Single Parameter Inference of Non-sparse Logistic Regression Models

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Abstract

This paper infers a single parameter in non-sparse logistic regression models. By transforming the null hypothesis into a moment condition, we construct the test statistic and obtain the asymptotic null distribution. Numerical experiments show that our method performs well.

Keywords: Logistic models Non-sparse Single parameter hypothesis test Moment condition

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1 Introduction

The logistic regression models have been widely used in finance and genetics analysis, which increasingly rely on high-dimensional observations. In other words, the dimension $p$ is high, and the sample size $n$ is relatively small, i.e. $n \rightarrow \infty$ and $p/n \rightarrow \infty$, therefore, modeling, inference, and prediction become more challenging than in traditional environments.

Hypothesis test and confidence intervals in high-dimensional generalized linear models have been widely studied. Van de Geer et al (2014) constructed confidence intervals and statistical tests for single or low-dimensional components of regression coefficients. Ning and Liu (2017) proposed a general framework for hypothesis testing and confidence intervals for low-dimensional
components based on general penalized M-estimators. Cai et al (2019) constructed a debiased estimator based on Lasso estimator and consistently established its asymptotic normality for future observations of arbitrary high dimensions. In the logistic regression models, Sur et al (2019) studied the likelihood ratio test under $p/n \to k$ for some $k < \frac{1}{2}$. Shi et al (2021) focused on the logistic link and imposed certain stringent assumptions. Ma et al (2021) constructed a test statistic for testing the global null hypothesis using a generalized low-dimensional projection for bias correction. Guo et al (2021) proposed a novel bias-corrected estimator through linearization and variance enhancement techniques.

The above methods are sensitive to the sparsity assumption, which leads to the easy loss of error control when this assumption is violated. Statistical inference in non-sparse linear models has been studied extensively. Lin et al (2011b) proposed semiparametric re-modeling and inference method. Lin et al (2011a) introduced a simulation-based procedure to reformulate a new model, with no need to estimate high-dimensional nuisance parameter. Dezeure et al (2017) proposed a residual and wild bootstrap methodology for individual and simultaneous inference. By transforming the null hypothesis into a testable moment condition, Zhu and Bradic (2018) proposed an asymptotically sparse CorrT method to solve the single-parameter testing problem. By convolving the variables from the two samples and combining the moment method, Zhu and Bradic (2016) conducted the homogeneity test of the global parameters in two populations. Zhu and Bradic (2017) further extended this moment method to test linear functionals of the regression parameters, and proposed Modified Dantzig Selector (MDS) to estimate model parameters. Bradic et al (2022) developed uniform and essentially uniform nontestability which identified a collection of alternatives such that the power of any test was at most equal to the nominal size.

In this paper, we consider single parameter significance test in high-dimensional non-sparse logistic regression models, which is of great importance in practice, and is a prerequisite to statistical analysis. For example, we study the effect of a treatment/drug on response after controlling for the impact of high-dimensional non-sparse genetic markers. This problem of statistical inference has not been solved in the existing literature. First, we linearize the regression function based on the logistic Lasso estimator. Then, the approximate linear model is reconstructed according to the hypothesis, which is transformed into a testable moment condition. Finally, we use MDS estimators to construct the test statistics and prove the asymptotic null distribution and power property. Besides its applicability in logistic regression, this method can be extended to other nonlinear regression models.

The remainder of this paper is organized as follows. In Section 2, We present a significance test method for single parameter in non-sparse logistic regression model, and introduce a new moment construction method. Section 3 shows the size and power properties of the proposed test. Section 4 shows the numerical experiments and compares them with the results of another advanced method.
2 Single parameter significance test

2.1 Notations
For a vector $V \in \mathbb{R}^k$, $v_i$ represents the $i$-th element of $V$. $\|V\|_\infty = \max_{1 \leq i \leq k} |v_i|$ and $\|V\|_0 = \sum_{i=1}^k I(v_i \neq 0)$, where $I(\cdot)$ denotes the indicator function. For matrix $A$, its $(i,j)$ entry is denoted by $A_{i,j}$, and the $i$-th row is denoted by $A_i$. For two sequences $a_n, b_n > 0$, $a_n \asymp b_n$ means that there exist constants $C_1, C_2 > 0$ such that $\forall n$, $a_n \leq C_1 b_n$ and $b_n \leq C_2 a_n$.

