ON UNIMODULAR TRANSFORMATIONS OF CONSERVATIVE L-SYSTEMS

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Dedicated with great pleasure to Heinz Langer on the occasion of his 80-th birthday

Abstract. We study unimodular transformations of conservative L-systems. Classes $M^Q$, $M^Q_\kappa$, $M^{-1}Q$, $M^{-1}(-Q)$ that are impedance functions of the corresponding L-systems are introduced. A unique unimodular transformation of a given L-system with impedance function from the mentioned above classes is found such that the impedance function of a new L-system belongs to $M(-Q)$, $M(-Q)_\kappa$, $M^{-1}$, respectively. As a result we get that considered classes (that are perturbations of the Donoghue classes of Herglotz-Nevanlinna functions with an arbitrary real constant $Q$) are invariant under the corresponding unimodular transformations of L-systems. We define a coupling of an L-system and a so called F-system and on its basis obtain a multiplication theorem for their transfer functions. In particular, it is shown that any unimodular transformation of a given L-system is equivalent to a coupling of this system and the corresponding controller, an F-system with a constant unimodular transfer function. In addition, we derive an explicit form of a controller responsible for a corresponding unimodular transformation of an L-system. Examples that illustrate the developed approach are presented.

1. Introduction

This paper is yet another part of an ongoing project studying the connections between various subclasses of Herglotz-Nevanlinna functions and conservative realizations of L-systems with one-dimensional input-output space (see [3], [6], [7], [15], [16]).

Let $T$ be a densely defined closed operator in a Hilbert space $\mathcal{H}$ such that its resolvent set $\rho(T)$ is not empty. We also assume that $\text{Dom}(T) \cap \text{Dom}(T^*)$ is dense and that the restriction $T|_{\text{Dom}(T) \cap \text{Dom}(T^*)}$ is a closed symmetric operator with finite equal deficiency indices. Let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be the rigged Hilbert space associated with $\tilde{A}$.

One of the main objectives of the current paper is the study of the L-system

\[
\Theta = \begin{pmatrix}
\mathbb{A} & K \\
\mathbb{H}_+ \subset \mathbb{H} \subset \mathbb{H}_- & E
\end{pmatrix},
\]

where the state-space operator $\mathbb{A}$ is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ such that $\mathbb{A} \subset T \subset \mathbb{A}$, $\mathbb{A}^* \subset T^* \subset \mathbb{A}$, $K$ is a bounded linear operator from the finite-dimensional Hilbert space $E$ into $\mathcal{H}_-$, $J = J^* = (J^{-1}$ is a self-adjoint

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isometry on $E$ such that $\text{Im} \, A = KJ^*$. Due to the facts that $\mathcal{H}_\pm$ is dual to $\mathcal{H}_\mp$ and that $A^*$ is a bounded linear operator from $\mathcal{H}_+ \to \mathcal{H}_-$, $\text{Im} \, A = (A - A^*)/2i$ is a well defined bounded operator from $\mathcal{H}_+ \to \mathcal{H}_-$. Note that the main operator $T$ associated with the system $\Theta$ is uniquely determined by the state-space operator $A$ as its restriction onto the domain $\text{Dom}(T) = \{ f \in \mathcal{H}_+ \mid A \, f \in \mathcal{H} \}$. A detailed description of the $L$-systems together with their connections to various subclasses of Herglotz-Nevanlinna functions can be found in [3] (see also [1], [2], [5], [6], [7], [9]).

Recall that the operator-valued function given by
$$W_\Theta(z) = I - 2iK^*(A - zI)^{-1}KJ, \quad z \in \rho(T),$$
is called the transfer function of the $L$-system $\Theta$ and
$$V_\Theta(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\text{Re} \, A - zI)^{-1}K, \quad z \in \rho(T) \cap \mathbb{C}_\pm,$$is called the impedance function of $\Theta$.

In addition to $L$-systems we also recall (see [12], [3]) the definition of $F$-systems of the form
$$\Theta_F = \begin{pmatrix} M & F & J \\ H & K & E \end{pmatrix},$$
that will play an auxiliary role in our development.

The main goal of the paper is to study the effect of a unimodular transformation applied to an $L$-system with one-dimensional input-output space. A new twist in our exposition is introducing the concept of $LF$-coupling of systems and a controller. Applying the latter to an $L$-system has an effect equivalent to a corresponding unimodular transformation.

The paper is organized as follows.

In Section 2 we recall the definitions of $L$- and $F$-systems, their transfer and impedance functions, and provide necessary background.

In Section 3 we introduce the concept of an $LF$-coupling that is a coupling of an $L$-system and an $F$-system. We also obtain a multiplication theorem of relating transfer functions of LF-coupling and both individual $L$- and $F$-system being coupled this way.

In Section 4 we present the “perturbed” classes $M_Q$, $M_Q^{\kappa}$, and $M_{-1,Q}^{\kappa}$ of impedance functions of $L$-systems with one-dimensional input-output space.

Section 5 contains the definition of a unimodular transformation of an $L$-system of the type considered in Section 4 and main results of the paper. Here we construct a unique unimodular transformation of a given $L$-system with impedance function from $M_Q$, $M_Q^{\kappa}$, and $M_{-1,Q}^{\kappa}$ classes such that the impedance function of a new $L$-system belongs to $M(-Q)$, $M(-Q)^{\kappa}$, $M_{-1}(-Q)^{\kappa}$, respectively.

In Section 6 we put forward a concept of a controller that is a special form of an $F$-system with a constant unimodular transfer function. We show that any unimodular transformation of a given $L$-system is equivalent to a coupling of this system with the corresponding controller. In the end of the section we also present an analog of the “absorption property” for the Donoghue class $M$ that was discussed in [7].

We conclude the paper by providing several examples that illustrate all the main results and concepts. Connections of the considered systems and the corresponding differential equations are pointed out in Appendix A.
2. Preliminaries

For a pair of Hilbert spaces $H_1, H_2$ we denote by $[H_1, H_2]$ the set of all bounded linear operators from $H_1$ to $H_2$. Let $\hat{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $H$ with inner product $(f, g), f, g \in H$. Any non-symmetric operator $T$ in $H$ such that

$$\hat{A} \subset T \subset \hat{A}^*$$

is called a quasi-self-adjoint extension of $\hat{A}$.

Consider the rigged Hilbert space (see [1], [3]) $H_+ \subset H \subset H_-$, where $H_+ = \text{Dom}(\hat{A}^*)$ and

$$(f, g)_+ = (f, g) + (\hat{A}^*f, \hat{A}^*g), \ f, g \in \text{Dom}(\hat{A}^*).$$

Let $\mathcal{R}$ be the Riesz-Berezansky operator $\mathcal{R}$ (see [1], [3]) which maps $H_-$ onto $H_+$ such that $(f, g)_+ = (f, \mathcal{R}g)_+$ $(\forall f \in H_+, \ g \in H_-)$ and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $H_\pm$ with $H_\mp$, we get that if $\mathcal{A} \in [H_+, H_-]$, then $\mathcal{A}^* \in [H_+, H_-]$. An operator $\mathcal{A} \in [H_+, H_-]$ is called a self-adjoint bi-extension of a symmetric operator $\hat{A}$ if $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A} \supset \hat{A}$. Let $\mathcal{A}$ be a self-adjoint bi-extension of $\hat{A}$ and let the operator $\hat{A}$ in $H$ be defined as follows:

$$\text{Dom}(\hat{A}) = \{ f \in H_+ : \hat{A}f \in H \}, \ \hat{A} = \mathcal{A} | \text{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a quasi-kernel of a self-adjoint bi-extension $\mathcal{A}$ (see [1], [3], Section 2.1). According to the von Neumann Theorem (see [3], Theorem 1.3.1) the domain of $\hat{A}$, a self-adjoint extension of $\hat{A}$, can be expressed as

$$(3) \quad \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) \oplus (I + U)\mathcal{N}_i,$$

where $U$ is a (-) and (+)-isometric operator from $\mathcal{N}_i$ into $\mathcal{N}_{-i}$ and

$$\mathcal{N}_{\pm i} = \text{Ker}(\hat{A}^* \mp iI)$$

are the deficiency subspaces of $\hat{A}$. A self-adjoint bi-extension $\mathcal{A}$ of a symmetric operator $\hat{A}$ is called t-self-adjoint (see [3], Definition 3.3.5) if its quasi-kernel $\hat{A}$ is self-adjoint operator in $H$. An operator $\mathcal{A} \in [H_+, H_-]$ is called a quasi-self-adjoint bi-extension of a non-symmetric operator $T$ if $\mathcal{A} \supset T \supset \hat{A}$ and $\mathcal{A}^* \supset T^* \supset \hat{A}$. We will be mostly interested in the following type of quasi-self-adjoint bi-extensions.

**Definition 1** ([3]). Let $T$ be a quasi-self-adjoint extension of $\hat{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathcal{A}$ of an operator $T$ is called a $(\ast)$-extension of $T$ if $\text{Re} \mathcal{A}$ is a t-self-adjoint bi-extension of $\hat{A}$.

In what follows we assume that $\hat{A}$ has equal finite deficiency indices and will say that a quasi-self-adjoint extension $T$ of $\hat{A}$ belongs to the class $\Lambda(\hat{A})$ if $\rho(T) \neq \emptyset$, $\text{Dom}(\hat{A}) = \text{Dom}(T) \cap \text{Dom}(T^*)$, and hence $T$ admits $(\ast)$-extensions. The description of all $(\ast)$-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [3], Section 4.3.

