Differentiability of a two-parameter family of self-affine functions

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Abstract

This paper highlights an unexpected connection between expansions of real numbers to noninteger bases (so-called $\beta$-expansions) and the infinite derivatives of a class of self-affine functions. Precisely, we extend Okamoto’s function (itself a generalization of the well-known functions of Perkins and Katsuura) to a two-parameter family $\{F_{N,a} : N \in \mathbb{N}, a \in (0,1)\}$. We first show that for each $x$, $F'_{N,a}(x)$ is either 0, $\pm\infty$, or undefined. We then extend Okamoto’s theorem by proving that for each $N$, depending on the value of $a$ relative to a pair of thresholds, the set $\{x : F'_{N,a}(x) = 0\}$ is either empty, uncountable but Lebesgue null, or of full Lebesgue measure. We compute its Hausdorff dimension in the second case.

The second result is a characterization of the set $\mathcal{D}_\infty(a) := \{x : F'_{N,a}(x) = \pm\infty\}$, which enables us to closely relate this set to the set of points which have a unique expansion in the (typically noninteger) base $\beta = 1/a$. Recent advances in the theory of $\beta$-expansions are then used to determine the cardinality and Hausdorff dimension of $\mathcal{D}_\infty(a)$, which depends qualitatively on the value of $a$ relative to a second pair of thresholds.

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1 Introduction

The aim of this paper is to investigate the differentiability of a two-parameter family of self-affine functions, constructed as follows. Fix a positive integer $N$ and a real parameter $a$ satisfying $1/(N+1) < a < 1$, and let $b$ be the number such that $(N+1)a - Nb = 1$. Note that $0 < b < a$. Let $x_i := i/(2N+1)$, $i = 0, 1, \ldots, 2N+1$, and for $j = 0, 1, \ldots, N$, put $y_{2j} := j(a-b)$, and $y_{2j+1} := (j+1)a - jb$. Now set $f_0(x) := x$, and for $n = 1, 2, \ldots$, define $f_n$ recursively on each interval $[x_i, x_{i+1}]$ ($i = 0, 1, \ldots, 2N$) by

$$f_n(x) := y_i + (y_{i+1} - y_i)f_{n-1}((2N+1)(x-x_i)), \quad x_i \leq x \leq x_{i+1}. \quad (1)$$

Each $f_n$ is a continuous, piecewise linear function from the interval $[0, 1]$ onto itself, and it is easy to see that the sequence $(f_n)$ converges uniformly to a limit function which we denote by $F_{N,a}$. This function $F_{N,a}$ is again continuous and maps $[0, 1]$ onto itself. It may be viewed as the self-affine function “generated” by the piecewise linear function with interpolation points $(x_i, y_i)$, $i = 0, 1, \ldots, 2N+1$. When $N = 1$ we have Okamoto’s family of self-affine functions [22], which includes Perkins’ function [24] for $a = 5/6$ and the Katsuura function [14] for $a = 2/3$; see also Bourbaki [4]. Figure 1 illustrates the above construction for $N = 1$; graphs of $F_{1,a}$ for two values of $a$ are shown in Figure 2 and Figure 3 illustrates the case $N = 2$.

The restriction $a > 1/(N+1)$ is not necessary; when $a = 1/(N+1)$ we have $b = 0$ and $F_{N,a}$ is a generalized Cantor function. When $0 < a < 1/(N+1)$, we have $b < 0$ and $F_{N,a}$ is strictly singular. Since the differentiability of such functions has been well-studied (e.g. [6, 7, 8, 10, 12, 15, 26]), we will focus exclusively on the case $a > 1/(N+1)$, when $F_{N,a}$ is of unbounded variation. However, see [1] for a detailed

![Figure 1: The first two steps in the construction of $F_{1,a}$](image-url)
Figure 2: Graph of $F_{1,a}$ for $a = 5/6$ (Perkins’ function; left) and $a = \hat{a}_\infty \doteq .5598$ (right).

Figure 3: The generating pattern and graph of $F_{2,a}$, shown here for $a = 0.6$. 
analysis of the case \( N = 1, a \leq 1/2 \).

Our main goal is to study the finite and infinite derivatives of \( F_{N,a} \), thereby extending results of Okamoto [22] and Allaart [1]. We first show that for each \( N \) the differentiability of \( F_{N,a} \) follows the trichotomy discovered by Okamoto [22] for the case \( N = 1 \): There are thresholds \( \tilde{a}_0 \) and \( a_0^* \) (depending on \( N \)) such that \( F_{N,a} \) is nowhere differentiable for \( a \geq a_0^* \); nondifferentiable almost everywhere for \( \tilde{a}_0 \leq a < a_0^* \); and differentiable almost everywhere for \( a < \tilde{a}_0 \). We moreover compute the Hausdorff dimension of the exceptional sets implicit in the above statement.

In [1] a surprising connection was found between the infinite derivatives of \( F_{1,a} \) and expansions of real numbers in noninteger bases. Our aim here is to generalize this result. We first give an explicit description of the set \( D_\infty(a) \) of points \( x \) for which \( F'_{1,a}(x) = \pm \infty \), and then show that this set is closely related to the set \( A_\beta \) of real numbers which have a unique expansion in base \( \beta \) over the alphabet \( \{0, 1, \ldots, N\} \), where \( \beta = 1/a \). This allows us to express the Hausdorff dimension of \( D_\infty(a) \) directly in terms of the dimension of \( A_\beta \), which is known to vary continuously with \( \beta \) and can be calculated explicitly for many values of \( \beta \); see [17, 20]. We also take advantage of other recent breakthroughs in the theory of \( \beta \)-expansions to obtain a complete classification of the cardinality of \( D_\infty(a) \).

To end this introduction, we point out that the box-counting dimension of the graph of \( F_{N,a} \) is given by

\[
\dim_B \text{Graph}(F_{N,a}) = 1 + \frac{\log (2(N + 1)a - 1)}{\log(2N + 1)}, \quad \frac{1}{N + 1} < a < 1.
\]

This follows easily from the self-affine structure of \( F_{N,a} \), for instance by using Example 11.4 of Falconer [9]. The Hausdorff dimension, on the other hand, does not appear to be known for any value of \( a > 1/(N + 1) \), even when \( N = 1 \); see the remark in the introduction of [1].

