Inverse square law of gravitation in (2+1)-dimensional space-time as a consequence of Casimir energy

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Abstract: The gravitational effect of vacuum polarization in space exterior to a particle in (2+1)-dimensional Einstein theory is investigated. In the weak field limit this gravitational field corresponds to an inverse square law of gravitational attraction, even though the gravitational mass of the quantum vacuum is negative. The paradox is resolved by considering a particle of finite extension and taking into account the vacuum polarization in its interior.

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1 Introduction

Even though space-time is the arena of all physics, space-time by itself is not fully understood. The main remaining theoretical problem is the unification of General Relativity with quantum theory. A possible way to reach a deeper understanding of the problem and maybe find the key to quantum gravity is to study simpler theories that have some of the characteristics of General Relativity. The present study is meant to resolve a paradox encountered in semi-classical (2+1)-dimensional Einstein theory, namely the problem that the Casimir energy outside a point particle gives rise to an attractive gravitational field even though the gravitational mass of this field is negative. For this purpose a general expression for the (2+1)-dimensional Tolman mass is derived.

The metric is algebraically determined by the energy-momentum tensor in (2+1)-dimensional Einstein theory [1-5]. On account of this, space-time is flat in the empty classical vacuum, and there is no Newtonian limit to Einstein’s theory in ‘flatland’ [4].

Yet gravitation does exist also in empty two dimensional space, not in terms of local curvature but in terms of global topological effects. Thus, space outside a static point particle is characterized by an angle deficit of an otherwise Minkowskian geometry [1-5]. A static dust planet has the same exterior geometry.

This nontrivial geometry induces non-vanishing vacuum expectation values of the energy-momentum tensor of quantum fields. In a semi-classical approach one should take this energy density into account when calculating the gravitational field. This problem has recently been addressed by Souradeep and Sahni [7] who computed the vacuum expectation value of the energy-momentum tensor of massless fields exterior to a point mass and derived the gravitational backreaction induced by this energy density.

Here we find an exact solution to Einstein’s field equations for an energy-momentum tensor of the same algebraic form as the one induced by massless conformally coupled quantum fields. It is pointed out that the backreaction calculation for a point mass gives a paradoxical result: there is a negative gravitational mass outside the conical singularity but still the gravitational field corresponds to gravitational attraction. By using Tolman’s [8] method of effective gravitational mass, and deriving a formula for gravitational mass in (2+1)-dimensional gravity, the paradox is resolved by assuming an extended particle and by taking into account vacuum polarization also in the interior of the particle.
2 The gravitational field of the quantum vacuum

Let a static rotational symmetric space-time be described by the space-time metric

\[ ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 r^2 d\theta^2 \] (1)

where \( A, B \) and \( C \) are functions of the radial coordinate \( r \) only. The solution for classical vacuum exterior to a particle is \( A = B = 1 \) and \( C = C_0 \equiv 1 - G\mu \), where \( \mu \) is the mass of the particle and where \( G \) is the \((2+1)\)-dimensional Newton’s constant. We shall employ units of measurements so that the speed of light is unity. The deviation of \( C \) from unity describes the angle deficit of the cone geometry. By a suitable redefinition of the radial coordinate we may eliminate one of the metric coefficients \( B \) or \( C \). Here we choose to fix \( C \) to its classical value, \( C_0 \).

The conical topology induces a nonvanishing vacuum expectation value of the energy momentum tensor. Its form may be found by symmetry arguments. In odd-dimensional space-times there is no conformal anomaly. Besides, the background geometry is flat. Hence, the trace of the energy-momentum tensor of a massless conformally coupled scalar vanishes. Then, conformal invariance and conservation of the energy-momentum tensor in the classical background metric \((A = B = 1)\) gives an energy-momentum tensor of the form

\[ \langle T^{\mu\nu} \rangle = \frac{a}{r^3} \text{diag}[1, 1, -2] \] (2)

where the coordinates are \( x^\mu \in \{t, r, \theta\} \) and \( a \) is a constant.

