Frequency Domain Statistical Inference for High-Dimensional Time Series

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Abstract: Analyzing time series in the frequency domain enables the development of powerful tools for investigating the second-order characteristics of multivariate stochastic processes. Parameters like the spectral density matrix and its inverse, the coherence or the partial coherence, encode comprehensively the complex linear relations between the component processes of the multivariate system. In this paper, we develop inference procedures for such parameters in a high-dimensional, time series setup. In particular, we first focus on the derivation of consistent estimators of the coherence and, more importantly, of the partial coherence which possess manageable limiting distributions that are suitable for testing purposes. Statistical tests of the hypothesis that the maximum over frequencies of the coherence, respectively, of the partial coherence, do not exceed a pre-specified threshold value are developed. Our approach allows for testing hypotheses for individual coherences and/or partial coherences as well as for multiple testing of large sets of such parameters. In the latter case, a consistent procedure to control the false discovery rate is developed. The finite sample performance of the inference procedures proposed is investigated by means of simulations and applications to the construction of graphical interaction models for brain connectivity based on EEG data are presented.

Keywords and phrases: Partial Coherence, Testing, False Discovery Control, Graphical Model, De-biased estimator.

1. Introduction

Spectral analysis concerns a number of powerful tools for analyzing the second-order properties of multiple time series. Parameters like the coherence or the partial coherence, describe comprehensively the linear relations between the components of the vector time series by taking into account all lead and lag relations as well as the distinction between direct and indirect effects. Coherence
and partial coherence are process parameters that can be expressed as functions of the spectral density, respectively, of the inverse spectral density matrix; see among other the classical textbooks to multivariate time series analysis by Hannan (1970), Koopmans (1995) and Brillinger (2001). More specifically, partial coherence, which is a measure of the strength of the linear relations between two time series after eliminating the indirect linear effects caused by all other time series of the system, plays a crucial role, for instance, in extending the concept of graphical models to time dependent data; see Brillinger (1996), Dahlhaus (2000) and Eichler (2012). In graphical models, each time series represents a vertex of a network while edges between two vertices describe conditional (on all other time series) linear dependence of the corresponding time series. If the underlying process is Gaussian, then a zero partial coherence even reflects conditional independence between the pair of time series considered. In this special case, an edge between two vertices exists if the partial coherence is non-zero. The corresponding network describes then a Gaussian graphical model for the multivariate times series at hand. Clearly, if the underlying distribution is not specified or if it is non-Gaussian, then statements about independence cannot be made. However, even in these cases, visualizing the linear dependence structure of a multivariate time series system can be helpful in providing valuable information about existing linear relations. Analyzing linear dependencies is a useful tool in many areas of applied research like, for instance, finance, (Gray, 2014), signal processing, (Bach and Jordan, 2004), or medicine. Especially in medicine, there exists a a rich literature devoted to investigations of brain connectivity problems and spectral analysis of EEG and fMRI data; see among others Sun, Miller and D’esposito (2004), Medkour, Walden and Burgess (2009), Fiecas et al. (2011), Ryali et al. (2012); Bowyer (2016); Fiecas and Ombao (2016); Schneider-Luftman (2016) and Walden and Zhuang (2019).

When the dimension of the time series at hand is small, statistical inference for frequency-domain parameters like coherence or partial coherence is a well-developed area of multiple time series analysis. We refer among others to Hannan (1970), Koopmans (1995) and Brillinger (2001). Testing hypothesis about parameters of the spectral density matrix or of its inverse, also have been considered in the literature; see for instance Eichler (2007), Eichler (2008) and Dette and Paparoditis (2009) for $L_2$-type tests. Schneider-Luftman (2016) in-
investigated various combinations of p-values for testing partial coherences and considered their applicability to EEG data. However, when the dimension of the time series is moderate or large compared to the sample size, one needs to somehow restrict dependencies between the component processes to make statistical inference possible. For this, several (not necessarily exclusive) approaches exist in the literature.

One approach is to impose sparsity assumptions directly in the frequency domain and in particular to restrict the number of nonzero elements of the spectral density or of the inverse spectral density matrix. In such a context, Sun et al. (2018) considered estimators of large spectral density matrices, while Fiecas et al. (2019), obtained non-asymptotic results for smoothed (kernel-type) estimators of the spectral density matrix and of its inverse. Zhang and Wu (2021) established convergence rates of nonparametric estimators of the same parameters also allowing for a time-varying autocovariance structure. Rosuel, Loubaton and Vallet (2021) considered high-dimensional Gaussian time series, derived the asymptotic distribution of the maximum of smoothing-based estimators of coherences, and considered tests for independence. In the same context, Jung (2015), Schneider-Luftman and Walden (2016) and Tugnait (2021) use shrinkage methods to estimate the inverse spectral density matrix and to construct Gaussian graphical models.

An alternative approach is to use high-dimensional time series models in order to estimate the spectral density matrix or its inverse. Time series models used in this context are dynamic factor models or other types of low rank models, (Forni et al., 2000; Bai and Ng, 2019); sparse vector autoregressive (VAR) models, (Basu and Michailidis, 2015; Krampe and Paparoditis, 2021); as well as combinations thereof, (Basu, Li and Michailidis, 2019; Krampe and Margaritella, 2021). It should be stressed here that a sparse inverse spectral density matrix does not necessarily imply a sparse spectral density matrix, a sparse autocovariance structure, or more general, a sparse time-domain representation of the underlying high-dimensional process. The reverse argument also is true. Hence, depending on the particular structure of the high-dimensional time series data at hand, different approaches may be used to estimate the parameters of interest, like for instance, the inverse spectral density matrix.

This paper develops statistical inference procedures for coherences and partial coherences in a high-dimensional, time series setup under mild assump-
tions on the underlying process and presents applications of these procedures in the construction of graphical models. Toward this goal, consistent estimators of the parameters of interest are needed as well as powerful procedures for testing whether coherences and/or partial coherences exceed, across all frequencies, some user-specified threshold value. Moreover, testing multiple hypotheses about these parameters, for instance in the construction of graphical models, calls for suitable procedures to control the false discovery rate (FDR).

To tackle the aforementioned inference problems we first develop consistent, nonparametric estimators of the parameters of interest, which possess manageable limiting distributions suitable for inferring properties of their population counterparts. Notice that especially for partial coherences, the development of such estimators in the high-dimensional context is challenging and much more involved than in the finite-dimensional setup. This is mainly due to difficulties associated with the derivation of distributional results for regularized estimators. To overcome such difficulties, we develop so-called de-biased estimators of the partial coherence that use appropriate, regularized regression-type estimators which involve the finite Fourier transform of the vector time series at hand. Asymptotic normality of the nonparametric estimators proposed is established under mild conditions on the underlying high-dimensional system. In contexts and for inference problems different from those considered in this paper, de-biased or de-sparsified estimators, have been found to be useful tools for making statistical inference in a high-dimensional context possible; we refer here to, among others, Javanmard and Montanari (2014), van de Geer et al. (2014), Zhang and Zhang (2014) and Krampe, Kreiss and Paparoditis (2021).

After introducing the estimators of coherence and partial coherence used and establishing their asymptotic properties, the focus of the paper is directed towards the development of powerful testing procedures for the null hypothesis that the aforementioned frequency domain parameters do not exceed some user-specified threshold value within a frequency band of interest. For this, a max-type test statistic is introduced which evaluates coherences and partial coherences over a (with sample size increasing) number of frequencies within the frequency band of interest. The testing procedure developed allows for testing hypotheses for a single pair of component time series as well as for multiple testing, i.e., for testing a large set of such hypotheses. For the latter case, a procedure to control the false discovery rate is proposed. The procedure pro-
vides a screening of the test statistics considered using an appropriate threshold value and is based on an adaptation to the high-dimensional time series setup of thresholding procedures to control the FDR proposed for the construction of Gaussian graphical models for i.i.d. data, Liu (2013); also see Cai et al. (2016) for the application of such an approach to multiple testing for zero correlations. The procedure to control the false discovery rate proposed in this paper is theoretically justified by showing that it achieves (asymptotically) the desired level of false discovery.

We tweak the test statistic to computational feasibility and present semi-parametric approaches to improve the finite sample performance of the estimators involved. The test statistic introduced as well as the testing procedure developed, are flexible enough in that they allow for the use of different approaches to estimate the underlying inverse spectral density matrix provided the corresponding estimators satisfy certain consistency properties. As already mentioned, we do not only consider the case of testing for non-zero (partial) coherence but also allow for testing whether these parameters exceed some user-specified positive threshold value δ. The latter case is of particular importance in some applications. For instance in finance, analyzing connectivity among firms based on stock market data is an important problem, see among others Demirer et al. (2018). The conditional connectivity between two firms might be quite small and practically irrelevant. In such a context, allowing for a positive threshold δ > 0 turns out to be important in order to distinguish between “irrelevant” and “relevant” connections among firms.

Applications of the inference procedure developed to the construction of graphical models for high-dimensional time series use the max-type testing procedure introduced in this paper to determine whether partial coherences exceed a pre-specified threshold value over a frequency band of interest and at the same time control the overall error rate using the FDR procedure proposed. Notice that some alternatives to the approach proposed in this paper also exist in the literature. To elaborate, Jung (2015) and Tugnait (2021) considered regularization based procedures to directly estimate graphical models. Compared to such regularizing approaches, testing approaches have two main advantages. First, they allow for greater flexibility which is due to different possible choices of the (inverse) spectral density matrix estimators involved and can, therefore, be applied to a much broader setup. Second, the tuning parameters involved in the
direct estimation of graphical models using regularizing procedures, have a great
ing impact on the number of edges and therefore on the dependencies discovered.
Although several data-adaptive approaches to select such tuning parameters ex-
ist that might be quite effective in terms of prediction, it is not clear how these
parameters should be chosen in order to achieve a desired level of FDR control.
In contrast to this, testing approaches with FDR control achieve this goal by
construction in a more direct way and they do not rely on the selection of any
additional tuning parameters.

The remaining of the paper is organized as follows. Section 2 discusses some
useful preliminary concepts, presents the estimators for the coherence and, es-
pecially for the partial coherence used in this paper and derives the ir limiting
properties. Section 3 deals with testing problems for coherence and partial co-
herence and it focuses on testing single as well as multiple hypotheses for partial
coherences. The procedure used for false discovery rate control also is presented
in this section. Section 4 investigates via simulations the finite sample behav-
or of the testing procedures proposed and Section 5 is devoted to an empirical ap-
plication, that is, to the construction of a graphical model for brain connectivity.
Auxiliary lemmas as well as technical proofs are deferred to Section 6 and to
the supplementary material of this paper.

Convention. Throughout the paper the following notation is used. For a
vector \( x \in \mathbb{R}^p \), \( \| x \|_1 = \sum_{j=1}^p |x_j| \), \( \| x \|_2^2 = \sum_{j=1}^p |x_j|^2 \) and \( \| x \|_\infty = \max_j |x_j| \).
Furthermore, for a \( r \times s \) matrix \( B \) with elements \( b_{i,j} \), \( i = 1, \ldots, r \) and \( j = 1, \ldots, s \),
\( \| B \|_1 = \max_{1 \leq j \leq s} \sum_{i=1}^r |b_{i,j}| = \max_j \| B e_j \|_1 \), \( \| B \|_\infty = \max_{1 \leq i \leq r} \sum_{j=1}^s |b_{i,j}| = \max_i \| e_i^T B \|_1 \) and \( \| B \|_{\max} = \max_{i,j} |e_i^T B e_j| \), where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \)
is the unit vector of appropriate dimension with the one appearing in the \( j \)th posi-
tion. Denote the largest absolute eigenvalue of a square matrix \( B \) by \( \lambda_{\max}(B) \),
the smallest absolute eigenvalue by \( \lambda_{\min}(B) \) and let \( \| B \|_2 = \lambda_{\max}(BB^*) \). Fur-
thermore, for two matrices \( A, B \), \( A \otimes B \) denotes their Kronecker product. The
\( p \)-dimensional identity matrix is denoted by \( I_p \) while \( I_{p,-J} \in \mathbb{R}^{p \times (p-|J|)} \) is the
same matrix after deleting all columns \( j \in J \), where \( J \) is a nonempty subset
of \( \{1, 2, \ldots, p\} \). For a matrix \( A \) we denote by \( A_{u,v} \) its \((u,v)\)th element and for
a vector \( x \) we write \( x_v \) for its \( v \)th element. For a complex matrix \( A \in \mathbb{C}^{r \times s} \),
we write \( \text{Re}(A) \) for its real and \( \text{Im}(A) \) for its imaginary part. \( A^{(C)} \) denotes the
complex conjugate and \( A^H = (A^{(C)})^\top \) the complex conjugate and transpose
of \( A \). Finally, \( C \) denote some positive generic constant and \( C_\tau \) indicates some
positive generic constant which depends on $\tau$.

2. Estimation of Frequency Domain Parameters in High Dimensions

Consider a $p$-dimensional, zero mean stochastic process $\{X_t, t \in \mathbb{Z}\}$, where for each $t \in \mathbb{Z}$, the random vector $X_t = (X_{j,t}, j = 1, 2, \ldots, p)\top$, is generated as

$$X_t = R(\varepsilon_t, \varepsilon_{t-1}, \ldots).$$

(1)

Here $R : \mathbb{R}^{P \times \infty} \to \mathbb{R}^p$ is some measurable function and $\{\varepsilon_t, t \in \mathbb{Z}\}$ a $p$-dimensional sequence of independent and identically distributed (i.i.d.) random vectors with mean zero and covariance matrix $\Sigma_{\varepsilon}$. Denote by $\Gamma(u) = E(X_uX_0\top)$, $u \in \mathbb{Z}$, the lag $u$ autocovariance matrix and assume that $\{X_t, t \in \mathbb{Z}\}$ possesses a spectral density matrix denoted by $f(\omega)$, $\omega \in \mathbb{R}$. Let $f^{-1}(\omega)$ be the inverse spectral density matrix of the process which we assume that it exists for all frequencies $\omega$.

For an observed stretch $X_1, X_2, \ldots, X_n$ stemming from $\{X_t, t \in \mathbb{Z}\}$, let $\hat{\Gamma}(u)$ be the estimator of $\Gamma(u)$ given by

$$\hat{\Gamma}(u) = \frac{1}{n} \sum_{t=\max\{1,1-u\}}^{n-u} X_t X_t\top,$$

(2)

where $M$ is the truncation lag used and which determines the number of sample autocovariances effectively taken into account for estimating $f(\omega)$. Furthermore, $K(\cdot)$ is a lag-window kernel which satisfies certain conditions to be specified later on. Recall that the lag-window estimator $\hat{f}_M(\omega)$ also has a (discrete) smoothed periodogram analogue given by

$$\hat{f}_M(\omega) = \frac{1}{2\pi} \frac{1}{2\pi} \sum_{u=-n+1}^{n-1} K(u/M)\hat{\Gamma}(u) \exp(-iu\omega),$$

(3)

where $M$ is the truncation lag used and which determines the number of sample autocovariances effectively taken into account for estimating $f(\omega)$. Furthermore, $K(\cdot)$ is a lag-window kernel which satisfies certain conditions to be specified later on. Recall that the lag-window estimator $\hat{f}_M(\omega)$ also has a (discrete) smoothed periodogram analogue given by

$$\hat{f}_M(\omega) = \frac{1}{n} \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)Z(\omega_k)Z^H(\omega_k),$$

(4)
where \( \kappa_M(\cdot) = 1/M \sum_{u=-n+1}^{n-1} K(u/M) \exp(-iu\cdot) \) is the discrete Fourier transform of \( K \).

2.1. Estimation of Coherence and Partial Coherence

The coherence \( \sigma_{u,v}(\omega) \) between the components \( u \) and \( v \) of the vector process \( \{X_t, t \in \mathbb{Z}\} \) at any frequency \( \omega \in [0, 2\pi] \) is given by

\[
\sigma_{u,v}(\omega) = |s_{u,v}(\omega)|, \quad \text{where} \quad s_{u,v}(\omega) = f_{u,u}(\omega)\sqrt{f_{v,v}(\omega)}.
\] (5)

Here, \( f_{r,s}(\omega) \) denotes the \((r,s)\)th element of the spectral density matrix \( f(\omega) \).

Analogue to the partial correlation in the i.i.d. context, the partial coherence \( R_{u,v}(\omega) \) at any frequency \( \omega \) between any two component processes \( \{X_u,t\} \) and \( \{X_v,t\} \), describes the direct linear relation between these components, that is, their cross-correlation structure, after taking into account the linear effects due to all other component processes. Using the inverse spectral density matrix \( f^{-1}(\omega) \), a useful expression of the partial coherence is given by

\[
R_{u,v}(\omega) = |\rho_{u,v}(\omega)|, \quad \text{where} \quad \rho_{u,v}(\omega) = -f_{u,v}^{-1}(\omega)/\sqrt{f_{u,u}^{-1}(\omega)f_{v,v}^{-1}(\omega)}; \quad \text{(6)}
\]

see Dahlhaus (2000), where \( f_{r,s}^{-1}(\omega) \) denotes the \((r,s)\)th element of \( f^{-1}(\omega) \).

Complex coherence, \( s_{u,v} \), and complex partial coherence, \( \rho_{u,v} \), respectively, can be approximated using the correlation and partial correlation of the discrete Fourier transform \( Z_n(\omega) \). More specifically, we have

\[
\text{Corr}(Z_{n,u}(\omega), Z_{n,v}(\omega)) = \Sigma_{n;u,v}(\omega)/\sqrt{\Sigma_{n,v,v}(\omega)\Sigma_{n,u,u}(\omega)} = s_{u,v}(\omega) + O(1/n),
\] (7)

where \( \Sigma_{n;r,s}(\omega) \) denotes the \((r,s)\)th element of the matrix \( \Sigma_n(\omega) \); see (2). An analogue expression for \( \rho_{u,v}(\omega) \) also can be derived. For this, consider for \( v \in \{1,2,\ldots,p\} \) the regression of the component \( Z_{n,v}(\omega) \) of the vector \( Z_n(\omega) \) on all other components of the same vector, that is, the regression

\[
Z_{n,v}(\omega) = \beta_v^H(\omega)Z_{n,-v}(\omega) + E_v(\omega).
\] (8)

Notice that \( E_v(\omega) = e_v^\top \Sigma_n^{-1}(\omega)Z_n(\omega)/\Sigma_n^{-1,v,v}(\omega) \), while

\[
\beta_v^\top = \text{Var}(Z_{n,-v}(\omega))^{-1}\text{Cov}(Z_{n,-v}(\omega), Z_{n,v}(\omega)) = -I_{p,-v}^\top \Sigma_n^{-1}(\omega)e_v/\Sigma_n^{-1,v,v}(\omega).
\]
We can then write for $\rho_{u,v}(\omega)$,

$$
\frac{e_u^\top I_{p,-v}\beta_v(\omega)}{\sqrt{\frac{\Sigma_{n,u,v}(\omega)}{\Sigma_{n;u,u}(\omega)}}} = -\Sigma_{n;u,v}(\omega)/\sqrt{\Sigma_{n;v,v}(\omega)\Sigma_{n;u,u}(\omega)} = \rho_{u,v}(\omega) + O(1/n).
$$

(9)

Here $\Sigma_{n;r,s}^{-1}(\omega)$ denotes the $(r,s)$th element of the inverse matrix $\Sigma_n^{-1}(\omega)$.

Expressions (7) and (9) suggest that estimates of the functions $s_{u,v}()$ and $\rho_{u,v}()$ can be obtained by replacing the matrices $\Sigma_n$ and $\Sigma_n^{-1}$, by appropriate sample estimators. Moreover, to ensure consistency, smoothing over frequencies is necessary.

The estimation of the complex coherence is straightforward. In particular, from (7) and using the estimator $\hat{f}_M$ given in (4), the following, commonly used, estimator of $s_{u,v}(\omega)$, is obtained,

$$
\hat{s}_{u,v}(\omega) = e_u^\top \hat{f}_M(\omega)e_v / \sqrt{e_v^\top \hat{f}_M(\omega)e_v \cdot e_u^\top \hat{f}_M(\omega)e_u}.
$$

(10)

The deviation of an estimator for the complex partial coherence $\rho_{u,v}(\omega)$, is more involved. This is due to the fact that some form of regularization is required in order to obtain a consistent estimator of the inverse spectral density matrix $f^{-1}$, respectively, of the inverse matrix $\Sigma_n^{-1}$ used in (9). However, regularization-based estimators have largely unknown distributional properties which makes the derivation of distributional results for the corresponding estimators of partial coherences a difficult task. Furthermore, the use of regularization introduces some bias in estimating the coefficients $\beta_v(\omega), v = 1, \ldots, p$, which calls for a bias correction in the construction of an estimator of the complex partial coherence. Last but not least, we want the estimator of the complex partial coherence used to possess a manageable limiting distribution which is suitable for testing purposes and more specifically, for properly implementing statistical tests for the corresponding population parameters. Taking all these considerations into account, we introduce in the following an estimator of $\rho_{u,v}$ which builds upon a bias correction of a regularized estimator of the component $\beta_{v,\hat{u}}(\omega) = e_{\hat{u}}^\top I_{p,-v}\beta_v(\omega) = e_{\hat{u}}^\top \beta_v(\omega)$ of the vector $\beta_v(\omega)$; see (9).

