ON IRREDUCIBLE REPRESENTATIONS OF A CLASS OF QUANTUM SPHERES

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1. INTRODUCTION

Throughout the paper we shall let $0 < q < 1$ be a deformation parameter and $n$ be a positive integer. We denote by $\mathcal{A}(S_q^{4n-1})$ the complex unital $*$-algebra generated by elements $\{x_i, y_i\}_{i=1}^n$ and their adjoints, subject to the relations in Definition 1 below. This sphere is a comodule algebra for the quantum symplectic group $\mathcal{A}(\text{Sp}_q(n))$, with coaction $\mathcal{A}(S_q^{4n-1}) \to \mathcal{A}(\text{Sp}_q(n)) \otimes \mathcal{A}(S_q^{4n-1})$. In fact this is a quantum homogeneous space and the algebra $\mathcal{A}(S_q^{4n-1})$ sits as a subalgebra of the algebra $\mathcal{A}(\text{Sp}_q(n))$. Representations of $\mathcal{A}(S_q^{4n-1})$ can be obtained as restrictions of representations of $\mathcal{A}(\text{Sp}_q(n))$, see e.g. [3].

Let $\mathcal{A}(\Sigma_q^{2n+1})$ be the quotient of $\mathcal{A}(S_q^{4n-1})$ by the two-sided $*$-ideal generated by the elements $\{x_i\}_{i=1}^n$. As customary, we interpret a quotient algebra as consisting of “functions” on a quantum subspace and think of $\Sigma_q^{2n+1}$ as a quantum subsphere of $S_q^{4n-1}$. If we further quotient by the ideal generated by the coordinate $x_n$, we get a $(2n-1)$-dimensional Vaksman-Soibelman quantum sphere [4], whose representation theory is well known (see e.g. [11]). In this short letter we give an independent derivation of the bounded irreducible $*$-representations of the algebra $\mathcal{A}(\Sigma_q^{2n+1})$ that do not annihilate the generator $x_n$.

Definition 1. We denote by $\mathcal{A}(S_q^{4n-1})$ the complex unital $*$-algebra generated by elements $\{x_i, y_i\}_{i=1}^n$ and their adjoints, subject to the following relations. Firstly, one has:

$$x_ix_j = q^{-1}x_jx_i \quad (i < j), \quad y_iy_j = q^{-1}y_jy_i \quad (i > j), \quad x_ix_j = q^{-1}y_jx_i \quad (i \neq j),$$

(1)

$$y_ix_i = q^2x_iy_i + (q^2 - 1) \sum_{k=1}^{i-1} q^{i-k} x_k y_k,$$

Next, one has

$$x_i^2 = x_i\{x_i, x_i\} + (1 - q^2) \sum_{k=1}^{i-1} x_k^2 x_k$$

(3)

$$y_i^2 = y_i\{y_i, y_i\} + (1 - q^2) \left\{ q^{2(n+1-i)} x_i^2 + \sum_{k=1}^n x_k^2 x_k + \sum_{k=i+1}^n y_k^2 y_k \right\}$$

(4)

$$x_ix_i^* = q^2 y_i^* x_i$$

(5)

$$x_ix_i^* = q x_i^* x_i \quad (i \neq j)$$

(6)

$$y_iy_i^* = q y_i^* y_i - (q^2 - 1) q^{2n+2-i} x_i^* x_j \quad (i \neq j)$$

(7)

$$x_iy_i^* = q x_i^* y_i \quad (i < j)$$

(8)

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\[
x_i y_j^* = q y_j^* x_i + (q^2 - 1) q^{1-j} y_i^* x_j \quad (i > j)
\]

Finally, one has the sphere relation:

\[
\sum_{i=1}^{n} (x_i^* x_i + y_i^* y_i) = 1.
\]

One passes to the notations of [2] by setting \( y_i = x_{2n+1-i} \) and replacing \( q \) by \( q^{-1} \).

Let \( A(\Sigma^2_q) \) be the quotient of \( A(S^4_q) \) by the two-sided \(*\)-ideal generated by \( \{x_i\}_{i=1}^{n-1} \).

Let us write down explicitly the relations in this quotient algebra. If we rename \( y_{n+1} := x_n \), it follows from [3] that \( y_{n+1} \) is normal. The remaining relations become:

\[
y_i y_j = q^{-1} y_j y_i \quad (i > j \land (i, j) \neq (n + 1, n)) \quad (11)
\]

\[
y_i^* y_j = q^{-1} y_j^* y_i^* \quad (i > j \land (i, j) \neq (n + 1, n)) \quad (12)
\]

\[
y_{n+1} y_n = q^{-2} y_n y_{n+1} \quad y_{n+1}^* y_n = q^{-2} y_n^* y_{n+1} \quad (13)
\]

\[
[y_i, y_i^*] = (1 - q^2) \sum_{k=1}^{n+1} y_k y_k \quad (i \neq n) \quad (14)
\]

\[
[y_n, y_n^*] = (1 - q^4) y_{n+1}^* y_{n+1} \quad (15)
\]

plus the ones obtained by adjunction and the sphere relation:

\[
\sum_{i=1}^{n+1} y_i^* y_i = 1. \quad (16)
\]

Using these relations, it is straightforward to check the following statement.

