Form factors of exclusive $b \to u$ transitions

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We present the form factors of the $B \to \pi, \rho$ transitions induced by the $b \to u$ quark currents at all kinematically accessible $q^2$. Our analysis is based on the spectral representations of the form factors within the constituent quark picture: we fix the soft meson wave functions and the constituent quark masses by fitting $A_1(q^2)$ and $T_2(q^2)$ to the lattice results at small recoils ($17 \lesssim q^2 \lesssim 20$ GeV$^2$). We then calculate the $B \to \pi, \rho$ transition form factors down to $q^2 = 0$. For the $B \to \pi$ case the region $q^2 \lesssim 20$ GeV$^2$ however does not cover the whole kinematically accessible range. Due to the smallness of the pion mass the region of small recoils is close to the nearby $B^*(5234)$ resonance. We develop a parametrization which includes the $B^*$ dominance of the form factors $f_+$ and $f_-$ at small recoils and numerically reproduces the results of calculations at $q^2 \lesssim 20$ GeV$^2$. We find $\Gamma(B \to \pi \ell \nu) = 8.0^{+0.8}_{-0.2} |V_{ub}|^2$ ps$^{-1}$ and $\Gamma(B \to \rho \ell \nu) = 15.8 \pm 2.3 |V_{ub}|^2$ ps$^{-1}$.

First measurements of the semileptonic (SL) $B \to (\pi, \rho) \ell \nu$ branching fractions by CLEO$^{[1,2]}$ opened a possibility to determine $|V_{ub}|$. Precise knowledge of this element of the Cabibo-Kobayashi-Maskawa matrix which describes the quark mixing in the Standard Model (SM) is necessary both for understanding the dynamics of the SM and the origin of CP violation. However, for a proper extraction of $|V_{ub}|$ from the SL decays one needs a reliable knowledge of the meson transition form factors which encode the long-distance (LD) contributions to the exclusive $b \to u$ transitions.

Various nonperturbative theoretical frameworks have been applied to the description of the meson transition form factors induced by the $b \to u$ weak transition: among them are the constituent quark models$^{[3,12]}$, QCD sum rules$^{[13,14]}$, lattice QCD$^{[15]}$, and analytical constraints$^{[17,18]}$.

Lattice QCD simulations provide the most fundamental nonperturbative approach and thus should lead to the most reliable results. Still, some restrictions remain to be solved in the context of heavy-to-light transitions. One of them is the necessity to extrapolate the transition form factors in the heavy quark mass from the values of order $m_c$ utilized in the lattice approach to $m_b$. Another problem is that lattice calculations provide the form factors only in a region excluding large recoils. Therefore to obtain form factors in the whole kinematical decay region one has to rely on some extrapolation procedures.

QCD sum rules give a complementary information on the form factors as they allow one to determine the form factors at not very large momentum transfers and therefore also do not cover the whole kinematically accessible $q^2$-range$^{[13]}$. In practice, however, various versions of QCD sum rules give rather uncertain predictions dependent on the technical subtleties of the particular version$^{[13,15]}$.

Various models based on the constituent quark picture have been used for considering meson decays (see, e.g. a talk of A. Le Yaouanc for a detailed review$^{[12]}$). An attractive feature of the approaches based on the concept of constituent quarks is that these approaches provide a physical picture of the process. However, a long-standing problem of the constituent quark model (QM) applications to meson decays is a strong dependence of the predictions on the QM parameters.

Although none of these approaches is able at the moment to provide the form factors in the whole accessible kinematical region of the $B$ decay, a combination of different approaches might be fruitful. For instance, in Ref.$^{[16]}$ a simple lattice-constrained parametrization based on approximate relations obtained within the constituent quark picture and pole dominance have been proposed. However, within this approach the $B$ meson decays induced by the different quark transitions, e.g. $b \to u$ and $b \to s$, remain largely disconnected. In Ref.$^{[20]}$ it was noticed that determining the soft meson wave functions by matching the quark model calculations of the transition form factors to the lattice results at small recoils allows one to connect many decay processes to each other. In this letter we apply such an approach to a study of the $B \to \pi, \rho$ transition form factors.