**Definition 2.** A system of equations

$$\begin{cases} (\mathcal{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases},$$

or an array

$$(4) \quad \Theta = \left( \begin{array}{cc} \mathcal{A} & K \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & J \end{array} \right)$$
is called an L-system if:
(1) $\mathcal{A}$ is a $(\ast)$-extension of an operator $T$ of the class $\Lambda(\hat{A})$;
(2) $J = J^* = J^{-1} \in [E, E]$, $\dim E < \infty$;
(3) $\text{Im} \mathcal{A} = KJK^*$, where $K \in [E, \mathcal{H}]$, $K^* \in [\mathcal{H}, E]$, and $\text{Ran}(K) = \text{Ran}(\mathcal{A})$.

In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and $x$ is a state space vector in $\mathcal{H}$. The operator $\mathcal{A}$ is called the state-space operator of the system $\Theta$, $T$ is the main operator, $J$ is the direction operator, and $K$ is the channel operator. A system $\Theta$ in (iv) is called minimal if the operator $\hat{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator.

We associate with an L-system $\Theta$ the operator-valued function

$$W_\Theta(z) = I - 2izK^*(\mathcal{A} - zI)^{-1}KJ, \quad z \in \rho(T),$$

which is called the transfer function of the L-system $\Theta$. We also consider the operator-valued function

$$V_\Theta(z) = K^*(\text{Re} \mathcal{A} - zI)^{-1}K, \quad z \in \rho(\hat{A}).$$

It was shown in [1, 16, Section 6.3] that both (i) and (ii) are well defined. The transfer operator-function $W_\Theta(z)$ of the system $\Theta$ and an operator-function $V_\Theta(z)$ of the form (iii) are connected by the following relations valid for $\text{Im} z \neq 0$, $z \in \rho(T)$,

$$V_\Theta(z) = i[W_\Theta(z) + 1]^{-1}|W_\Theta(z) - 1|J,$$
$$W_\Theta(z) = (I + iV_\Theta(z))J^{-1}(I - iV_\Theta(z))J.$$

The function $V_\Theta(z)$ defined by (iv) is called the impedance function of an L-system $\Theta$ of the form (iii). The class of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space $E$, that can be realized as impedance functions of an L-system, was described in [1, 16, Definition 6.4.1].

Let $A$ be a closed linear operator in a Hilbert space $\mathcal{H}$ and let $F$ be an orthogonal projection in $\mathcal{H}$. Associated to the pair $(A, F)$ is the resolvent set $\rho(A, F)$, i.e., the set of all $z \in \mathbb{C}$ for which $A - zF$ is boundedly invertible in $\mathcal{H}$ and $(A - zF)^{-1}$ is defined on entire $\mathcal{H}$. The corresponding resolvent operator is defined as $(A - zF)^{-1}$, $z \in \rho(A, F)$. Following [16, Chapter 12], [18] we put forward the following

**Definition 3.** Let $\mathcal{H}$ and $E$ be Hilbert spaces with $\dim E < \infty$. A system of equations

$$\begin{align*}
(M - zF)x &= KJ\varphi_-, \\
\varphi_+ &= \varphi_- - 2iK^*x,
\end{align*}$$

or an array

$$\Theta_F = \begin{pmatrix} M & F & K & J \\ \mathcal{H} & E & E & E \end{pmatrix},$$

is called an $F$-system if:

(i) $M \in [\mathcal{H}, \mathcal{H}]$;
(ii) $J = J^* = J^{-1} \in [E, E]$;
(iii) $\text{Im} M = KJK^*$, where $K \in [E, \mathcal{H}]$;
(iv) $F$ is an orthogonal projection in $\mathcal{H}$;
(v) the resolvent sets $\rho(\text{Re} M, F)$ and $\rho(M, F)$ are nonempty.
To each $F$-system in Definition 3, one can associate the following transfer function
\begin{equation}
W_{\Theta_F}(z) = I - 2iK^*(M - zF)^{-1}KJ, \quad z \in \rho(M, F),
\end{equation}
and the impedance function
\begin{equation}
V_{\Theta_F}(z) = K^*(Re M - zF)^{-1}K, \quad z \in \rho(Re M, F).
\end{equation}

Consider the two $F$-systems $\Theta_{F_1}$ and $\Theta_{F_2}$ of the form (9), defined by
\begin{equation}
\Theta_{F_1} = \left( \begin{array}{ccc}
M_1 & F_1 & K_1 \\
H_1 & J & E
\end{array} \right),
\end{equation}
and
\begin{equation}
\Theta_{F_2} = \left( \begin{array}{ccc}
M_2 & F_2 & K_2 \\
H_2 & J & E
\end{array} \right).
\end{equation}

Define the Hilbert space $\mathcal{H}$ by
\begin{equation}
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,
\end{equation}
and let $P_j$ be the orthoprojections from $\mathcal{H}$ onto $\mathcal{H}_j$, $j = 1, 2$. Define the operators $M, F,$ and $K$ by
\begin{equation}
M = M_1P_1 + M_2P_2 + 2iK_1JK_2^*P_2, \quad F = F_1P_1 + F_2P_2, \quad K = K_1 + K_2.
\end{equation}

It is shown in [3, Theorem 12.2.1], [12] that if $\Theta_{F_1}$ is the $F_1$-system in (12) and let $\Theta_{F_2}$ is the $F_2$-system in (13), then the aggregate
\begin{equation}
\Theta = \left( \begin{array}{ccc}
M & F & K \\
\mathcal{H} & J & E
\end{array} \right),
\end{equation}
with $\mathcal{H}, M, F,$ and $K$, defined by (14) and (15), is also an $F$-system. This $F$-system $\Theta$ in (16) is called the coupling of the $F_1$-system $\Theta_{F_1}$ and the $F_2$-system $\Theta_{F_2}$. It is denoted by
\begin{equation}
\Theta = \Theta_{F_1} \cdot \Theta_{F_2}.
\end{equation}

It is also shown in [3, Theorem 12.2.2], [12] that if an $F$-system $\Theta$ is the coupling of the $F_1$-system $\Theta_{F_1}$ and the $F_2$-system $\Theta_{F_2}$, then the associated transfer functions satisfy
\begin{equation}
W_{\Theta}(z) = W_{\Theta_{F_1}}(z)W_{\Theta_{F_2}}(z), \quad z \in \rho(M_1, F_1) \cap \rho(M_2, F_2).
\end{equation}

3. Mixed coupling of $L$-systems and $F$-systems

Consider an $L$-system $\Theta_L$ and an $F$-system $\Theta_F$ of the forms (4) and (9), respectively, and defined by
\begin{equation}
\Theta_L = \left( \begin{array}{ccc}
A & K_1 \\
\mathcal{H}_+ \subset \mathcal{H}_1 \subset \mathcal{H}_{-1} & J & E
\end{array} \right),
\end{equation}
and
\begin{equation}
\Theta_F = \left( \begin{array}{ccc}
M & F & J \\
\mathcal{H}_2 & K_2 & E
\end{array} \right),
\end{equation}
where $M$ is a bounded in $\mathcal{H}_2$ operator. Define the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ by
\begin{equation}
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- = \mathcal{H}_+ \oplus \mathcal{H}_2 \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}_{-1} \oplus \mathcal{H}_2.
\end{equation}
Define the operators $M \in [\mathcal{H}_+, \mathcal{H}_-], F : \mathcal{H} \to \mathcal{H}_2,$ and $K : E \to \mathcal{H}_-$ by
\begin{equation}
M = \begin{pmatrix}
A & 2iK_1JK_2^*
\end{pmatrix}, \quad F = \begin{pmatrix}
I & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
K_1 \\
K_2
\end{pmatrix}.
\end{equation}

**Definition 4.** Let $\Theta_L$ be the L-system in (22) and let $\Theta_F$ be the F-system in (22). Then the aggregate
\begin{equation}
\Theta_{LF} = \Theta_L \cdot \Theta_F = \frac{M}{\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-} F K J E,
\end{equation}
with $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, $M$, $F$, and $K$, defined by (22) and (21), is called an LF-coupling of systems $\Theta_L$ and $\Theta_F$.

Taking adjoints in (22) gives
\begin{equation}
M^* = \begin{pmatrix}
A^* & 0 \\
-2iK_2JK_1^*
\end{pmatrix}, \quad K^* = \begin{pmatrix}
K_1^* \\
K_2
\end{pmatrix}^T, \quad KJ = \begin{pmatrix}
K_1J \\
K_2J
\end{pmatrix},
\end{equation}
and therefore,
\begin{equation}
M - M^* = \begin{pmatrix}
A - A^* & 2iK_1JK_2^* \\
2iK_2JK_1^* & M - M^*
\end{pmatrix} = 2i \begin{pmatrix}
K_1JK_1^* \\
K_2JK_1^*
\end{pmatrix} = 2iKK^*.
\end{equation}

A function
\begin{equation}
W_{\Theta_{LF}}(z) = 1 - 2iK^* (M - zF)^{-1} KJ, \quad z \in \rho(M, F),
\end{equation}
will be associated with LF-coupling and called the transfer function of LF-coupling.

