On the issue of imposing boundary conditions on quantum fields

E. Elizalde
Consejo Superior de Investigaciones Científicas (ICE/CSIC)
Institut d’Estudis Espacials de Catalunya (IEEC)
Edifici Nexus, Gran Capità 2-4, 08034 Barcelona, Spain

Abstract

An interesting example of the deep interrelation between Physics and Mathematics is obtained when trying to impose mathematical boundary conditions on physical quantum fields. This procedure has recently been re-examined with care. Comments on that and previous analysis are here provided, together with considerations on the results of the purely mathematical zeta-function method, in an attempt at clarifying the issue. Hadamard regularization is invoked in order to fill the gap between the infinities appearing in the QFT renormalized results and the finite values obtained in the literature with other procedures.

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
A. Einstein
1 Introduction

The question, most beautifully phrased by Eugene Wigner as that of the unreasonable effectiveness of mathematics in the natural sciences [1] is an old and intriguing one. It certainly goes back to Pythagoras and his school ("all things are numbers"), even probably to the sumerians, and maybe to more ancient cultures, which left no trace. In more recent times, I. Kant and A. Einstein, among others, contributed to this idea with extremely beautiful and profound reflections. Also, mathematical simplicity, and beauty, have remained for many years, and to many important theoretical physicists, crucial ingredients when having to choose among different plausible possibilities.

A crude example of unreasonable effectiveness is provided by the regularization procedures in quantum field theory based upon analytic continuation in the complex plane (dimensional, heat-kernel, zeta-function regularization, and the like). That one obtains a physical, experimentally measurable, and extremely precise result after these weird mathematical manipulations is, if not unreasonable, certainly very mysterious. No wonder that for more one highly honorable physicist, the taming of the infinities that plague the calculations of modern quantum physics and the ensuing regularization/renormalization issue, were always unacceptable, illegal practices. Those methods are presently full justified and blessed with Nobel Prizes, but more because of the many and very precise experimental checkouts (the effectiveness) than for their intrinsic reasonableness.

As this note is meant not just for specialists, a simple example may be clarifying. Consider the calculation of the zero point energy (vacuum to vacuum transition, also called Casimir energy [2]) corresponding to a quantum operator, $H$, with eigenvalues $\lambda_n$,

$$E_0 = \langle 0|H|0\rangle = \frac{1}{2} \sum_n \lambda_n, \quad (1)$$

where the sum over $n$ is a sum over the whole spectrum (which will involve, in general, several continuum and several discrete indices.) The last appear typically when compactifying the space coordinates (much in the same way as time compactification gives rise to finite-temperature field theory). In turns out, in fact, that only in very special cases will this sum yield a convergent series. Generically one has to deal with a divergent sum, which is then regularized by different appropriate means. The zeta-function method [3] —which stands on solid and flourishing mathematical grounds [4]— will interprete this sum as being the value obtained for the zeta function of the operator $H$, namely

$$\zeta_H(s) = \sum_n \lambda_n^{-s}, \quad (2)$$

at the point $s = -1$ (we set $\hbar = c = 1$). It turns out generically that $\zeta_H(s)$ is well defined as an absolutely convergent series for Re $s > a_0$ ($a_0$ is a certain abscissa of convergence), and that it can be analytically continued (in a perfectly defined way) to the whole complex plane, with the possible appearance of poles as only singularities. If, as it often happens, $\zeta_H(s)$ has no pole at $s = -1$ we are done; if it hits a pole there, then further elaboration of the method is necessary. That the mathematical result one gets through this process
of analytical continuation in the complex plane, does coincide with the experimental result, constitutes the specific example of unreasonable effectiveness of mathematics we are referring to.

As one should expect, things do not turn out to be so simple and clear-cut in practice. Actually, one cannot assign a meaning to the absolute value of the zero-point energy, and any physically measurable effect is to be worked out as an energy difference between two situations, such as a quantum field in curved space as compared with the same field in flat space, or a field satisfying boundary conditions on some surface as compared with the same field in the absence of the surface, etc. This energy difference is the Casimir energy:

\[ E_C = E_{0C} - E_0 = \frac{1}{2} \left( \text{tr} \ H^{BC} - \text{tr} \ H \right). \]  

(3)

