Operads up to Homotopy and Deformations of Operad Maps

Pepijn van der Laan

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1 Introduction

Abstract

From the ‘cofree’ cooperad $T'(A[-1])$ on a collection $A$ together with a differential, we construct an $L_\infty$-algebra structure on the total space $\bigoplus_n A(n)$ that descends to coinvariants. We use this construction to define an $L_\infty$-algebra controlling deformations of the operad $P$ under $Q$ from a cofibrant resolution for an operad $Q$, and an operad map $Q \to P$. Starting from a different cofibrant resolution one obtains a quasi isomorphic $L_\infty$-algebra. This approach unifies Markl’s cotangent cohomology of operads and the approaches to deformation of $Q$-algebras by Balavoine, and Kontsevich and Soibelman.

Motivation

We recall that the total space $\bigoplus_n P(n)$ of any (in $k$-$\text{dgVect}$) operad $P$ has a natural Lie algebra structure that descends to the coinvariants $\bigoplus_n P(n)_{S_n}$. Since bar/cobar duality identifies operad structures on the collection $P$ with differentials on the ‘cofree’ cooperad $T'(P[-1])$, it is natural to ask what happens on $P$ if we start from an arbitrary differential on $T'(P[-1])$. We call the structure obtained on $P$ an operad up to homotopy. The explanation of this terminology in terms of Quillen model categories is postponed to [13]. Our notion is manifestly different from the lax operads described by Brinkmeier [3]. In our context the compatibility with the symmetric group actions is strict. This makes the notion more applicable.

We show that an operad up to homotopy $P$ gives rise to a natural $L_\infty$-algebra structure on the total space $\bigoplus_n P(n)$, that descends to coinvariants. This is the natural analogue of the Lie algebra structure on the total space of an operad. Morphisms of operads up to homotopy (i.e. morphisms of the corresponding free cooperads with differential) yield morphisms between the corresponding $L_\infty$-algebras.
Applying the Lie algebra construction to the convolution operad $P^A$ of a cooperad $A$ and an operad $P$, solutions to the Maurer-Cartan equation correspond to operad maps $B^*A \rightarrow P$ from the cobar construction $B^*A$ of $A$ to $P$. Now suppose that $B^*A$ is a cofibrant resolution for an operad $Q$, then the Lie algebra $\bigoplus_n P^A(n) \Sigma_n$ controls deformations of operads $Q \rightarrow P$ under $Q$. We show the Lie algebras for different cofibrant models are quasi isomorphic. We use the model category of operads and the functoriality of the $L_\infty$-structure on the total space to do this. We observe that this construction extends to more general models for $Q$, as to include Markl’s cotangent cohomology of operads, and the approaches to deformations of $Q$-algebras by Balavoine, and Kontsevich and Soibelman.

The material in this paper is intended to constitute the core of a chapter in the author’s dissertation. The author aims to include more applications and examples there. Please note the paper is written in a way that allows generalisation to coloured operads.

**Plan of the Paper**

Section 2 gives a short overview of the necessary standard results on operads. Section 3 introduces operads up to homotopy and shows how to construct $L_\infty$ algebras from these. We show how morphisms of operads up to homotopy induce morphisms of $L_\infty$-algebras. Section 5 defines a cohomology theory using convolution operads. The $L_\infty$-algebra constructed in Section 3 serves to show that the cohomology is independent of the necessary choice of cofibrant resolution. Section 5 continues to show the relation the known approaches to deformations of algebras over an operad. To fulfill these tasks we need the results we recall in Section 4.

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**2 Operads and Cooperads**

This section is purely expositorial. We describe relevant structures on operads and cooperads such as the Bar/Cobar construction and the convolution operad.
Definitions

Throughout this paper, let \( k \) be a field of characteristic 0. We work in the category \( k\text{-dgVect} \) of dg vector spaces over \( k \) (by convention \( \mathbb{Z} \)-graded with a differential of degree +1). This is a symmetric monoidal category with the usual tensor product \( \otimes \) and the symmetry \( A \otimes B \ni a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \in A \otimes B \) for homogeneous elements \( a \) and \( b \) of degrees \( |a| \) and \( |b| \) is dg vector spaces \( A \) and \( B \).

We will make extensive use of the shift functor on \( k\text{-dgVect} \), given by \( V[p] = V[n] - p \) with differential \( d[p] = (-1)^p d[n] - p \) (superscripts denoting degrees here, not powers). We have the usual Koszul sign convention. In particular, for graded maps \( f \) and \( g \) we write \( (f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b) \). These conventions allow us to hide most of the signs.

Let \( \text{Fin}_* \) be the category of pointed finite sets and bijections preserving the basepoint. For \( (X,x_0),(Y,y_0) \in \text{Fin}_* \) and \( x \in X - \{x_0\} \) define \( Y \cup_x X \) to be the pushout (in \( \text{Set} \), the category of sets)

\[
\begin{array}{c}
\ast \\
Y \\
\downarrow \\
Y \cup_x X
\end{array}
\xrightarrow{x}
\begin{array}{c}
X \\
Y \cup_x Y
\end{array}
\]

with the basepoint \( x_0 \). Let \( \tau \in \text{Aut}(Y,y_0) \) and \( \sigma \in \text{Aut}(X,x_0) \). The map \( \sigma \circ_x \tau : X \cup_x Y \longrightarrow X \cup_{\tau(x)} Y \) in \( \text{Fin}_* \) is the unique dotted arrow that makes the diagram below commute

\[
\begin{array}{c}
Y \\
\downarrow \sigma \\
Y \cup_{\tau(x)} X \\
\downarrow \tau \\
X
\end{array}
\xleftarrow{x} \begin{array}{c}
X \\
Y \cup_x X
\end{array}
\]

(2.1)

A collection in is a contravariant functor \( C : \text{Fin}_* \longrightarrow k\text{-dgVect} \). A pseudo operad is a collection together with a map

\[
\circ_x : C(X,x_0) \otimes C(Y,y_0) \longrightarrow C(X \cup_x Y,x_0),
\]

for each pair of pointed sets \( (X,x_0) \) and \( (Y,y_0) \) and each \( x \in X \) such that \( x \neq x_0 \), such that

(i). With respect to the automorphisms of \( X \) and \( Y \) the map \( \circ_x \) behaves as

\[
(\tau \circ_x q \sigma = (p \circ_{\tau^{-1}x} q)(\tau \circ_{\tau^{-1}x} \sigma)
\]

(cf. equation (2.1)) for \( p \in C(X), q \in C(Y) \) and \( x \in X - \{x_0\}, \tau \in \text{Aut}(X,x_0), \sigma \in \text{Aut}(Y,y_0) \).
(ii). The squares

\[
\begin{align*}
C(X) \otimes C(Y) \otimes C(Z) & \xrightarrow{\circ_x \otimes \text{id}} C(X \cup_x Y) \otimes C(Z) \\
\downarrow \quad (\text{id} \otimes \circ_{x'}) & \quad \downarrow \circ_{x'} \\
C(X \cup_{x'} Z) \otimes C(Y) & \xrightarrow{\circ_x} C(X \cup_x Y \cup_{x'} Z) \\
\downarrow \quad \text{id} \otimes \circ_x & \quad \downarrow \circ_x \\
C(X \cup_x Y) \otimes C(Z) & \xrightarrow{\circ_y} C(X \cup_x Y \cup_y Z)
\end{align*}
\]

commute for \( y \in Y \) and \( x, x' \in X \). We omit the basepoints from the notation and the map \( s \) is the symmetry of the tensor product.

An operad is a pseudo operad together with a map \( \text{id} : I \rightarrow C(\{x, x_0\}) \) for a two element set \( \{x, x_0\} \), such that \( \text{id} \) is a right/left identity with respect to any well defined right/left composition. Together with the natural notion of morphism this defines the categories of collections, pseudo operads and operads.

In some cases (though not always) a different description is convenient. The description is obtained by the inclusion of the symmetric groupoid as a skeleton in \( \text{Fin}_n \) with objects \( n = \{0, \ldots, n\} \) and 0 as basepoint. We write \( P(n) = P(n) \).

At times we will use the equivalent description of operads in terms of the maps

\[
\gamma : P(n) \otimes S_n (P(k_1) \otimes \ldots \otimes P(k_n)) \rightarrow P(k_1 + \ldots + k_n)
\]

and an identity map in \( P(1) \). Here \( \gamma(p, q_1, \ldots, q_n) = (\cdots ((p \circ_n q_n) \circ_{n-1} q_{n-1}) \cdots \circ_1 q_1) \) for \( p \in P(n) \), and \( q_i \in P(k_i) \).

