Boundedness of Littlewood-Paley Operators in Variable Morrey Spaces

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Abstract: In this paper, the authors prove norm inequalities for the intrinsic square functions and commutators generated by this class operator and BMO function in variable Morrey spaces. This implies that the same norm inequalities for the Lusin area integrals, the Littlewood-Paley operators and the continuous square functions. As application, we get the boundedness for convolution Calderón-Zygmund operators in generalized Morrey spaces.

Keywords: Littlewood-Paley Operators, Singular Integrals, Morrey Spaces, Commutator

1. Introduction

There is a greatly interested to study variable exponent spaces and operators with variable parameters in last two decade. Some researches are monographs about Lebesgue spaces with variable exponents, for example, [1, 2]. Let \( p(\cdot) \) be a measurable function : \( \Omega \to [1, \infty) \). We suppose that

\[
1 < p_- \leq p(\cdot) \leq p_+ < \infty,
\]

where \( p_- = \text{ess inf}_{x \in \Omega} p(x) \), \( p_+ = \text{ess sup}_{x \in \Omega} p(x) \).

We let \( \mathcal{L}^{p(\cdot)}(\Omega) \) be the set of functions \( f \) such that

\[
\rho_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty.
\]

It is a Banach space equipped with the norm

\[
\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : \rho_{p(\cdot)}(f/\eta) \leq 1 \right\},
\]

we denote the conjugate exponent by \( p'(x) = \frac{p(x)}{p(x) - 1} \) for \( x \in \Omega \).

The Hölder inequality is valid in the form

\[
\int_{\Omega} |f(x)g(x)| dx \leq \left( \frac{1}{p_-} + \frac{1}{p'_+} \right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.
\]

Definition 1.1. Let \( \Omega \) is open bounded set. We say that \( p(\cdot) \) satisfies the log-Hölder condition, and denote this by \( p(\cdot) \in LH(\Omega) \), if there exists a constant \( C \) such that for all \( x, y \in \Omega \), \( |x - y| < 1/2 \),

\[
|p(x) - p(y)| \leq \frac{C}{-\ln |x - y|}.
\]

Let \( M \) be the Hardy-Littlewood maximal operator, i.e., for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \)

\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,
\]

where the supremum is taken over all cubes \( Q \) containing \( x \), and the \( |Q| \) is the Lebesgue measure of \( Q \subset \mathbb{R}^n \). Let \( \mathcal{B}(\Omega) \) be the set such that \( M \) is bounded on \( \mathcal{L}^{p(\cdot)}(\Omega) \). Diening [3] proved that \( p(\cdot) \in BMO(\Omega) \) if \( p(\cdot) \in LH(\Omega) \), i.e. \( M \) is bounded on \( \mathcal{L}^{p(\cdot)}(\Omega) \). When \( \Omega \) is unbounded to see [4].

Morrey spaces play an important role in study of local properties of solutions of partial differential equations. These spaces were introduced by Morrey [11] in 1938. The norm of Morrey space is defined as follows:

\[
\|f\|_{p,\lambda} = \sup_{x, r > 0} \left( \frac{1}{r^\lambda} \int_{B_r(x)} |f(y)|^p dy \right)^{1/p},
\]
where $Q$ is the cube with the center $x$ and with side-length $r$ and its sides parallel to the coordinate axes, $0 < \lambda < n$ and $1 \leq p < \infty$. Variable exponent Morrey spaces were firstly introduced in [13] and also studied in [12]. Generalized variable exponent Morrey spaces $\mathcal{L}^{p(\cdot)}(\omega)(\Omega)$ were introduced in [6]. They proved the boundedness of maximal, singular, potential operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot)}(\omega)(\Omega)$ in [8].

At the same time, the authors of [6] introduced another generalized variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ in [7].

In this paper, we denote $Q = Q \cap \Omega$. Let $\omega(x, r) = r^{\frac{n}{\lambda}}\omega(x, r) = r^{\frac{n}{\lambda}}$ be a positive measurable function on $\Omega \times (0, \ell)$, $\ell = \text{diam} \Omega$ and $1 \leq p < \infty$.

The generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is equipped with the norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, r > 0} \omega(x, r)^{\frac{n}{\lambda}} \|f\|_{\mathcal{L}^{p(\cdot)}(\tilde{Q}(x, r))},$$

where we assume that

$$\inf_{x \in \Omega, r > 0} \omega(x, r) > 0. \quad (2)$$

It is the variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x, r) = r^\frac{\lambda(x)-n}{p(x)}$, where $\lambda(x)$ is a measurable function on $\Omega$ with values in $[0, n]$. When $p$ is constant, the norm is defined by

$$\|f\|_{p,\omega} = \sup_{x \in \Omega, r > 0} \omega(x, r)^{\frac{1}{p}} \|f\|_{\mathcal{L}^{p}(\tilde{Q}(x, r))},$$

In this paper we mainly consider the boundedness of intrinsic square functions in variable Morrey spaces. Let $u(x, t) = P_t * f(x)$ be the Poisson integral of $f$, where

$$P_t(x) = c_n \frac{t^{n+1}}{(t^2 + |x|^2)^{(n+1)/2}}$$

with $c_n = \frac{n!}{2^nn^{n/2}}$.

