Bayesian Estimation of the Degrees of Freedom Parameter of the Student-t Distribution—A Beneficial Re-parameterization

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Abstract

In this paper, conditional data augmentation (DA) is investigated for the degrees of freedom parameter $\nu$ of a Student-t distribution. Based on a restricted version of the expected augmented Fisher information, it is conjectured that the ancillarity DA is progressively more efficient for MCMC estimation than the sufficiency DA as $\nu$ increases; with the break even point lying at as low as $\nu \approx 4$. The claim is examined further and generalized through a large simulation study and an application to U.S. macroeconomic time series. Finally, the ancillarity-sufficiency interweaving strategy is empirically shown to combine the benefits of both DAs. The proposed algorithm may set a new standard for estimating $\nu$ as part of any model.

1 Introduction

The Student-t distribution is a standard item in the statistician’s toolbox. Utilized as the regression error distribution, it robustifies parameter estimation against extreme observations in the data set (Gelman et al. 2013). Furthermore, the properties of Student-t are classical building blocks of hierarchical models for financial (Bollerslev 1987; Harvey, Ruiz, and Shephard 1994) and polychotomous data (Albert and Chib 1993), among others.

Maximum likelihood (ML) estimation of the degrees of freedom ($\nu$) parameter can be cumbersome due to the complexity of the likelihood function; Lange, Little, and Taylor (1989) give an overview and develop a quasi-Newton, an expectation-maximization (EM), and a score function-based approach to ML. Liu (1997) reviews extensions to the EM algorithm for the multivariate Student-t distribution. In the non-asymptotic branch of frequentist literature, Fraser (1976) discusses structural inference for the parameters, and Sutradhar and Ali (1986) and Singh (1988) provide method of moments estimators for $\nu$.

Bayesian treatments of robust regression date back to de Finetti (1961), who explores scale-mixtures of normal distributions as a tool for handling outliers and presents the Cauchy distribution as an example, which corresponds to $\nu = 1$. Still with a fixed $\nu$, Zellner (1976) discusses posterior inference for the parameters of a multivariate linear Student-t regression using conjugate prior distributions. Ramsay and Novick (1980) and West (1984) further extend de Finetti’s framework and provide more examples including the Student-t distribution. Chib, Osiewalski, and Steel (1991) demonstrate conditions, when this type of robust regression can not work due to a non-informative likelihood for $\nu$.

In terms of Bayesian inference for $\nu$ using Markov chain Monte Carlo methods (MCMC; Tierney 1994), the Gibbs sampler (Geman and Geman 1984) by Geweke (1993) has been established as the standard method and it comprises the main focus in this paper. The Gibbs sampler relies on the scale-mixture representation of the Student-t distribution to draw a posterior sample using a model with data augmentation (DA; van Dyk and Meng 2001). There exist many options for DA; however, to our knowledge, only the one outlined by Geweke (1993), termed sufficient augmentation (SA), has been in use for Student-t.

Multiple examples demonstrate that SA may not work well for sampling $\nu$; see, for instance, Geweke (1992, efficiency values in Appendix C), or Nakajima and Omori (2009), who
blame the hierarchical structure and the lack of marginalization for the low sampling efficiency. However, inspired by the general framework of Papaspiliopoulos, Roberts, and Sköld (2007), an alternative way to higher efficiency is found. They demonstrate through multiple example models that the transformation of SA to the ancillary augmentation (AA) often boosts the efficiency already quite drastically; unfortunately, whether AA or SA is superior depends on the data set. Papaspiliopoulos and Roberts (2008) propose as the way out of this data-dependence a hybrid Gibbs-sampler, which randomly picks AA or SA in every pass; they conclude that a combination of sufficiency and ancillarity may be desirable in any setting. Finally, Yu and Meng (2011) introduce the ancillarity-sufficiency interweaving strategy (ASIS), which exploits the complementary roles of AA and SA in any hierarchical model and integrates them into one generally effective recipe. The applicability and utility of ASIS is documented in the literature (see, e.g., Kastner and Frühwirth-Schnatter 2014; Bitto and Frühwirth-Schnatter 2019; Monterrubio-Gómez et al. 2020).

The contribution of this paper is two-fold: the ancillary parameterization for the Student-$t$ distribution is introduced, and it is shown through simulations that the AA sampler explores the posterior space more efficiently than the usual SA sampler for a large set of data generating processes; furthermore, an implementation of ASIS is proposed that can be applied as a plug-and-play substitute in any MCMC procedure for models that include the Student-$t$ distribution with unknown degrees of freedom.

The remaining part of the paper is structured as follows: Section 2 introduces the model and the notation. Section 3 provides details on measurement of efficiency, the Gibbs sampling algorithm, and the corresponding conditional distributions. Section 4 showcases the efficiency of the introduced method on a large grid of simulated data sets, and Section 5 is an application inspired by Geweke (1994). Section 6 concludes.