To introduce the estimator we propose, recall that $\beta_v(\omega)$ is obtained by regressing $Z_{n,v}(\omega)$ onto the set $Z_{n,-v}(\omega)$; see (8). To implement a bias correction, we proceed as follows. We first transform the regressor $Z_{n,-v}(\omega)$ so that this vector becomes orthogonal to the vector $Z_{n,-v,u}(\omega)$. For this let $\hat{W}_{-v,\hat{u}}(\omega) =$
\[ \gamma_{-v,\tilde{u}}(\omega)^H Z_{n,-v}(\omega) \text{ be a rotation of } Z_{n,-v}(\omega) \text{ which will act as a new regressor.} \]

Our goal is to choose \( \gamma_{-v,\tilde{u}}(\omega) \) such that \( \text{Cov}(\gamma_{-v,\tilde{u}}(\omega)^H Z_{n,-v}(\omega), e_r^T Z_{n,-v}(\omega)) = 0 \) for all \( r \neq \tilde{u} \) and at the same time \( \text{Cov}(\gamma_{-v,\tilde{u}}(\omega)^H Z_{n,-v}(\omega), Z_{n,u}(\omega)) \neq 0 \). If this can be achieved, then \( \beta_{v,\tilde{u}} \) can be obtained by regressing \( Z_{n,v}(\omega) \) onto \( W_{-v,\tilde{u}}(\omega) \) only. Towards this goal, we choose \( \gamma_{-v,\tilde{u}}(\omega) \) equal to

\[
\gamma_{-v,\tilde{u}}(\omega)^H = e_{\tilde{u}}^T c(\Sigma_{n,-v}(\omega))^{-1} \\
= e_{\tilde{u}}^T (\Sigma_{n,v,\omega}^{-1}) \Sigma_{n,-v,\omega}^{-1} - \Sigma_{n,-v,\omega}^{-1}(\Sigma_{n,v,\omega})\Sigma_{n,-v,\omega}^{-1})c/(\Sigma_{n,v,\omega}^{-1}).
\]

The second equality above follows by Lemma 20 in the supplementary material and the constant \( c \) appearing in the displayed equations, is a scaling factor. In principle, we could select \( c \) so that \( e_{\tilde{u}}^T \gamma_{-v,\tilde{u}}(\omega) = 1 \). However, since \( c \) cancels out in the de-biased estimator we will introduce later on, we set for simplicity \( c = \Sigma_{n,v,\omega}^{-1} \). Assume that we have a regularized estimator \( \hat{f}^{-1}(\omega) \) of \( f^{-1}(\omega) \) which satisfies certain consistency properties to be specified later (see Assumption 4 in Section 2.2). Plugging in the estimator \( \hat{f}^{-1} \) for \( \Sigma_n^{-1} \) leads to the following regularized estimator \( \hat{\gamma}_{-v,\tilde{u}}^H(\omega) = \hat{f}^{-1}_v(\omega)\hat{f}^{-1}_{u,v}(\omega) - \hat{f}^{-1}_{u,v}(\omega)\hat{f}^{-1}_{v,\omega}(\omega) \).

Consistency of the estimator above is established in Lemma 10 in the supplementary material. Note that in the high-dimensional setup and because a regularized estimator of \( \Sigma_n^{-1} \) is used, the obtained estimator \( \hat{W}_{-v,\tilde{u}}(\omega) = \hat{\gamma}_{-v,\tilde{u}}^H(\omega) Z_{n,-v}(\omega) \) of \( W_{-v,\tilde{u}}(\omega) \) is not exactly orthogonal to \( e_r^T Z_{n,-v}(\omega) \) for \( r \neq \tilde{u} \), that is, the corresponding sample covariance matrix is not exactly zero. This means that when regressing \( Z_{n,v}(\omega) \) onto \( \hat{W}_{-v,\tilde{u}}(\omega) \) a biased estimator of \( \beta_{v,\tilde{u}} \) occurs. The corresponding bias can, however, be corrected using an estimator \( \hat{\beta}_v^H(\omega) = f^{-1}_{v,\omega}(\omega)/\hat{f}^{-1}_{v,\omega}(\omega) \) of \( \beta_v^H \).

Given the estimators \( \hat{\beta}_v^H(\omega) \) and \( \hat{\gamma}_{-v,\tilde{u}}^H(\omega) \), we can now introduce the de-biased estimator of \( \beta_{v,\tilde{u}}(\omega) \) we will use. Since, as previously mentioned, smoothing is a necessary step for consistency, the de-biased estimator is a smoothing based estimator and it is given by

\[
\hat{\beta}^{(de)}_{v,\tilde{u}}(\omega) = \hat{\beta}_{v,\tilde{u}}(\omega) + \frac{\sum_{k=1}^{\nu} \kappa_M(\omega - \omega_k) [Z_{n,v}(\omega_k) - \hat{\beta}_v^H(\omega) Z_{n,-v}(\omega_k)] Z_{n,-v}(\omega_k)^H \hat{\gamma}_{-v,\tilde{u}}(\omega)}{\sum_{k=1}^{\nu} \kappa_M(\omega - \omega_k) Z_{n,u}(\omega_k) Z_{n,-v}(\omega_k) \hat{\gamma}_{-v,\tilde{u}}(\omega)}.
\]

Observe that \( \hat{\beta}^{(de)}_{v,\tilde{u}}(\omega) \) in (11) is constructed in such a way that only an estimator of \( f^{-1} \) at frequency \( \omega \) is used. Now, using the estimators \( \hat{\beta}^{(de)}_{v,\tilde{u}}(\omega) \) and \( \hat{\beta}^{(de)}_{v,\tilde{u}}(\omega) \), where the latter estimator is obtained as \( \hat{\beta}^{(de)}_{v,\tilde{u}}(\omega) \) in (11) but by replacing \( v \) and
by $\tilde{v}$ and $u$, respectively, the estimator of the complex partial coherence we finally propose is given by

$$
\hat{\rho}_{u,v}^{(de)}(\omega) := \frac{1}{2} \left( \hat{\beta}_{u,\tilde{v}}^{(de)}(\omega) \sqrt{\frac{\hat{f}_{u,u}^{-1}(\omega)}{\hat{f}_{v,v}^{-1}(\omega)}} + \hat{\beta}_{v,\tilde{u}}^{(de)}(\omega) \sqrt{\frac{\hat{f}_{v,v}^{-1}(\omega)}{\hat{f}_{u,u}^{-1}(\omega)}} \right),
$$

(12)

Note the appropriate rescaling of the estimators $\hat{\beta}_{u,\tilde{v}}^{(de)}(\omega)$ and $\hat{\beta}_{v,\tilde{u}}^{(de)}(\omega)$ as well as the fact that $\hat{\rho}_{u,v}^{(de)}(\omega)$ fulfills the property $\hat{\rho}_{u,v}^{(de)}(\omega) = \left( \hat{\rho}_{v,u}^{(de)}(\omega) \right)^{C(\omega)}$, which also is fulfilled by the population partial coherence $\rho_{u,v}(\omega)$.

Recall that, so far, we have derived an estimator of $s_{u,v}(\omega)$ and of $\rho_{u,v}(\omega)$ rather than of the coherence $\sigma_{u,v}(\omega)$ and the partial coherence $R_{u,v}(\omega)$. This is due to the fact that our interest is primarily focused in inferring properties of the corresponding population parameters and that estimators of $s_{u,v}(\omega)$ and $\rho_{u,v}(\omega)$ can be obtained by using the relations of these parameters to the vector of discrete Fourier transforms. As we will see in the next subsections, this relation also allow for the estimators obtained to posses nice asymptotic properties and, therefore, to be useful for testing purposes.

### 2.2. Asymptotic Properties of Estimators

In order to derive the limiting distribution of $\hat{s}_{u,v}(\omega)$ and $\hat{\rho}_{u,v}^{(de)}(\omega)$, we have to impose some assumptions on the stochastic properties of the underlying high-dimensional process $\{X_t, t \in \mathbb{Z}\}$, the inverse spectral density matrix $f^{-1}$, the lag-window kernel $K$ and the consistency properties of the regularized estimator $\hat{f}^{-1}$ used.

To control the temporal dependence of the high-dimensional process $\{X_t, t \in \mathbb{Z}\}$, we make use of the concept of functional dependence, Wu (2005); also see Wu et al. (2016) and Zhang and Wu (2021). For this let for $k \leq t$, $X_{t,k} = R(\varepsilon_t, \ldots, \varepsilon_{k+1}, \varepsilon_k, \varepsilon_{k-1}, \ldots)$, be a coupled processes, where $\varepsilon_k$ is an i.i.d. copy of $\varepsilon_k$. For $\tau \in \mathbb{N}$ refering to the number of finite moments of $X_t$, define the so-called functional dependence measures $\delta_{1,\tau}^{[i]}$ and $w_{t,\tau}$ as

$$
\delta_{1,\tau}^{[i]} = \left\{ E |e_i^\top (X_t - X_{t,\{0\}})|^{\tau} \right\}^{1/\tau} \quad \text{and} \quad w_{t,\tau} = \left\{ E \left( \max_i |e_i^\top (X_t - X_{t,\{0\}})|^{\tau} \right) \right\}^{1/\tau}.
$$

Let $\delta_{t,\tau}^{[\max]} = \max_{i} \delta_{1,\tau}^{[i]}$ and for $\alpha \geq 0$, let $\|X_\cdot\|_{\tau,\alpha} = \sup_{t \geq 0} (m+1)^{\alpha} \sum_{t=m}^{\infty} w_{t,\tau}$.

The following assumption is imposed.
Assumption 1. For some $\tau \geq 8$ and for all $p \in \mathbb{N}$, it holds true that $\delta_{t,\tau}^{[\text{max}]} \leq C \rho t$ and $\sup_{\|v\|_2=1}(E|v^\top X|^{\tau})^{1/\tau} \leq C$, where $\rho \in (0, 1)$ is a constant. Furthermore, $\|X\|_{\infty,\alpha} \leq C p^{r(\tau)}$ for some $\alpha > 1/2 - 1/\tau$, where $r(\tau)$ is a positive number that depends on the number of finite moments $\tau$.

Stationary Markov chains as well as stationary linear processes, see Examples 2.1 and 2.2 in Chen et al. (2013), are examples of processes fulfilling the conditions of Assumption 1. Note that by this assumption, $\sup_{\|v\|_2=1}(E|v^\top (X - X_t, \{0\})|^{\tau})^{1/\tau} \leq C \rho t$ and $\sum_{j=m}^{\infty} \delta_{t,q}^{[\text{max}]} \leq C \rho^m$ for some $q \leq \tau$. Furthermore, it can be shown that vector linear processes satisfying some additional conditions regarding the coefficient matrices, they satisfy $\|X\|_{\infty,\alpha} = O(p^{1/\tau} \sqrt{\log(p)})$; see Example 2.2 in Zhang and Wu (2021). We mention here that the geometric decay of $\delta_{t,\tau}^{[\text{max}]}$ given in Assumption 1 can be relaxed to a polynomial decay; see condition (12) in Wu and Zaffaroni (2018). Such a polynomial decay would, however, require additional restrictions on the allowed increase of the dimension $p$ of the process compared to those stated in Assumption 5 below.

The following assumption summarizes our requirements regarding the kernel function used in the construction of the estimator $\hat{f}$.

Assumption 2. $K$ is an even and bounded kernel function with compact support $[-1, 1]$ satisfying $K(0) = 1$ and $C_K = \int_{-1}^{1} K^2(u) du < 1$. Furthermore, $K$ is Lipschitz continuous in $[-1, 1]$ with Lipschitz constant $L_K$ and $1/M \sum_{s=-M}^{M} K^2(\exp(-i\pi ks/M)) \leq C [(1/k)^r + 1/M]$ for all $k \geq 1$ and some $r \geq 1$.

The last condition in Assumption 2 specifies the decay behavior of the Fourier coefficients of the kernel and it affects the covariance of the lag-window estimator at different frequencies. Note that for the uniform kernel, $(1/k)^r$ vanishes while for the modified Bartlett kernel, $K(u) = (1 - |u|)1(|u| \leq 1)$, we have $r = 2$. Under Assumption 1 and 2, we have by Proposition 4.3 in Zhang and Wu (2021) that,

$$P(\sup_{\omega} |\hat{f}_M(\omega) - f(\omega)|_{\text{max}} \geq x) \leq g(x, p, n, M, \tau),$$

where the function $g$ is defined as

$$g(x, p, n, M, \tau) = C_{\tau} n \left( \frac{\max(1, \log(p) C_1) p^{2r(\tau)}}{nx/M} \right)^{\tau/2} + CM p^2 \exp \left( -C nx^2/M \right),$$

with $C_{\tau}, C_1$ and $C$ generic constants. Zhang and Wu (2021) obtained the above result for $C_1 = 5/2$. Observe that $r(\tau)$ in (13) is in some sense the price paid for
increasing dimension. As mentioned, in Example 2.2 of Zhang and Wu (2021), it is shown that under some additional conditions, for linear processes \( p^{r(\tau)} = p^{1/\tau} \sqrt{\log(p)} \).

Our next two assumptions deal with the conditions imposed on the inverse spectral density matrix \( f^{-1} \) as well as on its estimator \( \hat{f}^{-1} \).

**Assumption 3.** The inverse spectral density matrix \( f^{-1} \) exists for all \( p \in \mathbb{N} \) and satisfies for some positive constant \( C < \infty \),

\[
\sup_\omega \| f^{-1}(\omega) \|_2 < C \quad \text{and} \quad \sup_\omega \| f^{-1}(\omega) \|_1 \leq C \cdot s(p),
\]

where \( s(p) \) is an increasing sequence of the dimension \( p \in \mathbb{N} \). Furthermore, \( \sup_\omega \| \Sigma_n(\omega) - f(\omega) \|_{\max} = O(1/n) \), \( \sup_\omega \| \Sigma_n^{-1}(\omega) - f^{-1}(\omega) \|_1 = O(s(p)/n) \) and

\[
\sup_\omega \| \sqrt{M/n} \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)(\Sigma_n(\omega_k) - f(\omega)) \|_{\max} \leq C \cdot g_{\text{Bias}} \quad (14)
\]

with \( g_{\text{Bias}} = o(\log(n)) \).

The condition \( \sup_\omega \| f^{-1}(\omega) \|_2 < C \) can be interpreted as a lower bound condition for the eigenvalues of the spectral density matrix \( f \). The parameter \( s(p) \) restricts in fact the growth of the cross-sectional (conditional) dependence structure of the process and if \( f^{-1} \) is (weakly) sparse, see for instance \( (17) \), then \( s(p) \) also depends on the (weakly) sparsity parameters. Expression \( (14) \) restricts the order of the bias of the lag-window estimator \( \hat{f}_M \) to be \( g_{\text{Bias}} \), where the condition \( g_{\text{Bias}} = o(\log(n)) \) is in fact a lower bound condition on the truncation lag \( M \). For instance, for \( K(u) = \mathbb{1}(\{|u| \leq 1\}) \) the uniform kernel, we have \( g_{\text{Bias}} \leq C \sqrt{n/M} \sum_{|h| \geq M} \| \Gamma(h) \|_{\max} \). Hence, \( \sum_{h \in \mathbb{Z}} |h| \| \Gamma(h) \|_{\max} < \infty \) leads to \( M = n^a \) for some \( a \geq 1/5 \). Under Assumption 1, we have \( C \sqrt{n/M} \sum_{|h| \geq M} \| \Gamma(h) \|_{\max} \leq C \sqrt{n/M} p^M \) and \( a \geq \varepsilon > 0 \) will suffice. Similar results can be obtained for other kernels.

Since we do not want to restrict ourselves to a particular estimator \( \hat{f}^{-1} \) of \( f^{-1} \), the following assumption summarizes the conditions we impose on the estimator of the inverse spectral density matrix used.

**Assumption 4.** A (regularized) estimator \( \hat{f}^{-1} \) of the inverse spectral density matrix \( f^{-1} \) exists which satisfies

\[
P(\sup_\omega \| \hat{f}^{-1}(\omega) - f^{-1}(\omega) \|_{\max} \geq x) \leq g(x, p, n, M, \tau)
\]
and
\[ P(\sup_\omega \| \hat{f}^{-1}(\omega) - f^{-1}(\omega) \|_1 \geq x) \leq g(x/s(p), p, n, M, \tau), \]
where \( g(x, p, n, M, \tau) \) is given in (13).

If sparsity assumptions are imposed on \( f^{-1} \), known procedures for covariance matrix estimation (see Cai, Ren and Zhou (2016) for an overview) can be applied to obtain consistent estimators of the inverse spectral density matrix then satisfy Assumption 4. Using as starting point the lag-window estimator \( \hat{f}_M \), such estimators have been investigated by several authors; see among others, Sun et al. (2018); Fiecas et al. (2019); Zhang and Wu (2021); Jung (2015) and Tugnait (2021). The last two papers also consider regularization procedures applied not only locally, i.e., to each frequency, but also globally and group-wise, that is applied to all frequencies in the interval \([0, 2\pi]\). An alternative, semiparametric approach to estimate \( f^{-1} \) is to use sparse vector autoregressive models; see for instance Krampe and Paparoditis (2021) and Krampe and Margaritella (2021) for details.

To elaborate on an example of an estimator of \( f^{-1} \) satisfying the requirements of Assumption 4, we consider a CLIME type estimator. Similar results can be obtained using estimators based on graphical lasso or other types of regularization. Let \( \text{CLIME}_\lambda(A) \) denote the CLIME estimator applied to an input matrix \( A \) with tuning parameter \( \lambda \) and obtained as the solution of the optimization problem \( \min \| B \|_1 \) such that \( \| AB - I \|_{\text{max}} \leq \lambda \), plus a possible correction for symmetry. Then, an estimator of the inverse spectral density matrix at frequency \( \omega \) based on the lag-window estimator \( \hat{f}_M \) and using the tuning parameter \( \lambda \), is given by

\[ \hat{f}^{-1}(\omega) := (1, i) \left( \text{CLIME}_\lambda \left( \begin{pmatrix} \text{Re}(\hat{f}_M(\omega)) & \text{Im}(\hat{f}_M(\omega)) \\ -\text{Im}(\hat{f}_M(\omega)) & \text{Re}(\hat{f}_M(\omega)) \end{pmatrix} \right) \right) \begin{pmatrix} 1 \\ -i \end{pmatrix} / 2; \]

(15)

with \( \hat{f}_M(\omega) \) as given in (4). Zhang and Wu (2021), Theorem 5.1 and Remark 4, showed that the above estimator satisfies,

\[ P(\sup_\omega \| \hat{f}^{-1}(\omega) - f^{-1}(\omega) \|_{\text{max}} \geq x) \leq g(x/ \sup_\omega \| f^{-1}(\omega) \|_1^2, p, n, M, \tau). \]

(16)

For the class of so-called weak sparse inverse spectral density matrices this result also can be extended to other matrix norms. To elaborate, let \( \mathcal{G}_c(w(p)) \) be the
set of inverse spectral density matrices which are weakly sparse within a small $\ell_r$ ball, that is,

$$
G_r(w(p)) := \left\{ f^{-1} : [0, 2\pi] \to \mathbb{C}^{p \times p} \mid \sup_{\omega} \max_j \sum_{i=1}^p |f_i^{-1}(\omega)|^r \leq w(p) \right\}. \tag{17}
$$

Then, $\sup_{\omega} ||f^{-1}(\omega)||_1 \leq Cw(p)$ and by the arguments of Cai, Liu and Luo (2011), it can be shown that for all $f^{-1} \in G_r(w(p))$ and any $l \in [1, \infty]$,

$$
P(\sup_{\omega} ||f^{-1}(\omega) - f^{-1}(\omega)|| l \geq x) \leq g(x/(w(p) \sup_{\omega} ||f^{-1}(\omega)||_2^2), p, n, M, \tau)
$$

$$
\leq g(x/w(p))^3, p, n, M, \tau). \tag{18}
$$

**Assumption 5.** Let $M = Cn^a$ for $a \in [1/5, 4/7]$, $p \leq n^b$, $s(p) = n^s$ for $s \leq 3/(8(1-a))$ and $g(x, p, n, M, \tau)$ given as in (13). Assume that the following condition holds true:

$$
n^{2b+a}\left[ g\left(\frac{\log(n) - \log_{Bias}}{(s(p)(n/M)^{1/4})^{1/2}}\right) + \frac{\log(n)^{C_r}}{(n^{1/2}(1-1/4a)-1/2(2-a)-2(1+a)/\tau^2)} + \frac{\log(n)^{C_r}}{n^{r/4(1+a)-1}}\right] = o(1). \tag{19}
$$

The left hand side of (19) can be bounded by

$$
C_r \log(n)^{C_r} p^{r \times r(\tau)} s(p)^{r/2} n^{-7/16\tau(1-a) + 1 + 2b + a},
$$

which for $a \leq 1/2$, $r(\tau) \leq 1/\tau \log(n)^C$ and $s(p) \leq n^{1/8}$, implies that $b \leq \tau/16 - 1/2$.

We now consider the limiting distribution of the estimators $\hat{s}_{u,v}$ and $\hat{\rho}^{(de)}_{u,v}$ introduced in the previous section. For the coherence estimator, asymptotic normality can be established under validity of Assumption 1 and Assumption 2. For instance, Koopmans (1995), Section 8.4, establishes such a result under slightly different assumptions than those used in this paper. More precisely, we have for $u, v \in \{1, \ldots, p\}$ and $\omega \neq \pi\mathbb{Z}$, that, as $n \to \infty$,

$$
\sqrt{n/M}(\hat{s}_{u,v}(\omega) - s_{u,v}(\omega)) \xrightarrow{d} \psi_1^{(u,v)} + i\psi_2^{(u,v)},
$$

where

$$
\begin{pmatrix}
\psi_1^{(u,v)} \\
\psi_2^{(u,v)}
\end{pmatrix} \sim \mathcal{N}\left(0, \frac{CK_2(1 - |s_{u,v}(\omega)|^2)^2}{2} \begin{pmatrix}
1 - \text{Re}(s_{u,v}(\omega))^2 & -\text{Re}(s_{u,v}(\omega))\text{Im}(s_{u,v}(\omega)) \\
-\text{Re}(s_{u,v}(\omega))\text{Im}(s_{u,v}(\omega)) & 1 - \text{Im}(s_{u,v}(\omega))^2
\end{pmatrix}\right).
$$

Moreover, for $\omega \in \pi\mathbb{Z}$ we have, $\sqrt{n/M}(\hat{s}_{u,v}(\omega) - s_{u,v}(\omega)) \xrightarrow{d} \mathcal{N}(0, C_{K_2}^{1/2}(1 - |s_{u,v}(\omega)|^2)).$
Deriving the limiting distributional properties of the de-biased estimator \( \hat{\rho}_{u,v}(\omega) \) introduced in Section 2.1 is more involved. The following theorem shows that this distribution has an analogue structure as that of the coherence estimator \( \hat{s}_{u,v}(\omega) \).

