On Strict Higher C*-categories

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Abstract

We provide definitions for strict involutive higher categories (a vertical categorification of dagger categories), strict higher C*-categories and higher Fell bundles (over arbitrary involutive higher topological categories). We put forward a proposal for a relaxed form of the exchange property for higher (C*)-categories that avoids the Eckmann-Hilton collapse and hence allows the construction of explicit non-trivial "non-commutative" examples arising from the study of hypermatrices and hyper-C*-algebras, here defined. Alternatives to the usual globular and cubical settings for strict higher categories are also explored. Applications of these non-commutative higher C*-categories are envisaged in the study of morphisms in non-commutative geometry and in the algebraic formulation of relational quantum theory.

Keywords: C*-category, Fell Bundle, Involutive Category, Higher Category, Hypermatrix.

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1 Introduction

The usage of categorical methods in functional analysis is probably going back to A. Grothendieck, but category theory began to be applied to the theory of operator algebras in the seventies, with the pioneering work of J.E. Roberts [GLR] that introduced the definition of C*-categories mainly in view of applications to algebraic quantum field theory. Since then category theory has been extensively used by J.E. Roberts and S. Doplicher [DR] in the theory of superselection sectors in algebraic quantum field theory (see R. Haag’s monograph [H] and M. Müger’s review [Mu]). Operator categories and C*-categories have been further studied by S. Yamagami [Y], P. Mitchener [M], T. Kajiwara-C. Pinzari-Y. Watatani [KPW] and Y. Sawada-S. Yamagami [SY] to name a few. The closely related and more general notion of a Fell bundle over a topological group was first defined by J. Fell [FD] (under the name of a Banach ∗-algebraic bundle) and later extended respectively to: topological groupoids, by S. Yamagami and then A. Kumjian [Ku]; topological inverse semigroups, by N. Seiben (see R. Excel [E]); and topological involutive inverse categories in [BCL4, BCL6].

The study of higher n-categories, at least in their strict versions, can be traced back to the work of C. Ehresmann [E] on structured categories (the notion of ∞-groupoid predating actually to a paper of D. Kan on simplicial complexes [Ka]). Strict ω-categories were proposed by J.E. Roberts [R1] (in his work on local cohomology in algebraic quantum field theory) and were independently developed by R. Brown-P. Higgins [BH] and A. Grothendieck [G] with strong motivations from homotopy theory in algebraic topology. Weak higher category theory, starting from the notion of bicategory of J. Benabou [Be], subsequently formalized by R. Street [S], recently developed into a wide area of extremely active research (see for example as references T. Leinster [L1, L2], E. Cheng-A. Lauda [CL] and the wiki-resources at http://ncatlab.org/nlab).

Surprisingly, despite their quite close initial historical developments and the current widespread usage of categorical methods/techniques in recent research, a satisfactory interplay between higher categories and operator algebra theory has never been achieved and higher category theory has more recently evolved along lines, much closer to classical higher homotopy, that in our opinion further prevent a direct interaction between the two subjects.

Although monoidal C*-categories (2-C*-categories with one object) have been systematically used since the inception of the theory of superselection sectors [DR, H], a first notion of 2-C*-category appears only in the paper by R. Longo-J. E. Roberts [LR] and the topic has been later reconsidered by P. Zito [Z]. Bicategories (weak 2-categories) of von Neumann algebras have also been investigated by N. Landsman [Lan] and R. M. Brouwer [Br]. Anyway, no hint of operator algebraic structures capable of climbing up, in a non-trivial way, the ladder of n-C*-categories for n bigger than 2 has been produced,

1As can be seen for example in the works by J.E. Roberts, G. Ruzzi, E. Vasselli [V1, V2, V3, V5], [RRV1, RRV2] in algebraic quantum field theory; by B. Coecke, S. Abramski, C. Heunen and collaborators [AC1, AC2, CP, ColAH] in categorical quantum mechanics; and by A. Buss-R. Meyer-C. Zhu [BMZ1, BMZ2], S. Mahanta [Ma], among others, in non-commutative topology and geometry.
so that higher C*-categories, and with them the full development of a comprehensive theory of “higher functional analysis”, have remained elusive. We announced a tentative definition of strict (globular) $n$-C*-category (still based on the usual axioms for strict higher categories) in [BCL2] section 4.2.2 with details in [BCL3] section 3.3 and in the following paper we propose a much wider notion of strict $n$-C*-category, able to encompass several quite interesting non-trivial and natural examples of non-commutative operator theoretic constructs.

In very general terms, with the terminology that we introduced in [BCL2] section 4.2], the efforts presented here can be seen as the first attempt for the development of a full vertical categorification of functional analysis and operator algebra, in the same way as the transition from C*-algebras to C*-categories can be described as a horizontal categorification of functional analysis.

We stress, as a disclaimer, that the main inspiration for our proposed set of C*-categorical axioms stems from the attempt to vertically categorify Gel’fand-Naimark dualities; in particular it is not our intention here to define “higher C*-categorical settings” for a discussion of Tannaka-Krein dualities.

In section 2, we briefly recall the main C*-algebraic definitions and results that constitute the background for our work. Here the reader who is not already familiar with operator algebras will find a detailed definition of C*-algebras, their horizontal categorification (C*-categories) and their “bundled” generalizations (Fell bundles also on general involutive categories) as well as previously available definitions of monoidal (tensor) C*-category (Doplicher-Roberts) and 2-C*-category (Longo-Roberts).

In section 3 the standard notions of strict globular higher $n$-category are introduced making use of “partial $n$-monoids”, an equivalent definition in term of properties of $n$ composition operations $(\alpha_0, \ldots, \alpha_{n-1})$ partially defined on a family of $n$-cells. The Eckmann-Hilton collapse argument is presented in detail, explaining how it prevents any inclusion of non-commutative “diagonal blocks” at depth higher than 1. In order to avoid this fatal degeneration (that is ultimately responsible for the lack of reasonable examples of higher C*-categories that exhibit non-commutative features), we propose here to relax the exchange property and substitute it with a weaker condition of left/right $\alpha_p$-functoriality of the compositions by $\alpha_q$-identities, for $0 \leq q < p < n$. We are fully aware of the fact that this modified “non-commutative” exchange property is not fitting with most of the current developments in higher category theory, but we stress that ultimately its relevance in higher category theory will be vindicated by the abundance of quite natural examples available. In this same section, for later use, we also discuss examples of strict $n$-categories (mainly Cartesian products of 1-categories) whose $n$-cells naturally admit compositions that do not fit with the usual globular or cubical picture of strict higher $n$-categories now available: relaxing the exchange property not only allows more non-commutativity for the compositions, but also more freedom in the “composability” of cubical $n$-cells.

In section 4 we describe a full vertical categorification of P.Selinger’s dagger categories, via strict involutions defined as endo-functors that can be covariant or contravariant with respect to any of the partial compositions of a strict globular $n$-category. This is not the only way to introduce notions of “duality” for $n$-cells, but it is in perfect agreement with the tradition of J.E.Roberts’ $\ast$-categories, where involutions are treated on the same footing as compositions. The resulting notion of a (partially/fully) involutive higher category should be interesting on its own. A much more detailed study of higher involutions for globular and cubical $n$-categories appears in our companion paper [BCM].

Although the introduction of involutions with mixed covariance properties might seem to invalidate the non-commutativity gained via the relaxed exchange property (see remark 5.28), its effects still allow the existence of non-trivial non-commutative examples as long as the “diagonal-blocks” are equipped with “more products/involutions” as will be described in section 5 (see proposition 5.29 and theorem 5.37).

The definition of strict higher (globular) C*-categories rests on several additional pieces of structure that are considered in section 5. As the first step, we define higher $\ast$-algebroids (of minimal depth) introducing complex linear structures on each family of globular $n$-arrows with a common $(n-1)$-sources/targets and imposing conditions of bilinearity for compositions and conjugate-linearity for involutions. This is just the easiest form of vertical categorification of the usual notion of $\ast$-category used by J.E.Roberts (and later reconsidered by P.Mitchener): in principle (as already suggested in the axioms presented in [BCL3]) one might provide, for all $0 \leq p < n$, completely different linear structures
on the sets of $n$-arrows with common source/targets at depth-$p$; for simplicity we decided to avoid here this further generalization, that will be discussed in more details elsewhere. Next, if a Banach norm is placed on each of the previous linear spaces one can impose suitable axioms of submultiplicativity for the compositions $\sigma_0, \ldots, \sigma_{n-1}$ and $\text{C}^*$-norm and positivity conditions for some or for all pairs $(\sigma_p, \sigma_q)$ of depth-$p$ composition/involution. In this way one obtains a vertical categorification of the notion of $\text{C}^*$-category (of minimal linear depth) that can in principle be fully involutive and that also generalizes Longo-Roberts 2-$\text{C}^*$-categories, where only the $\sigma_{n-1}$ involution is present. A definition of $n$-Fell bundle is easily obtained, whenever the pair $n$-groupoid $\mathcal{C}/\mathcal{C}$ of linear spaces in the strict globular fully involutive $n$-$\text{C}^*$-category $\mathcal{C}$ is replaced by a more general fully involutive $n$-category.

In order to provide many non-trivial examples of fully involutive strict higher $\text{C}^*$-categories, always in section 5 we look at the usual algebra $\mathbb{M}_{N\times N}(\mathbb{C})$ of square complex matrices of order $N$ as an algebra of sections of a Fell line-bundle over the pair groupoid of a set of $N$ points and we simply substitute $\mathbb{C}$ with an arbitrary (possibly non-commutative) unital $\text{C}^*$-algebra $\mathcal{A}$ and $N \times N$ with an arbitrary finite discrete (fully) involutive $n$-category $\mathcal{X}$. The Cartesian bundle $\mathcal{X} \times \mathcal{A}$ is a natural example of a strict globular $n$-category (see theorem 5.24) and the non-commutative exchange property is absolutely necessary to give “citizenship rights” to the structure in the case of non-commutative algebras $\mathcal{A}$. Whenever $\mathcal{A}$ is a commutative $\text{C}^*$-algebra, $\mathcal{X} \times \mathcal{A}$ becomes an $n$-Fell bundle (a fully involutive $n$-$\text{C}^*$-category, when $\mathcal{X}$ is a $n$-groupoid); unfortunately (as explained in remark 5.28 and proposition 5.29) this result cannot hold for the case of non-commutative $\text{C}^*$-algebras $\mathcal{A}$, but it can be recovered (see theorem 5.37) with a more complex system of non-commutative coefficients in place of the $\text{C}^*$-algebra $\mathcal{A}$.

The resulting family of sections $\mathbb{M}_\mathcal{X}(\mathcal{A})$, the “enveloping $n$-convolution algebra” of the $n$-Fell bundle $\mathcal{X} \times \mathcal{A}$, is the first example of what we call a hyper-$\text{C}^*$-algebra: a complete topolinear space equipped with $N$ different $\text{C}^*$-algebraic structures $(\sigma_p, \sigma_p, \| \cdot \|_p)$, whose norms are equivalent. We finally provide further natural examples of such hyper-$\text{C}^*$-algebras, via nested hypermatrices and we also show how these hyper-$\text{C}^*$-algebras can be seen as higher-convolution algebras . . . as long as we allow cubical sets (in place of globular sets) and we consider, over the set of $n$-cells, $2^n$ different pairs $(\sigma_p, \sigma_p)$, $p = 0, \ldots, n - 1$, of composition/involution. This is quite a strong hint for the relevance of higher categorical constructs that do not find place in present-day axiomatizations of $n$-categories and where the relevance of our non-commutative exchange property is even more evident.

In section 6 we informally discuss some wilder speculations on the possible applications of the formalism of non-commutative higher $\text{C}^*$-categories to non-commutative geometry and quantum theory. A quite strong motivation for the consideration of higher $\text{C}^*$-categories comes from the need to formulate general categorical environments for non-commutative geometry. Morphisms between usual “commutative” spaces are given by families of 1-arrows (a relation or more generally a 1-quiver) connecting points of the spaces, so that “dually” a morphism corresponds to a bimodule over the commutative algebra of functions over the graph of a relation. In that “classical” context, as suggested in [BJ], there is no problem at all in performing a vertical categorification. On the contrary, vertical categorifications of morphisms between non-commutative spaces (dually described by bimodules over non-commutative algebras) are quite difficult to achieve, since the usual exchange property imposes strong commutativity conditions. Taking inspiration from our previous work on the spectral theorem for commutative full $\text{C}^*$-categories [BCL4], we are led to think of the spectrum of a non-commutative algebra as a “family” of Fell line-bundles (spaceoids), so that morphisms of non-commutative spaces appear to be naturally described by 2-quivers with a cubical structure, and hence dually, by suitable higher bimodules.

Since non-commutative spaces (in the language of A.Connes’ spectral triples) are essentially very specific quantum dynamical systems, it does not come as a surprise that higher operator category theory becomes relevant in the description of “quantum channels” and “correlations” between quantum systems (at least when these are described in the language of algebraic quantum theory as $\text{C}^*$-algebras). Actually, since the very beginning of this investigation in higher $\text{C}^*$-category theory, the mathematical formalization of relational quantum theory has been one of the basic goals of our research in view of its potential impact on our ongoing efforts in modular algebraic quantum gravity [BCL3, B].

We finally collect in section 6 some further indication on possible extensions of this work, also in directions that we plan to explore in the future.
2 C*-algebras and C*-categories

The theory of operator algebras (see for example B.Blackadar [Bl] for an overview of the subject and further references) is a quite developed area of functional analysis with extremely important applications to the mathematical approaches to quantum theory (see for example the books by F.Strocchi [St], R.Haag [H], G.Emch [Em], O.Bratteli-D.Robinson [BR] and J.E.Roberts’ lectures [R2, R3]). Since our main purpose is to examine some possible routes for a vertical categorification of such a theory (with some non-trivial examples), we start here with a brief review, recalling the basic notion of C*-algebra, its horizontal categorified and “bundlified” versions (C*-categories and Fell bundles), as well as the few instances of already available axioms for monoidal and 2-C*-categories.

The readers that are not already familiar with the notions of category theory mentioned here, will find all the references and required definitions in detail in the following section 3.

2.1 C*-algebras, C*-categories, Fell Bundles, Spaceoids

C*-algebras, originally defined by I.Gel’fand-M.Naimark [GN], are the most basic gadget in the theory of operator algebras and non-commutative geometry [C], where they play the role of non-commutative topological spaces and it is natural to start from them in any attempt to categorify functional analysis. A C*-algebra is a rigid blend of algebraic and topological structures: an associative algebra over \( \mathbb{C} \) equipped with an associative unital bilinear multiplication \( \circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and conjugate-linear antimultiplicative involution \( \ast : \mathcal{C} \to \mathcal{C} \), a complex unital C*-algebra.

Definition 2.1. A complex unital C*-algebra \((\mathcal{C}, \circ, \ast, +, \cdot, \|\|)\) is given by the following data:

- a complex associative unital involutive algebra i.e. a complex vector space \((\mathcal{C}, +, \cdot)\) over \( \mathbb{C} \), equipped with an associative unital bilinear multiplication \( \circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and conjugate-linear antimultiplicative involution \( \ast : \mathcal{C} \to \mathcal{C} \),
- a norm \( \|\| : \mathcal{C} \to \mathbb{R} \) such that the following properties are satisfied:
  - completeness: \((\mathcal{C}, +, \cdot, \|\|)\) is a Banach space,
  - submultiplicativity of the norm: \(\|x \circ y\| \leq \|x\| \cdot \|y\|\) for all \(x, y \in \mathcal{C}\),
  - C*-property: \(\|x^\ast \circ x\| = \|x\|^2\) for all \(x \in \mathcal{C}\).

Basic non-commutative examples are the families \(\mathcal{B}(\mathcal{H})\) of linear continuous maps on a Hilbert space \(\mathcal{H}\) (and all the norm-closed unital involutive subalgebras of them); commutative examples are essentially algebras \(\mathcal{C}(X; \mathbb{C})\) of complex-valued continuous functions on a compact Hausdorff topological space \(X\).

Horizontal categorifications of C*-algebras have been developed a long time ago by J.E.Roberts and used in the theory of superselection sectors in algebraic quantum field theory. The formal definition first appeared in P.Ghez-R.Lima-J.E.Roberts [GLR] and it has been revisited more recently in greater details in P.Mitchener [M].

Definition 2.2. A C*-category \((\mathcal{C}, \circ, \ast, +, \cdot, \|\|)\) is given by the following data:

- an involutive algebroid \((\mathcal{C}, \circ, \ast, +, \cdot)\) over \(\mathbb{C}\):
  - a category \((\mathcal{C}, \circ)\), with objects (partial identities) \(\mathcal{C}^0 \subset \mathcal{C}\),
  - a contravariant functor \(\ast : \mathcal{C} \to \mathcal{C}\) acting trivially on \(\mathcal{C}^0\),
  - for all pairs of objects \(A, B \in \mathcal{C}^0\), a complex vector space structure \((\mathcal{C}_{AB}, +, \cdot)\) on the blocks \(\mathcal{C}_{AB} := \text{Hom}_c(\mathcal{B}(\mathcal{H}), \mathcal{A})\), on which the composition \(\circ : \mathcal{C}_{BC} \times \mathcal{C}_{AB} \to \mathcal{C}_{AC}\), \((y, x) \mapsto x \circ y\) is bilinear and the involution \(\ast : \mathcal{C}_{AB} \to \mathcal{C}_{BA}, x \mapsto x^\ast\) is conjugate-linear,
- a norm function \(\|\| : \mathcal{C} \to \mathbb{R}\) such that:
  - completeness: \((\mathcal{C}_{AB}, +, \cdot)\) are Banach spaces, \(\forall A, B \in \mathcal{C}^0\),
objects contains only one element. Basic examples of C*-categories are provided by the family
involutive categorical bundle, i.e.
\[\|x^\ast \circ x\|=\|x\|^2\text{ for all }x\in\mathcal{C},\]
positive: for all \(x \in\mathcal{C}\), the element \(x^\ast \circ x\) is positive in the unital C*-algebra \(\mathcal{C}_{(x^\ast \circ x)}\) where \(s(x)\rightarrow t(x)\).

Remark 2.3. The axiom of positivity, in the case of C*-algebras, is redundant.
In the statement of this positivity property, we make use of the fact that \(\mathcal{C}_{(x^\ast \circ x)}\) is a unital C*-algebra, for all \(x \in\mathcal{C}\), where \(s(x)\) denotes the source partial identity of the element \(x\).
In fact it is immediately implied by the definition that, for all objects \(A,B \in\mathcal{C}\), the diagonal blocks \(\mathcal{C}_{AA}\) are unital C*-algebras and the off-diagonal blocks \(\mathcal{C}_{AB}\) are unital Hilbert C*-bimodules, over the C*-algebra \(\mathcal{C}_{BB}\) to the right, and over the C*-algebra \(\mathcal{C}_{AA}\) to the left, with right and left inner products given respectively by \(\langle x \mid y\rangle := x \circ y^\ast\) and \(\langle x \mid y\rangle: = x^\ast \circ y\) that satisfy the associative property \(\langle x \mid y\rangle z = x \langle y \mid z\rangle\), for all \(x,y,z \in\mathcal{C}\).

As we can expect from horizontal categorification, a C*-algebra is just a C*-category whose class of objects contains only one element. Basic examples of C*-categories are provided by the family \(\mathcal{B}(\mathcal{H})\) of linear bounded operators between Hilbert spaces belonging to a given class \(\mathcal{H}\) (every C*-category can be seen as a norm-closed unital involutive sub-algebra of \(\mathbb{B}(\mathcal{H})\) for a given family \(\mathcal{H}\)).

A C*-category \(\mathcal{C}\) can immediately be seen as a bundle, with Banach fibers \(\mathcal{C}_{AB}\) over the pair groupoid \(\mathcal{G}^0 \times \mathcal{G}^0 := \{AB \mid A,B \in \mathcal{G}\}\) of its objects with the discrete topology. Allowing more than a single arrow connecting two objects \(A,B\) of the base category and adding the possibility of a non-trivial topology, leads to the definition of a Fell bundle, that plays a fundamental role in spectral theory (in a way that further elaborates on the tradition of the celebrated Dauns-Hofmann theorem [DH]).

Definition 2.4. A Banach bundle is a bundle \((\mathcal{E}, \pi, \mathcal{X})\), i.e. a continous open surjective map \(\pi : \mathcal{E} \rightarrow \mathcal{X}\), whose total space is equipped with:

- a partially defined continuous binary operation of addition \(+ : \mathcal{E} \times_{\mathcal{X}} \mathcal{E} \rightarrow \mathcal{E}\), with domain the set \(\mathcal{E} \times_{\mathcal{X}} \mathcal{E} := \{(x,y) \in \mathcal{E} \times \mathcal{E} \mid \pi(x) = \pi(y)\}\),
- a continuous operation of multiplication by scalars \(\cdot : \mathbb{K} \times \mathcal{E} \rightarrow \mathcal{E}\),
- a continuous “norm” \(\|\| : \mathcal{E} \rightarrow \mathbb{R}\), such that:
  - for all \(x \in \mathcal{X}\), the fiber \(\mathcal{E}_x := \pi^{-1}(x)\) is a complex Banach space \((\mathcal{E}_x,+,\cdot)\) with the norm \(\|\|\).
  - for all \(x_0 \in \mathcal{X}\), the family \(U_{x_0}^0 = \{e \in \mathcal{E} \mid \|e\| < \epsilon, \pi(e) \in O\}\), where \(O \subset \mathcal{X}\) is an open set containing \(x_0 \in \mathcal{X}\) and \(\epsilon > 0\), is a fundamental system of neighborhoods of \(0 \in \mathcal{E}_{x_0}\).

A Hilbert bundle is a Banach bundle whose norm is induced fiberwise by inner products.

A Fell bundle over a topological involutive category \(\mathcal{X}\), is a Banach bundle \((\mathcal{E}, \pi, \mathcal{X})\) that is also an involutive categorical bundle, i.e. \(\pi : \mathcal{E} \rightarrow \mathcal{X}\) is a continuous *-functor between topological involutive categories \(\mathcal{E}, \mathcal{X}\), and such that:

- \(\|x \circ y\| \leq \|x\| \cdot \|y\|\) for all composable \(x,y \in \mathcal{E}\),
- \(\|x^\ast \circ x\| = \|x\|^2\) for all \(x \in \mathcal{E}\),
- \(x^\ast \circ x\) is positive whenever \(\pi(x^\ast \circ x)\) is an idempotent in \(\mathcal{X}\).[]

See, for example, J.Fell-R.Doran [FD] Section I.13] or N.Waer [W] Chapter 9.1] and the references therein.

For Fell bundles over topological groups see J.Fell [FD] Section II.16]; for Fell bundles over groupoids (originally introduced by S.Yamagami) see K.Kumjian [Ko]; for Fell bundles over inverse semigroups (defined by N.Seiben) see R.Exel [E] Section 2]; Fell bundles over involutive inverse categories (involutive categories \(\mathcal{X}\) such that \(x \circ x^\ast \circ x = x\) for all \(x \in \mathcal{X}\) appeared in [BCL6].

The condition is meaningful because the fiber \(\mathcal{E}_{(x^\ast \circ x)} \in \mathcal{E}\) is a C*-algebra if and only if \(\pi(x^\ast \circ x) \in \mathcal{E}\) is an idempotent.
Remark 2.5. The positivity condition in the previous definition requires some care: the axioms preceding it already imply that every fiber $\mathcal{E}_P$ is a C*-algebra, whenever $P \in \mathcal{X}$ is an idempotent in the involutive category $\mathcal{X}$, hence it is perfectly possible to require the positivity of $x^* \circ x$ if it belongs to such a fiber (this is the usual condition in the case of Fell bundles over groupoids and C*-categories).

It might seem suspicious that no additional positivity requirement is necessary for an arbitrary $x \in \mathcal{E}$. Since $\mathcal{E}_{\mathcal{X}(P \circ x)}$ is generally only a Hilbert C*-bimodule, the only reasonable option would be to ask the positivity of $x^* \circ x$ as an element of a suitable convolution C*-algebra “generated” by $\mathcal{E}_{\mathcal{X}(P \circ x)}$. The positivity axiom in the previous definition of Fell bundle is a necessary condition for the existence of such a C*-algebra; anyway, if such a C*-algebra exists, all the elements $x^* \circ x$ would always be already positive, making further requirements redundant.

Although we will not enter here into this very interesting topic, using a variant of the construction of the C*-algebra of multipliers via double centralizers, it is actually possible to show that convolution C*-algebras for fibers of a Fell bundle (as defined here) always exist. As already anticipated, a C*-category $(\mathcal{E}, \circ, *, \|\|)$ is itself a special case of a Fell bundle over the pair groupoid $\mathcal{E}_0 \times \mathcal{E}_0$ with the discrete topology and with fibers $\mathcal{E}_{AB}$, for $(A, B) \in \mathcal{E}_0 \times \mathcal{E}_0$.

Other elementary examples of Fell bundles (over groupoids) are given by the “tautological” bundles with base any strict groupoid of imprimitivity Hilbert C*-bimodules (in the category of strong Morita equivalences of complex unital C*-algebras), with fibers the Hilbert C*-bimodules themselves. Other notable examples of Fell bundles are given by spectra of commutative full C*-categories defined, used and studied in [BCL4, BCL6].

The relevance of these structures for spectral theory can be fully appreciated considering the following theorem by A. Takahashi [T1, T2] (originally proved via the Dauns-Hofmann theorem), that simultaneously subsumes the Gel’fand-Na˘ımark duality and (the Hermitian version of) the Serre-Swan equivalence, and from the horizontal categorification of the Gel’fand-Na˘ımark duality described in [BCL4].

