Path-dependent SDEs in Hilbert spaces

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Abstract

We study path-dependent SDEs in Hilbert spaces. By using methods based on contractions in Banach spaces, we prove existence and uniqueness of mild solutions, continuity of mild solutions with respect to perturbations of all the data of the system, Gâteaux differentiability of generic order \( n \) of mild solutions with respect to the starting point, continuity of the Gâteaux derivatives with respect to all the data. The analysis is performed for generic spaces of paths that do not necessarily coincide with the space of continuous functions.

Key words: stochastic functional differential equations in Hilbert spaces, Gâteaux differentiability, contraction mapping theorem.

AMS 2010 subject classification: 37C25, 34K50, 37C05, 47H10, 47J35, 58C20, 58D25, 60G99, 60H10.

1 Introduction

In this paper we deal with mild solutions to path-dependent SDEs evolving in a separable Hilbert space \( H \), of the form

\[
\begin{aligned}
    dX_t &= (AX_t + b((\cdot,t),X))dt + \sigma((\cdot,s),X)dW_s \\
    X_s &= Y_s \\
\end{aligned}
\]

\( s \in [0,t], \forall s \in (t,T] \)

where \( t \in [0,T) \), \( Y \) is a \( H \)-valued adapted process defined on a filtered probability space \((\Omega,\mathcal{F},(\mathcal{F}_t)_{t \in [0,T]},\mathbb{P})\), \( W \) is a cylindrical Wiener process on \((\Omega,\mathcal{F},(\mathcal{F}_t)_{t \in [0,T]},\mathbb{P})\) taking values in a separable Hilbert space \( U \), \( b((\omega,s),X) \) is a \( H \)-valued random variable depending on \( \omega \in \Omega \), on the time \( s \), and on the path \( X \), \( \sigma((\omega,s),X) \) is a \( L_2(U,H) \)-valued random variable depending on \( \omega \in \Omega \), on the time \( s \), and on the path \( X \), and \( A \) is the generator of a \( C_0 \)-semigroup \( S \) on \( H \). By using methods based on implicit functions associated to contractions in Banach spaces, we study continuity of the mild solution \( X^{t,Y} \) of (1.1) with respect to \( t,Y,A,b,\sigma \) under standard Lipschitz conditions on \( b,\sigma \), Gâteaux differentiability of generic order \( n \geq 1 \) of \( X^{t,Y} \) with respect to \( Y \) under Gâteaux differentiability assumptions on \( b,\sigma \), and continuity with respect to \( t,Y,A,b,\sigma \) of the Gâteaux differentials \( \partial^n_{Y} X^{t,Y} \).

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Path-dependent SDEs in finite dimensional spaces are studied in [16]. The standard reference for SDEs in Hilbert spaces is [10]. More generally, in addition to SDEs in Hilbert spaces, also the case of path-dependent SDEs in Hilbert spaces is considered in [13, Ch. 3], but for the path-dependent case the study is there limited mainly to existence and uniqueness of mild solutions. Our framework generalize the latter one by weakening the Lipschitz conditions on the coefficients, by letting the starting process \( Y \) belong to a generic space of paths contained in \( B_b([0,T],H) \) \(^1\) obeying few conditions, but not necessarily assumed to be \( C([0,T],H) \), and by providing results on differentiability with respect to the initial datum and on continuity with respect to all the data.

In the literature on mild solutions to SDEs in Hilbert spaces, differentiability with respect to the initial datum is always proved only up to order \( n = 2 \), in the sense of Gâteaux ([9, 10]) or Fréchet ([13, 15]). In [9, Theorem 7.3.6] the case \( n > 2 \) is stated but not proved. There are no available results regarding differentiability with respect to the initial condition of mild solutions to SDEs of the type (1.1). One of the contributions of the present work is to fill this gap in the literature, by extending to a generic order \( n \), in the Gâteaux sense, and to the path-dependent case the results so far available.

In case (1.1) is not path-dependent, the continuity of \( X^{t,Y}_t \), \( \partial_Y X^{t,Y}_t \), and \( \partial^2_{YY} X^{t,Y}_t \), separately with respect to \( t, Y \) and \( A, b, \sigma \), is considered and used in [9, Ch. 7]. We extend these previous results to the path-dependent case and to Gâteaux derivatives \( \partial^n_Y X^{t,Y}_t \) of generic order \( n \), proving joint continuity with respect to all the data \( t, Y, A, b, \sigma \).

Similarly as in the cited literature, we obtain our results for mild solutions (differentiability and continuity with respect to the data) starting from analogous results for implicit functions associated to Banach space-valued contracting maps. Because of that, the first part of the paper is entirely devoted to study parametric contractions in Banach spaces and regularity of the associated implicit functions. In this respect, regarding Gâteaux differentiability of implicit functions associated to parametric contractions and continuity of the derivatives under perturbation of the data, we prove a general result, for a generic order \( n \) of differentiability, extending the results in [1, 9, 10], that were limited to the case \( n = 2 \).

In a unified framework, our work provides a collection of results for mild solutions to path-dependent SDEs which are very general, within the standard case of Lipschitz-type assumptions on the coefficients, a useful toolbox for starting dealing with path-dependent stochastic analysis in Hilbert spaces. For example, the so called “vertical derivative” in the finite dimensional functional Itô calculus ([5, 11]) of functionals like \( F(t, \mathbf{x}) = \mathbb{E}[\varphi(X^{t,\mathbf{x}})] \), where \( \varphi \) is a functional on the space \( \mathbb{D} \) of càdlàg functions and \( \mathbf{x} \in \mathbb{D} \), is easily obtained starting from the partial derivative of \( X^{t,\mathbf{x}} \) with respect to a step function, which can be treated in our setting by choosing \( \mathbb{D} \) as space of paths (we refer to Remark 3.11 for further details). Another field in which the tools here provided can be employed is the study of stochastic representations of classical solutions to path dependent Kolmogorov equations, where second order derivatives are required. Furthermore, the continuity of the mild solution and of its derivatives with respect to all the data, including the coefficients, can be used e.g. when merely continuous Lipschitz coefficients need to be approximated by smoothed out coefficients, which is in general helpful when dealing with Kolmogorov equations in Hilbert spaces (path- or non-path-dependent) for

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\(^1\) \( B_b([0,T],H) \) denotes the space of bounded Borel functions \([0,T] \rightarrow H \).
which notions other than classical solutions are considered, as strong-viscosity solutions ([6, 7]) or strong solutions ([1]).

The contents of the paper are organized as follows. First, in Section 2, we recall some notions regarding strongly continuous Gâteaux differentiability and some basic results for contractions in Banach spaces. Then we provide the first main result (Theorem 2.12): the strongly continuous Gâteaux differentiability up to a generic order $n$ of fixed-point maps associated to parametric contractions which are differentiable only with respect to some subspaces. We conclude the section with a result regarding the continuity of the Gâteaux differentials of the implicit function with respect to the data (Proposition 2.14).

In Section 3 we consider path-dependent SDEs. After a standard existence and uniqueness result (Theorem 3.6), we move to study Gâteaux differentiability with respect to the initial datum up to order $n$ of mild solutions, in Theorem 3.9, which is the other main result and justifies the study made in Section 2. We conclude with Theorem 3.16, which concerns the continuity of the Gâteaux differentials with respect to all the data of the system (coefficients, initial time, initial condition).

## 2 Preliminaries

In this section we recall the notions and develop the tools that we will apply to study path-dependent SDEs in Section 3. We focus on strongly continuous Gâteaux differentiability of fixed-point maps associated to parametric contractions in Banach spaces.

### 2.1 Strongly continuous Gâteaux differentials

We begin by recalling the basic definitions regarding Gâteaux differentials, mainly following [12]. Then we will define the space of strongly continuously Gâteaux differentiable functions, that will be the reference spaces in the following sections.

If $X, Y$ are topological vector spaces, $U \subset X$ is a set, $f : U \rightarrow Y$ is a function, $u \in U$, $x \in X$ is such that $[u - \varepsilon x, u + \varepsilon x] \subset U$ \footnote{If $x, x' \in X$, the segment $[x, x']$ is the set $\{x + (1 - \zeta)x' | \zeta \in [0, 1]\}$.} for some $\varepsilon > 0$, the directional derivative of $f$ at $u$ for the increment $x$ is the limit

$$\partial_x f(u) := \lim_{t \to 0} \frac{f(u + tx) - f(u)}{t}$$

whenever it exists. Also in the case in which the directional derivative $\partial_x f(u)$ is defined for all $x \in X$, it need not be linear.

Higher order directional derivatives are defined recursively. For $n \geq 1$, $u \in U$, the $n$th-order directional derivative $\partial^n_{x_1, \ldots, x_n} f(u)$ at $u$ for the increments $x_1, \ldots, x_n \in X$ is the directional derivative of $\partial^{n-1}_{x_1, \ldots, x_{n-1}} f$ at $u$ for the increment $x_n$ (notice that this implies, by definition, the existence of $\partial^n_{x_1, \ldots, x_{n-1}} f(u')$ for $u'$ in some neighborhood of $u'$ in $U \cap (u + \mathbb{R}x_n)$.

If $Y$ is locally convex, we denote by $L_s(X, Y)$ the space $L(X, Y)$ endowed with the coarsest topology which makes continuous the linear functions of the form

$$L(X, Y) \rightarrow Y, \Lambda \mapsto \Lambda(x),$$
for all \( x \in X \). Then \( L_s(X, Y) \) is a locally convex space.

Let \( X_0 \) be a topological vector space continuously embedded into \( X \). If \( u \in U \), if \( \partial_x f(u) \) exists for all \( x \in X_0 \) and \( X_0 \to Y, x \mapsto \partial_x f(u) \), belongs to \( L(X_0, Y) \), then \( f \) is said to be Gâteaux differentiable at \( u \) with respect to \( X_0 \) and the map \( X_0 \to Y, x \mapsto \partial_x f(u) \), is the Gâteaux differential of \( f \) at \( u \) with respect to \( X_0 \). In this case, we denote the Gâteaux differential of \( f \) at \( u \) by \( \partial_{X_0} f(u) \) and its evaluation \( \partial_x f(u) \) by \( \partial_{X_0} f(u)x \). If \( \partial_{X_0} f(u) \) exists for all \( u \in U \), then we say that \( f \) is Gâteaux differentiable with respect to \( X_0 \), or, in case \( X_0 = X \), we just say that \( f \) is Gâteaux differentiable and we use the notation \( \partial f(u) \) in place of \( \partial_{X_0} f(u) \).

A function \( f : U \to Y \) is said to be strongly continuously Gâteaux differentiable with respect to \( X_0 \) if it is Gâteaux differentiable with respect to \( X_0 \) and

\[
U \to L_s(X_0, Y), u \mapsto \partial_{X_0} f(u)
\]

is continuous. If \( n > 1 \), we say that \( f \) is strongly continuously Gâteaux differentiable up to order \( n \) with respect to \( X_0 \) if it is strongly continuously Gâteaux differentiable up to order \( n-1 \) with respect to \( X_0 \) and

\[
\partial_{X_0}^{n-1} f : U \to L_s(X_0, L_s(X_0, \ldots, L_s(X_0, Y), \ldots))
\]

exists and is strongly continuously Gâteaux differentiable with respect to \( X_0 \). In this case, we denote \( \partial_{X_0}^n f := \partial_{X_0} \partial_{X_0}^{n-1} f \) and \( \partial^n f := \partial \partial^{n-1} f \).

Let \( X, X_0 \) be topological vector spaces, with \( X_0 \) continuously embedded into \( X \), let \( U \) be an open subset of \( X \), and let \( Y \) be a locally convex space.

We denote by \( \mathcal{G}^n(U, Y; X_0) \) the space of functions \( f : U \to Y \) which are continuous and strongly continuously Gâteaux differentiable up to order \( n \) with respect to \( X_0 \). In case \( X_0 = X \), we use the notation \( \mathcal{G}^n(U, Y) \) instead of \( \mathcal{G}^n(U, Y; X) \).

Let \( L_s^n(X_0^n, Y) \) the vector space of \( n \)-linear functions from \( X_0^n \) into \( Y \) which are continuous with respect to each variable separately, endowed with the coarsest vector topology making continuous all the linear functions of the form

\[
L_s^n(X_0^n, Y) \to Y, \Lambda \to \Lambda(x_1, \ldots, x_n)
\]

for \( x_1, \ldots, x_n \in X_0 \). Then \( L_s^n(X_0^n, Y) \) is a locally convex space. Through the canonical identification (as topological vector spaces)

\[
L_s^n(X_0, \ldots, L_s(X_0, Y), \ldots)) \equiv L_s^n(X_0^n, Y),
\]

we can consider \( \partial_{X_0}^n f \) as taking values in \( L_s^n(X_0^n, Y) \), whenever \( f \in \mathcal{G}^n(U, Y; X_0) \).

If \( X_0, X, Y \) are normed spaces, \( U \) is an open subset of \( X \), \( \partial_x f(u) \) exists for all \( u \in U \), \( x \in X_0 \), \( \partial_x f(u) \) is continuous with respect to \( u \), for all \( x \in X_0 \), then \( \partial_x f(u) \) is linear in \( x \) (see [12, Lemma 4.1.5]).

The following proposition is a characterization for the continuity conditions on the directional derivatives of a function \( f \in \mathcal{G}^n(U, Y; X_0) \), when \( X_0, X, Y \) are normed spaces.
Proposition 2.1. Let \( n \geq 1 \), let \( X_0, X, Y \) be normed spaces, with \( X_0 \) continuously embedded into \( X \), and let \( U \) be an open subset of \( X \). Then \( f \in \mathcal{G}^n(U, Y; X_0) \) if and only if \( f \) is continuous, the directional derivatives \( \partial_{x_1} f(u) \) exist for all \( u \in U, x_1, \ldots, x_j \in X_0, \) \( j = 1, \ldots, n \), and the functions

\[
U \times X^j_0 \rightarrow Y, \; (u, x_1, \ldots, x_j) \mapsto \partial_{x_1} f(u)
\]

are separately continuous in each variable. In this case,

\[
\partial_{x_0}^j f(u)(x_1, \ldots, x_j) = \partial_{x_1} f(x_1, \ldots, x_j) f(u) \quad \forall u \in U, \forall x_1, \ldots, x_j \in X_0, \quad j = 1, \ldots, n.
\]

Proof. Suppose that the derivatives \( \partial_{x_1} f(u) \) exists for all \( u \in U, x_1, \ldots, x_j \in X_0, \) \( j = 1, \ldots, n \), separately continuous in \( u, x_1, \ldots, x_j \). We want to show that \( f \in \mathcal{G}^n(U, Y; X_0) \).

We proceed by induction on \( n \). Let \( n = 1 \). Since \( \partial_x f(u) \) is continuous in \( u \), for all \( x \in X_0 \), we have that \( X_0 \rightarrow Y, x \mapsto \partial_x f(u) \) is linear ([12, Lemma 4.1.5]). By assumption, it is also continuous. Hence \( x \rightarrow \partial_x f(u) \in L(X_0, Y) \) for all \( u \in U \). This shows the existence of \( \partial_{X_0} f \). The continuity of \( U \rightarrow L_s(X_0, Y), u \mapsto \partial_{X_0} f(u) \), comes from the separate continuity of (2.1) and from the definition of the locally convex topology on \( L_s(X_0, Y) \). This shows that \( f \in \mathcal{G}^1(U, Y; X_0) \).

Let now \( n > 1 \). By inductive hypothesis, we may assume that \( f \in \mathcal{G}^{n-1}(U, Y; X_0) \) and

\[
\partial_{X_0}^j f(u)(x_1, \ldots, x_j) = \partial_{x_1} f(x_1, \ldots, x_j) f(u) \quad \forall u \in U, \forall j = 1, \ldots, n-1, \forall (x_1, \ldots, x_j) \in X_0^j.
\]

Let \( x_n \in X_0 \). The limit

\[
\lim_{t \to 0} \frac{\partial_{x_0}^{n-1} f(u + tx_n) - \partial_{X_0}^{n-1} f(u)}{t} = \Lambda
\]

exists in \( L_s^{(n-1)}(X_0^{n-1}, Y) \) if and only if \( \Lambda \in L_s^{(n-1)}(X_0^{n-1}, Y) \) and, for all \( x_1, \ldots, x_{n-1} \in X_0 \), the limit

\[
\lim_{t \to 0} \frac{\partial_{x_0}^{n-1} f(u + tx_n) - \partial_{X_0}^{n-1} f(u)}{t} = \Lambda(x_1, \ldots, x_{n-1})
\]

holds in \( Y \). By assumption, the limit (2.4) is equal to \( \partial_{x_1} f(u) \), for all \( x_1, \ldots, x_{n-1} \). Since, by assumption, \( \partial_{x_1} f(u) \) is separately continuous in \( u, x_1, \ldots, x_{n-1}, x_n \), we have that the limit (2.3) exists in \( L_s^{(n-1)}(X_0^{n-1}, Y) \) and is given by

\[
\partial_{x_n} \partial_{X_0}^{n-1} f(u)(x_1, \ldots, x_{n-1}) = \Lambda(x_1, \ldots, x_{n-1}) = \partial_{x_n}^{n-1} f(u) \quad \forall x_1, \ldots, x_{n-1} \in X_0.
\]

Since \( u \) and \( x_n \) were arbitrary, we have proved that \( \partial_{x_n} \partial_{X_0}^{n-1} f(u) \) exists for all \( u, x_n \). Moreover, for all \( x_1, \ldots, x_n \in X_0 \), the function

\[
U \rightarrow Y, \; u \mapsto \partial_{x_n} \partial_{X_0}^{n-1} f(u)(x_1, \ldots, x_{n-1}) = \partial_{x_n}^{n-1} f(u)
\]

is continuous, by separate continuity of (2.1). Then \( \partial_{x_1}^{n-1} f(u) \) is linear in \( x_n \). The continuity of

\[
X_0 \rightarrow L_s^{(n-1)}(X_0^{n-1}, Y), \; x \mapsto \partial_x \partial_{X_0}^{n-1} f(u)
\]

comes from the continuity of \( \partial_{x_1}^{n-1} f(u) \) in each variable, separately. Hence (2.5) belongs to \( L_s(X_0, L_s^{(n-1)}(X_0^{n-1}, Y)) \) for all \( u \in U \). This shows that \( \partial_{X_0}^{n-1} f \) is Gâteaux differentiable with respect to \( X_0 \) and that

\[
\partial_{X_0}^{n} f(u)(x_1, \ldots, x_n) = \partial_{x_1}^{n} f(u) \quad \forall u \in U, \forall x_1, \ldots, x_n \in X_0.
\]
and shows also the continuity of
\[ U \rightarrow L^{(n)}_g(X^n_0, Y), \ u \mapsto \partial_{X^0}^n f(u), \]
due to the continuity of the derivatives of \( f \), separately in each direction. Then we have proved that \( f \in \mathcal{G}^n(U, Y; X_0) \) and that \((2.2)\) holds.

