Bootstrap Approximating the Mean of Long Memory Time Series

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Abstract. This paper evaluated the finite sample performances of three bootstrap methods, sieve AR bootstrap (SARB), fractional differencing sieve bootstrap (FDSB) and fractional differencing block bootstrap (FDBB), in approximating the mean of long memory time series. Extensive simulations show that the FDBB method has more stable approximate results in most cases and can more accurately approximate the mean distribution of long memory time series than the other two methods.

Keywords: Long memory time series; Sieve bootstrap; Block bootstrap.

1. Introduction

Since Efron [5] provide the bootstrap, the method has been developing and extending unceasingly in nearly thirty years and it is widely used in various fields of statistics. Bootstrapping techniques can be quite useful as it overcomes the limitations of insufficient data size or unknown theoretical distribution. However, in addition to being computationally intensive, IID bootstrap [5] has its limitations when applied to dependent data such as time series data. These barriers can be overcome through methods such as sieve bootstrap and block bootstrap. Most studies about these bootstrap methods are concentrating on short memory time series. It is broadly accepted that many dependent variables exhibit long-range dependencies that cannot be covered by the short memory processes. This leads long memory time series also is an important and well used model in practice. No doubt, studying the performance of sieve bootstrap and block bootstrap in long memory time series is an important issue.

Sieve Bootstrap has been first provided by Buhlmann [2]. The idea of sieve bootstrap is resampling from residuals of a fitted model as opposed to resampling the original data itself. Since then, many authors have studied this method, see Park [13] and related literature. In spite of there are such research of bootstrap for long memory processes, there is less study applying the sieve bootstrap to long memory processes. Hidalgo [6] presented a bootstrap method based on resampling in the frequency and Lazarova [10] indicated it can valid apply to stationary long memory process. The research provided by Andrews et. al. [1] showed that the bootstrap could effectively approximate the probability distribution of covariance parameter estimation for stationary long memory processes. Poskitt [14] pointed out that the SARB could present a superior result of the stationary long memory processes on theoretical analysis. Rupasinghe and Samaranayake [16] established an improved Sieve Bootstrap which is more useful in ARFIMA(p,d,q) processes. Preuss and Vetter [15], Davision and Rambacussing [4] provided a FDSB technique be suitable for analysis of non-stationary long memory time series.

In term of block bootstrap, Carlstein [3] first introduced this concept. The block bootstrap method, where the idea is to preserve the dependency structural in the data by resampling from continuous blocks of data as opposed to selecting individual points. Kunsch [8] introduced a moving-block
bootstrap, which is an extension of previous research by Efon [5]. Murphy and Izzeldin [15] investigated the moving-block bootstrap size and power properties of six long memory tests via simulation. Lahiri [9] showed that the moving-block bootstrap will fail in approximating sample means for a category of long-range dependent processes generated by transformations of Gaussian series. Tewses [17] showed to validate the block bootstrap for means under significantly weakened assumptions in many existing (and some new) dependence settings. Papailias and Kapetanios [12] introduced a FDBB algorithm for long memory time series, and showed that it can provide better bootstrap resamples than the SARB used in Poskitt [14]. However, the SARB [14] does not depend on the long memory parameter estimator while the FDBB needs. Obviously, in this point, compare these two methods is not fair. A parallel sieve bootstrap which needs estimating long memory parameter is FDSB [15, 4]. Does this sieve bootstrap still has worse performance than the FDBB is an interesting topic.

This paper aims to compare the SARB [14], the FDSB [4, 7], and the FDBB [12] under long memory time series. We compare the distribution approximation performance of these three methods by estimating the bootstrap sample mean. In changing point test problem, the theory of sequence distribution is usually unknown makes the test statistics of asymptotic critical value is not easy to get. By the Bootstrap method to approximate the critical value of test statistics makes the test process is simple. In this paper, numerical simulation was used to compare the approximate effects of three Bootstrap methods on the mean distribution of sequences under different long-memory parameter assumptions, the distribution approximation performance was illustrated based on the simulation results.

