Arithmetic progressions of three prime numbers with two primes of the form 

\[ p = x^2 + y^2 + 1 \]

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Abstract

In the present paper we prove that there exist infinitely many arithmetic progressions of three different primes \( p_1, p_2, p_3 = 2p_2 - p_1 \) such that \( p_1 = x_1^2 + y_1^2 + 1 \), \( p_2 = x_2^2 + y_2^2 + 1 \).

Keywords: Arithmetic progression, Prime numbers, Circle method.

Notations. The letter \( p \), with or without subscript, will always denote prime numbers. By \( \varepsilon \) we denote an arbitrary small positive number, not the same in all appearances. \( X \) is a sufficiently large positive number and \( \mathcal{L} = \log X \). We denote by \( J \) the set of all subintervals of the interval \((X/2, X]\) and if \( J_1, J_2 \in J \) then \( J = \langle J_1, J_2 \rangle \) is the corresponding ordered pair. Respectively \( d = \langle d_1, d_2 \rangle \) is a two-dimensional vector with integer components \( d_1, d_2 \) and in particular \( 1 = \langle 1, 1 \rangle \). We denote by \( (m, n) \) the greatest common divisor of \( m \) and \( n \). As usual \( \phi(d) \) is Euler’s function; \( \tau(d) \) is the number of positive divisors of \( d \); \( r(d) \) is the number of solutions of the equation \( d = m_1^2 + m_2^2 \) in integers \( m_j \); \( \chi(d) \) is the non-principal character modulo 4 and \( L(s, \chi) \) is the corresponding Dirichlet’s \( L \)-function.

1 Introduction and statement of the result.

In 1939 Van der Corput \[9\] proved that there exist infinitely many arithmetic progressions of three different primes. On the other hand Linnik \[3\] has proved that there exist infinitely many prime numbers of the form \( p = x^2 + y^2 + 1 \), where \( x \) and \( y \) – integers. More precisely he has proved the asymptotic formula

\[
\sum_{p \leq X} r(p - 1) = \pi X \mathcal{L}^{-1} \prod_{p > 2} \left( 1 + \frac{\chi(p)}{p(p - 1)} \right) + \mathcal{O} \left( X \mathcal{L}^{-1/2 + \epsilon} (\log \mathcal{L})^3 \right),
\]

(1)
where
\[
\theta_0 = \frac{1}{2} - \frac{1}{4} \log 2 = 0.0289...
\] (2)

We couple these two theorems by proving the following theorem.

Define
\[
R(X) = \sum_{\substack{x/2 < p_1, p_2, p_3 \leq X \\ p_1 + p_3 = 2p_2}} r(p_1 - 1) r(p_2 - 1) \log p_1 \log p_2 \log p_3
\] (3)

and
\[
\sigma_0 = 2 \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right),
\] (4)
\[
\mathcal{S}_R = \pi^2 \sigma_0 \prod_p \left( 1 + \chi(p) \frac{2p^2 + p\chi(p) - 4p + 2\chi(p)}{p^2(p-1)(p-2)} \right).
\] (5)

**Theorem 1.** We have the following asymptotic formula
\[
R(X) = \frac{1}{8} \mathcal{S}_R X^2 + O \left( X^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^7 \right),
\] (6)

where \( \theta_0 \) is denoted by (2).

We use sieve methods to pick out primes of the form \( x^2 + y^2 + 1 \) and the circle method to pick out primes satisfying \( p_1 + p_3 = 2p_2 \). Recently Matomäki \[4\] and Tolev \[7\] have obtained a similar results related to a binary Goldbach problem. Tolev \[8\] has also proved that every sufficiently large odd integer can be represented as a sum of three primes, two of which of the form \( x^2 + y^2 + 1 \). Our argument is a modification of Tolev’s \[8\] argument.

