Deriving accurate peculiar velocities (even at high redshift)

Tamara M. Davis\textsuperscript{1,*} and Morag I. Scrimgeour\textsuperscript{2,3}

\textsuperscript{1}School of Mathematics and Physics, University of Queensland, QLD 4072, Australia
\textsuperscript{2}Department of Physics and Astronomy, University of Waterloo, Waterloo, ON N2L 3G1, Canada
\textsuperscript{3}Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada

Accepted 2014 May 5. Received 2014 May 1; in original form 2014 April 1

\begin{abstract}
The way that peculiar velocities are often inferred from measurements of distances and redshifts makes an approximation, \( v_p = cz - H_0 D \), that gives significant errors even at relatively low redshifts (overestimates by \( \Delta v_p \sim 100 \text{ km s}^{-1} \) at \( z \sim 0.04 \)). Here, we demonstrate where the approximation breaks down, the systematic offset it introduces, and how the exact calculation should be implemented.

\textbf{Key words:} cosmology: theory.
\end{abstract}

1 INTRODUCTION

Peculiar velocities can be a useful cosmological probe, as they are sensitive to the matter distribution on large scales, and can test the link between gravity and matter. As peculiar velocity measurements are getting more numerous, more accurate, and stretching to higher redshift, it is important that we examine the assumptions made in the derivation of peculiar velocity from measurements of redshift and distance. Some of the derivations used in the past have used approximations that are inadequate at the redshifts modern telescopes are able to reach.

The starting point of many peculiar velocity papers is the equation,

\[ v_p = cz - H_0 D, \tag{1} \]

where radial peculiar velocity \( v_p \) is the difference between \( cz \), the observed redshift \( z \), multiplied by the speed of light, \( c \); and the velocity given by Hubble’s law, \( \bar{v} = H_0 D \), where \( H_0 \) is Hubble’s constant and \( D \) is the proper distance.

This appears in the classic papers such as Kaiser (1988, equation 1.1), Dekel (1994, equation 21), and Strauss & Willick (1995, equations 1, 2, and 147), through to more recent peculiar velocity papers such as Masters et al. (2006, equation 5), Springob et al. (2007, equation 1), Sarkar, Feldman & Watkins (2007, equation 1), Abate et al. (2008, section 1, who note it is low-\( z \) only), and Lavaux et al. (2008, equation 1), Nusser & Davis (2011, equations 5 and 10),\textsuperscript{1}

\footnotetext{1}{Private communication with some groups indicates that although this equation appears in this paper, they do not use it in their code (e.g. Turnbull et al. 2012); others (e.g. Abate & Erdogdu 2009) do not specify how they derive velocities, but use data sets that have used the approximation; and yet others (e.g. the COMPOSITE sample of Watkins, Feldman & Hudson 2009; Feldman, Watkins & Hudson 2010) use a combination of data sets that have used the approximation (e.g. Springob et al. 2007, 2009) with other data sets in which they have converted distances to velocities themselves without using the approximation.}

and is used in major compilations of peculiar velocity data such as Cosmicflows-2 (Tully et al. 2013).

This formula contains the approximation that \( v_{\text{approx}} = cz \), which fails at high redshift. This has been pointed out in the past (e.g. Harrison 1974; Faber & Dressler 1977; Lynden-Bell et al. 1988; Harrison 1993; Colless et al. 2001), but since many recent papers still use the approximation, and because several major peculiar velocity surveys are imminent, it is timely to revisit this issue. Most of the analyses listed above used data at low enough redshift that this approximation gives only small biases, but that will not remain true for future surveys.\textsuperscript{2} In what follows, we will demonstrate how the approximation causes one to overestimate peculiar velocities, and how that should be corrected for.

