The exponentially truncated q-distribution: A generalized distribution for real complex systems.

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Abstract

To know the statistical distribution of a variable is an important problem in management of resources. Distributions of the power law type are observed in many real systems. However power law distributions have an infinite variance and thus can not be used as a standard distribution. Normally professionals in the area use normal distribution with variable parameters or some other approximate distribution like Gumbel, Wakeby, or Pareto, which has limited validity.

Tsallis presented a microscopic theory of power law in the framework of non-extensive thermodynamics considering long-range interactions or long memory. In the present work, we consider softing of long-range interactions or memory and presented a generalized distribution which have finite variance and can be used as a standard distribution for all real complex systems with power law behaviour. We applied this distribution for a financial system, rain precipitation and some geophysical and social systems. We found a good agreement for entire range in all cases for the probability density function (pdf) as well as the accumulated probability. This distribution shows universal nature of the size limiting in real systems.

I. Introduction
To know the statistical distribution of a variable is an important problem in management. For example, the distribution of the variation of a share price is important for financial management, while the distribution of the water flux or water level in a river or rain precipitation is important for water and flood management.

Recently physicists started to study the natural systems as a whole rather than in parts [1-6] and are interested in holistic properties of these systems normally called “Complex Systems”. This also include financial, social, biological, economical and geophysical systems as they have the same characteristics.

Power law scaling [7,8] is observed in many such systems [2,9-29] and it is now considered an important property of these systems. In general, power law exists in the central part of the distribution. It deviates from power law for very small and very large steps.

Long-range interactions and memory effects are present in all real systems including social and economical systems [30,31] and are important for a statistical distribution. Tsallis through non-extensive thermodynamics gives a microscopic basis for power law [32,33] considering long-range interactions and long memory effects. The distribution also explain the initial deviation from power law

Power law have infinite variance which discourage a physical approach and an unavoidable cut-off is always present. Mantega and Stanley [34] introduced the truncated Lévy flight in which the probability of taking a step is abruptly cut to zero at a certain critical step size. Koponen [35], gradually truncated the probability distribution from the begining. This violate power law distribution in central part. Gupta and Campanha [36-38] proposed the gradually truncated Lévy flight in which the probability distribution is cut-off gradually only after a certain critical value. This distribution have an undesirable discontinuity at the critical length. Tsallis et. al. [39] consider a cross-over behaviour to explain deviation from power law for extreme values. This distribution do not explain a sharp cut-off as observed in many cases [40]. Thus, in absence of a standard distribution for these systems, in practice, normal distribution with variable parameters is used. When one is interested in extreme value distributions, some other distribution like Wakely, Gumbel, exponential, log-normal etc [41] are used, which are valid only in a particular range and generally do with provide a physical basis.

A statistical distribution for these systems must have:

(i) finite variance,
(ii) continuous distribution,
(iii) power law in the central part,
(iv) can explain all kind of cut-off from very sharp to very slow for extreme values,
(v) must have a physical basis for the truncation of the power law.

In the present paper, we propose softing of long-range interactions or long memory with increase of the variable size under consideration. This avoid infinite variance. Finally we present a generalised statistical distribution based on this concept. With the availability of computer programs to numerically integrate an function, this distribution can be used to calculate the probability distribution function and accumulated probability in any range of the distribution with fixed parameters. In Section II, we present the model and the distribution. In Section III, we present a method to estimate the parameters of the distribution. In Section IV, we apply this statistics for many problems in diverse areas and finally in Section V, we discuss the results.

II. The model and the statistics

In 1998, Tsallis [32], presented non-extensive thermodynamics in which he incorporated long range interactions and long memory effects. He proposed a generalized definition of entropy ($S_q$):

$$S_q = C \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}$$

where $C$ is a positive constant, and $W$ is the total number of microscopic possibilities of the system. $q$ is an entropic index, which plays a central role and is related to long range interactions and long memory effect in a network. This expression recovers the usual Boltzmann-Gibbs entropy ($-C \sum_{i=1}^{W} p_i \ln p_i$) in the limit $q \to 1$, i.e. in short range interactions. In this case, the size frequency distribution function $N(x)$ is given through
\[ \frac{dN(x)}{dx} = -\lambda N(x) \]  

(2)

where \( \lambda \) is a positive constant. \( N(x) \) is the frequency probability of size \( x \).

