AN INFINITE BRANCH IN A DECIDABLE TREE

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Abstract. We consider a structure $M = \langle \mathbb{N}, \{Tr, <\} \rangle$, where the relation $Tr(a, x, y)$ with a parameter $a$ defines a family of trees on $\mathbb{N}$ and $<$ is the usual order on $\mathbb{N}$. We show that if the elementary theory of $M$ is decidable then (1) the relation $Q(a) \Leftrightarrow "there is an infinite branch in the tree Tr(a, x, y)"$ is definable in $M$, and (2) if there is an infinite branch in the tree $Tr(a, x, y)$, then there is a definable in $M$ infinite branch.

1. Preliminaries

Let $Tr(x, y)$ be a tree on the $\mathbb{N}$, we are interested in whether there is an infinite branch in this tree. If the tree is locally finite then, according König’s lemma [1], an infinite branch exists iff the tree is infinite. It is easy to notice, that in this case an infinite branch can be defined in the structure $\langle \mathbb{N}, \{Tr, <\} \rangle$. The question is more complicated for an arbitrary tree.

We show that if a family $Tr(a, x, y)$ of trees with a parameter $a$ such, that the elementary theory of $M = \langle \mathbb{N}, \{Tr, <\} \rangle$ is decidable then (1) the relation $Q(a) \Leftrightarrow "there is an infinite branch in the tree Tr(a, x, y)"$ is definable in $M$, and (2) if there is an infinite branch in the tree $Tr(a, x, y)$, then there is a definable in $M$ infinite branch in the tree $Tr(a, x, y)$.

For simplicity hereinafter we write $a$ instead of $\bar{a}$ in parameters though all parameters could be vectors as well as numbers.

The proof consists of two steps. First we show, that if a tree is in some sense complicated, then the theory of the corresponding structure is undecidable. Second we show, that if a tree is not complicated, then (1) and (2) holds. To demonstrate undecidability we use an interpretation of fragments of the arithmetic in the structure [2].

2. Interpretation

In this section we consider a structure $M = \langle \mathbb{N}, \Sigma \rangle$, the usual order $<$ belongs to $\Sigma$. Suppose that subset $S \subset \mathbb{N}$ is finite and a relation $B(x, y)$ is definable in $M$. By $s^B_i$ we denote $S \cap \{x|B(x, i)\}$ and say, that $B$ realise the number $k$ on $S$ ($k \leq |S|$) if $\{s^B_i|i \in \mathbb{N}\} = \{s \subset S|\|s\| = k\}$. The property to realise a number can be expressed by the statement:

$$\forall i, j(s^B_i \subset s^B_j \rightarrow s^B_i = s^B_j) \wedge (\forall i, a, b)(a \in s^B_i \wedge b \in S \setminus s^B_i \rightarrow (\exists j)(s^B_j = s^B_i \cup \{b\} \setminus \{a\}))$$

We say that a relation $C(x, y, z)$ realise the arithmetic on $S$ if for any $k \leq |S|$ there is such $a$, that the relation $B_a(x, y) \Leftrightarrow C(x, y, a)$ realises the number $k$ on $S$. 1
The property to realise the arithmetic can be expressed by the statement:

\[(\exists z)(\forall i)(s_i^{B_z} = \emptyset) \land \]
\[\land (\forall z)((B_z \text{ realises a number on } S) \land (\exists i)(s_i^{B_z} \neq S) \rightarrow \]
\[\rightarrow (\exists u)(B_u \text{ realises a number on } S) \land \]
\[\land (\exists i,j,a)(a \in S \setminus s_i^{B_z} \land s_j^{B_z} = s_i^{B_z} \cup \{a\}))\]

Note that if a relation \(C\) realises the arithmetic on \(S\), then we can define addition and multiplication on the segment \([0,|S|]\). Addition \(S(n,m,l)\) may be defined as

\[
B_n \text{ realises a number on } S \land B_m \text{ realises a number on } S \land B_{l} \text{ realises a number on } S \land
\]
\[(\exists i,j,k)(s_k^{B_{l}} = s_i^{B_{n}} \cup s_j^{B_{m}} \land s_i^{B_{n}} \cap s_j^{B_{m}} = \emptyset)\]

