Linearized gravity with matter time

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Received 11 January 2016, revised 22 February 2016
Accepted for publication 8 March 2016
Published 19 April 2016

Abstract
We study general relativity with pressureless dust in the canonical formulation, with the dust field chosen as a matter time gauge. The resulting theory has three physical degrees of freedom in the metric field. The linearized canonical theory reveals two graviton modes and a scalar mode. We find that the graviton modes remain Lorentz covariant despite the time gauge, and that the scalar mode is ultralocal. We also discuss a modification of the theory to include a parameter in the Hamiltonian that is analogous to that in Horava–Lifshitz models. In this case the scalar mode is no longer ultralocal and it acquires a propagation speed that is dependent on the deformation parameter.

Keywords: canonical gravity, linearized theory, quantum gravity

1. Introduction

In the Hamiltonian formulation of general relativity (GR) there is an understanding of how graviton modes arise. This parallels the more familiar covariant analysis of linearized theory. The analysis starts with the expansion of the canonical Arnowitt–Deser–Misner (ADM) phase space variables and the constraints of GR around the flat spacetime solution, followed by imposition of the canonical transverse traceless (TT) gauge condition [1]. The solution of the constraints in this gauge gives the two (unconstrained) graviton degrees of freedom. (See e.g. [2] for a pedagogical review of ADM analysis.)

ADM analysis of the TT gauge is for vacuum GR. If there is matter coupling, it is possible to set up a linearized canonical theory with matter perturbations, as is done, for example, in the theory of cosmological perturbations [3]. The standard analysis, however, still uses the TT gauge, which, taken together, constitutes four conditions that fix four coordinates. Importantly, these gauge choices are conditions only on the gravitational phase space variables; there are no restrictions on the matter sector as far as the fixing of general coordinate invariance is concerned.
Although the TT gauge is very useful and meshes well with the covariant analyses of gravitational perturbation theory, it is only one choice. In a Hamiltonian setting with GR coupled to matter, it is clearly possible to use matter degrees of freedom in making coordinate gauge choices. This is not usually done because the interpretation of gravitational waves as spin 2 fields on a background spacetime is lost. Furthermore, it is apparent that if matter degrees of freedom (‘matter reference systems’) are used to fix spacetime diffeomorphism freedom, then the gauge fixed theory has additional local degrees of freedom in the geometry sector, which makes it harder to interpret physically.

We nevertheless consider this possibility in the specific setting of GR coupled to pressureless dust, and any other matter field. This is a special case of the matter reference system used by Brown and Kuchar [4], who used a more general four-component dust field to set all four coordinate conditions. Our analysis uses the dust time gauge, where the dust scalar is set as the time coordinate. We develop this idea in part because it leads to some novel results that have a bearing on discussions of Lorentz violation and quantum gravity; after a complete gauge fixing, we find that the scalar field degree of freedom is manifest as a scalar mode in the spatial metric, and the interpretation of gravitational waves as spin 2 fields on the background is preserved.

There is recent related work in two directions. Firstly, there are the Einstein-aether models [5], where a dynamical vector field of timelike norm is added to the GR action. A linearized analysis of these models has been performed, with the result that the graviton modes decouple from the aether modes [6]. Secondly, there is the so-called mimetic gravity model [7], where the conformal mode of the spacetime metric is encoded as a scalar field with an arbitrary potential. This extra mode in the gravitational field represents self-interacting matter with arbitrary potential [8, 9], and has been used to model inflationary and bouncing cosmologies. When the potential vanishes, the canonical theory is equivalent to GR coupled to pressureless dust, which is studied as a quantum gravity model in [10]. Given these analogies, it is potentially useful to consider this paper in the larger context of Einstein-aether [11] and mimetic gravity theories.