Now suppose that \( f \in \mathcal{G}^n(U, Y; X_0) \). By the very definition of \( \partial_{X^0} f \), \( \partial_x f(u) \) exists for all \( x \in X_0 \) and \( u \in U \), it is separately continuous in \( u, x \), and coincides with \( \partial_{X^0} f(u).x \). By induction, assume that \( \partial_{x_1 \ldots x_{n-1}}^{n-1} f(u) \) exists and that
\[
\partial_{x_1 \ldots x_{n-1}}^{n-1} f(u). (x_1, \ldots, x_{n-1}) = \partial_{x_1 \ldots x_{n-1}}^{n-1} f(u) \quad \forall u \in U, \ \forall x_1, \ldots, x_{n-1} \in X_0. \quad (2.6)
\]
Since \( \partial_{X^0}^{n-1} f(u) \) is Gâteaux differentiable, the directional derivative \( \partial_{x_n} \partial_{X^0}^{n-1} f(u) \) exists. Hence, by \((2.6)\), the derivative \( \partial_{x_1 \ldots x_{n-1} x_n}^{n-1} f(u) \) exists for all \( x_1, \ldots, x_{n-1}, x_n \in X_0 \). The continuity of \( \partial_{X^0}^{n-1} f(u) \) with respect to \( u \) comes from the continuity of \( \partial_{X^0}^n f \). The continuity of \( \partial_{x_1 \ldots x_j x_{n-j}}^{n-1} f(u) \) with respect to \( x_j \) comes from the fact that, for all \( x_{j+1}, \ldots, x_n \in X_0 \), \( u \in U \),
\[
X^j_0 \rightarrow Y, \ (x'_1, \ldots, x'_j) \mapsto \partial_{x^j_0}^n f(u). (x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n)
\]
belongs to \( L^{(j)}(X^j_0, Y) \). \[\blacksquare\]

**Remark 2.2.** If \( X_0 \) is Banach, \( X \) is normed, \( Y \) is locally convex, and \( f \in \mathcal{G}^n(X, Y; X_0) \), then, by Proposition 2.1 and the Banach-Steinhaus theorem, if follows that the map
\[
U \times X^n_0 \rightarrow Y, \ (u, x_1, \ldots, x_n) \mapsto \partial_{X^0}^n f(u). (x_1, \ldots, x_n)
\]
is continuous, jointly in \( u, x_1, \ldots, x_n \).

**Remark 2.3.** Under the assumption of Proposition 2.1, by Schwarz’ theorem,
\[
y^* (\partial^2_{zw} f(u)) = \partial^2_{zw} (y^* f)(u) = \partial^2_{wz} (y^* f)(u) = y^* (\partial^2_{wz} f(u)), \ \forall u \in U, \ \forall w, x \in X_0, \ \forall y^* \in Y^*.
\]
Hence \( \partial^2_{wz} f = \partial^2_{wz} f \) for all \( w, z \in X_0 \).

### 2.1 Chain rule

In this subsection, we show the classical Faà di Bruno formula, together with a corresponding stability result, for derivatives of order \( n \geq 1 \) of compositions of strongly continuously Gâteaux differentiable functions. We will use this formula in order to prove the main results of Section 2.3 (Theorem 2.12 and Proposition 2.14).

In \([2]\), a version of Proposition 2.6 is provided for the case of “chain differentials”. We could prove that the strongly continuously Gâteaux differentiable functions that we consider satisfy the assumptions of \([2, \text{Theorem 2}]\). This would provide Proposition 2.6 as a corollary of \([2, \text{Theorem 2}]\). Since the direct proof of Proposition 2.6 is quite concise, we prefer to report it, and avoid introducing other notions of differential. Besides, we give the related stability results.
Lemma 2.4. Let \( k \geq 0 \), let \( X_1, X_2, X_3 \) be Banach spaces, let \( U \) be an open subset of \( X_1 \), and let \( X_0 \) be a subspace of \( X_1 \). Let \( f, f_1, \ldots, f_k : U \to X_2 \) be functions having directional derivatives \( \partial_x f, \partial_x f_1, \ldots, \partial_x f_k \) with respect to all \( x \in X_0 \) and let \( g \in \mathcal{G}^{k+1}(X_2, X_3) \). Then

\[
\gamma : U \to X_3, \quad u \mapsto \partial^k_{f_1(u) \ldots f_k(u)} g(f(u)) \quad (2.7)
\]

has directional derivatives \( \partial_x \gamma \) with respect to all \( x \in X_0 \) and

\[
\partial_x \gamma(u) = \partial^{k+1}_{\partial_x f(u)f_1(u) \ldots f_k(u)} g(f(u)) + \sum_{i=1}^k \partial^k_{f_i(u) \ldots \partial_x f_i(u) \ldots f_k(u)} g(f(u)) \quad \forall u \in U, \forall x \in X_0. \quad (2.8)
\]

If \( X_0 \) is a Banach space continuously embedded in \( X_1 \) and if \( f, f_1, \ldots, f_k \in \mathcal{G}^1(U, X_2; X_0) \), then \( \gamma \in \mathcal{G}^1(U, X_3; X_0) \).

Proof. Let \( u \in U, x \in X_0 \), and let \( [u-\varepsilon x, u+\varepsilon x] \subset U \), for some \( \varepsilon > 0 \). Let \( h \in [-\varepsilon, \varepsilon] \setminus \{0\} \). By strong continuity of \( \partial^{k+1} g \) and by \( k \)-linearity of \( \partial^k_{x_1 \ldots x_k} g \) with respect to \( x_1, \ldots, x_k \), we can write

\[
\frac{\gamma(u+hx) - \gamma(u)}{h} = \left. \frac{1}{h} \left( \partial^k_{f_1(u+hx)f_2(u+hx) \ldots f_k(u+hx)} g(f(u+hx)) - \partial^k_{f_1(u+hx)f_2(u+hx) \ldots f_k(u+hx)} g(f(u)) \right) \right| \quad (2.8)
\]

By continuity of \( f, f_1, \ldots, f_k \) on the set \( (u+\mathbb{R}x) \cap U \) and by joint continuity of \( \partial^{k+1} g \), the integrand function is uniformly continuous in \( (h, \theta) \in \{[-\varepsilon, \varepsilon] \setminus \{0\} \} \times [0,1] \). Then we can pass to the limit \( h \to 0 \) and obtain (2.8).

If \( f, f_1, \ldots, f_k \in \mathcal{G}^1(U, X_2; X_0) \), then the strong continuity of \( \partial x_0^k \gamma \) comes from Proposition 2.1 and formula (2.8), by recalling also Remark 2.2.

Lemma 2.5. Let \( n \in \mathbb{N} \). Let \( X_0, X_1, X_2, X_3 \) be Banach spaces, with \( X_0 \) continuously embedded in \( X_1 \), and let \( U \subset X_1 \) be an open set. Let

\[
\begin{align*}
& \begin{cases}
  f_0, \ldots, f_n \in \mathcal{G}^1(U, X_2; X_0) \\
  f^{(k)}_0, \ldots, f^{(k)}_n \in \mathcal{G}^1(U, X_2; X_0) \quad \forall k \in \mathbb{N} \\
  g \in \mathcal{G}^{n+1}(X_2, X_3) \\
  g^{(k)} \in \mathcal{G}^{n+1}(X_2, X_3) \quad \forall n \in \mathbb{N}.
\end{cases}
\end{align*}
\]
Suppose that, for $i = 0, \ldots, n$,

\[
\begin{align*}
\lim_{k \to \infty} f_i^{(k)}(u) &= f_i(u) \\
\lim_{k \to \infty} \partial_x f_i^{(k)}(u) &= \partial_x f_i(u),
\end{align*}
\]

uniformly for $u$ on compact subsets of $U$ and $x$ on compact subsets of $X_0$, and that

\[
\lim_{k \to \infty} \partial_{x_1 \ldots x_j} g^{(k)}(x_0) = \partial_{x_1 \ldots x_j} g(x_0) \quad j = n, n+1,
\]

uniformly for $x_0, x_1, \ldots, x_j$ on compact subsets of $X_2$. Define

\[
\begin{align*}
\gamma & : U \to X_3, \quad u \mapsto \partial_{f_1(u) \ldots f_n(u)} g(f_0(u)) \\
\gamma^{(k)} & : U \to X_3, \quad u \mapsto \partial_{f_1^{(k)}(u) \ldots f_n^{(k)}(u)} g^{(k)}(f_0^{(k)}(u)), \quad \forall k \in \mathbb{N}.
\end{align*}
\]

Then

\[
\lim_{k \to \infty} \partial_x \gamma^{(k)}(u) = \partial_x \gamma(u)
\]

uniformly for $u$ on compact subsets of $U$ and $x$ on compact subsets of $X_0$.

**Proof.** Since the composition of sequences of continuous functions uniformly convergent on compact sets is convergent to the composition of the limits, uniformly on compact sets, it is sufficient to recall Remark 2.2, apply Lemma 2.4, and consider (2.8). \qed

Let $X_0, X_1$ be Banach spaces, with $X_0$ continuously embedded in $X_1$, and let $U$ be an open subset of $X_1$. Let $n \in \mathbb{N}, n \geq 1$, $x_n := (x_1, \ldots, x_n) \subset X_0^n, j \in \{1, \ldots, n\}$. Then

- $P^j(x_n)$ denotes the set of partitions of $x_n$ in $j$ non-empty subsets.
- If $f \in \mathcal{G}^n(U, X_1; X_2)$ and $q := \{y_1, \ldots, y_j\} \subset x_n$, then $\partial_q f(u)$ denotes the derivative $\partial_{y_1 \ldots y_j} f(u)$ \footnote{By Remark 2.3, there is no ambiguity due to the fact that $q$ is not ordered.}.
- $|q|$ denotes the cardinality of $q$.

**Proposition 2.6** (Faà di Bruno’s formula). Let $n \geq 1$. Let $X_0, X_1, X_2, X_3$ be Banach spaces, with $X_0$ continuously embedded in $X_1$, and let $U$ be an open subset of $X_1$. If $f \in \mathcal{G}^n(U, X_2; X_0)$ and $g \in \mathcal{G}^n(X_2, X_3)$, then $g \circ f \in \mathcal{G}^n(U, X_3; X_0)$. Moreover

\[
\partial_{x_j} g \circ f(u) = \sum_{i=1}^j \sum_{P^j \in P^j(x_j)} \partial_{[p_1^j, \ldots, p_j^j]} g(f(u)).
\]

for all $u \in U$, $j = 1, \ldots, n$, $x_j = (x_1, \ldots, x_j) \subset X_0^j$.

**Proof.** The proof is standard and is obtained by induction on $n$ and by making use of Lemma 2.4 at each step of the inductive argument. The case $n = 1$ is obtained by applying Lemma 2.4 with $k = 0$. Now consider the case $n \geq 2$. By inductive hypothesis, formula
(2.11) holds true for \( j = 1, \ldots, n - 1 \), and we need to prove that it holds for \( j = n \). Let \( u \in U \), \( x_1, \ldots, x_n \in X_0 \), \( x_{n-1} := \{x_1, \ldots, x_{n-1}\} \). Then, by (2.11),

\[
\partial_{x_1, \ldots, x_n}^{n-1} g \circ f(u) = \sum_{i=1}^{n-1} \sum_{[\mathbf{p}_1, \ldots, \mathbf{p}_n] \in P^i(X_{n-1})} \partial_{x_1, \ldots, x_n}^{n-1} g(f(u)).
\]

By applying Lemma 2.4, with \( k = i \) and \( f_j = \partial_{\mathbf{p}_j}^{[p]} f \), for \( j = 1, \ldots, i \), to each member of the sum over \( P^i(X_{n-1}) \), we obtain, for all \( x_n \in X_0 \),

\[
\partial_{x_1, \ldots, x_{n-1}}^{n} g \circ f(u) = \sum_{i=1}^{n-1} \sum_{[\mathbf{p}_1, \ldots, \mathbf{p}_n] \in P^i(X_{n-1})} \left( \partial_{x_1, \ldots, x_n}^{n} g(f(u)) \right) + \sum_{i=1}^{n} \partial_{\mathbf{p}_1}^{[p]} \partial_{\mathbf{p}_1}^{[p]} \partial_{\mathbf{p}_1}^{[p]} g(f(u))
\]

\[
= \sum_{i=1}^{n-1} \sum_{[\mathbf{p}_1, \ldots, \mathbf{p}_n] \in P^i(X_{n-1})} \left( \partial_{x_1, \ldots, x_n}^{n} g(f(u)) \right) + \sum_{[\mathbf{p}_1, \ldots, \mathbf{p}_n] \in P^i(X_{n-1})} \partial_{\mathbf{p}_1}^{[p]} \partial_{\mathbf{p}_1}^{[p]} \partial_{\mathbf{p}_1}^{[p]} g(f(u))
\]

\[
= \sum_{i=1}^{n} \sum_{[\mathbf{p}_1, \ldots, \mathbf{p}_n] \in P^i(X_{n-1})} \partial_{\mathbf{p}_1}^{[p]} g(f(u)).
\]

This concludes the proof of (2.11).

**Proposition 2.7.** Let \( n \geq 1 \). Let \( X_0, X_1, X_2, X_3 \) be Banach spaces, with \( X_0 \) continuously embedded in \( X_1 \), and let \( U \) be an open subset of \( X_1 \). Let

\[
\begin{align*}
&\{ f \in \mathcal{G}^n(U, X_2; X_0) \\
&\{ f^{(k)} \in \mathcal{G}^n(U, X_2; X_0) \quad \forall k \in \mathbb{N} \\
&g \in \mathcal{G}^n(X_2, X_3) \\
&g^{(k)} \in \mathcal{G}^n(X_2, X_3) \quad \forall k \in \mathbb{N}.
\end{align*}
\]

Suppose that

\[
\begin{align*}
\lim_{k \to \infty} f^{(k)}(u) &= f(u) \\
\lim_{k \to \infty} \partial_{x_1, \ldots, x_j} f^{(k)}(u) &= \partial_{x_1, \ldots, x_j} f(u) \quad \text{for } j = 1, \ldots, n,
\end{align*}
\]

uniformly for \( u \) on compact subsets of \( U \) and \( x_1, \ldots, x_j \) on compact subsets of \( X_0 \), and that

\[
\begin{align*}
\lim_{k \to \infty} g^{(k)}(x) &= g(x) \\
\lim_{k \to \infty} \partial_{x_1, \ldots, x_j} g^{(k)}(x) &= \partial_{x_1, \ldots, x_j} g(x) \quad \text{for } j = 1, \ldots, n,
\end{align*}
\]

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uniformly for \( x, x_1, \ldots, x_j \) on compact subsets of \( X_2 \). Then

\[
\begin{align*}
\lim_{k \to \infty} g^{(k)} \circ f^{(k)}(u) &= g \circ f(u) \\
\lim_{k \to \infty} \partial_{x_{i_1}, \ldots, x_{i_j}} g^{(k)}(u) &= \partial_{x_{i_1}, \ldots, x_{i_j}} g \circ f(u) \quad \text{for } j = 1, \ldots, n,
\end{align*}
\]

uniformly for \( u \) on compact subsets of \( U \) and \( x_1, \ldots, x_j \) on compact subsets of \( X_0 \).

**Proof.** Use recursively formula (2.11) and Lemma 2.5. \( \blacksquare \)

### 2.2 Contractions in Banach spaces: survey of basic results

In this section, we assume that \( X \) and \( Y \) are Banach spaces, and that \( U \) is an open subset of \( X \). We recall that, if \( \alpha \in [0,1) \) and \( h : U \times Y \to Y \), then \( h \) is said to be a parametric \( \alpha \)-contraction if

\[ |h(u, y) - h(u, y')|_Y \leq \alpha |y - y'| \quad \forall u \in U, \forall y, y' \in Y. \]

By the Banach contraction principle, to any such \( h \) we can associate a uniquely defined map \( \varphi : U \to Y \) such that \( h(u, \varphi(u)) = \varphi(u) \) for all \( u \in U \). We refer to \( \varphi \) as to the fixed-point map associated to \( h \). For future reference, we summarize some basic continuity properties that \( \varphi \) inherits from \( h \).

The following lemma can be found in [14, p. 13].

**Lemma 2.8.** Let \( \alpha \in [0,1) \) and let \( h(u, \cdot) : U \times Y \to Y \), \( h_n(u, \cdot) : U \times Y \to Y \), for \( n \in \mathbb{N} \), be parametric \( \alpha \)-contractions. Denote by \( \varphi \) (resp. \( \varphi_n \)) the fixed-point map associated to \( h \) (resp. \( h_n \)).

(i) If \( h_n \to h \) pointwise on \( U \times Y \), then \( \varphi_n \to \varphi \) pointwise on \( U \).

(ii) If \( A \subset U \) is a set and if there exists an increasing concave function \( w \) on \( \mathbb{R}^+ \) such that \( w(0) = 0 \) and

\[ |h(u, y) - h(u', y)|_Y \leq w(|u - u'|_X) \quad \forall u, u' \in A, \forall y \in Y, \tag{2.12} \]

then

\[ |\varphi(u) - \varphi(u')|_Y \leq \frac{1}{1 - \alpha} w(|u - u'|_X) \quad \forall u, u' \in A. \]

(iii) If \( h \) is continuous, then \( \varphi \) is continuous.

**Proof.** Since \( h \) and \( h_n \) are \( \alpha \)-contractions, we have

\[ |\varphi_n(u) - \varphi(u')| \leq \frac{|h_n(u_n, \varphi(u)) - h(u', \varphi(u'))|}{1 - \alpha}, \tag{2.13} \]

\[ |\varphi(u) - \varphi(u')| \leq \frac{|h(u_n, \varphi(u')) - h(u', \varphi(u'))|}{1 - \alpha}, \tag{2.14} \]

for all \( u, u' \in U \). Then (2.13) yields (i) by taking \( u = u' \) and letting \( n \to \infty \), and (2.14) yields (ii) by using (2.12).

Regarding (iii), let \( u' \in U \), \( u_n \to u' \) in \( U \), let \( V \subset U \) be an open set containing \( u' \), and let \( n \in \mathbb{N} \) such that \( u_n - u' + V \subset U \) for all \( n \geq n \). Define \( h_n : V \times Y \to Y \) by \( h_n(u, y) := \)
in the following $h(u + u_n - u', y)$ for all $(u, y) \in V \times Y$. Then $h_n$ is a parametric $\alpha$-contraction. Denote by $\varphi_n$ its associated fixed-point map. Then, by continuity of $h$ and by (i), $\varphi_n(u) \to \varphi(u)$ for all $u \in V$. In particular, $\varphi(u_n) = \varphi_n(u') \to \varphi(u')$, hence $\varphi$ is continuous. 

