HYPERBOLICITY OF BASES OF LOG CALABI-YAU FAMILIES

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Abstract. In this paper, we prove that the quasi-projective base of any maximally variational smooth family of Calabi-Yau klt pairs is both of log general type, and pseudo Kobayashi hyperbolic. Moreover, such a base is Brody hyperbolic if the family is effectively parametrized.

0. Introduction

The goal of this paper is to prove the hyperbolicity of bases of maximally variational smooth families of log Calabi-Yau pairs.

Theorem A. Let $f^o : (X^o, D^o) \to V$ be a smooth family of log pairs (cf. Definition 1.1.(iv)) over a quasi-projective manifold $V$. Assume that each fiber $(X_y, D_y)$ of $f^o$ is Kawamata log terminal (klt for short) and $K_{X_y} + D_y \equiv_Q 0$, and the family is of maximal variation, i.e. the logarithmic Kodaira-Spencer map $T_V \to R^1f^o_*(T_{X^o/V}(−\log D^o))$ is generically injective. Then

(i) $V$ is of log general type.
(ii) $V$ is pseudo Kobayashi hyperbolic, i.e. there exists a proper subvariety $Z \subsetneq V$ so that Kobayashi pseudo distance $d_V(p, q) > 0$ for any two distinct points $(p, q)$ not both contained in $Z$. In particular, any non-constant holomorphic map $\gamma : \mathbb{C} \to V$ has image $\gamma(\mathbb{C}) \subset Z$.

Theorem A.(i) is often referred to the Viehweg hyperbolicity in the literatures. Although Theorem A.(i) and Theorem A.(ii) are conjecturally to be equivalent by the tantalizing Lang’s conjecture [Lan91, Chapter VIII. Conjecture 1.4], we cannot conclude one from the other directly at the present time. Theorem A can be seen as some sort of Shafarevich hyperbolicity conjecture for families of log Calabi-Yau pairs. As first formulated by Viehweg and Kovács, Shafarevich’s conjecture for higher dimensional fibers and parametrizing spaces states that a family of canonically polarized manifolds of maximal variation has as its base a variety of log general type. Shafarevich hyperbolicity conjecture as well as its generalized formulations drew a lot of attention for a long time, and much progress has been achieved during the last two decades, cf. [Kov00, VZ01, VZ02, Kov02, VZ03, KK08a, KK08b, KK10, JK11, Sch12, Pat12, TY15, CP15a, CP15b, CP16, PS17, BPW17, Sch17, PTW18, Den18a, Den18b, TY18, Taj18, WW18], to quote only a few.

The proof of Theorem A is reduced to the construction of certain negatively twisted Higgs bundles (which we call Viehweg-Zuo Higgs bundle in Definition 2.1) over the base $V$ (see Theorem 2.2 below), following the general strategies in [VZ01, VZ02, VZ03, PS17, PTW18]. Indeed, once the Viehweg-Zuo (VZ for short) Higgs bundle is established, as is well-known to the experts, Theorem A.(i) follows from the celebrated theorem of Campana-Păun [CP15b, Theorem 4.1] on the vast generalization of generic semipositivity result of Miyaoka, and Theorem A.(ii) can be deduced from the author’s recent work [Den18a, Theorem 3.8] and
[Den18b, Theorem C] on the construction of \textit{generically non-degenerate} Finsler metrics over the base (up to a birational model) with the holomorphic sectional curvatures bounded above by a negative constant.

A complex manifold $V$ is said to be \textit{Kobayashi hyperbolic} if the Kobayashi pseudo distance $d_V$ is a metric, and in particular, $V$ is \textit{Brody hyperbolic}: there exists no non-constant holomorphic map $\mathbb{C} \to V$. The following result is a direct consequence of Theorem A.

\textbf{Theorem B.} For the log Calabi-Yau family $f^\circ : (X^\circ, D^\circ) \to V$ as in Theorem A, if $f^\circ$ is further assumed to be effectively parametrized, i.e. the logarithmic Kodaira-Spencer map

$$T_{V,y} \hookrightarrow H^1(X_y, T_{X_y}(- \log D_y))$$

is injective for any $y \in V$, then

(i) every irreducible subvariety of $V$ is of log general type.
(ii) $V$ is Brody hyperbolic.

According to another conjecture of Lang [Lan86, Conjecture 5.6], Theorem B.(i) and Theorem B.(ii) are also expected to be equivalent. One might expect that the base $V$ in Theorem B should be moreover \textit{Kobayashi hyperbolic}, which is indeed the case when fibers of the effectively parametrized family are

- compact Riemann surfaces of genus $g \geq 2$ [Ahl61, Roy75, Wol86];
- projective manifolds with ample canonical bundles [TY15];
- Calabi-Yau manifolds (orbifolds) [BPW17, Sch17, TY18];
- projective manifolds with big and nef canonical bundles [Den18a].

We conclude the introduction by briefly explaining the proof of Theorem 2.2. Recall that the starting point in the work [VZ02, VZ03, PS17, PTW18, Den18a, WW18] is the Viehweg’s $Q_{n,m}$ conjecture on the strong positivity of direct images of pluricanonical bundles, which is known to us when general fibers are of (log) general type or admit good minimal models, cf. [Vie83b, Kol87, Kaw85, KP17]. When the fibers are Calabi-Yau klt pairs, Viehweg’s $Q_{n,m}$ conjecture was only proved recently by Cao-Guenancia-P˘ aun (see Theorem 1.5 below) using very delicate analysis of singular Kähler-Einstein metrics. Based on this theorem of Cao-Guenancia-P˘ aun, we apply Abramovich’s $Q$-mild reduction for slc families as in [Den18a], to find a good compactification of $V$ so that after replacing the original family by Viehweg’s fiber product, one gains enough positivity for the relative dualizing sheaves, which enables us to perform the Viehweg’s cyclic cover technique. To construct the desired negatively twisted Higgs bundles, we mainly follow the general strategies by Viehweg-Zuo [VZ02, VZ03] to generalize their Hodge theoretical methods to the logarithmic setting. However, different Higgs bundles defined in the proof of Theorem 2.2 are related in a more direct manner inspired by the recent work of Popa-Schnell [PS17] on the alternative construction via \textit{tautological sections of cyclic coverings} (see also the more recent ones [Wei17, Taj18, WW18]). Our work is also influenced by the work [PTW18]. Indeed, since the divisor used for the cyclic cover might not be \textit{generically smooth over the base}, we have to perform some “a priori” birational modification of the base so that certain desingularization of the cyclic cover is smooth over an open set of the base whose complement is a simple normal crossing divisor, by applying an important technique in [PTW18, Proposition 4.4].

In a forthcoming paper we will study the hyperbolicity of bases of log canonical Calabi-Yau families using quite different approaches.

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1. \textbf{Positivity of direct image sheaves}

Let us begin this section with the following definitions.
Definition 1.1. (i) A quasi-projective irreducible variety $V$ is of log general type, if for any desingularization $\tilde{V} \to V$, and any smooth compactification $Y$ of $\tilde{V}$ with the boundary $D := Y \setminus \tilde{V}$ simple normal crossing, $K_Y + D$ is a big line bundle\(^1\).

(ii) Let $X$ be a quasi-projective variety, and let $D$ be an effective divisor on $X$. $(X, D)$ is called a smooth log pair if $X$ is smooth and $D_{\text{red}}$ is simple normal crossing.

(iii) $(X, D)$ is called a smooth log pair, if $(X, D)$ is a smooth log pair and all the coefficients of $D$ are in $(0, 1) \cap \mathbb{Q}$.

(iv) Let $\mathcal{X}$ and $V$ be quasi-projective manifolds, and let $D = \sum_{i=1}^\ell a_iD_i$ be a divisor on $\mathcal{X}$ with $\sum_{i=1}^\ell D_i$ simple normal crossing. A surjective projective morphism $(\mathcal{X}, D) \to V$ with connected fibers is called smooth family of log pairs if both $\mathcal{X}$ and every stratum $D_{a_1} \cap D_{a_2} \cap \ldots \cap D_{a_m}$ are smooth and dominant onto $V$. Such a $D$ is called relatively normal crossing over $V$.

(v) Let $(X, D)$ be a smooth log pair, let $f : (X, D) \to Y$ be a surjective projective morphism and let $V \subset Y$ be a Zariski open set of $Y$. Say $f$ is smooth over $V$ if the restriction $f : (X, D)|_{f^{-1}(V)} \to V$ is a smooth family of log pairs in the sense of (iv).

As in [VZ03, PTW18, Den18a, WW18], we have to find a good compactification of $V$ and replace the original family by its fiber product to construct a cyclic cover, along some divisor in the linear system of relative pluri-canonical bundles with a sufficiently negative twist. Since the direct images of pluri-canonical bundles are rank 1 coherent sheaves, the proof can be made much simpler than the general setting, which requires delicate analysis of weak positivity, log canonical threshold and vanishing theorems (or $L^2$-extension theorems).

