Momentum-space calculation of four-boson recombination

A. Deltuva
Centro de Física Nuclear da Universidade de Lisboa, P-1649-003 Lisboa, Portugal
(Received December 12, 2011)

The system of four identical bosons with large two-boson scattering length is described using momentum-space integral equations for the four-particle transition operators. The creation of Efimov trimers via ultracold four-boson recombination is studied. The universal behavior of the recombination rate is demonstrated.

PACS numbers: 34.50.-s, 31.15.ac

I. INTRODUCTION

In 1970 V. Efimov predicted that with vanishing two-particle binding an infinite number of weakly bound states with total orbital angular momentum \( L = 0 \) may exist in the three-particle system \(^1\). Such a situation takes place if at least two pairs have infinite scattering length \( a \to \infty \). Away from this limit the number of trimers is finite but, depending on \( a \), may be large \(^2\). This three-particle Efimov effect has an impact on the behavior of more complex few-particle systems. In particular, for each Efimov trimer the existence of two tetramers was predicted in Refs. \(^3\) \(^4\). The signatures of these states were observed recently in experiments with ultracold atoms where the formation of the tetramers led to a resonant enhancement of the recombination or relaxation processes \(^5\) \(^7\).

These first steps in exploring the four-body Efimov physics experimentally also call for accurate theoretical studies of the four-body systems with large \( a \). Following our recent works devoted to the bosonic atom-trimer \(^8\) and dimer-dimer \(^9\) scattering, here we focus on the four-atom recombination process in the system of four identical bosons. The description is based on the exact four-particle equations for the transition operators \(^10\) derived by Alt, Grassberger, and Sandhas (AGS). For the recombination process we need the transition operator connecting two- and four-cluster channel states; its relation to the two-cluster AGS operators calculated in Refs. \(^8\) \(^4\) \(^11\) \(^12\), will be given in the present work. We solve the integral AGS equations using momentum-space framework. In previous works we demonstrated its reliability for reactions involving many very weakly bound trimers as it happens in the universal regime where the accuracy of the coordinate-space methods may be more limited \(^4\) \(^13\) \(^14\). For example, in Ref. \(^15\) we predicted universal intersections of shallow tetramers with the corresponding atom-trimer thresholds while the adiabatic hyperspherical calculations of Refs. \(^4\) \(^16\) were not sufficiently precise to find this remarkable feature that leads to a resonant behavior in the ultracold atom-trimer collisions. Thus, we expect that also in the case of the four-boson recombination we will be able to achieve the universal limit with higher accuracy compared to the existing coordinate-space calculations \(^4\) \(^10\). We note that approximate semi-analytical recombination results have been obtained in Ref. \(^17\) for a system of three identical bosons plus a distinguishable particle.

An extension of the four-body scattering calculations above the four-cluster breakup threshold is a very serious theoretical challenge both in atomic and nuclear physics. In the momentum-space framework the difficulties arise due to complicated singularity structure in the kernel of the integral equations. Not all but some of these difficulties are present already at the four-cluster breakup threshold. In this work we restrict ourselves to a latter case which is sufficient to calculate the four-atom recombination in the ultracold limit. Nevertheless, the present work is an important intermediate step towards solving the momentum-space four-body scattering equations at positive energies.

We use a system of units where \( \hbar = 1 \) and therefore is omitted in the equations unless needed for dimensional analysis. In Sec. \( \text{II} \) we recall the AGS equations and derive the relation between the two- and four-cluster transition operators. In Sec. \( \text{III} \) we present results for the four-boson recombination. We summarize in Sec. \( \text{IV} \).

II. FOUR-BOSON SCATTERING

We consider a system of four spinless particles with Hamiltonian

\[
H = H_0 + \sum_{i=1}^{6} v_i
\]

where \( H_0 \) is the kinetic energy operator for the relative motion and \( v_i \) the short-range potential acting within the pair \( i \); there are six pairs. For the description of the scattering process in such a system we use the AGS equations \(^10\). They are exact quantum-mechanical equations of the Faddeev-Yakubovsky (FY) type. However, in contrast to the original FY equations \(^18\) for the wave-function components, the AGS equations are formulated for the transition operators in the integral form and therefore are better suited to be solved in the momentum-space framework preferred by us.

We aim to determine the amplitude for the recombination of four free particles into a two-cluster state. Due to time reversal symmetry it is equal to the amplitude for the four-cluster breakup of the initial two-cluster state.
The latter is more directly related to the AGS transition operators calculated in our previous works \cite{[8, 9, 11, 12]}.