2 Main results

2.1 Finite derivatives

Following standard convention, we consider a function \( f \) to be differentiable at a point \( x \) if \( f \) has a well-defined finite derivative at \( x \). For each \( N \in \mathbb{N} \), let \( a_{\min} := a_{\min}(N) := 1/(N + 1) \), let

\[
a_0^* := a_0^*(N) := \frac{3N + 1}{(N + 1)(2N + 1)},
\]

(2)
and let $\tilde{a}_0 := \tilde{a}_0(N)$ be the unique root in $(a_{\min}, 1)$ of the polynomial equation
\[(2N + 1)^{2N+1}a^{N+1}((N + 1)a - 1)^N = N^N.\] (3)

The first ten values of $\tilde{a}_0(N)$ are shown in Table 1 below. Asymptotically, $\tilde{a}_0(N) \sim (1 + \sqrt{2})/2N$ as $N \to \infty$; see Proposition 2.7 below.

The case $N = 1$ of the following theorem is due to Okamoto [22], with the exception of the boundary value $a = \tilde{a}_0$, which was addressed by Kobayashi [16].

**Theorem 2.1.** (i) If $a_0^* \leq a < 1$, then $F_{N,a}$ is nowhere differentiable;

(ii) If $\tilde{a}_0 \leq a < a_0^*$, then $F_{N,a}$ is nondifferentiable almost everywhere, but is differentiable at uncountably many points;

(iii) If $a_{\min} < a < \tilde{a}_0$, then $F_{N,a}$ is differentiable almost everywhere, but is nondifferentiable at uncountably many points.

Moreover, if $F_{N,a}$ is differentiable at a point $x \in (0, 1)$, then $F_{N,a}'(x) = 0$. Statements (ii) and (iii) of the above theorem involve uncountably large sets of Lebesgue measure zero; it is of interest to determine their Hausdorff dimension. Let
\[D_0(a) := D_0^{(N)}(a) := \{x \in (0, 1): F_{N,a}'(x) = 0\}.\]

Define the functions
\[\phi_N(a) := \log((2N + 1)a)/\log(Na) - \log((N + 1)a - 1), \quad a_{\min} < a < 1,\]
and
\[h_N(p) := -\frac{1}{\log(2N + 1)} \left[ p \log\left( \frac{p}{N} \right) + (1 - p) \log\left( \frac{1 - p}{N + 1} \right) \right], \quad 0 < p < 1.\]

For a set $E \subset \mathbb{R}$, we denote the Hausdorff dimension of $E$ by $\dim_H E$.

**Theorem 2.2.** (i) If $a_0^* \leq a < a_0^*$, then $\dim_H D_0^{(N)}(a) = h_N(\phi_N(a))$;

(ii) If $a_{\min} < a \leq \tilde{a}_0$, then $\dim_H ((0, 1) \setminus D_0^{(N)}(a)) = h_N(\phi_N(a))$;

(iii) The function $a \mapsto h_N(\phi_N(a))$ is concave on $a_{\min} < a < a_0^*$, takes on its maximum value of 1 at $a = \tilde{a}_0$, and its limits as $a \downarrow a_{\min}$ and as $a \uparrow a_0^*$ are $\log(N + 1)/\log(2N + 1)$ and $\log N/\log(2N + 1)$, respectively.

Observe that for $N \geq 2$, $\dim_H D_0(a)$ has a discontinuity at $a_0^*$, where it jumps from $\log N/\log(2N + 1) > 0$ to 0; see Figure 4.
Figure 4: Graph of the function $h_N(\phi_N(a))$ from Theorem 2.2 for $N = 1$ (left) and $N = 2$ (right). While not clearly visible, the limits at the left endpoints are $\log 2/\log 3$ and $\log 3/\log 5$, respectively.

| $N$ | $a_{\text{min}}(N)$ | $\tilde{a}_0(N)$ | $a_0^*(N)$ | $\hat{a}_\infty(N)$ | $a_\infty^*(N)$ |
|-----|-----------------|-----------------|-------------|-----------------|-----------------|
| 1   | $\frac{1}{2}$  | .5000           | .5592       | $\frac{2}{3}$  | .5598           | .6180           |
| 2   | $\frac{1}{3}$  | .3333           | .3835       | $\frac{7}{15}$ | .4047           | $\frac{1}{2}$  | .5000           |
| 3   | $\frac{1}{4}$  | .2500           | .2914       | $\frac{5}{11}$ | .3444           | .3660           |
| 4   | $\frac{1}{5}$  | .2000           | .2349       | $\frac{13}{45}$| .2728           | $\frac{1}{3}$  | .3333           |
| 5   | $\frac{1}{6}$  | .1667           | .1967       | $\frac{8}{33}$ | .2534           | .2638           |
| 6   | $\frac{1}{7}$  | .1429           | .1692       | $\frac{19}{91}$| .2104           | $\frac{1}{4}$  | .2500           |
| 7   | $\frac{1}{8}$  | .1250           | .1484       | $\frac{11}{63}$| .2014           | .2071           |
| 8   | $\frac{1}{9}$  | .1111           | .1321       | $\frac{25}{153}$| .1724           | $\frac{1}{5}$  | .2000           |
| 9   | $\frac{1}{10}$ | .1000           | .1191       | $\frac{14}{95}$| .1674           | .1708           |
| 10  | $\frac{1}{11}$ | .0909           | .1084       | $\frac{31}{231}$| .1463           | $\frac{1}{6}$  | .1667           |

Table 1: Five important thresholds for $a$ determining differentiability of $F_{N,a}$
2.2 Infinite derivatives

For $x \in [0, 1)$, let

$$x = 0.\xi_1\xi_2\xi_3\cdots = \sum_{i=1}^{\infty} \xi_i(2N+1)^{-i}$$

denote the expansion of $x$ in base $2N+1$, so $\xi_i \in \{0, 1, \ldots, 2N\}$ for each $i$. When $x$ has two such expansions, we take the one ending in all zeros. Let

$$M(x) := \#\{i \in \mathbb{N} : \xi_i \text{ is odd}\}.$$ 

We also write $\omega_i := \xi_i/2$, and note that when $\xi_i$ is even, $\omega_i \in A := \{0, 1, \ldots, N\}$. For $d \in A$, write $\bar{d} := N - d$.