The cone of aperture $\beta$ for any $\beta > 0$ is defined by

$$\Gamma_{\beta}(x) = \{y(t) \in \mathbb{R}^{n+1} : |y - x| < \beta t\}.$$

The corresponding Lusin area integral $S_{\beta}$ is defined by

$$S_{\beta}(f)(x) = \left( \int_{\Gamma_{\beta}(x)} |\nabla u(y, t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}.$$

The Littlewood-Paley $g$-function and $g_1^\alpha$-function are defined respectively by

$$g(f)(x) = \left( \int_0^\infty t|\nabla u(x, t)|^2 dt \right)^{1/2}$$

and

$$g_1^\alpha(f)(x) = \left( \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |\nabla u(y, t)|^2 \frac{dtdy}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1.$$

for all $x$ and $x’$.

$$|\phi(x) - \phi(x’)| \leq |x - x’|^\alpha.$$

For $f \in L^1_{loc}(\mathbb{R}^n)$ and $(y, t) \in \mathbb{R}^{n+1}$, set

$$A_{\alpha}(f)(y, t) = \sup_{\phi \in C_0^\infty} |f * \phi_t(y)|.$$

In [16], the intrinsic square function is defined by

$$G_{\beta,\alpha}(f)(x) = \left( \int_{\Gamma_{\beta}(x)} (A_{\alpha}(f)(y, t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

when $\beta = 1$, denote $G_{1,\alpha}$ by $G_{\alpha}$.

The intrinsic Littlewood-Paley $g$-function and the intrinsic $g_1^\alpha$-function are defined respectively by

$$g_{\alpha}(f)(x) = \left( \int_0^\infty (A_{\alpha}(f)(x, t))^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g_{\alpha}^\alpha(f)(x) = \left( \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |\nabla u(y, t)|^2 \frac{dtdy}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1.$$
where \( b \) condition on \( \omega \) parallel to the coordinate axes. We can impose the following the cube with the center \( x \) increasing and there is a constant \( D \), for any \( r > 0 \),

\[
\begin{align*}
|x| &\leq \frac{r}{2} + \frac{D}{2}, \\
&\text{where the constant } C \text{ depends on } n, \alpha, \beta, \text{ and } \omega.
\end{align*}
\]

We recall several properties of the intrinsic square function, the proofs of them to see [16, 17].

(1) \( G_\alpha \) is of weak type \((1,1)\):

\[
\{ x \in \mathbb{R}^n : G_\alpha(f)(x) > \sigma \} \leq \frac{C(n, \alpha)}{\sigma} \| f \|_{L^1(\mathbb{R}^n)};
\]

(2) If \( \beta \geq 1 \), then for all \( x \in \mathbb{R}^n \),

\[
G_{\beta, \alpha}(f)(x) \leq C(\alpha, \beta, n)G_\alpha(f)(x);
\]

(3) If \( S \) is anyone the Littlewood-Paley operators defined above, then

\[
S(f)(x) \leq CG_\alpha(f)(x),
\]

where the constant \( C \) is independent of \( f \) and \( x \);

(4) The function \( G_\alpha \) and \( g_\alpha \) are pointwise comparable with comparability constants only depending on \( \alpha \) and \( n \).

A locally integrable function \( b \) belongs to \( \text{BMO} \) if

\[
\| b \|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty,
\]

where \( b_Q = \frac{1}{|Q|} \int_Q b(x) dx \) and the supremum is taken over all cube \( Q \) whose sides parallel to the coordinate axes in \( \mathbb{R}^n \).

The commutators generated by \( \text{BMO} \) function \( b \) and intrinsic square functions are respectively defined by

\[
[b, G_\alpha](f)(x) = \left( \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y - z)f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]

\[
[b, g_\alpha](f)(x) = \left( \int_0^\infty \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(x - z)f(z) dz \right|^2 \frac{dt}{t} \right)^{1/2}
\]

and

\[
[b, g^*_{\alpha, \omega}](f)(x) = \left( \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y - z)f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.
\]

In order to study the commutators, we need the following properties of \( \text{BMO} \). For \( b \in \text{BMO} \) and \( 1 < p < \infty \), we get

\[
\| b \|_{\text{BMO}} \sim \sup_Q \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p}
\]

For all nonnegative integers \( l \), we obtain

\[
|b^{2^{l+1}}_x - b_Q| \leq C(l + 1)\| b \|_{\text{BMO}}.
\]

Wang [19] has estimated this class operator for \( \omega \) is increasing and there is a constant \( D \), \( 1 \leq D \leq 2^n \), such that, for any \( r > 0 \), \( \omega(2Q) \leq D\omega(Q) \), where \( \omega(Q) = \omega(x, r) \), \( Q \) is the cube with the center \( x \) and with side-length \( r \) and its sides parallel to the coordinate axes. We can impose the following condition on \( \omega(x, r) \) as in [10].

Assume that there is a constant \( C \) such that, for any \( x \in \mathbb{R}^n \) and for any \( r > 0 \),

\[
\frac{1}{C} \leq \omega(x, r)/\omega(x, r) \leq C, \quad \text{where } r \leq t \leq 2r
\]
\(\int_r^\infty \frac{\omega(x,t)}{t^{n+1}} \, dt \leq C \frac{\omega(x,r)}{r^n}. \) 

(9)

**Theorem 1.1.** Assume that \(\omega(x,r)\) satisfy the conditions (8) and (9). Then (i) there is a constant \(C_p > 0\) such that

\[ \|S(f)\|_{p,\omega} \leq C_p \|f\|_{p,\omega} \]

(10)

for \(f \in L^{p,\omega}\) and \(1 < p < \infty\):

\(T f(x) = \lim_{\epsilon \to 0} T_{\epsilon} f(x) = \lim_{\epsilon \to 0} f \ast K_\epsilon(x)\)

and \(T^* f(x) = \sup_{\epsilon > 0} |T_{\epsilon} f(x)|\).

