Reliability polynomials of consecutive-$k$-out-of-$n$:F systems have unbounded roots

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Abstract
This article studies the roots of the reliability polynomials of linear consecutive-$k$-out-of-$n$:F systems. We prove that these roots are unbounded in the complex plane, for any fixed $k \geq 2$. In the particular case $k = 2$, we show that the reliability polynomials have only real roots and highlight the closure of these roots by establishing their explicit formulas. We also point out that in this case, for any fixed $n$, the nonzero roots of the reliability polynomial are distinct numbers.

KEYWORDS
Beraha-Kahane-Weiss theorem, consecutive-$k$-out-of-$n$:F systems, Fibonacci polynomials, Jacobstahl polynomials, polynomial roots, two-terminal node reliability

1 INTRODUCTION

Computer reliability was properly established around the mid-50s when John von Neumann [24] on one side, and Moore and Shannon [18] on the other side, set up the two main lines of thought for “building reliable systems from unreliable components.” One line of research flourished from networks of gates [24] (the components being the well-known logical gates), while the other one from networks of devices [18] (the components being relays—hence equivalent to present day transistors). Moore and Shannon’s approach [18] founded a probabilistic graph model where nodes (devices) are perfectly reliable, while the undirected edges (connections/wires) fail independently with a probability $q = 1 - p$. The fundamental problem was (and still is) that of finding the probability that two (or more) nodes are connected, the solution being represented by the reliability polynomial $\text{Rel}(N, p)$ of a network $N$. The most common particular cases are represented by the all-terminal reliability polynomial (which gives the probability that any two nodes are connected), and the two-terminal reliability polynomial (which gives the probability that two fixed, particular nodes $s$ and $t$ called terminals, are connected, also known as $s$-$t$ connectivity).

Alternatively, one could consider that the nodes are the ones failing with some probability $q = 1 - p$, with edges being perfectly reliable. This case is known as node reliability, which can be either residual node connectedness [11] (the probability that at least one node is operational and all the operational nodes form a connected subgraph), or of the two-terminal type [4] (the probability that two terminals $s$ and $t$ are connected, supposing that the nodes $s$ and $t$ are always operational).

Although the reliability of a network $\text{Rel}(N, p)$ is defined for $p \in [0, 1]$, being a polynomial, it is completely characterized by its roots in the complex plane $\mathbb{C}$ [19]. As reliability of networks (as well as circuits, be they CMOS or quantum based) has lately taken center stage, the location of the roots of various reliability polynomials has become a highly explored topic in recent decades (see, for instance, [5-7, 13]).
The roots of all-terminal reliability polynomials were analyzed for the first time by Brown and Colbourn [5], who conjectured (in 1992) that all these roots lie inside the closed unit disk centered at $z = 1$. The conjecture was shown to be false 12 years later by very shy margins [20], but the original question posed by Brown and Colbourn [5] ”Can we find a bounded region that is guaranteed to contain all the roots of the [all-terminal] reliability polynomials?” is still open now [8, 14]. Since 2006 the roots of two-terminal reliability polynomials have also started to be scrutinized by Christian Tanguy, in a series of papers tackling particular networks [21-23]. Recently, Brown and DeGagné [6] proved that the closure of two-terminal roots contains the closed unit disks centered at $z = 0$ and $z = 1$, but the question if the two-terminal roots are bounded also remains an open problem [6, 14].

The roots of node reliability polynomials received far less attention. Still, Brown and Mol [7] proved that the roots of residual node connectedness reliability are unbounded and are dense in the entire complex plane, which is in stark contrast with the roots of all-terminal and two-terminal reliability polynomials (which, expectedly, are bounded).

As far as we know, the present article is the first one studying the roots of two-terminal node reliability polynomials. The particular networks we are going to consider here are known as linear consecutive-$k$-out-of-$n$: F systems. The linear consecutive-$k$-out-of-$n$: F system was introduced by Kontoleon [17] under the name of $r$-successive-out-of-$n$: F, being rebranded consecutive-$k$-out-of-$n$: F one year later [10]. It is defined as a system formed by $n$ components placed in a row (i.e., sequentially), which fails iff at least $k$ consecutive components fail. Linear consecutive-$k$-out-of-$n$: F systems are of interest as they can achieve high reliability [2] at reasonably low costs [1], and have been used for street lights and oil/gas pipelines, as well as in biology [12].