In the next section we consider the GR+dust theory and review the use of the dust time gauge in the ADM canonical framework [10]. In section 3 we present an analysis of the linearized theory about flat spacetime in the Hamiltonian theory. This section contains the main result: of the three physical degrees of freedom in the metric, two are the graviton modes, and the third degree is ultralocal. We also comment on an extension of the theory to include the Horava–Lifshitz (HL) parameter [12, 13] in the physical Hamiltonian. (For a Hamiltonian formulation of HL gravity and a review see [14, 15].) We find that if this parameter is different from its GR value, the two graviton mode equations are unaffected, but in the third the propagation speed is determined by this parameter. We conclude in section 4 with a summary and discussion of the role of the dust time gauge for quantization.

2. Action and Hamiltonian theory

We consider GR coupled to pressureless dust and any other arbitrary matter field,

\[
S = \frac{1}{2\pi} \int d^4x \sqrt{-g} R - \frac{1}{4\pi} \int d^4x \sqrt{-g} \ m(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1) + \int d^4x \ L_m(\chi). \tag{1}
\]

The second term is the dust action, and the last is an arbitrary matter Lagrangian. Variation with respect to \( m \) gives the condition that the dust field \( \phi \) has a timelike gradient.
The ADM canonical theory obtained from this action is

$$S = \frac{1}{2\pi} \int dt \, d^3x \left( \pi_{ab} \dot{q}_{ab} + p_\phi \dot{\phi} + p_\chi \dot{\chi} - N \mathcal{H} - N^a \mathcal{C}_a \right),$$

where the pairs \((q_{ab}, \pi_{ab})\) and \((\phi, p_\phi)\) are respectively the phase space variables of gravity and dust. The matter fields are symbolically denoted by \((\chi, p_\chi)\), although the number of fields and their tensorial structures will depend upon the choice of the matter Lagrangian. The lapse and shift functions, \(N\) and \(N^a\), are the coefficients of the Hamiltonian and diffeomorphism constraints

$$\mathcal{H} = \mathcal{H}^G + \mathcal{H}^D + \mathcal{H}^M,$$

$$\mathcal{C}_a = C_a^G + C_a^D + C_a^M$$

$$= -2D_b \pi_{ab}^b + p_\phi \partial_\phi \phi + c_a^M,$$

where

$$\mathcal{H}^G = -\sqrt{q} R^{(3)} + \frac{1}{\sqrt{q}} \left( \pi_{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right),$$

$$\mathcal{H}^D = \frac{1}{2} \left( \frac{p_\phi^2}{m \sqrt{q}} + m \sqrt{q} (q_{ab} \partial_a \phi \partial_b \phi + 1) \right).$$

The trace of the gravitational momentum is \(\pi = q_{ab} \pi^{ab}\), \(R^{(3)}\) is the scalar curvature of the spatial hypersurfaces, \(D_b\) is the metric compatible covariant derivative associated with \(q_{ab}\), and \(\mathcal{H}^M\) is the (non-dust) matter Hamiltonian.

The momentum conjugate to the field \(m\) is zero, since it appears as a Lagrange multiplier in the covariant action. At this point one could enlarge the phase space to treat \(m\) and its conjugate momentum as independent degrees of freedom, subsequently eliminating them by gauge fixing. However, it is more straightforward to vary the term \(\mathcal{H}^D\) in the canonical action with respect to \(m\) and use the resulting equation of motion:

$$m = \pm \frac{p_\phi}{\sqrt{q} (q_{ab} \partial_a \phi \partial_b \phi + 1)},$$

This can then be substituted back into \(\mathcal{H}^D\) to give

$$\mathcal{H}^D = \pm p_\phi \sqrt{q} (q_{ab} \partial_a \phi \partial_b \phi + 1),$$

leaving a canonical action for \((q_{ab}, \pi^{ab}), (\phi, p_\phi)\) and the (non-dust) matter phase space variables. It is readily verified that the constraints remain first class with this elimination of \(m\). We will see in the gauge fixing below how the sign is selected.

