THE PLASTICITY OF SOME MASS TRANSPORTATION NETWORKS IN THE THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We obtain an important generalization of the inverse weighted Fermat-Torricelli problem for tetrahedra in $\mathbb{R}^3$ by assigning at the corresponding weighted Fermat-Torricelli point a remaining positive number (residual weight). As a consequence, we derive a new plasticity principle of weighted Fermat-Torricelli trees of degree five for boundary closed hexahedra in $\mathbb{R}^3$ by applying a geometric plasticity principle which lead to the plasticity of mass transportation networks of degree five in $\mathbb{R}^3$. We also derive a complete solution for an important generalization of the inverse weighted Fermat-Torricelli problem for three non-collinear points and a new plasticity principle of mass networks of degree four for boundary convex quadrilaterals in $\mathbb{R}^2$. The plasticity of mass transportation networks provides some first evidence in a creation of a new field that we may call in the future Mathematical Botany.

1. Introduction

Let $A_1A_2A_3A_4A_5$ be a closed hexahedron in $\mathbb{R}^3$, $B_i$ be a non-negative number (weight) which corresponds to each vertex $A_i$, $A_0$ be a point in $\mathbb{R}^3$ and $a_{ij}$ be the Euclidean distance of the linear segment $A_iA_j$, for $i, j = 0, 1, 2, 3, 4, 5$ respectively.

The weighted Fermat-Torricelli problem for a closed hexahedron $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$ states that:

Problem 1. Find a point $A_0$ which minimizes the sum of the lengths of the linear segments that connect every vertex $A_i$ with $A_0$ multiplied by the positive weight $B_i$:

$$\sum_{i=1}^{5} B_i a_{0i} = \text{minimum}.$$  \hspace{1cm} (1.1)
For \( B_1 = B_2 = B_3 \) and \( B_4 = B_5 = 0 \) we derive the classical Fermat-Torricelli problem which has been introduced by Fermat in 1643 and Torricelli discover the first geometrical construction in \( \mathbb{R}^2 \). In 1877, Engelbrecht extended Torricelli’s construction in the weighted case. In 2014, Uteshev succeeded in finding an elegant algebraic solution of the weighted Fermat-Torricelli problem in \( \mathbb{R}^2 \) in [8]. A detailed history of the weighted Fermat-Torricelli problem is given in [7], [2] and [5].

In 1997, Y. Kupitz and H. Martini gave a complete study concerning the existence, uniqueness and a characterization of the weighted Fermat-Torricelli point for \( n \) non-collinear points in \( \mathbb{R}^m \) in [7] (see also in [2, Theorem 18.37, p. 250]).

**Theorem 1.** Let there be given \( n \) non-collinear points in \( \mathbb{R}^m \), with corresponding positive weights \( B_1, B_2, ..., B_n \).

(i) Then the weighted Fermat-Torricelli point \( A_0 \) of \( \{A_1 A_2 A_3 ... A_n\} \) exists and is unique.

(ii) If

\[
\| \sum_{j=1}^{n} B_j \bar{u}(A_i, A_j) \| > B_i, \ i \neq j.
\]

for \( i,j=1,2,3,4,5 \), then

(a) the weighted Fermat-Torricelli point does not belong in \( \{A_1 A_2 A_3 ... A_n\} \) (Weighted Floating Case).

(b)

\[
\sum_{i=1}^{n} B_i \bar{u}(A_0, A_i) = 0
\]

(Weighted Floating Case).

(iii) If there is some \( i \) with

\[
\| \sum_{j=1}^{n} B_j \bar{u}(A_i, A_j) \| \leq B_i, \ i \neq j.
\]

for \( i,j=1,2,3,4,5 \), then the weighted Fermat-Torricelli point is the vertex \( A_i \) (Weighted Absorbed Case),

where \( \bar{u}(A_i, A_j) \) is the unit vector with direction from \( A_i \) to \( A_j \), for \( i, j = 0, 1, 2, 3, ..., n \) and \( i \neq j \).

The inverse weighted Fermat-Torricelli problem for tetrahedra in \( \mathbb{R}^3 \) states that:
**Problem 2.** Given a point $A_0$ which belongs to the interior of $A_1A_2A_3A_4$ in $\mathbb{R}^3$, does there exist a unique set of positive weights $B_i$, such that

$$B_1 + B_2 + B_3 + B_4 = c = \text{const},$$

for which $A_0$ minimizes

$$f(A_0) = \sum_{i=1}^{4} B_i a_{0i}.$$

By letting $B_4 = 0$ and $c = 1$ in the inverse weighted Fermat-Torricelli problem for tetrahedra we obtain the (normalized) inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$. In 2002, S. Gueron and R. Tessler introduce the normalized inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ who also gave a positive answer in [5].

In 2009, a positive answer with respect to the inverse weighted Fermat-Torricelli problem for tetrahedra is given in [10] and recently, Uteshev also obtain a positive answer in [8] by using the Cartesian coordinates of the four non-collinear and non-coplanar fixed vertices. In 2011, a negative answer with respect to the inverse weighted Fermat-Torricelli problem for tetragonal pyramids in $\mathbb{R}^3$ is derived in [11]. This negative answer lead to an important dependence of the five variable weights, such that the corresponding weighted Fermat-Torricelli point remains the same, which we call a plasticity principle of closed hexahedra. In 2013, we prove a plasticity principle of closed hexahedra in $\mathbb{R}^3$ and a plasticity principle for convex quadrilaterals in [12] and [13], respectively.

