Non-perturbative Quantization of Phantom and Ghost Theories: Relating Definite and Indefinite Representations

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Abstract

We investigate the non-perturbative quantization of phantom and ghost degrees of freedom by relating their representations in definite and indefinite inner product spaces. For a large class of potentials, we argue that the same physical information can be extracted from either representation. We provide a definition of the path integral for these theories, even in cases where the integrand may be exponentially unbounded, thereby removing some previous obstacles to their non-perturbative study. We apply our results to the study of ghost fields of Pauli-Villars and Lee-Wick type, and we show in the context of a toy model how to derive, from an exact non-perturbative path integral calculation, previously ad hoc prescriptions for Feynman diagram contour integrals in the presence of complex energies. We point out that the pole prescriptions obtained in ghost theories are opposite to what would have been expected if one had added conventional $i\epsilon$ convergence factors in the path integral.
1 Introduction

In this article we discuss the non-perturbative quantization of bosonic phantom degrees of freedom. A phantom is defined as a degree of freedom for which the sign of the kinetic term is opposite to that of an ordinary bosonic degree of freedom. Phantoms may often be quantized using either an ordinary Hilbert space representation or an indefinite inner product representation. In the latter case, the phantom is usually called a ghost.

It is often physically required that we represent phantom degrees of freedom on an indefinite inner product space. Examples include the timelike components of gauge fields, Faddeev-Popov ghosts, Pauli-Villars regulator ghosts, and the regulator ghost fields postulated in Lee-Wick [1, 2] ultraviolet completions of field theories.

To describe such degrees of freedom non-perturbatively, Boulware and Gross [4] constructed a functional integral from an indefinite inner product representation of the canonical commutation relations. The resulting functional integral had two main problems, both of which we address in this article.

The first problem occurs if the potential has odd terms. The integrand then becomes exponentially unbounded and the functional integral naively diverges, seemingly casting the non-perturbative existence of the theory in doubt [4]. This problem already arises in the presence of source terms that are linear in the fields [5]. We show in this article that such linear – and in certain cases even quadratic – exponentially unbounded integrands can be treated non-perturbatively by defining path integrals in the distributional sense on appropriate test function spaces of a type studied by Gel’fand and Shilov in their theory of generalized functions [6, 7].

The second problem occurs when quantizing relativistic fields using the indefinite metric representation, since the region of integration may not be manifestly covariant. This happens for the electromagnetic field, as discussed by Arisue et al. [8], and for the Dirac boson discussed in the present article. In particular, the region of integration for each ghost degree of freedom is effectively the imaginary axis. In general, there is no coordinate-independent way of distinguishing ordinary and ghost degrees of freedom, so that the functional integral is not manifestly covariant, though it may still be implicitly covariant [8]. The main result of the present article overcomes this problem by showing that information appropriate to the indefinite inner product representation can in fact be extracted in the usual way from the usual Hilbert...
space representation, where covariance of the integrand is manifest.

For gauge fields, the existence of a well-behaved analytic continuation to Euclidean time makes these issues less urgent. However, for fields such as the Dirac boson, which are of particular interest as Pauli-Villars and Lee-Wick ghosts, the Euclidean functional integrand is exponentially unbounded. For these fields, the real-time formalism works fine, as we illustrate. Astonishingly, for fields that appear at most quadratically in the action, even the exponentially unbounded Euclidean functional integral can be calculated in an appropriate distributional sense, as we discuss in section 5.

It is in fact common practice to derive results based on heuristic functional integrals, in theories containing indefinite-metric gauge field components or ghosts, by using the classical covariant action. One may justifiably ask why not. In the real-time formalism, the integrand is $\exp(iS)$. As long as $S$ is real, negative terms in $S$ are not necessarily worse than positive terms in $S$, and the oscillatory integral may be equally well defined in either case.

In this article we justify the use of the ordinary covariant action and integration region for potentials sufficiently general to cover the applications mentioned in the introductory paragraphs. In particular, we show that the indefinite-metric ground state expectation values can be obtained from the definite-metric transition function using the usual $T \to -i\infty$ prescription.

Because they lacked a non-perturbative definition of the indefinite inner product theories corresponding to the unbounded path integrands that we are now able to treat, various authors [1] 2 3 studied the consistency of ad hoc prescriptions for defining the propagators, and more complex diagrams, order by order in perturbation theory. Lee and Wick [1 2] introduced one such prescription, requiring the contour integrations defining various diagrams to be continuously deformed to avoid the movement of poles in the complex plane as one changes the parameters of the theory. Cutkosky, Landshoff, Olive and Polkinghorne [3] attempted to generalize this prescription to coalescing singularities not covered by Lee and Wick.

The framework of this article allows a non-perturbative functional integral to be calculated in the distributional sense for various kinds of unbounded path integrands, including the ones studied by Lee and Wick. In principle, we expect the ad hoc prescriptions of the above authors to be either reproduced or corrected in our non-perturbative framework. We motivate this statement with a couple of simple exact calculations in a toy model, where we succeed in deriving Lee-Wick-type pole prescriptions from first principles via explicitly non-perturbative calculations.
2 Related work and applications

In addition to the seminal references discussed in the introduction, we also
point out the following related work.

The study of phantoms has recently been of interest in cosmological model
building. They are used, for example, in some models of cosmological dark
energy. The literature on the subject is too large to cite representatively, but
see for example [9]. There are open questions regarding the consistency and
stability of these models [10] [11]. It is our hope that the non-perturbative
methods introduced in the current article may throw further light on the
subject. For example, in section 5 we learn how to non-perturbatively treat
unbounded and complex potentials, which may be useful in studying, from
a quantum-mechanical point of view, the stability questions discussed in, for
example, [12] and [13].

In [14] [15] Erdem, and in [16] ’t Hooft and Nobbenhuis discuss a novel
kind of symmetry transformation consisting of a rotation of real positional
coordinates to the imaginary axis, with the aim of ruling out a cosmolog-
cal constant. The rotated representation that these authors use for their
non-relativistic particle toy model is identical to the one we discuss in sec-
tion 3 for the indefinite inner product theory. These authors are particularly
interested in the relationship between the real and imaginary coordinate rep-
resentations. Since we study this relationship in detail in the present article,
it is conceivable that our mathematical framework may have further appli-
cations in this direction.

Also related to the cosmological constant problem is the paper [17], which
introduces phantom fields to cancel the ordinary matter contribution to the
vacuum energy. Again, our non-perturbative approach to phantom fields
may be useful in the study of these models.

In the paper [18], Antoniadis et al. promote the study of theories with
ghosts in the real-time formalism, as opposed to the Euclidean formalism. In
the real-time formalism, they study the issue of \( i\epsilon \) prescriptions for resolving
ambiguities in Feynman diagrams. However, we disagree with the assump-
tion of their approach, which derives pole prescriptions by introducing \( i\epsilon \)
terms to provide convergence factors in path integrals, and states (we believe
incorrectly) that the convergence terms are necessary and that theories with-
out such convergence terms are ambiguous. As we discuss in section 9 the
pole prescription for ground state expectation values in ghost theories is in
fact opposite to what one would expect from such a convergence factor. Our
derivation is unambiguous, and does not require a convergence factor, since our path integral is well-defined without it. As a result of their assumptions, the authors of [18] find opposite $i0$ prescriptions for the two-point functions of a particle and a corresponding ghost, in conflict with our results, which agree with many authors starting with Pauli and Villars [19]. In the light of these remarks, we believe that the conclusions of [18] may have to be revised or reinterpreted. In this regard, we believe that the expectation values calculated in [18] may be valid in a representation based on a vacuum that is not a ground state. It is unclear to us how such a vacuum can be defined invariantly.

In section 12, we provide further non-perturbative calculations demonstrating what happens when the poles may travel in the complex plane.

There is a large body of work in scattering theory that includes resonances in the description of quantum systems via complex-coordinate or complex-momentum methods that displace either the position or the momentum representation into the complex plane [20, 21, 22, 23, 24, 25]. In these cases, the original and displaced representations are not equivalent, yet they are expected to encode the same physical information. What we do in the present article may be seen as a special case of this method, made rigorous for a specific class of potentials. We show that the same information is encoded by the representation based on the definite inner product, whose configuration space is the real line, and the representation based on an indefinite inner product whose configuration space is the imaginary axis.

In scattering theory, so-called Siegert or Gamow states may be used to represent resonances. These are states with complex momentum [20], which may be given precise mathematical meaning in the framework of section 5 of the current article, where such states are defined as distributions on test function spaces of Gel’fand-Shilov type. Since we are mainly interested in field theory applications where interactions are polynomial, we do not treat sufficiently general potentials for our results to be directly applicable to many traditional non-relativistic scattering problems, but perhaps the mathematical machinery can be generalized.

The author of [26] describes a treatment of these states in the context of rigged Hilbert spaces. His construction should be very closely related to the one of the current article.

We also mention the following recent work on alternative physical interpretations of theories of Lee-Wick type, which differs from the approach of the original authors [27].
3 Relating definite and indefinite quantizations: The free particle

We will show that certain quantum systems can be quantized in either a definite or an indefinite inner product representation. We will also argue that, under certain conditions, the choice is a matter of convenience, since physical quantities appropriate to one representation can be obtained from results calculated in the other.

To introduce the ideas, in this section we start with the ordinary free particle in one dimension with Hamiltonian

\[ H = \frac{p^2}{2m}. \]

The transition function is given by

\[ \langle y | e^{-iHt} | x \rangle = \int \frac{dk}{2\pi} \langle y | e^{-iHt} | k \rangle \langle k | x \rangle \]

\[ \begin{align*}
&= \int \frac{dk}{2\pi} e^{ik(y-x)-itk^2/2m} \\
&= e^{-i\pi/4} \sqrt{\frac{m}{2\pi t}} e^{im(y-x)^2/2t},
\end{align*} \]

where the last line is calculated for positive \( t \). Here we have summed over a complete generalized basis

\[ \langle x | k \rangle = e^{ikx}, \quad p | k \rangle = k | k \rangle, \quad \langle k | k' \rangle = 2\pi \delta(k' - k), \quad k \in \mathbb{R}, \]

of a Hilbert space with positive definite inner product. The energy spectrum is positive, and the oscillatory integral has a rigorous interpretation as the Fourier transform of a distribution.

Notice that the form (1) of the spectral representation may be continued to a function analytic in \( t \) on the lower half of the complex plane.

The same Hamiltonian can be quantized in an indefinite inner product representation. We will recast the representation used by Boulware and Gross [4] in a more convenient notation similar to the one used by Arisue et al [8]. This proceeds by representing \( q \) and \( p \) as Hermitian operators in an indefinite inner product space spanned by generalized bases

\[ p | ik \rangle = ik | ik \rangle, \quad \langle -ik | k' \rangle = 2\pi \delta(k' - k), \quad k \in \mathbb{R}, \]

\[ q | ix \rangle = ix | ix \rangle, \quad \langle -ix | x' \rangle = \delta(x' - x), \quad x \in \mathbb{R}, \]
where

\[ \langle \langle ix|k \rangle \rangle = e^{-ikx}. \]

Note that the eigenvalues of the Hermitian\(^1\) operators \(q\) and \(p\) occur in complex conjugate pairs, which is allowed in an indefinite inner product space \(^\text{[28]}\). The corresponding pairs of generalized eigenstates are dual null states. The configuration space is now the imaginary axis, where wave functions take the values

\[ \phi(ix) \equiv \langle \langle ix|\phi \rangle \rangle. \]

Thus, a generic wave function in the position representation has as its domain the imaginary axis. There is no assumption of analyticity – we cannot analytically extend a generic wave function away from the imaginary axis, so that \(\langle \langle z|\phi \rangle \rangle\) is in general undefined unless \(z \in i\mathbb{R}\). We will, however, have reason below to investigate subspaces consisting of test functions that are analytic.

The completeness relations in terms of these sets of states are

\[ \int dx \ |ix\rangle \langle ix| = 1 = \int \frac{dk}{2\pi} \ |-ik\rangle \langle ik| \]

 corresponding to the inner product

\[ \langle \langle \psi|\phi \rangle \rangle = \int dx \ (\psi(-ix))^* \phi(ix). \]

satisfying

\[ \langle \langle \psi|\phi \rangle \rangle = \langle \langle \phi|\psi \rangle \rangle^*. \]

This inner product is indefinite. The state space is not a Hilbert space but a Krein space, which may be constructed by completing the space of square-integrable functions on the imaginary axis with respect to an auxiliary \(L^2\) inner product to obtain a topologically complete vector space, which is then equipped with the above indefinite inner product, discarding the auxiliary \(L^2\) inner product \(^\text{[28]}\). We emphasize that all that matters about the auxiliary \(L^2\) inner product used in the intermediate step is the resulting topological vector space. The auxiliary inner product is not unique and has

\(^1\)Hermitian operators on indefinite inner product spaces are sometimes also called pseudo-Hermitian. We will not use this qualifier.
no physical meaning. The construction is analogous to using an arbitrary and non-physical Euclidean auxiliary inner product to define the topology of Minkowski space.

Our goal is to investigate the relationship between definite and indefinite quantizations of the same Hamiltonian. We will therefore compare the ordinary Hilbert space transition function $\langle y | e^{-iHt} | x = 0 \rangle$ to the quantity $\langle\langle y | e^{-iHt} | x = 0 \rangle\rangle$ calculated in the indefinite representation, where a suitable meaning for $\langle\langle y |, y \in \mathbb{R}$, will be assigned below.

At first glance, this might not seem like a reasonable thing to do. The two quantities are distributions on different test spaces. As distributions, they are therefore incomparable. However, as long as there exists a range of $t$ for which the distributions are representable by continuous functions in $y$, we can compare these functions.

The first observation is that we need a final state $\langle\langle y |, y \in \mathbb{R}$, whereas generic states in the indefinite representation are functions $\phi(ix) \equiv \langle\langle ix | \phi \rangle\rangle$ defined on the imaginary axis. Generalized functions such as $\langle\langle y |$ can, however, be defined provided we can find a subspace of the state space consisting of test functions that may be analytically continued from the imaginary axis into the complex plane and are, at the same time, invariant under time evolution. For the free particle, a simple test function space satisfying these conditions is the Gel’fand-Shilov space $S^{1/2}_{1/2}(ix)$, to be discussed in more detail in section 5. This test space consists of entire functions satisfying the growth condition

$$|y^k \phi(x + iy)| \leq C_k \exp \left(-a|y|^2 + b|x|^2\right),$$

where $C_k$, $a$, and $b$ are positive real numbers that are allowed to depend on $\phi$.