**Theorem 1.** Under validity of Assumptions 1-5 and for \( u, v \in \{1, \ldots, p\} \), we have for \( \omega \neq \pi \mathbb{Z} \), that, as \( n \to \infty \),

\[
\frac{\sqrt{n/M}(\hat{\rho}_{u,v}(\omega) - \rho_{u,v}(\omega))}{C_{K^2}^{1/2}} \overset{d}{\to} \xi^{(u,v)}_1 + i \xi^{(u,v)}_2,
\]

where \( \left( \xi^{(u,v)}_1 \right) \sim \mathcal{N}(0, \Sigma_{(u,v)}(\omega)) \) and

\[
\Sigma_{(u,v)}(\omega) = \frac{C_{K^2}(1 - |\rho_{u,v}(\omega)|^2)}{2} \begin{pmatrix}
1 - \text{Re}(\rho_{u,v}(\omega))^2 & -\text{Re}(\rho_{u,v}(\omega)) \text{Im}(\rho_{u,v}(\omega)) \\
-\text{Re}(\rho_{u,v}(\omega)) \text{Im}(\rho_{u,v}(\omega)) & 1 - \text{Im}(\rho_{u,v}(\omega))^2
\end{pmatrix}.
\]

Furthermore, for \( \omega \in \pi \mathbb{Z} \),

\[
\frac{n/M}{C_{K^2}^2} \text{Cov}(\hat{\rho}_{u_1,v_1}(\omega), \hat{\rho}_{u_2,v_2}(\omega)) \overset{d}{\to} \mathcal{N}(0, 1(C_{K^2}(1 - |\rho_{u,v}(\omega)|^2)).
\]

The following corollary refers to the limiting covariance between the estimators \( \hat{\rho}_{u_i,v_j}(\omega) \) for different pairs of indices \( (u_i, v_j) \), \( i, j = 1, 2 \).

**Corollary 2.** Let \( u_1, u_2, v_1, v_2 \in \{1, \ldots, p\} \) and \( \omega \in [0, 2\pi] \). Under the assumptions of Theorem 1, we have, as \( n \to \infty \),

\[
\frac{n/M}{C_{K^2}^2} \text{Cov}(\hat{\rho}_{u_1,v_1}(\omega), \hat{\rho}_{u_2,v_2}(\omega)) \overset{d}{\to} \rho_{u_1,u_2}(\omega)\rho_{v_1,v_2}(\omega) + 1(\omega \in \pi \mathbb{Z})\rho_{u_1,v_2}(\omega)\rho_{u_2,v_1}(\omega)
\]

\[
+ \frac{1 + 1(\omega \in \pi \mathbb{Z})}{2} \left[ -\rho_{v_2,u_2}(\omega)|\rho_{u_1,v_2}(\omega)|\rho_{v_2,v_1}(\omega) + \rho_{u_1,u_2}(\omega)\rho_{u_2,v_1}(\omega) \right]
\]

\[
- \rho_{u_1,v_1}(\omega)|\rho_{v_2,u_2}(\omega)|\rho_{v_1,u_2}(\omega) + \rho_{v_2,u_1}(\omega)\rho_{u_1,u_2}(\omega)
\]

\[
+ 1/2\rho_{u_1,v_1}(\omega)|\rho_{v_1,v_2}(\omega)| + |\rho_{v_1,u_2}(\omega)|^2 + |\rho_{u_1,v_2}(\omega)|^2 + |\rho_{u_1,u_2}(\omega)|^2
\].

In many applications the focus is not on the behavior of \( \rho_{u,v} \) at a particular frequency \( \omega \) but rather on a set of frequencies belonging to a frequency band, say \( \mathcal{W} \subset [0, \pi] \), of interest. In such a case and in order to handle the behavior of the de-biased coherence \( \hat{\rho}_{u,v}(\omega) \) for all \( \omega \in \mathcal{W} \), distributional results are needed which hold true simultaneously over the corresponding set of frequencies. The following theorem deals with the asymptotic distribution of the maximum of the standardized real and imaginary parts of the estimator \( \hat{\rho}_{u,v} \) evaluated over
a growing grid of frequencies belonging to a frequency band \( W \). Liu and Wu (2010) and Wu and Zaffaroni (2018) derived an asymptotic, Gumbel type approximation for the maximum deviation of the spectral density over growing sets of frequencies. The following theorem extends this result to the case of the proposed de-biased estimator of the partial coherence. Furthermore, it gives a uniform convergence result for the ratio of the upper tails of the distribution of the maximum deviation of the standardized, de-biased partial coherence and its asymptotic \( \chi^2 \) approximation. Such a result is of particular importance if interest is focused on the behavior of the upper quantiles of the aforementioned distributions which is the case if one deals with multiple testing. It is the main ingredient in proving the false discovery rate control associated with the thresholding-based, multiple testing procedure proposed in this paper.

Before stating the announced theoretical result, let us mention that our aim is to obtain an approximation which is independent of the parameters of the underlying process. Hence, for establishing the desired uniform approximation, it is important that the estimators \( \hat{\rho}_{u,v}^{(de)}(\omega) \) involved, are asymptotically independent at nearby frequencies in order for a maximum of independent \( \chi^2 \) random variables to serve as a valid approximation. Toward this goal, we consider a grid of frequencies \( \omega'_l = \pi l N / M, l = 1, 2, \ldots, M / N - 1, \) where neighborhood frequencies are \( \pi N / M \) apart from each other and the term \( N \) depends on the particular kernel used. For the uniform kernel, this statement holds true for \( N = 1 \). For other kernels, we can choose \( N = \log(M)^2 / r \), where \( r \) is determined by the decay behavior of the Fourier coefficients of the kernel used; see Assumption 2.

**Theorem 3.** Let \( \omega'_l = \pi l N / M, l = 1, \ldots, M / N - 1, M = C n^a \) and \( \mathcal{L} = \{ l = 1, \ldots, M / N - 1 : \omega'_l \in W \} \) with \( d = |\mathcal{L}| \). Then, under Assumptions 1 to 5, we have for \( u, v \in \{ 1, \ldots, p \} \), that,

\[
\sup_{0 \leq t \leq 2(a+2b) \log(n)} \frac{\Pr(n/M \max_{l} \chi_{(u,v)}(\omega'_l) \geq t)}{G_d(t)} - 1 = o(1),
\]

where \( G_d(t) = 1 - (1 - \exp(-t^2/2))^d = \Pr(\max_{1 \leq l \leq d} Z_l \geq t), Z_l \sim \chi^2_2, \text{iid}, \)

\[
\chi_{(u,v)}(\omega) = \begin{pmatrix} \Re(\hat{\rho}_{u,v}^{(de)}(\omega) - \rho_{u,v}(\omega)) \\ \Im(\hat{\rho}_{u,v}^{(de)}(\omega) - \rho_{u,v}(\omega)) \end{pmatrix}^\top \Sigma_{(u,v)}^{-1}(\omega) \begin{pmatrix} \Re(\hat{\rho}_{u,v}^{(de)}(\omega) - \rho_{u,v}(\omega)) \\ \Im(\hat{\rho}_{u,v}^{(de)}(\omega) - \rho_{u,v}(\omega)) \end{pmatrix},
\]

and

\[
\Sigma_{(u,v)}^{-1}(\omega) = \frac{2}{C K_2(1 - |\hat{\rho}_{u,v}(\omega)|^2)^2} \begin{pmatrix} 1 - \Im(\hat{\rho}_{u,v}(\omega))^2 & \Re(\hat{\rho}_{u,v}(\omega)) \Im(\hat{\rho}_{u,v}(\omega)) \\ \Re(\hat{\rho}_{u,v}(\omega)) \Im(\hat{\rho}_{u,v}(\omega)) & 1 - \Re(\hat{\rho}_{u,v}(\omega))^2 \end{pmatrix}.
\]
Note that instead of $G_d(t) = 1 - (1 - \exp(-t/2))^d$ we could also use $	ilde{G}_d(t) = 1 - \exp(-\exp(-(x + 2 \log(d))/2))$, where $G_d$ is the Gumbel distribution with scaling factor 2 and shifted by $2 \log(d)$. It holds true that $\sup_t \frac{|G_d(t)/\tilde{G}_d(t) - 1|}{\exp(-1)/d}$.

3. Testing

We consider the problem of testing hypotheses about the frequency domain parameters discussed so far. For this, we build upon the estimators and their asymptotic properties derived in the previous section. For brevity of presentation, we only concentrate on the more involved case, that is the case of testing hypotheses about the partial coherences for frequencies belonging to a frequency band $W \subset [0, \pi]$ of interest. Applications of the testing procedures presented in this section to a particular frequency $\omega$ and coherences are straightforward and will be omitted.

3.1. Testing Single Hypothesis

We begin our discussion with the problem of testing a single hypothesis, that is of testing that the values of a particular partial coherence are over a frequency band $W \subset [0, \pi]$, below some desired threshold level. More specifically, let $W$ be a subset of $[0, \pi]$ having positive Lebesgue measure, $\delta \in [0, 1)$ be a user specified threshold and $(u, v) \in \{(u, v) | 1 \leq u < v \leq p\}$ a pair of indices. Consider the testing problem

$$H_{0}^{(u,v)}: \sup_{\omega \in W} |\rho_{u,v}(\omega)| \leq \delta,$$

vs.

$$H_{1}^{(u,v)}: |\rho_{u,v}(\omega)| > \delta \text{ for all } \omega \in A,$$

where $A \subset W$ has positive Lebesgue measure.

To implement the above test, let $\omega' = \pi N l / M, l = 1, \ldots, M/N - 1$, $\mathcal{L} = \{l = 1, \ldots, M/N - 1 : \omega'_l \in W\}$ and set $d := |\mathcal{L}|$. We refer to the discussion preceding Theorem 3 for the value of $N$. The following test statistic is then used,

$$T_{n}^{(u,v)} = 1 \left( \max_{l \in \mathcal{L}} |\hat{\rho}_{u,v}^{(de)}(\omega'_l)| > \delta \right) \times$$

$$\max_{l \in \mathcal{L}} \left\{ \frac{n}{M} \left( \frac{\Re(\hat{\rho}_{u,v}^{(de)}(\omega'_l) - \delta \exp(i\hat{\omega}_l))}{\hat{\Sigma}_{(u,v)}(\omega'_l)} \right) \right\},$$

(20)
where \( \tilde{\omega}_l = \arg(\hat{\rho}_{(u,v)}^{(de)}(\omega'_l)) \) and \( \hat{\Sigma}_{(u,v)}^{-1} \) is defined as in Theorem 3.

Note that for a given frequency, the real and imaginary parts are asymptotically normal which implies that the above transformation leads to an asymptotic \( \chi^2 \) distributed random variable. The frequency \( \omega = 0 \) is excluded from the calculation of the maximum above for two reasons. First, for \( \omega = 0 \), the imaginary part is zero and therefore the \( \chi^2 \)-approximation does not hold true. Second, deterministic trends in the time domain lead to peaks of the spectral density at frequencies close to zero. Hence, excluding this frequency makes the test results more robust with respect to such distortions.

For any desired level \( \alpha \in (0,1) \), the null hypothesis \( H_0^{(u,v)} \) given above, is rejected if

\[
T_n^{(u,v)} \geq G(1 - \alpha),
\]

where \( G(1 - \alpha) \) is the upper \( \alpha \) quantile of the distribution function \( 1 - G_d(t) = (1 - \exp(-t/2))^d \). The following theorem ensures that the test (21) has asymptotically the desired level \( \alpha \).

**Theorem 4.** Suppose that \( H_0^{(u,v)} \) is true. Then, under the conditions of Theorem 1, the test (21) satisfies

\[
\lim_{n \to \infty} P\left( T_n^{(u,v)} \geq G(1 - \alpha) \right) \leq \alpha.
\]

### 3.2. Testing Multiple Hypothesis with False Discovery Rate Control

Let \( Q \) denote a set of pairs of indices for which we want to test the hypothesis that the corresponding partial coherences exceed some threshold value. Let \( q = \vert Q \vert \) and observe that if all possible pairs of partial coherences are of interest, then \( Q = \{(u,v) : u, v = 1, \ldots, p, u < v\} \) and \( q = (p^2 - p)/2 \). For any \( (u,v) \in Q \), let \( H_0^{(u,v)} \) and \( H_1^{(u,v)} \) be the corresponding null and alternative hypothesis specified as in the previous section.

The main problem to be solved in implementing the above multiple testing procedure is that of controlling the false discovery rate (FDR). That is, ensuring that (at least asymptotically), the expected ratio of false rejections to the total number of rejections, does not exceed some desired level \( \alpha \). Controlling the FDR in multiple testing problems has a long-standing history in statistics, and different approaches have been proposed under a variety of assumptions; see
among others Benjamini and Hochberg (1995); Benjamini and Yekutieli (2001); Barber and Candès (2015). In our setting, more appropriate seems to be a thresholding based approach applied to the test statistics $T^{(u,v)}_n$ (see (20)), for all $(u, v) \in Q$. In this context, the null hypothesis $H_0^{(u,v)}$ is rejected if and only if the test statistic $T^{(u,v)}_n$ exceeds a specified threshold value. Such an approach for FDR control has been introduced in the i.i.d. context by Liu (2013). Clearly, the key issue here is how to select the threshold value so that the desired FDR control is achieved. Toward this goal, we introduce the threshold

$$\hat{t} = \min \left\{ 0 \leq t \leq 2 \log(dq) : \frac{G_d(t)q}{\max(1, \sum_{(u,v) \in Q} 1(T^{(u,v)}_n \geq t))} \leq \alpha \right\},$$  \hspace{1cm} (22)

The multiple testing procedure proposed in this paper, rejects then for every $(u, v) \in Q$ the corresponding null hypothesis if and only if $T^{(u,v)}_n > \hat{t}$. The idea behind the construction of $\hat{t}$ is that $G_d(t)q$ approximates the expected number of falsely rejected nulls for a given threshold $t$. Since a sparse signal setting is considered, the total number of nulls can be approximated by $q$, i.e., the total number of conducted tests. At the same time and by Theorem 3, $G_d(t)$ approximates well enough the upper tail of the distribution of a single test statistic under the null.

To state our next result which deals with the theoretical properties of the described thresholding procedure, we need to fix some additional notation. Let $H_0 = \{(u,v) \in Q : \sup_{\omega \in [0,2\pi]} |\rho_{u,v}(\omega)| \leq \delta, \}$ be the set of true null hypotheses and let for $\mu > 0$, $H(\mu) = \{(u,v) \in Q : \sup_{\omega} |\rho_{u,v}(\omega)| > \delta + \mu \}$ be the set of alternative hypotheses for which the corresponding partial coherences exceed $\delta$ by $\mu$. Recall the definition of the false discovery rate which is given by

$$FDR = E\left( \frac{\sum_{(u,v) \in H_0} 1(T^{(u,v)}_n \geq \hat{t})}{\max(\sum_{(u,v) \in Q} 1(T^{(u,v)}_n \geq \hat{t}), 1)} \right).$$

The following result shows that the proposed threshold-based, multiple testing procedure, succeeds in properly controlling the FDR.

**Theorem 5.** Suppose that the assumptions of Theorem 3, $q/|H_0| = 1 + o(1)$, and

$$|H\left(2\sqrt{M/n \log(dq)}\right)| \geq \log(\log(n)).$$

hold true. Then, it holds true for any $\alpha \in (0,1)$, that, as $n \to \infty$,

$$FDR \cdot \frac{q}{|H_0|} \leq \alpha + o(1).$$
4. Implementation issues and Simulations

4.1. Implementation Issues

Before summarizing the main steps involved in the practical implementation of our procedure, we first elaborate on the problem of spectral density estimation using a prewhitening approach.

It is known that, prewhitening or prefiltering can improve the finite sample behavior of nonparametric spectral density estimators; see for instance Section 5.8 in Brillinger (2001). The basic idea is that an appropriate filter applied to the time series at hand, can lead to a smoother spectral density of the filtered time series and consequently, to a less biased nonparametric estimator. In the multivariate set-up, an additional benefit is that prefiltering homogenizes (to a certain extend) the spectral densities of the filtered component processes, allowing, therefore, for the application of the same bandwidth to estimate the spectral density matrix. Note that using individual bandwidths in multivariate spectral density estimation has the disadvantage that the estimated spectral density matrix is not equivariant and that positive definiteness is not guaranteed. To elaborate more on the prefiltering procedure for spectral density estimation, let \( \Phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^j \) be a linear filter and define \( \Phi(\omega) := \Phi(\exp(-i\omega)) \) for \( \omega \in [0, 2\pi] \). Autoregressive filters form one important class of such filters. In the high-dimensional context, applying such a filter leads to the fit of a sparse vector autoregressive model to the time series at hand. Note that the idea of prefiltering also can be used in the construction of the estimator \( \hat{\beta}^{(de)}_{v, u}(\omega) \). More precisely, note that \( \hat{\beta}^{(de)}_{v, u}(\omega) \) also can be written as

\[
\hat{\beta}^{(de)}_{v, u}(\omega) = \hat{\beta}_{v, u}(\omega) + \frac{(e_v - I_{p,v,\hat{\gamma}_{v,u}}) H (\sum_{k=1}^n \kappa_M(\omega - \omega_k) Z_n(\omega_k) Z_n(\omega_k)^H) I_{p,\hat{\gamma}_{v,u}}(\omega)}{e_u \sum_{k=1}^n \kappa_M(\omega - \omega_k) Z_n(\omega_k) Z_n(\omega_k)^H I_{p,\hat{\gamma}_{v,u}}(\omega)}.
\]

Prefiltering in the frequency domain implies then that in the expression above, the term \( \sum_{k=1}^n \kappa_M(\omega - \omega_k) Z_n(\omega_k) Z_n(\omega_k)^H \) will be replaced by

\[
\Phi(\omega)^{-1} \left( \sum_{k=1}^n \kappa_M(\omega - \omega_k) \Phi(\omega_k) Z_n(\omega_k) Z_n(\omega_k)^H \Phi(\omega_k)^H \right)^{-1} (\Phi(\omega)^H)^{-1}.
\]

For the special case of using an autoregressive filter, this filter also can be applied in the time domain. Let \( \Phi(z) = 1 - \sum_{j=1}^m \phi_j z \) and \( Y_t = X_t - \sum_{j=1}^m \phi_j X_{t-j}, t = \ldots, 0, 1, \ldots, n \).
For $m + 1, m + 2, \ldots, n$. For

$$\tilde{Z}_n(\omega) = \frac{1}{\sqrt{2\pi(n-m)}} \sum_{t=1}^{n-m} Y_{t+m} \exp(-i\omega t),$$

the term $\sum_{k=1}^{n} \kappa_M(\omega - \omega_k)Z_n(\omega_k)Z_n(\omega_k)^H$ appearing in $\hat{\beta}_{u,v}^{(de)}(\omega)$ has to be replaced by

$$\Phi(\omega)^{-1}(\sum_{k=1}^{n-m} \kappa_M(\omega - \tilde{\omega}_k)\tilde{Z}_n(\tilde{\omega}_k)\tilde{Z}_n(\tilde{\omega}_k)^H)(\Phi(\omega)^H)^{-1},$$

where $\tilde{\omega}_k = 2\pi(k-1)/(n-m), k = 1, \ldots, n - m$ are the Fourier frequencies corresponding to $n - m$ observations.

The different steps involved in the practical implementation of our inference procedure discussed so far and designed to detect significant partial coherences, can be summarized as follows.

(i) Select a filter $\Phi(z)$ to prewhitening the vector time series at hand.

(ii) Select a global bandwidth (truncation lag) $M$ and a kernel $K$.

(iii) Use the grid of frequencies $L = \{\omega'_l = l\pi N/M : l \in 1, \ldots, M/N - 1\}$ to cover the frequency band $W$ of interest and set $N = \log(M)^{2/r}$.

(iv) Estimate the inverse spectral density matrix $f^{-1}$ at the frequencies $\omega'_l$ for every $l \in L$.

(v) Compute for all $(u, v) \in Q$ and all $l \in L$ the de-biased estimator $\hat{\rho}_{u,v}^{(de)}(\omega'_l)$ and the test statistic $T_n^{(u,v)}$ using (11) and (20), respectively.

(vi) Set $d = |L|$, $G_d(t) = 1 - (1 - \exp(-t/2))^d$ and compute the threshold

$$\hat{t} = \inf\{0 \leq t \leq 2 \log(dq) : \frac{G_d(t)q}{\max(1, \sum_{(u,v) \in Q} \mathbb{1}(T_n^{(u,v)} \geq t))} \leq \alpha\}.$$

(vii) For each $(u, v) \in Q$ reject $H_0^{(u,v)}$ if $T_n^{(u,v)} \geq \hat{t}$.

Some remarks on the above steps are in order. For (i), the use of a vector autoregressive filter is suggested which is implemented by fitting a sparse vector autoregressive model to the vector time series at hand. Regarding (ii), we suggest to use the simple rule proposed by Politis (2003) and adapted for the high-dimensional context. In particular, this rule is applied using the Frobenius norm of the matrices of sample autocorrelations calculated using the filtered time series obtained as in (i). We do not suggest the use of different bandwidths.
for each component time series as many expressions are obtained by using different component time series and this might destabilize the numerical results. Regarding the kernel, we suggest using kernels that ensure (semi-)positive definite results, for instance, the modified Bartlett Kernel \( K(u) = 1(|u| \leq 1)(1 - |u|) \).

Regarding (iv) we mention that depending on the time series at hand, different estimation procedures can be used. For instance, if the data does not have a factor structure, then a vector autoregressive filter together with a lag-window estimator applied to the filtered vector time series, that is a prewhitening type estimator, can be used. In this case, the inverse spectral density matrix can be estimated using, for instance, graphical lasso. However, if the data possesses a factor structure, then spectral density estimators which are designed for low-rank plus sparse structures can be applied; see Barigozzi and Farnè (2021) and Krampe and Margaritella (2021).

### 4.2. Simulations

We investigate the finite sample performance of the inference procedures developed in this paper. For this we consider the following six data generating processes where \( S_1 \) denotes the percentage of non-zero partial coherences and \( S_2 \) the percentage of partial coherences that exceed the value 0.2. The particular parameterization of the coefficient matrices of the corresponding vector autoregressive moving-average (VARMA) and vector moving-average (VMA) processes are reported as R-Data files in the supplementary material.

(a) A \( p = 50 \) dimensional sparse VARMA(1,1) model with \( S_1 = 7.5\% \) and \( S_2 = 6.3\% \).

(b) A \( p = 100 \) dimensional sparse VARMA(1,1) model with \( S_1 = 4.1\% \) and \( S_2 = 3.5\% \).