## 2 Simple Student-$t$-Model

The simplest setup with Student-$t$ errors is investigated:

$$y \sim t_\nu(0, 1),$$ (1)

where $t_\nu(\mu, \sigma^2)$ denotes the Student-$t$ distribution with $\nu$ degrees of freedom, location $\mu$, and variance $\sigma^2$; the kernel of its density function is $(1 + x^2/\nu)^{-(\nu+1)/2}$. Taking this simple model enables a pure analysis of sampling $\nu$, decoupled from other effects present in larger models; furthermore, Equation (1) is an ingredient of all models employing the Student-$t$ distribution. At the same time, the simplicity of Equation (1) is not restrictive either due to the modular nature of Gibbs-sampling: in case an econometric model is extended by Student-$t$ errors, its corresponding MCMC procedure shall merely include the additional steps for drawing $\nu$. This modularity is demonstrated in Section 5 through a more complex model.

As Zellner (1976) points out, the density function of the Student-$t$ distribution can be formulated as a scale mixture of normal distributions, which inspires the introduction of missing data $\tau$ and the data augmented representation

$$y \sim N(0, \tau),$$

$$\tau^{-1} \sim G(\nu/2, \nu/2),$$ (2)

where $N(\mu, \sigma^2)$ denotes the univariate normal distribution with mean $\mu$ and variance $\sigma^2$, and $G(\alpha, \beta)$ is the gamma distribution with shape $\alpha$ and rate $\beta$. Papaspiliopoulos, Roberts, and Sköld (2007) call Equation (2) the centered parameterization, while, following Yu and Meng (2011), we prefer the term sufficient parameterization. This terminology derives from the fact that $\tau$ is a sufficient statistic for $\nu$.

Based on Equation (2), it is possible to derive other data augmentation schemes, where the auxiliary variable and $\nu$ are a priori independent; these are called ancillary parameterization. As Papaspiliopoulos, Roberts, and Sköld (2007) point out, one such option is to replace $\tau$ by $u = F(\tau; \nu)$, where $F(x; \nu)$ is the cumulative distribution function (CDF) for the prior distribution of $\tau$. Note that $u$ follows a standard uniform distribution a priori; therefore, the
ancillary data augmentation for Equation (1) is

\[
\begin{align*}
    y & \sim N(0, F^{-1}(u; \nu)), \\
    u & \sim U(0,1),
\end{align*}
\]

where \(F^{-1}(u; \nu)\) is the inverse CDF for the prior distribution of \(\tau\).

As demonstrated by Yu and Meng (2011), for improved sampling efficiency, it is beneficial to consider both a sufficient and an ancillary data augmentation scheme in the design of an MCMC sampling algorithm. The added value in using both stems in the differences between the joint distributions \(p(y, \nu, \tau)\) and \(p(y, \nu, u)\); two slices for each distribution are depicted in Figures 1 and 2, respectively. The figures show contour plots of unnormalized\(^1\) posterior densities \(p(\nu, \tau \mid y = y_0)\) and \(p(\nu, u \mid y = y_0)\) for small \((y_0 = 0)\) and large \((y_0 = 4)\) observation error. The top panel of Figure 1 depicts rotating shapes, which signals that the dependence between \(\nu\) and \(\tau\) varies less in space than the dependence between \(\nu\) and \(u\). In conclusion, SA seems beneficial for fat tailed observations; on the other hand, AA may excel for lighter tailed observations.

The Bayesian description of the model is completed by the prior distribution \(p(\nu)\). Following Geweke (1993), an exponential distribution is assumed;

\[
\nu \sim \mathcal{E}(\lambda),
\]

where \(\lambda\) denotes the rate parameter. This choice is motivated by the availability of a suitable sampling procedure for SA using this prior (Geweke 1992). It is to be noted, that the adaptation of the proposed methods to alternative priors, such as the uniform distribution over an interval \([a, b]\) (Chib, Nardari, and Shephard 2002) or a gamma distribution (Nakajima and Omori 2012), is uncomplicated. This is discussed in Section 3.2.

3 Bayesian Estimation

Two Gibbs-samplers are implemented that correspond to AA and SA, respectively; both of them consist of sequentially drawing random values from two conditional posterior distributions. Above all, primary interest lies in how quickly the two resulting Markov chains mix, given different sets of observations; i.e., how close the posterior sample for \(\nu\) is to a set of independent draws. In particular, fast mixing is a desirable requirement for any MCMC procedure. van Dyk and Meng (2001) discuss several measures for mixing and propose to minimize the expected augmented Fisher information matrix

\[
I_*(y, \nu) = E \left[ \frac{\partial^2 \log p(\nu \mid y, \star)}{\partial \nu \cdot \partial \nu} \right| y, \nu]
\]

for choosing between DAs, where \(\star \equiv \tau\) for SA and \(\star \equiv u\) for AA. Although a simple derivation gives \(I_*(y, \nu) = n(\psi(\nu/2) - 1/\nu)/2\), where \(n\) denotes the number of observations and \(\psi(z) = d \log(\Gamma(z))/dz\) is the digamma function, we have been unsuccessful at computing \(I_*\) analytically. Fortunately, Monte Carlo (MC) estimation is possible. Figure 3 exemplifies the difference between the estimated \(I_u\) and \(I_*\) for data sets of size 1. The break even point (BEP) between AA and SA is at \(\nu \approx 4\) almost independently from \(y\); that is where AA overtakes SA at this benchmark as \(\nu\) increases.\(^2\)

\(^1\) There is no closed form for these distributions, and normalization is done over the shown window, which contains most of the probability mass.

