CHARACTER FORMULAE AND A REALIZATION OF TILTING MODULSES FOR $\mathfrak{sl}_2[t]$

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Abstract. In this paper we study the category of graded modules for the current algebra associated to $\mathfrak{sl}_2$. The category enjoys many nice properties, including a tilting theory which was established in [2]. We show that the indecomposable tilting modules for $\mathfrak{sl}_2[t]$ are the exterior powers of the fundamental global Weyl module and give the filtration multiplicities in the standard and costandard filtration. An interesting consequence of our result (which is far from obvious from the abstract definition) is that an indecomposable tilting module admits a free right action of the ring of symmetric polynomials in finitely many variables. Moreover, if we go modulo the augmentation ideal in this ring, the resulting $\mathfrak{sl}_2[t]$–module is isomorphic to the dual of a local Weyl module.

1. Introduction

The current algebra $\mathfrak{g}[t]$ associated to a simple Lie algebra $\mathfrak{g}$ is the algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$, or equivalently, it is the complex vector space $\mathfrak{g} \otimes \mathbb{C}[t]$ and the commutator is the $\mathbb{C}[t]$–bilinear extension of the Lie bracket of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}[t]$ is graded by the non–negative integers where the $r$–th graded component is $\mathfrak{g} \otimes t^r$. We are interested in the (non–semisimple) category $\mathcal{I}$ of $\mathbb{Z}$–graded representation of this Lie algebra with the restrictions that the graded components are finite–dimensional and, only finitely many negatively graded components are non–zero. The category contains a number of well–known and interesting objects: the $\mathfrak{g}$–stable Demazure modules occurring in the highest weight integrable representations of the affine Lie algebra associated to $\mathfrak{g}$ (see [8], [14]) and also the graded limits of many important families of finite–dimensional representations of quantum affine algebras (see for instance [9], [10], [13], [21], [22]). Moreover, the category $\mathcal{I}$ is one of the motivating examples of affine highest weight categories introduced recently in [16] and [17].

The isomorphism classes of simple objects in $\mathcal{I}$ are indexed by pairs $(\lambda, r)$ where $\lambda$ is a dominant integral weight of $\mathfrak{g}$ and $r \in \mathbb{Z}$. It is not hard to see that an irreducible module $V(\lambda, r)$ in the corresponding class is just isomorphic to the finite–dimensional irreducible $\mathfrak{g}$–module associated to $\lambda$. The category is well–behaved in the sense that it has enough projective objects (see [3]) although these are never finite–dimensional. The projective cover $P(\lambda, r)$ of $V(\lambda, r)$ has two other important quotients: the first one is called the global Weyl module $W(\lambda, r)$ and is the maximal quotient of $P(\lambda, r)$ whose weights are all bounded above by $\lambda$. The global Weyl modules are infinite–dimensional (if $\lambda \neq 0$) and the second quotient which is of interest to us, is the unique maximal finite–dimensional quotient of $W(\lambda, r)$. These are called

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the local Weyl modules and denoted as $W_{\text{loc}}(\lambda, r)$ and they are indecomposable but usually reducible objects of $\mathcal{I}$. The definition of the global Weyl module parallels the definition of standard modules which arise in different contexts in the literature [11],([12],) for instance. An important result conjectured in [4] and eventually proved in [7] in complete generality is: the projective module $P(\lambda, r)$ has a filtration where the successive quotients are global Weyl modules with multiplicities given by Jordan–Holder series of the dual local Weyl modules. This implies that the analog of a costandard object in this category should be the dual local Weyl module. (We remark here that the category $\mathcal{I}$ is not closed under taking duality in general although the full subcategory of $\mathcal{I}$ consisting of finite–dimensional objects is closed under duality).

With the definitions of standard and costandard objects in place, it is reasonable to ask if the category contains tilting objects; namely an object which has a filtration where the successive quotients are standard modules and another filtration where the successive quotients are costandard modules. In [2], the authors introduced the notion of an $\sigma$–canonical filtration of an object of $\mathcal{I}$ and gave an Ext–vanishing criterion for the existence of a standard filtration. A similar result for the costandard filtration does not follow since the standard and costandard modules are not dual to each other and in fact such a criterion is not known. This, along with the fact that the index set of simple objects is infinite causes some difficulty. However, the main result of that paper, using the ideas in [18] for algebraic groups, shows very abstractly, the existence of a unique (up to isomorphism) indecomposable tilting object $T(\lambda, r)$ for $r \in \mathbb{Z}$ and $\lambda$ a dominant integral weight of $\mathfrak{g}$. It was also proved that any other tilting object in $\mathcal{I}$ is the direct sum of the indecomposable tilting objects.

In this paper we turn to the question of determining the character of the indecomposable tilting module in the simplest case of $\mathfrak{sl}_2[t]$. In fact, we are able to realize the indecomposable tilting modules explicitly and to compute the filtration multiplicities in the standard and costandard filtration. In the case of $\mathfrak{sl}_2$, a dominant integral weight $\lambda$ is just a non–negative integer and we prove that the module $T(\lambda, 0)$ is just a grade shift of $\wedge^\lambda W(1, 0)$. This realization of the tilting module proves that it admits a free right action for the ring $A_\lambda$ of symmetric polynomials in $\lambda$–indeterminates and we show that

$$T(\lambda, 0) \otimes_{A_\lambda} A_\lambda/I_\lambda \cong_{\mathfrak{sl}_2[t]} W_{\text{loc}}(\lambda, r)^*,$$

for suitable $r \in \mathbb{Z}$ and $I_\lambda$ is the unique maximal graded ideal of $A_\lambda$. These results should be compared with the results on global Weyl modules (see [10]): $W(\lambda, 0)$ is the $\lambda$–th symmetric power of $W(1, 0)$ and hence a free right module for $A_\lambda$ with

$$W(\lambda) \otimes_{A_\lambda} A_\lambda/I_\lambda \cong_{\mathfrak{sl}_2[t]} W_{\text{loc}}(\lambda, 0).$$

In the case of higher rank simple Lie algebras, suitable analogs of the result for global Weyl modules is known through the work of [8], [14] and [20]. Based on our result on tilting modules for $\mathfrak{sl}_2[t]$ it is natural for us to conjecture that for the higher rank simple algebras, the tilting module $T(\lambda, 0)$ is a free $A_\lambda$–module and gives the dual local Weyl module on passing to the quotient by $I_\lambda$.

We conclude this introduction by making some comments on the conjecture which indicate that it is probably too simple in general. In the case of $\mathfrak{sl}_{n+1}[t]$, it is very plausible that the conjecture can be proved by our methods when $\lambda$ is a multiple of the first fundamental weight. However in the general case there are several obstructions. In the case of the global
Weyl modules for which one knows a presentation, the action is very natural (see [5]) even in the higher rank cases. However, this is not the case for tilting modules and the abstract construction does not indicate the existence of this action. The other difficulty is the following. In the case of $\mathfrak{sl}_2[t]$ we prove in this paper that the tensor product of global Weyl modules has a filtration by global Weyl modules. However, as Mark Shimozono pointed out to us, it is easy to check using the known character formulae for the global Weyl module that the tensor square of the second fundamental representation for $\mathfrak{sl}_n+1[t]$ cannot admit a filtration by global Weyl modules.

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2. Preliminaries

2.1. Throughout this paper we denote by $\mathbb{C}$ the field of complex numbers and by $\mathbb{Z}$ (resp., $\mathbb{Z}_+$, $\mathbb{N}$) the subset of integers (resp. non–negative, positive integers). We fix an indeterminate $u$ and let $\mathbb{Z}[[u, u^{-1}]]$ be the set of power series in $u$ and $u^{-1}$. We shall generally be interested in the case when the power series is infinite only in one direction. Given $n, s \in \mathbb{N}$, define elements $(1 : u)_n \in \mathbb{Z}[[u]]$ and $\begin{bmatrix} n \\ s \end{bmatrix} \in \mathbb{Z}[u]$ by

$$(1 : u)_n = \frac{1}{(1 - u)(1 - u^2) \cdots (1 - u^n)},$$

$$\begin{bmatrix} n \\ s \end{bmatrix} = \frac{(1 - u^n) \cdots (1 - u^{n-s+1})}{(1 - u) \cdots (1 - u^s)}.$$  

We understand also that $\begin{bmatrix} n \\ s \end{bmatrix} = 1$ if $s$ is zero.

