Non-proper complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

Examples of complete minimal surfaces properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ have been extensively studied and the literature contains a plethora of nontrivial ones. In this paper we construct a large class of examples of complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$, not necessarily proper, which are invariant by a vertical translation or by a hyperbolic or parabolic screw motion. In particular, we construct a large family of non-proper complete minimal disks embedded in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation and a hyperbolic screw motion and whose importance is twofold. They have finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by the isometry, thus highlighting a different behaviour from minimal surfaces embedded in $\mathbb{R}^3$ satisfying the same properties. They show that the Calabi-Yau conjectures do not hold for embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

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1 Introduction

Examples of complete minimal surfaces properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ have been extensively studied and the literature contains a plethora of nontrivial ones. In this paper we focus on complete embedded examples, not necessarily proper, which are invariant by either a vertical translation or a hyperbolic or parabolic screw motion. Some examples with these properties, but all of them properly embedded, have been constructed in [16, 14, 17, 20, 19, 15, 10, 13].

The key examples contained in this paper are complete minimal disks embedded in $\mathbb{H}^2 \times \mathbb{R}$ that are non-proper and invariant by a vertical translation and a hyperbolic screw motion, we call them helicoidal-Scherk examples. The importance of such helicoidal-Scherk examples is twofold in understanding the behaviour of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

In addition to being non-proper, a significant feature of these examples is that they have finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by the vertical translation or the hyperbolic screw motion. In [21] Toubiana proved that a complete embedded minimal annulus with finite total

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curvature in the quotient of $\mathbb{R}^3$ by a translation must the quotient of a helicoid. In [11] Meeks and Rosenberg proved that Toubiana’s result holds if the translation is replaced by a screw-motion. Moreover, in the same paper they also show that a complete embedded minimal surface with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation or a screw-motion must be proper. Our examples highlight a much different behaviour in $\mathbb{H}^2 \times \mathbb{R}$. Recently, Collin, Hauswirth and Rosenberg have studied the conformal type and the geometry of the ends of properly embedded minimal surfaces with finite total curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a vertical translation [5]. Our main examples are related to but not included in their study.

The same examples are also of interest in relation to the Calabi-Yau conjectures for embedded minimal surfaces [1, 2, 22]. In [3], Colding and Minicozzi showed that a complete minimal surface embedded in $\mathbb{R}^3$ with finite topology is proper. See [12] for a generalization of their result. Our helicoidal-Scherk examples show that Colding and Minicozzi’s result does not hold in $\mathbb{H}^2 \times \mathbb{R}$. Note that in [6] Coskunuzer has already constructed a complete embedded disk in $\mathbb{H}^3$ which is not proper, thus showing that Colding and Minicozzi’s result does not generalize to $\mathbb{H}^3$. The techniques that we use to construct our examples are completely different from his.

The helicoidal-Scherk examples are constructed in the next section. In the other sections we further generalize the construction and also give examples of properly embedded minimal surfaces that are invariant by a parabolic screw motion. These latter examples are included in the study in [5].

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2 Helicoidal-Scherk examples

In order to construct our examples we consider the Poincaré disk model of $\mathbb{H}^2$; i.e.

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \lvert z \rvert < 1 \},$$

with the hyperbolic metric

$$g_{-1} = \frac{4}{(1 - \lvert z \rvert^2)^2} dz^2.$$

We denote by $\partial_\infty \mathbb{H}$ the boundary at infinity of $\mathbb{H}^2$ and by $0$ the origin of $\mathbb{H}^2$. We use $t$ for the coordinate in $\mathbb{R}$. Finally, given any two points $p, q \in \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, we will denote by $pq$ the geodesic arc joining them.

