ON NONEXISTENCE OF GLOBAL SOLUTIONS FOR A SEMILINEAR EQUATION WITH HILFER-HADAMARD FRACTIONAL DERIVATIVE

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ABSTRACT. For the following semilinear equation with Hilfer-Hadamard fractional derivative
\[ \mathcal{D}^{\alpha_1,\beta}_{a^+} u - \Delta \mathcal{D}^{\alpha_2,\beta}_{a^+} u - \Delta u = |u|^p, \quad t > a > 0, \quad x \in \Omega, \]
where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)), \( p > 1 \), \( 0 < \alpha_2 < \alpha_1 < 1 \) and \( 0 < \beta < 1 \). \( \mathcal{D}^{\alpha_i,\beta}_{a^+} \) (\( i = 1, 2 \)) is the Hilfer-Hadamard fractional derivative of order \( \alpha_i \) and of type \( \beta \), we establish the necessary conditions for the existence of global solutions.

Keywords: Pseudo-parabolic problem, time fractional derivatives, structural damping, nonexistence

MSC 2010: Primary 26A33; Secondary 35B44

1. INTRODUCTION

We consider the following initial boundary value problem
\[
\begin{cases}
\mathcal{D}^{\alpha_1,\beta}_{a^+} u - \Delta \mathcal{D}^{\alpha_2,\beta}_{a^+} u - \Delta u = |u|^p, & t > a > 0, \quad x \in \Omega, \\
u(t, x) = 0, & t > a > 0, \quad x \in \partial \Omega, \\
\left( \mathcal{D}^{(\beta-1)(1-\alpha_1)}_{a^+} u \right) (a, x) = u_0(x), & x \in \Omega,
\end{cases}
\] (1)

where \( \Omega \) is a bounded domain \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) with smooth boundary \( \partial \Omega \), \( p > 1 \), \( 0 < \alpha_2 < \alpha_1 < 1 \), \( 0 < \beta < 1 \) and \( \Delta \) denotes the Laplacian operator with respect to the \( x \) variable. The operator \( \mathcal{D}^{\alpha_i,\beta}_{a^+} \) is the Hilfer-Hadamard fractional derivative of order \( \alpha_i \) and of type \( \beta \), which we will be defined carefully in a further part of this paper. The equation (1) is a generalization of the well-known pseudo-parabolic equation of first order. The integer derivative is replaced by a fractional derivative in the sense of Hilfer-Hadamard. The second Hilfer-Hadamard fractional derivative of the Laplacian is allowed to be different from the first one.

Our objective is to find the range of \( p \) for which nontrivial solutions cannot exist for all time. This leads us to shed some light on the interaction of the nonlinear source term with \( \Delta \mathcal{D}^{\alpha_2,\beta}_{a^+} u \). The analysis is based mainly on the test function method [10]. Let us first recall some works related to our problem.

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The semilinear pseudo-parabolic equation
\[
\begin{aligned}
&u_t - k\Delta u_t - \Delta u = |u|^p, \quad (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \\
u(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\] (2)
arises in many fields of science and engineering: the aggregation of population \[11\] and
the nonstationary processes in semiconductors \[7\]. Eq. (2) is also called a Sobolev type
equation, Sobolev Galpern type equation or the Benjamin Bona Mahony Burgers equation \[2\].
Many researchers have studied the existence and blow-up of solutions for problem (2) \[13, 3\],
by using different methods, such as the potential well method and the
Galerkin method combined with the compactness method. When \(k = 0\), Eq. (2) reduces
to the heat equation
\[
\begin{aligned}
&u_t - \Delta u = |u|^p, \quad (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \\
u(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\] (3)
Fujita \[4\] studied the global existence of mild solutions to (3), if \(p > 1 + \frac{2}{N}\) and small
initial data. In addition, he proved that the mild solution cannot exist globally when
\(1 < p < 1 + \frac{2}{N}\) and \(u_0 \neq 0\).
In \[14\], Weissler proved that if \(p = 1 + \frac{2}{N}\) (critical case) and \(u_0\) is small enough in
\(L^q(\mathbb{R}^N)\), \(q_c = N(p - 1)/2\), then the solution of (3) exists globally.
Xu and Su \[16\] showed that all nontrivial solutions \(u\) of the following problem
\[
\begin{aligned}
&u_t - \Delta u_t - \Delta u = |u|^p, \quad (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \\
u(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\] (4)
where \(1 < p < \infty\) if \(N = 1, 2\); \(1 < p \leq \frac{N+2}{N-2}\) if \(N \geq 3\), exist for all time under some
conditions and they obtained sufficient conditions for nonexistence of solutions. In 2017,
Xu and Zhou \[15\] gave new blow-up and lifespan conditions of problem (4).
The rest of the paper is organized as follows: In Section 2, we recall some definitions
about Hilfer-Hadamard fractional integrals and derivatives. In Section 3, we study the
absence of global nontrivial weak solutions.