The rest of the paper is organized as follows: Section 2 introduces the research model. In section 3, we illustrate the three bootstrap methods. Extensive simulation experiments will be given in Section 4. Section 5 concludes the paper.

2. Model

We consider the following long memory process

\[ Y_t = \mu + X_t, \quad \Phi(L)(1 - L)^d X_t = \Psi(L)\varepsilon_t, \quad t = 1, 2, \ldots, n. \]  

(1)

where \[ \mu = E(Y_t) \] is a deterministic component, \[ n \] is the sample size. Random component \[ X_t \] follows an ARFIMA (p, d, q) process, in which \[ \varepsilon_t \] are i.i.d random variables with mean zero and variance \[ \sigma^2 \], and \[ L \] is the lag operator. The AR- and MA- polynomials \[ \Phi(L) \] and \[ \Psi(L) \] are assumed to have all roots outside the unit circle. The long memory parameter \[ d \] is restricted to \[ 0 \leq d \leq 1 \]. Note that the process \[ Y_t \] is a stationary long memory process if \[ 0 \leq d < 1/2 \], and the process \[ Y_t \] is a non-stationary long memory process if \[ 1/2 < d \leq 1 \]. The fractional differencing operator defined by the binomial expansion can be decomposed as \[ (1 - L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(1+j)} \varepsilon_t \], where \[ \Gamma(\cdot) \] denotes the gamma function. We are interested in checking how well the available three bootstrap methods in the next section when estimating mean \[ \mu \] of model (1).

3. Bootstrap Methods

Suppose samples \[ y_1, \ldots, y_n \] are realization from model (1), and the probability distribution or the critical value of any interesting statistic \[ T_Y \] is our interesting quantity. In this section, we show three popular bootstrap methods to approximate the probability distribution or the critical value of statistic \[ T_Y \] based on the samples \[ y_1, \ldots, y_n \].

3.1. Sieve AR Bootstrap

The SARB algorithm is as follows:

Step 1. Fit an autoregressive model AR(p) based on the OLS residuals \[ \hat{\varepsilon} = y - \hat{\mu} \], i.e., \[ \hat{\varepsilon}_t = \beta_1 \hat{\varepsilon}_{t-1} + \beta_2 \hat{\varepsilon}_{t-2} + \cdots + \beta_p \varepsilon_{t-p} \] with fixed \[ p(n) = 10 \log_{10}(n) \], and choose an optimal \[ p \] using the AIC or BIC criterion. Then obtain the estimated coefficients \[ \hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p \] and residuals \[ \hat{\varepsilon}_{p+1}, \ldots, \hat{\varepsilon}_m \] via Yule-Walker equation or other estimation.
Step 1. Having observed the samples $y_t^*$, $x_t^* = \beta_1 x_{t-1}^* + \beta_2 x_{t-2}^* + \ldots + \beta_p x_{t-p}^* + \epsilon_t^*$, $t = 1, 2, \ldots, n$, where $\epsilon_t^*$ are randomly selected with replacement from the residuals $\tilde{\epsilon}_{p+1}, \ldots, \tilde{\epsilon}_n$, and the starting $p$ observations $x_{p-1}^*, x_{p-2}^*, \ldots, x_0^*$ can be set equal to $\widehat{x} = \frac{1}{n} \sum_{t=1}^{n} \hat{x}_t$.

Step 3. Calculate the statistic $T_{Y^*}$, which has same definition as $T_Y$, based on the SARB samples $y_t^*, y_t^*$. write as $T_{Y^*}$.

Step 4. Repeat Step 2 to Step 3 B times, approximate the probability distribution function of statistic $T_Y$ and it’s critical values by the empirical distribution function and empirical quantiles of $T_{Y^*}^{\beta}, \ldots, T_{Y^*}^B$.