And it is here where the problem appears. Imposing mathematical boundary conditions on physical quantum fields turns out to be a highly non-trivial act. This was discussed in much detail in a paper by Deutsch and Candelas a quarter of a century ago [5]. These authors quantized electromagnetic and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress-energy tensor, to find that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite physical energy was obtained if the mathematical boundary conditions were to be fulfilled. However, the authors argued that, in nature, surfaces have non-zero depth, and its value could be taken as a handy (dimensional) cutoff in order to regularize the infinities. This approach will be recovered later in the present paper. Just two years after Deutsch and Candelas’ work, Kurt Symanzik carried out a rigorous analysis of quantum field theory in the presence of boundaries [6]. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure, and also point splitting, to be applied. In any case, the issue was proven to be perfectly meaningful within the domains of renormalized quantum field theory. One should note that in this case the boundary conditions and the hypersurfaces themselves were always treated at the pure mathematical level (zero depth) by using delta functions.

More recently, a new approach to the problem has been postulated [7] which uses elements of the two methods above. Boundary conditions on a field, \( \phi \), are enforced on a surface, \( S \), by introducing a scalar potential, \( \sigma \), of Gaussian shape living on the surface (and near it). In the limit when the Gaussian becomes a delta function, the boundary conditions (Dirichlet, in this case) are enforced, in that the delta-shaped potential kills all the modes of \( \phi \) at the surface. For the rest, the quantum system undergoes a full-fledged quantum field theory renormalization, as in the case of Symanzik’s approach. The results obtained confirm those of [5], in the several models studied albeit they do not seem to agree with those of [6].
Such results are also in clear contradiction with the ones quoted in the usual textbooks and review articles dealing with the Casimir effect [8], where no infinite energy density when approaching any of the Casimir plates has been reported.

2 A zeta-function approach

Too often has it been argued that sophisticated regularization methods, as for instance the zeta-function regularization procedure, get rid of the infinities in a rather obscure way (through analytic continuation, as previously described), so that, contrary to what happens when a cut off is introduced, one cannot keep trace of the infinites, which are cleared up without control, leading sometimes to erroneous results.

There is no way to refute an statement of this kind rigorously, but it should be pointed out that in many occasions (if not always) the discrepancies between the result obtained by using the zeta procedure and other, say cut-off like, approaches have been proven to emerge from a \textit{misuse} of zeta regularization, and \textit{not} to stem from the method itself. When employed properly, the correct results have been recovered (for a good number of examples, see [3, 4, 9, 10]).

Take the most simple example of a scalar field in one dimension, $\phi(x)$, with a boundary condition (BC) of Dirichlet type imposed at a point (say $x = 0$), e.g. $\phi(0) = 0$. We would like to calculate the Casimir energy corresponding to this configuration, that is, the difference between the zero point energy corresponding to this field when the boundary condition is enforced, and the zero point energy in the absence of any boundary condition at all. Taken at face value, both energies are infinite. The regularized difference may still be infinite when the BC point is approached (this is the result in [7]) or might turn out to be finite (even zero, which is the result given in some standard books on the subject).

Let us try to understand this discrepancy. According to Eq. (1), we have to add up all energy modes. For the mode with energy $\omega$, the field equation reduces to:

$$-\phi''(x) + m^2 \phi(x) = \omega^2 \phi(x).$$

In the absence of a BC, the solutions to the field equation can be labeled by $k = +\sqrt{\omega^2 - m^2} > 0$, as

$$\phi_k(x) = Ae^{ikx} + Be^{-ikx},$$

with $A, B$ arbitrary complex (for the general complex), or as

$$\phi_k(x) = a \sin(kx) + b \cos(kx),$$

with $a, b$ arbitrary real (for the general real solution). Now, when the mathematical BC of Dirichlet type, $\phi(0) = 0$, is imposed, this does not influence at all the eigenvalues, $k$, which remain exactly the same (as stressed in the literature). However, the number of solutions corresponding to each eigenvalue is reduced by one half to:

$$\phi_k^{(D)}(x) = A(e^{ikx} - e^{-ikx}),$$
with an arbitrary complex (complex solution), and
\[ \phi_k^{(D)}(x) = a \sin(kx), \tag{8} \]
with an arbitrary real (real solution). In other words, the energy spectrum (for omega) that we obtain in both cases is the same, a continuous spectrum
\[ \omega = \sqrt{m^2 + k^2}, \tag{9} \]
but the number of eigenstates corresponding to a given eigenvalue is twice as big in the absence of the BC.\(^2\)