### 2.1. Dust time gauge

We now partially reduce the theory by fixing a time gauge and solving the Hamiltonian constraint to obtain a physical Hamiltonian. We use the dust time gauge [10, 16], which equates the physical time with level values of the scalar field,

$$\lambda \equiv \phi - t \approx 0.$$
Substituting $\phi = t$ into (7) gives $p_0 = \pm m \sqrt{q}$, which by the last relation leads to $N = \pm 1$. The sign of the lapse function determines whether the evolution is forward ($N = +1$) or backward ($N = -1$) in time. We select the positive sign, which fixes the above ambiguity in the Hamiltonian constraint, yielding $H^D = + p_0$.

With these results, solving the Hamiltonian constraint gives

$$p_0 = -(H^G + H^M).$$

Substituting this and the gauge condition (9) into (2) gives the gauge fixed action

$$S_{GF} = \frac{1}{2\pi} \int dt \, d^3x \, \left[ \dot{\bar{\chi}} \dot{q}_{ab} + p_\chi \dot{\bar{\chi}} - \left( H^G + H^M \right) - N^a \left( \dot{C}^G_a + \dot{C}^M_a \right) \right],$$

up to surface terms, which do not concern us here. Thus we see that in the dust time gauge, the sum of the gravitational and matter parts of the Hamiltonian constraint becomes the physical Hamiltonian, and the diffeomorphism constraint reduces to that with only the gravity and matter ($q_{ab}$) contributions. The ‘vacuum’ theory, with $\chi = p_\chi = 0$, has six configuration degrees of freedom in $q_{ab}$, subject to the diffeomorphism constraint, giving three local degrees of freedom. This is the action that we study in the remainder of this paper.

The corresponding spacetime metric is

$$ds^2 = -dt^2 + \left( dx^a + N^a dt \right) \left( dx^b + N^b dt \right) q_{ab}.$$ 

### 2.2. Deformation of the Hamiltonian

So far we have described a matter time gauge fixing of canonical GR, which results in the action (12). Taking the latter as a starting point for defining the theory, we introduce a deformation of the gravitational Hamiltonian

$$H^G_\alpha := -\sqrt{q} R^{(3)} + \frac{1}{\sqrt{q}} (\bar{\pi}^{ab} \bar{\pi}_{ab} - \alpha \bar{\pi}^2),$$

motivated by the HL models. In their original formulation, these models are also constructed from a first order action made from the spatial metric and extrinsic curvature; there is no covariant second order action as the starting point. These models also have higher derivative 3-metric self-interactions through terms such as $R_{ab}R^{ab}$, as well as a deformation of the ADM kinetic term. The generalization we consider, however, only introduces the latter through a parameter $\alpha$:

$$S_\alpha = \frac{1}{2\pi} \int dt \, d^3x \, \left[ \bar{\pi}^{ab} \dot{q}_{ab} + p_\chi \dot{\bar{\chi}} - \left( H^G_\alpha + H^M \right) - N^a \left( \dot{C}^G_a + \dot{C}^M_a \right) \right].$$

Although this generalization is motivated by HL theory, we emphasize that it is a different theory in a key aspect. There is no Hamiltonian constraint. Rather, there is a physical Hamiltonian that now has an additional coupling constant $\alpha$. The only constraint algebra is that of the spatial diffeomorphism constraints, which closes in the usual manner, and the physical Hamiltonian density transforms via the bracket.
\{ C^G(N), \mathcal{H}^G_a(x), \} = \mathcal{L}_N \mathcal{H}^G_a(x), \quad (16)

where $\mathcal{L}_N$ denotes the Lie derivative with respect to the vector field $N^a$.

In the following analysis we work with this deformation. We see that for its GR value ($\alpha = 1/2$), the additional degree of freedom in the metric is ultralocal, whereas for all other values it is a propagating scalar.