Theorem 2.6 (Takahashi [T2]). There is a duality between the bicategories $\mathcal{C}$ of homomorphisms of Hilbert C*-modules over commutative unital C*-algebras,

- $\mathcal{F}$ of Takahashi morphisms of Hilbert bundles over compact Hausdorff spaces.

Morphisms in $\mathcal{C}$ are pairs $(\phi, \Phi) : \mathcal{A} \mathcal{M} \to \mathcal{B} \mathcal{N}$ with $\Phi : \mathcal{M} \to \mathcal{N}$ adjointable map of Hilbert C*-modules and $\phi : \mathcal{A} \to \mathcal{B}$ a unital *-homomorphism such that $\Phi(\mathcal{M}) = \phi(\mathcal{A}) \cdot \Phi(\mathcal{M})$, $\forall a \in \mathcal{A}, x \in \mathcal{M}$.

Morphisms in $\mathcal{F}$ are pairs $(f, F) : (\mathcal{E}, \pi, X) \to (\mathcal{F}, \rho, Y)$, where $f : X \to Y$ is a continuous map and $F : \mathcal{F}(\mathcal{E}) \to \mathcal{E}$ is a continuous fiberwise-linear map of Hilbert bundles over $X$ defined on the total space $f^*(\mathcal{F})$ of the $f$-pull-back of the Hilbert bundle $(\mathcal{F}, \rho, Y)$.

Theorem 2.7 (Bertozzini, Conti, Lewkeeratiyutkul [BCL4]). There is a duality, via horizontally categorized Gel’fand evaluation natural transformations, between the categories:

- $\mathcal{C}$ of *-functors between full commutative C*-categories,

- $\mathcal{F}$ of Takahashi morphism of spectral spaceoids (that are Fell line-bundles over the Cartesian product of a pair groupoid and a compact Hausdorff topological space).

The two functors in duality are the:

- section functor $\mathcal{F} \overset{\Gamma}{\rightarrow} \mathcal{C}$ that to a spaceoid associates its C*-category of continuous sections,

- spectrum functor $\mathcal{C} \overset{\Sigma}{\rightarrow} \mathcal{F}$ that to a commutative full C*-category associates its spectral spaceoid.

5 Although A. Takahashi does not directly treat tensor products, the bicategorical version is immediate (see [BCL4, BCL6]).

6 With operations given by composition and tensor product.
2.2 Monoidal C*-categories, Longo-Roberts 2-C*-categories

Towards a full vertical categorification of C*-algebras, in this subsection we start with a discussion of those few already available notions that are directly related to higher C*-categories.

In S.Doplicher-J.E.Roberts [LR] a notion of monoidal (or tensor) C*-category has been developed. Since strict monoidal categories are strict 2-categories with only one object (the monoidal identity), such definition is the first available hint for the axioms of 2-C*-categories.

Definition 2.8. A strict monoidal C*-category is a C*-category \((\mathcal{C}, \circ, \ast, \cdot, \parallel\parallel)\) equipped with an additional binary operation \(\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) such that:

- \((\mathcal{C}, \otimes)\) is a monoid (a category with only one object),
- \(\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) is a bifunctor,\(^7\)
- \(\otimes\) is a bilinear map when restricted to pairs of composable 1-blocks,
- \(*: (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes)\) is a covariant functor.

Remark 2.9. The actual categories considered by S.Doplicher and J.E.Roberts for the theory of superselection sectors in algebraic quantum field theory are equipped with additional structures: they are symmetric monoidal C*-categories, closed under retracts, direct sums and (more important for us) with conjugates. Following R.Haag [H] section IV.4], if necessary to avoid confusion, we reserve the name Doplicher-Roberts C*-categories for such more specific cases. We will later return to a careful study of conjugates for strict monoidal C*-categories (and more generally for Longo-Roberts 2-C*-categories defined here below) in section 4.3 and example 5.18.

The notion of 2-C*-category was developed by R.Longo-J.E.Roberts [LR, section 7] and further studied by P.Zito [Z]. It is a horizontal categorification of a monoidal C*-category with the following definition:

Definition 2.10. A Longo-Roberts 2-C*-category is a strict 2-category \((\mathcal{C}, \circ, \otimes)\) such that

- for all objects \(A, B \in \mathcal{C}\) the block \(\mathcal{C}_{AB}\) is a C*-category with composition \(\circ\) and involution \(\ast\),
- the partial bifunctor \(\otimes\) is bilinear when restricted to \(\circ\)-composable 0-blocks,
- \(*: (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes)\) is a covariant functor.\(^8\)

The easiest examples of monoidal C*-categories are given by bounded linear maps between a family of Hilbert spaces (with the usual composition and tensor product). Similarly, examples of Longo-Roberts 2-C*-categories are given by adjointable maps between a family of right (respectively left) Hilbert C*-correspondences (i.e. unital bimodules \(A\mathcal{M}_B\) over complex unital C*-algebras with a \(B\)-valued (respectively \(A\)-valued) inner product).

Remark 2.11. Since algebraic tensor products of bimodules over rings (and similarly Rieffel internal tensor product of Hilbert C*-correspondences over C*-algebras) are only weakly associative and weakly unital, it would appear that the previous examples produce only 2-categories that are “weak” under \(\otimes\) and hence do not precisely comply with definition 2.10. This problem is easily eliminated via the following useful strictification procedure embedding all the given Hilbert C*-bimodules into their strictly associative tensor algebroid of paths (this is a horizontal categorification of a well-known C.Chevalley’s procedure [CH] and essentially consists of constructing the required tensor products of bimodules inside a strictly associative unital tensor ring: the free ring generated by the bimodules).

Consider a 1-quiver \(\mathcal{M} \rightrightarrows \mathcal{M}\) whose nodes are unital associative rings \(A, B \in \mathcal{M}\) and whose 1-arrows (for example with source \(B\) and target \(A\)) are unital bimodules of the form \(A\mathcal{M}_B \in \mathcal{M}\). Denote by

\[^7\text{Recall that a bifunctor from } (\mathcal{C}, \circ) \text{ to } (\mathcal{C}, \circ) \text{ is a functor } \otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C} \text{ defined on the product category } \mathcal{C} \times \mathcal{C}, \text{ with componentwise composition. This condition implies the exchange property.}
\[^8\text{For some aspects of the theory it is also required the triviality condition } \mathcal{C}_I = \mathcal{C}, \text{ where } I \text{ denotes the monoidal identity.}
\[^9\text{This property (that is true in all the examples) is actually missing in both LRZ, but this is probably just a careless omission, otherwise the definition would not reproduce that of monoidal C*-categories when there is only one object.}\]
whose

[\mathcal{M}] \Rightarrow \mathcal{A}

the fine graining of the previous 1-quiver \mathcal{M} consisting of the same nodes \mathcal{A} but with every element \( x \in \mathcal{A} \mathcal{M} \mathcal{B} \) considered as a different 1-arrow from \( \mathcal{B} \) to \( \mathcal{A} \) (and including, for all \( A \in \mathcal{A} \), all the elements \( a \in A \) as 1-loops based on \( A \)). Proceed to the construction of the free 1-category of paths ([\mathcal{M}]) generated by the fine grained 1-quiver [\mathcal{M}] \Rightarrow \mathcal{A} and then to \( \mathbb{Z}([\mathcal{M}]) \), its category ringoid with coefficient in \( \mathbb{Z} \). This is a horizontal categorification of the usual monoid ring \( \mathbb{Z}[X] \) with integer coefficients over the monoid \( X \): its elements are finite formal linear combinations with integer coefficients of 1-arrows belonging only to a given block ([\mathcal{M}]) \_A \B \_A (here each of these blocks is an abelian group). Bilinearly extending the composition, \( \mathbb{Z}([\mathcal{M}]) \) turns out to be a ringoid with the set of objects \( \mathcal{A} \). Finally we obtain the tensor ringoid \( \mathcal{T}(\mathcal{M}) \) quotienting the ringoid \( \mathbb{Z}([\mathcal{M}]) \) by the categorical ideal \( \mathcal{I} \) generated (block by block) by the elements of the form

\[
(\ldots, x, b, y, \ldots) - (\ldots, x, by, \ldots), \quad (\ldots, x, b, y, \ldots) - (\ldots, xb, y, \ldots),
\]

\[
(\ldots, x_1 + x_2, \ldots) - (\ldots, x_1, \ldots) - (\ldots, x_2, \ldots),
\]

for all 1-arrows \( x, x_1, x_2 \in \mathcal{A} \mathcal{M} \mathcal{B} \) and all 1-loops \( a \in \mathcal{A} \). Each one of the original bimodules \( \mathcal{A} \mathcal{M} \mathcal{B} \) (and each one of the rings \( \mathcal{A} \in \mathcal{A} \)) has an isomorphic copy inside \( \mathcal{T}(\mathcal{M}) \) via the inclusion \( x \mapsto [(x)] := (x) + \mathcal{I}_{\mathcal{A} \mathcal{B}} \) and the tensor product operation, defined by \( [(x)] \otimes [(y)] := [(x, y)] \), for all \( x \in \mathcal{A} \mathcal{M} \mathcal{B}, y \in \mathbb{Z} \mathcal{N} \mathcal{E} \), is now strictly associative and unital as required.

3 Strict Higher Categories and Non-commutative Exchange

In this section we introduce, with some detail, the basic definitions in the theory of strict n-categories with their usual “exchange property”. We then present the well-known Eckmann-Hilton collapse argument and, in order to avoid it, we propose a relaxed form of exchange property (quantum or non-commutative exchange) consisting in a request of p-identities, whenever \( q < p \). Finally, for later use, we also discuss examples (products of categories) whose n-cells admit compositions that do not fit with the usual globular or cubical situations.

Here, the (admittedly questionable) inspiring ideology is to view the current developments in higher category theory as heavily motivated by “classical homotopy theoretical” arguments (for example the exchange property) that might not be suitable for a formalization of non-commutative operator algebraic structures that are otherwise perfectly natural and fitting into a higher categorical context.

Although some of the most natural approaches to the definition of strict higher categorical environments are via “globular/cubical higher quivers” \([2] \) definition 1.4.8\] and either via “inductive enrichment of categories” \([2] \) definition 1.4.1\] (for the case of globular shaped cells) or via “inductive internal categories” \([2] \) definition 1.4.13\] (for the case of cubical shaped cells), for our discussion here, in view of its extreme compactness, we will use the algebraic definition of globular strict n-categories via axioms for their “n-cells”. We will mainly consider the case of “globular n-cells” and a more careful study of strict n-tuple categories, based on similar algebraic axioms for “cubical cells”, will be done elsewhere\([9]\)

3.1 Strict Globular Higher Categories (via partial higher monoids)

Among the equivalent definitions of strict 1-category, we choose a compact axiomatization formulated in terms of properties of 1-arrows under a partial binary operation of composition without any direct reference to objects, identities, source and target maps. The following, for example, is a variant of the definition provided in S. Mac Lane \([1]\) section 1, page 9]. The resulting notion of partial monoid is a horizontal categorification of the usual definition of monoid, obtained by “localization” of identities.

**Definition 3.1.** A 1-category \((\mathcal{E}, \odot)\) is a family \( \mathcal{E} \) of 1-cells (arrows) equipped with a partially defined binary operation of composition \( \odot \) that satisfies the following requirements:

\[\footnote{P.Bertozzi, R.Conti, R.Dawe-Martins, “Involutive Double Categories” (manuscript) and “Double C*-categories and Double Fell Bundles” (works in progress).}\]
the composition is associative i.e. whenever one of the two terms \( f \circ (g \circ h) \) and \((f \circ g) \circ h \) exists, the other one exists as well and they coincide,

for every arrow \( f \in \mathcal{C} \) there exist a right composable (source) arrow \( r \in \mathcal{C} \) and a left composable (target) arrow \( l \in \mathcal{C} \) that are partial identities (objects) i.e. for all arrows \( h_1, h_2, k_1, k_2 \in \mathcal{C} \):

\[
h_1 \circ r = h_1, r \circ h_2 = h_2 \text{ and } k_1 \circ l = k_1, l \circ k_2 = k_2,
\]

when the compositions exist,

if \( f \) has a right identity that is also a left identity for \( g \), the composition \( f \circ g \) exists.

A 1-functor \((\mathcal{C}_1, o_1) \xrightarrow{\phi} (\mathcal{C}_2, o_2)\) between two 1-categories is a map \( \phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) such that

- whenever \( x \circ_1 y \) exists, also \( \phi(x) \circ_2 \phi(y) \) exists and in this case \( \phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y) \),
- if \( e \) is a partial identity in \((\mathcal{C}_1, o_1)\), \( \phi(e) \) is a partial identity in \((\mathcal{C}_2, o_2)\).

The class of the partial identities (objects) will be denoted by \( \mathcal{C}^0 \subset \mathcal{C} \) with inclusion map \( \iota : A \mapsto \iota_A \).

From the axioms it follows immediately the every \( x \in \mathcal{C} \) has a unique right partially identity (its source) and a unique left partial identity (its target) that we will denote respectively by \( s(x) \) and \( t(x) \).

The following graphical representations are self-explicative:

- 0-cells (objects): \( \bullet \),
- 1-cells (arrows): \( \bullet_1 \xrightarrow{x} \bullet_2 \),
- sources / targets: \( s(x) \xrightarrow{x} t(x) \),
- identities: \( A \mapsto A \xrightarrow{1_A} \)
- composition: \( A \xrightarrow{g} B \xrightarrow{f} C \mapsto A \xrightarrow{f \circ g} C \).

Given \( A, B \in \mathcal{C}^0 \), we denote by \( \mathcal{C}_{AB} := \{ x \in \mathcal{C} | s(x) = B, t(x) = A \} \) the “block” of 1-arrows with source \( B \) and target \( A \). The category \((\mathcal{C}, o)\) is said to be locally small if every block \( \mathcal{C}_{AB} \) is a set and small if \( \mathcal{C} \) is also a set\(^{[1]}\).

Also for \( n \)-categories, we have an equivalent “\( n \)-arrows based”-definition. The following is essentially the following of J.E.Roberts as provided by J.E.Roberts-G.Ruzzi \([RR]\) and already used, for the case \( n = 2 \), in R.Longo-J.E.Roberts \([LR]\) and P.Zito \([Z]\):

**Definition 3.2.** A globular strict \( n \)-category \((\mathcal{C}, o_0, \ldots, o_{n-1})\) is a set \( \mathcal{C} \) equipped with a family of partially defined compositions \( o_p \) for \( p := 0, \ldots, n - 1 \), that satisfy the following list of axioms:

- for all \( p = 0, \ldots, n - 1 \), \((\mathcal{C}, o_p)\) is a 1-category, whose partial identities are denoted by \( \mathcal{C}^0 \)\(^{[2]}\),
- for all \( q < p \), a \( o_q \)-identity is also a \( o_p \)-identity, i.e. \( \mathcal{C}^q \subset \mathcal{C}^p \),
- for all \( p = 0, \ldots, n - 1 \), with \( q < p \), the \( o_q \)-composition of \( o_p \)-identities, whenever exists, is a \( o_q \)-identity, i.e. \( \mathcal{C}^q \circ_{o_q} \mathcal{C}^p \subset \mathcal{C}^q \),
- the exchange property holds for all \( q < p \): whenever \((x o_p y) o_q (w o_p z)\) exists also \((x o_q y) o_p (w o_q z)\) exists and they coincide\(^{[3]}\).

A covariant functor \((\mathcal{C}_1, \delta_0, \ldots, \delta_{n-1}) \xrightarrow{\phi} (\mathcal{C}_2, \delta_0, \ldots, \delta_{n-1})\) between two globular strict \( n \)-categories is a homomorphism for each of the partial 1-monomoids involved, i.e. a map \( \phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) such that

- whenever \( x \circ_2 y \) exists, also \( \phi(x) \circ_2 \phi(y) \) exists and in this case \( \phi(x \circ_2 y) = \phi(x) \circ_2 \phi(y) \),
- if \( e \) is a partial \( o_2 \)-identity in \((\mathcal{C}_2, o_2)\), \( \phi(e) \) is a \( \delta_2 \)-partial identity in \((\mathcal{C}_2, o_2)\).

More generally, a covariant relator between \( n \)-categories is a relation \( \mathcal{R} \subset \mathcal{C}_1 \times \mathcal{C}_2 \) such that for all \( p \):

- whenever \((x_1 o_p x_2, y_1 \delta_p y_2)\) exist and \((x_1, y_1), (x_2, y_2) \in \mathcal{R} \) we have \(((x_1 o_p x_2), (y_1 \delta_p y_2)) \in \mathcal{R} \).

\(^{[1]}\)For locally small categories this is equivalent to asking \( \mathcal{C}^0 \) to be a set.

\(^{[2]}\)We will of course use \( \mathcal{C}^0 \) to denote \( \mathcal{C} \).

\(^{[3]}\)By symmetry, the exchange property automatically holds for all \( q \neq p \).
\[ \text{if } (x, y) \in R \text{ and } e, f \in \mathcal{C}^p, \text{ we have } (e, f) \in R \text{ whenever } (x \circ_p e, y \circ_p f) \text{, or } (e \circ_p x, f \circ_p y), \text{ exists.} \]

The first and the third axioms, imply that, for all \( 0 \leq q < p \leq n \), \((\mathcal{C}^p, o_q)\) is a 1-category. It follows immediately that, for all \( p = 1, \ldots, n \), any \( p \)-cell \( x \in \mathcal{C}^p \) has a unique \( q \)-source and a unique \( q \)-target \( s_q^x(x), t_q^x(x) \in \mathcal{C}^q \). The second axiom allows us to define the inclusion maps \( i_q^p : \mathcal{C}^q \to \mathcal{C}^p \) such that, for all \( x \in \mathcal{C}^q \), \( s_q^x(i_q^p(x)) = x = t_q^x(i_q^p(x)) \). The third axiom also assures the functoriality of the maps \( i_q^p : (\mathcal{C}^p, o_q) \to (\mathcal{C}^m, o_q) \). It is particularly crucial to notice that the globular shape of the \( m \)-cells, for all \( 1 < m \leq n \), is actually implicitly determined by the specific form in which the exchange property is stated: for all \( x \in \mathcal{C}^m \), for all \( 0 \leq q < p < m \leq n \), \( s_q^p(x), x, t_p^m(x) \) are \( o_q \)-composable and from the fact that both \( s_q(s_p(x)), s_p(x), t_q(t_p(x)) \) and \( s_q(t_p(x)), t_p(x), t_q(t_p(x)) \) are \( o_q \)-composable, from the exchange property we obtain the \( o_q \)-composability of both \( s_q(s_p(x)), s_q(t_p(x)) \) and \( t_q(s_p(x)), t_q(t_p(x)) \) that, being \( o_q \)-identities, implies the globular property \( s_q(s_p(x)) = s_q(t_p(x)) \) and \( t_q(s_p(x)) = t_q(t_p(x)) \). In a perfectly similar way, the exchange property, implies the functoriality of the maps \( s_q^m, t_p^m : (\mathcal{C}^m, o_q) \to (\mathcal{C}^p, o_q) \), for all \( 0 \leq q < p < m \leq n \).

A graphical representation illustrates the combinatorial/geometrical meaning coded in the definition:

\[
\begin{align*}
\mathcal{C}^0 & \xrightarrow{i_0} \mathcal{C}^1 \xrightarrow{i_1} \cdots \xrightarrow{i_{q-1}} \mathcal{C}^q \xrightarrow{i_q} \cdots \xrightarrow{i_{m-1}} \mathcal{C}^m \\
q\text{-cells: } & \bullet, \quad p\text{-cells } (q < p): \quad \bullet_1 \xrightarrow{\cdots} \bullet_2 \quad m\text{-cells } (q < p < m): \quad \bullet \quad \downarrow \\
& \text{sources / targets: } s_q^m(x) \xrightarrow{\delta_q^m(x)} x \xrightarrow{\gamma_q^m(x)} t_q^m(x) , \quad \text{identities: } A \xrightarrow{\delta_q^m(A)} A \xrightarrow{\gamma_q^m(A)} A \\
& \text{\(o_q\)-composition: } A \xrightarrow{g} B \xrightarrow{f} C \xrightarrow{\delta_q^m} A \xrightarrow{\Phi_{o_q}^{f \circ g}} C \\
& \text{\(o_p\)-composition: } A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{f \circ g} A \xrightarrow{\Lambda_{o_p}^{f \circ g}} B \\
& \text{ functoriality of } i: A \xrightarrow{\phi} B \xrightarrow{\psi} C = A \xrightarrow{\psi \circ f} C \xrightarrow{f \circ \phi} A \\
& \text{ exchange property: } \\
\end{align*}
\]

We introduce, for all \( q < p \) and \( x, y \in \mathcal{C}^q \), the notation \( \phi_{xy}^p := \{ z \in \mathcal{C}^p \mid s_q^p(z) = y, \ t_q^p(z) = x \} \) to denote the \( q \)-block of \( p \)-arrows i.e. the class of \( p \)-arrows whose \( q \)-source is \( y \) and whose \( q \)-target is \( x \).

\[\text{\footnote{This is a vertical categorification of the (corrected) definition of relator for 1-categories that was presented in \cite{BCL1}, where this second unitality condition was mistakenly omitted.}}\]
Clearly, \( q_{xy}^{kp} \) \( \in \mathcal{C}_{xy}^p \), for all \( x,y,z \in \mathcal{C}^p \) and the family of pairs of \( \circ_p \)-composable \( p \)-cells \( \mathcal{C}^p \times \mathcal{C}^p := \{(a,b) \in \mathcal{C}^p \times \mathcal{C}^p \mid \xi_p^0(a) = \xi_p^0(b)\} \) is given by \( \mathcal{C}^p \times \mathcal{C}^p = \bigcup_{x,y \in \mathcal{C}^p} q_{xy}^{kp} \times q_{xy}^{kp}. \)

For all \( x,y \in \mathcal{C}^p \), whenever \( r < q < p \), we have \( \xi_q^0(\mathcal{C}^p_{xy}) \subseteq \xi_p^0(\mathcal{C}^p_{xy}) \), furthermore \( q_{xy}^{kp} \subseteq r_{xy}^{kp(3)}(\mathcal{C}^p_{xy}) \) and \( r_{xy}^{kp} = \bigcup_{a,b,c \in \mathcal{C}^p_{xy}} q_{ab}^{kp} \), where the union is disjoint.

The usual notion of natural transformation between functors can be immediately reframed, in the setting of our “n-arrows” definition of strict n-categories, via “intertwiners”. Furthermore, for the case of higher categories (\( n > 1 \)), following the terminology introduced by S.Crans [Cra] (see also the “transfor” page on the n-Lab1 page) and compare also with the works of C.Kachour [K1, K2] and G.V.Kondratiev [Ko], we can similarly introduce \( k \)-transfors, for \( 0 \leq k \leq n \), as vertically categorified analogs of natural transformations; in particular 0-transfors are functors, 1-transfors correspond to natural transformations, 2-transfors to modifications, 3-transfors to perturbations . . . (in the literature, for n-categories, with \( n \geq 2 \), there are already slightly different definitions in place, see the remarks in the n-Lab1 page).

The main idea behind the general definition of “higher natural transformations” between a pair of functors \( \phi, \psi : \mathcal{C} \to \mathcal{C} \) of n-categories, consists of introducing suitable “homotopies” between the different sources/targets of the \( n \)-cells \( \phi(x) \) and \( \psi(x) \), \( x \in \mathcal{C} \), and proceeding iteratively, imposing “intertwining conditions” that at the level \( n \) must consist of a usual commutative diagram of \( n \)-cells. The language of cubical n-categories [BH, BCM] is much more naturally adapted for the description of the \( (p + 1) \)-cells generated by homotopies of \( p \)-cells and, whenever necessary, we will conveniently translate the “intertwining conditions” and “compositions of transfors” in such cubical setting.

Definition 3.3. Let the \( 0 \)-transfors be just the covariant functors between strict globular n-categories.

Given two \( 0 \)-transfors \( \mathcal{C} \xrightarrow{\phi} \mathcal{C} \) of strict globular n-categories \( (\mathcal{C}, \circ_{0}, \ldots, \circ_{n-1}) \) and \( (\hat{\mathcal{C}}, \hat{\circ}_{0}, \ldots, \hat{\circ}_{n-1}) \), with \( n \geq 1 \), a \( 1 \)-transfor between \( \phi \) and \( \psi \) is a map \( \xi : \mathcal{C}^0 \to \hat{\mathcal{C}}^1 \), \( x \mapsto \xi(x) \) such that:

- \( \xi(x) \circ_0 \xi(x_0) = \xi(x_0) \circ_0 \phi(x) \) for all \( x \in \mathcal{C} \),

where \( x_0 \) and \( p \cdot x \) denote respectively the unique source and target partial \( \circ_p \)-identities of \( x \in \mathcal{C} \).

By recursion, suppose that we already defined (global) \((k-1)\)-transfors between \((k-2)\)-transfors.

