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The Tubby Torus as a Quotient Group

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Abstract: Let $E$ be any metrizable nuclear locally convex space and $\hat{E}$ the Pontryagin dual group of $E$. Then the topological group $\hat{E}$ has the tubby torus (that is, the countably infinite product of copies of the circle group) as a quotient group if and only if $E$ does not have the weak topology. This extends results in the literature related to the Banach–Mazur separable quotient problem.

Keywords: torus; tubby torus; separable quotient problem; locally convex space; nuclear space; Banach space; Pontryagin duality; weak topology

1. Introduction and Preliminaries

The Separable Quotient problem for Banach spaces has its roots in the 1930s and is due to Stefan Banach and Stanislaw Mazur. While a positive answer is known for various classes of Banach spaces [1], such as reflexive Banach spaces, weakly compactly generated Banach spaces, and more generally Banach-like spaces [2], the general problem remains unsolved.

Problem 1. (Separable quotient problem for Banach spaces) Does every infinite-dimensional Banach space have a quotient Banach space which is separable and infinite-dimensional?

The following problem stated in [3] is also unsolved, but a negative answer to it would give a negative answer to Problem 1.

Problem 2. Does every infinite-dimensional Banach space have a quotient topological group which is homeomorphic to the countably infinite product, $\mathbb{R}^\omega$, of copies of $\mathbb{R}$?

This suggests another question which we have not seen mentioned in the literature. We state the problem and answer it.

Question 1. Does every infinite-dimensional Banach space have a quotient topological space which is homeomorphic to $\mathbb{R}^\omega$?

Question 1 has a positive answer, although it uses very powerful machinery due to Toruńczyk. It is known [4] that every infinite-dimensional Fréchet space $F$ (that is, a complete metrizable locally convex space) is homeomorphic to an infinite-dimensional Hilbert space $H$. So an infinite-dimensional Banach space $B$ (indeed an infinite-dimensional Fréchet space) is homeomorphic to an infinite-dimensional Hilbert space $H$, which obviously has the infinite-dimensional separable Hilbert space $\ell_2$ as a quotient. Further, by the separable case of Toruńczyk’s theorem which is known as the Kadec–Anderson theorem, the separable Fréchet space $\mathbb{R}^\omega$ is homeomorphic to $\ell_2$, from which the positive answer to Question 1 follows.
Noting that Problem 2 remains open, it is natural to ask if every infinite-dimensional Banach space has a quotient topological group which is a separable metrizable topological group which is infinite-dimensional as a topological space. This was answered in the positive by the following theorem.

**Theorem 1.** [5] Every locally convex space $E$, which has a subspace which is an infinite-dimensional Fréchet space, has the tubby torus, $T^\omega$, as a quotient group, where $T$ is the compact circle group. In particular, this is the case if $E$ is an infinite-dimensional Banach space.

We should mention the following result.

**Theorem 2.** [6] If $E$ is any infinite-dimensional Fréchet space which is not a Banach space, then $E$ has the locally convex space $\mathbb{R}^\omega$ as a quotient vector space.

**Corollary 1.** If $E$ is any infinite-dimensional Fréchet space which is not a Banach space, then $E$ has the tubby torus $T^\omega$ as a quotient group.

One might suspect that every infinite-dimensional locally convex space has the tubby torus as a quotient group. This is shown to be false in [5] for the free locally convex space $\varphi$ on a countably infinite discrete space. Indeed in [7] it is shown that if $X$ is a countably infinite $k_\omega$-space, then the free topological vector space on $X$, which is a connected infinite-dimensional (in the topological sense) topological group, does not have the tubby torus as a quotient group or even any infinite-dimensional (in the topological sense) metrizable quotient group.

It was recently proved that free topological groups on infinite connected compact spaces also have the tubby torus as a quotient group.

**Theorem 3.** [7] Let $F_G(X)$ and $A_G(X)$ be the Graev free topological group and the Graev free abelian topological group, respectively, on an infinite connected compact Hausdorff space. Then the connected topological groups $F_G(X)$ and $A_G(X)$ have the tubby torus $T^\omega$ as a quotient group.

It follows from Theorem 2.5 of [3] that every non-metrizable connected locally compact abelian group has the tubby torus as a quotient group. But as a connected locally compact abelian group $G$ is isomorphic as a topological group to the product $\mathbb{R}^n \times K$, for some non-negative integer $n$ and compact abelian group $K$, and $\mathbb{R}^n$ and all compact metrizable groups are separable, we see that if $G$ is non-separable then it is non-metrizable. So we obtain the following result as a consequence.

**Theorem 4.** Every non-separable connected locally compact abelian group has the tubby torus as a quotient group.

As mentioned earlier, Problem 1 has been aswered for dual-like groups. In particular there is the following powerful and beautiful theorem.

**Theorem 5.** [8] If $B$ is the Banach space dual of any infinite-dimensional Banach space, then $B$ has a separable infinite-dimensional quotient Banach space.

**Corollary 2.** If $B$ is the Banach space dual of any infinite-dimensional Banach space, then $B$ has the tubby torus as a quotient group.

Recall that if $G$ is a (Hausdorff) abelian topological group, then we denote by $\hat{G}$ the group of all continuous homomorphisms of $G$ into the circle group $\mathbb{T}$, where $\hat{G}$ has the compact-open topology.
There is a natural homomorphism \( \alpha : G \to \hat{\hat{G}} \). The Pontryagin–van Kampen duality theorem is stated below and a discussion and proof appear in [9,10].

