Scalar Casimir effect between two concentric spheres

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Abstract

The Casimir effect giving rise to an attractive force between the closely spaced two concentric spheres that confine the massless scalar field is calculated by using a direct mode summation with contour integration in the complex plane of eigenfrequencies. We developed a new approach appropriate for the calculation of the Casimir energy for spherical boundary conditions. The Casimir energy for a massless scalar field between the closely spaced two concentric spheres coincides with the Casimir energy of the parallel plates for a massless scalar field in the limit when the dimensionless parameter \( \eta \), \( \eta = \frac{a - b}{\sqrt{ab}} \) where \( a \) (\( b \)) is inner (outer) radius of sphere, goes to zero. The efficiency of new approach is demonstrated by calculation of the Casimir energy for a massless scalar field between the closely spaced two concentric half spheres.

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1 INTRODUCTION

In 1948 Casimir [1] had shown that two uncharged perfectly conducting parallel plates attracted each other. The main idea under this attractive force is that changes in infinite vacuum energy of the quantized electromagnetic field can be finite and observable [2, 3, 4]. The sign of Casimir energy depends on manifolds of different topology, geometry, the shapes and compositions of objects as well as the boundary and curvature [5, 6, 7]. The Casimir effect has been studied in different areas of theoretical physics and an important results may be found in detailed reviews [8, 9, 10].

It is well-known fact that the negative sign of the Casimir energy between the plates for the electromagnetic field produces an attractive force. Stimulated

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by this event, Casimir hoped that one could construct an electron model as a perfectly conducting charged spherical shell. However, Boyer [11] showed that the quantum electromagnetic vacuum energy of a conducting spherical shell is positive which means that the force on the shell is outward and depends only on the radius of the spherical shell. This outward force was opposite to what Casimir had expected. Later, Milton et. al. [12] confirmed Boyer result by using Green’s functions. More recently, the direct mode summation technique and the Casimir energy calculations for the spherical shell has been advanced by [13, 14]. Nowadays, the Casimir effect between two objects has also been considered [15, 16, 17, 18, 19, 20, 21]. Several different methods have been developed to calculate the Casimir effect beyond the proximity force approximation. One of the pioneering study is the work by M. Bordag [15], where he developed to compute the small separation leading order terms of the Casimir interaction between a cylinder and a plane by using the path integral approach. Moreover, the exact result for the Casimir interactions between a finite number of compact objects of arbitrary shape and separation have been developed by using the functional determinant or multiple scattering approach [16, 17, 18, 19, 20, 21]. Besides, the Casimir force problems of two concentric spheres [22, 23, 24, 25, 26] and two concentric cylinders [27, 28, 29] have also been calculated by using different regularization method recently.

In this work we calculate the Casimir energy for the massless scalar field in a nontrivial two smoothly curved objects at close separation. The system we consider is made of two compact closely spaced spherical surfaces and the vacuum gap consists of between two concentric spheres with radii \( a \) and \( b \) \((b > a)\). There are two motivations for us to perform the Casimir energy calculation for a massless scalar field between the closely spaced two concentric spheres, and between the closely spaced two concentric half spheres. First it is interesting to find what similarities and difference of the Casimir energy of the topologically similar geometry and comparing the parallel plates and the spherical shell, respectively. The sign of the Casimir energy will decide whether the Casimir force will be attractive or repulsive. We hope that there will be an application of our result to special systems in nanotechnologies and nanoelectromechanical devices. Second from the mathematical point of view the direct mode summation approach to the our geometry has been developed. Our aim is to show the simplicity and efficiency of the direct mode summation by contour integration when calculating the Casimir energy for a difficult boundary as the closely spaced two concentric spheres. The direct evaluation of the infinite sum over all the vacuum energy eigenvalues of the massless scalar field modes implies that the translation matrices in multiple scattering approach is not needed in order for the finite values for the vacuum energy to be obtained for given spherical boundary conditions. Moreover, the improved value for the computation of the Casimir energy for a massless scalar field between the closely spaced two concentric half spheres we reconsider here is developed by using the Abel-Plana sum formula for evenly spaced frequency spectrum for large argument.