Theorem 1.2. For the log Calabi-Yau family $f^* : (X^o, D^o) \to V$ as in Theorem A, there exists a projective compactification $Y \supset V$, a smooth log pair $(X, D)$ and a surjective morphism $f : X \to Y$ so that

(i) each irreducible component of $D$ is dominant onto $Y$, and $(X, D) \to Y$ is smooth over $V$;

(ii) $B := Y \setminus V$ is simple normal crossing, and $\Delta := f^*B$ is normal crossing.

(iii) for $m$ sufficiently large and divisible, there exist an ample line bundle $\mathcal{A}$ on $Y$ with $\mathcal{A} - B$ also ample, and an effective divisor $B \in |mK_{X/Y} + mD - mf^*\mathcal{A}|$.

To find the smooth projective compactification $Y$ of $V$ in Theorem 1.2, we need the log version of the Q-mild reduction by Abramovich [Den18a, Corollary A.2]. Note that in [VZ02, VZ03] the weakly semi-stable reduction by Abramovich-Karu [AK00] was applied to achieve the same goal. This was substituted in [Den18a] by Q-mild reduction, i.e. compactifying the family into Kollár family of slc singularities after taking a finite base change, so that we can control the birational modification of the base precisely. In a recent preprint [WW18], Wei-Wu applied moduli of stable log pairs by Kovács-Patakfalvi [KP17] to obtain the similar reduction (which is called stable reduction in [WW18, §4]) for smooth families of klt pairs of log general type.

Theorem 1.3 (Q-mild reduction, Abramovich). Let $f_0 : (S_0, D_0) \to T_0$ be a projective smooth family of log canonical pairs with $T_0$ quasi-projective manifold. For any given finite subset $Z \subset T_0$, there exist

(i) a compactification $T_0 \subset \mathcal{I}$ with $\mathcal{I}$ a regular projective scheme,

(ii) a simple normal crossings divisor $\Delta \subset \mathcal{I}$ containing $\mathcal{I} \setminus T_0$ and disjoint from $Z$,

(iii) a finite morphism $W \to \mathcal{I}$ unramified outside $\Delta$, and

(iv) An slc family $g : (S_W, D_W) \to W$ extending the given family $(S_0, D_0) \times _{T_0} W$. That is, each fiber of $g$ is an slc pair, $\omega_{S_W/W}^m(mD_W)$ is flat over $W$, and commutes with arbitrary base changes for all $m \in \mathbb{Z}_{>0}$ (Kollár condition).

\(^1\)When $V$ is projective, this definition is equivalent to say that the Grauert-Riemenschneider canonical sheaf of $V$ is big.
We will not recall the basic properties of slc families, and we refer the readers to [AT16, Pat16] for further details. Let us mention that although Theorem 1.3 is only stated for the compact setting in [Den18a, Corollary A.2], i.e. $D_0 = \emptyset$, its proof also holds for the log cases, since [Den18a, Theorem A.3] holds for moduli of Alexeev stable map of slc pairs by [DR18, Theorem 1.5].

We will need the following lemma [Bou04, Proposition 5.2], whose proof is a standard application of $\mathbb{L}^2$-methods (see e.g. [Dem12]).

**Lemma 1.4.** Let $X$ be a projective manifold equipped with two line bundle $G$ and $L$, and let $\{x_1, \ldots , x_\ell\} \subset X \setminus \mathcal{B}_+ (L)$ be any finite set. If $L$ is big, then for any $m \gg 0$, one has the following surjection

$$H^0 (X, mL + G) \twoheadrightarrow \bigoplus_{j=1}^\ell J_{x_j} (mL + G)$$

where $J_{x_j} (mL + G) := (mL + G) \otimes \mathcal{O}_X / m_{x_j}$.

Now we recall a difficult theorem by Cao-Guenancia-Păun [CGP17] on the proof of Viehweg’s $Q_{n,m}$ conjecture for families of Calabi-Yau klt pairs, which is the starting point for the proof of Theorem 1.2. Their proof requires quite delicate analysis on the variation of singular Kähler-Einstein metrics.

**Theorem 1.5 ([CGP17, Corollary D]).** For $f^o : (X^o, D^o) \to V$ as in Theorem A, let $f : (X, D) \to Y$ be any projective compactification, so that $(X, D)$ is a smooth klt pair, and $Y$ is smooth. Then $f_* (mK_{X/Y} + mD)'^*_{\bullet}$ is a big line bundle for $m$ sufficiently large and divisible.

Let us begin to prove Theorem 1.2. We mainly follow the proof in [VZ03, Den18a], and our proof is also influenced by [WW18, Proposition 4.8].

**Proof of Theorem 1.2.** By the $\mathbb{Q}$-mild reduction in Theorem 1.3, we can take a projective compactification $Y \supset V$ with $B := Y \setminus V$ simple normal crossing, and a finite morphism $\tau : W \to Y$ so that $(X^o, D^o) \times_Y W$ extends to an slc family $(Z, D_Z) \to W$. We denote by $Z^r := Z \times_w \cdots \times_w Z$ the $r$-folded fiber product of $Z \to W$, and set $\text{pr}^r_j : Z^r \to Z$ to be the projection to the $j$-th factor. Write $D^r_Z := \sum_{j=1}^r \text{pr}_j^* D_Z$. As is well-known, $g^r : (Z^r, D^r_Z) \to W$ is also an slc family.

**Claim 1.6.** For some sufficiently large and divisible $m$, $\omega_{Z/W}^r [mD_Z]$ is invertible, $g_* (\omega_{Z/W}^r [mD_Z])$ is a big line bundle, and for any $r \in \mathbb{Z}_{>0}$, one has

$$g^* (\omega_{Z/W}^r [mD_Z]) = g_* (g^* (\omega_{Z/W}^r [mD_Z]))^\otimes r.$$  

**Proof of Claim 1.6.** By [AT16, Proposition 4.4], $(Z, D_Z)$ is a log canonical pair, and in particular $K_Z + D_Z$ is $\mathbb{Q}$-Cartier. Moreover, it was proved in [WW18, Proposition 4.1] that $(Z, D_Z)$ is even klt. Take a strong log resolution $\mu : \tilde{Z} \to Z$ of $(Z, D_Z)$ with

$$\mu^* (K_Z + D_Z) = K_{\tilde{Z}} + \tilde{D}_Z - \tilde{E},$$

where $\tilde{D}_Z$ and $\tilde{E}$ are both effective $\mathbb{Q}$-divisors, such that $(\tilde{D}_Z + \tilde{E})_{\text{red}}$ is simple normal crossing, $\tilde{E}$ is exceptional and the coefficients of irreducible components of $\tilde{D}_Z$ are all in $(0, 1)$. Take $W_0 := \tau^{-1} (V)$, and $Z_0 := g^{-1} (W_0)$. Then $Z_0$ is smooth, and $\mu : \mu^{-1} (Z_0) \to Z_0$ is isomorphism. By the assumption, the logarithmic Kodaira-Spencer map of $\tilde{Z} \to W$ is also generically injective, and by Theorem 1.5 of Cao-Guenancia-Păun, for $m \gg 0$ large and divisible enough $g_* (mK_{\tilde{Z}/W} + m\tilde{D}_Z)'^*_{\bullet}$ is a big line bundle. Here we denote by $\tilde{g} := g \circ \mu$. On the other hand, by (1.2) one has

$$\tilde{g}_* (mK_{\tilde{Z}/W} + m\tilde{D}_Z) = g_* (mK_{\tilde{Z}/W} + mD_Z),$$
which is a reflexive sheaf of rank 1, and thus invertible. Therefore, \( g_*(mK_{Z/W} + mD_Z) \) is a big line bundle as well. (1.1) follows from the base change property of slc families. \( \square \)

Claim 1.6 allows us to replace the original family \((X^0, D^0) \to V\) by its fiber product to increase the positivity of the direct images.

Set \( B_1 \subset Y \) be the proper analytic subset of \( Y \) so that \( \tau: \tau^{-1}(Y \setminus B_1) \to Y \setminus B_1 \) is étale. Since \( B_2 := \tau(B_+(g_*(mK_{Z/W} + mD_Z))) \) is a proper subset of \( Y \), \( V_0 := V \setminus (B_1 \cup B_2) \) is a non-empty open set of \( V \). Then for any fixed \( y \in V_0 \), \( \tau \) is unramified at \( y \), and \( \tau^{-1}(y) \subset W \setminus B_+g_*(mK_{Z/W} + mD_Z) \). It follows from Lemma 1.4 that one can take \( \tau \gg 0 \) so that

\[
H^0 \left( W, rg_*(\omega_{Z/W}^m(mD_Z)) \right) - \tau^*(m\mathcal{A} + mB) \]

generates jet of order 1 at all points in the finite set \( \tau^{-1}(y) \). In other words, the locally free sheaf \( \tau_*(g_*(\omega_{Z/W}^m(mD_Z)^r)) \otimes \mathcal{A}^{-m} \otimes \mathcal{O}_Y(-mB) \) is generated by global sections at \( y \), and thus generically over \( Y \). Now we replace \((X^0, D^0) \to V\) by its \( r \)-fold fiber product, and \((Z, D_Z) \to W\) will also be replaced by the \( r \)-fold fiber product automatically. We keep the same notation for simplicity. By Claim 1.6, the locally free sheaf

\[
\tau_*g_*(\omega_{Z/W}^m(mD_Z)) \otimes \mathcal{A}^{-m} \otimes \mathcal{O}_Y(-mB)
\]

is generically generated by global sections.