2. Preliminary

In this section, we present some results and basic properties of fractional calculus. For
more details, we refer to \[6, 5, 1, 12\].

**Definition 2.1.** The Hadamard fractional integrals of order \(\alpha > 0\) of a function \(\varphi \in L^q[a, b]\) \((1 \leq q < \infty, 0 < a \leq b \leq +\infty)\), are defined by
\[
(I_{a+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{t}{\tau} \right)^{a-1} \varphi(\tau) \frac{d\tau}{\tau}, \quad a < t < b,
\]
and
\[
(I_{b-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left( \log \frac{\tau}{t} \right)^{a-1} \varphi(\tau) \frac{d\tau}{\tau}, \quad a < t < b.
\]
\textbf{Definition 2.2.} Let $0 < a < t < b$ and $n - 1 < \alpha < n$. The Hadamard fractional derivatives of order $\alpha$ for a function $\varphi$ are defined by
\[
(D_{a+}^{\alpha} \varphi)(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{t}{d} \right)^n \int_a^t \left( \log \frac{t}{\tau} \right)^{n - \alpha - 1} \varphi(\tau) \frac{d\tau}{\tau} = \delta^n \left( I_{a+}^{n - \alpha} \varphi \right)(t),
\]
and
\[
(D_{b-}^{\alpha} \varphi)(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_t^b \left( \log \frac{\tau}{t} \right)^{n - \alpha - 1} \varphi(\tau) \frac{d\tau}{\tau} = (-1)^n \delta^n \left( I_{b-}^{n - \alpha} \varphi \right)(t),
\]
where $\delta = t \frac{d}{dt}$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number $\alpha$.

\textbf{Definition 2.3.} Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. The Hilfer- Hadamard fractional derivative of order $\alpha$ and type $\beta$ is defined by
\[
(D_{a+}^{\alpha,\beta} \varphi)(t) = \left( I_{a+}^{\beta(1-\alpha)} \delta I_{a+}^{(1-\beta)(1-\alpha)} \varphi \right)(t),
\]
that is,
\[
(D_{a+}^{\alpha,\beta} \varphi)(t) = I_{a+}^{\beta(1-\alpha)} \left( t \frac{d}{dt} \right) \left( I_{a+}^{(1-\beta)(1-\alpha)} \varphi \right)(t). \tag{5}
\]

\textbf{Definition 2.4.} Let $[a, b]$ be a finite interval of the half-axis $\mathbb{R}^+$ and $0 \leq \gamma < 1$. We introduce the weighted spaces of continuous functions
\[
C_{\gamma,\log}[a, b] = \left\{ \varphi : [a, b] \to \mathbb{R} : \left( \log \frac{t}{a} \right)^\gamma \varphi(t) \in C[a, b] \right\}, \tag{6}
\]
\[
C_{1-\gamma,\log}[a, b] = \left\{ \varphi \in C_{\gamma,\log}[a, b] : D_{a+}^{\gamma} \varphi \in C_{1-\gamma,\log}[a, b] \right\}, \tag{7}
\]
and
\[
C_{\delta,\gamma}[a, b] = \left\{ \varphi : [a, b] \to \mathbb{R} : \delta^k \varphi \in C[a, b], \ 0 \leq k \leq n - 1, \ \delta^n \varphi \in C_{\gamma,\log}[a, b] \right\}, \tag{8}
\]
where $\delta = t \frac{d}{dt}$ and $n \in \mathbb{N}$. In particular, when $n = 0$ we define
\[
C_{\delta,\gamma}[a, b] = C_{\gamma,\log}[a, b].
\]

\textbf{Definition 2.5.} The Banach space $X_{-1/p}^p(a, b)$ $(1 \leq p \leq \infty, c \in \mathbb{R})$ consists of those real-valued Lebesgue measurable functions $\varphi : (a, b) \to \mathbb{R}$ such that
\[
\| \varphi \|_{X_p} = \left( \int_a^b |t^c \varphi(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, \ p < \infty, \tag{9}
\]
\[
\| \varphi \|_{X_\infty} = \text{ess sup}_{a \leq t \leq b} \left| t^c \varphi(t) \right| < \infty. \tag{10}
\]
When $c = 1/p$, we see that $X_{-1/p}^p(a, b) = L^p(a, b)$.

