A New Type of Conformal Dynamics

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Abstract

We consider the Lagrangian particle model introduced in [1] for zero mass but nonvanishing second central charge of the planar Galilei group. Extended by a magnetic vortex or a Coulomb potential the model exhibits conformal symmetry. In the former case we observe an additional $SO(2,1)$ hidden symmetry. By either a canonical transformation with constraints or by freezing scale and special conformal transformations at $t = 0$ we reduce the six-dimensional phase-space to the physically required four dimensions. Then we discuss bound states (bounded solutions) in quantum dynamics (classical mechanics). We show that the Schrödinger equation for the pure vortex case may be transformed into the Morse potential problem thus providing us with an explanation of the hidden $SO(2,1)$ symmetry.

1 Introduction

In a recent paper [1] Lukierski and the present authors introduced a dynamical realisation of the two-fold centrally extended planar Galilei group [2] by means of the higher order particle Lagrangian

$$L_0 = \frac{m}{2} \dot{x}_i^2 - \kappa \epsilon_{ij} \dot{x}_i \dot{x}_j. \quad (1.1)$$

In (1.1) $m$ and $2\kappa$ are the two central charges of the group with $2\kappa$ being given by the nonvanishing Poisson-bracket between the boost generators

$$\{K_i, K_j\} = 2\kappa \epsilon_{ij}. \quad (1.2)$$
The Lagrangian (1.1) leads to a six-dimensional phase space which, for \( m \neq 0 \) may be split into a four-dimensional physical sector, described by noncommuting variables, and an auxiliary two-dimensional sector [1,3]. Both sectors are dynamically independent from each other.

The model (1.1) has been successfully extended by coupling it to either a scalar potential [1,3], electromagnetic and other gauge interactions [4,5] or by its supersymmetrization [6].

Furthermore, field-theoretic models which allow for the nonvanishing of the second central charge of the planar Galilei group have also recently been studied [7,8] by Horvathy, Martina and one of the present authors (PCS).

In the present paper we examine conformal symmetry of the model determined by (1.1) to which some further interactions are added.

However, let us point out at the onset that even in the noninteracting case we have a problem; namely, the two terms in (1.1) have different transformation properties with respect to dilatations and special conformal transformations. To see this we introduce a function
\[
f(t) := a + bt + ct^2
\]
with infinitesimal parameters \( a, b \) and \( c \). Then the first term, i.e. \( \dot{x}_i^2 \) in (1.1) is quasi-invariant with respect to [9,10]
\[
\delta x_i = f \dot{x}_i - \frac{1}{2} \ddot{f} x_i
\]
while the second term \( (\epsilon_{ij} \dot{x}_i \ddot{x}_j) \) is quasi-invariant with respect to
\[
\delta x_i = f \dot{x}_i - \ddot{f} x_i
\]
i.e. \( x_i \) has an anomalous scale dimension in this case.

If we keep only the first term in (1.1) we can easily extend our model to the well known models with either an inverse square potential [9] or a magnetic vortex [10]. In order to construct particle models with a nonvanishing second central charge of the planar Galilei group we must, therefore, put the first central charge, the mass \( m \), equal to zero.

Our paper is organized as follows. In section 2 we show that we preserve conformal invariance if we add to (1.1) a Coulomb potential or a magnetic vortex interaction. In the case of the pure vortex we find an additional hidden \( SO(2,1) \) symmetry. As for \( m = 0 \) the decomposition of the six-dimensional phase-space into a physical and an auxiliary sector is not possible so in section 3 we perform a canonical transformation with constraints giving us the reduction to a reduced phase-space with the physically required four dimensions. Freezing a symmetry turns out to be an alternative approach to that. In Section 4 we look for bounded solutions of the classical equations of motion in this reduced phase-space. Quantum dynamics is discussed in section 5. We show that the Schrödinger equation for the pure vortex can be transformed into the Morse potential problem [11]. This explains the \( SO(2,1) \) symmetry found in the classical case in the original phase space.
2 An Exotic Particle Model with a Coulomb and a Magnetic Vortex Interaction

We consider here a massless particle with a nonvanishing second central charge of the Galilei group in two dimensions, whose motion is governed by the equations of motion which follow from the Lagrangian

\[ L = L^0 - V(\vec{x}, \dot{\vec{x}}), \]  

where \[ L^0 = \frac{\theta}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j \]  
i.e. we have put \( \kappa = -\frac{\theta}{2} \) in the notation of [1].

The potential \( V \) consists of two terms (Coulomb + vortex) i.e. it is given by

\[ V = V_1 + V_2, \]  

where

\[ V_1 = \frac{\lambda}{|\vec{x}|} \]  

and

\[ V_2 = -\frac{g}{|\vec{x}|^2} \epsilon_{ij} \dot{x}_i x_j. \]  

Note that \( V_2 \) describes the potential of the magnetic vortex field located at the origin [10] with flux \( \phi = \frac{2\pi g}{r} \).

It is well known that \( V_2 \) may be written as a total time derivative of a function which is singular at the origin. Therefore \( V_2 \) does not contribute to the classical equations of motion in the punctured plane. However, in order for all the equations to be valid in the whole plane, and to have a smooth transition to the quantum case, in the rest of the paper we will leave \( V_2 \) in the form (2.5).

2.1 Hamiltonian and Symplectic Structure

2.1.1 First-order formalism

Here we use the first order formalism, which for \( L_0 \) was used in [1]. We have

\[ L = P_i (\dot{x}_i - y_i) + \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j - \frac{\lambda}{r} + \frac{g \epsilon_{ij} y_i x_j}{r^2} \]  

with \( r := |\vec{x}| \).

The equations of motion which follow from this \( L \) are given by

\[ i)\ \text{Var} \ P_i \rightarrow \dot{x}_i = y_i \]  
\[ ii)\ \text{Var} \ x_i \rightarrow \dot{P}_i = g \partial_i \left( \frac{\epsilon_{kl} y_k x_l}{r^2} \right) + \frac{\lambda x_i}{r^3} \]  
\[ iii)\ \text{Var} \ y_i \rightarrow \dot{y}_i = -\frac{1}{\theta} \left( \epsilon_{ij} P_j + \frac{g x_i}{r^2} \right). \]
2.1.2 Hamiltonian

The Hamiltonian which follows from (2.6) is then given by
\[ H = P_i y_i + \frac{\lambda}{r} + g \frac{\epsilon_{ij} x_i y_j}{r^2} \]  \hspace{1cm} (2.10)

2.1.3 Symplectic structure

From \( \dot{Y} = \{Y, H\} \), where \( Y \in (x_i, y_i, P_i) \) we have
\[ \{x_i, P_k\} = \delta_{ik}, \] \hspace{1cm} (2.11)
\[ \{y_i, y_k\} = -\frac{\epsilon_{ik}}{\theta} \] \hspace{1cm} (2.12)
with all other Poisson brackets vanishing.

