Statistics of the Microwave Background Anisotropies Caused by Cosmological Perturbations of Quantum-Mechanical Origin

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Abstract

The genuine quantum gravity effects can already be around us. It is likely that the observed large-angular-scale anisotropies in the microwave background radiation are induced by cosmological perturbations of quantum-mechanical origin. Such perturbations are placed in squeezed vacuum quantum states and, hence, are characterized by large variances of their amplitude. The statistical properties of the anisotropies should reflect the underlying statistics of the squeezed vacuum quantum states. In this paper, the theoretical variances for the temperature angular correlation function are described in detail. It is shown that they are indeed large and must be present in the observational data, if the anisotropies are truly caused by the perturbations of quantum-mechanical origin. Unfortunately, these large theoretical statistical uncertainties will make the extraction of cosmological information from the measured anisotropies a much more difficult problem than we wanted it to be. This contribution to the Proceedings is largely based on references [42,8]. The Appendix contains an analysis of the “standard” inflationary formula for density perturbations.

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I. INTRODUCTION

In the context of cosmology, the quantum gravitational physics is usually understood as the early Universe physics at the Planck scale. It is assumed that the gravitational field is fully quantized and the Universe itself is, in a sense, quantized. (For a comprehensive recent review of the structural issues in quantum gravity see [1]). However, there exists also another meaning in which we can speak of quantum gravity effects. To clarify the difference, let us consider an analogy from condensed matter physics. Imagine a crystal and various quantized excitations in it: phonons, rotons, excitons, etc.. The creation and annihilation operators that the condensed matter theorists write do not create and annihilate the crystal, they create and annihilate excitations in the crystal. The excitations should not necessarily be linear, one can take account of the higher-order corrections too. The theory of the crystal excitations is a fully legitimate quantum theory which does not attempt to quantize “everything in sight”. Similarly, in our study below, we do not write the creation and annihilation operators that create and destroy universes. Our operators create and destroy perturbations in the Universe. Nevertheless, the effects that we are studying are genuine quantum gravity effects in the sense that they inherently contain all the three fundamental constants. The gravitational constant and velocity of light enter because we deal with a gravitational field (its energy-momentum tensor), the Planck constant enters because we normalize the vacuum energy of the field to have “a half of the quantum” in each mode. All three fundamental constants combine in the Planck length $l_{Pl}$, and this quantity naturally appears as the coefficient in the most of our formulas. The sending of the $l_{Pl}$ to zero would eliminate the entire expression.

Now, why at all do we think that the anisotropies in the microwave background radiation may have something to do with gravitational quantum physics? Why, in the first place, is there such a considerable interest to the measured large-angular-scale anisotropies [2]? After all, we have always knew that the observed part of the Universe is homogeneous and isotropic only approximately, on average, and we are certainly aware of large deviations on smaller scales.

The point is that the photons of the microwave background have been traveling to us for almost all duration of that characteristic time which we call the age of the Universe and which is determined by the value of the measurable quantity — the Hubble parameter. The anisotropies on largest angular scales are specific in that they are produced by cosmological perturbations on largest spatial scales. In our context, the largest spatial scale means definitely larger than any directly studied distance, and comparable or larger than the characteristic cosmological distance — the Hubble radius — associated with the Hubble parameter. In terms of the crystal analogy, and assuming that the size of the crystal varies at some time scale, we are dealing with excitations whose wavelengths are comparable and longer than the light travel spatial scale associated with the time variation.

The next question is whether we will be attempting to explain the origin and nature of such long-wavelength cosmological perturbations or we will be happy to simply accept their existence. There is no logical contradiction in the second position. One can study the perturbations with whatever accessible observational accuracy one has, and then extrapolate back in time their evolution according to classical dynamical laws. Typically, we will end up with a very anisotropic and inhomogeneous universe at some very early times. As an expla-
nation, we will have nothing more to say except that this was the Universe that was given to us for our future life and study. A more appealing position, at least aesthetically, is to try and find out a universal and reliable mechanism which could be capable of generating the required perturbations in the originally homogeneous and isotropic Universe. The problem is not to increase the wavelengths of the perturbations up to the size of the present Hubble radius (in the expanding Universe, the wavelength is always growing with time) but to generate them. In principle, one can imagine that such a mechanism may have operated in the relatively recent Universe and may have not be related to quantum gravity. In practice, taking into account everything what we know about cosmology and the necessity to produce perturbations with extremely long wavelengths, it is difficult to suggest such a mechanism, especially if we do not want to make many additional hypotheses. It appears that we can only rely on the quantum processes in the very early Universe. The parametric interaction of quantized cosmological perturbations with strong variable gravitational field of the very early Universe provides us with such a possibility. The perturbations are generated as a result of amplification of their zero-point quantum oscillations. Returning to the laboratory physics analogy, one can recall that a cavity filled with a dielectric medium and initially free of electromagnetic radiation will eventually contain the radiation if the parameters of the dielectric medium vary properly in time. The quantum-mechanical (parametric) mechanism of generating cosmological perturbations relies only on the validity of the general relativity and the basic principles of quantum field theory. The observational consequences of this phenomenon we will study below.

The line of reasoning in this study can be summarized as follows.

We see the anisotropies in the microwave background radiation at the largest angular scales [2]. Observers convincingly argue that this is a genuine cosmological effect.

If the large-angular-scale anisotropy in the microwave background is really produced by cosmological perturbations (density perturbations, rotational perturbations, gravitational waves), then their today’s wavelengths are of the order and longer than today’s Hubble radius $l_H$. Strictly speaking, all wavelengths give contributions to the anisotropy at every given angular scale. But if the spectrum of the perturbations is not excessively “red” or “blue”, the dominant contribution is provided by wavelengths indicated above. For instance, the major contribution to the quadrupole anisotropy is provided by wavelengths somewhat longer than $l_H$.

In the expanding Universe, the wavelengths of perturbations increase in proportion to the cosmological scale factor. The wavelengths that are longer than some length scale today have always been longer than that scale in the past. Moreover, the wavelengths of the perturbations of our interest are much longer than the Hubble radius defined at the previous times, when one goes back in time up to the era of primordial nucleosynthesis — the earliest era of which we have observational data. It is hard to imagine (although it does not seem to be logically impossible) that cosmological perturbations of our interest, with such long wavelengths, could have been generated by local physical processes during the interval of time between the era of primordial nucleosynthesis and now. We are bound to conclude that these perturbations were generated in the very early Universe, before the era of primordial nucleosynthesis. There is still 80 orders of magnitude, in terms of energy density, to go from the era of primordial nucleosynthesis to the Planck era; a lot of things could have happened in between.
The law of evolution of the very early Universe is not known, but it is likely that it could have been significantly different from the law of expansion of the radiation-dominated Universe. If so, some amount of cosmological perturbations must have been generated quantum-mechanically, as a result of parametric interaction of the quantized perturbations with strong variable gravitational field of the very early Universe. Gravitational waves have been generated inevitably, while density and rotational perturbations — if we were lucky; see [3] and references therein. (If the cosmological scale factor has always been the one of the radiation-dominated Universe, we must stop here, because the parametric coupling vanishes in this case, and cosmological perturbations cannot be amplified classically and cannot be generated quantum-mechanically.) The amount and spectrum of the generated perturbations depend on the law of evolution of the very early Universe (the strength and variability of the gravitational pump field), and this is how we can learn about what was going on there. In particular, the law of evolution of the very early Universe could have been of inflationary type.

If the cosmological perturbations were generated quantum-mechanically, they should be placed in the squeezed vacuum quantum states [4] (for an introduction to squeezed states see, for example, Ref. [5,6] and the pioneering works quoted there). Squeezing of cosmological perturbations might have degraded by now at short wavelengths but should survive at long wavelengths, especially in the case of gravitational waves.

The squeezed vacuum quantum states can only be squeezed in the variances of phase which unavoidably means the increased variances in amplitude. The statistical properties of the squeezed vacuum quantum states are significantly different from the statistical properties of the “most classical” quantum states — coherent states. This is well illustrated by the fact that the variance of the number of quanta in a strongly squeezed vacuum quantum state is much larger than the variance of the number of quanta in the coherent state with the same mean number of quanta $\langle N \rangle$, $\langle N \rangle \gg 1$. For a squeezed vacuum state the variance is $\langle N^2 \rangle - \langle N \rangle^2 = 2\langle N \rangle(\langle N \rangle + 1) \gg \langle N \rangle$, while for a coherent state it is $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$. In cosmology, the mean number $\langle N \rangle$ is a characteristic of the expected mean square amplitude of cosmological perturbations, while the variance $\langle N^2 \rangle - \langle N \rangle^2$ is a characteristic of theoretical uncertainties in the amplitude. These two characteristics are independent properties of a quantum state or a stochastic process. Theoretical models may agree on $\langle N \rangle$ and disagree on variance or agree on variance and disagree on $\langle N \rangle$.

The statistical properties of squeezed cosmological perturbations will inevitably be reflected in statistical properties of the microwave background anisotropies caused by them. Squeezing is a phase-sensitive phenomenon, and to fully extract its properties the quantum optics experimenters use the phase-sensitive detecting techniques based on a local oscillator. In cosmology, we are very far from being able to build a local oscillator, except of maybe, in the distant future, for short gravitational waves. Besides, in our study of the microwave background anisotropies, we are interested in so long-wavelength perturbations that it would take billions of years to wait for seeing the time dependent oscillations of variances in the quadrature components of the perturbation field. On the other hand, the amount of cosmological squeezing is enormously greater than what is achieved in quantum optics laboratory experiments. In cosmology, we can only rely on the phase-insensitive, direct detection. One can expect that the underlying large variances of the amplitude of cosmological perturbations should result in large statistical deviations from the mean values for the microwave
anisotropies.

A detailed study and proof of this statement is the purpose of this work.

At this point it is necessary to comment on the possible numerical values of \( \langle N \rangle \) (and, hence, the amplitude) for cosmological perturbations of different nature which can be generated quantum-mechanically in one and the same cosmological model. A consistent quantum theory provides us, of course, with both, the mean value of \( N \) and its variance. According to the calculations of Ref. [3], the contribution of quantum-mechanically generated gravitational waves to the large-angular-scale anisotropy is somewhat greater (even in the limit of the de Sitter expansion) than the contribution of quantum-mechanically generated density perturbations. It is argued in [7,8] that the “standard” inflationary formula for density perturbations, which requires in this limit an arbitrarily large excess of density perturbations over gravitational waves, is based on errors. We additionally discuss this issue in some detail in the Appendix of this paper. However, the major emphasis of this paper is on the statistical properties of cosmological perturbations which are determined by their common origin — quantum mechanics and squeezing. These properties are related to the variance of \( N \) rather than to its mean value. Our discussion will be equally well applicable to the perturbations of any nature (density perturbations, rotational perturbations, gravitational waves) if they have the same origin.

II. THE GENERAL EQUATIONS FOR QUANTIZED COSMOLOGICAL PERTURBATIONS

Here we will briefly summarize some basic information about quantized cosmological perturbations (see [3,7] and references therein). The squeezed field operator derived in this Section is a basic mathematical construction for our further discussion of statistical properties.

The metric of the homogeneous isotropic universe can be written in the form

\[
\begin{align*}
\text{ds}^2 & = -a^2(\eta)\left(d\eta^2 - \gamma_{ij} \, dx^i \, dx^j\right), \tag{1}
\end{align*}
\]

where \( \gamma_{ij} \) is the spatial metric. For reasons of simplicity, we will be considering only spatially flat universes, that is \( \gamma_{ij} = \delta_{ij} \).