A (pseudo) cooperad is a (pseudo) operad in the opposite category \( (k\text{-dgVect})^{\text{op}} \).

That is, a covariant (!) functor \( \text{Fin}_n \rightarrow k\text{-dgVect} \) satisfying the dual axioms.

Here the directions of arrows is inverted, the coinvariants change to invariants, sums change into products, and the right action of \( S_n \) changes to a left action. For convenience of the reader we twist this action by an inverse and obtain a right action. Therefore we do not have to distinguish between left and right collections.

**Graphs and Trees**

This paragraph recalls the approach of Getzler and Kapranov \[7\] on graphs. A graph is a finite set of flags (or half edges) together with both an equivalence relation \( \sim \), and an automorphism \( \sigma \) of order \( \leq 2 \). The equivalence classes are called vertices, the orbits of length 2 (internal) edges, and the orbits of length 1 external edges or legs. We denote the vertices of a graph \( g \) by \( v(g) \), its (internal) edges by \( e(g) \) and its legs by \( l(g) \), especially we denote the half edges in a vertex \( v \) by \( l(v) \). The flags in a vertex are also called its legs. The
vertex containing a flag $f$ is called the vertex of $f$. To picture a graph, draw a node for each vertex with outgoing edges labeled by its flags. For each orbit of $\sigma$ of length 2 (edge) draw a line connecting the two flags. For each orbit of length one this leaves an external edge. A tree is a connected graph $t$ such that $|v(t)| - |e(t)| = 1$. A rooted tree is a tree together with a basepoint in the set of external edges. Note that on a rooted tree, the base point induces a canonical base point on the legs of any vertex. We denote by $T(X)$ the set of rooted trees with external edges labeled by the pointed set $X$. (An element $t$ of $T(X)$ is a tree $t$ together with a basepoint preserving bijection $X \to \mathbb{I}(t)$.)

A morphism of graphs is a morphism $\varphi$ of the set of flags, such that for all flags $f, f'$ both $\varphi(\sigma f) = \sigma \varphi(f)$, and $\varphi(f) \sim \varphi(f')$ iff $f \sim f'$ (where $\sigma$ and $\sim$ are the defining automorphisms and equivalence relations). If the set of external edges has a basepoint, we assume that morphisms preserve the basepoint. The group of automorphisms of a graph $g$ is denoted $\text{Aut}(g)$. Note that an automorphism defines an isomorphism of flags, an automorphism of edges, an automorphism of external edges, and an isomorphism of vertices. Moreover it defines an isomorphism of the legs of a vertex to the legs of its image under the map on vertices. For two rooted trees $s$ and $t$ and $x \in \mathbb{I}(t)$ define $s \circ_x t$ as the rooted tree obtained from $s$ and $t$ by grafting the root of $s$ on leg $x$ of $t$.

### Free Operads and ‘Cofree’ Cooperads

The forgetful functor from pseudo operads to collections in $k\text{-dgVect}$ has a left adjoint $T$, which is the free pseudo operad functor. We can give an explicit formula for $T$. Let

$$C(t) = \left( \bigotimes_{v \in v(t)} C(\mathbb{I}(v)) \right)_{\text{Aut}(t)}, \tag{2.3}$$

for any rooted tree $t$. The action of $\text{Aut}(t)$ is by permutation of the tensor factors according to the permutation of vertices, by the associated isomorphism of $\mathbb{I}(v) \to \mathbb{I}(\sigma(v))$ action inside the tensor factor associated to $v$, and by the composition of the labeling of $\mathbb{I}(t)$ by the induced isomorphism of $\mathbb{I}(t)$. Define the underlying collection $TC$ of the free pseudo operad by

$$TC(X) = \colim_{T(X)} (C(t)) \equiv \bigoplus_{t \in \bar{T}(X)} C(t),$$

where the colimit is over the groupoid with objects $t \in T(X)$ and isomorphisms of trees as maps, and the sum is over the isomorphism classes of rooted trees $\bar{T}(X)$ in $T(X)$. The pseudo operad structure on $TC$ is given by grafting trees. The forgetful functor from operads to pseudo operads has an left adjoint. This map is given by adjoining a unit with respect to composition in $P(X)$ for $X$ such that $|X| = 2$. By definition the free operad functor $T^+$ is the composition of both left adjoints.
We now define a cooperad that is cofree with respect to a restricted class of cooperads. Therefore we use the term ‘cofree’ when referring to this cooperad. The ‘cofree’ pseudo cooperad functor \( T' \) is defined by

\[
T' C(X, x_0) = \bigoplus_{t \in P(t)} \left( \bigotimes_{v \in V(t)} C(I(v)) \right)^{\text{Aut}(t)}.
\]

The pseudo cooperad structure is determined by cutting edges. Composition of the functor \( T' \) with the right adjoint to the forgetful functor from cooperads to pseudo cooperads defines the ‘cofree’ cooperad functor \( (T')^+ \).

Recall the monoidal structure \( (\square, I, a, i_r, i_l) \) on the category of collections in \( k\)-dgVect from Smirnov \[22\] (Shnider and Van Osdol \[21\] is a more accessible reference):

\[
C \square D(X) = \bigoplus_Y \bigoplus_{f: X \to Y} C(Y) \otimes_{\text{Aut}(Y)} \left( \bigotimes_{y \in Y - y_0} D(f^{-1}(y) \cup \{y\}) \right)
\]

where the sum is over one representing pointed sets \( Y \) for each isomorphism class, and where \( y \) is the basepoint of \( f^{-1}(y) \cup \{y\} \). Operads are algebras in this monoidal category. Getzler and Jones \[6\] define a cooperad as a coalgebra with respect to this monoidal structure. Since \( \text{char}(k) = 0 \), the natural isomorphism from coinvariants to invariants identifies both notions of cooperad.

**Convolution Operad**

For convenience the lemma below uses the description of operads and cooperads with an explicit choice of coordinates. That is, we use \( \gamma \) (cf. Formula \[22\]), and for cooperads the dual map \( \gamma^* \). Let \( C \) and \( P \) be collections. Consider the collection \( P^C(X) = \text{Hom}(C(X), P(X)) \), with the natural action (i.e. \( (\varphi \sigma)(c) := (\varphi(\sigma^{-1})) \sigma \) for \( \sigma \in \text{Aut}(X) \)) and the natural differential (i.e. \( d\varphi := \varphi \circ d - d \circ \varphi \)).

**2.1 Lemma (Berger and Moerdijk \[3\])** Let \( P \) be an operad and let \( C \) be a cooperad. The collection \( P^C \) enjoys an operad structure given by

\[
\gamma(\varphi; \psi_1, \ldots, \psi_n) = \gamma P \circ (\varphi \otimes (\psi \otimes \ldots \otimes \psi)) \circ \gamma^*_C,
\]

where we restrict \( \gamma^* \) to the suitable summand for homogeneous (wrt arity) \( \varphi \) and \( \psi_i \). The unit is given by the composition \( u \circ \varepsilon \) of the counit \( \varepsilon \) of \( C \) with the identity \( u \) of \( P \).

The operad defined in Lemma 2.1 is called the **convolution operad** of \( C \) and \( P \).
Bar and Cobar Construction

An operad $P$ is **augmented** if there is a projection of operads $\varepsilon : P \to k \cdot \text{id}$ such that the unit $k \cdot \text{id} \to P(1)$ composed with $\varepsilon$ is the identity on $k \cdot \text{id}$. The kernel of $\varepsilon$ is the augmentation ideal of $P$. The categories of augmented operads and pseudo operads are isomorphic. The dual notion is a **coaugmented cooperad**.

2.2 Convention In the sequel we use augmented operads exclusively. Likewise for cooperads. We denote by $T$ (resp. $T'$) the (co)free (co)operad in the (co)augmented category.

Let $C$ be a graded collection. A **differential** on the ‘cofree’ cooperad $T'C$ is a coderivation $\partial$ of degree +1 that satisfies $\partial^2 = 0$. A differential on the free cooperad $TC$ is a derivation $\partial$ of degree +1 that satisfies $\partial^2 = 0$. Recall that the ‘cofree’ cooperad is graded by the number of vertices in the trees.

