ON THE SPECTRUM OF RINGS OF FUNCTIONS

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Abstract. For $D$ a domain and $E \subseteq D$, we investigate the prime spectrum of rings of functions from $E$ to $D$, that is, of rings contained in $\prod_{e \in E} D$ and containing $D$. Among other things, we characterize, when $M$ is a maximal ideal of finite index in $D$, those prime ideals lying above $M$ which contain the kernel of the canonical map to $\prod_{e \in E} (D/M)$ as being precisely the prime ideals corresponding to ultrafilters on $E$. We give a sufficient condition for when all primes above $M$ are of this form and thus establish a correspondence to the prime spectra of ultraproducts of residue class rings of $D$. As a corollary, we obtain a description using ultrafilters, differing from Chabert’s original one which uses elements of the $M$-adic completion, of the prime ideals in the ring of integer-valued polynomials $\text{Int}(D)$ lying above a maximal ideal of finite index.

1. Introduction

Let $D$ be an integral domain, $E \subseteq D$, and $\mathcal{R}$ a subring of $\prod_{e \in E} D$, containing $D$. The elements of $\mathcal{R}$ can be interpreted as functions from $E$ to $D$ and, consequently, we call $\mathcal{R}$ a ring of functions from $E$ to $D$. We will investigate the prime spectra of such rings of functions. We obtain, for quite general $\mathcal{R}$, a partial description of the prime spectrum, cf. Theorems 3.7 and 5.3, and in special cases a complete characterization, cf. Corollary 6.5.

Our motivation is the spectrum of a ring of integer-valued polynomials: For $D$ an integral domain with quotient field $K$, let $\text{Int}(D) = \{ f \in K[x] \mid f(D) \subseteq D \}$ be the ring of integer-valued polynomials on $D$. More generally, when $K$ is understood, we let $\text{Int}(A,B) = \{ f \in K[x] \mid f(A) \subseteq B \}$ for $A,B \subseteq K$.

If $D$ is a Noetherian one-dimensional domain, a celebrated theorem of Chabert [1, Ch. V] states that every prime ideal of $\text{Int}(D)$ lying over a maximal ideal $M$ of finite index in $D$ is maximal and of the form $M_\alpha = \{ f \in \text{Int}(D) \mid f(\alpha) \in \hat{M} \}$, where $\alpha$ is an element of the $M$-adic completion $\hat{D}_M$ of $D$ and $\hat{M}$ the maximal ideal of $\hat{D}_M$. 

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In fact, Chabert showed two separate statements independently – both under the assumption that $D$ is Noetherian and one-dimensional and $M$ a maximal ideal of finite index of $D$:

1. Every maximal ideal of $\text{Int}(D)$ containing $\text{Int}(D, M)$ is of the form $M_\alpha$ for some $\alpha \in \hat{D}_M$.
2. Every maximal ideal of $\text{Int}(D)$ lying over $M$ contains $\text{Int}(D, M)$.

For a simplified proof of Chabert’s result, see [4], Lemma 4.4 and the remark following it.

We will show that a modified version of statement (1) holds in far greater generality, for rings of functions. The modification consists in replacing elements of the $M$-adic completion by ultrafilters.

Whether (2) holds or not for a particular $D$ and a particular subring of $D^E$ will have to be examined separately. It is, in some sense, a question of density of the subring in the product $\prod_{e \in E} D$.

We will work in the following setting:

**Definition 1.1.** Let $D$ be a commutative ring and $E \subseteq D$. Let $R$ be a commutative ring and $\varphi: R \to \prod_{e \in E} D$ a monomorphism of rings. $\varphi$ allows us to interpret the elements of $R$ as functions from $E$ to $D$.

If all constant functions are contained in $\varphi(R)$, we call the pair $(R, \varphi)$ a ring of functions from $E$ to $D$. We use $R = R(E, D)$ (where $\varphi$ is understood) to denote a ring of functions from $E$ to $D$.

**Remark 1.2.** For our considerations it is vital that $R = R(E, D)$ contain all constant functions, because we will make extensive use of the following fact: when $I$ is an ideal of $R = R(E, D)$, $f \in I$ and $g \in D[x]$ a polynomial with zero constant term, then $g(f) \in I$, and similarly, if $g$ is a polynomial in several variables over $D$ with zero constant term, and an element of $I$ is substituted for each variable in $g$, then, an element of $I$ results.