In this paper, we consider an important generalization of the inverse weighted Fermat-Torricelli problem for boundary tetrahedra in $\mathbb{R}^3$ which is derived as an application of the geometric plasticity of weighted Fermat-Torricelli trees of degree four for boundary tetrahedra in a two-way communication network (Section 3, Proposition 3). This new evolutionary approach gives a new type of plasticity of mass transportation networks of degree four for boundary tetrahedra and of degree five for boundary closed hexahedra in $\mathbb{R}^3$ (Section 3, Theorem 2, Proposition 4). As a corollary, we also derive an important generalization of the inverse weighted Fermat-Torricelli problem for three non-collinear points and a new type of plasticity for mass transportation networks of degree four for boundary weighted quadrilaterals in $\mathbb{R}^2$ (Section 4, Theorem 3, Proposition 5, Theorem 4). It is worth mentioning that this method provides a unified approach to deal with the inverse weighted Fermat-Torricelli problem for boundary triangle
invented by S. Gueron and R. Tessler which also includes the weighted absorbed case (Theorem 1 (iii) for \( n = 3 \)).

2. THE DEPENDENCE OF THE ANGLES OF A WEIGHTED FERMAT-TORRICELLI TREE HAVING DEGREE AT MOST FOUR AND AT MOST FIVE

We shall start with the definitions of a tree topology, a Fermat-Torricelli tree topology, the degree of a boundary vertex in \( \mathbb{R}^3 \) and the degree of the weighted Fermat-Torricelli point which is located at the interior of the convex hull of a closed hexahedron or tetrahedron, in order to describe the structure of a weighted Fermat-Torricelli tree of a boundary closed hexahedron or a boundary tetrahedron in \( \mathbb{R}^3 \).

Definition 1. [4] A tree topology is a connection matrix specifying which pairs of points from the list \( A_1, A_2, ..., A_m, A_{0,1}, A_{0,2}, ..., A_{0,m-2} \) have a connecting linear segment (edge).

Definition 2. [6] The degree of a vertex corresponds to the number of connections of the vertex with linear segments.
Definition 3. A Fermat-Torricelli tree topology of degree at most five is a tree topology with all boundary vertices of a closed hexahedron and one mobile vertex having at most degree five.

Definition 4. A tree of minimum length with a Fermat-Torricelli tree topology of degree at most five is called a Fermat-Torricelli tree.

Definition 5. A Fermat-Torricelli tree of weighted minimum length with a Fermat-Torricelli tree topology of degree at most five is called a weighted Fermat-Torricelli tree of degree at most five.

Definition 6. A Fermat-Torricelli tree of weighted minimum length having one zero weight is called a weighted Fermat-Torricelli tree of degree at most four.

Definition 7. A unique solution of the weighted Fermat-Torricelli problem for closed hexahedra is a unique weighted Fermat-Torricelli tree of degree at most five.

Definition 8. A unique solution of the weighted Fermat-Torricelli problem for tetrahedra is a unique weighted Fermat-Torricelli tree (weighted Fermat-Torricelli network) of degree at most four.

By following the methodology given in [12, Lemmas 1, 2 pp. 15-17] and [10, Solution of Problem 2, pp. 119-120], we shall show that the position of a weighted Fermat-Torricelli tree w.r. to a boundary tetrahedron is determined by five given angles.

We denote by $\alpha_{i0j} \equiv \angle A_iA_0A_j$ and $\alpha_{i,j0k}$ the angle which is formed by the linear segment that connects $A_0$ with the trace of the orthogonal projection of $A_i$ to the plane $A_jA_0A_k$ with $a_{0i}$, for $i,j,k,l = 1, 2, 3, 4$, and $i \neq j \neq k \neq i$.

Proposition 1. The angles $\alpha_{i,k0m}$ depend on exactly five given angles $\alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}$ and $\alpha_{204}$, for $i,k,m = 1, 2, 3, 4$, and $i \neq k \neq m$.

Proof of Lemma [7]: We shall use the same expressions used in [10, Solution of Problem 2, pp. 119-120] for the unit vectors $\vec{a}_i$ in terms of spherical coordinates, for $i = 1, 2, 3, 4$. We denote by

$$\vec{a}_1 = (1, 0, 0)$$

$$\vec{a}_2 = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0)$$

$$\vec{a}_3 = (\cos(\alpha_{3,102}) \cos(\omega_{3,102}), \cos(\alpha_{3,102}) \sin(\omega_{3,102}), \sin(\alpha_{3,102}))$$

$$\vec{a}_4 = (\cos(\alpha_{4,102}) \cos(\omega_{4,102}), \cos(\alpha_{4,102}) \sin(\omega_{4,102}), \sin(\alpha_{4,102}))$$

such that: $|\vec{a}_i| = 1$. 


The angles $\alpha_{3,102}$, $\alpha_{4,102}$, are calculated by the following two relations in [10, Formulas (10), (11), p. 120]:

$$\cos^2(\alpha_{3,102}) = \frac{\cos^2(\alpha_{203}) + \cos^2(\alpha_{103}) - 2 \cos(\alpha_{203}) \cos(\alpha_{103}) \cos(\alpha_{102})}{\sin^2(\alpha_{102})},$$

(2.5)

and

$$\cos^2(\alpha_{4,102}) = \frac{\cos^2(\alpha_{204}) + \cos^2(\alpha_{104}) - 2 \cos(\alpha_{204}) \cos(\alpha_{104}) \cos(\alpha_{102})}{\sin^2(\alpha_{102})},$$

(2.6)

The inner product of $\vec{a}_i$, $\vec{a}_j$ is given by:

$$\vec{a}_i \cdot \vec{a}_j = \cos(\alpha_{ij}).$$

(2.7)

By replacing (2.5) and (2.6) in (2.18), by eliminating $\omega_{3,102}$ and $\omega_{4,102}$ and by squaring both parts of the derived equation, we obtain a quadratic equation w.r. to $\cos \alpha_{304}$:

$$[\cos(\alpha_{103}) \cos(\alpha_{104}) - \cos(\alpha_{304})]^2 = (1 - \cos^2 \alpha_{3,102})(1 - \cos^2 \alpha_{4,102})$$

(2.8)

By solving (2.8) w.r. to $\cos \alpha_{304}$, we get:

$$\cos \alpha_{304} = -\frac{1}{4} \left[ 2b + 4 \cos \alpha_{102} \left( \cos \alpha_{104} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{204} \right) - 4 \left( \cos \alpha_{103} \cos \alpha_{104} + \cos \alpha_{203} \cos \alpha_{204} \right) \csc^2 \alpha_{102} \right]$$

