Power operations in the K"unneth spectral sequence and commutative $\mathbb{H}F_p$-algebras

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February 23, 2016

Abstract

In this paper, we prove the multiplicativity of the K"unneth spectral sequence. This is established by an analogue of the Comparison Theorem from homological algebra, which we suspect may be useful for other spectral sequences. This multiplicativity is then used to compute the action of the Dyer-Lashof algebra on $H_{\mathbb{H}F_p} \wedge \mathbb{K} \mathbb{H}F_p$, $H_{\mathbb{H}F_p} \wedge BP(2) \mathbb{H}F_p$, and part of the action on $H_{\mathbb{H}F_p} \wedge MU \mathbb{H}F_p$. We then relate these computations to the construction of commutative $R$-algebra structures on commutative $H_{\mathbb{H}F_p}$-algebras. In the case of $MU$, we obtain a necessary closure condition on ideals $I \subset MU_*$ such that $MU/I$ can be realized as a commutative $MU$-algebra.

1 Introduction

In modern algebraic topology, the theme of importing classical constructions in algebra has been very fruitful. Modern models of spectra have enabled topologists to exploit the intuition gained from homological and commutative algebra in a rigorous fashion. The goal of this paper is to continue in that vein and investigate structure present on relative smash products of commutative ring spectra. The relative smash product plays the role of the (derived) relative tensor product and so is a natural source of new examples of module and ring spectra. Our primary focus will be on understanding these relative smash products as commutative ring spectra. In particular, we will find that $A \wedge_R A$ contains interesting information about $R$ as a commutative ring spectrum.

The K"unneth spectral sequence (sometimes abbreviated as KSS) is the main tool for computing invariants of relative smash products. It is a natural extension of the classical K"unneth Theorem of algebraic topology which gives the short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(X;Z) \otimes H_j(Y;Z) \rightarrow H_n(X \times Y;Z) \rightarrow \text{Tor}_1^Z(H_i(X;Z), H_j(Y;Z))$$

where $X$ and $Y$ are spaces and $H_n(-;Z)$ is singular homology with coefficients in $\mathbb{Z}$. As is common in algebraic topology, this short exact sequence was generalized to a spectral sequence for arbitrary homology theories so that we can recover the above theorem from the following spectral sequence. In Chapter 4 section 6 of [10], they construct the K"unneth spectral sequence

$$E_2^{s,t} = \text{Tor}_s^R(\pi_* A, \pi_* B)_t \Rightarrow \pi_{s+t}(A \wedge_R B),$$

where $R$ is an $S$-algebra, and $A, B$ are right and left $R$-modules respectively. The K"unneth Theorem is a particular case of the above where $R = \mathbb{H}Z$, $A = \mathbb{H}Z \wedge \Sigma^\infty X_+$, and $B = \mathbb{H}Z \wedge \Sigma^\infty Y_+$. Previous work on the multiplicativity of the Eilenberg-Moore spectral sequence was done by [14]. They use the diagonal map of spaces in order to developed power operations and used their results to compute $H_*(B(G/\text{Top}))$. Our work here is inspired by that of Bruner in [8] on the Adams spectral sequence as well as the work of Baker and Richter in [5] on the K"unneth spectral sequence.
The Künneth spectral sequence is also a key tool for understanding relative smash products of module spectra. The main aim of this paper is to enhance this spectral sequence so that we can understand the multiplicative structure present on relative smash products of ring spectra. In particular, the multiplicativity of the above spectral sequence can provide collapse results, which are very useful in aiding computations. We establish the multiplicativity of the KSS by use of our Comparison Theorem 1.

**Theorem** (Comparison Theorem). Suppose that we have a map $f : Y \rightarrow A$ of $R$-module spectra, an exact filtration $A_\bullet \subset A$, and a free and exhaustive filtration $Y_\bullet \subset Y$. Also, suppose that there exist $f^{-1} : Y^{-1} \rightarrow A^{-1}$ such that

$$
\begin{array}{c}
Y^{-1} \xrightarrow{f^{-1}} Y \\
\downarrow \quad \downarrow f \\
A^{-1} \xrightarrow{f} A
\end{array}
$$

commutes. Then there is a map of filtrations $Y_i \xrightarrow{f_i} A_i$ such that colim $f_i \simeq f$ under the equivalences colim $Y_i \simeq Y$ and colim $A_i \simeq A$. Furthermore, the lift $f_\bullet$ of $f$ is unique up to homotopy of filtered modules, in the sense of Definition 2.14.

This result can be seen as both a lift of the classical Comparison Theorem of homological algebra to filtered $R$-module spectra and as a cellular approximation Theorem. Using this result we can derive the multiplicativity of the KSS. This multiplicativity is precisely an example of lifting properties of an $R$-module to a filtration of that $R$-module.

**Theorem** (Multiplicativity of the Künneth spectral sequence). Let $R$ be a commutative $S$-algebra, and $A$ and $B$ be $R$-algebras. The Künneth spectral sequence is multiplicative. Further, the filtration of the KSS has an $H_\infty$-structure that filters the commutative $S$-algebra structure of $A \wedge_R B$.

With this new-found multiplicative structure we are able to compute the homotopy of $H\mathbb{F}_p \wedge_R H\mathbb{F}_p$ for various commutative $S$-algebras $R$. We then take advantage of the additional $H_\infty$-structure to compute the action of the Dyer-Lashof algebra on various relative smash products. For example, we have the following computation.

**Proposition 1.1.** The Künneth spectral sequence

$$\text{Tor}_{*}^{MU_*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_* H\mathbb{F}_p \wedge_{MU} H\mathbb{F}_p$$

collapses at the $E_2$ page. The resulting algebra $\pi_* H\mathbb{F}_p \wedge_{MU} H\mathbb{F}_p$ is an exterior algebra $E_{\mathbb{F}_p}[p, \pi_1, \pi_2, \ldots]$ with Dyer-Lashof operations given by $Q^p(i)(p) = \pm \pi_{p^{-1} - 1}$ and $Q^p(i)(\pi_{p^{-1} - 1}) = \pm \pi_{p^{-1} - 2}$ where $|p| = 1$ and $|\pi_n| = 2n + 1$.

The classes $x_{p^{-1}} \in \pi_{2p^{-1} - 2}MU$ are classes that hit the Hazewinkel generators under the $p$-localization map

$$MU \rightarrow MU_{(p)}.$$

The classes $\pi_i \in \pi_{2i+1} (H\mathbb{F}_p \wedge_{MU} H\mathbb{F}_p)$ are induced classes, see Proposition 6.1 for an explicit construction. This result gives the following necessary condition on an ideal in order for the quotient to be realizable as a commutative $S$-algebra. This is result is comparable to the work of Strickland in [17].

**Corollary 1.2.** Let $I$ be an ideal of $MU_*$ generated by a regular sequence. If $I$ contains a finite non-zero number of the $x_{p^{-1}}$, then the quotient map $MU \rightarrow MU/I$ cannot be realized as a map of commutative $S$-algebras.
This result is not the only application of our computations. The classes in these relative smash products can frequently be realized as differences of null-homotopies so that one obtains a function

$$\pi_n(A \wedge R A) \times CAlg_R(A, X) \times \pi_n X$$

$$(\pi, f, g) \mapsto d(f, g)_*(\pi)$$

that measures differences between commutative $R$-algebra maps $f$ and $g$. The map

$$d(f, g) : A \wedge R A \rightarrow X$$

is the coproduct of $f, g \in CAlg_R(A, X)$, see section 6. It is a map of commutative ring spectra and therefore respects power operations. So if we think of $\pi_*(A \wedge_R A)$ as giving functions on $CAlg_R(A, -)$ we can then hope to interpret the power operations we compute as giving dependence relations between such functions. We believe this perspective will be fruitful and hope to study it in future work with Markus Spitzweck on fundamental groups of derived affine schemes. We will also use the multiplicativity of the KSS to compute the homology of connective Morava $E$-theory with Lukas Katthän.

## 1.1 Outline

We begin in section 2 by introducing some basic properties of the category of filtered $R$-modules and the category of complexes of $R$-modules for a commutative $S$-algebra $R$. The category of filtered $R$-modules, its tensoring over spaces, and its monoidal structure are what we will take advantage of in order to obtain structural results about spectral sequences. We will explain the relationship between the categories of filtered $R$-modules and complexes of $R$-modules as well as relating their monoidal structures. That structures on filtrations induce analogous structures on spectral sequences can be found in [9] or [19]. We neglect to include it here as it is well known to the experts.

In section 3, we prove our Comparison Theorem 1. This result is what allows us to show that the KSS is multiplicative. This result is essentially a cellular approximation argument. We think it may be of use in other settings, just as the Comparison Theorem of homological algebra has many applications.

In section 4, we begin studying the KSS. This is the main tool for computing invariants of relative smash products. Using the results of the previous section, we are able to show that many examples of KSSs are multiplicative. This fact should be thought of as a mirror of the analogous result regarding the Adams and Adams-Novikov spectral sequences. After proving that the KSS has an $H_\infty$-filtration, Theorem 1, we utilize this extra structure to compute the action of the Dyer-Lashof algebra on various relative smash products of $HF_p$ in section 5. While we only discuss relative smash products over $ku$, $BP(2)$, and $MU$ here, we hope to discuss relative smash products over less regular (in the technical sense) ring spectra in the future.

In section 6, we interpret our computations in terms of differences of null-homotopies. This interpretation yields a necessary condition for determining if a map between commutative $S$-algebras over $HF_p$ is a map of commutative $S$-algebras in terms of its effect on homotopy. Strickland [17] obtained comparable, and stronger, results.

We expect similar results when one works over more complicated commutative $S$-algebras than $HF_p$. However, our methods here do not extend to Landweber exact spectra. This is due to the observation of Gerd Laures that Landweber exact spectra are flat over $MU$, which was related to us by Charles Rezk.

## 1.2 Conventions

We work with $S$-modules as developed in [10]. The main reason for this choice is that the model category structures developed in [10] is well adapted to computation and categorical manipulations. In particular, every $S$-module is fibrant and there exist functorial factorization and cofibrant replacement functors. Thus, when we restrict attention to cofibrant objects and consider diagrams of cofibrations we can work as if in the
homotopy category with the added benefit that our constructions are natural. This will be obvious from our work in section 2 as our arguments feel like those made in triangulated categories but are a bit stronger. We will also not distinguish between fiber and cofiber sequences as we are working in a stable model category.

We will frequently go back and forth between $R$-module spectra and their graded homotopy groups. If we have denoted an $R$-modules spectrum by $F_i$ then by $F$ we will mean $\pi_* F_i$ as a $\pi_* R$-module. We will reserve the letter $F$ for free $R$-modules, and $A$ and $B$ for commutative $S$-algebras. By a free $R$-module $F$ we mean a wedge of $S^n R$, which are cofibrant replacements of $\Sigma^n R$ as an $R$-module. This is necessary as one drawback of the model of spectra presented in [10] is that $R$ may not be a cofibrant $R$-module. Our primary reason for working with free $R$-modules is that $R$-module maps out of free $R$-modules are determined by their effect in homotopy. Most importantly, if a map out of a free $R$-module is 0 in homotopy then it is null homotopic.

Acknowledgments

There are many people who deserve thanks and acknowledgment for their support, criticisms, and advice during the production of this work. However, I will be brief as a more thorough list of acknowledgements can be found in my thesis. This document contains much of the work I did for my thesis at Wayne State University under the guidance of Robert Bruner. As such, I must thank him at the outset for being right about things so frequently and being patient when I didn’t believe him. I would also like to thank the members of my committee Daniel Isaksen, John Klein, James McClure, and Andrew Salch. In particular, I learned a great deal from Andrew in the last couple years of my program. Michael Mandell suggested what became the Comparison Theorem 1, and for that I am grateful. I would similarly like to thank Andrew Baker and Tyler Lawson for their encouragement, insight, and interest in various aspects of this project. Lastly, this would have been impossible without the support of Jim Veneri and Mike Catanzaro.

2 Filtrations and Complexes

In this section we develop some preliminary machinery and terminology for filtered $R$-module spectra, referred to frequently as filtrations. The goal of this section is the necessary background for our Comparison Theorem, Theorem 1, and working with spectral sequences. We believe this section may also be of use to those wishing to establish similar structures on other spectral sequences.

The main objects discussed in this section are filtered $R$-modules and complexes of $R$-modules. There is a correspondence between these two categories, but it is only partially defined on complexes of $R$-modules. While we can always associate the associated graded complex to any filtration, we are not always able to invert this procedure. This is possible when the complex is a free resolution, in the classical sense, which will be proved in section 2.2.

2.1 Basic definitions

Let $R$ be a commutative $S$-algebra that is cofibrant and connective. We begin by giving a few definitions.

Definition 2.1. A filtered $R$-module $A_\bullet$ is a diagram of $R$-modules

\[ A_{-1} = \ast \to A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \to \cdots \]

where each $\alpha_i$ is a cofibration of $R$-modules. We call a filtered $R$-module $A_\bullet$ a filtration of an $R$-module $A$ if for every $i$ there is a map $j_i : A_i \to A$ of $R$-modules such that
commutes. We call the filtration free if the filtration quotients, $\Sigma^i F_i := \text{Cof}(A_{i-1} \to A_i)$, are free $R$-modules. We call the filtration exhaustive if the $j_i$ induce an equivalence $\text{hocolim}_i A_i \simeq A$. Let $\Sigma^i K_i := \text{Fiber}(A_i \xrightarrow{j_i} A)$, and let $\pi_i$ denote maps induced by the $\alpha_i$ via the following diagram.

\[
\begin{array}{ccccccc}
\Sigma^i K_i & \longrightarrow & A_i & \xrightarrow{j_i} & A \\
\pi_i & \downarrow & \alpha_i & \downarrow & 1_A \\
\Sigma^{i+1} K_{i+1} & \longrightarrow & A_{i+1} & \xrightarrow{j_{i+1}} & A
\end{array}
\]

We call the filtration exact if the maps $\pi_i$ are all 0 in homotopy. We will call a filtration $A_\bullet$ cellular if each $R$-module $A_i$ is a cellular $R$-module and each pair $(A_{i+1}, A_i)$ is a relative cell $R$-module. If there is a morphism of filtrations $f_\bullet : A_\bullet \to B_\bullet$ such that each $f_i : A_i \to B_i$ is a weak equivalence and $B_\bullet$ is a cellular filtration, then we say that $A_\bullet$ has the homotopy type of a cellular filtration.

