PSEUDOCONFORMAL STRUCTURES AND THE EXAMPLE OF FALBEL’S CROSS–RATIO VARIETY

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Abstract. We introduce pseudoconformal structures on 4–dimensional manifolds and study their properties. Such structures are arising from two complex operators which commute in a 2–dimensional subbundle of the tangent bundle; this subbundle thus forms a codimension 2 CR structure. A non trivial example of a manifold endowed with a pseudoconformal structure is Falbel’s cross–ratio variety \( X \); this variety is isomorphic to the \( \text{PU}(2,1) \) configuration space of quadruples of pairwise distinct points in \( S^3 \). We prove that there are two complex structures that appear naturally in \( X \); these give \( X \) a pseudoconformal structure which coincides with its well known CR structure.

1. Introduction and Statement of Results

In this article we are aiming to reveal the ties between the rich structures of Falbel’s cross–ratio variety, a set which is isomorphic to the \( \text{PU}(2,1) \) configuration space of four pairwise distinct points in the sphere \( S^3 \). All these structures are encoded within the property of pseudoconformality, a notion which we also introduce in this work and we shall explain below. Our motivation comes from the classical case: to any quadruple \( p = (p_1, p_2, p_3, p_4) \) of four pairwise distinct points in the Riemann sphere \( S^2 = \mathbb{C} \cup \{\infty\} \), there is associated their (complex) cross–ratio which is the complex number \( X(p) \) defined by

\[
X(p) = \left[ p_1, p_2, p_3, p_4 \right] = \frac{p_4 - p_2}{p_4 - p_1} \cdot \frac{p_3 - p_1}{p_3 - p_2},
\]

with the obvious modifications if one of the points is \( \infty \). There are 24 cross–ratios associated to each such quadruple \( p \), but due to symmetries it turns out that all possible cross–ratios depend complex analytically on \( X(p) \). Letting the group of Möbius transformations \( \text{PSL}(2, \mathbb{C}) \) of the Riemann sphere act diagonally on the set of quadruples of \( S^2 \), it is a classical result that the quotient set, that is, the \( \text{PSL}(2, \mathbb{C}) \) configuration space of four points in \( S^2 \), is isomorphic to \( \mathbb{C} \setminus \{0,1\} \) via \( [p] \mapsto X(p) \). In this manner, it follows that the configuration space admits the structure of a 1–complex dimensional complex manifold, which is inherited from \( \mathbb{C} \setminus \{0,1\} \).

Situation is far more complicated in the case of \( F \), the \( \text{PU}(2,1) \) configuration space of quadruples in \( S^3 \). \( F \) is the space \( \mathcal{C} \) of quadruples of pairwise distinct points in \( S^3 \), factored by the diagonal action of the projective unitary group \( \text{PU}(2,1) \). The sphere \( S^3 \) is identified via stereographic projection to \( H \cup \{\infty\} \), where \( H \) is the Heisenberg group. Recall that \( H \) is the 1–step nilpotent Lie group with underlying manifold \( \mathbb{C} \times \mathbb{R} \) and multiplication

\[
(z, t) * (w, s) = (z + w, t + s + 2\text{Im}(tz)) ,
\]

for each \( (z, t), (w, s) \in \mathbb{C} \times \mathbb{R} \). The set \( H \cup \{\infty\} \) is also the boundary of complex hyperbolic plane \( \mathbb{H}^2_{\mathbb{C}} \) and the projective unitary group \( \text{PU}(2,1) \) is the group of holomorphic isometries of \( \mathbb{H}^2_{\mathbb{C}} \); it acts doubly transitively on \( \partial \mathbb{H}^2_{\mathbb{C}} = S^3 \). Korányi and Reimann defined in [9] a complex cross–ratio \( X(p) \)
associated to a quadruple $p$ of four pairwise distinct points $p_i = (z_i, t_i)$, $i = 1, \ldots, 4$ in $S^3 = \mathbb{H} \cup \{\infty\}$ in the following manner: If $p \in S^3$, let $\mathcal{A}(z, t) = |z|^2 - it$ if $p = (z, t) \in \mathcal{H}$ and $\mathcal{A}(p) = \infty$ if $p = \infty$. Then the complex cross–ratio $X(p)$ is

$$X(p) = \frac{\mathcal{A}(p_4 \ast p_2^{-1})}{\mathcal{A}(p_3 \ast p_1^{-1})},$$

with the obvious modifications if one of the points is $\infty$. This cross–ratio is invariant under the diagonal action of $\text{PU}(2, 1)$ in $\partial \mathbb{H}^2$. Falbel showed in [5], that the 24 cross–ratios associated to a quadruple $p$ satisfy symmetries analogous to those in the classical case. However, in this case all possible cross–ratios corresponding to a given quadruple depend real analytically on the following three:

- $X_1 = X_1(p) = [p_1, p_2, p_3, p_4],
- X_2 = X_2(p) = [p_1, p_3, p_2, p_4]$ and
- $X_3 = X_3(p) = [p_2, p_3, p_1, p_4],

which satisfy the next two conditions:

$$|X_2| = |X_3||X_1|,$$

$$2|X_1|^2 \cdot \Re(X_3) = |X_1|^2 + |X_2|^2 - 2\Re(X_1 + X_2) + 1.$$

These two equations define a 4–dimensional real subvariety $X$ of $\mathbb{C}^3$, which we call (Falbel’s) cross–ratio variety. It has been shown in [6] that $\mathfrak{J}$ is isomorphic to $X$ via the isomorphism

$$\varpi : \mathfrak{J} \ni [p] \mapsto (X_1(p), X_2(p), X_3(p)) \in X,$$

see also Section 3.2 for further details. Besides Falbel’s own results, cross–ratio variety $X$ has been studied in [12], [13], see also [3] for a different approach. An extensive study of the geometric structures of $X$ is [6]; here are the main results:

First, there is 4–real dimensional manifold structure on a subset $X'$ of $X$ (see Theorem 3.7 below). The inverse image $\varpi^{-1}(X')$ comprises equivalent classes of a quadruples $(p_1, p_2, p_3, p_4)$ such that not all $p_i$ lie in a $\mathbb{C}$–circle (for the definition of a $\mathbb{C}$–circle, see Section 3.1.2).

Secondly, there is a $\mathbb{CR}$ structure of codimension 2 defined on a subset $X''$ of $X$ (see Theorem 3.8 below). The inverse image $\varpi^{-1}(X'')$ comprises equivalent classes of a quadruples $(p_1, p_2, p_3, p_4)$ such that $p_1, p_2, p_3$ do not all lie in a $\mathbb{C}$–circle. We note that this structure is actually a $\mathbb{CR}$ submanifold structure; $X''$ is considered as a real submanifold of $\mathbb{C}^3$.

Thirdly, there is a structure of a 2–complex dimensional disconnected complex manifold biholomorphic to $\mathbb{C}P^1 \times (\mathbb{C} \setminus \mathbb{R})$, defined on a subset $X^*$ of $X$ (see Theorem 3.10 below); we denote this structure by $J$. The inverse image $\varpi^{-1}(X^*)$ comprises equivalent classes of a quadruples $(p_1, p_2, p_3, p_4)$ such that $p_2, p_3$ do not lie in the same orbit of the stabiliser of $p_1, p_4$.

In this paper we show that apart from the complex structure $J$, there also exists another complex structure defined on $X^*$ which we shall denote by $I$. This structure is inherited from two copies of a Levi strictly pseudoconvex subset $\mathcal{P}$ of $\mathbb{C}^2$, see Section 4.1. A natural question arising here is the following: Does there exist any relation between $I$ and $J$, and if yes, what is the nature of this relation? To answer this question, we introduce the notions of pseudoconformal (psc) and strictly pseudoconformal (spsc) manifolds, which we study in detail in Section 2. It turns out that psc and spsc manifolds form intermediate objects between totally real, $\mathbb{CR}$ and hypercomplex 4–real dimensional manifolds. The real dimension of ambient manifolds we are only interested in here is 4 and we discuss below these notions in some extent. We shall see that they are very much related to $\mathbb{CR}$ manifolds; details about $\mathbb{CR}$ structures are in Section 2.1. We remark at this point that the term pseudoconformal which inspired our definition, is found for instance in p. 138 of [1]; there it is used as an alternative for $\mathbb{CR}$ mappings.
Let $M$ be a 4–dimensional real manifold and suppose that it is endowed with two complex structures $I$ and $J$. Suppose also that there exist 1–complex dimensional subbundles $H^{(1,0)}(M,I)$, $H^{(1,0)}(M,J)$ of the $(1,0)$–tangent bundles $T^{(1,0)}(M,I)$, $T^{(1,0)}(M,J)$ respectively, such that

$$(id_\ast)H^{(1,0)}(M,I) = H^{(1,0)}(M,J),$$

where $(id_\ast)$ is the differential of the identity mapping $id.: (M,I) \to (M,J)$. We then call the triple $(M,I,J)$ a psc manifold.

Let $H(M)$ be the underlying real subbundle of $H^{(1,0)}(M,I)$ and $H^{(1,0)}(M,J)$ and consider $M$ as a real manifold with the complex operators $I$ and $J$ acting as bundle automorphisms on $H(M)$. Then $(H(M),I)$ and $(H(M),J)$ are $\mathbb{C}R$ structures of codimension 2 in $M$, and the main observation is that by the very definition of the psc manifold we have:

$$(id_\ast)_*(H(M),I) = (H(M),J).$$

Thus the map $id.: M \to M$ is also a $\mathbb{C}R$ diffeomorphism; therefore the two $\mathbb{C}R$ structures of codimension 2 in $M$ are equivalent. It also follows that psc manifolds have the property that the complex structures $I$ and $J$ commute in the underlying real bundle $H(M)$. Clearly, a psc structure on $M$ induces a unique $\mathbb{C}R$ structure in $M$ by identifying the two equivalent $\mathbb{C}R$ structures. But the converse does not hold in general; for this to happen, the bundle automorphisms $I$ and $J$ have to be extended to integrable almost complex automorphisms of $M$.

Suppose now $(M,I,J)$ be a psc manifold as before. Suppose also that there exist splittings of $T^{(1,0)}(M,I), T^{(1,0)}(M,J)$ into direct sums $V^{(1,0)}(I)(M,I) \oplus V^{(1,0)}(M,I)$ and $H^{(1,0)}(M,I) \oplus V^{(1,0)}(M,I)$, respectively, such that

$$(id_\ast)_*V^{(1,0)}(M,I) = V^{(1,0)}(M,J) = V^{(0,1)}(M,J).$$

Then $(M,I,J)$ shall be called a spsc manifold. Let $V(M)$ be the underlying real subbundle of $V^{(1,0)}(M,I)$ and $V^{(1,0)}(M,J)$. The pairs $(V(M),I)$ and $V(M),J)$ may be also considered as $\mathbb{C}R$ structures of codimension 2 in $M$ (in a broad sense, where maximality is not required), but now we have:

$$(id_\ast)_*(V(M),I) = (V(M),-J).$$

Thus the identity map is anti–$\mathbb{C}R$; therefore the two $\mathbb{C}R$ structures of codimension 2 are anti–equivalent. All in all, strictly psc manifolds have the following property: Their real 4–dimensional tangent bundle admits a decomposition

$$T(M) = H(M) \oplus V(M),$$

where $H(M)$ and $V(M)$ are the underlying 2–dimensional real subbundles of the horizontal and the vertical bundles of $M$ respectively, and are such that:

1. the complex structures $I$ and $J$ commute in $H(M)$ and
2. the complex structures $I$ and $J$ skew–commute in $V(M)$.

We mention that they may exist singular sets in a psc (resp. spsc) structure $(M,I,J)$: These comprise points in $p \in M$ such that

1. $(id_\ast)_s H(M) = \{0\}$ in the psc case and
2. $(id_\ast)_s H(M) = \{0\}$ or $(id_\ast)_s V(M) = \{0\}$ in the spsc case.

Our first main result provides a concrete, non trivial example of a psc as well as of a spsc manifold. In fact we have (see Section 4.2):

**Theorem 1.1.** Away from certain singular sets, the triple $(X^*, I, J)$ is a psc and a spsc manifold.
Which is the relation of the psc and the spsc structures of $X^*$ with its known $\mathbb{CR}$ structure? At this point, enter the involution $T : X \to X$ given by

$$T(X_1, X_2, X_3) = (X_1, X_2, \bar{X_3}).$$

This involution was first observed by the authors of [12] and its geometric action was characterised there as very mysterious; $T$ is not induced by a permutation of the points of $\pi^{-1}(X_1, X_2, X_3)$. Here, we notice first that the fixed point set of $T$ is the set of points $X \setminus X^*$, i.e. the set where both complex structures $I$ and $J$ are not defined. In Section 3.5.2 we show that way from this set we may define a distribution $\mathcal{V}$, which is complementary to the distribution $H$ which defines the $\mathbb{CR}$ submanifold structure of $X$; that distribution is $\mathcal{V} = T_\ast H$, where $T_\ast$ is the derivative of $T$. If $\mathbb{J}$ denotes the natural complex structure of $\mathbb{C}^3$, then $\mathbb{J} \cap T(X^*) = \{0\}$ and therefore the $\mathbb{CR}$ structure is antiholomorphic (see Theorems 3.8 and 3.9).

Now in terms of the embedding of a spsc manifold $(M, I, J)$ into $\mathbb{C}^3$, $M$ inherits from its strict pseudoconformality the structure of a codimension 2 antiholomorphic $\mathbb{CR}$ submanifold of $\mathbb{C}^3$, see Section 2.2.4. Our second main result follows:

**Theorem 1.2.** Away from certain singular sets, the $\mathbb{CR}$ and antiholomorphic $\mathbb{CR}$ submanifold structures of $(X^*, I, J)$ induced by its pseudoconformality and its strict pseudoconformality, coincide with the $\mathbb{CR}$ and antiholomorphic $\mathbb{CR}$ structure respectively, as these are defined in Section 3.5.2.

This paper is organised as follows. In Section 2 we revise some well known facts about $\mathbb{CR}$ structures and we introduce pseudoconformal and strict pseudoconformal structures in 4-dimensional manifolds. Section 3 is a broad revision of the well known manifold, $\mathbb{CR}$ and complex structures of $X$. Due to the different conventions about $X$ considered in [6], for clarity we repeat the proofs of these results. The proofs of our main theorems lie in Section 4. Finally, we provide in an appendix (Section 5) further details about the cross-ratio variety, especially about its singular sets and we interpret geometrically the involution $T$.

## 2. Pseudoconformal Structures

The material in this section is divided in two parts. The content of the first part is well known; there is a vast bibliography about $\mathbb{CR}$ structures which we revise in Section 2.1, see for instance [1], [2], [4]. In Section 2.2 we study the notions of what we shall call pseudoconformal mappings and pseudoconformal manifolds. Both these notions are very much alike $\mathbb{CR}$ mappings and $\mathbb{CR}$ manifolds; the case we are interested in is that of codimension 2, but there are direct generalisations.

### 2.1. Preliminaries: $\mathbb{CR}$ structures.

There are two equivalent definitions of an abstract $\mathbb{CR}$ structure. Suppose first that $M$ is a $(2p + s)$-dimensional real manifold. A $\mathbb{CR}$ structure of codimension $s$ in $M$ is a pair $(\mathcal{D}, J)$ where $\mathcal{D}$ is a $2p$-dimensional smooth subbundle of $T(M)$ and $J$ is a bundle automorphism of $\mathcal{D}$ such that:

(i) $J^2 = -id.$ and

(ii) if $X$ and $Y$ are sections of $\mathcal{D}$ then the same holds for $[X, Y] - [JX, JY]$, $[JX, Y] + [X, JY]$ and moreover $J ([X, Y] - [JX, JY]) = [JX, Y] + [X, JY]$.

On the other hand, let $M$ be a $(2p + s)$-dimensional real manifold and let $T_\mathbb{C}(M)$ be its complexified tangent bundle. A $\mathbb{CR}$ structure of codimension $s$ in $M$ is a complex $p$-complex dimensional smooth subbundle $\mathcal{H}$ of $T_\mathbb{C}(M)$ such that:

(i) $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$ and

(ii) $\mathcal{H}$ is involutive, that is for any vector fields $Z$ and $W$ in $\mathcal{H}$ we have $[Z, W] \in \mathcal{H}$. 
The two definitions are equivalent; see for instance Theorem 1.1, Chpt. VI of [1]. A manifold endowed with a CR structure is called a CR manifold. A special class of CR manifolds are the CR submanifolds: Suppose that $N$ be a complex manifold of complex dimension $n$ with complex structure $J$, and let $M$ be a submanifold of $N$ of real dimension $m$. Then

$$\mathcal{H} = T(M) \cap J(T(M)),$$

is the maximal invariant subspace of $T(M)$ under the action of $J$, it is also a smooth subbundle on $M$ and $M$ is called a CR submanifold of $(N, J)$. A CR submanifold is in fact a CR manifold (see for instance Theorem 2.1, p.135 of [1]). The CR structure is $(\mathcal{H}, J)$, where here by $J$ we denote the bundle automorphism induced by the restriction of $J$ in $\mathcal{H}$. The corresponding complex subbundle is

$$\mathcal{H}^{(1,0)} = \{Z \in T^C(M) \mid Z = X - iJX, \ X \in \mathcal{H}\},$$

and we have

$$X \in \mathcal{H} \quad \text{if and only if} \quad Z = X - iJX \in \mathcal{H}^{(1,0)}.$$ 

Suppose now that $M$ is a CR submanifold of the $n$–complex dimensional complex manifold $N$ with $n = p + s$, such that $\dim_{\mathbb{R}}(M) = 2p + s$, where $2p = \dim_{\mathbb{R}} \mathcal{H}$; that is, $M$ is a codimension $s$ CR submanifold of $N$. Let $\mathcal{V}$ be a complementary to $\mathcal{H}$ subbundle of $M$:

$$T(M) = \mathcal{H} \oplus \mathcal{V},$$

Note that $\dim_{\mathbb{R}} \mathcal{V}_x = s$. If

$$J(\mathcal{V}) \cap T(M) = \{0\},$$

we call $M$ an antiholomorphic CR submanifold of $N$.