Let us note that considerable research has been done on the spectrum of a power of a ring $D^E = \prod_{d \in E} D$ or a product of rings $\prod_{e \in E} D_e$. Gilmer and Heinzer [5, Prop. 2.3] have determined the spectrum of an infinite product of local rings, and Levy, Loustaunau and Shapiro [8] that of an infinite power of $\mathbb{Z}$. Our focus here is not on the full product of rings, but on comparatively small subrings and the question of how much information about the spectrum of a ring can be obtained from its embedding in a power of a domain.

One ring can be embedded in different products: $\text{Int}(D)$ can be seen as a ring of functions from $K$ to $K$ as well as a ring of functions from $D$ to $D$. We will glean a lot more information about the spectrum of $\text{Int}(D)$ from the second interpretation than from the first.
2. Prime ideals corresponding to ultrafilters

Let \( \mathcal{R} = \mathcal{R}(E, D) \) be a ring of functions from \( E \) to \( D \) as in Definition 1.1. We will now make precise the concept of ideals corresponding to ultrafilters, and the connection to ultraproducts \( \prod_{s \in S} D/M \), where \( M \) is a maximal ideal of \( D \), and \( \mathcal{U} \) an ultrafilter on \( E \). First a quick review of filters, ultrafilters and ultraproducts:

**Definition 2.1.** Let \( S \) be a set. A non-empty collection \( F \) of subsets of \( S \) is called a filter on \( S \) if

1. \( \emptyset \not\in F \).
2. \( A, B \in F \) implies \( A \cap B \in F \).
3. \( A \subseteq C \subseteq S \) with \( A \in F \) implies \( C \in F \).

A filter \( F \) on \( S \) is called an ultrafilter on \( S \) if, for every \( C \subseteq S \), either \( C \in F \) or \( S \setminus C \in F \).

Let \( S \) be a fixed set and \( \mathcal{P}(S) \) its power-set. For \( C \in \mathcal{P}(S) \), a superset of \( C \) is a set \( D \in \mathcal{P}(S) \) with \( C \subseteq D \subseteq S \). A collection \( C \) of subsets of \( S \) is said to have the finite intersection property if the intersection of any finitely many members of \( C \) is non-empty.

**Remark 2.2.** Clearly, a necessary and sufficient condition for \( C \subseteq \mathcal{P}(S) \) to be contained in a filter on \( S \) is that \( C \) satisfies the finite intersection property. If the finite intersection property is satisfied, then the supersets of finite intersections of members of \( C \) form a filter.

Although, strictly speaking, we do not need ultraproducts to prove our results, we will nevertheless introduce them, because they provide context, in particular to Lemma 2.6 and to sections 3 and 5.

**Definition 2.3.** Let \( S \) be an index set and \( \mathcal{U} \) an ultrafilter on \( S \). Suppose we are given, for each \( s \in S \), a ring \( R_s \). Then the ultraproduct of rings \( \prod_{s \in S} R_s \) is defined as the direct product \( \prod_{s \in S} R_s \) modulo the congruence relation

\[
(r_s)_{s \in S} \sim (t_s)_{s \in S} \iff \{ s \in S \mid r_s = t_s \} \in \mathcal{U}.
\]

Ultraproducts of other algebraic structures are defined analogously. The usefulness of ultraproducts is captured by the Theorem of Loš (cf. [6, Chpt. 3.2] or [7, Prop 1.6.14]) which states that an ultraproduct \( \prod_{s \in S} R_s \) satisfies a first-order formula if and only if the set of indices \( s \) for which \( R_s \) satisfies the formula is in \( \mathcal{U} \). Here first-order formula means a formula in the first-order language whose only non-logical symbols (apart from the equality sign) are symbols for the algebraic operations; for instance, + and \( \cdot \) in the case of an ultraproduct of rings.

**Definition 2.4.** Let \( D \) be a domain, \( E \subseteq D \), \( \mathcal{R} = \mathcal{R}(E, D) \) a ring of functions, \( I \) an ideal of \( D \) and \( \mathcal{F} \) a filter on \( E \).