A (global) \( k \)-transfor \( \mathcal{C} \xrightarrow{\Phi^{(k-1)}} \hat{\mathcal{C}} \), between \((k-1)\)-transfors \( \Phi^{(k-1)}, \Psi^{(k-1)}, \ldots \), between two functors \( \Phi^{(0)}, \Psi^{(0)} : \mathcal{C} \to \hat{\mathcal{C}} \), for two strict globular n-categories, \((\mathcal{C}, \circ_{0}, \ldots, \circ_{n-1})\), \((\hat{\mathcal{C}}, \hat{\circ}_{0}, \ldots, \hat{\circ}_{n-1})\), with \( n \geq k \), is a map \( \xi^{(k)} : \mathcal{C}^0 \to \hat{\mathcal{C}}^k \), \( x \mapsto \xi^{(k)}(x) \) such that:

- \( \Phi^{(k-1)}(A) \xrightarrow{\xi^{(k)}(A)} \Psi^{(k-1)}(A) \) for all \( A \in \mathcal{C}^0 \),

- \( \Psi^{(0)}(x) \circ_0 \xi^{(0)}(x_0) = \xi^{(1)}(0, x) \circ_\phi \Phi^{(0)}(x) \) for all \( x \in \mathcal{C} \).

Functors \((k = 0)\) between small strict n-categories constitute a strict 1-category and natural transfors \((k = 1)\) constitute a strict 2-category. Similarly, by induction, we have the following result (a sketch of the proof is presented in [Ko] proposition 1.4) and, for the case \( k = 2 \), in F.Borceux [B] section 7.3.1) that provides a nice class of examples of strict higher categories constructed inductively.

Theorem 3.4. The family of (global) \( n \)-transfors between small strict globular n-categories becomes a strict \((n+1)\)-category [14]

Variants of this result can be explored also for the case of “lax” transfors (C.Kachour [K2]).

---

[13] We are using here a strict notion for 1-transfors (as in the treatment provided by F.Borceux [B] section 7.3) and G.V.Kondratiev [Ko] definitions 1.6,1.7,1.8); more generally, one can consider lax natural transfors (see for example T.Leinster [Le] definition 1.5.10) and introduce classes of “lax” higher natural transfors (see S.Crans [Cra] and C.Kachour [K1, K2]).

[14] More generally, in the same way, the family of (global) \( k \)-transfors (for fixed \( k \leq n \)), of small strict globular n-categories, becomes a strict globular \((k+1)\)-category.
3.2 Exchange Property and Eckmann-Hilton Collapse

Whenever \( o \in \mathcal{C} \) and \( 0 \leq q < p \leq m \leq n \), we define the \( q \)-diagonal \( p \)-blocks of \( m \)-arrows over \( o \)

\[
P_{\mathcal{O}^o_n} \colon \{ x \in \mathcal{O}^o_\mathcal{C} | s^p_k(x) = t^q_k(x), \ \forall k = q, \ldots, p \} \]

as the family \( P_{\mathcal{O}^o_n} \subset q \mathcal{O}^o_n \) of \( m \)-arrows that share a common source and target \( k \)-arrow \( t^q_k(o) \) for all \( q \leq k \leq p \) and we note that, as a consequence, on such diagonal blocks all the compositions \( o_q, \ldots, o_p \) are well defined global binary operations.

The following proposition, that is fundamental for the discussion about non-commutativity in the context of \( n \)-categories, is just a higher-categorical version of the well-known Eckmann-Hilton argument \([EH]\) (see for example T.Leinster \([L] \) proposition 1.2.4 [Z] in the case of 2-categories); it follows immediately from the exchange law and assures a strong trivialization of the categorical structure.

**Proposition 3.5.** If \( 0 \leq q \leq p < m \leq n \) and \( o \in \mathcal{C} \) is a \( q \)-arrow in an \( n \)-category \((\mathcal{C}, o_0, \ldots, o_{n-1})\), the \( q \)-diagonal \( p \)-block \( P_{\mathcal{O}^o_n} \) is a \((m - q)\)-category \((P_{\mathcal{O}^o_n} o_q, \ldots, o_{m-1})\) and a monoid for all the operations \( o_q, \ldots, o_p \). If \( q < p \), all the operations \( o_q, \ldots, o_p \) coincide and they are commutative, hence \( P_{\mathcal{O}^o_n} \) actually collapses to a \((m - p)\)-category that is a commutative monoid for \( o_p \).

**Proof.** If \( o \in \mathcal{C} \), for all \( q \leq k \leq p \) for all \( x, y \in \mathcal{O}^o_n \), if \( x \circ_k y \) exists, it always belongs to \( \mathcal{O}^o_n \) and so we have a category \((\mathcal{O}^o_n o_q, \ldots, o_{p-1})\). Since \( \mathcal{O}^o_n \subset \mathcal{C} \), if \( x, y \in \mathcal{O}^o_n \), their compositions \( x \circ_k y \) are always defined and so \((\mathcal{O}^o_n o_q, \ldots, o_k)\) is a \((k - q + 1)\)-monoid.

Suppose now that \( q \leq i < j \leq p - 1 \), since \( o \in \mathcal{C} \subset \mathcal{O}^o_n \subset \mathcal{C} \), we have that \( o \) is an identity for both the compositions \( o_i \) and \( o_j \). From the exchange property it follows that, for all \( x, y \in \mathcal{O}^o_n \),

\[
(x \circ_i y) = (x \circ_j o_i (o \circ_j y) = (x \circ_i o) \circ_j (o \circ_j y) = x \circ_j y,
\]

\[
x \circ_i y = (o \circ_i x) = y \circ_i (x \circ_i o) = (o \circ_i y) \circ_j (x \circ_i o) = y \circ_j x
\]

that shows that \( o_i = o_j \) when restricted to the set \( i \circ^p_\mathcal{O} \).

If \( q < p - 1 \) and \( o \in \mathcal{C} \) is a \( q \)-arrow in an \( n \)-category \((\mathcal{C}, o_0, \ldots, o_{n-1})\) the \( q \)-diagonal block \( \mathcal{O}^o_{\mathcal{C} o_q} \), with \( q < p - 1 \), is a commutative \( p \)-category \((\mathcal{O}^o_{\mathcal{C} o_q} o_q, \ldots, o_{p-1})\) and all the operations coincide and are commutative when restricted to the block \((p-1)\mathcal{O}^o_{\mathcal{C} o_q}\).

Again a much better intuition comes from the following graphical explanation of the proof. If \( q < p < n \) and \( n \)-arrows have a common \( q \)-source \( q \)-target \( \bullet : o^p_q = o^q_q \) and they are commutative operations.

\[
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array}
\end{array}

where \( i \) is \( \begin{array}{ccc}
\Phi & \Psi & \Phi \\
\| & \| & \|
\end{array} \).

\[\text{3.3 Non-commutative Exchange Property}\]

As a possible solution in order to avoid the Eckmann-Hilton collapse of the algebraic structure of \( q \)-diagonal \( p \)-blocks for \( q < p \), we propose to relax the form of the exchange property for globular strict \( n \)-categories and we put forward the following definition \[\] \[17\] We stress that the unique change from definition \([Z]\) consists in the modified exchange property (the fourth item below).
Definition 3.6. A globular strict \( n \)-category with non-commutative exchange \((\mathcal{C}, o_0, \ldots, o_{n-1})\) is a class \(\mathcal{C}\) equipped with a family \(o_0, \ldots, o_{n-1}\) of \(n\) partially defined binary operations \(o_p : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), for \(p = 0, \ldots, n-1\), such that:

- \((\mathcal{C}, o_p)\) is a 1-category for all \(p = 0, \ldots, n-1\),
- \(\mathcal{C}^q \subseteq \mathcal{C}^p\) for all \(q < p\), i.e. a \(o_q\)-identity is also a \(o_p\)-identity,
- \(\mathcal{C}^p \circ \mathcal{C}^q \subseteq \mathcal{C}^p\) for all \(p, q = 0, \ldots, n-1\) with \(q < p\), i.e. the \(o_q\)-composition of \(o_p\)-identities, whenever exists, is a \(o_p\)-identity,
- non-commutative (quantum) exchange: for all \(p\)-identities \(i\), for all \(q < p\), the partially defined maps \(i \circ_q : (\mathcal{C}, o_p) \to (\mathcal{C}, o_p)\) and \(- \circ_q i : (\mathcal{C}, o_p) \to (\mathcal{C}, o_q)\) are functorial (homomorphisms of partial 1-monoids).

The graphical representation of the non-commutative exchange property (here \(q < p\), \(A, B, C \in \mathcal{C}^q\), \(e, f, g, h \in \mathcal{C}^p\), \(\Phi, \Psi \in \mathcal{C}\)), makes immediately clear that this is just the original exchange axiom, required to hold only for the special situation when two of the \(n\)-cells involved coincide with a given \(o_p\)-identity:

An acute reader will remember that the globularity of the \(n\)-cells was actually encoded in the specific form of the exchange property and might at this point question if our relaxed non-commutative exchange property still implies the globularity condition. Indeed this is true: the right-functoriality in the quantum exchange property assures that for \(x \in \mathcal{C}\), the \(o_p\)-identity \(s_p(x)\) is \(o_p\)-composable (on the right) with both the \(o_q\)-identities \(s_q(s_p(x))\) and \(s_q(t_p(x))\) and so the three of them must coincide. A similar argument using the \(o_q\)-identity \(t_q(x)\) and the left-functoriality provides the second globular condition.

The Eckmann-Hilton collapse (proposition \(\Box\)) is avoided: in the strict non-commutative \(n\)-category \((\mathcal{C}, o_0, \ldots, o_{n-1})\), for all \(q < n - 1\), for all \(o \in \mathcal{C}^q\), the \(q\)-diagonal \((n-1)\)-blocks of \(n\)-cells \(\mathcal{C}_{n\to}\) can all be non-commutative monoids with respect to the restriction of any one of the operations \(o_{n-1}, \ldots, o_q\), and these restrictions are not forced anymore to coincide. This will be clear from the examples that are provided in the context of higher \(C^n\)-categories.

We are well aware that the proposed non-commutative exchange property is somehow going against the usual lines of development of the subject as inspired by higher homotopy theory. The full justification for such a questionable, apparently arbitrary, modification of the usual notion of \(n\)-category, comes from the richness of natural examples of operator structures perfectly fitting with this framework as well as from quite elementary discussion of higher relational environments (higher categories of \(n\)-quivers) that will be presented further on. We stress that some of our examples of strict \(n\)-categories with non-commutative exchange do not necessarily fall within the scope of weak \(n\)-categories, since they do not satisfy the usual exchange property even up to higher isomorphisms.
3.4 Product Categories as Full-depth Strict Higher Categories

In this subsection we propose a generalization of the previous notion of strict higher category with non-commutative exchange that later on will turn out to be essential for a complete description of the operations between hypercategories as “higher convolutions”.

It is well-known (see for example S. Mac Lane [ML section II.3]) that the Cartesian product $X_1 \times \cdots \times X_n$ of a family of $n$ different 1-categories $(X_k, o_k)$, for $k = 1, \ldots, n$, can be seen either as another 1-category, with componentwise composition, or as an $n$-tuple category, with $n$ different directional compositions.

In reality, the strict cubical $n$-category obtained from the Cartesian product $X_1 \times \cdots \times X_n$ of 1-categories has a much richer structure and its cubical $n$-cells can generally be composed over $q$-arrows, for all $0 \leq q \leq n$, via $2^n = \sum_{q=0}^{n} \binom{n}{q}$ compositions, $\circ_{j_1, \cdots, j_q}$ of which are at depth-$q$, one for each subset \( \{j_1, \ldots, j_q\} \subset \{1, \ldots, n\} \), as specified by the following definition:

**Definition 3.7.** Let $(X_1, o_1), \ldots, (X_n, o_n)$ be strict 1-categories and $x_k, y_k \in X_k$, for all $k = 1, \ldots, n$. We say that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are $\circ_{j_1, \cdots, j_q}$-composable if and only if for all $p \in \{j_1, \ldots, j_q\}$, $x_p = y_p$ and $(x_p, y_p)$ are composable in $X_p$, for all $p \notin \{j_1, \ldots, j_q\}$ and in this case

\[
(x_1, \ldots, x_n) \circ_{j_1, \cdots, j_q} (y_1, \ldots, y_n) := (t_1, \ldots, t_n)
\]

with $t_p := \begin{cases} x_p = y_p, & \text{for all } p \in \{j_1, \ldots, j_q\}, \\ x_p \circ y_p, & \text{for all } p \notin \{j_1, \ldots, j_q\}. \end{cases}$

The usual componentwise composition making $X_1 \times \cdots \times X_n$ into the 1-category Cartesian product of the family $(X_k, o_k)_{k=1, \ldots, n}$ corresponds to the unique case of operation $\circ_{(n-1)}$ of depth-$n$.

A graphical representation of the $4 = 2^2$ composition for the case $n = 2$ is here illustrated:

- cubical 2-cells:

- $\circ_{(n)}$-composition:

- $\circ_{1}$-composition:

- $\circ_{0}$-composition:

Motivated by this elementary Cartesian product example, we tentatively put forward an axiomatization of a higher strict categorical structure suitable for treating such situations.
Definition 3.8. A full-depth strict cubical n-category \((\mathcal{C}, \{\circ_\gamma\}_{\gamma \in \Pi_{1\ldots n}})\) is a class \(\mathcal{C}\) equipped with a family of \(2^n\) partially defined compositions \(\circ_\gamma\), one for each subset \(\gamma \subset \{1, \ldots, n\}\) satisfying:

- \((\mathcal{C}, \circ_\gamma)\) is a strict 1-category for all \(\gamma \subset \{1, \ldots, n\}\),
- \(\mathcal{C}^\alpha \subset \mathcal{C}^\beta\), whenever \(\beta \subset \alpha\) and \(\mathcal{C}^\gamma\) denotes the partial identities of \((\mathcal{C}, \circ_\gamma)\),
- \(\mathcal{C}^\beta \circ_\alpha \mathcal{C}^\gamma \subset \mathcal{C}^\beta\), for all \(\beta \subset \alpha\),
- non-commutative (quantum) exchange: for all \(\gamma\)-identities \(\iota \in \mathcal{C}^\gamma\) and for all \(\gamma \subset \alpha\), the left/right \(\circ_\alpha\)-compositions \(\iota \circ_\alpha - : (\mathcal{C}, \circ_\gamma) \to (\mathcal{C}, \circ_\gamma)\) and \(- \circ_\alpha \iota : (\mathcal{C}, \circ_\gamma) \to (\mathcal{C}, \circ_\gamma)\) are functorial.

The above definition essentially works for any arbitrary ordered set of indices. Remarkably, it is the specific choice of the ordered set of indices that (via such axioms) determines the geometrical / combinatorial shape of the \(n\)-cells: when the set of indices is the power set of \(\{1, \ldots, n\}\) we get cubical cells; when the index set is the set of cardinals less than \(n\), we get globular \(n\)-cells. Other choices different from these two will produce more exotic shapes... and this is another even stronger departure from the usual world of higher categories inspired by higher homotopy theory.

4 Strict Involutive Higher Categories

In this section we discuss a possible notion of strict involution in the context of strict higher categories. From the case of 1-categories, we know that there are actually several different ways in which involutions and dualities have been introduced in the categorical context, either via involutive endofunctors, or via dualities implemented via adjointability, or dualizable objects. A full comparison between these different notions deserves a separate treatment elsewhere; for our purpose here, involutions (dualities) will be defined as involutive functors with specific covariance and contravariance properties with respect to the several compositions. This point of view is directly inspired by the notion of \(\ast\)-category introduced (with additional linearity assumptions) by P.Ghez-R.Lima-J.E.Roberts and P.Mitchener [GLR, M] as a horizontal categorification of the involution of a \(\ast\)-algebra. Categories equipped with involutive (contravariant) endofunctors have been studied since the works of M.Burgin [Bu] and J.Lambeck [La]; with the denomination “dagger categories” they have been axiomatized by P.Selinger [Se] and are now systematically used by S.Abramsky-B.Coecke [AC1] and collaborators in their works on categorical quantum mechanics via compact closed categories. Weak forms of such involutive categories appear in our definition of \(\ast\)-monoidal category [EGL6] and similar structures were independently developed by J.Egger [EG] and B.Jacobs [J]. Weak higher involutions, for weak \(\omega\)-categories (in Penon’s approach [P],[KH]) are introduced in [BB].

4.1 Strict Higher Involutions

A graphical display of “duals” of \(n\)-cells helps to grasp the intuition behind the formal definitions:

- dual cells for 1-arrows (usual involution): \(A \xrightarrow{f} B \quad \mapsto \quad A \xleftarrow{f^\ast} B\)
- dual cells for globular \(n\)-arrows \((A, B \in \mathcal{C}^q, f, g \in \mathcal{C}^p, \Phi \in \mathcal{C}^m, 0 \le q < p < m \le n)\):

\[
\begin{align*}
\ast_p : \quad & A \xleftarrow{f} \Phi^g B \mapsto A \xleftarrow{f_{\Phi^g}} B, \\
\ast_q : \quad & A \xleftarrow{f} \Phi^g B \mapsto A \xleftarrow{f_{\Phi^g}} B,
\end{align*}
\]

\[
\begin{align*}
\ast_q : \quad & A \xleftarrow{f} \Phi^g B \mapsto A \xleftarrow{f_{\Phi^g}} B, \\
\ast_\circ : \quad & A \xleftarrow{f} \Phi^g B \mapsto A \xleftarrow{f_{\Phi^g}} B.
\end{align*}
\]

\(^{19}\)For a more detailed discussion of other approaches to “categorical involutivity/duality” we suggest J.Baez-M.Stay [BS], the n-lab page [Categories with Duals], and references therein.
For the general case of globular $n$-cells we have $2^n$ duals $*_{\alpha}$ (including the identity $*_{\emptyset}$) exchanging $q$-sources / $q$-targets for $q$ in an arbitrary set $\alpha \subset \{0, \ldots, n - 1\}$. In the previous diagrams, with some abuse of notation, we wrote $*_{p}$ for $*_{\{p\}}$ and $*_{q}$ for $*_{\{p,q\}}$; this last duality can be realized as composition of $*_{p}$ and $*_{q}$.

With the ideological point of view that an involution/duality in category theory should be considered, on the same level of the binary operations of composition, as a “1-ary operation” of the structure, we introduce the following definition of a strict involutive globular $n$-category via strict $n$-functors.

**Definition 4.1.** If $n \in \mathbb{N}_0$ and $\alpha \subset \{0, \ldots, n - 1\}$, an $\alpha$-contravariant functor\footnote{Notice that for $\alpha = \emptyset$ we recover the definition of covariant functor.} between two strict globular $n$-categories $(\mathcal{C}_1, \alpha_0, \ldots, \alpha_{n-1}) \xrightarrow{\phi} (\mathcal{C}_2, \delta_0, \ldots, \delta_{n-1})$, is a map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ such that:

- for all $q \notin \alpha$, whenever $x \circ_q y$ exists, $\phi(x) \circ_q \phi(y)$ also exists and in this case $\phi(x \circ_q y) = \phi(x) \circ_q \phi(y)$,
- for all $q \in \alpha$, whenever $x \circ_q y$ exists, $\phi(y) \circ_q \phi(x)$ also exists and in this case $\phi(x \circ_q y) = \phi(y) \circ_q \phi(x)$,
- if $e \in \mathcal{C}_i^q$ is a $\circ_q$-identity, $\phi(e) \in \mathcal{C}_i^q$ is a $\circ_q$-identity.

An $\alpha$-involution $*_\alpha$ on $(\mathcal{C}_0, \alpha_0, \ldots, \alpha_{n-1})$ is an $\alpha$-contravariant endofunctor such that $(x^\ast)^\ast = x$, $\forall x \in \mathcal{C}_i$.

If $\{\alpha \in \Lambda\}$, with $\Lambda \subset \mathbb{P}(\{0, \ldots, n - 1\})$ (the power-set of $\{0, \ldots, n - 1\}$) is a family of commuting $\alpha$-involutions, the strict globular $n$-category is said to be $\Lambda$-involutive.

In practice, an $\alpha$-involution is an involution that is a unital homomorphism for all $\circ_q$-compositions with $q \notin \alpha$ and a unital anti-homomorphism for $\circ_q$-compositions with $q \in \alpha$.

Whenever the family $\alpha \subset \{0, \ldots, n - 1\}$ consists of a singlet $\alpha = \{q\}$, we will simply use the notation $*_q := *_{\{q\}}$ and in this particular case we will make use of the following terminology:

**Definition 4.2.** Let $(\mathcal{C}, \alpha_0, \ldots, \alpha_{n-1})$ be a strict $n$-category and let $q \in \{0, \ldots, n - 1\}$. We say that $(\mathcal{C}, \alpha_0, \ldots, \alpha_{n-1})$ is equipped with an **involution over $q$-arrows**, if there exists an $\{q\}$-involution i.e. a map $*_q : \mathcal{C} \to \mathcal{C}$ such that:

- for all $p \neq q$ if $(x \circ_p y)^\ast$ exists, $x^\ast \circ_p y^\ast$ also exists and they coincide,
- for $p = q$, if $(x \circ_p y)^\ast$ exists, $y^\ast \circ_p x^\ast$ also exists and they coincide,
- for all $p$, if $x$ is a $p$-identity, $x^\ast$ is also a $p$-identity,
- $(x^\ast)^\ast = x$ for all $x \in \mathcal{C}$.

The involution $*_q$ is said to be **Hermitian** if:

- $x^\ast = x$ whenever $x$ is a $\circ_q$-identity.

If the strict $n$-category is $\Lambda$-involutive and $\{q\}, \{p\} \subset \Lambda$, we further impose the commutativity condition:

- $(x^\ast)^\ast = (x^\ast)^\ast$, for all $x \in \mathcal{C}$.

A **fully involutive strict $n$-category** is a strict $n$-category that is equipped with a $q$-involution for every $q = 0, \ldots, n - 1$. A strict $n$-category is **partially involutive** if it is equipped with only a proper subset of the family of involutions $*_q$, for $q = 0, \ldots, n - 1$.

It is immediate to check that, for all $x \in \mathcal{C}_i^0$, $s^0_p(x^\ast) = t^0_p(x^\ast) = t^0_p(x)$ and $s^0_p(x) = s^0_p(x^\ast)$, $t^0_p(x^\ast) = t^0_p(x)$, when $p \neq q$. Similarly $t^p_q(x^\ast) = (t^p_q(x))^\ast$. If $x \in \mathcal{C}_i^p$ and the $\{q\}$-involution $*_q$ is Hermitian, we also have $x^\ast = x$, for all $q \geq p$. 
Remark 4.3. As a specific illustration of the previous definitions and also in view of a more direct comparison with the already existing literature on 2-C*-categories (in section 4.3), we present here a detailed list of the properties required on a fully involutive 2-category $(\mathcal{C}, \otimes, \alpha, \ast, \ast^\ast) = (\mathcal{C}, 0, \alpha_1, \ast_0, \ast^\ast)$. This is a 2-category $(\mathcal{C}, \otimes, \alpha)$ with two involutions, $\ast$ over objects, and $\ast^\ast$ over 1-arrows, such that:

\[
\begin{align*}
(x^\ast)^\ast &= x, & (x \otimes y)^\ast &= x^\ast \otimes y^\ast, & (x \circ y)^\ast &= y^\ast \circ x^\ast, \\
\ast(x) &= x, & (x \otimes y) &= x \otimes y, & (x \circ y) &= y \circ x, \\
\ast^\ast(x) &= x, & (x \otimes y)^\ast &= x^\ast \otimes y^\ast, & (x \circ y)^\ast &= y^\ast \circ x^\ast.
\end{align*}
\]

For all $\circ$-identities $e \in \mathcal{C}^1$, $\ast(e) \in \mathcal{C}^1$, for all $\circ$-identities $e \in \mathcal{C}^0$, and $\ast^\ast(e) \in \mathcal{C}^0$.

The Hermitianity of $\ast$ means $e^\ast = e$, for all $e \in \mathcal{C}^1$; the Hermitianity of $\ast^\ast$ means $\ast^\ast(e) = e$, for all $e \in \mathcal{C}^0$.

The Hermitianity of $\ast$ is trivially satisfied when $\mathcal{C}^0$ consists of only one element, i.e. in the case of a monoidal (tensorial) category; furthermore, in this case, for all $e \in \mathcal{C}^0$, $e^\ast = e$.

Remark 4.4. If a strict $n$-category is $\Lambda$-involutive and $\alpha, \beta \in \Lambda$, then it is also $(\alpha \Lambda \beta)$-involutive with involution $\ast_{\alpha \Lambda \beta} := \ast_\alpha \circ \ast_\beta$ and hence the strict $n$-category is actually $< \Lambda >$-involutive, where the symbol $< \Lambda > \subset P(\{0, \ldots, n-1\})$ denotes the family of sets generated by the symmetric difference of sets of $\Lambda$. This is actually an abelian group under set-difference that is isomorphic to the group of “automorphisms” generated by $\{\ast_\alpha \mid \alpha \in \Lambda\}$. The maximal abelian group obtainable in this way consists of $P(\{0, \ldots, n-1\})$, $\Lambda$, it has cardinality $2^n$ and has a very convenient set of generators given by $\{\ast_q \mid q = 0, \ldots, n-1\}$ corresponding to the involutions $\{\ast_q \mid q = 0, \ldots, n-1\}$ described here above.

A strict $n$-category is fully involutive if it is equipped with a family of involutions $\{\ast_\alpha \mid \alpha \in \Lambda\}$ that generates such a maximal abelian group with $n$-generators (that is always isomorphic to $\mathbb{Z}^n_\alpha$).

Remark 4.5. In principle it is perfectly possible for a strict $n$-category to be equipped with different (commuting) involutions $\ast_\alpha, \ast_\beta$ with the same covariance $\alpha$. In this case we say that the involutive strict $n$-category has involutive multiplicity. In our treatment here, we assume that $\alpha \mapsto \ast_\alpha$ is a map, since we are only interested in the internal self-duality of the strict $n$-category, rather than its “dual-multiplicity”.

