A Quantum Interface to Electrons in a Vacuum

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Abstract

An electron detector capable of transferring the quantum state of an electron to a set of superconducting qubits is proposed. The key element of the detector, which is insensitive to the kinetic energy of the electron, is a superconducting flux qubit with a ring-shaped magnetic flux quantum. The scheme enables quantum state transfer back to an electron as well. A single such qubit will already allow for non-destructive charged particle detection, leading to a potentially useful application. Major causes of measurement errors are also examined.
Quantum information processing methods involving both the flying qubits and fixed qubits are often desirable. While photons are almost synonymous to flying qubits at present [1, 2], we propose to use flying electrons with fixed superconducting qubits [3]. A known potential application of such quantum information processing is entanglement-assisted electron microscopy for radiation sensitive biological specimens [4]. The use of a superconducting charge qubit [5] and a flux qubit [6] has been suggested to reduce the noise level down to the Heisenberg limit. In what follows, we show that bidirectional quantum information transfer between electrons and superconducting qubits are possible, opening ways to a wider range of applications. Since the flux qubit is more convenient in practice than charge qubits [6], we will concentrate on flux qubits. Before proceeding, we note the recent proposal on the use of trapped electrons with superconducting qubits [7]. We also note that the spin degree of the electron will be ignored in this paper.

Figure 1 (a) shows a qubit based on a radio frequency superconducting quantum interference device (rf-SQUID) [8] with an unusual hollow-ring geometry [6]. We will refer to it as a hollow-ring qubit (HRQ). The design allows for interaction with electron waves passing through the hollow ring via the Aharonov-Bohm (AB) effect [9]. It has been argued that fabrication of such a device is feasible [6] and moreover structures such as one shown in Fig. 1 (b) may suffice for a proof-of-principle experiment. The rf-SQUID loop is magnetic-flux-biased with a half magnetic flux quantum $\phi_0/2 = h/4e$. Hence, the two qubit states $|0\rangle_q$ and $|1\rangle_q$, respectively associated with the trapped flux $\phi \approx -\phi_0/2$ and $\phi_0/2$, have the same energy. For simplicity, we first assume that these values are exactly $\pm \phi_0/2$ and quantum mechanically well-defined. We will use the Coulomb gauge $\text{div} A = 0$ for the vector potential $A$. The structure of the qubit is designed to shift the phase of an electron wave passing through the hollow ring by a phase angle $\pm \pi/2$ depending on the qubit state, while not affecting the phase of the electron waves passing outside the ring. Note that this type of interaction remains the same for any single-charged particles including ions, regardless of kinetic energy of the particle. Let the quantum state of the electron wave passing through the hollow ring be $|a\rangle = (|0\rangle - |1\rangle) / \sqrt{2}$ and the wave passing outside the ring be $|s\rangle = (|0\rangle + |1\rangle) / \sqrt{2}$, where we also introduced the states $|0\rangle, |1\rangle$. We assume that there is zero or only negligible amount of electron wave component impinging on the body of the hollow ring, which may be done deliberately in certain cases [6]. Now, consider a process comprising 2 steps. First, the state $|a\rangle$, and not the state $|s\rangle$, receives a phase shift $\pi/2$ by
a classical electron optical component, which we will call a $\pi/2$ phase shifter \[10\]. Second, the electron wave passes the HRQ. Clearly, this whole process flips the sign of the state $|a\rangle$ if and only if the HRQ is in a certain state, say $|1\rangle_q$. Hence, the process represents a controlled-not (CNOT) gate \[6\], where the HRQ with its logical value represented by the basis states \{$|0\rangle_q, |1\rangle_q$\} controls the electron’s value in terms of the basis states \{$|0\rangle, |1\rangle$\}.

It is well-known that the qubits in a quantum CNOT gate swap their roles upon basis change by the Hadamard transform. Specifically, the state of the HRQ in terms of the symmetric/antisymmetric states \{$|s\rangle_q, |a\rangle_q$\} is flipped only when the electron wave is in the state $|a\rangle$. This immediately suggests a use of a HRQ as a non-destructive electron counter, because its quantum state flips only when an electron flies through it.

