Stability estimates for the relativistic Schrödinger equation from partial boundary data

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Abstract
We derive stability estimates for the determination of time-dependent vector and scalar potentials in the relativistic Schrödinger equation from partial boundary data. For the case of space dimensions at least 3, we obtain log–log stability estimates for the determination of vector potentials (modulo gauge equivalence) and log–log–log stability estimates for the determination of scalar potentials from partial boundary data assuming suitable \textit{a priori} bounds on these potentials.

Keywords: hyperbolic inverse problem, stability estimate, time-dependent coefficient, partial boundary data

1. Introduction and statement of the main result
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \geq 3)\) with smooth boundary \( \Gamma \). Let \( \nu(x) \) be the outward unit normal to \( \Gamma \) and \( Q = (0, T) \times \Omega \) be the finite cylindrical domain where \( T > \text{diam}(\Omega) \). We denote the lateral boundary of \( Q \) by \( \Sigma := (0, T) \times \Gamma \). The relativistic Schrödinger operator on \( Q \) denoted by \( \mathcal{L}_{A,q} \) is defined as follows

\[
\mathcal{L}_{A,q} = (\partial_t + A_0(t,x))^2 - \sum_{k=1}^n (\partial_{x_k} + A_k(t,x))^2 + q(t,x).
\]

Here \( A \equiv (A_i)_{0 \leq i \leq n} \) is the vector potential and \( q \) is the scalar potential. We assume that \( q \in L^\infty(Q) \) and all the components of the vector potential are real valued functions belonging to \( C^\infty_c(Q) \). The question of unique recovery of two time-dependent coefficients \( A \) and \( q \) has been addressed in prior works [12, 14, 15]. We are interested in deriving stability estimates for
the recovery of these coefficients from partial boundary measurements of the solutions of the following IBVP:

\[
\begin{align*}
    &L_{A\beta}(u) = 0 & \text{in} & \quad Q, \\
    &u|_{t=0} = 0 & \text{on} & \quad \partial Q, \\
    &\partial_u u|_{t=0} = 0 & \text{on} & \quad \Sigma.
\end{align*}
\]

Let us first discuss the well-posedness and some relevant results of the above IBVP. From [11, 16] it is well known that if \( u_0 \in H^1(\Omega), u_1 \in L^2(\Omega) \) and \( f \in H^1(\Sigma) \) satisfying the compatibility condition \( u_0|_{\Gamma} = f|_{[0,T]} \), then the above IBVP has a unique solution in \( C^1([0,T];L^2(\Omega)) \cap C([0,T];H^1(\Omega)) \) and there exists a constant \( C > 0 \) such that for any \( t \in [0,T] \) we have

\[
\|\partial_u u\|_{L^2(\Sigma)} + \|\partial u(t,\cdot)\|_{L^2(\Omega)} + \|u(t,\cdot)\|_{H^1(\Omega)} \leq C \left( \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Sigma)} \right).
\]

(1)

The relativistic Schrödinger operator comes up in the study of quantum mechanics and general relativity [23, chapter 12]. The inverse problem we address here focuses on the stable determination of time-evolving physical properties \( A \) and \( q \) of a medium by probing it with disturbances made at the initial time and at the lateral boundary and measuring the response on a part of the boundary. To state the problem precisely, let us define the following subsets of \( \Gamma \) and \( \Sigma \) for some \( \omega \in S^{n-1} \) and \( \epsilon > 0 \)

\[
\begin{align*}
    &\Gamma_+ (\omega) = \{ x \in \Gamma; \nu(x) \cdot \omega > 0 \}, \quad \Gamma_- (\omega) = \{ x \in \Gamma; \nu(x) \cdot \omega < 0 \}, \\
    &\Gamma_{+,\epsilon} (\omega) = \{ x \in \Gamma; \nu(x) \cdot \omega > \epsilon \}, \quad \Gamma_{-,\epsilon} (\omega) = \{ x \in \Gamma; \nu(x) \cdot \omega < \epsilon \}, \\
    &\Sigma_{+,\epsilon} (\omega) = (0,T) \times \Gamma_{+,\epsilon} (\omega), \quad \Sigma_{-,\epsilon/2} (\omega) = (0,T) \times \Gamma_{-,\epsilon/2} (\omega).
\end{align*}
\]

(2)

For a fixed \( \omega_0 \in S^{n-1} \) and small \( \epsilon > 0 \), let us define the input–output operator \( \Lambda \) as follows

\[
\Lambda : H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \mapsto L^2(\Sigma_{-,\epsilon/2}(\omega_0)) \times H^1(\Omega)
\]

(3)

\[
\Lambda(u_0, u_1, f) = \left( \partial_u u|_{\Sigma_{-,\epsilon/2}(\omega_0)}, u(T,\cdot) \right).
\]

(4)

From (1), we see that the operator \( \Lambda \) given by (4) is continuous from \( \mathcal{S}_1 := H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \) to \( \mathcal{S}_2 := L^2(\Sigma_{-,\epsilon/2}(\omega_0)) \times H^1(\Omega) \), and we denote by \( \| \cdot \| \) its norm in \( L(\mathcal{S}_1, \mathcal{S}_2) \). Given \( C_0, \alpha > 0 \) we introduce the admissible set of potentials \( (A, q) \) defined by

\[
\mathcal{M}(C_0, \alpha) = \left\{ (A, q) \in C_c^\infty(Q)^{n+1} \times L^\infty(Q); \|A\|_{H^{\alpha+1+\alpha}(Q)} \right\}
\]

\[
\leq C_0, \|q\|_{H^{\alpha+1+\alpha}(Q)} \leq C_0 \right\}.
\]

We emphasize that the Neumann measurements \( \partial_u u \) are taken on a part which is slightly more than the half of the lateral boundary. Let us point out that there is an obvious obstruction to uniqueness in terms of gauge invariance for the vector potential term. More precisely, one can uniquely recover only a component of the vector potential term. We refer to [15, 22] for a detailed discussion on this gauge equivalence. Therefore some additional assumptions on the vector potential term must be made. Here we impose the divergence free assumption. With this,
we aim to establish stability estimates for the determination of vector and scalar potentials in this divergence free set-up. Now we state the main result of this paper as follows.

**Theorem.** Let $(A_i, q_i) \in \mathcal{M}(C_0, \alpha)$, $i = 1, 2$, and $T > \text{diam}(\Omega)$. We denote by $\Lambda_i$ the input–output operator corresponding to $\mathcal{L}_{A_i,q_i}$. Further, we assume that $\text{div}_{(t,0)} A_1 = \text{div}_{(t,0)} A_2$. Then there exist $C$, $\mu_1$, $\mu_2$, $\alpha_1$ and $\alpha_2 > 0$ depending on $C_0$, $\alpha$ and $Q$ such that

$$
\|A_2 - A_1\|_{L^\infty(\Omega)} \leq C \left( \|\Lambda_1 - \Lambda_2\|^{\mu_1} + \|\log \|\Lambda_1 - \Lambda_2\|\|\Lambda_1 - \Lambda_2\|^{-\mu_2}\right),
$$

(5)

$$
\|q_2 - q_1\|_{L^\infty(\Omega)} \leq C \left( \|\Lambda_1 - \Lambda_2\|^{\mu_1} + \|\log \|\Lambda_1 - \Lambda_2\|\|\Lambda_1 - \Lambda_2\|^{-\mu_2}\right).
$$

(6)

Let us cite some existing papers treating the inverse problem associated to hyperbolic equations. Motivated by the construction of complex geometric optics solutions for the Calderón problem by Sylvester and Uhlman [27], Rakesh and Symes [18] proved unique determination of the time-independent scalar potential in the wave equation from full Neumann–Dirichlet data. It was extended by Isakov [10] to the recovery of time-independent time derivative perturbations but in the absence of any space derivative perturbation. In [20], Ramm and Sjöstrand dealt uniqueness problems for time-dependent potential in an infinite cylinder. This was later generalized by Salazar; see [21, 22]. For finite time, Rakesh and Ramm in [19] showed that time-dependent potential can be recovered in some specific set outside which they are assumed to be known. We should also mention the unique recovery of the time-dependent potential from scattering data by Stefanov [24]. Kian in [12–14] considered the problem of unique determination and stability of the time-derivative perturbation and scalar potential from full Dirichlet to Neumann data. Using properties of light-ray transform from [17, 25], Krishnan and Vashisth [15] proved uniqueness of all coefficients (up to a gauge invariance for the vector potential term) appearing in relativistic Schrödinger equation from partial boundary data. Similar coefficient recovery problems in various settings were extensively studied by Yamamoto, Bellassoued, Choulli and Ben Aïcha in numerous papers; see [1–3, 5–9]. Bellassoued and Ben Aïcha [2] stably recovered both the time-dependent vector field and scalar potential term from full input–output operator but in the absence of time-derivative perturbation. In a recent work by Bellassoued and Fräj [4], stable determination of zeroth order time-dependent perturbation was shown from Neumann measurements made on an arbitrary part of the lateral boundary. We should also mention a recent paper of Stefanov and Yang [26] where they proved unique recovery of time-dependent first order perturbations form Cauchy data in a Lorentzian setting. In this work, we deal with full first order perturbations and partial boundary measurements. To the best of our knowledge, this has not been studied before. The novelty here lies in stably inverting the light-ray transform of the vector potential term in a partial data setting. More precisely, since we are working with partial boundary data, we only have light-ray transform information along directions $(1, -\omega)$ where $\omega \in S^{n-1}$ is near a fixed $\omega_0 \in S^{n-1}$. Due to this reason, we have to work in space dimension at least 3. However the method used here immediately shows that one can stably recover a vector field term if it has no time-derivative perturbation, that is when $\Lambda_0 = 0$. In other words, there is no gauge equivalence. Thus the current paper in fact strengthens the result by Bellassoued and Ben Aïcha [2] even in the full data case. In all of our discussion, we consider smooth coefficients vanishing on the boundary for vector potentials for simplicity. One can use the approximation argument presented in [12] for more general coefficients.
2. Carleman estimates and geometric optics solutions

We start by showing the existence of geometric optics solutions to the relativistic Schrödinger equation depending on a large parameter. It will be shown using weighted $L^2$-coercivity of some conjugated operators also known as the Carleman estimate. Then to bound certain boundary terms, we will need a boundary Carleman estimate. In [13] Kian proved those estimates where he convexified the linear Carleman weight introducing another parameter to absorb the first order derivatives present in the relativistic Schrödinger operator. We state the results by Krishnan and Vashisth [15] which were motivated by the one in [13].