Namely, we fix the meson soft wave functions and the constituent quark masses by fitting the lattice results to the form factors $A_1(q^2)$ and $T_2(q^2)$ at small recoils$^{[16]}$, and then calculate the form factors in the region $0 < q^2 \lesssim 20$ GeV$^2$ through the spectral representations of the quark model$^{[10]}$. These spectral representations respect rigorous QCD constraints in the limit of heavy meson decays both to heavy and light mesons and thus we expect them to supply a reliable continuation of the lattice results to the lower $q^2$ region.

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Thus, for the $B \to \rho$ transition we calculate the form factors at all kinematically accessible $q^2$. For the $B \to \pi$ case this range given above does not cover the whole kinematically accessible region. Also, no lattice points are provided for $f_+(q^2)$ above $q^2 > 20 \text{GeV}^2$. To extrapolate the form factors $f_+$ and $f_-$ to larger $q^2$, note that the momentum transfers become rather close to the $B^*(5234)$ resonance. We therefore propose a parametrization which takes into account the $B^*$ dominance in the region of small recoils and reproduces the results of calculations at $q^2 \lesssim 20 \text{GeV}^2$.

The form factors of interest are connected with the meson transition amplitudes induced by the vector $V_\mu = \bar{q} \gamma_\mu q_1$, axial-vector $A_\mu = \bar{q} \gamma_\mu \gamma_5 q_1$, and tensor $T_{\mu\nu} = \bar{q} \sigma_{\mu\nu} q_1$, $q_1 \to q_2$ quark transition currents as follows (see notations in Ref. [20])

\[
< P(M_2, p_2)|V_\mu(0)|P(M_1, p_1) > = f_+(q^2)P_+ + f_-(q^2)P_-, \\
< V(M_2, p_2, \epsilon)|V_\mu(0)|P(M_1, p_1) > = 2g(q^2)\epsilon_{\mu\nu\rho\sigma}\epsilon^{\epsilon\nu} P_1^\rho P_2^\sigma, \\
< V(M_2, p_2, \epsilon)|A_\mu(0)|P(M_1, p_1) > = i\epsilon^{\alpha\beta\gamma} [f_+(q^2)g_{\alpha\mu} + a_+(q^2)p_{1\alpha}P_\mu + a_-(q^2)p_{1\alpha}q_\mu], \\
< P(M_2, p_2)|T_{\mu\nu}(0)|P(M_1, p_1) > = -2i s(q^2) (p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu}), \\
< V(M_2, p_2, \epsilon)|T_{\mu\nu}(0)|P(M_1, p_1) > = i\epsilon^{\alpha\beta\gamma} [g_+(q^2)\epsilon_{\mu\nu\rho\sigma} P^\beta + g_-(q^2)\epsilon_{\mu\nu\rho\sigma} q^\beta + g_0(q^2) (p_{1\alpha}q_{\mu}p_{2\nu} - q_{\alpha}p_{2\nu}p_{1\mu})],
\]

where $q = p_1 - p_2$, $P = p_1 + p_2$.

The dispersion approach of Refs. [10,19] gives the transition form factors of the meson $M_1$ to the meson $M_2$ as double relativistic spectral representations through the soft wave functions of the initial and final mesons, $\psi_1(s_1)$ and $\psi_2(s_2)$, respectively

\[
f_i(q^2) = \int ds_1 \psi_1(s_1) ds_2 \psi_2(s_2) f_i(s_1, s_2, q^2),
\]

where $s_1$ ($s_2$) is the invariant mass of the initial (final) $\bar{q}q$ pair. The double spectral densities $\tilde{f}_i$ of the representation [2] for the $0^- \to 0^-, 1^+$ meson decays induced by the vector, axial-vector and tensor quark currents have been calculated in [10,19]. The representation [2] is valid for $q^2 \leq (m_2 - m_1)^2$.