(2.9)

or

$$\cos \alpha_{304} = \frac{1}{4} \left[ 4 \cos \alpha_{103} (\cos \alpha_{104} - \cos \alpha_{102} \cos \alpha_{204}) + 2 (b + 2 \cos \alpha_{203} (\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204})) \csc^2 \alpha_{102} \right]$$

(2.10)

where

$$b \equiv \prod_{i=3}^{4} (1 + \cos (2\alpha_{102}) + \cos (2\alpha_{10i}) + \cos (2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i})$$

(2.11)

Therefore, $\alpha_{304}$ depends exactly on $\alpha_{102}$, $\alpha_{103}$, $\alpha_{104}$, $\alpha_{203}$ and $\alpha_{204}$.
By projecting the vector \( a_i \) w.r. to the plane defined by \( \triangle A_1A_0A_3 \) or \( \triangle A_2A_0A_3 \) or \( \triangle A_1A_0A_4 \) or \( \triangle A_2A_0A_4 \) or \( \triangle A_3A_0A_4 \), we get:

\[
\cos^2(\alpha_{i,k0m}) = \frac{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0i}) - \cos^2(\alpha_{k0i}) + 2 \cos(\alpha_{m0i}) \cos(\alpha_{k0i}) \cos(\alpha_{k0m})}{\sin^2(\alpha_{k0m})}
\] (2.12)

Hence, taking into account (2.12) and (2.9) or (2.10) we derive that \( \alpha_{i,k0m} \) depends on \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204} \) and \( \alpha_{205} \).

\[ \square \]

**Proposition 2.** The angles \( \alpha_{i,k0m} \) depend on exactly seven given angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204} \) and \( \alpha_{205} \), for \( i, k, m = 1, 2, 3, 4, 5 \) and \( i \neq k \neq m \).

**Proof.** We consider the directions of five unit vectors which meet a fixed point \( A_0 \).

For instance, we get:

\[
\vec{a}_1 = (1, 0, 0) \quad (2.13)
\]

\[
\vec{a}_2 = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0) \quad (2.14)
\]

\[
\vec{a}_3 = (\cos(\alpha_{3,102}) \cos(\omega_{3,102}), \cos(\alpha_{3,102}) \sin(\omega_{3,102}), \sin(\alpha_{3,102})) \quad (2.15)
\]

\[
\vec{a}_4 = (\cos(\alpha_{4,102}) \cos(\omega_{4,102}), \cos(\alpha_{4,102}) \sin(\omega_{4,102}), \sin(\alpha_{4,102})) \quad (2.16)
\]

\[
\vec{a}_5 = (\cos(\alpha_{5,102}) \cos(\omega_{5,102}), \cos(\alpha_{5,102}) \sin(\omega_{5,102}), \sin(\alpha_{5,102})) \quad (2.17)
\]

such that: \( |\vec{a}_i| = 1 \). The inner product of \( \vec{a}_i, \vec{a}_j \) is:

\[
\vec{a}_i \cdot \vec{a}_j = \cos(\alpha_{i0j}).
\] (2.18)

By following a similar process with the proof of Proposition 1, we obtain that \( \cos(\alpha_{304}) \), \( \cos(\alpha_{305}) \) and \( \cos(\alpha_{405}) \) derived by (2.18) are given by the following six relations which depend on exactly seven angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204} \) and \( \alpha_{205} \):

\[
\cos \alpha_{304} = -\frac{1}{4}[2b_{304} + 4 \cos \alpha_{102} (\cos \alpha_{104} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{204}) -
-4 (\cos \alpha_{103} \cos \alpha_{104} + \cos \alpha_{203} \cos \alpha_{204})] \csc^2 \alpha_{102}
\] (2.19)

or

\[
\cos \alpha_{304} = \frac{1}{4}[4 \cos \alpha_{103} (\cos \alpha_{104} - \cos \alpha_{102} \cos \alpha_{204}) +
+2 (b_{304} + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204}))] \csc^2 \alpha_{102}
\] (2.20)
where

\[ b_{304} \equiv \prod_{i=3}^{4} (1 + \cos (2\alpha_{102}) + \cos (2\alpha_{10i}) + \cos (2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i}) \]

(2.21)

\[ \cos \alpha_{305} = -\frac{1}{4} \left[ 2b_{305} + 4 \cos \alpha_{102} (\cos \alpha_{105} \cos \alpha_{203} + \cos \alpha_{103} \cos \alpha_{205}) - 4 (\cos \alpha_{103} \cos \alpha_{105} + \cos \alpha_{203} \cos \alpha_{205}) \right] \csc^{2} \alpha_{102} \]

or

\[ \cos \alpha_{305} = \frac{1}{4} \left[ 4 \cos \alpha_{103} (\cos \alpha_{105} - \cos \alpha_{102} \cos \alpha_{205}) + 2 \left( b_{305} + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{105} + \cos \alpha_{205}) \right) \right] \csc^{2} \alpha_{102} \]

(2.23)

where

\[ b_{305} \equiv \prod_{i=3, i \neq 4}^{5} (1 + \cos (2\alpha_{102}) + \cos (2\alpha_{10i}) + \cos (2\alpha_{20i}) - 4 \cos \alpha_{102} \cos \alpha_{10i} \cos \alpha_{20i}) \]

(2.24)

and

\[ \cos \alpha_{405} = -\frac{1}{4} \left[ 2b_{405} + 4 \cos \alpha_{102} (\cos \alpha_{105} \cos \alpha_{204} + \cos \alpha_{104} \cos \alpha_{205}) - 4 (\cos \alpha_{104} \cos \alpha_{105} + \cos \alpha_{204} \cos \alpha_{205}) \right] \csc^{2} \alpha_{102} \]

(2.25)

or

\[ \cos \alpha_{405} = \frac{1}{4} \left[ 4 \cos \alpha_{104} (\cos \alpha_{105} - \cos \alpha_{102} \cos \alpha_{205}) + 2 \left( b_{405} + 2 \cos \alpha_{204} (-\cos \alpha_{102} \cos \alpha_{105} + \cos \alpha_{205}) \right) \right] \csc^{2} \alpha_{102} \]

(2.26)

where
\[ b_{305} = \sqrt{\prod_{i=4}^{5} (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4 \cos(2\alpha_{102}) \cos(2\alpha_{10i}) \cos(2\alpha_{20i}))}. \]  

(2.27)

Remark 1. We note that the calculations of formulas of \( \cos(2\alpha_{304}), \cos(2\alpha_{305}), \) and \( \cos(2\alpha_{405}), \) which are derived in \([12, \text{Lemma 1, pp. 16]}\) are corrected and replaced by (2.19), (2.20), (2.22), (2.23), (2.25) and (2.26).