Of course these considerations are elementary, but they seem to have been put aside sometimes. They are crucial when trying to calculate (or just to give sense to) the Casimir energy density and force. More to the point, just as the traces of the two matrices \(M_1 = \text{diag} (\alpha, \beta)\) and \(M_2 = \text{diag} (\alpha, \alpha, \beta, \beta)\) are not equal in spite of having “the same spectrum \(\alpha, \beta\),” it also turns out that, in the problem under discussion, the traces of the Hamiltonian with and without the Dirichlet BC imposed yield different results, both of them divergent, namely
\[ \text{tr } H = 2 \text{tr } H^{BC} = 2 \int_0^\infty dk \sqrt{m^2 + k^2}. \tag{10} \]
By using the zeta function, we define
\[ \zeta^{BC}(s) := \int_0^\infty d\kappa (\nu^2 + \kappa^2)^{-s}, \quad \nu := \frac{m}{\mu}, \tag{11} \]
with \(\mu\) a regularization parameter with dimensions of mass.\(^3\) We get
\[ \zeta_{BC}(s) = \frac{\sqrt{\pi}}{2 \Gamma(s)} \frac{\Gamma(s - 1/2)}{\Gamma(s)} (\nu^2)^{1/2-s}, \tag{12} \]
and consequently,
\[ \text{tr } H^{BC} = \frac{1}{2} \zeta_{BC}(s = -1/2) = \left. \frac{m^2}{4 \sqrt{\pi}} \left[ \frac{1}{s + 1/2} + 1 - \gamma - \log \frac{m^2}{\mu^2} - \Psi(-1/2) + \mathcal{O}(s + 1/2) \right] \right|_{s=-1/2}. \tag{13} \]
As is obvious, this divergence is not cured when taking the difference of the two traces, Eq. (3), in order to obtain the Casimir energy:
\[ E_C/\mu = E_0^{BC}/\mu - E_0/\mu = -E_0^{BC}/\mu = \frac{\Gamma(-1)m^2}{8\mu^2}. \tag{14} \]

\(^2\)To understand this point even better (by making recourse to what is learned in the maths classes at high school), consider the fact that further, by imposing Cauchy BC: \(\phi(0) = 0, \phi'(0) = 0\), the eigenvalues remain the same, but for any \(k\) the family of eigenfunctions shrinks to just the trivial one: \(\phi_k(x) = 0, \forall k\) (the Cauchy problem is an initial value problem, which completely determines the solution).

\(^3\)Always necessary in zeta regularization, since the complex powers of the spectrum of a (pseudo-) differential operator can only be defined, physically, if the operator is rendered dimensionless, what is done by introducing this parameter. That is also an important issue, which is sometimes overlooked.
We just hit the pole of the zeta function, in this case.

How is this infinite to be interpreted? What is its origin? Just by making recourse to the pure mathematical theory (\textit{reine Mathematik}), we already get a perfect description of what happens and understand well where does this infinite energy\(^4\) come from. It clearly originates from the fact that imposing the boundary condition has reduced to one-half the family of eigenfunctions corresponding to any of the eigenvalues which constitute the spectrum of the operator. And we can also advance that, since this dramatic reduction of the family of eigenfunctions takes place precisely at the point where the BC is imposed, the physical divergence (infinite energy) will originate right there, and nowhere else.

While the analysis above cannot be taken as a substitute for the actual modelization of Jaffe et al. [7] —where the BC is explicitly enforced through the introduction of an auxiliary, localized field, which probes what happens at the boundary in a much more precise way—it certainly shows that \textit{pure mathematical considerations}, which include the use of analytic continuation by means of the zeta function, are in no way blind to the infinites of the physical model and \textit{do not} produce misleading results, when the mathematics are used properly. And it is very remarkable to realize how close the mathematical description of the appearance of an infinite contribution is to the one provided by the physical realization [7].