It is important to notice that the form factors [2] develop the correct structure of the heavy-quark expansion in accordance with QCD in the leading and next-to-leading $1/m_Q$ orders if the soft wave functions $\psi_i$ are localized in the momentum space in a region of the order of the confinement scale. The spectral densities for all the form factors [2] have been calculated in [10,19].

The spectral representations [2] take into account LD contributions connected with the meson formation in the initial and final channels. At large $q^2$ the LD effects in the $q^2$-channel become more essential and thus one should properly replace

\[
f_{M_1 \to M_2}(q^2) \to f_{q_1 \to q_2}(q^2) f_{M_1 \to M_2}(q^2),
\]

where the quark transition form factor $f_{q_1 \to q_2}(q^2)$ is introduced that accounts for the LD effects at large $q^2$ given by the relevant hadronic resonances and continuum states. The form factor $f_{q_1 \to q_2}(q^2)$ equals unity at $q^2$ far below the resonance region and contains poles at $q^2 = M_{\text{res}}^2$. Notice that the particular form of the quark transition form factor does not depend on the initial and final mesons involved but rather depends on the set of the relevant hadronic resonances and is different for the vector, axial-vector etc channels.

I. $B \to \rho$ TRANSITIONS

We consider the meson wave functions and the constituent quark masses as variational parameters and determine them from fitting the lattice results to reproduce $T_2(q^2)$ and $A_1(q^2)$ at $q^2 = 19.6$ and $17.6 \text{GeV}^2$ [14] by the double spectral representations [2], and assuming $f_{b \to u} = 1$ in the region $q^2 \lesssim 20 \text{GeV}^2$.

The soft wave function of a meson $M$ [$\bar{q}(m_\bar{q})q(m_q)$] can be written as

\[\text{1 One comment on the previous application of the dispersion quark model to meson decays is in order. In [10] it was shown that the form factors calculated with the QM parameters of the ISG2 model [3] (which differs considerably from the ISG2 model for the transition form factors) provide a good description of all experimental data on semileptonic $B$ and $D$ decays. However, the form factors of [10] have a much flatter $q^2$-dependence and do not match the lattice results at large $q^2$.}\]
where $k^2 = \lambda(s, m_q^2, m_{q'}^2)/4s$ with $\lambda(s, m_q^2, m_{q'}^2) = (s - m_q^2 - m_{q'}^2)^2 - 4m_q^2m_{q'}^2$, and the ground-state radial $S$-wave function $w(k^2)$ is normalized as $\int w^2(k^2)dk = 1$. For the functions $w(k^2)$ we assume a simple gaussian form

$$w(k^2) \propto \exp(-k^2/\beta^2)$$

where $\beta$ to be obtained by a fit.

The ranges of the $B$ and $\rho$ are shown in Table II. The values of the constituent quark masses and the slope parameter $\beta_\rho$ are fixed rather tightly by the $\chi^2$ fit to the lattice data, whereas $\beta_B$ cannot be fixed with a good accuracy. We determine the ranges of $\beta_B$ such that the leptonic decay constant $f_B$ calculated through the relation

$$f_B = \sqrt{Nc(m_q + m_{q'})} \int ds \psi(s) \frac{\lambda^{1/2}(s, m_q^2, m_{q'}^2)}{8\pi^2 s} \frac{s - (m_q - m_{q'})^2}{s}.$$ 

lies in the interval $f_B = 170 \pm 30$ MeV in accordance with the lattice estimates [16]. Once the wave functions and the quark masses are determined, we use the spectral representations (2) for calculating all the form factors for the $B \to \rho$ transition in the whole kinematically accessible region. Fig. 1 illustrates the calculated form factors versus the lattice data. Table II gives parameters of a convenient interpolation of the results of the calculation in the form

$$f(q^2) = \frac{f(0)}{1 - \sigma_1 q^2 + \sigma_2 q^4},$$

where we have introduced $q^2 = q^2/M_B^2$. With $M_B^2 = 5.324$ GeV. Since we have calculated the form factors at all kinematically accessible $q^2$ the particular form of the fit function is not important. The interpolations [7] deviate from the results of calculation by less than 1%. The calculated decay rates are given in Table II.

