POLYNOMIAL MINIMAL SURFACES OF DEGREE FIVE

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Abstract. The problem of finding all minimal surfaces presented in parametric form as polynomials of certain degree is discussed by many authors. It is known that the classical Enneper surface is (up to position in space and homothety) the only polynomial minimal surface of degree 3 in isothermal parameters. In higher degrees the problem is quite more complicated. Here we find a general form for the functions that generate a polynomial minimal surface of arbitrary degree via the Weierstrass formula and prove that any polynomial minimal surface of degree 5 in isothermal parameters may be considered as belonging to one of three special families.

1. Introduction

The minimal surfaces are a topic of great interest in many areas as mathematics, computer science, physics, medicine, architecture. The reason is that in small areas they have a minimizing property.

For the applications of minimal surfaces, in particular in computer graphic researches, it is important to use minimal surfaces in polynomial form and hence to know all such surfaces in small degrees. In this direction Cosín and Monterde [1] proved that up to position in space and homothety the only polynomial minimal surface of degree three in isothermal parameters is the classical Enneper surface. The case of degree four is considered in [6]. Polynomial minimal surfaces of degrees five and six are studied in [8] and [7], respectively. There are found theorems about their coefficients-vectors and some examples are considered. Unfortunately the systems for the coefficients are very complicated and the general solution is difficult to be found. In [9] are studied polynomial minimal surfaces of arbitrary degree, constructed on some special functions, so some special surfaces are proposed. It is remarked that in degrees 3 and 5 these surfaces coincide with the Enneper surface and the surfaces from [8], respectively.

In the present paper we first show that a polynomial minimal surface in isothermal parameters must be generated via the Weierstrass formula with a polynomial and a rational function (Section 3). Then in Section 4 we determine all polynomial minimal surface of degree five but we do not try to solve the system for the coefficients. Instead we use the result from Section 3 and we obtain a list of functions that generate via the Weierstrass formula all such surfaces. As a special case of the surfaces in the obtained families are included the surfaces introduced in [8], but these families contain many other surfaces as well.

It is natural to ask whether these families contain different surfaces. In general the question to compare surfaces given in different parametric form is very complicated. For minimal surfaces such a method is proposed in [5]. It is based on the canonical parameters

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introduced in [3] and respectively on solving an ordinary differential equation for finding these parameters.

When trying to investigate the relation between the obtained in Section 4 families we cannot use directly the method from [5], because we cannot find a simple form of the surface in canonical parameters. So we change a little the approach and we escape the solving of the differential equation for the transition to canonical parameters. As a result we find that the obtained three families contain different minimal surfaces except in a special case.

2. Preliminaries

Let $S$ be a regular surface in the Euclidean space defined by the parametric equation
\[ \mathbf{x} = \mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)), \quad (u, v) \in U \subset \mathbb{R}^2. \]
The derivatives of the vector function $\mathbf{x} = \mathbf{x}(u, v)$ are usually denoted by $\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu},$ etc. Then the coefficients of the first fundamental form are equal to the scalar products
\[ E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v. \]

The unit normal to the surface is the vector field
\[ \mathbf{U} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}}. \]

In particular, if $E = G$, $F = 0$, then the parameters $(u, v)$ of the surface are called isothermal.

The coefficients of the second fundamental form of $S$ are given by
\[ L = \mathbf{U} \cdot \mathbf{x}_{uu}, \quad M = \mathbf{U} \cdot \mathbf{x}_{uv}, \quad N = \mathbf{U} \cdot \mathbf{x}_{vv}. \]

The Gauss curvature $K$ and the mean curvature $H$ of $S$ are defined respectively by
\[ K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \]
Recall that the surface $S$ is called minimal if its mean curvature vanishes identically. In this case it follows easily that the Gauss curvature is nonpositive.

The study of minimal surfaces is closely related with some complex curves - those with isotropic tangent vectors. They are called minimal curves. Indeed we have the following construction.