3. Linearized theory

For the remainder of this paper we assume that $\chi = p_\chi = \mathcal{H}^M = \mathcal{C}_a^M = 0$, that is we consider the action (12) without matter. It is easy to check that Minkowski spacetime, $q_{ab} = \delta_{ab}$, $\pi^{ab} = 0 = N^a$, is a solution of equations of motion in the dust time gauge. We linearize the theory on this background by writing

$$q_{ab}(x, t) = \delta_{ab} + h_{ab}(x, t), \quad \pi^{ab} = 0 + \rho^{ab}(x, t), \quad N^a = 0 + \xi^a(x, t). \quad (17)$$

It is convenient to work in 3–momentum space by expanding the perturbations $h_{ab}, \rho^{ab}, \xi^a$ in modes of the flat space Laplacian (plane waves) as

$$h_{ab}(x, t) = \frac{1}{(2\pi)^3} \int d^3k \ e^{ikx} \tilde{h}_{ab}(k, t),$$
$$\rho^{ab}(x, t) = \frac{1}{(2\pi)^3} \int d^3k \ e^{ikx} \tilde{\rho}^{ab}(k, t)$$
$$\xi^a(x, t) = \frac{1}{(2\pi)^3} \int d^3k \ e^{ikx} \tilde{\xi}^a(k, t). \quad (18)$$

This allows us to write the Hamiltonian and equations of motion in Fourier space. The background solution $\delta_{ab}$ and $k^a$ may be used to define an orthonormal basis of symmetric $3 \times 3$ matrices $M^I$ so that the perturbations can be decomposed as

$$\tilde{h}_{ab} = h_I(k, t)M^I_{ab}, \quad \tilde{\rho}^{ab} = \rho^I(k, t)M^I_{ab}, \quad I = 1, 2 \ldots 6. \quad (19)$$

As we see below, the coefficients $(h_I, \rho^I)$ provide a natural separation of the perturbations into scalar, vector and tensor modes. Furthermore, if the chosen basis is static and orthonormal in the inner product

$$\text{Tr}(M^I M^J) = \delta^{IJ}, \quad (20)$$

the symplectic form decomposes as

$$\int d^3k \ \tilde{\rho}^{ab} \tilde{h}_{ab} = \int d^3k \ p^I h_I. \quad (21)$$

This identifies the six canonically conjugate degrees of freedom $(h_I(k, t), p^I(k, t))$.

A basis that fulfills these requirements is obtained using an orthonormal basis of vectors

$$\tilde{k}^a = k^a / |k|, \quad e_1^a, \quad e_2^a, \quad (22)$$

where the latter pair span the plane orthogonal to $k^a$. By considering rotations $J_\sigma$ by angle $\sigma$ about the $k^a$-axis, one obtains a definition of ‘helicity’ for the eigenvectors of these rotations; see e.g. appendix A.2.1 in [17]. The eigenvectors of $J_\sigma$ are the linear combinations $e_\pm = (e_1^a \pm ie_2^a) / \sqrt{2}$. These satisfy $J_\sigma e_\pm = e^{\pm i\sigma} e_\pm$, $\delta_{ab} e_a^e e_b^e = 0$ and $\delta_{ab} e_a^e e_b^e = 1$. The matrices
\[ \delta_{ab}, \ \hat{k}^a k^b, \ \epsilon_4^{(a} k^{b)} , \ \epsilon_5^{(a} k^{b)}. \]  

(23)

are the eigentensors of \( J_\sigma \): the first two are rotationally invariant and so are (helicity 0) scalars, the next pair are (helicity ±1) vectors, and the last pair are (helicity ±2) tensors.

A basis \( M^i \) with the above properties may be made as a linear combination of these elements. We choose the scalar, tensor, and vector bases, respectively, as

\[
\begin{align*}
M_1^{ab} &= \frac{1}{\sqrt{3}} \delta^{ab}, \quad M_2^{ab} = \frac{\sqrt{3}}{2} \left( \hat{k}^a k^b - \frac{1}{3} \delta^{ab} \right), \\
M_3^{ab} &= \frac{1}{\sqrt{2}} (\epsilon^a e^b - \epsilon^b e^a), \quad M_4^{ab} = \frac{1}{\sqrt{2}} (\epsilon^a e^b + \epsilon^b e^a), \\
M_5^{ab} &= i (\epsilon^a k^b - \epsilon^b k^a), \quad M_6^{ab} = (\epsilon^a k^b + \epsilon^b k^a).
\end{align*}
\]