II. $B \to \pi$ TRANSITIONS

For the transition $B \to \pi$ a new wave function parameter $\beta_\pi$ appears. It is not independent and strongly correlates with $m_u$ through $f_\pi$ given by eq (3). Requiring $f_\pi = 132$ MeV this implicitly determines $\beta_\pi$ once $m_u$ is fixed.

With the wave functions given, we calculate the $B \to \pi$ transition form factors at $0 < q^2 \lesssim 20$ GeV$^2$. The form factors versus the lattice results shown in Fig. 1 are found to be in perfect agreement. This confirms our assumption $f_{B \to u} = 1$ at $q^2 \lesssim 20$ GeV$^2$. This region however does not cover the whole kinematically accessible range. To find the form factors at larger $q^2$ we must use some extrapolation procedure.

In the region of small recoils the form factors are dominated by the neighbouring $B^* \pi$ poles and one finds

$$f_+(q^2) = \frac{g_{B^* B \pi} f_{B^*}}{2M_{B^*}(1 - q^2/M_{B^*}^2)} + \text{regular terms at } q^2 = M_{B^*}^2,$$ 

$$f_-(q^2) = \frac{g_{B^* B \pi} f_{B^*}}{2M_{B^*}(1 - q^2/M_{B^*}^2)} \frac{M_B^2 - M_{B^*}^2}{M_{B^*}^2} + \text{regular terms at } q^2 = M_{B^*}^2,$$

where the $B^* B \pi$ coupling constant $g_{B^* B \pi}$ is defined through $\langle \pi(p_2)B^*(q)|B(p_1) \rangle = g_{B^* B \pi} \epsilon^*_\pi(q)p_{12}$. The regular terms here stand for the contribution of other resonances and continuum hadronic states. It should be noted, that both the vector $1^-$ and scalar $0^+$ resonances contribute to $f_-$ whereas only vector $1^-$ states contribute to $f_{+}$ (see e.g. [21]).

Regular terms might be taken into account by assuming a single-pole form for the form factors with a modified $q^2$-dependent ‘residue’ as follows

$$f_\pm(q^2) = \frac{\hat{f}_\pm(q^2)}{1 - \hat{q}^2},$$

where

$$\hat{f}_+ (1) = \frac{g_{B^* B \pi} f_{B^*}}{2M_{B^*}},$$

$$\hat{f}_- (1) = -\frac{\lambda M_{B^*}^2 - M_{B^*}^2}{M_{B^*}^2}.$$
Using the PCAC prescription for the pion field, the $B^*B\pi$ coupling constant can be estimated at the unphysical point $g_{B^*B\pi}(p_1^2 = M_B^2, q^2 = M_B^2, p_2^2 = 0)$. At this point the coupling constant is represented through the meson transition form factor $f_{P(M_B^2)\to V(M_B^2)}$ which can be calculated within the same dispersion approach. Namely, we find

$$\langle \pi(p_2)V(q)|P(p_1)\rangle = \lim_{p_2^2 \to 0, q^2 \to p_1^2} \frac{1}{f_\pi} e_\sigma(q)p_2^2 \left[ f(p_2^2, p_1^2, q^2) + a+(p_2^2, p_1^2, q^2)(p_1^2 - q^2) + a-(p_2^2, p_1^2, q^2)p_2^2 \right]$$

$$= \frac{1}{f_\pi} e_\sigma(q)p_2^2 f(0, M_B^2, M_B^2), \quad (12)$$

and the form factor $f(0, M_B^2, M_B^2)$ of the $B \to B^*$ transition is calculated through the spectral representation assuming identical radial wave functions of $B^*$ and $B$ mesons. In the heavy quark limit this is a rigorous property, and we expect this approximation to work well for real $B$ and $B^*$ mesons. To get to the physical point $g_{B^*B\pi}(m_\pi^2, M_B^2, M_{B^*}^2)$ one needs to perform a continuation which is not unique. However due to the small difference of the $B$ and $B^*$ meson masses we expect $g_{B^*B\pi}(m_\pi^2, M_B^2, M_{B^*}^2) \simeq g_{B^*B\pi}(0, M_B^2, M_B^2)$.