Note that the interaction terms in (2.6) do not change the symplectic structure based on \( L_0 \) (cp. [1]).

2.2 Symmetries

2.2.1 Conformal transformations

Define the functions
\[ D := tH - x_i P_i \] \hspace{1cm} (2.13)
and
\[ K := -t^2 H + 2tD + 2\theta \epsilon_{ij} x_i y_j. \] \hspace{1cm} (2.14)

Note that due to (2.7-9) both these functions are conserved, i.e.
\[ \frac{d}{dt} D = \frac{d}{dt} K = 0. \] \hspace{1cm} (2.15)

Moreover, note also that \( D \) and \( K \), together with \( H \), build the operators of the conformal algebra, i.e. we have
\[ \{D, H\} = -H \] \hspace{1cm} (2.16)
\[ \{K, H\} = -2D \] \hspace{1cm} (2.17)
\[ \{D, K\} = K \] \hspace{1cm} (2.18)
as can be checked by using the Poisson brackets (2.11-12).

The Casimir of the conformal algebra (2.16-18) is then given by
\[ C = HK - D^2 = 2\theta H \epsilon_{ij} x_i y_j - (x_i P_i)^2, \] \hspace{1cm} (2.19)
where the last equality follows from (2.13) and (2.14).
Observe that $D$ and $K$ are, respectively, the generators of dilatations and of special conformal transformations. The general conformal transformation of a phase-space element $Y$ is thus given by

$$\delta Y = a\{Y, H\} + b\{Y, D\} + c\{Y, K\},$$  \hfill (2.20)

where $a$, $b$ and $c$ are infinitesimal numbers.

Remark: For $Y = x_i$ we have from (2.20)

$$\delta x_i = fy_i - \dot{f}x_i,$$  \hfill (2.21)

where (cp. (1.3))

$$f(t) = a + bt + ct^2.$$  \hfill (2.22)

Note that (2.21) differs from the corresponding expression in a conventional massive theory with an inverse square potential

$$\hat{L} = \frac{m}{2} \dot{x}_i^2 - \frac{\lambda}{r^2}$$  \hfill (2.23)

where the symmetry transformation is given by [9]

$$\hat{\delta} x_i = f \dot{x}_i - \frac{1}{2} \dot{f} x_i.$$  \hfill (2.24)

### 2.2.2 Rotations

Note that the conserved generator of planar rotations is given by (cp. [1])

$$J = \epsilon_{ij} x_i P_j + \frac{\theta}{2} y_i^2.$$  \hfill (2.25)

### 2.2.3 Hidden Symmetry of the pure vortex case

Let us observe that when $\lambda = 0$, i.e. when we have a pure vortex case, our equations (2.7-9) possess two additional conserved quantities, namely:

\[
\frac{2}{\dot{\theta}} J_1 = \cos 2\varphi \left( \frac{(\epsilon_{ij} x_i y_j)^2}{r^2} - \frac{J + g}{\theta} \right) + \zeta \sin 2\varphi
\]

and

\[
\frac{2}{\dot{\theta}} J_2 = \sin 2\varphi \left( \frac{(\epsilon_{ij} x_i y_j)^2}{r^2} - \frac{J + g}{\theta} \right) - \zeta \cos 2\varphi,
\]

where $r, \varphi$ are polar coordinates of the vector $\{x_i\}$ and $\zeta$ is defined by

$$\zeta := \frac{x_i y_i}{r^2} \epsilon_{kl} x_k y_l - \frac{1}{\theta} x_i P_i.$$  \hfill (2.28)

Let us define

$$J_{\pm} := J_1 \pm iJ_2 \quad \text{and} \quad J_0 := \frac{1}{2}(J + g).$$  \hfill (2.29)
Then, as is easy to check, these quantities satisfy a $SO(2,1)$ Poisson bracket algebra

$$\{J_0, J_\pm\} = \mp iJ_\pm$$  \hspace{1cm} (2.30)

and

$$\{J_+, J_-\} = 2iJ_0,$$  \hspace{1cm} (2.31)

where we have used

$$\{\zeta, x_k\} = \frac{x_k}{\theta}$$  \hspace{1cm} (2.32)

and

$$\{\zeta, \epsilon_{ij}x_iy_j\} = 0.$$  \hspace{1cm} (2.33)

Surprisingly, the algebra’s Casimir $J_0^2 - J_+J_-$ is proportional to the Casimir $C$ of the conformal algebra given by (2.19), i.e.

$$I := J_0^2 - J_+J_- = \frac{1}{4}C.$$  \hspace{1cm} (2.34)

In deriving (2.34) we have used the planar vector identity

$$y_i = x_i \frac{x_ky_k}{r^2} - \epsilon_{ij}x_j \frac{\epsilon_{kl}x_ky_l}{r^2}.$$  \hspace{1cm} (2.35)

Each term in (2.26-27) has a vanishing Poisson bracket with $x_iP_i$, i.e.

$$\{x_iP_i, \varphi\} = 0$$

$$\{x_iP_i, \frac{(\epsilon_{kl}x_ky_l)^2}{r^2}\} = 0,$$

and

$$\{x_iP_i, \zeta\} = 0$$  \hspace{1cm} (2.36)

leading to

$$\{D, J_\pm\} = 0.$$  \hspace{1cm} (2.37)

Using (2.14) and (2.33) we see that $J_\pm$ are invariant also with respect to the special conformal transformations, i.e.

$$\{K, J_\pm\} = 0.$$  \hspace{1cm} (2.38)

As the conformal algebra (2.16-18) is isomorphic to the $SO(2,1)$ algebra [9] we conclude that in the pure vortex case the total symmetry group $G$ is given by

$$G = SO(2,1) \otimes SO(2,1).$$  \hspace{1cm} (2.39)
2.3 Classification of Solutions

From (2.10) and (2.7-9) we see that

\[
\frac{E}{\theta} = \frac{d^2}{dt^2}(\epsilon_{ij} x_i \dot{x}_j).
\]  

(2.40)

Integrating this we get

\[
\epsilon_{ij} x_i \dot{x}_j = \frac{E}{2\theta} t^2 + A_1 t + A_2,
\]

where \(A_{1,2}\) are constants. Let us note that, in polar coordinates, the left hand side of (2.41) is given by \(r^2 \dot{\phi}\).

