Transition Complexity of Incomplete DFAs *

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In this paper, we consider the transition complexity of regular languages based on the incomplete deterministic finite automata. A number of results on Boolean operations have been obtained. It is shown that the transition complexity results for union and complementation are very different from the state complexity results for the same operations. However, for intersection, the transition complexity result is similar to that of state complexity.

1 Introduction

Many results have been obtained in recent years on the state complexity of individual and combined operations of regular languages and a number of sub-families of regular languages [5, 8, 11, 12, 15, 16]. The study of state complexity has been mostly based on the model of complete deterministic finite automata (DFAs). When the alphabet is fixed, the number of states of a complete DFA determines the number of transitions of the DFA. Note that a description of a DFA consists of a list of transitions, which determine the size of the DFA. Incomplete DFAs are implied in many publications [1, 14]. In quite a number of applications of finite automata, incomplete rather than complete DFAs are more suitable for those applications [9, 13]. For example, in natural language and speech processing, the input alphabet of a DFA commonly includes at least all the ASCII symbols or the UNICODE symbols. However, the number of useful transitions from each state is usually much smaller than the size of the whole alphabet, which may include only a few symbols [9]. Although the state complexity of such an incomplete DFA can still give a rough estimate of the size of the DFA, the number of transitions would give a more precise measurement of its size.

In this paper, we consider the descriptional complexity measure that counts the number of transitions in an incomplete DFA. It is clear that for two DFAs with an equal number of states, the size of the description may be much smaller for a DFA where many transitions are undefined. Especially, for applications that use very large or possibly non-constant alphabets, or DFAs with most transitions undefined, it can be argued that transition complexity is a more accurate descriptional complexity measure than state complexity. Before this paper, transition complexity was investigated only on nondeterministic finite automata [3, 4, 6, 7] and Watson-Crick finite automata [10].

We consider operational transition complexity of Boolean operations. The transition complexity results for union and complementation turn out to be essentially different from the known state complexity results [16, 17] that deal with complete DFAs. Perhaps, as expected, the results for intersection are more similar with the state complexity results. For union we have upper and lower bounds that differ, roughly, by a multiplicative constant of 2. We conjecture that worst-case examples for transition complexity of

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union need to be based on two DFAs that for some symbol of the alphabet have all transitions defined. For DFAs of this type we have a tight transition complexity bound for union.

We can note that for union the state complexity results are also different for complete and incomplete DFAs, respectively. When dealing with incomplete DFAs the state complexity of the union of an $n_1$ state and an $n_2$ state language is in the worst case $n_1 \cdot n_2 + n_1 + n_2$.

## 2 Preliminaries

In the following, $\Sigma$ denotes a finite alphabet, $\Sigma^*$ is the set of strings over $\Sigma$ and $\varepsilon$ is the empty string. A language is any subset of $\Sigma^*$. When $\Sigma$ is known, the complement of a language $L \subseteq \Sigma^*$ is denoted as $L^c = \Sigma^* - L$.

A deterministic finite automaton (DFA) is a tuple $A = (\Sigma, Q, q_0, F, \delta)$ where $\Sigma$ is the input alphabet, $Q$ is the finite set of states, $F \subseteq Q$ is the set of accepting states and the transition function $\delta$ is a partial function $Q \times \Sigma \rightarrow Q$. The transition function is extended in the usual way to a (partial) function $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$ and also $\hat{\delta}$ is denoted simply by $\delta$. The language recognized by $A$ is $L(A) = \{ w \in \Sigma^* \mid \delta(q_0, w) \in F \}$.

Unless otherwise mentioned, by a DFA we mean always an incomplete DFA, that is, some transitions may be undefined. For more knowledge in incomplete automata, the reader may refer to [1]. The state complexity of a regular language $L$, $\text{sc}(L)$, is the number of states of the minimal incomplete DFA recognizing $L$.

The Myhill-Nerode right congruence of a regular language $L$ is denoted $\equiv_L$ [17]. The number of equivalence classes of $\equiv_L$ is equal to $\text{sc}(L)$ if the minimal DFA for $L$ has no undefined transitions, $\text{sc}(L) + 1$ otherwise.

If $A = (\Sigma, Q, q_0, F, \delta)$ is as above, the number of transitions of $A$ is the cardinality of the domain of $\delta$, $|\text{dom}(\delta)|$. In the following the number of transitions of $A$ is denoted $\#_\text{tr}(A)$. We note that if $A$ is connected (that is, all states are reachable from the start state), then

$$ |Q| - 1 \leq \#_\text{tr}(A) \leq |\Sigma| \cdot |Q|. \quad (1) $$

For $b \in \Sigma$, the number of transitions labeled by $b$ in $A$ is denoted $\#_\text{tr}(A, b)$.

