A premier analysis of supersymmetric closed string tachyon cosmology

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Abstract. From a previously found worldline supersymmetric formulation for the effective action of the closed string tachyon in a FRW background, the Hamiltonian of the theory is constructed, by means of the Dirac procedure, and written in a quantum version. Using the supersymmetry algebra we are able to find solutions to the Wheeler-DeWitt equation via a more simple set of first order differential equations. Finally, for the \(k = 0\) case, we compute the expectation value of the scale factor with a suitably potential also favored in the present literature. We give some interpretations of the results and state future work lines on this matter.

1. Introduction
Tachyons correspond to the lowest mode of string theory, which for the lack of knowledge on how to handle unstable configurations were ignored for many years. With the inclusion of supersymmetry they can be consistently eliminated from the spectrum by the GSO truncation. Some years ago it has been seen that the evolution of these tachyonic instabilities can be described by the condensation of the tachyonic modes. This was first performed in the simpler case of open strings, resumed by the well known Sen conjectures [1]. Whereas for closed strings the situation is more complicated, because it involves the structure of space-time. A key element for the development of the present work is the interesting fact that closed string tachyons, which are nonsupersymmetric in the target space, can have worldsheet supersymmetry [2]. Supersymmetric cosmology has been studied in a variety of different schemes, we refer the interested reader to the well known books by D’Eath and Moniz for a review [3, 4]. In [5] a Lagrangian of supersymmetric tachyons in the framework of a FRW background was given in a worldline superspace where, the time variable is extended to the superspace of supersymmetry [6, 7], this is done considering the covariant formulation of one-dimensional supergravity of the so called ‘new’ \(\Theta\) variables [8, 9], which allows in a straightforward way to write supergravity invariant actions. The present work is performed along such approach, we depart from the Hamiltonian given in [5], then we give a particular, and normalizable, solution to the Wheeler-DeWitt equation via the superalgebra constraints, which act as square roots of the WDW equation. We compute the scale factor expectation value as customary. Plots of the results are given, follow up by some conclusions and final remarks on future work.
2. Closed String Tachyon Effective Action

The bosonic closed string tachyon effective action is given according to [10] as

\[ S = \frac{1}{2\kappa'^2} \int \sqrt{-g} e^{-2\phi} \left[ R + 4 (\partial \phi)^2 - (\partial T)^2 - 2V(T) \right] d^D x, \]  

(1)

where \( T \) is the closed string tachyon field, \( V(T) \) is the tachyon potential and \( \phi \) is the dilaton field. We write this action in the Einstein frame, by means of \( g^{string}_{\mu\nu} = e^\phi g^{Einstein}_{\mu\nu} \), which is more suitable for our cosmological approach, in this frame and for a 4-dimensional FRW metric the action takes the form

\[ S = \int \left[ -\frac{3\alpha'^2}{\kappa'^2 N} + \frac{3Nk}{\kappa'^2} - \frac{a^3}{2\kappa'^2 N} \dot{T}^2 - \frac{a^3 N e^{2\phi} V(T)}{\kappa'^2} \right] dt, \]  

(2)

where \( N \) is the lapse function and \( a \) is the scale factor. The natural invariance of this Lagrangian under time reparametrizations is extended to supersymmetry introducing a Grassmann superspace associated to the bosonic time coordinate \( t \) (see Tkach et al. in [6, 7]).

3. Supersymmetric Closed String Tachyon Model

Superspace is the natural framework for a geometrical formulation of supersymmetry and supergravity [11], it extends spacetime by anticommuting Grassmann variables, \( x^m \to (x^m, \theta^\mu) \), and the field content of the superfields is given by the Grassmann power expansion in the anticommuting variables \( \phi(z) = \sum_n 1/n! \theta^{\mu_1} \cdots \theta^{\mu_n} \phi_{\mu_1 \cdots \mu_n}(x) \).