The result of the calculation of $f(0, M_B^2, M_B^2)$ is weakly sensitive to the values of the quark masses but mostly depends on the $B$ wave function. The value $f(0, M_B^2, M_B^2)$ strongly correlates with $f_B$ such that the relation

$$g_{B^*B\pi} = \frac{9 \pm 0.4 \text{ GeV}}{f_B} \quad (13)$$

is fulfilled for the range of the QM parameters which reproduce $f_B = 170 \pm 30 \text{ MeV}$. The Sum Rule analysis of the $g_{B^*B\pi}$ and references to other results can be found in [22].

Finally, the residue of the form factor $f_+$ at the $B^*$ pole takes the value

$$\hat{f}_+(1) = (0.8 \pm 0.04) f_{B^*}/f_B, \quad (14)$$

and for further numerical estimates we use $f_{B^*}/f_B = 1.2 \pm 0.1$.

For the quantities $f_\pm$ we assume a smooth parametrization

$$\hat{f}_\pm(\hat{q}^2) = \frac{f_\pm(0)}{(1 - \sigma_1^+ \hat{q}^2 + \sigma_2^+ \hat{q}^4)}, \quad (15)$$

where the coefficients $\sigma_{1,2}$ are not independent: Eq. (15) gives

$$\frac{f_+(0)}{1 - \sigma_1^+ + \sigma_2^+} = (0.8 \pm 0.04) \frac{f_{B^*}}{f_B} \quad (16)$$

and the relation (15) leads to

$$\frac{f_+(0)}{1 - \sigma_1^+ + \sigma_2^+} + \frac{f_-(0)}{1 - \sigma_1^- + \sigma_2^-} \frac{M_{B^*}^2}{M_B^2 - M_{B^*}^2} = 0. \quad (17)$$

The parameters $\sigma_{1,2}$ are determined from the $\chi^2$-fit to the results of the calculation at $q^2 \lesssim 20 \text{ GeV}^2$. Table [1] presents the relevant numbers. At $q^2 \geq 20 \text{ GeV}^2$ the parametrizations are used for extrapolation of the form factors $f_\pm$ to all kinematically accessible $q^2$ (see Fig. 1).

For the form factor $f_0(q^2) = f_+(q^2) + \alpha_2 f_-(q^2)/Pq$ a combination of PCAC and current algebra yields the relation

$$f_0(M_B^2) = f_B/f_{\pi} \quad (18)$$

Using the value $f_B = 170 \pm 30 \text{ MeV}$ we obtain

$$f_0(M_B^2) = 1.35 \pm 0.3$$

which is found to be in a reasonable agreement with the results of our extrapolating formulas.

The calculated $B \to \pi\ell\nu$ decay rate is given in Table [11]. Notice that the details of the high-$q^2$ behavior of the form factors which depend on the extrapolation procedure do not affect considerably the decay rate. The latter is mostly determined by the region $q^2 \lesssim 20 \text{ GeV}^2$ where the form factors are calculated directly.
Fig. 1 compares our results with recent light-cone sum rule calculations available at \( q^2 \lesssim 16 \text{GeV}^2 \) \cite{15} and lattice-constrained parametrizations of ref. \cite{16}. One can see that the results of different approaches to the form factors do not differ significantly. However, it should be taken into account that in the case of the \( B \to \rho \) transition this minor difference in the form factors provides rather sizeable spread of predictions for the decay rates.

