (4.4) and $\| p_h \|_{0,2,\Omega} \lesssim h^{-\beta}$, we have
\[
|I_3| \leq \| p_h \|_{0,2,\Omega} \| \det \nabla \Pi_h^2 \tilde{u} - 1 \|_{0,2,\Omega} \lesssim h^{2-\beta}.
\]
It follows from (4.3) and (4.5) that $|I_2| \lesssim h^3$ and $|I_4| \lesssim h^3$. Thus, we see that the second relationship in (4.6) holds, and as a consequence we also have $E(u_h) \lesssim 1$, which implies (4.7), since $\int_{\Omega_h} p_h(\det \nabla u_h - 1) \, dx = 0$, $\int_{\Omega_h} u_h \, dx = 0$ and $d_1(\det \nabla u_h) > 0$.

Secondly, due to $(\tilde{u}, \tilde{p}) = \text{arg sup}_{q_h \in L^2(\Omega_h)} \text{inf}_{v \in \mathcal{A}} E(v, q)$, we see that $\tilde{u}$ minimizes $E(v, p)$ in $\mathcal{A}$. Recall $\int_{\Omega_h} q_h(\det \nabla u_h - 1) \, dx = 0$, $\forall q_h \in \mathcal{M}_h$ (see 3.4), we have

$$E(\tilde{u}, \tilde{p}) \leq E(u_h, p_h) = E(u_h, p_h) - \int_{\Omega_h} (\tilde{p} - P^1_h \tilde{p})(\det \nabla u_h - 1) \, dx,$$

where $P^1_h : H^{\gamma+1}(\Omega) \to \mathcal{X}_h$, $\gamma = 0$ or 1 is an orthogonal projection operator with $P^1_h |_T = P^1_T : H^{\gamma+1}(T) \to \tilde{P}_T(T)$ being defined by

$$\int_{T} q(P^1_T p - p) \det \nabla \tilde{x} \, dx = 0, \quad \forall q \in \tilde{P}_T(T), \quad \forall T \in T_h.$$

Since $T_h$ satisfies (H4), we have $\| P^1_h \tilde{p} - \tilde{p} \|_{0,2,\Omega} \lesssim h^{\gamma+1} \| \tilde{p} \|_{\gamma+1,2,\Omega}$ (see 3.5). In addition

$$\left| \int_{\Omega_h} (\tilde{p} - P^1_h \tilde{p})(\det \nabla u_h - 1) \, dx \right| \lesssim \| \tilde{p} - P^1_h \tilde{p} \|_{0,2,\Omega} \| \det \nabla u_h - 1 \|_{0,2,\Omega}.$$

Thus, we are led to the first relationship in (4.6). \qed

**Remark 3.** $O(h^{\gamma+1})$ in the energy error bounds (4.6) is not optimal and can be improved at least to $o(h^{\gamma+1})$, since $\| \det \nabla u_h - 1 \|_{0,2,\Omega} \to 0$ (see (4.8)).

**Theorem 2.** Suppose problem (2.3) has a solution $(\tilde{u}, \tilde{p}) \in (\mathcal{A} \cap \mathcal{U}(\gamma)) \times H^1(\Omega)$ with $\tilde{u}$ being an absolute minimizer of $E(\cdot)$ in $\mathcal{A}_I$. Let $T_h$ and $(u_h, p_h) \in \mathcal{X}_h \times \mathcal{M}_h$ satisfy the same conditions as in Lemma 7. Then, there exist a subsequence $\{u_{h}\}_{h>0}$ (not relabeled) and an absolute energy minimizer $\tilde{u}$ of $E(\cdot)$ in $\mathcal{A}_I$, such that

$$u_h \to \tilde{u} \quad \text{in} \quad W^{1,s}(\Omega; \mathbb{R}^2), \quad \det \nabla u_h \to 1 \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad h \to 0. \quad (4.8)$$

