A coupled Meshless-FEM method based on strong form of Radial Point Interpolation Method (RPIM)

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Abstract. In this present work, we are implementing a novel hybrid method based on the coupling of RPIM in strong form and Finite Element Method (FEM). The basic idea is to ensure the coupling between the two methods through the collocation technique based on RPIM interpolation. This technique is used to evaluate the local equations of the problem at the interface between FEM and RPIM regions. We can avoid numerical integrations of a big part of nodes using the strong form of RPIM. Numerical studies show that this method gives reasonably accurate results consistent with the theory.

1. Introduction

Although FEM was the dominant technique in structural calculation for decades, this method is sensitive to the element distortion and often encountered difficulties in mesh progress which leads to erroneous results without physical significance. However as an alternative for over thirty years, the meshless methods have been the subject of significant research as they allow to implement the numerical simulation of a set of field nodes arbitrarily scattered [1-6]. The meshless approximation to the unknown functions is constructed by nodal approximations without connectivity between the nodes. Indeed, the absence of the concept of element makes it possible to overcome distortion problems in case of large deformations. Meshless methods not require additional efforts to construct the mesh, which make them the most appropriate solution for structures with complex geometry. These are theoretical advantages of Meshless methods. However, FEM is more favorable compared to meshless methods in many aspects, such as the imposition of boundary conditions and computational efficiency. To benefit the advantages of each method the solution in the literature is the coupling of meshless methods with FEM as [7-15] and soon. The aim of FEM-Meshless coupling is to treat multi-physical problems, large deformations and to overcome the difficulty of imposing boundary conditions encountered by Meshless methods, etc. Now we present some works on FEM-Meshless coupling. In papers [10, 11] the interface elements-based coupling method based on affine functions and Element Free Galerkin (EFG) method is proposed, in works [12-14] the extends weak forms-based
coupling method using Lagrange multipliers is presented. Later, Huerta and Fernández-Méndez [15] propose a general coupling formulation for a mixed FEM-EFG interpolation. There are several FEM-Meshless coupling methods that meet the mathematical and physical requirements of the coupling which are based on weak form meshless methods. Note that weak form meshless methods require the numerical integration and the creation of elements to overcome this problem.

In the present work, a collocation technique is used to couple FEM with the strong form RPIM method [16]. The basic idea is to evaluate the local equations of the problem at the interface between FEM and RPIM regions using the collocation approximation based on RPIM interpolation. In this paper, the strong form RPIM interpolation is applied inside the domain and FEM is applied near the edges to satisfy the boundary conditions.

2. Equilibrium equations
Consider a problem described by equations (1) in the domain $\Omega$, with complementary boundary conditions in the boundary $\Gamma$. $\Gamma_U$ and $\Gamma_F$ are complementary parts of $\Omega$ on which the Dirichlet and Neumann boundary conditions are applied respectively (see Fig.1).

To validate our coupling approach, we consider the following equations:

$$
\begin{align*}
\Delta U(x,y) &= f(x,y) \quad (x,y) \in \Omega \\
\nabla U \cdot n &= F^d_i \quad \text{on} \quad \Gamma_F \\
U_i &= U^d_i \quad \text{on} \quad \Gamma_U
\end{align*}
$$

(1)

where $U$ is the unknown function, $n_j$ is the unit outward normal vector to the boundary $\Gamma_F$, $F^d_i$ and $U^d_i$ represent the values applied on $\Gamma_F$ and $\Gamma_U$ respectively.
3. RPIM interpolation

Using a set of arbitrarily scattered nodes localized in a local support domain, the RPIM interpolation of the unknown \( u(X) \) can be written as:

\[
\begin{align*}
u(X) &= \sum_{i=1}^{n} R_i(X) a_i + \sum_{j=1}^{m} p_j(X) b_j \\
&= \langle R(X) \ P(X) \rangle \begin{bmatrix} a \\ b \end{bmatrix}
\end{align*}
\]

where \( R(X) \) and \( p(X) \) are Radial Basis Functions (RBFs) and Polynomial Basis Functions (PBFs) respectively, \( n \) and \( m \) are the numbers of Radial Basis Functions RBFs and polynomial basis functions respectively, \( a_i \) and \( b_j \) are new unknowns yet to be determined. In the equations (2), \( R_i(X) \) is the RBF calculated from the Euclidean distance \( r_i = \|X - X^*\| \) between the interest point \( X \) and its neighbors \( X^* \) as follow [17, 18, 19]:

\[
R_i(X) = \begin{cases} (c^2 + r_i^2)^{\frac{q}{2}} & \text{Multiquadrics} \\ e^{-c^2r_i^2} & \text{Gaussian} \end{cases}
\]

where \( c \) and \( q \) are parameters that can be selected from numerical tests or using algorithms designed for their determination to stabilize the solution. The polynomial basis functions \( p_j(X) \) is constructed using Pascal’s triangle while choosing a complete basis. In the two-dimensional frame work, the basis functions can be given as follow:

\[
P^T(X) = \begin{cases} <1, x, y> & \text{Linear basis functions} \\ <1, x, y, x^2, xy, y^2> & \text{Quadratic basis functions} \end{cases}
\]

By enforcing the interpolation function through all \( n \) scattered nodal points within the support domain of the interest point \( X \), we can construct for each interest point a support domain containing \( n \) nearest neighbors points, which allows us to determine the unknown coefficients \( a_i \) and \( b_j \) of the equation (2). This leads to \( n \) linear equations which can be expressed in the matrix form as:

\[
\{U_n\} = [R_n] \{a\} + [P_m] \{b\}
\]

where \( \{U_n\} \) is the nodal vector at the support domain, [\( R_n \)] and [\( P_m \)] are the momentum matrix assigned to the RBF and the polynomial moment matrix respectively which are defined by:

\[
[R_n] = \begin{bmatrix}
R_1(r_1) & \ldots & R_n(r_1) \\
\vdots & \ddots & \vdots \\
R_1(r_n) & \ldots & R_n(r_n)
\end{bmatrix}
\]

\[
[P_m] = \begin{bmatrix}
1 & x_1 & y_1 \ldots & P_m(x_1) \\
1 & x_2 & y_2 \ldots & P_m(x_2) \\
\vdots & \ddots & \vdots & \vdots \\
1 & x_n & y_n \ldots & P_m(x_n)
\end{bmatrix}
\]

The unknown vectors of RBFs coefficients \( \{a\} \) and polynomial coefficients \( \{b\} \) are respectively:

\[
\{a\} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}^T
\]

and

\[
\{b\} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}^T
\]
The additional $m$ equations can be added using the following $m$ constraint conditions:

$$\sum_{i=1}^{n} p_j(X) a_i ; \quad j = 1, ..., m$$

(10)

By combining Eqs.(5) and (10), we obtain a following compacted linear system:

$$U = \begin{bmatrix} U_n \\ 0 \end{bmatrix} = \begin{bmatrix} R_n & \cdots & P_m \\ \vdots & \ddots & \vdots \\ P_m^T & \cdots & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [G]\{d\}$$

(11)

The matrix $[G]$ is positive, symmetric and theoretically non-singular [18, 19] then, the vector $\{d\}$, containing the unknown coefficients $a_i$ and $b_j$, can be obtained by triangulating the matrix $[G]$ as follows:

$$\{d\} = [G]^{-1}\{U\}$$

(12)

By replacing $\{d\}$ in the Eq.(2), the approximation of the function $u(X)$ can be written in the following form:

$$u(X) = <R(X), P(X) > [G]^{-1}\{U\}$$

$$= <\phi_1(X), \cdots, \phi_n(X), \phi_{n+1}(X), \cdots, \phi_{n+m}(X) > \{U\}$$

(13)

The first $n$ components of this vector correspond the RPIM shape functions representing the approximation of the function $u(X)$ which is given by:

$$u(X) = \sum_{i=1}^{n} \phi_i(X) U_i$$

(14)

We note that the RPIM shape functions satisfy the Kronecker delta property [20, 21, 22], i.e:

4. A collocation approach for coupling of FEM and strong form RPIM

The proposed Meshless-FEM coupling can be used in different situations depending on the problem. In this paper, we are interested the Meshless-FEM coupling to overcome the difficulty of imposing boundary conditions encountered by Meshless methods. An array of elements in FEM region and points in RPIM region is shown in Fig.2.

