A CESÀRO AVERAGE OF GENERALISED HARDY-LITTLEWOOD NUMBERS

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Abstract. We continue our recent work on additive problems with prime summands: we already studied the average number of representations of an integer as a sum of two primes, and also considered individual integers. Furthermore, we dealt with representations of integers as sums of powers of prime numbers. In this paper, we study a Cesàro weighted partial explicit formula for generalised Hardy-Littlewood numbers (integers that can be written as a sum of a prime power and a square) thus extending and improving our earlier results.

1. Introduction

We continue our recent work on additive problems with prime summands. In [8], we studied the average number of representations of an integer as a sum of two primes, whereas in [9] we considered individual integers. In [11] we treated the case of \( p_1^\ell + p_2^\ell = n \) thus generalising and improving the result in [8]. In this paper, we study a Cesàro weighted partial explicit formula for generalised Hardy-Littlewood numbers (integers that can be written as a sum of a prime power and a square) thus extending and improving our result in [10]. Let \( \ell \geq 1 \) be an integer and

\[
 r_{\ell,2}(n) = \sum_{m_1^\ell + m_2^\ell = n} \Lambda(m_1),
\]

and also use the following abbreviations for the terms of the development:

\[
 \mathcal{M}_{1,\ell,k}(N) = \pi^{1/2} \frac{\Gamma(1/\ell)}{2\ell} N^{1/2 + 1/\ell} \frac{\Gamma(1/\ell)}{2\ell} \frac{N^{1/\ell}}{\Gamma(k + 1 + 1/\ell)},
\]

\[
 \mathcal{M}_{2,\ell,k}(N) = -\pi^{1/2} \frac{\Gamma(1/\ell)}{2\ell} \sum_\rho \frac{\Gamma(\rho/\ell)}{\Gamma(k + 3/2 + \rho/\ell)} N^{\rho/\ell + 1/2},
\]

\[
 \mathcal{M}_{3,\ell,k}(N) = \frac{1}{2\ell} \sum_\rho \frac{\Gamma(\rho/\ell)}{\Gamma(k + 1 + \rho/\ell)} N^{\rho/\ell},
\]

\[
 \mathcal{M}_{4,\ell,k}(N) = -\pi^{1/2} \log(2\pi) N^{1/2},
\]

\[
 \mathcal{M}_{5,\ell,k}(N) = \frac{N^{1-k/2 + 1/2\ell}}{\pi^{k+1/\ell}} \frac{\Gamma(1/\ell)}{\ell} \sum_{j \geq 1} \frac{J_{k+1/2+1/\ell}(2\pi j N^{1/2})}{j^{k+1/2+1/\ell}},
\]

\[
 \mathcal{M}_{6,\ell,k}(N) = -\frac{N^{1-k/2}}{\pi^k} \sum_\rho \frac{\Gamma(\rho/\ell)}{\ell} \frac{N^{\rho/(2\ell)}}{\pi^{\rho/\ell}} \sum_{j \geq 1} \frac{J_{k+1/2+\rho/\ell}(2\pi j N^{1/2})}{j^{k+1/2+\rho/\ell}},
\]

\[
\]

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Here \( \rho \) runs over the non-trivial zeros of the Riemann zeta-function \( \zeta(s) \), \( \Gamma \) is Euler’s function and \( J_v(u) \) denotes the Bessel function of complex order \( v \) and real argument \( u \). The main result of the paper is the following theorem.

**Theorem 1.** Let \( \ell \geq 1 \) be an integer and \( N \) be a sufficiently large integer. For \( k > 1 \) we have

\[
\sum_{n \leq N} r_{\ell,2}(n) \frac{(1-n/N)^k}{\Gamma(k+1)} = \sum_{j=1}^7 M_{j,\ell,k}(N) + O(k^j(1).
\]

Clearly, depending on the size of \( \ell \), some of the previously listed terms should be included in the error term. Theorem \( \boxed{11} \) generalises and improves our Theorem 1 in \( \boxed{10} \) in which the error term should be read as \( O_k(N^{1/2}) \), see Theorem 2.3 of \( \boxed{7} \), which corresponds to the case \( \ell = 1 \). In fact, in this case we are now able to detect the terms \( M_{4,1,k} \) and \( M_{7,1,k} \). Moreover Theorem \( \boxed{11} \) covers another interesting and classical case like the sum of a prime square and a square (\( \ell = 2 \)).

As in our previous papers on this subject, the method we will use in this additive problem is based on a formula due to Laplace \( \boxed{12} \), namely

\[
\frac{1}{2\pi i} \int_{(\alpha)} v^{-s} e^{\nu v} \, dv = \frac{1}{\Gamma(s)}, \tag{2}
\]

where \( \Re(s) > 0 \) and \( a > 0 \); see Formula 5.4(1) on page 238 of \( \boxed{3} \). In fact, we will need the general case of \( \boxed{2} \), which can be found in de Azevedo Pribitkin \( \boxed{11} \), formulae (8) and (9):

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^s} \, du = \begin{cases} 
D^{s-1} e^{-aD}/\Gamma(s) & \text{if } D > 0, \\
0 & \text{if } D < 0,
\end{cases} \tag{3}
\]

which is valid for \( \sigma = \Re(s) > 0 \) and \( a \in \mathbb{C} \) with \( \Re(a) > 0 \), and

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(a+iu)^s} \, du = \begin{cases} 
0 & \text{if } \Re(s) > 1, \\
1/2 & \text{if } s = 1,
\end{cases} \tag{4}
\]

for \( a \in \mathbb{C} \) with \( \Re(a) > 0 \). Formulae \( \boxed{3} \)-\( \boxed{4} \) enable us to write averages of arithmetical functions by means of line integrals as we will see in \( \Box \boxed{2} \) below.