This gives

\[ N(x) = N_0 \exp(-\lambda x) \]  

(3)

In general, the frequency density distribution function \( N(x) \) is given through:

\[ \frac{dN(x)}{dx} = -\lambda N^q(x) \]  

(4)

hence

\[ N(x) = \frac{N_0}{[1 + (q-1)\lambda x]^{\frac{1}{q-1}}} \]  

(5)

where \( N_0 \) is a normalization constant. This expression recovers the usual Boltzmann distribution in the limit \( q \to 1 \) i.e. in short-range interactions as shown in Equation (3). For \( q > 1 \), this expression gives power law for relatively large values of the step \( x \).

The power law distribution can not continue forever in real systems. It has to be truncated in some way to avoid infinite variance and have a finite size.

In order to consider long range departure, Tsallis et. al. [39] assume a crossover to another type of behavior and modify Equation (4) as

\[ \frac{dN(x)}{dx} = -\mu_r N^r(x) - (\lambda - \mu_r)N^q(x) \]  

(6)

\( \mu_r \) is very small compared to \( \lambda \). That gives a crossover between two different power laws (respectively characterized by \( q \) and \( r \)) or from power law to normal distribution within a nonextensive scenario.
Although cross-over behavior as suggested by Tsallis can avoid an infinite variance, in the present work, we are looking for another possibility, i.e., the truncation of power law due to softing of long-range interaction or memory which gives finite size in real systems. This is not a cross-over behavior. We consider that entropy factor $q$ decreases with step size $(x)$ due to the softening of long-range interactions or memory effects which arises because of the physical limitations of the components or the system itself. Thus $q$ depends on the step size. This is similar as anharmonic terms are important for calculating potential energy in lattice vibrations.

The size limiting factor is of a very small importance for small steps, while it is necessary for larger steps. Entropy index $q$ is equal to 1 in the absence of long memory or long range interactions. Thus the information about these interactions are given through $(q - 1)$. We consider that this factor approaches to zero for very large values of $x$. In general, for this:

$$ (q(x) - 1) = \frac{(q_0 - 1)}{1 + \sum_i \theta_i x^i} $$

where $q_0$ and $q(x)$ are values of entropy index $q$ for step size zero and step size $x$ respectively. $\theta_i$ and $i$ are adjustable parameters depending on the softing of long-range interactions or memory.

To simplify, we propose an exponential decay i.e.

$$ q(x) - 1 = (q_0 - 1) \exp(-\theta x^i) $$

where $\theta$ and $i$ show the rate of decrease of the importance of these interactions with the increase of step size $x$. The higher value of $i$ indicates a sharper cut-off. $\theta$ is a scaling factor for cut-off.

For very large values of $x$, $q(x)$ approaches to 1 and thus gives normal distribution as required through central limit theorem. In the present model the distribution function is given through:

$$ N(x) = N_0[1 + (q_0 - 1)\lambda x \exp(-\theta x^i)]^{-\frac{\exp((\theta x^i)/(q_0 - 1))}{(q_0 - 1)}} $$
For simplification we replace \((q_0 - 1)\lambda\) by another constant \(\beta\), and \(1/(q_0 - 1)\) through the power exponent \(\alpha\). Finally the frequency density distribution is given by

\[
N(x) = N_0[1 + \beta x \exp(-(\theta x)^i)]^{-\alpha \exp((\theta x)^i)}
\]  

(10)

In Figure 1 we compare \(N(x)\) vs. \(x\) in present approach through Equation (10) for very large steps. Under the present model, the gradual truncation of the power law can be adjusted from very sharp to very slow through the value of \(i\) without interfering in power law behavior in the central part of the distribution. In cross-over behaviour we can not explain very sharp cut-off without violating power law in the central part [40].

Figure 1 – Theoretical distribution of present Model in log-log scale. We consider: \(N_0 = 1.10^6\); \(\beta = 0.0025\); \(\alpha = 2.0\). Curves A, B, C, D, E, and F are through considering \(i = 1/2\) and \(\theta = 3.10^{-6}\) (Curve A), \(i = 1\) and \(\theta = 3.10^{-5}\) (Curve B), \(i = 2\) and \(\theta = 1.10^{-4}\) (Curve C), \(i = 3\) and \(\theta = 1.5.10^{-4}\) (Curve D), \(i = 4\) and \(\theta = 1.8.10^{-4}\) (Curve E) and \(i = 5\) and \(\theta = 2.10^{-4}\) for Curve F.