To evaluate the above crude idea more quantitatively, we need more detailed understanding of its physics. Figure 1 (c) shows a ring-shaped magnetic flux $B$ associated with the state $|0\rangle_q$, around which the vector potential $A$ swirls. The roles played by current density $j$ and magnetic flux $B$ in an ordinary solenoid are instead played respectively by $B$ and $A$ in our device, which we refer to as an A-solenoid when we emphasize this aspect. To the first approximation, appreciable vector potential exists only inside the A-solenoid. Let us consider how the electron in the state $|a\rangle$, flying through the qubit, flips the qubit state. First, the initial qubit state is $|s\rangle_q = (|0\rangle_q + |1\rangle_q)/\sqrt{2}$. With an electron in the state $|a\rangle$, the initial state of the combined system is $|\psi_0\rangle = |a\rangle|s\rangle_q = (|a\rangle|0\rangle_q + |a\rangle|1\rangle_q) / \sqrt{2}$. The $\pi/2$ phase shifter changes nothing except the overall phase, which we ignore. Upon interaction with the qubit, the electron wave follows or goes against the $A$-field, resulting in a phase shift $\pm \pi/2$. Thus, the state of the combined system after interaction is $(e^{-i\pi/2}|a\rangle|0\rangle_q + e^{i\pi/2}|a\rangle|1\rangle_q) / \sqrt{2}$, which equals, up to an overall phase factor, $|\psi_1\rangle = (|a\rangle|0\rangle_q - |a\rangle|1\rangle_q) / \sqrt{2}$. Passing the phase factor to the qubit, the electron stays in the initial state as $|\psi_1\rangle = |a\rangle (|0\rangle_q - |1\rangle_q) / \sqrt{2} = |a\rangle |a\rangle_q$.

The above consideration identifies the first cause of error with respect to electron detection. Specifically, since there is an $A$-field outside the A-solenoid, the difference $\Delta \theta$ between the two phase shifts associated with two qubit states $|0\rangle_q, |1\rangle_q$ is generally less than $\pi$. This is in contrast to experiments demonstrating the AB effect \[11\], where the phase shift difference between two paths, together encircling the magnetic flux, is measured. Writing $\Delta \theta = \pi - \delta$, the error probability for electron detection is $\approx \delta^2/4$ if $\delta \ll 1$. To estimate $\delta$, let the length of the inner bore of the A-solenoid be
Figure 1: The hollow-ring qubit device. (a) The overall structure. A cross section is shown on the right. The persistent current flows through a double-walled tube, i.e. a hollow-ring with a slit. A Josephson junction ("X" symbol) is inserted across the slit. A ring-shaped magnetic flux is trapped inside the hollow ring. (b) An approximate realization of the device. A planer rf-SQUID is sandwiched between two superconducting blocks, forcing the trapped magnetic flux to loop. (c) An electron, represented by a particle with a negative sign, follows the flow of the $A$-field. (d) The potential energy landscape of an rf-SQUID that is biased with an external magnetic flux $\phi_0/2$. 
and the radius of the bore be \( r \ll l \). Let the magnitude of the \( A \)-field inside the bore be \( A_{\text{bore}} \), which we assume to be highly uniform. Analogous to the fact that an ordinary solenoid can be seen as two ‘magnetic charges’ of opposite signs placed at both the ends, the \( A \)-solenoid can be seen as a pair of ‘vector potential charges’ (VPCs) \( q_A = \pm \pi r^2 A_{\text{bore}} \) located at the two ends, disregarding the \( A \)-field inside the bore. Treating these two VPCs as well-separated point charges, \( A \)-field distribution outside the \( A \)-solenoid can be computed in the same manner the \( E \)-field is computed in electrostatics. The ‘potential difference’ \( \Delta \phi_A \) between the two VPCs would be infinity if these were indeed point charges, but the VPCs have the ‘size’ \( \approx r \). Hence the potential difference is approximately \( \Delta \phi_A \approx 2q_A/4\pi r \approx r A_{\text{bore}} \). Now, consider an integral of \( A \) along a closed path interlinking the HRQ. The integral given as \( \oint_C A \cdot dl \approx A_{\text{bore}} (l + r) \) equals \( \phi_0/2 \). Since the electron flies only inside the bore, the error is \( \delta \approx r/l \). This suggests that an aspect ratio of \( l/r \approx 10 \) would be sufficient to achieve \( \lesssim 1\% \) error for this particular error source.

