Selection of measure and a Large Deviation Principle for the general one-dimensional XY model

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Abstract

We consider $(M, d)$ a connected and compact manifold and we denote by $X$ the Bernoulli space $M^\mathbb{N}$. The shift acting on $X$ is denoted by $\sigma$.

We analyze the general XY model, as presented in a recent paper by A. T. Baraviera, L. M. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza. Denote the Gibbs measure by $\mu_c := h_c \nu_c$, where $h_c$ is the eigenfunction, and, $\nu_c$ is the eigenmeasure of the Ruelle operator associated to $c f$. We are going to prove that any measure selected by $\mu_c$, as $c \to +\infty$, is a maximizing measure for $f$. We also show, when the maximizing probability measure is unique, that it is true a Large Deviation Principle, with the deviation function $R_+^\infty = \sum_{j=0}^{\infty} R_+ (\sigma^j)$, where $R_+ := \beta (f) + V \circ \sigma - V - f$, and, $V$ is any calibrated subaction.

1 Introduction

We consider $(M, d)$ a connected and compact manifold and we denote by $X$ the Bernoulli space $M^\mathbb{N}$. The shift acting on $X$ is denoted by $\sigma$.

We point out that the number of preimages by $\sigma$ of each point is not countable.

Let $f : X \to \mathbb{R}$ be a fixed Holder potential defined in the Bernoulli space $X$. We denote by $m$ the Lebesgue probability on $M$. We suppose without

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lost of generality that the diameter of the manifold $M$ is smaller than one. This distance induces another one, in the usual fashion, on $M^N$ \cite{4}.

We are interested in the Gibbs state (for finite and zero temperature) associated to the potential $f$. This model is called the general $XY$ model in \cite{4}. We refer the reader to such work for a detailed explanation about the motivation for considering such kind of problems. We point out that in the literature in Physics what is called the $XY$ model is the case when $M = S^1$, and, the potential depends on a finite number of coordinates. In \cite{4} and here the hypothesis are more general.

Classical references in the $XY$ model are \cite{15}, \cite{24} and \cite{28}. A nice reference for general results in Statistical Mechanics is \cite{13}.

In order to define a transfer operator we need a probability a priori on $M$ which we will denote by $d\mu$. In the case $M = S^1$ it usually consider the Lebesgue measure $dx$ \cite{28}.

First we will recall some definitions and results from \cite{4}.

**Definition 1.** Let $\mathcal{C}$ be the space of continuous functions from $X = M^N$ to $\mathbb{R}$. We define the Ruelle operator on $\mathcal{C}$, associated to the Holder potential $f : M^N \rightarrow \mathbb{R}$, which is the linear operator that gets $w \in \mathcal{C}$, and sends to $L_f(w) \in \mathcal{C}$, defined for any $x = (x_0, x_1, x_2, \ldots) \in X$, by

$$L_f(w)(x) = \int e^{f(ax)} w(ax) \, dm(a),$$

where $ax$ represents the sequence $(a, x_0, x_1, x_2, \ldots) \in X$, and $dm(a)$ is the Lebesgue probability on $M$.

Following \cite{4}, for a real value $c$ we consider $\beta_c$ the eigenvalue, $h_c$ the eigenfunction, and $g_c = cf + \log(h_c) - \log(h_c \circ \sigma) - \log(\beta_c)$ the normalized function associated to the Ruelle operator $L_{cf}$ obtained from $cf$. We also denote $\nu_c$ the eigenmeasure of $L_{cf}^*$, and, $\mu_c := h_c \nu_c$, the Gibbs probability of the potential $cf$.

As usual, by notation $f^n(x) = \sum_{j=0}^{n-1} f(\sigma^j(x))$, for any $n \in \mathbb{N}$, $x \in X$.

**Remark on notation:** the iterated Ruelle Operator $L^n_f w(x)$, $n = 1, 2, 3, \ldots$, can be written as

$$\int_{a_n \ldots a_1} e^{f^n(a_n \ldots a_1 x)} w(a_n \ldots a_1 x) \, da_1 \ldots da_n, \text{ or } \int_{\sigma^n z = x} e^{f^n(z)} w(z) \, dm.$$
We consider the following problem: for the given \( f : X \to \mathbb{R} \), we want to find probability measures that maximize, over \( \mathcal{M}_\sigma \), the value \( \int f(x) \, d\mu(x) \).

**Definition 2.** We define

\[
\beta(f) = \max_{\mu \in \mathcal{M}_\sigma} \left\{ \int f \, d\mu \right\}.
\]

Any of the measures which attains the maximal value will be called a maximizing probability measure for \( f \), which is sometimes denoted by \( \mu_\infty \).

In Section 2 we are going to prove the following

**Theorem 3.** Any weak*\(-\)limit of subsequence of \( \mu_c \), \( (c \to +\infty) \), is a maximizing probability measure to \( f \).