### A. AGS equations

In a compact notation the AGS equations can be written as 18-component matrix equations \cite{[10, 14]}. The components are distinguished by the chains of partitions, i.e., by the two-cluster partition and by the three-cluster partition. Obviously, the two-cluster partitions may be of 3+1 or 2+2 type; we will denote them by Greek subscripts. All three-cluster partitions are of 2+1+1 type and are specified by the pair of particles that we will denote by Latin superscripts. For our consideration we need the explicit form of the AGS equations for the transition operators, i.e.,

$$U_{\rho\sigma}^i = (G_0 t_{ij} G_0)^{-1} \delta_{\rho\sigma} \delta_{ji} + \sum_{\gamma k} \delta_{\rho\gamma} U_{\gamma k}^i G_0 t_k G_0 U_{\gamma p}^{ki}. \quad (2)$$

Here $\delta_{\rho\sigma} = 1 - \delta_{\rho\sigma}$,

$$G_0 = (E + i0 - H_0)^{-1} \quad (3)$$

is the free resolvent with the available four-particle energy $E$,

$$t_i = v_i + v_i G_0 t_i \quad (4)$$

is the two-particle transition matrix for the pair $i$, and

$$U_{\gamma}^k = G_0^{-1} \delta_{jk} + \sum_{i} \delta_{ji} t_i G_0 U_{\gamma}^{ik} \quad (5)$$

is the subsystem transition operator of 3+1 or 2+2 type, depending on $\gamma$. In the summations over the pairs only the ones internal to the respective two-cluster partition contribute.

The transition operators $U_{\rho\sigma}^i$ contain full information on the scattering process. Their on-shell matrix elements $\sum_{ji} \langle \phi_{\sigma,n}^j | U_{\rho\sigma}^i | \phi_{\rho,n}^i \rangle$ are the amplitudes $\langle \Phi_{\sigma,n}^i | T_{\rho\sigma} | \Phi_{\rho,n}^i \rangle$ for the two-cluster reactions. Here

$$|\phi_{\rho,n}^i\rangle = G_0 \sum_{j} \delta_{ij} t_j |\phi_{\rho,n}^j\rangle \quad (6)$$

are the Faddeev components of the initial and final channel states

$$|\Phi_{\rho,n}\rangle = \sum_i |\phi_{\rho,n}^i\rangle, \quad (7)$$

and $n$ distinguishes between the different channel states in the same partition. $|\Phi_{\rho,n}\rangle$ is given by the $n$th bound state wave function in partition $\rho$ times the plane wave with momentum $p_{\rho,n}$ between the clusters, and is normalized to $\langle \Phi_{\sigma,n}^i | \Phi_{\rho,n}^i \rangle = \delta_{\rho\sigma} \delta_{n'n} \delta(p_{\sigma,n'} - p_{\rho,n})$. The on-shell condition for each channel state relates its binding energy, relative two-cluster momentum and reduced mass as $\epsilon_{\rho,n} + p_{\rho,n}^2/2\mu_\rho = E$.

The full wave function corresponding to the initial two-cluster channel state $|\Phi_{\rho,n}\rangle$ is also determined by the AGS transition operators, i.e.,

$$|\Psi_{\rho,n}\rangle = |\Phi_{\rho,n}\rangle + \sum_{\gamma k} G_0 t_{jk} G_0 U_{\gamma k}^{ki} G_0 U_{\gamma p}^{ki} |\phi_{\rho,n}^i\rangle. \quad (8)$$

The amplitude for the four-cluster breakup reaction can be obtained as

$$\langle \Phi_{\rho,n}^i | T_{0\rho} | \Phi_{\rho,n}^i \rangle = \sum_{i=1}^{6} v_i |\Psi_{\rho,n}\rangle. \quad (9)$$

The wave function $\langle \Phi_{\rho,n}^i | T_{0\rho} | \Phi_{\rho,n}^i \rangle$ satisfies also the Schrödinger equation, thus $\sum_{i=1}^{6} v_i |\Psi_{\rho,n}\rangle = G_0^{-1} |\Phi_{\rho,n}\rangle$. Furthermore, $G_0^{-1} |\Phi_{\rho}\rangle$ is an eigenstate of $H_0$ with eigenvalue $E$. In fact, $|\Phi_{\rho}\rangle$ is a product of three plane waves (each is normalized to the Dirac $\delta$-function) corresponding to the relative motion of four free particles. Thus, the amplitude for the four-cluster breakup of the two-cluster initial state is

$$\langle \Phi_{0} | T_{0\rho} | \Phi_{\rho,n} \rangle = \sum_{\gamma k i} \langle \Phi_{0} | t_{jk} G_0 U_{\gamma k}^{ki} G_0 t_k G_0 U_{\gamma p}^{ki} |\phi_{\rho,n}^i\rangle. \quad (10)$$

