Bosonic color-flavor transformation for the special unitary group

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Abstract

We extend Zirnbauer’s color-flavor transformation in the bosonic sector to the color group SU\((N_c)\). Because the flavor group U\((N_b, N_b)\) is non-compact, the algebraic method by which the original color-flavor transformation was derived leads to a useful result only for \(2N_b \leq N_c\). Using the character expansion method, we obtain a different form of the transformation in the extended range \(N_b \leq N_c\). This result can also be used for the color group U\((N_c)\). The integrals to which the transformation can be applied are of relevance for the recently proposed boson-induced lattice gauge theory.

1 Introduction

In 1996, Zirnbauer [1] invented a generalized Hubbard-Stratonovitch transformation which trades an integration over a “color” gauge group for an integration over a certain supersymmetric coset space, or “flavor” space. Although the transformation was originally derived to study disordered systems in condensed matter physics, the terminology comes from lattice gauge theory because the integral over the gauge group to which the color-flavor transformation is applied is precisely of the form of a one-link integral in lattice gauge theory at infinite coupling.

The fields that appear in the transformation carry two types of indices that will be referred to as color and flavor indices. The number of colors is denoted by \(N_c\), and the numbers of bosonic and fermionic flavors are denoted by \(N_b\) and \(N_f\), respectively. Zirnbauer derived versions of the color-flavor transformation for the three color groups U\((N_c)\), O\((N_c)\), and Sp\((2N_c)\) [1]. In his original work, the flavor space contained an equal number of bosonic and fermionic degrees of freedom, but it is possible to relax this constraint [2]. Convergence
requirements then place an upper bound on the difference between the number of bosons and fermions. For the case of U($N_c$), this bound is given by $2(N_b - N_f) \leq N_c$.

The color-flavor transformation has been used in a number of physical applications, e.g., in the derivation of a field theory for the random flux model by Altland and Simons [3] and in the construction of chiral Lagrangians for lattice gauge theories by Nagao and Nishigaki [4]. In the latter paper, the calculations were done for the above-mentioned color groups. However, in quantum chromodynamics (QCD) the color group is SU(3). To be able to apply the color-flavor transformation to this very important physical case, a variant of the transformation for the special unitary group needed to be derived. Following earlier work by Budczies and Shnir [5,6], this was done in Refs. [7,8] in the fermionic sector, i.e. for $N_b = 0$. The result was then applied to lattice QCD in Refs. [8,9].

As mentioned above, the color-flavor transformation can only be applied to the one-link integral of lattice gauge theory if the gauge coupling is infinite. Fortunately, it is possible to go beyond the infinite-coupling limit. A gauge action (Yang-Mills action in the continuum or Wilson action on the lattice) can be induced by coupling auxiliary fields to the gauge field (still at infinite coupling) and integrating out these extra fields. This idea, which is known as “induced QCD”, has been considered in various forms in the literature [10–15]. For example, the job can be done by a number $N_h$ of heavy auxiliary fermions with common mass $m_h$ in the combined limit $N_h \to \infty$, $m_h \to \infty$ such that the ratio $N_h/m_h^4$ is constant, and this constant can then be related to the strength of the induced gauge coupling.

So far, the color-flavor transformation for SU($N_c$) has only been derived in the fermionic sector, for three reasons: (a) physical quarks are fermions, (b) a gauge action can be induced by auxiliary fermions alone, and (c) the calculation is somewhat easier for fermions than for bosons. There seemed to be no special need for a bosonic variant of the transformation until Budczies and Zirnbauer suggested a new method to induce the gauge action using auxiliary bosons [16]. Their approach has the advantage of requiring only a small number (essentially equal to $N_c$) of auxiliary fields. However, to be able to use their method in lattice QCD, one requires an SU($N_c$)-variant of the color-flavor transformation that accommodates both fermions (the physical quarks) and bosons (to induce the gauge action).

As a first step towards this goal, we found it useful to consider the purely bosonic case with $N_f = 0$ for which we will present results in this paper. Our results can be applied to study a boson-induced gauge theory without physical quarks. The concentration on the bosonic sector allows us to separate the complications due to the fact that the fields are bosonic from those due to
the supersymmetric framework. This is the main motivation for the present paper. The supersymmetric case will be addressed in a separate publication.

The convergence requirement mentioned above is irrelevant for the purely fermionic case in which \( N_b = 0 \) since the inequality is always satisfied. However, it becomes relevant for the purely bosonic case. We obtain a “standard” form of the color-flavor transformation if the condition \( 2N_b \leq N_c \) is satisfied. In the extended range \( N_b \leq N_c \), we derive a different form of the transformation. Interestingly, for \( N_b < N_c \) our results for SU\((N_c)\) are identical to those for gauge group U\((N_c)\), and we therefore obtain new results for U\((N_c)\) in the range \( N_b < N_c < 2N_b \). For \( N_b > N_c \) we have not been able to simplify our formal result to be useful in applications.

This paper is organized as follows. We first state our results in Sec. 2. In Sec. 3, we use the algebraic method of Refs. [1,7] to derive our general result for the bosonic color-flavor transformation. In Sec. 4, we use a different approach, the character expansion method, to derive an alternative form of the bosonic color-flavor transformation. We close with a brief discussion of possible applications and open problems. The appendix contains derivations of several intermediate results as well as a number of examples for the results from both approaches.

2 Statement of results

Let \( \psi^i_a, \bar{\psi}^i_a, \varphi^i_a, \) and \( \bar{\varphi}^i_a \) be complex bosonic variables that carry a color index (superscript \( i \) running from 1 to \( N_c \)) and a flavor index (subscript \( a \) running from 1 to \( N_b \)). The bar denotes complex conjugation. Summation over repeated indices is implied here and throughout the paper unless indicated otherwise. Using the algebraic method of Refs. [1,7], we obtain

\[
\int_{\text{SU}(N_c)} dU \exp \left( \bar{\psi}^i_a U^{ij} \psi^j_a + \varphi^i_a U^{ij} \varphi^j_a \right) \tag{2.1}
\]

\[
= \int_{|ZZ^\dagger| \leq 1} \frac{dZ dZ^\dagger}{\det^{2N_b-N_c}(1 - ZZ^\dagger)} \exp \left( \bar{\psi}^i_a Z_{ab} \varphi^j_b + \varphi^i_a Z^\dagger_{ab} \psi^j_b \right) \sum_{Q=0}^\infty \chi_Q ,
\]

where

\[
\chi_0 = C_0 , \quad \chi_{Q>0} = C_Q \left( \det^Q M + \det^Q N \right) ,
\]

the \( N_c \times N_c \) matrices \( M \) and \( N \) are defined by \( M^{ij} = \bar{\psi}^i_a (1 - ZZ^\dagger)_{ab} \psi^j_b \) and \( N^{ij} = \varphi^i_a (1 - Z^\dagger Z)_{ab} \varphi^j_b \), and the coefficients \( C_Q \) are computed in App. B.

The integration on the left-hand side (LHS) of Eq. (2.1) is over SU\((N_c)\) matrices \( U \) distributed according to the Haar measure \( dU \), normalized such that the group volume is unity. The integration on the right-hand side (RHS) of that equation is over complex \( N_b \times N_b \) matrices \( Z \), with the restriction that all
eigenvalues of \( ZZ^\dagger \) are less than or equal to one. These matrices parameterize the non-compact coset space \( U(N_b, N_b)/[U(N_b) \times U(N_b)] \). The corresponding invariant integration measure is given by \[17\]

\[ d\mu(Z, Z^\dagger) = \frac{dZdZ^\dagger}{\det(1 - ZZ^\dagger)^{2N_b}} \quad \text{with} \quad dZdZ^\dagger = \prod_{a,b=1}^{N_b} d\text{Re}Z_{ab}d\text{Im}Z_{ab}. \quad (2.3) \]

Note that the \( N_c \times N_c \) matrices \( \mathcal{M} \) and \( \mathcal{N} \) can be thought of as products of three matrices of dimension \( N_c \times N_b, N_b \times N_b, \) and \( N_b \times N_c, \) respectively. The resulting matrix is of full rank only if \( N_b \geq N_c. \) For \( N_b < N_c, \) we therefore have \( \det \mathcal{M} = \det \mathcal{N} = 0 \) \[18\], and the transformation simplifies to

\[
\int_{SU(N_c)} dU \exp \left( \bar{\psi}_a^i U^i j \psi_j^a + \bar{\varphi}_a^i U^{ij} \varphi_j^a \right) = C_0 \int_{|ZZ^\dagger| \leq 1} \frac{dZdZ^\dagger}{\det^{2N_b-N_c}(1 - ZZ^\dagger)} \exp \left( \bar{\psi}_a^i Z_{ab} \varphi_b^j + \bar{\varphi}_a^i Z_{ab}^\dagger \psi_b^j \right) \quad (2.4)
\]

with a constant \( C_0 \) given in Eq. (B.5). This agrees with the result for the color group \( U(N_c) \) in Ref. [2].

Equation (2.1) looks similar to the result for the fermionic sector presented in Ref. [7]. There are two major differences, however. First, due to the nilpotency of Grassmann variables the sum over \( Q \) in Eq. (2.1) only went up to \( N_f \) in the fermionic case, whereas it extends to infinity now. Second, the invariant measure of the coset space \( U(N_b, N_b)/[U(N_b) \times U(N_b)] \) in Eq. (2.3) diverges at the boundary, giving rise to convergence issues which we discuss now.

For \( 2N_b \leq N_c, \) the divergence of the measure is canceled by the factor of \( \det^{N_c}(1 - ZZ^\dagger) \) in the integrand of Eq. (2.1). In this case the result (2.4) applies and is free from divergences. For \( N_b < N_c < 2N_b, \) the integral over \( Z \) in Eq. (2.4) diverges. For \( N_b = N_c, \) the integral over \( Z \) in Eq. (2.1) diverges for \( Q < N_b, \) whereas for \( N_b > N_c, \) it diverges for all \( Q. \) Of course, the final result for the RHS of Eqs. (2.1) and (2.4) must be finite, so whatever divergence arises from the integration over \( Z \) will be canceled by a similar divergence in the integral for the corresponding (inverse) constant \( C_Q^{-1}, \) see Eq. (3.49). A finite ratio could in principle be obtained by a limiting procedure, but it is not clear to us whether this would lead to a simple final result.

Instead, we have used the character expansion method [19–21] to derive a different form of the color-flavor transformation and obtain for \( N_b < N_c \)

\[
\int_{SU(N_c)} dU \exp \left( \bar{\psi}_a^i U^i j \psi_j^a + \bar{\varphi}_a^i U^{ij} \varphi_j^a \right) = \prod_{n=0}^{N_c-N_b-1} \frac{(N_b + n)!}{n!} \int_{U(N_b)} dV \det^{N_b-N_c}(VB) \exp \left( \bar{\psi}_a^i V_{ab} \varphi_b^j + \bar{\varphi}_a^i V_{ab}^\dagger \psi_b^j \right), \quad (2.5)
\]
where the $N_b \times N_b$ matrix $B$ is defined by $B_{ab} = \varphi_a^i \bar{\psi}_b^i$. Note that the integration on the RHS is over the unitary group with the normalized Haar measure $dV$. Equation (2.5) is also valid if the integration on the LHS is over the color group $U(N_c)$ and, to the best of our knowledge, represents a new result for this case.

The corresponding result for $N_b = N_c$ reads

$$
\int_{SU(N_c)} dU \exp \left( \bar{\psi}_a^i U_{ij}^a \psi_b^j + \varphi_a^i U_{ij}^a \bar{\varphi}_b^j \right)
$$

$$
= \sum_{Q=0}^{\infty} \tilde{\chi}_Q \int_{SU(N_b)} dV \text{det}^{-Q}(V B) \exp \left( \bar{\psi}_a^i V_{ab}^i \varphi_b^j + \varphi_a^i V_{ab}^j \bar{\psi}_b^j \right)
$$

(2.6)

with

$$
\tilde{\chi}_0 = 1, \quad \tilde{\chi}_{Q>0} = \text{det}^Q M + \text{det}^Q N,
$$

(2.7)

the matrix $B$ as defined above, and $N_c \times N_c$ matrices $M$ and $N$ defined by $M^{ij} = \psi_a^i \bar{\psi}_a^j$ and $N^{ij} = \varphi_a^i \bar{\varphi}_a^j$. Note that we are not allowed to change the order of summation and integration in Eq. (2.6), see Sec. 4.3. If the integration on the LHS of Eq. (2.6) is over $U(N_c)$, only the $Q = 0$ term contributes on the RHS, see Eq. (4.20). The integral over $U(N_b)$ in Eqs. (2.5) and (2.6) can be done analytically [22–24], resulting in a determinant involving modified Bessel functions, but we do not display this result here because Eqs. (2.5) and (2.6) are to be viewed as transformations.