We will also need Bessel functions of complex order \( v \) and real argument \( u \). For their definition and main properties we refer to Watson \( \boxed{15} \). In particular, equation (8) on page 177 gives the Sonine representation:

\[
J_v(u) := \frac{(u/2)^v}{2\pi i} \int_{(\alpha)} s^{-v-1} e^{s} e^{-u^2/4s} \, ds, \tag{5}
\]

where \( a > 0 \) and \( u, v \in \mathbb{C} \) with \( \Re(v) > -1 \). We will also use the Poisson integral formula

\[
J_v(u) := \frac{2(u/2)^v}{\pi^{1/2} \Gamma(v + 1/2)} \int_{0}^{1} (1 - t^2)^{v-1/2} \cos(ut) \, dt, \tag{6}
\]

which holds for \( \Re(v) > -1/2 \) and \( u \in \mathbb{C} \). (See eq. (3) on page 48 of \( \boxed{15} \).) An asymptotic estimate we will need is

\[
J_v(u) = \left( \frac{2}{\pi u} \right)^{1/2} \cos(u - \frac{\pi v}{2} - \frac{\pi}{4}) + O(|u|^{-5/2}), \tag{7}
\]

which follows from eq. (1) on page 199 of Watson \( \boxed{15} \).
As in [11], we combine this approach with line integrals with the classical methods dealing with infinite sums over primes, exploited by Hardy and Littlewood (see [5] and [6]) and by Linnik [13]. The main difficulty here is, as in [10], that the problem naturally involves the modular relation for the complex theta function. (See eq. (9).) The presence of the Bessel functions in our statement strictly depends on such modularity relation. It is worth mentioning that it is not clear how to get such “modular” terms using the finite sums approach for the function $r_{\ell,2}(n)$. The previously mentioned improvement we get in Theorem 1 follows by using Lemma 3 below (which is proved in [11]).

2. Settings

We need $k > 0$ in this section. Let $z = a + iy$ with $a > 0$,

$$\tilde{S}_\ell(z) = \sum_{m \geq 1} \Lambda(m)e^{-m^2z} \quad \text{and} \quad \omega_2(z) = \sum_{m \geq 1} e^{-m^2z}. \quad (8)$$

Letting further $\theta(z) = \sum_{m=\infty}^{+\infty} e^{-m^2z}$, we notice that $\theta(z) = 1 + 2\omega_2(z)$ and, recalling the functional equation for $\theta$ (see, e.g., Proposition VI.4.3 of Freitag and Busam [4, page 340]):

$$\theta(z) = \left(\frac{\pi}{z}\right)^{1/2}\theta\left(\frac{\pi^2}{z}\right) \quad \text{for } \Re(z) > 0, \quad (9)$$

we immediately get

$$\omega_2(z) = \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right) \quad \text{for } \Re(z) > 0. \quad (10)$$

Recalling (9), we can write

$$\tilde{S}_\ell(z)\omega_2(z) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \Lambda(m_1)e^{-(m_1^2+m_2^2)z} = \sum_{n \geq 1} r_{\ell,2}(n)e^{-nz}$$

and, by (3)-(4), we see that

$$\sum_{n \leq N} r_{\ell,2}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \sum_{n \geq 1} r_{\ell,2}(n) \left(\frac{1}{2\pi i} \int_{(a)} e^{(N-n)z}z^{-k-1} \, dz\right). \quad (11)$$

Our first goal is to exchange the series with the line integral in (11). To do so we have to recall that the Prime Number Theorem (PNT) is equivalent to the statement

$$\tilde{S}_\ell(a) \sim \frac{\Gamma(1/\ell)}{\ell a^{1/\ell}} \quad \text{for } a \to 0+. \quad (12)$$

We will also use the inequality

$$|\omega_2(z)| \leq \omega_2(a) \leq \int_0^\infty e^{-at^2} \, dt \leq a^{-1/2} \int_0^\infty e^{-v^2} \, dv \ll a^{-1/2}, \quad (12)$$

from which we immediately get

$$\sum_{n \geq 1} |r_{\ell,2}(n)e^{-nz}| = \sum_{n \geq 1} r_{\ell,2}(n)e^{-na} = \tilde{S}_\ell(a)\omega_2(a) \ll_{\ell} a^{-1/\ell-1/2}. \quad (13)$$

Taking into account the estimates

$$|z|^{-1} = \begin{cases} a^{-1} & \text{if } |y| \leq a, \\ |y|^{-1} & \text{if } |y| \geq a, \end{cases} \quad (13)$$
where \( f \sim g \) means \( g \ll f \ll g \), and

\[
|e^{Nz} z^{-k-1}| \leq e^{N \lambda a} \begin{cases} \lambda^{-k-1} & \text{if } |y| \leq a, \\ |y|^{-k-1} & \text{if } |y| \geq a, \end{cases}
\]

we have

\[
\int_{(a)} |e^{Nz} z^{-k-1}| |\widetilde{S}_\ell(z)\omega_2(z)| \, |dz| \ll \ell \lambda^{-1/2} e^{N \lambda a} \left( \int_{-a}^{a} \lambda^{-k-1} \, dy + 2 \int_{a}^{+\infty} y^{-k-1} \, dy \right)
\]

\[
\ll \ell \lambda^{-1/2} e^{N \lambda a} \left( \lambda^{-k} + \lambda^{-k} \right).
\]

The last estimate is valid only if \( k > 0 \). So, for \( k > 0 \), we can exchange the line integral with the sum over \( n \) in (13), thus getting

\[
\sum_{n \leq N} r_{\ell, n}(N-n)^k \frac{1}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \widetilde{S}_\ell(z)\omega_2(z) \, dz.
\]  

This is the fundamental relation for the method.