**Theorem 2.3.** Let $a_{\min} < a < 1$. A point $x \in (0, 1)$ satisfies $F'_{N,a}(x) = \pm \infty$ if and only if $M(x) < \infty$ and the following two limits hold:

$$\lim_{n \to \infty} ((2N+1)a)^n \left(1 - \sum_{j=1}^{\infty} a^j \omega_{n+j}\right) = \infty, \tag{4}$$

and

$$\lim_{n \to \infty} ((2N+1)a)^n \left(1 - \sum_{j=1}^{\infty} a^j \bar{\omega}_{n+j}\right) = \infty. \tag{5}$$

Assuming all these conditions are satisfied, $F'_{N,a}(x) = \infty$ if $M(x)$ is even, and $F'_{N,a}(x) = -\infty$ if $M(x)$ is odd.

While the conditions (4) and (5) may look complicated at first sight, readers familiar with $\beta$-expansions will recognize the summations appearing in them. For a real number $1 < \beta < N + 1$ and $x \in \mathbb{R}$, we call an expression of the form

$$x = \sum_{j=1}^{\infty} \frac{\omega_j}{\beta^j}, \quad \text{where } \omega_1, \omega_2, \cdots \in A \tag{6}$$

an expansion of $x$ in base $\beta$ over the alphabet $A$ (or simply, a $\beta$-expansion). Clearly such an expansion exists if and only if $0 \leq x \leq N/(\beta - 1)$. It is well known (see [25]) that almost every $x$ in this interval has a continuum of $\beta$-expansions. For the purpose of this article, we reduce the interval a bit further and consider the so-called univoque set

$$A_\beta := \{x \in ((N - \beta + 1)/(\beta - 1), 1) : x \text{ has a unique expansion of the form } (6)\}.$$
Let $\Omega := A^N$. For $\beta > 1$, let $\Pi_\beta : \Omega \to \mathbb{R}$ denote the projection map given by

$$\Pi_\beta(\omega) = \sum_{j=1}^{\infty} \frac{\omega_j}{\beta^j}, \quad \omega = \omega_1\omega_2\cdots \in \Omega,$$

so that (6) can be written compactly as $x = \Pi_\beta(\omega)$. Let $\sigma$ denote the left shift map on $\Omega$; that is, $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$. Define the set

$$U_\beta := \Pi_{\beta^{-1}}(A_\beta).$$

It is essentially due to Parry [23] (see also [1, Lemma 5.1]) that

$$\omega \in U_\beta \iff \Pi_\beta(\sigma^n(\omega)) < 1 \text{ and } \Pi_\beta(\sigma^n(\bar{\omega})) < 1 \text{ for all } n \geq 0,$$

and this, together with Theorem 2.3, suggests a close connection between the set

$$D_\infty(a) := D_\infty^{(N)}(a) := \{x \in (0, 1) : F_{N,a}'(x) = \pm \infty\}$$

and the univoque set $A_\beta$, where $\beta = 1/a$. The size of $A_\beta$ has been well-studied, starting with the remarkable theorem of Glendinning and Sidorov [11]. There are two pertinent thresholds, which we now define. First, for $N \in \mathbb{N}$, let

$$G(N) := \begin{cases} m + 1 & \text{if } N = 2m, \\ m + \sqrt{m^2 + 4m} & \text{if } N = 2m - 1. \end{cases}$$

Baker [3] calls $G(N)$ a \textit{generalized golden ratio}, because $G(1) = (1 + \sqrt{5})/2$.

Next, recall that the \textit{Thue-Morse sequence} is the sequence $(\tau_j)_{j=0}^\infty$ of 0’s and 1’s given by $\tau_j = s_j \mod 2$, where $s_j$ is the number of 1’s in the binary representation of $j$. Thus,

$$(\tau_j)_{j=0}^\infty = 0110\ 1001\ 1001\ 0110\ 0110\ 0110\ 1001\ \ldots$$

For each $N \in \mathbb{N}$, define a generalized Thue-Morse sequence $\tau^{(N)} = (\tau_i^{(N)})_{i=1}^\infty$ by

$$\tau_i^{(N)} := \begin{cases} m - 1 + \tau_i & \text{if } N = 2m - 1, \\ m + \tau_i - \tau_{i-1} & \text{if } N = 2m. \end{cases}$$

Finally, let $\beta_c := \beta_c(N)$ be the \textit{Komornik-Loreti constant} [18, 19]; that is, $\beta_c$ is the unique positive value of $\beta$ such that

$$\Pi_\beta \left(\tau^{(N)}\right) = 1.$$

The following theorem is due to Glendinning and Sidorov [11] for $N = 1$, and to Kong et al. [21] and Baker [3] for $N \geq 2$. 8
Theorem 2.4. The set $A_\beta$ is:

(i) empty if $\beta \leq G(N)$;
(ii) nonempty but countable if $G(N) < \beta < \beta_c(N)$;
(iii) uncountable but of Hausdorff dimension zero if $\beta = \beta_c(N)$;
(iv) of positive Hausdorff dimension if $\beta > \beta_c(N)$.

(In case (ii), there is a further threshold between $G(N)$ and $\beta_c(N)$ that separates finite and infinite cardinalities of $A_\beta$, but this is not relevant to our present aims.)

Now let $a^* := a^*(N) := 1/G(N)$, and $\hat{a}_\infty := \hat{a}_\infty(N) := 1/\beta_c(N)$. For a finite set $S$, let $S^*$ denote the set of all finite sequences of elements of $S$, including the empty sequence.