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2The space $S^{1/2}_{1/2}$ defined by Gel’fand and Shilov is based on imposing growth conditions on the real axis and corresponds, in the general notation introduced in the section 5, to $S^{1/2}_{1/2}(x)$. The space $S^{1/2}_{1/2}(ix)$ used here is simply obtained by imposing the same growth conditions on the imaginary axis, exchanging the roles of $x$ and $y$.

3Although the powers of $x$ and $y$ in the growth conditions are preserved under time evolution, it should be noted that the sign of $b$ is not, so that the space $S^{1/2}_{1/2}(ix)$ is not quite closed under $e^{-iHt}$. This makes it necessary to consider instead families of generalized functions on subspaces denoted by $S^{1/2}_{1/2,A}(ix)$ that are mapped into each other under time evolution, where $A$ and $B$ are related to $a$ and $b$ in 2. The method is developed rigorously by Gel’fand and Shilov in 7, and can be applied as long as the powers of $x$ and $y$ in the
For arbitrary $z \in \mathbb{C}$, we may now define the generalized function $\langle \langle z \rangle \rangle$ as

$$\langle \langle z|\phi \rangle \rangle \equiv \phi(z), \quad \phi \in S^{1/2}_{1/2}(ix).$$

Acting on an element of $S^{1/2}_{1/2}(ix)$, we have, purely formally,

$$\langle \langle y|e^{-iHt}|\phi \rangle \rangle = \int dk \langle \langle y|e^{-iHt}|-ik \rangle \rangle \langle \langle ik|\phi \rangle \rangle \equiv \int dk \langle \langle y|e^{-iHt}|-ik \rangle \rangle \int dx \langle \langle ik|-ix \rangle \rangle \langle \langle ix|\phi \rangle \rangle.$$ 

However, neither of the supposed kets $|−ik\rangle \rangle$ nor $|−i\tilde{x}\rangle \rangle$ is a test function, and the above should be read as a shorthand for the rigorously defined

$$\langle \langle y|e^{-iHt}|\phi \rangle \rangle = \int dk e^{ik^2t+ky} \langle \langle ik|\phi \rangle \rangle \equiv \int dk e^{ik^2t+ky} \int dx e^{-ikx} \phi(ix).$$

It is here that the utility of $S^{1/2}_{1/2}(ix)$ comes to the fore. Despite the exponentially growing factor $e^{ky}$, the integral over $k$ converges, since $S^{1/2}_{1/2}(ix)$ is closed under the Fourier transform [6], schematically $\mathcal{F}(S^{1/2}_{1/2}(ix)) = S^{1/2}_{1/2}(ik)$, which means that $\langle \langle ik|\phi \rangle \rangle$ decreases as $e^{-bk^2}$ for some positive $b$. Then

$$\langle \langle y|e^{-iHt}|\phi \rangle \rangle = \lim_{\epsilon \to 0} \int dk e^{ik^2t+ky-\epsilon k^2} \int dx e^{-ikx} \phi(ix)$$

$$= \int dx \left( \lim_{\epsilon \to 0} \int dk e^{ik^2t+k(y-ix)-\epsilon k^2} \right) \phi(ix), \quad (3)$$

where the added convergence factor, entirely superfluous in the convergent integral on the first line, has allowed us to exchange integrations in the second line.

We see from the second line that the generalized function $\langle \langle y|e^{-iHt}$ can
be represented by the kernel

$$\langle \langle y | e^{-iHt} | -ix \rangle \rangle \equiv \lim_{\epsilon \to 0} \int dk \langle \langle y | e^{-iHt} | -ik \rangle \rangle e^{-\epsilon k^2} \langle \langle ik | -ix \rangle \rangle$$

$$= \lim_{\epsilon \to 0} \int dk e^{ik^2t + k(y - ix) - \epsilon k^2},$$

$$= e^{i\pi/4} \sqrt{\frac{m}{2\pi t}} e^{im(y - ix)^2/2t},$$

where again the formally undefined inner products on the first line should be taken as a useful mnemonic for the well-defined expression on the second line, copied from $\langle \langle y | e^{-iHt} | -ix \rangle \rangle$. Again, the last line was calculated for positive $t$.

We have obtained a spectral representation of the kernel $\langle \langle y | e^{-iHt} | -ix \rangle \rangle$ in the indefinite inner product representation. The expression (4) in fact defines a function analytic in $t$ on the upper half plane.

For $x = 0$, we may now compare this with the definite representation, and we find

$$\langle \langle y | e^{-iHt} | x = 0 \rangle \rangle = i \langle y | e^{-iHt} | x = 0 \rangle$$

for positive $t$. For negative $t$, one may check that

$$\langle \langle y | e^{-iHt} | x = 0 \rangle \rangle = -i \langle y | e^{-iHt} | x = 0 \rangle.$$

To summarize, we have obtained the following structure:

- The function $i \langle y | e^{-iHt} | x = 0 \rangle$, calculated in the definite inner product representation, has a positive-energy spectral representation that defines a function analytic in $t$ on the lower half plane.

- The function $\langle \langle y | e^{-iHt} | x = 0 \rangle \rangle$, calculated in the indefinite inner product representation, has a negative-energy spectral representation that defines a function analytic in $t$ on the upper half plane.

- On the positive real axis in $t$, these two functions coincide. Together, they therefore define a single function analytic on the entire complex plane except for a cut on the negative real axis, where the value of $\langle \langle y | e^{-iHt} | x = 0 \rangle \rangle$ is obtained by approaching the cut from above, and $i \langle y | e^{-iHt} | x = 0 \rangle$ is obtained by approaching the cut from below.
We see that, for positive real $t$, the results calculated using the two representations coincide. If we knew beforehand that the two spectral representations determined the two halves of a single function analytic in $t$, we could have used either representation to infer the result for the other. In section 5, we will discuss under what conditions this can be done.

4. The phantom free particle

We briefly discuss the quantization of the particle with opposite sign Hamiltonian

$$H = -\frac{p^2}{2m}.$$ 

Now the positive-definite inner product momentum eigenstates have negative energy

$$E_k = -\frac{k^2}{2m},$$

and

$$\langle y | e^{-iHt} | x \rangle = \int \frac{dk}{2\pi} \langle y | e^{-iHt} | k \rangle \langle k | x \rangle$$

$$= \int \frac{dk}{2\pi} e^{ik(y-x)+itk^2/2m}$$

$$= e^{i\pi/4} \sqrt{\frac{m}{2\pi t}} e^{-im(y-x)^2/2t}.$$

Note that, although the energy is unbounded below, the real-time representation exists. The spectral representation now directly defines an analytic function in $t$ on the upper half plane.

The same transition function can be written in terms of a positive energy spectral representation using the indefinite inner product space defined in section 3. The states

$$|ik\rangle, \quad k \in \mathbb{R},$$

Since the inner product in the indefinite representation has no canonical choice of overall sign, we could have made these two quantities coincide for negative $t$, and positioned the cut at positive $t$, by defining the inner product with an opposite sign.
are now eigenstates of $H$ with positive energy $k^2/2m$. Similarly to section 3, we find the spectral representation

$$
\langle\langle y | e^{-iHt} | -ix \rangle \rangle \equiv \lim_{\epsilon \to 0} \int dk \langle\langle y | e^{-iHt} | -ik \rangle \rangle e^{-\epsilon k^2} \langle\langle ik | -ix \rangle \rangle 
= \lim_{\epsilon \to 0} \int dk e^{-ik^2 t + k(y - ix) - \epsilon k^2},
\tag{7}
$$

and (7) extends analytically to the lower half plane.

We have found the structure:

- The function $-i \langle y | e^{-iHt} | x = 0 \rangle$, calculated in the definite inner product representation, has a negative-energy spectral representation that defines a function analytic in $t$ on the upper half plane.

- The function $\langle\langle y | e^{-iHt} | x = 0 \rangle \rangle$, calculated in the indefinite inner product representation, has a positive-energy spectral representation that defines a function analytic in $t$ on the upper half plane.

- On the positive real axis in $t$, these two functions coincide. Together, they therefore define a single function analytic on the entire complex plane except for a cut on the negative real axis, where the value of $\langle\langle y | e^{-iHt} | x = 0 \rangle \rangle$ is obtained by approaching the cut from below, and $-i \langle y | e^{-iHt} | x = 0 \rangle$ is obtained by approaching the cut from above.

Again, for positive real $t$, the results calculated using the two representations coincide.

## 5 General potentials

We now generalize the above analysis to a wider class of potentials. We shall see that the transition functions in the definite and indefinite quantizations typically describe a single analytic function for different regions of complex time that overlap for an initial interval of real time. In section 6 we will apply this correspondence to the extraction of ground state correlation functions.

The general problem will be as follows. Consider a one-dimensional Hamiltonian

$$
H \equiv -\frac{p^2}{2m} + \mu p + V(q).
$$
that is Hermitian with respect to the ordinary Hilbert space inner product on the real line, which requires that \( \mu \) be real and that \( V(x) \) be a real-valued function of \( x \). Our discussion will include the case that is of interest in phantom and ghost theories, where the spectrum of \( H \), which is real in the ordinary Hilbert space representation, may be unbounded below. As a result, the usual rotation to Euclidean time may not exist\(^5\) but typically there is no obstacle to calculating the real-time transition function, which will be denoted by

\[
\langle y | e^{-iHt} | x \rangle. \tag{8}
\]

In the indefinite inner product representation constructed in section\(^3\) where the configuration space of \( q \) is the imaginary axis, the operators \( p \) and \( q \) are still Hermitian, as is the Hamiltonian. However, the spectrum of the Hamiltonian may be complex\(^6\) Of most interest will be those phantom or ghost theories for which, in the indefinite representation, the real part of the spectrum becomes bounded below\(^7\) We will assume that the imaginary part of the spectrum, if any, is sufficiently well-behaved that \( e^{-iHt} \) exists for all real \( t \). The transition function in the indefinite representation is denoted by

\[
\langle \langle iy | e^{-iHt} | -ix \rangle \rangle. \tag{9}
\]

Due to the different configuration spaces, the quantities (8) and (9) are not directly comparable. For a comparison, we need states of the form \( \langle \langle y \rangle \rangle \) instead of \( \langle iy \rangle \), whereas general states in the indefinite representation are functions \( \phi(iy) \equiv \langle \langle iy | \phi \rangle \rangle \) that are only defined on the imaginary axis. Generalized functions such as \( \langle iy \rangle \) can, however, be defined on suitable test function spaces that are invariant under time evolution and consist of functions that may be analytically continued away from the imaginary axis into the complex plane. The choice of test function space is dependent on the Hamiltonian.

Gel’fand and Shilov\(^6, 7\) introduced families \( S^\alpha_0(x) \) of test function spaces on the real line that are suitable for our purposes. These are defined, for

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\(^5\)Although we shall see that the relevant unbounded Euclidean path integral may in fact be calculable as a Gel’fand-Shilov generalized function.

\(^6\)We remind the reader Hermitian operators in indefinite inner product spaces may have complex eigenvalues. These always occur in complex conjugate pairs. The corresponding eigenstates are dual null states\(^2\).

\(^7\)More generally, our arguments remain valid even in cases where the real part of the spectrum is unbounded below, as long as the density of generalized eigenvalues decreases sufficiently rapidly in the negative real direction.
$\alpha, \beta \leq 0$, as consisting of all infinitely differentiable functions $\phi(x)$ on the real line satisfying the growth conditions

$$|x^k \phi^{(q)}(x)| \leq CA^k B^q k^{\alpha k} q^{\beta q},$$

where $A$, $B$ and $C$ may depend on $\phi$. Writing the independent variable $x$ in parentheses is not part of the original notation but will be necessary for later disambiguation when we discuss similar spaces based on functions $\phi(ix)$ defined on the imaginary axis. There exists an interpretation of the limit of infinite $\alpha$ or $\beta$, and it is conventional to omit either index when it is infinite.

The familiar Schwartz space is the same as $S = S^{\infty}_{\infty}$. For other values of $\alpha$ and $\beta$, the space is more restricted than Schwartz space since the growth conditions are stricter, and in fact for $\beta \leq 1$ these conditions are so strict that the functions $\phi(x)$ can be extended to analytic functions $\phi(x + iy)$ on some complex domain.

The domain depends on $\beta$. Specifically, for $\beta = 1$, the test functions $\phi(x)$ may be analytically extended to analytic functions on a strip

$$\{x + iy | x \in \mathbb{R}, |y| \leq B\},$$

where the width $B$ of the strip may depend on $\phi$. For $\beta < 1$, the test functions can in fact be continued an entire analytic function on the whole complex plane. In this case, the spaces $S^\beta_\alpha$ are characterized completely by the growth conditions

$$|x^k \phi(x + iy)| \leq C_k \exp \left(-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\right), \quad (10)$$

where $C_k$, $\alpha$, and $b$ are positive real numbers that are allowed to depend on $\phi$. Smaller $\alpha$ or $\beta$ correspond to smaller spaces of test functions, and therefore larger dual spaces of generalized functions. The restrictions become so stringent that the test spaces are trivial for

$$\alpha + \beta < 1.$$ 

Under the Fourier transform, it may be shown that [6]

$$\mathcal{F} \left(S^\beta_\alpha(x)\right) = S^\alpha_\beta(k). \quad (11)$$

In the indefinite representation, the wave functions are not defined on the real line but rather on the imaginary axis, and we will denote the spaces
where the same growth conditions are imposed in the imaginary direction by \(S_\alpha^3(ix)\). As above, for \(\beta < 1\) these test functions can be continued away from the imaginary axis to obtain entire functions. For \(\beta = 1\), the strip of analyticity will now be a vertical one that includes the whole imaginary axis.