The supersymmetric cosmological model for the action (2) is obtained upon such extension of the time coordinate into a supermultiplet \( t \to (t, \Theta, \bar{\Theta}) \). Due to this generalization the fields of the theory also are generalized as superfields, the power expansion mentioned above is given explicitly by

\[ \mathcal{A}(t, \Theta, \bar{\Theta}) = a(t) + i\Theta \lambda(t) + i\bar{\Theta} \lambda(t) + B(t) \Theta \bar{\Theta}, \]
\[ \mathcal{T}(t, \Theta, \bar{\Theta}) = T(t) + i\Theta \eta(t) + i\bar{\Theta} \eta(t) + G(t) \Theta \bar{\Theta}, \]
\[ \Phi(t, \Theta, \bar{\Theta}) = \phi(t) + i\Theta \chi(t) + i\bar{\Theta} \chi(t) + F(t) \Theta \bar{\Theta}, \]  

(3)

where, \( \mathcal{A}, \mathcal{T} \) and \( \Phi \) are the superfields corresponding to \( a, T \) and \( \phi \), respectively.

The supersymmetric generalization of the action is

\[ S = S_{Rsusy} + S_{Msusy}, \]  

(4)

where, \( S_{Rsusy} \) is the cosmological supersymmetric generalization of the free FRW model

\[ S_{Rsusy} = \frac{1}{\kappa'^2} \int \left( 3\mathcal{E} \nabla_{\phi} \mathcal{A} \nabla_{\phi} \mathcal{A} - 3\sqrt{k} \mathcal{A}^2 \right) d\Theta d\bar{\Theta} dt, \]  

(5)

and the supersymmetric matter term is

\[ S_{Msusy} = \frac{1}{\kappa'^2} \int \left[ -\mathcal{E} \mathcal{A} \nabla_{\phi} \Phi \nabla_{\phi} \Phi - \frac{1}{2} \mathcal{E} \mathcal{A} T \nabla_{\phi} \mathcal{T} + \mathcal{E} \mathcal{A} W(\Phi, \mathcal{T}) \right] d\Theta d\bar{\Theta} dt, \]  

(6)

where \( W(\Phi, \mathcal{T}) \) is the superpotential, and \( \mathcal{E} \) is an invariant density playing the role of a supersymmetric \( \sqrt{-g} \) in the action and given by

\[ \mathcal{E} = -e - \frac{i}{2}(\Theta \bar{\Psi} + \bar{\Theta} \psi), \]  

(7)

we refer the interested reader to [5, 8, 9] for full details of its derivation.
The superpotential is also written as a power series expansion, given as

$$W(\Phi, T) = W(\phi, T) + \frac{\partial W}{\partial \phi}(\Phi - \phi) + \frac{\partial W}{\partial T}(T - T) + \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2}(\Phi - \phi)^2 + \frac{1}{2} \frac{\partial^2 W}{\partial T^2}(T - T)^2 (\Phi - \phi).$$

This expansion is finite because the terms $(T - T)$ and $(\Phi - \phi)$ are purely Grassmannian and nilpotency holds for these kind of variables.

The total Lagrangian is found integrating over the Grassmann parameters $\Theta$ and $\bar{\Theta}$. After we perform the variation of the Lagrangian with respect to the fields $B$, $F$ and $G$ they can be eliminated from the Lagrangian, as usual their equations of motion are algebraic constraints, that is $B$, $F$ and $G$ are auxiliary fields. Upon solving for the auxiliary fields and making the further rescalings for convinience $\lambda \rightarrow \kappa \alpha^{-1/2} \lambda$, $\bar{\lambda} \rightarrow \kappa \alpha^{-1/2} \bar{\lambda}$, $\eta \rightarrow \kappa \alpha^{-3/2} \eta$, $\bar{\eta} \rightarrow \kappa \alpha^{-3/2} \bar{\eta}$, $\chi \rightarrow \kappa \alpha^{-3/2} \chi$, $\bar{\chi} \rightarrow \kappa \alpha^{-3/2} \bar{\chi}$, the Lagrangian is written as