2 THE BASICS

Here, we base our calculations in the Robertson–Walker metric, in which distance is given by \( D = R\chi \), where \( R(t) \) is the scalefactor at time \( t \), denoted by \( R_0 \) at the present day (dimensions of distance), and \( \chi(\bar{z}) \) is the comoving coordinate of an object at cosmological redshift \( \bar{z} \). (Throughout, we use overbars to denote quantities that would be measured in a perfectly homogeneous and isotropic universe without peculiar velocities.)

Differentiating distance with respect to time (denoted by an over-dot) gives the total velocity \( v = R\dot{\chi} + R\chi \). Thus, it is convenient to separate motion into recession velocities, due only to the Hubble

\footnotetext{2}{For example, the TAIPAN survey soon to start on the UK Schmidt telescope in Australia will measure distances to \( \sim 45\,000 \) elliptical galaxies out to \( z \sim 0.1 \) (Koda et al. 2013); supernova surveys such as SkyMapper (Keller et al. 2007) and Palomar Transient Factory (http://www.ptf.caltech.edu/) will be finding hundreds of supernovae in wide fields out to \( z \sim 0.1 \) and \( z \sim 0.25 \), respectively; and Tully–Fisher measurements from radio surveys such as WALLABY on the Australian Square Kilometre Array Pathfinder will deliver \( \sim 32\,000 \) galaxy distances out to \( z \sim 0.1 \) (Duffy et al. 2012; Koda et al. 2013). At \( z \sim 0.1 \), the approximation overestimates peculiar velocities by \( \sim 700 \text{ km s}^{-1} \).}
flow, \( v = R \chi \), and peculiar velocities, \( v_p = R \dot{\chi} \), which encapsulate all motion other than the homogeneous and isotropic expansion, so

\[ v = \bar{v} + v_p. \]  

(2)

Peculiar velocities can be distinguished from recession velocities because an observer with a non-zero peculiar velocity sees a dipole in the cosmic microwave background (CMB). The comoving distance, \( \chi \), to a galaxy is related to its cosmological redshift, \( \bar{z} \), by

\[ \chi(\bar{z}) = \frac{c}{H_0} \int_0^{\bar{z}} \frac{dz}{H(z)}, \]  

(3)

where \( c \) is the speed of light and \( H(\bar{z}) \) is the Hubble parameter as a function of cosmological redshift (assuming a homogeneous universe and smooth expansion, about which peculiar velocities are a small perturbation). This distance is the cornerstone of any of the distances we measure using our distance probes, such as Type Ia supernovae, Fundamental Plane, and Tully–Fisher distances. The present-day values of proper distance, \( D_p \), luminosity distance, \( D_L \), and angular diameter distance, \( D_A \), are all built from comoving distance according to, respectively,

\[ D(\bar{z}) = R_0 \chi(\bar{z}), \]  

(4)

\[ D_L(\bar{z}) = R_0 S_0(\chi)(1 + \bar{z}), \]  

(5)

\[ D_A(\bar{z}) = R_0 S_0(\chi)(1 + \bar{z})^{-1}, \]  

(6)

where \( S_0(\chi) = \sin(\chi), \chi, \sinh(\chi) \) for closed, flat, and open universes, respectively.

When an object has a peculiar velocity, it acquires an additional redshift component \( z_p \). The relationship between the peculiar velocity and the ‘peculiar’ redshift, \( \bar{z}_p \), is

\[ v_p = c z_p, \]  

(7)

when the velocities are non-relativistic, or

\[ v_p = c \frac{(1 + z_p)^2 - 1}{(1 + z_p)^2 + 1}. \]  

(8)

when the velocities are relativistic.\(^3\) In almost all practical situations, the non-relativistic approximation is adequate for peculiar velocities. The jets being ejected from active galactic nuclei would be one exception; cosmic rays would be another.