Take a smooth projective compactification \( X \supset X^0 \), and define \( D \) to be the closure of \( D^0 \) in \( X \). After passing to a blow-up of \( X \) with the center in \( X \setminus X^0 \), we can assume that that \( f: (X, D) \to Y \) is a surjective morphism, and \( f^*B + D \) is normal crossing. In particular, \((X, D)\) is a smooth klt pair, which is smooth over \( V \). Set \( Z_1 := X \times_Y W \), and \( Z_2 \) denotes to be its normalization. Since \( Z_2 \) is birational to \( Z \), one can even assume that the log resolution \( \tilde{Z} \to Z \) in the proof of Claim 1.6 resolves the birational map \( Z \dashrightarrow Z_2 \), and \( \tilde{g}^*(W \setminus W_0) \) is normal crossing.

\[
\begin{aligned}
X & \xrightarrow{h} \quad Z_1 & \xrightarrow{\nu} \quad \tilde{Z} & \xrightarrow{\mu} \quad Z \\
\downarrow f & \quad \downarrow g_1 & \quad \downarrow \tilde{g} & \quad \downarrow g \\
Y & \xleftarrow{\tau} \quad W & \xleftarrow{\iota} \quad W & \xrightarrow{\iota} \quad W
\end{aligned}
\]

Since \( \tau : W \to Y \) is flat, by [Vie83a, Proof of Lemma 3.3] or [Mor87, (4.10)], \( Z_1 \) is irreducible Gorenstein, \( h^*\omega_{X/Y} = \omega_{Z_1/W} \) and \( \nu_\ast \omega_{Z/W} \subset \omega_{Z_1/W} \). By flat base change and the projection formula, one has

\[
\tilde{g}_*(mK_{\tilde{Z}/W} + m\phi^*(D + f^*B)) \to (g_1)_*(\omega_{Z_1/W}^m \otimes h^*\mathcal{O}_X(mD + mf^*B))
\]

which is an isomorphism over \( W_0 \). Since \( \mu : \tilde{g}^{-1}(W_0) \to Z_0 \) is an isomorphism, then \( \tilde{g}^{-1}(W_0) \simeq X_0 \times_Y W \), and thus \( \phi^*(D + f^*B) - \tilde{D}_Z \) is supported on \( \tilde{g}^*(W \setminus W_0) \). Since \( \tilde{D}_Z \) is klt, and \( \phi^*f^*B = \tilde{g}^*\tau^*B \geq \tilde{g}^*(W \setminus W_0) \), \( \phi^*(D + f^*B) - \tilde{D}_Z \) is thus effective, as also observed in [WW18, Proof of Proposition 4.8]. One thus has the following morphism

\[
g_*(mK_{Z/W} + mD_Z) = \tilde{g}_*(mK_{\tilde{Z}/W} + mD_Z) \to \tau_*g_*(mK_{X/Y} + mD + mf^*B)
\]

which is an isomorphic over \( W_0 \). Hence the morphism

\[
\tau_*g_*(mK_{Z/W} + mD_Z) \to \tau_*f_*(mK_{X/Y} + mD + mf^*B) = f_*(mK_{X/Y} + mD + mf^*B) \otimes \tau_*\mathcal{O}_W
\]

is isomorphic over \( V \). Thanks to the generically global generation of (1.3), \( f_*(mK_{X/Y} + mD) \otimes \mathcal{A}^{-m} \otimes \tau_*\mathcal{O}_W \) is also generically generated by global sections. Since the trace map
\( \tau_* \mathcal{O}_W \to \mathcal{O}_Y \) is a splitting surjective morphism, \( f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m} \) is an effective line bundle. Hence any non-zero section \( s \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}) \) will be sent to an effective divisor \( \Gamma_s \in [mK_{X/Y} + mD - mf^* \mathcal{O}] \) via the following natural morphism
\[
(1.5) \quad f^*(f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}) \to mK_{X/Y} + mD - mf^* \mathcal{O}.
\]

\[\square\]

**Remark 1.7.** (i) Since \( mK_{X/Y} + mD \) is relatively trivial over \( f^{-1}(V) \), the morphism (1.5) is thus an isomorphism over \( V \). In particular, the effective divisor \( \Gamma_s \in H^0(X, mK_{X/Y} + mD - mf^* \mathcal{O}) \) induced by \( s \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}) \) is supported on \( f^{-1}(\{s = 0\} \cup B) \).

(ii) Assume that \( \mathcal{O} = 2L \), where \( L \) is another very ample line bundle on \( Y \). Then for any non-zero \( s \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}) \), we can take a general smooth hypersurface \( H \in [mL] \) which does not contain any prime divisor of \( (s = 0) \), and \( H \cap V \neq \emptyset \). Define \( \sigma := s \cdot s_H \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes L^{-m}) \), where \( s_H \in H^0(Y, mL) \) is the canonical divisor defining \( H \). By Remark 1.7(i), for the effective divisor \( \Gamma_{\sigma} \in H^0(X, mK_{X/Y} + mD - mf^* L) \) induced by \( \sigma \), it has at least one irreducible component \( P \) with multiplicity one, so that
\[
P \cap f^{-1}(V) = f^*H_{f^{-1}(V)}.
\]

Since \( D \) and \( \Gamma_{\sigma} \) do not have common prime divisors, \( P \) is also an irreducible component of \( \Gamma_{\sigma} + D \) with multiplicity one.

(iii) By the proof of Theorem 1.2 one can show that we have a canonical morphism
\[
\Psi : \tau_* g_*(mK_{Z/W} + mD_Z) \otimes \mathcal{O}^{-m} \otimes \mathcal{O}_Y(-mB) \to f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}
\]

which does not depend on the choice of the intermediate birational model \( \tilde{Z} \) of \( Z \) in (1.4). Moreover, the linear map
\[
\Phi : H^0(Z, mK_{Z/W} + mD_Z - mg^* \tau^* \mathcal{O} - mg^* \tau^* B) \to H^0(X, mK_{X/Y} + mD - mf^* \mathcal{O})
\]

induced by \( \Psi \) is non-trivial.

As we mentioned in the end of § 0, since we require the discriminant locus of the the new family obtained by the desingularization of certain cyclic cover to be simple normal crossing, we need some sort of “base change property” of direct images. The following proposition follows the ideas in [PTW18, Proposition 4.4], and it can be seen as the log version of [Den18a, Theorem 1.23].

**Proposition 1.8.** For \((X, D) \to Y \) and \( V \subset Y \) as in Theorem 1.2, let \( V_0 \subset V \) be any non-empty Zariski open set. Assume that \( s \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes \mathcal{O}^{-m}) \) is a non-zero section induced by \( s = \Phi(\sigma) \) with \( \sigma \in H^0(Z, mK_{Z/W} + mD_Z - mg^* \tau^* \mathcal{O} - mg^* \tau^* B) \) and \( \Phi \) defined in Remark 1.7(iii). Then there exists a birational morphism \( \mu : \tilde{Y} \to Y \) from a projective manifold \( \tilde{Y} \) and a new birational model \( \tilde{f} : (\tilde{X}, \tilde{D}) \to \tilde{Y} \) where \((\tilde{X}, \tilde{D}) \) is a smooth klt pair satisfying
\[
(\tilde{X}, \tilde{D}) \xrightarrow{\phi} (X \times_Y \tilde{Y})^\sim \xrightarrow{(X, D)}
\]

\[
\tilde{Y} \xrightarrow{\mu} Y
\]

where \((X \times_Y \tilde{Y})^\sim \) is the main component of the normalization of \( X \times_Y \tilde{Y} \), \( \tilde{X} \to (X \times_Y \tilde{Y})^\sim \) is some desingularization, and
\[
(1) \ \tilde{V}_0 := \mu^{-1}(V_0) \xrightarrow{\tilde{f}} V_0 \text{ is an isomorphism.}
\]
(2) \( T_0 := \tilde{Y} \setminus \tilde{V}_0 \) and \( \tilde{B} := \mu^*(B)_{\text{red}} \) are both simple normal crossing.

(3) Set \( \tilde{X}_0 := (\mu \circ \tilde{f})^{-1}(V) \), and \( \tilde{D}_0 := D \cap \tilde{X}_0 \). Then \( (\tilde{X}_0, \tilde{D}_0) = (X^\circ, D^\circ) \times Y \tilde{Y} \). In particular, \( \phi : (\tilde{X}, \tilde{D})_{\tilde{f}^{-1}(V_0)} \to (X, D)_{f^{-1}(V_0)} \) is also an isomorphism.