Following Lifshitz, it is convenient to write the perturbed metric in the form

\[
\begin{align*}
\text{ds}^2 & = -a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij}) \, dx^i \, dx^j], \tag{2}
\end{align*}
\]

where \( h_{ij} \) are functions of \( \eta \)-time and spatial coordinates. By writing the perturbed metric in this form we do not lose anything in the physical content of the problem, but we gain considerably in the mathematical tractability of the perturbed Einstein equations. The one who is interested in solving equations will certainly be interested in their simpler form. Those who prefer “gauge-invariant formalisms” are welcome to take the found solution and compute with its help whichever gauge-invariant quantity they like. These quantities, being gauge-invariant, have the same values in all gauges.

The components \( h_{ij} \) of the perturbed gravitational field can be classified in terms of scalar, vector, and tensor eigenfunctions of the Laplace differential operator. The components of the perturbed energy-momentum tensor can also be classified in the same manner.
After that, the linearized Einstein equations reduce to a set of ordinary differential equations, separately for scalar (density perturbations), vector (rotational perturbations), and tensor (gravitational waves) parts.

The number of independent unknown functions of time that can potentially be present (on grounds of the classification scheme) in the perturbed Einstein equations is always greater than the number of independent equations. It is 6 functions and 4 equations for density perturbations, 3 functions and 2 equations for rotational perturbations, and 2 functions and 1 equation for gravitational waves. In order to make the system of equations closed, it is necessary to say something about the perturbed components of the energy-momentum tensor or to specify from the very beginning the form of the energy-momentum tensor. The popular choices are perfect fluids and scalar fields. Even for gravitational waves, it is not a totally trivial question what their definition is (see, for example, Ref. [9]). However, after everything is being set, and as soon as the scale factor $a(\eta)$ (the background solution) is known, the general solution to the perturbed equations can be found. In practice, exact solutions are being found piecewise, at the intervals of evolution where the energy-momentum tensor has simple prescribed forms.

We can now write the quantum-mechanical operator for the perturbations of the gravitational field $h_{ij}$ in the following universal form:

$$h_{ij} = \frac{C}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 n \sum_{s=1}^{2} \hat{s}_{ij}(n) \frac{1}{\sqrt{2n}} \left[ \hat{c}^s_n(\eta)e^{\text{i}nx} + \hat{c}^s_{\text{n}}(\eta)e^{-\text{i}nx} \right]. (3)$$

We will start the explanation of Eq. (3) from the polarization tensors $\hat{s}_{ij}$. Let us introduce, in addition to the unit wave-vector $n/n$, two more unit vectors $l_i, m_i$, orthogonal to each other and to $n$:

$$n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad l_i = (\sin \phi, -\cos \phi, 0),$$

$$m_i = \pm(\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),$$

+ for $\theta < \pi/2$, − for $\theta > \pi/2$.

The two independent polarization tensors, $s = 1, 2$, for each class of perturbations, can be written as follows. For gravitational waves:

$$\hat{1}_{ij}(n) = (l_il_j - m_im_j), \quad \hat{2}_{ij}(n) = (l_im_j + ljm_i).$$

For rotational perturbations:

$$\hat{1}_{ij}(n) = \frac{1}{n}(l_in_j + l_jn_i), \quad \hat{2}_{ij}(n) = \frac{1}{n}(m_in_j + m_jn_i).$$

For density perturbations:

$$\hat{1}_{ij}(n) = \sqrt{\frac{2}{3}} \delta_{ij}, \quad \hat{2}_{ij}(n) = -\sqrt{3} \frac{n_in_j}{n^2} + \frac{1}{\sqrt{3}} \delta_{ij}.$$  

The polarization tensors of each class satisfy the conditions $\hat{s}_{ij} \hat{s}_{ij} = 2\delta_{ss'}$, $\hat{s}_{ij}(-n) = \hat{s}_{ij}(n)$. In practical handling of the density perturbations it proves convenient to use sometimes, in
addition to the scalar polarization component $\hat{p}_{ij}$, the longitudinal-longitudinal component (proportional to $n_i n_j$) instead of $\hat{p}_{ij}$. The explicit functional dependence of the polarization tensors is needed for the calculation of various angular correlation functions.

The evolution of the creation and annihilation operators $\hat{s}_n(\eta), \hat{s}_n^\dagger(\eta)$, for each class of perturbations and for each polarization state, is defined by the Heisenberg equations of motion:

$$\frac{dc_n(\eta)}{d\eta} = -i[c_n(\eta), H], \quad \frac{dc_n^\dagger(\eta)}{d\eta} = -i[c_n^\dagger(\eta), H].$$  \tag{5}

The dynamical content of the problem is determined by the Hamiltonian $H$. Its form depends on the class of perturbations and additional assumptions about the energy-momentum tensor which we have to make, as was discussed above.

Under the simplest assumptions about gravitational waves (waves interact only with the background gravitational field, there is no anisotropic material sources) the Hamiltonian for each polarization component takes on the form

$$H = nc_n^\dagger c_n + nc_{-n}^\dagger c_{-n} + 2\sigma(\eta)c_n^\dagger c_{-n} + 2\sigma^*(\eta)c_n c_{-n}$$  \tag{6}

where the coupling function $\sigma(\eta)$ is $\sigma(\eta) = \frac{i}{2} a' a^\dagger$.

For rotational perturbations, assuming that the primeval matter is capable of supporting torque oscillations, assuming that the oscillations are minimally coupled to gravity, and assuming that the torsional velocity of sound is equal to the velocity of light, the Hamiltonian for each polarization component reduces to exactly the same form (6) with the same coupling function $\sigma(\eta)$.

For density perturbations, we consider specifically a minimally coupled scalar field with arbitrary scalar field potential as a model for matter in the very early Universe, and perfect fluids at the later eras. The quantization is based on the scalar polarization component (the function of time responsible for another polarization state is not independent). There is only one independent sort of creation and annihilation operators in this case. The operators $\hat{s}_n(\eta), \hat{s}_n^\dagger(\eta)$ are expressible in terms of the operators $d_n(\eta), d_n^\dagger(\eta)$ for which the Hamiltonian has again the same form (6) but with the coupling function $\sigma(\eta)$.

For density perturbations, it is the operators $d_n(\eta), d_n^\dagger(\eta)$ that participate in Eqs. (5), (6).

Now, let us turn to the constant $C$ in Eq. (3). Its value is determined by the normalization of the field of each class to the “half of the quantum in each mode”. Under the assumptions listed above, one derives $C = \sqrt{16\pi} l_{Pl}$ for gravitational waves, $C = \sqrt{32\pi} l_{Pl}$ for rotational perturbations, and $C = \sqrt{24\pi} l_{Pl}$ for density perturbations, where $l_{Pl}$ is the Planck length, $l_{Pl} = (G\hbar/c^3)^{1/2}$.

The form of the Hamiltonian (6) dictates the form of the solution (Bogoliubov transformation) to Eq. (5):

$$c_n(\eta) = u_n(\eta)c_n(0) + v_n(\eta)c_{-n}^\dagger(0)$$
$$c_n^\dagger(\eta) = u_n^*(\eta)c_n^\dagger(0) + v_n^*(\eta)c_{-n}(0)$$  \tag{7}
where \(c_n(0), \ c^\dagger_n(0)\) are the initial values of the operators taken long before the interaction with the pump field became important \((\sigma(\eta)/n \to 0)\) and which define the vacuum state \(c_n(0)|0\rangle = 0\). The complex functions \(u_n(\eta), \ v_n(\eta)\) obey coupled first-order differential equations following from Eq. (4) and satisfy the condition \(|u_n|^2 - |v_n|^2 = 1\) which guarantees that the commutator relationship \([c_n(0), \ c^\dagger_m(0)] = \delta^3(\mathbf{n} - \mathbf{m})\) is satisfied at all times, \([c_n(\eta), \ c^\dagger_m(\eta)] = \delta^3(\mathbf{n} - \mathbf{m})\). If one introduces the function \(\mu_n(\eta) = u_n(\eta) + v_n^*(\eta)\), one recovers from the equations for \(u_n(\eta), \ v_n(\eta)\) the classical equations of motion.

For gravitational waves:

\[
\mu_n'' + \left[n^2 - \frac{a''}{a}\right] \mu_n = 0 .
\]

(8)

For rotational perturbations:

\[
\mu_n'' + \left[n^2 \frac{v_t^2}{c^2} - \frac{a''}{a}\right] \mu_n = 0 .
\]

(9)

where \(v_t\) is the torsional velocity of sound which we assumed above to be \(c\). For the scalar field density perturbations:

\[
\mu_n'' + \left[n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}}\right] \mu_n = 0 .
\]

(10)

If the pump field is such that the \(\gamma\) function is independent of time, Eq. (10) reduces to exactly the same form as Eq. (8) for gravitational waves.

In the Schrödinger picture, the initial vacuum quantum state \(|0_n\rangle \ |0_{-n}\rangle\) evolves into a two-mode squeezed vacuum quantum state. In our problem, each of the two-mode squeezed vacuum quantum states is a product of two identical one-mode squeezed vacuum quantum states which correspond to the decomposition of the real field \(h_{ij}\) over real spatial harmonics \(\sin nx\) and \(\cos nx\). In the Heisenberg picture, the initial vacuum quantum state does not evolve in time and is the same now.

By using Eq. (7) one can present the field (3) in the form

\[
h_{ij}(\eta, \mathbf{x}) = C \sqrt{\frac{1}{2\pi^{3/2}}} \int_{-\infty}^{\infty} d^3 n \sum_{s=1}^{2} \hat{p}_{ij}(\mathbf{n}) \frac{1}{\sqrt{2n}} \left[\hat{s}_n(\eta)e^{inx}\hat{c}_n(0) + \hat{s}^*_n(\eta)e^{-inx}\hat{c}^\dagger_n(0)\right] ,
\]

(11)

where the functions \(\hat{s}_n(\eta)\) are \(\hat{s}_n(\eta) = \frac{1}{a(\eta)}[\hat{\mu}_n(\eta) + \hat{v}_n(\eta)]\). For gravitational waves and rotational perturbations, the functions \(\hat{s}_n\) are simply \(\hat{s}_n = \hat{\mu}_n/a\) where \(\hat{\mu}_n\) are solutions to Eqs. (8), (9) with appropriate initial conditions. For density perturbations, the functions \(\hat{s}_n\) are derivable from solutions to Eq. (10) in accord with the relationship between \(c\) and \(d\) operators. Besides, for density perturbations, we should regard \(\hat{c}_n(0) = \frac{1}{a(\eta)}[\hat{c}_n(\eta) + \hat{v}_n^*(\eta)]\) in Eq. (11). In all cases, for a given cosmological model, that is for a model in which the scale factor \(a(\eta)\) is known from the very early times and up to now, the functions \(\hat{s}_n\) can be found from the classical equations of motion with appropriate initial conditions.

It follows from Eq. (11) that the mean quantum-mechanical value of the field \(h_{ij}\) is zero at every spatial point and at every moment of time, \(\langle 0| h_{ij} |0\rangle = 0\). One can also
calculate variances of the field, that is the expectation values of its quadratic combinations. One useful quantity is $h_{ij}h^{ij}$. By manipulating with the product of two expressions (11), using the summation properties of the polarization tensors, and remembering that the only nonvanishing correlation function is

$$\langle 0 | c_n^\dagger(0)c_n^\dagger(0) | 0 \rangle = \delta_{ss'}\delta^3(n - n') \ ,$$

one can derive the formula

$$\langle 0 | h_{ij}(\eta, x)h^{ij}(\eta, x) | 0 \rangle = \frac{C^2}{2\pi^2} \int_0^\infty n \sum_{s=1}^2 |\tilde{h}_n(\eta)|^2 dn \ . \ (12)$$

Equation (12) shows that the variance is independent of the spatial point $x$ but does depend on time.