The organization of the paper is as follows. In Sec. 2. The Casimir energy of a massless scalar field subjected to spherical boundary conditions on between the
closely spaced two concentric spheres is calculated without any approximation techniques. This approach will be employed for the Casimir energy between the closely spaced two concentric half spheres in Sec. 3. Concluding remarks and discussion of the Casimir energy for a massless scalar field in an annular region of our geometry is presented in section 4.

The units are such that $\hbar = c = 1$.

2 CASIMIR ENERGY BETWEEN TWO CONCENTRIC SPHERES

We start with the spacetime metric

$$ds^2 = dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

(1)

in spherical coordinates. Where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The wave equation that the massless scalar field satisfies in these coordinates is given by

$$\Box \Psi (t, r, \theta, \phi) = 0.$$  

(2)

Where $\Box$ is the D’Alembertian operator associated with the metric given by the line element Eq. (1). Solution of equation (2) could be easily found by using the method of separation of variables and is given as

$$\Psi_{\omega \ell m}(t, r, \theta, \phi) = \sum_{\ell, m} e^{-i \omega t} r^{-\frac{\Delta}{2}} [A_{\ell} J_{\ell+1/2}(\omega r) + B_{\ell} N_{\ell+1/2}(\omega r)] Y_{\ell m}(\theta, \phi).$$

(3)

Where $c_0$ is the normalization constant and $\omega$ is the root of the following transcendental equation

$$J_{\ell+1/2}(\omega b) N_{\ell+1/2}(\omega a) - J_{\ell+1/2}(\omega a) N_{\ell+1/2}(\omega b) = 0.$$  

(6)
The Casimir energy for the massless scalar field between two concentric spheres is

\[ E_C = \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=1}^{\infty} \omega_{n\ell}, \]

\[ = \sum_{\ell=0}^{\infty} \nu \sum_{n=1}^{\infty} \omega_{n\ell}, \quad \text{where } \nu = \ell + \frac{1}{2}. \quad (7) \]

Where \( \omega_{n\ell} \) are eigenfrequencies which are determined by solving the frequency equation given in Eq. (6). We need to describe the eigenfrequencies spectrum for the Casimir energy in a nontrivial smoothly curved objects at close separation. We note that the frequency equation involves an infinite series since the complete solution of Bessel's equation has series of ascending powers of \( \omega \). Bessel's series equation are convergent for all values of argument. But, when \( |\omega| \) is large, the series converge slowly. Although the series has the converge slowly, the initial terms of such a series gives no information about the sum. Thus Bessel functions are needed to describe the transition for large argument.

To overcome this difficulty, we need the rapidly convergent evaluation of the Bessel's function formula. The meaning of rapidly convergent is that the series rapidly approaches a constant, taking the limit as \( \ell \to \infty \). To this aim, we will use the uniform asymptotic expansions of the Bessel's functions. The uniform asymptotic expansions are useful in describing the transition of behaviour. Moreover, after taking the limit \( \omega \to \infty \) at fixed \( \ell \) the spectrum will consist of a discrete set embedded in a continuum part. Hence, we should examine the behavior of the eigenfrequency spectrum for large arguments at fixed \( \ell \). Thus, to carry out the summation with respect to \( \ell \) in \( E_C \), the sum \( \sum_{n=1}^{\infty} \omega_{n\ell} \) given in equation (7) replaced by \( \sum_{n=1}^{\infty} \omega_{n\ell} + \sum_{n=1}^{\infty} \tilde{\omega}_{n\ell} \) where \( \tilde{\omega}_{n\ell} \) is the eigenvalue spectrum of the limit \( \omega \to \infty \) at fixed \( \ell \) \cite{11}. Then, Casimir energy which is defined by the eigenfrequency spectrum for large arguments at fixed \( \ell \) and large order as \( \ell \to \infty \) can be written as

\[ E_C = \sum_{\ell=0}^{\infty} \nu \sum_{n=1}^{\infty} \omega_{n\ell} + \sum_{\ell=0}^{\infty} \nu \sum_{n=1}^{\infty} \tilde{\omega}_{n\ell}. \quad (8) \]