### 3. Inserting Zeros and Modularity

We need \( k > 1/2 \) in this section. The treatment of the integral at the right hand side of (14) requires Lemma 1. We split \( \widetilde{S}_\ell(z) \) according to its statement as \( \widetilde{S}_\ell(z) = E(a, y, \ell) \) where \( E \) satisfies the bound in (10) and

\[
S_\ell(z) := \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\left(\frac{\rho}{\ell}\right) - \log(2\pi),
\]

where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \). Formula (14) becomes

\[
\sum_{n \leq N} r_{\ell, n}(N-n)^k \frac{1}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} S_\ell(z)\omega_2(z)e^{Nz} z^{-k-1} \, dz
\]

\[
+ \Theta \left( \int_{(a)} |E(a, y, \ell)| |e^{Nz}| |z|^{-k-1} |\omega_2(z)| |dz| \right).
\]

Using (12)-(13) and (10), we see that the error term is

\[
\ll \ell \lambda^{-1/2} e^{N \lambda a} \left( \int_{-a}^{a} \lambda^{-k-1} \, dy + \int_{a}^{+\infty} y^{-k-1/2} \log^2(y/a) \, dy \right)
\]

\[
\ll k, \ell \, e^{N \lambda a} \lambda^{-k} \left( 1 + \int_{1}^{+\infty} y^{-k-1/2} \log^2 y \, dy \right) \ll_{k, \ell} e^{N \lambda a} \lambda^{-k},
\]

provided that \( k > 1/2 \). Choosing \( a = 1/N \), the previous estimate becomes \( \ll_{k, \ell} N^k \). Summing up, for \( k > 1/2 \), we can write

\[
\sum_{n \leq N} r_{\ell, n}(N-n)^k \frac{1}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} S_\ell(z)\omega_2(z)e^{Nz} z^{-k-1} \, dz + \Theta_{k, \ell}(N^k).
\]  

We now insert (10) into (16), so that the integral on the right-hand side of (16) becomes

\[
\frac{1}{2\pi i} \int_{(a)} S_\ell(z) \left( \frac{1}{2} \left( \frac{\pi}{z} \right)^{1/2} - \frac{1}{2} \right) e^{Nz} z^{-k-1} \, dz + \frac{1}{2\pi i} \int_{(a)} \left( \frac{\pi}{z} \right)^{1/2} S_\ell(z)\omega_2 \left( \frac{\pi^2}{z} \right) e^{Nz} z^{-k-1} \, dz
\]

\[
= \mathcal{F}_1 + \mathcal{F}_2,
\]  

(17)
say. We now proceed to evaluate \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \).

4. Evaluation of \( \mathcal{J}_1 \)

We need \( k > 1/2 \) in this section. By a direct computation we can write that

\[
\mathcal{J}_1 = \frac{1}{4\pi i} \frac{\Gamma(1/\ell)}{\ell} \int_{(\phi)} \left( \frac{\pi^{1/2}}{z^{1/2} - 1} \right) e^{Nz/z^{1/2} - 1} \, dz - \frac{\pi^{1/2}}{4\ell \pi i} \int_{(\phi)} \sum_{\rho} \Gamma\left( \frac{\rho}{\ell} \right) e^{Nz/z^{1/2} - 2} \, dz
\]

\[
+ \frac{1}{4\ell \pi i} \sum_{\rho} \Gamma\left( \frac{\rho}{\ell} \right) e^{Nz/z^{1/2} - 2} \, dz - \frac{\log(2\pi)}{4\pi i} \int_{(\phi)} \left( \frac{\pi^{1/2}}{z^{1/2} - 1} \right) e^{Nz/z^{1/2} - 1} \, dz
\]

\[
= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4,
\]

say. We see now how to evaluate \( \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \) and \( \mathcal{J}_4 \).

4.1. Evaluation of \( \mathcal{J}_1 \). Using the substitution \( s = Nz \), by (2) we immediately have

\[
\mathcal{J}_1 = \frac{\pi^{1/2}}{2} \frac{\Gamma(1/\ell)}{\ell} N^{k+1/2+1/\ell} \frac{1}{2\pi i} \int_{(1)} e^{s}s^{k-3/2-1/\ell} ds - \frac{\Gamma(1/\ell)}{2 \ell} \frac{1}{2\pi i} \int_{(1)} e^{s}s^{k-1-1/\ell} ds
\]

\[
= \frac{\pi^{1/2}}{2} \frac{\Gamma(1/\ell)}{\ell} \frac{N^{k+1/2+1/\ell}}{\Gamma(k + 3/2 + 1/\ell)} - \frac{\Gamma(1/\ell)}{2 \ell} \frac{N^{k+1/\ell}}{\Gamma(k + 1 + 1/\ell)}.
\]

(18)

4.2. Evaluation of \( \mathcal{J}_2 \). Exchanging the sum over \( \rho \) with the integral (this can be done for \( k > 0 \); see \( \S \)) and using the substitution \( s = Nz \), we have

\[
\mathcal{J}_2 = -\frac{\pi^{1/2}}{2 \ell} \sum_{\rho} \Gamma\left( \frac{\rho}{\ell} \right) \frac{1}{2\pi i} \int_{(\phi)} e^{Nz/z^{1/2} - 2} \, dz
\]

\[
= -\frac{\pi^{1/2}}{2 \ell} \sum_{\rho} \Gamma\left( \frac{\rho}{\ell} \right) N^{k+1/2+1/\ell} \frac{1}{2\pi i} \int_{(1)} e^{s}s^{k-3/2-1/\ell} ds
\]

\[
= -\frac{\pi^{1/2}}{2 \ell} \sum_{\rho} \frac{\Gamma(\rho/\ell)}{\Gamma(k + 3/2 + \rho/\ell)} N^{k+1/\ell+1/2},
\]

(19)

again by (2). By the Stirling formula (29), we remark that the series in \( \mathcal{J}_2 \) converges absolutely for \( k > -1/2 \).

4.3. Evaluation of \( \mathcal{J}_3 \). Arguing as in \( \S \) with \( -k - 1 \) which plays the role of \( -k - 3/2 \) there, we see that we can exchange the sum with the integral provided that \( k > 1/2 \). Hence, performing again the usual substitution \( s = Nz \), we can write

\[
\mathcal{J}_3 = \frac{1}{2 \ell} \sum_{\rho} \Gamma\left( \frac{\rho}{\ell} \right) N^{k+1/\ell} \frac{1}{2\pi i} \int_{(1)} e^{s}s^{k-1-1/\ell} ds = \frac{1}{2 \ell} \sum_{\rho} \frac{\Gamma(\rho/\ell)}{\Gamma(k + 1 + \rho/\ell)} N^{k+1/\ell}.
\]

(20)

By the Stirling formula (29), we remark that the series in \( \mathcal{J}_3 \) converges absolutely for \( k > 0 \).