As mentioned in the introduction, for $N_b > N_c$ we have not been able to obtain a simple form of the color-flavor transformation in which the divergences have been eliminated.

3 Bosonic color-flavor transformation: algebraic method

The basic idea of the algebraic approach to the color-flavor transformation is to construct two projection operators onto the subspace of Fock space (to be defined below) which is invariant under the action of the color group $SU(N_c)$. One such projector is implemented by integrating over the color group. The other one is obtained by integrating over a certain coset of the flavor group $U(N_b, N_b)$. Identification of the two projection operations then leads to Eq. (2.1). In this section, we shall use this algebraic approach to derive the bosonic color-flavor transformation. We closely follow Refs. [1,7,8] whenever possible.
3.1 Fock space, Lie algebras, and Lie groups

We introduce two sets of bosonic creation and annihilation operators \{\bar{c}^i_a, c^i_a\} and \{\bar{d}^i_a, d^i_a\}, where \(i = 1, \ldots, N_c\) and \(a = 1, \ldots, N_b\). As mentioned above, we shall refer to the upper index as “color” and to the lower index as “flavor”. The Fock vacuum \(|0\rangle\) is defined by the requirement that \(\bar{c}^i_a|0\rangle = d^i_a|0\rangle = 0\) for all combinations of \(i\) and \(a\), and the Fock space is generated by acting on \(|0\rangle\) with the \(\bar{c}^i_a\) and \(\bar{d}^i_a\). In the following, the two sets of particles created by the \(\bar{c}^i_a\) and \(\bar{d}^i_a\) will be referred to as particles and antiparticles, respectively.

For simplicity of notation, we also introduce the unified operators \{\bar{c}^i_A, c^i_A\} defined by

\[
\begin{align*}
\bar{c}^i_A & = \begin{cases} 
  \bar{c}^i_A & \text{for } 1 \leq A \leq N_b, \\
  \bar{d}^i_{A-N_b} & \text{for } N_b < A \leq 2N_b,
\end{cases} \\
\bar{c}^i_A & = \begin{cases} 
  \bar{c}^i_A & \text{for } 1 \leq A \leq N_b, \\
  -\bar{d}^i_{A-N_b} & \text{for } N_b < A \leq 2N_b.
\end{cases}
\end{align*}
\]  

(3.1)

They satisfy the usual commutation relations for bosonic operators,

\[
[c^i_A, \bar{c}^j_B] = \delta^{ij} \delta_{AB}.
\]  

(3.2)

Next we define operators \(E^{ij}_{AB} = \bar{c}^i_A c^j_B\). Simple algebra reveals that they satisfy the commutation relations

\[
[E^{ij}_{AB}, E^{k\ell}_{CD}] = \delta_{BC} \delta^{k\ell} E^{ij}_{AD} - \delta_{AD} \delta^{k\ell} E^{ij}_{CB},
\]  

(3.3)

and hence they are generators of the Lie algebra \(\text{gl}(2N_cN_b)\).

The Lie algebra \(\text{gl}(2N_cN_b)\) has two commuting subalgebras that are important for our proof, namely \(\text{gl}(2N_b)\), which is generated by the color-singlet operators

\[
\left\{ G_{AB} \equiv \sum_{i=1}^{N_c} E^{ii}_{AB} \right\},
\]  

(3.4)

and \(\text{sl}(N_c)\), which is generated by the flavor-singlet operators

\[
\left\{ E^{ij} \equiv \sum_{A=1}^{2N_b} E^{ij}_{AA}; \ i \neq j \right\} \quad \text{and} \quad \left\{ H^i \equiv \sum_{A=1}^{2N_b} E^{ii}_{AA} - \frac{1}{N_c} \sum_{j=1}^{N_c} \sum_{A=1}^{2N_b} E^{jj}_{AA}; \ i = 1, \ldots, N_c \right\}.
\]  

(3.5)

Note that only \(N_c - 1\) of the generators \(H^i\) are linearly independent.
The action of the group $GL(2N_cN_b)$ and its subgroups on the Fock space is defined by exponential mapping, i.e. for all $g \in GL(2N_cN_b)$ we define a map $g \mapsto T_g$ from group elements to operators by \[ T_g = \exp \left[ \bar{c}^i_A (\ln g)^{ij}_{AB} c^j_B \right]. \] (3.6)

Following Zirnbauer [1], one can show that the map $g \mapsto T_g$ is well-defined and a homomorphism of $GL(2N_cN_b)$,

\[ T_g T_h = T_{gh}. \] (3.7)

Therefore it furnishes a (reducible) representation of $GL(2N_cN_b)$.

In the following, we will consider the action of the subgroups $SU(N_c)$ (the color group) and $U(N_b, N_b)$ (the flavor group) of $GL(2N_cN_b)$ on the Fock space. The corresponding subalgebras $sl(N_c)$ and $gl(2N_b)$ have been given above. What are the reasons to single out these two subgroups? For the color group the reason is simple: The integration on the LHS of Eq. (2.1) is over $SU(N_c)$. For the flavor group, the choice of the non-compact subgroup $U(N_b, N_b)$ is not immediately obvious at this point but will become clear as we proceed. We shall see below that the color-neutral sector of Fock space is non-compact in the bosonic case, as opposed to the fermionic case in which it was compact. Attempts to work with the compact subgroup $U(2N_b)$ do not lead to useful results. Also, when Eq. (2.1) is used in applications, one wants the resulting integrals over the bosonic variables to converge, and this requirement necessitates a non-compact integration domain on the RHS of that equation [25–27,1].

Under the action of the subgroups $SU(N_c)$ and $U(N_b, N_b)$, the operators $c^i_A$ and $\bar{c}^i_A$ transform as follows,

\[ g \in SU(N_c) : \quad T_g c^i_A T_g^{-1} = (g^{-1})^{ij} c^j_A, \quad T_g \bar{c}^i_A T_g^{-1} = \bar{c}^j_B g^{ji}, \] (3.8)

\[ g \in U(N_b, N_b) : \quad T_g c^i_A T_g^{-1} = g_{AB} c^i_B, \quad T_g \bar{c}^i_A T_g^{-1} = \bar{c}^j_B g_{BA}. \] (3.9)

which can be shown using the Baker-Campbell-Hausdorff formula.

3.2 Bose coherent states and projection onto the color-neutral sector

We call a vector $|\mathcal{N}\rangle$ in the Fock space color-neutral if it is invariant under $SU(N_c)$ transformations, i.e. $T_U |\mathcal{N}\rangle = |\mathcal{N}\rangle$ for all $U \in SU(N_c)$. The subspace of Fock space spanned by these invariant vectors is called the color-neutral subspace or sector.

The following argument closely parallels Ref. [8]. With the complex bosonic variables $\psi^i_a$, $\bar{\psi}^i_a$, $\varphi^i_a$, and $\bar{\varphi}^i_a$ introduced in Sec. 2, Bose coherent states are
defined as
\[ |\Psi\rangle = \exp(\bar{c}^i \psi^i_a + \bar{d}^i \varphi^i_a) |0\rangle, \quad \langle \Psi | = \langle 0 | \exp(\bar{\psi}^i \bar{c}^i_a + \bar{\varphi}^i \bar{d}^i_a). \tag{3.10} \]
They span the entire Fock space (or its dual). Using Eq. (3.8) we find
\[ \langle \Psi | T_U | \Psi \rangle = \exp(\bar{\psi}^i U^i_j \psi^j_a + \varphi^i U^{i\dagger} \varphi^j_a). \tag{3.11} \]
The LHS of Eq. (2.1) can therefore be written as
\[ Z = \int_{SU(N_c)} dU \exp(\bar{\psi}^i U^i_j \psi^j_a + \varphi^i U^{i\dagger} \varphi^j_a) = \langle \Psi | P | \Psi \rangle, \tag{3.12} \]
where we have introduced the operator \( P \) defined by
\[ P = \int_{SU(N_c)} dU \; T_U. \tag{3.13} \]
This operator annihilates all states that are not color-neutral, while leaving color-neutral states invariant (recall that the volume of \( SU(N_c) \) is unity). Therefore, it is a projector onto the color-neutral sector. As advertised above, it is one possible representation of such a projector, and we will now derive an alternative form.

### 3.3 Action of the flavor group in the color-neutral sector

By definition, color-neutral vectors \( |\mathcal{N}\rangle \) are annihilated by all generators of \( \text{sl}(N_c) \), i.e. \( \mathcal{E}^i |\mathcal{N}\rangle = 0 \) and \( \mathcal{H}^i |\mathcal{N}\rangle = 0 \). Using Eq. (3.5) and the commutation relations (3.2), this requirement leads to
\[ \left( \sum_{a=1}^{N_b} \bar{c}^i_a c^j_a - \sum_{a=1}^{N_b} \bar{d}^i_a d^j_a \right) |\mathcal{N}\rangle = \delta^{ij} Q |\mathcal{N}\rangle, \tag{3.14} \]
where \( Q \) is an integer. Clearly, the color-neutral sector contains the vacuum. For \( i = j \), the operator on the LHS of Eq. (3.14) counts the difference in the number of particles and antiparticles for each color, the difference being equal to \( Q \).

The color-neutral sector can be generated by acting on the vacuum state with three types of operators,
\[
\begin{align*}
\text{type-1a:} & \quad \bar{c}^i_a \bar{d}^j_b, \\
\text{type-2a:} & \quad \epsilon_{i_1 \ldots i_{N_c}} \bar{c}^{i_1}_{a_1} \bar{c}^{i_2}_{a_2} \cdots \bar{c}^{i_{N_c}}_{a_{N_c}}, \\
\text{type-2b:} & \quad \epsilon_{i_1 \ldots i_{N_c}} \bar{d}^{i_1}_{b_1} \bar{d}^{i_2}_{b_2} \cdots \bar{d}^{i_{N_c}}_{b_{N_c}},
\end{align*}
\tag{3.15}
\]
where $\epsilon$ denotes the totally antisymmetric Levi-Civita symbol which ensures that the resulting state is invariant under SU($N_c$) transformations. In addition, the following types of operators make transformations in the color-neutral sector,

\begin{align}
\text{type-1b: } & \ c^i_a d^i_b, \\
\text{type-1c: } & \ c^i_a c^i_b, \\
\text{type-1d: } & \ d^i_a d^i_b.
\end{align}

(3.16)

When acting on a color-neutral state, type-1 operators do not change the $Q$-value of that state, whereas type-2a (type-2b) operators increase (decrease) the $Q$-value by one. Note, however, that for $N_b < N_c$ the type-2 operators do not exist, which makes it impossible to generate a vector in a non-zero $Q$-sector. The range of $Q$ is therefore given by

\begin{equation}
Q = \begin{cases}
-\infty, \ldots, \infty & \text{for } N_b \geq N_c, \\
0 & \text{for } N_b < N_c.
\end{cases}
\end{equation}

(3.17)

(In the case of $N_b < N_c$, we are back to the bosonic color-flavor transformation for the group U($N_c$) [2]. The Lie algebra of U($N_c$) has an extra U(1) generator, and by requiring invariance under U($N_c$), this U(1) generator eliminates all non-zero $Q$-sectors.)

The action of the flavor group on the Fock space is defined by Eq. (3.6) with $g \in U(N_b, N_b)$. We now choose the color-neutral sector to be the carrier space of this representation. The type-1 operators are the generators of the flavor group and do not change the $Q$-value of a given state. Therefore, under the action of the flavor group, the color-neutral sector decomposes into invariant subspaces labeled by $Q$, which we shall call “$Q$-sectors”. As mentioned above, a $Q$-sector contains $Q$ more particles than antiparticles for each color.

We now proof that the flavor group acts irreducibly in a given $Q$-sector [1]. For this we need to show that from any given state in the $Q$-sector we can reach any other state by the action of the flavor group. Equivalently, we can single out a particular state $|\psi_Q\rangle$, defined by

\begin{align}
|\psi_{Q>0}\rangle &= (\epsilon_{i_1 \cdots i_{N_c}} c^{i_1} c^{i_2} \cdots c^{i_{N_c}}) Q |0\rangle, \\
|\psi_{Q=0}\rangle &= |0\rangle, \\
|\psi_{Q<0}\rangle &= (\epsilon_{i_1 \cdots i_{N_c}} d^{i_1} d^{i_2} \cdots d^{i_{N_c}}) Q |0\rangle,
\end{align}

(3.18)

and show that (i) starting from this state, we can reach any other state, and (ii) from that state we can return to $|\psi_Q\rangle$, using type-1 operators only.

