Existence and disappearance of conical singularities in Gleyzes-Langlois-Piazza-Vernizzi theories

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In a class of Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories, we derive both vacuum and interior Schwarzschild solutions under the condition that the derivatives of a scalar field φ with respect to the radius r vanish. If the parameter αH characterizing the deviation from Horndeski theories approaches a non-zero constant at the center of a spherically symmetric body, we find that the conical singularity arises at r = 0 with the Ricci scalar given by R = −2αH/r². This originates from violation of the geometrical structure of four-dimensional curvature quantities. The conical singularity can disappear for the models in which the parameter αH vanishes in the limit that r → 0. We propose explicit models without the conical singularity by properly designing the classical Lagrangian in such a way that the main contribution to αH comes from the field derivative φ'(r) around r = 0. We show that the extension of covariant Galileons with a diatonic coupling allows for the recovery of general relativistic behavior inside a so-called Vainshtein radius. In this case, both the propagation of a fifth force and the deviation from Horndeski theories are suppressed outside a compact body in such a way that the model is compatible with local gravity experiments inside the solar system.

I. INTRODUCTION

Over the past few decades, there have been numerous attempts for extending General Relativity (GR) to more general gravitational theories [1]. This is motivated by the ultra-violet completion of gravity [2, 3] or by the observational evidence of early and late phases of cosmic acceleration [4, 5]. In particular, the problem of dark energy implies that the today’s acceleration of the Universe may be related to some infra-red modification of gravity.

If we extend GR in such a way that only one additional scalar degree of freedom (DOF) arises, Horndeski theories [6] are known as the most general scalar-tensor theories with second-order equations of motion [7]. Theories with derivatives higher than second order can be prone to instabilities due to the appearance of extra radiative DOF [8]. Nevertheless, it is possible to generalize Horndeski theories to those with only one propagating scalar DOF. In Hořava-Lifshitz gravity [8], for example, addition of higher-order spatial derivatives does not increase the number of the scalar DOF.

Another possibility for the extension of second-order gravitational theories is to violate two conditions Horndeski theories obey [9]. One of such conditions is related to a geometric modification of the Einstein-Hilbert Lagrangian $L_{EH} = M_p^2 R/2$, where $R$ is the four-dimensional Ricci scalar and $M_p$ is the reduced Planck mass. In terms of the 3+1 Arnowitt-Deser-Misner (ADM) decomposition of space-time [10], this Lagrangian can be expressed in the form $L_{EH} = A_4 (K^2 - K_{μν}K^{μν}) + B_4 R$ with $−A_4 = B_4 = M_p^2/2$, where $K$ and $R$ are the traces of three-dimensional extrinsic and intrinsic curvatures respectively. On the isotropic cosmological background Horndeski theories contain the same form of Lagrangian as $L_{EH}$, but $A_4$ and $B_4$ are functions of the scalar field $φ$ and its kinetic term $X$. These functions satisfy the particular relation $A_4 = 2XB_{4,X} - B_4$ [11], where $B_4,X ≡ ∂B_4/∂X$.

GLPV theories do not obey this particular relation as well as another condition associated with the Horndeski Lagrangian $L_5$ [8]. In the unitary gauge, where the constant field hypersurfaces are identified with the constant time hypersurfaces, the Hamiltonian analysis shows that GLPV theories possess only one scalar DOF [12]. The simplest class of GLPV theories corresponds to functions $A_4$ and $B_4$ depending on $φ$ alone, in which case the difference between $−A_4$ and $B_4$ characterizes the deviation from Horndeski theories. This deviation gives rise to several interesting observational signatures such as the mixing of scalar and matter propagation speeds [9, 13, 14], the large gravitational slip parameter [15], and the realization of weak gravity [16].

For the consistency with local gravity experiments, the interaction with matter mediated by the scalar DOF should be suppressed inside the solar system [17]. In Horndeski theories, nonlinear scalar-field self interactions can screen the fifth force [18, 19] through the Vainshtein mechanism [20]. In GLPV theories it was recently claimed that the Vainshtein mechanism tends to break down around the vicinity of a compact object [21, 22]. Whether this conclusion holds or not for all classes of GLPV theories remains to be an open question.

In this paper, we revisit the analysis of spherically symmetric solutions in GLPV theories to study how much deviation from Horndeski theories can be allowed. On using the background equations of motion derived in Sec. [11] we first obtain vacuum solutions around a point source ($r = 0$) in Sec. [11] and show that the Ricci scalar is given by $R = −2αH/r²$ in the absence of the cosmological constant Λ. Since $αH ≠ 0$ in GLPV theories, there
is a conical singularity at the origin. In Sec. [16] we also derive solutions inside a compact object for constant \( \alpha_H \) under the condition that radial derivatives of \( \phi(r) \) vanish everywhere and show that \( R \) is again divergent at \( r = 0 \). Since the boundary condition \( \phi'(r) = 0 \) at \( r = 0 \) is most natural for the regularity of solutions, the conical singularity inevitably arises for the theories in which \( \alpha_H \) approaches a non-zero constant as \( r \to 0 \).

A possible way out for avoiding the conical singularity is to consider the theories with a vanishing \( \alpha_H \) at the origin by designing the Lagrangian such that the main contribution to \( \alpha_H \) corresponds to \( X \)-dependent terms. The exact form of choice for the Lagrangian is essential to remove such a conical singularity, and the absence of a symmetry which preserves its form might be a problem, once one considers, e.g., quantum corrections. In Sec. [16] we discuss conditions for eliminating the conical singularity and present an explicit Lagrangian which allows for this possibility. In Sec. [17] we derive the field profile and gravitational potentials inside and outside the body under the approximation of weak gravity for a concrete model without the conical singularity. We show that the Vainshtein mechanism is efficient enough to suppress the fifth force and the deviation from Horndeski theories. We conclude in Sec. [VII].

II. BACKGROUND EQUATIONS OF MOTION

We consider the theories described by the action [1]

\[
S = \int d^4x \sqrt{-g} \sum_{i=2}^{4} L_i + \int d^4x \sqrt{-g} L_m(g_{\mu \nu}, \Psi_m), \tag{2.1}
\]

where \( g \) is the determinant of metric \( g_{\mu \nu} \), and \( L_m \) is the Lagrangian of matter fields \( \Psi_m \) with the energymomentum tensor \( T_{\mu \nu} = \text{diag}(-\rho_m, P_m, P_m, P_m) \). We assume that matter is minimally coupled to the metric \( g_{\mu \nu} \). The Lagrangians \( L_2, L_3, \) and \( L_4 \) are given by

\[
L_2 = A_2(\phi, X), \tag{2.2}
\]

\[
L_3 = (C_3 + 2XC_{3,X}) \Box \phi + XC_{3,\phi}, \tag{2.3}
\]

\[
L_4 = B_4 R - \frac{B_4 + A_4}{X} \left[ (\Box \phi)^2 - \nabla \mu \nabla \nu \phi \nabla \mu \phi \nabla \nu \phi \right] + \frac{2(B_4 + A_4 - 2XB_4X)}{X^2} \left( \nabla \mu \phi \nabla \nu \phi \nabla \mu \phi \nabla \nu \phi \Box \phi \right. \\
\left. + \nabla \mu \phi \nabla \nu \phi \nabla \sigma \phi \nabla \nu \phi \nabla \sigma \phi \right), \tag{2.4}
\]