(c) A \( p = 200 \) dimensional sparse VARMA(1,1) model with \( S_1 = 1.9\% \) and \( S_2 = 1.5\% \).

(d) A \( p = 50 \) dimensional sparse VMA(5) model with \( S_1 = 3.9\% \) and \( S_2 = 3.8\% \).

(e) A \( p = 100 \) dimensional sparse VMA(5) model with \( S_1 = 4\% \) and \( S_2 = 3.6\% \).

(f) A \( p = 200 \) dimensional sparse VMA(5) model with \( S_1 = 4\% \) and \( S_2 = 2.8\% \).

Three sample sizes, namely \( n = 512, 2048 \) and \( n = 4096 \), are considered and the results presented are based on \( B = 255 \) replications and on implementations in R (R Core Team, 2021). For this implementation we follow the steps described
in Subsection 4.1. The frequency band $\mathcal{W}$ of interest is set equal to $[0, \pi]$ and the set of hypotheses tested equals $Q = \{(u,v) | 1 \leq u < v \leq p\}$. Two different threshold values $\delta$ are considered, $\delta = 0$ and $\delta = 0.2$. For the prefiltering step, we use a sparse vector autoregressive model of order $\lceil \log_{10}(n) \rceil$ estimated by a row-wise adaptive lasso with tuning parameter selected by BIC; see for instance Krampe and Paparoditis (2021) for details. The BIC is adapted to handle a diverging number of parameters using the approach in Wang, Li and Leng (2009) and the parameter $C_n$ therein is set equal to $C_n = \log_2(n)$. To determine the bandwidth of the lag-window estimator, we use the rule described in Section 4.1 adapted from Politis (2003) with tuning parameters $K_n = 5$ and $c_{\text{thres}} = 1.5$.

We use the modified Bartlett kernel, i.e. $K(u) = 1(u \in (-1,1))(1 - |u|)$, which has Fourier representation $k_{1/M}(\omega) = (2\pi M)^{-1} \sin^2(M\omega/2)/\sin^2(\omega/2)$ and is positive definite. Furthermore, we set $N = \log(M)$. To estimate the inverse spectral density matrix, the aforementioned prefiltering procedure is used in combination with the lag-window estimator (3) applied to the filtered vector time series. The inverse spectral density matrix is then calculated applying a frequency-by-frequency graphical lasso procedure with tuning parameter chosen by BIC.

Table 1 presents the empirical false discovery rates and powers for the case $\delta = 0$ and for the different models, levels, and sample sizes considered in the simulation study. To compare the results of our inference procedure we use as a benchmark the selection of non-zero partial coherences obtained using a regularized estimator of the inverse spectral density matrix. For this, we estimate the inverse spectral density matrix $f^{-1}$ using the frequency-by-frequency graphical lasso. The tuning parameter is selected again by BIC and additionally, the outcome is thresholded at the tuning parameter to get a sparse result. This approach is called in the following Regularizing, while the procedure proposed in this paper and which uses statistical tests based on the de-biased estimators of partial coherences together with FDR control, is called Testing.

For the case $\delta = 0$, the empirical FDR of Testing is for all processes, sample sizes, and dimensions close to the nominal level. In terms of power, Testing outperforms Regularizing in all situations. The dimension affects the power only slightly for the case of the VARMA(1,1) model whereas the same effect is larger for the case of the VMA(5) model. The power improves as the sample size increases. Results for the case $\delta = 0.2$ are given in the Supplementary File.
We additionally, visualize in Figure 1 the results for the VARMA(1, 1) process given in (a) and the VMA(5) process given in (d) for the sample sizes $n = 512$ and $n = 4096$. In the context of this figure, a blank plot would describe the best situation while colored dots indicate some form of error. Highly blue-saturated dots indicate that the underlying positive partial coherence is not detected in most of the cases. Highly red-saturated dots indicate a systematic false detection error. Let us discuss first the results for the VARMA(1, 1) process. Here, we clearly see the benefits of the increased sample size. The higher power of Testing is marked by far fewer blue dots appearing in the left part of Figure 1 compared to the right panel of the same figure. Since Testing has no red dots, this method seems to have no systematic error. To elaborate, the three highest false detection rates for an individual partial coherence in both sample sizes together are around 12%, 3.5% and 2.7%. By contrast, Regularizing has systematic errors where zero partial coherences are identified as non zero. Even for the sample size $n = 4096$, zero partial coherences are wrongly detected in 100% of the cases.
For the sample size $n = 512$, the false detection rate behind the four red dots presented in the upper right panel of Figure 1, are 55%, 72%, 82% and 94%. For the VMA(5) processes, both approaches give reasonable results and no approach has a systematic error. However, Testing outperforms Regularizing in terms of power.

To summarize the findings of our simulation study, Testing successfully controls the false discovery rate at a desired level $\alpha$ and in contrast to Regularizing has larger power and it is not prone to systematic errors.

5. Applications: A Graphical Model for Brain Connectivity

In this section, we study brain connectivity based on electroencephalography (EEG) data. For this, we use the coherence and partial coherence to measure the connectivity and conditional connectivity, respectively, of several regions of the brain. We display non-zero coherences and partial coherences in terms of graphs keeping in mind that a graph based on partial coherences is the time series analog of a graphical model based on partial correlations for i.i.d. data.

We use the data set provided by Trujillo, Stanfield and Vela (2017). The data set consists of EEG recordings of 22 undergraduate students, where for each student, 72 channels with a sampling rate of 256Hz are used. The students were recorded in a resting state with 4 minutes of eyes open and 4 minutes of eyes closed. Trujillo, Stanfield and Vela (2017) preprocess the data. Among other steps, they split the data into epochs of one second, i.e., each epoch consists of 256 observations. The epochs are constructed with 50% overlap. Epochs with artifacts caused, for instance, by blinks or muscles were removed automatically and by hand. Additionally, they apply band-pass filters to remove linear trends and other noise effects. Finally, they focus on the alpha band ($4 - 13\text{Hz}$) and the beta band ($14 - 25\text{Hz}$). In our application, we use the preprocessed data and focus on the beta band, which is available for download from https://doi.org/10.18738/T8/CNVLAM. We consider two students (student with number 19 and 20, respectively). For these two students, no channels were interpolated and 209 to 387 epochs are available. Then, for each epoch (i.e. $n = 256, p = 72$), we compute the brain connectivity for the beta band. That is we use the multiple testing procedure described in Section 3 to test whether coherences and partial coherences having indices belonging to the set $Q = \{(u, v) : u, v = 1, \ldots, 72, u < v\}$ are zero ($\delta = 0$) in the frequency band.
$W = [14\text{Hz}, 25\text{Hz}]$. We use the same implementation with prewhiteting, automatic bandwidth selection, and the modified Bartlett kernel as in the simulation study. For the FDR control, we set $\alpha = 0.1$. The results obtained are averaged for each student and each state (eyes open and eyes closed). To display the results, we only keep those edges which are present in more than 50% of the epochs.

In Figure 2, we display the graphs obtained using coherences and partial coherences for student 19 in the same state of eyes open. As it is seen that the brain connectivity based on coherences is quite dense with 35% of all possible connections identified as non-zero. For other states and students, we can observe values up to 55%. This is in stark contrast to the results obtained using partial coherences where the focus is on the direct effects only. Here, only 6% of all possible connections are identified as non-zero. Furthermore, connections based on partial coherences are mainly between neighboring brain regions. This is not the case for the results of brain connectivity analysis based on coherences, where connections are strong not only between neighboring regions but also between regions which are far apart.

Concerning the differences in brain connectivity between the two states, eyes open and eyes closed, we focus on the partial coherence results only. Figure 3 presents the corresponding graphical model in the two aforementioned states. We observe that in the state of eyes closed, brain connectivity is higher. In particular, measuring brain connectivity by the number of edges divided by the number of all possible edges, we have for student 19 a connectivity of 6% in the state of open eyes and of 9% in the state of closed eyes. The corresponding percentages for student 20, are 11% and 14%, respectively.

Acknowledgments. The research of the first author was supported by the Research Center (SFB) 884 “Political Economy of Reforms” (Project B6), funded by the German Research Foundation (DFG). Furthermore, the first author acknowledges support by the state of Baden-Württemberg through bwHPC. The research of the second author has been supported by a University of Cyprus research grant.

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6. Auxiliary Results and Proofs

In this section, we present some auxiliary lemmas used to proof the main results of this paper. Lemma 6 is the key lemma used to express the de-biased partial coherence as a quadratic form and uses results given in Subsection B.1 of the supplementary file. The expression as quadratic form is the starting point to obtain a Gaussian approximation, where the steps involved are described in more detail in Remark 1. The corresponding lemmas can be found in Subsection B.2 of the supplementary file. Lemma 7 deals with the uniform convergence of the maximum of the de-biased partial coherence over a growing grid of frequencies. To establish the corresponding results, convergences rates of the covariance structure of the lag-window estimator and of the quadratic forms involved are required. The results used in this context are stated as auxiliary lemmas in Subsection B.3 of the supplementary file. Since Lemma 7 is the key ingredient to proof Theorem 3, Theorem 4 and Theorem 5, we present its proof here. The proofs of the other lemmas as well as the proof of Theorem 5 are deferred to Section A of the supplementary file.

We first fix some additional notation used in the proofs. \( \{Z_n\} \) denotes a sequence of random variables and \( \{f_n(\cdot)\} \) a sequence of functions. If \( P(|Z_n| \geq x) \leq Cf_n(x) \), for some constant \( C > 0 \) and for all \( n \), then the notation \( Z_n = \tilde{O}_p(f_n(x)) \) is used. Note that for \( A_n = \tilde{O}_p(f_n(x)), B_n = \tilde{O}_p(g_n(x)) \), we have \( A_n + B_n = \tilde{O}_p((f_n(x/2) + g_n(x/2))) \) because \( P(|A_n + B_n| \geq x) \leq P(|A_n| \geq x/2) + P(|B_n| \geq x/2) \). Similarly, \( A_n \cdot B_n = \tilde{O}_p((f_n(\sqrt{X}) + g_n(\sqrt{X}))) \) and for \( Z_n = \tilde{O}_p(f_n(x)) \) and a sequence \( x_n \), we have \( Z_n + x_n = \tilde{O}_p(f_n(x - x_n)) \).

**Lemma 6.** Let \( \omega_l = \pi l N/M, l = 1, \ldots, M/N - 1 \). Under Assumption 1, 2, 3, and 4, we have

\[
\sqrt{n/M} \sup_l \left| (\hat{\beta}_{u,v}^{(de)}(\omega_l') - \rho_{u,v}(\omega_l')) - \left[ \frac{1}{n} T_{n,u,v}(l) - \frac{1}{2n} \rho_{u,v}(\omega_l') [T_{n,v,v}(l) + T_{n,u,u}(l)] \right] \right| = \tilde{O}_p(g_{\beta(\tilde{e})} (x, p, n, M, \tau, s(p))),
\]

where

\[
g_{\beta(\tilde{e})} (x, p, n, M, \tau, s(p)) = (g((x-g_{Bias})/(s(p)(n/M)^{1/4}))^{1/2}, p, n, M, \tau) + g(1, p, n, M, \tau),
\]
\[
T_{n,u,v}(l) = \frac{1}{2\pi} \sum_{u=-n+1}^{n-1} K(uh) \sum_{t=\max(1,1-u)}^{\min(n,n-u)} [U_{t+u,l}V_{t,l} - EU_{t+u,l}V_{t,l}] \exp(-iu\omega_l^t),
\]
and, for \(l = 1, \ldots, M/N - 1,\)
\[
U_{t,l} = c_u^\top f^{-1}(\omega_l^t)X_t(f^{-1}_u(\omega_l^t))^{-1/2} \quad \text{and} \quad V_{t,l} = c_v^\top f^{-1}(\omega_l^t)X_t(f^{-1}_v(\omega_l^t))^{-1/2}.
\]

Under Assumption 3 we have that the processes \(\{U_{t,l}\}\) and \(\{V_{t,l}\}\) posses the same functional dependence as \(\{X_t\}\).

**Remark 1.** The strategy used to proof the results presented in Lemma 7, is to first approximate the random variables \(U_{t,l}\) and \(V_{t,l}\) defined in Lemma 6, by \(m\)-dependent and bounded random variables. The corresponding sums are then split into chunks of big and small blocks so that the big blocks are independent from each other. For this, we follow ideas used in Liu and Wu (2010). We can approximate \(\{X_t\}\) by the \(m\)-dependent process \(\{X_{t,m} = E(X_t|\varepsilon_{t-m}, \ldots, \varepsilon_t)\}\) and denote by \(d_{m,\tau}\) the approximation error. By the assumptions made, we have \(d_{m,\tau} = \sum_{t=0}^{\infty} \min(\delta_{t,\tau}^{(m)}, \Psi_{m+1,\tau}) = O(|\rho|^m)\) for some \(\rho < 1\); see Lemma 12. Consequently, for \(\varepsilon > 0\) small and \(d_\xi > 0\), we set \(m = \left\lceil \log((d_{\xi} + \varepsilon) \log(n))/(\log(\rho)) \right\rceil\), which implies that \(d_{m,\tau} = o(\log(n)^{-d_\xi})\).

Replacing \(X_t\) by \(X_{t,m}\) in the construction of \(U_{t,l}\) and \(V_{t,l}\) used in Lemma 6, leads to the random sequences \(\{U_{t,l,m}\}\), \(\{V_{t,l,m}\}\) and \(T_{n,u,v,m}(l)\). We further define
\[
g_n^{[u,v]}(\omega_l^t) = T_{n,u,v}(l) - E(T_{n,u,v}(l)) - \sum_{t=1}^{n} (U_{t,l}V_{t,l} - EU_{t,l}V_{t,l})
\]
and
\[
g_n^{[u,v,m]}(\omega_l^t) = T_{n,u,v,m}(l) - E(T_{n,u,v,m}(l)) - \sum_{t=1}^{n} (U_{t,l,m}V_{t,l,m} - EU_{t,l,m}V_{t,l,m})
\]

The next step in our proof is to truncate \(U_{t,l,m}\) and \(V_{t,l,m}\) in order to obtain bounded random variables. For this we consider
\[
U'_{t,l,m} = U_{t,l,m} \mathbb{1}(|U_{t,l,m}| \leq (Mn)^\kappa) \quad \text{and} \quad V'_{t,l,m} = V_{t,l,m} \mathbb{1}(|V_{t,l,m}| \leq (Mn)^\kappa),
\]
where \(\kappa = 1/\tau\). We then focus on the centered random variables
\[
\bar{U}_{t,l,m} = U'_{t,l,m} - EU'_{t,l,m} \quad \text{and} \quad \bar{V}_{t,l,m} = V'_{t,l,m} - EV'_{t,l,m}.
\]
For the construction of the big and small blocks, let \( \rho_n = \lfloor (M)^{1+\beta} \rfloor, q_n = M+m, \) and \( k_n = \lfloor n/(\rho_n + q_n) \rfloor. \) Note that \( k_n = O(n^{1-\alpha(1+\beta)}) \) and recall that \( M = n^\alpha. \) We need \( k_n = n^\varepsilon \) for some \( \varepsilon > 0, \) so we set \( \beta = 1/4 - 2\alpha \) and \( \alpha \leq 4/7. \)

To proceed we split the interval \([1, n]\) into alternating big and small blocks of lengths \( H_j \) and \( I_j, \) respectively, where for \( 1 \leq j \leq k_n, \)

\[
H_j = [(j-1)(p_n + q_n) + 1, jp_n + (j-1)q_n], \quad I_j = [jp_n + (j-1)q_n + 1, j(p_n + q_n)],
\]

and \( I_{k_n+1} = [k_n(p_n + q_n) + 1, n] \) is the remaining block. We set

\[
\tilde{Y}_{t,m}^{[u,v]}(l) = U_{t,m} \sum_{s=1}^{t-1} K((t-s)/M) \exp(-i(t-s)\omega'_l) \tilde{V}_{s,m} + V_{t,m} \sum_{s=1}^{t-1} K((t-s)/M) \exp(i(t-s)\omega'_l) \tilde{U}_{s,m}. \tag{23}
\]

The above lag-window estimator uses the sample autocovariances between \( \tilde{U}_{t,m}, \tilde{V}_{t,m} \) and \( \tilde{U}_{t,m}, \tilde{V}_{t,m} \) for lags \( u = -n+1, \ldots, n+1. \) The first term on the right hand side of the above equation corresponds to such sample autocovariances for lags \( u \geq 1, \) while the second term on the right hand side of (23), to lags \( u \leq -1. \) Observe that the sample autocovariance at lag \( u = 0 \) is removed; also see the construction of \( g_{n}^{[u,v]}(\omega'_l). \) Using \( \tilde{Y}_{t,m}^{[u,v]}(l) \) we further introduce the random variables

\[
u_j(l)^{[u,v]} = \sum_{t \in H_j} (\tilde{Y}_{t,m}^{[u,v]} - E\tilde{Y}_{t,m}^{[u,v]})
\]

for \( 1 \leq j \leq k_n, \) and

\[
v_j(l)^{[u,v]} = \sum_{t \in I_j} (\tilde{Y}_{t,m}^{[u,v]} - E\tilde{Y}_{t,m}^{[u,v]})
\]

for \( 1 \leq j \leq k_n+1. \) We then have

\[
\frac{1}{2\pi} \sum_{u=-n+1}^{n-1} K(\omega) \min(n,n-u) \sum_{t=\max(1,1-u)}^{\min(n,n-u)} [\tilde{U}_{t+u,m} \tilde{V}_{t,m} - E\tilde{U}_{t+u,m} \tilde{V}_{t,m}] \exp(-iu\omega'_l) \]

\[
- \frac{1}{2\pi} \sum_{u=-n+1}^{n-1} K(\omega) \min(n,n-u) \sum_{t=\max(1,1-u)}^{\min(n,n-u)} [\tilde{U}_{t+u,m} \tilde{V}_{t,m} - E\tilde{U}_{t+u,m} \tilde{V}_{t,m}] \exp(-iu\omega'_l) \]

\[
- \sum_{t=1}^{n} (\tilde{U}_{t,m} \tilde{V}_{t,m} - E\tilde{U}_{t,m} \tilde{V}_{t,m}) \]

\[
= \sum_{j=1}^{k_n} u_j(l)^{[u,v]} + \sum_{j=1}^{k_n+1} v_j(l)^{[u,v]} =: g_{n,m}(l)^{[u,v]}.
\]
Proof of Lemma 7. Let ω'_l = πlN/M, l ∈ L and T_{n,u,v}(l) as defined in Remark 1, and Z_l i.i.d. with Z_l ∼ N(0, I_2). We have under Assumption 1 to 5 and for h_n > 0, that,

\[ \sup_{0 ≤ t ≤ h_n} \left| \frac{P(\max_l 1/(Mn)χ(l)[u,v] ≥ t)}{P(\max_l ||Z_l||_2 ≥ t)} - 1 \right| ≤ C((\log(n^{d_\xi}))^{-\zeta_\xi} + 3/2 + \max(2, \exp(h_n/2)) \times \log(n)^C \left[ g_{\beta(\omega')} (\log(n), p, n, M, \tau, s(p)) + (n^{1/2(1-1/4a)-1/\tau(2-\alpha)-2(1+\alpha)/\tau^2})^{\tau} \right], \]

where ζ(l) := χ_{n,u,v}(ω'_l).

Proof of Lemma 7. Without loss of generality, we set L = \{1, \ldots, M/N - 1\}. We first establish the following equality

\[ P(\max_l χ(l)[u,v] ≥ t) = P(\max_l ||\tilde{Z}_l||^2 + W_n ≥ t), \]
where $\tilde{Z}_t$ are 2-dimensional standard Gaussians and $W_n$ denotes an error term satisfying for some $\delta > 0$, $P(|W_n| > \delta) \leq \Delta(\delta)$. To elaborate, let

$$
R_t := \sqrt{n/M}((\Re(\hat{\rho}_{u,v}^{(de)}(\omega_i^t) - \rho_{u,v}(\omega_i^t)), \Im(\hat{\rho}_{u,v}^{(de)}(\omega_i^t) - \rho_{u,v}(\omega_i^t)))^T
$$

such that $\chi(l) =: R_t^T \hat{\Sigma}^{-1}(\omega_i^t) R_t$. Also let $\tilde{Z}_t = (\Sigma^{-1}(\omega_i^t))^{1/2} \tilde{Z}_t$. We can then write

$$
\chi(l)^{u,v} = \tilde{Z}_t^T \Sigma^{-1}(\omega_i^t) \tilde{Z}_t + (\tilde{Z}_t - R_t)^T \Sigma^{-1}(\omega_i^t) (\tilde{Z}_t - R_t) - (\tilde{Z}_t - R_t)^T \Sigma^{-1}(\omega_i^t) R_t \tilde{Z}_t + R_t^T (\Sigma^{-1}(\omega_i^t) - \Sigma^{-1}(\omega_i^t)) R_t
$$

$$=: \|\tilde{Z}_t\|^2 + W_n.
$$

That is, we need to determine the order of $W_n$, i.e. of $\Delta(\cdot)$. Note that by Lemma 6 we have the following

$$
\max_i \|R_t\|_2 = \tilde{O}_p(g_{\beta(x,p,n,M,\tau,s(p))} + g(x/\sqrt{n/M}, p, n, M, \tau)),
$$

$$
\|\tilde{\Sigma}(\omega_i^t) - \Sigma(\omega_i^t)\|_2 \leq C\sqrt{M/n} R_t \|_2 = \tilde{O}_p(g_{\beta(x,p,n,M,\tau,s(p))}) + g(x/\sqrt{n/M}, p, n, M, \tau)),
$$

and

$$
\|\Sigma^{-1}(\omega_i^t) - \Sigma^{-1}(\omega_i^t)\|_2 \leq C\|\tilde{\Sigma}(\omega_i^t) - \Sigma(\omega_i^t)\|_2/\sqrt{1 - C\|\tilde{\Sigma}(\omega_i^t) - \Sigma(\omega_i^t)\|_2}.
$$

Furthermore, since $(\Sigma^{-1}(\omega_i^t))^{1/2} Z_t \sim N(0, I_2)$ and $d \leq M$, we have

$$
P(\max_i \|\Sigma^{-1}(\omega_i^t))^{1/2} Z_t \|_2 \geq 2) \leq MP(\|\Sigma^{-1}(\omega_i^t))^{1/2} Z_t \|_2 \geq 2) = M \exp(-\sqrt{2}/2).
$$

It remains to establish the order of $\max_i \|\Sigma^{-1}(\omega_i^t))^{1/2} (Z_t - R_t)\|_2$. We will show that $(\tilde{Z}_t - R_t)^T \Sigma^{-1}(\omega_i^t) (\tilde{Z}_t - R_t)$ is the dominating term. To determine the order of $\max_i \|\Sigma^{-1}(\omega_i^t))^{1/2} (Z_t - R_t)\|_2$ we split it up into three approximation terms and obtain for each of them an order which is denoted $\Delta_i(\cdot), i = 1, 2, 3$.