Every cofibrant $R$-module $A$ can be thought of as the filtered $R$-module $c(A)_\bullet$ by taking all of the maps to be the identity. The associated graded of this filtration is the complex of $R$-modules that is just $A$ in degree 0 and the trivial $R$-module in every other degree. Frequently, we will be considering filtrations $A_\bullet$ of $A := \text{hocolim}_\bullet A_\bullet$ and so we will not mention the $R$-module that $A_\bullet$ is a filtration of, as it is implicit. We will be primarily concerned with cellular filtrations. As every $R$-module $M$ has a cellular filtration this will impose no real restriction. This will follow from our construction of a free and exact filtration from a free resolution, see Proposition 2.6. Therefore, filtrations will be assumed to be cellular without comment. As the maps in the filtration are assumed to be cofibrations $\text{colim} A_\bullet \simeq \text{hocolim} A_\bullet$.

**Lemma 2.2.** In an exact filtration, each map

\[\Sigma^{-1} A \xrightarrow{\partial_i} \Sigma^i K_i,\]

$i \geq 0$, induces the 0 map in homotopy. Thus $\pi_i \Sigma^i K_i$ is the kernel of $j_i : \pi_\ast A_i \to \pi_\ast A$. Further, every exact filtration is exhaustive.

As mentioned above, we also work with complexes of $R$-module spectra.

**Definition 2.3.** A complex $A \leftarrow F_\ast (\text{of } R\text{-modules})$ is a diagram

\[
A \leftarrow \varepsilon \begin{array}{cccccccc}
F_0 & \xleftarrow{d_0} & F_1 & \xleftarrow{d_1} & F_2 & \xleftarrow{d_2} & F_3 & \leftarrow \cdots
\end{array}
\]

such that

$\pi_\ast A \leftarrow \varepsilon \begin{array}{cccccccc}
\pi_\ast F_0 & \xleftarrow{d_0} & \pi_\ast F_1 & \xleftarrow{d_1} & \pi_\ast F_2 & \xleftarrow{d_2} & \pi_\ast F_3 & \leftarrow \cdots
\end{array}$

is a complex in the category of $\pi_\ast R$-modules. A complex is exact if taking homotopy groups induces an exact sequence in the category of $\pi_\ast R$-modules. We will call such a complex a resolution. We call a resolution a free if the $F_i$ are free $R$-modules.

We realize this use of the asterisk may conflict with the convention that $M_\ast$ denotes $\pi_\ast M$, but the alternative is less desirable.
Lemma 2.4. Given a free resolution of $\pi_* A$ by $\pi_* R$-modules

$$\pi_* A \leftarrow \varepsilon \leftarrow F_0 \leftarrow d_0 \leftarrow F_1 \leftarrow d_1 \leftarrow F_2 \leftarrow d_2 \leftarrow F_3 \leftarrow \cdots$$

there is a free resolution of $R$-modules

$$A \leftarrow \varepsilon \leftarrow F_0 \leftarrow d_0 \leftarrow F_1 \leftarrow d_1 \leftarrow F_2 \leftarrow d_2 \leftarrow F_3 \leftarrow \cdots$$

that realizes the above algebraic resolution.

Proof. First, we construct free $R$-modules $F_i$ such that $\pi_* F_i = F_i$ which is straightforward. To construct maps between the free $R$-modules $F_i$ we use the fact that maps out of free $R$-modules exist whenever they exist in the homotopy category. \qed

Note that while we do not claim that an exact complex of $R$-modules has that $d^2 \simeq 0$ but that $\pi_*(d^2) = 0$. These two conditions do coincide when the complex is of free $R$-modules.

2.2 Filtration to Resolution and back again

Every filtration has a corresponding complex constructed as the "associated graded" of the filtration. However, it is not possible to construct a filtration for any given complex. In case that complex is a free resolution, we can construct an associated filtration. The associated filtration will be free and exact, and have the resolution we began with as its associated graded complex. The assumption that a filtration or complex be both free and exact is exactly the condition necessary in order to have a correspondence between filtrations and complexes. In section 3, we will extend this correspondence to morphisms.

Here we construct the associated graded complex of a filtration.

Proposition 2.5. Given a free and exact filtration

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\text{colim}(A_i)}$$

there is an associated free resolution

$$A \leftarrow \varepsilon \leftarrow F_0 \leftarrow d_0 \leftarrow F_1 \leftarrow d_1 \leftarrow F_2 \leftarrow d_2 \leftarrow F_3 \leftarrow \cdots$$

where $\text{Cof}(A_{i-1} \rightarrow A_i) = \Sigma^i F_i$.

Proof. To construct the desired resolution, we first need to determine the cofiber of $\Sigma^i K_i \rightarrow \Sigma^{i+1} K_{i+1}$, where $\Sigma^i K_i := \text{Fiber}(A_i \xrightarrow{j_i} A)$. We make use of Verdier’s axiom to identify this cofiber.
The cofiber sequences $\Sigma^{i+1}K_{i+1} \to \Sigma^{i+1}F_{i+1} \to \Sigma^{i+1}K_i$ induce short exact sequences in homotopy as $\Sigma^{i+1}K_i \to \Sigma^{i+2}K_{i+1}$ is $0$ in homotopy as the filtration is assumed to be exact. Now we splice together the short exact sequences $0 \to K_i \to F_i \to K_{i-1} \to 0$ in the usual way to get

$$
\begin{array}{ccccccccc}
A_\ast & \to & F_0 & \to & F_1 & \to & F_2 & \to & F_3 & \to & \cdots \\
& \downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
& K_0 & \to & K_1 & \to & K_2 & \to & & & \\
\end{array}
$$

an $R_\ast$-free resolution of $A_\ast := \pi_\ast A$, where $F_i := \pi_\ast(F_i)$ and $K_i := \pi_\ast(K_i)$. As each $F_i$ is a free $R$-module, this is enough to construct the desired free resolution in $R$-modules.

We now have the following realization result which will be used in the construction of the KSS. It is equivalent to the construction in [10].

**Proposition 2.6.** Given a free resolution of $\pi_\ast A$ by $\pi_\ast R$-modules

$$
A_\ast \xleftarrow{\varepsilon} F_0 \xleftarrow{d_0} F_1 \xleftarrow{d_1} F_2 \xleftarrow{d_2} F_3 \xleftarrow{\cdots}
$$

there is an exhaustive, exact, and free filtration of $A$ by $R$-modules $A_i$

$$
A_{-1} = * \xrightarrow{\alpha_{-1}} A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \to \text{colim}(A_i)
$$

such that $\Sigma^iF_i = \pi_\ast \text{Cof}(A_{i-1} \to A_i)$.

**Proof.** The resolution can be factored into SESs of $R_\ast$-modules

$$
\begin{array}{cccccccccc}
0 & \to & K_{i-1} & \xleftarrow{p_{i-1}} & F_i & \xleftarrow{\ell_i} & K_i & \to & 0 \\
& \quad & \downarrow & \quad & \quad & \quad & \quad & \quad & \quad \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}
$$

We can realize this free $R_\ast$ resolution geometrically as a resolution by free $R$-modules $F_i$ as in [4] and Chapter 4 section 5 of [10]. Here, define $F_i$ to be the appropriate wedge of $S^n_R$’s so that $\pi_\ast(F_i) = F_i$ as an $R_\ast$-module. We define $K_0$ to be the fiber of the map $F_0 \xrightarrow{\varepsilon} A$. Since the composite

$$
A \xleftarrow{\varepsilon} F_0 \xleftarrow{d_0} F_1
$$

is $0$ in homotopy it is null homotopic and we can lift $d_0$ over $\iota_0$ to get the following diagram.

$$
\begin{array}{cccccccccc}
F_1 & \to & F_0 & \xleftarrow{d_0} F_1 & \xleftarrow{\ell_0} & K_0 & \to & \Sigma^{-1}A \\
& \downarrow & \quad & \downarrow & \quad & \quad & \quad & \quad \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}
$$

Now, define $K_1$ to be the fiber of $F_1 \xrightarrow{p_0} K_0$. Next, we repeat this process inductively. As $F_{i-1} \xleftarrow{K_{i-1}}$ is always an injection,

$$
K_{i-1} \xleftarrow{F_i} \xleftarrow{F_{i+1}}
$$
is zero since

\[ F_{i-1} \leftarrow F_i \leftarrow F_{i+1} \]

is zero as well. Therefore,

\[ K_{i+1} \leftarrow F_i \leftarrow F_{i+1} \]

is null homotopic, and we proceed as above. This gives us cofiber sequences

\[ K_{i-1} \leftarrow F_i \leftarrow F_{i+1} \]

which induce short exact sequences in homotopy. This implies that the boundary maps \( K_{i-1} \xrightarrow{\pi_i} \Sigma K_i \) are zero in homotopy. The factorization

\[ F_i \leftarrow F_{i-1} \leftarrow F_i \leftarrow \cdots \]

is realized geometrically by

\[ A_i \xrightarrow{\partial_i} K_i \]

since \( \pi_*(K_i) \cong K_i \cong \text{Ker}(d_{i-1}) \cong \text{Ker}(p_{i-1}) \). We define \( A_i \) to be the cofiber of the map \( \Sigma^{-1}A \xrightarrow{\partial_i} \Sigma^iK_i \). This is the composite of the boundary \( \Sigma^{-1}A \xrightarrow{\pi_0} K_0 \) (the initial fiber sequence above) with \( i \) composites of the \( j \)-th suspensions of the \( i \) boundary maps \( \Sigma^jK_{i-1} \xrightarrow{\pi_i} \Sigma^jK_i \). These give cofiber sequences \( \Sigma^{-1}A \xrightarrow{\pi_0} \Sigma^iK_i \xrightarrow{q_0} A_i \) or rather \( \Sigma^iK_i \xrightarrow{q_0} A_i \xrightarrow{\partial_j} A \). As we already know that the boundary maps \( K_{i-1} \xrightarrow{\pi_i} \Sigma K_i \) are zero in homotopy, we see that the filtration \( A_\bullet \) is exact.

The following diagram and summary may help in visualizing what is going on. The top row is the lift of the algebraic resolution of \( A_* \) by free \( R_* \) modules to a resolution of \( A \) by free \( R \)-modules. Then there is the “epi-mono” factorization of this resolution is then the zig zag featuring \( F_i \)’s and \( K_i \)’s. We rotate these cofiber sequences and obtain a tower of \( \Sigma^iK_i \)’s from which we obtain our desired filtration. This filtration \( A_\bullet \) is obtained by taking the levelwise cofiber of the map of towers \( \Sigma^{-1}A \rightarrow \Sigma^*K_\bullet \) where the domain is the constant tower.

To see that this filtration \( A_\bullet \) is also free, we compute \( \text{Cofiber}(A_{i-1} \xrightarrow{\alpha_{i-1}} A_i) \) using the Verdier braid diagram that analyzes the composition

\[ \partial_i : \Sigma^{-1}A \xrightarrow{\pi_0} \Sigma^iK_{i-1} \xrightarrow{\pi_i} \Sigma^iK_i. \]
Thus the filtration is exact and free as claimed.
We also want for the maps $A_i \to A_{i+1}$ to be cofibrations. To see this note that $A_{i+1}$ is the pushout of

$$\Sigma F_{i+1} \xrightarrow{q_i \circ p_{i+1}} A_i \quad \Sigma C(F_{i+1})$$

therefore the map $A_i \to A_{i+1}$ is a cofibration since the inclusion of $X \to CX$ is always a cofibration. □

This is the general construction of a filtration. It is carried out in more detail and with a bit of a different focus in [10].

### 2.3 Monoidal Structures

In this subsection, we define various multiplicative structures in filtered $R$-modules. All of our definitions will induce the familiar structure in complexes of $R$-modules. The goal of these constructions is to provide our spectral sequences with multiplicative structures and power operations whenever the relevant filtration is multiplicative or is an $H_\infty$-filtration, respectively. We first define a monoidal structure on filtered $R$-modules and then proceed to $H_\infty$-filtrations. Our definition of $H_\infty$-filtration is as an $H_\infty$-object in filtrations as opposed to a filtration by $H_\infty$-objects in the underlying category. The latter leads to a different theory. For earlier examples of the former, see the work of Bruner [7], and Hackney [11]. Note that we use a potentially unconventional “union” notation for iterated pushouts in $R$-modules.

**Definition 2.7.** The smash product of two filtrations $X_\bullet$ and $Y_\bullet$ is denoted by $\Gamma_\bullet(X_\bullet, Y_\bullet)$. The $n$th term in the filtration is

$$\Gamma_n(X_\bullet, Y_\bullet) := \bigcup_{i+j=n} X_i \wedge Y_j = X_0 \wedge Y_0 \cup X_0 \wedge Y_{n-1} \cup X_1 \wedge Y_{n-1} \cup \ldots \cup X_{n-1} \wedge Y_0 \cup X_n \wedge Y_0.$$ 

We will also denote iterated smash products of $r$-filtrations $X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet$ by $\Gamma^r(X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet)_\bullet$. Here, the $n$th term in the filtration is

$$\Gamma^r_n(X_1^\bullet, X_2^\bullet, \ldots, X_r^\bullet) := \bigcup_{\sum_{i=1}^r \alpha_i = n} X_{\alpha_1}^1 \wedge X_{\alpha_2}^2 \wedge \ldots \wedge X_{\alpha_r}^r.$$ 

If all filtrations are the same, we will use the symbol $\Gamma^r_\bullet$ or simply $\Gamma_\bullet$ when $r = 2$.

**Remark 2.8.** We will use relative smash products of $R$-modules. We will use $\Gamma^r_\bullet$ to denote the relative construction where every occurrence of $- \wedge -$ is replaced by $\wedge_R -$ when $R$ is understood from context, otherwise we will use the notation $\Gamma^R_\bullet$. This will only come up when $R$ is a commutative $S$-algebra.
The above definition allows us to consider monoid objects in the category of filtered modules, in exact analogy with DGA’s being monoids in chain complexes. In fact, the definition of a (strictly) multiplicative filtration is precisely what is necessary to ensure that the associated graded complex of a filtration is a DGA (in spectra) as we will see in Lemma 2.12.