where \( \nabla \mu \) denotes covariant derivatives, \( A_2, C_3, A_4, B_4 \) are functions of the scalar field \( \phi \) and its kinetic energy \( X = g^{\mu \nu} \nabla \mu \phi \nabla \nu \phi \). Introducing the function \( F_4 \) defined by

\[
-F_4 X^2 = B_4 + A_4 - 2XB_4X, \tag{2.5}
\]

one can write the Lagrangian \( L_4 \) of the form [21]

\[
L_4 = B_4 R - 2B_4X \left[ (\Box \phi)^2 - \nabla \mu \nabla \nu \phi \nabla \mu \phi \nabla \nu \phi \right] + F_4 \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu' \nu' \rho' \sigma'} \nabla \mu \phi \nabla \nu \phi \nabla \mu' \phi \nabla \nu' \phi \nabla \rho \phi \nabla \sigma \phi. \tag{2.6}
\]

where \( \epsilon_{\mu \nu \rho \sigma} \) is the totally antisymmetric Levi-Civita tensor. Horndeski theories satisfy the condition \( F_4 = 0 \), i.e., \( B_4 + A_4 - 2XB_4X = 0 \), under which the terms after the second lines of Eq. (2.4) vanish.

The full action of GLPV theories contains the Lagrangian \( L_5 \) associated with the Einstein tensor, but we do not take into account such a contribution in this paper. In fact, inclusion of \( L_5 \) tends to prevent the recovery of GR in the solar system even in Horndeski theories [18, 19].

Using the 3+1 ADM formalism, we can construct several scalar quantities such as \( K = g^{\mu \nu} K_{\mu \nu} \) and \( R = g^{\mu \nu} R_{\mu \nu} \) from the extrinsic curvature \( K_{\mu \nu} \) and the intrinsic curvature \( R_{\mu \nu} \) [11]. In the unitary gauge \( (\phi = \phi(t)) \), the Lagrangian \( L = L_2 + L_3 + L_4 \) is equivalent to \( L = A_2 + A_3 K + A_4(K^2 - K_{\mu \nu} K^{\mu \nu}) + B_4 R \) with the relation [0, 24, 25]

\[
A_3 = 2|X|^{3/2} \left( C_{3,X} + \frac{B_4, \phi}{X} \right), \tag{2.7}
\]

where the sign of \( X \) is different depending on the given space-time. On the spherically symmetric and static background we have that \( X > 0 \), whereas on the isotropic and homogenous background \( X < 0 \).

The deviation from Horndeski theories can be quantified by the parameter [2]

\[
\alpha_H \equiv \frac{2XB_4X - B_4}{A_4} - 1 = \frac{F_4 X^2}{A_4}. \tag{2.8}
\]

We also define [20]

\[
\alpha_t \equiv -\frac{B_4}{A_4} - 1. \tag{2.9}
\]

For linear perturbations on the spherically symmetric background, \( \alpha_t \) is related to the tensor propagation speed squared \( c_s^2 \), as \( \alpha_t = 1/c_s^2 - 1 \) [25, 27]. Then, \( \alpha_t \) characterizes the deviation of \( c_s^2 \) from 1. If the function \( B_4 \) depends on \( \phi \) alone, \( \alpha_H \) is equivalent to \( \alpha_t \).

We consider the spherically symmetric background described by the metric

\[
ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Phi(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{2.10}
\]

where the gravitational potentials \( \Psi(r) \) and \( \Phi(r) \) are functions of the distance \( r \) from the center of symmetry. Variation of the action (2.1) leads to the equations...
of motion

\[
\left( \frac{4e^{-2\Phi}A_4}{r} - C_1 + 4C_2 \right) \Phi' - A_2 + C_3 - C_4 = -(\rho - a_0), \tag{2.11}
\]

\[
- \frac{2A_4}{r^2} \left( e^{-2\Phi} - 1 - a_0 \right) = -(\rho - a_0),
\]

\[
\left( \frac{4e^{-2\Phi}A_4}{r} - C_1 + 4C_2 \right) \Psi' + A_2 - 2X_2A_2X - 2C_1 = 0 \tag{2.12}
\]

\[
+ \frac{2}{r^2} \left[ 4(e^{-2\Phi} - 1 - a_0) + rC_2 \right] = -P_m, \tag{2.12}
\]

\[
2e^{-2\Phi}A_4 \left( \Psi'' + \Psi^2 \right) + \left( \frac{2e^{-\Phi}A_4}{r} + \frac{1}{2}C_4 \right) \Phi' = 0 \tag{2.13}
\]

\[
P' + \Psi (\rho_m + P_m) = 0, \tag{2.14}
\]

where \( X = e^{-2\Phi} \phi^2 \), a prime represents a derivative with respect to \( r \), and

\[
C_1 = 2e^{-\Phi}X_2A_2X, \quad C_2 = \frac{2e^{-2\Phi}X_2A_4X}{r},
\]

\[
C_3 = e^{-\Phi} \phi' \left( A_4, \phi + 2e^{-2\Phi} \phi'' A_3X \right),
\]

\[
C_4 = \frac{4e^{-2\Phi} \phi' \left( A_4, \phi + 2e^{-2\Phi} \phi'' A_4X \right)}{r}. \tag{2.15}
\]

In GR we have \(-A_4 = B_4 = M^2_p/2 \) and \( \alpha_H = \alpha_0 = 0 \). In GLPV theories, the parameter \( \alpha_0 \) gives rise to a contribution comparable to the dominant term \( A_4\Phi/r^2 \) in Eq. (2.12) for \( |\alpha_0| \) larger than the order of \( \Phi \).

In the following we will assume that both \( A_4 \) and \( B_4 \) also remain finite as \( r \to 0 \). For this class of theories (without shift symmetry), on a static spherically symmetric background, the field \( \phi \) is required to be only \( r \)-dependent, and \( \phi' (r \to 0) \) for the regularity in compact objects. In this case, we have that \( \alpha_0 \) at the origin.

It should be noticed that the action (2.1) remains finite in the limit \( X \to 0 \). In fact, for analytic profiles for the field, one can verify that \( \sqrt{-g}L_4 \to \) constant around the origin. This should not come as a surprise as, in fact, the same limit is well-defined also for Horndeski theories, whose Lagrangian reduces to the first line of Eq. (2.1). We also note that the equations of motion (2.11)-(2.13) do not contain the \( X \)-dependent term in the denominators. It is clear that finite values of \( A_4 \) and \( B_4 \) are allowed as long as \( \Phi', \Psi' \to 0 \), as requested for standard boundary conditions for compact objects.