2.1. Carleman estimates

**Theorem (Boundary Carleman estimate; theorem 3.1 of [15]).** For $\omega \in \mathbb{S}^{n-1}$, $(A, q) \in \mathcal{M}(C_0, \alpha)$ and $u \in C^2(\bar{Q})$ satisfying $u|\Sigma=0 = \partial_t u|\Sigma=0 = u|_{\Sigma} = 0$, there 1, 0 exist $\lambda_0, C > 0$ both depending only on $C_0, \alpha$ and $Q$ such that for $\lambda \geq \lambda_0$ we obtain

$$
\int_{\bar{Q}} e^{-2\lambda(T+t-x \cdot \omega)} \left( \lambda^2 |u(t,x)|^2 + |\nabla_{(t,x)}u(t,x)|^2 \right) \, dx \, dt 
+ \lambda \int_{\Sigma_+} e^{-2\lambda(T+t-x \cdot \omega)} |\omega \cdot \nu(x)| |\partial_\nu u|^2 \, ds
+ \lambda \int_{\Sigma_-} e^{-2\lambda(T+t-x \cdot \omega)} |\partial_\nu u(T,x)|^2 \, ds 
\leq C \left( \int_{\bar{Q}} e^{-2\lambda(T+t-x \cdot \omega)} |\mathcal{L}_{Aq} u(t,x)|^2 \, dx \, dt 
+ \int_{\Omega} e^{-2\lambda(T+t-x \cdot \omega)} \left( \lambda^2 |u(T,x)|^2 + \lambda |\nabla_x u(T,x)|^2 \right) \, dx
+ \lambda \int_{\Sigma_-} e^{-2\lambda(T+t-x \cdot \omega)} |\omega \cdot \nu(x)| |\partial_\nu u|^2 \, ds \right),
$$

(7)

**Corollary 1 (Interior Carleman estimate).** Given $(A, q) \in \mathcal{M}(C_0, \alpha)$, there exist $C, \lambda_0 > 0$ both depending only on $C_0, \alpha$ and $Q$ such that the following estimate holds for $u \in C^\infty_c(Q)$ and $\lambda \geq \lambda_0$

$$
\int_{\bar{Q}} e^{-2\lambda(T+t-x \cdot \omega)} (\lambda^2 |u(t,x)|^2 + |\nabla_{(t,x)}u(t,x)|^2) \, dx \, dt 
\leq C \| e^{-\lambda(T+t-x \cdot \omega)} \mathcal{L}_{Aq} u \|_{L^2(Q)}^2.
$$

2.2. Construction of geometric optics solutions

We make use of the interior Carleman estimate and Hahn–Banach extension theorem to find the following parameter dependent solutions.

**Theorem.** Let $\phi \in C^\infty_c(\mathbb{R}^n)$ and $(A, q) \in \mathcal{M}(C_0, \alpha)$. There exist $C, \lambda_0 > 0$ both depending only on $\phi, C_0, \alpha$ and $Q$ such that for $\lambda \geq \lambda_0$

$$
\phi(t,x) = e^{\lambda(T+t-x \cdot \omega)} \left( \phi(x + t\omega) e^{-\int_0^\lambda (1 - \omega \cdot A(x,u(x+s\omega)) \, ds) + R_\lambda(t,x)} \right)
$$

solves $\mathcal{L}_{Aq} u = 0$, where $\| R_\lambda \|_{H^2(Q)} \leq C \lambda^{-1+k} \| \phi \|_{H^3(\mathbb{R}^n)}$ for $k \in \{0, 1, 2\}$.
The result is quite standard. However we give the proof for the sake of completeness.

**Proof.** Let us first define the following conjugated operator

\[ P_{A,\lambda,\omega} = e^{-\lambda(x,\omega)\cdot t}(\mathcal{L}_{A,q} - q)e^{\lambda(x,\omega)\cdot t}. \]

We search for solutions of the relativistic Schrödinger equation of the following form

\[ u(t,x) = e^{\lambda(x,\omega)\cdot t}(B(t,x) + R_{\lambda}(t,x)). \]

That is, we find \( B \) and \( R_{\lambda} \) such that, 

\[ e^{-\lambda(x,\omega)\cdot t}\mathcal{L}_{A,q}e^{\lambda(x,\omega)\cdot t}(B(t,x) + R_{\lambda}(t,x)) = 0, \]

where, \( P_{A,\lambda,\omega}R_{\lambda} = -qR_{\lambda} - P_{A,\lambda,\omega}B - qB = -qR_{\lambda} - 2\lambda(1, -\omega)\cdot (\nabla_{(t,x)}B + AB) - \mathcal{L}_{A,q}B. \)

If we take \( B(t,x) = \phi(x + t\omega)e^{-\int_{0}^{t}(1, -\omega)\cdot (\mathcal{L}_{A,q} + t\omega)\cdot \mathrm{d}x}, \) then it satisfies \( (1, -\omega)\cdot (\nabla_{(t,x)}B + AB)(t,x) = 0. \)

Thus it suffices to find \( R_{\lambda} \in H^2(Q) \) which satisfies

\[ (P_{A,\lambda,\omega} + q)R_{\lambda} = -\mathcal{L}_{A,q}B. \]  

(8)

To complete the construction of geometric optics solutions let us state the following Carleman estimate in Sobolev spaces with negative order but with an additional index shift by \(-1\) compared to the one in [12] or [15]. This can be proved following exactly the same set of arguments presented in [12] or [15].

**Lemma 1 ([12, lemma 5.4]).** For \((A, q) \in \mathcal{M}(C_{0}, \alpha), \) there exists \( C, \lambda_{0} > 0 \) both depending only on \( C_{0}, \alpha \) and \( Q \) such that for \( u \in C^\infty_{c}(Q) \) and \( \lambda \geq \lambda_{0} \)

\[ \|u\|_{H^{-1}_{\lambda}([R^{n+1}])} \leq C\|P_{A,\lambda,\omega}u\|_{H^{\lambda}_{\omega}([R^{n+1}])}. \]

For \( f \in H^{1}_{\lambda}(Q), \) we define the linear map \( S: \{P_{A,\lambda,\omega}v; \ v \in C^\infty_{c}(Q)\} \to \mathbb{R} \) by

\[ S(P_{A,\lambda,\omega}v) = \int_{Q}vf \, \mathrm{d}x \, \mathrm{d}t. \]

(9)

Now we use lemma 1 to conclude \( S \) is continuous, that is

\[ \|S(P_{A,\lambda,\omega}v)\|_{H^{1}_{\lambda}(Q)} \leq \|P_{A,\lambda,\omega}v\|_{H^{\lambda}_{\omega}([R^{n+1}])}\|f\|_{H^{1}_{\lambda}(Q)}. \]

(10)

Now by Hahn–Banach theorem we can extend \( S \) as a continuous functional on \( H^{-1}_{\lambda}([R^{n+1}]) \) still denoted as \( S \) which satisfies \( \|S\| \leq \|f\|_{H^{1}_{\lambda}(Q)}. \) But by Reisz representation theorem, we have a unique \( u \in H^{1}_{\lambda}([R^{n+1}]) \) such that the following relation holds

\[ S(v) = (v, u)_{H^{-1}_{\lambda}([R^{n+1}]), H^{1}_{\lambda}([R^{n+1}])(\cdot, \cdot)}. \]

(11)

Combining (9) and (11) we get for all \( v \in C^\infty_{c}(Q) \)

\[ S(P_{A,\lambda,\omega}v) = \int_{Q}vf \, \mathrm{d}x \, \mathrm{d}t = (P_{A,\lambda,\omega}v, u)_{H^{-1}_{\lambda}([R^{n+1}]), H^{1}_{\lambda}([R^{n+1}])(\cdot, \cdot)}. \]

(12)
From (12) we observe
\[ u \in H^1_\lambda(\mathbb{R}^{n+1}) \text{ satisfies } P_{A_{\lambda}u}u = f \text{ with } \|u\|_{H^1_\lambda(\mathbb{R}^{n+1})} \leq \|f\| \leq \|f\|_{H^1_\lambda(\mathbb{Q})}. \]

Now we define \( T(f) = u \) where \( P_{A_{\lambda}u}u = f \). We see then \( \|T\|_{H^1_\lambda(\mathbb{Q}) \rightarrow H^1_\lambda(\mathbb{Q})} \leq \frac{C}{\lambda} \). Thus the problem (8) is reduced to finding \( \tilde{f} \in H^1_\lambda(\mathbb{Q}) \) such that
\[ (I + qT)\tilde{f} = -\mathcal{L}_{A_{\lambda}}B. \]

For \( \lambda \) large enough, we have the invertibility of \((I + qT)\) in \( H^1_\lambda(\mathbb{Q}) \). So we can find \( \tilde{f} \in H^1_\lambda(\mathbb{Q}) \) and hence \( R_{\lambda} \in H^1_\lambda(\mathbb{Q}) \), which satisfies for some \( C_1, C_2 \) and \( C_3 > 0 \)
\[ \|R_{\lambda}\|_{H^1_\lambda(\mathbb{Q})} \leq C_1\|f\|_{H^1_\lambda(\mathbb{Q})} \leq C_2\|\mathcal{L}_{A_{\lambda}B}\|_{H^1_\lambda(\mathbb{Q})} \leq C_3\lambda\|\phi\|_{H^1_\mathbb{Q}}. \] (13)

For future purposes we write (13) in the following way
\[ \|R_{\lambda}\|_{H^1_\mathbb{Q}} \leq C\lambda^{-1+k}\|\phi\|_{H^1_\mathbb{R}^{n+1}} \text{ for } k \in \{0, 1, 2\}. \]

This ends the construction of geometric optics solutions. \( \square \)

3. Proof of the main theorem

3.1. Stability estimate for the vector potential

We outline the proof as follows. Using Green’s formula and geometric optics solutions constructed earlier we establish estimates connecting vector potential and the input–output operator. We crucially use the boundary Carleman estimate for this purpose. Then we estimate the line integrals of a component of the vector potential along some direction of light-rays. We include these in lemma 3 and end up deriving a Fourier estimate as a corollary. Then under the divergence free condition of the vector potential, we stably recover all components of the vector potential. We obtain a Fourier estimate of the vector potential term over large enough balls using Vessella’s conditional stability result [28].