Next we substitute this expression, together with (2.9), into (2.19) and find that

\[
C = 2E\theta A_2 - \theta^2 A_1^2.
\]

(2.42)

Thus we see that each solution of the equations of motion is characterised by the three constants: \(E, A_1\) and \(A_2\).

3 Canonical Transformation

Our phase space is six-dimensional. In order to reduce it to the physically required four dimensions we consider, in the following, an appropriate canonical transformation with constraints.

Let us return to (2.41) and define

\[
h(t) := \epsilon_{ij} x_i \dot{x}_j = \frac{E}{2\theta} t^2 + A_1 t + A_2.
\]

(3.1)

Next we look for a transformation

\[
t \rightarrow t', \quad x_i(t) \rightarrow x'_i(t')
\]

(3.2)

which brings (3.1) to the form

\[
\epsilon_{ij} x'_i \dot{x}'_j = 1.
\]

(3.3)

This is achieved by putting

\[
\frac{dt'}{dt} = \frac{1}{h(t)} x_i(t) = h(t) x'_i(t').
\]

(3.4)

Integrating we find

\[
t'(t) = \begin{cases} 
-\frac{2}{\sqrt{-C}} \arctanh \frac{A_1 + \frac{E}{\theta} t}{\sqrt{-C}}, & C < 0 \\
-\frac{2}{A_1 + \frac{E}{\theta} t}, & C = 0 \\
\frac{2}{\sqrt{C}} \arctan \frac{A_1 + \frac{E}{\theta} t}{\sqrt{C}}, & C > 0,
\end{cases}
\]

(3.5)
Note that here $C = 2E\theta A_2 - \theta^2 A_1^2$.

Next we extend (3.4) to a canonical transformation:

$$\left(\bar{x}, \bar{y}, \bar{P}\right) \rightarrow \left(\bar{x}', \bar{y}', \bar{P}'\right)$$

(3.6)

by putting

$$y'_i(t') = y_i(t) + f_1(t) x_i(t),$$

(3.7)

$$P'_i(t') = f_2(t) P_i(t) + f_3(t) \epsilon_{ij} y_j(t) + f_4(t) \epsilon_{ij} x_j(t).$$

(3.8)

Using the Poisson bracket structure (2.11)

$$\{x'_i, P'_j\} = \delta_{ij}$$

(3.9)

we find

$$f_2(t) = h(t).$$

(3.10)

Moreover,

$$\{y'_i, y'_j\} = \frac{1}{\theta} \epsilon_{ij}$$

(3.11)

$$\{x'_i, y'_j\} = \{x'_i, x'_j\} = 0$$

(3.12)

which are all automatically satisfied and

$$\{y'_i, P'_j\} = 0 \rightarrow f_1 h - \frac{1}{\theta} f_3 = 0$$

(3.13)

$$\{P'_i, P'_j\} = 0 \rightarrow 2 f_2 f_4 - \frac{1}{\theta} f_3^2 = 0$$

(3.14)

Furthermore, we require that, on shell,

$$\dot{x}'_i = y'_i,$$

(3.15)

which gives us

$$f_1 = -\frac{\dot{h}}{h}$$

(3.16)

while the other conditions are satisfied if we put

$$f_3 = -\theta \dot{h}$$

(3.17)

$$f_4 = \frac{\theta \dot{h}^2}{2h^2}.$$ (3.18)

Next we solve (3.7-8) with the functions given above. We find (all primed variables are functions of $t'$)

$$x_i = h x'_i, \quad y_i = y'_i + \dot{h} x'_i,$$

(3.19)

$$\hbar P_i = P'_i + \dot{h} \epsilon_{ij} y'_j + \frac{1}{2} \theta \dot{h}^2 \epsilon_{ij} x'_j.$$ (3.20)
The new Lagrangian is then given by

\[ L' = L_{\dot{\mathbf{x}}_i - \dot{y}_j} + \frac{\theta}{2} \epsilon_{ij} y_i' y_j' - \frac{1}{2\theta} C \epsilon_{ij} x_i' y_j' - \frac{\lambda}{r^3} + g \frac{\epsilon_{ij} y_i' x_j'}{r^2}. \] (3.21)

where we have neglected the total time-derivative term

\[ \frac{d}{dt'} \left( \frac{\theta}{2} \epsilon_{ij} x_i' y_j' \right). \] (3.22)

The new total Hamiltonian is then given by

\[ H' = P_i y_i' + \frac{\lambda}{r^3} + C \epsilon_{ij} x_i' y_j' + g \frac{\epsilon_{ij} x_i' y_j'}{r^2}. \] (3.23)

When compared to the original Lagrangian (2.6) we note that (3.21) and so (3.23) contains an additional interaction term corresponding to the interaction with a uniform magnetic field of strength \(-\frac{C}{\theta^2}\).

It is easy to derive the equations of motion which follow from our (new) Lagrangian. They are

\[ \theta \epsilon_{ij} \dot{x}_j + \frac{C}{\theta} \epsilon_{ij} \dot{x}_i = \frac{\lambda x_i'}{r^3}, \] (3.24)

where \( \vec{x}' \in R_0^2 := R^2 \setminus \{0\} \).

However, the solutions of these equations will not, in general, satisfy (3.3). In order to overcome this problem we note that the transformation (3.19-20) depends, through the constants \( E, A_1 \) and \( A_2 \), for the initial conditions for the old (unprimed) variables:

\[ h(0) = \epsilon_{ij} x_i(0) \dot{x}_j(0) = A_2, \] (3.25)

\[ \dot{h}(0) = \epsilon_{ij} x_i(0) \dot{x}_j(0) = A_1, \] (3.26)

\[ E = \theta \epsilon_{ij} \dot{x}_i(0) \dot{x}_j(0) + \frac{\lambda}{r(0)}. \] (3.27)

Thus we need to transform these relations into the corresponding combinations of initial conditions on the new variables \( x_i(t') \) at \( t'_0 = t'(0) \).