Let us consider $p_1 = 1$ and $p_2 = e^{i\pi/n}$, for some $n \in \mathbb{N}$. Let $\Omega$ be the region bounded by the ideal geodesic triangle with vertices $0, p_1$ and $p_2$ and edges $0p_1, 0p_2$ and $p_1p_2$. By Theorem 4.9 in [9], there exists a minimal graph over $\Omega$ with boundary values $0$ over $0p_1$, $h$ over $0p_2$ and $+\infty$ over $p_1p_2$, for any constant $h > 0$ (see Figure 1). We call this graph the fundamental piece. Using Schwarz reflection principle, after considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, namely
Figure 1: Fundamental piece of a helicoidal-Scherk example for \( n=2 \).

\( \overline{0p_1 \times \{0\}, \overline{0p_2 \times \{h\}} \subset \mathbb{H}^2 \times \mathbb{R} } \), we obtain a simply-connected minimal surface \( \hat{M}_{n,h} \) with boundary the vertical line \( \{0\} \times \mathbb{R} \) and invariant by the vertical translation \( T \) by \( (0,4nh) \) and by the hyperbolic screw motion \( S \) obtained by composing the rotation by angle \( \frac{\pi}{n} \) around \( 0 \) and the vertical translation by \( (0,2h) \). After reflecting across the line \( \{0\} \times \mathbb{R} \), we obtain a simply-connected complete minimal surface \( M_{n,h} \) invariant by both the vertical translation \( T \) and the hyperbolic screw motion \( S \). We call these surfaces helicoidal-Scherk examples.

We observe that if we consider \( h = 0 \) in this construction, we obtain a Scherk graph over a symmetric ideal polygonal domain, see [4, 16].

As a consequence of the Gauss-Bonnet Theorem applied on a fundamental piece (see [1] page 1896 for a similar argument), \( M_{n,h} \) has finite total curvature in its quotient by both \( T \) and \( S \).

We next show that \( M_{n,h} \) is embedded. Let us denote by \( \Omega_i, i = 1, \ldots, 4n \), the domain obtained by rotating \( \Omega \) around the origin by an angle \( \frac{\pi}{2n}(i-1) \) so that \( \Omega_1 = \Omega \), and let \( \overline{\Omega_i} = \overline{\Omega_i} \times \mathbb{R} \), where \( \overline{\Omega_i} \) is the closure of \( \Omega_i \). Note that the domain \( \overline{\Omega_{2n+i}} \) can also be obtained by reflecting \( \Omega_i \) across the origin. We are going to prove that

\[
M_{n,h} \cap \overline{\Omega_{2n+1}}
\]

has no self-intersections. After this, repeating the same argument shows that \( M_{n,h} \) is embedded. Let \( p_1^*, p_2^* \) be the reflection across the origin of \( p_1 \) and \( p_2 \). Recall that \( \hat{M}_{n,h} \cap \overline{\Omega_1} \) consists of a graph with boundary values 0 over \( \overline{0p_1} \), \( h \) over \( \overline{0p_2} \) and \( +\infty \) over \( \overline{p_1p_2} \), together with its vertical translates by the vector \( k(0,4nh) \), \( k \in \mathbb{Z} \). By construction, since we have reflected an even
number of times, \( \widehat{M}_{n,h} \cap \widehat{\Omega}_{2n+1} \) consists of a union of graphs with boundary values

\[
\begin{cases}
2nh + 4knh & \text{over } \overline{0p_1} \\
(2n + 1)h + 4knh & \text{over } \overline{0p_2} \\
+\infty & \text{over } \overline{p_1p_2}
\end{cases}
\]

with \( k \in \mathbb{Z} \). The reflection of \( \widehat{M}_{n,h} \cap \widehat{\Omega}_1 \) across \( \{0\} \times \mathbb{R} \) instead consists of a union of graphs with boundary values

\[
\begin{cases}
4knh & \text{over } \overline{0p_1} \\
h + 4knh & \text{over } \overline{0p_2} \\
+\infty & \text{over } \overline{p_1p_2}
\end{cases}
\]

with \( k \in \mathbb{Z} \). In fact, \( M_{n,h} \cap \widetilde{\Omega}_{2n+1} \) consists of the reflected fundamental piece together with its vertical translates by the vector \( k(0, 2nh) \), \( k \in \mathbb{Z} \). In particular \( M_{n,h} \cap \widetilde{\Omega}_{2n+1} \) is embedded.