Lemma 2 (Integral identity and estimates). For \( i = 1, 2 \) let \((A_i, q_i) \in \mathcal{M}(C_0, \alpha)\) with \( \phi \in C^{2\alpha}_c(\mathbb{R}^n) \) and \( \omega \in S^{n-1} \) satisfying \( |\omega - \omega_0| < \frac{\lambda}{2} \). Then there exist \( \beta > 0, C > 0 \) and \( \lambda_0 > 0 \) depending on \( C_0, \alpha, Q \) such that for all \( \lambda \geq \lambda_0 \)
\[ \int_{0}^{T} \int_{\mathbb{R}^n} (1, -\omega) \cdot (A_2 - A_1)(t, y - t\omega)e^{-\int_{0}^{t}(1, -\omega)(A_2 - A_1)(y - s\omega)ds}|\phi(y)|^2dydt \leq C \left( \frac{1}{\sqrt{\lambda}} + e^{-\beta\lambda}\|A_1 - A_2\| \right) \|\phi\|^2_{L^2(\mathbb{R}^n)}. \]

Proof. For \( u, v \in H^2(\mathbb{Q}) \) with \( u|_{x=0} = \frac{\partial}{\partial t}u|_{x=0} = u|_{\Sigma} = 0 \) and \( A_1, A_2 \in C^{\infty}_c(\mathbb{Q})^{n+1} \), Green’s formula gives
\[ \int_{\mathbb{Q}} (\mathcal{L}_{A_1}u_1)(u)v - u\mathcal{L}_{A_1}(u_1)v \, dx \, dt = \int_{\Omega} (\partial_t u(T, x)v(T, x) - u(T, x)\partial_t v(T, x)) \, dx + \int_{\Sigma} \partial_n u(t, x)v(t, x) \, dS. \] (14)
Now we consider geometric optics solutions corresponding to $L_{A_2,q_2}$ and $L_{-A_1,q_1}$, which are given by $u_2(t, x)$ and $v(t, x)$ respectively. To cancel the exponential terms, we simultaneously consider the exponentially growing and decaying geometric optics solutions. That is, there exist $R_1$ and $R_3$ in $H^2(Q)$ such that for all $\lambda \geq \lambda_0$

$$u_2(t, x) = e^{\lambda(t + x - \omega)} \left( \phi(x + t\omega) e^{-\int_0^t (1 - \omega) A_2(t, x + (t - s)\omega) ds} + R_3^2(t, x) \right),$$

$$v(t, x) = e^{-\lambda(t + x - \omega)} \left( \phi(x + t\omega) e^{\int_0^t (1 - \omega) A_1(t, x + (t - s)\omega) ds} + R_1^2(t, x) \right),$$

and, $\|R_i^k\|_{H^2(Q)} \leq C \lambda^{1+k} \|\phi\|_{H^1(\gamma)}$; for $i = 1, 2$ and $k \in \{0, 1, 2\}$.

Now taking the initial and boundary data same as that of $u_2$, we solve the following IBVP and denote the unique solution by $u_1$, i.e.

$$\begin{cases}
L_{A_1,q_1}u_1 = 0,

u_1(0, \cdot) = u_2(0, \cdot), \quad \partial_t u_1(0, \cdot) = \partial_t u_2(0, \cdot) \quad \text{and} \quad u_1|_{\Sigma} = u_2|_{\Sigma}.
\end{cases}$$

Let $A_i(t, x) \equiv (A_{i,k}(t, x))_{0 \leq k \leq n}$ for $i = 1, 2$, and define $u = u_1 - u_2$ in $Q$. Then $u$ solves the following problem

$$\begin{align*}
L_{A_1,q_1} u(t, x) &= (2A \cdot (\partial_t, -\nabla_x)u_2 + \bar{u}_2(t, x),

u(0, \cdot) &= \partial_t u(0, \cdot) = 0 \quad \text{and} \quad u|_{\Sigma} = 0.
\end{align*}$$

where, $A_i(t, x) = (A_2 - A_1)(t, x) \equiv (A_{i,k}(t, x))_{0 \leq k \leq n}$.

$$\begin{align*}
\bar{q}_i(t, x) &= \left( \partial_i A_1 - \sum_{k=1}^n \partial_{x_k} A_{i,k} + |A_{i,0}|^2 - \sum_{k=1}^n |A_{i,k}|^2 + q_i \right)

\times (t, x), \quad \text{for } i \in \{1, 2\}.

\bar{q}(t, x) &= (\bar{q}_2 - \bar{q}_1)(t, x).
\end{align*}$$

Now we make use of (14) and (18) to get the following integral identity

$$\int_Q (2A \cdot (\partial_t, -\nabla_x)u_2 + \bar{u}_2)\Phi \varpi dx dt = \int_\Omega \left( \partial_t u(T, x)\varphi(T, x) - u(T, x)\partial_t \varphi(T, x) \right) dx + \int_\Sigma \partial_t u(T, x)\varphi(T, x) dS.$$  (19)

We wish to modify the integral identity (19) into an estimate connecting line integrals and the input–output operator. We substitute (15) and (16) into (19). We also see that

$$(\partial_t, -\nabla_x)u_2(t, x) = e^{\lambda(t + x - \omega)} \left( \lambda(1 - \omega) \phi(x + t\omega) e^{-\int_0^t (1 - \omega) A_2(t, x + (t - s)\omega) ds} + R_3^2(t, x) \right),$$

(20)
Also we have
\[ \partial_t v(t, x) = e^{-\lambda(t+x)} \left( \lambda \phi(x + t\omega) e^{\int_0^t (1 - \omega) A_i(t, x + (t - s) \lambda) ds} + w_i^*(t, x) \right). \] (21)

For \( i = 1, 2 \) we see the term \( w_i^* \) involves derivatives of \( \phi \) and \( R_i^* \). For some \( C > 0 \), we arrive at the following estimate using (17)
\[ \| w_i^* \|_{H^k(\Omega)} \leq C \lambda^k \| \phi \|_{H^l(\mathbb{R}^n)} \] for \( k \in \{0, 1\} \) and \( i \in \{1, 2\} \).

We use (20) and (21) to get the following relation which will be helpful later as well
\[ A(t, x) \cdot (\partial_t - \nabla_x) u_2(t, x) = \left( \lambda A(t, x) \cdot (1 - \omega)(\phi(x + t\omega))^2 e^{\int_0^t (1 - \omega) A_i(t, x + (t - s) \lambda) ds} + p_A(t, x) \right). \] (22)

where, \( \| p_A \|_{H^1(\Omega)} \leq C \| \phi \|_{H^1(\mathbb{R}^n)} \) (using a priori bounds on \( A \)).

Now to estimate rhs of (19), we use explicit bounds for \( v \) and the boundary Carleman estimate (7) corresponding to \( L_{\Lambda_1, q_1} \) applied to \( u \). We also have
\[ (\Lambda_1 - \Lambda_2) (u_2|_{\Gamma = 0}, \partial_n u_2|_{\Gamma = 0}, 0) = \left( \partial_n |_{\Sigma} \omega \lambda \omega, u(T, \cdot) \right). \]

From (15) and (17) and using the trace theorem we get the existence of \( \beta > 0 \) and \( C > 0 \) such that
\[ \| u_2 \|_{H^1(\Omega)} \| \partial_n u_2 \|_{H^1(\Omega)} \| u \|_{H^1(\Omega)} \leq C e^{\beta x} \| \phi \|_{H^1(\mathbb{R}^n)}. \] (23)

From (1) we have,
\[ \| \partial_n u \|_{L^2(\Sigma_{\omega \lambda \omega})}, \| u \|_{L^2(\Omega)} \leq C e^{\beta x} \| \Lambda_1 - \Lambda_2 \| \| \phi \|_{H^1(\mathbb{R}^n)} \]. (24)

Let \( K \) be the rhs of the boundary Carleman estimate (7) corresponding to \( L_{\Lambda_1, q_1} \) applied to \( u \)
\[ K = \int_Q e^{-2\lambda(t+x)} |L_{\Lambda_1, q_1} u(t, x)|^2 dxdt \]
\[ + \lambda \int_{\Sigma_{\omega \lambda \omega}} e^{-2\lambda(t+x)} |\omega \cdot \nu(x)||\partial_n u|^2 dS \]
\[ + \int_{\Omega} e^{-2\lambda(t+x)} \left( \lambda^2 |u(T, x)|^2 + \lambda |\nabla_x u(T, x)|^2 \right) \ dx. \]

We use (18), (20) and a priori bounds of the potentials to get
\[ \int_Q e^{-2\lambda(t+x)} |L_{\Lambda_1, q_1} u|^2 dxdt \leq C \lambda^2 \| \phi \|^2_{H^1(\mathbb{R}^n)}. \] (25)

We use continuity of the input–output operator defined in (4) and estimates from (24) to get
\[ \int_{\Omega} e^{-2\lambda(t+x)} (\lambda^2 |u(T, x)|^2 + \lambda |\nabla_x u(T, x)|^2) dx \leq C e^{\beta x} \| \Lambda_1 - \Lambda_2 \|^2 \| \phi \|^2_{H^1(\mathbb{R}^n)}. \] (26)
Since $\Sigma_-(\omega) \subseteq \Sigma_{-\epsilon/2}(\omega_0)$, using (24) we get

$$\int_{\Sigma_-(\omega)} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u|^2\,dS \leq \int_{\Sigma_{-\epsilon/2}(\omega_0)} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u|^2\,dS$$

$$\leq Ce^{\lambda\lambda}||\Lambda_1 - \Lambda_2||_2^2 \Vert \phi \Vert^2_{L^2(\mathbb{R}^n)}. \tag{27}$$

Hence $K$ can be bounded by $C(\lambda^2 + e^{3\lambda}||\Lambda_1 - \Lambda_2||_2^2)\Vert \phi \Vert^2_{L^2(\mathbb{R}^n)}$. Using it with the boundary Carleman estimate, we bound each term present in the rhs of (19). We use Holder’s inequality and trace theorem to get from (16)

$$\left| \int_{\Omega} \partial_\nu u(T,x)\overline{u}(T,x)\,dx \right| = \left| \int_{\Omega} e^{-\lambda(t+\epsilon\omega)}\partial_\nu u(T,x)(\phi(x+T\omega)$$

$$\times e^{-\lambda}(x,T)\,\mathcal{A}_1(t,x+(T-x)\omega)\,dx + R_1(T,x)\right|$$

$$\leq C\Vert \phi \Vert_{H^1(\mathbb{R}^n)} \sqrt{\int_{\Omega} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(T,x)|^2\,dx},$$

$$\leq C\Vert \phi \Vert_{H^1(\mathbb{R}^n)} \sqrt{\int_{\Omega} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(T,x)|^2\,dx} \leq C\Vert \phi \Vert_{H^1(\mathbb{R}^n)} \left( \sqrt{\lambda} + e^{\lambda\lambda}||\Lambda_1 - \Lambda_2||_2 \right). \tag{28}$$

Proceeding similarly, we get from (16) and (21)

$$\left| \int_{\Omega} u(T,x)\partial_\nu u(T,x)\,dx \right| \leq C\lambda \Vert \phi \Vert_{H^1(\mathbb{R}^n)} \sqrt{\int_{\Omega} e^{-2\lambda(t+\epsilon\omega)}|u(T,x)|^2\,dx},$$

$$\leq Ce^{\lambda\lambda}||\Lambda_1 - \Lambda_2||_2 \Vert \phi \Vert^2_{H^1(\mathbb{R}^n)}. \tag{29}$$

and.

$$\int_{\Sigma} \partial_\nu u(t,x)\overline{u}(t,x)\,dS = \frac{1}{\sqrt{\lambda}} \left| \int_{\Sigma} \sqrt{\lambda}\partial_\nu u(t,x)\overline{u}(t,x)\,dS \right|$$

$$\leq C\Vert \phi \Vert_{H^1(\mathbb{R}^n)} \sqrt{\int_{\Sigma} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(t,x)|^2\,dS}. \tag{30}$$