## 2 Some lemmas.

First we consider the binary Goldbach problem with one prime variable lying in a given interval and belonging to an arithmetic progression. Suppose that \( n \leq 2X \), let \( d \) and \( l \) be integers with \( (d, l) = 1 \) and let \( J \in \mathcal{J} \).
Denote
\[ \lambda(n) = \prod_{\substack{p \mid n \\mid \text{p}^2 > \text{p}\geq 2}} \frac{p - 1}{p - 2}; \]  
\[ \mathcal{S}_{d,l}(n) = \begin{cases} \sigma_0 \lambda(nd) & \text{if } (d, n - l) = 1 \text{ and } 2 \mid n, \\ 0 & \text{otherwise}; \end{cases} \]
\[ I_{d,l}^{(1)}(n, X, J) = \sum_{\substack{X/2 < p_1 \leq X \\ 2p_2 - p_1 = n \\ p_2 \equiv l \pmod{d}}} \log p_1 \log p_2; \]
\[ \Phi^{(1)}(n, X, J) = \sum_{\substack{X/2 < m_1 \leq X \\ 2m_2 - m_1 = n \\ m_2 \in J}} 1; \]
\[ \Delta_{d,l}^{(1)}(n, X, J) = I_{d,l}^{(1)}(n, X, J) - \mathcal{S}_{d,l}(n) \frac{\phi(d)}{\varphi(d)} \Phi^{(1)}(n, X, J); \]
\[ I_{d,l}^{(2)}(n, X, J) = \sum_{\substack{X/2 < p_1 \leq X \\ p_1 + p_2 = n \\ p_2 \equiv l \pmod{d}}} \log p_1 \log p_2; \]
\[ \Phi^{(2)}(n, X, J) = \sum_{\substack{X/2 < m_1 \leq X \\ m_1 + m_2 = n \\ m_2 \in J}} 1; \]
\[ \Delta_{d,l}^{(2)}(n, X, J) = I_{d,l}^{(2)}(n, X, J) - \mathcal{S}_{d,l}(n) \frac{\phi(d)}{\varphi(d)} \Phi^{(2)}(n, X, J). \]

If \( J = (X/2, X] \) then we write for simplicity \( I_{d,l}^{(j)}(n, X), \Phi^{(j)}(n, X) \) and \( \Delta_{d,l}^{(j)}(n, X) \) for \( j = 1, 2 \).

The following Bombieri-Vinogradov type result gives the arithmetical information needed for the applications of the sieve.

**Lemma 1.** For any constant \( A > 0 \) there exists \( B = B(A) > 0 \) such that
\[ \sum_{d \leq \sqrt{X\mathcal{L}^{-B}}} \max_{(d,l)=1} \max_{J \in J} \sum_{n \leq X} |\Delta_{d,l}^{(j)}(n, X, J)| \ll X^2 \mathcal{L}^{-A}, \ j = 1, 2. \]

This lemma is a very similar to the result of Laporta [2]. This author studies the equation \( p_1 - p_2 = n \) and without the condition \( p_1 \in J \). However, inspecting the arguments presented in [2], the reader will readily see that the proof of Lemma 1 can be obtained in the same way.

Next we consider the equation \( p_1 + p_3 = 2p_2 \) with two primes from arithmetic progressions and belonging to given intervals. Suppose that \( d = \langle d_1, d_2 \rangle \) and \( l = \langle l_1, l_2 \rangle \) are
two-dimensional vectors with integer components such that \((d_i, l_i) = 1, \ i = 1, 2\) and let \(J = (J_1, J_2)\) be a pair of intervals \(J_1, J_2 \in \mathcal{J}\).

Denote

\[
I_{d, l}^{(3)}(X, J) = \sum_{\frac{X}{2} < p_3 \leq X \atop \frac{p_1 + p_2}{d_i} = l_i \ (\text{mod} \ d_i) \atop p_i \in J_i, i = 1, 2} \log p_1 \log p_2 \log p_3;
\]  

(15)

\[
\Phi^{(3)}(X, J) = \sum_{\frac{X}{2} < m_3 \leq X \atop m_1 \in J_1, i = 1, 2} 1.
\]  

(16)

Next we define \(\mathcal{G}_{d, l}^{(3)}\) in the following way. Consider the sets of primes

\[\begin{align*}
A &= \{p : p \nmid d_1 d_2\}; \\
B &= \{p : p \mid d_1, p \nmid d_2\} \cup \{p : p \nmid d_1, p \mid d_2\}; \\
C &= \{p : p \mid d_1, p \mid d_2, p \mid (l_1 - 2l_2)\}; \\
D &= \{p : p \mid d_1, p \mid d_2, p \nmid (l_1 - 2l_2)\}.
\end{align*}\]

If \(C \neq \emptyset\) then we assume that

\[
\mathcal{G}_{d, l}^{(3)} = 0.
\]  

(17)

If \(C = \emptyset\) then we put

\[
\mathcal{G}_{d, l}^{(3)} = 2 \prod_{p \in A \cup B} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{p \in D} \left(1 + \frac{1}{p - 1}\right).
\]  

(18)

We also define

\[
\Delta_{d, l}^{(3)}(X, J) = I_{d, l}^{(3)}(X, J) - \frac{\mathcal{G}_{d, l}^{(3)}}{\phi(d_1) \phi(d_2)} \Phi^{(3)}(X, J).
\]  

(19)

If \(J_1 = J_2 = (X/2, X]\) then we write for simplicity \(I_{d, l}^{(3)}(X), \Phi^{(3)}(X)\) and \(\Delta_{d, l}^{(3)}(X)\).