From (3.4) and (3.1) we get

\[ x_i(0) = A_2 x_i(t'_0), \] (3.28)

\[ \dot{x}_i(0) = A_1 x_i(t'_0) + \dot{x}_i(t'_0), \] (3.29)

\[ \ddot{x}_i(0) = \frac{E}{\theta} x_i(t'_0) + \frac{A_1}{A_2} \dot{x}_i(t'_0) + \frac{1}{A_2} \ddot{x}_i(t'_0). \] (3.30)

Therefore, defining

\[ h'(t') := \epsilon_{ij} x_i(t') \dot{x}_j(t') \] (3.31)

we see that

\[ h'(t'_0) = 1, \] (3.32)
\[
\dot{h}'(t'_0) = 0 \tag{3.33}
\]

and using (3.29)
\[
\ddot{h}'(t'_0) = 0. \tag{3.34}
\]

It is now easy to see that the solutions of the equations of motion, which satisfy the above given initial conditions, automatically satisfy the subsidiary condition \(h'(t') = 1\).

To see this we note that the equations of motion for \(x_i\) and \(y_i\) take the form
\[
\dot{x}'_i = y'_i \tag{3.35}
\]
and
\[
P'_i = \theta \epsilon_{ij} \dot{y}'_j + \frac{C}{2\theta} \epsilon_{ij} x'_j + g \frac{\epsilon_{ij} x'_i}{r^2}, \tag{3.36}
\]
and the 'new' energy \(E'\) may be rewritten as
\[
E' = \theta \ddot{h}'(t') + \frac{C}{\theta} h'(t'). \tag{3.37}
\]
When we use this expression at \(t' = t'_0\), together with (3.32,34), we get
\[
E'\theta = C \tag{3.38}
\]
while solving the remaining differential equation, with the initial conditions (3.32-33), gives us, as required
\[
h'(t') = 1. \tag{3.39}
\]

3.1 Constraint Analysis and the Reduced Phase Space

Our Lagrangian is given by
\[
L' = P_i (\dot{x}_i - y_i) + \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j - \frac{\lambda}{r} - \frac{C}{2\theta} \epsilon_{ij} x_i y_j - g \frac{\epsilon_{ij} x_i y_j }{r^2}, \tag{3.40}
\]
where we have dropped all the primes. As the system is subject to a subsidiary condition
\[
\epsilon_{ij} x_i y_j = 1, \tag{3.41}
\]
which is a primary constraint, we have to, for reasons of consistency, following Dirac, require also that
\[
\frac{d}{dt} (\epsilon_{ij} x_i y_j) = 0 \tag{3.42}
\]
which, due to (3.35-36), gives us a secondary constraint
\[
x_i P_i = 0. \tag{3.43}
\]
Our constraints (3.41,43) are solved by
\[
y_i = zx_i - \frac{1}{r^2} \epsilon_{ij} x_j \tag{3.44}
\]
and
\[ P_i = u \epsilon_{ij} x_j, \quad (3.45) \]

Solving (3.44) and (3.45) for \( z \) and \( u \) we find
\[ z = \frac{y_i x_i}{r^2}, \quad u = \epsilon_{ik} \frac{P_i x_k}{r^2}. \quad (3.46) \]

So the reduced phase-space variables are given by \( x_i, u \) and \( z \).

### 3.2 Dynamics in the reduced phase space

Following Jackiw [12] we insert the solutions for the constraints (3.44,45) into \( L' \) (modulo a total divergence term) and obtain
\[ L' = u \epsilon_{ij} \dot{x}_i x_j + u - \frac{\lambda}{r} \left( \frac{C}{2\theta} - \frac{g}{r^2} \right) + \frac{\theta}{2} \left( z^2 + \frac{1}{r^4} \right) \epsilon_{ij} \dot{x}_i x_j \]
\[ + \frac{\theta}{2} \left( \frac{z^2}{r^2} (x_i \dot{x}_i) - \frac{2z}{r^2} (x_i \dot{x}_i) \right). \quad (3.47) \]

Next we perform the point transformation
\[ (x_i, u, z) \rightarrow (r, \varphi, J, z), \quad (3.48) \]
where \( r \) and \( \varphi \) are the polar coordinates in the plane, and \( J \) is initially given by
\[ J = \epsilon_{ij} x_i P_j + \frac{\theta}{2} y_i^2 \quad (3.49) \]

and so becomes, due to (3.44-45)
\[ J = r^2 (-u + \frac{\theta}{2} z^2) + \frac{\theta}{2r^2}. \quad (3.50) \]

Then the Lagrangian takes the form
\[ L' = J \dot{\varphi} - \frac{\lambda}{r} - \frac{C}{2\theta} - \frac{J + g}{r^2} + \frac{\theta}{2} \left( z^2 + \frac{1}{r^4} \right) - \theta z \frac{\dot{r}}{r}. \quad (3.51) \]

To find the equations of motion we vary \( J \) and find
\[ \dot{\varphi} = \frac{1}{r^2}. \quad (3.52) \]

Next we vary \( z \) and then \( \varphi \) and obtain, respectively
\[ \dot{r} = rz \quad (3.53) \]
and
\[ J = 0. \quad (3.54) \]

Finally, varying \( r \) we get
\[ \dot{z} = -\frac{\lambda}{\theta r} - \frac{2(J + g)}{\theta r^2} + \frac{2}{r^4}. \quad (3.55) \]
3.3 Hamiltonian, Canonical EoM and PB algebra

The Hamiltonian becomes
\[ H' = \frac{J + g}{r^2} - \frac{\theta}{2} \left( z^2 + \frac{1}{r^4} \right) + \frac{\lambda}{r} + \frac{C}{2\theta}. \] (3.56)

Hence the canonical EoM
\[ \dot{A} = \{A, H'\} \] (3.57)
become identical (when used for \( A \in (r, \varphi, J, z) \)) to the EoM (3.52-55) if we require the following Poisson bracket structure
\[ \{\varphi, J\} = 1, \quad \{z, r\} = \frac{r}{\theta}. \] (3.58)

3.4 Phase-space reduction by freezing a symmetry

Instead of considering the phase-space reduction by means of a canonical transformation with constraints we can take a different approach.

We start with the following description of the 6-dimensional phase space:
\[ \{A_i\}_{i=1}^{4} \oplus \{B_i\}_{i=1}^{2}, \] (3.59)
where
\[ A_i \in \{\frac{r}{\epsilon_{ij}x_iy_j}, \varphi, J, \zeta\} \] (3.60)
and
\[ B_i \in \{\epsilon_{ij}x_iy_j, x_kP_k\}. \] (3.61)

The Poisson brackets of the decomposition (3.59) decouple from each other
\[ \{A_i, B_k\} = 0, \quad i = 1,..,4, k = 1,2 \] (3.62)
as can be seen from (2.33) and (2.36) respectively.

