Kaluza–Klein multi-black holes in five-dimensional Einstein–Maxwell theory

Hideki Ishihara, Masashi Kimura, Ken Matsuno and Shinya Tomizawa

Department of Mathematics and Physics, Graduate School of Science, Osaka City University,
3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan

E-mail: ishihara@sci.osaka-cu.ac.jp, mkimura@sci.osaka-cu.ac.jp, matsuno@sci.osaka-cu.ac.jp
and tomizawa@sci.osaka-cu.ac.jp

Received 4 August 2006, in final form 12 September 2006
Published 20 October 2006
Online at stacks.iop.org/CQG/23/6919

Abstract
We construct Kaluza–Klein multi-black hole solutions on the Gibbons–Hawking multi-instanton space in the five-dimensional Einstein–Maxwell theory. We study the geometric properties of the multi-black hole solutions. In particular, unlike the Gibbons–Hawking multi-instanton solutions, each nut-charge is able to take a different value due to the existence of a black hole on it. The spatial cross section of each horizon is admitted to have the topology of a different lens space $L(n; 1) = S^3/\mathbb{Z}_n$ in addition to $S^3$.

PACS numbers: 04.50.+h, 04.70.Bw

1. Introduction

It is well known that the Majumdar–Papapetrou solution [1] describe an arbitrary number of extremely charged static black holes in four dimensions. The construction of the solution is possible because of a force balance between the gravitational and Coulomb forces. These solutions have been extended to multi-black holes in arbitrary higher dimensions [2] and to multi-black $p$-branes [3]. All multi-black objects are constructed on the Euclid space which is transverse to the objects. The metric and the gauge potential 1-form are given by a solution of the Laplace equation in the Euclid space with point sources.

In the context of supersymmetry, five-dimensional Einstein–Maxwell theory gathers much attention. It is possible to construct black hole solutions by using the four-dimensional Euclidean self-dual Taub-NUT space [4–7] instead of the Euclid space. The black hole solutions on the Taub-NUT space have $S^1$ horizons, and have an asymptotic structure of four-dimensional locally flat spacetime with a twisted $S^1$. Such black hole solutions associated with a compact dimension, so-called Kaluza–Klein black holes, are interesting because the solutions would connect the higher-dimensional spacetime with the usual four-dimensional world.
Actually, if the Euclid space is replaced with any Ricci flat space, we can obtain solutions to the Einstein–Maxwell equations by use of a harmonic function on the space. Indeed, since the Laplace equation is linear, we can construct solutions describing a multi-object by superposition of harmonic functions with a point source. However, it should be clarified whether the objects are really black holes, i.e., all singularities are hidden behind horizons.

In this paper, as the generalization of the Kaluza–Klein black hole solutions [4–7] to multi-black hole solutions, we construct Kaluza–Klein multi-black holes on the Gibbons–Hawking multi-instanton space explicitly which are solutions to the five-dimensional Einstein–Maxwell system. Then, we study the geometric properties of the multi-black hole solutions.

2. Construction of solutions

2.1. A single extreme black hole

We start with the five-dimensional Einstein–Maxwell system with the action
\[ S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} (R - F_{\mu\nu}F^{\mu\nu}), \]
where \( G, R \) and \( F = dA \) are the five-dimensional gravitational constant, the Ricci scalar curvature and the Maxwell field with the gauge potential \( A \). From this action, we write down the Einstein equation
\[ R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 2(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}), \]
and the Maxwell equation
\[ F_{\mu\nu;\nu} = 0. \]

