Uniqueness for fractional nonsymmetric diffusion equations and an application to an inverse source problem

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In this article, we discuss a solution to time-fractional diffusion equation $\partial_t^\alpha (u - u_0) + Au = 0$ with the homogeneous Dirichlet boundary condition, where an elliptic operator $-A$ is not necessarily symmetric. We prove that the solution $u$ is identically zero if its normal derivative with respect to the operator $A$ vanishes on an arbitrarily chosen subboundary of the spatial domain over a time interval. The proof is based on the Laplace transform and the spectral decomposition for a nonsymmetric elliptic operator. As a direct application, we prove the uniqueness result for an inverse problem on determining the spatial component in the source term by Neumann boundary data on subboundary.

KEYWORDS
fractional partial differential equations, inverse problems, unique continuation

1 | INTRODUCTION AND MAIN RESULTS

Throughout this paper, we assume that $T > 0$, $0 < \alpha < 1$, and $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary $\partial \Omega$, and let $\nu = (\nu_1, \ldots, \nu_d)$ denote the unit outwards normal vector to the boundary $\partial \Omega$. Let the operator $A$ be defined by

$$-A\varphi(x) := \sum_{j,k=1}^d \partial_j (a_{jk}(x) \partial_k \varphi(x)) + \sum_{j=1}^d b_j(x) \partial_j \varphi(x) + c(x) \varphi(x), \ x \in \Omega$$

for $\varphi \in D(A) := H_0^1(\Omega) \cap H^2(\Omega)$, where we assume that $a_{jk} = a_{kj}, b_j, c \in C^1(\overline{\Omega}), 1 \leq j, k \leq d$ and

$$a_\alpha \sum_{j=1}^d \xi_j^2 \leq \sum_{j,k=1}^d a_{jk}(x) \xi_j \xi_k, \ x \in \overline{\Omega}, \xi_1, \ldots, \xi_d \in \mathbb{R}, \quad (1.1)$$
where $a_0 > 0$ is a constant independent of $x, \xi$. Here, we set the normal derivative with respect to the operator $A$ as

$$\partial_w u = \sum_{j,k=1}^d a_{jk} v_j \partial_k u$$

for any $u \in H^2(\Omega)$.

A classical definition of a fractional derivative is given by

$$\frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{dv}{ds}(s) ds,$$

where $\Gamma$ is the gamma function. However, this definition is not convenient for a fundamental formulation of initial boundary value problem with initial value and source term whose spatial regularity is in $L^2(\Omega)$. Indeed, according to the above classical definition, we have to define $\frac{dv}{ds}(x,s)$ in $L^1(0,T)$ or in some space of distributions in $(0,T)$, but for such initial value and source with the $L^2(\Omega)$-regularity, the justification is not trivial. The main reason for the difficulty is that we cannot expect sufficient differentiability of the solution, but the above pointwise definition of the fractional derivative of order $a$ requires the derivative of order $1 > a$.

Thus, here we will adopt the formulation in Gorenflo et al.\(^1\) and Kubica et al.\(^2\) and we define the time-fractional derivative $\partial_t^\alpha$ in Sobolev spaces, which are defined as follows. First, let $H^\alpha(0, T)$ be the fractional Sobolev space with the norm

$$\|v\|_{H^\alpha(0, T)} = \left( \|v\|^2_{L^2(0, T)} + \int_0^T \int_0^T \frac{|v(t) - v(s)|^2}{|t - s|^{1 + 2\alpha}} ds dt \right)^{1/2}$$

(e.g., Adams\(^3\)). We set

$$H_\alpha(0, T) = \begin{cases} 
\{ v \in H^\alpha(0, T); v(0) = 0 \}, & 0 < \alpha < 1, \\
\{ v \in H^{1/2}(0, T); \int_0^T \frac{|v(t)|^2}{t} dt < \infty \}, & \alpha = 1/2, \\
H^\alpha(0, T), & 0 < \alpha < 1/2.
\end{cases}$$

and

$$\|v\|_{H_\alpha(0, T)} = \begin{cases} 
\|v\|_{H^\alpha(0, T)}, & 0 < \alpha < 1, \alpha \neq 1/2, \\
\left( \|v\|^2_{H^\alpha(0, T)} + \int_0^T \frac{|v(t)|^2}{t} dt \right)^{1/2}, & \alpha = 1/2.
\end{cases}$$

We set

$$J^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v(s) ds.$$

Then it is known

$$J^\alpha L^2(0, T) = H_\alpha(0, T), \quad 0 < \alpha \leq 1$$

and there exists a constant $C > 0$ such that

$$C^{-1} \|J^\alpha v\|_{H_\alpha(0, T)} \leq \|v\|_{L^2(0, T)} \leq C \|J^\alpha v\|_{H_\alpha(0, T)}$$

for all $v \in L^2(0, T)$ (e.g., Gorenflo et al.\(^1\)).

We define the time-fractional derivative $\partial_t^\alpha$ in $H_\alpha(0, T)$ by

$$\partial_t^\alpha v := (J^\alpha)^{-1} v, \quad v \in H_\alpha(0, T).$$
Remark 1. We recall the Caputo derivative

$$d_t^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv}{ds}(s)ds$$

for $v \in C^1[0, T]$. Regarding $d_t^\alpha$ as an operator with the domain $\_0C^1[0, T] := \{v \in C^1[0, T]; v(0) = 0\}$, we can see that the minimum closed extension of $d_t^\alpha$ coincides with $\partial_t^\alpha$ (Kubica et al.). It is seen that

$$\partial_t^\alpha v(t) = d_t^\alpha v(t) = \frac{1}{(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} v(s)ds, \quad 0 < t < T$$

for $v \in W^{1,1}(0, T)$ satisfying $v(0) = 0$ in the trace sense. Here, the right-hand side is called the Riemann–Liouville fractional derivative.