Due to time reversal symmetry it describes also the four-particle recombination into a two-cluster state, i.e.,

$$\langle \Phi_{\rho,n}^i | T_{0\rho} | \Phi_{\rho,n}^i \rangle = \langle \Phi_{0} | T_{0\rho} | \Phi_{\rho,n} \rangle.$$

If all four particles are identical, there are only two distinct two-cluster partitions, one of 3+1 type and one of 2+2 type. We choose those partitions to be (12)34 and (12)(34) and denote them in the following by $\alpha = 1$ and 2, respectively. The AGS equations \cite{[10, 14]} for the symmetrized transition operators $U_{\alpha\beta}$ become

$$U_{11} = P_{34}(G_0 t_{34})^{-1} + P_{34} U_{1} G_0 t_{34} U_{11} + U_{2} G_0 t_{34} U_{21}, \quad (11a)$$

$$U_{21} = (1 + P_{34})(G_0 t_{34})^{-1} + (1 + P_{34}) U_{1} G_0 t_{34} U_{11}, \quad (11b)$$

$$U_{12} = (G_0 t_{34})^{-1} + P_{34} U_{1} G_0 t_{34} U_{12} + U_{2} G_0 t_{34} U_{22}, \quad (11c)$$

$$U_{22} = (1 + P_{34}) U_{1} G_0 t_{34} U_{12}. \quad (11d)$$

Here the two-particle transition matrix $t$ acts within the pair (12) and the symmetrized operators for the 3+1 and 2+2 subsystems are obtained from the integral equations

$$U_{\alpha} = P_{\alpha} G_0^{-1} + P_{\alpha} t_{34} U_{\alpha}. \quad (12)$$

The employed basis states have to be symmetric under exchange of two particles in subsystem (12) for 3+1 partition and in (12) and (34) for 2+2 partition. The correct symmetry of the four-boson system is ensured by the operators $P_{34}$, $P_{1} = P_{12} P_{23} + P_{13} P_{23}$, and $P_{2} = P_{13} P_{24}$ where $P_{ab}$ is the permutation operator of particles $a$ and $b$. For each two-cluster channel state $|\Phi_{\alpha,n}\rangle = |\Phi_{\rho,n}\rangle$.
\((1 + P_\alpha)|\phi_{\alpha,n}\rangle\) there is only one independent Faddeev amplitude obtained from the integral equation

\[
|\phi_{\alpha,n}\rangle = G_0 t P_\alpha |\phi_{\alpha,n}\rangle.  \tag{13}
\]

The symmetrized amplitudes for the two-cluster reactions are given in Refs. [8, 9, 11, 12]. The four-cluster breakup amplitude in terms of the symmetrized AGS operators \([11]\) becomes

\[
\langle \Phi_0|T_{0\alpha}|\Phi_{\alpha,n}\rangle = S_{0\alpha} \langle \Phi_0|(1 + P_1)(1 + P_1)\langle \Phi_0|12 t G_0 U_1 G_0 t G_0 U_{\alpha}\rangle |\phi_{\alpha,n}\rangle,
\]

\[
+ 6 t G_0 U_2 G_0 t G_0 U_{\alpha}\rangle |\phi_{\alpha,n}\rangle.  \tag{14}
\]

The factors \(S_{0\alpha}\) depend on the symmetry of the final channel state \(|\Phi_0\rangle\). For example, \(S_{01} = \sqrt{6}\) and \(S_{02} = 2\) for nonsymmetrized \(|\Phi_0\rangle\), while \(S_{01} = \sqrt{3}\) and \(S_{02} = \sqrt{2}\) for \(|\Phi_0\rangle\) that is symmetrized within the pair (12).

We denote by \(\Phi_0(\alpha)\) the four-cluster channel state at threshold, where \(E = 0\) and all momenta vanish. Since all particles have equal momenta, the channel state \(|\Phi_0\rangle\) is invariant under all permutations, i.e., \(P_{\alpha\beta}|\Phi_0\rangle = |\Phi_0\rangle\).