**Theorem 6.** [9,10] If \( G \) is any locally compact abelian group then the map \( \alpha \) is an isomorphism of topological groups of \( G \) onto \( \hat{\hat{G}} \). Also, if \( H \) is a closed subgroup of the locally compact abelian group \( G \), then \( \hat{H} \) is a quotient group of \( G \), and if \( A \) is a quotient group of \( G \), then \( \hat{A} \) is isomorphic as a topological group to a closed subgroup of \( \hat{G} \). Further, the map \( \alpha \) restricted to \( H \) is an isomorphism of topological groups of \( H \) onto the subgroup \( \alpha(H) \) of \( \hat{G} \).

The following is less well-known.

Let \( E \) be a locally convex space. As \( E \) is a topological group, the topological group \( \hat{\hat{E}} \) consisting of all continuous group homomorphisms of \( E \) into \( T \) with the compact-open topology is a topological group, as is \( \hat{E} \). As mentioned above, there is a natural homomorphism of \( E \) into \( \hat{\hat{E}} \).

**Theorem 7.** [11] Proposition 15.2. Let \( E \) be a complete metrizable locally convex space (that is a Fréchet space). Then \( \alpha \) is an isomorphism of topological groups of \( E \) onto \( \hat{\hat{E}} \).

We note that Theorem 7 does not tell us whether, for example a restricted to a closed subgroup \( H \) of \( E \) is an isomorphism of topological groups of \( H \) onto the subgroup \( \alpha(H) \) of \( \hat{\hat{E}} \). In fact this is not always true. §11 of [12] gives an example of a closed subgroup \( H \) of a Fréchet space \( E \) such that \( \alpha \) restricted to \( H \) is not an isomorphism of topological groups of \( H \) onto its image in \( \hat{\hat{E}} \). To see how badly things can go “wrong”, we note Theorem 6.1 of [11]: Let \( E \) be a metrizable locally convex space. If \( E \) is not a nuclear space, then it has a discrete subgroup \( H \) such that there are no non-trivial continuous homomorphisms from \( \text{span}(H)/H \) into \( T \), where \( \text{span}(H) \) denotes the linear span in \( E \) of \( H \).

Theorem 5 leads us then to the natural question:

**Problem 3.** If \( E \) is any infinite-dimensional Fréchet space which does not have the weak topology and \( \hat{E} \) is its dual topological group, does \( \hat{\hat{E}} \) have the tubby torus as a quotient group? In particular, is this the case for \( E \) a Banach space or a Schwartz space?

This question is open, however a positive answer is given for nuclear spaces in the next section.

2. The Main Result

**Definition 1.** A topological group \( G \) is said to be reflexive if the natural mapping \( \alpha \) from \( G \) to \( \hat{\hat{G}} \) is an isomorphism of topological groups. The topological group \( G \) is said to be strongly reflexive if every closed subgroup and every Hausdorff quotient group of \( G \) is reflexive.

**Theorem 8.** [12] (Theorem 20.35) Every complete metrizable nuclear locally convex space is strongly reflexive.

**Proposition 1.** [11] (Proposition 17.1(c)) Let \( H \) be a closed subgroup of a strongly reflexive topological group \( G \). Then \( \hat{H} \) is isomorphic as a topological group to a quotient group of \( \hat{G} \).

**Theorem 9.** Let \( E \) be a metrizable nuclear locally convex space. Then \( \hat{\hat{E}} \) has the tubby torus \( T^\omega \) as a quotient group if and only if \( E \) does not have the weak topology.

**Proof.** By Theorem 2 of [13], if \( H \) is a dense subgroup of the metrizable topological group \( G \), then \( \hat{G} \) is isomorphic as a topological group to \( \hat{H} \). So the dual group \( \hat{\hat{E}} \) of \( E \) is isomorphic as a topological group to the dual group of the completion of \( E \). So there is no loss of generality in assuming that \( E \) is complete. Further, the completion of a metrizable nuclear locally convex space is a metrizable nuclear locally convex space by Theorems 20.34 and 20.20 of [12].
The theorem in [14] says that a locally convex space $E$ has the weak topology if and only if every discrete subgroup of $E$ is finitely generated. However, its proof there gives rather more. Namely, the locally convex space $E$ does not have the weak topology if and only if $E$ contains a discrete free abelian subgroup $S$ which is not finitely generated.

So if the metrizable locally convex space $E$ does not have the weak topology, then it has a subgroup $S$ isomorphic as a topological group to a restricted direct product of $\mathbb{Z}_i$, $i = 1, 2, \ldots, n, \ldots$, where each $\mathbb{Z}_i$ is isomorphic as a topological group to the discrete $\mathbb{Z}$ of integers. Noting §3 of [15], we see that the dual group of this restricted direct product of $\mathbb{Z}_i$ is the tubby torus $\mathbb{T}^\omega$, and it then follows from Theorem 8 and Proposition 1 that $\hat{E}$ has the tubby torus as a quotient group, as required.

On the other hand if the complete metrizable locally convex space $E$ has the weak topology, then it is isomorphic as a locally convex space to $\mathbb{R}^\omega$. So its dual group $\hat{E}$ is isomorphic as a topological group to the locally convex space $\varphi$. However, as mentioned earlier, it is proved in [5] (and generalized in [7]), that $\varphi$ does not have the tubby torus as a quotient group, which completes the proof.

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