The relation (11) remains valid if we define the Fourier transform on the imaginary axis as

\[
\mathcal{F}[\phi](ik) = \int dx e^{-ikx} \phi(ix).
\]

We will restrict our attention to potentials for which a test space can be selected on which the Trotter product formula

\[
e^{-iHt} = \lim_{N \to \infty} (e^{-i\epsilon p^2/2m + i\epsilon \mu p} e^{-i\epsilon V})^N, \quad \epsilon \equiv \frac{t}{N},
\]

holds at least for a finite time interval. This formula can be taken a basis for the construction of a path integral representation. If it holds, a generalized path integral may be defined by writing the (generalized) transition function \(\langle\langle y | e^{-iHt} \rangle\rangle\) as

\[
\lim_{N \to \infty} \langle\langle y | \left(\mathcal{F}^{-1} \cdot e^{-i\epsilon p^2/2m + i\epsilon \mu p} \cdot \mathcal{F} \cdot e^{-i\epsilon V}\right)^N \rangle\rangle.
\]  

(12)

If one writes the Fourier transforms, each acting to the left on a generalized function, as formal integrals, this can be seen to coincide with the introductory textbook definition of the path integral. For related definitions of path integrals in terms of transforms, see for example [29] and the review in [30].

This expression will exist as a generalized function, but may not have a representation as an ordinary kernel. In other words

\[
\langle\langle y | e^{-iHt} | -ix \rangle\rangle
\]

may not exist as an ordinary function of \(y\) and \(ix\). It is only when the kernel can be represented by a function continuous in \(y\) and in \(x\) at \(x = 0\) that we can compare

\[
\langle\langle y | e^{-iHt} | -ix \rangle\rangle
\]

It should be remarked that even the simplest textbook real-time path integral calculations give meaning to the oscillatory integrals involved in a way that is essentially equivalent to calculating Fourier transforms of distributions as we do here. By using smaller invariant test spaces, our spaces of distributions are larger, allowing us to calculate a larger class of path integrals.

15
and

$$\langle y | e^{-iHt} | x \rangle,$$

in the point $x = 0$. Otherwise, as distributions, the objects

$$\langle \langle y | e^{-iHt}$$

and

$$\langle y | e^{-iHt}$$

are incomparable, since they act on different test spaces.

For the above construction, we saw that it is necessary to consider test spaces consisting of analytic functions that are invariant under all of $e^{-i\epsilon V(q)}$, $e^{-i\epsilon p^2/2m}$ and $e^{i\epsilon \mu p}$.

The appropriate test spaces will depend on the potential. We will choose suitable candidates from the family of spaces $S^\beta_\alpha(ix)$, whose elements are either entire functions for $\beta < 1$, or analytic functions in a vertical strip including the imaginary axis for $\beta = 1$.

We start with the spaces of entire functions $S^\beta_\alpha(ix)$ for $\beta < 1$. These spaces will be suitable only for potentials such that $e^{-i\epsilon V(ix)}$ can be extended to an entire function of $z$, since the space must be closed under this multiplier.

The indices $\alpha$ and $\beta$ are then further restricted by the growth properties of $e^{-i\epsilon V(z)}$. We will consider the class of potentials satisfying the inequality

$$|e^{-i\epsilon V(z)}| \leq C e^{b|z|^p},$$

on the complex plane, which is true in particular if $V$ is a polynomial of order $p$. Then a sufficient condition for $S^\beta_\alpha(ix)$ to be closed under this multiplier is that $p \leq 1/\alpha$ and $p \leq 1/(1-\beta)$, or

$$\alpha \leq \frac{1}{p}, \quad \beta \geq \frac{p - 1}{p}.$$  \hspace{1cm} (14)

Next, consider invariance of these spaces $S^\beta_\alpha(ix)$, still for $\beta < 1$, under $e^{\pm i\epsilon p^2/2m + i\mu p}$, which is equivalent to invariance of the Fourier transformed

\footnote{When the inequality is saturated, the space $S^\beta_\alpha$ is not necessarily closed under the action of the multiplier. In this case, the appropriate test function spaces are families of subspaces $S^\beta_\alpha,B$ of $S^\beta_\alpha$. As noted in a previous footnote, this technical complication does not affect the general argument and will be glossed over in this article. For full details on the technique, see \cite{7}.}
space $S_{B}^{\alpha}(ik)$ under $e^{i\epsilon(ik)^{2}/2m+i\epsilon\mu(ik)}$. We will restrict attention to the family (14) selected by the growth condition (13). First, if the order $p$ of the potential satisfies $p > 1$, then the first inequality constrains $\alpha \leq 1/p < 1$, so that the Fourier-transformed space also consists of entire functions. This space will be invariant if

$$\beta \leq \frac{1}{2}, \quad \alpha \geq \frac{1}{2}. \quad (15)$$

Solutions to both sets of conditions (14) and (15) may be found for potentials of order $1 < p \leq 2$. We may choose any such $\alpha$ and $\beta$ satisfying the additional condition $\alpha + \beta \geq 1$ necessary for non-triviality of the test space. The smallest test space satisfying all these conditions for all $p$ in this range is given by $S_{1/2}^{1/2}(ix)$, which happens to also have the nice property of being invariant under the Fourier transform. In fact, it is easily seen that $S_{1/2}^{1/2}(ix)$ satisfies the conditions for the extended range $0 \leq p \leq 2$, so that we may use as a test space for the whole range

$$S_{1/2}^{1/2}(ix), \quad 0 \leq p \leq 2.$$ 

In fact, the invariance conditions are unchanged for arbitrary complex coefficients of $p, p^2, x$ and $x^2$ in the Hamiltonian, enabling us to calculate path integrals as Gel’fand-Shilov distributions on $S_{1/2}^{1/2}(ix)$ even when the integrand may be exponentially unbounded.

We note that (14) and (15) cannot be satisfied for polynomial potentials of order $p > 2$. As a result, none of the Gel’fand-Shilov spaces of entire functions may be used for these potentials. However, for our later arguments, it will be good enough to have test functions that are analytic in a vertical strip containing the imaginary axis, a property satisfied by the spaces $S_{1}^{1}(ix)$. This family of spaces, with suitable restrictions on $\alpha$, will prove to be suitable for even potentials of order $p > 2$.

According to [6], a sufficient condition for a function $f(ix)$, defined on the imaginary axis, to be a multiplier with respect to a specific family of subspaces $S_{B}^{\alpha, B}(ix)$ of $S_{A}^{\beta}(ix)$, is that it satisfy the estimate

$$|f(q)(ix)| \leq C_{\epsilon} B^{q} q^{\beta q} e^{\epsilon|x|^{1/\alpha}}, \quad (16)$$

for any $\epsilon > 0$. Consider, then, the term $V(q) = cq^{p}$ in a polynomial potential, where $c$ is real. Its contribution to the multiplier $e^{iV(ix)}$ is

$$f(ix) = e^{i\rho+1cix^{p}}.$$
We see that \( p \) must be even to satisfy the sufficient condition above. By writing down the first few derivatives, it is easily seen that
\[
|f^q(ix)| \leq q! p^q (1 + |x|^{pq}).
\]
This is not the tightest possible estimate, but it will suffice for our purposes. Using the Stirling estimate for large \( q \)
\[
\ln(q!) \sim q \ln q - q,
\]
we find
\[
|f^q(ix)| \leq C q^q e^{-q} p^q (1 + |x|^{pq})
\leq C q^q p^q e^{\epsilon |x|^{1/\alpha}}.
\]
Comparing with the estimate (16), we read off that any
\[
\beta \geq 1, \quad \alpha > 0,
\]
will do, for any even power \( p \) in the potential. Of these, only \( \beta = 1 \) gives a space of analytic functions. Applying the same argument to the multiplier \( e^{\pm i\epsilon (ik)^2/2m} \) on the Fourier transformed space \( S^\alpha_{\beta} \) gives the (sufficient) condition
\[
\beta > 0, \quad \alpha \geq 1.
\]
The smallest space of analytic functions satisfying both these conditions is
\[
S^1_1(ix),
\]
which, again, has the nice property of invariance under the Fourier transform.

In fact, while the above conditions are sufficient, it may easily be seen that the linear potential or source term \( e^{i\lambda z} \) is also a multiplier on families of subspaces of \( S^1_1(ix) \), despite its exponential growth in the imaginary direction if \( \lambda \) is real, so that we may allow also source terms in our potential. This is not true for higher odd powers.

In fact, it may be checked that both test spaces that we have chosen are invariant under general source terms of the form
\[
\mu p + \lambda q,
\]
where \( \mu \) and \( \lambda \) may be complex.

To summarize, we find that the following test spaces are suitable:
\[ H = \nu p^2 / 2m + \mu p + V(q), \] where \( \mu, \nu \) and \( V(q) \) may be complex, as long as \( |e^{-i\lambda z}| \leq C e^{b|z|^p} \) for \( 0 \leq p \leq 2 \): A suitable test space is \( S_{1/2}(ix) \).

Elements of the test space are entire functions.

\[ H = \pm p^2 / 2m + \mu p + \lambda q + V(q), \] where \( \mu \) and \( \lambda \) may be complex and \( V(q) \) is an arbitrary even real polynomial: A suitable test space is \( S_{1/2}(ix) \).

Elements of the test space are analytic on a vertical strip including the imaginary axis.

A useful feature of the current formalism is that it allows the study of source terms in phantom or ghost theories. These caused some difficulty in prior treatments [4, 5]. In particular, while the exponential of a source term \( e^{i\lambda z} \) blows up on the imaginary axis where the states of the ghost theory are defined, it is still a legitimate multiplier on \( S_{1/2}(ix) \), where its exponential growth is dominated by the faster rate of decrease of the test functions, and on families of subspaces of \( S_{1}(ix) \), where it just modifies the coefficient, and not the exponential power, of decrease of test functions.

In both cases, the path integral may be defined, at least for a finite initial time interval, via the formula (12), which defines a legitimate distribution despite the fact that the integrand in the naive path integral may be exponentially unbounded.

A similar analysis can be made for the definite inner product version whose configuration space is the real line by replacing \( ix \) by \( x \) in the arguments.

Potentials for which both \( S_{1/2}(ix) \) and \( S_{1}(ix) \) are suitable include the free particle and the harmonic oscillator, as well as the linear potential, of interest in some cosmological models, and the inverted harmonic potential that appears in two-dimensional string theory, despite the fact that the potentials in the latter two cases are unbounded below. The path integral may be defined for these potentials via the formula (12).

While \( S_{1}(ix) \) allows the treatment of higher polynomial potentials that cannot be treated using \( S_{1/2}(ix) \), we note that \( S_{1/2}(ix) \) allows us to treat some potentials that cannot be treated using \( S_{1}(ix) \). In particular, while \( S_{1}(ix) \) is good for real polynomials, the growth condition \( |e^{-iV(z)}| \leq C e^{b|z|^p} \) for \( S_{1/2}(ix) \) does not require the momentum terms or \( V(q) \) to be real.

In this case, the expression (12) defines a good path integral even though the integrand may exponentially unbounded with growth of order \( e^{ak^2 + bx^2} \).
Given a test space consisting of analytic functions, a generalized function \( \langle \langle z | \rangle \rangle \) can be defined as
\[ \langle \langle z | \phi \rangle \rangle \equiv \phi(z), \]
for any \( z \) within the common domain of analyticity of the test functions \( \phi \), even though the full space of states is defined only on the imaginary axis.

We will consider the phantom Hamiltonian introduced at the beginning of the section
\[ H \equiv -\frac{p^2}{2m} + \mu p + V(q). \] (17)
and argue that under certain assumptions the quantities
\[ \langle y | e^{-iHt} | x = 0 \rangle \]
and
\[ i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle, \]
coincide for some range of \( y \) and for some initial time interval \( 0 < t < T \). This will hold for any real \( y \) when we use the test space \( S_{1/2}^{1/2}(ix) \), whose elements are entire functions, or for \( y \) in a neighbourhood of \( 0 \) in the case of \( S_{1}^{1}(ix) \), whose elements are analytic only on a strip including the imaginary axis.

Our first assumption is that the distribution \( \langle \langle z | e^{-iHt} \rangle \rangle \) can be represented by a kernel, denoted by
\[ \langle \langle z | e^{-iHt} | -ix \rangle \rangle, \]
that is a differentiable function in \( \Re(z) \), \( \Im(z) \) and \( x \) and a differentiable function of \( t \) for some initial time interval \( 0 < t < T \)\(^{10}\). Given this assumption, we may write
\[ \phi(z, t) \equiv \langle \langle z | e^{-iHt} | \phi \rangle \rangle = \int dx \langle \langle z | e^{-iHt} | -ix \rangle \rangle \phi(ix, 0) \]
for test functions \( \phi \in S_{1/2}^{1/2}(ix) \) or \( \phi \in S_{1}^{1}(ix) \). Here \( \phi(z, t) \) is analytic, since the space of test functions was chosen to be closed under time evolution. Differentiating on both sides with respect to \( \partial \bar{z} \), and using the fact that the spaces \( S_{\alpha}^{\beta}(ix) \) are “sufficiently rich” \(^{9}\), we conclude that
\[ \partial \bar{z} \langle \langle z | e^{-iHt} | -ix \rangle \rangle = 0. \]
\(^{10}\)The example of the harmonic oscillator, for which the transition function refocuses to a delta distribution after a half period, shows the need for the time interval restriction.
In other words, $\langle\langle z | e^{-iHt} | ix \rangle\rangle$ is analytic in $z$.

Due to this analyticity of $\langle\langle z | e^{-iHt} | ix \rangle\rangle$, the differential equation

$$(i \partial_t - H (-\partial_y, iy)) \langle\langle iy | e^{-iHt} | x = 0 \rangle\rangle = 0,$$

which is satisfied by assumption for $0 < t < T$ by the transition function in the indefinite representation ($q \to ix$, $p \to -\partial_x$) on the imaginary axis, is in fact true on the whole domain of analyticity in $z$. We find

$$(i \partial_t - H (-i\partial_z, z)) \langle\langle z | e^{-iHt} | x = 0 \rangle\rangle = 0,$$

on the entire real axis in the case of $S_{1/2}^1(iy)$, or some real neighbourhood of zero in the case of $S_1^1(iy)$. But for real values of $z$, this just coincides with the ordinary Schrödinger equation giving the time evolution of $\langle y | e^{-iHt} | x = 0 \rangle$ in the ordinary Hilbert space representation.