The extreme Kaluza–Klein black hole [4–7] is an exact solution of equations (2) and (3) constructed on the Taub-NUT space. The metric and the gauge potential 1-form are written as
\[ ds^2 = -H^{-2} dT^2 + H ds_{TN}^2, \]
\[ A = \pm \frac{\sqrt{3}}{2} H^{-1} dT. \]
When the Taub-NUT space is described in the form
\[ ds_{TN}^2 = V^{-1} (dR^2 + R^2 d\Omega_5^2) + V (d\xi + N \cos \theta d\phi)^2, \]
\[ d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\phi^2, \]
\[ V^{-1}(R) = 1 + \frac{N}{R}, \]
with
\[ 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \xi \leq 2\pi L, \]
the function \( H \) is given by
\[ H(R) = 1 + \frac{M}{R}, \]
where \( L, M \) and \( N \) are positive constants. Regularity of the spacetime requires that the nut charge, \( N \), and the asymptotic radius of \( S^1 \) along \( \zeta, L \), are related by
\[ N = \frac{L}{2} n, \]
where \( n \) is a natural number.
We can see that a degenerated horizon exists at \( R = 0 \), where the nut singularity of the Taub-NUT space was located. Although the Taub-NUT space is regular only when \( n = 1 \) on conditions (9), the nut singularity with \( n \geq 2 \) is resolved by the event horizon in the black hole solution [5].

In the case of \( n = 1 \), the horizon has the shape of round \( S^3 \) in a static time-slice in contrast to the non-degenerated case, where the horizon is squashed [7]. The spacetime is asymptotically locally flat, i.e., a constant \( S^1 \) fibre bundle over the four-dimensional Minkowski spacetime at \( R \to \infty \). Therefore, the spacetime behaves as a five-dimensional black hole near the horizon, while the dimensional reduction to four dimension is realized in a far region. In the case of \( n \geq 2 \), the horizon is in the shape of the lens space \( L(n; 1) = S^3/\mathbb{Z}_n \) [5]. The mass which is defined by the Komar integral at spatial infinity and the electric charge satisfy the extremality condition

\[
M_{\text{Komar}} = \frac{3\pi}{2G} LM = \frac{\sqrt{3}}{2} |Q|.
\]

The black hole solution (4) with \( n = 1 \) contains two popular spacetimes as limits: the five-dimensional extreme Reissner–Nordström black hole as \( L \to \infty \) and \( M \to 0 \) with \( LM \) finite, and the Gross–Perry–Sorkin monopole [8] as \( M \to 0 \).

### 2.2. Multi-black holes

When we generalize the single black hole solution (4) to the multi-black holes, it is natural to generalize the Taub-NUT space to the Gibbons–Hawking space [9] which has multi-nut singularities. The metric form of the Gibbons–Hawking space is

\[
d s^2_{GH} = V^{-1} (dx \cdot dx) + V (d\zeta + \omega)^2,
\]

\[
V^{-1} = 1 + \sum_i \frac{N_i}{|x - x_i|},
\]

where \( x_i = (x_i, y_i, z_i) \) denotes the position of the \( i \)th nut singularity with nut charge \( N_i \) in the three-dimensional Euclid space, and \( \omega \) satisfies

\[
\nabla \times \omega = \nabla \frac{1}{V}.
\]

We can write down a solution \( \omega \) explicitly as

\[
\omega = \sum_i \frac{N_i}{|x - x_i|} \frac{(z - z_i)(x - x_i) dy - (y - y_i) dx}{(x - x_i)^2 + (y - y_i)^2}.
\]

If we assume the metric form with the Gibbons–Hawking space instead of the Taub-NUT space in (4), the Einstein equation (2) and the Maxwell equation (3) reduce to

\[
\Delta_{GH} H = 0,
\]

where \( \Delta_{GH} \) is the Laplacian of the Gibbons–Hawking space. In general, it is difficult to solve this equation, but if one assumes \( \partial_\zeta \) to be a Killing vector, as it is for the Gibbons–Hawking space; then equation (17) reduces to the Laplace equation in the three-dimensional Euclid space,

\[
\Delta_E H = 0.
\]

We take a solution with point sources to equation (18) as a generalization of equation (10), and we have the final form of the metric
\[ ds^2 = -H^{-2}dT^2 + H \, ds_{\text{Gri}}^2, \quad (19) \]
\[ H = 1 + \sum_i \frac{M_i}{|x - x_i|}, \quad (20) \]

where \( M_i \) are constants\(^1\).

