Price of Anarchy with Heterogeneous Latency Functions

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Abstract

In this paper we consider the price of anarchy (PoA) in multi-commodity flows where the latency or delay function on an edge has a heterogeneous dependency on the flow commodities, i.e. when the delay on each link is dependent on the flow of individual commodities, rather than on the aggregate flow. An application of this study is the performance analysis of a network with differentiated traffic that may arise when traffic is prioritized according to some type classification. This study has implications in the debate on net-neutrality. We provide price of anarchy bounds for networks with \( k \) (types of) commodities where each link is associated with heterogeneous polynomial delays, i.e. commodity \( i \) on edge \( e \) faces delay specified by \( g_{i1}(e)f_{\theta 1}(e) + g_{i2}(e)f_{\theta 2}(e) + \ldots + g_{ik}(e)f_{\theta k}(e) + c_i(e) \), where \( f_i(e) \) is the flow of the \( i \)th commodity through edge \( e \), \( \theta \in \mathbb{N} \), \( g_{i1}(e) \), \( g_{i2}(e) \), \ldots, \( g_{ik}(e) \) and \( c_i(e) \) are nonnegative constants. We consider both atomic and non-atomic flows.

For networks with decomposable delay functions where the delay induced by a particular commodity is the same, i.e. delays on edge \( e \) are defined by \( a_1(e)f_{\theta 1}(e) + a_2(e)f_{\theta 2}(e) + \ldots + c(e) \) where \( \forall j, \forall e : g_{1j}(e) = g_{2j}(e) = \ldots = a_j(e) \), we show an improved bound on the price of anarchy.

Further, we show bounds on the price of anarchy for uniform latency functions where each edge of the network has the same delay function.

1 Introduction

The problem of selfish routing in networks is a well-studied topic. The inefficiency of Nash equilibrium flows, with respect to the total latency or delay of the flow, has been quantified in [14, 12]. In these works, the latency or delay on edges is typically a function of the aggregate flow on the edge.

We consider non-atomic and atomic \( k \)-commodity network flows where edges are associated with a heterogeneous delay function, an example being polynomial delays of the form \( a_1(e)f_1^\theta(e) + a_2(e)f_2^\theta(e) + \ldots + a_k(e)f_k^\theta(e) + c(e) \) where \( f_i \) represents the load contribution of the \( i \)th commodity and \( \theta \) the degree of the polynomial. Here, \( a_1, a_2, \ldots, a_k \) and \( c(e) \) are non-negative integers. This model has applications in transport networks [16]. As an application to the internet, we can use this model in the context of the net-neutrality debate [18, 19] where the impact of favoring a particular type of flow, say \( j \), can be modeled by increasing its contribution to the delay via the co-efficient \( a_j(e) \). We study the price of anarchy (PoA) in this context and show an increased PoA as compared to networks where the delay is simply \( f^\theta(e) \) where \( f(e) \) is the total flow on the edge. Our results indicate that type differentiation amongst traffic can lead to a worse price of anarchy as compared to the equal treatment of traffic, thus bolstering the defense of net-neutrality and motivating further similar studies.

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The notion of commodity dependent costs was first studied in Dafermos \cite{16, 17} where the author considers a transportation network where travel time on a link depends crucially on the delays of the types of traffic. One example of such a delay function, on edge $e$ for commodity $i$, is provided by $\Phi_{ij}^e(f_1, \ldots, f_k) = \sum_j g_{ij}(e) f_i(e) f_j(e) + h_i(e) f_i(e)$, where $g_{ij}$ and $h_i$ are constants and $i, j$ represent commodities. Dafermos \cite{16, 17} establishes a condition (positive definite matrix $[g_{ij}]$) under which the flow satisfies both Nash Equilibrium conditions and also optimizes an aggregate objective function.

In this paper we consider the price of anarchy in networks with the following total cost function.

$$\sum_i \sum_e f_i(e) \Phi_i^e(f_1, \ldots, f_k) = \sum_i \sum_e f_i(e) \left( \sum_j g_{ij}(e) f_j^e(e) + h_i(e) \right).$$

where $f_i(e)$ and $\Phi_i^e()$ represent the flow and the delay of commodity $i$ on edge $e$, respectively and $\theta \in \mathcal{N}$ is a positive constant. This total delay function is termed as a heterogenous delay function.

If the delay function, $\Phi_e()$, is expressible as $(f_1 + \ldots + f_k)(a_1(e)f_1^e(e) + \ldots + a_k(e)f_k^e(e) + c(e))$, i.e., $g_{ij}(e) = a_j(e)$ then we call this delay function decomposable. Note that in the case of decomposable function, $\forall e \colon \Phi_{ij}^e(f_1, \ldots, f_k) = \ldots = \Phi_i^e(f_1, \ldots, f_k)$. Further, we define uniform delay function as decomposable delay functions of the form $a_1 f_1^e + a_2 f_2^e + \ldots + a_k f_k^e + c(e)$ where the coefficients $a_i, 1 \leq i \leq k$ are independent of the edges. We consider both atomic and non-atomic flows. In atomic flows, each flow request is required to be satisfied on one path and in non-atomic networks the flow requests arise from multiple infinitesimal demands.

We provide bounds on the price of anarchy when there are $k$ types of commodities in networks with heterogeneous and decomposable (including uniform) delay functions. We also show almost tight price of anarchy bounds when the edges are associated with affine decomposable delay functions.

**Previous Work:** We review prior work, that is often cited in the study of equilibria of flow routing problems, initiated by Pigou \cite{20}, and furthered by Wardrop \cite{11}. In \cite{13}, Dafermos and Sparrow study the relationship between flows that are at Nash Equilibrium and flows that optimize a social welfare function. They consider a multi-commodity flow network with a delay function on the edges and establish a condition under which the flow satisfies both Nash Equilibrium conditions and also optimizes an aggregate objective function. Importantly the flows are assumed to be aggregated uniformly. The assumption is that units traveling along a link uniformly share the cost \cite{15}. The important and interesting aspect of this work is that for homogeneous polynomial costs over the aggregate variable $f(e)$, Nash equilibrium and social optimum solutions coincide. This is not so in a general setting. The study of the inefficiency of Nash Equilibria \cite{13} in network flows led to an analysis of price of anarchy for load dependent delays \cite{14}. Roughgarden and Tardos initiated the study of bounds on the price of anarchy \cite{12} in general networks followed by subsequent results by Roughgarden \cite{23, 24} and Correa et al. \cite{8, 27, 10} that studied the price of anarchy with respect to the optimum total latency. For the case where the price of anarchy is considered with respect to min-max delays, results of Weitz \cite{7} and Roughgarden \cite{12} show that the price of anarchy is dependent on the size of the network.

One closely related work is the player-specific congestion game \cite{6, 5} which allows for player specific delays but which still uses the aggregate flow to compute delays. Other congestion games include malicious players \cite{4, 3} and more recently result on the price of anarchy have been illustrated when the delay function is defined by the $L_p$ norm \cite{2}.

For atomic flows, the price of anarchy \cite{11} has been investigated for an un-splittable model with aggregate delay functions, which are independent of types of commodities. Further, in
an exact bound was proved on the worst-case PoA for unweighted and weighted in atomic un-splittable congestion games.

To our knowledge, the price of anarchy for heterogeneous latency functions has not been studied before.

For notational convenience, throughout this paper, we let $a_{\max} = \max_{i,e} a_i(e)$ (and correspondingly $a_{\min} = \min_{i,e} a_i(e)$ for the decomposable delay function case)

Our Results and Techniques for Non-atomic Network Flows: We will consider heterogeneous delay functions that are convex. The existence of Nash Equilibrium follows from the results in [12] which show the existence of a Nash Equilibrium flow in $k$-commodity non-atomic networks with convex delay functions.

- We provide a bound on the Price of Anarchy in networks with $k$ (types of) commodities where each edge is associated with polynomial delay functions that are heterogeneous. This bound is $a_{\max}(2k^2)^{\theta+1}$ in the most general form of the network. In the case of polynomial (heterogeneous) decomposable delay functions, the bound is $\max\{\theta + 1, a_{\max}k^{\theta-1}\}$. When we consider networks with affine (i.e. $\theta = 1$) and heterogeneous delay functions, the bound is $\frac{k^4a_{\max} + (k^4a_{\max})^{1/2}(k^4a_{\max}+4)^{1/2}}{2} + 1$.

- Examples illustrate that our results are almost tight for the case of decomposable delay functions. Furthermore, for affine functions we get tight bounds.

- For the general case, the techniques we use are based on variational inequalities combined with Holder’s inequality. To get sharper bounds for the decomposable case we introduce a multi-dimensional version of the Pigou bound [12] as well as another bound which we term as the Gamma bound. Combining the two provides us a tight bound for the affine case.

Our contributions for the price of anarchy in non-atomic selfish routing problems is shown in the following table, Table 1.

| k commodities | PoA |
|---------------|-----|
| Aggregate      | $\frac{1}{2\theta}$ (\cite{Lai}) |
| Decomposable   | $\max(2, a_{\max})^\theta$ [\text{Thm. 4.4}] | $\max(\theta + 1, a_{\max}k^{\theta-1})$ [\text{Thm. 4.4}] |
| Heterogeneous  | $a_{\max}(2k^2)^{\theta+1}$ [\text{Thm. 3.7}] | $a_{\min}(2k^2)^{\theta+1}$ [\text{Thm. 3.4}] |

Table 1: PoA for Nonatomic (indicates a tight bound and * indicates an almost tight bound)

Our Results and Techniques for Atomic Network Flows: In this section we present results regarding atomic flows. Wherever necessary we will assume that the instance provided has a Nash Equilibrium. Our results on the price of anarchy will be modulo this assumption.