**Remark 2.9.** If $h : U \times Y \to Y$ is a parametric $\alpha$-contraction ($\alpha \in [0, 1]$) belonging to $\mathcal{G}^1(U \times Y, Y; [0] \times Y)$, then

$$|\partial_y h(u, y)|_{L(Y)} \leq \alpha \quad \forall u \in U, \ y \in Y,$$

(2.15)

where $| \cdot |_{L(Y)}$ denotes the operator norm on $L(Y)$. Hence $\partial_y h(u, y)$ is invertible and the family $\{(I - \partial_y h(u, y))^{-1}(u, y)\}_{(u, y) \in U \times Y}$ is uniformly bounded in $L(Y)$. For what follows, it is important to notice that, for all $y \in Y$,

$$U \times Y \to Y, \ (u, y') \mapsto (I - \partial_y h(u, y'))^{-1} y$$

is continuous, hence, because of the formula

$$(I - \partial_y h(u, y'))^{-1}y = \sum_{n \in \mathbb{N}} (\partial_y h(u, y'))^n y$$

and of Lebesgue's dominated convergence theorem (for series), $(I - \partial_y h(u, y'))^{-1}y$ is jointly continuous in $u, y', y$.

The following proposition shows that the fixed-point map $\varphi$ associated to a parametric $\alpha$-contraction $h$ inherits from $h$ the strongly continuous Gâteaux differentiability.

**Proposition 2.10.** If $h \in \mathcal{G}^1(U \times Y, Y)$ is a parametric $\alpha$-contraction and if $\varphi$ is the fixed-point map associated to $h$, then $\varphi \in \mathcal{G}^1(U, Y)$ and

$$\partial_x \varphi(u) = (I - \partial_y h(u, \varphi(u)))^{-1} (\partial_y h(u, \varphi(u))) \quad \forall u \in U, \forall x \in X.$$  

(2.17)

**Proof.** For the proof, see [10, Lemma 2.9], or [1, Proposition C.0.3], taking into account also [1, Remark C.0.4], Lemma 2.8(iii), Remark 2.9. 

**2.3 Gâteaux differentiability of order $n$ of fixed-point maps**

In this section we provide a result for the Gâteaux differentiability up to a generic order $n$ of a fixed-point map $\varphi$ associated to a parametric $\alpha$-contraction $h$, under the assumption that $h$ is Gâteaux differentiable only with respect to some invariant subspaces of the domain.

The main result of this section is Theorem 2.12, which is suitable to be applied to mild solutions of SDEs (Section 3.2). When $n = 1$, Theorem 2.12 reduces to Proposition 2.10. In the case $n = 2$, Theorem 2.12 is also well-known, and a proof can be found in [1, Proposition C.0.5]. On the other hand, when the order of differentiability $n$ is generic, the fact that the parametric $\alpha$-contraction is assumed to be differentiable only with respect to certain subspaces makes non-trivial the proof of the theorem. To our knowledge, a reference for the case $n \geq 3$ is not available in the literature. The main issue consists in providing a precise formulation of the statement, with its assumptions, that can be proved by induction.

For the sake of readability, we collect the assumptions of Theorem 2.12 in the following
Assumption 2.11.

(1) \( n \geq 1 \) and \( \alpha \in [0, 1) \);

(2) \( X \) is a Banach space and \( U \) is an open subset of \( X \).

(3) \( Y_1 \supset Y_2 \supset \ldots \supset Y_n \) is a decreasing sequence of Banach spaces, with norms \( |\cdot|_1, \ldots, |\cdot|_n \), respectively.

(4) For \( k = 1, \ldots, n \) and \( j = 1, 2, \ldots, k \), the canonical embedding of \( Y_k \) into \( Y_j \), denoted by \( i_{k,j} : Y_k \to Y_j \), is continuous.

(5) \( h_1 : U \times Y_1 \to Y_1 \) is a function such that \( h_1(U \times Y_k) \subset Y_k \) for \( k = 2, \ldots, n \). For \( k = 2, \ldots, n \), we denote by \( h_k \) the induced function

\[
h_k : U \times Y_k \to Y_k, \quad (u, y) \mapsto h_1(u, y).
\] (2.18)

(6) For \( k = 1, \ldots, n \), \( h_k \) is continuous and satisfies

\[
|h_k(u, y) - h_k(u, y')|_k \leq \alpha |y - y'|_k \quad \forall u \in U, \forall y, y' \in Y_k.
\] (2.19)

(7) For \( k = 1, \ldots, n \), \( h_k \in \mathcal{G}^a(U \times Y_k, Y_k ; X \times \{0\}) \).

(8) For \( k = 1, \ldots, n - 1 \), \( h_k \in \mathcal{G}^a(U \times Y_k, Y_k ; X \times Y_{k+1}) \).

(9) For \( k = 1, \ldots, n \), \( j = 1, \ldots, n - 1 \), for all \( u \in U \), \( z_1, \ldots, z_j \in X \), \( y, z_{j+1} \in Y_k \), and for all permutations \( \sigma \) of \( \{1, \ldots, j + 1\} \), the directional derivative \( \partial^{j+1}_{z_{\sigma(1)} \ldots z_{\sigma(j+1)}} h_k(u, y) \) exists, and

\[
U \times Y_k \times X^j \times Y_k \to Y_k, \quad (u, y, z_1, \ldots, z_j, z_{j+1}) \mapsto \partial^{j+1}_{z_{\sigma(1)} \ldots z_{\sigma(j+1)}} h_k(u, y)
\] (2.20)

is continuous.

Theorem 2.12. Let Assumption 2.11 be satisfied and let \( \varphi : U \to Y_1 \) denote the fixed-point function associated to the parametric \( \alpha \)-contraction \( h_1 \). Then, for \( j = 1, \ldots, n \), we have \( \varphi \in \mathcal{G}^j(U, Y_{n-j+1}) \) and, for all \( u \in U \), \( x_1, \ldots, x_j \in X \), \( \partial^j_{x_1 \ldots x_j} \varphi(u) \) is given by the formula

\[
\partial^j_{x_1 \ldots x_j} \varphi(u) = (I - \partial Y_1 h_1(u, \varphi(u)))^{-1} \partial^j_{x_1 \ldots x_j} h_1(u, \varphi(u))
\]

\[+ \sum_{x \in 2^{[x_1, \ldots, x_j]} \atop x \neq \emptyset} \sum_{i = \max\{1, 2, \ldots, j + |x|\}}^{|x|} \sum_{\psi \in \mathcal{P}^i(x) \atop \psi = (p_1, \ldots, p_i)} (I - \partial Y_1 h_1(u, \varphi(u)))^{-1} \partial^i_{x^c, \psi} h_1(u, \varphi(u)) \] (2.21)

where \( 2^{[x_1, \ldots, x_j]} \) is the power set of \( \{x_1, \ldots, x_j\} \), \( \mathcal{P}^i(x) \) is the set of partitions of \( x \) in \( i \) non-empty parts, \( x^c := \{x_1, \ldots, x_j\} \setminus x \), and \( \partial^i_{x^c, \psi} := \partial^{j-|x|}_{x^c} \partial^{p_1}_{p_1} \varphi(u) \ldots \partial^{p_i}_{p_i} \varphi(u) \).
Proof. The proof is by induction on \( n \). The case \( n = 1 \) is provided by Proposition 2.10.

Let \( n \geq 2 \). Clearly, it is sufficient to prove that \( \varphi \in \mathcal{G}^n(U, Y_n) \) and that (2.21) holds true for \( j = n \). Since we are assuming that the theorem holds true for \( n - 1 \), we can apply it with the data

\[
\tilde{h}_1: U \times \tilde{Y}_2 \to \tilde{Y}_2, \ldots, \tilde{h}_{n-1}: U \times \tilde{Y}_n \to \tilde{Y}_n,
\]

where \( \tilde{h}_k := h_{k+1}, \tilde{Y}_k := Y_{k+1} \), for \( k = 1, \ldots, n - 1 \). According to the claim, the fixed-point function \( \tilde{\varphi} \) of \( \tilde{h}_1 \) belongs to \( \mathcal{G}^j(U, \tilde{Y}_{(n-1)-j+1}) \), for \( j = 1, \ldots, n - 1 \), and formula (2.21) holds true for \( \tilde{\varphi} \) and \( j = 1, \ldots, n - 1 \). Since \( \varphi(u) = (i_{2,1} \circ \tilde{\varphi})(u) \), for \( u \in U \), we have \( \varphi \in \mathcal{G}^j(U, \tilde{Y}_{n-j}) = \mathcal{G}^j(U, Y_{n-j+1}) \), for \( j = 1, \ldots, n - 1 \), and

\[
\frac{\partial^j_{x_1 \ldots x_j}}{\partial x_1 \ldots x_j} \varphi(u) = \tilde{\frac{\partial^j_{x_1 \ldots x_j}}{\partial x_1 \ldots x_j}} \tilde{\varphi}(u) \in \tilde{Y}_{n-j} = Y_{n-j+1}, \quad \forall u \in U, \forall x_1, \ldots, x_j \in X.
\]

Then (2.21) holds true for \( \varphi \) up to order \( j = n - 1 \). In particular \( \varphi \in \mathcal{G}^{n-1}(U, Y_2) \), hence, for \( x_1, \ldots, x_n \in X, \varepsilon > 0 \), we can write

\[
\begin{align*}
&\frac{\partial^j_{x_1 \ldots x_{n-j}}}{\partial x_1 \ldots x_{n-j}} \varphi(u + \varepsilon x_n) - \frac{\partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u) \\
= & (\partial_{y_1} h_1(u + \varepsilon x_n, \varphi(u + \varepsilon x_n)), \frac{\partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \varepsilon x_n) - \partial_{y_1} h_1(u, \varphi(u))^j_{x_1 \ldots x_{n-j-1}} \varphi(u)) \\
&+ (\mathcal{J}(u + \varepsilon x_n) - \mathcal{J}(u)) \\
= : & I + II,
\end{align*}
\]

where \( \mathcal{J}(\cdot) \) denotes the sum

\[
\mathcal{J}(v) := \frac{\partial^j_{x_1 \ldots x_{n-j}}}{\partial x_1 \ldots x_{n-j}} h_1(v, \varphi(v)) + \sum_{x \in \mathcal{E}_{1 \ldots n-j-1} \cup \mathcal{E}_{1 \ldots n-j-1}} \left( \sum_{p = (p_1, \ldots, p_j)} \frac{\partial^j_{\sum_{i=0}^{j} p_i} \varphi}{\partial \sum_{i=0}^{j} p_i} h_1(v, \varphi(v)) \right)
\]

for \( v \in U \). By recalling that \( \varphi \in \mathcal{G}^j(U, Y_{n-j+1}), j = 1, \ldots, n - 1 \), hence by taking into account with respect to which space the derivatives of \( \varphi \) are continuous, we write

\[
\begin{align*}
I =& \frac{\partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \varepsilon x_n) h_1(u + \theta x_n, \varphi(u + \varepsilon x_n)) - \frac{\partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u) h_1(u, \varphi(u)) \\
&= \int_0^1 \frac{\partial_{x_1} \partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \theta x_n, \varphi(u + \varepsilon x_n)) h_1(u, \varphi(u)) \theta \, d\theta \\
&+ \int_0^1 \frac{\partial_{x_1} \partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \varepsilon x_n, \varphi(u + \varepsilon x_n)) h_1(u, \varphi(u)) \theta \, d\theta \\
&+ \frac{\partial_{x_1} \partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \varepsilon x_n, \varphi(u + \varepsilon x_n)) h_1(u, \varphi(u)) h_1(u, \varphi(u)) \\
&= I_1 + I_2 + I_3 + I_4 + I_5,
\end{align*}
\]

with (5)

\[
\lim_{\varepsilon \to 0} \frac{I_1}{\varepsilon} = \frac{\partial_{x_1} \partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u) h_1(u, \varphi(u)) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{I_2}{\varepsilon} = \frac{\partial_{x_1} \partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u) h_1(u, \varphi(u)).
\]

The limits should be understood in the suitable spaces \( Y_k \). For instance, when computing \( \lim_{\varepsilon \to 0} \frac{I_1}{\varepsilon} \), the object \( \frac{\partial^j_{x_1 \ldots x_{n-j-1}}}{\partial x_1 \ldots x_{n-j-1}} \varphi(u + \varepsilon x_n) \) should be considered in the space \( Y_2 \), which can be done thanks to the inductive hypothesis.
In a similar way,

\[
\lim_{\varepsilon \to 0} \mathbb{P} = \partial_{x_n} \partial_{x_1 \ldots x_{n-1}}^n h_1(u, \varphi(u)) + \partial_{x_n, \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^n h_1(u, \varphi(u)) + \sum_{x \in 2^{i_1 \ldots i_n}} \left( \sum_{j=1}^{i} \partial^p \left[ x^{\ast}, p \right] h_1(u, \varphi(u)) \right)
\]

\[
+ \sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-(n-1)+|x|\}} \sum_{p \in P^\prime(x)} \partial_{x_n} \partial_{p_i}^n \left[ x^c, p \right] h_1(u, \varphi(u))
\]

\[
+ \sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-(n-1)+|x|\}} \sum_{p \in P^\prime(x)} \left( \partial_{x_n, \varphi(u)} \partial_{p_i}^n \left[ x^c, p \right] h_1(u, \varphi(u)) \right)
\]

\[
+ \sum_{j=1}^{i} \partial^p \left[ x^{\ast}, p_1 \right] \varphi(u) \ldots \partial^p \left[ x^{\ast}, p_{j-1} \right] \varphi(u) \partial_{p_j}^n \left[ x^c, p \right] \varphi(u) \partial_{p_{j+1}} \varphi(u) \ldots \partial^p \left[ x^c, p \right] \varphi(u) h_1(u, \varphi(u))
\]

(2.24)

Notice that

\[
\sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-(n-1)+|x|\}} \sum_{p \in P^\prime(x)} \partial_{x_n} \partial_{p_i}^n \left[ x^c, p \right] h_1(u, \varphi(u))
\]

\[
= \sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-n+|x|\}} \sum_{p \in P^\prime(x)} \partial^p \left[ x^{\ast}, p \right] h_1(u, \varphi(u)) - \partial_{x_n} \partial_{p_i}^n \varphi(u) h_1(u, \varphi(u))
\]

(2.25)

and

\[
\sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-(n-1)+|x|\}} \sum_{p \in P^\prime(x)} \partial_{x_n, \varphi(u)} \partial_{p_i}^n \left[ x^c, p \right] h_1(u, \varphi(u))
\]

\[
= \sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-n+|x|\}} \sum_{p \in P^\prime(x)} \partial^p \left[ x^{\ast}, p \right] h_1(u, \varphi(u)) - \partial_{x_n, \varphi(u)} \partial_{p_i}^n \varphi(u) h_1(u, \varphi(u))
\]

(2.26)

and

\[
\sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-(n-1)+|x|\}} \sum_{p \in P^\prime(x)} \sum_{j=1}^{i} L(p, j; u)
\]

\[
= \sum_{x \in 2^{i_1 \ldots i_n}, i = \max\{1,2-n+|x|\}} \sum_{p \in P^\prime(x)} \sum_{j=1}^{i} \partial^p \left[ x^{\ast}, p \right] h_1(u, \varphi(u))
\]

(2.27)

where

\[
L(p, j; u) := \partial^p \left[ x^{\ast}, p \right] \varphi(u) \ldots \partial^p \left[ x^{\ast}, p_{j-1} \right] \varphi(u) \partial_{p_j} \varphi(u) \partial_{p_{j+1}} \varphi(u) \ldots \partial^p \varphi(u) h_1(u, \varphi(u)).
\]
By collecting (2.24), (2.25), (2.26), (2.27), we obtain

\[
\lim_{\varepsilon \to 0} \frac{\Pi}{\varepsilon} = \partial_{\partial_{x_n} \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^{n-1} h_1(u, \varphi(u)) + \partial_{x_1 \ldots x_n}^{n} h_1(u, \varphi(u)) - \partial_{\partial_{x_n} \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u) h_1(u, \varphi(u)) \\
+ \sum_{\mathbf{x} \in \mathbb{Z}^{n}} \sum_{i = \max(1, 2 - n + |\mathbf{x}|)}^{\varepsilon} \partial_{\mathbf{x}^i, \mathbf{p}}^{n} h_1(u, \varphi(u)) - \partial_{\partial_{x_n} \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u) h_1(u, \varphi(u)) \\
= \partial_{x_1 \ldots x_n}^{n} h_1(u, \varphi(u)) - \partial_{\partial_{x_n} \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u) h_1(u, \varphi(u)) \\
+ \sum_{\mathbf{x} \in \mathbb{Z}^{n}} \sum_{i = \max(1, 2 - n + |\mathbf{x}|)}^{\varepsilon} \partial_{\mathbf{x}^i, \mathbf{p}}^{n} h_1(u, \varphi(u)) - \partial_{\partial_{x_n} \varphi(u)} \partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u) h_1(u, \varphi(u)).
\]

Hence

\[
\lim_{\varepsilon \to 0} \left( \frac{I_1}{\varepsilon} + \frac{I_2}{\varepsilon} + \frac{\Pi}{\varepsilon} \right) = \sum_{\mathbf{x} \in \mathbb{Z}^{n}} \sum_{i = \max(1, 2 - n + |\mathbf{x}|)}^{\varepsilon} \partial_{\mathbf{x}^i, \mathbf{p}}^{n} h_1(u, \varphi(u)) + \partial_{x_1 \ldots x_n}^{n} h_1(u, \varphi(u)),
\]

and, by recalling (2.22), (2.23), we obtain

\[
\lim_{\varepsilon \to 0} \left( I - \partial_{\partial_{1}} h_1(u, \varphi(u)) \right) \frac{\partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u + \varepsilon x_n) - \partial_{x_1 \ldots x_{n-1}}^{n-1} \varphi(u)}{\varepsilon} \\
= \sum_{\mathbf{x} \in \mathbb{Z}^{n}} \sum_{i = \max(1, 2 - n + |\mathbf{x}|)}^{\varepsilon} \partial_{\mathbf{x}^i, \mathbf{p}}^{n} h_1(u, \varphi(u)) + \partial_{x_1 \ldots x_n}^{n} h_1(u, \varphi(u)).
\]

Finally, we can conclude the proof by recalling that \( I - \partial_{\partial_{1}} h_1(u, \varphi(u)) \) is invertible with strongly continuous inverse. 

Theorem 2.12 says that \( \varphi \) is \( Y_n \)-valued, continuous as a map from \( U \) into \( Y_n \), and, for \( j = 1, \ldots, n \), for all \( u \in U, x_1, \ldots, x_j \in X \), the directional derivative \( \partial_{x_1 \ldots x_j} \varphi(u) \) exists, it belongs to \( Y_{n-j+1} \), the map

\[
U \times X^j \to Y_{n-j+1}, (u, x_1, \ldots, x_j) \mapsto \partial_{x_1 \ldots x_j} \varphi(u)
\]

is continuous, and (2.21) holds true.