The expression under the integral in formulas such as Eq. (12) is usually called the power spectrum (in this case, it is the power spectrum of the quantity $h_{ij}h^{ij}$):

$$P(n) = \frac{C^2}{2\pi^2} n \sum_{s=1}^2 |\tilde{h}_n(\eta)|^2 \ . \ (13)$$

In cosmology, it is common to use the power spectrum defined in terms of the logarithmic frequency interval, that is the function

$$P_Z(n) = \frac{C^2}{2\pi^2} n^2 \sum_{s=1}^2 |\tilde{h}_n(\eta)|^2 \ . \ (14)$$

($Z$ from Zeldovich). We are mostly interested in the power spectrum of cosmological perturbations in the present Universe, at the matter-dominated stage. This spectrum is never smooth as a function of frequency (wave-number) $n$. Squeezing and associated standing wave pattern of the field make the spectrum an oscillating function of $n$ for each moment of time. In their turn, the oscillations in the power spectrum will produce oscillations in the distribution of the higher-order multipoles of the angular correlation function for the temperature anisotropies. However, the spectrum is smooth for sufficiently long waves. At a given moment of time, it applies to all perturbations whose wavelengths are of the order and longer than the Hubble radius defined at that time. Moreover, the smooth part of the spectrum is power-law dependent on $n$ if the scale factor $a(\eta)$ of the very early Universe (the pump field) was power-law dependent on $\eta$-time.

Let us assume that the scale factor at the initial stage of expansion was

$$a(\eta) = l_o |\eta|^{1+\beta} \ . \ (15)$$

where $l_o$ and $\beta$ are constants. If the evolution is governed by a scalar field, the Einstein equations require the constant $\beta$ to be $\beta \leq -2$. The value $\beta = -2$ corresponds to the de Sitter expansion. At later times, the scale factor changed to the laws of the radiation-dominated and matter-dominated universes. From solutions for $\tilde{h}_n(\eta)$ traced up to the matter-dominated stage, one can find
\[
\sum_{s=1}^{2} |h_n(\eta)|^2 \sim \frac{1}{l_o^2} n^{2\beta+2} \quad \text{and} \quad P_Z(n) \sim \frac{l_{Pl}^2}{l_o^2} n^{2(\beta+2)}.
\]

It is convenient to introduce the characteristic amplitude \(h(n)\) of the metric perturbations defining this amplitude as the standard deviation (square root of variance) of the perturbed gravitational field per logarithmic frequency interval. In the long-wavelength limit under discussion, this quantity is universally expressed (both, for gravitational waves and density perturbations) by the formula \[21\]:

\[
h(n) \sim \frac{l_{Pl}}{l_o} n^{\beta+2}.
\] (16)

Note that the functional form of \(h(n)\) is the same for gravitational waves and density perturbations, the difference is in the numerical coefficient (omitted in this discussion) which is somewhat in favor of gravitational waves \[3,7\]. The numerical level of \(h(n)\) is mainly controlled by the constant \(l_o\).

The spectra of other quantities can be found in the same manner. For instance, in case of density perturbations, one can derive the spectrum of perturbations in the matter density \(\delta \rho/\rho\) since the relationship between \(\delta \rho/\rho\) and the metric perturbations involves the factor \((n\eta)^2\) and, hence, involves two extra powers of \(n\), \(\delta \rho/\rho(n) \sim n^2 h(n)\), one finds

\[
\langle 0 | \frac{\delta \rho}{\rho} \frac{\delta \rho}{\rho} | 0 \rangle \sim \int_0^\infty P^\rho_Z(n) \frac{dn}{n}
\]

where \(P^\rho_Z(n) \sim (l_{Pl}^2/l_o^2) n^{2(\beta+4)}\) and

\[
\frac{\delta \rho}{\rho}(n) \sim \frac{l_{Pl}}{l_o} n^{\beta+4}.
\] (17)

It follows from Eq. (16) that \(h(n)\) is independent of \(n\) if \(\beta = -2\). This independence corresponds to the original Zeldovich’s definition of the “flat” spectrum: all waves enter the Hubble radius with the same amplitude. If the gravitational field perturbations \(h(n)\), regardless of their wavelength, have equal amplitudes upon entering the Hubble radius, the matter density perturbations \(\delta \rho/\rho(n)\) do also have equal amplitudes (the extra factor \((n\eta)^2\) is of the order of 1 at the time when a given wave \(n\) enters the Hubble radius). For models of the very early Universe governed by a scalar field, the spectral index \(\beta + 2\) in Eq. (16) can never be positive.

Formula (16) and the associated formula (17) should be compared with the “standard” inflationary formula which requires that the amplitudes of density perturbations taken at the time of entering the Hubble radius should go to infinity in the limit of the de Sitter inflation, \(\beta \rightarrow -2\). It should also be noted that a “disgusting convention” (the term is borrowed from Ref. \[10\]) is often being used according to which one and the same Harrison-Zeldovich spectrum is described by the spectral index \(n_t = 0\) for gravitational waves and by the spectral index \(n_s = 1\) for density perturbations. Of course, there is no need in this convention. In both cases, the metric perturbations with the Harrison-Zeldovich spectrum are described by the same spectral index (zero), see Eq. (16).

We will finish this section with a short discussion of coherent states. There is no natural mechanism for the generation of cosmological perturbations in coherent states, but if there
were one it would be reflected in many parts of the theory. The interaction part of the Hamiltonian (6) would be linear (not quadratic) in the creation and annihilation operators. The analogue of Eq. (7) would read

\[ c_n(\eta) = e^{-in\eta}c_n(0) + \alpha_n(\eta) \]
\[ c_n^\dagger(\eta) = e^{in\eta}c_n^\dagger(0) + \alpha_n^*(\eta) \]

(18)

where the complex function \[ \alpha_n(\eta) \] is determined by the coupling function in the Hamiltonian. On the position — momentum diagram, the evolution (18) of the field operators corresponds to \textit{displacing} the vacuum state without \textit{squeezing} whereas the evolution (7) corresponds to \textit{squeezing} the vacuum state without \textit{displacing}. In terminology of mechanics, coherent states are produced by a force acting on the oscillator whereas squeezed vacuum states are produced by a parametric influence. In coherent states, the mean value of the field is not zero. The correlation functions (at least, for some quantities) would also be different from the squeezed state case. In cosmological context, this would eventually be reflected in the differing statistical properties of perturbations and induced microwave background anisotropies.

The calculations of the next Section are based in an essential way on the field operator (11) for the squeezed vacuum perturbations.

### III. QUANTUM-MECHANICAL EXPECTATION VALUES FOR THE MICROWAVE BACKGROUND ANISOTROPIES

The microwave background anisotropies are a subject of intense study [22].

In absence of cosmological perturbations, the temperature of the microwave background radiation seen in all directions on the sky would be the same, \[ T \]. Let us denote a direction on the sky by a unit vector \( e \). The presence of cosmological perturbations makes the temperature seen in the direction \( e \) differing from \( T \). The temperature perturbation produced by density perturbations or gravitational waves can be described by the formula [11]:

\[ \frac{\delta T}{T}(e) = \frac{1}{2} \int_0^{w_1} \frac{\partial h_{ij}}{\partial \eta} e^i e^j \, dw \]

(19)

where \( \partial h_{ij}/\partial \eta \) is taken along the integration path \( x^i = e^i w, \eta = \eta_R - w \), from the event of reception \( w = 0 \) to the event of emission \( w = w_1 = \eta_R - \eta_E \). The formula for rotational perturbations is more complicated than (19) [11], and we will leave rotational perturbations aside.

For quantized cosmological perturbations, the \( \frac{\delta T}{T}(e) \) becomes a quantum-mechanical operator. Using Eq. (11) we can write this operator as

\[ \frac{\delta T}{T}(e) = \frac{C}{2(2\pi)^{3/2}} \int_0^{w_1} dw \int_{-\infty}^{\infty} d^3n \sum_{s=1}^{2} \hat{p}_{ij}(n) e^i e^j 
\]
\[ \times \left[ s_c n(0) f_n(w) e^{iw e} + s_c^* n(0) f_n^*(w) e^{-iw e} \right] \]

(20)

where
\begin{equation}
{s_f_n}(w) \equiv \frac{1}{\sqrt{2n}} \frac{dh_n}{d\eta}\bigg|_{\eta = \eta_R - w} .
\end{equation}

Having defined the observable $\delta T(T(e))$ and knowing the quantum state $|0\rangle$ we can compute various quantum-mechanical expectation values. In the laboratory quantum mechanics, the verification of theoretical predictions expressed in terms of the expectation values would require experiments on many identical systems. An immediate generalization of this principle to cosmology would require speculations about outcomes of experiments performed in “many identical universes”. Without having access to “many universes” we can only rely on the mean (expected) values of the observables and on the probability distribution functions as indicators of what is likely or not to be observed in our own single Universe. We will return to this point in Sec. IV.

The expected value of the temperature perturbation to be observed in every fixed direction on the sky is zero:

\begin{equation}
\langle 0 | \frac{\delta T}{T}(e) | 0 \rangle = 0 .
\end{equation}

One particular measured temperature map is the result of the measurement performed over one particular realization of the random process describing cosmological perturbations of quantum-mechanical origin. For this realization, the temperature perturbations may, should, and in fact are, present. Many measurements will not help (except of reducing the instrumental noises) in the sense that they all should give identical results, because the timescale of the perturbations under discussion is so enormously larger than an interval of time between the experiments. If the COBE’s map is correct, we will have to live with this map practically forever.

Let us now compute the expected angular correlation function for the temperature perturbations seen in two given directions on the sky, $e_1$ and $e_2$. This correlation function is defined as the mean value for the product of $\delta T(T(e_1))$ and $\delta T(T(e_2))$:

\begin{equation}
K(e_1, e_2) = \langle 0 | \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) | 0 \rangle .
\end{equation}

By manipulating with the product of two expressions (20) one can derive the formula

\begin{equation}
K(e_1, e_2) = \frac{1}{4} C^2 \frac{1}{(2\pi)^3} \int_0^{\omega_1} dw \int_0^{\omega_1} d\bar{w} \int_{-\infty}^{\infty} d^3 n e^{in(e_1 w - e_2 \bar{w})} \times \sum_{s=1}^{2} \left( p_{ij}(n) e^{i \bar{n} e_1} \right) \left( p_{ij}(n) e^{i \bar{n} e_2} \right) f_n(w) f_n(\bar{w}) .
\end{equation}

The next step is the formidable task of taking the integrals over angular variables in 3-dimensional wave-vector $n$ space. However, it can be done (see Ref. [12] for gravitational waves, Ref. [13] for rotational perturbations, and Ref. [3] for density perturbations). The final expression reduces, without making any additional assumptions whatsoever, to the form

\begin{equation}
K(e_1, e_2) = K(\delta) = \frac{\omega^2}{\omega_{\Pi}} \sum_{l=l_{\text{min}}}^{\infty} K_l P_l(\cos \delta) .
\end{equation}
We see that the correlation function depends only on the angle $\delta$ between the directions $e_1, e_2$ not directions themselves. The coefficient $l^2_P$ is taken from $C^2$, other numerical coefficients are included in $K_l$. The quantities $K_l$ involve the integration of $f_n(w)$ over the parameter $w$ and the remaining integration over the wave-numbers $n$. The numerical values of $K_l$ depend on a chosen sort of cosmological perturbations and a chosen cosmological model; so far, the formula (23) is totally general. $P_l(cos \delta)$ are the Legendre polynomials. The lowest multipole $l_{\text{min}}$ follows automatically from the theory and it turns out to be, not surprisingly, $l_{\text{min}} = 0$ for density perturbations, $l_{\text{min}} = 2$ for gravitational waves (and $l_{\text{min}} = 1$ for rotational perturbations). For the separation angle $\delta = 0$, Eq. (23) reduces to the variance of $\delta T_T(e)$, that is

$$\langle 0 \mid \delta T_T(e) \delta T_T(e) \mid 0 \rangle = K(0) = l^2_P \sum_{l=l_{\text{min}}}^{\infty} K_l .$$ (24)