In this way one can say that any convergent subsequence of Gibbs states at positive temperature selects maximizing probabilities. In this result we do not assume uniqueness of the maximizing probability.

The similar result for the Classical Thermodynamical Formalism considers the shift acting on the Bernoulli space \( \{1, 2, \ldots, d\}^\mathbb{N} \). In this case one can consider entropy and pressure and the proof is trivial (see [11] [10]). Here we can not take advantage of this and the proof requires other methods.

**Definition 4.** A continuous function \( V : X \to \mathbb{R} \) is called a calibrated subaction for \( f : X \to \mathbb{R} \), if, for any \( y \in X \), we have

\[
V(y) = \max_{\sigma(x) = y} \left[ f(x) + V(x) - \beta(f) \right].
\]

This can be also be expressed as

\[
\beta(f) = \max_{a \in \mathcal{M}} \{ f(ay) + V(ay) - V(y) \}.
\]

One can show that for any \( x \) in the support of the maximizing probability measure for \( f \) we have that

\[
V(\sigma(x)) - V(x) - f(x) + \beta(f) = 0.
\]

In this way if we know the value \( \beta(A) \), then a calibrated subaction \( V \) for \( f \) helps to identify the union of the supports of maximizing probabilities \( \mu_\infty \) for \( f \). The above equation can be eventually true outside the union of the supports of the maximizing probabilities \( \mu \).
If the maximizing probability is unique, then the calibrated subaction is unique up to an additive constant \([3] [4] [12]\).

It is known \([4]\) that \(\frac{1}{c} \log(h_c), c \in \mathbb{R}\), is an equicontinuous family. Any limit of subsequence \(V = \lim_{n \to \infty} \frac{1}{c_n} \log(h_{c_n}), c_n \to \infty\), is a calibrated subaction \([4]\).

We denote in the following \(R^\infty_+ = \sum_{j=0}^{\infty} R_+(\sigma^j),\) with \(R_+ := \beta(f) + V \circ \sigma - V - f\), where \(V\) is any calibrated subaction.

We say that the potential \(A\) depends on two variables if for any \(x = (x_0, x_1, x_2, x_3, \ldots) \in X\) we have that the value \(A(x_0, x_1, x_2, x_3, \ldots)\) is independent of \((x_2, x_3, x_4, \ldots)\). This case is also known as the "nearest-neighbour" interaction".

The next theorem was shown to be true in the case the potential \(A\) depends on two coordinates in \([22]\).

In section 3 we consider the case where the maximizing measure for \(f\) is unique, and prove that the family \(\mu_c\) satisfies the following Large Deviation Principle:

**Theorem 5.** Suppose the maximizing probability for \(f\) is unique. Then, for any closed set \(F\), and any open set \(A\):

\[
\limsup_{c \to \infty} \frac{1}{c} \log(\mu_c(F)) \leq - \inf_{x \in F} R^\infty_+(x),
\]

\[
\liminf_{c \to \infty} \frac{1}{c} \log(\mu_c(A)) \geq - \inf_{x \in A} R^\infty_+(x).
\]

The function \(R^\infty_+\) is lower semicontinuous, and can attain the value \(\infty\) in some points.

The above theorem will be a consequence of a more general result:

**Theorem 6.** Suppose the maximizing probability for \(f\) is unique. Consider any point \(x \in X\), then, for any closed set \(F\), and any open set \(A\):

\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L^n_{g_c} \chi_F)(x)) = - \inf_{z \in F} (R^\infty_+(z)),
\]

\[
\liminf_{c,n \to \infty} \frac{1}{c} \log((L^n_{g_c} \chi_A)(x)) = - \inf_{z \in A} (R^\infty_+(z)).
\]

Theorem 5 is a consequence of the above just by taking first \(n \to \infty\) \([27]\), and, then, making \(c \to \infty\).

We point out that the reasoning which proves this last result can also be applied to the Classical Thermodynamic Formalism setting \([27]\), where the
Bernoulli space is $\{1, 2, ..., d\}^N$, to get the analogous result. This proof of the L. D. P. does not use the involution kernel as in [3].

In [21] is presented another kind of Large Deviation Principle: the setting of zeta measures. In this case the proof do not require that the maximizing probability is unique.

## 2 The selection of measure

**Lemma 7.** Let $V$ be a calibrated subaction, such that, $V = \lim_{c \to \infty} \frac{1}{c} \log(h_c)$, and, $R_- = f + V - V \circ \sigma - \beta(f)$, which is the limit function of the $g_c/c$ associated. For each $\epsilon > 0$ there exists a constant $\psi_\epsilon$ such that for any $x \in X$

$$m(\{a \in M : R_-(ax) > -\epsilon\}) > \psi_\epsilon > 0.$$  

**Proof.** Suppose $g_c$ converges to $R_-$, and then write $g_c = cR_- + \delta_c$, where $|\delta_c|_{\infty}/c \to 0$. Using that $V$ is a calibrated subaction, we have $R_- \leq 0$.