2.3 Theorem (Getzler and Jones [6]) Let $P$ be a graded collection. There is a 1-1 correspondence between operad structures on $P$ and differentials on $T'(P[-1])$ that are of degree $\geq -1$ in vertices.

Dually, there is a 1-1 correspondence between cooperad structures on $P$ and differentials on $T(P[1])$ that are of degree $\leq +1$ in vertices.

Let $P$ be an operad. Define the **bar construction** $B^*P$ to be the corresponding cooperad $(T'(P[1]), \partial)$ of Theorem 2.3. Let $C$ be a cooperad. Define the **cobar construction** $BC$ to be the corresponding operad $(T(P[1]), \partial)$ of Theorem 2.3. These constructions define two functors

$$B : \text{Opd} \to \text{Coopd} \quad \text{and} \quad B^* : \text{Coopd} \to \text{Opd}.$$  

2.4 Theorem (Ginzburg and Kapranov [7]) Let $P$ be an operad. The natural projection $B^*BP \to P$ is a quasi isomorphism of operads (i.e. the induced morphisms in cohomology are isomorphisms). Dually, let $C$ be a cooperad. The natural inclusion $C \to BB^*C$ is a quasi isomorphism of cooperads.

3 Operads up to homotopy

This section relaxes the operad axioms and obtains thus the definition of an operad up to homotopy. It shows an operad up to homotopy gives rise to an $L_\infty$-algebra, and that a morphism of operads up to homotopy gives rise to an $L_\infty$-morphism. The author will discuss the characterization of operads up to homotopy in terms of homotopy algebras for the Koszul self-dual $\Sigma$-colored Koszul operad of non-$\Sigma$ operads in [13].
Definitions

Following Convention 2.2 we use (co)augmented (co)operads throughout the sequel of this paper.

3.1 Definition An operad up to homotopy $P$ is a collection $P$ in $k\text{-}g\text{Vect}$ together with a differential $\partial$ on the tree cooperad $T'(P[1])$. Dually, a cooperad up to homotopy $C$ is a collection $C$ in $k\text{-}g\text{Vect}$ together with a differential $\partial$ on the free operad $T(C[-1])$. Analogous to the $A_\infty$-case one could discuss several notions of units for operads up to homotopy. We ignore this point since it is beyond our aims.

3.2 Remark In the sequel of this section we restrict to operads up to homotopy. The dual statements are readily deduced.

3.3 Proposition (Markl [17]) Let $C$ be a collection. There is a 1-1 correspondence between coderivations of $T' C$ and collection morphisms $T' C \rightarrow C$.

The differential on $(T' P[-1], \partial)$ defines for each rooted tree $t$ an operation

$$\circ_t : P[-1](t) \rightarrow P[-1](\text{I}(t))$$

(cf. Equation 2.3), that preserves both internal and collection degree. Conversely, $\partial$ is completely determined by the operations $\circ_t$. Part of the structure is an internal differential $d$ on the collection $P$ (corresponding to trees with one vertex). The condition on $\partial^2 = 0$ on the differential is equivalent to a sequence of relations on these operations. For each rooted tree $t$, we obtain a relation of the form

$$\sum_{s \subset t} \pm (\circ_{t/s}) \circ (\circ_s) = 0,$$

(3.4)

where the sum is over connected subtrees $s$ of $t$ and $t/s$ is the tree obtained from $t$ by contracting the subtree $s$ to a point. The signs involved are induced by a choice of ordering on the vertices of the trees $t$ and $s$ and the Koszul convention.

If $P$ is an operad up to homotopy, then the cohomology $H^* P$ with respect to the internal differential $d$ induced by $\partial$ is a graded operad.

3.4 Definition An operad up to homotopy is called minimal if the composition of $\partial|_{P[1]}$ with the projection to the cogenerators of $T'(P[1])$ is zero. That is, if the differential on the collection $P$ as induced by $\partial$ vanishes, or equivalently such that the differential on $T'(P[1])$ is of degree $\leq -1$ with respect to vertices. An operad up to homotopy is called strict if the differential $\partial$ is of degree $\geq -1$ in the number of vertices. Through Theorem 2.3 there is an obvious 1-1 correspondence between operads and strict operads up to homotopy.
Planar Rooted Trees

In order to show that for any operad up to homotopy $P$ the total space $\oplus P$ is an $L_\infty$-algebra we need planar trees. These will serve to keep track of the $\text{Aut}(I(v))$-actions, for $v \in v(t)$ for a rooted tree $t$.

3.5 Convention Let the functor $T'$ from $k$-$\text{dgVect}$ to cooperads denote the ‘cofree’ coassociative coalgebra. On a vector space $V$ we have $T'V = \bigoplus_{n \geq 0} V^\otimes n$. Confusion with the ‘cofree’ cooperad functor on collections should not be possible. Let $S'$ denote the ‘cofree’ non-counital cocommutative coalgebra functor, on a vector space $V$ given by $S'V = \bigoplus_{n \geq 0} (V^\otimes n)^{S_n}$. Introduce two functors $\oplus$ and $\oplus_S$ from collections to $k$-$\text{dgVect}$. On an object $C$ these are given by

$$\oplus(C) = \bigoplus_{n \in \mathbb{N}} C(n) \quad \text{and} \quad \oplus_S(C) = \bigoplus_{n \in \mathbb{N}} (C(n))_{S_n}.$$ 

For $n \in \mathbb{N}$, denote by $n$ the pointed set $\{0, \ldots, n\}$ with basepoint 0. A planar rooted tree is a rooted tree together with an isomorphism of pointed sets $n_v \rightarrow I(v)$ for each $v \in v(t)$, that sends 0 to the base point, where $n_v = |I(v)| - 1$. The planar structure induces a linear ordering on the vertices (we write $v(t) = \{v_1, \ldots, v_{|I(t)|}\}$) and a labeling of $I(t)$ by the pointed set $n_t = \{0, \ldots, |I(t)| - 1\}$. To see this, draw a planar tree with half edges of $v \in v(t)$ from left to right with respect to the ordering of $I(v)$.

Let $C$ be a collection. We define an inclusion of vector spaces from the tensor coalgebra $T'(\oplus C)$ on $\oplus C$ into a big space

$$T'(\oplus C) \xrightarrow{i} \bigoplus_{t \text{ planar}} \bigotimes_{v \in v(t)} C(I(t)). \quad (3.5)$$

The map $i$ on the summand $C(n_1) \otimes \ldots \otimes C(n_m) \subset T'(\oplus C)$ has a value in the summand corresponding to a planar tree $t$ (at the right hand side) for each planar tree $t$. The image in the summand of $t$ is non-zero only if $I(v_i) = n_i$ for all $i$. If $i$ is non-zero, it is given by the identification of $C(n_1) \otimes \ldots \otimes C(n_m)$ with $C(I(v_1)) \otimes \ldots \otimes C(I(v_m))$, where the identification $C(n_i) \rightarrow C(I(v_i))$ is the planar structure on $t$ (i.e. the isomorphism $n_i \rightarrow I(v_i)$).

Symmetrization

We can identify the symmetric coalgebra $S'(\oplus C)$ with the invariant part of the tensor coalgebra $T'(\oplus C)$ (with respect to permutation of tensor factors). Note that this does not involve the action of tree automorphisms. The restriction of the map $i$ in Formula (3.5) to $S'(\oplus C)$ is denoted $i$ as well.

3.6 Proposition The map $i$ induces a map (again denoted by $i$) from $S'(\oplus C)$ into the total space $\oplus(T'C)$ of the cofree pseudo cooperad on $C$. That is, $i$ defines a map

$$S'(\oplus C) \xrightarrow{i} \oplus(T'C)$$
into the total space of $T'C$.