A galaxy with \( \bar{z} \) and \( z_p \) will appear to the observer to have \( z \) where

\[ (1 + z) = (1 + \bar{z})(1 + z_p). \]  

(9)

Note that the approximation \( z = \bar{z} + z_p \) works only for small redshifts.\(^4\) Although equation (9) is a standard result, there is some confusion over this in the community so we run through the derivation in Appendix A.

\[ \textit{3} \text{ Equation (8) is only strictly true if the velocities are entirely radial. Converting between reference frames in special relativity depends on the total velocity. So when there is also a tangential component \( v_t \), such that the peculiar velocity is broken up into \( v_p^2 = v_t^2 + v_r^2 \), the relationship becomes:} \]

\[ v_p = c \frac{1 - t^2 + z_p^2 \sqrt{1 + z_p^2 (1 - t^2) - (1 - t^2)}}{1 + z_p^2 \sqrt{1 + z_p^2 (1 - t^2) - t^2 + z_p^2 (1 - t^2)}}. \]

\[ \textit{4 Note also that the NASA/IPAC Extragalactic Database (NED) provides a velocity calculator that correctly uses equation (2), but if the user calculated} \]

\[ \bar{v} = c \bar{z}, \text{ NED will give a biased result (see Section 6).} \]

### 3 THE USUAL APPROXIMATION AND WHERE IT FAILS

To measure a peculiar velocity, one needs

(i) the observed redshift, \( z \),

(ii) the observed distance, usually \( D_L \) or \( D_A \), from which one can infer \( \chi(\bar{z}) \), and thus \( \bar{z} \) (given a cosmological model).

With \( z \) and \( \bar{z} \) known, equation (9) gives \( z_p \), from which peculiar velocity can be inferred.

Sticking with redshifts for this calculation avoids some of the approximations below, but conceptually, researchers have tended to prefer to work in velocities. So the technique often used goes as follows.

(i) The observed redshift is used to infer the total velocity, \( v \), which includes the recession velocity due to expansion of the universe \( \bar{v} \), and the peculiar velocity \( v_p \).

(ii) The distance measurement is used to calculate the recession velocity as per Hubble’s law \( \bar{v} = H_0 D \).

The difference between these two velocities is the peculiar velocity (and measurement error), \( v_p = v - \bar{v} \).

So far, this technique would in principle be fine. However, calculating total velocity from observed redshift is tricky (as it needs knowledge of the peculiar velocity, which is what we are trying to measure). So, in the literature it is commonly approximated by \( v_{\text{approx}} = c \bar{z} \), and thus

\[ v_{\text{approx}} = v_{\text{approx}} - \bar{v}. \]  

(10)

\[ v_{\text{approx}} = c \bar{z} - H_0 D. \]  

(11)

However, the relationship between redshift and recession velocity does not follow that form, \( \bar{v} \neq c \bar{z} \). Fig. 1 shows that deviations are visible even at \( z \lesssim 0.05 \), and that the approximation causes one to overestimate peculiar velocities. Note that all the velocities that appear in Eq. 2 should be evaluated at the same cosmic time. Since we are measuring the peculiar velocity at the time of emission, \( t_e \),
we should be using the total and recession velocities at the time of emission too. However, using $\bar{v} = H(t_i)D(t_i)$ actually makes the approximation in Eq. 11 even worse. As you will see, we recommend always working in redshifts, never velocities, so this issue does not arise.

4 HOW BAD IS THE APPROXIMATION?

To assess the effect of the approximation on our estimates of peculiar velocities, we will calculate the peculiar velocity with and without this approximation. Let us start with an array of recession redshifts, $z_i$, and for each one calculate

(i) $D$ the proper distance corresponding to that redshift (using equations 3 and 4) – this is the distance we can infer from luminosity or angular diameter distances, and

(ii) $z$, the redshift we would observe if the galaxy had a peculiar velocity, $v_p$ (using equations 8 and 9); initially we will set $v_p = 0$ km s$^{-1}$.