2.2 Model and hypothesis
We consider the non-sparse logistic regression model:

$$y_i = f(\beta^T X_i) + \epsilon_i, i = 1, 2, ..., n$$  \hspace{1cm} (2.1)

where $f(u) = e^u / (1 + e^u)$, and $\beta = (\beta_*, \theta_*) \in \mathbb{R}^p$ is a non-sparse regression vector with single parameter $\beta_*$ and redundant parameter $\theta_* \in \mathbb{R}^{p-1}$. The observations are i.i.d. samples $(X_i, y_i) \in \mathbb{R}^p \times \{0, 1\}$ for $i = 1, 2, ..., n$, and $y_i \mid X_i \sim Bernoulli(f(\beta^T X_i))$ independently for each $i = 1, 2, ..., n$. We assume $X_i \sim N(0, \Sigma)$. In fact, this result can be extended to sub-Gaussian distribution. The $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_n)' \in \mathbb{R}^n$ is the error term, which is not correlated with $X = (X_1, X_2, ..., X_n)' \in \mathbb{R}^{n \times p}$.

In this paper, we focus on the significance test of single parameter $\beta_*$, i.e.

$$H_0 : \beta_* = \beta_0, \hspace{0.5cm} versus \hspace{0.5cm} H_1 : \beta_* \neq \beta_0.$$  \hspace{1cm} (2.2)

where $\beta_0$ is a given value. As a preliminary, we first give an estimator of the global parameter $\beta$. For technical reasons, we split the samples into two independent subsets $D_1$ and $D_2$. The $L_1$-regularized M-estimator $\hat{\beta}$ of $\beta$ is obtained from $D_1$:

$$\hat{\beta} = \arg\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^n [-y_i \beta^T X_i + \log(1 + e^{\beta^T X_i})] + \lambda \|\beta\|_1 \right\},$$  \hspace{1cm} (2.3)

which is the minimizer of a penalized log-likelihood function with $\lambda \asymp \sqrt{\frac{\log p}{n}}$.

Although $\hat{\beta}$ can achieve the optimal rate of convergence (Sahand et al, 2012; Huang and Zhang, 2012), it’s not suitable to construct confidence intervals and hypotheses test directly because of its biases.

In the following, we reconstruct the regression model based on $\hat{\beta}$ and the samples from $D_2$. We consider the Taylor expansion of $f(u_i)$ at $\hat{u}_i$ for $u_i = \beta^T X_i$ and $\hat{u}_i = \hat{\beta}^T X_i$,

$$f(u_i) = f(\hat{u}_i) + \hat{f}(\hat{u}_i)(u_i - \hat{u}_i) + Re_i, \hspace{0.5cm} i = 1, 2, ..., n,$$  \hspace{1cm} (2.4)
where \( \dot{f}(\cdot) \) is the derivative of \( f(\cdot) \), and \( Re(i) \) is the reminder term. Plugging (2.4) into (2.1), we have

\[
y_i - f(\hat{\beta}^T X_i) + \dot{f}(\hat{\beta}^T X_i)X_i^T \hat{\beta} - Re_i = \dot{f}(\hat{\beta}^T X_i)X_i^T \hat{\beta} + \varepsilon_i, \quad i = 1, 2, \ldots, n. \tag{2.5}
\]