1. We show the existence of Nash Equilibrium for uniform heterogeneous affine delay functions. Further, we show that for non-uniform heterogeneous affine functions, where edges have different delay functions, Nash Equilibrium need not exist. The second result is indeed surprising, especially since it only requires affine functions to generate the examples.
2. We consider the price of anarchy for atomic flows in networks with \( k \) (types of) commodities where each edge is associated with polynomial delay functions that are decomposable. When we consider networks with affine and decomposable delay functions this bound is 
\[
\frac{a_{\text{max}} + (a_{\text{max}^2} + 4a_{\text{max}})^{1/2}}{2} + 1. 
\]
In the case of polynomial decomposable functions, the bound is 
\[
a_{\text{max}} (2k)^d + 1. 
\]

3. Further, for atomic flows in networks with two different types of commodities and uniform delay functions, i.e. delays defined by \( a_1 f_1 + a_2 f_2 + c(e) \), we show an improved bound on the price of anarchy. This is achieved by considering the cycles that arise when the Nash equilibrium and social optimum flows are overlaid, canceling common flows. We are not aware of this technique being used before. We provide a bound of \( \sqrt{a_{\text{max}}} + 2 \). This bound is almost tight.

Our contributions for the price of anarchy in atomic selfish routing problems is shown in Table 2. For 2-commodity networks with uniform affine delay functions, we show (Table 3) improved bounds which are different from the bounds on general decomposable affine delay functions. Note that bound provided by Thm. 5.8 is an almost tight bound compared with a lower bound of \( \Omega(\sqrt{a_{\text{max}}}) \).

We present preliminaries in Section 2 and discuss relevant properties. We first consider PoA for non-atomic networks flows in the case of \( k \)-commodity network flows. The results are presented in Section 3 but all proofs are shown in the Appendix due to the page limitation. In subsection 4, we consider improved bounds for the price of anarchy for nonatomic networks with decomposable delay functions. Further, we present our results for atomic flows and strengthen our results on the price of anarchy in two-commodity network flows with uniform delay functions in Section 5.3.

| PoA | affine | polynomial |
|-----|--------|------------|
| Aggregate | \((3 + \sqrt{5})/2\) | \(\Theta\left(\frac{\vartheta}{\log \vartheta}\right)^{\theta + 1}\) |
| Decomposable | \((a_{\text{max}} + \sqrt{a_{\text{max}^2} + 4a_{\text{max}}})/2 + 1^*\) | \(a_{\text{max}} (2k)^{d + 1}\) |

Table 2: PoA for \( k \)-commodity Atomic Flows(* indicates an almost tight bound.)

| PoA | uniform | decomposable |
|-----|---------|-------------|
| Affine | \(\sqrt{a_{\text{max}}} + 2^*\) | \((a_{\text{max}} + \sqrt{a_{\text{max}^2} + 4a_{\text{max}}})/2 + 1\) |

Table 3: PoA for 2-commodity Atomic Flows

of \( \Omega(\sqrt{a_{\text{max}}}) \).

We consider a directed network \( G = (V, E) \) with a vertex set \( V \) and edge set \( E \), and a set \( K \) of \( k \) source-destination pairs \( \{s_1, t_1\}, \ldots, \{s_k, t_k\} \). We do not allow self-loops for any vertex but we accept parallel edges between any pair of vertices.

For commodity \( i \), denote the set of (simple) \( s_i-t_i \) paths by \( \mathcal{P}_i \), and define a set of paths \( \mathcal{P} = \bigcup \mathcal{P}_i \). For any path \( P \in \mathcal{P} \), we define a flow, \( f_P \in \mathbb{R}^+ \). The set of all flows is represented...
by the flow vector \( f = (f_P)_{P \in \mathcal{P}} \). For a fixed flow \( f \), commodity \( i \) and an edge \( e \) \( \in \mathcal{E} \) we denote the flow for commodity \( i \) through edge \( e \) by \( f_i(e) = \sum_{P \in \mathcal{P}^i} f_P \) and the flow vector on the edge by \( f(e) = (f_i(e))_{i \in \mathcal{K}} \). The total aggregate flow will also denoted by \(||f(e)||_1 = \sum_i f_i(e)\). The demand requirement of commodity \( i \) is denoted by \( r_i \), i.e., the amount of flow required to be pushed from source \( s_i \) to destination \( t_i \). We call \( f \) as a feasible flow if all demand requirements from each commodity are satisfied, i.e., \( \sum_{P \in \mathcal{P}^i} f_P = r_i, \forall i \).

We let \( \Phi_e : \mathbb{R}^k \rightarrow \mathbb{R}^+ \) be a heterogeneous delay function on edge \( e \), and assume that \( \Phi_e() \) is nonnegative and convex. The delay function of a path \( P \in \mathcal{P}^i \) is defined to be

\[
\Phi_{P}(f) = \sum_{e \in P} \Phi^i_e(f_1(e), \ldots, f_k(e))
\]

where \( \Phi^i_e(f_1(e), \ldots, f_k(e)) \) represents delay on edge \( e \) faced by commodity \( i \) under flow pattern \( f \).

A heterogeneous delay function is a generalization of the standard aggregate delay function \( \tilde{\Phi} \). In the most general case, we consider polynomial heterogeneous delay functions on \( e \) represented by \( g_{1}(e) f^0_1(e)+ g_{2}(e) f^0_2(e)+ \ldots + g_{k}(e) f^0_k(e)+ c(e) \) where \( f_i(e) \) is the flow of commodity \( i \) on edge \( e \), \( g_{j}(e) \in \mathcal{N} \) and nonnegative \( c(e) \) are constants as considered by Dafermos [17]. Here \( \theta \) represents the degree of the polynomial in the latency function. And the total delay on edge \( e \) that commodity \( i \) incurs is as follows:

\[
f_i(e)\Phi^i_e(f_1, \ldots, f_k) = f_i(e)(\sum_{j} g_{ij}(e)f^\theta_{ij}(e) + c(e))
\]

In the case of heterogeneous delay functions, each commodity might have different commodity related delay functions on the same edge \( e \). However, under the condition that \( g_{1}(e) = g_{2}(e) = \ldots = g_{k}(e) = a_i, 1 \leq \forall i \leq k \), i.e. the total delay is expressible as

\[
(f_1(e) + \ldots + f_k(e))(a_1(f^0_1(e) + \ldots + a_k(f^0_k(e) + c(e))),
\]

and the delay function is termed as decomposable.

Further, we define uniform functions as decomposable delay functions if the delay is of the form: \( a_1 f_1(e) + a_2 f_2(e) \ldots f_k e_k + c(e) \) such that \( a_1, \ldots, a_k \) are non-negative integers. We may assume w.l.o.g. that \( a_1 \geq a_2 \geq \ldots \geq a_k \).

Given an instance \( (G, r, \Phi) \), we define the social cost to be the sum total of all delays on edges over flows of all commodities, i.e. \( TC(f) = \sum_{e \in \mathcal{E}} \sum_{i} f_i(e)\Phi^i_e(f_1(e), f_2(e), \ldots, f_k(e)) \). Let \( f \) and \( \hat{f} \) be a Nash Equilibrium and a social optimum flow, respectively. Then the worst case Nash Equilibrium cost is denoted by \( C(f) = \max_{f \in \mathcal{E}} TC(f) \), where \( \mathcal{E} \) is the set of all Nash Equilibrium solutions and the social optimum cost \( C(\hat{f}) = \min_{\hat{f}} TC(\hat{f}) \). The price of anarchy is defined as follows:

\[
PoA = \frac{C(f)}{C(\hat{f})}.
\]

With respect to an instance \( (G, r, \Phi) \), a feasible flow minimizing \( TC(f) \) is said to be optimal or minimum-delay flow; such a flow always exists because the space of all feasible flows is a compact set and our cost function is continuous.

### 3 Price of Anarchy for k-Commodity Non-Atomic Network Flows

In this section we show bounds on the price of anarchy in \( k \)-commodity nonatomic networks with heterogeneous affine and polynomial delay functions. We consider lower bounds in section 3.2 and upper bounds in subsection 3.3 and 3.2.
We need the following variational inequality, the proof of which is simple.

**Lemma 3.1** Let \( f \) be a feasible flow for the non-atomic instance \((G, R, \Phi)\). The flow \( f \) is a Nash equilibrium flow if and only if

\[
\sum_{e} \sum_{i} f_i(e) \Phi_e(f_1, \ldots, f_k) \leq \sum_{e} \sum_{i} \hat{f}_i(e) \Phi_e(f_1, \ldots, f_k)
\]

for every flow \( \hat{f} \) feasible for \((G, R, \Phi)\).

### 3.1 A Lower Bound on PoA

We consider an example of a \( k \)-commodity, with \( k \) edges, network illustrated in figure 1. In this network, the top edge \( e_1 \) is associated with the delay function \( \Phi_{e_1}(f_1(e_1), \ldots, f_k(e_1)) = af_{\theta_1} + f_{\theta_2} + \ldots + f_{\theta_k} \) and in general the \( i \)th edge, \( e_i \) is associated with the delay function \( \Phi_{e_i}(f_1(e_i), \ldots, f_k(e_i)) = f_{\theta_1} + \ldots + f_{\theta_{i-1}} + af_{\theta_i} + \ldots f_{\theta_k} \). Demand requirements are defined as \( r_1 = \ldots = r_k = 1 \). The worst-case Nash equilibrium flow vector \( f \) is achieved when \( f_j(e_j) = 1 \), \( 1 \leq j \leq k \) and consequently the NE cost \( TC(f) = ak \). Conversely the social optimum flow \( \hat{f} \) can be obtained from \( \hat{f}_i(e_j) = 1/(k-1) \) when \( \ell \neq j, 1 \leq \ell, j \leq k \) and the social optimum is cost \( TC(\hat{f}) = k/(k-1)^{\theta-1} \).