CR diffeomorphisms are in order and are defined as follows: Let $M$ and $M'$ be CR manifolds of the same dimension $m = 2p + s$ with CR structures $\mathcal{H}$ and $\mathcal{H}'$ respectively, of the same dimension $s$. A diffeomorphism $F : M \to M'$ is a CR diffeomorphism if it preserves CR structures; that is $F_* \mathcal{H} = \mathcal{H}'$. In other words, $F$ is a CR diffeomorphism if and only if for each $Z \in \mathcal{H}$ we have $F_*Z \in \mathcal{H}'$. In terms of the corresponding real distributions $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ we may say that $F$ is CR if for each $X \in \mathcal{D}$ we have $F_* (JX) = J'(F_*X)$.

In this paper we are concerned in particular with CR structures in subvarieties of $\mathbb{C}^n$. We consider the manifold $\mathbb{C}^n$, $n > 1$, with the natural complex coordinates $(\zeta_1, \ldots, \zeta_n)$, $\zeta_i = x_i + iy_i$, $i = 1, \ldots, n$. Denote also by $J$ the natural complex structure of $\mathbb{C}^n$. An $m$–real dimensional smooth subvariety of $\mathbb{C}^n$ is defined by a set of equations

$$F_i(\zeta_1, \ldots, \zeta_n) = 0, \quad i = 1, \ldots, k = 2n - m.$$

The set $M$ consisting of points of the subvariety at which the matrix

$$D = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial y_n}
\end{bmatrix}$$

is of constant rank $k$ is a real submanifold of $\mathbb{C}^n$ with $\dim(M) = m$. Its tangent space $T_x(M)$ at a point $x \in M$ is identified to the set

$$T_x(M) = \{X \in T_x(\mathbb{C}^n) \mid (dF_i)_x(X) = 0, \ i = 1, \ldots, k\}.$$ 

The maximal complex subspace $\mathcal{H}_x$ at each $x \in M$ comprises of $X \in T_x(\mathbb{C}^n)$ such that

$$(dF_i)_x(X) = 0 \quad \text{and} \quad (d^c F_i)_x(X) = 0, \quad i = 1, \ldots, k, \quad \text{where} \quad (d^c F_i)_x(X) = -(dF_i)_x(JX).$$
Let
\[ \mathcal{H}_x^{(1,0)} = \{ Z = X - i \mathbb{J} X \in T_x^{(1,0)}(\mathbb{C}^n) \mid X \in \mathcal{H}_x \}. \]

Then \( \mathcal{H}_x^{(1,0)} \) comprises of \( Z \in T_x^{(1,0)}(\mathbb{C}^n) \) such that
\[ (\partial F_i)_x(Z) = 0, \quad i = 1, \ldots k, \]
and one verifies that
\[ X \in \mathcal{H}_x \text{ if and only if } Z = X - i \mathbb{J} X \in \mathcal{H}_x^{(1,0)}. \]

Denote by \( \mathcal{H}^{(1,0)} \) the complex subbundle comprising of \( \mathcal{H}_x^{(1,0)} \), \( x \in M \). At points \( x \in M \) consider the matrix
\[
D^{(1,0)} = \begin{bmatrix}
\frac{\partial F_1}{\partial \zeta_1} & \ldots & \frac{\partial F_1}{\partial \zeta_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_k}{\partial \zeta_1} & \ldots & \frac{\partial F_k}{\partial \zeta_n}
\end{bmatrix},
\]
and let \( M' \subset M \) be the set at which \( D^{(1,0)} \) is of constant rank \( l \leq k \). Then \( \mathcal{H}^{(1,0)} \) is defined at \( M' \), \( \dim \mathcal{H}^{(1,0)} = n - l = p \) and therefore, if the integrability condition
\[ [Z_i, Z_j] \in \mathcal{H}^{(1,0)} \text{ for every } i, j = 1, \ldots p, \ i \neq j, \]
holds, then \( \mathcal{H}^{(1,0)} \) is a \( \mathbb{CR} \) structure of codimension \( s = 2l - k \) since \( m = 2n - k = 2p + s \). We call the set \( S = M \setminus M' \) the singular set of the \( \mathbb{CR} \) structure.

In the particular case when \( k = l = n - 1 \), that is \( \dim \mathcal{H}^{(1,0)} = 1 \), the single vector field generating the \( \mathbb{CR} \) structure is
\[ Z = D_{\zeta_2, \ldots, \zeta_n} \frac{\partial}{\partial \zeta_1} + D_{\zeta_3, \ldots, \zeta_n, \zeta_1} \frac{\partial}{\partial \zeta_2} + \cdots + D_{\zeta_1, \ldots, \zeta_{n-1}} \frac{\partial}{\partial \zeta_n}, \]
where
\[ D_{\zeta_1, \ldots, \zeta_{n-1}} = \begin{vmatrix}
\frac{\partial (F_1, \ldots, F_{n-1})}{\partial (\zeta_1, \ldots, \zeta_{n-1})}
\end{vmatrix} \]
are the \( (n - 1) \)-minor subdeterminants of \( D^{(1,0)} \). Note that in this case, the above integrability condition holds vacuously. Also in this case, the Levi form \( (L)_p : \mathcal{H}_p^{(1,0)} \rightarrow \mathbb{R}^n \) is defined in \( M' \) by
\[ Z_p \mapsto (L_1(p), \ldots, L_{n-1}(p)) = (dd^c F_1(Z, \overline{Z})_p, \ldots, dd^c F_{n-1}(Z, \overline{Z})_p), \]
where
\[
L_i(p) = \left[ D_{\zeta_2, \ldots, \zeta_n} \ldots D_{\zeta_1, \ldots, \zeta_{n-1}} \right]_p \cdot \begin{bmatrix}
\frac{\partial^2 F_1}{\partial \zeta_1 \partial \zeta_1} & \ldots & \frac{\partial^2 F_1}{\partial \zeta_1 \partial \zeta_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 F_k}{\partial \zeta_n \partial \zeta_1} & \ldots & \frac{\partial^2 F_k}{\partial \zeta_n \partial \zeta_n}
\end{bmatrix}_p \cdot \left[ D_{\zeta_1}, \ldots, \zeta_{n-1} \right]_p.
\]

2.2 Pseudoconformal Structures. Pseudoconformal structures that we are about to define are quite relative to \( \mathbb{CR} \) structures. The main difference is that whether in the \( \mathbb{CR} \) case there is no prescribed complex structure, in the pseudoconformal case complex structures are ubiquitous.
2.2.1. Pseudoconformal mappings and submanifolds. We start with a definition.

**Definition 2.1.** Let \((M, I)\) and \((N, J)\) be complex manifolds with complex structures \(I\) and \(J\) respectively, \(\dim_{\mathbb{C}}(M) = 2\) and \(\dim_{\mathbb{C}}(N) = n \geq 2\). A smooth immersion \(F : M \to N\) shall be called \textit{pseudoconformal (psc)} if there exists a 1–complex dimensional subbundle \(\mathcal{H}^{(1,0)}(M, I)\) of \(T^{(1,0)}(M, I)\) such that

\[
F^* \mathcal{H}^{(1,0)}(M, I) \subset T^{(1,0)}(N, J),
\]

and \(\mathcal{H}^{(1,0)}(M, I)\) is the the maximal subbundle with this property: If \((\mathcal{H}')^{(1,0)}(M, I)\) is any subbundle of \(T^{(1,0)}(M, I)\) of such that \(F^* (\mathcal{H}')^{(1,0)}(M, I) \subset T^{(1,0)}(N, J)\), then \(\mathcal{H}^{(1,0)}(M, I) = (\mathcal{H}')^{(1,0)}(M, I)\).

We call \(\mathcal{H}^{(1,0)}(M, I)\) (resp. \(\mathcal{H}^{(1,0)}(F(M), J)\)) the horizontal bundle of \(M\) (resp. of \(N\)). In case where \(n > 2\) and \(M\) is an immersed submanifold of \(N\), then the manifold \((M, I)\) is called a \textit{psc submanifold (of codimension 2)} of \((N, J)\).

When there is no risk of confusion, the underlying 2–real dimensional real subbundles \(\mathcal{H}(M)\) and \(\mathcal{H}(F(M))\) of \(\mathcal{H}^{(1,0)}(M, I)\) and \(\mathcal{H}^{(1,0)}(F(M), J)\) respectively, shall be also called horizontal bundles (of \(M\) and \(N\) respectively).

Some remarks on Definition 2.1. First, if the restriction \(\dim_{\mathbb{C}} \mathcal{H}^{(1,0)}(N, J) = 1\) is replaced by \(\dim_{\mathbb{C}} \mathcal{H}^{(1,0)}(N, J) = 2\), then by the Newlander–Nirenberg Theorem \(F\) is holomorphic and \(N\) is a complex submanifold of \(M\). By putting the maximality condition in Definition 2.1 we do not allow such a case. Second, in case where \(m = 2\) and \(F\) is a smooth diffeomorphism, it is clear that \(F\) is psc if and only if \(F^{-1}\) is psc; such diffeomorphisms are studied below. But perhaps the most important observation is given by the next proposition whose proof is following directly from Definition 2.1.

**Proposition 2.2.** Let \((M, I)\) and \((N, J)\) as above and let also \(F\) be a psc immersion with horizontal bundle \(\mathcal{H}^{(1,0)}(M, I)\). Denote by \(\mathcal{H}(M)\) the underlying 2–real dimensional real subbundle of \(\mathcal{H}^{(1,0)}(M, I)\). Then the following hold:

1. Consider \(M\) and \(N\) as real manifolds and their complex structures as bundle automorphisms such that \(I^2 = J^2 = -\text{id.}\), acting only on \(\mathcal{H}(M)\) and \(\mathcal{H}(F(M))\), respectively. Then \(M\) and \(N\) are \(\mathbb{CR}\) manifolds of codimension 2 and the \(\mathbb{CR}\) structures are \((\mathcal{H}(M), I)\) and \((F^* \mathcal{H}(M), J)\) respectively.

2. \(F : M \to N\) is a \(\mathbb{CR}\) map; the immersed submanifold \(F(M)\) is a \(\mathbb{CR}\) submanifold of codimension 2 in \(M\).

Hence psc mappings are \(\mathbb{CR}\) mappings; the converse is of course not generally true. The following proposition gives a local but useful description of psc mappings.

**Proposition 2.3.** The smooth immersion \(F : (M, I) \to (N, J)\) is psc if and only if at each point \(p \in M\), there exists a local parametrisation \((\zeta_1, \zeta_2) \mapsto (\xi_1, \ldots, \xi_n)\) of \(F\), \((\zeta_1, \zeta_2)\) are local \(I–\)holomorphic coordinates around \(p\) and \((\xi_1, \ldots, \xi_m)\) are local \(J–\)holomorphic coordinates around \(F(p)\), such that

\[
\text{rank}(DF^{(0,1)}) = 1, \quad \text{where} \quad DF^{(0,1)} = \begin{bmatrix}
\frac{\partial \xi_1}{\partial \zeta_1} & \frac{\partial \xi_1}{\partial \zeta_2} \\
\frac{\partial \xi_2}{\partial \zeta_1} & \frac{\partial \xi_2}{\partial \zeta_2} \\
\vdots & \vdots \\
\frac{\partial \xi_m}{\partial \zeta_1} & \frac{\partial \xi_m}{\partial \zeta_2}
\end{bmatrix}.
\]

**Proof.** We first prove that \(F\) is psc if and only if condition (2.1) holds for each local representation \((\zeta_1, \zeta_2) \mapsto (\xi_1, \ldots, \xi_m)\) of \(F\). For this, let \(Z \in T^{(1,0)}(M, I)\) with a local representation
\[ Z = \sum_{i=1}^{2} a_i (\partial / \partial \zeta_i). \] Then
\[
F_* Z = \sum_{i=1}^{2} a_i F_* \left( \frac{\partial}{\partial \zeta_i} \right)
= \sum_{i=1}^{2} a_i \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial \zeta_i} \left( \frac{\partial}{\partial \xi_j} \right) + \sum_{i=1}^{2} a_i \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial \zeta_i} \left( \frac{\partial}{\partial \xi_j} \right).
\]

Therefore, and due to linear independence, \( F_* Z \in T^{(1,0)}(N, J) \) if and only if the linear system
\[
\frac{\partial \xi_1}{\partial \zeta_1} a_1 + \frac{\partial \xi_2}{\partial \zeta_2} a_2 = 0,
\]
\[
\vdots
\]
\[
\frac{\partial \xi_m}{\partial \zeta_1} a_1 + \frac{\partial \xi_m}{\partial \zeta_2} a_2 = 0,
\]
adopts non zero solutions, which is equivalent to condition 2.1.

For the converse, fix \( p \in M \) and let \( (\zeta_1, \zeta_2) \mapsto (\xi_1, \ldots, \xi_m) \) be the local representation of \( F \) around \( p \) in which (2.1) holds. If \( (\xi'_1, \xi'_2) \mapsto (\xi'_1, \ldots, \xi'_m) \) is another local representation around \( p \) with corresponding matrix \( D\tilde{F}^{(0,1)} \), consider the holomorphic change of coordinates \( (\zeta_1, \zeta_2) \mapsto (\zeta'_1, \zeta'_2) \) and \( (\xi_1, \ldots, \xi_m) \mapsto (\xi'_1, \ldots, \xi'_m) \). Then, from the chain rule we have
\[
D\tilde{F}^{(0,1)} = \frac{\partial (\overline{\xi'_1}, \ldots, \overline{\xi'_m})}{\partial (\overline{\xi_1}, \ldots, \overline{\xi_m})} D\tilde{F}^{(0,1)} \frac{\partial (\xi_1, \xi_2)}{\partial (\zeta'_1, \zeta'_2)},
\]
and our assertion is proved. \( \square \)

We comment here that \( \text{rank}(DF^{(0,1)}) = 1 \) is equivalent to say that all minor \( 2 \times 2 \) subdeterminants \( \left| \frac{\partial (\xi_k, \xi_j)}{\partial (\zeta_k, \zeta_j)} \right| \) vanish for each \( j, k = 1, \ldots, m, j \neq k \) and the partial derivatives do not vanish simultaneously. The set of points of \( M \) where the opposite case occurs is defined below.

**Definition 2.4.** Let \( F : (M, I) \to (N, J) \) be a psc mapping. The set
\[
S = \left\{ p \in M \mid (F_*, p) H^{(1,0)}_p (M, I) = 0 \right\}
\]
is called the **singular set of** \( F \).

**2.2.2. Antiholomorphic pseudoconformal submanifolds.** The following proposition connects psc immersions of a certain nature with antiholomorphic \( \mathbb{CR} \) submanifold structures.

**Proposition 2.5.** Suppose that \( (M, I) \) is a \( 2 \)-complex dimensional manifold which is pseudoconformally immersed into the \( n \)-complex dimensional complex manifold \( (N, J) \), \( n > 2 \). Let \( i : M \hookrightarrow N \) be the inclusion and let \( \mathcal{H}(M) \) be the underlying real subbundle of the horizontal bundle \( \mathcal{H}^{(1,0)}(M, I) \). Suppose in addition that there exists an \( 2 \)-real dimensional subbundle \( \mathcal{V}(M) \) of the tangent bundle \( T(M) \) such that:
\[
T(M) = \mathcal{H}(M) \oplus \mathcal{V}(M),
\]
Then \( M \) inherits from \( i \) an antiholomorphic \( \mathbb{CR} \) submanifold structure of codimension 2.