3. A generalization of the inverse weighted Fermat-Torricelli problem in \( \mathbb{R}^3 \).

In this section, we consider mass transportation networks which deal with weighted Fermat-Torricelli networks of degree at most four (or five), in which the weights correspond to an instantaneous collection of images of masses and satisfy some specific conditions.

We denote by \( h_{0,ik} \) the length of the height of \( \triangle A_0A_iA_k \) from \( A_0 \) with respect to \( A_iA_j \), by \( A_{0,ij} \) the intersection of \( h_{0,ij} \) with \( A_iA_j \), and by \( h_{0,ijk} \) the distance of \( A_0 \) from the plane defined by \( \triangle A_iA_jA_k \).

We denote by \( \alpha \), the dihedral angle which is formed between the planes defined by \( \triangle A_1A_2A_3 \) and \( \triangle A_1A_2A_0 \), with \( \alpha_{ij} \), the dihedral angle which is formed by the planes defined by \( \triangle A_1A_2A_i \) and \( \triangle A_1A_2A_0 \), and by \( \alpha_{i,rs} \), the angle which is formed by \( a_{0i} \) and the linear segment which connects \( A_0 \) with the trace from the orthogonal projection of \( a_{0i} \) to the plane defined by \( \triangle A_0A_iA_s \), for \( i, k, l, m, r, s = 0, 1, 2, 3, 4, 5 \).

We proceed by mentioning a fundamental result which we call a geometric plasticity principle of mass transportation networks for boundary closed hexahedra and it is proved in \([11, \text{Appendix A.II}]\) for closed polyhedra in \( \mathbb{R}^3 \).

Proposition 3. \([11, \text{Appendix A.II}]\) Suppose that there is a closed polyhedron \( A_1A_2A_3A_4A_5 \) in \( \mathbb{R}^3 \) and each vertex \( A_i \) has a non-negative weight \( B_i \) for \( i = 1, 2, 3, 4, 5 \). Assume that the floating case of the generalized weighted Fermat-Torricelli point \( A_0 \) point is valid: for each \( A_i \in \{ A_1, A_2, A_3, A_4, A_5 \} \)

\[ \| \sum_{j=1}^{5} B_j \overline{u}(A_i, A_j) \| > B_i, i \neq j. \]

If \( A_0 \) is connected with every vertex \( A_i \) for \( i = 1, 2, 3, 4, 5 \), and a point
A′ is selected with a non-negative weight B$_i$ of the line that is defined by the linear segment $A_0A_i$ and a closed hexahedron $A'_1A'_2...A'_n$ is constructed such that:

$$\| \sum_{j=1}^{5} B_j \vec{u}(A'_i, A'_j) \| > B_i, i \neq j.$$ 

Then the generalized weighted Fermat-Torricelli point $A'_0$ is identical with $A_0$ (geometric plasticity principle).

The geometric plasticity principle of closed hexahedra connects the weighted Fermat-Torricelli problem for closed hexahedra with the modified weighted Fermat-Torricelli problem for boundary closed hexahedra by allowing a mass flow continuity for the weights, such that the corresponding weighted Fermat-Torricelli point remains invariant in $\mathbb{R}^3$.

The modified weighted Fermat-Torricelli problem for closed hexahedra states that:

**Problem 3. Modified weighted Fermat-Torricelli problem**

Let $A_1A_2A_3A_4A_5$ be a closed hexahedron in $\mathbb{R}^3$, $B_i$ be a non-negative number (weight) which corresponds to each linear segment $A_0A_i$, respectively. Find a point $A_0$ which minimizes the sum of the lengths of the linear segments that connect every vertex $A_i$ with $A_0$ multiplied by the positive weight $B_i$:

$$\sum_{i=1}^{5} B_i a_{0i} = \text{minimum.} \quad (3.1)$$

By letting $B_i = B_i$, for $i = 1, 2, 3, 4, 5$ the weighted Fermat-Torricelli problem for closed hexahedra (Problem 1) and the corresponding modified weighted Fermat-Torricelli problem (Problem 3) are equivalent by collecting instantaneous images of the weighted Fermat-Torricelli network via the geometric plasticity principle.

We note that various generalizations of the modified Fermat-Torricelli problem for weighted minimal networks of degree at most three in the sense of the Steiner tree Problem are given in the classical work of A. Ivanov and A. Tuzhilin in [6].

We introduce a mixed weighted Fermat-Torricelli problem in $\mathbb{R}^3$ which may give some new fundamental results in molecular structures and mass transportation networks in a new field that we may call in the future Mathematical Botany and possible applications in the geometry of drug design.

We state the mixed Fermat-Torricelli problem for closed hexahedra in $\mathbb{R}^3$, considering a two way communication weighted network.
Problem 4. Given a boundary closed hexahedron \( A_1A_2A_3A_4A_5 \) in \( \mathbb{R}^3 \) having one interior weighted mobile vertex \( A_0 \) with remaining positive weight \( \bar{B}_0 \) find a connected weighted system of linear segments of shortest total weighted length such that any two of the points of the network can be joined by a polygon consisting of linear segments:

\[
f(X) = \bar{B}_1a_1 + \bar{B}_2a_2 + \bar{B}_3a_3 + \bar{B}_4a_4 + \bar{B}_5a_5 = \text{minimum},
\]

where

\[
B_i + \tilde{B}_i = \bar{B}_i
\]

under the following condition:

\[
\bar{B}_i + \bar{B}_j + \bar{B}_k + \bar{B}_l = \bar{B}_0 + \bar{B}_m
\]

for \( i, j, k, l = 1, 2, 3, 4, 5 \) and \( i \neq j \neq k \neq l \).