Now, to calculate the eigenfrequencies for large arguments at fixed \( \nu \), we use Hankel's asymptotic expansion \cite{31} when \( \nu \) is fixed, \( \omega a \gg 1 \) and \( \omega b \gg 1 \), we get

\[ J_{\nu} (\tilde{\omega} a) \simeq \sqrt{\frac{2}{\pi \tilde{\omega} a}} \left[ \cos \left( \frac{\tilde{\omega} a - \nu}{2} - \frac{\pi}{4} \right) - \frac{(4\nu^2 - 1)}{8\tilde{\omega} a} \sin \left( \frac{\tilde{\omega} a - \nu}{2} - \frac{\pi}{4} \right) \right]. \quad (9) \]

\[ N_{\nu} (\tilde{\omega} a) \simeq \sqrt{\frac{2}{\pi \tilde{\omega} a}} \left[ \cos \left( \frac{\tilde{\omega} a - \nu}{2} - \frac{\pi}{4} \right) + \frac{(4\nu^2 - 1)}{8\tilde{\omega} a} \sin \left( \frac{\tilde{\omega} a - \nu}{2} - \frac{\pi}{4} \right) \right]. \quad (10) \]
and similar expressions for \( J_\nu (\tilde{\omega}b) \) and \( N_\nu (\tilde{\omega}b) \) with \( a \) interchanged for \( b \).

Putting (9) and (10) in the frequency equation given by (6), we obtain the zeros of frequency equation are almost evenly spaced

\[
\tilde{\omega}_{n\ell}^2 \simeq \left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab} \quad \text{where } n = 1, 2, 3, 4, 5, 6, \ldots .
\]  

(11)

The frequencies equation \( \omega_{n\ell} \) the first sum is given in Eq. (8) as the uniform asymptotic expansions of the Bessel functions at large \( \ell \) as \( \ell \to \infty \) can be written as

\[
f_\nu (\nu \omega, \lambda) = J_\nu (\nu \omega) \ N_\nu (\nu \omega \lambda) - J_\nu (\nu \omega \lambda) \ N_\nu (\nu \omega)
\]  

(12)

Where \( \lambda = \frac{a}{b} \) (\( b > a \)).

Then, the scalar Casimir energy between the closely spaced two concentric spheres can be written as

\[
E_C = \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} \omega_{n\ell} + \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}}
\]

\[
= \mathcal{E}_C + \tilde{E}_C .
\]  

(13)

Where \( \omega_{n\ell} \) is the root of the frequency equation given in Eq. (12).

We consider the first sum defined in Eq. (13). This divergent expression can be rendered finite by the use of a cutoff or convergence factor. Then we define the first sum,

\[
\mathcal{E}_C = \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right) \sum_{n=1}^{\infty} \omega_{n\ell} \ e^{-\alpha \omega_{n\ell}}
\]

\[
= \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right) \ S_\ell ,
\]  

(14)

where the factor of \( e^{-\alpha \omega_{n\ell}} \) plays the role of an exponential cutoff function, and \( S_\ell = \sum_{n=1}^{\infty} \omega_{n\ell} \ e^{-\alpha \omega_{n\ell}} \) is generated by the frequency equation (12). To evaluate the sum \( S_\ell \), we use the integral representation from the Cauchy’s theorem [14] [20] [29] that for two functions \( f_\ell(z) \) and \( \phi(z) \) analytic within a closed contour \( C \) in which \( f_\ell(z) \) has isolated zeros at \( x_1, x_2, x_3, \ldots, x_n \),

\[
\frac{1}{2\pi i} \oint_C dz \ \phi(z) \ \frac{d}{dz} \ln f_\ell(z) = \sum_j \phi(x_j) .
\]  

(15)