Then we use equation (11) to calculate the $v_p^{\text{approx}}$ that we would infer for that galaxy. So in essence, this test is taking information we do not have (the intrinsic redshift and true peculiar velocity of the galaxy), calculating what we would observe (the observed redshift and distance), and then calculating what the inferred peculiar velocity would be if we used the approximation in equation (11). Comparing this to the true peculiar velocity that we inputted shows how good (or bad) the approximation is. The results are shown in Fig. 2.

From Fig. 2 it is clear that the approximation is a poor one, even as close as $z = 0.04$, where we would overestimate $v_p$ by about 100 km s$^{-1}$.

5 Modulo knowledge of the cosmological model – i.e. matter and dark energy densities and properties – so $S_1(z)$ and $H(z)/H_0$ can be calculated. At low redshifts ($z \ll 0.1$), this tends towards $D = cz/H_0$, and no cosmological model is required, which is one reason that approximation is so popular.
the cosmological model, the full expression for velocity using any fiducial model will always be better than the linear approximation in $z$, and the cosmology dependence of the resulting $H_0$ measurement is weak.

6 DISCUSSION AND CONCLUSIONS

Until recently, most peculiar velocity surveys have been performed at $z \ll 0.02$, for which $v^{\text{approx}} = cz$ gives less than a $50 \text{ km s}^{-1}$ error (see Fig. 2), less than the measurement uncertainty (but still biased on the high side). However, as current and future peculiar velocity surveys probe ever deeper, and become more accurate, using the full formula for deriving peculiar velocities will be crucial. This is particularly true for supernova surveys as they have the most precise distance measurements and can reach to high redshift. For example, Dai, Kinney & Stojkovic (2011) use a second-order approximation in their measurement of a bulk flow using supernova peculiar velocities in two bins on either side of $z = 0.05$, while Colin et al. (2011) fit for a cosmological model, before comparing the magnitude residuals, and Rathaus, Kovetz & Itzhaki (2013) use a first-order expansion with a fiducial model to analyse the Union 2 supernova sample out to $z < 0.2$. Analyses using the kinetic Sunyaev–Zeldovich effect, which measure peculiar velocities at $z \sim 0.1$ (e.g. Kashlinsky et al. 2008), should not be susceptible to this effect (which would be $\sim 700 \text{ km s}^{-1}$ at $z = 0.1$) because the temperature variation measured reflects only the peculiar velocity of the high-redshift cluster, and not the recession velocity. \(^8\)

Peculiar velocities can also affect the cosmological parameters estimated from the magnitude–redshift diagram of supernovae. In Davis et al. (2011), we showed how to optimally de-weight low-redshift supernovae that potentially have correlated peculiar velocities that could bias cosmological measurements.

The discussion in this paper is also relevant when correcting for our own motion. Our Sun moves at approximately $300 \text{ km s}^{-1}$ with respect to the Hubble flow frame, as measured by our motion with respect to the CMB. If the $\bar{z} = z - z_\odot$ approximation is used when correcting observed redshifts into the CMB frame, then it introduces a systematic redshift error that increases with cosmological redshift. Luckily that error will be small (see Fig. 3). We checked the potential effect on supernova cosmology results, for example, and it is negligible.

We also note here that peculiar velocities have a second-order effect, in that they perturb the observed magnitude in addition to the redshift (Hui & Greene 2006). In brief, the observed luminosity distance, $D_\text{L}$, at the observed redshift, $z$, is related to the luminosity distance we would have seen in the absence of peculiar velocities, $D_\text{L}_0$ at the cosmological redshift $\bar{z}$, by

$$D_\text{L}(z) = D(\bar{z})(1 + z_\text{obs})(1 + z_\odot)^2,$$

where the two factors of $(1 + z_\odot)$ come from Doppler shifting and relativistic beaming, and $z_\text{obs}$ is the redshift due to our own motion along the line-of-sight direction. \(^9\) Since there is no factor of $(1 + \bar{z})$ in that equation, the correction is almost always small – of the order of 0.3 per cent for peculiar velocities of both observer and emitter of $300 \text{ km s}^{-1}$. We refer readers to Davis et al. (2011, equation 18) for additional details of how to take that effect into account.

