Invisible obstacles *†

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Abstract

It is proved that one can choose a control function on an arbitrary small open subset of the boundary of an obstacle so that the total radiation from this obstacle for a fixed direction of the incident plane wave and for a fixed wave number will be as small as one wishes. The obstacle is called "invisible" in this case.

1 Introduction

Consider a bounded domain $D \subset \mathbb{R}^n$, $n = 3$, with a connected Lipschitz boundary $S$. Let $F$ be an arbitrary small, fixed, open subset on $S$, $F^\prime = S \setminus F$, and $N$ be the outer unit normal to $S$. The domain $D$ is the obstacle. Consider the scattering problem:

$$\nabla^2 u + k^2 u = 0 \text{ in } D^\prime := \mathbb{R}^3 \setminus D, \quad u = w \text{ on } F, \quad u_N + hu = 0 \text{ on } F^\prime. \quad (1)$$

Here $w$ is the function we can set up at will, the control function, $h$ is a piecewise-continuous function, $\text{Im} h \geq 0$, and $k > 0$ is a fixed constant. The function $u$ satisfies the following condition:

$$u = u_0 + v, \quad u_0 = e^{i k_0 x}, \quad (2)$$

and

$$v = \frac{e^{ikr}}{r} A(\beta, \alpha) + o\left(\frac{1}{r}\right) \quad r := |x| \to \infty, \quad \beta := \frac{x}{r}. \quad (3)$$

The function $A(\beta, \alpha)$ is called the scattering amplitude, $\alpha, \beta \in S^2$ are the unit vectors, $S^2$ is the unit sphere, $\alpha$, the direction of the incident wave $u_0$, is assumed fixed, so $A(\beta, \alpha) = A(\beta)$. Problem (1)-(3) has a unique solution ([1]).

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Define the cross section $\sigma$, or the total radiation from the obstacle, as

$$\sigma = \int_{S^2} |A(\beta)|^2 d\beta. \quad (4)$$

The problem is:

Given an arbitrary small $\epsilon > 0$, can one choose $w$ so that $\sigma < \epsilon$?

If this choice is possible, we call the obstacle "invisible" for the fixed $\alpha$ and $k$.

Our basic result is the following theorem:

**Theorem 1.** Given an arbitrary small $\epsilon > 0$ and an arbitrary small open subset $F \in S$, one can find $w \in C^\infty_0(F)$ such that $\sigma < \epsilon$. The same result holds for the boundary conditions $u|_F = w$, $u|_{F'} = 0$.

A similar problem was first posed and solved in [2], where the Neumann boundary condition was assumed and the control function was not $u$ on $F$, but $u_N$ on $F$. The boundary conditions in this paper allow one to consider impedance obstacles, so it broadens the possible applications of our theory. Inverse problems for scattering by obstacles are considered in [1] and [3].

In Section 2 proofs are given.

## 2 Proofs.

**Proof of Theorem 1.**

By Green's formula we get

$$v(x) = \int_{F'} G(x, s)(u_0N + hu_0)ds + \int_{F} G_N(x, s)vds, \quad (5)$$

where $G$ is the Green's function:

$$\nabla^2 G + k^2 G = -\delta(x - y) \quad \text{in} \quad D', \quad \lim_{|x| \to \infty} |x|(\frac{\partial G}{\partial |x|} - ikG) = 0, \quad (6)$$

and

$$G_N + hG = 0 \quad \text{on} \quad F', \quad G = 0 \quad \text{on} \quad F. \quad (7)$$

By Ramm’s lemma ([1], p.46), one gets:

$$G(x, y) = \frac{e^{ikr}}{4\pi r}\psi(y, \nu) + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} = -\nu. \quad (8)$$