We fix $\epsilon > 0$, and we define

$$A_\epsilon := \{a : R_-(ax) \leq -\epsilon\}$$

$$B_\epsilon := \{a : R_-(ax) > -\epsilon\}.$$

$V$ is Holder, so $R_-$ is Holder, then, it is a continuous function on the first symbol. In this way, $A_\epsilon$ and $B_\epsilon$ are measurable sets. We have:

$$1 = L_{g_c}1(x) = \int e^{g_c(ax)} da = \int e^{cR_-(ax) + \delta_c(ax)} da.$$  

Then,

$$1 = \int_{A_\epsilon} e^{cR_-(ax) + \delta_c(ax)} da + \int_{B_\epsilon} e^{cR_-(ax) + \delta_c(ax)} da$$

$$\leq \int_{A_\epsilon} e^{-c\epsilon + \delta_c(ax)} da + \int_{B_\epsilon} e^{0 + \delta_c(ax)} da$$

$$\leq \int_{A_\epsilon} e^{-c\epsilon + |\delta_c|_{\infty}} da + \int_{B_\epsilon} e^{0 + |\delta_c|_{\infty}} da$$

$$= e^{-c\epsilon + |\delta_c|_{\infty}} m(A_\epsilon) + e^{|\delta_c|_{\infty}} m(B_\epsilon)$$

$$\leq e^{-c\epsilon + |\delta_c|_{\infty}} + e^{|\delta_c|_{\infty}} m(B_\epsilon).$$

Let $c_0 > 0$ be such that $e^{-c_0\epsilon + |\delta_{c_0}|_{\infty}} \leq 1/2$. Then, it follows that

$$1/2 \leq e^{|\delta_{c_0}|_{\infty}} m(B_\epsilon),$$
so,

\[ m(B_ε) \geq \frac{1}{2ε|h_0|}. \]

Then, we just take \( ψ_ε = \frac{1}{3ε|h_0|} \), and the result follows.

**Proof of Theorem**

**Proof.** Let \( ν \) be an accumulation point of \( \mu,c \) given by a certain subsequence \( c_j \to \infty \). Let \( c_i \) be a subsequence of this one such that there exists the limit \( V \) of the sequence \( \frac{1}{c_j} \log h_{c_i} \). Let \( R_- := f + V - V \circ \sigma - β(f) \) be the function associated to such limit. Then, \( g_{c_i}/c_i \to \infty \). Define \( a := \lim c_i \to \infty \mu_{c_i}(R_-) = \lim_{c_j \to \infty} \mu_{c_j}(R_-) \). Then, it follows that \( a \leq 0 \). We are going to show that for any fixed \( \epsilon > 0 \), we define the sets:

\[ A_n := \{a_n \cdots a_1 : R_-(a_n \cdots a_1) < -\epsilon \}, \]

\[ B_n := \{a_n \cdots a_1 : R_-(a_n \cdots a_1) \geq -\epsilon \}, \]

\[ C_n := \{a_n \cdots a_1 : R_-(a_n \cdots a_1) > 0 \}. \]
Clearly, we have that $C_n \subseteq B_n$.

As $V$ is a calibrated subaction, then $C_n$ is not empty. We remark that $A_n \cup B_n = [0, 1]^n$, and, by the Lemma above, for each $a_{n-1} \ldots a_1$, we have that $m\{a_n : a_n \ldots a_1 \in C_n\} > \psi > 0$.

Then:

$$\int_{A_n} e^{c_1 R_n^+ + \delta} da_1 \ldots da_n = \int_{A_n} e^{c_1 R_n^+ + \delta_1} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\leq \int_{A_n} e^{-c_1 \varepsilon + |\delta_1|} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$= e^{-c_1 \varepsilon + |\delta_1|} \int_{A_n} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\leq e^{-c_1 \varepsilon + |\delta_1|} \int_{M_n} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$= e^{-c_1 \varepsilon + |\delta_1|} \int_{M_n-1} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n,$$

and,

$$\int_{B_n} e^{c_1 R_n^+ + \delta} da_1 \ldots da_n = \int_{B_n} e^{c_1 R_n^+ + \delta_1} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\geq \int_{C_n} e^{c_1 R_n^+ + \delta} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\geq \int_{C_n} e^{-c_1 \varepsilon - |\delta_1|} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\geq e^{-c_1 \varepsilon - |\delta_1|} \int_{C_n} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$\geq e^{-c_1 \varepsilon - |\delta_1|} \int_{M_n-1} \int_{\{a_n : a_n \ldots a_1 \in C_n\}} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n$$

$$= e^{-c_1 \varepsilon - |\delta_1|} \int_{M_n-1} e^{c_1 R_n^+ - \sigma} \int_{\{a_n : a_n \ldots a_1 \in C_n\}} da_1 \ldots da_n$$

$$= e^{-c_1 \varepsilon - |\delta_1|} \psi / 2 \int_{M_n-1} e^{c_1 R_n^+ - \sigma} da_1 \ldots da_n.$$

