One-dimensional long-range percolation: a numerical study

G. Gori, 1 M. Michelangeli, 2 N. Defenu, 2, 1 and A. Trombettoni 1, 2, 3

1 CNR-IOM DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy
2 SISSA, Via Bonomea 265, I-34136 Trieste, Italy
3 INFN, Sezione di Trieste, Via Bonomea 265, I-34136 Trieste, Italy

In this paper we study bond percolation on a one-dimensional chain with power-law bond probability \( C/r^{1+\sigma} \), where \( r \) is the distance length between distinct sites. We introduce and test an order \( N \) Monte Carlo algorithm and we determine as a function of \( \sigma \) the critical value \( C_c \) at which percolation occurs. The critical exponents in the range \( 0 < \sigma < 1 \) are reported and compared with mean-field and \( \epsilon \)-expansion results. Our analysis is in agreement, up to a numerical precision \( \approx 10^{-3} \), with the mean field result for the anomalous dimension \( \eta = 2 - \sigma \), showing that there is no correction to \( \eta \) due to correlation effects.

PACS numbers:

I. INTRODUCTION

Percolation is a paradigmatic model of statistical mechanics [1]. Despite its simplicity it is able to capture many features of real world phenomena ranging from coffee brewing to oil industry (see [2] for a recent review) and at the same time it allows for various exact results [3, 4]. One of the most remarkable features of percolation is to display phase transition phenomena with nontrivial geometric properties [5] prototyping in a simplified and essential stripped-to-the-bone setting features found in more sophisticated models.

A significant bulk of the work regarding percolation deals with models where probability of connecting two elements is zero beyond a given range, i.e. short-range (SR) models. The typical SR percolation refers to a non-vanishing probability to connect nearest neighbour elements or sites of a lattice [3, 4]. In SR percolation the one-dimensional limit is trivial: if one denotes by \( C \) the probability to connect with a bond two nearest neighbour sites one immediately sees that in one dimension the critical value \( C_c \) equals 1 (otherwise the two ends of the chain cannot be connected). This result parallels the well known results that for classical Ising and \( O(N) \) chains the critical temperature is vanishing for SR interactions [6, 7]. However it is well known that adding long-range (LR) couplings in the one-dimensional Ising model one may have a non vanishing critical temperature [8, 9]. Therefore in the literature it has been explored the percolation in presence of long-range (LR) interactions, for which one expects a percolative transition also in one dimension [10].

Systems where the elementary constituents affect each other with laws that decay slowly as a function of their mutual distance play a central role in physics including, among others, gravitational systems, unscreened plasmas and (di)polar systems [11]. A typical form of the interaction is \( \propto 1/r^{d+\sigma} \), where \( r \) is the distance between elements or sites of the system having dimension \( d \), and \( \sigma \) quantifies the range of the interactions.

The inclusion of LR interactions into percolation models, with bond probability of the form \( \propto 1/r^{d+\sigma} \), proves valuable also for modeling of several phenomena belonging to the wide arena of complex systems, such as disease spreading [12], social aggregation [13] and financial transactions [14]. A very interesting example of such studies is provided by epidemic spreading processes [15]. In the presence of SR infection, when no cooperative effects are considered and there is immunization or death after the infection, the epidemic process gives rise to ordinary percolation clusters [16]. The generalization to cases where the dynamical process takes place in lattices where at least some of the infections are LR was considered [17,19]. Further interest in the study of LR percolation was also triggered by papers giving some more exact results and its realization on finite graphs [20,21].

The consideration of LR systems presents in general many interesting features, with theoretical and numerical challenges. Indeed in a renormalization group (RG) setting the SR Wilson-Fisher fixed point can be altered by sufficiently strong LR interactions, leading at criticality to different universality classes. In particular the effect of LR interactions on a system in \( d \) dimensions has been related (even though the mapping should not be exact) to a system living in an effective dimension \( d_{\text{eff}} \geq d \).

The existence of such effective dimension relation between LR and SR systems comes from the low energy behavior of the critical propagator for LR interactions. Indeed using the conventional definition that the two point correlation
functions at criticality scales as $G(r) \sim r^{d-2+\eta}$, for a LR interacting system the anomalous dimension correction can be rewritten as $\eta = 2 - \sigma + \delta \eta$, where $2 - \sigma$ is the result obtained in the mean-field approximation and $\delta \eta$ is the correction due to correlation effects.

The expected anomalous dimension correction $\delta \eta$ was at the center of an long-lasting, intense debate. Classical early-day RG results on the Ising model conjectured $\delta \eta = 0$ to be valid at all orders [22]. Successive investigations complemented this result introducing the threshold value $\sigma_* = 2 - \eta_{SR}$, where $\eta_{SR}$ is the anomalous dimension of a SR model in the same dimension of the LR one under study [23]. At $\sigma = \sigma_*$, LR interactions become irrelevant with respect to SR ones [23]. These results, in agreement with Monte Carlo findings [24], have been questioned by numerical simulations on the two-dimensional LR Ising model indicating $\delta \eta \neq 0$ and $\sigma_* = 2 - \eta_{SR}$ [25, 26]. We also observe that Monte Carlo simulations support the presence of SR behaviour into a finite amount of the region $\sigma < 2$ for the 2D percolation [27], in agreement with the Sak’s result [23]. However recently presented results for susceptible-infected-removed epidemic processes with LR infection on a two-dimensional lattice appear to be in contradiction with the Sak estimate for $\sigma_*$, pointing out to the possibility that at least some of the critical exponents are different for all $\sigma < 2$ from those of ordinary SR percolation [19].