4.4. Evaluation of \( \mathcal{J}_4 \). Performing again the usual substitution \( s = Nz \), we can write

\[
\mathcal{J}_4 = -\frac{\pi^{1/2} \log(2\pi)}{4\pi i} N^{k+1/2} \int_{(1)} e^{s}s^{k-3/2} ds + \frac{\log(2\pi)}{4\pi i} N^{k} \int_{(1)} e^{s}s^{k-1} ds
\]

\[
= -\frac{\pi^{1/2} \log(2\pi)}{2\Gamma(k + 3/2)} N^{k+1/2} + \frac{\log(2\pi)}{2\Gamma(k + 1)} N^{k}.
\]

(21)
5. Evaluation of $J_2$ and Conclusion of the Proof of Theorem I

We need $k > 1$ in this section. Using the definition of $\omega_2(\pi^2/z)$ (see (8)) we have

\[
J_2 = \frac{1}{2\pi i} \Gamma(1/\ell) \int_{\mathbb{R}} \left( \frac{\pi}{\ell} \right)^{1/2} \left( \sum_{j \geq 1} e^{-j^2\pi^2/\ell} \right) e^{Nz^{-k-1}/\ell} dz
\]

\[
- \frac{1}{2\ell\pi i} \int_{\mathbb{R}} \left( \frac{\pi}{\ell} \right)^{1/2} \left( \sum_{j \geq 1} e^{-j^2\pi^2/\ell} \right) \left( \sum_{\rho} e^{-\rho j/\ell} \right) e^{Nz^{-k-1}/\ell} dz
\]

\[
+ \frac{\log(2\pi)}{2\pi i} \int_{\mathbb{R}} \left( \frac{\pi}{\ell} \right)^{1/2} \left( \sum_{j \geq 1} e^{-j^2\pi^2/\ell} \right) e^{Nz^{-k-1}} dz = J_5 + J_6 + J_7, \tag{22}
\]

say. We see now how to evaluate $J_5$, $J_6$ and $J_7$.

5.1. Evaluation of $J_5$. By means of the substitution $s = Nz$, since the exchange is justified in §8 for $k > 1/2 - 1/\ell$, we get

\[
J_5 = \pi^{1/2} \Gamma(1/\ell) N^{k+1/2+1/\ell} \sum_{j \geq 1} \frac{1}{2\pi i} \int_{(1)} e^{s} e^{-j^2\pi^2 N/s} s^{-k-3/2-1/\ell} ds.
\]

Setting $u = 2\pi j N^{1/2}$ in (5), we obtain

\[
J_5 = \left( \frac{\pi j N^{1/2}}{2\pi i} \right)^{\nu} \int_{(1)} e^{s} e^{-j^2\pi^2 N/s} s^{-\nu-1} ds, \tag{23}
\]

and hence we have

\[
J_5 = \frac{N^{k+1/2+1/\ell} \Gamma(1/\ell)}{\pi^{k+1/\ell}} \sum_{j \geq 1} \frac{J_{k+1/2+1/\ell}(2\pi j N^{1/2})}{j^{k+1/2+1/\ell}}. \tag{24}
\]

The absolute convergence of the series in $J_5$ is studied in §10.

5.2. Evaluation of $J_6$. With the same substitution used before, since the double exchange between sums and the line integral is justified in §9 for $k > 1$, we see that

\[
J_6 = \pi^{1/2} \sum_{\rho} \Gamma(\rho/\ell) N^{k+1/2+\rho/\ell} \sum_{j \geq 1} \frac{1}{2\pi i} \int_{(1)} e^{s} e^{-j^2\pi^2 N/s} s^{-k-3-2\rho-1/\ell} ds.
\]

Using (23), we get

\[
J_6 = \frac{N^{k+1/2+1/4}}{\pi^{k}} \sum_{\rho} \Gamma(\rho/\ell) N^{\rho/2(2\ell)} \sum_{j \geq 1} \frac{J_{k+1/2+\rho/\ell}(2\pi j N^{1/2})}{j^{k+1/2+\rho/\ell}}. \tag{25}
\]

In this case, the absolute convergence of the series in $J_6$ is more delicate; such a treatment is again described in §10.

5.3. Evaluation of $J_7$. With the same substitution used before, since the exchange between sum and the line integral is justified in §8 for $k > 1/2$, we see that

\[
J_7 = \pi^{1/2} \log(2\pi) N^{k+1/2} \sum_{j \geq 1} \frac{1}{2\pi i} \int_{(1)} e^{s} e^{-j^2\pi^2 N/s} s^{-k-3/2} ds
\]

Using (23), we get

\[
J_7 = \frac{\log(2\pi)}{\pi^{k}} N^{k+1/2+1/4} \sum_{j \geq 1} \frac{J_{k+1/2}(2\pi j N^{1/2})}{j^{k+1/2}}. \tag{26}
\]
The absolute convergence of the series in \( J \) is studied in \( \S 10 \).

Finally, inserting \((18)\)–\((20)\) into \((17)\) and \((16)\), we obtain

\[
\sum_{n \leq N} r_{\ell}z(n) \frac{(N-n)^k}{\Gamma(k+1)} = N^k \sum_{j=1}^{7} M_{j,\ell,k}(N) + O_{k,\ell}(N^k),
\]

for \( k > 1 \). Theorem 1 follows dividing \((27)\) by \( N^k \).

6. Lemmas

We recall some basic facts in complex analysis. First, if \( z = a + iy \) with \( a > 0 \), we see that for complex \( w \) we have

\[
z^{-w} = |z|^{-w} \exp(-iw \arctan(y/a))
\]

so that

\[
|z^{-w}| = |z|^{-\Re(w)} \exp(\Im(w) \arctan(y/a)).
\]

We also recall that, uniformly for \( x \in [x_1, x_2] \), with \( x_1 \) and \( x_2 \) fixed, and for \( |y| \to +\infty \), by the Stirling formula we have

\[
|\Gamma(x + iy)| \sim \sqrt{2\pi} e^{-\pi |y|/2} |y|^{x-1/2};
\]

see, e.g., Titchmarsh [14, §4.42].