2.2. We say that a complex vector space $V$ is $\mathbb{Z}$–graded and locally finite–dimensional if it is a direct sum of finite–dimensional subspaces $V[r], r \in \mathbb{Z}$ Set

$$\mathbb{H}(V) = \sum_{r \in \mathbb{Z}} \dim V[r]u^r.$$  

For $s \in \mathbb{Z}$ we let $\tau_s^* V$ be the grade shift of $V$, i.e. the vector space $V$ with the grading given by declaring $V[r + s]$ to be of grade $r$. The one–dimensional vector space $\mathbb{C}$ will be regarded as being $\mathbb{Z}$–graded and located in grade zero. We adopt the convention that all unadorned tensor products of complex vector spaces are over $\mathbb{C}$. The tensor product of $\mathbb{Z}$–graded vector spaces $V_1$ and $V_2$ is again graded, with grading given by,

$$(V_1 \otimes V_2)[r] = \bigoplus_{k \in \mathbb{Z}} V_1[k] \otimes V_2[r - k].$$

If $\dim V_j[r] = 0$ for all $r$ sufficiently large (resp. sufficiently small) for $j = 1, 2$ then $V_1 \otimes V_2$ is locally finite–dimensional if $V_1$ and $V_2$ are locally finite–dimensional and in this case we have

$$\mathbb{H}(V_1 \otimes V_2) = \mathbb{H}(V_1)\mathbb{H}(V_2).$$
A linear map \( f : V_1 \to V_2 \) is graded of degree \( p \) if \( f(V_1[r]) \subset V_2[r+p] \) for some fixed \( p \in \mathbb{Z} \). The subspace \( \text{Hom}_{gr}(V_1, V_2) \) of \( \text{Hom}_{\mathbb{C}}(V_1, V_2) \) spanned by graded maps is again a graded vector space. In the special case when \( V_1 \) is locally finite–dimensional, and \( V_2 \) is one–dimensional, i.e. \( V_2 = \tau_s^* \mathbb{C} \) for some \( s \in \mathbb{Z} \), we have
\[
\mathbb{H}(\text{Hom}_{gr}(V_1, \tau_s^* \mathbb{C})) = \sum_{r \in \mathbb{Z}} \dim \text{Hom}_{\mathbb{C}}(V[r], \mathbb{C}) u^{s-r}.
\]

In the rest of the paper, we shall denote by \( V^* \) the vector space \( \text{Hom}_{gr}(V, \mathbb{C}) \) in which case, \( \tau_s^* V^* \cong \text{Hom}_{gr}(V, \tau_{-s}\mathbb{C}) \).

2.3. Fix an indeterminate \( t \), and let \( a[t] = a \otimes \mathbb{C}[t] \) be the Lie algebra with bracket
\[
[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}, \quad x, y \in a, \quad r, s \in \mathbb{Z}_+.
\]
The Lie algebra \( a[t] \) has a natural grading given by the powers of \( t \) and the enveloping algebra \( U(a[t]) \) acquires a canonical grading; thus \( U(a[t])[k] \) is the subspace of \( U(a[t]) \) spanned by elements of the form \((a_1 \otimes t^{r_1}) \cdots (a_s \otimes t^{r_s})\) with \( s \in \mathbb{Z}_+ \) and \( r_1 + \cdots + r_s = k \). We identify \( a \) with the graded subalgebra \( a \otimes 1 \subset a[t] \).

2.4. By a graded representation of \( a[t] \), we mean a \( \mathbb{Z} \)-graded vector space \( V \) such that
\[
(a \otimes t^r)V[s] \subset V[r+s], \quad r \in \mathbb{Z}_+, \quad s \in \mathbb{Z}.
\]
If \( V_j, j = 1, 2 \) are graded representations of \( a[t] \), then so are \( V_1 \otimes V_2 \) and \( V_j^* \) with \( j = 1, 2 \). Let \( \text{ev}_0 : a[t] \to a \) be the homomorphism of Lie algebras which is the identity on \( a \) and zero on \( a \otimes t^s \) if \( s > 0 \). The pull–back of a representation \( V \) of \( a \) by \( \text{ev}_0 \) gives a graded representation \( \text{ev}_0^* V \) of \( a[t] \).

2.5. Given a representation \( V \) of \( a \), we define a representation of \( a[t] \) on \( V \otimes \mathbb{C}[t] \), by
\[
(x \otimes t^r)(v \otimes t^s) = xv \otimes t^{r+s}.
\]
It is clear that the obvious grading on \( V \otimes \mathbb{C}[t] \) makes it a graded \( a[t] \)-module. For \( r \in \mathbb{N} \), consider the graded left \( a[t] \)-module \((V \otimes \mathbb{C}[t])^{\otimes r} \). Let \( \mathbb{C}[t_1, \ldots, t_r] \) be the polynomial ring in \( r \)-variables and regard it being \( \mathbb{Z}_+ \) graded by requiring the grade of \( t_j \) to be one for \( 1 \leq j \leq r \). Using the identification,
\[
(V \otimes \mathbb{C}[t])^{\otimes r} \cong V^{\otimes r} \otimes \mathbb{C}[t_1, \ldots, t_r],
\]
we see that there exists a right action of \( \mathbb{C}[t_1, \ldots, t_r] \) on \((V \otimes \mathbb{C}[t])^{\otimes r} \) given by right multiplication. This action however does not commute with the action of \( a[t] \). However, the obvious left action of the symmetric group \( S_r \) on \( r \)-letters on \((V \otimes \mathbb{C}[t])^{\otimes r} \) does commute with the action of both the left action of \( a[t] \) and the right action of \( \mathbb{C}[t_1, \ldots, t_r] \). Let \( A_r \) be the graded subalgebra of invariants of the action of \( S_r \) on \( \mathbb{C}[t_1, \ldots, t_r] \), in which case
\[
\mathbb{H}(A_r) = (1 : u)_r.
\]
It is now clear that the left actions of \( S_r, a[t] \) commute with each other and also that the right action of \( A_r \) commutes with the action of \( a[t] \) and all actions are grade. Since \( \mathbb{C}[t_1, \ldots, t_r] \) is a free right \( A_r \)-module it also follows that the right action of \( A_r \) on \((V \otimes \mathbb{C}[t])^{\otimes r} \) is free. In the rest of the paper, we shall use freely that \( A_r \) is a polynomial algebra and hence by the Quillen–Suslin theorem that a summand of a free module for \( A_r \) is free. In particular, every
isotypical component of the action of $S_r$ on $(V \otimes \mathbb{C}[t])^\otimes t$ is a left $\mathfrak{sl}_2$–module and a right $A_\lambda$–module. We shall be interested in two particular isotypical components, namely the one corresponding to the trivial and the sign representation of $S_r$, i.e., the symmetric and wedge powers of $V \otimes \mathbb{C}[t]$. Let $I_r$ be the maximal ideal defined by taking the sum of all the positively graded components of $A_r$.

2.6. In the rest of the paper we take $a$ to be the Lie algebra $\mathfrak{sl}_2$ of $2 \times 2$ complex matrices of trace zero and $V = V(1)$ to be the defining two dimensional representation of $\mathfrak{sl}_2$. We let $x, y, h$ be the standard basis of $\mathfrak{sl}_2$ and take $e_j, j = 1, 2$ to be the standard basis of $V(1)$. We have

$$xe_1 = ye_2 = 0, \quad he_j = (3 - 2j)e_j, \quad j = 1, 2, \quad ye_1 = e_2, \quad xe_2 = e_1.$$ 

For any $\lambda \in \mathbb{Z}_+$, let $V(\lambda)$ be the unique (up to isomorphism) irreducible representation of $\mathfrak{sl}_2$ with dimension $\lambda + 1$. Any finite–dimensional $\mathfrak{sl}_2$–module $V$ is isomorphic to a direct sum of the $V(\lambda), \lambda \in \mathbb{Z}_+$ and

$$V \cong \bigoplus_{\mu \in \mathbb{Z}} V_{\mu}, \quad V_\mu = \{v \in V : hv = \mu v\}.$$ 

The character of $V$, denoted as $\text{ch} V$ is the function $\mathbb{Z} \to \mathbb{Z}_+$ sending $\mu \to \dim V_\mu$. Then, 

$$V \cong \bigoplus_{\lambda \in \mathbb{Z}_+} V(\lambda)^\otimes m_\lambda \implies \text{ch} V = \sum_{\lambda \in \mathbb{Z}_+} m_\lambda \text{ch} V(\lambda),$$

where,

$$\text{ch} V(\lambda)(\mu) = \begin{cases} 1, & \mu = \lambda - 2p, 0 \leq p \leq \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

3. The main results

3.1. Let $\mathcal{I}_{\text{bdd}}$ be the category whose objects are $\mathbb{Z}$–graded representations $V$ of $\mathfrak{sl}_2[t]$ with $V[p] = 0$ for all $p < 0$, $\dim V[p] < \infty$ for all $p \in \mathbb{Z}$ and $V_\mu = 0$ for all but finitely many $\mu \in \mathbb{Z}$.

