Exact and optimal controllability for scalar conservation laws with discontinuous flux

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Abstract
This paper describes the reachable set and resolves an optimal control problem for the scalar conservation laws with discontinuous flux. We give a necessary and sufficient criteria for the reachable set. A new backward resolution has been described to obtain the reachable set. Regarding the optimal control problem we first prove the existence of a minimizer and then the backward algorithm allows us to compute it. The same method also applies to compute the initial data control for an exact control problem. Our methodology for the proof relies on the explicit formula for the conservation laws with the discontinuous flux and finer properties of the characteristics curves.

Keywords: Scalar conservation laws, discontinuous flux, exact control, reachable sets, optimal control, Hamilton-Jacobi equation.

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1. Introduction

The goal of this paper is to study the reachable sets, optimal controllability and exact controllability of the following scalar conservation laws with discontinuous flux

\[
\begin{align*}
    u_t + F(x,u)_x &= 0, \quad \text{if } x \in \mathbb{R}, \; t > 0, \\
    u(x,0) &= u_0(x), \quad \text{if } x \in \mathbb{R},
\end{align*}
\]

where the flux \( F \) is given by, \( F(x,u) = H(x)f(u) + (1-H(x))g(u), \) \( H \) is the Heaviside function. Through out this article we assume the fluxes \( f,g \) to be \( C^1(\mathbb{R}) \), strictly convex with superlinear growth (i.e., \( \lim \frac{f(p)}{|p|}, \frac{g(p)}{|p|} = (\infty, \infty) \)) and initial data \( u_0 \in L^\infty(\mathbb{R}) \). We denote by \( \theta_f, \theta_g \) the unique minima of the fluxes \( f,g \) respectively. In this article, by entropy solution we mean a weak solution to (1.1) satisfying interface entropy condition as in [5].

Here we explore three aspects of control theory in conservation laws with discontinuous flux: (i) characterization of reachable set, (ii) exact controllability and (iii) optimal controlability. Above three problems are classical and they are answered for (1) unlike the scalar conservation laws, for (1.1), \( L^1 \)-contraction is still unknown in general setting even if \( f,g \) are convex.

(2) Entropy solutions do not admit rarefaction waves from the interface \( \{x = 0\} \) (see subsection 3.1).

(3) Reflected characteristic curves (see definition 2.4) from the boundary can occur in the structure of entropy solution to (1.1).

We resolve the above difficulties by introducing a new backward construction to characterize the reachable sets. Then we adopt this to obtain the optimal control result.

The scalar conservation laws with discontinuous flux of type (1.1) has a huge variety of applications in several fields, namely traffic flow modeling, modeling gravity, modeling continuous sedimentation in clarifier-thickener units, ion etching in the semiconductor industry and many more. In the past two decades the first order model of type (1.1) has been extensively studied from both the theoretical and numerical point of view. Concerning the uniqueness it is worth to mention that the following Kružkov type entropy inequalities, in both the two upper quarter-planes are not sufficient to guarantee the uniqueness,

\[
\begin{align*}
    \int_0^\infty \int_0^\infty \left( \phi_1(u) \frac{\partial s}{\partial t} + \psi_1(u) \frac{\partial s}{\partial x} \right) dt &\geq - \int_0^\infty \psi_1(u(0+,t)) s(0,t) dt, \\
    \int_0^\infty \int_0^\infty \left( \phi_2(u) \frac{\partial s}{\partial t} + \psi_2(u) \frac{\partial s}{\partial x} \right) dt &\geq \int_0^\infty \psi_2(u(0+,t)) s(0,t) dt.
\end{align*}
\]

Here \( (\phi_1, \psi_1) \) denote the entropy pair corresponding to the flux \( f \), \( (\phi_2, \psi_2) \) denote the entropy pair corresponding to the flux \( g \), and \( s \in C^1_0(\mathbb{R} \times \mathbb{R}_+) \), a non-negative test function. Consequently one need an extra criteria on the interface called “interface entropy condition” (see [5]) given by

\[
\text{meas}\{t : f'(u(0+,t)) > 0, g'(u(0-,t)) < 0\} = 0.
\]

Using this extra entropy along with the above Kružkov type inequalities the uniqueness result has been obtained in [5]. On the other hand, the existence result has been proved in several ways, namely via Hamilton-Jacobi, convergence of numerical schemes, vanishing viscosity method, for further details we refer the reader to [5, 6, 7, 8, 11, 16, 17, 24, 26, 32] and the references therein. The present article uses the
explicit formula obtained in [5], via the Hamilton-Jacobi Cauchy problem. By using this formula it can be shown that if the initial data $v_0$ is uniformly Lipschitz then the viscosity solution $v(\cdot, t)$ is also uniformly Lipschitz, for all $t > 0$. Let $u := \frac{\partial v}{\partial x}$, then $u$ is the unique weak solution (see [5], Theorem 2.2) of (1.1), enjoys (1.3) near interface and satisfies the following Rankine-Hugoniot condition on the interface.

$$\text{meas}\{ t : f(u(0+, t)) \neq g(u(0-, t)) \} = 0. \quad (1.4)$$

Note that in general TV of entropy solution to (1.1) can blow up [1, 25] at finite time even for BV initial data which makes the current article more technical while obtaining the compactness. Regarding the well-posedness theory to $f = g$ case, we refer the reader to [23] for Cauchy problem and for the initial boundary value problem to [31].

Concerning the exact controllability for the scalar convex conservation laws the first work has been done in [13], where they considered the initial boundary value problem in a quarter plane with $u_0 = 0$ and by using one boundary control they investigated the reachable set. As in [2], they considered $u_0 \in L^\infty$ and three possible cases, namely pure initial value problem with initial data control outside any domain, initial boundary value problem in a quarter plane with one boundary control and initial boundary problem in a strip with two boundary controls to get the reachable sets in a complete generality. In both the articles the Lax-Oleinik type formulas has been exploited. An alternative approach has been provided in [30] by using the return method (see [20, 21]). For the viscous Burgers equation any non-zero state can be reached in finite time by two boundary controls [29], recently, it has been proved [12] that there exist many pairs $(C, T)$ so that the state $C$ is not reachable from zero state at time $T$ for the viscosity 1. Control theory for the system of conservation laws is still largely open. We refer to [10, 14, 15, 22, 27, 28] and references therein for controllability results on system of hyperbolic conservation laws.

Let us briefly discuss the optimal controllability results for the case $f = g$. Assume the target function $k \in L^2_{loc}(\mathbb{R})$, support of $f'(k)$ is compact and $T > 0$. We denote by $J_{f=g}(u_0)$, a cost functional, defined in the following way

$$J_{f=g}(u_0) = \int_{-\infty}^{\infty} |f'(u(x, T)) - f'(k(x))|^2 dx, \quad (1.5)$$

where $u_0 \in L^\infty(\mathbb{R})$, $u_0 \equiv \theta_f$ outside a compact set, $\theta_f$ being the only critical point of the flux $f$. Here $u(\cdot, T)$ denotes the unique weak solution at $t = T$ to the Cauchy problem (1.1), in the case $f = g$ with initial datum $u_0$. Then in this case, the optimal control reads like: find a $w_0$ such that $J_{f=g}(w_0) = \min_{u_0 \in L^\infty (\mathbb{R})} J_{f=g}(u_0)$. In [18, 19], they considered the above optimal control problem for the Burgers’ equation and proved such minimizer exists and proposed a numerical scheme called “alternating descent algorithm”, although the convergence of these scheme still remains open. Whereas in [3], they made use of the Lax-Oleinik formula and derived a numerical backward construction which converges to a solution of the above problem. The latter method can be applied also to general convex fluxes as long as a Lax-Oleinik type formula is available. It has to be noticed that even for the case $f = g$, due to the occurrence of the shocks in the solution of (1.1), one may have several minimizers of the optimal control problem (1.5).

One if the main results in this paper is to characterize the reachable set (see subsection 1.1) and then we prove the exact and optimal controllability (subsections 1.2 and 1.3 respectively) for (1.1). In order to do so, we divide $\mathbb{R} \times (0, T)$ into three sub domains: $D_1$, $D_2$ and $D_3$ (see subsection 4.1). These three domains correspond to the solution (a) with reflected characteristics, (b) having interface interactions and (c) solving pure initial value problem (i.e. $f = g$ case) respectively. Now we define a reachable set at $t = T$ in such a way that a given solution corresponds to an element in the reachable set. This imposes a constraint on the elements of the reachable set. Then for any element in the reachable set, using this constraint, we first construct a data in $D_1 \cap \{(x, t) : t = 0\}$ and the solution. Using the ‘no forward interface rarefaction’ from the $t$-axis (see lemma 3.2), we construct another initial data and a solution in $D_2$. Construction of solution in $D_3$ is as in the $f = g$ case. Using the R-H condition, we glue all the three solutions to form a single solution which corresponds to the given element in the reachable set. In
deals with the non existence of forward rarefaction and section 3 proves the main results. We indicate how to extend the above results for the \((A,B)\)-connection [8].

The paper is organized as follows. In the next three subsections we state our main results. In section 2, we recall some known results from [5]. Section 3 deals with the non existence of forward rarefaction from the interface, backward construction for shock and continuous solution. Also the construction of \((\tau_0,\xi_0)\) which is used to define the reachable set \(\mathcal{R}(T)\). Subsections 3.3 and section 4 deals with the backward construction when the reflected characteristics exist. Sections 4 and 5 proves the main results of the paper. In section 6 we indicate how to extend the above results for the \((A,B)\) entropy solution. In section 7, we prove a stability lemma which is used to prove the main results.

Throughout this article we assume that \(f(\theta_f) \leq g(\theta_f)\). The other case \(f(\theta_f) > g(\theta_f)\) follows in a similar way.

**Authors declaration:** It is to be noted most of the ideas and technical details was in the arxiv version but that there was a gap in the proof in [9] because the reflected characteristics was not being considered in the proof and hence the definition of reachable set was incomplete. In this article we fill this gap in subsection 3.3 and present a modified version of [9].

1.1. Reachable Set

Let \(\bar{\theta}_g \leq \theta_f \leq \bar{\theta}_g\) such that \(f(\bar{\theta}_g) = f(\bar{\theta}_g) = g(\theta_g)\). Then we define:

**Reachable set:** Let \(T > 0, 0 \leq R_2 \leq R_1, y : (-\infty, R_2) \cup (R_1, \infty) \to \mathbb{R}\) be a function be given. Then \((T,R_1,R_2,y(\cdot))\) is called an element in the reachable set if the following holds:

1. \(y(\cdot)\) is an non-decreasing function such that
   - (i) \(y(x) \leq 0\) for all \(x \in (-\infty, R_2)\).
   - (ii) \(y(x) \geq 0\) for all \(x \in (R_1, \infty)\).
   - (iii) \(\sup\{|x - y(x)| : x \in (-\infty, R_2) \cup (R_1, \infty)\} < \infty\).

2. Let \(\gamma_0(t) = R_1 + f'(\bar{\theta}_g)(t - T)\) and we denote \((0, T(R_1))\) to be the point of intersection of lines \(t\)-axis and \(\gamma_0\), i.e., \(\gamma_0(T(R_1)) = 0\). Suppose \(T(R_1) \geq 0\), let \((\tau_0, \xi_0)\) be as in lemma 3.8 with \(\bar{\alpha} = \bar{\theta}_g\), \(T_1 = T(R_1)\), then
   \[\xi_0 \leq y(R_1+).\]

3. If \(R_2 < R_1\), then \(T(R_1) \geq 0\).

Denote
\[\mathcal{R}(T) = \{(T,R_1,R_2,y(\cdot)) : \text{they satisfy (1), (2) and (3)}\}\] (1.6)

is called the reachable set.

Then we have the following theorem.

**Theorem 1.1.** Let \((T,R_1,R_2,y(\cdot)) \in \mathcal{R}(T)\) if and only if there exists a \(u_0 \in L^\infty(\mathbb{R})\) and the solution \(u\) of (1.1) such that for \(i = 1, 2\), \(R_i = R_i(T), y(x) = y(x,T)\), where \(R_i\) and \(y(x,T)\) defined in theorem 2.1.

1.2. Exact controllability

**Theorem 1.2.** Let \((T,R_1,R_2,y(\cdot)) \in \mathcal{R}(T)\) where \(\mathcal{R}(T)\) is defined as in (1.6) and \(C_1 < 0 < R_1 < C_2, B_1 < 0 < B_2\) be given. Assume that
\[y(C_1+) > B_1,\] (1.7)
\[y(C_2-) < B_2\] (1.8)
and \( u_{1,0} \in L^\infty(\mathbb{R} \setminus (B_1, B_2)) \), then there exist a \( \tilde{u}_0 \in L^\infty(B_1, B_2) \) and the solution \( u \) of (1.1) with initial data \( u_0 \) satisfying

\[
u_0(x) = \begin{cases} u_{1,0}(x) & \text{if } x \notin (B_1, B_2), \\ \tilde{u}_0(x) & \text{if } x \in (B_1, B_2). \end{cases} \tag{1.9}
\]

Let \( (T, R_1(T), R_2(T), y(\cdot, T)) \) be the element in \( \mathcal{R}(T) \) corresponds to \( u(\cdot, T) \), then

\[
R_i = R_i(T) \text{ for } i = 1, 2, \tag{1.10}
\]

\[
y(x) = y(x, T) \text{ for all } x \in (C_1, R_2) \cup (R_1, C_2). \tag{1.11}
\]

### 1.3. Optimal control

Let \( u_0 \in L^\infty(\mathbb{R}) \) and \( u \) be the solution of (1.1) with initial data \( u_0 \). Let \( T > 0 \). Let \( k \in L^\infty(\mathbb{R}) \) and \( c > 0 \) such that

\[
k(x) = \begin{cases} \theta_g & \text{if } x < -c, \\ \theta_f & \text{if } x > c. \end{cases}
\]

Define

\[
K(x) = \begin{cases} f'(k(x)) & \text{if } x > 0, \\ g'(k(x)) & \text{if } x < 0. \end{cases}
\]

Note that \( K \in L^\infty(\mathbb{R}) \) and support of \( K \subset [-c, c] \). Denote \( g_+^{-1} \) to be the inverse of \( g \) on \( [g(\theta_g), \infty) \). Let \( u_0 \in L^\infty(\mathbb{R}) \) and \( u \) be the corresponding solution of (1.1). Define the cost functional \( J : L^\infty(\mathbb{R}) \to \mathbb{R} \) by

\[
J(u_0) = \int_{-\infty}^{0} |g'(u(x, T)) - K(x)|^2 dx + \int_{0}^{R_2(T)} |g' \circ g_+^{-1} \circ f(u(x, T)) - K(x)|^2 dx \\
+ \int_{R_2(T)}^\infty |f'(u(x, T)) - K(x)|^2 dx. \tag{1.12}
\]

Then we have the following result on optimal control problem.

**Theorem 1.3.** Let \( \mathcal{A} \) be the admissible class of functions defined by

\[
\mathcal{A} = \left\{ u_0 \in L^\infty(\mathbb{R}) : \exists M > 0 \text{ such that } u_0(x) = \begin{cases} \theta_g & \text{if } x < -M, \\ \theta_f & \text{if } x > M. \end{cases} \right\},
\]

Then there exists a \( u_0 \in \mathcal{A} \) such that

\[
J(u_0) = \min_{u_0 \in \mathcal{A}} J(u_0). \tag{1.13}
\]

We prove the Theorem 1.3 via an explicit construction and hence can be adopted to numerical computation.

**Remark 1.1.** We can obtain the similar results, when one of the flux is concave and another one is convex in the equation (1.1). One can use the explicit formulas as in [8] and similar analysis in the present paper.

### 2. Preliminaries

In order to make the paper self contained we recall some results, definitions and notations from [5].

**Definition 2.1.** **Control curve:** (See figure 1 for illustration) We say \( \gamma \in C([0, t], \mathbb{R}) \) is a control curve if it verifies the following conditions:

1. \( \gamma \) is piece-wise affine and it can have at most 3 affine segments such that each affine part lies completely in either \( [0, \infty) \times [0, \infty) \) or \( (-\infty, 0] \times [0, \infty) \).
that the figures (c), (f) represents the reflected control curves.

Figure 1: Figures (a), (b) and (c) is representing control curves for the case $x > 0$, figures (d), (e) and (f) is for $x < 0$. Note that the figures (c), (f) represents the reflected control curves.

2. If $\gamma$ has three affine segments $\{\gamma_i; i = 1, 2, 3\}$ defined as $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$ where $0 = t_0 \leq t_1 \leq t_2 \leq t_3 = t$, then $\gamma_2(s) = 0$ for all $s \in (t_1, t_2)$ and for all $s \in (t_0, t_1) \cup (t_2, t)$, either $\gamma_1(s), \gamma_3(s)$ are in $(-\infty, 0)$ or in $(0, \infty)$.

Let $0 < t, x \in \mathbb{R}$ and let $c(x, t)$ be the set of all control curves such that $\gamma(t) = x$. The set $c(x, t)$ can be partitioned into three categories:

1. $c_0(x, t) \subset c(x, t)$ consists of control curves $\gamma$ which have only one affine segment and satisfies $x\gamma(s) \geq 0$ for $s \in [0, t]$.
2. $c_r(x, t) \subset c(x, t)$ consists of control curves $\gamma$ which have exactly 3 affine segments and satisfies $x\gamma(s) \geq 0$ for $s \in [0, t]$. Here we say $c_r(x, t)$ to be the set of all reflected control curves.
3. $c_b(x, t) = c(x, t) \setminus \{c_0(x, t) \cup c_r(x, t)\}$.

Definition 2.2. Convex dual: Let $f$ be a $C^1$ convex function with superlinear growth, that is $f$ satisfies $\lim_{|s| \to \infty} \frac{f(s)}{|s|} = \infty$. Then we denote the convex dual of $f$ by $f^*$ and defined by $f^*(p) = \sup_q \{pq - f(q)\}$. Observe that $(f^*)' = (f')^{-1}$.

Definition 2.3. Cost function: Let $f^*, g^*$ be the respective convex duals of the fluxes $f$ and $g$. Let us assume that $v_0 : \mathbb{R} \to \mathbb{R}$ be an uniformly Lipschitz continuous function. Let $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $\gamma \in c(x, t)$.