An arbitrary vector in a given $Q$-sector, which we should be able to reach from $|\psi_Q\rangle$, is obtained by acting on the vacuum with the appropriate number
of type-1a operators and $Q$ more type-2a than type-2b operators. There are already $Q$ ($-Q$) unpaired type-2a (type-2b) operators associated with $|\psi_Q\rangle$, so what remains are pairs consisting of a type-2a and a type-2b operator. Such a pair can be expanded in terms of type-1a operators as

$$
\epsilon_{i_1 \cdots i_{N_c}} \bar{c}_{a_1}^i c_{b_1} \cdots \bar{c}_{a_{N_c}}^i c_{b_{N_c}} = \sum_{\sigma \in S_{N_c}} \text{sgn}(\sigma) \bar{c}_{a_1}^1 c_{b_1} \cdots \bar{c}_{a_{N_c}}^{N_c} c_{b_{N_c}}
$$

(3.19)

and is therefore in the algebra of the flavor group. The type-1c and type-1d operators obey the commutation relations

$$
[\bar{c}_{a_1}^i c_{b_1}, \epsilon_{i_1 \cdots i_{N_c}} \bar{c}_{a_1}^{i_1} c_{b_1} \cdots \bar{c}_{a_{N_c}}^{i_{N_c}}] = \epsilon_{i_1 \cdots i_{N_c}} \bar{c}_{a_1}^{i_1} c_{b_1} \cdots \bar{c}_{a_{N_c}}^{i_{N_c}},
$$

$$
[d_{a_1}^{\bar{i}} b_1, \epsilon_{i_1 \cdots i_{N_c}} \bar{c}_{a_1}^{i_1} c_{b_1} \cdots \bar{c}_{a_{N_c}}^{i_{N_c}}] = \epsilon_{i_1 \cdots i_{N_c}} \bar{c}_{a_1}^{i_1} c_{b_1} \cdots \bar{c}_{a_{N_c}}^{i_{N_c}}.
$$

(3.20)

Thus, they enable us to change the flavor indices of the type-2 operators. We can thus reach any state in a given $Q$-sector by acting with the corresponding number and subtypes of type-1 operators on $|\psi_Q\rangle$. Furthermore, type-1a operators can be undone by type-1b operators, while type-1c and type-1d operators can undo themselves. We can therefore go from an arbitrary state in the $Q$-sector back to $|\psi_Q\rangle$. In other words, $|\psi_Q\rangle$ is a cyclic vector in the $Q$-sector under the action of the flavor group, which implies irreducibility.

### 3.4 Generalized coherent states and projection onto the color-neutral sector

Generalized coherent states are described in detail in Ref. [28]. They are useful for our purposes because they allow a resolution of the identity operator. For a Lie group $G$ and an irreducible representation $T_g$, a set of generalized coherent states is obtained by acting on a state $|\psi_T\rangle$ in the carrier space of $T_g$ with all elements of $T_g$. This results in the set $\{T_g|\psi_T\}$ which, in general, is overcomplete. If $H$ is the maximal subgroup of $G$ such that $T_h|\psi_T\rangle \propto |\psi_T\rangle$ for all $h \in H$, the subgroup $H$ is called the isotropy subgroup of $|\psi_T\rangle$, and the set of generalized coherent states can be parameterized without overcounting by the elements of the coset space $G/H$. (If the subgroup $H$ is not maximal, some overcounting remains.)

We now set $G = U(N_b, N_c)$ and consider the representation $T_g$ of Eq. (3.6) (with $g \in G$) which acts irreducibly in a given $Q$-sector. For the starting vector we choose the vector $|\psi_Q\rangle$ defined in Eq. (3.18), resulting in the (overcomplete) set of generalized coherent states $\{T_g|\psi_Q\rangle\}$. The identity operator in this $Q$-sector is then given by [28]

$$
1_Q = C_Q \int_G dg\, T_g|\psi_Q\rangle\langle\psi_Q|T_g^{-1},
$$

(3.21)
where \( dg \) is the invariant measure of \( \text{U}(N_b, N_b) \) and \( C_Q \) is a normalization factor defined by
\[
C_Q^{-1} = \frac{1}{N_Q} \int_G dg \langle \psi_Q| T_g|\psi_Q\rangle \langle \psi_Q| T_g^{-1}|\psi_Q\rangle . \tag{3.22}
\]
Here, \( N_Q \) is the norm of \( |\psi_Q\rangle \),
\[
N_Q = \langle \psi_Q|\psi_Q\rangle = \prod_{n=0}^{N_c-1} \frac{(|Q| + n)!}{n!} . \tag{3.23}
\]
A detailed calculation of \( N_Q \) is given in App. A.

The operator in Eq. (3.21) annihilates all states that are not color-neutral, as well as color-neutral states corresponding to a different value of \( Q \). Thus, it is a projector onto the \( Q \)-sector. We can therefore write the projector onto the color-neutral sector as
\[
P = \bigoplus_Q 1_Q , \tag{3.24}
\]
where the sum runs over the values of \( Q \) given in Eq. (3.17). Identifying this projection operator with the one in Eq. (3.13) yields
\[
Z = \sum_Q Z_Q \quad \text{with} \quad Z_Q = \langle \Psi|1_Q|\Psi\rangle = \langle 0|\exp(\overline{\psi}_i^a c_a^i + \psi_a^i d_a^i) 1_Q \exp(\overline{c}_a^i \psi_a^i + \overline{d}_a^i \overline{\psi}_a^i)|0\rangle . \tag{3.25}
\]

### 3.5 Parameterization of the coherent states

The maximal compact subgroup of the flavor group \( G = \text{U}(N_b, N_b) \) is \( H = H_+ \times H_- = \text{U}(N_b) \times \text{U}(N_b) \) with elements \( h = \text{diag}(h_+, h_-) \), where \( h_{\pm} \in \text{U}(N_b) \). The corresponding Fock operators are
\[
T_h = \exp \left[ \overline{c}_a^i (\ln h_+)_{ab} c_b^i - d_a^i (\ln h_-)_{ab} \overline{d}_b^i \right] . \tag{3.26}
\]
For \( Q = 0 \), these operators stabilize the vacuum,
\[
T_h|0\rangle = \exp (-N_c \text{tr} \ln h_-)|0\rangle = (\det^{N_c} h_-)|0\rangle , \tag{3.27}
\]
and therefore the set of coherent states for \( Q = 0 \) can be parameterized without overcounting by the elements of the non-compact coset space \( S = G/H = \text{U}(N_b, N_b)/[\text{U}(N_b) \times \text{U}(N_b)] \). (We will use the same coset space also for \( Q \neq 0 \), see Sec. 3.7.)

To arrive at this parameterization, we first use the canonical projection \( \pi : G \rightarrow G/H \) which assigns to each \( g \in G \) the corresponding equivalence class
We then choose a representative group element \( s(\pi(g)) \) from each equivalence class and write an arbitrary group element \( g \) as the product \( g = s(\pi(g))h(g) \). The coset element \( s(\pi(g)) \) can be parameterized using projective coordinates \( Z \), see Eqs. (5.8n) and (5.28) in Ch. 9 of Ref. [29],

\[
s(\pi(g)) \equiv s(Z, Z^\dagger) = \\
\begin{pmatrix}
(1 - ZZ^\dagger)^{-1/2} & Z(1 - Z^\dagger Z)^{-1/2} \\
Z^\dagger(1 - ZZ^\dagger)^{-1/2} & (1 - Z^\dagger Z)^{-1/2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\tag{3.28}
\]

Here, \( Z \) is an \( N_b \times N_b \) complex matrix with the constraint \( |ZZ^\dagger| \leq 1 \). We have \( s = s^\dagger \) and \( s^{-1} = s(-Z, -Z^\dagger) \). Also, \( s(Z, Z^\dagger) \) satisfies the pseudo-unitarity condition \( s \text{ diag}(1_{N_b}, -1_{N_b})s^\dagger = \text{ diag}(1_{N_b}, -1_{N_b}) \). Using the decomposition (3.28), the Fock space operator corresponding to \( s(Z, Z^\dagger) \) becomes

\[
T_{s(Z,Z^\dagger)} = \exp(c^a_Z Z_{ab} d_b^\dagger) \exp \left[ \frac{1}{2} \overline{c}^a_Z \ln(1 - ZZ^\dagger)_{ab} c^b_Z + \frac{1}{2} d^a_b \ln(1 - Z^\dagger Z)_{ab} d^b_a \right] \\
\times \exp(-d^a_b Z_{ab}^\dagger c^b_Z).
\tag{3.29}
\]

The coset space \( G/H \) has a \( G \)-invariant measure [17] that has already been given in Eq. (2.3). We can therefore use Eq. (3.7) to rewrite the integral (3.21) over \( G \) as an integral over \( H \) and \( S = G/H \),

\[
1_Q = C_Q \int_G dg T_g |\psi_Q\rangle \langle \psi_Q| T_g^{-1}
= C_Q \int_{G/H} d\mu(Z, Z^\dagger) \int_H dh T_s T_h |\psi_Q\rangle \langle \psi_Q| T_h^{-1} T_s^{-1}.
\tag{3.30}
\]

The Haar measure \( dh \) of \( H \) is normalized to unity.

### 3.6 Calculation of \( Z_0 \)

For \( Q = 0 \), Eq. (3.27) tells us that the integration over \( H \) in Eq. (3.30) is trivial, and we are left with

\[
Z_0 = C_0 \int_{G/H} d\mu(Z, Z^\dagger) \langle 0 | \exp \left( \overline{\psi}^a_Z c^a_Z + \varphi^a_Z d^a_Z \right) T_s |0\rangle \langle 0 | T_s^{-1} \exp \left( \overline{c}^a_Z \psi^a_Z + \overline{d}^a_Z \varphi^a_Z \right) |0\rangle.
\tag{3.31}
\]

Defining the notation \( |Z\rangle = \exp(\overline{c}^a_Z Z_{ab} d_b^\dagger) |0\rangle \) and \( \langle Z| = \langle 0 | \exp(d^a_b Z_{ab}^\dagger c^b_Z) \), we find

\[
T_s |0\rangle = \text{det}^{N_c/2}(1 - ZZ^\dagger) |Z\rangle
\tag{3.32}
\]
and

\[ \langle 0 | \exp \left( \bar{\psi}^i_a c^i_a + \varphi^i_a d^i_a \right) T_s |0 \rangle \langle 0 | T_s^{-1} \exp \left( c^i_a \psi^i_a + \bar{d}^i_a \bar{\varphi}^i_a \right) |0 \rangle = \det N_c (1 - ZZ^\dagger) \langle 0 | \exp \left( \bar{\psi}^i_a Z_{ab} \varphi^i_b + \bar{\varphi}^i_a Z_{ab}^\dagger \psi^i_b \right) |0 \rangle \]

\[ \] (3.33)

Thus

\[ Z_0 = C_0 \int_{|Z| \leq 1} D(Z, Z^\dagger) \exp \left( Z^i a Z_{ab} \varphi^i_b + \bar{Z}^i a \bar{Z}_{ab}^\dagger \psi^i_b \right) \] (3.34)

with

\[ D(Z, Z^\dagger) = \frac{dZ dZ^\dagger}{\det 2N_b - N_c (1 - ZZ^\dagger)} \] (3.35)

Not surprisingly, this is the same result as in Ref. [2] for the color group U(N_c).

From Eq. (3.22) we have

\[ C_0^{-1} = \frac{1}{N_0} \int_G d\eta \langle 0 | T_s |0 \rangle \langle 0 | T_s^{-1} |0 \rangle = \int_{|Z| \leq 1} D(Z, Z^\dagger) . \] (3.36)

An explicit calculation of C_0 is given in App. B.

Note that for 2N_b > N_c, D(Z, Z^\dagger) becomes divergent at the boundary of the integration domain. This divergence is due to the non-compactness of the symmetric space U(N_b, N_b)/[U(N_b) × U(N_b)] and is a feature of the bosonic color-flavor transformation. In Ref. [1], this divergence is canceled by the measure of the fermionic degrees of freedom, and the non-compact supersymmetric coset space has a flat measure if there is an equal number of bosons and fermions. In the fermionic case [7], the integral on the RHS is over the compact symmetric space U(2N_f)/[U(N_f) × U(N_f)], and there is no divergence problem.