We choose \( \phi(z) = z \ e^{-\alpha z} \) where \( \alpha \) is a real positive constant thus leads to
\[
\frac{1}{2\pi i} \oint_C dz \ e^{-\alpha z} \ z \frac{d}{dz} \ln f_\ell(z) = \sum_j z_j \ e^{-\alpha z_j} \ .
\] (16)

Using this result to replace the sum \( S_\ell \) by a contour integral, the first term of the Casimir energy becomes

\[
\mathcal{E}_C = \sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right) \frac{1}{2\pi i} \oint_C dz \ e^{-\alpha z} \ z \frac{d}{dz} \ln f_\nu(\nu z) \ ,
\] (17)

where the frequency function \( f_\nu(\nu z) \) is given Eq. (12). The contour \( C \) encloses all the positive roots of the equation \( f_\nu(\nu z) = 0 \). This contour can be conveniently broken into three parts [14, 26, 29]. These consist of a circular segment \( C_\Gamma \) and two straight line segments \( \Gamma_1 \) and \( \Gamma_2 \) forming an angle \( \phi \) and \( \pi - \phi \) with respect to the imaginary axis. When the radius \( \Gamma \) is fixed, the contour \( C_\Gamma \) encloses a finite number of roots of the equation \( f_\nu(\nu z) = 0 \). Since the sum of these roots is obviously infinite, the radius \( \Gamma \) is a regularization parameter, taking the limit \( \Gamma \to \infty \) (when \( \alpha > 0 \) ) means the removal of the regularization, the contribution of \( C_\Gamma \) vanishes provided that \( \phi \neq 0 \). Hence the exponential cutoff function in the Cauchy integral plays the role of the elimination of the contribution to the circular part of the contour integral. Taking the contributions along \( \Gamma_1 \) and \( \Gamma_2 \) which are complex conjugates of each other, then Eq. (16) becomes [13, 14]

\[
\mathcal{E}_C = \frac{1}{\pi \ b} \lim_{\alpha \to 0} \sum_{\ell=0}^{\infty} \nu \ Re \ e^{-i\phi} \int_0^{\infty} dy \ e^{-i\alpha y e^{-i\phi}} \ \frac{d}{dy} \ln f_\nu(\nu y e^{-i\phi}; a, b) \ .
\] (18)

Where

\[
f_\nu(\nu y e^{-i\phi}; a, b) = -\frac{2}{\pi} \left[ I_\nu(\nu y e^{-i\phi}) K_\nu(\nu y e^{-i\phi}) - I_\nu(\nu y e^{-i\phi} a) K_\nu(\nu y e^{-i\phi} b) \right] \]

(19)

Now we calculate the integral given in Eq. (18). Defining \( \lambda = \frac{\phi}{\pi} (b > a) \) and after rescaling integral variable with \( y b \to y \) in equation \( \mathcal{E}_C \), one obtains

\[
\mathcal{E}_C = -\frac{1}{\pi b} \lim_{\alpha \to 0} \sum_{\ell=0}^{\infty} \nu \ Re \ e^{-i\phi} \int_0^{\infty} dy \ e^{-i\alpha y e^{-i\phi}/b} \ y \frac{d}{dy} \ln f_\nu(\nu y e^{-i\phi}; \lambda) \ .
\] (20)

Where \( f_\nu(\nu y e^{-i\phi}; \lambda) = -\frac{2}{\pi} \left[ I_\nu(\nu y e^{-i\phi}) K_\nu(\nu y e^{-i\phi}) - I_\nu(\nu y e^{-i\phi} \lambda) K_\nu(\nu y e^{-i\phi} \lambda) \right] \) stems from Eq. (19).