**Lemma 3.2** Let \( C \) be a set of polynomial heterogeneous delay functions. Let \((G, R, \Phi)\) be a nonatomic \( k \)-commodity flow instance such that \( \Phi \in C \). The price of anarchy of nonatomic \( k \)-commodity flow routing on \((G, R, \Phi)\) is \( \Omega(a_{\max}(k-1)^{\theta-1}) \).

![Figure 1: Polynomial and Heterogeneous Latency](image)

### 3.2 \( k \)-commodities nonatomic heterogeneous polynomial delay functions

**Lemma 3.3** Let \( f \) be a Nash equilibrium flow and \( \hat{f} \) be any feasible flow vector in the instance with heterogeneous polynomial delay functions. Then

\[
TC(f) \leq k a_{\max}^{\theta} (TC(f))^{\theta-1} (TC(\hat{f}))^{\frac{1}{\theta-1}} + TC(\hat{f}).
\]
Proof: We use the variational inequality defined above. Then we apply Hölder’s inequality by letting \( x = (\theta + 1)/\theta \) and \( y = \theta + 1 \). Thus

\[
\mathcal{TC}(f) \leq \sum_e (f_1(e)(g_{11}(e)f_1^\theta(e) + \ldots + g_{1k}(e)f_k^\theta + c_1(e)) + \ldots + \\
(f_k(e)(g_{k1}(e)f_1^\theta(e) + \ldots + g_{kk}(e)f_k^\theta + c_k(e)))
\]

\[
\leq \sum_e (f_1^\theta(e))^{\frac{1}{\theta + 1}} \left( \sum_e (g_{11}(e)f_1^\theta(e))^{\frac{\theta + 1}{\theta}} \right) + \ldots + \\
(f_k^\theta(e))^{\frac{1}{\theta + 1}} \left( \sum_e (g_{kk}(e)f_k^\theta(e))^{\frac{\theta + 1}{\theta}} \right) + \ldots + \\
\sum_e (f_1(e)c_1(e) + \ldots + f_k(e)c_k(e))
\]

\[
\leq \mathcal{TC}(\hat{f})^{\frac{1}{\theta + 1}} \times \sum_e (g_{11}(e)f_1^\theta(e))^{\frac{1}{\theta}} \left( \Phi_e^1(f) \right) + \ldots + \\
\mathcal{TC}(\hat{f})^{\frac{1}{\theta + 1}} \times \sum_e (g_{kk}(e)f_k^\theta(e))^{\frac{1}{\theta}} \left( \Phi_e^k(f) \right) + \ldots + \mathcal{TC}(\hat{f})
\]

\[
\leq \mathcal{TC}(\hat{f})^{\frac{1}{\theta + 1}} \times \sum_e (g_{11}(e)f_1(e))^{\frac{1}{\theta}} \left( \Phi_e^1(f) \right) + \ldots + \\
\mathcal{TC}(\hat{f})^{\frac{1}{\theta + 1}} \times \sum_e (g_{kk}(e)f_k(e))^{\frac{1}{\theta}} \left( \Phi_e^k(f) \right) + \ldots + \mathcal{TC}(\hat{f})
\]

which leads to \( \mathcal{TC}(f) \leq k^2 a_{max} \mathcal{TC}(\hat{f})^{\frac{1}{\theta + 1}} \mathcal{TC}(f)^{\theta/\theta + 1} + \mathcal{TC}(\hat{f}) \). The first inequality holds due to variational inequality, and the second inequality can be obtained by applying Hölder’s inequality. The third inequality is true since all coefficients are at least 1, i.e., \( \forall i, j, e : g_{ij}(e) \geq 1 \). Because \( \forall i, j, e : (g_{ij}(e)f_j^\theta(e))^{\frac{1}{\theta}} = b_{ij}^\theta(e)f_j(e) \) when all coefficients are at least 1 and \( \theta \geq 1 \), the second last inequality also holds. The second last inequality leads to the last inequality, since we define \( a_{max} \) as a maximum coefficient.

\[ \square \]

**Theorem 3.4** Let \( C \) be a set of polynomial \( \theta \)-degree and heterogeneous delay functions. If
\((G, R, \Phi)\) is a \(k\)-commodity non-atomic network flow instance with delay functions in \(C\), then the price of anarchy of congestion games on \((G, R, \Phi)\) is at most \(a_{\text{max}}(2k^2)^{\theta+1}\).

We note that in a network with affine and heterogeneous latency functions, the total cost incurred by traffic at Nash equilibrium is at most the cost of a minimum-latency flow forced to route \((a_{\text{max}} + 1)\) factor more traffic between each source-destination pair as shown in \([12]\). We omit the corresponding proof since it is similar to the proof in \([12]\).

**Theorem 3.5** If \(f\) is a flow at Nash equilibrium for a \(k\)-commodity non-atomic flow problem \((G, R, \Phi)\), in networks with affine and heterogeneous delay functions, and \(\hat{f}\) is feasible for \((G, (a_{\text{max}} + 1)R, \Phi)\) then \(TC(f) \leq TC(\hat{f})\).

### 3.3 \(k\)-commodity nonatomic - heterogeneous affine delay functions

We show the price of anarchy in \(k\)-commodity nonatomic networks with heterogeneous affine delay functions. Based on Lemma 3.3, we have the following corollary:

**Corollary 3.6** Let \(f\) and \(\hat{f}\) be a Nash equilibrium flow and any feasible flow vector, respectively, in an instance with heterogeneous affine delay functions. Then

\[
TC(f) \leq k^2 a_{\text{max}}^{1/2} (TC(f)TC(\hat{f}))^{1/2} + TC(\hat{f}).
\]

Also, in a fashion similar to Theorem 3.4, we have the following theorem.

**Theorem 3.7** Let \(C\) be a set of heterogeneous, affine functions. If \((G, R, \Phi)\) is a \(k\)-commodity nonatomic instance with delay functions in \(C\), then the price of anarchy of congestion games on \((G, R, \Phi)\) is at most \(k^4 a_{\text{max}} + \frac{k^4 a_{\text{max}}}{2} + 1\).

### 4 Improved PoA for \(k\)-Commodity non-atomic Networks; the decomposable case

In this section we consider the price of anarchy for \(k\)-commodity non-atomic network flow congestion games where each edge is associated with a decomposable delay function.

As discussed in the introduction, it has been shown that Nash equilibrium flow is equivalent to a social optimum flow under homogeneous aggregate delay functions. However, generally this does not hold, and even with an affine delay function, PoA is bounded by \(4/3\). The PoA is shown to be worse for heterogeneous degree-\(\theta\) polynomial delay functions.

We consider degree-\(\theta\) polynomial functions (affine cases are similar to polynomial cases). In this case the delay functions are characterized by \(\Phi(x(e)) = a_1(e)x_1^\theta(e) + \ldots + a_k(e)x_k^\theta(e) + c(e)\) where \(c(e)\) indicates a nonnegative constant. We omit the edge notation \((e)\) to simplify notations.

For the delay functions that we consider, the standard Pigou bound arguments do not apply. We thus consider a multi-dimensional version of the Pigou bound, which in itself does not suffice. In the first subsection below we establish that the price of anarchy is bounded by the worse of two bounds, the (multi-dimensional) Pigou bound and the Gamma bound, which we describe in detail below. In the second subsection we analyze these bounds in the context of the delay functions defined by degree-\(\theta\) polynomial functions.
4.1 Upper Bounds on PoA

We will determine PoA for decomposable delay functions described by degree-$\Theta$ polynomials. To obtain the bounds we require the notion of the Pigou bound [12] as applied to vectors of flow that represent the $k$ types of commodities.

**Definition 1 (Pigou bound)** Let $\mathcal{C}$ be a nonempty set of cost functions. The Pigou bound $\alpha(\mathcal{C})$ for $\mathcal{C}$ is

$$\alpha(\mathcal{C}) = \sup_{\Phi \in \mathcal{C}} \sup_{x, r \geq 0} \frac{||r||_1 \Phi(r_1, \ldots, r_k)}{||x||_1 \Phi(x_1, \ldots, x_k) + (||r||_1 - ||x||_1) \Phi(r_1, \ldots, r_k)};$$

where $r = (r_1, \ldots, r_k)$ and $x = (x_1, \ldots, x_k)$ and $r, x \in \mathbb{R}^k$ with the understanding that $0/0 = 1$. $r_1, \ldots, r_k$ and $x_1, \ldots, x_k$ are nonnegative.

We will evaluate the Pigou bound in a restricted setting by using the following convex program (SC):

**SC:** $\alpha(\mathcal{C}) = \sup_{\Phi \in \mathcal{C}} \sup_{x, r \geq 0} \frac{||r||_1 \Phi(r_1, \ldots, r_k)}{||x||_1 \Phi(x_1, \ldots, x_k) + (||r||_1 - ||x||_1) \Phi(r_1, \ldots, r_k)}$ subject to

$$\Phi(x_1, \ldots, x_k) \leq \Phi(r_1, \ldots, r_k)$$

$$||x||_1 \leq ||r||_1$$

Note that when $\Phi(x_1, \ldots, x_k) > \Phi(r_1, \ldots, r_k)$, $||x||_1(\Phi(x_1, \ldots, x_k) - \Phi(r_1, \ldots, r_k))$ becomes positive and decreases the value of $\alpha(\mathcal{C})$: thus we assume that $\Phi(x_1, \ldots, x_k) \leq \Phi(r_1, \ldots, r_k)$.