These results eventually led to several numerical and analytic investigations over LR spin models. However up to date numerical simulations on the two-dimensional LR Ising model confirmed the traditional scenario with $\delta \eta = 0$ and $\sigma_* = 2 - \eta_{SR}$ [28], also indicating an extremely slow convergence as a function of the system size for the critical amplitude of the standard Binder ratio of magnetization. Logarithmic corrections at the boundary value $\sigma_*$ are a possible source of error in numerical approaches. The occurrence of such logarithmic corrections at the threshold value $\sigma_*$ were confirmed by analytical results, which also indicated the results $\delta \eta = 0$ and $\sigma_* = 2 - \eta_{SR}$ [29]. Numerical results for the two-dimensional LR percolation were presented in [29]. The conformal invariance at criticality of the two-dimensional LR Ising model in dimension $d$ and in quantum LR chains [30] has been as well recently addressed. Moreover using both scaling arguments [31] and functional RG techniques [32] the results for the critical exponents of a LR interacting model in dimension $d$ with decay exponent $\sigma$ were related to the ones of a SR model in dimension $d = (2 - \eta_{SR})d/\sigma$. Such effective dimension relation was discussed in [33], finding very good agreement with numerical Monte Carlo results. One can also see that the effective dimension is valid only at low approximation level in the RG treatment, even though the error introduced by considering it exact has been quantified to be of $\lesssim 1\%$ for the critical exponent $\nu$ for the two-dimensional LR Ising models [30]. We observe that in one-dimensional LR Ising and percolation models Sak’s results implies $\sigma_* = 1$ (since $\eta_{SR} = 1$). A Berezinskii-Kosterlitz-Thouless (BKT) transition may be expected at $\sigma = 1$ for both one-dimensional Ising and percolation models, which has been deeply investigated in Ising chains [35] and also observed for susceptible-infected-removed epidemic processes with LR infection in one dimension [30].

Despite the wide appeal of the LR percolation model some important information is still missing. Indeed, for the equilibrium properties, no estimate of the critical threshold and of critical exponents for different values of the decay parameter $\sigma$ is available to date, at the best of our knowledge, for the LR one-dimensional percolation. The goal of the present paper is to give numerical estimates of these quantities. To this aim we develop a Monte Carlo algorithm allowing to efficiently simulate one-dimensional LR bond percolation. As a side result we can numerically check the validity of the Sak’s prediction [23] for the value of $\sigma_*$ in the one-dimensional LR percolation.

The paper is organized as follows: in Section II we present the model and briefly remind previous results relevant for our work. In Section III we present a novel order-$N$ algorithm for the study of LR percolation models and discuss its application to a LR version of Swendsen-Wang algorithm. In Sections IV and V we respectively present our findings for the critical thresholds and the critical exponents. In Section VII conclusions are drawn.

II. THE MODEL

We will consider a bond percolation model on a one-dimensional lattice having $N$ sites with occupation probability $p_{i,j}$ of the bond connecting sites $i$ and $j$ given by (with $i \neq j$):

$$p_{i,j} = p_{|i-j|} = \frac{C}{|i-j|^{1+\sigma}}$$

(1)

where $C \leq 1$. See Figure 1 for a graphical depiction of the model. When periodic boundary conditions are used, an appropriate periodic distance will be chosen, defined by the minimum distance among periodic images of the sites $i$ and $j$.

It is clear that if $\sigma \to \infty$ then the SR bond percolation with a bond probability $C$ between nearest neighbour sites is obtained. Notice that in the SR limit it is $C_c = 1$.

It is known that for the bond probability [1] one has $C_c = 1$ for $\sigma \geq 2$ [10]. Notice that if $p_{|i-j|}$ decays for large
\[|i - j| \propto \frac{1}{|i - j|^{1 + \sigma}},\] then one has \(C_c = 1\) for \(\sigma > 1\), but it is not necessarily \(C_c = 1\) for \(\sigma = 1\): in other words, the short distance behaviour of the probability \([1]\) may give rise to a percolation transition also at \(\sigma = 1\) \([10]\).

For future convenience we also mention the exact result by Schulman \([37]\): comparing the LR percolation model to a related percolation model on the Bethe lattice a rigorous lower bound for the critical threshold \(C_c\) was obtained

\[C_c \geq C_{\text{Bethe}}^c = \frac{1}{2\zeta(1 + \sigma)},\]

where \(\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}\) is the Riemann zeta function.

### III. THE ALGORITHM

The numerical study of percolation has largely benefitted from efficient clustering algorithms. Such algorithms are able to track the cluster to which the local degrees of freedom belong to. Since the inception of the efficient Hoshen-Kopelman \([38]\) relabeling scheme these algorithms have been brought closer to the optimal rigorous bounds predicted for union and find algorithm class to which they belong.