\textbf{Lemma 2.1 \cite{12}.} Let $\alpha > 0$, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\varphi \in L^p(a, b)$ and $\Psi \in X_{-1/p}^q(a, b)$, then
\[
\int_a^b \varphi(t)(I_{a+}^{\alpha} \Psi)(t) \frac{dt}{t} = \int_a^b (I_{b-}^{\alpha} \varphi)(t) \Psi(t) \frac{dt}{t}. \tag{11}
\]
Lemma 2.2 ([I]). Assume \( \varphi \in C_{1,\log}[a,b] \), for \( a < t < b \), \( 0 < \gamma < 1 \) and \( 0 < \alpha < 1 \). Then \( D^\alpha_{a^+} \) exists on \((a,b]\) and \( D^\alpha_{b^-} \) on \([a,b)\) and can be represented as
\[
(D^\alpha_{a^+}\varphi)(t) = \frac{\varphi(a)}{\Gamma(1-\alpha)} \left( \frac{t-a}{\log t} \right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \frac{\log \frac{t}{\tau}}{\tau} \right)^{-\alpha} \varphi'(\tau) d\tau,
\]
\[
(D^\alpha_{b^-}\varphi)(t) = \frac{\varphi(b)}{\Gamma(1-\alpha)} \left( \frac{b-t}{\log b} \right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)} \int_t^b \left( \frac{\log \frac{\tau}{t}}{\tau} \right)^{-\alpha} \varphi'(\tau) d\tau,
\]
respectively.

Lemma 2.3 ([I]). Let \( 0 \leq \gamma < 1 \) and \( 0 < \alpha \). If \( \gamma \leq \alpha \), then the operator \( I_a^\alpha \) is bounded from \( C_{\gamma,\log}(a,b) \) into \( C(a,b) \). In particular, it is bounded in \( C_{\gamma,\log}(a,b) \).

Lemma 2.4 ([I]). Let \( \alpha > 0 \), \( \beta > 0 \) and \( 0 < \gamma < 1 \). Assume \( 0 < a < b < \infty \), then for \( \varphi \in C_{\gamma,\log}(a,b) \),
\[
I_a^\alpha I_a^\beta \varphi = I_a^{\alpha+\beta} \varphi,
\]
for each \( t \in (a,b) \). In particular, if \( \varphi \in C[a,b] \) the relation \( (III) \) is valid at any point \( t \in [a,b] \).

Throughout the next section, we take \( 0 < \gamma = \alpha_1 + \beta - \alpha_1 \beta < 1 \).

3. Blow-up of solutions

First, we give the definition of weak solution of \( (I) \). After we prove the non-existence of nontrivial solutions.

Definition 3.1. Let \( u_0 \in C_0(\Omega) \) and \( 0 < \alpha_2 < \alpha_1 < 1 \). The function \( u \in C^{\gamma}_{1-\gamma,\log}([a,b], C_0(\Omega)) \) is a weak solution of problem \( (I) \), if
\[
\int_\Omega \int_a^T \tilde{\varphi} D^{\alpha_1,\beta}_{a^+} u dt dx - \int_\Omega \int_a^T \Delta \tilde{\varphi} D^{\alpha_2,\beta}_{a^+} u dt dx - \int_\Omega \int_a^T \tilde{\varphi} u dt dx = \int_\Omega \int_a^T |u|^p \tilde{\varphi} dt dx,
\]
for all compactly supported test function \( \tilde{\varphi} \in C^{1,2}_{0,\Omega}([a,T] \times \Omega) \).

Theorem 3.1. Let \( u_0 \in C_0(\Omega) \) and \( u_0 \geq 0 \). If
\[
1 < p < \frac{\alpha_2 N + 1}{(\alpha_2 N + 1 - 2\alpha_2)},
\]
then the problem \( (I) \) does not admit global nontrivial solutions in the space \( C^{\gamma}_{1-\gamma,\log}([a,b], C_0(\Omega)) \).

Proof. We assume the contrary. Let \( \Phi \in C_0^\infty([0,\infty)) \) be a decreasing function satisfying
\[
\Phi(\sigma) = \begin{cases} 1, & 0 \leq \sigma \leq 1, \\ 0, & \sigma \geq 2. \end{cases}
\]
We define the function \( \tilde{\varphi}(t,x) \) as follows
\[
\tilde{\varphi}(t,x) = \frac{\varphi_1(t)}{t} \varphi_2(x),
\]
with \( \varphi_1(t) \in C([a,\infty)) \), \( \varphi_1(t) \geq 0 \) and \( \varphi_1(t) \) is non-increasing such that
\[
\varphi_1(t) = \begin{cases} 1, & 0 < a \leq t \leq \theta T, \\ 0, & t \geq T, \end{cases}
\]
(14)
for $T > a > 0$ and we choose

$$
\varphi_2(x) = \left[ \Phi \left( \frac{\|x\|}{T^{\alpha_2}} \right) \right]^{\mu}, \quad \mu \geq \frac{2p}{p - 1}.
$$