Note also that for 2N_b > N_c, there are divergences in both numerator and denominator of the above formula. Apart from D(Z, Z^\dagger), the integrands are analytic on the entire coset space, therefore the divergences in numerator and denominator are of the same degree and the ratio must be finite. We will show this for a simple example in App. D.3. However, in the general case it is not obvious how to cancel the divergences, and even if it were possible, the resulting expressions might not be simple enough to be useful in applications. That is why in Sec. 4 we will use another method to extend the range of N_b in which all terms in the transformation are finite.

### 3.7 Calculation of Z_Q for Q ≠ 0

Let us start with the case of Q > 0; the case of Q < 0 follows analogously. Similar to Ref. [7], the idea is to relate the state \( |\psi_Q \rangle \) to \( |0 \rangle \) by the action of
the creation operators. Starting from the integrand of Eq. (3.25), we perform the following manipulations,

$$
\begin{align*}
\langle 0 | \exp(\tilde{\psi}_a^i \phi_a^i + \varphi_a^i d_a^i) T_g | \psi_Q \rangle & \langle \psi_Q | T_g^{-1} \exp(\tilde{\psi}_a^i \phi_a^i + \bar{d}_a^i \bar{\varphi}_a^i) | 0 \rangle \\
= \langle 0 | \exp(\tilde{\psi}_a^i \phi_a^i + \varphi_a^i d_a^i) (\epsilon_{i_1...i_{Nc}} T_{g_1}^i T_{g_2}^i ... T_{g_{Nc}}^i T_g^{-1}) T_g | 0 \rangle \\
\times \langle 0 | T_g^{-1} (\epsilon_{i_1...i_{Nc}} T_{g_1}^i T_{g_2}^i ... T_{g_{Nc}}^i T_g^{-1}) Q \exp(\tilde{\psi}_a^i \phi_a^i + \bar{d}_a^i \bar{\varphi}_a^i) | 0 \rangle \\
= \det_{Nc} (1 - ZZ^\dagger) \exp(\tilde{\psi}_a^i Z_a^b \varphi_b^i + \bar{\varphi}_a^i Z^a_b \psi_b^i) \\
\times \left[ (\epsilon_{i_1...i_{Nc}} \tilde{\psi}_a^i \psi_a^i) ... \left( \epsilon_{j_1...j_{Nc}} \tilde{\psi}_a^i \psi_a^i \right) \right] \Gamma_{(a_1...a_{Nc})...\{a_1^Q...a_{Nc}^Q\}} \\
\times \left[ (\epsilon_{i_1'...i_{Nc}'} \tilde{\psi}_b^i \psi_b^i) ... \left( \epsilon_{j_1'...j_{Nc}'} \tilde{\psi}_b^i \psi_b^i \right) \right] \Gamma_{(b_1'...b_{Nc}')...(b_1^Q...b_{Nc}^Q)} .
\end{align*}
\tag{3.37}
$$

In the first step, we have used Eq. (3.18) and inserted \( T_g^{-1} T_g \) between each pair of creation and annihilation operators. In the second step, which involves a tedious but straightforward calculation, we have used the transformation properties of the Fock space operators given in Eq. (3.9), the Baker-Campbell-Hausdorff formula, and the coset decomposition of the group elements, \( g = s(Z, Z^\dagger) h(g) \). We have also defined

$$
\tilde{\psi}_a^i = \tilde{\psi}_b^i (1 - ZZ^\dagger)^{\frac{1}{2}} , \quad \tilde{\psi}_a^i = (1 - ZZ^\dagger)^{\frac{1}{2}} \tilde{\psi}_b^i
\tag{3.38}
$$

and

$$
\Gamma_{(a_1...a_{Nc})...\{a_1^Q...a_{Nc}^Q\}} = (h + a_1^1 b_1^1 \cdots h + a_{Nc}^1 b_{Nc}^1) \cdots (h + a_1^Q b_1^Q \cdots h + a_{Nc}^Q b_{Nc}^Q) .
\tag{3.39}
$$

The hypermatrix \( \Gamma \) represents the direct product of \( Nc \cdot Q \) fundamental representations of \( H_+ = U(Nc) \), see Fig. 1. Note that \( H_- \) does not appear here. Inserting Eq. (3.37) into Eq. (3.25), we obtain

$$
\begin{align*}
\mathcal{Z}_Q = C_Q \int_{G/H} \, d\mu(Z, Z^\dagger) \int_H \, dh \langle 0 | \exp(\tilde{\psi}_a^i c_a^i + \varphi_a^i d_a^i) T_{s(Z, Z^\dagger)} T_h | \psi_Q \rangle \\
\times \langle \psi_Q | T_h^{-1} T_{s(Z, Z^\dagger)}^{-1} \exp(\tilde{\psi}_a^i \phi_a^i + \bar{d}_a^i \bar{\varphi}_a^i) | 0 \rangle \\
= \int_{|ZZ^\dagger| \leq 1} \mathcal{D}(Z, Z^\dagger) \exp \left( \tilde{\psi}_a^i Z_a^b \varphi_b^i + \bar{\varphi}_a^i Z^a_b \psi_b^i \right) \chi_Q ,
\end{align*}
\tag{3.40}
$$

where we have defined

$$
\chi_Q = C_Q \left\{ (\epsilon_{i_1...i_{Nc}} \tilde{\psi}_a^i \psi_a^i) ... \left( \epsilon_{j_1...j_{Nc}} \tilde{\psi}_a^i \psi_a^i \right) \right\} \\
\times \left\{ (\epsilon_{i_1'...i_{Nc}'} \tilde{\psi}_b^i \psi_b^i) ... \left( \epsilon_{j_1'...j_{Nc}'} \tilde{\psi}_b^i \psi_b^i \right) \right\} \\
\times \int_{U(Nc)} \, dh_+ \left[ \Gamma_{(a_1...a_{Nc})...\{a_1^Q...a_{Nc}^Q\}} - \Gamma_{(b_1...b_{Nc})...\{b_1^Q...b_{Nc}^Q\}} \right] .
\tag{3.41}
$$

It follows from the definition of \( \Gamma \) that the term in square brackets is totally symmetric under the exchange of \( a_a^i \) with \( a_a^j \) and of \( b_b^i \) with \( b_b^j \). Because of
Fig. 1. The (reducible) product of fundamental representations of $U(N_b)$ contains an irreducible representation of $U(N_b)$ with symmetric color indices and antisymmetric flavor indices. Here $Q > 0$.

the contraction with the totally antisymmetric tensor $\epsilon$, the terms in curly brackets are totally antisymmetric under the exchange of $a_a^i$ with $a_a'a'$ and of $b_a^i$ with $b_a'a'$. Therefore, after the contractions of the $a_a^i$'s and $b_a^i$'s, only terms with the correct symmetry properties survive, i.e. symmetric in color and antisymmetric in flavor. In other words, when the (reducible) direct-product representation $\Gamma$ is decomposed into irreducible representations, only the irreducible representation $\hat{\Gamma}$ shown in Fig. 1 contributes to $\chi_Q$.

This observation enables us to perform the integration over $H_+$ in the same way as in Ref. [7]. We use the group theoretic result that for irreducible unitary representations $\Gamma^r$ and $\Gamma^{r'}$ of a compact Lie group $G$,

$$
\int_G dg \bar{\Gamma}^r_{ab} \Gamma^{r'}_{a'b'} = \frac{1}{d_r} \delta_{aa'} \delta_{bb'} \delta^{rr'} \int_G dg ,
$$

where $d_r$ is the dimension of $\Gamma^r$. The group volume of $U(N_b)$ is normalized to unity as mentioned above.

Comparing Eq. (3.41) with Eq. (41) of Ref. [7], we realize that we can use (a slightly modified version of) the result in Eq. (46) of that reference and thus obtain

$$
\chi_Q = \frac{(N_c!)^Q}{d_{N_c,Q}(Q!)} C_Q \det^Q \mathcal{M} ,
$$

where we have defined the $N_c \times N_c$ matrix $\mathcal{M}$ by $\mathcal{M}^{ij} = \bar{\psi}_a^i (1 - ZZ^\dagger)_{ab} \psi_b^j$. The symbol $d_{N_c,Q}$ denotes the dimension of the irreducible representation of $U(N_b)$ specified by a Young diagram with $N_c$ rows and $Q$ columns, i.e. of the representation $\hat{\Gamma}$ in Fig. 1. (For $N_b \leq N_c$, this dimension is equal to one.) We will see in a moment that the prefactor of $C_Q \det^Q \mathcal{M}$ in Eq. (3.43) is in fact irrelevant.

We now use Eq. (3.22) to calculate the normalization factor $C_Q$. Using similar
methods as in the calculation of $\chi_Q$, we obtain

$$\langle \psi_Q | T_g | \psi_Q \rangle$$

$$= \det \frac{N_c}{1} (1 - ZZ^\dagger) \langle \psi_Q | \left\{ (\varepsilon_{i_1, j_N c} c_{i_1} \cdots \varepsilon_{j_N c} c_{j_N}) \cdots (\varepsilon_{j_1, j_N c} c_{j_1} \cdots \varepsilon_{j_N c} c_{j_N}) \right\} | 0 \rangle$$

$$\times \left[ (1 - ZZ^\dagger) c_{i_1}^\dagger \cdots (1 - ZZ^\dagger) c_{j_N}^\dagger \right] \cdots \left[ (1 - ZZ^\dagger) c_{i_1}^\dagger a_1 \cdots (1 - ZZ^\dagger) c_{j_N}^\dagger a_{Nc} \right]$$

$$\times \left[ h_{a_1} \cdots h_{a_{Nc}} \right] \cdots \left[ h_{a_1}^\dagger \cdots h_{a_{Nc}}^\dagger \right]$$

$$= N_Q \det \frac{N_c}{1} (1 - ZZ^\dagger) \Gamma(1 \cdots 1 | 1 \cdots 1)$$

$$\times \sum_{\sigma_1 \cdots \sigma_Q} \left[ \text{sgn}(\sigma_1)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_1(1) a_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_1(Nc) a_{Nc}} \right]$$

$$\cdots \left[ \text{sgn}(\sigma_Q)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_Q(1) a_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_Q(Nc) a_{Nc}} \right],$$

(3.44)

where we have used $N_Q = \langle \psi_Q | \psi_Q \rangle$. The symbol $\sigma$ and the symbol $\rho$ in the equation below denote elements of the permutation group $S_{Nc}$. Analogously, we find

$$\langle \psi_Q | T_g^{-1} | \psi_Q \rangle = N_Q \det \frac{N_c}{1} (1 - ZZ^\dagger) \Gamma(1 \cdots 1 | 1 \cdots 1)$$

$$\times \sum_{\rho_1 \cdots \rho_Q} \left[ \text{sgn}(\rho_1)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_1(1) b_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_1(Nc) b_{Nc}} \right]$$

$$\cdots \left[ \text{sgn}(\rho_Q)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_Q(1) b_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_Q(Nc) b_{Nc}} \right].$$

(3.45)

Combining these two results, we have from Eq. (3.22)

$$C_Q^{-1} = \frac{1}{N_Q} \int_{U(N_b, N_c)} d\mu(Z, Z^\dagger) \det \frac{N_c}{1} (1 - ZZ^\dagger)$$

$$\times \sum_{\sigma_1 \cdots \sigma_Q} \left[ \text{sgn}(\sigma_1)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_1(1) a_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_1(Nc) a_{Nc}} \right]$$

$$\cdots \left[ \text{sgn}(\sigma_Q)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_Q(1) a_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\sigma_Q(Nc) a_{Nc}} \right]$$

$$\times \sum_{\rho_1 \cdots \rho_Q} \left[ \text{sgn}(\rho_1)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_1(1) b_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_1(Nc) b_{Nc}} \right]$$

$$\cdots \left[ \text{sgn}(\rho_Q)(1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_Q(1) b_1} \cdots (1 - ZZ^\dagger)^{\frac{1}{2}}_{\rho_Q(Nc) b_{Nc}} \right]$$

$$\times \int_{U(N_b)} dh \Gamma(1 \cdots 1 | 1 \cdots 1) \Gamma(1 \cdots 1 | 1 \cdots 1).$$

(3.46)

The integration over $U(N_b)$ is of the same type as in the calculation of $\chi_Q$,
and using the same method we obtain

\[ C_Q^{-1} = \frac{N_Q(N_c!)^Q}{d_{N_c,Q}(Q)!} N_c \int_{|ZZ^\dagger| \leq 1} D(Z, Z^\dagger) \, \det^Q(1 - ZZ^\dagger)_{[N_c]} , \quad (3.47) \]

where \((1 - ZZ^\dagger)_{[N_c]}\) denotes the upper left \(N_c \times N_c\) block of the \(N_b \times N_b\) matrix \((1 - ZZ^\dagger)\). Recall that non-zero \(Q\)-sectors only exist for \(N_b \geq N_c\), so this notation always makes sense.