Here $\psi$ is the scattering solution:

$$\nabla^2 \psi + k^2 \psi = 0 \quad \text{in} \quad D', \quad \psi_N + h\psi = 0 \quad \text{on} \quad F', \quad \psi = 0 \quad \text{on} \quad F, \quad (9)$$

and

$$\psi = e^{ik\nu \cdot x} + \eta, \quad \lim_{|x| \to \infty} |x|(\eta_r - ik\eta) = 0. \quad (10)$$
Using (4), (5) and (8), we get:

\[ A(\beta) = \frac{1}{4\pi} \int_{F'} \psi(s, -\beta)(u_{0N} + hu_0) ds + \frac{1}{4\pi} \int_F (w - u_0) \psi_N(s, -\beta) ds, \]  

(11)

and

\[ \sigma = \int_{S^2} |A_0(\beta) - A_1(\beta)|^2 d\beta, \]  

(12)

where

\[ A_0(\beta) := \frac{1}{4\pi} \int_{F'} \psi(s, -\beta)(u_{0N} + hu_0) ds - \frac{1}{4\pi} \int_F u_0 \psi_N(s, -\beta) ds, \]  

(13)

and

\[ A_1(\beta) := \frac{1}{4\pi} \int_F w(s) \psi_N(s, -\beta) ds. \]  

(14)

The conclusion of Theorem 1 follows immediately from Lemma 1.

**Lemma 1.** Given an arbitrary function \( f \in L^2(S^2) \) and an arbitrary small \( \epsilon > 0 \), one can find \( w \in C_0^\infty(F) \), such that \( ||f(\beta) - A_1(\beta)|| < \epsilon \), where \( || \cdot || := || \cdot ||_{L^2(S^2)} \).

Indeed, one can take \( f(\beta) = A_0(\beta) \) and use Lemma 1.

Let us prove Lemma 1.

If this lemma is false, then there is an \( f \in L^2(S^2), \ f \neq 0 \), such that

\[ \int_{S^2} \! d\beta f(\beta) \int_{F'} \! dsw(s) \psi_N(s, -\beta) = 0 \ \forall w \in C_0^\infty(F). \]  

(15)

This implies

\[ \int_{S^2} \! d\beta f(\beta) \psi_N(s, -\beta) = 0 \ \forall s \in F. \]  

(16)

Define the function

\[ z(x) := \int_{S^2} \! d\beta f(\beta) \psi(x, -\beta). \]  

(17)

This function solves equation

\[ \nabla^2 z + k^2 z = 0 \text{ in } D' \]

and satisfies the boundary conditions:

\[ z = z_N = 0 \text{ on } F. \]

By the uniqueness of the solution to the Cauchy problem for elliptic equations, this implies

\[ z(x) = 0 \text{ in } D'. \]  

(18)

It follows from (18) that \( f = 0 \). This contradiction proves Lemma 1 and, consequently, Theorem 1.
To complete the proof, let us derive from (18) that \( f = 0 \). The function
\[
\psi(x, \beta) = Te^{ik\beta \cdot x},
\]
where \( T \) is a linear boundedly invertible operator, acting on the \( x \) variable only (see [1]).
The specific form of \( T \) is not important for our argument. Applying the inverse operator \( T^{-1} \) to (17) and taking into account (18), one gets:
\[
\int_{S^2} d\beta f(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in D'.
\] (19)
The left-hand side in (19) is an entire function of \( x \). Therefore (19) implies
\[
\int_{S^2} d\beta f(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in \mathbb{R}^3.
\] (20)
Equation (20) means that the Fourier transform of the distribution \( f(\beta)\delta(|\xi| - k)|\xi|^2 \) equals to zero. Here \( \xi = |\xi|\beta \) is the dual to \( x \) Fourier transform variable. By the injectivity of the Fourier transform, it follows that this distribution equals to zero, so \( f = 0 \), and the proof is completed. The last statement of Theorem 1 is proved similarly.

3 Conclusion

The basic result of this note is the proof of the following statement:

*By choosing a suitable control function on an arbitrarily small open subset of the boundary of a bounded obstacle, one can make the total radiation from this obstacle, although positive, but as small as one wishes, for a fixed wave number and a fixed direction of the incident wave. Thus, the obstacle can be made practically invisible.*

References

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[3] Ramm, A.G., *Inverse Problems*, Springer, New York, 2005.