Equality (12) actually reads

$$\int_{\Omega_1} \int_a^T \varphi_2(x) \varphi_1(t) D_{a+}^{\alpha_1, \beta} u \frac{dt}{t} dx - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \varphi_1(t) D_{a+}^{\alpha_2, \beta} u \frac{dt}{t} dx - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \varphi_1(t) u \frac{dt}{t} dx
$$

$$= \int_{\Omega_1} \int_a^T |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{t} dx.
$$

where $\Omega_1 := \{ x \in \Omega : \|x\| \leq 2T^{\alpha_2} \}$. From the definition of $D_{a+}^{\alpha, \beta} u$, we can re-write the above equation as

$$\int_{\Omega_1} \int_a^T \varphi_2(x) \varphi_1(t) T_{a+}^{1-\alpha_1} \left( t \frac{d}{dt} \right) \left( T_{a+}^{1-\beta} T_{a+}^{1-\alpha_1} \right) u \frac{dt}{t} dx
$$

$$- \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \varphi_1(t) T_{a+}^{1-\alpha_2} \left( t \frac{d}{dt} \right) \left( T_{a+}^{1-\beta} T_{a+}^{1-\alpha_2} \right) u \frac{dt}{t} dx - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \varphi_1(t) u \frac{dt}{t} dx
$$

$$= \int_{\Omega_1} \int_a^T |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{t} dx.
$$

By Definition (2.4), we have

$$\left( \log \frac{t}{a} \right)^{1-\gamma} D_{a+}^{\gamma} u \text{ is continuous on } [a, T] \text{ implies that}
$$

$$\left| \left( \log \frac{t}{a} \right)^{1-\gamma} D_{a+}^{\gamma} u \right| \leq M, \quad \forall t \in [a, T],
$$

for some positive constant $M$ (the constant $M$ will be a generic constant which may change at different places). Therefore

$$\int_a^T t^{-1/p} \left( D_{a+}^{\gamma} u \right)(t) \left( \log \frac{t}{a} \right)^{-p'(1-\gamma)} \frac{dt}{t} \leq M^{p'} \int_a^T t^{1-p'} \left( \log \frac{t}{a} \right)^{-p'(1-\gamma)} \frac{dt}{t},
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We introduce the following scaled variable

$$w = (p' - 1) \log \left( t/a \right).
$$

Then

$$\int_a^T t^{-1/p} \left( D_{a+}^{\gamma} u \right)(t) \left( \log \frac{t}{a} \right)^{-p'(1-\gamma)} \frac{dt}{t} \leq \frac{M^{p'} a^{1-p'}}{(p' - 1)^{1-p'(1-\gamma)}} \int_0^\infty w^{-p'(1-\gamma)} e^{-w} dw
$$

$$\leq \frac{M^{p'} a^{1-p'}}{(p' - 1)^{1-p'(1-\gamma)}} \Gamma(1 - p'(1 - \gamma)) < \infty.
$$

(18)
Consequently, \((D_a^+ u)(t) \in X_{1/p}^\prime\). Thus it follows from Lemma\(2.1\) that
\[
\begin{align*}
\int_{\Omega_1} \int_a^T \phi_2(x) T_{T_+}^{(1-\alpha_1)} \phi_1(t) \frac{d}{dt} T_{T_+}^{(1-\alpha_1)} u dt dx \\
- \int_{\Omega_1} \int_a^T \Delta \phi_2(x) T_{T_+}^{(1-\alpha_2)} \phi_1(t) \frac{d}{dt} T_{T_+}^{(1-\alpha_2)} u dt dx - \int_{\Omega_1} \int_a^T \Delta \phi_2(x) \phi_1(t) u \frac{dt}{t} dx \\
= \int_{\Omega_1} \int_a^T |u|^p \phi_2(x) \phi_1(t) \frac{dt}{t} dx.
\end{align*}
\] (19)