**Theorem 5.** Let $\Theta$ be the LF-coupling of an L-system $\Theta_L$ and the F-system $\Theta_F$. Then the associated transfer functions satisfy
\begin{equation}
W_{\Theta_{LF}}(z) = W_{\Theta_L}(z)W_{\Theta_F}(z), \quad z \in \rho(T) \cap \rho(M, F).
\end{equation}

**Proof.** Let $z \in \rho(T) \cap \rho(M, F)$. Observe that
\begin{equation}
M - zF = \begin{pmatrix}
A & 2iK_1JK_2^* \\
0 & M - zF
\end{pmatrix} - z \begin{pmatrix}
I & 0 \\
0 & F
\end{pmatrix} = \begin{pmatrix}
A - zI & 2iK_1JK_2^* \\
0 & M - zF
\end{pmatrix},
\end{equation}
and hence
\begin{equation}
(M - zF)^{-1} = \begin{pmatrix}
(A - zI)^{-1} & -2i(A - zI)^{-1}K_1JK_2^*(M - zF)^{-1} \\
0 & (M - zF)^{-1}
\end{pmatrix}.
\end{equation}

Indeed, by direct check
\begin{align}
(M - zF)(M - zF)^{-1} &= \begin{pmatrix}
A - zI & 2iK_1JK_2^* \\
0 & M - zF
\end{pmatrix} \begin{pmatrix}
(A - zI)^{-1} & -2i(A - zI)^{-1}K_1JK_2^*(M - zF)^{-1} \\
0 & (M - zF)^{-1}
\end{pmatrix} \\
&= \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} = I.
\end{align}

Consequently,
\begin{equation}
(M - zF)^{-1}K = \begin{pmatrix}
(A - zI)^{-1} & -2i(A - zI)^{-1}K_1JK_2^*(M - zF)^{-1} \\
0 & (M - zF)^{-1}
\end{pmatrix} \begin{pmatrix}
K_1 \\
K_2
\end{pmatrix} = \begin{pmatrix}
(A - zI)^{-1}K_1 - 2i(A - zI)^{-1}K_1JK_2^*(M - zF)^{-1}K_2 \\
(M - zF)^{-1}K_2
\end{pmatrix}.
\end{equation}
and
\[ K^* (\mathbb{M} - z \mathbb{F})^{-1} K = (K_1^* \ K_2^*) \left( (\mathbb{A} - z \mathbb{I})^{-1} K_1 - 2i(\mathbb{A} - z \mathbb{I})^{-1} K_1 J K_2^*(M - z \mathbb{F})^{-1} K_2 \right) (M - z \mathbb{F})^{-1} K_2 \]

Furthermore, \((\mathbb{V})(26)\) follows from 
\[ W_{\Theta,L,F}(z) = I - 2i K^* (\mathbb{M} - z \mathbb{F})^{-1} K J \]

will be associated with \(LF\)-coupling and called the **impedance function of \(LF\)-coupling**. First, let us show that the impedance function of \(LF\)-coupling is well defined. It follows from (24) and (26) that 
\[ \text{Re} \mathbb{M} - z \mathbb{I} = \begin{pmatrix} \text{Re} \mathbb{A} - z \mathbb{I} & iK_1 J K_2^* \\ -iK_2 J K_1^* & \text{Re} M - z \mathbb{F} \end{pmatrix} \]

Let \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), where \( x_1 \in \mathcal{H}_1, \ x_2 \in \mathcal{H}_2 \). Consider an equation 
\[ (\text{Re} \mathbb{M} - z \mathbb{I}) x = \begin{pmatrix} (\text{Re} \mathbb{A} - z \mathbb{I}) x_1 & iK_1 J K_2^* \\ -iK_2 J K_1^* & \text{Re} M - z \mathbb{F} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\text{Re} \mathbb{A} - z \mathbb{I}) x_1 + iK_1 J K_2^* x_2 \\ -iK_2 J K_1^* x_1 + (\text{Re} M - z \mathbb{F}) x_2 \end{pmatrix} = \begin{pmatrix} K_1 e \\ K_2 e \end{pmatrix}, \]

for some \( e \in E \). Then 
\[ (\text{Re} \mathbb{A} - z \mathbb{I}) x_1 + iK_1 J K_2^* x_2 = K_1 e, \]
\[ -iK_2 J K_1^* x_1 + (\text{Re} M - z \mathbb{F}) x_2 = K_2 e. \]

Applying \((\text{Re} \mathbb{A} - z \mathbb{I})^{-1}\) to the first equation and solving the result for \( x_1 \) yields 
\[ x_1 = (\text{Re} \mathbb{A} - z \mathbb{I})^{-1} [K_1 e - iK_1 J K_2^* x_2], \]

Substituting this value of \( x_1 \) in to the second equation, we have 
\[ -iK_2 J K_1^* (\text{Re} \mathbb{A} - z \mathbb{I})^{-1} [K_1 e - iK_1 J K_2^* x_2] + (\text{Re} M - z \mathbb{F}) x_2 = K_2 e, \]
or 
\[ [\text{Re} M - z \mathbb{F} - K_2 J K_1^* (\text{Re} \mathbb{A} - z \mathbb{I})^{-1} K_1 J K_2^*] x_2 = K_2 [I + iJ V_{\Theta,L}(z)] e. \]

Taking into account that the impedance function of our L-system \( \Theta_L \) is given by 
\[ V_{\Theta,L}(z) = K_1^* (\text{Re} \mathbb{A} - z \mathbb{I})^{-1} K_1, \]
we have 
\[ (27) \quad [\text{Re} M - z \mathbb{F} - K_2 J V_{\Theta,L}(z) J K_2^*] x_2 = K_2 [I + iJ V_{\Theta,L}(z)] e. \]
Multiplying both sides of (27) by $K_2^*(\text{Re } M - z F)^{-1}$ yields

\[ [K_2^* - K_2^*(\text{Re } M - z F)^{-1} K_2 J V_{\Theta_L}(z) J K_2^*] x_2 = K_2^*(\text{Re } M - z F)^{-1} K_2 [I + i J V_{\Theta_L}(z)] e. \]

We recall that

\[ V_{\Theta_F}(z) = K_2^*(\text{Re } M - z F)^{-1} K_2, \]

and obtain

\[ [I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J] K_2^* x_2 = V_{\Theta_F}(z) [I + i J V_{\Theta_L}(z)] e. \]

Let us assume that in addition to $\rho(\text{Re } M, F) \neq 0$ we have that the operator-function $[I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]$ is invertible at some point $z_0 \in \mathbb{C}_+$. Then applying the theorem on holomorphic operator-function \[ \text{[1, Appendix 2]} \] we have that $[I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]$ is invertible on the entire $\mathbb{C}_+$. Then

\[ K_2^* x_2 = [I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]^{-1} V_{\Theta_F}(z) [I + i J V_{\Theta_L}(z)] e. \]

Consequently, (27) can be modified into

\[ (\text{Re } M - z F) x_2 - K_2 J V_{\Theta_L}(z) J [I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]^{-1} V_{\Theta_F}(z) [I + i J V_{\Theta_L}(z)] e = K_2 [I + i J V_{\Theta_L}(z)] e, \]

which can be solved for $x_2$ as

\[ x_2 = (\text{Re } M - z F)^{-1} \times (K_2 J V_{\Theta_L}(z) J [I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]^{-1} V_{\Theta_F}(z) [I + i J V_{\Theta_L}(z)] e). \]

Thus, under the assumptions that $\rho(\text{Re } M, F) \neq 0$ and $[I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]$ is invertible at some point $z_0 \in \mathbb{C}_+$, the impedance function $V_{\Theta_{LF}}(z)$ is well defined by (26).

The impedance function $V_{\Theta_{LF}}(z)$ defined in (26) and the transfer function $W_{\Theta_{LF}}(z)$ defined in (24) are closely connected.

**Lemma 6.** Let $\Theta_{LF}$ be an LF-coupling of the form (22). Let also $\rho(\text{Re } M, F) \neq 0$ and $[I - V_{\Theta_F}(z) J V_{\Theta_L}(z) J]$ be invertible at some point $z_0 \in \mathbb{C}_+$. Then for all $z \in \rho(\mathbb{M}, \mathbb{F}) \cap \rho(\text{Re } \mathbb{M}, \mathbb{F})$

\[ V_{\Theta_{LF}}(z) = i [W_{\Theta_{LF}}(z) - I] [W_{\Theta_{LF}}(z) + I]^{-1} J = i [W_{\Theta_{LF}}(z) + I]^{-1} [W_{\Theta_{LF}}(z) - I] J, \]

and

\[ W_{\Theta_{LF}}(z) = [I - i V_{\Theta_{LF}}(z) J] [I + i V_{\Theta_{LF}}(z) J]^{-1} = [I + i V_{\Theta_{LF}}(z) J]^{-1} [I - i V_{\Theta_{LF}}(z) J]. \]

**Proof.** The following identity with $z \in \rho(\mathbb{M}, \mathbb{F}) \cap \rho(\text{Re } \mathbb{M}, \mathbb{F})$

\[ (\text{Re } \mathbb{M} - z \mathbb{F})^{-1} - (\mathbb{M} - z \mathbb{F})^{-1} = i (\mathbb{M} - z \mathbb{F})^{-1} \text{Im } (\text{Re } \mathbb{M} - z \mathbb{F})^{-1}, \]

leads to

\[ K^*(\text{Re } \mathbb{M} - z \mathbb{F})^{-1} K - K^*(\mathbb{M} - z \mathbb{F})^{-1} K^* = i K^*(\mathbb{M} - z \mathbb{F})^{-1} K J K^*(\text{Re } \mathbb{M} - z \mathbb{F})^{-1} K. \]

Now in view of (24) and (20)

\[ 2 V_{\Theta_{LF}}(z) + i (I - W_{\Theta_{LF}}(z)) J = (I - W_{\Theta_{LF}}(z)) V_{\Theta_{LF}}(z), \]

or equivalently,

\[ [I + W_{\Theta_{LF}}(z)] [I + i V_{\Theta_{LF}}(z) J] = 2 I. \]
Similarly, the identity
\[(\text{Re} \mathcal{M} - z \mathcal{F})^{-1} - (\mathcal{M} - z \mathcal{F})^{-1} = i(\text{Re} \mathcal{M} - z \mathcal{F})^{-1} \text{Im} \mathcal{M}(\mathcal{M} - z \mathcal{F})^{-1}\]
with \(z \in \rho(\mathcal{M}, \mathcal{F}) \cap \rho(\text{Re} \mathcal{M}, \mathcal{F})\) leads to
\[(31) \quad [I + iV_{\Theta_{LF}}(z)J][I + W_{\Theta_{LF}}(z)] = 2I.\]
The equalities (30) and (31) show that the operators are boundedly invertible and consequently one obtains (28) and (29).