Therefore, on the interval $0 < t < T$, the objects

$$i \langle\langle y | e^{-iHt} | x = 0 \rangle\rangle$$

and

$$\langle y | e^{-iHt} | x = 0 \rangle$$

are ordinary functions satisfying the the same differential equation in $y$. They would therefore coincide if they satisfied the same initial conditions. However, the initial conditions at $t = 0$ in the two representations are not ordinary functions, but rather delta distributions on two different test spaces. These distributions are not comparable.

We therefore recast the problem in a form that gives an ordinary function as initial condition. This is done by considering the quotient

$$G(z, t) \equiv \frac{\langle\langle z | e^{-iHt} | x = 0 \rangle\rangle}{\langle\langle z | e^{-iH_0t} | x = 0 \rangle\rangle},$$

where $H_0$ is the free phantom Hamiltonian considered in section IV and the denominator is the analytic continuation to all $z \in \mathbb{C}$

$$\langle\langle z | e^{-iH_0t} | x = 0 \rangle\rangle = e^{-i\pi/4} \sqrt{\frac{m}{2\pi t}} e^{-imz^2/2t}. \quad (18)$$

of the transition function

$$\langle\langle iy | e^{-iH_0t} | x = 0 \rangle\rangle = e^{-i\pi/4} \sqrt{\frac{m}{2\pi t}} e^{imy^2/2t} \to \delta(y) \quad \text{as} \quad t \to 0.$$
of the phantom particle on its imaginary axis configuration space. Given the 
form (17) of the Hamiltonian, the differential equation satisfied by $G(z,t)$ is

$$
(it\partial_t - tH (-i\partial_z, z) + iz\partial_z - \mu mz) G(z,t) = 0,
$$

for $0 < t < T$, with initial condition the ordinary function

$$
G(iy,t) \rightarrow 1, \quad t \to 0,
$$
on the imaginary axis, assuming that the effect of the interaction becomes 
negligible in the limit as $t \to 0$, so that the behaviour of the numerator 
approaches that of the denominator in this limit. For the class of smooth 
potentials under discussion, we expect this to hold, although we do not have 
general proof.\[11\]

To compare this solution with the one obtained in the definite representation, 
we need to know what happens to $G(z,t)$ for real values of $z$ as $t \to 0$. 
To this end, we note that the solution of this differential equation gives 
a family of functions, indexed by $t$, that are analytic on an open domain in the 
complex plane in $z$ including either a vertical strip containing the imaginary 
axis or the whole complex plane, depending on the choice of test space. This 
family approaches to the constant function 1 on the imaginary axis. If this 
family is uniformly bounded on some open neighbourhood of the origin for 
small enough $t$, we may use Vitali’s theorem \[31, 32\] to conclude that the 
family converges to the constant function 1 not only on the imaginary axis 
but on this whole neighbourhood and, by implication, also on the segment 
of the real axis included in the neighbourhood.

For the class of potentials under discussion, we expect the condition of 
uniform boundedness to be true, although we have not found a proof. We 
will in the following assume this assertion without proof.

We would like to compare $G(y,t)$ to the corresponding quantity

$$
\tilde{G}(y,t) \equiv \frac{\langle x| e^{-iHt} |x = 0 \rangle}{\langle y| e^{-iH_0 t} |x = 0 \rangle},
$$
in the definite representation, which satisfies the differential equation

$$
(it\partial_t - tH (-i\partial_y, y) + iy\partial_y - \mu my) \tilde{G}(y,t) = 0,
$$

\[20\]

\[11\] It is physically reasonable that the potential requires time to make its presence felt, 
so that the behaviour approaches that of a free particle in the zero time limit.
for $0 < t < T$, with the initial condition
\[ \tilde{G}(y, t) \to 1, \quad t \to 0. \]
Since the equations (19) and (20) coincide for real $z$ and have the same initial condition on the real line, their solutions coincide on the indicated time range. For this range, we conclude
\[ \tilde{G}(y, t) = G(y, t), \quad (21) \]
From their explicit expressions, the free transition functions in the two representations can be compared to obtain
\[ \langle \langle y | e^{-iH_0 t} | x = 0 \rangle \rangle = e^{-i\pi/4} \sqrt{m/(2\pi t)} e^{-imy^2/2t} = -i \langle y | e^{-iH_0 t} | x = 0 \rangle, \quad t > 0, \quad (22) \]
and from (21) we find the result
\[ \langle y | e^{-iH t} | x = 0 \rangle = i \langle \langle y | e^{-iH t} | x = 0 \rangle \rangle, \quad 0 < t < T. \quad (23) \]
This is the main result allowing us to relate quantities calculated in the definite and indefinite representations. It states that, for an initial time interval, certain transition functions calculated in the definite and indefinite representations coincide up to a phase factor. The upper limit of coincidence $T$ is the leftmost point on the positive time axis where the transition function becomes singular (a proper distribution not representable by an ordinary function of $y$).

We now discuss the structure of the transition functions $\langle y | e^{-iH_0 t} | x = 0 \rangle$ and $\langle \langle y | e^{-iH t} | x = 0 \rangle$ as functions of $t$ extended to the complex plane. The above results do not depend on any bounds on the real part of the spectrum of $H$ in either representation. As noted in the introduction to this section, however, the most interesting case occurs when the real part of the spectrum in the indefinite representation is bounded below (or if not, the spectral density decreases sufficiently fast) so that
\[ \langle \langle y | e^{-iH t} | x = 0 \rangle \rangle \]
can be analytically extended to the lower half plane in $t$. We will discuss this case.\[ \text{12} \]
12The discussion for other permutations of conditions on the definite or indefinite spectra would proceed similarly.
Since we are not assuming any conditions on the spectrum of $H$ in the definite representation, the quantity
\[ \langle y | e^{-iHt} | x = 0 \rangle , \]
is not necessarily the boundary value of an analytic function in $t$ on the range of $t$ for which it exists. However, since
\[ i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle = \langle y | e^{-iHt} | x = 0 \rangle , \]
on an initial interval $0 < t < T$, we may recover by analytic continuation the full
\[ \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle \]
for all $t$ in the lower half of the complex plane from the values of $\langle y | e^{-iHt} | x = 0 \rangle$ on this initial interval.

In specific cases, the spectrum of $H$ may, in addition, be bounded above in the definite representation. This is true for the free phantom particle and the phantom harmonic oscillator, but not of the two-particle system described in section 10. In this case, the function
\[ \langle y | e^{-iHt} | x = 0 \rangle \]
can be analytically extended to the upper half plane in $t$. Together with
\[ i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle , \]
it therefore defines a single analytic function on the complex plane, with possible singularities on the real axis. The value for real $t$ of $\langle y | e^{-iHt} | x = 0 \rangle$ is obtained by approaching the real axis from above, while the value of $i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle$, is obtained by approaching the real axis from below. These singularities may, among other possibilities, take the form of branch points and cuts, as some of the examples in this article illustrate.

There will always be at least one singularity on the real axis at the origin $t = 0$, where the transition function approaches the delta distribution. Repeating the analysis of the differential equation for negative times, and using the fact that the free phantom particle transition function for $t < 0$ requires the opposite sign on the right of (22), we see that $i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle$ will

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13 For an example where this function is not the boundary value of an analytic function, see section 10.
instead coincide with $-\langle y | e^{-iHt} | x = 0 \rangle$ for small negative $t$. As a result, in the special case where there are cuts on the real axis, the origin will be a branch point with the corresponding branch cut extending to the left.

In general, there will be a singularity in $t$ wherever the solution becomes a proper distribution not representable by an ordinary function. For example, the harmonic oscillator, due to the common periodicity of all the modes, will have singularities in $t$ at all multiples of the half period. In this case, the upper limit $T$ of the range of validity of

$$\langle y | e^{-iHt} | x = 0 \rangle = i \langle \langle y | e^{-iHt} | x = 0 \rangle \rangle, \quad 0 < t < T.$$  \hspace{1cm} (24)

is one half of the period. The fact that there is such an upper limit appears to be a special consequence of the even spacing of the harmonic oscillator energy levels, which causes the wave function to be refocused to a delta function after a half period. In general, perturbations that are not linear or harmonic are expected to destroy this even spacing, erasing the singularity that forces us to take $T$ finite. Therefore, for generic potentials, except at exceptional values of the constants in the potential, the period may be expected to become infinite, so that $T$ will be infinite and the above result should be valid for all positive $t$.

6 Ground state expectation values

In this section we will argue that ground state expectation values appropriate to the indefinite representation can be extracted from the corresponding calculation in the definite representation.

Since a theory is completely determined by its ground state expectation values, one may conclude that the definite representation encodes all the physical information relevant to the indefinite representation, so that we can use whichever representation is most convenient to perform calculations.

14 The definite representation may be preferred for functional integral calculations in field theories, since the functional integral obtained from the definite representation is in many cases manifestly covariant (modulo boundary conditions), whereas the functional integral obtained from the indefinite representation is not manifestly covariant due to an unusual integration region for the phantom components of the fields [4, 8]. Examples where this happens include the time-like component of a gauge field [5] and the Dirac boson. Dirac bosons appear in Pauli-Villars regularizations of field theories containing fermions and in Lee-Wick field theories, and are discussed in section [11]. The results of
As in section 5, we will discuss the case that is of most interest in phantom and ghost theories, where the spectrum of $H$, which is real in the definite representation, may be unbounded above and below. In the indefinite representation, the spectrum may be complex \[28\]. In the cases of interest to us, the real part of the indefinite representation spectrum will be bounded below, which we will assume. We also assume that the indefinite representation has a unique state, called the ground state, for which the real part of the energy spectrum attains this lowest bound.

We will now argue that, under these assumptions, either representation may be used to calculate the ground state expectation values relevant to the indefinite representation.

General expectation values may be generated by adding a time-dependent source term to the action. The argument is then based on a generalization of the result of section 5 whose arguments and conclusions remain valid, with slight modifications, if the Hamiltonian contains time-dependent linear source terms of the form

$$\mu(t) p + \lambda(t) q,$$

where $\mu$ and $\lambda$ are real-valued functions of $t$. We already know that the relevant test spaces admit time-independent linear terms of this type. However, the argument determining the choice of test space was based on the space being closed under time evolution in the limit of vanishingly small time increments, and therefore remains equally valid when the source terms are allowed to depend on time.

The further modifications needed are as follows: Since $H$ in (19) may now depend on time, we replace $e^{-iHt}$ in (19) and subsequently. The correspondingly modified conclusion (24) of section 5 may now be written as

$$i \langle\langle y | U(t_f, t_i) | x = 0 \rangle\rangle = \langle y | U(t_f, t_i) | x = 0 \rangle, \quad -T < t_i < t_f < T. \quad (25)$$

By taking functional derivatives with respect to $\mu(t)$ and $\lambda(t)$, we may generate expectation values of insertions of powers of $p$ and $q$ in the two representations. These will then coincide up to a factor $i$ if all insertions are in the region $-T < t < T$ where (25) is valid.

this section may be seen as a justification for the covariant treatment of these fields in the functional integral.
For simplicity, we specialize our argument to the two-point function of $q$. Similar arguments can be made for other correlation functions. The above result implies that

\[
\langle \langle y, T \mid Tq(t_f)q(t_i) \mid x = 0, -T \rangle \rangle = \langle \langle y \mid e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} \mid x = 0 \rangle \rangle = -i \langle y \mid e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} \mid x = 0 \rangle
\]

provided we choose $T$ small enough and provided that

\[-T < t_i < t_f < T.\]

First, we would like to get rid of the restriction on the range of $t_i$ and $t_f$ to obtain a useful relation also outside the interval $(-T, T)$. Second, we would like to extract the ground-state expectation value from either representation.

Although we do not know of a general argument to get rid of the restriction on the range of $t_i$ and $t_f$, we may do so in specific cases by one of at least two methods.

First, as we remarked in section 5, for generic non-harmonic potentials one expects $T$ to be infinite. In this case, the restriction on $t_i$ and $t_f$ disappears and we are done.

Second, in the cases where $T$ cannot be chosen infinite, the expressions

\[
\langle \langle y \mid e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} \mid x = 0 \rangle \rangle,
\]

and

\[-i \langle y \mid e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} \mid x = 0 \rangle,
\]

generally do define functions of $t_f$ and $t_i$ also outside the range $(-T, T)$. Although now these expressions have singularities on the real axis at finite values of $T$, in various cases of interest they do describe analytic functions of $t_f$ and $t_i$ on the entire real line. Indeed, there are often separate considerations from which analyticity of the two-point functions may be inferred. Since we have shown that these expressions coincide for sufficiently small $t_f$ and $t_i$, in the case where they define analytic functions they will coincide on the entire region of analyticity in $t_i$ and $t_f$.

\footnote{See for example the phantom harmonic oscillator of section 5.}
When a ground state exists in the indefinite representation and has overlap with the state \( |x = 0\rangle \), the ground state expectation value may be extracted in the standard way by writing

\[
\langle \langle x = 0 | e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} | x = 0 \rangle \rangle = \sum_{m,n,k} \langle \langle x = 0 | m \rangle \rangle e^{-iE_m(T-t_f)} \langle \langle m | n \rangle \rangle \langle \langle m | q | n \rangle \rangle e^{-iE_n(t_f-t_i)} \langle \langle n | k \rangle \rangle \times \langle \langle n | q | k \rangle \rangle e^{-iE_k(t_i+T)} \langle \langle k | x = 0 \rangle \rangle,
\]

and

\[
\langle \langle x = 0 | e^{-iH(2T)} | x = 0 \rangle \rangle = \sum_{m} \langle \langle x = 0 | m \rangle \rangle e^{-iE_m(2T)} \langle \langle m | m \rangle \rangle \langle \langle k | x = 0 \rangle \rangle.
\]

and taking the limit \( T \to -i\infty \) of the analytic continuation in \( T \) of their quotient to the lower half plane. This analytic continuation exists given the bound on the indefinite representation spectrum that we assumed in the introduction to this section.