## III. VACUUM SCHWARZSCHILD SOLUTIONS

First of all, we derive exact solutions to Eqs. (2.11) and (2.12) in the absence of matter for a point source with mass \( M \). Since the field derivatives \( \phi'(r) \) and \( \phi''(r) \) vanish in this case, the terms \( C_i \) defined in Eq. (2.15) do not contribute to the equations of motion. The terms \( A_4 \) and \( B_4 \) are regarded as constants, with the particular relation \( \alpha_H = \alpha_0 = -B_4/A_4 - 1 \). The term \( A_2 \) is related to the cosmological constant \( \Lambda > 0 \), as \( A_2 = -\Lambda \). Then, Eqs. (2.11) and (2.12) reduce to

\[
\frac{4e^{-2\Phi}A_4}{r} \Phi' - \Lambda - \frac{2A_4}{r^2} (e^{-2\Phi} - 1 - \alpha_H) = 0 \tag{3.1}
\]

\[
\frac{4e^{-2\Phi}A_4}{r} \Psi' - \Lambda - \frac{2A_4}{r^2} (e^{-2\Phi} - 1 - \alpha_H) = 0 \tag{3.2}
\]

The solutions to these equations are given by

\[
e^{2\Phi} = \left( 1 + \alpha_H + \frac{\Lambda}{6A_4} r^2 + \frac{c_1}{2A_4} \right)^{-1}, \tag{3.3}
\]

\[
e^{2\Psi} = -6A_4c_2 \left( 1 + \alpha_H + \frac{\Lambda}{6A_4} r^2 + \frac{c_1}{2A_4} \right), \tag{3.4}
\]

where \( c_1 \) and \( c_2 \) are integration constants. The deviation from Horndeski theories leads to the contribution to gravitational potentials. The cosmological constant gives rise to the contribution \( \Lambda r^2/6A_4 \), as it happens for the Schwarzschild-de Sitter solution. The two coefficients \( c_1 \) and \( c_2 \) also arise in GR. The constant \( c_1 \) is chosen as \( c_2 \), where \( c_2 \) is not restricted to \( -4A_4GM \) to obtain the standard term \( -2GM/r \) in Eqs. (3.3) and (3.4). To recover the usual relation \( e^{2\Psi} = e^{-2\Phi} \), the constant \( c_2 \) is chosen to be \( c_2 = -1/(6A_4) \). In this way, we have fixed two freedoms (Schwarzschild mass \( M \) and the time reparametrization) as in the context of GR.

Then, we obtain the following solution

\[
e^{2\Psi} = e^{-2\Phi} = 1 + \alpha_H + \frac{\Lambda}{6A_4} r^2 - \frac{2GM}{r}. \tag{3.5}
\]

This is the extension of the Schwarzschild-de Sitter solution with the additional factor \( \alpha_H \). Since \( g^{\theta\theta}R_{\theta\theta} = g^{\theta\phi}R_{\theta\phi} = g^{\phi\phi}R_{\phi\phi} = -\Lambda/(2A_4) - \alpha_H/r^2 \), the Ricci scalar is given by

\[
R = -2\Lambda/A_4 - \frac{2\alpha_H}{r^2}, \tag{3.6}
\]

which diverges at \( r = 0 \) for \( \alpha_H \neq 0 \). Provided that \( \alpha_H \neq 0 \), the divergence of \( R \) occurs independent of the choices of \( c_1 \) and \( c_2 \), so it cannot be eliminated by the change of boundary conditions.

The divergence of curvature at \( r = 0 \) can be interpreted as a conical singularity originating from the \( \theta, \phi \) contributions to \( R \). This singularity cannot be eliminated even for \( \Lambda \to 0 \) and \( M \to 0 \). In this limit, the three-dimensional spatial metric is given by

\[
ds_{(3)}^2 = \left( 1 + \alpha_H \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

Defining \( \tilde{r} = r/\sqrt{1 + \alpha_H} \), the two-dimensional metric in the \( \theta = \pi/2 \) plane is of the form

\[
ds_{(2)}^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2 ,
\]

where \( \tilde{\phi} = \sqrt{1 + \alpha_H} \tilde{\phi} \). Since the angle \( \tilde{\phi} \) is not restricted to be between 0 and \( 2\pi \), the resulting space-time is a cone with a geodesically incomplete point at \( r = 0 \).
IV. INTERIOR SCHWARZSCHILD SOLUTIONS FOR CONSTANT $\alpha_H$

We derive interior and exterior Schwarzschild solutions under the conditions that the density $\rho_m$ inside a compact body is constant and that $\phi'(r) = 0$ and $\phi''(r) = 0$ everywhere. Again, we deal with $\alpha_H$ as a constant satisfying the relation $\alpha_H = \alpha_1 = -B_4/A_4 - 1$.

Integration of Eq. (4.1) gives

$$P_m = -\rho_m + Be^{-\Phi},$$

where $B$ is a constant. Solving Eq. (4.11) for $\Phi$, we obtain the same solution as Eq. (3.3) with the replacement $\Lambda \rightarrow \Lambda + \rho_m$. For the regularity of metric at $r = 0$ we need to choose $c_1 = 0$, so the solution reduces to

$$e^{2\Phi} = (1 + \alpha_H - A r^2)^{-1},$$

where

$$A \equiv -\frac{\Lambda + \rho_m}{6A_4}.$$  (4.3)

There exists the term $\alpha_H$ in Eq. (4.2), which cannot be eliminated by any boundary condition. As we will see below, this again leads to the conical singularity at $r = 0$.

Solving Eq. (4.12) for $\Psi$, we obtain

$$e^\Psi = \frac{3B}{2(\Lambda + \rho_m)} - D \sqrt{1 + \alpha_H - Ar^2},$$

where $D$ is a constant. There are two free integration constants $B$ and $D$ in addition to the mass $M$ of the compact object. These constants can be fixed by matching interior and exterior solutions of $\Phi$ and $\Psi$ at the radius $r_0$ of the body determined by the condition $P_m(r_0) = 0$. Analogous to the derivation of Eq. (3.5) the exterior vacuum solution for $r > r_0$ is given by

$$e^{2\Psi} = e^{-2\Phi} = 1 + \alpha_H - \frac{\Lambda}{\Lambda + \rho_m} A r^2 - \frac{2GM}{r}.$$  (4.5)

Matching the solutions at $r = r_0$, it follows that

$$B = \rho_m \sqrt{1 + \alpha_H - A r_0^2},$$  (4.6)

$$A = \frac{2GM}{r_0^2} \frac{\Lambda + \rho_m}{\rho_m},$$  (4.7)

$$D = \frac{\rho_m - 2\Lambda}{2(\Lambda + \rho_m)}.$$  (4.8)

After substitutions of Eqs. (4.6)-(4.8) into Eqs. (4.2) and (4.3), we obtain the interior Schwarzschild solution with appropriate matching conditions. Then, we can find the behavior of three curvature scalars $R, S = R_{\mu\nu}R^{\mu\nu}$, and $T = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. In the limit where $r \rightarrow 0$, it follows that

$$R \rightarrow -\frac{2\alpha_H}{r^2}, \quad S \rightarrow \frac{2\alpha_H^2}{r^4}, \quad T \rightarrow \frac{4\alpha_H^3}{r^6}.$$  (4.9)

We have dropped next-to-leading order corrections, which appear as a constant for $R$ and as the terms involving $\alpha_H/r^2$ for $S$ and $T$. Taking into account these corrections, the Gauss-Bonnet term $R^2_{\mu\nu\lambda\rho} = R^2 - 4S + T$ is proportional to $\alpha_H/r^2$ around $r = 0$. The integration constants have been fixed as Eqs. (4.6)-(4.8), but the singular behavior of $R, S, T$ is independent of the choice of boundary conditions. Thus, for $\alpha_H = \text{constant}$, the conical singularity similar to that discussed in Sec. III is present at the center of a compact object.

It should be noted that the metric and matter fields are all regular at the origin, whereas the singularity appears only at the level of the Ricci scalar, i.e., for tidal forces and geodesics deviation. Although we have found the presence of this singularity for exact solutions (which always give a clear insight of theories), one may worry that it is merely a consequence of the fact we have assumed a specific constant field profile $\phi(r) = \text{constant}$ everywhere. But this “special” solution is actually “the” solution which is approached at the origin for compact objects (assuming the analytic profile for each field).