For \( f \in \mathcal{R}(E, D) \), we let \( f^{-1}(I) = \{ e \in E \mid f(e) \in I \} \) and define

\[
I_{\mathcal{F}} = \{ f \in \mathcal{R}(E, D) \mid f^{-1}(I) \in \mathcal{F} \}.
\]
Remark 2.5. Let everything as in Definition 2.4, \(I, J\) ideals of \(D\) and \(F, G\) filters on \(E\). Some easy consequences of Definition 2.4 are:

1. If \(I \neq D\) then \(I_F \neq R\).
2. \(I_F\) is an ideal of \(R\) containing \(R(E, I) = \{ f \in R \mid f(E) \subseteq I\}\).
3. \(I \subseteq J \implies I_F \subseteq J_F\)
4. \(F \subseteq G \implies I_F \subseteq I_G\)

Lemma 2.6. Let \(D\) be a domain, \(E \subseteq D\), and \(R = R(E, D)\) a ring of functions from \(E\) to \(D\).

Then for every prime ideal \(P\) of \(D\) and every ultrafilter \(U\) on \(E\), \(P_U\) is a prime ideal of \(R\).

Proof. Easy direct verification: let \(fg \in P_U\); because \(P\) is a prime ideal of \(D\), the inverse image of \(P\) under \(f \cdot g\) is the union of \(f^{-1}(P)\) and \(g^{-1}(P)\). If the union of two sets is in an ultrafilter, then one of them must be in the ultrafilter. Therefore, \(f \in P_U\) or \(g \in P_U\). Also, \(P_U\) cannot be all of \(R\) because it doesn’t contain the constant function 1. \(\square\)

One way of looking at \(P_U\) is by considering the following commuting diagram of ring-homomorphisms, where \(\pi\) and \(\pi_1\) mean applying the canonical projection in each factor of the product, and \(\sigma\) and \(\sigma_1\) mean factoring through the defining congruence relation of an ultraproduct.

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\varphi} & \prod_{e \in E} D \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
\prod_{e \in E} (D/P) & \xrightarrow{\sigma} & \prod_{e \in E} (D/P)
\end{array}
\]

\(P_U\) is the kernel of the following composition of ring homomorphisms:

\[
\varphi: \mathcal{R} \to \prod_{e \in E} D
\]

followed by the canonical projection

\[
\pi: \prod_{e \in E} D \to \prod_{e \in E} (D/P)
\]

and the canonical projection

\[
\sigma: \prod_{e \in E} (D/P) \to \prod_{e \in E} (D/P)
\]

Since \(D/P\) is an integral domain, any ultraproduct of copies of \(D/P\) is also an integral domain, by the Theorem of Loś. Therefore \((0)\) is a prime ideal of \(\prod_{e \in E} (D/P)\) and hence \(P_U\) a prime ideal of \(\mathcal{R}\). We also see that \(P_U\) is the inverse
image of a prime ideal of \( \prod_{e \in E} D \) under \( \varphi \), and further, of a prime ideal of the ultraproduct \( \prod_{\sigma \in E} D \) under \( \sigma_1 \circ \varphi \).

3. The set of zero-loci mod \( M \) of an ideal of the ring of functions

As before, \( D \) is a domain with quotient field \( K \), \( E \subseteq D \) and \( R = R(E, D) \) a ring of functions from \( E \) to \( D \) as in Def. \[1.1\]. Especially, recall from Def. \[1.1\] that \( R \) is assumed to contain all constant functions.

**Definition 3.1.** For \( M \subseteq D \) and \( f \in R = R(E, D) \), let

\[
    f^{-1}(M) = \{ e \in E \mid f(e) \in M \}.
\]

For an ideal \( M \) of \( D \) and an ideal \( \mathcal{I} \) of \( R \), let

\[
    \mathcal{Z}_M(\mathcal{I}) = \{ f^{-1}(M) \mid f \in \mathcal{I} \}
\]

Recall from Def. \[2.4\] that for a filter \( \mathcal{F} \) on \( E \),

\[
    M_\mathcal{F} = \{ f \in R(E, D) \mid f^{-1}(M) \in \mathcal{F} \}
\]

**Remark 3.2.** Note that the above definition implies

1. \( \mathcal{I} \subseteq \mathcal{J} \implies \mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{Z}_M(\mathcal{J}) \)
2. \( \mathcal{I} \subseteq M_\mathcal{F} \iff \mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{F} \)

**Lemma 3.3.** Let \( M \) be an ideal of \( D \) and \( \mathcal{I} \) an ideal of \( R \). The following are equivalent:

(a) There exists a filter \( \mathcal{F} \) on \( E \) such that \( \mathcal{I} \subseteq M_\mathcal{F} \).
(b) \( \mathcal{Z}_M(\mathcal{I}) \) satisfies the finite intersection property.