\(^2\) For the evaluation of \(I_u(y_0, \nu_0)\), a MC sample \(\tilde{u}_{MC} = (u^{(1)}, \ldots, u^{(L)})\) of length \(L = 10000\) was simulated from \(p(u \mid y = y_0, \nu = \nu_0)\) using the algorithm in Section 3.2; next, \(\partial^2 \log p(\nu \mid y = y_0, u = u^{(i)})/\partial \nu \cdot \partial \nu\) is
Figure 1: Contour plot of the bivariate density functions $p(\nu, \tau \mid y = 0)$ (top) and $p(\nu, u \mid y = 0)$ (bottom), where $\nu \in [1, 30]$, $\tau \in (0, 20]$, and $u \in (0, 1]$.

The comparison of $I_\tau$ and $I_u$ through MC quickly becomes impractical for larger data sets. Therefore, an extensive computer simulation is conducted and empirical measurement of mixing is computed. Ideally, the latter is based on exact sampling from the four conditional posterior distributions.

### 3.1 Conditional Posterior Distributions

Given data points $\vec{y} = (y_1, \ldots, y_n)^\top$, the specification for $\vec{\tau} = (\tau_1, \ldots, \tau_n)^\top$ is conditionally conjugate (Geweke 1993), and, therefore, sampling from $p(\vec{\tau} \mid \vec{y}, \nu)$ is straightforward. Next, evaluated at points $i = 1, \ldots, L$; finally, the average of the values is taken to be the estimator for $I_u(y_0, \nu_0)$. The numerical second derivative of $\log p(\nu \mid y, u)$ is approximated using Richardson’s extrapolation with 6 steps, which is implemented in the hessian function from the software package numDeriv (Gilbert and Varadhan 2019).
Figure 2: Contour plot of the bivariate density functions \( p(\nu, \tau \mid y = 4) \) (top) and \( p(\nu, u \mid y = 4) \) (bottom), where \( \nu \in [1, 30] \), \( \tau \in (0, 4] \), and \( u \in (0, 1) \).

as Papaspiliopoulos, Roberts, and Sköld (2007) mention, sampling from \( p(\mathbf{\nu} \mid \mathbf{y}, \tau) \) can be reduced to sampling \( \tau \).

The rejection sampling (RS) method by Geweke (1993) is employed for \( p(\nu \mid \mathbf{y}, \mathbf{\tau}) \): denote by \( \xi^* \) the unique solution to

\[
(\log(\xi/2) + 1 - \Psi(\xi/2))n/2 + \xi - 1 - \eta = 0,
\]

where \( \eta = 0.5 \sum_{i=1}^{n} (\log(\tau_i) + 1/\tau_i) - \lambda \) is a sufficient statistic. By finding \( \xi^* \), the normalizing constant in \( p(\nu \mid \mathbf{y}, \mathbf{\tau}) \) is estimated indirectly, which is necessary for RS. Then, a proposal \( \nu_p \) is drawn from an exponential distribution with rate \( 1/\xi^* \), and it is retained with probability

\[
\exp(\eta \xi^* - 1) \left( \frac{(\xi^*/2)^{\xi^*/2}}{\Gamma(\xi^*/2)} \right)^{-n} \exp(\nu_p(1/\xi^* - \eta)) \left( \frac{(\nu_p/2)^{\nu_p/2}}{\Gamma(\nu_p/2)} \right)^{n}.
\]
Figure 3: Contour plot of the estimated difference $I_u(y, \nu) - I_\tau(y, \nu)$ in the region $(y, \nu)^T \in [0, 10] \times [0.1, 100]$. The depicted function is symmetric about $y = 0$. Note that the horizontal axis is on a logarithmic scale.

Last, we are unaware of exact sampling procedures for $p(\nu \mid \vec{y}, \vec{u})$. A modern and easy-to-implement substitute is to employ adaptive Metropolis (AM; Haario, Saksman, and Tamminen 2001) within Gibbs, which can be used to sample from any unnormalized density that can be evaluated. However, contrary to RS, AM produces an auto-correlated Markov chain, and therefore it is inferior to RS in our setup. As a simple improvement, independent sampling is approximated by repeating AM in one step of the Gibbs-sampler. Our implementation of AM proposes $\log(\nu_p)$ for the logarithm of $\nu$ through a Gaussian random walk with variance $\sigma^2$. Next, $\nu_p$ is accepted or rejected according to the Metropolis ratio $\alpha$. The proposal distribution is tuned to achieve the famous $\alpha = 0.44$ (Gelman, Roberts, and Gilks 1994; Gelman, Gilks, and Roberts 1997; Roberts and Rosenthal 2001) by increasing or decreasing $\sigma$ after batches of several hundred draws; this is similar to the examples by Roberts and Rosenthal (2009).

### 3.2 MCMC Algorithm

Following Geweke (1993), Equation (2) naturally translates into the following MCMC scheme, henceforth termed algorithm SA, which simulates from the posterior distribution $p(\nu, \vec{\tau} \mid \vec{y})$:

1. Initialize $\nu$;
2. Draw $\tau_i^{-1} \sim \mathcal{G} \left( \frac{x_i+1}{2}, \frac{\nu_i+\vec{y}_i^2}{2} \right)$ independently for each $i = 1, \ldots, n$;
3. Draw $\nu \sim p(\nu \mid \vec{y}, \vec{\tau})$, where $\vec{\tau} = (\tau_1, \ldots, \tau_n)^T$ using RS;
4. Repeat Steps 2 and 3.