Remark 4.6. For $\alpha \neq \emptyset$, a strict globular $n$-category $\mathcal{C}$ is $\alpha$-involutive with $\alpha$-involution $\ast_\alpha$ if and only if $(\mathcal{C}, \ast_\alpha)$ is an $\alpha$-dual of $\mathcal{C}$, i.e. $\mathcal{C}$ is “$\alpha$-self-dual”.22 It is in this sense that $(\alpha)$-involutions on a strict globular category provide a way to “internalize” the $(\alpha)$-dualities.

The previous remark is fundamental to understanding our choice of formalization of the definition of “fully involutive higher category”: we are requesting the self-dualizability of the category for all possible choices of $\alpha$-duals, selecting a minimal family of $\alpha$-involutions that are adequate for the purpose.

4.2 Examples of Strict Involutive Higher Categories

The most elementary examples of strict involutive $n$-categories come from a strictification of the usual (weak) $n$-categories of higher bipartite quivers (also called spans $\mathbf{B}^n$ or $\mathbf{C}^n$) between sets.

Example 4.7. The strict 1-category of relations between sets, with the operation of composition of relations, is involutive when we define, for every relation $f \subset A \times B$ from the set $A$ to the set $B$, its reciprocal relation $f^\ast := \{(b, a) \mid (a, b) \in f\} \subset B \times A$ from $B$ to $A$.

To generalize this basic example to arbitrary level-$n$, it is convenient to be able to possibly consider different “links” between the same pair of elements $a \in A$ and $b \in B$. For this purpose we consider a bipartite 1-quiver from $A$ to $B$ i.e. a pair of maps $A \overset{s}{\rightarrow} R \overset{t}{\rightarrow} B$. Each element $r \in R$ is interpreted as an arrow connecting its source point $s(r) \in A$ to its target point $t(b) \in B$.

20Here $\Delta$ denotes the set-theoretic symmetric difference.
21This means that the pair $(\mathcal{C}, \ast_\alpha)$ satisfies the following universal factorization property: for every $\alpha$-contravariant functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$ into another strict globular $n$-category $\mathcal{D}$, there exists a unique covariant functor $\check{\phi} : \mathcal{C} \rightarrow \mathcal{D}$ such that $\phi = \check{\phi} \circ \ast_\alpha$.
22The 1-category of functions between sets is not involutive, since the reciprocal relation of a function, usually is not a function.
Construct now the free 1-category \([Q]\) generated by the 1-graph \(Q\) consisting of a certain family of bipartite 1-quivers between sets. The 1-arrows in \([Q]\) are finite sequences \((a_1, r_1, \ldots, a_k, t_k)\), such that for all \(j = 2, \ldots, k \in \mathbb{N}\), \(s(r_j) = t(r_{j-1})\), with source \(a_k := s(r_k)\) and target \(a_1 := t(r_1)\) and with composition given by the concatenation of finite sequences defined as follows:

\[(a_1, r_1, \ldots, r_j, a_k) \circ (a_k, r_{k+1}, \ldots, r_{j+k}, a_{j+k}) := (a_1, r_1, \ldots, r_j, r_{k+1}, \ldots, r_{j+k}, a_{j+k}), \text{ only when } a_k = a_{j+k}.\]

The strict 1-category \([Q]\) contains a disjoint copy of each 1-quiver of the original family and can be used to "strictify" the usual composition of spans obtaining a strict 1-category of 1-quivers.

Such a strict 1-category is not yet involutive. To obtain an involutive strict category, notice that every bivariant 1-quiver \(A \Rightarrow \overleftarrow{R} \Rightarrow B\) has a dual 1-quiver \(B \Rightarrow \overleftarrow{R} \Rightarrow A\), with \(\overleftarrow{R} : \{r \in R\}\) a disjoint copy of \(R\) and where \(R(r) := t(r)\). The strict 1-category \([Q \cup \overline{Q}]\) generated by the union of the original family of bivariant 1-quivers and their dual 1-quivers is now naturally equipped with an involution:

\[(a_1, r_1, \ldots, r_j, a_k) \stackrel{x}{\Rightarrow} (a_k, x_1, \ldots, x_j, a_1), \text{ where } x_i := \overline{a}_i \text{ if } x \in Q, \text{ and } x_i := \overline{x}_i \text{ if } x \in \overline{Q}.\]

Since the strict involutive 1-category \([Q \cup \overline{Q}]\) contains disjoint copies of the original bipartite quivers (and their duals), it can be used to define a "strictified" notion of involution in the category of spans obtaining a strict involutive 1-category of 1-quivers.

A bipartite n-quiver is a sequence of 1-quivers \(R^{(n)} \Rightarrow R^{(n-1)} \Rightarrow \cdots \Rightarrow R^{(0)}\), where at each level \(q = 0, \ldots, n - 1\), \(R^{(q)} := A^{(q)} \cup B^{(q)}\) and \(B^{(q)} \leftarrow \overleftarrow{R}^{(q+1)} \overleftarrow{R}^{(q)} \rightarrow A^{(q)}\) is a bipartite 1-quiver from \(A^{(q)}\) to \(B^{(q)}\). We restrict the attention to globular bipartite n-quivers that are those n-quivers that satisfy the globularity condition \(q^{(q)}\) of \(q^{(q+1)}\) and \(q^{(q+1)}\) of \(q^{(q+2)}\), for \(q = 0, \ldots, n - 2\).

Every bipartite n-quiver \(R\) admits \(2^n\) different \(\alpha\)-dual n-quivers \(R \Rightarrow \overleftarrow{R}\), for \(\alpha \in \{1, \ldots, n\}\), consisting of a disjoint copy \(R^{(m)}\) of the sets \(R^{(m)}\), for \(m = 0, \ldots, n\), with sources and targets maps given, whenever \(q \in \alpha\) by \(\overleftarrow{R}^{(q)} := \overleftarrow{R}^{(q)}\); and whenever \(q \in \alpha\) by \(\overleftarrow{R}^{(q)} := \overleftarrow{R}^{(q)}\).

Given again a family of globular bipartite n-quivers, we consider the globular n-graph \(Q \cup \overline{Q}\), whose globular n-arrows belong to at least one of the n-quivers in the given family or to one of their duals, and construct the free globular n-category \(Q \cup \overline{Q} \Rightarrow \overleftarrow{Q} \Rightarrow \overleftarrow{Q}\) generated by the globular n-quiver \(Q \cup \overline{Q}\).

The strict globular n-category \([Q \cup \overline{Q}]\) is naturally equipped with an \(\alpha\)-involution, for all \(\alpha \in \{0, \ldots, n\}\), obtained by the universal factorization property for the free n-category \(Q \cup \overline{Q} \Rightarrow \overleftarrow{Q} \Rightarrow \overleftarrow{Q}\), from the covariant morphism of n-quivers \(Q \cup \overline{Q} \Rightarrow \overleftarrow{Q} \Rightarrow [Q \cup \overline{Q}]\) into the underlying n-quiver of the abstract \(\alpha\)-dual category by \(\gamma_\alpha := \overleftarrow{R}^{(q)}\) of \(\theta \circ \overleftarrow{R}^{(q)}\) is the composition of the \(\alpha\)-duality morphism of n-quivers \(Q \cup \overline{Q} \Rightarrow \overleftarrow{Q} \Rightarrow \overleftarrow{Q}\), first with the covariant inclusion morphism \(\theta\) into the free n-category, and then with the co-contravariant isomorphism of n-categories \([Q \cup \overline{Q}] \Rightarrow [Q \cup \overline{Q}]\).

The strict globular fully involutive n-category \([Q \cup \overline{Q}]\) contains a (disjoint) copy of every globular bipartite n-quiver of the original family (and of each of their \(\alpha\)-duals) and can now be used to obtain a strictification of the usual weak (involutive) n-category of globular spans of sets. For this purpose, one simply considers the coarse graining of \([Q \cup \overline{Q}]\) i.e. the strict globular n-category \(\mathcal{P}([Q \cup \overline{Q}]\)) whose elements are the subsets of \([Q \cup \overline{Q}]\) under term-by-term compositions (and term-by-term involutions) and finally selects inside \(\mathcal{P}([Q \cup \overline{Q}]\)) the strict globular (involutive) n-category generated by the disjoint family of the original bipartite n-quivers (and their \(\alpha\)-duals) embedded into \([Q \cup \overline{Q}]\).

The fully involutive category \(\mathcal{R}\) of globular n-relations is just a special case of the construction above, since every relation \(R \subset A \times B\) canonically determines a bipartite 1-quiver from \(A\) to \(B\) via the restriction

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23 If \((K_m, i_m, t_m)\), for \(m \in \Lambda\) is a family of bipartite 1-quivers, \(Q := \amalg_{m \in \Lambda} K_m\) is the index set of edges of an oriented 1-multigraph, possibly with loops, whose source and targets are the unique maps \(s, t\) with restrictions on \(R_m\) coinciding with \(i_m, t_m\), for all \(m \in \Lambda\) and with vertex set \(\amalg_{m \in \Lambda} (K_m \cup B_m)\).

24 We include here, for \(k = 0\), one "empty" sequence \((a, a)\), for all \(a \in Q^0\).

25 A construction of the free globular strict n-category of a globular n-graph can be found in T. Leinster [L2, appendix F]: a left-adjoint functor to the forgetful functor from globular \(\omega\)-categories to globular \(\omega\)-graphs is also described in J. Penon [P] proposition 1. Quotient constructions of free (involutive) \(\omega\)-categories over a globular \(\omega\)-graph are presented in [BB].

26 The abstract \(\alpha\)-dual n-category of a \(\alpha\)-category \(C\) is an \(\alpha\)-contravariant functor \(C \Rightarrow \overleftarrow{C}\) into a strict n-category \(\overleftarrow{C}\) that uniquely factorizes, via covariant functors, any \(\alpha\)-contravariant functor into another n-category.
of the Cartesian projection to $R$. Unfortunately, such strict globular $n$-category of relations is degenerate above $k = 1$ because $R^k = R^1$ for $1 < k \leq n$. In fact the globularity condition imposed on the $n$-cells in $[R^n]$ implies that if $((a_1, b_1), (a_2, b_2)) \in R \in R^3$, necessarily $a_1 = a_2, b_1 = b_2$ and so 2-cells (and similarly higher cells) in $[R^n]$ are identities. This justifies the need to consider globular $n$-quivers. Eliminating the globularity condition is not going to solve the problem: for non-globular $n$-quivers the (non-commutative) exchange property is not satisfied.

Example 4.8. Strict globular $n$-groupoids (see the papers by R.Brown-P.Higgins [BH] and the their book with R.Sivera [BHS]) are of course a special case of fully involutive strict globular $n$-categories, where the role of the involutions is taken by the inverse maps.

Example 4.9. Consider now the 2-category whose 1-arrows are the unital morphisms (or even more specifically unital algebras) and the family of unital, not necessarily involutive, homomorphisms between them, with the operation of functional composition. The composition of unital homomorphisms is a unital homomorphism, the composition is associative and every monoid is equipped with an identity map that is a unital homomorphism that satisfies the identity property and hence we have a 1-category.

The involution on the monoids can be used to introduce a covariant involution $\phi \mapsto \phi^*$ of 1-arrows: given two unital involutive monoids $(A_1, \cdot_1, \cdot_1^\dagger)$, $(A_2, \cdot_2, \cdot_2^\dagger)$ and a unital homomorphism $\phi : A_1 \to A_2$, define $\phi^*(x) := \phi(x)^{\dagger}$, for all $x \in A_1$, note that $\phi^* : A_1 \to A_2$ is another unital homomorphism (it coincides with $\phi$ if and only if $\phi$ is a $*$-homomorphism) and that the map $\phi \mapsto \phi^*$ is a covariant involution. This is an example of 1-involution that is Hermitian (since it does not move the objects).

Example 4.10. Consider now the 2-category whose 1-arrows are the unital $*$-homomorphisms $A_1 \xrightarrow{\phi} A_2$ of unital involutive monoids $(A_j, \cdot_j, \cdot_j^\dagger)$, $j = 1, 2$, and whose 2-arrows $\phi \xrightarrow{\psi} \psi$, for $\phi, \psi : A_1 \to A_2$, are the intertwiners of pairs of unital homomorphisms of unital involutive monoids, i.e. those elements $e \in A_2$ such that $e_2 \phi(x) = \psi(x) e_2 e$, for all $x \in A_1$. Given three 1-arrows $\phi, \psi, \eta : A_1 \to A_2$ and two intertwiners $\phi \xrightarrow{e_1} \psi \xrightarrow{e_2} \eta$, their composition over 1-arrows is given by $e_1 \circ_1 e_2 := e_{1,2} e_2$ that is an intertwiner from $\phi$ to $\eta$. Given instead $\phi_2 \xrightarrow{e_2} \psi_2$ and $\phi_1 \xrightarrow{e_1} \psi_1$, where $\phi_1, \psi_1 : A_2 \to A_3$ and $\phi_2, \psi_2 : A_1 \to A_2$, the composition over objects is given by $e_1 \circ_0 e_2 := e_{1,0} \phi_1(e_2)$ that is an intertwiner from $\phi_1 \circ \phi_2$ to $\psi_1 \circ \psi_2$. An involution of 2-arrows over 1-arrows is obtained as follows: if $\phi \xrightarrow{e} \psi$ is an intertwiner from $\phi$ to $\psi$, the element $e^{\dagger} \in A_2$ is an intertwiner from $\psi$ to $\phi$. In general there is no involution of 2-arrows over objects and so this is an example of a partially involutive strict 2-category, whose only involution is $*_\alpha$ with $\alpha = \{1\}$.

Restricting to the case of intertwiners between unital $*$-isomorphisms of $*$-monoids, it is not difficult to check that one obtains a fully involutive 2-category, where the additional involutions over objects $*_\alpha$, with $\alpha = \{0\}$, is given, for every $A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_1$, with $e^{\dagger} := \phi^{-1}(e^{\dagger})$.

Example 4.11. There is a horizontal categorification of example 4.10. An involutive 1-category $(C, \circ_0, \circ_0^\dagger)$ is a horizontal categorification (a “many objects” version) of an involutive monoid. A family $\mathcal{C}^{(2)} := \{(C_k, \circ_k, \circ_k^\dagger) \mid k \in A\}$ of involutive 1-categories becomes a strict 1-category with 1-arrows consisting of the $*$-functors $C_1 \xrightarrow{\phi} C_2$, and a strict globular 2-category with 2-arrows $\phi \xrightarrow{\psi} \psi$ consisting of natural transformations (the horizontal categorification of intertwiners) $(e_o) : o \mapsto e_o$, with $o \in C_0$ and $e_o \in C_1$. The category $\mathcal{C}^{(2)}$ is partially involutive, with involution of 2-arrows over 1-arrows given by $(e_o)^* = (e^{\dagger})_o$ for all $o \in C_0$, where $C_1 \xrightarrow{\phi} C_2$ is a natural transformation.

As in example 4.10 restricting to the case of invertible $*$-functors, the category $\mathcal{C}^{(2)}$ becomes a fully involutive strict globular 2-category.

---

\[^{27}\text{In practice all the } n\text{-cells } x \in \mathcal{R}^n \text{ coincide with higher identities corresponding to } 1\text{-cells: } x = r^n(x).\]
It is natural to ask whether it is possible to embed (as an involutive 2-subcategory) the fully involutive 2-category of intertwiners of unit \(*\)-isomorphisms of \(*\)-monoids, in the previous example 4.10 into a fully involutive 2-category including (as a non-fully involutive 2-subcategory) the 2-category of intertwiners between unital \(*\)-homomorphisms of \(*\)-monoids. In order to introduce the missing \(\alpha\)-involutions, for \(\alpha = [0]\), we will have to generalize the notion of a unital \(*\)-homomorphism between monoids, along the same lines leading from functions to relations and to spans in example 4.4.

**Example 4.12.** Every \(*\)-homomorphism \(\phi: A_1 \to A_2\) between involutive monoids uniquely determines a congruence (and hence a span) of involutive monoids i.e. \(\phi = \{(a_1, a_2) \in A_1 \times A_2 \mid a_2 = \phi(a_1)\}\) is a unital involutive submonoid of the product involutive monoid \(A_1 \times A_2\). Since the reciprocal relation \(\phi^* \subset A_2 \times A_1\) of any congruence \(\phi \subset A_1 \times A_2\) is another congruence, we immediately obtain an involutive 1-category of congruences of involutive monoids.

More generally one has an involutive 1-category of spans \(A_1 \xleftarrow{\phi} \xrightarrow{\psi} A_2\) of involutive monoids (cf. example 4.7), where the source and target maps are involutive morphisms of involutive monoids.

One might wonder whether it is possible to identify in this setting a notion of “intertwiner” between monoids, along the same lines leading from functions to relations and to spans in example 4.7.

**Remark 4.13.** Let \(A_1 \xleftarrow{j_1} \xrightarrow{j_0} A_2\) and \(A_1 \xleftarrow{\phi} \xrightarrow{\psi} A_2\) be spans between a pair of \(*\)-monoids \(A_1, A_2\).

A bivariant intertwiner \(A_1 \xleftarrow{\phi} \xrightarrow{\psi} A_2\) between \(\phi\) and \(\psi\) is any family \(\Xi\) consisting of some quadruples \((e, \psi, \phi, f)\) with \((e, f) \in A_2 \times A_1\) such that the following intertwining conditions hold\(^2\):

\[
f \circ J_{\phi}(x) = J_{\psi}(y) \circ f, \quad e \circ J_{\phi}(x) = J_{\psi}(y) \circ e, \quad \forall x \in \phi, \forall y \in \psi. \tag{4.1}
\]

Note that, given a collection \(\mathcal{Q}\) of spans of involutive monoids, the family of all quadruples \(f \xleftarrow{\phi} \xrightarrow{\psi} e\), with \(\xi := (e, y, x, f) \in A_2 \times \psi \times \phi \times A_1\), such that the following intertwining conditions hold:

\[
f \circ J_{\phi}(x) = J_{\psi}(y) \circ f, \quad e \circ J_{\phi}(x) = J_{\psi}(y) \circ e,
\]

becomes a strict double category (cubical 2-category) \([\mathcal{Q}]^2\) under the following compositions:

\[
(e_1, y_1, x_1, f_1) \circ_1 (e_2, y_2, x_2, f_2) := (e_1 \circ e_2, y_1, x_2, f_1 \cdot f_2), \quad \text{whenever } y_2 = x_1,
\]

\[
(e_3, y_3, x_3, f_3) \circ_0 (e_4, y_4, x_4, f_4) := (e_3, (y_3, y_4), (x_3, x_4), f_4), \quad \text{whenever } f_3 = e_4,
\]

where \((y_3, y_4), (x_3, x_4)\) denote the concatenations of composable paths in the fine-grained category \([\mathcal{Q}]\). The category \([\mathcal{Q}]^2\) is a strict fully involutive double category (see the manuscript [BCM] for definitions and a detailed treatment), with involutions given by:

\[
(e, y, x, f)^{\circ_1} := (e, x, y, f), \quad (e, y, x, f)^{\circ_0} := (f, y, x, e), \quad \forall (e, y, x, f) \in [\mathcal{Q}]^2.
\]

Making the harmless identification between \(\Xi\) and \(\{(e, y, x, f) \in [\mathcal{Q}]^2 \mid (e, \psi, \phi, f) \in \Xi, \ x \in \phi, \ y \in \psi\}\), the strict fully involutive 2-category \([\mathcal{Q}]^{(2)}\) of bivariant intertwiners between globular spans in \(\mathcal{Q}\) is obtained by “coarse graining” the previous fully involutive double category \([\mathcal{Q}]^2\) i.e., considering, for all pairs \(\phi, \psi \in \mathcal{Q}\) in globular position, those subsets \(\Xi \subset [\mathcal{Q}]^2\) consisting of quadruples \((e, y, x, f) \in [\mathcal{Q}]^2\).

---

\(^{2}\)The usage of quadruples \((e, \psi, \phi, f) \in \Xi\) instead of just pairs \((e, f)\) in the definition of \(\Xi\) is necessary to avoid the possibility that the same bivariant intertwiner \(\Xi\) might have different spans as source and/or target.
with \( x \in \phi \) and \( y \in \psi \), satisfying property (4.1) and defining all the compositions and involutions “elementwise”, whenever such compositions exist:

\[
\Xi_1, \Xi_2 := \{ \xi_1, \xi_2 | (\xi_1 \in \Xi_1, \xi_2 \in \Xi_2) \}, \quad \Xi_3, \xi_0, \Xi_4 := \{ \xi_3, \xi_4 | (\xi_3 \in \Xi_3, \xi_4 \in \Xi_4) \},
\]

\[
(\Xi)^{\ast} := \{ \xi^{\ast} | \xi \in \Xi \}, \quad (\Xi)^{\ast /} := \{ \xi^{\ast /} | \xi \in \Xi \}.
\]

Of course such a notion of bivariant intertwiner between spans of involutive monoids admits a horizontal categorification in the context of example 4.11.

A relational 1-transfor \( \mathcal{C}_1 \xrightarrow{\phi} \mathcal{C}_2 \) between a pair \( \phi, \psi \) of spans of involutive 1-categories \( \mathcal{C}_1, \mathcal{C}_2 \)
consists of a family \( \Xi \) of quadruples \( (e, \psi, \phi, f) \), with \( (e, f) \in \mathcal{C}_2 \times \mathcal{C}_1 \) satisfying intertwining conditions of the form (4.7) in the respective 1-categories \( (\mathcal{C}_1, \overset{\sim}{\varphi}) \), \( (\mathcal{C}_2, \overset{\sim}{\varphi}) \):

\[
\forall x \in \phi \text{ such that } a_0(s_\phi(x)) = f_0, \ a_0(t_\phi(x)) = e_0, \quad \forall y \in \psi \text{ such that } (s_\psi(y))_0 = 0 f, \ (t_\psi(y))_0 = 0 e,
\]

\[
f \circ_1 s_\phi(x) = s_\psi(y) \circ_1 f, \quad e \circ_2 t_\phi(x) = t_\psi(y) \circ_2 e.
\]

The construction of a fully involutive 2-category of such relational 1-transfors follows the same lines indicated above for the one-object case.

Relational 1-transfors are a quite vast generalization of natural transformations: every natural transformation provide a relational 1-transfor, but even if \( \phi, \psi : \mathcal{C}_1 \to \mathcal{C}_2 \) are a pair of usual functors, a relational 1-transfor \( \Xi : \phi \to \psi \) is not necessarily a natural transformation. Natural transformations are recovered if and only if, for all \( A \in \mathcal{C}_0 \), there exists one and only one \( e_A \in \mathcal{C}^2 \) such that \( (e_A, \psi, \phi, t_A) \in \Xi \).

For a general relational 1-transfor \( \Xi \) between functors, it is neither assured that such an element \( e_A \)
exists, nor that it is unique.

Recursively, in the case of strict globular \( n \)-categories, relational \( n \)-transfors can be similarly defined and one can recover versions of theorem 4.14 for spans, without restricting to invertible \( \ast \)-functors.

In the subsequent treatment (for simplicity) we will confine the discussion to the case of bijective spans, i.e. \( \ast \)-isomorphisms of \( \ast \)-monoids, and further proceed to horizontal (natural transformations) and vertical categorification (\( n \)-transfors) in this particular case, see theorem 4.14. In remark 5.17 we will further generalize the notion of intertwining via morphisms of bimodules (over involutive monoids) and we will mention how (at least for intertwiners between \( \ast \)-isomorphisms) the present 2-categories are embedded in categories of bimodules.

The main motivation for the introduction of such generalized forms of intertwiners can be found in the attempt to prove an involutive analogue of theorem 3.4 leading to a recursive construction of (fully) involutive higher categories via “relational \( n \)-transfors” (cf. 5.9 for a partially involutive case). For now, we examine the vertical categorification of example 4.11 restricting our considerations to the case of isomorphisms (further generalizations will be dealt with elsewhere).

**Theorem 4.14.** The family of small totally involutive \( n \)-categories with strict \( n \)-transfors, between invertible \( \ast \)-functors, constitutes a fully involutive \( (n + 1) \)-category.

**Proof.** We already know that taking (as objects) involutive 1-categories \( (\mathcal{C}, \cdot, \overset{\sim}{\cdot}), (\hat{\mathcal{C}}, \cdot, \hat{\overset{\sim}{\cdot}}) \), with invertible \( \ast \)-functors \( \Phi, \Psi : \mathcal{C} \to \hat{\mathcal{C}} \) (as 1-arrows), natural transformations (1-transfors) consist of intertwiners \( \Xi : \mathcal{C}^0 \to \hat{\mathcal{C}}^1 \). In this way, the family of 1-transfors \( \mathcal{C}^{(1)} \) constitutes a 2-category \( (\mathcal{C}^{(1)}, \mathcal{C}^0, \overset{\ast}{\mathcal{C}}^0, \hat{\mathcal{C}}^0, \hat{\mathcal{C}}^1) \) that is fully involutive with involutions given by:

\[
\mathcal{C} \xrightarrow{\Phi, \Psi} \hat{\mathcal{C}} \quad \Xi^\ast(A) := \Xi(A)^\hat{\overset{\sim}{\ast}}, \quad \forall A \in \mathcal{C}^0,
\]

\[
\mathcal{C} \xrightarrow{\Phi, \Psi} \hat{\mathcal{C}} \quad \Xi^{\ast /}(A) := \Phi^{-1}(\Xi(A)^\hat{\overset{\sim}{\ast}}) \quad \forall A \in \mathcal{C}^0.
\]
Suppose now, by induction, that we have a fully involutive $n$-category $(\mathcal{C}^{(n)}, \circ_0, \ldots, \circ_{n-1}, *_0, \ldots, *_{n-1})$ whose objects are small totally involutive $n$-categories $(\mathcal{C}_i, \circ_i, \ldots, \circ_i^{n-1}, *_0, \ldots, *_{n-1})$, with 1-arrows the invertible $*$-functors (the $*$-isomorphisms of involutive $n$-categories) and $n$-arrows the strict $n$-transfers.