(4) \( \tilde{f}^*(T_0) + \tilde{D} \) is normal crossing.

(5) Let \( V' \subset V \) be the big open set so that \( \mu : \mu^{-1}(V') \to V' \) is an isomorphism. Then \( V' \supset V_0 \), and there exists a section \( \tilde{s} \in H^0(\tilde{Y}, \tilde{f}_*(mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E}) \otimes (\mu^*\mathscr{A})^{-m}) \) which coincides with \( s \) when restricted to \( \tilde{V}' := \mu^{-1}(V') \). Here \( \tilde{E} \) is an effective \( \tilde{f} \)-exceptional divisor with \( \tilde{f}(E) \cap \tilde{V}' = \emptyset \).

Proof. Take a birational morphism \( \mu : \tilde{Y} \to Y \) so that (1) and (2) are satisfied. Let \( \tilde{X} \to (X \times_Y \tilde{Y}) \) be a blowing-up with center in \( \psi^{-1}(\tilde{B}) \), so that \( \phi^*D, \tilde{f}^*\tilde{B} \) and \( \phi^*D \) are all normal crossing. Set \( \tilde{D} \) to be the strict transform of \( D \) under \( \phi \). Hence \( (\tilde{X}, \tilde{D}) \) is also a smooth klt pair, which is smooth over \( \tilde{V} := \mu^{-1}(V) \). One can easily see that (3) is satisfied automatically, and \( \tilde{X}_0 \cap \tilde{f}^*T_0 \) is normal crossing. One can take further blow-up of \( \tilde{X} \) with centers in \( \tilde{f}^*\tilde{B} \) so that (4) is satisfied.

Take \( \tilde{W} \) to be a strong desingularization of \( W \times_Y \tilde{Y} \), where \( \tau : W \to Y \) is the flat finite morphism introduced in the proof of Theorem 1.2. \( \tau : \tilde{W} \to \tilde{Y} \) is a generically finite to one surjective morphism, which is finite over \( \tilde{V}' \). Hence we can leave out a codimension at least two closed subvariety on \( \tilde{Y} \setminus \tilde{V}' \) so that \( \tau \) is finite. Set \( (\tilde{Z}, \tilde{D}_{\tilde{Z}}) := (Z, D_Z) \times W \tilde{W} \tilde{W} \to \tilde{W}W \), which is also an slc family. By the base change property,

\[
\delta^*\sigma \in H^0(\tilde{Z}, mK_{\tilde{Z}/\tilde{W}} + m\tilde{D}_{\tilde{Z}} - m(\tau \circ \delta \circ \tilde{g})^*\mathscr{A} - m(\tau \circ \delta \circ \tilde{g}^*)B)
\]

where \( \delta : \tilde{W} \to W \). Since \( \mu^*B \supset \tilde{B} \), and \( \tau \circ \delta = \mu \circ \tilde{\tau} \), then \( \delta^*\sigma \) induces a section

\[
\tilde{s} \in H^0(\tilde{Z}, mK_{\tilde{Z}/\tilde{W}} + m\tilde{D}_{\tilde{Z}} - m\tilde{g}^*\tilde{\tau}^*(\mu^*\mathscr{A}) - m\tilde{g}^*\tilde{\tau}^*\tilde{B})
\]

which is isomorphic to itself when restricted to \( \tilde{\tau}^{-1}(\tilde{V}) \). By the functorial property stated in Remark 1.7.(iii), \( \tilde{\sigma} \) gives rise to a section

\[
\tilde{s} \in H^0(\tilde{Y}, \tilde{f}_*(mK_{\tilde{X}/\tilde{Y}} + m\tilde{D}) \otimes \mu^*\mathscr{A}^{-m}),
\]

which is isomorphic to \( s \) when restricted over \( \tilde{V}' \cong V' \). Note that \( \tilde{s} \) only defined over a big open set of \( \tilde{Y} \). It extends to a global section, and the \( \tilde{f} \)-exceptional divisor \( \tilde{E} \) might appear.

Remark 1.9. (i) When \( V_0 \subset Y \setminus (s = 0) \), for the non-zero section \( \tilde{s} \in H^0(\tilde{Y}, \tilde{f}_*(mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E}) \otimes \mu^*\mathscr{A}^{-m}) \) defined in (5) of Proposition 1.8, \( (\tilde{s} = 0) \cap \mu^{-1}(V_0) = \emptyset \). Let \( \tilde{\Gamma} \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - m(\mu \circ \tilde{f})^*\mathscr{A}| \) be zero divisor defined by \( \tilde{s} \). By Remark 1.7.(i), \( \tilde{\Gamma} \subset \tilde{f}^{-1}(T_0) \). By (4), \( \tilde{f}^*(T_0) + \tilde{D} \) is normal crossing, and thus \( \tilde{\Gamma} + \tilde{D} \) is also normal crossing.

(ii) By Remark 1.7.(ii), we can assume there exists an \( s \in H^0(Y, f_*(mK_{X/Y} + mD) \otimes \mathscr{A}^{-m}) \) so that its zero divisor \( \Gamma_1 \in |mK_{X/Y} + mD - mf^*\mathscr{A}| \) contains an irreducible component \( P \) with multiplicity one, and

\[
P \cap f^{-1}(V) = f^*H_{f^{-1}(V)}
\]

for some smooth hypersurface \( H \) on \( Y \) with \( H \cap V' \neq \emptyset \). Since \( V' \) in (5) of Proposition 1.8 is a big open set of \( V \), one thus has \( H \cap V' \neq \emptyset \). By (5) in Proposition 1.8, there exists \( \tilde{s} \in H^0(\tilde{Y}, \tilde{f}_*(mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E}) \otimes (\mu^*\mathscr{A})^{-m}) \) which coincides with \( s \) when restricted to \( \tilde{V}' := \mu^{-1}(V') \). Hence for the induced zero divisor \( \tilde{\Gamma}_1 \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - \tilde{f}^*\mu^*\mathscr{A}| \), it also contains at least one irreducible component \( \tilde{P} \) with multiplicity one, and \( \tilde{P} \cap \tilde{f}^{-1}(V') \neq \emptyset \). Such \( \tilde{P} \) is indeed the strict transform of \( P \) under the birational morphism \( \phi : \tilde{X} \to X \).
2. CONSTRUCTION OF THE VIEWHEG-ZUO HIGGS BUNDLE

This section is devoted to the construction of certain negatively curved Higgs bundles on the base. This type of Higgs bundles, first introduced by Viehweg-Zuo [VZ01, VZ02, VZ03] and later developed by Popa-Schnell [PS17], has proven to be a powerful tool in studying the hyperbolicity of moduli spaces. Let us give the definition in an abstract way, following [Den18a, Definition 3.1].

Definition 2.1 (Abstract Viehweg-Zuo Higgs bundles). Let $V$ be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of $V$ with the boundary $B := Y \setminus V$ simple normal crossing. A Viehweg-Zuo Higgs bundle over $(Y, B)$ is a logarithmic Higgs bundle $(\tilde{E}, \tilde{\theta})$ over $Y$ consisting of the following data:

(i) a divisor $T$ on $Y$ so that $B + T$ is simple normal crossing.
(ii) A big and nef line bundle $\mathcal{B}$ over $Y$ with $\mathcal{B}^{-1}(\mathcal{B}) \subset B \cup T$.
(iii) A logarithmic Higgs bundle $(\tilde{E}, \tilde{\theta}) := (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ induced by the Deligne extension of a polarized variation of Hodge structure defined over $Y \setminus (B \cup T)$ with eigenvalues of residues lying in $[0, 1) \cap \mathbb{Q}$.
(iv) A sub-Higgs sheaf $(\mathcal{F}, \eta) \subset (\tilde{E}, \tilde{\theta})$ satisfying

1. $(\tilde{E}, \tilde{\theta}) := (\mathcal{B}^{-1} \otimes \tilde{E}, 1 \otimes \tilde{\theta})$. In particular, $\tilde{\theta} : \tilde{E} \to \tilde{E} \otimes \Omega_Y(\log(B + T))$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
2. $(\mathcal{F}, \eta)$ has log poles only on the boundary $B$, i.e. $\eta : \mathcal{F} \to \mathcal{F} \otimes \Omega_Y(\log B)$.
3. Write $\tilde{\xi}_k := \mathcal{B}^{-1} \otimes \mathcal{E}^{n-k,k}$, and denote by $\mathcal{F}_k := \tilde{\xi}_k \cap \mathcal{F}$. Then the first stage $\mathcal{F}_0$ of $\mathcal{F}$ is an effective line bundle. In other words, there exists a non-trivial morphism $\mathcal{O}_X \to \mathcal{F}_0$.

Now we are able to state our main result in this section.

Theorem 2.2. For the $(X^0, D^0) \to V$ as in Theorem A, after replacing $V$ by a birational model, there exists a VZ Higgs bundle over some smooth projective compactification $Y$ of $V$.

Let us first show how the above theorem implies Theorem A.