3 \hspace{1em} \textbf{The case of two-point Dirichlet boundary conditions}

A similar analysis can be done for the case of a two-point Dirichlet BC:

\[ \phi(a) = 0, \quad \phi(-a) = 0. \]  \hspace{1em} \text{(15)}

Straightforward algebra shows, in this situation, that the eigenvalues \(k\) are quantized, as \(k = \frac{\pi}{2a}\), so that:

\[ \omega_\ell = \sqrt{m^2 + \frac{\ell^2\pi^2}{4a^2}}, \quad \ell = 0, 1, 2, \ldots \]  \hspace{1em} \text{(16)}

The family of eigenfunctions corresponding to a given eigenvalue, \(\omega_\ell\), is of continuous dimension 1, exactly as in the former case of a one-point Dirichlet BC, namely,

\[ \phi_\ell(x) = b \sin \left( \frac{\ell \pi}{2a} (x - a) \right), \]  \hspace{1em} \text{(17)}

where \(b\) is an arbitrary, real parameter.\(^5\) To repeat, the act of imposing Dirichlet BC on two points has the effect of discretizing the spectrum (as is well known) but there is no further shrinking in the number of eigenfunctions corresponding to a given (now discrete) eigenvalue.

The calculation of the Casimir energy, by means of the zeta function, proceeds in this case as follows [3, 4, 9, 10]. To begin with, it may be interesting to recall that the zeta-‘measure’

\(^4\)In mathematical terms, this infinite value for the trace of the Hamiltonian operator.

\(^5\)The contribution of the zero-mode (\(\ell = 0\)) is controverted, but we are not going to discuss this issue here (see e.g. [11] and references therein.)
of the continuum equals twice the zeta-‘measure’ of the discrete. In fact, just consider the following regularizations:

\[
\sum_{n=1}^{\infty} \mu = \mu \sum_{n=1}^{\infty} n^{-s} \bigg|_{s=0} = \mu \zeta_R(0) = -\frac{\mu}{2},
\]

and

\[
\int_{\mu}^{\infty} dk = \int_{0}^{\infty} dk (k + \mu)^{-s} \bigg|_{s=0} = \frac{\mu^{1-s}}{s-1} \bigg|_{s=0} = -\mu.
\]

The result is, as advanced, that the \( \zeta \) ‘measure’ of the discrete is half that of the continuum.

The trace of the Hamiltonian corresponding to the quantum system with the BC imposed, in the massive case, is obtained by means of the zeta function

\[
\zeta^{BC}(s) := \sum_{\ell=1}^{\infty} \left( \frac{m^2}{\mu^2} + \frac{\pi^2 \ell^2}{4 \mu^2 a^2} \right)^{-s}
= \left( \frac{\mu}{m} \right)^{2s} \left[ -\frac{1}{2} + \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{am}{\sqrt{\pi}} + \frac{2\pi^s}{\Gamma(s)} \left( \frac{2am}{\pi} \right)^{1/2+s} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(4amn) \right],
\]

being \( K_\nu \) a modified Bessel function of the third kind (or MacDonald’s function). Thus, for the zero point energy of the system with two-point Dirichlet BC, we get

\[
\text{tr } H^{BC}/\mu = \frac{1}{2} \zeta^{BC}(s = -1/2) = -\frac{\Gamma(-1)m^2}{8\mu^2} - \frac{m}{2\pi \mu} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi nm/\mu),
\]

where \( \mu \) is, in this case, \( \mu := \pi/(2a) \) (\( a \) fixes the mass scale in a natural way here). As in the previous example, we finally obtain an infinite value for the Casimir energy, namely

\[
E_C/\mu = E_0^{BC}/\mu - E_0/\mu = \frac{\Gamma(-1)m^2}{8\mu^2} - \frac{m}{2\pi \mu} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2\pi nm/\mu).
\]

It is, therefore, not true that regularization methods using analytical continuation (and, in particular, the zeta approach) are unable to see the infinite energy that is generated on the boundary-conditions surface [5, 6, 7] (see Eq. (34) later). The reason is still the same as in the previous example: imposing a two-point Dirichlet BC amounts again to halving the family of eigenfunctions which correspond to any given eigenvalue (all are discrete, in the present case, but this makes no difference). In physical terms, that means having to apply an infinite amount of energy on the BC sites, in order to enforce the BC. In absolute analogy, from the mathematical viewpoint, halving the family of eigenfunctions immediately results in the appearance of an infinite contribution, under the form of a pole of the zeta function.