Theorem 2.5. (i) For all $a \in (\hat{a}_\infty, 1)$ and for almost all $a \in (a_{\min}, \hat{a}_\infty)$,

$$D_\infty(a) = \{\Pi_{2N+1}(v \cdot 2\omega) : v \in \{0,1,\ldots,2N\}^*, \omega \in U_{1/a}\},$$  \hfill (7)

where $2\omega := (2\omega_1,2\omega_2,\ldots)$ and $v \cdot 2\omega$ denotes the concatenation of $v$ with $2\omega$;

(ii) For all $a \in (a_{\min}, 1)$, $D_\infty(a)$ is a subset of the set in the right-hand-side of (7), and the inclusion is proper for infinitely many $a \in (a_{\min}, \hat{a}_\infty]$, including $\hat{a}_\infty$ itself;

(iii) For all $a \in (a_{\min}, 1)$,

$$\dim_H D_\infty(a) = \frac{\log(1/a)}{\log(2N+1)} \dim_H A_{1/a}. \hfill (8)$$

Theorem 2.5(i) says that for almost all $a$, the set $D_\infty(a)$ consists precisely of those points whose base $2N+1$ expansion is obtained by taking an arbitrary point $x$ having a unique expansion in base $\beta$ (where $\beta = 1/a$), doubling the base $\beta$ digits of $x$, and appending the resulting sequence to an arbitrary finite prefix of digits from $\{0,1,\ldots,2N\}$.

Corollary 2.6. The set $D_\infty(a)$ is:

(i) empty if $a \geq a^*_\infty$;
(ii) countably infinite, containing only rational points, if $\hat{a}_\infty < a < a^*_\infty$;
(iii) uncountable with Hausdorff dimension zero if $a = \hat{a}_\infty$;
(iv) of strictly positive Hausdorff dimension if \( a_{\text{min}} < a < \hat{a}_\infty \).

Moreover, on the interval \((a_{\text{min}}, \hat{a}_\infty)\), the function \( a \mapsto \dim_H D_\infty(a) \) is continuous and nonincreasing, and its points of decrease form a set of Lebesgue measure zero.

Regarding the relative ordering and the asymptotics of the five thresholds in Table 1, we have the following:

**Proposition 2.7.** (i) For each \( N \geq 5 \), we have

\[
a_{\text{min}}(N) < \tilde{a}_0(N) < a_0^*(N) < a_\infty^*(N).
\]

(ii) As \( N \to \infty \), we have

\[
Na_{\text{min}}(N) \to 1, \quad Na_0(N) \to \frac{1 + \sqrt{2}}{2} = 1.207 \ldots,
\]

\[
Na_0^*(N) \to \frac{3}{2}, \quad Na_\infty(N) \to 2, \quad Na_\infty^*(N) \to 2.
\]

It is interesting to observe that for \( N = 1 \), there is an interval of \( a \)-values (namely, \( a_\infty^* < a < a_0^* \)) for which \( \dim_H D_0(a) > 0 \) but \( D_\infty(a) = \emptyset \). In other words, for such \( a \) there are uncountably many points where \( F_{N,a} \) is differentiable, but no points where it has an infinite derivative. For \( 2 \leq N \leq 4 \) there is no such \( a \), but there is still an interval (namely, \( \hat{a}_\infty < a < a_0^* \)) for which \( \dim_H D_0(a) > 0 \) but \( D_\infty(a) \) is only countable. For all \( N \geq 5 \), however, \( \dim_H D_\infty(a) > 0 \) whenever \( \dim_H D_0(a) > 0 \).

### 3 Proofs of Theorems 2.1 and 2.2

Recall that for \( x \in [0,1) \), \( x = 0.\xi_1\xi_2\xi_3 \cdots \) denotes the expansion of \( x \) in base \( 2N+1 \). We first introduce some additional notation. For \( n \in \mathbb{N} \), let \( i(n) := i(n; x) \) denote the number of odd digits among \( \xi_1, \ldots, \xi_n \). Let \( x_{n,j} := j/(2N+1)^n \), and put \( I_{n,j} := [x_{n,j}, x_{n,j+1}) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( j \in \mathbb{Z} \). For \( x \in (0,1) \) and \( n \in \mathbb{N} \cup \{0\} \), let \( I_n(x) \) denote that interval \( I_{n,j} \) which contains \( x \).

The first important observation is that

\[
F_{N,a}(x_{n,j}) = f_n(x_{n,j}), \quad \text{for } j = 0, 1, \ldots, (2N+1)^n.
\]

Next, recall that \( b \) is the number such that \( (N+1)a - Nb = 1 \). The recursive construction of the piecewise linear approximants \( f_n \) implies that

\[
f_n'(x) = (2N+1)^na^n(-b)^i(n),
\]
at all $x$ not of the form $x_{n,j}$, $j \in \mathbb{Z}$. As a result,

$$\frac{f'_{n+1}(x)}{f'_n(x)} \in \{(2N+1)a, -(2N+1)b\},$$

and since neither of these two values equals 1, $f'_n(x)$ cannot converge to a nonzero finite number. Clearly, if $F'_{N,a}(x)$ exists, it must be equal to $\lim_{n \to \infty} f'_n(x)$ in view of (10). The only possible finite value of $F'_{N,a}(x)$, therefore, is zero.

**Lemma 3.1.** For $x \in (0, 1)$, $F'_{N,a}(x) = 0$ if and only if $\lim_{n \to \infty} f'_n(x) = 0$.

**Proof.** Only the “if” part requires proof. For simplicity, write $F := F_{N,a}$. Let $s_{n,j}$ denote the slope of $f_n$ on the interval $I_{n,j}$. An easy induction argument shows that

$$\frac{s_{n,j+1}}{s_{n,j}} \in \left\{ \frac{-a}{b}, \frac{b}{a} \right\}, \quad j = 0, 1, \ldots, (2N+1)^n - 2. \tag{12}$$

Furthermore,

$$x \in I_{n,j} \implies \min\{F(x_{n,j}), F(x_{n,j+1})\} \leq F(x) \leq \max\{F(x_{n,j}), F(x_{n,j+1})\}. \tag{13}$$