The cost functional $\Gamma$ associated to $v_0$ is defined by

$$\Gamma_{v_0, \gamma}(x, t) = v_0(\gamma(0)) + \int_{\{\theta \in [0, t] : \gamma(\theta) > 0\}} f^*(\gamma) d\theta + \int_{\{\theta \in [0, t] : \gamma(\theta) < 0\}} g^*(\gamma) d\theta + \text{meas}\{\theta \in [0, t] : \gamma(\theta) = 0\} \min\{f^*(0), g^*(0)\}.$$ 

Then we define the value function $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$v(x, t) = \inf_{\gamma \in c(x, t)} \{\Gamma_{v_0, \gamma}(x, t)\}.$$
Definition 2.4. Let us define by $\text{ch}(x,t) = \{\gamma : \Gamma_{\gamma_0}\gamma(x,t) = v(x,t)\}$, the set of characteristics curves. We say an element of the set $\text{ch}(x,t) \cap c_r(x,t)$ to be a reflected characteristics curves.

Let $t > 0$, define (see figure 2 for further illustration)

$$R_1(t) = \inf\{x \geq 0 : \text{ch}(x,t) \subset c_0(x,t)\}.$$  

$$R_2(t) = \begin{cases} \inf\{x : 0 \leq x \leq R_1(t), \text{ch} x(t) \cap c_r(x,t) \neq \emptyset\}, \\
R_1(t) \text{ if the above set is empty.} \end{cases}$$  

$$L_1(t) = \sup\{x \leq 0 : \text{ch}(x,t) \subset c_0(x,t)\}.$$  

$$L_2(t) = \begin{cases} \sup\{x : L_1(t) \leq x \leq 0, \text{ch} x(t) \cap c_r(x,t) \neq \emptyset\}, \\
L_1(t) \text{ if the above set is empty.} \end{cases}$$  

$$y(x,t) = \inf\{\gamma(0) : \gamma \in \text{ch} x(t), x \in (-\infty, L_1(t)) \cup (R_1(t), \infty)\}.$$  

Let $0 \leq x \leq R_1(t)$, define

$$t_+(x,t) = \inf\{t_1 : \gamma(t_1) = 0, \gamma(\theta) > 0, \forall \theta \in (t_1, t), \gamma \in \text{ch} x(t)\}.$$  

$$t_+(R_1(t), t) = \lim_{x \to R_1(t)} t_+(x,t), \quad i = 1, 2.$$  

For $t_+(R_2(t)) \leq s \leq t$, define

$$y_-(s) = \inf\{\gamma(0) : \gamma \in \text{ch}(0,s)\}.$$  

Let $L_1(t) \leq x \leq 0$, define

$$t_-(x,t) = \inf\{t_1 : \gamma(t_1) = 0, \gamma(\theta) < 0, \forall \theta \in (t_1, t), \gamma \in \text{ch} x(t)\}.$$  

$$t_-(L_i(t), t) = \lim_{x \to L_i(t)} t_-(x,t), \quad i = 1, 2.$$  

For $t_-(L_2(t), t) \leq s \leq t$, define

$$y_+(s) = \inf\{\gamma(0) : \gamma \in \text{ch}(0,s)\}.$$  

Definition 2.5. Let $(X,d)$ be a metric space and $A_k, A$ are subsets of $X$ for each $k \geq 1$. We say that $\lim_{k \to \infty} A_k \subset A$ if for every sequence $\{x_k\}$ with $x_k \in A_k$, there exists a subsequence $\{x_{k_i}\}$ converges to some $x \in A$. 

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Let $f, g$ be in $C^1(\mathbb{R})$ and strictly convex function with $F(x, u) = H(x)f(u) + (1-H(x))g(u)$. Let \{f_k\}, \{g_k\} in $C^1(\mathbb{R})$ be sequences of strictly convex functions and $F_k(x, u) = H(x)f_k(u) + (1-H(x))g_k(u)$ such that
\[
\lim_{k \to \infty} (f_k(u), g_k(u)) = (f(u), g(u)) \text{ in } C^1_{\text{loc}}(\mathbb{R}),
\]
and
\[
\lim_{|p| \to \infty} \left( \inf_k \frac{f_k(p)}{|p|}, \inf_k \frac{g_k(p)}{|p|} \right) = (\infty, \infty).
\]
Let $u_0 \in L^\infty(\mathbb{R})$ and \{u_{0,k}\} $\subset L^\infty(\mathbb{R})$ be such that
\[
\lim_{k \to \infty} u_{0,k} = u_0, \text{ in } L^\infty(\mathbb{R}) \text{ - weak * }.
\]
Let $v_0(x) = \int_0^x u_0(\theta)d\theta$ and $v_{0,k}(x) = \int_0^x u_{0,k}(\theta)d\theta$ be the associated primitives of $u_0$ and $u_{0,k}$. Notice that in [5], it was assumed that $u_0 \in C(\mathbb{R})$. Now it is easy to show that results in [5] continue to hold for $u_0 \in L^\infty(\mathbb{R})$. In order to prove this result, we need the following stability lemma.

**Lemma 2.1 (Stability lemma).** With the data as above, let $v,v_k$ be the corresponding value functions associated to the fluxes $F$ and $F_k$ and initial data $v_0$ and $v_{0,k}$ respectively. Let $c_h(x,t)$ and $c_{h,k}(x,t)$ be the respective characteristic sets. Let $u = \frac{\partial v}{\partial x}$ and $u_k = \frac{\partial v_k}{\partial x}$, then

1. $\lim_{k \to \infty} v_k = v$ in $C^1_{\text{loc}}(\mathbb{R} \times [0, \infty))$,
2. $\lim_{k \to \infty} u_k = u$ in $D'(\mathbb{R} \times [0, \infty))$,
3. $\lim_{k \to \infty} c_{h,k}(x,t) \subset c_h(x,t)$.

Proof of this lemma is given in the appendix (section 7).

**Theorem 2.1 ([5]).** Let $u_0 \in L^\infty(\mathbb{R})$ and $v$ be the corresponding value function defined in definition 2.3. Then $u(x,t) = \frac{\partial v}{\partial x}(x,t)$ exists for $t > 0$, a.e. $x \in \mathbb{R}$ and is a solution to (1.1). Furthermore there exist Lipschitz continuous curves $R_i(t), L_i(t)$, for $i = 1, 2$ such that for each $t > 0$ a.e. $x \in \mathbb{R}$, we have

1. No two characteristics intersects in the region $\{(x,t): x \neq 0, t > 0\}$, i.e., if $\gamma_i \in c_h(x_i,t_i), i = 1, 2$, then if for some $\theta \in (0, \min(t_1,t_2))$, $\gamma_1(\theta) \neq 0, \gamma_2(\theta) \neq 0$, then $\gamma_1(\theta) \neq \gamma_2(\theta)$, provided $\gamma_1$ and $\gamma_2$ are two different characteristic curves.
2. Let $T > 0$, then one of the following holds:
   (i) If $R_1(T) > 0$, then $L_1(T) = 0$ and for all $t \in (t_+(R_1(T)-,T), R_1(t) > 0$.
   (ii) If $L_1(T) < 0$, then $R_1(T) = 0$ and for all $t \in (t_-(L_1(T)+,T), L_1(t) < 0$.
   (iii) $R_1(T) = L_1(T) = 0$.
3. The following properties are true:
   (i) If $f^*(0) \geq g^*(0)$ (equivalently $f(\theta_f) \leq g(\theta_g)$), then $L_1(t) = 0$ and if $f^*(0) \leq g^*(0)$ (equivalently $f(\theta_f) \geq g(\theta_g)$), then $R_1(t) = 0$.
   (ii) $x \mapsto y(x,t)$ is a non decreasing function and $x \mapsto t_+(x,t)$ is a non increasing function on the domain of definitions.
   (iii) For $R_2(t) < x < R_1(t)$, $\frac{x}{t-t_+(x,t)} > 0$ and for a.e. $x$,
   \[
   g(\theta_g) = f \circ f^*(\frac{x}{t-t_+(x,t)}) .
   \]
(iv) $s \mapsto y_{-0}(s)$ is non increasing function.
(v) $x \mapsto t_{-}(x, t)$ is non decreasing function.
(vi) For $L_1(t) < x < L_2(t)$, $\frac{x}{t - t_{-}(x, t)} < 0$ and for a.e. $x$,

$$f(\theta_f) = g \circ g' \left( \frac{x}{t - t_{-}(x, t)} \right).$$

(vii) $s \mapsto y_{+0}(s)$ is non decreasing function.
(viii) $u(0+, t)$, $u(0-, t)$ exist and RH condition holds, i.e., $f(u(0+, t)) = g(u(0-, t))$ for a.e. $t > 0$.

Interface entropy condition: Let $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure, then

$$\mathcal{L}^1 \{ t : f'(u(0+, t)) > 0, g'(u(0-, t)) < 0 \} = 0. \quad (2.1)$$

(ix) The entropy solution $u$ is explicitly given by the following Lax-Oleinik type formula, for $t > 0$, a.e., $x \in \mathbb{R}$,

$$u(x, t) = \begin{cases} 
  f' \left( \frac{x - y(x,t)}{t} \right) & \text{if } x > R_1(t), \\
  f' \left( \frac{x}{t - t_{+}(x,t)} \right) & \text{if } 0 < x < R_1(t), \\
  g' \left( \frac{x - y(x,t)}{t} \right) & \text{if } x < L_1(t), \\
  g' \left( \frac{x}{t - t_{-}(x,t)} \right) & \text{if } L_1(t) < x < 0.
\end{cases} \quad (2.2)$$

(x) For a.e., $x \in (0, R_2(T))$,

$$\frac{x}{t - t_{+}(x,t)} = f'(u(x,t)) = f'(u(0+, t_{+}(x,t))), g'(u(0-, t_{+}(x,t))) = -\frac{y_{-0}(t_{+}(x,t))}{t_{+}(x,t)}.$$

(xi) For a.e., $x \in (L_2(T), 0)$,

$$\frac{x}{t - t_{-}(x,t)} = g'(u(x,t)) = g'(u(0-, t_{-}(x,t))), f'(u(0+, t_{-}(x,t))) = -\frac{y_{+0}(t_{-}(x,t))}{t_{-}(x,t)}.$$

(xii) $L^1$- Contractivity: Let $u_0$, $v_0 \in L^\infty(\mathbb{R})$ and $u, v$ be the solution of (1.1) with corresponding initial data $u_0$, $v_0$ respectively. Assume that the set of discontinuity of $u$ and $v$ are discrete set of Lipschitz curves. Then

$$\int_a^b |u(x, t) - v(x, t)| dx \leq \int_{a-Mt}^{b+Mt} |u_0(x) - v_0(x)| dx,$$

where with $M_1 = \max(||u_0||_\infty, ||v_0||_\infty)$

$$M = \max \left\{ \frac{|f(a) - f(b)|}{|a - b|}, : a \neq b, a, b \in (-M_1, M_1) \right\}.$$

We remark that there is a change in the notation used here and in [5] and is as follows:

See equations (4.13), (4.10), page 51 in [5]:

$$y(x, t) := \begin{cases} 
  y_{+}(x, t) & \text{if } x > R_1(t), \\
  y_{-}(x, t) & \text{if } x < L_1(t). \end{cases} \quad (2.3)$$

$$t_{+}(x, t) := y_{+}(x, t) \text{ if } 0 < x < R_1(t),$$

$$t_{-}(x, t) := y_{-}(x, t) \text{ if } L_1(t) < x < 0. \quad (2.5)$$
See equation (4.25), page 54 in [5]

\[
y_{-0} \left( t - \frac{x}{q_1} \right) := y_{-} \left( 0, t - \frac{x}{q_1} \right) \quad \text{if } 0 < x < R_2(t), \quad (2.6)
\]

\[
y_{+0} \left( t - \frac{x}{q_1} \right) := y_{+} \left( 0, t - \frac{x}{q_1} \right) \quad \text{if } L_2(t) < x < 0. \quad (2.7)
\]

Now comes to the identification of (i) to (xii) in [5]. (i) follows from (i) of [5, Lemma 4.9, page 51], (ii), (iv), (v), (vii) and (viii) to follows from the non intersecting proved in [5, Lemma 4.8, 4.9, page 50 and page 51]. (iii) follows from (4.20) to (4.25) of page 53, (4.26) and last 4 lines of page 54 and first 3 lines in [5, Lemma 4.10 in page 55]. (ix) follows from [5, Lemma 4.10 in page 55]. (x) follows from [5, Theorem 3.2]. (x) and (xii) follows from (4.10) page 55 and (ix), (xii) in [5, Theorem 2.2, page 30].

3. Some technical lemmas

First observe that \( \eta \in \text{ch}(x,t) \), then \( \eta \) is a curve consists of atmost three line segments and denote \( \dot{\eta}(\theta) = (p_1, p_2, p_3) \), where \( p_i \) is the slope of \( i \)th line segment. By abuse of notations we denote \( p_i = \emptyset \) if the \( i \)th line segment does not exist. Note that if \( 0 < x < R_2(t) \) or \( L_2(t) < x < 0 \), then for any \( \eta \in \text{ch}(x,t) \), \( \dot{\eta}(\theta) = (p_1, \emptyset, p_3) \), \( p_1 > 0, p_3 > 0 \) if \( x > 0 \) and \( p_1 < 0, p_3 < 0 \) if \( x < 0 \).

**Definition 3.1.** For \( 0 < x < R_2(t) \) or \( L_2(t) < x < 0 \), define

\[
\text{ch}_+(x,t) = \{ p_1 : \exists \eta \in \text{ch}(x,t) \text{ such that } \dot{\eta}(\theta) = (p_1, \emptyset, p_3) \text{ for some } p_3 \}.
\]

\[
\text{ch}_-(x,t) = \{ p_3 : \exists \eta \in \text{ch}(x,t) \text{ such that } \dot{\eta}(\theta) = (p_1, \emptyset, p_3) \text{ for some } p_1 \}.
\]

**Lemma 3.1.** Let \( 0 < x_1 < x_2 < R_2(t) \) or \( L_2(t) < x_1 < x_2 < 0 \), \( p_1 \in \text{ch}_+(x_1,t), q_1 \in \text{ch}_+(x_2,t), p_3 \in \text{ch}_-(x_1,t), q_3 \in \text{ch}_-(x_2,t) \), then

\[
\begin{cases}
    \frac{x_1}{p_1} \leq \frac{x_2}{q_1} & \text{if } x_1 > 0, \\
    \frac{x_2}{q_1} \leq \frac{x_1}{p_1} & \text{if } x_2 < 0,
\end{cases}
\]

\[
- p_3 \left( t - \frac{x_1}{p_1} \right) \leq - q_3 \left( t - \frac{x_2}{q_1} \right).
\]

**Proof.** Let \( 0 < x_1 < x_2 < R_2(t) \) and

\[
\gamma_1(\theta) = \begin{cases}
    x_1 + p_1(\theta - t) & \text{if } t - \frac{x_1}{p_1} \leq \theta < t, \\
    p_3(\theta - t + \frac{x_1}{p_1}) & \text{if } 0 \leq \theta \leq t - \frac{x_1}{p_1}.
\end{cases}
\]

\[
\gamma_2(\theta) = \begin{cases}
    x_2 + q_1(\theta - t) & \text{if } t - \frac{x_2}{p_2} \leq \theta < t, \\
    q_3(\theta - t + \frac{x_2}{q_1}) & \text{if } 0 \leq \theta \leq t - \frac{x_2}{p_2}.
\end{cases}
\]

Then by dynamic programming principle, \( \gamma_1(\theta) \in \text{ch}(x_1,t), \gamma_2(\theta) \in \text{ch}(x_2,t) \). Hence from (1) of theorem 2.1 we have \( \gamma_1 \) and \( \gamma_2 \) do not intersect in \( x \neq 0 \). Hence if \( \theta_1, \theta_2 \) be such that \( \gamma_i(\theta_i) = 0 \), then for \( x_1 > 0, \theta_2 \leq \theta_1 \) and \( \gamma_1(0) \leq \gamma_2(0) \). That is

\[
\begin{cases}
    0 \leq t - \frac{x_2}{q_1} \leq t - \frac{x_1}{p_1}, \\
    - p_3 \left( t - \frac{x_1}{p_1} \right) \leq - q_3 \left( t - \frac{x_2}{q_1} \right).
\end{cases}
\]

If \( x_2 < 0 \), then \( \theta_1 \leq \theta_2 \) and \( \gamma_1(0) \leq \gamma_2(0) \). This prove the lemma. \( \square \)
3.1. No rarefaction from the interface

One of the key factors of this article is that there exists no rarefaction from the interface for the solution of (1.1). This is useful for backward construction (section 4).

**Definition 3.2** (Forward rarefaction from the interface). *We say that the solution \( u \) admits a forward rarefaction on the interface if \( \exists 0 < x_1 < x_2 < R_2(t) \) or \( L_2(t) < x_1 < x_2 < 0 \), \( t_0 \in (0, t) \) and \( p_1 \in ch_+(x_1, t), q_1 \in ch_+(x_2, t) \) such that \( t_0 = t - x_1/p_1 = t - x_2/q_1 \).

**Lemma 3.2.** There does not exist forward rarefaction from the interface (see figure 3).

**Proof.** Suppose not, without loss of generality we can assume that there exist \( 0 < x_1 < x_2 < R_2(t) \), \( p_1 \in ch_+(x_1, t), q_1 \in ch_+(x_2, t) \), \( t_0 \in (0, t) \) such that \( t_0 = t - x_1/p_1 = t - x_2/q_1 \). Therefore from (1) of theorem 2.1, if \( x_1 < x < x_2, \gamma \in ch(x, t) \), then \( \exists p_1(x) \in ch_+(x, t), p_3(x) \in ch_-(x, t) \) with

\[
\gamma(\theta) = \begin{cases} 
 x + p_1(x)(\theta - t) & \text{if } t - \frac{x}{p_1(x)} \leq \theta < t, \\
 p_3(x)\left(\theta - t + \frac{x}{p_1(x)}\right) & \text{if } 0 \leq x < t - \frac{x}{p_1(x)}, 
\end{cases}
\]

and \( t_0 = t - x/p_1(x) \). Hence for each \( x_1 < x < x_2 \), \( p_1(x) \) is unique.