Simulation of LR models has always presented major challenges on its own. The fact that when we perform a move/update of a system degree of freedom we have to deal with the state of all the other degrees of freedom, unlike SR systems, typically brings the time for a global update of the system from \(O(N)\) to \(O(N^2)\) severely reducing performance.

Recent advances in this field has been brought by the introduction of algorithms scaling linearly in the size of the system \([39]\). The efficiency of these algorithms relies upon the fact that only a small fraction of local degrees of freedom are actually interacting. Indeed even if every local variable in principle interacts with all the other degrees of freedom it effectively does it only with a \(O(N)\) fraction of them (in the case of non-frustrated interacting systems this condition often translates in the request for an extensive energy).

Similarly, in our LR percolation model, the number of connected bonds, among the \(N^2\) available, is of the order \(O(N)\). More specifically consider the class of bonds of length \(l\): the probability \(\pi(n)\) of having \(n\) connected bonds out of the \(N\) available (with periodic boundary conditions) is a binomial

\[\pi_l(n) = \binom{N}{n} p_l^n (1 - p_l)^{N-n}\]

which average to \(Np_l\). If we sum all these probabilities we get a total number of connected links, in the thermodynamic limit, of \(N2C\zeta(1 + \sigma)\) for our model. The above consideration suggests the structure of an efficient \(O(N)\) algorithm that is \([40]\):

- extract, for each bond length \(l\) class, the connected bonds according to \([3]\). This can be done in a rejection-free fashion by sampling the skip \(s_l\) among two connected bonds which turn out to be distributed geometrically \(p(s_l) \propto (1 - p_l)^{s_l}\). The number of random numbers to be extracted is on average \(1 + p_l\)
- turn on the selected bond and cluster the sites according to the new connections. This can be done in constant time.

Note that the speedup can be traced back to having large classes of bonds with the same probability, which is the case for translational invariant systems (also with open boundary conditions). The above algorithm, albeit its simplicity, has not been discussed nor applied previously in the literature. Another remark is in order: the rejection free extraction of connected bonds plays a central role, since a more naive sampling (i.e. scanning all the elements) would spoil the overall order-\(N\) efficiency.
Interestingly this algorithm for LR bond percolation lends itself to effectively implement the Swendsen-Wang algorithm [41] in LR Ising or similar models. Indeed the Swendsen-Wang algorithm [41] the cluster construction can be dealt with the above scheme. In contrast to what is usually done one should first extract a number of tentatively connected bonds of length \( l \) via our algorithm with the appropriate probability, that is \( p_l = 1 - e^{-\beta J_l} \) where \( J_l \) is the interaction among spins \( l \) site apart, and later connect them just if they have the same sign. The cluster flipping part would be not altered. This provides an \( O(N) \) algorithm for LR Ising models. Although LR adaptations of Swendsen-Wang algorithms scaling as \( O(N \log N) \) [39] and also \( O(N) \) [42] have already appeared in the literature, we believe our formulation to be very concise and clear since the mentioned algorithms rely on binary search algorithms and Walker alias method respectively which, albeit well established, are more complicated to implement. Please notice however that our algorithm is well suited only for translational invariant systems, while other available algorithms, mentioned before, apply also to non translational invariant systems.

### IV. OBSERVABLES

Our estimates for the critical thresholds and critical exponents are based on a finite size scaling analysis of two quantities: where possible the analysis is done with both.

The first observable we have monitored consists of the average cluster size \( S \):

\[
S = \left\langle \frac{\sum_C \#_C^2}{N} \right\rangle.
\]

This quantity is the analogue of the susceptibility of an Ising model in the paramagnetic phase. Indeed it is expected to scale as

\[
S \propto N^{\gamma/\nu}
\]

at the critical point [24]. Since according to scaling relations, \( \gamma/\nu = 2 - \eta \) and \( \eta = 2 - \sigma + \delta \eta \) for a LR system, the scaling of \( S \) will be used to evaluate the correction \( \delta \eta \) to the power law decay of the correlation functions at criticality. In the following it will be shown that our results are in agreement with the widely accepted result \( \delta \eta = 0 \), which, as already mentioned, has been recently confirmed by extensive numerical simulations and theoretical investigations on the LR Ising model [28, 30, 34].

