SINGULAR RENORMALIZATION GROUP APPROACH TO SIS PROBLEMS

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(Communicated by Yuan Lou)

Abstract. In this paper, we consider the boundary value problems of a one-dimensional steady-state SIS epidemic reaction-diffusion-advection system in the following two cases: (i) the advection rate is relatively large comparing to the diffusion rates of infected and susceptible populations; (ii) the diffusion rate of the susceptible population approaches zero. By introducing a singular parameter, the system can be viewed as a singularly perturbed problem. By the renormalization group method, we construct the first-order approximate solutions and obtain error estimates.

1. Introduction. In recent years, it has been commonly recognized that environmental heterogeneity and individual motility are two significant factors that should be considered in studying disease dynamics. In order to capture the impact of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease, there are a lot of works have been devoted into this investigation[1, 4, 11, 16, 17, 25, 26, 27, 28]. In [1], Allen et al. proposed a frequency-dependent Susceptible-Infected-Susceptible (SIS) reaction-diffusion model for a population inhabiting a continuous spatial habitat. Peng and Yi[25, 27] considered the effects of large and small diffusion rates of the susceptible and infected individuals on the persistence and extinction of the epidemic disease. However, in some heterogeneous environments, the external environmental forces, such as wind[10] and water flow[19, 20, 21, 22], could be added into the existing reaction-diffusion models as an advection term, which is significant to predict the patterns of disease occurrence and design optimal control strategies. Cui and Lou[9] considered the effects of diffusion and advection on asymptotic profiles of steady states of an epidemic reaction-diffusion-advection model.

2020 Mathematics Subject Classification. Primary: 34C60, 34D15, 34E10, 34E15, 34K26.
Key words and phrases. Renormalization group method, SIS epidemic model, error estimate.

This work is supported by NSFC grant (No. 11771177, 11301210), China Automobile Industry Innovation and Development Joint Fund (No. U1664257), Program for Changbaishan Scholars of Jilin Province and Program for JLU Science, Technology Innovative Research Team (No. 2017TD-20), NSF grant (No. 20190201132JC) and ESF grant (No. JJKH20170776KJ) of Jilin, China.

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We consider the following parameterized second order boundary valued problem:

\[
\begin{aligned}
\frac{d^2 S}{dx^2} - q \frac{dS}{dx} - \beta(x) \frac{SI}{S+I} + \gamma(x) I &= 0, \quad 0 < x < L, \\
\frac{d^2 I}{dx^2} - q \frac{dI}{dx} + \beta(x) \frac{SI}{S+I} - \gamma(x) I &= 0, \quad 0 < x < L, \\
\frac{dS}{dx} - qS &= d_I \frac{dI}{dx} - qI = 0, \quad x = 0, L, \\
\end{aligned}
\]

where \(S(x)\) and \(I(x)\) denote the density of susceptible and infected individuals at equilibrium at \(x \in [0, L]\), respectively. (1) can be, in fact, viewed as the equilibrium equations of the one dimensional SIS reaction-diffusion-advection model [9]:

\[
\begin{aligned}
\frac{dS}{dx} = d_S \frac{d^2 S}{dx^2} - qS - \beta(x) \frac{SI}{S+I} + \gamma(x) I, \quad 0 < x < L, t > 0, \\
I_t = d_I \frac{d^2 I}{dx^2} - qI + \beta(x) \frac{SI}{S+I} - \gamma(x) I, \quad 0 < x < L, t > 0, \\
dS_x - qS = d_I d_I - qI = 0, \quad x = 0, L, t > 0, \\
S(x, 0) = S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \quad 0 < x < L,
\end{aligned}
\]

where \(S(x, t)\) and \(I(x, t)\) respectively denote the density of susceptible and infected individuals at time \(t\) and position \(x\) in the interval \([0, L]\), the positive constants \(d_S\) and \(d_I\) are diffusion rates for the susceptible and infected populations, the positive constant \(q\) presents the advection rate of a stream or wind which carries the susceptible and infected populations from the upstream \(x = 0\) to the downstream \(x = L\). The functions \(\beta(x)\) and \(\gamma(x)\) are assumed to be positive Hölder continuous functions on \([0, L]\) which represent the rates of disease transmission and recovery at \(x\), respectively. The no-flux boundary conditions indicate there is no population flux across the upstream and downstream ends, so that both susceptible and infected populations live in a closed environment. Summing two equations of (2) and integrating over \((0, L)\) gives

\[
\begin{aligned}
\frac{\partial}{\partial t} \int_0^L (S + I) dx = \int_0^L (d_S S_{xx} + d_I I_{xx}) dx - q \int_0^L (S_x + I_x) dx = 0, \quad t > 0,
\end{aligned}
\]

which means the total population size is constant in time, i.e.,

\[
\int_0^L S(x, t) dx + I(x, t) dx = \int_0^L S(x, 0) + I(x, 0) dx = N > 0, \quad t \geq 0,
\]

and hence

(\(H\)) \[
\int_0^L S(x) + I(x) dx = N.
\]

It is well-known that, from the ecological point of view, only solutions \((S(x), I(x))\) satisfying \(S(x) \geq 0\) and \(I(x) \geq 0\) on \([0, L]\) are of interest. Therefore, we only concern the non-negative solution of (1) and (\(H\)) with \(I(x) > 0\) for some \(x \in (0, L)\), we call it endemic equilibrium(EE). Let

\[
\mathcal{R}_0(d_I, q) = \sup_{\varphi \in H^1((0, L))} \left\{ \frac{\int_0^L \beta(x) e^{\frac{q}{4} x^2} \varphi^2 dx}{d_I \int_0^L e^{\frac{q}{4} x^2} \varphi^2 dx + \int_0^L \gamma(x) e^{\frac{q}{4} x^2} \varphi^2 dx} \right\}.
\]

Cui, Lam and Lou[8] proved that \(\mathcal{R}_0 > 1\) is a necessary and sufficient condition for the existence of EE, furthermore, they studied the qualitative properties of EE of
(1) and (H) when diffusion and advection rates $d_S$, $d_I$ and $q$ vary in the following two cases:

- $\beta(L) > \gamma(L)$ and $\frac{d_I}{q}$ is sufficiently small;
- $\beta(L) < \gamma(L)$ and $d_S$ approaches zero with $R_0 > 1$.

In the first case, they obtained an exponential decay result by constructing the upper solutions and lower solutions, and showed that two populations persist and concentrate at the downstream end. In the second case, they proved susceptible individuals concentrate at the downstream end and $I(x)$ decays exponentially for positive advection rate. Similar results were also obtained by Kuto, Matsuzawa and Peng [15].

The main purpose of this paper is to provide a new perspective to understand system (1) in the above two cases. In fact, it is easily found that the steady state system (1) can be treated as a singularly perturbed system by introducing a new singular parameter. Therefore, we can study system (1) from the point of singular perturbation. Our main idea is the so called renormalization group (RG) method developed by Chen, Goldenfeld and Oono[5], which has been showed to be effective for singular perturbation systems, and been applied successfully to many fields, such as quantum kinetics[3], invariant manifold[14, 6, 7], ODEs[30, 12, 29, 18], PDEs[23, 24] and SDEs[2, 13]. Ziane[30] considered one-order autonomous initial value problems and proved that the approximation results obtained by RG method are valid over long time intervals. Chiba[6, 7] treated differential equations on manifolds and proved that RG method could provide approximate vector fields and invariant manifolds as well as approximate solutions. Zhou et al. [29] presented a new formulation of the RG method, as well as applications to second-order boundary layer problems and to boundary layer problems with delay. In [13], the authors investigated a class of stochastic differential equations and proved that the approximate solutions constructed by RG method remain valid with high probability on large time scales. Here, we will firstly present a RG strategy for a kind of integro-differential perturbed problem, and then use the obtained RG formula to deal with (1) and (H), construct the approximate solution of EE, and give the estimate of the error of the exact EE and the approximate EE and make a comparison with the previous results.

The rest of the paper is organized as follows. In section 2, we give the RG formula for a class of integro-differential equations. In section 3, we rewrite the system (1) as the form introduced in section 2 by introducing new parameters and variables. Then we apply RG method to construct the first-order approximate solutions and obtain error estimates for the actual solution and the approximate solutions in two cases respectively. The last section is a short conclusion.

2. RG formula. In order to use RG method to study system (1) systematically, we firstly consider the following integro-differential equations

$$u' + Au = \varepsilon \int_{\tau_0}^{\tau} \tilde{F}(u)dt, \quad u \in \mathbb{R}^n,$$

where $\varepsilon > 0$ is a positive small parameter, $'$ denotes the derivative to $\tau$, $A$ is a nonsingular $n \times n$ matrix, which is assumed for simplicity to be a diagonalizable matrix, and $\tilde{F}(u)$ is a nonlinear vector valued functions on $\mathbb{R}^n$. Without loss of generality, we may suppose $A$ is a diagonal matrix.
Now we construct an approximate solution of (4) with initial condition \( u(0) \).