The transition complexity of a regular language $L$, $\text{tc}(L)$, is the minimal number of transitions of any DFA recognizing $L$. In constructions establishing bounds for the number of transitions it is sometimes useful to restrict consideration to transitions corresponding to a particular alphabet symbol and we introduce the following notation. For $b \in \Sigma$, the $b$-transition complexity of $L$, $\text{tc}_b(L)$ is the minimal number of $b$-transitions of any DFA recognizing $L$. The following lemma establishes that for any $b \in \Sigma$, the state minimal DFA for $L$ has the minimal number of $b$-transitions of any DFA recognizing $L$.

**Lemma 1** Suppose that $A = (\Sigma, Q, q_0, F, \delta)$ is the state minimal DFA for a language $L$. For any $b \in \Sigma$,

$$ \text{tc}_b(L) = \#_\text{tr}(A, b). $$

Since the result is expected, we omit the proof. Lemma[1] means, in particular, that for any given $b \in \Sigma$ we cannot reduce the number of $b$-transitions by introducing additional states or transitions for other input symbols. From Lemma[1] it follows that

$$ \text{tc}(L) = \sum_{b \in \Sigma} \text{tc}_b(L). $$

As a corollary of Lemma[1] we have also:
Corollary 1 Let A be the minimal DFA for a language L. For any $b \in \Sigma$, the number of undefined $b$-transitions in A is $\text{sc}(L) - \text{tc}_b(L)$.

To conclude this section, we give a formal asymptotic definition of the transition complexity of an operation on regular languages. Let $\odot$ be an m-ary operation, $m \geq 1$, on languages and let $f : \mathbb{N}^m \rightarrow \mathbb{N}$. We say that the transition complexity of $\odot$ is f if

(a) for all regular languages $L_1, \ldots, L_m$,

$$\text{tc}(\odot(L_1, \ldots, L_m)) \leq f(\text{tc}(L_1), \ldots, \text{tc}(L_m)),$$

(2)

(b) for any $(n_1, \ldots, n_m) \in \mathbb{N}^m$ there exist $n_i' \geq n_i$, $i = 1, \ldots, m$, and regular languages $L_i$ with $\text{tc}_i(L_i) = n_i'$, $i = 1, \ldots, m$, such that the equality holds in (2).

The above definition requires that there exist worst case examples with arbitrarily large transition complexity matching the upper bound, however, we do not require that matching lower bound examples exist where the argument languages have transition complexity exactly $n_i$ for all positive integers $n_i$, $i = 1, \ldots, m$.

3 Transition complexity of union

We first give bounds for the number of transitions corresponding to a particular input symbol $b \in \Sigma$. These bounds will be used in the next subsection to develop upper bounds for the total number of transitions needed to recognize the union of two languages.

3.1 Number of transitions corresponding to a fixed symbol

The upper bound for the $b$-transition complexity of union of languages $L_1$ and $L_2$ depends also on the number of states of the minimal DFAs for $L_i$, $i = 1, 2$, for which a $b$-transition is not defined. From Corollary 1 we recall that this quantity equals to $\text{sc}(L_i) - \text{tc}_b(L_i)$.

Lemma 2 Suppose that $\Sigma$ has at least two symbols and $L_1, L_2$ are regular languages over $\Sigma$. For any $b \in \Sigma$,

$$\text{tc}_b(L_1 \cup L_2) \leq \text{tc}_b(L_1) \cdot \text{tc}_b(L_2) + \text{tc}_b(L_1)(1 + \text{sc}(L_2) - \text{tc}_b(L_2)) + \text{tc}_b(L_2)(1 + \text{sc}(L_1) - \text{tc}_b(L_1)).$$

(3)

If $n_1, n_2 \geq 2$ are relatively prime, for any $1 \leq k_i < n_i$, $i = 1, 2$, there exist regular languages $L_i$ with $\text{sc}(L_i) = n_i$, and $\text{tc}_b(L_i) = k_i$, $i = 1, 2$, such that the inequality (3) is an equality.

