Some basic properties of Lagrangians

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Abstract Consider $L$ a regular Lagrangian, $S$ the canonical semispray, and $h$ the horizontal projector of the canonical nonlinear connection. We prove that if the Lagrangian is constant along the integral curves of the Euler-Lagrange equations then it is constant along the horizontal curves of the canonical nonlinear connection. In other words $S(L) = 0$ implies $dhL = 0$. If the Lagrangian $L$ is homogeneous of order $k \neq 1$ then $L$ is a conservation law and hence $dhL = 0$. We give an example of nonhomogeneous Lagrangians for which $dhL \neq 0$.

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Introduction

For a Lagrangian $L$ its energy $E_L = \mathbb{C}(L) - L$ is constant along the solution curves of the Euler-Lagrange equations. In other words, we can say that the energy $E_L$ is constant along the integral curves of the canonical semispray $S$, and this property can be written as $S(E_L) = 0$. For the particular case when the Lagrangian $L$ is homogeneous of order $k \neq 1$, with respect to the fibre coordinates, its energy is proportional with the Lagrangian, which means that $E_L = (k - 1)L$. If this is the case, the horizontal curves of the canonical nonlinear connection coincide with the integral curves of the canonical semispray. Hence, the Lagrangian is constant along the horizontal curves of the canonical nonlinear connection. We can express this property as $dhL = 0$, where $h$ is the horizontal projector that corresponds to the canonical nonlinear connection. a similar result has been obtained by J. Grifone in [Gri72] and J. Szenthe in [Sze96].
In two papers [Sze93] and [SM93] it is stated that for any Lagrangian $L$ we have that $d_h L = 0$, which is not true if $L$ is not homogeneous of order one. For nonhomogeneous Lagrangians the integral curves of the canonical semispray do not coincide with the horizontal curves of the nonlinear connection. We prove that if the Lagrangian is constant along the integral curves of the canonical semispray then it is constant also along the horizontal curves of the canonical nonlinear connection. In other words we have that $S(L) = 0$ implies $d_h (L) = 0$. In general for nonhomogeneous Lagrangian we have that $S(L) \neq 0$ and $d_h L \neq 0$ and we give examples of such Lagrangians. However, we don’t know if the converse implication, $d_h L = 0$ implies $S(L) = 0$, is true.

1 Geometric structures on tangent bundle

In this section we introduce the geometric objects we are going to deal with in this paper such as: Liouville vector field, tangent structure, semispray and nonlinear connection. Since most of them live on the total space of the tangent bundle of a differentiable manifold, we briefly recall some basic properties of the tangent bundle.

Let $M$ be a real, $n$-dimensional manifold of $C^\infty$-class and denote by $(TM, \pi, M)$ its tangent bundle. We denote by $\widetilde{TM} = TM \setminus 0$ the tangent bundle with zero section removed. If $(U, \phi = (x^i))$ is a local chart at $p \in M$ from a fixed atlas of $C^\infty$-class, then we denote by $(\pi^{-1}(U), \Phi = (x^i, y^i))$ the induced local chart at $u \in \pi^{-1}(p) \subset TM$. The linear map $\pi_{*,u} : T_u TM \to T_{\pi(u)} M$, induced by the canonical submersion $\pi$, is an epimorphism of linear spaces for each $u \in TM$. Therefore, its kernel determines a regular, $n$-dimensional, integrable distribution $V : u \in TM \mapsto V_u TM := \ker \pi_{*,u} \subset T_u TM$, which is called the vertical distribution. For every $u \in TM$, $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$ is a basis of $V_u TM$, where $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$ is the natural basis of $T_u TM$ induced by a local chart. Denote by $\mathcal{F}(TM)$ the ring of real-valued functions over $TM$ and by $\mathcal{X}(TM)$ the $\mathcal{F}(TM)$-module of vector fields on $TM$. We also consider $\mathcal{X}^v(TM)$ the $\mathcal{F}(TM)$-module of vertical vector fields on $TM$. An important vertical vector field is $C = y^i(\frac{\partial}{\partial y^i})$, which is called the Liouville vector field.

The mapping $J : \mathcal{X}(TM) \to \mathcal{X}(TM)$ given by $J = (\frac{\partial}{\partial y^i}) \otimes dx^i$ is called the tangent structure and it has the following properties: $\ker J = \im J = \mathcal{X}^v(TM)$; rank $J = n$ and $J^2 = 0$. The cotangent structure is defined as $J^* = dx^i \otimes (\frac{\partial}{\partial y^i})$ and it has similar properties.
A vector field $S \in \chi(TM)$, which is differentiable of $C^\infty$-class on $\tilde{TM}$ and only continuous on the null section, is called a *semispray*, or a second order vector field, if $JS = C$. In local coordinates a semispray can be represented as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$  \hspace{1cm} (1)

We refer to the functions $G^i(x, y)$ as to the local coefficients of the semispray $S$.