We consider

$$\left\{ \begin{array}{l}
\partial_t^\alpha (u - u_0) + Au = 0 \quad \text{in } \Omega \times (0, T), \\
u(x, \cdot) - u_0(x) \in H_d(0, T), \quad \text{for almost all } x \in \Omega, \\
u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T).
\end{array} \right. \quad (1.2)$$

We assume that $u_0 \in H^1_d(\Omega)$. Then for arbitrarily given $T > 0$, there exists a unique solution

$$u \in L^2(0, T; H^1_d(\Omega) \cap H^2(\Omega)) \quad (1.3)$$

such that $u - u_0 \in H_d(0, T; L^2(\Omega))$ and $u(\cdot, t) \in H^1_d(\Omega)$ for almost all $t \in (0, T)$. As for the proof, we can refer to Gorenflo et al., Kubica et al., Kubica and Yamamoto, and Zacher. By (1.3), we see that $\partial_{\alpha} u \in L^2(\partial\Omega \times (0, T))$.

The first equation in (1.2) is a time-fractional diffusion equation with the advection term $\sum_{j=1}^d b_j \partial_j u$, where $(b_1, \ldots, b_d)$ is a velocity field describing for example an underground water flow in the case of diffusion in soil. In other words, such an equation with advection term is a more feasible model for considering diffusion phenomena in such heterogeneous media.

We are ready to state the first main result.

Theorem 1.1. Let $\Gamma \neq \emptyset$ be an arbitrarily chosen subboundary of $\partial\Omega$ and let $T > 0$. For $u_0 \in H^1_d(\Omega)$, let $u$ be the solution to (1.2). If $\partial_{\alpha} u = 0$ on $\Gamma \times (0, T)$, then $u = 0$ in $\Omega \times (0, T)$ and $u_0 = 0$ in $\Omega$.

This uniqueness result is known to be equivalent to the approximate controllability for the adjoint system to (1.1) (e.g., Fujishiro and Yamamoto). For evolution equations with natural number order time-derivative, see, for example, Schmidt and Weck and Triggiani. Here, we do not discuss about the approximate controllability.

To the best knowledge of the authors, Theorem 1.1 is the first uniqueness for a general nonsymmetric elliptic operator $A$. In the case where $-A$ is symmetric, that is, $b_j = 0$ for $j = 1, \ldots, d$, the uniqueness has been already known. For example, we refer to Sakamoto and Yamamoto. For not necessarily symmetric $A$, see also Jiang et al.

The works of Jiang et al. and Sakamoto and Yamamoto adopt additional data $u|_{\omega \times (0, T)}$ where $\omega \subset \Omega$ is an arbitrarily chosen subdomain, in place of $\partial_{\alpha} u|_{\Gamma \times (0, T)}$. The work of Sakamoto and Yamamoto relies directly on the eigenfunction expansion thanks to the symmetry of $A$, and the argument in Jiang et al. is based on the transformation of the problem to the determination of $u_0$ of the corresponding parabolic equation through the Laplace transform and the well-known unique continuation for a parabolic equation (e.g., Isakov and Yamamoto). The argument in Jiang et al. works in the case of nonsymmetric $A$ but assumes that the zeroth-order coefficient $c(x)$ of $A$ should be nonpositive in $\Omega$, and the removal of this assumption is not trivial, while our method does not require it. Moreover in Jiang et al., one needs more arguments in estimating $\|u(\cdot, t)\|_{H^2(\Omega)}$ for $t > 0$ in order to justify the Laplace transform of $Au(x, t)$ in $t$. However, in our article, by uniform treatments based on $\partial_t^\alpha$, we do not need such an extra estimate in $H^2(\Omega)$ (Lemmata 2.3 and 2.4).

As one application of Theorem 1.1, we show the uniqueness for an inverse source problem. We consider

$$\left\{ \begin{array}{l}
\partial_t^\alpha y + Ay = \mu(t) f(x) \quad \text{in } \Omega \times (0, T), \\
y(x, \cdot) \in H_d(0, T) \quad \text{for almost all } x \in \Omega, \\
y(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T).
\end{array} \right. \quad (1.4)$$
We assume that \( f \in H^1_0(\Omega) \) and \( \mu \in C^1[0, T] \). Then we know (e.g., Kubica et al.\(^2\)) that there exists a unique solution \( y \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega) \cap H^1_0(0, T; L^2(\Omega))) \) to (1.4). Now for given \( \mu \), we discuss an inverse source problem of determining \( f \) in \( \Omega \) by \( \partial_\nu y|_{\Gamma \times (0, T)} \). This inverse problem is concerned with the determination of density of a source causing the diffusion provided that we know temporal change \( \mu(t) \). We emphasize that we take into consideration some advection that makes our fractional diffusion equation a feasible model and consequently the inverse source problem becomes more realistically formulated.

We can state the main result for the inverse problem.

**Theorem 1.2.** Let \( \Gamma \subset \partial \Omega \) be an arbitrarily chosen nonempty set and let \( f \in H^1_0(\Omega), \mu \in C^1[0, T], \neq 0 \) in \([0, T]\). If \( \partial_\nu y = 0 \) on \( \Gamma \times (0, T) \), then \( f = 0 \) in \( \Omega \).

For inverse problems, the first theoretical topic is the uniqueness in determining an unknown function \( f(x), x \in \Omega \) by the adopted observation data \( \partial_\nu y|_{\Gamma \times (0, T)} \). The second topic is the stability that asserts that small errors in data cause only small deviations in solutions with adopted topologies. For \( \alpha = 1 \) or \( \alpha = 2 \), the inverse source problems are classical and there have been many results even though limited to theoretical works. As early works we can refer to Yamamoto\(^{13,14}\) for \( \alpha = 1 \) and \( \alpha = 2 \), and as for generally applicable methodologies by Carleman estimates, one can refer to Yamamoto\(^{12}\) and Bellassoued and Yamamoto.\(^{15}\) Here, we are limited to a few works and one can consult the references therein. These works proved the uniqueness and the following stability results:

- Case \( \alpha = 1 \): Conditional stability with a priori boundedness assumption on \( f \) which is of logarithmic rate.\(^{13}\)
- Case \( \alpha = 2 \): Unconditional Lipschitz stability.\(^{14}\)

For \( 0 < \alpha < 1 \), Theorem 1.2 is the first uniqueness result for a general nonsymmetric elliptic operator \( A \) without assumptions on the sign of \( c(x) \).