Consider the globular $n$-cells $\Phi^{(n)}(A) = \downarrow \frac{\Xi}{\Psi^{(n)}(A)} \Psi^{(n)}(A)$ in $\hat{\mathcal{C}}$, for all $A \in \mathcal{C}^{(n)}$, where $\Xi: \mathcal{C}^{(n)} \to \hat{\mathcal{C}}^{(n)}$ is an $n$-transfer between $k$-transfers $\Phi^{(k)}$, $\Psi^{(k)}$, for $k = 0, \ldots, n-1$, between invertible $*$-functors $\Phi^{(0)}$, $\Psi^{(0)}$ from $\mathcal{C}$ to $\hat{\mathcal{C}}$. The $(n+1)$-cells in $\mathcal{C}^{(n)}$ are defined as $c^* \downarrow \frac{\Xi}{\Psi^{(n-1)}(A)} \Psi^{(n-1)}(A)$, for $k = 0, \ldots, n-1$.

By theorem [5.3] we know that $\mathcal{C}^{(n)}$ with the $n$-transfers between (invertible) $*$-functors is already a strict globular $(n+1)$-category. Since, by induction we already have $n$-involutions of $n$-transfers $*_0, \ldots, *_{n-1}$ in $\mathcal{C}^{(n)}$, to complete the proof, we only need to provide an involution over objects $*_0$, commuting with the previous involutions (and verify its covariance/contravariance properties).

For this purpose, define $*_0: \mathcal{C}^{(0)} \to \mathcal{C}^{(0)}$

\[
*_0: \hat{\mathcal{C}} \downarrow \frac{\Phi^{(n)}(A)}{\Psi^{(n)}(A)} \Psi^{(n)}(A) \Rightarrow \mathcal{C} \downarrow \frac{\Phi^{(n)}(A)}{\Psi^{(n)}(A)} \Psi^{(n)}(A),
\]

where $\Xi^n: \hat{\mathcal{C}}^{(0)} \to \mathcal{C}^{(0)}$ is given by $\Xi^n(B) := (\Phi^{(0)})^{-1}(\Psi^{(k)}(\Psi^{(0)}(B))^{(k)})$, for all $B \in \mathcal{C}^{(0)}$ (for $k = 1, \ldots, n-1$, $\Phi^{(k)}$ is the $k$-source and $\Psi^{(k)}$ is the $k$-target of $\Xi^n$).

\[\Box\]

**Example 4.15.** Anticipating somehow the material developed later on in section 5.13, when $(A, \circ, \circ)$ is a commutative $*$-monoid and $(\mathcal{X}, \circ_0, \ldots, \circ_{n-1}, *_0, \ldots, *_{n-1})$ is a finite $n$-groupoid (or more generally a fully involutive $n$-category), the set $\hat{\mathcal{C}} := A \times \mathcal{X}$ becomes a fully involutive $n$-category with the following compositions and involutions, for $k = 0, 1$:

\[
a_x \circ_k b_y := (a \cdot b)_{x \circ_0 y}, \quad (a^*_y)^* := (a^*_y)_C^1,
\]

where we use the notation $a_x := (a, x) \in \hat{\mathcal{C}}$.

### 4.3 Strict Involutive 2-Categories and Conjugations

As promised in remarks 2.9 and 4.3, we are going to discuss here in some more detail how fully involutive categories are related to the well-known notion of “conjugation” introduced in algebraic quantum field theory and constantly used in the theory of superselection sectors (see [DHR] section III, [PR] section 3 and [LR] section 2 and 7]. The study of fully involutive 2-C*-categories obtained in this way, will be completed later in example 5.13. Several of the properties of the conjugation maps (and of their associated involutions over objects) that are mentioned in this section, appeared already in [LR] and have also been used in previous works by C.Pinzari-J.E.Roberts [PR1] [PR2] [PR3].

For a more straightforward comparison with the formulas in the literature on superselection theory, we are using the “reversed notation” for the composition over objects in a 2-category $(\mathcal{C}, \otimes, \circ, \oplus)$ and the usual notation for the composition over 1-arrows: $x \otimes y := y \circ_0 x$ and $x \circ y := x \circ_1 y$ for $x, y \in \mathcal{C}$.

A generalized notion of right (left) conjugation for a pair $(x, y)$ of 1-arrows could actually be defined in the setting of (strict) 2-categories without involutions (or C*-structure) and is equivalent to requiring the adjointability condition $(\sim \otimes x) \circ (\sim \otimes y)$ between the partially defined functors $\sim \otimes x$ and $\sim \otimes y$; this is further investigated in detail in the companion paper [BCM] section 3.2. Here we will assume (as it is always the case in superselection theory) the existence of a strict involution $*$ over 1-arrows, hence $(\Phi \otimes \Psi)^* = \Phi^* \otimes \Psi^*$ and $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$ for all $\Phi, \Psi \in \mathcal{C}$.

In a strict 2-category (in particular in a strict monoidal category) equipped with a strict involution over 1-arrows $(\mathcal{C}, \otimes, \circ, *)$, a pair $x, \tau \in \mathcal{C}^1$ of 1-arrows $\tau \downarrow \frac{\tau}{s(x)} s(x)$, are said to be conjugate
if there exist a pair of 2-arrows $R_x, \overline{R}_x$ that satisfy these conjugate equations:

$$i^2(x) \circ (i^2(x) \otimes R_x) = i^2(x), \quad (R_x \otimes i^2(x)) \circ i^2(x) = i^2(x).$$

If, as we assume, the involution $\ast$ is Hermitian, the conjugation equations are equivalently rewritten as:

$$(i^2(x) \otimes R_x) \circ (i^2(x) \otimes R_x) = i^2(x), \quad (R_x \otimes i^2(x)) \circ (R_x \otimes i^2(x)) = i^2(x).$$

If $x, \overline{x}$ are conjugates, there are in general several pairs $(R_x, \overline{R}_x)$ of 2-arrows that satisfy the conjugate equations; on the other side, any pair $(R_x, \overline{R}_x)$ that satisfies the conjugate equations determines a unique pair $(x, \overline{x})$ of conjugate 1-arrows. The relation of conjugation is symmetric.

If we assume all the 1-arrows in $\mathcal{C}$ to be conjugable (or alternatively we consider the full subcategory $\mathcal{C}_f$ of those 2-arrows in $\mathcal{C}$ with conjugable source and target), we can always choose a specific conjugation map $x \mapsto (R_x, \overline{R}_x)$. Under this choice, we can define two folding maps on 2-arrows $\Phi \in \mathcal{C}_f^2$:

- $A \xrightarrow{x, \overline{x}} B \xrightarrow{\overline{\phi}, \phi} A$,\[ \Phi (\Phi) := \Phi \ast = (i^2(\overline{x}) \otimes \overline{R}_{\overline{y}}) \circ (i^2(x) \otimes \Phi \otimes \overline{i^2(y)}) \circ (R_y \otimes i^2(y)), \]
- $A \xrightarrow{x, \overline{x}} B \xrightarrow{\overline{\phi}, \phi} A$,\[ \Phi (\Phi) := \Phi \ast = (i^2(x) \otimes R_y) \circ (i^2(y) \otimes \Phi \otimes i^2(x)) \circ (i^2(x) \otimes i^2(x)). \]

and hence two additional “pseudo-involutions” of 2-arrows over objects: $B \xrightarrow{\overline{\phi}, \phi} A$, $A \xrightarrow{\overline{\phi}, \phi} A$.

$$\Phi^\dagger := (\Phi \ast) = (i^2(\overline{y}) \otimes \overline{R}_{\overline{y}}) \circ (i^2(y) \otimes \Phi \ast \otimes i^2(x)) \circ (R_y \otimes i^2(x)), \quad \Phi^\ddagger := (\Phi \ast) \ast = (i^2(y) \ast \ast \ast \otimes R_y) \circ (i^2(y) \ast \ast \ast \otimes \Phi \ast \ast \ast \ast) \circ (i^2(y) \ast \ast \ast \otimes i^2(x)).$$

The map $\Phi \mapsto \Phi^\dagger$ is the one actually considered by R. Longo-I.E. Roberts [LR, lemma 2.3] and here we would like to further explore under which additional conditions $\dagger$ (and similarly $\ddagger$) can be taken as an involution over objects and hence provide a further example of fully involutive 2-category ($\mathcal{C}, \otimes, \circ, \ast, \dagger$).

The folding maps always satisfy the following $\circ$-contravariant properties:

$$(\Phi \circ \Phi)_\ast = \Psi \ast \ast \ast \Phi \ast \ast \ast \ast, \quad \ast (\Phi \circ \Psi) = \ast \Psi \ast \Phi, \quad \text{for} \quad A \xrightarrow{x, \overline{x}} B.$$

As a consequence: $\Phi \circ \Psi$ and $\Psi \circ \Phi$ have the same involution, and, in a perfectly similar way, $\Phi \circ \Psi = \Phi \circ \Psi$ and $\Psi \circ \Phi = \Psi \circ \Phi$ and, in a perfectly similar way, $\Phi \circ \Psi = \Phi \circ \Psi$ and $\Psi \circ \Phi = \Psi \circ \Phi$. Hence $\dagger$ and $\ddagger$ are both $\circ$-covariant maps.

\[\text{Notice that, for this statement, the existence of the involution $\ast$ over 1-arrows is crucial.}\]
Property (4.2) is proved by usage of the exchange property (3) and the conjugate equations (2):

$$\Psi_* \circ \Phi_* = (\tau \otimes \overline{R}_x) \circ (\tau \otimes \Psi \otimes \overline{\tau}) \circ (R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \otimes \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

$$= (\tau \otimes \overline{R}_x) \circ ((\tau \otimes \Psi) \circ R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \circ \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

$$= (\tau \otimes \overline{R}_x) \circ ((\tau \otimes \Psi) \circ R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \circ \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

$$= (\tau \otimes \overline{R}_x) \circ ((\tau \otimes \Psi) \circ R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \circ \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

$$= (\tau \otimes \overline{R}_x) \circ ((\tau \otimes \Psi) \circ R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \circ \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

$$= (\tau \otimes \overline{R}_x) \circ (\tau \otimes \Psi) \circ (\tau \otimes \Phi \circ \overline{\tau}) \circ (R_x \otimes \overline{\tau})$$

If $\Phi = \iota^2(A) \in \mathcal{C}^0$ is an object, since $\Phi^* = \Phi$, we get $\Phi^! = (\Phi^*)_* = \Phi_!$. If we assume now that the conjugation map $x \mapsto (R_x, \overline{R}_x)$ satisfies the additional unitality condition

$$R_x = \iota^2(x) = \overline{R}_x, \quad \forall x \in \mathcal{C}^0, \quad (4.3)$$

(such a choice is not restrictive and can always be done), in this case we necessarily have $\tau = x$, furthermore $\Phi_* = (\iota^2(A) \otimes \iota^2(A)^*) \circ (\Phi \otimes \iota^2(A)) \circ (\iota^2(A) \otimes \iota^2(A)^* ) = \Phi$ and hence $\Phi^! = \Phi$. Similarly, under condition (4.3), $\Phi^! = \Phi$, for $\Phi \in \mathcal{C}^0$, so that $\tau$ and $\Phi$ are covariant ($\mathcal{C}_c$, o) endofunctors.

The two foldings interchange under the $*$-involution i.e. $(\Phi_*)^* = \Phi^*$ (and similarly $(\Phi_!)^* = (\Phi^!)_*$):

$$(\Phi_*)^* = [(\tau \otimes \overline{R}_x) \circ (\tau \otimes \Phi \otimes \overline{\tau}) \circ (R_x \otimes \overline{\tau})]^* = (R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi \otimes \overline{\tau}) \circ (\tau \otimes \overline{R}_x)^*$$

$$= (R_x \otimes \overline{\tau}) \circ (\tau \otimes \Phi^* \otimes \overline{\tau}) \circ (\tau \otimes \overline{R}_x) = \Phi^!.$$
Without further requirements for conjugations, the maps \( \dagger \) and \( \ddagger \) are not usually involutive. For this purpose, let us assume that the conjugation map \( x \mapsto (R_x, \overline{R}_x) \) satisfies the involutivity condition\(^{32}\)

\[
(R_x, \overline{R}_x) = (\overline{R}_x, R_x), \quad \forall x \in \mathcal{C}^1.
\]  

(4.4)

Whenever a conjugation map satisfies \( 4.4 \), we have an induced involution \( x \mapsto \overline{x} \) on 1-arrows \( x \in \mathcal{C}^1 \) and, when unitality \( 4.3 \) also holds, such involution is Hermitian on objects: \( \overline{A} = A \), for \( A \in \mathcal{C}^0 \).

If the involutivity condition \( 4.4 \) holds, the two previous folding maps are mutually inverses. Here below we show \( \ast (\Phi_x) = \Phi_x \), the proof of \( \ast (\Phi_x) \) is similarly obtained in a “specular way”\(^{33}\)

\[
\ast (\Phi_x) = (R_x \otimes y) \circ (x \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ (R_x \otimes \overline{\mathcal{T}}) \circ y) \circ (x \otimes \overline{R}_x)
\]

\[
= (R_x \otimes y) \circ (x \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ (R_x \otimes \overline{\mathcal{T}}) \circ y) \circ (x \otimes R_x)
\]

\[
= (R_x \otimes y) \circ (x \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ (x \otimes \overline{R}_x) \circ y) \circ (x \otimes R_x)
\]

\[
= (R_x \otimes y) \circ (x \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x ) \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= \left[ (A \circ \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \overline{R}_x) \circ (\overline{\mathcal{T}} \otimes \Phi \otimes \overline{\mathcal{T}}) \circ x \right] \circ (x \otimes R_x)
\]

\[
= y \circ \Phi \circ x = \Phi. \quad (4.5)
\]

The maps \( \dagger, \ddagger \) are not necessarily involutive since, in general, \( \ast (\Phi_x) \neq \Phi \) and \( \ast (\ast (\Phi_x)) \neq \Phi \); but, when the conjugation map \( x \mapsto (R_x, \overline{R}_x) \) satisfies the involutivity condition \( 4.4 \), \( \dagger \) and \( \ddagger \) are indeed involutions:

\[
(\Phi^\dagger)^\ddagger = (((\Phi^\dagger)^\ast)_x)_x = (\ast (\Phi^\dagger))_x = \Phi_x = \Phi,
\]

\[
(\Phi^\ddagger)^\dagger = ((\ast (\Phi^\ddagger))_x)_x = (\ast (\Phi^\ast))_x = (\ast (\Phi^\ast))_x = \Phi_x = \Phi.
\]

In general the maps \( \dagger, \ddagger \) do not necessarily coincide:

\[
\Phi^\dagger = \Phi^\ddagger \iff (\Phi^\ast)^\dagger = (\Phi^\ast)^\ddagger \iff (\Phi^\ast)^\dagger = (\Phi^\ast)^\ddagger \iff \Phi_x = \ast (\Phi_x), \quad \forall \Phi \in \mathcal{C}. \quad (4.8)
\]

\(^{32}\)Notice that the unitality \( 4.3 \) does not conflict with the involutivity \( 4.4 \), so that both conditions can be required together.

\(^{33}\)Notice that, we have used the exchange properties involving at least a pair of arrows in \( \mathcal{C}^1 \), with the exception of the three passages leading to equations \( 4.3, 4.7 \), where at least one object and at least one adjunction unit/counit was involved.
This follows immediately from \((\Phi^*)^\dagger = (\Phi^*)^\ast \Phi^* \ast \Phi = \Phi \ast \Phi\), and from \((\Phi^\dagger)^\ast = ((\Phi^\ast)^\dagger)^\ast = (\Phi^\dagger)^\ast = \Phi\). (even in absence of unitarity and involutivity conditions for the conjugation map). When the involutivity property \((\Phi^\ast)^\dagger = \Phi \ast \Phi\) is assumed, conditions \((4.8)\) are further equivalent to the involutivity of the folding maps: \((\Phi^\ast)^\dagger = \Phi \ast \Phi\).

The validity of any of the equivalent properties \((4.8)\) is (in the monoidal category case) implied by the “traceability condition” described by R.Longo-J.E.Roberts \([LR]\) lemma 2.3 c], but is in general false.

In general the equations \((\Phi \otimes \Psi)^\dagger = \Psi^\dagger \otimes \Phi^\dagger \) and \((\Phi \otimes \Psi^\dagger) = \Psi^\dagger \otimes \Phi \) do not hold.

In those specific cases where it is possible to globally select a conjugation map \(x \mapsto (R_x, \overrightarrow{R}_x)\) that satisfies the tensorial conditions \([LR]\) proof of theorem 2.4 \([34]\)

\[
\begin{align*}
R_{\text{obj}} &= (i^2(\overrightarrow{\gamma}) \otimes R_x \otimes i^2(y)) \circ R_y, \\
\overrightarrow{R}_{\text{obj}} &= (i^2(x) \otimes \overrightarrow{R}_y \otimes i^2(\overrightarrow{\gamma})) \circ \overrightarrow{R}_x,
\end{align*}
\]

(4.9)

we observe (see \([LR]\) theorem 2.4) for the monoidal case) that conjugation becomes a \(\gamma\)-congruence relation on 1-arrows, i.e. whenever \((x, \overrightarrow{\gamma})\) and \((y, \overrightarrow{\gamma})\) are conjugate pairs, via \((R_x, \overrightarrow{R}_x)\) and \((R_y, \overrightarrow{R}_y)\), if \(x \otimes y\) exists (so also \(\overrightarrow{\gamma} \otimes \overrightarrow{\gamma}\) exists), \(x \otimes y = \overrightarrow{\gamma} \otimes \overrightarrow{\gamma}\), because \((x \otimes y, \overrightarrow{\gamma} \otimes \overrightarrow{\gamma})\) is a conjugate pair via \((R_{\text{obj}}, \overrightarrow{R}_{\text{obj}})\):

\[
\begin{align*}
(R_{\text{obj}} \otimes x \otimes y) \circ (x \otimes y \otimes R_{\text{obj}}) &= x \otimes y, \\
(R_{\text{obj}} \otimes \overrightarrow{\gamma} \otimes \overrightarrow{\gamma}) \circ (\overrightarrow{\gamma} \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_{\text{obj}}) &= (R_{\text{obj}} \otimes (\overrightarrow{\gamma} \otimes \overrightarrow{\gamma} \otimes \overrightarrow{y})) \circ (\overrightarrow{\gamma} \otimes \overrightarrow{\gamma} \otimes (x \otimes \overrightarrow{R}_y \otimes \overrightarrow{\gamma}) \circ \overrightarrow{R}_x),
\end{align*}
\]

Under the same tensorial conditions \((4.9)\) we obtain also the \(\overrightarrow{\gamma}\)-contravariance of the folding maps

\[
(\Phi \otimes \Psi)^* = \Psi^* \otimes \Phi^*, \quad (\Phi \otimes \Psi)^\dagger = \Psi^\dagger \otimes \Phi^\dagger, \quad \text{for} \quad A \xrightarrow{\phi} B \xrightarrow{\psi} C,
\]

via the following computation, using again the exchange property \([35]\) and the conjugate equations:

\[
\begin{align*}
\Psi^\ast \otimes \Phi^* &= (\overrightarrow{\gamma} \otimes \Phi^*) \circ (\Phi^* \otimes \overrightarrow{\gamma}) \circ (\overrightarrow{\gamma} \otimes \Phi^*) \circ (\Phi^* \otimes \overrightarrow{\gamma}) \circ (\overrightarrow{\gamma} \otimes \Phi^*) \circ (\Phi^* \otimes \overrightarrow{\gamma}), \\
&= (\overrightarrow{\gamma} \otimes [(\overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{\gamma} \otimes \Phi^*) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma})]) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y), \\
&= (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y), \\
&= (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y) \circ (\overrightarrow{R}_y \otimes \overrightarrow{\gamma} \otimes \overrightarrow{R}_y).
\end{align*}
\]

\([34]\) In \([BCM]\), a strict 2-category with such property (and unitarity) is said to be equipped with an “internal adjunction structure”.  
\([35]\) Again, also here we only use the exchange property whenever at least two of the four 2-arrows involved are in \(\mathfrak{C}\).
process of “bundlification”, trading categories for bundles and further generalizing to cases when the base is not simply a discrete pair groupoid, is quite useful and admits a vertical categorification.

We will eventually make use of such notions for the specific case of vertical categorification of Fell bundles that will be treated in the subsequent section 5.2.

\[ \pi \]

Every 1-category \((C, \odot)\) in independent linear structures are imposed for \(0 \leq n \leq 1\) and \(X \leq 1\), we will later restrict mostly to the case of depth-\((n-1)\).

Here, for simplicity, we will usually assume that the blocks \((C, \odot, n-1)\) are complex vector spaces.

As a consequence of the previous discussion, we see that a strict 2-category \((C, \odot, +, \cdot)\) can be seen as a bundle over the discrete pair groupoid \(X\) can be required the involution to be linear or conjugate-linear when restricted to the linear spaces. In this case we say that we have a \(\ast\)-n-algebroid. In the case of fully involutive n-categories, we will simply use the term \(\ast\)-n-algebroid.

Remark 5.2. In general the blocks \(C_{xy}\), for \(x, y \in \mathcal{O}\) might even be abelian groupoids \((C_{xy}, +)\).

Here, for simplicity, we will usually assume that the blocks \((C_{xy}, +, \cdot)\) are complex vector spaces.

Furthermore, we will later restrict mostly to the case of depth-(\(n-1\)), i.e. \(p = n-1\), so that no other independent linear structures are imposed for \(0 \leq p < n-1\).

Every 1-category \((C, \odot)\) can be seen as a bundle over the discrete pair groupoid \(X := \mathcal{O} \times \mathcal{O}\), with projection functor \(\pi : C \to X\) given by \(\pi(x) := (t(x), s(x))\) and with fibers \(C_{AB}\), for all \((A, B) \in X\). This process of “bundlification”, trading categories for bundles and further generalizing to cases when the base is not simply a discrete pair groupoid, is quite useful and admits a vertical categorification.

We introduce here, in a slightly more general form than needed, those notions of bundles that are compatible with functorial projections and with the involutions and fiberwise linear structures present in algebroids. We will eventually make use of such notions for the specific case of vertical categorification of Fell bundles that will be treated in the subsequent section 5.2.
Definition 5.3. A categorical \( n \)-bundle \((E, \pi, X)\) is a continuous open surjective \( n \)-functor \( \pi : E \to X \) between topological \( n \)-categories such that \( E \) is equipped with a “fiberwise uniform structure” i.e. a family of sets \( U \subset E \times E \), with \( U_e := U \cap E_{\pi(e)} \subset E_{\pi(e)} \), for all \( e \in E \), such that the sets \( \bigcup_{e \in O} U_{\pi(e)} \) with \( O \) an open set in \( X \) and \( \sigma : X \to E \) a continuous section of \( E \), form a base of neighborhoods of the topology of \( E \).\(^{36}\)

In a perfectly similar way we have bundifications of the definitions of strict globular (involutive) \( n \)-algebroid at level-\( p \).

Definition 5.4. A level-\( p \) algebroidal \( n \)-bundle is a categorical \( n \)-bundle \( \pi : E \to X \) such that \((E, \sigma_0, \ldots, \sigma_{n-1})\) is an \( n \)-algebroid at level-\( p \) and such that \( \sigma_p \) is bilinear when restricted to composable \( k \)-blocks.\(^{37}\) Algebroidal bundles, with the fiberwise uniformities induced via a choice of norms on the fibers, are said to be normed (and Banach when fiberwise complete).

Similarly, whenever \( X \) and \( E \) are (partially/totally) involutive topological categories, we define involutive \( n \)-algebroidal bundles as \( n \)-algebroidal bundles such that \( \pi : E \to X \) is a \(*\)-functor (for all the relevant involutions) and requiring that the involutions are linear or conjugate-linear when restricted to the linear spaces in every fiber.

The description of a 1-category \( C \) as a bundle over the pair groupoid \( \mathcal{C}^0 \times \mathcal{C}^0 \) admits a vertical categorification.

Remark 5.5. Every \( n \)-category \((C, \sigma_0, \ldots, \sigma_{n-1})\) (with usual exchange or non-commutative exchange) can be equivalently described as an \( n \)-categorical bundle \( \mathcal{E} \xrightarrow{(\sigma^{-1}, \sigma^{-1})} \mathcal{X} \) over the (discrete) \( n \)-category \( \mathcal{X} := (\mathcal{C}^{n-1} \times \mathcal{C}^{n-1}, \sigma_0, \ldots, \sigma_{n-1}, \sigma_n) \) with operations defined as pairwise compositions, if \( 0 \leq p < n \), and as “concatenations”, if \( p = n \).

\[
(x_1, y_1)\overset{p}{\circ}(x_2, y_2) := (x_1 \sigma_p x_2, y_1 \sigma_p y_2), \quad 0 \leq p < n, \quad (x_1, x_2), (y_1, y_2) \in \mathcal{C}^{n-1} \times \mathcal{C}^{n-1}
\]

\[
(x, y)\overset{n}{\circ}(y, z) := (x, z), \quad x, y, z \in \mathcal{C}^{n-1}
\]

5.2 Higher C*-categories and Higher Fell Bundles

After this quite long preparation on involutions and linear structures on strict globular \( n \)-categories, we are now ready to deal with the main subject of our investigation. We start with the following vertical categorification of the Longo-Roberts \( 2 \)-\( C^* \)-categories presented in definition 2.10.

