On some $\mathcal{D}$-modules in dimension 2

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June 30, 2000

Abstract.- We prove a duality formula for two $\mathcal{D}$-modules arising from logarithmic derivations w.r.t. a plane curve. As an application we give a differential proof of a logarithmic comparison theorem in [4].

Keywords: $\mathcal{D}$-modules, Differential Operators, Gröbner Bases, Logarithmic Comparison Theorem.

Math. Classification: 32C38, 13N10, 14F40, 13P10.

1 Introduction

Let $\mathcal{O} = \mathbb{C}\{x, y\}$ be the ring of convergent power series in two variables and $\mathcal{D}$ the ring of linear differential operators with coefficients in $\mathcal{O}$. For each reduced power series $f \in \mathcal{O}$, with $f(0, 0) = 0$, we will denote by $I^\log$ the left ideal of $\mathcal{D}$ generated by the logarithmic derivations (see [1]) with respect to $f$. We denote by Der$_\mathbb{C}(\mathcal{O})$ the Lie algebra of $\mathbb{C}$-derivations on $\mathcal{O}$. Recall that a derivation $\delta \in$ Der$_\mathbb{C}(\mathcal{O})$ is logarithmic if there exists $a \in \mathcal{O}$ such that $\delta(f) = af$. We denote by $\tilde{I}^\log$ the left ideal of $\mathcal{D}$ generated by the operators of the form $\delta + a$ where $\delta(f) = af$.

We first prove that the $\mathcal{D}$-modules $M^\log = \mathcal{D}/I^\log$ and $\tilde{M}^\log = \mathcal{D}/\tilde{I}^\log$ are dual each to the other and then that both $\mathcal{D}$-modules are regular holonomic.

Let $\mathcal{O}[1/f]$ be the $\mathcal{D}$-module of (the germs of) the meromorphic functions in two variables with poles along $f$. There exists a natural surjective
morphism \( \psi : \tilde{M}_{\log} \to \mathcal{O}[1/f] \). Using [4] we prove that \( \text{Ext}_D^2(\tilde{M}_{\log}, \mathcal{O}) = 0 \) if and only if \( f \) is quasi-homogeneous and then we obtain that the morphism \( \psi \) is an isomorphism if and only if \( f \) is quasi-homogeneous (see 2.3). As a consequence we give a new “differential” proof of the logarithmic comparison theorem of [4].

These results are susceptible to be generalized to the case of higher dimensions but no general results are known up to now. See [6] for a proof of the duality formula in higher dimension. Nevertheless we give a complete example showing that some results of the present work are true in dimension 3.

We wish to thank Prof. L. Narváez for giving us useful suggestions.

2 The module \( \tilde{M}_{\log} \) in the general case.

Let us consider any reduced \( f \in \mathcal{O} = \mathbb{C}\{x, y\} \) with a singular point at the origin. It is possible to obtain, from the logarithmic derivations, an ideal inside \( \text{Ann}_D(1/f) \): if \( \delta(f) = af \) then \( \delta + a \in \text{Ann}_D(1/f) \). This fact suggested us a general way to present the annihilating ideal of \( 1/f \) for a constructive proof of the equality \( \text{Ext}_D^2(\mathcal{O}[1/f], \mathcal{O}) = 0 \) for any “polynomial” curve (see [14]).

We have \( \tilde{I}_{\log} \subset \text{Ann}_D(1/f) \), where \( \tilde{I}_{\log} \) is the left ideal in \( \mathcal{D} \) generated by the operators \( \delta + a \) for \( \delta \in \text{Der}_C(\mathcal{O}) \) and \( \delta(f) = af \). Then we have a surjective morphism \( \psi : \tilde{M}_{\log} = \mathcal{D}_{\log} / \mathcal{D}\text{Ann}_D(1/f) \simeq \mathcal{O}[1/f] \) (for the last isomorphism we use that the Bernstein polynomial of \( f \) has no integer roots smaller than \(-1\) (see [13])). It is well known that around each smooth point of \( f = 0 \) the morphism \( \psi \) is in fact an isomorphism. So, the kernel \( K \) of \( \psi \) is a \( \mathcal{D} \)-module concentrated at the origin. Then \( K \) is a direct sum of “couches-multiples” modules [8], and this type of modules are regular holonomic [9]. In particular \( \tilde{M}_{\log} \) is regular holonomic because \( \mathcal{O}[1/f] \) and \( K \) are.