$$L = -\frac{3\alpha a^2}{\kappa^2} - \frac{3\sqrt{a}a^3}{\kappa} \left( \psi \lambda - \bar{\psi} \bar{\lambda} \right) - \frac{3\kappa a^3}{\kappa^2} + \frac{\bar{\eta}^2 a^3}{\kappa^2} - \frac{\sqrt{a}a^3}{\kappa^2} \left( \psi \eta - \bar{\psi} \bar{\eta} \right) + \frac{3i\kappa a^3}{2} \left( \lambda \bar{\eta} + \bar{\lambda} \eta \right)$$

where the subscripts in $W$ denote partial differentiation with respect to $\phi$ and $T$, respectively.

As usual, when we compute the canonical momenta, in order to obtain the Hamiltonian of the theory, we have the appearance of fermionic constraints, which are second class. Thus, we must follow Dirac formalism to obtain the dynamics of the system. Using the standard definition for the Hamiltonian and according to Dirac’s procedure we can write

$$H = NH_0 + \frac{1}{2} \psi S - \frac{1}{2} \bar{\psi} \bar{S},$$

where

$$H_0 = -\frac{3\alpha a^2}{2\kappa^2} + \frac{\kappa^2 a^2}{2\kappa} - \frac{3\alpha^2 \pi_0}{4\alpha^2} \left( \lambda \bar{\eta} + \bar{\lambda} \eta \right) + \frac{\kappa^2 a^2}{4\alpha^2} - \frac{3\alpha^2 \pi_0}{4\alpha^2} \left( \lambda \bar{\chi} + \bar{\lambda} \chi \right) - \frac{3\alpha^2 \pi_0}{4\alpha^2} W^2$$

and

$$S = \frac{\kappa a^2}{\sqrt{a}^2 \kappa^2} \lambda + \frac{\kappa \pi_0}{\sqrt{a}^2 \kappa^2} \eta + \frac{\alpha^2 \pi_0}{\sqrt{a}^2 \kappa^2} \chi - \frac{6i\sqrt{a} \pi_0}{\kappa} \lambda + \frac{3\alpha \sqrt{a} \pi_0}{\kappa} \lambda \bar{\chi} + \frac{3\alpha \sqrt{a} \pi_0}{\kappa} \bar{\lambda} \chi,$$

$$\bar{S} = \frac{\kappa \sqrt{a}}{\kappa \sqrt{a}} \bar{\lambda} + \frac{\kappa \pi_0}{\kappa \sqrt{a}} \bar{\eta} + \frac{\alpha \pi_0}{\kappa \sqrt{a}} \bar{\chi} + \frac{6i\sqrt{a} \pi_0}{\kappa} \bar{\lambda} - \frac{3\alpha \sqrt{a} \pi_0}{\kappa} \bar{\lambda} \bar{\chi},$$

are the set of first class constraints satisfying the Dirac algebra $\{S, \bar{S}\}_D = 2H_0$, $\{H_0, S\}_D = \{H_0, \bar{S}\}_D = 0$. 

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4. Wave function of the universe

In the quantum version the constraints (9)-(11) become conditions on the wave function of the universe, i.e., \( H\Psi = \bar{S}\Psi = S\Psi = 0 \), which due to the operator superalgebra imply that any state wave function, \( \Psi \), satisfying \( \bar{S}\Psi = S\Psi = 0 \) is also solution of the Wheeler-DeWitt equation, \( H\Psi = 0 \). We have for the quantum constraints:

\[
S = \frac{\kappa \pi a}{\sqrt{a}} \lambda + \frac{\kappa \pi \phi}{\sqrt{a^3}} \chi + \frac{\kappa \pi T}{\sqrt{a^3}} \eta + \frac{3i\sqrt{a^3}W}{\kappa} \lambda + \frac{i\sqrt{a^3}W_T}{\kappa} \eta + \frac{i\sqrt{a^3}W_\phi}{\kappa} \chi \\
+ \frac{3i\kappa}{4a^{3/2}} \lambda [\eta, \bar{\eta}] + \frac{3i\kappa}{2a^{3/2}} \lambda [\chi, \bar{\chi}] - \frac{6i\sqrt{a}k}{\kappa} \lambda. 
\]