In fact, if the fields are analytic around $r = 0$, then we can Taylor-expand them as: $\phi = \phi_\infty + \zeta_\sigma r^2/2 + O(r^3), \quad \Phi = \Phi_\infty + \zeta_\Phi r^2/2 + O(r^3), \quad \Psi = \Psi_\infty + \zeta_\Psi r^2/2 + O(r^3)$, and $\rho = \rho_\infty + \zeta_\rho r^2/2 + O(r^3)$. The coefficients $\zeta$'s are numerical constants which can be determined by the equations of motion or boundary conditions. Here, the crucial point to be shown is whether the variation of $\phi(r)$ induced by the term $\zeta_\rho r^2/2$ can modify the existence of the conical singularity.

We do not specify the functional forms of $A_2, A_3, A_4$, but the leading-order contributions to the terms $A_2, A_3, A_4$ are regarded as constants. Using the above Taylor-expansions of $\phi$ and $\Phi$, the leading-order terms in Eq. (4.16) are given, respectively, by $C_1 = \tilde{c}_1 r^2, \quad C_2 = \tilde{c}_2 r, \quad C_3 = \tilde{c}_3 r$, and $C_4 = \tilde{c}_4$, where $\tilde{c}_i$’s are constants associated with $\zeta_\rho$. In the limit that $r \rightarrow 0$ the terms $C_1, C_2$, and $C_3$ vanish, so only the non-vanishing contributions to Eqs. (4.11) and (4.13) are $-C_4$ in Eq. (4.11) and $C_4/2$ in Eq. (4.13). These $C_4$ terms behave in the same way as another constant $A_2 = -\Lambda$ already studied above. This discussion shows that the variation of $\phi(r)$ around $r = 0$ does not give rise to any new contribution to modify the existence of the conical singularity.

More concretely, one can verify from the integration of Eq. (4.11) that the gravitational potential $e^{-2\Phi}$ acquires the terms proportional to $\tilde{c}_1 r^4, \quad \tilde{c}_2 r^3, \quad \tilde{c}_3 r^2$, and $\zeta_\rho r^4$ in addition to those given in Eq. (4.11). In the limit that $r \rightarrow 0$ all these additional terms vanish, so the gravitational potential again approaches the value $e^{-2\Phi} \rightarrow 1 + \alpha_H$. Similarly the additional terms $C_1$ and $C_2$...
appearing in Eq. (2.12) only give rise to vanishing contributions to $\Psi$ as $r \to 0$, such that $e^{2\Phi} \to 1 + \alpha_H$ for $r \to 0$. In addition to the fact that the integrated solution $\varphi$ of Eq. (2.11) is used in this procedure, the above solutions are consistent with Eq. (2.13). Hence they are the solutions of the full equations of motion (2.11)-(2.14) around $r = 0$. The existence of the term $\alpha_H$ in $\Phi$ and $\Psi$ shows that the variation of the field $\phi(r)$ around $r = 0$ cannot eliminate the conical singularity.

The robustness of the behavior $e^{-2\Phi} = e^{2\Psi} = 1 + \alpha_H$ for $r \to 0$ can be also confirmed by directly substituting the above expansions into the background equations of motion. On expanding Eq. (2.11) around the origin, we find that the first non-trivial contribution which needs to be canceled is given by

$$
\frac{2}{r^2} \left[ B_1(\phi_c, 0) + e^{-2\Phi_c} A_4(\phi_c, 0) \right],
$$

(4.10)

from which we obtain $e^{-2\Phi_c} = 1 + \alpha_H(\phi_c, 0)$. Applying the same procedure to Eq. (2.12) with Eq. (2.11), it follows that $e^{2\Phi} = 1 + \alpha_H(\phi_c, 0)$. Thus, the theories approaching a non-zero value of $\alpha_H$ as $r \to 0$ lead to the appearance of the conical singularity.

It should be noticed that we have focused on static solutions and that we have not imposed the shift symmetry of the theory. Abandoning any of the two assumptions could lead to a different behavior for the theory, and possibly, to the disappearance of the singularity. We leave this interesting topic for a future study.

V. CONDITIONS FOR EXISTENCE AND DISAPPEARANCE OF THE CONICAL SINGULARITY

The results in Sec. III and IV have been derived under the conditions that (i) $\phi'(r) = 0$ for $r \geq 0$ and (ii) $\alpha_H = $ constant. Since the boundary condition $\phi'(0) = 0$ is generic for the regularity of solutions [23, 29], this is consistent with the assumption (i) around $r = 0$. The remaining possibility for eliminating the conical singularity is to construct models in which the parameter $\alpha_H$ approaches 0 as $r \to 0$.

Let us consider this second possibility in detail. In general, around the center of a compact object with spherical symmetry and staticity, we should expect that $X \to 0$ for regularity. Therefore, the parameter $\alpha_H$ may be expressed in the form

$$
\alpha_H = \alpha_H(\phi_c, 0) + \mathcal{O}(X).
$$

(5.1)

We need to impose $\alpha_H(\phi_c, 0) = 0$ for removing the conical singularity. To fulfill this condition, there are two possibilities: 1) to solve $\alpha_H(\phi_c, 0) = 0$ for $\phi_c$ if a solution (or even more than one) exists; 2) to impose that the first term in the Taylor expansion (i.e., the constant term) in $X$ identically vanishes.

The first possibility cannot be viable in general, as this corresponds to setting two different boundary conditions for the field at the origin, that is $\phi'(r = 0) = 0$ (standard boundary condition), $\phi(r = 0) = \phi_c$ (non-standard additional boundary condition). Reflecting the fact that the field equation of motion is of second order, these two boundary conditions completely fix the dynamics of the field even at infinity. This would make the system over-constrained, in general, and would reduce the freedom of solutions.

The second possibility, which will be explored in the following, consists of imposing that, in a Taylor expansion of $\phi(r)$ around the origin, the constant term $\alpha_H(\phi_c, 0)$ identically vanishes for any $\phi_c$. In general this condition cannot be imposed by any symmetry, so that we need to tune the choice of the function $\alpha_H$. Moreover, it is not clear whether such a functional form of $\alpha_H$ is valid in the presence of quantum corrections and whether such corrections spoil (or not) the classical picture by making the conical singularity reappear. The analysis of this important issue is beyond the scope of our paper, but we would like to discuss it in a future project.

To explore the second possibility further, we consider a canonical scalar field $\phi$ without the potential, i.e., $A_2 = -X/2$ and $C_3 = 0$. To go beyond the Horndeski domain, we need to choose functions $A_2$ and $B_4$ which violate the condition $A_4 = 2XB_1, X - B_4$. One way is to extend the scalar-curvature coupling appearing in Brans-Dicke theories [30] or dilatation gravity [31] to the forms $A_4 = -M^2_{\text{pl}}F_1(\phi)/2$ and $B_4 = M^2_{\text{pl}}F_2(\phi)/2$, where $F_1(\phi)$ and $F_2(\phi)$ are functions with respect to $\phi$. Another way is to take into account the $X$ dependence in the functions $A_4$ and $B_4$, as it happens for covariant Galileons [32] and extended Galileons [33].