(24)

(25)

(26)

The subset \( M_I, I = 1 \cdots 6 \) are trace free, \( M_I^a \delta_{ab} = 0 \), and satisfy the traverse properties \( k_a M_I^{ab} = k_b M_I^{ab} = 0 \) and \( k_a k_b M_I^{ab} = k_a k_b M_I^{ab} = 0 \). (We note that the tensors of definite helicity in (23) have zero norm in the inner product (20) and lead to a degenerate reduction of the symplectic form. For this reason the above linear combinations of helicity tensors are necessary as basis elements in order to derive canonical equations of motion.)

Our goal now is to write the linearized canonical Einstein equations in the dust time gauge in \( k \)-space, fix three phase space gauge conditions, and solve the spatial diffeomorphism constraint. This will identify the three local physical degrees of freedom. As we will see, two of these turn out to be the usual polarizations of the graviton, and the third is the manifestation in the metric of the dust degree of freedom. The details of these steps follow.

3.1. Linearized equations of motion

The linearized equations about the flat background solution are

\[
\begin{align*}
\dot{h}_{ab} &= 2 (p_{ab} - \alpha \delta_{ab} p) + \mathcal{L}_x \delta_{ab}, \\
\dot{\rho}^{ab} &= -\partial^c \partial^{(a} h_{c)} + \frac{1}{2} \partial^c \partial_{cd} h^{ab} + \frac{1}{2} \delta^{ab} (\partial^c \partial_{cd} h - \partial^c \partial_c h),
\end{align*}
\]

(27)

which in \( k \)-space are

\[
\begin{align*}
\dot{\tilde{h}}_{ab} &= 2 (\tilde{p}_{ab} - \alpha \delta_{ab} \tilde{p}) + 2i k (\tilde{\xi}_b), \\
\dot{\tilde{\rho}}^{ab} &= k \delta^{(b} h_a) - \frac{1}{2} k \delta^{ab} h - \frac{1}{2} \delta^{ab} (k \delta^{bc} h_{cd} - k \delta_{cd} h). 
\end{align*}
\]

(28)

From these, the equations for the phase space pairs \((h_I, p^I)\) are obtained by projecting onto each basis element \( M^I \). The shift vector can be decomposed as

\[
\tilde{\xi}^a = \xi_1 \hat{k}^a + \xi_2 e_1^a + \xi_2 e_2^a. 
\]

(29)

The scalar mode equations are

\[
\begin{align*}
\dot{h}_1 &= 2(1 - 3\alpha) p_1 + \frac{2i}{\sqrt{3}} |k| \xi_1, \\
\dot{h}_2 &= 2p_2 + 2i \frac{2}{\sqrt{3}} |k| \xi_1.
\end{align*}
\]

(30)

(31)
\begin{equation}
\dot{p}_1 = \frac{1}{3} |k|^2 h_1 - \frac{1}{3\sqrt{2}} |k|^2 h_2, \tag{32}
\end{equation}

\begin{equation}
\dot{p}_2 = -\frac{1}{3\sqrt{2}} |k|^2 h_1 + \frac{1}{6} |k|^2 h_2. \tag{33}
\end{equation}

The tensor mode equations are
\begin{equation}
\dot{h}_3 = 2p_3, \quad \dot{p}_3 = -\frac{1}{2} |k|^2 h_3, \tag{34}
\end{equation}

\begin{equation}
\dot{h}_4 = 2p_4, \quad \dot{p}_4 = -\frac{1}{2} |k|^2 h_4, \tag{35}
\end{equation}

and the vector mode equations are
\begin{equation}
\dot{h}_5 = 2p_5 + 1i\sqrt{2} |k|\xi_2, \quad \dot{p}_5 = 0, \tag{36}
\end{equation}

\begin{equation}
\dot{h}_6 = 2p_6 + i\sqrt{2} |k|\xi_1, \quad \dot{p}_6 = 0. \tag{37}
\end{equation}

These equations are supplemented by the linearized diffeomorphism constraint, which we discuss next.