It was shown in [3, Theorem 12.2.4], [4] that each constant \(J\)-unitary operator \(B\) on a finite-dimensional Hilbert space \(E\) can be realized as a transfer function of some \(F\)-system of the form (9). Let us recall the construction of the realizing \(F\)-system. Assume that \((\pm 1)\) belongs to the resolvent set of the \(J\)-unitary operator \(B\), and define
\[C = i[B - I][B + I]^{-1}J.\]
As it was shown in the proof of [3, Theorem 12.2.4], \(C\) is a self-adjoint operator. Let also \(K : E \to E\) be any bounded and boundedly invertible operator. Then the aggregate
\[(32) \quad \Theta_0 = \begin{pmatrix} KC^{-1}(I + iCJ)K^* & 0 & K \\ 0 & J \\ E & E \end{pmatrix},\]
is an \(F\)-system with \(F = 0\). By construction, \(W_{\Theta_0}(z) \equiv B\). Let \(\Theta_L\) be an \(L\)-system of the form (18). If we compose the \(LF\)-coupling \(\Theta_{L0}\) of \(\Theta_L\) and \(\Theta_0\) of the form (22)
\[\Theta_{L0} = \Theta_L \cdot \Theta_0,\]
then according to Theorem 4
\[(33) \quad W_{\Theta_{L0}}(z) = W_{\Theta_{L}}(z)W_{\Theta_0}(z) = W_{\Theta_{L}}(z)B.\]

As it was also shown in the proof of [3, Theorem 12.2.4], the condition of \((\pm 1) \in \rho(B)\) can be released since \(E\) is finite-dimensional. In this case it is easy to see that \(B\) can be represented in the form \(B = B_1B_2\), where \(B_j\) is a \(J\)-unitary operator in \(E\) and \((\pm 1) \in \rho(B_j), j = 1, 2\). Each of the operators \(B_1\) and \(B_2\) can be realized (see [3, Theorem 12.2.4]) as transfer functions of two \(F\)-systems \(\Theta_{F_1}\) and \(\Theta_{F_2}\), respectively, i.e.,
\[W_{\Theta_{F_1}}(z) = B_1, \quad W_{\Theta_{F_2}}(z) = B_2.\]
Consider the coupling \(\Theta_F = \Theta_{F_1} \cdot \Theta_{F_2}\) of these \(F\)-systems as defined in (10) and apply the multiplication formula (17). Then
\[W_{\Theta_F}(z) = W_{\Theta_{F_1}}(z)W_{\Theta_{F_2}}(z) = B_1B_2 = B.\]

4. Systems with one-dimensional input-output and Donoghue classes

In this Section we are going to apply the concepts and results covered in Section 4 to \(L\)- and \(F\)-systems with one-dimensional input-output space \(\mathbb{C}\). Let
\[(34) \quad \Theta_L = \begin{pmatrix} \mathcal{H}_{+1} & \mathcal{H}_1 \subset \mathcal{H}_{-1} & K_1 \\ 1 \end{pmatrix}.\]
be a minimal scattering $L$-system of the form $[13]$ with one-dimensional input-output space $\mathbb{C}$ with the main operator $T$ and the quasi-kernel $\hat{A}$ of $\text{Re} \hat{A}$. Let also

$$
\Theta_F = \begin{pmatrix} M & F \\ \mathcal{H}_2 & K_2 & 1 \\ \mathbb{C} \end{pmatrix}.
$$

be a minimal $F$-system of the form $[13]$ also with one-dimensional input-output space $\mathbb{C}$ and $J = 1$. Then the $LF$-coupling $\Theta_{LF} = \Theta_L \cdot \Theta_F$ of the form $[22]$ takes the reduced form

$$
\Theta_{LF} = \Theta_L \cdot \Theta_F = \begin{pmatrix} M & F \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & K & 1 \\ \mathbb{C} \end{pmatrix}.
$$

Let us observe that in the case under consideration the conditions of Lemma $[6]$ can be weakened since $[1 - V_{\Theta_F}(z)V_{\Theta_L}(z)]$ is always invertible at some point $z_0 \in \mathbb{C}_+$. Indeed, suppose $z_1 \in \mathbb{C}_+$ is a point where $1 - V_{\Theta_F}(z_1)V_{\Theta_L}(z_1) = 0$. Then

$$
V_{\Theta_L}(z_1) = \frac{1}{V_{\Theta_F}(z_1)}.
$$

We know (see $[3]$) that both $V_{\Theta_F}(z)$ and $V_{\Theta_L}(z)$ are Herglotz-Nevanlinna functions mapping $\mathbb{C}_+$ into itself. Then left hand side of $[37]$ belongs to the upper half-plane while the right hand side clearly must lie in $\mathbb{C}_-$ which is a contradiction. Therefore $[1 - V_{\Theta_F}(z)V_{\Theta_L}(z)]$ is invertible at any $z \in \mathbb{C}_+$.

Now we recall the definitions of Donoghue classes of scalar functions (see $[6], [7], [10]$).

Denote by $\mathfrak{M}$ the Donoghue class of all analytic mappings $M$ from $\mathbb{C}_+$ into itself that admits the representation (see $[7], [11], [13]$)

$$
M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu,
$$

where $\mu$ is an infinite Borel measure and

$$
\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1, \quad \text{equivalently,} \quad M(i) = i.
$$

We say (see $[8]$) that an analytic function $M$ from $\mathbb{C}_+$ into itself belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$, $(0 \leq \kappa < 1)$ if it admits the representation $[38]$ and

$$
\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = \frac{1 - \kappa}{1 + \kappa}, \quad \text{equivalently,} \quad M(i) = i \frac{1 - \kappa}{1 + \kappa},
$$

and to the generalized Donoghue class $\mathfrak{M}_\kappa^{-1}$, $(0 \leq \kappa < 1)$ if it admits the representation $[38]$ and

$$
\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = \frac{1 + \kappa}{1 - \kappa}, \quad \text{equivalently,} \quad M(i) = i \frac{1 + \kappa}{1 - \kappa}.
$$

Clearly, $\mathfrak{M}_0 = \mathfrak{M}_1^{-1} = \mathfrak{M}$, the (standard) Donoghue class introduced above.

It is shown in $[8]$ Theorem 11] that the impedance function $V_{\Theta}(z)$ of an $L$-system $\Theta$ of the form $[24]$ belongs to the class $\mathfrak{M}$ if and only if the von Neumann parameter $\kappa$ of the main operator $T$ of $\Theta$ is zero. Similar descriptions were given to $L$-systems $\Theta$ whose impedance functions belong to classes $\mathfrak{M}_\kappa$, $\mathfrak{M}_\kappa^{-1}$ (see $[8]$ Theorem 12] and $[7$, Theorem 5.4]).
Let us introduce the “perturbed” versions of the Donoghue classes above. We say that a scalar Herglotz-Nevanlinna function \( V(z) \) belongs to the class \( \mathcal{M}^Q \) if it admits the following integral representation

\[
V(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu, \quad Q = \bar{Q},
\]

and has condition (39) on the measure \( \mu \). Similarly, we introduce perturbed classes \( \mathcal{M}^Q_{\kappa} \) and \( \mathcal{M}^{-1}_{\kappa} \) if normalization conditions (40) and (41), respectively, hold on measure \( \mu \) in (42).

Let us note that it was shown in [3] that every function of a Donoghue class mentioned above (standard, generalized, or perturbed) belongs to the class of Krein-Langer \( Q \)-functions introduced in [14].

5. A unimodular transformation of an L-system

Consider an L-system \( \Theta \) of the form (34) with a main operator \( T \) and transfer function \( W_\Theta(z) \). Let \( B \) be a complex number such that \( |B| = 1 \). It was shown in [3], Theorem 8.2.3 (see also [4]) that there exists another L-system \( \Theta_B \) of the form (34) with the same main operator \( T \) and such that \( W_{\Theta_B}(z) = W_\Theta(z)B \). We rely on this result to put forward the following definition.

**Definition 7.** An L-system \( \Theta_\alpha \) is called a unimodular transformation of an L-system \( \Theta \) of the form (34) for some \( \alpha \in [0, \pi) \) if

\[
W_{\Theta_\alpha}(z) = W_\Theta(z) \cdot (-e^{2i\alpha}),
\]

where \( W_\Theta(z) \) and \( W_{\Theta_\alpha}(z) \) are transfer functions of the corresponding L-systems.