Denoting the ground state by \( |0\rangle \), we obtain

\[
\langle \langle 0 | q(t_f) q(t_i) | 0 \rangle \rangle = \lim_{T \to -i\infty} \frac{\langle \langle x = 0 | e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} | x = 0 \rangle \rangle}{\langle \langle x = 0 | e^{-iH(2T)} | x = 0 \rangle \rangle}.
\]

By (24) and (26), we know that the definite and indefinite representations of the numerator and denominator on the right hand side coincide up to a common factor \( i \) for sufficiently small real \( T \), and will therefore have the same analytic continuation to the lower half plane. We may therefore just as well write

\[
\langle \langle 0 | q(t_f) q(t_i) | 0 \rangle \rangle = \lim_{T \to -i\infty} \frac{\langle \langle x = 0 | e^{-iH(T-t_f)} q e^{-iH(t_f-t_i)} q e^{-iH(t_i+T)} | x = 0 \rangle \rangle}{\langle \langle x = 0 | e^{-iH(2T)} | x = 0 \rangle \rangle}.
\]

16 We have assumed here for notational convenience that the spectra are discrete and do not contain zero norm states, so that a basis can be chosen satisfying \( \langle \langle m | n \rangle \rangle = \pm \delta_{mn} \). The generalizations when this is not the case are straightforward.

17 It is important to note that in general we have to take \( x = 0 \) in the final state \( \langle \langle x = 0 | \rangle \rangle \), since the eigenstates \( | m \rangle \rangle \) do not necessarily belong to the subspace of analytically continuable test functions. Since the indefinite representation configuration space is the imaginary axis, \( \langle \langle x | m \rangle \rangle \) is only guaranteed to be meaningful for \( x = 0 \). This restriction on the final boundary condition may be relaxed in cases where all the eigenstates \( | m \rangle \rangle \) belong to the test space.
This formula expresses the ground state expectation value relevant to the indefinite representation in terms of quantities calculated in the definite representation, which may not have a ground state.

To make the connection with the path integral, we note that it is only when \(-T < t_i, t_f < T\) that the expressions have a simple path integral representation. With this provision, we may express the result as

\[
\langle\langle 0| Tq(t_f) q(t_i) |0\rangle\rangle = \lim_{T \to -i\infty} \frac{\langle x = 0, T | Tq(t_f) q(t_i) | x = 0, -T \rangle}{\langle x = 0, T | x = 0, -T \rangle}, \quad -T < t_i < t_f < T.
\]

(29)

We call the reader’s attention to the fact that this formula enables us to extract the indefinite ground state expectation value from the path integral obtained from the definite representation, where there is no ground state.

Once again, we repeat that for sufficiently generic potentials we expect \(T\) to be infinite, so that there is no need to restrict the ranges of \(t_i\) and \(t_f\). When \(T\) is finite, the path integral calculation may be done assuming \(-T < t_i, t_f < T\), and the general two-point-function for \(t_i\) and \(t_f\) outside this range must in general be obtained from separate analyticity considerations in \(t_i\) and \(t_f\).

Even if the upper limit for \(T\) is finite, the restriction on \(T\) may still be overcome even without such analyticity considerations if the definite spectrum is bounded above, as in the case of the phantom harmonic oscillator discussed below. When there is such a bound, the expectation values on the right hand sides of (27) and (28) describe the boundary values of analytic functions defined respectively on the lower and upper half planes in \(T\). Since these coincide for small enough \(T\), they in fact describe a single analytic function with singularities or cuts on the real axis, but which can be analytically continued from the upper to the lower half plane past the above interval of coincidence. This analytic function, and its limit for \(T \to -i\infty\), can therefore be recovered from the value of the definite representation for any range of \(T\).

After this rather technical discussion, it may be helpful to remind the reader why the result (29) is useful to us. As we remarked, in concrete applications the definite representation is often the one where covariance is manifest, a property that becomes obscured in the indefinite representation. If we do this, the absence of a ground state in the definite representation may
make a Euclidean time quantization problematic, whereas the real-time quantization considered in this paper remains well-defined. In a non-perturbative path integral approach, we do not know the ground state a priori, but the fixed boundary conditions in quantities such as

$$\langle x = 0, T | Tq(t_f) q(t_i) | x = 0, -T \rangle$$

occurring in (29) are naturally and easily imposed.

We note in conclusion that, since in both representations we have to perform an analytic continuation to extract the ground state expectation values, the indefinite representation provides no relative advantage compared to the definite representation for the calculation of these quantities.

7 Inverting the construction

So far we have argued that, for certain classes of Hamiltonians, ground state expectation values in an indefinite inner product representation can be obtained from transition amplitudes calculated in the positive-definite Hilbert space quantization of the Hamiltonian.

It is conceivable that, for some Hamiltonians that do not belong to the classes we discussed, the transition amplitudes in the positive-definite quantization on a given initial time interval may be analytically extensible to the lower half plane. In this case, one would still expect this analytic continuation to display properties of the indefinite inner product representation.

Studying this would require developing the argument of the previous sections in reverse, and is beyond the scope of this article.

8 The phantom harmonic oscillator

In this section we study a harmonic oscillator with opposite sign action. It is now the indefinite representation that has positive energies and a ground state. We show that ground state expectation values for the indefinite representation may be extracted from the definite-representation path integral in the usual way, leading to an unambiguous $i0$ prescription for the two-point function.
Let us first review the canonical quantization of the phantom Lagrangian
\[ L = -\frac{1}{2}m\dot{x}^2 + \frac{m\omega^2}{2}x^2. \]  
(30)

We read off that
\[ p = -m\dot{x}, \]
and canonical quantization instructs us to take \([x, p] = i\). Defining as usual
\[ a = \sqrt{\frac{m\omega}{2}} \left( x + \frac{i}{m\omega} p \right), \]
(31)
\[ a^\dagger = \sqrt{\frac{m\omega}{2}} \left( x - \frac{i}{m\omega} p \right), \]
(32)
where by convention \(\omega > 0\),
we obtain
\[ [a, a^\dagger] = 1 \]
and the Hamiltonian is given by
\[ H = -\frac{p^2}{2m} - \frac{m\omega^2}{2} x^2 = -\frac{\omega}{2} (aa^\dagger + a^\dagger a). \]

Note that we would have obtained the same Hamiltonian if we had started from the first-order form
\[ L = \frac{i}{2} (a^\dagger \dot{a} - \dot{a}^\dagger a) + \omega a^\dagger a, \]
(33)
of the original Lagrangian.

We now have two choices of representation. The first is the familiar positive inner product representation generated from a state \(|0\rangle\) satisfying
\[ a |0\rangle = 0. \]
The resulting states
\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \]
have positive inner products and negative energies
\[ E_n = -\omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots, \]
and there is no ground state. In position space, the normalized states are

$$\langle x | n \rangle = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\pi} \right)^{1/4} H_n(\sqrt{m\omega} x) e^{-m\omega x^2/2}. \tag{34}$$

The other representation lives in the indefinite inner product space introduced in section 3 and is generated from a state $$|0\rangle$$ satisfying

$$a^\dagger |0\rangle = 0, \quad \langle 0 | 0 \rangle = 1.$$

For notational convenience, we define

$$b \equiv a^\dagger,$$

so that

$$[b, b^\dagger] = -1, \quad b |0\rangle = 0.$$ An easy consequence of the negative sign in the commutation relation is that the inner products of the states

$$| n \rangle \equiv \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle$$

alternate in sign, and that these states have positive energy

$$E_n = +\omega \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots,$$

so that the state $$|0\rangle$$ is in this case a ground state. From the differential equation implied by $$b |0\rangle = 0$$, we may construct these states in position space. We find that they are

$$\langle i x | n \rangle = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\pi} \right)^{1/4} H_n(\sqrt{m\omega} (-x)) e^{-m\omega x^2/2}, \tag{35}$$

on the imaginary axis where the wave functions take their values according to the discussion of section 3. These are normalized states with respect to the indefinite inner product $$\langle i x | \cdots \rangle$$ defined in section 3. In fact, we have

$$\langle m | n \rangle = (-)^m \delta_{mn}.$$
This indeed coincides with the inner product implied by the commutation relations of \( b \) and \( b^\dagger \). The completeness relation is

\[
1 = \sum_{n=0}^{\infty} |n\rangle \langle n| |n\rangle \langle n| = \sum_{n=0}^{\infty} (-)^n |n\rangle \langle n|.
\]

We can straightforwardly evaluate the ground state expectation value

\[
\langle \langle Tb(t) b^\dagger(0) \rangle \rangle = -\theta(t) e^{-i\omega t}.
\] (36)

Expressing

\[
x = \frac{1}{\sqrt{2m\omega}} (a + a^\dagger) = \frac{1}{\sqrt{2m\omega}} (b^\dagger + b),
\]

this implies

\[
\langle \langle Tx(t) x(0) \rangle \rangle = -\frac{1}{2m\omega} e^{-i\omega |t|}.
\] (37)

Note that the sign is opposite to that of the ordinary harmonic oscillator.

We now consider the path integral quantization of the theory. We start from the Hamiltonian formulation of the path integral corresponding to the positive-definite inner product representation, which will enable us to carefully compute overall normalization factors important for the subsequent discussion. We need to calculate

\[
\int [dp] [dx] \exp i \int dt (p \dot{x} - H)
\] (38)

\[
= \int [dp] [dx] \exp i \int dt (p \dot{x} + \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2)
\]

\[
\equiv \lim_{\Delta t \to 0} \int \prod_i \frac{dp_i}{2\pi} \prod_i dx_i \exp i \sum_i \Delta t \left( p_i (x_{i+1} - x_i) + \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 \right)
\]

\[
= \lim_{\Delta t \to 0} e^{i\pi/4} \sqrt{\frac{m}{2\pi \Delta t}} \int \prod_i \left( e^{i\pi/4} \sqrt{\frac{m}{2\pi \Delta t}} dx_i \right)
\]

\[
\times \exp i \sum_i \Delta t \left( -\frac{1}{2} m \left( \frac{x_{i+1} - x_i}{\Delta t} \right)^2 + \frac{m\omega^2}{2} x_i^2 \right),
\]

33
where we have integrated over the \( p_i \) in the last step. The prefactor comes from the integration over the final \( p_i \), while the final \( x_i \) is fixed.

Note that the last line contains the expected discretization of the Lagrangian \( \mathcal{L} \) in equation (30). However, it is important to note that the measure factors

\[ e^{i\pi/4} \sqrt{\frac{m}{2\pi \Delta t}} \]

are different from the ones that would have been obtained from the usual harmonic oscillator \(-\mathcal{L}\), which has a factor \( e^{-i\pi/4} \) instead of \( e^{i\pi/4} \).

We will be interested in calculating ground state expectation values appropriate for the indefinite inner product representation, whereas the above path integral is the one we would obtain from the positive-definite representation. However, in section 6 we argued that the former can be extracted from the latter via the usual procedure of calculating the path integral for real time intervals \( T \) and then taking the \( T \to -i\infty \) limit of its analytic continuation, as long as we take the initial and final boundary conditions fixed at

\[ x_0 = x_N = 0, \]

and provided these states have some overlap with the ground state. This is exactly the same procedure one would use to extract ground state expectation values, if one existed, for the positive-definite inner product representation.

The change of variables

\[ x_j = \sum_p q_p \sin \frac{p\pi j \Delta t}{T} \]

(39)

can be checked to have Jacobian

\[ \sqrt{\frac{N}{2}}^{N-1}, \]

and we obtain

\[ e^{i\pi/4} \lim_{N \to \infty} \sqrt{\frac{Nm}{2\pi T}} \prod_{p=1}^{N-1} dq_p e^{i\pi/4} \sqrt{\frac{Nm}{2\pi T}} \sqrt{\frac{N}{2}} \]

\[ \times \exp \left( -i \right) \frac{mN^2}{2T} \left( 1 - \cos \frac{p\pi}{N} - \frac{\omega^2T^2}{2N^2} \right) q_p^2 \]

\[ \text{In other words, the } T \to -i\infty \text{ limit automatically chooses whichever representation has a ground state.} \]
The exponent here has sign opposite to that of the usual harmonic oscillator. It bears repeating that this does not cause any problems in the real time formalism used here.\(^{19}\) The general definition in section 5 gives a well-defined meaning to the path integral as a distribution. More concretely, the above integrals have an unambiguous meaning as the Fourier transforms of distributions, and can be operationally evaluated in a non-perturbative setting by approximating the integrand by a convenient family of test functions. More precisely, the Fourier transform of the integrand gives a distribution that is locally equivalent to a unique continuous function in a neighbourhood of zero. The above integral is then by definition the value of this function at zero.\(^{20}\)

According to the discussion of the section 5, the answer, here calculated for real \(T\), is expected to be analytically extensible to a function on the lower half plane that may have singularities and cuts on the real axis. The path integral expression will give the exact result as long as we keep careful track of overall factors of \(i\).

To this end, note that for each \(p\) such that
\[
1 - \cos \frac{p \pi}{N} - \frac{\omega^2 T^2}{2N^2} > 0,
\]
the corresponding integral gives a factor
\[
1 \sqrt{2 \left| 1 - \cos \frac{p \pi}{N} - \frac{\omega^2 T^2}{2N^2} \right|},
\]
the absolute value being superfluous here, whereas for
\[
1 - \cos \frac{p \pi}{N} - \frac{\omega^2 T^2}{2N^2} < 0, \quad (40)
\]
\(^{19}\) In fact, as discussed in section 5, the integral could even have been calculated as a distribution in the Euclidean formalism despite the exponentially unbounded behaviour of the integrand in that case.