For this note first that as outlined in Remark 1, we can approximate our random variables by bounded, $m$-dependent variables. That is we approximate $R_t$ by $\sum_{j=1}^{k_n} (\Re(\hat{w}_j(l)^{[u,v]}), \Im(\hat{w}_j(l)^{[u,v]}))$. Putting the results of Lemma 11 to 16 together gives

$$
\text{Cov}_{l}(n)^{1/2} |\text{max}_l \Sigma^{-1}(\omega_i^t))^{1/2} [R_t - \sum_{j=1}^{k_n} (\Re(\hat{w}_j(l)^{[u,v]}), \Im(\hat{w}_j(l)^{[u,v]}))]|_2 = \tilde{O}_p(g_{\beta(x,p,n,M,\tau,s(p))} + C\log(n)|X_n^{1/2}(1-\tau)/\text{Cov}_{l}(n)^{1/2} - 1\tau/2(1+\tau)^{1/2})^{-\tau} + x^{C\tau(Mn)^{-\tau}} + \frac{n}{x^{\tau/2}(Mn)^{\tau/2}} = \tilde{O}_p(\Delta_1(x))
$$
In the second approximation, we introduce the Gaussian approximation. We rewrite the statistic by using bounded, $m$-dependent variables as follows

$$\max_l (\Sigma^{-1}(\omega'_l))^{1/2} \frac{1}{k_n} \sum_{j=1}^{k_n} (\text{Re}(\hat{w}_j(l)^{[u,v]}), \text{Im}(\hat{w}_j(l)^{[u,v]})) = \max_l (\frac{1}{k_n} \sum_{j=1}^{k_n} \hat{e}_l^\top \tilde{A}_j),$$

where $\hat{e}_j = (e_j \otimes (1,1)^\top) \in \mathbb{R}^{2d}$, and $\tilde{A}_j$ is a $2d$-dimensional vector given by

$$\tilde{A}_j = \begin{pmatrix}
(\sigma_1(l) \text{Re}(\hat{w}_j(1)^{[u,v]})) + \sigma_3(l) \text{Im}(\hat{w}_j(1)^{[u,v]})) \\
(\sigma_3(l) \text{Re}(\hat{w}_j(1)^{[u,v]})) + \sigma_2(l) \text{Im}(\hat{w}_j(1)^{[u,v]})) \\
\vdots \\
(\sigma_3(d) \text{Re}(\hat{w}_j(d)^{[u,v]})) + \sigma_2(d) \text{Im}(\hat{w}_j(d)^{[u,v]})) \\
(\sigma_1(d) \text{Re}(\hat{w}_j(d)^{[u,v]})) + \sigma_3(d) \text{Im}(\hat{w}_j(d)^{[u,v]}))
\end{pmatrix}^\top,$$

and

$$(\Sigma^{-1}(\omega'_l))^{1/2} = \begin{pmatrix}
\sigma_1(l) & \sigma_3(l) \\
\sigma_3(l) & \sigma_2(l)
\end{pmatrix}.$$
i.e., we have \( Z \sim \mathcal{N}(0, H) \). Let \( \tilde{Z} = H^{-1/2}Z \) such that \( \tilde{Z} \sim \mathcal{N}(0, I_{2d}) \). Then, we have further for \( x > 0 \)

\[
P(\max(\|e_i^T (H^{1/2} - I_{2d}) \tilde{Z}\|_2 \geq x)) \leq \sum_{l=1}^{d} P(\|e_{2(l-1)+1}^T + e_{2(l-1)+2}^T (H^{1/2} - I_{2d}) \tilde{Z}\|_2 \geq x)
\]

\[
\leq \sum_{l=1}^{d} (P(\|e_{2(l-1)+1}^T (H^{1/2} - I_{2d}) \tilde{Z}\|_2 \geq x/2) + P(\|e_{2(l-1)+2}^T (H^{1/2} - I_{2d}) \tilde{Z}\|_2 \geq x/2)).
\]

The two probabilities can be bounded by the same arguments. We focus here on the first. Note that \( e_{2(l-1)+1}^T (H^{1/2} - I_{2d}) \tilde{Z} \sim N(0, \|e_{2(l-1)+1}^T (H^{1/2} - I_{2d})\|_2^2) \).

Additionally, we have

\[
\|e_{2(l-1)+1}^T (H^{1/2} - I_{2d})\|_2 \leq \|e_{2(l-1)+1}^T (H - I_{2d})\|_2 \|(H^{1/2} + I_{2d})^{-1}\|_2.
\]

Since \( H \) is a variance matrix and by definition positive semi-definite, we have \( \|(H^{1/2} + I_{2d})^{-1}\|_2 \leq 1 \). Furthermore,

\[
\|e_{2(l-1)+1}^T (H - I_{2d})\|_2^2 = \sum_{r=1, r \neq 2(l-1)+1}^{2d} (e_{2(l-1)+1}^T H e_r)^2 + (e_{2(l-1)+1}^T H e_{2(l-1)+1})^2
\]

and \( e_{2(l-1)+1}^T H e_{2(l-1)+1} = e_1^T \text{Var}(\tilde{Z} \sum_{j=1}^{k_n} \tilde{A}_j) e_1 \). The term \( \sum_{r=1, r \neq 2(l-1)+1}^{2d} (e_{2(l-1)+1}^T H e_r)^2 \) contains the covariance between different frequencies and we use Lemma 18 to determine its order. For other kernels than the uniform one, the additional terms \( 1/((r \pm (2l - 1) + 1)|N|^{-1}) \) appear. Note that the terms with \( r \) are \( \ell_2 \)-summable and that no periodicity in terms of the Fourier coefficients occur.

\[
\sum_{r=1, r \neq 2(l-1)+1}^{2d} (e_{2(l-1)+1}^T H e_r)^2 = O(M(M^{-2} + k_n^{-2}) + N^{-2r}) = O(n^{-a + n(3+2\beta) - 2} + N^{-2r}).
\]

For \( a < 4/7 \) and \( \beta < 1/4 \) we have \( O(n^{-a + n(3+2\beta) - 2}) = o(n^{-\tilde{v}}) \) for some \( \tilde{v} > 0 \). For the variance term, we have

\[
\tilde{e}_l^T \sum_{j=1}^{k_n} \tilde{A}_j = (\Sigma^{-1}(\omega_l))^{1/2} \sum_{j=1}^{k_n} \left( \text{Re} \left( \tilde{\mu}_j(l)^{[u,v]} - \rho_{u,v}(\omega_l)'/2[\tilde{\mu}_j(l)^{[u,u]} + \tilde{\mu}_j(l)^{[v,v]}] \right) \right)
\]

\[
+ \left( \text{Im} \left( \tilde{\mu}_j(l)^{[u,v]} - \rho_{u,v}(\omega_l)'/2[\tilde{\mu}_j(l)^{[u,u]} + \tilde{\mu}_j(l)^{[v,v]}] \right) \right).
\]

Additionally, we have by Lemma 12 to 16

\[
\max_{1 \leq l \leq d} 1/(Mn) \text{Var} \left( \sum_{j=1}^{k_n} \left[ \tilde{\mu}_j(l)^{[u,v]} - \rho_{u,v}(\omega_l)'/2[\tilde{\mu}_j(l)^{[u,u]} + \tilde{\mu}_j(l)^{[v,v]}] \right] \right)
\]

\[
- \left( 1/nT_{n,u,v}(l) - 1/2\rho_{u,v}(\omega_l)'[1/nT_{n,u,v}(l) + 1/nT_{n,u,u}(l)] \right) = o(n^{-\tilde{v}'}),
\]
where $0 < \nu' < \beta\alpha$. The same rate is also obtained for the real and imaginary part. Additionally, we have by Lemma 9

$$\max_l \|\text{Var} \left( \text{Re} \left( \frac{1}{\sqrt{n}} T_{n,u,v}(l) - 1/2 \rho_{u,v}(\omega_j') \left[ 1/2 \left( T_{n,v,v}(l) + 1/2 T_{n,u,u}(l) \right) \right] \right), \text{Im} \left( \frac{1}{\sqrt{n}} T_{n,u,v}(l) - 1/2 \rho_{u,v}(\omega_j') \left[ 1/2 \left( T_{n,v,v}(l) + 1/2 T_{n,u,u}(l) \right) \right] \right) \right) - \Sigma(\omega_j') \|_2 = O(M^{-1}).$$

Thus, we get $\max_1 \|\text{Var}(\tilde{\varepsilon}_l^T \sum_{j=1}^{k_n} \tilde{A}_j) - I_2 \|_2 = O(1/M) + o(n^{-\nu'}) = o\left(n^{-\nu'}\right)$. Consequently, we have with $N = \log(M)^2/r$ $\max_{l=2(l-1)+1} \|e_{2(l-1)+1}^T (H - I_2d)\|_2 = O(\log(M)^{-2}).$

With this bound and additional tail bounds for Gaussian random variables, see among others Appendix A in Chatterjee (2014), we have

$$P(\max_l (|\tilde{\varepsilon}_l^T (H^{1/2} - I_2d) \tilde{Z}_l|_2 \geq x)) \leq C(\exp(-x \log(M)^2) + \log(d)) \frac{1}{x \log(M)^2} =: \Delta_3(x).$$

Now we set $x = (\log(n^{d\varepsilon}))^{-\zeta\varepsilon + 3/2}$ with $d\varepsilon, \zeta\varepsilon$ as in Remark 1, especially $\zeta\varepsilon \geq 2$ and $\zeta\varepsilon < 3/2 + 2$. Then, $\Delta_3 = O(\exp(-C\log(n)^{1+\varepsilon})$, where $\varepsilon > 0$ and especially, we have that $\Delta_1$ obeys the slowest rate between the three terms $\Delta_1, \Delta_2, \Delta_3$. Hence, $P(\max_l (|\tilde{\varepsilon}_l^T (H^{1/2} - I_2d) \tilde{Z}_l - R_l|_2 \geq x)) \leq C\Delta_3(x)$ and we get

$$P(\max_l \chi(l)^{[u,v]} \geq t) = P(\max_l \|\tilde{Z}_l\|^2 + W_n \geq t),$$

where $W_n$ denotes the error and

$$P(|W_n| > x) \leq C P(||\Sigma^{-1}(\omega_j')||^{1/2} (Z_l - R_l)\|_2^2 > x) \leq C\Delta_1(\sqrt{x}).$$

Now we apply Lemma 19 and then, we obtain with $G_d(t) = (1 - (1 - \exp(-t/2))^d$,

$$\sup_{0 \leq t \leq h_n} \left| \frac{P(\max_l \chi(l)^{[u,v]} \geq t)}{G_d(t)} - 1 \right| \leq C \Delta_1 \left( \log(n^{d\varepsilon})^{-\zeta\varepsilon/2 + 3/4} \right)$$

$$\times \max(2, \exp(h_n/2)) + 2 \log(n^{d\varepsilon})^{-\zeta\varepsilon + 3/2}$$

$$\leq C \max(2, \exp(h_n/2)) \log(n)^{C\varepsilon} \left( g_{\beta\alpha}(\log(n), p, n, M, \tau, s(p)) \right)$$

$$+ \left( n^{1/2(1-1/4a+1/4+1)} - 1/2(a+1) - 2(1+a)/\tau + n^{-\tau/4(1+a)+1} \right) + 2 \log(n^{d\varepsilon})^{-\zeta\varepsilon + 3/2}.$$

\[ \square \]

Lemma 8. Let $\hat{f}_{M,c}(\omega) = \frac{M}{n} \sum_{k=1}^n \kappa_M(\omega - \omega_k)(Z(\omega_k)Z^H(\omega_k) - \Sigma_n(\omega_k))$ and $u_1, u_2, v_1, v_2 = 1, \ldots, p$. Under Assumption 1, 2, 3, we have

$$\sup_{\omega} |n/M \text{Cov}(e_{v_1}^T \hat{f}_{M,c}(\omega)e_{v_2}, e_{u_1}^T \hat{f}_{M,c}(\omega)e_{u_2})|$$
we have
\[ -C_{K_2}(f_{v_1,u_1}(\omega)f_{u_2,v_2}(\omega) + 1(\omega \in \pi \mathcal{Z})f_{v_1,u_2}(\omega) f_{v_2,u_1}(\omega))| = O(1/M) \]
and
\[
\sup_{\omega} \frac{n}{M} \text{Cov}(e_{v_1}^T f^{-1}(\omega) \hat{f}_{M,c}(\omega) f^{-1}(\omega) e_{v_2}, e_{u_1}^T f^{-1}(\omega) \hat{f}_{M,c}(\omega) f^{-1}(\omega) e_{u_2})
\]
\[ -C_{K_2}(f_{v_1,u_1}(\omega)f_{u_2,v_2}(\omega) + 1(\omega \in \pi \mathcal{Z})f_{v_1,u_2}(\omega) f_{v_2,u_1}(\omega))| = O(1/M). \]

**Lemma 9.** Under the conditions of Lemma 6 we have
\[
\max_l \left\| \frac{n}{M} \text{Var} \left( \frac{\text{Re}(1/nT_{n,u,v}(l) - 1/2\rho_{u,v}(\omega_l))}{\text{Im}(1/nT_{n,u,v}(l) - 1/2\rho_{u,v}(\omega_l))} \right) \right\|_2 = O(1/M),
\]
where
\[
\Sigma(\omega) = \frac{C_{K_2}(1 - |\rho_{u,v}(\omega)|^2)}{2} \begin{pmatrix} 1 - \text{Re}(\rho_{u,v}(\omega))^2 & -\text{Re}(\rho_{u,v}(\omega)) \text{Im}(\rho_{u,v}(\omega)) \\ -\text{Re}(\rho_{u,v}(\omega)) \text{Im}(\rho_{u,v}(\omega)) & 1 - \text{Im}(\rho_{u,v}(\omega))^2 \end{pmatrix}.
\]

**Proof of Theorem 1.** The assertion follows by Lemma 6, Lemma 9 and following the arguments of Lemma 7 for \(d = 1\). \(\Box\)

**Proof of Theorem 3.** The assertion follows by Lemma 7 with \(h_n = 2(a+2b) \log(n)\) and Assumption 5. \(\Box\)

**Proof of Corollary 2.** The assertions follows by Lemma 6 and Lemma 8. \(\Box\)

**Proof of Theorem 4.** Under \(H_0\), we have
\[
P(T_{n,u,v}^{[u,v]} \geq x) \leq P\left( \max_l \frac{n}{M} \begin{pmatrix} \text{Re}(\rho_{u,v}^{(de)}(\omega_l) - \rho_{u,v}(\omega_l)) \\ \text{Im}(\rho_{u,v}^{(de)}(\omega_l) - \rho_{u,v}(\omega_l)) \end{pmatrix} \hat{\Sigma}^{-1}(\omega_l) \begin{pmatrix} \text{Re}(\rho_{u,v}^{(de)}(\omega_l) - \rho_{u,v}(\omega_l)) \\ \text{Im}(\rho_{u,v}^{(de)}(\omega_l) - \rho_{u,v}(\omega_l)) \end{pmatrix} \geq x \right).
\]

To see this, let \(\tilde{\omega} \in \{\omega_l : l \in \mathcal{L}\}\) such that \(|\rho_{u,v}^{(de)}(\tilde{\omega})| > \delta\). If such a \(\tilde{\omega}\) does not exist, then \(T_{n,u,v}^{[u,v]} = 0\) and the above statement holds true. Since \(\hat{\Sigma}^{-1}(\omega_l)\) is positive (semi)-definite for all \(l\), it suffices to show that \(|\rho_{u,v}^{(de)}(\tilde{\omega}) - \delta \exp(i \text{arg}(\rho_{u,v}^{(de)}(\tilde{\omega})))| \leq |\rho_{u,v}^{(de)}(\tilde{\omega}) - \rho_{u,v}(\tilde{\omega})|\). Let \(\rho_{u,v}^{(de)}(\tilde{\omega}) = \delta' \exp(i \lambda')\) and \(\rho_{u,v}(\tilde{\omega}) = \delta \exp(i \lambda)\), where \(\delta' > \delta\). Under \(H_0\) and since \(|\rho_{u,v}^{(de)}(\tilde{\omega})| > \delta\), we have \(|\rho_{u,v}^{(de)}(\tilde{\omega})| > \delta\). Then, \(|\rho_{u,v}^{(de)}(\tilde{\omega}) - \delta \exp(i \text{arg}(\rho_{u,v}^{(de)}(\tilde{\omega})))| = |\delta' \exp(i \lambda') - \delta \exp(i \lambda')| = \delta' - \delta.\) Furthermore, \(|\rho_{u,v}^{(de)}(\tilde{\omega}) - \rho_{u,v}(\tilde{\omega})| \geq \rho_{u,v}^{(de)}(\tilde{\omega})| - |\rho_{u,v}(\tilde{\omega})| = \delta' - \delta \geq \delta' - \delta.\) This implies (24). The assertion follows then by Lemma 7. \(\Box\)
Proof of Theorem 5. Let \( H_0 = |\mathcal{H}_0| \leq q \) and set \( b_n = 2 \log(qd) \). Note first that by the inequalities \((1 + x)^d \leq 1/(1 - dx)\) for \( x \in [-1, 0] \) and \((1 + x)^d \geq 1 + dx\) for \( x \geq -1\), we have the following bounds: \( 1 \leq qG_d(b_n) = q(1 - (1 - 1/(qd))^d) \geq q/(q + 1) \geq 1/2\). Recall that

\[
\hat{t} = \inf \left\{ 0 \leq t \leq 2 \log(dq) : \frac{G_d(t)q}{\max(1, \sum_{(u,v) \in \mathcal{Q}} 1(T_{n}^{(u,v)} \geq t))} \leq \alpha \right\},
\]

Our goal is to show that

\[
E \left( \frac{\sum_{(u,v) \in \mathcal{H}_0} 1(T_{n}^{(u,v)} \geq \hat{t})}{\max(\sum_{(u,v) \in \mathcal{Q}} 1(T_{n}^{(u,v)} \geq \hat{t}), 1)} \right) = E \left( \frac{\sum_{(u,v) \in \mathcal{H}_a} 1(T_{n}^{(u,v)} \geq \hat{t})G_d(\hat{t})q}{\max(\sum_{(u,v) \in \mathcal{Q}} 1(T_{n}^{(u,v)} \geq \hat{t}), 1)G_d(\hat{t})q} \right) \leq \alpha.
\]

Consider the expression \( \max(\sum_{(u,v) \in \mathcal{Q}} 1(T_{n}^{(u,v)} \geq \hat{t}), 1) \). Let \( \mathcal{H}(2\sqrt{M}/nb_n) = \mathcal{H}_a \). We have by assumption \( |\mathcal{H}_a| \geq \log\log(n) \). Furthermore,

\[
\sum_{(u,v) \in \mathcal{Q}} 1(T_{n}^{(u,v)} \geq b_n) \geq \sum_{(u,v) \in \mathcal{H}_a} 1(T_{n}^{(u,v)} \geq b_n).
\]

We then have, for \((u, v) \in \mathcal{H}_a\), that

\[
P(T_{n}^{(u,v)} \geq b_n) = P((T_{n}^{(u,v)})^{1/2} \geq \sqrt{b_n}) \\
\geq P(\sqrt{n/M} \max_{l} \lambda \exp(\sum_{(\omega')_j \in (\omega')_i} - \delta |\rho_{u,v}(\omega')|)) \geq \sqrt{b_n}) \\
\geq P(\sqrt{n/M} \max_{l} \lambda \exp(\sum_{(\omega')_j \in (\omega')_i} - \delta |\rho_{u,v}(\omega')|)) \\
\geq P(\sqrt{n/M} \max_{l} \lambda \exp(\sum_{l} \chi_{[u,v]})) \geq \sqrt{b_n}) \\
\geq 1 - P(\max_{l} \chi_{[u,v]} \geq b_n) \\
\geq 1 - (1 - (1 + o(1))/q),
\]

where the second to last inequality follows by Lemma 7. Note that \( \Sigma(\omega'_j) \) has the eigenvalues \( C_{K_2}(1 - \rho_{u,v}(\omega'_j)^2)/2 \) and \( C_{K_2}(1 + \rho_{u,v}(\omega'_j)^2)/2 \), which imply that \( \lambda_{\min}(\Sigma(\omega'_j)) = 2(C_{K_2}(1 + \rho_{u,v}(\omega'_j)^2))^{-1} \geq 2 \).

This means that for \( n \) large enough, we have for some generic constant \( C > 0 \)

\[
P\left( \sum_{(u,v) \in \mathcal{H}_a} 1(T_{n}^{(u,v)} \geq b_n) \geq \log(\log(n)) \right) \geq 1 - |\mathcal{H}_a|/C/q = 1 - C(1 - |\mathcal{H}_0|/q).\]
By assumption $1 - |\mathcal{H}_0|/q = o(1)$ and therefore, it yields with high probability that $\sum_{(u,v) \in \mathcal{H}_0} \mathbb{I}(T_n^{[u,v]} \geq b_n) \geq \log(\log(n))$. Together with $qG_d(b_n) \leq 1$ we get for $n$ large enough with high probability, that,

$$\frac{G_d(b_n)q}{\max(1, \sum_{(u,v) \in Q} \mathbb{I}(T_n^{[u,v]} \geq b_n))} \leq C \log(\log(n))^{-1} < \alpha.$$ 

Consequently, $P(\hat{t} \in [0, b_n)) \to 1$ for $n \to \infty$.