Proof. Consider regular languages $L_i$ and let $A_i = (\Sigma, Q_i, q_{0,i}, F_i, \delta_i)$ be a DFA recognizing $L_i$, $i = 1, 2$. From $A_1$ and $A_2$ we obtain a DFA for $L_1 \cup L_2$ using the well-known cross-product construction modified to the case of incomplete automata. We define

$$Q'_i = \begin{cases} Q_i \cup \{d\} & \text{if } A_i \text{ has some undefined transitions}, \\ Q_i, & \text{otherwise}, \end{cases} \quad i = 1, 2.$$

Now let

$$B = (\Sigma, Q'_1 \times Q'_2, (q_{0,1}, q_{0,2}), (F_1 \times Q'_2) \cup (Q'_1 \times F_2), \gamma),$$

(4)
where for $b \in \Sigma$, $q_i' \in Q_i$, $i = 1, 2$,

$$
\gamma((q'_1, q'_2), b) = \begin{cases} 
(\delta_1(q'_1, b), \delta_2(q'_2, b)) & \text{if } \delta_1(q'_1, b) \text{ and } \delta_2(q'_2, b) \text{ are both defined,} \\
(\delta_1(q'_1, b), d) & \text{if } \delta_1(q'_1, b) \text{ is defined and } \delta_2(q'_2, b) \text{ is undefined,} \\
(d, \delta_2(q'_2, b)) & \text{if } \delta_1(q'_1, b) \text{ is undefined and } \delta_2(q'_2, b) \text{ is defined,} \\
\text{undefined, otherwise.} & 
\end{cases}
$$

(5)

Note that above $\delta_i(d, b)$, $i = 1, 2$, is always undefined for any $b \in \Sigma$.

We note that for $b \in \Sigma$,

$$
\#_{tr}(B, b) = \#_{tr}(A_1, b) \cdot \#_{tr}(A_2, b) + \#_{tr}(A_1, b) + \#_{tr}(A_2, b) + \\
\#_{tr}(A_1, b) \cdot (|Q_2| - \#_{tr}(A_2, b)) + \#_{tr}(A_2, b) \cdot (|Q_1| - \#_{tr}(A_1, b)).
$$

(6)

Here

- $\#_{tr}(A_1, b) \cdot \#_{tr}(A_2, b)$ is the number of transitions in (5) where both $\delta_i(q'_i, b)$, $i = 1, 2$, are defined,
- $\#_{tr}(A_i, b) \cdot (|Q_j| - \#_{tr}(A_j, b))$, $\{i, j\} = \{1, 2\}$, is the number of transitions in (5) where $\delta_i(q'_i, b)$ is defined, $\delta_j(q'_j, b)$ is undefined and $q'_j \in Q_j$, and,
- $\#_{tr}(A_i, b)$ is the number of transitions in (5) where $\delta_i(q'_i, b)$ is defined and $q'_j = d$, $\{i, j\} = \{1, 2\}$.

By choosing $A_i$ as the minimal DFA for $L_i$, $i = 1, 2$, and using Lemma[1] and Corollary[1] the right side of equation (6) gives the right side of inequality (3). Since $B$ recognizes $L_1 \cup L_2$, $tc_\delta(L_1 \cup L_2) \leq \#_{tr}(B, b)$.

We give a construction for the lower bound. Fix $b \in \Sigma$. Let $n_1, n_2 \geq 1$ be relatively prime, $1 \leq k_i < n_i$, $i = 1, 2$, and let $c \in \Sigma$ be a symbol distinct from $b$. Define

$$
C_i = (\Sigma, \{q_0,i, q_1,i, \ldots, q_{n_i-1}, i\}, q_0,i, \{q_0,i\}, \delta_i),
$$

where the transitions defined by $\delta_i$ are as follows:

- $\delta_i(q_{j,i}, c) = q_{j+1,i}$, $j = 0, \ldots, n_i - 2$,
- $\delta_i(q_{n_i-1,i}, c) = q_0,i$,
- $\delta_i(q_{j,i}, b) = q_{j,i}$, $j = 0, \ldots, k_i - 1$.

![Figure 1: The transition diagram of the witness DFA $C_i$ of Lemma[2]](image)

The transition diagram of $C_i$ is shown in Figure[1]. We note that $\#_{tr}(C_i, b) = k_i$ and

$$
L(C_i) = ((b^* c)^k c^{n_i-k})^* b^*, \quad i = 1, 2.
$$

Clearly $C_i$ is minimal and hence $sc(L(C_i)) = n_i$, $i = 1, 2$. In the following we denote $L_i = L(C_i)$, $i = 1, 2$, for short. Choose $m_1, m_2 \in \mathbb{N}$ such that

$$
m_1 \equiv 0 \pmod{n_1}, \quad m_1 \equiv -1 \pmod{n_2}, \quad m_2 \equiv 0 \pmod{n_2}, \quad m_2 \equiv -1 \pmod{n_1}.
$$
Since $n_1$ and $n_2$ are relatively prime, the numbers $m_i$, $i = 1, 2$, exist. The intuitive idea is that we want that the string $c^m_1$ takes the automaton $C_1$ to a state where the $b$-transition is defined and the same string $c^m_2$ takes the automaton $C_2$ to a state where the $b$-transition is not defined. Recall that $k_2 < n_2$ and $\delta_2(q_{n_2-1,2}b)$ is undefined. Also, a similar property holds for $c^{m_2}$ with $C_1$ and $C_2$ interchanged.

