Some exact solutions of the heat conduction equation in parallelepiped obtained by the fast expansions method

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Abstract. An analytical solution to the boundary value problem for the Poisson equation in a parallelepiped was obtained in general form using the fast expansions method. The construction of exact solutions of the heat conduction equation for the cases of an internal source depending on coordinates was shown in the work. The study of the influence of the variability of the internal source on the propagation of heat in the body at the same initial temperatures at the parallelepiped vertices was carried out. In this case, the heat in the body will be transferred only in those directions in which the internal source is changed.

1. Introduction
Various numerical methods are used to solve spatial problems of heat conduction. The possibility of using the method of fictitious canonical domains for solving problems of stationary heat conduction in complex three-dimensional domains is shown in [1]. An example of a numerical solution of a boundary value problem for a three-dimensional heat conduction equation with a moving boundary in a continuous medium with discontinuous thermophysical parameters using adaptive spatial hexahedral grids is given in [2]. The method of least squares with T-elements for solving linear boundary value problems with the Laplace and Poisson equations was developed in [3]. Authors use discontinuous high-order basic approximation functions from special functional spaces. An algorithm for solving the general inhomogeneous Dirichlet boundary value problem for the three-dimensional Poisson equation on a parallelepiped with the sixth order of error and with a minimum 27-point pattern was proposed in [4]. Among the numerical methods, the collocation method [5], the quadrature element method [6], the modified cubic B-spline differential-quadrature method [7] and the method based on the use of Haar wavelets [8], generalized finite difference method [9, 10], differential element expansion method [11] should be noted as well.

Along with the numerical solutions of the Poisson equation, there are also analytical solutions. Thus, in [12], an approach was proposed to obtain some exact solutions of the Poisson equation based on the introduction of terms into the Poisson equation that contain the first derivatives of the desired function. To solve the equation obtained in such a way in a system of ordinary differential equations, a related system of two partial differential equations is considered. However, in [12] exact solutions of boundary value problems are not examined. In [13], the Fourier transform method was used to solve the Dirichlet boundary value problem for the Poisson equation in a domain bounded by two parallel hyperplanes in $\mathbb{R}^n$. The solution is presented as a sum of integrals, the kernels of which are found in a finite form. A general approach to solving problems of heat conduction with internal heat sources...
using the concepts of resistance and quadrupole is shown in [14]. The method for obtaining analytical solutions of boundary value problems of heat conduction with time-varying internal heat sources which makes it possible to obtain solutions of satisfactory accuracy in the entire range of the Fourier number variation, is studied in [15] using the integral method of heat balance based on the introduction of a front of temperature perturbation and additional boundary conditions.

In this paper, we will use the fast expansions method [16] developed by Professor A.D. Chernyshov. This method makes it possible to obtain not only new approximate analytical solutions of problems [17–19], but also new exact ones [20]. With the help of fast expansions [16], a general solution of the problem of heat conduction in a parallelepiped with a variable internal source, which exactly satisfies the differential equation and boundary conditions, will be obtained, and the construction of exact solutions for the cases of an internal source depending on one or two and three coordinates.

2. Materials and methods

Let us consider the problem of stationary heat conduction for a parallelepiped \( \Omega = \{(x, y, z) \in \mathbb{R}, 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \} \). We write the heat conduction equation in the Poisson form as a second-order partial differential equation with respect to variables \( x, y, z \) with a given internal source \( F(x, y, z) \):

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + F(x, y, z) = 0, \ (x, y, z) \in \Omega, \ 0 \leq x \leq a, \ 0 \leq y \leq b, \ 0 \leq z \leq c. \tag{1}
\]

The boundary conditions on each parallelepiped plane can be written as

\[
T_{\mid x=0} = f_1(y, z), \ \ T_{\mid y=0} = f_2(x, z), \ \ T_{\mid z=0} = f_3(x, y),
\]

\[
T_{\mid x=a} = f_4(y, z), \ \ T_{\mid y=b} = f_5(x, z), \ \ T_{\mid z=c} = f_6(x, y). \tag{2}
\]

Functions \( f_1, \ldots, f_6 \) in (2) must satisfy the compatibility conditions, since the temperature \( T(x, y, z) \) must take the same values, for example, when approaching an edge \((x = 0, z = 0)\) along two adjacent planes. Ensuring similar conditions on all the parallelepiped edges, we will have the following functional equalities

\[
f_1(0, z) = f_2(0, z), \ \ f_1(b, z) = f_2(0, z), \ \ f_2(a, z) = f_3(0, z), \ \ f_1(b, z) = f_3(a, z),
\]

\[
f_2(x, 0) = f_3(x, 0), \ \ f_2(x, c) = f_4(x, 0), \ \ f_3(x, 0) = f_4(x, b), \ \ f_2(x, c) = f_5(x, b),
\]

\[
f_3(y, 0) = f_4(0, y), \ \ f_3(y, c) = f_5(0, y), \ \ f_4(y, 0) = f_6(a, y), \ \ f_3(y, c) = f_6(a, y). \tag{3}
\]