Figure 1 (d) shows the potential energy of an rf-SQUID biased with an external flux \( \phi_0/2 \), in terms of the magnetic flux \( \phi \) through the SQUID ring. The separation \( \Delta \phi \) between the two potential minima is somewhat less than \( \phi_0 \), hence bringing in another source of phase error. The error probability in terms of electron detection is \( \approx \epsilon^2/4 \) where \( \epsilon = \pi (\phi_0 - \Delta \phi)/\phi_0 \). To evaluate the error, we introduce the inductance of the SQUID loop \( L \), the critical current of the Josephson junction \( i_0 \), and the junction capacitance \( C \). The potential energy of the rf-SQUID is \( U(\phi) = \phi^2/2L - E_J \cos(2\pi\phi/\phi_0 + \pi) \), where \( E_J = i_0\phi_0/2\pi \). From this, \( \pi - \epsilon = \beta \sin \epsilon \) follows, where \( \beta/2\pi = Li_0/\phi_0 \). Numerical calculation reveals that \( Li_0 \gtrsim 2.4\phi_0 \) (or \( \epsilon < 0.03 \)) should be satisfied to have \( \lesssim 1\% \) detection error.

The flux qubit is not strictly discrete in the sense that a spin 1/2 is discrete, and therein lies yet another source of an error. For example, the wavefunction \( \psi_q(\phi) = q \langle \phi|0 \rangle_q \) of the HRQ, where \( |\phi \rangle_q \) is an eigenstate of the magnetic flux \( \phi \), is not fully localized at the potential minimum at \( \phi = \pm (1/2 - \epsilon/2\pi) \phi_0 \). To see the effect of it, notice a relation \( |0 \rangle_q = \int \psi_q(\phi) |\phi \rangle_q d\phi \) for a properly normalized basis system \( \{|\phi \rangle_q \} \). Since the state \( |\phi \rangle_q \) induces a phase shift \( \pi \phi/\phi_0 \) to the electron wave, after interaction with the electron the qubit is left in the state \( |0' \rangle_q = \int \psi_q(\phi) e^{i\pi\phi/\phi_0} |\phi \rangle_q d\phi \). This state is no longer exactly \( |0 \rangle_q \), meaning that there is a finite probability \( p_t = 1 - |q\langle 0'|0 \rangle_q|^2 \) that the qubit state is leaked out of its logical Hilbert space. To estimate the wavefunction spread, first use the standard method to
obtain the Hamiltonian $H = q^2/2C + \phi^2/2L + E_J \cos (2\pi \phi/\phi_0)$ and the commutation relation $[\phi, q] = i\hbar$. To focus on one of the two potential minima, consider a purely harmonic potential that fits one of the two minima of $U(\phi)$. Differentiating $U(\phi)$ twice, the effective inductance $(d^2U(\phi)/d\phi^2)^{-1}$ at the potential minimum $\phi = (1/2 - \varepsilon/2\pi) \phi_0$ is $L_e = LL_J/\{L(1-\varepsilon^2/2) + L_J\} = \{\beta/[(\beta(1-\varepsilon^2/2) + 1)]\} L_J \approx \beta/((\beta + 1)) L_J$, where $L_J = \phi_0/2\pi i_0 = L/\beta$ is the Josephson inductance at zero phase difference across it. Using this, the original Hamiltonian is approximated with a Hamiltonian of a harmonic oscillator, i.e. $H' = q^2/2C + (\phi - \phi_0/2)^2/2L_e$. The ground state is $\psi_q(\phi) = \left(1/\sqrt{\pi \phi_1^2}\right) e^{-\pi \phi/\pi \phi_1^2}$, where $\phi_1^2 = \hbar \sqrt{L_e/C}$. Hence we obtain

$$|q\langle 0' |0 \rangle_q|^2 = \left| \int_{-\infty}^{\infty} d\phi |\psi_q(\phi)|^2 \cos \frac{\pi (\phi - \phi_0/2)}{\phi_0} \right|^2 = e^{-\pi^2 \phi_1^2},$$

and the leakage probability is $p_l \approx (\pi^2/2) (\phi_1/\phi_0)^2$, or $p_l \approx \sqrt{\beta}/((\beta + 1) \sqrt{E_C/8E_J})$ in terms of $E_J$ and $E_C = e^2/2C$. Assuming $\sqrt{\beta}/(\beta + 1) \approx 1$, the ratio $E_C/E_J$ should be $\approx 10^{-3}$ to achieve $\approx 1\%$ detection error, which is not unusual for an rf-SQUID qubit [8].