After solving the primary theoretical issue, we should next proceed to the stability. However, to the best knowledge of the authors, the stability is an open problem for \( 0 < \alpha < 1 \), although one can try to apply methods similar to the cases of \( \alpha = 1, 2 \).

Our total discussions are carried out on the basis of the formulation of \( \partial^\alpha_t \) in the space \( H^\alpha(0, T) \), and through each step of the proof, the arguments within \( H^\alpha(0, T) \) are consistent with the Laplace transform of \( \partial^\alpha_t \) and the regularity of the solutions \( u \) and \( y \) to (1.2) and (1.4), respectively.

We can relax the regularity conditions \( f \in H^1_0(\Omega) \) and \( \mu \in C^1[0, T] \). However, for it, we need a lot of technicalities, and we omit details.

This article is outlined as follows. In Section 2, we first provide several preliminary results from spectral theory and prove some auxiliary lemmas for the formula of the Laplace transform for the fractional derivative \( \partial^\alpha_t \) in \( H^\alpha(0, T) \), which plays crucial roles in the proof of the main theorem. In Section 3, by the Laplace transform argument, Theorem 1.1 is proved. In Section 4, based on Theorem 1.1, from the Duhamel principle (see Lemma 4.1), we finish the proof of Theorem 1.2. Finally, a concluding remark is given in Section 5.

### 2 | PRELIMINARIES

#### 2.1 | Properties of solution to an initial boundary value problem

In this subsection, for a solution to the initial boundary value problem (1.2), we establish exponential growth as \( t \to \infty \) and the analyticity in time \( t \), which allows us to apply the Laplace transforms.

First, we show the following:

**Lemma 2.1 (Coercivity).** Let \( v - c_0 \in H^\alpha(0, T) \) with \( \alpha \in (0, 1) \) and \( c_0 \in \mathbb{R} \). Then

\[
\frac{2}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} v(s) \partial^\alpha_t (v(s) - c_0) ds \geq v^2(t) - c_0^2 \quad \text{for almost all } t \in (0, T).
\]

For \( v \in H^\alpha(0, T) \), that is, \( c_0 = 0 \), we can refer to Theorem 3.2 in Kubica et al.\(^2\)
Proof. First, we assume $v - c_0 \in C^1(0, T)$. Then from Lemma 1 in Alikhanov,\textsuperscript{16} it follows that

$$v(t)d_t^\alpha v(t) \geq \frac{1}{2}d_t^\alpha (v^2)(t), \quad t > 0. \quad (2.1)$$

Here, we recall that $d_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv(s)}{ds} ds$ for $v \in C^1[0, T]$. Now, noting that $v(0) = c_0$, along with the formula

$$J^\alpha d_t^\alpha v(t) = v(t) - c_0, \quad t \in (0, T),$$

which can be verified directly by the definitions of $d_t^\alpha$ and $J^\alpha$, we have

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)d_t^\alpha v(s) ds \geq v^2(t) - c_0^2, \quad t \in (0, T).$$

Since $d_t^\alpha c_0 = 0$, we further arrive at the inequality

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)d_t^\alpha (v(s) - c_0) ds \geq v^2(t) - c_0^2 \quad \text{for almost all} \quad t \in (0, T).$$

Moreover, from Remark 1, it follows that the Caputo derivative $d_t^\alpha$ coincides with the fractional derivative $\partial_t^\alpha$ under the domain $0C^1[0, T]$, and then we see that

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\partial_t^\alpha v(s) ds \geq v^2(t) - c_0^2 \quad \text{for almost all} \quad t \in (0, T).$$

Setting $w := v - c_0$, we can rewrite as

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (w(s) + c_0)\partial_t^\alpha w(s) ds \geq (w(t) + c_0)^2 - c_0^2 \quad \text{for almost all} \quad t \in (0, T) \quad (2.2)$$

if $w \in 0C^1[0, T]$.

Next, let $w \in H_0(0, T)$ be arbitrarily given. Since $0C^1[0, T] = H_0(0, T)$,\textsuperscript{1} we can choose $w_n \in 0C^1[0, T]$ such that $\lim_{n \to \infty} w_n = w$ in $H_0(0, T)$. Then by (2.3), we see that

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (w_n(s) + c_0)\partial_t^\alpha w_n(s) ds \geq (w_n(t) + c_0)^2 - c_0^2 \quad \text{for almost all} \quad t \in (0, T) \quad (2.3)$$

for all $n \in \mathbb{N}$. Since $w_n \to w$ in $H_0^\prime(0, T)$ as $n \to \infty$, there exists a subsequence $\{n'\} \subset \mathbb{N}$ such that $w_{n'}(t) \to w(t)$ for almost all $t \in (0, T)$. Therefore, the right-hand side of (2.3) tends to $(w(t) + c_0)^2 - c_0^2$ for almost all $t \in (0, T)$ by choosing the subsequence $\{n'\}$ and letting $n' \to \infty$.