If $\hat{t}$ is chosen so that $\hat{t} < b_n$, then $(G_d(\hat{t})q)/(\sum_{(u,v) \in Q} \mathbb{I}(T_n^{[u,v]} \geq \hat{t})) \leq \alpha$. Hence, we have by using the definition of $\hat{t}$

$$E \left( \frac{\sum_{(u,v) \in \mathcal{H}_0} \mathbb{I}(T_n^{[u,v]} \geq \hat{t})}{\max(\sum_{(u,v) \in Q} \mathbb{I}(T_n^{[u,v]} \geq \hat{t}), 1)} \right)$$

$$= E \left( \frac{1(\hat{t} < b_n) \sum_{(u,v) \in \mathcal{H}_0} \mathbb{I}(T_n^{[u,v]} \geq \hat{t})}{G_d(\hat{t})q} \frac{G_d(\hat{t})q}{\max(\sum_{(u,v) \in Q} \mathbb{I}(T_n^{[u,v]} \geq \hat{t}), 1)} \right)$$

$$+ E \left( \frac{1(\hat{t} \geq b_n) \sum_{(u,v) \in \mathcal{H}_0} \mathbb{I}(T_n^{[u,v]} \geq \hat{t})}{\max(\sum_{(u,v) \in Q} \mathbb{I}(T_n^{[u,v]} \geq \hat{t}), 1)} \right)$$

$$\leq \alpha/q \sum_{(u,v) \in \mathcal{H}_0} \sup_{0 \leq t < b_n} \frac{P(T_n^{[u,v]} \geq \hat{t})}{G_d(t)} + P(\hat{t} = b_n)$$

$$\leq \alpha/q \sum_{(u,v) \in \mathcal{H}_0} \sup_{0 \leq t < b_n} \frac{P(\max_{1/(Mn)} \chi(l)^{[u,v]} \geq t)}{G_d(t)} + P(\hat{t} = b_n)$$

$$= \alpha/q |\mathcal{H}_0| \sup_{0 \leq t < b_n} \frac{P(\max_{1/(Mn)} \|Z_t\|_2 + W \geq t)}{G_d(t)} + o(1) = \alpha/q |\mathcal{H}_0| + o(1),$$

where for the second to last inequality note that we have, by the arguments of the proof of Theorem 4, that, for $1 \leq u < v \leq p$

$$P_{\mathcal{H}_0}(T_n^{[u,v]} \geq t) \leq P\left( \max_{1 \leq l \leq M} \frac{n}{\bar{M}} \left( \frac{\text{Re}(\hat{\rho}_{u,v}(\omega'_l) - \rho_{u,v}(\omega'_l))}{\text{Im}(\hat{\rho}_{u,v}(\omega'_l) - \rho_{u,v}(\omega'_l))} \right) \bar{\Sigma}^{-1}(\omega'_l) \left( \frac{\text{Re}(\hat{\rho}_{u,v}(\omega'_l) - \rho_{u,v}(\omega'_l))}{\text{Im}(\hat{\rho}_{u,v}(\omega'_l) - \rho_{u,v}(\omega'_l))} \right) \geq t \right)$$

$$= P\left( \max_{1 \leq l \leq M} \frac{1}{\bar{M}} \chi(l)^{[u,v]} \geq t \right).$$

Furthermore, for the last equality note that by Lemma 7

$$\sup_{0 \leq t \leq b_n} \left| \frac{P(\max_{1/(Mn)} \chi(l)^{[u,v]} \geq t)}{G_d(t)} - 1 \right| \leq C((\log(n^{d_q}))^{-\xi_q + 3/2} + qd \log(n)^{C_T}$$

$$\times \left[ g_{\beta(q)}(\log(n), p, n, M, \tau, s(p)) + (n^{1/2(1-1/d_q)} - 1/\tau(2-a) - 2(1+a)/\tau)^{2-\tau} \right],$$

because of Assumption 5. The assertion of the theorem follows then because $q/|\mathcal{H}_0| = 1 + o(1)$. \qed
Proof of Lemma 6. Let \( g(x_1, x_2, x_3, x_4, x_5, x_6) = 1/2((x_1 + ix_2)\sqrt{x_3/x_6} + (x_3 - ix_4)\sqrt{x_5/x_6}) \). Then,

\[
\hat{\rho}_{u,v}(\omega) = g(Re(\hat{\beta}_{u,\hat{u}}^{(de)}(\omega)), Im(\hat{\beta}_{u,\hat{u}}^{(de)}(\omega)), Re(\hat{\beta}_{v,\hat{u}}^{(de)}(\omega)), Im(\hat{\beta}_{v,\hat{u}}^{(de)}(\omega)), \hat{f}_{v,v}(\omega), \hat{f}_{u,u}(\omega)).
\]

Using a Taylor expansion of \( \hat{\rho}_{u,v}(\omega) \) around \( \rho_{u,v}(\omega) \), we get that the terms given by the derivatives of \( x_5 \) and \( x_6 \) are of order \( O(1/n) \) due to \( \beta_{v,u}(\omega)\sqrt{\Sigma_{n,v,v}(\omega)\Sigma_{n,u,u}(\omega)} = (\beta_{u,\hat{u}}(\omega))^{(C)} \sqrt{\Sigma_{n,u,u}(\omega)/\Sigma_{n,v,v}(\omega)\Sigma_{n,u,u}(\omega)} \) and sup \( \omega \| \Sigma_n(\omega) - f(\omega) \|_{\max} = O(1/n) \).

Therefore,

\[
\hat{\rho}_{u,v}(\omega) - \rho_{u,v}(\omega) = \frac{1}{2}
\left[
\left(\hat{\beta}_{u,\hat{u}}^{(de)}(\omega) - \beta_{u,\hat{u}}(\omega)\right)^2 + \left(\hat{\beta}_{v,\hat{u}}^{(de)}(\omega) - \beta_{v,\hat{u}}(\omega)\right)^2
\right]
+ \text{Error}_1(\omega), \tag{25}
\]

where \( \text{sup}_\omega \text{Error}_1(\omega) = O(\text{sup}_\omega \| F_M(\omega) - E\hat{F}_M(\omega) \|_{\max}^2) = \tilde{O}_p(g(\sqrt{x}, p, n, M, \tau)) \).

Furthermore, we have by Lemma 11 for \( u, v = 1, \ldots, p, \)

\[
\sqrt{n/M}(\hat{\beta}_{v,u}(\omega) - \beta_{v,u}(\omega)) = \delta(\omega) + \sum_{k=1}^{n} \frac{\kappa_M(\omega - \omega_k)}{\sqrt{n/M}(f_{v,v}(\omega))^2} e_v^T f^{-1}(\omega)[Z(\omega_k)Z^H(\omega_k) - f(\omega)]f^{-1}(\omega)[e_u f^{-1}_{v,u}(\omega) - e_v f^{-1}_{v,u}(\omega)]
\]

\[
= \delta(\omega) + \frac{\sqrt{n/M}}{(f_{v,v}(\omega))^2} e_v^T f^{-1}(\omega)[\hat{F}_M(\omega) - f(\omega)]f^{-1}(\omega)[e_u f^{-1}_{v,u}(\omega) - e_v f^{-1}_{v,u}(\omega)],
\]

where \( \text{sup}_\omega |\delta(\omega)| = \tilde{O}_p(g(\beta_{\omega}(x, p, n, M, \tau, s(p)))) \). Define \( \hat{h}_{u,v}(\omega) := e_v^T f^{-1}(\omega)[\hat{F}_M(\omega) - f(\omega)]f^{-1}(\omega)e_u \). Inserting this into (25) we obtain

\[
\sqrt{n/M}(\hat{\rho}_{u,v}(\omega) - \rho_{u,v}(\omega)) = \sqrt{n/M}\left[\frac{\hat{h}_{u,v}(\omega)}{f^{-1}_{v,v}(\omega)f^{-1}_{u,u}(\omega)} - \frac{\rho_{u,v}(\omega)\hat{h}_{u,v}(\omega)}{2f^{-1}_{v,v}(\omega)f^{-1}_{u,u}(\omega)} + \frac{\hat{h}_{u,v}(\omega)}{f^{-1}_{v,v}(\omega)}\right]
+ \sqrt{n/M}\text{Error}_1(\omega) + \delta(\omega).
\]

We have

\[
\text{sup}_\omega \sqrt{n/M}\text{Error}_1(\omega) = \tilde{O}_p(g(\sqrt{x/(n/M)^{1/4}}, p, n, M, \tau)) = \tilde{O}_p(g(\beta_{\omega}(x, p, n, M, \tau, s(p))).
\]

Now, we take a closer look at \( \hat{h}_{u,v}(\omega') \). For \( l = 1, \ldots, M/N - 1 \) we define \( U_{l,\ell} = e_v^T f^{-1}(\omega')X_l(f^{-1}_{u,u}(\omega'))^{1/2} \) and \( V_{l,\ell} = e_v^T f^{-1}(\omega')X_l(f^{-1}_{v,v}(\omega'))^{1/2} \). Since by Assumption 1 \( \text{sup}_\omega \| f^{-1} \|_2 \) is bounded and \( \max_{\|v\|_2 = 1} E|v^TX|^r < \infty \), we have that \( E|U_{l,\ell}|^r < \infty \) and \( E|V_{l,\ell}|^r < \infty \) and that \( \{U_{l,\ell}\} \) and \( \{V_{l,\ell}\} \) posses the
same functional dependence as \{X_t\}. Furthermore, let
\[
T_{n,u,v}(l) = \frac{1}{2\pi} \sum_{u=-n+1}^{n-1} K(uh) \sum_{t=\max(1,1-u)}^{\min(n,n-u)} \left| U_{t+u,l}V_{t,l} - EU_{t+u,l}V_{t,l} \right| \exp(-iu\omega')
\]
Then, by (14) we have \(\sqrt{n/M} |\hat{h}_{n,v}(\omega') - 1/nT_{n,u,v}(l)| = o(g_{Bias})\). Thus
\[
\sqrt{n/M} \sup_{\omega'} \left| \langle \hat{\rho}_{u,v}(\omega') \rangle - \rho_{u,v}(\omega') \right| = \tilde{O}_p(g_{\beta(d_e)}(x,p,n,M,\tau,s(p))).
\]

**Proof of Lemma 8.** Let \(\hat{f}_{x_1,y_1} = (ae_{v_1} + be_{v_1})\hat{f}_{M,c}(\omega)(c(e_{v_2} + d(e_{u_2})).\)
Note that \(E\hat{f}_{M,c}(\omega) = 0\) and \(\text{Cov}(\hat{f}_{x_1,y_2}, \hat{f}_{y_1,y_2}) = Ee_{v_1}^T\hat{f}_{M,c}(\omega)e_{v_2}e_{u_2}\hat{f}_{M,c}(\omega)e_{u_1} \)
Furthermore, we have
\[
4(\text{Var}\hat{f}_{x_1,y_2} - \text{Var}\hat{f}_{y_1,y_2} + \text{Var}\hat{f}_{x_1,y_2} - \text{Var}\hat{f}_{y_1,y_2}) -
\]
\[
\text{Var}\hat{f}_{x_1-y_1,x_2+y_2} - \text{Var}\hat{f}_{x_1+y_1,x_2-y_2} - \text{Var}\hat{f}_{x_1+y_1,x_2+y_2} - \text{Var}\hat{f}_{x_1-y_1,x_2-y_2} -
\]
\[
\text{Var}\hat{f}_{x_1-y_1,x_2+y_2} + \text{Var}\hat{f}_{x_1+y_1,x_2-y_2} - \text{Var}\hat{f}_{x_1+y_1,x_2+y_2} - \text{Var}\hat{f}_{x_1-y_1,x_2-y_2}
\]
\[
= 8Ee_{v_1}^T\hat{f}_{M,c}(\omega)e_{v_2}e_{u_1}^T\hat{f}_{M,c}(\omega)e_{u_2} = 8\text{Cov}(\hat{f}_{x_1,y_2}, \hat{f}_{y_1,y_2})
\]

The sesquilinearity of the spectral density implies \(\text{Var}(\hat{f}_{x_1+y_1,x_2+y_2}) =
\]
\[
CK_2[(|a|^2\hat{f}_{x_1,x_1} + |b|^2\hat{f}_{y_1,y_1} + ab(c)\hat{f}_{x_1,y_1} + ba(c)\hat{f}_{x_1,x_1})(|c|^2\hat{f}_{x_2,x_2} + |d|^2\hat{f}_{y_2,y_2} +
\]
\[
cd(c)\hat{f}_{x_2,y_2} + dc(c)\hat{f}_{y_2,x_2}) + \mathbb{I}(\omega \in \pi Z)(ac(c))\hat{f}_{x_1,x_2} + bd(c)\hat{f}_{y_1,y_2} +
\]
\[
b(c)\hat{f}_{x_1,y_2} +
\]
\[
ad(c)\hat{f}_{y_1,x_2})^2].\)
Then, by Lemma 17 and some algebra, we get \(\text{Cov}(\hat{f}_{x_1,y_2}, \hat{f}_{y_1,y_2}) =
\]
\[
\hat{f}_{x_1,y_1}\hat{f}_{y_2,y_2} + \mathbb{I}(\omega \in \pi Z)\hat{f}_{x_1,y_2}\hat{f}_{x_2,y_1},\)
from which the first assertion of the lemma follows. For the second assertion, note that
\[
\text{Cov}(\hat{f}_{x_1,y_2}, \hat{f}_{y_1,y_2}) =
\]
\[
\sum_{n_1,n_2=1}^{\infty, \infty} \hat{f}_{n_1,n_2}^\dagger \hat{f}_{n_2,n_1}^\dagger \hat{f}_{n_1,n_2} \hat{f}_{n_2,n_1} \text{Cov}(e_{v_1}^T\hat{f}_{M,c}(\omega)e_{v_2}, e_{u_1}^T\hat{f}_{M,c}(\omega)e_{u_2})
\]

Inserting in the above expression the first assertion of the lemma leads to its
second assertion. \(\square\)

**Proof of Lemma 9.** By Lemma 8 and the definition of \(T_{n,u,v}(l)\) for \(l, l_2 = 1, \ldots, M/N - 1,\) we have that
\[
\frac{1}{Mn} \text{Cov}(T_{n,u_1,v_1}(l), T_{n,u_2,v_2}(l_2)) - \mathbb{I}(l = l_2)CK_2\hat{f}_{x_1,y_1}(-1)(\hat{f}_{x_1,x_1}(-1)\hat{f}_{y_1,y_1}(-1))^{1/2} = O\left(\frac{1}{M}\right).
\]
Both assertions of the lemma then follow by sesquilinearity of the covariance. \(\square\)
Appendix B: Auxiliary Results

We split this section into four subsections. The first is devoted to error bounds related to the construction of the de-biased partial coherence, the second contains useful lemmas for the Gaussian approximation, the third deals with the covariance structure of the lag-window estimator and the last subsection gives some additional useful lemmas.

B.1. Error bounds in the construction of de-biased partial coherences

Lemma 10. Under Assumption 3 and 4 we have \( \max_v \sup_\omega \| \hat{\beta}_v(\omega) \|_1 = O(s(p)) \),
\[
\max_v \sup_\omega \| \hat{\beta}_v - \beta_v \|_1 = \tilde{O}_p(g((x - g_{Bias})/s(p), p, n, M, \tau)),
\]
and
\[
\max_v \sup_\omega \| \hat{\gamma}_v - \gamma_v(\omega) \|_1 = \tilde{O}_p(g((x - g_{Bias})/s(p), p, n, M, \tau)).
\]

Proof. The first assertion follows immediately by Assumption 3. Note that
\[
\hat{\beta}_v(\omega) - \beta_v(\omega) = - I^\top_{p, v} (\hat{f}^{-1}(\omega) - \Sigma^{-1}_n(\omega)) e_v (\Sigma^{-1}_{n, v, v}(\omega))^{-1}
- I^\top_{p, v} \Sigma^{-1}_n(\omega) e_v [(\hat{f}^{-1}_{n, v, v}(\omega))^{-1} - (\Sigma^{-1}_{n, v, v}(\omega))^{-1}]
- I^\top_{p, v} (\hat{f}^{-1}(\omega) - \Sigma^{-1}_n(\omega)) e_v [(\hat{f}^{-1}_{n, v, v}(\omega))^{-1} - (\Sigma^{-1}_{n, v, v}(\omega))^{-1}]
\]
from which the second assertion follows by (13) and since \( \| \cdot \|_1 \) is sub-multiplicative. The third assertion follows by similar arguments. \( \square \)

Lemma 11. Under Assumption 1, 2, 3, and 4, we have for \( u, v = 1, \ldots, p \) and \( \omega \in [0, 2\pi] \)
\[
\sqrt{n/M} (\hat{\beta}^{(de)}_{v, u}(\omega) - \beta_{v, u}(\omega)) = \delta(\omega) + \frac{\sum_{k=1}^n \kappa M (\omega - \omega_k) e_v \hat{f}^{-1}(\omega) [Z(\omega_k) Z^H(\omega_k) - f(\omega)] [f^{-1}(\omega) e_u f^{-1}_{v, v}(\omega) - f^{-1}(\omega) e_v f^{-1}_{v, u}(\omega)]}{\sqrt{nM(f^{-1}_{v, v}(\omega))^2}},
\]
where \( \sup_\omega | \delta(\omega) | = \tilde{O}_p(g((x - g_{Bias})/(s(p)(n/M)^{1/4}))^{1/2}, p, n, M, \tau) + g(1, p, n, M, \tau) + g_{j(\omega)}(x, p, n, M, \tau, s(p)) =: \tilde{O}_p(g_{j(\omega)}(x, p, n, M, \tau, s(p))) \).
Proof. To simplify notation let
\[ DN = 1/n \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)Z_u(\omega_k)Z_{-v}^H(\omega_k)\gamma_{-v,u}(\omega). \]

By Lemma 10 we have \( \sup_{\omega} |DN - \tilde{\gamma}_{v,u}(\omega)| \leq \tilde{O}_p(\epsilon((x-g_{Bias})/s(p), p, n, M, \tau)). \)
Furthermore, note that for some vector \( c \in \mathbb{R}^p \), \( e_v^\top \Sigma_n^{-1}(\omega)\Sigma_n(\omega)I_{p,v}c = 0 \).
Then, using \( e_v^\top - \beta^H(\omega)I_{p,v} = e_v^\top \Sigma_n^{-1}(\omega)^{-1}/\Sigma_n(\omega) \) we have
\[
\sqrt{n/M(\beta^{(de)}_{v,u}(\omega) - \beta_{v,u}(\omega))} = \frac{\sum_{k=1}^{n} \kappa_M(\omega - \omega_k)[Z_v(\omega_k) - \beta^H_{v,u}(\omega)Z_{-v,u}(\omega_k)]Z_{-v}^H(\omega_k)\gamma_{-v,u}(\omega)}{\sqrt{MnDN}}
\]
\[
- \frac{\sqrt{n/M\beta_{v,u}(\omega)DN}}{DN}
\]
\[
= \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)[Z_v(\omega_k) - \beta^H_{v,u}(\omega)Z_{-v,u}(\omega_k)]Z_{-v}^H(\omega_k)\gamma_{-v,u}(\omega)
\]
\[
+ \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)e_v^\top e_v^\top \Sigma_n^{-1}(\omega)Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)]I_{p,v}\gamma_{-v,u}(\omega)
\]
\[
+ \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)e_v^\top e_v^\top \Sigma_n^{-1}(\omega)Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)]I_{p,v}\gamma_{-v,u}(\omega)
\]
\[
+ \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)[\beta_{v,u}(\omega) - \tilde{\beta}_{v,u}(\omega)]^H[Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)]\gamma_{-v,u}(\omega)
\]
\[
+ \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)[\beta_{v,u}(\omega) - \tilde{\beta}_{v,u}(\omega)]^H[Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)]\gamma_{-v,u}(\omega)
\]
\[
= \frac{\sum_{k=1}^{n} \kappa_M(\omega - \omega_k)e_v^\top f^{-1}(\omega)[Z_v(\omega_k)Z_{-v,u}(\omega_k) - f(\omega)]I_{p,v}\gamma_{-v,u}(\omega)}{\sqrt{Mn(f^{-1}(\omega))}} + \delta(\omega).
\]

We investigate the remainder \( \delta(\omega) \). This remainder consists of the errors caused by replacing \( \tilde{\beta}_{v,u}(\omega) \) by \( \beta_{v,u}(\omega) \), \( \gamma_{-v,u}(\omega) \) by \( \gamma_{-v,u}(\omega) \), \( DN \) by \( f^{-1}(\omega) \), and \( \Sigma_n(\omega) \) by \( f(\omega) \). Denote these errors by \( I \) to \( IV \). By Lemma 10 that I, II, III
are of order \( \tilde{O}_p(\epsilon((x-g_{Bias})/s(p), p, n, M, \tau)) \) in \( \| \cdot \|_1 \) norm and uniformly in \( \omega \). Furthermore, we have by Assumption 3 \( \| \Sigma_n(\omega) - f(\omega) \|_{\text{max}} = O(1/n) \) and \( \| \Sigma_n^{-1}(\omega) - f^{-1}(\omega) \|_1 = O(s(p)/n) \). Additionally, note that we have by (13)
\[
\sup_{\omega} ||1/(nh) \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)||_{\text{max}} = \sup_{\omega} \| \tilde{f}_M(\omega) - E\tilde{f}_M(\omega) \|_{\text{max}} = \tilde{O}_p(g(x,p,n,M,\tau)) \text{ and } \sup_{\omega} \| \Sigma_n^{-1}(\omega) \|_1 = O(s(p)).
\]
From this we have
\[
\frac{\sum_{k=1}^{n} \kappa_M(\omega - \omega_k)e_v^\top e_v^\top \Sigma_n^{-1}(\omega)Z_v(\omega_k)Z_{-v,u}(\omega_k) - \Sigma_n(\omega)]I_{p,v}\gamma_{-v,u}(\omega)}{\sqrt{MnDN\Sigma_n^{-1}(\omega)}}
\]
\[ \leq \sqrt{n/M} \| \Sigma_{n}^{-1}(\omega) \| \| \hat{f}_{M}(\omega) - E\hat{f}_{M}(\omega) \|_{\infty} \| [\hat{\gamma}_{v,u}(\omega) - \gamma_{v,u}(\omega)]_{1} |DN\Sigma_{n,v,v}^{-1}(\omega)| \]
\[ \leq \left( \frac{n}{M} \right)^{1/4} s(p) \| \hat{f}_{M}(\omega) - E\hat{f}_{M}(\omega) \|_{\infty} \left( \frac{n}{M} \right)^{1/4} \| [\hat{\gamma}_{v,u}(\omega) - \gamma_{v,u}(\omega)]_{1} |DN\Sigma_{n,v,v}^{-1}(\omega)| \]
\[ = \tilde{O}_{p}(g(((x-g_{Bias})/(s(p))(n/M)^{1/4}))^{1/2}, p, n, M, \tau) + g(1, p, n, M, \tau), \]

where the above equality holds uniformly in \( \omega \). The other terms of \( \delta(\omega) \) are of the same order and, therefore,
\[ \sup_{\omega} |\delta(\omega)| = \tilde{O}_{p}(g(((x-g_{Bias})/(s(p))(n/M)^{1/4}))^{1/2}, p, n, M, \tau) + g(1, p, n, M, \tau)). \]

\[ \square \]

**B.2. Error bounds for Gaussian approximations**

In all Lemmas of this subsection it is assumed Assumption 1 to 5 are satisfied.