(ii) there is a constant \(C_p > 0\) such that for any \(\sigma > 0\) and any cube \(Q\),

\[ |\{x \in Q : S(f)(x) > \sigma\}| \leq C_p \frac{\|f\|_{p,\omega}}{\sigma^p} \]

(11)

for \(f \in L^{p,\omega}\) and \(1 < p < \infty\).

Based on Theorem 1.1, (10) holds for the classical Calderón-Zygmund operators along with the Littlewood-Paley technique in [15, 16, 17] developed by Wilson. Let

\(K_\epsilon(x) = K(x)\chi_{\{|x|>\epsilon\}}\) and \(K\) is the standard kernel.

**Corollary 1.2** If \(\omega\) satisfies the conditions (8) and (9). Then for \(1 < p < \infty\), there is a constant \(C_p > 0\) such that (10) holds for \(T\).

Corollary was also proved in [10].

**Theorem 1.2.** If \(\omega\) satisfies the conditions (8) and (9). Then (i) there is a constant \(C_p > 0\) such that (10) holds for \(g_{N,\alpha}\) where \(0 < \alpha \leq 1\), \(\lambda > 3\) and \(1 < p < \infty\);

(ii) there is a constant \(C_p > 0\) such that any cube \(Q\), (11) holds for \(g_{N,\alpha}\), where \(0 < \alpha \leq 1\), \(\lambda > \frac{3n - \alpha n}{n}\) and \(1 < p < \infty\).

**Theorem 1.3.** If \(\omega\) satisfies the conditions (8) and (9). Then for \(0 < \alpha \leq 1\) and \(1 < p < \infty\), there is a constant \(C_p > 0\) such that (10) holds for \(g_{\alpha}\).

**Theorem 1.4.** If \(\omega\) satisfies the conditions (8) and (9). Then for \(0 < \alpha \leq 1\), \(\lambda > 3\) and \(1 < p < \infty\), there is a constant \(C_p > 0\) such that (10) holds for \(g_{\alpha}\).

We can obtain the following results by property (4), Theorem 1.1 and Theorem 1.3.

**Corollary 1.5** If \(\omega\) satisfies the conditions (8) and (9). Then (i) for \(0 < \alpha < 1\) and \(1 < p < \infty\), there is a constant \(C_p > 0\) such that (10) holds for \(g_{\alpha}\);

\[ \|f\|_{L^{p,\omega}(\lambda(x),\omega(\cdot))(\Omega)} = \sup_{x \in \Omega} \|\frac{\omega(x,r)}{r^{n+1}}\|_\lambda \left( \frac{\|f\|_{L^{p,\omega}(\lambda(x),\omega(\cdot))(\Omega)}}{r^{n+1}} \right) \|_{L^{\lambda}(0,\infty)}. \]

The generalized Morrey space \(L^{p,\omega}(\Omega)\) with constant exponent first appeared in [11]. If \(\theta(t) = \infty\), we can write

\[ L^{p,\omega}(\Omega) = L^{p,\omega}(\Omega). \]

In addition, we assume that \(\omega(x,r)\) satisfies the condition

\[ \sup_{x \in \Omega} \|\omega(x,.\|_{L^\infty(0,t)} < \infty. \]

(12)

**Theorem 1.6** Assume \(p \in LH(\Omega)\) satisfy (1), \(\ell = \text{diam } \Omega < \infty\) and Let

\[ 1 < \theta_1^- \leq \theta_1(t) \leq \theta_1^+ < \infty, 0 < t < \ell, \]

\[ 1 < \theta_2^- \leq \theta_2(t) \leq \theta_2^+ < \infty, 0 < t < \ell. \]
if there exists \( \delta > 0 \) such that

\[
\theta_1(t) \leq \theta_2(t)
\]

for \( t \in (0, \delta) \), \( \theta_1 \) and \( \omega_1 \) satisfy (12) and

\[
\sup_{0 < t < \delta, x \in \Omega} \int_0^t \omega_2(x, \xi)^{\theta_2(t)} \left( \int_t^\xi \frac{dr}{r^\omega_1(x, r)} \right)^{\frac{p_2(t)}{p_1(t)}} \, d\xi < \infty,
\]

where \( \tilde{\theta} = \inf_{s \in (\xi, t)} \theta_1(s) \), then \( G_\alpha \) is bounded from \( L^{p_1(\cdot), \omega_1(\cdot)}(\Omega) \) to \( L^{p_2(\cdot), \omega_2(\cdot)}(\Omega) \).

2. Preliminaries

A nonnegative locally integrable function \( w \) belongs to \( A_p \) (\( p > 1 \)) if

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{p-1} < \infty,
\]

where \( p' \) is the conjugate index of \( p \) i.e. \( 1/p + 1/p' = 1 \).

**Lemma 2.1.** [18] The intrinsic square function \( G_\alpha \), the Lusin area integral \( S_\beta \), the Littlewood-Paley function \( g \), the continuous square functions \( S_\psi \) and \( g_\psi \) are bounded on \( L^p(w) \) for \( w \in A_p \) (\( 1 < p \leq \infty \)).

**Lemma 2.2.** [20] Let \( 0 < \alpha \leq 1 \), \( 1 < p \leq \infty \) and \( \omega \in A_p \). Then the commutators \( [b, G_\alpha] \) and \( [b, g_\psi] \) are bounded from \( L^p(w) \) into itself whenever \( b \in BMO \).

**Lemma 2.3.** [19] Let \( 0 < \alpha \leq 1 \) and \( \lambda > (3n + 2\alpha)/n \). Then for any \( \sigma > 0 \), there exists a constant \( C > 0 \) independent of \( f \) and \( \sigma > 0 \), such that

\[
|\{ x \in \mathbb{R}^n : g_{\lambda, \alpha}(f)(x) > \sigma \}| \leq \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(x)| \, dx.
\]

An important extrapolation theorem was introduced in [5].