Formula (2.21) can be useful e.g. when considering the boundedness of the derivatives of \( \varphi \), or when studying convergences of derivatives under perturbations of \( h \), as Corollary 2.13 and Proposition 2.14 show.

**Corollary 2.13.** Let Assumption 2.11 be satisfied. Suppose that there exists \( M > 0 \) such
2.15 holds true for a given \( \varphi \) that uniformly for \( n \) and let

\[
|\partial_{x_1 \ldots x_j} h_k(u, y)|_k \leq M \prod_{l=1}^j |x_l|_X
\]

\( k = 1, \ldots, n \), on compact subsets of \( X \).

Then, for \( k = 1, \ldots, n \),

\[
\sup_{u \in U, \; x_1, \ldots, x_k \in X, \; |x_1|_X = \ldots = |x_k|_X = 1} |\partial^i_{x_1 \ldots x_k} \varphi(u)|_{n-k+1} \leq C(\alpha, M),
\]

where \( C(\alpha, M) \in \mathbb{R} \) depends only on \( \alpha, M \).

**Proof.** Reason by induction taking into account (2.21) and (2.15). \( \blacksquare \)

**Proposition 2.14.** Suppose that Assumption 2.11 holds true for a given \( h_1 \) and that \( h_1^{(1)}, h_1^{(2)}, h_1^{(3)} \ldots \) is a sequence of functions, each of which satisfies Assumption 2.11, uniformly with respect to the same \( n, \alpha \). Let \( h_k^{(m)} \) denote the map associated to \( h_k^{(m)} \) according to (2.18) and let \( \varphi^{(m)} \) denote the fixed-point map associated to the parametric \( \alpha \)-contraction \( h_k^{(m)} \).

Suppose that the following convergences occur:

(i) For \( k = 1, \ldots, n \), \( y \in Y_k \),

\[
\lim_{m \to \infty} h_k^{(m)}(u, y) = h_k(u, y) \quad \text{in } Y_k
\]

uniformly for \( u \) on compact subsets of \( U \);

(ii) for \( k = 1, \ldots, n \),

\[
\begin{align*}
\lim_{m \to \infty} \partial_{x} h_k^{(m)}(u, y) &= \partial_{x} h_k(u, y) \quad \text{in } Y_k \\
\lim_{m \to \infty} \partial_{y} h_k^{(m)}(u, y') &= \partial_{y} h_k(u, y') \quad \text{in } Y_k
\end{align*}
\]

uniformly for \( u \) on compact subsets of \( U \), \( x \) on compact subsets of \( X \), and \( y, y' \) on compact subsets of \( Y_k \);

(iii) for all \( k = 1, \ldots, n - 1 \), \( u \in U \), \( j, i = 0, \ldots, n \), \( 1 \leq j + i \leq n \),

\[
\lim_{m \to \infty} \partial^{j+i}_{x_1 \ldots x_j y_1 \ldots y_i} h_k^{(m)}(u, y) = \partial^{j+i}_{x_1 \ldots x_j y_1 \ldots y_i} h_k(u, y) \quad \text{in } Y_k
\]

uniformly for \( u \) on compact subsets of \( U \), \( x_1, \ldots, x_j \) on compact subsets of \( X \), \( y \) on compact subsets of \( Y_k \), \( y_1, \ldots, y_i \) on compact subsets of \( Y_{k+1} \).

Then \( \varphi^{(m)} \to \varphi \) uniformly on compact subsets of \( Y_n \) and, for all \( j = 1, \ldots, n \)

\[
\lim_{m \to \infty} \partial_j^{j}_{x_1 \ldots x_j} \varphi^{(m)}(u) = \partial_j^{j}_{x_1 \ldots x_j} \varphi(u) \quad \text{in } Y_{n-j+1}
\]

uniformly for \( u \) on compact subsets of \( U \) and \( x_1, \ldots, x_j \) on compact subsets of \( X \).
Proof. Notice that (2.29) and the fact that each \( h_k^{(m)} \) is a parametric \( a \)-contraction (with the same \( a \)) imply the uniform convergence \( h_k^{(m)} \to h_k \) on compact subsets of \( Y_k \). In particular, the sequence \( h_k^{(1)}, h_k^{(2)}, h_k^{(3)} \ldots \) is uniformly equicontinuous on compact sets. Then, by Lemma 2.8(i), (ii), \( q_k^{(m)} \to q \) in \( Y_k \) uniformly on compact subsets of \( Y_k \), for \( k = 1, \ldots, n \). Moreover, by (2.15), that holds for all \( h_k^{(m)} \) uniformly in \( m \), we have the boundedness of \( (I - \partial Y_1 h_1^{(m)})^{-1} \), uniformly in \( m \). Convergence (2.32) is then obtained by reasoning by induction on (2.21), taking into account the strong continuity of \( (I - \partial Y_1 h_1)^{-1} \). \( \blacksquare \)

3 Path-dependent SDEs in Hilbert spaces

In this section we study mild solutions of path-dependent SDEs in Hilbert spaces. In particular, by applying the results of the previous section, we address differentiability with respect to the initial datum and stability of the derivatives. By emulating the arguments of [9, Ch. 7] for the Markovian case and for differentiability up to order 2, we extend the results there provided to the following path-dependent setting and to differentiability of generic order \( n \).

Let \( H \) and \( U \) be real separable Hilbert spaces, with scalar product denoted by \( \langle \cdot, \cdot \rangle_H \) and \( \langle \cdot, \cdot \rangle_U \), respectively. Let \( e := \{e_n\}_{n \in \mathbb{N}} \) be an orthonormal basis of \( H \), where \( \mathcal{N} = \{1, \ldots, N\} \) if \( H \) has dimension \( N \in \mathbb{N} \setminus \{0\} \), or \( \mathcal{N} = \mathbb{N} \) if \( H \) has infinite dimension, and let \( e' := \{e'_m\}_{m \in \mathcal{M}} \) be an orthonormal basis of \( U \), where \( \mathcal{M} = \{1, \ldots, M\} \) if \( U \) has dimension \( M \in \mathbb{N} \setminus \{0\} \), or \( \mathcal{M} = \mathbb{N} \) if \( U \) has infinite dimension. If \( x : [0, T] \to \mathcal{S} \) is a function taking values in any set \( \mathcal{S} \) and if \( t \in [0, T] \), we denote by \( x_{t \wedge} \) the function defined by

\[
\begin{align*}
\{ & x_{t \wedge}(s) := x(s) \quad s \in [0, t] \\
& x_{t \wedge}(s) := x(t) \quad s \in (t, T].
\end{align*}
\]

For elements of stochastic analysis in infinite dimension used hereafter, we refer to [10, 13].

We begin by considering the SDE

\[
\begin{align*}
& dX_s = (AX_s + b((s, X))\, dt + \sigma((s, X))\, dW_s \quad s \in (t, T] \\
& X_s = Y_s \quad s \in [0, t],
\end{align*}
\] (3.1)

where \( t \in [0, T] \), \( Y \) is a \( H \)-valued adapted process defined on a complete filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}) \), \( W \) is a \( U \)-valued cylindrical Wiener process defined on \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), \( b((\omega, s), X) \) is a \( H \)-valued random variable depending on \( \omega \in \Omega \), on the time \( s \), and on the path \( X, \sigma((\omega, s), X) \) is a \( L_2(U, H) \)-valued random variable depending on \( \omega \in \Omega \), on the time \( s \), and on the path \( X \), and \( A \) is the generator of a \( C_0 \)-semigroup \( S \) on \( H \).

We introduce the following notation:

- \( \mathcal{S} \) denotes a closed subspace of \( B_b([0, T], H) \)\(^6\) such that

\[
\begin{align*}
(a) \quad & C([0, T], H) \subset \mathcal{S} \\
(b) \quad & x_{t \wedge} \in \mathcal{S}, \forall x \in \mathcal{S}, \forall t \in [0, T] \\
(c) \quad & \text{for all } T \in L(H) \text{ and } x \in \mathcal{S}, \text{ the map } [0, T] \to H, \ t \mapsto T x_{t \wedge}, \text{ belongs to } \mathcal{S}.
\end{align*}
\] (3.2)

\(^6\)We recall that \( B_b([0, T], H) \) is endowed with the norm \( \| \cdot \|_\infty \).
Hereafter, unless otherwise specified, $S$ will be always considered as a Banach space endowed with the norm $|\cdot|_\infty$. For example, $S$ could be $C([0,T],H)$, the space of càdlàg functions $[0,T] \to H$, or $B_b([0,T],H)$ itself.

- $\Omega_T$ denotes the product space $\Omega \times [0,T]$ and $\mathcal{P}_T$ denotes the product measure $\mathbb{P} \otimes m$ on $(\Omega_T,\mathcal{F}_T \otimes \mathcal{B}_{[0,T]})$, where $m$ is the Lebesgue measure and $\mathcal{B}_{[0,1]}$ is the Borel $\sigma$-algebra on $[0,1]$.

- $\mathcal{L}^0_{\mathcal{P}_T}(S)$ denotes the space of functions $X : \Omega_T \to H$ such that

  \[
  (a) \ \forall \omega \in \Omega, \text{ the map } [0,T] \to H, \ t \mapsto X_t(\omega), \text{ belongs to } S \\
  (b) (\Omega_T,\mathcal{P}_T) \to S, \ (\omega,t) \mapsto X_{t\omega}(\omega) \text{ is measurable.}
  \]

Two processes $X, X' \in \mathcal{L}^0_{\mathcal{P}_T}(S)$ are equal if and only if $\mathbb{P}(|X - X'|_\infty = 0) = 1$.

- For $p \in [1,\infty)$, $\mathcal{L}^p_{\mathcal{P}_T}(S)$ denotes the space of equivalence classes of functions $X \in \mathcal{L}^0_{\mathcal{P}_T}(S)$ such that $\Omega_T \to S, (\omega,t) \mapsto X_{t\omega}(\omega)$ has separable range and

  \[
  |X|_{\mathcal{L}^p_{\mathcal{P}_T}(S)} := \left( \mathbb{E} \left[ |X|_\infty^p \right] \right)^{1/p} < \infty.
  \]

- For $p, q \in [1,\infty)$ and $\beta \in [0,1)$, $\Lambda^{p,q,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$ denotes the space of functions $\Phi : \Omega_T \to L(U,H)$ such that

  \[
  \left\{ \Phi u : (\Omega_T,\mathcal{P}_T) \to H, (\omega,t) \mapsto \Phi_t(\omega)u, \text{ is measurable, } \forall u \in U \right\}
  \]

  \[
  |\Phi|_{p,q,S,\beta} := \left( \int_0^T \left( \int_0^t (t-s)^{-\beta q} \mathbb{E} \left[ |S_{t-s} \Phi_s|_{L^p(U,H)} \right] \right)^{q/p} ds \right)^{1/p} < \infty.
  \]

The space $\Lambda^{p,q,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$ is normed by $|\cdot|_{p,q,S,\beta}$ (see Remark 3.1 below).

- $\overline{\Lambda^{p,q,p}_{\mathcal{P}_T,S,\beta}}(L(U,H))$ denotes the completion of $\Lambda^{p,q,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$. We keep the notation $|\cdot|_{p,q,S,\beta}$ for the extended norm.

It can be seen that $(\mathcal{L}^p_{\mathcal{P}_T}(S), |\cdot|_{\mathcal{L}^p_{\mathcal{P}_T}(S)})$ is a Banach space ($F$ is supposed to be complete).

**Remark 3.1.** To see that $|\cdot|_{p,q,S,\beta}$ is a norm and not just a seminorm, suppose that $|\Phi|_{p,q,S,\beta} = 0$. In particular, for $u \in U$,

\[
\int_{[0,T]^2} 1_{(0,T)}(t-s)(t-s)^{-\beta} \mathbb{E}[|S_{t-s} \Phi_s u|_H] ds \otimes dt = 0,
\]

which entails, for $\mathbb{P} \otimes m$-a.e. $(\omega,s) \in \Omega_T$,

\[
S_{t-s} \Phi_s(\omega)u = 0 \quad \text{m-a.e. } t \in (s,T].
\]

Since $S$ is strongly continuous, (3.5) gives

\[
\Phi_s(\omega)u = 0 \quad \mathbb{P} \otimes m$-a.e. $(\omega,s) \in \Omega_T,
\]

which provides $\Phi = 0 \mathbb{P} \otimes m$-a.e., since $U$ is supposed to be separable and $\Phi_s(\omega) \in L(U,H)$ for all $\omega,s$. 

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Remark 3.2. The space \( L_{\mathcal{P}_T, S, \beta}^p(L(U, H)) \) can be naturally identified with a closed subspace of the space of all those measurable functions

\[ \zeta : (\Omega_T \times [0, T], \mathcal{P}_T \otimes \mathcal{B}_T) \to L^2(U, H) \]

such that

\[
\begin{cases}
\zeta((\omega, s), t) = 0, \forall ((\omega, s), t) \in \Omega_T \times [0, T], s > t, \\
|\zeta|_{p, q, p} := \left( \int_0^T \left( \int_0^t \mathbb{E} \left[ |\zeta((\cdot, s), t)|_{L^2(U, H)}^p \right]^q ds \right)^{p/q} dt \right)^{1/p} < \infty.
\end{cases}
\]

Indeed, if we denote by \( L_{\mathcal{P}_T \otimes \mathcal{B}_T}^p(L_2(U, H)) \) such a space, then \( L_{\mathcal{P}_T \otimes \mathcal{B}_T}^p(L_2(U, H)) \) endowed with \(|:\|_{p, q, p} \) is a Banach space and the map

\[ i : L_{\mathcal{P}_T, S, \beta}^p(L(U, H)) \to L_{\mathcal{P}_T \otimes \mathcal{B}_T}^p(L_2(U, H)) \]

defined by

\[ i(\Phi)(\omega, s, t) := \begin{cases} 
(t-s)^{-\beta} S_{t-s} \Phi_s(\omega) & \forall ((\omega, s), t) \in \Omega_T \times [0, T], s \leq t, \\
0 & \text{otherwise.}
\end{cases} \]

is an isometry.

The reason to introduce the space \( L_{\mathcal{P}_T, S, \beta}^p(L(U, H)) \) is related to the existence of a continuous version of the stochastic convolution and to the factorization method used to construct such a version. Let \( p > \max(2, 1/\beta) \), \( t \in [0, T] \), and \( \Phi \in L_{\mathcal{P}_T, S, \beta}^p(L(U, H)) \). If we consider the two stochastic convolutions

\[ Y_t' := 1_{[t, T]}(t') \int_t^{t'} S_{t-s} \Phi_s dW_s, \quad Z_t' := 1_{[t, T]}(t') \int_t^{t'} (t'-s)^{-\beta} S_{t-s} \Phi_s dW_s, \quad (3.6) \]

then \( Y_t' \) is well-defined for all \( t' \in [0, T] \), \( Z_t' \) is well-defined for \( m \)-a.e. \( t' \in [0, T] \), and \( Y_{t'} \), \( Z_{t'} \) belong to \( L^p((0, \infty), \mathcal{F}_t, \mathbb{P}) \), \( H \). By using the stochastic Fubini theorem and the factorization method (see [10]), we can find a predictable process \( \tilde{Z} \) such that:

(a) for \( m \)-a.e. \( t \in [0, T] \), \( \tilde{Z}_t = Z_t \ \mathbb{P}\)-a.e.;

(b) for all \( t' \in [0, T] \), the following formula holds

\[ Y_{t'} = c_\beta 1_{[t, T]}(t') \int_t^{t'} (t'-s)^{-\beta-1} \tilde{Z}_s ds \quad \mathbb{P}\text{-a.e.}, \quad (3.7) \]

where \( c_\beta \) is a constant depending only on \( \beta \).

By (3.6), (a), [8, Lemma 7.7], it follows that \( \tilde{Z}(\omega) \in L^p((0, T), H) \) for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \), hence, by [13, Lemma 3.2], the right-hand side of (3.7) is continuous in \( t' \).

This classical argument shows that there exists a pathwise continuous process \( S_{\ast t} \Phi \) such that, for all \( t' \in [0, T] \), \( (S_{\ast t} \Phi)_{t'} = Y_{t'} \ \mathbb{P}\)-a.e.. In particular, \( S_{\ast t} \Phi \in L_{\mathcal{P}_T}^{0, 0} \) \((\mathbb{C}([0, T], H))\). By (3.6), (3.7), Hölder’s inequality, and [8, Lemma 7.7], we also have

\[
\mathbb{E} \left[ |S_{\ast t} \Phi|_\infty^p \right] \leq c_\beta \left( \int_0^T \int_0^{t'} (s^{-\beta-1})^p d\nu \right)^{1/p} \mathbb{E} \left[ \int_0^T |\tilde{Z}_s|_{H}^p ds \right] \leq c'_\beta, (3.8)
\]

where \( c_\beta \) is a constant depending only on \( \beta \).
where \( c'_{\beta,T,p} \) is a constant depending only on \( \beta, T, p \). This shows that the linear map \( S_{*t}^w \), defined as

\[
\Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)) \to \mathcal{L}^p_\mathcal{P}(C([[0,T],H))), \Phi \mapsto S_{*t}^w \Phi,
\]

is well-defined and continuous. Then, we can uniquely extend (3.9) to a continuous linear map on \( \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)) \), that we can see as \( \mathcal{L}^p_\mathcal{P}(\mathbb{S}) \)-valued, since, by assumption, \( C([0,T],H) \subset \mathbb{S} \). We end up with a continuous linear map, again denoted by \( S_{*t}^w \),

\[
S_{*t}^w: \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)) \to \mathcal{L}^p_\mathcal{P}(\mathbb{S}).
\]

Summarizing,

1. the map \( S_{*t}^w \) is linear, continuous, \( \mathcal{L}^p_\mathcal{P}(C([0,T],H)) \)-valued;

2. the operator norm of \( S_{*t}^w \) depends only on \( \beta, T, p \);

3. if \( \Phi \in \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L_2(U,H)), S_{*t}^w \Phi \) is a continuous version of the process \( Y \) in (3.6).

Within the approach using the factorization method, the space \( \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)) \) is then naturally introduced if we want to see the stochastic convolution as a continuous linear operator acting on a Banach space and providing pathwise continuous processes, and this perspective is useful when applying to SDEs the results based on parametric \( \alpha \)-contractions obtained in the first part of the paper.