**Proof.** Let \( \tilde{T}(M) \) be the tangent bundle of \( M \) in \( N \) and let also \( \tilde{\mathcal{H}}(M) = i_* \mathcal{H}(M) \) and \( \tilde{\mathcal{V}}(M) = i_* \mathcal{V}(M) \). We have
\[
\tilde{T}(M) = \tilde{\mathcal{H}}(M) \oplus \tilde{\mathcal{V}}(M),
\]
where the partial derivatives involved do not vanish simultaneously.

\[ J\tilde{T}(M) = J\tilde{H}(M) \oplus J\tilde{V}(M) \]
\[ = \iota_*H(M) \oplus J\tilde{V}(M) \]
\[ = \iota_*H(M) \oplus J\tilde{V}(M) \]
\[ = \tilde{H}(M) \oplus J\tilde{V}(M) \]

and thus
\[
\tilde{H}(M) = \tilde{T}(M) \cap J\tilde{T}(M)
\]
\[
= \left(\tilde{H}(M) \oplus \tilde{V}(M)\right) \oplus \left(\tilde{H}(M) \oplus J\tilde{V}(M)\right)
\]
\[
= \tilde{H}(M) \oplus \left(\tilde{V}(M) \cap J\tilde{V}(M)\right).
\]

Hence we must have \(J\tilde{V}(M) \cap J\tilde{V}(M) = \{0\}\) and we conclude that
\[ J\tilde{V}(M) \cap \tilde{T}(M) = J\tilde{V}(M) \cap \left(\tilde{H}(M) \oplus \tilde{V}(M)\right) = \{0\}. \]

Therefore \(M\) is an antiholomorphic \(\mathbb{C}\mathbb{R}\) submanifold of codimension 2 of \(N\).

Since \((M, I)\) is a complex manifold we may always assume that \(V(M)\) is \(I\)-invariant. In this case, we also have a splitting
\[ T^{(1,0)}(M, I) = H^{(1,0)}(M, I) \oplus \mathcal{V}^{(1,0)}(M, I), \]
where \(\mathcal{V}^{(1,0)}(M, I) = \{W - iW | W \in V(M)\}\). We call \(V(M)\) (and \(\mathcal{V}^{(1,0)}(M, I)\)) the vertical bundle of \(M\).

**Remark 2.6.** Let \(\tilde{\mathcal{V}}^C = \iota_*\mathcal{V}^{(1,0)}(M, I)\). Then
\[ \tilde{\mathcal{V}}^C \cap \tilde{T}^{(1,0)}(M, J) = \{0\} \quad \text{and} \quad \tilde{\mathcal{V}}^C \cap \tilde{T}^{(0,1)}(M, J) = \{0\}. \]

Indeed, left relation of (2.2) holds because of pseudoconformality. To show the right relation, suppose that there exists a \(W \in V(M)\) and an \(X \in \tilde{T}(M)\) such that
\[ \iota_*W - \iota_*(iW) = X + iJX. \]

Therefore \(X = \tilde{W} = \iota_*W \in \tilde{V}(M)\) and also \(\iota_*(iW) = -JX \in J\tilde{V}(M)\). Applying \(J\) to the last relation we have \(X = J(\iota_*(iW)) \in J\tilde{V}(M)\). From the proof of Proposition 2.5 we deduce \(X = 0\).

We conclude that \(\tilde{\mathcal{V}}^C\) consists of complex vector fields of mixed type with respect to \(J\), i.e., if \(\tilde{W} \in \tilde{\mathcal{V}}^C\) then \(\tilde{W} = \tilde{W}_1 + \tilde{W}_2\) where \(\tilde{W}_1\) is of type \((1, 0)\), \(\tilde{W}_2\) is of type \((0, 1)\) and neither of which is zero.

2.2.3. **Pseudoconformal diffeomorphisms and manifolds.** Suppose that \((N, J)\) and \((M, I)\) are as before but now \(\dim \mathbb{C}(N) = \dim \mathbb{C}(M) = 2\) and \(F : (M, I) \to (N, J)\) is a psc diffeomorphism. We call \((M, I)\) and \((N, J)\) pseudoconformally equivalent and the following corollary is evident.

**Corollary 2.7.** A smooth diffeomorphism \(F : (M, I) \to (N, J)\) is psc if and only if at each point \(p \in M\) there exists a local representation \((\zeta_1, \zeta_2) \mapsto (z_1, z_2)\) of \(F, (\zeta_1, \zeta_2)\) local \(I\)-holomorphic coordinates around \(p\) and \((z_1, z_2)\) are local \(J\)-holomorphic coordinates around \(F(p)\), such that
\[ \frac{\partial(z_1, z_2)}{\partial(\zeta_1, \zeta_2)} = 0 \quad \text{(which is equivalent to) \quad \frac{\partial(\zeta_1, \zeta_2)}{\partial(z_1, z_2)} = 0}, \]

where the partial derivatives involved do not vanish simultaneously.
We are mostly interested in the case when $M = N$ is a 4-dimensional real manifold endowed with complex structures $I$ and $J$ arising from atlantes $A_I$ and $A_J$ respectively. If these structures are psc equivalent, i.e., the identity mapping $id: (M, I) \to (M, J)$ is psc, then we call $(M, I, J)$ a psc manifold. From the previous discussion it follows that the horizontal bundles $\mathcal{H}^{(1,0)}(M, I)$ and $\mathcal{H}^{(1,0)}(N, J)$ are holomorphically identified via the derivative $id_*$. The resulting bundle which we shall denote simply by $\mathcal{H}^{(1,0)}(M)$, is a 1-complex dimensional subbundle of $T(M) \otimes \mathbb{C}$; it is also a $\mathbb{CR}$ structure of codimension 2 in $M$ in the usual sense when the action of the complex structures are considered only in $\mathcal{H}^{(1,0)}(M)$.

The next proposition follows directly from Corollary 2.7 and shows that in a psc manifold $(M, I, J)$ the unified atlas $A = A_I \cup A_J$ is such that the transition maps from charts of $A_I$ to charts of $A_J$ are psc diffeomorphisms of open sets in $\mathbb{C}^2$.

**Proposition 2.8.** Let $(M, I, J)$ be a 4-dimensional real manifold with complex structures $J$ and $I$ arising from the atlantes $A_J = \{(U_j, \phi_j)\}$ and $A_I = \{(V_i, \psi_i)\}$ respectively. Then $(M, I, J)$ is psc if and only if for each $p \in M$ there exist $(U_p, \phi_p) \in A_J$ and $(V_p, \psi_p) \in A_I$ so that the map $\psi \circ \phi^{-1} : \phi_p(U_p \cap V_p) \to \psi_p(U_p \cap V_p)$ is a psc diffeomorphism. Explicitly, if $\phi_p : q \mapsto (\zeta_1, \zeta_2)$ for each $q \in U_p$ and $\psi_p : r \mapsto (z_1, z_2)$ for each $r \in V_p$ and $\psi \circ \phi^{-1}$ is given by $(\zeta_1, \zeta_2) \mapsto (z_1, z_2)$, then Condition (2.3) holds.

We also have the following proposition which is quite expected: Equivalent psc structures immersed in $\mathbb{C}^n$ produce equivalent $\mathbb{CR}$ submanifold structures.

**Proposition 2.9.** Let $M$ and $N$ be 2-complex dimensional pseudoconformally equivalent complex manifolds with complex structures $J$ and $I$ respectively. Let also $\mathcal{H}^{(1,0)}(M, I)$ and $\mathcal{H}^{(1,0)}(N, J)$ be the horizontal bundles of $(M, I)$ and $(N, J)$ respectively. We suppose that there exists a psc immersion $F_M: (M, I) \to \mathbb{C}^n$ so that

$$ (F_M)_* \mathcal{H}^{(1,0)}(M, I) = \mathcal{H}^{(1,0)} $$

is a $\mathbb{CR}$ structure of codimension 2 on $M$.

Then the map $F_N: (N, J) \to \mathbb{C}^n$ defined by $F_N = F_M \circ F^{-1}$ is psc and $(F_N)_* \mathcal{H}^{(1,0)}(N, J) = \mathcal{H}^{(1,0)}$.

**Proof.** From psc equivalence of $(M, I)$ and $(N, J)$ we have that there exists a smooth psc diffeomorphism $F: (M, I) \to (N, J)$ so that

$$ F_* \mathcal{H}^{(1,0)}(M, I) = \mathcal{H}^{(1,0)}(N, J), $$

Since $F_M$ is an immersion of $M$ into $\mathbb{C}^3$ and $F$ is a diffeomorphism from $M$ onto $N$, we have that $F_N = F_M \circ F^{-1}$ is an immersion of $N$ into $\mathbb{C}^3$. Now $F_N$ is a psc embedding, since

$$ (F_N)_* \mathcal{H}^{(1,0)}(N, J) = ((F_M)_* \circ F_*^{-1}) F_* \mathcal{H}^{(1,0)}(M, I) = (F_M)_* \mathcal{H}^{(1,0)}(M, I) = \mathcal{H}^{(1,0)}. $$

This completes the proof. $\square$

**Corollary 2.10.** Let $(M, I, J)$ be a 2-complex dimensional psc manifold with horizontal bundle $\mathcal{H}^{(1,0)}(M)$. If there exists a psc embedding $\iota: (M, I) \to \mathbb{C}^n$ so that $\iota_* \mathcal{H}^{(1,0)}(M)$ constitutes a $\mathbb{CR}$ submanifold structure of codimension 2 of $M$, then there exists a psc embedding $j: (M, J) \to \mathbb{C}^n$ so that $j_* \mathcal{H}^{(1,0)} = \iota_* \mathcal{H}^{(1,0)}$. Therefore, the $\mathbb{CR}$ submanifold structure of $M$ may be identified to both $\mathbb{CR}$ structures arising from the psc structure of $(M, I, J)$. 

\[ \]
2.2.4. **Strictly pseudoconformal diffeomorphisms and manifolds.** Among the class of psc diffeomorphisms we distinguish one which is defined as follows.

**Definition 2.11.** Let \((M, I)\) and \((N, J)\) be 2–complex dimensional complex manifolds which are psc equivalent via the psc diffeomorphism \(F : (M, I) \rightarrow (N, J)\) and let \(H^{(1,0)}(M, I)\) be the horizontal bundle of \((M, I)\). We assume additionally the existence of a 1–complex dimensional complex subbundle \(V^{(1,0)}(M, I)\) of \(T^{(1,0)}(M, I)\) such that

i) \(H^{(1,0)}(M, I) \oplus V^{(1,0)}(M, I) = T^{(1,0)}(M, I)\) and

ii) the \(F_\ast\)–image of \(V^{(1,0)}(M, I)\) is an 1–complex dimensional complex subbundle of \(T^{(0,1)}(N, J)\).

Such an \(F\) shall be called **strictly pseudoconformal (spsc)**, the subbundle \(V^{(1,0)}(M, I)\) shall be called **vertical bundle** and the manifolds \((M, I)\) and \((N, J)\) shall be called **strictly pseudoconformally equivalent**.

Working analogously as in the previous paragraph, we can prove the counterparts of Proposition 2.3 and Corollary 2.7.

**Proposition 2.12.** Let \((M, I)\) and \((N, J)\) be 2–complex dimensional complex manifolds. The smooth diffeomorphism \(F : (M, I) \rightarrow (N, J)\) is strictly psc if and only if at each point \(p \in M\) there exists a local representation \((z_1, z_2) \mapsto (\zeta_1, \zeta_2)\) of \(F\), \((z_1, z_2)\) are local \(J\)–holomorphic coordinates around \(p\) and \((\zeta_1, \zeta_2)\) are local \(I\)–holomorphic coordinates around \(F(p)\), such that

\[
\text{(4.4)} \quad \text{rank}(DF^{(1,0)}) = \text{rank}(DF^{(0,1)}) = 1, \quad DF^{(1,0)} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial z_1} & \frac{\partial \zeta_2}{\partial z_1} \\ \frac{\partial \zeta_2}{\partial z_1} & \frac{\partial \zeta_2}{\partial z_1} \end{bmatrix}, \quad DF^{(0,1)} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial z_2} & \frac{\partial \zeta_2}{\partial z_2} \\ \frac{\partial \zeta_2}{\partial z_2} & \frac{\partial \zeta_2}{\partial z_2} \end{bmatrix}.
\]

Equivalently,

\[
\text{(4.5)} \quad \left| \frac{\partial (\zeta_1, \zeta_2)}{\partial (z_1, z_2)} \right| = \left| \frac{\partial (\bar{\zeta}_1, \bar{\zeta}_2)}{\partial (\bar{z}_1, \bar{z}_2)} \right| = 0,
\]

where not all partial derivatives at each one of the above determinants vanish simultaneously.

It is clear that Equation (4.5) is equivalent to

\[
\left| \frac{\partial (z_1, z_2)}{\partial (\zeta_1, \zeta_2)} \right| = \left| \frac{\partial (\bar{z}_1, \bar{z}_2)}{\partial (\bar{\zeta}_1, \bar{\zeta}_2)} \right| = 0.
\]

A **strictly psc manifold** \((M, I, J)\) is a psc manifold with the property that the identity mapping \(id. : (M, I) \rightarrow (N, J)\) is strictly psc. In this case, besides the holomorphic identification of horizontal bundles there is an antiholomorphic identification of vertical bundles \(V^{(1,0)}(M, I)\) and \(V^{(0,1)} = id. \ast V^{(1,0)}\) respectively. The resulting underlying real bundle shall be denoted by \(V(M)\).

**Proposition 2.13.** Let \((M, I, J)\) be a psc manifold with complex structures \(J\) and \(I\) arising from the atlantes \(A_J = \{(U_j, \phi_j)\}\) and \(A_I = \{(V_i, \psi_i)\}\) respectively. Then \((M, I, J)\) is strictly psc if and only if for each \(p \in M\) there exist \((U_p, \phi_p) \in A_J\) and \((V_p, \psi_p) \in A_I\) so that the map \(\psi \circ \phi^{-1} : \phi_p(U_p \cap V_p) \rightarrow \psi_p(U_p \cap V_p)\) is a strictly psc diffeomorphism. Explicitly, if \(\phi_p : q \mapsto (\zeta_1, \zeta_2)\) for each \(q \in U_p\) and \(\psi_p : r \mapsto (z_1, z_2)\) for each \(r \in V_p\) and \(\psi \circ \phi^{-1}\) is given by \((\zeta_1, \zeta_2) \mapsto (z_1, z_2)\) then Condition (2.5) holds.

The next proposition is the spsc counterpart of Proposition 2.9.
Proposition 2.14. Let $M$ and $N$ be 2–complex dimensional complex manifolds with complex structures $J$ and $I$ respectively, which are strictly pseudoconformally equivalent via the spsc map $F : (M, I) \to (N, J)$. We also suppose that there exists an antiholomorphic psc immersion $F_M : (M, I) \to \mathbb{C}^n$ so that

$$(F_M)_* \mathcal{H}^{(1,0)}(M, I) = \mathcal{H}^{(1,0)}(N, J), \quad (F_M)_* \mathcal{V}^{(1,0)}(M, I) = \mathcal{V}^C,$$

and the underlying real subbundles $\mathcal{H}$ and $\mathcal{V}$ of $\mathcal{H}^{(1,0)}$ and $\mathcal{V}^C$ respectively, form an antiholomorphic CR submanifold structure of codimension 2 of $M$.

Then the map $F_N : (N, J) \to \mathbb{C}^n$ defined by $F_N = F_M \circ F^{-1}$ is an antiholomorphic psc immersion and $(F_N)_* \mathcal{H}^{(1,0)}(N, J) = \mathcal{H}^{(1,0)}$, $(F_N)_* \mathcal{V}^{(0,1)}(N, J) = \mathcal{V}^C$.

Proof. From spsc equivalence, there exists a smooth strictly psc diffeomorphism $F : (M, I) \to (N, J)$ so that

$$F_* \mathcal{H}^{(1,0)}(M, I) = \mathcal{H}^{(1,0)}(N, J) \quad \text{and} \quad F_* \mathcal{V}^{(1,0)}(M, I) = \mathcal{V}^{(0,1)}(N, J),$$

where $\mathcal{H}^{(1,0)}(M, I)$, $\mathcal{H}^{(1,0)}(N, J)$ are the horizontal bundles and $\mathcal{V}^{(1,0)}(M, I)$, $\mathcal{V}^{(0,1)}(N, J)$ are the vertical bundles of $(M, I)$ and $(N, J)$, respectively. We only have to prove the last equation and we do so by proving the equivalent relation:

$$(F_N)_* \mathcal{V}^{(0,1)}(N, J) = \mathcal{V}^C.$$

We indeed have

$$(F_N)_* \mathcal{V}^{(0,1)}(N, J) = ((F_M)_* \circ F^{-1}) F_* \mathcal{V}^{(1,0)}(M, I) = (F_M)_* \mathcal{V}^{(1,0)}(M, I) = \mathcal{V}^C$$

and the proof is complete. \qed

Corollary 2.15. Let $(M, I, J)$ be a 2–complex dimensional strictly psc manifold. If there exists an antiholomorphic psc embedding $\iota : (M, I) \to \mathbb{C}^n$ giving $M$ the structure of an antiholomorphic CR submanifold of codimension 2, then there exists an antiholomorphic psc embedding $j : (M, J) \to \mathbb{C}^n$ which gives $M$ the same antiholomorphic CR submanifold structure.

3. Cross–Ratio Variety

This section contains a revision of all but one well known results about Falbel’s cross–ratio variety. This revision is quite extended, partly for clarity and partly due to the different conventions considered for $\mathcal{X}$ in [5, 6]. As preliminaries to cross–ratio variety we discuss complex hyperbolic plane and its boundary in Section 3.1. We define the cross–ratio variety $\mathcal{X}$ and we discuss its relation with $\mathfrak{F}$ the $\text{PU}(2,1)$–configuration space of four points in $S^3$ (Section 3.2). Singular sets and the involution $\mathcal{T}$ of $\mathcal{X}$ are in Sections 3.3 and 3.4, respectively. Manifold, CR and complex structures in $\mathcal{X}$ are in Section 3.5.