**Proof.** If \( \mathcal{I} \subseteq M_\mathcal{F} \), then \( \mathcal{Z}_M(\mathcal{I}) \) is contained in \( \mathcal{F} \) and hence satisfies the finite intersection property. Conversely, if \( \mathcal{Z}_M(\mathcal{I}) \) satisfies the finite intersection property then, by Remark \[2.2\], the supersets of finite intersections of sets in \( \mathcal{Z}_M(\mathcal{I}) \) form a filter \( \mathcal{F} \) on \( E \) for which \( \mathcal{Z}_M(\mathcal{I}) \subseteq \mathcal{F} \) and hence \( \mathcal{I} \subseteq M_\mathcal{F} \). \( \Box \)

In the case where \( R(E, D) = \prod_{e \in E} D \) is the ring of all functions from \( E \) to \( D \), much more can be said; see the papers by Gilmer and Heinzer \[5\] Prop. 2.3\] (concerning local rings) and Levy, Loustaunau and Shapiro \[8\] (concerning \( D = \mathbb{Z} \)).

For a field \( K \) that is not algebraically closed, we will need, for an arbitrary \( n \geq 2 \), an \( n \)-ary form that has no zero but the trivial one. For this purpose, recall how to define a norm form: if \( L : K \) is an \( n \)-dimensional field extension, multiplication by any \( w \in L \) is a \( K \)-endomorphism \( \psi_w \) of \( L \). For a fixed choice of a \( K \)-basis of \( L \), map every \( w \in L \) to the determinant of the matrix of \( \psi_w \) with respect to the chosen basis. This mapping, regarded as a function of the coordinates of \( w \) with respect to the chosen basis, is easily seen to be an \( n \)-ary form that has no zero but the trivial one.
Lemma 3.4. Let $M$ be a maximal ideal of $D$ such that $D/M$ is not algebraically closed. Then for every ideal $I$ of $R = R(E,D)$, $Z_M(I)$ is closed under finite intersections.

Proof. Given $f, g \in I$, we show that there exists $h \in I$ with
\[ h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M). \]
Consider any finite-dimensional non-trivial field extension of $D/M$, and let $n$ be the degree of the extension. The norm form of this field extension is a homogeneous polynomial in $n \geq 2$ indeterminates whose only zero in $(D/M)^n$ is the trivial one. By identifying $n - 1$ variables, we get a binary form $\bar{s} \in (D/M)[x,y]$ with no zero in $(D/M)^2$ other than $(0,0)$. Let $s \in D[x,y]$ be a binary form that reduces to $\bar{s}$ when the coefficients are taken mod $M$.

Now, given $f$ and $g$ in $I$, we set $h = s(f,g)$. By the fact that $R$ contains all constant functions, $h$ is in $I$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. \qed

Lemma 3.5. Let $M$ be a maximal ideal of $D$ and $\mathcal{R} = R(E,D)$ a ring of functions such that every $f \in \mathcal{R}$ takes values in only finitely many residue classes mod $M$.

Then for every ideal $I$ of $\mathcal{R}$, $Z_M(I)$ is closed under finite intersections.

Proof. Again, given $f, g \in I$, we show that there exists $h \in I$ with
\[ h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M). \]
Let $A, B \subseteq D/M$ be finite sets of residue classes of $D$ mod $M$ such that $f(A)$ is contained in the union of $A$ and $g(E)$ in the union of $B$.

We can interpolate any function from $(D/M)^2$ to $(D/M)$ at any finite set of arguments by a polynomial in $(D/M)[x,y]$. Pick $\bar{s} \in (D/M)[x,y]$ with $\bar{s}(0,0) = 0$ and $\bar{s}(a,b) = 1$ for all $(a,b) \in (A \times B) \setminus \{(0,0)\}$. Let $s \in D[x,y]$ be a polynomial with zero constant coefficient that reduces to $\bar{s}$ when the coefficients are taken mod $M$.

Now, given $f$ and $g$ in $I$, we set $h = s(f,g)$. By the fact that $\mathcal{R}$ contains all constant functions, $h$ is in $I$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. \qed

Definition 3.6. Let $\mathcal{R} = R(E,D)$ be a ring of functions and $M$ an ideal of $D$.
We call $f \in \mathcal{R}$ an $M$-unit-valued function if $f(e) + M$ is a unit in $D/M$ for every $e \in E$.

Theorem 3.7. Let $M$ be a maximal ideal of $D$ and $I$ an ideal of $\mathcal{R} = R(E,D)$.
Assume that either $D/M$ is not algebraically closed or that each function in $\mathcal{R}$ takes values in only finitely many residue classes mod $M$.