The morphisms are degree zero maps of $\mathfrak{sl}_2[t]$–modules and $\mathcal{I}_{\text{bdd}}$ is an abelian category which is closed under tensor products. The dual (see Section 2.2) of a finite–dimensional object of $\mathcal{I}_{\text{bdd}}$ is again an object of $\mathcal{I}_{\text{bdd}}$. Given an object $V$ of $\mathcal{I}_{\text{bdd}}$, we have

$$V \cong_{\mathfrak{sl}_2} \bigoplus_{r \in \mathbb{Z}} V[r], \quad V_\mu = \bigoplus_{r \in \mathbb{Z}} V[r]_\mu, \quad \mu \in \mathbb{Z}.$$ 

The graded character is,

$$\text{ch}_{\text{gr}} V = \sum_{r \in \mathbb{Z}} \text{ch} V[r] u^r = \sum_{r \in \mathbb{Z}} \left( \sum_{\lambda \in \mathbb{Z}_+} \text{dim} \text{Hom}_{\mathfrak{sl}_2}(V(\lambda), V[r]) \text{ch} V(\lambda) \right) u^r.$$ 

For brevity, we shall also write 

$$[V[r] : V(\lambda)] = \text{dim} \text{Hom}_{\mathfrak{sl}_2}(V(\lambda), V[r]).$$

Graded characters are additive on short exact sequences and multiplicative on tensor products.
3.2. We now introduce the main objects of interest and summarize the necessary results. We refer the reader to [10, Section 2] (see also [8], [13]) for details. The simple objects of $I_{\text{bdd}}$ are

$$V(\lambda, r) = \tau_*^{r} \text{ev}_0^* V(\lambda), \quad r \in \mathbb{Z}, \quad \lambda \in \mathbb{Z}_+.$$  

For $\lambda \in \mathbb{Z}_+$, the global Weyl module $W(\lambda)$ is defined to be the $\mathfrak{sl}_2[\mathbb{C}[t]]$–module generated by an element $w_\lambda$ with relations:

$$(x \otimes \mathbb{C}[t])w_\lambda = 0, \quad hw_\lambda = \lambda w_\lambda, \quad (y \otimes 1)^{\lambda + 1} w_\lambda = 0. \quad (3.1)$$

Since the defining relations of $W(\lambda)$ are graded, it follows that $W(\lambda)$ is a graded $\mathfrak{sl}_2[\mathbb{C}[t]]$–module once we define the grade of $w_\lambda$ to be zero. Moreover, $W(\lambda)[-r] = 0$ if $r \in \mathbb{N}$ and $W(\lambda) \in I_{\text{bdd}}$.

For $r \in \mathbb{Z}$, we denote the grade shifted module $W(\lambda, r) = \tau_*^r W(\lambda)$.

The local Weyl module $W_{\text{loc}}(\lambda)$ is the graded quotient of $W(\lambda)$ obtained by imposing the additional relation: $(h \otimes t^s)w_\lambda = 0$, $s \in \mathbb{N}$ and we write $W_{\text{loc}}(\lambda, r) = \tau_*^r W_{\text{loc}}(\lambda)$. It is elementary to see that $\dim W(\lambda, r)[r] = \dim W_{\text{loc}}(\lambda, r) = 1$,

and we shall denote by $w_\lambda$ any non–zero element of the corresponding space. Note that $\text{ch}_{gr} W(\lambda, r) = u^r \text{ch}_{gr} W(\lambda)$ etc. We use the notation and results of Section 2.5 freely in the next proposition.

**Proposition.** (i) We have an isomorphism of $\mathfrak{sl}_2[\mathbb{C}[t]]$—modules:

$$W(\lambda) \cong S^\lambda (V(1) \otimes \mathbb{C}[t]).$$

In particular $W(1) \cong V \otimes \mathbb{C}[t]$. Moreover, $W(\lambda)$ is a free right module for $A_\lambda$ of rank $2^\lambda$ and

$$W_{\text{loc}}(\lambda) \cong W(\lambda) \otimes_{A_\lambda} A_\lambda / I_\lambda.$$  

(ii) For $0 \leq p \leq \lambda$, we have,

$$\mathbb{H}(W(\lambda)_{\lambda - 2p}) = (1 : u)_{\lambda} \mathbb{H}(W_{\text{loc}}(\lambda)_{\lambda - 2p}) = (1 : u)_{\lambda} \binom{\lambda}{p},$$

$$\text{ch}_{gr} W_{\text{loc}}(\lambda) = \sum_{r \in \mathbb{Z}} \left( \binom{\lambda}{\lambda - 2r} - \binom{\lambda}{\lambda - 2r + 2} \right) \text{ch} V(\lambda - 2r),$$

$$\text{ch}_{gr} W(\lambda) = (1 : u)_{\lambda} \text{ch}_{gr} W_{\text{loc}}(\lambda).$$

(iii) The module $W_{\text{loc}}(\lambda, r)$ has a unique irreducible quotient which is isomorphic to $V(\lambda, r)$. Equivalently, the dual module $W_{\text{loc}}(\lambda, -r)^* \text{ has an irreducible socle which is isomorphic to } V(\lambda, r)$.

\[\square\]
3.3. We say that an object $M$ of $\mathcal{I}_{\text{bdd}}$ has a filtration by global Weyl modules if there exists a decreasing family of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_k \supset \{0\}, \quad M_s/M_{s+1} \cong \bigoplus_{p \in \mathbb{Z}} W(\lambda_s, p)^{\oplus m_{s,p}},$$

for some $1 \leq s \leq k$ and some $\lambda_s, m_{s,p} \in \mathbb{Z}_+$. It was shown in [2], [4] that the multiplicity of a global Weyl module in such a filtration is independent of the choice of the filtration.

Remark. It is worth emphasizing that for a fixed $s$, one could have $m_{s,p} > 0$ for infinitely many $p \in \mathbb{Z}_+$.

We shall say that $M$ has a filtration by dual local Weyl modules, if there exists a decreasing family of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots, \quad \cap_{s \geq 0} M_s = \{0\}, \quad M_s/M_{s+1} \cong W_{\text{loc}}(\lambda_s, p_s)^*,$$

for some $\lambda_s \in \mathbb{Z}_+, p_s \in \mathbb{Z}$. Again, the multiplicity of the dual local Weyl module in such a filtration was proved in [2] to be independent of the filtration. Since the local Weyl modules are finite–dimensional, we see that the filtration is of infinite length unless $\dim M < \infty$.

Say that $T \in \text{Ob} \mathcal{I}_{\text{bdd}}$ is tilting if it has a filtration by global Weyl modules and a (possibly different) filtration by dual local Weyl modules. The dual of the following was proved in [2, Theorem 2.7].

Theorem. Given $\lambda \in \mathbb{Z}_+$, there exists a unique (upto isomorphism) indecomposable tilting module $T(\lambda)$ such that

$$\text{wt } T(\lambda) \subset \{\lambda, \lambda - 2, \cdots, -\lambda\}, \quad \text{dim } T(\lambda)[0]_\lambda = 1, \quad \text{dim } T(\lambda)[-s]_\lambda = 0, \quad s \in \mathbb{N}.$$ (3.2)

Moreover any tilting module in $\mathcal{I}_{\text{bdd}}$ is isomorphic to a direct sum of modules $T(\lambda, r) = \tau^r_\lambda T(\lambda)$, $\lambda \in \mathbb{Z}_+, r \in \mathbb{Z}$.

3.4. The main result of this paper is the following.

Theorem. Let $\lambda \in \mathbb{Z}_+$ and recall that $W(1) \cong V(1) \otimes \mathbb{C}[t]$.