**Proof** Forgetting the planar structure defines a map of $\text{Aut}(I(t))$-modules

$$S'(\oplus C) \xrightarrow{i} \bigoplus_{t \text{planar } v \in \psi(t)} C(I(v_i)) \xrightarrow{\psi} \bigoplus_{t \in \psi(t)} C(I(v_i)) . \quad (3.6)$$

It now suffices to show that the image of the composition in Equation (3.6) is invariant under the $\text{Aut}(t)$-action for each rooted tree $t$. We study $\psi \circ i$ on $\sum_{\sigma \in S_m} (p_1, \ldots, p_m) \in S^m(\oplus C)$ with $p_j \in C(n_j)$. The map (3.6) satisfies

$$\sum_{\sigma \in S_m} (p_1, \ldots, p_m) \mapsto \sum_{j=1}^m \sum \sum \left((\cdots (v(P_j) \circ_{n_j} s_{n_j}(P_{n_j})) \ldots) \circ_1 s_1(P_1), \right) \quad (3.7)$$

where the second sum is over $n_j$-tuples of planar trees $(s_1, \ldots, s_{n_j})$ such that $\sum_k |v(s_k)| = m - 1$ (for the sake of simplicity of the formula a planar tree might be empty here) and the third sum is over ordered partitions $(P_1, \ldots, P_{n_j})$ of $\{1, \ldots, m\} - \{j\}$ such that $|P_k| = |v(s_k)|$ for all $k$. The symbolic notation $((\cdots (v(P_j) \circ_{n_j} s_{n_j}(P_{n_j})) \ldots) \circ_1 s_1(P_1))$ denotes the sum of all terms where the vertices of $s_k$ are labeled labeled by $p_l$ with $l \in P_k$, and the root vertex $v_0$ labeled by $p_j$. The summand of $(s_1, \ldots, s_{n_j})$ corresponds to the tree with root $r$ such that $I(r) = \{0, \ldots, n_j\}$ and with planar tree $s_k$ grafted to the root along leg $k$.

We prove the invariance by induction on the number of vertices of a tree. Let $n \in \mathbb{N}$ and suppose that the image of (3.6) is contained in the invariants for each rooted tree with $< n$ vertices. Let $s$ be a rooted tree such that $|v(s)| = n$. One can write $\text{Aut}(s)$ as a crossed product of $\bigoplus_i \text{Aut}(s_i)$ with the subgroup of $\text{Aut}(I(v_0))$ that permutes those legs $l_i$ such that the corresponding $s_i$ are isomorphic. Consequently, it suffices (by the induction hypothesis) to show that the image is invariant with respect to elements $\hat{\tau} \in \text{Aut}(s)$ defined by a permutation $\tau \in \text{Aut}(I(v_0))$. This will be done for each summand $j$ in Formula (3.7). Let $\tau \in \text{Aut}(I(p_j))$. Then, $\tau$ applied to a summand of the right hand side of Formula (3.7) gives

$$((\cdots (v(P_j) \circ_{n_j} s_{\tau^{-1}n_j}(P_{\tau^{-1}n_j})) \ldots) \circ_1 s_{\tau^{-1}1}(P_{\tau^{-1}1})) , \quad (3.8)$$

where we compensate for the omitted action of $\tau$ on $p_j$ by not permuting the legs of the tree according to the induced permutation. This gives a permutation of summands and equivariance follows. QED

**3.7 Proposition** The map $i$ of Proposition 3.4 descends to the coinvariants with respect to the action on the collection $C$. That is, $i$ defines a map

$$S'((\oplus_s C) \xrightarrow{i} \oplus_s (\hat{T}'C).$$
Proof This follows by the equivariance property of cocompositions in $\bar{T}'C$, which we can express by compensating for the action of $\tau$ on the label of a vertex of a tree $t$ by a permutation of $l(t)$, as in Formula (3.8). QED

$L_\infty$-Algebras

An $L_\infty$-algebra (Lada and Stasheff [15], et. al.) is a dg vector space $g$, together with a differential $\partial$ (i.e. a coderivation s.t. $\partial^2 = 0$) on the free cocommutative coalgebra $S'(g[-1])$. The differential is completely determined by its restrictions $l_n : S^n(g[-1]) \to g[-1]$, where $S^n(g[-1])$ of course denotes the elements of tensor degree $n$. A notemorphism of $L_\infty$-algebras (or $L_\infty$-map) $f : g \to h$ is a morphism of the cofree coalgebras compatible with the differentials. It is determined by its restrictions $f_n : S^n(g[-1]) \to h[-1]$. An $L_\infty$-map is a quasi isomorphism if $f_1 : g[-1] \to h[-1]$ induces an isomorphism in cohomology.

3.8 Lemma Let $C$ be a collection and let $D$ be a coderivation on $T'C$. Then the diagram

$$
\begin{array}{ccc}
S'(\bigoplus C) & \xrightarrow{i} & \bigoplus(T'C) \\
D \downarrow & & \downarrow \oplus D \\
S'(\bigoplus C) & \xrightarrow{i} & \bigoplus(T'C)
\end{array}
$$

commutes, where $D : S' \bigoplus C \to S' \bigoplus C$ is the coderivation defined by the composition

$$
\begin{array}{ccc}
S'(\bigoplus C) & \xrightarrow{i} & \bigoplus(T'C) \\
\bigoplus C & \xrightarrow{\oplus D} & \bigoplus(T'C)
\end{array}
$$

Proof This is a direct corollary of the definition of coderivation for coalgebras and cooperads. QED

3.9 Theorem Let $P$ be an operad up to homotopy. Then

(i). there exists a natural $L_\infty$-algebra structure on $\bigoplus P$.

(ii). this structure descends to the quotient $\bigoplus S P$.

Proof Observe that the differential $\partial$ on $T'(P[-1])$ induces a derivation $d$ on $S'(\bigoplus P[-1])$ according to Lemma 3.8. Regarding part (i), we need to show that
\(d^2 = 0\). Since \(d\) and \(i\) commute, we have a commutative diagram

\[
\begin{align*}
S'(\oplus P[-1]) & \xrightarrow{i} \oplus(T'(P[-1])) \\
S'(\oplus P[-1]) & \xrightarrow{i} \oplus(T'(P[-1])) \\
S'(\oplus P[-1]) & \xrightarrow{i} \oplus(T'(P[-1])) \xrightarrow{\pi} \oplus P[-1]
\end{align*}
\]

with the composition from \(S' (\oplus P[-1])\) at the top left to \(\oplus P[-1]\) (dotted arrow) equal to 0. Therefore, \(d^2 = 0\) on \(S'(\oplus P[-1])\). This shows (i).

Since the differential on \(T'(P[-1])\) preserves the symmetric group action (cf. \(\circ_t\) operations), it induces a differential on \(\oplus T'(P[-1])\). The map \(i\) is well defined as a map \(i : S'(\oplus P[-1]) \rightarrow \oplus T'(P[-1])\). The induced differential on \(S'(\oplus P[-1])\) commutes with the map \(i\). Part (iii) follows by the commutation relation of the \(\circ_t\) with the \(\text{Aut}(\text{I}(t))\)-actions.

3.10 Remark It is instructive to write the higher operations \(l_n\) in the \(L_\infty\)-algebra structure of the previous theorem.

\[l_n = \sum_{t \text{ planar}, \ |v(t)| = n} (\circ_t) \circ i : S^n (\oplus P[-1]) \rightarrow \oplus P[-1], \quad (3.9)\]

The signs are induced by the Koszul convention for the \(|v(t)|\)-ary operations \(\circ_t\).

**Examples: \(A_\infty\)-Algebras and Operads**

This paragraph treats some special cases of the \(L_\infty\)-algebras constructed in the previous section. Firstly we restrict our attention to \(A_\infty\)-algebras, where we obtain a well known result as a corollary to Theorem 3.9. This example is based on the observation that \(A_\infty\)-algebras are exactly operads up to homotopy concentrated in arity 1. Secondly, we recover the Lie algebra structure on the total space \(\oplus P\) of an operad \(P\).

An \(A_\infty\)-algebra (Stasheff [23]) is a vector space \(A\), together with a differential \(\partial\) (i.e. a coderivation s.t. \(\partial^2 = 0\)) on the free coalgebra \(T'(A[-1])\). The differential is completely determined by its restrictions

\[m_n : T^n(A[-1]) \rightarrow A[-1],\]

where \(T^n(A[-1])\) of course denotes the elements of tensor degree \(n\).

3.11 Lemma Let \((T'(P[1]), d)\) be an operad up to homotopy. Then the restriction of \(d\) to the part of arity 1 makes \(P(1)\) an \(A_\infty\)-algebra.
Proof Let $A = P(1)$. The differential on the tree cooperads $T'(P[-1])$ restricts to a differential on the subspace of $T'(P[-1])(1)$ consisting of the summands corresponding to trees in which $|I(v)| = 2$ for each vertex $v$. This sub-cooperad equals the cofree coalgebra $T'(A[-1])$ on $A[-1]$. Moreover, the differential is a coderivation since it is a coderivation on $T'(P[-1])$. This proves the result. QED

3.12 Example (Lada and Markl [14]) When we start out from $P$ concentrated in arity 1, the previous results shows that each $A_{\infty}$-algebra becomes an $L_{\infty}$-algebra with what one might call higher order commutators. This generalizes the construction of the commutator Lie algebra with the bracket $[a,b] = ab - ba$ of an associative algebra $A$. One can write the operations $l_n$ as the symmetrization of $m_n$.