Multiplication \(P(n,m,l)\) may be defined as

\[
B_n \text{ realises a number on } S \land B_m \text{ realises a number on } S \land B_{l} \text{ realises a number on } S \land
\]
\[(\exists i,j)(s_i^{B_{n}} \subset s_j^{B_{m}} \land \max(s_i^{B_{n}}) = \max(s_j^{B_{m}}) \land \min(s_i^{B_{n}}) = \min(s_j^{B_{m}}) \land
\]
\[(\forall a,b \in s_i^{B_{n}})(a < b \land (\forall c \in s_i^{B_{n}})(a < c \rightarrow b \leq c) \rightarrow (\exists k)(s_k^{B_{m}} = \{x \in s_j^{B_{m}}| a \leq x < b\}))\]

(It is not exactly \(l = n \cdot m\) but rather \(l = n \cdot m + 1\) which is not important)

**Lemma 1.** If there are definable in \(\mathcal{M}\) relations \(S(b,x), D(b,x,y,z)\) such that for any natural \(n\) for some \(b_n\) the relation \(D(b_n,x,y,z)\) realises the arithmetic on \(\{x|S(b_n,x)\}\) and \(n = |\{x|S(b_n,x)\}|\), then the elementary theory of \(\mathcal{M}\) is undecidable.

**Proof.** Consider an arithmetic formula \((\exists n)Q(n)\) where \(Q(x)\) is a bounded quantifiers formula. Under the assumptions of the lemma we can construct the equivalent formula in the structure \(\mathcal{M}\), so the elementary theory of \(\mathcal{M}\) is undecidable. \(\square\)

### 3. Rank of nodes

Without loss of generality we suppose that a tree \(Tr\) on \(\mathbb{N}\) is a family of finite subsets \(\mathbb{N}\) such that if \(s \in Tr\) then any initial segment of \(s\) belongs to \(Tr\) as well.

There is the order \(s \preceq s' \Rightarrow s\text{ is initial segment of } s'\) on the tree. We say that a relation \(Tr(x,y)\) defines the tree, if \(\{s_i|s_i = \{x|Tr(x,i)\}\}\) is a tree.

We are going to define the main notion of the article: the rank of a tree node. But before the definition of rank we need the supporting partial mapping \(g: Tr \rightarrow \mathbb{N}\). We describe the mapping \(g\) in the terms of the \(s\)-game assigned to a node \(s\) of the tree. **Game:** There are 2 players. In the starting position all items of the node \(s\) are drawn on the natural numbers line (red dots):

First player mark a boundary \(a \geq \max(s)\) (black dot).
The second player has to choose a finite set $s', \min(s') > a$ of numbers (pink dots) in such a way, that $s \cup s'$ form new node:

Now it is first player turn, and so on.

We set $g(s) = \max k$ [there is a strategy for second player not to lose the game in $k$ moves]. It’s easy to note that $g(s) = 0 \iff s$ has finite number of sons. We say that a node is $k$-regular if $g(s) = k$ and regular if it is $k$-regular for some $k$.

Now we define a rank of nodes: a partial mapping $rk: Tr \to \mathbb{N}$: $rk(s) = n \iff$

\begin{enumerate}
  \item any $s' \supset s$ is regular and (2) $n = \max\{g(s') | s' \supset s\}$. Note that $rk(s) = 0$ if the subtree $\{s'| s \preceq s'\}$ is locally finite.
\end{enumerate}

We say that a node $s$ of finite rank ($rk(s) < \infty$) if $rk(s)$ is defined, otherwise we say that $s$ of infinite rank ($rk(s) = \infty$).

**Lemma 2.** For any node $s$

(i) if $rk(s) = n, s_1 \supset s$, then $s_1$ has finite rank and $rk(s_1) \leq rk(s)$.

(ii) if $rk(s) = n, s_1 \supset s$, then $s_1$ has finite rank and $rk(s_1) \leq rk(s)$.

(iii) $rk(s) \geq g(s)$.