Definition 2.9. We say that a filtration $X_\bullet$ is multiplicative if there is a map of filtrations

$$
\xymatrix{
\Gamma_\bullet 
\ar[r]^-{\mu_\bullet} & X_\bullet
}
$$

In particular, we require maps $\mu_n : \Gamma_n \to X_n$ such that

$$
\xymatrix{
\Gamma_{n-1} 
\ar[r]^-{\mu_{n-1}} \ar[d] & \Gamma_n 
\ar[d]^-{\mu_n}
\ar[r] & X_{n-1} \ar[r]^-{\mu_n} & X_n
}
$$

commutes. We also require that $\mu_\bullet$ satisfy the obvious associativity condition up to homotopy. If the map $\mu_\bullet$ satisfies the associativity condition on the nose, then we call it strictly multiplicative.

The associativity condition is not unreasonable as there is a natural equivalence

$$
\Gamma_\bullet(X_\bullet, \Gamma_\bullet(Y_\bullet, Z_\bullet)) \cong \Gamma_\bullet^R(X_\bullet, Y_\bullet, Z_\bullet) \cong \Gamma_\bullet(\Gamma_\bullet(X_\bullet, Y_\bullet), Z_\bullet)
$$

which is induced by the associativity of the smash product in the category of $S$-modules. If $A$ is an $R$-algebra then the constant filtration $c(A)_\bullet$ is strictly multiplicative. If a filtration is multiplicative then the (homotopy) colimit of the filtration has a product that is compatible with the filtration. This follows from the fact that $\text{colim}_\bullet \Gamma_\bullet(X_\bullet, Y_\bullet) \simeq (\text{colim}_\bullet X_\bullet) \wedge (\text{colim}_\bullet Y_\bullet)$ (under suitable finiteness assumptions). The reason for only requiring a weak form of associativity is that we will produce such maps of filtrations by a general construction that produces maps uniquely only up to homotopy and that our applications do not require strictness. This will be spelled out in the proof of 3.1.

Lemma 2.10. If $X_\bullet$ and $Y_\bullet$ are two multiplicative filtrations then $\Gamma_\bullet(X_\bullet, Y_\bullet)$ is multiplicative.

Here we use that the category of $S$-modules is symmetric monoidal and so the related result for $\Gamma^R(X_\bullet, Yb)$ holds when $R$ is a commutative $S$-algebra. This result will only be applied in the case that $Y_\bullet$ is a constant filtration of a commutative $S$-algebra.

Notice that there is a natural action of $\pi \subset \Sigma_r$ on the filtration $\Gamma_r\bullet$ given by permuting factors. This action, along with the tensoring of the category of $S$-modules over spaces, will be used in the definition of $H_\infty$-filtration below. When necessary, we consider the space $E\Sigma_n$ as a filtered spectrum in $S$-modules via its skeletal filtration.

Definition 2.11. We say that a filtration $X_\bullet$ is $H_\infty$ or has an $H_\infty$-structure if there are maps of filtrations

$$
\xi_\bullet : \Gamma_{\bullet (\cdot, \Sigma_r)} := \Gamma_\bullet(\text{ES}_{r+}^{(\cdot)}, \Gamma_\bullet(X_\bullet))_{\Sigma_r} \to X_\bullet
$$

that are compatible in the sense of Chapter 1 definition 3.1 of [8]. In particular, we have maps

$$
\xi^n_r : \Gamma_n(\text{ES}_{r+}^{(\cdot)}, \Gamma_\bullet)_\Sigma_r = \bigcup_{i+j=n} E\Sigma_r^{(i)} \wedge \left( \bigcup_{\sum_{j=1}^{i} \alpha_i = j} X_{\alpha_1} \wedge X_{\alpha_2} \wedge \ldots \wedge X_{\alpha_i} \right) \to X_n.
$$

Or more explicitly, we require

$$
\xi^{n,m}_r : E\Sigma_r^{(m)} \wedge \Gamma_\bullet^n \to X_{n+m}.
$$

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for every \(r, m, \) and \(n\). The \(\xi^{n,m}_r\)'s must be compatible in the sense that

\[
E\Sigma_r^{(n)} \wedge_{\Sigma_r} \Gamma_{m-1}^r \cup E\Sigma_r^{(n-1)} \wedge_{\Sigma_r} \Gamma_m^r \rightarrow E\Sigma_r^{(n)} \wedge_{\Sigma_r} \Gamma_m^r
\]

commutes for all \(r, n\) and \(m\).

We will frequently abbreviate \(E\Sigma_r^{(k)} \wedge_{\pi} \Gamma_m^r\) as \(\Gamma_{k,n}^r\). The above definitions are motivated by the following lemmas. They were chosen so that the associated graded of the filtrations will be exactly the familiar object in chain complexes.

**Lemma 2.12.** Let \(X_\bullet \subset X\) and \(Y_\bullet \subset Y\) be bounded below cellular filtrations. Let the associated graded complexes be denoted by \(E^0(X_\bullet) = F_\ast\) and \(E^0(Y_\bullet) = G_\ast\). Then, \(\Gamma_\ast(X_\bullet, Y_\bullet)\) is a cellular filtration of \(X \wedge Y\).

The associated graded complex of \(\Gamma_\ast(X_\bullet, Y_\bullet)\) is the smash product of the two complexes; that is,

\[
E^0(\Gamma_\ast(X_\bullet, Y_\bullet)) = E^0(X_\bullet) \wedge E^0(Y_\bullet) = F_\ast \wedge G_\ast
\]

where the \(n\)th term in the complex \(F_\ast \wedge G_\ast\) is given by \(\bigvee_{i+j=n} F_i \wedge G_j\). Also, the associated graded complex of \(\Gamma_\ast'(X_\bullet, Y_\bullet)\) is the relative smash product of the two complexes

\[
E^0(\Gamma_\ast'(X_\bullet, Y_\bullet)) = E^0(X_\bullet) \wedge_R E^0(Y_\bullet) = F_\ast \wedge_R G_\ast
\]

where the \(n\)th term in the complex \(F_\ast \wedge_R G_\ast\) is given by \(\bigvee_{i+j=n} F_i \wedge_R G_j\).

**Lemma 2.13.** Let \(X_\bullet \subset X\) be a bounded below cellular filtration with associated graded complex \(E^0(X_\bullet) = F_\ast\). We then have the following regarding the filtration \(\overline{\Gamma}_m^\pi(X_\bullet)\) of \(E\pi^+_\ast \wedge_{\pi} X^\wedge\tau\) where

\[
\overline{\Gamma}_m^\pi(X_\bullet) := \Gamma_m(\overline{E\pi^+_\ast} \wedge_{\pi} \overline{X^\wedge\tau}) = \bigcup_{k+n=m} \overline{\Gamma}_{k,n}^r.
\]

- \(\Gamma_\ast^\pi\) and \(\overline{\Gamma}_m^\pi(X_\bullet)\) are both bounded below cellular filtrations.

- The associated graded complex of \(\overline{\Gamma}_m^\pi(X_\bullet)\) is given by the smash products of the associated graded complexes of \(E\pi^+_\ast\) and \(\Gamma_\ast^\pi\). In particular,

\[
\frac{\overline{\Gamma}_m^\pi(X_\bullet)}{\Gamma_{m-1}^\pi(X_\bullet)} \simeq \bigvee_{i+j=n} \frac{B_{\pi}^{(i)}}{B_{\pi}^{(i-1)}} \wedge_{\pi} \bigvee_{\sum_{k=1}^r \alpha_k = j} \left( \bigwedge_{k=1}^r \Sigma_{\alpha_k} F_{\alpha_k} \right)
\]

where \(B_{\pi}^{(i)} = E_{\pi}^{(i)}/\pi\).

As before, if \(A\) is a commutative \(R\)-algebra then the filtration \(c(A)_{\pi}\) is a genuinely commutative multiplicative filtration. We use the term \(H_\infty\)-filtrations as opposed to \(E_\infty\)-filtrations because while we will be able to construct the relevant maps for an \(E_\infty\)-structure we will only be able to show that the relevant diagrams commute up to homotopy. This will not surprise those familiar with the algebraic approach to Steenrod and Dyer-Lashof operations of Peter May in [15]. This inability to construct genuinely commuting diagrams is due to our tool for constructing such structures being the Comparison Theorem, Theorem 1, which only constructs a unique map up to homotopy. This definition is equivalent to the one used by Bruner in [8] and [7]. The apparent difference is due to the filtration happening in “negative” degrees in [8] and so the skeletal filtration degree of \(E\Sigma_\ast\) decreases the Adams filtration.

The existence of a monoidal structure also implies that the category of filtered modules is tensored over \(R\)-modules, and therefore any other category that \(R\)-modules happens to be tensored over. In particular, this allows us to define a notion of homotopy.
**Definition 2.14.** Let \( f_\bullet, g_\bullet : A_\bullet \to B_\bullet \) be two maps of filtered \( R \)-modules. We call \( H_\bullet : I^R_\bullet \land_R A_\bullet \to B_\bullet \) a homotopy from \( f_\bullet \) to \( g_\bullet \) if the following diagram commutes.

\[
\begin{array}{ccc}
\Gamma'(c(R \land 0_+), A_\bullet) & \simeq & A_\bullet \\
& f_\bullet \searrow & \downarrow \\
& & \Gamma'(I^R_\bullet, A_\bullet) \\
& & \searrow H_\bullet \\
& & B_\bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma'(c(R \land 1_+), A_\bullet) & \simeq & A_\bullet \\
& g_\bullet \swarrow & \uparrow \\
& & \Gamma'(I^R_\bullet, A_\bullet) \\
& & \nearrow H_\bullet \\
& & B_\bullet \\
\end{array}
\]

Here, \( I^R_\bullet \) is the filtered \( R \)-module coming from the cellular structure on the standard unit interval. In filtration 0 we have that \( I^R_0 := R \land \{0, 1\}_+ \simeq R \lor R \) where the first wedge summand comes from \( 0 \subset I \) and the second from \( 1 \subset I \). In filtration \( n \) we have that \( I^R_n := R \land I_+ \) and the maps in the filtration are the obvious ones.

Note that such a homotopy will induce a chain homotopy on homotopy groups of the associated graded complex of a filtration. This notion of homotopy is also an equivalence relation, the standard proofs that rely on nice properties of the unit interval apply.

### 3 Comparison Theorem

In this section we state and prove our Comparison Theorem 1 as well as provide some of its corollaries regarding multiplicative structures. This result is a filtered version of the Comparison Theorem of homological algebra. In fact, the Comparison Theorem of homological algebra can be recovered from our result by applying the “associated graded complex” functor everywhere. One can also view this result as a “cellular approximation” type result.

**Theorem 1.** Suppose that we have a map \( f : Y \to A \) of \( R \)-modules, an exact filtration \( A_\bullet \subset A \), and a free and exhaustive filtration \( Y_\bullet \subset Y \). Also, suppose that there exist \( f_{-1} : Y_{-1} \to A_{-1} \) such that

\[
\begin{array}{ccc}
Y_{-1} & \to & Y \\
\downarrow f_{-1} & & \downarrow f \\
A_{-1} & \to & A
\end{array}
\]

commutes. Then there is a map of filtrations \( Y_i \xrightarrow{f_i} A_i \) such that \( \colim f_i \simeq f \) under the equivalences \( \colim Y_i \simeq Y \) and \( \colim A_i \simeq A \). Furthermore, the lift \( f_\bullet \) of \( f \) is unique up to homotopy of filtered modules, in the sense of Definition 2.14.

**Proof.** We first prove the existence of such a lift by induction and then show that uniqueness is a special case of existence. The base case is handled by the assumption that we have an \( f_{-1} \) making the above diagram commute. In practice, \( Y_{-1} \) and \( A_{-1} \) will be a point and so having just a map \( f : Y \to A \) is sufficient.
Suppose we have already constructed \( f_j : Y_j \to A_j \) so that the diagram

\[
\begin{array}{ccc}
Y_{j-1} & \xrightarrow{s_{j-1}} & Y_j \\
\downarrow f_{j-1} & & \downarrow y_j \\
A_{j-1} & \xrightarrow{\alpha_{j-1}} & A_j
\end{array}
\]

commutes for all \(-1 \leq j < i\). We now wish to construct \( f_i : Y_i \to A_i \) such that

\[
\begin{array}{ccc}
Y_{i-1} & \xrightarrow{s_{i-1}} & Y_i \\
\downarrow f_{i-1} & & \downarrow y_i \\
A_{i-1} & \xrightarrow{\alpha_{i-1}} & A_i
\end{array}
\]

commutes. We have the cofiber sequences

\[
Y_{i-1} \xrightarrow{s_{i-1}} Y_i \xrightarrow{\delta_i} \Sigma^i G_i.
\]

with \( G_i \) a free \( R \)-module as \( Y_* \) is a free filtration. For the above diagram to commutate we must construct

\[
f_i : Y_i \to A_i
\]

so that

\[
f_i \circ s_{i-1} = \alpha_{i-1} \circ f_{i-1} \text{ and } j_i \circ f_i = y_i.
\]

The map

\[
Y_{i-1} \xrightarrow{y_{i-1}} A
\]

factors through \( Y_i \) so that \( y_{i-1} = y_i \circ s_{i-1} \), as indicated by the above diagram. Therefore the composite \( \Sigma^{i-1} G_i \to Y_{i-1} \xrightarrow{\Sigma^{i-1} y_i} A \) is 0 in homotopy as \( \Sigma^{i-1} G_i \) is the fiber of \( s_{i-1} \). From the following commutative diagram

\[
\begin{array}{ccc}
\pi_* \Sigma^{i-1} G_i & \xrightarrow{\pi_* y_{i-1}} & \pi_* Y_{i-1} \\
\downarrow g_i & & \downarrow \pi_* s_{i-1} \\
Ker(\pi_* j_{i-1}) = \pi_* \Sigma^{i-1} K_{i-1} & \xrightarrow{\pi_* \alpha_{i-1}} & \pi_* A_{i-1} \\
\downarrow & & \downarrow \pi_* \alpha_{i-1} \\
\pi_* A_i & & \pi_* A_i
\end{array}
\]

we can deduce that the composite

\[
\Sigma^{i-1} G_i \to Y_{i-1} \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{\alpha_{i-1}} A_i
\]
is null-homotopic. As the top row of that commutative diagram comes from a cofiber sequence, we know that the image of \( \pi \Sigma^{i-1} G_i \) in \( \pi_* A \) is 0 and therefore contained in the kernel of

\[
\pi_* j_{i-1} : \pi_* A_{i-1} \to \pi_* A.
\]

The map \( \pi_* \alpha_{i-1} : \pi_* A_{i-1} \to \pi_* A \) annihilates the kernel of \( j_{i-1} \) as the cofiber sequence

\[
\Sigma^k K_k \to A_k \to A
\]

induces a short exact sequence in homotopy for all \( k \) by Lemma 2.2. Thus the composite

\[
\Sigma^{i-1} G_i \to A_{i-1} \to A
\]

is 0 in homotopy and is therefore null-homotopic since \( G_i \) is free.