It follows that

$$0 \leq \liminf_{i,n \to \infty} \frac{\int_{A_n} e^{c_1 R_n^+ + \delta} da_1 \ldots da_n}{\int_{B_n} e^{c_1 R_n^+ + \delta} da_1 \ldots da_n}$$
\[ \begin{align*}
&\leq \limsup_{i,n \to \infty} \frac{\int_{A_n} e^{c_i R_n + \delta_i} \, da_1 \ldots da_n}{\int_{B_n} e^{c_i R_n + \delta_i} \, da_1 \ldots da_n} \\
&\leq \limsup_{i,n \to \infty} e^{-c_i \varepsilon + |\delta_i|_{\infty}} \\
&\leq \limsup_{i,n \to \infty} e^{-c_i \varepsilon/2 + 2|\delta_i|_{\infty} \psi^{-1}/2} \\
&= \limsup_{i,n \to \infty} e^{c_i (-\varepsilon/2 + 2|\delta_i|_{\infty} \psi^{-1}/2)} = 0.
\end{align*} \]

In the same way

\[ \begin{align*}
0 &\leq \liminf_{i,n \to \infty} \frac{\int_{A_n} e^{c_i R_n + \delta_i} R_- da_1 \ldots da_n}{\int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-\varepsilon)} \\
&\leq \limsup_{i,n \to \infty} \frac{\int_{A_n} e^{c_i R_n + \delta_i} R_- da_1 \ldots da_n}{\int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-\varepsilon)} \\
&\leq \limsup_{i,n \to \infty} \frac{\int_{A_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-|R_-|_{\infty})}{\int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-\varepsilon)} \\
&\leq \limsup_{i,n \to \infty} e^{-c_i \varepsilon + |\delta_i|_{\infty}(-|R_-|_{\infty})} e^{-c_i \varepsilon/2 - |\delta_i|_{\infty}} = 0.
\end{align*} \]

From the above, and writing \( \int_{M^n} da_1 \ldots da_n = \int_{A_n} da_1 \ldots da_n + \int_{B_n} da_1 \ldots da_n \), we have:

\[ \begin{align*}
\liminf_{c_i,n \to \infty} L^n_{g_{c_i}} (R_-)(x) &= \liminf_{c_i,n \to \infty} \frac{\int_{a_n \ldots a_1} e^{c_i R_n (a_n \ldots a_1 x) + \delta_i (a_n \ldots a_1 x)} R_- (a_n \ldots a_1 x) da_1 \ldots da_n}{\int_{a_n \ldots a_1} e^{c_i R_n (a_n \ldots a_1 x) + \delta_i (a_n \ldots a_1 x)} da_1 \ldots da_n} \\
&\geq \liminf_{c_i,n \to \infty} \frac{\int_{A_n} e^{c_i R_n + \delta_i} R_- da_1 \ldots da_n + \int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-\varepsilon)}{\int_{A_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n + \int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n} \\
&= \liminf_{c_i,n \to \infty} \frac{\int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n(-\varepsilon)}{\int_{B_n} e^{c_i R_n + \delta_i} da_1 \ldots da_n} \\
&\geq -\varepsilon.
\end{align*} \]

Taking \( \varepsilon \to 0 \), we get our claim. \( \square \)

This ends the proof of our first main result.

The bottom line is: a convergent subsequence of Gibbs states at positive temperature selects maximizing probabilities (eventually, different limits of subsequences can localize different probabilities).
3 On the Large Deviation Principle

On this section we are going to prove Theorem 5. The proof will follow from some lemmas.

We suppose that the maximizing measure for $f$ is unique, and we denote $\mu_\infty$ the maximal one.

Under this assumption, two calibrated subactions differ by a constant. This follows from proposition 5 in [3]. In particular the function $R_+ := \beta(f) + V \circ \sigma - V - f$ is well defined. The function $R_- := -R_+$ is the unique accumulation point of $g_c/c$, on the uniform topology, so $g_c/c \to R_-$ uniformly.

Given a double indexed sequence $z_{c,n}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$, we say that
\[ \lim_{c,n \to \infty} z_n = w, \] for any given $\epsilon > 0$, there exists an $M > 0$, such that, if $c,n > M$, then $|z_{c,n} - w| < \epsilon$.

We are going to prove a stronger result than Theorem 5.

**Theorem 8.** Fixed any point $x \in X$, for any closed set $F$ and open set $A$:
\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L^n g_c \chi_F)(x)) \leq \sup_{z \in F} R_\infty(z) = -\inf_{z \in F} (R_\infty(z)),
\]
\[
\liminf_{c,n \to \infty} \frac{1}{c} \log((L^n g_c \chi_A)(x)) \geq \sup_{z \in A} R_\infty(z) = -\inf_{z \in A} (R_\infty(z)).
\]

The function $R_\infty^+$ is lower semi-continuous.