We can treat \( y_i - f(\hat{\beta}^T X_i) + \dot{f}(\hat{\beta}^T X_i)X_i^T \hat{\beta} - Re_i \) as a new response variable \( y_{\text{new},i} \), and \( Y_{\text{new}} = (y_{\text{new},1}, y_{\text{new},2}, \ldots, y_{\text{new},n})^T \in \mathbb{R}^n \), whereas \( \dot{f}(\hat{\beta}^T X_i)X_i^T \) as the new covariate \( X_{\text{new},i} = (z_i, W_i^T)^T \) with \( z_i \in \mathbb{R} \) and \( W_i \in \mathbb{R}^{p-1} \). Consequently, \( \beta \) can be considered as the regression coefficient of this approximate linear model. Then (2.5) is transformed into

\[
y_{\text{new},i} = X_{\text{new},i}^T \beta + \varepsilon_i, \quad i = 1, 2, \ldots, n. \tag{2.6}
\]

Since the null hypothesis is \( H_0 : \beta_* = \beta_0 \), the above equation can be rewritten as:

\[
y_{\text{new},i} = z_i \beta_* + W_i^T \theta_* + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{2.7}
\]

where \( Z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n \) and \( W = (W_1, W_2, \ldots, W_n)^T \in \mathbb{R}^{n \times (p-1)} \) are the design matrices. Subtracting \( z_i \beta_0 \) from both sides of model (2.7), we build the following reconstructed model

\[
y_{\text{new},i} = z_i \gamma_* + W_i^T \theta_* + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{2.8}
\]

where \( \gamma_* = \beta_* - \beta_0 \) is of main interest, and the original \( H_0 \) in (2.2) is equivalent to

\[
H_0 : \gamma_* = 0. \tag{2.9}
\]

Thus, we define a pseudo-response \( V = Y_{\text{new}} - Z\beta_0 \) and a pseudo-error \( e = Z\gamma_* + \varepsilon \), which satisfy that

\[
V = W\theta_* + e, \tag{2.10}
\]

Since \( X \) and \( \varepsilon \) are unrelated, \( W \) and \( \varepsilon \) are unrelated. Therefore, when the null hypothesis is true, we have \( E(e^T W) = E(\varepsilon^T W) = 0 \). Otherwise, \( e \) and \( W \) may be linear dependent through \( z \), which is caused by the confounding effects of \( W \) and \( Z \). Next, we establish a linear correlation model between \( Z \) and \( W \):

\[
Z = W \pi + U, \tag{2.11}
\]

where \( \pi \in \mathbb{R}^{p-1} \) is an unknown regression coefficient vector and \( U \in \mathbb{R}^n \) is the error term, internally independent of each other, which follows Gaussian distribution with zero mean, and \( U \) is uncorrelated to \( (V, W) \). It is worth mentioning that we assume that \( \pi \) is sparse to decouple the correlation between \( Z \) and \( W \).

We consider the correlation between \( e \) in (2.10) and \( U \) in (2.11):

\[
E(U^T e) = E(U^T Z\gamma_* + U^T \varepsilon) = E(U^T U)\gamma_* . \tag{2.12}
\]
2.3 Modified Dantzig Selector

Therefore, the original test problem in (2.2) is equivalent to

\[ H_0 : E((V-W\theta_*)^T(Z-W\pi)) = 0 \quad \text{versus} \quad H_1 : E((V-W\theta_*)^T(Z-W\pi)) \neq 0. \]

(2.13)

Since \( \pi \) is sparse, consistent estimator \( \tilde{\pi} \) is easy to obtain. However, it is difficult to obtain the consistent estimator of non-sparse parameter \( \theta_* \). For any estimator \( \tilde{\theta}_* \) of \( \theta_* \), we have

\[ E((V-W\tilde{\theta}_*)^T(Z-W\tilde{\pi})) \to E((V-W\theta_*)^T(Z-W\pi)) + E((Z-W\pi)^TW(\theta_*-\tilde{\theta}_*)). \]