We make some observations that will be useful later. Let \( \hat{S} \) be another \( C_0 \)-semigroup on \( H \), and let \( \Phi \in \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)), \hat{\Phi} \in \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},\hat{S},\beta(L(U,H)) \). Then, by using the factorization formula (3.7) both with respect to the couples \( (S,\Phi) \) and \( (\hat{S},\hat{\Phi}) \), and by an estimate analogous to (3.8), we obtain

\[
\mathbb{E}\left[ |S_{*t}^w \Phi - \hat{S}_{*t}^w \Phi|_\infty^p \right] \leq \int_0^T \left( \int_0^t (t-s)^{-2\beta} \mathbb{E}\left[ |S_{t-s} \Phi_s - \hat{S}_{t-s} \hat{\Phi}_s|_\mathcal{L}^p(L_2(U,H)) |^2 \right] ds \right)^{p/2} dt.
\]

(3.11)

For \( 0 \leq t_1 \leq t_2 \leq T \) and \( \Phi \in \Lambda^{p,2,p}_\mathcal{P}_\mathcal{T},S,\beta(L(U,H)) \), we also have

\[
(S_{*t_1}^{dW} \Phi - S_{*t_2}^{dW} \Phi)_s = 1_{[t_1,t_2]}(s)(S_{*t_1}^{dW} \Phi)_s + 1_{[t_2,T]}(s)S_{s-t_2}(S_{*t_1}^{dW} \Phi)_{t_2} \quad \forall s \in [0,T].
\]

(3.12)

Since

\[
\sup_{s \in [t_1,t_2]} |(S_{*t_1}^{dW} \Phi)_s|_H \leq |S_{*t_1}^{dW} (1_{[t_1,t_2]}(\cdot)\Phi)|_\infty \quad \text{\( \mathbb{P} \)-a.e.,}
\]

we obtain, by (3.8),

\[
\lim_{t_2 \to t_1 \to 0^+} \mathbb{E}\left[ \sup_{s \in [t_1,t_2]} |(S_{*t_1}^{dW} \Phi)_s|_H^p \right] \leq \lim_{t_2 \to t_1 \to 0^+} c'_{\beta,T,p} |1_{[t_1,t_2]}(\cdot)\Phi|_{p,2,S,\beta}^p = 0,
\]

(3.13)
where the latter limit can be seen by applying Lebesgue’s dominated convergence theorem three times, to the three integrals defining $| \cdot |_{p,2,S,\beta}$. Actually, since the linear map
\[
\tilde{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)) \to \tilde{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)), \Phi \to 1_{[t_1,t_2]}(\cdot)\Phi
\]
is bounded, uniformly in $t_1, t_2$, the limit (3.13) is uniform for $\Phi$ in compact subsets of $\tilde{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$ and $t_1, t_2 \in [0,T]$, $t_2 - t_1 \to 0^+$. Then, by (3.12) and (3.13), we finally obtain
\[
\lim_{|t_2-t_1|\to 0} |S_{\ast t_1} \Phi - S_{\ast t_2} \Phi|_{\mathcal{L}^p(\mathbb{S})} = 0 \quad (3.14)
\]
uniformly for $\Phi$ in compact subsets of $\tilde{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H))$. In particular, thanks to the uniform boundedness of $\{S_{\ast t_1} \Phi\}_{t_1 \in [0,T]}$ (see (3.8)), the map
\[
[0,T] \times \tilde{\Lambda}^{p,2,p}_{\mathcal{P}_T,S,\beta}(L(U,H)) \to \mathcal{L}^p(\mathbb{S}), (t, \Phi) \to S_{\ast t} \Phi \quad (3.15)
\]
is continuous.

### 3.1 Existence and uniqueness of mild solution

The following assumption will be standing for the remaining part of this manuscript. We recall that, if $E$ is a Banach space, then $\mathcal{B}_E$ denotes its Borel $\sigma$-algebra.

**Assumption 3.3.**

(i) $b : (\Omega_T \times \mathbb{S}, \mathcal{P}_T \otimes \mathcal{B}_E \to (H, \mathcal{B}_H)$ is measurable;

(ii) $\sigma : (\Omega_T \times \mathbb{S}, \mathcal{P}_T \otimes \mathcal{B}_E \to L(U,H)$ is strongly measurable, that is $\Omega_T \times \mathbb{S}, \mathcal{P}_T \otimes \mathcal{B}_E) \to H$, $((\omega,t), \mathbf{x}) \mapsto \sigma((\omega,t), \mathbf{x})$ is measurable, for all $\mathbf{u} \in U$;

(iii) (non-anticipativity condition) for all $((\omega, t), \mathbf{x}) \in \Omega_T \times \mathbb{S}$, $b((\omega, t), \mathbf{x}) = b((\omega, t), \mathbf{x}_{t\wedge})$ and $\sigma((\omega, t), \mathbf{x}) = \sigma((\omega, t), \mathbf{x}_{t\wedge})$;

(iv) there exists $g \in L^1((0,T), \mathbb{R})$ such that
\[
\begin{cases}
|b((\omega, t), \mathbf{x})|_H \leq g(t)(1 + |\mathbf{x}|_\infty) & \forall ((\omega, t), \mathbf{x}) \in \Omega_T \times \mathbb{S}, \\
|b((\omega, t), \mathbf{x}) - b((\omega, t), \mathbf{x}')|_H \leq g(t)|\mathbf{x} - \mathbf{x}'|_\infty & \forall (\omega, t) \in \Omega_T, \forall \mathbf{X}, \mathbf{X}' \in \mathbb{S}.
\end{cases}
\]

(v) there exist $M > 0$, $\gamma \in (0,1/2)$ such that
\[
\begin{cases}
|\mathbf{S}_t \sigma((\omega, s), \mathbf{x})|_{L^2(U,H)} \leq Mt^{-\gamma}(1 + |\mathbf{x}|_\infty) & \forall ((\omega, s), \mathbf{x}) \in \Omega_T \times \mathbb{S}, \forall t \in (0,T], \\
|\mathbb{S}_t \sigma((\omega, s), \mathbf{x}) - \mathbb{S}_t \sigma((\omega, s), \mathbf{x}')|_{L^2(U,H)} \leq Mt^{-\gamma}|\mathbf{x} - \mathbf{x}'|_\infty & \forall (\omega, s) \in \Omega_T, \forall t \in (0,T], \forall \mathbf{x}, \mathbf{x}' \in \mathbb{S}.
\end{cases}
\]

**Remark 3.4.** Assumption 3.3(iv) could be generalized to the form
\[
\begin{cases}
|\mathbf{S}_t b((\omega, s), \mathbf{x})|_H \leq t^{-\gamma}g(s)(1 + |\mathbf{x}|_\infty) & \forall ((\omega, s), \mathbf{x}) \in \Omega_T \times \mathbb{S}, \forall t \in (0,T] \\
|\mathbb{S}_t (b((\omega, s), \mathbf{x}) - b((\omega, s), \mathbf{x}'))|_H \leq t^{-\gamma}g(s)|\mathbf{x} - \mathbf{x}'|_\infty & \forall (\omega, s) \in \Omega_T, \forall t \in (0,T], \forall \mathbf{x}, \mathbf{x}' \in \mathbb{S},
\end{cases}
\]
with $g$ suitably integrable, and similarly for Assumption 3.3(v). The results obtained and the methods used hereafter can be adapted to cover these more general assumptions.
Definition 3.5 (Mild solution). Let $Y \in \mathcal{L}^0_{\mathcal{P}_T}(\mathbb{S})$ and $t \in [0,T)$. A function $X \in \mathcal{L}^0_{\mathcal{P}_T}(\mathbb{S})$ is a mild solution to (3.1) if, for all $t' \in [t,T]$,
\[
\mathbb{P}\left( \int_t^{t'} |S_{t-s}b(\cdot,s,X)|_H ds + \int_t^{t'} |S_{t-s}\sigma(\cdot,s,X)|_{L_2(U,H)}^2 ds < \infty \right) = 1,
\]
and
\[
\begin{align*}
\forall t' \in [0,t], \quad & X_{t'} = Y_{t'} \mathbb{P} \text{-a.e.}, \\
\forall t' \in (t,T), \quad & X_{t'} = S_{t'-t}Y_t + \int_t^{t'} S_{t'-s}b(\cdot,s,X) ds + \int_t^{t'} S_{t'-s}\sigma(\cdot,s,X) dW_s \mathbb{P} \text{-a.e.}.
\end{align*}
\]

Using a classical contraction argument, we are going to prove existence and uniqueness of mild solution in the space $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, when the initial datum $Y$ belongs to $\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, for $p$ large enough. This will let us apply the theory developed in Section 2.

For $t \in [0,T]$ and
\[
p > p^* := \frac{2}{1 - 2\gamma}, \quad \beta \in (1/p,1/2-\gamma),
\]
we define the following maps:
\[
\begin{align*}
id^S_t : \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) & \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), \quad Y \mapsto 1_{[0,t]}(\cdot)Y + 1_{(t,T)}(\cdot)S_{t-t}Y_t, \\
F_b : \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) & \to L^{p,1}_{\mathcal{P}_T}(H), \quad X \mapsto b(\cdot,\cdot),X, \\
F_\sigma : \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}) & \to L^{p,2p}_{\mathcal{P}_T}(L(U,H)), \quad X \mapsto \sigma(\cdot,\cdot),X, \\
S *_{t} : L^{p,1}_{\mathcal{P}_T}(H) & \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), \quad X \mapsto 1_{[t,T]}(\cdot) \int_t^{t'} S_{t'-s}X_s ds,
\end{align*}
\]
and we recall the map
\[
S *_{t}^dW : L^{p,2p}_{\mathcal{P}_T,S,\beta}(L(U,H)) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S}), \quad \Phi \mapsto S *_{t}^dW \Phi.
\]

Then $id^S_t$ is well-defined, due to (a) and (b) in (3.2), because we can write
\[
id^S_t(Y) = Y_{t\wedge} + 1_{(t,T]}(\cdot)(S_{t-t} - I)Y_t. \tag{3.17}
\]
As regarding $F_b$, by Assumption 3.3(i),(iii), and by (b) in (3.2), the map
\[
\Omega_T \to H, \quad (\omega,t) \mapsto b((\omega,t),X(\omega)) = b((\omega,t),X_{t\wedge}(\omega))
\]
is predictable. Moreover, by Assumption 3.3(iv), we have
\[
\int_0^T \left[ \mathbb{E}\left[ |b(\cdot,t,X_{t\wedge})|^p \right] \right]^{1/p} dt \leq \int_0^T g(t) \left( \mathbb{E}\left[ (1 + |X_{t\wedge}|^p) \right] \right)^{1/p} dt \leq |g|_{L^1((0,T),\mathcal{P})}(1 + |X|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})}),
\]
which shows that $F_b(X) \in L^{p,1}_{\mathcal{P}_T}(H)$. By Assumption 3.3(iv), we also have that $F_b$ is Lipschitz, with Lipschitz constant dominated by $|g|_{L^1((0,1),\mathcal{P})}$. Similarly as done for $F_b$, by using Assumption 3.3(ii), one can see that, for $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathbb{S})$, the map
\[
(\Omega_T,\mathcal{P}_T) \to L(U,H), \quad (\omega,t) \mapsto \sigma((\omega,t),X_{t\wedge}(\omega))
\]

is strongly measurable. Moreover, by Assumption 3.3(v), we have

\[
|F_\sigma(X)_{p,2,S,\beta}| = \left( \int_0^T \left( \int_0^t (t-s)^{\beta/2} \left[ \mathbb{E} \left\{ |S_{t-s}\sigma((\cdot,s),X_{s\wedge\cdot})| L_{L_2(U,H)} \right\} \right]^{2/p} ds \right)^{p/2} dt \right)^{1/p} \leq M \left( \int_0^T \left( \int_0^t u^{-(\beta+\gamma)/2} du \right)^{p/2} dt \right)^{1/p} \left( 1 + |X|_{L_{\mathcal{F}_T}(\mathbb{S})} \right)
\]

and the latter term is finite because \( \beta < 1/2 - \gamma \) and \( Y \in L_{\mathcal{F}_T}(\mathbb{S}) \). Then \( F_\sigma \) is well-defined. With similar computations, we have that \( F_\sigma \) is Lipschitz, with Lipschitz constant depending only on \( M, \beta, \gamma, p \). Regarding \( S * t \# \), if \( X \in L_{\mathcal{F}_T}^{p,1}(H) \), then \( X(\omega) \in L^1((0,T),H) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), hence it is easily checked that

\[
[0,T] \rightarrow H, \ t' \rightarrow 1_{[0,t]}(t') \int_0^{t'} S_{t'-s}X_s(\omega) ds
\]

is continuous, and then it belongs to \( \mathbb{S} \). Since \( \mathcal{F} \) is complete, we can assume that \( S * t X(\omega) \) is continuous for all \( \omega \), hence it is predictable, because it is \( \mathcal{F} \)-adapted. Since the trajectories are continuous, we also have the measurability of

\[
(\Omega_T, \mathcal{F}_T) \rightarrow C([0,T],H) \subset \mathbb{S}, \ (\omega,t') \rightarrow (S * t X)_{t',\omega}(\omega).
\]

Then, to show that \( S * t X \in L_{\mathcal{F}_T}^{p}(\mathbb{S}) \), it remains to verify the integrability condition. We have

\[
|S * t X|_{L_{\mathcal{F}_T}^{p}(\mathbb{S})} \leq M' \left( \mathbb{E} \left[ \left( \int_0^T |X_s| H ds \right)^p \right] \right)^{1/p} \leq M' \int_0^T \mathbb{E} \left[ |X_s|_{H}^p \right]^{1/p} ds = M'|X|_{p,1},
\]

where

\[
M' \text{ is any upper bound for } \sup_{t \in [0,T]} |S_t|_{L(H)}.
\]

The good definition of \( S * t \# \) was discussed above (observe that \( p > \max(2,1/\beta) \)).

We can then build the map

\[
\psi : L_{\mathcal{F}_T}^p(\mathbb{S}) \times L_{\mathcal{F}_T}^p(\mathbb{S}) \rightarrow L_{\mathcal{F}_T}^p(\mathbb{S}), \ (Y,X) \mapsto \text{id}_t^\p(Y) + S * t F_b(X) + S * t F_{\sigma}(X). \quad (3.18)
\]

In what follows, whenever we need to make explicit the dependence of \( \psi(Y,X) \) on the data \( t,S,b,\sigma \), we write \( \psi(Y,X,t,S,b,\sigma) \).

We first show that, for each \( Y \in L_{\mathcal{F}_T}^p(\mathbb{S}) \), \( \psi(Y,\cdot) \) has a unique fixed point \( X \). Such a fixed point is a mild solution to (3.1).

The advantage of introducing the setting above is that it permits to see \( \psi \) as a composition of maps that have different regularity and that can be considered individually when studying the regularity of the mild solution \( X^{t,Y} \) with respect to \( Y \) or the dependence of \( X^{t,Y} \) with respect to a perturbation of the data \( Y,t,S,b,\sigma \).

For \( \lambda > 0 \), we consider the following norm on \( L_{\mathcal{F}_T}^p(\mathbb{S}) \)

\[
|X|_{L_{\mathcal{F}_T}^p(\mathbb{S})}^{\lambda} := \left( \mathbb{E} \left[ \sup_{t \in [0,T]} e^{-\lambda pt} |X_t|^p \right] \right)^{1/p}, \quad \forall X \in L_{\mathcal{F}_T}^p(\mathbb{S}).
\]

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Then $|\cdot|_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}$ is equivalent to $|\cdot|_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}$.

We proceed to show that there exists $\lambda > 0$ such that $\psi$ is a parametric contraction. For $X, X' \in \mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})$, $\lambda > 0$, and $t' \in [0, T]$, we have

$$e^{-\lambda pt'}|(S \ast_t F_\sigma(X))_{t'} - (S \ast_t F_\sigma(X'))_{t'}|_H^p \leq (M')^p \left( \int_0^{t'} e^{-\lambda (t' - s)} |b((s, s), X) - b((s, s), X')|_H ds \right)^p \leq (M')^p \left( \int_0^{t'} e^{-\lambda (t' - s)} g(s) e^{-\lambda s} |X_{s\lambda} - X'_{s\lambda}|_{\infty} ds \right)^p \leq C_{\lambda, g, M'} \sup_{s \in [0, T]} \left\{ e^{-\lambda ps} |X_s - X'_{s\lambda}|_H^p \right\},$$

where $C_{\lambda, g, M'} := M' \sup_{t' \in [0, T]} \int_0^{t'} e^{-\lambda (t' - s)} g(t') dv$. We then obtain

$$|S \ast_t F_\sigma(X) - S \ast_t F_\sigma(X')|_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}_\lambda \leq C_{\lambda, g, M'} |X - X'|_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}_\lambda.$$

(3.19)

It is not difficult to see that $C_{\lambda, g, M'} \to 0$ as $\lambda \to \infty$.