**Step 1.** The naive expansion.

Substituting the naive perturbation expansion
\[
u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots
\]
into equations (4) yields
\[
\varepsilon^0: \quad u'_0 + Au_0 = 0, \tag{5}
\]
\[
\varepsilon^1: \quad u'_1 + Au_1 = \int_{t_0}^{T} AF(u_0)dt, \tag{6}
\]
here \( F(u_0) = A^{-1}\tilde{F}(u_0) \). Solve equations (5) as \( u_0(\tau) = e^{-\tau A}a \), where \( a \) is an arbitrary constant vector. Taking \( u_0(\tau) \) into equations (6) and solving it with \( u_1(0) = 0 \), we obtain
\[
u_1(\tau) = e^{-\tau A} \int_{0}^{T} e^{t A} \int_{t_0}^{t} AF(e^{-t A}a)d\tau dt,
\]
which becomes into
\[
u_1(\tau) = e^{-\tau A} \left( \int_{0}^{T_0} F(e^{-t A}a)dt - \int_{0}^{T} e^{t A} F(e^{-t A}a)dt \right) + \int_{T_0}^{T} F(e^{-t A}a)dt,
\]
by using integration by parts. Thus, the naive expansion to first order is
\[
\dot{\nu}(\tau) = e^{-t A}a + \varepsilon e^{-\tau A} \left( \int_{0}^{T_0} F(e^{-t A}a)dt - \int_{0}^{T} e^{t A} F(e^{-t A}a)dt \right)
\]
\[
\quad + \varepsilon \int_{T_0}^{T} F(e^{-t A}a)dt. \tag{7}
\]

**Step 2.** The introduction of a free parameter.

The solution \( u_1(\tau) \) can be split into two parts. The first is the resonant part, generated in our case by the time-constant part of the integrand \( e^{t A} F(e^{-t A}a) \) and \( F(e^{-t A}a) \), which is called secular term. The secular term is proportional to \( \tau \), causing the asymptotic property of (7) to be lost on time scales of \( O(\frac{1}{\varepsilon}) \). The second is the non-resonant part corresponding to the time-dependent part of the integrand \( e^{t A} F(e^{-t A}a) \) and \( F(e^{-t A}a) \). Thus, we decompose
\[
e^{t A} F(e^{-t A}a) = R_1(a) + F_1(t,a), \quad F(e^{-t A}a) = R_2(a) + F_2(t,a). \tag{8}
\]
Taking (8) into (7) leads to
\[
\dot{\nu}(\tau) = e^{-\tau A}( a + \varepsilon b - \varepsilon R_1(\tau) + \varepsilon R_2(\tau - \tau_0) + \varepsilon G(\tau, a), \tag{9}
\]
where
\[
b = \int_{0}^{T_0} F(e^{-t A}a)dt,
\]
\[
G(\tau, a) = -e^{-\tau A} \int_{0}^{\tau} F_1(t,a)dt + \int_{\tau_0}^{\tau} F_2(t,a)dt.
\]

In order to remove the secular term to get the effective asymptotic solution, the key step of RG method is to introduce a free parameter \( \zeta \) to split \( \tau \) as \( \tau - \zeta + \zeta \), i.e.,
\[
R_i(\tau) = R_i(a)\zeta + R_i(a)(\tau - \zeta), \quad i = 1, 2.
\]
Set
\[
V(\zeta) = a + \varepsilon b - \varepsilon R_1(a)\zeta + \varepsilon e^{\tau A} R_2(a)(\zeta - \tau_0)
\]
subsection, we investigate the asymptotic profiles of EE when \( d \) elements

the following singularly perturbed BVP

for matrix \( A \) in (9) to obtain

\[
e^{\tau A} \tilde{u}(\tau) = V(\xi) - \varepsilon R_1(V(\xi)) (\tau - \xi) + \varepsilon e^{\tau A} R_2(V(\xi)) (\tau - \xi) + \varepsilon e^{\tau A} G(\tau, V(\xi)) + O(\varepsilon^2).
\]

Differentiating the renormalized expansion (10) with respect to \( \xi \), we derive the following renormalization group equations up to \( O(\varepsilon) \)

\[
\frac{dV(\xi)}{d\xi} = -\varepsilon R_1(V(\xi)) + \varepsilon e^{\tau A} R_2(V(\xi)).
\]

Suppose now that we can solve RG equation (11) as \( V(\xi, C) \), where \( C \) is determined by \( V(0, C) = u(0) \). Then, setting \( \xi = \tau \) in (10), we obtain the approximate solution of (4)

\[
\hat{u}(\tau) = e^{-\tau A} V(\tau, C) + \varepsilon G(\tau, V(\tau, C)).
\]

3. Main results. In this section, we construct the approximation for EE of boundary value problem (BVP) (1) for two different parameter settings and investigate the asymptotic profiles of EE respectively. For simplicity of presentation, we rewrite BVP (1) as:

\[
\begin{cases}
    d_S \frac{d^2 S}{dx^2} - q_S \frac{dS}{dx} - f(x, S, I) = 0, & 0 \leq x \leq L, \\
    d_I \frac{d^2 I}{dx^2} - q_I \frac{dI}{dx} + f(x, S, I) = 0, & 0 \leq x \leq L, \\
    d_S \frac{dS}{dx} - qS = d_I \frac{dI}{dx} - qI = 0, & x = 0, L,
\end{cases}
\]

where \( f(x, S, I) = \beta(x) S \frac{dI}{dx} - \gamma(x) I \). It was shown in [8] that if \( R_0 > 1 \), there exists at least one EE. Moreover, if \( d_S = d_I \), the EE is unique.

In this paper, we define

\[
\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^m |a_{ij}|
\]

for matrix \( A = (a_{ij})_{n \times m} \), and denote the \( n \)-order diagonal matrix with diagonal elements \( \lambda_1, \ldots, \lambda_n \) by \( \text{diag}(\lambda_1, \ldots, \lambda_n) \). And for ease of notations, we set \( \sigma = \max_{0 \leq x \leq L} \{ \beta(x) + \gamma(x) \} \).

3.1. The asymptotic profiles of EE when \( \frac{d_I}{q} \ll 1 \) and \( \beta(L) > \gamma(L) \). In this subsection, we investigate the asymptotic profiles of EE when \( \frac{d_I}{q} \) is sufficiently small and the downstream site is high risk, i.e., \( \frac{d_I}{q} \ll 1 \) and \( \beta(L) > \gamma(L) \), which ensures the existence of EE by the above analysis.

Let \( \varepsilon = \frac{d_I}{q} \), \( \eta = \frac{d_S}{d_I} \) and introduce \( \tau = \frac{L-x}{\varepsilon} \), then BVP (13) can be rewritten as the following singularly perturbed BVP

\[
\begin{align*}
    u'' + Au' &= \frac{\varepsilon A}{q} F(\varepsilon \tau, u), \\
    u'(0) + Au(0) &= 0, \\
    u'(L) + Au(L) &= 0,
\end{align*}
\]

where \( \varepsilon \) is a positive small parameter, \( u = (S, I)^T, A = \text{diag}(\eta, 1) \) and \( F(\varepsilon \tau, u) = (1, -1)^T f(\varepsilon \tau, u) \).
Let Lemma 1. \( u \) intergo-differential equations

\[ u' + Au = \frac{\varepsilon A}{q} \int_{\frac{L}{\varepsilon}}^{\tau} F(\varepsilon t, u)dt, \quad 0 \leq \tau \leq \frac{L}{\varepsilon}. \]  

(17)

It is clear that the solution \( u(\tau) \) of (17) with (15) is a solution of (14)-(16). We first give a prior estimation of \( u(\tau) \).

**Lemma 1.** Let \( u(\tau) = (S(\tau), I(\tau))^T \) be the EE of (17) with (15), then

\[ \|u(\tau)\| \leq 2\|u(0)\|e^{-\delta(1-\frac{\varepsilon}{q})\tau}, \quad 0 \leq \tau \leq \frac{L}{\varepsilon}, \]  

where \( \delta = \min\{1, \eta\} \).