On the other hand, since $\lim_{n \to \infty} w_n = w$ and $\lim_{n \to \infty} \partial_t^\alpha w_n = \partial_t^\alpha w$ in $L^2(0, T)$, we obtain

$$(w_n + c_0)\partial_t^\alpha w_n \to (w + c_0)\partial_t^\alpha w \quad \text{in} \quad L^1(0, T)$$

by the Hölder inequality. Therefore, the Young inequality (e.g., Lemma A.1 in Kubica et al.\textsuperscript{2}) yields that

$$\int_0^t (t-s)^{\alpha-1} (w_{n'}(s) + c_0)\partial_t^\alpha w_{n'}(s) ds = (s^{\alpha-1} * (w_{n'} + c_0)\partial_t^\alpha w_{n'})(t) \to (s^{\alpha-1} * (w + c_0)\partial_t^\alpha w)(t) \quad \text{in} \quad L^1(0, T)$$

as $n' \to \infty$. Hence, again choosing a subsequence $\{n''\}$ of $\{n'\}$, we see that

$$\frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (w_{n''}(s) + c_0)\partial_t^\alpha w_{n''}(s) ds \to \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (w(s) + c_0)\partial_t^\alpha w(s) ds$$
When we want to specify that a constant depends on \( u \) we write \( C_u \).

We finish the proof of the lemma by changing \( w(t) + c_0 \) back to \( v(t) \).

Henceforth for simplifying descriptions, by \( C > 0 \), we denote generic constants which are independent of \( u(x, t), u_0(x), f(x) \) and choices of \( t \in [0, T] \), but dependent on other quantities \( \alpha, T, \mu(t) \), and the coefficients of \( A \) and \( \Omega \). When we want to specify that a constant depends on \( u \), we write \( C_u \).

Next we will prove

**Lemma 2.2.** Let \( u_0 \in H^1_0(\Omega) \). Then the unique solution \( u : (0, T) \to H^2(\Omega) \) to (1.2), is \( t \)-analytic and can be analytically extended to \((0, \infty)\). Moreover, there exists a constant \( C > 0 \) such that

\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq Ce^{Ct}\|u_0\|_{L^2(\Omega)}, \quad t > 0.
\]

**Proof.** For the proof of the \( t \)-analyticity of the solution, one can refer to Li et al.\(^{17}\) It is sufficient to show the solution \( u \) admits an exponential growth. For this, we multiply both sides of Equation (1.2) by \( u \) and integrate over \( \Omega \) to derive that

\[
(\partial_t^2(u(\cdot, t) - u_0), u(\cdot, t))_{L^2(\Omega)} + (Au(\cdot, t), u(\cdot, t))_{L^2(\Omega)} = 0.
\]

Here and henceforth, we set \((w, v)_{L^2(\Omega)} := \int_\Omega w(x)v(x)dx\).

Now applying \( J^\alpha \) on both sides of the above equation, by Lemma 2.1 and the integration by parts in \( x \), we see that

\[
\frac{1}{2}\|u(\cdot, t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + J^\alpha \left( \sum_{j,k=1}^d a_{jk}(x)\partial_j u(x, t)\partial_k u(x, t)dx \right)
\leq J^\alpha \left( \int_\Omega \sum_{j=1}^d b_j(x)\partial_j u(x, t) + c(x)u(x, t) u(x, t)dx \right).
\]

From the ellipticity (1.1) and the Cauchy–Schwarz inequality, for a sufficiently small \( \epsilon > 0 \), we can further derive

\[
\frac{1}{2}\|u(\cdot, t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + a_0 J^\alpha(\|\nabla u(\cdot, t)\|^2_{L^2(\Omega)}) \leq J^\alpha \left( \epsilon \|u(\cdot, t)\|^2_{H^1(\Omega)} + \frac{C}{\epsilon}\|u(\cdot, t)\|^2_{L^2(\Omega)} \right).
\]

Noting the Poincaré inequality \( \|u(\cdot, t)\|_{L^2(\Omega)} \leq C\|\nabla u(\cdot, t)\|_{L^2(\Omega)} \), and taking \( \epsilon > 0 \) small enough, we have

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} + J^\alpha(\|u(\cdot, t)\|^2_{H^1(\Omega)}) \leq C\|u_0\|^2_{L^2(\Omega)} + CJ^\alpha\|u(\cdot, t)\|^2_{L^2(\Omega)},
\]

which implies

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq C\|u_0\|^2_{L^2(\Omega)} + \frac{C}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\|u(\cdot, s)\|^2_{L^2(\Omega)}ds.
\]

Therefore, we conclude from the generalized Gronwall inequality (e.g., Lemma A.2 in Kubica et al.\(^{2}\)) that

\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq Ce^{Ct}\|u_0\|_{L^2(\Omega)}, \quad t \geq 0.
\]

Thus, the proof of the lemma is complete. \( \square \)
2.2 Laplace transform of $\partial_t^\alpha$

We define the Laplace transform $L(v)(p)$ by

$$L(v)(p) := \int_0^\infty e^{-pt}v(t)dt$$

for $\text{Re} \ p > p_0$; some constant.

The formula of the Laplace transforms for fractional derivatives is well known (e.g., Podlubny\textsuperscript{18}):

$$L(d_t^\alpha v)(p) = p^\alpha L(v)(p) - p^{\alpha-1}v(0) \quad \text{(2.4)}$$

for $\text{Re} \ p > p_0$, which is some constant. The formula (2.4) is convenient for explicitly solving fractional differential equations. However, formula (2.4) requires some regularity for $v$ such as the continuity at $t = 0$. However, in view of the well-posedness of the initial boundary value problem, we are working with the time regularity $H_\alpha(0, T)$ and for $0 < \alpha < \frac{1}{2}$, the trace $v(0)$ does not make a sense. Our proof is based on the Laplace transform, so that we should establish a formula corresponding to (2.4) within $H_\alpha(0, T)$.