Let $F_{T_Y}$ and $F_{T_{Y^*}}$ denote the marginal distributions of statistics $T_Y$ and $T_{Y^*}$, and $\eta(F_{T_{Y^*}}, F_{T_{Y^*}}) = \inf\{E \parallel T_{Y^*} - T_Y \parallel^2 \}^{1/2}$ denote Mallow’s measure of the distance between two probability distributions $F_{T_Y}$ and $F_{T_{Y^*}}$. Under some mild conditions, the Theorem 2 of Postkitt [14] has proved that $\eta(F_{T_{Y^*}}, F_{T_{Y^*}}) = o\left(\left(\frac{n^\beta}{\ln(n-d^2)}\right)^{\frac{1}{2}}\right)$ for any $\beta > 0$ and $d' = \max\{0, d\}$, that implies that $F_{T_{Y^*}}$ converges to $F_{T_Y}$ with probability one and establishes the validity of SARB method to analysis the long memory time series.

### 3.2. Fractional Differencing Sieve Bootstrap

The FDSB algorithm is as follows:

Step 1. Having observed the samples $y_1, \ldots, y_n$, we start computing the OLS residuals $\hat{x}_t = y_t - \hat{\mu}$, $t = 1, 2, \ldots, n$, and then estimating long memory parameter $\hat{d}$ based $\hat{x}_t$, $t = 1, 2, \ldots, n$, through some estimation. The estimator writes as $\hat{d}$.

Step 2. Fit an autoregressive model AR(p) based on the $\hat{d}$-order fractional differencing data $\hat{\epsilon}_t = (1 - B)^{\hat{d}} \hat{x}_t$, i.e. $\hat{\epsilon}_t = \beta_1 \hat{\epsilon}_{t-1} + \beta_2 \hat{\epsilon}_{t-2} + \ldots + \beta_p(\hat{\sigma}) \hat{\epsilon}_{t-p}(\hat{\sigma})$ with fixed $p(n) = 10 \log_{10} n$, and then chose the optimal $p$ using the AIC or BIC criterion. Then, we obtain the estimated coefficients $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$ and residuals $\hat{\epsilon}_{p+1}, \ldots, \hat{\epsilon}_n$ via Yule-Walker equation or other estimation.

Step 3. Generate the bootstrap samples $\hat{y}_t^*$ according to $\hat{y}_t^* = \hat{\mu} + \hat{x}_t^* = (1 - B)^{\hat{d}} \hat{x}_t^*$, $\hat{\epsilon}_t^* = \sum_{j=1}^{\hat{d}} \hat{\beta}_j \hat{\epsilon}_{t-j}^* + \hat{\epsilon}_t^*$, $t = 1, 2, \ldots, T$, Where $\hat{\epsilon}_t^*$ are randomly selected with replacement from the residuals $\hat{\epsilon}_{p+1}, \ldots, \hat{\epsilon}_n$, and starting $p$ observations $\hat{\epsilon}_{t-p}^*, \hat{\epsilon}_{t-p-1}^*, \ldots, \hat{\epsilon}_0^*$ can be set equal to $\frac{1}{n-p} \sum_{i=p+1}^{n} \hat{\epsilon}_i$.

Step 4. Calculate the statistic $T_{Y^*}$, which has same definition as $T_Y$, based on the FDSB samples $y_t^*, y_t^*$, write as $T_{Y^*}$.

Step 5. Repeat Step 2 to Step 4 B times, approximate the probability distribution function of statistic $T_Y$ and it’s critical values by the empirical distribution function and empirical quantiles of $T_{Y^*}^{\beta}, \ldots, T_{Y^*}^B$. Compared to the SARB method, the main difference of FDSB method is fitting autoregressive model AR(p) based on the $\hat{d}$-order differenced data, and generates bootstrap samples based on $\hat{d}$-order fractional accumulative data. The asymptotic validity of this bootstrap method to analysis the long memory time series has been proved by Preuss and Vetter [15].

### 3.3. Fractional Differencing Block Bootstrap

The FDBB algorithm is as follows:

Step 1. Having observed the samples $y_1, \ldots, y_n$, we start computing the OLS residuals $\hat{x}_t = y_t - \hat{\mu}$, $t = 1, 2, \ldots, n$, and then estimating long memory parameter $\hat{d}$ based on $\hat{x}_t$, $t = 1, 2, \ldots, n$, through some estimation. The estimator writes as $\hat{d}$.