3.1. Preliminaries to Cross–Ratio Variety. The material in this section is well known; for details we refer the reader to the standard book of Goldman [7]. Complex hyperbolic plane is treated in Section 3.1.1 and its boundary in Section 3.1.2. Definitions of Cartan’s invariant and complex cross–ratio are in Section 3.1.3.
3.1.1. Complex Hyperbolic Plane. We consider $\mathbb{C}^{2,1}$, the vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by
\[
(z, w) = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1,
\]
and consider the following subspaces of $\mathbb{C}^{2,1}$:
\[
V_- = \left\{ z \in \mathbb{C}^{2,1} : \langle z, z \rangle < 0 \right\}, \quad V_0 = \left\{ z \in \mathbb{C}^{2,1} \setminus \{0\} : \langle z, z \rangle = 0 \right\}.
\]
Denote by $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{C}P^2$ the canonical projection onto complex projective space. Then complex hyperbolic plane $H^2_C$ is defined to be $\mathbb{P}V_-$ and its boundary $\partial H^2_C$ is $\mathbb{P}V_0$. Hence we have
\[
H^2_C = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0 \right\},
\]
and in this manner, $H^2_C$ is the Siegel domain in $\mathbb{C}^2$.

There are two distinguished points in $V_0$ which we denote by $o$ and $\infty$:
\[
o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Let $\mathbb{P}o = o$ and $\mathbb{P}\infty = \infty$. Then
\[
\partial H^2_C \setminus \{\infty\} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 = 1 \right\},
\]
and in particular, $o = (0, 0) \in \mathbb{C}^2$.

If conversely we are given a point $z = (z_1, z_2)$ of $\mathbb{C}^2$, then the point
\[
z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix},
\]
is called the standard lift of $z$. Therefore the standard lifts of points of the complex hyperbolic plane and its boundary (except infinity) are vectors of $V_-$ and $V_0$ respectively with the third inhomogeneous coordinate equal to 1.

Complex hyperbolic plane $H^2_C$ is a Kähler manifold, its Kähler structure is given by the Bergman metric. The holomorphic sectional curvature equals to $-1$ and its real sectional curvature is pinched between $-1$ and $-1/4$. The full group of holomorphic isometries is the projective unitary group
\[
\text{PU}(2,1) = \text{SU}(2,1)/\{I, \omega I, \omega^2 I\},
\]
where $\omega$ is a non real cube root of unity (that is $\text{SU}(2,1)$ is a 3-fold covering of $\text{PU}(2,1)$). Real hyperbolic plane can be embedded in two ways into complex hyperbolic plane, namely as $H^1_C$ as well as $H^2_R$. These embeddings give rise to complex lines, i.e., isometric images of the embedding of $H^1_C$ into $H^2_C$ and Lagrangian planes, i.e., isometric images of $H^2_R$ into $H^2_C$, respectively.

3.1.2. The boundary–Heisenberg group. There is an identification of the boundary of the Siegel domain with the one point compactification of $\mathbb{C} \times \mathbb{R}$: A finite point $z$ in the boundary of the Siegel domain has a standard lift of the form
\[
z = \begin{bmatrix} -|z|^2 + it \\ \sqrt{2z} \\ 1 \end{bmatrix}.
\]
The action of the stabiliser of infinity $\text{Stab}(\infty)$ gives to the set of these points the structure of a 1–step nilpotent Lie group; that is the Heisenberg group $\mathbf{H}$ which is $\mathbb{C} \times \mathbb{R}$ with group law:
\[(z, t) \ast (w, s) = (z + w, t + s + 2\Im(\overline{w}z)).\]
The Heisenberg norm (Korányi gauge) is given by
\[ |(z,t)|_H = |A(z,t)|^{1/2}, \quad \text{where} \quad A(z,t) = |z|^2 - it. \]
From this norm arises a metric, the Korányi–Cygan (K–C) metric, on \( \mathfrak{h} \) by the relation
\[ d_H ((z,t), (w,s)) = |(z,t)^{-1} * (w,s)|_H. \]
The K–C metric is invariant under
1. the left action of \( \mathfrak{h} \), \( (z,t) \mapsto (w,s) * (z,t) \);
2. the rotations \( (z,t) \mapsto (ze^{i\phi}, t), \phi \in \mathbb{R} \).
These form the group \( \text{Isom}(\mathfrak{h}, d_K) \) of Heisenberg isometries. The K–C metric is also scaled up to multiplicative constants by the action of Heisenberg dilations \( (z,t) \mapsto (rz, r^2 t), r \in \mathbb{R}_+ \) and there is also an inversion \( R \), defined for each \( p = (z,t) \in \mathfrak{h} \), \( p \neq o \) by \( (z,t) \mapsto \left( \frac{-\bar{z}}{|z|^2 + t^2}, \frac{-t}{|z|^2 + t^2} \right) \) which satisfies
\[ d_H(R(p), R(p')) = \frac{d_H(p, p')}{d_H(p, o)d_H(p', o)}. \]
All the above transformations are extended to infinity in the obvious way and the action of \( \text{PU}(2,1) \) in the boundary is given by compositions of these transformations.
\( \mathbb{R} \)–circles are boundaries of Lagrangian planes and \( \mathbb{C} \)–circles are boundaries of complex lines. They come in two flavours, infinite ones (i.e. containing the point at infinity) and finite ones. We refer to [7] for more details about these curves.

3.1.3. **Invariants: Cartan’s Invariant and Complex Cross–Ratio.** Given a triple \((p_1, p_2, p_3)\) of points at the boundary \( \partial \mathbb{H}^2 \), the Cartan’s invariant \( \mathbb{A}(p_1, p_2, p_3) \) is defined by
\[ \mathbb{A}(p_1, p_2, p_3) = \text{arg}(-\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle) \]
where \( p_i \) are lifts of \( p_i \), \( i = 1, 2, 3 \). The Cartan’s invariant is lying in \([-\pi/2, \pi/2]\), is independent of the choice of the lifts and remains invariant under the diagonal action of \( \text{PU}(2,1) \). Any other permutation of points produces Cartan’s invariants which differs from the above possibly up to sign. The following propositions are in [7] to which we also refer the reader for further details:

**Proposition 3.1.** Let \((p_1, p_2, p_3)\) be a triple of points lying in \( \partial \mathbb{H}^2 \) and let also \( \mathbb{A} = \mathbb{A}(p_1, p_2, p_3) \) be their Cartan’s invariant. Then
1. All points lie in an \( \mathbb{R} \)–circle if and only if \( \mathbb{A} = 0 \).
2. All points lie in a \( \mathbb{C} \)–circle if and only if \( \mathbb{A} = \pm \pi/2 \).

**Proposition 3.2.** Suppose that \( p_i \) and \( p'_i \), \( i = 1, 2, 3 \) are points in \( \partial \mathbb{H}^2 \) such that neither all \( p_i \) nor all \( p'_i \) lie in a \( \mathbb{C} \)–circle. Then there exists a holomorphic isometry \( g \) of \( \mathbb{H}^2 \) such that \( g(p_i) = p'_i \), \( i = 1, 2, 3 \) if and only if \( \mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3) \). If all \( p_i \) and all \( p'_i \) lie in \( \mathbb{C} \)–circles, there exists an isometry \( g \) of \( \mathbb{H}^2 \) such that \( g(p_i) = p'_i \), \( i = 1, 2, 3 \) if and only if either \( \mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3) \), or \( \mathbb{A}(p_1, p_2, p_3) = -\mathbb{A}(p'_1, p'_2, p'_3) \).

Given a quadruple of pairwise distinct points \( p = (p_1, p_2, p_3, p_4) \) in \( \partial \mathbb{H}^2 \), we define their complex cross–ratio as follows:
\[ \mathcal{X}(p_1, p_2, p_3, p_4) = \frac{\langle p_3, p_1 \rangle \langle p_4, p_2 \rangle}{\langle p_4, p_1 \rangle \langle p_3, p_2 \rangle}, \]
where \( p_i \) are lifts of \( p_i \), \( i = 1, 2, 3, 4 \). Note that this definition coincides with the one given in the introduction. The cross–ratio is independent of the choice of lifts and remains invariant under the
diagonal action of $\text{PU}(2,1)$. We stress here that for points in the Heisenberg group, the square root of its absolute value is

$$|X(p_1, p_2, p_3, p_4)|^{1/2} = \frac{d_K(p_4, p_2) \cdot d_K(p_3, p_1)}{d_K(p_4, p_1) \cdot d_K(p_3, p_2)}$$

3.2. Cross–ratio variety and the configuration space. As noted in the introduction, given a quadruple $p = (p_1, p_2, p_3, p_4)$ of distinct points in the boundary $\partial H^2$, all possible permutations of points gives us 24 complex cross–ratios corresponding to $p$. Due to symmetries, see [5], Falbel showed that all cross–ratios corresponding to a quadruple of points depend on three cross–ratios which satisfy two real equations. The following proposition holds; for its proof, see for instance in [12].

**Proposition 3.3.** Let $p = (p_1, p_2, p_3, p_4)$ be any quadruple of distinct points in $\partial H^2$. Let

$$X_1(p) = X(p_1, p_2, p_3, p_4), \quad X_2(p) = X(p_1, p_3, p_2, p_4), \quad X_3(p) = X(p_2, p_3, p_1, p_4).$$

Then

(3.1) \quad |X_3|^2 = |X_2|^2/|X_1|^2,

(3.2) \quad 2|X_1|^2 \Re(X_3) = |X_1|^2 + |X_2|^2 - 2\Re(X_1) - 2\Re(X_2) + 1.

**Definition 3.4.** Equations (3.1) and (3.1) define a 4–dimensional real subvariety of $\mathbb{C}^3$ which we call the cross–ratio variety $\mathcal{X}$.

We now discuss in brief but in a little more detail than we did in the introduction, the relation between cross–ratio variety $\mathcal{X}$ and $\mathcal{F}$, the space of $\text{PU}(2,1)$ configurations of four points in $S^3$. Recall that $\mathcal{F}$ consists of equivalence classes of quadruples $[p]$ where $p$ is a quadruple of distinct points in $\partial H^2$. Two quadruples $p = (p_1, p_2, p_3, p_4)$ and $p' = (p'_1, p'_2, p'_3, p'_4)$ belong to the same equivalence class, if there exists an element $g \in \text{PU}(2,1)$ such that $g(p_j) = p'_j$ for each $j = 1, 2, 3, 4$.

Consider the map $\varpi : \mathcal{F} \to \mathcal{X}$ given by

$$[p] \mapsto (X_1(p), X_2(p), X_3(p)).$$

This map is a surjection as the following proposition (Proposition 5.5, [12]) shows.

**Proposition 3.5.** Let $x_1, x_2$ and $x_3$ be three complex numbers satisfying

$$|x_3|^2 = |x_2|^2/|x_1|^2 \quad \text{and} \quad 2|x_1|^2 \Re(x_3) = |x_1|^2 + |x_2|^2 - 2\Re(x_1) + \Re(x_2) + 1.$$

Then there exist a quadruple of points $p = (p_1, p_2, p_3, p_4), p_i \in \partial H^2, i = 1, \ldots, 4$ so that

$$X_1(p) = x_1, \quad X_2(p) = x_2, \quad X_3(p) = x_3.$$

By Proposition 5.10 in [12], of which we state the corrected version here, the map $\varpi$ is also 1–1 in a large subspace of $\mathcal{F}$.

**Proposition 3.6.** Let $p = (p_1, p_2, p_3, p_4)$ and $p' = (p'_1, p'_2, p'_3, p'_4)$ two quadruples of distinct points in $\partial H^2$ such that $p_i$ and $p'_i$ do not all lie in the same $\mathbb{C}$–circle. There exists an element $g \in \text{PU}(2,1)$ such that $g(p_j) = p'_j$ for each $j = 1, 2, 3, 4$ if and only if $X_i(p) = X_i(p')$ for each $i = 1, 2, 3$. 
Therefore, to each point \([p]\) of \(\mathcal{F}\) such that not all \(p_i\) lie in a \(\mathbb{C}\)–circle, there is associated a unique point \(\varpi(p) = (X_1(p), X_2(p), X_3(p))\) of the cross–ratio variety \(\mathcal{X}\). In the degenerate case where all \(p_i \in \mathbb{R}\) lie in a \(\mathbb{C}\)–circle, surjection of \(\varpi\) still holds, but injection fails as this was shown in [3]. Following Lemma 5.5 of the corrected version of [5], the map \(\varpi\) is 2–1 from the space \(\mathcal{F}_R\) of configurations of points lying in a \(\mathbb{C}\)–circle to a subset \(\mathcal{X}_R\) of \(\mathcal{X}\) called the real singular set of \(\mathcal{X}\). Besides \(\mathcal{X}_R\) there are also other singular sets which we discuss below.

3.3. Singular Sets. The structures we are about to study are not defined in the whole of the cross–ratio variety \(\mathcal{X}\). There are singular sets; in this section we state the definitions of these sets and describe their properties in brief. For the proof of those properties as well as for a more detailed discussion on singular sets, see Section 5.2.

We mentioned above the real singular set \(\mathcal{X}_R\), which is in 2–1 correspondence with the subset \(\mathcal{F}_R\) of the configuration space consisting of classes of quadruples of points such that all lie in a \(\mathbb{C}\)–circle. It turns out that

\[
\mathcal{X}_R = \left\{ (X_1, X_2, X_3) \in \mathcal{X} \mid X_1, X_2, X_3 \in \mathbb{R}, X_1 + X_2 = 1, \frac{1}{X_2} + \frac{1}{X_3} = 1, X_3 + \frac{1}{X_1} = 1 \right\}.
\]

As a manifold, \(\mathcal{X}_R\) is a straight line with two points removed. Next, we have the \(\mathbb{C}\mathbb{R}\) singular set:

\[
\mathcal{X}_{\mathbb{C}\mathbb{R}} = \left\{ (X_1, X_2, X_3) \in \mathcal{X} \mid X_1 + X_2 = 1, \frac{1}{X_2} + \frac{1}{X_3} = 1, X_3 + \frac{1}{X_1} = 1 \right\}.
\]

\(\mathcal{X}_{\mathbb{C}\mathbb{R}}\) is the singular set of the codimension 2 \(\mathbb{C}\mathbb{R}\) submanifold structure of cross–ratio variety. This set is a 1–complex dimensional complex manifold biholomorphic to \(\mathbb{C} \setminus \{0, 1\}\) and it is in 1–1 correspondence with the subset \(\mathcal{F}_{\mathbb{C}\mathbb{R}}\) of \(\mathcal{F}\) consisting of equivalence classes of quadruples \(p = (p_1, p_2, p_3, p_4)\) such that \(p_1, p_2, p_3\) lie in a \(\mathbb{C}\)–circle. Finally, we consider the complex singular set:

\[
\mathcal{X}_{\mathbb{C}} = \left\{ (X_1, X_2, X_3) \in \mathcal{X} \mid \Re(X_3) = 0 \right\}.
\]

Complex structures in \(\mathcal{X}\) may be defined away from \(\mathcal{X}_{\mathbb{C}}\) which is 1–1 correspondence with the subset \(\mathcal{F}_{\mathbb{C}}\) of \(\mathcal{F}\) consisting of equivalence classes of quadruples \(p = (p_1, p_2, p_3, p_4)\) such that \(p_2, p_3\) lie in the same orbit of the stabiliser of \(p_1, p_4\). Complex singular set \(\mathcal{X}_{\mathbb{C}}\) has rich structures itself; besides a small subset of dimension one it can be endowed with the structure of a 3–dimensional submanifold of \(\mathbb{C}^2\). Additionally, it has a \(\mathbb{C}\mathbb{R}\) structure of codimension 1 which is simply the restriction of the \(\mathbb{C}\) structure of \(\mathcal{X}\) in \(\mathcal{X}_{\mathbb{C}\mathbb{R}}\).

Besides the above singular sets, we are going to consider another one which is obtained by a natural involution of \(\mathcal{X}\).

3.4. The Involution \(\mathcal{T}\). We introduce the involution \(\mathcal{T}\) of \(\mathcal{X}\); this is given by

\[
\mathcal{T}(X_1, X_2, X_3) = (X_1, X_2, \overline{X_3}), \quad (X_1, X_2, X_3) \in \mathcal{X}.
\]

A geometric interpretation of this involution is given in the Appendix. Here, we will also consider some rather obvious properties of \(\mathcal{T}\). First, it is clear that \(\mathcal{T}(\mathcal{X}^*) = \mathcal{X}^*\) and moreover it leaves pointwise invariant the singular set \(\mathcal{X}_{\mathbb{C}}\). We will see below that for the complex structures we define for \(\mathcal{X}\), involution \(\mathcal{T}\) plays the role of the natural conjugation \(z \to \overline{z}\) in the trivial case of \(\mathbb{C}\). (see Sections 3.6 and 4.1). Moreover, the antiholomorphic nature of the \(\mathbb{C}\mathbb{R}\) structure we define in 3.5.2 is arising from \(\mathcal{T}\).