Furthermore, the subspace \( \{B_i\}_{i=1}^{2} \) is invariant with respect to the scale and special conformal transformations at \( t = 0 \):
\[ \{\epsilon_{ij}x_iy_j, D\}_{t=0} = -\epsilon_{ij}x_iy_j, \quad \{\epsilon_{ij}x_iy_j, K\}_{t=0} = 0 \]
\[ \{x_kP_k, D\}_{t=0} = 0 \quad \text{and} \quad \{x_kP_k, K\}_{t=0} = -2\theta\epsilon_{ij}x_iy_j. \] (3.63)
Thus the choice of the constraints (3.41) and (3.43) is equivalent to the fixing of the components of the second part \( \{B_i\}_{i=1}^{2} \) in (3.59) in a particular way and may be understood as freezing the scale and the special conformal transformation at \( t = 0 \).

The remaining phase space \( \{A_i\}_{i=1}^{4} \) is then identical to our reduced phase space \( \{r, \varphi, J, z\} \) because, as we can see from (2.28) and (3.46), we have
\[ \zeta = z \] (3.64)
on the constraint surface (3.41,43).

Note that in this procedure we do not obtain the additional interaction with a uniform magnetic field (as in (3.21) and (3.23)).
4 Solutions of the Classical Equations of Motion

Next we consider the solutions of the equations of motion, i.e., the equations (3.52-55). First, we note that these equations have two integrals of motion, namely

\[ E' = \frac{C}{2\theta} + \frac{J + g}{r^2} + \frac{\lambda}{r} - \frac{\theta}{2} \left( \frac{\dot{r}}{r} \right)^2 + \frac{1}{r^4}, \]  

(4.1)

and the angular momentum \( J \).

By using (3.38) we may rewrite (4.1) as

\[ \frac{J + g}{\theta} = V(r) + \frac{\dot{r}^2}{2}, \]  

(4.2)

where

\[ V(r) := \frac{1}{2r^2} - \frac{\lambda}{\theta} r + \frac{C}{2\theta^2} r^2. \]  

(4.3)

We see that we have reduced the problem to that of a motion of a particle of "mass" \( m = 1 \) moving in the potential \( W(r) = \frac{\lambda}{\theta} r + \frac{C}{2\theta^2} r^2 \) with "angular momentum" \( \mathcal{L} = 1 \) and "energy" \( \mathcal{E} = \frac{J}{\theta} + g \).

Note, that in this way our "nonstandard" problem has become a "standard" one.

We have to consider three separate cases: \( C = 0 \), \( C > 0 \) and \( C < 0 \).

4.1 The case of \( C = 0 \)

Thus

\[ V(r) := \frac{1}{2r^2} - \frac{\lambda}{\theta} r. \]  

(4.4)

Let us concentrate our attention on the case of

\[ \frac{\lambda}{\theta} < 0 \]  

(4.5)

i.e., of an attractive potential.

Then, we note that \( \frac{J + g}{\theta} > 0 \). At the same time \( V(r) \) has a minimum at \( r_0 = \left( -\frac{\theta}{\lambda} \right)^{\frac{1}{3}} \), which, given that \( r_0 > 0 \) tells us that \( V(r_0) = \frac{3}{2} \left( \frac{\lambda}{\theta} \right)^{\frac{2}{3}} \). Thus we have

\[ \frac{J + g}{\theta} \geq \frac{3}{2} \left( \frac{\lambda}{\theta} \right)^{\frac{2}{3}}. \]  

(4.6)

Note that when

\[ \frac{J + g}{\theta} = \frac{3}{2} \left( \frac{\lambda}{\theta} \right)^{\frac{2}{3}} \]  

(4.7)

we have

\[ r(t) = r_0 \]  

(4.8)
i.e. we have a steady motion along a circle of radius \( r_0 \).

For \( \frac{J+g}{\theta} > \frac{3}{2} \left( \frac{\theta}{\lambda} \right)^{\frac{3}{2}} \), we have a bounded motion. To see this we note that the integration of (4.2) gives us

\[
t = \pm \int_{r(0)}^{r(t)} dr' \left( 2 \left( \frac{J+g}{\theta} - V(r') \right) \right)^{-\frac{1}{2}}.
\]  (4.9)

When \( \frac{J+g}{\theta} = V(r) \) has at least 2 positive roots there is a region for the bounded motion

\[
0 < r_{min} \leq r \leq r_{max}.
\]  (4.10)

4.2 The case of \( C > 0 \)

When \( C > 0 \) we have

\[
\frac{J + g}{\theta} = \frac{\dot{r}^2}{2} + \frac{1}{2r^2} - \frac{\lambda r}{\theta} + \frac{C r^2}{\theta^2 r^2}
\]  (4.11)

and so it is clear that the motion is bounded.

Thus we are left with having to discuss the \( C < 0 \) case.

4.3 The case of \( C < 0 \)

Now we have

\[
\dot{r}^2 = \frac{1}{r^2} \left[ \frac{2(J + g)}{\theta} r^2 - 1 + \frac{2\lambda}{\theta} r^3 + \frac{B r^4}{\theta^2} \right]
\]  (4.12)

where \( B = |C| \).

Next we define

\[
F(r) = \frac{2(J + g)}{\theta} r^2 - 1 + \frac{2\lambda}{\theta} r^3 + \frac{B r^4}{\theta^2}
\]  (4.13)

which we rewrite as

\[
F(r) = \frac{B}{\theta^2} \left( r^4 + \alpha r^3 + \beta r^2 + \gamma \right),
\]  (4.14)

where \( \alpha = \frac{2\lambda}{B} \), \( \beta = \frac{2(J+g)}{B} \) and \( \gamma = -\frac{\theta^2}{B} \).

Then all the properties of the solution depend on the values of the parameters \( \alpha \), \( \beta \) and \( \gamma \).

Clearly \( F(r) \to \infty \) as \( r \to \infty \). Moreover \( F(0) = \gamma < 0 \). As \( r = 0 \) is an extremum of \( F(r) \), in order to have a bounded motion we need to have \( F(r_1) > 0 \) and \( F(r_2) < 0 \) where \( r_1 \) and \( r_2 \) are two further extrema of \( F \), which should lie at \( r > 0 \) with \( r_1 < r_2 \).

Then the bounded motion would involve \( r \) changing between two roots of \( F \) lying between \( r = 0 \) and \( r_1 \), and, \( r_1 \) and \( r_2 \) respectively.

So we look at the extrema of \( F \). Clearly \( r_1 \) and \( r_2 \) are given by

\[
r_i = \frac{-3\alpha \pm \sqrt{9\alpha^2 - 32\beta}}{8},
\]  (4.15)
with the lower (upper) sign for \( r_1(r_2) \).