The first part of the following proposition has been around for a long time. It suffices to use an argument of Gerstenhaber [3] (predating the very definition of operads!). The second part first appeared in Kapranov and Manin [16], though for example Balavoine [1] already gives a sufficient computation. But he failed to state a result in this generality.

3.13 Corollary Let $P$ be an operad. Then

(i). The total space $\oplus P$ is a Lie algebra with respect to the commutator of the (non-associative) multiplication $p \circ q = \sum_{i=1}^{n} p \circ_i q$ for $p \in P(n)$ and $q \in P(m)$.

(ii). The Lie algebra structure descends to quotient $\oplus S P$.

Homotopy Homomorphisms

This section studies morphisms of (co)operads up to homotopy. We are interested in the interaction between these maps and the $L_{\infty}$-structures described in the previous paragraph.

3.14 Definition A morphism of operads up to homotopy $\varphi : P \rightarrow Q$ is a morphism of dg cooperads $\varphi : T'(P[-1], d) \rightarrow T'(Q[-1], d)$.

Dually, a morphism of cooperads up to homotopy $\psi : A \rightarrow C$ is a morphism of dg operads $\psi : T(A[1], d) \rightarrow T(C[1], d)$.

The sequel of this section focuses on the operad up to homotopy case and leaves the formulation of the dual statements as an exercise. The adjunction of $T'$ and the forgetful functor assures that $\varphi$ is determined by maps $\varphi(t) : (P[-1])(t) \rightarrow P[-1]$. The condition that $\varphi$ is compatible with the differential can be described
in terms of conditions about compatibility with the $o_t$ operations:

$$\sum_{s \subset t} \pm \varphi(t/s) \circ (o_s) - (o_{t/s}) \circ \varphi(s) = 0,$$

again with the signs depending on the trees as in Formula (3.4).

3.15 Lemma Let $C$ and $E$ be collections, and let $\varphi : T'C \to T'E$ be a morphism of cooperads. Then the diagram

$$\begin{array}{ccc}
S' \oplus C & \xrightarrow{i} & \oplus T'C \\
\text{\hat{\varphi}} \downarrow & & \downarrow \oplus \varphi \\
S' \oplus E & \xrightarrow{i} & \oplus T'E
\end{array}$$

is commutative, where $\text{\hat{\varphi}} : S' \oplus C \to S' \oplus E$ is the coalgebra morphism induced by

$$\text{\hat{\varphi}}(S') \oplus (\oplus Q) \xrightarrow{i} \oplus (T'C) \oplus \varphi,$$

$$\text{\hat{\varphi}}(S') \oplus (\oplus E) \leftrightarrow \oplus (T'E)$$

Proof This is clear from the definition of coalgebra and cooperad morphism. QED

3.16 Theorem Let $\varphi : P \to Q$ be a morphism of operads up to homotopy. Then $\varphi$ induces morphisms of $L_\infty$-algebras

$$\oplus P \to \oplus Q \quad \text{and} \quad \oplus_s P \to \oplus_s Q.$$  

Proof The universal property of $S'(\oplus Q[-1])$ defines a coalgebra morphism $\varphi$ as in Lemma 3.15. It suffices to check that the coalgebra map $\varphi$ commutes with the differential. Due to the coderivation property (Lemma 3.8) it suffices to show that $\pi \circ d \circ \varphi = \pi \circ \varphi \circ d$, where $\pi : S'(\oplus Q[-1]) \to \oplus Q$ is the natural projection. The result now follows since $i$ and $\varphi$ commute by Lemma 3.15 and since $\varphi$ is equivariant with respect to the $S_n$-actions. QED

4 Auxiliary Results

In order to proceed, we need to recall some results. Most results in this section are well known. Nevertheless we feel inclined to spell out some details on $L_\infty$-algebras and the Maurer-Cartan equation.
Model Category of Operads

We recall the model category (cf. Quillen [19] and Hovey [10]) of operads first defined in Hinich [9]. The weak equivalences are quasi isomorphisms, fibrations are surjections, and the generating cofibrations are extensions by free operads with a differential.

We recall some facts on minimal models of 1-reduced operads, following Markl [17]. Let \( P \) be a 1-reduced operad. A model \( A \) of \( P \) is a cooperad up to homotopy \( A \), together with a quasi isomorphism \( T(A[-1], \partial) \rightarrow P \). A model \( M = (T(A[-1]), \partial) \) of \( P \) is a minimal model if the differential \( \partial \) is minimal in the sense that the induced internal differential on \( A \) vanishes. A model \( M = (T(A[-1]), \partial) \) of \( P \) is a strict model if \( A \) is strict. That is, the differential \( \partial \) defines a cooperad structure on \( A \).

4.1 Theorem (Markl [17]) Every 1-reduced operad has a minimal model. Moreover, the minimal model is unique up to isomorphism of cooperads up to homotopy.

4.2 Remark (Hinich [9]) Operads in \( k \)-dgVect form a model category. Cofibrant objects in the model category are exactly retracts of operads associated to homotopy cooperads. Models and minimal models are cofibrant objects (of a special kind!) in this model category. To see this, use that fact that the generating cofibrations are free operads on a cofibrant collection, together with a differential. Since the characteristic of \( k \) is 0, every collection is cofibrant. A minimal model \( (T(A[-1]), \partial) \) of an operad \( P \) is a cofibrant replacement for \( P \) if the map \( TA[1] \rightarrow P \) is a fibration (i.e. is surjective). It suffices to ask the internal differential of \( P \) to be 0.

Maurer-Cartan Equation

Throughout this section, let \( (S'(g[-1]), \partial) \) be an \( L_\infty \)-algebra and let \( \varphi \in g^1 \) be a solution of the (generalized) Maurer-Cartan equation

\[
\sum_{n \geq 1} l_n(\varphi^{\otimes n}) = 0.
\] (4.11)

Note that in the case of a Lie algebra this equation takes the form

\[
d\varphi + \frac{1}{2}[\varphi, \varphi] = 0,
\]

where the factor \( \frac{1}{2} \) comes from using coinvariants instead of invariants. We will need some lemmas on these structures. The first lemma shows how a solution of the Maurer-Cartan equation can be interpreted as a perturbation of the \( L_\infty \)-structure. The second lemma describes how a solution of the Maurer-Cartan equation can be transported along an \( L_\infty \)-map. The third lemma shows that
in this situation there exists a morphism of the perturbed $L_\infty$-algebras. These results should be well known to the experts in the field. Our formulas differ a bit from the usual ones due to consequent identification of $S'(\mathfrak{g}[-1])$ with invariants (as opposed to coinvariants).

4.3 Convention In this section we work formally. That is, the formulas contain a priori infinite series and we do not worry about convergence. All results are understood to hold modulo convergence of the relevant series. A sufficient condition for convergence will be stated in the next section.

First we elaborate a bit on the exact definition of the operations in $L_\infty$-algebras. The differential $\partial$ on $(S'(\mathfrak{g}[-1]), \partial)$ can be described in terms of the maps $l_n^k : S^{n+k} \rightarrow S^{k+1}$, where

$$l_n^k = Sh_{1,k} \circ (l_n \otimes \text{id}^k).$$

Here $Sh_{p,q}$ denotes the sum of all $(p, q)$-shuffles permuting tensor factors. These serve to assure that the result is again invariant under the action of the symmetric group. The square-zero equation reads

$$\sum_{n+k=m} l_{k+1} \circ l_n^k = 0 \quad \text{for all } m. \quad (4.12)$$

An $L_\infty$-map $f$ satisfies

$$\sum_m \sum_{(n_1, \ldots, n_m)} l_m \circ (f_{n_1} \otimes \cdots \otimes f_{n_m}) = \sum_{k+p=n} f_{k+1} \circ l_p^k, \quad (4.13)$$

where the sum over $(n_1, \ldots, n_m)$ assumes $n_1 + \cdots + n_m = n$. The only difficulty in establishing the results below is in careful bookkeeping with respect to these maps. To do this in a satisfactory way we need some properties of the shuffle maps. The shuffles product is associative and graded commutative in the sense that

$$Sh_{p+q,r} \circ (Sh_{p,q} \otimes \text{id}^r) = Sh_{p,q+r} \circ (\text{id}^p \otimes Sh_{q,r}),$$

$$Sh_{p,q} = Sh_{q,p} \circ \tau,$$

where $\tau : V^\otimes p \otimes V^\otimes q \rightarrow V^\otimes q \otimes V^\otimes p$ is the twist of tensor factors. Note that the usual signs are hidden in the action of the shuffles.