**Lemma 2.4.** Let \( f \) and \( g \) be non-negative and measurable functions, assume that there exists a constant \( C \) such that

\[
\int_{\Omega} f(x)^q w(x) \, dx \leq C \int_{\Omega} g(x)^q w(x) \, dx
\]

holds for some \( 1 < q < \infty \) and for every \( w \in A_q \). Then

\[
\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}
\]

for \( p(\cdot) \in LH(\Omega) \).

We can obtain the following conclusion by Lemma 2.1 and Lemma 2.4.

**Theorem 2.1.** (13) holds for the intrinsic square function \( G_\alpha(f) \) (\( 0 < \alpha \leq 1 \)), the Lusin area integral \( S_\beta(f) \), the Littlewood-Paley function \( g(f) \), the continuous square functions \( S_\psi(f) \) and \( g_\psi(f) \) for \( f \in \mathcal{L}^{p(\cdot)}(\Omega) \) and \( p(\cdot) \in LH(\Omega) \).

The Littlewood-Paley \( g_\psi^* \) function has been discussed in [5].

For commutators, we have the following theorem by using Lemma 2.2 and Lemma 2.4.

**Theorem 2.2.** (13) holds for \( [b, G_\alpha](f) \) and \( [b, g_\psi^*](f) \) for \( f \in \mathcal{L}^{p(\cdot)}(\Omega) \) and \( p(\cdot) \in LH(\Omega) \).

We will use the following estimate.

**Lemma 2.5.** [9] Suppose that \( 1 \leq p_- \leq p(x) \leq p_+ < \infty \), \( p(\cdot) \in LH(\Omega) \) and

\[
\sup_{x \in \Omega} \nu(x) < \infty, \quad \inf_{x \in \Omega} [n + \nu(x)p(x)] > 0.
\]

Then

\[
\|x - \nu(x)\chi_{Q(x,r)}\|_{p(\cdot)} \leq C r^{\nu(x)+\frac{1}{p(\cdot)}}, \quad x \in \Omega, \ 0 < r < \ell = \text{diam} \Omega,
\]

where \( C \) is independent of \( x \) and \( r \).

[6, 7] assert that the constant \( C \) in Lemma 2 can be express as

\[
C = C_0 \ell^{n\left(\frac{1}{p_--1}\right)}.
\]
where $C_0$ is independent of $Ω$.

Lemma 2.5. [21] Let $θ(τ)$ and $r(τ)$ be measurable functions on $I = (0, ℓ)$ such that

$$1 < \inf_{τ \in (0, ℓ)} θ(τ), \sup_{τ \in (0, ℓ)} r(τ) < \infty, \quad θ(τ) \leq r(τ), \quad τ \in (0, ℓ).$$

if

$$\sup_{0 < t < ℓ} \int_0^t v(ξ)^{(τ)} \left( \int_t^ξ ω(ξ)^{(r)} dξ \right)^{\frac{τ}{r(t)}} dξ < \infty,$$

then $H_{v,ω}$ is bounded from $L^{θ(·)}(0, ℓ)$ to $L^{r(·)}(0, ℓ)$, where $θ = \inf_{s \in (ξ, ℓ)} θ(s)$ and $H_{v,ω}$ is the weighted Hardy type operator

$$H_{v,ω}(f)(t) = v(t) \int_t^ξ f(ξ) ω(ξ) dξ.$$

Lemma 2.6. [7] For $p(·) \in LH$ the embeddings

$$L^∞(Ω) \hookrightarrow L^{p(·)}(0, ℓ) \hookrightarrow L^{p(·)}(Ω)$$

holds if (12) is satisfied and there exist a $δ \in (0, ℓ)$ such that

$$\sup_{x \in Ω} ||ω(x, ·)||_{L^{p(·)}(δ, ℓ)} > 0.$$

Lemma 2.7. [10] Assume that $ω(x, r)$ satisfy the conditions (8) and (9). Then for $1 \leq q < p < \infty$, there is a constant $C_{p, q} > 0$ such that

$$∥M_q f∥_{p, ω} \leq C_{p, q} ∥f∥_{p, ω} \quad \text{for } f \in L^{p, ω},$$

where $(M([f])^q(x))^{1/q} = M_q f(x)$ and $M$ is the Hardy-Littlewood maximal operator.

3. Proof of the Main Results

Proof of Theorem 1.2: By (4) and (5), it is enough to prove Theorem 1.1 for $G_α(f)$. (i) Let $Q$ be a cube of $\mathbb{R}^n$, with the center $x_0$ and side-length $r$. Decompose $f = f_{χ_{2Q}} + f_{χ_{2Q}^c}$ and denote $f_{χ_{2Q}}$ by $f_1$ and $f_{χ_{2Q}^c}$ by $f_2$, $χ_{2Q}$ denotes the characteristic function of set $2Q$. Since $G_α(f)$ is a sublinear, we have

$$\left( \frac{1}{ω(x_0, r)} \int_Q |G_α(f)(x)|^p dx \right)^{1/p} \leq \left( \frac{1}{ω(x_0, r)} \int_Q |G_α(f_1(x)|^p dx \right)^{1/p} + \left( \frac{1}{ω(x_0, r)} \int_Q |G_α(f_2(x)|^p dx \right)^{1/p}$$

$$= I_1 + I_2.$$

Lemma 2.1 and (8) imply

$$I_1 = \frac{1}{ω(x_0, r)^{1/p}} \left( \int_{\mathbb{R}^n} |G_α(f_1(x)|^p dx \right)^{1/p} \leq C_{ω(x_0, 2r)}^{1/p} \left( \int_{2Q} |f(x)|^p dx \right)^{1/p} \leq C ∥f∥_{p, ω}$$