\(^{20}\) The integral in each variable can easily be checked to exist as an improper Riemann integral, though it does not have meaning as a Lebesgue integral. However, in the present case where the integral is multi-dimensional, an iterated Riemann integration would not be invariant under general changes of variables.\(^{20}\) This, and the fact that in general the integrand in the real-time formalism is a proper distribution, forces us to define the integral in terms of a transform of a distribution.
we obtain a factor
\[ i \sqrt{2 \left| 1 - \cos \frac{p\pi}{N} - \frac{\omega^2 T^2}{N^2} \right|} \, . \]

One can check that (40) is satisfied for no values of \( p \) as \( N \to \infty \) when
\[ 0 < T < \frac{\pi}{\omega} \, , \]
for the single value \( p = 1 \) when
\[ \frac{\pi}{\omega} < T < \frac{2\pi}{\omega} \, , \]
for the two values \( p = 1 \) and \( p = 2 \) when
\[ \frac{2\pi}{\omega} < T < \frac{3\pi}{\omega} \, , \]
and so on. We obtain
\[
i^{[T\omega/\pi]} e^{i\pi/4} \lim_{N \to \infty} \sqrt{\frac{Nm}{2\pi T}} \left/ \sqrt{\prod_{p=1}^{N-1} 2 \left( 1 - \cos \frac{p\pi}{N} - \frac{\omega^2 T^2}{N^2} \right)} \right/ \frac{\prod_{p=1}^{N-1} \left( 1 - \frac{\omega^2 T^2}{2N^2 \left( 1 - \cos \frac{p\pi}{N} \right)} \right)}{\prod_{p=1}^{N-1} \left( 1 - \frac{\omega^2 T^2}{2N^2 \left( 1 - \cos \frac{p\pi}{N} \right)} \right)} \right/ \sqrt{\prod_{p=1}^{N-1} \left( 1 - \frac{\omega^2 T^2}{2N^2 \left( 1 - \cos \frac{p\pi}{N} \right)} \right)} \, .
\]

Using the identity
\[
\prod_{p=1}^{N-1} \left( 1 - \cos \frac{p\pi}{N} \right) = \frac{N}{2^{N-1}} ;
\]
the factor in braces is seen to be unity. Furthermore, it can be shown that, as \( N \to \infty \), one gets
\[
\prod_{p=1}^{N-1} \left( 1 - \frac{\omega^2 T^2}{2N^2 \left( 1 - \cos \frac{p\pi}{N} \right)} \right) \to \prod_{p=1}^{N-1} \left( 1 - \frac{\omega^2 T^2}{p^2\pi^2} \right) = \frac{\sin \omega T}{\omega T} ,
\]
and we obtain the final result

$$\langle x = 0, T | x = 0, 0 \rangle = i^{T\omega/\pi} e^{i\pi/4} \frac{m\omega}{2\pi |\sin \omega T|}. \quad (41)$$

This is the physical result, valid for purely real $T$. We can express it as the boundary value when approaching the real axis from above of a function analytic on the whole complex plane with cuts on the real axis as

$$\langle x = 0, T | x = 0, 0 \rangle = e^{i\pi/4} \frac{m\omega}{2\pi \sin \omega (T + i0)}, \quad (42)$$

Here the meaning of the square root is defined by specifying the cuts to be on the intervals

$$(2n - 1) \pi/\omega < T < 2n\pi/\omega, \quad n \in \mathbb{Z}.$$ 

The endpoints of the cuts are precisely integer multiples of the half-period. At these times the original position state is refocused and again becomes a delta distribution, so that singularities are indeed expected according to the general analysis of section 5.

To see that the overall powers of $i$ are correctly reproduced, note that as one travels horizontally from $x + iy$ to $x + 2\pi + iy$ for positive $y$, the value of $\sin (x + iy) = \sin x \cosh y + i \cos x \sinh y$ describes a clockwise half circle around the origin in $\mathbb{C}$, so that $\sqrt{\sin (x + iy)}$ picks up a factor $-i$, and $1/\sqrt{\sin (x + iy)}$ picks up a factor $i$, as required.

This has the spectral representation

$$e^{i\pi/4} \frac{m\omega}{2\pi \sin \omega (T + i0)} = \sqrt{\frac{m\omega}{\pi}} \left( e^{i\left(\frac{\omega}{2}\right)(T+i0)} + \frac{1}{2} e^{i\left(\frac{\omega}{2}+2\omega\right)(T+i0)} + \cdots \right), \quad (43)$$

which has the operator interpretation

$$\langle x = 0, T | x = 0, 0 \rangle = \langle x = 0 | e^{-iHT} | x = 0 \rangle$$

$$= \sum_{n=0}^{\infty} \langle x = 0 | e^{-iHT} | n \rangle \langle n | x = 0 \rangle$$

$$= \sum_{n=0}^{\infty} e^{-iE_nT} |\langle x = 0 | n \rangle|^2,$$

37
in terms of the states (34) with positive-definite inner product \( \langle \cdot | \cdot \rangle \) and negative energies \( E_n = -\frac{\omega^2}{4} - n\omega \). Notice that only the states for even \( n \) have nonzero overlap with \( | x = 0 \rangle \), and that the path integral gives exactly the correct factors \( | \langle x = 0 | n \rangle |^2 \). Also note that the powers of \( i \) required for correspondence with (41) are correctly reproduced due to the common factor \( e^{\omega T/2} \).

The series converges in the sense of Abel summation, which is equivalent to having the \( +i0 \) in (43).

The positive-inner-product, negative energy spectral representation (43) is analytically extensible to the upper half plane. As expected from our general discussion in section 5, this function can be analytically extended, crossing the real axis on an interval to the right of the origin, to the lower half plane, where it can be used to obtain the indefinite inner product, positive energy representation. This function has cuts on the real axis and is given by

\[
e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega z}}
\]

with the constraint that the cut starting at \( z = 0 \) is taken to the left of the origin.

The values of the indefinite inner product representation inner product for real \( T \) are given by approaching the real axis from below, where we get the spectral representation

\[
e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T - i0)}}
= i \sqrt{\frac{m\omega}{\pi}} \left( e^{-i\left(\frac{\omega}{2}\right)(T - i0)} + \frac{1}{2} e^{-i\left(\frac{\omega}{2} + 2\omega\right)(T - i0)} + \ldots \right), \quad (44)
\]

for all real \( T \). This has the operator interpretation

\[
i \langle x = 0, T | x = 0, 0 \rangle = i \langle x = 0 | e^{-iHT} | x = 0 \rangle
= i \sum_{n=0}^{\infty} \langle x = 0 | e^{-iHT} | n \rangle \langle n | n \rangle \langle n | x = 0 \rangle
= i \sum_{n=0}^{\infty} e^{-iE_nT} \langle n | n \rangle | \langle x = 0 | n \rangle |^2, \quad (45)
\]
in terms of the states (35) with indefinite inner product \( \langle \langle \cdot \cdot \cdot \cdot \cdot \rangle \rangle \) and positive energies \( E_n = \frac{\omega^2}{2} + n\omega \). Again, the \(-i0\) in (44) is equivalent to Abel summation. We see that the path integral has exactly supplied the correct prefactor \( i \) to satisfy the relation

\[
\langle x = 0, T | x = 0, 0 \rangle = i \langle \langle x = 0, T | x = 0, 0 \rangle \rangle ,
\]
described in our general analysis of section 5.

Note that the path integral gives the correct factors

\[
| \langle \langle x = 0 | 0 \rangle \rangle |^2 = \sqrt{\frac{m\omega}{\pi}},
\]

and

\[
\langle \langle n | n \rangle \rangle = +1, \quad n \text{ even},
\]
since only the states for even \( n \) have nonzero overlap with \( \langle \langle x = 0 | \rangle \rangle \).

We now calculate the two-point function in the path integral approach. We will first do the calculation for finite time \( T \) and fixed boundary conditions \( x(0) = x(T) = 0 \), and then show that the correct ground state expectation value is unambiguously extracted from the result.

Consider the two-point function first in momentum space

\[
e^{i\pi/4} \lim_{N \to \infty} \sqrt{\frac{Nm}{2\pi T}} \left\{ \prod_{p=1}^{N-1} \int dp \, e^{i\pi/4} \sqrt{\frac{Nm}{2\pi T}} \sqrt{\frac{N}{2}} \times \exp \left(-i\frac{mN^2}{2T} \left( 1 - \cos \frac{p\pi}{N} - \frac{\omega^2 T^2}{2N^2} \right) q_p^2 \right) \right\}
\]

\[
\times \int dq \, e^{i\pi/4} \sqrt{\frac{Nm}{2\pi T}} \sqrt{\frac{N}{2}} \times q_p^2 \exp \left(-i\frac{mN^2}{2T} \left( 1 - \cos \frac{p\pi}{N} - \frac{\omega^2 T^2}{2N^2} \right) q_p^2 \right).
\]

The integrands, including the expression on the last line, are distributions. Again, the integrals have a rigorous and unambiguous interpretation in terms of Fourier transforms of distributions. Evaluating these, we obtain the mo-
mentum space two-point function

\[- e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T+i0)}} \lim_{N \to \infty} \left( \frac{iT}{mN^2} \right) \left( 1 - \cos \frac{\tilde{p}\pi}{N} - \frac{\omega^2 T^2}{2N^2} \right) \]

\[= - e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T+i0)}} \left( \frac{2i}{mT} \right) \frac{1}{(\tilde{p}\pi/T)^2 - \omega^2}. \]

Transforming to position space using (39), we obtain

\[\langle x = 0, T \mid T x(t_2) x(t_1) \mid x = 0, 0 \rangle = - e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T+i0)}} \left( \frac{2i}{mT} \right) \sum_{p=1}^{\infty} \frac{\sin \frac{p\pi t_1}{T} \sin \frac{p\pi t_2}{T}}{(\tilde{p}\pi/T)^2 - \omega^2} \]

\[= - e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T+i0)}} \left( \frac{2i}{mT} \right) \sum_{p=1}^{\infty} \frac{\cos \frac{p\pi (t_1-t_2)}{T} - \cos \frac{p\pi(t_1+t_2-T)}{T}}{(\tilde{p}\pi/T)^2 - \omega^2} \]

\[= - e^{i\pi/4} \sqrt{\frac{m\omega}{2\pi \sin \omega (T+i0)}} \frac{i}{2m\omega} \left( \frac{-\cos \omega (T - |t_2 - t_1|) + \cos \omega (t_2 + t_1)}{\sin \omega T} \right), \]

where we have used [4], formula 1.445 (7) to evaluate the sums explicitly. The sum converges as long as \(\omega T/\pi\) is not an integer. At these values we obtain poles, corresponding to the refocusing of the initial delta function by the harmonic oscillator potential after an integral number of half-periods, which makes the overlap with the final state diverge.

It is important to note that the above summations would not converge for imaginary \(T\). The path integral result must be fully evaluated for real \(T\) before taking the analytic continuation in \(T\) to extract the ground state expectation values below.

We have obtained the two-point function for real time and fixed boundary conditions in the positive-definite inner product representation, which has no ground state. As discussed in section 6, we may therefore extract the ground state expectation value appropriate to the indefinite inner product representation from the analytic continuation to the lower half plane in \(T\) as

\[\langle 0 \mid T x(t_2) x(t_1) \mid 0 \rangle = \lim_{T \to -i\infty} \frac{\langle x = 0, T \mid T x(t_2) x(t_1) \mid x = 0, -T \rangle}{\langle x = 0, T \mid x = 0, -T \rangle} \]

\[= - \frac{1}{2m\omega} e^{-\omega |t_2 - t_1|}, \]
which agrees with the operator formalism result (37) in the indefinite inner product representation.

From the above and

\[ b = \sqrt{\frac{m\omega}{2}} \left( x + \frac{i\dot{x}}{\omega} \right), \quad b^\dagger = \sqrt{\frac{m\omega}{2}} \left( x - \frac{i\dot{x}}{\omega} \right), \]

we easily obtain the desired

\[ \langle \langle 0 | T b(t) b^\dagger(0) | 0 \rangle \rangle = -\theta(t) e^{-i\omega t} = \frac{1}{i} \int \frac{dk}{2\pi} \frac{e^{-ikt}}{k - \omega + i0}. \]

9 The pole prescription is not derivable from a convergence factor

It is important to note that the two-point function (47), which we have unambiguously calculated starting from the real-time path integral, does not coincide with the naive result one would obtain if one were to replace

\[ \omega \to \omega + i\epsilon \]

in the path integral expression (38), as is sometimes incorrectly argued to be necessary to make the integral converge. In fact, such a replacement is not necessary since, as we have pointed out, the integral already exists in the distributional sense without such an \( i\epsilon \) factor. Instead, the correct and unambiguous result (47) so obtained for the two-point function can be written as

\[ -\frac{1}{2m\omega} e^{-i\omega|t_2 - t_1|} = \frac{1}{im} \int \frac{dk}{2\pi} \frac{e^{-ikt}}{-k^2 + \omega^2 + i0}, \]

which is what one would obtain by inverting the quadratic part of an action with the opposite replacement

\[ \omega \to \omega - i0. \]

Although little practical advantage would have been gained in this example by doing so, we could in fact have reproduced the correct two-point function by adding a term \( -i\epsilon x^2 \) to the original action (38). Adding such a term would make the originally bounded integrand exponentially unbounded of quadratic order, so that this is not a convergence factor. As explained in
section [5], despite its exponential growth, such a path integral is calculable as a Gel’fand-Shilov distribution, and it will indeed give the correct result as $\epsilon \to 0$. However, the fact that this is not a “convergence factor” shows that the usual naive convergence argument will not work here.

This observation is closely related to the fact that the ordinary Euclidean rotation of the path integral (38) produces an integrand of quadratic exponential growth. In other words, in contrast to the usual case, the Euclidean path integral is worse than the real-time path integral. Again, despite this bad behaviour, such a Euclidean path integral can be calculated as a distribution using the methods of section [5] and the correct results will be obtained.

10 Combining harmonic and phantom harmonic oscillators

In this section we briefly describe the two-particle system consisting of a harmonic oscillator and a phantom harmonic oscillator, with Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{m\omega^2}{2} x_1^2 - \frac{p_2^2}{2m} - \frac{m\omega^2}{2} x_2^2.$$  

This system is instructive in exhibiting some features not seen in the examples we have considered so far. In particular, in the positive definite Hilbert space quantization

- The spectrum is neither bounded above nor bounded below.

- The transition amplitude exists for all real $T$ except for simple pole singularities (not cuts), but is not the boundary value of any analytic function.

- Even so, the restriction of the amplitude to a suitable initial interval in $T$ can be extended to an analytic function on the lower half plane, and indeed to the entire complex plane except for the poles on the real axis.

In the indefinite representation, on the other hand

- The spectrum has a ground state.
• The transition amplitude is the boundary value of an analytic function extensible to the lower half plane, and indeed to the entire complex plane except for poles on the real axis.