Let $S$ be a minimal surface defined in isothermal parameters. Then it can be considered as the real part of a minimal curve. More precisely, let $f(z)$ and $g(z)$ be two holomorphic functions. Define the Weierstrass complex curve $\Psi(z)$ by
\[ (2.1) \quad \Psi(z) = \int_{z_0}^z \left( \frac{1}{2} f(z)(1 - g^2(z)), \frac{i}{2} f(z)(1 + g^2(z)), f(z)g(z) \right) dz. \]
Then $\Psi(z)$ is a minimal curve and its real and imaginary parts $\mathbf{x}(u, v)$ and $\mathbf{y}(u, v)$ are harmonic functions that define two minimal surfaces in isothermal parametrizations. We say that these two minimal surfaces are conjugate. Moreover, every minimal surface can be obtained at least locally as the real (as well as the imaginary) part of a Weierstrass minimal curve.
For any two conjugate minimal surfaces \( x(u,v) \) and \( y(u,v) \) it is defined the associated family

\[
S_t : \quad x_t(u,v) = x(u,v) \cos t + y(u,v) \sin t.
\]

Then for any real number \( t \) the surface \( S_t \) is also minimal and has the same first fundamental form as \( S = S_0 \).

**Example.** Taking \( f(z) = 1, g(z) = z \), we obtain a Weierstrass minimal curve whose real part is the classical Enneper surface

\[
x(u,v) = \left( \frac{u}{2}(1 + v^2 - \frac{u^2}{3}), -\frac{v}{2}(1 + u^2 - \frac{v^2}{3}), \frac{1}{2}(u^2 - v^2) \right).
\]

It is well known that the Enneper surface coincide (up to position in space) with any surface in its associated family, see e.g. [4].

In [3] Ganchev introduce canonical principal parameters. If a surface is parametrized with them, the coefficients of its fundamental forms are given by

\[
E = \frac{1}{\nu}, \quad F = 0, \quad G = \frac{1}{\nu},
\]

\[
L = 1, \quad M = 0, \quad N = -1.
\]

where \( \nu = \sqrt{-K} \) is the normal curvature of the surface. Actually a surface in canonical principal parametrization is the real part of a Weierstrass minimal curve generated by some functions \( f(z), g(z) \) with \( f(z) = -1/g'(z) \), i.e. it is the real part of the special Weierstrass curve

\[
\Phi(z) = -\int_{z_0}^{z} \left( \frac{1 - g^2(z)}{2g'(z)}, \frac{i(1 + g^2(z))}{2g'(z)}, \frac{g(z)}{g'(z)} \right) dz
\]

The canonical principal parameters and the normal curvature play a role similar to that of the natural parameters and the curvature and torsion of a space curve. Namely the following theorem holds:

**Theorem 2.1.** [3] If a surface is parametrized with canonical principal parameters, then its normal curvature satisfies the differential equation

\[
\Delta \ln \nu + 2\nu = 0.
\]

Conversely, for any solution \( \nu(u,v) \) of this equation (with \( \nu_u \nu_v \neq 0) \), there exists an unique (up to position in space) minimal surface with normal curvature \( \nu(u,v), (u,v) \) being canonical principal parameters. Moreover, the canonical principal parameters \( (u,v) \) are determined uniquely up to the following transformations

\[
u = \varepsilon \bar{u} + a, \quad \varepsilon = \pm 1, \quad a = \text{const.}, \quad b = \text{const}.
\]

We will also use the following results:

**Theorem 2.2.** [5] Let the minimal surface \( S \) be defined by the real part of the Weierstrass minimal curve (2.1). Any solution of the differential equation

\[
(2.2) \quad \left( z'(w) \right)^2 = -\frac{1}{f(z(w))g'(z(w))}
\]
defines a transformation of the isothermal parameters of $S$ to canonical principal parameters. Moreover the function $\tilde{g}(w)$ that defines $S$ via the Ganchev formula is given by $\tilde{g}(w) = g(z(w))$.