We observe

$$\int_{\Sigma} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(t,x)|^2\,dS = \int_{\Sigma_{+\epsilon/2}(\omega_0)} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(t,x)|^2\,dS$$

$$+ \int_{\Sigma_{-\epsilon/2}(\omega_0)} \lambda e^{-2\lambda(t+\epsilon\omega)}|\partial_\nu u(t,x)|^2\,dS.$$
We collect the estimates (27)–(31) to obtain from (19) the following estimate
\[
\left| \int_{Q} (2A \cdot (\partial_{t} - \nabla \cdot v) \b u_{2} + \tilde{q} u_{2}(t, x) v(t, x)) \,dx \,dt \right| \leq C \| \phi \|_{H^{1}(\mathbb{R}^{n})}^{2} \left( \sqrt{\lambda} + e^{\beta \lambda} \| \Lambda_{1} - \Lambda_{2} \|_{*} \right). \tag{32}
\]

We use expressions of the geometric optics solutions (15) and (16) to get
\[
\left| \int_{Q} \tilde{q}(t, x) u_{2}(t, x) v(t, x) \,dx \,dt \right| \leq C \| \phi \|_{H^{1}(\mathbb{R}^{n})}^{2}.
\]
Thus by recalling (32) and dividing by \( \lambda \) we obtain
\[
\left| \int_{0}^{T} \int_{\mathbb{R}^{n}} (1, -\omega) \cdot \mathcal{A}(t, x, y - t \omega) e^{-\int_{0}^{T} (1, -\omega) \cdot \mathcal{A}(s, x + t(s - t) \omega, t - s \omega) \,ds} |\phi(y)\|^2 \,dy \,dt \right| \leq C \left( \frac{1}{\sqrt{\lambda}} + e^{\beta \lambda} \| \Lambda_{1} - \Lambda_{2} \|_{*} \right) \| \phi \|_{H^{1}(\mathbb{R}^{n})}^{2}.
\]

Now, we consider the change of variables given by \( y = x + t \omega \) and use the fact that \( \mathcal{A} \in C_{c}^{\infty}(Q^{r+1}) \) to obtain
\[
\left| \int_{0}^{T} \int_{\mathbb{R}^{n}} (1, -\omega) \cdot \mathcal{A}(t, y - t \omega) e^{-\int_{0}^{T} (1, -\omega) \cdot \mathcal{A}(s, y + t(s - t) \omega) \,ds} |\phi(y)\|^2 \,dy \,dt \right| \leq C \left( \frac{1}{\sqrt{\lambda}} + e^{\beta \lambda} \| \Lambda_{1} - \Lambda_{2} \|_{*} \right) \| \phi \|_{H^{1}(\mathbb{R}^{n})}^{2}.
\]

This completes the proof of the lemma. \( \square \)

To get estimates for the light-ray transform of the vector potential term from (33) we adapt the arguments presented in [6] or [8]. Basically the proof relies on a limit passing argument using an approximate identity.
Lemma 3. For all $x \in \mathbb{R}^n$ and $\omega \in \mathbb{S}^{n-1}$ satisfying $|\omega - \omega_0| < \frac{\delta}{2}$, there exist $\delta > 0$, $\lambda_0 > 0$ and $C > 0$ such that following estimate holds whenever $\lambda \geq \lambda_0$:

$$\left| \int_{\mathbb{R}} (1, -\omega) \cdot \mathcal{A}(s, x - s\omega) \, ds \right| \leq C \left( \frac{1}{\sqrt{\lambda}} + e^{2\lambda} \|A_1 - A_2\| \right).$$

Proof. We can write

$$\int_0^T \int_{\mathbb{R}^n} (1, -\omega) \cdot \mathcal{A}(t, y - t\omega) e^{-\int_0^t (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds} |\phi(y)|^2 \, dy \, dt = - \int_{\mathbb{R}^n} \int_0^T |\phi(y)|^2 \partial_t \left( e^{-\int_0^t (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds} \right) \, dy \, dt = - \int_{\mathbb{R}^n} |\phi(y)|^2 \left( e^{-\int_0^T (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds} - 1 \right) dy. \quad (34)$$

Fix $x \in \mathbb{R}^n$ and choose $\phi \in C_c^\infty(B(0, 1))$ with $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, where $B(0, 1)$ is the open unit ball in $\mathbb{R}^n$. Let us define

$$\phi_h(y) = h^{-\frac{n}{2}} \phi \left( \frac{y - x}{h} \right) \text{ in } \mathbb{R}^n \text{, for } h > 0.$$ 

Then $\phi_h \in C_c^\infty(\mathbb{R}^n)$ and there exists $C > 0$ depending on $\phi$ such that following relations hold

$$\text{supp}(\phi_h) \subseteq B(x, h), \|\phi_h\|_{L^2(\mathbb{R}^n)} = 1 \quad \text{and} \quad \|\phi_h\|_{H^1(\mathbb{R}^n)} \leq Ch^{-\frac{1}{2}}.$$ \quad (35)

Now using (34), we get from (33) and (35)

$$\left| \int_{\mathbb{R}^n} |\phi_h(y)|^2 \left( e^{-\int_0^T (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds} - 1 \right) dy \right| \leq C h^{-\frac{n}{2}} \left( \frac{1}{\sqrt{\lambda}} + e^{2\lambda} \|A_1 - A_2\| \right). \quad (36)$$

As $\mathcal{A} \in C_c^\infty(Q_n^{n+1})$, using mean value theorem twice we get $C > 0$ depending on $Q$ and a priori bounds of $\mathcal{A}$, such that the following estimate holds

$$\left| e^{-\int_0^T (1, -\omega) \cdot \mathcal{A}(s, x - s\omega) \, ds} - e^{-\int_0^T (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds} \right| \leq C \left| \int_0^T (1, -\omega) \cdot \mathcal{A}(s, x - s\omega) \, ds - \int_0^T (1, -\omega) \cdot \mathcal{A}(s, y - s\omega) \, ds \right| \leq C|x - y|. \quad (37)$$

Now consider the following positive continuous function on $\mathbb{R}$

$$f(x) = \begin{cases} 
\frac{e^x - 1}{x} & \text{for } x \neq 0, \\
1 & \text{for } x = 0.
\end{cases} \quad (38)$$

For $M > 0$ we use continuity of $f$ on the compact interval $[-M, M]$ to get $C > 0$ depending on $M$ such that

$$|x| \leq C|e^x - 1| \quad \forall \ x \in [-M, M].$$
Since \( A \in C_c^\infty(Q)^{n+1} \), we get \( C > 0 \) depending on \( Q \) and the \textit{a priori} bounds of \( A \), such that the following estimate holds

\[
\left| \int_0^T (1, -\omega) \cdot A(s, x - s\omega) ds \right| \leq C \left| e^{-\int_0^T (1, -\omega) \cdot A(t, x - t\omega) dt} - 1 \right|.
\]

Now by (35)–(37), we obtain

\[
\left| e^{-\int_0^T (1, -\omega) \cdot A(t, x - t\omega) dt} - 1 \right| = \left| \int_\mathbb{R}^n |\phi_h(y)|^2 (e^{-\int_0^T h \cdot A(t, x - t\omega) dt} - 1) dy \right|,
\]

\[
\leq C \left( h + h^{-6} \left( \frac{1}{\sqrt{\lambda}} + e^{\beta\lambda}\|A_1 - A_2\|_\ast \right) \right),
\]

(using (37) and (36)) (39).

We choose \( h > 0 \) small enough such that \( h \) and \( \frac{h}{\sqrt{\lambda}} \) are comparable. It can be done by taking \( h = \lambda^{-\frac{1}{2}} \). Then for \( \lambda \geq \lambda_0 \) and \( |\omega - \omega_0| < \frac{\tau}{\sqrt{\lambda}} \) we get from (39) the existence of \( \delta < 1 \) such that

\[
\left| \int_\mathbb{R} (1, -\omega) \cdot A(s, x - s\omega) ds \right| = \left| \int_0^T (1, -\omega) \cdot A(s, x - s\omega) ds \right|,
\]

\[
\leq C \left( \frac{1}{\lambda^\delta} + e^{\beta\lambda}\|A_1 - A_2\|_\ast \right).
\]

This completes the proof. \( \square \)

Now we use ideas from [7] to get Fourier estimates of some specific components of the vector potential along light-rays. Henceforth, we denote by \( f \) the Fourier transform of \( f \in L^1(\mathbb{R}^{n+1}) \). That is, for \( (\tau, \xi) \in \mathbb{R}^{n+1} \) we have

\[
\tilde{f}(\tau, \xi) = \int_{\mathbb{R}^{n+1}} e^{-i\tau \cdot t - i\xi \cdot x} f(t, x) dt dx.
\]

**Corollary 2.** There exists an open cone \( C \) in \( \mathbb{R}^{n+1} \) such that for \( (\tau, \xi) \in C \), the following estimate holds for all \( \omega(\tau, \xi) \in S^{n-1} \) satisfying \( |\omega(\tau, \xi) - \omega_0| < \frac{\tau}{\sqrt{\lambda}} \) and \( \omega(\tau, \xi) \cdot \xi = \tau \).

\[
| (1, -\omega(\tau, \xi)) \cdot \tilde{A}(\tau, \xi) | \leq C \left( \frac{1}{\lambda^\delta} + e^{\beta\lambda}\|A_1 - A_2\|_\ast \right).
\]
Proof. Consider \( x \in \mathbb{R}^n \) and \( \omega \in \mathbb{S}^{n-1} \) satisfying \( |\omega - \omega_0| < \epsilon/2 \). Then we have
\[
| (1, -\omega) \cdot \tilde{A}(\omega, \xi, \xi) | = \left| \int_{\mathbb{R}^{n+1}} e^{-i\omega \cdot \xi} \xi (1, -\omega) \cdot \mathcal{A}(s, y, z) \, ds \right|,
\]

\[
= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}} (1, -\omega) \cdot \mathcal{A}(s, z - \omega s) \, ds \, dz \right|,
\]

\[
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}} (1, -\omega) \cdot \mathcal{A}(s, z - \omega s) \, dz, \quad \leq C \left( \frac{1}{\lambda^8} + e^{\beta \lambda} \| \Lambda_1 - \Lambda_2 \|_s \right). \quad \text{(using (40) and } \mathcal{A} \in C^\infty_c(Q^{n+1}) \text{) (41)}
\]

Now as in [7], we characterize points of \( \mathbb{R}^{n+1} \) for which estimate (41) holds. We define
\[
K_\epsilon = \bigcup_{|\omega - \omega_0| < \epsilon/2} \omega \quad \text{and} \quad E_\epsilon = \{ (\tau, \xi) \in \mathbb{R} \times K_{1/2}; |\tau| < \frac{\epsilon}{8} \}.
\]

Since the interior of \( K_\epsilon \) is nonempty, \( E_\epsilon \) contains an open cone say \( C \). Now we will show that if \( (\tau_0, \xi_0) \in E_\epsilon \), then there exists \( \omega \in \mathbb{S}^{n-1} \) such that \( |\omega - \omega_0| < \frac{\epsilon}{8} \) and \( t_0 = \omega \cdot x_0 \).