We now combine Eqs. (3.40), (3.43), and (3.47) to obtain for \(Q > 0\)

\[ Z_Q = C_Q \int_{|ZZ^\dagger| \leq 1} D(Z, Z^\dagger) \, \exp \left( \bar{\psi}_a^i Z_{ab} \varphi_b^i + \bar{\varphi}_a^i Z_{ab}^\dagger \psi_b^i \right) \, \det^Q \mathcal{M} , \quad (3.48) \]

where we have defined

\[ C_Q^{-1} = N_Q \int_{|ZZ^\dagger| \leq 1} D(Z, Z^\dagger) \, \det^Q(1 - ZZ^\dagger)_{[N_c]} . \quad (3.49) \]

The explicit calculation of this integral is performed in App. B. As anticipated, the nontrivial prefactors in Eqs. (3.43) and (3.47) have dropped out.

For \(Q < 0\), the calculation proceeds in exact analogy, and we obtain

\[ Z_Q = C_Q \int_{|ZZ^\dagger| \leq 1} D(Z, Z^\dagger) \, \exp \left( \bar{\psi}_a^i Z_{ab} \varphi_b^i + \bar{\varphi}_a^i Z_{ab}^\dagger \psi_b^i \right) \, \det^{|Q|} \mathcal{N} , \quad (3.50) \]

where the \(N_c \times N_c\) matrix \(\mathcal{N}\) is defined by \(\mathcal{N}^{ij} = \bar{\varphi}_a^i (1 - Z^\dagger Z)_{ab} \varphi_b^j\). This completes the derivation of Eq. (2.1).

A number of concrete examples illustrating the transformation are given in App. D.1. In particular, in App. D.3 we consider an example where the integration measure diverges, and show how this problem can be solved in a simple case.

### 3.8 Generalization to unequal flavor numbers

So far we only considered the case in which particles and antiparticles have equal flavor numbers, i.e. \(N_{b+} = N_{b-} = N_b\), where \(N_{b+}\) (\(N_{b-}\)) denotes the number of flavors of the particles (antiparticles). In practice this constraint may not be present. It is not difficult to see how our method can be extended to the general case in which \(N_{b+} \neq N_{b-}\). The flavor group is then \(U(N_{b+}, N_{b-})\), and the integral on the RHS of Eq. (2.1) is over the non-compact symmetric space \(U(N_{b+}, N_{b-})/[U(N_{b+}) \times U(N_{b-})]\). All results derived in earlier parts of this section are still valid with some minor changes: 1. the complex matrix \(Z\) has dimension \(N_{b+} \times N_{b-}\), 2. replace \(2N_b\) by \(N_{b+} + N_{b-}\) in the integration
measure, and 3. choose the range of \( Q \) accordingly. For example, if \( N_{b-} < N_c \) and \( N_{b+} \geq N_c \), there are no \( Q_- \) sectors, and we sum over \( Q \geq 0 \) and set \( N = 0 \) in our results. We give a concrete example with \( N_{b+} \neq N_{b-} \) in App. D.2.

4 Bosonic color-flavor transformation: character expansion method

In this section, we use the character expansion method [19] to derive an alternative form of the bosonic color-flavor transformation which is free from divergences in the range \( N_b \leq N_c \). We will also make use of the results of Refs. [20,21].

4.1 Setup of the calculation

In the last section, we have traded the integral over the compact color group for an integral over a non-compact manifold parameterized by an \( N_b \times N_b \) complex matrix \( Z \). Employing a singular value decomposition, this matrix can be written as

\[
Z = U \Lambda V, \tag{4.1}
\]

where \( U \in U(N_b), V \in U(N_b)/U_{N_b}(1) \), and \( \Lambda \) is a diagonal matrix with real entries, the so-called radial coordinates, satisfying \( 0 \leq \Lambda_a \leq 1 \). The divergence problem we met in the last section is caused by the integration over the sub-manifold spanned by the radial coordinates. Specifically, the divergence of highest degree occurs at the boundary, \( \Lambda_a = 1 \) for all \( a \), and the entire information that is needed to complete the color-flavor transformation resides in the boundary. A natural question to ask at this point is whether the integration over the radial coordinates can be avoided. To answer this question, we integrate over the two compact unitary groups first. At the same time, we relax the constraints on the radial coordinates by replacing \( \Lambda \) with an arbitrary complex matrix.

Our strategy is as follows. We first perform the integration over the color group on the LHS of the transformation explicitly using the character expansion method. Next we compute an integral over a compact subgroup of the flavor group with a manifestly color-invariant integrand. We then complete the transformation by observing that the two integrals are equal.

We define two rectangular \( N_c \times N_b \) matrices \( \Psi \) and \( \Phi \) by

\[
\Psi = (\psi_a^i), \quad \Phi = (\phi_a^i). \tag{4.2}
\]
The integrand on the LHS of Eq. (2.1) can then be rewritten as
\[ \exp \left( \bar{\psi}^i U^{ij} \psi^j + \bar{\varphi}^i U^{ij} \varphi^j \right) = \exp \left( \text{tr} \, M + \text{tr} \, \bar{N} \right), \]
where we have defined two \( N_c \times N_c \) matrices \( M \) and \( N \) by
\[ M = (M^{ij}) = (\bar{\psi}^i \psi^j) = \Psi \Psi^\dagger, \quad N = (N^{ij}) = (\varphi^i \varphi^j) = \Phi \Phi^\dagger. \]

In the following, we consider irreducible representations of \( \text{GL}(m) \) (for various values of \( m \)) labeled by
\[ r = (r_1, r_2, \ldots, r_m) \quad \text{with integers} \quad r_1 \geq r_2 \geq \ldots \geq r_m \geq 0, \]
where \( r_j \) is the number of boxes in row \( j \) of the corresponding Young diagram. Using Eq. (3.5) of Ref. [20], we have
\[ \exp (\text{tr} \, U M) = \sum_r \alpha_r^{(0)} \chi_r(U M), \quad \exp (\text{tr} \, U^\dagger N) = \sum_{r'} \alpha_{r'}^{(0)} \chi_{r'}(U^\dagger N), \]
where the sums are over all irreducible representations of \( \text{GL}(N_c) \) of the form (4.5). For a given representation \( r \), we have [20]
\[ \alpha_r^{(\nu)} = \det \left[ \frac{1}{(r_j - \nu + i - j)!} \right] = \left[ \prod_{i=1}^{N_c} (N_c - i)! \right] d_r \quad \text{with} \quad k_i = N_c + r_i - i, \]
where \( i \) and \( j \) run from 1 to \( N_c \), \( \nu \) is an additional integer which we shall need later on, \( d_r \) is the dimension of the representation \( r \), given by Weyl’s formula
\[ d_r = \left[ \prod_{n=1}^{N_c-1} n! \right] \Delta(k_1, \ldots, k_{N_c}), \]
and \( \Delta(k_1, \ldots, k_{N_c}) = \prod_{i<j} (k_i - k_j) \) is the Vandermonde determinant. We then obtain
\[ \int_{\text{SU}(N_c)} dU \exp \left( \text{tr} \, U M + \text{tr} \, U^\dagger N \right) = \sum_{rr'} \alpha_r^{(0)} \alpha_{r'}^{(0)} \int_{\text{SU}(N_c)} dU \, \chi_r(U M) \chi_{r'}(U^\dagger N) \]
\[ = \sum_{rr'} \alpha_r^{(0)} \alpha_{r'}^{(0)} \int_{\text{SU}(N_c)} dU \, U_r^{ij} M_r^{ij} \bar{U}_{r'}^{kl} \bar{N}_{r'}^{kl}, \]
where we use the notation \( U_r \) for the matrix corresponding to the representation \( r \) specified by a given Young diagram, an example of which is shown in the left part of Fig. 2. In the right part of that figure, we show a Young diagram that has \( Q \) more columns, each containing \( N_c \) boxes, than the Young diagram for \( r \). (Here we assume \( Q \geq 0 \).) We denote the corresponding representation by \( r + N_c Q \). Note that for \( \text{SU}(N_c) \), these two representations are identical. We
have the orthogonality relation

$$\int_{SU(N_c)} dU \ U_r^{ij} \bar{U}_{r'}^{kl} = \frac{1}{d_r} \delta^{ik} \delta^{jl} \delta_{r', r+N_cQ}. \quad (4.10)$$

From Eq. (4.7) it is clear that

$$\alpha^{(0)}_{r+N_cQ} = \alpha^{(-Q)}_r. \quad (4.11)$$

Furthermore, for all $g \in \text{GL}(N_c)$ we have [30]

$$g_r^{ij} = g_r^{ij} \det^Q g. \quad (4.12)$$

Rewriting the sum over $r$ and $r'$ in Eq. (4.9) as a sum over $r$ and $Q$, we obtain

$$\int_{SU(N_c)} dU \ \exp \left( \bar{\psi}^i_a U^{ij} \psi^j_a + \bar{\varphi}^i_a U^{ij} \varphi^j_a \right) \quad (4.13)$$

$$= \sum_r \frac{\alpha^{(0)}_r \alpha^{(0)}_r}{d_r} \chi_r(MN) + \sum_{Q=1}^\infty \left( \det^Q M + \det^Q N \right) \sum_r \frac{\alpha^{(0)}_r \alpha^{(-Q)}_r}{d_r} \chi_r(MN).$$

The sums over $r$ can be done analytically, resulting in an expression involving determinants of modified Bessel functions [21], but we shall not need this explicit result and therefore do not quote it here.

Note that in the case of color group $U(N_c)$ only the $Q = 0$ term is non-zero in Eq. (4.10), and therefore only the first term contributes on the RHS of Eq. (4.13). This observation will allow us to read off results for $U(N_c)$ from those for $SU(N_c)$ in Secs. 4.2 and 4.3.

We now turn to the RHS of Eq. (2.1) and first define two $N_b \times N_b$ matrices

$$B = (B_{ab}) = (\varphi^i_a \bar{\psi}^i_b) = [\Psi^\dagger \Phi]^T, \quad C = (C_{ab}) = (\psi^i_a \bar{\varphi}^i_b) = [\Phi^\dagger \Psi]^T. \quad (4.14)$$
We then consider the integral
\[
\int dU \int dV \det^{-Q}(UAVB) \exp \left[ \tilde{\psi}_a^i(UAV)_{ab} \tilde{\varphi}_b^i + \varphi_a^i(V^\dagger DU^\dagger)_{ab} \psi_b^i \right]
\]
\[
= \int dU \int dV \det^{-Q}(UAVB) \exp \left[ \text{tr}(UAVB) + \text{tr}(CV^\dagger DU^\dagger) \right], \tag{4.15}
\]
where \(A\) and \(D\) are two arbitrary \(N_b \times N_b\) matrices. Using Ref. [21], we have for \(AD = I\), i.e. \(A\) is the inverse of \(D\),
\[
(4.15) = \sum_s \alpha_s^{(0)}(0) \alpha_s^{(-Q)} \chi_s(AD) \chi_s(BC) = \sum_s \alpha_s^{(0)}(0) \alpha_s^{(-Q)} \frac{d_s}{d_s} \chi_s(BC), \tag{4.16}
\]
where the sum is over all irreducible representations \(s\) of GL(\(N_b\)) of the form (4.5), and we have used \(d_s = \chi_s(I)\).

In App. C we prove the following identity for \(\delta = N_c - N_b \geq 0\),
\[
\sum_r \alpha_r^{(0)}(0) \alpha_r^{(-Q)} \chi_r(MN) = \prod_{n=0}^{\delta-1} \frac{(N_b + n)!}{(Q + n)!} \sum_s \alpha_s^{(0)}(0) \alpha_s^{(-Q)} \frac{d_s}{d_s} \chi_s(BC), \tag{4.17}
\]
where the sums on the LHS and the RHS are over all irreducible representations of GL(\(N_c\)) and GL(\(N_b\)), respectively, that are of the form (4.5). Using this identity, we can now relate Eqs. (4.13) and (4.16). We consider separately the cases \(N_b < N_c\), \(N_b = N_c\), and \(N_b > N_c\).