**Theorem 2.3.** [5] Let the holomorphic function $g(z)$ generate a minimal surface in canonical principal parameters, i.e. via the Ganchev formula. Then, for an arbitrary complex number $\alpha$, and for an arbitrary real number $\varphi$, any of the functions

\[ e^{i\varphi} \frac{\alpha + g(z)}{1 - \alpha g(z)}, \quad e^{i\varphi} \frac{g(z)}{\alpha} \]

generates the same surface in canonical principal parameters (up to position in space). Conversely, any function that generates this surface (up to position in space) in canonical principal parameters has one of the above forms.

In sections 4 and 5 we shall consider minimal polynomial surfaces of degree five. An interesting study of such surfaces is presented in [8]. First of all it is proved that the harmonic condition implies that such a surface must have the form

\[
\begin{align*}
\mathbf{r}(u,v) &= a(u^5 - 10u^3v^2 + 5uv^4) + b(v^5 - 10u^2v^3 + 5u^4v) \\
&\quad + c(u^4 - 6u^2v^2 + v^4) + d uv(u^2 - v^2) + e u(u^2 - 3v^2) \\
&\quad + f v(v^2 - 3u^2) + g (u^2 - v^2) + h uv + i u + j v + k
\end{align*}
\]

(2.3)

where $a, b, c, d, e, f, g, h, i, j, k$ are coefficient vectors. For these coefficients the following holds, see [8]:

**Theorem 2.4.** The harmonic polynomial surface (2.1) is minimal if and only if its coefficient vectors satisfy the following system of equations

\[
\begin{align*}
\begin{cases}
\alpha^2 &= b^2 \\
a.b &= 0 \\
4a.c - b.d &= 0 \\
a.d + 4b.c &= 0 \\
16e^2 - d^2 + 30a.e + 30b.f &= 0 \\
4d.c + 15b.e - 15a.f &= 0 \\
9e^2 - 9f^2 + 16c.g - 2d.h + 10a.i - 10b.j &= 0 \\
9e.f - 4c.h - 2d.g - 5b.i - 5a.j &= 0 \\
4g^2 - h^2 + 6e.i + 6f.j &= 0 \\
2g.h - 3f.i + 3e.j &= 0 \\
5a.h + 10b.g - 12c.f + 3d.e &= 0 \\
5b.h - 10a.g - 3d.f - 12c.e &= 0 \\
6e.g + 3f.h + 4c.i - d.j &= 0 \\
6f.g - 3e.h - d.i - 4c.j &= 0 \\
h.i + 2g.j &= 0 \\
2g.i - h.j &= 0 \\
i^2 &= j^2 \\
i.j &= 0
\end{cases}
\end{align*}
\]

(2.4)

It seems impossible to find the general solution of the system (2.4). So in [8] some special solutions are considered and several interesting properties are proved for the obtained surfaces. Using a different approach we shall find all polynomial minimal surfaces of degree five.
3. POLYNOMIAL MINIMAL SURFACES OF ARBITRARY DEGREE

As is said in Introduction, polynomial minimal surfaces of arbitrary degree are studied in [9]. The following construction is proposed. Consider the functions

\[ P_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k} u^{n-2k} v^{2k} \]

\[ Q_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k+1} u^{n-2k-1} v^{2k+1} \]

Then it is proved that for any real number \( \omega \) the polynomial surface of degree \( n \) defined by

\[ \mathbf{x}(u, v) = (-P_n + \omega P_{n-2}, Q_n + \omega Q_{n-2}, \frac{2\sqrt{n(n-2)\omega}}{n-1} P_{n-1}) \]

is minimal. Of course this large family is very interesting. But it is important also to know are these all the possible polynomial minimal surfaces and if not to find other families. To resolve the last problem we propose the following approach.