The invariance of the mixed weighted Fermat-Torricelli tree of degree at most five is obtained by the inverse mixed weighted Fermat-Torricelli problem for closed hexahedra in \( \mathbb{R}^3 \):

Problem 5. Given a point \( A_0 \) which belongs to the interior of \( A_1A_2A_3A_4A_5 \) in \( \mathbb{R}^3 \), does there exist a unique set of positive weights \( B_i \), such that

\[
B_1 + B_2 + B_3 + B_4 + B_5 = c = \text{const},
\]

for which \( A_0 \) minimizes

\[
f(A_0) = \sum_{i=1}^{5} \bar{B}_ia_{0i}
\]

and

\[
B_i + \tilde{B}_i = \bar{B}_i
\]

under the condition for the weights:

\[
\bar{B}_i + \bar{B}_j + \bar{B}_k + \bar{B}_l = \bar{B}_0 + \bar{B}_m
\]

for \( i, j, k, l, m = 1, 2, 3, 4, 5 \), and \( i \neq j \neq k \neq l \neq m \) (Inverse mixed weighted Fermat-Torricelli problem for closed hexahedra).

Letting \( \bar{B}_5 = 0 \) in Problem 5 we obtain the inverse mixed weighted Fermat-Torricelli problem for tetrahedra.

Theorem 2. Given the mixed weighted Fermat-Torricelli point \( A_0 \) to be an interior point of the tetrahedron \( A_1A_2A_3A_4 \) with the vertices lie on four prescribed rays that meet at \( A_0 \) and from the five given values of \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204} \), the positive real weights \( \bar{B}_i \) given by the formulas

\[
\bar{B}_i + \tilde{B}_i = \bar{B}_i
\]
\[ \bar{B}_1 = \left( \frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}} \right) \frac{c - \bar{B}_0}{2}, \tag{3.7} \]
\[ \bar{B}_2 = \left( \frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} \right) \frac{c - \bar{B}_0}{2}, \tag{3.8} \]
\[ \bar{B}_3 = \left( \frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}} \right) \frac{c - \bar{B}_0}{2} \tag{3.9} \]

and
\[ \bar{B}_4 = \frac{c - \bar{B}_0}{2} \tag{3.10} \]

give a negative answer w.r. to the inverse mixed weighted Fermat-Torricelli problem for tetrahedra for \( i, j, k, m = 1, 2, 3, 4 \) and \( i \neq j \neq k \neq m \).

**Proof.** We denote by \( B_i \) a mass flow which is transferred from \( A_i \) to \( A_0 \) for \( i = 1, 2, 3 \) by \( B_0 \) a residual weight which remains at \( A_0 \) and by \( B_4 \) a mass flow which is transferred from \( A_0 \) to \( A_4 \).

We denote by \( \bar{B}_i \) a mass flow which is transferred from \( A_0 \) to \( A_i \) for \( i = 1, 2, 3 \) by \( \bar{B}_0 \) a residual weight which remains at \( A_0 \) and by \( \bar{B}_4 \) a mass flow which is transferred from \( A_4 \) to \( A_0 \).

Hence, we get:
\[ B_1 + B_2 + B_3 = B_4 + B_0 \tag{3.11} \]

and
\[ \bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_0 = \bar{B}_4. \tag{3.12} \]

By adding (3.11) and (3.12) and by letting \( \bar{B}_0 = B_0 - \bar{B}_0 \) we get:
\[ \bar{B}_1 + \bar{B}_2 + \bar{B}_3 = \bar{B}_4 + \bar{B}_0 \tag{3.13} \]

such that:
\[ \bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = c, \tag{3.14} \]
where \( c \) is a positive real number.

Therefore, the objective function takes the form:
\[ \sum_{i=1}^{4} B_i a_{0i} + \sum_{i=1}^{4} \bar{B}_i a_{0i} = \text{minimum}, \tag{3.15} \]

which yields
\[ \sum_{i=1}^{4} \bar{B}_i a_{0i} = \text{minimum}. \] (3.16)

We start by expressing the lengths \( a_{0i} \), w.r. to \( a_{0j}, a_{0k}, a_{0l} \).

For instance, the lengths \( a_{03} \) and \( a_{04} \) are expressed w.r. to \( a_{01}, a_{02} \) and the dihedral angle \( \alpha \) taking into account the two formulas given in [10], Formulas (2.14), (2.20) p. 116:

\[
a_{03}^2 = a_{02}^2 + a_{23}^2 - 2a_{23} \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{123} + h_{0,12} \sin \alpha_{123} \cos \alpha} \] (3.17)

and

\[
a_{04}^2 = a_{02}^2 + a_{24}^2 - 2a_{24} \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{124} + h_{0,12} \sin \alpha_{124} \cos(\alpha_{g4} - \alpha)} \] (3.18)

By eliminating \( \alpha \) from (3.17) and (3.18) we get:

\[
a_{04}^2 = a_{02}^2 + a_{24}^2 - 2a_{24} \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{124} + h_{0,12} \sin \alpha_{124} \cos(\alpha_{g4} - \alpha)} \] (3.19)

By differentiating (3.19) w.r. to \( a_{01}, a_{02} \) and \( a_{03} \), we obtain:

\[
\frac{\partial a_{04}}{\partial a_{01}} = -\frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}} \] (3.20)

\[
\frac{\partial a_{04}}{\partial a_{02}} = -\frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} \] (3.21)

\[
\frac{\partial a_{04}}{\partial a_{03}} = -\frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}}. \] (3.22)