Note that \( \Theta_{\pi/2} = \Theta \). It is known (see [3], Theorem 8.3.1) that if \( \Theta_\alpha \) is a unimodular transformation of \( \Theta \) and \( V_{\Theta_\alpha}(z) \) is its impedance function then

\[
V_{\Theta_\alpha}(z) = \frac{\cos \alpha + (\sin \alpha)V_\Theta(z)}{\sin \alpha - (\cos \alpha)V_\Theta(z)}, \quad z \in \mathbb{C}_+.
\]

The following theorem shows that the class \( \mathcal{M} \) is in some sense invariant under a unimodular transformation.

**Theorem 8.** Let \( \Theta_\alpha \) be a unimodular transformation of an L-system \( \Theta \) with the impedance function \( V_\Theta(z) \) that belongs to class \( \mathcal{M} \). Then \( V_{\Theta_\alpha}(z) \in \mathcal{M} \).

**Proof.** Since \( \Theta_\alpha \) is a unimodular transformation of \( \Theta \), then for any \( \alpha \in [0, \pi) \) relation (14) takes place. It was shown in [4], Theorem 8.3.2 that in this case the function \( V_{\Theta_\alpha}(z) \) admits integral representation (12). Thus, all we need to show is that \( V_{\Theta_\alpha}(i) = i \). Indeed,

\[
V_{\Theta_\alpha}(i) = \frac{\cos \alpha + (\sin \alpha)V_\Theta(i)}{\sin \alpha - (\cos \alpha)V_\Theta(i)} = \frac{\cos \alpha + (\sin \alpha)i}{\sin \alpha - (\cos \alpha)i} = \frac{1}{-i} = i.
\]

\[\Box\]

Now we study how a unimodular transformation affects the class \( \mathcal{M}^Q \).

**Theorem 9.** Let \( \Theta_\alpha \) be a non-trivial (\( \alpha \neq \pi/2 \)) unimodular transformation of an L-system \( \Theta \) with the impedance function \( V_\Theta(z) \) that belongs to class \( \mathcal{M}^Q \). Then \( V_{\Theta_\alpha}(z) \in \mathcal{M}^{-Q} \) if and only if \( \tan \alpha = Q/2 \).
Proof. Since $V_{\Theta}(z) \in \mathbb{M}^Q$, then it has integral representation (12) with $Q \neq 0$ and $V_{\Theta}(i) = Q + i$. Then

\[
V_{\Theta_a}(i) = \frac{\cos \alpha + (\sin \alpha) V_{\Theta}(i)}{\sin \alpha - (\cos \alpha) V_{\Theta}(i)} = \frac{\cos \alpha + (\sin \alpha)(Q + i)}{\sin \alpha - (\cos \alpha)(Q + i)}
\]

\[
= \frac{(\cos \alpha + Q \sin \alpha) + i \sin \alpha}{(\sin \alpha - Q \cos \alpha) - i \cos \alpha} = \frac{-Q \cos 2\alpha - (1/2)Q^2 \sin 2\alpha}{(\sin \alpha - Q \sin \alpha)^2 + \cos^2 \alpha}
\]

\[
+ i \frac{1}{(\sin \alpha - Q \cos \alpha)^2 + \cos^2 \alpha} = Q_\alpha + i \int_R \frac{d \mu_\alpha(\lambda)}{1 + \lambda^2} = Q_\alpha + ia_\alpha,
\]

where $Q_\alpha$ and $\mu_\alpha$ are the elements of integral representation (12) of the function $V_{\Theta_a}(z)$ and $a_\alpha = \int_R \frac{d \mu_\alpha(\lambda)}{1 + \lambda^2}$. Thus,

\[
Q_\alpha = \frac{-Q \cos 2\alpha - (1/2)Q^2 \sin 2\alpha}{(\sin \alpha - Q \sin \alpha)^2 + \cos^2 \alpha},
\]

and

\[
a_\alpha = \int_R \frac{d \mu_\alpha(\lambda)}{1 + \lambda^2} = \frac{1}{(\sin \alpha - Q \cos \alpha)^2 + \cos^2 \alpha}.
\]

If we would like to derive necessary and sufficient conditions on $V_{\Theta_a}(z) \in \mathbb{M}^{-Q}$, then we need to see when $a_\alpha = 1$ and $Q_\alpha = -Q$. Setting $a_\alpha = 1$ in (14) yields

\[
(\sin \alpha - Q \cos \alpha)^2 + \cos^2 \alpha = 1,
\]

or

\[
(\sin \alpha - Q \cos \alpha)^2 - \sin^2 \alpha = 0 \iff (2 \sin \alpha - Q \cos \alpha) \cdot (Q \cos \alpha) = 0,
\]

implying that either $Q = 0$ or $\alpha = \frac{\pi}{2}$ or $\tan \alpha = Q/2$. Discarding first two options as contradicting to the definition of class $\mathbb{M}^Q$ or producing trivial transformation, we focus on the third option

\[
\tan \alpha = \frac{Q}{2}.
\]

Clearly, under the current set of assumptions, (17) implies that $a_\alpha = 1$ if and only if $\tan \alpha = Q/2$. We observe that in this case (17) transforms into

\[
Q_\alpha = -Q \cos 2\alpha - (1/2)Q^2 \sin 2\alpha.
\]

Applying trigonometric identities to (17) yields

\[
\cos^2 \alpha = \frac{4}{Q^2 + 4} \quad \text{and} \quad \sin^2 \alpha = \frac{Q^2}{Q^2 + 4},
\]

and hence

\[
\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{4 - Q^2}{Q^2 + 4}.
\]

Moreover,

\[
\cos \alpha = \frac{\pm 2}{\sqrt{Q^2 + 4}} \quad \text{and} \quad \sin \alpha = \frac{|Q|}{\sqrt{Q^2 + 4}}.
\]

The sign of $\cos \alpha$ above depends on whether $\alpha \in [0, \pi/2)$ (positive) or $\alpha \in (\pi/2, \pi)$ (negative). We also notice that (17) implies that if $Q > 0$, then $\alpha \in [0, \pi/2)$ and if $Q < 0$, then $\alpha \in (\pi/2, \pi)$. Therefore,

\[
\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{\pm 4|Q|}{Q^2 + 4} = \frac{4Q}{Q^2 + 4}.
\]
Substituting the above values for \( \cos 2\alpha \) and \( \sin 2\alpha \) into \([30]\), we have
\[
Q_\alpha = \frac{-Q(4 - Q^2)}{Q^2 + 4} - \frac{4Q^2Q}{2(Q^2 + 4)} = \frac{Q^3 - 4Q - 2Q^3}{Q^2 + 4} = -\frac{Q(4 + Q^2)}{Q^2 + 4} = -Q.
\]
This completes the proof. \(\square\)

Let us make one important observation. Clearly, every function \( V_i(z) \) of the perturbed class \( \mathcal{W}^Q \) can be represented as

\[
V_i(z) = Q + V_{i,0}(z),
\]
where \( V_{i,0}(z) \in \mathcal{W} \). Theorem 9 above shows that for \( V_1(z) = V_{\Theta}(z) \in \mathcal{W}^Q \) a unimodular transformation with \( \tan \alpha = Q/2 \) is such that \( V_2(z) = V_{\Theta,\alpha}(z) \in \mathcal{W}^{-Q} \) and hence

\[
V_2(z) = -Q + V_{2,0}(z),
\]
where \( V_{2,0}(z) \in \mathcal{W} \). However, the theorem does not provide a connection between \( V_{2,0}(z) \) and \( V_{1,0}(z) \) that is not difficult to obtain. Indeed, for \( \tan \alpha = Q/2 \)
\[
V_2(z) = \frac{\cos \alpha + (\sin \alpha)V_1(z)}{\sin \alpha - (\cos \alpha)V_1(z)} = \frac{1 + (\tan \alpha)V_1(z)}{\tan \alpha - V_1(z)} = 1 + \frac{(Q/2)V_1(z)}{Q/2 - V_1(z)} = \frac{2 + QV_1(z)}{Q - 2V_1(z)} = -\frac{2 + Q^2 + QV_1(z)}{Q + 2V_1(z)} = -Q + \frac{QV_1(z) - 2}{Q + 2V_1(z)}.
\]
A direct substitution into the above formula yields that \( V_2(i) = -Q + i \) which immediately confirms that \( V_{2,0}(z) \in \mathcal{W} \). Thus, we have established a formula relating \( V_{2,0}(z) \) and \( V_{1,0}(z) \)
\[
V_{2,0}(z) = \frac{QV_1(z) - 2}{Q + 2V_1(z)}.
\]

A similar to Theorem 9 result takes place for the other two classes \( \mathcal{W}^{Q}_\kappa \) and \( \mathcal{W}^{Q-1}_\kappa \).