Still the arguments of section 5 will hold. The transition amplitude for the two representations agree for initial real $T$, so that ground state expectation values appropriate to the indefinite representation can be extracted from either quantization.

To obtain the transition amplitudes, we use the following known result for the ordinary harmonic oscillator

$$\langle x_1 = 0, T | x_1 = 0, 0 \rangle = e^{-i \pi / 4} \sqrt{\frac{m \omega}{2 \pi \sin \omega (T - i0)}}$$

which, combined with the result (43)

$$\langle x_2 = 0, T | x_2 = 0, 0 \rangle = e^{i \pi / 4} \sqrt{\frac{m \omega}{2 \pi \sin \omega (T + i0)}}$$

for the phantom harmonic oscillator in the positive definite representation, gives

$$\langle x_1 = 0, x_2 = 0, T | x_1 = 0, x_2 = 0, 0 \rangle = (-)^{|T\omega/\pi|} \frac{m \omega}{2 \pi \sin \omega T}.$$  

We emphasize that, even though the system does not have a ground state, the real-time transition amplitude is well-defined. However, due to the factor $(-)^{|T\omega/\pi|}$, we do not obtain the boundary value of any analytic function – the function cannot be analytically extended to either the upper or lower half plane in $T$, which is as expected in the absence of either bound on the energy. We also point out that the singularities on the real line are not cuts now, but poles.

On the other hand, choosing the indefinite representation for the phantom oscillator, while keeping the definite representation for the ordinary oscillator, the combined system has a ground state. Since

$$i \langle \langle x_2 = 0, T | x_2 = 0 \rangle = e^{i \pi / 4} \sqrt{\frac{m \omega}{2 \pi \sin \omega (T - i0)}}$$

we find

$$i \langle \langle x_1 = 0, x_2 = 0, T | x_1 = 0, x_2 = 0, 0 \rangle = \frac{m \omega}{2 \pi \sin \omega T}.$$
where
\[ \langle \langle \phi, \psi | \phi', \psi' \rangle \rangle \equiv \langle \phi | \phi' \rangle \langle \langle \psi | \psi \rangle \rangle'. \]
The transition function can be analytically extended to the lower half plane, as expected in the presence of a lower bound on the energy.

Consistent with the arguments of section 5, the two results agree for an initial real interval, concretely \( 0 < T < \pi/\omega \), which allows us to obtain the result for the indefinite representation from the calculation in the definite representation.

11 The Dirac boson

In this section we discuss the quantization of a Dirac field with bosonic statistics. This kind of field arises, for example, as regulator fields in Pauli-Villars and Lee-Wick regularizations of theories containing Dirac fermions.

This field is particularly interesting, since its Euclidean functional integrand is exponentially unbounded. Despite this, both the real-time and the Euclidean functional integrals may be computed using the methods of section 5. We will do the computation in the real-time formalism.

Relativistic field theory requires locality (causality) and existence of a ground state. These conditions, imposed on a bosonic Dirac field, requires that we choose an indefinite inner product representation for anti-particle modes. However, we shall see that the indefinite representation is unsuitable for formulating a covariant functional integral, since there is no covariant way of separating these modes when we are integrating over all configurations, including those that are not solutions to the equations of motion.

For this reason, it will be very useful to us that, as we have argued, we can extract the same results from the manifestly covariant functional integral based on the definite representation.

Our starting point is the Dirac action
\[ S = \int \bar{\phi} (i\gamma^\mu \partial_\mu - m) \phi, \]
where we take the field \( \phi \) to be bosonic. We first note that, because the spinor field is bosonic, any Euclidean version of this action \( 33 \) is unbounded below, so that the Euclidean path integrand is exponentially unbounded. Since the field appears quadratically, such an unbounded integrand could be treated
using the methods of section 5. However, it will be easier to work in the real-time formalism.

We may expand the action in modes as

\[
S = \int dt \sum_s \int \frac{d^3p}{(2\pi)^3} \left( i \left( a_p^s \dot{a}_p^s + b_p^s \dot{b}_p^s \right) - E_p \left( a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right) \right),
\]

where \(a_p^s\) and \(b_p^s\) are bosonic. This leads to the commutation relations

\[
[a_p^s, a_q^{s\dagger}] = (2\pi)^3 \delta^3(p - q) \delta^{rs}, \quad [b_p^s, b_q^{s\dagger}] = (2\pi)^3 \delta^3(p - q) \delta^{rs}.
\]

with respect to which the field satisfies locality [34]. However, if we define the vacuum state by

\[
a_p^s |0\rangle = b_p^{s\dagger} |0\rangle = 0,
\]

obtaining a positive-definite state space, it is easy to see that we may create states with arbitrary negative energy by applying suitable anti-particle creation operators \(b_p^{s\dagger}\).

We can obtain a local (causal), positive-energy representation by keeping the commutation relations of the modes the same but changing the ground state to satisfy

\[
a_p^s |0\rangle = b_p^{s\dagger} |0\rangle = 0.
\]

Since this does not affect the commutation relations of the fields \(\phi\) and \(\phi^{\dagger}\), the desired locality is preserved. However, as in section 8 the representation now has an indefinite inner product. This is a general feature of local fields with the “wrong” statistics [21] as discussed, for example, in [35]. This choice of ground state is equivalent to interchanging what we mean by anti-particle creation/annihilation, and we could, if we wished, reflect that in the notation by calling \(b_p^s \leftrightarrow b_p^{s\dagger}\) everywhere.

Nothing prevents us from constructing a functional integral based on the indefinite representation (50) and (51). However, this cannot be done covariantly. To understand this, note that in the indefinite representation (51), each \(a_p^s\) mode generates a positive-definite factor in the state space, while each \(b_p^s\) mode generates an indefinite factor. As a result, the two sets of modes would have be represented differently in the functional integral constructed from this representation. Specifically, as we saw in section 3 the

\[21\] But note that Pauli-Villars regulator fields are allowed to have the “wrong” statistics.
configuration space of a ghost degree of freedom is the imaginary axis, so the range of integration for the real and imaginary parts of $b_p^s$ would become the imaginary axis $[1,8,5]$, whereas this would not happen for $a_p^s$. However, the mode expansion (49) depends on a particular space-time decomposition and is not Lorentz-invariant. While the particle-antiparticle distinction can be made covariantly on the space of solutions, a covariant distinction is not possible in the space of arbitrary time-dependent configurations of the field used in the functional integral. The path integral resulting from the indefinite representation, while correct, would not be manifestly covariant.

Since the definite representation treats all modes the same as far as their range of integration is concerned, the definite representation can be used to generate an ordinary functional integral, based on the usual sum over configurations using the Dirac action (48), that does not suffer from this lack of covariance. Fortunately, we have learned how to relate indefinite and definite representations.

To calculate, note that, due to the sign of the energy term, the action for each $b_p^s$ mode in (49) is equivalent to that of the phantom harmonic oscillator (33). As we did in section 8, we may then extract the ground state expectation values for the physically relevant indefinite representation from the functional integral performed in the definite representation. We find

$$\langle\langle 0 \mid T b_p^s(t) b_p^s(0) \mid 0 \rangle\rangle = \lim_{T \to -i \infty} \frac{\langle \phi = 0, T \mid T b_p^s(t) b_p^s(0) \mid \phi = 0, -T \rangle}{\langle \phi = 0, T \mid \phi = 0, -T \rangle}$$

$$= -\theta(t) e^{-iE_p t},$$

where the calculation on the right was done in section 8 in the definite representation. From this, we obtain the correct Dirac two-point function, appropriate for the indefinite inner product theory, as

$$\langle\langle 0 \mid T \phi(x) \bar{\phi}(y) \mid 0 \rangle\rangle = \lim_{T \to -i \infty} \frac{\langle \phi = 0, T \mid T \phi(x) \bar{\phi}(y) \mid \phi = 0, -T \rangle}{\langle \phi = 0, T \mid \phi = 0, -T \rangle}$$

$$= i \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma \mu p_\mu + m}{p^2 - m^2 + i0} e^{-ip(x-y)}$$

where the quantities on the right hand side are calculated using the conventional, manifestly covariant, path integral constructed from the definite

\[\text{22}
\]

Here $b_p^s$ corresponds to $a$ in (33), which corresponds to $b^\dagger$ in the notation of (36).
representation. This two-point function coincides with that of a fermionic Dirac particle. As required, the field satisfies locality.

We note again that we could have reproduced this by replacing \( m \to m - i\epsilon \) in the original action \( (48) \), obtaining an exponentially unbounded integrand that could still be calculated in the distributional sense according to the discussion of section 5. Note, however, that the correct replacement \( m \to m - i\epsilon \) is opposite to the naive convergence factor, and that our above real-time calculation showed such a replacement to be unnecessary.

## 12 Complex energies and non-perturbative Lee-Wick type pole prescriptions

We consider a simple example where an interaction gives rise to complex energies in the indefinite representation. For this example, we show that the path integral exists and provides a unique pole prescription for the relevant contour integrals.

The example consists of a harmonic oscillator \( a \) and a phantom harmonic oscillator \( b \) satisfying
\[
[a, a^\dagger] = 1, \quad [b, b^\dagger] = -1,
\]
with hermitian Hamiltonian
\[
H = \omega (a^\dagger a - b^\dagger b) + \gamma (a^\dagger b + b^\dagger a).
\]
in an indefinite inner product representation based on a state \(|0\rangle\rangle\), where
\[
a |0\rangle\rangle = b |0\rangle\rangle = 0, \quad \langle\langle 0|0\rangle\rangle = 1.
\]
This system may be solved in the operator formalism by defining operators
\[
\bar{a}^\dagger = a^\dagger + ib^\dagger, \quad \bar{a} = a - ib
\]
\[
\bar{b}^\dagger = a^\dagger - ib^\dagger, \quad \bar{b} = a + ib,
\]
that have commutation relations
\[
[\bar{a}, \bar{a}^\dagger] = 0, \quad [\bar{b}, \bar{b}^\dagger] = 0
\]
\[
[\bar{a}, \bar{b}^\dagger] = 2, \quad [\bar{b}, \bar{a}^\dagger] = 2.
\]

\(^{23}\)We should qualify this by noting that while the action used in the functional integrand is covariant, the boundary conditions are not.
Here $\bar{a}^\dagger$ and $\bar{b}^\dagger$ create null states from the vacuum $|0\rangle$. The Hamiltonian becomes

$$H = \frac{1}{2} (\omega - i\gamma) \bar{a}^\dagger \bar{b} + \frac{1}{2} (\omega + i\gamma) \bar{b}^\dagger \bar{a},$$

so that the energy eigenvalues are

$$E_{m,n} = (\omega - i\gamma) m + (\omega + i\gamma) n, \quad m, n = 0, 1, 2, \ldots$$

The spectrum includes complex eigenvalues. Since the Hamiltonian is hermitian, these come in complex conjugate pairs, as expected in an indefinite inner product space, and the corresponding eigenstates are null.

We now apply the considerations of section 5 to construct the path integral representation of this system. Since all terms in the action are at most quadratic in the position and momentum of either oscillator, and since no terms of the form $x_1p_1$ or $x_2p_2$ appear, we may obtain the ground state expectation values appropriate to the indefinite inner product representation from a path integral calculated as a distribution either on the test space

$$S_{1/2}^{1/2}(x_1) \otimes S_{1/2}^{1/2}(ix_2),$$

corresponding to the indefinite representation used above, or on the test space

$$S_{1/2}^{1/2}(x_1) \otimes S_{1/2}^{1/2}(x_2)$$

corresponding to a positive-definite inner product representation for the $b$-oscillator. It was the content of section 6 that both representations encode the same ground state expectation values.

As we previously discussed, the definite representation gives a simpler path integral. With this choice, the action used in the path integral

$$S = \frac{1}{2} \int dt \left\{ \bar{a}^* (i\partial_t - \omega + i\gamma) \bar{b} + \bar{b}^* (i\partial_t - \omega - i\gamma) \bar{a} \right\}.$$
However, despite this, it was the content of section 5 that the corresponding path integrals still exist as distributions on the chosen test spaces. Because they lacked a non-perturbative definition of the indefinite inner product theories corresponding to such unbounded path integrands, various authors [1, 2, 3] studied the consistency of ad hoc prescriptions for defining the two-point functions, and more complex diagrams, order by order in perturbation theory. Lee and Wick [1, 2] introduced one such prescription, requiring the contour integrations defining various diagrams to be continuously deformed to avoid the movement of poles in the complex plane as we increase parameters such as $\gamma$ starting from zero. Cutkosky, Landshoff, Olive and Polkinghorne [3] attempted to generalize this prescription to coalescing singularities not covered by Lee and Wick. Boulware and Gross [4] discussed the problem from the path integral point of view but did not succeed in defining these theories non-perturbatively, mainly due to the unboundedness of the path integrand.

The framework of this article allows a non-perturbative functional integral to be calculated in the distributional sense for various kinds of unbounded path integrands, including the ones studied by Lee and Wick. In principle, we expect the ad hoc prescriptions of the above authors to be either reproduced or corrected in our non-perturbative framework.

We start with the simplest such calculation, where we derive a prescription of Lee-Wick type for the two-point function from the path integral. Consider the problem of obtaining the two-point function

$$\langle \langle T \bar{a}(t) \bar{b}^\dagger(0) \rangle \rangle$$

from the above action in the path-integral approach. If we were to naively invert the relevant quadratic term, we would obtain

$$\frac{2}{i} \int \frac{dk}{2\pi} \frac{e^{-ikt}}{k - (\omega + i\gamma)} = -2 \theta(-t) e^{-i(\omega+i\gamma)t}.$$

A glance at the operator representation confirms that this is wrong. The correct result is reproduced by

$$\langle \langle T \bar{a}(t) \bar{b}^\dagger(0) \rangle \rangle = 2 \theta(t) e^{-i(\omega+i\gamma)t} = \frac{2}{i} \int_{\gamma} \frac{dk}{2\pi} \frac{e^{-ikt}}{k - (\omega + i\gamma)}.$$

(52)

where the integration contour $\Gamma$ has been displaced from the real line to pass above the pole at $1 + i\gamma$. This prescription, in the spirit of Lee and Wick,
seems ad hoc from a naive path integral point of view, but can be rigorously derived in our formalism.