Thus, in this subsection, we state the formula of the Laplace transform for the fractional derivative $\partial_t^\alpha$ in $H_\alpha(0, T)$. We set

$$V_\alpha(0, \infty) := \{v \in L^1_{\text{loc}}(0, \infty); v|_{(0, T)} \in H_\alpha(0, T) \text{ for any } T > 0, \text{ there exists a constant } C_v > 0 \text{ such that } |v(t)| \leq C_v e^{C_v t} \text{ for } t \geq 0\}.$$  \hfill (2.5)

Here, the constant $C_v > 0$ depends on $v$, and we set

$$L^1_{\text{loc}}(0, \infty) := \{v; v|_{(0, T)} \in L^1(0, T) \text{ for any } T > 0\}.$$  \hfill (2.6)

Then, we can state the following:

**Lemma 2.3.** For $v \in V_\alpha(0, \infty)$, the Laplace transform $L(\partial_t^\alpha v)(p)$ defined by

$$L(\partial_t^\alpha v)(p) = \lim_{T \to \infty} \int_0^T e^{-pt} \partial_t^\alpha u(t)dt, \ p > C_v$$

exists, and

$$L(\partial_t^\alpha v)(p) = p^\alpha L(v)(p), \ p > C_v.$$  \hfill (2.7)

**Proof.** We can refer to Kubica et al\textsuperscript{2} for the proof, but for completeness, we here provide the proof. First for $v \in H_\alpha(0, T)$, by Theorem 2.3 in Kubica et al\textsuperscript{2} we can see that

$$J^{1-\alpha} v \in H_1(0, T) = \{w \in H^1(0, T); w(0) = 0\}.$$  \hfill (2.6)

Theorem 2.4 from Kubica et al\textsuperscript{1} yields $\partial_t^\alpha v = \frac{d}{dt} J^{1-\alpha} v$ for $v \in H_\alpha(0, T)$. Let $T > 0$ be arbitrarily fixed. Then, in terms of (2.6), we integrate by parts to obtain

$$\int_0^T e^{-pt} \partial_t^\alpha v(t)dt = \int_0^T e^{-pt} \frac{d}{dt} (J^{1-\alpha} v)(t)dt$$

$$= [J^{1-\alpha} v(t)e^{-pt}]_{t=0}^{t=T} + p \int_0^T e^{-pt} J^{1-\alpha} v(t)dt$$

$$= (J^{1-\alpha} v)(T)e^{-pT} + p \int_0^T e^{-pt} J^{1-\alpha} v(t)dt =: I_1 + I_2.$$  \hfill (2.8)
Indeed, by taking into consideration that 

\[ |I_1| \leq \frac{C_v}{\Gamma(1-\alpha)} e^{-pT} \int_0^T (T-s)^{-\alpha} e^{C_v s} ds = \frac{C_v}{\Gamma(1-\alpha)} e^{-pT} \int_0^T s^{-\alpha} e^{C_v (T-s)} ds \]

\[ = \frac{C_v}{\Gamma(1-\alpha)} e^{-(p-C_v)p} \int_0^1 T^{-\alpha} e^{C_v s} ds \leq \frac{C_v}{\Gamma(1-\alpha)} e^{-(p-C_v)p} \int_0^\infty s^{-\alpha} e^{C_v s} ds \]

\[ = C_v e^{-(p-C_v)p} \frac{1}{C_v^{1-\alpha}}. \]

Hence, if \( p > C_v \), then \( \lim_{T \to \infty} I_1 = 0 \).

As for \( I_2 \), exchanging the orders of the integrals, we see that

\[ I_2 = p \int_0^T e^{-pt} \frac{1}{\Gamma(1-\alpha)} \left( \int_0^t (t-s)^{-\alpha} v(s) ds \right) dt \]

\[ = \frac{p}{\Gamma(1-\alpha)} \int_0^T \left( \int_0^T e^{-pt}(t-s)^{-\alpha} dt \right) v(s) ds \]

\[ = \frac{p}{\Gamma(1-\alpha)} \int_0^T \left( \int_0^{T-s} e^{-pt} \eta^{-\alpha} d\eta \right) e^{-pt} v(s) ds. \]

For \( p > C_v \), since \( |v(s)| \leq C_v e^{C_v s} \) for \( s \geq 0 \), we have

\[ \left| \left( \int_0^{T-s} e^{-pt} \eta^{-\alpha} d\eta \right) e^{-pt} v(s) \right| \leq C_v \left( \int_0^\infty e^{-pt} \eta^{-\alpha} d\eta \right) e^{-pt} v(s) \]

for all \( s > 0 \) and \( T > 0 \), and the Lebesgue dominated convergence theorem yields

\[ \lim_{T \to \infty} I_2 = \frac{p}{\Gamma(1-\alpha)} \int_0^\infty \left( \int_0^\infty e^{-pt} \eta^{-\alpha} d\eta \right) e^{-pt} v(s) ds \]

\[ = \frac{p}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)}{p^{1-\alpha}} \int_0^\infty e^{-pt} v(s) ds = p^\alpha L(v)(p) \]

for \( p > C_v \). Thus the proof of the lemma is complete. \( \square \)

We conclude this subsection with the following:

**Lemma 2.4.** Let \( u_0 \in H^1_0(\Omega) \) be arbitrarily given, and let \( u \in L^2(0, T; H^2(\Omega)) \) satisfy (1.2) and \( u - u_0 \in H_0(0, T; L^2(\Omega)). \) Then \( (u(\cdot, t) - u_0, \varphi)_{L^2(\Omega)} \in V_a(0, \infty) \) for any \( \varphi \in C_0^\infty(\Omega) \), and there exists a constant \( C(u) > 0 \) depending on \( u \), such that

\[
(L(u)(\cdot, p), A\varphi)_{L^2(\Omega)} + (p^\alpha L(u)(\cdot, p), \varphi)_{L^2(\Omega)} = p^{\alpha-1}(u_0, \varphi)_{L^2(\Omega)}, \quad p > C(u). \tag{2.7}
\]

We can rewrite (2.7) as

\[
(A + p^\alpha L(u)(x, p) = p^{\alpha-1} u_0(x) \quad \text{for almost all } x \in \Omega \text{ and all } p > C(u). \tag{2.8}
\]

Indeed, by taking into consideration that \( \varphi \in C_0^\infty(\Omega) \) is arbitrary, Equation (2.7) implies

\[
AL(u)(\cdot, p) + p^\alpha L(u)(\cdot, p) = p^{\alpha-1} u_0 \quad \text{in } (C_0^\infty(\Omega))^\prime \text{ for all } p > C(u).
\]

Therefore, (2.8) follows from \( p^{\alpha-1} u_0 \in L^2(\Omega) \) with \( p > C(u) \).