**Lemma 9.** The function $R_\infty^+$ is lower semi-continuous.

**Proof.** We take $z, z_j \in X$, with $z_j \to z$. We are going to show that
\[
\liminf_{j \to \infty} R_\infty^+(z_j) \geq R_\infty^+(z).
\]

In the case $R_\infty^+(z) = 0$ the result is true.

**First case:** $R_\infty^+(z) = \infty$.

Given $M > 0$, let $n$ be such that $R_\infty^+(z) > 2M$. We fix $n_1$, such that,
\[
\frac{(1-\theta)^{n_1} M}{|R_\theta|} < 1.\]
Let $n_0$ be such that for $j \geq n_0$, we have $d(z_j, z) < \frac{(1-\theta)^{n_1} M \theta^n}{|R_\theta|}$.
Then, for $j \geq n_0$:
\[
R_\infty^+(z_j) \geq R_\infty^+(z) - |R_\theta| d(z_j, z) + \ldots + d(\sigma^{n-1}(z_j), \sigma^{n-1}z)) \\
\geq 2M - |R_\theta| \frac{\theta^{n_1} M}{|R_\theta|} \geq M.
\]
It follows that
\[ \liminf_{j \to \infty} R^{\infty}_+(z_j) \geq M. \]
Taking \( M \to +\infty \):
\[ \liminf_{j \to \infty} R^{\infty}_+(z_j) = +\infty. \]

**Second case**: \( R^{\infty}_+(z) = M > 0 \).
Fixed \( \varepsilon > 0 \), there exist \( n \), such that,
\[ R^n_+(z_j) > M - \varepsilon/2. \]
Let \( n_0 \) be such that for \( j \geq n_0 \), we have
\[ \frac{1}{c_j} \log((L^n_{cR}1)(x)) - \frac{n_j \varepsilon c}{c} < 0. \]
So, we have that
\[ \liminf_{j \to \infty} R^{\infty}_+(z_j) \geq M - \varepsilon. \]
Taking \( \varepsilon \to 0 \), we get:
\[ \liminf_{j \to \infty} R^{\infty}_+(z_j) \geq M. \]

**Remark**: Note that \( R := R_- = -R_+ \).

We note that in [4] it is proved that \( \frac{1}{c} \log(\beta_c) \to \beta(f) \). We denote \( \varepsilon_c = \log(\beta_c) - c\beta(f) \). Then, we have \( \frac{\varepsilon_c}{c} \to 0 \).

**Lemma 10.**
\[ \lim_{c,n \to \infty} \left( \frac{1}{c} \log((L^n_{cR}1)(x)) - \frac{n \varepsilon_c}{c} \right) = 0, \]
in particular, for a fixed \( k \):
\[ \lim_{c,n \to \infty} \frac{1}{c} \log((L^n_{cR}1)(x)) - \frac{1}{c} \log((L^{n+k}_{cR}1)(x)) = 0. \]

**Proof.** Let \( a \) an accumulation point of \( \frac{1}{c} \log((L^n_{cR}1)(x)) - \frac{n \varepsilon_c}{c} \), when \( c, n \to \infty \). Then, there exists \( c_j, n_j \to \infty \), such that,
\[ \lim_{j \to \infty} \left( \frac{1}{c_j} \log((L^n_{c_jR}1)(x)) - \frac{n_j \varepsilon c_j}{c_j} \right) = a. \]

Following [3] we can take a subsequence \( \{j_i\} \) such that \( \frac{1}{c_{j_i}} \log(h_{c_{j_i}}) \) converges uniformly to a calibrated subaction \( V \). So there exist sequences \( c_i, n_i \to \infty \) such that:
\[ \lim_{i \to \infty} \left( \frac{1}{c_i} \log((L^n_{c_iR}1)(x)) - \frac{n_i \varepsilon c_i}{c_i} \right) = a, \quad \text{and} \quad \lim_{i \to \infty} \frac{1}{c_i} \log(h_{c_i}) = V. \]
Denoting \( \log(h_{c_i}) = c_i V + \delta_{c_i} \) where \( |\delta_{c_i}| / c_i \to 0 \), we have:

\[
0 = \lim_{i \to \infty} \frac{1}{c_i} \log((L^i_{g_{c_i}} 1) (x)) = \lim_{i \to \infty} \frac{1}{c_i} \log(\int_{\sigma^i_{n}(z)=x} e^{c_i f^i(z) + \log(h_{c_i}(z)) - \log(h_{c_i}(x))} dm)
\]

This shows that any accumulation point have to be equal to zero. \( \square \)

The first inequality of Theorem 8

Proposition 11. For any closed set \( F \subseteq X \)

\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L^n_{g_{c}} \chi_F)(x)) \leq \sup_{z \in F} R\chi^{-}(z).
\]