**Lemma 12.** For \( l = 1, \ldots, M/N - 1 \) and if \( \Theta_{l,\tau} < \infty \) and for some sequence \( g_{n}, d_{m,\tau} = \sum_{t=0}^{\infty} \min(\delta_{l,\tau}, \Psi_{m+1,\tau}) = o(g_{n}) \), we have for some \( q \leq \tau \)
\[ \max_{l} \frac{1}{\sqrt{Mn}} |g_{n}^{[u,v]}(\omega_{l}) - g_{n,m}^{[u,v]}(\omega_{l})| = \tilde{O}_{p}(\rho^{n}/x)^{q/2} \]
and
\[ E(\frac{1}{\sqrt{Mn}} |g_{n}^{[u,v]}(\omega_{l}) - g_{n,m}^{[u,v]}(\omega_{l})|)^{q/2} = O(d_{m,q}^{q/2}). \]

**Proof.** Following the proof of Lemma A.1 in Liu and Wu (2010), we have by Lemma A.2 in Wu and Zaffaroni (2018) and some \( q \leq \tau \)
\[ E(\frac{1}{\sqrt{Mn}} |g_{n}^{[u,v]}(\omega_{l}) - g_{n,m}^{[u,v]}(\omega_{l})|)^{q/2} = O(d_{m,q}^{q/2}). \]

Under Assumption 1 we have \( d_{m,q} = \sum_{t=0}^{\infty} \min(\delta_{t,q}^{[\text{max}]}, (\sum_{j=m}^{\infty} (\delta_{j,q}^{[\text{max}]})^{2})^{1/2}) \leq C\rho^{n} \). Hence, by Markov’s inequality
\[ P(\max_{l} \frac{1}{\sqrt{Mn}} |g_{n}^{[u,v]}(\omega_{l}) - g_{n,m}^{[u,v]}(\omega_{l})| \geq x) \leq C(\rho^{n}/x)^{q/2}. \]

\[ \square \]

**Lemma 13.** For \( q \leq \tau/2 \) it holds true that
\[ E(\frac{1}{\sqrt{Mn}} \max_{l} |\bar{g}_{n,m}(l)^{[u,v]} - g_{n,m}(l)^{[u,v]}|)^{q} \leq C_{\tau} m^{2q} \left[ (Mn)^{-1+q/\tau} \right. \]
\[ + M^{-q/2}(Mn)^{-1+(2q-1)/2+q/\tau} \]

and
\[ \frac{1}{\sqrt{Mn}} \max_{l} |\bar{g}_{n,m}(l)^{[u,v]} - g_{n,m}(l)^{[u,v]}| = \tilde{O}_{p}(C_{\tau} m^{3/2\tau-2} ((x\sqrt{Mn})^{-\tau/2} + M^{\tau/2} x^{-\tau} (Mn)^{-\tau/2+1})). \]
Proof. First note that since the kernel is bounded, \( \tilde{g}_{n,m}(l) \) can be written as
\[
\tilde{g}_{n,m}(l) = \sum_{t=2}^{\tau} \tilde{U}_{t,l,m} \sum_{s=1}^{t-1} a_{n,t-s} \tilde{V}_{s,l,m}
\]
for some bounded coefficients \( a_{n,i} \in \mathbb{C} \).
A similar expression for \( g_{n,m}(l) \) holds true. We then obtain for some constant \( C > 0 \), that,
\[
\max_l |g_{n,m}(l)[u,v] - g_{n,m}(t)[u,v]| \leq C \max_l \sum_{t=2}^{\tau} |\tilde{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |V_{s,l,m}|
\]
\[
+ C \max_l \sum_{t=2}^{\tau} |U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |V_{s,l,m} - \tilde{V}_{s,l,m}|
\]
\[
+ C \max_l \sum_{t=2}^{\tau} |\tilde{V}_{t,l,m} - V_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |U_{s,l,m}|
\]
\[
+ C \max_l \sum_{t=2}^{\tau} |V_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |U_{s,l,m} - \tilde{U}_{s,l,m}|.
\]
The four terms of the above inequality, can be treated by the same arguments. We focus here on the first only. Note that for all \( t \) it holds true that \( E[\max_t |U_{t,l,m}|^\tau < \infty, E[\max_t |V_{t,l,m}|^\tau < \infty. \) Hence, by Hölder’s inequality and Markov’s inequality
\[
E(\max_l |\tilde{U}_{t,l,m} - U_{t,l,m}|^q) = E \max_l |U_{t,l,m}|^q 1(|U_{t,l,m}| > (Mn)^\ell)\]
\[
\leq E |U_{t,l,m}|^\tau P(|U_{t,l,m}| > (Mn)^\ell)^{1-q/\tau} = O((Mn)^{-1+q/\tau}).
\]
Using \( m \)-dependency and Hölder’s inequality, we obtain
\[
(1/(Mn))^{q/2} E \left( \max_l \sum_{t=2}^{\tau} |\tilde{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |V_{s,l,m}| \right)^q \leq C_r (1/(Mn))^{q/2} E \left( \max_l \sum_{t=m+1}^{\tau} |\tilde{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-m} |V_{s,l,m}| \right)^q + C_r \left( \max_l \sum_{t=2}^{\tau} |U_{t,l,m} - \tilde{U}_{t,l,m}| \sum_{s=\max(1,t-M+1)}^{t-1} |V_{s,l,m}| \right)^q \leq C_r (1/(Mn))^{q/2} \left[ (Mn)^{q/2} m^{2(q-1)} (Mn)^{-1+q/\tau} + n^{q/2} m^{2q-1} (Mn)^{-1+(q-1)^{1+q/(\tau q - (q/(\tau q - 1))}} \right]
\[ \leq C_\tau m^{2q}[(Mn)^{-1+q/\tau} + M^{-q/2}(Mn)^{-1+(\tau/\rho-1)^{-1}+q/\tau-(\rho/\tau)/(\rho/\tau-1)}]. \]

This proves the first assertion of the lemma. For the second assertion, we use the same splitting arguments as above and get

\[
\max_l \sum_{t=2}^n |\bar{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-1} |V_{s,l,m}|
\]

\[
eq \sum_{t=m+1}^n \max_l |\bar{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-M)}^{t-m} |V_{s,l,m}|
\]

\[
+ \sum_{t=2}^n \max_l |\bar{U}_{t,l,m} - U_{t,l,m}| \sum_{s=\max(1,t-m+1)}^{t-1} |V_{s,l,m}|
\]

\[= : \sum_{t=m+1}^n W_t + \sum_{t=2}^n \tilde{W}_t, \]

with an obvious notation for \(W_t\) and \(\tilde{W}_t\). We obtain by Lemma 2 in Liu and Wu (2010) and for \(Q = \tau/2\) and the arguments used above, that

\[
P(\sum_{t=m+1}^n W_t \geq x) \leq C_Q(m/x^2E(\sum_{t=m+1}^n W_t)^2)^Q + C_Qm^{\tau-1}/x^\tau \sum_{t=m+1}^n EW_t^\tau
\]

\[
\leq C_Q(m/x^2m^2(Mn)^2/\tau)^Q + C_Qm^{\tau-1}x^{-\tau}nm^{\tau-1}
\]

\[\leq C_\tau m^{2\tau-2}x^{-\tau}nM. \]

Furthermore, we have for \(\tilde{W}_t\)

\[
P(\sum_{t=m+1}^n \tilde{W}_t \geq x) \leq C_Q(m/x^2E(\sum_{t=m+1}^n \tilde{W}_t)^2)^Q + C_Qm^{\tau/2-1}/x^\tau/2 \sum_{t=m+1}^n E\tilde{W}_t^{\tau/2}
\]

\[
\leq C_Q(m/x^2m^2M^{-1}(Mn)^{(2\tau-1)^{-1}})^Q + C_Qm^{\tau/2-1}x^{-\tau/2}nm^{\tau-1}
\]

\[\leq C_\tau m^{3/2\tau-2}x^{-\tau/2}n. \]

Putting this together with \(x = x\sqrt{Mn}\) gives the second assertion.

**Lemma 14.** We have

\[E(\max_l \sum_{j=1}^{k_n+1} u_j(l)^{[u,v]}^2) \leq Cn(M)^{1-\beta}m^4 \]

and

\[E(\max_l \sum_{j=1}^{k_n} u_j(l)^{[u,v]}^2) \leq CMnm^4. \]
Proof. First note that by independence
\[
E\left(\sum_{j=1}^{k_n+1} \max_l v_j(l)^{[u,v]}\right)^2 = \sum_{j=1}^{k_n+1} E(v_j(l)^{[u,v]})^2.
\]

For \(j = 1, \ldots, k_n\) and similarly to the proof of Lemma 13, the properties of the kernel \(K\) and the \(m\)-dependence of \(\bar{U}_{t,l,m}, \bar{V}_{t,l,m}\), we have
\[
E(\max_l \sum_{j=1}^{k_n+1} \max_{u,v} u_j(l)\bar{U}_{t,l,m})^2 \leq C\|I\| m^3 \|I\| m^{2q-1} \leq C q |I| m^3 (M + m).
\]

Similar arguments apply to \(u_j(l)\) with \(I_j\) replaced by \(H_j\). We have \(|I_j| = CM + m, j = 1, \ldots, k_n, |I_{k_n+1}| < C(M)^{1+\beta} + M + m, \beta > 0, |H_j| = C(M)^{1+\beta}\), and \(k_n = \lfloor n/(|I_1| + |H_1|)\rfloor\). Thus,
\[
\max_l E(\sum_{j=1}^{k_n+1} v_j(l)^{[u,v]})^2 \leq C(n/(1 + |H_1|/|I_1|)m^3(M + m)) \leq C q n(M)^{1-\beta} m^4
\]
and
\[
\max_l E(\sum_{j=1}^{k_n+1} u_j(l)^{[u,v]})^2 \leq C M n m^4.
\]

Lemma 15. It holds true that
\[
\sqrt{1/(Mn)} \max_l |\Gamma^{[u,v]}(l)| = \tilde{O}_p(C \log(n)^C \tau (x/\tau)^{-2(1+\alpha)/\tau^2} - x^{-C\tau (Mn)^{-\tau}}),
\]
where \(\Gamma^{[u,v]}(l) = \sum_{j=1}^{k_n+1} v_j^{[u,v]}(l)\).

Proof. Note first that
\[
P(\max_l \sqrt{1/(Mn)}|\Gamma^{[u,v]}(l)| > x) \leq \sum_l P(|\Gamma^{[u,v]}(l)| > \sqrt{Mnx}) \leq MP(|\Gamma^{[u,v]}(l)| > \sqrt{Mnx}).
\]
We focus on the term on the RHS of the last inequality. For this, we use Lemma 2 in Liu and Wu (2010) and obtain for any $Q$ and some positive constants $C_Q$

$$P(\Gamma(l)^{[u,v]} \geq \sqrt{Mnx}) \leq C \left( \sum_{j=1}^{k_n + 1} \frac{E[|v_j^{[u,v]}(l)|^2]}{Mnx} \right)^Q + C \sum_{j=1}^{k_n + 1} P(|v_j^{[u,v]}(l)| \geq C_q \sqrt{Mnx}).$$

For the first term on the RHS we have by Lemma 14

$$\sum_{j=1}^{k_n + 1} \frac{E[|v_j^{[u,v]}(l)|^2]}{Mnx} \leq C(n^{-\beta_a}m^2x).$$

Hence, for some $Q = C\tau$ we have that this term is of order $O(x^{-C\tau}(Mn)^{-\tau})$.

Furthermore, by Lemma A.5 in Wu and Zaffaroni (2018), see also Proposition 3 in Liu and Wu (2010) (note that we do not use the replacement of $m$ by $n$ for the second and third term on the RHS of Proposition 3), we have with $M = (Mn)^t$, $k = M + m$ and $x = \sqrt{Mnx}$, $y = 4\log((nxM)^\tau)$, that

$$P(|v_j^{[u,v]}(l)| \geq C_q \sqrt{Mnx}) \leq 2(Mnx)^{-\tau} + Cn^2(Mn)^{2t}(\frac{Mnx2m^{(\log((Mn)^\tau)^{-2}m^3[(Mn)^{2t} + M + m]M^2}{(Mnx)^{2m^2}})^Q$$

$$+ Cn^2m^2(Mn)^{2t} \max_{Z \in \{U,V\}} P\left( |Z_{0,t,m}| \geq C \frac{\log((nxM)^\tau)^{-1}x\sqrt{Mn}}{m^2((Mn)^t + (M + m))^{1/2}} \right)$$

$$= I + II + III.$$

Consider the second term on the RHS and recall that $M = n^a$.

$$\frac{m^3[(Mn)^{2t} + M + m]M}{Mnx^2\log((Mn)^\tau)^2} \leq x^{-2}\log(n)^{C\tau(n^{-1+a} + n^{(1-a)(-1+2/\tau)+a(4/\tau-1)})}.$$

Hence, for $Q = C\tau$ we obtain $II = O(x^{-C\tau}(Mn)^{-\tau})$.

For the third term we use Markov’s inequality and obtain for $\tau \geq 8$

$$III \leq Cn^2m^2(Mn)^{2t} \max_{Z \in \{U,V\}} E[|Z_{0,t,m}|^\tau \left( \frac{m^2((Mn)^t + (M + m))^{1/2}}{\log((Mn)^\tau)x\sqrt{Mn}} \right)^\tau$$

$$\leq C \log(n)^{C\tau}n^{-\tau/2+2+2(1+a)^2/\tau}x^{-\tau} = C \log(n)^{C\tau}x^{1/2-2/\tau-2(1+a)/\tau^2-\tau}.$$

Lemma 16. Let $\xi$ be defined as in Remark 1 of the main paper and let

$$\hat{u}_j(l)^{[u,v]} = u_j(l)^{[u,v]} \mathbb{1}(|u_j(l)^{[u,v]}| \leq \sqrt{Mn}\xi) - E(u_j(l)^{[u,v]}\mathbb{1}(|u_j(l)^{[u,v]}| \leq \sqrt{Mn}\xi)$$
be a truncated version of $u_j(l)^{[u,v]}$, $j = 1, \ldots, k_n, l = 1, \ldots, M/N - 1$. We have
\[
\frac{1}{\sqrt{Mn}} \max_l \sum_{j=1}^{k_n} |\hat{u}_j(l)^{[u,v]} - u_j(l)^{[u,v]}| = O_p\left(\frac{\log(n)^{C_\tau}}{(\sqrt{2(1-\frac{1}{4\alpha})-1/\tau(2-\alpha)-2(1+\alpha)/\tau^2})^\tau} + x^{-C\tau} \frac{1}{(Mn)^\tau}\right)
\]
and
\[
\frac{1}{Mn} \max_l \sum_{j=1}^{k_n} E(\|\hat{u}_j(l)^{[u,v]} - u_j(l)^{[u,v]}\|^2 = O\left(\frac{\log(n)^{C_\tau}}{(\sqrt{2(1-\frac{1}{4\alpha})-1/\tau(2-\alpha)-2(1+\alpha)/\tau^2})^\tau} + x^{-C\tau} \frac{1}{(Mn)^\tau}\right)
\]

**Proof.** Let $\hat{Z}_j(l) = u_j(l)^{[u,v]} 1(\|u_j(l)^{[u,v]}\| > \sqrt{Mn})$. We follow the lines of proof of Lemma 15. The main difference is that $u_j$ consist of $p_n = (M)^{1+\beta}$ elements whereas in Lemma 15, $v_j$ consist of only $M$ elements. We obtain
\[
P(|\sum_{j=1}^{k_n} \hat{Z}_j(l)| \geq \sqrt{Mnx}) \leq C\left(\frac{\sum_{j=1}^{k_n} E(\hat{Z}_j(l))^2}{Mnx}\right)^Q + C \sum_{j=1}^{k_n+1} P(|\hat{Z}_j(l)| \geq C_q \sqrt{Mnx}).
\]
We have by Cauchy-Schwarz’s inequality, that $E(\hat{Z}_j(l))^2 \leq (E(u_j(l)^{[u,v]})^4 P(|u_j(l)^{[u,v]}| \geq C_q \sqrt{Mnx})^{1/2}$. Furthermore, we have with similar arguments as in Lemma 14 $E(u_j(l)^{[u,v]})^4 \leq C(|H_j| M)^{2m^8}$. This means that $(\frac{\sum_{j=1}^{k_n} E(\hat{Z}_j(l))^2}{Mnx})^Q \leq (P(|u_j(l)^{[u,v]}| \geq C_q \sqrt{Mnx})^{Q/2}$. In the above derivation eight moments are required. Note that since $Q$ can be chosen large, it is also possible to use H"older’s inequality so that only $4 + \delta$ for some $\delta > 0$ moments are required.

We first investigate $P(|\hat{Z}_j(l)| \geq C_q \sqrt{Mnx})$. Similar to the arguments used for $P(|v_j^{[u,v]}(l)| \geq C_q \sqrt{Mnx})$ in the proof of Lemma 15, we have by Lemma A.5 in Wu and Zaffaroni (2018), see also Proposition 3 in Liu and Wu (2010), with $M = (M)^1, k = (M)^{1+\beta}, x = \sqrt{Mnx}, y = 4\log((nxM)^\tau)$
\[
P(|u_j^{[u,v]}(l)| \geq C_q \sqrt{Mnx}) \leq 2(nx/(h))^{-7}
\]
\[
+ Cnm^2(Mn)^{2t} (Mn)^{-1-2\log((nxM)^\tau)} - 2m^3(Mn)^{2t} + (M)^{1+\beta}) \left[ M \log((nxM)^\tau) x \sqrt{Mn} \right]^{Q/2}
\]
\[
+ Cn^2 m^2(Mn)^{2t} \max_{Z \in \{U,V\}} P \left( |\hat{Z}_{0,l,m}| \geq C \log((nxM)^\tau) x \sqrt{Mn} \right) \left[ m^2(M) + (M)^{1+\beta})^{1/2} \right]
\]
\[
= I + II + III.
\]
Consider the second term on the RHS above. Recall that \( M = n^a \) and \( a \leq 2/3 \).

\[
(Mn)^{-1} x^{-2} \log((Mn)\tau^{-2} m^2[\tau^2 + (M)^{1+\beta}]M \\
\leq x^{-2} \log(n) C_\tau (n^{-1+a(1+\beta)} + n^{1-a(-1+2/\tau)+a(4/\tau-1)}).
\]

Note that \( n^{-1+a(1+\beta)} = O(k_n) = n^{8/(5\tau)} \). Hence, for \( Q = C_\tau \) we obtain \( II = O(x^{-C_\tau} (Mn)^{-\tau}) \).