To accommodate these two cases, we focus on the models characterized by

$$
A_2 = -\frac{1}{2}X, \quad C_3 = 0,
$$

$$
A_4 = -\frac{1}{2}M^2_{\text{pl}}F_1(\phi) + f_1(X),
$$

$$
B_4 = \frac{1}{2}M^2_{\text{pl}}F_2(\phi) + f_2(X),
$$

(5.2)

where $f_1(X)$ and $f_2(X)$ are functions of $X$. From Eq. (2.7) the function $A_3$ is given by

$$
A_3 = M^2_{\text{pl}}\sqrt{X}F_{2, \phi}(\phi).
$$

(5.3)

The parameters $\alpha_H$ and $\alpha_t$ read

$$
\alpha_H = \frac{1}{A_4} \left[ \frac{M^2_{\text{pl}}}{2} (f_1 - f_2) - (f_1 + f_2 - 2Xf_{2, X}) \right],
$$

(5.4)

$$
\alpha_t = \alpha_H - \frac{2Xf_{2, X}}{A_4}.
$$

(5.5)

The difference between $F_1(\phi)$ and $F_2(\phi)$ gives rise to a non-zero contribution to $\alpha_H$. The term $f_1 + f_2 - 2Xf_{2, X}$ in Eq. (5.3) does not generally vanish for theories beyond the Horndeski domain. The covariant Galileon corresponds to $f_1(X) = a_4X^2$ and $f_2(X) = b_4X^2$ with
the parameters $\alpha, \phi$ the field of the origin of a compact object, so the conical singularity $f_i(X)$ around $X = 0$, i.e., $f_i(X) = f_i(0) + f_i'(0) X + \mathcal{O}(X^2)$, the constants $f_i(0)$ have been absorbed to the functions $F_i(\phi)$. As we will see below, the theories with $F_1(\phi) \neq F_2(\phi)$ are plagued by the existence of the conical singularity. To avoid this problem, we need to choose the two constants appearing in $-A_4$ and $B_4$ are exactly the same as each other. This corresponds to the aforementioned tuning of functions required to eliminate the conical singularity at $r = 0$.

A. Theories with $F_1(\phi) \neq F_2(\phi)$

Let us first consider the theories with $F_1(\phi) \neq F_2(\phi)$ and $f_1(X) = f_2(X) = 0$. In this case we have $\alpha_1 = \alpha_2 = F_2(\phi)/F_1(\phi) - 1$. For the boundary condition $\phi'(0) = 0$, the field $\phi$ approaches a constant value $\phi_c$ as $r \to 0$. Then the parameters $\alpha_H$ and $\alpha_t$ behave as constants around the origin of a compact object, so the conical singularity cannot be avoided at $r = 0$.

Next, we proceed to the theories with $F_1(\phi) \neq F_2(\phi)$ and non-vanishing functions of $f_1(X)$ and $f_2(X)$. An example of functions $f_1(X)$ and $f_2(X)$, which encompasses both covariant and extended Galileons, is given by

$$f_1(X) = a_4 X^m, \quad f_2(X) = b_4 X^n,$$

where $m, n$ are positive integers ($m, n \geq 1$), and $a_4, b_4$ are constants. Since $f_1 + f_2 - 2X f_2, X = a_4 X^m + (1-2n) b_4 X^n$ and $-2X f_2, X / A_t = 4b_4 n X^n / [M_{pl}^2 F_1(\phi) - 2a_4 X^m]$ in Eqs. (5.1) and (5.2), the parameters $\alpha_H$ and $\alpha_t$ again reduce to $\alpha_H = \alpha_4 = F_2(\phi_c)/F_1(\phi_c) - 1$ constant around the origin for the boundary condition $\phi'(0) = 0 = \phi_c$. This constant behavior of $\alpha_H$ and $\alpha_t$ corresponds to the case studied in Sec. IV.

We caution, however, that the result in Sec. IV has been derived by assuming that all the terms $C_i$ defined in Eq. (2.15) vanish. Let us consider the effect of these terms around the center of a compact object. To satisfy the regular boundary condition $\phi'(0) = 0$, we expand the field derivative of the form

$$\phi'(r) = \sum_{p=1} \kappa_p r^p,$$

where $\kappa_p$ is a constant and $p$ is a positive integer. We substitute Eq. (5.7) into Eq. (2.15) by using the functions given by Eqs. (5.2) and (5.3) with Eq. (5.6), under the condition that $\Phi$ approaches a constant as $r \to 0$. It then follows that

$$C_1 \to 0, \quad C_2 \to 0,$$

$$C_3 \to \kappa_1 M_{pl}^2 e^{-3\Phi} F_2, \phi,$$

$$C_4 \to -2\kappa_1 M_{pl}^2 e^{-2\Phi} F_1, \phi + 8\kappa_2^2 a_4 e^{-4\Phi},$$

where the second term in $C_4$ survives only for $m = 1$.

Since the terms $C_3$ and $C_4$ in Eq. (5.8) approach constants as $r \to 0$, we can absorb these terms into the cosmological constant $\Lambda$ appearing in $A_2$. Then, in the limit that $r \to 0$, Eqs. (2.11) and (2.12) reduce to the same forms as those studied in Sec. IV with non-vanishing constants $\alpha_H = \alpha_t = F_2(\phi_c)/F_1(\phi_c) - 1$. Hence the conical singularity is present at $r = 0$ for the theories with $F_1(\phi) \neq F_2(\phi)$.

We note that the non-analytic function $\phi'(r) = \kappa r^p$ ($0 < p < 1$) satisfies the boundary condition $\phi'(0) = 0$, but the second field derivative diverges for $r \to 0$. This gives rise to the divergent terms $r^{p-1}$ in $C_3$ and $C_4$, whose functional dependence is different from that of the term $2A_4 \alpha_t r^2$ in Eq. (2.11). This means that such non-analytic field derivatives do not eliminate the conical singularity.

B. Theories with $F_1(\phi) = F_2(\phi)$

We proceed to the theories satisfying $F_1(\phi) = F_2(\phi)$ with $f_1(X)$ and $f_2(X)$ given by Eq. (5.6). In this case we have

$$\alpha_H = \frac{a_4 X^m + (1-2n) b_4 X^n}{M_{pl}^2 F_1(\phi)/2 - a_4 X^m},$$

$$\alpha_t = \frac{a_4 X^m + b_4 X^n}{M_{pl}^2 F_1(\phi)/2 - a_4 X^m}.$$

For the boundary condition $\phi'(0) = 0$, both $\alpha_H$ and $\alpha_t$ approach 0 in the limit that $r \to 0$.

For the field profile (5.7) around $r = 0$, the quantities $C_i$ behave in the same way as Eq. (5.8) with the relation $F_1 = F_2$. Since $C_3$ and $C_4$ approach constants as $r \to 0$, the solutions to Eqs. (2.11) and (2.12) around the center of a compact body are of the same forms as Eqs. (4.2) and (4.3), respectively, with $\alpha_H = 0$ and $\Lambda$ subject to modifications arising from constant $C_3$ and $C_4$. Since the term $\alpha_H$ vanishes for $r \to 0$, we can avoid the conical singularity at the origin.

In the above argument the equation of $\phi$ has not been solved explicitly, but we employed the Taylor expansion of the form (5.7) with the boundary condition $\phi'(0) = 0$. In Sec. IV we shall derive solutions to the scalar-field equation of motion in concrete models under the approximation of weak gravity and show the existence of the field profile of the form (5.7) around the origin. Provided that the conical singularity is absent, the solutions derived under the weak-gravity approximation should not cause any divergent behavior for curvature quantities.