We will denote by \( \text{Der}(\log f) \) the Lie algebra of logarithmic derivations with respect to \( f \). By [11] \( \text{Der}(\log f) \) is a free \( \mathcal{O} \)-module of rank two. Let \( \{\delta_1, \delta_2\} \) be a basis of \( \text{Der}(\log f) \),

\[
\begin{align*}
\delta_1 &= b_1 \partial_x + c_1 \partial_y, \\
\delta_2 &= b_2 \partial_x + c_2 \partial_y.
\end{align*}
\]
We can suppose that
\[
\begin{vmatrix}
  b_1 & c_1 \\
  b_2 & c_2 \\
\end{vmatrix} = f
\]
We will take into account the following results for any (reduced) curve \( f \):

- Every basis \( \delta_1, \delta_2 \) of \( \text{Der}(\log f) \) verifies that
  \[
  \langle \sigma(\delta_1), \sigma(\delta_2) \rangle = \text{gr}^F(I^\log) = \text{gr}^F(\tilde{I}^\log),
  \]
  because \( \{\sigma(\delta_1), \sigma(\delta_2)\} \) is a regular sequence (see [2] and ([3], Corollary 4.2.2)). Here \( \sigma(\cdot) \) denotes the principal symbol of the corresponding operator and \( \text{gr}^F(I^\log) \) is the graded ideal associated to the order filtration on \( \mathcal{D} \). Therefore,
  \[
  CCh(\tilde{M}^\log) = CCh(M^\log),
  \]
  where \( CCh(\cdot) \) represents the characteristic cycle of the \( \mathcal{D} \)-module (see, for example, [7]). Of course both \( M^\log \) and \( \tilde{M}^\log \) define coherent \( \mathcal{D} \)-modules in some neighborhood of the origin and then we can properly speak of characteristic varieties and characteristic cycles. Since \( \tilde{M}^\log \) is holonomic then \( M^\log \) is holonomic.

- For any curve,
  \[
  Sol(M^\log) \overset{\text{q.i.}}{\cong} \Omega^\bullet(\log f) \xrightarrow{\varphi} \Omega^\bullet[1/f] \cong DR(\mathcal{O}[1/f]),
  \]
  where \( Sol(\cdot) \) and \( DR(\cdot) \) are the solutions complex and the De Rham complex (see, for example, [3]) and where \( \Omega^\bullet(\log f) \) (resp. \( \Omega^\bullet([1/f]) \)) is the complex of logarithmic differential forms (resp. meromorphic differential forms). The first quasi-isomorphism appears in [3] and \( \varphi \) is the natural morphism.

**Proposition 2.1** Let \( f \) be a (reduced) curve and let \( \{\delta_1, \delta_2\} \) be a basis of \( \text{Der}(\log f) \) with \( [\delta_1, \delta_2] = \alpha_1 \delta_1 + \alpha_2 \delta_2 \) and \( \delta_i(f) = a_i f, \ i = 1, 2 \). Then
  \[
  \mathcal{D}\{\delta_1' + \alpha_1, \delta_2' - \alpha_2\} = \mathcal{D}\{\delta_1 + a_1, \delta_2 + a_2\}
  \]
  where \( \delta_i' \) is the transposed of \( \delta_i \).
Proof: First we find an expression of the $\alpha_i$ from the $a_j, b_k, c_i$:

$$[\delta_1, \delta_2] = \alpha_1(b_1 \partial_x + c_1 \partial_y) + \alpha_2(b_2 \partial_x + c_2 \partial_y) =$$

$$= (\alpha_1 b_1 + \alpha_2 b_2) \partial_x + (\alpha_1 c_1 + \alpha_2 c_2) \partial_y =$$

$$= b_1 \partial_x(b_2) \partial_x - b_2 \partial_x(b_1) \partial_x + b_1 \partial_x(c_2) \partial_y - c_2 \partial_y(b_1) \partial_x +$$

$$+ c_1 \partial_y(b_2) \partial_x - b_2 \partial_x(c_1) \partial_y + c_1 \partial_y(c_2) \partial_y - c_2 \partial_y(c_1) \partial_y =$$

$$= (c_1 \partial_y(b_2) - b_2 \partial_x(b_1) - c_2 \partial_y(b_1) + c_1 \partial_y(b_2)) \partial_x +$$

$$+ (b_1 \partial_x(c_2) - b_2 \partial_x(c_1) - c_2 \partial_y(c_1) + c_1 \partial_y(c_2)) \partial_y.$$ 