Repeating this argument proves that \( M_{n,h} \) is embedded.

Observe that the previous argument also shows that we can consider the quotient of \( M_{n,h} \) by \( (0, 2nh) \), obtaining a non-orientable complete non-proper embedded minimal surface.

Finally we remark that \( M_{n,h} \cap \widetilde{\Omega}_1 \) accumulates to \( p_1 \times \mathbb{R} \), and therefore \( M_{n,h} \) is a simply-connected (in particular, with finite topology) minimal surface embedded in \( \mathbb{H}^2 \times \mathbb{R} \) which is complete but not proper.

Let us now describe a generalization of these examples. Instead of considering a geodesic triangle, let \( \Omega \) be the region bounded by an ideal geodesic polygon constructed in the following way. As before, let \( p_1 = 1 \) and \( p_2 = e^{i\pi/2n} \) and let \( \text{arc}(p_1p_2) \) denote the shortest arc in \( \partial_{\infty} \mathbb{H}^2 \) with end points \( p_1, p_2 \). In this construction, the geodesics \( \overline{0p_1} \) and \( \overline{0p_2} \) are the same but, instead of connecting the two with the geodesic \( \overline{p_1p_2} \), we consider \( k \geq 1 \) points \( q_1, \ldots, q_k \), cyclically ordered in \( \text{arc}(p_1p_2) \). We define \( \Omega \) as the region bounded by the ideal geodesic polygon with vertices 0, \( p_1, q_1, \ldots, q_k \) and \( p_2 \). Assuming that \( \Omega \) satisfies the Jenkins-Serrin condition of Theorem 4.9 in \( \mathbb{H}^2 \), we can find a graph over \( \Omega \) with boundary values 0 on \( \overline{0p_1} \), \( h > 0 \) on \( \overline{0p_2} \) and alternating \( \pm\infty \) on the remaining geodesic arcs \( \overline{p_1q_1}, \overline{q_1q_2}, \ldots, \overline{q_{k-1}q_k}, \overline{q_kp_2} \). After considering successive symmetries with respect to the horizontal and vertical geodesics contained in the boundary of such a graph, we obtain a simply-connected complete embedded minimal surface in \( \mathbb{H}^2 \times \mathbb{R} \) invariant by the same vertical translation \( T \) and the same hyperbolic screw motion \( S \) previously defined. Again, this surface is non-proper and has finite total curvature when considered in the quotient by \( T \) or \( S \). Moreover, it admits a non-orientable quotient by the vertical translation given by the vector \( (0, 2nh) \). We also refer to such examples as helicoidal-Scherk examples.

It is easy to show that the class of these more general examples is rather large. Here is an easy way to construct domains as previously described. If for \( j = 1, \ldots, k \), we let \( q_j = e^{i\pi j/2n} \), then we recover the symmetric helicoidal-Scherk examples for a smaller choice of \( h \), by the generalized maximum principle for such minimal graphs (see [4, Theorem 2] or [9, Theorem 4.13]). However,
after slightly perturbing one such $q_i$, we would obtain a domain satisfying the Jenkins-Serrin condition of Theorem 4.9 in [9]. In particular, we observe that when $k = 1$ and $q_1$ is any point in $\text{arc}(p_1p_2)$ then the Jenkins-Serrin condition is satisfied.

As mentioned in the introduction, the importance of these examples is twofold.

- In [21] Toubiana proved that a complete embedded minimal annulus with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation must the quotient of a helicoid. In [11] Meeks and Rosenberg proved that Toubiana’s result holds if the translation is replaced by a screw-motion. Moreover, in the same paper they also show that a complete embedded minimal surface with finite total curvature in the quotient of $\mathbb{R}^3$ by a translation or a screw-motion must be proper. Our examples highlight a much different behaviour in $H^2 \times \mathbb{R}$.