To equalities (3) we add the matching conditions at the parallelepiped vertices

\[
f_1(0, 0) = f_2(0, 0) = f_3(0, 0), \ \ f_1(b, 0) = f_2(0, b) = f_3(0, 0),
\]

\[
f_1(0, c) = f_2(0, c) = f_3(0, c) = f_4(0, c) = f_5(0, 0), \ \ f_2(a, 0) = f_3(a, 0) = f_4(0, 0), \ \ f_3(a, c) = f_4(0, c) = f_5(a, 0), \ \ f_2(b, c) = f_3(b, c) = f_4(b, c) = f_6(a, b). \tag{4}
\]

Equalities (4) follow from the independence of the temperature \( T(x, y, z) \) on the direction of approach to the parallelogram vertices.

In addition to conditions (3) and (4), we write down the conditions for performing the differential equation (1) at 8 parallelepiped vertices:
\begin{align}
T_{xx}(0,0,0) + T_{yy}(0,0,0) + T_{zz}(0,0,0) + F(0,0,0) &= 0,
T_{xx}(a,0,0) + T_{yy}(a,0,0) + T_{zz}(a,0,0) + F(a,0,0) &= 0,
T_{xx}(0,b,0) + T_{yy}(0,b,0) + T_{zz}(0,b,0) + F(0,b,0) &= 0,
T_{xx}(0,0,c) + T_{yy}(0,0,c) + T_{zz}(0,0,c) + F(0,0,c) &= 0,
T_{xx}(a,b,0) + T_{yy}(a,b,0) + T_{zz}(a,b,0) + F(a,b,0) &= 0,
T_{xx}(a,0,c) + T_{yy}(a,0,c) + T_{zz}(a,0,c) + F(a,0,c) &= 0,
T_{xx}(0,b,c) + T_{yy}(0,b,c) + T_{zz}(0,b,c) + F(0,b,c) &= 0,
T_{xx}(a,b,c) + T_{yy}(a,b,c) + T_{zz}(a,b,c) + F(a,b,c) &= 0.
\end{align}

(5)

The requirement of temperature continuity in the vicinity of edges and vertices leads to the need to meet conditions (3) – (5). If conditions (3) – (5) are not met, then the temperature will undergo a rupture, which is physically contradictory.

The assumed equation for the exact solution $T(x,y,z)$ will be represented by the final equation as the sum of the zero-order boundary function and the Fourier series in sines, in which two Fourier coefficients are taken into account [16]

\begin{align}
T(x,y,z) &= A_i(y,z) \left(1 - \frac{x}{a}\right) + A_2(y,z) \frac{x}{a} + A_3(y,z) \sin \frac{\pi x}{a} + A_4(y,z) \sin 2\pi \frac{x}{a}, \quad 0 \leq x \leq a, \\
A_i(y,z) &= A_{i,1}(z) \left(1 - \frac{y}{b}\right) + A_{i,2}(z) \frac{y}{b} + A_{i,3}(z) \sin \frac{\pi y}{b} + A_{i,4}(z) \sin 2\pi \frac{y}{b}, \quad i = 1 + 4, \quad 0 \leq y \leq b, \\
A_{i,j}(z) &= A_{i,j,1}(z) \left(1 - \frac{z}{c}\right) + A_{i,j,2}(z) \frac{z}{c} + A_{i,j,3}(z) \sin \frac{\pi z}{c} + A_{i,j,4}(z) \sin 2\pi \frac{z}{c}, \quad j = 1 + 4, \quad 0 \leq z \leq c.
\end{align}

(6)

The equation (6) can be used to describe a sufficiently wide class of practically important dependencies. Let us introduce the notation

\begin{align}
P_i(x) &= 1 - \frac{x}{a}, \quad P_2(x) = \frac{x}{a}, \quad P_i(y) = 1 - \frac{y}{b}, \quad P_2(y) = \frac{y}{b}, \quad P_i(z) = 1 - \frac{z}{c}, \quad P_2(z) = \frac{z}{c}, \\
P_i(x) &= \sin \frac{\pi x}{a}, \quad P_i(y) = \sin \frac{\pi y}{b}, \quad P_i(z) = \sin \frac{\pi z}{c}, \\
P_i(y) &= \sin 2\pi \frac{y}{b}, \quad P_i(z) = \sin 2\pi \frac{z}{c}.
\end{align}

(7)

Equation (6) can be written more briefly if we take into account (7),

\begin{align}
T(x,y,z) &= \sum_{i=1}^{4} \left( \sum_{j=1}^{4} \sum_{k=1}^{4} A_{i,j,k} P_i(z) P_j(y) \right) P_i(x), \quad 0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c.
\end{align}

(8)

Thus, the required function $T(x,y,z)$ is represented as a finite triple sum containing 64 unknown coefficients

$$A_{i,j,k}, \quad i = 1 + 4, \quad j = 1 + 4, \quad k = 1 + 4.$$  

(9)

We represent the internal source $F(x,y,z)$ as a finite sum by analogy with the equation (8):

\begin{align}
F(x,y,z) &= \sum_{i=1}^{4} \left( \sum_{j=1}^{4} \sum_{k=1}^{4} F_{i,j,k} P_i(z) P_j(y) \right) P_i(x), \quad 0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c.
\end{align}

(10)
All coefficients $F_{i,j,k}$ in (10) for the source are assumed to be known.