We split our analysis into two cases: 1) $||x||_1 \leq ||r||_1$ and 2) $||x||_1 > ||r||_1$. In the case when $||x||_1 \leq ||r||_1$, we consider the Pigou bound as described above. For the second case, we will consider a different ratio, termed the Gamma Bound, which is defined as follows:

$$\gamma(\mathcal{C}) = \sup_{\Phi \in \mathcal{C}} \sup_{f, \hat{f}} \sup_{e \in E} \frac{\Phi_e(f_1(e), \ldots, f_k(e)) ||f(e)||_1}{\Phi_e(\hat{f}_1(e), \ldots, \hat{f}_k(e)) ||\hat{f}(e)||_1}$$

where, in our proof, $\hat{f}$ and $f$ will indicate optimal and Nash equilibrium flows on edge $e \in E$ termed as $E_{||f(e)|| > ||\hat{f}(e)||}$, which is the set of all edges with the property that $||f(e)|| > ||\hat{f}(e)||$.

To show a bound on PoA we use the variational inequality characterization [22] described before.

**Theorem 4.1** Let $\mathcal{C}$ be a set of decomposable delay functions and $\alpha(\mathcal{C})$ the Pigou bound for $\mathcal{C}$. If $N = (G, R, \Phi)$ is a non-atomic $k$-commodity instance of the network congestion game with delay function $\Phi \in \mathcal{C}$, then the price of anarchy of congestion games in the network $N$ is at most $\max\{\alpha(\mathcal{C}), \gamma(\mathcal{C})\}$.

**Proof:** Let $\hat{f}$ and $f$ be optimal and equilibrium flows, respectively, for a non-atomic instance $(G, R, \Phi)$ with delay functions in the set $\mathcal{C}$. We let the total cost of flow $f$ be $C(f)$. To prove
the theorem we note that:

\[
C(\hat{f}) = \sum_{e \in E} \Phi_e(\hat{f}_1(e), \ldots, \hat{f}_k(e))\|\hat{f}(e)\|_1 \\
\geq \frac{1}{\alpha(C)} \sum_{e \in E_{|x|| \leq |r|}} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 + \sum_{e \in E_{|x|| > |r|}} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 - \|f(e)\|_1 + \frac{1}{\gamma(C)} \sum_{e \in E_{|x|| > |r|}} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 \\
\geq \frac{1}{\alpha(C)} \sum_{e \in E_{|x|| \leq |r|}} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 + \frac{1}{\gamma(C)} \sum_{e \in E_{|x|| > |r|}} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 \\
\geq \min\{\frac{1}{\alpha(C)}, \frac{1}{\gamma(C)}\} \sum_{e \in E} \Phi_e(f_1(e), \ldots, f_k(e))\|f(e)\|_1 = \frac{C(f)}{\max\{\alpha(C), \gamma(C)\}}.
\]

The first inequality follows from the definition of the Pigou bound applied to each edge in \(E_{|x|| \leq |r|}\) and the Gamma bound w.r.t. edges in \(E_{|x|| > |r|}\). Here we split the edge set into two cases as mentioned before: \(E_{|x|| \leq |r|}\) and \(E_{|x|| > |r|}\) indicates a set of edges with \(\|x(e)\|_1 \leq \|r(e)\|_1\) and \(\|x(e)\|_1 > \|r(e)\|_1\), respectively. The second term in the first inequality can be ignored due to the variational inequality characterization.

\[\square\]

### 4.2 k-commodity non-atomic flow with polynomial and decomposable delay function

In this subsection we evaluate the Pigou and Gamma bounds for network instances where the delay function of each path is defined as a decomposable polynomial function of degree \(\theta\). All proofs are in the appendix due to the page limitation.

**Lemma 4.2** Let \(C\) be a set of polynomial \(\theta\)-degree decomposable delay functions and \(\alpha(C)\) the Pigou bound for \(C\). Let \((G, R, \Phi)\) be a non-atomic \(k\)-commodity flow instance such that \(\Phi \in C\). The Pigou bound for congestion games on \((G, R, \Phi)\) is \(\alpha(C) = \theta + 1\).

**Proof:** We consider the value of the Pigou bound and find a bound on the infimum of the divisor, \(\beta(C)\), instead of supremum of \(\alpha(C)\) for simplicity. W.l.o.g. let us assume that \(|\{i \in K^\ell : \frac{\partial \beta(C)}{\partial x_i} = 0\}| = \ell\) and \(\forall i \in \{1, 2, \ldots, \ell\}, \frac{\partial \beta(C)}{\partial x_i} = 0\). Note that by solving system of equations, \(\frac{\partial \beta(C)}{\partial x_i} = 0\) for \(i, 1 \leq i \leq \ell\), we can obtain

\[
a_i \theta x_i^{\theta - 1} ||x||_1 - \Phi(r_1, \ldots, r_k) + \Phi(x_1, \ldots, x_k) = 0
\]

which can be written as

\[
a_i \theta x_i^{\theta - 1} ||x||_1 = \Phi(r_1, \ldots, r_k) - \Phi(x_1, \ldots, x_k).
\]

Let us consider multiply with \(x_i\). Then

\[a_i \theta x_i^{\theta} ||x||_1 = x_i(\Phi(r_1, \ldots, r_k) - \Phi(x_1, \ldots, x_k)).\]

and let us further sum over all \(1 \leq i \leq \ell\),

\[
\sum_{i=1}^{\ell} a_i \theta x_i^{\theta} ||x||_1 = \sum_{i=1}^{\ell} x_i(\Phi(r_1, \ldots, r_k) - \Phi(x_1, \ldots, x_k))
\]
which is equivalent to
\[ ||x||_1 \sum_{i=1}^{\ell} a_i x_i^\theta = ||x||_1 (\Phi(r_1, \ldots, r_k) - \Phi(x_1, \ldots, x_k)) \]

and
\[ \sum_{i=1}^{\ell} a_i x_i^\theta = \Phi(r_1, \ldots, r_k) - \Phi(x_1, \ldots, x_k). \]  (1)

By adding \( c(e) \theta \) in both sides, we can obtain
\[ (\theta + 1)\Phi(x_1, \ldots, x_k) = \Phi(r_1, \ldots, r_k) + c(e) \theta. \]

Further,
\[ \beta(C) = ||r||_1 \Phi(r_1, \ldots, r_k) + ||x||_1 \frac{c(e) \theta + \Phi(r_1, \ldots, r_k)}{\theta + 1} - ||x||_1 \Phi(r_1, \ldots, r_k) \]  (2)
\[ = ||r||_1 \Phi(r_1, \ldots, r_k) + ||x||_1 \frac{c(e) \theta - \theta \Phi(r_1, \ldots, r_k)}{\theta + 1} \]  (3)
\[ \geq ||r||_1 \Phi(r_1, \ldots, r_k) + ||x||_1 \frac{c(e) \theta - \theta \Phi(r_1, \ldots, r_k)}{\theta + 1} \]  (4)
\[ = ||r||_1 (\Phi(r_1, \ldots, r_k) + \frac{c(e) \theta}{\theta + 1} - \frac{\theta}{\theta + 1} \Phi(r_1, \ldots, r_k)) \]  (5)
\[ \geq \frac{1}{\theta + 1} ||r||_1 \Phi(r_1, \ldots, r_k) \]  (6)

Thus,
\[ \alpha(C) = \frac{||r||_1 \Phi(r_1, \ldots, r_k)}{||x||_1 \Phi(x_1, \ldots, x_k) + ||r||_1 - ||x||_1 \Phi(r_1, \ldots, r_k)} \]  (8)
\[ \alpha(C) = \frac{||r||_1 \Phi(r_1, \ldots, r_k)}{\beta(C)} \]  (9)
\[ \leq \frac{||r||_1 \Phi(r_1, \ldots, r_k)}{||r||_1 \frac{\Phi(r_1, \ldots, r_k)}{\theta + 1}} \]  (10)
\[ = \theta + 1 \]  (11)
\[ \square \]

Now let us consider when \( ||x||_1 > ||r||_1 \).

**Lemma 4.3** Let \( C \) be a set of degree-\( \theta \) polynomial decomposable delay functions. If \((G, R, \Phi)\) is a non-atomic instance with delay functions \( \Phi \in C \), then \( \gamma(C) = a_{\max}k^{\theta-1} \).

**Proof:** When \( ||x||_1 \geq ||r||_1 \),
1. When \( x_1 + \ldots + x_\ell > r_1 + \ldots + r_k \) on edge \( e \), social optimum delay can be minimized by considering \( x_1 + \ldots + x_\ell = ||r||_1 \).

\[
\gamma(C) = \frac{\Phi(r_1, \ldots, r_k)}{\Phi(x_1, \ldots, x_\ell)} \leq \frac{a_1 r_1^\theta + \ldots + a_k r_k^\theta}{a_1 x_1^\theta + \ldots + a_\ell x_\ell^\theta} \leq \frac{\max_i a_i r_i^\theta + \ldots + r_k^\theta}{\min_i a_i x_i^\theta + \ldots + x_\ell^\theta} \leq \frac{\max_i a_i}{\min_i a_i} \left( \frac{r_1^\theta + \ldots + r_k^\theta}{\ell^{(r_1 + \ldots + r_k)/\ell}} \right) \leq \frac{\max_i a_i}{\min_i a_i} \ell^{\theta - 1} \frac{r_1^\theta + \ldots + r_k^\theta}{(r_1 + \ldots + r_k)^\theta} \leq a_{\max} k^{\theta - 1} \tag{17}
\]

In (14) to minimize the divisor, i.e. \( x_1^\theta = \ldots = x_\ell^\theta \), we have \( x_1 = \ldots = x_\ell \). Since \( x_1 + \ldots + x_\ell = ||r||_1 \), it holds that \( x_1 = \ldots = x_\ell = ||r||_1/\ell \). (17) can be obtained by \( \ell \leq k \).