Using integration by parts in (19), we obtain
\[
\begin{align*}
\int_{\Omega_1} \phi_2(x) \left[ T_{T_+}^{(1-\alpha_1)} \phi_1(t) (T_{T_+}^{(1-\alpha_1)} u)(t, x) \right]_{t=a}^T dx \\
- \int_{\Omega_1} \int_a^T \phi_2(x) \frac{d}{dt} T_{T_+}^{(1-\alpha_1)} \phi_1(t) (T_{T_+}^{(1-\alpha_1)} u) dt dx \\
- \int_{\Omega_1} \Delta \phi_2(x) \left[ T_{T_+}^{(1-\alpha_2)} \phi_1(t) (T_{T_+}^{(1-\alpha_2)} u)(t, x) \right]_{t=a}^T dx \\
+ \int_{\Omega_1} \int_a^T \Delta \phi_2(x) \frac{d}{dt} T_{T_+}^{(1-\alpha_2)} \phi_1(t) (T_{T_+}^{(1-\alpha_2)} u) dt dx - \int_{\Omega_1} \int_a^T \Delta \phi_2(x) \phi_1(t) u \frac{dt}{t} dx \\
= \int_{\Omega_1} \int_a^T |u|^p \phi_2(x) \phi_1(t) \frac{dt}{t} dx.
\end{align*}
\] (20)

Since \(\phi_1 \in C^1[a, b]\), then there exists a constant \(M > 0\) such that \(|\phi_1(t)| \leq M\). Hence
\[
|T_{T_+}^{(1-\alpha)} \phi_1(t)| \leq \frac{M}{\Gamma(\beta(1-\alpha))} \int_t^\tau \left( \log \frac{\tau}{t} \right)^{\beta(1-\alpha_i)-1} \frac{d\tau}{\tau}
\leq \frac{M}{\Gamma(\beta(1-\alpha_i)+1)} \left( \log \frac{T-t}{t} \right)^{\beta(1-\alpha_i)},
\]
where \(i = 1, 2\). We see that \((T_{T_+}^{(1-\alpha)} \phi_1)(T) = \operatorname{lim}_{t \to T} T_{T_+}^{(1-\alpha)} \phi_1(t) = 0\) and
\[
(T_{T_+}^{(1-\beta)(1-\alpha_1)} u)(a^+, x) = (D_{a^+}^{(\beta-1)(1-\alpha_1)} u)(a^+, x) = u_0(x).
\] (21)

It appears from Lemma\(2.3\) that
\[
\left| \left( \log \frac{t}{a} \right)^{1-\gamma} u(., x) \right| \leq M,
\]
for \(1 - \gamma < (1 - \beta)(1 - \alpha_2)\). We can deduce
\[
|T_{a^+}^{(1-\beta)(1-\alpha_2)} u| \leq \frac{1}{\Gamma((1-\beta)(1-\alpha_2))} \int_a^t \left| \left( \log \frac{\tau}{t} \right)^{(1-\beta)(1-\alpha_2)-1} \frac{d\tau}{\tau} \right|
\leq M \left( \log \frac{t}{a} \right)^{\gamma-1} \int_a^t \left( \log \frac{\tau}{t} \right)^{(1-\beta)(1-\alpha_2)-1} \frac{d\tau}{\tau}
\leq M \left( \log \frac{t}{a} \right)^{\gamma-1+(1-\beta)(1-\alpha_2)}
\]
(22)
Therefore

\[
\left( \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u \right)(a, x) = 0.
\]  

(23)

Taking into account the above relations (21) and (23) in (20), we find

\[
- \int_{\Omega_1} \int_a^T \varphi_2(x) \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_1)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_1)} u dt dx \\
+ \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_2)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u dt dx - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \varphi_1(t) u \frac{dt}{t} dx
\]

\[
= \int_{\Omega_1} \int_a^T |u| \varphi_2(x) \varphi_1(t) \frac{dt}{t} dx + \int_{\Omega_1} \varphi_2(x) (\mathcal{I}_{T^-}^{\beta(1-\alpha_1)} \varphi_1)(a) u_0(x) dx.
\]  

(24)

Let

\[
A_1 = - \int_{\Omega_1} \int_a^T \varphi_2(x) \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_1)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_1)} u dt dx,
\]

and

\[
A_2 = \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_2)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u dt dx.
\]

Multiplying \( A_1 \) and \( A_2 \) by \( t/t \), we see that

\[
A_1 = \int_{\Omega_1} \int_a^T \varphi_2(x) \left( -t \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_1)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_1)} u \right) \frac{dt}{t} dx, 
\]

(25)

and

\[
A_2 = - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \left( -t \frac{d}{dt} \mathcal{I}_{T^-}^{\beta(1-\alpha_2)} \varphi_1(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u \right) \frac{dt}{t} dx.
\]

(26)

Definition 2.2 allows us to write

\[
A_1 = \int_{\Omega_1} \int_a^T \varphi_2(x) \left( \mathcal{D}_{T^-}^{1-\beta(1-\alpha_1)} \varphi_1 \right)(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_1)} u \frac{dt}{t} dx,
\]

(27)

and

\[
A_2 = - \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \left( \mathcal{D}_{T^-}^{1-\beta(1-\alpha_2)} \varphi_1 \right)(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u \frac{dt}{t} dx.
\]

(28)