Let us define the functions $f_1(y,z), f_2(x,z), f_3(x,y), f_4(y,z), f_5(x,z), f_6(x,y)$ included in the boundary conditions (2) as follows

$$
f_i(y,z) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{i,j,k} P_i(z) \right) P_j(y), \quad f_2(x,z) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{2,j,k} P_i(z) \right) P_j(x),$$

$$
f_3(x,y) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{3,j,k} P_i(z) \right) P_j(x), \quad f_4(y,z) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{4,j,k} P_i(z) \right) P_j(y), \quad f_5(x,z) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{5,j,k} P_i(z) \right) P_j(x), \quad f_6(x,y) = \sum_{j=1}^{4} \left( \sum_{k=1}^{4} f_{6,j,k} P_i(z) \right) P_j(x),$$

(11)

Here we consider $f_{i,j,k}$ to be the known constants.

Thus, it is required to find a solution to equation (1) with a given internal source in the form (10), which exactly satisfies the boundary conditions (2) and the matching conditions (3) - (5).

To find the unknown coefficients $A_{i,j,k}$ from (9), we apply the fast expansions method, according to which we substitute the triple fast expansion of the function $T(x,y,z)$ into the boundary conditions (2), the matching conditions (3) – (5), and the differential equation (1). Then we set up a system of linear algebraic equations for the unknown (9). The resulting system has an analytical solution

$$A_{i,j,k} = f_{i,j,k}, \quad j = 1 \div 4, \quad k = 1 \div 4; \quad A_{2,i,k} = f_{2,i,k}, \quad k = 1 \div 4; \quad A_{2,i,k} = f_{2,i,k}, \quad j = 2,3,4, \quad k = 1 \div 4;$$

$$A_{3,1,k} = f_{3,1,k}, \quad k = 1 \div 4; \quad A_{3,2,k} = f_{3,2,k}, \quad k = 1 \div 4; \quad A_{3,3,1} = f_{3,3,1}, \quad A_{3,3,2} = f_{3,3,2},$$

$$A_{3,3,3} = F_{3,3,3} \left( \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} \right), \quad A_{3,4,3} = F_{3,4,3} \left( \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{4\pi^2}{c^2} \right), \quad A_{3,4,4} = f_{3,4,4}, \quad A_{3,4,2} = f_{6,4,4},$$

$$A_{3,4,4} = F_{3,4,4} \left( \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{4\pi^2}{c^2} \right), \quad A_{4,3,i} = f_{4,3,i}, \quad A_{4,3,2} = f_{6,4,3}, \quad A_{4,3,3} = F_{4,3,3} \left( \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} \right),$$

$$A_{4,3,4} = F_{4,3,4} \left( \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{4\pi^2}{c^2} \right), \quad A_{4,4,i} = f_{4,4,i}, \quad A_{4,4,2} = f_{6,4,4}, \quad A_{4,4,3} = F_{4,4,3} \left( \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{4\pi^2}{c^2} \right), \quad A_{4,4,4} = F_{4,4,4} \left( \frac{4\pi^2}{a^2} + \frac{4\pi^2}{b^2} + \frac{4\pi^2}{c^2} \right).$$

(12)

Putting coefficients (12) into equation (8), we will have an exact solution to the problem.

When setting the boundary conditions and the internal source, the following conditions must be met:

$$f_{1,1,1} = f_{2,1,1}, \quad f_{2,1,2} = f_{2,2,1}, \quad f_{1,1,2} = f_{2,1,2}, \quad f_{1,1,3} = f_{2,1,3}, \quad f_{1,2,1} = f_{2,2,1},$$

$$f_{1,2,2} = f_{2,2,2}, \quad f_{1,2,3} = f_{2,2,3}, \quad f_{1,2,4} = f_{2,2,4}, \quad f_{1,3,1} = f_{2,3,1}, \quad f_{1,3,2} = f_{2,3,2}, \quad f_{1,3,3} = f_{2,3,3}, \quad f_{1,3,4} = f_{2,3,4},$$

$$f_{2,2,1} = f_{4,1,1}, \quad f_{2,2,2} = f_{4,2,2}, \quad f_{2,2,3} = f_{4,2,3}, \quad f_{2,2,4} = f_{4,2,4},$$