For the moment, we focus of the \(\mathcal{T}\)–image of the singular set \(\mathcal{X}_{\mathbb{C}\mathbb{R}}\). One shows that \(\mathcal{T}(\mathcal{X}_{\mathbb{C}\mathbb{R}})\) is the set

\[
\mathcal{X}^\perp_{\mathbb{C}\mathbb{R}} = \left\{ (X_1, X_2, X_3) \in \mathcal{X} \mid X_1 + X_2 = 1, \frac{1}{X_2} + \frac{1}{X_3} = 1, X_3 + \frac{1}{X_1} = 1 \right\}.
\]
We show in Section 5.2 that $X_{\text{CR}}^\perp$ is isomorphic to the subset of $\mathcal{F}$ consisting of classes of quadruples $p = (p_1, p_2, p_3, p_4)$ such that $p_2, p_3, p_4$ lie in a $C$-circle. As a manifold, it is isomorphic to $X_{\text{CR}}$ and it is also quite obvious that

$$X_{\text{CR}} \cap X_{\text{CR}}^\perp = X_{\text{R}}.$$  

3.5. Manifold, $CR$ and Complex Structures. We consider the following subsets of $X$.

$$(3.8) \quad X' = X \setminus X_{R},$$

$$(3.9) \quad X'' = X \setminus X_{\text{CR}},$$

$$(3.10) \quad X^* = X' \setminus X_{C}.$$  

It has been proven in [6] that:

(1) $X'$ is a 4–dimensional real submanifold of $C^3$,

(2) $X''$ is a codimension 2 $CR$ submanifold of $C^3$ and

(3) $X^*$ is a 2–complex dimensional complex manifold.

Below we are going to reprove these results, see Theorems 3.7, 3.8 and 3.10, respectively. There is also a new result here: the $CR$ structure is antiholomorphic, see Theorem 3.9.

3.5.1. Manifold structure.

Theorem 3.7. The subset $X' = X \setminus X_{R}$, where $X'$ and $X_{R}$ are as in (3.8) and (3.3) respectively, can be endowed with a structure of a 4–dimensional smooth real regular submanifold of $C^3$.

Proof. The proof is calculative; one considers the equations defining $X$, those are

$$F_1(\zeta_1, \zeta_2, \zeta_3) = |\zeta_2|^2 - |\zeta_1|^2 |\zeta_3|^2 = 0,$$

$$F_2(\zeta_1, \zeta_2, \zeta_3) = |\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1 + \zeta_2) + 1 - 2|\zeta_1|^2 \Re(\zeta_3) = 0,$$

where $\zeta_i = x_i + iy_i$, and calculates the rank of the Jacobian matrix

$$D = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \frac{\partial F_1}{\partial y_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \frac{\partial F_2}{\partial y_3} \end{bmatrix}$$

at points of $X$. We have

$$\frac{\partial F_1}{\partial x_1} = -2x_1 |\zeta_3|^2, \quad \frac{\partial F_1}{\partial x_2} = 2x_2, \quad \frac{\partial F_1}{\partial x_3} = -2x_3 |\zeta_1|^2,$$

$$\frac{\partial F_1}{\partial y_1} = -2y_1 |\zeta_3|^2, \quad \frac{\partial F_1}{\partial y_2} = 2y_2, \quad \frac{\partial F_1}{\partial y_3} = -2y_3 |\zeta_1|^2,$$

$$\frac{\partial F_2}{\partial x_1} = -4x_1 \Re(\zeta_3) + 2x_2 - 2, \quad \frac{\partial F_2}{\partial x_2} = 2x_2 - 2, \quad \frac{\partial F_2}{\partial x_3} = 2 - |\zeta_1|^2,$$

$$\frac{\partial F_2}{\partial y_1} = -4y_1 \Re(\zeta_3) + 2y_2, \quad \frac{\partial F_2}{\partial y_2} = 2y_2, \quad \frac{\partial F_2}{\partial y_3} = 0,$$

and from the $2 \times 2$ minor subdeterminants it eventually turns out that the rank is 2 everywhere except at points of $X_{R}$. The result now follows from the regular level set theorem.  

We stress here that $X'$ is maximal in the following sense: the diagonal action of $PU(2,1)$ is not free outside the subset $\mathcal{F}' = \mathcal{F} \setminus \mathcal{F}_{R}$ of the configuration space $\mathcal{F}$, therefore a natural (with respect to the group action) manifold structure can be given only in open subsets of $\mathcal{F}'$. Maximality now is in the sense that in fact $\mathcal{F}'$ is in bijection with $X'$ and thus inherits a manifold structure itself which is exactly the one defined in Theorem 3.7. For details about the group action, see [6].
3.5.2. $CR$ structure.

**Theorem 3.8.** There is a $CR$ structure of a codimension 2 defined on $\mathcal{X}$. Its singular set is $\mathcal{X}_{CR}$.

*Proof.* Consider the defining equations 3.11 and 3.11 of $\mathcal{X}$. Following the discussion in Section 2.1, we examine whether the matrix

$$
D^{(1,0)} = \begin{bmatrix}
\frac{\partial F_1}{\partial \zeta_1} & \frac{\partial F_1}{\partial \zeta_2} & \frac{\partial F_1}{\partial \zeta_3} \\
\frac{\partial F_2}{\partial \zeta_1} & \frac{\partial F_2}{\partial \zeta_2} & \frac{\partial F_2}{\partial \zeta_3}
\end{bmatrix}
$$

is of rank 2. We have

$$
\frac{\partial F_1}{\partial \zeta_1} = -\zeta_1|\zeta_3|^2, \quad \frac{\partial F_1}{\partial \zeta_2} = \zeta_2, \quad \frac{\partial F_1}{\partial \zeta_3} = -\zeta_3|\zeta_1|^2,
$$

$$
\frac{\partial F_2}{\partial \zeta_1} = -2\zeta_1 \Re(\zeta_3) + \zeta_1 - 1, \quad \frac{\partial F_2}{\partial \zeta_2} = \zeta_2 - 1, \quad \frac{\partial F_2}{\partial \zeta_3} = -|\zeta_1|^2.
$$

Calculating the $2 \times 2$ minor subdeterminants $D_{\zeta_i, \zeta_j} = \frac{\partial (F_1, F_2)}{\partial (\zeta_i, \zeta_j)}$ at points of $\mathcal{X}$ we obtain

\begin{align*}
D_{\zeta_2, \zeta_3} &= \frac{|\zeta_2|^2}{|\zeta_3|^2} (\zeta_2 \zeta_3 - \zeta_2 - \zeta_3), \\
D_{\zeta_3, \zeta_1} &= \frac{|\zeta_3|^2}{|\zeta_1|^2} (1 + \overline{\zeta_1} \overline{\zeta_3} - \overline{\zeta_1}), \\
D_{\zeta_1, \zeta_2} &= \frac{\zeta_2}{\zeta_1} (1 - \overline{\zeta_1} - \overline{\zeta_2}).
\end{align*}

The $(1,0)$-vector field of $\mathbb{C}^3$

$$
Z = D_{\zeta_2, \zeta_3} \frac{\partial}{\partial \zeta_1} + D_{\zeta_3, \zeta_1} \frac{\partial}{\partial \zeta_2} + D_{\zeta_1, \zeta_2} \frac{\partial}{\partial \zeta_3},
$$

declared at points of $\mathcal{X''}$, is the generator of $\mathcal{H}^{(1,0)}$, that is, $\mathcal{H}^{(1,0)} = \langle Z \rangle$. The singular set $S$ of $\mathcal{H}^{(1,0)}$ comprises of points of $\mathcal{X'}$ at which $Z$ is identically zero; it is clear that this happens only at points of $\mathcal{X}_{CR}$ and the proof is complete. 

**Theorem 3.9.** The $CR$ structure of Theorem 3.8 is antiholomorphic.

*Proof.* Consider the involution $\mathcal{T}$ as in (3.6) and let $W = \mathcal{T}(Z)$. If $\omega = (X_1, X_2, X_3) \in \mathcal{X}$ we have

$$
W_\omega = \mathcal{T}_* (Z)_\omega = (\mathcal{T}_* T^{-1}(\omega)) Z_{T^{-1}(\omega)}
$$

and thus

$$
W = D_{\zeta_2, \zeta_3} \frac{\partial}{\partial \zeta_1} + D_{\zeta_3, \zeta_1} \frac{\partial}{\partial \zeta_2} + D_{\zeta_1, \zeta_2} \frac{\partial}{\partial \zeta_3}.
$$

The vector field $W$ is by definition in $T(\mathcal{X'})$ and is nowhere zero at points of $\mathcal{X} \setminus \mathcal{X}_{CR}$. We write

$$
Z = \frac{1}{2} (X - i[JX]), \quad W = \frac{1}{2} (\mathcal{T}_*(X) - i\mathcal{T}_*(JX)) = \frac{1}{2} (U - iV),
$$

where $J$ is the natural complex structure of $\mathbb{C}^3$. Let also $\mathcal{H} = \{X, Y = JX\}$ and $V = \{U, V\}$; clearly $\mathcal{H} \cap V = \{0\}$. From this, we also have that $\{X, Y, U, V\}$ is a basis for $\mathcal{H} \oplus V$.

Finally, we show that $J\mathcal{V} \cap T(\mathcal{X'}) = \{0\}$. Indeed, for $i = 1, 2$ relations $dF_i(JU) = 0$ would imply $U - iJU \in \mathcal{H}^{(1,0)}$ and therefore $U \in \mathcal{H}$, a contradiction. In the same manner we prove that $J\mathcal{V}$ is not in the tangent space and the proof is complete. 

□
Proof. Pick a point \((p_1, p_2, p_3, p_4)\) such that either \(p_1, p_2, p_3\) or \(p_2, p_3, p_4\) lie in the same \(\mathbb{C}\)-circle.

To complete this section, we calculate the Levi forms of the above defined \(\mathbb{CR}\) structure. We have:

\[
\frac{\partial^2 F_1}{\partial \zeta_1 \partial \zeta_1} = -|\zeta_3|^2, \quad \frac{\partial^2 F_1}{\partial \zeta_1 \partial \zeta_2} = 0, \quad \frac{\partial^2 F_1}{\partial \zeta_1 \partial \zeta_3} = -\zeta_1 \zeta_3,
\]
\[
\frac{\partial^2 F_1}{\partial \zeta_2 \partial \zeta_2} = 1, \quad \frac{\partial^2 F_1}{\partial \zeta_2 \partial \zeta_3} = 0,
\]
\[
\frac{\partial^2 F_1}{\partial \zeta_3 \partial \zeta_3} = -|\zeta_1|^2
\]

and also

\[
\frac{\partial^2 F_2}{\partial \zeta_1 \partial \zeta_1} = 1 - 2\Re(\zeta_3), \quad \frac{\partial^2 F_2}{\partial \zeta_1 \partial \zeta_2} = 0, \quad \frac{\partial^2 F_2}{\partial \zeta_1 \partial \zeta_3} = -\zeta_1,
\]
\[
\frac{\partial^2 F_2}{\partial \zeta_2 \partial \zeta_2} = 1, \quad \frac{\partial^2 F_2}{\partial \zeta_2 \partial \zeta_3} = 0,
\]
\[
\frac{\partial^2 F_2}{\partial \zeta_3 \partial \zeta_3} = 0.
\]

Therefore,

\[
L_1 = |D_{\zeta_3, \zeta_1}|^2 - |\zeta_1 D_{\zeta_1, \zeta_2} + \zeta_3 D_{\zeta_2, \zeta_3}|^2,
\]
\[
L_2 = (1 - 2\Re(\zeta_3))|D_{\zeta_2, \zeta_3}|^2 + |D_{\zeta_3, \zeta_1}|^2 - 2\Re(\zeta_1 D_{\zeta_1, \zeta_3} D_{\zeta_2, \zeta_3}).
\]

Now, analysing Equation 3.11 gives the following symmetric condition:

\[
(3.14) \quad \frac{1}{\zeta_1} D_{\zeta_2, \zeta_3} - \frac{1}{\zeta_2} D_{\zeta_3, \zeta_1} + \frac{1}{\zeta_3} D_{\zeta_1, \zeta_2} = 0.
\]

From this, and together with Eqs. 3.11 and 3.11 we deduce

\[
L_1 \equiv 0, \quad L_2 = |\zeta_1|^2|\zeta_2 - \zeta_3 \zeta_1|^2.
\]

At points of \(X''\) we have \(L_2 > 0\). Indeed, the only case where \(L_2 = 0\) is when \(\zeta_2 = \zeta_3 \zeta_1\). But this happens only at points of \(X_{\mathbb{CR}}\), see Proposition 4.4 of [6].

3.6. Complex Structure \(J\). The first one of the complex structures of \(X^*\) we encounter in this work is revealed in the following theorem which has been proved in [6]. For clarity, we repeat here the proof.

**Theorem 3.10.** The set \(X^*\) can be endowed with the structure of a 2–complex dimensional complex manifold. With this structure \(X^*\) is biholomorphic to \(\mathbb{CP}^1 \times (\mathbb{C} \setminus \mathbb{R})\).

**Proof.** Pick a point \((X_1, X_2, X_3)\) \(\in X^*\) and consider the unique class \([p]\) of quadruples \(p = (p_1, p_2, p_3, p_4)\) which is such that \(X_i(p) = X_i, \ i = 1, 2, 3\). We normalise so that

\[
p_1 = (z_1, t_1), \quad p_2 = \infty, \quad p_3 = (0, 0), \quad p_4 = (z_4, t_4), \quad |z_1| + |z_4| \neq 0.
\]
We have
\[
X_1 = -\frac{|z_1|^2 - it_1}{-|z_1|^2 - |z_4|^2 + i(t_4 - t_1) + 2z_1z_4},
\]
\[
X_2 = -\frac{|z_4|^2 + it_4}{-|z_1|^2 - |z_4|^2 + i(t_4 - t_1) + 2z_1z_4},
\]
\[
X_3 = -\frac{|z_4|^2 + it_4}{-|z_1|^2 + it_1},
\]
and the map
\[
\mathcal{N} : \mathcal{X}^* \ni (X_1, X_2, X_3) \mapsto \left(\left|z_1\right|^2, \frac{|z_4|^2 - it_4}{|z_1|^2 - it_1}\right) \in \mathbb{C}P^1 \times (\mathbb{C} - \mathbb{R})
\]
is a homeomorphism; with brackets we denote homogeneous coordinates in \(\mathbb{C}P^1\). To define an atlas, we first observe that
\[
z = \frac{z_1}{z_4} = \frac{X_1 + X_2/X_3}{X_1 + X_2 - 1}, \quad w = \frac{|z_4|^2 - it_4}{|z_1|^2 - it_1} = X_3,
\]
(the right equation is obvious; the left is obtained by straightforward calculations). Hence by considering \(N_0 : \mathcal{X}^* \to \mathbb{C}P^1 \times (\mathbb{C} \setminus \mathbb{R})\) and \(N_{\infty} : \mathcal{X}^* \to \mathbb{C}P^1 \times (\mathbb{C} \setminus \mathbb{R})\) given respectively by
\[
N_0(X_1, X_2, X_3) = (z, w), \quad \text{and} \quad N_{\infty}(X_1, X_2, X_3) = (1/z, w),
\]
we obtain an atlas \(\mathcal{A}_J\) for \(\mathcal{X}^*\), consisting of the charts \((\mathcal{X}^*, N_0)\) and \((\mathcal{X}^*, N_{\infty})\). Moreover, the complex manifold structure of \(\mathcal{X}^*\) which we shall denote by \(J\), arises from this atlas.

\[ \square \]

We next prove a formula for the inverse mapping \(N_0^{-1}\):
\[
N_0^{-1}(z, w) = \left(\frac{w|z|^2 - 1}{w - 1 + (w - |w|^2)|z|^2 - (w - \bar{w})z}, \frac{w - |w|^2|z|^2}{w - 1 + (w - |w|^2)|z|^2 - (w - \bar{w})z}, w\right).
\]
To prove this, we first observe that by taking real and imaginary parts in both sides of the equation
\[
|z_4|^2 - it_4 = w(|z_1|^2 - it_1)
\]
we get
\[
t_1 = \frac{|z_4|^2 - \Re(w)|z_1|^2}{\Im(w)}, \quad t_4 = \frac{\Re(w)|z_4|^2 - |w|^2|z_1|^2}{\Im(w)}.
\]

Thus
\[
X_1 = \frac{|z_1|^2/|z_4|^2 + it_1/|z_4|^2}{|z_1|^2/|z_4|^2 + 1 + i(t_1 - t_4)/|z_4|^2 - 2z_1/z_4}
\]
\[
= \frac{|z|^2 + i(1 - \Re(w))|z|^2/\Im(w)}{|z|^2 + 1 + i(1 - \Re(w))/\Im(w) + i(|w|^2 - \Re(w))|z|^2/\Im(w) - 2z}
\]
\[
= \frac{w|z|^2 - 1}{w - 1 + (w - |w|^2)|z|^2 - (w - \bar{w})z},
\]
and analogously for \(X_2\).