Now assume $f'_n(x) \to 0$. Fix $h > 0$, let $n$ be the integer such that $(2N+1)^{n-1} < h \leq (2N+1)^{-n}$, and let $j$ be such that $I_{n,j} = I_n(x)$. Then $x_{n,j} \leq x < x_{n,j+1}$ and $x_{n,j} \leq x + h < x_{n,j+2}$. If $x + h > x_{n,j+1}$, (13) gives

$$|F(x + h) - F(x)| \leq |F(x + h) - F(x_{n,j+1})| + |F(x_{n,j+1}) - F(x)| \leq (2N+1)^{-n} (|s_{n,j+1}| + |s_{n,j}|) \leq (1 + C)(2N+1)^{-n} |s_{n,j}|.$$

where $C := \max\{a/b, b/a\}$, and the last inequality follows from (12). If $x + h \leq x_{n,j+1}$, the same bound follows even more directly. Thus, we obtain the estimate

$$\left|\frac{F(x + h) - F(x)}{h}\right| \leq (2N+1)(1 + C)|f'_n(x)|,$$

showing that $F$ has a vanishing right derivative at $x$. By a similar argument, $F$ has a vanishing left derivative at $x$ as well, and hence, $F'(x) = 0$. \hfill \Box

Now define

$$l(x) := \liminf_{n \to \infty} \frac{i(n; x)}{n}, \quad x \in (0, 1),$$
and use (11) to write

\[ |f'_n(x)| = \left( (2N + 1)a \left( \frac{b}{a} \right)^{i(n)/n} \right)^n. \]

The significance of the function \( \phi_N(a) \) is that

\[ (2N + 1)a \left( \frac{b}{a} \right)^{\phi_N(a)} = 1. \]

Since \( 0 < b/a < 1 \), this last equation together with Lemma 3.1 implies

\[ \{ x \in (0, 1) : l(x) > \phi_N(a) \} \subseteq D_0(a) \subseteq \{ x \in (0, 1) : l(x) \geq \phi_N(a) \}. \tag{14} \]

**Proof of Theorem 2.1.** (i) Assume first that \( a \geq a_0^* \). Since \( a > b > 0 \), (11) yields

\[ |f'_n(x)| \geq \left( (2N + 1)b \right)^n = \left[ \frac{2N + 1}{N}((N + 1)a - 1) \right]^n \geq 1 \]

for every \( x \) not of the form \( k/(2N + 1)^n \), so \( f'_n(x) \neq 0 \) for such \( x \). Hence, \( F_{N,a} \) is nowhere differentiable.

(ii) and (iii): By Borel’s normal number theorem,

\[ \lim_{n \to \infty} \frac{i(n; x)}{n} = \frac{N}{2N + 1} \quad \text{for almost every} \ x \in (0, 1). \tag{15} \]

By definition of \( \tilde{a}_0 \), we have \( \phi_N(\tilde{a}_0) = N/(2N + 1) \). Moreover, \( \phi_N \) is monotone increasing on \( (a_{\min}, a_0^*) \). From these observations, it follows via (14) and (15) that \( D_0(a) \) has Lebesgue measure one if \( a_{\min} < a < \tilde{a}_0 \), and Lebesgue measure zero if \( a > \tilde{a}_0 \). Finally, the law of the iterated logarithm implies that for almost every \( x \in (0, 1) \), \( i(n; x)/n < N/(2N + 1) = \phi_N(\tilde{a}_0) \), and therefore \( |f'_n(x)| \geq 1 \), for infinitely many \( n \). (See [16] for more details in the case \( N = 1 \). Thus, \( D_0(\tilde{a}_0) \) has measure zero as well. The remaining statements follow from Theorem 2.2, which is proved below. \[ \square \]

In view of the relations (14), we define for \( p \in (0, 1) \) the sets

\[ R_p := \{ x \in (0, 1) : l(x) > p \}, \quad \bar{R}_p := \{ x \in (0, 1) : l(x) \geq p \}, \]

\[ S_p := \{ x \in (0, 1) : l(x) < p \}, \quad \bar{S}_p := \{ x \in (0, 1) : l(x) \leq p \}. \]
Lemma 3.2. We have
\[
\dim_H R_p = \dim_H \bar{R}_p = \begin{cases} 
1 & \text{if } 0 < p \leq N/(2N + 1), \\
\h_N(p) & \text{if } N/(2N + 1) \leq p < 1,
\end{cases}
\tag{16}
\]
and
\[
\dim_H S_p = \dim_H \bar{S}_p = \begin{cases} 
\h_N(p) & \text{if } 0 < p \leq N/(2N + 1), \\
1 & \text{if } N/(2N + 1) \leq p < 1.
\end{cases}
\tag{17}
\]
Proof. We prove (16); the proof of (17) is analogous. Since \(\bar{R}_p \supseteq R_p \supseteq \bar{R}_{p-\varepsilon}\) for all \(p > \varepsilon > 0\) and \(h_N\) is continuous in \(p\), it suffices to compute \(\dim_H \bar{R}_p\). First define, for nonnegative real numbers \(p_0, \ldots, p_{2N}\) with \(p_0 + \cdots + p_{2N} = 1\), the set
\[
F(p_0, \ldots, p_{2N}) := \left\{ x \in (0, 1) : \lim_{n \to \infty} \frac{\# \{ j \leq n : \xi_j(x) = i \}}{n} = p_i, \quad i = 0, 1, \ldots, 2N \right\}.
\]
It is well known (see, for instance, [9, Proposition 10.1]) that
\[
\dim_H F(p_0, \ldots, p_{2N}) = -\frac{1}{\log(2N + 1)} \sum_{i=0}^{2N} p_i \log p_i.
\tag{18}
\]
If \(0 < p \leq N/(2N + 1)\), then \(R_p\) has Lebesgue measure one by (15). Suppose \(N/(2N + 1) < p < 1\). Then \(\bar{R}_p\) contains the set
\[
F \left( \frac{1 - p}{N + 1}, \frac{p}{N + 1}, \ldots, \frac{1 - p}{N + 1} \right),
\]
which has Hausdorff dimension \(h_N(p)\) by (18). Therefore, \(\dim_H \bar{R}_p \geq h_N(p)\).