$$f_{2,3,1} = f_{4,3,1}, \quad f_{2,3,2} = f_{4,3,2}, \quad f_{2,3,3} = f_{4,3,3}, \quad f_{2,3,4} = f_{4,3,4}, \quad f_{3,1,1} = f_{4,1,1}, \quad f_{3,1,2} = f_{4,1,2}, \quad f_{3,1,3} = f_{4,1,3}, \quad f_{3,1,4} = f_{4,1,4},$$

$$f_{3,2,1} = f_{4,2,1}, \quad f_{3,2,2} = f_{4,2,2}, \quad f_{3,2,3} = f_{4,2,3}, \quad f_{3,2,4} = f_{4,2,4},$$

$$f_{3,3,1} = f_{4,3,1}, \quad f_{3,3,2} = f_{4,3,2}, \quad f_{3,3,3} = f_{4,3,3}, \quad f_{3,3,4} = f_{4,3,4},$$

$$f_{4,1,1} = f_{5,1,1}, \quad f_{4,1,2} = f_{5,1,2}, \quad f_{4,1,3} = f_{5,1,3}, \quad f_{4,1,4} = f_{5,1,4},$$

$$f_{4,2,1} = f_{5,2,1}, \quad f_{4,2,2} = f_{5,2,2}, \quad f_{4,2,3} = f_{5,2,3}, \quad f_{4,2,4} = f_{5,2,4},$$

$$f_{4,3,1} = f_{5,3,1}, \quad f_{4,3,2} = f_{5,3,2}, \quad f_{4,3,3} = f_{5,3,3}, \quad f_{4,3,4} = f_{5,3,4},$$

$$f_{4,4,1} = f_{5,4,1}, \quad f_{4,4,2} = f_{5,4,2}, \quad f_{4,4,3} = f_{5,4,3}, \quad f_{4,4,4} = f_{5,4,4},$$

(13)
\[ f_{1, i, j} = f_{3, i, j}, \quad f_{2, i, j} = f_{6, i, j}, \quad f_{5, i, j} = f_{3, i, j}, \quad f_{3, i, j} = f_{6, i, j}. \]

\[ F_{1, i, j} = F_{4, i, j}, \quad F_{2, i, j} = F_{6, i, j}, \quad F_{3, i, j} = F_{6, i, j}, \quad F_{3, i, j} = F_{6, i, j} = 0. \]  

(14)

\[ f_{i, j, k} = \frac{c^2}{\pi} F_{i, j, k}, \quad f_{i, j, k} = \frac{c^2}{4\pi^2} F_{i, j, k}, \quad j = 1, 2; \quad f_{i, j, k} = \frac{b^2}{\pi^2} F_{i, j, k}, \quad f_{i, j, k} = \frac{b^2}{4\pi} F_{i, j, k}, \quad k = 1, 2; \]

(15)

3. Results and discussion

Let us define an internal source acting in a parallelepiped as follows. We take \( F_3(y, z) = F_4(y, z) = 0 \) in (10), thus the source will have the form

\[ F(x, y, z) = F_3(y, z) P_1(x) + F_4(y, z) P_2(x). \]

(16)

In the functions \( F_1(y, z) \) and \( F_2(y, z) \) from (16), we take \( F_{1, i, j}(z) = F_{4, i, j}(z) = F_{2, i, j}(z) = 0. \)

As a result, we get

\[ F_1(y, z) = F_{1, i, j}(z) P_1(y) + F_{2, i, j}(z) P_2(y), \quad F_2(y, z) = F_{2, i, j}(z) P_1(y) + F_{2, i, j}(z) P_2(y). \]

In the functions \( F_1(i, j), F_{1, 2}(z), F_{2, 4}(z), F_{2, 2}(z) \) we define \( F_{1, i, j} = F_{2, i, j} = F_{2, i, j} = F_{2, i, j} = 0 \) and, taking into account equalities (14), these functions take the form
Using equalities (7) and introducing the notation

\[
F_{1,1,3} = Q_1, \quad F_{1,2,3} = Q_2, \quad F_{2,1,3} = Q_3, \quad F_{2,2,3} = Q_4
\]

we can finally write

\[
F(x,y,z) = \left[ Q_1 \left( 1 - \frac{y}{b} \right) \sin \pi \frac{z}{c} + Q_2 \left( 1 - \frac{y}{b} \right) \sin \pi \frac{z}{c} \right] \left( 1 - \frac{x}{a} \right) + \left[ Q_3 \left( 1 - \frac{y}{b} \right) \sin \pi \frac{z}{c} + Q_4 \left( 1 - \frac{y}{b} \right) \sin \pi \frac{z}{c} \right] \frac{x}{a}. \tag{17}
\]