Let

\[ \Phi : \mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)) \]

be a polynomial minimal surface of degree \( n \) in isothermal parameters. Then \( x_i(u, v) \) are polynomials of degree \( \leq n \), and at least for one \( i = 1, 2, 3 \) there is an equality. Suppose that the parametrization is isothermal and \( \Phi \) is defined in an open subset of \( \mathbb{R}^2 \), containing \( (0, 0) \). From Lemma 22.25 in [4] it follows that (up to translation) \( \mathbf{x}(u, v) \) is the real part of the minimal curve

\[ \Psi(z) = 2\mathbf{x}\left(\frac{z}{2}, \frac{z}{2i}\right) \]

So this minimal curve is also polynomial of degree \( n \). Then

\[ \Psi'(z) = (\phi_1(z), \phi_2(z), \phi_3(z)) = \left( \frac{1}{2} f(z)(1 - g^2(z)), \frac{i}{2} f(z)(1 + g^2(z)), f(z)g(z) \right) \]

for some functions \( f(z), g(z) \) and the coordinate functions \( \phi_i(z) \) are polynomials of degree at least \( \leq n - 1 \) and at least for one \( i \) the degree of \( \phi_i(z) \) is exactly \( n - 1 \). Hence every one of the functions

\[ f(1 - g^2) = 2\phi_1 \quad f(1 + g^2) = \frac{2}{i} \phi_2 \quad fg = \phi_3 \]

is a polynomial and so

\[ f = \phi_1 - i\phi_2 \quad fg^2 = \frac{1}{i} \phi_2 - \phi_1 = -(\phi_1 + i\phi_2) \quad fg = \phi_3 \]

are polynomials of degree \( \leq n - 1 \) and at least for one \( i \) the degree is exactly \( n - 1 \). So \( f \) is a polynomial of degree \( \leq n - 1 \). Now (3.1) implies that \( g(z) \) is a rational function of the form

\[ g(z) = \frac{P_p(z)}{Q_q(z)} \]

where \( P_p(z) \) and \( Q_q(z) \) are polynomials (of degrees \( p \) and \( q \), respectively) with no common zeros. According to (3.1)\(_2\) the function \( fg^2 \) is also a polynomial, so

\[ f(z) = (Q_q(z))^2 R_r(z) \]
with a polynomial $R_r$ (of degree $r$). Moreover since the polynomials
\[ f(z) = (Q_q(z))^2 R_r(z) \quad f(z)g^2(z) = P^2_p(z) R_r(z) \quad f(z)g(z) = P_p(z) Q_q(z) R_r(z) \]
are of degree $\leq n-1$, then $2q + r \leq n - 1$, $2p + r \leq n - 1$, $p + q + r \leq n - 1$ and at least once there is an equality.

Conversely it is easy to see that for arbitrary polynomials $P_p(z), Q_q(z), R_r(z)$ with the above restrictions on $p, q, r$, the functions $\text{(3.2)}$ and $\text{(3.3)}$ generate a minimal polynomial surface of degree $n$ via the Weierstrass formula.

So we have

**Theorem 3.1.** Any polynomial minimal surface of degree $n$ in isothermal parameters is generated via the Weierstrass formula by two functions of the form
\[ f(z) = (Q_q(z))^2 R_r(z) \quad g(z) = \frac{P_p(z)}{Q_q(z)} \]
where $P_p(z), Q_q(z), R_r(z)$ are polynomials of degree $p, q, r$, respectively and $2q + r \leq n - 1$, $2p + r \leq n - 1$, $p + q + r \leq n - 1$ with at least one equality.

4. Consequences for Polynomial Minimal Surfaces of Degree Five

With the notations of the previous section we assume $n = 5$. Then
\[ 2q + r \leq 4, \quad 2p + r \leq 4, \quad p + q + r \leq 4 \]
with at least one equality. According to the first two equations in Theorem 2.4 we may assume that (up to a change of the coordinate system) $a = (a_1, a_2, 0)$, $b = (-a_2, a_1, 0)$ and hence
\[ (4.1) \quad 2q + r \leq 4, \quad 2p + r \leq 4, \quad p + q + r \leq 3 \]

Then (4.1) and (4.1) imply $q \leq 2$, $p \leq 2$, so the following cases can appear:

1. $p = 2$. Then $r = 0$.