Assume \( (\tau_0, \xi_0) \in E_\epsilon \), then \( \xi_0 \cdot \omega_1 = 0 \) for some \( \omega_1 \in \mathbb{S}^{n-1} \) with \( |\omega_1 - \omega_0| < \frac{\epsilon}{8} \). Now we take, \( \omega = \frac{\tau_0}{|\xi_0|^2} \xi_0 + \frac{1}{|\xi_0|^2} \omega_1 \). We see then \( \tau_0 = \xi_0 \cdot \omega \).

By our choice of \( \omega_0 \in \mathbb{S}^{n-1} \) we observe
\[
|\omega - \omega_0| \leq \frac{\tau_0}{|\xi_0|^2} |\xi_0| + \sqrt{1 - \frac{\tau_0^2}{|\xi_0|^2}} |\omega_1 - \omega_0| + \left( \sqrt{1 - \frac{\tau_0^2}{|\xi_0|^2}} - 1 \right) \omega_0 \]
\[
\leq \frac{\epsilon}{8} + \frac{\epsilon}{4} + \frac{\epsilon}{8} = \frac{\epsilon}{2}.
\]

Thus for \( (\tau, \xi) \in E_\epsilon \), we obtain \( (1, -\omega(\tau, \xi)) \cdot \tilde{A}(\tau, \xi) \leq C \left( \frac{1}{\lambda^8} + e^{\beta \lambda} \| \Lambda_1 - \Lambda_2 \|_s \right) \), where \( \omega(\tau, \xi) \in \mathbb{S}^{n-1} \) with \( |\omega(\tau, \xi) - \omega_0| < \epsilon/2 \) and \( \tau = \omega(\tau, \xi) \cdot \xi \).

Now we obtain some uniform norm estimate of the vector potential over a cone so that we can take advantage of Vessella’s analytic continuation argument for estimating the Fourier transform of vector potentials over large balls. We give a partial data version of the argument in lemma 2.5 of [22]. In the proof of lemma 5, we provide an explanation as to why the method does not work for \( n = 2 \).

Lemma 4. For a fixed \( (\tau, \xi) \in \mathbb{R}^{n+1} \) satisfying \( |\tau| < \frac{1}{2} |\xi| \), consider the following set of equations
\[
(1, -\omega(\tau, \xi)) \cdot \tilde{A}(\tau, \xi) = G(\xi, \omega(\tau, \xi)), \quad (42)
\]
\[
(\tau, \xi) \cdot \tilde{A}(\tau, \xi) = 0, \quad \text{(This is because } A \text{ is divergence – free)} \quad (43)
\]
where \( \omega(\tau, \xi) \in \mathbb{S}^{n-1} \) satisfies \( \omega(\tau, \xi) \cdot \xi = \tau \).

Then there exist some choice of \( \{ \omega^k(\tau, \xi) \}_{1 \leq k \leq n} \) satisfying (44) and \( C > 0 \) independent of the cone \( \{ (\tau, \xi) \in \mathbb{R}^{n+1}; |\tau| < \frac{1}{2} |\xi| \} \) such that we have \( \forall j \in \{ 0, 1, \ldots, n \} \)
\[
| \tilde{A}_j(\tau, \xi) | \leq C \max_{1 \leq k \leq n} | G(\xi, \omega^k(\tau, \xi)) |.
\]
Proof. We can assume $\xi = e_n$. Otherwise, one can consider appropriate rotation in $\mathbb{R}^n$ which will not change the end result as orthogonal transformations respect the inner product. Under this assumption, the unit vector $\omega(\tau, \xi) \equiv (\omega_j(\tau, \xi))_{1 \leq j \leq n}$ in (44) must satisfy $\omega_n(\tau, \xi) = \tau$. We observe also

$$\sum_{k=1}^{n-1} \omega_k^2(\tau, \xi) = 1 - \omega_n^2(\tau, \xi) = 1 - \tau^2 > \frac{3}{4}. $$

Hence all those $\omega(\tau, \xi)$ can be parametrized by $\mathbb{R}^{n-2}$ where $\frac{\sqrt{3}}{4} < r < 1$. Let us choose $n-1$ orthogonal vectors from $\mathbb{R}^{n-2}$ and denote them by $\tilde{\omega}^i$ for $i \in \{1, 2, \ldots, n-1\}$. We define

$$\tilde{\omega}^n = \frac{1}{\sqrt{2}}(\tilde{\omega}^{n-2} + \tilde{\omega}^{n-1}).$$

Now we consider (42) and (43) for $\omega'(\tau, \xi)$ where $\omega'(\tau, \xi) = (\tilde{\omega}', \tau)$ for $i \in \{1, 2, \ldots, n\}$. The system of equations we are interested is given by

$$\hat{A}_0(\tau, \xi) - \sum_{j=1}^{n} \omega_j'(\tau, \xi)\hat{A}_j(\tau, \xi) = G(\xi, \omega'(\tau, \xi)), \quad i \in \{1, 2, \ldots, n\}. $$

$$\frac{1}{\sqrt{\tau^2 + |\xi|^2}}(\tau\hat{A}_0(\tau, \xi) + \sum_{j=1}^{n} \xi_j \hat{A}_j(\tau, \xi)) = 0. $$

Unique solvability of the above system follows from the fact that we have taken divergence free vector potential and the orthogonal complement of $\{(1, -\omega(\tau, \xi)); \omega(\tau, \xi) \in S^{n-1} \text{ and } \tau + \xi \cdot \omega(\tau, \xi) = 0\}$ is one dimensional (see appendix of [21]). For stable recovery of the vector potentials, we want to obtain a positive lower bound on the absolute value of the determinant of the matrix $M(\tau, \xi)$ defined by

$$M(\tau, \xi) = \begin{pmatrix}
  1 & -\omega_1'(\tau, \xi) & \cdots & -\omega_n'(\tau, \xi) \\
  1 & -\omega_1'(\tau, \xi) & \cdots & -\omega_n'(\tau, \xi) \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & -\omega_1'(\tau, \xi) & \cdots & -\omega_n'(\tau, \xi) \\
  \sqrt{\tau^2 + |\xi|^2} & \sqrt{\tau^2 + |\xi|^2} & \cdots & \sqrt{\tau^2 + |\xi|^2}
\end{pmatrix}. \quad (45)$$

If $V(\tau, \xi)$ is the $n$ dimensional volume generated by the vectors $\{(1, -\omega'(\tau, \xi))\}_{1 \leq i \leq n}$, then we see the following relation

$$\det M(\tau, \xi) = V(\tau, \xi) \cdot P(\tau, \xi).$$

Here $P(\tau, \xi)$ is the length of the component of $\left(\frac{\tau}{\sqrt{\tau^2 + |\xi|^2}}, \frac{\xi}{\sqrt{\tau^2 + |\xi|^2}}\right)$ which is perpendicular to the subspace generated by $\{(1, -\omega'(\tau, \xi))\}_{1 \leq i \leq n}$. Since $(\tau, \xi)$ is at least $\frac{\pi}{8}$ angle away from the light cone and the vectors $\{(1, -\omega'(\tau, \xi))\}_{1 \leq i \leq n}$ lie on the boundary of light cone, we have

$$|V(\tau, \xi) \sin \left(\frac{\pi}{8}\right)| \leq |\det M(\tau, \xi)|.$$


To compute $V(\tau, \xi)$ we consider the Gramian of $\{(1, -\omega^j(\tau, \xi))\}_{1 \leq i \leq n}$ denoted by $G(\tau, \xi)$. For convenience we denote the unit vectors satisfying (44) as $\tilde{\omega}$ only. We see then

$$G(\tau, \xi) = \begin{vmatrix} (1, -\omega^1) \cdot (1, -\omega^1) & (1, -\omega^1) \cdot (1, -\omega^2) & \ldots & (1, -\omega^1) \cdot (1, -\omega^n) \\ (1, -\omega^2) \cdot (1, -\omega^1) & (1, -\omega^2) \cdot (1, -\omega^2) & \ldots & (1, -\omega^2) \cdot (1, -\omega^n) \\ \vdots & \vdots & \ddots & \vdots \\ (1, -\omega^n) \cdot (1, -\omega^1) & (1, -\omega^n) \cdot (1, -\omega^2) & \ldots & (1, -\omega^n) \cdot (1, -\omega^n) \end{vmatrix}$$

$$= 1 + \tau^2 + \|	ilde{\omega}^1\|^2 \begin{vmatrix} 1 + \tau^2 + \tilde{\omega}^1 \cdot \tilde{\omega}^1 & \ldots & 1 + \tau^2 + \tilde{\omega}^1 \cdot \tilde{\omega}^n \\ \tilde{\omega}^2 \cdot \tilde{\omega}^1 & \ldots & \tilde{\omega}^2 \cdot \tilde{\omega}^n \\ \vdots & \vdots & \vdots \\ \tilde{\omega}^n \cdot \tilde{\omega}^1 & \ldots & \tilde{\omega}^n \cdot \tilde{\omega}^n \end{vmatrix}$$

(46)

We now use multi-linearity property of determinants to expand (46) and the fact that the Gramian of $\{\tilde{\omega}^j\}_{1 \leq i \leq n}$ is zero as they were chosen from $\mathbb{R}^{n-1}$. Then we have

$$G(\tau, \xi) = \sum_{k=1}^{n} (1 + \tau^2) B_k(\tau, \xi),$$

(47)

where $B_k(\tau, \xi)$ has $k$th column as $(1, 1, \ldots, 1)'$ and $j$th as $(\tilde{\omega}^1 \cdot \tilde{\omega}^1, \tilde{\omega}^2 \cdot \tilde{\omega}^1, \ldots, \tilde{\omega}^n \cdot \tilde{\omega}^1)'$ for $j \neq k$.