Proof. For a fixed \( k \), we have that

\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L^{n+k}_{g_{c}} \chi_F)(x)) = \limsup_{c,n \to \infty} \frac{1}{c} \log\left( \frac{(L^{n+k}_{g_{c}} \chi_F)(x)}{(L^{n+k}_{g_{c}})(x)} \right)
\]

\[
= \limsup_{c,n \to \infty} \frac{1}{c} \log\left( \frac{\int_{\sigma^{n+k}(z)=x} e^{c R^{n+k}_{c}(z)} \chi_F(z) \, dm}{\int_{\sigma^{n+k}(z)=x} e^{c R^{n+k}_{c}(z)} \, dm} \right)
\]

Note, however that

\[
\int_{\sigma^k(z)=y} e^{c R^k_{c}(z)} \chi_F(z) \, dm \leq e^{c \sup_{z \in F} R^k_{c}(z)},
\]

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then,
\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L_n^g \chi_F)(x)) \leq \limsup_{c,n \to \infty} \left( \sup_{z \in F} R^k(z) \right) = \sup_{z \in F} R^k_\infty(z).
\]

For each \( k \) fixed, we have that \( R^k_\infty \) is a continuous function, and \( F \subset X \) is a compact set, then, there exist \( y_k \in F \), such that, \( \sup_{z \in F} R^k(z) = R^k_\infty(y_k) \).

Define
\[
Y_k := \{ y \in F : \limsup_{c,n \to \infty} \frac{1}{c} \log((L_n^g \chi_F)(x)) \leq R^k_\infty(y) \}.
\]
Then \( Y_k \) is closed (because \( R^k_\infty \) is a continuous function) and not empty (because \( y_k \in Y_k \)). Using \( R^- \leq 0 \) we have
\[
Y_1 \supseteq Y_2 \supseteq \ldots
\]
These sets are closed and not empty, then there exist some \( x_0 \in \bigcap_{k \geq 1} Y_k \). So, for each \( k \):
\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L_n^g \chi_F)(x)) \leq R^k_\infty(x_0).
\]
Using the fact that \( R^k_\infty(x_0) \to R_\infty^\infty(x_0) \), we conclude that
\[
\limsup_{c,n \to \infty} \frac{1}{c} \log((L_n^g \chi_F)(x)) \leq R_\infty^\infty(x_0) \leq \sup_{z \in F} R_\infty^\infty(z).
\]

The second inequality of Theorem \( \S \)

Suppose that \( A \) is open. So, there exists \( n_0 \), such that, for \( n \geq n_0 \), and, \( x \in X \), there exists \( y \in A \), such that, \( \sigma^n(y) = x \). More precisely, given \( y = y_1y_2\ldots \) in \( A \), let \( \epsilon > 0 \), such that, \( B(y, \epsilon) \subset A \). Let \( n_0 \) such that \( \frac{\theta^{n_0}}{1-\theta} < \epsilon \). If \( z \in X \) coincide with \( y_1\ldots y_{n_0} \) in its firsts symbols, then, clearly, 
\[
d(z, y) \leq \theta^{n_0} + \theta^{n_0+1} + \ldots \leq \frac{\theta^{n_0}}{1-\theta} < \epsilon,
\]
and, so \( z \in A \). We conclude that given \( x \in X \), we have that \( y_1\ldots y_{n_0}x \in A \).

Lemma 12. There exist \( y_0 \in X \) such that
\[
\liminf_{c,n \to \infty} \frac{1}{c} \log((L_n^g \chi_A)(x)) \geq \limsup_{k \to \infty} \left( \sup_{z \in A, \sigma^k(z) = y_0} R^k_\infty(z) \right).
\]

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Proof. For a fixed $k \geq n_0$, we have

$$\liminf_{c,n \to \infty} \frac{1}{c} \log((L_{a,c}^{n+k}(x,\mu_A)(x)) = \liminf_{c,n \to \infty} \frac{1}{c} \log \left( \frac{\int_{\sigma^n(y) = x} e^{cR_+^c(y)} (\int_{\sigma^k(z) = x} e^{cR_{-}^k(z)} \mu_A(z) \, dm) \, dm}{\int_{\sigma^n(y) = x} e^{cR_+^c(y)} \, dm} \right) \geq \liminf_{c,n \to \infty} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right) = \liminf_{c \to \infty} \inf_{y \in X} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right).$$

Then, we get

$$\liminf_{c,n \to \infty} \frac{1}{c} \log((L_{a,c}^{n}(x,\mu_A)(x)) \geq \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right).$$

Let $y_{c,k}$ be such that

$$\inf_{y \in X} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right) > \frac{1}{c} \log \left( \int_{\sigma^k(z) = y_{c,k}} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right) - \frac{1}{k}.$$