We define $S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{c^i | 0 \leq i < n_1 \cdot n_2\}$, $S_2 = \{c^m_1bc^i | 0 \leq i < n_1\}$ and $S_3 = \{c^m_2bc^i | 0 \leq i < n_2\}$. We verify that all strings of $S$ are pairwise in different equivalence classes of the right congruence $\equiv_{L_1 \cup L_2}$. First consider $c^i \in S_1$, $0 \leq i < j < n_1 \cdot n_2$. Since $n_1$ and $n_2$ are relatively prime, there exists $k \in \{1, 2\}$ such that $n_k$ does not divide $j-i$. Denote by $k'$ the element of $\{1, 2\}$ distinct from $k$. Select $z \in \mathbb{N}$ such that $i + z \equiv 0 \pmod{n_k}$ and $j + z \equiv 1 \pmod{n_k'}$. Now $c^i c^z \in L_k$ and $c^j c^z \notin L_1 \cup L_2$. ($j \neq 0 \mod{n_k}$ because $n_k$ does not divide $j-i$.)

Consider $c^m_1 bc^i, c^m_1 bc^j \in S_2$, $0 \leq i < j < n_1$. We note that strings of $S_2$ are not prefixes of any string in $L_2$ and hence for $z \in \mathbb{N}$ such that $m_1 + i + z \equiv 0 \pmod{n_1}$ we have $c^m_1 bc^i c^z \in L_1$ and $c^m_1 bc^j c^z \notin L_1 \cup L_2$. Similarly we see that any two elements of $S_3$ are not in the same $\equiv_{L_1 \cup L_2}$-class.

Next consider $c^i \in S_1$ and $c^m_1 bc^j \in S_2$, $0 \leq i < j < n_1$, $0 \leq j < n_1$. Choose $z \in \mathbb{N}$ such that $i + z \equiv 0 \pmod{n_2}$ and $m_1 + j + z \equiv 1 \pmod{n_1}$. Now $c^i c^z \in L_2$ and, since no string of $S_2$ is a prefix of a string of $L_2$, it follows that $c^m_1 bc^i c^z \notin L_1 \cup L_2$. Completely similarly it follows that a string of $S_1$ is not equivalent with any string of $S_3$.

As the last case consider $c^m_1 bc^i \in S_2$ and $c^m_1 bc^j \in S_3$, $0 \leq i < n_1$, $0 \leq j < n_2$. Choose $z \in \mathbb{N}$ such that $m_1 + i + z \equiv 0 \pmod{n_1}$ and $m_2 + j + z \equiv 1 \pmod{n_2}$. Now $c^m_1 bc^i c^z \in L_1$ and, since no string of $S_3$ is a prefix of a string in $L_1$, $c^m_1 bc^i c^z \notin L_1 \cup L_2$.

Now we are ready to give a lower bound for the $b$-transition complexity of $L_1 \cup L_2$. Let $D$ be the minimal DFA for $L_1 \cup L_2$. By Lemma[1] we know that $\#_D(D, b) = tc_b(L_1 \cup L_2)$.

For $w \in S$, let $q_w$ be the state of $D$ corresponding to $w$. We have verified that $q_w \neq q_w'$ when $w \neq w'$. For $c^i \in S_1$, $0 \leq i < n_1 \cdot n_2$, the string $c^i b$ is a prefix of some string in $L_1 \cup L_2$ if and only if

$$i \equiv j \pmod{n_i}$$

for some $0 \leq j < k_i$ and some $x \in \{1, 2\}$. (7)

The number of integers $0 \leq i < n_1 \cdot n_2$ that satisfy (7) with value $x \in \{1, 2\}$ is equal to $k_x \cdot n_y$, where $\{x, y\} = \{1, 2\}$, and the number of integers $0 \leq i < n_1 \cdot n_2$ that satisfy (7) with both values $x = 1$ and $x = 2$ is $k_1 \cdot k_2$. Thus, the number of states $q_w$, $w \in S_1$ for which the $b$-transition is defined is $k_1 n_2 + k_2 n_1 - k_1 k_2$.