For the reverse inequality, we introduce a probability measure \(\mu\) on \((0, 1)\) as follows. Set
\[
\mu(I_n(x)) := \left( \frac{p}{N} \right)^{i(n;x)} \left( \frac{1 - p}{N + 1} \right)^{n-i(n;x)}.
\]
This defines \(\mu(I_{n,j})\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(j = 0, 1, \ldots, (2N + 1)^n - 1\) in such a way that \(\mu(I_{n,j}) = \sum_{\nu=0}^{2N} \mu(I_{n+1,(2N+1)^j+\nu})\) for all \(n\) and \(j\), and hence \(\mu\) extends uniquely to a Borel probability measure on \((0, 1)\), which we again denote by \(\mu\). It is a routine exercise that \(\mu\) concentrates its mass on the set \(\{ x : \lim_{n \to \infty} i(n;x)/n = p \}\), so in particular \(\mu(\bar{R}_p) = 1\). It now follows just as in the proof of [1] Lemma 4.2 that if \(s > h_N(p)\), then
\[
\limsup_{n \to \infty} \frac{\mu(I_n(x))}{|I_n(x)|^s} = \infty,
\]
where \(|I_n(x)|\) denotes the length of \(I_n(x)\). Using [9, Proposition 4.9], we conclude that \(\dim_H \bar{R}_p \leq h_N(p)\). \(\square\)
Proof of Theorem 2.2. Statements (i) and (ii) follow immediately from Lemma 3.2 and (14). The proof of (iii) is a straightforward calculus exercise.

4 Proof of Theorem 2.3

To avoid notational clutter we again write $F := F_{N,a}$. In order for $F$ to have an infinite derivative at $x$, it is clear that $f_n'(x)$ must tend to $\pm\infty$. By (11), this is the case if and only if $\xi_n$ is even for all but finitely many $n$. However, it turns out that this condition is not sufficient.

We begin with an infinite-series representation of $F$ (see [16] for a proof when $N = 1$):

$$F(x) = \sum_{n=1}^{\infty} a^{n-1-i(n-1)}(-b)^{i(n-1)}y_\xi_n,$$

where $y_0, y_1, \ldots, y_{2N}$ are the numbers used in the introduction to define $f_n$. In the special case when $\xi_n$ is even for every $n$, this reduces to

$$F(x) = \frac{1}{N} \sum_{n=1}^{\infty} a^{n-1}(1-a)\omega_n,$$

where $\omega_n := \xi_n/2$, and we have used that $i(a-b) = i(1-a)/N$ for $i = 0, 1, \ldots, N$. Let $F'_+$ and $F'_-$ denote the right-hand and left-hand derivative of $F$, respectively.

Lemma 4.1. Assume $\xi_n(x)$ is even for every $x$, and let $\omega_n := \xi_n(x)/2$. Then $F'_+(x) = \infty$ if and only if

$$\lim_{n \to \infty} \left( (2N+1)a \right)^n \left( 1 - \sum_{k=1}^{\infty} a^k \omega_{n+k} \right) = \infty. \quad (20)$$

Proof. For each $n \in \mathbb{N}$, let $j_n$ be the integer such that $x \in I_{n,j_n}$, and put $z_n := x_{n,j_n+2}$ (the right endpoint of $I_{n,j_n+1}$). In order that $F'_+(x) = \infty$, it is clearly necessary that

$$\lim_{n \to \infty} \frac{F(z_n) - F(x)}{z_n - x} = \infty. \quad (21)$$

The slope of $f_n$ on $I_{n,j_n}$ is $((2N+1)a)^n$, and by (12), the slope of $f_n$ on $I_{n,j_n+1}$ is $-(2N+1)^n a^{n-1}b$, independent of $\xi_n$. Therefore, the difference $F(z_n) - F(x)$ does not depend on $\xi_n$, and we may assume $\xi_n = 0$. Then $z_n = 0, \xi_1 \xi_2 \cdots \xi_{n-1} 1200 \cdots$, and
applied to \( x \) and \( z_n \) (noting that \( \omega_n(x) = 0 \) and \( \omega_n(z_n) = 1 \)) gives

\[
F(z_n) - F(x) = a^{n-1} \frac{1-a}{N} - \sum_{k=n+1}^{\infty} a^{k-1} \frac{(1-a)\omega_k}{N}
\]

\[
= a^{n-1}(1-a) \left( 1 - \sum_{k=1}^{\infty} a^k \omega_{n+k} \right).
\]

Since \( 1/(2N+1)^n < z_n - x \leq 2/(2N+1)^n \), it follows that (21) is equivalent to (20), showing that (20) is necessary. We now demonstrate that it is also sufficient.

Assume (20), or equivalently, (21). Then \( F(z_n) > F(x) \) for all large enough \( n \). Given \( h > 0 \), let \( n \in \mathbb{N} \) such that \( (2N+1)^{-n-1} < h \leq (2N+1)^{-n} \), let \( j := j_n \), and as before, \( z_n := x_{n,j+2} \). If \( x + h \geq x_{n,j+1} \), then, since \( f'_n < 0 \) on \( (x_{n,j+1}, x_{n,j+2}) \), we have by (13),

\[
\frac{F(x+h) - F(x)}{h} \geq \frac{F(z_n) - F(x)}{h} \geq \frac{F(z_n) - F(x)}{z_n - x} \to \infty,
\]

where the last inequality holds for all sufficiently large \( n \).