**Proposition 2.** For every \( \lambda \in \mathbb{U}(1) \), an irreducible bounded \(*\)-representation \( \pi_\lambda \) of \( A(\Sigma^2_q) \) on \( \ell^2(\mathbb{N}^n) \) is given by the formulas:

\[
\pi_\lambda(y_i)|_k = q^{k_i} \cdots q^{k_{i-1}} \sqrt{1 - q^{2k_i}} | k - e_i \rangle \quad (1 \leq i \leq n - 1)
\]

\[
\pi_\lambda(y_n)|_k = q^{k_1} \cdots q^{k_{n-1}} \sqrt{1 - q^{4k_n}} | k - e_n \rangle,
\]

\[
\pi_\lambda(y_{n+1})|_k = \lambda q^{k_1 + k_{n-1}} | k \rangle,
\]

where \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \), \( |k\rangle := k_1 + \ldots + k_n \), \( \{ |k\rangle \}_{k \in \mathbb{N}^n} \) is the canonical orthonormal basis of \( \ell^2(\mathbb{N}^n) \) and \( e_i \) the \( i \)-th row of the identity matrix of order \( n \).

Our aim now is to prove the next proposition.

**Proposition 3.** Any irreducible bounded \(*\)-representation of \( A(\Sigma^2_q) \) that does not annihilate \( x_n \) is unitarily equivalent to one of the representations in Proposition 2.

We need a few preliminary lemmas.

**Lemma 4.** For all \( m \geq 1 \) and all \( 1 \leq i < n \):

\[
y_i y_i^m = q^{2m} y_i^m + (1 - q^{2m}) y_i^m - 1 \left( \sum_{k=1}^{n} y_k^* y_k \right)
\]

\[
y_n (y_n^*)^m = q^{4m} y_n^m + (1 - q^{4m}) y_n^m - 1 \left( \sum_{k=n}^{n} y_k^* y_k \right)
\]
Proof. When \( m = 1 \), these follow from (14) and (15), and can be rewritten using (16) as:

\[
y_i y_i^* = q^2 y_i^* y_i + (1 - q^2) \left( 1 - \sum_{k < i} y_k^* y_k \right) \quad \text{if } i < n
\]

\[
y_n y_n^* = q^4 y_n^* y_n + (1 - q^4) \left( 1 - \sum_{k < n} y_k^* y_k \right)
\]

The general result easily follows using the latter relations, induction on \( m \) and the fact that \( y_i^* y_k \) commutes with \( y_i \) for all \( k < i \leq n \).

\[\Box\]

Lemma 5. Let \( A \geq 0 \) and \( B \) be bounded operators on a Hilbert space \( \mathcal{H} \) satisfying

\[
[B, B^*] = (1 - \mu)A \quad A + B^*B = 1
\]

with \( 0 < \mu < 1 \). Then, \( \ker(B) = \{0\} \) if and only if \( A = 0 \).

Proof. If \( A = 0 \), from (17) it follows that \( B \) is unitary, so that \( \ker(B) = \{0\} \). We have to prove the opposite implication. From (17) we deduce that \( \|A\| \leq 1 \) and:

\[
BB^* = B^*B + (1 - \mu)A = 1 - \mu A.
\]

Since \( \|\mu A\| < 1 \), the operator \( BB^* \) has bounded inverse. Therefore \( U := (BB^*)^{-1/2}B \) is well defined. Notice that \( UU^* = 1 \) and \( \ker(B) = \ker(U) \). From (18), it follows that

\[
\mu A = 1 - BB^* = U(1 - B^*B)U^* = UAU^*.
\]

Assume that \( \ker(U) = \{0\} \). The identity \( U(1 - U^*U) = 0 \) implies \( U(1 - U^*U)v = 0 \) for all \( v \in \mathcal{H} \). Therefore \( U^*U = 1 \) and \( U \) is a unitary operator.