1.1. $q = 0$, i.e. $f(z) = a$, $g(z) = Az^2 + Bz + C$, where $a, A \neq 0$. To obtain the same surface in an other parametric form we can use the following assertions:

**Proposition 4.1.** Suppose the pairs $(\tilde{f}(z), \tilde{g}(z))$ and $(f(w), g(w))$ generate two minimal surfaces via the Weierstrass formula. Then these surfaces coincide (up to translation) iff there exists a function $w = w(z)$, such that
\[ \tilde{f}(z) = f(w(z))w'(z) \quad \text{and} \quad \tilde{g}(z) = g(w(z)). \]

**Corollary 4.2.** Suppose the pair $(f(z), g(z))$ generates a minimal surface via the Weierstrass formula. Then for arbitrary numbers $\alpha (\alpha \neq 0)$, $\beta$ the pair
\[ \tilde{f}(z) = \alpha f(\alpha z + \beta), \quad \tilde{g}(z) = g(\alpha z + \beta) \]
generates the same minimal surface (up to translation).

So using Corollary 4.2 with $\alpha = \sqrt{A}$, $\beta = \frac{B}{2\sqrt{A}}$ we can say that the surface is generated by two functions of the form

1.1. $f(z) = a$, $g(z) = z^2 + b$, $a \neq 0$.

Analogously we obtain the cases:

1.2. $p = 2, q = 1, r = 0$ and $f(z) = a(z + b)^2$, $g(z) = \frac{cz^2 + d}{z + b}$, $a, c \neq 0$. 


2.1. $p = 0, q = 2, r = 0$ and $f(z) = a(z^2 + b)^2, \quad g(z) = \frac{1}{z^2 + b}, \quad a \neq 0$;

2.2. $p = 1, q = 2, r = 0$ and $f(z) = a(bz^2 + c)^2, \quad g(z) = \frac{z + d}{bz^2 + c}, \quad a, b \neq 0$;

3. $p = 1, q = 0, r = 2$ and $f(z) = az^2 + b, \quad g(z) = z + c, \quad a \neq 0$;

4. $p = 0, q = 1, r = 2$ and $f(z) = (az^2 + b)(z + c)^2, \quad g(z) = \frac{1}{z + c}, \quad a \neq 0$.

We will denote the corresponding surfaces $r_{11}[a, b](u, v), \ r_{12}[a, b, c, d](u, v)$ etc., respectively.

Now we remark that the following can be easily proved:

**Proposition 4.3.** Consider the surfaces

$$S_1 : \quad \mathbf{x}_1(u, v) = \text{Re} \int_{z_0}^z \left( \frac{f_1(z)}{2}(1 - g_1^2(z)), \frac{i f_1(z)}{2}(1 + g_1^2(z)), f_1(z)g_1(z) \right) \, dz$$

$$S_2 : \quad \mathbf{x}_2(u, v) = \text{Re} \int_{w_0}^w \left( \frac{f_2(w)}{2}(1 - g_2^2(w)), \frac{i f_2(w)}{2}(1 + g_2^2(w)), f_2(w)g_2(w) \right) \, dw$$

Denote by

$$S_2^s : \quad \mathbf{x}_2^s(u, v) = \text{Re} \int_{w_0}^w \left( -\frac{f_2(w)}{2}(1 - g_2^2(w)), \frac{i f_2(w)}{2}(1 + g_2^2(w)), f_2(w)g_2(w) \right) \, dw$$

the surface, symmetric of $S_2$ about the plane $Oyz$. Then $S_1$ and $S_2^s$ coincide if and only if

$$f_2(w) = f_1(Z(w))g_1^2(Z(w))Z'(w) \quad g_2(w) = \frac{1}{g_1(Z(w))}$$

for some function $Z(w)$.

Using this proposition we see that the surfaces from cases 2.1, 2.2 and 4 can be viewed as symmetric to those in cases 1.1, 1.2 and 3, respectively. Consequently we have

**Theorem 4.4.** Any polynomial minimal surface of degree 5 in isothermal parameters coincides up to position in space and symmetry with a surface generated via the Weierstrass formula with the pair of functions

1.1. $f(z) = a, \quad g(z) = z^2 + b, \quad a \neq 0$.

