SHIFTED CONVOLUTION OF DIVISOR FUNCTION $d_3$
AND RAMANUJAN $\tau$ FUNCTION

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1. Introduction

This note can be viewed as a bridge between the work of Pitt [5] and my recent paper [3]. In [5] Pitt considers the sum

$$\Psi(f, x) = \sum_{1 \leq n \leq x} d_3(n)a(rn - 1)$$

where $d_3(n)$ is the divisor function of order 3 (the coefficients of the Dirichlet series $\zeta(s)^3$), $a(m)$ is the normalized Fourier coefficients of a holomorphic cusp form $f$, and $r$ is a positive integer. Without loss one may take $a(m) = \tau(m)/m^{1/2}$ where $\tau$ is the Ramanujan function, as one does not expect any new complication to arise while dealing with Fourier coefficients of a general holomorphic cusp form. The trivial bound is given by $O(x^{1+\varepsilon})$. Pitt [5] proved

$$\Psi(f, x) \ll x^{71/72+\varepsilon}$$

where the implied constant is uniform with respect to $r$ in the range $0 < r \ll X^{1/24}$. This sum is intrinsically related to the generalized Titchmarsh divisor problem, where one seeks to estimate the sum (see [5], [6])

$$\sum_{p < x \text{ prime}} a(p - 1).$$

In this paper we will use our method from [3], [4] to prove the following (improved bound).

**Theorem 1.** For $r \ll X^{1/10}$ we have

$$\Psi(\Delta, x) \ll r^{3/2}X^{34/35+\varepsilon},$$

where the implied constant depends only on $\varepsilon$.

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2. Twisted Voronoi summation formulae

Let

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$$

be the Ramanujan’s $\Delta$ function. Here we are using the standard notation $e(z) = e^{2\pi iz}$. The function $\Delta(z)$ is a cusp form for $SL(2, \mathbb{Z})$ of weight 12. The Ramanujan $\tau$ function is defined as the Fourier coefficients of $\Delta(z)$, namely

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e(nz).$$

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Ramanujan conjectured, and later Deligne proved, that \(|\tau(p)| \leq 2p^{11/2}\) for any prime number \(p\). In light of this bound it is natural to define the normalized \(\tau\) function as
\[
\tau_0(n) = \tau(n)/n^{11/2}.
\]
Using the modularity of \(\Delta(z)\) one can establish the following Voronoi type summation formula for \(\tau_0(n)\).

**Lemma 1.** Let \(q\) be a positive integer, and \(a\) be an integer such that \((a, q) = 1\). Let \(g\) be a compactly supported smooth function on \(\mathbb{R}_+\). We have
\[
\sum_{m=1}^{\infty} \tau_0(m) e_q(\bar{a}m) g(m) = \frac{2\pi}{q} \sum_{m=1}^{\infty} \tau_0(m) e_q(-\bar{a}m) G \left( \frac{m}{q^2} \right)
\]
where \(\bar{a}\) is the multiplicative inverse of \(a\) mod \(q\), \(e_q(z) = e(z/q)\) and
\[
G(y) = \int_0^\infty g(x) J_{11}(4\pi \sqrt{xy}) \, dx.
\]
Here \(J_{11}(z)\) is the Bessel functions in standard notations.

If \(g\) is supported in \([AY, BY]\) (with \(0 < A < B\)), satisfying \(y'\phi(\gamma(y)) \ll \epsilon\) for \(\epsilon\) small, then the sum on the right hand side of (1) is essentially supported on \(m \ll q^2(qY)^\epsilon/Y\). The contribution from the tail \(m \gg q^2(qY)^\epsilon/Y\) is negligibly small. For smaller values of \(m\) we will use the trivial bound \(G(m/q^2) \ll Y\).