A linear consecutive-$k$-out-of-$n$: F system can be represented as an undirected graph $G = (V, E)$, with $n + 2$ vertices, $V = \{S = 0, 1, 2, \ldots , n, n + 1 = T\}$, and having the set of edges $E = \{ij, 0 \leq i < j \leq n + 1, j - i \leq k\}$ (Figure 1). The nodes $1, 2, \ldots , n$ are identical and statistically independent, failing with probability $q = 1 - p$, while the terminals $S = 0, T = n + 1$, as well as the edges, are always operational. The reliability of the system is the probability that the system is working, being expressed as a polynomial in $p$, denoted by $\text{Rel}(k, n; p)$.

For any fixed $k \geq 2$, the reliability polynomials of a linear consecutive-$k$-out-of-$n$ system can be computed by a recurrence relation [9], for all $n \geq k$. Using this recurrence relation and Beraha-Kahane-Weiss theorem, we prove, in Section 2, that the set of the roots of the polynomials $\text{Rel}(k, n; p)$ is unbounded, for any $k \geq 2$. In Section 3, we study the particular case of linear consecutive-$2$-out-of-$n$: F systems and prove that all the nonzero roots are real, distinct numbers belonging to $(-\infty, 0) \cup \left[\frac{4}{7}, \infty\right)$, which is the smallest closed set containing them (i.e., the closure of these roots).

2 CONSECUTIVE-k-OUT-OF-n:F SYSTEMS HAVE UNBOUNDED ROOTS

For any fixed $k \geq 2$, the reliability polynomials $R_n(p) := \text{Rel}(k, n; p)$ of linear consecutive-$k$-out-of-$n$: F systems satisfy the following recurrence relation [9]:

$$R_n(p) = pR_{n-1}(p) + pqR_{n-2}(p) + \cdots + pq^{k-1}R_{n-k}(p), \quad (1)$$

for any $n \geq k$. Since the system fails only if at least $k$ consecutive components fail, the first $k$ polynomials are equal to 1:

$$R_n(p) = 1, \quad n = 0, 1, \ldots , k - 1. \quad (2)$$

For any $n \geq 0$, the polynomial $R_n(p)$ can be written

$$R_n(p) = p^nP_n(z), \quad (3)$$

where

$$z = \frac{q}{p} = \frac{1 - p}{p}. \quad (4)$$

From (1) we find that the polynomials $P_n(z)$ satisfy the following recurrence relation, for any $n \geq k$:

$$P_n(z) = P_{n-1}(z) + zP_{n-2}(z) + \cdots + z^{k-1}P_{n-k}(z), \quad (5)$$

FIGURE 1  Linear consecutive-3-out-of-10:F system.
and from (2) we obtain the initial conditions:

\[ P_n(z) = (1 + z)^n, \quad n = 0, 1, \ldots, k - 1. \tag{5} \]

In the following, we present some basic results on the theory of recursively-defined polynomials. We consider an arbitrary sequence of polynomials \( \{P_n\}, n = 0, 1, \ldots \), defined by a linear recurrence relation of order \( k \):

\[ P_n(z) = \sum_{j=1}^{k} f_j(z) P_{n-j}(z), \quad n \geq k, \tag{6} \]

where \( f_j(z) \) and \( P_{j-1}(z), j = 1, 2, \ldots, k \) are given polynomials.

We say that \( x \in \mathbb{C} \) is a limit of zeros of \( \{P_n\} \) if there exists a sequence of complex numbers \( \{z_n\} \) such that \( P_n(z_n) = 0 \) and \( \lim_{n \to \infty} z_n = x \).

Let \( \lambda_j(z), j = 1, 2, \ldots, k \) be the roots of the characteristic equation

\[ \lambda^k - \sum_{j=1}^{k} f_j(z) \lambda^{k-j} = 0. \tag{7} \]

Then, for any \( z \in \mathbb{C} \) such that \( \lambda_i(z) \neq \lambda_j(z) \) for \( i \neq j \), one can write:

\[ P_n(z) = \sum_{j=1}^{k} a_j(z) \lambda_j(z)^n, \tag{8} \]

where \( a_j(z), j = 1, \ldots, k \) are found from the linear system of equations obtained by writing (8) for \( n = 0, \ldots, k - 1 \).