Now we use the Lommel’s expansions or the multiplication theorem for the function of \( f_\nu(\nu y e^{-i\phi}; \lambda) \) [26, 32]. Thus, we have
\[ f_\nu(z, \lambda) = I_\nu(z) K_\nu(z\lambda) - I_\nu(z\lambda) K_\nu(z) \]

\[ = \lambda^{-\nu} \sum_{k=0}^{\infty} \left( \frac{\lambda^2 - 1}{k! 2^k} \right)^k z^{2k-\nu} \]

\[ \times \left\{ \begin{array}{l}
I_\nu(z) \left( \frac{d}{dz} \right)^k \{ z^{\nu} K_\nu(z) \} - K_\nu(z) \left( \frac{d}{dz} \right)^k \{ z^{\nu} I_\nu(z) \} \end{array} \right\} . \]

(21)

Where \(|\lambda^2 - 1| < 1\). Applying the uniform with respect to \(z\) asymptotics for the modified Bessel functions at large \(\nu\), after long calculations [26], we obtain

\[ \nu y \frac{d}{dy} \ln f_\nu(\nu ye^{-i\phi}, \lambda) = \frac{(1 - \lambda^2)^2}{12} (\nu ye^{-i\phi})^2 + \frac{(1 - \lambda^2)^3}{24} (\nu ye^{-i\phi})^2 \]

\[ + \frac{(1 - \lambda^2)^4}{720} \left[ (\nu ye^{-i\phi})^2 (-\nu^2 + 19) - (\nu ye^{-i\phi})^4 \right] \]

\[ - \frac{(1 - \lambda^2)^5}{1440} \left[ 3 (\nu ye^{-i\phi})^2 (\nu^2 - 9) + 2 (\nu ye^{-i\phi})^4 \right] \]

\[ + \frac{(1 - \lambda^2)^6}{120960} \left[ (\nu ye^{-i\phi})^2 (4\nu^4 - 290\nu^2 + 1726) + (\nu ye^{-i\phi})^4 (8\nu^2 - 149) \right. \]

\[ + 4 (\nu ye^{-i\phi})^6 \left] + \text{[Terms in even powers of } (\nu ye^{-i\phi})\text{]} , \right. \]

(22)

Inserting Eq. (22) into Eq. (20) and using the following integral result

\[ I(2n) = e^{-i\phi} \int_0^\infty dy e^{-i\alpha ye^{-i\phi}/b} (\nu ye^{-i\phi})^{2n} \]

\[ = i (-)^{n+1} (2n)! \nu^{2n} \left( \frac{b}{\alpha} \right)^{2n+1} , \quad \text{where } n = 0, 1, 2, 3... \]  

(23)

then Eq. (20) becomes
$$\overline{E_C} = \frac{-1}{\pi b} \lim_{\alpha \to 0} \sum_{\ell=0}^{\infty} \Re \left\{ i \frac{1 - \lambda^2}{6} \nu^2 \left( \frac{b}{\alpha} \right)^3 + i \frac{(1 - \lambda^2)^3}{12} \nu^2 \left( \frac{b}{\alpha} \right)^3 \right\}$$

$$+ i \frac{(1 - \lambda^2)^4}{720} \left[ 2(-\nu^4 + 19\nu^2) \left( \frac{b}{\alpha} \right)^3 + 24\nu^4 \left( \frac{b}{\alpha} \right)^5 \right]$$

$$+ i \frac{(1 - \lambda^2)^5}{720} \left[ -6(\nu^4 - 9\nu^2) \left( \frac{b}{\alpha} \right)^3 + 48\nu^4 \left( \frac{b}{\alpha} \right)^5 \right]$$

$$+ i \frac{(1 - \lambda^2)^6}{120960} \left[ (8\nu^6 - 580\nu^4 + 3452\nu^2) \left( \frac{b}{\alpha} \right)^3 - 24(8\nu^6 - 149\nu^4) \left( \frac{b}{\alpha} \right)^5 + 2880\nu^6 \left( \frac{b}{\alpha} \right)^7 \right]$$

$$+ \{\text{Terms in imaginary number and even powers of } \nu\}.$$ \hspace{1cm} (24)

All terms in the above equation have the singular term in the regulator parameter \(\alpha\) and purely imaginary. Taking the real part of the parenthesis, thus it leaves the zero result.

$$\overline{E_C} = 0.$$ \hspace{1cm} (25)