Assume now that \( x+h < x_{n,j+1} \). Then \( \xi_{n+1} = 2i \) for some \( i < N \), so \( x \in I_{n+1,(2N+1)j+2i} \) and \( z_{n+1} = x_{n+1,(2N+1)j+2i+2} < x_{n,j+1} \). Moreover, \( x_{n+1,(2N+1)j+2i+1} < x+h < x_{n,j+1} \). In view of (13) and the zig-zag pattern in the graph of \( f_{n+1} \), it follows that \( F(x+h) \geq F(z_{n+1}) \). Finally, \( h \leq (2N+1)(z_{n+1} - x) \). Combining these facts, we obtain, for all sufficiently large \( n \),

\[
\frac{F(x+h) - F(x)}{h} \geq \frac{F(z_{n+1}) - F(x)}{h} \geq \frac{F(z_{n+1}) - F(x)}{(2N+1)(z_{n+1} - x)}.
\]

Thus, by (21), \( F'_+(x) = \infty \). \( \square \)

**Proof of Theorem 2.3**. If \( x = j/(2N+1)^n \) for some \( n \in \mathbb{N} \) and \( j \in \mathbb{Z} \), then it follows immediately from (12) that \( F'_+(x) \) and \( F'_-(x) \) are of opposite signs (in fact, one \( +\infty \), the other \( -\infty \)), so \( F \) does not have an infinite derivative at \( x \). And since \( \omega_n = \xi_n = 0 \) for all but finitely many \( n \) in this case,

\[
\sum_{j=1}^{\infty} a^j \omega_{n+j} = \sum_{j=1}^{\infty} a^j N = \frac{aN}{1-a} > 1
\]

for all sufficiently large \( n \), so (5) fails.

Now assume that \( x \) is not of the form \( j/(2N+1)^n \), and let \( m := M(x) \). As already observed earlier, \( F \) does not have an infinite derivative at \( x \) if \( m = \infty \), so assume \( m < \infty \). Since \( F(1-x) = 1 - F(x) \), it follows that \( F'_-(x) = F'_+(1-x) \) when
at least one of these two quantities exists, and moreover, \( \xi_n(1 - x) = 2N - \xi_n(x) \)
and so \( \omega_n(1 - x) = \omega_n(x) \) when \( \xi_n(x) \) is even. Therefore, it suffices to show that
\[
F'(x) = \pm \infty \text{ if and only if (4) holds. If } m = 0, \text{ this is immediate from Lemma 4.1, so assume } m > 0. \text{ Choose } n_0 \text{ so that } \\
\xi_n \text{ is even for all } n \geq n_0, \text{ let } j \text{ be the integer such that } x \in I_{n_0,j} = [x_{n_0,j}, x_{n_0,j+1}). \text{ Then } x = x_{n_0,j} + (2N + 1)^{-n_0} \hat{x}, \text{ where } \hat{x} \in [0,1) \\
satisfies the hypothesis of Lemma 4.1, and } \\
M(x_{n_0,j}) = m. \text{ Note that (20) holds for } \hat{x} \text{ if and only if it holds for } x, \text{ since the condition is invariant under a shift of the sequence } (\xi_n). \text{ The graph of } F \text{ above } I_{n_0,j} \text{ is an affine copy of the full graph of } F, \text{ scaled horizontally by } (2N + 1)^{-n_0} \text{ and vertically by } a^{n_0 - mb}, \text{ and reflected top-to-bottom if } m \text{ is odd. Thus, } \\
F'(x) \text{ is infinite if and only if } F'(\hat{x}) \text{ is, with the same sign when } m \text{ is even, and the opposite sign when } m \text{ is odd.} \qed

5 Proofs of Theorem 2.5 and Corollary 2.6

Proof of Theorem 2.5. We will need the auxiliary sets
\[
\tilde{U}_\beta := \{ \omega \in U_\beta : \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\omega)) < 1 \text{ and } \limsup_{n \to \infty} \Pi_\beta(\sigma^n(\bar{\omega})) < 1 \}, \tag{22}
\]
for \( 1 < \beta < N + 1 \), as well as the family of affine maps
\[
\psi_{n,k}(x) := (2N + 1)^{-n}(x + k), \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, (2N + 1)^n - 1,
\]
and the function \( \Phi : \Omega \to [0,1] \) given by
\[
\Phi(\omega) := 2\Pi_{2N+1}(\omega), \quad \omega \in \Omega. \tag{23}
\]
Since \( a > a_{\min} \) implies that \( (2N + 1)a)^n \to \infty \), it follows from Theorem 2.3 that
\[
\bigcup_{n,k} \psi_{n,k}(\Phi(\tilde{U}_{1/a})) \subset D_{\infty}(a) \subset \bigcup_{n,k} \psi_{n,k}(\Phi(U_{1/a})), \tag{24}
\]
where the unions are over \( n \in \mathbb{N} \) and \( k = 0, 1, \ldots, (2N + 1)^n - 1 \). Note that the set on the far right of (24) is precisely the set on the right hand side of (17). It was shown in [2] that \( \tilde{U}_\beta = U_\beta \) for all \( 1 < \beta < \beta_c(N) \) and almost all \( \beta_c(N) < \beta < N + 1 \), so (24) yields (i) and the first part of (ii). It was further shown in [2] that there are infinitely many values of \( \beta \), including \( \beta_c(N) \), for which \( \tilde{U}_\beta \) is a proper subset of \( U_\beta \), and that for each such \( \beta \) and any given sequence \( (\theta_n) \) of positive numbers, there are in fact uncountably many \( \omega \in U_\beta \setminus \tilde{U}_\beta \) such that
\[
\liminf_{n \to \infty} \theta_n(1 - \Pi_\beta(\sigma^n(\omega))) < \infty.
\]
Taking $\theta_n = (2N + 1)a^n$ we obtain the second part of statement (ii).

To prove (iii), we consider Hausdorff dimension in the sequence space $\Omega$. For each $\beta > 1$, define a metric $\rho_\beta$ on $\Omega$ by $\rho_\beta(\omega, \eta) := \beta^{-\inf\{i: \omega_i \neq \eta_i\}}$ for $\omega = \omega_1\omega_2 \cdots$ and $\eta = \eta_1\eta_2 \cdots$. To avoid confusion, we reserve the notation $\dim_H$ for Hausdorff dimension in $\mathbb{R}$ and write $\dim_H^{(\beta)}$ for Hausdorff dimension in $\Omega$ induced by the metric $\rho_\beta$. Since for any two numbers $\beta_1, \beta_2 \in (1, \infty)$ we have

$$\rho_{\beta_2}(\omega, \eta) = (\rho_{\beta_1}(\omega, \eta))^{\log \beta_2 / \log \beta_1},$$

it follows in a straightforward manner that

$$\dim_H^{(\beta_1)} E = \frac{\log \beta_2}{\log \beta_1} \dim_H^{(\beta_2)} E, \quad E \subset \Omega. \quad (25)$$

We now make two important observations:

1. The restriction of $\Pi_\beta$ to $U_\beta$ is bi-Lipschitz with respect to the metric $\rho_\beta$ (see [13, Lemma 2.7] or [2, Lemma 2.2]);

2. The map $\Pi_{2N+1}$ is bi-Lipschitz on all of $\Omega$ with respect to $\rho_{2N+1}$. (This follows because $\Pi_{2N+1}$ maps the Cantor space $\Omega$ onto a geometric Cantor set in $[0, 1]$.)