Now, if $\Phi \in \Lambda^{p, 2, p}_{\mathcal{G}_T, \mathcal{S}, \beta}(L(U, H))$, then $e^{-\lambda \Phi} \in \Lambda^{p, 2, p}_{\mathcal{G}_T, \mathcal{S}, \beta}(L(U, H))$ for all $\lambda \geq 0$ and, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $e^{-\lambda pt'}(S \ast_t \Phi)_{t'} = ((e^{-\lambda S} \ast_t (e^{-\lambda \Phi}))_{t'}) \forall t' \in [0, T].$ (3.20)

For $X \in \mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})$, we have

$$\int_t^{t'} \mathbb{E} \left[ |e^{-\lambda (t' - s)} S_{t' - s} (e^{-\lambda F_\sigma(X))}_s|_H^2 \right] ds < \infty \quad \forall t' \in [t, T].$$

Then, for $X, X' \in \mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})$, $\lambda \geq 0$, and for all $t' \in [t, T]$, formula (3.7) provides

$$((e^{-\lambda S} \ast_t (e^{-\lambda F_\sigma(X)))_{t'}} - ((e^{-\lambda S} \ast_t (e^{-\lambda F_\sigma(X'))}_{t'} = c_{\beta} \int_t^{t'} (t' - s)^{p-1} \hat{Z}_s ds \quad \mathbb{P}$-a.e.,$$

where $\hat{Z}$ is an $H$-valued predictable process such that, for a.e. $t' \in [t, T]$,

$$\hat{Z}_{t'} = \int_t^{t'} (t' - s)^{-p} e^{-\lambda (t' - s)} S_{t' - s} (e^{-\lambda F_\sigma(X))} - e^{-\lambda F_\sigma(X')} ds \quad \mathbb{P}$-a.e..$$

By collecting the observations above, we can write, for $\lambda \geq 0$ and for all $t' \in [t, T]$, $e^{-\lambda pt'}|(S \ast_t F_\sigma(X))_{t'} - (S \ast_t F_\sigma(X'))_{t'}|_H^p \leq C_{\beta, T, p} \int_0^{T} \int_{\mathbb{R}^{p-1}} |\hat{Z}_{s\lambda}|_H^p ds$.

then, by applying [8, Lemma 7.7],

$$|S \ast_t F_\sigma(X) - S \ast_t F_\sigma(X')|_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}_\lambda \leq c_{\beta, T, p} e^{-\lambda F_\sigma(X)} - e^{-\lambda F_\sigma(X')} \mathfrak{p}_{\mathcal{L}^p_{\mathcal{G}_T}(\mathcal{S})}.$$
where $c'_{\beta,T,p}$ is a constant depending only on $\beta, T, p$. Now, by using Assumption 3.3(v), we have

$$|e^{-\lambda t} F_{\sigma}(X) - e^{-\lambda t} F_{\sigma}(X')|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda}^{p} \leq M^{p} \left( \int_{0}^{T} \left( \int_{0}^{t} v^{-(\beta+\gamma)/2} e^{-\lambda v} dv \right)^{p/2} dt \right) |X - X'|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda}.$$  

We finally obtain

$$|S_{\ast t}^{dW} F_{\sigma}(X) - S_{\ast t}^{dW} F_{\sigma}(X')|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda} \leq c''_{\beta,\gamma,T,p,M,\lambda} |X - X'|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda},$$  \hspace{1cm} (3.21)

where $c''_{\beta,\gamma,T,p,M,\lambda}$ is a constant depending only on $\beta, \gamma, T, p, M, \lambda$, and is such that

$$\lim_{\lambda\to\infty} c''_{\beta,\gamma,T,p,M,\lambda} = 0.$$  

By (3.19) and (3.21), we have, for all $Y, X, Y', X'$,

$$|\psi(Y, X) - \psi(Y', X')|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda} \leq M|Y - Y'|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda} + C'_{\lambda,g,\gamma,M',\beta,T,p,M} |X - X'|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda},$$  \hspace{1cm} (3.22)

where $C'_{\lambda,g,\gamma,M',\beta,T,p,M}$ is a constant depending only on $\lambda, g, \gamma, M', \beta, T, p, M$, such that

$$\lim_{\lambda\to\infty} C'_{\lambda,g,\gamma,M',\beta,T,p,M} = 0.$$  \hspace{1cm} (3.23)

**Theorem 3.6.** Let Assumption 3.3 hold and let $t \in [0,T]$, $p > p^*$. Then there exists a unique mild solution $X^{t,Y} \in L^{p}_{\mathcal{P}_{T}}(S)$ to SDE (3.1). Moreover, there exists a constant $C$, depending only on $g, \gamma, M, M', T, p$, such that

$$|X^{t,Y} - X^{t,Y'}|_{L^{p}_{\mathcal{P}_{T}}(S)} \leq C|Y - Y'|_{L^{p}_{\mathcal{P}_{T}}(S)} \hspace{1cm} \forall Y, Y' \in L^{p}_{\mathcal{P}_{T}}(S).$$

**Proof.** Let us fix any $\beta \in (1/p, 1/2 - \gamma)$ and let $\psi$ be defined by (3.18). It is clear that any fixed point of $\psi(\cdot, \cdot)$ is a mild solution to SDE (3.1). Then, it is sufficient to apply Lemma 2.8 to $\psi$, taking into account (3.22) and (3.23), and recalling the equivalence of the norms $\| \cdot \|_{L^{p}_{\mathcal{P}_{T}}(S)}$ and $\| \cdot \|_{L^{p}_{\mathcal{P}_{T}}(S),\lambda}$.  

**Remark 3.1.** Since, for $p^* < p < q$, we have $L^{q}_{\mathcal{P}_{T}}(S) \subset L^{p}_{\mathcal{P}_{T}}(S)$, then, if $Z \in L^{q}_{\mathcal{P}_{T}}(S)$, the associated mild solution $X^{t,Z} \in L^{q}_{\mathcal{P}_{T}}(S)$ is also a mild solution in $L^{p}_{\mathcal{P}_{T}}(S)$ and, by uniqueness, it is the solution in that space. Hence the solution does not depend on the specific $p > p^*$ chosen.

### 3.2 Gâteaux differentiability with respect to the initial datum

We now study the differentiability of the mild solution $X^{t,Y}$ with respect to the initial datum $Y$.

**Assumption 3.7.** Let $b, \sigma, g, \gamma$ be as in Assumption 3.3. Let $n \in \mathbb{N}$, $n \geq 1$.
(i) For all \((\omega, t) \in \Omega_T\) and \(u \in U\), \(b((\omega, t), \cdot) \in \mathcal{G}^n(\mathbb{S}, H)\), \(\sigma((\omega, t), \cdot)u \in \mathcal{G}^n(\mathbb{S}, H)\).

(ii) There exists \(M''\) and \(c := \{c_m\}_{m \in \mathcal{M}} \in \ell^2(\mathcal{M})\) such that

\[
\sup_{j=1, \ldots, n} \sup_{x \in \Omega} |\partial_{y_1 \ldots y_j} b((\omega, s), x)|_H \leq M'' g(s), \quad (3.24)
\]

\[
\sup_{j=1, \ldots, n} \sup_{x \in \Omega} |S_j \partial_{y_1 \ldots y_j} (\sigma((\omega, s), x)e'_m)|_H \leq M'' t^{-\gamma} c_m, \quad (3.25)
\]

for all \(s \in [0, T], t \in (0, T], m \in \mathcal{M}\).

In accordance with Assumption 3.7(i), by writing \(\partial_{y_1 \ldots y_j} (\sigma((\omega, s), x)u)\), we mean the Gâteaux derivative of the map \(x \rightarrow \sigma((\omega, s), x).u\), for fixed \(u \in U\).

**Lemma 3.8.** Suppose that Assumption 3.3 and Assumption 3.7 are satisfied. Let \(p > p^*, \beta \in (1/p, 1/2 - \gamma)\). Then, for \(j = 1, \ldots, n\),

\[
F_b \in \mathcal{G}^j(\mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S}), \mathcal{L}^{p,1}_{\mathcal{F}_T}(H), \mathcal{L}^{p,1}_{\mathcal{F}_T}(\mathbb{S})), \quad F_\sigma \in \mathcal{G}^j(\mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S}), \mathcal{L}^{2,p}_{\mathcal{F}_T}, \mathcal{L}^{p,1}_{\mathcal{F}_T}(L(U, H)), \mathcal{L}^{p,1}_{\mathcal{F}_T}(\mathbb{S})).
\]

and, for \(X \in \mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S}), Y_1, \ldots, Y_j \in \mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S}), u \in U, \mathbb{P} \otimes m\text{-a.e. } (\omega, t) \in \Omega_T,\)

\[
\begin{aligned}
\partial_{Y_1 \ldots Y_j} F_b(X)(\omega, t) &= \partial_{Y_1(\omega) \ldots Y_j(\omega)} b((\omega, t), X(\omega)) \\
\partial_{Y_1 \ldots Y_j} F_\sigma(X)(\omega, t)u &= \partial_{Y_1(\omega) \ldots Y_j(\omega)} (\sigma((\omega, t), X(\omega)))u.
\end{aligned}
\]

Moreover,

\[
\sup_{j=1, \ldots, n} \sup_{X \in \mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S})} \left(|\partial_{Y_1 \ldots Y_j} F_b(X)|_{L^{p,1}_{\mathcal{F}_T}(H)} + |\partial_{Y_1 \ldots Y_j} F_\sigma(X)|_{p,2,p,2,S,\beta}\right) \leq M''',
\]

where \(M'''\) depends only on \(T, p, \beta, \gamma, |g|_{L^1((0, T), \mathbb{R})}, M'', |c|_{\ell^2(\mathcal{M})}\).

**Proof.** We prove the lemma by induction on \(n\).

**Case \(n = 1\).** Let \(X, Y \in \mathcal{L}^p_{\mathcal{F}_T}(\mathbb{S})\). First notice that the function

\((\Omega_T, \mathcal{F}_T) \rightarrow H, (\omega, t) \rightarrow \partial_Y b((\omega, t), X(\omega))\)

is measurable. Let \(\varepsilon \in \mathbb{R} \setminus \{0\}\). Since \(b((\omega, t), \cdot) \in \mathcal{G}^1(\mathbb{S}, H)\) for all \((\omega, t) \in \Omega_T\), we can write

\[
\Delta_X F_b(X)(\omega, t) := \varepsilon^{-1}(F_b(X + \varepsilon Y)(\omega, t) - F_b(X)(\omega, t))
\]

\[
= \varepsilon^{-1} (b((\omega, t), X(\omega) + \varepsilon Y(\omega)) - b((\omega, t), X(\omega)))
\]

\[
= \int_0^1 \partial_Y b((\omega, t), X(\omega) + \varepsilon \theta Y(\omega))d\theta \quad \mathbb{P} \otimes m\text{-a.e. } (\omega, t) \in \Omega_T.
\]

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By (3.24), we also have
\[
|\partial Y_{(\omega)} b((\omega, t), X(\omega) + \varepsilon Y(\omega))|_{H} \leq M^{\gamma} g(t)|Y(\omega)|_{\infty} \quad \forall (\omega, t) \in \Omega_{T}, \forall \varepsilon \in \mathbb{R}. \tag{3.28}
\]

By (3.27) and (3.28), we can apply Lebesgue’s dominated convergence theorem and obtain
\[
\lim_{\varepsilon \to 0} \int_{0}^{T} \left( \mathbb{E} \left[ |\Delta_{\varepsilon} Y F_{b}(X(\cdot, t) - \partial Y b((\cdot, t), X)|_{L}^{p} \right] \right)^{1/p} dt = 0.
\]
This proves that \( F_{b} \) has directional derivative at \( X \) for the increment \( Y \) and that
\[
\partial Y F_{b}(X(\omega, t)) = \partial Y_{(\omega)} b((\omega, t), X(\omega)) \quad \mathbb{P} \otimes m\text{-a.e.} \ (\omega, t) \in \Omega_{T}. \tag{3.29}
\]

We now show that \( \partial Y F_{b}(X) \) is continuous in \((X, Y) \in \mathcal{L}^{p}_{\mathcal{P}_{T}}(\mathbb{S})\). Notice that, by (3.24), the linear map \( \mathcal{L}^{p}_{\mathcal{P}_{T}}(\mathbb{S}) \to L^{p,1}_{\mathcal{P}_{T}}(H), Y \to \partial Y F_{b}(X) \), is bounded, uniformly in \( X \). Then it is sufficient to verify the continuity of \( \partial Y F_{b}(X) \) in \( X \), for fixed \( Y \). Let \( X_{k} \to X \) in \( \mathcal{L}^{p}_{\mathcal{P}_{T}}(\mathbb{S}) \). By (3.24), (3.29), and Lebesgue’s dominated convergence theorem, we have
\[
\lim_{k \to \infty} \partial Y F_{b}(X_{k}) = \partial Y F_{b}(X) \text{ in } L^{p,1}_{\mathcal{P}_{T}}(H).
\]
This concludes the proof that \( F_{b} \in \mathcal{G}^{1}(\mathcal{L}^{p}_{\mathcal{P}_{T}}(\mathbb{S}), L^{p,1}_{\mathcal{P}_{T}}(H)) \) and that the differential is uniformly bounded.

Similarly, as regarding \( F_{\sigma} \), we have that, for all \( u \in U \), the function
\[
(\Omega_{T}, \mathcal{P}_{T}) \to H, (\omega, t) \mapsto \partial Y_{(\omega)}(\sigma(t, X(\omega))u)
\]
is measurable, and
\[
\Delta_{\varepsilon} Y(\sigma(X)u)(\omega, t) := \varepsilon^{-1}((\sigma(X + \varepsilon Y(\omega))u(\omega, t) - (\sigma(X)u(\omega, t)))
\]
\[
= \varepsilon^{-1}(\sigma((\omega, t), X(\omega) + \varepsilon Y(\omega))u - \sigma((\omega, t), X(\omega))u)
\]
\[
= \int_{0}^{1} \partial Y_{(\omega)}(\sigma((\omega, t), X(\omega) + \varepsilon Y(\omega))u) d\theta \quad \mathbb{P} \otimes m\text{-a.e.} \ (\omega, t) \in \Omega_{T}. \tag{3.30}
\]

By (3.25), for all \( 0 \leq s < t \leq T, \omega \in \Omega, \varepsilon \in \mathbb{R}, m \in \mathcal{M}, \)
\[
|S_{t-s} \Delta_{\varepsilon} Y_{(\omega)}(\sigma((\omega, s), X(\omega) + \varepsilon Y(\omega))e_{m}^{\prime})|_{H} \leq M^{\gamma}(t-s)^{-\gamma} c_{m}|Y(\omega)|_{\infty}. \tag{3.31}
\]

By repeatedly applying Lebesgue’s dominated convergence theorem, we have that
\[
\int_{0}^{T} \left( \int_{0}^{t} (t-s)^{-2\beta} \mathbb{E} \left[ \sum_{m \in \mathcal{M}} |S_{t-s} \left( \Delta_{\varepsilon} Y F_{\sigma}(X)\cdot(s), e_{m}^{\prime} - \partial Y(\sigma((\cdot, s), X), e_{m}^{\prime}) \right)|_{H}^{2} \right]^{p/2} \right)^{2} dt ds^{p/2} \rightarrow 0 \quad \text{as} \ \varepsilon \to 0.
\]
This proves that \( F_{\sigma} \) has directional derivative at \( X \) for the increment \( Y \) and, taking into account the separability of \( U \), that
\[
\partial Y F_{\sigma}(X)(\omega, t) = \partial Y_{(\omega)}(\sigma((\omega, t), X(\omega))u) \quad \mathbb{P} \otimes m\text{-a.e.} \ (\omega, t) \in \Omega_{T}. \tag{3.32}
\]
fixed $Y$. Let $X_k \to X$ in $\mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$. By (3.25), (3.32), and Lebesgue’s dominated convergence theorem, we have

$$\lim_{k \to \infty} \partial_Y F_\sigma(X_k) = \partial_Y F_\sigma(X) \text{ in } \mathcal{L}_{\mathcal{F}_T}^{p,2,p}(L(U,H)).$$

This shows that $F_\sigma \in \mathcal{G}^1(\mathcal{L}_{\mathcal{F}_T}^p(\Sigma), \mathcal{L}_{\mathcal{F}_T}^{p,2,p}(L(U,H)))$ and that the differential is uniformly bounded.

**Case $n > 1$** Let $X \in \mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$ and $Y_1, \ldots, Y_n \in \mathcal{L}_{\mathcal{F}_T}^{n,p}(\Sigma)$. By inductive hypothesis, we can assume that $\partial_{Y_1 \ldots Y_{n-1}} F_b(X) \in \mathcal{L}_{\mathcal{F}_T}^{p-1}(H)$ exists, jointly continuous in $X \in \mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$ and $Y_1, \ldots, Y_{n-1} \in \mathcal{L}_{\mathcal{F}_T}^{(n-1,p)}(H)$, and that

$$\partial_{Y_1 \ldots Y_{n-1}} F_b(X)(\omega, t) = \partial_{Y_1(\omega) \ldots Y_{n-1}(\omega)} b((\omega, t), X(\omega)) \quad \mathbb{P} \otimes \text{m-a.e. } (\omega, t) \in \Omega_T.$$

The argument goes like the case $n = 1$. Let $\varepsilon \in \mathbb{R} \setminus \{0\}$. Since $b((\omega, t), \cdot) \in \mathcal{G}^n(\mathcal{S}, H)$ for $(\omega, t) \in \Omega_T$, we can write, for $\mathbb{P} \otimes \text{m-a.e. } (\omega, t) \in \Omega_T$,

$$\Delta_{\varepsilon} Y_n \partial_{Y_1 \ldots Y_{n-1}} F_b(X)(\omega, t) = \varepsilon^{-1} \left( \partial_{Y_1 \ldots Y_{n-1}} F_b(X + \varepsilon Y_n(\omega) t) - \partial_{Y_1 \ldots Y_{n-1}} F_b(X)(\omega, t) \right)$$

$$= \varepsilon^{-1} \left( \partial_{Y_1(\omega) \ldots Y_{n-1}(\omega)} b((\omega, t), X(\omega) + \varepsilon Y_n(\omega)) \right)$$

$$= \int_0^1 \partial_{Y_1(\omega) \ldots Y_{n-1}(\omega)} b((\omega, t), X(\omega) + \varepsilon \theta Y_n(\omega)) d\theta.$$ 

By (3.24) we have

$$| \partial_{Y_1(\omega) \ldots Y_n(\omega)} b((\omega, t), X(\omega) + \varepsilon Y_n(\omega))|_H \leq M''(t) \prod_{j=1}^n |Y_j(\omega)|_\infty \quad \forall (\omega, t) \in \Omega_T, \ \forall \varepsilon \in \mathbb{R}.$$

Since $Y_j \in \mathcal{L}_{\mathcal{F}_T}^{n,p}(H)$, by the generalized Hölder inequality $\prod_{j=1}^n |Y_j|_\infty \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$. Then we can apply Lebesgue’s dominated convergence theorem twice to obtain

$$\lim_{\varepsilon \to 0} \int_0^T \mathbb{E} \left[ | \Delta_{\varepsilon} Y_n \partial_{Y_1 \ldots Y_{n-1}} F_b(X)(\cdot, t) - \partial_{Y_1 \ldots Y_n} b((\cdot, t), X)|_H^p \right]^{1/p} dt = 0.$$

This proves that $\partial_{Y_1 \ldots Y_{n-1}} F_b$ has directional derivative at $X$ for the increment $Y_n$ and that

$$\partial_{Y_1 \ldots Y_{n-1} Y_n} F_b(X)(\omega, t) = \partial_{Y_1(\omega) \ldots Y_n(\omega)} b((\omega, t), X(\omega)) \quad \mathbb{P} \otimes \text{m-a.e. } (\omega, t) \in \Omega_T. \quad (3.33)$$

The continuity of $\partial_{Y_1 \ldots Y_{n-1}} F_b(X) \in \mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$, $Y_1, \ldots, Y_n \in \mathcal{L}_{\mathcal{F}_T}^p(H)$, is proved similarly as for the case $n = 1$, again by invoking the generalized Hölder inequality. This concludes the proof that $F_b \in \mathcal{G}^n(\mathcal{L}_{\mathcal{F}_T}^p(\Sigma), \mathcal{L}_{\mathcal{F}_T}^{p,p}(H); \mathcal{L}_{\mathcal{F}_T}^{p,p}(H))$. The uniform boundedness of the differentials is obtained by (3.24), (3.33), and the generalized Hölder inequality.