For alternative prior distributions, Step 3 can be replaced by AM as long as it is possible to
evaluate the prior density function $p(\nu)$. This change may affect the mixing speed for two reasons: autocorrelation is introduced into the Markov chain, and $I_r$ might be altered.

Next, the two novel algorithms are introduced. Corresponding to Equation (3), algorithm AA samples from the posterior distribution $p(\nu, u | \bar{y})$:

1. Initialize $\nu$;
2. Draw $u_i \sim p(u_i | y_i, \nu)$ independently for each $i = 1, \ldots, n$ by first drawing $\tau_i^{-1} \sim G \left( \frac{\nu+1}{2}, \frac{\nu y_i^2}{2} \right)$ and then computing $u_i = F(\tau_i; \nu)$ (inspired by Papaspiliopoulos, Roberts, and Sköld 2007);
3. Draw $\nu \sim p(\nu | \bar{y}, \bar{u})$, where $\bar{u} = (u_1, \ldots, u_n)^\top$ using AM; this step includes the evaluation of $F^{-1}(u_i; \nu)$; repeat this step $k_{AA}$ times;
4. Repeat Steps 2 and 3.

This is algorithm is trivially modified for other prior distributions: the Metropolis ratio $\alpha$ needs to be adapted to the new prior density.

Finally, algorithm ASIS combines SA and AA:

1. Initialize $\nu$;
2. Do Steps 2 and 3 of SA;
3. Compute $u_i = F(\tau_i; \nu)$ for $i = 1, \ldots, n$;
4. Do Step 3 of AA;
5. Repeat Steps 2 through 4.

Notice that ASIS is not a sequence of SA and AA; $\bar{u}$ is not sampled separately. Rather, focus lies on the differences between $p(\nu | \bar{y}, \bar{f})$ and $p(\nu | \bar{y}, \bar{u})$, which are exploited in a minimal setup.

Correctness of the implementations for AA, SA, and ASIS is confirmed using a variant on Geweke’s test (Geweke 2004). Specifically, if a new vector of observations is simulated at the end of every round of the particular algorithm, then the sampling distribution of $\nu$ is equivalent to its prior; and this is verified visually with the help of quantile-quantile-plots.

## 4 Simulation Study

In order to make a general comparison between the mixing of AA, SA, and ASIS, an extensive simulation study is conducted. To this end, the algorithms in Section 3.2 are implemented in the R language (R Core Team 2021, version 4.1.0). Then, multiple independent posterior Markov chains are generated for data simulated from Equation (1); for AA, $k_{AA} = 20$ is set. Next, convergence of the Markov chains is automatically validated by ensuring that their $\hat{R}$ value (Vehtari et al. 2021) is smaller than the recommended 1.1; this is done using the rhat function of the R package posterior (Bürkner et al. 2021). Finally, relative numerical efficiencies (RNE; Geweke 1989) of the chains are reported. RNE is defined for an identically distributed but serially correlated Monte Carlo sample \{${\theta}^{(j)}$\} and a function of interest $g$ as $\text{vár}(g)/S(0)$, where $\text{vár}(g)$ is a consistent estimator for the variance of the distribution $p(\theta)$ and $\hat{S}(\omega)$ is a consistent estimator for the spectral density $S(\omega)$ of \{${g}(\theta^{(j)})$\}, $j = 1, \ldots, M$.

This computation is implemented in function `effectiveSize` from the coda package (Plummer et al. 2006).

### 4.1 Setup of Data Generating Process

A grid of true values $\nu_{true} = 1, 1.5, 2, 2.5, 3, 4, 5, 10, 20, 50, 100$ and data set lengths $n = 1, 3, 10, 30, 100, 300, 1000, 3000, 10000$ is considered for the comparison. This covers “realistic” values ($\nu_{true} > 2$; Gelman et al. 2013) and both problematic regions of low and high degrees of freedom. Moreover, dependence on the size of the data set can be evaluated. Then, for each of the 99 setups, five data sets are simulated from Equation (1). Furthermore, to examine prior sensitivity, values $\lambda = 0.05, 0.1, 0.2, 0.5, 1$ are considered, which correspond

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3. RNE is the reciprocal of the inefficiency factor reported in other works (Nakajima and Omori 2009).
to prior expectations of 1 through 20 for $\nu$. In addition, following the recommendations for the computation of $\hat{R}$, four independent Markov chains are simulated with initial values $\nu = 0.5, 2, 10, 100$ (these receive the same $\hat{R}$ during post-processing), for each of the three algorithms and each setup. Overall, 29700 samples of length $M = 10000$ are drawn after a burn-in phase of length 1000, and samples with $\hat{R} > 1.1$ are removed.