Definition 5.6. A strict \( n \)-\( C^* \)-category of Longo-Roberts type \((\mathcal{C}, \sigma_{n-1}, \ldots, \sigma_0, *_{n-1}, \circ, *, \|\cdot\|)\) is a strict globular \( n \)-category \((\mathcal{C}, \sigma_{n-1}, \ldots, \sigma_0)\) that satisfies the following additional properties:

- \((\mathcal{C}, \sigma_{n-1}, \circ_{n-1}, +, \cdot, \|\cdot\|)\) is a \( C^* \)-category, with involution \(*_{n-1}\).
- \(*_{n-1}\) is a covariant functor on \((\mathcal{C}, \sigma_k)\), for all \( 0 \leq k < n - 1\).
- all the partial bifunctors \( \sigma_k\), for \( 0 \leq k < n - 1\), when restricted to \( \sigma_k\)-composable \( k \)-blocks, are bilinear and norm submultiplicative.

\[A *-functor \mathcal{C} \xrightarrow{\varphi} \mathcal{D} \text{ between n-C*-categories of Longo-Roberts type is just a functor between the underlying n-categories such that } \varphi(x^{*p}) = \varphi(x)^{*p}, \text{ for all } x \in \mathcal{C}.\]

A natural transformation between *-functors (and inductively a \( k \)-transfors) \( \Phi : \mathcal{C}^0 \to \mathcal{D}^k \) is bounded i.e. \( \sup_{x \in \mathcal{C}^0} \|\Phi(x)\| < \infty.\]

\(^{36}\)In most cases of practical interest \( X \) will be locally compact.

Completeness of the fibers can be imposed via the induced uniformity.

\(^{37}\)Such bilinearity is equivalent to the \( \sigma_p \)-bifunctoriality with respect to the additive structures.

\(^{38}\)The bundle defined here can have empty fibers (whenever the pair \((x, y) \in \mathcal{C}^{n-1} \times \mathcal{C}^{n-1}\) is not in globular position and there is no problem in restricting the base of the bundle to such “globular” pairs of \((n - 1)\)-arrows in \( \mathcal{C} \))
Remark 5.7. For $n = 1$, since the second and third properties are “vacuous” (there are no compositions $o_k$ with $k < n - 1 = 0$), the previous definition reproduces $C^*$-categories and for $n = 2$ we reobtain Longo-Roberts 2-$C^*$-categories in definition 2.10. Notice that $(\hat{C}, o_0, \ldots, o_{n-1}, +, \cdot)$ is an $n$-algebroid at level-$p$, for all $p = 0, \ldots, n - 1$, that is Banach with respect to the unique norm $\| \cdot \|$, and the category $\hat{C}$ is only partially involutive (with only one involution $\sigma_{n-1}$).

In the previous definition we have for now assumed that the strict globular $n$-category $(\hat{C}, o_0, \ldots, o_{n-1})$ satisfies the usual exchange property, so the partially defined compositions $o_k$, $0 \leq k < n$ are bifunctors on $(\hat{C}, o_0)$, for all $q \neq k$. This requirement can be relaxed as can be seen in the more general definition 5.10.

As natural transformations are 1-transfers, their boundedness requirement is $\sup_{\alpha, \beta \in C^{(1)}} \| \Phi(\alpha) \| < \infty$.

Examples of $n$-$C^*$-categories of Longo-Roberts type essentially reduce to a 1-$C^*$-category living “on the top” of a commutative $(n-1)$-$C^*$-category with only one involution; anyway, the following examples are interesting and naturally occur in the theory.

Example 5.8. The family of (bounded) natural transformations of $*$-functors of small 1-$C^*$-categories is a Longo-Roberts 2-$C^*$-category, cf. [LR] section 7. Let $\hat{C}^{(2)}$ denote the family of such natural transformations. We construct a 2-$C^*$-category $(\hat{C}^{(2)}, o_0, o_1, +, \cdot, \| \|)$ as follows:

- the vertical composition $\phi \circ \xi$ of bounded natural transformations $\Psi, \Phi \in \hat{C}^{(2)}$ between the

$*$-functors $\hat{C}$ of the $C^*$-categories $(\hat{C}, o, \ast, +, \cdot, \| \|)$, and $(\hat{C}, \hat{o}, \hat{\ast}, \hat{+}, \hat{\cdot}, \| \|)$, is the natural transformation from $\phi$ to $\xi$ defined, for all $o \in \hat{A}^{(1)}$, by $(\Phi \circ \Psi)_o := (\Phi_o \circ \Psi_o)$; the boundedness follows from $\|(\Phi \circ \Psi)_o\| \leq \|\Phi_o\| \cdot \|\Psi_o\|$.\n
- the horizontal composition $\hat{\theta} \circ \Omega$ of bounded natural transformations $\Theta, \Omega \in \hat{C}^{(2)}$ is defined, for all $o \in \hat{C}^{(0)}$, by $\psi_2(\Theta_o) \circ \hat{\varphi}_2(\hat{\varphi}_1(\hat{\varphi}_2(\hat{\varphi}_1(\Theta_o)))) = \hat{\varphi}_2(\hat{\varphi}_1(\hat{\varphi}_2(\hat{\varphi}_1(\Theta_o))))$. This expression yields indeed a natural transformation between the $*$-functors $\varphi_2 \circ \phi_1$ and $\psi_2 \circ \psi_1$ whose boundedness readily follows from $\|\varphi_2(\hat{\varphi}_1(\hat{\varphi}_2(\hat{\varphi}_1(\Theta_o))))\| \leq \|\varphi_2(\Theta_o)\| \leq \|\Theta_o\| \leq \|\Theta_o\|$. Since every $*$-functor between $C^*$-categories is automatically bounded [CLR].

- Given a natural transformation $\hat{\phi} \in \hat{C}^{(2)}$, the map $\Theta^{\hat{\phi}} : o \mapsto (\Phi_o)^{\hat{\phi}}$, for $o \in \hat{C}^{(0)}$, defines a natural transformation $\hat{\psi} \hat{\phi} \in \hat{C}^{(2)}$: for all $x \in \hat{C}$, since $\Phi_{\hat{\psi}(x)} \ast \phi(x) = \psi(x) \ast \Phi_{\hat{\phi}(x)}$, we have $\Phi_{\hat{\psi}(x)} \ast \phi(x) = (\Phi_{\hat{\phi}(x)} \ast \phi(x')) \hat{\phi}(x) = (\Phi_{\hat{\phi}(x)} \ast \phi(x')) \hat{\phi}(x) \ast \Phi_{\hat{\phi}(x)}$. Furthermore since the identities $\Phi^{\hat{\psi}(\hat{\phi}(x))} = \Phi_{\hat{\phi}(x)} \ast \phi(x)$ and $\Psi^{\hat{\psi}(\hat{\phi}(x))} = \Psi_{\hat{\phi}(x)} \ast \phi(x)$ are satisfied, we have that $\Phi \mapsto \Phi^{\hat{\psi}(\hat{\phi})}$ provides an involution over 1-arrows for the 2-category $(\hat{C}^{(2)}, o_0, o_1)$.

- Pointwise linear combinations of bounded natural transformations $\alpha \cdot \Phi + \Psi : o \mapsto \alpha^{\hat{\psi}} \Phi_o + \Psi_o$, with $\alpha \in \hat{C}$, are bounded natural transformations, since $\|\alpha^{\hat{\psi}} \Phi_o + \Psi_o\| \leq \|\alpha\| \cdot \|\Phi_o\| + \|\Psi_o\|$, and hence the blocks $\hat{C}^{(2)}$ are vector spaces with such linear structure. Since $\Phi \circ (\alpha \cdot \Phi + \Psi) = \alpha \cdot (\Phi \circ \Phi) + (\Phi \circ \Psi)$, for $j = 0, 1$, and similarly for the first argument, the previously defined compositions $o_0, o_1$ are blockwise bilinear maps. The involution $\alpha_0$ is blockwise conjugate linear: $(\alpha \cdot \Phi + \Psi)^{\hat{\psi}} = \overline{\alpha} \cdot \Phi^{\hat{\psi}} + \Psi^{\hat{\psi}}$.\n
• Every block $C^{(2)}_{\phi\circ\theta}$ becomes a normed space with the norm $||\Phi|| := \sup_{o\in\text{Ob}(A)}||\Phi_o||_\phi$. Since every Cauchy net $(\Phi^{(k)})_{k\in\mathbb{N}}$ induces, for all objects $o \in \text{Ob}(0)$, a Cauchy net $(\Phi^{(k)})_{k\in\mathbb{N}}$ in the Banach space $C^{(2)}_{\phi\circ\theta}$, the pointwise limit $\Phi: o \mapsto \lim_{k\to\infty}(\Phi^{(k)})$ exists. $\Phi$ is a natural transformation (as can be seen passing to the limit in the expression $\psi(x) \in C^{(4)}_{\phi\circ\theta} = \Phi^{(4)}_{\phi\circ\theta}$, using the continuity of $\hat{\delta}$) and is bounded: $||\Phi|| \leq ||\Phi^{(2)}||_\phi + ||\Phi^{(2)}||_\theta + \sup_{o\in\text{Ob}(A)}||\Phi^{(2)}_o||_\phi \leq ||\Phi^{(2)}_\phi||_\phi + \sup_{o\in\text{Ob}(A)}||\Phi^{(2)}_o||_\theta$, eventually in $\lambda$, for all $o \in \text{Ob}(0)$. As a consequence $C^{(2)}_{\phi\circ\theta}$ is a Banach space.

• The inequality $||\Phi \circ_1 \Psi||_\theta \leq ||\Phi||_\phi \cdot ||\Psi||_\theta$, is obtained taking the supremum of the pointwise submultiplicativity of the norms in $\hat{\psi}$ and similarly, for the $C^*$-property.

Hence $(\Phi^{(2)}, \phi, \circ_1, +, *, || \cdot ||_\phi)$ is a 1-$C^*$-category and so $C^{(2)}_{\phi\circ\theta}$ is a $C^*$-algebra for all $\ast$-functors $\phi$.

Finally observe that, for each $\mathcal{C} \xrightarrow{\phi} \hat{\mathcal{C}}$, $C^{(2)}_{\phi\circ\theta}$ can be isometrically embedded into $\bigoplus_{o\in\text{Ob}(0)} C^{(2)}_{\phi\circ\theta(o)}$ in the obvious way, from which the positivity of every $\phi^* \circ_{\theta} \Phi$ follows at once.

Proposition 5.9. The category of small strict globular $n$-$C^*$-categories of Longo-Roberts type with strict bounded $n$-transfers is an $(n+1)$-$C^*$-category of Longo-Roberts type.

Proof. For $n = 1$, the statement is described in the previous example [5.8] Inductively, assuming the result for $n$, we prove it for $n + 1$. Let $\mathcal{E}^{(n)}$ be a family of small globular $n$-$C^*$-categories of Longo-Roberts type and, for all $\mathcal{C}, \hat{\mathcal{C}} \in \mathcal{E}^{(n)}$, consider the family of bounded $k$-transfers $C^{(k)}_{\phi\circ\theta} \circ_{\psi} C^{(k)}_{\phi\circ\theta}$, for $k = 1, \ldots, n$, between $\ast$-functors $\Phi, \Psi: \mathcal{C} \rightarrow \hat{\mathcal{C}}$. By theorem [3.4], $\mathcal{E}^{(n)}$, with such bounded $n$-transfers, is already a strict globular $(n + 1)$-category. For all $A \in \text{Ob}(0)$, the component $C^{(n+1)}_A \circ_{\psi} C^{(n+1)}_A$, of a bounded $n$-transfer, between $k$-transfers $\Theta^{(k)}$, $\Omega^{(k)}$, for $k = 0, \ldots, n - 1$, is a globular $n$-cell $\Phi(A) \circ_{\psi} C^{(n+1)}_A \circ_{\psi} C^{(n+1)}_A : \text{Ob}(A) \rightarrow \text{Ob}(A)$, in the $n$-$C^*$-category $(\hat{\mathcal{C}}, \Phi_0, \ldots, \Phi_{n-1}, \hat{\tau}_{n-1}, \hat{\otimes}, || \cdot ||_{n})$ of Longo-Roberts type, hence $C^{(n)}_A$ belongs to the Banach space $\hat{\mathcal{E}}^{(n+1)}_{\Phi\circ_{\theta}}$. Addition and multiplication by scalars for $n$-transfers are defined “componentwise” by $(\Xi^{(n)} + \Xi^{(n)})_{A} := \Theta^{(n)} + \Theta^{(n)}$ and $(\alpha \cdot \Xi^{(n)})_{A} := \alpha \cdot \Xi^{(n)}$, and hence the box $\mathcal{E}^{(n+1)}_{\psi \circ_{\theta}}$ of $n$-transfers, between $\Omega^{(n-1)}$ and $\Theta^{(n-1)}$, with the supremum norm $||\Xi^{(n)}|| := \sup_{A \in \text{Ob}(0)} ||\Xi^{(n)}_A||_{\psi \circ_{\theta}}$, is a Banach space. Compositions of $n$-transfers over objects of $\mathcal{E}^{(n)}$ are defined componentwise by the formula $C^{(n)}_{\phi \circ_{\theta}} \circ_{\psi} C^{(n)}_{\phi \circ_{\theta}} \circ_{\psi} C^{(n)}_{\phi \circ_{\theta}} = (C^{(n)}_{\phi \circ_{\theta}} \circ_{\psi} C^{(n)}_{\phi \circ_{\theta}})_{A} := (C^{(n)}_{\phi \circ_{\theta}})_{A} \circ_{\psi} (C^{(n)}_{\phi \circ_{\theta}})_{A}$, for $A \in \text{Ob}(0)$.

Similarly, for $k = 1, \ldots, n$, the remaining compositions $C^{(n)}_{\phi \circ_{\theta}}$ of $\mathcal{E}^{(n)}$ and the unique involution $*_{n}$ of $\mathcal{E}^{(n)}$ are also defined componentwise by $(\Xi^{(n)} \circ_{\psi} \Xi^{(n)})_{A} := \Theta^{(n)} \circ_{\psi} \Theta^{(n)}$, and $(\Xi^{(n)})^{*}_{A} := (\Xi^{(n)})^{*}_{A-1}$. It follows that distributivity of compositions, the $C^*$-property $||\Xi^{(n)}||_{\phi \circ_{\theta}} \circ_{\psi} \Xi^{(n)} = ||\Xi^{(n)}||^2_{\phi \circ_{\theta}}$ and the positivity of $(\Xi^{(n)})^{*}_{\phi \circ_{\theta}} \circ_{\psi} \Xi^{(n)}$ in the $n$-$C^*$-algebra $(n-1)\mathcal{E}^{(n+1)}_{\psi \circ_{\theta}}$, are all derived by direct componentwise calculations, making use of the fact that $\hat{\mathcal{C}}$ is a $C^*$-category of Longo-Roberts type.

We now state our main definition of $n$-$C^*$-category with non-commutative exchange.

Definition 5.10. A fully involutive strict globular $n$-$C^*$-category with non-commutative exchange, denoted as $(\mathcal{C}, o_0, \ldots, o_{n-1}, *_{o_0, \ldots, o_{n-1}}, +, \cdot, || \cdot ||$), is a fully involutive strict $n$-category, with non-commutative exchange, such that:

• $(\mathcal{C}, o_0, \ldots, o_{n-1}, +, \cdot)$ is a $n$-algebroid at every level $p = 0, \ldots, n-1$,

• for all $a, b \in C^{n-1}_{\phi\circ\theta}$, the block $C_{ab}$ is a Banach space with the norm $|| \cdot ||$.

[3] Usually in examples $C_{ab}$ is actually a Banach space, but it might also be a disjoint union of Banach spaces or possibly even a horizontal categorification of a $C$-vector space (where only the diagonal additive blocks are Banach spaces).
for all $0 \leq p < n$, $\|x \circ_p y\| \leq \|x\| \cdot \|y\|$, whenever $x \circ_p y$ exists,

- for all $0 \leq p < n$, $\|x^{\circ_p} \circ_p y\| = \|x\|^2$, holds for all $x \in C_p$,

- for all $0 \leq p < n$, $x^{\circ_p} \circ_p y$ is positive in $(n-1)C_{p\epsilon}$, where $e$ is the $p$-source of $x$.

A partially involutive strict globular $n$-C*-category with non-commutative exchange will be equipped with only a subfamily of the previous involutions and will satisfy only those properties that can be formalized using the existing involutions.

**Remark 5.11.** Of course we can state the previous definition imposing the more restrictive exchange property (in which case we will omit the “non-commutative” in the denomination). $n$-C*-categories of Longo-Roberts type are special cases of partially involutive strict globular $n$-C*-categories.

We now examine the most natural elementary examples of strict globular $n$-C*-categories. We start with a C*-categorical version of examples 4.10 and 4.11. From example 5.12 we have a fully involutive 2-C*-category $\mathcal{C}_2$ of intertwiners of a strict bounded $n$-C*-category $\mathcal{C}_n$, which is a globular $\mathcal{C}_0$-category. By theorem 4.14, we know that $\mathcal{C}_n$ is a family of fully involutive $n$-C*-categories with bounded strict $n$-transfors, for $k = 0, \ldots, n$, equipped with bounded $k$-transfors, for $k = 1, \ldots, n$, $\mathcal{A} \xrightarrow{\Phi} \mathcal{B}$, between invertible $*$-functors $\Phi, \Psi$.

Since every $n$-C*-category is an $(n-1)$-C*-category (by truncating to its $(n-1)$-arrows), $\mathcal{C}_n$ can be seen as a family of fully involutive $(n-1)$-C*-categories. Together with the family of bounded $k$-transfors, for $k = 0, \ldots, n - 1$, $\mathcal{C}_n$ becomes a fully involutive $n$-C*-category, by the inductive hypothesis. We need to show that $\mathcal{C}_n$, with the bounded $k$-transfors, for $k = 0, \ldots, n$, is also a fully involutive $(n+1)$-C*-category. By theorem 4.14 we know that $\mathcal{C}_n$, with the family of bounded $k$-transfors, for $k = 0, \ldots, n$, is a fully involutive $(n+1)$-C*-category. Recall that each component $\Xi^{(n)}_A$, $A \in \mathcal{A}^0$, of a strict bounded $n$-transfor $\Xi^{(n)} : \mathcal{A}^0 \rightarrow \mathcal{A}^{n}$, between $k$-transfors $\Theta^{(k)}$, $\Omega^{(k)}$, for $k = 0, \ldots, n - 1$, between invertible $*$-functors $\Phi, \Psi$ from $A$ to $\hat{A}$, is a globular

---

2 Note that the conditions here assumed already imply that for all level $p$ identities $e \in \mathcal{C}_p$ the block $(n-1)C_{p\epsilon}$ is a C*-algebra $(\mathcal{(n-1)}C_{p\epsilon}, \circ_p, +_p, \| \cdot \|)$, with composition $\circ_p$ and involution $\ast_p$.

3 Imposing the C*-property only whenever $x^{\circ_p} \circ_p y$ is positive in $(n-1)C_{p\epsilon}$, where $e$ is the $p$-source of $x$, is certainly possible, but it is a less restrictive condition that would result in a much more general structure (as soon as $n > 1$).
In order to define the "horizontal" composition of 2-arrows, consider first the operations of addition and multiplication by scalars for n-transforms: \((\Xi^{(n)})_A := \Xi_A^{(n)} \oplus \Xi_A^{(n)}\) and \((\alpha \cdot \Xi^{(n)})_A := \alpha \cdot \Xi_A^{(n)}\); furthermore, with the norm \(\|\Xi^{(n)}\| := \sup_{A \in \mathcal{A}^n} \|\Xi_A^{(n)}\|_A\), the family \(\Theta_A^{(n)}\) of n-transforms, between \(\Omega_A^{(n-1)}\) and \(\Theta_A^{(n-1)}\) is a Banach space. Since compositions \(\circ\) and involutions \(*_k\) in the \((n + 1)\)-category \(\mathcal{C}^{(n)}\), for \(k = 1, \ldots, n\) are similarly componentwise defined by \((\Xi^{(n)})_k \Xi^{(n)} = \Xi_\Delta^{(n)} \delta_k \Phi_k (\Xi^{(n)}), \) the distributivity with respect to addition of \(\circ_1, \ldots, \circ_n\) is valid and, for the positivity of \((\Xi^{(n)})^{\circ_k} \circ_0 \Xi^{(n)}\) and the \(C^*\)-property \(\|\Xi^{(n)}\|^{\circ_k} \circ_0 \Xi^{(n)}\|_A = \|\Xi^{(n)}\|^2\), the distributivity follows immediately from the definition of \(\circ_0\)-composition of n-transforms:

\[
\Xi_A^{(n)} \circ_0 \Xi_B^{(n)} := (\Xi_A^{(n)})^{\circ_0} \circ_0 \Xi_B^{(n)}, \quad \text{for } A \in \mathcal{A}^n,
\]

and from the distributivity of compositions \(\circ\) objects in the small \(n\)-\(C^*\)-categories belonging to \(\mathcal{C}^{(n)}\). For the \(C^*\)-property and the positivity, we see that, for all \(A \in \mathcal{A}^n\):

\[
\|((\Xi^{(n)})^{\circ_k} \circ_0 \Xi^{(n)})(\cdot)\|_A = \sup_{A \in \mathcal{A}^n} \|\Xi^{(n)}(\cdot)\|_A = \sup_{A \in \mathcal{A}^n} \|\Phi^{-1}(\Xi^{(n)})(\cdot)\|_A = \sup_{A \in \mathcal{A}^n} \|\Phi^{-1}(\Xi^{(n)})(\cdot)\|^2 = \|\Xi^{(n)}\|^2
\]

and \(\Phi^{-1}(\Xi^{(n)})(\cdot)\) is positive, for all \(A \in \mathcal{A}^n\), in the \(C^*\)-algebra \((n-1)\)-cell.

\(\square\)

**Example 5.14.** We consider as objects a family \(\mathcal{C}^0 := \{A, B, \ldots\} \) of unital \(C^*\)-algebras and as 1-arrows the family \(\mathcal{C}^1\) of all the imprimitivity bimodules between them (this means that if two algebras \(A, B\) are not Morita equivalent, \(\mathcal{C}^1_{A,B} = \emptyset\)); specifically we denote by \(\mathcal{M} \rightarrow \mathcal{M}\) the morphisms of such bimodules (i.e. those additive maps such that \(\Phi(a \cdot x \cdot b) = a \cdot \Phi(x) \cdot b\), for all \(a \in A, b \in B, x \in M\) that are bi-adjointable in the sense that there exists a necessarily unique homomorphism of bimodules \(\Phi^* : M \rightarrow M\) that is simultaneously right-adjoint (i.e. with respect to the \(B\)-valued inner product) and left-adjoint (i.e. with respect to the \(A\)-valued inner product) to \(\Phi\). Of course, \((\Phi^*)^* = \Phi\) and we take \(*\) as the involution \(*_1\) over 1-arrows. As "vertical" composition of 2-arrows, we consider the usual composition of adjointable homomorphisms of imprimitivity bimodules (this is necessarily biadjointable since the composition of right/left adjointable maps between right/left-correspondences is right/left adjointable) and \((\Phi \circ_1 \Psi) = \Psi \circ_1 \Phi^*\). In order to define the "horizontal" composition of 2-arrows, consider first the quotient of the 1-category of the imprimitivity bimodules in \(\mathcal{C}^1\) under tensor products: this constitutes a Fell bundle whose total space is the free involutive category generated by all the elements \(x \in M\) with \(M \in \mathcal{C}^1\), where the fiberwise composition strictly implements the tensor product of the fibers (that are arbitrary tensor products of the bimodules in \(\mathcal{C}^1\) and their Rieffel duals) and the fiberwise conjugation implements the Rieffel dual. The "horizontal" composition of 2-arrows \(B_1 \mathcal{M}^2_{B_0} \rightarrow B_2 \mathcal{N}^2_{B_1} \rightarrow \mathcal{M}^2_{B_0} \rightarrow \mathcal{N}^2_{B_1}\) is obtained, via the universal factorization property for inner tensor products of imprimitivity bimodules, as the unique homomorphism of bimodules satisfying \((\Phi \circ_0 \Psi)(x \otimes_{B_0} y) := \Phi(x) \otimes_{B_0} \Psi(y),\) for all \(x \in \mathcal{M}^1\) and \(y \in \mathcal{N}^1\) for commutative \(C^*\)-algebras, if a homomorphism of bimodules that is both right and left adjointable, the right and left adjoints must coincide.
\( y \in M^2 \) and verifying that \( \Phi_0 \Psi : \mathcal{A} M^1 \otimes_B M^2_\mathcal{A} \to \mathcal{A} N^1 \otimes_B N^2_\mathcal{A} \) is bi-adjointable when \( \Phi, \Psi \) are such, and also that \( \Phi^* \Psi^* = \Phi^* \Psi^* \).