Finally, as regarding $F_\sigma$, let again $X \in \mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$ and $Y_1, \ldots, Y_n \in \mathcal{L}_{\mathcal{F}_T}^{n,p}(\Sigma)$. By inductive hypothesis, we can assume that $\partial_{Y_1 \ldots Y_{n-1}} F_\sigma(X) \in \mathcal{L}_{\mathcal{F}_T}^{p,2,p}(L(U,H))$ exists, that it is continuous in $X \in \mathcal{L}_{\mathcal{F}_T}^p(\Sigma)$, $Y_1, \ldots, Y_{n-1} \in \mathcal{L}_{\mathcal{F}_T}^{(n-1,p)}(\Sigma)$, and that, for all $u \in U$,

$$\partial_{Y_1 \ldots Y_{n-1}} F_\sigma(X)(\omega, t) u = \partial_{Y_1(\omega) \ldots Y_{n-1}(\omega)} \sigma((\omega, t), X(\omega)) u \quad \mathbb{P} \otimes \text{m-a.e. } (\omega, t) \in \Omega_T.$$
For $\varepsilon \in \mathbb{R} \setminus \{0\}$, by strongly continuous Gâteaux differentiability of
\[ x \rightarrow \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1}(\sigma(t,x)u), \]
we can write,
\[
\Delta \epsilon \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X)(\omega,t)u := \varepsilon^{-1} \left( \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X + \epsilon Y_n)(\omega,t)u - \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X)(\omega,t)u \right)
\]
\[ = \varepsilon^{-1} \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1}(\sigma((\omega,t),X(\omega) + \epsilon Y_n(\omega))u) - \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1}(\sigma((\omega,t),X(\omega))u)
\]
\[ = \int_0^1 \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon}(\sigma((\omega,t),X(\omega) + \epsilon \theta Y_n(\omega))u) d\theta. \]

By (3.25) we have, for all $\omega \in \Omega$, $\epsilon \in \mathbb{R}$, $0 \leq s < t \leq T$, $m \in \mathcal{M}$,
\[
|S_{t-s} \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon}(\sigma((\omega,s),X(\omega) + \epsilon Y_n(\omega))e_m')|_H \leq M(t-s)^{-\gamma} c_m \prod_{j=1}^n |Y_j(\omega)|_{\infty}. \]

By the generalized Hölder inequality and by Lebesgue’s dominated convergence theorem, we conclude
\[
\lim_{\varepsilon \to 0} \int_0^T \left( \int_0^t (t-s)^{-2\beta} \left( \mathbb{E} \left[ \sum_{m \in \mathcal{M}} |S_{t-s} \left( \Delta \epsilon \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X)(\omega,s)e_m'ight. \right. \\
- \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon}(\sigma((),X)e_m')|_H \right) \left. \left. \right|_H \right]^{2/p} \right)^{2/p} ds \right) dt = 0. \tag{3.34}
\]

Then $\partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}$ has directional derivative at $X$ for the increment $Y_n$, given by, for all $u \in U$,
\[
\partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X)(\omega,t)u = \partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon}(\sigma((\omega,t),X(\omega))u) \quad \mathbb{P} \otimes m\text{-a.e. } (\omega,t) \in \Omega_T.
\]

The continuity of $\partial_{\gamma(\omega)\ldots\gamma(\omega)}^{\varepsilon-1} F_{\sigma}(X)$ with respect to $X \in \mathcal{L}_p(\mathcal{S})$, $Y_1, \ldots, Y_n \in \mathcal{L}_p(\mathcal{H})$, is proved as for the case $n = 1$. Then $F_{\sigma} \in \mathcal{G}^n(\mathcal{L}_p(\mathcal{S}), \mathcal{L}_p(\mathcal{H}))$. The uniform boundedness of the differentials is obtained by (3.25), (3.34), and the generalized Hölder inequality. 

Due to the fact that $X^{t,Y}$ is the fixed point of $\psi(Y, \cdot)$ and due to the structure of $\psi$, the previous lemma permits to easily obtain the following

**Theorem 3.9.** Suppose that Assumption 3.7 is satisfied. Let $t \in [0,T]$, $p > p^*$, $p \geq n$. Then the map
\[
\mathcal{L}_p^n(\mathcal{S}) \to \mathcal{L}_p^n(\mathcal{S}), \ Y \to X^{t,Y} \tag{3.35}
\]
belongs to $\mathcal{G}(\mathcal{L}_p^n(\mathcal{S}), \mathcal{L}_p^n(\mathcal{S}))$ and the Gâteaux differentials up to order $n$ are uniformly bounded by a constant depending only on $T, p, \gamma, g, M, M', M''$, $|c|_{\mathcal{E}(\mathcal{M})}$. 

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Proof. Let $\beta \in (1/p, 1/2 - \gamma)$. We have $p^k > p^*$ and $\beta \in (1/p^k, 1/2 - \gamma)$ for all $k = 1, \ldots, n$. Then, for $k = 1, \ldots, n$, the map

$$\psi_k : \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \to \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}), (Y, X) \mapsto \text{id}_t^S(Y) + S \ast_t F_b(X) + S \ast_t F_\sigma(X)$$

is well-defined, where we have implicitly chosen the space $L^{p^k-1}_{\mathcal{T}_T}(H)$ as codomain of $F_b$ and $\overline{N}^{p^k,2,p^k}_{\mathcal{T}_T,\beta}(L(U,H))$ as codomain of $F_\sigma$. Since the functions

$$\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \to \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})$$

$$S \ast_t # : L^{p^k-1}_{\mathcal{T}_T}(H) \to \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})$$

$$S \ast_t # : \overline{N}^{p^k,2,p^k}_{\mathcal{T}_T,\beta}(L(U,H)) \to \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})$$

are linear and continuous, with an upper bound for the operator norms depending only on $\beta, M', T, p$, we have, by applying Lemma 3.8, for $k, j = 1, \ldots, n$,

$$\psi_k \in C^1(\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}), \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}); \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})),$$

with differentials bounded by a constant depending only on $g, \gamma, M, M', M'', |\cdot|_{C^1(\mathcal{U})}, T$, on $p^k$ (hence on $p$), and on $\beta$, which depends on $p, \gamma$. In particular, since $np^k \leq p^{k+1}$, we have, for the restrictions $\psi_{k|\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})}$ of $\psi_k$ to $\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})$,

$$\left\{ \begin{array}{l}
\psi_{k|\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})} \in C^1(\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}), \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})) \\
\psi_{k|\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S})} \in C^0(\mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}), \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}); \mathcal{L}^{p^k}_{\mathcal{T}_T}(\mathbb{S}) \times \mathcal{L}^{p^k+1}_{\mathcal{T}_T}(\mathbb{S}))
\end{array} \right.$$
The discussion above, together with the smooth dependence of \( h_k \) on the first variable, shows that Assumption 2.11 is verified. We can then apply Theorem 2.12, which provides

\[
(3.35) = \mathcal{L}_P^p(\mathbb{S}) \to \mathcal{L}_P^p(\mathbb{S}), \quad Y \mapsto X^{t,y}, \quad \mathcal{G}(n)(\mathcal{L}_P^p(\mathbb{S}), \mathcal{L}_P^p(\mathbb{S})).
\]

Finally, by applying Corollary 2.13, we obtain the uniform boundedness of the Gâteaux differentials up to order \( n \) of (3.35), with a bound that depends only on \( T, \gamma, \mu, M, M', M'' \), \(|c|^{2(\gamma, l)} \).

**Remark 3.10.** As said in the introduction, we obtain the Gâteaux differentiability of \( x \to X^{t,x} \) by studying the parametric contraction providing \( X^{t,x} \) as its unique fixed point, similarly as done in [9] for the non-path-dependent case. A different approach consists in studying directly the variations \( \lim_{h \to 0} X^{t,x+h_{t,x}} - X^{t,x} \), showing that the limit exists (under suitable smooth assumptions on the coefficients) and is continuous with respect to \( v \), for fixed \( t, x \). This would provide the existence of the Gâteaux differential \( dX^{t,x} \). Usually, in this way one shows also that \( \partial X^{t,x}.v \) solves an SDE. By using this SDE, one could go further and prove that the second order derivative \( \partial^2 X^{t,x}(v, w) \) exists, and that it is continuous in \( v, w \), for fixed \( t, x \). This would provide the second order Gâteaux differentiability of \( x \to X^{t,x} \). In this way, it is possible also to study the continuity of the Gâteaux differentials, by considering the SDEs solved by the directional derivatives, and to obtain Fréchet differentiability (under suitable assumptions on the coefficients, e.g. uniformly continuous Fréchet differentiability). By doing so, first- and second-order Fréchet differentiability are proved in [15]. But if one wants to use these methods to obtain derivatives of a generic order \( n \geq 3 \), then a recursive formula providing the SDE solved by the \((n-1)\)th-order derivatives is needed, hence we fall back to a statement like Theorem 2.12. One could also try to prove the Fréchet differentiability of \( x \to X^{t,x} \) by studying directly the Fréchet differentiability of the parametric contractions providing the mild solution \( X^{t,x} \). This is the approach followed in [13, Theorem 3.9], for orders \( n = 1, 2 \). Nevertheless, we notice that the proof of [13, Theorem 3.8], on which [13, Theorem 3.9] relies, contains some inaccuracy: it is not clear why the term \( |\eta(s)|^2_{H}/|\eta|_{\tilde{H}_2}^2 \) is bounded by 1, uniformly in \((\omega, s)\), when \( \eta \) is only supposed to be a process such that \( |\eta|_{\tilde{H}_2}^2 := \sup_{s \in [0,t]} \mathbb{E}[|\eta(s)|^2_H] < \infty \).

Let \( n = 2 \) and let \( h_1 \) as in the proof of Theorem 3.9. By continuity and linearity of \( id_1^S \), \( S \ast_t \#, S \ast_t dW \#, \) and by recalling Lemma 3.8, we have, for \( Y, Y_1, Y_2 \in \mathcal{L}_P^p(\mathbb{S}) \) (the space of the first variable of \( h_1 \)), \( X, X_1, X_2 \in \mathcal{L}_P^p(\mathbb{S}) \) (the space of the second variable of \( h_1 \)),

\[
\begin{align*}
\partial Y_1 h_1(Y, X) &= id_1^S(Y_1) \\
\partial X_1 h_1(Y, X) &= S \ast_t \partial X_1 F_b(X) + S \ast_t \partial X_1 F_\sigma(X) \\
\partial^2 Y_1 Y_2 h_1(Y, X) &= \partial^2_{Y_1 Y_2} h_1(Y, X) = 0 \\
\partial^2_{X_1 X_2} h_1(Y, X) &= S \ast_t \partial^2_{X_1 X_2} F_b(X) + S \ast_t \partial^2_{X_1 X_2} F_\sigma(X).
\end{align*}
\]

Then, by Theorem 2.12, we have

\[
\partial Y_1 X^{t,y} = id_1^S(Y_1) + S \ast_t \partial Y_1 X^{t,y} F_b(X^{t,y}) + S \ast_t \partial Y_1 X^{t,y} F_\sigma(X^{t,y}) \quad (3.36a)
\]
\[ \partial_{Y_1 Y_2}^2 X_{t,Y} = S \ast \partial_{Y_1 Y_2}^2 X_{t,Y} F_b(X) + S \ast \partial_{Y_1 Y_2}^2 X_{t,Y} F_{\sigma}(X) \]

where the equality (3.36a) holds in the space \( \mathcal{L}^p(\mathbb{S}) \) and the equality (3.36b) holds in the space \( \mathcal{L}^p(\mathbb{S}) \). Formulæ (3.36a) and (3.36b) generalize to the present setting the well-known SDEs for the first- and second-order derivatives with respect to the initial datum of mild solutions of non-path-dependent SDEs \((10, \text{Theorem 9.8 and Theorem 9.9})\).

**Remark 3.11.** Suppose that \( \mathbb{S} = \mathbb{D} \), where \( \mathbb{D} \) is the space of right-continuous left-limited functions \([0, T] \to H\). Notice that \( \mathbb{D} \) satisfies all the properties required at p. 17. Then our setting applies and (3.36a)-(3.36b) provide equations for the first- and second-order directional derivatives of \( X_{t,Y} \) with respect to vectors belonging to \( \mathcal{L}^p(\mathbb{D}) \). In particular, if \( \varphi : \mathbb{D} \to \mathbb{R} \) is a suitably regular functional, then the so-called “vertical derivatives” in the sense of Dupire of \( F(t, \mathbf{x}) := \mathbb{E}[\varphi(X_{t,Y})] \), used in the finite dimensional Itô calculus developed by \([3, 4, 5, 11]\) to show that \( F \) solves a path-dependent Kolmogorov equation associated to \( X \), can be classically obtained by the chain rule starting from the Gâteaux derivatives \( \partial_{Y_1} X_{t,Y}, \partial_{Y_2} X_{t,Y} \), where \( Y_1, Y_2 \in H \) and \( Y_1 := 1_{[t,T]}(\cdot) y_1, Y_2 := 1_{[t,T]}(\cdot) y_2 \).

### 3.3 Perturbation of path-dependent SDEs

In this section we study the stability of the mild solution \( X_{t,Y} \) and of its Gâteaux derivatives with respect to perturbations of the data \( t, Y, S, b, \sigma \).

Let us fix sequences \( t := \{ t_j \}_{j \in \mathbb{N}} \subset [0, T], \{ S_j \}_{j \in \mathbb{N}} \subset L(H), \{ b_j \}_{j \in \mathbb{N}}, \{ \sigma_j \}_{j \in \mathbb{N}} \), satisfying the following assumption.

**Assumption 3.12.** Let \( b, \sigma, g, \gamma, M \), be as in Assumption 3.3. Assume that

1. \( \{ t_j \}_{j \in \mathbb{N}} \) is a sequence converging to \( t \) in \([0, T]\);
2. for all \( j \in \mathbb{N} \), \( b_j : (\Omega_T \times \mathbb{S}, \mathcal{P}_T \otimes \mathcal{B}_\mathbb{S}) \to (H, \mathcal{B}_H) \) is measurable;
3. for all \( j \in \mathbb{N} \), \( \sigma_j : (\Omega_T \times \mathbb{S}, \mathcal{P}_T \otimes \mathcal{B}_\mathbb{S}) \to L(U, H) \) is strongly measurable;
4. for all \( j \in \mathbb{N} \) and all \((\omega, t), \mathbf{x}) \in \Omega_T \times \mathbb{S}, b_j((\omega, t), \mathbf{x}) = b_j((\omega, t), \mathbf{x}_{\Omega}), \sigma_j((\omega, t), \mathbf{x}) = \sigma_j((\omega, t), \mathbf{x}_{\Omega});\)
5. for all \( j \in \mathbb{N} \),
   \[
   \begin{align*}
   |b_j((\omega, t), \mathbf{x})|_H & \leq g(t)(1 + |\mathbf{x}|_\infty) \quad \forall ((\omega, t), \mathbf{x}) \in \Omega_T \times \mathbb{S}, \\
   |b_j((\omega, t), \mathbf{x}) - b_j((\omega, t), \mathbf{x}')|_H & \leq g(t)|\mathbf{x} - \mathbf{x}'|_\infty \quad \forall (\omega, t) \in \Omega_T, \forall \mathbf{x}, \mathbf{x}' \in \mathbb{S};
   \end{align*}
   \]
6. for all \( j \in \mathbb{N} \),
   \[
   \begin{align*}
   |(S_j)_{t}(\sigma_j((\omega, s), \mathbf{x}))|_{L_2(U,H)} & \leq M t^{-\frac{\gamma}{2}}(1 + |\mathbf{x}|_\infty) \quad \forall ((\omega, s), \mathbf{x}) \in \Omega_T \times \mathbb{S}, \forall t \in (0, T], \\
   |(S_j)_{t}(\sigma_j((\omega, s), \mathbf{x}) - (S_j)_{t}(\sigma_j((\omega, s), \mathbf{x}'))|_{L_2(U,H)} & \leq M t^{-\frac{\gamma}{2}}|\mathbf{x} - \mathbf{x}'|_\infty \quad \forall (\omega, s) \in \Omega_T, \forall \mathbf{x}, \mathbf{x}' \in \mathbb{S}, \forall t \in (0, T];
   \end{align*}
   \]
(vii) for all \( t \in [0, T] \), \( (S_j)_t \in \mathbb{N} \) converges strongly to \( S_t \), that is
\[
\lim_{j \to \infty} (S_j)_t x = S_t x \quad \forall x \in H;
\]

(viii) the following convergences hold true:
\[
\begin{align*}
\lim_{j \to \infty} |b((\omega, t), x) - b_j((\omega, t), x)|_H &= 0 \quad \forall (\omega, t) \in \Omega_T, \forall x \in \mathbb{S} \\
\lim_{j \to \infty} |S_t \sigma((\omega, s), x) - (S_j)_t \sigma_j((\omega, s), x)|_{L_d(U, H)} &= 0 \quad \forall (\omega, s) \in \Omega_T, \forall t \in (0, T], \forall x \in \mathbb{S}.
\end{align*}
\]

Under Assumption 3.12, for \( p > p^* \) and \( \beta \in (1/p, 1/2 - \gamma) \), we define \( \text{id}_{t_j}^S, F_{b_j}, F_{\sigma_j}, S_j \ast t_j, \#, S_j \ast_{t_j}^d, \#_j, \), similarly as done for \( \text{id}_t^S, F_{b,}, F_{\sigma}, S \ast_t, \#_t, \#_t^d, \), \( \psi_j \), that is
\[
\begin{align*}
\text{id}_{t_j}^S : & \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \to \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), Y \mapsto 1_{[0, t_j]}(\cdot) Y + 1_{(t_j, T]}(\cdot) (S_j)_{t_j} Y_{t_j} \\
F_{b_j} : & \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \to L^{p,1}_{\mathcal{D}'T}(H), X \mapsto b_j((\cdot), X) \\
F_{\sigma_j} : & \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \to \mathcal{N}^{2,p}_{\mathcal{D}'T, S_j, \beta}(L(U, H)), X \mapsto \sigma_j((\cdot), X) \\
S_j \ast_{t_j} : & L^{1,p}_{\mathcal{D}'T}(H) \to \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), X \mapsto 1_{[t_j, T]}(\cdot) \int_{t_j}^X (S_j)_{s} X_s d s \\
S_j \ast_{t_j}^d : & \mathcal{N}^{2,p}_{\mathcal{D}'T, S_j, \beta}(L(U, H)) \to \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), \Phi \to (S_j)_{t_j}^d \Phi.
\end{align*}
\]

\( \psi(j) : \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \times \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \to \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), (Y, X) \mapsto \text{id}_{t_j}^S(Y) + S_j \ast_{t_j} F_{b_j}(X) + S_j \ast_{t_j} F_{\sigma_j}(X) \).