Furthermore, if $\| p_h \|_{0,2,\Omega} \lesssim 1$, then there exist a subsequence $\{p_{h}\}_{h>0}$ (not relabeled) and a function $\tilde{p} \in L^2(\Omega)$, such that $(\tilde{u}, \tilde{p})$ solves problem (1.3) and

$$p_h \to \tilde{p} \quad \text{in} \quad L^2(\Omega). \quad (4.9)$$

**Proof.** Since $1 < s < 2$, it follows from (4.7) that there exist a subsequence $\{u_{h}\}_{h>0}$ (not relabeled) and functions $\tilde{u} \in W^{1,s}(\Omega; \mathbb{R}^2)$, $\vartheta \in L^2(\Omega)$ such that

$$u_h \to \tilde{u} \quad \text{in} \quad W^{1,s}(\Omega; \mathbb{R}^2), \quad u_h \to \tilde{u} \quad \text{and} \quad \det \nabla u_h \to \vartheta \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad h \to 0. \quad (4.10)$$
Thanks to some prominent results for the cavitation problems (see Theorem 3 in \[12\], Theorem 2 and Theorem 3 in \[11\]) that, in our case (see also in \[21\] for more general cases), (4.11) together with the continuity of \(u_h\) actually lead to
\[
\vartheta = \det \nabla \bar{u}, \text{ a.e. in } \Omega, \text{ and } \bar{u} \text{ is 1-to-1 a.e. in } \Omega.
\] (4.11)

On the other hand, since \(\int_{\Omega} q_h(\det \nabla u_h - 1) \, dx = 0, \forall q_h \in \mathcal{M}_h, \) we have
\[
\int_{\Omega} q(1 - \vartheta) \, dx = \int_{\Omega} q(\det \nabla u_h - \vartheta) \, dx - \int_{\Omega} (q - P^1_h q)(\det \nabla u_h - 1) \, dx, \forall q \in \mathcal{O}_0^\infty(\Omega).
\]
Since \(\|P^1_h q - q\|_{0,2,\Omega} \lesssim h^2|q|_{1,2,\Omega},\) it follows from (4.1) that
\[
\int_{\Omega} (q - P^1_h q)(\det \nabla u_h - 1) \, dx \leq \|q - P^1_h q\|_{0,2,\Omega} \|\det \nabla u_h - 1\|_{0,2,\Omega} \to 0, \text{ as } h \to 0.
\]
This combined with (4.10) and (4.11) yields \(\det \nabla \bar{u} = \vartheta = 1, \) a.e. in \(\Omega.\) Furthermore, since \(u_h \to \bar{u}\) in \(L^2(\Omega)\) and \(\int_{\Omega} \bar{u}_h \, dx = 0,\) we also have \(\int_{\Omega} \bar{u} \, dx = 0.\) Thus, recalling that \(\bar{u}\) is 1-to-1 a.e. in \(\Omega\) by (4.11), we conclude that \(\bar{u} \in \mathcal{A}.\)

Next, we claim \(\bar{u}\) is an absolute energy minimizer of \(E(\cdot)\) in \(\mathcal{A}.\) In fact, due to the convexity of both \(d(\xi)\) and \(|\nabla u|^8\), as a consequence of (4.10) and (4.11), we have
\[
\int_{\Omega} |\nabla \bar{u}|^8 \, dx \leq \liminf_{h \to 0} \int_{\Omega} |\nabla u_h|^8 \, dx, \quad (4.12)
\]
\[
\int_{\Omega} d(\det \nabla \bar{u}) \, dx \leq \liminf_{h \to 0} \int_{\Omega} d(\det \nabla u_h) \, dx. \quad (4.13)
\]
In addition, according to the trace theorem in Sobolev spaces, \(u_h \to \bar{u}\) in \(W^{1,8}(\Omega)\) also implies \(\lim_{h \to 0} \int_{\partial \Omega} t \cdot u_h \, ds = \int_{\partial \Omega} t \cdot \bar{u} \, ds.\) Hence, by Lemma 5 (see (4.6)), we obtain
\[
\inf_{v \in \mathcal{A}} E(v) \leq E(\bar{u}) \leq \liminf_{h \to 0} E(u_h) = \liminf_{h \to 0} E(u_h, p_h) = E(\bar{u}, \bar{p}) = \inf_{v \in \mathcal{A}} E(v). \quad (4.14)
\]
Now, we are going to show (4.8). Since (4.14) also implies that \(E(\bar{u}) = \lim_{h \to 0} E(u_h),\) it follows from (4.12), (4.13) and \(\lim_{h \to 0} \int_{\partial \Omega} t \cdot u_h \, ds = \int_{\partial \Omega} t \cdot \bar{u} \, ds\) that
\[
E(\bar{u}) + \int_{\partial \Omega} t \cdot \bar{u} \, ds - \int_{\Omega} \mu|\nabla \bar{u}|^8 \, dx = \int_{\Omega} d(\det \nabla \bar{u}) \, dx
\]
\[
\leq \liminf_{h \to 0} \int_{\Omega} d(\det \nabla u_h) \, dx = \liminf_{h \to 0} (E(u_h) + \int_{\partial \Omega} t \cdot u_h \, ds - \int_{\Omega} \mu|\nabla u_h|^8 \, dx)
\]
\[
= E(\bar{u}) + \int_{\partial \Omega} t \cdot \bar{u} \, ds - \limsup_{h \to 0} \int_{\Omega} \mu|\nabla u_h|^8 \, dx, \quad (4.15)
\]
i.e. \(\limsup_{h \to 0} |u_h|_{1,8,\Omega} \leq |\bar{u}|_{1,8,\Omega} \). This together with (4.12) yields \(\lim_{h \to 0} |u_h|_{1,8,\Omega} = |\bar{u}|_{1,8,\Omega} \). Hence, it follows from (4.10) that \(u_h \to \bar{u}\) in \(W^{1,8}(\Omega; \mathbb{R}^2)\), since \(W^{1,8}(\Omega; \mathbb{R}^2)\) enjoys the Radon-Riesz property (see [34]).
As a byproduct of $u_h \to \bar{u}$ in $W^{1,s}(\Omega_\rho; \mathbb{R}^2)$, we see that the inequality in (4.15) is in fact an equality, and consequently $\lim_{h \to 0} \int_{\Omega_\rho} d(\det \nabla u_h) \, dx = \int_{\Omega_\rho} d(\det \nabla \bar{u}) \, dx$. Hence, it follows from $\det \nabla u_h \to \det \nabla \bar{u}$ in $L^2(\Omega_\rho)$ and the convexity of $d_1(\cdot)$ that