By differentiating (3.16) w.r. to \( a_{01}, a_{02} \) and \( a_{03} \), and taking into account (3.20), (3.21) and (3.22), we obtain:

\[
\frac{\bar{B}_1}{B_4} = \frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}}. \] (3.23)
\[
\frac{\bar{B}_2}{\bar{B}_4} = \frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} \tag{3.24}
\]

and

\[
\frac{\bar{B}_3}{\bar{B}_4} = \frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}} \tag{3.25}
\]

By following a similar process and by expressing \(a_{0i}\) as a function w.r. to \(a_{0j}\), \(a_{0k}\) and \(a_{0l}\), for \(i, j, k, l = 1, 2, 3, 4\) and \(i \neq j \neq k \neq l\), we get:

\[
\frac{\bar{B}_i}{\bar{B}_j} = \frac{\sin \alpha_{j,k0l}}{\sin \alpha_{i,k0l}} \tag{3.26}
\]

By subtracting (3.13) from (3.14) we obtain (3.10). By replacing (3.10) in (3.23), (3.24) and (3.25) and taking into account Lemma 1 we derive (3.7), (3.8) and (3.9). Therefore, the weights \(\bar{B}_i\) depend on the residual weight \(\bar{B}_0\) and the five given angles \(\alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}\) and \(\alpha_{204}\).

\[\square\]

**Corollary 1.** If \(\alpha_{102} = \alpha_{103} = \alpha_{104} = \alpha_{203} = \alpha_{204} = \arccos \left( -\frac{1}{3} \right) \), \(\bar{B}_0 = \frac{1}{2}\) and \(\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = 1\), then \(\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_4 = \frac{1}{4}\).

**Proof.** By letting

\[
\alpha_{102} = \alpha_{103} = \alpha_{104} = \alpha_{203} = \alpha_{204} = \arccos \left( -\frac{1}{3} \right)
\]

in (2.9) and (2.9), we derive that \(\cos \alpha_{304} = -\frac{1}{3}\) or \(\cos \alpha_{304} = 1\) which yield \(\alpha_{304} = -\arccos \left( \frac{1}{3} \right)\). By replacing \(\bar{B}_0 = \frac{1}{2}\) in (3.7), (3.8), (3.9) and (3.10) we derive \(\bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_4 = \frac{1}{4}\).

\[\square\]

**Corollary 2.** For

\[
\bar{B}_0 = c \left( 1 - \frac{2}{1 + \frac{\sin \alpha_{4,203}}{\sin \alpha_{1,203}} + \frac{\sin \alpha_{4,103}}{\sin \alpha_{2,103}} + \frac{\sin \alpha_{4,102}}{\sin \alpha_{3,102}}} \right), \tag{3.27}
\]

we derive a unique solution

\[
\bar{B}_i = \frac{c}{1 + \frac{\sin \alpha_{j,0k}}{\sin \alpha_{i,0k}} + \frac{\sin \alpha_{j,0l}}{\sin \alpha_{i,0l}} + \frac{\sin \alpha_{k,0l}}{\sin \alpha_{j,0l}}}. \tag{3.28}
\]
for \(i, j, k, l = 1, 2, 3, 4\) and \(i \neq j \neq k \neq l\), which coincides with the unique solution of the inverse weighted Fermat-Torricelli problem for tetrahedra.

**Proof.** By replacing \((3.27)\) in \((3.7), (3.8), (3.9)\) and \((3.10)\) we obtain \((4.8)\), which yields a positive answer to the inverse weighted Fermat-Torricelli problem for tetrahedra in \(\mathbb{R}^3\). □

We proceed by generalizing the equations of (dynamic) plasticity for closed hexahedra, taking into account the residual weight \(B_0\) which exist at the knot \(A_0\), by following the method used in [12, Proposition 1, p. 17].

We set \(\text{sgn}_{i,j,k,l} = \begin{cases} +1, & \text{if } A_i \text{ is upper from the plane } A_j A_0 A_k, \\ 0, & \text{if } A_i \text{ belongs to the plane } A_j A_0 A_k, \\ -1, & \text{if } A_i \text{ is under the plane } A_j A_0 A_k, \end{cases}\)

with respect to an outward normal vector \(N_{j,k,l}\) for \(i, j, k = 1, 2, 3, 4, 5\), \(i \neq j \neq k\). We remind that the position of an arbitrary directed plane is determined by the outward normal and the distance from the weighted Fermat-Torricelli point \(A_0\).

**Proposition 4.** The following equations point out a new plasticity of mixed weighted closed hexahedra with respect to the non-negative variable weights \((B_i)_{12345}\) in \(\mathbb{R}^3\):

\[
\begin{align*}
\left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{12345} &= -\left(\frac{\text{sgn}_{4,203}}{\text{sgn}_{1,203}}\right)\left(\frac{\bar{B}_1}{\bar{B}_4}\right)_{12345}(1 + \frac{\text{sgn}_{5,203}}{\text{sgn}_{4,203}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_1}{\bar{B}_5}\right)_{12345}) \\
\left(\frac{\bar{B}_2}{\bar{B}_4}\right)_{12345} &= -\left(\frac{\text{sgn}_{4,103}}{\text{sgn}_{2,103}}\right)\left(\frac{\bar{B}_2}{\bar{B}_4}\right)_{12345}(1 + \frac{\text{sgn}_{5,103}}{\text{sgn}_{4,103}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_1}{\bar{B}_5}\right)_{12345}) \\
\left(\frac{\bar{B}_3}{\bar{B}_4}\right)_{12345} &= -\left(\frac{\text{sgn}_{4,102}}{\text{sgn}_{3,102}}\right)\left(\frac{\bar{B}_3}{\bar{B}_4}\right)_{12345}(1 + \frac{\text{sgn}_{5,102}}{\text{sgn}_{4,102}}\left(\frac{\bar{B}_5}{\bar{B}_4}\right)_{12345}\left(\frac{\bar{B}_1}{\bar{B}_5}\right)_{12345})
\end{align*}
\]

under the conditions

\[
\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 + \bar{B}_5 = c = \text{constant} \quad (3.32)
\]

and

\[
\bar{B}_1 + \bar{B}_2 + \bar{B}_3 + \bar{B}_4 = \bar{B}_0 + \bar{B}_1 \quad (3.33)
\]

where the weight \((B_i)_{12345}\) corresponds to the vertex that lies on the ray \(A_0 A_i\), for \(i = 1, 2, 3, 4, 5\), and the weight \((B_j)_{j,k,m}\) corresponds to the vertex \(A_j\) that lies in the ray \(A_0 A_j\) regarding the tetrahedron \(A_jA_kA_iA_m\), for \(j, k, l, m = 1, 2, 3, 4, 5\) and \(j \neq k \neq l \neq m\).
Proof. By eliminating $\bar{B}_1, \bar{B}_2, \bar{B}_3$ and $\bar{B}_5$ from (3.32) and (3.34) we get:

$$\bar{B}_4 = \frac{c - \bar{B}_0}{2}$$

(3.34)