**Corollary 3.11.** The involution \(\mathcal{T}\) of Equation (3.6) is an antiholomorphic mapping of the complex manifold \((\mathcal{X}^*, J)\).
Proof. One verifies that the coordinate expression of $T$ is 
\[(z, w) \mapsto \left( \frac{1}{\overline{w}z}, w \right),\]
which is clearly antiholomorphic. \qed

4. Pseudoconformality of Cross–Ratio Variety

In this section we prove that away from certain singular sets, cross–ratio variety can be given a psc as well as a spsc structure. In Section 4.1 we define the second complex operator for $X^*$ and in Sections 4.2 and 4.3 we prove our main Theorems 1.1 and 1.2 respectively.

4.1. Complex Structure I.

Let 
\[X^+_\pm = \{(X_1, X_2, X_3) \in X^* \mid \Im(X_3) > 0\}, \quad X^-\pm = \{(X_1, X_2, X_3) \in X^* \mid \Im(X_3) < 0\}.\]

We consider the subset $P$ of $\mathbb{C}^2$ defined as follows:
\[P = \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid (|\zeta_1| - |\zeta_2|)^2 < 2\Re(\zeta_1) + 2\Re(\zeta_2) - 1 \}. \]

The set $P$ is a Levi strictly pseudoconvex domain; the proof of this lies in Proposition 5.9.

We have the following Lemma; our result will follow as an immediate corollary.

**Lemma 4.1.** Let $M : X \to \mathbb{C}^2$ be the projection
\[X \ni (X_1, X_2, X_3) \mapsto (X_1, X_2) \in \mathbb{C}^2,\]
and denote by $M_\pm$ the restrictions of $M$ to $X^\pm_\pm$ respectively. Then, $M_\pm$ are bijections of $X^\pm_\pm$ onto $P$. If $(\zeta_1, \zeta_2) \in P$, then
\[(4.1) \quad M^{-1}_\pm(\zeta_1, \zeta_2) = \left( \zeta_1, \zeta_2, \frac{\zeta_2}{|\zeta_1|} e^{\pm i\theta} \right) \in X^\pm_\pm,\]
where
\[(4.2) \quad \theta = \arccos \left( \frac{|\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1) - 2\Re(\zeta_2) + 1}{2|\zeta_1||\zeta_2|} \right).\]

Proof. Let $(X_1, X_2, X_3)$ be any point in $X^*$. Since $\Im(X_3) \neq 0$, from the obvious inequality 
\[-|X_3| < \Re(X_3) < |X_3|\]
and Equation (3.2) we get 
\[(|X_1| - |X_2|)^2 < 2\Re(X_1) + 2\Re(X_2) - 1 < (|X_1| + |X_2|)^2,\]
where the right inequality holds vacuously. Thus $M_\pm$ are from $X^\pm_\pm$ to $P$. Moreover $\theta$ is well defined, that is, 
\[-1 < \frac{|\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1) - 2\Re(\zeta_2) + 1}{2|\zeta_1||\zeta_2|} < 1.\]
Writing
\[X_1 = \zeta_1, \quad X_2 = \zeta_2, \quad X_3 = \frac{\zeta_2}{|\zeta_1|} e^{\pm i\theta},\]
we have \( \Im(X_3) \neq 0, |X_2| = |X_1||X_3| \) and
\[
2|X_1|^2\Re(X_3) = 2|X_1||X_2|\cos \theta \\
= |\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1) - 2\Re(\zeta_2) + 1 \\
= |X_1|^2 + |X_2|^2 - 2\Re(X_1) - 2\Re(X_2) + 1.
\]

The proof is complete. \(\square\)

**Theorem 4.2.** The set \( \mathcal{X}^* \) can be endowed with the structure of a disconnected 2–complex dimensional complex manifold. The respective complex analytic atlas \( \mathcal{A}_I \) consists of the two non overlapping charts
\[
(\mathcal{X}^*_+, \mathcal{M}_+) \quad \text{and} \quad (\mathcal{X}^*_-, \overline{\mathcal{M}_-} = \iota \circ \mathcal{M}_-),
\]
where \( \mathcal{M}_\pm \) are the restrictions of \( \mathcal{M} \) to \( \mathcal{X}^*_\pm \) respectively and \( \iota \) is the complex conjugation in \( \mathcal{P} \). In this manner, the complex structure in \( \mathcal{X}^*_+ \) is that of \( \mathcal{P} \) whereas the complex structure in \( \mathcal{X}^*_- \) is that of \( \overline{\mathcal{P}} \).

The atlas \( \mathcal{A}_I \) of Theorem 4.2 helps us to visualise the subset of cross–ratio variety \( \mathcal{X}^* \) as a disconnected set comprising of two 4–dimensional connected components, i.e., the sets \( \mathcal{X}^+ \) and \( \mathcal{X}^- \). The first set is identified biholomorphically to \( \mathcal{P} \) and the second to \( \overline{\mathcal{P}} \). From now on, the complex structure of \( \mathcal{X}^* \) induced from the complex analytic atlas above, will be denoted by \( I \).

**Corollary 4.3.** The involution \( \mathcal{T} \) given by Equation (3.6) is an antiholomorphic mapping of the complex manifold \( (\mathcal{X}^*, I) \).

**Proof.** The coordinate expression for \( \mathcal{T} \) is \( (\zeta_1, \zeta_2) \mapsto (\overline{\zeta_1}, \overline{\zeta_2}) \). \(\square\)

**4.2. Proof of Theorem 1.1.** We are now able to prove that the manifold \( (\mathcal{X}^*, I, J) \) is pseudoconformal with singular set \( \mathcal{X}_{\mathrm{CR}} \cap \mathcal{X}^* \) and strictly pseudoconformal with singular set \( \mathcal{X}_{\mathrm{CR}} \cup \mathcal{X}_{\mathrm{CR}}^\perp \cap \mathcal{X}^* \).

**Proof.** We prove that the identity map \( \text{id} : (\mathcal{X}^*, I) \rightarrow (\mathcal{X}^*, J) \) is strictly pseudoconformal. Working in the coordinate charts \( (\mathcal{X}^+, \mathcal{M}^+) \) and \( (\mathcal{X}^*, \mathcal{N}_0) \), we have the following representation of the identity map:
\[
(\zeta_1, \zeta_2) \mapsto (z, w) = \left( \frac{\zeta_1 + \zeta_2/\zeta_3}{\zeta_1 + \zeta_2 - 1}, \frac{\zeta_1 + \zeta_2 - 1}{\zeta_1 + \zeta_2 - 1/\zeta_3} \right),
\]
where \( \zeta_3 = \frac{|\zeta_2|}{|\zeta_1|}e^{i\theta} \) with \( \theta \) as defined in Equation (4.2).

Calculating straightforwardly we have:
\[
\begin{align*}
\frac{\partial z}{\partial \zeta_1} &= -\frac{\zeta_2/\zeta_3^3}{\zeta_1 + \zeta_2 - 1} \cdot \frac{\partial \zeta_3}{\partial \zeta_1} + \frac{\zeta_2 - \zeta_3}{(\zeta_1 + \zeta_2 - 1)^2} \cdot \frac{\partial z}{\partial \zeta_2} \\
\frac{\partial w}{\partial \zeta_1} &= \frac{\partial \zeta_3}{\partial \zeta_1}, \quad \frac{\partial w}{\partial \zeta_2} = \frac{\partial \zeta_3}{\partial \zeta_2}, \\
\frac{\partial \overline{z}}{\partial \zeta_1} &= -\frac{\zeta_2/\zeta_3^3}{\zeta_1 + \zeta_2 - 1} \cdot \frac{\partial \zeta_3}{\partial \zeta_1} + \frac{\zeta_2 - \zeta_3}{(\zeta_1 + \zeta_2 - 1)^2} \cdot \frac{\partial \overline{z}}{\partial \zeta_2} \\
\frac{\partial \overline{w}}{\partial \zeta_1} &= \frac{\partial \zeta_3}{\partial \zeta_1}, \quad \frac{\partial \overline{w}}{\partial \zeta_2} = -\frac{\zeta_2/\zeta_3^3}{\zeta_1 + \zeta_2 - 1} \cdot \frac{\partial \overline{w}}{\partial \zeta_2}.
\end{align*}
\]
Thus it remains to calculate the partial derivatives $\partial \zeta_3 / \partial \zeta_i$ and $\overline{\partial \zeta_3} / \partial \zeta_i$, $i = 1, 2$. To do so, we take partial derivatives with respect to $\zeta_1$ and $\zeta_2$ in the equations

$$|\zeta_3|^2 = \frac{|\zeta_2|^2}{|\zeta_1|^2},$$

$$2|\zeta_1|^2 \Re(\zeta_3) = |\zeta_1|^2 + |\zeta_2|^2 - 2\Re(\zeta_1 + \zeta_2) + 1.$$

We find the implicit expressions:

\begin{align*}
\frac{\partial \zeta_3}{\partial \zeta_1} &= \frac{\zeta_3 (\zeta_1 - \overline{\zeta_1} \zeta_3 - 1)}{2i \Im(\zeta_3) \cdot |\zeta_1|^2} = \frac{-D_{\zeta_3, \zeta_1}}{2i \Im(\zeta_3) \cdot |\zeta_1|^2}, \\
\frac{\partial \zeta_3}{\partial \zeta_2} &= \frac{-\zeta_3 + \overline{\zeta_2} - \overline{\zeta_3} \zeta_2}{2i \Im(\zeta_3) \cdot |\zeta_1|^2} = \frac{D_{\zeta_3, \zeta_2}}{2i \Im(\zeta_3) \cdot |\zeta_1|^2},
\end{align*}

\begin{align*}
\frac{\partial \overline{\zeta_3}}{\partial \zeta_1} &= -\frac{\overline{\zeta_3} (\zeta_1 - \overline{\zeta_1} \zeta_3 - 1)}{2i \Im(\zeta_3) \cdot |\zeta_1|^2} = \frac{-D_{\overline{\zeta_3}, \zeta_1}}{2i \Im(\zeta_3) \cdot |\zeta_1|^2}, \\
\frac{\partial \overline{\zeta_3}}{\partial \zeta_2} &= \frac{-\overline{\zeta_3} + \overline{\zeta_2} - \overline{\zeta_3} \zeta_2}{2i \Im(\zeta_3) \cdot |\zeta_1|^2} = \frac{D_{\overline{\zeta_3}, \zeta_2}}{2i \Im(\zeta_3) \cdot |\zeta_1|^2}.
\end{align*}

Here $D_{\zeta_3, \zeta_1}$, $D_{\zeta_3, \zeta_2}$ are as in Equations (3.11) and (3.12), respectively, and $D_{\overline{\zeta_3}, \zeta_1}$, $D_{\overline{\zeta_3}, \zeta_2}$ are as the respective previous ones, but with $\zeta_3$ replaced by $\overline{\zeta_3}$. Let us now consider the matrices:

$$\text{Did.}^{(0,1)} = \begin{bmatrix} \frac{\partial \zeta_3}{\partial \zeta_1} & \frac{\partial \zeta_3}{\partial \zeta_2} \\ \frac{\partial \overline{\zeta_3}}{\partial \zeta_1} & \frac{\partial \overline{\zeta_3}}{\partial \zeta_2} \end{bmatrix} \quad \text{and} \quad \text{Did.}^{(1,0)} = \begin{bmatrix} \frac{\partial \zeta_3}{\partial \zeta_1} & \frac{\partial \zeta_2}{\partial \zeta_2} \\ \frac{\partial \overline{\zeta_3}}{\partial \zeta_1} & \frac{\partial \overline{\zeta_2}}{\partial \zeta_2} \end{bmatrix}.$$ 

We have

$$\det \left( \text{Did.}^{(0,1)} \right) = \frac{\partial \overline{\zeta_3}}{\partial \zeta_1} \cdot \frac{\partial \overline{\zeta_3}}{\partial \zeta_2} \begin{vmatrix} \zeta_3 & \frac{\zeta_3 \overline{\zeta_3}}{\zeta_1 + \zeta_2 - 1} \\ \frac{\zeta_3 \overline{\zeta_3}}{\zeta_1 + \zeta_2 - 1} & 1 \end{vmatrix} = 0.$$

Also, by setting $c = (2i \Im(\zeta_3) \cdot |\zeta_1|^2 (\zeta_1 + \zeta_2 - 1)^2)^{-1}$ we have:

\begin{align*}
\det \left( \text{Did.}^{(1,0)} \right) &= \frac{\zeta_2 - \zeta_2 / \zeta_3 - 1}{(\zeta_1 + \zeta_2 - 1)^2} \cdot \frac{\partial \zeta_3}{\partial \zeta_2} - \frac{\zeta_1 / \zeta_3 - \zeta_1 - 1 / \zeta_3}{(\zeta_1 + \zeta_2 - 1)^2} \cdot \frac{\partial \zeta_3}{\partial \zeta_1} \\
&= c \cdot (\zeta_2 - \zeta_2 / \zeta_3 - 1)(\zeta_3 \overline{\zeta_3} - \overline{\zeta_2} - \zeta_3) - (\zeta_1 - \zeta_1 \zeta_3 - 1)(\zeta_1 - \zeta_1 \overline{\zeta_3} - 1),
\end{align*}

Thus

$$c^{-1} \det \left( \text{Did.}^{(1,0)} \right) = \zeta_3 |\zeta_2|^2 - |\zeta_2|^2 - \zeta_2 \zeta_3 + |\zeta_2|^2 \overline{\zeta_3} + \overline{\zeta_2} - \zeta_3 \overline{\zeta_2} + \zeta_3 - |\zeta_1|^2 \overline{\zeta_3} - |\zeta_1|^2 \zeta_3 + \zeta_1 + |\zeta_1|^2 \zeta_3 - \zeta_1 \zeta_3 - \zeta_1 \overline{\zeta_3} - 1.$$

The sum of the second, the tenth and the last term of the right hand side is by Equation (3.2) equal to

$$-2\Re(\zeta_1 + \zeta_2) - 2|\zeta_1|^2 \Re(\zeta_3).$$

Plugging this to the preceding equation and writing the fifth term as $|\zeta_1|^2 \overline{\zeta_3}$, where we have used Equation (3.1), gives

$$c^{-1} \det \left( \text{Did.}^{(1,0)} \right) = \zeta_3 |\zeta_2|^2 - \zeta_2 \zeta_3 - |\zeta_2|^2 + |\zeta_1|^2 \overline{\zeta_3} - \zeta_3 \overline{\zeta_2} + \zeta_3 + |\zeta_1|^2 \zeta_3 - |\zeta_1|^2 \overline{\zeta_3} - |\zeta_1|^2 \zeta_3 + \zeta_1 - \zeta_1 \zeta_3 - \zeta_1 \overline{\zeta_3}.$$
Now the fourth and the last term cancel out; we next factor \( \zeta_3 \) out of all but the eighth terms to get by using Equation (3.2) again:

\[
c^{-1} \det \left( \text{Did.}^{(1,0)} \right) = 2\zeta_3 |\zeta_1|^2 \Re(\zeta_3) - |\zeta_2|^2 - |\zeta_1|^2 \zeta_3^2 \\
= \zeta_3^2 |\zeta_1|^2 + |\zeta_3|^2 |\zeta_1|^2 - |\zeta_2|^2 - |\zeta_1|^2 \zeta_3^2 \\
= 0
\]
as it follows from Equation (3.1). Now it is clear that all partial derivatives \( \frac{\partial \zeta_i}{\partial z_j} \) vanish simultaneously at points of \( X^* \cap X_{\CR} \) and all partial derivatives \( \frac{\partial \zeta_i}{\partial z_j}, i, j = 1, 2 \), vanish simultaneously at points of \( X^* \cap X_{\CR}^\perp \). Therefore, away from these points we have

\[
\text{rank} \left( \text{Did.}^{(0,1)} \right) = 1 = \text{rank} \left( \text{Did.}^{(1,0)} \right),
\]

and our assertion is proved. \( \square \)

**Remark 4.4.** From Equations (4.3) we also have:

1. The horizontal bundle \( \mathcal{H}^{(1,0)}(X^*, I) \) is generated by the vector field

\[
Z^I = D_{\zeta_2, \zeta_3} \frac{\partial}{\partial \zeta_1} + D_{\zeta_3, \zeta_1} \frac{\partial}{\partial \zeta_2},
\]

away from points of \( X^* \cap \overline{X}_{\CR} \).

2. The vertical bundle \( \mathcal{V}^{(1,0)}(X^*, I) \) is generated by the vector field

\[
W^I = D_{\zeta_2, \zeta_3} \frac{\partial}{\partial \zeta_1} + D_{\zeta_3, \zeta_1} \frac{\partial}{\partial \zeta_2},
\]

away from points of \( X^* \cap \overline{X}_{\CR} \).

4.3. **Proof of Theorem 1.2.** In this section we shall prove Theorem 1.2. Explicitly we will show that:

1. \( (X^*, I) \) is a pseudoconformal submanifold of \( \mathbb{C}^3 \) with singular set \( X_{\CR} \cap X^* \).
2. This psc structure is antiholomorphic with singular set \( (X_{\CR} \cup X_{\CR}^\perp) \cap X^* \).
3. The induced \( \CR \) and antiholomorphic \( \CR \) submanifold structure coincide with the \( \CR \) and antiholomorphic \( \CR \) structure respectively, defined in Section 3.5.2 for the set \( X'' \).