In this particular case the relation (11) simplifies to

\[
\langle \Phi_0|T_{0\alpha}|\Phi_{\alpha,n}\rangle = S_{0\alpha} \langle \Phi_0|(12 t G_0 U_1 G_0 t G_0 U_{\alpha}\rangle |\phi_{\alpha,n}\rangle.  \tag{15}
\]

We use the momentum-space partial-wave framework to solve the AGS equations (11). Two different types of basis states \(|k_x k_y k_z (l_x l_y l_z) J_{\alpha\alpha}|JM\rangle_{\alpha} = |k_x k_y k_z \rangle_{\alpha}\) with \(\alpha = 1\) and 2 are employed. Here \(k_x, k_y,\) and \(k_z\) denote magnitudes of the Jacobi momenta. For \(\alpha = 1\) the Jacobi momenta describe the relative motion in the 1+1, 2+1, and 3+1 subsystems and are expressed in terms of single particle momenta \(k_\alpha\) as

\[
k_x = \frac{1}{2}(k_2 - k_1),  \tag{16a}
\]

\[
k_y = \frac{1}{3}[2k_3 - (k_1 + k_2)],  \tag{16b}
\]

\[
k_z = \frac{1}{4}[3k_4 - (k_1 + k_2 + k_3)],  \tag{16c}
\]

while for \(\alpha = 2\) they describe the relative motion in the 1+1, 1+1, and 2+2 subsystems, i.e.,

\[
k_x = \frac{1}{2}(k_2 - k_1),  \tag{17a}
\]

\[
k_y = \frac{1}{2}(k_1 - k_3),  \tag{17b}
\]

\[
k_z = \frac{1}{2}[k_4 + k_3] - (k_1 + k_2)].  \tag{17c}
\]

The respective orbital angular momenta \(l_x, l_y,\) and \(l_z\) are coupled to the total angular momentum \(J\) with the projection \(M\); all discrete quantum numbers are abbreviated by \(\nu\). An explicit form of integral equations is obtained by inserting the respective completeness relations

\[
1 = \sum_{\nu} \int_0^\infty |k_x k_y k_z \nu\rangle_{\alpha} k_x^2 dk_x k_y^2 dk_y k_z^2 dk_z \langle k_x k_y k_z \nu|,
\]

between all operators in Eqs. (11). Due to rotational symmetry all operators are diagonal in \(J\) and independent of \(M\). In addition, \(U_\alpha\) are diagonal in \(k_x,\) \(J\) and \(l_z,\) \(P_{34}\) is diagonal in \(k_x\) and \(l_z,\) \(t\) is diagonal in \(k_y,\) \(k_z\) and all \(\nu,\) and \(G_0\) is diagonal in all quantum numbers. In this representation the AGS equations for each \(J\) become a system of coupled integral equations in three continuous variables \(k_x, k_y,\) and \(k_z\). Such three-variable equations have been solved in Refs. [11, 12] for the four-nucleon scattering below the three-cluster breakup threshold. The kernel of integral equations (11) contains integrable singularities arising from each sub-system bound state pole of \(U_\alpha\); we isolate these poles in different subintervals and treat them by the subtraction technique when integrating over \(k_z\) \([11]\). However, in the energy regime of the four-cluster breakup, \(E \geq 0,\) the AGS equations (11) contain additional singularities arising from \(G_0,\) namely,

\[
\alpha\langle k_x k_y k_z \nu|G_0|k_x k_y k_z \nu\rangle_{\alpha} = \delta_{\nu,0} \delta(k_x - k_z) \delta(k_x - k_y) \delta(k_y - k_z) / [k_x^2 k_y^2 k_z^2 (E + 0 - k_x^2 / 2 \mu_{\alpha x} - k_y^2 / 2 \mu_{\alpha y} - k_z^2 / 2 \mu_{\alpha z})],
\]

where \(\mu_{\alpha x}\) and \(\mu_{\alpha y}\) are the respective reduced masses. The operator products like \(P_{34} P_1 G_0\) render the singularity structure of Eqs. (11) very complicated. It is considerably simpler at \(E = 0\) where there is only one singular point \(k_x = k_y = k_z = 0\).