A similar Voronoi type summation formula for the divisor function \(d_3(n)\) is also known (see Ivic [1]). Let \(f\) be a compactly supported smooth function on \(\mathbb{R}_+\), and let \(\hat{f}(s) = \int_0^\infty f(x)x^s \, dx\). We define
\[
F_\pm(y) = \frac{1}{2\pi i} \int_{\{\pm\}} (\pi^3 y)^{-s} \frac{\Gamma^3 \left( \frac{1\pm 1+2s}{4} \right)}{\Gamma^3 \left( \frac{3\pm 1-2s}{4} \right)} \hat{f}(-s) \, ds.
\]

**Lemma 2.** Let \(f\) be a compactly supported smooth function on \(\mathbb{R}_+\), we have
\[
\sum_{n=1}^{\infty} d_3(n) e_q(an) f(n) = \frac{1}{q} \int_0^\infty P(\log y, q) f(y) dy
\]
\[
+ \frac{\pi^{3/2}}{2q^3} \sum_{a,q=1}^{\infty} D_{3,\pm}(a; q) F_\pm \left( \frac{n}{q^2} \right),
\]
where \(P(y, q) = A_0(q)y^2 + A_1(q)y + A_2(q)\) is a quadratic polynomial whose coefficients depend only on \(q\) and satisfy the bound \(|A_j(q)| \ll q^\epsilon\). Also \(D_{3,\pm}(a; q)\) are given by
\[
\sum_{n_1n_2n_3=n} \sum_{b,c,d=1}^q \{ e_q(bn_1 + cn_2 + dn_3 + abcd) \mp e_q(bn_1 + cn_2 + dn_3 - abcd) \}.
\]

Suppose \(f\) is supported in \([AX, BX]\), and \(x^3 f'(x) \ll H^2\). Then the sums on the right hand side of (3) are essentially supported on \(n \ll q^3H(qX)^\epsilon/X\). The contribution from the tail \(n \gg q^3H(qX)^\epsilon/X\) is negligibly small. This follows by estimating the integral \(F_\pm(y)\) by shifting the contour to the right. For smaller values of \(n\) we shift the contour to left up to \(\sigma = \epsilon\).

### 3. Setting up the circle method

As in [3], we will be using a variant Jutila’s version of the circle method. For any set \(S \subset \mathbb{R}\), let \(\mathbb{I}_S : \mathbb{R} \to \{0, 1\}\) be defined by \(\mathbb{I}_S(x) = 1\) for \(x \in S\) and 0 otherwise. For any collection of positive integers \(Q \subset [1, Q]\) (which we call the set of moduli), and a positive real number \(\delta\) in the range \(Q^{-2} \ll \delta \ll Q^{-1}\), we define the function
\[
\mathbb{I}_{Q, \delta}(x) = \frac{1}{2\delta L} \sum_{q \in Q} \sum_{\psi \mod q} \mathbb{I}_{[0, 1]} \rho_{[\delta, \delta + \delta]}(x),
\]
where \(L = \sum_{q \in Q} \phi(q)\). This is an approximation for \(\mathbb{I}_{[0, 1]}\) in the following sense (see [3]):
Lemma 3. We have

\[ \int_0^1 \left|1 - I_{Q,\delta}(x)\right|^2 dx \ll \frac{Q^{2+\varepsilon}}{\delta L}. \]

Instead of studying the sum in Theorem 1 we examine the related smoothed sum over dyadic segment

\[ D = \sum_{n=1}^{\infty} d_3(n)\tau_0(rn - 1)W(n/X) \]

where \( W \) is a non-negative smooth function supported in \([1 - H^{-1}, 2 + H^{-1}]\) (we will chose \( H = X^\theta \) optimally later), with \( W(x) = 1 \) for \( x \in [1, 2] \) and satisfying \( W^{(j)}(x) \ll_j H^j \). Clearly we have

\[ \Psi(\Delta, x) = D + O(X^{1+\varepsilon}/H). \]

In the rest of the paper we will prove a compatible bound for \( D \).

Let \( V \) be a smooth function supported in \([1/2, 3]\) satisfying \( V(x) = 1 \) for \( x \in [3/4, 5/2] \), \( V^{(j)}(x) \ll_j 1 \), and let \( Y = rX \). Then we have

\[ D = \sum_{n,m=1}^{\infty} d_3(n)\tau_0(m)W(x) V(x) \delta(rn - 1, m) \]

\[ = \int_0^1 e(-x) \left[ \sum_{n=1}^{\infty} d_3(n)e(xrn)W(x) \right] \left[ \sum_{m=1}^{\infty} \tau_0(m)e(-xm)V(x) \right] dx. \]