The involution over objects of a 2-arrow \( \mathcal{A} M_\mathcal{B} \to \mathcal{A} N_\mathcal{B} \) is obtained as the bi-adjointable homomorphism of bimodules \( \mathcal{A} M_\mathcal{B} \to \mathcal{A} M_\mathcal{A} \) (where \( \mathcal{B} \mathcal{A} \) denotes the Rieffel conjugate of \( \mathcal{A} M_\mathcal{B} \)), as fibers in the previous strictification Fell bundle, defined by \( \Phi(x) := \Phi(x) \), for all \( x \in \mathcal{M} \).

The identities \( \Phi_0 \Psi \) and \( \Phi \Psi \) are complete by a direct computation. Each block \( \mathcal{C}_{\mathcal{N}M} \), with \( \mathcal{A} M_\mathcal{B}, \mathcal{A} N_\mathcal{B} \in \mathcal{C}_{\mathcal{A} \mathcal{B}}^1 \) becomes a normed space with linear structure given by \( \Phi + \lambda \cdot \Psi(x) := \Phi(x) + \lambda \cdot \Psi(x) \), for all \( x \in \mathcal{M}, \lambda \in \mathcal{C} \) and norm \( \| \Phi \| := \sup_{x \in \mathcal{M}, \| \|, \| \leq 1} \| \Phi(x) \| \) (that, by closed graph theorem, is bounded for an adjointable operator). With respect to such spaces, all compositions are blockwise bilinear and all the involutions are conjugate linear.

The \( \Phi \)-submultiplicativity of the norm \( \| \Phi \| \leq \| \Phi \| \), for any functional composition of bounded linear maps. Since right (respectively left) adjointable maps of right (left) Hilbert \( \mathcal{C}^\ast \)-modules are a \( \mathcal{C}^\ast \)-category (and in our case the right and left norms of bi-adjointable homomorphisms coincide) the \( \Phi \)-property \( \| \Phi \| \) holds.

For a proof of the \( \Phi \)-submultiplicativity, given the pair of 2-arrows \( \mathcal{A} M^1 \to \mathcal{A} N^1 \) and \( \mathcal{A} M^2 \to \mathcal{A} N^2 \), denoting by \( M_M, I_M \) the identities of the respective bimodules and by \( \mathcal{C}_{\mathcal{N}M} \) the \( \mathcal{C}^\ast \)-algebra \( \mathcal{C}_{\mathcal{N}M} \), for \( \mathcal{M} \in \mathcal{C}^1 \), we notice that since the map \( \mathcal{Z} \mapsto \mathcal{Z}_0 I_M \mathcal{C} \) is a \( + \)-homomorphism between the \( \mathcal{C}^\ast \)-algebras \( \mathcal{C}_{\mathcal{N}M} \) and \( \mathcal{C}_{\mathcal{M} \otimes \mathcal{M}^\ast} \), and similarly the map \( \Theta \mapsto I_M \mathcal{C} \) is a \( + \)-homomorphism between the \( \mathcal{C}^\ast \)-algebras \( \mathcal{C}_{\mathcal{N}M} \) and \( \mathcal{C}_{\mathcal{M} \otimes \mathcal{M}^\ast} \), they are both contractive; and hence we have:

\[
\| \Phi_0 \| \leq \| \Phi \| \leq \| \Phi \| \leq \| \Phi \|
\]

The second inequality is obtained, for \( \Phi : \mathcal{M} \to \mathcal{N} \), as follows:

\[
\| \Phi_0 \| \leq \sup_{x \in \mathcal{M}, \| x \| \leq 1} \| \Phi(x) \| \leq \| \Phi \| \leq \| \Phi \|
\]

For the completeness of \( \mathcal{C}_{\mathcal{N}M} \), given a Cauchy net \( \Phi_\mu \in \mathcal{C}_{\mathcal{N}M} \), for all \( x \in \mathcal{M} \), we see that the net \( \Phi_\mu(x) \) is Cauchy in the Banach space \( \mathcal{M} \) and converges to \( \Phi(x) \). From the \( \Phi \)-property and the \( \Phi \)-submultiplicativity of the norm, we obtain the isometry of the \( \ast \)-involution hence \( \| \Phi_\mu \| = \| \Phi \| \) and so \( \Phi_\mu(x) \) is a Cauchy net as well, for all \( y \in \mathcal{N} \). Passing to the limit in the bi-adjointability conditions \( A(\Phi_\mu(x) | y) = \mathcal{A}(x | \Phi_\mu(y)) \) and \( \Phi_\mu(x) | y) = \mathcal{A}(x | \Phi_\mu(y)) \) for \( \Phi_\mu \) we immediately obtain that the map \( x \mapsto \Phi(x) \) is bi-adjointable, hence linear and bounded, and the convergence in \( \mathcal{C}_{\mathcal{N}M} \) of the net \( \Phi_\mu \).

By Beckmann-Hilton collapse, the \( \mathcal{C}^\ast \)-algebra of intertwiners of the Morita identity bimodule \( \mathcal{A} \) is a commutative \( \mathcal{C}^\ast \)-algebra under the common product \( \mathcal{0} = \mathcal{1} \) (see also TZ). Since two involutions that satisfy the \( \mathcal{C}^\ast \)-property, for a common product and the same norm, necessarily coincide (see H.F.Bohnenblust-S.Karlin [BK, theorem 9]) \( \Phi^\ast = \Phi \), for all \( \Phi \in \mathcal{C}^\ast \mathcal{A} \mathcal{A} \). Hence, for all such intertwiners \( \Phi \), we have \( \Phi_0 \Phi = \Phi^\ast \), that is a positive element in \( \mathcal{C}^\ast \mathcal{A} \mathcal{A} \).

**Example 5.15.** As a particular case of example 5.14, if \( \mathcal{C} \) is a full 1-\( \mathcal{C}^\ast \)-category, the family of bi-adjointable endomorphisms \( B \subseteq \mathcal{C}^\ast \mathcal{A} \mathcal{A} \) of each one of the imprimitivity Hilbert \( \mathcal{C}^\ast \)-bimodules \( \mathcal{C}_{\mathcal{A} \mathcal{B}} \), \( \mathcal{A}, \mathcal{B} \in \mathcal{C}^\ast \), is a fully involutive 2-\( \mathcal{C}^\ast \)-category (where all the 2-arrows are loops over 1-arrows).
We describe here a horizontal categorification of example 5.14. For this purpose, we recall (see P. Mitchener [M, section 8]) a preliminary definition of Hilbert C*-bimodule between 1-C*-categories: this is just a “C*-operator algebraic” version of the usual notion of “categorical bimodule” (the horizontal categorification of a bimodule over a monoid).

**Example 5.16.** We have a fully involutive 2-C*-category of bi-adjointable maps between imprimitivity bimodules of full 1-C*-categories.

The following remark goes in the direction of a vertical categorification of C*-Morita theory.

**Remark 5.17.** Although here we are not entering into further details, there is little doubt that it is possible to produce a vertical categorification of example 5.14 providing a recursive construction of (fully involutive) higher C*-categories (with non-commutative exchange) and an “operator categorical” analog of theorem 4.14, namely, given a family $\mathcal{C}$ of (fully involutive) $n$-C*-categories, the bi-adjointable morphisms between pairs of imprimitivity bimodules $\mathbf{c}_1, M_{\mathbf{c}_2} \xrightarrow{\Phi} \mathbf{c}_1, N_{\mathbf{c}_2} \in \mathcal{C}$, between $n$-C*-categories $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$, are the $(n+1)$-arrows of a (fully involutive) $(n+1)$-C*-category $\mathcal{C}$. To deal with such a construction, one needs to introduce the notion of (imprimitivity) higher-C*-bimodules between $n$-C*-categories along the lines already mentioned in [BCL5].

There is a 2-categorical immersion of the strict fully involutive 2-C*-category of example 5.14 into the strictified fully involutive 2-C*-category of example 5.14 that to every isomorphism $\Phi : \mathcal{C} \to \mathcal{D}$ of C*-categories associates the C*-categorical imprimitivity Hilbert C*-bimodule $\phi : \mathcal{B} \to \mathcal{C}$, obtained by left-twisting by $\Phi$, the identity bimodule $\mathcal{I} : \mathcal{B} \to \mathcal{C}$ and associating to every intertwiner $A \xrightarrow{\Phi} B$ the bi-adjointable morphism of C*-categorical bimodules $\phi : \mathcal{B} \to \mathcal{C}, M_{\mathbf{c}_1}(b) := \mathcal{I}_A \cdot b, \mathcal{I}_A \in \mathcal{C}(\mathbf{c}_1, \mathbf{c}_2), b \in (\mathcal{B})(\mathbf{c}_1, \mathbf{c}_2)$ defined as $M_{\mathbf{c}_1}(b) := \mathcal{I}_A \cdot b, \mathcal{I}_A \in \mathcal{C}(\mathbf{c}_1, \mathbf{c}_2), b \in (\mathcal{B})(\mathbf{c}_1, \mathbf{c}_2)$. 

As a particular case, we mention a vertical categorification of example 5.14 if $\mathcal{C}$ is a fully involutive $n$-C*-category, the family of bi-adjointable endomorphisms of the blocks $\mathcal{C}_{\mathbf{c}_1}$, with $x, y \in \mathcal{C}_{x y}$, is a fully involutive $(n+1)$-C*-category.

**Example 5.18.** We continue here, examining the C*-categorical properties, the study of involutions induced by conjugations already started in example 4.3.

If $(\mathcal{C}, \otimes, \circ, \ast, +, ||, ||)$ is a Longo-Roberts 2-C*-category equipped with a unital involutive tensorial conjugation map that satisfies the traciability condition, the resulting fully involutive strict 2-category $(\mathcal{C}, \otimes, \circ, \ast, +, ||, ||)$ is an example of a fully involutive 2-category.

Under our previous conditions the (unique) folding map is an involutive endofunctor of the C*-category $(\mathcal{C}, \otimes, \circ, \ast, +, \cdot, ||, ||)$ and hence it is a norm contractive map. Since it is involutive we obtain $||\Phi_0|| = ||\Phi||$ and we immediately get the isometric property of the $\dagger$-involution: $||\Phi_0|| = ||\Phi\dagger||, ||\Phi^0|| = ||\Phi||$.

Consider the following unitarity condition for the conjugations maps $x \mapsto (R_x, \overline{R_x})$, for all $x \in \mathcal{C}$ such that $\mathbf{T} \otimes x, x \otimes \mathbf{T} \in \mathcal{C}_0$, $R_x$ and $\overline{R_x}$ are unitary elements of the C*-category $(\mathcal{C}, \otimes, \circ, \ast, +, ||, ||)$, i.e. for $x \in \mathcal{C}_0$, such that $\mathbf{T} \otimes x = \iota^2(B)$ and $x \otimes \mathbf{T} = \iota^2(A)$, $R_x$ is a unitary element of the C*-algebra $\mathbf{c}_{BB}$ and $\overline{R_x}$ is a unitary element in the C*-algebra $\mathbf{c}_{AA}$.

Under such unitarity condition, a 2-C*-category of Longo-Roberts type, with unital involutive tensorial conjugations $(R_x, \overline{R_x})$, that satisfy traciability, becomes a fully involutive 2-C*-category.

In order to prove this statement, we recall that from section 4.3 $(\mathcal{C}, \otimes, \circ, \ast, +, ||, ||)$ is naturally equipped with a structure of fully involutive 2-category. We only need to show the C*-property $||\Phi_0 \otimes \Phi|| = ||\Phi||^2$ and the positivity of $\Phi_0 \otimes \Phi$, whenever $\Phi_0 \otimes \Phi$ belongs to the C*-algebra $\mathbf{c}_{AA}$, where $A = \delta_0(\Phi) \in \mathcal{C}_0$.

Whenever $A \xrightarrow{\Phi_0} B$ is such that $\Phi_0 \otimes \Phi \in \mathbf{c}_{AA}$, we always have $\mathbf{T} \otimes x = \iota^2(B), \mathbf{T} \otimes y = \iota^2(B)$ and hence $R_x, R_y \in \mathbf{c}_{BB}$ and $\overline{R_x}, \overline{R_y} \in \mathbf{c}_{AA}$ are unitary elements in the respective C*-algebras.
By the fact that left/right tensorization with elements of $\mathcal{C}^1$ is a C*-functor and C*-functors are always norm contractive in a C*-category, we have $|\Phi| \leq |\mathcal{T} \otimes (\Phi^* \circ \Phi)| \leq ||x \otimes \mathcal{T} \otimes \Phi|| = ||\mathcal{L}(\Lambda) \circ \Phi|| = ||\Phi||$. Hence, since conjugation by unitary element is a norm preserving operation, we immediately obtain:

$$||\Phi \cdot \Phi|| = ||R_x \circ (\mathcal{T} \otimes (\Phi^* \circ \Phi)) \circ R_x^*|| = ||\mathcal{T} \otimes (\Phi^* \circ \Phi)|| = ||\Phi^* \circ \Phi|| = ||\Phi||^2. \quad (5.1)$$

A direct computation of $\Phi^\dagger \otimes \Phi$ gives:

$$\Phi^\dagger \otimes \Phi = (\Phi^\dagger) \circ \Phi = [(R^x \otimes \mathcal{T}) \circ (\mathcal{T} \otimes \Phi^\dagger \otimes \mathcal{T}) \circ (\mathcal{T} \otimes R_x^\dagger)] \otimes \Phi$$

$$= (R^x \otimes \mathcal{T} \otimes y) \circ [(\mathcal{T} \otimes \Phi^* \otimes \mathcal{T}) \circ (\mathcal{T} \otimes y \otimes \mathcal{T} \otimes \Phi)] \circ (\mathcal{T} \otimes R_x^\dagger \otimes x)$$

$$= R^x \circ [(\mathcal{T} \otimes (\Phi^* \circ \Phi)] \circ (\mathcal{T} \otimes R_x^\dagger \otimes x).$$

Making use of unitarity and tensoriality of the conjugations:

$$||\Phi^\dagger \otimes \Phi|| = ||(R^x \circ (\mathcal{T} \otimes (\Phi^* \circ \Phi)) \circ (\mathcal{T} \otimes R_x^\dagger \otimes x)|| = ||[(\mathcal{T} \otimes (\Phi^* \circ \Phi)) \circ (\mathcal{T} \otimes R_x^\dagger \otimes x)]||$$

$$= ||[(\mathcal{T} \otimes (\Phi^* \circ \Phi)] \circ (\mathcal{T} \otimes R_x^\dagger \otimes x) \circ R_x|| = ||[(\mathcal{T} \otimes (\Phi^* \circ \Phi)) \circ (\mathcal{T} \otimes R_x^\dagger \otimes x)]||$$

and the C*-property follows comparing to equation (5.1).

Notice that, by Eckmann-Hilton argument, the C*-algebras $\mathcal{C}^1_{BB}$ are commutative and for elements $\Phi, \Psi \in \mathcal{C}^1_{BB}$, $\Phi \circ \Psi = \Phi \otimes \Psi$. Furthermore, for $\Phi \in \mathcal{C}^1_{BB}$, by the $\otimes$-C*-property and $\mathcal{B} = \Phi^\dagger \otimes \Phi$.

Regarding positivity, under the requirement of triviality of conjugations ($R_x = R_x^\dagger$), essentially says that $R_x$ is a continuous function with modulus one on the spectrum of the commutative C*-algebra $\mathcal{C}^1_{(x) \times (x)}$ and hence constant $\pm 1$ on each connected component. In such a context, it is likely that in many cases of interest the standard choice of triviality for conjugations (up to scalars) is forced from the other requirements.

**Remark 5.19.** For $n = 2$, when $\mathcal{C}^1$ consists of only one object, our definition of partially involutive strict globular 2-C*-category with non-commutative exchange is compatible with the generalization of monoidal C*-categories recently described by R.Blute-M.Comeau [BC].

**Remark 5.20.** For $n = 2$, again when $\mathcal{C}^1$ consists of only one object, our definition of a partially involutive strict globular 2-C*-category with non-commutative exchange is a special case of the semitensor C*-categories introduced by S.Doplicher-C.Pinzari-R.Zuccante [DPZ] section 2.

The usual process of “bundlification” can be applied to our definition of strict quantum $n$-C*-categories:

**Definition 5.21.** An $n$-Fell bundle (with non-commutative exchange) is given by a Banach bundle $(\mathcal{E}, \pi, \mathcal{X})$ where $\pi : \mathcal{E} \to \mathcal{X}$ is an $n$-functor between fully involutive topological strict $n$-categories (with non-commutative exchange), such that:

- the compositions $\circ_p$ are bilinear whenever defined;
- the involutions $\ast_p$ are fiberwise conjugate-linear;
- $||x \circ_p y|| \leq ||x|| \cdot ||y||$, for all $\circ_p$-composable $x, y \in \mathcal{E}$;
- $||x^{\ast_p} \circ_p x|| = ||x||^2$ holds, for all $p$, for all $x \in \mathcal{E}$;
- for all $x \in \mathcal{E}$, $x^{\ast_p} \circ_p x$ is positive whenever $\pi(x^{\ast_p} \circ_p x)$ is an idempotent in $\mathcal{X}$.

---

3. This means that $(\mathcal{E}, \pi, \mathcal{X})$ is an $n$-algebroid at all levels.
4. As in the definition of higher C*-category, imposing the C*-property only whenever $\pi(x^{\ast_p} \circ_p x)$ is an idempotent (or a $p$-identity) in $\mathcal{X}$, is for sure possible, but it results in a much more general structure (even in the case of ordinary Fell bundles).
5. This condition is meaningful since the previous axioms already assure that the fiber $\mathcal{E}_{(x^{\ast_p} \circ_p x)}$ is a C*-algebra.
5.3 Hypermatrices, Hyper-C*-algebras and Higher Convolutions

In this subsection we finally start to provide the long awaited direct examples of strict (fully) involutive higher C*-categories with non-commutative exchange. We will discuss mostly discrete finite cases, that are already of great interest.

The first step consists in reformulating the usual “innocent” definition of complex square matrix, making it apparently quite “convoluted”, but ready for generalizations.

**Proposition 5.22.** A complex square matrix \([x^i_j]\) \(\in \mathcal{M}_{N \times N}(\mathbb{C})\) of order \(N \in \mathbb{N}_0\) is a section of the Fell line-bundle \(E := X \times \mathbb{C}\) over the discrete finite pair groupoid \(X : X^1 \rightrightarrows X^0\) of the set \(X^0 := \{1, \ldots, N\}\).

**Proof.** To justify the statement, it is sufficient to consider the finite set \(X^0 := \{1, \ldots, N\}\) together with the finite set of 1-arrows (ordered pairs) \((i, j) \in X^1 := X^0 \times X^0\), with source \(j\) and target \(i\) and note that \(X^1\) is naturally a groupoid (actually an equivalence relation with only one equivalence class) under the usual composition \((i, j) \circ (j, k) := (i, k)\), for all \(i, j, k \in X^0\), with inverse given by \((i, j)^{-1} = (j, i)\), for all \(i, j \in X^0\) and partial identities \((j, j)\), for all \(j \in X^0\). The trivial Fell line-bundle \(E := X \times \mathbb{C}\) over the pair groupoid \(X\) is simply obtained by attaching a complex line \(\mathcal{E}(i, j) := \mathbb{C}\) to each of the 1-arrows \((i, j) \in X^1\). A section of such Fell line-bundle, being a function \(x : X^1 \to E := \bigcup_{(i, j) \in X^1} \mathcal{E}(i, j)\) such that \(x^j_i = x(i, j) \in \mathcal{E}(i, j)\), for all \((i, j) \in X^1\), is immediately seen to correspond to a complex square matrix \([x^i_j]\) with entries \(x^i_j\), for all \(i, j \in X^0\).

An alternative way to construct the previous Fell line-bundle consists in considering the complex line \(\mathbb{C}\) as fiber over the space \(\langle \bullet, \bullet \rangle\), consisting of a unique loop \(\bullet \bigcirc \cdots \bigcirc \bullet\) with source and target \(\bullet\), and the \(T\)-pull-back \(E := T^\bullet(\mathbb{C})\) of such trivial one-point Fell bundle, via the unique functor \(T : X \to \langle \bullet, \bullet \rangle\) that collapses every 1-arrow of the pair groupoid \(X\) to the unique loop \(\langle \bullet, \bullet \rangle\). Here is an intuitive picture of the Fell line-bundle \(E\) (restricted to the base pair subgroupoid generated by the two points 1 and \(N\)):

![Diagram](attachment:image.png)

Clearly for the family of continuous sections of \(E := T^\bullet(\mathbb{C})\) we have \(\Gamma(X; E) = \mathcal{M}_{N \times N}(\mathbb{C})\) and this construction can be applied in the same way, taking an arbitrary associative complex unital *-algebra \(A\) in place of \(\mathbb{C}\), obtaining the *-algebra \(\Gamma(X; T^\bullet(A)) \equiv \mathcal{M}_{N \times N}(A) \equiv \mathcal{M}_{N \times N}(\mathbb{C}) \otimes_{\mathbb{C}} A\) of \(A\)-valued matrices.

As a second step, we stress that there is no obstacle in generalizing the previous construction, starting with other finite groupoids, or even a finite involutive category, \(X\) in place of the previous pair groupoid of the set with \(N\) points.

**Proposition 5.23.** Given a finite involutive category \((X, \circ, \ast)\) and a complex unital *-algebra \((A, \cdot, -)\), the family \(\Gamma(X; T^\bullet(A))\) of sections of the Fell bundle \(T^\bullet(A)\) over \(X\), obtained by \(T\)-pull-back of the fiber \(A\) via the terminal functor \(T\) from \(X\) to the 1-loop space \(\langle \bullet, \bullet \rangle\), is a *-algebra with the operations:

\[
(\sigma \circ \rho)_z := \sum_{x \in \Xi} \sigma_x \cdot \rho_y, \quad \forall \sigma, \rho \in \Gamma(X; T^\bullet(A)), \quad \forall z \in X,
\]

\[
(\sigma^*_z) := \overline{\sigma_z}, \quad \forall \sigma \in \Gamma(X; T^\bullet(A)), \quad \forall z \in X.
\]

The resulting *-algebra of sections \(\Gamma(X; E)\) is just the convolution algebra of the groupoid (respectively of the finite involutive category) \(X\) and it is usually denoted by \(\mathbb{C}[X]\). In the case of the pair groupoid of a set of \(N\) elements, the previous operations reduce exactly to the usual row-by-column multiplication and transpose conjugate involution of matrices in \(\mathcal{M}_{N \times N}(A)\). Hence we just proved that:

*the *-algebra of matrices is just a special case of the convolution *-algebra of a finite *-category \(X\).*

Finally, as the last step, we examine what happens when, in place of a finite involutive 1-category, we allow a strict finite globular (fully involutive) \(n\)-category (with or without non-commutative exchange).
Theorem 5.24. Given a finite strict globular n-category \((X, \sigma_0, \ldots, \sigma_{n-1})\) (with usual exchange law or with non-commutative exchange) and an associative unital algebra \((A, \cdot)\), the family \(\Gamma(X; T^*(A))\) of sections of the bundle \(T^*(A)\) over \(X\), obtained by \(T\)-pull-back of the fiber \(A\) via the terminal functor \(T\) from \(X\) to the strict globular n-category with only one n-arrow is a unital associative algebra with respect to each one of the following convolution operations \(\delta_p\), for \(p = 0, \ldots, n-1\):

\[
(\sigma \delta_p \rho)_t := \sum_{\sigma \rho \gamma \in Z} \sigma_x \cdot \rho_y, \quad \forall \sigma, \rho \in \Gamma(X; T^*(A)), \quad \forall z \in X.
\]

The bundle \(T^*(A) = A \times X\) over \(X\) embeds into \(\Gamma(T^*(A))\) via the fiberwise linear maps:

\[
a_x \mapsto a \cdot (\delta^x)_y \quad \text{where} \quad a \in A, \ x \in X \quad \text{and for all} \quad x, y \in X \quad (\delta^x)_y := \begin{cases} 1_A, & \text{if} \ x = y \\ 0_A, & \text{if} \ x \neq y \end{cases}
\]

and becomes a strict globular n-category with the restriction of the convolution operations \(\delta_0, \ldots, \delta_{n-1}\). Whenever the algebra \(A\) fails to be commutative, the resulting n-category \((T^*(A), \delta_0, \ldots, \delta_{n-1})\) does not satisfy the usual exchange law (even when \(X\) does), but satisfies the non-commutative exchange.

Proof. With the notations introduced above, the results amount to a direct algebraic verification of associativity unitality and non-commutative exchange for the convolution operations \(\delta_p, \ldots, \delta_{n-1}\). The fact that non-commutative exchange is necessary whenever \(A\) is not abelian follows from the Eckmann-Hilton collapse and the fact that for \(x \in X^p\) and \(p < n\), the fibers \((T^*(A), \sigma_p)\) are isomorphic to \((A, \cdot)\) as unital associative algebras.

We would like to spend a few words to investigate here those algebraic properties making \(A\) eligible as a “system of coefficients” for a convolution n-category \(E := T^*(A) \subset M_X(A) := \Gamma(X; T^*(A))\) over an n-category \(X\) with usual, or with non-commutative exchange.

First of all we notice that for any convolution n-category \(E \subset M_X(A)\), the fibers \(E_x\) over the n-identities of an object \(\bullet \in X^0\) are isomorphic to \(A\) and hence we can infer the necessary properties of \(A\) from the study of these fibers. Secondly, for any n-categorical bundle \((E, \pi, X)\), the fibers \(E_x\), for \(\bullet \in X^0\) are themselves n-categories with all the sets of \(\sigma_p\)-identities of cardinality one, for all \(p = 0, \ldots, n-1\).