3. Properties

3.1. Regularity

In equations (19) with (20), a point source labelled by \( x_i \) with \( M_i > 0 \) and \( N_i > 0 \) is a black hole since the point becomes the Killing horizon and the cross section of it with a static time-slice has a finite area, as is seen below. A point source of \( M_i > 0 \) and \( N_i = 0 \), however, becomes a naked singularity. This is the reason why we need to generalize the Taub–NUT space to the Gibbons–Hawking space\(^2\). For a point with \( M_i < 0 \) or \( N_i < 0 \), a naked singularity also appears. Therefore, in the case of \( M_i > 0 \) and \( N_i > 0 \) for all \( i \), the metric describes multi-black holes, on which we focus our attention.

On the other hand, a point source with \( M_i = 0 \) and \( N_i > 0 \) corresponds to a Gross–Perry–Sorkin-type monopole with a nut charge \( N_i \). In the case of \( M_i = 0 \) and \( N_i > 0 \) for all \( i \), the metric becomes the Gross–Perry–Sorkin multi-monopole solution [8]. This multi-monopole solution is regular when all of the nut charges \( N_i \) have the same value \( L/2 \) [9], since the nut charges with different values from it yield the cone-singularities. However, the existence of black holes drastically changes this situation because the nut singularity with \( N_i = L/2 \times n_i \) (natural number) converts into the black hole whose topology is a lens space \( L(n_i; 1) = S^3/Z_n \), as mentioned below. Therefore, all \( N_i \) can take different values associated with the different \( n_i \).

Now, we investigate the regularity on the black hole horizon. For simplicity, we restrict ourselves to the solution with two black holes. In order to examine that the geometry near the horizon \( x = x_1 \), we make the coordinate transformation such that the point source \( x_1 \) is the origin of the three-dimensional Euclid space and \( x_2 = (0, 0, -a) \). In this case, from equations (19) and (20) the metric can take the following simple form:

\[ ds^2 = -H(R, \theta)^{-2}dT^2 + H(R, \theta) \left[ V(R, \theta)^{-1} (dR^2 + R^2 d\Omega_2^2) + V(R, \theta) (d\xi + \omega_\phi(R, \theta) \, d\phi)^2 \right], \quad (21) \]

with

\[ H(R, \theta) = 1 + \frac{M_1}{R} + \frac{M_2}{\sqrt{R^2 + a^2 + 2aR \cos \theta}}, \quad (22) \]
\[ V(R, \theta)^{-1} = 1 + \frac{N_1}{R} + \frac{N_2}{\sqrt{R^2 + a^2 + 2aR \cos \theta}}, \quad (23) \]
\[ \omega_\phi(R, \theta) = N_1 \cos \theta + \frac{N_2(a + R \cos \theta)}{\sqrt{R^2 + a^2 + 2aR \cos \theta}}, \quad (24) \]

where the parameter \( a \) denotes the separation between two point sources \( x_1 \) and \( x_2 \) in the three-dimensional Euclid space.

\(^1\) For the special case \( H = 1/V \), the metric (19) reduces to the four-dimensional Majumdar–Papapetrou multi-black holes with a twisted constant \( S^1 \), which is seen in [4]. In this special case, \( M_i \) are also quantized as well as nut charges.

\(^2\) Multi-black hole solutions in Taub-NUT space are constructed [5] by using a black ring in the space.
If we take the limit \( R = |x - x_1| \to 0 \), we can see the leading behaviour of the metric as follows:

\[
\begin{align*}
\text{d}s^2 &\simeq - \left( \frac{R}{M_1} \right)^2 \text{d}T^2 + \frac{R}{N_1} \left( \text{d}\zeta + N_1 \cos \theta \text{d}\phi \right)^2 + \frac{N_1}{R} \left( \text{d}R^2 + R^2 \text{d}\Omega_2^2 \right) .
\end{align*}
\]