For $c^m_1 bc^i \in S_2$, $0 \leq i < n_1$, the string $c^m_1 b c^i b$ is a prefix of some string of $L_1 \cup L_2$ if and only if

$$i \equiv j \pmod{n_1}$$

for some $0 \leq j < k_1$. This means that the number of states $q_w$, $w \in S_2$, for which the $b$-transition is defined is $k_1$. Similarly, $S_3$ contains $k_2$ strings $w$ such that the $b$-transition is defined for the state $q_w$.

Putting the above together we have seen that

$$\#_D(D, b) \geq k_1 n_2 + k_2 n_1 - k_1 k_2 + k_1 + k_2.$$  

Recalling that $sc(L_i) = n_i$, $tc_b(L_i) = k_i$, $i = 1, 2$, the right side of the above inequality becomes the right side of (3). Hence $tc_b(L_1 \cup L_2)$ is exactly $k_1 n_2 + k_2 n_1 - k_1 k_2 + k_1 + k_2$. ■

Note that the above lower bound construction does not work with $k_i = n_i$, $i \in \{1, 2\}$, because the proof relies on the property that some $b$-transitions of $C_i$ are undefined.

For given relatively prime integers $n_1$ and $n_2$, the construction used in Lemma[2] gives the maximum lower bound for $tc_b(L_1 \cup L_2)$ as a function of $n_i (= sc(L_i))$, $i = 1, 2$, by choosing $tc_b(L_i) = n_i - 1$, $i = 1, 2$. In this case also $sc(L_1) = tc_b(L_1) = 1$.

On the other hand, by choosing $k_1 = k_2 = 1$, Lemma[2] establishes that the $b$-transition complexity of $L_1 \cup L_2$ can be arbitrarily larger than the $b$-transition complexity of the languages $L_1$ and $L_2$. These observations are stated in the below corollary.
Corollary 2 Suppose that the alphabet $\Sigma$ has at least two symbols and let $b \in \Sigma$ be a fixed symbol of $\Sigma$.

(i) For any relatively prime integers $n_1$ and $n_2$, there exist regular languages $L_i$ with $sc(L_i) = n_i$, $tc_b(L_i) = n_i - 1$, $i = 1, 2$, such that
$$tc_b(L_1 \cup L_2) = n_1 n_2 + n_1 + n_2 - 3.$$ 

(ii) For any constants $h_i$, $i = 1, 2$, and $M \geq 1$ there exist regular languages $L_i$, $i = 1, 2$, such $tc_b(L_i) = h_i$, $i = 1, 2$, and $tc_b(L_1 \cup L_2) \geq M$.

For a given $b \in \Sigma$, Corollary 2(i) gives a lower bound for $tc_b(L_1 \cup L_2)$. The construction can be extended for more than one alphabet symbol as indicated in Corollary 3, however, it cannot be extended to all the alphabet symbols.

In the lower bound construction of the proof of Lemma 2, the language $L_i$ was defined by a DFA that has a $c$-cycle of length $n_i$, and where exactly $k_i$ of the states had self-loops on symbol $b$. We can get a simultaneous lower bound for the number of $d$-transitions for any $d \in \Sigma - \{c, b\}$ by adding, in a similar way, self-loops on the symbol $d$.

Corollary 3 Suppose that $\Sigma$ has at least two letters and fix $c \in \Sigma$. Let $n_1$ and $n_2$ be relatively prime and for each $b \in \Sigma - \{c\}$ fix a number $1 \leq k_{i,b} < n_i$, $i = 1, 2$.

Then there exist regular languages $L_1$ and $L_2$ such that
$$sc(L_i) = n_i, \quad tc_b(L_i) = k_{i,b}, \quad b \in \Sigma - \{c\}, \quad i = 1, 2,$$
and the equality holds in (3) for all $b \in \Sigma - \{c\}$.

Finally we note that the proof of Lemma 2 gives also the worst-case bound for the state complexity of union for incomplete DFAs.

Corollary 4 If $sc(L_i) = n_i$, $i = 1, 2$, the language $L_1 \cup L_2$ can be recognized by a DFA with at most $n_1 \cdot n_2 + n_1 + n_2$ states. For relatively prime numbers $n_1, n_2 \geq 2$ the upper bound is tight.