Let \( u_{0,k} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) be satisfying \( u_{0,k} \to u_0 \) in \( L_{loc}^1(\mathbb{R}) \) as \( k \to \infty \). Let \( x_1 < x < y < x_2 \), then from (3) of theorem 2.1, for large \( k \in \mathbb{N} \), there exist \( \gamma_k \in ch_k(x, t) \), \( \gamma_k(\theta) = (p_{1,k}, \emptyset, p_{3,k}), \eta_k \in ch_k(y, t), \eta_k(\theta) = (q_{1,k}, \emptyset, q_{3,k}) \) such that \( \lim_{k \to \infty} ((p_{1,k}, \emptyset, p_{3,k}), (q_{1,k}, \emptyset, q_{3,k})) = ((\tilde{p}_1, \emptyset, \tilde{p}_3), (\tilde{q}_1, \emptyset, \tilde{q}_3)) \). From lemma 2.1 and by the uniqueness of \( p_1(x), p_1(y) \), it follows that \( \tilde{p}_1 = p_1(x), \tilde{q}_1 = p_1(y) \). Since \( v_{0,k} \in C^1(\mathbb{R}) \), hence by minimizing property, we have

\[
\frac{\partial}{\partial p_3} \Gamma_{v_{0,k}, \lambda}(x, t)|_{\lambda = \gamma} = 0, 
\]

(3.1)

\[
\frac{\partial}{\partial p_1} \Gamma_{v_{0,k}, \lambda}(x, t)|_{\lambda = \gamma} = 0, 
\]

(3.2)

\[
g(g'(p_{3,k})) = f(f'(p_{1,k})), 
\]

(3.3)

\[
g(g'(q_{3,k})) = f(f'(q_{1,k})), 
\]

(3.4)

and from lemma 3.1, we have

\[
-p_{3,k}\left(\frac{t - x}{p_1(x)}\right) \leq -q_{3,k}\left(\frac{t - y}{q_1(x)}\right). 
\]
Letting $k \to \infty$ to obtain
\[ g(g'(p_3)) = f(f'(p_1(x))), \]
\[ g(g'(q_3)) = f(f'(p_1(y))), \] (3.5)

hence
\[ -p_3 \left( t - \frac{x}{p_1(x)} \right) \leq -q_3 \left( t - \frac{y}{p_1(y)} \right). \]

Since $t - \frac{x}{p_1(x)} = t - \frac{y}{p_1(y)} = t_0$, we have $q_3 \leq p_3$. Due to $q_3 \geq 0$, we obtain $0 \leq q_3 \leq p_3$ and $\theta_g \leq g'(q_3), \theta_f \leq f'(p_3)$. As $g$ is an increasing function on $(\theta_g, \infty)$, we get $g(g'(p_3)) \geq g(g'(q_3))$. This implies that $f(f'(p_1(x)))) \geq f(f'(p_1(y))))$. Because of the fact that $p_1(x) \geq 0, q_1(y) \geq 0$ and $f$ is an increasing function on $(\theta_f, \infty)$, it follows that $p_1(x) \geq p_1(y)$. Therefore we have
\[ \frac{x}{p_1(x)} < \frac{y}{p_1(x)} \leq \frac{y}{p_1(y)} = \frac{x}{p_1(x)}, \]
which is contradiction. This proves the Lemma.

As an immediate consequence of this lemma, we have the following:

**Lemma 3.3.** Let $u, R_2(T), L_2(T), t_+, t_-$ be as in theorem 2.1. Then for all $t > 0$, $x \mapsto t_+(x,t), x \in (0, R_2(t))$ is a strictly decreasing function and $x \mapsto t_-(x,t), x \in (0, L_2(t))$ is strictly increasing function.

**Proof.** We prove this for $t_+(x,t)$ and similarly proof holds for $t_-(x,t)$. Suppose $x \mapsto t_+(x,t)$ is not strictly decreasing function, then there exist $0 < x_1 < x_2 < R_2(t)$ and a $t_0 \in (t_+(R_2(t)−, t), t)$ such that $t_0 = t_+(x_1,t) = t_+(x_2,t)$. Then there exist $p_1$ and $q_1$ such that $t_+(x_1,t) = t - \frac{x_1}{p_1}, t_+(x_2,t) = t - \frac{x_2}{p_2}$. Hence $u$ admits a forward rarefaction from the interface and from lemma 3.2 we get a contradiction. This proves the lemma.

**Definition 3.3.** Let \( I_+ = [f'(\theta_g), \infty), g_+ = g|_{[\theta_g, \infty)}, \) then define \( h_+ : I_+ \to [0, \infty) \) by
\[ h_+(p) = g' \circ g_+^{-1} \circ f \circ (f')^{-1}(p). \]

**Lemma 3.4.** Let $T > 0$ and denote $t_+(x,T) = t_+(x)$. Then For a.e., $x \in (0, R_2(T))$, \[-\frac{y_0(t_+(x))}{t_+(x)} = h_+ \left( \frac{x}{T - t_+(x)} \right). \]

**Proof.** Let $R_2(T) > 0$ and $u_0 \in C_0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $x \in (0, R_2(T))$, then from non-intersecting of characteristics, $L_1(T) = 0$. Let $\eta \in ch(x(T), \), $\eta = (q_1, 0, q_3)$, then we have \[ 0 = \frac{\partial}{\partial q_3} \Gamma_{v_0, \eta}(x,T) = \frac{\partial}{\partial q_1} \Gamma_{v_0, \eta}(x,T). \]

This implies that
\[ v_0'(-q_3(x - \frac{T}{q_1})) = g'(q_3) \quad \text{and} \quad v_0'(-q_3(x - \frac{T}{q_1})) = q_1 f'(q_1) - f'(q_1) + g'(q_3). \]

As $f(f'(q)) = q f'(q) - \frac{T}{q_1}$, therefore we have from the above identities $f(f'(q_1)) = g'(q_3)$. Due to $q_1 \geq 0, q_3 \geq 0$, we get $q_3 = h_+(q_1)$. Since $h_+$ is an increasing function, hence if $ch_+(x,T) = \{q_1\}$, then $\{\eta\} = ch(x(T)$ and $q_1 = \frac{T}{T - t_+(x)}, q_3 = \frac{y_0(t_+(x))}{t_+(x)}$ and
\[ -\frac{y_0(t_+(x))}{t_+(x)} = h_+ \left( \frac{x}{T - t_+(x)} \right). \] (3.6)

Let $u_0 \in L^\infty(\mathbb{R})$ and $u_{0,k} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $u_{0,k} \to u_0$ in $L^1_{loc}(\mathbb{R})$ as $k \to \infty$. Then from lemma 2.1, $\lim_{k \to \infty} ch_k(x,T) \subset ch(x,T)$. 

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Let $u_0 \in L^\infty(\mathbb{R})$ and $u_{0,k} \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $u_{0,k} \to u_0$ in $L^1_{\text{loc}}(\mathbb{R})$ as $k \to \infty$. Then from lemma 2.1, $R_{2,k}(T) \to R_2(T)$ and $ch_k(x,t) \to ch(x,t)$ as $k \to \infty$. From lemma 3.3, for $x \in (0, R_2(T))$, $x \mapsto t_+(x)$ is a strictly decreasing function. Hence from (3)-(iv) of theorem 2.1, $x \mapsto y_{-0}(t_+(x))$ is a non decreasing function. Therefore there exists a countable set $N \subset (0, R_2(T))$ such that for $x \notin N$, $ch(x,T)$ is a singleton set. Therefore from (3) of lemma 2.1, for $x \notin N$, $\lim_{k \to \infty} (t_{+,k}(x), y_{-,k}(t_{+,k}(x))) = (t_+, y_{-0}(t_+(x)))$. Hence from (3.6) for $x \notin N, x \in (0, R_2(T))$

$$-y_{-0}(t_+(x)) \over t_+(x) = h_+ \left( {x \over T - t_+} \right).$$

This proves the lemma. \hfill \qed

**Lemma 3.5.** Let $[\alpha, \beta] \subset [0, \infty)$ and $\rho : [\alpha, \beta] \to (-\infty, 0]$ be a non decreasing function such that if $\rho(x) = 0$, then $T - \frac{f'(\theta_g)}{\alpha} \geq 0$.

1. Then there exists a strictly decreasing function $t : (\alpha, \beta] \to [0, T]$ such that for all $x \in (\alpha, \beta)$

   (i) $\frac{x}{T - t_+(x)} \in I_+$.

   (ii) $-\frac{\rho(x)}{t_+(x)} = h_+ \left( {x \over T - t_+(x)} \right)$.

   (iii) If $\rho$ is continuous, then $t_+(x)$ is continuous.

   (iv) If $\alpha > 0$, then $\lim_{x \to \alpha} t_+(x) = t(\alpha)$ exist and satisfies $\frac{\rho(\alpha)}{t(\alpha)} = h_+ \left( {\alpha \over T - t(\alpha)} \right)$.

   (v) If $\alpha = 0$, then $t(0+) = T$ and $\lim_{x \to 0} \frac{x}{T - t_+(x)} = f'(p_0)$ exist and satisfies $-\frac{\rho(0)}{T} = h_+ (f'(p_0))$.

2. Let $x_0 > 0, 0 < t_2 < t_1 < T$ such that $\frac{x_0}{T - t_i} \in I_+$ for $i = 1, 2$. Define for $i = 1, 2$,

$$f'(a_i) = \frac{x_0}{T - t_i}, \quad \frac{-\rho_i}{t_i} = h_+ \left( \frac{x_0}{T - t_i} \right).$$

Suppose $\rho_1 < \rho_2 \leq 0$, then there exist $t_3 \in (t_2, t_1)$, $\rho_3 \in (\rho_1, \rho_2)$, $b_1, b_2$ with $a_2 < a_1, b_1 < b_2$ such that

$$g(b_1) = f(a_1), g' \geq 0, \quad (3.9)$$

$$-x_0 = (T - t_3) \left( {f(a_1) - f(a_2) \over a_1 - a_2} \right), \quad (3.10)$$

$$-x_0 = t_3 \left( {g(b_1) - g(b_2) \over b_1 - b_2} \right). \quad (3.11)$$

**Proof.** Now $\frac{x}{T - t} \in I_+$ if and only if $t \geq T - \frac{x}{f'(\theta_g)}$. For fixed $x \in (\alpha, \beta]$ define $F(t) := h_+ \left( \frac{x}{T - t} + {\rho(x) \over t} \right)$ for $t \in \left[ \max \left\{ 0, T - \frac{x}{f'(\theta_g)} \right\}, T \right]$. If $\rho(x) = 0$, then by the hypothesis, $T - \frac{x}{f'(\theta_g)} \geq 0$. Hence for $t_+(x) = T - \frac{x}{f'(\theta_g)}$, $F(t_+(x)) = 0$. Let us consider the case $\rho(x) \neq 0$. If $T - \frac{x}{f'(\theta_g)} \leq 0$, then take $t_0 = 0$ to obtain $F(0) = -\infty$. If $T - \frac{x}{f'(\theta_g)} > 0$, then take $t_0 = T - \frac{x}{f'(\theta_g)}$ to obtain $F(t_0) < 0$.  

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As $x \neq 0$, we take $t = T$ to obtain $F(T) = \infty$. Since $t \mapsto F(t)$ is continuous, hence there exists a $t_+(x) \in \left(\max \left\{ 0, T - \frac{x}{f'(\theta_g)} \right\}, T \right)$ such that $F(t_+(x)) = 0$.

Let $\alpha < x_1 < x_2 < \beta$ and suppose $t(x_1) \leq t(x_2)$. Then $\frac{x_1}{T - t(x_1)} \leq \frac{x_1}{T - t(x_2)} < \frac{x_2}{T - t(x_2)}$. Subsequently, we have

$$-\rho(x_1) = t(x_1)h_+\left(\frac{x_1}{T - t(x_1)}\right) < t(x_1)h_+\left(\frac{x_2}{T - t(x_2)}\right) \leq t(x_2)h_+\left(\frac{x_2}{T - t(x_2)}\right) = -\rho(x_2),$$

contradicting the non-decreasing hypothesis on $\rho$. This proves (i) and (ii). If $\alpha > 0$, then $T - \frac{\alpha}{f'(\theta_g)} < T$ and hence (i) and (ii) together imply (iv). Let $\alpha = 0$ and $t_0 = \lim_{x \to 0} t(x)$. Suppose $t_0 < T$, then

$$\frac{x}{T - t_+(x)} \to 0 \text{ as } x \to 0. \text{ As } -\frac{\rho(0)}{t_0} = h_+(0), \text{ we have } 0 \in I_+. \text{ Therefore, } f'(\theta_g) = 0 \text{ and } h_+(0) = 0. \text{ This implies } \rho(0) = 0. \text{ Due to the fact that } \rho \text{ is a non-decreasing function we obtain for all } x \in [0, \beta], 0 = \rho(0) \leq \rho(x) \leq 0 \text{ and therefore } \rho(x) = 0 \text{ for all } x \in [0, \beta]. \text{ But for } x \neq 0, x \in (0, \beta), -\infty = T - \frac{x}{f'(\theta_g)} \geq 0,$

which is a contradiction. Hence $t_0 = T$ and $\frac{x}{T - t_+(x)} = p(x)$ is bounded. Let $p_0 = \lim_{x \to 0} p(x)$, then

$$-\frac{\rho(0)}{T} = h_+(p_0).$$

This proves (v).

Proof of (2): As $t_2 < t_1$, we have $\frac{x_0}{T - t_2} < \frac{x_0}{T - t_1}$. Thus, $a_2 < a_1$ and $b_2 < b_1$. By the choice of $a_1$ and $a_2$, $x_0$ satisfies $x_0 + (t_i - T)f'(a_i) = 0$, for $i = 1, 2$. Since $f$ is convex, we get $f'(a_2) < f'(a_1)$, hence the line $r(\theta) = x_0 + (\theta - T)f'(a_2) - f'(a_1)$ meets the $t$-axis at $t_3 \in (t_2, t_1)$, that is $r(t_3) = 0$. This proves (34). Again from the convexity of $g$, $g'(b_2) < \frac{g(b_2) - g(b_1)}{b_2 - b_1} < g'(b_1)$ and thus $-t_3g'(b_1) < -t_3\frac{g(b_2) - g(b_1)}{b_2 - b_1} < -t_3g'(b_2)$. From (31), we have, $-\rho_i = t_ih_+(f'(a_i)) = t_ig'(b_i)$ and $t_3 \in (t_2, t_1)$ implies that $\rho_3 = -t_3\left(\frac{g(b_2) - g(b_1)}{b_2 - b_1}\right) \in (\rho_1, \rho_2)$. This proves the lemma.

3.2. Building blocks: Construction of shock solution and continuous solution

**Lemma 3.6.** (Shock solution) Let $T > 0$, $x_0 > 0$, $\rho_1 < \rho_2 \leq 0$. Assume that for $t \in [0, T], \frac{x_0}{T - t} \geq f'(\theta_g)$ and if $\rho_2 = 0$, then $T - \frac{x_0}{f'(\theta_g)} = 0$. Let $a_1, a_2, b_1, b_2, t_1, t_2, t_3$ and $\rho_3$ be as in lemma 3.5. Define

$$u_0(x) = \begin{cases} b_1 & \text{if } x < \rho_3, \\ b_2 & \text{if } \rho_3 < x < 0, \\ a_2 & \text{if } x > 0, \end{cases}$$

(3.12)

then the solution $u$ of (1.1) in $\mathbb{R} \times [0, T]$ with initial data $u_0$ is given by (see figure 4)

$$u(x, t) = \begin{cases} b_1 & \text{if } x < 0, x < \rho_3 + \frac{g(b_1) - g(b_2)}{b_1 - b_2}t, \\ b_2 & \text{if } x < 0, x > \rho_3 + \frac{g(b_1) - g(b_2)}{b_1 - b_2}t, \\ a_1 & \text{if } x > 0, x < f'(a_1) - f'(a_2)(t - t_3), \\ a_2 & \text{if } x > 0, x > \frac{f'(a_1) - f'(a_2)}{a_1 - a_2}(t - t_3). \end{cases}$$

(3.13)
Proof. From lemma 3.5, \( \rho_3 = -t_3 \left( \frac{g(b_1) - g(b_2)}{b_1 - b_2} \right) \) and \( f(a_i) = g(b_i) \), hence \( u \) is a weak solution satisfying the interior and interface entropy condition with initial data \( u_0 \). This proves the lemma.

Remark 3.1. Suppose \( x_0 = 0 \), then by (1)-(v) of lemma 3.5, we have \( f'(p_0) = \lim_{x \to 0} \frac{x}{T - t_+(x)} \), hence \( f'(p_0) \in I_+ \) implies that \( p_0 \geq \theta_g \). Let \( q_0 \geq \theta_g \) such that \( f(p_0) = g(q_0) \). Hence

\[
-\frac{\rho(0)}{T} = h_+ \left( f'(p_0) \right) = g'(q_0).
\]

Define

\[
u_0(x) = \begin{cases} 
q_0 & \text{if } x \leq 0, \\
p_0 & \text{if } x \geq 0,
\end{cases}
\]

then \( u(x,t) = u_0 \) is the solution of (1.1), (1.3).

In lemma 3.6, under suitable hypothesis on \( x_0, \rho_i, i = 1, 2 \), we constructed a solution which admits shocks. Next we consider the case where \( 0 \leq x_1 < x_2 \) and \( \rho_0 < 0 \). Under a suitable hypothesis, we construct a continuous solution to (1.1).

Let \( T > 0, 0 \leq x_1 < x_2, \rho_0 < 0 \). From lemma 3.6, let \( 0 < t_i < T, i = 1, 2 \), be such that

\[
h_+ \left( \frac{x_i}{T - t_i} \right) = -\frac{\rho_0}{t_i}.
\]

Let \( f'(a_i) = \frac{x_i}{T - t_i}, f(a_i) = g(b_i), g'(b_i) > 0 \). Again from lemma 3.6, let \( t_+(x) : [x_1, x_2] \to [t_2, t_1] \) be the unique continuous strictly decreasing function satisfying

\[
h_+ \left( \frac{x}{T - t_+(x)} \right) = -\frac{\rho_0}{t_+(x)}, \quad x \in [x_1, x_2].
\]

By the uniqueness of \( t_+(x), t_+(x_i) = t_i \) and \( t_+(\cdot) \) is a homeomorphism.