VI. MODELS WITHOUT THE CONICAL SINGULARITY ($F_1 = F_2$) AND VAINSHTEIN MECHANISM

In this section, we solve the equations of motion for the field $\phi$ and gravitational potentials $\Phi, \Psi$ on the
weak gravitational background characterized by $|\Phi| \ll 1$, $|\Psi| \ll 1$ in concrete models without the conical singularity ($F_1 = F_2$).

We consider the situation in which the dominant contributions to Eqs. (2.11) and (2.12) are of the orders of $A_4 \Phi/r^2$ and $A_4 \Psi/r^2$ to recover the general relativistic behavior in the solar system (see Refs. [15, 29] for the similar approximation). From Eq. (2.14) the pressure $P_m$ of non-relativistic matter is of the same order as $\rho_m \Phi$, so $P_m$ is second-order in $\Psi$. We deal with other terms (including $\alpha_H$ and $\alpha_1$) as next-order corrections to the leading-order terms.

Eliminating the terms $\Phi'$ and $\Psi'$ in Eq. (2.15) by using Eqs. (2.11) and (2.12), we obtain the equation for $\Box \Psi$ coupled to $\Box \phi$, where $\Box = d^2/dr^2 + (2/r)(d/dr)$. Differentiating Eq. (2.19) with respect to $r$ and substituting it into Eq. (2.16), we can derive the equation of motion containing the terms $\Box \Psi$ and $\Box \phi$. Eliminating the term $\Box \Psi$ by using these two equations, the resulting scalar-field equation of motion reads

$$\Box \phi \simeq \mu_1 \rho_m + \mu_2, \quad (6.1)$$

where

$$\mu_1 = -\frac{\phi' r (A_{4,X} + \beta)}{4A_4^3}, \quad (6.2)$$

$$\mu_2 = \frac{1}{\beta r} \left[ \frac{A_{2,\phi}}{2} - X A_{2,\phi X} \right] r^2 + (A_{3,\phi} - 2X A_{3,\phi X})$$

$$+ 4\phi' X A_{2,XX} r^2 + 2\phi' (A_{3,X} + 4X A_{3,XX}) - 2A_{4,\phi}$$

$$+ 2X A_{4,\phi X} - \frac{8\phi' X A_{4,XX} + r \alpha_1 - 4\phi' \alpha_2}{r}, \quad (6.3)$$

and

$$\beta = (A_{2,X} + 2X A_{2,XX}) r + 2(A_{3,X} + 2X A_{3,XX})$$

$$- \frac{4X A_{4,XX}}{r} + 2\alpha_2, \quad (6.4)$$

$$\alpha_1 = 2X B_{4,\phi X} - B_{3,\phi} - A_{4,\phi}, \quad (6.5)$$

$$\alpha_2 = 2X B_{4,XX} + B_{4,XX} - A_{4,X}. \quad (6.6)$$

Here and in the following, the kinetic term $X$ should be understood as $\phi'^2$. Under the above scheme of approximation, Eqs. (2.11) and (2.12) reduce, respectively, to

$$r \Phi' \simeq -\frac{\rho_m r^2}{4A_4} + \frac{r}{A_4} \left[ A_2 - \phi' (A_{3,\phi} + 2\phi'' A_{3,X}) \right]$$

$$+ \frac{4\phi'}{r} (A_{4,\phi} + 2\phi'' A_{4,\phi}) \left[ \frac{1}{2} \alpha_1 \right], \quad (6.7)$$

$$\Psi' \simeq \frac{\Phi}{r} - \frac{r}{4A_4} \left[ A_2 - 2\phi'^2 A_{2,XX} \right] + \frac{\phi'^2 A_{3,X}}{A_4}$$

$$- \frac{\phi'^2 A_{2,X}}{A_4 r} + \frac{\alpha_1}{2 r}. \quad (6.8)$$

For concreteness, we study the model described by the Lagrangian (5.2) with the functions

$$F_1(\phi) = F_2(\phi) \equiv F(\phi) = e^{-\alpha \phi / M_{pl}},$$

$$f_1(X) = a_4 X^n, \quad f_2(X) = b_4 X^n, \quad (6.9)$$

where $q, a_4, b_4$ are constants and $n (\geq 1)$ is a positive integer. Then, the quantities $\mu_1$ and $\mu_2$ appearing in Eq. (6.1) are given, respectively, by

$$\mu_1 = -\frac{(qM_{pl}^2 - \beta \phi') r}{2\beta (M_{pl}^2 F - 2a_4^2 \phi'^2)}, \quad (6.10)$$

$$\mu_2 = \frac{4(a_4 - b_4) n (2n - 1) \phi'^{2n-1}}{r^2}, \quad (6.11)$$

where

$$\beta = -\frac{1}{2} r \left[ 1 + 4(a_4 - b_4) n (2n - 1) \phi'^{2(n-1)} \right]. \quad (6.12)$$

A. Field profile and gravitational potentials around the origin

We derive the solution to Eq. (6.1) around the center of a compact body whose density approaches a constant $\rho_m$ as $r \to 0$. We search for regular solutions described by Eq. (5.14) around $r = 0$ for non-zero values of $q$.

When $n > 1$ the terms $\beta \phi'$ and $2a_4 \phi'^{2n}$ approach 0 in the limit that $r \to 0$, so the quantity $\mu_1$ reduces to $\mu_1 \to -q r / (2M_{pl} \beta)$. Integrating Eq. (6.1) with respect to $r$, we obtain the following implicit solution

$$r^2 \phi' + 4n(a_4 - b_4) \phi'^{2n-1} = \frac{q \rho_m r^3}{3M_{pl}}, \quad (6.13)$$

where the integration constant is set to 0 to satisfy the boundary condition $\phi'(0) = 0$. From Eq. (6.13) it is clear that there is a dominant solution of the form

$$\phi'(r) = cr, \quad (6.14)$$

where $c$ is a constant. The coefficient $c$ is different depending on the values of $n$, i.e.,

$$c + 8(a_4 - b_4) c^3 = \frac{q \rho_m}{3M_{pl}} \quad (\text{for } n = 2), \quad (6.15)$$

$$c = \frac{q \rho_m}{3M_{pl}} \quad (\text{for } n > 2). \quad (6.16)$$

When $n = 1$, the solution to the field equation can be written in terms of an error function. Expanding it around $r = 0$ and using the boundary condition $\phi'(0) = 0$, the dominant solution is given by

$$\phi'(r) = \frac{q \rho_m}{12M_{pl} (a_4 - b_4)^3}. \quad (6.17)$$

Since both Eqs. (6.14) and (6.17) are of the form (5.1), the discussion in Sec. (5.3) to show the absence of the conical singularity is valid.