Note that it is also important to use proper distance, not luminosity distance, when deriving velocities (this may have been done incorrectly in some early data sets). Using luminosity distance would result in an offset from the true velocity of $\Delta v = \bar{v}$, i.e. much larger than any error from the $v = cz$ approximation (about $3000 \text{ km s}^{-1}$ at $z = 0.1$).

Another subtle aspect we may need to consider when analysing future large data sets is that peculiar velocities tend to grow with cosmic time (at least until virialized). Since we observe along a past light cone, we measure peculiar velocities at a range of cosmic times. Therefore, the measured peculiar velocities will typically be lower than their velocities at the present day, which can cause problems if we average those velocities and try to measure a present-day bulk flow. In order to correct peculiar velocities to the values they would take at the present day, we would need to assume a model and use a value of $\Omega_m$, which is one of the quantities we are trying to measure. So to compare velocity measurements to theory in detail, we should make our theoretical predictions along the past light cone.

To conclude, we note that while it is tempting to apply a correction factor to the velocities published in papers that used the approximation, it is not quite that simple. Peculiar velocity measurements are made complex by selection effects and calibration of these surveys. It is therefore crucial to account for peculiar velocities when trying to measure the cosmological parameters.

Figure 3. When correcting for our own Sun’s motion of $v_\odot \sim 300 \text{ km s}^{-1}$ with respect to the CMB frame, the errors due to the approximation are small. This plot shows the maximum difference between the true cosmological redshift and (a) the observed redshift (blue), (b) the redshift after correcting using the approximation $\bar{z} = z - z_\odot$ (red dashed), and (c) the redshift after the complete correction has been applied $\bar{z} = (1 + \bar{z})/(1 + z_\odot) - 1$. Note that deviations are more significant at high redshift, where the magnitude–redshift diagram has a shallow slope, and therefore an incorrect $\bar{z}$ has only a small effect on cosmology inferred from supernovae or large-scale structure. The calculations for this plot are done for the maximum peculiar velocity (when the direction of interest lies directly along our direction of motion with respect to the CMB).

\(^8\) Colin et al. (2011) still seem to measure their bulk flow by their residual in distance versus the observed $z$, rather than $\bar{z}$, which could potentially induce a small bias, but one that is well below the uncertainty of the measurement.

\(^9\) However, Kashlinsky et al. (2008) compare their results to a cosmological model in their fig. 1f, using $\bar{z} = cz$. Although that part is subtly incorrect, it does not alter their conclusion that they measured overly large peculiar velocities, even though that has since been disputed for other reasons (Keisler 2009).
issues, which need to be dealt with carefully for each data set. Since bulk flow measurements are typically a conglomeration of distance measurements across a range of redshifts, the magnitude of the systematic error would be dependent on the redshift distribution of the catalogue being used, and if zero-point calibration has been done after the approximation has been made, this may ameliorate some of the bias the approximation could cause.

Moreover, it is not entirely clear that the approximation, \( v_p^{\text{approx}} = cz - H_0 D \), that appears in so many papers has actually been applied in the codes that are used to calculate the peculiar velocities. Equation (9) is widely known, despite not being implemented in velocity calculators such as NED, and the fact that \( cz = H_0 D \) is an approximation is popular knowledge (e.g. Harrison 1974, 1993). Private communication with some groups indicates that their code does not reflect the derivation given in their paper (e.g. Turnbull et al. 2012).

So as alarming as a 100 km s\(^{-1}\) error at \( z = 0.04 \) would be (or a 700 km s\(^{-1}\) error at \( z = 0.1 \)), some of the peculiar velocity analyses listed in the introduction may not suffer from much bias. Nevertheless, it would be worth checking this point, especially in light of the fact that larger-than-expected peculiar velocities have been reported and claimed to challenge the standard \( \Lambda \)CDM cosmological model.