As the above mentioned composite is null-homotopic, we can extend \( f_{i-1} \) along \( s_{i-1} \) to all of \( Y_i \)

\[
\Sigma^{i-1} G_i 
\]

so that \( \tilde{f}_i \circ s_{i-1} = \alpha_{i-1} \circ f_{i-1} \). We now must determine if \( y_i = j_i \circ \tilde{f}_i \).

Consider the map

\[
\varphi := j_i \circ \tilde{f}_i - y_i : Y_i \to A.
\]

We have that

\[
\varphi \circ s_{i-1} = (j_i \circ \tilde{f}_i - y_i) \circ s_{i-1} \\
= j_i \circ \alpha_{i-1} \circ f_{i-1} - y_{i-1} \\
= j_{i-1} \circ f_{i-1} - y_{i-1} \\
= 0.
\]

since by assumption

\[
y_i \circ s_{i-1} = y_{i-1} \\
j_{i-1} \circ f_{i-1} = y_{i-1} \\
j_i \circ \alpha_{i-1} = j_{i-1}.
\]

Therefore, \( \varphi : Y_i \to A \) induces a map \( d : \Sigma^i G_i \to A \)

\[
Y_i \xrightarrow{s_{i-1}} Y_{i-1} \xrightarrow{\delta_i} \Sigma^i G_i \\
\downarrow \varphi \downarrow d \\
A
\]

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so that \( d \circ \delta_i = \varphi \). Since \( \partial_i \) is 0 in homotopy and \( G_i \) is free, we have that

\[
\partial_i \circ d : \Sigma^i G_i \to \Sigma^{i+1} K_i
\]

is null homotopic. Therefore, there is a lift

\[
\begin{array}{ccc}
A_i & \xrightarrow{j_i} & A \\
\downarrow{\partial_i} & & \downarrow{\partial_i} \\
\Sigma^i G_i & \xrightarrow{d} & \Sigma^{i+1} K_i
\end{array}
\]

\( d' \) of \( d \), and so we have \( j_i \circ d' = d \) and \( d \circ \delta_i = \varphi \). We now define \( f_i = \tilde{f}_i - d' \circ \delta_i \). All that remains is to check that \( f_i \circ s_{i-1} = \alpha_{i-1} \circ f_{i-1} \) and \( j_i \circ f_i = y_i \).

\[
\begin{align*}
f_i \circ s_{i-1} &= (\tilde{f}_i - d' \circ \delta_i) \circ s_{i-1} \\
&= \tilde{f}_i \circ s_{i-1} \\
&= \alpha_{i-1} \circ f_{i-1}
\end{align*}
\]

and

\[
\begin{align*}
j_i \circ f_i &= j_i \circ (\tilde{f}_i - d' \circ \delta_i) \\
&= j_i \circ \tilde{f}_i - d \circ \delta_i \\
&= j_i \circ \tilde{f}_i - \varphi \\
&= y_i
\end{align*}
\]

as desired.

To prove that any two such lifts are homotopic we will construct a homotopy using the above existence result. So assume that we are given two different lifts of \( f \), call them \( f_1 \) and \( f_2 \). As \( f \simeq \text{colim} \ f_i \) we have that \( f^1 \simeq f^2 \) where \( f^i := \text{colim} \ f^i \). Specifically, we have a map

\[
H : I^R_+ \wedge_R Y \longrightarrow A
\]

which is a homotopy from \( f^1 \) to \( f^2 \). We can now lift this homotopy by applying the above argument to the map \( H \) and the filtration \( \Gamma'(I^R, Y) \), which is still a free filtration. The desired homotopy of maps of filtered \( R \)-modules follows.

This key lemma will allow us to construct multiplicative and \( H_\infty \)-filtrations. The uniqueness up to homotopy will be necessary to show that certain diagrams commute up to homotopy.

One application of the above result is the lifting of maps from the category of \( R \)-modules to the category of filtered \( R \)-modules. Given a map of \( R \)-modules

\[
\varphi : A \longrightarrow B
\]

there is a unique homotopy class of lift

\[
\varphi_* : A_* \longrightarrow B_*
\]

where the filtrations \( A_* \) and \( B_* \) are those constructed from a free \( R_* \)-resolution of \( A_* \) and \( B_* \) as described in section 2.2. This fact will provide for the naturality of the following corollaries.

**Corollary 3.1.** If \( A \) is an \( R \)-algebra, then the filtration of \( A \) associated with a free resolution of \( A \) is also multiplicative. The construction of this multiplicative filtration is natural.
Proof. Let $A_\bullet \subset A$ be the filtration associated with the free resolution. It is free and exact by Proposition 2.6. There is a natural map of filtrations

$$\Gamma_\bullet \to \Gamma'_\bullet$$

where $\Gamma_n := \bigcup_{i+j=n} A_i \wedge A_j \subset A \wedge A$ and $\Gamma'_n := \bigcup_{i+j=n} A_i \wedge_R A_j \subset A \wedge_R A$. The filtration quotients of $\Gamma_\bullet$ are $\bigvee_{i+j=n} F_i \wedge_R F_j$ by Lemma 2.12. The filtration quotients are free as each $F_i$ and $F_j$ are free $R$-modules. To show that this filtration is multiplicative, we need to construct maps $\mu_n : \Gamma_n \to A_n$ that are compatible with the filtrations and are restrictions of the product on $A$. We obtain the desired map as the composition

$$\Gamma_\bullet \to \Gamma'_\bullet \to \mu_\bullet A_\bullet$$

where the second map is constructed by applying the Comparison Lemma to the filtration $\Gamma'_\bullet$ and the map $\mu' : A \wedge_R A \to A$.

To show that the product is homotopy associative we note that both $\mu'_\bullet \circ (\mu'_\bullet \wedge 1)$ and $\mu'_\bullet \circ (1 \wedge_R \mu'_\bullet)$ are solutions to the problem of lifting $\mu' \circ (\mu' \wedge_R 1) = \mu' \circ (1 \wedge_R \mu')$

$$A \wedge_R A \wedge_R A \to A$$

to a filtered map. By the uniqueness part of the Comparison Theorem we get the desired homotopy.

To see the naturality suppose we are given a map of $R$-algebras

$$\varphi : A \to B$$

that we have lifted to

$$\varphi_\bullet : A_\bullet \to B_\bullet.$$ We obtain a homotopy commutative diagram of filtered $R$-modules

$$\begin{array}{ccc}
\Gamma_\bullet(A_\bullet, A_\bullet) & \xrightarrow{\mu^A_\bullet} & A_\bullet \\
\varphi \wedge \varphi & \downarrow & \varphi_b \downarrow \\
\Gamma_\bullet(B_\bullet, B_\bullet) & \xrightarrow{\mu^B_\bullet} & B_\bullet.
\end{array}$$

The commutativity of the diagram up to homotopy follows as argued above, both composites solve the same lifting problem and are therefore homotopic.

The proof of Theorem 1 can also be used when we are mapping out of a cofiber sequence with free fiber. A particular case of interest is the skeletal filtration on the extended power construction, see Definition 2.11.

Corollary 3.2. Using the notation in Definition 2.11, the filtration $A_\bullet$ associated with a free resolution of a commutative $S$-algebra has a $H_\infty$-structure. Explicitly, there are maps $\Gamma^r_{n,k} = E^r_{\pi_+}(k) \wedge_{\pi} \Gamma_n \to A_{k+n}$ that refine

$$A^{\wedge r} \to A$$

$$\xrightarrow{E_{\pi_+} \wedge_{\pi} A^{\wedge r}}$$

$$A.$$
and fit into the following commutative diagrams:

\[ \tilde{\Gamma}_{k,n-1} \cup_{\tilde{\Gamma}_{k-1,n-1}} \tilde{\Gamma}_{k-1,n} \to \tilde{\Gamma}_{k,n} \]

\[ \tilde{\Gamma}_{k,n-1} \cap \tilde{\Gamma}_{k-1,n} \to A_{n+k-1} \to A_{n+k} \]

where \( \pi \subseteq \Sigma_r \). Further, maps of commutative \( R \)-algebras induce maps of \( H_\infty \)-filtrations.

**Proof.** This is an application of the methods of the proof of Theorem 1. As in the case of Corollary 3.1, there is an obvious map of filtrations

\[ \tilde{\Gamma}_r \to \tilde{\Gamma}'_r \]

where

\[ \tilde{\Gamma}_r = \bigcup_{i+j=n} E\pi_+^{(i)} \wedge \pi \bigcup_{\sum_{i=1}^{\alpha_i} = j} X_{\alpha_1} \wedge X_{\alpha_2} \wedge \ldots \wedge X_{\alpha_r} \]

and

\[ \tilde{\Gamma}'_r = \bigcup_{i+j=n} E\pi_+^{(i)} \wedge \pi \bigcup_{\sum_{i=1}^{\alpha_i} = j} X_{\alpha_1} \wedge_R X_{\alpha_2} \wedge_R \ldots \wedge_R X_{\alpha_r}. \]

We want to construct the map \( \xi_{k,n} \)

\[ \tilde{\Gamma}'_{k-1,n-1} \to \tilde{\Gamma}'_{k,n} \]

extending \( \xi_{k,n-1} \cup \xi_{k-1,n} \) to all of \( \tilde{\Gamma}'_{k,n} \). Theorem 1 outlines how we do this when the cofiber of the “inclusion” is free and we are mapping into an exact filtration. Observe that the cofiber of

\[ \tilde{\Gamma}'_{k,n-1} \cup \tilde{\Gamma}'_{k-1,n} \to \tilde{\Gamma}'_{k,n} \]

is just

\[ \frac{B\pi^{(k)}}{B\pi^{(k-1)}} \wedge \tilde{\Gamma}'_n \]

This is a free module as \( B\pi^{(k)}/B\pi^{(k-1)} \) is a wedge of spheres and \( \Gamma'_n / \tilde{\Gamma}'_{n-1} \) is free by Lemma 2.12. We construct the desired map by induction. The map on \( \tilde{\Gamma}'_{0,n} = \tilde{\Gamma}'_n \) is constructed by the previous corollary.

By induction, we assume we have already constructed \( \tilde{\Gamma}'_{i,j} \) for \( i + j < k \) that restricts to \( \mu'_j \) on \( \tilde{\Gamma}'_{0,j} \) (the zero skeleton) and filters the structure map

\[ E\pi_+ \wedge \pi A' \xrightarrow{\xi} A. \]

The rest follows from the argument in the proof of Theorem 1 applied to the cofiber sequence

\[ \tilde{\Gamma}'_{k-1,n-1} \to \tilde{\Gamma}'_{k-1,n} \cup \tilde{\Gamma}'_{k-1,n} \to \tilde{\Gamma}'_{k,n} \]
To show that the relevant diagrams commute so that we obtain an $H_{\infty}$-filtration we use the same argument as above when we showed that the filtration was multiplicative. One demonstrates that there are two different solutions to the relevant lifting problem and the homotopy uniqueness of the lift provided by Theorem 1 provides a homotopy. This same argument is also used to show that maps of commutative $R$-algebras induce maps of $H_{\infty}$-filtrations.

This argument is very similar to that used by May in [15] when one needs to construct the relevant homotopy commutative diagrams which we learned from the retelling in [8] by Bruner in Chapter 4 section 2.

**Corollary 3.3.** The Künneth spectral sequence
\[ \text{Tor}^R_{p,q}(A, B,)_q \Rightarrow \pi_{p+q}(A \wedge_R B) \]
is multiplicative whenever $R$ is a commutative $S$-algebra, and both $A$ and $B$ are $R$-algebras. When $A$ and $B$ are also commutative $R$-algebras then the above KSS has a $H_{\infty}$-filtration.

We postpone the proof until after our construction of the KSS.

## 4 The Künneth spectral sequence

Here we construct the Künneth spectral sequence. Our construction follows that of [10] although our notation is slightly different.

### 4.1 Construction of the Künneth spectral sequence

First we construct the KSS using the techniques for constructing filtrations in 2. Then we show that this filtration is multiplicative.

**Theorem** ([10], Chapter 4 section 6). Let $R$ be an $S$-algebra and $A, B$ be right and left $R$-modules respectively. Further assume that $(R, S)$ has the homotopy type of a relative CW $S$-module and that $A$ and $B$ are cell $R$-modules (right and left respectively). Then there is a upper half plane spectral sequence
\[ E_2^{s,t} = \text{Tor}^\pi_{s} R(\pi_s A, \pi_s B)_t \Rightarrow \pi_{s+t}(A \wedge_R B). \]

The grading $t$ comes from the internal grading as we are working with graded modules over graded algebras. This result can be found in Chapter 4 section 6 of [10], but we recall the construction here.

**Proof.** We wish to construct a filtration of $A \wedge_R B$ such that the homotopy of the associated graded of the filtration is the complex that computes $\text{Tor}^\pi_{s} R(\pi_s A, \pi_s B)$. Recall that in section 2.2 we constructed an exhaustive free and exact filtration of a given $R$-module $A$ from a free $R$-resolution of $A$. As before, let us denote the free $R$-module that resolution by $F_i$, and the $i$th term of the resulting filtration by $A_i$. Recall that $A_{-1} = \ast$ and $A_0 = F_0$. Since the filtration is exhaustive (Lemma 2.2), we have that $\text{holim}_i A_i \simeq A$ as an $R$-module. The Künneth filtration is given by
\[ A_0 \wedge_R B \to A_1 \wedge_R B \to \cdots \to A_i \wedge_R B \to A_{i+1} \wedge_R B \to \cdots. \]

We will abbreviate $A_i \wedge_R B$ as $X_i$. This filtration is also exhaustive as smashing with a cell $R$-module commutes with taking homotopy colimits; see Chapter 10 section 3 and Chapter 3 sections 2 and 3 of [10].