In terms of probability density distribution \((p(x))\), the distribution is:
\[ p(x) = k [1 + \beta x \exp(-(\theta x)^i)]^{-\alpha \exp((\theta x)^i)} \]  \hspace{1cm} (11)

where \( k \) is another normalization constant and equal to \( N_0/N_{total} \), where \( N_{total} \) is the total number of observations.

Further

\[ \int_{-\infty}^{\infty} p(x) dx = 1 \]  \hspace{1cm} (12)

Many times, the maximum frequency is not at \( x = 0 \). In this case we need to shift the origin to \( x_m \), where \( x_m \) is the most frequent value of the variable \( x \). At \( x_m \) the frequency is maximum, however it is not necessary a mean value of the variable. In this case the frequency density distribution is given through:

\[ N(x) = N_0 [1 + \beta |x - x_m| \exp(-(\theta |x - x_m|^i))]^{-\alpha \exp((\theta |x - x_m|^i))} \]  \hspace{1cm} (13)

The physical mechanisms behind the distribution for \( x > x_m \) and \( x < x_m \) may be different and thus the parameters of the distribution may also be different. Thus the two cases must be treated separately.

For \( \theta = 0 \), the present distribution turn out to be Tsallis distribution [33,39] as is expected.

The accumulated probability in between \( a \) and \( b \) is given through

\[ P(a \leq x \leq b) = \int_{a}^{b} p(x) dx \]  \hspace{1cm} (14)

The value of \( p(x) \) at \( x = 0 \) must be carefully studied, particularly when \( x_m = 0 \). In many cases it may be a discrete number and include many cases apart from the mechanism under discussion. For example, in financial market, the variation zero in the price of a particular share also include cases
in which it is not at all traded along with cases in which it is traded but
with zero variation. Thus in these cases $p(0)$ should be separately estimated
through

$$p(0) = \frac{Events \ x = 0}{Total \ events}$$  \hspace{1cm} (15)$$

and thus

$$\int_{-\infty}^{\infty} p(x)dx = p(0) + \int_{-\infty}^{0^-} p(x)dx + \int_{0^+}^{\infty} p(x)dx = 1$$  \hspace{1cm} (16)$$

where $0^-$ and $0^+$ are respectively the values of $x$ below and above zero.

We define $N(x)$ as events in the range $((x - \Delta x) < x < (x + \Delta x))$ divided by
$\Delta x$ and thus eliminate $p(0)$ when $x = \frac{\Delta x}{2}$.

There is very little mass for extreme values of $x$ and thus it is difficult to
compare a theoretical distribution with available data. The technique known
as a Zipf plot [38] is very important in this case. Suppose we ordered our
observations from largest to smallest so that the index $i$ is the rank of $x_i$. Then

$$i = N \int_{x_i}^{\infty} p(x)dx$$  \hspace{1cm} (17)$$

Thus the rank is simply a transformation of the accumulative distribution
function. The empirical accumulated probability above $x_i$ is its rank $i$ divided
by total number of observations. The accumulated probability accentuates
the upper tail of the distribution and therefore makes it easier to detect the
deviations in the extreme tails from the theoretical predictions of a particular
distribution.

III. Estimation of parameters.
We use the following steps to estimate the parameters of the distribution. Let \((x_1, x_2, ..., x_N)\) be the set of \(N\) observations of a random variable \(x\) for which the probability density function is \(p(x)\). We select a proper bin size \(\Delta x\) and make a frequency table \(f(x)\) vs \(x\). \(f(x)\) gives the number of observations in between \((x - \frac{\Delta x}{2})\) and \((x + \frac{\Delta x}{2})\). From this table we observe the value of \(x_m\), i.e. the value of \(x\), for which we have the maximum value of the frequency \(\left(N_0\right)\). We separate the observations in two groups, one for \(x > x_m\) and the other for \(x < x_m\). Each group may have different values of parameters because of different mechanisms in two cases.

The frequency density function \(N(x)\) is given by:

\[
N(x) = \left| \frac{f(x)}{\Delta x} \right|_x
\]

(18)

In the case of extreme values of \(x\), we have very little mass i.e. very few observations and thus, we have many zeros in \(f(x)\) vs \(x\) table for the extreme values because of the limited and random nature of the observations. Thus to make a physical significance of observations, we increase the value of the interval \(\Delta x\) for the extreme values to avoid zero values of \(N(x)\).