In a similar way as done for \( \psi \), we can obtain (3.22) for each \( \psi(j) \), with a constant \( C_{\lambda, s, \gamma, M, \beta, T, p, M} \) independent of \( j \). In particular, there exists \( \lambda_0 \) large enough such that, for all \( \lambda > \lambda_0 \) and all \( Y, X \in \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \),
\[
|\psi(j)(Y, X) - \psi(j)(Y', X')|_{\mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), \lambda} \leq M' |Y - Y'|_{\mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), \lambda} + \frac{1}{2} |X - X'|_{\mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}), \lambda}, \quad \forall j \in \mathbb{N}, \tag{3.38}
\]
where
\[
M' \text{ is any upper bound for } \sup_{t \in [0, T]} \sup_{j \in \mathbb{N}} |(S_j)_t|_{L(H)}.
\]

Let \( A_j \) denotes the infinitesimal generator of \( S_j \). By arguing as done in the proof of Theorem 3.6, we have that, for each \( j \in \mathbb{N} \), there exists a unique mild solution \( X_{j,t}^Y \) in \( \mathcal{L}^p_{\mathcal{D}'T}(\mathbb{S}) \) to
\[
\begin{align*}
\begin{cases}
  d(X_j)_s = (A_j(X_j)_s + b_j((\cdot, s), X_j)) dt + \sigma_j((\cdot, s), X_j) dW_s & s \in (t_j, T) \\
  (X_j)_s = Y_s & s \in [0, t_j],
\end{cases}
\tag{3.39}
\end{align*}
\]
and that, due to the equivalence of the norms $|\cdot|_{\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})}$, the map $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})$, $Y \mapsto X_j^{t_j,Y}$ is Lipschitz, with Lipschitz constant bounded by some $C''_{g,\gamma,M,M',T,p}$ depending only on $g, \gamma, M, M', T$, $p$ and independent of $j$.

For a given set $B \subset [0,T]$, let us denote

$$\mathcal{S}_B := \{x \in \mathcal{S} : \forall t \in B, \ x \text{ is continuous in } t\}.$$ 

Then $\mathcal{S}_B$ is a closed subspace of $\mathcal{S}$ and it satisfies all the three conditions required for $\mathcal{S}$ at p. 17. Moreover, if $t \in [0,T]$ and $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$, then $X_t^Y, Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$, because $X_t^Y$ is continuous on $[t,T]$ (recall that $S \ast_t^#$ and $S \ast_t^\#$ are $\mathcal{L}^p_{\mathcal{P}_T}(C([0,T],H))$-valued) and coincides with $Y$ on $[0,t]$.

**Proposition 3.13.** Suppose that Assumption 3.3 and Assumption 3.12 are satisfied and let $p > p^*$. Then

$$\lim_{j \to \infty} X_j^{t_j,Y} = X_t^Y$$

(3.40) in $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$, uniformly for $Y$ on compact subsets of $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$.

**Proof.** Let $\psi^{(j)}$ be defined as above (p. 33). It is clear that, if $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$ and $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})$, then $\psi(Y, X) \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$, because it is continuous on $[\hat{t}, T]$ and coincides with $Y$ on $[0, \hat{t}]$. Similarly, $\psi^{(j)}(Y, X)$ is continuous on $[t_j, T]$ and coincides with $Y$ on $[0, t_j]$, than also $\psi^{(j)}(Y, X) \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$. Then, if the claimed convergence occurs, it does in $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$.

In order to prove the convergence, we consider the restrictions

$$\begin{cases}
\hat{\psi}^{(j)} := \psi^{(j)} |_{\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B) \times \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})}, & \forall j \in \mathbb{N} \\
\hat{\psi} := \psi |_{\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B) \times \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})},
\end{cases}$$

which are $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$-valued, as noticed above. Clearly (3.38) still holds true with $\hat{\psi}^{(j)}, \hat{\psi}$ in place of $\psi^{(j)}, \psi$, respectively, and then

$$\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B) \to \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B), \ Y \mapsto X_j^{t_j,Y}$$

is Lipschitz in $Y$, uniformly in $j$. We then need only to prove the convergence

$$X_j^{t_j,Y} \to X_t^Y \text{ in } \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B), \forall Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B).$$

Thanks to Lemma 2.8(i), the latter convergence reduces to the pointwise convergence

$$\hat{\psi}^{(j)} \to \hat{\psi}.$$ 

Let $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$. Due to the continuity of $Y(\omega)$ in $\hat{t}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, the strong continuity of $S_j$ and $S$, and the strong convergence $S_j \to S$, we have $\mathcal{W}_{t_j} S_j(Y) \to \mathcal{W}_{t_j} S(Y)$ in $\mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S}_B)$ for all $Y \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})$ (this can be seen by (3.17)).

We show that $S_j \ast_{t_j}^\# F_{\sigma}(X) \to S \ast_t^\# F_{\sigma}(X)$, for all $X \in \mathcal{L}^p_{\mathcal{P}_T}(\mathcal{S})$. Write

$$S_j \ast_{t_j}^\# F_{\sigma} - S \ast_t^\# F_{\sigma} = (S_j \ast_{t_j}^\# F_{\sigma} - S \ast_{t_j}^\# F_{\sigma}) + (S \ast_{t_j}^\# F_{\sigma} - S \ast_t^\# F_{\sigma}).$$

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By Lebesgue’s dominated convergence theorem and by Assumption 3.12, we have, for \( \beta \in (1/p, 1/2 - \gamma) \),
\[
\lim_{j \to \infty} \int_0^T \left( \int_0^{t_j} (t - s)^{2\beta} \left[ \left( |(S_j)_{t-s} \sigma_j((\cdot, s), X)) - S_{t-s} \sigma((\cdot, s), X)|_{L^2(U,H)}^p \right) / \right]^{p/2} \right) \, ds \, dt = 0
\]
Then, by (3.11) (which holds uniformly in \( t \)),
\[
S_j \overset{dW}{*} t_j F_\sigma(X) - S \overset{dW}{*} t_j F_\sigma(X) \to 0 \text{ in } L^p_p(\mathbb{S}).
\]
By (3.14), we also have
\[
S \overset{dW}{*} t_j F_\sigma(X) - S \overset{dW}{*} t_j F_\sigma(X) \to 0 \text{ in } L^p_p(\mathbb{S}).
\]
Then, we conclude
\[
S_j \overset{dW}{*} t_j F_\sigma - S \overset{dW}{*} t_j F_\sigma \to 0 \text{ in } L^p_p(\mathbb{S}).
\]
By arguing in a very similar way as done for \( S_j \overset{dW}{*} t_j F_\sigma - S \overset{dW}{*} t_j F_\sigma \), one can prove that
\[
\forall X \in L^p_p(\mathbb{S}), S_j \overset{dW}{*} t_j F_{b_j}(X) - S \overset{dW}{*} t_j F_{b}(X) \to 0 \text{ in } L^p_p(\mathbb{S}).
\]
Then \( \hat{\psi}^{(j)} \to \hat{\psi} \) pointwise and the proof is complete.

The following result provides continuity of the mild solution with respect to perturbations of all the data of the system.

**Theorem 3.14.** Suppose that Assumption 3.3 and Assumption 3.12 are satisfied, let \( p > p^* \), \( Y \in L^p_p(\mathbb{S}[i]) \), and let \( \{Y_j\}_{j \in \mathbb{N}} \subset L^p_p(\mathbb{S}) \) be a sequence converging to \( Y \) in \( L^p_p(\mathbb{S}) \). Then
\[
\lim_{j \to \infty} X^{t_j, Y_j} = X^{t, Y} \text{ in } L^p_p(\mathbb{S}).
\]

**Proof.** Write
\[
X^{t, Y} - X^{t_j, Y_j} = (X^{t_j, Y} - X^{t_j, Y_j}) + (X^{t_j, Y} - X^{t_j, Y_j}), \quad (3.41)
\]
The term \( X^{t, Y} - X^{t_j, Y_j} \) tends to 0 by Proposition 3.13, whereas the term \( X^{t_j, Y} - X^{t_j, Y_j} \) tends to 0 by uniform equicontinuity of the family
\[
\left\{ L^p_p(\mathbb{S}) \to L^p_p(\mathbb{S}), Y \to X^{t_j, Y}_j \right\}_{j \in \mathbb{N}}.
\]

We end this chapter with a result regarding stability of Gâteaux differentials of mild solutions.

**Assumption 3.15.** Let \( b, \sigma, g, \gamma, n, c, \mu \) be as in Assumption 3.7, and let \( \{b_j\}_{j \in \mathbb{N}} \) \( \{\sigma_j\}_{j \in \mathbb{N}} \) \( \{S_j\}_{j \in \mathbb{N}} \) be as in Assumption 3.12. Assume that

(i) For all \( j \in \mathbb{N}, (\omega, t) \in \Omega_T \), and \( u \in U, b_j((\omega, t), \cdot) \in \mathcal{G}^n(\mathbb{S}, H) \) and \( \sigma_j((\omega, t), \cdot) u \in \mathcal{G}^n(\mathbb{S}, H)\);
(ii) for all $s \in [0,T]$, 
\[
\sup_{i=1,\ldots,n} \sup_{j \in \mathbb{N}} \sup_{\omega \in \Omega} |\partial^i_{y_1 \ldots y_i} b_j((\omega, s), x)|_H \leq M'' g(s), 
\] 
(3.42) and, for all $s \in [0,T]$, $t \in (0,T)$, and all $m \in \mathcal{M}$, 
\[
\sup_{i=1,\ldots,n} \sup_{j \in \mathbb{N}} \sup_{\omega \in \Omega} |(S_j)_i \partial^i_{y_1 \ldots y_i} (\sigma_j((\omega, s), x)e^j_m)|_H \leq M'' t^{-\gamma} c_m, \] 
(3.43)

(iii) for all $X \in \mathfrak{S}$,
\[
\begin{align*}
\lim_{j \to \infty} |S_j \partial^i_{y_1 \ldots y_i} (\sigma((\omega, s), x)e^j_m) - (S_j)_i \partial^i_{y_1 \ldots y_i} (\sigma_j((\omega, s), x)e^j_m)|_H & = 0 & \forall \omega \in \Omega, \\
\lim_{j \to \infty} |S_j \partial^i_{y_1 \ldots y_i} (\sigma((\omega, s), x)e^j_m) - (S_j)_i \partial^i_{y_1 \ldots y_i} (\sigma_j((\omega, s), x)e^j_m)|_H & = 0 & \forall s \in [0,T], \forall t \in (0,T), \\
& \forall m \in \mathcal{M}.
\end{align*}
\]

Theorem 3.16. Suppose that Assumption 3.3 and Assumption 3.12 are satisfied, and that, for some $n \in \mathbb{N}$, $n \geq 1$, Assumption 3.7 and Assumption 3.15 are satisfied. Let $p > p^*$, $p \geq n$. Then, for $i = 1, \ldots, n$,
\[
\partial^i_{y_1 \ldots y_i} X_j^{t_j, Y} \to \delta^i_{y_1 \ldots y_i} X^{t_j, Y} \text{ in } \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]),
\] 
(3.44) uniformly for $Y, Y_1, \ldots, Y_i$ in compact subsets of $\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])$.

Proof. By Theorem 3.9, $\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}) \to \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S})$, $Y \to X_j^{t_j, Y}$ belongs to $\mathcal{B}(\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}))$.

Then, since $X_j^{t_j, Y} \in \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])$ if $Y \in \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])$, the map $\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]) \to \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])$, $Y \to X_j^{t_j, Y}$ belongs to $\mathcal{B}(\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]))$.

To prove (3.44), we wish to apply Proposition 2.14. In the proof of Theorem 3.9, we associated the map $\psi$ and the spaces $\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S})$ to Assumption 2.11. In the same way, here, we associate the restrictions
\[
\psi^{(1)}_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])}, \psi^{(2)}_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])}, \psi^{(3)}_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}])}, \ldots ,
\]
respectively to the functions $h^{(1)}_1, h^{(2)}_1, h^{(3)}_1, \ldots$ appearing in the assumption of Proposition 2.14, and, to each $h^{(m)}_1$, we associate the functions $h^{(m)}_k$, for $k = 1, \ldots, n$, defined by
\[
h^{(m)}_k := \psi_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]), \mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}[\hat{t}]})
\] 
and considered as $\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S})$-valued functions.

As argued several times above, we can choose $\lambda > 0$ such that, for $m = 1, 2, \ldots$ and $k = 1, \ldots, n$, each function $h^{(m)}_k$ is a parametric 1/2-contractions with respect to the norm $|\cdot|_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}), \lambda}$. With respect to this equivalent norm, for each $h^{(m)}_1$, Assumption 2.11 can be verified in exactly the same way as it was verified for the function $h_1$ appearing in the proof of Theorem 3.9. Then, in order to apply Proposition 2.14, it remains to verify hypotheses (i), (ii), (iii) appearing in the statement of that proposition. Since the norms $|\cdot|_{\mathcal{L}^p_{\mathcal{D}_T}(\mathfrak{S}), \lambda}$, $\lambda \geq 0$, are equivalent, the three hypotheses reduce to the following convergences:
(i) for all \( k = 1, \ldots, n \), \( X \in \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \),

\[
\psi^{(j)}(Y, X) \to \psi(Y, X) \text{ in } (\mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})), \| \cdot \|_{\mathcal{L}^b_{\mathcal{P}}(\mathbb{S})})
\]  

(3.45)

uniformly for \( Y \) on compact subsets of \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \);

(ii) for \( k = 1, \ldots, n \)

\[
\begin{align*}
&\lim_{j \to \infty} \partial_Y \psi^{(j)}(Y, X) = \partial_Y \psi(Y, X) \text{ in } (\mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})), \| \cdot \|_{\mathcal{L}^b_{\mathcal{P}}(\mathbb{S})}) \\
&\lim_{j \to \infty} \partial_X \psi^{(j)}(Y, X) = \partial_X \psi(Y, X) \text{ in } (\mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})), \| \cdot \|_{\mathcal{L}^b_{\mathcal{P}}(\mathbb{S})})
\end{align*}
\]  

(3.46)

uniformly for \( Y, Y' \) on compact subsets of \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \) and \( X, X' \) on compact subsets of \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \);

(iii) for all \( k = 1, \ldots, n-1 \), \( Y \in \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \), \( l, i = 0, \ldots, n \), \( 1 \leq l + i \leq n \),

\[
\lim_{j \to \infty} \partial_{Y_1 \ldots Y_l X_1 \ldots X_i} \psi^{(j)}(Y, X) = \partial_{Y_1 \ldots Y_l X_1 \ldots X_i} \psi(Y, X) \text{ in } (\mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})), \| \cdot \|_{\mathcal{L}^b_{\mathcal{P}}(\mathbb{S})})
\]  

(3.47)

uniformly for \( Y, Y_1, \ldots, Y_l \) on compact subsets of \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \), \( X, X_1, \ldots, X_i \) on compact subsets of \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \).

Taking into account the equicontinuity of the family \( \{\psi^{(j)}\}_{j \in \mathbb{N}} \) with respect to the second variable, (i) is contained in the proof Proposition 3.13. As regarding (ii) and (iii), since the linear term \( \text{id}^S_{t_j} \) is easily treated in \( \mathcal{L}^b_{\mathcal{P}}(\mathbb{S}(\mathfrak{t})) \) (as shown in the proof of Proposition 3.13), the only comments to make are about the convergences of the derivatives

\[
\begin{align*}
\partial_Y^+ (S_j \ast t_j F_{b_j})(X) \\
\partial_X^+ (S_j \ast t_j F_{b_j})(X) \\
\partial_Y^- (S_j \ast t_j F_{\sigma_j})(X) \\
\partial_X^- (S_j \ast t_j F_{\sigma_j})(X)
\end{align*}
\]

and

\[
\begin{align*}
\partial_Y^+ (S_j \ast t_j F_{b_j})(X) \\
\partial_X^+ (S_j \ast t_j F_{b_j})(X) \\
\partial_Y^- (S_j \ast t_j F_{\sigma_j})(X) \\
\partial_X^- (S_j \ast t_j F_{\sigma_j})(X)
\end{align*}
\]

Due to linearity and continuity of the convolution operators, to the independence of the first variable of \( F_b \) and \( F_\sigma \), and to Lemma 3.8, the above derivatives are respectively equal to

\[
\begin{align*}
0 \\
S_j \ast t_j (\partial_X F_{b_j})(X) \\
0 \\
S_j \ast t_j (\partial_X F_{\sigma_j})(X)
\end{align*}
\]  

and

\[
\begin{align*}
S_j \ast t_j (\partial_X F_{b_j})(X) & \quad \text{if } l = 0 \\
0 & \quad \text{otherwise} \\
S_j \ast t_j (\partial_X F_{\sigma_j})(X) & \quad \text{if } l = 0 \\
0 & \quad \text{otherwise}
\end{align*}
\]  

(3.48)

Let us consider, for example, the difference

\[
S_j \ast t_j (\partial_{X_1 \ldots X_i} F_{\sigma_j}(X)) - S \ast t_j (\partial_{X_1 \ldots X_i} F_{\sigma}(X))
\]  

(3.49)
for some sequence \((X_j)_{j \in \mathbb{N}}\) converging to \(X\) in \(L_{\mathcal{P}}^{p_k}(\mathbb{S})\). We can decompose the above difference as done in (3.41), and then use the same arguments, together with expressions (3.26), the bounds (3.42) and (3.42), the generalized Hölder inequality, the pointwise convergences in Assumption 3.15(iii), and Lebesgue's dominated convergence theorem, to conclude
\[
S_j * t_j (\partial_{X_1 \ldots X_i}^i F_\sigma)(X_j) - S * t_i (\partial_{X_1 \ldots X_i}^i F_\sigma)(X) \to 0
\]
in \(L_{\mathcal{P}}^{p_k}(\mathbb{S}_{[\hat{t}]})\), for all \(X_1, \ldots, X_i \in L_{\mathcal{P}}^{p_{k+1}}(\mathbb{S}_{[\hat{t}]})\). By recalling the continuity of \(X \mapsto \partial_{X_1 \ldots X_i}^i F_\sigma(X)\) (Lemma 3.8), this shows the convergence
\[
S_j * t_j (\partial_{X_1 \ldots X_i}^i F_\sigma)(X) - S * t_i (\partial_{X_1 \ldots X_i}^i F_\sigma)(X) \to 0,
\] (3.50)
uniformly for \(X\) on compact sets of \(L_{\mathcal{P}}^{p_k}(\mathbb{S}_{[\hat{t}]})\), for fixed \(X_1, \ldots, X_i \in L_{\mathcal{P}}^{p_{k+1}}(\mathbb{S}_{[\hat{t}]})\). But, since by Lemma 3.8 the derivatives (3.48) are jointly continuous in \(X, X', X_1, \ldots, X_i\), and uniformly bounded, the convergence (3.50) occurs uniformly for \(X\) on compact sets of \(L_{\mathcal{P}}^{p_k}(\mathbb{S}_{[\hat{t}]})\) and \(X_1, \ldots, X_i\) on compact sets of \(L_{\mathcal{P}}^{p_{k+1}}(\mathbb{S}_{[\hat{t}]})\). The arguments for the other derivatives are similar. This shows that we can apply Proposition 2.14, which provides (3.44).

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