In this case the integrals involving \(G_0\) over any of the Jacobi momenta \(k_1\) (except for the trivial integrals involving momentum \(\delta\)-functions) are of type

\[
\int_0^\infty f(k_x, k_y, k_z) k_x^2 dk_x \frac{dk_y}{i 0 - k_y^2 / 2 \mu_{\alpha y} - k_z^2 / 2 \mu_{\alpha z}}  \tag{19}
\]

where the function \(f(k_x, k_y, k_z)\) is regular at \(k_1 = 0\). Thus, the the singularity of \(G_0\) is cancelled by \(k_1^2\) making the integral numerically harmless. Nevertheless, \(G_0\) varies very rapidly near \(k_x = k_y = k_z = 0\) and therefore one should avoid interpolation of \(G_0\); this will be explained in Sec. [11].

The four free boson channel state \(|\Phi_0\rangle\) has \(k_x = k_y = k_z = 0\) and \(l_x = l_y = l_z = J = J_0 = 0\). Due to the threshold law the only nonvanishing partial-wave component of the breakup amplitude (15) is the one with \(J = 0,\) i.e.,

\[
\langle \Phi_0|T_{0\alpha}|\Phi_{\alpha,n}\rangle = \langle \Phi_0|T_{0\alpha}^{J=0}|\Phi_{\alpha,n}\rangle / (4\pi)^2.  \tag{20}
\]

Although \(|\Phi_0\rangle\) has only the \(l_j = J = J_0 = 0\) component, we emphasize that angular momenta \(l_j\) and \(J\) are not conserved. Nonzero \(l_j\) and \(J\) basis states must be included when solving the AGS equations (11).
The form factor
\[ \langle k_x | g \rangle = [1 + c_2 (k_x / \Lambda)^2] e^{-(k_x / \Lambda)^2} \]  
(22)
is taken over from Ref. [8]. Two very different choices of \( c_2 \), namely, \( c_2 = 0 \) and \( c_2 = -9.17 \), are used in order to confirm the universality of the obtained results. The strength
\[ \lambda = \frac{2}{\pi m} \left[ 1 + \frac{c_2}{2} \left( 1 + \frac{3c_2}{8} \right) \frac{\Lambda}{\sqrt{2\pi}} \right]^{-1} \]  
(23)
is constrained to reproduce the given value of the scattering length \( a \) for two particles of mass \( m \). The potential \( \langle k_x | g \rangle \) supports one shallow two-boson bound state at \( a > 0 \) and no one at \( a < 0 \). There are no deeply bound dimers.

The resulting two-boson transition matrix
\[ t = | g \rangle \tau (| g \rangle) \]  
(24)
is separable as well with \( \tau = (\lambda^{-1} - (| g \rangle G_0 | g \rangle)^{-1} \). This allows to reduce the AGS equations \([11]\) to a system of two-variable \( (k_y \text{ and } k_z) \) integral equations and thereby simplifies considerably the numerical calculations. The AGS equations with separable interactions are solved in the form
\[ \langle g | G_0 U_{11} | \phi_{1,n} \rangle = P_{34} \langle g | P_1 | \phi_{1,n} \rangle \]
\[ \quad + P_{34} \langle g | G_0 U_{1} G_0 | g \rangle \tau (| g \rangle G_0 U_{11} | \phi_{1,n} \rangle) \]  
(25a)
\[ \langle g | G_0 U_{21} | \phi_{1,n} \rangle = (1 + P_{34}) \langle g | P_1 | \phi_{1,n} \rangle \]
\[ \quad + (1 + P_{34}) \langle g | G_0 U_{1} G_0 | g \rangle \tau (| g \rangle G_0 U_{21} | \phi_{1,n} \rangle), \]  
(25b)
\[ \langle g | G_0 U_{12} | \phi_{2,n} \rangle = \langle g | G_0 U_{2} | g \rangle \tau (| g \rangle G_0 U_{12} | \phi_{2,n} \rangle) \]
\[ \quad + P_{34} \langle g | G_0 U_{1} G_0 | g \rangle \tau (| g \rangle G_0 U_{12} | \phi_{2,n} \rangle) \]  
(25c)
\[ \langle g | G_0 U_{22} | \phi_{2,n} \rangle = (1 + P_{34}) \langle g | G_0 U_{1} G_0 | g \rangle \tau (| g \rangle G_0 U_{22} | \phi_{2,n} \rangle). \]  
(25d)