A variable internal source is widely used in heat and mass transfer processes modeling [15, 21]. Let us write down the boundary conditions that will be met for the source (17). Let some functions in equalities (11) be equal to zero

\[
f_{1,j,k}(z) = f_{2,j,k}(z) = f_{3,j,k}(z) = f_{4,j,k}(z) = f_{5,j,k}(z) = f_{6,j,k}(y) = 0, \quad j = 3; 4
\]

as well as the coefficients are equal to zero

\[
f_{1,1,4} = f_{1,2,4} = f_{2,1,4} = f_{2,2,4} = f_{3,1,4} = f_{3,2,4} = f_{4,1,4} = f_{4,2,4} = f_{5,1,4} = f_{5,2,4} = f_{6,1,4} = f_{6,2,4} = 0.
\]

In this case, taking into account equalities (13) boundary conditions (2) take the form

\[
T|_{y=0} = f_1(y,z) = \left[ f_{1,1,1}(z) + f_{1,2,1}(z) + f_{1,3,1}(z) \right] P_1(y) + \left[ f_{1,2,1}(z) + f_{1,3,1}(z) \right] P_1(y),
\]

\[
T|_{x=0} = f_2(x,z) = \left[ f_{2,1,1}(z) + f_{2,2,1}(z) + f_{2,3,1}(z) \right] P_2(x) + \left[ f_{2,2,1}(z) + f_{2,3,1}(z) \right] P_2(x),
\]

\[
T|_{z=0} = f_3(x,y) = \left[ f_{3,1,1}(y) + f_{3,2,1}(y) \right] P_3(x) + \left[ f_{3,2,1}(y) + f_{3,3,1}(y) \right] P_3(x),
\]

\[
T|_{y=0} = f_4(y,z) = \left[ f_{4,1,1}(z) + f_{4,2,1}(z) + f_{4,3,1}(z) \right] P_4(y) + \left[ f_{4,2,1}(z) + f_{4,3,1}(z) \right] P_4(y),
\]

\[
T|_{x=0} = f_5(x,z) = \left[ f_{5,1,1}(z) + f_{5,2,1}(z) + f_{5,3,1}(z) \right] P_5(x) + \left[ f_{5,2,1}(z) + f_{5,3,1}(z) \right] P_5(x),
\]

\[
T|_{z=0} = f_6(x,y) = \left[ f_{6,1,1}(y) + f_{6,2,1}(y) \right] P_6(x) + \left[ f_{6,2,1}(y) + f_{6,3,1}(y) \right] P_6(x). \tag{18}
\]

The choice of the coefficients values included in (18) was carried out taking into account (13) and (15). Hence, applying equalities (7) and introducing the following notation

\[
\begin{align*}
f_{1,1,1} = f_{2,1,1} = f_{3,1,1} = T_1, & \quad f_{1,2,1} = f_{2,2,1} = f_{6,1,1} = T_2, & \quad f_{1,3,1} = f_{2,3,1} = \frac{c^2}{\pi} Q_1, \\
f_{1,2,1} = f_{3,1,2} = f_{3,2,1} = T_3, & \quad f_{1,1,2} = f_{3,1,2} = f_{6,1,2} = T_4, & \quad f_{1,2,2} = f_{3,2,2} = \frac{c^2}{\pi} Q_2, \\
f_{2,2,1} = f_{4,1,1} = f_{4,2,1} = T_5, & \quad f_{2,2,2} = f_{4,2,2} = f_{6,2,1} = T_6, & \quad f_{2,2,2} = f_{4,2,3} = \frac{c^2}{\pi} Q_3, \\
f_{4,2,1} = f_{5,2,1} = f_{5,2,2} = T_7, & \quad f_{4,2,2} = f_{5,2,2} = f_{6,2,2} = T_8, & \quad f_{4,2,3} = f_{5,2,3} = \frac{c^2}{\pi} Q_4.
\end{align*} \tag{19}
\]
we will get the boundary conditions for the temperature

\[ T|_{x=0} = \left( T_1 \left( 1 - \frac{z}{c} \right) + T_2 \frac{z}{c} + \frac{c^2}{\pi^2} Q \sin \frac{\pi z}{c} \right) \left( 1 - \frac{y}{b} \right) + \left( T_3 \left( 1 - \frac{z}{c} \right) + T_4 \frac{z}{c} + \frac{c^2}{\pi^2} Q_1 \sin \frac{\pi z}{c} \right) \frac{y}{b}, \]

\[ T|_{y=0} = \left( T_1 \left( 1 - \frac{z}{c} \right) + T_2 \frac{z}{c} + \frac{c^2}{\pi^2} Q_2 \sin \frac{\pi z}{c} \right) \left( 1 - \frac{x}{a} \right) + \left( T_3 \left( 1 - \frac{z}{c} \right) + T_4 \frac{z}{c} + \frac{c^2}{\pi^2} Q_3 \sin \frac{\pi z}{c} \right) \frac{x}{a}, \]

\[ T|_{z=0} = \left( T_1 \left( 1 - \frac{y}{b} \right) + T_2 \frac{y}{b} \right) \left( 1 - \frac{x}{a} \right) + \left( T_3 \left( 1 - \frac{y}{b} \right) + T_4 \frac{y}{b} \right) \frac{x}{a}, \]