Besides,

$$-\delta_1^t + \alpha_2 = \partial_x b_1 + \partial_y c_1 + \alpha_2 = \delta_1 + \alpha_2 + \partial_x(b_1) + \partial_y(c_1).$$

To prove that $\alpha_2 + \partial_x(b_1) + \partial_y(c_1) = a_1$, we will establish that

$$\alpha_2 f = a_1 f - \partial_x(b_1) f - \partial_y(c_1) f$$

We have

$$a_1 f - \partial_x(b_1) f - \partial_y(c_1) f =$$

$$= (b_1 \partial_x + c_1 \partial_y) - \partial_x(b_1) - \partial_y(c_1)(b_1 c_2 - b_2 c_1) =$$

$$= b_1(b_1 \partial_x(c_2) - c_1 \partial_x(b_2) - b_2 \partial_x(c_1)) +$$

$$+ c_1(c_2 \partial_y(b_1) + b_1 \partial_y(c_2) - c_1 \partial_y(b_2)) +$$

$$+ b_2 c_1 \partial_x(b_1) - b_1 c_2 \partial_y(c_1).$$

Therefore

$$\left(\alpha_1, \alpha_2\right) \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = (\gamma_1, \gamma_2),$$

where

$$\gamma_1 = c_1 \partial_y(b_2) - b_2 \partial_x(b_1) - c_2 \partial_y(b_1) + c_1 \partial_y(b_2),$$

$$\gamma_2 = b_1 \partial_x(c_2) - b_2 \partial_x(c_1) - c_2 \partial_y(c_1) + c_1 \partial_y(c_2)) .$$

Multiplying by the transposed adjoint matrix and by $f$ we obtain

$$(\alpha_1 f, \alpha_2 f) = (\gamma_1, \gamma_2) \begin{pmatrix} c_2 & -c_1 \\ -b_2 & b_1 \end{pmatrix} ,$$

and hence the equality follows. In a similar way $\delta_2^t + \alpha_1 = -\delta_2 - a_2$. Then both ideals are equal. $\blacksquare$
Prof. Narváez pointed us to consider, instead of the Lie algebra $\text{Der}(\log f)$, the Lie algebra
\[ L = \{ \delta + a | \delta(f) = af \}, \]
and try to construct of a free resolution (of “Spencer type”) of $\tilde{M}^{\log}$. In fact, we have

**Proposition 2.2** A free resolution of $\tilde{M}^{\log}$ is

\[ 0 \rightarrow D \xrightarrow{\varphi_2} D^2 \xrightarrow{\varphi_1} D \rightarrow \tilde{M}^{\log} \rightarrow 0, \]

where $\varphi_2$ is defined by the matrix
\[ (-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2), \]
and $\varphi_1$ by \( \left( \begin{array}{c} \delta_1 + a_1 \\ \delta_2 + a_2 \end{array} \right) \).

Proof: To check the exactness of the resolution above, it is enough to consider a discrete filtration on that complex and to verify the exactness of the resulting resolution (see [1], chapter 2, lemma 3.13). The same argument is used in [2], [3] (proposition 4.1.3) to prove that the complex $D \otimes_{V_0^f(D)} Sp^\bullet(log f)$ is a free resolution of $M^{log}$ (as a left $D$-module)\(^1\). But, for $n = 2$, the exact graded complex in the proof of [3] is precisely

\[ 0 \rightarrow \text{gr}^F(D) \xrightarrow{M_1} \text{gr}^F(D)^2[-2] \xrightarrow{M_2} \text{gr}^F(D)[-1] \rightarrow \text{gr}^F(M^{log}) \rightarrow 0, \]

where the matrices are
\[ M_1 = (\sigma^F(\delta_2), \sigma^F(\delta_1)), \quad M_2 = \left( \begin{array}{c} \sigma^F(\delta_1) \\ \sigma^F(\delta_2) \end{array} \right). \]