Proof. The upper bound follows from the construction used in the proof of Lemma 2. The upper bound is reached by the automata $A_1$ and $A_2$ used there for the lower bound construction (with any values $1 \leq k_i < n_i$, $i = 1, 2$).

3.2 Total number of transitions

Here we give upper and lower bounds for the transition complexity of union of two regular languages.

With respect to the total number of transitions for all input symbols, the lower bound construction of the proof of Lemma 2 maximizes $tc(L_1 \cup L_2)$ as a function of $tc(L_i)$, $i = 1, 2$, by choosing $k_1 = k_2 = 1$. In this case it can be verified that $tc(L_1 \cup L_2) = tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2) - 2$. However, when the alphabet has at least three symbols we can increase the lower bound by one, roughly as in Corollary 2 by observing that $tc(L_i)$ can be chosen to be zero as long as for each $1 \leq i \leq 2$ there exists $b \in \Sigma$ such that $tc(L_i) \geq 1$. This is verified in the below lemma.

Lemma 3 Let $\Sigma = \{a, b, c\}$. For any relatively prime numbers $n_1$ and $n_2$ there exist regular languages $L_i \subseteq \Sigma^*$, such that $tc(L_i) = n_i + 1$, $i = 1, 2$, and
$$tc(L_1 \cup L_2) = tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2) - 1.$$ (8)
Lemma 4 can be proved with a construction similar to the construction of the proof of Lemma 3 and witness languages \(L_1 = a^*(a^*c^{n_1})^*\) and \(L_2 = b^*(b^*c^{n_2})^*\). Due to the page limitation, we omit the proof.

Next we give an upper bound for transition complexity of union. In the following lemma let \(A_1\) and \(A_2\) be arbitrary DFAs and \(B_{A_1,A_2}\) denotes the DFA constructed to recognize \(L(A_1) \cup L(A_2)\) as in the proof of Lemma 2 (The definition of \(B_{A_1,A_2}\) is given in equation (4).)

**Lemma 4** If \(A_i\) is connected, \(i = 1, 2\), then

\[
\#_t(B_{A_1,A_2}) \leq 2 \cdot (\#_t(A_1) \cdot \#_t(A_2) + \#_t(A_1) + \#_t(A_2)).
\]

**Proof.** We use induction on \(\#_t(A_1) + \#_t(A_2)\). First consider the case where \(\#_t(A_1) = \#_t(A_2) = 0\). In this case also \(B_{A_1,A_2}\) has no transitions.

Now assume that \(\#_t(A_1) + \#_t(A_2) = m\), and the claim holds when the total number of transitions is at most \(m - 1\). Without loss of generality, \(\#_t(A_1) \geq 1\), and let \(A'_1\) be a connected DFA obtained from \(A_1\) by deleting one transition and possible states that became disconnected as a result. We can choose the transition to be deleted in a way that at most one state becomes disconnected.

By the inductive hypothesis,

\[
\#_t(B_{A'_1,A_2}) \leq 2 \cdot (\#_t(A'_1) \cdot \#_t(A_2) + \#_t(A'_1) + \#_t(A_2)).
\]

The DFA \(A_1\) is obtained by adding one transition \(t_1\) and at most one state \(q_1\) to \(A'_1\). Let \(Q_2\) be the set of states of \(A_2\). The construction of \(B_{A_1,A_2}\) is the same as the construction of \(B_{A'_1,A_2}\), except that

(i) we add for \(t_1\) a new transition corresponding to each state of \(Q_2\) and a new transition corresponding to the dead state \(d\) in the second component, and,

(ii) we add a new transition corresponding to \(q_1\) and each transition of \(A_2\).

Thus,

\[
\#_t(B_{A_1,A_2}) \leq \#_t(B_{A'_1,A_2}) + |Q_2| + 1 + \#_t(A_2) \leq \#_t(B_{A'_1,A_2}) + 2(\#_t(A_2) + 1).
\]

The last inequality relies on (1) and the fact that \(A_2\) is connected. Thus using (9) and \(\#_t(A'_1) = \#_t(A_1) - 1\) we get

\[
\#_t(B_{A_1,A_2}) \leq 2((\#_t(A_1) - 1) \cdot \#_t(A_2) + \#_t(A_1) - 1 + \#_t(A_2)) + 2(\#_t(A_2) + 1).
\]

With arithmetic simplification this gives the claim for \(A_1\) and \(A_2\). □

From Lemma 3 and Lemma 4 we get now:

**Theorem 1** For all regular languages \(L_i\), \(i = 1, 2\),

\[
tc(L_1 \cup L_2) \leq 2 \cdot (tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2)).
\]

For any relatively prime numbers \(n_1\) and \(n_2\) there exist regular languages \(L_i\) over a three-letter alphabet, \(tc(L_i) = n_i + 1\), \(i = 1, 2\), such that

\[
tc(L_1 \cup L_2) = tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2) - 1.
\]
The upper and lower bound of Theorem 1 differ, roughly, by a multiplicative constant of two. We believe that the upper bound could be made lower (when $tc(L_i) \geq 2$, $i = 1, 2$), but do not have a proof for this in the general case.