As both \( r_i \) have to be positive we require that (note that due to (4.5) \( \alpha < 0 \))

\[
9\alpha^2 > 32\beta > 0 \quad \Rightarrow \quad \frac{\lambda^2}{\theta^2} > \frac{16(J + g)B}{9\theta}, \quad J + g > 0. \tag{4.16}
\]

It is easy to show that at \( r = r_i \quad F \) takes the values

\[
F(r_i) = \left( 1 + \frac{9}{64} \alpha^3 \right) r_i + \gamma - \frac{\beta^2}{4} + \frac{3}{32} \alpha^2 \beta \right) \frac{B}{\theta^2}. \tag{4.17}
\]

Now we want to show that there is a nontrivial region of the parameters \( \alpha, \beta \) and \( \gamma \) such that \( F(r_1) > 0 \) while \( F(r_2) < 0 \).

For this purpose we take, in accordance with (4.16), a particular value of \( \beta \), namely

\[
\beta_0 = \frac{3}{4} \frac{9\alpha^2}{32}. \tag{4.18}
\]

For this value we have

\[
r_i = \frac{3}{16} |\alpha|(2 \mp 1) \tag{4.19}
\]

and so

\[
F(r_i) = \left( \frac{27}{8^4} \alpha^4(\pm 1 + 1) - \frac{27^2}{2 \times 8^6} \alpha^4 + \gamma \right) \frac{B}{\theta^2} \tag{4.20}
\]

with the upper (lower) sign for \( r_1(r_2) \).

Then, obviously

\[
F(r_2) < 0 \tag{4.21}
\]

and also we have \( F(r_1) > 0 \) iff

\[
\left( 2 - \frac{27}{16} \right) \frac{27}{8^4} \alpha^4 + \gamma > 0 \tag{4.22}
\]

or equivalently

\[
\left| \frac{\lambda}{\theta} \right| > 2.347 \left( \frac{B}{\theta^2} \right)^{\frac{1}{2}}. \tag{4.23}
\]

Due to the continuity of \( F(r_i) \), as a function of \( \beta \) we will have (4.21-22), and therefore bounded motion, for a range of \( \beta \) around \( \beta_0 \) if we choose the numerical factor in (4.23) as an appropriate function of \( \beta \).

Thus we see that for \( C < 0 \) we can have solutions that describe bounded motion too.
4.4 Hidden Symmetry of the pure vortex case

The canonical transformation (3.19-20) supplemented by the constraints (3.41,43) leads to the following transformation rules

\[ \varphi \rightarrow \varphi, \quad \frac{(\epsilon_{ij}x_iy_j)^2}{r^2} \rightarrow \frac{1}{r^2} \]

and \[ \zeta \rightarrow z. \] (4.24)

Thus, the conserved quantities \( J_{1,2} \) given by (2.26-27) in the original phase-space become

\[ \frac{2J_1}{\theta} = \cos 2\varphi \left( \frac{1}{r^2} - \frac{J + g}{\theta} \right) + z \sin 2\varphi \] (4.25)

and

\[ \frac{2J_2}{\theta} = \sin 2\varphi \left( \frac{1}{r^2} - \frac{J + g}{\theta} \right) - z \cos 2\varphi \] (4.26)

in the reduced phase space. It is self-evident that, together with \( J_0 \), they still satisfy the \( SO(2,1) \) Poisson bracket algebra (2.30-31). The algebra’s Casimir (2.34) is now given by

\[ I = \frac{\theta}{2} H, \] (4.27)

where we have defined \( H := H' - \frac{C}{2\theta} \).

For the classical orbit we have from (4.25-26)

\[ \frac{2}{\theta} (J_1 \cos 2\varphi + J_2 \sin 2\varphi) = \frac{1}{r^2} - \frac{J + g}{\theta} \] (4.28)

thus showing that the classical trajectory is then given by

\[ r(\varphi) = \left( \frac{2J_1}{\theta} \cos 2\varphi + \frac{2J_2}{\theta} \sin 2\varphi + \frac{J + g}{\theta} \right)^{-\frac{1}{2}}. \] (4.29)

An interesting case then arises when

\[ J_1 = J_2 = 0 \] (4.30)

realised by means of the initial conditions

\[ z(t = 0) = 0, \quad r(t = 0) = \left( \frac{J + g}{\theta} \right)^{-\frac{1}{2}}. \] (4.31)

Then, due to \( J_{1,2} \) being conserved, (4.31) hold at an arbitrary time \( t \). In this particular case the EoM (3.52-53) reduce to

\[ \dot{\psi} = \frac{1}{r^2}, \quad \dot{r} = 0, \] (4.32)
which is equivalent to
\[ \dot{x}_i = -\frac{\epsilon_{ij}x_j}{r^2}. \] (4.33)

Note that (4.33) describes the relative motion of two fluid vortices (see [13] and the literature cited therein). Note also that (4.33) may be derived as a canonical EoM from any Hamiltonian of the form
\[ H = H(r) \] (4.34)
if we have the symplectic structure
\[ \{x_i, x_j\} = -\frac{\epsilon_{ij}}{rH'(r)}. \] (4.35)

In our case we would have
\[ H(r) = \frac{\theta}{2r^4} \] (4.36)
derived from (3.51)|\lambda=0 by means of the primary constraint \( z = 0 \).

## 5 Quantum Dynamics

### 5.1 Schrödinger equation in reduced phase space

We return to our Hamiltonian (3.56) and proceed to its quantization. Thus our Poisson brackets (3.58) now become the commutation relations (\( \hbar = 1 \))
\[ [z, r] = i \frac{r}{\theta}, \quad [J, \varphi] = \frac{1}{i}. \] (5.1)

Representing \( J \) and \( z \) by symmetric differential operators
\[ J = \frac{1}{i} \partial_\varphi, \quad z = \frac{i}{2\theta} (r\partial_r + \partial_r r) = \frac{i}{\theta} (r\partial_r + \frac{1}{2}). \] (5.2)
gives us the radial Schrödinger equation
\[ \left( \frac{1}{2\theta} \left( r\partial_r + \frac{1}{2} \right)^2 + \frac{\lambda}{r} + \frac{\bar{m}}{r^2} - \frac{\theta^2}{2r^4} - E \right) \varphi_{E,m}(r) = 0 \] (5.3)
with \( \bar{m} := m + g, m \in Z \) and \( E := E' - \frac{C^2}{2\theta} \).