For any $x \in Q$, $(y, t) ∈ Γ(x)$ and $z ∈ (2^l+1Q \setminus 2^lQ) \cap Q(y, t)$, $l ∈ Z^+$, then

$$2t ≥ |x - y| + |y - z| ≥ |x - z| ≥ |z - x_0| - |x - x_0| ≥ 2^{l-1}r.$$
By the Minkowski inequality and Hölder inequality, we have
\[ |G_\alpha(f_2)(x)| \]
\[ = \left( \int_{\Gamma_1(x)} (A_\alpha(f_2)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \]
\[ = \left( \int_{\Gamma_1(x)} \left( \sup_{\phi \in L_\alpha} |f \ast \phi_t(y)| \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \]
\[ \leq C \left( \int_{2^{-2}r}^{1} \int_{|x-y|<t} \left( t^{-n} \int_{2Q \cap \{ |z|: |y-z| \leq t \}} |f(z)|dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \]
\[ \leq C \sum_{l=1}^{\infty} \omega(x, 2^{l+1}r) \left( \frac{1}{\omega(x, 2^{l+1}r)} \int_{2^{l+1}Q \setminus 2^lQ} |f(z)|dz \right)^{1/p} \]

It is easy to know that
\[ \frac{\omega(x, 2^{l+1}r)}{2^{l+1}Q} \text{ is comparable to } \int_{2^{l+1}r}^{\infty} \frac{\omega(x, t)}{t^{n+1}} dt \]
by (8) and (9). So we get
\[ |G_\alpha(f_2)(x)|^p \leq C \| f \|_{p,\omega}^p \sum_{l=1}^{\infty} \int_{2^{l+1}r}^{\infty} \frac{\omega(x, t)}{t^{n+1}} dt \]
\[ \leq C \| f \|_{p,\omega}^p \int_{r}^{\infty} \frac{\omega(x, t)}{t^{n+1}} dt \]
\[ \leq C \| f \|_{p,\omega}^p \frac{\omega(x, r)}{r^n}. \]

Hence
\[ I_2 \leq C \| f \|_{p,\omega}. \]

Combining the estimate for $I_1$ and $I_2$, we obtain the (10).

(ii) For $\sigma > 0$, by (3) and (10), we get (11).

In order to prove 1.2 we need the following estimate.

**Proposition 3.1.** Let $0 < \theta \leq 1$. Assume that $\omega$ satisfies
\[ 1/C \leq \omega(x, t)/\omega(x, r) \leq C, \text{ for } r \leq t \leq 2r \]
and
\[ \int_{r}^{\infty} \frac{\omega(x, t)}{t^{n+1+\theta}} dt \leq C \frac{\omega(x, r)}{r^{\theta n}}. \]

Then for $1 \leq p < \infty$ there is a constant $C > 0$ such that
\[ \int_{Q} |G_N(f)(x)|^p dx \leq C \omega(Q) \| S_{\psi,\beta}(f)(x) \|_{p,\omega}^p \]
for $f \in L^{p, \infty}$, any cube $Q$, $\beta$ and $N$ large enough, where $G_N$ is the grand maximal function.

The following important weight norm inequality is proved in [15]:

$$
\int_{\mathbb{R}^n} |G_N(f)(x)|^p V(x) dx \leq C \int_{\mathbb{R}^n} (S_{\psi, \beta}(f)(x))^p \tilde{M} V(x) dx.
$$

(14)

That is for $0 < p < \infty$, there exits a “Maximal operator” $\tilde{M}$ such that (14) is valid for $f \in L^p(\mathbb{R}^n)$, any nonnegative $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, appropriate $\psi$ and large enough $\beta$ and $N$ with a constant $C$ which is independent $V$ and $f$. For every $p$, we can take $\tilde{M} = M^k (k > 1)$; if $k = 1$, (14) holds for $p \in (0, 2)$.

**Proof of Proposition 3.1:** Taking $V = \chi_Q$ in (14), we have

$$
\int_{\mathbb{R}^n} |G_N(f)(x)|^p \chi_Q(x) dx \leq C \int_{\mathbb{R}^n} (S_{\psi, \beta}(f)(x))^p \tilde{M} \chi_Q(x) dx.
$$

Since $\chi_Q$ is the characteristic function of $Q$, $\tilde{M} \chi_Q(x) \leq 1$. And for $x \in 2^{l+1} Q \setminus 2^l Q$, $\tilde{M} \chi_Q(x)$ is comparable to $2^{-ln}$, $l \in Z^+$.

Therefore

$$
\int_Q |G_N(f)(x)|^p dx 
\leq C \left[ \int_{2^l Q} (S_{\psi, \beta}(f)(x))^p dx + \sum_{l=1}^{\infty} \int_{(2^{l+1} Q \setminus 2^{l} Q)} 2^{-ln} \chi_Q(x) dx \right] 
\leq C \left[ \omega(2Q) + \sum_{l=1}^{\infty} 2^{-ln} \omega(2^{l+1} Q) \right] \|S_{\psi, \beta}(f)\|_{p, \omega}^p 
\leq C r^{nq} \sum_{l=0}^{\infty} \frac{\omega(x_0, 2^l r)}{(2^{l} r)^{nq}} \|S_{\psi, \beta}(f)\|_{p, \omega}^p.
$$