A *nonlinear connection* $N$ on $TM$ is an $n$-dimensional distribution, which is also called the horizontal distribution, $N : u \in TM \mapsto N_u TM \subset T_u TM$ that is supplementary to the vertical distribution $VTM$. This means that for every $u \in TM$ we have the direct decomposition:

$$T_u TM = N_u TM \oplus V_u TM.$$ \hspace{1cm} (2)

We denote by $h$ and $v$ the horizontal and the vertical projectors that correspond to the above decomposition and by $\mathcal{X}^h(TM)$ the $\mathcal{F}(TM)$-module of horizontal vector fields on $TM$. For every $u = (x, y) \in TM$ we denote by $\delta/\delta x^i|_u = h(\partial/\partial x^i|_u)$. Then $\{\delta/\delta x^i|_u, \partial/\partial y^i|_u\}$ is a basis of $T_u TM$ adapted to the decomposition \(\square\). With respect to the natural basis $\{\partial/\partial x^i|_u, \partial/\partial y^i|_u\}$ of $T_u TM$, the horizontal components of the adapted basis have the expression:

$$\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i}|_u - N^i_j(u) \frac{\partial}{\partial y^j}|_u, \quad u \in TM.$$ \hspace{1cm} (3)

The functions $N^i_j(x, y)$, defined on domains of induced local charts, are called the *local coefficients* of the nonlinear connection. The dual basis of the adapted basis is $\{dx^i, \delta y^i = dy^i + N^i_j dx^j\}$.

Every semispray $S$ determines a nonlinear connection. The horizontal projector that corresponds to this nonlinear connection is given by \cite{Gri72}:

$$h(X) = \frac{1}{2}(Id - L_S J)(X) = \frac{1}{2}(X - [S, JX] + J[S, X]).$$ \hspace{1cm} (4)

Local coefficients of the induced nonlinear connection are defined on domains of induced local charts and they are given by $N^i_j = \partial G^i/\partial y^j$, \cite{Cra71}, \cite{Gri72}.

### 2 Geometric structures on Lagrange space

The presence of a regular Lagrangian on the tangent bundle $TM$ determines the existence of some geometric structures one can associate to it such as:
semispray, nonlinear connection and symplectic structure.

Consider $L^n = (M, L)$ a Lagrange space. This means that $L : TM \rightarrow \mathbb{R}$ is differentiable of $C^\infty$-class on $\tilde{T}M$ and only continuous on the null section. We also assume that $L$ is a regular Lagrangian. In other words, the $(0,2)$-type, symmetric, d-tensor field with components

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \text{ has rank } n \text{ on } \tilde{T}M.$$  

(5)

The Cartan 1-form $\theta_L$ of the Lagrange space can be defined as follows:

$$\theta_L = J^*(dL) = d_JL = \frac{\partial L}{\partial y^i} dx^i.$$  

(6)

For a vector field $X = X^i(\partial/\partial x^i) + Y^i(\partial/\partial y^i)$ on $TM$, the following formulae are true:

$$\theta_L(X) = dL(JX) = d_JL(X) = (JX)(L) = \frac{\partial L}{\partial y^i} X^i.$$  

(7)

The Cartan 2-form $\omega_L$ of the Lagrange space can be defined as follows:

$$\omega_L = d\theta_L = d(J^*(dL)) = d^2JL = d\left(\frac{\partial L}{\partial y^i} dx^i\right).$$  

(8)

In local coordinates, the Cartan 2-form $\omega_L$ has the following expression:

$$\omega_L = 2g_{ij} dy^j \wedge dx^i + \frac{1}{2} \left(\frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j}\right) dx^i \wedge dx^j.$$  

(9)

We can see from expression (9) that the regularity of the Lagrangian $L$ is equivalent with the fact that the Cartan 2-form $\omega_L$ has rank $2n$ on $\tilde{T}M$ and hence it is a symplectic structure on $\tilde{T}M$.