As $X$ is a compact set, let $y_0$ be an accumulation point of $y_{c,k}$, when $c, k \to \infty$. Then, we have:

$$\liminf_{c,n \to \infty} \frac{1}{c} \log((L_{a,c}^{n}(x,\mu_A)(x)) \geq \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y_{c,k}} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right) - \frac{1}{k} \geq \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log \left( \int_{\sigma^k(z) = y_{c,k}} e^{cR_{-}^k(z)} \mu_A(z) \, dm \right).$$

(2)

For $k$ sufficiently large, let $z_k \in A$, such that: $\sigma^k(z_k) = y_0$, and

$$R_{-}^k(z_k) > \sup_{z \in A, \sigma^k(z) = y_0} R_{-}^k(z) - \frac{1}{2k}.$$ 

We denote $z_k := x_k \ldots x_1 y_0$, $x_i \in M$. We can take $\epsilon > 0$ sufficiently small such that the ball $A_{k,\epsilon} := \{a_k \ldots a_1 \in m^k : |(a_k \ldots a_1) - (x_k \ldots x_1)| \leq \epsilon \}$ satisfies:

1. $a_k \ldots a_1 y_0 \in A$,
2. $a_k \ldots a_1 y_{c,k} \in A$, for $k, c \gg 0$
3. $R_{-}^k(a_k \ldots a_1 y_0) > \sup_{z \in A, \sigma^k(z) = y_0} R_{-}^k(z) - \frac{1}{k}$

Then we have:

$$\limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log \left( \int_{A_{k,\epsilon}} e^{cR_{-}^k(a_k \ldots a_1 y_0)} \, dm \right)$$
\[
\geq \limsup_{k \to \infty} \liminf_{c \to \infty} \left( \sup_{z \in A, \sigma^k(z) = y_0} R^k(z) - \frac{1}{k} \right) + \frac{1}{c} \log(m(A_{k, \epsilon})) \\
= \limsup_{k \to \infty} \left( \sup_{z \in A, \sigma^k(z) = y_0} R^k(z) \right) + \limsup_{c \to \infty} \liminf_{k \to \infty} \frac{1}{c} \log(c) \\
= \limsup_{k \to \infty} \left( \sup_{z \in A, \sigma^k(z) = y_0} R^k(z) \right). \tag{3}
\]

By other hand, on \(A_{k, \epsilon}\) we have:

\[
R^k(a_k \ldots a_1 y_{c,k}) \geq R^k(a_k \ldots a_1 y_0) - |R_-| \theta + \theta^2 + \ldots + \theta^k) d(y_{c,k}, y_0) \\
\geq R^k(a_k \ldots a_1 y_0) - \frac{|R_-| \theta d(y_{c,k}, y_0)}{1 - \theta}.
\]

Then, we get

\[
\limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log\left( \int_{\sigma^k(z) = y_{c,k}} e^{R^k(z)} \chi_A(z) \, dm \right) \\
\geq \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log\left( \int_{A_{k, \epsilon}} e^{R^k(a_k \ldots a_1 y_{c,k})} \, dm \right) \\
\geq \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log\left( \int_{A_{k, \epsilon}} e^{R^k(a_k \ldots a_1 y_0) - \frac{|R_-| \theta d(y_{c,k}, y_0)}{1 - \theta}} \, dm \right) \\
= \limsup_{k \to \infty} \liminf_{c \to \infty} \frac{1}{c} \log\left( \int_{A_{k, \epsilon}} e^{R^k(a_k \ldots a_1 y_0)} \, dm \right). \tag{4}
\]

Using (2), (4) and (3), we finish the proof. \(\square\)

Now we fix the point \(y_0\) given above. The next result is basically contained in the proof of proposition 5 in [3].

**Lemma 13.** Let \(p\) be a point on the support of \(\mu_{\infty}\). Let \(y_n\) a sequence satisfying \(\sigma(y_n) = y_{n-1}, n = 1, 2, 3, \ldots\), and, \(0 = R_-(y_1) = R_-(y_2) = \ldots\) (it follows from the property of the calibrated subaction). Then \(p\) is an accumulation point of \(\{y_n\}\).

**Proof.** Let \(B\) be the set of accumulation points of \(\{y_n\}\). \(B\) is closed and \(\sigma(B) = B\). Then there exists an invariant probability \(\nu\) with support on \(B\). The inclusion \(B \subseteq X\) implies the existence of an extension of \(\nu\) to \(X\) by the rule: \(\nu(\phi) := \nu(\phi \chi_B)\). Using the fact that \(R_-\) is a continuous function, and, that \(R_-(y_n) = 0, n = 1, 2, \ldots\), we conclude that \(\chi_B \cdot R_- = 0\). So, \(\nu(R_-) = 0\), and then, \(\nu = \mu_{\infty}\). From this we get that the support of \(\mu_{\infty}\) is contained on \(B\). \(\square\)

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The next lemma follows the same reasoning of Lemma 18 in \[22\].