**Proof.** We consider the inclusion map \( \iota : (X^*, I) \hookrightarrow \mathbb{C}^3 \); this is given in the chart \( (X^*_+ , M^*_+ ) \) by

\[
\iota(\zeta_1, \zeta_2) = \left( \zeta_1, \zeta_2, \frac{|\zeta_2|}{|\zeta_1|} e^{i\theta} \right) = (\xi_1, \xi_2, \xi_3).
\]

We will show first that \( \iota_* \mathcal{H}^{(1,0)}(M, I) \) is the \( \CR \) structure defined in Section 3.5.2. One checks that

\[
D_{\iota}^{(0,1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial \xi_2}{\partial \xi_1} & \frac{\partial \xi_2}{\partial \xi_2} \end{bmatrix}
\]
is clearly of rank 1, except at points where the partial derivatives of \( \overline{\zeta_3} \) vanish. From Equations (4.3) we have that this is happening at points of \( X^* \cap X_{\CR} \) and thus \( \iota \) is psc away from points of this set.
Now from Remark 4.4 recall that $\mathcal{H}^{(1,0)}(\mathfrak{X}^*, I)$ is spanned in the chart $(\mathfrak{X}^+, M^+)$ by the vector field $Z^I$, where $Z^I$ is as in Equation (4.5). Using Equations (3.11), (3.12) and (3.13) we may verify the formulae:

\begin{align}
\frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} &= D_{\xi_1, \xi_2}, \\
\frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} &= 0.
\end{align}

We next calculate:

$$
\iota_* Z^I = D_{\xi_2, \xi_3} \frac{\partial}{\partial \xi_1} + D_{\xi_3, \xi_1} \frac{\partial}{\partial \xi_2} + \left( \frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} \right) \frac{\partial}{\partial \xi_3}
$$

using (4.7) and (4.8) = $D_{\xi_2, \xi_3} \frac{\partial}{\partial \xi_1} + D_{\xi_3, \xi_1} \frac{\partial}{\partial \xi_2} + D_{\xi_1, \xi_2} \frac{\partial}{\partial \xi_3},$

which proves our assertion. Considering the vertical bundle $V^{(1,0)}(M, I)$, recall again from Remark 4.4 that in the chart $(\mathfrak{X}^+, M^+)$ it is spanned by the vector field $W^I$, where $W^I$ is as in Equation (4.6). The following equations hold:

\begin{align}
\frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} &= 0, \\
\frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} &= D_{\xi_1, \xi_2}.
\end{align}

We therefore have:

\begin{align}
\iota_* W^I &= D_{\xi_2, \xi_3} \frac{\partial}{\partial \xi_1} + D_{\xi_3, \xi_1} \frac{\partial}{\partial \xi_2} + \left( \frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} \right) \frac{\partial}{\partial \xi_3} \\
&\quad + \left( \frac{\partial \xi_3}{\partial \xi_1} D_{\xi_2, \xi_3} + \frac{\partial \xi_3}{\partial \xi_2} D_{\xi_3, \xi_1} \right) \frac{\partial}{\partial \xi_3}
\end{align}

using (4.9) and (4.10) = $D_{\xi_2, \xi_3} \frac{\partial}{\partial \xi_1} + D_{\xi_3, \xi_1} \frac{\partial}{\partial \xi_2} + D_{\xi_1, \xi_2} \frac{\partial}{\partial \xi_3},$

and the proof is complete. □

5. Appendix: Further Comments on Cross–Ratio Variety

This section contains further supplementary information about the cross–ratio variety $\mathfrak{X}$. Using Cartan’s invariants associated to certain triples of a given quadruple $p$ of pairwise distinct points, we are able to give an alternative description of $\mathfrak{X}$ in Section 5.1. Section 5.2 is concerned with the analytic description of singular sets; the complex singular set $\mathfrak{X}_C$ which is the largest of all singular sets and the one with the richer structures, is studied in Section 5.2.2. Finally, in Section 5.3 we give a geometric interpretation of the involution $\mathcal{T}$.

5.1. Cross–ratios and Cartan’s invariants. An alternative description of $\mathfrak{X}$. Our aim in this section is to give an alternative description of $\mathfrak{X}$. We do so, by using Cartan’s invariants of certain triples of points of a given quadruple.

Associated to a quadruple of points $p = (p_1, p_2, p_3, p_4)$ are the following Cartan’s invariants:

$A_1(p) = A_1(p_2, p_3, p_4), \ A_2(p) = A_1(p_1, p_3, p_4), \ A_3(p) = A_1(p_1, p_2, p_4), \ A_4(p) = A_1(p_1, p_2, p_3).$
Proposition 5.1. Let $p = (p_1, p_2, p_3, p_4)$ a quadruple of distinct points in $\partial \mathbb{H}^2_\mathbb{C}$, $X_i = X_i(p)$, $i = 1, 2, 3$ and $\lambda_i = \lambda_i(p)$. Then
\[
\arg(X_1) = \lambda_1 - \lambda_2, \quad \arg(X_2) = -\lambda_2 - \lambda_4, \quad \arg(X_3) = \lambda_4 - \lambda_1, \quad \lambda_3 = \lambda_2 - \lambda_1 + \lambda_4.
\]

Proof. We only prove the first of these identities; the proof of the rest is similar and it is left to the reader. We have:
\[
\arg X_1 = \arg \left( \frac{p_4, p_2}{p_4, p_1} \frac{p_3, p_1}{p_3, p_2} \right) = \arg \left( \frac{p_4, p_2}{p_4, p_1} \frac{p_3, p_1}{p_3, p_2} \right)^2 = \arg \left( \frac{p_2, p_3}{p_2, p_1} \frac{p_3, p_1}{p_1, p_3} \frac{p_4, p_1}{p_4, p_3} \right) = \arg \left( -\frac{p_2, p_3}{p_2, p_1} \frac{p_3, p_1}{p_1, p_3} \frac{p_4, p_1}{p_4, p_3} \right) - \arg \left( -\frac{p_1, p_3}{p_1, p_2} \frac{p_3, p_2}{p_2, p_3} \frac{p_4, p_2}{p_4, p_3} \right) = \lambda_1 - \lambda_2.
\]

Plugging into Equation (3.2), we immediately have a counterpart of Proposition 3.3, as an alternative definition of the cross–ratio variety $X$.

Proposition 5.2. Let $p = (p_1, p_2, p_3, p_4)$ be any quadruple of distinct points in $\partial \mathbb{H}^2_\mathbb{C}$. Let $X_1(p) = X(p_1, p_2, p_3, p_4)$, $X_2(p) = X(p_1, p_3, p_2, p_4)$, and $\lambda_1(p) = \lambda(p_2, p_3, p_4)$, $\lambda_2(p) = \lambda(p_1, p_3, p_4)$, $\lambda_4(p) = \lambda(p_1, p_2, p_3)$. Then
\[
|X_1|^2 + |X_2|^2 = 2|X_1||X_2| \cos(\lambda_1 - \lambda_4) + 2|X_1| \cos(\lambda_2 - \lambda_1) + 2|X_2| \cos(\lambda_2 + \lambda_4) - 1.
\]

In this manner we obtain a description of $X$ as a 4–dimensional real subset of $\mathbb{R}^2_+ \times [-\pi/2, \pi/2]^3$, compare to the one in [3]. We underline that the choice of $\lambda_1, \lambda_2$ and $\lambda_4$ is arbitrary; Proposition 5.2 can be modified analogously for any other choice of three Cartan’s invariants among $\lambda_i(p)$, $i = 1, 2, 3, 4$.

Closing this section, we prove a lemma which relates cross–ratios and Cartan’s invariants and will be useful in our subsequent discussion.

Lemma 5.3. The following formulae fold.
\[
|X_1 + X_2 - 1|^2 = 4|X_1||X_2| \cos(\lambda_1) \cos(\lambda_4),
\]
\[
|X_1 + \overline{X_2} - 1|^2 = 4|X_1||X_2| \cos(\lambda_2) \cos(\lambda_3),
\]
\[
|X_3 + \frac{1}{X_1} - 1|^2 = 4|X_3| \cos(\lambda_1) \cos(\lambda_4),
\]
\[
|X_3 + \frac{1}{X_1} - 1|^2 = 4|X_3| \cos(\lambda_2) \cos(\lambda_3),
\]

\[
\left| \frac{1}{X_2} + \frac{1}{X_3} - 1 \right|^2 = \frac{4}{|X_2||X_3|} \cos(A_1) \cos(A_4),
\]
\[
\left| \frac{1}{X_2} + \frac{1}{X_3} - 1 \right|^2 = \frac{4}{|X_2||X_3|} \cos(A_1) \cos(A_2).
\]

**Proof.** We only prove the first identity; the proof of the rest is similar: one has to rearrange Equations (3.1) and (3.2) to take the other two equivalent pairs of defining equations of $\mathcal{X}$.

We have
\[
|X_1 + X_2 - 1|^2 = |X_1|^2 + |X_2|^2 - 2R(X_1 + X_2) + 1 + 2R(X_1X_2)
\]
\[
= 2|X_1|^2R(X_3) + 2R(X_1X_2)
\]
\[
= 2|X_1||X_2|\cos(A_4 - A_1) + 2|X_1||X_2|\cos(A_1 + A_4)
\]
\[
= 4|X_1||X_2|\cos(A_1) \cos(A_4),
\]
where for the second equality we have used Equation (3.2) and for the third equality we have used Proposition 5.1. \qedh

5.2. Properties and Structures of Singular Sets.

5.2.1. Real and $\mathbb{CR}$ singular sets. Lemma 5.3 induces two corollaries from which we obtain the manifold description of the real singular set $\mathcal{X}_R$ and the $\mathbb{CR}$ singular sets $\mathcal{X}_\mathbb{CR}$ and $\mathcal{X}_\mathbb{CR}^\perp$.

**Corollary 5.4.** Let $p = (p_1, p_2, p_3, p_4)$ be a quadruple of points in $\partial \mathbb{H}_C^2$ and let $X_i(p)$, $i = 1, 2, 3$ be its cross–ratios. The following hold:

i) $X_1(p) + X_2(p) = 1$ if and only if either $p_1, p_2, p_3$ or $p_2, p_3, p_4$ lie in the same $\mathbb{C}$–circle; in the first case $X_3(p) + \frac{1}{X_1(p)} = 1$ and $\frac{1}{X_2(p)} + \frac{1}{X_3(p)} = 1$, and in the second case $\frac{1}{X_3(p)} + \frac{1}{X_1(p)} = 1$ and $\frac{1}{X_2(p)} + \frac{1}{X_3(p)} = 1$.

ii) $X_3(p) + \frac{1}{X_1(p)} = 1$ if and only if either $p_1, p_3, p_4$ or $p_1, p_2, p_3$ lie in the same $\mathbb{C}$–circle; in the first case $X_1(p) + X_2(p) = 1$ and $\frac{1}{X_2(p)} + \frac{1}{X_3(p)} = 1$, and in the second case $\frac{1}{X_2(p)} + \frac{1}{X_3(p)} = 1$ and $\frac{1}{X_1(p)} + \frac{1}{X_3(p)} = 1$.

iii) $\frac{1}{X_3(p)} + \frac{1}{X_1(p)} = 1$ if and only if either $p_1, p_2, p_3$ or $p_1, p_2, p_4$ lie in the same $\mathbb{C}$–circle; in the first case $X_1(p) + X_2(p) = 1$ and $X_3(p) + \frac{1}{X_1(p)} = 1$, and in the second case $X_1(p) + X_2(p) = 1$ and $\frac{1}{X_3(p)} + \frac{1}{X_1(p)} = 1$.

In the specific case where all points of $p$ lie in a $\mathbb{C}$–circle we have:

**Corollary 5.5.** Let $p = (p_1, p_2, p_3, p_4)$ be a quadruple of points in $\partial \mathbb{H}_C^2$. The following are equivalent:

i) All $p_i$ lie in the same $\mathbb{C}$–circle;

ii) $X_i(p) \in \mathbb{R}$, $i = 1, 2$ and $X_1(p) + X_2(p) = 1$;

iii) $X_i(p) \in \mathbb{R}$, $i = 1, 3$ and $X_3(p) + \frac{1}{X_1(p)} = 1$;

iv) $X_i(p) \in \mathbb{R}$, $i = 2, 3$ and $\frac{1}{X_2(p)} + \frac{1}{X_3(p)} = 1$;

v) $X_i(p) \in \mathbb{R}$, $i = 1, 2, 3$ and $X_3 = -X_2/X_1$.

We note that condition i) $\iff$ v) is Proposition 5.13 of [12]. The next proposition describes the differentiable structure of $\mathcal{X}_R$, $\mathcal{X}_\mathbb{CR}$ and $\mathcal{X}_\mathbb{CR}^\perp$. Note that $\mathcal{X}_R$ is actually very small.
Proposition 5.6. Consider the singular sets $\mathcal{X}_R$, $\mathcal{X}_{CR}$ and $\mathcal{X}^\perp_{CR}$ of cross-ratio variety $\mathcal{X}$. The following hold:

i) The real singular set $\mathcal{X}_R$ is a 1–dimensional disconnected real manifold, isomorphic to the real line $x + y = 1$ with the points $(0,1)$ and $(1,0)$ removed.

ii) The $\mathcal{CR}$ singular set $\mathcal{X}_{CR}$ and the singular set $\mathcal{X}^\perp_{CR}$ are 1–complex dimensional complex manifolds, both biholomorphic to $\mathbb{C} - \{0, 1\}$. Moreover, the real singular set $\mathcal{X}_R$ is contained in $\mathcal{X}_{CR}$ (and in $\mathcal{X}^\perp_{CR}$) as a disconnected real submanifold.

5.2.2. The complex singular set. Finally, we turn our attention to the complex singular set $\mathcal{X}_C$. First, we make the following observation: From the defining Equation (3.2) of $\mathcal{X}$ and the obvious inequality $-\Re(X_3) \leq X_3 \leq \Re(X_3)$ we have

\[(5.1) \quad (|X_1| - |X_2|)^2 \leq 2\Re(X_1 + X_2) - 1 \leq (|X_1| + |X_2|)^2.\]

If $\Im(X_3) = 0$, then $\Re(X_3) = |X_3|$ or $\Re(X_3) = -|X_3|$ and thus we have either

\[(5.2) \quad (|X_1| - |X_2|)^2 = 2\Re(X_1 + X_2) - 1 \quad \text{or} \quad 2\Re(X_1 + X_2) - 1 = (|X_1| + |X_2|)^2,\]

respectively. Therefore, excluding all points of $\mathcal{X}$ at which $\Im(X_3) = 0$, we obtain strict inequalities in (5.1).

Proposition 5.7. The following holds.

$\mathcal{X}_C = \{(X_1, X_2, X_3) \in \mathcal{X} \mid X_3 > 0\} \cup \mathcal{X}_R$.

Moreover, the set of triples of cross–ratios corresponding to quadruples that lie in an $\mathbb{R}$–circle or in a $\mathbb{C}$–circle is contained in $\mathcal{X}_C$.

Proof. We only prove the second assertion of our proposition. All triples of cross–ratios corresponding to quadruples that lie in an $\mathbb{R}$–circle are contained in $\mathcal{X}_C$. This is because in that case (cf. Proposition 5.14 of [12]) $X_i > 0$ for all $i = 1, 2, 3$ and moreover

\[(X_1 - X_2)^2 = 2X_1 + X_2 - 1.\]

Next, consider triples of cross–ratios corresponding to quadruples that lie in a $\mathbb{C}$–circle. For such a triple we have that $X_i \in \mathbb{R}$ and $X_1 + X_2 = 1$. The following possibilities can occur:

1. $X_1X_2 > 0$, $X_3 = -X_2/X_1 < 0$ and

2. $X_1X_2 < 0$, $X_3 = -X_2/X_1 > 0$.

In case (1) the right Equation (5.2) is satisfied; the left Equation (5.2) is satisfied in case (2). \qed

We may now write the complex singular set $\mathcal{X}_C$ of the cross–ratio variety $\mathcal{X}$ as the disjoint union of $\mathcal{X}_C^1$ and $\mathcal{X}_C^2$ where

\[(5.3) \quad \mathcal{X}_C^1 = \{(X_1, X_2, X_3) \in \mathcal{X} \mid (|X_1| - |X_2|)^2 = 2\Re(X_1) + 2\Re(X_2) - 1\},\]

\[(5.4) \quad \mathcal{X}_C^2 = \{(X_1, X_2, X_3) \in \mathcal{X} \mid (|X_1| + |X_2|)^2 = 2\Re(X_1) + 2\Re(X_2) - 1\}.\]

Proposition 5.8. Let $\mathcal{X}_C$ be the complex singular set of $\mathcal{X}$ and let also $\mathcal{X}_C = \mathcal{X}_C^1 \cup \mathcal{X}_C^2$ where $\mathcal{X}_C^1$ and $\mathcal{X}_C^2$ are as in (5.3) and (5.4), respectively.

1. $\mathcal{X}_C^1 = \mathcal{X}_C^1 - \{(X_1, X_2, X_3) \mid X_1 + X_2 = 1, X_1X_2 > 0\}$ admits the structure of a 3–dimensional submanifold of $\mathbb{C}^2$.