**Theorem 10.** Let \( \Theta_\alpha \) be a non-trivial (\( \alpha \neq \pi/2 \)) unimodular transformation of an L-system \( \Theta \) with the impedance function \( V_\Theta(i) \) that belongs to class \( \mathcal{W}^Q \). Then \( V_{\Theta,\alpha}(z) \in \mathcal{W}^{-Q}_\kappa \) if and only if

\[
\tan \alpha = \frac{b}{2Q},
\]
where

\[
b = Q^2 + a^2 - 1 \quad \text{and} \quad a = \frac{1 - \kappa}{1 + \kappa}.
\]

**Proof.** Since \( V_\Theta(z) \in \mathcal{W}^Q \), then it has integral representation \([30]\) with \( Q \neq 0 \) and \( V_\Theta(i) = Q + ai \), where \( a \) is defined in \([32]\). Then
\[
V_{\Theta,\alpha}(i) = \frac{\cos \alpha + (\sin \alpha)V_\Theta(i)}{\sin \alpha - (\cos \alpha)V_\Theta(i)} = \frac{\cos \alpha + (\sin \alpha)(Q + ai)}{(\sin \alpha + Q \sin \alpha) + ia \sin \alpha} = \frac{(1/2)(1 - Q^2 - a^2) \sin 2\alpha - Q \cos 2\alpha}{(\sin \alpha - Q \sin \alpha) + ia \cos \alpha} + \frac{a}{(\sin \alpha - Q \cos \alpha)^2 + a^2 \cos^2 \alpha} = Q_\alpha + i \int_\mathbb{R} \frac{d\mu_\alpha(\lambda)}{1 + \lambda^2} = Q_\alpha + ia_\alpha,
\]
where $Q_\alpha$ and $\mu_\alpha$ are the elements of integral representation (42) of the function $V_{\Theta_\alpha}(z)$ and $a_\alpha = \int R \frac{d\mu_\alpha(\lambda)}{1 + \lambda^2}$. Thus,

\[(53) \quad Q_\alpha = \frac{(1/2)(1 - Q^2 - a^2) \sin 2\alpha - Q \cos 2\alpha}{(\sin \alpha - Q \cos \alpha)^2 + a^2 \cos^2 \alpha},\]

and

\[(54) \quad a_\alpha = \frac{a}{(\sin \alpha - Q \cos \alpha)^2 + a^2 \cos^2 \alpha}.\]

If we would like to derive necessary and sufficient conditions on $V_{\Theta_\alpha}(z) \in M_{-\kappa}$, then we need to see when $a_\alpha = a$ and $Q_\alpha = -Q$. Setting $a_\alpha = a$ in (46) yields

\[(55) \quad (\sin \alpha - Q \cos \alpha)^2 + a^2 \cos^2 \alpha = 1,\]

that is equivalent to

\[(Q^2 + a^2 - 1) \cos^2 \alpha - 2Q \sin \alpha \cos \alpha = 0.\]

Using (52) we get

\[\cos \alpha (b \cos \alpha - 2Q \sin \alpha) = 0.\]

Since $\alpha \neq \pi/2$ by the condition of our theorem, then we have

\[b \cos \alpha - 2Q \sin \alpha = 0,\]

or $\tan \alpha = \frac{b}{2Q}$. Thus we have just proven that (41) is equivalent to $a_\alpha = a$. All we need to show than that in the case when (53) holds, $Q_\alpha = -Q$. We observe that if $a_\alpha = a$, (53) transforms into

\[(55) \quad Q_\alpha = (1/2)(1 - Q^2 - a^2) \sin 2\alpha - Q \cos 2\alpha = \frac{-b}{2} \sin 2\alpha - Q \cos 2\alpha.\]

Applying trigonometric identities to (53) yields

\[\cos^2 \alpha = \frac{4Q^2}{4Q^2 + b^2} \quad \text{and} \quad \sin^2 \alpha = \frac{b^2}{4Q^2 + b^2},\]

and hence

\[\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{4Q^2 - b^2}{4Q^2 + b^2}.\]

Moreover,

\[(56) \quad \cos \alpha = \frac{2|Q|}{\sqrt{4Q^2 + b^2}} \quad \text{and} \quad \sin \alpha = \frac{|b|}{\sqrt{4Q^2 + b^2}},\]

Assume that $\alpha \in (0, \pi/2)$. Then $\tan \alpha > 0$ and (55) implies that $|b/2Q| > 0$ which means that either: (i) $b > 0$ and $Q > 0$ or (ii) $b < 0$ and $Q < 0$. Since both $\cos \alpha$ and $\sin \alpha$ are positive in the first quadrant, then (55) will turn into

\[(57) \quad \cos \alpha = \frac{\pm 2Q}{\sqrt{4Q^2 + b^2}} \quad \text{and} \quad \sin \alpha = \frac{\pm b}{\sqrt{4Q^2 + b^2}},\]

where (+) sign in both formulas is taken in the case (i) and (−) sign, respectively, in the case of (ii).

Now assume that $\alpha \in (\pi/2, \pi)$. Then $\tan \alpha > 0$ and (55) implies that $|b/2Q| < 0$ which means that either: (iii) $b > 0$ and $Q < 0$ or (iv) $b < 0$ and $Q > 0$. But this time we are in the second quadrant and hence $\cos \alpha < 0$ while $\sin \alpha > 0$. 
Consequently, formula (57) is true again in the sense that (+) sign in both formulas is taken in the case (iii) and (−) sign in the case (iv). Thus in all the possible cases (i)–(iv) the signs in the numerators in (57) match.

We have then
\[ Q_\alpha = \frac{b}{2} \sin 2\alpha - Q \cos 2\alpha = -b \sin \alpha \cos \alpha - Q \cos 2\alpha \]
\[ = \frac{-2b(\pm Q)(\pm b)}{4Q^2 + b^2} - \frac{Q(4Q^2 - b^2)}{4Q^2 + b^2} = -\frac{2bQ + Q(4Q^2 - b^2)}{4Q^2 + b^2} \]
\[ = (-Q) \frac{2b^2 + 4Q^2 - b^2}{4Q^2 + b^2} = -Q. \]

This completes the proof. □

A similar result takes place for the class \( \mathcal{M}_\kappa^{-1,Q} \).

**Theorem 11.** Let \( \Theta_{\alpha} \) be a non-trivial \((\alpha \neq \pi/2)\) unimodular transformation of an L-system \( \Theta \) with the impedance function \( V_\Theta(z) \) that belongs to class \( \mathcal{M}_\kappa^{-1,Q} \). Then \( V_{\Theta_{\alpha}}(z) \in \mathcal{M}_\kappa^{-1,-Q} \) if and only if (51) holds true for
\[
(58) \quad b = Q^2 + a^2 - 1 \quad \text{and} \quad a = \frac{1 + \kappa}{1 - \kappa}.
\]

**Proof.** The proof has similar to the one of Theorem 10 structure. Performing the same set of derivations as we did in the proof of Theorem 10 we show that (51) holds if and only if \( a_\alpha = a \). The main difference in what follows is that since \( V_\Theta(z) \in \mathcal{M}_\kappa^{-1,Q} \), then \( a > 1 \) and consequently \( b > 0 \) for any real \( Q \). As a result, if we assume that \( \alpha \in (0, \pi/2) \), then we can immediately conclude that \( Q > 0 \) or otherwise we will arrive at a contradiction to \( \tan \alpha > 0 \) in the first quadrant. Similarly, the assumption \( \alpha \in (\pi/2, \pi) \) yields \( Q < 0 \). Consequently, (57) becomes
\[
(59) \quad \cos \alpha = \frac{2Q}{\sqrt{4Q^2 + b^2}} \quad \text{and} \quad \sin \alpha = \frac{b}{\sqrt{4Q^2 + b^2}}.
\]

for any \( \alpha \in (0, \pi/2) \cup (\pi/2, \pi) \). Evaluating \( Q_\alpha \) as we did in the proof of Theorem 10 we obtain
\[ Q_\alpha = \frac{b}{2} \sin 2\alpha - Q \cos 2\alpha = -b \sin \alpha \cos \alpha - Q \cos 2\alpha \]
\[ = \frac{-2b^2Q}{4Q^2 + b^2} - \frac{Q(4Q^2 - b^2)}{4Q^2 + b^2} = -Q. \]

Thus, \( V_{\Theta_{\alpha}}(z) \in \mathcal{M}_\kappa^{-1,-Q} \) and the proof is complete. □

We make another observation similar to the one we made after Theorem 10. Clearly, every function \( V_1(z) \) of the perturbed class \( \mathcal{M}_\kappa^Q \) (or \( \mathcal{M}_\kappa^{-1,Q} \)) can be written as
\[ V_1(z) = Q + V_{1,0}(z), \]
where \( V_{1,0}(z) \in \mathcal{M}_\kappa \) (or \( V_{1,0}(z) \in \mathcal{M}_\kappa^{-1} \)). Theorems 10 and 11 show that for \( V_1(z) = V_\Theta(z) \in \mathcal{M}_\kappa^Q \) (or \( V_1(z) = V_\Theta(z) \in \mathcal{M}_\kappa^{-1,Q} \)) a unimodular transformation with \( \tan \alpha = b/2Q \) is such that \( V_2(z) = V_\Theta(z) \in \mathcal{M}_\kappa^{-Q} \) (or \( V_2(z) = V_\Theta(z) \in \mathcal{M}_\kappa^{-1,-Q} \)) and hence
\[ V_2(z) = -Q + V_{2,0}(z), \]
where \( V_{2,0}(z) \in \mathfrak{M}_\kappa \) (or \( V_{2,0}(z) \in \mathfrak{M}^{-1}_\kappa \)). However, the theorems do not provide a connection between \( V_{2,0}(z) \) and \( V_{1,0}(z) \) that is not difficult to obtain. Following (49) for \( \tan \alpha = b/2Q \) we get

\[
V_2(z) = \frac{\cos \alpha + (\sin \alpha)V_1(z)}{\sin \alpha - (\cos \alpha)V_1(z)} = \frac{1 + (\tan \alpha)V_1(z)}{\tan \alpha - V_1(z)} = \frac{1 + (b/2Q)V_1(z)}{b/2Q - V_1(z)}
\]

\[
= \frac{2Q + bV_1(z)}{b - 2QV_1(z)} = \frac{Q^2 + Q^2V_1(z) - bQ - Q - (b/2)V_1(z)}{Q^2 + QV_1(z) - (b/2)}
\]

Thus, we have established a formula relating \( V_{2,0}(z) \) and \( V_{1,0}(z) \)

\[
V_{2,0}(z) = \frac{Q^3 + Q^2V_1(z) - bQ - Q - (b/2)V_1(z)}{Q^2 + QV_1(z) - (b/2)}
\]

The result below immediately follows from Theorems 10.