**Theorem 1** (Beraha-Kahane-Weiss theorem [3]). Suppose that \( \{P_n\} \) is a sequence of polynomials defined by a relation of the form (6) such that \( \{P_n\} \) satisfies no recursion of order less than \( k \), and there is no constant \( \omega \), with \( |\omega| = 1 \), such that \( \lambda_i(z) = \omega \lambda_j(z) \) for some \( i \neq j \).

Then \( x \) is a limit of zeros of \( \{P_n\} \) if and only if the roots of the characteristic equation (7) can be ordered such that one of the following holds:

(i) \( |\lambda_2(x)| > |\lambda_j(x)| \) for every \( j = 2, \ldots, k \), and \( a_1(x) = 0 \),

(ii) \( |\lambda_i(x)| = |\lambda_2(x)| = \cdots = |\lambda_{i+l}(x)| > |\lambda_j(x)| \), \( j = i + 1, \ldots, k \), for some \( l \geq 2 \).

The sequence of polynomials \( P_n(z) \) defined by the recurrence (4) of order \( k \), with the initial conditions (2), satisfies the conditions from Theorem 1. In our case, the characteristic equation (7) is written

\[ \lambda^k - \sum_{j=1}^{k} \omega_j^{k-j} \lambda^{k-j} = 0. \tag{9} \]

For \( z = -1 \), the characteristic equation (9) becomes

\[ \lambda^k - \lambda^{k-1} + \cdots + (-1)^{k-1} \lambda + (-1)^k = 0 \tag{10} \]

and its roots are the roots \( \neq -1 \) of the equation

\[ \lambda^{k+1} = (-1)^{k+1}. \]

Hence, the roots of the characteristic equation (10) are all distinct complex numbers, of modulus 1:

\[ |\lambda_1(-1)| = \cdots = |\lambda_k(-1)| = 1. \]

By Theorem 1 case (ii), there exists a sequence of complex numbers \( \{z_n\} \) such that \( P_n(z_n) = 0 \) and \( \lim_{n \to \infty} z_n = -1 \).

From the recurrence relation (4) one can easily deduce that

\[ P_n(z) = (1 + z)P_{n-2}(z) + (z + z^2)P_{n-3}(z) + \cdots + (z^{k-2} + z^{k-1})P_{n-k}(z) + z^{k-1}P_{n-k-1}(z). \tag{11} \]

hence,

\[ P_n(-1) = (-1)^{k-1}P_{n-k-1}(-1), \]

for every \( n \geq k + 1 \). Since \( P_0 = 1 \), we find that all the polynomials of the subsequence \( \{P_{m(k+1)}\}_{m \geq 1} \) do not have the root \( z = -1 \). [The same is true for the polynomials of the subsequence \( \{P_{m(k+1)-1}\}_{m \geq 1} \), while all the others have the root \( -1 \).] Therefore, the subsequence \( \{z_{m(k+1)}\}_{m \geq 1} \) of \( \{z_n\} \) tends to \(-1\) as \( m \to \infty \) and \( z_{m(k+1)} \neq -1 \). Since \( z_m = \frac{1-P_m}{P_m} \), it follows that

\[ p_{m(k+1)} = \frac{1}{z_{m(k+1)} + 1}, \quad m = 1, 2, \ldots \]

is a sequence of roots of the reliability polynomial \( \text{Rel}(k, n; p) \), with \( \lim_{m \to \infty} |p_{m(k+1)}| = \infty \) and the following theorem is proved.
3 CONSECUTIVE-2-OUT-OF-n:F SYSTEMS HAVE REAL, DISTINCT ROOTS

The two-terminal reliability polynomial of a linear consecutive-2-out-of-n: F system is [10]

\[ \text{Rel}(2, n; p) = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-j+1}{j} p^{n-j} (1-p)^j = p^\alpha \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-j+1}{j} \left( \frac{1-p}{p} \right)^j = p^\alpha P_n(z), \]

where

\[ z = \frac{1-p}{p} \]

and

\[ P_n(z) = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-j+1}{j} z^j, \quad n = 0, 1, \ldots \]

(see also (3)).