(i) We have an isomorphism of objects of $\mathcal{I}_{\text{bdd}}$,

$$T(\lambda) \cong \tau^\lambda_{-r_\lambda} (\wedge^\lambda W(1)), \quad r_\lambda = \left(\frac{\lambda}{2}\right).$$

In particular, $T(\lambda)$ has the structure of a free right $A_\lambda$–module of rank $2^\lambda$.

(ii) We have an isomorphism of objects of $\mathcal{I}_{\text{bdd}}$,

$$T(\lambda) \otimes_{A_\lambda} A_\lambda/I_\lambda \cong_{\mathfrak{sl}_2[t]} T^*_{-r_\lambda} W_{\text{loc}}(\lambda)^*.$$

(iii) The character of the tilting module is,

$$\text{ch}_{\text{gr}} T(\lambda) = u^{r_\lambda}(1 : u)_\lambda \text{ch}_{\text{gr}} W_{\text{loc}}(\lambda)^* = \sum_{k=0}^{[\lambda/2]} u^{(\lambda - k)/2 + (k/2)}(1 : u)_k \text{ch}_{\text{gr}} W(\lambda - 2k).$$
3.5. It is trivial to see that the module \( \tau^*_{\lambda} (\wedge^2 W(1)) \) satisfies equations (3.2) and (3.3). Hence Theorem 3.4(1) follows if we establish that \( \wedge^2 W(1) \) is indecomposable and tilting. In Sections 4 we prove that \( \wedge^2 W(1) \) is indecomposable, establish the existence of a filtration by dual local Weyl modules and prove part (ii) of the theorem. The first equality of Theorem 3.4(iii) is immediate from (ii) and the second equality is a simple checking using the explicit formulae for the characters of the local and global Weyl modules. In Section 5 we complete the proof that \( \wedge^2 W(1) \) has a filtration by global Weyl modules. The key ingredients are the homological criterion for the existence of such a filtration which was proved in [2] and the following proposition.

**Proposition.** Suppose that \( M \) has a filtration by global Weyl modules then so does \( M \otimes W(1) \). In particular, \( W(1)^{\otimes \lambda} \) has a filtration by global Weyl modules.

**Corollary.** If \( M, N \in \text{Ob} I_{\text{bdd}} \) admit a finite filtration by global Weyl modules then so does \( M \otimes N \).

**Remark.** The corollary is false for higher rank simple Lie algebras. In fact in the case of \( \mathfrak{sl}_4 \), it is not hard to check by computing the characters that the tensor square of the global Weyl module associated to the second fundamental representation cannot admit a filtration by global Weyl modules.

4. **Indecomposability of \( \wedge^2 W(1) \) and the Dual Weyl Filtration**

The first part of the section is devoted to proving that \( \wedge^2 W(1) \) is indecomposable.

4.1. The following was proved in [3] and will play an important role in this section.

**Proposition.** Suppose that \( \mu \in \mathbb{Z}^+ \), \( r \in \mathbb{Z} \) and that \( \psi : \tau^* W(\mu) \rightarrow W(1)^{\otimes \lambda} \) is any non-zero map. Then \( r \geq 0 \), \( \lambda = \mu \) and \( \psi \) is injective.

**Remark.** The proof in the special case of \( \mathfrak{sl}_2 \) is relatively easy. The module \( W(1) \) has basis \( \{ e_j \otimes t^r : j = 1, 2, r \in \mathbb{Z}_+ \} \) where we recall that \( e_j, j = 1, 2 \) is the standard basis of \( V(1) \). To prove the proposition, one first shows that the only vectors in \( W(1)^{\otimes \lambda} \) which are annihilated by \( x \otimes \mathbb{C}[t] \) must be of weight \( \lambda \). To see that the map is injective, one uses the fact that both \( W(\lambda, r)_\lambda \) and \( W(1)^{\otimes \lambda} \) are free right \( A_\lambda \)-modules, with the former module being of rank one.

4.2. Let \( \text{Vand}(\lambda) \) be the Vandermonde determinant in \( t_1, \ldots, t_\lambda \). Then

\[
\wedge^\lambda (W(1))_\lambda = e^{\otimes \lambda} \otimes \text{Vand}(\lambda) A_\lambda, \quad \mathbb{H} (\wedge^\lambda W(1))_\lambda = u^{\tau^\lambda}(1 : u)_\lambda. \tag{4.1}
\]

More generally the space \( (\wedge^\lambda W(1))_{\lambda - 2k} \) has basis \( e_1 t^{r_1} \wedge \cdots \wedge e_1 t^{r_{\lambda-k}} \wedge e_2 t^{s_1} \wedge \cdots \wedge e_2 t^{s_k} \) with \( 0 \leq r_1 < \cdots < r_{\lambda-k} \) and \( 0 \leq s_1 < \cdots < s_k \). It follows that

\[
\mathbb{H}(\wedge^\lambda W(1))_{\lambda - 2k} = u^{(\lambda-k)+(\ell/2)}(1 : u)_{\lambda-k}(1 : u)_k \tag{4.2}
\]

We isolate the following case for later use.

\[
\wedge^\lambda W(1)[s]_{\lambda - 2[\lambda/2]} = \begin{cases} 0, & s < \left(\frac{[\lambda/2]}{2}\right)(\frac{[\lambda/2]}{2}), \\ 1, & s = \left(\frac{[\lambda/2]}{2}\right)(\frac{[\lambda/2]}{2}). \end{cases} \tag{4.3}
\]
Note that
\[\left(\frac{\lfloor \lambda/2 \rfloor}{2}\right) \left(\frac{\lceil \lambda/2 \rceil}{2}\right) = r_\lambda - \lfloor \lambda/2 \rfloor \lceil \lambda/2 \rceil.\]

The following is now immediate using Proposition 3.2(ii), equation (4.2) and
\[\sum_{j=0}^{k} u^{(\lambda-k)+\left(\frac{j}{2}\right)}(1 : u)_{\lambda-k-j} = u^{(\lambda-k)+\left(\frac{k}{2}\right)}(1 : u)_{\lambda-k}(1 : u)_{k}.\]

In particular, assuming Theorem 3.4(i), it establishes the second identity in Theorem 3.4(iii).

**Lemma.** We have
\[\text{ch}_{\mathfrak{g}} \Lambda^\lambda W(1,0) = \sum_{k=0}^{\lfloor \lambda/2 \rfloor} u^{(\lambda-k)+\left(\frac{j}{2}\right)}(1 : u)_{k} \text{ch}_{\mathfrak{g}} W(\lambda-2k,0).\]

4.3. We now prove,

**Proposition.** Let \(\lambda \in \mathbb{Z}_+\).

(i) If \(M\) is a non-zero graded \(\mathfrak{sl}_2[t]\)-submodule of \(\Lambda^\lambda W(1)\) then \(M_\lambda \neq 0\).

(ii) The subspace \((\Lambda^\lambda W(1))_\lambda\) is generated as a \((h \otimes \mathbb{C}[t])\)-module by the element \(e_1^{\otimes \lambda} \otimes \text{Vand}(\lambda)\).

(iii) The module \(\Lambda^\lambda W(1)\) is an indecomposable \(\mathfrak{sl}_2[t]\)-module.

**Proof.** Suppose that \(M\) is a graded \(\mathfrak{sl}_2[t]\)-submodule of \(\Lambda^\lambda W(1)\). Since \(\text{wt} M \subset \lambda - \mathbb{Z}_+\) and \(M\) is a sum of finite-dimensional \(\mathfrak{sl}_2\)-modules, there exists \(\mu \in \mathbb{Z}_+\) and \(p \in \mathbb{Z}\) and \(M[p]_\mu \neq 0\), \((r \otimes \mathbb{C}[t])M[p]_\mu = 0\).

It follows from the defining relations of \(W(\mu)\) that there exists a non-zero map \(\psi: \tau^*_p W(\mu) \rightarrow M \rightarrow \Lambda^\lambda W(1) \rightarrow W(1)^{\otimes \lambda}\).

By Proposition 4.1 implies that \(\psi\) is injective and \(\mu = \lambda\), i.e. \(M_\lambda \neq 0\) as required and proves part (i) of the proposition.