We had taken $\tilde{\omega}^1$ as a linear combination of $\tilde{\omega}^{n-1}$ and $\tilde{\omega}^{n-2}$. Thus $B_k(\tau, \xi) = 0$, for $1 \leq k \leq n - 3$. To calculate the rest of the terms in (47), we observe

$$B_{n-2}(\tau, \xi) = \begin{vmatrix} \|	ilde{\omega}^1\|^2 & \ldots & 1 \tilde{\omega}^1 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^1 \cdot \tilde{\omega}^n \\ \tilde{\omega}^2 \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^2 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^2 \cdot \tilde{\omega}^n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\omega}^n \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^n \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^n \cdot \tilde{\omega}^n \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \|	ilde{\omega}^1\|^2 & \ldots & 1 \tilde{\omega}^1 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^1 \cdot (\tilde{\omega}^{n-2} + \tilde{\omega}^{n-1}) \\ \tilde{\omega}^2 \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^2 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^2 \cdot (\tilde{\omega}^{n-2} + \tilde{\omega}^{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\omega}^n \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^n \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^n \cdot (\tilde{\omega}^{n-2} + \tilde{\omega}^{n-1}) \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \|	ilde{\omega}^1\|^2 & \ldots & 1 \tilde{\omega}^1 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^1 \cdot \tilde{\omega}^{n-2} \\ \tilde{\omega}^2 \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^2 \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^2 \cdot \tilde{\omega}^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\omega}^n \cdot \tilde{\omega}^1 & \ldots & 1 \tilde{\omega}^n \cdot \tilde{\omega}^{n-1} & \tilde{\omega}^n \cdot \tilde{\omega}^{n-2} \end{vmatrix}$$

$$= -\frac{1}{\sqrt{2}} B_{n}(\tau, \xi).$$

Similarly we can have $B_{n-1}(\tau, \xi) = -\frac{1}{\sqrt{2}} B_{n}(\tau, \xi)$. Consequently we get

$$G(\tau, \xi) = (1 + \tau^2)(1 - \sqrt{2})B_{n}(\tau, \xi).$$

(48)
Now,

\[ B_\epsilon(\tau, \xi) = \begin{vmatrix} ||\tilde{\omega}\|^2 & \cdots & \tilde{\omega}^{n-1} \cdot \tilde{\omega}^{n-2} & \cdots & \tilde{\omega} \cdot \tilde{\omega}^{n-1} & 1 \\ \tilde{\omega}^2 \cdot \tilde{\omega} & \cdots & \tilde{\omega}^3 \cdot \tilde{\omega}^{n-2} & \cdots & \tilde{\omega}^2 \cdot \tilde{\omega}^{n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{\omega}^n \cdot \tilde{\omega} & \cdots & \tilde{\omega}^{n+1} \cdot \tilde{\omega}^{n-2} & \cdots & \tilde{\omega}^n \cdot \tilde{\omega}^{n-1} & 1 \end{vmatrix} = r^{2(n-1)}(1 - \sqrt{2}). \] (49)

We combine (48) and (49) to conclude $0 < c \leq |V(\tau, \xi)|$ where $c$ is independent of $(\tau, \xi)$. By cofactor expansion of the matrix (45) we solve the linear system given by (42) and (43) as

\[ \tilde{A}_J(\tau, \xi) = \sum_{k=1}^{n} c_{j,k}(\tau, \xi) G(\xi, \omega^j(\tau, \xi)), \quad j \in \{0, 1, \ldots, n\}, \] (50)

where $c_{j,k}(\tau, \xi) = \frac{1}{\det M(\tau, \xi)} C_{j,k}(\tau, \xi)$ and $C_{j,k}(\tau, \xi)$ is the $(j,k)$th cofactor of $M(\tau, \xi)$.

Each of the entries in $M(\tau, \xi)$ has absolute values less than or equal to one and $C_{j,k}(\tau, \xi)$ consists of the sum of products of these terms. Therefore there exists $C > 0$ independent of $(\tau, \xi) \in \mathbb{R}^{n+1}; |\tau| < \frac{1}{2} |\xi|$ such that the following estimate holds

\[ |\tilde{A}_J(\tau, \xi)| \leq C \max_{1 \leq k \leq n} |G(\xi, \omega^k(\tau, \xi))|, \quad \forall j \in \{0, 1, \ldots, n\}. \]

We now make a modification in the arguments presented above to invert the light-ray transform for the partial data case. Before that we make the following observations.

**Lemma 5.** For a fixed $\epsilon > 0$ and $(\tau, \xi) \in \mathbb{R}^{n+1}$ satisfying $|\tau| < \frac{1}{2} |\xi|$, the following set

\[ B_{\tau, \xi} = \{ \omega \in S^{n+1}; \omega \cdot \xi = \tau \text{ and } |\omega - \omega_0| < \epsilon \} \]

if non-empty has at least two linearly independent vectors when $n \geq 3$.

**Proof.** Consider the following map on the spherical cap $\{ \omega \in S^{n-1}; |\omega - \omega_0| < \epsilon \}$

\[ f(\omega) = \omega \cdot \xi - \tau. \]

We assume the zero-set of $f$ is non-empty and wish to show that it contains at least two linearly independent vectors. If possible let it have only one element say $\omega$. We observe then

\[ \{ \omega \in S^{n-1}; \omega \neq \tilde{\omega} \text{ and } |\omega - \omega_0| < \epsilon \} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty). \] (51)

For $n \geq 3$ we notice the set in the lhs of (51) is connected whereas image of it under $f$ is not. Hence $B_{\tau, \xi}$ must have at least two linearly independent vectors. This argument does not hold for the case $n = 2$, as punctured spherical caps are not connected there. For $n = 2$, we see that $B_{\tau, \xi}$ has at most one vector when $\epsilon > 0$ is small enough. ⧫
For $n \geq 3$, we show below that $B_{\tau,\xi}$ has enough linearly independent vectors so that one can invert the matrix $M(\tau,\xi)$.

**Lemma 6.** For $n \geq 3$, the set $B_{\tau,\xi}$ if non-empty has $n$ linearly independent vectors.

**Proof.** Without loss of generality we may take $\xi = e_n$, otherwise one can consider $x_n$-axis along the vector $\xi$. Our representation of the unit vector $\omega$ in spherical coordinates is as follows

$$\omega_1 = \sin \theta_1 \sin \theta_2 \ldots \sin \theta_n,$$
$$\omega_2 = \sin \theta_1 \sin \theta_2 \ldots \cos \theta_n,$$
$$\vdots$$
$$\omega_n = \cos \theta_1.$$

For $\omega \in B_{\tau,\xi}$, we represent the respective angles in bar. We see then $\bar{\theta}_1 \in (0, \frac{\pi}{2})$. We want to show that $B_{\tau,\xi}$ contains $n$ linearly independent vectors. If not, then $B_{\tau,\xi}$ should lie on a plane in $\mathbb{R}^n$. Hence we must have a non-zero vector say $\alpha \in \mathbb{R}^n$ such that for $(\theta_2, \ldots, \theta_n)$ varying in a small neighborhood of $(\theta_2, \ldots, \theta_n)$ we have

$$\alpha \cdot (\sin \bar{\theta}_1 \sin \theta_2 \ldots \sin \theta_n, \sin \bar{\theta}_1 \sin \theta_2 \ldots \cos \theta_n, \ldots, \sin \bar{\theta}_1 \cos \theta_2, \cos \bar{\theta}_1) = 0$$

(52)

We now differentiate (52) with respect to $\theta_2$ twice to get the following relation

$$-\alpha \cdot (\sin \bar{\theta}_1 \sin \theta_2 \ldots \sin \theta_n, \sin \bar{\theta}_1 \sin \theta_2 \ldots \cos \theta_n, \ldots, \sin \bar{\theta}_1 \cos \theta_2, 0) = 0$$

(53)

We now combine (52) and (53) to conclude $\alpha_n = 0$, because $\bar{\theta}_1 \in (0, \frac{\pi}{2})$. Now we differentiate (52) with respect to $\theta_3$ twice to get

$$-\alpha \cdot (\sin \bar{\theta}_1 \sin \theta_2 \ldots \sin \theta_n, \sin \bar{\theta}_1 \sin \theta_2 \ldots \cos \theta_n, \ldots, \sin \bar{\theta}_1 \sin \theta_2 \cos \theta_3, 0, 0) = 0$$

(54)

We add (54) and (52) to get $\alpha_{n-1} \sin \bar{\theta}_1 \cos \theta_2 = 0$ which in turn implies $\alpha_{n-1} \sin \bar{\theta}_1 = 0$, since $\bar{\theta}_2$ varies in a small neighborhood of $\bar{\theta}_2$. This yields $\alpha_{n-1}$ is actually zero. Proceeding similarly we can show $\alpha = 0$ which contradicts our assumption on $\alpha$. □

**Lemma 7.** Let $\mathcal{C}$ be the open cone as described in corollary 2. Then there exists an open cone $\mathcal{C}_0 \subseteq \mathcal{C}$ such that the following estimate holds

$$|\tilde{A}(\tau, \xi)| \leq C \left( \frac{1}{\lambda^3} + e^{\beta \lambda \|A_1 - A_2\|} \right), \quad \text{for } (\tau, \xi) \in \mathcal{C}_0.$$

**Proof.** For $(\tau_0, \xi_0) \in \mathcal{C} \cap S^n$, we have from corollary 2 that the set $B_{\tau,\xi}$ is non-empty and thus has a set of $n$ linearly independent vectors from lemma 6 which are denoted by $\{\omega(k(\tau_0, \xi_0))\}_{k \in \mathbb{N}}$ which implies that the vectors $\{(1, -\omega(k(\tau_0, \xi_0)))\}_{k \in \mathbb{N}}$ are linearly independent too. Since $(\tau_0, \xi_0)$ is perpendicular to each of those vectors, we get that the matrix $M(\tau_0, \xi_0)$ is invertible. Now as we change $(\tau, \xi)$ continuously in a neighborhood of $(\tau_0, \xi_0)$ in $\mathbb{R}^n$ say $\tilde{\mathcal{C}}$, we notice that the hyperplane $\{x \in \mathbb{R}^n; x \cdot \xi = \tau\}$ moves in a continuous way. Hence we get $n$
linearly independent vectors from $B_{r,\xi}$ denoted by $\{\omega_k(\tau, \xi)\}_{1 \leq k \leq n}$ depending continuously on $(\tau, \xi)$. We proceed as in Lemma 4 and see that $M_{r,\xi}$ is invertible for $(\tau, \xi) \in \tilde{C}$. For a compactly contained open set in $\tilde{C}$ say $\tilde{C}$, we have $0 < c \leq |\det M_{r,\xi}|$. Now for $(\tau, \xi) \in \tilde{C}$, we consider the system of equations

$$
(1, -\omega_k(\tau, \xi)) \cdot \tilde{A}(\tau, \xi) = G(\xi, \omega_k(\tau, \xi)), \text{ for } k \in \{1, 2, \ldots, n\}, \tag{55}
$$

$$
\frac{1}{\sqrt{\tau^2 + \xi^2}}(\tau, \xi) \cdot \tilde{A}(\tau, \xi) = 0, \quad \text{(This is because $A$ is divergence free)}.
$$