$$\int_{\Omega_\rho} d(\det \nabla \bar{u}) - \kappa(\det \nabla \bar{u} - 1)^2 \, dx = \int_{\Omega_\rho} d_1(\det \nabla \bar{u}) \, dx$$

$$\leq \liminf_{h \to 0} \int_{\Omega_\rho} d(\det \nabla u_h) \, dx = \liminf_{h \to 0} \int_{\Omega_\rho} d(\det \nabla u_h) - \kappa(\det \nabla u_h - 1)^2 \, dx$$

$$= \int_{\Omega_\rho} d(\det \nabla \bar{u}) \, dx - \limsup_{h \to 0} \int_{\Omega_\rho} \kappa(\det \nabla u_h - 1)^2 \, dx,$$

which yields $\limsup_{h \to 0} \int_{\Omega_\rho} (\det \nabla u_h - 1)^2 \, dx \leq \int_{\Omega_\rho} (\det \nabla \bar{u} - 1)^2 \, dx = 0$, i.e. $\det \nabla u_h \to 1$ in $L^2(\Omega_\rho)$.

Finally, $\|p_h\|_{0,2,\Omega_\rho} \leq 1$ implies that there exist a subsequence $\{p_h\}_{h > 0}$ (not relabeled) and a function $\bar{p} \in L^2(\Omega_\rho)$, such that (4.9) holds. Thus, by $\det \nabla u_h \to \det \nabla \bar{u} = 1$ in $L^2(\Omega_\rho)$ and (4.14), we have $E(\bar{u}, \bar{p}) = \lim_{h \to 0} E(u_h, p_h) = E(\bar{u}, \bar{p})$, this completes the proof of the theorem.

\[\square\]

**Theorem 3.** Suppose problem (4.3) has a solution $(\bar{u}, \bar{p}) \in (A \cap U(\gamma)) \times H^1(\Omega_\rho)$ with $\bar{u}$ being an absolute minimizer of $E(\cdot)$ in $A_1$. Let $T_h$ and $(u_h, p_h) \in X_h \times M_h$ satisfy the same conditions as in Lemma 2. Let $(\bar{u}, \bar{p})$ be given by Theorem 2. If, in addition, $\bar{p} \in H^1(\Omega_\rho)$, $\|p_h\|_{0,2,\Omega_\rho} \leq 1$, $\|u_h\|_{1,\zeta,\Omega_\rho} \leq 1$ and $c \leq \det \nabla u_h \leq C$, a.e. in $\Omega_\rho$, where $\zeta > 2$ and $0 < c < 1 < C$ are constants independent of $h$. Then

$$p_h \to \bar{p} \text{ in } L^2(\Omega_\rho), \text{ as } h \to 0. \quad (4.16)$$