And the last complex is the result of applying the same graduation to the resolution of $\tilde{M}^{log}$ too, because
\[ \sigma^F(\delta_i) = \sigma^F(\delta_i + a_i). \]

\[^1\]Here $Sp^\bullet(log f)$ is the Logarithmic Spencer complex and $V_0^f(D)$ is the ring of degree zero differential operators w.r.t. $V$-filtration relative to $f$. See [2], [3], section 1.2.
Proposition 2.3 Given $f \in C\{x,y\}$, $\tilde{M}^{\log} \simeq (M^{\log})^*$ where $(\cdot)^*$ is the dual in the sense of $\mathcal{D}$-modules. In particular $\tilde{M}^{\log}$ and $M^{\log}$ are regular $\mathcal{D}$-modules.

Proof: We take the free resolution of $M^{\log}$ (see [2], (3, Th. 3.1.2))

$$0 \rightarrow \mathcal{D} \xrightarrow{\psi_2} \mathcal{D}^2 \xrightarrow{\psi_1} \mathcal{D} \rightarrow M^{\log} \rightarrow 0,$$

where $\{\delta_1, \delta_2\}$ is a basis of the $\mathcal{O}$-module $\text{Der}(\log f)$, where

$$[\delta_1, \delta_2] = \alpha_1 \delta_1 + \alpha_2 \delta_2,$$

$$\psi_1 = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

and, on the other hand, $\psi_2$ is the syzygy matrix

$$\psi_2 = (-\delta_2 - \alpha_1, \delta_1 - \alpha_2).$$

Applying the $\text{Hom}_{\mathcal{D}}(-, \mathcal{D})$ functor to calculate the dual module, we obtain the sequence

$$0 \rightarrow \mathcal{D} \xrightarrow{\psi_2^*} \mathcal{D}^2 \xrightarrow{\psi_1^*} \mathcal{D} \rightarrow (M^{\log})^* \rightarrow 0,$$

where $\psi_2^*$ is the right product by $\begin{pmatrix} -\delta_2 - \alpha_1 \\ \delta_1 - \alpha_2 \end{pmatrix}$. Hence, $(M^{\log})^*$ is the left $\mathcal{D}$-module associated to the right $\mathcal{D}$-module $\mathcal{D}/(\delta_2 + \alpha_1, \delta_1 - \alpha_2)\mathcal{D}$, that is to say,

$$(M^{\log})^* \simeq \mathcal{D}/\mathcal{D}(\delta_2 + \alpha_1, \delta_1 - \alpha_2).$$

Using the proposition 2.1, we deduce that $(M^{\log})^* \simeq \tilde{M}^{\log}$. The regularity of $M^{\log}$ follows from the regularity of $\tilde{M}^{\log}$ (c.f. [9]).}

Proposition 2.4 If $f$ is a non quasi homogeneous (reduced) curve, then

$$\text{Ext}^2_{\mathcal{D}}(\tilde{M}^{\log}, \mathcal{O}) \neq 0.$$
where $\phi_2$ is the matrix

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2).$$

Hence, $\text{Ext}_2^D(\tilde{M}^\log, \mathcal{O}) \simeq \mathcal{O}/\text{Img} \phi_2^*$. To guarantee that this vector space has dimension greater than zero, it is enough to show that a pair of functions $h_1, h_2 \in \mathcal{O}$ such that

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2) \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right) = 1,$$

does not exist, that is to say, that $1 \notin \text{Img} \phi_2^*$.

Let us take $\delta_1 = b_1 \partial_x + c_1 \partial_y$. As $a_1 - \alpha_2 = \partial_x(b_1) + \partial_y(c_1)$, (proposition 2.1) we will prove that, or $b_1$ and $c_1$ have no lineal parts, or that after derivation those lineal parts become 0.

Of course $f$ has no quadratic part: in that case, because of the classification of the singularities in two variables, $f$ would be equivalent to a quasi homogeneous curve $x^2 + y^{k+1}$, for some $k$. Then we can suppose that

$$f = f_n + f_{n+1} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j = k} a_{ij} x^i y^j,$$

where $n \geq 3$ and $f_n \neq 0$.