Let \( Q \), which we choose carefully later, be a collection of moduli of size \( Q \). Suppose \( |Q| \gg Q^{1-\varepsilon} \), so that \( L = \sum_{q \in Q} \phi(q) \gg Q^{2-\varepsilon} \). Let \( \delta = Y^{-1} \), and define

\[ \hat{D} := \int_0^1 I_{Q,\delta}(x)e(-x) \left[ \sum_{n=1}^{\infty} d_3(n)e(xrn)W(x) \right] \left[ \sum_{m=1}^{\infty} \tau_0(m)e(-xm)V(x) \right] dx. \]

It follows that

\[ \hat{D} = \frac{1}{2\delta} \int_{-\delta}^{\delta} \hat{D}(\alpha)e(-\alpha) d\alpha, \]

where

\[ \hat{D}(\alpha) = \frac{1}{L} \sum_{q \in Q} \sum_{\alpha \mod q} e_q(-\alpha) \left[ \sum_{n=1}^{\infty} d_3(n)e(qrn)e(qrn)W(x) \right] \]

\[ \times \left[ \sum_{m=1}^{\infty} \tau_0(m)e(-am)e(-\alpha m)V(x) \right]. \]

In circle method we approximate \( D \) by \( \hat{D} \), and then try to estimate the latter sum. Lemma 3 gives a way to estimate the error in this process. More precisely we have

\[ |D - \hat{D}| \ll \int_0^1 \left[ \sum_{n=1}^{\infty} d_3(n)e(xrn)W(x) \right] \left[ \sum_{m=1}^{\infty} \tau_0(m)e(-xm)V(x) \right] \left|1 - I_{Q,\delta}(x)\right| dx, \]

Using the well-known point-wise uniform bound

\[ \sum_{m=1}^{\infty} \tau_0(m)e(-xm)V(x) \ll Y^{1+\varepsilon} \]

it follows that the right hand side of (5) is bounded by

\[ \ll Y^{1+\varepsilon} \int_0^1 \left[ \sum_{n=1}^{\infty} d_3(n)e(xrn)W(x) \right] \left|1 - I(x)\right| dx. \]
Now we consider the case where $q = Y X^{-\frac{1}{2} + \delta}$ for any $\delta > 0$, to arrive at the following:

**Lemma 4.** We have

$$D = \hat{D} + O \left(X^{1-\delta + \varepsilon}\right).$$

We can now decide what will be the optimal choice for $H$. Naturally we wish to take $H$ as small as possible to aid in our analysis of $\hat{D}$. Matching the error term in Lemma 4 and that in (3) we pick $H = X^{\delta}$.

4. Estimation of $\hat{D}$

Now we apply Voronoi summations on the sums over $m$ and $n$. This process gives rise to several terms as noted in Section 2, Lemma 1 and Lemma 2. As far as our analysis is concerned we can focus our attention on two such terms, namely

$$\hat{D}_0(\alpha) = \frac{2\pi}{L} \sum_{q \in \mathbb{Q}} \frac{1}{q^2} \sum_{m=1}^{\infty} \tau_0(m) S(1, m; q) \left(\frac{m}{q^2}\right) \int_{0}^{\infty} P(\log x) f(x) dx,$$

which is the zero frequency contribution ($S(1, m; q)$ is the Kloosterman sum), and

$$\hat{D}_1(\alpha) = \frac{\pi^{5/2}}{L} \sum_{q \in \mathbb{Q}} \frac{1}{q^4} \sum_{m=1}^{\infty} \tau_0(m) \sum_{n=1}^{\infty} S^*(m, n; q) \left(\frac{m}{q^2}\right) F_+ \left(\frac{n}{q^3}\right),$$

where the character sum is given by

$$S^*(m, n; q) := \sum_{\alpha \mod q} \varepsilon_q(-\alpha m) \sum_{n_1 n_2 n_3 = n} \sum_{b, c, d = 1}^q \varepsilon_q(b n_1 + c n_2 + d n_3 + abc dr)$$

Also here we are taking

$$g(y) = V \left(\frac{y}{Y}\right) e(-\alpha y), \quad \text{and} \quad f(x) = W \left(\frac{x}{X}\right) e(-\alpha x).$$

The functions $G$ and $F_+$ are defined in Lemma 1 and Lemma 2 respectively. It follows that in both the sums (10) and (11), the sum over $m$ essentially ranges up to $m \ll Q^2 Y^{-1+\varepsilon} = Y X^{-1+2\delta+\varepsilon}$. The tail contribution is negligibly small. So using the Weil bound for the Kloosterman sums it follows that

$$\hat{D}_0(\alpha) \ll X^{1+\varepsilon}/\sqrt{Q}$$

which is smaller than the bound in Lemma 4 (as $Q = Y X^{-\frac{1}{2} + \delta} > X^{2\delta}$ or $Y > X^{\frac{1}{2} + \delta}$). One can use Deligne’s theory to show that there is square root cancellation in the character sum (12). But this is not enough to establish a satisfactory bound for $\hat{D}_1(\alpha)$.