Next, we wish to identify the $E_2$ page of the spectral sequence. The associated graded complex of this filtration is
\[ \text{Cof}(X_i \to X_{i+1}) = \text{Cof}(A_i \wedge_R B \to A_{i+1} \wedge_R B) \simeq \text{Cof}(A_i \to A_{i+1}) \wedge_R B. \]
The associated graded complex of the filtration $A_i$ is a free $R$-resolution of $A$ by $R$-modules denoted $F_i$. Therefore the homotopy of $E^{\delta}(X_i)_s$ is given by $\pi_s(F_i \wedge_R B) \cong \pi_s F_i \otimes_{\pi_s R} \pi_s B$, see Proposition 3.9 in [10]. As $\pi_s \Sigma F_i$ is a free $\pi_s$-$R$-resolution of $\pi_s A$, by construction, the $E_2$ page of the spectral sequence is $\text{Tor}^\pi_{s} R(\pi_s A, \pi_s B)_s$.
It should be noted that there are potentially other constructions of this spectral sequence. One in particular where a proof of the multiplicative structure was claimed can be found in [4]. The proof they supply is, unfortunately, incorrect. See Subsection 4.1.1 for more retails.

**Theorem 2** (Multiplicativity of the Künneth spectral sequence). Let \( R \) be a commutative \( S \)-algebra, and \( A \) and \( B \) be \( R \)-algebras. The Künneth spectral sequence is multiplicative. Further, the filtration of the KSS has an \( H_\infty \)-structure that filters the commutative \( S \)-algebra structure of \( A \wedge_R B \).

**Proof.** We will use corollaries 3.1 and 3.2 to show that the Künneth filtration is not only multiplicative but has an \( H_\infty \)-structure. As \( A \) is a commutative \( R \)-algebra and \( A_\bullet \) is exact and free, there exists a map of filtrations 

\[ \mu_\bullet : \Gamma(A_\bullet, A_\bullet)_\bullet \to \Gamma'(A_\bullet, A_\bullet)_\bullet \to A_\bullet \]

by 3.1. The commutative \( R \)-algebra structure on \( A \wedge_R B \) is the composition

\[ (A \wedge_R B) \wedge_R (A \wedge_R B) \xrightarrow{1 \wedge_R 1} (A \wedge_R A) \wedge_R (B \wedge_R B) \to A \wedge_R B. \]

We can use \( \mu_\bullet \) above to obtain a filtered version of this map. Using the fact that there is a filtered/levelwise twist map \( \tau_\bullet \) and that \( \Gamma_\bullet(X_\bullet, Y_\bullet \wedge Z) = \Gamma_\bullet(X_\bullet, Y_\bullet) \wedge Z \), we obtain the

\[ \Gamma_\bullet(A_\bullet \wedge_R B, A_\bullet \wedge_R B) \to A_\bullet \wedge_R B \]

as the composition of \( \mu_\bullet \) smashed with the product of \( B \) and the following

\[ \Gamma_\bullet(A_\bullet \wedge_R B, A_\bullet \wedge_R B) = \Gamma_\bullet(A_\bullet, B \wedge_R A_\bullet \wedge_R B) \xrightarrow{1 \wedge_R 1} \Gamma_\bullet(A_\bullet, A_\bullet \wedge_R B \wedge_R B) = \Gamma_\bullet(A_\bullet, A_\bullet) \wedge_R B \wedge_R B. \]

More explicitly, the maps

\[ A_i \wedge A_j \to A_i \wedge_R A_j \to A_{i+j} \]

can be smashed with the product of \( B \) to obtain maps

\[ A_i \wedge_R B \wedge A_j \wedge_R B \to A_i \wedge_R B \wedge A_j \wedge_R B \to A_i \wedge_R A_j \wedge_R B \wedge_R B \to A_{i+j} \wedge_R B \]

That the associated spectral sequence to such a multiplicative filtration is multiplicative is due to Dugger, see [9]. For a slightly different proof see [19].

Showing that the filtration is an \( H_\infty \)-filtration is similar. As \( B \) is a commutative \( R \)-algebra it has a natural \( H_\infty \)-structure. We can then construct

\[ \Gamma'_\bullet(A_\bullet, c(B)) \to A_\bullet \wedge_R B \]

using the \( H_\infty \)-structure on \( A_\bullet \) constructed in 3.2, \( H_\infty \)-structure on \( B \), and a cellular approximation of the diagonal map on \( ES_\gamma \).

**Remark 4.1.** Relaxing the assumptions on the commutativity of \( R \) is an active area of interest that we are pursuing. Of particular interest is the case when \( A \) and \( B \) are commutative \( S \)-algebras under \( R \) but \( R \) is only an \( E_1 \)-algebra. See our forthcoming joint work with Katthän for more details.

**4.1.1 Previous work**

Previous work of Baker-Lazarev has appeared that claims to prove that the KSS is multiplicative. Their proof relied on an alternate construction of the KSS. This construction does not yield the KSS as it does not give a filtration. However, if we supposed it did give rise to a filtration then that filtration, due to the nature of its construction, would have a trivial homotopy colimit whenever our base ring spectrum had homotopy groups of finite homological dimension. Their filtration is constructed in two steps. First they realize the resolution \( F_\bullet \to \pi_*A \) used to compute \( \text{Tor}^{\bullet,R}(A, B) \) as a resolution of \( R \)-modules \( F_\bullet \to A \). They then propose to filter \( A \) by the sequence of spectra \( A_i := \text{cof}(F_i \to F_{i-1}) \) for \( i > 0 \) and \( A_0 := F_0 \). If \( \pi_*R \) has finite homological dimension then the \( F_i \) are eventually trivial \( R \)-modules and so the \( A_i \) must be trivial \( R \)-modules in the same range. This ensures that \( \text{hococolim}_i A_i \) can not be weakly equivalent to \( A \) whenever \( \pi_*A \) has a finite length projective resolution as a \( \pi_*R \)-module.
4.2 Algebraic operations

Since we have shown that the Künneth filtration has an $H_\infty$-structure we might expect that there is a theory of algebraic operations in Tor that arise from the structure induced on the associated graded complex of the Künneth filtration. This is the case in the Adams spectral sequence, see [8] Chapter 4. However, in our situation the operations are rather uninteresting as they vanish unless they happen to be the $p$th power. This is due to the following Theorem of Tate.

\textbf{Theorem 3} (Tate [18]). If $R$ is a Noetherian local ring and $I$ is an ideal of $R$, then there exists a free $R$-resolution of $R/I$ that is a graded commutative DGA.

Tate proves the above by explicitly constructing a graded commutative DGA. It also can be extended as follows.

\textbf{Corollary 4.2.} For $R$ as above and any finitely generated commutative $R$-algebra, there exists a free $R$-resolution of it that is a graded commutative DGA.

\textit{Proof.} Suppose that $A$ is a finitely generated $R$-algebra, then $A$ is a quotient of $R[x_1,x_2,\ldots,x_n]$ by an ideal. The result of Tate gives a free $R[x_1,x_2,\ldots,x_n]$-resolution of $A$ that is graded commutative as a DGA. However, since $R[x_1,x_2,\ldots,x_n]$ is free over $R$, so is the above resolution and we have the desired result. \hfill \Box

The operations above come from lifting null-homotopies of the difference of ways of ordering $p$-fold products of classes in the free resolution. If the free resolution is graded commutative, then all of the $p$-fold products are equal and we can take the constant null-homotopy. This lift will only give trivial operations. It can be shown that any other choice of lift is homotopic to this one, see [19]. We do not include that result here in order to be brief, but the argument is the appropriate analog of the argument given in [8] by Bruner in Chapter 4 section 2. However, the filtration has the requisite structure, so we compute operations in $\pi_*A \wedge R B$ using the naturality of the $H_\infty$-structure and detect them in the KSS

$$\text{Tor}^R_*(A_*,B_*) \Rightarrow \pi_*(A \wedge R B).$$

It might be said that the spectral sequence hides the operations in a lower filtration. Even when the KSS collapses at $E_2$, there is a difference between operations in Tor and operations in homotopy as we will see in 5. Any reference to operations will of course be to homotopical operations that only exist on the spectral sequence.

5 Computations

In this section we use the above results to compute the action of the Dyer-Lashof algebra on various relative smash products of $Hk$ where $k$ is a field of prime order. Later, in Subsection 5.4, we will address the difference between the prime 2 and odd primes. The main difference is in the structure of the dual Steenrod. There is also a difference between $ku(p)$ and the Adams summand at odd primes.

The computation proceeds in three steps

- Use the Koszul complex to compute the $E_2$-pages of the KSS’s converging to $\pi_*Hk \wedge R \wedge_R Hk$ and $\pi_*Hk \wedge_R Hk$.
- Determine the collapse of the spectral sequence as well as the product structure from the multiplicativity of the KSS.
- Use step 1 to compute the map of $E_2$-pages and deduce the action of the Dyer-Lashof algebra using this map and the computation of Steinberger [8].

To do this, it will help to have some notation in place. This notation will also be used in section 6 when we interpret these computations.
Definition 5.1. Given a diagram of commutative $S$-algebras

$$
\begin{array}{ccc}
R & \xrightarrow{\psi} & R' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
Hk & & Hk
\end{array}
$$

we denote

- the induced map of relative smash products $\tilde{\psi} : Hk \wedge_R Hk \to Hk \wedge_{R'} Hk$,
- the induced map of KSSs $\hat{\psi}_* : \text{Tor}_s^R(k, k)_t \to \text{Tor}_s^R(k, k)_t$,
- the induced map of KSSs $\check{\varphi}_* : \text{Tor}_s^R(H_*(R; k), k)_t \to \text{Tor}_s^R(k, k)_t$.

We fix a map of commutative $S$-algebras $\varphi : R \to Hk$. The map $\varphi$ makes $Hk$ into a commutative $R$-algebra. Therefore the action map of $R$ on $Hk$ coming from said algebra structure is a map of commutative $S$-algebras. This ensures that the map $\check{\varphi}$ of spectral sequences commutes with all of the extra structure present on the filtration, such as power operations. To compute the action of the Dyer-Lashof algebra on $\pi_* Hk \wedge_R Hk$, consider the map of spectral sequences induced by $\varphi$:

$$
\begin{array}{ccc}
\text{Tor}_s^R((H_*(R; k), k)_t) & \xrightarrow{\check{\varphi}} & \pi_{s+t}(Hk \wedge_R Hk) \\
\downarrow{\varphi} & & \downarrow{\check{\varphi}} \\
\text{Tor}_s^R(k, k)_t & \xrightarrow{\check{\varphi}} & \pi_{s+t}(Hk \wedge_R Hk).
\end{array}
$$

When $k$ is a quotient of $R$, by a regular sequence, we can use a Koszul complex to compute the $E_2$-page of the spectral sequence. The spectral sequence is multiplicative and the $E_2$-page is multiplicatively generated by classes on the 1-line of the spectral sequence. As $d_r = 0$ when restricted to the 1-line for every $r \geq 2$ we obtain a collapsing spectral sequence. In our examples, $\varphi_* : R_* \to k$ is a regular quotient, and so both spectral sequences collapse at the $E_2$ page leaving us with

$$
\text{Tor}_s^R((H_*(R; k), k)_t) \cong H_{s+t}(Hk; k)
$$

and

$$
\text{Tor}_s^R(k, k)_t \cong \pi_{s+t} Hk \wedge_R Hk.
$$

We will also need the following computational ingredients.

Theorem 4 (Milnor). At the prime 2, the dual Steenrod algebra is $H_*(H\mathbb{F}_2; \mathbb{F}_2) \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$. The conjugation is given by

$$
\xi_i := \chi_*(\xi_i) = \sum_{\alpha \in \text{Part}(i)} \prod_{n=1}^{l(\alpha)} \xi_{\alpha(n)}^{2^r(i)}
$$

where $l(\alpha)$ is the length of the ordered partition $\alpha$ of $i$.

The map $\chi : Hk \wedge_R Hk \to Hk \wedge_R Hk$ is the twist map of $S$-modules.

The following result gives us the action of the Dyer-Lashof algebra on $H_*(H\mathbb{F}_2; \mathbb{F}_2)$ and is due to Steinberger; see Chapter 3 Thm 2.2 of citeHRS.

Theorem 5 (Steinberger). In the dual Steenrod algebra, $H_*(H\mathbb{F}_2; \mathbb{F}_2) \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$, we have that $Q^{2^{i-2}}(\xi_1) = \xi_i$. The dual Steenrod algebra, $H_*(H\mathbb{F}_2; \mathbb{F}_2)$ is generated as an algebra over the Dyer-Lashof algebra by $\xi_1$. Further, we have that $Q^{2^i}(\xi_i) = \xi_{i+1}$ for $i \geq 1$. 

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Recall that the Dyer-Lashof algebra acts on the homotopy of any commutative $HF_2$ algebra. We will compute the action of $Q^i$ on elements $\bar{\tau} \in im(\bar{\varphi} : H_*(HF_2; \mathbb{F}_2) \to \pi_*HF_2 \wedge_K HF_2)$ as follows. First, we represent both $\bar{\tau}$ and its lift $\bar{x} \in H_*(HF_2; \mathbb{F}_2)$ as elements in the $E_2$-page of their respective KSSs. We then use Steinberger’s computation to identify the action of $Q^i$ in the spectral sequence converging to $H_*(HF_2; \mathbb{F}_2)$. Lastly, we push forward $Q^i(\bar{x})$ along the map of spectral sequences induced by $\varphi$. We can compute what the map of spectral sequences induced by $\varphi$ does as we will use the same resolution to compute the $E_2$-pages, which we will demonstrate below.