Integrating the solution $\phi'(r) = cr^p$ (where $p = 1$ for $n \geq 2$ and $p = 3$ for $n = 1$), it follows that $\phi(r) = \phi_0 + cr^{p+1}/(p+1)$ with $\phi_0$ an integration constant. Then, in the limit that $r \to 0$, the function $F(\phi)$ converges to a
constant value $F_c \equiv F(\Phi)$. In this case the parameters $\alpha_1$ and $\alpha_0$ are in proportion to $r^{2n}$, so they vanish at
the origin. Employing these results and picking up the dominant contributions to Eqs. (6.17) and (6.8) around $r = 0$, they are integrated to give
\[
\Phi = \frac{\rho_m r^2}{6 M_p^2 F_c} - \frac{q c r^{p+1}}{M_p^2} \left(\frac{1 + 8 q^2 F_c c^2 r^{2(p+1)}}{4(2p+3) M_p^2 F_c} \right) + \frac{c_F}{2m+1} \frac{M_p^2 F_c}{M_p^2} + \frac{c_F}{r}, \quad (6.18)
\]
\[
\Psi = \frac{\rho_m r^2}{12 M_p^2 F_c} + \frac{q c r^{p+1}}{(1 + p) M_p^2} \left(\frac{p + 2 + 4 q^2 F_c c^2 r^{2(p+1)}}{4(p+1)(2p+3) M_p^2 F_c} \right) + \frac{1}{p(2m+1) M_p^2 F_c} \left(\frac{c_F}{r} + c_F \right), \quad (6.19)
\]
where $c_F$ and $c_F$ are integration constants. We need to choose $c_F = 0$ for the regularity of $\Phi$ at $r = 0$. The constant $c_F$, which can be derived by matching solutions at the radius of a compact body, is finite as in the case of the interior Schwarzschild solution. The contributions coming from $\alpha_0$ and $\alpha_H$ give rise to terms proportional to $r^{2n}$ in Eqs. (6.18) and (6.19), which vanish as $r \to 0$.

Evaluating the Ricci scalar $\tilde{R}$ by using the solutions (6.18) and (6.19) with $c_F = 0$, it follows that $\tilde{R}$ approaches a finite constant as $r \to 0$. Thus, the model (6.9) is free from the problem of the conical singularity as a consequence of vanishing parameters $\alpha_0$ and $\alpha_1$ at $r = 0$.

**B. Vainshtein mechanism outside the compact body**

Since there is no conical singularity for the model (6.5) at the center of a compact object, we proceed to the derivation of solutions outside the body. In covariant Galileons (62) with the dilatonic coupling $F(\phi) = e^{-2q\phi/M_p}$, it is known that the presence of the terms $f_1(X) = a_4 X^2$ and $f_2(X) = b_4 X^2$ with $a_4 = 3b_4$ leads to the recovery of GR inside the so-called Vainshtein radius $r_V$, even for $|q|$ of the order of 1 (12, 19, 34). Now, we are going to discuss the Vainshtein mechanism for the model (6.9) without imposing the Horndeski condition $a_4 + (1 - 2n)b_4 = 0$. For simplicity, we focus on the theory with $n = 2$ and $a_4 \neq 3b_4$.

The Vainshtein radius is characterized by the distance at which the field derivative in Eq. (6.12) becomes comparable to the first term in Eq. (6.12), i.e.,
\[
r_V^2 \simeq 24 |a_4 - b_4| \phi^2 (r_V), \quad (6.20)
\]
where $r_V$ can be explicitly known by solving the field equation (6.1). For the distance $r \gg r_V$ the field self-interaction term $\mu_2$ is suppressed relative to the term $\mu_1 \rho_m$, with $\mu_1 / M_p \approx 0$ and $\beta \approx -r/2$. Then, Eq. (6.1) is integrated to give
\[
\phi'(r) = \frac{q M_p r g}{r^2}, \quad \text{for } r \gg r_V, \quad (6.21)
\]
where $r_g$ is the Schwarzschild radius of the source defined by
\[
r_g \equiv \frac{1}{M_p^2} \int_0^r \rho_m(\tilde{r}) \tilde{r}^2 d\tilde{r}. \quad (6.22)
\]
Substituting the solution (6.21) into Eq. (6.20), we obtain the Vainshtein radius
\[
r_V = \left(\frac{|q| M_p r_g^{1/3}}{M} \right), \quad M \equiv (24 |a_4 - b_4|)^{-1/6}, \quad (6.23)
\]
where $M$ is a constant having a dimension of mass.

For $r_g \ll r \ll r_V$ the field-derivative term $\mu_2$ corresponds to the dominant contribution to Eq. (6.1), such that $\mu_1 \rho_m \ll \mu_2 \simeq 2\phi'/r$. Since the integrated solution in this regime is $\phi'(r) = \text{constant}$, matching this solution with Eq. (6.21) at $r = r_V$ gives
\[
\phi'(r) = \frac{q M_p r g}{r_V}, \quad \text{for } r_g \ll r \ll r_V, \quad (6.24)
\]
which is of the same form as that derived in Ref. [19] in Horndeski theories. This reflects the fact that, provided $a_4 \neq b_4$, the factor $\beta$ is independent of the values $a_4$, $b_4$ and $n$. Hence the Vainshtein mechanism can be at work outside the compact body for the model (6.9) satisfying the condition $a_4 \neq b_4$.

The integrated solution to Eq. (6.24) is given by $\phi(r) = \phi_0 + q M_p r_g r/r_V$, where $\phi_0$ is a constant. Since the second term in $\phi(r)$ is much smaller than $M_p$ for $|q| \lesssim 1$ and $r_g \ll r \ll r_V$, the quantity $F(\phi) = e^{-2q\phi/M_p}$ is close to 1 (i.e., the value of GR) for $|\phi_0| \ll M_p$. In the following we shall focus on the case $|\phi_0| \ll M_p$ and employ the approximation $A_4^{-1} \approx -2M_p^2 (1 + 2a_4 X^2/M_p^2)$. Integrating Eqs. (6.17) and (6.8) after substitution of the solution (6.24), we obtain
\[
\Phi \simeq \frac{r_g}{2r} \left[1 - 2q^2 \left(\frac{r}{r_V}\right)^2 + \frac{q^2 (1 + 8 q^2)}{6} \frac{r_g}{r_V} \left(\frac{r}{r_V}\right)^3 \right] - \frac{2(a_4 + b_4)}{r_V} \frac{M_p^2 q r_g^3}{r_V} \frac{r}{r_V}, \quad (6.25)
\]
\[
\Psi \simeq \frac{r_g}{2r} \left[1 - 2q^2 \left(\frac{r}{r_V}\right)^2 - \frac{q^2 (1 + 2 q^2)}{3} \frac{r_g}{r_V} \left(\frac{r}{r_V}\right)^3 \right] - \frac{8(a_4 - b_4)}{3} \frac{M_p^2 q r_g^3}{r_V} \frac{r}{r_V}, \quad (6.26)
\]
where $r_\ast$ is an integration constant. We have dropped the contribution coming from the term $2a_4 X^2/M_p^2$ in $A_4^{-1}$, as this gives rise to corrections much smaller than the last terms of Eqs. (6.25) and (6.26). For the distance $r \ll r_V$ the contributions other than the first terms in the square brackets of Eqs. (6.25) and (6.26) are much smaller than 1, under which $\Phi$ and $-\Psi$ are close to the value $r_g/(2r)$ of GR.