**Lemma 14.** If \( R^\infty_\varepsilon(z) > -\infty \), then the family of probabilities \( \nu_n \) (also called empirical measures), given by \( \phi \to \frac{1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j(z)) \), converges to \( \mu_\infty \) weakly*, when \( n \to \infty \).

**Proof.** Any accumulation measure of \( \nu_n \) is an invariant probability. We are going to show that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \mu_\infty(f).
\]

Let \( M = R^\infty_\varepsilon(z) \). Then, for each \( n \) we have \( R^a_n(z) \geq M \), so:

\[
V(z) - V(\sigma^n(z)) - n\mu_\infty(f) + \sum_{j=0}^{n-1} f(\sigma^j(z)) = \sum_{j=0}^{n-1} R^-_\varepsilon(\sigma^j(z)) = R^a_n(z) \geq M.
\]

Then, we get

\[
\frac{1}{n} \sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \frac{M}{n} - \frac{2|V|^\infty}{n} + \mu_\infty(f).
\]

Finally, taking \( \liminf \) in the above we show the claim. \(\blacksquare\)

**Corollary 15.** If \( R^\infty_\varepsilon(z) > -\infty \), and, \( p \in supp(\mu_\infty) \), then \( p \) is an accumulation point of \( \sigma^n(z) \).

**Proof.** Let \( p \in supp(\mu_\infty) \), and \( \varepsilon > 0 \). Consider the ball \( B(p, \varepsilon) := \{ x \in X : d(x, p) < \varepsilon \} \). Using the fact that \( p \in supp(\mu_\infty) \), we have that \( \mu_\infty(B(p, \varepsilon)) > 0 \). So, by the above lemma, we have that \( \{ \sigma^n(z) \} \) is in the ball for infinite values of \( n \). \(\blacksquare\)

**Lemma 16.**

\[
\sup_{z \in A} R^\infty_\varepsilon(z) \leq \limsup_{k \to \infty} \left( \sup_{z \in A, \sigma^k(z) = y_0} R^-_\varepsilon(z) \right).
\]

**Proof.** We fix a point \( p \in supp(\mu_\infty) \), and, we denote \( p = p_1p_2... \). For \( n \geq n_0 \), there exists \( y \in A \), with \( \sigma^n(y) = p \). Note that \( R^\infty_\varepsilon(y) = R^n_\varepsilon(y) > -\infty \). So, \( \sup_{z \in A} R^\infty_\varepsilon(z) > -\infty \). Let \( z_0 \in A \) be such that, \( R^\infty_\varepsilon(z_0) > -\infty \), and, denote \( z_0 = x_1x_2... \). Given \( t \in \mathbb{N} \), let \( n(t) \) be such that, \( d(\sigma^{n(t)}(z_0), p) \leq \theta^t \), and moreover, such that, the choice \( x_1x_2...x_{n(t)} \), determines that \( z_0 \in A \) (open). By lemma 13 there exist a pre-image of \( y_0 \) (we suppose the point \( y_{l(t)} \) of the form \( a_{l(t)}...a_1(y_0) \)), such that, \( R^-_\varepsilon(y_{l(t)}) = 0 \), and, \( d(y_{l(t)}, p) < \theta^t \). Define \( z(t) \) by

\[
z(t) := x_1...x_{n(t)}a_{l(t)}...a_1(y_0).
\]
Then, we have:

\[
\sup_{z \in A: \sigma^l(t) + n(t)(z) \geq z_0} R^{(l(t)+n(t))}_-(z) \geq R^{(l(t)+n(t))}_-(z(t))
\]

\[
= R^{n(t)}_-(z(t)) + R^{l(t)}_-(y_l(t)) = R^{n(t)}_-(z(t))
\]

\[
\geq R^{n(t)}_-(z_0) - |R_-| \theta (\theta^t + \theta^{t+1} + \ldots + \theta^{t+n(t)})
\]

\[
\geq R^{n(t)}_-(z_0) - 2 \frac{\theta^t |R_-| \theta}{1 - \theta} \geq R^\infty_-(z_0) - 2 \frac{\theta^t |R_-| \theta}{1 - \theta}.
\]

So, when \( t \to \infty \)

\[
\limsup_{k \to \infty} \left( \sup_{z \in A, \sigma^k(z) = z_0} R^k_-(z) \right) \geq \limsup_{t \to \infty} \sup_{z \in A: \sigma^l(t) + n(t)(z) = z_0} R^{(l(t)+n(t))}_-(z)
\]

\[
\geq R^\infty_-(z_0).
\]

Using that \( z_0 \) is arbitrary, and satisfies \( R^\infty_-(z_0) > -\infty \), we conclude the proof.

From this result and lemma 12 we get:

**Proposition 17.**

\[
\liminf_{c,n \to \infty} \frac{1}{c} \log(\langle L^n_{g_0} \chi_A \rangle(x)) \geq \sup_{z \in A} R^-_\infty(z).
\]

This conclude the proof of Theorem 8.

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