### 3.2. Diffeomorphism constraint

The position space diffeomorphism constraint \( D_a \tilde{\pi}^{ab} = 0 \) linearizes about the flat background to \( \partial_a p^{ab} = 0 \). In \( k \)-space this is
\begin{equation}
k_a \rho^{ab} = k_a p^l (k, t) M_j^{ab} = 0 \implies \left( \frac{1}{\sqrt{3}} p_1 + \frac{2}{\sqrt{3}} p_2 \right) k^b + \frac{|k|}{\sqrt{2}} (p_5 e_2^b + p_6 e_1^b) = 0. \tag{38}\end{equation}

It is evident that this constraint has transverse and longitudinal components, and furthermore, that a partial solution of this constraint must come from setting \( p_5 = p_6 = 0 \), since these are the only coefficients in the transverse directions \( e_1^a \) and \( e_2^a \).

More systematically, the vector modes are eliminated in three steps: (i) imposing the gauge conditions
\begin{equation}
h_5 = 0, \quad h_6 = 0, \tag{39}
\end{equation}

which are second class with the linearized diffeomorphism constraint, (ii) solving the transverse component of the diffeomorphism constraint by setting \( p_5 = p_6 = 0 \), and (iii) using the conditions that the gauge be dynamically preserved to fix the transverse components of the shift perturbation \( (29) \),
\begin{equation}
\dot{h}_5 = i\sqrt{2} |k|\xi_2 = 0, \tag{40}
\end{equation}

\begin{equation}
\dot{h}_6 = i\sqrt{2} |k|\xi_1 = 0. \tag{41}
\end{equation}

This fixes \( \xi_1 = \xi_2 = 0 \). The longitudinal component of the shift \( \dot{\xi}^a = \xi^b \dot{k}_b^a \) remains undetermined at this stage.

This leaves the scalar and tensor mode equations for \((h_I, p_I)\), \( I = 1 \cdots 4 \), and the longitudinal part of the diffeomorphism constraint.
This remaining constraint is on the two scalar degrees of freedom. After one more gauge fixing, the last of the three necessary to fully gauge fix the theory, only one scalar mode and the transverse traceless graviton modes \((h_3, p_3)\) and \((h_4, p_4)\) remain. The former may be chosen as the canonical pair \((h_1, p_1)\) or \((h_2, p_2)\).

Let us consider the gauge \(h_1 = 0\) and solve the remaining diffeomorphism constraint, giving \(p_2 = -p_1 / \sqrt{2}\). The corresponding evolution equation gives

\[
\dot{q} = 2p_2 + 2|k|i \frac{\sqrt{2}}{3} \xi = 0 \implies \xi = -i \frac{\sqrt{3}}{2|k|} p_1,
\]

and the \(p_1\) and \(p_2\) equations become identical. The remaining scalar mode equations reduce, using the above expression for \(\xi\), to

\[
\dot{p}_1 = 3(1 - 2\alpha)p_1, \quad \dot{p}_1 = \frac{1}{3} |k|^2 h_1,
\]

or equivalently,

\[
\dot{h}_1 = (1 - 2\alpha)|k|^2 h_1.
\]

These are equivalent to the position space wave equation

\[
\dot{h}_1 = (2\alpha - 1) \delta^{ab} \partial_a \partial_b h_1,
\]

so the propagation speed is \(v = \sqrt{2\alpha - 1}\). It is therefore evident that for the GR value \(\alpha = 1/2\), this scalar mode is ultralocal: there are no spatial derivatives in the equation, so \(h_1\) evolves independently at each space point. For \(\alpha > 1/2\) the propagation speed varies from subluminal to superluminal, whereas for \(\alpha < 1/2\) the equation becomes a 4d Laplacian. Had we chosen the gauge \(h_2 = 0\) (instead of \(h_2 = 0\)), a similar analysis would reveal the same wave equation for the scalar mode \(h_2\).