**Corollary 12.** Let \( \Theta \) be an L-system of the form (44) with the impedance function \( V_\theta(z) \). Then there exists a unique (for a given \( Q \)) unimodular transformation \( \Theta_\alpha \) of \( \Theta \) such that its impedance function \( V_{\theta_\alpha}(z) \) belongs to exactly one of the disjoint classes \( \mathfrak{M}^{-Q}, \mathfrak{M}^{-Q}_\kappa \), or \( \mathfrak{M}^{-1,-Q}_\kappa \).

### 6. Control of L-systems

In this section we are going to formalize the procedure of unimodular transformation of an L-system. We start off with the following definition.

**Definition 13.** An L-system \( \Theta \) of the form (44) is called **equivalent** to an LF-system \( \Theta_{LF} \) of the form (43) if the transfer mappings \( W_\theta(z) \) and \( W_{\theta_{LF}}(z) \) of both systems coincide on the intersection of their domains of definitions.

In Section 4 we mentioned that any constant \( J \)-unitary operator \( B \) on a finite-dimensional Hilbert space \( E \) can be realized as a transfer function of an F-system \( \Theta_0 \) of the form (32). Now we apply this result to the situation treated in Section 4. We set

\[
B = -e^{2i\alpha}, \quad E = \mathbb{C}, \quad J = 1, \quad \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right).
\]

Then the operator \( C \) involved in the construction of \( \Theta_0 \) is

\[
C = i[B - I][B + I]^{-1}J = i\frac{e^{2i\alpha} - 1}{e^{2i\alpha} + 1} = \frac{e^{i\alpha} + e^{-i\alpha}}{e^{i\alpha} - e^{-i\alpha}} = \cot \alpha.
\]

Also, the main operator of the F-system \( \Theta_0 \) of the form (32) is

\[
KC^{-1}(I + iCJ)K^* = K(C^{-1} + i)K^* = K(\tan \alpha + i)K^*.
\]

By construction, the operator \( K \) in F-system \( \Theta_0 \) can be chosen as any bounded and boundedly invertible operator from \( E \) to \( E \). In our case \( E = \mathbb{C} \) and hence we can chose \( K = 1 \). As a result, the F-system \( \Theta_0 \) of the form (32) in our case boils down to

\[
\Theta_{0,\alpha} = \begin{pmatrix}
\tan \alpha + i & 0 \\
0 & 1 \\
\end{pmatrix}, \quad \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right).
\]

We know that \( W_{\Theta_{0,\alpha}}(z) \equiv -e^{2i\alpha} \).
In the case when $\alpha = \pi/2$, $B = 1$ and parameter $C^{-1}$ is undefined. We utilize the approach explained in Section 5. Namely, we represent

$$B = 1 = (\cos \frac{\pi}{2}) (\cos \frac{\pi}{4}) = (\cos \frac{\pi}{4}) (\cos \frac{\pi}{4}) = B_1 \cdot B_2.$$ 

The corresponding $C_1 = \cot \frac{\pi}{4} = 1$ and $C_2 = \cot \frac{3\pi}{4} = -1$ and

$$W_{\Theta_0, \frac{\pi}{4}}(z) = -i$$ and $W_{\Theta_0, \frac{3\pi}{4}}(z) = i$ are F-systems of the form (63) that realize $B_1$ and $B_2$.

Similarly, in the case when $\alpha = 0$, $B = -1$ and parameter $C$ is undefined. We proceed as above and represent

$$B = -1 = i^2 = (\cos \frac{3\pi}{4}) (\cos \frac{3\pi}{4}) = B_2 \cdot B_2.$$ 

The corresponding $C_2 = \cot \frac{3\pi}{4} = -1$ and $\Theta_0, \frac{3\pi}{4}$ is given by (63).

**Definition 14.** An F-system $\Theta_{0, \alpha}$ of the form (62) is called a **controller** to an L-system $\Theta_L$ of the form (34) corresponding to a unimodular transformation $\Theta_{\alpha}$ for $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$.

In “trivial” cases when $\alpha = 0$ and $\alpha = \pi/2$ the controller is respectively defined as a coupling of the corresponding F-systems

$$\Theta_{0, 0} = \Theta_{0, \frac{\pi}{4}} \cdot \Theta_{0, \frac{3\pi}{4}} \quad \text{and} \quad \Theta_{0, \frac{\pi}{2}} = \Theta_{0, \frac{3\pi}{4}} \cdot \Theta_{0, \frac{\pi}{4}}.$$ 

The following result follows directly from the above discussion.

**Theorem 15.** Let $\Theta_{LF}$ be an LF-coupling of an L-system $\Theta_L$ of the form (34) and a controller $\Theta_{0, \alpha}$ for $\alpha \in (0, \pi)$, that is

$$\Theta_{LF} = \Theta_{L} \cdot \Theta_{0, \alpha}.$$ 

Then $\Theta_{LF}$ is equivalent to a unimodular transformation $\Theta_{\alpha}$ of $\Theta_L$ for the same value of $\alpha$ and hence $W_{\Theta_{LF}}(z) = W_{\Theta_{\alpha}}(z)$ on the intersection of their domains of definitions.

Theorem 15 is illustrated on Figure 1. The following theorem is an analogue of the “absorption property” of the class $\mathfrak{M}$ that was discussed in details in 7.

**Theorem 16.** Let $\Theta_L$ be an L-system of the form (34) such that $V_{\Theta_L} \in \mathfrak{M}$ and let $\Theta_{0, \alpha}$ be a controller with an arbitrary value of $\alpha \in (0, \pi)$. If $\Theta_{LF}$ is an LF-coupling such that $\Theta_{LF} = \Theta_L \cdot \Theta_{0, \alpha}$, then $V_{\Theta_{LF}}(z) \in \mathfrak{M}$.

**Proof.** The proof of this result follows from the invariance of the Donoghue class $\mathfrak{M}$ under a unimodular transformation (see 3, 4, 5) and Theorem 15. □
7. Examples

Example 1. Consider an L-system

\[ \Theta^{(\xi)} = \begin{pmatrix} A^{(\xi)} & K^{(\xi)} \\ iW_1^2 \subset L^2_{[0,1]} \subset (W_2^1)_{-} & 0 \end{pmatrix}, \]

where

\[ A^{(\xi)} x = \frac{1}{i} \frac{dx}{dt} + i x(t) \left[ \delta(t-l) - e^{-i\xi \ell} \delta(t) \right], \]

\[ A^{(\xi)}^* x = \frac{1}{i} \frac{dx}{dt} + i x(0) \left[ e^{i\xi \ell} \delta(t-l) - \delta(t) \right], \]

and

\[ K^{(\xi)} c = c \cdot \frac{1}{\sqrt{2}} [e^{i\xi \ell} \delta(t-l) - \delta(t)], \quad (c \in \mathbb{C}), \]

\[ K^{(\xi)}^* x = \left( x, \frac{1}{\sqrt{2}} [e^{i\xi \ell} \delta(t-l) - \delta(t)] \right) = \frac{1}{\sqrt{2}} [e^{-i\xi \ell} x(l) - x(0)], \]

with \( x(t) \in W_2^1 \). Here \( A^{(\xi)} \) is a \((*)\)-extension of the operator

\[ Tx = \frac{1}{i} \frac{dx}{dt}, \]

with

\[ \text{Dom}(T) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,1]}, x(0) = 0 \right\}. \]

The system of this type was described in details in \[ \text{[1]}, \text{Section 8.5}. \] It can also be shown based on this reference that

\[ W_{\Theta^{(\xi)}}(z) = 1 - 2i K^{(\xi)}(A^{(\xi)} - z I)^{-1} K^{(\xi)} = e^{i(\xi - z)l} = e^{-izl} \cdot e^{i\xi \ell}. \]

Set \( B^{(\xi)} = e^{i\xi \ell}. \) Then applying \[ \text{[1]} \] we obtain

\[ V_{\Theta^{(\xi)}}(z) = i \frac{W_{\Theta^{(\xi)}}(z) - 1}{W_{\Theta^{(\xi)}}(z) + 1} = i \frac{B^{(\xi)} e^{-izl} - 1}{B^{(\xi)} e^{-izl} + 1} = i \frac{B^{(\xi)} - e^{izl}}{B^{(\xi)} + e^{izl}}. \]

Note that when \( \xi = 0 \), then \( B^{(0)} = 1, W_{\Theta^{(0)}}(z) = e^{-izl} \), and

\[ V_{\Theta^{(0)}}(z) = i \frac{1 - e^{izl}}{1 + e^{izl}} \quad \text{with} \quad V_{\Theta^{(0)}}(i) = \frac{1 - e^{-l}}{1 + e^{-l}}. \]

Therefore, \( V_{\Theta^{(0)}}(z) \in \mathbb{M}_\kappa \) for \( \kappa = e^{-l} \). Comparing \[ \text{[1]} \] to \[ \text{[1]} \] lets us interpret \( B^{(\xi)} = e^{i\xi \ell} \) as a unimodular transformation of the L-system \( \Theta^{(0)} \). In order to find the angle \( \alpha \) that corresponds to this unimodular transformation we set \((-e^{2i\alpha}) = e^{i\xi \ell}\) and solve for \( \alpha \) to get

\[ \alpha = \frac{\xi \ell - \pi}{2}. \]