By putting coefficients from (19) into (12) and, taking into account the coefficients taken equal to zero from the boundary conditions and the internal source, we obtain

\[ A_{1,1,1} = T_1, \quad A_{1,1,2} = T_2, \quad A_{1,1,3} = \frac{c^2}{\pi^2} Q_1, \quad A_{1,2,1} = T_3, \quad A_{1,2,2} = T_4, \quad A_{1,2,3} = \frac{c^2}{\pi^2} Q_2, \]

\[ A_{2,1,1} = T_5, \quad A_{2,1,2} = T_6, \quad A_{2,1,3} = \frac{c^2}{\pi^2} Q_2, \quad A_{2,2,1} = T_7, \quad A_{2,2,2} = T_8, \quad A_{2,2,3} = \frac{c^2}{\pi^2} Q_4. \]

The exact solution of equation (1), corresponding to conditions (2), with the internal source (17) takes the form

\[ T(x, y, z) = \left( T_1 \left( 1 - \frac{z}{c} \right) + T_2 \frac{z}{c} + \frac{c^2}{\pi^2} Q \sin \frac{\pi z}{c} \right) \left( 1 - \frac{y}{b} \right) \]

\[ + \left( T_3 \left( 1 - \frac{z}{c} \right) + T_4 \frac{z}{c} + \frac{c^2}{\pi^2} Q_1 \sin \frac{\pi z}{c} \right) \frac{y}{b} \left( 1 - \frac{x}{a} \right) \]

\[ + \left( T_5 \left( 1 - \frac{z}{c} \right) + T_6 \frac{z}{c} + \frac{c^2}{\pi^2} Q_2 \sin \frac{\pi z}{c} \right) \left( 1 - \frac{y}{b} \right) \]

\[ + \left( T_7 \left( 1 - \frac{z}{c} \right) + T_8 \frac{z}{c} + \frac{c^2}{\pi^2} Q_3 \sin \frac{\pi z}{c} \right) \frac{x}{a} \frac{y}{b}. \]

One can calculate the temperature \( T(x, y, z) \) at any point of the parallelepiped by equation (21). For example, substituting in (21) \( x = a/2 \), \( y = b/2 \) and \( z = c/2 \), we calculate the temperature value at the center of the body

\[ T \left( \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \right) = \frac{T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + \frac{c^2}{\pi^2} Q_1 + Q_2 + Q_3 + Q_4}{8}. \]

From (22) it follows that the temperature at the parallelepiped center is equal to the sum of the arithmetic temperatures value at its vertices (figure 1) and the arithmetic value of the coefficients of the internal source multiplied by \( c^2/\pi^2 \).
We calculate heat fluxes by the equations
\[ q_x(x,y,z) = -\lambda \frac{\partial T(x,y,z)}{\partial x}, \quad q_y(x,y,z) = -\lambda \frac{\partial T(x,y,z)}{\partial y}, \quad q_z(x,y,z) = -\lambda \frac{\partial T(x,y,z)}{\partial z}, \] where \( \lambda \) is material heat conductivity coefficient.

Putting the exact solution (21) into (23) we will have the following after transformations
\[ q_x(x,y,z) = -\lambda \left( (T_x - T_7) + \frac{c}{b}((T_4 - T_8) - (T_2 - T_6)) + \frac{x}{a}((T_1 - T_5) - (T_3 - T_7)) \right) \]
\[ + \frac{c^2}{\pi} \sin \frac{\pi}{a} \left( (Q_1 - Q_2) + ((Q_4 - Q_5) - (Q_2 - Q_3)) \frac{x}{a} \right). \] (24)

\[ q_y(x,y,z) = -\lambda \left( (T_y - T_1) + \frac{a}{b}((T_2 - T_6) - (T_3 - T_7)) + \frac{y}{c}((T_2 - T_8) - (T_1 - T_5)) \right) \]
\[ + \frac{a^2}{\pi} \sin \frac{\pi}{c} \left( (Q_2 - Q_3) + ((Q_4 - Q_5) - (Q_2 - Q_3)) \frac{y}{c} \right). \] (25)

\[ q_z(x,y,z) = -\lambda \left( (T_z - T_1) + \frac{b}{a}((T_3 - T_7) - (T_5 - T_9)) + \frac{z}{c}((T_3 - T_7) - (T_5 - T_9)) \right) \]
\[ + \frac{b^2}{\pi} \cos \frac{\pi}{c} \left( Q_1 + (Q_2 - Q_3) \frac{y}{b} + \left( (Q_4 - Q_5) + ((Q_4 - Q_5) - (Q_2 - Q_3)) \frac{y}{b} \right) \frac{z}{c} \right). \] (26)