In the above equation, \( \tilde{\theta}_* \) is a function of \((V,W)\), while \( U \) is uncorrelated to \((V,W)\), so \( Z-W\pi \) and \( W(\theta_*-\tilde{\theta}_*) \) are uncorrelated. Then

\[ E((Z-W\pi)^TW(\theta_*-\tilde{\theta}_*)) = E(Z-W\pi)^TE(W(\theta_*-\tilde{\theta}_*)) = 0. \]

Therefore,

\[ E((V-W\tilde{\theta}_*)^T(Z-W\tilde{\pi})) \to E((V-W\theta_*)^T(Z-W\pi)). \]

The inner product structure in (2.13) alleviates the reliance on a good estimator of \( \theta_* \). We will estimate the unknown parameters \( \pi \) and \( \theta_* \) in the next subsection.

2.3 Modified Dantzig Selector

MDS is used to estimate the unknown parameter \( \theta_* \) and error variance \( \sigma^2_e \) simultaneously,

\[
\hat{\theta}_* = \arg \min_{\theta_* \in \mathbb{R}^{p-1}} \| \theta_* \|_1 \\
\text{s.t. } \| W^T (V - W\theta_*) \|_\infty \leq \eta \rho_1 \sqrt{n} \| V \|_2 \\
V^T (V - W\theta_*) \geq \rho_0 \rho_1 \| V \|_2^2 / 2 \\
\rho_1 \in [\rho_0, 1],
\]

(2.14)

where \( \rho_1 = \sigma_e / \sqrt{E(v_1)^2} \), and \( \rho_0 \in (0,1) \) is a lower bound for this ratio. \( \eta \approx \sqrt{n^{-1} \log p} \) is a tuning parameter.

Similarly, the estimator \( \tilde{\pi} \in \mathbb{R}^{p-1} \) of \( \pi \) is

\[
\tilde{\pi} = \arg \min_{\pi \in \mathbb{R}^{p-1}} \| \pi \|_1 \\
\text{s.t.} \| W^T (Z - W\pi) \|_\infty \leq \eta \rho_2 \sqrt{n} \| Z \|_2 \\
Z^T (Z - W\pi) \geq \rho_0 \rho_2 \| Z \|_2^2 / 2 \\
\rho_2 \in [\rho_0, 1],
\]

(2.15)
where $\rho_2 = \sigma_u / \sqrt{E(z_i)^2}$.

### 2.4 Test statistic

By plugging in the estimators $\tilde{\pi}$ and $\tilde{\theta}_*$, we construct the following test statistic

$$T_n = n^{-\frac{1}{2}} \hat{\sigma}_e^{-1} (Z - W \tilde{\pi})^T (V - W \tilde{\theta}_*),$$

(2.16)

where $\hat{\sigma} = \sqrt{\|V - W \tilde{\theta}_*\|_2 / \sqrt{n}}$. Obviously, under the null hypothesis and the sparsity assumption of $\pi$, we have

$$T_n = n^{-\frac{1}{2}} \hat{\sigma}_e^{-1} (Z - W \tilde{\pi})^T (V - W \tilde{\theta}_*) = \Delta \cdot \hat{\sigma}_e^{-1} + n^{-\frac{1}{2}} \sum_{i=1}^{n} u_i \hat{e}_i \hat{\sigma}_e^{-1},$$

where $\Delta = n^{-\frac{1}{2}} \sum_{i=1}^{n} (\pi - \tilde{\pi}) W^T \hat{e}_i$. Thus, $\Delta \cdot \hat{\sigma}_e^{-1}$ is $o_p(1)$. So the statistical properties of $T_n$ are determined by $n^{-\frac{1}{2}} \sum_{i=1}^{n} u_i \hat{e}_i \hat{\sigma}_e^{-1}$.