The constructions of Lemma 2 and Lemma 3 use languages $L_i$, $i = 1, 2$, such that for one particular alphabet symbol $c \in \Sigma$, the minimal DFA for $L_i$, $i = 1, 2$, has all $c$-transitions defined. It seems likely that worst-case examples need to be based on cycles of transitions on a particular alphabet symbol, in order to reach the maximal state complexity blow-up with as small number of transitions as possible. Below we establish that for this type of constructions the right side of (8) is also an upper bound for $tc$.

**Lemma 5** Let $L_1$ and $L_2$ be regular languages over $\Sigma$. If there exists $c \in \Sigma$ such that in the minimal DFA for $L_i$, $i = 1, 2$, all $c$-transitions are defined, then

$$tc(L_1 \cup L_2) \leq tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2) - 1. \quad (10)$$

The idea of the proof of Lemma 5 is similar to that of the proof of Lemma 4. The crucial difference is that we have one symbol for which all transitions are defined and the inductive argument is with respect to the number of the remaining transitions. Thus, in the inductive step when replacing $A_1$ with a DFA $A'_1$ with one fewer transition, we know that $A'_1$ is connected and the inductive step does not need to add transitions corresponding to a state that would be added to $A'_1$.

Lemma 5 establishes that the bound given by Lemma 3 cannot be exceeded by any construction that is based on automata that both have a complete cycle defined on the same alphabet symbol. Usually it is easier to establish upper bounds for descriptional complexity measures, and finding matching lower bounds is a relatively harder question. In the case of transition complexity of union we have a lower bound and only indirect evidence, via Lemma 5, that this lower bound cannot be exceeded.

**Conjecture 1** For any regular languages $L_1$ and $L_2$ where $tc(L_i) \geq 2$, $i = 1, 2$,

$$tc(L_1 \cup L_2) \leq tc(L_1) \cdot tc(L_2) + tc(L_1) + tc(L_2).$$

Note that the conjecture does not hold for small values of $tc(L_i)$, $i = 1, 2$. For example, $tc(\{\varepsilon\}) = 0$, $tc(a^* b^{m-1}) = m$, but $tc(a^* b^{m-1} + \varepsilon) = m + 2$.

### 3.3 Transition complexity of union of unary languages

For languages over a unary alphabet, the transition complexity of union of incomplete DFAs turns out to coincide with the known bound for state complexity of union of complete DFAs. However, the proof is slightly different.

Recall that a DFA with a unary input alphabet always has a “tail” possibly followed by a “loop” [2, 11]. Note that an incomplete DFA recognizing a finite language does not need to have a loop.

**Theorem 2** Let $L_1, L_2$ be unary languages over an alphabet $\{b\}$. If $tc(L_i) \geq 2$, $i = 1, 2$, then

$$tc(L_1 \cup L_2) \leq tc(L_1) \cdot tc(L_2). \quad (11)$$

For any relatively prime $n_1 \geq 3$, $n_2 \geq 2$, there exist regular languages $L_i \subseteq \{b\}^*$, $tc(L_i) = n_i$, $i = 1, 2$, such that (11) is an equality.

This theorem can be proved by separately considering the cases where $L_1$ and $L_2$ are finite or infinite. The detailed proof is omitted. Note that the upper bound of Theorem 2 does not hold when $n_1 < 3$ or $n_2 < 2$. For example, $tc(b) = 1$, $tc((b^*)^*) = n$ and $tc(b \cup (b^*)^*) = n + 1$ when $n \geq 2$. 


4 Intersection and complementation

As can, perhaps, be expected the worst-case transition complexity bounds for intersection are the same as the corresponding state complexity results based on complete DFAs. When dealing with intersection, worst-case examples can be constructed using a unary alphabet and a complete DFA. On the other hand, state complexity of complementation of complete DFAs is the identity function whereas the bound for transition complexity of complementation is significantly different.