The meaning of this result is that there is no contribution from \(\ell \to \infty\) modes for the Casimir energy between two concentric spheres \[26\]. Now we return to the second sum given in Eq. (13), included high eigenfrequency modes i.e. \(\omega \to \infty\) at fixed \(\ell\).

$$\tilde{E_C} = \sum_{\ell=0}^{\infty} \nu \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}}.$$ \hspace{1cm} (26)

Where \(\nu = (\ell + \frac{1}{2})\). This divergent sum can be regularized by using the Abel-Plana sum formula. Before we proceed, we recall the Abel-Plana formula gives an expression for the difference between the sum and corresponding integral, which could be given as \[9, 10\]

$$\text{Reg} \left[ \sum_{n=1}^{\infty} f(n) \right] =$$

$$\sum_{n=0}^{\infty} f(n) - \int_{0}^{\infty} f(x)dx = \frac{1}{2} f(0) + i \int_{0}^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt.$$ \hspace{1cm} (27)

Where \(f(z)\) is an analytic function in the right half plane and \(\text{Reg}\) refers to the regularized value of the sum. The other useful Abel-Plana sum formula for the half integer number is
\[
\sum_{n=0}^{\infty} f \left( n + \frac{1}{2} \right) = \int_0^\infty f(x)dx - i \int_0^\infty \frac{f(it) - f(-it)}{e^{2\pi t} + 1} dt. \tag{28}
\]

We can rewrite the sum given in Eq. (26) using the Abel-Plana sum formula, which leads to

\[
\tilde{E}_C = \sum_{\ell=0}^{\infty} \nu \ \text{Reg} \left[ \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}} \right]
= -\frac{1}{2\sqrt{ab}} \sum_{\ell=0}^{\infty} \nu^2 - 2 \sum_{\ell=0}^{\nu} \nu \int_{\frac{\nu\pi}{b-a}}^{\frac{\nu\pi}{b-a}} \left[ \left( \frac{t\pi}{b-a} \right)^2 - \frac{\nu^2}{ab} \right] \frac{1}{e^{2\pi t} - 1}. \tag{29}
\]

where \( \xi = \frac{2d}{\sqrt{ab}} \) and \( d = b-a \). The first divergent sum in Eq. (29) can be removed by using the Hurwitz zeta function \[33\]

\[
\sum_{\ell=0}^{\infty} \left( \ell + \frac{1}{2} \right)^2 = \zeta \left( -2, \frac{1}{2} \right)
= 0. \tag{30}
\]

Thus Eq. (29) becomes

\[
\tilde{E}_C = -2 \sum_{\ell=0}^{\infty} F(\nu)
\]

where 

\[
F(\nu) = \nu \int_{\frac{\nu\pi}{b-a}}^{\frac{\nu\pi}{b-a}} \left[ \left( \frac{t\pi}{b-a} \right)^2 - \frac{\nu^2}{ab} \right] \frac{1}{e^{2\pi t} - 1}. \tag{31}
\]

Using the half integer Abel-Plana sum formula given in Eq. (28), one obtains

\[
\tilde{E}_C = -\frac{1}{4\pi} \left( \frac{\sqrt{ab}}{d^3} \right) \zeta(4) + 2i \int_0^\infty \frac{F(it) - F(-it)}{e^{2\pi t} + 1} dt. \tag{32}
\]

Where \( \zeta(s) \) is the Riemann zeta function. The divergent term occurs in the second term of the Eq. (32). We avoid this problem by substituting \( \nu - i \in \) for \( \nu \) with the understanding that we will let \( \in \to 0 \) at the end. Thus the Casimir energy for a massless scalar field between the closely spaced two concentric spheres is obtained
\[ E_C = \bar{E}_C + E_C \]

\[ = -\frac{1}{4\pi} \left( \frac{\sqrt{ab}}{d^3} \right)^2 \zeta(4) \left[ 1 + \frac{1}{12} \eta^2 \zeta(2) \right], \]

\[ E_C = -\frac{\pi^3 ab}{360 d^3} \left[ 1 + \frac{5}{4\pi^2 ab} \right]. \]