**Proof.** Note that (17) is equivalent to the following integral equations

\[ u(\tau) = e^{-\tau A}u(0) + \frac{\varepsilon A}{q} \int_{0}^{\tau} e^{-(\tau-t)A} \int_{\frac{L}{\varepsilon}}^{t} F(\varepsilon s, u(s))dsdt. \]  

(19)

Applying (15) to (17) yields \( \int_{0}^{\tau} F(\varepsilon t, u(t))dt = 0 \), so (19) becomes into

\[ u(\tau) = e^{-\tau A}u(0) + \frac{\varepsilon}{q} \int_{0}^{\tau} F(\varepsilon t, u(t))dt - \frac{\varepsilon}{q} \int_{0}^{\tau} e^{-(\tau-t)A}F(\varepsilon t, u(t))dt \]

by using integration by parts. It follows from the expression of \( F(\varepsilon t, u(t)) \) that

\[ S(\tau) + I(\tau) = S(L)e^{-\eta \tau} + I(L)e^{-\tau} + \frac{\varepsilon}{q} \int_{0}^{\tau} (e^{-\eta \tau} - e^{-\eta \eta \tau})f(\varepsilon \tau, u(t))dt. \]  

(20)

Note that \( S(\tau), I(\tau) \geq 0 \) for \( 0 \leq \tau \leq \frac{L}{\varepsilon} \), and \( \|f(\varepsilon \tau, u(t))\| \leq \sigma\|u(t)\| \), therefore

\[ \|u(\tau)\| \leq \|S(\tau) + I(\tau)\| \leq 2\|u(0)\|e^{-\delta \tau} + \frac{\varepsilon \sigma}{q} \int_{0}^{\tau} e^{-\delta \tau} \|u(t)\|dt, \]

or equivalently,

\[ \|e^{\delta \tau}u(\tau)\| \leq 2\|u(0)\| + \frac{\varepsilon \sigma}{q} \int_{0}^{\tau} \|e^{\delta \tau}u(t)\|dt. \]

By the Gronwall’s inequality, we obtain \( \|e^{\delta \tau}u(\tau)\| \leq 2\|u(0)\|e^{\frac{\varepsilon \sigma}{\delta} \tau} \), which leads to (18).

Let \( u(\tau) = (S(\tau), I(\tau))^T \) be the EE of (17) with (15) and

\[ \varepsilon \int_{0}^{\frac{L}{\varepsilon}} S(t) + I(t)dt = N, \]

then the following lemma can be straighly obtained from [8].

**Lemma 2.** There exists a positive constant \( K_0 \) such that if \( \frac{L}{\varepsilon} \) is sufficiently small, then

\[ \|u(0)\| \leq \frac{\eta K_0}{\varepsilon}. \]  

(21)

Furthermore, when \( \varepsilon \to 0 \), \( u(0) = (S(0), I(0))^T \) satisfies

\[ \int_{0}^{1} \frac{dz}{1 + \frac{\gamma(L)}{S(0)}z^{1-\eta}} = \frac{\gamma(L)}{\beta(L)}. \]  

(22)
Case 1. \(0 < \eta < 1\)

Following the same RG procedure mentioned in Section 2, we obtain the naive expansion of (17) up to order \(O(\varepsilon)\)

\[
\tilde{u}(\tau) = e^{-\tau A} (a + \varepsilon b + \varepsilon R a \tau) + \varepsilon G(\tau, a),
\]

where \(a = (a_1, a_2)^T\) is a constant vector with \(a_1, a_2 > 0\) (since we only focus on the positive solutions of (17)), and

\[
b = \frac{1}{q} \int_0^{\frac{\tau}{q}} F(0, e^{-t A} a) dt,
\]

\[
R = \frac{1}{q} \text{diag}(0, \beta(L) - \gamma(L)),
\]

\[
G(\tau, a) = h(\tau, a)e^{-\tau A} a + H(\tau) a + \int_0^{\frac{\tau}{q}} F(0, e^{-t A} a) dt
\]

with

\[
h(\tau, a) = \frac{\beta(L)}{q(\eta - 1)} \ln \left( \frac{a_1 + a_2}{a_1 + a_2 e^{(\eta - 1)\tau}} \right),
\]

\[
H(\tau) = \frac{\gamma(L)}{q(\eta - 1)} \left( \begin{array}{cc} e^{-\tau} - e^{-\eta \tau} \\ 0 & 0 \end{array} \right).
\]

As in Section 2, we obtain the following ansatz approximate EE of (17)

\[
\tilde{\tilde{u}}(\tau) = e^{-\tau A} V(\tau) + \varepsilon G(\tau, V(\tau)),
\]

where \(V(\tau) = e^{e R \tau} u(0)\).

A straightforward calculation gives

\[
\tilde{\tilde{u}}' + A \tilde{\tilde{u}} = \frac{\varepsilon A}{q} \int_{\frac{\tau}{q}}^{\tau} F(0, e^{-t A} V(\tau)) dt + \varepsilon DG(\tau, V(\tau))V'(\tau), \quad 0 \leq \tau \leq \frac{L}{\varepsilon},
\]

where \(DG(\tau, V(\tau))\) denotes the Jacobian matrix of \(G(\tau, V(\tau))\) with respect to \(V\).

To obtain the error between the exact EE and the approximate EE (24), we need the following lemma.

**Lemma 3.** For \(\tau \in [0, \frac{L}{\varepsilon}]\),

\[
\|G(\tau, V(\tau))\| \leq \frac{\|u(0)\|}{q(1 - \eta)} K_1^{1-\eta (1 - \frac{\varepsilon \sigma}{q}) \tau},
\]

\[
\|DG(\tau, V(\tau))V'(\tau)\| \leq \frac{\varepsilon \sigma}{q^2(1 - \eta)} \|u(0)\| K_1^{1-\eta (1 - \frac{\varepsilon \sigma}{q}) \tau},
\]

where \(K_1 = (2 - \eta)(\beta(L) + \gamma(L)) + \beta(L) \ln (1 + \frac{I(0)}{S(0)} e^{\frac{\varepsilon \sigma}{q}})\).

**Proof.** Let \(V(\tau) = (V_1(\tau), V_2(\tau))^T\). By the expression of \(V(\tau)\) and \(G(\tau, V(\tau))\), we conclude that for \(\tau \in [0, \frac{L}{\varepsilon}]\),

\[
\|V(\tau)\| \leq \|u(0)\| e^{\frac{\varepsilon \sigma \tau}{q}}, \quad \frac{V_2(\tau)}{V_1(\tau)} \leq \frac{I(0)}{S(0)} e^{\frac{\varepsilon \sigma \tau}{q}}, \quad \|V'(\tau)\| \leq \frac{\varepsilon \sigma}{q} \|u(0)\| e^{\frac{\varepsilon \sigma \tau}{q}},
\]

(26)
and
\[
|h(\tau, V(\tau))| \leq \frac{\beta(L)}{q(1-\eta)} \ln(1 + \frac{V_2(\tau)}{V_1(\tau)}),
\]
(27)
\[
\|H(\tau)\| \leq \frac{\gamma(L)}{q(1-\eta)} e^{-\eta \tau},
\]
\[
\left\| \int_{\frac{\tau}{2}}^{\tau} F(0, e^{-tA}V(\tau)) dt \right\| \leq \int_{\frac{\tau}{2}}^{\tau} \left| f(0, e^{-tA}V(\tau)) \right| dt \leq (\beta(L) + \gamma(L)) V_2(\tau) e^{-\tau}.
\]
(28)

Therefore
\[
\|G(\tau, V(\tau))\| \leq \left| h(\tau, V(\tau)) \right| \left\| e^{-\tau A} V(\tau) \right\| + \left\| H(\tau) \right\| \left\| V(\tau) \right\|
+ \frac{1}{q} \left\| \int_{\frac{\tau}{2}}^{\tau} F(0, e^{-tA}V(\tau)) dt \right\|
\leq \frac{\|C\| K_1}{q(1-\eta)} e^{-\eta (1-\frac{\eta}{4}) \tau}.
\]

On the other hand, it is not difficult to obtain that
\[
DG(\tau, V(\tau)) V'(\tau) = h(\tau, V(\tau)) e^{-\tau A} V'(\tau) + e^{-\tau A} V(\tau) Dh(\tau, V(\tau)) V'(\tau)
+ H(\tau) V'(\tau) + \frac{1}{q} \int_{\frac{\tau}{2}}^{\tau} DF(0, e^{-tA}V(\tau)) e^{-tA} V'(\tau) dt,
\]
where
\[
Dh(\tau, V(\tau)) V'(\tau) = \frac{\beta(L)}{q(\eta-1)} \left( \frac{V_1'(\tau) + V_2'(\tau)}{V_1(\tau) + V_2(\tau)} - \frac{V_1'(\tau) + V_2'((\tau)e^{(\eta-1)\tau})}{V_1(\tau) + V_2(\tau)e^{(\eta-1)\tau}} \right),
\]
\[
DF(0, e^{-tA}V(\tau)) e^{-tA} V'(\tau)
= \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \left( \frac{\beta(L)(V_1(\tau)e^{-\eta \tau})^2 + V_2'(\tau) V_2(\tau)}{V_2'(\tau)(V_1(\tau)e^{-\eta \tau} + V_2(\tau)e^{-t\tau})^2} - \gamma(L) \right) V_2'(\tau) e^{-t}.
\]