For small steps, cut-off parameters are of negligible importance. We therefore put \(\theta = 0\) and \(i = 0\) for initially 50% steps and estimate \(\alpha\) and \(\beta\). \(N_0\) and \(x_m\) are estimated through frequency tables. Knowing these parameters, we estimate \(\theta\) and \(i\) for the best fit for the entire curve. Some time it may be necessary to re-estimate \(\alpha\) in this stage.

IV. Applications.

Now we apply this model to describe the distribution of a parameter in some geophysical complex systems of interest

(A) Water level of a river:

For water, flood and agriculture management, it is extremely important to know the distribution of water level in a river of interest. It is therefore regularly registered by water management department. In the present case we took the water level in Paraná River, one of the important river in Brazil,
at the São Paulo-Paraná border. The water level is measured daily by the Agencia Nacional de Agua and can be obtained at the site www.ana.gov.br. We analyze water level in the period of 1\textsuperscript{st} of January of 1964 to 30\textsuperscript{th} of June of 2005, in total having 15,067 observations. This river receives water from many sources and the water level depends on rainfall at different places at different times and thus present a complex system with long term memory and long-range interactions.

Through frequency distribution of the the empirical data, we observe maximum frequency density $N_0 = 10,032$ days/m at the height of 2.87 m ($x_m$). For $x \geq x_m$, we found $\alpha = 1.66$, $\beta = 1.25$, $\theta = 0.185$ and $i = 2.85$ for best fit. For $x \leq x_m$, we found $\alpha = 3.2$, $\beta = 0.54$, $\theta = 0.5$ and $i = 9.0$.

In Figure 2a we compare $\log(\text{frequency})$ vs water level(a) distribution. Plotting log(frequency) we can compare cases of even very small frequency at extremely higher water levels. In Figure 2b, we compare accumulated probability distribution density above water level $a$ $P(x > a)$ vs $a$. The agreement is good for the entire curve including extreme cases up to four orders of magnitude in accumulated probability density as well as in frequency density distribution.

![Figure 2a](image)

**Figure 2a** - Frequency density vs water level (m) in semi log scale. The continuous line is through present model. The dotted points are empirical
Figure 2b - Accumulated probability density above water level \( a \) \((P(x > a))\) vs water level \((a)\) in semi-log scale. The dotted points are empirical, while the continuous line is through present model.

(B) Water flux in a river:

Another problem of interest in water management is the distribution of water flux in a river at a particular point of interest. We have 5,428 observations in the period of 23\(^{th}\) of October of 1969 to 31\(^{th}\) of August of 1984 in Paraná river at São Paulo-Paraná border. Through frequency distribution of the observations data, we find \( N_0 = 24.4 \) days./seg./m\(^3\) at \( x_m = 155 \) m\(^3\)/s. For \( x \geq x_m \), we found \( \alpha = 0.84, \beta = 0.013, \theta = 0.00177 \) and \( i = 3.0 \). For \( x \leq x_m \), we found \( \alpha = 0.8, \beta = 0.045 \). Due to the small number of observations on this side, we did not observe any truncation of q-values and thus consider \( \theta = 0 \) and \( i = 0 \). In Figure 3a we compare log(frequency) vs water flux while in Figure 3b we compare accumulated probability distribution density \( P(x > a) \) above water flux \((a)\) vs water flux \((a)\). Again the agreement is good throughout the curve including extreme values.
Figure 3a - Frequency density vs water flux ($m^3/s$) in semi log scale
To compare the distribution of rain precipitation, an importante problem for agriculture and water control, we took a time series of daily rain precipitation at Campinas city in São Paulo State, Brazil, in the period of 1950 to 1980 at station prefix D4-044. These data were obtained from the Departamento de Aguas e Energia Eletrica of São Paulo State and are available at [http://www.sigrh.sp.gov.br](http://www.sigrh.sp.gov.br). In total we have 21,549 observations.

In case of rain precipitation, probability of precipitation zero is a singular point as it also includes days when there is no rain at all. We are only considering days when there is rain precipitation more than zero. We have $N_0 = 598$ days/mm for $x_m = 1.0$ mm. The values of the parameters are $\alpha = 1.7$, $\beta = 0.074$, $\theta = 0.08$ and $i = 0.12$. The number of days when there is no rain precipitation is 15.558 days, thereby giving $p(0) = 0.722$.