### 4.2 \(N_b < N_c\)

For \(N_b < N_c\), the matrices \(M\) and \(N\) are not of full rank, i.e. we have \(\det M = \det N = 0\). The terms multiplied by \(\det M\) and \(\det N\) in Eq. (4.13) are finite [21]. Thus, only the \(Q = 0\) term in Eq. (4.13) survives, and we obtain
\[
(4.13) = \sum_r \alpha_r^{(0)}(0) \chi_r(MN) = \prod_{n=0}^{\delta-1} \frac{(N_b + n)!}{n!} \sum_s \alpha_s^{(0)}(0) \alpha_s^{(-\delta)} \frac{d_s}{d_s} \chi_s(BC), \tag{4.18}
\]
where we have used Eq. (4.17). This is already in the form of Eq. (4.16) with \(Q = \delta = N_c - N_b\). We can further simplify the integral in Eq. (4.15) by choosing \(A = D = I\) and using the invariance of the Haar measure to eliminate \(U\) from the integrand. This yields
\[
\int_{\text{SU}(N_c)} dU \exp \left( \tilde{\psi}_a^i U^{ij} \psi_a^j + \tilde{\varphi}_a^i U^{ij} \varphi_a^j \right) \tag{2.5}
\]
\[
= \prod_{n=0}^{N_c-N_b-1} \frac{(N_b + n)!}{n!} \int_{U(N_b)} dV \det^{N_b-N_c}(V^B) \exp \left( \tilde{\psi}_a^i V_{ab} \varphi_b^i + \tilde{\varphi}_a^i V_{ab}^\dagger \psi_b^i \right)
\]
as advertised in Sec. 2. If in the above expression one wants to take the limit of det $B \rightarrow 0$, the integral over $V$ needs to be done first. This procedure yields a finite result, see the example in App. E.3.

The result (2.5) is also valid if the integration on the LHS is over the color group $U(N_c)$. This follows immediately from the remarks made after Eq. (4.13) and from the fact that the terms with $Q > 0$ do not contribute on the RHS of Eq. (4.13).

4.3 $N_b = N_c$

For $N_b = N_c$ the matrices $M$ and $N$ are of full rank, and all terms in Eq. (4.17) contribute. Eq. (4.19) now becomes trivial,

$$\sum_r \frac{\alpha_r(0)\alpha_r(-Q)}{d_r} \chi_r(MN) = \sum_s \frac{\beta_s(0)\beta_s(-Q)}{d_s} \chi_s(BC).$$

We again simplify the integral in Eq. (4.15) by choosing $A = D = I$ and using the invariance of the Haar measure to arrive at

$$\int_{SU(N_c)} dU \exp \left( \bar{\psi}_a^i U_{ij} \psi_a^j + \bar{\varphi}_a^i U_{ij}^\dagger \varphi_a^j \right) = \sum_{Q=0}^{\infty} \tilde{\chi}_Q \int_{U(N_b)} dV \det^{-Q}(VB) \exp \left( \bar{\psi}_a^i V_{ab} \varphi_b^j + \bar{\varphi}_a^i V_{ab}^\dagger \psi_b^j \right)$$

with $\tilde{\chi}_0 = 1$ and $\tilde{\chi}_{Q>0} = \det^Q M + \det^Q N$ as stated in Sec. 2. If the det $B \rightarrow 0$ limit is desired, the integral over $V$ needs to be done first as mentioned at the end of the previous subsection. If det $^{-Q}(VB)$ is combined with the terms in $\tilde{\chi}_{Q>0}$, we obtain $\beta^Q + 1/\beta^Q$ with $\beta = \det \Psi / \det(\Phi V)$. This shows that we are not allowed to change the order of summation and integration in Eq. (2.6), since the resulting geometric series would diverge for one of the two terms.

If the integration on the LHS of Eq. (2.6) is over the color group $U(N_c)$, only the $Q = 0$ term contributes on the RHS as explained after Eq. (4.13), and we obtain for $N_b = N_c$

$$\int_{U(N_c)} dU \exp \left( \bar{\psi}_a^i U_{ij} \psi_a^j + \bar{\varphi}_a^i U_{ij}^\dagger \varphi_a^j \right) = \int_{U(N_b)} dV \exp \left( \bar{\psi}_a^i V_{ab} \varphi_b^j + \bar{\varphi}_a^i V_{ab}^\dagger \psi_b^j \right),$$

see also Sec. 6 of Ref. [16].
4.4 $N_b > N_c$

In this case the $N_c \times N_c$ matrices $M$ and $N$ are of full rank, whereas the $N_b \times N_b$ matrices $B$ and $C$ are of rank $N_c$ with $N_b - N_c$ eigenvalues equal to zero so that $\det B = \det C = 0$. We now have $\delta = N_c - N_b < 0$. Using similar arguments as in App. C but in the reverse direction, we obtain instead of Eq. (4.17)

\[
\sum_r \frac{\alpha_r^{(0)} \alpha_r^{(-Q)}}{d_r} \chi_r(MN) = C_{|\delta|} \sum_s \frac{\alpha_s^{(0)} \alpha_s^{(|\delta|-Q)}}{d_s} \chi_s(BC)
\]

\[
= C_{|\delta|} \int_{U(N_b)} dU \int_{U(N_b)} dV \det^{|\delta|-Q}(UAVB) \exp \left[ \text{tr}(UAVB) + \text{tr}(CV^\dagger DU^\dagger) \right]
\]

\[
= C_{|\delta|} \int_{U(N_b)} dV \det^{|\delta|-Q}(VB) \exp \left[ \text{tr}(VB) + \text{tr}(CV^\dagger) \right]
\]

(4.21)

with

\[
C_{|\delta|} = \prod_{n=1}^{|\delta|} \frac{(Q - n)!}{(N_b - n)!},
\]

(4.22)

where in the last step in Eq. (4.21) we have again set $A = D = I$ and used the invariance of the Haar measure to eliminate $U$ from the integrand. Note that this expression is only valid for $Q \geq |\delta| = N_b - N_c$. Although $\det B = 0$ appears with a non-positive power, the RHS of Eq. (4.21) must be finite because the LHS is. This fact can be established explicitly by a suitable limiting procedure.

For $0 \leq Q < |\delta|$, the integral over $U(N_b)$ in Eq. (4.21) is zero because $\det B = 0$ appears with a positive power. For this range of $Q$, we cannot replace the corresponding terms in Eq. (4.13) by integrals over $U(N_b)$ and therefore cannot complete the transformation. Thus, it seems that the character expansion method does not yield a useful result for $N_b > N_c$.

5 Conclusions and outlook

We have generalized Zirnbauer’s color-flavor transformation in the bosonic sector to the special unitary group SU($N_c$). Because the flavor group $U(N_b, N_b)$ is non-compact, divergences arise if the number of bosonic flavors is too large. This has already been noted in Refs. [2,16] where the gauge group was $U(N_c)$ and results were given for $2N_b \leq N_c$. We have found a “standard” result for the color-flavor transformation in the same range, and an alternative form of the transformation in the extended range $N_b \leq N_c$. (A special case of this result for $N_b = N_c$ and color group $U(N_c)$ has already been given in Ref. [16].) For $N_b < N_c$, the results for SU($N_c$) are identical to those for $U(N_c)$ because only the sector with $Q = 0$ contributes.
The results of the present paper can be applied to study a boson-induced SU($N_c$) lattice gauge theory analogous to the U($N_c$) gauge theory discussed in Ref. [16]. We hope that other applications will arise, e.g., in the field of disordered and/or chaotic systems.

One obvious open problem is to obtain a manifestly convergent result for $N_b > N_c$. While the divergences that appear in numerator and denominator of our formal result (2.1) can always be canceled in special cases, see App. D.3, the general case is difficult to deal with. The character expansion method, which led to a convergent result in the extended range $N_b \leq N_c$, fails for $N_b > N_c$ since it cannot generate the terms with $0 \leq Q < N_b - N_c$ on the RHS of Eq. (4.13) in terms of integrals over (a subgroup of) the flavor group. However, as stated earlier and in Ref. [16], all necessary information resides in the boundary of the coset space $\text{U}(N_b, N_b)/[\text{U}(N_b) \times \text{U}(N_b)]$, so it is conceivable that an explicit result in terms of an integration over this boundary might yet be obtained.

The other open problem is the extension of the present results to the supersymmetric case in which both fermionic and bosonic flavors are present. In this case the divergence of the integration measure due to the bosonic degrees of freedom can be canceled by the contribution of the fermions to the measure, as long as sufficiently many fermions are included. The physically interesting case is $N_c = 3$ (the gauge group of QCD), $N_f \geq 2$ (the number of physical quark flavors), and $N_b = N_c$ (the lower bound for $N_b$ so that the bosons induce the correct gauge action [16]). In this case, the convergence requirement $2(N_b - N_f) \leq N_c$ is satisfied. However, the supersymmetric case raises other issues which will be addressed in a separate publication.

Acknowledgements

This work was supported in part by the U.S. Department of Energy (contract no. DE-FG02-91ER40608), by the RIKEN-BNL Research Center, and by Deutsche Forschungsgemeinschaft (project no. FOR 465). We would like to thank B. Schlittgen for many helpful conversations and M.R. Zirnbauer for a stimulating discussion.

A Normalization of generalized Slater states

In this section, we calculate the norm, $N_Q = \langle \psi_Q | \psi_Q \rangle$, of the state $| \psi_Q \rangle$ defined in Eq. (3.18). We first assume $Q \geq 0$ and discuss the case of $Q < 0$ at the end.
of this section. The vacuum is normalized by definition so that \( N_0 = 1 \). For \( Q = 1 \), \(|\psi_Q\rangle\) is the Slater state with the well-known norm \( N_1 = N_c \).

To prove Eq. (3.23), we study a different version of the color-flavor transformation in which the flavor group is \( U(N_b) \) with \( N_b = N_c \). Note that the flavor group is compact now. We follow the same method as in Sec. 3 but keep only the particles created by the \( \bar{c}_a^i \) and discard the antiparticles created by the \( \bar{d}_a^i \). The flavor group \( U(N_b) \) is then generated by \( \{ \tilde{G}_{ab} = \bar{c}_a^i c_b^j \} \). The state \(|\psi_Q\rangle\) is still defined as in Eq. (3.18). The projector onto the \( Q \)-sector is now

\[
1_Q = \tilde{C}_Q \int_{U(N_b)} dg \, \tilde{T}_g |\psi_Q\rangle\langle\psi_Q| \tilde{T}_g^{-1}
\]  

(A.1)

with the normalized Haar measure \( dg \) of \( U(N_b) \) and \( \tilde{T}_g = \exp(\bar{c}_a^i (\ln g)_{ab} c_b^j) \).