Proof of Theorem A. Once the VZ Higgs bundle is constructed, the proof of Theorem A.(i) should be well-known to the experts, and we briefly recall the proof for completeness sake. The first step is the construction of Viehweg-Zuo (big) sheaf, which is due to Viehweg-Zuo in [VZ02]. Since $(\bigoplus_{q=0}^n \mathcal{F}_q, \bigoplus_{q=0}^n \eta_q)$ is a sub-Higgs sheaf of $(\tilde{E}, \tilde{\theta})$, as initialed in [VZ01], for any $q = 1, \ldots, n$, the morphism $\eta^q : \mathcal{F} \to \mathcal{F} \otimes \Omega_Y(q \log B)$ defined by iterating $\eta : \mathcal{F} \to \mathcal{F} \otimes \Omega_Y(\log B)$ $q$-times induces

\begin{equation}
\mathcal{F}_0 \to \mathcal{F}_q \otimes \mathcal{O}_Y(q \log B).
\end{equation}

(2.6) factors through $\mathcal{F}_q \otimes \mathcal{O}_Y(q \log B)$ by $\eta \wedge \eta = 0$. Recall that $\mathcal{F}_0$ is effective, one thus has a morphism

\begin{equation}
\mathcal{O}_Y \to \mathcal{F}_0 \to \mathcal{F}_q \otimes \mathcal{O}_Y(q \log B).
\end{equation}

(2.7) We denote by $N_q$ and $K_q$ the kernels of $\eta_q : \mathcal{F}_q \to \mathcal{F}_q+1 \otimes \mathcal{O}_Y(q \log B)$ and $\theta_{n-q,q} : E_{n-q,q} \to E_{n-q+1,q+1} \otimes \mathcal{O}_Y(q \log B + T)$ respectively, which are both torsion free sheaves. Then $N_q = (\mathcal{B}^{-1} \otimes K_q) \cap \mathcal{F}_q$. By the work of Zuo [Zuo00] (see also [PW16, Bru17, FF17, Bru18] for various generalizations) on the negativity of kernels of Kodaira-Spencer maps of Hodge bundles, $K_q^*$ is weakly positive in the sense of Viehweg\footnote{By [Bru17], one can even prove that the Hodge metric induces a semi-negatively curved singular hermitian metric for $K_q^*$ in the sense of Raufi [Rau15] (cf. also [PT18, HP16]).}, cf. [VZ02, Lemma 4.4.(v)]. Hence there exists a morphism

$$\mathcal{B} \otimes K_q^* \to N_q^*.$$
which is generically surjective. Let \( k \in \mathbb{Z}_{\geq 0} \) the minimal non-negative integer so that \( \eta^k \neq 0 \) and \( \eta^{k+1} = 0 \). As proved in [VZ02, Corollary 4.5], \( k \) must be positive. Indeed, if this is not the case, one has \( \mathcal{O}_Y \subset K_0 \otimes B^{-1} \), which is not possible. Hence there exists a non-trivial morphism
\[
\mathcal{O}_Y \to \mathcal{F}_0 \to N_k \otimes S^k \Omega_Y(\log B).
\]
In other words, there exists a non-trivial morphism
\[
(2.8) \quad B \otimes K^*_k \to N^*_k \to S^k \Omega_Y(\log B).
\]
Since \( B \) is big and nef, \( B \otimes K^*_k \) is big in the sense of Viehweg [VZ02, Definition 1.1.(c)], and thus for any ample line bundle \( \mathcal{A} \) there exists \( \alpha \gg 0 \) so that \( S^\alpha(B \otimes K^*_k) \otimes \mathcal{A}^{-1} \) is generated by global sections over a Zariski open set. By (2.8) there is a non-zero morphism
\[
\mathcal{A} \to S^\alpha \Omega_Y(\log B).
\]
Such \( \mathcal{A} \) is called the Viehweg-Zuo big sheaf in literatures.

Once the Viehweg-Zuo sheaf \( \mathcal{A} \) is constructed, it follows from [CP15b, Theorem 4.1] that \( K_Y + B \) is big. Theorem A.(i) is thus proved.

The proof of Theorem A.(ii) is exactly the same as [Den18b, Proof of Theorem B]. In [Den18a, §3] we establish an algorithm to construct a Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant via VZ Higgs bundles. By proving some sort of infinitesimal generic Torelli theorem (cf. [Den18b, Theorem C]) for VZ Higgs bundles, in [Den18b, Theorem B] we show that this Finsler metric is generically non-degenerate. The pseudo Kobayashi hyperbolicity of \( V \), which is indeed a bimeromorphic property, follows from the Ahlfors-Schwarz lemma (cf. [Dem97]) immediately. \( \square \)

**Remark 2.3.** Let us mention that although the Lang conjecture [Lan91, Chapter VIII, Conjecture 1.4] on the equivalence between pseudo Kobayashi hyperbolicity and being of log general type is quite open at the present time, we know that it holds for Hilbert modular varieties by [Rou16, CRT17] and subvarieties of Abelian varieties [Yam19].

We show how Theorem A implies Theorem B, whose proof is quite standard.

**Proof of Theorem B.** We first prove (i). Pick any positively dimensional irreducible subvariety \( Z \subset V \), and take a desingularization \( \tilde{Z} \to Z \). Define a base change \( f_\tilde{Z} : (X^0, D^0) \times_V \tilde{Z} \to \tilde{Z} \), which is also a smooth family of Calabi-Yau klt pairs. Since the logarithmic Kodaira-Spencer map is functorial under the base change, by the effective parametrization assumption, the logarithmic Kodaira-Spencer map of \( f_\tilde{Z} \) is generically injective. Hence \( \tilde{Z} \) is of log general type by Theorem A.(i), and (i) is proved.

We will prove (ii) by contradiction. Suppose that there exists a non-constant holomorphic map \( \gamma : \mathbb{C} \to V \). Then \( \tilde{Z} \to Z \), which is positively dimensional, smooth and irreducible. Then \( \gamma : \mathbb{C} \to Z \) lifts to a Zariski dense entire curve \( \tilde{\gamma} : \mathbb{C} \to \tilde{Z} \). By the above arguments, \( \tilde{Z} \) can be realized as the base of a maximally variational smooth family of Calabi-Yau klt pairs. This is a contradiction, since by Theorem A.(ii), \( \tilde{Z} \) cannot have any Zariski dense entire curve. \( \square \)

Now let us construct the VZ Higgs bundles on some birational modification of the base \( V \) in Theorem A.

**Proof of Theorem 2.2.** For the effective divisor \( \Gamma \in |mK_{X/Y} + mD - mf^*\mathcal{A}| \) defined in Theorem 1.2.(iii), set \( H := \Gamma + m[D] - mD \in |mK_{X/Y} + m[D] - mf^*\mathcal{A}| \). By Remark 1.7.(ii), we can assume that \( H \) contains an irreducible component \( P \) with multiplicity one. Let \( Z_{\text{cyc}} \) be the cyclic cover of \( X \) obtained by taking the \( m \)-th roots along \( H := \Gamma + m[D] - mD \in |mK_{X/Y} + m[D] - mf^*\mathcal{A}| \), and let \( Z_{\text{nor}} \) be the normalization of \( Z_{\text{cyc}} \), which is irreducible by [EV92, Lemma 3.15.(a)]. Take a desingularization \( Z \to Z_{\text{nor}} \), so that the inverse image of \( \psi^*H \) is normal crossing, where \( \psi : Z \to X \) is the composition map. Then \( Z \) is connected.
By Remark 1.7(i) there exists a Zariski open set $V_1 \subset V$ so that $\Gamma \cap f^{-1}(V_1) = \emptyset$. Since $[D]$ is relatively normal crossing over $V$, then $H$ is relatively normal crossing over $V_1$. Take a smaller Zariski open set $V_0 \subset V_1$ so that the composition morphism $g : Z \to Y$ is smooth over $V_0$.