By using this energy-momentum tensor as a source in Einstein’s field equations we find the quantum corrections to the classical cone geometry. With the metric on the form (1) and \( C \) constant, the Einstein tensor has the following non-vanishing components

\[ G^t_t = -\frac{B'}{Br} \frac{1}{B^2}, \] (3)

\[ G^r_r = \frac{A'}{Ar} \frac{1}{B^2}, \] (4)

\[ G^\theta_\theta = \left[ \frac{A''}{A} - \frac{A'B'}{AB} \right] \frac{1}{B^2}. \] (5)

\(^1\)For later convenience the \((2+1)\)-dimensional Einstein constant is chosen to be \( 2\pi G \), where \( G \) is a coupling constant of dimension \( 1/\text{mass} \).
With an energy-momentum tensor of the algebraic form of (2), the semi-classical field equations

$$G^\mu_\nu = 2\pi G \langle T^\mu_\nu \rangle$$

are reduced to the following two field equations

$$G^t_t = G^r_r \quad \text{and} \quad 2G^t_t = -G^\theta_\theta.$$ (7)

By introducing the new variable $u$ defined by $u \equiv r A'/A$ one obtains the following set of field equations

\begin{align*}
    du &= -u(2u + 1) \frac{dr}{r}, \\
    dA &= Au \frac{dr}{r}, \\
    dB &= -Bu \frac{dr}{r}.
\end{align*}

There are two solutions with constant $u$. The first one is the Minkowski solution $u = 0$. The second solution corresponding to $u = -1/2$, is a power law solution for $A$ and $B$ as functions of $r$. This is a solution with a positive energy density, which does not smoothly reduce to the classical solution in the limit $T^\mu_\nu \rightarrow 0$, and hence it is not the solution sought here. Having thus eliminated the solutions with constant $u$, we may use $u$ as a new radial variable instead of $r$. Hence, integrating $A$ and $B$ as functions of $u$ we get

$$A = (1 + 2u)^{-1/2} \quad \text{and} \quad B = (1 + 2u)^{1/2}.$$ (11)

Two integration constants have been determined by demanding that the metric asymptotically approaches the classical one. The solution for $r$ is

$$r = \frac{1 + 2u}{u} \ell,$$ (12)

where $\ell$ is an arbitrary integration constant of dimension length. Using these solutions to calculate Einstein’s tensor, we find the energy-momentum tensor

$$2\pi G \langle T^t_t \rangle = \frac{\ell}{r^3}.$$ (13)

Thus, the $1/r^3$-dependence of the energy-momentum tensor follows from its algebraic form when we use Einstein’s equations. Because of the Bianchi identity, energy momentum conservation also holds in the quantum corrected geometry. The integration constant $\ell$ is determined by the vacuum energy density of equation (2) as

$$\ell = 2\pi Ga.$$ (14)
Note that since the Casimir energy density, $\rho = -a/r^3$, is negative, $\ell$ is positive. Solving equation (12) for $u$ we get

$$1 + 2u = \left(1 - \frac{2\ell}{r}\right)^{-1}$$

which gives us the metric

$$ds^2 = -\left(1 - \frac{2\ell}{r}\right)dt^2 + \left(1 - \frac{2\ell}{r}\right)^{-1}dr^2 + C_0^2r^2d\theta^2.$$  

(15)

This is the equatorial section of the (3+1)-dimensional Schwarzschild metric minus a wedge. The Schwarzschild metric has a horizon which would pose a serious problem of interpretation if this were the exact solution of the backreaction problem. But here one could argue that one should expand the metric and keep only first order terms in $\ell$, since a semi-classical approximation is not believed to be valid when the perturbations of the metric become of order unity. To this it could be objected that the expansion in Schwarzschild mass is coordinate dependent: If one uses Eddington coordinates the exact solution is already of first order in $\ell$. Instead one could require the corrections to the curvature scalars to be bounded by the Planck curvature. Then one finds the constraint $r^3 \gg \ell^2\ell_P$. Now, the global geometry of the (2+1)-dimensional universe is conical only if the sum of the angle deficits of all the particles is less than $2\pi$. One might therefore argue that a single particle should have a mass much smaller than the total mass of the universe, and consequently that its angle deficit is much less than $2\pi$. In this case $\ell \ll \ell_P$, and the curvature constraint implies that only the region $r \gg \ell$ is well described by semi-classical Einstein theory. In this way, according to both cut off schemes the semi-classical approximation breaks down before the horizon is reached. If particles are to be described semi-classically, they must have radii much larger than $\ell_P$, and then the horizon problem is avoided.