![Figure 2](image-url)

Figure 2. Discretization of the domain $\Omega$ by the points domain $\Omega^{RPIM}$ and the elements domain $\Omega^{FEM}$
For a quadratic element in the FEM region $\Omega^{FEM}$, the interpolation function for the unknown field $U_I$ is:

$$U_h^I = \sum_{I=1}^{4} N_I(x) U_{iI} \quad (15)$$

where $N_I(x)$ are shape functions. We denote the element discretized equations in the general FEM as:

$$\int_{\Omega}^{} T[H][H] d\Omega \{U^{FEM}\} = -\int_{\Omega}^{} T[N] f d\Omega + \int_{\Gamma_{pd}}^{} T[N] \{F^d\} d\Gamma \quad (16)$$

where $\{U^{FEM}\}$ is the nodal values, $[N]$ is the matrix contains the shape functions and $[H]$ is the matrix contains derivatives of the shape functions.

For a point in RPIM region $\Omega^{RPIM}$, the approximation of the unknown field is based on RPIM approximation (8) and it is given by:

$$\{U\} = \sum_{l=1}^{n} \phi_I \{U_I\} \quad (17)$$

After substitution of the approximation (17), the problem (1) for each point $j$ in the interior region $\Omega^{RPIM}$ can be written in the following strong form:

$$\sum_{l=1}^{n} B_I U_I = f_I \quad (18)$$

where

$$B_I = \phi_{I,xx} + \phi_{I,yy} \quad (19)$$

The weak form (16) and the strong form (18) lead the elementary matrix $[K^e]$ and the vector $\{F^e\}$ as follows:

$$[K^e] = \begin{cases} \int_{\Omega}^{} T[H][\lambda][H] d\Omega & \text{for each element in region } \Omega^{FEM} \\ \sum_{l=1}^{n} B_I U_I = f_I & \text{for each point } j \text{ in region } \Omega^{RPIM} \end{cases} \quad (20)$$

$$\{F^e\} = -\int_{\Omega}^{} T[N] f d\Omega + \int_{\Gamma_{pd}}^{} T[N] \{F^d\} d\Gamma \quad (21)$$

We note that for RPIM points at and near the interface, RPIM interpolation consider the nodes of FEM region as neighbors of RPIM points ((see Fig.3).

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**Figure 3.** Domain of influence of points $I$ and $J$ at and near the interface $\Gamma^{int}$.
The idea to satisfy the coupling is based on the adjacent finite elements of the interface. In these elements, we just take into consideration the contribution of the active finite element nodes for adjacent finite elements of the interface as shown in Fig.4. Collocation technique based on RPIM interpolation is used for other nodes at the interface (called also interface points) which allow us to ensure the coupling between the two methods and to satisfy the unknown field continuity across the nodes on the interface.

Technically, we replace the lines corresponding to degrees of freedom (DOF) of each inactive finite element node at the interface (see Fig.4) by the strong form RPIM approximation of the problem. As example, for a point \( I \) at the interface \( \Gamma^{int} \) as shown in Fig.4, we modify the matrix \( [K^e] \) as follows:

\[
K^e(\text{DOF}(J)) = \sum_{I=1}^{n} B_I U_I
\]

(22)

where \( n \) is the neighbors points in the domain of influence of point \( J \), which the nodes of FEM region are also considered as neighbors of points in the interface as shown Fig.4. After substitution and assembly, we obtain the following global system:

\[
[K] \{U\} = \{F\}
\]

(23)

where \( \{U\} \) is the vector of the unknown field for all nodes in the entire problem domain.

5. Numerical analysis to study of the convergence of the MEF-RPIM coupling method

In this section, numerical examples are presented to verify the effectiveness of the proposed coupling method. We consider two examples commonly used; poisson’s equation and heat equation. These equations allow us to verify the convergence of the MEF-RPIM coupling method by comparing the numerical solution with that obtained analytically. For FEM, a quadratic element \( Q_4 \) and four gauss points in each element are used. For strong form RPIM, a quadratic basis functions and Multiquadric functions are used for PBFs and RBFs respectively with \( c = 1.42 \) and \( q = 1.03 \) [16].