\[
\bar{S} = \frac{\kappa \pi a}{\sqrt{a}} \bar{\lambda} + \frac{\kappa \pi \phi}{\sqrt{a^3}} \bar{\chi} + \frac{\kappa \pi T}{\sqrt{a^3}} \bar{\eta} - \frac{3i\sqrt{a^3}W}{\kappa} \bar{\lambda} - \frac{i\sqrt{a^3}W_T}{\kappa} \bar{\eta} - \frac{i\sqrt{a^3}W_\phi}{\kappa} \bar{\chi} \\
- \frac{3i\kappa}{4a^{3/2}} \bar{\lambda} [\eta, \bar{\eta}] - \frac{3i\kappa}{2a^{3/2}} \bar{\lambda} [\chi, \bar{\chi}] + \frac{6i\sqrt{a}k}{\kappa} \bar{\lambda}. 
\]

For details on this calculations we refer the reader to [12] where we computed a wave function of the universe for a supersymmetric tachyon in a FRW background, providing a normalizable solution in which the unstable nature of the tachyon can be appreciated; in such study the action (2) was used with the dilaton turned off, whereas in [13] the solution of the Wheeler-DeWitt equation for the complete theory was addressed, where it is introduced a representation for the Grassmann variables in terms of suitable linear combinations of 8-dimension Dirac matrices, therefore the wave function will be an eight-component spinor \( \Psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8) \).

Upon application of the operators \( S \) and \( \bar{S} \) on this state, we get two sets of eight equations:

\[
\left( \frac{\hbar}{\kappa} \frac{\partial}{\partial \phi} - \frac{a^3}{\kappa} \frac{\partial W}{\partial \phi} \right) \psi_1 = 0, \\
\left( \frac{\hbar}{\kappa} \frac{\partial}{\partial T} - \frac{a^3}{\kappa} \frac{\partial W}{\partial T} \right) \psi_1 = 0, \\
\left( \hbar \kappa^2 \frac{\partial}{\partial a} - 3a^3W + 6a^2k + \frac{9}{8} \hbar \right) \psi_1 = 0, 
\]

and

\[
\left( \hbar \kappa^2 \frac{\partial}{\partial \phi} + \frac{a^3}{\kappa} \frac{\partial W}{\partial \phi} \right) \psi_6 = 0, \\
\left( \hbar \kappa \frac{\partial}{\partial T} + \frac{a^3}{\kappa} \frac{\partial W}{\partial T} \right) \psi_6 = 0, \\
\left( \hbar \kappa^2 \frac{\partial}{\partial a} + 3a^3W - 6a^2k + \frac{9}{8} \hbar \right) \psi_6 = 0, 
\]

where we have chosen the particular solution for which \( \psi_2 = \psi_3 = \psi_4 = \psi_5 = \psi_7 = \psi_8 = 0 \).

Solving these equations we find that

\[
\psi_1 = C_1 a^x \exp \left[ - \frac{a^{3/2} \left( 3\sqrt{ak} - \sqrt{a^3W(\phi,T)} \right)}{\kappa^2 \hbar} \right]
\]
and

\[ \psi_0 = C_2 a^c \exp \left[ \frac{a^{3/2} \left( 3\sqrt{a}k - \sqrt{a^3}W(\phi, T) \right)}{\kappa^2 \hbar} \right] , \]

with \( c = -\left( \frac{c}{2} + \frac{7}{8} \right) \) being a free parameter arising from the operator ordering. These componentes are normalizable provided that \( W(\phi, T \to \pm \infty) > 0 \) for \( \psi_0 \) and \( \psi_1 \) is non-normalizable in this case. On the other hand, if the superpotential \( W(\phi, T \to \pm \infty) < 0 \) then \( \psi_0 \) is non-normalizable and \( \psi_1 \) is normalizable, therefore these are non-degenerate eigenstates of the Hamiltonian.