In view of Lemma 2.2 and (14), we get

\[
\left( \mathcal{D}_{T^-}^{1-\beta(1-\alpha_i)} \varphi_1 \right)(t) = \frac{-1}{\Gamma(\beta(1-\alpha_i))} \int_t^T \left( \log \frac{s}{t} \right)^{\beta(1-\alpha_i)-1} \varphi_1'(s) ds \\
= - \left( \mathcal{I}_{T^-}^{\beta(1-\alpha_i)} \delta \varphi_1 \right)(t), \quad i = 1, 2.
\]

(29)

According to (29), we have

\[
A_1 = - \int_{\Omega_1} \int_a^T \varphi_2(x) \left( \mathcal{I}_{T^-}^{\beta(1-\alpha_1)} \delta \varphi_1 \right)(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_1)} u \frac{dt}{t} dx,
\]

(30)

and

\[
A_2 = \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \left( \mathcal{I}_{T^-}^{\beta(1-\alpha_2)} \delta \varphi_1 \right)(t) \mathcal{I}_{a+}^{(1-\beta)(1-\alpha_2)} u \frac{dt}{t} dx.
\]

(31)
Note that $\delta \varphi_1 \in L^p([a,T])$ and by the same arguments as in the proof of $(D_{a+}^\gamma u)(t) \in X^{-1/p}$ we may show that $I_{a+}^{(1-\beta)(1-\alpha)}u \in X^{-1/p}$ since $I_{a+}^{1-\gamma}u \in C_{1-\gamma,\log}[a,T]$.

Therefore, we see that Lemma 2.1 is satisfied

$$A_1 = - \int_{\Omega_1} \int_a^T \varphi_2(x) \delta \varphi_1(t) \left( I_{a+}^{1-\alpha_1} u \right) (t, x) \frac{dt}{t} dx,$$

$$A_2 = \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \delta \varphi_1(t) \left( I_{a+}^{1-\alpha_2} u \right) (t, x) \frac{dt}{t} dx.$$  

Lemma 2.4 yields

$$A_1 = - \int_{\Omega_1} \int_a^T \varphi_2(x) \delta \varphi_1(t) \left( I_{a+}^{1-\alpha_1} u \right) (t, x) \frac{dt}{t} dx,$$

$$A_2 = \int_{\Omega_1} \int_a^T \Delta \varphi_2(x) \delta \varphi_1(t) \left( I_{a+}^{1-\alpha_2} u \right) (t, x) \frac{dt}{t} dx.$$  

By Definition 2.1 and the property of $\varphi_1$, we have

$$A_1 \leq \frac{1}{\Gamma(1 - \alpha_1)} \int_{\Omega_1} \int_a^T \varphi_2(x) \delta \varphi_1(t) \left( \log \frac{t}{s} \right)^{\alpha_1 - 1} |u(s, x)| \frac{ds}{s} dt dx,$$

$$\leq \frac{1}{\Gamma(1 - \alpha_1)} \int_{\Omega_1} \int_a^T \varphi_2(x) \delta \varphi_1(t) \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( \log \frac{t}{s} \right)^{\alpha_1 - 1} |u(s, x)| \frac{\varphi_1^{1/p}(s)}{s} ds dt dx,$$

$$\leq \int_{\Omega_1} \int_a^T \varphi_2(x) \delta \varphi_1(t) \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( \log \frac{t}{s} \right)^{\alpha_1 - 1} |u(s, x)| \frac{\varphi_1^{1/p}(s)}{s} ds \frac{dt}{t} dx.$$  

A similar analysis for $A_2$, we get

$$A_2 \leq \frac{1}{\Gamma(1 - \alpha_2)} \int_{\Delta \Omega_1} \int_{\theta T} |\Delta \varphi_2(x)| \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( \log \frac{t}{s} \right)^{\alpha_2 - 1} |u(s, x)| \frac{\varphi_1^{1/p}(s)}{s} ds \frac{dt}{t} dx,$$

$$\leq \int_{\Delta \Omega_1} \int_{\theta T} |\Delta \varphi_2(x)| \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( \log \frac{t}{s} \right)^{\alpha_2 - 1} |u(s, x)| \frac{\varphi_1^{1/p}(s)}{s} \frac{dt}{t} dx,$$  

where $\Delta \Omega_1 := \{ x \in \Omega : T^{\alpha_2} \leq \| x \| \leq 2T^{\alpha_2} \}$. We obtain from (24), (36) and (37)

$$\int_{\Omega_1} \int_a^T |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{t} dx + \int_{\Omega_1} \varphi_2(x) (I_{a+}^{\beta(1-\alpha_1)} \varphi_1(a) u_0(x)) dx$$