The next lemma is analogous to Lemma 1 and also gives the arithmetical information needed for the applications of the sieve.

**Lemma 2.** For any constant \(A > 0\) there exists \(B = B(A) > 0\) such that

\[
\sum_{d_1 \leq \sqrt{XL^{-C}}} \sum_{d_2 \leq \sqrt{XL^{-C}}} \max_{(d_i, l_i) = 1 \atop i = 1, 2} \max_{J_i \in \mathcal{J}} \left| \Delta_{d, l}^{(3)}(X, J) \right| \ll X^2 L^{-A}.
\]

This assertion is a slight generalization to the theorem of Peneva and Tolev [5]. In their theorem \((d_1d_2, 2) = 1, l_1 = l_2 = -2\) and there are no conditions \(p_i \in J_i\). But it is not hard to verify that the method used in this paper implies also the correctness of Lemma 2.

Several times we shall use the following
Lemma 3. Suppose that $j \in \{-1, 1\}$ and let $d, l, m$ be natural numbers. Then the quantities $S_{4m,1+jm}(n)$ and $S_{(d,4m),l,1+jm}^{(3)}$ do not depend on $j$.

The proof is a immediate consequence from the definitions of $S_{d,l}(n)$ and $S_{d,l}^{(3)}$.

The next two lemmas are due to C. Hooley.

Lemma 4. For any constant $\omega > 0$ we have

$$\sum_{p \leq X} \left| \sum_{\sqrt{X}L^{-\omega} < d < \sqrt{X}L^\omega} \chi(d) \right| \ll XL^{-1-\theta_0}(\log L)^5,$$

where $\theta_0$ is defined by (2). The constant in the Vinogradov symbol depends only on $\omega > 0$.

Lemma 5. For any constant $\omega > 0$ we have

$$\sum_{p \leq X} \left| \sum_{\sqrt{X}L^{-\omega} < d < \sqrt{X}L^\omega} \chi(d) \right|^2 \ll XL^{-1}(\log L)^7,$$

where the constant in the Vinogradov symbol depends on $\omega > 0$.

The proofs of very similar results are available in [1], Ch.5.

The next lemma is due to Tolev [8, Lemma 6].

Lemma 6. Let $n$ be an integer satisfying $1 \leq n \leq X$. Suppose that $\omega > 0$ is a constant and let $\mathcal{P} = \mathcal{P}_\omega(X)$ be the set of primes $p \leq X$ such that $p - 1$ has a divisor lying between $\sqrt{X}L^{-\omega} < d < \sqrt{X}L^\omega$. Then we have

$$\sum_{p_1 + p_2 = n \atop p_1 \in \mathcal{P}} 1 \ll XL^{-2-2\theta_0}(\log L)^6,$$

where $\theta_0$ is defined by (2). The constant in the Vinogradov’s symbol depends only on $\omega > 0$.

Arguing as in Lemma 6 we obtain similar result.

Lemma 7. Let $n$ be an integer satisfying $1 \leq n \leq X$. Suppose that $\omega > 0$ is a constant and let $\mathcal{P} = \mathcal{P}_\omega(X)$ be the set of primes $p \leq X$ such that $p - 1$ has a divisor lying between $\sqrt{X}L^{-\omega} < d < \sqrt{X}L^\omega$. Then we have

$$\sum_{2p_1 - p_2 = n \atop p_1 \in \mathcal{P}} 1 \ll XL^{-2-2\theta_0}(\log L)^6,$$

where $\theta_0$ is defined by (2). The constant in the Vinogradov’s symbol depends only on $\omega > 0$. 
3 Proof of Theorem 1.

Beginning. Denote

\[ D = \sqrt{X} \mathcal{L}^{-B(10)-C(10)}, \]  

(20)

where \( B(A) \) and \( C(A) \) are specified respectively in Lemma 1 and Lemma 2. Obviously

\[ r(m) = 4 \sum_{d|m} \chi(d) = 4\left(r_1(m) + r_2(m) + r_3(m)\right), \]  

(21)

where

\[ r_1(m) = \sum_{d|m, \, d \leq D} \chi(d), \quad r_2(m) = \sum_{d|m, \, D < d < X/D} \chi(d), \quad r_3(m) = \sum_{d|m, \, d \geq X/D} \chi(d). \]  

(22)

Using (3) and (21) we find

\[ R(X) = 16 \sum_{1 \leq i, j \leq 3} R_{i,j}(X), \]  

(23)

where

\[ R_{i,j}(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} r_i(p_1 - 1)r_j(p_2 - 1) \log p_1 \log p_2 \log p_3. \]  

(24)

We shall prove that the main term in (6) comes from \( R_{1,1}(X) \) and the other sums \( R_{i,j}(X) \) contribute only to the remainder term.