Following 3 we will now make an appropriate choice for the set of moduli. We choose $Q$ to be the product set $Q_1 Q_2$, where $Q_i$ consists of primes in the dyadic segment $[Q_{i-1}, 2Q_i]$ (and not dividing $r$) for $i = 1, 2$, and $Q_1 Q_2 = Q$. Also we pick $Q_1$ and $Q_2$ (whose optimal sizes will be determined later) so that the collections $Q_1$ and $Q_2$ are disjoint.

Suppose $q = q_1 q_2$ with $q_i \in Q_i$. The character sum $S^*(m, n; q)$ splits as a product of two character sums with prime moduli. The one modulo $q_1$ looks like (after a change of variables)

$$S^t(m, n, q_2; q_1) = \sum_{\alpha \mod q_1} \varepsilon_{q_1}(-q_2 \alpha m) \sum_{n_1 n_2 n_3 = n} \sum_{b, c, d = 1}^q \varepsilon_q(b n_1 + c n_2 + d n_3 + abc dr).$$

Now let us consider the case where $q_1 | n$. Suppose $q_1 | n_1$, then summing over $b$ we arrive at

$$q_1 \sum_{d=1}^{q_1} e_{q_1}(dn_3) + q_1 \sum_{c=1}^{q_1} e_{q_1}(cn_2) - q_1.$$

This sum is bounded by $q_1 (q_1, n_2 n_3)$. Then using Weil bound for Kloosterman sums we conclude that

$$S^t(m, q_1 n, q_2; q_1) \ll q_1^{3/2}(q_1, n)d_3(n).$$
On the other hand if \( q_1 \nmid n \) then we arrive at the following expression for the character sum after a change of variables

\[
S^\dagger(m, n, q_2; q_1) = d_3(n) \sum_{a \mod q_1}^* e_q((-q_2^3a - q_2\bar{a}m) \sum_{b,c,d=1}^{q_1} e_q((b + c + d + \bar{n}abcdr)).
\]

Summing over \( b \) we arrive at

\[
S^\dagger(m, n, q_2; q_1) = d_3(n)q_1 \sum_{a \mod q_1}^* e_q((-q_2^3a - q_2\bar{a}m)S(1, -n\bar{a}r; q_1).
\]

This can be compared with the character sums which appear in [3] and [5]. Strong bounds (square root cancellation) have been established for this sums using Deligne’s result. In the light of this, it follows that to estimate the contribution of those \( n \) in (11) with \((n, q) \neq 1\) it is enough to look at the sum

\[
\frac{1}{L} \sum_{q \in \mathbb{Q}} Q^3/ \sum_{0 < m \in \mathbb{Q}^2/Y} Q \sum_{0 < n \in \mathbb{Q}^3H/\min\{Q_1, Q_2\}X} Q^{3/2} \sqrt{\max\{Q_1, Q_2\}} \cdot Y \cdot X\cdot H.
\]

The last sum is bounded by \(O(Q^{2+\varepsilon}H^{3/2}\min\{Q_1, Q_2\}^{-3/2})\), and we get

\[
\tilde{D}_1(\alpha) = \frac{\pi^{5/2}}{L} \sum_{q \in \mathbb{Q}} Q^3/ \sum_{m=1}^{M} \tau_0(m) \sum_{n=1}^{N} d_3(n)S^\dagger(m, n; q)G \left( \frac{m}{q^2} \right) F_+ \left( \frac{n}{q^3} \right) + O \left( \frac{r^2 X^{1+7\delta/2+\varepsilon}}{\min\{Q_1, Q_2\}^{3/2}} \right),
\]

where

\[
S^\dagger(m, n; q) = \sum_{a \mod q}^* e_q(-a - \bar{a}m)S(1, -n\bar{a}r; q),
\]