For the third term we use Markov’s inequality and obtain for \( \tau \geq 8 \)

\[
III \leq C n^2 m^2 (Mn)^{2a} \max_{Z \in \{U, V\}} E|\tilde{Z}_{0,1,m}|^\tau \left( m^2 [(Mn)^\tau + (M)^{1+\beta}]^{1/2} \right) \tau \\
\leq C \log(n) C_\tau n^{-\tau/2+\beta a/2+2+1/2} \tau = C \log(n) C_\tau (xn^{1/2(1-1/4a)-1/2(2-a)-2(1+a)/\tau})^{-\tau}.
\]

Since \( 1/\xi = \log(n) C_\tau \), \( Q \) can chosen as \( Q = C_\tau \) such that \( C \left( \sum_{k=0}^{\infty} |E|\tilde{Z}_{0,1}(k)^2|/M_n \right) = O(x^{-C_\tau} (Mn)^{-\tau}) \).

\[ \square \]

**B.3. Covariance structure of lag-window estimators**

**Lemma 17.** Let \( \hat{f}_{M,c}(\omega) = \frac{4\pi}{n} \sum_{k=1}^{n} \kappa_M(\omega - \omega_k)(Z(\omega_k)Z^H(\omega_k) - \Sigma_n(\omega_k)) \) and \( u, v \in \{1, \ldots, p\} \). Under Assumption 1,2,3, we have

\[
\operatorname{sup}_{\omega} |n/M \operatorname{Var}(e_v^\top \hat{f}_{M,c}(\omega)e_u) - C_{K_2} f_{u,v}(\omega) f_{u,u}(\omega) + \mathbb{I}(\omega \in \pi Z) f_{v,u}(\omega^2)| = O(1/M).
\]

**Proof.** We express the occurring fourth order moments in terms of covariances and cumulants, see among others Section 5.1 in Rosenblatt (1985). To elaborate, we have

\[
n/M \operatorname{Var}(e_v^\top \hat{f}_M(\omega)e_u) = I_1 + I_2 + I_3,
\]

\[
I_1 = \frac{M}{n} \sum_{k_1,k_2=1}^{n} \kappa_M(\omega - \omega_{k_1}) \kappa_M(\omega - \omega_{k_2}) \operatorname{Cov}(e_v^\top Z_n(\omega_{k_1}), e_v^\top Z_n(\omega_{k_2})) \operatorname{Cov}(Z_n(\omega_{k_1})^H e_u, Z_n(\omega_{k_2})^H e_u)
\]

\[
I_2 = \frac{M}{n} \sum_{k_1,k_2=1}^{n} \kappa_M(\omega - \omega_{k_1}) \kappa_M(\omega - \omega_{k_2}) \operatorname{Cov}(e_v^\top Z_n(\omega_{k_1}), Z_n(\omega_{k_2})^H e_u) \operatorname{Cov}(Z_n(\omega_{k_1})^H e_u, e_v^\top Z_n(\omega_{k_2}))
\]

\[
I_3 = \frac{M}{n} \sum_{k_1,k_2=1}^{n} \kappa_M(\omega - \omega_{k_1}) \kappa_M(\omega - \omega_{k_2}) \operatorname{cum}(e_v^\top Z_n(\omega_{k_1}), e_v^\top Z_n(\omega_{k_1}), e_v^\top Z_n(\omega_{k_2}), e_v^\top Z_n(\omega_{k_2})).
\]
By Theorem 4.1 in Shao and Wu (2007) and the summability of fourth order cumulants, i.e.,

$$\max_{a_1,\ldots,a_4} \sum_{\tau_1,\tau_2,\tau_3 \in \mathbb{Z}} |\text{cum}(e_{a_1}^T X_0, e_{a_2}^T X_{\tau_1}, e_{a_3}^T X_{\tau_2}, e_{a_4}^T X_{\tau_3})| < \infty.$$  

This implies uniformly for all $\omega_1, \ldots, \omega_4 \in [0, 2\pi]$

$$\max_{a_1,\ldots,a_4} |\text{cum}(e_{a_1}^T Z_n(\omega_1), e_{a_2}^T Z_n(\omega_2), e_{a_3}^T Z_n(\omega_3), e_{a_4}^T Z_n(\omega_4))| = O(n^{-1}),$$

due to Assumption 1, we have that $I_1 = O(1/M)$. Following the arguments of Rosenblatt (1985), Section 5.1, see also Section 5.4 in Brillinger (2001), we obtain uniformly for all $\omega \in [0, 2\pi]$

$$I_1 = \frac{M}{n} \sum_{k_1,k_2=1}^{n} \kappa_M(\omega - \omega_{k_1})\kappa_M(\omega - \omega_{k_2}) \frac{1}{(2\pi n)^2} \sum_{t_1,t_2=1}^{n} \Gamma_{u,v}(t_1 - t_2) \exp(i(t_1\omega_{k_1} - t_2\omega_{k_2}))$$

$$\times \sum_{\tau_1,\tau_2=1}^{n} \Gamma_{u,u}(\tau_1 - \tau_2) \exp(-i(\tau_1\omega_{k_1} - \tau_2\omega_{k_2}))$$

$$= \frac{M}{n} \sum_{k_1}^{n} \kappa_M(\omega - \omega_{k_1})^2 (f_{u,u}(\omega_{k_1}) + O(1/n))(f_{u,u}(\omega_{k_1}) + O(1/n))$$

$$= C_{K_2} f_{u,u}(\omega) f_{v,v}(\omega) + O(1/M).$$

$$I_2 = \frac{M}{n} \sum_{k_1,k_2=1}^{n} \kappa_M(\omega - \omega_{k_1})\kappa_M(\omega - \omega_{k_2}) \frac{1}{(2\pi n)^2} \sum_{t_1,t_2=1}^{n} \Gamma_{u,u}(t_1 - t_2) \exp(i(t_1\omega_{k_1} + t_2\omega_{k_2}))$$

$$\times \sum_{\tau_1,\tau_2=1}^{n} \Gamma_{u,v}(\tau_1 - \tau_2) \exp(i(\tau_1\omega_{k_1} + \tau_2\omega_{k_2}))$$

$$= 1(\omega \in \pi \mathbb{Z}) C_{K_2} f_{u,u}(\omega) f_{v,v}(\omega) + O(1/M).$$

\[\square\]

**Lemma 18.** Under Assumption 1 to 5 the following assertions hold true. If $K$ is the uniform kernel, then

$$\frac{1}{M n} \max_{t_1,t_2=1,\ldots,M,t_1 \neq t_2} |\text{Cov}(\sum_{j=1}^{k_n} \tilde{u}_j(l_1)^{[u,v]}, \sum_{j=1}^{k_n} \tilde{u}_j(l_2)^{[u,v]}\text{)}| = O(1/M + 1/k_n).$$

For other kernels, we have

$$\frac{1}{M n} |\text{Cov}(\sum_{j=1}^{k_n} \tilde{u}_j(l_1)^{[u,v]}, \sum_{j=1}^{k_n} \tilde{u}_j(l_2)^{[u,v]}\text{)}| = O(1/M + 1/k_n + (M(\omega_1' + \omega_2') - r + (M(\omega_1' - \omega_2')) - r).$$
Proof. First note that by independence \( \text{Cov}(\sum_{j=1}^{k_n} \hat{u}_j(l_1)_{[u,v]}, \sum_{j=1}^{k_n} \hat{u}_j(l_2)_{[u,v]}) = \sum_{j=1}^{k_n} \text{Cov}(\hat{u}_j(l_1)_{[u,v]}, \hat{u}_j(l_2)_{[u,v]}) \). Furthermore,

\[
\sum_{j=1}^{k_n} \text{Cov}(\hat{u}_j(l_1)_{[u,v]}, \hat{u}_j(l_2)_{[u,v]}) = \\
\sum_{j=1}^{k_n} \left[ \text{Cov}(u_j(l_1)_{[u,v]}, u_j(l_2)_{[u,v]}) - \text{Cov}(u_j(l_1)_{[u,v]}, \hat{u}_j(l_2)_{[u,v]} - u_j(l_2)_{[u,v]}) \\
- \text{Cov}(\hat{u}_j(l_1)_{[u,v]} - u_j(l_1)_{[u,v]}, u_j(l_2)_{[u,v]} + \text{Cov}(\hat{u}_j(l_1)_{[u,v]} - u_j(l_1)_{[u,v]}, \hat{u}_j(l_2)_{[u,v]} - u_j(l_2)_{[u,v]}) \\
= I + II + III + IV
\]

For II, III and IV note that by Lemma 16 and \( \tau \geq 8, Mn^{-1} \max \sum_{j=1}^{k_n} \text{Var}(\hat{u}_j(l)_{[u,v]} - u_j(l)_{[u,v]}) = O(M^{-2}). \) Hence, we obtain by Cauchy-Schwarz’s and Hölder’s inequality, that

\[
II = \frac{1}{Mn} \sum_{j=1}^{k_n} \text{Cov}(\hat{u}_j(l_1)_{[u,v]} - u_j(l_1)_{[u,v]}, u_j(l_2)_{[u,v]}) \\
\leq \left( \sum_{j=1}^{k_n} \frac{1}{Mn} \text{Var}(\hat{u}_j(l_1)_{[u,v]} - u_j(l_1)_{[u,v]}) \right)^{1/2} \left( \sum_{j=1}^{k_n} \frac{1}{Mn} \text{Var}(u_j(l_2)_{[u,v]}) \right)^{1/2} = O(M^{-1}).
\]

Similar arguments apply to III and IV.

For I note that

\[
u_j(l)_{[u,v]} = \sum_{t \in H_j} \sum_{s=1}^{t-1} K((t-s)/M) \exp(-i(t-s)\omega_1^j)[\hat{U}_{t,l,m} \hat{V}_{s,l,m} - EU_{t,l,m} \hat{V}_{s,l,m} + \sum_{s=1}^{t-1} K((t-s)/M) \exp(i(t-s)\omega_1^j)[\hat{V}_{t,l,m} \hat{U}_{s,l,m} - EV_{t,l,m} \hat{U}_{s,l,m}]
\]

Evaluating \( \text{Cov}(u_j(l_1)_{[u,v]}, u_j(l_2)_{[u,v]}) \) using the above decomposition leads to four terms, i.e.,

\[
\text{Cov}(u_j(l_1)_{[u,v]}, u_j(l_2)_{[u,v]}) = \\
\sum_{t \in Z} \sum_{s_1, s_2 = -M}^{M} K(\frac{s_1}{M})K(\frac{s_2}{M})\text{Cov}(\hat{U}_{0,t,1,m} \hat{V}_{s_1,l_1,m}, \hat{U}_{t,l_2,m} \hat{V}_{s_2,l_2,m}) \exp(-is_1\omega_1^j + s_2\omega_2^j) + O(\frac{1}{k_n})
\]

\[
= \sum_{t \in Z} \sum_{s_1, s_2 = -M}^{M} K(\frac{s_1}{M})K(\frac{s_2}{M}) \exp(-is_1\omega_1^j + s_2\omega_2^j) [\text{Cov}(\hat{U}_{0,t,1,m}, \hat{U}_{t,l_2,m})\text{Cov}(\hat{V}_{s_1,l_1,m}, \hat{V}_{s_2,l_2,m})
\]

\[+ \text{Cov}(\hat{U}_{0,t,1,m}, \hat{V}_{s_2,l_2,m})\text{Cov}(\hat{V}_{s_1,l_1,m}, \hat{U}_{t,l_2,m}) + \text{cum}(\hat{U}_{0,t,1,m}, \hat{V}_{s_1,l_1,m}, \hat{U}_{t,l_2,m}, \hat{V}_{s_2,l_2,m})] + O(\frac{1}{k_n}).
\]
By the summability of the fourth order cumulants, see also Theorem 4.1 in Shao and Wu (2007), we have that the cumulant term is of order $O(1/M)$. Furthermore, we have with the decay conditions of the autocovariance, smoothness of the kernel and the decay condition of the Fourier coefficients of the kernel

$$I.1 = f_{\tilde{u}_{1,m},\tilde{u}_{2,m}}(\omega_{l}^{'},s)\sum_{s=-M}^{M}K^{2}(s/M)\exp(-is(\omega_{l} - \omega_{l}^{'})) + O(1/M)$$

$$\leq C(M(\omega_{l} - \omega_{l}^{'})^{-r} + O(1/M)$$

and

$$I.2 = f_{\tilde{u}_{1,m},\tilde{u}_{2,m}}(\omega_{l}^{'},s)\sum_{s=-M}^{M}K^{2}(s/M)\exp(-is(\omega_{l} + \omega_{l}^{'}) + O(1/M)$$

$$\leq C(M(\omega_{l} + \omega_{l}^{'})^{-r} + O(1/M)$$

For $K$ being the uniform kernel and $\omega_{l}^{'} = \pi l/M$, we have $\frac{1}{M}\sum_{s=-M}^{M}K^{2}(s/M)\exp(-is(\omega_{l} - \omega_{l}^{'}) = 1(\omega_{l} = \omega_{l}^{'}) + O(1/M)$ and $\frac{1}{M}\sum_{s=-M}^{M}K^{2}(s/M)\exp(-is(\omega_{l} + \omega_{l}^{'}) = 1(\omega_{l} + \omega_{l}^{'}) \in 2MZ) + O(1/M)$.

**B.4. Additional Lemmas**

**Lemma 19.** Let $Z_{l}, l = 1, \ldots, d, d \geq 1$ be independent Gaussian random vectors and $Z_{l} \sim \mathcal{N}(0, I_{2})$. Furthermore, let $\gamma_{n}, g_{n}, h_{n}^{-1} > 0, \gamma_{n} \leq 2\log(d)$ and $W_{n}$ be a random variable with $P(|W_{n}| > \gamma_{n}) = g_{n}$. Set $G_{d}(t) = (1 - (1 - \exp(-\max(0, t/2)/2))^{d}$. Then,

$$\sup_{0 \leq t \leq h_{n}}\left|\frac{P(\max_{l} ||Z_{l}||^{2} + W_{n} \geq t)}{G_{d}(t)} - 1\right| \leq (g_{n}(\max(2, \exp(h_{n}/2)) + \gamma_{n})(\max(2, \exp(\gamma_{n}/2))).$$

(26)

**Proof.** We compute derivatives under the condition $d \geq 2$. The same arguments with simpler derivatives can also be applied in the case $d = 1$. We have $P(\max_{l} ||Z_{l}||^{2} + W_{n} \geq t) \leq P(\max_{l} ||Z_{l}||^{2} \geq t - \gamma_{n}) + P(|W_{n}| > \gamma_{n})$ and by Lemma 21, $P(\max_{l} ||Z_{l}||^{2} + W_{n} \geq t) \geq P(\max_{l} ||Z_{l}||^{2} \geq t + \gamma_{n}) - P(|W_{n}| > \gamma_{n})$.

Thus,

$$\left|\frac{P(\max_{l} ||Z_{l}||^{2} + W_{n} \geq t)}{G_{d}(t)} - 1\right| \leq g_{n}/G_{d}(t) + G_{d}(t)^{-1}|G_{d}(t + \gamma_{n}) - G_{d}(t)|.$$
Note that \((G_d(t))^{-1}\) is monotonic increasing in \(t\), where \(G_d(0)^{-1} = 1\). Additionally, we have with

\[
(1 + x)^d \leq 1/(1 - dx) \text{ for } x \in [-1, 0]
\]  

\[(27)\]

\[G_d(t)^{-1} \leq \exp(t/2)/d + 1.\]  

\[(28)\]

We consider two cases \(t \leq 2 \log(d)\) and \(t > 2 \log(d)\). In the first case, we further have \(G_d(t)^{-1} \leq 2\) and in the second, \(G_d(t)^{-1} \leq 2\exp(t/2)/d\). With this and using the monotonicity of \((G_d(t))^{-1}\), we get the first summand of the bound given in \((26)\), that is, \(g_n \max(2, \exp(h_n/2)/d)\). Regarding the second summand of the same bound, note first that we have a positive and a negative case. Only in the case \(t \leq \gamma_n\) the two cases need to be treated separately. We begin with this case and consider \(t \leq \gamma_n\). Then, we have \(G_d(t)^{-1}|G_d(t - \gamma_n) - G_d(t)| \leq 2(G_d(0) - G_d(\gamma_n))\). Further, we have by the mean value theorem for some \(\varepsilon \in [0, \gamma_n]\) and the inequality \((27)\)

\[2(G_d(0) - G_d(\gamma_n)) \leq |\gamma_n|d(1 - \exp(-\varepsilon/2))^{d-1}\exp(-\varepsilon/2)\]

\[\leq |\gamma_n|\frac{1}{\exp(\varepsilon/2)/d + (d - 1)/d} \leq \frac{\gamma_n}{(d - 1)/d} \leq 2\gamma_n.\]

With similar arguments, we obtain for the case \(\gamma_n \leq t \leq 2 \log(d)\) that for \(\varepsilon \in [0, \gamma_n]\) (not necessarily the same \(\varepsilon\) for the positive and negative case)

\[G_d(t)^{-1}|G_d(t + \gamma_n) - G_d(t)| \leq 2|\gamma_n|2d(1 - \exp(-(t + \varepsilon)/2))^{d-1}\exp(-(t + \varepsilon)/2)\]

\[\leq |\gamma_n|\frac{1}{\exp((t + \varepsilon)/2)/d + (d - 1)/d} \leq |\gamma_n|d/(d - 1) \leq 2\gamma_n.\]

The same arguments applied to the case \(t \leq \gamma_n\) and we obtain \(G_d(t)^{-1}|G_d(t + \gamma_n) - G_d(t)|\). Now, let \(t > 2 \log(d)\). Then, by the mean value theorem we have for some \(\varepsilon \in [0, \gamma_n]\) (not necessarily the same \(\varepsilon\) for the positive and negative case)

\[G_d(t)^{-1}|G_d(t + \gamma_n) - G_d(t)| \leq |\gamma_n|\exp(t/2)(1 - \exp(-(t + \varepsilon)/2))^{d-1}\exp(-(t + \varepsilon)/2)\]

\[\leq |\gamma_n|\exp(|\gamma_n|/2)\]

Hence,

\[G_d(t)^{-1}|G_d(t + \gamma_n) - G_d(t)| \leq |\gamma_n| \max(2, \exp(|\gamma_n|/2)),\]

where this bound is independent of \(t\) and holds for all \(t \geq 0\).
Lemma 20. Let $A \in \mathbb{R}^{p \times p}$ be a positive definite matrix and $v \in \{1, \ldots, p\}$. Then, we have for the inverse of the sub-matrix $A_{-v,-v} = I_{p-v}^T A I_{p-v}$ the following

$$
(I_{p-v}^T A I_{p-v})^{-1} = I_{p-v}^T A^{-1} I_{p-v} - \frac{I_{p-v}^T A^{-1} e_v e_v^T A^{-1} I_{p-v}}{e_v^T A^{-1} e_v} 
$$

(29)

Proof. Note that $I_p = I_{p-v}^T I_{p-v} + e_v e_v^T$ and $I_{p-v} e_v = 0$. (29) is equivalent to

$$
I_{p-v}^T (I_{p-v}^T A_{-v,-v} e_v^T) A^{-1} e_v = I_{p-v}^T (A_{-v,-v} I_{p-v}^T e_v e_v^T A^{-1} e_v) - I_{p-v}^T A_{-v,-v} I_{p-v}^T (A^{-1} e_v e_v^T A^{-1}) e_v
$$

$$
= I_{p-v}^T (I_{p-v}^T A_{-v,-v} A^{-1} e_v) + I_{p-v}^T (I_{p-v}^T A e_v e_v^T A^{-1}) e_v - I_{p-v}^T A_{-v,-v} I_{p-v}^T (A^{-1} e_v e_v^T A^{-1}) e_v
$$

$$
= I_{p-v}^T (A_{-v,-v} A^{-1} e_v) + I_{p-v}^T (A e_v e_v^T A^{-1} e_v)
$$

$$
= I_{p-v}^T I_{p-v} e_v^T A^{-1} e_v.
$$

Lemma 21. For random variables $X, Y$ and $t \in \mathbb{R}, \varepsilon > 0$, we have

$$
P(X + Y \geq t) \geq P(X \geq t + \varepsilon) - P(|Y| > \varepsilon)
$$

Proof.

$$
P(X \geq t + \varepsilon) = P(X \geq t + \varepsilon, |Y| > \varepsilon) + P(X \geq t + \varepsilon, |Y| \leq \varepsilon)
$$

$$
\leq P(|Y| > \varepsilon) + P(X \geq t - B)
$$

Appendix C: Additional Simulation Results
Figure 1. Results for detecting non-zero partial coherences of the VARMA(1, 1) process given in (a) and the VMA(5) process given in (d). In each figure, the upper triangular part presents results for the sample size $n = 512$ and the lower triangular part for the sample size $n = 4096$. A bluish dot represents a not certain discovery of a non-zero partial coherence, where the darker the blue color is the lower is the correct detection rate. A reddish dot represents a false discovery of a zero partial coherence and here the darker the red color is the higher is the false detection rate. The left figure presents the results of the testing procedure, while the right figure presents the results obtained using regularizing.
Figure 2. Graphs representing the brain connectivity for student 19 in the state of eyes open. The left-hand-side figure displays the unconditional connectivity based on coherence and the right-hand-side figure displays the conditional connectivity based on partial coherence. The vertex labels denote the channel labels of the EEG recording and the patient’s nose is located at the top.

Figure 3. Graphical model representing the conditional brain connectivity for Student 20 in the two states, eyes open and eyes closed. The vertex labels denote the channel labels of the EEG recording and the patient’s nose is located at the top.
Table 2 reports size and power results obtained using the simulation set up described in Section 4.2 for the case $\delta = 0.2$.

| Model  | p  | n  | $\alpha = 0.05$ | $\alpha = 0.1$ |
|--------|----|----|-----------------|-----------------|
|        | FDR | Power | FDR | Power | FDR | Power |
| VARMA(1,1) | 50  | 512 | 0.00 | 0.62 | 0.00 | 0.63 | 0.03 | 0.45 |
|        | 2048 | 0.00 | 0.82 | 0.00 | 0.82 | 0.06 | 0.66 |
|        | 4096 | 0.00 | 0.89 | 0.00 | 0.89 | 0.10 | 0.70 |
| VARMA(1,1) | 100 | 512 | 0.00 | 0.30 | 0.00 | 0.31 | 0.06 | 0.20 |
|        | 2048 | 0.00 | 0.58 | 0.00 | 0.59 | 0.08 | 0.54 |
|        | 4096 | 0.00 | 0.69 | 0.00 | 0.69 | 0.09 | 0.67 |
| VARMA(1,1) | 200 | 512 | 0.00 | 0.63 | 0.00 | 0.64 | 0.01 | 0.35 |
|        | 2048 | 0.00 | 0.81 | 0.00 | 0.81 | 0.06 | 0.64 |
|        | 4096 | 0.00 | 0.86 | 0.00 | 0.86 | 0.13 | 0.68 |
| VMA(5) | 50  | 512 | 0.00 | 0.46 | 0.00 | 0.46 | 0.00 | 0.41 |
|        | 2048 | 0.00 | 0.68 | 0.00 | 0.69 | 0.00 | 0.51 |
|        | 4096 | 0.00 | 0.81 | 0.00 | 0.81 | 0.00 | 0.70 |
| VMA(5) | 100 | 512 | 0.00 | 0.16 | 0.00 | 0.16 | 0.00 | 0.07 |
|        | 2048 | 0.00 | 0.37 | 0.00 | 0.38 | 0.00 | 0.19 |
|        | 4096 | 0.00 | 0.51 | 0.00 | 0.51 | 0.00 | 0.27 |
| VMA(5) | 200 | 512 | 0.00 | 0.08 | 0.00 | 0.08 | 0.00 | 0.04 |
|        | 2048 | 0.00 | 0.18 | 0.00 | 0.18 | 0.00 | 0.12 |
|        | 4096 | 0.00 | 0.24 | 0.00 | 0.25 | 0.00 | 0.16 |

Table 2

Empirical false discovery rates and powers for the case $\delta = 0.2$ and the different models, levels, and sample sizes considered.

As it is seen from the above table, for the case $\delta = 0.2$, Testing has in all situations an empirical FDR of 0%. The reason for this lies in the construction of the test statistic and the behavior of $G_d(t)q$ used in the determination of the threshold value $\hat{t}$. To elaborate, if $\delta > 0$ and due to the indicator function appearing in the definition of $T^{(u,v)}_n$, the test statistic accumulates under the null hypothesis, more point mass around zero. In fact, the larger $\delta$ is the less test statistics $T^{(u,v)}_n$ will exceed $\delta$, i.e., the more $T^{(u,v)}_n$ values will be equal to zero. The asymptotic distribution used to determine critical values of this test, however, only forms an upper bound; also see the Proof of Theorem 4. Hence for $\delta > 0$, $G_d(t)q$ becomes a crude estimator of the expected number of falsely rejected null hypotheses and as consequence, the FDR is forced towards zero. Nevertheless, even under such a conservative behavior of the empirical FDR, the empirical power of Testing outperforms in all cases the corresponding power of
Regularizing.