Formula (23) gives the expected value of the observable $\delta T_T(e_1) \delta T_T(e_2)$. If the experimenter measured this observable in “many universes” and averaged the measured numbers, he/she would get the result (23). Moreover, formula (23) says that if the experimenter made the measurements at any other pair of directions, but with the same separation angle $\delta$, he/she would again get, after the averaging over “many universes”, the result (23). Without having access to “many universes”, we can ask what is the theoretical standard deviation of the quantity $\delta T_T(e_1) \delta T_T(e_2)$. (In practice, for deriving $K(\delta)$ we need a kind of ergodic hypothesis allowing us to replace the averaging over “universes” by the averaging over pixels on a single map.) The variance $V(e_1, e_2)$ of this quantity is, by definition,

$$V(e_1, e_2) = \langle 0 \mid \delta T_T(e_1) \delta T_T(e_2) \delta T_T(e_1) \delta T_T(e_2) \mid 0 \rangle - \left[ \langle 0 \mid \delta T_T(e_1) \delta T_T(e_2) \mid 0 \rangle \right]^2 .$$ (25)

The standard deviation is the square root of this number.

The calculation of $V(e_1, e_2)$ requires us to deal with the product of four expressions (20). However, the mean values of the products of four creation and annihilation operators are easy to handle. One can show that $V(e_1, e_2)$ depends only on the separation angle $\delta$ and

$$V(e_1, e_2) = V(\delta) = \left[ \langle 0 \mid \delta T_T(e_1) \delta T_T(e_2) \mid 0 \rangle \right]^2 + \left[ \langle 0 \mid \delta T_T(e) \delta T_T(e) \mid 0 \rangle \right]^2 ,$$ (26)

that is

$$V(\delta) = K^2(\delta) + K^2(0) .$$ (27)

The standard deviation for the observable $\delta T_T(e_1) \delta T_T(e_2)$ is

$$\sigma(\delta) = [V(\delta)]^{1/2} = \sqrt{K^2(\delta) + K^2(0)} .$$ (28)

In a similar fashion one can derive the higher order correlation functions for two directions $e_1, e_2$ and the correlation functions for larger number of directions, but we will not need this information.

In the limit $\delta = 0$ Eqs. (25), (26) say that
\[
\langle 0 | \left[ \frac{\delta T}{T}(\mathbf{e}) \right]^4 | 0 \rangle = 3 \left[ \langle 0 | \left[ \frac{\delta T}{T}(\mathbf{e}) \right]^2 | 0 \rangle \right]^2.
\]  

(29)

The familiar factor 3 relating the fourth-order moment with the square of the second-order moment (given that the first-order moment is equal to zero) is the reflection of the underlying Gaussian nature of the squeezed vacuum wavefunctions associated with the Hamiltonian (6).

By examining Eq. (28) one can conclude that for each separation angle \( \delta \) the standard deviation of the angular correlation function is very big. Even at those separation angles at which \( K(\delta) \) vanishes, the standard deviation is as big as the variance for \( \delta T_T(e) \) itself. However, the value of the standard deviation for a given variable is not very informative per se, as long as the probability density function for this variable is not known. If the probability density function (p.d.f.) were normal, we could say that the probability to find a result outside of \( 1\sigma \) interval is 32\%. Without knowing the p.d.f. we could resort to the Chebyshev inequality, but it would only tell us that this probability is less than 1. To get more information about possible deviation of the angular correlation function from its mean values we will consider in Sec. IV a classical random model which will reproduce the expectation values calculated above and will allow us to construct the p.d.f. for the variable \( \frac{\delta T}{T}(\mathbf{e}_1) \) \( \frac{\delta T}{T}(\mathbf{e}_2) \). On the other hand, the quantum-mechanical calculations of this Section will shed light on the classical model. As is known, “quantum mechanics helps us understand classical mechanics”, see on this subject a paper of Zeldovich signed by the pseudonym Paradoksov [14].

IV. CLASSICAL MODEL FOR THE STATISTICS OF THE MICROWAVE BACKGROUND ANISOTROPIES

A distribution of the microwave background temperature over the sky is a real function of the angular coordinates. Assuming that \( \frac{\delta T}{T} \) is a sufficiently smooth function on a sphere, one can expand it over the set of orthonormal complex spherical harmonics \( Y_{lm}(\theta, \phi) \) [15]:

\[
\frac{\delta T}{T}(\mathbf{e}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ a_{lm} Y_{lm}(\mathbf{e}) + a_{lm}^* Y_{lm}^*(\mathbf{e}) \right].
\]

(30)

We want to formulate a statistical hypothesis about the coefficients \( a_{lm} \), so it is better to write them first in terms of real \( (r) \) and imaginary \( (i) \) components:

\[
a_{lm} = a_{lm}^r + ia_{lm}^i, \quad a_{lm}^* = a_{lm}^{r*} - ia_{lm}^{i*},
\]

\[
Y_{lm} = Y_{lm}^r + iY_{lm}^i, \quad Y_{lm}^{*} = Y_{lm}^{r*} - iY_{lm}^{i*},
\]

\[
\frac{\delta T}{T}(\mathbf{e}) = 2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ a_{lm}^r Y_{lm}^r(\mathbf{e}) - a_{lm}^{i} Y_{lm}^{i}(\mathbf{e}) \right].
\]

(31)

Our statistical hypothesis is as follows: (i) all members of the set of random variables \( \{a_{lm}^r, a_{lm}^i\} \) are statistically independent, (ii) each individual variable is normally distributed and has a zero mean, (iii) all variables with the same index \( l \) have the same standard deviation \( \sigma_l \). All said is expressed by the probability density function (p.d.f.) for individual variables:
\[ f(a_{lm}^r) = \frac{1}{\sqrt{2\pi}\sigma_{l}} e^{-\frac{(a_{lm}^r)^2}{2\sigma_{l}^2}}, \quad f(a_{lm}^i) = \frac{1}{\sqrt{2\pi}\sigma_{l}} e^{-\frac{(a_{lm}^i)^2}{2\sigma_{l}^2}}, \quad (32) \]

defines the p.d.f. for the entire set of variables, which is simply a product of all p.d.f.’s for all individual variables:

\[ f \{ (a_{lm}^r, a_{lm}^i) \} = \cap_{l,m} f(a_{lm}^r) f(a_{lm}^i). \quad (33) \]

Having postulated the p.d.f.’s, we can now compute the expectation values of certain functions of the random variables. Below, the angular brackets will denote the expectation values calculated with the help of the p.d.f. (33), unless other definition is stated.

Obviously, all linear functions have a zero mean:

\[ \langle a_{lm}^r \rangle = 0, \quad \langle a_{lm}^i \rangle = 0. \quad (34) \]

For quadratic combinations we have

\[ \begin{align*}
\langle a_{lm1}^r a_{lm2}^r a_{lm3}^i a_{lm4}^i \rangle &= \sigma_{l1}^2 \delta_{l1} \delta_{m1} \delta_{m2} \delta_{m3} \delta_{m4}, \\
\langle a_{lm1}^i a_{lm2}^r a_{lm3}^i a_{lm4}^r \rangle &= \sigma_{l1}^2 \delta_{l1} \delta_{m1} \delta_{m2} \delta_{m3} \delta_{m4}.
\end{align*} \quad (35) \]

All triple products have zero means. Among quartic combinations, only those can survive which have four indices (r), or four indices (i), or two indices (r) and two indices (i). Two representative expressions are:

\[ \begin{align*}
\langle a_{lm1}^r a_{lm2}^r a_{lm3}^i a_{lm4}^i \rangle &= \sigma_{l1}^2 \sigma_{l3}^2 \delta_{l1} \delta_{l3} \delta_{m1} \delta_{m2} \delta_{m3} \delta_{m4}, \\
\langle a_{lm1}^r a_{lm2}^r a_{lm3}^i a_{lm4}^i \rangle &= \sigma_{l1}^2 \sigma_{l3}^2 \delta_{l1} \delta_{l3} \delta_{m1} \delta_{m2} \delta_{m3} \delta_{m4}.
\end{align*} \quad (36) \]

Other quartic combinations can be obtained by the replacement (r) \(\leftrightarrow\) (i) in Eqs. (36), (37) (or by permutation of pairs (lm) in case of Eq. (37)). The higher-order correlations can be derived in a similar way, but we will not need them.

In our further calculations related to the random variables \(\delta T/e\) and \(\delta T/(e_1)\delta T/(e_2)\) it is easier to deal with the complex coefficients \(a_{lm}\), so we will first translate the above relationships to them. By using the available information one can derive

\[ \begin{align*}
\langle a_{lm} \rangle &= 0, \quad \langle a_{lm1}^* a_{lm2}^* \rangle = 2\sigma_{l1}^2 \delta_{l1} \delta_{m1} \delta_{m2}, \\
\langle a_{lm1}^* a_{lm2}^* a_{lm3}^* a_{lm4}^* \rangle &= 4\sigma_{l1}^2 \sigma_{l2}^2 \delta_{l1} \delta_{l2} \delta_{m1} \delta_{m2} \delta_{m3} \delta_{m4}.
\end{align*} \quad (38) \]

The mean values of the complex conjugated quantities are given by the same formulas (38). Other nonvanishing quartic combinations can be obtained from the one in Eq. (38) by the permutation of pairs (lm).

Now, even before deriving the p.d.f.’s for the random variables \(\delta T/e\) and \(\delta T/(e_1)\delta T/(e_2)\), we can find some expectation values. It is clear from the definition (30) and Eq. (38) that

\[ \langle \frac{\delta T}{T} e \rangle = 0. \quad (39) \]

When calculating the angular correlation function one should remember that
\[ \sum_{m=-l}^{l} Y_{lm}(e_1) Y_{lm}^*(e_2) = \frac{2l+1}{4\pi} P_l(\cos \delta) \] (40)

(note the origin of the factor \(2l+1\) which will accompany us often). By taking the product of two expressions (30) and using Eqs. (38), (40) one can find the angular correlation function

\[ \left\langle \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \right\rangle = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma_l^2 (2l+1) P_l(\cos \delta) \] . (41)

If the separation angle \(\delta\) is zero, we obtain

\[ \left\langle \frac{\delta T}{T}(e) \frac{\delta T}{T}(e) \right\rangle = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma_l^2 (2l+1) \] . (42)

[One may notice an incidental fact that the mean value of the random variable \(a_l^2\) defined as \(a_l^2 = \sum_{m=-l}^{l} a_{lm} a_{lm}^*\) is \(<a_l^2> = 2(2l+1)\sigma_l^2\), that is the same expression which enters Eq. (41). This may suggest an interpretation of the quantity \(<a_l^4> - <a_l^2>^2\) as the variance of the multipole moments. One should be careful with this interpretation, see Sec. V.]