Direct calculation shows that for \( \tau \in [0, \frac{\tau}{2}] \),
\[
\left| Dh(\tau, V(\tau)) V'(\tau) \right| \leq \frac{\varepsilon \sigma \beta(L)}{q^2(1-\eta)},
\]
\[
\left\| \int_{\frac{\tau}{2}}^{\tau} DF(0, e^{-tA}V(\tau)) e^{-tA} V'(\tau) dt \right\| \leq \int_{\frac{\tau}{2}}^{\tau} (\beta(L) + \gamma(L)) \left| V_2'(\tau) \right| e^{-t} dt
\leq \frac{\varepsilon \sigma}{q} \| u(0) \| (\beta(L) + \gamma(L)) e^{-(1-\frac{\eta}{4}) \tau},
\]
thus
\[
\|DG(\tau, V(\tau))\| \leq \left| h(\tau, V(\tau)) \right| \left\| e^{-\tau A} V'(\tau) \right\| + \left\| e^{-\tau A} V(\tau) \right\| \left\| Dh(\tau, V(\tau)) V'(\tau) \right\|
+ \left\| H(\tau) \right\| \left\| V'(\tau) \right\|
+ \frac{1}{q} \left\| \int_{\frac{\tau}{2}}^{\tau} DF(0, e^{-tA}V(\tau)) e^{-tA} V'(\tau) dt \right\|
\leq \frac{\varepsilon \sigma}{q} \| u(0) \| K_1 \frac{e^{-\eta (1-\frac{\eta}{4}) \tau}}{q^2(1-\eta)}.
\]
The proof is completed.\(\square\)
Proposition 1. Assume that $\beta(L) > \gamma(L)$, $0 < \eta < 1$ and $\frac{\varepsilon\sigma}{q} < 1$. Let $u(\tau)$ be the EE of (17) with (15) and $\bar{u}(\tau)$ be given by (24), then

$$\|u(\tau) - \bar{u}(\tau)\| \leq \frac{\varepsilon}{q} \|u(0)\| (K_2 + \frac{K_2}{\eta(1 - \eta)}) e^{-\eta \tau}, \quad 0 \leq \tau \leq \frac{L}{\varepsilon},$$

where

$$K_2 = \left| \int_0^\tau \beta(L) \frac{S(0)e^{(1 + \eta)t} - \gamma(L)e^{-t}}{S(0)e^{-nt} + I(0)e^{-t} - \gamma(L)e^{-t}} dt \right|,$$

$$K_3 = \sigma \left( \frac{2\gamma}{1 - \frac{\varepsilon\sigma}{q}} + \eta(1 - \eta) + \frac{\varepsilon\eta}{q} K_1 \right) e^{\frac{\varepsilon\eta}{2}}.$$

Proof. Let $\omega(\tau) = u(\tau) - \bar{u}(\tau)$, then by (17) and (25), we obtain for $\tau \in [0, \frac{L}{\varepsilon}]$,

$$\omega' + A\omega = \frac{\varepsilon A}{q} \int_\frac{\varepsilon}{q}^\tau \left( F(\varepsilon t, u(t)) - F(0, e^{-tA}V(\tau)) \right) dt - \varepsilon DG(\tau, V(\tau))V'(\tau),$$

or equivalently,

$$\omega(\tau) = e^{-tA} \omega(0) + \varepsilon (\Phi_1 + \Phi_2 + \Phi_3),$$

where

$$\Phi_1 = \frac{A}{q} \int_0^\tau e^{-(\tau-t)A} \int_\frac{\varepsilon}{q}^t F(\varepsilon s, u(s)) ds dt,$$

$$\Phi_2 = -\frac{A}{q} \int_0^\tau e^{-(\tau-t)A} \int_\frac{\varepsilon}{q}^t F(0, e^{-sA}V(t)) ds dt,$$

$$\Phi_3 = -\int_0^\tau e^{-(\tau-t)A} DG(t, V(t))V'(t) dt.$$

By (24) and the expression of $F(0, e^{-tA}u(0))$, we obtain

$$\|\omega(0)\| \leq \left\| \frac{\varepsilon}{q} \int_0^\frac{\varepsilon}{q} F(0, e^{-tA}u(0)) dt \right\|$$

$$= \frac{\varepsilon}{q} \int_0^\frac{\varepsilon}{q} \left( \beta(L) \frac{S(0)e^{-(1 + \eta)t}}{S(0)e^{-nt} + I(0)e^{-t} - \gamma(L)e^{-t}} I(0) dt \right)$$

$$\leq \frac{\varepsilon}{q} K_2 \|u(0)\|.$$  

Since $\|F(\tau, u(t))\| \leq \sigma \|u(t)\|$, by Lemma 1, for $\tau \in [0, \frac{L}{\varepsilon}]$,

$$\|\Phi_1\| \leq \left\| \frac{\varepsilon A}{q} \int_0^\tau \|e^{-(\tau-t)A}\| \int_\frac{\varepsilon}{q}^t \sigma \|u(s)\| ds dt \right\|$$

$$\leq \frac{1}{q} \int_0^\tau e^{-\eta(\tau-t)} \int_\frac{\varepsilon}{q}^t 2\sigma \|u(0)\| e^{-\eta(1 - \frac{\varepsilon\sigma}{q})} ds dt$$

$$\leq \frac{2\sigma}{q \eta(1 - \frac{\varepsilon\sigma}{q})} \int_0^\tau e^{-\eta t} ds dt$$

$$\leq \frac{2\sigma e^{\frac{\varepsilon\eta}{q}}}{q \eta(1 - \frac{\varepsilon\sigma}{q})} \int_0^\tau e^{-\eta t} dt$$

$$\leq \frac{2\sigma e^{\frac{\varepsilon\eta}{q}}}{q \eta(1 - \frac{\varepsilon\sigma}{q})} \tau e^{-\eta \tau}.$$  

(32)
By (26) and (28), for \( \tau \in [0, \frac{L}{\varepsilon}] \),
\[
\| \Phi_2 \| \leq \frac{\| A \|}{q} \int_{0}^{\tau} \| e^{-(\tau-t)A} \| \int_{s}^{t} F(0, e^{-A(t-s)})ds \| dt \\
\leq \frac{1}{q} \int_{0}^{\tau} e^{-\eta(\tau-t)}(\beta(L) + \gamma(L)) \| u(0) \| e^{-(1-\frac{\varepsilon}{q})t} \| dt \\
\leq \frac{\sigma \| u(0) \|}{q} e^{-\eta \tau} \int_{0}^{\tau} e^{\frac{\varepsilon}{q} t} dt \\
\leq \frac{\sigma e^{\frac{\varepsilon}{q}}}{q} \| u(0) \| \tau e^{-\eta \tau}.
\]

By Lemma 3, for \( \tau \in [0, \frac{L}{\varepsilon}] \),
\[
\| \Phi_3 \| \leq \int_{0}^{\tau} \| e^{-(\tau-t)A} \| \| D G(t, V(t)) V'(t) \| dt \\
\leq \int_{0}^{\tau} e^{-\eta(\tau-t)} \frac{\varepsilon \sigma \| u(0) \| K_1}{q^2(1-\eta)} e^{-\eta(1-\frac{\varepsilon}{q})t} \| dt \\
= \frac{\varepsilon \sigma \| u(0) \| K_1}{q^2(1-\eta)} \int_{0}^{\tau} e^{\frac{\varepsilon}{q} t} dt \\
\leq \frac{\varepsilon \sigma e^{\frac{\varepsilon}{q}}}{q^2(1-\eta)} \| u(0) \| K_1 \tau e^{-\eta \tau}.
\]

Combining (30)-(34) yields (29), the proof is completed. \( \Box \)

Remark 1. We should point out here that \( \lim_{\varepsilon \to 0} K_\varepsilon = 0 \). In fact, by the transformation \( z = e^{-t} \) and (22),
\[
\lim_{\varepsilon \to 0} K_\varepsilon = \lim_{\varepsilon \to 0} \beta(L) \int_{e^{-\frac{1}{2}}}^{1} \frac{dz}{1 + \frac{\beta(0)}{S(0)} z^{1-\eta}} - \frac{\gamma(L)}{\beta(L)}(1-e^{-\frac{\varepsilon}{4}}) = 0.
\]

To obtain the approximation for EE of BVP (14)-(16) with (H), we should take boundary value condition (15) and the additional hypothesis (H) into consideration.