**Proof.** Firstly, we see that $\|u_h\|_{1,\zeta,\Omega_\rho} \leq 1$ implies $\bar{u} \in W^{1,\zeta}(\Omega_\rho; \mathbb{R}^2)$, and thus it follows from (see the interpolation inequality on page 125 in [37])

$$\|\nabla u_h - \nabla \bar{u}\|_{0,2,\Omega_\rho} \leq \|\nabla u_h - \nabla \bar{u}\|_{0,\zeta,\Omega_\rho}^{1-\alpha} \|\nabla u_h - \nabla \bar{u}\|_{0,s,\Omega_\rho}^{\alpha}, \quad (4.17)$$

where $0 < \alpha < 1$ is determined by $\frac{1}{2} = \frac{1-\alpha}{\zeta} + \frac{\alpha}{s}$, that $u_h \to \bar{u}$ in $W^{1,s}(\Omega_\rho; \mathbb{R}^2)$ in (4.8) can be strengthened to $u_h \to \bar{u}$ in $H^1(\Omega_\rho; \mathbb{R}^2)$.

We will repeatedly use below the facts that $|A : B| \leq |A||B|$, $\forall A, B \in M^{n \times n}$, and $|\nabla u_h| \geq \det \nabla u_h \geq c$ a.e. in $\Omega_\rho$, $|\nabla \bar{u}| \geq 1 > c$ a.e. in $\Omega_\rho$, which follow as consequences of $\det A \leq |A|$, $\forall A \in M^{n \times n}$.

Secondly, we claim that

$$\lim_{h \to 0} \sup_{v_h \in X_h} \int_{\Omega_\rho} \mu_s(\frac{|\nabla \bar{u}|^{s-2} \nabla \bar{u} - |\nabla u_h|^{s-2} \nabla u_h}{|v_h|_{1,2,\Omega_\rho}} : \nabla v_h) \, dx = 0, \quad (4.18)$$
where \( \nabla \) and \( \nabla' \).

It follows from (2.3), (3.4), (4.18) and (4.19) that

\[
\lim_{h \to 0} \sup_{\mathbf{v}_h \in \mathcal{X}_h} \frac{\int_{\Omega_p} (d'(\det \nabla \mathbf{u}) \cof \nabla \mathbf{u} - d'(\det \nabla \mathbf{u}_h) \cof \nabla \mathbf{u}_h) : \nabla \mathbf{v}_h \, dx}{|\mathbf{v}_h|_{1,2,\Omega_p}} = 0, \tag{4.19}
\]

In fact, since \( \mathbf{u}_h \to \mathbf{u} \) in \( H^1(\Omega_p; \mathbb{R}^2) \), (4.18) follows as a consequence of

\[
\left| \int_{\Omega_p} |\nabla \mathbf{u}|^{s-2} (\nabla \mathbf{u} - \nabla \mathbf{u}_h) : \nabla \mathbf{v}_h \, dx \right| \leq c^{s-2} \|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_{0,2,\Omega_p} \|\nabla \mathbf{v}_h\|_{0,2,\Omega_p},
\]

and

\[
\left| \int_{\Omega_p} (|\nabla \mathbf{u}|^{s-2} - |\nabla \mathbf{u}_h|^{s-2})(\nabla \mathbf{u}_h : \nabla \mathbf{v}_h) \, dx \right|
= \left| \int_{\Omega_p} (s - 2)|\nabla \mathbf{u}_\xi|^{s-4}(\nabla \mathbf{u}_\xi : (\nabla \mathbf{u} - \nabla \mathbf{u}_h))(\nabla \mathbf{u}_h : \nabla \mathbf{v}_h) \, dx \right|
\leq c^{s-4} (s - 2) \|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_{0,2,\Omega_p} \|\nabla \mathbf{v}_h\|_{0,2,\Omega_p},
\]

where \( \nabla \mathbf{u}_\xi := \nabla \mathbf{u} + \xi(\nabla \mathbf{u}_h - \nabla \mathbf{u}) \) with \( \xi \in (0, 1) \).