Let \( \lambda \in \mathbb{C} \) and suppose \( A - \mu^{-1}\lambda \) has bounded inverse. Then

\[
A - \lambda = \mu U^*(A - \mu^{-1}\lambda)U
\]

has bounded inverse as well. Thus, \( \lambda \in \sigma(A) \) implies that \( \mu^{-1}\lambda \in \sigma(A) \) and hence, by induction, that \( \mu^{-k}\lambda \in \sigma(A) \) for all \( k \geq 0 \). Since \( A \) is bounded and the sequence \( \{\mu^{-k}\lambda\}_{k \geq 0} \) is divergent when \( \lambda \neq 0 \), it follows that \( \sigma(A) = \{0\} \). Hence \( A = 0 \).

\[\Box\]

Lemma 6. Let \( \pi \) be an irreducible bounded \(*\)-representation of \( A(\Sigma q^{2n+1}) \) with \( \pi(y_{n+1}) \neq 0 \). Then:

(i) \( \pi(y_{n+1}) \) is injective;

(ii) there exists a vector \( \xi \neq 0 \) such that \( \pi(y_i)\xi = 0 \) for all \( i \neq n + 1 \).

Proof. (i) If \( a \) is any generator other than \( y_{n+1} \), since \( y_{n+1}a \) is a scalar multiple of \( ay_{n+1} \), the operator \( \pi(a) \) maps the kernel of \( \pi(y_{n+1}) \) to itself. Hence, \( \ker\pi(y_{n+1}) \) carries a subrepresentation of the irreducible representation \( \pi \), so that either \( \ker\pi(y_{n+1}) = \{0\} \) or \( \ker\pi(y_{n+1}) = \mathcal{H} \). The latter implies \( \pi(y_{n+1}) = 0 \), contradicting the hypothesis, so that the former must hold.

(ii) Given \( 1 \leq k \leq n \), let \( \mathcal{H}_k := \cap_{i=1}^k \ker\pi(y_i) \). We prove by induction on \( k \) that \( \mathcal{H}_k \neq \{0\} \). When \( k = 1 \), this follows from Lemma 5 applied to the operators \( A = \sum_{i=1}^{n+1} \pi(y_i^*y_i) \) and \( B = \pi(y_1) \). Since \( \pi(y_{n+1}) \neq 0 \), it follows that \( A \neq 0 \), and hence \( \ker(B) \neq \{0\} \).

Now assume that \( \mathcal{H}_{k-1} \neq \{0\} \). Let \( A = \sum_{i=k}^{n+1} \pi(y_i^*y_i) \) and \( B = \pi(y_k) \), and note that the operators \( \pi(y_{n+1}), A, B, B^* \) map \( \mathcal{H}_{k-1} \) to itself, since \( y_i y_j \) is a scalar multiple of \( y_j y_k \), and \( y_i y_i^* \) is a scalar multiple of \( y_j^* y_i \) for all \( i \neq j \). It follows from point (i) that \( \pi(y_{n+1})|_{\mathcal{H}_{k-1}} \neq 0 \), so that \( A \) is non-zero on \( \mathcal{H}_{k-1} \). The operator \( A + B^*B \) restricts to the identity on \( \mathcal{H}_{k-1} \) and
\[ B, B^* = (1 - \mu)A \] with \( \mu = q^2 \) if \( k < n \) and \( \mu = q^4 \) if \( k = n \). From Lemma 6(ii) applied to the restrictions of \( A, B, B^* \) to \( \mathcal{H}_{k-1} \) it follows that \( \ker(B) \cap \mathcal{H}_{k-1} = \mathcal{H}_k \neq \{0\} \). \[ \square \]

**Proof of Prop. 3** Let \( \pi \) be a bounded irreducible \( \sigma \)-representation of \( A(\Sigma_q^{2n+1}) \) on a Hilbert space \( \mathcal{H} \) such that \( \pi(y_{n+1}) \neq 0 \). With an abuse of notation, we suppress the map \( \pi \). We know from Lemma 5(ii) that \( V := \bigcap_{i=1}^n \ker(y_i) \neq \{0\} \). From the commutation relations we deduce that \( y_{n+1} V \subset V \), so that \( V \) carries a bounded \( \sigma \)-representation of the commutative \( C^* \)-algebra \( C^*(y_{n+1}, y_{n+1}^*) \) generated by \( y_{n+1} \) and \( y_{n+1}^* \).