**Proof.** Let \( \varphi \in C_0^\infty(\Omega) \) be arbitrarily chosen. We set \( w(t) = (u(\cdot, t) - u_0, \varphi)_{L^2(\Omega)} \) for \( t > 0 \). Then Lemma 2.2 yields

\[
|w(t)| \leq \|u(\cdot, t) - u_0\|_{L^2(\Omega)} \||\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)} (C e^{C t} \|u_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}).
\]
By \( u - u_0 \in H_a(0, T; L^2(\Omega)) \), we see that \( w \in H_a(0, T) \) for all \( T > 0 \). Therefore, \( w = w(t) = (u(\cdot, t) - u_0, \varphi)_{L^2(\Omega)} \in V_a(0, \infty) \).

Next, since \( \partial_t^p = (J^p)^{-1} \) by the definition, we have

\[
J^p(\partial_t^p(u(\cdot, t) - u_0), \varphi)_{L^2(\Omega)} = (u(\cdot, t) - u_0, \varphi)_{L^2(\Omega)} = w(t),
\]

so that using again \( \partial_t^p = (J^p)^{-1} \), we reach \( \partial_t^p w(t) = (\partial_t^p(u(\cdot, t) - u_0), \varphi)_{L^2(\Omega)} \) for \( t > 0 \). Hence, in view of Lemma 2.3, we obtain

\[
(L(\partial_t^p(u - u_0))(\cdot, p), \varphi)_{L^2(\Omega)} = L(\partial_t^p w)(p) = p^s L(w)(p) \quad \text{for } p > C(u). \tag{2.9}
\]

By the Fubini theorem, we see

\[
p^s L(w)(p) = p^s \int_0^\infty e^{-pt} \left( \int_{\Omega} (u(x, t) - u_0(x)) \varphi(x) \, dx \right) \, dt \tag{2.10}
\]

\[= p^s(L(u - u_0)(\cdot, p), \varphi)_{L^2(\Omega)} = p^s(\mathcal{L}(u(\cdot, p) - p^{-1}u_0), \varphi)_{L^2(\Omega)} \quad \text{for } p > C(u).
\]

Here, we used

\[
L(u_0)(p) = \int_0^\infty e^{-pt} u_0(x) \, dx = p^{-1}u_0(x).
\]

Therefore, (2.9) and (2.10) imply

\[
(L(\partial_t^p(u - u_0))(\cdot, p), \varphi)_{L^2(\Omega)} = (p^sL(u)(\cdot, p) - p^{s-1}u_0, \varphi)_{L^2(\Omega)} \tag{2.11}
\]

for \( p > C(u) \). Since \( \partial_t^p(u - u_0) + A u = 0 \) in \( \Omega \times (0, \infty) \) by the \( \tau \)-analyticity of \( u \), integrating by parts in \( x \), we obtain \( (\partial_t^p(u(\cdot, t) - u_0), \varphi)_{L^2(\Omega)} = (u(\cdot, t), -A\varphi)_{L^2(\Omega)} \) for \( t > 0 \), and taking the Laplace transforms of both sides and using (2.11), we reach

\[
(p^sL(u)(\cdot, p) - p^{s-1}u_0, \varphi)_{L^2(\Omega)} = (L(\partial_t^p(u - u_0))(\cdot, p), \varphi)_{L^2(\Omega)} = (L(u)(\cdot, p), -A\varphi)_{L^2(\Omega)}
\]

for \( p > C(u) \). Thus, the proof of Lemma 2.4 is complete. \( \square \)

### 2.3 Some results from the spectral theory

We recall that

\[
-A\varphi(x) = \sum_{j, k=1}^{d} \partial_j(a_{kj}(x) \partial_k \varphi(x)) + \sum_{j=1}^{d} b_j(x) \partial_j \varphi(x) + c(x) \varphi(x)
\]

with the domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). Then it is known that the spectrum \( \sigma(A) \) consists entirely of countably many eigenvalues in \( \mathbb{C} \) such that \( \infty \) is only an accumulation point. We set

\[
\sigma(A) := \{ \lambda_m \}_{m \in \mathbb{N}}.
\]

Then

\[
\inf \{ \Re \lambda_m; m \in \mathbb{N} \} > -\infty \tag{2.12}
\]

(e.g., Agmon\(^{19} \)). For each \( m \in \mathbb{N} \), we can choose a sufficiently small circle \( \gamma_m \subset \mathbb{C} \) centered at \( \lambda_m \) and not surrounding \( \lambda_n \) if \( n \neq m \), and we define operators \( P_m \) and \( D_m \) by

\[
P_m \varphi := \frac{1}{2\pi i} \int_{\gamma_m} (\eta - A)^{-1} \varphi \, d\eta, \quad \varphi \in L^2(\Omega)
\]

and

\[
D_m \varphi := \frac{1}{2\pi i} \int_{\gamma_m} (\eta - \lambda_m)(\eta - A)^{-1} \varphi \, d\eta, \quad \varphi \in L^2(\Omega)
\]
for each \( m \in \mathbb{N} \) (e.g., Agmon\(^1\)). Moreover, for each \( m \in \mathbb{N} \) (e.g., Agmon\(^1\)), and \( \varphi \in P_m L^2(\Omega) \), \( \neq 0 \) is called a generalized eigenfunction of \( A \) for \( \lambda_m \).