\section{PoA for atomic flows}

In this section we first show negative and positive results on the existence of Nash equilibrium. While in general, Nash equilibrium does not exist, even for affine functions, as shown in subsection 5.1, we show that Nash equilibrium exists when delay functions are affine and uniform.

\subsection{Existence of Atomic NE flow?}

We first show that there may be no pure Nash equilibrium in the case of atomic, decomposable and affine delay functions with unweighted demand requirement. In figure 2 there are two

![Figure 2: Affine and Decomposable Latency](image)

users with \( r_1 = r_2 = 1 \) (say, unweighted) and the users have the same set of strategies (paths) from source \( s \) to destination \( t \). The delay function associated with an edge is shown in the
Let us define four paths $P_1 = \{(s,t)\}$, $P_2 = \{(s,u),(u,t)\}$, $P_3 = \{(s,u),(u,v),(v,t)\}$ and $P_4 = \{(s,v),(v,t)\}$. Since coefficient of user 2 edges in $P_1$ and $P_3$, user 2 will not utilize these paths. Thus, there are totally eight pairs of paths each of which are chosen by user 1 and user 2, respectively as shown in Table 4. The strategies of the two users will be represented by the pair of strategies $(P, P')$ where $P$ and $P'$ belong to the set of 4 paths $\{P_1, P_2, P_3, P_4\}$. There exists a sequence of strategy pairs that cycle $\{(P_2, P_3), (P_3, P_4), (P_4, P_1), (P_1, P_2)\}$. Also, other strategies are not stable. Strategy pairs $(P_4, P_2)$ and $(P_2, P_3)$ will shift to the strategy pair $(P_3, P_2)$ because $P_3$ is more beneficial to user 1. And lastly, strategy pairs $(P_2, P_4)$ and $(P_4, P_3)$ are unstable since the strategy pair $(P_3, P_3)$ is preferred by user 1. This example shows that there may be no pure Nash equilibrium in 2-commodity atomic networks where each edge is associated with affine, decomposable delay functions and each user have one unit of demand requirement (a case usually referred to as unweighted). Note that this is somewhat surprising since it well known that any unweighted congestion game with homogenous delays has at least one pure Nash equilibrium.\[23\, 24\]

However, in two-commodity atomic networks when each edge is associated with affine, uniform delay functions, we show that there exists at least one pure Nash equilibrium. This proof applies to both unweighted and weighted demands and is based on the existence of a potential function.

**Theorem 5.1** Let $C$ be a set of affine, uniform delay functions. If $(G, R, \Phi)$ is a $k$-commodity atomic instance with delay functions in $C$, then $(G, R, \Phi)$ admits at least one equilibrium.

**Proof:** Let us define a potential function $\Psi(f) = \sum_{e}(d^2(e) + \sum_{j \in S(e)}(a_j(e)r_j(e))^2)$ where $d(e) = \sum_{i=1}^{k}a_i(e)r_i(e) + c(e)$ for every feasible flow $f$. We claim that a global minimum $f$ of the potential function $\Psi$ is also an equilibrium flow for $(G, R, \Phi)$. Assume, for a contradiction, that deviating from path $P_i$ to path $P_i$ by user $i$ strictly decrease its delay. In other words,

$$\Delta(f, \bar{f}) = \sum_{e \in P_i \backslash P_i} \Phi_e(f_1(e), \ldots, f_k(e)) - \sum_{e \in P_i \backslash P_i} \Phi_e(f_1(e), \ldots, f_k(e)) < 0.$$ 

On the other hand, let us consider the potential function $\Psi$ w.r.t. user $i$ deviation. For any edge $\bar{e}$ in $P_i \backslash P_i$, we have extra term $d(\bar{e}) + a_i(\bar{e})r_i(\bar{e}) = a_i^2(\bar{e})r_i^2(\bar{e}) - d^2(\bar{e})$ which leads to $2d(\bar{e})a_i(\bar{e})r_i(\bar{e}) + 2a_i^2(\bar{e})r_i^2(\bar{e})$. For any edge $e$ in $P_i \backslash P_i$, we lose term $d^2(e) - (d(e) - a_i(e)r_i(e))^2 = a_i^2(e)r_i^2(e)$ which leads to $2d(e)a_i(e)r_i(e)$. Thus, $\Psi(f) - \Psi(\bar{f}) = \sum_e 2d(e)a_i(e)r_i(e) + 2a_i^2(e)r_i^2(e) - \sum_e 2d(e)a_i(e)r_i(e) = 2a_i(\bar{f}) \Delta(f, \bar{f})$. Note that for all edges $e, a_i(e)$ is the same because we consider uniform delay functions. Since $\Delta(f, \bar{f})$ is negative and $a_i, r_i > 0$, the potential function value of $\bar{f}$ is strictly less than the potential value of $f$, which contradicts that $f$ is a global minimum. \[\square\]

![Table 4: Non-existence of Pure Nash Equilibrium](image)
5.2 \textit{k-commodity atomic - decomposable polynomial delay functions}

In this section we provide bounds on the price of anarchy in \(k\)-commodity network flows with decomposable delay functions.

5.2.1 A Lower Bound on PoA

We show an example network in figure 3. In this network, the top edge \(e\) is associated with the delay function \(\Phi_e(f_1(e), f_2(e)) = f_1 + af_2\) and the bottom edge \(h\) is associated with the delay function \(\Phi_h(f_1(h), f_2(h)) = af_1 + f_2\). Demand requirements are defined as \(r_1 = 1\) and \(r_2 = 1\).

![Figure 3: Affine and Decomposable Latency](image)

The worst-case Nash equilibrium flow vector \(f\) is achieved when \(f_1(e) = 1\) and \(f_2(h) = 1\) and consequently the NE cost \(TC(f) = 2a\). Conversely the social optimum flow \(\hat{f}\) can be obtained from \(f_1(e) = 1\) and \(f_2(h) = 1\) and the SO cost \(\overline{TC}(\hat{f}) = 2\).

\textbf{Lemma 5.2} Let \(C\) be a set of affine decomposable heterogeneous delay functions. Let \((G, R, \Phi)\) be an atomic two-commodity flow instance such that \(\Phi \in C\). The price of anarchy of atomic flow routing on \((G, R, \Phi)\) is \(\Omega(a_{\text{max}})\) when \(a_{\text{max}} \geq 2\).

Note that when \(a = 1\), it has been shown that PoA = 4/3 [12].

5.2.2 An Upper Bound on PoA

In this section we provide an upper bound on the price of anarchy for \(k\)-commodity network flows. We first consider affine delay functions. We use the Cauchy-Schwartz inequality to prove the following:

\textbf{Lemma 5.3} Let \(f\) and \(\hat{f}\) be a Nash equilibrium and a social optimum flow, respectively. Then

\[
C_I(f) - a_{\text{max}}^{1/2} \sqrt{a_{\text{max}}(C_I(f))^{1/2} + C_I(\hat{f})}.\]

The proofs of the above lemma and the following theorems are in the appendix due to page limitations. Further,

\textbf{Theorem 5.4} Let \(C\) be a set of affine, decomposable delay functions. If \((G, R, \Phi)\) is a \(k\)-commodity atomic instance with delay functions in \(C\), then the price of anarchy of atomic flow routing on \((G, R, \Phi)\) is at most \(\frac{a_{\text{max}} + \sqrt{a_{\text{max}}^2 + 4a_{\text{max}}}}{2} + 1\).

\textbf{Proof:} From Lemma 5.3

\[
C_I(f) - a_{\text{max}}^{1/2} \left( \frac{C_I(f)C_I(\hat{f})}{C_I(\hat{f})} \right)^{1/2} - C_I(\hat{f}) \leq 0,
\]

and thus

\[
\left( \frac{C_I(f)}{C_I(\hat{f})} \right) - a_{\text{max}}^{1/2} \left( \frac{C_I(f)}{C_I(\hat{f})} \right)^{1/2} - 1 \leq 0,
\]

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by dividing by \( C_I(\hat{f}) \). Then, we obtain a quadratic equation \( x^2 - \frac{a_{\max}^2}{2a_{\max} + 4}x - 1 \leq 0 \) where
\[ x = \left( \frac{C_I(f)}{C_I(\hat{f})} \right)^{1/2} \]
which leads to \( \text{PoA} = \left( \frac{(a_{\max}^2 + 4)^{1/2} + (a_{\max})^{1/2}}{2} \right)^2 = a_{\max} + (a_{\max}^2 + 4a_{\max})^{1/2} + 1. \)

We can extend the above proof to find an upper bound on the price of anarchy in \( k \)-commodity atomic networks with decomposable polynomial delay functions. The proof technique is the same as the proof shown in Section in [1].

**Theorem 5.5** Let \( C \) be a set of degree \( \theta \) polynomial, decomposable delay functions. If \((G, R, \Phi)\) is a \( k \)-commodity atomic instance with delay functions in \( C \), then the price of anarchy of atomic flow routing on \((G, R, \Phi)\) is \( O(2^\theta \theta^\theta + 1) \).

### 5.3 Improved PoA for Uniform Two-Commodity Networks, Affine Delays

In this subsection we provide improved bounds for two-commodity atomic flows when the delay function is affine and uniform, i.e. all edges in the network have the same delay function.

### 5.4 Lower Bounds on PoA for Uniform Delay Functions

For the class of affine functions of the form \( af_1 + f_2 \), we provide a simple example which shows PoA is \( (a)^{1/2} \). In this network, the top edge \( e \) is associated with the delay function \( \Phi_e(f_1(e), f_2(e)) = a \) and the bottom edge \( h \) is associated with the delay function \( \Phi_h(f_1(h), f_2(h)) = af_1 + f_2 \). Demand requirements are defined as \( r_1 = 1 \) and \( r_2 = (a)^{1/2} \). The Nash equilibrium (NE) flow vector \( f \) has the flow \( f_1(h) = 1 \) and \( f_2(e) = (a)^{1/2} \) and consequently the cost of NE \( TC(f) = a((a)^{1/2} + 1) \). Conversely the social optimum (SO) flow \( \hat{f} \) can be obtained from \( \hat{f}_1(e) = 1 \) and \( \hat{f}_2(h) = (a)^{1/2} \) and the SO cost \( TC(\hat{f}) = 2a \). The price of anarchy is thus \( 2a = \frac{(a)^{1/2} + 1}{a} \).