We observe that Corollary 2 gives an upper bound of $G$ in the cone $C$ and $M_{r,\xi}$ is a non-singular homogeneous matrix of degree zero, and so will be the cofactors and minors. We use these facts to obtain estimates for the vector potentials in an open cone in $\mathbb{R}^{n+1}$ from the estimates over $\tilde{C}$. We proceed as in (50) to get from (55) and (56) for $(\tau, \xi) \in \tilde{C}$ and $r > 0$

$$
\tilde{A}_j(\tau, r\xi) = \sum_{k=1}^{n} c_{k,j}(\tau, r\xi) G(r\xi, \omega_k(\tau, r\xi)), \quad j \in \{0, 1, \ldots, n\},
$$

where $c_{k,j}(\tau, \xi) = \frac{1}{\det M_{r,\xi}} C_{jk}(\tau, \xi)$ and $C_{jk}(\tau, \xi)$ is $(j,k)$th cofactor of $M(\tau, \xi)$. Hence for the open cone $C_0(= \cup_{\tau > 0} \tilde{C})$ in $\mathbb{R}^{n+1}$ we use Corollary 2 and the lower bound on the determinant considered for the points in $\tilde{C}$ to obtain $C > 0$ such that

$$
|\tilde{A}_j(\tau, \xi)| \leq C \left( \frac{1}{\lambda^{\gamma}} + e^{\beta \lambda} ||A_1 - A_2|| \right), \quad \text{for } j \in \{0, 1, \ldots, n\}. \tag{59}
$$

**Remark.** We make a remark here for the specific case when there is no time-derivative perturbation present that is $A_0(\tau, x) = 0$. Then we can get exactly the same Fourier estimates over an open cone as obtained in (59) for the rest of the components of the vector potential without the divergence-free assumption. This is because, for a fixed $(\tau, \xi) \in \tilde{C}$ we have $n$ linearly independent vectors $\{\omega_k(\tau, \xi)\}_{1 \leq k \leq n}$ as before which makes (55) an invertible system consisting of $n$ unknowns to be determined from $n$ equations. After that one carries out the arguments presented below to arrive at the stability results similar to (5) and (6). Thus we do not need any divergence-free assumption (with respect to the space variables) on the vector potential to obtain the stability results. This way our result generalizes the result by Bellassoued and Ben Aïcha [2].

As mentioned earlier, to get the Fourier estimate of vector potentials over arbitrary large balls we need to use Vessella’s analytic continuation argument (see [28]) into (59). To do so, we define

$$
f_k(t, x) = \tilde{A}(kt, kx) \text{ for } k > 0 \text{ and } (t, x) \in \mathbb{R}^{n+1}.
$$
Then $f_k$ is an analytic function satisfying the following estimate for any multi-index $\gamma$
\[
|\partial^{\gamma}_{(t,x)} f_k(t,x)| = |\partial^{\gamma}_{(t,x)} \hat{A}(kt,kx)|,
\]
\[
= \left| \int_{\mathbb{R}^{1+n}} e^{-i(ky)(t,x)} (-i)^{\gamma} k^{\gamma} (s^2 + |y|^2)^{\frac{1}{2}} A(s,y) \, ds \, dy \right|,
\]
\[
\leq (2T^2)^{\frac{1}{2}} k^{\gamma} \int_{\mathbb{R}^{1+n}} |A(s,y)| \, ds \, dy, \quad \text{(as, diam(\Omega) < T)}
\]
\[
\leq C \, (2T^2)^{\frac{1}{2}} k^{\gamma} = C (\sqrt{2T})^{\frac{1}{2}} |\gamma|^{\frac{1}{2}} k^{\gamma}.
\]
\[
\times \text{(using a priori estimates of } A)\]

Hence we get,
\[
|\partial^{\gamma}_{(t,x)} f_k(t,x)| \leq C e^{\delta k} \frac{|\gamma|^{\frac{1}{2}}}{(T-1)^{\frac{1}{2}} k^{\gamma}} \text{ for } (t, x) \in \mathbb{R}^{n+1} \text{ and any multi–index } \gamma. \quad (60)
\]

Now we appeal to Vessella’s conditional stability result [28] to $f_k$, since it satisfies (60). Then we have some $C > 0$ and $\theta \in (0, 1)$ such that the following estimate holds
\[
\|f_k\|_{L^\infty(B(0,1))} \leq C e^{k^{1-\theta}} \|f_k\|_{L^\infty(\mathbb{C} \setminus B(0,1))}^\theta. \quad (61)
\]

Since $\|f_k\|_{L^\infty(B(0,1))} = ||\hat{A}||_{L^\infty(B(0,\delta))}$, after using lemma 4 we get from (61)
\[
\|\hat{A}\|_{L^\infty(B(0,\delta))} \leq C e^{k^{1-\theta}} \left( \frac{1}{\lambda^d} + e^{\beta \lambda} \|A_1 - A_2\|_1 \right)^\theta. \quad (62)
\]

Now using (62) with the a priori assumption on the vector potential, we express the Sobolev norm of $\hat{A}$ in terms of the input–output operator as follows
\[
\|\hat{A}\|_{H^{-\gamma}(\mathbb{R}^{1+n})}^2 = \left( \int_{\mathbb{R}^{1+n}} (1 + s^2 + |y|^2)^{-\gamma} |\hat{A}(s,y)|^2 \, ds \, dy \right)^{\frac{1}{2}},
\]
\[
= \left( \int_{B(0,\delta)} (1 + s^2 + |y|^2)^{-\gamma} |\hat{A}(s,y)|^2 \, ds \, dy \right)^{\frac{1}{2}}
\]
\[
+ \left( \int_{B(0,\delta)^c} (1 + s^2 + |y|^2)^{-\gamma} |\hat{A}(s,y)|^2 \, ds \, dy \right)^{\frac{1}{2}},
\]
\[
\leq C \left( k^{\gamma+1} \|\hat{A}\|_{L^\infty(B(0,\delta))}^2 + \frac{1}{k^{2\gamma}} \right)^{\frac{1}{2}},
\]
\[
\leq C \left( k^{\gamma+1} e^{2(k^{1-\theta})^\delta} \left( \frac{1}{\lambda^d} + e^{\beta \lambda} \|A_1 - A_2\|_1^2 \right) + \frac{1}{k^{2\gamma}} \right), \quad \text{(using (62))}
\]
\[
\leq C \left( k^{\gamma+1} e^{2(k^{1-\theta})^\delta} \left( \frac{1}{\lambda^d} + e^{\beta \lambda} \|A_1 - A_2\|_1^2 + \frac{1}{k^{2\gamma}} \right) \right)^{\frac{1}{2}}. \quad (63)
\]
To make (I) and (III) of (63) comparable we need to choose large enough \( k \) such that

\[
\lambda = e^{-\frac{k(1-d)}{C_1}} k^{\frac{\mu+1}{C_1}}.
\]

Then (II) of (63) becomes

\[
k^{\frac{\mu+1}{C_1}} e^{\frac{k(1-d)}{C_1} + \frac{\mu+1}{C_1}} \frac{2^{k(1-d)}}{C_1} \| \Lambda_1 - \Lambda_2 \|^2.
\]

There exists \( L_{k_0} > 0 \) which depends on \( \theta > 0, \delta > 0, \lambda_0 > 0 \) only such that

\[
k^{\frac{\mu+1}{C_1}} e^{\frac{k(1-d)}{C_1} + \frac{\mu+1}{C_1}} \frac{2^{k(1-d)}}{C_1} \| \Lambda_1 - \Lambda_2 \|^2 \leq e^{L_{k_0} k} \| \Lambda_1 - \Lambda_2 \|^2.
\]  \hfill (64)

Now we choose \( k = \frac{1}{T_0} \log | \log \| \Lambda_1 - \Lambda_2 \|_\ast |. \) Then from (63) and (64)

\[
\| \mathcal{A} \|_{H^{-1}(\mathbb{R}^{n+1})}^\theta \leq C \left( \| \Lambda_1 - \Lambda_2 \|_\ast + (\log | \log \| \Lambda_1 - \Lambda_2 \|_\ast |) \right)^{\frac{\theta+1}{\theta}}.
\]  \hfill (65)

Since we need to satisfy \( \lambda \geq \lambda_0 \), validity of our above choice of \( k \) depends on the smallness of the input–output operator, that is when \( \| \Lambda_1 - \Lambda_2 \|_\ast \leq c_* \) (where \( c_* \) depends only on \( \lambda_0 \)). If it is not the case that \( \| \Lambda_1 - \Lambda_2 \|_\ast > c_* \), we can have the following estimate for some \( C > 0 \) depending on \( C_\alpha \) and \( \alpha \)

\[
\| \mathcal{A} \|_{H^{-1}(\mathbb{R}^{n+1})} \leq C \| \mathcal{A} \|_{L^\infty(\mathbb{R}^{n+1})} \frac{C}{c_*} \| \Lambda_1 - \Lambda_2 \|_\ast.
\]  \hfill (66)

Hence for both of the cases discussed above, we get from (65) and (66)

\[
\| \mathcal{A} \|_{H^{-1}(\mathbb{R}^{n+1})} \leq C \left( \| \Lambda_1 - \Lambda_2 \|_\ast^\theta + | \log | \log \| \Lambda_1 - \Lambda_2 \|_\ast | \right)^{-\frac{1}{\theta'}}.
\]  \hfill (67)

We can translate the above norm estimates into a much stronger Sobolev norms using a convexity argument.

**Corollary 3.** For some \( \mu_1, \mu_2, \kappa_1, \kappa_2 \in (0, 1) \) and \( C > 0 \) we have

\[
\| \mathcal{A} \|_{L^\infty(\mathbb{R}^{n+1})} \leq C \left( \| \Lambda_1 - \Lambda_2 \|_{\mu_1}^\mu_1 + | \log | \log \| \Lambda_1 - \Lambda_2 \|_\ast | \right)^{-\mu_1}.
\]

and,

\[
\| \mathcal{A} \|_{H^{\mu_2}(\mathbb{R}^{n+1})} \leq C \left( \| \Lambda_1 - \Lambda_2 \|_{\mu_2}^{\kappa_1} + | \log | \log \| \Lambda_1 - \Lambda_2 \|_\ast | \right)^{-\kappa_2}.
\]

**Proof.** There exists \( \eta \in (0, 1) \) such that \( \eta \frac{n+1}{2} + \frac{\alpha}{2} = \eta \frac{n+1}{2} + \alpha + (1 - \eta)(-1) \).