1. $I$ is contained in an ideal of the form $M_F$ for some filter $F$ on $E$ if and only if $I$ contains no $M$-unit-valued function.

2. Every ideal $Q$ of $\mathcal{R}$ that is maximal with respect to not containing any $M$-unit-valued function is of the form $M_U$ for some ultrafilter $U$ on $E$. 
In particular, every maximal ideal of $R$ that does not contain any $M$-unit-valued function is of the form $M_U$ for some ultrafilter $U$ on $E$.

Proof. Ad (1). If $I$ is contained in an ideal of the form $M_F$, $I$ cannot contain any $M$-unit-valued function, because $F$ doesn’t contain the empty set.

Conversely, suppose that $I$ does not contain any $M$-unit-valued function. Then $\emptyset \notin Z_M(I)$. By Lemmata 3.4 and 3.5 $Z_M(I)$ is closed under finite intersections.

$Z_M(I)$, therefore, satisfies the finite intersection property. By Remark 2.2, $Z_M(I)$ is contained in a filter $F$ on $E$. For this filter, $I \subseteq M_F$, by Remark 3.2.

Ad (2). Suppose $Q$ is maximal with respect to not containing any $M$-unit-valued function. By (1), $Q \subseteq M_F$ for some filter $F$. Refine $F$ to an ultrafilter $U$. Then, by Remark 2.5, $Q \subseteq M_F \subseteq M_U$, and $M_U$ doesn’t contain any $M$-unit-valued function. Since $Q$ is maximal with this property, $Q = M_U$.

(3) is a special case of (2). □

4. A dichotomy of maximal ideals

In what follows, $D$ is always a domain with quotient field $K$, $E \subseteq D$ and $R = R(E,D)$ a ring of functions from $E$ to $D$ as in Def. 1.1. When the interpretation of $R$ as a subring of $\prod_{e \in E} D$ is understood, then for $M \subseteq D$ we let

$$R(E,M) = \{f \in R \mid f(E) \subseteq M\}.$$ 

Proposition 4.1. Let $M$ be a maximal ideal of $D$ and $Q$ a maximal ideal of $R = R(E,D)$. Then exactly one of the following two statements holds:

1. $Q$ contains $R(E,M) = \{f \in R \mid f(E) \subseteq M\}$

2. $Q$ contains an element $f$ with $f(e) \equiv 1 \mod M$ for all $e \in E$.

Proof. The two cases are mutually exclusive, because any ideal $Q$ satisfying both statements must contain 1.

Now suppose $Q$ does not contain $R(E,M)$. Let $g \in R(E,M) \setminus Q$. By the maximality of $Q$, $1 = h(x)g(x) + f(x)$ for some $h \in R$ and $f \in Q$. We see that $f(x) = 1 - h(x)g(x) \in Q$ satisfies $f(e) \equiv 1 \mod M$ for all $e \in E$. □

Recall that a function $f \in R$ is called $M$-unit-valued if $f(e) + M$ is a unit in $D/M$ for every $e \in E$.

Lemma 4.2. Let $M$ be an ideal of $D$ and $Q$ an ideal of $R = R(E,D)$. The following are equivalent:

A. $Q$ contains an element $f$ with $f(e) \equiv 1 \mod M$ for all $e \in E$.

B. $Q$ contains an $M$-unit-valued function that takes values in only finitely many residue classes mod $M$.

Proof. To see that the a priori weaker statement implies the stronger, let $g \in Q$ be an $M$-unit-valued function taking only finitely many different values mod $M$. Let $d_1, \ldots, d_k \in D$ be representatives of the finitely many residue classes mod $M$ intersecting $g(E)$ non-trivially, and $u \in D$ an inverse mod $M$ of $(-1)^{k+1}d_1 \cdots d_k$. 

Then
\[ h(x) = \prod_{i=1}^{k} (g(x) - d_i) - (-1)^k d_1 \cdot \ldots \cdot d_k \]
is in \( Q \) and \( h(e) \equiv (-1)^{k+1} d_1 \cdot \ldots \cdot d_k \mod M \) for all \( e \in E \). Therefore \( f(x) = uh(x) \in Q \) satisfies \( f(e) \equiv 1 \mod M \) for all \( e \in E \). \( \square \)

**Proposition 4.3.** Let \( M \) be a maximal ideal of \( D \) and \( Q \) a maximal ideal of \( \mathcal{R} = \mathcal{R}(E, D) \). If each \( f \in \mathcal{R} \) takes values in only finitely many residue classes mod \( M \) (in particular, if \( D/M \) happens to be finite) then exactly one of the following statements holds:

1. \( Q \) contains \( \mathcal{R}(E, M) = \{ f \in \mathcal{R} \mid f(E) \subseteq M \} \)
2. \( Q \) contains an \( M \)-unit-valued function.