$$\leq \int_{\Omega_1} \int_{\theta T} \varphi_2(x) \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( I_{a+}^{1-\alpha_1} |u| \varphi_1^{1/p} \right) (t, x) \frac{dt}{t} dx$$

$$+ \int_{\Delta \Omega_1} \int_{\theta T} \left| \Delta \varphi_2(x) \right| \left( \frac{\delta \varphi_1(t)}{\varphi_1^{1/p}(t)} \right) \left( I_{a+}^{1-\alpha_2} |u| \varphi_1^{1/p} \right) (t, x) \frac{dt}{t} dx + \int_{\Delta \Omega_1} \int_a^T |\Delta \varphi_2(x)| \varphi_1(t) u \frac{dt}{t} dx.$$
The condition \( u_0 \geq 0 \) yields
\[
\int_{\Omega} \int_{a}^{T} |u|^p \varphi_2(x) \varphi_1(t) dt \, dx \\
\leq \int_{\Omega} \int_{a}^{T} \varphi_2(x) \frac{|\delta \varphi_1(t)|}{\varphi_1^{1/p'}(t)} \left( I_{a}^{-1-\alpha}|u| \varphi_1^{1/p}(t) \right) dt \, dx \\
+ \int_{\Delta \Omega} \int_{a}^{T} |\varphi_1(t)|^p \left( I_{a}^{-1-\alpha}|u| \varphi_1^{1/p}(t) \right) dt \, dx + \int_{\Delta \Omega} \int_{a}^{T} |\varphi_2(x)| \varphi_1(t) u \frac{dt}{t} \, dx.
\] (38)

It is easy to prove that \( |u \varphi_1^{1/p}| \in X_{-1/p}' \) since \( u(., x) \in C_{1-\gamma, 0}[a, T] \). Thus, we can apply Lemma 2.1 to obtain
\[
\int_{\Omega} \int_{a}^{T} |u|^p \varphi_2(x) \varphi_1(t) dt \, dx \\
\leq \int_{\Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) dt \, dx \\
+ \int_{\Delta \Omega} \int_{a}^{T} |\varphi_2(x)| \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) dt \, dx + \int_{\Delta \Omega} \int_{a}^{T} |\varphi_2(x)| \varphi_1(t) u \frac{dt}{t} \, dx.
\] (39)

Using Young inequality with parameters \( p \) and \( p' = \frac{p}{p-1} \), we have
\[
\int_{\Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) dt \, dx \\
\leq \frac{1}{6p} \int_{\Omega} \int_{a}^{T} |u|^p \varphi_1(t) \varphi_2(x) \frac{dt}{t} \, dx \\
+ \frac{6p-1}{p'} \int_{\Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) \frac{dt}{t} \, dx
\]
\[
\leq \frac{1}{6p} \int_{\Omega} \int_{a}^{T} |u|^p \varphi_1(t) \varphi_2(x) \frac{dt}{t} \, dx \\
+ \frac{6p-1}{p'} \int_{\Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) \frac{dt}{t} \, dx,
\] (40)

\[
\int_{\Delta \Omega} \int_{a}^{T} |\varphi_1(t)| \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) dt \, dx \\
\leq \frac{1}{6p} \int_{\Omega} \int_{a}^{T} |u|^p \varphi_1(t) \varphi_2(x) \frac{dt}{t} \, dx \\
+ \frac{6p-1}{p'} \int_{\Delta \Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) \frac{dt}{t} \, dx
\]
\[
\leq \frac{1}{6p} \int_{\Omega} \int_{a}^{T} |u|^p \varphi_1(t) \varphi_2(x) \frac{dt}{t} \, dx \\
+ \frac{6p-1}{p'} \int_{\Delta \Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) \frac{dt}{t} \, dx,
\] (41)

and
\[
\int_{\Delta \Omega} \int_{a}^{T} |\varphi_2(x)| \varphi_1(t) u \frac{dt}{t} \, dx \\
\leq \frac{1}{6p} \int_{\Omega} \int_{a}^{T} |u|^p \varphi_1(t) \varphi_2(x) \frac{dt}{t} \, dx \\
+ \frac{6p-1}{p'} \int_{\Delta \Omega} \int_{a}^{T} \varphi_2(x) \left( I_{a}^{-1-\alpha}|\delta \varphi_1(t)| \varphi_1^{1/p'}(t) \right) \frac{dt}{t} \, dx.
\] (42)
Using inequalities (39), (40), (41) and (42), we obtain the inequality

\[
\left(1 - \frac{1}{2p}\right) \int_{T_0}^{T} |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{t} dx \\
\leq \frac{6^{p-1}}{p'} \int_{T_0}^{T} \int_{\Omega} \varphi_1(t) \left| \int_{T_0}^{T} \varphi_1(t) \frac{dt}{t} \right| dx \\
+ \frac{6^{p-1}}{p'} \int_{\Omega} \varphi_1(t) \int_{T_0}^{T} \varphi_2(x) \left| \int_{T_0}^{T} \varphi_1(t) \frac{dt}{t} \right| dx \\
+ \frac{6^{p-1}}{p'} \int_{\Omega} \varphi_1(t) \int_{T_0}^{T} \varphi_2(x) \left| \int_{T_0}^{T} \varphi_1(t) \frac{dt}{t} \right| dx.
\]