The reason why these infinities (the one here and that in the previous section) do not usually show up in the literature on the Casimir effect is probably because the textbooks on the subject focus towards the calculation of the Casimir force, which is obtained by taking minus the derivative of the energy with respect to the plate (or point) separation (here w.r.t. \( 2a \)). Since the infinite terms do not depend on \( a \), they do not contribute to the force (as
is recognized explicitly in [7]). However, some erroneous statements have indeed appeared in the above mentioned classical references, stemming from the lack of recognition of the catastrophic implications of the act of halving the number of eigenfunctions, when imposing the BC. The persistence of the eigenvalues of the spectrum was probably misleading. We hope to have clarified this issue here.

4 How to deal with the infinities

Here, the infinite contributions have shown up at the regularization level, but a more careful study [7] was able to prove that they do not disappear even after renormalizing in a proper way. The important question is now: are these infinities physical? Will they be observed as a manifestation of a very large energy pressure when approaching the BC surface in a lab experiment? No doubt such questions will be best answered in that way, e.g. experimentally. If, on the contrary, that sort of large pressures fails to manifest itself, this might be a clear indication of the need for an additional regularization prescription. In principle, this seems to be forbidden by standard renormalization theory, since the procedure has been already carried out to the very end: there remains no additional physical quantity which could possibly absorb the divergences (see [7]).

In any case, there are circumstances —both in physics and in mathematics— where certain ‘non-orthodox’ regularization methods have been employed with promising success. In particular, Hadamard regularization in higher-post-Newtonian general relativity [12] and also in recent variants of axiomatic and constructive quantum field theory [13]. Among mathematicians, Hadamard regularization is nowadays a rather standard technique in order to deal with singular differential and integral equations with boundary conditions, both analytically and numerically (for a sample of references see [14]). Indeed, Hadamard regularization is a well-established procedure in order to give sense to infinite integrals. It is not to be found in the classical books on infinite calculus by Hardy or Knopp; it was L. Schwartz [15] who popularized it, rescuing Hadamard’s original papers. Nowadays, Hadamard convergence is one of the cornerstones in the rigorous formulation of QFT through micro-localization, which on its turn is considered by specialists to be the most important step towards the understanding of linear PDEs since the invention of distributions (for a beautiful, updated treatment of Hadamard’s regularization see [16]).

Let us briefly recall this formulation. Consider a function, \( g(x) \), expandable as

\[
g(x) = \sum_{j=1}^{k} \frac{a_j}{(x-a)^{\lambda_j}} + h(x),
\]

with \( \lambda_j \) complex in general and \( h(x) \) a regular function. Then, it is immediate that

\[
\int_{a+\epsilon}^{b} dx \, g(x) = P(1/\epsilon) + H(\epsilon),
\]

being \( P \) a polynomial and \( H(0) \) finite. If the \( \lambda_j \not\in \mathbb{N} \), then one defines the Hadamard
regularized integral as
\[ \int_a^b \, dx \, g(x) := \int_a^b \, h(x) \, dx - \sum_{j=1}^{k} \frac{a_j}{\lambda_j - 1} (b - a)^{1 - \lambda_j}. \] (25)

Alternatively, one may define, for \( \alpha \not\in \mathbb{N} \), \( p < \alpha < p + 1 \), and \( f^{(p+1)} \in C_{[-1,1]} \),
\[ K^\alpha f := \frac{1}{\Gamma(-\alpha)} \int_{-1}^{1} \frac{f(t)}{(1 - t)^{\alpha + 1}}, \] (26)
to obtain, after some steps,
\[ K^\alpha f = \sum_{j=0}^{p} \frac{f^{(j)}(-1)}{\Gamma(j + 1 - \alpha) 2^{\alpha - j}} + \frac{1}{\Gamma(p + 1 - \alpha)} \int_{-1}^{1} (1 - t)^{p - \alpha} f^{(p+1)}(t), \] (27)
where the last integral is at worst improper (Cauchy’s principal part). If \( \lambda_1 = 1 \), then the result is \( a_1 \ln(b - a) \), instead. If \( \lambda_1 = p \in \mathbb{N} \), calling
\[ H_p(f; x) := \int_{-1}^{1} \frac{f(t)}{(t - x)^{p+1}}, \quad |x| < 1, \] (28)
we get
\[ H_p(f; x) = \int_{-1}^{1} \left[ f(t) - \sum_{j=0}^{p} \frac{f^{(j)}(x)}{j!} (t - x)^{j} \right] \frac{dt}{(t - x)^{p+1}} + \frac{f^{(j)}(x)}{j!} \int_{-1}^{1} \frac{dt}{(t - x)^{p+1-j}}, \] (29)
where the first term is regular and the second one can be easily reduced to
\[ \frac{1}{(p - j)!} \frac{d^{p-j}}{dx^{p-j}} \int_{-1}^{1} \frac{dt}{t - x}, \] (30)
being the last integral, as before, a Cauchy PP.