Similarly, since \( \mathbf{u}_h \to \mathbf{u} \) in \( H^1(\Omega_p; \mathbb{R}^2) \) and \( \det \nabla \mathbf{u}_h \to \det \nabla \mathbf{u} = 1 \) in \( L^2(\Omega_p) \), (4.19) follows as a consequence of

\[
\left| \int_{\Omega_p} d'(\det \nabla \mathbf{u})(\cof \nabla \mathbf{u} - \cof \nabla \mathbf{u}_h) : \nabla \mathbf{v}_h \, dx \right| \leq |d'(1)| \|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_{0,2,\Omega_p} \|\nabla \mathbf{v}_h\|_{0,2,\Omega_p},
\]

and

\[
\left| \int_{\Omega_p} (d'(\det \nabla \mathbf{u}) - d'(\det \nabla \mathbf{u}_h)) \cof \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, dx \right|
\leq \|d''(\eta)\|_{0,\infty,\Omega_p} \|\det \nabla \mathbf{u} - \det \nabla \mathbf{u}_h\|_{0,2,\Omega_p} \|\nabla \mathbf{u}_h\|_{0,2,\Omega_p} \|\nabla \mathbf{v}_h\|_{0,2,\Omega_p},
\]

where \( \eta \) is between \( \det \nabla \mathbf{u} \) and \( \det \nabla \mathbf{u}_h \), hence \( \|d''(\eta)\|_{0,\infty,\Omega_p} \leq \max_{\epsilon \leq \xi \leq \zeta} d''(\xi)|\Omega_p| \).

Next, we are going to show that

\[
\lim_{h \to 0} \sup_{\mathbf{v}_h \in \mathcal{X}_h} \frac{\int_{\Omega_p} (\mathbf{P}_h^0 \mathbf{p} - p_h) \cof \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, dx}{|\mathbf{v}_h|_{1,2,\Omega_p}} = 0, \tag{4.20}
\]

where \( \mathbf{P}_h^0 : H^1(\Omega_p) \to \mathcal{X}_h \) is the operator defined in the proof of Lemma 5. In fact,

\[
\int_{\Omega_p} (\mathbf{P}_h^0 \mathbf{p} - p_h) \cof \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, dx = \int_{\Omega_p} (\bar{\mathbf{p}} \cof \nabla \mathbf{u} - p_h \cof \nabla \mathbf{u}_h) : \nabla \mathbf{v}_h \, dx
- \int_{\Omega_p} \bar{\mathbf{p}}(\cof \nabla \mathbf{u} - \cof \nabla \mathbf{u}_h) : \nabla \mathbf{v}_h \, dx - \int_{\Omega_p} (\mathbf{P}_h^0 \bar{\mathbf{p}}) \cof \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, dx = I_1 + I_2 + I_3.
\]

It follows from (2.3), (3.4), (4.18) and (4.19) that \( \lim_{h \to 0} \sup_{\mathbf{v}_h \in \mathcal{X}_h} \frac{|I_1|}{|\mathbf{v}_h|_{1,2,\Omega_p}} = 0 \). On the other hand,

\[
|I_2| \lesssim \|\bar{\mathbf{p}}\|_{0,\infty,\Omega_p} \|\mathbf{u} - \mathbf{u}_h\|_{1,2,\Omega_p} \|\mathbf{v}_h\|_{1,2,\Omega_p}, \quad \forall \mathbf{v}_h \in \mathcal{X}_h,
\]

\[
|I_3| \lesssim \|\bar{\mathbf{p}} - \mathbf{P}_h^0 \bar{\mathbf{p}}\|_{0,\xi,\Omega_p} \|\mathbf{u}_h\|_{1,\xi,\Omega_p} \|\mathbf{v}_h\|_{1,2,\Omega_p}, \quad \forall \mathbf{v}_h \in \mathcal{X}_h,
\]
where $\xi = 2\zeta/(\zeta - 2)$. Since $\bar{p} \in H^1(\Omega_p)$ implies that $\lim_{h \to 0} \|\bar{p} - P_h^0 \bar{p}\|_{0, \xi, \Omega_p} = 0$ and $\|\bar{p}\|_{0, \infty, \Omega_p} < \infty$, we are left to $\lim_{h \to 0} \sup_{\psi_h \in X_h} \frac{|I_2| + |I_3|}{\psi_h|_{1, 2, \Omega_p}} = 0$.