Under the null hypothesis, $U$ is uncorrelated with $(V, W)$, while $\tilde{\theta}_*$ is completely dependent on $(V, W)$, so $\hat{e}$ is also only related to $(V, W)$. Therefore, $U$ and $\hat{e}$ are independent. Because of this independence, we have

$$E[n^{-\frac{1}{2}} \sum_{i=1}^{n} u_i \hat{e}_i \hat{\sigma}_e^{-1} | \hat{e}_i] = \frac{1}{\|\hat{e}\|_2} \sum_{i=1}^{n} \hat{e}_i E(u_i) = 0,$$

$$Var[n^{-\frac{1}{2}} \sum_{i=1}^{n} u_i \hat{e}_i \hat{\sigma}_e^{-1} | \hat{e}_i] = n^{-1} \hat{\sigma}_e^{-2} \sum_{i=1}^{n} \hat{e}_i^2 Var(u_i) = E(u_i^2).$$

Therefore, according to the Gaussianity of $U$, the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^{n} u_i \hat{e}_i \hat{\sigma}_e^{-1}$ conditional on $\{\hat{e}_i\}, i = 1, 2, ..., n$ is $N(0, Q)$ and $Q = E(u_i^2)$.

That is,

$$n^{-\frac{1}{2}} \hat{\sigma}_e^{-1} \hat{e}^T U | \hat{e} \sim N(0, Q),$$

where $Q$ is unknown, which we want to replace with a natural estimator $\hat{Q} = \frac{1}{n} \sum_{i=1}^{n} \hat{u}_i^2$.

### 3 Theoretical results

#### 3.1 Size property

We now turn our attention to the property of the test, which is imposed under extremely weak conditions.

**Assumption 3.1** Consider the model (2.1). Suppose that the following hold:

(i) there exist constants $c, d \in (0, +\infty)$ such that the eigenvalues of covariance matrix $\Sigma$ lie in $(c, d)$;

(ii) $\pi$ is sparse, which means $s_\pi = o(\sqrt{n/\log^3 p})$, where $s_\pi = \|\pi\|_0$. 


3.2 Power property

Assumption 3.1 is reasonably weak. Assumption 3.1(i) is a common condition imposed in high-dimensional literature. Assumption 3.1(ii) imposes a sparsity condition on the regression coefficient vector \( \pi \), rather than on \( \beta \) or \( \Sigma \) of the model \( (2.1) \), which shows that the following conclusions are robust to dense models. Then we provide the following result for \( T_n \).

**Theorem 1** Let Assumption 3.1 be hold, when \( n, p \to \infty \) with \( \log p = o(\sqrt{n}) \), then under null hypothesis,

\[
P(|T_n| > \hat{Q}^{1/2} \Phi^{-1}(1 - \alpha/2)) \to \alpha, \forall \alpha \in (0, 1),
\]

(3.1)

where \( \Phi^{-1}(1 - \alpha/2) \) is the \( 1 - \alpha/2 \) quantile of standard normal distribution.

Theorem 1 shows that \( T_n \), under the null hypothesis, converges to \( N(0, \hat{Q}) \). Hence, a test with nominal size \( \alpha \in (0, 1) \) rejects null hypothesis if and only if \( |T_n| > \hat{Q}^{1/2} \Phi^{-1}(1 - \alpha/2) \). In particular, the test is robust to dense \( \theta^* \), in the sense that even under dense \( \theta^* \), our procedure does not generate false positive results. Instead of an inference on the basis of an estimator, it is a direct statistical conclusion on the basis of a null hypothesis. At the same time, we can construct confidence sets for \( \beta^* \) even when the nuisance parameter \( \theta^* \) is non-sparse.