We emphasize that with respect to angular momentum quantum numbers, the full complexity of the four-body problem is still present in Eqs. \([25]\). Basis states up to \( l_y, l_z, J < 3 \) are included to achieve the partial-wave convergence for the solutions of the AGS equations \([25]\). Regarding the continuous variables \( k_y \text{ and } k_z \), the treatment is taken over from Ref. \([11]\). It requires interpolation of \( (| g \rangle G_0 U_a G_0 | g \rangle) \) in initial and final momenta \( k_y \text{ and } k_z \). To take care of the \( G_0 \) singularity at \( k_j = 0 \) we factorize \( (k_y k'_y k_z k'_z G_0 U_a G_0 | k_y k_z k'_y k'_z \rangle = (k_y^2 / 2\mu_{a y} + k_z^2 / 2\mu_{a z})^{-1} U_0^{\alpha \nu} (k_y, k_z, k'_y, k'_z) (k_y^2 / 2\mu_{a y} + k_z^2 / 2\mu_{a z})^{-1} \delta (k'_y - k_y) (k'_z - k_z) / k_z^2 \) such that \( U_0^{\alpha \nu} \) is a very smooth function and is interpolated very reliably using spline functions as in Ref. \([11]\), while the factors of the type \( (k_y^2 / 2\mu_{a y} + k_z^2 / 2\mu_{a z})^{-1} \) are calculated explicitly wherever needed.

The discretization of integrals in Eqs. \([24]\) using Gaussian quadrature rules leads to a system of linear algebraic equations. We reduce its dimension even further by applying the so-called \( K \)-matrix technique \([14]\) converting the complex equations into the real ones, and by eliminating the \( \alpha = 2 \) component, i.e., by inserting Eq. \([25c]\) into \([25a]\) and Eq. \([25d]\) into \([25b]\). To avoid difficulties due to possibly slow convergence of iterative methods, the resulting linear system is solved using the direct matrix inversion. Further details of the calculations can be found in Refs. \([11, 13]\).

### III. FOUR-BOSON RECOMBINATION

The number of the four-atom recombination events in a non-degenerate atomic gas per volume and time is \( K_4 \rho^4 \) where \( \rho \) is the density of atoms and \( K_4 \) the four-atom recombination rate. In the ultracold limit the kinetic energies of initial atoms are much smaller than the final two-cluster kinetic or binding energies, and the initial momenta are much smaller than the momenta of the two resulting clusters, \( k_x, k_y, k_z \). Under these conditions \( K_4 \) can be approximated very well by the zero-temperature limit \( K_4^0 \) calculated at \( E = 0 \) with \( k_x = k_y = k_z = 0 \) in the initial state, i.e.,
\[ K_4^0 = 16 \pi^7 \sum_{\alpha, n'} \mu_{a, n'} \rho p_{a, n'} | \langle \Phi_0^0 | T_{00}^0 | \Phi_{\alpha, n'} \rangle |^2. \]  
(26)

It has contributions from all two-cluster channels \( (\alpha, n') \).

Concerning the four-atom recombination, the most interesting regime lies at large negative two-boson scattering length \( a \) where no shallow dimers exist but the Efimov trimers and tetratets cross the zero-energy threshold. In our nomenclature we characterize the trimers by one integer number \( n \), starting with \( n = 0 \) for the ground state, and the tetratets by two integers \((n, k)\), where \( n \) refers to the associated trimer and \( k = 1, 2 \) for a deeper (shallow) tetramer. The \( a \)-dependence of the \( n \)th family trimer \( (b_n) \) and tetramer \( (B_n, k) \) binding energies is shown in Fig. \([1]\) for \( a < 0 \). We denote by \( a_n^0 \) the specific negative value of \( a \) where the \( n \)th trimer binding energy vanishes, i.e., \( b_n = 0 \). In the universal limit, i.e., for sufficiently large \( n \), the ratio \( a_{n+1}^0 / a_n^0 = e^{\pi / s_0} \approx 22.6944 \) where \( s_0 \approx 1.00624 \) \([2]\). Trimmer binding energy \( b_n^0 \) taken in the unitary limit \( 1/a = 0 \), is used to build dimensionless ratios \( b_n / b_n^0 \) and \( B_n / b_n^0 \). These ratios become independent of the short-range details of the interaction provided that its range is small enough compared to the size of the few-boson states. This condition is fulfilled for high excited states, i.e., for sufficiently large \( n \). We checked this independence of the results in Fig. \([1]\) performing calculations with \( n = 3 \) and \( n = 4 \) for two choices of the potential form factor \([22]\).