Finally, by the interpolation error estimate of $P_h^0$ and the LBB condition (3.6),

$$\|\bar{p} - p_h\|_{0, 2, \Omega_p} \leq \|\bar{p} - P_h^0 \bar{p}\|_{0, 2, \Omega_p} + \|p_h - P_h^0 \bar{p}\|_{0, 2, \Omega_p} \lesssim h|\bar{p}|_{1, 2, \Omega_p} + \sup_{\psi_h \in X_h} \frac{\int_{\Omega_p} (p_h - P_h^0 \bar{p}) \nabla \psi_h : \nabla \psi_h \, dx}{\|\psi_h\|_{1, 2, \Omega_p}}.$$  (4.21)

Since $\int_{\Omega_p} \psi_h \, dx = 0$, we have $|\psi_h|_{1, 2, \Omega_p} \simeq \|\psi_h\|_{1, 2, \Omega_p}$ by the Poincaré inequality. Thus, (4.16) follows as a consequence of (4.20) and (4.21).

5 Numerical experiments and results

In our numerical experiments, the energy density is given by (2.1) with $s = 3/2$, $\mu = 1$, and $d(\xi) = (\xi - 1)^2/2 + 1/\xi$, and the reference configuration $\Omega_p = B_1(0) \setminus B_\rho(0)$.

The mesh $\mathcal{T}_h$ used in the numerical experiments must meet some basic requirements. For properly given constants $C \geq (2 - s)2^{s-1}$, $C_1$, $C_2 > 0$, and $h \leq \min\{\frac{2^{-s}}{2^{2-\alpha}C}, \frac{2^{-s}}{2^{3-s}C}\}$, they are summarized as: (1) Orientation preservation condition $\tau_T \leq C_1 \rho_T^{1/2}$ and $N \geq C_2 \rho_T^{-1/2}$ (see [20]); (2) Stability condition, which requires that (H4) holds (see §3); (3) Quasi-optimal convergence rate (see (4.1)–(4.5)) condition, which requires that, for a given constant $\alpha \in (0, 1)$, $N^{-1} \lesssim \rho^{(1-q)/4} h$ and $\tau_T \lesssim \rho_T^{(1-q)/4} h$ (see §4); (4) Relative error of elastic energy quasi-equidistribution condition, which requires that $(\rho_T + \tau_T)^{2-s} \leq \rho_T^{2-s} + Ch$ (see [20]).

| $h$ | $\min \tau_T$ | $\max \tau_T$ | layers | $N$ |
|-----|----------------|----------------|--------|-----|
| 0.05 | 0.0300         | 0.1900         | 8      | 20  |
| 0.04 | 0.0224         | 0.1376         | 11     | 26  |
| 0.03 | 0.0156         | 0.1164         | 14     | 34  |
| 0.02 | 0.0096         | 0.0736         | 22     | 50  |

| $h$ | $\min \tau_T$ | $\max \tau_T$ | layers | $N$ |
|-----|----------------|----------------|--------|-----|
| 0.05 | 0.0120         | 0.1720         | 9      | 24  |
| 0.04 | 0.0080         | 0.1360         | 12     | 28  |
| 0.03 | 0.0048         | 0.1056         | 16     | 38  |
| 0.02 | 0.0024         | 0.0728         | 22     | 56  |

Table 1: Data of two typical meshes produced by the meshing strategy.

Compared with the mesh conditions for the compressible materials (see [20]), the conditions for the incompressible materials here are certainly more restrictive, especially
when \( \rho \) is very small. More specifically, there is an additional stability condition \( h_T \approx h \) (see (H4)), and in condition (3), \( \rho_T^{(1-\alpha)/4}h \) instead of \( h \) is used. However, for practical defect size \( \rho \geq 10^{-6} \), our numerical experiments show that the mesh produced by the meshing strategy designed for compressible materials in [20] works well also for incompressible materials. In our numerical experiments below, we use the meshes produced with \( C = 2, C_1 = 1.0, N = 2\left\lceil \frac{1}{1.8h} \right\rceil \) for the radially-symmetric case, and \( C = 2, C_1 = 2.0, N = 2\left\lceil \frac{1}{1.8h} \right\rceil \) for the non-radially-symmetric case, where \( \lceil a \rceil \) represents the least integer \( \geq a \), while neglecting the condition \( N \geq C_2 \rho^{-1/2} \). In Table 1, some of the key data of two typical meshes produced by the meshing strategy are shown.