The lemmas below are obtained by applying the definitions above and the properties of the shuffle product.

4.4 Lemma Let $(S'(\mathfrak{g}[-1]), \partial)$ be a $L_\infty$-algebra and let $\varphi \in \mathfrak{g}^1$ be a solution of the Maurer-Cartan equation. Then $(S'(\mathfrak{g}[-1]), \partial_\varphi)$ is an $L_\infty$-algebra with the differential given by

$$\tilde{l}_j(x_1, \ldots, x_j) = \sum_{p \geq 0} l_{j+p}(Sh_{p,j}(\varphi^\otimes p; x_1, \ldots, x_j)).$$
4.5 Lemma Let $f : (S'(g[-1]), \partial) \rightarrow (S'(h[-1]), \partial)$ be a map of $L_{\infty}$-algebras and let $\varphi$ be a solution of the Maurer-Cartan equation in $(S'(g[-1]), \partial)$. Then

$$\psi = \sum_n f_n(\varphi \otimes n)$$

is a solution of the Maurer-Cartan equation in $(S'(h[-1]), \partial)$.

4.6 Lemma Let $f : (S'(g[-1]), \partial) \rightarrow (S'(h[-1]), \partial)$ be a map of $L_{\infty}$-algebras. Let $\varphi$ be a solution of the Maurer-Cartan equation in $(S'(g[-1]), \partial)$ and let $\psi$ be the induced solution of the Maurer-Cartan equation on $(S'(h[-1]), \partial)$. Then the formula

$$\tilde{f}_n(x_1, \ldots, x_n) = \sum_{p \geq 0} f_{n+p}(S_{p,n}(\varphi \otimes p; x_1, \ldots, x_n)).$$

defines a morphism of $L_{\infty}$-algebras $\tilde{f} = f_\varphi : (S'(g[-1]), \partial_\varphi) \rightarrow (S'(h[-1]), \partial_\psi)$.

Convergence and Quasi Isomorphisms

The lemmas above need some extension. We study how the constructions above behave under quasi isomorphism. In order for this to work we need some mild assumptions.

For an $L_{\infty}$-algebra $(S'(g[-1]), \partial)$ we denote by $\partial$ the internal differential induced by $\partial$. If $\varphi$ is a solution of the Maurer-Cartan equation we denote by $\partial_\varphi$ the internal differential of the $\varphi$-perturbed $L_{\infty}$-algebra.

An $L_{\infty}$-algebra is called bigraded if we can write $g = \bigoplus_{n \in \mathbb{N}} g_n$ as dg vector spaces, and the differential $\partial$ preserves the induced grading on the symmetric coalgebra. If $g$ is a bigraded $L_{\infty}$-algebra, we can construct the completion of $g$ in the usual way. The completion $\hat{g} = \bigoplus_n g_n = \prod_n g_n$ is an $L_{\infty}$-algebra with respect to the completed tensor product. We call completions of bigraded $L_{\infty}$-algebras bigraded complete $L_{\infty}$-algebras. A morphism of bigraded complete $L_{\infty}$-algebras is assumed to preserve the second grading.

4.7 Remark The completeness in the definition above assures that all the series in the previous section are well defined for bigraded complete $L_{\infty}$-algebras with a solution $\varphi$ of the Maurer-Cartan equation in $\bigoplus_{m \geq 0} g_m$.

4.8 Proposition Let $(S'(g[-1]), \partial)$ and $(S'(h[-1]), \partial)$ be bigraded $L_{\infty}$-algebras. Let $\varphi \in \bigoplus_{m \geq 0} g_m$ be a solution to the Maurer-Cartan equation in the completed $L_{\infty}$ algebra. Let $f : (g, d) \rightarrow (h, d)$ be a quasi isomorphism of bigraded $L_{\infty}$-algebras. Then the map $f_\varphi$ of Lemma 4.6 is a quasi isomorphism of the completed $L_{\infty}$-algebras.
This is a direct corollary of the two lemmas below, since the filtration \( F^{\geq n} \hat{g} \) defined in the proof of Lemma 4.9 is exhaustive, weakly convergent and complete (cf. McCleary [18]). QED

4.9 Lemma Let \((S'(\mathfrak{g}[-1], \partial)) \) be a bigraded \( L_\infty \)-algebra and let \( \varphi \in \bigoplus_{m>0} \mathfrak{g}_m \) be a solution of the Maurer-Cartan equation in the completed \( L_\infty \)-algebra. Then there exists a spectral sequence

\[
E^{pq}_1(\hat{g}) = (F^pH(\hat{g}, \partial)/F^{p+1}H(\hat{g}, \partial))^{p+q},
\]

that converges to \( H(\hat{g}, \tilde{\partial}) \).

Proof The spectral sequence is the spectral sequence defined by the filtration of \((\hat{g}, \tilde{\partial})\) given by

\[
F^{\geq n} \hat{g} = \bigoplus_{m \geq n} \mathfrak{g}_m.
\]

The term of \( \tilde{\partial}(x) = \sum_k l_{k+1}(Sh_{k,1}(\varphi^{\otimes k}; x)) \) that involves \( \varphi^{\otimes k} \) increases the grading and thus contributes to the differential only in \( E_r \), where \( r \geq k \). Then \( E^{pq}_0 = (F^p\hat{g}/F^{p+1}\hat{g})^{p+q} \), and \( d_0 \) is the internal differential \( \partial \) of \( \hat{g} \). Since \( \mathfrak{g} \) is bigraded, the result follows. QED

4.10 Lemma Let \( f : (\mathfrak{g}, \partial) \to (\mathfrak{h}, \partial) \) be a quasi isomorphism of bigraded \( L_\infty \)-algebras. Let \( \varphi \in \bigoplus_{m>0} \mathfrak{g}_m \) be a solution to the Maurer-Cartan equation and let \( \psi = \sum_n f_n(\varphi^{\otimes n}) \). Then the morphism of \( L_\infty \)-algebras \( \tilde{f} : \hat{\mathfrak{g}}_\varphi \to \hat{\mathfrak{h}}_\psi \) induces a morphism of spectral sequences \( E^*_k(\hat{\mathfrak{g}}_\varphi) \to E^*_k(\hat{\mathfrak{h}}_\psi) \), which is an isomorphism for \( k \geq 1 \).

Proof Since the \( f \) is a morphism of bigraded \( L_\infty \)-algebras, the morphism \( \tilde{f}_1 : \hat{\mathfrak{g}}_\varphi \to \hat{\mathfrak{h}}_\psi \) commutes with the internal differentials \( \tilde{\partial} \) in \( \hat{\mathfrak{g}}_\varphi \) and \( \hat{\mathfrak{h}}_\psi \), and preserves the filtration. Therefore \( \tilde{f}_1 \) defines a morphism of spectral sequences. The part of \( \tilde{f} \) which has degree 0 with respect to the second grading is \( f_1 \) since \( \varphi \in \bigoplus_{m \geq 1} \mathfrak{g}_m \).

Thus the map on \( E_0 \) is given by \( f_1 \). Since \( f_1 \) is a quasi isomorphism, the explicit form of \( E_1 \) given in Lemma 4.9 shows that the map on \( E_1 \) is

\[
E_1(\hat{g}) = H(\hat{g}, \partial) \xrightarrow{H(f_1)} H(\hat{h}, \partial) = E_1(\hat{h}).
\]

This suffices to prove the result (cf. McCleary [18]). QED

5 Cochains and Deformations

This section defines for any reduced operad \( Q \) and each operad \( \varphi : Q \to P \) under \( Q \) the operad cohomology \( H_Q(P) \) of \( Q \) with coefficients in \( P \). We identify
$H_Q(P)$ with the cotangent cohomology defined by Markl \cite{17} using an $L_\infty$-structure on the cotangent complex. We show how the approach of Balavoine \cite{1}, and Konstevich-Soibelman \cite{12} to deformations of algebras over an operad fit in this framework.