### 4.2 Posterior Estimates

A sample of interval estimators for $\nu$ is presented in Table 1 for $n = 1000$ and $\lambda = 0.2$. With the exception of two cells, the estimates by AA, SA, and ASIS are similar for all values of $\nu_{\text{true}}$. The twin of these exceptions is an example of a general caveat about AA: for a long vector of heavy-tailed observations, the numerical evaluation of $F^{-1}$ may fail within AA; especially for values $\nu$ at the outskirts of the posterior distribution. The issue materializes as a constant estimator (100, 100) in Table 1. Typically, the Markov chain is constant in these settings because the MH step always rejects; in all other settings, AA is reliable. Consequently, by choosing (our implementation of) AA, one assumes a complex restriction on the prior distribution $p(\nu, \tilde{\nu})$. Moreover, the three rows for $\nu_{\text{true}} = 100$ provide two further insights. First, obviously, even at sample size $n = 1000$, most of the posterior mass is closer to the prior expectation of $1/\lambda = 5$ than to 100; indeed, the prior choice is quite influential compared to the flat likelihood for light-tailed data. Second, the variation in the estimates from SA is the largest in this row both in relative and in absolute terms; the 90% quantile ranges between 37.9 and 40.7. This sheds light on the outstandingly deteriorating effectiveness of SA as $\nu_{\text{true}}$ increases; this is discussed in Section 4.3. As a result, its finite sample properties make SA less reliable for estimating a large $\nu$. Importantly, ASIS does not suffer from any of the aforementioned issues; indeed, it combines the best of both worlds.

### 4.3 Sampling Efficiency

Detailed results are presented in Table 2. In most cases of practical interest, i.e., where the observations have finite variance by assumption, the RNE of AA is at least comparable to that of the classical SA; and, for lighter-tailed inputs, AA is incomparably more efficient. Nevertheless, as discussed in Section 4.2, AA may fail; these cases are marked as missing values in Table 2. In particular, AA breaks down for the combination of a data set of length $n \gtrapprox 100$ simulated with $\nu_{\text{true}} \lesssim 3$, and a chain with initialization $\nu = 100$; however, apparent from the results, lighter-tailed data sets may be increasingly affected as their size grows. On the other hand, AA is reliable in all other settings. Luckily, ASIS proves to be a solution in this regard, as SA does not suffer from the issue; therefore, the chain can move out of this initial bad region. Moreover, ASIS outperforms both AA and SA in all setups. Even an RNE of 100% is reached for the small $n = 10$, which is measurably on par with an i.i.d. sample.

Some further results are highlighted using visualizations. First, Figure 4 is a demonstration of the differences between AA, SA, and ASIS using trace plots: these are shown for hand-picked but representative examples. The top three panels correspond to very heavy-tailed input data ($\nu_{\text{true}} = 1$) and a good initialization for AA; SA and ASIS mix quickly, but AA seems to slightly lag behind. In contrast, the bottom three panels represent the case of a light-tailed input vector ($\nu_{\text{true}} = 100$); now, AA and ASIS show good performance, but SA is highly ineffective.

Next, Figure 5 depicts a sample of RNE values conditional on the algorithm, $\nu_{\text{true}}$, $\lambda$, and the simulated data set; for improved readability, the plot is restricted to $n = 100$, $\nu_{\text{true}} \in \{1, 2, 3, 5, 10\}$, and four out of five data sets. Clearly, the sampling efficiency strongly depends on the exact properties of the observations. For instance, the data set with $DataID = 1$ is an outlier in the panel for $\nu = 5$; and the reason for the comparably poor performance of AA is the extreme outlier $\sim 15$ contained in the data. Similar behavior can be observed in the rightmost panel for $DataID = 1$, which is due to the same random seed being used for generating data sets with the same $DataID$. Further, at this sample size, $\lambda$ does not affect RNE. Indeed, there is no clustering of colors in any of the 15 columns. Since $\lambda$ cancels out in

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4. Our implementation for $F^{-1}$ is based on the built-in R function `qgamma`. 
Table 1: Estimated 10th and 90th percentiles for $\nu$. The same data of length $n = 1000$ is used inside each group of $\nu_{true}$; further, $\lambda = 0.2$ for each entry. Sets of chains with $R \geq 1.1$ are marked.

| $\nu_{true}$ | Initial $\nu = 0.5$ | 2 | 10 | 100 |
|----------|------------------|---|----|-----|
|          | $\nu_{true}$     |   |    |     |
| AA       | (0.894,1.01)*    | (0.895,1.01)* | (0.893,1.01)* | (100,100)* |
| 1        | (0.893,1.01)     | (0.894,1.01) | (0.893,1.01) | (0.895,1.01) |
| SA       | (0.893,1.01)     | (0.894,1.01) | (0.893,1.01) | (0.895,1.01) |
| ASIS     | (0.893,1.01)     | (0.893,1.01) | (0.893,1.01) | (0.894,1.01) |
| 2        | (1.74,2.05)*    | (1.75,2.05)* | (1.75,2.05)* | (100,100)* |
| AA       | (1.75,2.05)      | (1.74,2.05) | (1.75,2.05) | (1.74,2.04) |
| SA       | (1.75,2.05)      | (1.74,2.05) | (1.75,2.05) | (1.74,2.04) |
| ASIS     | (1.75,2.05)      | (1.75,2.05) | (1.75,2.05) | (1.74,2.04) |
| 5        | (3.99,5.25)      | (3.98,5.23) | (3.99,5.26) | (3.98,5.23) |
| AA       | (4.531           | (3.94,5.21) | (3.98,5.25) | (3.99,5.26) |
| SA       | (3.98,5.23)      | (3.98,5.24) | (3.98,5.24) | (3.99,5.24) |
| ASIS     | (10.1,19.1)      | (10.2,19.4) | (10.2,19.3) | (10.2,19.4) |
| 20       | (16.6,39.9)      | (16.5,40.1) | (16.7,39.9) | (16.7,40.2) |
| AA       | (16.7,40.7)      | (16.8,37.9) | (17.1,38.4) | (17.38.9) |
| SA       | (16.6,39.9)      | (16.7,40.1) | (16.7,39.8) | (16.7,40.2) |