Proposition 2. Assume that \( \beta(L) > \gamma(L) \) and \( 0 < \eta < 1 \). Let \( \tilde{u}(\tau) = (\tilde{S}(\tau), \tilde{I}(\tau)) \) be given by (24), then
\[
\lim_{\varepsilon \to 0} (\tilde{u}'(0) + A \tilde{u}(0)) = 0,
\]
\[
\varepsilon \int_{0}^{\frac{L}{\varepsilon}} \tilde{S}(t) + \tilde{I}(t) dt = N + O(\varepsilon).
\]

Proof. To begin with, we will prove (35). Setting \( \tau = 0 \) in (25) yields
\[
\tilde{u}'(0) + A \tilde{u}(0) = \frac{\varepsilon A}{q} \int_{\frac{1}{4}}^{0} F(0, e^{-A(t-s)})ds + \varepsilon DG(0, u(0))V'(0).
\]

By (21) and (31),
\[
\| \frac{\varepsilon A}{q} \int_{\frac{1}{4}}^{0} F(0, e^{-A(t-s)})ds \| \leq \frac{\varepsilon}{q} K_\varepsilon \| I(0) \| \leq \frac{\eta}{q} K_0 K_\varepsilon.
\]
By Lemma 3 and (31),
\[ \|DG(0, u(0))V'(0)\| \leq \frac{\varepsilon\sigma K_1}{q^2(1 - \eta)} \|u(0)\| \leq \frac{\eta\sigma K_0 K_1}{q^2(1 - \eta)} K_\varepsilon. \]  
(39)

Combining (37)-(39) and Remark 1 yields (35).

Next, we will prove (36). It follows from Proposition 1 that
\[ \int_0^\frac{\varepsilon}{q} \|u(\tau) - \tilde{u}(\tau)\|d\tau \leq \frac{\varepsilon}{q\eta} \|u(0)\| \left(K_\varepsilon + \frac{K_2}{\eta^2(1 - \eta)}\right). \]

Then, we obtain (36).

**Remark 2.** From the Proposition 1 and 2, we know that \( \tilde{u}(\tau) \) given by (24) satisfies the boundary value condition (15) and constraint condition (H), which means \( \tilde{u}(\tau) \) is actually an approximation for the EE of (17) with (15) and (H), i.e., \( \tilde{u}(\tau) \) is an approximation for the EE of (14)-(16) with (H).

In the rest of the article, we will use the same notations as in Case 1 for simplicity.

**Case 2.** \( \eta > 1 \)

Adopting the same procedure as in Case 1, we obtain the ansatz approximate solution
\[ \tilde{u}(\tau) = e^{-\tau A}V(\tau) + \varepsilon G(\tau, V(\tau)), \]
(40)
where \( V(\tau) = e^{\varepsilon R\tau}u(0), R = -\frac{1}{q}\text{diag}(\beta(L), \gamma(L)) \) and
\[ G(\tau, V(\tau)) = h(\tau, V(\tau))e^{-\tau A}V(\tau) + H(\tau)V(\tau) + \frac{1}{q} \int_{\frac{\varepsilon}{q}}^{\tau} F(0, e^{-tA}V(\tau))dt, \]
with
\[ h(\tau, V(\tau)) = \frac{\beta(L)}{q(\eta - 1)} \ln \frac{V_1(\tau) + V_2(\tau)}{V_1(\tau)e^{(1 - \eta)\tau} + V_2(\tau)}, \]
\[ H(\tau) = \frac{\gamma(L)}{q(\eta - 1)} \begin{pmatrix} 0 & e^{-\tau} - e^{-\eta\tau} \\ 0 & 0 \end{pmatrix}. \]

Using the same method as in the proof of Proposition 1 and Proposition 2 yields the following results.

**Proposition 3.** Assume that \( \beta(L) > \gamma(L), \eta > 1 \) and \( \frac{\varepsilon\sigma}{q} < 1 \). Let \( u(\tau) \) be the EE of (17) with (15) and \( \tilde{u}(\tau) \) be given by (40), then
\[ \|u(\tau) - \tilde{u}(\tau)\| \leq \frac{\varepsilon}{q} \|u(0)\| \left(K_\varepsilon + \frac{K_3}{\eta - 1}\right)e^{-\tau}, 0 \leq \tau \leq \frac{L}{\varepsilon}, \]
where
\[ K_3 = \sigma \left( \frac{2e^{\frac{x}{q}}}{1 - \frac{x}{q}} + 1 \right) + \frac{\varepsilon}{q} L, \]
\[ K_4 = \eta (\beta(L) + \gamma(L)) + \beta(L)(\ln(1 + \frac{S(0)}{I(0)}) + (\eta - 1)). \]

Furthermore, \( \hat{u}(\tau) \) satisfies \( (35)-(36) \).

**Case 3.** \( \eta = 1 \)

Proceeding as in Case 1, we obtain the ansatz approximate solution
\[ \hat{u}(\tau) = e^{-\tau}A V(\tau) + \frac{\varepsilon}{q} F(0, V(\tau)) \left( e^{-\tau} - e^{-\frac{\tau}{q}} \right), \]
where \( V(\tau) \) satisfies RG equations
\[ V'(\tau) = -\frac{\varepsilon}{q} F(0, V(\tau)), \quad V(0) = u(0), \]
which can be solved as
\[ V_1(\tau) = \frac{\gamma(L)}{\beta(L) + C_2 e^{\frac{x}{q}(\beta(L) - \gamma(L)) \tau}} C_1, \quad V_2(\tau) = \frac{\beta(L) - \gamma(L)}{\beta(L) + C_2 e^{\frac{x}{q}(\beta(L) - \gamma(L)) \tau}} \xi, \]
where \( C_1 = S(0) + I(0) \) and \( C_2 = (\beta(L) - \gamma(L))(1 + \frac{S(0)}{I(0)}) - \beta(L) \).

**Proposition 4.** Assume that \( \beta(L) > \gamma(L) \) and \( \eta = 1 \). Let \( u(\tau) \) be the EE of \( (17) \) with \( (15) \) and \( \hat{u}(\tau) \) be given by \( (41) \), then
\[ \| u(\tau) - \hat{u}(\tau) \| \leq \frac{\varepsilon}{q} \| u(0) \| K_5 \varepsilon^{1-\tau \varepsilon}, \quad 0 < \tau \leq L, \]
where \( K_5 = 4\sigma + \frac{2\sigma^2}{q} \). Furthermore, \( \hat{u}(\tau) \) satisfies \( (35)-(36) \).

We summarize what we have proved as the following theorem.

**Theorem 3.1.** Assume that \( \beta(L) > \gamma(L) \), \( \frac{dL}{dS} \) is sufficiently small and \( \frac{dL}{dS} < 1 \), then any EE \( (S_e(x), I_e(x)) \) of \( (14)-(16) \) with \( (H) \) satisfies
\[ |S_e(x) - \hat{S}(x)| \leq \frac{dL}{qds} K_0(K_e + \frac{q}{qL}(L - x)) e^{-\frac{dL}{q}(L - x)}, \quad 0 < x < L, \]
\[ |I_e(x) - \hat{I}(x)| \leq \frac{dL}{qds} K_0(K_e + \frac{q}{qL}(L - x)) e^{-\frac{dL}{q}(L - x)}, \quad 0 < x < L, \]
where
\[ K_e = \left| \int_0^{\frac{dL}{q}} \beta(L) \left( \frac{S(0)e^{-(1 + \frac{dL}{q})t}}{S(0)e^{-\frac{dL}{q}t} + I(0)e^{-t}} - \gamma(L)e^{-t} \right) dt \right|, \]
for \( 0 < \frac{dL}{qS} < 1 \), \( (\hat{S}, \hat{I})^T \) is given by \( (24) \) and \( K = \frac{K_5}{\frac{dL}{qS}(1 - \frac{dL}{qS})} \); for \( \frac{dL}{qS} > 1 \), \( (\hat{S}, \hat{I})^T \) is given by \( (40) \) and \( K = \frac{K_5}{\frac{dL}{qS} - 1} \); for \( \frac{dL}{qS} = 1 \), \( (\hat{S}, \hat{I})^T \) is given by \( (41) \) and \( K = \frac{dL}{qS} K_5 \), and \( \delta = \min\{1, \frac{dL}{qS}\} \).

**Remark 3.** In this subsection, we proved the error of the exact EE and the approximate EE is exponentially small, which is equivalent to the Theorem 1.2 in [8]. Moreover, the approximate EE is up to and including \( O(\varepsilon) \), which extends the results of [8].
3.2. The asymptotic profiles of EE when $d_S \ll 1$, $\beta(L) < \gamma(L)$ and $R_0 > 1$.

In this subsection, we study the asymptotic profiles of EE when $d_S$ approaches zero and the downstream site is low risk, i.e., $d_S \ll 1$ and $\beta(L) < \gamma(L)$, under the assumption $R_0 > 1$ which ensures the existence of EE.