(33)

Where \( \eta = \frac{d}{\sqrt{ab}} \). And known formula \( \zeta(4) = \frac{\pi^4}{90} \) and \( \zeta(2) = \frac{\pi^2}{6} \). The main contribution to the Casimir energy is given by the second term which represents the high frequency modes at fixed \( \ell \) between the closely spaced two concentric spheres for a massless scalar field. The Casimir energy per unit surface area on the inner sphere (the total surface area \( A = 4\pi a^2 \)) can be written as

\[ \frac{E_C}{A} = -\frac{1}{16\pi^2} \left( \frac{\sqrt{ab}}{a} \right)^2 \frac{\zeta(4)}{d^3} \left[ 1 + \frac{1}{12} \eta^2 \frac{\zeta(2)}{\zeta(4)} \right]. \]

(34)

This result is interest at the limiting case which is narrow slit is defined by \( \eta = \frac{d}{\sqrt{ab}} \ll 1 \) [22]. We easily analysis that in the limit \( b \to a \) (i.e. \( \eta \to 0 \) and \( \sqrt{ab}/a \to 1 \)) which means that the close separation between two spheres one finds that the leading term of the Casimir energy per unit area can be written as

\[ \frac{E_C}{A} = -\frac{1}{16\pi^2} \frac{\zeta(4)}{d^3}. \]

(35)

This result is exactly the same as the Casimir energy of the parallel plates for a massless scalar field [34]. Thus our approach developed here has been the satisfactory check.

### 3 CASIMIR ENERGY BETWEEN TWO CONCENTRIC HALF SPHERES

The our approach can be easily employed to the computation of the Casimir energy of a massless scalar field between the closely spaced two concentric half spheres. In this case of our spherical boundary [26] the sum in Eq. (13) can be written as

\[ E_C = \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \omega_{n\ell} + \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}}. \]

(36)

Where \( \nu = \ell + \frac{1}{2} \) and \( \omega_{n\ell} \) is the root of the transcendental equation given in Eq. (6). From Eq. (15) we obtain
\[ EC = \mathcal{E}_C + \tilde{E}_C \]

\[ = -\frac{1}{2\pi b} \sum_{\ell=1}^{\infty} \left( \nu - \frac{1}{2} \right) \Re \text{e}^{-i\phi} \int_0^\infty dy \text{e}^{-i\alpha ye^{-i\phi}/b} \frac{d}{dy} \ln f_\nu(\nu ye^{-i\phi}, \lambda) \]

\[ + \frac{1}{2} \sum_{\ell=1}^{\infty} \left( \nu - \frac{1}{2} \right) \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}} \]

(37)

Where \( y \frac{d}{dy} \ln f_\nu(\nu ye^{-i\phi}, \lambda) \) given in eq. (22). The first term (\( = \mathcal{E}_C \)) is equal to zero from the Eq. (25). Thus the Casimir energy between the closely spaced two concentric half spheres becomes

\[ EC = \frac{1}{2} \sum_{\ell=1}^{\infty} \left( \nu - \frac{1}{2} \right) \sum_{n=1}^{\infty} \sqrt{\left( \frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}} \]

(38)

Using the Abel-plana sum formula one obtains

\[ EC = -\frac{1}{4\sqrt{ab}} \sum_{\ell=1}^{\infty} \left( \nu^2 - \nu \right) - \frac{d}{\pi ab} \sum_{\ell=1}^{\infty} F(\nu), \]

(39)

where \( F(\nu) = \left( \frac{\nu^3}{2} \right) \int_1^\infty (y^2 - 1)^{1/2} \frac{dy}{e^{2\pi y} - 1}, \) and \( d = b - a. \)

Again using the half integer Abel-Plana sum formula for the second sum in Eq. (39), we obtain

\[ EC = -\frac{1}{8\pi} \left( \sqrt{ab} \right)^2 \zeta(4) \left[ 1 - \frac{\pi}{4} \zeta(3) + \frac{\pi^3}{24} \frac{1}{\zeta(4)} + \frac{1}{12} \eta^2 \zeta(2) \zeta(4) - \frac{1}{4\pi} \eta^3 \zeta(4) \right]. \]

(40)

We have used the Hurwitz zeta function i.e. \( \zeta(-m, q) = \frac{B_{m+1}(q)}{m+1}, \) \( m = 0, 1, 2, 3, ... \) where \( B_{m+1}(q) \) is the Bernoulli polynomials.