We can also find the 4th order expectation values. The product of 4 expressions (30) in conjunction with Eq. (38) gives

\[ \left\langle \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \right\rangle - \left\langle \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \right\rangle^2 \\
= \left\langle \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \right\rangle^2 + \left\langle \frac{\delta T}{T}(e) \frac{\delta T}{T}(e) \right\rangle^2 . \] (43)

If \(\delta = 0\), it follows from Eq. (43) that

\[ \left\langle \left[ \frac{\delta T}{T}(e) \right]^4 \right\rangle = 3 \left\langle \left[ \frac{\delta T}{T}(e) \right]^2 \right\rangle^2 \] . (44)

Up to difference in the meaning of the angular brackets, the formulas (39), (43), (44) reproduce the analogous results of the previous Section. Moreover, from comparison of Eqs. (23), (24) with Eqs. (41), (42) we can relate the quantities \(K_l\), derivable from a given cosmological model plus perturbations, with the abstract quantities \(\sigma_l\).

We can now engage in our major enterprise — the construction of the p.d.f. for the random variable \(v \equiv \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2)\). We will start from the p.d.f. for the random variable \(z \equiv \frac{\delta T}{T}(e)\). When it is necessary to distinguish directions \(e_1\) and \(e_2\), we will use the notations \(z_1\) and \(z_2\).

The variable \(z\) is a function of the variables \(\{a_{lm}^r, a_{lm}^i\}\) whose p.d.f.’s are known, Eqs. (31), (32). There exist regular methods (see, for example, an excellent book [16]) allowing to derive rigorously the p.d.f. of a function. However, in our case that the function is linear and all p.d.f.’s are normal, we can partially rely on a guesswork. Combining formulas and guessing we can write

\[ f(z) = \frac{1}{\sqrt{2\pi} \sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} , \] (45)
where

\[ \sigma_z^2 = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma_l^2 (2l + 1) . \]  

(46)

The p.d.f. (45) certainly leads to Eqs. (39), (42), (44). Moreover, it allows us to say that the probability to find \( z \) outside of \( 1 \sigma_z \) interval is approximately 32%:

\[ P(|z| > \sigma_z) \approx 0.32 . \]

We now introduce two variables, \( z_1 \) and \( z_2 \), and ask about the p.d.f. in the 2-dimensional space \((z_1, z_2)\). Again, partially relying on a guesswork, we find that

\[ f(z_1, z_2) = \frac{1}{2\pi \sigma_z^2 \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2\sigma_z^2(1-\rho^2)} \left( z_1^2 + z_2^2 - 2\rho z_1 z_2 \right) \right\} \]  

(47)

where

\[ \rho \sigma_z^2 = \frac{1}{\pi} \sum_{l=0}^{\infty} (2l + 1) \sigma_l^2 P_l(\cos \delta) , \quad |\rho| \leq 1 . \]  

(48)

(See Eq. (5.11.1) in Ref. [16]). First, we can check that the marginal distributions are correct. For \( f(z_1) \), one obtains

\[ f(z_1) = \int_{-\infty}^{\infty} f(z_1, z_2) dz_2 = \frac{1}{\sqrt{2\pi \sigma_z}} e^{-\frac{z_1^2}{2\sigma_z^2}} \]

and one obtains a similar expression for \( f(z_2) \). Second, one can check that

\[ \langle z_1^2 \rangle = \sigma_z^2 , \quad \langle z_2^2 \rangle = \sigma_z^2 , \quad \langle z_1 z_2 \rangle = \rho \sigma_z^2 \]

where the angular brackets mean the integration with the p.d.f. (47). These equalities are Eqs. (42), (41) which we must have obtained.

Finally, we shall derive the p.d.f. for the variable \( v = z_1 z_2 \). We will do this in some detail following the prescriptions of [16].

Let us introduce the two new variables \((z_1, v)\) instead of \((z_1, z_2)\) according to the transformation

\[ z_1 = z_1 , \quad z_2 = \frac{v}{z_1} . \]

The Jacobian of this transformation is \( J = 1/z_1 \). The p.d.f. \( f(v) \) is the result of the following integration:

\[ f(v) = \frac{1}{2\pi \sigma_z^2 \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \frac{1}{|z_1|} \exp \left\{ -\frac{1}{2\sigma_z^2(1-\rho^2)} \left[ z_1^2 + \frac{v^2}{z_1^2} - 2\rho v \right] \right\} dz_1 . \]

The integral over \( z_1 \) can be taken with the help of 3.471.9 from [17]. The resulting p.d.f. can be written in the form:
\[ f(v) = \begin{cases} \frac{1}{\pi \sigma^2} e^{\frac{v^2}{\sigma^2(1-\rho^2)}} K_0 \left( \frac{v}{\sigma^2(1-\rho^2)} \right), & \text{for } v > 0 \\ \frac{1}{\pi \sigma^2} e^{\frac{-v^2}{\sigma^2(1-\rho^2)}} K_0 \left( \frac{-v}{\sigma^2(1-\rho^2)} \right), & \text{for } v < 0 \end{cases} \]  

(49)

where \( K_0 \) is the modified Bessel function of its argument [18].

The function \( f(v) \) is quite complicated and the distribution is obviously not normal. The function \( f(v) \) goes to zero for \( v \to \pm \infty \) and diverges logarithmically at the point \( v = 0 \). Even a verification of the normalization condition

\[ \int_{-\infty}^{\infty} f(v) dv = 1 \]  

(50)

is not trivial. However, with the help of 6.621.3, 9.131.1, 9.121.7, 1.624.9 and 1.623.2 from [17] one can prove the validity of Eq. (50).

The mean value and the standard deviation of the variable \( v \) are known, see Eqs. (41), (43):

\[ \langle v \rangle = \rho \sigma^2 z, \quad \sigma_v = \left[ \langle v^2 \rangle - \langle v \rangle^2 \right]^{1/2} = \sigma^2 z \sqrt{\rho^2 + 1} \]  

(51)

We already knew that the standard deviation is big. We now see again that \( \sigma_v = \sigma^2 z \) at the separation angles at which the angular correlation function vanishes, \( \rho = 0 \), and \( \sigma_v = \sqrt{2} \sigma^2 z \) at zero separation angle, \( \rho = 1 \). For other separation angles, \( \sigma_v \) lies between these two numbers.

Now that we know the p.d.f., we can assign probabilities to the different ranges of the variable \( v \). For instance, we can calculate the probability that the measured \( v \) will be found, say, outside of the \( \lambda \sigma_v \) interval surrounding the mean value of \( v \), where \( \lambda \) is an arbitrary fixed number. The probability of our interest is

\[ P(|v - \langle v \rangle| > \lambda \sigma_v) = \int_{-\infty}^{\sigma^2 (\rho - \lambda \sqrt{\rho^2 + 1})} f(v) dv + \int_{\sigma^2 (\rho + \lambda \sqrt{\rho^2 + 1})}^{\infty} f(v) dv \]  

(52)

To get a qualitative estimate of the associated theoretical uncertainties for the observable \( v \), we will ask a slightly different question. What should the number \( \lambda \) be in order to have the 0.32 chance of finding \( v \) outside the \( \lambda \sigma_v \) interval and, hence, the 0.68 chance to find it inside the interval?

To evaluate the size of the disaster, we will start from the case \( \rho = 0 \). In this case, the p.d.f. (49) is symmetric with respect to the origin \( v = 0 \) (this is why \( \langle v \rangle \) is zero in this case) and

\[ P(|v| > \lambda \sigma^2 z) = \frac{2}{\pi} \int_{\lambda}^{\infty} K_0(x) dx \]  

(53)

We want this number to be approximately equal to 0.32. Judging from the Fig. 9.7 in Ref. [18], a half of the area under the \( K_0(x) \) function is accumulated when integrating from approximately \( x = 1/2 \) and up to infinity. This means that \( \lambda \) should approximately be equal to \( 1/2 \).

If \( \rho \neq 0 \) the evaluation of \( P \) is more complicated. For \( \rho \neq 0 \), the function (49) is not symmetric with respect to the origin \( v = 0 \). It has larger values at positive \( v \)'s if \( \rho > 0 \) (this
is why \( \langle v \rangle > 0 \) in this case) and it has larger values at negative \( v \)'s if \( \rho < 0 \) (this is why \( \langle v \rangle < 0 \) in this case). The graph of the function \( e^x K_0(x) \) plotted on Fig. 9.8 in Ref. [18] is helpful. A qualitative analysis shows again that \( \lambda \) is approximately equal to 1/2. (More accurate estimates can of course be reached by numerical methods.)

At any rate, the \( \frac{1}{2} \sigma_v \) interval gives approximately the same probability estimates as if the distribution (49) were normal.

V. ON THE “COSMIC VARIANCE”

The set of random variables \( \{a^r_{lm}, a^i_{lm}\} \) defined by Eqs. (32), (33) lives its own independent life regardless of whether or not the variables are considered random coefficients in the expansion of some function over spherical harmonics. Being such, it allows introduction of new functions and calculation of their expectation values. One interesting variable is defined by the equation

\[
a^2_l = \sum_{m=-l}^{l} a_{lm} a^*_{lm} = \sum_{m=-l}^{l} |a_{lm}|^2 = \sum_{m=-l}^{l} \left[ (a^r_{lm})^2 + (a^i_{lm})^2 \right].
\]

By using Eq. (38) one can calculate the expectation value of \( a^2_l \):

\[
\langle a^2_l \rangle = (2l + 1)2\sigma^2_l.
\]

The factor 2\((2l+1)\) reflects the number of independent “degrees of freedom” associated with the index \( l \). One can also introduce the variable \( a^4_l \) and calculate its expectation value:

\[
\langle a^4_l \rangle = (2l + 1)(l + 1)8\sigma^4_l = \langle a^2_l \rangle^2 \frac{2(l + 1)}{2l + 1}.
\]

The difference \( \langle a^4_l \rangle - \langle a^2_l \rangle^2 \) is, by definition, the variance of the variable \( a^2_l \). From Eqs. (56), (55) one finds

\[
\langle a^4_l \rangle - \langle a^2_l \rangle^2 = \frac{1}{2l + 1} \langle a^2_l \rangle^2.
\]

This formula, as it stands, expresses a well-known fact: the variance of the random variable \( \chi^2 \) defined as the sum of squares of \( n \) independent random variables (degrees of freedom) with the same normal density, is \( \frac{n}{2} \) times smaller than the square of the mean value of \( \chi^2 \) [16]. There is nothing “cosmological” or “inflationary” in this fact. Knowing the p.d.f.’s for the set \( \{a^r_{lm}, a^i_{lm}\} \) one can calculate the higher-order correlation functions for the variable \( a^2_l \) and its distribution function [23-29]. In the recent literature, formula (57) became known as the “cosmic variance”. Formula (57) and a possibility (or lack of) to extract complete information about a stochastic process from its single realization are, in general, different issues. For ergodic processes, the existence of a definitely true relationship (57) prevents in no way the extraction of complete information about the process from a single realization [30].

It is important to realize that it is the mean value of the random variable \( a^2_l \), not \( a^2_l \) itself, that enters the expected angular correlation function in front of the Legendre polynomials
and which is often called the multipole moment. The $a_l^2$ is a random variable and its variance has meaning, the $\langle a_l^2 \rangle$ is a number and its variance has no meaning. Specifically, one can notice that the angular correlation function (41) can be written in the form

$$\left\langle \frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) \right\rangle = \frac{1}{2\pi} \sum_{l=0}^{\infty} \langle a_l^2 \rangle P_l(\cos \delta) .$$

(58)

On this ground, there may be a temptation to write the random variable $\frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2)$ in the form

$$\frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) = \frac{1}{2\pi} \sum_{l=0}^{\infty} a_l^2 P_l(\cos \delta)$$

(59)

and to interpret Eq. (57) as the variance for the multipole moments of the correlation function. One should resist to this temptation.