A controller corresponding to this unimodular transformation is given via \[ \text{[1]} \] and is

\[ \Theta_{0,\alpha} = \begin{pmatrix} \tan \frac{\xi \ell - \pi}{2} + i & 0 & 1 \\ \frac{\xi \ell - \pi}{2} & 1 \end{pmatrix}, \]

where \( \alpha \) is given by \[ \text{[1]} \] and \( \xi \ell \neq 2\pi \). We also have an LF-system

\[ \Theta_{LF} = \Theta^{(0)} \cdot \Theta_{0,\alpha}, \]
that is equivalent to \( \Theta(\xi) \) in the sense of Definition 13 that is
\[
W_{\Theta_{\ell,F}}(z) = W_{\Theta(\xi)}(z).
\]
This LF-system takes form (22) and is explicitly written as
\[
\Theta_{\ell,F} = \left( \begin{array}{ccc} M & F & K \\ \mathcal{H}_+ & H & \mathcal{H}_- \\ 0 & i & 0 \end{array} \right),
\]
where
\[
\mathcal{H}_+ \subset H \subset \mathcal{H}_- = W_1^2 \oplus \mathbb{C} \subset L_{[0,\ell]} \oplus \mathbb{C} \subset (W_2^1)_- \oplus \mathbb{C},
\]
and
\[
M = \left( \begin{array}{cc} A^{(0)} & 2iK^{(0)} \\ 0 & \tan \frac{i(\ell-\pi)}{2} + i \end{array} \right), \quad F = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad K = \left( \begin{array}{c} K^{(0)} \\ 1 \end{array} \right).
\]

Example 2. Now we are going to perturb the function \( V_{\Theta(\xi)}(0)(z) \) in (67) so that it would fall in the class \( \mathcal{M}_\kappa^Q \) for \( Q = 1 \) and \( \kappa = e^{-l} \). We introduce
\[
V_1(z) = 1 + i \frac{1 - e^{izl}}{1 + e^{izl}}.
\]
Clearly, (69) implies that \( V_1(z) \) belongs to the class \( \mathcal{M}_1^1/2 \). It can be shown (and checked by direct yet tedious computations) that \( V_1(z) \) is the impedance function of an L-system of the form
\[
\Theta_{\rho \mu} = \left( \begin{array}{ccc} \mathbb{K}_{\rho \mu} & K & 1 \\ W_2^1 & L^2_{[0,\ell]} & (W_2^1)_- \\ 0 & i(\ell) & 0 \end{array} \right),
\]
where
\[
\mathbb{K}_{\rho \mu}x = i \frac{dx}{dt} + i \frac{1}{\rho + \mu} (\rho x(0) - \rho(t)) \left[ \mu\delta(t - \ell) + \delta(t) \right],
\]
\[
K_*x = i \frac{dx}{dt} + i \frac{\bar{\mu}}{\rho + \bar{\mu}} (x(0) - \rho x(t)) \left[ \mu\delta(t - \ell) + \delta(t) \right],
\]
\[
Kc = c \cdot \chi, \quad (c \in \mathbb{C}), \quad K_*x = (x, \chi), \quad x(t) \in W_2^1, \quad \chi = \sqrt{\frac{c_2^2 - 4c_1^2}{2c_1^2 i^2}} \left[ \mu\delta(t - \ell) - \delta(t) \right].
\]
For the sake of simplicity of further calculations we set \( l = \ln 2 \). Then the values of parameters \( \rho \) and \( \mu \) in (70)-(73) are given by
\[
\rho = -\frac{343 + 40\sqrt{13}}{18 + 45\sqrt{13}},
\]
and
\[
\mu = \frac{1291 + 25\sqrt{13} + (3087 + 360\sqrt{13})i}{1291 + 835\sqrt{13} + (162 + 405\sqrt{13})i}.
\]
For the above value of \( l = \ln 2 \) we have \( \kappa = \frac{3}{4} \). Moreover, our function \( V_1(z) \) in (69) takes form
\[
V_1(z) = 1 + i \frac{1 - 2iz}{1 + 2iz},
\]
and belongs to the class \( \mathcal{M}_1^{1/2} \). If we want to find a unimodular transformation (and the corresponding controller) that transforms the L-system \( \Theta_{\rho \mu} \) in (70) into the one
whose impedance function $V_2(z)$ belongs to the class $\mathcal{M}_{1/2}^{(-1)}$, we apply Theorem 1 and formulas (51)-(53). In our case $Q = 1$, and hence $b = a^2$, where
\[ a = \frac{1 - e^{-l}}{1 + e^{-l}} = \frac{e^l - 1}{e^l + 1} = \frac{1}{3}, \quad \text{for} \quad l = \ln 2. \]

Applying (52) gives
\[ \tan \alpha = \frac{b}{2Q} = \frac{a^2}{2} = \frac{1}{18}. \]

Thus, the value $\alpha = \arctan \frac{1}{18}$ defines the unimodular transformation we seek and provides a controller
\[ \Theta_{0, \alpha} = \begin{pmatrix} \frac{1}{18} & i & 0 & 1 \\ 1 & 1 \end{pmatrix}, \]
responsible for this transformation in the above sense. Using this value of tangent we obtain
\[ \cos \alpha = \frac{18}{5\sqrt{13}} \quad \text{and} \quad \sin \alpha = \frac{1}{5\sqrt{13}}. \]

Observe that
\[ V_2(i) = \frac{\cos \alpha + (\sin \alpha)V_1(i)}{\sin \alpha - (\cos \alpha)V_1(i)} = \frac{\frac{18}{5\sqrt{13}} + \frac{1}{5\sqrt{13}}(1 + \frac{i}{2})}{\frac{18}{5\sqrt{13}} - \frac{1}{5\sqrt{13}}(1 + \frac{i}{2})} = - \frac{57 + i}{51 + 18i} = -1 + \frac{1}{3}i. \]

This confirms that $V_2(z) \in \mathcal{M}_{1/2}^{(-1)}$. Finally,
\[ V_2(z) = \frac{\cos \alpha + (\sin \alpha)V_1(z)}{\sin \alpha - (\cos \alpha)V_1(z)} = \frac{\frac{18}{5\sqrt{13}} + \frac{1}{5\sqrt{13}}(1 + i\frac{1-2i\pi}{1+2i\pi})}{\frac{18}{5\sqrt{13}} - \frac{1}{5\sqrt{13}}(1 + i\frac{1-2i\pi}{1+2i\pi})} = - \frac{19 + i(19 - i)2i}{17 + 18i + (17 - 18i)2i} = -1 + 2 + 17i - (2 + 17i)2i. \]

We have shown that applying a unimodular transformation with $\tan \alpha = 1/18$ maps function $V_1(z) \in \mathcal{M}_{1/2}^{(-1)}$ into a function $V_2(z) \in \mathcal{M}_{1/2}^{(-1)}$ of the form (73).

**APPENDIX A. DIFFERENTIAL EQUATIONS AND L- AND F-SYSTEMS**

Let $T \in \Lambda$, $K$ be a bounded linear operator from a finite-dimensional Hilbert space $E$ into $\mathcal{H}_-$, $K^* \in [\mathcal{H}_+, E]$, and $J = J^* = J^{-1} \in [E, E]$. Consider the following singular system of equations
\[ \begin{cases} \frac{d\chi}{dt} + T\chi(t) = KJ\psi_-(t), \\ \chi(0) = x \in \text{Dom}(T), \\ \psi_+ = \psi_- - 2iK^*\chi(t). \end{cases} \]  

Given an input vector $\psi_- = \varphi_- e^{iz} \in E$, we seek solutions to the system (73) as an output vector $\psi_+ = \varphi_+ e^{iz} \in E$, and a state-space vector $\chi(t) = xe^{iz} \in \text{Dom}(T)$. Substituting the expressions for $\psi_\pm(t)$ and $\chi(t)$ allows us to cancel exponential terms and convert the system (73) to the form
\[ \begin{cases} (T - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^*x, \quad z \in \rho(T). \end{cases} \]
Consider the following system of equations

\[
\begin{align*}
\text{(80)} & \quad \begin{cases}
    iF \frac{d}{dt} + M \chi(t) = KJ \psi_-(t), \\
    \chi(0) = x \in \mathcal{H}, \\
    \psi_+ = \psi_- - 2iK^* \chi(t).
\end{cases}
\end{align*}
\]

Given an input vector \( \psi_- = \varphi_- e^{izt} \in E \), we seek solutions to the system (80) as an output vector \( \psi_+ = \varphi_+ e^{izt} \in E \) and a state-space vector \( \chi(t) = xe^{izt} \in \mathcal{H} \). Substituting the expressions for \( \psi_\pm(t) \) and \( \chi(t) \) allows us to cancel exponential terms and convert the system (80) to the stationary form

\[
\begin{align*}
\text{(81)} & \quad \begin{cases}
    (M - zI) x = KJ \varphi_-, \\
    \varphi_+ = \varphi_- - 2iK^* x,
\end{cases} \quad z \in \rho(M, F).
\end{align*}
\]

Both differential equation systems (81) and (80) are associated with the corresponding F-system \( \Theta_F \) of the form (80).

It can be shown in [4] that L-systems written in the form (78) (or (79)) and F-systems written in the form (81) (or (80)) obey appropriate conservation laws. For details the reader is referred to Sections 6.3 and 12.1 of [4].
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