(iv) if $rk(s) = n, s_1 \supset s$, then $s_1$ has finite rank and $rk(s_1) \leq rk(s)$.

(v) if $g(s) = n$ then for any $a > \max(s)$ there is a finite set $s', \min(s') > a$ such that $g(s \cup s') = \max(s \cup s') = n - 1$.

(vi) if $g(s) = n$ then there is such $a > \max(s)$ that $g(s \cup s') < n$ for any finite subset $s', \min(s') > a, s \cup s'$ is the tree node.

(vii) if $rk(s) = n$ then there are infinitely many pairwise incomparable $s' \supset s$, such that $rk(s') = g(s') = n - 1$.

**Proof.** (i)–(iv) obvious, due to definitions.

(v) for any move $a \geq \max(s)$ of the first player denote by $s'_a$ a best answer of the second player, by the definition of mapping $g$ holds (a) $g(s \cup s'_a) \geq n - 1$ and (b) there is such $a_0$ (best first move of the first player) such that $g(s \cup s'_a) = n - 1$ for all $a \geq a_0$. Due to (iv) $rk(s \cup s'_a) \geq n - 1$. If $s'_a$ is the best answer, then $g(s \cup s'_a) \geq g(s')$ for all $s' \supset s \cup s'_a$ so $rk(s \cup s'_a) = g(s \cup s'_a) = n - 1$. The existence of infinitely many pairwise incomparable $s'_a \supset s$ for different $a > a_0$ is obvious.

(vi) Denote by $a_0$ a best move of the first player in the $s$-game. Then for any replay $s', \min(s') > a_0$ of the second player holds $g(s \cup s') < n$.

(vii) if $rk(s) = n$, then there is such $s' \supset s$ that $g(s') = n$, so we use (v) here. \hfill \Box

**Lemma 3.** Consider a structure $\mathcal{M} = \langle \mathbb{N}, \{Tr, \prec\}\rangle$, where the relation $Tr(a, x, y)$ with a parameter $a$ defines a family of trees on $\mathbb{N}$ and $\prec$ is the usual order on $\mathbb{N}$. If the elementary theory of $\mathcal{M}$ is decidable, then there is such number $k$, that $rk(s) < k$ holds for all nodes $s$ of finite rank in all trees $Tr(a, x, y)$.

**Proof.** In the contrary: we suppose that there are nodes of arbitrary big finite rank and show that conditions of lemma 1 hold. We fix a value of the parameter $a_0$ and consider the tree $Tr = Tr(a_0, x, y)$.

We define functions $\varphi(x), \psi(x, y)$ on $\mathbb{N}$ in the following way: for a number $a \in \mathbb{N}$ consider the segment $[0, a]$ and choose a node $s \subset [0, a]$, let $g(s) = k$-regular for some
Lemma 4. Suppose that for $a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$ holds $b_i > \varphi(a_i), a_{i+1} > \psi(a_i, b_i)$. Choose $s \in [0, a_1], g(s) = k < n$. Then

(i) for any $u \in [1, n-1], |u| = k$ there is such node $s' > s$, that $s' \cap (a_i, b_i) = \emptyset$ for all $i < n$ and $\{i|s' \cap [b_i, a_{i+1}] \neq \emptyset\} = u$.

(ii) if $s' > s$ and $s' \cap (a_i, b_i) = \emptyset$ for all $i \leq n$, then $|\{i|s' \cap [b_i, a_{i+1}] \neq \emptyset\}| \leq k$.

Proof.

(i) induction on $k$. Let $i = \min(u)$. By definition of mapping $\psi$ and because $s \in [0, a_1]$, there is such $s' > s$, that $s' \cap (\max(s) + 1, b_i) = \emptyset, \max(s') < a_{i+1}, g(s') = k - 1$. So we can apply an inductive hypothesis to the collection $a_{i+1} < b_i + 1 < \cdots < b_n < a_{n+1}$, the node $s'$ an the set $u \setminus \{i\}$.