Since bi-Lipschitz maps preserve Hausdorff dimension, these observations and (25) imply that for any $\beta \in (1, N + 1)$,

$$\dim_H \Pi_{2N+1}(U_\beta) = \dim_H^{(2N+1)} U_\beta = \frac{\log \beta}{\log (2N + 1)} \dim_H^{(\beta)} U_\beta = \frac{\log \beta}{\log (2N + 1)} \dim_H A_\beta.$$

Now it was shown in [2] that $\dim_H^{(\beta)} U_\beta = \dim_H^{(\beta)} \tilde{U}_\beta$ for all $1 < \beta < N + 1$. Thus, taking $\beta = 1/a$ and using (23), (24) and the countable stability of Hausdorff dimension, we obtain (8).

Proof of Corollary 2.6. From Theorems 2.4 and 2.5 it follows immediately that $D_\infty(a)$ is empty when $a \geq a^*_\infty$, nonempty but countable when $a^*_\infty < a < a^*_\infty$, of Hausdorff dimension zero when $a = a^*_\infty$, and of positive Hausdorff dimension when $a_{\min} < a < a^*_\infty$. The “moreover” statement of the theorem is a consequence of Theorem 2.5(iii) and [20, Theorem 2.6]. It remains to show that $D_\infty(a)$ contains only rational points when $a^*_\infty < a < a^*_\infty$, and is uncountable when $a = a^*_\infty$. The former follows from Theorem 2.5(ii) since, as pointed out in [11, 21], $U_\beta$ contains only eventually periodic sequences when $\beta < \beta_c$; the latter is a direct consequence of Theorem 2.3 and [2, Theorem 1.3].

Proof of Proposition 2.7. From Theorems 2.4 and 2.5 it follows immediately that $D_\infty(a)$ is empty when $a \geq a^*_\infty$, nonempty but countable when $a^*_\infty < a < a^*_\infty$, of Hausdorff dimension zero when $a = a^*_\infty$, and of positive Hausdorff dimension when $a_{\min} < a < a^*_\infty$. The “moreover” statement of the theorem is a consequence of Theorem 2.5(iii) and [20, Theorem 2.6]. It remains to show that $D_\infty(a)$ contains only rational points when $a^*_\infty < a < a^*_\infty$, and is uncountable when $a = a^*_\infty$. The former follows from Theorem 2.5(ii) since, as pointed out in [11, 21], $U_\beta$ contains only eventually periodic sequences when $\beta < \beta_c$; the latter is a direct consequence of Theorem 2.3 and [2, Theorem 1.3].
Proof of Proposition 2.7. (i) For the first two inequalities, let
\[ g_N(x) := N^{-N}(2N + 1)^{2(N+1)} x^{N+1} ((N+1)x - 1)^N, \]
and observe that \( g_N \) is strictly increasing for \( x \geq 1/(N+1) \), with \( g_N(\tilde{a}_0(N)) = 1 \). Since \( g_N(\tilde{a}_{\min}(N)) = 0 \), this gives the first inequality. Now for a constant \( c > 1 \), we can write
\[ g_N(c/N) = 2c(4c(c-1))^{N/2} \left( 1 + \frac{c}{(c-1)N} \right)^N. \tag{26} \]
Let \( c_0 := (1 + \sqrt{2})/2 \). Then \( 4c_0(c_0 - 1) = 1 \), so (26) shows that \( g_N(c_0/N) \geq 2c_0 > 1 \), and hence, \( \tilde{a}_0(N) < c_0/N \). Straightforward algebra shows that \( c_0/N < a^*_0(N) \) for every \( N \geq 5 \), establishing the second inequality. (Of course, by direct calculation, \( \tilde{a}_0(N) < a_0(N) \) also for \( N < 5 \).)

For the third inequality, observe that \( \tau_i^{(2m)} \leq m + 1 \), so the definition of \( \beta_c(N) \) implies, by summing a geometric series, that \( \beta_c(2m) \leq m + 2 \). Routine algebra gives
\[ a^*_0(2m) = \frac{6m + 1}{(2m + 1)(4m + 1)} < \frac{1}{m + 2} \leq \hat{a}_\infty(2m), \]
for all \( m \geq 4 \). In addition, a direct calculation shows that \( a^*_0(6) < \hat{a}_\infty(6) \); see Table 1. Similarly, \( \tau_i^{(2m-1)} \leq m \), so that \( \beta_c(2m - 1) \leq m + 1 \), and again by routine algebra,
\[ a^*_0(2m - 1) = \frac{3m - 1}{m(4m - 1)} < \frac{1}{m + 1} \leq \hat{a}_\infty(2m - 1), \]
for all \( m \geq 3 \). Thus, the third inequality holds for all \( N \geq 5 \). Finally, the last inequality follows since \( G(N) < \beta_c(N) \); see Baker [3].

(ii) The limits involving \( a_{\min}, a^*_0 \) and \( a^*_\infty \) are obvious, and the limit involving \( \hat{a}_\infty \) was established by Baker [3], who gives a finer analysis of the asymptotics of the threshold \( \beta_c(N) \). The second limit follows since, by (26) and the aforementioned fact that \( 4c_0(c_0 - 1) = 1 \), \( g_N(c/N) \to 0 \) for \( 1 < c < c_0 \), and \( g_N(c/N) \to \infty \) for \( c > c_0 \). \( \square \)

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