We apply Proposition 1.8 to find the new birational model $\tilde{f} : (\tilde{X}, \tilde{D}) \to \tilde{Y}$ with respect to $V_0 \subset V$ and a section $\tilde{s} \in H^0(\tilde{Y}, \tilde{f}_*(mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E}) \otimes (\mu^*\mathcal{A})^{-m})$ satisfying all the properties in Proposition 1.8. For zero divisor $\tilde{\Gamma}_s \in |mK_{\tilde{X}/\tilde{Y}} + m\tilde{D} + m\tilde{E} - (\mu \circ \tilde{f})^*\mathcal{A}|$ defined by $\tilde{s}$, set $\tilde{H} := \tilde{\Gamma}_s + m[\tilde{D}] - m\tilde{D} \in |mK_{\tilde{X}/\tilde{Y}} + m[\tilde{D}] + m\tilde{E} - m(\mu \circ \tilde{f})^*\mathcal{A}|$. By Remark 1.9, $\tilde{H}$ is normal crossing and contains a prime divisor $\tilde{V}$ obtained by taking the $m$-th roots along $\tilde{H}$, and let $\tilde{Z}_{\text{nor}}$ be the normalization of $\tilde{Z}_{\text{cyc}}$, which is also irreducible. Then $\tilde{Z}_{\text{nor}} \to \tilde{Y}$ and $Z_{\text{nor}} \to Y$ is isomorphic when restricted over $\mu^{-1}(V_0) \simeq V_0$. Hence there exists a birational map $Z \dasharrow \tilde{Z}_{\text{nor}}$, which is regular over $g^{-1}(V_0)$. Take a strong resolution of the indeterminacy $\phi : \tilde{Z} \to \tilde{Z}_{\text{nor}}$ of the above birational map

$$
\begin{array}{c}
\tilde{Z} \\
\downarrow \phi \\
\tilde{Z}_{\text{nor}} \\
\downarrow \tilde{g} \\
\tilde{Y} \\
\mu \\
\downarrow \\
Y
\end{array}
$$

so that $\phi : \phi^{-1}(g^{-1}(V_0)) \to g^{-1}(V_0)$ is an isomorphism. By the Zariski’s Main theorem, $\tilde{g} : \tilde{Z} \to \tilde{Y}$ and $g : Z \to Y$ is isomorphic when restricted to $\mu^{-1}(V_0) \simeq V_0$. In particular, $\tilde{g} : \tilde{Z} \to \tilde{Y}$ is smooth over $\mu^{-1}(V_0)$, whose complement is a simple normal crossing divisor in $\tilde{Y}$.

To lighten the notation, we replace $(X, D) \to Y$ by the birational model $(\tilde{X}, \tilde{D}) \to \tilde{Y}$ and keep the same notation. In summary, it follows from the properties in Proposition 1.8 that we have the following data:

(a) a surjective morphism $(X, D) \to Y$ which is smooth over $V$, where $(X, D)$ is a smooth klt pair.

(b) $D + f^*B$ is normal crossing, where $B := Y \setminus V$.

(c) a normal crossing divisor $H := \Gamma + m[D] - m\tilde{D} \in |mK_{X/Y} + m[D] - mf^*\mathcal{A} + mE|$ which is relatively normal crossing over a Zariski open set $V_0 \subset V$. Here $\mathcal{A}$ and $\mathcal{B} := \mathcal{A} - B$ are both big and nef so that $(\mathcal{B}_+ + \mathcal{A}_+) \cap V_0 = \emptyset$; $E$ is an $f$-exceptional divisor with $f(E) \cap V_0 = \emptyset$.

(d) there exists a simple normal crossing divisor $T$ so that $V_0 = Y \setminus (B \cup T)$ and $B + T$ is normal crossing. Moreover, $f^*(B + T) + D$ is normal crossing.

(e) For some desingularization $\tilde{Z}$ of the normalization of the cyclic cover of $X$ by taking the $m$-th roots along $H$, $g : Z \to Y$ is smooth over $V_0$, where $g$ is the composition map. Moreover, $Z$ is connected.

(f) Denote by $\Pi := g^*(B + T)$ and $H' := \psi^*H$, where $\psi : Z \to X$ is the composition map. Then $\Pi + H'$ is normal crossing.

Write $D_{\text{red}} := \sum_{i=1}^{r} D_i$, $\Delta := f^*B$ and $\Sigma := f^*T$. Leaving out a codimension at least two subscheme, one can assume that

(1) both $T$ and $B + T$ are smooth;

(2) both $f : X \to Y$ and $g : Z \to Y$ are flat; in particular, the $f$-exceptional divisor $E$ disappears;

(3) $\Delta$ (resp. $\Pi$) is relatively normal crossing over $B$ (resp. $B + T$);
(4) for any $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, r\}$, the surjective morphism $D_I \to Y$ is also flat, and $D_I \cap \Delta$ is relatively normal crossing over $B$. Here we denote $D_I := D_{i_1} \cap D_{i_2} \cap \ldots \cap D_{i_t}$.

**Claim 2.4** (good partial compactification). Write $\Omega_{X/Y}(\log(\Delta + [D])) := \Omega_X(\log(\Delta + [D]))/f^*\Omega_Y(\log B)$, $\Omega_{Z/Y}(\log \Pi) := \Omega_Z(\log \Pi)/g^*\Omega_Y(\log (B + T))$. Then they are all locally free. In other words, one has the following short exact sequences of locally free sheaves

\[
\begin{align*}
0 & \to f^*\Omega_Y(\log B) \to \Omega_X(\log(\Delta + [D])) \to \Omega_{X/Y}(\log(\Delta + [D])) \to 0 \\
0 & \to g^*\Omega_Y(\log(B + T)) \to \Omega_Z(\log \Pi) \to \Omega_{Z/Y}(\log(\Pi)) \to 0.
\end{align*}
\]

**Proof of Claim 2.4.** As is well-known, the morphism $h : (Z, \Pi) \to (Y, B + T)$ is a “good partial compactification” of the smooth morphism $Z \setminus \Pi \to Y \setminus (B \cup T)$ in the sense of [VZ02, Definition 2.1.(c)], and one can easily show that $\Omega_{Z/Y}(\log \Pi)$ is locally free.

To prove the local freeness of $\Omega_{X/Y}(\log(\Delta + [D]))$, it suffices to show that for any $y_0 \in Y$ and $\sigma \in \Omega_Y(\log B)(U)$, where $U \ni y_0$ is some open set with $\sigma(y_0) \neq 0$, $f^*\sigma(x_0) \neq 0$ for any $x_0 \in f^{-1}(y_0)$.

Note that there is a unique $I \subset \{1, \ldots, r\}$ so that $x_0 \in D_I$ and $x_0 \notin D_J$ for any other $J \supset I$. After reordering, we can assume that $I = \{\ell + 1, \ldots, p\}$. Take local coordinates $(x_1, \ldots, x_m)$ around $x$ so that and $\Delta_{\text{red}} = (x_1 \cdots x_\ell = 0)$ and $D_i = (x_i = 0)$ for $i \in I$. Since $\Delta = f^*(B)$, the morphism $f^*\Omega_Y(\log B) \to \Omega_X(\log(\Delta + [D]))$ factors through $\Omega_X(\log \Delta)$. Note that $d \log x_1, \ldots, d \log x_\ell, dx_{\ell+1}, \ldots, dx_m$ and $d \log x_1, \ldots, d \log x_\ell, dx_{\ell+1}, \ldots, dx_m$ form the local basis for $\Omega_X(\log(\Delta + [D]))$ and $\Omega_X(\log \Delta)$. Hence $f^*\sigma = \sum_{j=1}^\ell a_j(x)d\log x_j + \sum_{i=\ell+1}^m a_i(x)dx_i$, where $a_i(x)$ are local holomorphic functions. When we write $f^*\sigma$ in terms of the basis of $\Omega_X(\log(\Delta + [D]))$, one has

\[
f^*\sigma = \sum_{j=1}^\ell a_j(x)d\log x_j + \sum_{i=\ell+1}^p a_i(x)dx_i + \sum_{k=p+1}^m a_k(x)dx_k.
\]

Since $x_0 \in D_I = (x_{\ell+1} = x_{\ell+2} = \cdots = x_p = 0)$, one has

\[
f^*\sigma(x_0) = \sum_{j=1}^\ell a_j(x_0)d\log x_j + \sum_{k=p+1}^m a_k(x_0)dx_k.
\]

When $f^*\sigma$ is seen as the local section in $\Omega_X(\log(\Delta + [D]))$, $f^*\sigma(x_0) = 0$ if and only if $a_1(x_0) = \cdots = a_\ell(x_0) = a_{\ell+1}(x_0) = \cdots = a_m(x_0) = 0$. On the other hand, since the projective morphism $f_1 : (D_I, D_I \cap \Delta) \to (Y, B)$ satisfies that $f_1$ is flat, $D_I \cap \Delta \to B$ is relatively normal crossing, we thus conclude that $f_1^*\sigma(x_0) \neq 0$. Since $d \log x_1, \ldots, d \log x_\ell, dx_{\ell+1}, \ldots, dx_m$ form the local basis for $\Omega_{D_I}(d \log D_I \cap \Delta)$, by (2.11), $f^*\sigma(x_0) = f_1^*\sigma(x_0) \neq 0$, and we conclude that $\Omega_{X/Y}(\log(\Delta + [D]))$ is also locally free. \hfill $\square$

By the above claim, such a partial compactification of the smooth family of log pairs $f : (X^p, D^p) \to V$ can thus be seen as the “good partial compactification” in the log setting.