Recall that \(A_\lambda\) is generated as a subalgebra of \(\mathbb{C}[t_1, \ldots, t_\lambda]\) by the elements \(t_1^s + \cdots + t_\lambda^s\), \(s \in \mathbb{Z}_+\). Part (ii) follows from 4.1
and the fact that
\[(h \otimes t^s) \left( e_1^{\otimes \lambda} \otimes \text{Vand}(\lambda) \right) = e_1^{\otimes \lambda} \otimes \text{Vand}(\lambda)(t_1^s + \cdots + t_\lambda^s).\]

To prove (iii), suppose that \(\Lambda^\lambda W(1)\) is decomposable and is a direct sum of graded \(\mathfrak{sl}_2[t]\)-submodules \(M_j, j = 1, 2\). Since \(W(1)|_{r_\lambda}\lambda\) is one-dimensional it must be contained in \(M_1\) or \(M_2\), say \(M_1\). Then, by part (ii) of the proposition we have \((\Lambda^\lambda W(1))_\lambda \subset M_1\) which proves that \(M_2_\lambda = 0\). Part (i) implies that \(M_2 = 0\) and the proof is complete.
The local Weyl modules $W_{\text{loc}}(\lambda, r)$ have a natural universal property which is straightforward from the defining relations. Namely, let $V \in \text{Ob } \mathcal{I}_{\text{bdd}}$ be such that
\[
\dim V_\lambda = \dim V[r]_\lambda = p, \quad V_\mu = 0, \quad \mu > \lambda,
\]
and $V$ is generated as an $\mathfrak{sl}_2[t]$-module by $V_\lambda$. There exists a surjective morphism of objects of $\mathcal{I}_{\text{bdd}}$,
\[
W_{\text{loc}}(\lambda, r)^{\oplus p} \rightarrow V \rightarrow 0,
\]
extending a vector space isomorphism $W_{\text{loc}}(\lambda, r)^{\oplus p}_\lambda \rightarrow V_\lambda$. We shall frequently need the analog of this for the dual local Weyl module which we isolate in the following.

**Lemma.** Suppose that $U \in \text{Ob } \mathcal{I}_{\text{bdd}}$ is such that
\[
\dim U_\lambda = \dim U[r]_\lambda = p, \quad U_\mu = 0, \quad \mu > \lambda,
\]
and assume that every non–zero submodule of $U$ intersects $U_\lambda$ non–trivially. There exists an injective morphism
\[
0 \rightarrow U \rightarrow (W_{\text{loc}}(\lambda, -r)^*)^{\oplus p},
\]
of objects of $\mathcal{I}_{\text{bdd}}$.

**Proof.** Since $U$ is such that any submodule intersects $U_\lambda$ non–trivially, it follows that the dual $U^*$ is generated by $U_\lambda^*$. Moreover, $U^*[-r]_\lambda = \dim U_\lambda^* = p$ and the proof of the Lemma is now immediate by using duality and the universal property of local Weyl modules. □

4.5. We now define the desired filtration. Set $M = M_0 = \wedge^1 W(1)$ and for $s \geq 1$, let $M_s$ be the maximal $\mathfrak{sl}_2[t]$-submodule of $M_{s-1}$ such that
\[
M_{s}[r_\lambda + s - 1] \cap M_\lambda = 0.
\]
Equivalently, $M_s$, $s \geq 1$ is the maximal submodule of $M_0$ such that
\[
M_{s}[r_\lambda + p] \cap M_\lambda = 0, \quad 0 \leq p \leq s - 1.
\]
In particular, we have
\[
M[r_\lambda + p]_\lambda \subset M_s, \quad p \geq s, \quad (M_s/M_{s+1})_\lambda \cong M[r_\lambda + s]_\lambda. \tag{4.4}
\]

**Lemma.** We have
\[
\bigcap_{s \in \mathbb{Z}_+} M_s = \{0\}.
\]

**Proof.** Let $V$ be a submodule of $\bigcap_{s} M_s$, in which case, we have by the definition of $M_s$ that $V[r_\lambda + s]_\lambda = 0$ for all $s \geq 0$, i.e., $V_\lambda = 0$. Proposition 4.3 (ii) implies that $V = 0$ and the proof is complete. □

4.6. We postpone the proof of the next result to the end of the section.

**Lemma.** (i) For $s \geq 0$, we have an injective morphism
\[
0 \rightarrow M_s/M_{s+1} \xrightarrow{\varphi_s} (\tau_{r_\lambda + s} W_{\text{loc}}(\lambda)^*)^{\oplus \dim M[r_\lambda + s]_\lambda}.
\]
(ii) The map $\varphi_0$ gives an isomorphism
\[
M/M_1 \cong \tau_{r_\lambda} W_{\text{loc}}(\lambda)^*.
\]
4.7. As a consequence of (4.4), Lemma 4.5 and Lemma 4.6 we get
\[ \operatorname{ch}_{\gr} M \leq u'^{r_s} \operatorname{ch}_{\gr} W_{\operatorname{loc}}(\lambda)^* \sum_{p \geq 0} u^p \dim M[p + r_s] = \operatorname{ch}_{\gr} W_{\operatorname{loc}}(\lambda)^* \mathbb{H}(M_\lambda). \]

Now equation (4.1) gives,
\[ \operatorname{ch}_{\gr} M \leq u'^{r_s}(1 : u)_\lambda \operatorname{ch}_{\gr} W_{\operatorname{loc}}(\lambda)^*. \]

To prove that the quotients \( M_s/M_{s+1}, \ s \geq 0 \) are isomorphic to dual local Weyl modules, it suffices to prove that equality holds in the preceding equation. For this, recall that \( M \) is a free \( A_\lambda \)-module and hence
\[ \operatorname{ch}_{\gr}(M) = (1 : u)_\lambda \operatorname{ch}_{\gr}(M \otimes_{A_\lambda} A_\lambda/I_\lambda), \ i.e., \ \operatorname{ch}_{\gr}(M \otimes_{A_\lambda} A_\lambda/I_\lambda) \leq u'^{r_s} \operatorname{ch}_{\gr} W_{\operatorname{loc}}(\lambda)^* \] (4.6)

The following proposition shows that equality holds in (4.5), i.e., that \( \wedge^\lambda W(1) \) has a filtration by dual local Weyl modules and together with (4.6) proves Theorem 3.4(ii).

**Proposition.** We have a surjective morphism \( \mathfrak{sl}_2[t] \)-modules,
\[ (M \otimes_{A_\lambda} A_\lambda/I_\lambda) \to \tau_{r_s} W_{\operatorname{loc}}(\lambda)^*. \]

**Proof.** Since \( M \) is a free \( A_\lambda \)-module, we have that \( M_\mu \) is also a free \( A_\lambda \)-module for all \( \mu \in \mathbb{Z} \), and
\[ \text{rk}_{A_\lambda} M_\mu = \dim_{\mathbb{C}} (M \otimes_{A_\lambda} A_\lambda/I_\lambda)_\mu. \]

Let \( K \) be the kernel of the map \( \pi : M \to (M \otimes_{A_\lambda} A_\lambda/I_\lambda) \to 0 \). By (4.1) \( M_\lambda \) has as basis any non–zero element of the one–dimensional space \( M[r_s]_\lambda \), which means that \( \pi(M[r_s]_\lambda) \neq 0 \), or equivalently,
\[ K \cap M[r_s]_\lambda = \{0\}. \]

The definition of \( M_1 \) implies that \( K \subset M_1 \) and hence we have a non–zero map \( M/K_1 \to M/M_1 \to 0 \). The proposition follows by using Lemma 4.6(ii). \( \square \)

4.8. **Proof of Lemma 4.7(i).** Set \( U = M_s/M_{s+1} \) and notice that \( U_\mu = 0 \) if \( \mu > \lambda \) and \( U_\lambda = U[r_s+s]_\lambda \). Let \( U_1 \) be a non–zero submodule of \( U \). Then its preimage \( \tilde{U} \) in \( M_s \) contains \( M_{s+1} \) properly. It follows from the definition of \( M_{s+1} \) that \( \tilde{U}_1 \cap M[r_s+s]_\lambda \neq 0 \). Equation (4.4) shows that \( \tilde{U}_1 \cap M[r_s+s]_\lambda \) has non–zero image in \( U \), i.e., \( (U_1)_\lambda \neq 0 \). Thus, have now proved that \( U \) satisfies the hypotheses of Lemma 4.4 which establishes the existence of the morphism \( \varphi_s \) with the desired properties