We will need the following lemmas from Languasco and Zaccagnini [11].

**Lemma 1** (See Lemma 1 of [11]). Let \( \ell \geq 1 \) be an integer, \( z = a + iy \), where \( a > 0 \) and \( y \in \mathbb{R} \) and let \( \mathcal{S}_{\ell}(z) \) be defined as in \((15)\). Then \( \mathcal{S}_{\ell}(z) = \mathcal{S}_{\ell}(c) + E(a, y, \ell) \) where

\[
E(a, y, \ell) \ll |z|^{1/2} \left\{ \begin{array}{ll}
1 & \text{if } |y| \leq a \\
1 + \log^2(|y|/a) & \text{if } |y| > a.
\end{array} \right.
\]

**Lemma 2** (See Lemma 2 of [11]). Let \( \ell \geq 1 \) be an integer, let \( \beta + iy \) run over the non-trivial zeros of the Riemann zeta-function and \( \alpha > 1 \) be a parameter. The series

\[
\sum_{\rho: \gamma > 0} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \int_{1}^{+\infty} \exp\left( -\frac{\gamma}{\ell} \arctan \frac{1}{u} \right) \frac{du}{u^{\alpha + \beta/\ell}}
\]

converges provided that \( \alpha > 3/2 \). For \( \alpha \leq 3/2 \) the series does not converge. The result remains true if we insert in the integral a factor \((\log u)^c\), for any fixed \( c \geq 0 \).

**Lemma 3** (See Lemma 3 of [11]). Let \( \ell \geq 1 \) be an integer, \( \alpha > 1 \), \( z = a + iy, a \in (0, 1) \) and \( y \in \mathbb{R} \). Let further \( \rho = \beta + iy \) run over the non-trivial zeros of the Riemann zeta-function. We have

\[
\sum_{\rho} \left( \frac{\gamma}{\ell} \right)^{\beta/\ell - 1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\left( \frac{\gamma}{\ell} \arctan \frac{y}{a} - \frac{\pi}{2} \frac{y}{\ell} \right) \frac{dy}{\gamma|z|^{\alpha + \beta/\ell}} \ll_{a,\ell} a^{-1-a-1/\ell},
\]

where \( \mathbb{Y}_1 = \{ y \in \mathbb{R}: yy \leq 0 \} \) and \( \mathbb{Y}_2 = \{ y \in [-a,a]: yy > 0 \} \). The result remains true if we insert in the integral a factor \((\log(|y|/a))^c\), for any fixed \( c \geq 0 \).
7. Interchange of the series over zeros with the line integral in $J_2, J_3$

We need $k > 1/2$ in this section. For $J_2$ we have to establish the convergence of

$$
\sum_\rho |\Gamma(\frac{\rho}{\ell})| \int_{\mathbb{R}} |e^{Nz}| |z|^{-k-3/2} |z^{-\rho/\ell}| \, |dz|,
$$

(31)

where, as usual, $\rho = \beta + iy$ runs over the non-trivial zeros of the Riemann zeta-function. By (28) and the Stirling formula (29), we are left with estimating

$$
\sum_\rho |\gamma|^{\beta_*-1/2} \int_{\mathbb{R}} \exp\left(\gamma \arctan(Ny) - \frac{\pi |\gamma|}{2|\ell|} \right) \frac{dy}{|z|^{k+3/2+\beta_*/\ell}}.
$$

(32)

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_k N^{k+1/2+1/\ell}$ by Lemma 3 with $\alpha = k + 3/2$ and $\alpha = 1/N$. By symmetry, we may assume that $\gamma > 0$. We have that the integral in (32) is

$$
\ll \ell \sum_\rho |\gamma|^{\beta_*-1/2} \int_{1/N}^{+\infty} \exp\left(-\gamma \arctan\left(\frac{1}{Ny}\right)\right) \frac{dy}{y^{k+3/2+\beta_*/\ell}}.
$$

(33)

For $k > 0$, this is $\ll_k \ell N^{k+1/2+1/\ell}$ by Lemma 2. This implies that the integrals in (32) and in (31) are both $\ll_k \ell N^{k+1/2+1/\ell}$, and hence this exchange step for $J_2$ is fully justified.

For $J_3$, we have to consider

$$
\sum_\rho |\Gamma(\frac{\rho}{\ell})| \int_{\mathbb{R}} |e^{Nz}| |z|^{-k-1} |z^{-\rho/\ell}| \, |dz|.
$$

(34)

We can repeat the same reasoning we used for $J_2$ just replacing $k + 3/2$ with $k + 1$. This means that we need $k > 1/2$ here to get that the integral in (34) is $\ll_k \ell N^{k+1/\ell}$, and that this exchange step for $J_3$ is fully justified too.

8. Interchange of the series over $j$ with the line integral in $J_5, J_7$

We need $k > 1/2$ in this section. For $J_5$ we have to establish the convergence of

$$
\sum_{j \geq 1} \int_{\mathbb{R}} |e^{Nz}| |z|^{-k-3/2-1/\ell} e^{-j^2 \beta z^2} \Re(1/z) \, |dz|.
$$

(35)

A trivial computation gives

$$
\Re(1/z) = \frac{N}{1 + N^2 y^2} \gg \begin{cases} N & \text{if } |y| \leq 1/N, \\ 1/(Ny^2) & \text{if } |y| > 1/N. \end{cases}
$$

(36)

By (35), we can write that the quantity in (34) is

$$
\ll \ell \sum_{j \geq 1} \int_0^{1/N} e^{-j^2 N/|z|^{k+3/2+1/\ell}} \, dy + \sum_{j \geq 1} \int_{1/N}^{+\infty} e^{-j^2/|N y^2|} \, dy = U_1 + U_2,
$$