We have also introduced and studied the ratio:

\[
Q_G = \left\langle \frac{\sum_C \#_C^4}{(\sum_C \#_C^2)^2} \right\rangle
\]

where \( \#_C \) is the size of cluster \( C \), the sum runs over all clusters and the brackets denote average over the different realizations of percolation. \( Q_G \) measures the spread in the cluster size since when they have a characteristic finite size (in the non percolating phase) it tends to zero as we approach the thermodynamic limit; for \( C > C_c \), when a macroscopic (percolating) cluster develops, the ratio \( Q_G \) tends to one thus behaving as the Binder cumulant [43], that is widely used in magnetic systems. if we had considered the ratio of cluster size moments \( \langle \sum_C \#_C^q \rangle / (\sum_C \#_C^2)^{q/2} \) as in [44] we would have not observed any crossing [40], as insted it occurs for \( Q_G \). Moreover since the value of \( Q_G \) is ruled by the correlation length of the system we can put forward the following scaling form in the vicinity of \( C_c \), for a finite system of size \( N \):

\[
Q_G = Q_{G;c} + a_1(C - C_c)N^{1/\nu} + a_2(C - C_c)N^{2/\nu} + b_1 N^{\omega_1} + b_2 N^{\omega_2} + \ldots
\]

Our numerical investigations showed that from the crossing of \( Q_G \) it is possible to have a good estimate of the critical thresholds \( C_c(\sigma) \): to substantiate the use of \( Q_G \) we report in Appendix A the results for \( Q_G \) for the two-dimensional SR percolation on a square lattice, showing that the correct result can be retrieved. Regarding the critical exponents, it turns out that using \( S \) gives good estimates for the critical exponents (we determined \( \nu \) and \( \eta \)) for all the range \( 0 < \sigma < 1 \), while \( Q_G \) can be used for \( \sigma \) close to 1, where by the way it gives more precise estimates than the ones obtained by using \( S \). Both methods, i.e. using \( S \) and \( Q_G \) give results in agreement between each others (where one obtains reliable findings). Finally, we observe that using \( Q_G \) one can conclude that \( C_c = 1 \) for \( \sigma = 1 \) while using \( S \) one numerically confirms that \( C_c = 0 \) for \( \sigma = 0 \), as it should be.

The runs have scanned different values of \( \sigma, N \) and \( C \). For each value of the parameters we used \( 4 \cdot 10^6 \) different realizations. The considered size of the system is up to \( N = 512 \cdot 10^3 \). The random number generator chosen is the MT19973 [45]. The bulk of the experiments have been carried out with open boundary conditions, while for selected values of the decay parameter \( \sigma \) we have also simulated periodic boundary conditions.
V. CRITICAL THRESHOLDS

To determine the critical thresholds $C_C$ as a function of $\sigma$ we study the quantity $SN^\sigma$ which should cross at $C_C$ (this is consistent with the fact that $\delta\eta = 0$ and therefore $\gamma = \sigma \nu$, as it will be shown in the next Section). Indeed as one can see in Figure 2, the curves for different sizes meet at an almost $N$ independent point.

Another way to locate the phase transition is to use the cumulant $Q_G$ defined in (6). The behavior of the ratio $Q_G$ as a function of $C$ is depicted in the left panel of Figure 2 for the value of $\sigma = 0.5$. Increasing the system size $Q_G$ approaches a step profile jumping at the critical value of $C$ allowing for a precise determination of the critical point $C_c$. In the right panel of Figure 2 we show the same plot for $\sigma = 1$. As we can see the curves, within numerical error (see inset), do not cross for $C < 1$ supporting the theoretical prediction that $C_c = 1$ for the threshold value $\sigma = 1$.

Both methods devised agree for most values of $\sigma$ but the $Q_G$ crossing fails for low $\sigma$ because the crossing happens in parts of the curve with large curvature (at the sizes considered). For our best estimates of $C_c$ we will then rely on crossings of $SN^\sigma$. 

FIG. 2: (Left panel) $SN^\sigma$ as a function of $C$ for $\sigma = 0.25$. The lines refer to increasing (small) sizes $N = 10^3, 2 \cdot 10^3, 4 \cdot 10^3, 8 \cdot 10^3$, and $16 \cdot 10^3$ in the sense of the black arrow. Errors are smaller than the size of the lines. In the inset a zoom of the figure around the crossing point is shown: the curves refer to sizes $N = 10^3, 2 \cdot 10^3, 4 \cdot 10^3, 8 \cdot 10^3, \ldots, 2^9 \cdot 10^3$ with the curves becoming steeper as the system size is enlarged. (Right panel) Same as left panel with $\sigma = 0$. As it is apparent, no crossing occurs.

FIG. 3: (Left panel) Ratio $Q$ as a function of $C$ for $\sigma = 0.5$. The lines refer to increasing (small) sizes $N = 10^3, 2 \cdot 10^3, 4 \cdot 10^3, 8 \cdot 10^3$, and $16 \cdot 10^3$ in the sense of the black arrow. Errors are smaller than the size of the lines. The inset shows the curves in the vicinity of $C = 1$ for larger system sizes (the same of the inset of Figure 2) as used to locate the transition. (Right panel) Same as left panel with $\sigma = 1$. In the inset a zoom around the $C = 1$ point is reported: one can notice that the curves at the level of precision available do not show any crossing.
TABLE I: Estimated critical thresholds $C_c$ different values of decay parameter $\sigma$. Values of the critical exponent $\nu$ are also reported. See text for details.