\( M = Q^{2+\varepsilon}Y^{-1} = rX^{2\delta+\varepsilon}\) and \( N = Q^{3+\varepsilon}H^{-1} = r^4X^{1/2+4\delta+\varepsilon}. \) Next we observe that we can now remove the coprimality restriction \((n, q) = 1\), without worsening the error term. Here we are using square root cancellation in the character sum \( S^\dagger(m, n; q) \). We get

\[
\tilde{D}_1(\alpha) = \frac{\pi^{5/2}}{L} \sum_{q \in \mathbb{Q}} Q^3/ \sum_{m=1}^{M} \tau_0(m) \sum_{n=1}^{N} d_3(n)S^\dagger(m, n; q)G \left( \frac{m}{q^2} \right) F_+ \left( \frac{n}{q^3} \right) + O \left( \frac{r^2 X^{1+7\delta/2+\varepsilon}}{\min\{Q_1, Q_2\}^{3/2}} \right),
\]

5. Estimation of \( \tilde{D}_1(\alpha) \): Final analysis

Applying Deligne’s bound for \( \tau(m) \), the problem now reduces to estimating

\[
\frac{1}{Q^5} \sum_{q_2 \in \mathbb{Q}_2} \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq N} \left| \sum_{q_1 \in \mathcal{Q}_1} S^\dagger(m, n; q)G \left( \frac{m}{q^2} \right) F_+ \left( \frac{n}{q^3} \right) \right|,
\]

where \( q = q_1q_2 \). Applying Cauchy inequality we get

\[
\tilde{D}_1(\alpha) \ll \sqrt{N} Q^\frac{1}{2} \sum_{q_2 \in \mathbb{Q}_2} \sum_{1 \leq m \leq M} \tilde{D}^\dagger(m, q_2)^\frac{1}{2},
\]

where

\[
\tilde{D}^\dagger(m, q_2) = \sum_{n \in \mathbb{Z}} h(n) \left| \sum_{q_1 \in \mathcal{Q}_1} S^\dagger(m, n; q_1q_2)G \left( \frac{m}{q_1q_2^2} \right) F_+ \left( \frac{n}{q_1q_2^3} \right) \right|^2.
\]

Here \( h \) is non-negative smooth function on \((0, \infty)\), supported on \([1/2, 2N]\), and such that \( h(x) = 1 \) for \( x \in [1, N] \) and \( x^j h^{(j)}(x) \ll 1. \)
Opening the absolute square and interchanging the order of summations we get
\[ \tilde{D}^2(m, q_2) = \sum_{q_1 \in \mathcal{Q}_1} \sum_{q_1 \in \mathcal{Q}_1} G \left( \frac{m}{q_1 q_2} \right) G \left( \frac{m}{q_1 q_2} \right) \times \sum_{n \in \mathbb{Z}} h(n) S^\dagger(m, n; q_1 q_2) \tilde{S}^\dagger(m, n; \tilde{q}_1 q_2) F_+ \left( \frac{n}{q_1 q_2} \right) F_+ \left( \frac{n}{q_1 q_2} \right). \]

Applying Poisson summation on the sum over \( n \) with modulus \( q_1 \tilde{q}_1 q_2 \), we get
\[ \frac{1}{q_2} \sum_{q_1 \in \mathcal{Q}_1} \sum_{\tilde{q}_1 \in \mathcal{Q}_1} \frac{1}{q_1 \tilde{q}_1} G \left( \frac{m}{q_1 \tilde{q}_1 q_2} \right) \tilde{G} \left( \frac{m}{\tilde{q}_1 q_2} \right) \sum_{n \in \mathbb{Z}} T(m, n; q_1, \tilde{q}_1, q_2) \mathcal{I}(n; q_1, \tilde{q}_1, q_2). \]

The character sum is given by
\[ T(m, n; q_1, \tilde{q}_1, q_2) = \sum_{\alpha \text{ mod } q_1 \tilde{q}_1 q_2} S^\dagger(m, \alpha; q_1 q_2) \tilde{S}^\dagger(m, \alpha; \tilde{q}_1 q_2) e_{q_1 \tilde{q}_1 q_2}(n \alpha), \]
and the integral is given by
\[ \mathcal{I}(n; q_1, \tilde{q}_1, q_2) = \int_{\mathbb{R}} h(x) F_+ \left( \frac{x}{q_1 \tilde{q}_1 q_2} \right) e_{q_1 \tilde{q}_1 q_2}(-nx) dx. \]