- In [3], Colding and Minicozzi showed that a complete minimal surface embedded in $\mathbb{R}^3$ with finite topology is proper. Thus showing that the Calabi-Yau conjectures hold for complete minimal surfaces embedded in $\mathbb{R}^3$, see [11] [22]. Our helicoidal-Scherk examples show that Colding and Minicozzi’s result does not hold in $H^2 \times \mathbb{R}$. Note that in [6] Coskunuzer has already constructed a complete embedded disk in $H^3$ which is not proper, thus showing that Colding and Minicozzi’s result does not generalize to $H^3$. The techniques that we have used to construct the helicoidal-Scherk examples are completely different from his. Note also that in [12], Meeks and Rosenberg generalized the result in [3] to complete minimal surfaces with positive injectivity radius and, among other things, showed that the closure of a complete minimal surface with positive injectivity radius embedded in a 3-manifold has the structure of a minimal lamination. The closure of a helicoidal-Scherk example is the minimal lamination given by the union of such helicoidal-Scherk example with the related totally geodesic vertical planes.

3 Helicoidal examples

Let us now consider $p_1 = 1$ and $p_2 = e^{\frac{2i\pi}{m}}$, with $m \in \mathbb{N}$. Let $\Omega$ be the region bounded by $\overline{0p_1}, \overline{0p_2}$ and $\text{arc}(p_1p_2)$, see Figure 2. By Theorem 4.9 in [9], there exists a minimal graph over $\Omega$ with boundary values 0 over $\overline{0p_1}$, $h$ over $\overline{0p_2}$ and $f$ over $\text{arc}(p_1p_2)$, for any $h > 0$ and any continuous function $f$ on $\text{arc}(p_1p_2)$ (in fact, finitely many points of discontinuity for $f$ are allowed). Again, after considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we get a minimal surface $\tilde{M}_{m,h,f}$ bounded by the vertical line $\{0\} \times \mathbb{R}$ and invariant by the vertical translation $T$ by $(0, 2mh)$ and by the hyperbolic screw motion $S$ obtained by composition of the rotation by angle $\frac{2\pi}{m}$ around $0$ and the vertical translation by $(0, 2h)$. Considering a final symmetry with respect to $\{0\} \times \mathbb{R}$, we obtain a simply-connected complete minimal surface $M_{m,h,f} \subset H^2 \times \mathbb{R}$ which is invariant by the vertical translation $T$ and by the hyperbolic screw motion $S$. We call these surfaces helicoidal examples. These examples
Figure 2: Fundamental piece of a helicoidal example with $m = 4$.

have infinite total curvature in the quotient, since their normal vectors do not become horizontal when we approach points in $\text{arc}(p_1p_2) \times \mathbb{R}$ (see \cite[Theorem 3.1]{7}).

Let us show that $M_{m,h,f}$ is embedded when $m$ is even and, if $f$ satisfies certain conditions, when $m$ is odd. Using the same notation as in the previous section, we know that $\hat{M}_{m,h,f} \cap \tilde{\Omega}_{m+1}$ consists of the union of the minimal graph with boundary values

$$
\begin{cases}
mh, & \text{over } \overline{0p_1^*} \\
(m+1)h, & \text{over } \overline{0p_2^*} \\
f_m, & \text{over } \text{arc}(p_1^*p_2^*)
\end{cases}
$$

where

$$f_m = \begin{cases}
mh + f, & \text{if } m \text{ is even} \\
(1+m)h - f, & \text{if } m \text{ is odd}
\end{cases}
$$

together with its vertical translates by the vector $k(0,2mh)$, $k \in \mathbb{Z}$. The reflection of $\hat{M}_{m,h,f} \cap \tilde{\Omega}_1$ across $\{0\} \times \mathbb{R}$ instead consists of the graph with boundary values

$$
\begin{cases}
0, & \text{over } \overline{0p_1^*} \\
h, & \text{over } \overline{0p_2^*} \\
f, & \text{over } \text{arc}(p_1^*p_2^*)
\end{cases}
$$

together with its vertical translates by the vector $k(0,2mh)$, $k \in \mathbb{Z}$. Hence, using the general maximum principle for minimal graphs \cite[Theorem 4.16]{9}, we get that $M_{m,h,f} \cap \tilde{\Omega}_{m+1}$ is embedded when $m$ is even or when $m$ is odd and

$$(1-m)h \leq 2f \leq (1+m)h.$$
Thus \( M_{m,h,f} \) is embedded under the same conditions.