It will be important to determine which class in $H_*(HF_2; \mathbb{F}_2)$ is being detected by given class $\alpha \in \text{Tor}_1^{ ku}(HF_2, R, F_2)$. In the cases of interest, the class $\alpha$ will be detecting either $\xi_{i+1}$ or $\bar{\xi}_{i+1}$ in $H_*(HF_2; F_2)$. The relationship between the $v_i \in \pi_1BP$ and $\xi_i, \bar{\xi}_i$ in $H_*(HF_2; \mathbb{F}_2)$ discussed by Ravenel in [16], see Chapter 4 section 2 starting on page 114, implies that $\bar{\tau}_i$ detects either $\xi_{i+1}$ or $\bar{\xi}_{i+1}$. For our purposes, however, it will not matter if $\bar{\tau}_i$ detects $\xi_{i+1}$ or $\bar{\xi}_{i+1}$.

**Lemma 5.2.** The image of $\xi_i$ agrees with $\bar{\xi}_i$ modulo decomposables.

*Proof.* The only indecomposable element in the sum $\tau_*(\xi_i) = \sum_{\alpha \in \text{part}(i)} \Pi^{l(\alpha)}\phi_{\alpha}^{\tau_*(\xi_i)}$ is $\xi_i$. Here, $l(\alpha)$ is the length of the ordered partition $\alpha$ of $i$. While the formula may be complicated, the only indecomposable elements on the right hand side occur when $l(\alpha) = 1$, and this forces $\alpha$ to be the indiscrete partition of $i$ that is just $i$ itself.

### 5.1 $ku \longrightarrow HF_2$

Consider the map $ku \to HF_2$ whose effect in homotopy is reduction of $ku_* \cong \mathbb{Z}[v]$ modulo $(2, v)$, where $v \in \pi_2(ku)$ is the Bott class. This example is the simplest one with nontrivial Dyer-Lashof operations (of which we are aware). The computations in the following sections will be very similar to the computation here.

**Proposition 5.3.** The KSS $\text{Tor}_{ku}^{ ku}(F_2, F_2) \Rightarrow \pi_*HF_2 \wedge_{ku} HF_2$ collapses at the $E_2$ page. $\pi_*HF_2 \wedge_{ku} HF_2$ is an exterior algebra $E_{\pi_2}[\mathbb{F}_2, v]$ with $Q^2(\mathbb{F}_2) = v$ where $|\mathbb{F}_2| = 1$ and $|v| = 3$.

*Proof.* We give this proof in full as the later examples will follow the method closely. To compute the relevant $\text{Tor}$ group we use the Koszul complex $\Lambda_{ku}(2, v)$ associated with the regular sequence $(2, v) \subset ku_*$. The resolution is

\[
ku_* \leftarrow \begin{array}{c}
\begin{array}{c}
2 \\
v
\end{array}
\end{array} \quad ku_* \oplus \Sigma^2ku_* \leftarrow \begin{array}{c}
\begin{array}{c}
v \\
-2
\end{array}
\end{array} \quad \Sigma^2ku_*
\]

which becomes

\[
\begin{array}{c}
F_2 \leftarrow 0 \\
F_2 \oplus \Sigma^2F_2 \leftarrow 0 \\
\Sigma^2F_2
\end{array}
\]

after applying $F_2 \otimes_{ku} -$ . We see that $\text{Tor}_{ku}^{ ku}(F_2, F_2)_* \Rightarrow \pi_*HF_2 \wedge_{ku} HF_2$ is an exterior algebra over $F_2$ generated by classes by $\mathbb{F}_2 / \Sigma^2F_2$ and $\tau \in \text{Tor}_{ku}^{ ku}(F_2, F_2)_0$ and $v \in \text{Tor}_{ku}^{ ku}(F_2, F_2)_2$. 

\[
\begin{array}{c}
\begin{array}{c}
F_2 \leftarrow 0 \\
F_2 \oplus \Sigma^2F_2 \leftarrow 0 \\
\Sigma^2F_2
\end{array}
\end{array}
\]
The spectral sequence is multiplicatively generated by $\bar{2}$ and $\bar{\pi}$ which are on the 1-line of the spectral sequence, and are therefore permanent cycles.

To compute the action of the Dyer-Lashof algebra, we use the induced map of spectral sequences. We compare the above spectral sequence to

$$\text{Tor}_s^{ku}(H_\ast(ku; F_2), F_2) \to \pi_s(HF_2 \wedge ku, HF_2) \cong H_\ast(HF_2; F_2).$$

By Steinberger’s Theorem, we know the action of the Dyer-Lashof algebra on $H_\ast(HF_2; F_2)$, the target of the spectral sequence. As $H_\ast(ku; F_2) \cong F_2[\xi_1, \xi_2, \xi_3, \xi_4, \ldots]$ and both $2$ and $v$ act trivially on $H_\ast(ku; F_2)$ we can compute Tor using the same Koszul complex as above. This complex becomes

$$H_\ast(ku; F_2) \leftarrow H_\ast(ku; F_2) \oplus \Sigma^2 H_\ast(ku; F_2) \leftarrow \Sigma^2 H_\ast(ku; F_2)$$

after applying $H_\ast(ku; F_2) \otimes_{ku} -$ to $\Lambda_{ku}(2, v)$. Therefore we have that

$$\text{Tor}_s^{ku}(H_\ast(ku; F_2), F_2) \cong H_\ast(ku; F_2) \otimes E_{\bar{2}, \bar{\pi}} F_2$$

with $\bar{2}$ and $\bar{\pi}$ as above. This spectral sequence collapses for the same reasons as before. We know that $\bar{2}$ detects $\xi_1 \in H_1(HF_2; F_2)$ as there are no other nonzero classes in those degrees and so $\bar{2} \xi_1$ detects $\bar{\xi}_1$. The class $\bar{\pi}$ detects either $\xi_2$ or $\bar{\xi}_2 = \xi_2 + \xi_1$ in $H_2(HF_2; F_2)$. It will not matter which, as they agree modulo $\ker(\bar{\varphi})$ by 5.2. See the computation of $H_\ast(ku; F_2)$ in Chapter 3 of [16] for the relationship between $v$ and $\bar{\xi}_2$, specifically Theorem 3.1.16. From Steinberger’s computation, we know that there is an operation $Q^2(\bar{\xi}_1) = \bar{\xi}_2$, this lifts to $Q^2(\bar{2}) = \bar{\pi} + \epsilon \bar{2} \xi_1$ where $\epsilon \in \{0, 1\}$ in the $E_2$ page of the spectral sequence.

The map of spectral sequences induced by $\varphi : ku \to HF_2$ is obtained from $\bar{\varphi} : HF_2 \wedge ku \to HF_2$ which induces quotient map $H_\ast(ku; F_2) \to F_2$. We then obtain the map of Koszul complexes

$$H_\ast(ku; F_2) \leftarrow H_\ast(ku; F_2) \oplus \Sigma^2 H_\ast(ku; F_2) \leftarrow \Sigma^2 H_\ast(ku; F_2)$$

This induces the map of $E_2$-pages $H_\ast(ku; F_2) \otimes E_{\bar{2}} F_2(\bar{2}, \bar{\pi}) \to F_2 \otimes E_{\bar{2}} F_2(\bar{2}, \bar{\pi})$. This map preserves the action of the Dyer-Lashof algebra since the construction of the $H_\infty$-structure of the filtration in the KSS is natural. Regardless of the value of $\epsilon$, $\bar{\varphi}(\bar{2} \xi_1) = 0$. We then have that $\pi_1 HF_2 \wedge_{ku} HF_2 \cong E_{\bar{2}}(\bar{2}, \bar{\pi})$ with $Q^2(\bar{2}) = \bar{\pi}$ as claimed.
5.2 \( BP(2) \rightarrow HF_2 \)

It was recently shown by Lawson and Naumann in [13] that the \( S \)-module \( BP(2) \) can be modeled as a commutative \( S \)-algebra. Given this, it is easy to construct a map of commutative \( S \)-algebras \( BP(2) \rightarrow HF_2 \) whose effect in homotopy is reduction of \( BP(2) \), \( \cong \mathbb{Z}(2)[v_1, v_2] \) modulo \( (2, v_1, v_2) \), where \( v_1 \in \pi_2 BP(2) \) and \( v_2 \in \pi_6 BP(2) \). Using this map, we proceed as in the previous section.

**Proposition 5.4.** The \( KSS \) \( Tor_\ast^{BP(2)}(F_2, F_2) \Rightarrow \pi_\ast HF_2 \wedge_{BP(2)} HF_2 \) collapses at the \( E_2 \) page. \( \pi_\ast HF_2 \wedge_{BP(2)} HF_2 \) is an exterior algebra \( E_{\bar{3}}[\bar{2}, v_1, v_2] \) with \( Q^2(\bar{2}) = v_1, Q^6(\bar{2}) = v_2, Q^4(\pi_1) = v_2, \) and \( Q^6(\bar{v}_1) = \bar{v}_1 \bar{v}_2 \) where \( |\bar{2}| = 1, |v_1| = 3 \) and \( |v_2| = 7 \).

**Proof.** To compute the relevant Tor group, we use the Koszul complex associated with the regular sequence \( (2, v_1, v_2) \subset \pi_\ast BP(2) \). The resulting complex that computes \( Tor_\ast^{BP(2)}(F_2, F_2) \) is

\[
\begin{align*}
F_2 & \leftarrow F_2 \oplus \Sigma^2 F_2 \oplus \Sigma^6 F_2 & \leftarrow 0 & \Sigma^2 F_2 \oplus \Sigma^6 F_2 \oplus \Sigma^8 F_2 & \leftarrow 0 & \Sigma^8 F_2
\end{align*}
\]

We see that \( Tor_\ast^{BP(2)}(F_2, F_2) \) is an exterior algebra over \( F_2 \) generated by classes \( \bar{2} \in Tor_1^{BP(2)}(F_2, F_2)_0, \bar{v}_1 \in Tor_1^{BP(2)}(F_2, F_2)_2 \) and \( \bar{v}_2 \in Tor_1^{BP(2)}(F_2, F_2)_6 \).

\[
\begin{array}{cccc}
\bar{2} & \bar{v}_1 & \bar{v}_2 & \bar{v}_1 \bar{v}_2 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
s & 2 & 3 \\
t + s
\end{array}
\]

For exactly the same reasons as in the case of \( ku \), this spectral sequence collapses at the \( E_2 \)-page.

We compute the action of the Dyer-Lashof algebra just as in the case of \( ku \), by computing the induced map of spectral sequences. Consider KSS

\[
Tor_\ast^{BP(2)}(H_\ast(BP(2); F_2), F_2) \Rightarrow \pi_\ast(HF_2 \wedge BP(2) \wedge BP(2) HF_2) \cong H_\ast(HF_2; F_2).
\]

By Steinberger’s Theorem, we know the action of the Dyer-Lashof algebra on \( H_\ast(HF_2; F_2) \), the target of the spectral sequence. As \( H_\ast(BP(2); F_2) \cong F_2[\xi_2, \xi_4, \xi_5, \ldots] \) with 2, \( v_1 \), and \( v_2 \) all acting trivially, we compute Tor using same Koszul complex as above. After applying \( H_\ast(BP(2); F_2) \otimes_{BP(2)} \sim \), it becomes

\[
\begin{align*}
H_\ast(BP(2); F_2)_\ast & \oplus \Sigma^2 H_\ast(BP(2); F_2)_\ast & \oplus \Sigma^6 H_\ast(BP(2); F_2)_\ast & \oplus \Sigma^8 H_\ast(BP(2); F_2)_\ast \\
\Sigma^2 H_\ast(BP(2); F_2)_\ast & \oplus \Sigma^4 H_\ast(BP(2); F_2)_\ast & \oplus \Sigma^8 H_\ast(BP(2); F_2)_\ast
\end{align*}
\]

We see that

\[
Tor_\ast^{BP(2)}(H_\ast(BP(2); F_2), F_2)_\ast \cong H_\ast(BP(2); F_2) \otimes_{F_2} (\bar{2}, v_1, v_2)
\]

with degrees as described above. This spectral sequence also collapses. As before, \( \bar{2} \) detects \( \bar{\xi}_1 \in H_1(HF_2; F_2) \), and so \( \bar{F}_1^{2} \) detects \( \bar{\xi}_1 \). We also have that \( \bar{v}_1 \) detects either \( \xi_2 \) or \( \bar{\xi}_2 = \bar{\xi}_2 + \bar{\xi}_1 \) in \( H_2(HF_2; F_2) \). Similarly, \( \bar{v}_2 \) detects either \( \xi_3 \) or \( \bar{\xi}_3 \) in \( H_7(HF_2; F_2) \). Details relating \( v_1 \) and \( \xi_{i+1} \) can be found in Ravenel’s [16] where
Ravenel discusses the computation of $H^*(k(n); F_2)$, see Chapter 4 section 2 starting on page 114. From Steinberger’s computation we know that $Q^2(\xi_1) = \xi_2$, $Q^6(\xi_1) = \xi_3$, and $Q^4(\xi_2) = \xi_3$

The map of spectral sequences induced by $\varphi$ is obtained from $HF_2 \wedge BP(2) \to HF_2$ which induces the reduction map $H_*(BP(2); F_2) \to F_2$. This induces the map of complexes

$$H_*(BP(2); F_2) \leftarrow \Sigma H_*(BP(2); F_2) \leftarrow \Sigma^2 H_*(BP(2); F_2) \leftarrow \Sigma^6 H_*(BP(2); F_2) \leftarrow \Sigma^8 H_*(BP(2); F_2)$$

This map of complexes induces the map of $E_2$-pages $H_*(BP(2); F_2) \otimes E_{\varphi}(Z, \varpi_1, \varpi_2) \to F_2 \otimes E_{\varphi}(Z, \varpi_1, \varpi_2)$.