| $\sigma$ | $C_c$ | $\nu$ | $\nu(Q_G$-est.) | $\sigma$ | $C_c$ | $\nu$ | $\nu(Q_G$-est.) |
|----------|-------|-------|------------------|----------|-------|-------|------------------|
| 0.05     | 0.0243486(7) | 0.0497(2) | -               | 0.5     | 0.25482(5) | 0.3503(12) | -               |
| 0.1      | 0.047685(8)  | 0.1006(5) | -               | 0.05    | 0.289410(13) | 0.3532(13) | -               |
| 0.15     | 0.070482(2)  | 0.1503(3) | -               | 0.6     | 0.327098(6)  | 0.3512(15) | -               |
| 0.2      | 0.093211(16) | 0.2016(7) | -               | 0.65    | 0.368333(13) | 0.3495(5)  | -               |
| 0.25     | 0.11638(3)   | 0.2474(7) | -               | 0.7     | 0.413752(14) | 0.341(2)   | -               |
| 0.3      | 0.140546(17) | 0.305(8)  | -               | 0.75    | 0.464202(19) | 0.3293(11) | -               |
| $1/3$    | 0.15736(6)   | 0.340(4)  | -               | 0.8     | 0.521001(14) | 0.319(2)   | 0.3161(4)       |
| 0.35     | 0.16610(5)   | 0.344(3)  | -               | 0.85    | 0.586264(10) | 0.299(2)   | 0.2953(7)       |
| 0.4      | 0.193471(15) | 0.355(9)  | -               | 0.9     | 0.66408(7)   | 0.258(4)   | 0.262(2)        |
| 0.45     | 0.22293(6)   | 0.363(12) | -               | 0.95    | 0.76501(9)   | 0.200(8)   | 0.207(7)        |

FIG. 4: Critical thresholds $C_c$ as a function of $\sigma$. The dotted line is the rigorous lower bound \[2\].

The results of the above analysis yield the critical threshold as a function of $\sigma$ reported in the first column of Table I and for convenience in Figure 4 where we also report the Bethe lower bound \[2\]. We notice that as $\sigma$ approaches zero the estimated $C_c$ follows $C_{Bethe}^c$ closer and closer; this is especially true in the classical region $\sigma < d/3$. On the other side as $\sigma \to 1$ the critical threshold $C_c$ approaches the expected value of 1.

VI. CRITICAL EXPONENTS

For the LR percolation model under exam we expect the existence of a SR effective dimension $d_{eff}$ such that the universal quantities of the LR model can be related to the ones of the corresponding SR in $d_{eff}$ dimensions \[30, 34, 46\]. Such effective dimension relation reproduces standard scaling arguments and it reproduces correctly the qualitative features of phase diagrams \[30\]. Also effective dimensional approach are very accurate close to the mean field region even if generally not exact.

We have that for $\sigma < d/3$ the effective dimension relation is $d_{eff} > 6$ which is the mean-field region for the SR percolation problem \[47\]. Moreover, in $d = 1$ for $\sigma > 1$ we have $d_{eff} < 2$ and the phase transition vanishes. These results are in agreement with the ones already present in LR percolation literature \[37, 48\].

Our numerical results for $\delta \eta$ and $\nu$ as a function of $\sigma$ are reported in Figures 5 and 6 respectively. It is clear that the Sak prediction is rather clearly confirmed. The values of $\delta \eta$ have been obtained by looking for crossing of the curves $SN^{-\gamma/\nu} = SN^{-\sigma+\delta \eta}$ allowing for a nonzero value of $\delta \eta$. The best value has been chosen as the one giving the most constant value of the crossing value $C$ for the largest sizes $N \geq 16 \cdot 10^3$ considered. Since we found values of $\delta \eta$ consistent with zero up to 1.5 standard deviations we assumed $\eta = 2 - \sigma$ in the subsequent analysis. Our best estimates of $\nu$ (reported in the third column of table I) rely on the behaviour of $SN^{-\sigma}$ near $C_c$. Standard scaling arguments lead us to expect the following scaling: $SN^{-\sigma} = \text{const.} + a_1(C - C_c)N^{1/\nu} + a_2(C - C_c)N^{2/\nu} + a_3(C - C_c)N^{3/\nu} \ldots$;
the analysis of the data indeed confirms this expectation. Notice that in contrast to the scaling for $Q_G$ (see (7)) no subleading exponents $\omega_i$ were required to explain the data. The region around the value $\sigma = 1/3$ requires a higher number of powers of $(C - C_c)$ to be included in the fit in order to keep $\chi^2_{3,0,\omega} < 1.5$ yielding a higher error on the estimates for $1/\nu$. These data are given in the third column of Table I. Finally an independent estimate of $1/\nu$ has been calculated via the more involved scaling (7) of $Q_G$ (reported in the fourth column of Table I). The values obtained are reliable only in a small region around $0.8 \leq \sigma < 1$ but do confirm the values retrieved with the observable $S$.

In order to benchmark our numerical results for $\nu$ it would be useful to have analytical predictions for it. However RG (and functional RG) calculations in LR percolation problems are less straightforward than the corresponding ones in LR Ising and $O(N)$ models [30]. The derivation of perturbative or non-perturbative approximated formulas goes beyond the scope of the present work. Nevertheless, it exists a shorthand to obtain approximated reference results to control the validity of our findings, relying on the effective dimension relation between LR and SR models. Indeed scaling arguments used to derive effective dimension relation $d_{\text{eff}}$ are based on the field theory description and are thus also valid for the percolation problem [17].