The estimation of \( R_{1,1}(X) \). Using (16), (19), (22) and (24) we obtain

\[ R_{1,1}(X) = \sum_{d_1, d_2 \leq D} \chi(d_1)\chi(d_2)I_{d,1}^{(3)}(X) = R_{1,1}'(X) + R_{1,1}^*(X), \]  

(25)

where

\[ R_{1,1}'(X) = \Phi^{(3)}(X) \sum_{d_1, d_2 \leq D} \frac{\chi(d_1)\chi(d_2)}{\varphi(d_1)\varphi(d_2)}S_{d,1}^{(3)}, \]  

(26)

\[ R_{1,1}^*(X) = \sum_{d_1, d_2 \leq D} \chi(d_1)\chi(d_2)\Delta_{d,1}^{(3)}(X). \]  

(27)

From (20), (27) and Lemma 2 we have that

\[ R_{1,1}^*(X) \ll X^2\mathcal{L}^{-1}. \]  

(28)

Consider \( R_{1,1}'(X) \). From (16) and (26) it follows that

\[ R_{1,1}'(X) = \frac{1}{8}X^2\Gamma(X) + \mathcal{O}\left(X^{1+\epsilon}\right), \]  

(29)
where

\[ \Gamma(X) = \sum_{d_1, d_2 \leq D} \frac{\chi(d_1)\chi(d_2)}{\varphi(d_1)\varphi(d_2)} S_{d_1, d_2}^{(3)}. \]  

(30)

We shall find an asymptotic formula for \( \Gamma(X) \).

Using (17), (18) and (30) we get

\[ \Gamma(X) = \sigma_0 \sum_{d \leq D} \frac{\chi(d)}{\varphi(d)} \sum_{t \leq D} f_d(t), \]

(31)

where

\[ f_d(t) = \frac{\chi(t)}{\varphi(t)} \prod_{p | (d, t)} \frac{p - 1}{p - 2}. \]

(32)

First we estimate the sum over \( t \) in (31). From (32) we have that

\[ f_d(t) \ll (\log L)t^{-1}\log\log(10t) \]

(33)

with absolute constant in the Vinogradov’s symbol. Thus the corresponding Dirichlet series

\[ F_d(s) = \sum_{t=1}^{\infty} f_d(t)t^{-s} \]

is absolutely convergent in \( \Re(s) > 0 \). On the other hand \( f_d(t) \) is a multiplicative with respect to \( t \) and applying Euler’s identity we find

\[ F_d(s) = \prod_p T_d(p, s), \quad T_d(p, s) = 1 + \sum_{l=1}^{\infty} f_d(p^l)p^{-ls}. \]

(34)

From (32) and (34) it follows that

\[ T_d(p, s) = (1 - \chi(p)p^{-s-1})^{-1} (1 + \chi(p)p^{-s-1}Y_d(p)), \]

where

\[ Y_d(p) = \begin{cases} (p - 1)^{-1} & \text{if } p \nmid d, \\ 2(p - 2)^{-1} & \text{if } p | d. \end{cases} \]

(35)

Hence we obtain

\[ F_d(s) = L(s + 1, \chi) \prod_p (1 + \chi(p)p^{-s-1}Y_d(p)). \]

(36)

From this formula it follows that \( F_d(s) \) has an analytic continuation to \( \Re(s) > -1 \). Using (35), (36) and the simplest bound for \( L(s + 1, \chi) \) we find

\[ F_d(s) \ll X^{1/6} \quad \text{for } \Re(s) \geq -\frac{1}{2}, \quad |\text{Im}(s)| \leq X. \]

(37)
We apply Perron’s formula given at Tenenbaum ([6], Chapter II.2) and also (33) to get

\[ \sum_{t \leq D} f_d(t) = \frac{1}{2\pi i} \int_{\kappa-iX}^{\kappa+iX} F_d(s) \frac{D^s}{s} ds + O \left( \sum_{t=1}^{\infty} X^\epsilon D^\kappa \log \log(10t) \right). \] (38)

where \( \kappa = 1/10. \) It is easy to see that the error term above is \( O \left( X^{-1/20} \right). \) Applying the residue theorem we see that the main term is equal to

\[ F_d(0) + \frac{1}{2\pi i} \left( \int_{1/10-iX}^{1/10+iX} + \int_{-1/2-iX}^{-1/2+iX} + \int_{-1/2+iX}^{-1/2-iX} \right) F_d(s) \frac{D^s}{s} ds. \]