(37)

say, since the $\pi^2$ factor in the exponential function is negligible. Using (112)-(113), we have

$$
U_1 \ll \ell N^{k+1/2+1/\ell} \omega_2(N) \ll \ell N^{k+1/\ell}
$$

(38)
and

\[ U_2 \ll_{\ell} \sum_{j \geq 1} \int_{1/N}^{+\infty} \frac{e^{-j^2/(Ny^2)}}{y^{k+3/2}} \, dy \ll_{\ell} N^{k/2+1/4+1/(2\ell)} \sum_{j \geq 1} \frac{1}{j^{k+1/2+1/\ell}} \int_{0}^{j^2 N} u^{k/2-3/4+1/(2\ell)} e^{-u} \, du \]

\[ \leq \Gamma\left(\frac{2k + 1 + 2\ell}{4}\right) N^{k/2+1/4+1/(2\ell)} \sum_{j \geq 1} \frac{1}{j^{k+1/2+1/\ell}} \ll_{k,\ell} N^{k/2+1/4+1/(2\ell)}, \tag{38} \]

provided that \( k > 1/2 - 1/\ell \), where we used the substitution \( u = j^2/(Ny^2) \). Inserting (37)–(38) into (36) we get, for \( k > 1/2 - 1/\ell \), that the quantity in (34) is \( \ll N^{k+1/\ell} \) and so it is for \( f_5 \).

For \( f_7 \) we have to establish the convergence of

\[ \sum_{j \geq 1} \int_{1/N}^{+\infty} |e^{Nz}| |z|^{-k-3/2} e^{-\pi j^2 N (1/z)} |dz|. \tag{39} \]

We can repeat the same reasoning we used for \( f_5 \) just replacing \( k + 3/2 + 1/\ell \) with \( k + 3/2 \). This means that we need \( k > 1/2 \) here to get that the integral in (39) is \( \ll_{k,\ell} N^k \), and that this exchange step for \( f_7 \) is fully justified too.

9. Interchange of series with the line integral in \( f_6 \)

We need \( k > 1 \) in this section. We first have to establish the convergence of

\[ \sum_{\rho} \sum_{j \geq 1} \int_{1/N}^{+\infty} \left| \Gamma\left(\frac{\rho}{\ell}\right) z^{-\rho/\ell} \right| e^{Nz} |z|^{-k-3/2} e^{-\pi j^2 N (1/z)} |dz|. \tag{40} \]

Using the Prime Number Theorem and (30), we first remark that

\[ \left| \sum_{\rho} \Gamma\left(\frac{\rho}{\ell}\right) z^{-\rho/\ell} \right| \ll_{\ell} N^{1/\ell} + |z|^{1/2} \log(2N|y|). \tag{41} \]

By (35) and (41), we can write that the quantity in (40) is

\[ \ll_{\ell} N^{1/\ell} \sum_{j \geq 1} \int_{1/N}^{1/|z|^{k+3/2}} e^{-j^2 N} \, dy + N^{1/\ell} \sum_{j \geq 1} \int_{|z|^{k+3/2}}^{+\infty} \frac{e^{-j^2 N}}{|z|^{k+3/2}} \, dy \]

\[ + \sum_{j \geq 1} \int_{1/N}^{+\infty} \log(2N|y|) \frac{e^{-j^2 N}}{|z|^{k+1}} \, dy = V_1 + V_2 + V_3, \tag{42} \]

say. \( V_1 \) can be estimated exactly as \( U_1 \) in Section 8 and we get \( V_1 \ll_{k,\ell} N^{k+1/\ell} \). For \( V_2 \) we can work analogously to \( U_2 \) thus obtaining

\[ V_2 \ll_{k,\ell} N^{1/\ell} \sum_{j \geq 1} \int_{1/N}^{+\infty} \frac{e^{-j^2 N}}{y^{k+3/2}} \, dy \ll_{k,\ell} N^{k/2+1/4+1/\ell} \sum_{j \geq 1} \frac{1}{j^{k+1/2}} \int_{0}^{j^2 N} u^{k/2-3/4} e^{-u} \, du \]

\[ \ll_{k,\ell} \Gamma\left(\frac{2k + 1 + 2\ell}{4}\right) N^{k/2+1/4+1/\ell} \sum_{j \geq 1} \frac{1}{j^{k+1/2}} \ll_{k,\ell} N^{k/2+1/4+1/\ell}, \]

provided that \( k > 1/2 \), where we used the substitution \( u = j^2/(Ny^2) \). Hence, we have

\[ V_1 + V_2 \ll_{k,\ell} N^{k+1/\ell}, \tag{43} \]

provided that \( k > 1/2 \).
Using the substitution \( u = j^2/(Ny^2) \), we obtain
\[
V_3 \ll_{k,\ell} \sum_{j \geq 1} \int_{1/N}^{+\infty} \log^2(2Ny) \frac{e^{-j^2/(Ny^2)}}{y^{k+1}} dy = \frac{N^{k/2}}{8} \sum_{j \geq 1} \frac{1}{j} \int_0^{j^2N} u^{k/2-1} \log^2(4j^2N/u) e^{-u} \, du.
\]
Hence, a direct computation shows that
\[
V_3 \ll_{k,\ell} N^{k/2} \sum_{j \geq 1} \frac{\log^2(jN)}{j^k} \int_0^{j^2N} u^{k/2-1} e^{-u} \, du + N^{k/2} \sum_{j \geq 1} \frac{1}{j} \int_0^{j^2N} u^{k/2-1} \log^2(u) e^{-u} \, du
\]
\[
\ll_{k,\ell} \Gamma(k/2)N^{k/2} \sum_{j \geq 1} \frac{\log^2(\ell N)}{j^k} + N^{k/2} \ll_{k,\ell} N^{k/2} \log^2 N \tag{44}
\]
provided that \( k > 1 \). Inserting (43)–(44) into (42) we get, for \( k > 1 \), that the quantity in (40) is \( \ll_{k,\ell} N^{k+1/\ell} \).