In a very recent paper [49] $\varepsilon$-expansion formulas on $\phi^3$ theory up to order $\varepsilon^4$ were derived and numerical estimates for the critical exponents in integer dimensions of SR percolation were given. We then can use the previously mentioned LR-SR effective dimension relation. According to scaling arguments and low order RG approximations the critical exponents of a LR model in dimension $d$ with exponent $\sigma$ are related to the ones of a SR model in dimension $d_{\text{eff}}$ using the relation

$$d_{\text{eff}} = \frac{(2 - \eta_{SR})d}{\sigma}. \quad (8)$$

The latter result was successfully applied in the case of two-dimensional LR Ising [30, 34] and LR $O(N)$ models [30]. In these works the correlation length exponent $\nu$ of the LR model was compared to the one of a SR model in dimension $d_{\text{eff}}$ using the relation

$$\nu = \frac{(2 - \eta_{SR})}{\sigma} \nu_{SR}(d_{\text{eff}}). \quad (9)$$

However we expect our case to be rather different. Indeed in the two-dimensional case both the LR and SR systems are characterized by a spontaneous symmetry breaking of a $\mathbb{Z}_2$ symmetry. On the other hand the one-dimensional LR Ising model shows spontaneous symmetry breaking only for $\sigma < 1$, while for $\sigma = 1$ it undergoes a topological phase transition of the BKT type [8], where kink excitations having logarithmic interactions [50] play a role similar to the vortices in the standard SR XY model [51].

The presence of BKT type phase transition is normally associated with a divergence of the correlation length exponent $\nu$ and then we expect the correlation length exponent of the $d = 1$ LR Ising model to go to infinity as a function of $\sigma$ in the $\sigma \to 1$ limit, as it happens for the SR $O(N)$ models at $d = 2$ in the $N \to 2$ limit [52]. This result should not be reproduced by the effective dimension relation (9) since the SR Ising model should have finite correlation length exponent in any dimension.

The case of LR percolation in one dimension is analogous to the case of the one-dimensional Ising model, indeed the percolation is the $q \to 1$ limit of the Potts model [17]. Using the results of [50] in the limit $q \to 1$ one expects BKT behavior for LR percolation at $\sigma = 1$. Then we expect our data to agree with the ones obtained from the SR results via relation (9) only far from $\sigma = 1$ and close to the mean-field threshold $\sigma = \frac{1}{2}$. We can then discard anomalous dimension corrections in the SR models and use the simplified mapping

$$\nu = \frac{2}{\sigma} \nu_{SR}(d_{\text{eff}}). \quad (10)$$

with $d_{\text{eff}} = \frac{2d}{\sigma}$ which is the $N \to \infty$ result [30].

Therefore perturbative expressions for the critical exponents of LR percolation can be obtained imposing the relation between $d_{\text{eff}}$ and $\varepsilon$ in the second order $\varepsilon$-expansion formulas obtained in [5] and contained in [49] (notice that in [49] it is $d_{\text{eff}} \equiv 6 - 2\varepsilon$). Defining $\tilde{\varepsilon} = \sigma - 1/3$, valid in the one-dimensional system, and expanding to second order in $\tilde{\varepsilon}$ we obtain

$$\nu^{-1} \equiv \frac{1}{3} + \frac{2}{7} \tilde{\varepsilon} - \frac{653}{343} \tilde{\varepsilon}^2 + O(\tilde{\varepsilon}^3) \quad (11)$$

which is plotted in Figure 6 as a gray line.

Moreover in [49] results for the critical exponents are presented for integer dimensions using a Padé approximant of $O(\varepsilon^4)$ expansion results constrained to reproduce the known exact values of the two-dimensional SR percolation. Interpolating such results a curve $\nu_{SR}(d)$ for the correlation length exponent of the SR model as a function of the dimension $d$ is obtained. This interpolation function is then transformed according to relation (10) with $d_{\text{eff}} = \frac{2d}{\sigma}$
yielding the results shown in Figure 5 (green line). We observe that if we instead use the relations (8) and (9) we obtain practically the same results for $1/3 \lesssim \sigma \lesssim 0.6$ and worst results approaching $\sigma = 1$ for the reasons explained above, while as expected poor results are obtained if non-resummed series for $\nu$ are used.

A reasonable agreement is found with the numerical results even if it does not match with the numerical errors. This issue was expected due to the presence of logarithmic corrections at $\sigma = 1/3$, which have not been included in our scaling assumptions and then lead to lack of accuracy close to the mean-field region. It is worth noting that logarithmic effects rapidly vanish since for $\sigma > \sim 0.4$ our results coincide with the RG ones, at least for $\sigma \lesssim 0.5$ where perturbative expansion is expected to be still reliable.