1.2. $f(z) = a(z + b)^2, \quad g(z) = \frac{cz^2 + d}{z + b}, \quad a, c \neq 0$.

3. $f(z) = az^2 + b, \quad g(z) = z + c, \quad a \neq 0$

where $a, b, c$ are complex numbers.

Note that the surface from case 3 with $b = c = 0$ (or, which is the same, the surface from case 2.1 with $c = 1, \ b = d = 0$) is the same, introduced in [8] and which may be obtained from the method presented in [9].

5. **Relations among the families in Theorem 4.4**

For some special values of the parameters the surfaces from Theorem 4.4 obviously coincide. Namely if $d = -b^2c$ in $r_{12}[a, b, c, d](u, v)$, the surface is of type 3. On the other hand even when this equality is not satisfied, the corresponding surfaces may look very similar, as Fig. 5.1 and Fig. 5.2 show.
We will see that despite the resemblance these two surfaces are different as well as that in general the families $r_{11}, r_{12}$ and $r_3$ give different surfaces.

Suppose that a surface $r_{12}[a, b, c, d](u, v)$ generated via the Weierstrass formula by the functions

\begin{alignat}{2}
  f_{12}(z) &= a(z + b)^2 & \quad g_{12}(z) &= \frac{cz^2 + d}{z + b} \\
\end{alignat}

coincides with $r_3[A, B, C](u, v)$ generated by

\begin{alignat}{2}
  f_3(z) &= Az^2 + B & \quad g_3 &= z + C.
\end{alignat}

Denote $z_{12}(w), z_3(w)$ solutions of the respective equations (2.2), so that the generating functions in canonical principal parameters

\begin{alignat}{2}
  \tilde{g}_{12}(w) &= g_{12}(z_{12}(w)) & \quad \tilde{g}_3(w) &= g_3(z_3(w))
\end{alignat}

are related by

\begin{alignat}{2}
  \tilde{g}_3(w) &= e^{i\varphi} \frac{\alpha + \tilde{g}_{12}(w)}{1 - \alpha \tilde{g}_{12}(w)} & \quad \text{or} & \quad \tilde{g}_3(w) &= \frac{e^{i\varphi}}{\tilde{g}_{12}(w)}.
\end{alignat}

We will consider only the first possibility. The second can be considered analogously. Note that according to the equation (2.2) the functions $z_{12}(w)$ and $z_3(w)$ are related by

\begin{alignat}{2}
  f_{12}(z_{12}(w))g'_{12}(z_{12}(w))(z'_{12})^2 &= f_3(z_3(w))g'_3(z_3(w))(z'_3)^2.
\end{alignat}

From the last equality, using (5.1)–(5.4) and comparing the coefficients of $z_{12}(w)$ (note that $z_{12}(w)$ may not be constant) we may derive

\begin{alignat}{2}
  \alpha &= 0 & \quad a &= Ac^3e^{4i\varphi} & \quad C + 2bc e^{i\varphi} &= 0 & \quad B &= 0 & \quad b^2c + d &= 0.
\end{alignat}

So the surfaces can coincide only if $b^2c + d = 0$. In particular the surfaces in Fig. 5.1 and Fig 5.2 are different despite the resemblance in Figures 5.1 and 5.2. Actually in a smaller neighborhood of $(u, v) = (0, 0)$ (with the same viewpoint as for Figures 5.1 and 5.2) the difference is clear, see Figures 5.3 and 5.4.
It is not convenient to use the same method to investigate an eventual coincidence of two surfaces from the cases 1.1 and 1.1, so we will change a little the idea. Let us take a surface $S_{11}: r_{11}[A, B](u, v)$ generated via the Weierstrass formula by

$$f_{11}(z) = A \quad g_{11} = z^2 + B,$$

and suppose that it coincides with $S_{12}: r_{12}[a, b, c, d](u, v)$. Then some functions $\tilde{g}_{11}$ and $\tilde{g}_{12}$ that generate them in canonical parameters are related by

$$\tilde{g}_{11}(w) = e^{i\varphi} \frac{\alpha + \tilde{g}_{12}(w)}{1 - \alpha \tilde{g}_{12}(w)}.$$