In equations (24) – (26), the temperatures \( T_i, \ i = 1...8 \) at the parallelepiped vertices are grouped so as to highlight the temperature difference along the parallelepiped edges (figure 1). Thus, (24) – (26) show that the flow description \( q_x(x,y,z) \) contains temperature differences along the edges collinear...
only on the OX axis (equation (24)). For the description \( q_y(x, y, z) \), temperature differences along the edges collinear only on the OY axis (equation (25)) are taken into account, and in equation (26) the record \( q_z(x, y, z) \) contains temperature differences along the edges collinear only on the OZ axis.

Let us consider the distributions of temperature fields and heat fluxes corresponding to the case when the temperature values \( T_i, i = 1...8 \) at all vertices of the parallelepiped and the internal source coefficients values \( Q_i, i = 1...4 \) are different. We will choose the corrosion-resistant heat-resistant steel 08X17T, used for products operating in oxidizing environments and in atmospheric conditions as the parallelepiped material [22]. Let

\[
T_1 = 1^\circ \text{C}, T_2 = 5^\circ \text{C}, T_3 = 3^\circ \text{C}, T_4 = 7^\circ \text{C}, T_5 = 10^\circ \text{C}, T_6 = 20^\circ \text{C}, T_7 = 30^\circ \text{C}, T_8 = 50^\circ \text{C},
\]
\[
Q_1 = 12^\circ \text{C}/\text{m}^2, Q_2 = 40^\circ \text{C}/\text{m}^2, Q_3 = 16^\circ \text{C}/\text{m}^2, Q_4 = 30^\circ \text{C}/\text{m}^2,
\]
\[a = 0.1 \text{ m}, b = 0.2 \text{ m}, c = 0.3 \text{ m}, \lambda = 25 \text{ W/(m} \cdot ^\circ \text{C}).\] (27)

For the data (27), the temperature fields in the parallelepiped calculated by equation (21) in the sections by the planes \( x = a/2 \), \( y = b/2 \) and \( z = c/2 \) are shown in figure 2a, figure 2b and figure 2c, respectively. Heat fluxes calculated by equations (24) – (26) for values (27) are shown in figure 3. The distributions of \( q_x(x, y, z) \) and \( q_y(x, y, z) \) are shown in figure 3a and figure 3b and in figure 3c \( q_z(x, y, z) \) is in section \( z = c/2 \). Figure 2 and figure 3 show that when setting different temperatures at the parallelepiped vertices, these profiles of temperatures and heat fluxes of the planes do not have symmetry.

Figure 2. Temperature fields in a parallelepiped in sections:
(a) \( x = a/2 \), (b) \( y = b/2 \), (c) \( z = c/2 \).
Figure 3. Heat fluxes in the parallelepiped:
(a) $q_x(x, y, z)$, (b) $q_y(x, y, z)$, (c) $q_z(x, y, z)$ in section $z = c/2$.

If we take all temperatures at the parallelepiped vertices the same in (21)

$$T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 = T$$

(28)

the internal source coefficients

$$Q_1 = Q_2 = Q_3 = Q_4 = Q$$

(29)

then in this case we obtain the propagation of temperatures and heat fluxes having symmetry planes passing through the parallelepiped centre.

Let us study the influence of the internal source variability $F(x, y, z)$ on the propagation of heat in the body at the same initial temperatures at the parallelepiped vertices (fulfillment of conditions (28)). From equations (24) – (26) it follows that the analysis of those cases is of interest when all $Q_i$, $i = 1...4$ take the same values (condition (29)), all $Q_i$, $i = 1...4$ take different values and the values of $Q_i$, $i = 1...4$ will be the same in pairs, for example, $Q_1 = Q_2 \neq Q_3 = Q_4$ or $Q_1 = Q_3 \neq Q_2 = Q_4$.

Let us first consider the case corresponding to conditions (29). The internal source will be determined by the equation
The temperature and heat fluxes in the body for the source \((30)\) when equalities \((28)\) are satisfied will be calculated by the equations

\[
T(x,y,z) = T + \frac{c^2}{\pi^2} Q \sin \frac{\pi z}{c}, \quad q_s(x,y,z) = 0, \quad q_s(x,y,z) = -\lambda \frac{c}{\pi} \cos \frac{\pi z}{c}. \quad (31)
\]