For any $p \in \mathbb{Z}_{\geq 0}$, let

\[
\Omega^p_X(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^p \supset \mathcal{F}^{p+1} = 0
\]

be the Koszul filtration associated to (2.9) twisted with $\mathcal{L}^{-1} := -K_{X/Y} - [D] + f^*\mathcal{A}$, defined by

\[
\mathcal{F}^i := \text{Im} \left( f^*\Omega_Y(\log B) \otimes \Omega^p_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \to \Omega^p_X(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \right)
\]

so that the associated graded objects are given by

\[
\text{gr}^i \mathcal{F}^\bullet := \mathcal{F}^i / \mathcal{F}^{i+1} = f^*\Omega_Y(\log B) \otimes \Omega^p_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}
\]
The tautological exact sequence associated to (2.12) is defined by

\[
\begin{align*}
\ell^*\Omega_Y(\log B) \otimes \Omega_{X/Y}^{p-1}(\log(\Delta + [D])) & \otimes \mathcal{L}^{-1} \quad \Omega_{X/Y}^{p}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \\
0 & \rightarrow \text{gr}^1\mathcal{F}^\bullet \rightarrow \mathcal{F}^0/\mathcal{F}^2 \rightarrow \text{gr}^0\mathcal{F}^\bullet \rightarrow 0
\end{align*}
\]

By taking higher direct images \( Rf_* \) of (2.14), the connecting morphisms of the associated long exact sequences induce \( F^{p,q} \rightarrow F^{p-1,q+1} \otimes \Omega_Y(\log B) \), where we denote

\[
F^{p,q} := Rf_*\left(\Omega_{X/Y}^{p}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right)/\text{torsion}.
\]

Recall that \( \psi : Z \rightarrow X \) denotes to be the composition map, and \( H' := \psi^*H \) is normal crossing. As introduced by Popa-Schnell [PS17] and developed in [WW18], the tautological section of \( \mathcal{L} \) induces a morphism \( \psi^*\mathcal{L}^{-1} \rightarrow \mathcal{O}_Z(-H'_{\text{red}}) \). Since \( \psi^*(\Delta + [D])_{\text{red}} \subset (\Pi + H'_{\text{red}}, \) for every \( i \in \mathbb{Z}_{>0} \) one has a morphism

\[
\psi^*\Omega_X^i(\log(\Delta + [D])) \rightarrow \Omega_Z^i(\log(\Pi + H')).
\]

Hence there exists a natural morphism

\[
(2.15) 
\Xi : \psi^*\left(\Omega_X^i(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right) \rightarrow \Omega_Z^i(\log(\Pi + H')) \otimes \mathcal{O}_Z(-H'_{\text{red}}) \subset \Omega_Z^i(\log\Pi),
\]

as similarly shown in [PS17, Wei17, Taj18, WW18].

Pulling back (2.9) by \( \psi^* \), we have a short exact sequence of locally free sheaves

\[
(2.16) 
0 \rightarrow g^*\Omega_Y(\log B) \rightarrow \psi^*\Omega_X(\log(\Delta + [D])) \rightarrow \psi^*\Omega_{X/Y}(\log(\Delta + [D])) \rightarrow 0
\]

In a similar way as (2.12), we associate (2.10) and (2.16) with two filtrations

\[
(2.17) 
\Omega_Z^p(\log\Pi) = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^p \supset \mathcal{G}^{p+1} = 0
\]

\[
(2.18) 
\psi^*\left(\Omega_X^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right) = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^p \supset \mathcal{F}^{p+1} = 0
\]

defined by

\[
\begin{align*}
\mathcal{G}^i & := \text{Im}\left(g^*\Omega_Y^i(\log(B + T)) \otimes \Omega_Z^{p-i}(\log\Pi) \rightarrow \Omega_Z^p(\log\Pi)\right), \\
\mathcal{F}^i & := \text{Im}\left(g^*\Omega_Y^i(\log B) \otimes \psi^*\left(\Omega_X^{p-i}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right) \rightarrow \psi^*\left(\Omega_X^p(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right)\right).
\end{align*}
\]

Their associated graded objects are thus given by

\[
(2.19) 
\text{gr}^i\mathcal{G}^\bullet := \mathcal{G}^i/\mathcal{G}^{i+1} = g^*\Omega_Y^i(\log(B + T)) \otimes \Omega_Z^{p-i}(\log\Pi)
\]

\[
\text{gr}^i\mathcal{F}^\bullet := \mathcal{F}^i/\mathcal{F}^{i+1} = g^*\Omega_Y^i(\log B) \otimes \psi^*\left(\Omega_X^{p-i}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right) = \psi^*\text{gr}^i\mathcal{F}^\bullet.
\]

One can easily show that \( \Xi \) defined in (2.15) is compatible with the filtration structures \( \Xi : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \) in (2.17) and (2.18). It thus induces a morphism between their graded terms...
and in particular, a morphism between the following short exact sequences (2.20)
\[
g^{*}\Omega_{Y}(\log B) \otimes \psi^{*}\left(\Omega^{p-1}_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right) \rightarrow \psi^{*}\left(\Omega^{p}_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right)
\]

\[
0 \rightarrow \text{gr}^{i}\tilde{\mathcal{F}}^{*} \rightarrow \tilde{\mathcal{F}}^{0}/\tilde{\mathcal{F}}^{2} \rightarrow \text{gr}^{i}\tilde{\mathcal{F}}^{*} \rightarrow 0
\]

Pushing forward (2.20) by $Rg_{*}$, the edge morphisms induce
\[
\tilde{F}_{p,q}^{0} \xrightarrow{\varphi_{p,q}} \tilde{F}_{p-1,q+1}^{0} \otimes \Omega_{Y}(\log B)
\]
(2.21)
\[
E_{p,q}^{0} \xrightarrow{\varphi_{p,q}} E_{p-1,q+1}^{0} \otimes \Omega_{Y}(\log(B + T))
\]
where we denote by $E_{p,q}^{0} := R^{q}g_{*}((\Omega^{p}_{Z/Y}(\log \Pi))$, which is locally free by a theorem of Steenbrink [Ste77] (see also [Zuc84, Kol86, Kaw02, KMN02] for various generalizations), and
\[
\tilde{F}_{p,q}^{0} := R^{q}g_{*}\left(\psi^{*}\left(\Omega^{p}_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1}\right)\right)/\text{torsion}
\]
\[\iota : \Omega_{Y}(\log B) \rightarrow \Omega_{Y}(\log(B + T))\]
denotes the natural inclusion. Let us mention that the similar construction as \((\bigoplus_{p+q=\ell}F_{p,q}, \bigoplus_{p+q=\ell}\varphi_{p,q})\) is made by Taji in his work [Taj18] on a conjecture of Kebekus-Kovács.

Recall that $Z$ is a desingularization of the normalization $Z_{\text{nor}}$ of the cyclic cover of $X$ by taking the $m$-th roots along the normal crossing divisor $H$. Write $\psi : Z \xrightarrow{\delta} Z_{\text{nor}} \xrightarrow{\phi} X$. As is well-known, $Z_{\text{nor}}$ has rational singularities (see e.g. [EV92, §3]), and one thus has $R^{q}\delta_{*}\mathcal{O}_{Z} = 0$ for any $q > 0$. By the projection formula and the degeneration of relative Leray spectral sequences, for any locally free sheaf $\mathcal{E}$ on $X$, one has
\[
R^{q}\psi_{*}(\mathcal{E} \otimes \mathcal{O}_{Z}) = \mathcal{E} \otimes R^{q}\phi_{*}(\mathcal{O}_{Z}) = 0, \quad \forall q > 0
\]
(2.22)

thanks to the finiteness of $\phi$. Applying (2.22) to (2.19), for any $q > 0$, we have $R^{q}\psi_{*}(\text{gr}^{i}\tilde{\mathcal{F}}^{*}) = 0$, and therefore, the exactness of the tautological short exact sequence of $\tilde{\mathcal{F}}^{*}$ is preserved under the direct images $\psi_{*}$ as follows:
\[
0 \rightarrow \psi_{*}(\text{gr}^{i}\tilde{\mathcal{F}}^{*}) \rightarrow \psi_{*}(\tilde{\mathcal{F}}^{0}/\tilde{\mathcal{F}}^{2}) \rightarrow \psi_{*}(\text{gr}^{i}\tilde{\mathcal{F}}^{*}) \rightarrow 0
\]
(2.23)
\[
0 \rightarrow \text{gr}^{i}\tilde{\mathcal{F}}^{*} \otimes \psi_{*}\mathcal{O}_{Z} \rightarrow \tilde{\mathcal{F}}^{0}/\tilde{\mathcal{F}}^{2} \otimes \psi_{*}\mathcal{O}_{Z} \rightarrow \text{gr}^{i}\tilde{\mathcal{F}}^{*} \otimes \psi_{*}\mathcal{O}_{Z} \rightarrow 0
\]

By the collapse of relative Leray spectral sequences, one has
\[
R^{q}g_{*}(\text{gr}^{i}\tilde{\mathcal{F}}^{*}) = R^{q}g_{*}(\psi^{*}\text{gr}^{i}\tilde{\mathcal{F}}^{*}) = R^{q}f_{*}(\psi_{*}(\psi^{*}\text{gr}^{i}\tilde{\mathcal{F}}^{*})) = R^{q}f_{*}(\text{gr}^{i}\tilde{\mathcal{F}}^{*} \otimes \psi_{*}\mathcal{O}_{Z})
\]
\[= \Omega_{Y}(\log B) \otimes R^{q}f_{*}\left(\Omega^{p-1}_{X/Y}(\log(\Delta + [D])) \otimes \mathcal{L}^{-1} \otimes \psi_{*}\mathcal{O}_{Z}\right).
\]