As discussed earlier, the electron state is fully disentangled from the HRQ state and remains in the initial state after going through the HRQ. Hence, neither spatial nor temporal coherence of the electron wave is important in this case. This remarkable robustness suggests that a classical electron will suffice to explain the device operation. Figure 2 (a) shows a cross section of a HRQ, in which an electron goes through. The HRQ is electrically grounded to a conducting wall with a wire. The electron induces positive charge on the surface of nearby conductors. As the electron goes through the hollow ring, the positive charge moves along. The reader will see that a total of charge $e$ will flow through the Josephson junction from the bottom to up, especially when the hollow ring is such that all the electric field lines from the electron terminates on the surface of the HRQ at one point of time during the electron’s passage. (Things can be complicated when the HRQ is electrically isolated, because its electrostatic potential changes during the passage.) Note that the current generated by movement of the induced charge flows on the outer surface of the HRQ, whereas the current keeping the magnetic flux inside the hollow ring flows on the inner surface of the device. Since these two currents must essentially only at the Josephson junction, we model the rf-SQUID circuit as in Fig. 2 (b), where a current source, producing a current that integrates to $e$, is attached to near the both sides of the Josephson
junction. To make the analysis easier, we replace the current source by a large inductor $L_L$ that traps a large magnetic flux $\phi_L$, generating a bias current $i_b = \phi_L / L_L$. (See Fig. 2 (c). This inductor-based current source is for our thought experiment and need not exist. One is free to imagine changing $i_b$ at will by mechanically deforming the inductor $L_L$ etc.) Since the total magnetic flux $\phi_t = \phi + \phi_L$ (besides the bias flux $\phi_0/2$) is firmly trapped within a superconductor, it is a constant. Hence, the potential energy of the system is

$$U'(\phi) = \frac{\phi^2}{2L} + \frac{(\phi_t - \phi)^2}{2L_L} - E_J \cos (2\pi\phi/\phi_0 + \pi) \approx \frac{\phi^2}{2L} - i_b \phi + E_J \cos (2\pi\phi/\phi_0) + \text{const.,}$$

where we used $\phi_t \approx \phi_b \gg \phi$ and assumed the absence of mutual inductance between $L$ and $L_b$. (Incidentally, we found that the bias flux $\phi_0/2$ may be applied by this type of current biasing. Further thoughts reveals that $i_0$ can also be manipulated using a pair of Josephson junctions with the dc-SQUID configuration, each accepting current biasing.) We set $i_b = e/T$ for a time duration $T$. Then, one potential minimum goes up by an energy amount $\Delta E = i_b \phi_0 / 2 = \hbar/4T$, whereas the other minimum goes down by the same amount. As expected, this results in the phase difference $2\Delta ET/\hbar = \pi$ between the states $|0\rangle_q, |1\rangle_q$, at least for sudden and adiabatic changes of the potential. The same idea goes through with more general shapes of $i_b(t)$.