**Maurer-Cartan for Convolution Operads**

The associated Lie algebra of an operad $R$ always has a second grading given by the arity as $(\mathfrak{B}(R))_m = R(m-1)$. Let $A$ be a cooperad and let $P$ be an operad. This section examines dg operad maps from the cobar construction $B^*A$ to $P$ in relation to the convolution operad $P^A$ and the Maurer-Cartan equation in the completion $\hat{\mathfrak{B}}S^P A$, of $\mathfrak{B}S^P A$.

5.1 **Theorem** Let $A$ be a cooperad and $P$ an operad. There is a one-one correspondence between elements $\varphi \in (\hat{\mathfrak{B}}S^P A)^1$ that satisfy the Maurer-Cartan equation $\frac{1}{2}[[\varphi, \varphi]] + d(\varphi) = 0$ and dg operad morphisms $\tilde{\varphi} : B^*(A) \rightarrow P$.

**Proof** Identify the collection morphisms from $A$ to $P$ of degree $n$ with the objects of $(\mathfrak{B}S^P A)^n$ (i.e. elements of internal degree $n$) using the section $\frac{1}{m} \sum_\sigma \sigma$. By the universal property of the free operad, an operad morphism is uniquely determined by a collection morphism $\tilde{\varphi} : A[1] \rightarrow P$ in $k$-gVect. To complete the proof we need to see that the condition $\frac{1}{2}[[\varphi, \varphi]] + d(\varphi) = 0$ in $\mathfrak{B}S^P A$ is equivalent to the induced map $\tilde{\varphi} : T(A[1]) \rightarrow P$ being compatible with the differentials. Of course, here $d(\varphi)$ denotes the differential induced by the convolution operad $P^A$ (cf. Lemma 2.1).

Denote the total differential on $B^*A$ induced by the cooperad structure by $\partial_A$ and the internal differential of $A$ by $d$. Let $\partial = \partial_A + d$ be the total differential. Compatibility of $\tilde{\varphi}$ with the differential can be stated as

$$\tilde{\varphi} \circ \partial_A = d \circ \tilde{\varphi} - \tilde{\varphi} \circ d,$$

where the internal differential of $P$ is denoted $d$ as well. The restriction of $\tilde{\varphi} \circ \partial_A$ to $A$ is by definition of the Lie bracket $\frac{1}{2}[[\varphi, \varphi]]$. QED

5.2 **Remark** As Martin Markl observed, Theorem 5.1 gives a Lie algebraic framework for the twisting functions of Getzler and Jones \cite{6}.

We can deduce an invariance property of these convolution operads from generalities on model categories and our previous results.

5.3 **Proposition** Let $A$ and $C$ be cooperads. Suppose that $B^*A$ and $B^*C$ are cofibrant replacements for the same operad $Q$, then there exists a pair of quasi isomorphisms of $L_\infty$-algebras

$$\hat{\mathfrak{B}}S^P A \rightarrow \hat{\mathfrak{B}}S^P C \quad \text{and} \quad \hat{\mathfrak{B}}S^P C \rightarrow \hat{\mathfrak{B}}S^P A.$$
that are inverse in cohomology.

Proof We first construct \( \eta : B^*A \to B^*C \) and \( \zeta : B^*C \to B^*A \) of dg operads such that \( H\eta \) and \( H\zeta \) are each others inverse. Both models are cofibrant in the model category of operads. In the solid square

\[
\begin{array}{c}
0 \\
B^*C \\
\downarrow \\
Q
\end{array}
\begin{array}{c}
\downarrow \\
B^*A \\
\nearrow \\
\nearrow
\end{array}
\]

the top and left arrows are cofibrations and the bottom and right arrows are trivial fibrations. Consequently, the two dotted diagonals \( \eta \) and \( \zeta \) exist. The maps \( \eta \) and \( \zeta \) induce homotopy homomorphisms \( \eta : P^C \to P^A \) and \( \zeta : P^A \to P^C \). These are quasi isomorphisms by the Künneth formula. The induced quasi isomorphisms of \( L_\infty \)-algebras (cf. Theorem 3.16) are the desired maps.

QED

Operad Cohomology

This section introduces a cohomology theory that we will show to contain the approaches of Markl [17], Balavoine [1], and Kontsevich and Soibelman [12] on the deformation of algebras over operads. The present construction is done purely at the operad level and thus generalizes to a dg Lie algebra controlling deformations of operad maps.

A straightforward application of Lemma 4.4 in the case of convolution operads as studied in the previous section yields the following:

5.4 Corollary Let \( A \) be a cooperad and let \( P \) be an operad. Let \( \hat{\varphi} : B^*A \to P \) be a map of dg operads. Then \( D = d + [\varphi, -] \) makes \( L_A(P) = \langle \oplus P^A, D \rangle \) a dg Lie algebra, where the correspondence between \( \varphi \) and \( \hat{\varphi} \) is the correspondence of Theorem 5.1.

5.5 Definition An operad \( Q \) is 1-reduced if \( Q(0) = 0 \) and \( Q(1) \) is a 1-dimensional space spanned by the identity. Let \( A \) be a cooperad, let \( Q \) and \( P \) be operads, such that \( B^*A \) is a cofibrant resolution of \( Q \) and \( Q \) is reduced. Let \( \varphi : Q \to P \) be a map of operads. The cohomology of \( Q \) with coefficients in \( P \) is the cohomology of the dg Lie algebra \( L_A(P) \). The cohomology of \( L_A(P) \) is called the operad cohomology of \( Q \) with coefficients in \( P \) and is denoted \( H_Q^*(P) \).

We now show that the cohomology \( H_Q^*(P) \) is independent of the cofibrant resolution \( A \). This justifies the notation that suppresses \( A \).
5.6 Theorem Let $Q$ be a 1-reduced operad, and let $\hat{\varphi} : Q \to P$ be an operad under $Q$. Any two complexes of the form $L_A(P)$ as in 5.5 are quasi isomorphic. Consequently, the operad cohomology $H_Q(P)$ is independent of the choice of a strict model $A$.

Proof Let $\pi_A : B^*A \to Q$ and $\pi_C : B^*C \to Q$ be two strict cofibrant replacements for $Q$. Consider the convolution operads $P_C$ and $P_A$. These are quasi isomorphic by Proposition 5.3. Let $\varphi_A$ (resp. $\varphi_C$) be the solution of the Maurer-Cartan equation in the completed $\bigoplus S P_A$ (resp. $\bigoplus S P_C$) defined as in Theorem 5.1 by the composition of $\hat{\varphi}$ with $\pi_A$ (resp. $\pi_C$). Since $Q$ is reduced, these solutions of the Maurer-Cartan equation strictly increase the arity-grading. The result now follows from Lemma 4.6 and Proposition 4.10. QED

5.7 Example It might be good to point out that we recover well known results an special cases of the theorem above. For example, let $Q$ be a quadratic Koszul (!) operad (cf. Ginzburg and Kapranov [8]), and let $P = \text{End}_V$, the endomorphism operad of a $Q$-algebra $V$. Using the strict model $B^*A = B^*(Q^\perp)$ in Definition 5.5, the cohomology $H_Q(P)$ is easily seen to be the cohomology of the $P$-algebra $V$ (cf. Balavoine [1], Ginzburg and Kapranov [8]).

5.8 Example If $Q = \text{Ass}$ the operad of associative algebras, and $P = \text{End}_V$ the endomorphism operad of an associative algebra $V$ Example 5.7 recovers the Hochschild cohomology complex with coefficients in $V$. If $Q = \text{Ass}$ and $P$ is any operad with multiplication, $H_Q(P)$ is the Hochschild cohomology of the operad $P$.

5.9 Example If $Q = \text{Lie}$ the operad of Lie algebras, and $P = \text{End}_V$ the endomorphism operad of a Lie algebra $V$, Example 5.7 recovers the (Chevalley-Eilenberg) Lie algebra cohomology complex with coefficients in $V$.

**Cotangent Cohomology**

This section describes how we can use non-strict (and in particular minimal) models to compute the operad cohomology $H_Q(P)$. The previous section only shows how to do this if the minimal model is strict. The advantage of minimal models is of course that they tend to be small. Moreover, the vanishing internal differential simplifies computations. We first construct a non-strict analogue of the convolution operad. The proof is the operad-analogue of the $A_\infty$-algebra structure on the tensor product of an $A_\infty$-algebra with an associative algebra.