Let us show that the definition (59) is incorrect despite the fact that it gives correct expectation value (58). It follows from the definition (59) that

$$\frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) = \frac{1}{2\pi} \sum_{l=0}^{\infty} a_l^2 P_l(\cos \delta)$$

(60)

Using (56), (58) and remembering that $a_l^2$ and $a_l^2$ are statistically independent for $l \neq l'$, one can find the expectation value of the quantity (60):

$$\left\langle \frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) \frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) \right\rangle$$

$$= \frac{1}{4\pi^2} \sum_{l=0}^{\infty} \langle a_l^4 \rangle [P_l(\cos \delta)]^2 + \frac{1}{4\pi^2} \sum_{l,l'=0,l\neq l'} \langle a_l^2 \rangle \langle a_{l'}^2 \rangle P_l(\cos \delta) P_{l'}(\cos \delta)$$

$$= \frac{1}{4\pi^2} \sum_{l=0}^{\infty} \langle a_l^4 \rangle [P_l(\cos \delta)]^2 + \frac{1}{4\pi^2} \sum_{l=0}^{\infty} \left[ \sum_{l=0}^{\infty} \langle a_l^2 \rangle P_l(\cos \delta) \right]^2 - \frac{1}{4\pi^2} \sum_{l=0}^{\infty} \langle a_l^2 \rangle^2 [P_l(\cos \delta)]^2$$

$$= \left\langle \frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2) \right\rangle^2 + \frac{1}{4\pi^2} \sum_{l=0}^{\infty} \frac{1}{2l+1} \langle a_l^2 \rangle^2 [P_l(\cos \delta)]^2 .$$

(61)

It follows from (61) that the variance of the variable $\frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2)$ would read (if (59) were correct):

$$\frac{1}{\pi^2} \sum_{l=0}^{\infty} (2l+1) \sigma_l^2 [P_l(\cos \delta)]^2$$

(62)

This expression should be compared with the correct variance following from Eq. (43):

$$\frac{1}{\pi^2} \left[ \sum_{l=0}^{\infty} (2l+1) \sigma_l^2 P_l(\cos \delta) \right]^2 + \frac{1}{\pi^2} \left[ \sum_{l=0}^{\infty} (2l+1) \sigma_l^2 \right]^2 .$$

(63)

Formulas (62), (63) disagree even for $\delta = 0$, and even in their first, $l = 0$ term. This shows that the ad hoc definition (59) is incorrect. The correct definition of the random variable $\frac{\delta T}{T} (e_1) \frac{\delta T}{T} (e_2)$ is the one following from the definition (30) and which we have used in this paper.
VI. CONCLUSIONS

A particular cosmological model plus perturbations gives unambiguous predictions with regard to the expectation values of the measurable quantities. Differing models give different predictions. We want to distinguish them observationally and to learn about physics of the very early Universe. However, the quantum-mechanical origin of the cosmological perturbations is reflected in the theoretical statistical uncertainties surrounding the expectation values. One important measurable quantity is the angular correlation function of the microwave background anisotropies. Its mean value at the zero separation angle was denoted $\sigma_z^2$ in this paper. It was shown that the standard deviation for the correlation function is very big. The 68% confidence level corresponds, approximately, to $\frac{1}{2}\sigma_z^2$ at the separation angles where the correlation function vanishes, to $\frac{\sqrt{2}}{2}\sigma_z^2$ at the zero separation angle, and to intermediate numbers for other separation angles.

The angular correlation function has actually been measured. It is presented at Fig. 3 in the paper [19]. The authors surround the measured points by a narrow shaded region which they address as follows: “The shaded region is the 68% confidence region .... including cosmic variance and instrument noise”. It is not quite clear what the authors of Ref. [19] (see also Ref. [20]) mean by “cosmic variance”, but if they mean the theoretical statistical uncertainties for the correlation function variable $v$, these uncertainties are significantly larger than what is plotted. According to the calculations presented above, the half-width of the shaded region should be approximately 600 ($\mu K)^2$ near the points where the correlation function vanishes and approximately 840 ($\mu K)^2$ near the point marking the zero separation angle.

The conclusion is a bit disappointing. Apparently, God is telling us something important about the very early Universe by exhibiting the microwave background anisotropies, but the channel of information is so noisy that it will be hard to understand the message.
APPENDIX. ON THE “STANDARD” INFLATIONARY FORMULA FOR DENSITY PERTURBATIONS

In the body of this paper we have studied the statistical properties of cosmological perturbations of quantum-mechanical origin and related anisotropies in the microwave background. These properties are essentially universal, they are equally well true for perturbations of any nature - density perturbations, rotational perturbations, or gravitational waves. However, in practice, it is very important to know which sort of perturbations we are actually dealing with, assuming that the observed large-angular-scale anisotropy is indeed caused by perturbations of this origin. In other words, what does theory say about the comparative values of the amplitudes, if we agree on the generating mechanism, equations, and a class of cosmological models of the very early Universe, say, governed by the scale factors (15)? In addition to gravitational waves, density perturbations can also be generated by the same mechanism, if one makes favorable assumptions about the dominant matter in the very early Universe (scalar field) and its coupling to gravity (minimal, the same as for gravitational waves). According to calculations of Ref. [3], density perturbations and gravitational waves will have amplitudes of the same order of magnitude, whereas according to the “standard” inflationary formula the amplitude of density perturbations will be many orders of magnitude larger than the amplitude of gravitational waves, if the expansion rate of the very early Universe was sufficiently close to the archetype inflationary model - the de Sitter expansion.

It is necessary to say that the quantum-mechanical generating mechanism has become very popular in the context of the inflationary hypothesis. Inflationary literature often speaks about cosmological perturbations being generated “from quantum fluctuations”. However, this literature associates the explanation of the phenomenon with such things as ambiguity in the choice of time in the de Sitter universe, horizon temperature, tremendous inflation of scales, and so on. The basic concepts are adjusted accordingly. Instead of amplification, with the emphasis on a nonvanishing parametric coupling, increase of amplitude at the expense of energy of the pump field, quantum-mechanical generation of waves (particles) in strictly correlated pairs, etc., inflationary literature speaks about magnification, with the emphasis on “stretching the waves” and “crossing the horizons”. Apparently for these reasons, inflationists did not get puzzled with their “standard” formula for density perturbations which states that one can produce arbitrarily large amount of density perturbations by practically doing nothing.

The “standard” formula relates the amplitude of density perturbations today with the values of the scalar field during inflation. Let us consider, for definiteness, perturbations of the matter density \( \delta \rho / \rho \) with today’s wavelengths of the order of today’s Hubble radius \( l_H \). The “standard” formula says that

\[
\left. \frac{\delta \rho}{\rho} \right|_H \sim \frac{H^2}{\dot{\phi}(t_i)}
\]

where the right hand side of this formula is supposed to be evaluated at the time \( t_i \) when the wavelengths of our interest were “crossing the horizon” during inflation. Let us agree with the so-called “slow-roll” approximation and assume that the Hubble parameter \( H \) was almost constant during that epoch, \( |\dot{H}| \ll H^2 \). Let us take the numerical value of \( H \) during that epoch at the level, say, 20 orders of magnitude smaller than the Planck value of \( H \).
For quantum-mechanically generated gravitational waves, this would result in today’s amplitude $h \approx 10^{-20}$ and the induced anisotropies of the microwave background $\delta T/T \approx 10^{-20}$ which are much much lower than the level currently discussed in the experiment. However, for density perturbations, according to the “standard” inflationary formula, the situation is totally different. Without changing anything in the curvature of the space-time responsible for the generating process (that is, leaving $\dot{H}$ almost constant and at the same numerical level 20 orders of magnitude smaller than the Planck value), but simply sending $\dot{\phi}(t_i)$ to zero (which corresponds, due to the Einstein equations, to sending $\dot{H}$ to zero, i.e., making the “slow-roll” approximation better and better, making the expansion law closer and closer to the de Sitter expansion and making the shape of the generated spectrum closer and closer to the scale-invariant form) one produces arbitrarily large $(\delta \rho/\rho)|_H$. Inflationists love to stress that the de Sitter gravitational pump field generates perturbations with the Harrison-Zeldovich (scale-invariant, flat) spectrum. What they do not stress is that, according to the “standard” formula for density perturbations, the amplitudes of the scale-invariant spectrum are infinite, and the amplitudes of the almost scale-invariant spectrum are almost infinite. Instead of blaming their own formula, inflationists blame the scalar field potentials. This formula is the reason for rejecting certain scalar field potentials on the grounds that they generate “too much” of density perturbations, for claims that the contribution of gravitational waves to $\delta T/T$ is “negligibly small” in the limit of the de Sitter expansion, and even for claims about copious production of black holes during inflation.

Recently, the “standard” formula has been claimed to be confirmed [31] and reconfirmed [32]. In Ref. [31], this formula has been formulated, essentially, as the following “standard result”: “... we see that the scalar perturbations can be very strongly amplified” [the increase of numerical value from (almost) zero to (almost) infinity] “in the course of the transition” [the instantaneous change of the cosmological scale factor from one power-law behavior to another power-law behavior]. The authors of the paper [31] assure the trusting reader: “We think that there is nothing strange about this ...”.

Here, we will try to understand the origin and mathematical justification for the “standard” inflationary formula.

The early papers which are usually quoted in this connection are the papers [33-35]. We will start from the paper of Hawking [33] which seems to be clearer than others in expressing the basic idea and intentions. The papers [33-35] are similar in many respects.

Hawking considers a scalar field $\phi$ running slowly down an effective scalar field potential. He discusses the inhomogeneous fluctuations $\phi_1(t, x)$ in the field $\phi = \phi_0(t) + \phi_1(t, x)$ which mean that on a surface of constant time there will be some regions where the $\phi$ field has run further down the hill than in other regions. He introduces a new time coordinate $\tilde{t} = t + \delta t(t, x)$ in such a way that the variations of the field are removed and the surfaces of constant time are surfaces of constant $\phi$. Since the scalar field transforms as $\phi_0 + \phi_1 \rightarrow \phi_0 + \phi_1 - \dot{\phi}_0 \delta t$, the required condition is achieved by the time coordinate shift $\delta t = \phi_1/\dot{\phi}_0$. Then Hawking says that the change of time coordinate will introduce inhomogeneous fluctuations in the rate of expansion $\dot{H}$. He and other authors take (apparently, on the grounds of dimensionality only) $\delta \dot{H} \sim H^2 \delta t$. From here they come, implicitly or explicitly, to the dimensionless amplitude of density perturbations

$$\frac{\delta \rho}{\rho} \sim \frac{\delta H}{H} \sim H \delta t \sim \frac{H \phi_1}{\phi_0}.$$  

(65)
Some authors write explicitly $\phi_1 \sim H$ and $\delta \rho/\rho \sim H^2/\dot{\phi}_0$.

The analysis has been done at the inflationary stage. To obtain the today’s amplitude of density perturbations in wavelengths, say, of the order of the today’s Hubble radius, it is recommended to calculate the right hand side of Eq. (65) at the moments of time when the scales of our interest were “crossing horizon” during inflationary epoch. In one or another version this formula appears in the most of inflationary literature and because of numerous repetitions it has grown to the “standard” one. According to this formula, the amplitude of density perturbations becomes larger if one takes the $\dot{\phi}_0$ smaller.