Test particles in this geometry will feel an inverse square law of gravitational attraction corresponding to the gravitational potential\footnote{Souradeep and Sahni have obtained the same result in the linearized approximation to the semi-classical Einstein equations. They also calculate $\langle T^\mu_\nu \rangle$ explicitly. The results are in agreement with the symmetry considerations presented here.}

$$\Phi = -\frac{\ell}{r}.$$  

(17)
Despite its appearance, this is not the Newtonian potential. Newton’s theory gives a logarithmic potential [4] in 2+1 dimensions. On the other hand, it is something of a paradox that by adding a quantum vacuum having a negative energy density to a particle which in the classical limit produces no gravitational field, we get a positive gravitational mass and gravitational attraction.

3 Resolving the positive mass problem

By assuming a point particle at the origin one encounters the paradox that negative energy gives positive mass. Moreover, the vacuum polarization energy diverges at the origin. Since a point particle is a singularity where space-time curvature diverges, it should not be surprising that such a model gives unphysical results. To avoid this problem one has to allow for a finite radius of the source. Note that the exterior geometry remains the same, so at distances much larger than \( \ell \) one expects the same quantum field.

With a finite source it is clear that one must take into account the effect of vacuum polarization not only in the exterior of the particle but also in its interior. In connection with cosmic strings this point was stressed by Frolov et al. [11] who argued that “it is not permissible to ignore the . . . interior of the string . . . when assessing the integrated effect of vacuum polarization”. In the case of a cosmic string we also have negative vacuum energy outside the source, yet there is an attractive gravitational force there [12-14].

In general it would be difficult to calculate this effect from first principles, but in this case we know the unique exterior solution to Einstein’s field equations. Therefore, we may try to match the exterior solution to an interior one corresponding to the classical source. The misfit may be identified with a singular string on the junction according to Israel’s formalism [15], and in turn one may identify this extra term in the energy-momentum tensor with the integrated effect of vacuum polarization in the interior [14].

By matching the exterior solution

\[
\begin{align*}
  ds^2 &= - \left( 1 - \frac{2\ell C_0}{\rho} \right) dt^2 + \left( 1 - \frac{2\ell C_0}{\rho} \right)^{-1} \frac{d\rho^2}{C_0^2} + \rho^2 d\theta^2 \\
  \end{align*}
\]  

(18)

where \( \rho \equiv C_0 r \) to the interior solution of Giddings et al. [4]

\[
\begin{align*}
  ds^2 &= -dt^2 + (1 - \lambda \rho^2)^{-1} d\rho^2 + \rho^2 d\theta^2, \\
\end{align*}
\]  

(19)
where $\lambda$ is the energy-density, and using Israel’s formulae, the energy-
momentum density of the junction string is given by the discontinuity of the exterior curvature at the junction.

Defining

$$K_{ij} \equiv -\frac{1}{2} \frac{d}{dR} g_{ij} \quad \text{where} \quad dR^2 \equiv g_{\rho\rho}d\rho^2,$$

the Lanczos string energy-momentum tensor is

$$S^i_j = [K^i_j] - \delta^i_j[K]$$

where the square brackets signify the discontinuity at the junction string. It follows that the string mass per length, $\mu$, and the string tension, $\tau$, are

$$2\pi G \mu = [K^\theta_\theta] \quad \text{and} \quad 2\pi G \tau = [K^i_i].$$

Using that $C_0 = (1 - \lambda \rho_0^2)^{1/2}$ where $\rho_0$ is the radius of the particle, and $\ell = 2\pi Ga$, the mass per length and the tension of the junction string derived from equations (22) are

$$\mu = -\tau = \frac{a}{r_0},$$

to first order in $\ell$. Because the junction energy is identified with the integrated effect of the vacuum polarization energy of a massless conformally coupled field, it is reasonable that the string energy-momentum is traceless.