We assume that the residual weight $\bar{B}_0$ could be split at the mixed weighted Fermat-Torricelli trees of degree four at $A_0$, such that the residual weights $\bar{B}_0,2345$ and $\bar{B}_0,1345$ and $\bar{B}_0,1245$ correspond to the boundary tetrahedra $A_2A_3A_4A_5$, $A_1A_3A_4A_5$ and $A_1A_2A_4A_5$.

We select five initial (given) values $(\bar{B}_i)_{12345}(0)$ concerning the weights $(\bar{B}_i)_{12345}$ for $i = 1, 2, 3, 4, 5$ such that the mixed weighted Fermat-Torricelli point $A_0$ exists and it is located at the interior of $A_1A_2A_3A_4A_5$.

By applying the method used in the proof of Theorem 2, the length of the linear segments $a_{04}$, $a_{05}$ can be expressed as functions of $a_{01}$, $a_{02}$ and $a_{03}$:

$$a_{0i} = a_{02}^2 + a_{03}^2 - 2a_{01}a_{02}\sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{12}}$$

$$+ h_{0,12} \sin \alpha_{12} \cos \alpha_{g_i} \left( \frac{a_{02}^2 + a_{03}^2 - a_{0i}^2}{2a_{02}} - \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{12}} \right) +$$

$$+ \sin \alpha_{g_i} \sin \arccos \left( \frac{a_{02}^2 + a_{03}^2 - a_{0i}^2}{2a_{02}} - \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{12}} \right) \right)$$

(3.35)

for $i = 4, 5$.

From (3.35), we get:

$$B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04}(a_{01}, a_{02}, a_{03}) + B_5a_{05}(a_{01}, a_{02}, a_{03}) = \text{minimum}.$$  

(3.36)

By differentiating (3.36) with respect to $a_{01}, a_{02}$ and $a_{03}$ we get:

$$B_1 + \frac{\partial a_{04}}{\partial a_{01}} + \frac{\partial a_{05}}{\partial a_{01}} = 0.$$  

(3.37)

$$B_2 + \frac{\partial a_{04}}{\partial a_{02}} + \frac{\partial a_{05}}{\partial a_{02}} = 0.$$  

(3.38)

$$B_3 + \frac{\partial a_{04}}{\partial a_{03}} + \frac{\partial a_{05}}{\partial a_{03}} = 0.$$  

(3.39)

By differentiating (3.35) w.r. to $a_{03}$ and by replacing $\frac{\partial a_{04}}{\partial a_{03}}$ for $i = 4, 5$ in (3.39), we obtain:
\[
\frac{\vec{B}_3}{\vec{B}_4}_{12345} = -\frac{\text{sgn}_{4,102}}{\text{sgn}_{3,102}} \sin(\alpha_{4,102}) \left(1 + \frac{\vec{B}_5}{\vec{B}_4}_{12345} \frac{\text{sgn}_{5,102}}{\text{sgn}_{4,102}} \sin(\alpha_{5,102})\right).
\]

Taking into account the solution of the inverse mixed weighted Fermat-Torricelli problem for boundary tetrahedra we derive (3.31). Following a similar evolutionary process, we derive (3.30) and (3.29).

From Lemma 1, the variable weights \((\vec{B}_1)_{12345}, (\vec{B}_2)_{12345},\) and \((\vec{B}_3)_{12345},\) depend on the weight \((\vec{B}_5)_{12345},\) the residual weight \(\vec{B}_0\) and the seven given angles \(\alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{105}, \alpha_{203}, \alpha_{204}\) and \(\alpha_{205}\).

**Remark 2.** We note that numerical examples of the plasticity of tetragonal pyramids are given in [11, Examples 3.4, 3.7, p. 844-847].

4. **A generalization of the inverse weighted Fermat-Torricelli problem in \(\mathbb{R}^2\)**

The inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in \(\mathbb{R}^2\) states that:

**Problem 6.** Given a point \(A_0\) which belongs to the interior of \(\triangle A_1A_2A_3\) in \(\mathbb{R}^2,\) does there exist a unique set of positive weights \(\vec{B}_i,\) such that

\[
\vec{B}_1 + \vec{B}_2 + \vec{B}_3 = c = \text{const},
\]

for which \(A_0\) minimizes

\[
f(A_0) = \sum_{i=1}^{3} \vec{B}_i a_{0i}
\]

and

\[
\vec{B}_i + \tilde{\vec{B}}_i = \vec{B}_i
\]

under the condition for the weights:

\[
\vec{B}_i + \vec{B}_j = \vec{B}_0 + \vec{B}_k
\]

for \(i, j, k = 1, 2, 3\) and \(i \neq j \neq k\) (Inverse mixed weighted Fermat-Torricelli problem for three non-collinear points).