An alternative form of Hadamard’s regularization, which is more fashionable for physical applications (as is apparent from the expression itself) is the following [12]. For the case of two singularities, at \( \vec{x}_1, \vec{x}_2 \), after excising from space two little balls around them, \( \mathbb{R}^3 \setminus (B_{r_1}(\vec{x}_1) \cup B_{r_2}(\vec{x}_2)) \), with \( B_{r_1}(\vec{x}_1) \cap B_{r_2}(\vec{x}_2) = \emptyset \), one defines the regularized integral as being the limit
\[ \int d^3 x \, F(\vec{x}) := \text{FP}_{\alpha, \beta \to 0} \int d^3 x \left( \frac{r_1}{s_1} \right)^{\alpha} \left( \frac{r_2}{s_2} \right)^{\beta} F(\vec{x}), \] (31)
where \( s_1 \) and \( s_2 \) are two (dimensionfull) regularization parameters [12]. This is the version that will be employed in what follows.

5 Hadamard regularization of the Casimir effect

We now use Hadamard’s regularization as an additional tool in order to make sense of the infinite expressions encountered in the boundary value problems considered before. As it
turns out from a detailed analysis of the results in [7] (which we shall not repeat here, for conciseness), the basic integrals which produce infinities, in the one-dimensional and two-dimensional cases there considered, are the following.

In one dimension, with Dirichlet BC imposed at one \((x = 0)\) and two \((x = \pm a)\) points, respectively, by means of a delta-background of strength \(\lambda\) (see [7]), one encounters the two divergent integrals:

\[
E_1(\lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{dt}{\sqrt{t^2 - m^2}} \left[ t \log \left( 1 + \frac{\lambda}{2t} \right) - \frac{\lambda}{2} \right], \tag{32}
\]

\[
E_2(a, \lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{dt}{\sqrt{t^2 - m^2}} \left\{ t \log \left[ 1 + \frac{\lambda}{t} + \frac{\lambda^2}{4t^2} \left( 1 - e^{-4at} \right) \right] - \lambda \right\}. \tag{33}
\]

Using Hadamard’s regularization, as described before, we obtain for the first one, Eq. (32),

\[
E_1(m) = \frac{\lambda}{4\pi} \left( 1 - \ln \frac{\lambda}{m} \right) \bigg|_{\lambda \to \infty} + \int f, \tag{34}
\]

where the first term is the singular part when the limit \(\lambda \to \infty\) is taken, and the second—which is Hadamard’s finite part—yields in this case

\[
f = -\frac{m}{4}. \tag{35}
\]

Such result is coinciding with the classical one \((0, \text{ for } m = 0)\). Note in particular, that the further \(\ln m\) divergence as \(m \to \infty\) is hidden in the \(\lambda\)-divergent part, and that behavior does explain why the classical results which are obtained using hard Dirichlet BC—which corresponds as we just prove here to the Hadamard’s regularized part—cannot see anything of it.

In the case of a two-point boundary at \(x = \pm a\) (separation \(2a\)), Eq. (33), we get a similar Eq. (34) but now the regularized integral is as follows. For the massless case, we obtain

\[
f = -\frac{\pi}{48a}, \tag{36}
\]

which is the regularized result to be found in the classical books. In the massive case, \(m \neq 0\), after some additional work the following fast convergent series turns up [cf. Eq. (22)]

\[
f = -\frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} K_1(4akm). \tag{37}
\]

Thus Eq. (34) yields strictly the same result (22) that was already obtained by imposing the Dirichlet BC \(ab initio\). What has now been \(gained\) is a more clear identification of the singular part, in terms of the strength of the delta potential at the boundary. This will be the general conclusion, common to all the other cases here considered.