This map preserves the action of the Dyer-Lashof algebra and so we have that $\pi_*HF_2 \wedge BP(2)$ HF$_2 \cong E_{\varphi}(Z, \varpi_1, \varpi_2)$ with $Q^2(\varpi) = \varpi_1$, $Q^6(\varpi) = \varpi_2$, and $Q^4(\varpi_1) = \varpi_2$ as desired. The operation $Q^6(\varpi_1) = \varpi_1 \varpi_2$ follows from the Cartan formula.

\[\begin{array}{cccccc}
3 & & & & & \\
2 & s & & & & \\
1 & & Q^2 & & Q^6 & \\
0 & 1 & 2 & 3 & 4 & 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \\
\end{array}\]

\[\begin{array}{cccccc}
\varpi_1 \rightarrow Q^2 \varpi_2 \rightarrow Q^6 \varpi_1 \varpi_2 \\
\varpi_2 \leftarrow Q^2 \varpi_2 \leftarrow Q^6 \varpi_1 \varpi_2 \\
\end{array}\]

5.3 $MU \to HF_2$

In this section we compute some of the action of the Dyer-Lashof algebra on $\pi_*HF_2 \wedge MU$ HF$_2$. As before, we will need to know the Hurewicz image for $MU$; for this see section 2 of [5] or Chapter 3 of [16]. Recall that $\pi_*MU \cong Z[x_1, x_2, ...]$ where $x_i \in \pi_{2i}MU$ and $H_*(MU; Z) \cong Z[b_1, b_2, ...]$ where $b_i \in H_{2i}(MU; Z)$. Our choice of generators for $\pi_*MU$ is such that $\rho(x_{2i-1}) = v_i$, where

$$\rho : MU \to MU_{(2)}$$

is the localization map and we think of the $v_i$ as elements of $\pi_*MU_{(2)}$ via a preferred retraction of $MU_{(2)}$ onto $BP$. This choice of generators for $MU_*$ will determine our choice of generators for $H_*(MU; F_2)$, but this choice can be made so that our Hurewicz map behaves as desired; see [5] section 2. The Hurewicz map $h_{HZ} : \pi_*MU \to H_*(MU; Z)$ modulo decomposables is given by

$$h(x_i) = \begin{cases} -pb_i & i = p^k - 1 \\ -b_i & else. \end{cases}$$

When the Hurewicz map

$$h_E : \pi_*MU \to E_*MU$$
As before, the spectral sequence collapses, and we have that
\[ S \text{number of the } x \]
\[ \text{Corollary 5.6. Let } I \text{ be an ideal of } MU_* \text{ generated by a regular sequence. If } I \text{ contains a finite nonzero number of the } x_{2i-1}, \text{ then the quotient map } MU \to MU/I \text{ cannot be realized as a map of commutative } S\text{-algebras.} \]
Proof. Let \( I \) be as above and let us use \( E \) to denote \( MU/I \). Suppose that the quotient map
\[
\varphi : MU \to E
\]
is a map of commutative \( S \)-algebras. Consider the induced map of the spectral sequences converging to
\[
\tilde{\varphi} : HF_2 \wedge MU \xrightarrow{\varphi} HF_2 \wedge E \wedge HF_2.
\]
This map must preserve the action of the Dyer-Lashof algebra as it is a induced by a morphism of commutative \( HF_2 \)-algebras. However, by assumption we have that some \( x_{2i-1} \in ker(\varphi) \) and so \( \beta x_{2i-1} \in ker \tilde{\varphi} \). For each \( k \), \( \beta x_{2i+k-1} \) is a power operation on \( \beta x_{2i-1} \). Therefore, each \( \beta x_{2i+k-1} \) must also be in the kernel, but they are not by construction of \( I \).

This should be compared with the results of Strickland, see [17], where he realizes regular quotients of \( MU \). In his work, the ideal in question must satisfies a similar condition in order for the realization to be a commutative ring spectrum, see Proposition 6.2 in [17] for the relevant formula.

5.4 Odd primary case

When \( p \neq 2 \) the dual Steenrod algebra is no longer polynomial. It is the tensor product of a polynomial algebra and an exterior algebra. These exterior generators are what the \( v_i \) detect. In this subsection, we give the analogues of the above computations when \( p \neq 2 \). First, we recall some facts about the dual Steenrod algebra and its structure as an algebra over the Dyer-Lashof algebra.

Theorem 6 (Milnor). The dual steenrod algebra is
\[
H_\ast(HF_p; \mathbb{F}_p) \cong \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \ldots] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \tau_2, \ldots)
\]
(where \( \Lambda \) denotes that the \( \tau_i \)'s are exterior generators). The degree of \( |\xi_i| = 2p^i - 2 \) and \( |\tau_i| = 2p^i - 1 \). The conjugation is given by
\[
\xi_i := \chi^\ast(\xi_i) = \sum_{\alpha \in Part(i)} \Pi_{n=1}^{l(\alpha)} \chi^p_{\alpha(n)}
\]
where \( l(\alpha) \) is the length of the ordered partition \( \alpha \) of \( i \) for all primes \( p \).

The map \( \chi : HF_p \wedge HF_p \to HF_p \wedge HF_p \) is induced by the twist map. We also need Steinberger’s computation of the action of the Dyer-Lashof algebra on the dual Steenrod algebra.

Theorem 7 (Steinberger). For all \( p > 2 \), the dual Steenrod algebra is generated as an algebra over the Dyer-Lashof algebra by \( \tau_0 \). The action is generated by the formulas
\[
Q^{\rho(i)} \tau_0 = (-1)^i \tau_i
\]
\[
\beta Q^{\rho(i)} \tau_0 = (-1)^i \xi_i
\]
where \( \rho(i) := \frac{p^i - 1}{p - 1} \). In particular, we have that
\[
Q^{p^i - 1} \tau_{i-1} = \tau_i
\]
\[
\beta Q^{p^i} \xi_i = \xi_{i+1}
\]
for \( i > 0 \).

Before we proceed as above, we should mention that at an odd prime, \( BP(2) \) is only known to be a commutative \( S \)-algebra at the primes 3, which is due to [12]. Also, we work with the \( p \)-local Adams summand \( \ell \) as well as \( ku \). They have different homotopy and homology as
\[
ku(\ell) \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell.
\]
The computations in this section follow exactly the method carried out when $p = 2$. The difference between $\chi_\ast \tau_i$ and $-\tau_i$ is decomposable. This follows as the we have that

$$\tau_n + \Sigma_{i=0}^{n} c^{p^i}_{n-1} \chi_\ast \tau_i = 0.$$  

While this means that the sign is not determined in the below formulas this will not be so concerning. Our main use of these results is as obstructions, as we will see in 6, and so we will see that their vanishing is what is most relevant.

**Proposition 5.7.** The $KSS$ Tor$_{\ast}$($F_p$, $F_p$) $\Rightarrow$ $\pi_\ast HF_p \land_\ell HF_p$ collapses at the $E_2$ page. $\pi_\ast HF_p \land_\ell HF_p$ is an exterior algebra $E_{\ell} p$, $\ell_1$ with $Q^1(p) = \pm \ell_1$ where $|\ell_1| = 1$ and $|\ell_1| = 2p - 1$.

The proof of this follows the exact same lines as in the case for $ku$ and $p = 2$, which is not surprising as $ku(2) \simeq \ell$ at the prime 2. We have the following result regarding $ku$ at odd primes.

**Proposition 5.8.** The $KSS$ Tor$_{\ast}$($F_p$, $F_p$) $\Rightarrow$ $\pi_\ast HF_p \land_{ku} HF_p$ collapses at the $E_2$ page. $\pi_\ast HF_p \land_{ku} HF_p$ is an exterior algebra $E_{\ell} v$, $\ell_1$ where $|\ell_1| = 1$ and $|\ell_1| = 3$. The Dyer-Lashof algebra acts trivially for degree reasons.

The first possible nontrivial Dyer-Lashof operation is $Q^1(v)$, but this must be zero because $Q^1(v) \in \pi_{2p-1}(HF_p \land_{ku} HF_p) = 0$.

**Proposition 5.9.** The $KSS$ Tor$_{\ast}$($F_3$, $F_3$) $\Rightarrow$ $\pi_\ast HF_3 \land_{BP(2)} HF_3$ collapses at the $E_2$ page. $\pi_\ast HF_3 \land_{BP(2)} HF_3$ is an exterior algebra $E_{\ell_1, \ell_2}$ with $Q^1(3) = \pm \ell_1$, $Q^4(3) = \pm \ell_2$, $Q^3(\ell_1) = \pm \ell_2$, and $Q^4(3\ell_1) = \pm \ell_1\ell_2$ where $|3| = 1$, $|\ell_1| = 5$ and $|\ell_2| = 17$.

Finally, we have the case of $MU$.

**Proposition 5.10.** The $KSS$ Tor$_{\ast}$($F_p$, $F_p$) $\Rightarrow$ $\pi_\ast HF_p \land_{MU} HF_p$ collapses at the $E_2$ page. $\pi_\ast HF_p \land_{MU} HF_p$ is an exterior algebra $E_{\ell_1, \ell_2, \ldots}$ with $Q_p^0(\ell) = \pm \ell_{p-1}$ and $Q_p^0(\ell_{p-1}) = \pm \ell_{p-1}$ where $|\ell| = 1$ and $|\ell| = 2n + 1$.

And again, we have a corollary regarding complex orientations that lift to maps of commutative $S$-algebras.

**Corollary 5.11.** Let $I$ be an ideal of $MU_\ast$ generated by a regular sequence. If $I$ contains a finite nonzero number of the $x_{p-1}$, then the quotient map $MU \rightarrow MU/I$ cannot be realized as a map of commutative $S$-algebras.

### 6 Applications and Interpretations

In this section we present some applications and interpretations of the computations in section 5. We must first develop ways of relating our computations to null-homotopies of maps. Then we interpret our power operations as giving a dependence between choices of null-homotopies of different elements. We first record some basic results relating classes in the KSS to differences of null-homotopies. This will be useful later. We begin with the following Theorem.

**Proposition 6.1.** Suppose that $\varphi : R \rightarrow A$ is a map of commutative $S$-algebras that is surjective in homotopy. Then $\forall x \in I := \ker(\varphi_\ast)$ with nonzero image in $I/I^2$ there is a nonzero class $x$ $\in$ $\Tor^{R_\ast}(A_\ast, A_\ast)$. If this class is not an “eventual” boundary in the KSS, then it can be realized as the difference of two null-homotopies of $\varphi_\ast(x)$ $\in$ $A_\ast$.  

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Proof. Represent \( x \in \pi_n R \) by a map \( x : S^n \to R \). Since \( x \in I \), we know that the map \( S^n \to R \) becomes null-homotopic when followed by \( \varphi \). Choose a null homotopy and represent it as the following commutative diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{x} & R \\
\downarrow i & & \downarrow \varphi \\
CS^n & \xrightarrow{x'} & A.
\end{array}
\]

As \( \varphi \) is onto in homotopy, we can take the first stage of the Künneth resolution (and filtration) to be \( F_0 = A_0 = R \). By assumption, \( x \in \pi_* F_0 = \mathbb{F}_0 \) is in the kernel of the augmentation. Therefore, \( x \in im(\mathbb{F}_1 \to \mathbb{F}_0) \), so we get the following commutative diagram:

\[
\begin{array}{cccc}
S^n & \xrightarrow{1} & S^n & \xrightarrow{i} \to CS^n \\
\downarrow \tilde{x} & & \downarrow x & \downarrow x'' \\
F_1 & \xrightarrow{d} & F_0 = R & \xrightarrow{\alpha_0} A_1.
\end{array}
\]

We can then suture the right side of the above diagram with the diagram expressing the null-homotopy of \( x \) along \( x \) to produce

\[
\begin{array}{cccc}
S^n & \xrightarrow{x} & CS^n \\
\downarrow R & & \downarrow A_1 \\
CS^n & \xrightarrow{x''} & A & \xrightarrow{\wedge_R A_1}
\end{array}
\]

as both \( A \) and \( A_1 \) map to \( A \wedge_R A_1 \). This induces a map from \( S^{n+1} \) to \( A \wedge_R A_1 \). Denote elements obtained in this way from \( x \in ker \varphi_* \) as \( \overline{x} \in \pi_{n+1} A \wedge_R A_1 \). To see that this element contributes to

\[
ker(\pi_*(A) \otimes \mathbb{R}_* \mathbb{F}_1 \to \pi_*(A) \otimes \mathbb{R}_* \mathbb{F}_0),
\]

notice that the above definition of \( \overline{x} \) gives the following commutative diagram

\[
\begin{array}{cccc}
CS^n & \xrightarrow{z} & S^{n+1} & \xrightarrow{z} \to S^{n+1} & \xrightarrow{z} \to CS^{n+1} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
A = A \wedge_R A_0 & \xrightarrow{1_A \wedge_R \alpha_0} & A \wedge_R A_1 & \xrightarrow{1_A \wedge_R \beta_1} & A \wedge_R \Sigma \mathbb{F}_1 \to \Sigma A = A \wedge_R \Sigma \mathbb{F}_0.
\end{array}
\]
We will now show that the image of \( \pi \) in \( \pi_* A \otimes_R \mathbb{F}_1 \) survives to a class in \( \text{Tor}_1^R(A_*, A_*) \). Since \( \varphi \) is surjective in homotopy, we have that \( A_* \cong R_*/I \). Applying \( \text{Tor}_n^R(A_*, -) \) to the short exact sequence

\[
0 \to I \to R_* \xrightarrow{\varphi} A_* \to 0
\]

gives a long exact sequence in \( \text{Tor} \). This long exact sequence degenerates to the isomorphisms

\[
\text{Tor}_n^R(A_*, A_*) \cong \text{Tor}_n^R(A_*, I).
\]

Therefore \( \text{Tor}_n^R(A_*, A_*) \cong A_* \otimes_{R_*} I \), which we compute by applying \(- \otimes_{R_*} I \) to the right exact sequence

\[
I \to R_* \xrightarrow{\varphi} A_* \to 0
\]

which gives the exact sequence

\[
I \otimes_{R_*} I \xrightarrow{\mu} I \to A_* \otimes_{R_*} I \to 0.
\]

Therefore,

\[
\text{Tor}_1^R(A_*, A_*) \cong A_* \otimes_{R_*} I \cong I/\mu(I) \cong I/I^2.
\]

All that remains is that this class survives to contribute to \( \pi_* A \wedge R A \), which it does by assumption. \( \square \)

We will apply the above result frequently to elements of \( \pi_* A \wedge_R A \) that are detected on the 1-line of the KSS. Later in Proposition 6.5, we will realize certain elements in \( \pi_* A \wedge_R A \) as differences of null-homotopies. The above result allows us to locate such elements in the KSS. The next proposition is a similar result about how to relate power operations on relative smash products to the maps of commutative \( S \)-algebras.