For \( N_b = N_c \), we have by explicit calculation

\[
\tilde{T}_g |\psi_Q\rangle = (\det^Q g) |\psi_Q\rangle.
\]

(A.2)

Using this equation, the normalization constant \( \tilde{C}_Q \), see also Eq. (3.22), simplifies to

\[
\tilde{C}_Q = N_Q \left[ \int_{U(N_b)} dg \, \langle \psi_Q | \tilde{T}_g |\psi_Q\rangle \langle\psi_Q| \tilde{T}_g^{-1} |\psi_Q\rangle \right]^{-1} = N_Q \left[ N_Q^2 \right]^{-1} = \frac{1}{N_Q}.
\]

(A.3)

Next we consider the following integral and perform manipulations similar to those in Sec. 3,

\[
\int_{SU(N_c)} dU \exp \left( \bar{\psi}_a^i U^{ij} \psi_a^j \right) = \int_{SU(N_c)} dU \, \langle 0 | \exp(\bar{\psi}_a^i c_a^j) \exp(\bar{c}_a^j U^{ij} \psi_a^i) | 0 \rangle
\]

\[
= \sum_{Q=0}^\infty \langle 0 | \exp(\bar{\psi}_a^i c_a^j) 1_Q \exp(\bar{c}_a^j \psi_a^i) | 0 \rangle
\]

\[
= \sum_{Q=0}^\infty \tilde{C}_Q \int_{U(N_b)} dg \, \langle 0 | \exp(\bar{\psi}_a^i c_a^j) \tilde{T}_g |\psi_Q\rangle \langle\psi_Q| \tilde{T}_g^{-1} \exp(\bar{c}_a^j \psi_a^i) | 0 \rangle
\]

\[
= \sum_{Q=0}^\infty \frac{1}{N_Q} \langle 0 | \exp(\bar{\psi}_a^i c_a^j) |\psi_Q\rangle \langle\psi_Q| \exp(\bar{c}_a^j \psi_a^i) | 0 \rangle
\]

\[
= \sum_{Q=0}^\infty \frac{1}{N_Q} \det^Q M,
\]

(A.4)

where \( M \) is an \( N_c \times N_c \) matrix, \( M^{ij} = \psi_a^i \bar{\psi}_a^j \), and we have again used Eq. (A.2). However, we can also use the character expansion method [19,20] to do this.
integral. Using the same notation as in Sec. 4, we obtain

\[
\int_{\text{SU}(N_c)} dU \exp \left( \overline{\psi}_a^i U^{ij} \psi^j_a \right) = \int_{\text{SU}(N_c)} dU \exp \left( \text{tr} U M \right) \\
= \int_{\text{SU}(N_c)} dU \sum_r \alpha_r^{(0)} \chi_r(U M) \\
= \sum_r \alpha_r^{(0)} M^{ij}_r \int_{\text{SU}(N_c)} dU U^{ij}_r \\
= \sum_{Q=0}^{\infty} \alpha_{r=N_cQ}^{(0)} \det^Q M. \quad (A.5)
\]

In the last step, we have used the facts that

\[
\int_{\text{SU}(N_c)} dU U^{ij}_r = \begin{cases} 
1, & r = N_c Q, \\
0, & \text{else}, 
\end{cases} \quad (A.6)
\]

where \( r = N_c Q \) denotes the (one-dimensional) irreducible representation of \( \text{GL}(N_c) \) specified by a Young diagram with \( N_c \) rows and \( Q \) columns, and that for all \( M \in \text{GL}(N_c) \) we have \( M^{ij}_{r=N_cQ} = \det^Q M \), see Eq. (4.12). (Note that in the one-dimensional representation \( r = N_c Q \) the indices \( i \) and \( j \) only take the value 1.) From Eq. (4.7) we obtain with \( d_r = N_c Q = 1 \)

\[
\alpha_{r=N_cQ}^{(0)} = \prod_{n=0}^{N_c-1} \frac{n!}{(Q+n)!}. \quad (A.7)
\]

Comparing Eqs. (A.4) and (A.5) we arrive at Eq. (3.23), valid for \( Q \geq 0 \).

The calculation for \( Q < 0 \) proceeds in complete analogy by working with the antiparticles instead of the particles, and the result for this case can be obtained by the replacement \( Q \to -Q \) in the expression for \( N_Q \).

### B Calculation of the \( C_Q \)

In this section, we do the integral in Eq. (3.49). This is an example of so-called Hua-type integrals that were studied by Hua a long time ago [17] and recently extended by Neretin [31]. Here, we follow the method introduced in Ref. [17]. We first consider the case of \( N_b \geq N_c \) and \( Q \geq 0 \), and then give a result for \( N_b < N_c \) and \( Q = 0 \).

Using Eq. (3.35), Eq. (3.49) becomes

\[
C_Q^{-1} = N_Q \int_{|ZZ^\dagger| \leq 1} \frac{dZdZ^\dagger}{\det^{2N_b-N_c}(1-Z Z^\dagger)} \det^Q (1-Z Z^\dagger)_{[N_c]} . \quad (B.1)
\]
We now write the matrix $Z$ as $Z = (Z_{N_b,N_b-1}, q)$, where $Z_{N_b,N_b-1}$ is an $N_b \times (N_b - 1)$ matrix and $q$ is a single column. We then have

$$1 - ZZ^\dagger = 1 - Z_{N_b,N_b-1}Z_{N_b,N_b-1}^\dagger - qq^\dagger. \quad \text{(B.2)}$$

Since $|1 - Z_{N_b,N_b-1}Z_{N_b,N_b-1}^\dagger| \geq 0$, i.e. the matrix has real and non-negative eigenvalues, we write $1 - Z_{N_b,N_b-1}Z_{N_b,N_b-1}^\dagger = \Gamma^\dagger$ and define $q = \Gamma w$. Then

$$dq dq^\dagger = |\det \Gamma|^2 dw dw^\dagger = \det \left(1 - Z_{N_b,N_b-1}Z_{N_b,N_b-1}^\dagger\right) dw dw^\dagger. \quad \text{(B.3)}$$

On the other hand,

$$\det(1 - ZZ^\dagger) = \det \left[\Gamma(1 - w w^\dagger)\Gamma^\dagger\right] = (1 - w^\dagger w) \det(1 - Z_{N_b,N_b-1}Z_{N_b,N_b-1}^\dagger),$$

where we have used $\det(1 - w w^\dagger) = (1 - w^\dagger w)$. Applying the same procedure to $Z_{N_b,N_b-1}$ and so on, we obtain

$$[C_Q N_Q]^{-1} = \int_{||Z|| \leq 1} \frac{dZ dZ^\dagger}{\det Z_{N_b,N_b} \det(1 - ZZ^\dagger)^Q} \det Q(1 - ZZ^\dagger)_{[N_c]}$$

$$= \prod_{i=1}^{N_c} \int_{w_i \leq 1} dw_i dw_i^\dagger (1 - w_i^\dagger w_i)^{N_c - N_b + Q - i} \prod_{j=N_c+1}^{N_b} \int_{w_j \leq 1} dw_j dw_j^\dagger (1 - w_j^\dagger w_j)^{N_c - N_b - j}$$

$$= \prod_{i=1}^{N_c} \pi^{N_b} \frac{(N_c - N_b + Q - i)!}{(N_c + Q - i)!} \prod_{j=N_c+1}^{N_b} \frac{\pi^{N_b} (N_c - N_b - j)!}{(N_c - j)!}. \quad \text{(B.4)}$$

From this equation, which was derived for $N_b \geq N_c$, we see that for $N_b > N_c$, the above integral diverges for all $Q \geq 0$, whereas for $N_b = N_c$, it diverges for $Q < N_b$.

For $N_b < N_c$, only the $Q = 0$ sector exists, and we obtain from a very similar and even simpler calculation

$$C_0^{-1} = C_0^{-1} = \pi^{N_b^2} \prod_{n=1}^{N_b} \frac{(N_c - N_b - n)!}{(N_c - n)!}, \quad \text{(B.5)}$$

which is finite for $2N_b \leq N_c$ but diverges for $N_b < N_c < 2N_b$.

### C Proof of identity (4.17)

We have $\text{tr}(MN) = \text{tr}(\Psi \Psi^\dagger \Phi \Phi^\dagger) = \text{tr}(\Phi \Psi \Psi^\dagger \Phi) = \text{tr}(\Phi \Psi \Psi^\dagger \Phi) = \text{tr}(BC)$, which proves the identity immediately for $N_b = N_c$. In the following, we assume $\delta = N_c - N_b > 0$ and prove the identity by an iterative procedure.
The semi-positive definite $N_b \times N_b$ matrix $BC = (\Phi^\dagger \Psi^\dagger \Phi)^T$ has $N_b$ eigenvalues that we denote by $\lambda_a^2$ with $a = 1, \ldots, N_b$. From linear algebra [18] we know that the $N_c \times N_c$ matrix $MN = \Psi \Psi^\dagger \Phi^\dagger \Phi$ has $N_b$ eigenvalues equal to those of $BC$ and $\delta = N_c - N_b$ eigenvalues equal to zero. We denote the eigenvalues of $MN$ by $\mu_i^2$, with $\mu_i = \lambda_i$ for $1 \leq i \leq N_b$. For the purpose of this proof, we start with nonzero values $\mu_i$ for $N_b < i \leq N_c$ and let them go to zero one by one, starting with $\mu_{N_c}$. Weyl’s character formula is [17]

$$
\chi_{(r_1, r_2, \ldots, r_{N_c})}(MN) = \frac{\det(\mu_i^{2(N_c+r_j-j)})}{\Delta(\mu^2)} = \frac{\det(\mu_i^{2k_j})}{\Delta(\mu^2)} ,
$$

(C.1)

where the $k_j$ have been defined in Eq. (4.7) and $\Delta$ again denotes the Vandermonde determinant. We have

$$
\lim_{\mu_{N_c} \to 0} \Delta_{N_c}(\mu^2) = \lim_{\mu_{N_c} \to 0} \prod_{i<j}^N (\mu_i^2 - \mu_j^2) = \Delta_{N_c-1}(\mu^2) \prod_{i=1}^{N_c-1} \mu_i^2 ,
$$

(C.2)

where the index on $\Delta$ denotes the number of eigenvalues involved. Similarly, we introduce the notation $\det_m(\mu_i^{2k_j})$ to denote the determinant of the $m \times m$ upper-left sub-matrix of the matrix $(\mu_i^{2k_j})$, i.e. $i$ and $j$ run from 1 to $m$ instead of from 1 to $N_c$. Note that for $\mu_{N_c} \to 0$, $\det_{N_c}(\mu_i^{2k_j})$ is non-vanishing only if $k_{N_c} = 0$ or, equivalently, $r_{N_c} = 0$, in which case we have

$$
\lim_{\mu_{N_c} \to 0} \det_{N_c}(\mu_i^{2k_j}) = \det_{N_{c}-1}(\mu_i^{2k'_j}) \prod_{i=1}^{N_{c}-1} \mu_i^2 ,
$$

(C.3)

where

$$
k'_j = r_j + (N_c - 1) - j \quad \text{with} \quad j = 1, \ldots, N_c - 1 .
$$

(C.4)

We conclude that for $\mu_{N_c} = 0$, $\chi_r(MN)$ is non-zero only for representations of $GL(N_c)$ with Young diagram $r = (r_1 \geq \cdots \geq r_{N_c} = 0)$. We thus obtain

$$
\lim_{\mu_{N_c} \to 0} \sum_r \frac{\alpha_r^{(0)}(\alpha_r^{(-Q)})}{d_r} \chi_r(MN) = \sum_{r_1 \geq \cdots \geq r_{N_c} = 0} \frac{\alpha_r^{(0)}(\alpha_r^{(-Q)})}{d_r} \frac{\det_{N_{c}-1}(\mu_i^{2k'_j})}{\Delta_{N_{c}-1}(\mu^2)}
$$

$$
= \prod_{n=1}^{N_{c}-1} n! \sum_{k_1 > \cdots > k_{N_{c}} = 0} \det_{N_{c}} \left( \frac{1}{k_j!(k_j - N_c + Q + i)!} \frac{\det_{N_{c}-1}(\mu_i^{2k'_j})}{\Delta_{N_{c}-1}(\mu^2)} \right) ,
$$

(C.5)

where we have used Eqs. (4.7) and (4.8). From Eq. (4.7) with $k_{N_c} = 0$ we have

$$
\det_{N_c} \left( \frac{1}{k_j!(k_j - N_c + Q + i)!} \right) = \frac{d^{GL(N_c)}_{(r_1, \ldots, r_{N_{c}-1}, 0)}}{\prod_{j=1}^{N_{c}-1} k_j!} \prod_{i=1}^{N_{c}} \frac{(N_c - i)!}{(k_i + Q)!} .
$$

(C.6)
From Eq. (4.8) we find that

$$d_{(r_1, \ldots, r_{N_c - 1}, 0)}^{GL(N_c)} = \frac{N_c - 1}{(N_c - 1)!} \prod_{j=1}^{N_c - 1} k_j d_{(r_1 + 1, \ldots, r_{N_c - 1} + 1)}^{GL(N_c - 1)}.$$  \hfill (C.7)

Hence, for $\mu_{N_c} = 0$, and thus $k_{N_c} = 0$, we obtain from (C.6), (C.7), and (4.7)

$$\det_{N_c} \left( \frac{1}{k_j!(k_j - N_c + Q + i)!} \right) = \frac{1}{Q!} \det_{N_c - 1} \left( \frac{1}{k'_j!(k'_j - (N_c - 1) + (Q + 1) + i)!} \right).$$  \hfill (C.8)

Putting everything together, we arrive at

$$\lim_{\mu_{N_c} \to 0} \sum_r \frac{\alpha_r^{(0)} \alpha_r^{(-Q)}}{d_r} \chi_r(MN) = \frac{(N_c - 1)!}{Q!} \sum_{r'} \frac{\alpha_{r'}^{(0)} \alpha_{r'}^{(-Q - 1)}}{d_{r'}} \chi_{r'}(MN),$$  \hfill (C.9)

where the sum on the RHS is over all irreducible representations $r' = (r_1 \geq \ldots \geq r_{N_c - 1} \geq 0)$ of $GL(N_c - 1)$. Repeating this procedure $\delta = N_c - N_b$ times, we obtain our identity

$$\sum_r \frac{\alpha_r^{(0)} \alpha_r^{(-Q)}}{d_r} \chi_r(MN) = \prod_{n=0}^{\delta - 1} \frac{(N_b + n)!}{(Q + n)!} \sum_s \frac{\alpha_s^{(0)} \alpha_s^{(-Q - \delta)}}{d_s} \chi_s(BC),$$  \hfill (4.17)

where the sum on the RHS is over all irreducible representations $s$ of $GL(N_b)$ of the form (4.5), and where we have again used $\text{tr}(MN) = \text{tr}(BC)$.