Close to the boundary region $\sigma = 1$ we can compare with the results of [49] which we generalized to the $\sigma \neq 1$ case following the procedure of [53]. Thus we obtain an analytic result for the correlation length exponent $\nu$ close to the BKT point $\sigma = 1$:

$$\nu^{-1} = \sqrt{\frac{1 - \sigma}{8q}} \left( q - 2 + \sqrt{4 + 12q + q^2 + (q - 2)\sqrt{2q(1 - \sigma)}} \right),$$  \hspace{1cm} (12)

where the limit $q \to 1$ has to be taken.

Our results for the exponent $\nu$ are summarized in Figure 6.

![Figure 5](image)

**FIG. 5:** Estimated values for $\eta$ together with predicted values from the RG prediction $\eta = 2 - \sigma$ [23]. Inset: deviation from the RG prediction $\eta = 2 - \sigma$.

**VII. CONCLUSIONS**

In this paper we developed, tested and used an order-N algorithm to study one-dimensional LR bond percolation. As a function of the power $\sigma$ of the decay of the bond probability we determined the critical threshold for percolation and the critical exponents $\eta$ and $\nu$. A precise estimate of the critical threshold is obtained by introducing a suitable geometric cumulant $Q_G$: comparison with exact results where available is reported. The value $\sigma_*$ above which no percolation occurs is retrieved to be 1. For the critical exponent $\eta$ we compared our findings with the RG result $\eta = 2 - \sigma$ by Sak [23], getting agreement at the level of $10^{-3}$ precision. The exponent $\nu$ is compared with mean-field results and $\varepsilon$-expansion for the short-range percolation at an approximate effective dimension. For $\sigma < 1/3$ agreement with the mean-field result for $\nu$ is found, while the $\varepsilon$-expansion provides a reasonable estimate in the region $1/3 \lesssim \sigma \lesssim 0.5$. Close to $\sigma = 1$, where a Berezinskii-Kosterlitz-Thouless (BKT) transition is expected, an expansion in $1 - \sigma$ has been derived.

An interesting future work would be to explore more in detail the region close to $\sigma = 1$ and in particular to consider different short-distance behaviour of the bond probability with non trivial percolation thresholds $C_c$ at $\sigma = 1$. The extension of the results presented in this paper to two-dimensional LR percolation may provide a worthwhile subject of future investigations: we expect that the critical exponents only depend on the ratio $d/\sigma$. Finally, we mention that with the algorithm discussed in this paper one can study the occurrence (or absence) of conformal invariance at criticality in two-dimensional LR percolation and the corresponding problem in one dimension.
FIG. 6: Estimated values for $1/\nu$ (circles) together with predicted values from RG in the classical region ($\sigma < 1/3$, blue continuous line) and the results for the nonclassical region ($1/3 < \sigma < 1$); the red dashed line on the right of the figure is the prediction (12) (dashed red) valid close to the BKT point, while the gray and green lines are obtained the effective dimension (9) with the second (dotted gray, bottom) and fourth (dash-dotted green, top) $\varepsilon$-expansion results (11) taken from [49].

Acknowledgement: We gratefully acknowledge discussions with Miguel Ibáñez Berganza, Luca Lepori, Stefano Ruffo, Slava Rychkov and Hirohiko Shimada. Hospitality in the Program "Conformal Field Theories and Renormalization Group Flows in Dimensions $d > 2$" at the Galileo Galilei Institute for Theoretical Physics, Florence (Italy), where part of this work was performed, is gratefully acknowledged. GG and AT acknowledge funding support from ”Progetto Premiale Anno 2012 ABNANOTECH - Atom-based technology”.

Appendix A: The ratio $Q_G$ for short-range two-dimensional percolation

In order to assess the validity of the ratio $Q_G$ introduced in Section [14] for characterizing a percolation transition we examine its behavior for a well known percolation problem. Bond percolation on a square lattice is known to occur, due to symmetry arguments, at an occupation probability of $p = p_c = 1/2$. In the left panel of figure [7] we plot $Q_G$ for different system sizes. As we can see the intersections approach the exact value. If we simply take the best value for our estimate of $p_c$ as the intersection of the two largest systems considered (i.e. $128 \times 128$ and $256 \times 256$) we get $p_c = 0.499997(4)$. If we consider the crossing of the curves $SN^{-\gamma/\nu}$ instead, shown in the right panel of figure [7] with the known value of $\gamma/\nu = 43/24$ we obtain the slightly less precise estimate $p_c = 0.4997(3)$.