Table 2: Mean RNE (%) for AA, SA, and ASIS, and different values of $n$ and $\nu_{true}$.

| $\nu_{true}$ | 1 | 1.5 | 2 | 2.5 | 3 | 4 | 5 | 10 | 20 | 50 | 100 |
|--------------|---|-----|---|-----|---|---|---|----|----|----|-----|
| $n = 10$     | 23.2 | 39.6 | 53.1 | 63.9 | 70.2 | 77.6 | 81.7 | 87.7 | 90.1 | 90.9 | 91.1 |
| 100          | 4.3 | 15.4 | 22.0 | 30.5 | 38.2 | 45.5 | 51.6 | 60.4 | 68.0 | 70.9 | 71.3 |
| 1000         | - | - | 23.3 | 29.9 | 36.3 | 43.7 | 48.1 | 45.1 | 39.3 | 44.4 | 46.6 |
| 10000        | - | - | 37.3 | 43.2 | 48.9 | 45.3 | 28.4 | 21.8 | 24.5 |    |    |
| 10           | 45.5 | 31.8 | 25.7 | 22.5 | 21.2 | 19.8 | 19.5 | 19.2 | 19.7 | 19.8 | 19.8 |
| 100          | 45.4 | 32.7 | 24.7 | 19.2 | 15.4 | 10.0 | 7.2 | 4.4 | 4.0 | 3.9 | 3.9 |
| 1000         | 44.4 | 31.3 | 23.2 | 17.8 | 14.2 | 9.5 | 6.8 | 2.2 | 1.0 | 0.9 | 0.9 |
| 10000        | 44.6 | 31.5 | 23.5 | 18.2 | 14.5 | 9.9 | 7.1 | 2.3 | 0.7 | 0.3 | 0.2 |
| 10           | 76.7 | 80.8 | 86.0 | 90.4 | 93.1 | 95.3 | 96.8 | 99.4 | 99.8 | 100.6 | 100.6 |
| 100          | 63.1 | 59.5 | 58.9 | 60.4 | 62.1 | 65.5 | 65.4 | 68.3 | 73.5 | 74.3 | 76.1 |
| 1000         | 63.0 | 60.3 | 60.6 | 61.6 | 63.1 | 64.1 | 64.1 | 49.6 | 40.9 | 45.3 | 48.0 |
| 10000        | 62.5 | 59.7 | 60.5 | 62.0 | 63.3 | 65.3 | 64.4 | 50.0 | 29.3 | 21.9 | 25.1 |

the Fisher information, this speaks for the similarity between RNE and $I_r$ or $I_m$. In summary, part of the conclusion of Figure 5 is consistent with that of Figure 3: SA outperforms AA
only for particularly heavy-tailed input data. However, the data do affect the BEP, which, in addition, comes at a lower $\nu_{true}$ than the aforementioned value 4: between 2 and 3. Finally and most importantly, ASIS is immune to both the numerical issues of AA and the quickly deteriorating efficiency of SA.

The significance of ASIS becomes even more apparent through Figure 6. A summary of RNE is displayed for AA, SA, and ASIS, as it decreases with increasing $n$ and $\nu_{true}$. Above all, the larger $\nu_{true}$ is, the greater the difference between the slopes for SA and ASIS. Based
Figure 5: RNE in dependence on four variables: the algorithm (AA, SA, or ASIS), $\nu$, $\lambda$, and the input data set; $n = 100$ is fixed. Points correspond to the individual Markov chains; therefore, four points belong to each setting. For increased visibility, only four out of the five simulated data sets are included, and horizontal jitter is applied to each point. Algorithm AA has only one chain for $\nu = 1$ with an $\hat{R} < 1.1$.

on the bottom panel, one may hypothesize that the RNE of SA is linearly proportional to $1/n$ for light-tailed input; this should be connected to the multiplicative term $n$ in $I_\tau$. Finally, for the last time, the dominance of ASIS is conspicuous.

5 Application

The efficiency of ASIS is evaluated on an extension of the 14 U.S. macroeconomic time series by Nelson and Plosser (1982) as part of the model specified by Geweke (1992, 1993, 1994); these three closely connected publications are followed in this section. The data set can be downloaded with the R package \textit{tseries} (R Core Team 2021; Trapletti and Hornik 2020).\footnote{The data set was formerly stored in the data archive of the Journal of Business and Economic Statistics, currently at http://korora.econ.yale.edu/phillips/data/nps\&enp.dat.}

The observation series are annual, they start between 1860 and 1909, and all end in 1988.