(25)

We should note that the other black holes do not contribute to this behaviour of the metric in the leading order. Therefore, the form of each black hole is equivalent with the single extremal black hole \(4\) in the vicinity of the horizon\(^3\). The Kretschmann invariant near the horizon \( R = 0 \) can be computed as

\[
R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{19}{4 M_1^2 N_1^2} + O \left( \frac{R}{a} \right) .
\]

(26)

which suggests the horizon \( R = 0 \) is regular. In fact, under the coordinate transformation,

\[
u = T - F(R) ,
\]

(27)

\[
\frac{dF(R)}{dR} = \left( 1 + \frac{M_1}{R} + \frac{M_2}{a} \right)^{3/2} \left( 1 + \frac{N_1}{R} + \frac{N_2}{a} \right)^{1/2} ,
\]

(28)

the metric near the horizon \( R = 0 \) has the following regular form:

\[
\begin{align*}
\text{d}s^2 &\simeq 2 \sqrt{\frac{N_1}{M_1}} \text{d}\nu \text{d}R + M_1 N_1 \left[ \left( \frac{\text{d}\zeta}{N_1} + \cos \theta \text{d}\phi \right)^2 + \text{d}\Omega_2^2 \right] .
\end{align*}
\]

(29)

Of course, from the similar discussion, the regularity of the other black hole horizon \( x = x_2 \) is also assured. Outside the black holes, there is evidently no place with a singular point from the explicit form of the metric. Even if we consider the situations with more than two black holes, these properties do not change in such spacetimes.

3.2. Geometry near horizons

The induced metric on an intersection of the \(i\)th black hole horizon with a static time-slice is

\[
\text{d}s^2_{\text{Horizon}} = \frac{L M_i n_i}{2} \left[ \left( \frac{\text{d}\psi}{n_i} + \cos \theta \text{d}\phi \right)^2 + \text{d}\Omega_2^2 \right] .
\]

(30)

where \(0 \leq \psi = 2\zeta/L \leq 4\pi\). In the case of \(n_i = 1\), it is apparent that the \(i\)th black hole is a round \(S^3\); however, in the case of \(n_i \geq 2\), the topological structure becomes a lens space \(L(n_i;1) = S^3/Z_{n_i}\).

In an asymptotically flat stationary five-dimensional black hole spacetime, the only possible geometric type of spatial cross section of horizon is restricted to \(S^3\) or \(S^1 \times S^2\) \([11, 12]\), which is the extension of Hawking’s theorem on event horizon topology \([13]\) to five dimensions. However, each Kaluza–Klein black hole horizon can have a topological structure of lens spaces \(S^3/Z_{n_i}\) besides \(S^3\). This fact does not contradict with the theorem in \([11, 12]\), where the boundary is assumed to be asymptotically flat since the Kaluza–Klein black holes are not asymptotically flat.

3.3. Asymptotic structure

Finally, we study the asymptotic behaviour of the Kaluza–Klein multi-black hole in the neighbourhood of the spatial infinity \( R \to \infty \). The functions \( H, V^{-1} \) and \( \omega \) behave as

\(^3\) The smoothness of the horizon in the higher-dimensional multi-black holes would be lost \([3, 10]\).
\[ H(R, \theta) \simeq 1 + \frac{1}{R} \sum_i \frac{M_i}{R} + O\left(\frac{1}{R^2}\right). \] (31)

\[ V(R, \theta)^{-1} \simeq 1 + \frac{1}{R} \sum_i \frac{N_i}{R} + O\left(\frac{1}{R^2}\right). \] (32)

\[ \omega(R, \theta) \simeq \left(\sum_i N_i\right) \cos \theta \, d\phi + O\left(\frac{1}{R}\right). \] (33)