From the argument above we deduce that, when \( m \) is even, \( M_{m,h,f} \cap \overline{\Omega}_{m+1} \) consists of the reflected fundamental piece together with its vertical translates by the vector \( k(0, mh), k \in \mathbb{Z} \). Thus \( M_{m,h,f} \) admits also a non-orientable quotient by \((0, mh)\).

In the case \( m \) is even, if we consider the sequence of functions \( \{f_k\} \), where \( f_k = k \) over \( \text{arc}(p_1 p_2) \), then we obtain the fundamental piece of the corresponding symmetric helicoidal-Scherk example as a limit of the fundamental piece of \( M_{m,h,f_k} \) and thus \( M_{\tilde{m},h} \) as a limit of the sequence of surfaces \( \{M_{m,h,f_k}\}_k \). We could take another choice of functions \( f_k \) with the same limit but in such a way that each \( M_{m,h,f_k} \) has a smooth boundary. In fact, any helicoidal-Scherk example can be recovered as a limit of some sequence \( \{M_{m,h,f_k}\}_k \) of helicoidal examples, by choosing appropriate functions \( f_k \).

Finally, observe that if we consider \( f(e^{i t}) = \frac{h m}{\pi} t, \) with \( t \in (0, \frac{\pi}{m}) \), we recover one of the helicoids given by Nelli and Rosenberg in [16], congruent to the Euclidean one. In fact, by varying \( h > 0 \) we re-obtain all of their examples. Hence the family of helicoidal examples contains the helicoids.

4 Helicoidal-Scherk examples with axis at infinity

We now take \( p_0 = -1, p_1 = 1 \) and \( p_2 = e^{i \theta} \), for some \( \theta \in (0, \pi) \). Let \( \Omega \) be the region bounded by the ideal geodesic triangle with vertices \( p_0, p_1 \) and \( p_2 \), see Figure [4] By Theorem 4.9 in [9], there exists a minimal graph over \( \Omega \) with boundary values \( 0 \) over \( \overline{p_0 p_1} \), \( h \) over \( \overline{p_0 p_2} \) and \(+\infty\) over \( \overline{p_1 p_2} \), for any constant \( h > 0 \). After considering successive symmetries with respect to \( \overline{p_0 p_1} \times \{0\}, \overline{p_0 p_2} \times \{h\} \subset \mathbb{H}^2 \times \mathbb{R} \), we obtain a properly embedded minimal surface \( M_{\theta,h} \) (in fact, it is a graph over an ideal polygonal domain with infinitely many boundary geodesic arcs) invariant by the parabolic screw motion \( P \) obtained by composition of the parabolic translation with fixed point \( p_0 \) which maps \( p_1 \) onto \( e^{i \theta} \) with the vertical translation by \((0, 2h)\). We call these examples helicoidal-Scherk examples with axis at infinity, since they can be obtained as a limit of helicoidal-Scherk examples whose axes go to infinity. As a consequence of the Gauss-Bonnet Theorem, \( M_{\theta,h} \) has finite total curvature in its quotient by \( P \).

We observe that if we consider \( h = 0 \) in this construction, we obtain a pseudo-Scherk graph considered by Leguil and Rosenberg in [8].