From (37) it follows that the contribution from the above integrals is \( O \left( X^{-1/20} \right). \) Hence

\[ \sum_{t \leq D} f_d(t) = F_d(0) + O \left( X^{-1/20} \right). \] (39)

Using (36) we obtain

\[ F_d(0) = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi(p)}{p} Y_d(p) \right). \] (40)

Having in mind (31), (35), (39) and (40) we establish that

\[ \Gamma(X) = \frac{\pi}{4} \sigma_0 \mathfrak{X} \sum_{d \leq D} g(d) + O \left( X^{\varepsilon-1/20} \right). \] (41)

where \( \sigma_0 \) is defined by (41),

\[ \mathfrak{X} = \prod_p \left( 1 + \frac{\chi(p)}{p(p-1)} \right), \] (42)

and

\[ g(d) = \frac{\chi(d)}{\varphi(d)} \prod_{p|d} \frac{1 + \frac{2\chi(p)}{p(p-2)}}{1 + \frac{\chi(p)}{p(p-1)}}. \] (43)

Obviously \( g(d) \) is multiplicative with respect to \( d \) and satisfies

\[ g(d) \ll d^{-1} \log \log(10d), \] (44)

where the constant in Vinogradov’s symbol is absolute. Thus the Dirichlet series

\[ G(s) = \sum_{d=1}^{\infty} g(d)d^{-s} \]
is absolutely convergent in $Re(s) > 0$ and applying the Euler’s identity we find

$$G(s) = \prod_p H(p, s), \quad H(p, s) = 1 + \sum_{l=1}^{\infty} g(p^l) p^{-ls}. \quad (45)$$

From (43) and (45) we get

$$H(p, s) = \left(1 - \chi(p)p^{-s-1}\right)^{-1}\left(1 + \chi(p)p^{-s-1}K(p)\right),$$

where

$$K(p) = \frac{p^2 + p\chi(p) - 2p + 2\chi(p)}{p^3 - 3p^2 + p\chi(p) + 2p - 2\chi(p)}. \quad (46)$$

This implies

$$G(s) = L(s + 1, \chi) \prod_p \left(1 + \chi(p)p^{-s-1}K(p)\right). \quad (47)$$

We see that $G(s)$ has an analytic continuation to $Re(s) > -1$ and

$$G(s) \ll X^{1/6} \quad \text{for} \quad Re(s) \geq -\frac{1}{2}, \quad |Im(s)| \leq X. \quad (48)$$

Applying Perron’s formula and proceeding as above we obtain

$$\sum_{d \leq D} g(d) = G(0) + \mathcal{O}(X^{-1/20}) = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi(p)}{p} K(p)\right) + \mathcal{O}(X^{-1/20}). \quad (49)$$

Using (41), (42), (46) and (49) we get

$$\Gamma(X) = \frac{1}{16} \mathcal{S}_R + \mathcal{O}(X^{\varepsilon-1/20}). \quad (50)$$

where $\mathcal{S}_R$ is defined by (5).

Now bearing in mind (25), (28), (29) and (50) we find

$$R_{1,1}(X) = \frac{1}{128} \mathcal{S}_R X^2 + \mathcal{O}(X^2 \mathcal{L}^{-1}). \quad (51)$$

**The estimation of $R_{1,2}(X)$.** Using (12) – (14), (22) and (24) we obtain

$$R_{1,2}(X) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{d \leq D} \chi(d) I_{d,1}^{(2)}(2p, X) = R'_{1,2}(X) + R^*_1,2(X), \quad (52)$$

where

$$R'_{1,2}(X) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{d \leq D} \frac{\chi(d)}{\phi(d)} \mathcal{S}_{d,1}(2p) \Phi^{(2)}(2p, X), \quad (53)$$

$$R^*_{1,2}(X) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{d \leq D} \chi(d) \Delta^{(2)}_{d,1}(2p, X). \quad (54)$$
From (22), (54) and Cauchy’s inequality it follows
\[ R^{*}_{1,2}(X) \ll \mathcal{L} \sum_{X/2 < p \leq X} \tau(p - 1) \sum_{d \leq D} |\Delta^{(2)}_{d,1}(2p, X)| \]
\[ \ll \mathcal{L} \sum_{X < n \leq 2X} \tau(n) \sum_{d \leq D} |\Delta^{(2)}_{d,1}(n, X)| \]
\[ \ll \mathcal{L} \left( \sum_{X < n \leq 2X} \sum_{d \leq D} \tau^{2}(n)|\Delta^{(2)}_{d,1}(n, X)| \right)^{1/2} \left( \sum_{X < n \leq 2X} \sum_{d \leq D} |\Delta^{(2)}_{d,1}(n, X)| \right)^{1/2} \]
\[ = \mathcal{L} U^{1/2} V^{1/2}, \quad (55) \]
say. We use the trivial estimation \( \Delta^{(2)}_{d,1}(n, X) \ll \mathcal{L}^{2} X d^{-1} \) and the well-known inequality \( \sum_{n \leq y} \tau^{2}(n) \ll y \log^{3} y \) to get
\[ U \ll X^{2} \mathcal{L}^{6}. \quad (56) \]
In order to estimate \( V \) we use (20) and Lemma 1 to obtain
\[ V \ll X^{2} \mathcal{L}^{-10}. \quad (57) \]
From (55) – (57) it follows that
\[ R^{*}_{1,2}(X) \ll X^{2} \mathcal{L}^{-1}. \quad (58) \]