Now let us consider the case when all \(Q_i, \, i = 1...4\) take different values (the internal source will be described by equation \((17)\)). For this option, the temperature and heat fluxes in the body are described by the following equations

\[
T(x,y,z) = T + \left( Q_1 \left( 1 - \frac{y}{b} \right) + Q_2 \frac{y}{b} \right) \left( 1 - \frac{x}{a} \right) + \left( Q_3 \left( 1 - \frac{y}{b} \right) + Q_4 \frac{y}{b} \right) \frac{x}{a} \frac{c^2}{\pi^2} \sin \frac{\pi z}{c}. \quad (32)
\]

\[
q_s(x,y,z) = -\frac{\lambda c^2}{a \pi^2} \sin \frac{\pi z}{c} \left( (Q_1 - Q_1) + ((Q_4 - Q_3) - (Q_2 - Q_2)) \frac{y}{b} \right). \quad (33)
\]

\[
q_s(x,y,z) = -\frac{\lambda c^2}{b \pi^2} \sin \frac{\pi z}{c} \left( (Q_2 - Q_1) + ((Q_4 - Q_3) - (Q_2 - Q_2)) \frac{x}{a} \right). \quad (34)
\]

\[
q_s(x,y,z) = -\lambda \frac{c}{\pi} \cos \frac{\pi z}{c} \left( Q_1 + (Q_2 - Q_1) \frac{y}{b} + (Q_3 - Q_2) \frac{x}{a} \right). \quad (35)
\]

If the values of \(Q_i, \, i = 1...4\) are such that \(Q_1 = Q_2 = Q_3 = Q_4\), then the internal source will not depend on the coordinate \(y\)

\[
F(x,y,z) = \left( Q_1 + (Q_1 - Q_1) \frac{y}{a} \right) \sin \frac{\pi z}{c}
\]

and we obtain the following equalities for temperature fields and heat fluxes

\[
T(x,y,z) = T + \left( Q_1 \left( 1 - \frac{y}{b} \right) + Q_2 \frac{y}{b} \right) \left( 1 - \frac{x}{a} \right) \frac{c^2}{\pi^2} \sin \frac{\pi z}{c}. \quad (36)
\]

\[
q_s(x,y,z) = -\frac{\lambda c^2}{a \pi^2} \sin \frac{\pi z}{c} (Q_3 - Q_1). \quad (37)
\]

\[
q_s(x,y,z) = 0. \quad (38)
\]

\[
q_s(x,y,z) = -\lambda \frac{c}{\pi} \cos \frac{\pi z}{c} \left( Q_1 + (Q_3 - Q_2) \frac{x}{a} \right). \quad (39)
\]

For values \(Q_1 = Q_3 \neq Q_2 = Q_4\), the internal source will not depend on the \(x\) coordinate

\[
F(x,y,z) = \left( Q_1 + (Q_2 - Q_1) \frac{y}{b} \right) \sin \frac{\pi z}{c}.
\]

In this case, the temperature fields and heat fluxes are given by the equalities

\[
T(x,y,z) = T + \left( Q_1 \left( 1 - \frac{y}{b} \right) + Q_2 \frac{y}{b} \right) \frac{c^2}{\pi^2} \sin \frac{\pi z}{c}. \quad (40)
\]
\begin{align*}
q_i(x,y,z) &= 0. \\
q_x(x,y,z) &= -\frac{\lambda c^2}{\pi} \sin \pi \frac{z}{c} (Q_z - Q_i). \\
q_z(x,y,z) &= -\frac{\lambda c^2}{\pi} \cos \pi \frac{z}{c} \left( Q_i + (Q_z - Q_i) \frac{y}{b} \right).
\end{align*}

Analyzing equations (31) – (43), we can conclude that when setting the same initial temperature \( T_i, i = 1...8 \) at the parallelepiped vertices (condition (28)), the heat in the body will be transferred only in those directions in which the internal source is variable. For example, when specifying the internal source by equation (30) it can be seen from (31) that fluxes \( q_i(x,y,z) \) and \( q_y(x,y,z) \) are equal to zero, and only heat flow \( q_z(x,y,z) \) is nonzero and depends only on the coordinate \( z \).

4. Conclusion

To obtain new three-dimensional exact solutions of the heat conduction equation, the selection of the numerical values of the coefficients of the functions included in the boundary conditions and source \( F(x,y,z) \) should be carried out taking into account equalities (13) – (15). The article presents equations for calculating temperature and heat fluxes at any point in the body.

In case of different values of temperature \( T_i, i = 1...8 \) at all parallelepiped vertices, it is shown that the flow description \( q_i(x,y,z) \) contains temperature differences along the edges collinear only on the \( OX \) axis. To describe \( q_y(x,y,z) \), the temperature differences along the edges collinear only on the \( OY \) axis are taken into account, and there are temperature differences along the edges collinear only on the \( OZ \) axis in record \( q_z(x,y,z) \).

The study of the influence of the internal source \( F(x,y,z) \) variability on the heat propagation in the body at the same initial temperatures \( T_i, i = 1...8 \) at the parallelepiped vertices was carried out. In the course of the study, it was found out that heat in the body will be transferred only in those directions in which the internal source is changed.

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