Just like in section [2] these examples can be generalized by taking an ideal geodesic polygon \( \Omega \) with vertices \( p_0 = -1, p_1 = 1, p_2 = e^{i \theta} \) and \( k \geq 1 \) points \( q_1, \ldots, q_k \) in \( \text{arc}(p_1 p_2) \), such that \( \Omega \) satisfies the Jenkins-Serrin condition of Theorem 4.9 in [9]. One such polygonal domain is called pseudo-Scherk polygon in [8]. We start with the graph over \( \Omega \) with boundary values \( 0 \) on \( \overline{p_0 p_1} \), \( h > 0 \) on \( \overline{p_0 p_2} \) and alternating \( \pm \infty \) on the remaining geodesics. After considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we obtain a properly embedded minimal surface in \( \mathbb{H}^2 \times \mathbb{R} \) invariant by the parabolic screw motion \( P \) described above (again, it is a graph over an ideal polygonal domain with infinitely
Figure 3: Fundamental piece of a helicoidal-Scherk example with axis at infinity.

many boundary geodesic arcs). In the quotient by $P$, such a surface has finite total curvature. We also refer to these generalized surfaces as helicoidal-Scherk examples with axis at infinity.

5 Helicoidal examples with axis at infinity

Let us now consider $p_0 = -1$, $p_1 = 1$ and $p_2 = e^{i\theta}$, with $\theta \in (0, \pi)$. Let $\Omega$ be the region bounded by $p_0p_1, p_0p_2$ and arc$(p_1p_2)$, see Figure 4. By Theorem 4.9 in [9], there exists a minimal graph over $\Omega$ with boundary values 0 over $p_0p_1$, $h$ over $p_0p_2$ and $f$ over arc$(p_1p_2)$, for any $h > 0$ and any continuous function $f$ on arc$(p_1p_2)$ (again, finitely many points of discontinuity for $f$ are allowed).

After considering successive symmetries with respect to the horizontal geodesics contained in the boundary of such a graph, we get a properly embedded minimal surface $M_{\theta,h,f}$ (in fact, it is an entire graph) invariant by the parabolic screw motion $P$ obtained by composition of the parabolic translation with fixed point $p_0$ which maps $p_1$ onto $e^{i2\theta}$ with the vertical translation by $(0, 2h)$. In the quotient by $P$, $M_{\theta,h,f}$ has infinite total curvature. We call these surfaces helicoidal examples with axis at infinity.

We observe that, arguing as in section 3, any helicoidal-Scherk example with axis at infinity can be recovered as a limit of helicoidal examples $M_{\theta,h,f_k}$, by choosing appropriate functions $f_k$.

Finally, if we consider $f(e^{i\theta}) = \frac{h}{\theta} t$, for any $t \in (0, \theta)$, we recover one of the examples invariant by the 1-parametric isometry group generated by $P$, founded by Onnis [17] and Sa Earp [19] independently.
6 Non-periodic examples

In this last section, we point out how this method can be used to construct a lot of simply-connected examples which cannot be written as graphs. We now let \( p_1 = 1 \) and \( p_2 = e^{i\theta} \), for some fixed \( \theta \in (0, \pi) \), and define \( \Omega \) as the domain bounded by \( \overline{0p_1} \), \( \overline{0p_2} \) and arc\((p_1p_2)\). By Theorem 4.9 in \([9]\), we know there exists a minimal graph over \( \Omega \) with boundary values \( +\infty \) on \( \overline{0p_1} \), \( 0 \) on \( \overline{0p_2} \) and \( f \) on arc\((p_1p_2)\), for any continuous function \( f \) (again, finitely many discontinuity points are allowed). By rotating such a graph by an angle \( \pi \) about the horizontal geodesic \( \{0\} \times \mathbb{R} \) contained in its boundary, we obtain a minimal graph whose boundary consists of the vertical line \( \{0\} \times \mathbb{R} \). After extending such a graph by symmetry about its boundary, we obtain a properly immersed simply-connected minimal surface. When \( \theta \leq \pi/2 \) or \( f \) is positive, the obtained surface is embedded. And its asymptotic boundary curve is smooth if \( f = 0 \) on \( p_2 \).

When \( \theta \leq \pi/2 \) and \( f \) diverges to \( +\infty \) at any point (or to \( \pm \infty \) alternately over a finite number of arcs contained in arc\((p_1p_2)\), with some additional restrictions to where the endpoints of such arcs are placed in order to satisfy the Jenkins-Serrin condition in the limit) we get the simply-connected minimal examples with finite total curvature constructed by Pyo and the first author in \([18]\), called \textit{twisted Scherk examples}.

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