Consider now \( R'_{1,2}(X) \). Having in mind (7), (8) and (53) we find
\[ R'_{1,2}(X) = \sigma_{0} \sum_{X/2 < p \leq X} r_{2}(p - 1) \Phi^{(2)}(2p, X) \lambda(2p) \log p \sum_{(d, 2p - 1) = 1}^{\chi(d) \lambda(d)} \varphi(d) \lambda((d, 2p)). \quad (59) \]
For the sum over \( d \) according to (7, Section 3.2) we have the bound \( \sum_{d} \ll \log \mathcal{L} \).
Therefore, using also (4) and (13) we get
\[ R'_{1,2}(X) \ll X \mathcal{L}^{2} \log \mathcal{L}^{2} \sum_{X/2 < p \leq X} |r_{2}(p - 1)|. \quad (60) \]
Bearing in mind (20), (22), (60) and Lemma 4 we obtain
\[ R'_{1,2}(X) \ll X^{2} \mathcal{L}^{\theta_{0}} \log \mathcal{L}^{7}. \quad (61) \]
Finally from (52), (58) and (61) we find
\[ R_{1,2}(X) \ll X^{2} \mathcal{L}^{\theta_{0}} \log \mathcal{L}^{7}. \quad (62) \]
The estimation of $R_{1,3}(X)$. From \([15]\), \((22)\) and \((24)\) we have

$$R_{1,3}(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X \atop p_1 + p_2 = 2p_3} \log p_1 \log p_2 \log p_3 \sum_{d | p_1 - 1 \atop d \leq D} \sum_{m | p_2 - 1 \atop m \geq X/D} \chi(d) \chi\left(\frac{p_2 - 1}{m}\right)$$

$$= \sum_{d \leq D \atop m \leq D \atop 2 | m} \chi(d) \sum_{j = \pm 1} \chi(j) I^{(3)}_{(d,4m), (1,1+jm)}(X, \langle (X/2, X], J_m \rangle),$$

where $J_m = \max\{1 + mX/D, X/2\}, X\}$. From \((19)\) we find

$$R_{1,3}(X) = R'_{1,3}(X) + R^*_{1,3}(X), \quad (63)$$

where

$$R'_{1,3}(X) = \sum_{d \leq D \atop m \leq D \atop 2 | m} \chi(d) \Phi^{(3)}(X, \langle (X/2, X], J_m \rangle) \frac{\varphi(d) \varphi(4m)}{\varphi(4m)} \sum_{j = \pm 1} \chi(j) \Theta^{(3)}_{(d,4m), (1,1+jm)}, \quad (64)$$

$$R^*_{1,3}(X) = \sum_{d \leq D \atop m \leq D \atop 2 | m} \chi(d) \sum_{j = \pm 1} \chi(j) \Delta^{(3)}_{(d,4m), (1,1+jm)}(X, \langle (X/2, X], J_m \rangle). \quad (65)$$

From \((20)\) and Lemma \([2]\) we obtain

$$R^*_{1,3}(X) \ll X^2 \mathcal{L}^{-1}. \quad (66)$$

Consider $R'_{1,3}(X)$. According to Lemma \([3]\) the expression $\Theta^{(3)}_{(d,4m), (1,1+jm)}$ does not depend on $j$. Therefore from \((64)\) we get

$$R'_{1,3}(X) = 0. \quad (67)$$

From \((63)\), \((66)\) and \((67)\) it follows that

$$R_{1,3}(X) \ll X^2 \mathcal{L}^{-1}. \quad (68)$$

The estimation of $R_{2,2}(X)$. Let $\mathcal{P}$ be the set of primes $X/2 < p \leq X$ such that $p - 1$ has a divisor lying between $\sqrt{X} \mathcal{L}^{-\omega} < d < \sqrt{X} \mathcal{L}^{\omega}$, (with $\omega = B(10) + C(10) + 1$). Using \((20)\), \((22)\) and \((24)\) and the inequality $uv \leq u^2 + v^2$ we obtain