5.1 Radially symmetric case

In our numerical experiments, we take \( u(x; \lambda) = \sqrt{R^2 + \lambda^2 - 1} \mathbf{x} \) with \( \lambda > 1 \) as the analytical cavitation solution to the radially symmetric dead-load traction problem with

\[
t(x) = t\mathbf{n}(x), \quad \forall x \in \partial B_1(0), \quad \text{and} \quad t(x) = 0, \quad \forall x \in \partial B_\rho(0), \tag{5.1}
\]

where \( \mathbf{n} \) is the unit outward normal to \( \partial B_1(0) \), and \( t = t_{\rho, \lambda} \) is uniquely determined by \( \rho \) and \( \lambda \) [4]. For example \( t_{0.1,2} \approx 3.00487, \ t_{0.01,2} \approx 3.94237, \ t_{0.0001,2} \approx 4.21590 \). The convergence behavior of the numerical cavitation solutions with \( \lambda = 2 \) obtained by the DP-Q2-P1 mixed finite element method is shown in Fig 5-Fig 7, where \( N_d \) is the total degrees of freedom of \( u_h \). Fig 8 shows \( N_d \) as a function of the mesh size \( h \) in the radially symmetric case.

![Figure 5: Convergence behavior of the energy and deformation in symmetric case.](image)

(a) Energy error \( \Delta E = |E(u_h) - E(u)| \). (b) Error in \( W^{1,s} \)-seminorm \( |u_h - u|_{1,s,\Omega_\rho} \).
Since $N_d \sim h^{-2}$ for our mesh, it is clearly seen that the optimal convergence rates are obtained by the DP-Q2-P1 cavitation solutions in the radially symmetric case.

### 5.2 Non-radially symmetric case

Consider the non-radially-symmetric dead-load traction problem with

$$\mathbf{t}(\mathbf{x}) = (1 + \eta |\cos \theta|) \mathbf{n}(\mathbf{x}), \quad \forall \mathbf{x} \in \partial B(0), \quad \text{and} \quad \mathbf{t}(\mathbf{x}) = \mathbf{0}, \quad \forall \mathbf{x} \in \partial B_\rho(0), \quad (5.2)$$

where $\theta = \arctan(x_2/x_1)$, $\eta$ and $t$ being given parameters. In our numerical experiments, we take $\eta = 1/10$, and $t = t_{\rho,2}$ as is given in the radially-symmetric case for various $\rho$.

The convergence behavior of the numerical cavitation solutions obtained by the DP-Q2-P1 mixed finite element method is shown in Fig. [9]-Fig [11]. Fig [12] shows $N_d$ as a function of $h$ in the non-radially-symmetric case.
We see that, in the non-radially-symmetric case, again $N_d \sim h^{-2}$ for the meshes
produced by the meshing strategy, and the convergence rates obtained by the DP-Q2-P1 cavitation solutions, though dropped a little bit, are still close to the optimal rates.

**Remark 4.** In cavitation computation, even for $h$ generally considered to be reasonably small, the error on the Neumann boundary condition, i.e., $T_R(u_h, p_h)n(x) - t(x)$, where $T_R(u_h, p_h)$ is the Piola-Kirchhoff stress tensor of finite element solutions, can still be quite big and thus present a serious problem to the overall accuracy of the numerical results. To reduce such an error, we adaptively adjust the traction $t_h$ so that the resulted cavitation solution is such that $T_R(u_h, p_h)n(x) - t(x)$ is reasonably small on the boundary nodes.

### 6 Conclusion

A locking free and convergent DP-Q2-P1 mixed finite element method combined with a Newton iteration scheme is introduced in this paper to numerically solve the cavitation problem in incompressible nonlinear elasticity. Even though, compared with the DP-Q2 finite element method for the compressible case, the meshes are required in theory to satisfy certain more restrictive conditions imposed by the stability and convergence analysis, our numerical experiments show that, for practical initial defect size $\rho \geq 10^{-6}$, these conditions do not really take effect, and the method works well on the meshes produced by the meshing strategy designed for compressible materials in [20]. On the other hand, our numerical experiments also show that the errors of the numerical solutions tend to increase as the initial defect size $\rho$ decreases, which well reflects our numerical analysis on the stability and convergence of the method.

It is worth pointing out here that our method and analysis can be extended to the multi-prescribed-defects case, by introducing meshes $T_h$ consisting of circular ring layered meshes in a neighborhood of each of the prescribed defects and a regular mesh elsewhere, and noticing that the cavitation solution subjects to finite deformation independent of the defect sizes except in the neighborhood of the defects.

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