**Lemma 5.6** Let \( C \) be a set of affine uniform delay functions. Let \((G, R, \Phi)\) be an atomic two-commodity flow instance such that \( \Phi \in C \). The price of anarchy of atomic flow routing on \((G, R, \Phi)\) is at least \( \frac{(a)^{1/2} + 1}{2} \), i.e., \( \Omega(\sqrt{a}) \).

### 5.5 Almost tight upper bound for Uniform Functions

Consider \( F_\otimes = (F - \hat{F}) \cup (\hat{F} - F) \) where \( F \) and \( \hat{F} \) are a Nash equilibrium flow and a social optimum flow, respectively. Note that \( \otimes \) eliminates common flows in \( F \) and \( \hat{F} \). We let edges in \( F_1 \otimes \hat{F}_1 \) be colored red and edges in \( F_2 \otimes \hat{F}_2 \) be colored blue.

By reversing the flow in \( \hat{F} \) and partitioning \( F_\otimes \), we obtain a set of cycles. We let the red cycles be denoted by \( CC^1 \) and blue cycles by \( CC^2 \). Note that an edge could be utilized by commodity 1 as well as commodity 2, and it can occur in cycles of both colors. We need a lemma which
show that there exists a good partition of the cycles so that we can obtain an almost tight upper bound.

**Lemma 5.7**

\[
\frac{a_1 + a_2 + \ldots + a_d}{b_1 + b_2 + \ldots + b_d} \leq \max_{\pi \in \Pi} \frac{\sum_{e \in \pi} a^j_e}{\sum_{e \in \pi} b^j_e}.
\]

Let \((a_i, b_i)\) be a collection of pairs where \(1 \leq i \leq d\) : \(a_i, b_i \in \mathbb{R}_+\). Let \(\Pi\) be a set of partitions of \(I = \{1, \ldots, d\}\) such that

- If \(\pi = (\pi^1, \ldots, \pi^d) \in \Pi\) then \(|\pi^j| \leq 2\),
- If \(\pi = (\pi^1, \ldots, \pi^d) \in \Pi\) then \(\bigcup_j \pi^j = I\),
- If \(\pi^{j_1}, \pi^{j_2} \in \pi\) and \(\pi^{j_1} \neq \pi^{j_2}\) then \(\pi^{j_1} \cap \pi^{j_2} = \emptyset\).

Let \(\hat{\pi} \in \Pi\) be an optimum partition of \(I\) defined as follows.

\[
\forall \pi \in \Pi : \max_{\pi^j \in \pi} \frac{\sum_{e \in \pi^j} a^j_e}{\sum_{e \in \pi^j} b^j_e} \geq \max_{\pi^j \in \hat{\pi}} \frac{\sum_{e \in \pi^j} a^j_e}{\sum_{e \in \pi^j} b^j_e}
\]

where \(\hat{\pi} = (\hat{\pi}^1, \ldots, \hat{\pi}^q)\) and \(\hat{\pi}^j\) corresponds to the \(j\)-th part and is a subset of \(t_j\) (at most 2) elements of \(I\), i.e. \(\{x_1, \ldots, x_{t_j}\}\).

### 5.6 Analyzing PoA bounds for Cycles

We next show that PoA can be estimated by considering at most one cycle per commodity instead of all cycles. Throughout this subsection, let \(a = a_1/a_2\).

Let \(CC\) be the set of all cycles. Let partition \(\bar{\pi}\) of \(CC\) where each part is of size either 1 or of size 2 (in which case it contains 1 cycle from each commodity, \(C^1 \in CC^1\) and \(C^2 \in CC^2\)). Then this partition \(\bar{\pi}\) guarantees an almost tight upper bound.

Let the pair \((f_1, f_2)\) and \((\bar{f}_1, \bar{f}_2)\) be the NE flow of commodity 1 and commodity 2 and social optimum flow of commodity 1 and commodity 2, respectively. Then,

\[
\frac{\sum_{e \in E} (f_1(e) + f_2(e)) \Phi(f_1(e), f_2(e))}{\sum_{e \in E} (\bar{f}_1(e) + \bar{f}_2(e)) \Phi(\bar{f}_1(e), \bar{f}_2(e))} \leq \max_{\pi^j \in \bar{\pi}} \left( \frac{\text{cost of NE flow in cycles in } \bar{\pi}^j}{\text{cost of SO flow in cycles in } \bar{\pi}^j} \right) \leq \max_{\pi^j \in \Pi} \left( \frac{\text{cost of NE flow in cycles in } \pi^j}{\text{cost of SO flow in cycles in } \pi^j} \right) \leq \max_{\pi^j = (C^1, C^2)} \left( \frac{f_1 \Phi_{C^1}(f_1, f_2) + f_2 \Phi_{C^2}(f_1, f_2)}{f_1 \Phi_{C^1}(\bar{f}_1, \bar{f}_2) + f_2 \Phi_{C^2}(\bar{f}_1, \bar{f}_2)} \right) = \frac{f_1 \Phi_{C^1}(f_1, f_2)}{f_1 \Phi_{C^1}(\bar{f}_1, \bar{f}_2)}
\]

where \(\Phi_{C^1}(\cdot)\) and \(\Phi_{C^2}(\cdot)\) represent cost incurred in \(C^1\) and \(C^2\), respectively. The first inequality holds due to Lemma 5.7. In the last inequality, we have \(|\bar{\pi}| = 2\) for the first factor and \(|\bar{\pi}| = 1\) for the second and third factors. Again, let \(\bar{\pi} = \{\bar{\pi}^1, \ldots, \bar{\pi}^q\}\) be a partition which guarantees the minimum of the maximum cost amongst partitions in \(\Pi\). In Lemma 5.7 we show that the maximum cost element in \(\bar{\pi}\) provides an upper bound for \(C_1(f)/C_1(\bar{f})\). So, the partition \(\bar{\pi}\)
considered here may provide an upper bound for $C_1(f)/C_1(\hat{f})$. Later we will show that this partition is good enough to obtain an almost tight bound for $C_1(f)/C_1(\hat{f})$.

As shown in Figure 4 we have two paths, termed as top path and bottom path. Let $P$ be either top path or bottom path in any cycle. Throughout this section, let $\Phi_P(f_1, f_2)$ be cost of path $P$ when flow on path $P$ of commodity 1 is $f_1$ on and that on path $P$ of commodity 2 is $f_2$. For $P' \subseteq P$, we define $\Phi'_P(f_1, f_2)$ as cost function of path $P'$ which is utilized by both $f_1$ and $f_2$. Similarly for $P'' \subseteq P$ let $\Phi''_P(f_1, 0)$ (or $\Phi''_P(0, f_2)$) be defined as the cost function over path $P''$ which is utilized by $f_1$ (or $f_2$), but not both. Note that $P'' \cap P'' = \emptyset$.

To analyze the price of anarchy, we consider the structure of the cycles obtained when the optimal flow is reversed. We then consider all possible cases when commodity 1 flow is considered as a primary flow as shown in Figure 6. The cases are categorized into three classes: i) cases 1, 5, 7 and 13, ii) cases 2, 3, 9 and 16 and iii) cases 4, 6, 8, 10, 11, 12, 14 and 15. The details of the cycle structure and the case analysis is in the appendix due to the page limitation.

5.6.1 Cycle Structure

In this subsection we consider the structure of the cycles obtained when the optimal flow is reversed. One of the cycle obtained is illustrated in figure 5. Here $u$ and $v$ are starting and end nodes such that reversing of $\hat{f}_1$ leads to a cycle. Though there are cases for commodity 1 and commodity 2, we consider cycles for commodity 1. We consider flow from commodity 1 as a primary flow and flow from commodity 2 as a secondary flow, respectively. The case when flow from commodity 2 is a primary flow is symmetric to this case. Note that though we consider cycles from commodity 1, it is possible that those might be intersected with cycles from commodity 2. We show an example of a cycle from commodity 1 intersecting with commodity 2 flows in Figure 6. In figure 6 the blue and real line represent $f_2$; while the blue and dashed lines represent $\hat{f}_2$. As shown in figure 6, $f_2$ and $\hat{f}_2$ are intersected with a cycle of commodity 1 and also $f_2$ and $\hat{f}_2$ can be overlapped to each other.

Dependong on how NE flow and social optimum flow uses the top and bottom path in the cycle, we have sixteen possibilities. We show that three cases dominate all other cases and thus we provide an upper bound of the PoA for these three cases. Throughout this subsection we denote by $\ell_1, \ell_2, \ell_3$ and $\ell_4$ the number of edges in path $P_1, P_2, P_3$ and $P_4$, respectively. Remember that $f_1 = \hat{f}_1 = r_1$ and $f_2 = \hat{f}_2 = r_2$ due to the definition of atomic network model in this chapter.