5.1 Model Specification

The model is a re-parameterized auto-regressive linear model of lag five with a time trend and Student-$t$ increments, and, as highlighted by (Geweke 1993), it is an alternative to the specifications employed by Nelson and Plosser (1982) and others. To underline the time-series nature of the data set, index $t$ is used in this section instead of $i$. Formally, observations $\{y_t\}$
for \( t = 1, \ldots, T \) are assumed to be explained through
\[
y_t = \gamma + \delta t + u_t, \\
A^*(L)u_t = \varepsilon_t, \\
A^*(L) = (1 - \rho L) + (1 - L) \sum_{j=1}^{4} a_j L^j,
\]
\( \{\varepsilon_t\} \sim \text{i.i.d. } t_\nu(0, \sigma^2), \)

where \( L \) is the lag operator, \( \gamma \in \mathbb{R} \) is the intercept, \( \delta \in \mathbb{R} \) is the slope of the main time trend, and the series \( \{u_t\} \) stands for the cyclic deviations from the trend. The cycle is assumed to vary according to the fifth degree polynomial specification for \( A^* \) with parameters \( 0 \leq \rho < 1 \) and \( a_1, a_2, a_3, a_4 \in \mathbb{R} \).

It is possible to eliminate the lag operator (Geweke 1993) and write down Equation (5) as
\[
y_t = \gamma(1 - \rho) + \delta \left( \rho - \frac{4}{j=1} a_j \right) + \delta (1 - \rho) t + \rho y_{t-1} + \sum_{j=1}^{4} a_j (y_{t-j} - y_{t-j-1}) + \varepsilon_t.
\]

Following Geweke (1993), the following prior distributions are assumed;
\[
p(\rho) = 5\rho^4 I_{[0,1]}(\rho), \\
\delta \sim \mathcal{N}(0, 0.05^2), \\
a_j \sim \mathcal{N}(0, 0.731 \cdot 0.342^{j-1}), \\
\gamma \sim \mathcal{N}(y_0, 10^2), \\
p(\sigma) \propto \sigma^{-1} I_{(0, \infty)}(\sigma)
\]

where \( y_0 \) is the first observation and it is not included in (5), and \( I_A(x) \) is the indicator function that takes 1 for \( x \in A \) and 0 otherwise. These prior distributions are in line with
the hypotheses in the original work, where the model is employed with Gaussian increments ($\nu = \infty$) to show that business cycles deviate from a linear time trend in a non-stationary manner. Finally, Equation (4) is adopted for $p(\nu)$ with $\lambda = 0.333$.

A variant on the algorithm by Geweke (1993) is implemented;

1. Initialize $(\gamma, \delta, \rho, a_1, a_2, a_3, a_4)^\top$ as the ordinary least squares (OLS) estimate (enforcing the bounds for $\rho$ if needed), $\sigma^2$ as the sample variance given the OLS estimates, $\tau \equiv 1$, and $\nu = 4$;
2. Draw $(\gamma, \delta)^\top$ given all other variables from a bivariate normal distribution;
3. Draw $(a_1, a_2, a_3, a_4)^\top$ given all other variables from a multivariate normal distribution;
4. Draw $\rho$ given all other variables from a non-standard distribution using rejection sampling;
5. Draw $\nu$ and $\bar{\tau}$ (for SA), $\bar{u}$ (for AA), or both (for ASIS) given all other variables as in Section 3.2; note that $\{\epsilon_t\}$ is a sufficient statistic;
6. Draw $\sigma^2$ given all other variables from a scaled inverse-chi-square distribution with $T$ degrees of freedom.

Our modification only affects Step 5, where the method described in Section 3 is applied. The exact distributions and methods of the remaining steps can be found in Geweke (1993) and in the software code as part of the Supplementary Material.

Similarly to Section 3.2, correctness of our implementations for Steps 1-5 is ensured through Geweke’s test (Geweke 2004); the prior distributions are recovered by simulating data from Equation (5) at the end of each pass. This does not work for Step 6 due to $p(\sigma)$ being improper. There, the empirical quantile function of the conditional posterior distribution is compared to the theoretical quantile function.

5.2 Posterior Estimates

For all 14 variables, a posterior sample of size 10000 is drawn after a burnin of 1000 passes using AA, SA, and ASIS, resulting in 42 Markov chains; after the verification of convergence through trace plots, these numbers are considered to be adequate. There were no numerical issues with the computation of $F^{-1}$ as part of AA.

Estimated posterior distributions are summarized in Table 3. The three algorithms largely agree on the results, and there do not seem to be systematic biases present. The figures suggest highly heavy-tailed posterior distributions for $\{\epsilon_t\}$ in all cases, given the prior information. Median values range between approximately 1.2 and 6, and, therefore, they fall on both sides of the previously measured BEP’s.

5.3 Sampling Efficiency

RNE of all chains is reported in Table 4. Parallels can be found to Table 2: AA is most efficient compared to SA for large posterior medians. Although other parts of the Gibbs sampler also affect the figures, the median $\nu$ seems to be a strong factor in the ratio between the RNE’s of AA and SA. In both columns, the lowest value belongs to Interest Rate, and one of the largest to Stock Prices, respectively. Moreover, the two columns predominantly stay close in magnitude.