**Proposition 1** For any regular languages $L_i$, $i = 1, 2$,

$$\text{tc}(L_1 \cap L_2) \leq \text{tc}(L_1) \cdot \text{tc}(L_2).$$  \hfill (12)

Always when $n_1$ and $n_2$ are relatively prime there exist regular languages $L_i$, $i = 1, 2$, such that equality holds in (12).

**Proof.** Let $A_i = (\Sigma, Q_i, q_{0,i}, F_i, \delta_i)$ be a DFA recognizing $L_i$, $i = 1, 2$. We define

$$B = (\Sigma, Q_1 \times Q_2, (q_{0,1}, q_{0,2}), F_1 \times F_2, \gamma),$$

where for $b \in \Sigma$, $q_i \in Q_i$, $i = 1, 2$,

$$\gamma((q_1, q_2), b) = \begin{cases} (\delta_1(q_1, b), \delta_2(q_2, b)) & \text{if } \delta_1(q_1, b) \text{ and } \delta_2(q_2, b) \text{ are both defined}, \\ \text{undefined, otherwise.} & \end{cases}$$

Clearly $B$ recognizes $L_1 \cap L_2$ and $\#_u(B) = \#_u(A_1) \cdot \#_u(A_2)$.

The lower bound follows from the observation that if $n_1$ and $n_2$ are relatively prime and $A_i$ is the minimal DFA for $(b^{n_i})^*$, $i = 1, 2$, then the DFA $B$ in (13) is also minimal and $\#_u(B) = n_1n_2$. \hfill $\blacksquare$

The proof of Proposition 1 gives for $b \in \Sigma$ the same tight bound for the number of $b$-transitions needed to recognize the intersection of given languages.

**Corollary 5** For any regular languages $L_i$ over $\Sigma$, $i = 1, 2$, and $b \in \Sigma$,

$$\text{tc}_b(L_1 \cap L_2) \leq \text{tc}_b(L_1) \cdot \text{tc}_b(L_2).$$ \hfill (14)

For relatively prime integers $n_1$ and $n_2$ there exist regular languages $L_i$ with $\text{tc}_b(L_i) = n_i$, $i = 1, 2$, such that equality holds in (14).

To conclude this section we consider complementation. If $A$ is an $n$-state DFA, a DFA to recognize the complement of $L(A)$ needs at most $n + 1$ states. The worst-case bound for transition complexity of complementation is significantly different.

**Proposition 2** Let $L$ be a regular language over an alphabet $\Sigma$. The transition complexity of the complement of $L$ is upper bounded by

$$\text{tc}(L^c) \leq |\Sigma| \cdot (\text{tc}(L) + 2).$$

The bound is tight, that is, for any $n \geq 1$ there exists a regular language $L$ with $\text{tc}(L) = n$ such that in the above inequality the equality holds.
**Proof.** Let $A = (\Sigma, Q, q_0, F, \delta)$ be a DFA for $L$. The complement of $L$ is recognized by the DFA

$$B = (\Sigma, Q \cup \{d\}, q_0, (Q - F) \cup \{d\}, \gamma),$$

where for $b \in \Sigma$

$$\gamma(p, b) = \begin{cases} 
\delta(p, b) & \text{if } p \in Q \text{ and } \delta(p, b) \text{ is defined}, \\
d & \text{if } \delta(p, b) \text{ is undefined.}
\end{cases}$$

Note that when $p = d$, $\delta(p, b)$ is undefined for all $b \in \Sigma$.

The DFA $B$ has $(|Q| + 1) \cdot |\Sigma|$ transitions. If $A$ is minimal, $A$ has at least $|Q| - 1$ transitions, and this gives the upper bound.

We establish the lower bound. Choose $b \in \Sigma$ and for $n \geq 1$ define $L_n = \{b^n\}$. Now $tc(L_n) = n$. Denote $S = \{\varepsilon, b, \ldots, b^{n+1}\}$. All strings of $S$ are pairwise inequivalent with respect to the right congruence $\equiv_{L_n^c}$, and

$$(\forall x \in S)(\forall c \in \Sigma)(\exists y \in \Sigma^*) xcy \in L_n^c.$$ 

This means that the minimal DFA for $L_n^c$ has (at least) $n + 2$ states for which all transitions are defined. Thus, $tc(L_n^c) \geq |\Sigma| \cdot (n + 2).$ $\blacksquare$

From the construction of the proof of Proposition 2 we see that if $\Sigma$ contains at least two symbols then for $a \in \Sigma$ and any $M \geq 1$ there exists a regular language $L$ over $\Sigma$ such that $tc_a(L) = 0$ and $tc_a(L_c^c) \geq M$.

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