(ii) suppose that $s' > s$ and $s' \cap (a_i, b_i) = \emptyset$ for all $i < n$ and $\{i|s' \cap [b_i, a_{i+1}] \neq \emptyset\} = \{c_1 < c_2 < \cdots < c_n\}$. By induction on $i$ show that $r(s' \cap [0, a_{c_i+1}]) \leq k - i$. If $r(s' \cap [0, a_{c_i+1}]) = m$, then by definition of mapping $\varphi$ for any $s'' > s'$, $s'' \cap [a_{c_i}, b_{c_i}] = \emptyset$ holds $r(s'') < r_k(s'') \leq m$, i.e. $r(s' \cap [0, a_{c_i+1}]) < r(s' \cap [0, a_{c_i}])$.

Lemma 5. For any $u, v > s$ we denote $A_{u,v} \equiv \{\max(s) \cup \{a \in u \setminus s| (\forall \alpha' < a, \alpha' \in u)(([\alpha', a] \cap v \neq \emptyset)\}, B_{u,v} \equiv \{b \in v \setminus s| (\forall \beta' < b)(([\beta', b] \cap u \neq \emptyset)\}$. If $r(s) = n+1$ then there are such $u, v > s$ that sets $A_{u,v} = \{a_1 < \cdots < a_{n+1}\}$ and $B_{u,v} = \{b_1 < \cdots < b_n\}$ meet the conditions of lemma 3.

Proof. We will construct collections $u_0 < u_1 < \cdots < u_n = v_0 < v_1 < \cdots < v_{n-1} = v$ such that $u_0 = v_0 = s, r(u_i) = r(v_i) = n + 1 - i$. Suppose that $u_i, v_i$ are already constructed, $\max(v_i) \leq \max(u_i)$. Choose such $v_{i+1} > v_i$ that $g(v_{i+1}) = g(v_i) - 1, v_{i+1} \cap (\max(v_i) + 1, \varphi(\max(u_i)) = \emptyset$. Since $\min(v_{i+1} \setminus v_i) > u_i$, so $A_{u_{i+1}, v_{i+1}} = A_{u_i, v_i} \cup \{\min(v_{i+1}) \setminus v_i\}$ and $\min(v_{i+1} \setminus v_i) > \varphi(\max(u_i)) > \varphi(\max(u_{i+1}))$.

Now we in the same way choose the node $u_{i+1}$ considering the node $v_{i+1}$ instead of $u_i$ and the number $\psi(\max(u_i), \max(v_{i+1}))$ instead of $\varphi(\max(u_i))$.

Continue the proof of lemma 3. Suppose that there exist nodes of arbitrary big finite rank. Fix some $n \in \mathbb{N}$. According the lemma 3 there are such $u, v, v_0 = s, u_0 = v_0 = s, r(u_i) = r(v_i) = n + 1 - i$. Suppose that $u_i, v_i$ are already constructed, $\max(v_i) \leq \max(u_i)$. Choose such $v_{i+1} > v_i$ that $g(v_{i+1}) = g(v_i) - 1, v_{i+1} \cap (\max(v_i) + 1, \varphi(\max(u_i)) = \emptyset$. Since $\min(v_{i+1} \setminus v_i) > u_i$, so $A_{u_{i+1}, v_{i+1}} = A_{u_i, v_i} \cup \{\min(v_{i+1}) \setminus v_i\}$ and $\min(v_{i+1} \setminus v_i) > \varphi(\max(u_i)) > \varphi(\max(u_{i+1}))$.

Now we in the same way choose the node $u_{i+1}$ considering the node $v_{i+1}$ instead of $u_i$ and the number $\psi(\max(u_i), \max(v_{i+1}))$ instead of $\varphi(\max(u_i))$. 

\[\square\]
Consequence 1. Let a relation \( \text{Tr}(y, x) \) defines a tree on \( \mathbb{N} \), and elementary theory of the structure \( \mathcal{M} = \langle \mathbb{N}, \{\text{Tr}, <\} \rangle \) is decidable. Then

(i) the relation "\( s \) is a node of finite rank" is definable (in \( \mathcal{M} \)).
(ii) the functions \( \varphi, \psi \) are definable.
(iii) if the set of nodes of infinite rank is not empty, then it contains a definable subtree isomorphic to \( \mathbb{N}^\omega \).
(iv) there is \( k \in \mathbb{N} \) such that for any node \( s \) of finite rank, \( s = \{a_1 < a_2 < \cdots < a_n\} \),
holds \( k \geq |\{a_i \in s | a_{i+1} > \varphi(a_i)\}| \).