Therefore, \((\bigoplus_{p+q=\ell}F_{p,q}, \bigoplus_{p+q=\ell}\varphi_{p,q})\) can also be defined alternatively by pushing forward (2.23) via $Rf_{*}$, with $\varphi_{p,q}$ the edge morphisms.
By [EV92, Corollary 3.11], the cyclic group \( G := \mathbb{Z}/m\mathbb{Z} \) acts on \( \psi_* \mathcal{O}_Z \), and one has the decomposition

\[
\psi_* \mathcal{O}_Z = \mathcal{O}_X \oplus \bigoplus_{i=1}^{m-1} (\mathcal{L}^{(i)})^{-1}, \quad \text{where} \quad \mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_X(-\lfloor \frac{iH}{m} \rfloor).
\]

In particular, the \( G \)-invariant part \( (\psi_* \mathcal{O}_Z)^G = \mathcal{O}_X \). Hence one can easily show that the cyclic group \( G \) acts on (2.23), whose \( G \)-invariant part is

\[
0 \longrightarrow \psi_* (\text{gr}^1 \mathcal{F}^\bullet)^G \longrightarrow \psi_* (\mathcal{F}^0/\mathcal{F}^2)^G \longrightarrow \psi_* (\text{gr}^0 \mathcal{F}^\bullet)^G \longrightarrow 0
\]

Therefore, \( (\bigoplus_{p+q=\ell} F_{p,q}, \bigoplus_{p+q=\ell} \tau_{p,q}) \) is a direct factor of \( (\bigoplus_{p+q=\ell} \tilde{F}_{p,q}, \bigoplus_{p+q=\ell} \varphi_{p,q}) \). Combining (2.21), we have

\[
\begin{align*}
E^{p,q}_{0} & \xrightarrow{\theta^{p,q}_{0}} E^{p-1,q+1}_{0} \otimes \Omega_Y(\log(B + T)) \\
\xi^{p,q}_{0} & \xrightarrow{\ni^{p-1,q+1} \otimes \ell} \\
\tilde{F}^{p,q} & \xrightarrow{\varphi^{p,q}} \tilde{F}^{p-1,q+1} \otimes \Omega_Y(\log B) \\
\tilde{F}^{p,q} & \xrightarrow{\tau^{p,q}} \tilde{F}^{p-1,q+1} \otimes \Omega_Y(\log B) \\
\end{align*}
\]

(2.24)

Set \( n \) to be the relative dimension of \( X \to Y \). Note that

\[
F^{n,0} = f_*(K_{X/Y} - \Delta + \Delta_{\text{red}} + \lfloor D \rfloor + \mathcal{L}^{-1}) = f_*(\Delta_{\text{red}} + f^*(\mathcal{A} - B)) \supset \mathcal{B},
\]

where \( \mathcal{B} := \mathcal{A} - B \) is a big and nef line bundle. Define \( F^{n-q,q}_0 := \mathcal{B}^{-1} \otimes F^{n-q,q} \), and

\[
\tau''_{n-q,q} : \mathcal{B}^{-1} \otimes F^{n-q,q} \xrightarrow{1 \otimes \tau_{n-q,q}} \mathcal{B}^{-1} \otimes F^{n-q-1,q+1} \otimes \Omega_Y(\log B)
\]

By (2.24), one has the following diagram:

\[
\begin{align*}
\mathcal{B}^{-1} \otimes E^{n-q,q}_0 & \xrightarrow{1 \otimes \theta'_{n-q,q}} \mathcal{B}^{-1} \otimes E^{n-q-1,q+1}_0 \otimes \Omega_Y(\log(B + T)) \\
\rho'_{n-q,q} & \xrightarrow{\iota : \Omega_Y(\log B) \to \Omega_Y(\log(B + T))} \iota \otimes \Omega_Y(\log(B + T)) \\
F^{n-q,q}_0 & \xrightarrow{\tau'_{n-q,q}} F^{n-q-1,q+1}_0 \otimes \Omega_Y(\log B)
\end{align*}
\]

where \( \iota : \Omega_Y(\log B) \to \Omega_Y(\log(B + T)) \) is the natural inclusion, and \( F^{n,0}_0 \) is an effective line bundle. Note that all the objects in (2.26) are only defined over a big open set \( Y' \) of \( Y \).

Write \( Z_0 := g^{-1}(V_0) \), which is smooth over \( V_0 \). The local system \( R^ng_* \mathcal{C}|_{Z_0} \) extends to a locally free sheaf \( \mathcal{V} \) on \( Y \) (here \( Y \) is projective rather than the big open set!) equipped with the logarithmic connection

\[
\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_Y(\log(B + T)),
\]

whose eigenvalues of the residues lie in \([0, 1) \cap \mathbb{Q}\) (the so-called lower canonical extension in [Kol86]). By [Sch73, CKS86, Kol86], the Hodge filtration of \( R^ng_* \mathcal{C}|_{Z_0} \) extends to a filtration \( \mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n \) of subbundles so that their graded sheaves \( E^{n-q,q} := \mathcal{F}^{n-q} / \mathcal{F}^{n-q+1} \) are also locally free, and there exists

\[
\theta_{n-q,q} : E^{n-q,q} \to E^{n-q-1,q+1} \otimes \Omega_Y(\log(B + T)).
\]

As mentioned above, \( E^{n-q,q}_0 \) is locally free by Steenbrink’s theorem. By a theorem of Katz [Kat71], we know that \( (\bigoplus_{q=0} E^{n-q,q}_0, \bigoplus_{q=0} \theta'_{n-q,q}) = (\bigoplus_{q=0} E^{n-q,q}_0, \bigoplus_{q=0} \theta_{n-q,q}) \mid_{Y'} \), hence it
can be extended to the whole projective manifold $Y$ defined by $(\bigoplus_{q=0}^{n} F_{q}^{-n-q}, \bigoplus_{q=0}^{n} \theta_{n-q,q})$. For every $q = 0, \ldots, n$, we replace $F_{0}^{-n-q}$ by its reflexive hull and thus the morphisms $\tau_{n-q,q}, \rho_{n-q,q}$ and the diagram (2.26) extends to the whole $Y$.

To finish the construction, we have to introduce the sub-Higgs sheaf of $(\bigoplus_{q=0}^{n} \mathcal{B}^{-1} \otimes E^{-n-q}, \bigoplus_{q=0}^{n} \mathcal{B}^{-1} \otimes \theta_{n-q,q})$ in Definition 2.1. For each $q = 0, \ldots, n$, we define a coherent torsion-free sheaf $\mathcal{F}_{q} := \rho_{n-q,q}(F_{0}^{-n-q}) \subset \mathcal{B}^{-1} \otimes E^{-n-q}$. By (2.26), one has

$$1 \otimes \theta_{n-q,q} : \mathcal{F}_{q} \to \mathcal{F}_{q+1} \otimes \Omega Y(\log B),$$

and let us denote $\eta_{q}$ by the restriction of $1 \otimes \theta_{n-q,q}$ to $\mathcal{F}_{q}$. Then $(\bigoplus_{q=0}^{n} \mathcal{F}_{q}, \bigoplus_{q=0}^{n} \eta_{q})$ is a sub-Higgs sheaf of $(\bigoplus_{q=0}^{n} \mathcal{B}^{-1} \otimes E^{-n-q}, \bigoplus_{q=0}^{n} 1 \otimes \theta_{n-q,q})$. By (2.25), there exists a morphism $\mathcal{G}_{Y} \to \mathcal{F}_{0}$ which is an isomorphism over $V_{0}$. The VZ Higgs bundle is therefore constructed.

Remark 2.5. As mentioned above, the morphism $\Xi$ defined in (2.15) was first introduced by Popa-Schnell [PS17], and was later generalized to the log setting in [Wei17, WW18]. This morphism inspires us to construct an intermediate Higgs bundle $(\bigoplus_{q=0}^{n} \mathcal{F}_{q}, \bigoplus_{q=0}^{n} \tau_{n-q,q})$, which relates $(\bigoplus_{q=0}^{n} F_{q}^{-n-q}, \bigoplus_{q=0}^{n} \tau_{n-q,q})$ with $(\bigoplus_{q=0}^{n} F_{q}^{-n-q}, \bigoplus_{q=0}^{n} \eta_{n-q,q})$ in a more direct manner. In the above proof, we do not require the divisor $H$ for cyclic cover to be generically smooth over the base$^{3}$, which is more flexible than the original construction in [VZ02, VZ03].

One can also see that we reduce the construction of VZ Higgs bundles over a log pair $(Y, B)$ to the existence of the data $(a)$–$(f)$.

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$^{3}$As pointed out by Zuo, when the base is a curve, it has already been studied in [VZ06, §3].
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