Let $\varepsilon = \frac{d_s}{q}$, $\xi = \frac{d_s}{I}$, $B = \text{diag}(1, \xi)$ and introduce $\tau = \frac{L - x}{\varepsilon}$, then BVP (13) can be rewritten as the following singularly perturbed BVP

$$
\begin{align*}
\begin{cases}
   u'' + Bu' &= \frac{\varepsilon B}{q} F(\varepsilon \tau, u), \\
   u'(0) + Bu(0) &= 0, \\
   u'\left(\frac{L}{\varepsilon}\right) + Bu\left(\frac{L}{\varepsilon}\right) &= 0.
\end{cases}
\end{align*}
$$

Integrating (42) from $\frac{L}{\varepsilon}$ to $\tau$ and applying boundary condition (44) leads to the intergo-differential equations

$$
\begin{align*}
   u' + Bu &= \frac{\varepsilon B}{q} \int_{\frac{L}{\varepsilon}}^{\tau} F(\varepsilon t, u) dt, 0 \leq \tau \leq \frac{L}{\varepsilon}.
\end{align*}
$$

The solution of (45) with (43) is obviously a solution of (42)-(44).

The following lemma can be found in [8].

**Lemma 4.** Assume that $\beta(L) < \gamma(L)$, $R_0 > 1$ and $\varepsilon$ is sufficiently small, then there exist positive constants $K_0$ and $\kappa$ such that

$$
S(0) = \frac{N}{\varepsilon(1 - e^{-\frac{\tau}{\varepsilon}})} + O\left(\frac{1}{\varepsilon e^{-\frac{\tau}{\varepsilon}}}\right), I(0) \leq K_0 e^{-\frac{\tau}{\varepsilon}}.
$$

**Case 1.** $\xi > 1$

Following the same procedure in subsection 3.1, we obtain the ansatz approximate solution of (45)

$$
\tilde{u}(\tau) = e^{-\tau R} V(\tau) + \varepsilon G(\tau, V(\tau)),
$$

where $V(\tau) = e^{\xi R} u(0)$, $R = -\frac{1}{q} \text{diag}(0, \gamma(L) - \beta(L))$ and

$$
G(\tau, V(\tau)) = h(\tau, V(\tau)) e^{-\tau B V(\tau)} + H(\tau) V(\tau) + \frac{1}{q} \int_{\frac{L}{\varepsilon}}^{\tau} F(0, e^{-t B} V(\tau)) dt,
$$

with

$$
\begin{align*}
   h(\tau, V(\tau)) &= \frac{\beta(L)}{q(1 - \xi)} \ln \frac{V_1(\tau) + V_2(\tau)}{V_1(\tau) + V_2(\tau) e^{(1 - \xi) \tau}}, \\
   H(\tau) &= \frac{\gamma(L)}{q(1 - \xi)} \begin{pmatrix} 0 & e^{-\xi \tau} - e^{-\tau} \\ 0 & 0 \end{pmatrix}.
\end{align*}
$$

A straight calculation gives

$$
\tilde{u}' + B\tilde{u} = \frac{\varepsilon B}{q} \int_{\frac{L}{\varepsilon}}^{\tau} F(0, e^{-t B} V(\tau)) dt + \varepsilon D G(\tau, V(\tau)) V'(\tau), 0 \leq \tau \leq \frac{L}{\varepsilon}.
$$


Lemma 5. Let \( G(\tau, V(\tau)) = (G_1(\tau, V(\tau)), G_2(\tau, V(\tau)))^T \), then for \( \tau \in [0, \frac{L}{\varepsilon}] \),
\[
|G_1(\tau, V(\tau))| \leq \frac{K_1}{q(\xi - 1)} S(0)e^{-\tau},
\]
\[
|G_2(\tau, V(\tau))| \leq \frac{K_1}{q(\xi - 1)} I(0)e^{-\tau},
\]
\[
|DG_1(\tau, V(\tau))V'(\tau)| \leq \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} S(0)e^{-\tau},
\]
\[
|DG_2(\tau, V(\tau))V'(\tau)| \leq \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} I(0)e^{-\tau},
\]
where \( K_1 = \beta(L)(1 + \ln(1 + \frac{L(0)}{S(0)})) + \xi \gamma(L) \).

Proof. We note that
\[
G_1(\tau, V(\tau)) = h(\tau, V(\tau)) \frac{\gamma(L)}{q(\xi - 1)} (e^{-\xi \tau} - e^{-\tau})V_2(\tau) + \frac{1}{q} \int_0^\tau f(0, e^{-tB}V(\tau))dt,
\]
therefore,
\[
DG_1(\tau, V(\tau))V'(\tau) = V_1(\tau)e^{-\tau} Dh(\tau, V(\tau))V'(\tau) + \frac{\gamma(L)}{q(\xi - 1)} (e^{-\xi \tau} - e^{-\tau})V_2'(\tau) + \frac{1}{q} \int_0^\tau Df(0, e^{-tB}V(\tau))e^{-tB}V'(\tau)dt.
\]

Adopting the same procedure as in the proof of Lemma 3, for \( \tau \in [0, \frac{L}{\varepsilon}] \), we obtain
\[
\left\| \int_0^\tau f(0, e^{-tB}V(\tau))dt \right\| \leq \int_0^\tau \gamma(L)|V_2(\tau)|e^{-\tau}dt \leq \gamma(L)I(0)e^{-\tau},
\]
\[
\left\| \int_0^\tau Df(0, e^{-tB}V(\tau))e^{-tB}V'(\tau)dt \right\| \leq \int_0^\tau \gamma(L)|V_2'(\tau)|e^{-\xi \tau}dt \leq \frac{\varepsilon \gamma^2(L)}{q} I(0)e^{-\tau}.
\]

Therefore, by (50), for \( \tau \in [0, \frac{L}{\varepsilon}] \),
\[
|G_1(\tau, V(\tau))| \leq |h(\tau, V(\tau))||V_1(\tau)e^{-\tau}| + \|H(\tau)||V_2(\tau)| + \frac{1}{q} \int_0^\tau f(0, e^{-tB}V(\tau))dt|
\]
\[
\leq \frac{\beta(L)}{q(\xi - 1)} \ln(1 + \frac{V_2(\tau)}{V_1(\tau)}) S(0)e^{-\tau} + \frac{\gamma(L)}{q(\xi - 1)} f(0)e^{-\tau} + \gamma(L)I(0)e^{-\tau}
\]
\[
\leq \frac{K_1}{q(\xi - 1)} S(0)e^{-\tau},
\]
since \( I(0) \leq S(0) \) by Lemma 4. Furthermore, by (49) and (51), for \( \tau \in [0, \frac{L}{\varepsilon}] \),
\[
|DG_1(\tau, V(\tau))V'(\tau)| \leq \frac{\varepsilon \gamma(L)\beta(L)}{q^2(\xi - 1)} S(0)e^{-\tau} + \frac{\varepsilon \gamma^2(L)}{q^2(\xi - 1)} I(0)e^{-\tau} + \frac{\varepsilon \gamma^2(L)}{q^2} I(0)e^{-\tau}
\]
\[
\leq \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} S(0)e^{-\tau}.
\]
Similarly, we can obtain the estimates of \( G_2(\tau, V(\tau)) \) and \( DG_2(\tau, V(\tau))V'(\tau) \). \( \Box \)
Proposition 5. Assume that $\beta(L) < \gamma(L)$, $R_0 > 1$ and $\xi > 1$. Let $u(\tau) = (S(\tau), I(\tau))^T$ be the EE of (45) with (43) and $\bar{u}(\tau) = (\bar{S}(\tau), \bar{I}(\tau))^T$ be given by (47), then

$$
|S(\tau) - \bar{S}(\tau)| \leq \frac{\varepsilon^2 \sigma}{q} K_3 S(0), \quad 0 \leq \tau \leq \frac{L}{\varepsilon}, \quad (52)
$$

$$
|I(\tau) - \bar{I}(\tau)| \leq \frac{\varepsilon \sigma}{q} K_3 I(0), \quad 0 \leq \tau \leq \frac{L}{\varepsilon}, \quad (53)
$$

where $K_3 = (3K_2 + 1)e^{\frac{K_1}{L}}$ with $K_2 = \frac{2\xi - 1}{\xi - 1} + \frac{K_1}{\eta(\xi - 1)}$.