$$R_{2,2}(X) \ll \mathcal{L}^3 \sum_{X/2 < p_1, p_2, p_3 \leq X \atop p_1 + p_2 = 2p_3 \atop p_1 \not\in \mathcal{P}} \left| \sum_{d | p_2 - 1 \atop D < d < X/D} \chi(d) \chi\left(\frac{p_2 - 1}{D}\right) \right|^2 + \mathcal{L}^3 \sum_{X/2 < p_1, p_2, p_3 \leq X \atop p_1 + p_2 = 2p_3 \atop p_2 \not\in \mathcal{P}} \left| \sum_{d | p_1 - 1 \atop D < d < X/D} \chi(d) \chi\left(\frac{p_1 - 1}{D}\right) \right|^2$$

$$= \mathcal{L}^3 \sum_{X/2 < p_2 \leq X \atop D < d < X/D} \left| \sum_{d | p_2 - 1} \chi(d) \right|^2 + \mathcal{L}^3 \sum_{X/2 < p_1 \leq X \atop D < d < X/D} \left| \sum_{d | p_1 - 1} \chi(d) \right|^2 \sum_{2p_2 - p_3 = p_1 \atop p_2 \not\in \mathcal{P}} 1.$$
In order to estimate the sums over $p_1, p_3$ and $p_2, p_3$ we apply respectively Lemma 6 and Lemma 7 and we get

$$R_{2,2}(X) \ll X \mathcal{L}^{1-2\theta_0} (\log \mathcal{L})^6 \sum_{X/2 < p \leq X} \left| \sum_{d \mid p-1 \atop D < d < X/D} \chi(d) \right|^2.$$

Using Lemma 5 we find

$$R_{2,2}(X) \ll X^2 \mathcal{L}^{-2\theta_0} (\log \mathcal{L})^{13}. \quad (69)$$

**The estimation of $R_{2,3}(X)$.** From (9), (22) and (24) we have

$$R_{2,3}(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} r_2(p_1 - 1) \log p_2 \log p_3 \sum_{m \mid p_2 - 1 \atop \nu_m^2 \geq X/D} \chi \left( \frac{p_2 - 1}{m} \right) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{m < D \atop \nu_m^2 \geq X/D} \sum_{j=\pm 1} \chi(j) I_{4m,1+jm}^{(1)}(p, X, J_m),$$

where $J_m = \left\{ \max\{1 + mX/D, X/2\}, X \right\}$. Using (11) we obtain

$$R_{2,3}(X) = R'_{2,3}(X) + R^*_{2,3}(X), \quad (70)$$

where

$$R'_{2,3}(X) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{m < D \atop \nu_m^2 \geq X/D} \frac{\Phi^{(1)}(p, X, J_m)}{\varphi(4m)} \sum_{j=\pm 1} \chi(j) \mathcal{S}_{4m,1+jm}^{(1)}(p, X, J_m), \quad (71)$$

$$R^*_{2,3}(X) = \sum_{X/2 < p \leq X} r_2(p - 1) \log p \sum_{m < D \atop \nu_m^2 \geq X/D} \sum_{j=\pm 1} \chi(j) \Delta^{(1)}_{4m,1+jm}(p, X, J_m). \quad (72)$$

Consider $R'_{2,3}(X)$. From Lemma 3 we have that $\mathcal{S}_{4m,1+jm}(p)$ does not depend on $j$. Therefore using (11) we find

$$R'_{2,3}(X) = 0. \quad (73)$$

Next we consider $R^*_{2,3}(X)$. From (22), (72) and Cauchy’s inequality we get

$$R^*_{2,3}(X) \ll \mathcal{L} \sum_{X/2 < p \leq X} \tau(p - 1) \sum_{m < D \atop \nu_m^2 \geq X/D} \sum_{j=\pm 1} |\Delta^{(1)}_{4m,1+jm}(p, X, J_m)|$$

$$\ll \mathcal{L} \sum_{X/2 < n \leq X} \tau(n) \sum_{m < D \atop \nu_m^2 \geq X/D} \sum_{j=\pm 1} |\Delta^{(1)}_{4m,1+jm}(n, X, J_m)|$$

$$\ll \mathcal{L} \mathcal{U}_1^{1/2} \mathcal{V}_1^{1/2}, \quad (74)$$
The estimation of $R$

From (74) – (76) it follows that

We use the trivial estimate $\Delta^{(1)}$ and the inequality $\sum_{n \leq y} \tau^2(n) \ll y \log^3 y$ to obtain

$$U_1 \ll X^2 \mathcal{L}^6.$$  \hfill (75)