For \( i = 1, 2 \), let

\[
\eta_i(t) = -\frac{\rho_0}{t_i}(t - t_i), \quad \gamma_i(t) = f'(a_i)(t - t_i), \quad u_0(x) = \begin{cases} 
b_1 & \text{if } x < \rho_0, \\
b_2 & \text{if } \rho_0 < x < 0, \\
a_2 & \text{if } x > 0.
\end{cases}
\]
For $x \geq 0$, let $t(x, t)$ be the unique solution of
\[
\frac{x}{t - t_+(x, t)} = -\frac{\rho_0}{t_+(x, t)}.
\]
Then we have following:

**Lemma 3.7.** (Continuous solution) (See figure 5) Let $\rho_0 < 0$, $0 < t_2 < t_1$, $a_i, b_i, \eta_i, \gamma_i$, $i = 1, 2$ be as above. Let $u(x, t)$ be the solution of (1.1) with initial data $u_0$ as above. Then
\[
\begin{aligned}
\quad
u(x, t) = \begin{cases}
    b_1 & \text{if } x < \min\{\eta_1(t), 0\}, \\
    (g')^{-1}\left(\frac{x - \rho_0}{t}\right) & \text{if } \min\{\eta_1(t), 0\} < x < \min\{\eta_2(t), 0\}, \\
    b_2 & \text{if } \eta_2(t) < x < 0, \\
    a_2 & \text{if } \max\{0, \gamma_2(t)\} < x, \\
    (f')^{-1}\left(\frac{x}{t - t_+(x, t)}\right) & \text{if } \max\{\gamma_1(t), 0\} < x < \gamma_2(t), \\
    a_1 & \text{if } 0 < x < \gamma_1(t).
\end{cases}
\end{aligned}
\]

**Proof.** Define the regions in $\mathbb{R} \times (0, T)$ by
\[
\begin{aligned}
\quad
\Omega_1 &= \{(x, t) : 0 < x < \gamma_1(t)\}, \\
\Omega_2 &= \{(x, t) : \max\{0, \gamma_1(t)\} < x < \gamma_2(t)\}, \\
\Omega_3 &= \{(x, t) : \max\{0, \gamma_2(t)\} < x\}.
\end{aligned}
\]
Let $x > 0$, $0 < t \leq T$ and $w \in c_b(x, t) \cap ch(x, t)$. Then $w = (w_1, \emptyset, w_3)$ is given by
\[
\begin{aligned}
\quad
w_1(\theta) &= x + \frac{x}{t - \tau}(\theta - t), \ \tau \leq \theta \leq t, \\
\quad
w_3(\theta) &= -\frac{w_3(0)}{\tau}(\theta - \tau),
\end{aligned}
\]
where $\tau > 0$ satisfies $w_3(\tau) = w_1(\tau) = 0$. Since $w \in ch(x, t)$, hence $\frac{\partial}{\partial \tau}\Gamma_{w_0,w}(x, t) = 0$. That is
\[
0 = -f^*\left(\frac{x}{t - \tau}\right) + \left(\frac{x}{t - \tau}\right)(f^*)'(\frac{x}{t - \tau}) + g^*\left(\frac{-w_3(0)}{\tau}\right) + \frac{w_3(0)}{\tau}(g^*)'\left(\frac{-w_3(0)}{\tau}\right).
\]
Let $f'(p_1) = \frac{x}{t - \tau}$, $g'(q_1) = -\frac{w_3(0)}{\tau}$, then from the convexity of $f$ and $g$ and the above relation gives
\[
f\left((f^*)'\left(\frac{x}{t - \tau}\right)\right) = g\left((g^*)'\left(\frac{-w_3(0)}{\tau}\right)\right).
\]
That is $f(p_1) = g(q_1)$. Observe that $(x, t) \in \Omega_3$ if and only if $x > \gamma_2(t) = f'(a_2)(t - t_2).$ If $t - t_2 \leq 0$, then $\tau < t \leq t_2$. Now $\tilde{\gamma}(t) = g'(b_2)(t - \tau)$ is the only characteristic of $u$ in $x < 0$ and $\tilde{\gamma}(\tau) = 0$. Since $w_3$ is a characteristic in $x < 0$ and $w_3(\tau) = 0$, hence $w_3(t) = g'(b_2)(t - \tau)$. This implies $\rho_0 < w_3(0) < 0$. If $t > t_2$, then $\frac{x}{t - t_2} > f'(a_2)$. Suppose $\tau > t_2$, then $\frac{x}{t - t_2} < \frac{x}{t - \tau}$ and we get

$$-\frac{w_3(0)}{\tau} = h_+ \left( \frac{x}{t - t_2} \right) > h_+ \left( \frac{x}{t - t_2} \right).$$

As $y \mapsto t_+(y, t)$ is an increasing continuous function, constant on the line $\gamma_2(t)$ and hence for $x > \gamma_2(t)$, it follows that $t_+(x, t) < t_2$. Due to $t > t_2$, we therefore have

$$-\frac{w_3(0)}{\tau} > h_+ \left( \frac{x}{t - t_2} \right) > h_+ \left( \frac{x}{t - t_2} \right) = -\frac{\rho_0}{\tau} + h_+ \left( \frac{x}{t - t_2} \right) > -\frac{\rho_0}{\tau} - \frac{\rho_0}{\tau}$$

and therefore $w_3(0) < \rho_0$. Since no two characteristic intersects, hence $\tau > t_1$ and $-\frac{w_3(0)}{\tau} = g'(b_1)$. Note that $x > \gamma_2(t) = f'(a_2)(t - t_2)$, and subsequently, we obtain

$$-\frac{w_3(0)}{\tau} = g'(b_1) > h_+ \left( \frac{x}{t - t_2} \right) > h_+ \left( \frac{x}{f'(a_2)} \right) = g'(b_2)$$

and therefore $b_1 > b_2$. But $b_1 < b_2$ which is a contradiction. Hence $\tau < t_2$ and from the non intersecting of characteristics, it follows that $\rho_0 < w_3(0) < 0$ and $w_3(0) = -g'(b_2)\tau$. Conversely if, $w = (w_1, 0, w_3) \in ch(x, t)$ and $\rho_0 < w_3(0) < 0$ then $x > \gamma_2(t)$. For let $w_3(\tau) = 0$, then $w_3(0) = -g'(b_2)\tau$ and $-\frac{w_3(0)}{\tau} = g'(b_2) = h_+ \left( \frac{x}{t - \tau} \right)$, implies that $\frac{x}{t - \tau} = f'(a_2)$. Thus $(x, t) \in \Omega_3$ and $u(x, t) = a_2$. Similarly $(x, t) \in \Omega_1$ if and only if $\forall w \in ch(x, t)$ with $w = (w_1, 0, w_3), w_3(0) < \rho_0, u(x, t) = a_1$.

As a consequence of this if $\max \{\gamma_1(t), 0\} < x < \gamma_2(t)$ and $w = (w_1, 0, w_3) \in ch(x, t)$. Then $w_3(0) = \rho_0$ and $-\frac{\rho_0}{\tau} = h_+ \left( \frac{x}{t - \tau} \right), \text{ where } w_3(\tau) = 0$. Hence $\tau = t_+(x, t)$ and $t_2 < t_+(x, t) < t_1$. Since $x \mapsto t_+(x, t)$ is an increasing function, hence at the point of differentiability of $t(. , t)$, we have $u(x, t) = \frac{\partial}{\partial x} \Gamma_v(w)(x, t) = (f')^{-1} \left( \frac{x}{t - t_+(x, t)} \right)$. This proves the lemma.

\[\square\]

Previous construction (as in lemma 3.6 and 3.7) of solutions corresponds to the case when no reflected characteristic occurs. Now we deal with the case when reflected characteristics occur and the definition of $(\tau_0, \xi_0)$ (see subsection 1.1) which is needed to define the reachable set $\mathcal{R}(T)$. We do it in the region $x > 0$. Similar construction follows for $x < 0$.

Let $\bar{\alpha} \leq \theta_f \leq \bar{\alpha}$, such that

(i) $f(\bar{\alpha}) = f(\bar{\alpha})$.

(ii) $D = \{(R, T): T \geq 0, 0 \leq R < f'(\bar{\alpha})T\}$ and for $(R_1, T) \in D$, define $0 \leq T_1 < T$ by $f'(\bar{\alpha}) = \frac{R_1}{T - T_1}$.

(iii) $\xi_1 = -f'(\bar{\alpha})T_1$.

For $0 \leq \xi \leq \xi_1$, define $\beta = \beta(\xi), \tau = \tau(\xi)$ by,

(iv) $\xi = -f'(\bar{\alpha})\tau$.

(v) $f'(\beta) = \frac{R_1 - \xi}{T}$. 

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In fact $f'(\bar{\alpha})$ and $f'(\beta)$ are the inverse of the slopes of line joining between $(0, \tau)$, $(\xi, 0)$ and $(R_1, T)$, $(\xi, 0)$ respectively. Since $0 \leq \xi \leq \xi_1$, we have
\[
\begin{cases}
\bar{\alpha} \leq \beta \leq \alpha, \\
\beta = \bar{\alpha} \text{ if and only if } T_1 = 0, \xi_1 = 0.
\end{cases}
\]
For $u \in \mathbb{R}$, define
\[
L_u(t) = \xi + f'(u)t, \\
\Omega(\xi) = \{(x, t) : t > 0, L_\alpha(t) < x < L_\beta(t)\},
\]
\[
v(x, t) = (f')^{-1}\left(\frac{x - \xi}{t}\right) \text{ for } (x, t) \in \Omega(\xi),
\]
the rarefaction wave in the $\Omega(\xi)$, which satisfies the equation
\[
v_t + f(v)_x = 0 \text{ in } \Omega(\xi).
\]
Let $k \geq 1$, $\bar{\alpha} = u_0 < u_1 < \cdots < u_k = \beta$ such that
\[(vi) \mid u_{i+1} - u_i \mid \leq \frac{\beta - \bar{\alpha}}{k+1}.
\]
For $0 \leq i \leq k$, define the lines passing through $(\xi, 0)$ by
\[(vii) l_i(t) = L_{u_i}(t) = \xi + f'(u_i)t.
\]
\[(viii) v_k(x, t) = u_i \text{ if } l_{i-1}(t) \leq x < l_i(t), \quad 1 \leq i \leq k.
\]
Observe that for $l_{i-1}(t) \leq x < l_i(t)$, $u_{i-1} \leq (f')^{-1}\left(\frac{x - \xi}{t}\right) < u_i$ and hence,
\[(ix) \mid v_k(x, t) - (f')^{-1}\left(\frac{x - \xi}{t}\right) \mid \leq \mid u_i - u_{i-1} \mid \leq \frac{\beta - \bar{\alpha}}{k+1}.
\]
\[(x) \lim_{k \to \infty} v_k(x, t) = (f')^{-1}\left(\frac{x - \xi}{t}\right) \text{ uniformly in } \Omega(\xi).
\]
Then we have the following:

**Lemma 3.8.** Let $T > 0$, $\bar{\alpha} \leq \theta f \leq \alpha$, $0 \leq T_1 < T$, $\xi_1 \geq 0$ and $L_u$ as defined above. Then for all $0 \leq \xi \leq \xi_1$, there exists a $0 \leq t_0(\xi) \leq T$ and a Lipschitz curve $s_\xi : [t_0(\xi), T] \to [0, T]$ (See figure 6 for illustration) such that

1. $s_\xi(T) = R_1$ and either $s_\xi(t_0(\xi)) = L_\bar{\alpha}(t_0(\xi))$ or $s_\xi(t_0(\xi)) = 0$ and $L_\bar{\alpha}(t_0(\xi)) \leq 0$.
2. $t \mapsto s_\xi(t)$ is a non decreasing convex function with
\[
\frac{ds_\xi}{dt} = \frac{f(\bar{\alpha}) - f((f')^{-1}\left(\frac{s_\xi(t) - \xi}{t}\right))}{\bar{\alpha} - (f')^{-1}\left(\frac{s_\xi(t) - \xi}{t}\right)}.
\]
3. For $0 \leq \xi \leq \eta \leq \xi_1$, $s_\xi(t) \leq s_\eta(t)$ if $t \in [t_0(\eta), T]$. Hence $s_\xi$ is unique.
4. $\xi \mapsto t_0(\xi)$ is continuous.
5. For $(x, t) \in \Omega(\xi)$, let
\[
w_\xi(x, t) = \begin{cases}
\bar{\alpha} & \text{if } x > s_\xi(t), \\
(f')^{-1}\left(\frac{x - \xi}{t}\right) & \text{if } x < s_\xi(t).
\end{cases}
\]
Then $w_\xi$ is an entropy solution of
\[
u_t + f(u)_x = 0 \text{ in } \Omega(\xi).
\]
(6) There exists $(\xi_0, \tau_0)$ such that $\xi_0 = -f'(\bar{\alpha}) \tau_0$, $t_0(\xi_0) = \tau_0$ and $s_{\xi_0}(\tau_0) = 0$.

(7) For $(R_1, T) \in D$, denote $\xi_0 = \xi_0(R_1, T)$, $\tau_0 = \tau_0(R_1, T)$, $s_{\xi_0} = s_{\xi_0}(R_1, T)$ be as in (6). Then $(R_1, T) \rightarrow (\xi_0(R_1, T), \tau_0(R_1, T), s_{\xi_0}(R_1, T))$ is continuous with

(i) $\tau_0(f'(\bar{\alpha})T, T) = \xi_0(f'(\bar{\alpha})T, T) = 0$.

(ii) $s_{\xi_0}(f'(\bar{\alpha})T; T)(t) = f'(\bar{\alpha})t$ for $0 \leq t \leq T$.

(iii) $\tau_0(0, T) = T, \xi_0(0, T) = -f'(\bar{\alpha})\tau_0(0, T), s_{\xi_0}(0, T) = 0$.

Proof. If $T_1 = 0$, then $\xi_1 = 0$ and take $s_k(t) = L_\alpha(t)$. Let $T_1 > 0$, $0 \leq \xi \leq \xi_1$, $\beta$ and $\tau$ be as in (iv) and (v). Let $k \geq 1$ and $\bar{\alpha} = \alpha_0 < \alpha_1 < \cdots < \alpha_k = \beta$ be a discretization of $[\bar{\alpha}, \beta]$ satisfying (vi). Let $l_i$ and $v_k$ be defined as in (vii) and (viii). Define $t_p < t_1 < \cdots < t_k = T$ and $s_k$ inductively by

(a) $s_k(t_k) = s_k(T) = R_1$.

(b) $s_k$ is linear in $[t_{i-1}, t_i]$ and for $t \in (t_{i-1}, t_i)$,

$$
\frac{ds_k}{dt} = \frac{f(\bar{\alpha}) - f(u_i)}{\bar{\alpha} - u_i},
$$

$$
\quad s_k(t_{i-1}) = l_{i-1}(t_{i-1}).
$$

(c) Either $s_k(t_p) = 0$ or if $s_k(t_{p}) > 0$, then $p = 0$ and $l_0(t_p) = s_k(t_p)$.

From the convexity, we prove this by induction on $i$. As $\bar{\alpha} \leq \beta < \bar{\alpha}$, by the convexity of $f$, we have

$$
f'(\bar{\alpha}) \leq f'(\beta) < \frac{f(\bar{\alpha}) - f(\beta)}{\bar{\alpha} - \beta} < f'(\bar{\alpha}).
$$

Hence integrating from $\theta$ to $T$ to obtain

$$
R_1 + f'(\bar{\alpha})(\theta - T) \leq R_1 + \frac{f(\bar{\alpha}) - f(\beta)}{\bar{\alpha} - \beta} (\theta - T),
$$

$$
\leq R_1 + f'(\beta)(\theta - T).
$$

Now choose $t_{k-1}$ by

$$
R_1 + \frac{f(\bar{\alpha}) - f(\beta)}{\bar{\alpha} - \beta}(t_{k-1} - T) = s_k(t_{k-1}) = l_{k-1}(t_{k-1}) = \xi + f'(u_{k-1})t_{k-1}.
$$
Since $R_1 - \xi = Tf'(\beta)$ and $u_{k-1} < u_k = \beta$ to obtain
\[ t_{k-1} = T \left( \frac{f(\bar{\alpha}) - f(u_k)}{\bar{\alpha} - u_k} - f'(\beta) \right) > 0 \]
and $t_{k-1} < T$ since $u_{k-1} < u_k$. For $t \in (t_{k-1}, t_k)$, define $s_k(t) = R_1 + \frac{f(\bar{\alpha}) - f(u_k)}{\bar{\alpha} - u_k}(t - T)$. Now by induction on $i$ $(a_1), (a_2)$ and $(a_3)$ holds.
Let $v_k(x, t)$ be as in $(vii)$, then for $t \notin \{t_p, t_1, \cdots, t_k\}$,
\[ \frac{ds_k}{dt} = \frac{f(\bar{\alpha}) - f(v_k(s_k(t), t))}{\bar{\alpha} - v_k(s_k(t), t)}. \]
Hence
\[ s_k(t) = s_k(T) + \int_{t}^{T} \left( \frac{f(\bar{\alpha}) - f(v_k(s_k(\theta), \theta))}{\bar{\alpha} - v_k(s_k(\theta), \theta)} \right) d\theta. \]
From Arzela-Ascoli, we can find a subsequence still denoted by $\{s_{\xi_k}\}$ such that $s_{\xi_k} \to s_{\xi}$ uniformly and $t_p(\xi_k) \to t_0(\xi)$ as $k \to \infty$. Since $s_k$ is convex for each $k$, hence $s_{\xi}$ is convex and satisfies $(1)$ and $(2)$.
Let $0 \leq \xi \leq \eta \leq \xi_1$. Since $s_{\xi}(T) = s_{\eta}(T) = R_1$, hence if $(3)$ is not true, then there exists $a < b \leq T$ such that
\[ \begin{cases} s_\eta(t) < s_\xi(t), & \text{for all } t \in (a, b), \\ s_\eta(b) = s_\xi(b). \end{cases} \]
Now $\xi \leq \eta$ and hence $-\eta \leq -\xi$ and for $t \in (a, b)$,
\[ (f')^{-1} \left( \frac{s_\eta(t) - \eta}{t} \right) < (f')^{-1} \left( \frac{s_\xi(t) - \xi}{t} \right) \leq \bar{\alpha}. \]
Therefore by the convexity of $f$, we have for $t \in (a, b)$,
\[ \frac{ds_\eta(t)}{dt} < \frac{ds_\xi(t)}{dt}. \]
Integrating from $t$ to $b$ in $(a, b)$ to obtain
\[ s_\eta(b) - s_\eta(t) < s_\xi(b) - s_\xi(t) \]
and hence $s_\xi(t) < s_\eta(t) < s_\xi(t)$ which is a contradiction. Thus, $s_\xi(t) \leq s_\eta(t)$ for all $t \in [t_0(\xi), T]$. This also shows that $s_\xi$ satisfying $(a_1)$ and $(a_2)$ is unique. This proves $(3)$. From the uniqueness of $s_\xi$, $\xi \mapsto t_0(\xi)$ is continuous and hence $(4)$ follows. From Rankine-Hugoniot condition, $w_{\xi}$ is an entropy solution in $\Omega(\xi)$. This proves $(5)$.