The authors of [33-35] work with a specific scalar field potential, so the numerical value of $\dot{\phi}_0$ and the numerical value of $\delta \rho/\rho$ following from Eq. (65) turn out to be dependent on the self-coupling constant in the potential. These authors are concerned about the unacceptably large amplitude of density perturbations that they have produced. But this is not a concern about the fact that the Einstein equations play no role in this argumentation, it is a specific detail in the scalar field potential that the authors of [33-35] do not like.

Now let us show the shortcomings of the argumentation in [33-35]. Let us consider a scalar field $\phi$ with arbitrary potential. Write the field as $\phi = \phi_0(t) + \phi_1(t)Q$ where $Q$ is the $n$-th spatial harmonic, $Q^i_i + n^2Q = 0$. Write the perturbed metric in the form

$$ds^2 = -dt^2 + a^2(t)(1 + h(t)Q)\delta_{ij} + h_1(t)n^{-2}Q_{,ij}dx^idx^j.$$  

The de Sitter solution corresponds to $\dot{\phi}_0 = 0$, $a(t) \sim e^{Ht}$, and $H(t) = \dot{a}/a = \text{const}$. It follows from the Einstein equations that the (linear) contribution $\epsilon_\phi$ of the scalar field perturbations to the total energy density $\epsilon = \epsilon_0 + \epsilon_\phi$ can be written as

$$\epsilon_\phi = \dot{\phi}_0 \left\{ \dot{\phi}_1 - \phi_1 [\ln(a^3\dot{\phi}_0)] \right\} Q.$$  

The contribution $\epsilon_\phi$, as well as other components of the perturbed energy-momentum tensor, vanish in the de Sitter limit $\dot{\phi}_0 \to 0$.

Thus, the first conclusion we have to make is that in the de Sitter limit there is no linear density perturbations at all. The scalar field perturbations are uncoupled from gravity, they are not accompanied by linear perturbations of the energy-momentum tensor and they are not accompanied by linear perturbations of the gravitational field. The general solution to the perturbed Einstein equations is a set of purely coordinate solutions which can be produced or totally removed by appropriate coordinate transformations. The scalar field perturbations reduce to a test field whose only role is to identify events in the spacetime. One can still ask about a coordinate system such that the surfaces of constant time $\tau$, $\tau = \phi_1(t, x)$ are surfaces of constant $\phi$. But the perturbation of the expansion rate of this new coordinate system will have nothing to do with perturbations in the energy density. Despite the presence of the test scalar field, every space-like hypersurface is a surface of constant energy density.

Now let us assume that $\dot{\phi}_0$ is not zero. Transformation of time $\bar{t} = t + \chi(t)Q$ generates a Lie transformation of the scalar field:

$$\phi_0(t) + \phi_1(t)Q \to \phi_0(t) + [\phi_1(t) - \dot{\phi}_0(t)\chi(t)]Q.$$  

If one wants the transformed field to be homogeneous one takes $\chi(t) = \phi_1(t)/\dot{\phi}_0(t)$. The same transformation of time generates Lie transformations of the metric. The transformed
The transformed energy density is
\[ \bar{\epsilon} = \epsilon_0 + \epsilon_\phi - \dot{\epsilon}_0 \chi Q = \epsilon_0 + \dot{\phi}_0^2 \left( \frac{\phi_1}{\phi_0} \right) Q . \]

Thus, if one makes the \( \dot{\phi}_0 \) smaller, the energy density perturbation decreases according to the Einstein equations, and it increases according to the conjectures of Refs. [33-35]. The use of the correct expression \( \delta H \sim \dot{H} \delta t \) in Eq. (65) makes the \( \delta \rho / \rho \) decreasing when the \( \dot{\phi}_0 \) is decreasing.

The situation becomes even more disturbing if one recalls that the formula (64) has been seemingly confirmed and derived rigorously as a result of more detailed studies. People did really write the perturbed Einstein equations. Moreover, it was done in the framework of the so-called gauge-invariant formalism, the whole purpose of which is to eliminate coordinate solutions and to work exclusively with something “physical”. The basic mathematical tool in these studies is the gauge-invariant potentials \( \Phi \) and \( \Psi \) constructed from the components of the perturbed metric.

As we have seen above, density perturbations in the scalar field matter vanish when \( \dot{\phi}_0 \) goes to zero, and there is no density perturbations at all at the de Sitter stage. This is true irrespective of the wavelength of the perturbation. It can be shorter or much longer than the Hubble radius, that is, it can be “inside” or “outside” the Hubble radius. In particular, this is true of the perturbation whose wavelength is such that it will grow by today to the scale of today’s Hubble radius. The gauge-invariant potentials \( \Phi \) and \( \Psi \) are strictly zero at the de Sitter stage. Why does then the today’s amplitude of the perturbation go to infinity, according to the “standard” formula, in the limit \( \dot{\phi}_0 \to 0 \) at the “first horizon crossing”? Because, the inflationary literature explains, the amplitude will be almost infinitely enhanced in “the course of transition” of the background equation of state from the quasi-de Sitter one \( p \approx -\epsilon \) to the radiation-dominated \( p = \frac{1}{3} \epsilon \) or matter-dominated \( p = 0 \) one. The favorite concept in this argumentation is the “constancy of \( \zeta \)”. The often quoted papers are Ref. [37], which uses notations and equations of [38], and Ref. [39] which summarizes the
previous work and gives a clearer exposition. We will follow equations and notations of the paper [39]. The advantage of this paper is that it contains enough mathematical details to make it possible to follow the spirit and the letter of calculations.

Mukhanov et al. [39] work with the gauge-invariant potentials \( \Phi \) and \( \Psi \). According to one of the perturbed Einstein equations, \( \Phi = \Psi \). In terms of the \( \Phi \), the basic equation of [39] at the scalar field stage is

\[
\Phi'' + 2 \frac{(a/\phi'_0)'}{(a/\phi'_0)} \Phi' - \nabla^2 \Phi + 2 \phi'_0 \left( \frac{H}{\phi'_0} \right)' \Phi = 0 \tag{66}
\]

where \( ' = d/d\eta, \) \( \phi' = a/\eta, \) \( H = a'/a \). Equation (66) is exactly the same equation as the basic equation (2.23) of Ref. [37]. At the perfect fluid stage, the basic equation of [39] for “adiabatic” density perturbations is

\[
\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2 \nabla^2 \Phi + [2H' + (1 + 3 c_s^2) H^2] \Phi = 0 \tag{67}
\]

where \( c_s^2 = \delta p/\delta \epsilon = \dot{p}_0/\dot{\epsilon}_0 \). Equations (66) and (67) were derived from the original perturbed Einstein equations with the help of manipulations aimed at expressing the equations in terms of the gauge-invariant potentials.

The authors of [39] introduce a new quantity \( \zeta \) defined as

\[
\zeta \equiv \frac{2}{3} \frac{H^{-1} \Phi + \Phi}{1 + w} + \Phi \tag{68}
\]

where \( w = p_0/\epsilon_0 \). This quantity is simply a new letter. As soon as the function \( \Phi \) is known, the function \( \zeta \) can be calculated from the definition (68). Using the definition of \( \zeta \), Eq. (66) can be written in the form

\[
\frac{3}{2} \dot{\zeta} H (1 + w) = \frac{1}{a^2} \nabla^2 \Phi \tag{69}
\]

and Eq. (67) in the form

\[
\frac{3}{2} \dot{\zeta} H (1 + w) = \frac{1}{a^2} c_s^2 \nabla^2 \Phi \tag{70}
\]

Mukhanov et al. [39] consider perturbations with wavelengths “far outside the Hubble radius for which \( \nabla^2 \Phi \) can be neglected”. Neglecting the right hand sides of Eqs. (69), (70) they arrive, in this approximation, at the equation \( \dot{\zeta} \approx 0 \) and the “conservation law”

\[
\zeta \approx \text{const} \tag{71}
\]

It is important to note that the derivation of this conservation law did not require any knowledge about the initial data and solutions for \( \Phi \). The constant in the right hand side of Eq. (71) emerges as a universal number, irrespective of any particular solution for \( \Phi \). Although the “constancy of \( \zeta \)” is a favorite notion in the inflationary literature, it appears that inflationists have never asked what the origin and numerical value of this constant is. If this constant is a universal number, is it equal to a billion, or one, or zero? We will later show that this particular constant must be equal to zero. However, without specifying the value of this constant, it is often assumed that it is not zero.
The next step in this argumentation proceeds as follows. Imagine that the background equation of state changes during a short interval of time from the initial \( p \approx -\epsilon \) to the final \( p = \frac{1}{3}\epsilon \) or \( p = 0 \), so that \( 1 + w_i \ll 1 \) while \( 1 + w_f \approx 1 \). The authors of [39] consider the evolution of a long-wavelength perturbation from the initial Hubble radius crossing \( (t_i) \) to the final Hubble radius crossing \( (t_f) \). They make additional assumptions, they assume that \( \dot{\Phi} \) vanishes “at very early and very late times”. In the definition (68), they drop the term with \( \dot{\Phi} \) and simplify the quantity \( \zeta \):

\[
\zeta(t_i) \approx \frac{5 + 3w(t_i)}{3(1 + w(t_i))} \Phi(t_i)
\]

\[
\zeta(t_f) \approx \frac{5 + 3w(t_f)}{3(1 + w(t_f))} \Phi(t_f)
\]

Then they refer to the constancy of \( \zeta \), \( \zeta(t_i) = \zeta(t_f) \), and arrive at the formula

\[
\Phi(t_f) \approx \frac{1 + w(t_f)}{1 + w(t_i)} \frac{5 + 3w(t_i)}{5 + 3w(t_f)} \Phi(t_i) \approx \frac{1}{1 + w(t_i)} \Phi(t_i) \quad . \tag{72}
\]

(which, in fact, is in a conflict with the constancy of \( \zeta \), as we will show later).

According to this formula, and since \( 1 + w(t_i) \) can be arbitrarily close to zero, the \( \Phi(t_f) \) can be made arbitrarily large for any nonvanishing \( \Phi(t_i) \). Moreover, since the time of transition from one equation of state to another can be arbitrarily short, the tremendous increase of numerical value of the potential \( \Phi \) is supposed to happen almost instantaneously. (The author of [32], who reconfirms the “standard” results, has even plotted a graph for this jump.) The authors of Ref. [39] emphasize that their result (72) (the actually published [39] formula (6.67) contains a misprint: the position of symbols \( w(t_i) \) and \( w(t_f) \) should be interchanged) is in a full agreement with previous studies and Eq. (64). Formula (72) suggests an arbitrarily large production of density perturbations for no other reason but simply because the \( 1 + w(t_i) \) was very close to zero. There is something strange with this formula. [I realize well that what I qualify here as strange is certainly considered by others as perfectly alright. Otherwise somebody would raise a voice of protest against the ease with which inflationists generate tremendous amounts of various substances (some of the authors are even claiming that they can “overclose” our Universe). However, judging from the literature, it is not only that there are no voices of protest, but there is rather an element of competition as for who was the first to proclaim the “standard” inflationary results. For instance, the authors of [40] address the inflationary claims about density perturbations as “first quantitatively calculated in [33-35] [and which] have been successfully quantitatively confirmed by the COBE discovery”.]

Let us now try to sort out what we are dealing with.

The first point to realize is that the basic equations (66), (67) are, strictly speaking, incorrect. They are incorrect in the following sense: they are constructed from a correct equation and the first time derivative of the correct equation. Transforming the original Einstein equations, the authors of [38,39] have effectively raised the order of differential equations. If the correct equation is satisfied, the constructed equation is satisfied too, but not \textit{vise versa}. To get the feeling of the danger involved, consider a correct equation
\( \dot{x} = 0 \) and a constructed equation \( \ddot{x} - A \dot{x} = 0 \) where \( A \) is arbitrary constant. Solutions to the correct equation do satisfy the constructed equation, but the latter one admits exponentially growing solutions which are not allowed by the original equation.