The above argument, while not rigorous, is useful because it provides insights into 'backaction' to the electron. First, there is non-dissipative influence on the electron motion because of the induced positive charge moving on the inductive HRQ surface, although similar effect should be present with any conductor. Second, there is energy loss of the electron if the HRQ is excited from the state $|s\rangle_q$ to $|a\rangle_q$ when these state have different amounts of energy. (Here we put aside the possibility of excitation within a potential minimum, which we have already discussed.) This is the case if the states $|0\rangle_q$ and $|1\rangle_q$ are coupled through the tunneling barrier in $U(\phi)$. In the excited state $|a\rangle_q$, the phase of the HRQ wavefunction $\psi_q(\phi)$ rotates by $\pi$, mostly within the tunnel barrier when the intra-potential-well wavefunctions are projected onto the respective lowest-energy states. Since the charge operator $q$ may be represented as $-i\hbar \partial / \partial \phi$, momentarily there is electric charge stored in the Josephson junction when the phase is $\approx \pi/2$, albeit with a small probability amplitude corresponding to the small $\psi_q(\phi)$ in the tunnel barrier. This electrostatic charge decelerates the outgoing electron, which thus loses a small amount of energy. Ideally, one would exponentially suppress such energy loss by keeping the tunnel barrier height large by manipulating $i_0$ of the Josephson junction. [12].
Figure 2: Non-destructive charge counting operation. (a) Induced positive charge flows on conductor surfaces as the electron flies through the HRQ. (b) The flying electron may be modeled as a current source, which in turn may be modeled as another superconducting ring as in (c).
A possible application of non-invasive charged particle counting is nano-scale assembly of molecules on a substrate, because the above scheme can count any single-charged object. It is natural to consider an instrument similar to the low energy electron microscope (LEEM), to which ionized molecules are introduced. The HRQ is placed somewhere in the instrument to count the ions going through it. The density of ions are made sufficiently low that only zero or one ions are counted within a suitable time window. The ionized single molecule is then decelerated at the LEEM objective lens and lands on a substrate. The landing energy should be sufficiently low that the molecule would not be damaged, but at the same time minute charging of the substrate should not significantly affect the trajectory of the molecule. There are ways to produce large ionized biological molecules [13]. Thus, when combined with a cryogenic substrate etc., this instrument could be useful in synthetic biology. In actual implementations, however, keeping the HRQ at the dilution-refrigerator temperature while maintaining electron/ion optical access is an issue, since infrared radiation is known to affect superconducting qubit performance [14]. This scheme does not require coherence of charged particle waves and hence may be easier to realize than entanglement-assisted electron microscopy [5, 6].

Because of linearity of quantum mechanics, the above single-qubit scheme can be extended to form a multiple-qubit electron area detector. Consider a 2-dimensional array of $N$ HRQs, which we label 0, 1, \ldots, $N-1$. The initial quantum state of all HRQs are $|s\rangle_q$. The state of the HRQ array $A$, in which only the $k$-th qubit is excited to $|a\rangle_q$, is written as $|k\rangle_A$. Let the electron state going through the $k$-th HRQ be $|k\rangle$, and the initial (unknown) electron state be $\sum_{k=0}^{N-1} c_k |k\rangle$. Because of linearity, after going through the 2D array the state of the system becomes $\sum_{k=0}^{N-1} c_k |k\rangle_A$. The transmitted electron is then detected in the far field with a conventional single electron area detector. Suppose that the electron is detected in a diffracted state $N^{-1/2} \sum_{k=0}^{N-1} e^{i\theta_k} |k\rangle$, where phases $\theta_k$ are known from the electron optical geometry. This leaves the 2D detector in the state $\sum_{k=0}^{N-1} c_k e^{-i\theta_k} |k\rangle_A$. After suitable single-qubit phase manipulations, one will have transferred the electron quantum state to the HRQ array.

Transferring quantum information back to an electron is more involved and we assume the availability of a quantum computer (QC). The basic idea is to use the qudit-version of quantum teleportation [15]. The quantum state to be transferred to an electron is prepared in a register $R$ of the QC as
First, an electron is generated in a plane wave state, which is \( N^{-1/2} \sum_{k=0}^{N-1} |k\rangle \). The electron wave then goes through a HRQ array, which we assume to be 1-dimensional for now. We call the electron optical plane of the array, or any plane conjugate to it, an image plane. The ‘far field’ with respect to the array, or any plane conjugate to it, will be called a diffraction plane. After interaction, the state of the electron and the HRQ array is \( N^{-1/2} \sum_{k=0}^{N-1} |k\rangle |k\rangle_A \). The next step is to use the QC to perform a Bell measurement on the combined system of the HRQ array and the register. (Meanwhile, the electron may go through an electron-optical version of a delay line.) Specifically, the state is measured with respect to basis states \( |\psi_{n,m}\rangle_{AR} = N^{-1/2} \sum_{k=0}^{N-1} e^{2\pi i kn/N} |k\rangle_A |(k + m) \mod N\rangle_R \), where the labels \( n \) and \( m \) run through 0, 1, \ldots, \( N - 1 \). If the measurement outcome is \( (n,m) \), then the electron state is \( \sum_{k=0}^{N-1} e^{-2\pi i (k-m)n/N} d_k |(k-m) \mod N\rangle \).