Let $h(\xi) = -\frac{\xi}{f'(\bar{\alpha})} - t_0(\xi)$, then we have $h(0) = -t_0(0) \leq 0$ and $h(\xi_1) = -\frac{\xi_1}{f'(\bar{\alpha})} - t_0(\xi_1) = T_1 - t_0(\xi_1) \geq 0$. Therefore there exists $(\xi_0, \tau_0)$ with $t_0(\xi_0) = \tau_0(\xi_0) = \tau_0$ such that $L_\alpha(\tau_0) = 0$ and $\xi_0 = -f'(\bar{\alpha})\tau_0$. This proves $(6)$. From the uniqueness of $s_{\xi(0,T)}$, it follows that $(R_1, T) \to (\xi_0(0,T), \tau_0(0,T), s_{\xi_0(0,T)})$ is continuous in $D$. Suppose $\tau_0 = \tau_0(0,T) < T$, then $s_{\xi_0(0,T)}(\tau_0) = s_{\xi_0(0,T)}(T) = 0$ and $s_{\xi_0(0,T)}(T)$ is convex, hence $s_{\xi_0(0,T)}(T) \equiv 0$. Integrating $\frac{ds_\xi}{dt}$ from $\tau_0$ to $T$ to obtain with $\xi(0,T) = \xi_0$, we have
\[ 0 = s_{\xi_0}(T) - s_{\xi_0}(\tau_0) = \int_{\tau_0}^{T} f(\bar{\alpha}) - f \left( (f')^{-1} \left( -\frac{\xi_0}{\bar{\alpha}} \right) \right) dt \neq 0, \]
since $(f')^{-1} \left( -\frac{\xi_0}{\bar{\alpha}} \right) \leq (f')^{-1}(0) \leq \bar{\alpha}$ and $f$ is convex. This is a contradiction and we get $\tau_0 = T$.

If $(R_1, T) = (f'(\bar{\alpha}), T)$, then $T_1 = 0$ and hence $\tau_0(R_1, T) = 0$, $\xi_0(R_1, T) = 0$. Therefore by uniqueness, $s_{\xi_0(R_1,T)}(t) = f'(\bar{\alpha})t$ is the given solution. This proves $(7)$ and hence the lemma.

**Lemma 3.9.** Let $u_0 \in L^\infty(\mathbb{R})$ and $u$ be the corresponding solution of $(1.1)$. Let $0 \leq T_1 \leq T$ be such that
\( f'(\tilde{\theta}_g) = \frac{R_1(T)}{T - T_1} \). Let \( \bar{\alpha} = \tilde{\theta}_g \) and \( (\xi_0, \tau_0, s_{\xi_0}) \) be as in lemma 3.8 at \( (R_1(T), T) \). Then
\[ \xi_0 \leq y(R_1(T) +, T). \]

**Proof.** First assume that \( R_1(T) > 0 \). Then \( f'(\tilde{\theta}_g) > 0 \). Suppose \( T_1 = 0 \), then from lemma 3.8, \( \xi_0 = 0 \) and hence the lemma is true. Therefore we assume that \( R_1(T) > 0, T_1 > 0 \) and \( y(R_1(T) +, T) < \xi_0 \).

**Step-1:** For a.e. \( x \in (0, R_1(t)) \), and \( t \in (0, T) \), \( u(x, t) \geq \tilde{\theta}_g \).

Suppose \( R_2(t) < x < R_1(t) \), then from (iii) and (ix) of theorem 2.1, we have for a.e. \( x \),
\[ f(u(x, t)) = g(\theta_g), \]
\[ f'(u(x, t)) = \frac{x}{t - t_+(x, t)} \geq 0. \]
Hence \( u(x, t) = \tilde{\theta}_g \). Suppose \( 0 < x < R_2(t) \), let \( \gamma \in ch(x, t) \) such that \( \gamma = (\gamma_1, \phi, \gamma_3) \), \( \tilde{\gamma} = (p_1, \phi, p_3) \) and \( p_1 \geq 0, p_3 \geq 0 \). From (3.5) we have \( g(g'(p_3)) = f(f'(p_1)) \). Therefore \( f'\tilde{\gamma}(p_1) \geq \tilde{\theta}_g \). Since for a.e. \( x \in (0, R_2(T)), p_1 = \frac{x}{t - t_+(x, t)} \), hence from (ix) of theorem 2.1, we have \( u(x, t) = (f'\tilde{\gamma})\left(\frac{x}{t - t_+(x, t)}\right) \geq \tilde{\theta}_g \).

This proves **Step-1**.

**Step-2:** For all \( t \in [\tau_0, T], R_1(t) \leq s_{\xi_0}(t) \).

Suppose not, since \( R_1(T) = s_{\xi_0}(T) \), there exist \( a < b \) such that for \( t \in (a, b) \), we have
\[ s_{\xi_0}(t) < R_1(t), \quad s_{\xi_0}(b) = R_1(b). \]

From the non-intersecting of characteristics, it follows that for \( t < T \), \( y(R_1(t) +, t) \leq y(R_1(T) +, T) < \xi_0 \). Hence for \( t \in (a, b) \), we have
\[ \frac{R_1(t) - y(R_1(t) +, t)}{t} > s_{\xi_0}(t) - \xi_0. \]

From **Step-1** and convexity of \( f \), we have for a.e., \( t \in (a, b) \),
\[ \frac{dR_1}{dt} = \frac{f(u(R_1(t) -, t)) - f\left((f')^{-1}\left(\frac{R_1(t) - y(R_1(t) +, t)}{t}\right)\right)}{u(R_1(t) -, t) - (f')^{-1}\left(\frac{R_1(t) - y(R_1(t) +, t)}{t}\right)} \geq \frac{f(\tilde{\theta}_g) - f\left((f')^{-1}\left(\frac{s_{\xi_0}(t) - \xi_0}{t}\right)\right)}{\tilde{\theta}_g - (f')^{-1}\left(\frac{s_{\xi_0}(t) - \xi_0}{t}\right)} = \frac{ds_{\xi_0}}{dt}. \]

Integrating from \( t \) to \( b \) to obtain
\[ R_1(b) - R_1(t) \geq s_{\xi_0}(b) - s_{\xi_0}(t). \]
Hence for \( t \in (a, b) \), \( s_{\xi_0}(t) \geq R_1(t) > s_{\xi_0}(t) \) which is a contradiction. This proves **Step-2**. From **Step-2**, we have
\[ R_1(\tau_0) = 0, y(0+, \tau_0) \leq y(R_1(T) +, T) < \xi_0. \]

**Step-3:** There exists an \( \epsilon_0 > 0 \) such that for all \( t \in (\tau_0 - \epsilon_0, \tau_0) \), \( R_1(t) = 0 \).

Since \( \xi_0(0, \tau_0) = \xi_0, \tau_0(0, \tau_0) = \tau_0 \), hence by continuity, there exist an \( \epsilon_1 > 0 \) such that \( \forall (y, t) \in \Omega = \{(\xi, s) : \xi \geq 0, s \geq 0 \} \cap B((0, \tau_0), \epsilon_1) \), we have
\[ \xi_0 + y(R_1(T) +, T) \leq \xi_0(y, t). \]

Suppose **Step-3** is not true. Then there exists a \( \tau_0 - \epsilon_1 < \tilde{\epsilon} < \tau_0 \) such that \( (R_1(t), t) \in \Omega \) for \( \tilde{\epsilon} \leq t \leq \tau_0 \) and \( R_1(\tilde{\epsilon}) > 0 \). Choose \( \tilde{\epsilon} < \hat{\epsilon}_1 < \tau_0 \) such that for \( t \in (\tilde{\epsilon}, \hat{\epsilon}_1) \), \( R_1(t) > 0 \) and \( R_1(\hat{\epsilon}_1) = 0 \). Note that
\( \tilde{t}_1 \) exist because \( R_1(\tau_0) = 0 \). Let \( (\xi_0(t), \tau_0(t), s_{\xi_0(t)}) \) be as in lemma 3.8 starting at \((R_1(t), t)\) for \( t \in (\tilde{t}, \tilde{t}_1) \). From (3.16) and Step-2 we obtain \( R_1(\tau_0(t)) = 0 \). From (7) of lemma 3.8, we have \( \tau_0(t) \to \tilde{t}_1 \) as \( t \to \tilde{t}_1 \) and \( t \mapsto \tau_0(t) \) is continuous. Since \( \tau_0(\tilde{t}_1) < \tilde{t}_1 \), hence by continuity, there exists \( t_2 \in (\tilde{t}_1, \tilde{t}_1) \) such that \( \tau_0(t_2) = \tilde{t}_1 \). Therefore \( 0 = R_1(\tau_0(t_2)) = R_1(\tilde{t}_1) > 0 \) which is a contradiction. Hence Step-3 holds.

Step-4: From R-H condition, we have for a.e., \( t \in (\tau_0 - \epsilon_0, \tau_0) \), \( f(u(0+, t)) = g(u(0-, t)) \). Since \( R_1(t) = 0 \), hence \( f'(u(0+, t)) \leq 0 \). Since \( f(\theta_g) \leq g(\theta_g) \), hence \( L_1(t) = 0 \), therefore \( g'(u(0-, t)) \geq 0 \), hence \( u(0+, t) \leq \tilde{\theta}_g \). Therefore \( f'(u(0+, t)) \leq f'(\tilde{\theta}_g) \). Letting \( t \to \tau_0 \) to obtain \( \lim_{t \to \tau_0} f'(u(0+, t)) \leq f'(\tilde{\theta}_g) \). This implies that \( y(0+, \tau_0+) \geq -\tau_0 f'(\tilde{\theta}_g) = \xi_0 \). But from the hypothesis we have \( y(0+, \tau_0+) \leq y(R_1(T)+, T) < \xi_0 \), which is a contradiction. This proves the lemma if \( R_1(T) > 0 \).

Step-5: If \( R_1(T) = 0 \), repeat Step-3, Step-4 to obtain a contradiction if \( y(0+, T) < \xi_0 \). Hence the lemma.

\[\square\]

3.3. Solution with reflected characteristics

Earlier we build two solutions via backward construction, namely one has shock and other is a continuous solution. Now we need to construct another solution by backward construction for the reflected case and is as follows:

Let \( (T, R_1, R_2, y(\cdot)) \in \mathcal{R}(T) \). Assume that there are constants \( y_- \leq 0 \leq y_+ \) such that

\[ y(x) = \begin{cases} y_- & \text{if } x \in (-\infty, R_2), \\ y_+ & \text{if } x \in (R_1, \infty). \end{cases} \]

Since \( (T, R_1, R_2, y(\cdot)) \in \mathcal{R}(T) \), we have to consider three cases. In each case we construct a \( u_{1,0} \in L^\infty(\mathbb{R}) \) and the description of the corresponding solution \( u \) such that for \( i = 1, 2, R_i(T) = R_i, y(\cdot, T) = y(\cdot) \).

Case 1: (see figure 7 for illustration) Let \( 0 \leq R_2 \leq R_1 \) and assume that there exist \( 0 \leq T_1 \leq T_2 \leq T \) such that \( f'(\tilde{\theta}_g) = \frac{R_1}{T - T_1} = \frac{R_2}{T - T_2} \) and \( (\tau_0, \xi_0, s_{\xi_0}) \) be as in lemma 3.8 for \((R_1, T)\).

Since \( (T_1, R_1, R_2, y(\cdot)) \in \mathcal{R}(T) \) hence \( y_+ \) must satisfy

\[ \xi_0 \leq y_+. \]

In this case define the following quantities: let \( T_2 \leq t_0 \leq T \) be the unique solution of \( h_+\left(\frac{R_2}{T - t_0}\right) = \frac{y_-}{t_0} \). Let

\[ g'(w_-) = \frac{y_-}{t_0}, \quad f'(\bar{w}_-) = \frac{R_2}{T - t_0}, \]

\[ \eta_2(t) = g'(w_-)(t - t_0), \quad 0 \leq t \leq t_0, \]

\[ \gamma_5(t) = f'(\bar{w}_-)(t - t_0), \quad t_0 \leq t \leq T, \]

\[ \gamma_6(t) = R_2 + \frac{f(\bar{w}_-) - f(\tilde{\theta}_g)}{w_- - \tilde{\theta}_g}(t - T), \]

\[ \eta_1(t) = \frac{g(w_-) - g(\tilde{\theta}_g)}{w_- - \tilde{\theta}_g}(t - t_1), \]

where \( t_1 \) be such that \( \gamma_6(t_1) = 0 \). Then by the definition of \( h_+, t_0 \) and convexity of \( f, g \), it follows
easily that \( \eta_2(0) = y_-, \gamma_5(T) = R_2, T_2 \leq t_1 \leq t_0 \leq T, y_- \leq \xi_2 = \eta_1(0) \leq 0 \). Define
\[
\begin{align*}
  f'(u_+) &= \frac{R_1 - y_+}{T}, \\
  f'(\beta_0) &= \frac{R_1 - \xi_0}{T}, \\
  \gamma_1(t) &= R_1 + f'(u_+)(t - T), \\
  \gamma_2(t) &= R_1 + \frac{f(u_+) - f(\beta_0)}{u_+ - \beta_0}, \\
  \gamma_3(t) &= R_1 + f'(\beta_0)(t - T), \\
  \gamma_4(t) &= f'(\beta_0)(t - \tau_0).
\end{align*}
\]

Since \( \xi_0 \leq y_+ \), hence from convexity of \( f \), we have

\[ \xi_0 \leq \xi_1 = \gamma_2(0) \leq y_+. \]

In this case define the initial data \( u_{1,0} \) by

\[
u_{1,0} = \begin{cases} 
  w_- & \text{if } x < \xi_2, \\
  \theta_g & \text{if } \xi_2 < x < 0, \\
  \bar{\theta}_g & \text{if } 0 < x < \xi_0, \\
  \beta_0 & \text{if } \xi_0 < x < \xi_1, \\
  u_+ & \text{if } x > \xi_1.
\end{cases}
\]

It is easy to verify that the solution \( u_1(\cdot, \cdot) \) of (1.1) with initial data \( u_{1,0} \) is given by

\[
u_1(x,t) = \begin{cases} 
  w_- & \text{if } x < \min\{\eta_1(t), 0\}, \\
  \theta_g & \text{if } \min\{\eta_1(t), 0\} < x < 0, \\
  \bar{\theta}_g & \text{if } 0 < x < \gamma_6(t), \\
  \beta_0 & \text{if } \max\{\gamma_6(t), 0\} < x < s_{\xi_0}(t), \\
  \gamma_2(t) & \text{if } 0 < x < \gamma_4(t), \\
  (f')^{-1}\left(\frac{x - \xi_0}{t}\right) & \text{if } \left\{ \begin{array}{ll}
    s_{\xi_0}(t) < x < \gamma_3(t), t \in (\tau_0, T) \\
    \text{or} \max\{\gamma_4(t), 0\} < x < \gamma_3(t)
  \end{array} \right.
  \\
  \beta_0 & \text{if } \gamma_3(t) < x < \gamma_2(t), \\
  u_+ & \text{if } x > \gamma_2(t).
\end{cases}
\]

In this case define the domain \( D_1 \) for \( t \leq T \) by

\[ D_1 = \{(x, t) : \min\{\eta_2(t), 0\} < x \leq 0\} \cup \{(x, t) : \max\{\gamma_5(t), 0\} < x < \gamma_1(t)\}. \]
Then \( u \) satisfies
\[
 u_1(\eta_2(t)+, t) = w_-, \quad u_1(\gamma_5(t)-, t) = \bar{w}_-, \quad u_1(\gamma_1(t)-, t) = u_+.
\]

Case 2: Let \( R_1 = R_2 > 0 \) and for all \( t \in [0, T] \), \( f'(\bar{\theta}_g) < \frac{R_1}{T-t} \).

Let \( 0 \leq t_0 \leq T \) be the unique solution of \( h_+ \left( \frac{R_1}{T-t_0} \right) = -\frac{y_-}{t_0} \). As in Case 1, define
\[
 g'(w_-) = -\frac{y_-}{t_0}, \quad f'(-\bar{w}_-) = \frac{R_1}{T-t_0}, \quad f'(u_+) = \frac{R_1 - y_+}{T},
\]
\[
 \eta_2(t) = g'(w_-)(t-t_0) \text{ if } 0 \leq t \leq t_0,
\]
\[
 \gamma_4(t) = f'(\bar{w}_-)(t-t_0) \text{ if } t_0 \leq t \leq T,
\]
\[
 \gamma_1(t) = R_1 + f'(u_+)(t-T),
\]
\[
 \gamma_2(t) = R_1 + f'(\bar{\theta}_g)(t-T),
\]
\[
 \gamma_3(t) = R_1 + \frac{f(u_+)-f(\bar{w}_-)}{u_+ - \bar{w}_-}(t-T),
\]
\[
 \gamma_6(t) = R_1 + \frac{f(u_+)-f(\bar{\theta}_g)}{u_+ - \bar{\theta}_g}(t-T),
\]
\[
 \gamma_5(t) = R_1 + \frac{f(\bar{\theta}_g)-f(\bar{w}_-)}{\theta_g - \bar{w}_-}(t-T).
\]

Now we have to consider four sub-cases:

Subcase 1: (see figure 8 for illustration) \( y_+ = \gamma_1(0) < \gamma_2(0) \) and \( \gamma_3(0) \).