Denote $z_{11}(w), z_{12}(w)$ the respective solutions of the equation (2.2). Then

$$\tilde{g}_{11}(w) = z_{11}(w) + B$$

and hence

$$2z_{11}(w)z'_{11} = \tilde{g}_{11}(w).$$

On the other hand analogously to (5.5) the equality

$$f_{11}(z_{11}(w))g'_{11}(z_{11}(w))(z'_{11})^2 = f_{12}(z_{12}(w))g'_{12}(z_{12}(w))(z'_{12})^2.$$

holds and hence

$$\left(f_{11}(z_{11}(w))g'_{11}(z_{11}(w))\right)^2(z'_{11})^4 = \left(f_{12}(z_{12}(w))g'_{12}(z_{12}(w))\right)^2(z'_{12})^4.$$

Now applying (5.6)–(5.9) and looking on the coefficients of $z_{12}(w)$ we may conclude that this is impossible. So a surface $S_{11}$ can not coincide with a surface $S_{12}$.

Applying the same argument we may prove that a surface $S_{11}$ can not coincide with a surface $S_{3}$. Actually, analogously we can see that for example a surface of the type 1.1 can not coincide with a surface of the type 2.2 or 4. So we have

**Theorem 5.1.** The families from Theorem 4.4 contain different surfaces except if $b^2c + d = 0$ in case 2.1, in which case the surface belongs also to the case 3.

**Remark.** The surfaces, generated via the Weierstrass formula by the pairs of functions

$$(f(z), g(z)) \quad \text{and} \quad (Cf(z), g(z))$$

are homothetic for any positive real number $C$. On the other hand if $C \neq 0$ is not real, these surfaces are different in general. More precisely let $C = |C|e^{i\varphi}$ for a real number $\varphi$. The pairs $(f(z), g(z))$ and $(|C|f(z), g(z))$ generate two homothetic surfaces, but the surface generated by $(e^{i\varphi}|C|f(z), g(z))$ belongs to the associated family of the surface, generated by $(|C|f(z), g(z))$. 
6. Conclusions

A general formula for polynomial minimal surfaces is found. All polynomial minimal surfaces of degree five are classified in three families, depending on the functions that generated them via the Weierstrass formula. The relation among these families is found.

References

[1] Cosín, C., Monterde, J.: Bézier surfaces of minimal area. Proc. Int. Workshop of Computer Graphics and Geom. Modelig. Lecture Notes in Comput. Sci. 2330, 2002, 72–81.
[2] Eisenhart, L.: A Treatise on the Differential Geometry of Curves and Surfaces. Ginn and company, Boston, New York, Chicago, London, 1909.
[3] Ganchev, G.: Canonical Weierstrass representation of minimal surfaces in Euclidean space. To appear. Available as arXiv:0802.2374.
[4] Gray, A., Abbena, E., Salomon, S.: Modern Differential Geometry of Curves and Surfaces with MATHEMATICA. Boca Raton, FL: CRC Press, 2006.
[5] Kassabov O.: Transition to Canonical Principal Parameters On Minimal Surfaces. Comput. Aided Geom. Design, 31(2014), 441-450.
[6] Kassabov O., Vlachkova, K: On polynomial bi-quartic minimal surfaces. To appear.
[7] Xu, G., Wang, G.: Parametric polynomial minimal surfaces of degree six with isothermal parameter. Lecture Notes in Comput. Sci. 4975, 2008, 329–343.
[8] Xu G., Wang G.: Quintic parametric polynomial minimal surfaces and their properties. Differential Geometry and its Applications, 28(2010), 697-704.
[9] Xu G., Wang G.: Parametric polynomial minimal surfaces of arbitrary degree. To appear. Available as arXiv:1008.0208v2.

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