Since

$$
\frac{\omega(x_0, 2^l r)}{(2^{l+1} Q)^{nq}} \text{ is comparable to } \int_{2^l r}^{2^{l+1} r} \frac{\omega(x_0, t)}{t^{nq+1}} dt,
$$

we get

$$
\int_Q |G_N(f)(x)|^p dx \leq C r^{nq} \int_{2^l r}^{\infty} \frac{\omega(x_0, t)}{t^{nq+1}} dt \|S_{\psi, \beta}(f)\|_{p, \omega}^p
\leq C \omega(Q) \|S_{\psi, \beta}(f)\|_{p, \omega}^p.
\qed
$$

**Proof of 1.2:** Wislon ([16, 17]) prove that there exists $\alpha \leq 1$ which depends on $T$ such that for all $x \in \mathbb{R}^n$,

$$
S_{\psi}(T f)(x) \leq c(T, \psi, n) G_\alpha(f)(x).
$$

By the same method yields

$$
S_{\psi, \beta}(T f)(x) \leq c(T, \psi, n, \beta) G_\alpha(f)(x),
$$

(15)

for $\beta \geq 1$.

The Proposition 3.1 tells us

$$
\|G_N(f)\|_{p, \varphi} \leq C \|S_{\psi, \beta}(f)\|_{p, \omega}
$$

(16)

for large enough $\beta$ and $N$.

It is well known that for all $x \in \mathbb{R}^n$ [22, pp. 67-68],

$$
T^* f(x) \leq C(n, T)(G_N(T f)(x) + M f(x)).
$$
Combining this with (15) and (16), we have
\[
\|T^* f\|_{p,\omega} \leq C \left( \|S_{\psi,\beta}(Tf)\|_{p,\omega} + \|M f\|_{p,\omega} \right) \\
\leq C \left( \|G_\alpha(f)\|_{p,\omega} + \|M f\|_{p,\omega} \right).
\]
This estimate along with Theorem 1.1 and Lemma 2.7, (10) yields for \( T \).

**Proof of Theorem 1.2** (i) As before, we set \( f = f_1 + f_2, f_1 = f \chi_{2Q} \). From the definition of \( g_{\lambda,\alpha} \), we can easily deduce that
\[
\left( \frac{1}{\omega(x_0, r)} \int_Q (g_{\lambda,\alpha}^*(f(x))^p \right) 1/p dx \right) 1/p
\leq \frac{C}{\omega(x_0, r)^{1/p}} \left[ (\int_Q (G_\alpha(f(x))^p dx)^{1/p} + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \left( \int_Q (G_{2j,\alpha}(f)(x))^p dx \right)^{1/p} \right]
\leq \frac{C}{\omega(x_0, r)^{1/p}} \left[ (\int_Q (G_\alpha(f(x))^p dx)^{1/p} + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \left( \int_Q (G_{2j,\alpha}(f_1)(x))^p dx \right)^{1/p} \right]
+ \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \left( \int_Q (G_{2j,\alpha}(f_2)(x))^p dx \right)^{1/p} \right]
\]
\[
= I_0 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} I_{1j}^1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} I_{1j}^2.
\]
By Theorem 1.1, we know that \( I_0 \leq C \|f\|_{p,\omega} \). We can get the following estimate for \( I_{1j}^1 \) from [19], i.e.
\[
I_{1j}^1 \leq C \|f\|_{p,\omega} (2^{jn/2} + 2^{jn/p}) \frac{\omega(x_0, 2r)^{1/p}}{\omega(x_0, r)^{1/p}}.
\]
We obtain
\[
I_{1j}^1 \leq C \|f\|_{p,\omega} (2^{jn/2} + 2^{jn/p})
\]
by (8).

The estimate of \( I_{1j}^{(2)} \) is similar to \( I_2 \),
\[
G_{2j,\alpha}(f_2)(x)dx
\leq C 2^{3jn/2} \sum_{l=1}^{\infty} \frac{\omega(x_0, 2l+1r)^{1/p}}{[2^{l+1}Q]^{1/p}} \left( \frac{1}{\omega(x_0, 2l+1r)} \int_{2^{l+1}Q} |f(z)|^p dz \right)^{1/p}
\leq C 2^{3jn/2} \|f\|_{p,\omega} \left( \int_{r}^{\infty} \frac{\omega(x_0, l)}{l^{n+1}} dt \right)^{1/p}
\leq C 2^{3jn/2} \|f\|_{p,\omega} \frac{\omega(x_0, r)^{1/p}}{r^{n/p}},
\]
hence
\[
I_{1j}^{(2)} \leq C 2^{3jn/2} \|f\|_{p,\omega}.
\]
Therefore
\[
\left( \frac{1}{\omega(x_0, r)} \int_Q (g_{\lambda,\alpha}^*(f(x))^p \right) 1/p dx \right) 1/p
\leq C \|f\|_{p,\omega} (1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} 2^{3jn/2} + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} 2^{jn/p})
\leq C \|f\|_{p,\omega}.
\]
for $\lambda > 3 > 2/p$.

(ii) For $\sigma > 0$, by Lemma 2.3 and the estimate of $I_j^{(2)}$, the (11) is valid to $g_{X,\sigma}$.