Clearly \( \bar{w}_- \geq u_+ \), then define the initial data \( u_{1,0} \) and the solution \( u_1 \) of (1.1) by
\[
u_{1,0}(x) = \begin{cases} 
  w_- & \text{if } x < 0, \\
  \bar{w}_- & \text{if } 0 < x < \gamma_3(0), \\
  u_+ & \text{if } x > \gamma_3(0) 
\end{cases}
\]
and the solution \( u_1 \) is given by
\[
 u_1(x,t) = \begin{cases} 
  w_- & \text{if } x < 0, \\
  \bar{w}_- & \text{if } 0 < x < \gamma_3(t), \\
  u_+ & \text{if } x > \gamma_3(t). 
\end{cases}
\]

Define for \( 0 \leq t \leq T \),
\[
 D_1 = \{(x,t) : \min\{\eta_2(t),0\} < x \leq 0 \} \cup \{(x,t) : \max\{\gamma_4(t),0\} < x < \gamma_1(t) \}. 
\]
Then \( u_1 \) satisfies
\[
u_1(\eta_2(t) + t) = w_-, \quad u_1(\gamma_4(t) - t) = \bar{w}_-, \quad u_1(\gamma_1(t) - t) = u_+.
\]

Subcase 2: (see figure 9 for illustration) Let \( y_+ = \gamma_1(0) < \gamma_2(0) \) and \( \gamma_3(0) < 0 \).

Let \( 0 < t_1 < t_0 \) be such that \( \gamma_3(t_1) = 0 \). Since \( \gamma_1(0) < \gamma_2(0) \), hence \( \bar{w}_- \geq \bar{u}_+ \geq \theta_g \), therefore there exists a unique \( w_- \geq \bar{u}_+ \geq \theta_g \) such that \( f(u_+) = g(\bar{u}_+) \). Let \( \eta_1(t) = \frac{g(w_-) - g(\bar{u}_+)}{w_- - \bar{u}_+}(t - t_1) \), then by convexity of \( g \), it follows that \( y_- \leq \eta_1(0) = \xi_2 \leq 0 \). Let
\[
u_{1,0}(x) = \begin{cases} w_- & \text{if } x < \xi_2, \\ \bar{u}_+ & \text{if } \xi_2 < x < 0, \\ u_+ & \text{if } x > 0, \end{cases}
\]
then the solution \( u_1 \) to (1.1) with initial data \( u_{1,0} \) is given by
\[
u_1(x, t) = \begin{cases} w_- & \text{if } x < \min\{\eta_1(t), 0\}, \\ \bar{u}_+ & \text{if } \eta_1(t) < x < 0, \\ \bar{w}_- & \text{if } 0 < x < \gamma_3(t), \\ u_+ & \text{if } x > \gamma_3(t). \end{cases}
\]

Let
\[
D_1 = \{(x, t) : \min\{\eta_2(t), 0\} < x \leq 0\} \cup \{(x, t) : \max\{0, \gamma_4(t)\} < x < \gamma_1(t)\},
\]
then \( u_1 \) satisfies
\[
u_1(\eta_2(t) + t) = w_-, \quad u_1(\gamma_4(t) + t) = \bar{w}_-, \quad u_1(\gamma_1(t) - t) = u_+.
\]

Subcase 3: \( 0 \leq \gamma_2(0) \leq \gamma_1(0) = y_+, \gamma_5(0) \geq 0 \).

Let
\[
u_{1,0} = \begin{cases} w_- & \text{if } x < 0, \\ \bar{w}_- & \text{if } 0 < x < \gamma_5(0), \\ \theta_g & \text{if } \gamma_5(0) < x < \gamma_6(0), \\ u_+ & \text{if } x > \gamma_6(0) \end{cases}
\]
and the corresponding solution \( u_1 \) in \( \mathbb{R} \times [0, T] \) is given by
\[
u_1(x, t) = \begin{cases} w_- & \text{if } x < 0, \\ \bar{w}_- & \text{if } 0 < x < \gamma_5(t), \\ \theta_g & \text{if } \gamma_5(t) < x < \gamma_6(t), \\ u_+ & \text{if } x > \gamma_6(t). \end{cases}
\]
Define for $0 < t \leq T$,

$$D_1 = \{(x,t) : \min\{\eta_2(t),0\} < x \leq 0\} \cup \{(x,t) : \max\{\gamma_4(t),0\} < x < \gamma_1(t)\},$$

then $u_1$ satisfies

$$u_1(\eta_2(t)+,t) = w_-, \quad u_1(\gamma_4(t)+,t) = \bar{w}_-, \quad u_1(\gamma_1(t)-,t) = u_+.$$

Subcase 4: (see figure 10 for illustration) $0 \leq \gamma_2(0) \leq \gamma_1(0)$, $\gamma_5(0) < 0$.

Let $t_1$ be such that $\gamma_5(t_1) = 0$. Let $\eta_1(t) = \frac{g(w_-) - g(\theta_g)}{w_- - \theta_g}$ and

$$u_{1,0}(x) = \begin{cases} 
    w_- & \text{if } x < \eta_1(0), \\
    \theta_g & \text{if } \eta_1(0) < x < 0, \\
    \bar{\theta}_g & \text{if } 0 < x < \gamma_6(0), \\
    u_+ & \text{if } x > \gamma_6(0)
\end{cases}$$

then the corresponding solution $u_1$ of (1.1) is given by

$$u_1(x,t) = \begin{cases} 
    w_- & \text{if } x < \min\{\eta_1(t),0\}, \\
    \theta_g & \text{if } \min\{\eta_1(t),0\} < x < 0, \\
    \bar{w}_- & \text{if } 0 < x < \gamma_5(t), \\
    \bar{\theta}_g & \text{if } \max\{\gamma_5(t),0\} < x < \gamma_6(t), \\
    u_+ & \text{if } x > \gamma_6(t).
\end{cases}$$

Define

$$D_1 = \{(x,t) : \min\{\eta_2(t),0\} < x \leq 0\} \cup \{(x,t) : \max\{0,\gamma_4(t)\} < x < \gamma_1(t)\},$$

then $u_1$ satisfies

$$u_1(\eta_2(t)+,t) = w_-, \quad u_1(\gamma_4(t)+,t) = \bar{w}_-, \quad u_1(\gamma_1(t)-,t) = u_+.$$

Case 3: $R_1 = 0$, $y_- \leq 0 \leq \xi_0 = -f'(\bar{\theta}_g)T \leq y_+$. 
Define

\[ g'(w-) = \frac{y_1 - T}{T}, \quad f'(u_+) = \frac{R_1 - y_1}{T}, \]

\[ \eta_2(t) = g'(w_-(t - T)), \]

\[ \eta_1(t) = \left( \frac{g(w_-) - g(\theta_g)}{w_- - \theta_g} \right)(t - T), \]

\[ \gamma_3(t) = f'(\bar{\theta}_g)(t - T), \]

\[ \gamma_2(t) = R_1 + \frac{f(\bar{\theta}_g) - f(u_+)}{\bar{\theta}_g - u_+}(t - T), \]

\[ \gamma_1(t) = R_1 + f'(u_+)(t - T). \]

Due to \( \xi_0 \leq y_+ = \gamma_1(0) \), we have \( u_+ \leq \bar{\theta}_g \). Hence by convexity of \( f \), \( \xi_0 \leq \xi_1 = \gamma_2(0) \leq y_+ \). Since \( w_- \geq \theta_g \), hence \( y_- \leq \xi_2 = \eta_1(0) \leq 0 \). Define

\[ u_{1,0}(x) = \begin{cases} w_- & \text{if } x < \xi_2, \\ \theta_g & \text{if } \xi_2 < x < 0, \\ \bar{\theta}_g & \text{if } 0 < x < \xi_1, \\ u_+ & \text{if } x > \xi_1 \end{cases} \]

and the corresponding solution \( u_1 \) is given by

\[ u_1(x,t) = \begin{cases} w_- & \text{if } x < \min\{0, \eta_1(t)\}, \\ \theta_g & \text{if } \min\{0, \eta_1(t)\} < x < 0, \\ \bar{\theta}_g & \text{if } 0 < x < \gamma_2(t), \\ u_+ & \text{if } x > \gamma_2(t). \end{cases} \]

Let

\[ D_1 = \{(x,t) : \min\{0, \eta_2(t)\} < x \leq 0\} \cup \{(x,t) : 0 \leq x < \max\{0, \gamma_1(t)\}\}, \]

then \( u_1 \) satisfies

\[ u_1(\eta_2(t)+, t) = w_-, \quad u_1(\gamma_1(t)-, t) = u_+. \]
4. Backward construction

Lemma 4.1. Let $0 < R_2$ and $y : [0, R_2] \to (-\infty, 0]$ be a non decreasing function. Define

$$
y_0 = y(0^+), y_1 = y(R_2^-),
$$

Let

$$
h_+ \left( \frac{R_2}{T - t_1} \right) = -\frac{y_1}{t_1},
$$

$$
g'(u_-) = -\frac{y_0}{T}, g'(w_-) = -\frac{y_1}{t_1}, f(w_-) = g(w_-), f'(\bar{w}_-) \geq 0,
$$

$$
\eta_3(t) = g'(u_-)(t - T), \eta_2(t) = g'(w_-)(t - t_1), \bar{\eta}_2(t) = f'(\bar{w}_-)(t - t_1).
$$

Then there exists a $u_{2,0} \in L^\infty(\mathbb{R})$ and the corresponding solution $u_2$ of (1.1) such that

$$
w_2(\eta_3(t)+, t) = u_-, u_2(\eta_2(t)−, t) = w_-, u_2(\bar{\eta}_2(t)−, t) = \bar{w}_-.
$$

Proof. Without loss of generality by approximation, we assume that $y$ is a strictly increasing continuous function and $N > 1$, let $k > 1$ and define a discretization by

$$
y_0 = z_0 < z_1 < \cdots < z_k = y_1,
$$

$$
|z_{i+1} - z_i| < \frac{1}{N},
$$

$$
y(x_i) = z_i \text{ with } y_0 = y(0) \text{ and } y_1 = y(R_2^-).
$$

Let $\tau_0 = T$ and define $\{\tau_i\}$ for $1 \leq i \leq 2k$, $\{a_i\}, \{b_i\}, \{\tau_i(x)\}$ for $1 \leq i \leq k$ by

$$
h_+ \left( \frac{x_i}{T - \tau_{2i-1}} \right) = -\frac{z_{i-1}}{\tau_{2i-1}},
$$

$$
h_+ \left( \frac{x_i}{T - \tau_{2i}} \right) = -\frac{z_i}{\tau_{2i}},
$$

$$
f'(a_{2i-1}) = \frac{x_i}{T - \tau_{2i-1}},
$$

$$
f'(a_{2i}) = \frac{x_i}{T - \tau_{2i}},
$$

$$
\alpha_i(t) = x_i + s_i(t - T), \beta_i(t) = S_i(t - \delta_i),
$$

where $\delta_i$ is defined by $\alpha_i(\delta_i) = 0$. Then from the convexity of $f$ and $g$, we have $\tau_{2i-1} < \delta_i < \tau_{2i}$,
Then the solution $u$.

**Convergence Analysis:**

First we show that

$$T - \tau_2k = t_1, \quad f'(a_{2k}) = \frac{x_k}{T - \tau_2k} = \frac{R_2}{T - t_1} = f'(\tilde{w}_-)$$

and

$$g'(b_{2k}) = -\frac{z_k}{\tau_2k} = -\frac{y_1}{t_1}, \text{ hence } b_{2k} = w_-.$$ Define

$$u_{2,0}^N = \begin{cases} u_ - & \text{if } x < y_0 = z_0, \\ b_{2i-1} & \text{if } z_{i-1} < x < \beta_i(0), 1 \leq i \leq k, \\ b_{2i} & \text{if } \beta_i(0) < x < z_i, \\ w_- & \text{if } z_{2k} < x < 0, \\ \tilde{w}_- & \text{if } x > 0. \end{cases} \quad (4.2)$$

Then the solution $u_2^N$ of (1.1) with initial data $u_{2,0}^N$ in $\mathbb{R} \times (0, T)$ is given by (see figure 12)

$$u_2^N(x, t) = \begin{cases} u_- & \text{if } x < r_0(t), \\ (g')^{-1}\left(\frac{x - z_i}{t}\right) & \text{if } r_{2i}(t) < x < \min\{r_{2i+1}(t), 0\}, \\ (f')^{-1}\left(\frac{x}{t - t_i(x, t)}\right) & \text{if } \max\{\tilde{r}_{2i}(t), 0\} < x < \tilde{r}_{2i+1}(t), \\ b_{2i-1} & \text{if } r_{2i-1}(t) < x < \min\{S_i(t), 0\}, \\ b_{2i} & \text{if } S_i(t) < x < \min\{r_{2i}(t), 0\}, \\ a_{2i-1} & \text{if } \max\{\tilde{r}_{2i+1}(t), 0\} < x < s_i(t), \\ a_{2i} & \text{if } \max\{s_i(t), 0\} < x < \tilde{r}_{2i}(t), \\ w_- & \text{if } \tilde{r}_{2k}(t) < x < 0, \\ \tilde{w}_- & \text{if } \max\{\tilde{r}_{2k}(t), 0\} < x, \end{cases} \quad (4.3)$$

where $t_i(x, t)$ is the unique solution of

$$h_+\left(\frac{x}{t - t_i(x, t)}\right) = -\frac{z_i}{t_i(x, t)}, \text{ for } x \in (x_i, x_{i+1}), i \leq k - 1.$$

Next we show that the above sequences converges.

**Convergence Analysis:**

First we show that $\{|u_{2,0}^N|\}$ is uniformly bounded. Let $i_0 = \sup\{i : \tau_i \leq T/2\}$. For $i \leq i_0$, we have

$$f'(a_i) = \begin{cases} \frac{x_l}{T - \tau_{2l}} & \text{if } l = i/2, \\ \frac{x_{2l}}{T - \tau_{2l-1}} & \text{if } l = (i + 1)/2. \end{cases}$$
Hence $f'(a_i) \leq \frac{2R_2}{T}$. For $i \geq i_0$, then
\[
g'(b_i) = \begin{cases} 
-\frac{z_{i-1}}{\tau_{2i-1}} & \text{if } l = (i + 1)/2, \\
-\frac{x_{i-1}}{\tau_{2i}} & \text{if } l = i/2.
\end{cases}
\]

Thus, we have $g'(b_i) \leq \frac{2|y_0|}{T}$. Since $f(a_i) = g(b_i)$, $g'(b_i) \geq 0$, we get $\{b_i\}$ is uniformly bounded in $\mathbb{R}$ and $\{u_{2,0}^N\}$ is uniformly bounded in $L^\infty(\mathbb{R})$.

First assume that $f$ and $g$ are uniformly convex. Then $h_+$ is a Lipschitz continuous function.
\[
TV(g'(u_{2,0}^N)) = \sum_{i=1}^{2k-1} |g'(b_{i+1}) - g'(b_i)|
\]
\[
= \sum_{i=1}^{k} |g'(b_{2i-1}) - g'(b_{2i})| + \sum_{i=1}^{k-1} |g'(b_{2i}) - g'(b_{2i+1})|
\]
\[
= \sum_{i=1}^{k} \left| \frac{z_{i-1}}{\tau_{2i-1}} - \frac{z_i}{\tau_{2i}} \right| + \sum_{i=1}^{k-1} \left| \frac{z_i}{\tau_{2i}} - \frac{z_{i+1}}{\tau_{2i+1}} \right|
\]
\[
= \sum_{\tau_{2i} \leq T/2} \left| \frac{z_{i-1}}{\tau_{2i-1}} - \frac{z_i}{\tau_{2i}} \right| + \sum_{\tau_{2i+1} \leq T/2} \left| \frac{z_i}{\tau_{2i}} - \frac{z_{i+1}}{\tau_{2i+1}} \right|
\]
\[
= I_1 + I_2,
\]
where
\[
I_1 = \sum_{\tau_{2i} \leq T/2} \left| \frac{z_{i-1}}{\tau_{2i-1}} - \frac{z_i}{\tau_{2i}} \right| + \sum_{\tau_{2i+1} \leq T/2} \left| \frac{z_i}{\tau_{2i}} - \frac{z_{i+1}}{\tau_{2i+1}} \right|
\]
\[
= \sum_{\tau_{2i} \leq T/2} h_+ \left( \frac{x_i}{T - \tau_{2i-1}} \right) - h_+ \left( \frac{x_i}{T - \tau_{2i}} \right) + \sum_{\tau_{2i+1} \leq T/2} h_+ \left( \frac{x_i}{T - \tau_{2i}} \right) - h_+ \left( \frac{x_{i+1}}{T - \tau_{2i+1}} \right).
\]

As $f, g$ are uniformly convex, we get $h_+$ is a locally Lipschitz function. Due to $\tau_{2i} \leq T/2$, $\tau_{2i+1} \leq T/2$, we obtain $T - \tau_{2i} \geq T/2$, $T - \tau_{2i+1} \geq T/2$, hence $\frac{x_i}{T - \tau_{2i}}, \frac{x_i}{T - \tau_{2i+1}}$ are bounded. Let $M = \text{Lipschitz constant of } h_+$ on $\left[ \frac{\bar{y}}{\theta g_+}, \frac{2R_2}{T} \right]$, then
\[
I_1 \leq M \left[ \sum_{\tau_{2i} \leq T/2} \left| \frac{x_i}{T - \tau_{2i-1}} - \frac{x_i}{T - \tau_{2i}} \right| + \sum_{\tau_{2i+1} \leq T/2} \left| \frac{x_i}{T - \tau_{2i}} - \frac{x_{i+1}}{T - \tau_{2i+1}} \right| \right]
\]
\[
\leq 4R_2 M \left[ \sum_{\tau_{2i} \leq T/2} |\tau_{2i} - \tau_{2i-1}| + \sum_{\tau_{2i+1} \leq T/2} |\tau_{2i+1} - \tau_{2i}| \right] + \frac{4M}{T^2} \sum_{\tau_{2i+1} \leq T/2} |x_i - x_{i+1}|
\]
\[
\leq \frac{4M}{T^2} (R_2 + 1).
\]
Since \( \{\tau_i\} \) is a decreasing sequence and \( \{x_i\} \) is an increasing sequence, we have

\[
I_2 = \sum_{\tau_{2i} > T/2} \left| \frac{\tau_{2i} - \tau_{2i-1}}{\tau_{2i-1} - \tau_{2i}} \right| + \sum_{\tau_{2i+1} > T/2} \left| \frac{\tau_{2i+1} - \tau_{2i}}{\tau_{2i+1} - \tau_{2i}} \right|
\]

\[
\leq 4 \frac{|y_0|}{T^2} \sum_{i=1}^{2k-1} |\tau_{i+1} - \tau_i| + \frac{4}{T^2} \sum_{i=1}^{k-1} |\tau_{i+1} - \tau_i| - 4 |y_0|.
\]