Integrating by parts repeatedly one shows that the integral is negligibly small for large values of \( |n| \), say \( |n| \geq X^{2013} \). Observe that differentiating under the integral sign in (2), one can show that \( y^j F_+^j(y) \ll_j XH \). So we have the bound
\[ \mathcal{I}(n; q_1, \tilde{q}_1, q_2) \ll \frac{X^2 HQ^3}{|n|}. \]

The following lemma now follows from (15).

**Lemma 5.** We have
\[ \tilde{D}^2(m, q_2) \ll (XY)^2 H \sum_{q_1 \in \mathcal{Q}_1} \sum_{\tilde{q}_1 \in \mathcal{Q}_1} \left\{ \sum_{1 \leq |n| \leq X^{2013}} \frac{H}{|n|} |T(m, n; q_1, \tilde{q}_1, q_2)| + \frac{N}{QQ_1} |T(m, 0; q_1, \tilde{q}_1, q_2)| \right\} + X^{-2013}. \]

It now remains to estimate the character sum. This has been done in [3]. We summarize the result in the following lemma.

**Lemma 6.** For \( q_1 \neq \tilde{q}_1 \), the character sum \( T(m, n; q_1, \tilde{q}_1, q_2) \) vanishes unless \( (n, q_1 \tilde{q}_1) = 1 \), in which case we have
\[ T(m, n; q_1, \tilde{q}_1, q_2) \ll q_1^{5/2} q_2^{5/2} (n, q_2)^{5/2}. \]

The character sum \( T(m, n; q_1, q_2) \) vanishes unless \( n \) is divisible by \( q_1 \), in which case we have
\[ T(m, q_1 n'; q_1, q_1, q_2) \ll q_1^5 q_2^{5/2} \sqrt{(n', q_1 q_2)}. \]

It follows from Lemma 6 that
\[ \sum_{q_1 \in \mathcal{Q}_1} \sum_{\tilde{q}_1 \in \mathcal{Q}_1} \sum_{1 \leq |n| \leq X^{2013}} \frac{|T(m, n; q_1, \tilde{q}_1, q_2)|}{|n|} \ll Q_1^5 Q_2^{5/2} \sum_{1 \leq |n| \leq X^{2013}} \sqrt{(n, q_2)} \ll Q_1^5 Q_2^{5/2} X^{\varepsilon}. \]

Again applying Lemma 6 it follows that
\[ H \sum_{q_1 \in \mathcal{Q}_1} \sum_{1 \leq |n| \leq X^{2013}} \frac{|T(m, q_1 n; q_1, q_1, q_2)|}{q_1 |n|} + \frac{N}{Q Q_1} \sum_{q_1 \in \mathcal{Q}_1} |T(m, 0; q_1, q_1, q_2)| \ll HQ^{5/2} X^{\varepsilon} + NQ^2 X^{\varepsilon}. \]
The above two bounds (16), (17) yield
\[ \hat{D}^t(m, q_2) \ll (HQ_1^5Q_2^{5/2} + NQ^2)H(XY)^{3+\varepsilon}. \]

Plugging this estimate in (14) we get the following:

**Lemma 7.** For \( Q_1Q_2 = Q \), we have
\[ \hat{D}_1(\alpha) \ll \frac{\sqrt{NQ_2M}}{Q^2}(\sqrt{H}Q_1^{5/4}Q_2^{5/4} + \sqrt{NQ})\sqrt{HYX^{1+\varepsilon}}. \]

The optimal breakup \( Q_1Q_2 = Q \) is now obtained by equating the two terms. We get that \( Q_2 = X^{2/5} \) and \( Q_1 = rX^{1/10+\delta} \). The optimal choice for \( \delta \) is now obtained by equating the resulting error term with the previous error term, namely \( X^{1-\delta} \). We get
\[ \delta = \frac{1}{35} - \frac{2}{7} \log \frac{r}{\log X}. \]

Finally one checks that the error term in (13) is satisfactory for the above choice of \( \delta \). This holds as long as \( r \ll X^{4/5} \).

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