Correspondingly, for the Casimir force we obtain the finite values\(^6\)

\[
F_2(a) = -\frac{\pi}{96a^2}, \tag{38}
\]

\(^6\)Note that the force \(F(a)\) is here given by minus the derivative of the total energy \(E(a)\) w.r.t. \(2a\), since this is the distance between the two Dirichlet points (not \(a\)).
in the massless case, and in the massive one

\[ F_2(a, m) = -\frac{m^2}{\pi} \sum_{k=1}^{\infty} \left[ K_0(4akm) + \frac{1}{4akm} K_1(4akm) \right]. \] (39)

Those expressions coincide with the ones derived in the above mentioned textbooks on the Casimir effect, and reproduced before by using the zeta function method (just take minus the derivative of Eq. (34) w.r.t. 2a).

The two dimensional case turns out to be more singular [7] —in part just for dimensional reasons— and requires additional wishful thinking in order to deal with the circular delta function sitting on the circumference where the Dirichlet BC are imposed. Here one encounters the basic singular integral, for the term contributing to the second Born approximation (we use the same notation as in [7]),

\[ \tilde{\sigma}(p) = \int_0^\infty dr r J_0(pr) \sigma(r), \quad \sigma(r) = b\lambda \exp\left[ -\frac{(r-a)^2}{2\omega^2} \right], \] (40)

with \( J_0 \) a Bessel function of the first kind, and

\[ \int_0^\infty dr \sigma(r) = \lambda, \quad \sigma(r) \xrightarrow{\omega \to 0} \lambda \delta(r-a). \] (41)

Hadamard’s regularization yields, in this case (the \( \tau \)’s replacing the \( \sigma \)’s in the regularized version),

\[ \tau(r, p) = c\lambda(rp + 1)^{-\omega/2} \exp\left[ -\frac{(r-a)^2}{2\omega^2} \right] \xrightarrow{\omega \to 0} \lambda \delta(r-a), \] (42)

with \( p \) a (dimensionfull) regularization parameter, being the constant \( c \) given by

\[ c^{-1} = \int_0^\infty dr r^{-\omega} \exp\left[ -\frac{(r-a)^2}{2\omega^2} \right], \] (43)

which exists and is perfectly finite; in particular, \( c^{-1}(\omega = .1, a = 1) = .25 \). Then,

\[ \tilde{\tau}(p) = 2\pi \int_0^\infty dr J_0(pr) \tau(r, p) = 2\pi \lambda a(ap + 1)^{-\omega/2} J_0(ap) \] (44)

It turns out that for the Casimir energy we get in this case (notation as in [7])

\[ E_{\lambda^2}^{(2)}[\tau] = \frac{\lambda^2 a^2}{8} \left. \int_0^\infty dp (ap + 1)^{-\omega} J_0(ap)^2 \arctan(p/2m) \right|_{\omega \to 0} \]

\[ = \frac{\lambda^2 a^2}{8} \left\{ \frac{1}{2\omega} + \frac{\gamma + 3 \ln 2}{2a} + 4m \left[ \gamma - \frac{2}{\sqrt{\pi}} [1 - \ln(am)] h(4a^2m^2) \right] \right\}, \] (45)

where \( h(z) := {}_2F_3((1/2, 1/2); (1, 1, 3/2); z) \) and \( \gamma \) is the Euler-Mascheroni constant; in particular, for instance \( h(1) = 1.186711 \), what is quite a nice value. Recall also that \( \omega \) is the width of the Gaussian \( \delta \), which is the very physical parameter considered in [5]. When this
width tends to zero an infinite energy appears (the width controls the formation of the pole). 
The rest of the result is the Hadamard regularization of the integral, e.g.\[ \int_0^\infty dp J_0(ap)^2 \arctan(p/2m). \] (46)

Again, the finite part reverts to the results obtained in the literature with Dirichlet BC \textit{ab initio}.

To summarize, it has been here proven —in some particular but rather non-trivial and representative examples— that the finite results derived through the use of Hadamard’s regularization exactly coincide with the values obtained using the more classical, less full-fledged methods to be found in the literature on the Casimir effect. Moreover, Hadamard’s prescription is able to separate and identify the singularities as physically meaningful cutoffs. Although the validity of this additional regularization is at present questionable, the fact that it bridges the two approaches is already remarkable, maybe again a manifestation of the unreasonable effectiveness of mathematics.

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\footnote{It should be pointed out that the computational program Mathematica [17] directly assigns the Hadamard regularized value to particular cases of integrals of this kind; but it does so without any hint on what is going on. This has often confused more one user, who fails to understand how it comes that an infinite integral gets a finite value out of nothing.}
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