**Corollary 1** Let Assumption 3.1 be hold and \( 1 - \alpha \) be the nominal coverage level. We define

\[
C_{1-\alpha/2} := \{ \beta : |T_n| \leq \hat{Q}^{1/2} \Phi^{-1}(1 - \alpha/2) \},
\]

(3.2)

which has the exact coverage asymptotically:

\[
\lim_{n, p \to \infty} P(\beta^* \in C_{1-\alpha}) = 1 - \alpha.
\]

(3.3)

### 3.2 Power property

To evaluate the power property of the test, we consider the following test problem:

\[
H_0 : \beta^* = \beta_0 \quad \text{versus} \quad H_1 : \beta^* = \beta_0 + h,
\]

(3.4)

where \( h \) is a given constant. It is clear that the difficulty in distinguishing \( H_0 \) from \( H_1 \) depends on \( h \).

**Assumption 3.2** Let Assumption 3.1 be hold. In addition, suppose

(i) \( \|\theta^*\|_0 = o(\sqrt{n}/\log p) \);

(ii) there exist constants \( \delta \) and \( \kappa_1 \in (0, +\infty) \) such that \( E|\varepsilon|^2 + \delta < \kappa_1 \).

Assumption 3.2 is relatively mild. The sparsity condition of \( \theta^* \) is used to guarantee the asymptotic power of high-dimensional tests in Assumption
3.2(i), which implies the sparsity of the model, and it is consistent with the traditional test (Cai et al, 2013; Van de Geer et al, 2014). Assumption 3.1(ii) is a regular moment condition. Then we provide the following result for $T_n$.

**Theorem 2** Let $H_1$ in (3.4) and Assumption 3.2 be hold. When $n, p \to \infty$, with $\log p = o(\sqrt{n})$, then there exist constants $K_1, K_2 > 0$ depending only on the constants in Assumption 3.2 such that, whenever

$$|\sigma_u^2(\beta_* - \beta_0)| \geq \sqrt{n^{-1}\log(p)}(K_1|\beta_* - \beta_0| + K_2),$$

where $\sigma_u^2 = E(u^2)$, the test is asymptotically powerful, i.e.

$$P(|T_n| > \hat{Q}^{1/2}\Phi^{-1}(1 - \alpha/2)) \to 1, \forall \alpha \in (0, 1).$$

Theorem 2 establishes the power property of the proposed test under the sparse model.

4 Numerical Examples

In this section, we evaluate the proposed method in the finite sample setting by observing its behavior in both simulated and real data.

4.1 Simulation Examples

We consider model (2.1). In all simulations, we set $n = 200$, $p = 500$ and the nominal size is $5\%$. The rejection probabilities are based on 100 repetitions. For application purposes, we recommend choosing the tuning parameters as $\eta = 0.5\sqrt{\log p / n}$ and $\rho_0 = 0.01$, which are commonly used options, and we will demonstrate in our simulations that it provides good results.

For the test problem (2.2), without loss of generality, we consider the test of the first component of the parameter, i.e.

$$H_0 : \beta_1 = \beta^0_1 \text{ versus } H_1 : \beta_1 = \beta^0_1 + h,$$

(4.1)

where $\beta^0_1$ is a given constant. We show the results for three different Gaussian designs as follows.

1. (Toeplitz) Here we consider the standard Toeplitz design where the rows of $X$ are drawn as an i.i.d random draws from a multivariate Gaussian distribution $N(0, \Sigma_X)$, with covariance matrix $(\Sigma_X)_{i,j} = 0.4|i-j|.$

2. (Noncorrelation) Here we consider uncorrelated design where the rows of $X$ are i.i.d draws from $N(0, \Sigma_X)$, where $(\Sigma_X)_{i,j}$ is 1 for $i = j$ and is 0 for $i \neq j$.

3. (Equal correlation) Here we consider a non-sparse design matrix with equal correlation among the features. Namely, the rows of $X$ are i.i.d draws from $N(0, \Sigma_X)$, where $(\Sigma_X)_{i,j}$ is 1 for $i = j$ and is 0.01 for $i \neq j$.

Let $s = \|\beta\|_0$ denotes model sparsity. To show the size property of our method for dense model, we vary $s$ from $s = 10$ to extremely large $s = p$. For
4.1 Simulation Examples

sparsity $s$, we set the model parameters as $\beta_j = \frac{3}{\sqrt{p}}$, $1 \leq j \leq s$ and $\beta_j = 0$, $j > s$.