We can see that the spatial infinity possesses the structure of a \( S^1 \) bundle over \( S^2 \) such that it is a lens space \( L(\sum n_i; 1) \). For an example, in the case of two Kaluza–Klein black holes which have the same topological structure of \( S^3 \), the asymptotic structure is topologically homeomorphic to the lens space \( L(2; 1) = S^3/\mathbb{Z}_2 \). From this behaviour of the metric near the spatial infinity, we can compute the Komar mass at spatial infinity of this multi-black hole system as

\[ M_{\text{Komar}} = \frac{3\pi}{2G} L \sum_i M_i. \] (34)

Since the total electric charge is given by

\[ Q_{\text{total}} = \sum_i Q_i = \pm \frac{\sqrt{3\pi}}{G} L \sum_i M_i, \] (35)

then the total Komar mass and the total electric charge satisfy

\[ M_{\text{Komar}} = \frac{\sqrt{3}}{2} |Q_{\text{total}}|. \] (36)

Therefore, from equation (12) we find that an observer located in the neighbourhood of the spatial infinity feels as if there were a single Kaluza–Klein black hole with the point source with the parameter \( M = \sum_i M_i \) and the nut charge \( N = \sum_i N_i \).

4. Summary and discussion

In conclusion, we constructed the Kaluza–Klein multi-black hole solutions on the nuts of the Gibbons–Hawking space as solutions in the five-dimensional Einstein–Maxwell theory. We also investigated the properties of these solutions, in particular, the regularity, the geometry near horizon and the asymptotic structure. In the solution equations (19) and (20) a point source labelled by \( x_i \) with \( M_i > 0 \) and \( N_i > 0 \) is a black hole. One of the most interesting properties is that the possible spatial topology of the horizon of each black hole is the lens space \( S^3/\mathbb{Z}_{n_i} \), where the natural number \( n_i \) is related to the value \( N_i \) of each nut charge by \( N_i = L/2 \times n_i \). The spatial infinity has the structure of a \( S^1 \) bundle over \( S^2 \) such that it is a lens space \( L(\sum n_i; 1) \).

This fact suggests that two black holes with \( S^3 \) horizons constructed in this paper coalesce into a black hole with a \( L(2; 1) = S^3/\mathbb{Z}_2 \) horizon. By the coalescences, black holes may change into a black hole with a different lens space \( L(n; 1) \). It is also expected that the area of a single black hole formed by such a process would be different from that in a topology-preserving process. Observing such difference might lead to the verification of the existence of extra dimensions. Therefore, from this viewpoint, the coalescence of black holes with the topologies of various lens spaces may be an interesting physical phenomenon.
Acknowledgments

We thank K Nakao and Y Yasui for useful discussions. This work is supported by the Grant-in-Aid for Scientific Research No 14540275 and No 13135208.

References

[1] Majumdar S D 1947 Phys. Rev. 72 390
   Papapetrou A 1947 Proc. R. Ir. Acad. A 51 191
[2] Myers R C 1987 Phys. Rev. D 35 455
[3] Gibbons G W, Horowitz G T and Townsend P K 1995 Class. Quantum Grav. 12 297
[4] Gauntlett J P, Gutowski J B, Hull C M, Pakis S and Reall H S 2003 Class. Quantum Grav. 20 4587
[5] Elvang H, Emparan R, Mateos D and Reall H S 2005 J. High Energy Phys. JHEP08(2005)042
[6] Gaiotto D, Strominger A and Yin X 2006 J. High Energy Phys. JHEP02(2006)024
[7] Ishihara H and Matsuno K 2006 Prog. Theor. Phys. 116 417
[8] Gross D J and Perry M J 1983 Nucl. Phys. B 226 29
   Sorkin R 1983 Phys. Rev. Lett. 51 87
[9] Gibbons G W and Hawking S W 1978 Phys. Lett. B 78 430
[10] Welch D L 1995 Phys. Rev. D 52 985
[11] Cai M I and Galloway G J 2003 Class. Quantum Grav. 18 2707
[12] Helfgott C, Oz Y and Yanay Y 2006 J. High Energy Phys. JHEP02(2006)025
[13] Hawking S W 1972 Commun. Math. Phys. 25 152