5.10 Lemma Let $A$ be a cooperad up to homotopy, and let $P$ be an operad. Then the collection $P^A$ has a natural structure of an operad up to homotopy.
5.11 Lemma Let $A$ be a cooperad up to homotopy, and let $P$ be an operad. There is a 1-1 correspondence between operad maps $\varphi : (T(A[1]), \partial) \rightarrow P$ and solutions of the Maurer-Cartan equation in the $L_\infty$-algebra $\hat{\otimes}_S P^A$.

Proof Same argument as in the proof of Theorem 5.1. QED

Let $Q$ be a reduced operad, let $A$ be a cooperad up to homotopy such that $\pi_A : T(A[-1]), \partial) \rightarrow Q$ is a quasi isomorphism, and let $\varphi : Q \rightarrow P$ be a morphism of operads. We use the above lemma to define the complex $L_A(P)$ as the $\varphi$-perturbed $L_\infty$-algebra with respect to the perturbed internal differential.

5.12 Theorem Let $Q$ be a reduced operad, and let $A$ be a cooperad up to homotopy such that $\pi_A : T(A[-1]), \partial) \rightarrow Q$ is a quasi isomorphism, and let $\varphi : Q \rightarrow P$ be an operad under $Q$. Then $L_A(P)$ is quasi isomorphic to any complex as in Definition 5.5.

Proof Let $C$ be a cooperad such that $B^*C$ is a cofibrant replacement for $Q$. We construct an $L_\infty$-morphism $\hat{\otimes}_S P^A \rightarrow \hat{\otimes}_S P^C$ and show that the the induced map in spectral sequences is an isomorphism in $E_k$ for $k \geq 1$.

Let $(T(A[-1]), \partial)$ be an arbitrary cofibrant replacement for $Q$. Then the solid diagram

\[
\begin{array}{ccc}
0 & \rightarrow & T(C[-1]) \\
\downarrow & & \downarrow \pi_C \\
T(A[-1]) & \xrightarrow{\pi_A} & Q,
\end{array}
\]

where left arrow is a cofibration and the right arrow is a acyclic fibration admits a lift (dotted arrow) compatible with $\partial$. Since bottom and right arrow are weak equivalences, so is the lift. This defines a quasi isomorphism of cooperads up to homotopy from $A$ to $C$ and thus a quasi isomorphism of $L_\infty$-algebras $L_C(P) \rightarrow L_A(P)$.

Since the differential on $L_A(P)$ is defined as the perturbed differential with respect to the solution of the Maurer-Cartan equation corresponding to the
composition \((T'(A[-1]), \partial) \to Q \to P\), we can again apply Lemma 4.6 and Proposition 4.10. QED

Remember that the cotangent cohomology \(H^\ast(Q; P)\) defined by Markl [17] is computed using a special kind of minimal model, a so called bigraded minimal model (do not confuse this notion with the bigraded \(L_\infty\)-algebras we defined before). Let \(Q\) be a reduced operad, let \(\varphi : Q \to P\) be an operad under \(Q\), and let \(T(M[-1])\) be a bigraded minimal model for \(Q\). Then the cotangent complex can be written as \((\oplus S^P M, d_\varphi)\), where \(d_\varphi\) is the internal differential of the \(\varphi \circ \pi_M\)-perturbed \(L_\infty\)-algebra \(L_A(P)\).

5.13 Corollary Let \(Q\) be a reduced operad, and let \(\varphi : Q \to P\) be an operad under \(Q\). Then

\[
H^p_Q(P) = \bigoplus_q H^{pq}(Q; P).
\]

Proof Apply Theorem 5.12 to \(A = M\), a bigraded minimal model. QED

Let \(Q\) be a 1-reduced operad. An operad under \(Q\) is an operad \(P\) together with an operad map \(\varphi : Q \to P\). This section gives an elegant purely operadic definition of an \(L_\infty\)-algebra \(L_A(P)\) controlling deformations of operads under \(Q\) (depending on a cofibrant resolution \(B^*A\)). That is, we have a deformation theory of operad maps. Deformations of the map \(\varphi : Q \to P\) correspond to solutions of the Maurer Cartan equation. There is a natural equivalence relation, given by a suitable notion of homotopy for coalgebras (cf. eg. Schlessinger and Stasheff [20]). The quotient of the space of solutions by this equivalence relation is invariant under quasi isomorphism. Therefore the quotient space is independant of the choice of model for \(Q\) in the construction of the \(L_\infty\)-algebra.

We call this quotient space the space of deformations of the map \(\varphi\). We will not be explicit about the equivalence relation here, though one might get some intuition from the next section.

Due to Corollary 5.13, this is a more precise statement of Markl’s observation: “The cotangent cohomology is the best possible cohomology controlling deformations of operads.”

In Kontsevich’s terminology [11], the \(L_\infty\)-algebra \(L_A(P)\), computing the operad cohomology \(H^\ast_Q(P)\), is a formal pointed manifold with an odd square-zero vector field controlling deformations of the operad map \(\varphi : Q \to P\). To phrase this in terms of formal deformations, pass to \(L_A(P) \otimes k[[t]]\), and consider solutions of Maurer-Cartan of strictly positive degree in \(t\).

5.14 Example In the situation described above, let \((T(A[-1]), \partial)\) be a resolution of \(Q\) and let \(P = \text{End}_V\). Then the perturbed \(L_\infty\)-algebra \(L_A(P)\) of
Theorem 5.12 defines an $L_\infty$-algebra. This defines the $L_\infty$-algebra controlling deformations of $Q$-algebras described by Kontsevich and Soibelman [12] by an elegant, purely operadic construction.

**Quadratic Operads**

Let $Q$ be an operad and $\varphi : Q \to P$ an operad map. Let $T^i(A[-1], \partial)$ be a cofibrant replacement for $Q$. In Kontsevich’s terminology the $L_\infty$-algebra $L_A(P)$ as in Theorem 5.12 is a formal pointed manifold with an odd square-zero vector field controlling deformations of the operad map $\varphi : Q \to P$. This is illustrated by Theorem 5.1, Corollary 5.13 and Example 5.14. This section intends to show that in the case of quadratic operads our point of view explains a known result of Balavoine [1].

Let $Q$ be a quadratic operad (cf. Ginzburg and Kapranov [8]) concentrated in degree 0, and let $\varphi : Q \to P$ be an operad under $Q$. Let $Q^\perp$ be the dual cooperad of $Q$ (cf. Getzler and Jones [6]). Then we can form the operad $PQ^\perp$. Let $\varphi : Q \to P$ be an operad under $Q$. We can use the natural inclusion $Q^\perp \to B^*Q$ to define a map $\varphi^\perp : B^*Q^\perp \to P$ of dg operads by

$$\varphi^\perp : B^*Q^\perp \to B^*BQ \to Q \to P.$$ (5.14)

By slight abuse of notation we denote the $\varphi^\perp$-perturbed $L_\infty$-algebra $\hat{\oplus}PQ^\perp$ by $L_{Q^\perp}(P)$. Balavoine [1] shows that the cohomology in degrees 1 and 2 of $L_{Q^\perp}$ controls formal deformations in the sense of Gerstenhaber [5]. The following result shows that Balavoine’s construction gives a nice approximation of operad cohomology in low degrees, which explains the result.

5.15 PROPOSITION Let $Q$ be a quadratic operad concentrated in degree 0. Let $\varphi : Q \to P$ be an operad under $Q$ and define $L_{Q^\perp}(P)$ as above. The map of complexes $L_{BQ}(P) \to L_{Q^\perp}(P)$ induced by $Q^\perp \to BQ$ induces an isomorphism

$$H^k_{Q^\perp}(P) \to H^k(L_{Q^\perp}(P)) \quad (k \leq 2).$$

Proof By definition of $Q^\perp$, the map $L_{BQ}(P) \to L_{Q^\perp}(P)$ (cf. Theorem 5.13) is an quasi isomorphism in arity $\leq 3$. The result now follows from $Q^\perp(n + 1) = (Q^\perp)^{-n}$ (cf. Ginzburg and Kapranov [8], Getzler and Jones [6]) together with Lemma 4.6 and Proposition 4.10 restricted to degree $\leq 2$. QED

5.16 REMARK Thus we have set up a deformation theory of operad maps and shown how this relates to some known approaches to deformation of algebras over an operad (Examples 5.7 and 6.14 and Proposition 5.15), and cotangent cohomology (Corollary 5.13).
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