We remind that only two lowest \( n = 0 \) tetratets are true bound states. All higher tetratets lie above the lowest atom-trimer threshold and therefore are unstable.
bound states with finite width and lifetime. Their positions $B_{n,k}$ shown in Fig. 1 are extracted from the atom-trimer scattering calculations at $E < 0$ as described in Refs. [8, 13]. The specific negative values of $a = a_{n,k}^0$ where the respective tetramers emerge at the four free particle threshold can be obtained by extrapolating the results in Fig. 1 to $B_{n,k} = 0$. This procedure yields universal ratios

$$a_{n,1}^0/a_n^0 = 0.4254(2), \quad a_{n,2}^0/a_n^0 = 0.9125(2). \quad (27a)$$

The uncertainties, apart from the numerical accuracy, arise due to the residual dependence on $n \geq 3$ and $c_2$ in Eq. (22).

We will show our results for the four-boson recombination rate as functions of the dimensionless ratio $a/a_n^0$. Then in the interval $(e^{-\pi/s_0}, 1)$ there are exactly $n$ trimer states with $0 \leq n' \leq n - 1$ that contribute to Eq. (20). Instead of $K_n^0$ we build a dimensionless quantity

$$\kappa_n = K_n^0 m/(\hbar|a_n^0|)^7. \quad (28)$$

With $1 \leq n \leq 4$ in Fig. 2, we explore a broad range of the two-boson scattering length $a$ and recombination channels $n'$ whereas $\alpha = 1$. The four-boson recombination rate varies over many orders of magnitude, but all $\kappa_n$ as functions of the respective $a/a_n^0$ show qualitatively the same behavior. For higher $n$, i.e., $n \geq 3$, there is also a very good quantitative agreement, indicating the universality of our $\kappa_n$ results. Indeed, while $\kappa_n$ in Fig. 2 are obtained with $c_2 = 0$ in Eq. (22), additional calculations with $c_2 = -9.17$ for $n = 3$ and $4$ agree very well with the corresponding predictions in Fig. 2, thereby confirming our conclusion on $\kappa_n$ universality. In contrast, $\kappa_1$ shows significant quantitative deviations from the universal behavior due to finite-range effects: in the regime $a/a_1^0 < 1$ the only available recombination channel leads to the ground state trimer whose size is comparable to the interaction range [13]. We note that our non-universal results at $n \leq 1$ depend not only on $c_2$ but also on $m$ and $\Lambda$; our calculations are performed with $m$ and $\Lambda$ being the mass of $^4$He atom and $\Lambda = 0.4 \, \text{Å}^{-1}$.

The four-boson recombination rate has resonant peaks at $a = a_{n,k}^0$ where the respective tetramers are at the four free atom threshold. The ratios $a_{n,k}^0/a_n^0$ extracted from the peak positions (from $B_{n,k} = 0$ in the case $n = 0$) are collected in Table I. For $n \geq 3$ they approach the universal limit with high accuracy and are fully consistent with the ones in Eqs. (27). Furthermore, the four-boson recombination rate has sharp cusps at $a = a_n^0$, or, equivalently, at $a = e^{-\pi/s_0} a_n^0$, where a new atom-trimer channel opens. We list in Table I also the ratios $a_{n+1}^0/a_n^0$ to prove that our numerical predictions converge towards analytical result $e^{\pi/s_0} \approx 22.6944$ [1, 12]. Furthermore, for $n \geq 3$ we obtain $a_n^0 \sqrt{mb_n^0}/\hbar = -1.5077(1)$.

At $a/a_1^0 > 1$ not only the ground state but also excited trimers are produced via four-atom recombination.
and contribute to the total rate \( \nu_{n,n'} \). In Fig. 3 we compare the relative weights \( \nu_{n,n'} \) for the production of the \( n' \)th state trimers via four-boson recombination. The results in Fig. 3 clearly demonstrate that more weakly bound trimers are produced more efficiently and the four-atom recombination process is strongly dominated by the \( n' = (n-1) \)th channel with the shallowest available trimer. We note that a similar conclusion was drawn in Ref. 9 about the trimer production in dimer-dimer collisions. Except for the vicinity of \( a/a_n^0 = e^{-\pi/\sigma_0} \) where the shallowest trimer disappears, the relative weights \( \nu_{n,n'} \) depend quite weakly on \( a \). Thus, the tetramer states around \( a = a_{n,k}^0 \) enhance the production of all trimers by equal factors.