In Figure 4a, we plotted $\log(frequncy) vs rain precipitation$. In Figure 4b we plotted accumulated probability density distribution $P(x > a)$ for rain precipitation above $a$ versus rain precipitation. Again the agreement is good.
Figure 4a - Frequency density vs rain precipitation in mm.

Figure 4b - Accumulated probability density for rain precipitation above a ($P(x > a)$) vs rain precipitation a in semi log scale. The dotted points are empirical, while a continuous line is through present model.
(D) Financial Systems: Variation of an economical index.

It has been shown recently that the variation of a share price in high frequency limit i.e. variation per minute is given through power law [36]. However for extreme values the predicted variation by power law is much more than what is observed and it must be truncated in some way or other. In the present case we took variation of the price of the share of Banco do Brasil, the biggest semi-government bank of Brazil. We consider variation per minute, i.e. in high frequency range. The period is from 1st of July of 2004 to 30th of June of 2007 in total of 329,489 observations and furnished by IBOVESPA - São Paulo. In this case the probability of the variation zero is a singular point as it also includes all those minutes, when the share is not at all traded along with minutes when the share is traded but with zero variation.

For positive variation we have frequency density \(N_0 = 29,640\) at \(x_m = 0.5 \times 10^{-3}\)%. The values of the parameters for this side is \(\alpha = 5.5, \beta = 0.197\). We did not observe the effect of gradual truncation in this period and so we put \(\theta = 0\) and \(i = 0\). For negative side we found \(N_0 = 29,640\) at \(x_m = -0.5 \times 10^{-3}\)%. The values of the parameters for this side are \(\alpha = 4.0, \beta = 0.3325, \theta = 1.10^{-7}\) and \(i = 0.12\). This means that for this side, the effect of the truncation although small, still is necessary. In Figure 5a, we shown log(frequency) vs percentage variation in unit of \(10^{-3}\) The agreement is good. In Figure 5b we only compare the accumulated probability for the 10,000 highest variations in frequency density as they are most important. Again we found a good agreement.
Figure 5a - Frequency density vs. percentage variation ($10^3$) in share price/min.

Figure 5b - Accumulated probability density for share price variation above a ($P(x > a)$) vs. share price variation for extreme values in semi log
scale. The dotted points are empirical, while a continuous line is through present model..

(E) Distribution of the sun spots.

The number of sun spots per month is a very old index and is available from 1611. This index measures the magnetic activity in the sun. Sun spot number data can be obtained from National Geophysical Data Center in Boulder, Colorado and is available at the site [http://www.ngdc.noaa.gov].

In the Figure 6a we show the distribution of monthly sun spots from 1749 to 2007 and compare with present model with parameters $N_0 = 58.4$ month$^2$/sun spot for $x_m = 2.5$, $\alpha = 0.8$, $\beta = 0.04$, $\theta = 0.0075$ and $i = 1.9$. In Figure 6b we compare the accumulated probability density for $x \geq a$ versus $a$. The agreement is good in both cases.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6a.png}
\caption{Figure 6a - Frequency density vs. sun spots/month}
\end{figure}
V. Discussion

In the present work, we presented a statistical distribution considering that the entropic index \((q - 1)\), which gives information about long range interactions and/or memory effects, decreases exponentially with size of the variable. This distribution automatically gives a power law in the central part and deviates for very small and very large values of the variable as normally is observed in most of the complex systems. It gives finite variance as required through central limit theorem. In the present work we applied this model for various geophysical and financial systems, and found a good agreement in all cases. We tried this model in some other cases like citation index of scientists and marks distribution in an entrance examination [40], citation index of scientific publications [42] where also we obtain a good agreement for eight order of magnitude. In certain cases, due to limited observations, we could not estimate the values of gradually truncation parameters and thus consider them equal to zero.
This distribution present an statistics for complex systems which is valid for entire range and can be used by geophysical and financial professionals. Thus it elliminate the necessity to use distribution with variable parameters or an approximate distribution which is valid only in a limited range. It has a strong physical basis and we have shown a good agreement up to four orders of magnitude or more. Thus it provide a confiable standard distribution for these systems. This model also present an universal nature of the truncation process in the distribution of a parameter of a complex system obeying power law.
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