Proposition 5.25. Let \((E, \sigma_0, \ldots, \sigma_{n-1})\) be a n-categorical bundle with non-commutative exchange over the n-category \(X\). For all \(p = 0, \ldots, n-1\), for all \(\bullet \in X^0\), the fibers \((E_x, \sigma_p)\) are a family of (possibly non-commutative) monoids with a common identity (i.e. such that \(E_x^0 = E_x^1 = \cdots = E_x^{p-1}\)).

By remark 5.26 the previous proposition can be directly applied to the case of an n-category \(E\) yielding conditions on the n-diagonal blocks \(\sigma_n\).

Remark 5.26. Recall that when \(E\) is an n-category with the usual exchange property, the Eckmann-Hilton collapse induce a strong trivialization, further imposing the coincidence of all the binary operations and their commutativity. As a consequence of the previous proposition, if \(A\) is a monoid with respect to \(n\)-operations, then \(A\) can be taken as a set of coefficients for a convolution n-category with non-commutative exchange if and only if, for all \(p = 0, \ldots, n-1\), all the \(\sigma_p\)-identities of the monoids coincide. Moreover, in that case, if \(A\) is a commutative monoid, then it can be taken as a set of coefficients for a convolution n-category. In particular this explains why we could immediately obtain examples of convolution n-categories \(E\) over an n-category \(X\) with non-commutative exchange with coefficients in a single algebra (monoid) \(A\), since in this case all the operations in the monoid \(A\) coincide \(\sigma_0 = \cdots = \sigma_{n-1}\) and so do their identities.

We proceed now to examine what happens when one attempts to define involutions on the convolution n-category \(E \subset M_X(A)\) over an involutive n-category \(X\) and which conditions must be imposed on the system of coefficients \(A\) in order to obtain such involutions on \(E\).

When the base category \(X\) has an involution that is contravariant with respect to all the compositions, we can immediately extend theorem 5.24 taking as a system of coefficients an involutive algebra \(A\).

---

\(\text{M.2} \) This is the terminal n-category in which all the operations coincide and that satisfies the usual exchange law.
Proposition 5.27. For a strict globular \( n \)-category \( (X, \circ_0, \ldots, \circ_{n-1}, *, \circ) \) equipped with an \( \alpha \)-involution, with \( \alpha = \{0, \ldots, n-1\} \), and a complex unital associative \(*\)-algebra \((\Lambda, \cdot, *\Lambda)\), the map
\[
(\sigma)_{\alpha} := (\sigma_{\alpha}^{\circ})_{\Lambda}, \quad \forall \sigma \in \Gamma(T^*(\Lambda)), \quad \forall \sigma \in X
\]
becomes an involution \( \hat{\circ} \) for all the unital associative algebras \((\Gamma(T^*(\Lambda)), \delta_p)\), for all \( p = 0, \ldots, n-1 \) and \((T^*(\Lambda), \delta_0, \ldots, \delta_{n-1}, \delta_n)\) is a partially involutive \( n \)-category with an \( \alpha \)-contravariant involution.

Remark 5.28. If the involutive unital associative algebra \( \Lambda \) is commutative, and the strict globular \( n \)-category \( X \) is \( \Lambda \)-involutive, formula \ref{5.2} can be used to define \( \hat{\circ} \)-involutions on \( \Gamma(T^*(\Lambda)) \), for all \( \alpha \in \Lambda \) and hence \( T^*(\Lambda) \subset \Gamma(T^*(\Lambda)) \) becomes a \( \Lambda \)-involutive category as well. Unfortunately, whenever \( \Lambda \) is not abelian, the antimultiplicativity of \( *\Lambda \)-involutions conflicts with the covariance/contravariance properties required to define \( \alpha \)-involutions on \( T^*(\Lambda) \) unless \( \alpha = \{0, \ldots, n-1\} \) (as already stated in the previous proposition). Hence, in order to construct examples of fully involutive strict globular \( n \)-categories with non-commutative exchange, as “convolution algebroids”, we need a more elaborate choice of “involutive algebra of coefficients” \( \Lambda \).

If \((E, \circ_0, \ldots, \circ_{n-1}, *, \circ)\) is an \( n \)-category with non-commutative exchange that is \(*\)-involutive, for \( \sigma \in \mathbb{N} \), the \( n \)-diagonal block \( E_\sigma \), corresponding to the object \( \bullet \in \mathbb{E}^\mathbf{d} \) (that we already know to be a monoid with respect to each one of the operations \( \circ_p \), \( p = 0, \ldots, n-1 \), sharing the same identity) is equipped with an involution \( \circ_\sigma \) maintaining the same covariance/contravariance properties with respect to the monoidal compositions. This introduces further complications in the study of the class of “systems of coefficients” for a convolution bundle over a (partially) involutive \( n \)-category \( X \) with non-commutative exchange, as explained in the following result.

Proposition 5.29. Let \((X, \circ_0, \ldots, \circ_{n-1}, \Lambda)\) be a (partially) involutive \( n \)-category, with non-commutative exchange, equipped with a family \( \Lambda \) of \( \alpha \)-involutions \( \circ^{\alpha} \in \Lambda \). Let \((A, \circ_0, \ldots, \circ_\mathbf{r}, \top)\) be such that, for all \( k = 0, \ldots, r \), the \( (\Lambda, \circ^{\alpha}) \) are monoids with a common identity, and let it be equipped with a family of involutions \( \top_j \) for all \( j = 0, \ldots, s \). The algebraic structure \( A \) can be a “system of coefficients” for a convolution (partially) involutive \( n \)-category \( E \) over \( X \) if and only if it is possible to find a function \( f : \{(\circ_p, *\circ)\} \mid p = 0, \ldots, n-1, *\circ \in \Lambda \} \rightarrow \{(\circ_i, \top_j)\} \mid k = 0, \ldots, r, j = 0, \ldots, s \) that is preserving the covariance properties of the pairs.

As a consequence, we see immediately that commutative monoids do not pose any further problem as “systems of coefficients” and that, even when the non-commutative exchange is assumed, non-commutative involutive monoids \((A, \cdot, \top)\) can be “systems of coefficients” only when all the involutions in the base category \( X \) have (with all the compositions) the same covariance of the pair \((\cdot, \top)\).

In order to exploit convolution \( n \)-categories \( E \) as a source of non-trivial examples of fully involutive \( n \)-categories with non-commutative \( n \)-diagonal blocks \( E_\sigma \) (and hence necessarily with non-commutative exchange), we must utilize an additional “sophisticated” system of coefficients \( \Lambda \).

Motivated from the previous discussion, we are naturally induced to propose the following notion:

Definition 5.30. A hyper-\( C^* \)-algebra \( \Lambda(A, \circ_0, \ldots, \circ_{n-1}, *\circ, \ldots, *\circ_{n-1}) \) is a complete topological space \( A \) equipped with pairs of multiplication/involution \((\circ_k, *\circ_k)\), for \( k = 0, \ldots, n-1 \), each inducing on \( A \) a \( C^* \)-algebra structure, via a necessarily unique \( C^* \)-norm \( ||\cdot||_k \), compatible with the given fixed topology.

In the same vein, we might introduce the notions of hyper-monoid and hyper-involutive-monoid to describe the more general abstract algebraic structures naturally arising from (involutive) convolutions of \( n \)-categories and (partially) involutive \( n \)-categories (with non-commutative exchange), but we will not elaborate on this any further.

Proposition 5.31. Given a unital commutative \( C^* \)-algebra \( A \) and a finite globular (cubical) higher (fully) involutive \( n \)-category \( X \), the \( X \)-convolution \( C^* \)-algebra \( \mathbb{M}_X(\Lambda) := \Gamma(T^*(\Lambda)) \) is a hyper-\( C^* \)-algebra with the operations of \( \circ_\sigma \)-convolution and \( *\circ_q \)-involutions, for \( q = 0, \ldots, n-1 \).

\( ^7 \) We warn the reader that there is a conflict of terminology with the usage of the term “hyper-algebra” in the area of universal algebra, where an “hyperalgebra” (also called multialgebra or polyalgebra) means an algebraic structure with set-valued operations.
Definition 5.32. A hypermatrix of depth-n is a multimatrix \( [x_{i_1...i_n}^{j_1...j_k}] \in \mathcal{M}_{N_1,...,N_k}(\mathbb{C}) \) having indices \( i_k, j_k = 1,...,N_k \), for all \( k = 1,...,n \).

Theorem 5.33. The family \( \mathcal{M}_X(\mathbb{C}) \) of \( \mathbb{C} \)-valued hypermatrices of depth-n is a hyper C*-algebra.

Proof. On \( \mathcal{M}_{N_1,...,N_k}(\mathbb{C}) \) there are \( 2^n \) different multiplications acting at every level either as convolution or as Schur product: \([x_{i_1...i_n}^{j_1...j_k}] \cdot_y [y_{j_1...j_k}^{k_1...k_l}] := [\sum_{\alpha} x_{i_1...i_n}^{j_1...j_k} y_{j_1...j_k}^{k_1...k_l}]\), where \( y \in \{1,...,n\} \) is the set of contracting indices. There are \( 2^n \) involutions taking the conjugate of all the entries and, at every level, either the transpose or the identity: \([x_{i_1...i_n}^{j_1...j_k}]^\ast := [x_{i_1...i_n}^{j_1...j_k}]^\ast\), for all \( y := \{k_1,...,k_m\} \subset \{1,...,n\} \).

There are \( 2^n \) *-nормs taking either the operator norm or the maximum norm at every level. Using the natural isomorphism \( \mathcal{M}_{N_1,...,N_k}(\mathbb{C}) \cong \mathcal{M}_{N_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{N_k}(\mathbb{C}) \), these norms can be defined as: \( \|x_{j_1}^{i_1}\| \otimes \cdots \otimes \|x_{j_k}^{i_k}\| := \prod_{i_1} \|x_{i_1}^{i_1}\| \otimes \cdots \otimes \prod_{i_k} \|x_{i_k}^{i_k}\| \), \( \forall y \subset \{1,...,n\} \), where \( \|x_{j_k}^{i_k}\| \) is the C*-norm on \( \mathcal{M}_{N_k}(\mathbb{C}) \) and \( \|x_{j_1}^{i_1}\| := \max_{i_1} |x_{j_1}^{i_1}| \).

With such ingredients \( \mathcal{M}_{N_1,...,N_k}(\mathbb{C}), \cdot_y, \gamma, \|\cdot\|, y \in \{1,...,n\} \) is a hyper C*-algebra.  

Remark 5.34. If in place of the complex numbers \( \mathbb{C} \) we consider an arbitrary non-commutative C*-algebra \( \mathcal{A} \), the family of \( \mathcal{A} \)-valued hypermatrices \( \mathcal{M}_X(\mathcal{A}) \) is still a hyper-C*-algebra.

Can we see all the \( 2^n \) operations in the hypermatrices \( \mathcal{M}_{N_1,...,N_k}(\mathcal{A}) \) as convolutions of some n-category?

Hypermartices \( \mathcal{M}_X(\mathbb{C}) \) obtained via convolution of globular n-categories \( X \) have only \( n \) compositions. The same is actually true for convolutions of cubical n-categories (see [BCM]). The C*-algebra \( \mathcal{M}_X(\mathbb{C}) \otimes \mathcal{M}_X(\mathbb{C}) \) coincides with the convolution C*-algebra \( \mathbb{C}[X_1 \times X_2] = \mathcal{M}_{X_1 \times X_2}(\mathbb{C}) \) of the Cartesian product \( X_1 \times X_2 \) of the finite pair groupoids, but the product of \( n \) finite pair groupoids \( X := \{1,...,N_1\}^2 \times \cdots \times \{1,...,N_n\}^2 \) has a richer structure of “full-depth n-tuple category” (via compositions on the “oriented borders”), as we described in section 48. Hence there are \( 2^n \) such possible compositions on \( X \) and we can recover \( \mathcal{M}_{N_1,...,N_k}(\mathbb{C}) \) as a convolution hyper C*-algebra of the “full depth n-tuple category” \( X \).

Theorem 5.35. Let \( \mathcal{A} \) be a commutative C*-algebra. The hyper C*-algebra \( \mathcal{M}_X(\mathcal{A}) \) of \( \mathcal{A} \)-valued hypermatrices, indexed by the Cartesian product \( X \) of \( n \) finite pair groupoids \( X := X_1 \times \cdots \times X_n \), is the convolution hyper C*-algebra of the fully involutive full-depth n-category \( X := X_1 \times \cdots \times X_n \).

Remark 5.36. When \( \mathcal{A} \) is commutative, the hyper C*-algebra \( \mathcal{M}_X(\mathcal{A}) \) is the envelope of the fully involutive full-depth Fell-bundle \( \mathcal{E} := \mathcal{A}^\ast \) with the usual exchange property in place. Unfortunately, when \( \mathcal{A} \) is a non-commutative C*-algebra, the hyper C*-algebra \( \mathcal{M}_X(\mathcal{A}) \) cannot be obtained as a convolution enveloping algebra of a Fell bundle, even if the non-commutative exchange property is assumed.

If we utilize hyper C*-algebras \( \mathcal{A} \) as systems of coefficients, we can finally obtain explicit examples of fully involutive convolution globular n-categories \( \mathcal{E} \).

Theorem 5.37. Let \( \mathcal{A} \) be a hyper C*-algebra with respect to \( n \) pairs of product/involution \((\lambda_k, \gamma_k)\), for \( k = 0,...,n-1 \). Let \((\mathcal{X}, \alpha_0,...,\alpha_{n-1}, \omega_0,...,\omega_{n-1})\) be a fully involutive globular n-category (with commutative or non-commutative exchange). The convolution n-bundle \( \mathcal{E} \subset \mathcal{M}_X(\mathcal{A}) \) is now a fully involutive globular n-category necessarily with non-commutative exchange, as soon as one of the products in \( \mathcal{A} \) is non-commutative.

\[ \text{Remark 5.36. The reason is again that the covariance conditions imposed by proposition 5.29 cannot in general be satisfied.} \]
6 Outlook and Applications

In this final section, we informally venture into uncharted territory, trying to suggest some intriguing connections between higher categories with non-commutative exchange and the study of “morphisms” of “non-commutative spaces” (and hence interactions of quantum systems [B]). We also provide a detailed list of several further interesting lines of development for the study of the categorical structures introduced in this paper.

6.1 Morphisms of Non-commutative Spaces

Several people have already advocated the existence of an interplay between (higher) category theory and quantization (and hence non-commutative algebras) notably: J.E.Roberts, C.Isham, J.Baez, B.Coecke, N.Landsman, . . . , but the leading ideas for us here are mainly coming from:

- L.Crane / R.Feynman [Cra]: in the suggestion to see quantization (non-commutativity) as a categorification effect (due to different paths between points),
- A.Connes / W.Heisenberg [C, chapter 1, section 1]: in their way to look at algebras of non-commutative spaces, such as matrix algebras, as convolution algebras of a category (groupoid).

These basic ideologies merge and are somehow strongly supported from our already mentioned results, theorem 2.7 on the spectral structure of commutative full C*-categories in terms of spaceoids that seem to indicate a direct route to a general spectral reconstruction of non-commutative C*-algebras as algebras of “sections” of complex line-bundles with a suitable categorical base space:

**Spectral Conjecture:** there is a spectral theory of non-commutative C*-algebras in terms of families of Fell complex line-bundles over involutive categories.

Quantum space \( \cong \) spectrum of C*-algebra \( \cong \) Fell line-bundle over an inverse involutive category

As it is stated above, without further details on the precise nature of the functors involved in such a non-commutative generalization of Gel’fand-Na˘ımark duality, the conjecture is “not even wrong” anyway this is not a serious issue for us here, because the conjecture surely holds for some sufficiently many interesting finite dimensional cases (such as matrix algebras) and our only goal for now is to make use of the spectral conjecture, in those “safe cases”, as a motivation to propose an alternative way to look at the notion of morphism of non-commutative spaces.

The usual Gel’fand-Na˘ımark duality, when recasted in the language of theorem 2.7 essentially says:

classical space \( \cong \) spectrum of abelian C*-algebra \( \cong \) trivial line bundle over space \( \cong \) Fell line-bundle over the space \( \Delta_X \) of “loops” of \( X \),

Abelian C*-algebra \( C(X) \cong \) algebra \( \Gamma(X; X \times \mathbb{C}) \) of sections of \( X \times \mathbb{C} \)

\( \cong \) convolution algebra \( \Gamma(\Delta_X; \Delta_X \times \mathbb{C}) \).

For the spectrum of a finite discrete space \( X \) consisting of \( N \) points, we have the following “transitions”:

\[ \begin{array}{cccccc}
\cdots & \bullet & \bullet & \cdots & \cdots & \Delta_X \\
\Delta_X \times \mathbb{C} & \cdots & \cdots & \cdots & \cdots & \Delta_X \times \mathbb{C}
\end{array} \]

---

49 P.Bertozzini, R.Conti, N.Pitiwan, Non-commutative Gel’fand-Na˘ımark Duality, work in progress.
50 For example, even in finite dimensional situations, there are “gauge redundancies” that allow to express in different ways the same C*-algebra as convolution algebra of different spaceoids: an algebra of linear operators on a finite dimensional vector space is isomorphic in many different ways, one for every alternative choice of an orthonormal base, to an algebra of square matrices.
For the case of morphisms between classical spaces, the first transition entails:

\[
morphism \text{ of classical spaces } X, Y \cong \text{ map relation } 1\text{-quiver } : X \to Y \\cong \text{ level-2 relation } : \Delta_X \to \Delta_Y,
\]

\[
x \quad y \quad \sim \quad x \quad \sim \quad y \quad \Rightarrow \quad y \quad \Rightarrow \quad x \quad \sim \quad y \quad \sim \quad x \quad \in X, \ y \in Y.
\]

The transition from \( \Delta_X, \Delta_Y \) to their associated Fell line-bundles \( \Delta_X \times \mathbb{C}, \Delta_Y \times \mathbb{C} \), (attaching a complex fiber to each 1-loop) seems to further suggest that also each 1-arrow \( x \mapsto y \) in the morphism from \( X \) to \( Y \) should have a complex fiber attached.

Dually, for a relation \( R \subset X \times Y \) (1-quiver) with reciprocal \( R^* \subset Y \times X \), the “convolution algebra” \( \mathcal{A} \) of the trivial Fell line-bundle with base \( \Delta_X \cup R \cup R^* \cup \Delta_Y \) is given by a linking C*-algebra

\[
\mathcal{A} = \begin{bmatrix} C(X) & \Gamma(R^* \times \mathbb{C}) \\ \Gamma(R \times \mathbb{C}) & C(Y) \end{bmatrix}
\]

that contains on the diagonal the C*-algebras \( C(X), C(Y) \), and off-diagonal the bimodule \( \Gamma(R, R \times \mathbb{C}) \) and its contragredient \( \Gamma(R^*, R^* \times \mathbb{C}) \). Hence, in a quite familiar way, the morphisms from \( X \) to \( Y \) are dually given by (Hilbert C*) bimodules, over the commutative C*-algebras \( C(Y) \) and \( C(X) \).

When we pass to the study of (finite discrete) non-commutative spaces, we see that the appearance of level-2 relations and 2-cells, becomes unavoidable and much more intriguing because, in light of the previous spectral conjecture, we have:

\[
\text{quantum space } \cong \text{ spectrum of non-commutative C*-algebra } \cong \text{ space of points with “linearized relations” } \cong \text{ Fell line-bundle over a 1-quiver } Q^1,
\]

algebra of functions on \( Q^1 \) = “convolution” algebra of \( Q^1 \).

As a consequence, proceeding as before, we claim that: at the “spectral level” a morphism between two (finite discrete) quantum spaces \( Q^1_X, Q^1_Y \) is a 2-quiver \( Q^2 \) with 2-cells like

\[
\begin{array}{c}
\begin{array}{c}
x_1 \rightarrow f \rightarrow y_1 \\
\end{array} \\
\begin{array}{c}
x_2 \rightarrow g \rightarrow y_2 \\
\end{array}
\end{array}
\]

\( f \in Q^1_X, \ g \in Q^1_Y \),

and so, at the “dual level”, a morphism of quantum spaces is a “level-2 bimodule” inside the convolution depth-2 hyper C*-algebra \( \Gamma(Q^2) \) of the involutive 2-category generated by the morphism 2-quiver \( Q^2 \).

The possible relevance of higher C*-categories and hyper-C*-algebras to formally describe, at least at the topological level, these situations should be self-evident and we plan to address such issues in the future.

### 6.2 Other Related Topics

Among the several lines of development directly related to the material introduced in this paper, we mention here only a few that are either already under study (and partially covered in other works) or that we deem particularly interesting or intriguing.

- Involutive double categories (with usual exchange property) and their relationship with involutive 2-categories are extensively studied in the preprint [BCM]. The study of involutions for general \( n \)-tuple cubical categories and versions of the non-commutative exchange for the cubical case should be the
next immediate goal also in view of the inevitable appearance of cubical structures both in the study of hypermatrices and morphisms of non-commutative spaces. The possibility of even further “exotic” type of $n$-cells can be considered. In [BP] we had a first look at the case of “hybrid” globular 2-categories.

- **Strict (involutive) $\omega$-categories** with quantum exchange and strict $\omega$-$C^*$-categories are immediately obtained omitting the finite bound on the number of binary operations of composition and the number of involutions involved in the definitions.

Weak (involutive) higher categories and weak higher $C^*$-categories are of course a much more involved and complex area of investigation. We are currently formally developing such notions, starting from those more “algebraic” definitions of J.Penon, M.Batanin, T.Leinster (see [C], [L1], [L2] and the more recent works by C.Kachour [K1], [K2]) that, being less motivated by classical homotopy theory, are more suitable for applications to operator theory. Weak involutive categories in Penon’s approach are studied in [BB]. Natural examples of weak higher $C^*$-categories can be found in the study of higher categories of “bimodules” over strict higher $C^*$-categories in the same way as the Morita weak 2-$C^*$-category originates from imprimitivity bimodules over $C^*$-algebras.

In this work we have only touched on some variants of higher categories, but we also have quite strong interest in the investigation of other possible higher involutive and $C^*$-algebraic structures in the wider contexts of polycategories, multicategories (operads) and their vertically categorified counterparts.

- **In the present paper**, we opted for a compact treatment of $n$-$C^*$-categories as “involutive partial $n$-monoids” i.e. via binary partial operations and involutions defined on $n$-cells. This can be too restrictive for the study of weak higher categories. A more immediate concern (also for the case of strict $C^*$-categories) is that, in the same spirit, we defined linear structures and norms only at the level of $n$-cells. It is actually possible to provide a definition of “iterated” (quantum) $n$-$C^*$-categories (and “iterated” $n$-Fell bundles), where different linear structures and different norms are introduced at each depth-level. This kind of approach has been already briefly presented (only for the case of usual exchange) in previous works [BCL5] and in the near future we plan to further comment on this point, clarifying the link between the two definitions.

- **Krein versions** of higher $C^*$-categories, as a vertical categorification of the Krein $C^*$-categories already defined in [BRu], can be produced and will be treated elsewhere.

Among the many issues that remain to be explored, once a viable theory of higher $C^*$-categories is in place, we mention:

- Developing a representation theory of (quantum) higher $C^*$-categories and higher $C^*$-algebras (higher Gel’fand-Naimark representation theorems; Hilbert higher bimodules (higher Hilbert spaces) and higher Morita theory (higher $K$-theory); higher spectral theory via higher $n$-Fell line-bundles . . . higher Gel’fand-Naimark duality and more generally higher functional analysis.

- In light of the already known connections between Fell bundles one one side and product systems [A] on the other, one would like to see if also higher categorifications of such notions retain similar connections.

- Further extension of the investigation on the role of higher categories in the study of morphisms for non-commutative geometries and the study of “higher non-commutative geometries” as suitable “spectral triples” on higher $C^*$-categories (as already suggested in [BCL5]).

- In view of the current general interest in homotopy type theory and higher $\infty$-groupoids in the foundations of mathematics and the attempts to reconsider in this light also the famous Hilbert sixth problem on the possible (categorical) mathematical foundations of physics (see U.Schreiber [Sc1], [Sc2] and A.Rodin [Ro1], [Ro2]), it is quite natural to speculate if the basic quantum nature of physics will give a more prominent role to quantum $\infty$-$C^*$-categories for its foundations.

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51 A small technical obstacle in the definition of $\omega$-$C^*$-categories, the triviality of the sets of $\omega$-arrows sharing the same $\omega$-cell, can be easily avoided (see for example P.Bejrakarbum’s thesis [Bej]).

52 Bejrakarbum P., Bertozzini P., “Weak Involutive Higher $C^*$-Categories” (work in progress).

53 Bertozzini P., “$C^*$-polycategories” (work in progress).
• The usage of higher C*-categories for the formalization of relational Rovelli’s quantum theory, and more generally “quantum cybernetics”, has been already touched in [B] and it is one of the main motivations for the development of such techniques.

• A possible definition of non-commutative homotopy theory.

• The development of non-commutative “higher measure theory” and higher categorical modular theory (vertically categorifying the results in P.Ghez-R.Lima-J.E.Roberts [GLR, section 3]) is one of our most immediate priorities also in view of the strong motivations coming from proposals in “modular algebraic quantum gravity” [BCL2, BCL3, B, Ra1, Ra2].

• The study of how non-commutative exchange will affect the usual notions of (higher) topoi, sites and Grothendieck categories (especially in situations where Cartesian closure is replaced by monoidal closure and suitable involutions/dualities are introduced). Possible links with the new notions of gleaves developed by F.Flori-F.Fritz [FF] are quite intriguing.

We are only taking the first steps into a vast landscape of vertically categorified functional analysis.

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