2. $\mathcal{X}_C^2$ is a 1–dimensional disconnected manifold diffeomorphic to the disjoint union of the two open open line segments given by $x_1 + x_2 = 1, x_1x_2 < 0$. 
Proof. We only sketch the proof of (1). Let \( \zeta_i = x_i + iy_i, i = 1, 2 \) and the equation
\[
F(\zeta_1, \zeta_2) = (|\zeta_1| - |\zeta_2|)^2 - 2\Re(\zeta_1 + \zeta_2) + 1 = 0.
\]
We have
\[
\frac{\partial F}{\partial x_1} = 2 \left(1 - \frac{|\zeta_2|}{|\zeta_1|}\right) x_1 - 1, \quad \frac{\partial F}{\partial x_2} = 2 \left(1 - \frac{|\zeta_1|}{|\zeta_2|}\right) x_2 - 1,
\]
\[
\frac{\partial F}{\partial y_1} = 2 \left(1 - \frac{|\zeta_2|}{|\zeta_1|}\right) y_1, \quad \frac{\partial F}{\partial y_2} = 2 \left(1 - \frac{|\zeta_1|}{|\zeta_2|}\right) y_2,
\]
and the reader may verify that all partial derivatives vanish at points where \( y_1 = y_2 = 0 \) and \( x_1 + x_2 = 1, x_1 x_2 > 0 \).

A codimension 1 \( \mathbb{C} \mathbb{R} \)–structure is called strictly pseudoconvex if the Levi form is strictly positive. We have the following:

**Proposition 5.9.** There is a strictly pseudoconvex \( \mathbb{C} \mathbb{R} \)–structure of codimension 1 defined on \( X_\zeta^1 \).

Proof. Consider the equation
\[
F(\zeta_1, \zeta_2) = (|\zeta_1| - |\zeta_2|)^2 - 2\Re(\zeta_1 + \zeta_2) + 1 = 0.
\]
We calculate
\[
\frac{\partial F}{\partial \zeta_1} = \left(1 - \frac{|\zeta_2|}{|\zeta_1|}\right) \zeta_1 - 1, \quad \frac{\partial F}{\partial \zeta_2} = \left(1 - \frac{|\zeta_1|}{|\zeta_2|}\right) \zeta_2 - 1,
\]
\[
\frac{\partial^2 F}{\partial \zeta_1^2} = 1 - \frac{|\zeta_2|}{2|\zeta_1|}, \quad \frac{\partial^2 F}{\partial \zeta_1 \zeta_2} = \frac{\zeta_1 \zeta_2}{2 |\zeta_1| |\zeta_2|}, \quad \frac{\partial^2 F}{\partial \zeta_2^2} = 1 - \frac{|\zeta_1|}{2 |\zeta_2|}.
\]
Observe that all partial derivatives of the first order vanish at points \((\zeta_1, \zeta_2)\) such that \( \zeta_1, \zeta_2 \in \mathbb{R} \) and \( \zeta_1 + \zeta_2 = 1 \). Now, straightforward calculations show that at points \((\zeta_1, \zeta_2)\) such that \( F(\zeta_1, \zeta_2) = 0 \) we have:
\[
L(\zeta_1, \zeta_2) = 1 + \frac{\Re(\zeta_1 \zeta_2)}{|\zeta_1| |\zeta_2|}.
\]
The above is in general greater or equal than zero; in the case where \( L(\zeta_1, \zeta_2) = 0 \) we have \( \Re(\zeta_1 \zeta_2) = -|\zeta_1| |\zeta_2| \) and then
\[
0 = (|\zeta_1| - |\zeta_2|)^2 - 2\Re(\zeta_1 + \zeta_2) + 1
= |\zeta_1|^2 + |\zeta_2|^2 + 2\Re(\zeta_1 \zeta_2) - 2\Re(\zeta_1 + \zeta_2) + 1
= |\zeta_1 + \zeta_2 - 1|^2
\]
which is not the case here. The proof is complete.

From Proposition 5.9 it follows that the set \( \mathcal{P} \) as in Section 4.1 is Levi strictly pseudoconvex with smooth boundary \( X_\zeta^1 \), see p.128 of [10] for the definition of Levi pseudoconvexity.

5.3. Geometric Interpretation of the Involution \( \mathcal{T} \). We are going to prove the following.

**Theorem 5.10.** Let \( p = (p_1, p_2, p_3, p_4) \) and \( p' = (p'_1, p'_2, p'_3, p'_4) \) be two quadruples of distinct points in \( \partial H_\zeta^2 \) with respective cross ratios \( X_i \) and \( X'_i \), \( i = 1, 2, 3 \). Assume that \( p_i \) and \( p'_i \) do not all lie in the same \( \mathbb{C} \)– circle and \( \Im(\mathbb{X}_3) \) and \( \Im(\mathbb{X}'_3) \) are both different from zero.

Then \( \mathcal{T}(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3) = (\mathbb{X}'_1, \mathbb{X}'_2, \mathbb{X}'_3) \) if and only if there exist holomorphic isometries \( g_1, g_4 \) of \( \partial H_\zeta^2 \) such that:

1. \( g_i(p_2) = p'_2, g_i(p_3) = p'_3 \) for \( i = 1, 4, g_1(p_1) = p'_4, g_4(p_4) = p'_1 \);
(2) the composition \( g_1 \circ g_4 \) is conjugate to the rotation in an angle \( \arg(X_3) \);
(3) the composition \( g_1 \circ g_4^{-1} \) is conjugate to the dilation by \( |X_3| \) followed by the rotation in an angle \( 2 \arg \left( \frac{1 - X_1 - X_2}{X_1' - X_2'} \right) \).

For the proof, we need the subsequent two lemmas.

**Lemma 5.11.** With the assumptions of Theorem 5.10, let also \( A_i \) and \( A_i' \), \( i = 1, 2, 3, 4 \), be the respective Cartan's invariants of \( p \) and \( p' \). Let also

\[
2\eta = \arg(1 - X_1 - X_2), \quad 2\eta' = \arg(1 - X_1' - X_2').
\]

Then, the following are equivalent:

i) \( T(X_1, X_2, X_3) = (X_1', X_2', X_3') \).

ii) \( |X_i| = |X_i'| \), \( i = 1, 2 \), and \( A_1 = A_1' \), \( A_2 = A_2' \), \( A_3 = A_3' \), \( A_4 = A_4' \).

iii) \( |X_3| = |X_3'| \) and \( \eta = \eta' \), \( A_1 = A_1' \), \( A_2 = A_2' \), \( A_3 = A_3' \), \( A_4 = A_4' \).

**Proof.** Recall from Proposition 5.1 that the following hold:

\[
\arg(X_1) = A_1 - A_2, \quad \arg(X_2) = -A_2 - A_4, \quad \arg(X_3) = A_3 - A_2, \quad A_1 = A_2 - A_3 + A_4.
\]

We first prove direction (1) \( \Rightarrow \) (3). Since \( X_i = X_i', \ i = 1, 2 \) and \( X_3 = \overline{X_3} \) we clearly have \( |X_3| = |X_3'| \) and also \( 2\eta = 2\eta' \). Now, from the relations \( \arg(X_i) = \arg(X_i') \), \( i = 1, 2 \) and \( \arg(X_3) = -\arg(X_3') \) we also have

\[
A_1 - A_2 = A_1' - A_2',
A_2 + A_4 = A_2' + A_4',
A_3 - A_2 = -A_3' + A_2.
\]

The third equation can be replaced by the equivalent \( A_4 - A_1 = A_4' - A_1' \); solving the \( 3 \times 3 \) system in \( A_1, A_2 \) and \( A_4 \) we get

\[
A_1 = A_4', \quad A_4 = A_1', \quad A_2 = A_2' + A_4' - A_1' = A_3'.
\]

Hence this also yields

\[
A_3 = A_2 + A_4 - A_1 = A_3' + A_4' - A_1' = A_2'.
\]

To prove direction (3) \( \Rightarrow \) (2) we first show that

\[
e^{2i\eta} = \frac{1 - X_1 - X_2}{|1 - X_1 - X_2|} \quad \text{and} \quad 2|X_1|^{1/2}|X_2|^{1/2} \sqrt{\cos(A_1) \cos(A_4)} = |1 - X_1 - X_2|.
\]

The left equation is following from the definition of \( \eta \). As for the right equation, observe that

\[
|1 - X_1 - X_2|^2 = 1 + |X_1|^2 + |X_2|^2 - 2\Re(X_1) - 2\Re(X_2) + 2\Re(X_1 \overline{X_2})
= 2|X_1|^2 \Re(X_3) + 2\Re(X_1 \overline{X_2})
= 2|X_1||X_2| \left( \cos(\arg(X_3)) + \cos(\arg(X_1 \overline{X_2})) \right)
= 2|X_1||X_2| \left( \cos(A_4 - A_1) + \cos(A_1 + A_4) \right)
= 4|X_1||X_2| \cos(A_1) \cos(A_4).
\]

Therefore, \( \eta = \eta' \) implies

\[
e^{2i\eta} = \frac{1 - X_1 - X_2}{2|X_1|^{1/2}|X_2|^{1/2} \sqrt{\cos(A_1) \cos(A_4)}} = \frac{1 - X_1' - X_2'}{2|X_1'|^{1/2}|X_2'|^{1/2} \sqrt{\cos(A_1') \cos(A_4')}} = e^{2i\eta'}.
\]
Since we additionally have \(|X_3| = |X_3'|\), \(A_1 = A_4'\) and \(A_4 = A_1'\), this is equivalent to
\[
\frac{1 - X_1 - X_2}{|X_1|} = \frac{1 - X_1' - X_2'}{|X_1'|}.
\]

From the Cartan’s invariants conditions we have \(\arg X_1 = \arg X_1', i = 1, 2\); the above equation then implies
\[
\frac{1}{|X_1|} - e^{i\arg(X_1)} - |X_3|e^{i\arg(X_3)} = \frac{1}{|X_1'|} - e^{i\arg(X_1)} - |X_3'|e^{i\arg(X_3')}.
\]
Hence \(|X_1| = |X_1'|\) and therefore also \(|X_2| = |X_2'|\).

We leave the proof of direction (2) \(\Rightarrow\) (1) to the reader. \(\square\)

**Lemma 5.12.** Let \(p = (p_1, p_2, p_3, p_4)\) a quadruple of distinct points of \(\partial \mathbb{H}_C^2\) with cross–ratios \(X_1, X_2, X_3\) and assume that \(p_i\) do not all lie in the same \(\mathbb{C}\)– circle and \(\Im(X_3)\) is different from zero. Then we can normalise so that
\[
p_1 = (z_1, t_1) = \left(\sqrt{\cos(A_4)}e^{-i\tilde{A}_3}, \sin(A_1)\right),
p_2 = \infty, \quad p_3 = (0, 0),
p_4 = (z_4, t_4) = \left(-\sqrt{\cos(A_1)}|X_3|^{1/2}e^{2i\eta}, \frac{|X_3|}{\sin(A_1)}\right).
\]

Here \(\tilde{A}_i, i = 1, \ldots, 4\) are the Cartan invariants of \(p\) and \(2\eta = \arg(1 - X_1 - X_2)\).

**Proof.** We normalise so that \(p_i\) have lifts
\[
p_1 = \begin{bmatrix} -e^{-i\tilde{A}_4} \\
\sqrt{2\cos(A_4)}e^{-i\tilde{A}_3} \\
1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix}, \quad p_4 = \begin{bmatrix} -\frac{|X_3|e^{-i\tilde{A}_1}}{\sqrt{2\cos(A_1)|X_3|^{1/2}e^{2i\eta}}} \\
0 \\
0 \end{bmatrix}.
\]

We have
\[
\langle p_1, p_2 \rangle = \langle p_2, p_3 \rangle = \langle p_2, p_4 \rangle = 1, \quad \langle p_1, p_3 \rangle = -e^{-i\tilde{A}_4}, \quad \langle p_3, p_4 \rangle = -|X_3|e^{i\tilde{A}_1}.
\]

Also,
\[
\langle p_1, p_4 \rangle = -e^{-i\tilde{A}_4} - |X_3|e^{i\tilde{A}_1} - 2\sqrt{\cos(A_1)}\cos(A_4)|X_3|^{1/2}e^{-2i\eta} \cdot e^{-i\tilde{A}_3}
\]
\[
= -e^{-i\tilde{A}_4} - \frac{|X_2|}{|X_1|}e^{i\tilde{A}_1} - 2\sqrt{\cos(A_1)}\cos(A_4)\frac{|X_2|^{1/2}}{|X_1|^{1/2}}e^{-2i\eta} \cdot e^{-i\tilde{A}_3}
\]
\[
= -e^{-i\tilde{A}_4} - \frac{|X_2|}{|X_1|}e^{i\tilde{A}_1} - \frac{|X_1 - X_2|}{|X_1|^{1/2}|X_2|^{1/2}} \cdot \frac{|X_2|^{1/2}}{|X_1|^{1/2}} \cdot \frac{1 - \frac{X_1 - X_2}{|X_1 - X_2|}}{1 - \frac{X_1 - X_2}{|X_1 - X_2|}} \cdot e^{-i\tilde{A}_3}
\]
\[
= -\frac{|X_1|e^{i(\tilde{A}_3 - \tilde{A}_4)}}{|X_1|} + \frac{|X_2|e^{i(\tilde{A}_1 + \tilde{A}_3)}}{|X_1|} + \frac{1 - \frac{X_1 - X_2}{|X_1 - X_2|}}{|X_1|e^{i\tilde{A}_3}}
\]
\[
= -\frac{X_1 + X_2 + 1 - \frac{X_1 - X_2}{|X_1 - X_2|}}{|X_1|e^{i\tilde{A}_3}}
\]
\[
= -|X_1|^{-1}e^{-i\tilde{A}_3},
\]
Lemma 5.11. Under our normalisation assumptions, we get:

\[
\begin{align*}
\frac{\langle p_1, p_2 \rangle}{\langle p_1, p_1 \rangle} &= \frac{-e^{i\lambda_4}}{-|x_1|e^{i\lambda_3}} = x_1, \\
\frac{\langle p_1, p_3 \rangle}{\langle p_1, p_2 \rangle} &= \frac{-|x_3|e^{-i\lambda_1}}{-|x_1|e^{i\lambda_3}} = x_2, \\
\frac{\langle p_1, p_3 \rangle}{\langle p_1, p_2 \rangle} &= \frac{-|x_3|e^{-i\lambda_4}}{-e^{-i\lambda_4}} = x_3.
\end{align*}
\]

Proof of Theorem 5.10.

Using Lemma 5.12 we normalise so that

\[
p_1 = (z_1, t_1) = \left(\sqrt{\cos(A_1)}e^{-i\lambda_3}, \sin(A_1)\right), \\
p_2 = \infty, \quad p_3 = (0, 0), \\
p_4 = (z_4, t_4) = \left(-\sqrt{\cos(A_4)}|x_3|^{1/2}e^{2i\eta}, |x_3|\sin(A_4)\right).
\]

Suppose first that \(T(X_1, X_2, X_3) = (X'_1, X'_2, X'_3)\). Then, using iii) of Lemma 5.11 we have the following normalisation for \(p'_i, i = 1, \ldots, 4\):

\[
p'_1 = (z'_1, t'_1) = \left(\sqrt{\cos(A'_1)}e^{-i\lambda_2}, \sin(A_1)\right), \\
p'_2 = \infty, \quad p'_3 = (0, 0), \\
p'_4 = (z'_4, t'_4) = \left(-\sqrt{\cos(A'_4)}|x_3|^{1/2}e^{2i\eta}, |x_3|\sin(A_4)\right).
\]

It follows after elementary calculations that the transformations

\[
g_1(z, t) = \left(-|x_3|^{1/2}e^{i(2\eta + \lambda_3)}z, |x_3|t\right), \quad g_4(z, t) = \left(-|x_3|^{-1/2}e^{-i(2\eta + \lambda_2)}z, |x_3|^{-1}t\right),
\]

satisfy conditions (1), (2) and (3) of the theorem.

Conversely, suppose that there exist \(g_i, i = 1, 4\) as in the conditions of the theorem. We write

\[
g_1(z, t) = \left(e^{l_1 + 3i\theta_1}z, e^{2l_1}t\right), \\
g_4(z, t) = \left(e^{l_4 + 3i\theta_4}z, e^{2l_4}t\right).
\]

From (1), and since \(g_1\) maps \((p_1, p_2, p_3)\) to \((p'_1, p'_2, p'_3)\) we have \(A_4 = A'_1\). Also, since \(g_4\) maps \((p_2, p_3, p_4)\) to \((p'_2, p'_3, p'_4)\) we have \(A_1 = A'_4\). By our assumptions,

\[
l_1 = \log(|x_3|^{1/2}), \quad l_4 = \log(|x_3|^{-1/2}), \\
3\theta_1 = 2\eta + A_3, \quad 3\theta_4 = -2\eta - A_2.
\]

Under our normalisation assumptions, \(g_1(z_1, t_1) = (z'_4, t'_4)\) gives \(2\eta = 2\eta' \pmod{\pi}\). Using this, \(g_4(z_4, t_4) = (z'_1, t'_1)\) gives \(A_2 = A'_3\) and therefore also \(A'_3 = A_2\). The result now follows from iii) of Lemma 5.11.

\[\square\]
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