In (3+1)-dimensional gravitation theory Tolman’s gravitational mass is a very useful tool to understand gravitational effects. In order to derive a formula for the (2+1)-dimensional effective gravitational mass, let us in analogy with the four dimensional case, consider the acceleration of gravity, $k^i$, as measured with standard rods and coordinate clocks

$$k^i = -\sqrt{-g_{tt}} \Gamma^i_{tt}.$$

Here, $\Gamma$ represents the connection coefficient and hats indicate that it refers to an orthonormal frame. The square root of the metric coefficient is introduced in order to measure acceleration with coordinate clocks. For a metric of the form (6) with $C =$ constant, the acceleration is radially directed and given by $k = -A'/B$. By use of the (2+1)-dimensional Einstein tensor (6)-(8), one finds for continuously differentiable metrics that

$$k = -\frac{1}{r} \int_0^r \left( G_{rr} + G_{\theta\theta} \right) \sqrt{-g} dr.$$
By relating the gravitational attraction to a concept of gravitational mass, $M$, one may define $GM \equiv -kr$. Substituting $-GM/r$ for $k$ and using Einstein’s equations to substitute the energy-momentum tensor for $G^\mu_\nu$ in equation (25), we derive the following equation

$$M(r) = \int_0^{2\pi} \int_0^r \left( T^r_r + T^\theta_\theta \right) \sqrt{-g} dr d\theta. \quad (26)$$

This is a (2+1)-dimensional analogue to Tolman’s mass formula. Since $T^t_t$ is missing from this expression, it is clear that (2+1)-dimensional gravity has no Newtonian limit.

In the present case, there is a gravitational mass contribution both from the exterior quantum vacuum and from the junction shell which represents the integrated effect of vacuum polarization in the interior of the particle. To first order in the angle deficit and $\ell$, the mass contribution from the exterior Casimir energy is

$$M_C(r) = \int_0^{2\pi} \int_{r_0}^r \left( \langle T^r_r \rangle + \langle T^\theta_\theta \rangle \right) r dr d\theta = 2\pi \left[ \frac{a}{r} - \frac{a}{r_0} \right]. \quad (27)$$

Note that $M_C(r) < 0$ for all $r > r_0$, i.e. outside the particle. The junction string has the effective mass

$$M_J = \int_0^{2\pi} (-\pi) r_0 d\theta = \frac{2\pi a}{r_0} \quad (28)$$

which is a positive mass. Physically we associate the positive mass of the junction string with the integrated effect of vacuum polarization in the interior of the particle. Adding these two terms we get the total vacuum polarization mass

$$M(r) = M_C(r) + M_J = \frac{2\pi a}{r}. \quad (29)$$

Hence, the gravitational potential, $\Phi = -\ell/r$, becomes

$$\Phi = -GM(r). \quad (30)$$

This expression is Newtonian only by appearance, since $M$ is a purely relativistic mass. Note also that $M$ is a function of $r$.

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3Since $\langle T^\mu_\nu \rangle$ is of first order in $\ell$, we may neglect the angle deficit in $\sqrt{-g}$ in the first order approximation.
4 Concluding remarks

Although Einstein’s theory in three-dimensional space-time does not give Newtonian gravitation as a weak field limit, it is amusing that quantum corrections and relativistic effects give an attractive inverse square law of gravitation. Note, however, that this also is a non-Newtonian law because in three dimensional space-time Newtonian gravitation implies a logarithmic potential.

In connection with straight cosmic strings it has been shown [17] that linearized $R + R^2$ gravity produces short range gravitational forces not present in pure Einstein theory. It was argued that such forces might have had an effect on the formation of structures in the early universe [17]. It is therefore interesting that such effects appear also in semi-classical Einstein theory, although the quantum corrections to cosmic strings in Einstein’s theory may be too small to produce significant cosmological effects [14].

It is important to realize that the semi-classical corrections cannot be self-consistently treated in the point mass approximation. In view of the pathological properties of point masses, it may be unfortunate that just point masses have been given so much attention in (2+1)-dimensional gravitation theory.

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