We will now show that the correct result can be unambiguously obtained, without any ad hoc prescription, from a careful evaluation of the path integral. We already discussed in section 9 how the unusual pole prescription in the free two-point function of b follows unambiguously from the path integration. The current discussion generalizes this to the case of complex energies. As we have seen, we need to evaluate the path integral for fixed boundary conditions between real times \( T \) and \(-T\), and then take the limit \( T \to -i\infty \) of the analytic continuation of the result to complex \( T \). Since we have already done this for the free oscillator in section 8, we will start from those results and evaluate the path integral by treating the quadratic interaction perturbatively in \( \gamma \) starting from the form

\[
S = \int dt \left( a^\dagger (i\partial_t - \omega) a - b^\dagger (i\partial_t - \omega) b + \gamma (a^\dagger b + b^\dagger a) \right)
\]

of the action. Nothing is lost in this approach, since the perturbation is quadratic, so that the series can be exactly summed, giving the exact result for the full path integral.

Our starting point is the result (46), describing the two-point function of the free \( b \)-field given the fixed boundary conditions. The two-point function for the \( a \)-field is identical except for an opposite overall sign. Performing perturbation theory with fixed boundary conditions at real \(-T\) and \(T\), analytically continuing the result in \( T \) and taking the limit as \( T \to -\infty \) as discussed, we find

\[
\langle \langle Ta(t) a^\dagger (0) \rangle \rangle = \frac{1}{2} \theta(t) e^{-i\omega t} \cosh \gamma t
\]

\[
= \frac{1}{2} \theta(t) \left( e^{-i(\omega+i\gamma)t} + e^{-i(\omega-i\gamma)t} \right)
\]

\[
= \frac{1}{2} \int \frac{dk}{2\pi} \left[ \frac{e^{-ikt}}{k - \omega - i\gamma} + \frac{e^{-ikt}}{k - \omega + i\gamma} \right],
\]
where the integral contour $\Gamma$ has been deformed from the real axis into the upper half plane to pass above the pole at $\omega + i\gamma$ to reproduce the exact path integral result on the previous line. There is no need to postulate this contour prescription ad hoc, since we have obtained it from an exact path integral calculation using the non-perturbatively defined path integral.\textsuperscript{24}

The expectation values

$$\langle \langle T b(t) b^\dagger(0) \rangle \rangle, \quad \langle \langle T b(t) a^\dagger(0) \rangle \rangle, \quad \langle \langle T a(t) b^\dagger(0) \rangle \rangle,$$

may be similarly computed and put together to reproduce the exact operator formalism result (52).

This method generalizes to more complex diagrams in interacting theories extending the above quadratic action. Consider, for example, an $a$-$b$ loop diagram, expected to be proportional to

$$\int \frac{dk}{2\pi \sqrt{\frac{1}{k - \omega + i0} + \frac{1}{k - \omega - i0}}} \left( \frac{1}{p - k - \omega - i0} + \frac{1}{p - k - \omega + i0} \right),$$

where the integration contour is so far unspecified. As above, we now do an exact, non-perturbative path integral calculation that determines the appropriate integration contour. The path integral may be unambiguously calculated for real $T$ and the expectation value extracted by continuing $T \rightarrow -i\infty$ is unique. Taking this limit term by term in the perturbation series around $\lambda = 0$ gives

$$\int \frac{dk}{2\pi} \frac{1}{k - \omega + i\delta} \frac{1}{p - k - \omega + i\delta} - \gamma^2 \int \frac{dk}{2\pi} \frac{1}{(k - \omega + i\delta)^3} \frac{1}{p - k - \omega + i\delta} - \gamma^2 \int \frac{dk}{2\pi} \frac{1}{k - \omega + i\delta} \frac{1}{(p - k - \omega + i\delta)^3} + \cdots,$$

where the integrals are over the real line. Again, we may not arbitrarily exchange sums and integrations. The integrals may be calculated by closing the contours in either the upper or lower half plane. Choosing the latter, only the poles contributed by the left factors contribute, and denoting

$$f(k) \equiv \frac{1}{p - k - \omega + i\delta},$$

Note that we would have obtained a wrong answer if we had interchanged the order if integration and summation, first formally summing the geometric series in momentum space and only afterwards taking the Fourier transform. We are prevented from doing this by the fact that the momentum-space geometric series does not converge for the full range of $k$ over which we integrate. The above answer is unambiguous.
the result is proportional to
\[
[f(\omega) - \gamma^2 f''(\omega) + \gamma^4 f'''(\omega) - \cdots] \\
- \gamma^2 [f^3(\omega) - \gamma^2 (f^3)'(\omega) + \gamma^4 (f^3)''(\omega) - \cdots] \\
+ \gamma^4 [f^5(\omega) - \gamma^2 (f^5)'(\omega) + \gamma^4 (f^5)''(\omega) - \cdots] \\
+ \cdots
\]
\[
\propto f(\omega + i\gamma) + f(\omega - i\gamma) - \gamma^2 [f^3(\omega + i\gamma) + f^3(\omega - i\gamma)] + \cdots
\]
\[
= \frac{f(\omega + i\gamma)}{1 + \gamma^2 f^2(\omega + i\gamma)} + \frac{f(\omega - i\gamma)}{1 + \gamma^2 f^2(\omega - i\gamma)}
\]
\[
\propto \frac{2}{p - 2\omega} + \frac{1}{p - 2(\omega + i\gamma)} + \frac{1}{p - 2(\omega - i\gamma)},
\]

since \(|\gamma^2 f^2(\omega \pm i\gamma)| \leq 1\) so that the geometric series converge. The poles are precisely at the positions one would expect from the various possible combinations of two intermediate particles with energies \(\omega \pm i\gamma\). This result is reproduced by the contour integral
\[
\int_{\Gamma} \frac{dk}{2\pi} \left( \frac{1}{k - \omega + i\gamma} + \frac{1}{k - \omega - i\gamma} \right) \left( \frac{1}{p - k - \omega - i\gamma} + \frac{1}{p - k - \omega + i\gamma} \right),
\]
where the integration path \(\Gamma\) stretches from \(-\infty\) to \(\infty\), but is deformed to pass below the two poles at \(p - \omega \pm i\gamma\) and above the two poles at \(\omega \pm i\gamma\). This coincides with Lee and Wick’s prescription that the contour should be deformed so that no poles cross it as we change \(\gamma\) continuously starting from \(\gamma = 0\). We have derived it unambiguously as a non-perturbative result in the path integral formulation.

We have seen that Lee-Wick type contour prescriptions can be obtained unambiguously from a non-perturbative path integral. Although we have only considered two very simple toy examples, there is every reason to expect that similar non-perturbative calculations can be used to either reproduce or correct the prescriptions of Cutkosky, Landshoff, Olive and Polkinghorne unambiguously.

13 Conclusion

In this article, we investigated the non-perturbative quantization of phantom and ghost degrees of freedom by relating their representations in definite and
indefinite inner product spaces. For a large class of potentials, we argued that the same physical information can be extracted from either representation, and provided a non-perturbative definition of the path integral for these theories. We applied the results to the study of ghost fields of Pauli-Villars and Lee-Wick type, calculating non-perturbatively previously ad hoc prescriptions for Feynman diagram contour integrals in the presence of complex energies.

The initial motivation for this work was to understand the nonperturbative path integral quantization of certain bosonic Pauli-Villars ghosts for which the Euclidean path integrand is unbounded [36]. These ghosts were an important ingredient in the construction of a class of functional integral measures that are generally covariant, background-independent and conformally invariant. The current article provides the missing ingredient to make the construction of [36] rigorous.

The mathematical framework of this article may have applications in the study of phantoms appearing in certain cosmological models of dark energy. It may also be of interest in the approaches to the cosmological constant problem based on phantoms, or in the approaches based on symmetries consisting of rotations of the configuration space to the imaginary axis.

We pointed out some pitfalls in trying to derive $i\epsilon$ prescriptions in Feynman integrals from naive convergence terms in the action. We showed that for ghost fields, the correct prescription is in fact opposite to what one would obtain from such convergence factors.

Finally, the methods introduced here may conceivably be useful in the complex-coordinate or complex-momentum approaches to the study of resonant scattering.

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References

[1] T.D. Lee and G.C. Wick, *Negative metric and the unitarity of the S-matrix*, Nuclear Physics B9 (1969) 209-243.
[2] T.D. Lee, *A finite theory of quantum electrodynamics*, in *Elementary processes at high energy, part A*, Ettore Majorana 1970 International School of Subnuclear Physics, Erice, July 1-19, Editor: A. Zichichi, Academic Press, New York and London, 1971.

[3] R.E. Cutkosky, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, *Nuclear Physics* **B12** (1969) 281-300.

[4] D.G. Boulware and D.G. Gross, *Lee-Wick indefinite metric quantization: A functional integral approach*, *Nuclear Physics* **B233** (1984) 1-23.

[5] Seiji Sakoda, *Euclidean path integral of the gauge field – holomorphic representation*, arXiv.org preprint [hep-th/0501205] 2005.

[6] I.M Gel’fand and G.E. Shilov, *Generalized functions: Spaces of fundamental and generalized functions*, Vol. 2, Academic Press, Orlando, 1958.

[7] I.M Gel’fand and G.E. Shilov, *Generalized functions: Theory of differential equations*, Vol. 3, Academic Press, Orlando, 1967.

[8] H. Arisue, T. Fujiwara, T. Inoue and K. Ogawa, *Generalized Schrödinger representation and its application to gauge field theories*, *Journal of Mathematical Physics* **22** (1981) 2055-2059.

[9] R. R. Caldwell, *A Phantom Menace? Cosmological consequences of a dark energy component with super-negative equation of state*, *Physics Letters* **B545** (2002) 23-29.

[10] James M. Cline and Sangyong Jeon and Guy D. Moore, *The phantom menaced: Constraints on low-energy effective ghosts*, *Physical Review* **D70** (2004) 043543.

[11] Antonio De Felice, Mark Hindmarsh and Mark Trodden, *Ghosts, instabilities, and superluminal propagation in modified gravity models*, *Journal of Cosmology and Astroparticle Physics* **08** (2006) 005.

[12] Jens Kujat, Robert J. Scherrer and A. A. Sen, *Phantom Dark Energy Models with Negative Kinetic Term*, arXiv.org preprint [astro-ph/0606735] 2006.
[13] Mariusz P. Dabrowski, Claus Kiefer and Barbara Sandhoefer, Quantum phantom cosmology, Physical Review D74 (2006) 044022.

[14] Recai Erdem, A symmetry for vanishing cosmological constant in an extra dimensional toy model, Physics Letters B621 (2005) 11-17.

[15] Recai Erdem, A symmetry for vanishing cosmological constant: Another realization, Physics Letters B639 (2006) 348-353.

[16] Gerard ’t Hooft and Stefan Nobbenhuis, Invariance under complex transformations, and its relevance to the cosmological constant problem, Classical and Quantum Gravity 23 (2006) 3819.

[17] David E. Kaplan and Raman Sundrum, A Symmetry for the Cosmological Constant, Journal of High Energy Physics 0607 (2006) 042.

[18] I. Antoniadis and E. Dudas and D. M. Ghilencea, Living with ghosts and their radiative corrections, arXiv.org preprint [hep-th/0608094] 2006.

[19] W. Pauli and F. Villars, On the invariant regularization in relativistic quantum field theory, Reviews of Modern Physics 21 (1949) 434-444.

[20] Tore Berggren, On the use of resonant states in eigenfunction expansions of scattering and reaction amplitudes, Nuclear Physics A109 (1968) 265-287.

[21] Roger, G. Newton, Analytic properties of radial wave functions, Journal of Mathematical Physics 1 (1960) 319-347.

[22] A. Buchleitner, B. Grémaud and D. Delande, Wavefunctions of atomic resonances, Journal of Physics B27 (1994) 2663-2679.

[23] Nimrod Moiseyev, Quantum theory of resonances: calculating energies, widths and cross-sections by complex scaling, Physics Reports 302 (1998) 211-293.

[24] G. Hagen, J.S. Vaagen and M. Hjort-Jensen, The contour deformation method in momentum space, applied to subatomic physics, Journal of Physics A37 (2004) 8991-9021.

[25] Y.K. Ho, The method of complex coordinate rotation and its application to atomic collision processes, Physics Reports 99 (1983) 1-68.
[26] R. de la Madrid, *Description of resonances within the rigged Hilbert space*, arXiv.org preprint quant-ph/0607168, 2006.

[27] Carl M. Bender, Sebastian F. Brandt, Jun-Hua Chen, Qinghai Wang, *Physical Review* D71 (2005) 025014.

[28] J. Bognár, *Indefinite inner product spaces*, Springer-Verlag Berlin, 1974; A.I. Mal’cev, *Foundations of linear algebra*, W.H. Freeman and Company, 1963.

[29] P. Cartier and C. DeWitt-Morette, *A rigorous mathematical foundation of functional integration*, in *Functional integration: Basics and applications*, eds. C. DeWitt-Morette, P. Cartier and A. Folacci, Proceedings of NATO Advanced Study Institute held September 1-14, 1996, Plenum Press, New York, 1997.

[30] G.W. Johnson and M.L. Lapidus, *The Feynman integral and Feynman’s operational calculus*, Oxford University Press, Oxford, 2000.

[31] G. Vitali, *Rend. R. Istor. Lombardo* (2), 36 (1903) 772-774.

[32] G. Vitali, *Ann. Mat. Pura Appl.* (3), 10 (1904) 73.

[33] P. van Nieuwenhuizen and A. Waldron, *On Euclidean spinors and Wick rotations*, *Physics Letters* B389 (1996) 29-36.

[34] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Perseus Books, Cambridge MA, 1995.

[35] N.N. Bogolubov, A.A. Logunov, A.I. Oksak and I.T. Todorov, *General principles of quantum field theory*, English edition, Kluwer Academic Publishers, Dordrecht, 1990.

[36] André van Tonder, *Worldsheet covariant path integral quantization of strings*, to appear in *International Journal of Modern Physics* A, arXiv.org preprint hep-th/0606017, 1996.