### 5.2 Asymptotic behaviour of bound state solutions

#### 5.2.1 \( r \to 0 \)

The dominant contribution of (5.3), in this limit, comes from
\[ \left( \left( r\partial_r + \frac{1}{2} \right)^2 - \frac{\theta^2}{r^4} \right) \varphi_{E,m}(r) = 0. \] (5.4)
We make the ansatz [14]
\[ \varphi(r) = r^{-b} e^{-\frac{\theta}{r^2}} \]  
(5.5)
and find that
\[ a = \frac{|\theta|}{2} \]  
(5.6)
while \( b \) is undetermined.

5.2.2 \( r \to \infty \)

In this limit the dominant part of (5.3) comes from
\[ \left( \left( r \partial_r + \frac{1}{2} \right)^2 - 2\theta E \right) \varphi_{E,m}(r) = 0, \]  
(5.7)
which shows that, as \( r \to \infty \),
\[ \varphi(r) \sim r^{-\left(\frac{1}{2} + \sqrt{2\theta E}\right)}. \]  
(5.8)

For \( \varphi \) to be normalisable, i.e. to satisfy
\[ \int_0^\infty dr r |\varphi|^2 < \infty \]
we have to require that
\[ E\theta > 0, \quad \text{and} \quad \sqrt{2E\theta} > \frac{1}{2}. \]  
(5.9)

Let us assume that \( \theta > 0 \) from now on. The general ansatz for \( \varphi \), respecting both limits (5.5, 5.8) is then given by (cp. [14])
\[ \varphi_{E,m}(r) = e^{-\frac{a}{r^2}} F(r) \]  
(5.10)
with
\[ F(r) \sim r^{-b}, \quad b := \frac{1}{2} + \sqrt{2\theta E} \]  
(5.11)
and
\[ r^N F(r) \sim o(1) \]  
(5.12)
for \( N \) larger than some \( N_0 \in N \).

Putting (5.10) into (5.3) we end up with a differential equation for \( F \), which is
\[ \frac{1}{2\theta} \left( r^2 \ddot{F} + 2 \left( \frac{\theta}{r} + r \right) \dot{F} \right) + \left( \bar{m} - \frac{\lambda}{r^2} + \frac{\lambda}{r} - E - \frac{1}{8\theta} \right) F = 0. \]  
(5.13)
We have not succeeded in solving this equation in full generality. In the next section we make some comments on its solutions obtained by a power series expansion.
5.3 Some comments about the solutions of (5.13)

First we perform a transformation

\[ r \to u := \frac{1}{r}, \quad G(u) := F(r) \]  (5.14)

and find that (5.13) has now become

\[ u^2 \ddot{G} - 2\theta u^3 \dot{G} + \mu u^2 G + (\nu u + \omega) G = 0 \]  (5.15)

where

\[ \mu := 2\theta (\bar{m} - \frac{1}{2}), \quad \nu := 2\theta \lambda, \quad \omega := \frac{1}{4} - 2\theta E. \]  (5.16)

Next we attempt to solve (5.15) by the generalised power series around \( u = 0 \). Hence we put

\[ G(u) = \sum_{n=0}^{\infty} a_n u^{n+\alpha}. \]  (5.17)

The indicial equation is then

\[ \alpha(\alpha - 1) + \omega = 0 \]  (5.18)

with solutions

\[ \alpha_{\pm} = \frac{1}{2} \pm \sqrt{2\theta E}. \]  (5.19)

Note that \( \alpha_+ = b \). Next we find that

\[ a_1 = -\frac{\nu}{2\alpha} a_0 \]  (5.20)

and that, for the general \( a_n \) \( (n \geq 2) \) we have the three point recurrence relation

\[ ((\alpha + n)(\alpha + n - 1) + \omega)a_n + \nu a_{n-1} + (\mu - 2\theta(\alpha + n - 2))a_{n-2} = 0. \]  (5.21)

Clearly, starting with \( a_0 \neq 0 \), we can determine \( a_1 \) and then, using (5.21), successively, all \( a_k \). This is true for either choice of \( \alpha \) in (5.19). Both series converge as (5.21) shows that as \( n \to \infty \) we have

\[ \frac{a_n}{a_{n-1}} \to \pm \sqrt{\frac{2\theta}{n}}. \]  (5.22)

We finish this section with a few comments

- Note that the solutions (5.17) corresponding to \( \alpha_- \) are non-normalisable (ie they describe scattering states).
• For $\nu \neq 0$ the series (5.17) cannot terminate for a generic value of $\nu$. Hence we have no discrete spectrum.

To show this we assume the contrary; i.e. $a_n = 0$ for $n \geq p + 1$ and $a_p \neq 0$. Then from (5.21) (with $n = p + 2$) we have $\mu - 2\theta(b + p) = 0$. Inserting this into (5.21) for $n = p + 1$ we find that

$$a_{p-1} = -\frac{\nu}{2\theta}a_p$$

(5.23)

and so we see that we have a linear system of $(p + 1)$ equations ((5.20), (5.23) and (5.21) for $2 \leq n \leq p$) for $(p + 1)$ unknowns $\{a_n\}_{n=0}^{p}$. This system has nontrivial solutions if and only if the corresponding determinant vanishes. But this is not the case for a generic value of $\nu$.

Example. Take $p = 1$. Then the system of two equations (5.20, 5.23) has the determinant $\det = 1 - \frac{\nu^2}{4\theta} \neq 0$.

Thus we see that the bound states (belonging to $\lambda_+ (5.19)$ have a continuous spectrum, which is characteristic of singular potentials (for more details see [15]).

• For $\nu = 0$, (i.e. $\lambda = 0$) we can show that $p=$even, $a_n = 0$ for odd $n$ and the spectrum is discrete as our system reduces to $\frac{p}{2}$ equations for $\frac{p}{2} + 1$ unknowns.

5.4 Hidden symmetry of the pure vortex problem

Let us now return to the case of the pure vortex. In the classical case we had two conserved quantities (4.25) and (4.26).