Lastly, we note that the graviton (TT) modes \((34)–(35)\) are independent of \(\alpha\) and satisfy the expected light speed wave equation

\[
\dot{h}_I = -|k|^2 h_I, \quad I = 3, 4,
\]

despite the dust time gauge fixing, which remarkably does not effect Lorentz invariance in the linearized theory. This demonstrates that ‘solving the problem of time’ by adding a dust field is compatible with Lorentz invariance, and that the dust time gauge leads to no pathologies.

### 3.3. Dust potential

We note in passing that it is possible to include a potential for the dust field in the starting theory [8]. This modifies the dust Lagrangian to

\[
S^D = \int d^4x \sqrt{-g} \left[ m (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1) - V(\phi) \right],
\]

and the dust contribution to the Hamiltonian constraint becomes

\[
\mathcal{H}^D = \frac{1}{2} \left( \frac{p^2}{m \sqrt{q}} + m \sqrt{q} (q^{ab} \partial_a \phi \partial_b \phi + 1) \right) + \sqrt{q} V(\phi).
\]
Now the dust time gauge canonical action (15) becomes

\[
S_\alpha = \frac{1}{2\pi} \int dt d^3x \left\{ \tilde{\pi}^{ab} \dot{q}_{ab} + p_\chi \dot{\chi} - \left( \mathcal{H}^G + \sqrt{q} V(t) + \mathcal{H}^M \right) - N^a (\mathcal{C}_a^G + \mathcal{C}_a^M) \right\}. \tag{50}
\]

This shows that the dust potential acts as a time dependent cosmological constant in the dust time gauge. It has been studied in explicit cosmological solutions in the context of mimetic gravity models and their extensions [7, 18].

The consequences of \( V(t) \) for constructing a linearized theory are interesting. The first question is to select a background solution on which to linearize the theory. Minkowski space is no longer a solution due to the change in the equation for the ADM momentum \( \tilde{\pi}^{ab} \). Rather, the simplest equations are cosmological for the given \( V(t) \), and the analysis differs significantly from the flat space linearized theory because of nonzero ADM momentum in the background solution. We leave this for future work, but note in particular that the time dependent potential would affect the graviton mode equation drastically by introducing an explicit time dependence into it. This would obviously violate Lorentz covariance, which might be recoverable in epochs where \( V(t) \) is chosen to be very slowly varying with \( t \).

4. Discussion

We studied GR coupled to pressureless dust in four spacetime dimensions, and analyzed in detail the linearized theory of perturbations about the flat spacetime solution in ADM canonical theory. This was done in the dust time gauge, which has the interesting feature that the physical Hamiltonian is particularly simple; it is the same function of phase space variables as the Hamiltonian constraint. We included a one-parameter generalization of this canonical theory that resembles the Horava–Lifshitz model.

We found a number of interesting and surprising features: (i) the graviton modes satisfy a Lorentz invariant wave equation, despite the time gauge fixing, for any value of \( \alpha \); (ii) the additional scalar mode in the metric is ultralocal for GR (\( \alpha = 1/2 \)), and so has no consequences associated with dynamical scalars; (iii) for \( \alpha > 1/2 \), the scalar mode is propagating, but for \( \alpha < 1/2 \), its equation becomes elliptic; and (iv) inclusion of a dust potential provides a dynamical cosmological constant.

We also note that despite the ‘intuition’ from the gravitational collapse of dust that the dust time gauge may break down due to shell crossing singularities, we have seen explicitly that linearized theory in this gauge shows no hint of pathology. Rather, it shows that the ultralocality of the scalar mode (for the GR value \( \alpha = 1/2 \)) shields propagating modes, and may serve as a model for dark matter, for the simple reason that ultralocal dynamics is dark.