Let us start our analysis from Eq. (66). Since the potential \( \Phi \) is an eigenfunction of the Laplace operator, \( \nabla^2 \Phi = -n^2 \Phi \), we will replace \( \nabla^2 \Phi \) with \( -n^2 \Phi \). We introduce also a new function of the scale factor \( a(\eta) \):

\[
\gamma = -\frac{\dot{H}}{H^2} = 1 + \left( \frac{a}{a'} \right)' \tag{73}
\]

So far, the variable \( \mu \) is simply a new variable replacing \( \Phi \), but the importance of \( \mu \) is in that the original perturbed Einstein equations require this variable to satisfy the equation

\[
\mu'' + \mu \left[ n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 0 \tag{74}
\]

(For those interested in gauge-invariant potentials, I may remark that \( \mu \) is a genuine gauge-invariant variable in the sense of [38,39].) With the help of Eq. (73), Eq. (66) identically transforms to

\[
\left[ \mu'' + \mu \left[ n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] \right]' - \frac{(a\sqrt{\gamma})'}{a\sqrt{\gamma}} \left[ \mu'' + \mu \left[ n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] \right] = 0 \tag{75}
\]

If Eq. (74) is satisfied, Eq. (75) is satisfied too, but not vice versa. Equation (75) is totally equivalent to

\[
\frac{1}{a^2 \gamma} \left[ a^2 \gamma \left( \frac{\mu}{a\sqrt{\gamma}} \right)' \right]' + n^2 \frac{\mu}{a\sqrt{\gamma}} = X \tag{76}
\]

where \( X \) is arbitrary constant. Thus, Eq. (66) involves an arbitrary constant \( X \) which, in fact, must be zero according to Eq. (74).

Let us now turn to the perfect fluid equation (67). The situation here is similar to the scalar field case. Introduce a new variable \( \nu \) according to the definition

\[
\Phi = \frac{1}{2n^2} \frac{a'}{a} \left( \frac{\nu}{a\sqrt{\gamma}c_s^2} \right)' \tag{77}
\]

The importance of \( \nu \) is in that the original perturbed Einstein equations require this variable to satisfy the equation

\[
Z \equiv \nu'' + \nu \left[ n^2 c_s^2 - \frac{(a\sqrt{\gamma}/c_s^2)''}{a\sqrt{\gamma}/c_s^2} \right] = 0 \tag{78}
\]
(The function \( \nu \) is a genuine gauge-invariant variable.) With the help of Eq. (77), Eq. (67) identically transforms to

\[
Z' - \frac{(a\sqrt{\gamma c_s^2})'}{a\sqrt{\gamma c_s^2}} Z = 0
\]  

(79)

This equation is totally equivalent to

\[
\frac{1}{a^2\gamma} \left[ \frac{1}{c_s^2} \left( \frac{\nu}{a\sqrt{\gamma/c_s^2}} \right)' \right]' + \frac{n^2}{a^2\gamma/c_s^2} \nu = Y
\]  

(80)

where \( Y \) is arbitrary constant. Thus, Eq. (80) does also involve an arbitrary constant \( Y \) which, in fact, must be zero according to Eq. (78). The role of these constants \( X \) and \( Y \) in the constancy of \( \zeta \) argument we will discuss shortly.

Equations (74), (78) is all we need to solve, in order to find all the perturbed components of the metric tensor and energy-momentum tensor. There is no other way to find the evolution of density perturbations except of doing the hard work of solving these equations, imposing appropriate initial conditions, joining the solutions, etc. (This is what is being done, convincingly or not, in Ref. [3]. It is only great physicists can afford having one or several the greatest mistakes of their life, we can not afford any.) Among other things, these solutions will tell you what is the value and time dependence of the functions \( \zeta \) introduced by the definitions (68), (73), (77). The correct equations do not show anything like enormously large amplitude of today’s density perturbations in the limit of the de Sitter expansion. However, we need to return to the concept of “constancy of \( \zeta \)” which pretends to answer important physical questions without solving any equations at all.

Combine the definitions (68), (73) to show that \( \zeta \) at the scalar field stage is

\[
\zeta = \frac{1}{2n^2} \frac{1}{a^2\gamma} \left[ \frac{1}{c_s^2} \left( \frac{\mu}{a\sqrt{\gamma}} \right)' \right]'.
\]

Combine the definitions (68), (77) to show that \( \zeta \) at the perfect fluid stage is

\[
\zeta = \frac{1}{2n^2} \frac{1}{a^2\gamma} \left[ \frac{1}{c_s^2} \left( \frac{\nu}{a\sqrt{\gamma/c_s^2}} \right)' \right]'.
\]

The term with the Laplacian neglected in Eqs. (66), (67) is the term with \( n^2 \) in equations (76) and (80). Let us drop this term as the authors of [39] do. Then, in this approximation, Eq. (76) gives \( \zeta \approx X/2n^2 = \text{const} \) and Eq. (80) gives \( \zeta \approx Y/2n^2 = \text{const} \). These relationships show that \( \zeta \) can be a universal constant, independent of any particular solution, only if it is supported by the constant \( X \) or, correspondingly, by the constant \( Y \). But, as we have shown above, these constants must be equal to zero, if one is willing to work with correct equations. The “constancy of \( \zeta \)” argument fails at the very beginning, regardless of validity or not of the additional assumptions that have been made on the route to Eq. (72). The conservation law \( \zeta(t_i) = \zeta(t_f) \) degenerates to an empty statement \( 0 = 0 \), and nothing can be derived from it.
The idea of a free lunch based on the “constancy of $\zeta$” seems to be so attractive that the following question is often being asked. Suppose that not for all solutions for $\Phi$, but for some of them, suppose that not in the leading order $n^{-2}$, but in the next order, suppose, nevertheless, that $\zeta$ calculated from this specially chosen solution for $\Phi$ is approximately a nonzero constant (which is indeed possible). Why cannot we return to Eq. (68) and repeat all the arguments that have seemingly led us from the definition (68) to the result (72)? For instance, the authors of [31] have even considered, along this line, a concrete model consisting of two consecutive power-law scale factors. They speak about an initially small potential $\Phi$ being “distributed” after the transition point “into two modes, both very large in amplitude, one which decays, the other yielding” the “standard” result (72). And this is how, they argue, the “scalar perturbations can be very strongly amplified in the course of the transition”.

So, we also need to consider this model and explore what is contained in the definition (68) treated as an equation for $\Phi$.

First of all, use the background equation

$$1 + w = -\frac{2}{3} \frac{\dot{H}}{H^2}$$

and write $\zeta$ in a more convenient form:

$$\zeta = -\frac{H^2}{aH} \left( \frac{a}{H} \Phi \right) .$$

Integrate this equation to produce the general solution

$$\Phi = \frac{H}{a} \left[ C + \int a \zeta \left( \frac{1}{H} \right) dt \right]$$

where $C$ is arbitrary integration constant. So far, $\zeta$ can be an arbitrary function. Now assume that $\zeta$ is a constant, $\zeta = \zeta_0 = \text{const}$, and write

$$\Phi = \zeta_0 \left[ 1 - \frac{H}{a} \int_{t_i}^{t} a dt \right] + C \frac{H}{a} .$$

[The constants $\zeta_0$ and $C$ can be related to the coefficients in front of two linearly independent solutions to Eq. (74). In the long wavelength approximation,

$$\frac{\mu}{H} = C_1 \left[ 1 - n^2 \int \frac{a^2}{a^2 \gamma} \left( \int a^2 \gamma d\eta \right) d\eta \right] + C_2 \int \frac{d\eta}{a^2 \gamma} + \ldots .$$

Using the definition (73) one can show that $\zeta_0 = -\frac{1}{2} C_1$.

As the second step, let us consider, together with Deruelle and Mukhanov [31], an expanding model which describes a transition from one power-law scale factor to another power-law scale factor. Let us write

$$a_i = a_1 t^{p_1}, \quad a_f = a_2 (t - t_*)^{p_2} .$$

The scale factor and its first time derivative are continuous functions at the transition point $t = t_1$. This requires

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\[ a_2 = a_1 \left( \frac{p_1}{p_2} \right)^{p_2} t_1^{p_1-p_2}, \quad t_* = t_1 \left( 1 - \frac{p_2}{p_1} \right). \]

To be closer to the inflationary results, one can keep in mind that \( p_1 \gg 1 \), while \( p_2 \) is 1/2 or 2/3.

Now we will follow the time evolution of \( \Phi \) specifically in this model. Let \( t_i \) be the time when our perturbation first crosses the Hubble radius: \( a(t_i)H(t_i) = n \). Let \( t_f \) be the time when the perturbation returns back inside the Hubble radius: \( a(t_f)H(t_f) = n \). The functions \( a(t) \) and \( H(t) \) are continuous all the way from \( t_i \) to \( t_f \), and so is the function (83). The initial value of the potential is

\[ \Phi(t_i) = \zeta_0 + C \frac{H(t_i)}{a(t_i)} \quad \text{(85)} \]

the final value is

\[ \Phi(t_f) = \zeta_0 I + C \frac{H(t_f)}{a(t_f)} \quad \text{(86)} \]

where

\[ I = 1 - \frac{H(t_f)}{a(t_f)} \int_{t_i}^{t_f} a(t) \, dt. \]

Combining (85) and (86) we find

\[ \Phi(t_f) = \Phi(t_i) I + C \left[ \frac{H(t_f)}{a(t_f)} - \frac{H(t_i)}{a(t_i)} I \right] \quad \text{(87)} \]

The integral \( I \) can be easily calculated for the scale factor (84). Since \( a(t_f) \gg a(t_i) \), \( p_1 \gg p_2 \), and \( a(t_f)H(t_f) \gg n \), we have approximately

\[ I \approx \frac{1}{p_2+1}. \]

Formula (87) reduces to

\[ \Phi(t_f) = \frac{1}{p_2+1} \Phi(t_i) - C \frac{H(t_i)}{a(t_i)} \frac{1}{p_2+1}. \]

The constant \( C \) could be set to zero from the very beginning. In any case, the term \( C(H(t_i)/a(t_i)) \) is of the same order or smaller than \( \Phi(t_i) \). By setting \( C = 0 \), we arrive at

\[ \Phi(t_f) \approx \frac{1}{p_2+1} \Phi(t_i). \]

There is nothing like tremendous jumps of \( \Phi \) at the transition point. The “constancy of \( \zeta \)” effectively translated into constancy of \( \Phi \). [The fact that \( \Phi \) is approximately constant for wavelengths longer than the Hubble radius follows also from the exact solution to Eq. (74) which can be found in case of power-law scale factors.]
The best what we can say about the “standard” inflationary formula is that it does not follow from correct equations.

It is important to recall that the “standard” inflationary results seem to be an indispensable tool in cosmology of our days (and possibly in the next Millennium too, see for example [41,10]). The expected relative contributions of density perturbations and gravitational waves to the observed microwave background anisotropies are a subject of active study. In the center of discussion are usually the “consistency relations” which state that the ratio of density to gravity-wave contributions goes to infinity when the spectrum of perturbations approaches the most favorite, Harrison-Zeldovich form. In reality, as we have shown above, these “consistency relations” are simply a manifestation of inconsistency of the “standard” inflationary theory from which they are derived.

In conclusion, if the “standard” inflationary results are incorrect and cannot be trusted, what is the amount of density perturbations that can be generated in the early Universe? My part of answer is formulated in Ref. [3,7].

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