5.1. The poisson problem

We consider the Poisson problem defined as follows:

\[
\begin{cases}
\Delta U(x, y) = -2(x^2 + y^2 - 2) & \text{if } (x, y) \in \Omega \\
U(x, y) = 0 & \text{if } (x, y) \in \Gamma
\end{cases}
\]

(24)
where $\Omega = [-1, 1] \times [-1, 1]$ and $\Gamma$ represents the four edges \((x = -1, y), (x = 1, y), (x, y = -1)\) and \((x, y = 1))\) of the domain $\Omega$. The analytical solution of the problem (24) is:

$$U(x, y) = (x^2 - 1)(y^2 - 1)$$  \hspace{1cm} (25)

The distribution of points in $\Omega^{FEM}$ and $\Omega^{RPIM}$ regions is presented in Fig.5. In this distribution, we choose to use FEM near the boundaries and the strong form RPIM inside the domain.

**Figure 5.** Distribution of 441 nodes in $\Omega^{FEM}$ and $\Omega^{RPIM}$

Fig.6 shows the distribution of the solution in the domain $\Omega$ obtained by the analytical solution and MEF-RPIM coupling method with a number of points in the influence domain equal to 13. The comparison between the two results gives a good agreement with a absolute error $|U_{exact} - U_{numerical}|$ that does not exceed $6 \times 10^{-5}\%$ (see Fig.6-(c)).

**Figure 6.** Distribution of the solution in the domain $\Omega$
Fig. 7 shows the solution along $x$ axis at $y = 0$, we can clearly see the part of the solution obtained by FEM in the region $\Omega^{FEM}$ and that obtained by RPIM in the region $\Omega^{RPIM}$. We can clearly see that the continuity is verified at the interface between the two regions.

![Figure 7](image)

**Figure 7.** The solution along $x$ axis at $y = 0$

To show the effect of the neighbors points in the domain of influence, we draw in Fig.7 the evolution of the global relative error $\frac{\|U_{\text{numerical}} - U_{\text{exact}}\|}{\|U_{\text{exact}}\|}$ versus number of points $N$ in the domain of influence. We observe that from $N = 13$ the global relative error stabilizes at the value 0.1%. Therefore, $N = 13$ is the optimal number of points in the domain of influence.

![Figure 8](image)

**Figure 8.** Evolution of the relative error versus number of points in the domain of influence

5.2. The thermal problem

We consider a problem of heat propagation in a plate defined by a domain as shown in Fig.5. In this study, we limit ourselves to a stationary heat problem given by:

$$\begin{cases} 
\Delta T(x,y) = 0 \\
T(-1,y) = 100, T(1,y) = 0, T(x,-1) = 0, T(x,1) = 0 & \text{if } (x,y) \in \Omega \\
\end{cases} \quad (26)$$

Fig.9 shows the temperature distribution in the domain $\Omega$ obtained by FEM method and MEF-RPIM coupling method respectively. The comparison between the two methods gives a good agreement with a relative error equal 0.2%.

![Figure 9](image)
6. Conclusion
A novel method, the coupled FEM-RPIM method, is developed based on a combined formulation of both weak form and strong form. We recommend using the meshfree strong form based on RPIM method in the entire structure except near the boundaries because there are no numerical integrations for nodes in the strong form region. FEM, which needs the numerical integration, is used in limited regions for nodes on or near the natural boundaries. In this proposed method, the boundary conditions can be easily imposed using FEM to produce stable and accurate solutions. Our mixed FEM-RPIM model showed a good capacity of prediction through the Poisson problem and the heat problem. During the comparative studies of solutions, we showed that our mixed model FEM-RPIM reproduces a fairly realistic behavior by comparing with analytical and FEM results. In this work we limited ourselves to the simples problems, but these first results are encouraging to develop other work in computational mechanics by integrating different sources of non-linearity: contact, large deformations, non-linear behavior, etc. An improvement of the developed formulation will also be considered in the future work to consider different type of structures [23, 24, 25].