Proof. Let $\omega(\tau) = (\omega_S(\tau), \omega_I(\tau))^T = u(\tau) - \bar{u}(\tau)$. Since $\int_0^\tau F(\varepsilon t, u(t))dt = 0$, by (45) and (48), for $\tau \in [0, \frac{L}{\varepsilon}]$,

$$
\omega' + B \omega = \frac{\varepsilon B}{q} \int_0^\tau \left( F(\varepsilon t, u(t)) - F(0, e^{-tB} V(\tau)) \right) dt - \varepsilon DG(\tau, V(\tau)) V'(\tau),
$$

or equivalently,

$$
\omega(\tau) = e^{-\tau B} \omega(0) + \varepsilon \left( \Phi_1 + \Phi_2 + (\phi_1, \phi_2)^T \right), \quad (54)
$$

where

$$
\Phi_1 = \frac{B}{q} \int_0^\tau e^{-(\tau - t)B} \int_t^\tau F(\varepsilon s, u(s))dsdt = \frac{1}{q} \int_0^\tau (1 - e^{-(\tau - t)B}) F(\varepsilon t, u(t))dt,
$$

$$
\Phi_2 = -\frac{B}{q} \int_0^\tau e^{-(\tau - t)B} \int_t^\tau F(0, e^{-sB} V(t))dsdt,
$$

$$
\phi_1 = -\int_0^\tau e^{-(\tau - t)} DG_1(t, V(t)) V'(t)dt,
$$

$$
\phi_2 = -\int_0^\tau e^{-\xi(\tau - t)} DG_2(t, V(t)) V'(t)dt.
$$

By (47) and (50), we obtain

$$
\|\omega(0)\| = \left\| \frac{\varepsilon}{q} \int_0^\tau F(0, e^{-tB} u(0))dt \right\| = \left\| \frac{\varepsilon}{q} \int_0^\tau f(0, e^{-tB} u(0))dt \right\| \leq \frac{\varepsilon \gamma(L)}{q} I(0).
$$

Since $\|F(\varepsilon t, u(t))\| \leq \sigma |\bar{I}(t)|$, it follows from (47) and Lemma 5 that for $\tau \in [0, \frac{L}{\varepsilon}]$,

$$
\|\Phi_1\| \leq \frac{1}{q} \int_0^\tau \|F(\varepsilon t, u(t))\|dt
$$

$$
\leq \frac{1}{q} \int_0^\tau \sigma (|\omega_I(t)| + |\bar{I}(t)|)dt
$$

$$
\leq \frac{\sigma}{q} \int_0^\tau |\omega_I(t)|dt + \frac{\sigma}{q} \int_0^\tau \left( 1 + \frac{\varepsilon K_1}{q(\xi - 1)} \right) I(0)e^{-t}dt
$$

$$
\leq \frac{\sigma}{q} \int_0^\tau |\omega_I(t)|dt + \frac{\sigma}{q} K_2 I(0). \quad (55)
$$
By (50), for $\tau \in [0, \frac{L}{\varepsilon}]$,
\[
\|\Phi_2\| \leq \frac{1}{q} \int_0^\tau \|Be^{-(\tau-t)}B\| \|f(0, e^{-sB}V(t))ds\|dt \\
\quad \leq \frac{1}{q} \int_0^\tau (\varepsilon e^{-\xi(\tau-t)} + e^{-(\tau-t)}\gamma(L)I(0)e^{-\tau})dt \\
\quad \leq \frac{\xi \gamma(L)I(0)}{q(\xi - 1)} e^{-\tau} + \frac{\gamma(L)I(0)}{q} \tau e^{-\tau} \\
\quad \leq \left(\frac{2(\xi - 1)}{}\right) \gamma(L)I(0) \left(\frac{1}{q(\xi - 1)}\right),
\]
(56)
since $\tau e^{-\tau} \leq 1$. By Lemma 5, for $\tau \in [0, \frac{L}{\varepsilon}]$,
\[
|\phi_1| \leq e^{-\tau} \int_0^\tau \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} S(0)dt \leq \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} S(0),
\]
(57)
\[
|\phi_2| \leq e^{-\tau} \int_0^\tau \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} I(0)dt \leq \frac{\varepsilon \gamma(L)K_1}{q^2(\xi - 1)} I(0).
\]
(58)
Noting $I(0) \leq \varepsilon S(0)$ and combining (54)-(58) yields for $\tau \in [0, \frac{L}{\varepsilon}]$,
\[
|\omega_S(\tau)| \leq |\omega_S(0)|e^{-\tau} + \varepsilon(|\|\Phi_1\| + \|\Phi_2\| + |\phi_1|) \\
\quad \leq \frac{\varepsilon \gamma(L)}{q} I(0)e^{-\tau} + \frac{\varepsilon}{q} \int_0^\tau |\omega_I(t)|dt + \frac{\varepsilon}{q} K_2 I(0) \\
\quad \quad + \frac{\varepsilon}{q} \int_0^\tau \frac{\varepsilon \gamma(L)}{q^2(\xi - 1)} S(0)dt \\
\quad \leq \frac{\varepsilon}{q} \int_0^\tau |\omega_I(t)|dt + \frac{\varepsilon}{q} (K_2 + 1) I(0) + \frac{\varepsilon}{q} K_2 S(0),
\]
(59)
\[
|\omega_I(\tau)| \leq |\omega_I(0)|e^{-\tau} + \varepsilon(|\|\Phi_1\| + \|\Phi_2\| + |\phi_2|) \\
\quad \leq \frac{\varepsilon}{q} \int_0^\tau |\omega_I(t)|dt + \frac{\varepsilon}{q} (2K_2 + 1) I(0).
\]
(60)
Applying Gronwall’s inequality to (60), we obtain for $\tau \in [0, \frac{L}{\varepsilon}]$,
\[
|\omega_I(\tau)| \leq \frac{\varepsilon}{q} (2K_2 + 1) I(0)e^{\frac{\varepsilon \tau}{q}},
\]
(61)
which leads to (53). By (59) and (61), for $\tau \in [0, \frac{L}{\varepsilon}]$,
\[
|\omega_S(\tau)| \leq \frac{\varepsilon}{q} (2K_2 + 1) I(0)e^{\frac{\varepsilon \tau}{q}} + \frac{\varepsilon \gamma(L)}{q} K_2 S(0) \leq \frac{\varepsilon}{q} K_3 S(0).
\]
The proof is completed. \(\square\)

Taking (44) and (H) into consideration yields the following result.

**Proposition 6.** Assume that $\beta(L) < \gamma(L)$, $R_0 > 1$ and $\xi > 1$. Let $\ddot{u}(\tau) = (\dot{S}(\tau), I(\tau))^T$ be given by (47), then
\[
\ddot{u}'(0) + B\ddot{u}(0) = O(e^{-\frac{\tau}{\varepsilon}}),
\]
(62)
\[
\varepsilon \int_0^{\frac{L}{\varepsilon}} \dot{S}(t) + \dot{I}(t)dt = N + O(\varepsilon).
\]
(63)
Proof. Let $\tau = 0$ in (48), we obtain
$$
\tilde{u}'(0) + B\tilde{u}(0) = \frac{\varepsilon B}{q} \int_0^\frac{\varepsilon}{q} F(0, e^{-tB}u(0))dt + \frac{\varepsilon}{q} \int_0^\frac{\varepsilon}{q} DF(0, e^{-tB}u(0))e^{-tB}V'(0)dt.
$$
It follows from (50), (51) and (46) that
$$
\|\tilde{u}'(0) + B\tilde{u}(0)\| \\
\leq \frac{\varepsilon\|B\|}{q} \int_0^\frac{\varepsilon}{q} f(0, e^{-tB}u(0))dt + \frac{\varepsilon}{q} \int_0^\frac{\varepsilon}{q} Df(0, e^{-tB}u(0)) e^{-tB}V'(0)dt \\
\leq \frac{\varepsilon\gamma(L)}{q} (1 + \frac{\varepsilon}{q}\gamma(L))I(0) \\
\leq \frac{\varepsilon\gamma(L)}{q} (1 + \frac{\varepsilon}{q}\gamma(L))K_0 e^{-\frac{\varepsilon}{q}},
$$
which leads to (62). We can obtain (63) by following the same proof of (36) in Proposition 2. \hfill \Box

Remark 4. (1) From the Proposition 5 and 6, we know that $\tilde{u}(\tau)$ be given by (47) is actually an approximate EE of (42)-(44) with (H).

(2) By Lemma 5 and Proposition 6, it can be obtained that for $\tau \in [0, \frac{\varepsilon}{q}]$,
$$
|I(\tau)| \leq \frac{\varepsilon\sigma}{q} K_3 I(0) + |\tilde{I}(\tau)| \\
\leq \frac{\varepsilon\sigma}{q} K_3 I(0) + (1 + \frac{\varepsilon}{q}\gamma(L))I(0)e^{-\tau} \\
\leq (1 + \frac{\varepsilon\sigma}{q} K_3 + \frac{\varepsilon}{q}\gamma(L))K_0 e^{-\frac{\varepsilon}{q}},
$$
which means that $I(\tau)$ decays exponentially.