Proof of Theorem 1.3: Fix a cube $Q$ of $\mathbb{R}^n$ whose center is $x_0$ and edges have length $r$. Decompose $f = f\chi_{2Q} + f\chi_{(2Q)^c}$ and denote $f\chi_{2Q}$ by $f_1$ and $f\chi_{(2Q)^c}$ by $f_2$.

\[
\left(\frac{1}{\omega(x_0, r)} \int_Q |[b, G_\alpha](f)(x)|^p dx\right)^{1/p} \leq \left(\frac{1}{\omega(x_0, r)} \int_Q |[b, G_\alpha](f_1)(x)|^p dx\right)^{1/p} + \left(\frac{1}{\omega(x_0, r)} \int_Q |[b, G_\alpha](f_2)(x)|^p dx\right)^{1/p} = II_1 + II_2.
\]

By Lemma 2.2 we get

\[
II_1 \leq C \frac{1}{\omega(x_0, r)^{1/p}} \left(\int_{\mathbb{R}^n} |[b, G_\alpha](f_1)(x)|^p dx\right)^{1/p} \leq C \frac{\|b\|_{BMO}}{\omega(x_0, 2r)^{1/p}} \left(\int_{2Q} |f(x)|^p dx\right)^{1/p} \leq C \|f\|_{p, \omega} \|b\|_{BMO}.
\]

Now we estimate $II_2$. For any $x \in Q$, $(y, t) \in \Gamma(x)$, since $|b(x) - b(z)| \leq |b(x) - b_Q| + |b(z) - b_Q|$, we have

\[
|[b, G_\alpha](f_2)(x)| \leq |b(x) - b(z)|G_\alpha(f_2)(x) + \left(\int_{\Gamma} \sup_{\phi \in \mathcal{C}_O} \left(\int_{\mathbb{R}^n} |b(z) - b_Q| \phi_s(y - z) f(z) dz\right)^{2} dydt \right)^{1/2}
\]

\[
= II_{11} + II_{21}.
\]

In the proof of Theorem 1.1, we get that for any $x \in Q$,

\[
|G_\alpha(f_2)(x)|^p \leq C \|f\|_{p, \omega}^p \frac{\omega(x_0, r)}{r^n}.
\]

And by (6), we obtain

\[
\frac{1}{\omega(x_0, r)} \int_Q II_{11}^p dx \leq C \|f\|_{p, \omega}^p \frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \leq C \|f\|_{p, \omega}^p \|b\|_{BMO}^p
\]

Next we deal with $II_{21}$. It is similar to $|G_\alpha(f_2)(x)|$,

\[
II_{21} \leq \left(\int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_O} \left(\int_{\mathbb{R}^n} |b(z) - b_Q| \phi_s(y - z) f(z) dz\right)^{2} dydt \right)^{1/2}
\]

\[
\leq C \left(\int_{\Gamma(x)} \left(\sum_{l=1}^{\infty} \int_{(2^{l+1}Q\setminus 2^lQ)\cap \{|z| : |z| \leq t\}} |b(z) - b_{2^{l+1}Q}| |f(z)| dz\right)^{2} dydt \right)^{1/2}
\]

\[
+ C \left(\int_{\Gamma(x)} \left(\sum_{l=1}^{\infty} \int_{(2^{l+1}Q\setminus 2^lQ)\cap \{|z| : |z| \leq t\}} |b_{2^{l+1}Q} - b_Q| |f(z)| dz\right)^{2} dydt \right)^{1/2}
\]

\[
= II_{211} + II_{212}.
\]
Applying (6), (7) and \( \omega(x_0, 2^{l+1}r) \sim \omega(x_0, r) \), we have

\[
\|f\|_{p, \omega}(Q) \lesssim \frac{\omega(x_0, r)^{1/p}}{r^{n/p}} + C'H \|f\|_{p, \omega}(Q) \lesssim C' \|f\|_{p, \omega}(Q).
\]

Hence

\[
\frac{1}{\omega(x_0, r)} \int_{Q} \|f\|_{p, \omega}(Q) \leq C \|f\|_{p, \omega}(Q).
\]

Combining \( II_{11} \) and \( II_{21} \), so

\[
II_2 \leq C \|f\|_{p, \omega}(Q).
\]

Thus (9) holds for \([b, G_\alpha]\).

**Proof of Theorem 1.4:** By Lemma 2.2, we can use the same arguments in the proof of Theorem 1.2 (i) and Theorem 1.3 to get the conclusion of Theorem 1.4.

We use the following estimate to show Theorem 1.5 and Theorem .

**Proposition 3.2** Let \( \Omega \) be bounded, \( \ell = \text{diam} \Omega \) and \( p \in HL(\Omega) \) satisfy (1). Then

\[
\|G_\alpha f\|_{L^p(Q, x, t)} \leq C t^{\frac{n-1}{p}} \|f\|_{L^p(Q, x, t)}, \quad 0 < t < \delta
\]

for \( f \in L^p(\Omega) \), where \( C \) is independent of \( f, x \in \Omega \) and \( t \).

**Proof Proposition 3.2:** Let \( Q \) be a cube of \( \mathbb{R}^n \), the center is \( x_0 \) and side length is \( t \). Decompose \( f = f_1 + f_2 \cdot \chi_{2Q} \) and denote \( f_1 \) by \( f_1 \) and \( f_2 \cdot \chi_{2Q} \) by \( f_2 \). \( \chi_{2Q} \) denotes the characteristic function of set \( 2Q \). Since \( G_\alpha \) is a sublinear, we have

\[
\|G_\alpha f_1\|_{L^p(Q, x, t)} \leq \|G_\alpha f_1\|_{L^p(Q, x, t)} + \|G_\alpha f_2\|_{L^p(Q, x, t)}
\]

We get

\[
\|G_\alpha f_1\|_{L^p(Q, x, t)} \leq \|G_\alpha f_1\|_{L^p(Q, x, t)} \leq C \|f_1\|_{L^p(Q, x, t)} = C \|f\|_{L^p(Q, x, t)}
\]

by Theorem 2.1.