\section{Examples for the algebraic result}

\subsection{N_c = 2, N_b = 1}

We parameterize elements of SU(2) as

$$U = \begin{pmatrix} e^{i\lambda} \cos \theta & -e^{i\eta} \sin \theta \\ e^{-i\eta} \sin \theta & e^{-i\lambda} \cos \theta \end{pmatrix} \quad \text{with} \ 0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq \lambda, \eta < 2\pi.$$  \hfill (D.1)
The corresponding normalized Haar measure is \( dU = (1/2\pi^2) \sin \theta \cos \theta d\theta d\lambda d\eta \). Performing the integral on the LHS of Eq. (2.4), we obtain

\[
\int_{SU(2)} dU \exp \left( \bar{\psi}^i U^{ij} \psi^j + \bar{\varphi}^i U^{ij} \varphi^j \right) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\bar{\psi}^1 \psi^1 \bar{\varphi}^1 \varphi^1 + \bar{\psi}^2 \psi^2 \bar{\varphi}^2 \varphi^2 + \bar{\psi}^1 \psi^2 \bar{\varphi}^2 \varphi^1 + \bar{\psi}^2 \psi^1 \bar{\varphi}^1 \varphi^2)^n . \tag{D.2}
\]

For the RHS of Eq. (2.4), we have with \( C_0 = 1/\pi \)

\[
\frac{1}{\pi} \int_{|z| \leq 1} dz d\bar{z} \exp \left[ (\bar{\psi}^1 \varphi^1 + \bar{\varphi}^1 \psi^1)z + (\bar{\varphi}^2 \psi^2 + \bar{\psi}^2 \varphi^2)\bar{z} \right]
= \frac{1}{\pi} \int_0^1 r dr \int_0^{2\pi} d\theta \exp \left[ (\bar{\psi}^1 \varphi^1 + \bar{\varphi}^2 \psi^2)re^{i\theta} + (\bar{\varphi}^1 \psi^1 + \bar{\psi}^2 \varphi^2)re^{-i\theta} \right]
= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\bar{\psi}^1 \psi^1 \bar{\varphi}^1 \varphi^1 + \bar{\psi}^2 \psi^2 \bar{\varphi}^2 \varphi^2 + \bar{\psi}^1 \psi^2 \bar{\varphi}^2 \varphi^1 + \bar{\psi}^2 \psi^1 \bar{\varphi}^1 \varphi^2)^n \tag{D.3}
\]

in agreement with Eq. (D.2).

\[D.2 \quad N_c = N_{b+} = 2, \quad N_{b-} = 0\]

In this example we check the argument we made in Sec. 3.8 for \( N_{b+} \neq N_{b-} \). To have \( N_{b-} = 0 \) we simply set \( \varphi^a_1 = \varphi^a_2 = 0 \). Using the same parameterization for SU(2) as in App. D.1, we calculate the integral on the LHS of Eq. (2.1),

\[
\int_{SU(2)} dU \exp \left( \bar{\psi}^i_a U^{ij} \psi^j_a \right)
= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\bar{\psi}^1_{a1} \psi^1_{a1} + \bar{\psi}^2_{a2} \psi^2_{a2} + \bar{\psi}^1_{a1} \psi^2_{a2} + \bar{\psi}^2_{a1} \psi^1_{a2})^n . \tag{D.4}
\]

In this case we have to sum over \( Q \) from zero to infinity on the RHS. The integral over the coset space \( U(2)/U(2) \) amounts to evaluating the integrand at the single point \( Z = 0 \). From Eqs. (3.49) and (3.23), we have \( C_Q = 1/N_Q = 1/[Q!(Q + 1)!] \) and thus obtain

\[
\text{RHS} = \sum_{Q=0}^{\infty} \frac{1}{Q!(Q + 1)!} \det^Q \mathcal{M}
= \sum_{Q=0}^{\infty} \frac{1}{Q!(Q + 1)!} (\bar{\psi}^1_{a1} \psi^1_{a1} + \bar{\psi}^2_{a2} \psi^2_{a2} + \bar{\psi}^1_{a1} \psi^2_{a2} + \bar{\psi}^2_{a1} \psi^1_{a2})^Q
\]

in agreement with Eq. (D.4), where we have used \( \mathcal{M}^{ij} = \bar{\psi}^i_a \psi^j_a \).
In this example, we run into the divergence problem discussed in Secs. 2 and 3. The LHS of Eq. (2.1) is simple because the integral over SU(1) reduces to evaluating the integrand at unity,

\[
\int_{\text{SU}(1)} dU \exp \left( \bar{\psi}^i U^{ij} \psi^j + \bar{\phi}^i U^{ij} \phi^j \right) = \exp(\bar{\psi} \psi + \bar{\phi} \phi). \tag{D.6}
\]

The RHS of Eq. (2.1) is a sum over \( Q \). For \( Q = 0 \) we have

\[
C_0^{-1} = \int_{|z| \leq 1} \frac{d z d \bar{z}}{1 - z \bar{z}} = 2\pi \int_0^1 \frac{r d r}{1 - r^2}, \tag{D.7}
\]

and

\[
\int_{|z| \leq 1} \frac{d z d \bar{z}}{1 - z \bar{z}} \exp(\bar{\psi} z \phi + \bar{\phi} \bar{z} \psi) = 2\pi \sum_{n=0}^{\infty} \frac{(\bar{\psi} \psi \bar{\phi} \phi)^n}{(n!)^2} \int_0^1 \frac{r^{2n+1} d r}{1 - r^2}. \tag{D.8}
\]

We now change the upper limit in the integral to \( 1 - \epsilon \) and let \( \epsilon \to 0 \) to obtain

\[
C_0 \int_{|z| \leq 1} \frac{d z d \bar{z}}{1 - z \bar{z}} \exp(\bar{\psi} z \phi + \bar{\phi} \bar{z} \psi) = \sum_{n=0}^{\infty} \frac{(\bar{\psi} \psi \bar{\phi} \phi)^n}{(n!)^2}. \tag{D.9}
\]

For \( Q \geq 1 \), there are no divergences, and we have

\[
\int_{|z| \leq 1} \frac{d z d \bar{z}}{(1 - z \bar{z})^{2N_b - N_c - Q}} \exp(\bar{\psi} z \phi + \bar{\phi} \bar{z} \psi) = \pi \sum_{n=0}^{\infty} \frac{(Q - 1)!}{n!(n + Q)!} (\bar{\psi} \psi \bar{\phi} \phi)^n. \tag{D.10}
\]

Collecting all terms and using \( C_Q = 1/[(\pi(Q - 1)!) \] , we find for the RHS of Eq. (2.1)

\[
\int_{|Z, Z^\dagger| \leq 1} D(Z, Z^\dagger) \exp \left( \bar{\psi}^i Z_a \phi_b^i + \bar{\phi}^i Z_a \psi_b^i \right) \sum_{Q=0}^{\infty} \chi_Q \tag{D.11}
\]

\[
= \sum_{n=0}^{\infty} \sum_{Q=0}^{\infty} \frac{1}{n!(n + Q)!} \left[ (\bar{\psi} \psi)^Q + (\bar{\phi} \phi)^Q - \delta_{Q0} \right] (\bar{\psi} \psi \bar{\phi} \phi)^n = \exp(\bar{\psi} \psi + \bar{\phi} \phi),
\]

where the last step requires some rearrangements of the terms in the sums. We see that the transformation works although for \( Q = 0 \) the normalization factor and the integral on the RHS of the transformation are divergent. After those infinities have been canceled, the finite ratio gives the correct result. However, as mentioned at the end of Sec. 3.6, it is not trivial to obtain a simple result for the general case.
E. Examples for the character expansion result

E.1 $N_c = N_b = 1$

Although this example was already considered in App. D.3, we revisit it here to check our result obtained using the character expansion method. Again, the LHS equals $\exp(\bar{\psi}\psi + \bar{\varphi}\varphi)$. From Eq. (2.6) we have

$$
\sum_{Q=0}^{\infty} \tilde{\chi}_Q \int_0^{2\pi} \frac{d\theta}{2\pi} (Be^{i\theta})^{-Q} \exp \left( \bar{\psi}e^{i\theta}\varphi + \bar{\varphi}e^{-i\theta}\psi \right)
$$

(E.1)

$$
= \sum_{Q=0}^{\infty} \left[ (\bar{\psi}\psi)^Q + (\bar{\varphi}\varphi)^Q - \delta_{Q0} \right] \sum_{n=0}^{\infty} \frac{1}{n!(n+Q)!} (\bar{\psi}\psi\bar{\varphi}\varphi)^n = \exp(\bar{\psi}\psi + \bar{\varphi}\varphi)
$$

as in Eq. (D.11), where we have used $M = \psi\bar{\psi}$, $N = \varphi\bar{\varphi}$, and $B = \varphi\bar{\psi}$. We see that the transformation (2.6) works and that, unlike in App. D.3, we do not have any divergence problem.

E.2 $N_c = N_b = 2$

In this case we also have $2N_b > N_c$, and therefore divergences would arise in Eq. (2.1). We now check our result (2.6) in which no divergence appears. Using the parameterization of SU(2) in App. D.1, we perform the integral on the LHS,

$$
\int_{SU(2)} dU \exp \left( \bar{\psi}^i_a U^{ij} \psi^j_a + \bar{\varphi}^i_a U^{ij} \varphi^j_a \right) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\det M + \det N + \text{tr} MN)^n
$$

(E.2)

with $M$ and $N$ given by Eq. (4.4) and $B$ and $C$ given by Eq. (4.14). To do the integral on the RHS, we parameterize U(2) by multiplying the matrix in Eq. (D.1), which we now call $V$, by a phase $e^{i\phi}$, with $0 \leq \phi < 2\pi$. The corresponding normalized Haar measure is $dV = (1/4\pi^3) \sin \theta \cos \theta d\theta d\lambda d\eta d\phi$. We then obtain

$$
\sum_{Q=0}^{\infty} \tilde{\chi}_Q \int_{U(2)} dV \det^{-Q}(VB) \exp \left( \bar{\psi}^i_a V_{ab} \varphi^j_b + \bar{\varphi}^i_a V^{ab} \psi^j_b \right)
$$

(E.3)

$$
= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\det M + \det N + \text{tr} BC)^n
$$

with $\tilde{\chi}_Q$ defined in Eq. (2.7). The two results agree. In the derivation of Eqs. (E.2) and (E.3) we have used $\det(BC) = \det(MN)$ and $\text{tr}(BC) = \text{tr}(MN)$.
Let us check our result (2.5). The LHS is given by Eq. (D.2). On the RHS we have

\[ \prod_{n=0}^{N_c-N_b-1} \frac{(N_b+n)!}{n!} \int_{\mathbb{U}(N_b)} dV \det^{N_b-N_c}(VB) \exp \left( \bar{\psi}^i_a V_{ab} \phi^i_b + \bar{\phi}^i_a V_{ab} \psi^i_b \right) \]

\[ = \int_0^{2\pi} \frac{d\theta}{2\pi} (Be^{i\theta})^{-1} \exp \left( Be^{i\theta} + Ce^{-i\theta} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\bar{\psi}^1 \psi^1 \bar{\phi}^1 \phi^1 + \bar{\psi}^2 \psi^2 \bar{\phi}^2 \phi^2 + \bar{\psi}^1 \psi^2 \bar{\phi}^1 \phi^1 + \bar{\psi}^2 \psi^1 \bar{\phi}^1 \phi^2)^n, \quad (E.4) \]

where we have used \( B = \phi^1 \bar{\psi}^1 + \phi^2 \bar{\psi}^2 \) and \( C = \psi^1 \bar{\phi}^1 + \psi^2 \bar{\phi}^2 \). Thus, the RHS agrees with the result in Eq. (D.2).

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