References
[1] Timesli A, Braikat B, Lahmann H and Zahrouni H 2013 An implicit algorithm based on continuous moving least square to simulate material mixing in friction stir welding process Model. Simul. Eng.. 2013 1-14
[2] Timesli A, Braikat B, Lahmann H and Zahrouni H 2015 A new algorithm based on Moving Least Square method to simulate material mixing in friction stir welding Eng. Anal. Bound. Elem.. 50 372-80
[3] Belaasilia Y, Timesli A, Braikat B and Jamal M 2017 A new algorithm based on Moving Least Square method to simulate material mixing in friction stir welding Eng. Anal. Bound. Elem.. 82 68-78
[4] Mesmoudi S, Timesli A, Braikat B, Lahmann H and Zahrouni H 2017 A 2D mechanical-thermal coupled model to simulate material mixing observed in Friction Stir Welding process Eng. Comput.. 33 885-95
[5] Timesli A, Zahrouni H, Braikat B, Moukli A and Lahmann H 2011 Numerical model based on meshless method to simulate FSW Computational Methods in Applied Sciences: Particle-Based Methods II - Fundamentals and Applications. 2 651-62
[6] Nguyen V, Rabczuk T, Bordas S and Duot M 2008 Meshless methods: A review and computer implementation aspects Math. Comput. Simulat.. 79 763-813
[7] Chen T and Raju I S 2003 A coupled finite element and Meshless local Petrov-Galerkin method for twodimensional potential problems Math. Comput. Simulat.. 192 4533-50
[8] Fernandez-Méndez S and Huerta A 2004 Imposing essential boundary conditions in mesh-free methods Comput. Methods Appl. Mech. Eng.. 193 1257-75
[9] Attaway S W, Heineston M W and Swegle J W 1994 Coupling of smooth particle hydrodynamics with the finite element method Nucl. Eng. Des.. 150 199–205
[10] Belytschko T, Organ D and Krongau Y 1995 A coupled finite element-element free Galerkin method Comput. Mech.. 17 186-95
[11] Rao B N 2011 Coupled meshfree and fractal finite element method for unbounded problems Comput. Geotech.. 38 697-708
[12] Hegen D 1996 Element-free Galerkin methods in combination with finite element approaches *Comput. Methods Appl. Mech. Eng.*, 135 143-66

[13] Krongauz Y and Belytschko T 1996 Enforcement of essential boundary conditions in Meshless approximations using finite elements *Comput. Methods Appl. Mech. Eng.*, 131 133-45

[14] Gu Y T and Zhang L C 2008 Coupling of the meshfree and finite element methods for determination of the crack tip fields *Eng. Fract. Mech.*, 75 986-1004

[15] Huerta A and Fernandez-Méndez S 2000 Enrichment and coupling of the finite element and Meshless methods *Int. J. Numer. Meth. Eng.*, 48 1615-36

[16] Wang J G and Liu G R 2002 A point interpolation meshless method based on radial basis functions *Int. J. Numer. Meth. Eng.*, 54 1623-48

[17] Li J, Feng X and He Y 2019 RBF-based meshless local Petrov Galerkin Method for the multi-dimensional convection-diffusion-reaction equation *Eng. Anal. Bound. Elem.*, 98 46-53

[18] Singh J and Shukla K K 2012 Nonlinear flexural analysis of laminated composite plates using RBF based meshless method *Compos. Struct.*, 94 1714-20

[19] Singh J and Shukla K K 2012 Nonlinear flexural analysis of functionally graded plates under different loadings using RBF based meshless method *Eng. Anal. Bound. Elem.*, 36 1819-27

[20] Kazemi Z, Hematiyan M and Vaghefi R 2017 Meshfree radial point interpolation method for analysis of viscoplastic problems *Eng. Anal. Bound. Elem.*, 82 172-84

[21] VanDo V N, Tran M T and Lee C H 2018 Nonlinear thermal buckling analyses of functionally graded plates by a mesh-free radial point interpolation method *Eng. Anal. Bound. Elem.*, 87 153-64

[22] Feng S Z, Cui X Y, Chen S Z and Meng D Y 2016 An edge/facet-based smoothed radial point interpolation method for static analysis of structures *Eng. Anal. Bound. Elem.*, 68 1-10

[23] Timesli A 2020 An efficient approach for prediction of the nonlocal critical buckling load of double-walled carbon nanotubes using the nonlocal Donnell shell theory *SN Appl. Sci.*, 2 407.

[24] Timesli A 2020 Prediction of the critical buckling load of SWCNT reinforced concrete cylindrical shell embedded in an elastic foundation *Comput. Concrete, Int. J.*, 26 53-62.

[25] Timesli A 2020 Buckling analysis of double walled carbon nanotubes embedded in Kerr elastic medium under axial compression using the nonlocal Donnell shell theory *Adv. Nano Res.*, 9 69-82.