Any nonzero root of Rel(2, n; p) corresponds to a root \( z \neq -1 \) of the polynomial \( P_n(z) \); hence, it follows naturally that we should study the roots of \( P_n(z) \) defined by (14).

The polynomial \( P_n(z) \) satisfies the recurrence relation (4) for \( k = 2 \):

\[ P_n(z) = P_{n-1}(z) + z P_{n-2}(z), \quad n = 2, 3, \ldots, \]

with the initial conditions

\[ P_0(z) = 1, \quad P_1(z) = 1 + z. \]

The characteristic equation corresponding to recurrence relation (15) is

\[ \lambda^z - \lambda - z = 0 \]

having the roots

\[ \lambda_{1,2}(z) = \frac{1 \pm \sqrt{1 + 4z}}{2}. \]

So, for any \( n \geq 0 \) and \( z \neq -\frac{1}{4} \), \( P_n(z) \) can be written as follows:

\[ P_n(z) = a_1(z)(\lambda_1(z))^n + a_2(z)(\lambda_2(z))^n, \]

where \( a_1(z) \) and \( a_2(z) \) are determined from the initial conditions (16).

Using (19) for \( n = 0, 1 \) and the initial conditions (16) we find that

\[ a_1(z) = \frac{(\lambda_1(z))^2}{\lambda_1(z) - \lambda_2(z)} \quad \text{and} \quad a_2(z) = \frac{(\lambda_2(z))^2}{\lambda_1(z) - \lambda_2(z)}, \]

hence,

\[ P_n(z) = \frac{(\lambda_1(z))^{n+2} - (\lambda_2(z))^{n+2}}{\lambda_1(z) - \lambda_2(z)} = \frac{1}{\sqrt{4z + 1}} \left[ \frac{1 + \sqrt{4z + 1}}{2} \right]^{n+2} - \left( \frac{1 - \sqrt{4z + 1}}{2} \right)^{n+2}, \]

for every \( z \neq -\frac{1}{4} \) and \( n \geq 0 \). If \( z = -\frac{1}{4} \), then \( \lambda_1 \left( -\frac{1}{4} \right) = \lambda_2 \left( -\frac{1}{4} \right) = \frac{1}{2} \), so \( P_n \left( -\frac{1}{4} \right) = \frac{n+2}{2^{n+1}} \).

Using the form (20) of the polynomial \( P_n(z) \) we obtain the following formula for the reliability polynomial Rel(2, n; p):

\[ \text{Rel}(2, n; p) = p^n \frac{2^{n+2} \sqrt{3-3p}}{2^{3-3p} p} \left[ \left( 1 + \frac{3-3p}{p} \right)^{n+2} - \left( 1 - \frac{4-3p}{p} \right)^{n+2} \right]. \]

The roots of \( P_n(z) \) are the roots \( z \neq -\frac{1}{4} \) of

\[ \left( \frac{1 + \sqrt{4z + 1}}{1 - \sqrt{4z + 1}} \right)^{n+2} = 1. \]

If we denote by

\[ \omega_{nj} = \cos \frac{2j\pi}{n} + i \sin \frac{2j\pi}{n}, \quad j = 0, 1, \ldots, n-1, \]
the \( n \)-th roots of unity, the roots of Equation (22) can be written as follows:

\[
\zeta_{n,j} = -\frac{\alpha_{n+2,j}}{(1 + \alpha_{n+2,j})^2}, \quad j = 1, \ldots, n, \quad j \neq \frac{n+2}{2}.
\]

Now, using (23) and the identity \( \cos^2x = \cos^2(\pi - x) \), we get

\[
\zeta_{n,j} = -\frac{1}{4\cos^2\frac{\pi j}{n+2}}, \quad j = 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor.
\]

It follows that all the roots of the polynomial \( P_n(z) \) are real, distinct numbers in the interval \( (-\infty, -\frac{1}{4}) \). The set of all the roots \( \{ \zeta_{n,j}, j = 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor, n \geq 1 \} \) is dense in this interval.