The canonical semispray of the Lagrange space $L^n$ is the unique vector field $S$ on $TM$ that satisfies the equation

$$i_S \omega_L = d(L - C(L)).$$  

(10)

The local coefficients $G^i$ of the canonical semispray $S$ are given by the following formula:

$$G^i = \frac{1}{4} g^{ik} \left(\frac{\partial^2 L}{\partial y^k \partial x^h} y^h - \frac{\partial L}{\partial x^k}\right).$$  

(11)
Using the canonical semispray $S$ we can associate to a regular Lagrangian $L$ a canonical nonlinear connection with the horizontal projector given by expression (4) and the local coefficients given by expression $N_j^i = \partial G^i / \partial y^j$.

The horizontal subbundle $NTM$ that corresponds to the canonical nonlinear connection is a Lagrangian subbundle of the tangent bundle $TTM$ with respect to the symplectic structure $\omega_L$. This means that $\omega_L(hX, hY) = 0$, $\forall X, Y \in \chi(TM)$. In local coordinates this implies the following expression for the symplectic structure $\omega_L$:

$$\omega_L = 2g_{ij} \delta y^j \wedge dx^i. \quad (12)$$

3 Basic properties of a Lagrange space

In this section we determine some basic properties for a Lagrangian $L$ using the Cartan forms, canonical semispray and nonlinear connection. If $h$ is the horizontal projector of the canonical nonlinear connection of a regular Lagrangian we determine a formula for the horizontal differential $d_hL$ of a regular Lagrangian. From this formula we conclude that there are non homogenous Lagrangians for which $d_hL \neq 0$.

Proposition 1 The following formulae regarding the Cartan 1-form $\theta_L$ of a Lagrange space are true:

$$\iota_S \theta_L = C(L),$$

$$\mathcal{L}_S \theta_L = dL. \quad (13)$$

Proof. First formula (13) follows from the following computation:

$$\iota_S \theta_L = \theta_L(S) = \left( \frac{\partial L}{\partial y^i} dx^i \right) \left( y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \right) = \frac{\partial L}{\partial y^i} y^i = C(L).$$

If we differentiate first formula (13) we obtain $d \iota_S \theta_L = dC(L)$. Using the expression of the Lie derivative $\mathcal{L}_S = d \iota_S + \iota_S d$ we obtain $\mathcal{L}_S \theta_L = \iota_S d \theta_L = dC(L)$. Using the defining formulae (8) for $\omega_L$ and (10) for the canonical semispray $S$ we obtain $\mathcal{L}_S \theta_L = \iota_S \omega_L + dC(L) = dL$ and hence second formula (13) is true. \hfill \blacksquare

Theorem 2 Consider $h$ the horizontal projector (4) of a Lagrange space. We have the following formula for the horizontal differential operator $d_h$:

$$d_hL = \frac{1}{2} d_J(S(L)). \quad (14)$$
In local coordinates, formula (14) is equivalent with the following expression for the horizontal covariant derivative of the Lagrangian $L$:

$$L_i := \frac{\delta L}{\delta x^i} = \frac{1}{2} \frac{\partial}{\partial y^i} (S(L)).$$

**(Proof.** In order to prove (14) we have to show that for every $X \in \chi(TM)$, we have that

$$(d_hL)(X) := dL(hX) = \frac{1}{2} (JX)(S(L)).$$

Using second formula (13) we obtain

$$0 = (\mathcal{L}_S \theta_L - dL)(X) = S\theta_L(X) - \theta_L[S, X] - dL(X)$$

$$= S((JX)(L)) - J[S, X](L) - dL(X)$$

$$= [S, JX](L) + (JX)(S(L)) - J[S, X](L) - dL(X)$$

$$= (JX)(S(L)) - dL(X - [S, JX] + J[S, X])$$

$$= (JX)(S(L)) - dL(2hX).$$

Consequently, formula (14) is true.

Due to the linearity of the operators involved in formula (14) we have that formulae (14) and (15) are equivalent. However, we give here an independent proof that formula (15) is true. The right hand side of formula (15) can be expressed as follows:

$$\frac{\partial}{\partial y^i} (S(L)) = \frac{\partial}{\partial y^i} \left( \frac{\partial L}{\partial x^j} y^j - 2 \frac{\partial L}{\partial y^j} G^j \right)$$

$$= \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial y^i \partial x^j} y^j - 4 g_{ij} G^j - 2 \frac{\partial L}{\partial y^j} N_i^j$$

$$= 2 \left( \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial y^j} N_i^j \right) = 2 \frac{\delta L}{\delta x^i} = 2 L_i.$$

In the above calculation we did use expression (11) for the local coefficients $G^i$ of the canonical semispray $S$.  