6 Discussion

In this paper, efficient Bayesian estimation of the degrees of freedom parameter for the Student-$t$ distribution is investigated. Specifically, the mixing speed of AA and SA is compared through various methods: an estimate for the expected augmented Fisher information, an extensive simulation study, and a suitable application. AA is found to be beneficial both for real-world data and in most synthetic setups. However, for extreme starting values, AA is found to be completely unreliable.
Table 3: Summary of posterior distributions for \( \nu \). Credible intervals show the 10th and the 90th posterior percentiles after 10000 draws.

| Metric                      | Median   | 80% Credible Interval |
|-----------------------------|----------|-----------------------|
|                             | AA       | SA        | ASIS      | AA       | SA        | ASIS      |
| CPI                         | 2.21     | 2.15      | 2.17      | (1.57, 3.31) | (1.55, 3.27) | (1.54, 3.21) |
| Employment                  | 2.3      | 2.34      | 2.29      | (1.47, 4.03) | (1.5, 4)    | (1.49, 4.09) |
| GNP Deflator                | 2.23     | 2.21      | 2.2       | (1.58, 3.27) | (1.54, 3.25) | (1.55, 3.26) |
| Industrial Production       | 2.88     | 2.81      | 2.89      | (1.96, 4.71) | (1.92, 4.49) | (1.92, 4.74) |
| Interest Rate               | 1.23     | 1.22      | 1.22      | (0.889, 1.72) | (0.891, 1.73) | (0.888, 1.75) |
| Money Stock                 | 3.55     | 3.63      | 3.56      | (2.26, 6.09) | (2.27, 6.46) | (2.27, 6.19) |
| Nominal GNP                 | 2.31     | 2.32      | 2.32      | (1.52, 3.79) | (1.53, 3.75) | (1.53, 3.77) |
| Real GNP                    | 3.0      | 3.09      | 3.05      | (1.8, 5.77)  | (1.82, 6.05) | (1.8, 5.85)  |
| Real per Capita GNP         | 2.79     | 2.84      | 2.87      | (1.68, 5.48) | (1.65, 5.68) | (1.69, 5.59) |
| Real Wages                  | 5.41     | 5.41      | 5.38      | (2.97, 10.4) | (2.97, 10.3) | (2.95, 10.2) |
| Stock Prices                | 6.11     | 5.99      | 5.99      | (3.74, 10.3) | (3.78, 10.1) | (3.72, 10.2) |
| Unemployment Rate           | 3.49     | 3.35      | 3.45      | (2.05, 6.56) | (2.626)     | (2.03, 6.55) |
| Velocity                    | 3.62     | 3.7       | 3.73      | (2.26, 6.94) | (2.28, 7.22) | (2.29, 7.06) |
| Wages                       | 1.84     | 1.84      | 1.81      | (1.25, 2.91) | (1.25, 2.95) | (1.24, 2.88) |

Table 4: RNE for estimating \( \nu \) using AA, SA, and ASIS, along with the ratio between AA and SA and the mean of the three corresponding posterior median estimates.

| RNE (%)                         | AA | SA | ASIS | AA / SA | Median |
|----------------------------------|----|----|------|---------|--------|
| CPI                              | 11.1| 4.7| 13.2 | 2.4     | \(~2.18\) |
| Employment                       | 10.6| 2.8| 10.3 | 3.7     | \(~2.31\) |
| GNP Deflator                     | 14.6| 8.0| 19.9 | 1.8     | \(~2.21\) |
| Industrial Production            | 12.7| 3.2| 12.6 | 3.9     | \(~2.86\) |
| Interest Rate                    | 7.7 | 6.6| 7.9  | 1.2     | \(~1.22\) |
| Money Stock                      | 20.3| 4.4| 23.9 | 4.6     | \(~3.58\) |
| Nominal GNP                      | 13.8| 7.3| 18.0 | 1.9     | \(~2.32\) |
| Real GNP                         | 13.8| 4.3| 13.1 | 3.2     | \(~3.05\) |
| Real per Capita GNP              | 10.7| 4.0| 11.6 | 2.7     | \(~2.84\) |
| Real Wages                       | 27.6| 4.5| 31.5 | 6.1     | \(~5.4\)  |
| Stock Prices                     | 35.4| 5.8| 40.4 | 6.1     | \(~6.03\) |
| Unemployment Rate                | 14.7| 4.2| 14.0 | 3.5     | \(~3.43\) |
| Velocity                         | 14.9| 3.0| 14.4 | 5.0     | \(~3.69\) |
| Wages                            | 9.3 | 4.6| 11.9 | 2.0     | \(~1.83\) |

As a solution, ASIS is proposed to combine the good parts of AA and SA. It is found to be as efficient as the better of AA and SA and also invulnerable compared to AA. Moreover, ASIS is easy to implement as soon as AA and SA have been derived. Lastly, it is to be highlighted that the prime interest of the present work is the simulation-based description of the mixing speed of algorithms AA, SA, and ASIS. In particular, the results are helpful for understanding the effect of re-parameterization on sampling efficiency.
which is independent from execution time. Therefore, optimization for effective sampling rate, which is a runtime-adjusted variant of RNE, and, granted, which is of most relevance to end users but is highly dependent on many factors (Kriegel, Schubert, and Zimek 2017), is not considered in the paper at hand. One would almost certainly turn away from rejection sampling nowadays and rather experiment with other modern methods. This remains for future research.

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