Therefore, by Helly’s Theorem, there exists a subsequence still denoting by \( \{y'(u_{2,0}^N)\} \) converges point-wise to \( g'(u_{2,0}) \). Hence \( \forall y \in \mathbb{R} \),

\[
\lim_{N \to \infty} u_{2,0}^N(y) = u_{2,0}(y),
\]

and \( u_{2,0} \in L^\infty(\mathbb{R}) \) with

\[
u_{2,0}(y) = \begin{cases} u_- & \text{if } y < y_0, \\ w_- & \text{if } y_1 < y < 0, \\ \bar{w}_- & \text{if } y > 0. \end{cases}
\]

Let

\[
y_N(x) = \sum_{i=0}^{k-1} \zeta_i \chi_{[x_i, x_{i+1})}(x).
\]

Then

\[
|y(x) - y_N(x)| = \left| \sum_{i=0}^{k-1} (\zeta_i - y(x)) \chi_{[x_i, x_{i+1})} \right| \leq \frac{1}{N}.
\]

Thus, \( y_N \to y \) in \( L^\infty[0, R_2] \). Let \( \tau_i(x) = \tilde{t}_i(x, T) \) for \( x \in [x_i, x_{i+1}] \). Define

\[
t_N(x) = \tau_i(x), \quad \text{if } x \in (x_i, x_{i+1}),
\]

then for a.e. \( x \in (0, R_2) \), we have, \( t_N \) is a strictly increasing function, \( t_1 \leq t_N(x) \leq T \) and for a.e., \( x \in (0, R_2) \), we have

\[
h_+ \left( \frac{x}{T - t_N(x)} \right),
\]

\[
u_2^N(x, T) = (f')^{-1} \left( \frac{x}{T - t_N(x)} \right),
\]

\[
u_2^N(\eta_3(t)^+, t) = u_-, \quad u_2^N(\eta_2(t)^-, t) = w_-, \quad u_2^N(\eta_2(t)^-, t) = \bar{w}_-.
\]

From the construction, set of discontinuities of \( u_2^N \) are discrete set of Lipschitz curves in \( \mathbb{R} \times [0, T] \), therefore, from (xii) of theorem 2.1

\[
\int_{\mathbb{R}} \left| u_{2,0}^N(x, t) - u_{2,0}^N(x, t) \right| \, dx \leq \int_{y_0}^{y_1} \left| u_{2,0}^N(x) - u_{2,0}^N(x) \right| \, dx.
\]
Subsequently, we have
\[
\int_0^T \int_\mathbb{R} |u_1^{N_1}(x,t) - u_2^{N_2}(x,t)| \ dx \ dt \leq T \int_{y_0}^{y_1} |u_{1,0}(x) - u_{2,0}(x)| \ dx
\]
\[
\rightarrow 0 \text{ as } N_1, N_2 \rightarrow \infty.
\]
Hence for a subsequence still denoted by \(\{u_2^N\}\) converges to \(u_2\), a solution of (1.1) with initial data \(u_{2,0}\). From Helly’s Theorem, again for a subsequence,
\[
\lim_{N \to \infty} t^N(x) = t_+(x).
\]
Then from (4.4), letting \(N \to \infty\) to obtain for a.e. \(x\)
\[
-\frac{y(x)}{t_+(x)} = h_+ \left( \frac{x}{T - t_+(x)} \right),
\]
\[
u_2(x, T) = (f')^{-1} \left( \frac{x}{T - t_+(x)} \right)
\]
and \(u_2\) satisfies the conclusion of the lemma. If \(f\) and \(g\) are not uniformly convex (and just strictly convex), then approximate \(f\) and \(g\) by \(f_\epsilon\) and \(g_\epsilon\) respectively which are uniformly convex and by stability lemma 2.1, the lemma follows as \(\epsilon \to 0\).

4.1. Proof of theorem 1.1

Proof of theorem 1.1. First we prove that if \(u_0 \in L^\infty(\mathbb{R})\) and \(u\) is the corresponding solution of (1.1), then \((T, R_1(T), R_2(T), y(\cdot; T)) \in \mathcal{R}(T)\). From lemma 3.9, if \(R_1(T) = 0\) or there exists a \(0 \leq T_1 \leq T\) such that \(f'(\bar{y}) = \frac{R_1(T)}{T - T_1}\), then \(y(R_1(T) + T, T) \geq \xi_0\). Hence \((T, R_1(T), R_2(T), y(T)) \in \mathcal{R}(T)\). Conversely, let \((T, R_1, R_2, y(\cdot)) \in \mathcal{R}(T)\), define \(y_0 = y(0+)\), \(y_- = y(R_2(T)-)\), \(y_+ = y(R_1+)\) and \(t_0\) by
\[
h_+ \left( \frac{R_2}{T - t_0} \right) = -\frac{y_-}{t_0}\]
and define
\[
f'(u_+) = \frac{R_1 - y_+}{T}, f'(\bar{w}_-) = \frac{R_2}{T - t_0}, g'(w_-) = -\frac{y_-}{t_0},
\]
\[
g'(u_-) = -\frac{y_0}{T},
\]
\[
\gamma_1(t) = R_1 + f'(u_+)(t - T),
\]
\[
\gamma_2(t) = R_2 + f'(\bar{w}_-)(t - T),
\]
\[
\eta_2(t) = g'(w_-)(t - t_0),
\]
\[
\eta_3(t) = g'(u_-)(t - T).
\]
Let for \(0 < t < T\), define
\[
D_1 = \{(x, t) : \min \{\eta_2(t), 0\} < x < \gamma_1(t)\} \cup \{(x, t) : \max \{\gamma_2(t), 0\} < x < \gamma_1(t)\},
\]
\[
D_2 = \{(x, t) : \min \{\eta_3(t), 0\} < x < \min \{\eta_2(t), 0\}\} \cup \{(x, t) : 0 < x < \max \{\gamma_2(t), 0\}\},
\]
\[
D_3 = \{(x, t) : x < \eta_3(t)\} \cup \{(x, t) : x > \gamma_1(t)\},
\]
\[
I_i = \bar{D}_i \cap \mathbb{R}, i = 1, 2, 3.
\]
From subsection 3.3, there exists a \(u_{1,0} \in L^\infty(\mathbb{R})\) and the corresponding solution \(u_1\) such that
\[
u_1(\eta_2(t)+, t) = w_-, u_1(\gamma_2(t)+, t) = \bar{w}_-, u_1(\gamma_1(t)-, t) = u_+.
\]
From lemma 4.1, there exists a \(u_{2,0} \in L^\infty(\mathbb{R})\) and the corresponding solution \(u_2\) of (1.1) satisfies:
\[
u_2(\eta_3(t)+, t) = u_-, u_2(\eta_2(t)-, t) = w_-, u_2(\gamma_2(t)-, t) = \bar{w}_-.
\]

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From the backward construction [2], there exists a $u_{3,0} \in L^\infty(\mathbb{R})$ and the corresponding solution $u_3$ of (1.1) such that

$$u_3(\eta_3(t) -, t) = u_-, u_3(\gamma_1(t) +, t) = u_+.$$ 

Therefore by R-H condition if we define

$$u_0(x) = \begin{cases} u_{1,0}(x) & \text{if } x \in \text{Interior of } I_1, \\ u_{2,0}(x) & \text{if } x \in \text{Interior of } I_2, \\ u_{3,0}(x) & \text{if } x \in \text{Interior of } I_3, \end{cases}$$

then $u$ is the solution of (1.1) with initial data $u_0$ given by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } (x, t) \in D_1, \\ u_2(x, t) & \text{if } (x, t) \in D_2, \\ u_3(x, t) & \text{if } (x, t) \in D_3, \end{cases}$$

satisfying $R_i(T) = R_i, i = 1, 2, y(\cdot, T) = y(\cdot)$. This proves the theorem. 

\[ \square \]

**4.2. Proof of theorem 1.2**

**Proof.** Define $\delta_1 = y(C_1) - B_1, \delta_2 = B_2 - y(C_2)$

$$\tilde{y}(x) = \begin{cases} y(x) & \text{if } x \in (C_1, R_2) \cup (R_1, C_2), \\ x & \text{if } x < C_1, C_1 < y(C_1 +), \\ x & \text{if } x < y(C_1 +) < C_1, \\ y(C_1 +) & \text{if } y(C_1 +) < x < C_1, \\ x & \text{if } x > C_2, \\ x & \text{if } x > y(C_2 -) > C_2, \\ y(C_2 -) & \text{if } C_2 < x < y(C_2 -). \end{cases} \quad (4.6)$$

Let $\tilde{u}_-, \tilde{u}_+$ be defined by

$$g'(\tilde{u}_-) = \frac{C_1 - y(C_1 +)}{T}, \quad (4.7)$$

$$f'(\tilde{u}_+) = \frac{C_2 - y(C_2 -)}{T} \quad (4.8)$$

and

$$\gamma_1(t) = C_1 + g'(\tilde{u}_-)(t - T), \quad (4.9)$$

$$\gamma_2(t) = C_2 + f'(\tilde{u}_+)(t - T). \quad (4.10)$$
Then from theorem 1.1, there exists a \( \tilde{u}_0 \in L^\infty(\mathbb{R}) \) and a solution \( \tilde{u} \) with initial data \( \tilde{u}_0 \) such that
\[
\tilde{u}(\gamma_1(t)+,t) = \tilde{u}_-,
\]
\[
\tilde{u}(\gamma_2(t)-,t) = \tilde{u}_+.
\]
Then the free region lemmas 2.2, 2.3 and 2.4 as in [2] (see figure 13 for illustration), one can find \( \lambda_1 \) large negative number and \( \lambda_2 \) large positive number, such that there exist solutions \( u_2 \) and \( u_3 \) of (1.1) with respective initial data \( u_2,0 \) and \( u_3,0 \) given by
\[
u_2,0 = \begin{cases} 
u_{1,0}(x) & \text{if } x < B_1, \\ \lambda_1 & \text{if } B_1 < x < B_1 + \delta_1, \\ \tilde{u}_- & \text{if } x > B_1 + \delta_1, \end{cases}
\]
\[
u_3,0 = \begin{cases} 
u_{1,0}(x) & \text{if } x > B_2, \\ \lambda_2 & \text{if } B_2 - \delta_2 < x < B_2, \\ \tilde{u}_+ & \text{if } x < B_2 - \delta_2. \end{cases}
\]
and satisfies
\[
\tilde{u}_2(\gamma_1(t)−,t) = \tilde{u}_−,
\]
\[
\tilde{u}_3(\gamma_2(t)+,t) = \tilde{u}_+.
\]
Hence define
\[
u_0(x) = \begin{cases} 
u_{1,0}(x) & \text{if } x < B_1, \\ \lambda_1 & \text{if } B_1 < x < B_1 + \delta_1, \\ \tilde{u}_0(x) & \text{if } B_1 + \delta_1 < x < B_2 - \delta_2, \\ \lambda_2 & \text{if } B_2 - \delta_2 < x < B_2, \\ u_{1,0}(x) & \text{if } B_2 < x, \end{cases}
\]
\[
u(x,t) = \begin{cases} u_{2}(x,t) & \text{if } x < \gamma_1(t), \\ \tilde{u}(x,t) & \text{if } \gamma_1(t) < x < \gamma_2(t), \\ u_{3}(x,t) & \text{if } x > \gamma_2(t). \end{cases}
\]
Then \((u_0,u)\) is the required solution satisfies the Theorem.

5. Optimal control

Let \( K \) be given, the associated cost functional \( J \) and admissible set \( \mathcal{A} \) are as in (1.12). Then we have the following:

Lemma 5.1. For \( u_0 \in \mathcal{A} \), \( J(u_0) \) is well defined.

Proof. Because of finite speed of propagation, it is immediate.

5.1. Proof of theorem 1.3

Proof of theorem 1.3. Proof involves several steps,

Step 1: Let
\[
\mathcal{\tilde{A}} = \{(T,R_1,R_2,y(\cdot)) \in R(T) : y(x) = x \text{ outside a compact set } \}.
\]
For \( \alpha = (T,R_1,R_2,y(\cdot)) \in \mathcal{\tilde{A}} \), define
\[
h_+ \left( \frac{x}{T-t_+(x)} \right) = -\frac{y(x)}{t_+(x)} \text{, for } x \in (0,R_2) \text{ and }
\]
\[
\hat{J}(\alpha) = \int_{-\infty}^{0} \left| \frac{x - y(x)}{T} - K(x) \right|^2 dx + \int_{0}^{R_2} \left| \frac{y(x)}{t_1(x)} + K(x) \right|^2 dx + \int_{R_2}^{R_3} |\tilde{\theta}_g - K(x)|^2 dx \\
+ \int_{R_1}^{\infty} \left| \frac{x - y(x)}{T} - K(x) \right|^2 dx.
\] (5.1)

\[R_0 = \frac{f(\tilde{\theta}_g) - f(\theta_f)}{\theta_g - \theta_f} T,\]

\[M_1 = \int_{-\infty}^{0} |K(x)|^2 dx + \int_{R_0}^{\infty} |K(x)|^2 dx + \int_{0}^{R_0} |\tilde{\theta}_g - K(x)|^2 dx.
\]

Then

\[\inf_{\alpha \in \mathcal{A}} \hat{J}(\alpha) \leq \inf_{u_0 \in \mathcal{A}} J(u_0) \leq M_1.
\]

**Proof of Step 1:** Let

\[w_0(x) = \begin{cases} 
\theta_g & \text{if } x < 0, \\
\theta_f & \text{if } x > 0.
\end{cases}
\]

Then \(w\) is the solution to (1.1) with data \(w_0\), where

\[w(x, t) = \begin{cases} 
\theta_g & \text{if } x < 0, \\
\bar{\theta}_g & \text{if } 0 < x < \frac{f(\bar{\theta}_g) - f(\theta_g)}{\theta_g - \theta_f} t, \\
\theta_f & \text{if } x > \frac{f(\bar{\theta}_g) - f(\theta_g)}{\theta_g - \theta_f} t.
\end{cases}
\]

here \(y(x, t) = x\) for \(x \in (-\infty, 0) \cup (R_0, \infty)\). Since from theorem 1.1 we have

\[\inf_{\alpha \in \mathcal{A}} \hat{J}(\alpha) \leq \inf_{u_0 \in \mathcal{A}} J(u_0) \text{ and } \inf_{u_0 \in \mathcal{A}} J(u_0) \leq J(w_0) = M_1.
\]

This proves Step 1. \(\square\)

Step 2: Let \(\tilde{\mathcal{A}}_1 = \{\alpha \in \tilde{\mathcal{A}} : \hat{J}(\alpha) \leq 2M_1\}\), then there exists a constant \(M_2 > 0, c_1 = \max\{M_2, c\}\) such that for all \(\alpha = (T,R_1,R_2,y(\cdot))\) with \(\hat{J}(\alpha) \leq 2M_1\),

(i) \(R_1 \leq M_2\),

(ii) \(|y(0^+)| \leq (18T^2M_1)^{1/3}\),

(iii) \(y(-c_1) \geq -(c_1 + (6T^2M_1)^{1/3})\),

(iv) \(y(c_1) \leq (c_1 + (12M_1T^2)^{1/3})\).

**Proof of Step 2.** Suppose \(R_1 > c + f'(\bar{\theta}_g)T\), then the line \(R_1 + f'(\bar{\theta}_g)(t-T)\) does not intersect the \(t\) axis for \(t > 0\). Subsequently, we have \(R_2 = R_1\) and \(t(R_1-) \leq t_+(x) \leq t(c+) < T\), for all \(x \in (c,R_1)\).
Since \( K(x) = 0 \) for \( x > c \), we get
\[
2M_1 \geq \tilde{J}(\alpha) \geq \int_{c}^{R_1} \left| \frac{y(x)}{t_+(x)} \right|^2 \, dx,
\]
\[
= \int_{c}^{R_1} \left| h_+ \left( \frac{x}{T-t_+(x)} \right) \right|^2 \, dx,
\]
\[
\geq \int_{c}^{R_1} \left| h_+ \left( \frac{x}{T-t_+(R_1-)} \right) \right|^2 \, dx \to \infty, \text{ as } R_1 \to \infty,
\]
which is a contradiction. Therefore, there exists \( M_2 > 0 \) such that \( R_1 \leq M_2 \).

Denote \( y(0+) = y(0) \) and for \( y(0) < x < 0 \), then we have \( y(x) \leq y(0) < x < 0 \) and \( 0 \leq x - y(0) \leq x - y(x) \). This gives
\[
2M_1 \geq \tilde{J}(\alpha) \geq \int_{y(0)}^{0} \left| \frac{x-y(x)}{T} - K(x) \right|^2 \, dx,
\]
\[
= \frac{1}{2} \int_{y(0)}^{0} \left| \frac{x-y(x)}{T} \right|^2 \, dx - \int_{y(0)}^{0} |K(x)|^2 \, dx,
\]
\[
\geq \frac{1}{2} \int_{y(0)}^{0} \left| \frac{x-y(0)}{T} \right|^2 \, dx - M_1.
\]
Hence
\[
18M_1 T^2 \geq 3 \int_{y(0)}^{0} (x-y(0))^2 \, dx = -y(0)^3.
\]
This proves (i).

Now we write \( y(-c_1) = y(-c_1+) \), \( y(c_1) = y(c_1-) \). If \( y(-c_1) \geq -c_1 \) then (iii) is obvious. Thus we assume that \( y(-c_1) < -c_1 \), then for \( y(-c_1) < x < -c_1 \), we have \( y(x) \leq y(-c_1) \) and \( x - y(x) \geq x - y(-c_1) \geq 0 \). Since \( K(x) = 0 \) for \( x < -c_1 \), we obtain
\[
2M_1 \geq \tilde{J}(\alpha) \geq \int_{y(-c_1)}^{-c_1} \left| \frac{x-y(x)}{T} \right|^2 \, dx,
\]
\[
\geq \frac{1}{T^2} \int_{y(-c_1)}^{-c_1} |x-y(-c_1)|^2 \, dx,
\]
\[
= \frac{1}{3T^2} (-c_1 - y(-c_1))^3.
\]
That is,
\[
y(-c_1) \geq -(c_1 + (16M_1 T^2)^{1/3}).
\]
This proves (iii).