4.9. **Proof of Lemma 4.7(ii).** It was proved in [8] and [10] that the local Weyl module for \( \mathfrak{sl}_2[t] \) is isomorphic to a Demazure module in a level one representation of the corresponding affine Lie algebra. A consequence of this result is that the local Weyl module has a simple socle. It is immediate that the socle is the maximal graded component of the local Weyl module. The character formula in Proposition 3.2(ii) shows that the socle is isomorphic to \( V(\lambda-2[\lambda/2],\lambda/2)[\lambda/2] \). By taking duals, we see that this is equivalent to the statement that \( W_{\operatorname{loc}}(\lambda)^* \) has simple head \( V(\lambda-2[\lambda/2],-\lambda/2)[\lambda/2] \) which is the minimal graded component of \( W_{\operatorname{loc}}(\lambda)^* \). Moreover, it also shows that \( W_{\operatorname{loc}}(\lambda)^* \) is generated by any non–zero element of weight \( \lambda - 2[\lambda/2] \) and of grade \((-\lambda/2)[\lambda/2]) \).
Consider the composite map
\[ M \to M/M_1 \xrightarrow{\varphi_0} \tau_r^\lambda W_{\text{loc}}(\lambda)^*. \]
To prove that \( \varphi_0 \) is surjective, it suffices from the preceding discussion to prove that the image of the element
\[ v = \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) \sigma(e_1 \otimes e_1 t \otimes \cdots \otimes e_1 t^{[\lambda/2]-1} \otimes e_2 \otimes e_2 t \otimes \cdots \otimes e_2 t^{[\lambda/2]-1}), \]
which is of weight weight \( \lambda - 2[\lambda/2] \) and of grade \((-[\lambda/2][\lambda/2])\) is non-zero. For this it suffices to prove that the image of \( w = (x \otimes t^{[\lambda/2]}[\lambda/2])v \) is non-zero. In fact, it is enough to prove that \( w \) is non-zero. For then, since \( w \) has weight \( \lambda \) and grade \( r_\lambda \), the image of \( w \) in \( M/M_1 \) is non-zero by definition of \( M_1 \) and since \( \varphi_0 \) is injective it follows that the image of \( w \) is non-zero. A trivial calculation and using the fact that the action of \( S_\lambda \) commutes with the \( \mathfrak{sl}_2[t] \)-action gives that
\[ w = \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) \sigma(e_1 \otimes e_1 t \otimes \cdots \otimes e_1 t^{[\lambda/2]-1} \otimes e_1 t^{[\lambda/2]} + 1 \otimes \cdots \otimes e_1 t^{[\lambda/2]+[\lambda/2]-1}), \]
which is just equal to \( e_1^{\otimes \lambda} \otimes \text{Vand}(\lambda) \) and the proof of Lemma 4.6(ii) is complete.

5. Proof of Proposition 3.3 and the Global Weyl Filtration of \( \wedge^\lambda W(1) \).

We begin this section by recalling an important homological criterion (established in [2] in the dual case) for an object of \( T_{\text{bdd}} \) to admit a filtration by global Weyl modules. We remark here that the dual objects lie in the category where the objects have only finitely many positively graded pieces and there is no difficulty in going between these categories using the duality functor.

5.1. Given \( M \in \text{Ob} T_{\text{bdd}} \) with \( M = \bigoplus_{\mu \in \mathbb{Z}} M_\mu \), and \( \lambda \in \mathbb{Z}_+ \), let \( M^\lambda \) be the \( \mathfrak{sl}_2[t] \)-submodule generated by the eigenspaces \( M_{\lambda+\nu} \) with \( \nu \in \mathbb{Z}_+ \). Then,
\[ M^0 = M, \quad M^\lambda \supset M^\lambda+1, \]
and \( M^\lambda = 0 \) for \( \lambda \) sufficiently large. This decreasing filtration is called the \( o \)-canonical filtration of \( M \). The dual of the following proposition was established in [2 Section 3].

Proposition. Let \( M \in \text{Ob} T_{\text{bdd}} \). For \( \lambda \in \mathbb{Z}_+ \) with \( M^\lambda \neq M^\lambda+1 \) and \( p \in \mathbb{Z} \), there exists a canonical homomorphism of object of \( T_{\text{bdd}} \),
\[ \bigoplus_{p \in \mathbb{Z}} W(\lambda, p)^{\otimes m_{p, k}} \to M^\lambda/M^\lambda+1 \to 0, \]
for some \( m_{p, \lambda} \in \mathbb{Z}_+ \). Moreover,
\[ m_{p, \lambda} \leq \dim \text{Hom}_{T_{\text{bdd}}}(M^\lambda/M^\lambda+1, V(\lambda, p)) = \dim \text{Hom}_{T_{\text{bdd}}}(M, W_{\text{loc}}(\lambda, -p)^*). \]
(5.1)
and so, we have
\[ \text{ch}_{gr} M = \sum_{\lambda \geq 0} \text{ch}_{gr} M^\lambda/M^\lambda+1 \leq \sum_{\lambda \geq 0} \sum_{p \in \mathbb{Z}} \dim \text{Hom}_{T_{\text{bdd}}}(M, W_{\text{loc}}(\lambda, -p)^*) \text{ch}_{gr} W(\lambda, p). \]
(5.2)
\[ \square \]
5.2. The relation between the \( o \)-canonical filtration of \( M \) and a global Weyl filtration on \( M \) was given in [2, Propositions 2.6 and 3.11] and is summarized as follows.

**Proposition.** Let \( M \in \text{Ob} \mathcal{I}_{\text{bdd}} \). The following are equivalent:

(i) \( M \) has a filtration by global Weyl modules.

(ii) The successive quotients in the \( o \)-canonical filtration are isomorphic to direct sums of global Weyl modules. and

\[
\text{ch}_{\text{gr}} M = \sum_{\lambda \geq 0} \sum_{s \in \mathbb{Z}} \text{dim} \text{Hom}_{\mathcal{I}_{\text{bdd}}} (M, W_{\text{loc}}(\lambda, -s)^*) \text{ch}_{\text{gr}} W(\lambda, s). \tag{5.3}
\]

(iii) For all \((\lambda, s) \in \mathbb{Z}_+ \times \mathbb{Z}\), we have \( \text{Ext}^1_{\mathcal{I}_{\text{bdd}}} (M, W_{\text{loc}}(\lambda, s)^*) = 0 \).

□

We note the following corollary. Even the first statement of the corollary is not trivial since the global Weyl filtrations involved are not finite (see Remark [3.3]) in the sense that \( m_{\lambda, p} \) can be non-zero for infinitely many \( p \).

**Corollary.** Suppose that \( M_j \in \text{Ob} \mathcal{I}_{\text{bdd}}, \ j = 1, 2, 3 \) and assume that we have a short exact sequence

\[
0 \to M_1 \to M_3 \to M_2 \to 0,
\]

of objects of \( \mathcal{I}_{\text{bdd}} \). If \( M_1 \) and \( M_2 \) admit a filtration by global Weyl modules then so does \( M_3 \). Moreover, if the sequence is split, then the converse is also true.

**Proof.** The first statement of the corollary follows by applying \( \text{Ext}^1_{\mathcal{I}_{\text{bdd}}} (\cdot, W_{\text{loc}}(s, r)^*) \) and using part (iii) of the proposition. The second statement is a consequence of part (ii) of the proposition since the graded character and the dimensions of the Hom spaces are additive on finite-direct sums. □

5.3. Using Corollary 5.2, we see that it suffices to establish Proposition 5.3 in the case when \( M = W(\lambda) \). A straightforward induction on \( \lambda \) then proves that \( W(1)^{\otimes (\lambda)} \) has a filtration by global Weyl modules. Since \( \wedge^\lambda W(1) \) is a summand of \( W(1)^{\otimes (\lambda)} \) it follows again by Corollary 5.2 that \( \wedge^\lambda W(1) \) admits a filtration by global Weyl modules which completes the proof of Theorem 3.4. To prove Corollary 3.5, it is enough to prove that \( W(\lambda) \otimes W(\mu) \) admits a filtration by global Weyl modules for \( \lambda, \mu \in \mathbb{Z}_+ \). But this follows now from Proposition 3.2(ii) which shows that \( W(\lambda) \otimes W(\mu) \) is a direct summand of \( W(1)^{\otimes (\lambda + \mu)} \).

The rest of the section is devoted to proving that \( W(\lambda) \otimes W(1) \) has a filtration by global Weyl modules.