Now we have to establish the convergence of
\[
\sum_{j \geq 1} \sum_{\rho} \left| \Gamma\left( \frac{\rho}{\ell} \right) \right| \int_{1/N}^1 \left| e^{Nz} \right| |z|^{-k-3/2} |z^{\rho/\ell}| e^{-\pi j^2 \mathbb{R}(1/2)} |dz|. \tag{45}
\]
By symmetry, we may assume that \( \gamma > 0 \). For \( y \in (-\infty, 0] \) we have \( \gamma \arctan(y/a) - \frac{\pi}{2} \gamma \leq -\frac{\pi}{2} \gamma \). Using (15), (13) and the Stirling formula (29), the quantity we are estimating becomes
\[
\ll \sum_{j \geq 1} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 2 \ell \right) \int_{-1/N}^{-1} N^{k+3/2+\beta/\ell} e^{-j^2N} dy + \int_{-\infty}^{-1/N} e^{-j^2/(Ny^2)} |y|^{k+3/2+\beta/\ell} \, dy
\]
\[
\ll_{k,\ell} N^{k+1/2+1/\ell} \sum_{j \geq 1} e^{-j^2N} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 2 \ell \right)
\]
\[
+ N^{k+1/2+1/4} \sum_{j \geq 1} \frac{1}{j^{k+1/2}} \sum_{\rho: \gamma > 0} N^{\beta/(2\ell)} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 2 \ell \right) \int_0^{j^2N} u^{k/2-3/4+\beta/(2\ell)} e^{-u} \, du
\]
\[
\ll_{k,\ell} N^{k+1/\ell} + \left( \max_{0 \leq b \leq 1} \Gamma\left( \frac{b}{2\ell} + \frac{k}{4} + \frac{1}{4} \right) \right) N^{k+1/4} \sum_{j \geq 1} \frac{1}{j^{k+1/2}} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 2 \ell \right)
\]
\[
\ll_{k,\ell} N^{k+1/\ell}, \tag{46}
\]
provided that \( k > 1/2 \), where we used the substitution \( u = j^2/(Ny^2) \), (12) and standard density estimates.

Let now \( y > 0 \). Using the Stirling formula (29) and (15), we can write that the quantity in (45) is
\[
\ll_{k,\ell} \sum_{j \geq 1} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 4 \ell \right) \int_0^{1/N} e^{-j^2N} \frac{1}{|z|^{k+3/2+\beta/\ell}} dy
\]
\[
+ \sum_{j \geq 1} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \int_{1/N}^{+\infty} \exp\left( \frac{\gamma}{\ell} (\arctan(Ny) - \frac{\pi}{2}) \right) e^{-j^2/(Ny^2)} \, dy = W_1 + W_2, \tag{47}
\]
say. Using (13) and (12), we have that
\[
W_1 \ll_{k,\ell} N^{k+1/2+1/\ell} \sum_{j \geq 1} e^{-j^2N} \sum_{\rho: \gamma > 0} \left( \gamma \right)^{\beta/\ell-1/2} \exp\left( -\pi \gamma 4 \ell \right) \ll_{k,\ell} N^{k+1/\ell}, \tag{48}
\]
by standard density estimates. Moreover, we get
\[
W_2 \ll_{k, \ell} \sum_{j \geq 1} \sum_{\rho : \gamma > 0} \int_{1/N}^{\infty} y^{-k + 3/2 - \beta} \exp\left(-\frac{\gamma}{\ell N y} - \frac{j^2}{N y^2}\right) dy
\]

\[
\ll_{k, \ell} N^{k/2 + 1/4} \sum_{j \geq 1} \frac{1}{j^{k-1/2}} \sum_{\rho : \gamma > 0} \frac{N^{\beta/(2 \ell)} y^{\beta/(\ell - 1/2)}}{y^{\beta/2}} \int_0^{j N \sqrt{\gamma}} v^{k-1/2 + \beta} \exp\left(-\frac{\gamma}{\ell j N \sqrt{\gamma}} - v^2\right) dv,
\]
in which we used the substitution \(v^2 = j^2/(N y^2)\). We remark that, for \(k > 1\), we can set \(\varepsilon = \varepsilon(k) = (k - 1)/2 > 0\) and that \(k - \varepsilon = (k + 1)/2 > 1\). We further remark that \(\max_{\nu}(v^k \varepsilon^{-v^2})\) is attained at \(v_0 = ((k - \varepsilon)/2)^{1/2}\), and hence we obtain, for \(N\) sufficiently large, that
\[
W_2 \ll_{k, \ell} N^{k/2 + 1/4} \sum_{j \geq 1} \frac{1}{j^{k-1/2}} \sum_{\rho : \gamma > 0} \frac{N^{\beta/(2 \ell)} y^{\beta/(\ell - 1/2)}}{y^{\beta/2}} \int_0^{j N \sqrt{\gamma}} v^{k-1/2 + \varepsilon} \exp\left(-\frac{\gamma}{\ell j N \sqrt{\gamma}} - v^2\right) dv.
\]

Making the substitution \(u = \gamma v/(j \sqrt{\gamma})\), we have
\[
W_2 \ll_{k, \ell} N^{k/2 + 1/2 + \varepsilon} \sum_{j \geq 1} \frac{1}{j^{k-1/2}} \sum_{\rho : \gamma > 0} \frac{N^{\beta/(2 \ell)} y^{\beta/(\ell - 1/2)}}{y^{\beta/2}} \int_0^{\gamma} u^{\beta/(\ell - 1/2 + \varepsilon)} e^{-u} du
\]

\[
\ll_{k, \ell} N^{k + \frac{5}{2} + \frac{1}{2} + \frac{\varepsilon}{2}} \sum_{j \geq 1} \frac{1}{j^{k-1/2}} \sum_{\rho : \gamma > 0} \frac{1}{y^{1+\varepsilon}} \left(\max_{0 \leq b \leq 1} \Gamma\left(\frac{b}{\ell} + \frac{1}{2} + \varepsilon\right)\right) \ll_{k, \ell} N^{\frac{3}{2} k + \frac{1}{2} + \frac{\varepsilon}{2}}, \quad (49)
\]
by standard density estimates and the definition of \(\varepsilon\). Inserting \((48)-(49)\) into \((47)\) and recalling \((46)\), we get, for \(k > 1\), that the quantity in \((45)\) is \(\ll_{k, \ell} N^{k/2 + 1/\ell}\).