**Theorem 3.** Given the mixed weighted Fermat-Torricelli point \(A_0\) to be an interior point of the triangle \(\triangle A_1A_2A_3\) with the vertices lie on three prescribed rays that meet at \(A_0\) and from the two given values of \(\alpha_{102}, \alpha_{103},\) the positive real weights \(\vec{B}_i\) given by the formulas
\[ B_1 = -\left( \frac{\sin(\alpha_{103} + \alpha_{102})}{\sin \alpha_{102}} \right) \frac{c - \bar{B}_0}{2}, \quad (4.4) \]

\[ \bar{B}_2 = \left( \frac{\sin \alpha_{103}}{\sin \alpha_{102}} \right) \frac{c - \bar{B}_0}{2}, \quad (4.5) \]

and

\[ \bar{B}_3 = \frac{c - \bar{B}_0}{2} \quad (4.6) \]

give a negative answer w.r. to the inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in \( \mathbb{R}^2 \).

**Proof.** Eliminating \( \bar{B}_1 \) and \( \bar{B}_2 \) from (4.1) and (4.3) we get (4.6). By setting \( \alpha_{g_i} = \alpha \) in (3.17), we derive that \( a_{03} = a_{03}(a_{01}, a_{02}) \). By differentiating \( a_{03} = a_{03}(a_{01}, a_{02}) \) w.r. to \( a_{0i} \) and by replacing \( \frac{\partial a_{03}}{\partial a_{0i}} \) for \( i = 1, 2 \) and setting \( B_4 = 0 \) in (3.16) we obtain (4.4) and (4.5).

\[ \Box \]

**Corollary 3.** If \( \alpha_{102} = \alpha_{103} = 120^\circ \), \( \bar{B}_0 = \frac{1}{3} \) and

\[ \bar{B}_1 + \bar{B}_2 + \bar{B}_3 = 1, \]

then \( \bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \bar{B}_0 = \frac{1}{3} \).

**Proof.** By letting

\[ \alpha_{102} = \alpha_{103} = 120^\circ \]

we have:

\[ \alpha_{203} = 2\pi - \alpha_{102} - \alpha_{103} = 120^\circ \]

By replacing \( \bar{B}_0 = \frac{1}{3} \) and \( c = 1 \) in (4.4), (4.5) and (4.6) we derive \( \bar{B}_1 = \bar{B}_2 = \bar{B}_3 = \frac{1}{3} \).

\[ \Box \]

**Corollary 4.** For

\[ \bar{B}_0 = c \left( 1 - \frac{2}{1 - \left( \frac{\sin(\alpha_{103} + \alpha_{102})}{\sin \alpha_{102}} + \frac{\sin \alpha_{103}}{\sin \alpha_{102}} \right)} \right), \quad (4.7) \]

we derive a unique solution

\[ \bar{B}_i = \frac{c}{1 + \frac{\sin \alpha_{j0k}}{\sin \alpha_{j0} + \frac{\sin \alpha_{j0k}}{\sin \alpha_{j0k}}}}, \quad (4.8) \]

for \( i, j, k = 1, 2, 3 \), and \( i \neq j \neq k \), which coincides with the unique solution of the inverse weighted Fermat-Torricelli problem for three non-collinear points.
Proof. By replacing (4.7) in (4.4), (4.5) and (4.6) we obtain (4.8), which yields a positive answer to the inverse weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$. □

Proposition 5. Let $\triangle A_1A_2A_3$ be a triangle in $\mathbb{R}^2$. If

$$\|\vec{B}_1\vec{u}(A_3, A_1) + \vec{B}_2\vec{u}(A_3, A_2)\| \leq \vec{B}_3$$

(4.9)

and

$$\vec{B}_1 + \vec{B}_2 = \vec{B}_3 + \vec{B}_0$$

(4.10)

holds, then the solution w.r. to the inverse mixed weighted Fermat-Torricelli problem for three non-collinear points in $\mathbb{R}^2$ for the weighted absorbed case is not unique.

Proof. Suppose that we choose three initial weights $\vec{B}_i(0) \equiv \vec{B}_i$, such that (4.9) holds. From Theorem 4 the weighted absorbed case occurs and the mixed weighted Fermat-Torricelli point $A_0 \equiv A_3$. Hence, if we select a new weight $\vec{B}_3 + \vec{B}_0$ which remains at the knot $A_3$, then (4.9) also holds and the corresponding mixed weighted Fermat-Torricelli point remains the same $A'_0 \equiv A_3$. □

Remark 3. Proposition 5 generalizes the inverse weighted Fermat-Torricelli problem for three non-collinear points in the weighted absorbed case.

Setting a condition with respect to the specific dihedral angles $\alpha_{g_3} = \alpha_{g_4} = \alpha$ we obtain quadrilaterals as a limiting case of tetrahedra on the plane defined by $\triangle A_1A_0A_2$. These equations are important, in order to derive a new plasticity for weighted quadrilaterals in $\mathbb{R}^2$, where the weighted floating case of Theorem 4 occurs.

Theorem 4. If $\alpha_{g_3} = \alpha_{g_4} = \alpha$ then the following four equations point out the mixed dynamic plasticity of convex quadrilaterals in $\mathbb{R}^2$:

$$\left(\frac{\vec{B}_2}{\vec{B}_1}\right)_{1234} = \left(\frac{\vec{B}_2}{\vec{B}_1}\right)_{123}\left[1 - \left(\frac{\vec{B}_1}{\vec{B}_1}\right)_{1234}\left(\frac{\vec{B}_1}{\vec{B}_1}\right)_{134}\right],$$

(4.11)

$$\left(\frac{\vec{B}_3}{\vec{B}_1}\right)_{1234} = \left(\frac{\vec{B}_3}{\vec{B}_1}\right)_{123}\left[1 - \left(\frac{\vec{B}_1}{\vec{B}_1}\right)_{1234}\left(\frac{\vec{B}_1}{\vec{B}_1}\right)_{124}\right],$$

(4.12)

$$(\vec{B}_1)_{1234} + (\vec{B}_2)_{1234} + (\vec{B}_3)_{1234} + (\vec{B}_4)_{1234} = c = \text{constant}.\quad (4.13)$$

and

$$(\vec{B}_1)_{1234} + (\vec{B}_2)_{1234} + (\vec{B}_3)_{1234} = (\vec{B}_4)_{1234} + (\vec{B}_0)_{1234}.$$

(4.14)
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