5.4. We shall need the following result.

**Lemma.** (i) For \( \lambda, \mu \in \mathbb{Z}_+ \), the module \( W(\lambda) \otimes W(\mu) \) is generated by the subspace \( w_{\lambda} \otimes W(\mu)_{-\mu} \).

(ii) The non-zero quotients occurring in the \( o \)-canonical filtration of \( W(\lambda) \otimes W(\mu) \) are quotients of direct sum of modules \( \tau_s^* W(\lambda + \mu - 2p) \) for \( s, p \in \mathbb{Z}_+ \) and \( 0 \leq p \leq \min \{ \lambda, \mu \} \).
Proof. Since \((x \otimes 1)W(\mu) = 0\), a standard \(\mathfrak{sl}_2\)-argument proves that \((x \otimes 1)^\mu(y \otimes 1)^\mu w\) is a non–zero multiple of \(w\) for all \(w \in W(\mu)_\mu\). Since \(\text{wt} W(\mu) \subset \{\mu, \mu - 2, \cdots, -\mu\}\) we get
\[
W(\mu) = U(x \otimes \mathbb{C}[t])W(\mu)_{-\mu}.
\]
Hence \(U(x \otimes \mathbb{C}[t])(w_\lambda \otimes W(\mu)_{-\mu}) = w_\lambda \otimes W(\mu)\). Applying elements of \(y \otimes \mathbb{C}[t]\) now proves that
\[
W(\lambda) \otimes W(\mu) = U(\mathfrak{sl}_2[t])(w_\lambda \otimes W(\mu)_{-\mu}).
\]

To prove (ii), set \(M = W(\lambda) \otimes W(\mu)\) and let \(M = M^0 \supseteq M^1 \supseteq \cdots M^k \supseteq \{0\}\) be a re-indexing of the \(o\)-canonical filtration, written so that \(M^p \neq M^{p+1}\) for \(0 \leq p \leq k\). By Proposition 5.1 the modules \(M^p/M^{p+1}\) are quotients of the direct sum of modules \(W(\lambda_\mu, r), r \in \mathbb{Z}, \lambda_\mu \in \lambda + \mu - 2\mathbb{Z}_+\) and \(\lambda_\mu > \lambda_{\mu-1}\). Assume that \(\mu \leq \lambda\) without loss of generality. To complete the proof of part (ii) of the Lemma we must show that \(M_{\lambda - \mu} = 0\) which proves that \(\lambda^0 \geq \lambda - \mu\) as required.

\[
\square
\]

5.5. Using Lemma 5.4 (i) and the fact that \(W(\lambda) \otimes W(1)\) has only non–negatively graded components, it is clear that
\[
\dim \text{Hom}_{\mathbb{Z}_{\text{bad}}}(W(\lambda) \otimes W(1), W_{\text{loc}}(\mu, -r)^*) \neq 0 \implies \mu = \lambda \pm 1, \ r \in \mathbb{Z}_{\geq 0}.
\]

Lemma. For \(\lambda \in \mathbb{Z}_+\), we have
\[
\text{ch}_{\text{gr}}(W(\lambda) \otimes W(1)) = [\lambda + 1] \text{ch}_{\text{gr}} W(\lambda + 1) + (1 : u)_1 \text{ch}_{\text{gr}} W(\lambda - 1), \quad (5.4)
\]
\[
\text{Hom}_{\mathbb{Z}_{\text{bad}}}(W(\lambda) \otimes W(1), W_{\text{loc}}(\lambda + 1, -r)^*) = 0, \quad r \notin \{0, 1 \cdots, \lambda\}, \quad (5.5)
\]
\[
\dim \text{Hom}_{\mathbb{Z}_{\text{bad}}}(W(\lambda) \otimes W(1), W_{\text{loc}}(\lambda \pm 1, -r)^*) \leq 1, \quad r \geq 0. \quad (5.6)
\]

Assuming the Lemma for the moment, we show that \(W(\lambda) \otimes W(1)\) admits a filtration by global Weyl modules. Substituting equations (5.5) and (5.6) in Proposition 5.1 we get
\[
\text{ch}_{\text{gr}}(W(\lambda) \otimes W(1)) \leq \sum_{r=0}^\lambda \text{ch}_{\text{gr}} W(\lambda + 1, r) + \sum_{r \geq 0} \text{ch}_{\text{gr}} W(\lambda - 1, r)
\]
i.e.,
\[
\text{ch}_{\text{gr}}(W(\lambda) \otimes W(1)) \leq \left( \sum_{r=0}^\lambda u^r \right) \text{ch}_{\text{gr}} W(\lambda + 1) + \left( \sum_{r \geq 0} u^r \right) \text{ch}_{\text{gr}} W(\lambda - 1).
\]

Equation 5.3 implies that we must have an equality. Proposition 5.2 now gives that that the \(o\)-canonical filtration is a filtration by global Weyl modules, more precisely, we have the following short exact sequence of objects of \(I_{\text{bad}}\):
\[
0 \rightarrow \bigoplus_{r=0}^\lambda W(\lambda + 1, r) \rightarrow W(\lambda) \otimes W(1) \rightarrow \bigoplus_{r \geq 0} W(\lambda - 1, r) \rightarrow 0,
\]
i.e., \(W(\lambda) \otimes W(1)\) admits a filtration by global Weyl modules.
5.6. We turn to the proof of Lemma 5.5. The proof of equation \((5.4)\) is a simple calculation using the explicit formulæ in Proposition 3.2(ii) for the graded character of the local Weyl module. To prove the remaining statements, we need to do some additional work. Define elements \(P_n \in \mathbf{U}(h \otimes \mathbb{C}[t])\), \(n \in \mathbb{Z}_+\) recursively by:

\[
P_0 = 1, \quad P_n = -\frac{1}{n} \sum_{s=0}^{n-1} (h \otimes t^{s+1})P_{n-s-1}.
\]

It is easily seen that monomials in the \(P_n\), \(n \in \mathbb{Z}_+\) are a basis for \(\mathbf{U}(h \otimes \mathbb{C}[t])\). Recall that the assignment \(\Delta(x) = x \otimes 1 + 1 \otimes x\), \(x \in \mathfrak{sl}_2[t]\) defines the comultiplication on \(\mathbf{U}(\mathfrak{sl}_2[t])\). A simple induction shows that

\[
\Delta(P_n) = \sum_{s=0}^{n} P_s \otimes P_{n-s} \tag{5.7}
\]

The following result is a consequence of [15, Lemma 7.16] (see [10] for this formulation).

**Lemma.** Suppose that \(V\) is a \(\mathfrak{sl}_2[t]\)-module and let \(v \in V\). Then,

\[
(x \otimes \mathbb{C}[t])v = 0 \implies (x \otimes t)^s(y \otimes 1)^sv = P_s v, \quad s \in \mathbb{Z}_+.
\]

\(\square\)

5.7. As a consequence of Lemma 5.6, we see that

\[
P_s w_{\lambda} = 0, \quad s \geq \lambda + 1, \tag{5.8}
\]

and hence \(W(\lambda)_\lambda\) is spanned by monomials in \(P_s, 0 \leq s \leq \lambda\). In particular, since \(P_1 = -(h \otimes t)\), we get

\[
W(1)[r] = \mathbb{C}(h \otimes t)^r w_1, \quad r \geq 0.
\]

**Lemma.** For \(\lambda \in \mathbb{Z}_+,\) the elements of the set \(\{w_{\lambda} \otimes (h \otimes t^s)w_1 : 0 \leq s \leq \lambda\}\) generate \(W(\lambda)_\lambda \otimes W(1)_1\) as a \((h \otimes \mathbb{C}[t])\)-module and hence also generate \(W(\lambda) \otimes W(1)\) as a module for \((\mathbb{C}g \oplus \mathbb{C}h) \otimes \mathbb{C}[t]\).