The same is true in the quantum case. We have

$$J_+ := \theta e^{2i\phi} \left( \frac{r \partial_r + \frac{1}{2}}{\theta} + \frac{1}{r^2} - \frac{J + g + 1}{\theta} \right)$$

(5.24)

and

$$J_- := J_+^\dagger = \theta e^{-2i\phi} \left( -\frac{r \partial_r + \frac{1}{2}}{\theta} + \frac{1}{r^2} - \frac{J + g - 1}{\theta} \right).$$

(5.25)

Then, as can be seen by straightforward calculation, we get, as in the classical case, the $SO(2,1)$ algebra

$$[J_0, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = -2J_0.$$  

(5.26)

Their Casimir is now

$$I := J_0(J_0 \mp 1) - J_{\pm}J_{\mp} = \frac{\theta}{2}H - \frac{1}{4}.$$  

(5.27)

Note that both $J_{\pm}$ and $I$ contain additional quantum correction terms when compared to their classical values.
A realization of the $SO(2,1)$ algebra (5.26) in terms of differential operators $J_{\pm}, J_0$ which commute with $H$, (potential algebra realization) has been considered already some time ago [16]. Such a realization allows only for the existence of the $D^+$ discrete principal series of $SO(2,1)$ for $\theta > 0$. Then the discrete spectrum of $H$ can be obtained by a standard procedure (cp. [16]):

We start with the ground state of the angular momentum ladder

$$J_- \Psi_{0,m} = 0, \quad (5.28)$$

where, in the coordinate representation,

$$\Psi_{0,m} = e^{im\varphi} \varphi_{E,m}(r). \quad (5.29)$$

Then (5.28) is solved by

$$\varphi_{E,m}(r) = r^{-(\bar{m} - \frac{1}{2})} e^{-\frac{\theta}{2r^2}}, \quad (5.30)$$

where, due to (5.8,9), resp. (5.26) and (5.27), we have

$$E = E_{0,m} := \frac{1}{2\theta}(\bar{m} - 1)^2 \quad (5.31)$$

with $m = m_0, m_0 + 2, m_0 + 4; m_0 := \{\frac{3}{2} - g\}$, where $\{\} \text{ denotes the Z-part.}$

We note an infinite-fold degeneracy of the energy spectrum, because the whole angular momentum ladder

$$\Psi_{p,m+2p} := (K_+)^p \Psi_{0,m} \quad p \in N \quad (5.32)$$

possesses the same energy (5.31).

Equivalently, at fixed angular momentum $m$, i.e. at fixed potential in the Schrödinger eq. (5.3), we have a finite number of bound states with energy

$$E_{p,m} = \frac{1}{2\theta}(m - 2p - 1)^2, \quad p = 0, 1, 2, ..\{\frac{m - m_0}{2}\}. \quad (5.33)$$

**5.5 Relation between the vortex problem and the Morse potential**

5.5.1 Transformation of the Schrödinger equation into the Morse potential problem

Let us perform the following change of variables

$$r \rightarrow \rho := 2ln r \quad (5.34)$$

and

$$\varphi_{E,m}(r) = e^{-\frac{\rho}{2}} \Phi_{E,m}(\rho). \quad (5.35)$$
Then our Schrödinger equation (5.3) transforms, for $\lambda = 0$, into

$$\left(\frac{2}{\theta} \partial^2_{\rho} + \bar{m} e^{-\rho} - \frac{\theta}{2} e^{-2\rho} - E\right) \Phi_{E,m}(\rho) = 0.$$ (5.36)

Define

$$e^{\rho_0} := \frac{\theta}{\bar{m}}$$ (5.37)

(we have chosen $\theta > 0$ and have $\bar{m} > 0$).

Then our equation becomes

$$\left(-\frac{2}{\theta} \partial^2_{\rho} + \frac{\bar{m}^2}{2\theta} \left(1 - e^{-(\rho-\rho_0)}\right)^2 - \frac{\bar{m}^2}{2\theta} + E\right) \Phi_{E,m} = 0 \quad -\infty < \rho < \infty.$$ (5.38)

This is the Schrödinger equation for the Morse potential [11] with ‘mass’ $= \frac{1}{4}\theta$ and $E_{\text{Morse}} = -E + \frac{\bar{m}^2}{2\theta}$. For the spectrum one has [11]

$$E_{\text{Morse}} = \frac{2\bar{m}}{\theta} \left(n + \frac{1}{2}\right) - \frac{2}{\theta} \left(n + \frac{1}{2}\right)^2, \quad n \in \mathbb{N}$$ (5.39)

which corresponds to (cp. (5.37))

$$E_{n,m} = \frac{1}{2\theta} (\bar{m} - 1 - 2n)^2.$$ (5.40)

### 5.5.2 Generalised hidden symmetry

The operators $I_\pm$ (see (5.24-5.25) shift the angular momentum, at fixed energy, by $\pm$ two units.

We may ask what are the operators which shift energy at fixed angular momentum. As, at fixed $\bar{m}$, the energies are not equidistant these shift operators are not elements of a linear algebra. However, a realisation in terms of a quadratic $SU(2)$-algebra is possible [17] for the Morse potential. This could then be translated to our case and would lead to the generalised form of the hidden symmetry (cp. also [18]).

An explicit construction of such shift operators, as functions of the parameters of the Schrödinger equation, is given in [19].

### 6 Concluding Remarks

The Lagrangian for a massive point particle with either an inverse square potential in arbitrary dimensions $D$ [9], a magnetic monopole in $D = 3$ [20] or a point magnetic vortex in $D = 2$ [10] exhibits conformal symmetry corresponding to the scale dimension $-\frac{1}{2}$ of $x_i$ (see (2.24)). But in the present exotic model (2.1) the potential is of the Coulomb type and $x_i$ has therefore anomalous scale dimension -1 (see (2.21)). We can still add a point magnetic vortex because the vortex allows an arbitrary scale
dimension of $x_i$ [10]. An additional hidden $SO(2,1)$ symmetry is observed in the pure vortex case.

The reduction of six-dimensional phase-space to the physically required four dimensions is performed in an unconventional way. We can use either a canonical transformation with constraints or a frozen symmetry at $t = 0$. Related to the frozen symmetry method, but different in detail, is the “geometric symmetry-breaking mechanism” introduced some time ago by Cho [21]. We believe that both our methods are of a general importance.

The solutions of the classical EOM in the reduced phase-space show for $C < 0$ a bounded motion for strong Coulomb coupling (see (4.23)). For the quantum analogue one would expect bound states embedded in the continuum. But it has been claimed [22] that these bounded solutions, which are due to potential barriers, would appear as scattering resonances in quantum dynamics.

In the quantum case we find a continuous spectrum of bound states for the generic values of the Coulomb coupling $\lambda$. This is due to the singular nature of the effective potential in (5.3). However, for $\lambda = 0$ we observe the hidden $SO(2,1)$ symmetry leading to the well known discrete spectrum of the equivalent Morse potential problem [11].

Further investigations of the present model are possible. In particular a supersymmetric extension is called for.

Acknowledgement: We are grateful to R. Jackiw for valuable suggestions, to P. Horvathy and J. Lukierski for comments and correspondence.

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