(43)

We introduce the following scaled variable

\[ \tau = \frac{t}{T}, \quad T \gg 1. \]

It appears that

\[
\int_{\theta T}^{T} |r_1^{1-a_1} |\delta \varphi_1(t)| |r' | dr = \frac{1}{\Gamma' (1 - a_1)} \int_{\theta T}^{T} \left( \int_{\Omega} \left( \log \frac{s}{T} \right) |r_1^{1-p} (s)| ds \right) \frac{dr}{r}. \\
= \frac{1}{\Gamma' (1 - a_1)} \int_{\theta}^{1} \left( \frac{\int_{\tau T}^{T} \left( \log \frac{s}{T} \right) |r_1^{1-p} (r)| dr}{\tau} \right) \frac{dr}{r},
\]

for \( i = 1, 2. \) Another change of variable \( r = \frac{r}{T} \) yields

\[
\int_{\theta}^{1} \left( \frac{\int_{\tau T}^{T} \left( \log \frac{s}{T} \right) |r_1^{1-p} (s)| ds}{\tau} \right) \frac{dr}{r} = \int_{\theta}^{1} \left( \frac{\int_{\tau}^{1} \left( \log \frac{r}{\tau} \right) |r_1^{1-p} (r)| dr}{\tau} \right) \frac{dr}{r}.
\]

(44)

Since \( \varphi_1 \in C^1[a, \infty) \), we assume without loss of generality that

\[
\int_{\tau}^{1} \left( \log \frac{r}{\tau} \right) |r_1^{1-p} (r)| dr \leq M,
\]

for \( M > 0. \) Then Eq (44) becomes

\[
\frac{1}{\Gamma' (1 - a_1)} \int_{\theta}^{1} \left( \frac{\int_{\tau}^{1} \left( \log \frac{r}{\tau} \right) |r_1^{1-p} (r)| dr}{\tau} \right) \frac{dr}{r} < C \int_{\theta}^{1} \frac{dr}{\tau}.
\]

Putting \( \theta = 1 - e^{-T} \) with \( T > a > 0 \), we obtain

\[
\int_{\theta T}^{T} |r_1^{1-a_1} |\delta \varphi_1(t)| |r' | dr \leq Ce^{-T},
\]

(45)

for some positive \( C \) independent of \( T \).
Next, using the change of variable $y = \frac{\|x\|}{T^{\alpha/2}}$, we get
\[
\int_{\Delta \Omega_1} \varphi_2(x) \frac{\partial^\prime}{\partial T} |\Delta \varphi_2(x)|^{p'} dx = T^{\alpha_2 N - 2\alpha_2 p'} \int_{1 \leq \|y\| \leq 2} \left[ \Phi(\|y\|) \right]^{-\frac{\mu}{p-1}} \left| \Delta \left[ \Phi(\|y\|) \right] \right|^\mu_{\frac{\mu}{p-1}} dy \\
\leq T^{\alpha_2 N - 2\alpha_2 p'} \int_{1 \leq \|y\| \leq 2} \left[ \Phi(\|y\|) \right]^{-\frac{\mu(p-1)-2\mu}{p-1}} dy \\
\leq T^{\alpha_2 N - 2\alpha_2 p'}.
\] (46)

Combining (43), (45) and (46), we get
\[
\frac{1}{2p} \int_{\Omega_1} \int_0^T |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{T} dx < C e^{-T T^{\alpha_2 N}} + C T^{\alpha_2 N - 2\alpha_2 p'} + C T^{\alpha_2 N - 2\alpha_2 p'+1},
\] (47)
when $T \to +\infty$, we obtain
\[
\lim_{T \to +\infty} e^{-T T^{\alpha_2 N}} = 0,
\]
and
\[
\lim_{T \to +\infty} T^{\alpha_2 N - 2\alpha_2 p'+1} = 0.
\]
Therefore
\[
\lim_{T \to +\infty} \int_{\Omega_1} \int_0^T |u|^p \varphi_2(x) \varphi_1(t) \frac{dt}{T} dx = 0.
\] (48)
This leads to a contradiction. □

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