**Proposition 6.2.** Suppose that \( \varphi : R \to R' \) is a map of commutative \( S \)-algebras over \( \text{HF}_p \), i.e. such that diagram of commutative \( S \)-algebras

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & R' \\
\downarrow & & \downarrow \\
\text{HF}_p & & \\
\end{array}
\]

commutes. If \( x \in \ker(\varphi_*) \) and \( \bar{y} \in \{ \ldots, \beta Q^i(\pi), Q^i(\pi), \ldots \} \subset \text{Tor}_1^R(\mathbb{F}_p, \mathbb{F}_p) \), then \( \varphi_*(y) \in \pi_* R' \) is decomposable.

**Proof.** Since \( \bar{y} = Q^i(\pi) \) and

\[
\hat{\varphi}_* : \text{Tor}_s^R(\mathbb{F}_p, \mathbb{F}_p)_t \xrightarrow{\text{Tor}_s^R(\mathbb{F}_p, \mathbb{F}_p)_t} \text{Tor}_s^R(\mathbb{F}_p, \mathbb{F}_p)_t
\]

commutes with the action of these operations, we have that \( \hat{\varphi}_*(\bar{y}) = 0 \) whenever \( \hat{\varphi}_*(\pi) = 0 \). Since \( \varphi_*(x) = 0 \), we have that \( \hat{\varphi}_*(\pi) = 0 \in \text{Tor}_1^R(\mathbb{F}_p, \mathbb{F}_p) \). Therefore, \( \hat{\varphi}_*(\bar{y}) = 0 \in \text{Tor}_1^R(\mathbb{F}_p, \mathbb{F}_p) \cong I/I^2 \) where \( I = \ker(R_*/I \to \mathbb{F}_p) \), and we have \( \varphi_*(y) \in I^2 \) as desired. \( \square \)

Notice that this result does not require that the classes \( \pi \) or \( \bar{y} \) survive the KSS and so it applies even when these classes may not give nullhomotopies as they do in Proposition 6.5.

The computations of section 5 have explicit interpretations. Let us work with a fixed map

\[
\varphi : R \to A
\]

of commutative \( S \)-algebras. Further, we will assume that \( A \) is a cofibrant \( R \)-module so that \( A \wedge_R A \) is \( A \wedge_R A \). Here we will interpret the computation of the action of the \( A \)-Dyer-Lashof algebra on \( \pi_* A \wedge_R A \). At the end of this section we will apply our results to the example of Subsection 5.1 on \( ku \to \text{HF}_2 \).

In commutative \( S \)-algebras, \( A \wedge_R A \) is the pushout. Therefore, a map of commutative \( S \)-algebras

\[
\varphi : A \wedge_R A \to X
\]

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is a map of commutative $S$-algebras from the diagram $A \leftarrow R \rightarrow A$ to $X$. Thus, $\varphi$ specifies two potentially different commutative $A$-algebra structures on $X$ with the same underlying commutative $R$-algebra structure. We wish to know, for example, how many different $A$-algebra structures on $X$ have the same underlying $R$-algebra structure. How different are the $A$-algebra structures on $X$ coming from $f$ and $g$?

**Definition 6.3.** We define the $d(f, g)$, the ‘difference’, of two maps of commutative $R$-algebras

$$f, g : A \rightarrow X$$

**Definition 6.3.** We define the $d(f, g)$, the ‘difference’, of two maps of commutative $R$-algebras

$$d(f, g) : A \wedge_R A \rightarrow X.$$  

The following lemma is illustrative of the role of the construction $d(f, g)$.

**Lemma 6.4.** The ‘difference’ $d(f, f) = f \circ \mu$ and factors through the product map $A \wedge_R A \rightarrow A$.

In particular, this implies that we can detect the difference between $f$ and $g$ by considering the effect in homotopy of $d(f, g)$. For example when $A$ is an Eilenberg-MacLane ring spectrum we see that $f \neq g$ whenever $d(f, g)$ is nonzero in homotopy outside degree zero.

**Proposition 6.5.** Let $x \in \ker \varphi : R_* \rightarrow A_*$ be such that the associated class $\bar{x} \in \operatorname{Tor}^R_1(A_*, A_*)$ survives the KSS and contributes $\bar{x} \in \pi_{n+1}A \wedge_R A$. Then the class $\bar{x} \in \pi_{n+1}A \wedge_R A$ can be represented as a “difference” of null-homotopies of $x$.

**Proof.** We can construct $x \in \ker \varphi : R_* \rightarrow A_*$ as a “difference” of null-homotopies as follows given that it comes from an element $x \in \ker \varphi : R_* \rightarrow A_*$. When the map

$$x : S^n \rightarrow R$$

is composed with

$$\varphi : R \rightarrow A$$

it becomes null-homotopic. After composing $\varphi \circ x$ with the right unit

$$\eta_r : A \simeq A \wedge_R R \xrightarrow{1 \wedge x} A \wedge_R A$$

we obtain the diagram

Composition with the left unit yields the diagram

$$S^n \xrightarrow{i} CS^n \xrightarrow{x'} A \xrightarrow{\eta_l} A \wedge_R A.$$
Suturing these diagrams together, we obtain

![Diagram](image)

where the back face is a pushout in $S$-modules. This constructs $\overline{\varphi} \in \pi_{n+1} A \wedge_R A$.

The class $\overline{\varphi}$ is a witness to the fact that $x$ has been made null-homotopic in $A \wedge_R A$ in two different ways. In fact, $d(f, g)_*(\overline{\varphi})$ is the obstruction to $f$ and $g$ giving the same null-homotopy of $x$.

**Lemma 6.6.** For $x \in \pi_n R$ as above, we have that $d(f, g)_*(\overline{\varphi}) = 0$ in $\pi_{n+1} X$ if and only if $f|_{CS^n} \simeq g|_{CS^n}$ relative to $S^n$.

**Proof.** This proof is elementary homotopy theory. Observe that $d(f, g)_*(\overline{\varphi}) = 0$ in $\pi_{n+1} X$ if and only if it can be extended over $S^{n+1} \subset D^{n+1}$. The element $d(f, g)_*(\overline{\varphi})\pi_{n+1}(A \wedge_R A)$ was defined as $f|_{CS^n}$ on one half of $S^{n+1}$, $g|_{CS^n}$ on the other half of $S^{n+1}$, and $f \circ \varphi \circ x = g \circ \varphi \circ x$ on the equatorial $S^n$. Therefore an extension of $d(f, g)_*(\overline{\varphi})$ to all of $D^{n+1}$ is equivalent to the desired homotopy of $f|_{CS^n} \simeq g|_{CS^n}$ relative to $S^n$.

We now apply the above discussion to interpret the computations made in the previous section. Let $f, g : HF_p \to X$ be two maps of commutative $S$-algebras which are equalized by the reduction map $HZ \to HF_p$. They then give a map $d(f, g) : HF_p \wedge_{HF_p} HF_p \to X$. The existence of $f$ and $g$ imply that we have a commutative $S$-algebra over $HZ$ which can be made a commutative $HF_p$-algebra in two potentially different ways, say $\overline{X}_f$ and $\overline{X}_g$. By the above result, Lemma 6.6, we see that if $d(f, g)_*(\overline{\varphi}) \neq 0$ then $f$ is not homotopic to $g$. It is in this way that we think of $\overline{\varphi}$ as an obstruction. It is not a complete obstruction in the sense that the vanishing of $d(f, g)_*(\overline{\varphi})$ does not guarantee that $f \simeq g$, therefore we will refer to such classes as witnesses.

Now suppose in addition that we have a map of commutative $S$-algebras $\varphi : R \to HF_p$ which also equalizes $f$ and $g$ and that $p \neq 0$ in $\pi_0 R$. Let $x \in \pi_0 R$ be as above, so that we have $\overline{\varphi}, \overline{\varphi} \in \pi_* HF_p \wedge_R HF_p$. If $\overline{\varphi}$ and $\overline{\varphi}$ are related in $\pi_* HF_p \wedge_R HF_p$ by a power operation, say $Q^i(\overline{\varphi}) = \overline{\varphi}$, then $d(f, g)_*(\overline{\varphi}) = 0$ implies $d(f, g)_*(\overline{\varphi}) = 0$ also. In this way, $\overline{\varphi}$ is a sort of indecomposable witness in that it generates an element in the cotangent complex of $\varphi$. This simple observation allows us to interpret the above computations of section 5 as saying that when we compare two different commutative $ku$, $BP(2)$, or $MU$-algebra structures on an $E_\infty$-dga, the null-homotopies of 2 that we choose effect the null-homotopies of $v$ and the $v_i$. This seems a strange fact, as all of these classes necessarily act trivially on $X$ or $\overline{X}$. However, their trivializations are related. One might hope that the above considerations can be strengthened to give a construction of an explicit null-homotopy of $v$ or $v_i$ respectively. For example, as $Q^2(\overline{2}) = \overline{2}$ for $2, v \in \pi_* ku$ we would like to use a null-homotopy of 2 to produce a null-homotopy of $v$ depending functorially on the null-homotopy of 2. The obvious construction fails as there is no operation $Q^2(\overline{2}) = v$ in $\pi_* HF_2 \wedge_{ku} HF_2$.

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7 Future Directions

This is the beginning of a larger program. It is our hope to understand why there are results such as those outlined in the previous section. That is to say “Why does the relative smash product $A \wedge_R A$ know so much about commutative $R$-algebra structures on commutative $A$-algebras?” To this end, we note that our computations can be used as input for the spectral sequence developed in [6] by Basterra, modulo some minor modifications. We spell this out below in Subsection 7.1.

Another reason for the “effectiveness” of the relative smash product is related to Tannakian formalism/Tannak-Krein duality. We explore this in future work with Markus Spitzweck. Roughly, the relative smash product is “functions” on the Tannakian fundamental group of $\text{Spec}(R)$ at the point $\text{Spec}(A)$ corresponding to the map of commutative $S$-algebras $\varphi : R \to A$. This result shouldn’t surprise those familiar with Tannakian formalism/Tannaka-Krein duality or derived algebraic geometry, but we think it is a very interesting perspective. We hope that this perspective can also help to shed light on the role of $A \wedge_R B$ in the above story through the construction of a Tannakian fundamental groupoid.

Another goal of ours is to do more computations. We would really like to have a family of results like 5.11 with the role of $H \mathbb{F}_p$ replaced by higher chromatic analogs such as the Morava $E$-theories. Unfortunately, this direction seems to be a bit harder to work in as the methods we have used here for computing power operations on $H \mathbb{F}_p \wedge_{MU} H \mathbb{F}_p$ do not work for computing the $\theta_p$-algebra structure on $K \mathbb{U}_p \wedge_{MU} K \mathbb{U}_p$ where $K \mathbb{U}_p$ is $p$-complete complex $K$-theory. We have tried other methods of trying to compute the $\theta_p$-algebra struture on $K \mathbb{U}_p \wedge_{MU} K \mathbb{U}_p$ but have not been successful.

7.1 The Cotangent Complex of a Map

We give one final application of our computations that we hope to pursue in future work. Our computation of the action of the Dyer-Lashof operations are input for a spectral sequence that computes the homotopy of the cotangent complex of the map $\varphi: R \to H \mathbb{F}_p$.

The cotangent complex of a map of commutative $S$-algebras is a generalization of the very useful classical concept. Basterra constructed this cotangent complex in [6]. This construction has been used by Baker, Gilmour, and Reinhard in [3]. Variants for $E_n$-algebras in $S$-modules have also been developed by Basterra-Mandell and then used to prove that $BP$ is an $E_1$-algebra. In her work, Basterra constructs a spectral sequence that converges to $TAQ^*(B/S; H \mathbb{F}_p)$ where $B$ is a connective commutative $S$-algebra. The construction of Basterra can easily be adapted to compute the cotangent complex of maps $R \to H \mathbb{F}_p$. We refer the reader to [6] as well as [3], [1], and [2] for specifics on the cotangent complex.

Basterra constructed a spectral sequence in [6] to compute the André-Quillen cohomology of $S \to B$ with coefficients in $H \mathbb{F}_p$ for a connective commutative $S$-algebra $B$. The spectral sequence is a bar spectral sequence associated with the bar construction for the comonad $F := \mathbb{P}U$ where $\mathbb{P}$ is the free non-unital $A$-algebra functor $\mathbb{P}(X) := \bigvee_{n \geq 1} X_{\Sigma_n}^{\wedge n}$ on an $A$-module $X$ and $U$ is the forgetful functor from non-unital commutative $A$-algebras to $A$-modules.

**Theorem 8** (Basterra). For a cofibrant connective commutative $S$-algebra $B$ with a cofibration $f: B \to H \mathbb{F}_p$ of $S$-algebras, there is a strongly convergent spectral sequence

$E_2^{s,t} = \text{Hom}_{\mathbb{F}_p}(L_{s}^{\mathbb{P}}(\mathbb{P} \otimes_{DL} Q^{alg})/(H \mathbb{F}_p,B), \mathbb{F}_p) \Rightarrow TAQ^{s+t}(B/S; H \mathbb{F}_p)$.

Here, $DL$ is the mod $p$ Dyer-Lashof algebra, $Q^{alg}$ is the algebraic indecomposables functor, $L^{\mathbb{P}}$ is comonad $\mathbb{P}$-left derived functor where $\mathbb{P}$ is the comonad associated with the free and forgetful adjunction between non-unital algebras with an allowable action of the Dyer-Lashof algebra and $\mathbb{F}_p$ vector spaces.

This spectral sequence is not specific to the category of commutative $S$-algebras, it only relies on formal properties of the model category of commutative $S$-algebras in $[10]$. For example, the above spectral sequence

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exists when $B$ is a cofibrant connective commutative $R$-algebra for $R$ a cofibrant connective commutative $S$-algebra. When $B$ is a cofibrant replacement of $H\F_p$ then the spectral sequence is

$$E_2^{s,t} = \text{Hom}_F\left(L^F_s(\F_p \otimes_{DL} Q^{alg})(\pi_t(\HF_p \wedge_R \HF_p), \F_p) \Rightarrow TAQ^{s+t}(B/R; H\F_p)\right).$$

To utilize the above spectral sequence we see that we need to know the action of the Dyer-Lashof algebra on $\pi_*\HF_p \wedge_R \HF_p$ which is exactly what is computed in section 5 for $ku$ and $BP(2)$ with partial information for $MU$. We hope to investigate this in future work with Andrew Salch.

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