(3) It follows from (46) that there exists a positive constant $K^*$ such that when $\varepsilon$ is sufficiently small,
$$
|S(0)| \leq \frac{2N}{\varepsilon} |S(0) - \frac{N}{\varepsilon}e^{-\tau}| = \left|\frac{Ne^{-\frac{\varepsilon}{q}}}{\varepsilon(1 - e^{-\frac{\varepsilon}{q}})} + O\left(\frac{1}{\varepsilon}e^{-\frac{\varepsilon}{q}}\right)\right| \leq \frac{K^*}{\varepsilon} e^{-\frac{\varepsilon}{q}},
$$
where $\kappa^* = \min\{\kappa, L\}$. According to Lemma 5 and Proposition 6, for $\tau \in [0, \frac{\varepsilon}{q}]$,
$$
|S(\tau) - \frac{N}{\varepsilon}e^{-\tau}| \leq |S(\tau) - \tilde{S}(\tau)| + |\tilde{S}(\tau) - \frac{N}{\varepsilon}e^{-\tau}| \\
\leq \frac{\varepsilon^2\sigma}{q} K_3 S(0) + |S(0) - \frac{N}{\varepsilon}e^{-\tau} + \frac{\varepsilon}{q}\gamma(L)G_1(\tau, V(\tau))| \\
\leq \frac{2\varepsilon^2\sigma}{q} NK_4 + \frac{K^*}{\varepsilon} e^{-\frac{\varepsilon}{q}} + \frac{2NK_1}{q(\xi - 1)} e^{-\tau},
$$
which means that
$$
\lim_{\tau \to \infty} |S(\tau) - \frac{N}{\varepsilon}e^{-\tau}| = 0.
$$

Case 2. $0 < \xi < 1$
In this case, we obtain the ansatz approximate solution
$$
\tilde{u}(\tau) = e^{-\tau B}V(\tau) + \varepsilon G(\tau, V(\tau)),
$$
where \( V(\tau) = e^{\sigma R\tau} u(0) \), \( R = -\frac{1}{q} \text{diag}(\beta(L), \gamma(L)) \) and

\[
G(\tau, V(\tau)) = h(\tau, V(\tau)) e^{-\tau B} V(\tau) + H(\tau) V(\tau) + \frac{1}{q} \int_{\tau}^{\tau_0} F(0, e^{-tB} V(\tau)) dt,
\]

with

\[
h(\tau, V(\tau)) = \frac{\beta(L)}{q(1 - \xi)} \ln \frac{V_1(\tau) + V_2(\tau)}{V_1(\tau)e^{(\xi - 1)\tau} + V_2(\tau)},
\]

\[
H(\tau) = \frac{\gamma(L)}{q(1 - \xi)} \begin{pmatrix} e^{-\xi\tau} - e^{-\tau} & 0 \\ 0 & 0 \end{pmatrix}.
\]

**Proposition 7.** Assume that \( \beta(L) < \gamma(L) \), \( R_0 > 1 \) and \( 0 < \xi < 1 \). Let \( u(\tau) = (S(\tau), I(\tau))^T \) be the solution of (45) with (43) and \( \bar{u}(\tau) = (\bar{S}(\tau), \bar{I}(\tau))^T \) be given by (65), then

\[
|S(\tau) - \bar{S}(\tau)| \leq \frac{\epsilon^2 q \sigma}{K_4} S(0), \quad |I(\tau) - \bar{I}(\tau)| \leq \frac{\epsilon \sigma}{q} K_4 I(0), \quad 0 \leq \tau \leq \frac{L}{\epsilon},
\]

where

\[
K_4 = \left( \frac{4(2 - \xi)}{\xi(1 - \xi)} + \frac{3K_3}{q\xi(1 - \xi)} \right) \left( \frac{\epsilon}{\xi} + 1 \right) e^{\frac{\epsilon}{4}},
\]

\[
K_3 = \frac{\epsilon^2 K_4}{q} \left( 1 + \ln(1 + \frac{S(0)}{\tau_0}) - \frac{(\gamma(L) - \beta(L))L}{\epsilon} \right) + \gamma(L).
\]

Furthermore, \( \bar{u}(\tau) \) satisfies (62)-(63).

**Case 3.** \( \xi = 1 \)

In this case, we obtain the ansatz approximate solution

\[
\bar{u}(\tau) = e^{-\tau B} V(\tau) + \frac{\epsilon}{q} F(0, V(\tau))(e^{-\tau} - e^{-\frac{\epsilon}{4}}),
\]

(66)

where \( V(\tau) \) satisfies RG equations

\[
V'(\tau) = -\frac{\epsilon}{q} F(0, V(\tau)), \quad V(0) = u(0),
\]

which can be solved as

\[
V_1(\tau) = \left( \frac{\gamma(L) - \beta(L)}{\beta(L) - \gamma(L)} \right) \frac{e^{\frac{\epsilon}{4}}} {\beta(L) - \gamma(L)} C_1 \bar{C}_2 + C_1 \bar{C}_2 (\gamma(L) - \beta(L)) C_1 e^{-\frac{\epsilon}{4}}, \quad V_2(\tau) = \left( \frac{\gamma(L) - \beta(L)}{\beta(L) - \gamma(L)} \right) \frac{e^{\frac{\epsilon}{4}}} {\beta(L) - \gamma(L)} C_1 \bar{C}_2 + C_1 \bar{C}_2 (\gamma(L) - \beta(L)) C_1 e^{-\frac{\epsilon}{4}},
\]

where \( \bar{C}_1 = S(0) + I(0) \) and \( \bar{C}_2 = \frac{S(0) + I(0)}{S(0) + I(0)} I(0) \).

**Proposition 8.** Assume that \( \beta(L) < \gamma(L) \), \( R_0 > 1 \) and \( \xi = 1 \). Let \( u(\tau) = (S(\tau), I(\tau))^T \) be the EE of (45) with (43) and \( \bar{u}(\tau) = (\bar{S}(\tau), \bar{I}(\tau))^T \) be given by (66), then

\[
|S(\tau) - \bar{S}(\tau)| \leq \frac{\epsilon \sigma}{q} K_6 I(0), \quad |I(\tau) - \bar{I}(\tau)| \leq \frac{\epsilon \sigma}{q} K_6 I(0), \quad 0 \leq \tau \leq \frac{L}{\epsilon},
\]

where \( K_6 = (3 + \frac{2\epsilon^2}{q}) e^{\frac{\epsilon}{4}} \). Furthermore, \( \bar{u}(\tau) \) satisfies (62)-(63).

Summarizing what we have proved yields the following theorem.
Theorem 3.2. Assume that $\beta(L) < \gamma(L)$, $R_0 > 1$ and $d_S$ is sufficiently small, then any EE $(S(x), I_x(x))$ of (42)-(44) with (H) satisfies

$$\left| S(x) - \bar{S}(x) \right| \leq \frac{d_S^2 \sigma}{q^2} K_S S(0), \quad \left| I_x(x) - \bar{I}(x) \right| \leq \frac{d_S \sigma}{q^2} K_I I(0), \quad 0 \leq x \leq L,$$

where for $\frac{d_S}{q^2} > 1$, $(\bar{S}, \bar{I})^T$ is given by (47) and $K_S = K_1 = K_3$; for $0 < \frac{d_S}{q^2} < 1$, $(\bar{S}, \bar{I})^T$ is given by (65), $K_S = \frac{d_S}{q^2} K_4$ and $K_I = K_4$; for $\frac{d_S}{q^2} = 1$, $(\bar{S}, \bar{I})^T$ is given by (66) and $K_S = K_I = K_6$.

Remark 5. By Remark 3.4, we can obtain that Theorem 3.2 leads to $I(x) \leq K_\tau e^{-\frac{x}{\bar{S}}}$ for some appropriate positive constant $K_\tau$. Furthermore, since we have constructed the approximate solution $\bar{S}(x)$ and obtained the error estimate between $S(x)$ and $\bar{S}(x)$, then we can adopt the same method as in the proof of (64) to obtain that for $x \in [0, L]$,

$$\lim_{d_S \to 0} \left| S(x) - \frac{qN}{d_S} e^{-\frac{q(x-s)}{d_S}} \right| = 0,$$

which is consistent with Theorem 4.1 in [8].

4. Conclusion. In this paper, we studied the SIS reaction-diffusion-advection model in a new perspective of singular perturbation theory by using RG method, which has not been done in previous studies to the best of our knowledge. In order to deal with system (1) and (H) with RG method, we introduced an intergo-differential equations using one boundary value condition, and applied RG method to construct an approximate solution of the intergo-differential equations. Then we proved that the approximate solution we constructed satisfying another boundary value condition is the approximate solution of (1) and (H) under some conditions. Our results extend the applicability of RG method, which is of great significance to study other BVPs.

Furthermore, we gave the error of the exact solution and the approximate solution we constructed, and compared our results with those in [8], which illustrates the effectiveness of our results.

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Received August 2019; revised November 2019.

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