**Proof.** This follows directly from Proposition 4.1 and Lemma 4.2. \( \square \)

The Propositions in this section partition the maximal ideals of \( \mathcal{R} \) lying over a maximal ideal \( M \) of \( D \) into two types: those containing \( \mathcal{R}(E, M) \) (the kernel of the restriction to \( \mathcal{R} \) of the canonical projection \( \pi: \prod_{e \in E} D \longrightarrow \prod_{e \in E} (D/M) \)), and the others.

In some cases, it is known that all maximal ideals of \( \mathcal{R} \) lying over \( M \) contain \( \mathcal{R}(E, M) \), notably if \( \mathcal{R} = \text{Int}(D) \) and \( M \) is finitely generated and of finite index in \( D \) \([1, \text{Ch. V}], [4, \text{Lemma 4.4}]\). We will find a sufficient condition for all maximal ideals of \( \mathcal{R} \) lying over \( M \) to contain \( \mathcal{R}(E, M) \) in Theorem 6.4.

We must not discount the possibility of a maximal ideal \( Q \) lying over \( M \) containing an \( M \)-unit-valued function, however. If \( D \) is an infinite domain, \( D[x] \) is embedded in \( D^D \) by mapping every polynomial to the corresponding polynomial function. When \( D/M \) is not algebraically closed, then there are certainly maximal ideals of \( D[x] \) lying over \( M \) that contain polynomials without a zero mod \( M \).

### 5. Prime ideals containing \( \mathcal{R}(E, M) \)

We are now in a position to characterize the prime ideals of \( \mathcal{R} \) containing \( \mathcal{R}(E, D) \) as being precisely the ideals of the form \( M_U \) for ultrafilters \( U \) on \( E \), under the following hypothesis: every \( f \in \mathcal{R} \) takes values in only finitely many residue classes of \( M \).

This hypothesis may seem only marginally weaker than the assumption that \( D/M \) is finite. Note however, that it is sometimes satisfied for infinite \( D/M \) under perfectly natural circumstances, for instance, when \( E \) intersects only finitely many residue classes of \( M^n \) for each \( n \in \mathbb{N} \), \( E \) precompact, and \( \mathcal{R} \) consists of functions that are uniformly \( M \)-adically continuous.

As in the case of integer-valued polynomials, we can show that every prime ideal of \( \mathcal{R}(E, D) \) containing \( \mathcal{R}(E, M) \) is maximal under certain conditions, notably if \( D/M \) is finite. The proof for \( \text{Int}(D) \), when \( D/M \) is finite \([1, \text{Lemma V.1.9}]\), carries over practically without change. Note that Definition 1.1 ensures that every ring
of functions \( R \) contains all constant functions – an essential requirement of the following proof.

**Lemma 5.1.** Let \( M \) be a maximal ideal of \( D \) such that every function in \( R = R(E, D) \) takes values in only finitely many residue classes mod \( M \), and \( Q \) a prime ideal of \( R(E, D) \) containing \( R(E, M) \). Then \( Q \) is maximal and \( R/Q \) is isomorphic to \( D/M \).

**Proof.** Let \( Q \) be a prime ideal of \( R(E, D) \) containing \( R(E, M) \), and \( A \) a system of representatives of \( D \) mod \( M \). It suffices to show that \( A \) (viewed as a set of constant functions) is also a system of representatives of \( R \) mod \( Q \). Let \( f \in R(E, D) \) and \( a_1, \ldots, a_r \in A \) the representatives of those residue classes of \( M \) that intersect \( f(E) \) non-trivially. Then \( \prod_{i=1}^{r}(f - a_i) \) is in \( R(E, M) \subseteq Q \) and, \( Q \) being prime, one of the factors \( f - a_i \) must be in \( Q \). This shows that \( f \) is congruent mod \( Q \) to one of the constant functions \( a_1, \ldots, a_r \), and, in particular, to an element of \( A \). Therefore, \( A \) is a system of representatives of \( R(E, D) \) mod \( Q \). \( \square \)

**Lemma 5.2.** Let \( R = R(E, D) \) a ring of functions and \( M \) a maximal ideal of \( D \) such that every \( f \in R \) takes values in only finitely many residue classes of \( M \). Let \( I \) be an ideal of \( R \).