[1] S. R. Broadbent and J. M Hammersley, Proc. Cambr. Phil. Soc. 53, 629 (1957).
[2] N. A. M. Araújo, P. Grassberger, B. Kahng, K. J. Schrenk, and R. M. Ziff, Eur. Phys. J. Spec. Top. 223, 2307 (2014).
[3] G. R. Grimmett, Percolation (Berlin, Springer-Verlag, 1999).
[4] B. Bollobás and O. Riordan, Percolation (Cambridge, Cambridge University Press, 2006).
[5] J. W. Essam, Rep. Prog. Phys. 43, 833 (1980).
[6] G. Mussardo, Statistical field theory: an introduction to exactly solved models in statistical physics (Oxford, Oxford University Press, 2010).
[7] H. Nishimori and G. Ortiz, Elements of phase transitions and critical phenomena (Oxford, Oxford University Press, 2011).
[8] D. J. Thouless, Phys. Rev. 187, 732 (1969).
[9] F. J. Dyson, Commun.Math. Phys. 21, 269 (1971).
[10] M. Aizenman, C.M. Newman, Commun. Math. Phys. 107, 611 (1986).
[11] A. Campa, T. Dauxois, D. Fanelli, and S. Ruffo, Physics of long-range interacting systems, (Oxford, Oxford University Press, 2014).
[12] P. Grassberger, Math. Biosci. 63, 157 (1983).
[13] S. Solomon, G. Weisbuch, L. de Arcangelis, N. Jan, and D. Stauffer, Physica A, 277, 239 (2000).
[14] D. Stauffer, Adv. Complex Syst. 04, 19 (2001).
[15] M. E. J. Newman, Phys. Rev. E 66, 016128 (2002).
FIG. 7: (Left panel) Ratio \( Q_G \) as a function of \( C \) for \( \sigma = 1/2 \) for the SR bond percolation on a square lattice. The lines refer to increasing sizes \( N = 32, 64, 128, \) and \( 256 \) in the sense of the black arrow. Errors are smaller than the size of the lines. (Right panel) \( SN^{\gamma/\nu} \) as a function of \( p \) for the same sizes in the left panel.

[16] D. Mollison, J. R. Stat. Soc. B 30, 283 (1977).
[17] M. Biskup, Ann. Prob. 19, 2938 (2004).
[18] T. Emmerich, A. Bunde, S. Havlin, L. Guanlian, and L. Daqing, arXiv:1206.5710
[19] P. Grassberger, J. Stat. Phys. 153, 289 (2013).
[20] I. Benjamini and N. Berger, Random Struct. Alg. 19, 102 (2001).
[21] D. Coppersmith, D. Gamarnik, and M. Sviridenko, The Diameter of a Long-Range Percolation Graph, in Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms (Philadelphia, Society for Industrial and Applied Mathematics, 2002).
[22] M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).
[23] J. Sak, Phys. Rev. B 8, 281 (1973).
[24] E. Luijten and H. W. J. Bl"ote, Phys. Rev. Lett. 89, 2 (2002).
[25] M. Picco, arXiv:1207.1018
[26] T. Blanchard, M. Picco, M. A. Rajabpour, Europhys. Lett. 101, 56003 (2013).
[27] F. Linder, J. Tran-Gia, S. R. Dahmen, and H. Hinrichsen, J. Phys. A 41, 185005 (2008).
[28] T. Horita, H. Suwa, and S. Todo, arXiv:1605.09496
[29] E. Brezin, G. Parisi, and F. Ricci-Tersenghi, J. Stat. Phys. 157, 855 (2014).
[30] N. Defenu, A. Trombettoni, and A. Codello, Phys. Rev. E 92, 052113 (2015).
[31] M. F. Paulos, S. Rychkov, B. C. van Rees, and B. Zan, Nucl. Phys. B 902, 246 (2016).
[32] L. Lepori, D. Vodola, G. Pupillo, G. Gori, and A. Trombettoni, Ann. Phys. 374, 35 (2016).
[33] L. Lepori, A. Trombettoni, and D. Vodola, arXiv:1607.05358
[34] M. C. Angelini, G. Parisi, and F. Ricci-Tersenghi, Phys. Rev. E 89, 062120 (2014).
[35] E. Luijten and H. Meffingfeld, Phys. Rev. Lett. 86, 5305 (2001).
[36] P. Grassberger, J. Stat. Mech P04004 (2013).
[37] L. S. Schulman, J. Phys. A 16, L639 (1983).
[38] J. Hoshen and R. Kopelman, Phys. Rev. B 14, 3438 (1976).
[39] E. Luijten and H. W. J. Blöte, Int. J. Mod. Phys. C 6, 359 (1995).
[40] M. Michelangeli, Graduation Thesis, Politecnico di Torino & Université Paris Diderot (Master in Complex Systems, 2015).
[41] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 58, 86 (1987).
[42] K. Fukui and S. Todo, J. Computat. Phys. 228, 2629 (2009).
[43] K. Binder, Z. Phys B - Condensed Matter 43, 119 (1981).
[44] Y. Deng, T. M. Garoni, and A. D. Sokal, Phys. Rev. Lett. 98, 030602 (2007).
[45] M. Matsumoto and T. Nishimura, ACM Trans. Mod. Comp. Sim. 8, 3 (1998).
[46] M. Ibañez Berganza and L. Leuzzi, Phys. Rev. B 88, 144104 (2013).
[47] D. J. Amit, J. Phys. A 9, 1441 (1976).
[48] Z. Q. Zhang, F. C. Pu, and B. Z. Li, J. Phys. A 16, L85 (1983).
[49] J. A. Gracey, Phys. Rev. D 92, 025012 (2015).
[50] J. L. Cardy, J. Phys. A 14, 1407 (1981).
[51] J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).
[52] A. Codello, N. Defenu, and G. D’Odorico, Phys. Rev. D 91, 105003 (2015).
[53] J. M. Kosterlitz, Phys. Rev. Lett. 37, 1577 (1976).