Then it is known that

\[
P_m^2 = P_m, \quad D_m = (A - \lambda_m)P_m, \quad D_m = D_m P_m = P_mD_m, \quad P_m L^2(\Omega) \subset D(A)
\]

(2.13)

\[
D_m^k \varphi := \frac{1}{2\pi i} \int_{T_m} (\eta - \lambda_m)^k (\eta - A)^{-1} \varphi d\eta, \quad \varphi \in L^2(\Omega), \quad k \in \mathbb{N}
\]

(2.14)

and

\[
D_m^d P_m = 0 \quad \text{for each} \quad m \in \mathbb{N}
\]

(2.15)

(e.g., Section 6 of Chapter III in Kato\(^2\)). Henceforth, we set \( P_m := D_m^0 \).

In view of (2.13) and (2.14), we can readily derive

**Lemma 2.5.** If \( \varphi \in L^2(\Omega) \) satisfies \( D_m^k P_m \varphi = 0 \) for some \( k_0 \in \mathbb{N} \), then

\[
(A - \lambda_m)D_m^{k_0 - 1} P_m \varphi = 0.
\]

**Proof.** By (2.13), we have \( P_m D_m^\varepsilon = D_m^\varepsilon P_m \) for each \( \varepsilon \in \mathbb{N} \) and so

\[
P_m D_m^{k_0 - 1} P_m \varphi = D_m^{k_0 - 1} P_m \varphi = D_m^{k_0 - 1} P_m \varphi.
\]

Therefore, again (2.13) yields

\[
(A - \lambda_m)(D_m^{k_0 - 1} P_m \varphi) = (A - \lambda_m)P_m(D_m^{k_0 - 1} P_m \varphi) = D_m(D_m^{k_0 - 1} P_m \varphi) = D_m D_m^{k_0} \varphi = 0.
\]

Hence, the assumption \( D_m^k P_m \varphi = 0 \) yields that \( (A - \lambda_m)D_m^{k_0 - 1} P_m \varphi = 0 \). This completes the proof. \( \square \)

## 3 | PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. Before giving the proof, we first employ the Laplace transform to show the uniqueness in determining the Neumann derivative of the initial value from the additional data of the solution on the subboundary, which plays crucial role in the proof of Theorem 1.1. We have

**Lemma 3.1.** For \( u_0 \in H^1_0(\Omega) \), let \( u \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) solve the initial boundary value problem (1.2) and let \( u - u_0 \in H_s(0, T; L^2(\Omega)). \) If \( \partial_s u = 0 \) on \( \Gamma \times (0, T) \), then \( \partial_s D_m^k P_m u_0 = 0 \) on \( \Gamma \) for any \( m, k \in \mathbb{N} \).

**Proof.** By (2.8), we obtain

\[
(A + p^s) L(u)(x, p) = p^{s - 1} u_0(x)
\]

for almost all \( x \in \Omega \) and \( p > C(\Omega) \); some constant. From (2.12), by choosing \( p_0 > 0 \) large, it follows that \( -p^s \in \rho(A) \): resolvent set of \( A \) for \( p > p_0 \). Hence,

\[
L(u)(\cdot, p) = p^{s - 1} (A + p^s)^{-1} u_0 \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)
\]

(3.1)

for \( p > p_0 \).

Moreover, the assumption \( \partial_s u = 0 \) on \( \Gamma \times (0, T) \) combined with the \( t \)-analyticity of the solution, it follows that \( \partial_s u(x, t) = 0 \) for \( x \in \Gamma \) and \( t > 0 \). Therefore,

\[
\partial_s L(u)(\cdot, p) = 0 \quad \text{on} \quad \Gamma \quad \text{for} \quad p > p_0.
\]
Hence, (3.1) yields
\[ \partial_{\nu}(A + p^n)^{-1}u_0 = 0 \quad \text{on } \Gamma \text{ for } p > p_0. \]  
(3.2)

Since \( \partial_{\nu}(A - \eta)^{-1}u_0 \) is holomorphic in \( \eta \in \mathbb{C}\setminus \{ \lambda_m \}_{m \in \mathbb{N}} \), the unicity theorem for holomorphic functions yields
\[ \partial_{\nu}(\eta - A)^{-1}u_0 = 0 \quad \text{on } \Gamma \text{ for } \eta \notin \{ \lambda_m \}_{m \in \mathbb{N}}. \]  
(3.3)

Let \( \gamma_m, m \in \mathbb{N} \) be a small circle around \( \lambda_m \), which does not surround \( \lambda_n \) with all \( n \neq m \). Then (3.3) implies
\[ \partial_{\nu} \left( \frac{1}{2\pi i} \int_{\gamma_m} (\eta - \lambda_m)^k (\eta - A)^{-1}u_0 d\eta \right) = \frac{1}{2\pi i} \int_{\gamma_m} (\eta - \lambda_m)^k \partial_{\nu}(\eta - A)^{-1}u_0 d\eta = 0 \quad \text{for each } k \in \mathbb{N}. \]

Hence, (2.14) yields \( \partial_{\nu} D^{d_m}_{m}u_0 = \partial_{\nu} D^{d_m}_{m}P_m u_0 = 0 \). This completes the proof of the lemma.

Now we are ready for the proof of our first result.

**Proof of Theorem 1.1.** Since \( D^{d_m}_{m}P_m u_0 = 0 \) by (2.15), from Lemma 2.5 with \( k_0 = d_m \), we derive
\[ (A - \lambda_m)(D^{d_m}_{m}P_m u_0) = 0 \quad \text{in } \Omega. \]  
(3.4)

Since \( D^{d_m}_{m}P_m u_0 \in D(A) = H^2(\Omega) \cap H^1(\Omega) \) by (2.13), we see that \( D^{d_m}_{m}P_m u_0 = 0 \) on \( \partial\Omega \). Moreover, \( \partial_{\nu} (D^{d_m}_{m}P_m u_0) = 0 \) on \( \Gamma \) for each \( m \in \mathbb{N} \) in view of Lemma 3.1. In particular, \( D^{d_m}_{m}P_m u_0 = \partial_{\nu} (D^{d_m}_{m}P_m u_0) = 0 \) on \( \Gamma \). Consequently, using (3.4), we conclude from the unique continuation for the elliptic operator \( \lambda_m - A \) that \( D^{d_m}_{m}P_m u_0 = 0 \) in \( \Omega \). Again a similar argument yields \( D^{d_m}_{m}P_m u_0 = 0 \) in \( \Omega \). Continuing this procedure, we obtain \( P_m u_0 = 0 \) in \( \Omega \) for any \( m \in \mathbb{N} \). Therefore, we must have \( u_0 = 0 \) from the completeness of the generalized eigenfunctions (see the last chapter of Agmon\textsuperscript{19}).