We will write

$$\delta_1 = b_1 \partial_x + c_1 \partial_y = \delta_0^1 + \delta_0^1 + \cdots = \sum_{k \geq 0} \sum_{i+j = k+1} (\beta_{ij}^1 x^i y^j \partial_x + \gamma_{ij}^1 x^i y^j \partial_y),$$

where the linear part $\delta_0^1$ is $(xy)A_0(\partial_x \partial_y)^t$, and $A_0$ is a matrix $2 \times 2$ with complex coefficients.

If $A_0 = 0$, we have finished. Otherwise, the possibilities of the Jordan form of $A_0$ are

$$A_0 = \left( \begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right), \quad A_0 = \left( \begin{array}{cc} \lambda_1 & 0 \\ 1 & \lambda_1 \end{array} \right).$$

As $\delta_1$ is not an Euler vector (because $f$ is not quasi homogeneous), we deduce:

- If we take the first Jordan form, then (see the cited demonstration of 4) $f_n = x^p y^q$ y $\delta_0 = qx \partial_x - py \partial_y$. After a sequence of changes of coordinates we have that $f = x^p y^q$ with $p + q = n \geq 3$, that contradicts that $f$ is reduced.
• For the second Jordan form with $\lambda_1 \neq 0$, it has to be $f_n = 0$, that contradicts that $f$ has its initial part of grade $n$.

• For the second option with $\lambda_1 = 0$ we have $\delta^1_0 = y\partial_x$ and, in this situation, the linear of $b_1$ is $y$. If we precisely apply $\partial_x$, we obtain 0.

In a similar way, you prove the same for $a_2 + \alpha_1$.

**Theorem 2.5** The natural morphism $\tilde{M}^{\log} \xrightarrow{\psi} \mathcal{O}[\frac{1}{f}]$ is an isomorphism if and only if $f$ is a quasi homogeneous (reduced) curve.

**Proof:** As we pointed, if $f$ is quasi homogeneous then $\tilde{I}^{\log} = \text{Ann}_D(1/f)$ and therefore $\psi$ is an isomorphism. Reciprocally, if $\psi$ is an isomorphism, then $\text{Ext}^2_D(\mathcal{O}[1/f], \mathcal{O}) \simeq \text{Ext}^2_D(\tilde{M}^{\log}, \mathcal{O})$. Because of a result of [9], we have $\text{Ext}^2_D(\mathcal{O}[1/f], \mathcal{O}) = 0$ and, if we take into account proposition 2.4, we obtain that $f$ has to be quasi homogeneous.

**Remark.** The following result can be obtain using [13]: if $f$ is not quasi-homogeneous curve then $\text{Ann}_D(1/f)$ could not be generated by elements of degree one in $\partial$ and then $\text{Ann}_D(1/f) \neq \tilde{I}^{\log}$.

Let us give a new “differential” proof of a version of the **Logarithmic Comparison Theorem** [4].

**Theorem 2.6** The complexes $\Omega^\bullet(\log f)$ and $\Omega^\bullet[1/f]$ are isomorphic in the correspondent derived category if and only if $f$ is quasi homogeneous.

**Proof:** If $f$ is quasi homogeneous we have pointed yet that $\tilde{M}^{\log}$ is isomorphic to $\mathcal{O}[\frac{1}{f}]$. By the proposition 2.3 ($M^{\log})^* \simeq \tilde{M}^{\log}$ and then we have

$$\Omega^\bullet(\log f) \simeq \text{Sol}(M^{\log}) \simeq \text{DR}((M^{\log})^*) \simeq \text{DR}(\tilde{M}^{\log}) \simeq \Omega^\bullet[1/f],$$

where the first isomorphism is obtained in [2] (see also [3]) and the second one could be found in [9]. Reciprocally, if $f$ is not quasi homogeneous then
\( \tilde{M}^{log} \not\cong \mathcal{O}[1/f] \) and, as both are regular holonomic, neither their De Rham complexes are isomorphic, that is

\[
DR(\tilde{M}^{log}) \not\cong \Omega^*[1/f],
\]

using the Riemann-Hilbert correspondence of Mebkhout-Kashiwara. []

3 Example in a constructive way of logarithmic comparison in surfaces.

We illustrate in this section an interesting example in dimension 3 of the situation presented in theorem 2.6 for curves. We consider the surface (see \( \mathbb{P} \)) \( h = 0 \) with

\[
h = xy(x + y)(xz + y),
\]

which is not locally quasi-homogeneous. We prove that:

- \( \text{Ann}_D(1/h) = \tilde{I}^{log} \).
- \( \tilde{M}^{log} \cong (M^{log})^* \)

and we conclude the logarithmic comparison theorem holds in this case. Although this example appears in [4], here the treatment is under an effective point of view.