To quantify the deviation from GR, we define the post-Newtonian parameter
\[
\gamma \equiv -\frac{\Phi}{\Psi}, \quad (6.27)
\]
The solar-system experiments placed the bound \[ |\gamma - 1| < 2.3 \times 10^{-5}. \] (6.28)

On using the solutions (6.29) and (6.26), it follows that \[
\gamma \simeq 1 + \frac{q^2(1 + 4q^2)}{2r_V^2} \left( \frac{r}{r_V} \right)^3 - \frac{2M_g^2q^2r_g^2}{r_V^3} \left[ a_4 + b_4 - 4(a_4 - b_4) \ln \frac{r}{r_q} \right].
\] (6.29)

The Vainshtein radius \( r_V \) depends on the mass scale \( M \) defined in Eq. (6.29). If the model (6.9) is responsible for \( M \) order of \( |c/r| \) around the origin, we have that \( r_V \sim (|q|r_g^2)_{1/3} \) from Eq. (6.23). Since \( r_g \approx 10^5 \text{cm} \) for the Sun, we can estimate \( r_V \approx 10^{25} \text{cm} \) for \( |q| \approx 1 \). Inside the solar system \( r \lesssim 10^{14} \text{cm} \), the second term on the rhs of Eq. (6.29) is smaller than \( 10^{-33} \). For \( r_s \) between \( r_g \) and \( r_V \) the term \( \ln r/r_s \) is at most of the order of 10, so the last term on the rhs of Eq. (6.29) does not exceed the order of \( 10^{-18} \). Hence the experimental bound (6.28) is well satisfied inside the solar system.

On using the relation \( |a_4 - 3b_4| \approx r_s^4M_{\text{pl}}^{-2} \) and the solution (6.23), the parameter \( \alpha_H \) inside the Vainshtein radius can be estimated as
\[
|\alpha_H| \approx \frac{r_s^4 \phi^4}{M_{\text{pl}}^4} \sim q^4 \frac{r_s^4}{r_V^4},
\] (6.30)
which is smaller than \( 10^{-28} \) for \( |q| \lesssim 1 \). Thus, the deviation from Horndeski theories is significantly suppressed for \( r \ll r_V \). If the Vainshtein mechanism is not at work and the solution is given by Eq. (6.24) outside a compact object, we have that \( |\alpha_H| \approx q^4 r_s^4 r_g^2 \) and hence \( |\alpha_H| \) exceeds the order of 1 for \( r \lesssim \sqrt{|q|r_V r_g} \). For the Sun and \( |q| \approx 1 \) this condition translates to \( r < 10^{16} \text{cm} \), so \( |\alpha_H| \) is much larger than 1 in the solar system. The above arguments show how the Vainshtein mechanism is efficient to suppress both the propagation of the fifth force and the deviation from Horndeski theories.

We recall that, for \( n = 2 \), the solution to Eq. (6.1) around \( r = 0 \) is given by \( \phi'(r) = cr \), where \( c \) obeys the relation (6.15). For the mass scale \( M \) of the order of \( M \sim |a_4|^{-1/6} \sim r_H^{-2/3} M_{\text{pl}}^{1/3} \) the term \( 8(a_4 - b_4)c^3 \) is the dominant contribution to the lhs of Eq. (6.15), so the solution around the origin is given by
\[
\phi'(r) \simeq \left( \frac{q \rho_m M_{\text{pl}}^6}{M_{\text{pl}}} \right)^{1/3} r,
\] (6.31)
where we have assumed \( q > 0 \). In fact, the ratio \( \xi = c/[8|a_4 - b_4|c^3] \) is of the order of \( \xi \sim (\rho_0/(q \rho_m))^{2/3} \ll 1 \) for \( q \sim 1 \), where \( \rho_0 \sim 10^{-29} \text{g/cm}^3 \) is the cosmological density and \( \rho_m \sim 1 \text{g/cm}^3 \) is the mean density of Sun.

The validity of the solution (6.31) is ensured around \( r = 0 \), but we can extrapolate it to the surface of the compact body (radius \( r_0 \)) to estimate the order of \( \phi'(r_0) \). On using the relations \( r_g \sim \rho_m r_3^3/M_{\text{pl}}^2 \) and \( M = (qM_{\text{pl}} r_g)^{1/3}/r_V \) the extrapolation of Eq. (6.31) gives \( \phi'(r_0) \sim qM_{\text{pl}} r_g/r_V^2 \), which is of the same order as Eq. (6.24). Thus, the two solutions (6.24) and (6.31) smoothly match each other around \( r = r_0 \). There are corrections to the solution (6.31), but they do not change the order of \( \phi'(r) \) inside the body. This situation is analogous to what was found in Ref. [17] in Horndeski theories.

The smallness of \( \alpha_H \) in our model comes from the fact that \( \alpha_H \) depends on \( X \), under the operation of the Vainshtein mechanism, the suppression of the field derivative leads to the small value of \( \alpha_H \). This is in contrast with the models where \( \alpha_H \) depends on \( \phi \) (i.e., \( F_1(\phi) \neq F_2(\phi) \)). In the latter case, the models suffer from not only the conical singularity problem, but also the breaking of the Vainshtein mechanism occurs as shown in Refs. [21, 22]. The fact that \( F_1(\phi) \) and \( F_2(\phi) \) are equivalent to each other is crucial for the success of the Vainshtein mechanism.

VII. CONCLUSIONS

In GLPV theories where the deviation from Horndeski theories is weighed by the parameter \( \alpha_H \), we have shown that the conical singularity arises at the origin of a spherically symmetric body for nonzero constant \( \alpha_H \) around the origin. For both vacuum and interior Schwarzschild solutions satisfying the boundary condition \( \phi'(r = 0) = 0 \), the Ricci scalar diverges as \( R = -2\alpha_H/r^2 \) around \( r = 0 \). In such cases, the spherically symmetric static body does not exist due to the singularity problem at its center. This divergence originates from the nonvanishing contribution \( \alpha_H \) to the gravitational potentials, which does not allow for the recovery of the Minkowski space-time.

For the theories in which \( \alpha_H \) vanishes as \( r \to 0 \), it is possible to avoid the appearance of the conical singularity. This requires the condition that the functions \( F_1(\phi) \) and \( F_2(\phi) \) appearing in Eq. (6.2) are equivalent to each other. The functions \( f_1(X) \) and \( f_2(X) \) need to be chosen such that they do not involve arbitrary constants which give rise to the difference between \( -A_2 \) and \( B_4 \). Violation of the condition \( F_1(\phi) = F_2(\phi) \) means that the geometric structure of the four-dimensional Ricci scalar \( R \) is modified, which causes a geodesically incomplete space-time at \( r = 0 \).

The model described by the functions (6.3), which corresponds to the extension of covariant Galileons with a dilatonic coupling, is free from the problem of the conical singularity. This model is designed to have the dependence of \( \alpha_H \) proportional to \( X^n \) around \( r = 0 \). We derived the field profile as well as the gravitational potentials around the center of a compact object under the approximation of weak gravity and showed that the Ricci scalar remains finite as a consequence of the vanishing \( \alpha_H \).
at the origin.

For the model (6.9) with $n = 2$ we also found that the Vainshtein mechanism is at work outside the body to suppress the fifth force and the parameter $\alpha_H$ in the solar system. The regular solution of the field derivative $\phi'(r)$ around the origin can be extrapolated to match the exterior solution around the surface of the body.

There are several issues we have not addressed in this paper. First, it will be of interest to study whether the similar properties for appearance and disappearance of conical singularities also hold for the theories involving the Lagrangian $L_5$ related to the Einstein tensor. Second, we showed that the functional form of Lagrangian must be properly chosen for eliminating the conical singularity, but the absence of a symmetry may give rise to quantum corrections which can modify the Lagrangian structure. If quantum corrections preserve the structures of four-dimensional curvature quantities like the Ricci scalar $R$, we expect that the modification is less harmful. If any difference of a constant between the functions $-A_4$ and $B_4$ arises by quantum corrections, this would lead to reappearance of the conical singularity. We leave these interesting issues for future works.

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