Proof. According the lemma \( \exists i \) there is such \( k \in \mathbb{N} \) that \( k \geq rk(s) \) for every node \( s \) of finite rank.

(i) the relation "\( rk(s) = 0 \)" is definable, so by induction the relation "\( rk(s) = i \)" is definable for any \( i \). Then the relation "\( s \) is a node of finite rank" is equivalent to \( \bigvee_{i=0}^k rk(s) = i \).
(ii) immediately follows from (i).
(iii) the relation \( \inf(s) = "s \) is a node of infinite rank" is definable. To proof the isomorphism to \( \mathbb{N}^\omega \) it is enough to show, that for any node \( s \) of infinite rank there is a node of infinite rank \( s' \succ s \), such that \((\forall a)(\exists s'' \succ s') (\inf(s'') \land s'' \cap \max(s'') + 1, a) = \emptyset \). On contrary suppose that \((\forall s' \succ s)(\forall s'' \succ s')(s'' \cap \max(s'') + 1, a) = \emptyset \) \(\Rightarrow rk(s') \leq k \), then, by definition the node \( s \) has finite rank.
(iv) from the definition of the function \( \varphi \) follows that \((a_{i+1} > \varphi(a_i)) \implies rk\{a_1, \ldots, a_{i+1}\} < rk\{a_1, \ldots, a_i\} \). \( \square \)

Consequence 2. Let a relation \( \text{Tr}(a, y, x) \) with the parameter \( a \) defines a family of trees on \( \mathbb{N} \), elementary theory of the structure \( \mathcal{M} = \langle \mathbb{N}, \{\text{Tr}, <\} \rangle \) is decidable. Then

(i) the relation \( Q(a) \equiv " \)there is an infinite branch in the tree \( \text{Tr}(a, y, x) \)" is definable.
(ii) if there is an infinite branch in the tree \( \text{Tr}(a, y, x) \) then there is a definable infinite branch.

Proof. According the lemma \( \exists i \) there is such number \( k \) that \( k \geq rk(s) \) holds for all nodes \( s \) of finite rank in all trees \( \text{Tr}(a, y, x) \). So the relation \( \inf(a, s) \equiv "s \) is a node of infinite rank in the tree \( \text{Tr}(a, y, x) \)" is definable. Consider two cases.
(i) There is a node of infinite rank in the tree \( \text{Tr}(a, y, x) \). Then due to sequence \( i \) there is a definable infinite branch in the tree.
(ii) All nodes of the tree \( \text{Tr}(a, y, x) \) are of finite rank. To any node \( s \in \text{Tr}(a) \) assign the subtree \( \text{Tr}_s = \{s' \succ s | rk(s') = rk(s)\} \). We show that there is an infinite branch in the tree \( \text{Tr}(a, y, x) \) if and only if the tree \( \text{Tr}_s \) is infinite for some \( s \). Note that the tree \( \text{Tr}_s \) is locally finite. Indeed, if a node \( s \) has an infinitely many sons \( s' \succ s \), \( rk(s') = rk(s) \), then \((\forall a)(\exists s' \succ s)(s' \cap \max(s') + 1, a) = \emptyset \land rk(s') = rk(s') \), and, by the definition of the function \( r \), holds \( rk(s) \leq r(s) < rk(s) \). So if the tree \( \text{Tr}_s \) is infinite, then there is a definable infinite branch, which is the branch in the tree \( \text{Tr}(a, y, x) \) as well.
Conversely, suppose that in the tree \( \text{Tr}(a, y, x) \) exists an infinite branch \( s_1 \prec \cdots \prec s_n \prec \cdots \). Because \( rk(s_1) \geq rk(s_{i+1}) \), so for some \( n \) and for all \( i > 0 \) holds \( rk(s_n) = rk(s_{n+i}) \) and the tree \( \text{Tr}_{s_n} \) is infinite. \( \square \)
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