We can compute a basis of \( \text{Der} (\log h) \) with a set of generators of the syzygies among \( h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \). We obtain

- \( \delta_1 = x\partial_x + y\partial_y \),
- \( \delta_2 = xz\partial_x + y\partial_z \),
- \( \delta_3 = x^2\partial_x - y^2\partial_y - xz\partial_z - yz\partial_z \),

with

\[
\delta_1(h) = 4h, \quad \delta_2(h) = xh, \quad \delta_3(h) = (2x - 3y)h,
\]

and

\[
\begin{vmatrix}
x & y & 0 \\
0 & 0 & xz + y \\
x^2 & -y^2 & -xz - yz
\end{vmatrix} = h.
\]
As a multiple of the $b$-function of $h$ in $\mathcal{D}$ is
\[ b(s) = (4s + 5)(2s + 1)(4s + 3)(s + 1)^3, \]
and this polynomial has no integer roots smaller than $-1$, we can assure that
\[ \mathcal{O}[[h]] \simeq \mathcal{D}_{1/h}. \]

It is easy to check that $Ann_{\mathcal{D}}(1/h)$ is equal to $\widetilde{I}^{\text{log}}$. The computations of the $b$-function and the annihilating ideal of $h^s$ have been made using the algorithms of [10], implemented in [12].

We calculate (using Gröbner bases) a free resolution of the module $\mathcal{D}/I^{\text{log}}$ where $I^{\text{log}} = (\delta_1, \delta_2, \delta_3)$ (see [3]). The first module of syzygies is generated (in this case) by the relations deduced from the expressions of the $[\delta_i, \delta_j]$ with $i \neq j$:

- $[\delta_1, \delta_2] = \delta_2$,
- $[\delta_1, \delta_3] = \delta_3$,
- $[\delta_2, \delta_3] = -x\delta_2$.

The second module of syzygies is generated by only one element $s = (s_1, s_2, s_3)$:

- $s_1 = -y^2\partial_y + x^2\partial_x - zy\partial_z - zx\partial_z - x$,
- $s_2 = -y\partial_z - xz\partial_z$,
- $s_3 = y\partial_y + x\partial_x - 2$.

The above calculations provide a free resolution of $M^{\text{log}}$. With a procedure similar to the used in [2,3] we obtain that $(M^{\text{log}})^*$ is the left $\mathcal{D}$-module associated to the right $\mathcal{D}$-module $\mathcal{D}/(s_1, s_2, s_3)\mathcal{D}$. Then
\[ (M^{\text{log}})^* \simeq \mathcal{D}/(s_1^t, s_2^t, s_3^t). \]

It is enough to compute $s_1^t, s_2^t, s_3^t$ and check (using Gröbner basis) that they span $\widetilde{I}^{\text{log}}$. Hence
\[ (M^{\text{log}})^* = (\mathcal{D}/\text{Der}(/(h))^*) \simeq \mathcal{D}/\widetilde{I}^{\text{log}} = \widetilde{M}^{\text{log}}. \]
At this point we have obtained that

\[ \text{Sol}(M^{\log}) \simeq DR((M^{\log})^*) \simeq DR(\widetilde{M}^{\log}) \simeq \Omega^*[1/h] \]

where the last two isomorphism are due to our computations (the first was used in the proof of \[2\]). Taking into account that \( \Omega^*(\log h) \simeq \text{Sol}(M^{\log}) \) (it was showed for this example in \[4,5\]) we can deduce that the logarithmic comparison theorem (i.e. \( \Omega^*(\log h) \simeq \Omega^*[1/h], \[4\]) holds without the “locally quasi homogeneous” hypothesis. It is interesting to remark too (see \[2\]) that \( \{\sigma^F(\delta_1), \sigma^F(\delta_2), \sigma^F(\delta_3)\} \) do not form a regular sequence in \( \text{gr}^F(D) \). We have

\[ (z\eta\zeta - \xi\zeta) \sigma^F(\delta_3) \in \langle \sigma^F(\delta_1), \sigma^F(\delta_2) \rangle. \]

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