We also remark that the polynomials \( J_1(z) = 1, J_n(z) = P_{n-2}(z) \) for \( n = 2, 3, \ldots \) are known as Jacobsthal polynomials, and are intimately related to the Fibonacci polynomials [16] defined by

\[
F_{n+2}(z) = z F_{n+1}(z) + F_n(z), \quad n = 1, 2, \ldots ,
\]

with the initial conditions \( F_1(z) = 1, \ F_2(z) = z \). Fibonacci and Jacobsthal polynomials have the same set of coefficients (see Table 1), being related as follows:

\[
F_n(z) = z^n J_n \left( \frac{1}{z^2} \right).
\]

It is known that the roots of the Fibonacci polynomials \( F_n(z) \) are given by the formula [15]:

\[
\zeta_{n,j} = 2i \cos \frac{j\pi}{n}, \quad j = 1, 2, \ldots, n - 1.
\]

Since \( P_n(z) = J_{n+2}(z), n = 0, 1, \ldots \), and using (27), we recover once again formula (25) for the roots \( \zeta_{n,j} \) of the polynomial \( P_n(z) \).

Now, by relying on Equations (13) and (25), the nonzero roots of \( \text{Rel}(2, n; p) \) can be written as follows:

\[
p_{n,j} = \frac{1}{1 + \zeta_{n+1,j}} = 1 + \frac{1}{4\cos^2\frac{\pi j}{n+2} - 1}, \quad j = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor, \quad j \neq \frac{n+2}{3},
\]

and so we have just proved Theorem 3.

**Theorem 3.** For any \( n \geq 2 \), the nonzero roots of the reliability polynomial \( \text{Rel}(2, n; p) \) are real, distinct numbers.

If \( \mathcal{R} \) denotes the set of all the roots of the polynomials \( \text{Rel}(2, n; p), n \geq 2 \), then the closure of \( \mathcal{R} \) is

\[
\overline{\mathcal{R}} = (-\infty, 0] \cup \left[ \frac{4}{3}, \infty \right).
\]

Figure 2 presents all the roots of the polynomials \( \text{Rel}(2, n; p) \) for the linear consecutive case, stacked on top of each other (as all of them are real numbers), the vertical axis being \( n = 2, \ldots, 100 \).

**Remark 1.** For each \( n \geq 2 \), the largest root of \( \text{Rel}(2, n; p) \) (in absolute value) is obtained for those \( j \neq \frac{n+2}{3} \) that minimize the denominator \( \cos^2\frac{j\pi}{n+2} - \frac{1}{4} \):

\[
j = \begin{cases} \left\lfloor \frac{n}{5} \right\rfloor + 1, & \text{if } n \not\equiv 1 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 1 \pmod{3} \end{cases}.
\]
4 | CONCLUSIONS

In this article, we have proved that the roots of linear consecutive-\( k \)-out-of-\( n \): \( F \) systems are unbounded. This implies that the roots of two-terminal node reliability polynomials are unbounded.

In particular, we have studied the reliability polynomials corresponding to linear consecutive-2-out-of-\( n \): \( F \) systems, showing that all their nonzero roots are distinct, real numbers contained into the union of intervals \((-\infty, 0] \cup [\frac{2}{3}, \infty)\), which is their closure (the smallest closed set containing them).

This property of having only real roots is specific to consecutive-2-out-of-\( n \): \( F \) systems. For \( k > 2 \) the reliability polynomials \( \text{Rel}(k, n; p) \) also exhibit complex roots, besides real ones. As an example, Figure 3 presents the roots of all the polynomials \( \text{Rel}(k, 16; p) \), \( k = 2, \ldots, 16 \). The green squares represent the (real) roots of \( \text{Rel}(2, 16; p) \).
The next step we plan to take is to have a closer look at circular consecutive-$k$-out-of-$n$: $F$ systems. This choice is motivated by simulations of both linear and circular consecutive-2-out-of-$n$: $F$ systems, which suggest that the roots of circular consecutive-2-out-of-$n$: $F$ systems are growing even faster than those of linear consecutive-2-out-of-$n$: $F$ systems (see Figure 4).

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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