**Corollary 3** If the Lagrangian $L$ is constant along the solution curves of the Euler-Lagrange equations then $L$ is constant along the horizontal curves of the canonical nonlinear connection.
Proof. It is immediate from (14) that \( S(L) = 0 \) implies \( d_h L = 0 \). ■

Proposition 4 Consider a regular Lagrangian \( L \) that is homogeneous of order \( k \), \( k \neq 1 \), with respect to \( y \). The horizontal differential of the Lagrangian \( L \) vanishes, which means that \( d_h L = 0 \).

Proof. Using Euler theorem for homogeneous functions we have that the Lagrangian \( L \) is homogeneous of order \( k \) if and only if \( C(L) = (\partial L / \partial y^i) y^i = kL \). Hence, formula (10), which defines the canonical semispray of the Lagrangian \( L \), can be written as \( \iota_S \omega_L = (1 - k)dL \). This implies that \( (1 - k)S(L) = (1 - k)dL(S) = \omega_L(S,S) = 0 \). Using expression (14) we obtain \( d_h L = 0 \). ■

Regular Lagrangians that are second order homogeneous with respect to \( y^i \) are encountered in Finsler geometry. For a Finsler space its geodesics with the arclength parameterization coincide with the integral curves of the canonical semispray, which are horizontal curves with respect to the nonlinear connection. The property \( d_h L = 0 \) tells us that the Lagrangian \( L \) is constant along the horizontal curves of the nonlinear connection and hence it is constant along the geodesics of the space.

Theorem 5 There exist regular Lagrangians \( L \) for which \( d_h L \neq 0 \).

Proof. Let \( a_{ij}(x) \) be a Riemannian structure on the base manifold \( M \) and \( \varphi \) a function on the manifold \( M \) that will be determined latter. The function \( L'(x,y) = a_{ij}(x)y^i y^j \) is a second order homogeneous regular Lagrangian on \( TM \). The following function is also a regular Lagrangian:

\[
L(x,y) = L'(x,y) + \frac{\partial \varphi}{\partial x^i}(x)y^i.
\] (17)

It is a straightforward calculation to check that \( L' \) and \( L = L' + \varphi \) have the same metric tensor \( a_{ij} \) and consequently since \( L' \) is a regular Lagrangian so is \( L \). One can also check that the Cartan 2-forms of the two Lagrangians coincide, which means that \( \omega_{L'} = \omega_L \). Therefore using expression (10) the two Lagrangians have the same semispray \( S \) and using (4) they have the same nonlinear connection with the same horizontal projector \( h \). Local coefficients of the canonical semispray \( S \) are \( 2G^i(x,y) = \gamma^i_{jk}(x)y^j y^k \), where \( \gamma^i_{jk}(x) \) are the Christoffel’s symbols of the second kind for the Riemannian metric \( a_{ij} \).
According to formula (14) and Proposition 4 we have

\[(d_h L)(X) = \frac{1}{2}(JX)(S(L' + \varphi^c)) = \frac{1}{2}(JX)(S(\varphi^c)) = \frac{1}{2} \left( 2 \frac{\partial^2 \varphi}{\partial x^i \partial x^j} y^i - 2 \frac{\partial \varphi}{\partial x^j} \frac{\partial G^j}{\partial y^i} \right) X^i = \left( \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \frac{\partial \varphi}{\partial x^k} \gamma^j_{ij} \right) y^i X^i, \tag{18} \]

where \(X = X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i}\) is a vector field on \(TM\). At this moment we can choose a function \(\varphi\) on \(M\) such that

\[\frac{\partial^2 \varphi}{\partial x^i \partial x^j} \neq \frac{\partial \varphi}{\partial x^k} \gamma^j_{ij}. \tag{19} \]

For example we can choose either the function \(\varphi\) to be linear in \(x\) and \(a_{ij}\) a non flat Riemannian metric or we can choose the Riemannian metric \(a_{ij}\) to be flat and \(\varphi\) a nonlinear function. For the first example, the left hand side of formula (19) is zero, while the right hand side is not. For the second example we have that the right hand side of formula (19) is zero, while the left hand side is not zero.

With a function \(\varphi\) and a Riemannian metric \(a_{ij}\) that satisfy (19), we have that for the Lagrangian \(L\) defined by (17) the horizontal differential is not zero, which means that \(d_h L \neq 0\).

We want to mention that for the Lagrangian (17) we have that \(S(L) = 0\) if and only if \(d_h (L) = 0\). However we don’t know if this result is true for an arbitrary Lagrangian \(L\). Hence we don’t know if the converse of Corollary 3 is true which means that \(d_h L = 0\) implies \(S(L) = 0\)?

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