Step 2. Dividing the $\hat{d}$-order fractional differencing data $\hat{\epsilon}_t = (1 - B)^{\hat{d}} \hat{x}_t$ into $M = \left\lfloor \frac{n}{\hat{d}} \right\rfloor$ disjoint blocks according to a given block size $b$, we write the $k$-th block as $\hat{\epsilon}_k = \{\hat{\epsilon}_{(k-1)b+1}, \ldots, \hat{\epsilon}_{kb}\}$
Step 3. We randomly choose \( M \) blocks by sampling with replacement from \( \{e_k, k = 1, \ldots, M\} \), and obtain block bootstrap sample \( \{e^*_k, k = 1, \ldots, M\} \). Then, we generate FDBB samples according to 
\[
\hat{y}^*_t = \hat{\mu} + \hat{\varphi}^*_t, (1 - B)^d \hat{x}^*_t = \epsilon^*_t.
\]
Step 4. Calculate the statistic \( T_y \), which has same definition as \( T_y \), based on the FDBB samples \( y^*_1, \ldots, y^*_T \) write as \( T_y^* \).
Step 5. Repeat Step 3 to Step 4 B times, approximate the probability distribution function of \( T_y \) and it’s critical values by the empirical distribution function and empirical quantiles of \( T_y^*, \ldots, T_y^B \).

Under some mild conditions, Papailias and Kapetanios [12] has proved similar result as (5), which guarantees the consistency of FDBB method to analysis the long memory time series. The blocks described in step 2 are not overlap one another, and the optimal block size \( b \) can be estimated by some data driven method. In fact, we can also use overlapping blocks or other block bootstrap methods dividing series \( \{e_k\} \). Papailias and Kapetanios [12] showed that the FDBB using stationary bootstrap in step 2, can provide more accurate approximations in finite samples when estimates the distribution of the sample mean and the Gaussian Semi Parametric Estimator. However, we found that the superiority of stationary bootstrap, especially in mean change point test, is not very significant. To simplify the algorithm, we still use un-overlap blocks with fixed block size \( b = [5 * n^{1/3}] \) in our simulation.

4. Simulation Studies
In this section, we perform simulation for a comparison of SARB, FDSB and FDBB methods. In the experiments, we compare the distribution approximation performance by estimating the bootstrap sample mean. The sample moment (SM) method which computes the result using the raw data as a benchmark method also will be considered. We use ARFIMA(p,d,q) model (1) to generate data by set \( \mu = 1 \), and consider AR coefficients \( \varphi_i \) and MA coefficients \( \theta_j \) set to be: \( (\varphi_1, \theta_1) = (0,0) \), \( (\varphi_1, \theta_1) = (0.5,0) \), \( (\varphi_1, \theta_1) = (-0.3,0.4) \), and \( (\varphi_1, \varphi_2, \theta_1) = (0.2,0.5,0) \), and the error term is standard normal in all cases. To save space, we only report the results for sample size \( n = 100,400 \), and the results for \( n = 1000 \) can be provided on request. All simulations are obtained via 1000 repeats.

The normalized sample mean by SM method is computed by 
\[
n^{1/2-d}(\bar{y}_n - \mu),
\]
and the bootstrap estimates are computed as averages of 199 bootstrap resamples of \( n^{1/2-d}(\bar{y}^*_n - \bar{y}_n) \). Here, \( \bar{y}_n \) and \( \bar{y}^*_n \) denote the sample means of observed data and generated data by bootstrap methods respectively. The long memory parameter \( d \) varying among \( \{0, 0.2, 0.4\} \).

Table 1-2 show the mean and standard deviations (St.d) of computed normalized sample mean in four different data generating cases. From these tables, we firstly notice that all means are closer to the zero but the St.d are far from each other. This indicates that all methods can well approximate the mean of true distribution, but the robustness of these methods are different. In table 1 we found that FDBB method has the best performance, and FDSB method has the worst performance in the most cases. An interesting founding is that the performance of SARB method becomes better as long memory parameter \( d \) increase while other three methods have opposite performance. In addition, the sample size has little influence to improve the accuracy of estimation for all methods. From table 2, we can get similar conclusions as in table 1 except SM method becomes the worst one in general.
5. Conclusions
This paper has studied finite sample performances of three popular used bootstrap methods in long memory time series via simulation experiments. Three Bootstrap methods were used to approximate the mean distribution of the samples and compared with the real mean distribution of the samples. By the degree of difference, we found the FDBB method has excellent performance in data reconstruction, that is, the FDBB method is superior to the other two methods in approximating the distribution of the sample mean.