Finally, from the uniqueness for the initial boundary value problem (1.2), it follows that \( u \equiv 0 \). This completes the proof of our first main theorem.

\[ \square \]

## 4 | PROOF OF THEOREM 1.2

The argument is based on Theorem 1.1 and the following Duhamel’s principle for a time-fractional diffusion equation.

**Lemma 4.1** (Duhamel’s principle). Let \( f \in H^1_0(\Omega) \) and \( \mu \in C^1[0, T] \). Then the weak solution \( y \) to the initial boundary value problem (1.4) allows the representation
\[ y(\cdot, t) = \int_0^t \theta(t - s)v(\cdot, s)ds, \quad 0 < t < T, \]  
(4.1)

where \( v \) solves the homogeneous problem
\[
\begin{aligned}
\partial_t^\alpha v - f &= Av = 0 & \text{in } & \Omega \times (0, T), \\
v(x, t) &= f(x) \in H_0(0, T) & \text{for almost all } & x \in \Omega, \\
v(x, t) &= 0, & \quad (x, t) \in & \partial\Omega \times (0, T)
\end{aligned}
\]  
(4.2)

and \( \theta \in L^1(0, T) \) is a unique solution to the fractional integral equation
\[ J^{1-\alpha}\theta(t) = \mu(t), \quad 0 < t < T. \]  
(4.3)

Lemma 4.1 is recast within our framework of \( \partial_t^\alpha \) in \( H_0(0, 1) \), and in that sense, it is different from Duhamel’s principle by Umarov (see a survey\textsuperscript{21} for example).
Although the formulation (4.2) for an initial boundary value problem is different and we treat the nonsymmetric elliptic operator $A$, we can prove Lemma 4.1 by the same way as in Liu et al. $^{22}$, Lemma 4.1. Hence, we omit the proof here.

**Proof of Theorem 1.2.** Let $y$ satisfy the initial boundary value problem (1.4) with $f \in H^1_0(\Omega)$ and $\mu \in C^1[0, T]$. Then $y$ takes the form of (4.1) in terms of Lemma 4.1. Performing the Riemann–Liouville fractional integral $J^{1-a}$ to (4.1), we deduce

$$J^{1-a}y(\cdot, t) = \frac{1}{\Gamma(1-a)} \int_0^t \frac{1}{(t - \tau)^a} \left( \int_0^\tau \theta(\tau - \xi) \nu(\cdot, \xi) d\xi \right) d\tau$$

$$= \frac{1}{\Gamma(1-a)} \int_0^t \nu(\cdot, \xi) \left( \int_0^\xi \frac{\theta(\tau - \xi)}{(t - \tau)^a} d\tau \right) d\xi$$

$$= \int_0^t \nu(\cdot, \xi) \frac{1}{\Gamma(1-a)} \left( \int_0^{\xi} \frac{\theta(\eta)}{(t - \xi - \eta)^a} d\eta \right) d\xi$$

$$= \int_0^t \nu(\cdot, \xi)J^{1-a}\theta(t - \xi) d\xi = \int_0^t \mu(t - \tau)\nu(\cdot, \tau) d\tau,$$

where we exchanged the orders of the integrals and used the relation (4.3). Since $\partial_\alpha u = 0$ on $\Gamma \times (0, T)$, we see

$$\int_0^t \mu(t - \tau)\partial_\alpha \nu(\cdot, \tau) d\tau = 0 \quad \text{on} \quad \Gamma, \quad 0 < t < T.$$

Therefore, the Titchmarsh convolution theorem (Titchmarsh$^{23}$) that there exist $T_1, T_2 \geq 0$ such that $T_1 + T_2 \geq T$, and $\mu(t) = 0$ for almost all $t \in (0, T_1)$ and $\partial_\alpha \nu(\cdot, t) = 0$ for almost all $t \in (0, T_2)$. For a simple proof of the Titchmarsh theorem, see also Doss.$^{24}$

First, we see that $T_1 < T$, because $\mu \neq 0$ in $(0, T)$ by the assumption. Since $T_1 + T_2 \geq T$, we must have $T_2 > 0$. Therefore, we employ Theorem 1.1 with $T = T_2$ to (4.2), so that $\nu = 0$ in $\Omega \times (0, T_2)$, implying $f = 0$ in $\Omega$. This completes the proof of Theorem 1.2.

5 | CONCLUDING REMARKS

In this article, we considered a time-fractional diffusion equation with advection. By taking the Laplace transforms, we changed the problem (1.2) to an elliptic equation. We then proved the uniqueness property of the solution to (1.2) by using the spectral decomposition of a general elliptic operator and the unique continuation for the elliptic equation. Theorem 1.1 concludes the uniqueness of solution of weaker type than the unique continuation asserting the uniqueness only by zero Cauchy data on arbitrarily chosen lateral subboundary. We remark that Theorem 1.1 requires the homogeneous Dirichlet boundary condition on the whole boundary.

For the unique continuation and related results for fractional diffusion equations, see Cheng et al.$^{25}$, Li and Yamamoto$^{26}$, Lin and Nakamura$^{27}$, and Xu et al.$^{28}$

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**CONFLICT OF INTEREST**

This work does not have any conflicts of interest.
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