The Casimir energy of a massless scalar field between the closely spaced two concentric half spheres per unit half surface area on the inner sphere (the half surface area \( A = 2\pi a^2 \)) can be written as

\[ \frac{E_C}{A} = -\frac{1}{16\pi} \left( \frac{\sqrt{ab}}{a} \right)^2 \frac{\zeta(4)}{d^3} \left[ 1 - \frac{\pi}{4} \eta \zeta(3) + \frac{\pi^3}{24} \frac{1}{\zeta(4)} + \frac{1}{12} \eta^2 \zeta(2) \zeta(4) - \frac{1}{4\pi} \eta^3 \zeta(4) \right]. \]

(41)

Where \( \zeta(s) \) is the Riemann zeta function.
Taking the small separation limit \((b \rightarrow a \text{ and } \eta \rightarrow 0)\) one obtains

\[
\frac{E_C}{A} = -\frac{1}{16\pi} \frac{\zeta(4)}{d^3},
\]

This result coincides with the Casimir energy of the parallel plates for the massless scalar field \([34]\).

4 CONCLUSION

In the present paper we considered the quantum vacuum energy for a massless scalar field between the closely spaced two concentric spheres, and between two concentric half spheres at small separations. The annular region in our geometry was considered since all massless scalar fields in the region \(r < a \text{ and } r > b\) are equal to zero. The Casimir energy for a massless scalar field in annular region was evaluated by a direct mode summation method. We have used the explicit expression for the frequency equations which include the product of Bessel functions. In order to evaluate the product of Bessel function expansion for large order in our approach, the Lommel's expansions has been used in the Casimir problem for the first time. These calculation is a direct application of the principle of argument from the complex integral with the cutoff function, in direct analogy to the Casimir calculation, together with a modification of the contour in order to ensure the convergence of the Cauchy integral expressions. Moreover, the product of Bessel function expansion for the large argument is defined by evenly spaced eigenfrequency spectrum. Although we use the Abel-Plana sum formula, divergent terms appear in our calculations. To remove this divergence we have applied the formal technique of the Zeta function regularization. Thus, our calculation implies that further regularization is not needed in order for the finite values for the vacuum energy to be obtained for given boundary conditions.

We are particularly interested in calculating the Casimir energy between spherical surfaces close to each other is that any approximation technique is not needed. The interesting point of our calculations is that all contributions in the Casimir energy for a massless scalar field comes from the higher frequencies for fixed \(\ell\) between two surfaces boundary conditions. \(\ell \rightarrow \infty\) frequency modes contribution in the Casimir energy is zero for close separation of annular region in present geometries.

Although the Casimir energy sign for the massless scalar field on spherical shell with the Dirichlet boundary condition is positive, our analysis reveals the negative sign of the Casimir energy between the closely spaced two concentric spheres, and between two concentric half spheres at small separations. Then, the Casimir force for a massless scalar field between two concentric spheres, and between two concentric half spheres are always attractive which is the same per unit area on a pair of parallel plates for a massless scalar field \([34]\). Moreover, the leading term of the Casimir energy between the closely spaced two concentric
spheres given in Eq. (33) agrees with that obtained the Casimir effect between two spheres at small separation by using the functional determinant or multiple scattering approach\cite{20,35}.

Closing, it is worth noting that, as far as we know, such boundary conditions with between the closely spaced two concentric spheres has been considered in the Casimir problem with the use mode sum technique for the first time. As far as we know this result is obtained here for the first time.

For future works, it would be interesting to consider the electrodynamics Casimir energy between the closely spaced two concentric spheres with those obtained.

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