Finally, the electron wave is manipulated classically, depending on the outcome \( (n,m) \). The phase factors \( e^{-2\pi i (k-m)n/N} \) can be compensated for by applying phase shifts pixelwise at the image plane, using e.g. a multi-pixel version of the obstruction-free phase shifter [16]. Another restoration step \( |(k-m) \mod N\rangle \to |k\rangle \) can be carried out similarly by another pixelwise phase shifter on a diffraction plane. This scheme can be extended to the 2-dimensional case if each pixel is numbered in the raster-scanning manner [17]. In this case, each of the two classical phase shifters, located respectively at an image plane and a diffraction plane, consists of two modified obstruction-free phase shifting devices oriented orthogonal to each other.

The above scheme is essentially universal in that anything programmable can be done, including generation of entangled electrons [18].

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[17] See Supplemental Material.
In a sense, any set of identical fermions are entangled because of antisymmetrization of the wavefunction. Needless to say, we do not mean it here.

Supplemental Material for "A Quantum Interface to Electrons in a Vacuum"

Here we spell out steps for quantum information transfer from the 2-dimensional HRQ array to an electron. We assume HRQs are on a square lattice on the $xy$ plane, which is perpendicular to the optical $z$-axis. Let the number of HRQs along the $x$ and $y$ axes be respectively $N_x$ and $N_y$, so that the total number of HRQ is $N = N_x N_y$. Each HRQ has a label $(k_x, k_y)$ comprising two integers, with the range $0 \leq k_x < N_x$ and $0 \leq k_y < N_y$. These two integers $k_x, k_y$ respectively specifies the HRQ’s position along the $x$ and $y$ axes. Next, we define a single-integer label $k \equiv k_x + N_x k_y$, which range from 0 to $N - 1$. With this label, the argument in the main text goes through without modification also in the present 2-dimensional case.

However, it may be useful to elaborate on the final classical electron-wave manipulation step. Let us write $k' = (k - m) \mod N$. The electron state before the final step is

$$
\sum_{k=0}^{N-1} e^{-2\pi i (k-m)n/N} d_k |(k - m) \mod N \rangle = \sum_{k'=0}^{N-1} e^{-2\pi i k'n/N} d_k |k' \rangle
$$

Note that $d_k = d_{(k'+m)\mod N}$. This can be written as

$$
\sum_{k'=0}^{N-1} e^{-2\pi i \frac{k'x}{N_x} k_x} e^{-2\pi i \frac{k'y}{N_y} k_y'} d_k |k' \rangle.
$$

Hence, the experimenter can first apply a phase shift $2\pi nk'_x/N$ to the $k'_x$-th row, and then another phase shift $2\pi nk'_y/N_y$ to the $k'_y$-th column to obtain

$$
\sum_{k'=0}^{N-1} d_k |k' \rangle.
$$
In terms of instrumentation, the first phase shift can be applied with the multi-pixel version of the obstruction-free phase shifter (See the main text) oriented along the \( x \)-axis, and the second phase shift can likewise be applied by another one aligned with the \( y \)-axis.

To further correct the above state, transformation \(|k'\rangle \rightarrow |k\rangle\) is carried out. To do so, we need to go to the Fourier space. This is done naturally in electron optics, as one can use a lens system to obtain the far-field wavefunction. Let us label the pixels in the far field with integers \( l = 0, 1, 2, \ldots, N - 1 \) and write the diffracted electron state going to the \( l \)-th pixel \(|d_l\rangle\). Following the numbering method used in the image plane, we write \( l = l_x + N_x l_y \), and also \( m = m_x + N_x m_y \). Since the transformation between \(|k'\rangle\) and \(|d_l\rangle\) is essentially the 2-dimensional Fourier transform, we write

\[
|k'\rangle = \frac{1}{\sqrt{N}} \sum_{l_x=0}^{N_x-1} \sum_{l_y=0}^{N_y-1} e^{2\pi i k'_x l_x/N_x} e^{2\pi i k'_y l_y/N_y} |d_l\rangle.
\]

Analogous to the image-plane case, the experimenter can first apply a phase shift \( 2\pi m_x l_x/N_x \) to the \( l_x \)-th row, and then another phase shift \( 2\pi m_y l_y/N_y \) to the \( l_y \)-th column in the diffraction plane to obtain \(|k\rangle\) back in the image plane. Because of the principle of superposition, this means that the above state \( \sum_{k'=0}^{N-1} d_k |k'\rangle \) transforms to \( \sum_{k=0}^{N-1} d_k |k\rangle \), which is what we wanted.