As mentioned in the introduction, the recombination of four identical bosons has been calculated in Refs. 4, 10 using the adiabatic hyperspherical framework. There is a good qualitative agreement between our results and those of Refs. 4, 10: the shape of the recombination rate, including its resonant peaks and cusps, is reproduced in both cases. However, the calculations of Refs. 4, 10 are limited to \( n = 1 \) where the finite-range effects are not entirely negligible and deviations from our universal results can be expected much like in the case of our \( n = 1 \) results in Fig. 2. Within this accuracy the predictions of Refs. 4, 10 for the peak positions, i.e., \( a_{n,1}^0/a_n^0 \approx 0.43 \) and \( a_{n,2}^0/a_n^0 \approx 0.90 \), are consistent with our results. On the other hand, the available experimental data 6 are obtained in the system of ultracold \(^{133}\)Cs atoms with deeply bound non-Efimov-like few-atom states and refer to \( n = 0 \). Therefore it is not surprising that the experimental results \( a_{n,1}^0/a_n^0 \approx 0.47 \) and \( a_{n,2}^0/a_n^0 \approx 0.84 \) deviate from the universal values even more. Nevertheless, they remain roughly consistent with the theoretical predictions. Thus, our calculations present no improvement over the ones of Refs. 4, 10 when comparing with the existing data, but serve as a theoretical benchmark and may be valuable for the future experiments performed in a truly universal regime.

IV. SUMMARY

We studied ultracold four-boson recombination. Exact four-particle scattering equations in the AGS version have been used. We related the four-cluster breakup and recombination operators to the standard two-cluster AGS operators. The scattering equations at the four-cluster breakup threshold have been precisely solved in the momentum-space framework. The zero-temperature limit of the four-boson recombination rate was calculated for negative values of the two-boson scattering length. We demonstrated its universal behavior in the regime with high excited Efimov trimers where the finite-range effects are negligible, and determined the positions of resonant recombination peaks caused by the tetramer states. We also have shown that most of the Efimov trimers produced via the four-boson recombination are in the state with weakest binding. Our results are consistent with existing experimental data 6 and theoretical predictions 4, 10, but exceed the latter ones with respect to the accuracy in the universal limit. In addition, the present work constitutes a step towards solving the momentum-space four-body scattering equations above the breakup threshold.

ACKNOWLEDGMENTS

The author thanks P. U. Sauer for discussions on few-body recombination problem.

[1] V. Efimov, Phys. Lett. B 33, 563 (1970).
[2] E. Braaten and H.-W. Hammer, Phys. Rep. 428, 259 (2006).
[3] H. W. Hammer and L. Platter, Eur. Phys. J. A 32, 113 (2007).
[4] J. von Stecher, J. P. D’Incao, and C. H. Greene, Nature Phys. 5, 417 (2009).
[5] F. Ferlaino, S. Knoop, M. Mark, M. Berninger, H. Schöbel, H.-C. Nägerl, and R. Grimm, Phys. Rev. Lett. 101, 023201 (2008).
[6] F. Ferlaino, S. Knoop, M. Berninger, W. Harm, J. P. D’Incao, H.-C. Nägerl, and R. Grimm, Phys. Rev. Lett. 102, 140401 (2009).
[7] S. E. Pollack, D. Dries, and R. G. Hulet, Science 326, 1683 (2009).
[8] A. Deltuva, Phys. Rev. A 82, 040701(R) (2010).
[9] A. Deltuva, Phys. Rev. A 84, 022703 (2011).
[10] P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967); E. O. Alt, P. Grassberger, and W. Sandhas, JINR report No. E4-6688 (1972).

[11] A. Deltuva and A. C. Fonseca, Phys. Rev. C 75, 014005 (2007).

[12] A. Deltuva and A. C. Fonseca, Phys. Rev. C 76, 021001(R) (2007).

[13] R. Lazauskas and J. Carbonell, Phys. Rev. A 73, 062717 (2006).

[14] A. Deltuva, R. Lazauskas, and L. Platter, Few-Body Syst. 51, 235 (2011).

[15] A. Deltuva, EPL 95, 43002 (2011).

[16] N. P. Mehta, S. T. Rittenhouse, J. P. D’Incao, J. von Stecher, and C. H. Greene, Phys. Rev. Lett. 103, 153201 (2009).

[17] Y. Wang and B. D. Esry, Phys. Rev. Lett. 102, 133201 (2009).

[18] O. A. Yakubovsky, Yad. Fiz. 5, 1312 (1967) [Sov. J. Nucl. Phys. 5, 937 (1967)].