Since \( \| \mathcal{A} \|_{H^{\frac{\eta(n+1)}{2} + \alpha}(\mathbb{R}^{n+1})} \leq C \), using Sobolev embedding and logarithmic convexity of Sobolev norms we obtain

\[
\| \mathcal{A} \|_{L^\infty(\mathbb{R}^{n+1})} \leq C \| \mathcal{A} \|_{H^{\frac{\eta(n+1)}{2} + \alpha}(\mathbb{R}^{n+1})} \leq C \| \mathcal{A} \|_{H^{\frac{\eta(n+1)}{2} + \alpha}(\mathbb{R}^{n+1})} \| \mathcal{A} \|_{H^{-1}(\mathbb{R}^{n+1})}^{1-\eta} \leq C \left( \| \Lambda_1 - \Lambda_2 \|_{\ast}^\eta + | \log | \log \| \Lambda_1 - \Lambda_2 \|_\ast | \right)^{-\frac{1}{\eta}}. \) (using (67))
Hence there exist $\mu_1, \mu_2 \in (0, 1)$ such that
\[
\|A\|_{L^{\infty}(\mathbb{R}^{n+1})} \leq C \left( \|\Lambda_1 - \Lambda_2\|_{\ast}^\mu_1 + |\log |\log \|\Lambda_1 - \Lambda_2\|\ast|^{-\mu_2} \right). \tag{68}
\]
Similarly one obtains the following estimate for some $\kappa_1, \kappa_2 > 0$
\[
\|A\|_{H^1(\mathbb{R}^{n+1})} \leq C \|A\|_{H^{-1}(\mathbb{R}^{n+1})}^{1-\eta}
\leq C \left( \|\Lambda_1 - \Lambda_2\|_{\ast}^\kappa_1 + |\log |\log \|\Lambda_1 - \Lambda_2\|\ast|^{-\kappa_2} \right). \tag{69}
\]

3.2. Stability estimate for the scalar potential

Proving stability of the scalar potentials will be slightly different from that of the vector potentials. In the process of getting integral identity and estimates we divided (32) by large enough $\lambda$ which made the scalar potential term disappear. We use here explicit uniform norm estimate for vector potentials rather than dividing by $\lambda$. So we have to make the necessary changes in lemma 2. However the light-ray transform and Fourier estimates of the scalar potential term are the same as before. In this section, using uniform norm estimates for the vector potential and Vessella’s analytic continuation argument, Fourier estimate of the scalar potential over arbitrary large balls will be shown resulting in a log–log–log stability of the scalar potential in terms of the input–output operator.

**Proof.** We have
\[
\mathcal{L}_{A_1,q_1} u(t,x) = (2A \cdot (\partial_t - \nabla_x)u_2 + \check{q}u_2)(t,x).
\]
Now using Hölder inequality, *a priori* bounds for the potentials and properties of geometric optics solutions from (15), (16), (20) and (22), we have
\[
\|e^{-\lambda t + x \cdot \omega} \mathcal{L}_{A_1,q_1} u\|_{L^2(Q)}^2 \leq C \left( \lambda^2 \|A\|_{L^\infty(Q)}^2 + 1 \right) \|\phi\|_{H^1(\mathbb{R}^n)}^2. \tag{70}
\]
As before, let $K$ be the rhs of the boundary Carleman estimate (7) corresponding to $\mathcal{L}_{A_1,q_1}$ applied to $u$. Then we obtain from (26), (27) and (70)
\[
K \leq C \left( \lambda^2 \|A\|_{L^\infty(Q)}^2 + 1 + e^{\lambda \beta} \|\Lambda_1 - \Lambda_2\|_{\aste}^2 \right) \|\phi\|_{H^1(\mathbb{R}^n)}^2. \tag{71}
\]
Now we proceed as before to have estimates of the light-ray transform of the scalar potential and in this situation it will include uniform norm of the vector potential in $Q$. We see
\[
\left| \int_Q (2A \cdot (\partial_t - \nabla_x)u_2 + \check{q}u_2)\nu dxdt \right|
\leq C \left( \sqrt{\frac{K}{\lambda}} \|\phi\|_{H^1(\mathbb{R}^n)} + e^{\lambda \beta} \|\Lambda_1 - \Lambda_2\|_{\aste} \|\phi\|_{H^1(\mathbb{R}^n)}^2 \right),
\]
or,
\[
\left| \int_Q \check{q}(t,x)u_2(t,x)\nu(t,x) dxdt \right|
\leq C \left( \lambda \|A\|_{L^\infty(Q)} + \frac{1}{\sqrt{\lambda}} + e^{\lambda \beta} \|\Lambda_1 - \Lambda_2\|_{\aste} \right) \|\phi\|_{H^1(\mathbb{R}^n)}^2.
\]
or, \[
\left| \int_Q \tilde{q}(t,x)|\phi(x + t\omega)|^2 e^{-\int_0^t \mathcal{A}(x,t+s)\omega ds} dx \right| \leq C \left( \lambda \|\mathcal{A}\|_{L^\infty(Q)} + \frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|A_1 - A_2\|_* \right) \|\phi\|^2_{L^1(\mathbb{R}^n)}. \tag{72}
\]

Using the mean value theorem and a priori bounds of the vector potentials we get \[
|e^{-\int_0^t \mathcal{A}(x,t+s)\omega ds} - 1| \leq C \|\mathcal{A}\|_{L^\infty(Q)} \quad \text{for all} \quad t \in [0,T]. \tag{73}
\]

Thus we use (72) and (73) to get \[
\left| \int_Q \tilde{q}(t,x)|\phi(x + t\omega)|^2 dx \right| \leq \left| \int_Q \tilde{q}(t,x)|\phi(x + t\omega)|^2 \left( e^{-\int_0^t \mathcal{A}(x,t+s)\omega ds} - 1 \right) dx \right| + \left| \int_Q \tilde{q}(t,x)|\phi(x + t\omega)|^2 e^{-\int_0^t \mathcal{A}(x,t+s)\omega ds} dx \right|
\leq C \left( \lambda \|\mathcal{A}\|_{L^\infty(Q)} + \frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|A_1 - A_2\|_* \right) \|\phi\|^2_{L^1(\mathbb{R}^n)}. \tag{74}
\]

Following similar steps as in lemma 3 and corollary 2, we have \(\gamma, \delta > 0\) and an open cone say \(C \subset \mathbb{R}^{n+1}\) such that \[
\left| \int_0^T \tilde{q}(s,x - ws) dx \right| \quad \text{and,} \quad \|\tilde{q}\|_{L^\infty(C)} \leq C \left( \lambda^\gamma \|\mathcal{A}\|_{L^\infty(Q)} + \frac{1}{\lambda^\delta} + e^{\beta\lambda} \|A_1 - A_2\|_* \right). \tag{75}
\]

Using (75) and Vessella’s analytic continuation result as done in (62), we get for \(k\) large \[
\|\tilde{q}\|_{L^\infty(B(0,k))} \leq Ce^{k(1-\theta)} \left( \lambda^\gamma \|\mathcal{A}\|_{L^\infty(Q)} + \frac{1}{\lambda^\delta} + e^{\beta\lambda} \|A_1 - A_2\|_* \right)^{\theta}. \tag{76}
\]

Then, \[
\|\tilde{q}\|_{\mathcal{H}^{-1}(\mathbb{R}^{n+1})} \leq C \left( k^{\frac{n+1}{2}} \frac{e^{k(1-\theta)}}{\lambda^{\delta/2}} \right) \times \left( \lambda^{2\gamma} \|\mathcal{A}\|_{L^\infty(Q)}^2 + \frac{1}{\lambda^{2\delta}} + e^{\beta\lambda} \|A_1 - A_2\|_*^2 \right)^{\theta} + k^{-\frac{\theta}{2}}. \tag{77}
\]

Now we choose \(\lambda = e^{-\frac{k(1-\theta)}{\delta}} k^\frac{n+1}{2}\) as in (63). For some \(L_0 > 0\), using (68) rhs of (77) can be dominated by the following term \[
e^{k L_0} \|A_1 - A_2\|^2_{L^p} + e^{k L_0 \log} \|A_1 - A_2\|^2_{L^p} \log \|A_1 - A_2\|_{L^p} + e^{k L_0} \|A_1 - A_2\|^2 + k^{-\frac{\theta}{2}}. \tag{78}
\]
Choose $e^{D_{\delta}^b} = (\log | \log | A_1 - A_2 | |, \ldots)^{\mu_k}$. That is, $k = \frac{\mu_k}{\mu_0} \log \log | \log | A_1 - A_2 | |$. Using the choice of $k$ above and $\mu_0 < 1$, rhs of (78) can have following upper bound

$$
\| A_1 - A_2 \|^2 \| \log | \log | A_1 - A_2 | | \| \mu_2 + \| \log | \log | A_1 - A_2 | | \|^{-\mu_22} \\
+ \| A_1 - A_2 \|^2 \| \log | A_1 - A_2 | | + \| \log | \log | A_1 - A_2 | | \|^{-\hat{\beta}}.
$$

(79)

Assume $\| A_1 - A_2 \| < c_\gamma$ small enough say $\| A_1 - A_2 \| < c_\gamma$ such that

a) choice of $k$ above is valid.

b) $\| A_1 - A_2 \|^2 \| \log | \log | A_1 - A_2 | | \|^{-2\mu_2}$ (for $\alpha > 0$, $\lim\limits_{x \to 0^+} x^\alpha \log | \log x | = 0$).

c) $(\log | \log | A_1 - A_2 | | \|^{-\mu_2} \leq (\log | \log | A_1 - A_2 | | \|^{-\mu_2}$ (as, $\log x \leq x$).

d) $\| A_1 - A_2 \|, \| \log | A_1 - A_2 | | \| \leq C$ (for $\alpha > 0$, $\lim\limits_{x \to 0^+} x^\alpha \log x = 0$).

(81)

Combining (80)–(83) we dominate (78) by the following term

$$
\| A_1 - A_2 \|^2 + (\log | \log | A_1 - A_2 | | \|^{-\beta}) \quad \text{for some } \nu_1, \nu_2 > 0.
$$

(84)

When $\| A_1 - A_2 \| \geq c_\gamma$, we may proceed as (66) to get similar estimates as in (84). Hence by Sobolev embedding and logarithmic convexity of Sobolev norms we get as before

$$
\| \hat{q} \|_{L_1^\infty(\mathbb{R}^{n+1})} \leq C (\| A_1 - A_2 \|^2 + | \log | \log | A_1 - A_2 | | \|^{-\nu_2}) \quad \text{for some } \nu_1, \nu_2 > 0.
$$

(85)

Our goal is to establish stability estimate for $q$. To do so, we notice the following relation

$$
q(t, x) = \hat{q}(t, x) - \left( \partial_t A_0 - \sum_{k=1}^n \partial_k A_k \right) + (|A_{1,0}|^2 - |A_{2,0}|^2) \quad (t, x)
$$

$$
+ \sum_{k=1}^n (|A_{2,k}|^2 - |A_{1,k}|^2) \quad (t, x).
$$

Hence we can write

$$
\| q \|_{L_1^\infty(\mathbb{R}^{n+1})} \leq C \left( \| \hat{q} \|_{L_1^\infty(\mathbb{R}^{n+1})} + \| A \|_{H^1(\mathbb{R}^{n+1})} \right) + \| A \|_{L_1^\infty(\mathbb{R}^{n+1})}.
$$

We combine (68), (69) and (85) to obtain

$$
\| q \|_{L_1^\infty(\mathbb{R}^{n+1})} \leq C (\| A_1 - A_2 \|^2 + | \log | \log | A_1 - A_2 | | \|^{-\nu_2}) \quad \text{for some } \alpha_1, \alpha_2 > 0.
$$

This shows stability of the scalar potential in terms of the input–output operator. \qed
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