5. Scale factor expectation value

We set \( W(\phi, T) = \frac{1}{4} e^\phi \left( \frac{m^2 T^4}{24} - \frac{m^2 T^3}{12} - \left( \frac{m^4}{160} + \frac{m^2}{240} \right) T^5 + e^T \right) \), this superpotential solution is consistent with the case \( k = 0 \) for the FRW metric as reported in [5]. Thus we have

\[ \psi_0(\phi, T) = a^c \exp \left[ \frac{a^3 \left( 3m^4 T^5 + 2m^2 T^5 - 20m^2 T^4 + 40m^2 T^3 - 480e^T \right)}{480\sqrt{2}\kappa^2 \hbar} \right] , \]

for the normalizable component with \( \phi = 0 \).

![Figure 1](image-url)  

**Figure 1.** Dependence on \( a \) and \( T \) of the wave function of the universe.

Regarding \( T \) as a time field we can normalize this wave function for \( a \in (0, \infty) \), this results in

\[ \psi_0(T) = N(T) \exp \left[ \frac{a^3 \left( 3m^4 T^5 + 2m^2 T^5 - 20m^2 T^4 + 40m^2 T^3 - 480e^T \right)}{480\sqrt{2}\kappa^2 \hbar} \right] , \]

where

\[ N(T) = \frac{2\frac{1}{2}(-3)(2c+1)3\frac{1}{2} - \frac{7}{5} - \frac{5}{8} - \frac{3}{8} a^c \left( \frac{-3m^4 T^5 - 2m^2 T^5 + 20m^2 T^4 - 40m^2 T^3 + 480e^T}{\kappa^2 \hbar} \right)^{\frac{1}{2}}}{\Gamma \left( \frac{2c}{3} + \frac{1}{3} \right)} \]

With this wave function we can find the expectation value of the scale factor, \( \langle a \rangle \), and the quantum fluctuations, \( \sigma_a \), explicitly we have

\[ \langle a \rangle = \frac{2\sqrt{2} \sqrt{15} \Gamma \left( \frac{2(c+1)}{3} \right)}{\Gamma \left( \frac{2c}{3} + \frac{1}{3} \right) \sqrt{480e^T - m^2 T^3((3m^2 + 2)T^2 - 20T + 40)}} \]

(17)
and

\[
\sigma_a = 2^{\frac{3}{15}} \sqrt{\frac{2\Gamma \left( \frac{2c}{3} + \frac{1}{3} \right) \Gamma \left( \frac{2c}{3} + 1 \right) - 2\Gamma \left( \frac{2(c+1)}{3} \right)^2}{\Gamma \left( \frac{2c}{3} + \frac{1}{3} \right)^2 \left( \frac{480e^{2} - m^2 T^3(T((3m^2+2)T-20)+40)}{\kappa^2\hbar} \right)^{2/3}}} \tag{18}
\]

The behavior of \( \langle a \rangle + \sigma_a \), \( \langle a \rangle \) and \( \langle a \rangle - \sigma_a \) can be seen in the figure below, where the collapse of the universe can be seen directly, which is consistent with the nature of this tachyon potential as stated in [10].

![Figure 2. Profile of \( \langle a \rangle \) plus fluctuations for \( \kappa = 1, \hbar = 1, m = 1, c = 1 \) and \( \phi = 0 \).](image)

6. Conclusions
We have been capable of extracting classical information from a quantum cosmological model from first principles, by means of the computation of the expectation value of the scale factor for a FRW model. The behavior that was found, for the potential used in this work, is consistent with the properties this potential exhibits as given in the analysis by Zwiebach et al., in which the final stage of the evolution of the Universe is a collapsing scenario as the tachyon rolls down. Although we have set \( \phi = 0 \) our study is complete since the roll played by the dilaton is, as expected, that of an overall scale factor. We are working towards obtaining a more detailed analysis of our results and some further predictions. Also, it would be interesting to probe some other proposals for the tachyon potential, specially those with suitable supersymmetric properties.

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