Then we easily obtain

\[
\|G_\alpha f\|_{L^p(Q, x, t)} \leq C t^{\frac{n-1}{p}} \|f\|_{L^p(Q, x, t)}
\]

\[
\leq C t^{\frac{n-1}{p}} \|f\|_{L^p(Q, x, t)}.
\]
Next we estimate $G_\alpha(f_2)$. It is similar to the second estimate in proof (i) of Theorem 1.1. We have
\[
|G_\alpha(f_2)(x)| \leq C \int_\mathbb{D} \sum_{k=1}^{\infty} \frac{1}{2^k+1} \int_{\mathbb{D}} |f(z)| dz \\
\leq C \int_{\mathbb{D}} [f(z)] dz.
\]
Choosing $\varepsilon > \frac{\alpha}{p-1}$, by the Hölder inequality of variable exponent and Lemma 2 we have
\[
\int_{\Omega(\mathbb{D},\varepsilon)} \frac{|f(z)|}{|x-z|^n} dz \\
\leq \varepsilon \int_{\Omega(\mathbb{D},\varepsilon)} \int_{|x-z|}^{\varepsilon} \frac{1}{s^{\varepsilon+1}} ds \\
\leq C \int_{t}^{\varepsilon} s^{-\frac{n}{p-1} - 1} \|f\|_{L^{p'}(\mathbb{D},\varepsilon)} ds.
\]
Hence
\[
|G_\alpha(f_2)(x)| \leq C \int_{t}^{\varepsilon} s^{-\frac{n}{p-1} - 1} \|f\|_{L^{p'}(\mathbb{D},\varepsilon)} ds.
\]
By Lemma 2 and the Hölder inequality of variable exponent again, we get
\[
\|G_\alpha(f_2)\|_{L^{p'}(\mathbb{D},\varepsilon)} \leq C \int_{t}^{\varepsilon} s^{-\frac{n}{p-1} - 1} \|f\|_{L^{p'}(\mathbb{D},\varepsilon)} ds. \quad \square
\]

**Proof of Theorem 1.5:**
\[
\|G_\alpha(f)\|_{L^{p'}(\mathbb{D},\varepsilon)} = \sup_{x \in \Omega, r > 0} \frac{r^{-\frac{n}{p-1}}}{\omega_2(x,r)} \|G_\alpha(f)\|_{L^{p'}(\mathbb{D},\varepsilon)}.
\]
We just consider $r \in (0, \delta)$ under the assumption of (2). Applying Proposition 3.2, we have
\[
\sup_{x \in \Omega, r \in (0, \delta)} \frac{r^{-\frac{n}{p-1}}}{\omega_2(x,r)} \|G_\alpha(f)\|_{L^{p'}(\mathbb{D},\varepsilon)} \leq C \int_{r}^{\delta} \frac{r^{-\frac{n}{p-1}}}{\omega_2(x,r)} ds.
\]
\[
\leq C \|f\|_{L^{p'}(\mathbb{D},\varepsilon)} \int_{r}^{\delta} \frac{1}{\omega_2(x,r)} ds \leq C \|f\|_{L^{p'}(\mathbb{D},\varepsilon)}.
\]
\square
Proof of Theorem:

\[ \|G_\alpha(f)\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} = \sup_{x \in \Omega} \left\| \frac{\omega(x,r)}{r^{n/p^*}} \left\| G_\alpha(f) \right\|_{L^{p(\cdot)}_x(\cdot,\cdot)} \right\|_{L^{\theta(\cdot)}(0,\delta)} \]
\[ \leq \sup_{x \in \Omega} \left\| \frac{\omega(x,r)}{r^{n/p^*}} \left\| G_\alpha(f) \right\|_{L^{p(\cdot)}_x(\cdot,\cdot)} \right\|_{L^{\theta(\cdot)}(0,\delta)} + \sup_{x \in \Omega} \left( \frac{\omega(x,r)}{r^{n/p^*}} \left\| f \right\|_{L^{p(\cdot)}_x(\cdot,\cdot)} \right) \cdot \sup_{x \in \Omega} \left\| G_\alpha(f) \right\|_{L^{p(\cdot)}_x(\cdot,\cdot)} \right\|_{L^{\theta(\cdot)}(0,\delta)} \]
\[ = I_1 + I_2 \]

The estimate of \( I_2 \) can directly follow from the Lemma 2.6.

\[ I_2 \leq C \|G_\alpha(f)\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} \omega_2(x,\cdot) \|_{L^{\theta(\cdot)}(\delta,\ell)} \]
\[ \leq C \|G_\alpha(f)\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} \|f\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} \cdot \omega_2(x,\cdot) \|_{L^{\theta(\cdot)}(\delta,\ell)} \]

For \( \ell \in (0,\delta) \), we find

\[ I_1 \leq C \omega_2(x,\ell) \int_\ell^\delta r^{-n/p^* - \frac{1}{p^*}} \|f\|_{L^{p(\cdot)}_x(Q(x,r))} dr \|_{L^{\theta(\cdot)}(\delta,\ell)} \]

We splitting the integral with respect to \( r \) into two integrals over \( (0,\delta) \) and \( (\delta,\ell) \). The estimate of integral over \( (\delta,\ell) \) is obtained as \( I_2 \) and the estimate of integral over \( (0,\delta) \) is used by Lemma 2.5. In addition, by the Lemma 2.6 we get

\[ I_1 \leq C \|f\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} \|\omega(x,r)\|_{L^{\theta(\cdot)}_x(Q(x,r))} \|_{L^{\theta(\cdot)}(0,\delta)} \]
\[ \leq C \|f\|_{L^{p(\cdot)}_x(\cdot,\cdot;\omega(\cdot))} \|\omega(x,r)\|_{L^{\theta(\cdot)}_x(Q(x,r))} \|_{L^{\theta(\cdot)}(0,\delta)} \]

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