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Table 1. Mean and St.d of normalized sample mean

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 100 | 0.039 | 0.983   | 0.014   | 0.968  | -0.014  | 1.407   |
|    | SAR   | -0.04   | 1.03    | -0.015 | 0.768   | 0.009   |
|    | FDSB  | -0.03   | 0.928   | -0.028 | 1.411   | 0.024   |
|    | FDBB  | -0.017  | 0.367   | 0.015  | 0.403   | -0.014  |

Table 2. Mean and St.d of normalized sample mean

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 100 | -0.022| 1.088   | 0.009   | 1.081  | 0.037   | 1.485   |
|    | SAR   | 0.024   | 1.105   | -0.007 | 0.775   | -0.025  |
|    | FDSB  | 0.011   | 0.949   | -0.009 | 1.216   | -0.051  |
|    | FDBB  | 0.362   | 0.009   | 0.392  | 0.024   | 0.569   |

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 400 | -0.047| 1.01    | 0.011   | 0.969  | -0.115  | 1.343   |
|    | SAR   | 0.052   | 1.03    | -0.008 | 0.62    | 0.036   |
|    | FDSB  | 0.043   | 0.96    | -0.011 | 1.095   | 0.148   |
|    | FDBB  | 0.012   | 0.239   | -0.002 | 0.253   | -0.038  |

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 100 | 0.114 | 3.183   | 0.114   | 3.129  | 0.114   | 3.958   |
|    | SAR   | 0.118  | 3.141   | 0.118  | 3.763   | 0.118   |
|    | FDSB  | -0.06  | 1.081   | -0.06  | 1.292   | -0.06   |
|    | FDBB  | 0.118  | 0.946   | 0.118  | 1.378   | 0.118   |

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 400 | 0.09   | 3.161   | 0.09    | 3.255  | 0.09    | 4.28    |
|    | SAR   | 0.118  | 3.141   | 0.118  | 3.763   | 0.118   |
|    | FDSB  | -0.026 | 1.103   | -0.026 | 1.555   | -0.026  |
|    | FDBB  | 0.118  | 0.946   | 0.118  | 1.378   | 0.118   |

Table 3. Mean and St.d of normalized sample mean

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 100 | 0.025 | 1.01    | 0.009   | 0.962  | 0.025   | 1.485   |
|    | SAR   | 0.024   | 1.03    | -0.007 | 0.775   | -0.025  |
|    | FDSB  | 0.011   | 0.949   | -0.009 | 1.216   | -0.051  |
|    | FDBB  | 0.362   | 0.009   | 0.392  | 0.024   | 0.569   |

| n  | d = 0 | d = 0.2 | d = 0.4 | d = 0 | d = 0.2 | d = 0.4 |
|----|-------|---------|---------|-------|---------|---------|
|    | Mean  | St. D.  | Mean    | St. D. | Mean    | St. D.  |
| 400 | 0.114 | 3.183   | 0.114   | 3.129  | 0.114   | 3.958   |
|    | SAR   | 0.118  | 3.141   | 0.118  | 3.763   | 0.118   |
|    | FDSB  | -0.06  | 1.081   | -0.06  | 1.292   | -0.06   |
|    | FDBB  | 0.118  | 0.946   | 0.118  | 1.378   | 0.118   |

5. Conclusions
This paper has studied finite sample performances of three popular used bootstrap methods in long memory time series via simulation experiments. Three Bootstrap methods were used to approximate the mean distribution of the samples and compared with the real mean distribution of the samples. By the degree of difference, we found the FDBB method has excellent performance in data reconstruction, that is, the FDBB method is superior to the other two methods in approximating the distribution of the sample mean.

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