We compare our method with the generalized low-dimensional projection (LDP) method for bias correction (Ma et al., 2021). The size results are collected in Table 1, where we can clearly see that the LDP method does not have the size property in the dense model, that is, the Type I error probabilities are much higher than the nominal level $\alpha$. This indicates that the LDP method fails to dense models. Conversely, when the sparsity of the model is equal to $s = p$, the Type I error probability of our method remains stable. That is true even if we change the correlation among the features.

| Method | Toeplitz | Noncorrelation | Equal correlation |
|--------|----------|----------------|-------------------|
| s=10   | 0.70     | 0.09           | 0.66              |
| s=20   | 0.69     | 0.02           | 0.69              |
| s=50   | 0.72     | 0.02           | 0.70              |
| s=100  | 0.81     | 0.03           | 0.72              |
| s=n    | 0.82     | 0.05           | 0.78              |
| s=p    | 0.90     | 0.04           | 0.89              |

For the first parameter component $\beta_1 = \frac{3}{\sqrt{p}}$, we construct its $1 - \alpha$ confidence intervals for different sparsity levels, and obtain the coverage probabilities (CP) based on 100 repetitions. According to Theorem 1, the asymptotic distribution of $T_n$ is $N(0, \hat{Q})$. Also by the analysis in Section 2.4, we have

$$T_n \rightarrow n^{-\frac{1}{2}} \bar{\sigma}_e^{-1} U^T \hat{e} \rightarrow N(0, \hat{Q}),$$

By inverting the solution $T_n \leq \hat{Q}^{\frac{1}{2}} \Phi^{-1}(1 - \alpha/2)$, the $1 - \alpha$ confidence interval of the parameter $\beta_1$ can be obtained as

$$[\beta_1^0 - \frac{n^{\frac{1}{2}} \hat{Q}^{\frac{1}{2}} \Phi^{-1}(1 - \alpha/2) \hat{e} + \hat{U}^T (W(\theta_* - \tilde{\theta}_*) + \varepsilon)}{\hat{U}^T Z},$$

$$\beta_1^0 + \frac{n^{\frac{1}{2}} \hat{Q}^{\frac{1}{2}} \Phi^{-1}(1 - \alpha/2) \hat{e} - \hat{U}^T (W(\theta_* - \tilde{\theta}_*) + \varepsilon)}{\hat{U}^T Z}].$$

The results for confidence intervals (CI), lengths and CP are collected in Table 2.

In addition, Theorem 2 gives the power property of the test under sparse models ($\|\theta_*\|_0 = o(\sqrt{n}/\log p)$). For simplicity, we observe the power property only for $s = 3$. The data is generated by the same model as in Table 1, except that the true value of $\beta_1 = \frac{3}{\sqrt{p}} + h$. The results are collected in Figure 1, which presents full power curves with various values of $h$. Therefore, the far left
presents Type I error \((h = 0)\) whereas other points on the curves correspond to Type II error \((h \neq 0)\). We clearly observe that our method outperforms LDP by providing firm Type I error and reaching full power quickly. Therefore, our proposed method provides a robust and more broadly applicable alternative to the existing inference process, achieving better error control.

4.2 Real Data

We illustrate our proposed method by analyzing ”Lee Silverman voice treatment” (LSVT) voice rehabilitation dataset (Athanasios et al., 2014). Vocal performance degradation is a common symptom for the vast majority of Parkinson’s disease (PD) subjects. The current study aims to investigate the
4.2 Real Data

Table 3 Significant variables selected by our method and the LDP method

| Dysphonia Measure | Number |
|-------------------|--------|
| Ours              | $x_3, x_{18}, x_{37}, x_{97}, x_{100}, x_{111}, x_{115}, x_{229}, x_{230}, x_{231}, x_{265}$ | 11 |
| LDP               | the above $+ x_4, x_6, x_7, x_8, x_9...$ | 109 |