In a similar study of gravity and dust in three spacetime dimensions [19], it was found that the scalar mode, the only local degree of freedom in 3D gravity, is ultralocal. It is interesting that the scalar perturbation is ultralocal in 4D as well, and that it leaves the usual graviton modes untouched. This decoupling of the scalar and tensor modes is a consequence of the special form of the gauge fixed action (12), which has the same spatial diffeomorphism constraint of GR coupled to matter fields. The structure of this constraint is such that at linear order the tensor modes remain unconstrained for any spatially constant metric. This also happens in the Einstein-aether theories [6]. Moreover, since the dust is a scalar, at linear order we expect the additional degree of freedom to be manifested only in the zero helicity modes of the metric perturbations, because of rotational invariance.

Although we did not study matter fields other than the dust, at the linearized level their inclusion is straightforward. For example, if one includes a standard scalar field in the action,
it is clear that perturbations about the zero solution will be Lorentz covariant. The same would hold for Fermi fields. The main point is that GR with dust, in the dust time gauge, appears to be perfectly consistent with standard Lorentz covariant field theory on Minkowski spacetime, while at the same time ‘solving the problem of time’ in quantum gravity.

At the quantum gravity level we can write a time dependent functional Schrödinger equation

\[ i\hbar \frac{\partial \psi[q, \chi]}{\partial t} = \int d^3x [\hat{H}^G + \hat{H}^M] \psi[q, \chi]. \]  

(51)

a point discussed in the loop quantum gravity context in [10], where a Hilbert space is available to write down this equation precisely; the cosmological setting is studied in [20].

Combining these observations, we note in concluding that the dust time gauge provides compatibility between Lorentz invariance for linearized fields on Minkowski spacetime and the quantum gravity problem. But it remains to explore the quantum gravity equation (51) in larger settings than cosmology. Perhaps of most interest is the problem of gravitational collapse, black hole formation, and subsequent evolution in a fully quantum setting.

Acknowledgments

This work was supported by NSERC of Canada and an AARMS Postdoctoral Fellowship to JZ. We thank Sanjeev Seahra for discussions.

References

[1] Arnowitt R L, Deser S and Misner C W 2008 Gen. Rel. Grav. 40 1997
[2] Hanson A J, Regge T and Teitelboim C 1976 Constrained Hamiltonian Systems (Rome: Accademia Nazionale dei Lincei)
[3] Langlois D 1993 Class. Quant. Grav. 11 389
[4] Brown J D and Kuchar K V 1995 Phys. Rev. D 51 5600
[5] Eling C, Jacobson T and Mattingly D 2004 Deserfest: a celebration of the life and works of Stanley Deser (Singapore: World Scientific) pp 163–79
[6] Jacobson T and Mattingly D 2004 Phys. Rev. D 70 024003
[7] Chamseddine A H and Mukhanov V 2013 J. High Energy Phys. JHEP11(2013)135
[8] Lim E A, Sawicki I and Vikman A 2010 J. Cosmol. Astropart. Phys. JCAP05(2010)012
[9] Chamseddine A H, Mukhanov V and Vikman A 2014 J. Cosmol. Astropart. Phys. JCAP14(2014)017
[10] Husain V and Pawlowski T 2012 Phys. Rev. Lett. 108 141301
[11] Jacobson T and Speranza A J 2015 Phys. Rev. D 92 044030
[12] Horava P 2009 Phys. Rev. D 79 084008
[13] Horava P and Melby-Thompson C M 2010 Phys. Rev. D 82 064027
[14] Donnelly W and Jacobson T 2011 Phys. Rev. D 84 104019
[15] Vagnozzi S 2011 J. Phys. Conf. Ser. 314 012002
[16] Świeżewski J 2013 Class. Quant. Grav. 30 237001
[17] Baumann D 2012 arXiv:0907.5424v2
[18] Myrzakulov R, Sebastiani L and Vagnozzi S 2015 Eur. Phys. J. C 75 444
[19] Husain V, Rahmati S and Ziprick J 2016 Phys. Rev. D 93 024039
[20] Husain V and Pawlowski T 2011 Class. Quant. Grav. 28 225014