Given \( k \in \mathbb{N}^n \) and \( \xi \in \mathcal{V} \) a unit vector, define:

\[
|k\rangle_{\xi} := \frac{1}{\sqrt{(q^2;q^2)_{k_1} \cdots (q^2;q^2)_{k_{n-1}} (q^4;q^4)_{k_n}}} (y_1^{k_1} \cdots (y_n^{k_n})^{k_n} \xi),
\]

where the \( q \)-shifted factorial is given by

\[
(a;b)_\ell := \prod_{i=0}^{\ell-1} (1 - ab^i).
\]

Given \( k \in \mathbb{Z}^n \), set \( |k\rangle_{\xi} := 0 \) if one of the components of \( k \) is negative. From the commutation relations we deduce:

\[
y_i^{\dagger} |k\rangle_{\xi} = q^{k_1 + \ldots + k_{i-1} - 1} \sqrt{1 - q^{2k_i + 2}} |k + e_i\rangle_{\xi}, \quad (i < n), \tag{19a}
\]

\[
y_n^{\dagger} |k\rangle_{\xi} = q^{k_1 + \ldots + k_{n-1} - 1} \sqrt{1 - q^{4k_n + 4}} |k + e_n\rangle_{\xi}, \tag{19b}
\]

\[
y_{n+1} |k\rangle_{\xi} = q^{k_1 + k_n} |k\rangle_{y_{n+1}\xi}.
\]

If \( W \subset V \) carries a subrepresentation of \( C^*(y_{n+1}, y_{n+1}^*) \), the Hilbert subspace of \( \mathcal{H} \) spanned by \( |k\rangle_{\xi} \) for \( \xi \in W \) and \( k \in \mathbb{N}^n \) carries a subrepresentation of \( A(\Sigma_q^{2n+1}) \). Since \( \mathcal{H} \) is irreducible, \( V \) carries an irreducible representation of \( C^*(y_{n+1}, y_{n+1}^*) \). This means that \( V \) is one-dimensional.

Let us fix a unit vector \( \xi \in \mathcal{V} \). Observe that the vectors

\[
\{ |k\rangle_{\xi} \}_{k \in \mathbb{N}^n}
\]

span \( \mathcal{H} \). Moreover \( y_{n+1}\xi = \lambda\xi \) for some \( \lambda \in \mathbb{R} \), and

\[
y_{n+1} |k\rangle_{\xi} = \lambda q^{k_1 + k_n} |k\rangle_{\xi}.
\]

From now on, instead of \( |k\rangle_{\xi} \) we shall simply write \( |k\rangle \). By (16), it follows that

\[
1 = \langle 0 | 0 \rangle = \langle 0 | \sum_{i=1}^{n+1} y_i y_i^* | 0 \rangle = |\lambda|^2,
\]

hence \( \lambda \in U(1) \). It remains to prove that the set (20) is orthonormal, so that by adjunction from (19) we get the formulas in Prop. 2.

Let \( W_i \) be the span of vectors \( |k\rangle_{\xi} \) with \( k_1 = \ldots = k_i = 0 \). It follows from the commutation relations that \( y_{k_i} \) is zero on \( W_i \) for all \( k \leq i \).

Applying the identities in Lemma 4 to a vector \( |k\rangle \in V_i \) we find that

\[
y_i (y_i^{\dagger})^m |0, \ldots, 0, k_{i+1}, \ldots, k_n\rangle = (1 - q^{2m}) (y_i^{\dagger})^{m-1} |0, \ldots, 0, k_{i+1}, \ldots, k_n\rangle
\]

\[
y_n (y_n^{\dagger})^m |0\rangle = (1 - q^{4m}) (y_n^{\dagger})^{m-1} |0\rangle
\]

for all \( m \geq 1 \) and all \( 1 \leq i < n \). Using (19) we find that

\[
y_i |0, \ldots, 0, m-1, k_{i+1}, \ldots, k_n\rangle = \sqrt{1 - q^{2m}} |0, \ldots, 0, m-1, k_{i+1}, \ldots, k_n\rangle
\]
\[ y_n |0, \ldots, 0, m - 1\rangle = \sqrt{1 - q^{4m}} |0, \ldots, 0, m - 1\rangle \]

Multiplying from the left by \( \langle j - e_i | \) and using (19) again, we find that
\[ \sqrt{1 - q^{2j}} \langle j | k \rangle = \sqrt{1 - q^{2k}} \langle j - e_i, k - e_i \rangle, \]
\[ \sqrt{1 - q^{4j}} \langle j_n e_n | k_n e_n \rangle = \sqrt{1 - q^{4k}} \langle (j_n - 1) e_n, (k_n - 1) e_n \rangle, \]
where the former is valid whenever \( j_1, \ldots, j_i = k_1 = \ldots = k_i = 0 \). From these relations, an obvious induction proves that the set (20) is orthonormal provided every vector \( |k\rangle \) with \( k \neq 0 \) is orthogonal to \( |0\rangle \). But this is obvious. If \( k_i \neq 0 \) for some \( i \), then
\[ \langle k | 0 \rangle \propto \langle k - e_i | y_i | 0 \rangle = 0 \]
since \( y_i \) annihilates \( \xi \).

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