Table 4 The predicted values of our method and the LDP method

| Measure | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    | 13    |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Original Value | 0     | 0     | 1     | 0     | 0     | 1     | 0     | 0     | 1     | 0     | 0     | 1     | 0     |
| Ours    | 0     | 0     | 0     | 0     | 1     | 1     | 0     | 1     | 0     | 0     | 0     | 0     | 0     |
| LDP     | 0     | 1     | 0     | 0     | 0     | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 0     |
| Measure | 14    | 15    | 16    | 17    | 18    | 19    | 20    | 21    | 22    | 23    | 24    | 25    | 26    |
| Original Value | 0     | 1     | 0     | 0     | 1     | 0     | 0     | 1     | 0     | 0     | 1     | 0     | 0     |
| Ours    | 0     | 1     | 0     | 1     | 1     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     |
| LDP     | 1     | 1     | 0     | 0     | 1     | 0     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |

potential of automatically assessing sustained vowel articulation as “acceptable” (a clinician would allow persisting in speech treatment) or “unacceptable” (a clinician would not allow persisting in speech treatment). We first standardized the data. The complete data includes 309 dysphonia measures, where each produces a single number per phonation, resulting in a design matrix of size $126 \times 309$. There are no missing entries in the design matrix. This is a high-dimensional logistic regression problem with $n = 126$ and $p = 309$. We try to determine ”which of the originally computed dysphonia measures matter in this problem.”

Results are reported in Table 3. Therein we report the significant variables identified using our approach and LDP that affect the assessments of speech experts, respectively. In addition to the above 11 dysphonia measures, the LDP method selects 98 measures as significant variables.

We divide the 126 samples into two parts, in which the first 100 samples are used as the training set and the last 26 samples are used as the testing set. The significant variables selected by the two methods are used to fit the logistic regression model on the training set. The logistic regression model obtained by our method is

$$
\hat{f}_{Ours} = \frac{e^{-1.8+52.8x_3+...+0.45x_{231}-25.73x_{265}}}{1 + e^{-1.8+52.8x_3+...+0.45x_{231}-25.73x_{265}}}.
$$

And LDP’s logistic regression model is

$$
\hat{f}_{LDP} = \frac{e^{-171.4-918.1x_3+...-14563x_{264}+30204x_{265}}}{1 + e^{-171.4-918.1x_3+...-14563x_{264}+30204x_{265}}}.
$$

The predicted values $y_i \mid X_i \sim Bernoulli(\hat{f})$, $i = 1, 2, ..., 26$, which are shown in Table 4.
According to the prediction results in Table 4, the prediction accuracies of our method and LDP method are 0.73 and 0.62, respectively. This shows that our method is more accurate. Such finding would indicate that this dataset likely does not follow a sparse model and that previous method was reporting false positives. In conclusion, our method identifies the 11 most representative significant variables, greatly simplifies the fitting model and presents more accurate results than existing methods. This finding provides a reference for improving the effectiveness of automatic rehabilitation speech assessment tools.

5 Conclusion

This paper considers the inference of single parameter in high-dimensional non-sparse logistic models. We first find the linearization of the regression model, and then construct the test statistics based on the moment method, which incorporates the null hypothesis. The proposed procedure is proved to have tight Type I error control even in the dense model. Our test also has desirable power property. Our test reaches full power quickly when the model is indeed sparse. It is worth mentioning that the method used in this paper can be extended to sub-Gaussian distribution design and other high dimensional generalized linear models. For these reasons, our method greatly complements existing literature.

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Supplementary information

Supplement of ”Single Parameter Inference of Non-sparse Logistic Regression Models”. The detailed proofs about the asymptotic distribution of test statistics are given. In addition, we also give detailed proofs of the power property of the test. Technical lemmas are also proved in the supplement.

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