Summing up, we have analyzed the form factors of the exclusive \( b \to u \) transition using the spectral representations based on constituent quark picture and obtain form factors in the whole kinematically accessible region.

The meson wave functions and the constituent quark masses have been determined by describing the results of lattice simulations of the form factors \( A_1(q^2) \) and \( T_2(q^2) \) at small recoils. This allowed us to calculate the form factors at \( q^2 \lesssim 20 \text{GeV}^2 \) which cover all kinematically accessible \( q^2 \) in the \( B \to \rho \) transition. In the \( B \to \pi \) case the interval \( q^2 \lesssim 20 \text{GeV}^2 \) does not cover the kinematically accessible region and an extrapolation to higher \( q^2 \) is necessary. To this end we take into account the dominance of the form factors at small recoils by the \( B^* \) pole. The calculated \( B \to \pi \ell \nu \) decay rate is found to be only slightly sensitive to the particular details of the extrapolation procedure.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\( m_b \) & \( m_u \) & \( \beta_B \) & \( \beta_\pi \) & \( \beta_\rho \) \\
\hline
4.85±0.03 & 0.23±0.01 & 0.54±0.04 & 0.36±0.02 & 0.31±0.03 \\
\hline
\end{tabular}
\caption{Quark masses and the slope parameters of the soft meson wave functions (in GeV).}
\end{table}
TABLE II. Parameters of the fits to the calculated $B \to \pi, \rho$ transition form factors in the form (11), (12) for $f_\pm$ and (9) for all other form factors. The numbers correspond to the central values of the QM parameters given in Table I.

|   | $f_+$ | $f_-$ | $s$ | $g$ | $f$ |
|---|---|---|---|---|---|
| $f(0)$ | 0.284 | -0.247 | 0.05 | 0.051 | 1.55 |
| $\sigma_1$ | 0.184 | 0.16 | 1.5 | 1.60 | 0.69 |
| $\sigma_2$ | -0.52 | -0.577 | 0.5 | 0.60 | 0.041 |

|   | $a_+$ | $a_-$ | $g_+$ | $g_-$ | $g$ |
|---|---|---|---|---|---|
| $f(0)$ | -0.04 | 0.044 | -0.27 | 0.25 | 0.00374 |
| $\sigma_1$ | 1.40 | 1.49 | 1.60 | 1.61 | 2.36 |
| $\sigma_2$ | 0.50 | 0.54 | 0.60 | 0.60 | 1.64 |

TABLE III. Decay rates in units $|V_{ub}|^2 \text{ps}^{-1}$.

| Ref. | $\Gamma(B \to \pi \ell \nu)$ | $\Gamma(B \to \rho \ell \nu)$ | $\Gamma_L/\Gamma_T$ |
|---|---|---|---|
| This work | $8.0^{+0.8}_{-0.2}$ | $15.8 \pm 2.3$ | $0.88 \pm 0.08$ |
| ISGW2 QM [4] | 9.6 | 14.2 | 0.3 |
| Lat [16] | $8.5^{+3.4}_{-0.9}$ | $16.5^{+3.5}_{-2.3}$ | $0.80^{+0.04}_{-0.03}$ |
| LCSR [15] | $13.5 \pm 4.0$ | | $0.52 \pm 0.08$ |

FIG. 1. The form factors of the $B \to \rho$ and $B \to \pi$ transitions through the $b \to u$ quark currents vs. lattice data [13] and calculations within different approaches. $A_1 = f/(M_B + M_\rho)$, $A_2 = -(M_B + M_\rho)a_+$, $A_0 = [q^2a_- + f + (M_B^2 - M_\rho^2)a_+]/2M_\rho$, $V = (M_B + M_\rho)g$, $T_1(q^2) = -g_+/2$, $T_2 = -1/2(g_+ + q^2g_-)/(M_B^2 - M_\rho^2))$. Solid lines - our QM results, dotted lines - lattice-constrained parametrizations of [13], dashed lines - LCSR [15].