10. Absolute convergence of \(\mathcal{F}_5, \mathcal{F}_6\) and \(\mathcal{F}_7\)

To study the absolute convergence of the series in \(\mathcal{F}_5\) we first remark that, by \((5)\) and \((24)\), we get
\[
\sum_{j \geq 1} \frac{|J_{k+1/2+1/\ell}(2\pi j N^{1/2}|)}{j^{k+1/2+1/\ell}} \ll_{k, \ell} N^{-k/2 - 1/4 - 1/(2 \ell)} \sum_{j \geq 1} \int_{(\infty)} |e^{Nz}| |z|^{-k-3/2-1/\ell} e^{-\pi |z|^2} |z|^{1} \ |dz|,
\]
which is the quantity in \((44)\). So the argument in \((8)\) also proves that the series in \(\mathcal{F}_5\) converges absolutely for \(k > 1/2 - 1/\ell\).

In fact, a more direct argument leads to a better estimate on \(k\). Using, for \(\nu > 0\) fixed, \(u \in \mathbb{R}\) and \(u \to +\infty\), the estimate
\[
|J_{\nu}(u)| \ll_{\nu} u^{-1/2} \quad (50)
\]
which immediately follows from \((7)\) (or from eq. (2.4) of Berndt \((22)\)), and performing a direct computation, we obtain that \(\mathcal{F}_5\) converges absolutely for \(k > -1/\ell\) (and for \(N\) sufficiently large) and that \(\mathcal{F}_5 \ll_{k, \ell} N^{k/2 + 1/(2 \ell)}\).

For the absolute convergence of the series in \(\mathcal{F}_7\) we argue analogously thus obtaining that, by \((5)\) and \((26)\), we get
\[
\sum_{j \geq 1} \frac{|J_{k+1/2}(2\pi j N^{1/2}|)}{j^{k+1/2}} \ll_{k, \ell} N^{-k/2 - 1/4} \sum_{j \geq 1} \int_{(\infty)} |e^{Nz}| |z|^{-k-3/2} e^{-\pi |z|^2} |z|^{1} \ |dz|,
\]
which is the quantity in \((49)\). So the argument in \((8)\) also proves that the series in \(\mathcal{F}_7\) converges absolutely for \(k > 1/2\). Using \((50)\) we can do better, as in the previous case. Performing a
direct computation, we obtain that $f_7$ converges absolutely for $k > 0$ (and for $N$ sufficiently large) and that $f_7 \ll_{k,\ell} N^{k/2}$.

For the study of the absolute convergence of the series in $f_6$ we have a different situation. In this case, the direct argument needs a more careful estimate of the Bessel functions involved, since both $\nu$ and $u$ are not fixed and, in fact, unbounded. In fact, it is easy to see that (7) can be used only if $\nu \in \mathbb{C}$ is bounded, but we are not in this case since $\nu = k + 1/2 + \rho/\ell$, where $\rho$ is a nontrivial zero of the Riemann zeta-function. On the other hand, (6) can be used only for $u$ bounded, but again this is not our case since $u = 2\pi j N^{1/2}$ and $j$ runs up to infinity. Moreover, the use of the asymptotic relations for $J_\nu(u)$ when $\nu \in \mathbb{C}$ and $u \in \mathbb{R}$ are both “large” seems to be very complicated in this setting.

So it turned out that the best direct approach we are able to perform is the following. By a double partial integration on (6), we immediately get

$$J_\nu(u) = \frac{2(u/2)^{(2\nu - 1)}}{\pi^{1/2}u^2 \Gamma(\nu + 1/2)} \int_0^1 \left(1 - \frac{(2\nu - 3)t^2}{1 - t^2}\right)(1 - t^2)^{\nu - 3/2} \cos(ut) \, dt$$

$$\ll_{\Re(\nu)} \frac{|u|^{\Re(\nu) - 1/2} |2\nu - 1|}{|\Gamma(\nu + 1/2)|} \int_0^1 \left(1 + |2\nu - 3|\right) |\cos(ut)| \, dt \ll_{\Re(\nu)} \frac{|u|^2 |u|^{3\Re(\nu) - 2}}{|\Gamma(\nu + 1/2)|} ,$$

(51)

where the last two estimates hold for $\Re(\nu) > 3/2$ and $u > 0$. Inserting (51) into (28) and using the Stirling formula (29), a direct computation shows the absolute convergence of the double sum in $f_6$ for $k > 2$ (and for $N$ sufficiently large).

Unfortunately, such a condition on $k$ is worse than the one we have in \S9. So, coming back to the Sonine representation of the Bessel functions (5) on the line $\Re(s) = 1$ and using the usual substitution $s = Nz$ to study the absolute convergence of the double sum in $f_6$, we are led to consider the quantity

$$\sum_{\rho} \left| \Gamma\left(\frac{\rho}{\ell}\right) \frac{N^{\rho/(2\ell)}}{|\rho/\ell|} \right| \sum_{j \geq 1} \frac{J_{k+1/2+\rho/\ell}(2\pi j N^{1/2})}{j^{k+1/2+\rho/\ell}}$$

$$\ll_k N^{-k/2 - 1/4} \sum_{\rho} \left| \Gamma\left(\frac{\rho}{\ell}\right) \right| \sum_{j \geq 1} \int_{1/\ell} \left| e^{Nz} |z|^{-k-3/2} |z|^{-\rho/\ell} |e^{-\pi^2 j^2 \Re(1/z)} |dz| ,$$

which is very similar to the one in (45) (the sums are interchanged). It is not hard to see that the argument used in (45)–(49) can be applied in this case too. It shows that the double series in $f_6$ converges absolutely for $k > 1$ and this condition fits now with the one we have in \S9.

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