Similarly if \( y(c_1) \leq c_1 \), there is nothing to prove. Hence we assume that \( c_1 < y(c_1) \). By the choice of \( c_1 \), we have \( c_1 > R_1 \) and subsequently we get \( y(c_1) \leq y(x) \) for \( c_1 < x < y(c_1) \). Now it follows
that \( x - y(x) \leq x - y(c_1) \leq 0 \). Due to \( c_1 > c \) we have \( K(x) = 0 \) for \( x \in (c_1, y(c_1)) \), therefore

\[ 2M_1 \geq \tilde{J}(\alpha) \geq \int_{c_1}^{y(c_1)} \left[ \frac{x - y(x)}{T} \right]^2 \, dx, \]

\[ = \frac{1}{T^2} \int_{c_1}^{y(c_1)} \left( x - y(x) \right)^2 \, dx, \]

\[ \geq \frac{1}{T^2} \int_{c_1}^{y(c_1)} \left( x - y(c_1) \right)^2 \, dx, \]

\[ = \frac{1}{3T^2} (y(c_1) - c_1)^3. \]

Thus \( y(c_1) \leq (c_1 + (6T^2M_1)^{1/3}) \) and it proves (iv). This completes the proof of Step 2.

**Step 3:** Define \( M_3 = c_1 + (18T^2M_1)^{1/3} \) and

\[ \tilde{A}_2 = \{ \alpha = (T, R_1, R_2, y(\cdot)) \in \tilde{A} : y(x) = x \text{ if } x \notin [-M_3, M_3], R_1 \leq M_2, |y(0 +)| \leq (18M_1T^2)^{1/3} \}. \]

Then

\[ \inf_{\alpha \in \tilde{A}_2} \tilde{J}(\alpha) \leq \inf_{\alpha \in \tilde{A}} \tilde{J}(\alpha). \]

**Proof of Step 3.** From Step 1, we have

\[ \inf_{\alpha \in \tilde{A}_1} \tilde{J}(\alpha) \leq \inf_{\alpha \in \tilde{A}} \tilde{J}(\alpha). \]

Let \( M_1, M_2, M_3 \) and \( c_1 \) defined as above. Let \( \alpha = (T, R_1, R_2, y(\cdot)) \in \tilde{A}_1 \). Then from Step 2, \( R_1 \leq M_2, |y(0 +)| \leq (18M_1T^2)^{1/3} \) and, \( y(-c_1) \geq -M_3, y(c_1) \leq M_3 \). Let \( c_1 > R_1 \) and define

\[ \hat{y}(x) = \begin{cases} y(-c_1), & \text{if } x \in (\min(-c_1, y(-c_1)), -c_1), \\ y(x), & \text{if } x \in (-c_1, R_2) \cup (R_1, c_1), \\ y(c_1), & \text{if } x \in (c_1, \max(c_1, y(c_1))), \\ x, & \text{otherwise.} \end{cases} \]

Then \( \hat{\alpha} = (T, R_1, R_2, \hat{y}(\cdot)) \in \tilde{A}_2 \) and

\[ \tilde{J}(\hat{\alpha}) - \tilde{J}(\alpha) \leq \int_{\min(-c_1, y(-c_1))}^{-c_1} \left| \frac{x - y(-c_1)}{T} \right|^2 \, dx - \int_{\min(-c_1, y(-c_1))}^{-c_1} \left| \frac{x - y(x)}{T} \right|^2 \, dx \]

\[ + \int_{c_2}^{\max(c_2, y(c_2))} \left| \frac{x - y(c_2)}{T} \right|^2 \, dx - \int_{c_1}^{\max(c_1, y(c_1))} \left| \frac{x - y(x)}{T} \right|^2 \, dx \]

\[ \leq 0. \]

Since \( y(x) \leq y(-c_1) \), for \( x \in (\min(-c_1, y(-c_1)), -c_1) \) and \( y(x) \geq y(c_1) \) for \( x \in (c_1, \max(c_1, y(c_1))) \), we get

\[ \tilde{J}(\hat{\alpha}) \leq \tilde{J}(\alpha). \]

Due to \( \tilde{A}_2 \subset \tilde{A} \), we have

\[ \inf_{\alpha \in \tilde{A}_2} \tilde{J}(\alpha) \leq \inf_{\alpha \in \tilde{A}} \tilde{J}(\alpha) \leq \inf_{\alpha \in \tilde{A}} \tilde{J}(\alpha) \leq \inf_{\alpha \in \tilde{A}_2} \tilde{J}(\alpha). \]

This proves

\[ \inf_{\alpha \in \tilde{A}_2} \tilde{J}(\alpha) = \inf_{\alpha \in \tilde{A}} \tilde{J}(\alpha). \]
Step 4: Let \( \{\alpha_k\} \subset \tilde{\mathcal{A}}_2 \) be a sequence such that

\[
\lim_{k \to \infty} \tilde{\mathcal{J}}(\alpha_k) = \inf_{\alpha \in \tilde{\mathcal{A}}} \tilde{\mathcal{J}}(\alpha).
\]

Let \( \alpha_k = (T, R_{1,k}, R_{2,k}, y_k(\cdot)) \), as \( \alpha_k \in \tilde{\mathcal{A}}_3 \), we have \( \{R_{1,k}\}, \{R_{2,k}\} \) are bounded and \( y_k|_{[-M_3, M_3]} \) is a bounded non-decreasing function. Hence for a subsequence still denoted by \( \alpha_k \) such that \( \alpha_k \to \alpha_0 = (T, R_1, R_2, y(\cdot)) \in \tilde{\mathcal{A}}_2 \subset \mathcal{A} \) and

\[
\tilde{\mathcal{J}}(\alpha_0) = \inf_{\alpha \in \mathcal{A}} \tilde{\mathcal{J}}(\alpha).
\]

Since \( \alpha_0 \in \tilde{\mathcal{A}} \) we get \( \alpha_0 = (T, R_1, R_2, y(\cdot)) \in R(T) \), therefore from Theorem 1.1 there exists a \( u_0 \in L^\infty(\mathbb{R}) \) and the corresponding solution \( u \) of (1.1) satisfying \( R_1 = R_1(T) \), \( R_2 = R_2(T) \) and \( y(x) = y(x, T) \). As \( y(x) = x \) for \( x \in (-M_3, M_3) \) we obtain

\[
u_0(x) = \begin{cases} \theta_g & \text{if } x < -M_3, \\ \theta_f & \text{if } x > M_3, \end{cases}
\]

then \( u_0 \in \mathcal{A} \). Hence

\[
\mathcal{J}(u_0) = \inf_{u_0 \in \mathcal{A}} \mathcal{J}(u_0)
\]

has a solution. This proves the theorem.

\[\square\]

6. Reachable set for \((A, B)\) connection

**Definition 6.1.** Let \( A \geq \theta_f, B \leq \theta_g \) is called a \((A, B)\) connection if \( f(A) = g(B) \).

So far in this article we considered the case \( A = \theta_f \) or \( B = \theta_g \). Therefore from now onwards we assume that \( f'(A) > 0, g'(B) < 0 \).

**Definition 6.2.** \( u \) is called a \((A, B)\) entropy solution of (1.1) with initial data \( u_0 \) if \( u \) is the solution obtained from the Hamilton-Jacobi method as in [8], associated to given \((A, B)\) connection.

Let \( L_1 \leq R_1 \) and \( 0 \leq T_1, T_2 \leq T \) be such that \( f'(B) = \frac{R_1}{T - T_1}, g'(A) = \frac{L_1}{T - T_2} \). Let \( \tilde{B} \leq \theta_f \leq B, \quad A \leq \theta_g \leq \tilde{\mathcal{A}} \) be such that \( f(B) = f(\tilde{B}), g(A) = g(\tilde{\mathcal{A}}) \). Let \( (\tau^+, \xi^+, s_{\xi^+}), (\tau^-, \xi^-, s_{\xi^-}) \) be constructed as in lemma 3.8 for \((R_1, T_1)\) with \( \tilde{\alpha} = \tilde{B} \) (for the flux \( f \)) and \((L_1, T_2)\) with \( \tilde{\alpha} = \tilde{\mathcal{A}} \) (for the flux \( g \)) respectively.

**Definition 6.3.** (Reachable set) Let \((T, L_1, R_1, y(\cdot))\) is called an element in the reachable set \( \mathcal{R}^{A, B}(T) \) if they satisfy one of the following conditions:

1. \( y : (-\infty, L_1) \cup (R_1, \infty) \to \mathbb{R} \) be a non decreasing function such that

\[
\begin{align*}
& y(x) \leq 0 \quad \text{if } x < L_1, \\
& y(x) \geq 0 \quad \text{if } x > R_1.
\end{align*}
\]

Suppose there exist \( 0 \leq T_1, T_2 \leq T \) such that

\[
f'(B) = \frac{R_1}{T - T_1}, g'(A) = \frac{L_1}{T - T_2},
\]

then

\[
y(L_1-,T) \leq \xi^-, \quad y(R_1+,T) \geq \xi^+.
\]

2. \( If R_1 \geq 0, L_1 = 0, \) then \( y : (-\infty, R_1) \cup (R_1, \infty) \to \mathbb{R} \) be a non decreasing function with

\[
\begin{align*}
& y(x) \leq 0 \quad \text{if } x < R_1, \\
& y(x) \geq 0 \quad \text{if } x > R_1, \\
& y(L_1-,T) \geq \xi^-.
\end{align*}
\]

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3. If $R_1 = 0, L_1 \leq 0$, then $y : (-\infty, L_1) \cup (L_1, \infty) \to \mathbb{R}$ be a non decreasing function with
   \[
   \begin{cases}
   y(x) \leq 0 & \text{if } x < L_1, \\
   y(x) \geq 0 & \text{if } x > L_1, \\
   y(R_1+, T) \geq \xi_0^+.
   \end{cases}
   \]

4. In all the cases, the following must hold:
   \[
   \sup_x |x - y(x)| < \infty.
   \]

Then we have the following:

**Theorem 6.1** (characterization of $\mathcal{R}^{A,B}(T)$). $(T, L_1, R_1, y(\cdot)) \in \mathcal{R}^{A,B}(T)$ if and only if there exist a $u_0 \in L^\infty(\mathbb{R})$ and the corresponding $(A, B)$ entropy solution $u$ of (1.1) satisfy
   \[
   (T, L_1, R_1, y(\cdot)) = (T, L_1(T), R_1(T), y(\cdot, T)),
   \]
   where $(L_1(T), R_1(T), y(\cdot, T))$ are defined by $u$.

As earlier we can decompose the domain $\mathbb{R} \times (0, T)$ into three disjoint regions $D_1, D_2$ and $D_3$. Here we only sketch the proof of backward construction and the rest follows as earlier.

I. Backward construction (continuous and shock solutions): Define
   
   (i) $h_+ : [f'(B), \infty) \to [g'(\bar{A}, \infty)$ by $h_+ = g' \circ g_1^{-1} \circ f \circ f'$.
   
   (ii) $h_- : (-\infty, g'(A)] \to (-\infty, f'(\bar{A})$ by $h_- = f' \circ f_1^{-1} \circ g \circ g'$.

Then $h_\pm$ are isomorphisms and by R-H condition across the interface, using $h_\pm$ it follows as in earlier case

(i) There are no forward rarefaction from the interface.

(ii) Continuous and shock solutions are constructed.

(iii) Using this and $L^1$-contractivity for solutions with discrete set of discontinuities of the solution, one can glue them to obtain a solution in $D_2 \cup D_3$, where $D_2$ and $D_3$ are described earlier.

II. Backward construction in $D_1$. This is the case where the $(A, B)$ entropy exist. Assume that $(T, L_1, R_1, y(\cdot)) \in \mathcal{R}^{A,B}(T)$ satisfies (6.1) with
   
   \[
   y(x) = \begin{cases}
   y_+ & \text{if } x > R_1, \\
   y_- & \text{if } x < L_1.
   \end{cases}
   \]

   $f'(B) = \frac{R_1}{T - T_1}, g'(A) = \frac{L_1}{T - T_2}$. Let $(\tau_0^\pm, \xi_0^\pm, s_0^\pm)$ be as defined earlier. Define

   \[
   \begin{align*}
   f'(\beta_-) &= \frac{R_1 - \xi_0^+}{T}, \\
   g'(\beta_-) &= \frac{L_1 - \xi_0^-}{T}, \\
   f'(u_+) &= \frac{R_1 - y_+}{T}, \\
   g'(u_-) &= \frac{L_1 - y_-}{T}, \\
   \gamma_1(t) &= R_1 + \frac{f(u_+) - f(\beta_+)}{u_+ - \beta_+}(t - T), \\
   \gamma_2(t) &= R_1 + f'(\beta_+)(t - T), \\
   \gamma_3(t) &= -f'(\bar{\beta})(t - \tau_0^+), \\
   \eta_3(t) &= -g'(\bar{A})(t - \tau_0^+), \\
   \eta_2(t) &= L_1 + g'(\beta_-)(t - T), \\
   \eta_1(t) &= L_1 + \frac{g(\beta_-) - g(u_-)}{\beta_- - u_-}(t - T).
   \end{align*}
   \]
Since \((T, L_1, R_1, y(\cdot)) \in \mathcal{R}_{A,B}(T)\), hence by condition (6.1) and convexity of \(f\) and \(g\), we have
\[y_- \leq \xi_-^1 \leq \xi_0^0 \leq 0 \leq \xi_0^+ \leq y_+\] where \(\xi_-^1 = \eta_2(0), \xi_0^+ = \xi_2(0)\). Define
\[
u_{1,0}(x) = \begin{cases} u_- & \text{if } x < \xi_-^1, \\ \beta_- & \text{if } \xi_-^1 < x < \xi_0^- \\ A & \text{if } \xi_0^- < x < 0, \\ B & \text{if } 0 < x < \xi_0^+ \\ \beta_+ & \text{if } \xi_0^+ < x < \xi_1^+ \\ u_+ & \text{if } x > \xi_1^+ \end{cases}
\]
and for \(0 < t \leq T\), (see figure 14 for illustration)
\[
u_1(x,t) = \begin{cases} u_- & \text{if } x < \eta_1(t), \\ \beta_- & \text{if } \eta_1(t) < x < \eta_2(t), \\ g'^{-1} \left( \frac{x - \xi_0^-}{t} \right) & \text{if } \eta_2(t) < x < \xi_0^-(t), \tau_0^- \leq t < T, \\ A & \text{if } \eta_2(t) < x < \eta_3(t), 0 < t < \tau_0^- \\ \bar{A} & \text{if } \eta_3(t) < x < 0, 0 < t < \tau_0^- \\ A & \text{if } s_{\xi_0}^-(t) < x < 0, \tau_0^- < x < 0, \\ B & \text{if } 0 < x < s_{\xi_0}^-(t), \\ \bar{B} & \text{if } 0 < x < \gamma_3(t), 0 < t < \tau_0^+ \\ f'^{-1} \left( \frac{x - \xi_0^+}{t} \right) & \text{if } s_{\xi_0}^+(t) < x < \gamma_2(t), \tau_0^+ \leq t < T, \\ \beta_+ & \text{if } \gamma_3(t) < x < \gamma_2(t), 0 < t < \tau_0^+ \\ u_+ & \text{if } x > \gamma_1(t). \end{cases}
\]

Then \(\nu_1\) is the \((A,B)\) entropy solution of (1.1) with \(\nu_{1,0}\) as the initial data.

7. Appendix

Proof of lemma 2.1. From the hypothesis on \(\{f_k\} \text{ and } \{g_k\}\), it follows that \(\lim_{k \to \infty} (f_k^*, g_k^*) \to (f^*, g^*)\) in \(C^1_{loc}(\mathbb{R})\). Since \(\{u_{0,k}\}\) is uniformly bounded in \(L^\infty(\mathbb{R})\) and converges to \(u_0\) in \(weak^*-L^\infty(\mathbb{R})\). Hence \(\{v_{0,k}\}\) converges to \(v_0\) uniformly on compact subsets of \(\mathbb{R}\) and having uniformly Lipschitz constant. Hence
\{v_k\} are having uniformly Lipschitz constant. Hence by Arzela-Ascoli theorem, there exists a subsequence still denoted by \{v_k\} converges to \(w\) in \(C^0_{\text{loc}}(\mathbb{R} \times [0, \infty))\).

Claim: \(\lim_{k \to \infty} ch_k(x, t) \subset ch(x, t)\), \(v = w\).

For \(\gamma_k \in ch_k(x, t)\), then from lemma 4.2 of [5] (page 38), \(\left\{\frac{d\gamma_k}{d\theta}\right\}\) is uniformly bounded and hence for subsequence \(\{\gamma_k\}\) converges to \(\tilde{\gamma} \in ch(x, t)\). In order to prove the claim we need to show that \(\tilde{\gamma} \in ch(x, t)\).

If \(\gamma \in c(x, t)\) then \(v_k(x, t) = \Gamma_{v_0, k, \gamma_k}(x, t) \leq \Gamma_{v_0, k, \gamma}(x, t)\). Letting \(k = k_i\) and \(k_i \to \infty\) to obtain

\[
\begin{align*}
  w(x, t) &= \lim_{k_i \to \infty} v_k_i(x, t) = \Gamma_{v_0, \tilde{\gamma}}(x, t) \\
  &\leq \Gamma_{v_0, \gamma}(x, t).
\end{align*}
\]

Hence \(\tilde{\gamma} \in ch(x, t)\) and

\[
  w(x, t) = \inf_{\gamma \in c(x, t)} \Gamma_{v_0, \gamma}(x, t) = v(x, t).
\]

This proves the claim. Hence by uniqueness of the limit, it follows that \(\lim v_k = v\) in \(C^0_{\text{loc}}(\mathbb{R})\) and \(\lim ch_k(x, t) \subset ch(x, t)\). Since Lipschitz constant of \(\{v_k\}\) are uniformly bounded, hence for any \(\varphi \in C^1_{\text{loc}}(\mathbb{R} \times (0, \infty))\), we have for \(\Omega = \mathbb{R} \times (0, \infty)\)

\[
\lim_{k \to \infty} \int_{\Omega} \frac{\partial v_k}{\partial x} \varphi dx dt = -\lim_{k \to \infty} \int_{\Omega} v_k \frac{\partial \varphi}{\partial x} dx dt = -\int_{\Omega} \lim_{k \to \infty} v_k \frac{\partial \varphi}{\partial x} dx dt \tag{7.1}
\]

\[
= -\int_{\Omega} v \frac{\partial \varphi}{\partial x} dx dt. \tag{7.2}
\]

Hence \(\frac{\partial v_k}{\partial x} \to \frac{\partial v}{\partial x}\) in \(D'(\Omega)\). This proves the lemma. \(\square\)

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