Then \( I \) is contained in an ideal of the form \( M_{\mathcal{F}} \) for a filter \( \mathcal{F} \) on \( E \) if and only if \( R(E, M) \subseteq I \).

**Proof.** \( R(E, M) \subseteq I \) is equivalent to \( I \) not containing an \( M \)-unit-valued function, by Proposition 4.3. The statement therefore follows from part (1) of Theorem 3.7. \( \square \)

**Theorem 5.3.** Let \( R = R(E, D) \) a ring of functions, and \( M \) a maximal ideal of \( D \). If every \( f \in R \) takes values in only finitely many residue classes of \( M \) (and, in particular, if \( D/M \) is finite), then the prime ideals of \( R \) containing \( R(E, M) \) are exactly the ideals of the form \( M_{\mathcal{U}} \) with \( \mathcal{U} \) an ultrafilter on \( E \). Each of them is maximal and its residue field isomorphic to \( D/M \).

**Proof.** Let \( Q \) be a prime ideal of \( R \) containing \( R(E, M) \). By Lemma 5.1, \( Q \) is maximal and \( R/Q \) is isomorphic to \( D/M \). By Lemma 5.2, \( Q \subseteq M_{\mathcal{F}} \) for some filter \( \mathcal{F} \) on \( E \). \( \mathcal{F} \) can be refined to an ultrafilter \( \mathcal{U} \) on \( E \), and then \( Q \subseteq M_{\mathcal{F}} \subseteq M_{\mathcal{U}} \neq R \), by Remark 2.5. Since \( Q \) is maximal, \( Q = M_{\mathcal{U}} \) follows.

Conversely, every ideal of the form \( M_{\mathcal{U}} \) for an ultrafilter \( \mathcal{U} \) on \( E \) is prime, by Lemma 2.6 and contains \( R(E, M) \), by Remark 2.5. \( \square \)

Note, in particular, that Theorems 3.7 and 5.3 apply to \( R = \text{Int}(E, D) \). In this way, we see, when \( M \) is a maximal ideal of finite index in \( D \), that prime ideals of \( \text{Int}(E, D) \) containing \( \text{Int}(D, M) \) are inverse images of prime ideals of \( D^E \), and ultimately come from ultrapowers of \( (D/M) \), as in the discussion after Lemma 2.6.
6. Divisible rings of functions

Let \( R \subseteq D^E \) be a ring of functions and \( M \) a maximal ideal of \( D \). We have seen that we can describe those maximal ideals of \( R \) lying over \( M \) that contain \( R(E, M) \). We would like to know under what conditions this holds for every maximal ideal of \( R \) lying over \( M \).

In the case where \( M \) is a maximal ideal of finite index in a one-dimensional Noetherian domain \( D \), Chabert showed that every maximal ideal of \( \text{Int}(D) \) lying over \( M \) contains \( \text{Int}(D, M) \), cf. [1, Prop. V.1.11] and [4, Lemma 3.3]. Once we know this, Theorem 5.3 is applicable. It can be used to give an alternative proof of the fact that every prime ideal of \( \text{Int}(D) \) lying over \( M \) is maximal and of the form \( M_\alpha = \{ f \in \text{Int}(D) \mid f(\alpha) \in \hat{M} \} \) for an element \( \alpha \) in the \( M \)-adic completion of \( D \).

We will now generalize Chabert’s argument from integer-valued polynomials to a class of rings of functions which we call divisible. Note that we do not have to restrict ourselves to Noetherian domains; we only require the individual maximal ideal for which we study the primes of \( R \) lying over it to be finitely generated. It is true that our questions only localize well when the domain is Noetherian, but we will pursue a different course, not relying on localization.

**Definition 6.1.** Let \( R \) be a commutative ring and \( E \subseteq R \). We call a ring of functions \( R \subseteq R^E \) **divisible** if it has the following property: If \( f \in R \) is such that \( f(E) \subseteq cR \) for some non-zero \( c \in R \), then every function \( g \in R^E \) satisfying \( cg(x) = f(x) \) is also in \( R \).

We call \( R \) **weakly divisible** if for every \( f \in R \) and every non-zero \( c \in R \) such that \( f(E) \subseteq cR \), there exists a function \( g \in R \) with \( cg(x) = f(x) \).