Despite these caveats, we can give a rough indication of the magnitude of the correction that should be applied to some of the data sets, that we do know use the approximation, by considering the correction that would be required at their characteristic redshifts. For example,

(i) SFI+ (Springob et al. 2007) has median redshift \( \sim 0.02 \) with a maximum at \( \sim 0.07 \), so may have overestimated velocities by a median of \( \sim 20 \text{ km s}^{-1} \) and a maximum of \( \sim 250 \text{ km s}^{-1} \) (modulo footnote 10).

(ii) The Cosmicflows-2 compendium of distance measurements (Tully et al. 2013), when reanalysed with equation (16), finds a shift of 57 km s\(^{-1}\) closer to a mean peculiar velocity of zero (the mean is still negative, but less so, now at \( \sim 221 \text{ km s}^{-1} \)), as well as a small reduction in the dispersion of their sample of 1% per cent (Tully, private communication).

Meanwhile, the formula that should become the starting point for future peculiar velocity papers is (from equation 12)

\[
v_p = c \left( \frac{\bar{z} - \bar{z}}{1 + \bar{z}} \right),
\]

where \( \bar{z} \) is calculated from a distance measurement, using a fiducial cosmological model.

**ACKNOWLEDGEMENTS**

We would like to thank the authors of many of the peculiar velocity papers mentioned here for checking their data sets in light of this discussion, including Alexandra Abate, Sarah Bridle, Hume Feldman, Chris Springob, Brent Tully, and Rick Watson, and in particular for extensive comments from Michael Hudson. We would also like to thank the 6dFGS peculiar velocities survey group for the discussions in our telecons that inspired this paper, and for feedback on the drafts, in particular Yin-Zhe Ma, Chris Blake, Matthew Colless, Andrew Johnson, John Lucey, Heath Jones, Jun Koda, Christina Magoulas, Jeremy Mould, and Chris Springob; and also Eric Linder for pointing out footnote 3. TMD acknowledges the support of the Australian Research Council through a Future Fellowship award, FT100100595. MIS acknowledges the support of a Jean Rogerson Scholarship, a top-up scholarship from the University of Western Australia, and a CSIRO Malcolm McInnost Lecture bankmecu scholarship. We both acknowledge the support of the ARC Centre of Excellence for All Sky Astrophysics, funded by grant CE110001020 and a UWA-UQ Bilateral Research Collaboration Award.