**Proof.** It clearly suffices to prove that for all \(s \in \mathbb{N}\), the element \(w_{\lambda} \otimes (h \otimes t^s)w_1\) is in the \((h \otimes \mathbb{C}[t])\)-submodule generated by the elements \(w_{\lambda} \otimes (h \otimes t^s)w_1, 0 \leq s \leq \lambda\). Recall that \(P_1 = -(h \otimes t)\). Using \((5.7)\) and \((5.8)\), we get

\[
P_{\lambda+1}(w_{\lambda} \otimes w_1) = P_{\lambda} w_{\lambda} \otimes P_1 w_1,
\]

\[
P_{\lambda}(w_{\lambda} \otimes P_1 w_1) = P_{\lambda} w_{\lambda} \otimes P_1 w_1 + P_{\lambda-1} w_{\lambda} \otimes P_1^2 w_1.
\]

This proves that \(P_{\lambda-1} w_{\lambda} \otimes P_1^2 w_1\) is in the \(\mathbf{U}((h \otimes \mathbb{C}[t]))\)-module generated by \(w_{\lambda} \otimes P_1^r w_1, r = 0, 1\). Now using,

\[
P_{\lambda-1}(w_{\lambda} \otimes P_1^2 w_1) = P_{\lambda-1} w_{\lambda} \otimes P_1^2 w_1 + P_{\lambda-2} \otimes P_1^3 w_1,
\]

we get \(P_{\lambda-2} \otimes P_1^3 w_1\) is in the \(\mathbf{U}((h \otimes \mathbb{C}[t]))\)-module generated by \(w_{\lambda} \otimes P_1^r w_1, r = 0, 1, 2\). Repeating this argument proves that \(w_{\lambda} \otimes P_1^{\lambda+1} w_1\) is in the \(\mathbf{U}((h \otimes \mathbb{C}[t]))\)-module generated by \(w_{\lambda} \otimes P_1^r w_1, r = 0, 1, \cdots, \lambda\). Repeating these steps with \(w_{\lambda} \otimes P_1^s w_{\lambda}\) for \(s \in \mathbb{N}\) now proves the Lemma.

\(\square\)
5.8. To prove equation (5.5) let \( \varphi : W(\lambda) \otimes W(1) \to W_{\text{loc}}(\lambda + 1, -r)^* \) be any non-zero map of objects of \( \mathcal{I}_{\text{bdd}} \). Since

\[
\text{soc} \ W_{\text{loc}}(\lambda + 1, -r)^* = V(\lambda + 1, r), \quad \text{dim} \ W_{\text{loc}}(\lambda + 1, -r)_{\lambda+1}^* = 1,
\]

we must have

\[
\varphi((W(\lambda)_\lambda \otimes W(1)_1)[r]) \neq 0, \quad \varphi((W(\lambda)_\lambda \otimes W(1)_1)[s]) = 0, \quad s \neq r.
\]

On the other hand Lemma 5.7 implies that \( \varphi(w_\lambda \otimes (h \otimes t)^p w_1) \neq 0 \) for some \( 0 \leq p \leq \lambda \). Hence we must have \( p = r \) and the proof is complete.

5.9. To prove, (5.6) for \( (\lambda + 1) \) observe that by equality in (5.1) it is enough to prove that

\[
\text{dim} \ \text{Hom}_{\mathcal{I}_{\text{bdd}}} (U(g[t])(W(\lambda) \otimes W(1)_1), V(\lambda + 1, s)) \leq 1, \quad s \geq 0.
\]

If \( \varphi \) is any non-zero element of this space, then it must be non-zero on \( W(\lambda)_\lambda \otimes W(1)_1 \). Lemma 5.7 implies that the map \( \varphi \) is determined by its values on the elements \( w_\lambda \otimes (h \otimes t)^p w_1 \) for \( 0 \leq p \leq \lambda \). If \( p \neq s \) the image of \( w_\lambda \otimes (h \otimes t)^p w_1 \) is zero since \( V(\lambda + 1, s) \) is concentrated in grade \( s \). Hence \( \varphi \) is determined by its value on \( w_\lambda \otimes (h \otimes t)^s w_1 \). Since \( \text{dim} \ V(\lambda + 1, s)_{\lambda+1} = 1 \) the result follows.

It remains to prove (5.6) for \( \lambda - 1 \). Let \( \varphi : W(\lambda) \otimes W(1) \to W_{\text{loc}}(\lambda - 1, -r)^* \) be any non-zero morphism of \( \mathfrak{sl}_2[t] \)-modules. By Lemma 5.4 \( \varphi \) is determined by its values on \( w_\lambda \otimes W(1)_{-1} \). Since \( xW(1)_1 = 0 \), and \( W(1) \) is isomorphic to the direct sum of finite-dimensional \( \mathfrak{sl}_2 \)-modules, it follows that \( y : W(1)_1 \to W(1)_{-1} \) is an isomorphism of graded spaces. It follows that

\[
\text{dim} \ W(1)_s \geq 1 = \text{dim} \ C(y(h \otimes t)^s w_\lambda) \leq 1.
\]

Since \( \text{dim} \ W_{\text{loc}}(\lambda - 1, r)[s]_{\lambda-1} \leq 1 \) with equality holding iff \( s = r \), we see that

\[
\varphi(w_\lambda \otimes W(1)_{-1}) = \varphi(w_\lambda \otimes y(h \otimes t') w_1) \subset W_{\text{loc}}(\lambda - 1, r)[r]_{\lambda-1},
\]

which completes the proof of (5.6) and so also the proof of Lemma 5.5.

5.10. We conclude the paper by determining explicitly the filtration multiplicities in the tensor product of global Weyl modules.

**Proposition.** For \( \lambda, \mu, \nu \in \mathbb{Z}_+ \), the multiplicity of \( W(\lambda + \mu - \nu) \) in a global Weyl filtration of \( W(\lambda) \otimes W(\mu) \) is we have

\[
\text{ch}_{\text{gr}} (W(\lambda) \otimes W(\mu)) = \sum_{\nu=0}^{\min(\lambda, \mu)} \begin{bmatrix} \lambda + \mu - 2\nu \\ \mu - \nu \end{bmatrix} (1 : u)_\nu, \quad \text{or equivalently,}
\]

\[
\text{ch}_{\text{gr}} (W(\lambda) \otimes W(\mu)) = \sum_{\nu=0}^{\min(\lambda, \mu)} \begin{bmatrix} \lambda + \mu - 2\nu \\ \mu - \nu \end{bmatrix} (1 : u)_{\nu} \text{ch}_{\text{gr}} W(\lambda + \mu - 2\nu). \quad (5.9)
\]

**Proof.** Notice first that since we know that \( W(\lambda) \otimes W(\mu) \) has a filtration by global Weyl modules, the two statements in the proposition are indeed equivalent. Assume without loss of generality that \( \lambda \geq \mu \) and note that the case when \( \mu = 1 \) was proved in Lemma 5.5. We complete the proof by induction on \( \mu \) and have only to establish the inductive step. Assuming
the result for $\mu$ we prove the result for $\mu + 1$ by using the induction hypothesis to compute the graded character of $W(\lambda) \otimes W(\mu) \otimes W(1)$ in two ways. Thus, we have
\[
ch_{gr} ((W(\lambda) \otimes W(\mu) \otimes W(1)) = \left( \sum_{\nu=0}^{\mu} \left[ \frac{\lambda + \mu - 2\nu}{\mu - \nu} \right] (1 : u)_\nu ch_{gr} W(\lambda + \mu - 2\nu) \right) ch_{gr} W(1)
\]
\[
= \left( \sum_{\nu=0}^{\mu} \left[ \frac{\lambda + \mu - 2\nu}{\mu - \nu} \right] (1 : u)_\nu \left[ \lambda + \mu + 1 - 2\nu \right] ch_{gr} W(\lambda + \mu + 1 - 2\nu) \right)
\]
\[
+ \left( \sum_{\nu=0}^{\mu} \left[ \frac{\lambda + \mu - 2\nu}{\mu - \nu} \right] (1 : u)_\nu (1 : u)_1 ch_{gr} W(\lambda + \mu - 1 - 2\nu) \right),
\]
and also,
\[
ch_{gr}(W(\lambda) \otimes (W(\mu) \otimes W(1)) = \left[ \mu \right] ch_{gr} W(\lambda)(\left[ \mu \right] ch_{gr} W(\mu + 1) + (1 : u)_1 ch_{gr}(W(\mu - 1))),
\]
\[
= \left[ \mu \right] ch_{gr}(W(\lambda) \otimes W(\mu)) + (1 : u)_1 \sum_{\nu=0}^{\mu-1} \left[ \frac{\lambda + \mu - 1 - 2\nu}{m - 1 - \nu} \right] (1 : u)_\nu W(\lambda + \mu - 1 - 2\nu).
\]
Equating the two solutions and a simple if tedious calculation gives the result.

\[\square\]

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