5.6.2 Three cases are enough to be considered

Note the $f$ and $\hat{f}$ correspond to a NE flow and a social optimum flow. We consider all possible cases when commodity 1 flow is considered as a primary flow as shown in Figure 6 (flow on top path vs. flow on bottom path): (1) $f_1, f_2, \hat{f}_2$ vs. $\hat{f}_1, \hat{f}_2$, (2) $f_1, f_2, \hat{f}_2$ vs. $f_1, f_2$, (3) $f_1, f_2$ vs. $\hat{f}_1, f_2, \hat{f}_2$, (4) $f_1, \hat{f}_2$ vs. $\hat{f}_1, f_2, \hat{f}_2$, (5) $f_1, f_2$ vs. $\hat{f}_1, f_2$, (6) $f_1, \hat{f}_2$ vs. $\hat{f}_1, f_2$, (7) $f_1, f_2$ vs. $\hat{f}_1$, (8)
\[ f_1, \hat{f}_2 \text{ vs. } \hat{f}_1, (9) \quad f_1, f_2, \hat{f}_2 \text{ vs. } \hat{f}_1, f_2, \hat{f}_2, (10) \quad f_1 \text{ vs. } \hat{f}_1, f_2, \quad (11) \quad f_1 \text{ vs. } \hat{f}_1, f_2, \quad (13) \quad f_1, f_2, \hat{f}_2 \text{ vs. } \hat{f}_1, (14) \quad f_1 \text{ vs. } \hat{f}_1, f_2, f_2, (15) \quad f_1, \hat{f}_2 \text{ vs. } f_1, f_2 \quad \text{and (16) } f_1, f_2 \text{ vs. } \hat{f}_1, f_2. \]

By reversing \( \hat{f}_1 \), each of the structures considered becomes a directed cycle. As mentioned before, \( f_2 \) or \( \hat{f}_2 \) (or both) use edges in these cycles. Table 5 lists the cases and a related variational inequality obtained by the flow being Nash equilibrium. Let us consider case 16 to show an example how to construct formulas for \( C_1(f) / C_1(\hat{f}) \) using the corresponding variational inequality. There are two NE flows \( f_1 \) and \( f_2 \) over the top path, and the cost of NE for commodity 1 is \( f_1 \Phi_1(f_1, f_2) \); while, the cost of SO for commodity 1 is incurred only by \( \hat{f}_1 \) which results in \( \hat{f}_1 \Phi_2(f_1, 0) \). Its corresponding variational inequality is \( \Phi_1(f_1, f_2) \leq \Phi_2(f_1, 0) \), \( \Phi_1(f_1, f_2) \leq \Phi_2(f_1, 0) \) since a shift of flow \( f_1 \) to the bottom path (note that \( f_2 \) already exists) does not decrease cost incurred by \( f_1 \) and \( f_2 \) over the top path.

### 5.6.3 cases 1, 5, 7 and 13

Note that case 1 and case 5 use the same formula and variational inequality, and case 7 and case 13 do the same formula and variational inequality. Then cost of case 1 and case 5 is upper bounded by case 7(case 13) since the divisor in case 7(case 13) is smaller than divisors in other two cases since \( f_1 = \hat{f}_1 = r_1 \) and \( f_2 = \hat{f}_2 = r_2 \).

In case 7 (or 13), note that there is no social optimum flow from commodity 2, and the NE cost incurred by commodity 2 can be considered from commodity 2 cycles. However, \( f_2 \) has
impact on latency of \( f_1 \), and we consider \( f_1 \Phi_1(f_1, f_2) \) as cost incurred by \( f_1 \) and \( f_2 \).

\[
\frac{C_I(f)}{C_I(\hat{f})} \leq \frac{f_1 \Phi_1(f_1, f_2)}{f_1 \Phi_2(f_1, 0)} \leq \frac{r_1 \Phi_1(r_1, r_2)}{r_1 \Phi_2(r_1, 0)} \leq \frac{r_1 \Phi_1(r_1, r_2)}{r_1 \Phi_1(r_1, r_2)} \leq 1.
\]

We have the first inequality since \( f_1 = \hat{f}_1 = r_1 \) and \( f_2 = \hat{f}_2 = r_2 \) due to the definition of atomic unsplittable flow. The last inequality holds since \( \Phi_1(f_1, f_2) \leq \Phi_2(f_1, 0) \) due to the variational inequality.

### 5.6.4 cases 2, 3, 9 and 16

Since the divisor for case 2 and case 16 is smaller than divisors in case 3 and case 9, the price of anarchy in case 3 and case 9 are dominated by other two cases. For either case 2 or case 16, we can have the following inequality:

\[
\frac{C_I(f)}{C_I(\hat{f})} \leq \frac{f_1 \Phi_1(f_1, f_2)}{f_1 \Phi_2(f_1, 0)} \leq \frac{f_1 \Phi_2(f_1, f_2) + f_1 \Phi_2'(f_1, 0)}{f_1 \Phi_2(f_1, 0)} = \frac{r_1 \Phi_2(r_1, r_2) + r_1 \Phi_2'(r_1, 0)}{r_1 \Phi_2(r_1, 0)} \leq \frac{\Phi_2(r_1, r_2)}{\Phi_2(r_1, 0)} \leq \frac{a_2 \ell_2(2 c_1 + r_2 + c_2')}{a_2 \ell_2(a_1 r_1 + c_2')} \leq 1 + \frac{r_2}{a_1 r_1}.
\]

In \([15]\), \( \Phi_1(f_1, f_2) \leq \Phi_2(f_1, f_2) + \Phi_2'(f_1, 0) \) due to the variational inequality. We have \( c_2' = c_2/a_2 \) in \([22]\). It seems that the price of anarchy for these cases are not bounded, but if \( r_2 \) is big enough (larger than \( a\sqrt{a r_1} \)) then \( r_2 \) play an important role. Otherwise, if \( r_2 \leq a\sqrt{a r_1} \), then the price of anarchy is bounded by \( \sqrt{a} + 1 \). Thus, we consider a cycle of commodity 2 which is intersected with this cycle when \( r_2 \) is larger than \( a\sqrt{a r_1} \) as shown in figure \(7\).

![Figure 7: Two cycles - one cycle per each commodity.](image)

\( P_1, P_2, P_3 \) and \( P_4 \) correspond to the (partially) top path and the (partially) bottom path from \( u_1 \) to \( v_1 \) and the (partially) top path and the (partially) bottom path from \( u_2 \) to \( v_2 \). Also, we define \( \ell_1 = |\{e \in P_1 \cap E\}|, \ell_2 = |\{e \in P_2 \cap E\}|, \ell_3 = |\{e \in P_3 \cap E\}| \) and \( \ell_4 = |\{e \in P_4 \cap E\}| \) throughout this section. In figure \(8\) the intersection of flows are shown. Note that the red and dashed line represents \( \hat{f}_1 \).

Let us consider the cycle of commodity 2. \( f_2 \) goes along top path either partially or totally overlapped with \( f_1 \) or \( \hat{f}_1 \) on the bottom path. This \( f_2 \) and another flow \( f_2 \) forms the cycle of
commodity 2 which is intersecting with a cycle shown in case 2. For this cycle, we derive the price of anarchy via the corresponding variational inequality. 

\[
\frac{C_I(f)}{C_I(f)} \leq \frac{f_1\Phi'_I(f_1, f_2) + f_1\Phi''_I(f_1, 0) + f_2\Phi'_I(f_1, f_2) + f_2\Phi_3(f_1, f_2) + f_2\Phi''_3(0, f_2)}{f_1\Phi_3(f_1, 0) + f_2\Phi_4(0, f_2)}
\leq \frac{f_1\Phi'_2(f_1, f_2) + f_1\Phi''_2(f_1, 0) + f_2\Phi_4(f_1, f_2)}{f_1\Phi_2(f_1, 0) + f_2\Phi_4(0, f_2)}
\leq \frac{f_1\Phi_2(f_1, f_2) + f_2\Phi_4(f_1, f_2)}{f_1\Phi_2(f_1, 0) + f_2\Phi_4(0, f_2)}
\leq \frac{f_1\Phi_2(f_1, 0) + f_2\Phi_4(0, f_2)}{f_1\Phi_2(f_1, 0) + f_2\Phi_4(0, f_2)}
= \frac{r_1\Phi_2(r_1, r_2) + r_2\Phi_4(r_1, r_2)}{r_1\Phi_2(r_1, 0) + r_2\Phi_4(0, r_2)}
= \frac{r_1\ell_2(\alpha r_1 + r_2 + c_2') + r_2\ell_4(\alpha r_1 + r_2 + c_4')}{r_1\ell_2(\alpha r_1 + c_2') + r_2\ell_4(\alpha r_1 + c_4')}
= \frac{r_1\ell(\alpha r_1 + r_2 + c_2') + r_2(\alpha r_1 + r_2 + c_4')}{r_1\ell(\alpha r_1 + c_2') + r_2(\alpha r_1 + c_4')}
= 1 + \frac{r_1\ell(\alpha r_1 + c_2') + r_2(\alpha r_1 + c_4')}{r_1\ell(\alpha r_1 + c_2') + r_2(\alpha r_1 + c_4')}.
\]

We have \(\Phi'_I(f_1, f_2) + \Phi_2(0, f_2) + \Phi''_I(f_1, f_2) + \Phi''_3(0, f_2) \leq \Phi_4(f_1, f_2)\) due to the variational inequality for commodity 2 and \(\Phi'_I(f_1, f_2) + \Phi''_I(f_1, 0) \leq \Phi'_2(f_1, f_2) + \Phi'_3(0, f_2)\) due to the variational inequality for commodity 1. By these observations, we can obtain (24) from the previous equation. Inequality (24) can be obtained summing over \(\Phi'_2(f_1, f_2)\) and \(\Phi''_I(f_1, 0)\) by making \(\Phi''_3(0, f_2)\) as \(\Phi'_3(f_1, 0)\). Let \(\ell = \ell_2/\ell_4\) where \(\ell_2\) and \(\ell_4\) represent the number of edges on path 2 and path 4, respectively. Also, let \(c_2' = c_2/a_2, c_4' = c_4/a_2\) throughout this chapter. From the above variational inequality \(\Phi'_I(f_1, f_2) + \Phi_2(0, f_2) + \Phi''_I(f_1, f_2) + \Phi''_3(0, f_2) \leq \Phi_4(f_1, f_2)\) we have \(\Phi_2(0, f_2) \leq \Phi_4(f_1, f_2)\) and further

\[\ell = \ell_2/\ell_4 \leq \frac{ar_1 + r_2 + c_4'}{r_2 + c_2'} \leq \frac{ar_1 + r_2 + c_4'}{r_2}.\]