If \( R \) is a domain, we note that \( g(x) \) in the above definition is unique and that, therefore, for domains, weakly divisible is equivalent to divisible.

**Example 6.2.**

1. \( \text{Int}(E, D) \) is divisible. - This is our motivation.

2. If \( D \) is a valuation domain with maximal ideal \( M \) then the ring of uniformly \( M \)-adically continuous functions from \( E \) to \( D \) (\( E \subseteq D \) equipped with subspace topology of \( M \)-adic topology) is a divisible ring of functions.

We now consider minimal prime ideals of non-zero principal ideals, that is, \( P \) containing some \( p \neq 0 \) such that there is no prime ideal strictly contained in \( P \) and containing \( p \). If \( D \) is Noetherian, this condition reduces to \( \text{ht}(P) = 1 \). In non-Noetherian domains, we find examples with \( \text{ht}(P) > 1 \), for instance, the maximal ideal of a finite-dimensional valuation domain.

**Lemma 6.3.** Let \( R \) be a domain, \( P \) a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal \( (p) \subseteq P \). Then there exist \( m \in \mathbb{N} \) and \( s \in R \setminus P \) such that \( sP^m \subseteq pR \).

**Proof.** In the localization \( R_P \), \( P_P \) is the radical of \( pR_P \). Therefore, since \( P \) (and hence \( P_P \)) is finitely generated, there exists \( m \in \mathbb{N} \) with \( P_P^m \subseteq pR_P \) and in
particular $P^m \subseteq pR_p$. The ideal $P^m$ is also finitely generated, by $p_1,\ldots,p_k$, say. Let $a_i \in R_p$ with $p_i = pa_i$. By considering the fractions $a_i = r_i/s_i$ (with $r_i \in R$ and $s_i \in R \setminus P$), and setting $s = s_1\ldots:s_k$, we see that $sP^m \subseteq pR$ as desired. \hfill \Box

**Theorem 6.4.** Let $D$ be a domain and $P$ a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal. Let $R \subseteq D^E$ be a divisible ring of functions from $E$ to $D$. Then every prime ideal $Q$ of $R$ with $Q \cap D = P$ contains $R(E,P)$.

**Proof.** Let $f \in R(E,P)$. Let $p \in P$ non-zero and such that there is no prime ideal $P_1$ with $(p) \subseteq P_1 \subsetneq P$. By Lemma 6.3 there are $m \in \mathbb{N}$ and $s \in D \setminus P$ such that $sP^m \subseteq pD$. Then $sf^m \in R(E,pD)$. Since $R$ is divisible, $sf^m = pg$ for some $g \in R(E,D)$. Therefore, $sf^m \in pR(E,D) \subseteq Q$. As $Q$ is prime and $s \notin Q$, we conclude that $f \in Q$. \hfill \Box

**Corollary 6.5.** Let $D$ be a domain, $M$ a finitely generated maximal ideal of height 1, and $E$ a subset of $D$. Let $R \subseteq D^E$ be a divisible ring of functions from $E$ to $D$, such that each $f \in R$ takes its values in only finitely many residue classes of $M$ in $D$.

Then the prime ideals of $R$ lying over $M$ are precisely the ideals of the form $M_U$ for an ultrafilter $U$ on $E$. Each $M_U$ is a maximal ideal and its residue field isomorphic to $D/M$.

**Proof.** This follows from Theorem 6.4 via Theorem 5.3. \hfill \Box

To summarize, we can, using ultrafilters, describe certain prime ideals of a ring of functions $R = R(E,D)$ lying over a maximal ideal $M$ pretty well: namely, those prime ideals that do not contain $M$-unit-valued functions (Theorem 5.7), or that contain $R(E,M)$ (Theorem 5.3).

We have, so far, little information about when all prime ideals of $R$ lying over $M$ are of this form, apart from the sufficient condition in Theorem 6.4.

If we restrict our attention to rings of functions $R$ with $D[x] \subseteq R(E,D) \subseteq D^E$, it would be interesting to find a precise criterion, perhaps involving topological density, for this property.

Note that in the “nicest” case, that of $\text{Int}(D)$, where $D$ is a Dedekind ring with finite residue fields, not only is $\text{Int}(D,M)$ contained in every prime ideal of $\text{Int}(D)$ lying over a maximal ideal $M$ of $D$, but also $\text{Int}(D)$ is dense in $D^D$ with product topology of discrete topology on $D$. \hfill [2][3]

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