**REFERENCES**

Abate A., Erdogdu P., 2009, MNRAS, 400, 1541
Abate A., Bridle S., Teodoro L. F. A., Warren M. S., Hendry M., 2008, MNRAS, 389, 1739
Carroll S. M., 2004, Spacetime and Geometry. Addison Wesley, San Francisco
Colin J., Mohayaee R., Sarkar S., Shaifeloo A., 2011, MNRAS, 414, 264
Colless M., Saglia R. P., Burstein D., Davies R. L., McMahan R. K., Wegner G., 2001, MNRAS, 321, 277
Dai D.-C., Kinney W. H., Stojkovic D., 2011, J. Cosmol. Astropart. Phys., 4, 15
Davis T. M. et al., 2011, ApJ, 741, 67
Dekel A., 1994, ARA&A, 32, 371
Duffy A. R., Meyer M. J., Staveley-Smith L., Bernyk M., Croton D. J., Koribalski B. S., Gerstmann D., Westerlund S., 2012, MNRAS, 426, 3385
Faber S. M., Dressler A., 1977, AJ, 82, 187
Feldman H. A., Watkins R., Hudson M. J., 2010, MNRAS, 407, 2328
Freedman W. L. et al., 2001, ApJ, 553, 47
Freedman W. L. et al., 2009, ApJ, 704, 1036
Harrison E. R., 1974, ApJ, 191, L51
Harrison E. R., 1993, ApJ, 403, 28
Hui L., Greene P. B., 2006, Phys. Rev. D, 73, 123526
Kaiser N., 1988, MNRAS, 231, 149
Kashlinsky A., Atrio-Barandela F., Kocevski D., Ebeling H., 2008, ApJ, 686, L49
Keisler R., 2009, ApJ, 707, L42
Keller S. C. et al., 2007, Publ. Astron. Soc. Aust., 24, 1
Koda J. et al., 2013, MNRAS, preprint (arXiv:1312.1022)
Lavaux G., Mohayaee R., Colombi S., Tully R. B., Bernardeau F., Silk J., 2008, MNRAS, 383, 1297
Lynden-Bell D., Faber S. M., Burstein D., Davies R. L., Dressler A., Terlevich R. J., Wegner G., 1988, ApJ, 326, 19
Masters K. L., Springob C. M., Haynes M. P., Giovanelli R., 2006, ApJ, 653, 861
Nusser A., Davis M., 2011, ApJ, 736, 93
Rathaus B., Kovetz E. D., Izhaki N., 2013, MNRAS, 431, 3676
Riess A. G. et al., 2009, ApJ, 699, 539
Riess A. G. et al., 2011, ApJ, 730, 119
Sarkar D., Feldman H. A., Watkins R., 2007, MNRAS, 375, 691
Springob C. M., Masters K. L., Haynes M. P., Giovanelli R., Marinoni C., 2007, ApJS, 172, 599
Springob C. M., Masters K. L., Haynes M. P., Giovanelli R., Marinoni C., 2009, ApJS, 182, 474
Strauss M. A., Willick J. A., 1995, Phys. Rep., 261, 271
Tully R. B. et al., 2013, AJ, 146, 86
APPENDIX A: DERIVATION OF OBSERVED REDSHIFT

Abbreviating the emitted and observed wavelengths as $\lambda_e$ and $\lambda_o$, respectively, the definition of redshift is

$$z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e} \quad \text{or equivalently} \quad 1 + z \equiv \frac{\lambda_o}{\lambda_e}. \quad (A1)$$

Imagine that you have three galaxies: (1) an emitter with peculiar velocity, which emits light at wavelength $\lambda_e$; (2) a local comoving galaxy (at the same position as the emitter), which sees the light from the emitter at $\lambda_c$; and (3) a distant comoving galaxy, which sees the light from the emitter at $\lambda_o$. (Assume that the peculiar velocity is along the line of sight between the distant comoving galaxy and the emitter.) See Fig. A1.

The redshift between the emitter and local comoving observer would be $1 + z_p = \lambda_c/\lambda_e$. This redshift occurs between two coincident observers (two observers at the same position, each with their own infinitesimal inertial frame); therefore, special relativity applies, and the redshift is related to peculiar velocity by equation (7) or (8). (For a technical discussion of infinitesimal inertial frames, see Carroll 2004, section 2.5.)

The redshift between the local comoving observer and the distant comoving observer would be $1 + \bar{z} = \lambda_o/\lambda_c$. This is the cosmological redshift, and is the redshift that all photons would experience en route from that position. It is related to the comoving distance by equation (3).

So the total redshift between the emitter and the distant comoving observer would be

$$(1 + z) = \frac{\lambda_o}{\lambda_e} = \frac{\lambda_o}{\lambda_c} \frac{\lambda_c}{\lambda_e} = (1 + \bar{z})(1 + z_p). \quad (A2)$$

That is the redshift that we would observe, if we were comoving. If we also have a peculiar velocity of our own (which we do), that adds an extra peculiar redshift at the point of observation, $z_{\text{obs}}^p$ (defined in footnote 9). So, following the same reasoning as above we find

$$(1 + z) = (1 + z_{\text{obs}}^p)(1 + \bar{z})(1 + z_p). \quad (A3)$$

This paper has been typeset from a TEX/LATEX file prepared by the author.