When \(\frac{ar_1 + r_2 + c_4'}{r_2 + c_2'} \leq 1\), let \(\ell = 0\) in the divisor and be 1 for the dividend. Then, equation (28) can be written as

\[\frac{C_I(f)}{C_I(f)} \leq 1 + \frac{r_1r_2 + ar_1r_2}{r_2(r_2 + c_2')} \leq 1 + \frac{r_1r_2 + ar_1r_2}{r_2^2} \leq 2 + \frac{1}{\sqrt{a}}.\]

The last inequality holds because \(r_2 \geq a\sqrt{ar_1}\), and \(a \geq 1\). When \(\frac{ar_1 + r_2 + c_4'}{r_2 + c_2'} \geq 1\), note that
\[ \frac{r_1 r_2 (a r_1 + r_2 + c_1')}{r_2} \geq 1 \text{ is true. Equation } (28) \text{ can be written as} \]
\[
\frac{C_1(f)}{C_1(f)} \leq 1 + \frac{r_1 r_2 (a r_1 + r_2 + c_1')}{r_1 \ell(a r_1 + c_2') + r_2 (r_2 + c_1')} \quad (30)
\leq 1 + \frac{r_1 (r_2/a + r_2 + c_1')} {r_1 \ell(a r_1 + c_2') + a \sqrt{a} r_1 (r_2 + c_1')} \quad (31)
\leq 1 + \frac{(a + 1 + 1/\sqrt{a}) r_2 + c_1'} {\ell(a r_1 + c_2') + a \sqrt{a} (r_2 + c_1')} \quad (32)
\leq 1 + \frac{(a + 1 + 1/\sqrt{a}) r_2 + c_1'} {a \sqrt{a} (r_2 + c_1')} \leq 1 + \frac{1} {\sqrt{a}} + \frac{2} {a \sqrt{a}} \quad (33)
\]

Observe that \( \ell(a r_1 + c_2') \) can be ignored to obtain the upper bound in (33). In equations, to maximize the price of anarchy we substitute \( r_1 \) with \( r_2/a \sqrt{a} \) in the dividend; while in the divisor we substitute \( r_2 \) with \( a \sqrt{a} r_1 \).

### 5.6.5 cases 4, 6, 8, 10, 11, 12, 14 and 15

Note that the divisors in case 6, 10 and 11 are smaller than others, and the dividend in case 6, 10 and 11 are bigger than others due to the variational inequality.

We consider one of the cases 6, 10 and 11. As shown in previous case, we need to consider commodity 1’s cycle and commodity 2’s cycle to estimate the price of anarchy. Note that the ratio of \( C_1(f) \) to \( C_1(f) \) in this case is
\[
\frac{f_1 \Phi_1(f_1, 0)} {f_1 \Phi_2(f_1, 0)} \leq \frac{\Phi_2'(f_1, f_2) + \Phi_2''(f_1, 0)} {f_1 \Phi_2(f_1, 0)}.
\]

Note that we consider only one cycle for commodity 1 if \( f_2 \) is partially utilizing a top path. In this case, the proof is similar to the previous case as shown in Sub-subsection 5.6.4 and we omit the corresponding proof.

In the case of totally utilizing the top path, the proof is not similar to the proof as shown in Sub-subsection 5.6.4. Instead of using the proof shown in as shown in Sub-subsection 5.6.4, we provide a different proof of a case when \( f_2 \) is totally utilizing the top path. Let us consider case 6 which upper bounds other two cases. we split into three sub-cases: (a) \( r_1 \leq r_2 \leq (a^{1/2} r_1 \), (b) \( r_1 > r_2 \) and (c) \( r_2 > (a^{1/2} r_1 \). In case (a),
\[
\frac{C_1(f)} {C_1(f)} \leq \frac{f_1 \Phi_1(f_1, 0) + f_2 \Phi_2(0, f_2)} {f_1 \Phi_2(f_1, 0) + f_2 \Phi_1(0, f_2)} \quad (34)
\leq \frac{f_1 \Phi_2(f_1, 0) + f_2 \Phi_2(0, f_2)} {f_1 \Phi_2(f_1, 0) + f_2 \Phi_1(0, f_2)} \quad (35)
\leq \frac{r_1 \Phi_2(r_1, r_2) + r_2 \Phi_2(0, r_2)} {r_1 \Phi_2(r_1, 0) + r_2 \Phi_2(0, r_2)} \quad (36)
\leq \frac{r_1 \Phi_2(r_1, r_2) + r_2 \Phi_2(0, r_2)} {r_1 \Phi_2(r_1, 0) + r_2 \Phi_2(0, r_2)} \quad (37)
\leq \frac{\Phi_2(r_1, r_2)} {\Phi_2(r_1, 0)} + \frac{r_2 \Phi_2(0, r_2)} {r_1 \Phi_2(r_1, 0)} \quad (38)
\leq 1 + \frac{\ell_2 \sqrt{a} r_1} {\ell_2 (a r_1 + c_2')} + \frac{r_2 \ell_2 (\sqrt{a} r_1 + c_2')} {r_1 \ell_2 (a r_1 + c_2')} \quad (39)
\leq \sqrt{a} + 1/\sqrt{a} + 1 \quad (40)
\]
Due to the variational inequality, we have (35) by replacing $\Phi_1(f_1, 0)$ with $\Phi_2(f_1, f_2)$. In inequality (39), we substitute $r_2$ with $\sqrt{ar_1}$. In case (b), we start at (36) to avoid duplicate formulas.

Due to Lemma 5.7, (43) holds. In equation (44), the divisor is minimized when $r_1/r_2$ is close to 1. Lastly, in case (c),

$$\frac{C_1(f)}{C_1(f)} \leq \frac{r_1\ell_1(a_1 + c_1') + r_2\ell_2(f_2 + c_2')}{r_1\ell_1(a_1 + c_1') + r_2\ell_2(f_2 + c_2')}$$

$$\leq 1 + \frac{\ell_1 r_1 r_2 + \ell_2 r_2 c_2'}{\ell_1 (a_1^2 + r_1 c_2') + \ell_2 (r_2^2 + r_2 c_1')}$$

$$\leq 1 + \max\left(\frac{r_1 r_2}{a_1^2 + r_2^2}, \frac{r_2 c_2'}{r_1 c_1' + r_2 c_1'}\right)$$

$$\leq 1 + \max\left(\frac{1}{a_1 + r_2/r_1}, 1\right) \leq 1$$

Due to Lemma 5.7, (43) holds. In equation (44), the divisor is minimized when $r_1/r_2$ is close to 1. Lastly, in case (c),

$$\frac{C_1(f)}{C_1(f)} \leq \frac{\ell_1 f_1(a_1 + c_1') + \ell_2 f_2(f_2 + c_2')}{\ell_1 f_1(a_1 + c_1') + \ell_2 f_2(f_2 + c_2')}$$

$$\leq \frac{\ell f_1(a_1 + c_1') + f_2(f_2 + c_2')}{\ell f_1(f_2 + c_1') + f_1(a_1 + c_2')}$$

$$\leq \frac{\ell f_2(f_2 + c_1') + f_1(a_1 + c_2')}{\ell f_1(a_1 + c_1') + \ell_2(f_2 + c_1')}$$

$$\leq \ell_2 r_2(r_2 + c_1') + r_1(a_1 + c_2')$$

$$\leq r_1(a_1 + c_1') + r_2(r_2 + c_2') + r_2 r_2 + c_2' + r_1(a_1 + c_2')$$

$$\leq (r_1 r_2 + c_2') + r_1(a_1 + c_2')$$

$$\leq \sqrt{a + 1}$$

By dividing by $\ell_2$ we obtain (46), and we further have (47) due to the definition of atomic unsplitable. Due to the variational inequality, we have $\Phi_1(f_1, 0) \leq \Phi_2(f_1, f_2)$ and $\Phi_2(0, f_2) \leq \ell \Phi_1(f_1, f_2)$ where $\ell = \ell_1/\ell_2$ and $\ell_1$ and $\ell_2$ represent the number of edges in path 1 and path 2 respectively. Inequality (48) is derived from the previous inequality by using the variational inequality. Further, by dividing by $r_2 + c_1'$ and substituting $r_1$ with $r_2/\sqrt{a}$, we have (49). In other words,

$$r_2 + c_1' \geq \frac{r_2 + c_1'}{\sqrt{a}} \geq \frac{1}{\sqrt{a} + 1}.$$
Theorem 5.8 Let $C$ be a set of uniform affine delay functions. If $(G, R, \Phi)$ is an atomic 2-commodity instance with delay functions in $C$, then the price of anarchy of $(G, R, \Phi)$ with partition $\pi$ is at most $\sqrt{a_{\text{max}}} + 2$ and this is almost tight by the lower bound as shown in Lemma 5.6.

6 Conclusion

In this paper we have studied the price of anarchy for $k$-commodity nonatomic and atomic network flows with heterogeneous and decomposable delay functions. We have also obtained improved bounds on the price of anarchy for 2-commodity atomic flows with heterogeneous uniform delay functions.

Further studies should include convex functions that model the behavior of network delays more accurately, in order to shed more light on the net-neutrality issue.

Acknowledgement

A preliminary version of these results also appears in [28].

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