We estimate $V_1$ using (20) and Lemma 1 and we find

$$V_1 \ll X^2 \mathcal{L}^{-10}.$$  \hfill (76)

From (74) – (76) it follows that

$$R_{2,3}^*(X) \ll X^2 \mathcal{L}^{-1}.$$  \hfill (77)

Now bearing in mind (70), (73) and (77) we obtain

$$R_{2,3}^*(X) \ll X^2 \mathcal{L}^{-1}.$$  \hfill (78)

The estimation of $R_{3,3}(X)$. From (15), (22) and (24) we have

$$R_{3,3}(X) = \sum_{X/2 < n \leq X} \log p_1 \log p_2 \log p_3 \sum_{m \leq D} \sum_{j=\pm 1} \Delta_{1+jm}(n, X, J_m),$$

where $J_m = \langle J_{m_1}, J_{m_2} \rangle; \ J_{m_\nu} = \max\{1 + m_\nu X/D, X/2\}, X, \nu = 1, 2$. We write

$$R_{3,3}(X) = R'_{3,3}(X) + R^*_3(X),$$

where

$$R'_{3,3}(X) = \sum_{m_1, m_2 < D} \frac{\Phi(3)(X, J_m)}{\varphi(4m_1)\varphi(4m_2)} \sum_{j_1=\pm 1} \sum_{j_2=\pm 1} \chi(j_1)\chi(j_2)\Theta(3)(4m_1, 4m_2, 1+j_1m_1, 1+j_2m_2),$$

$$R^*_3(X) = \sum_{m_1, m_2 < D} \sum_{j_1=\pm 1} \sum_{j_2=\pm 1} \chi(j_1)\chi(j_2)\Delta_{1+j_1m_1, 1+j_2m_2}(X, J_m).$$
Consider first \( R'_{3,3}(X) \). According to Lemma 3 the expression \( S^{(3)} \) does not depend on \( j_2 \). Therefore from (80) it follows that

\[
R'_{3,3}(X) = 0. \tag{82}
\]

Consider now \( R^*_{3,3}(X) \). Using (20) and Lemma 2 we find

\[
R^*_{3,3}(X) \ll X^2 L^{-1}. \tag{83}
\]

Now taking into account (79), (82) and (83) we obtain

\[
R_{3,3}(X) \ll X^2 L^{-1}. \tag{84}
\]

**The estimation of \( R_{2,1}(X) \).** Using (9), (11), (22) and (24) we write

\[
R_{2,1}(X) = \sum_{X/2 < p \leq X} r_2 (p - 1) \log p \sum_{d \leq D} \chi(d) I^{(1)}_{d,1}(p, X) = R'_{2,1}(X) + R^*_{2,1}(X),
\]

where

\[
R'_{2,1}(X) = \sum_{X/2 < p \leq X} r_2 (p - 1) \log p \sum_{d \leq D} \frac{\chi(d)}{\varphi(d)} \mathcal{S}_{d,1}(p) \Phi^{(1)}(p, X),
\]

\[
R^*_{2,1}(X) = \sum_{X/2 < p \leq X} r_2 (p - 1) \log p \sum_{d \leq D} \chi(d) \Delta^{(1)}_{d,1}(p, X).
\]

Further arguing as in \( R_{1,2}(X) \) we find

\[
R_{2,1}(X) \ll X^2 L^{-b_0} (\log L)^7. \tag{85}
\]

**The estimation of \( R_{3,2}(X) \).** From (12), (22) and (24) we have

\[
R_{3,2}(X) = \sum_{X/2 < p \leq X} r_2 (p_2 - 1) \log p_1 \log p_2 \log p_3 \sum_{m | p_1 - 1, m | p_2 = 2p_2} \chi \left( \frac{p_1 - 1}{m} \right) \\
= \sum_{X/2 < p \leq X} r_2 (p - 1) \log p \sum_{m | D} \sum_{j = \pm 1} \chi(j) I^{(2)}_{4m,1+jm}(p, X, J_m),
\]

where \( J_m = \left( \max \{1 + mX/D, X/2\}, X \right] \).

Further working as in \( R_{2,3}(X) \) we obtain

\[
R_{3,2}(X) \ll X^2 L^{-1}. \tag{86}
\]

**The estimation of \( R_{3,1}(X) \).** Arguing similar to \( R_{1,3}(X) \) we find

\[
R_{3,1}(X) \ll X^2 L^{-1}. \tag{87}
\]

**The end of the proof.** The asymptotic formula (6) follows from (23), (24), (51), (62), (68), (69), (72), (84), (85), (86) and (87).

The Theorem is proved.
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