Dispersion relation formalism for virtual Compton scattering and the generalized polarizabilities of the nucleon

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A dispersion relation formalism for the virtual Compton scattering (VCS) reaction on the proton, presented, which for the first time allows a dispersive evaluation of 4 generalized polarizabilities at a four-momentum transfer $Q^2 \leq 0.5 \text{GeV}^2$. The dispersive integrals are calculated using a state-of-the-art pion photo- and electroproduction analysis. The dispersion formalism provides a new tool to analyze VCS experiments above pion threshold, thus increasing the sensitivity to the generalized polarizabilities of the nucleon.

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Over the past years, the virtual Compton scattering (VCS) process on the proton, accessed through the $ep \rightarrow ep\gamma$ reaction, has become a powerful and precise tool to provide new information on the internal structure of the nucleon [1]. VCS has been shown to be of particular interest not only at low outgoing photon energies where one probes nucleon-core excitations and pion-cloud contributions to so-called generalized polarizabilities, but also at high energy and momentum transfers, where one is sensitive to a new type of parton distributions, generalizing the information obtained from inclusive deep-inelastic scattering.

In the low energy regime below pion threshold, the outgoing photon in the VCS process plays the role of a quasi-constant applied electromagnetic dipole field and, through electron scattering, one measures the spatial distribution of the nucleon response to this applied field [1]. The response is parametrized in terms of 6 generalized polarizabilities (GP’s) [2,3], which are functions of the square of the virtual photon four-momentum $Q^2$.

The first dedicated VCS experiment has been performed at MAMI [4] and two combinations of GP’s have been determined at $Q^2 = 0.33 \text{ GeV}^2$. Further VCS experiments are underway at lower $Q^2$ at MIT-Bates [5] and at higher $Q^2$ at JLab [6].

At present, VCS experiments at low outgoing photon energies are analyzed in terms of a low-energy expansion as proposed in [2], assuming that the non-Born response of the system to the quasi-constant electromagnetic field of the low energetic photon is proportional to the GP’s. As the sensitivity of the VCS cross sections to the GP’s grows with the photon energy, it is advantageous to go to higher photon energies, provided one can keep the theoretical uncertainties under control when crossing the pion threshold. The situation can be compared to real Compton scattering (RCS), for which one uses a dispersion relation formalism [7,8] to extract the polarizabilities at energies above pion threshold, with generally larger effects on the observables. The aim of the present work is to provide such a dispersion formalism for VCS, as a tool to analyze VCS experiments at higher energies in order to extract the GP’s from data over a larger energy range. It will be shown that the same formalism also provides for the first time a dispersive evaluation of 4 GP’s.

To calculate the VCS process, we start from the helicity amplitudes:

$$T_{\lambda^*; \lambda s} = -e^2 \varepsilon_{\mu}(q, \lambda) \varepsilon_{\mu}^*(q', \lambda') \bar{u}(p', s') M^{\mu\nu} u(p, s),$$

with $e$ the electric charge, $q$ ($q'$) the four-vectors of the virtual (real) photon in the VCS process, and $p$ ($p'$) the four-momenta of the initial (final) nucleons respectively. The nucleon helicities are denoted by $s, s' = \pm 1/2$, and $u, \bar{u}$ are the nucleon spinors. The initial virtual photon has helicity $\lambda = 0, \pm 1$ and polarization vector $\varepsilon_\mu$, whereas the final real photon has helicity $\lambda' = \pm 1$ and polarization vector $\varepsilon_\mu'$. The VCS process is characterized by 12 independent helicity amplitudes $T_{\lambda^*; \lambda s}$.

The VCS tensor $M^{\mu\nu}$ in Eq. (1) is then expanded into a basis of 12 independent gauge invariant tensors $\rho_{i}^{\mu\nu}$,

$$M^{\mu\nu} = \sum_{i=1}^{12} F_i(Q^2, \nu, t) \rho_{i}^{\mu\nu},$$

as introduced in [3] (starting from the amplitudes of [6]). The amplitudes $F_i$ in Eq. (2) contain all nucleon structure information and are functions of 3 invariants for the VCS process : $Q^2 \equiv -q^2$, $\nu = (s - u)/(4M_N)$ which is odd under $s \leftrightarrow u$ crossing, and $t$. The Mandelstam invariants $s, t$ and $u$ for VCS are defined by $s = (q + p)^2$, $t = (q - q')^2$, and $u = (q - p)^2$, with the constraint $s + t + u = 2M_N^2 - Q^2$, and $M_N$ is the nucleon mass.

Nucleon crossing combined with charge conjugation provides the following constraints on the amplitudes $F_i$:...
at arbitrary virtuality $Q^2$

$$ F_i(Q^2, -\nu, t) = F_i(Q^2, \nu, t) \quad (i = 1, \ldots, 12). \quad (3) $$

In a next step, the VCS tensor $M_{\mu\nu}$ at low outgoing photon energies is separated into Born ($B$) and non-Born ($NB$) parts, as described in [2]. In the Born process, the virtual photon is absorbed on a nucleon and the intermediate state remains a nucleon, whereas the non-Born process contains all nucleon excitations and meson-loop contributions, and is parametrized through 6 GP’s. With the choice of the tensor basis of [2], the resulting non-Born amplitudes $F_i^{NB} \ (i = 1, \ldots, 12)$ are free of all kinematical singularities and constraints.

Assuming further analyticity and an appropriate high-energy behavior, the non-Born amplitudes $F_i^{NB}(Q^2, \nu, t)$ fulfill unsubtracted dispersion relations (DR’s) with respect to the variable $\nu$ at fixed $t$ and fixed virtuality $Q^2$:

$$ \text{Re} F_i^{NB}(Q^2, \nu, t) = \frac{2}{\pi} \int_{0}^{+\infty} d\nu' \frac{\nu' \text{Im} F_i(Q^2, \nu', t)}{\nu'^2 - \nu^2}, \quad (4) $$

with $\text{Im} F_i$ the discontinuities across the $s$-channel cuts of the VCS process. Since pion production is the first inelastic channel, $\nu_{thr} = m_\pi + (m_\pi^2 + t/2 + Q^2/2)/(2M_N)$, where $m_\pi$ denotes the pion mass.

The unsubtracted DR’s of Eq. (4) require that at sufficiently high energies ($\nu \to \infty$ at fixed $t$ and fixed $Q^2$) the amplitudes $\text{Im} F_i(Q^2, \nu, t) \ (i = 1, \ldots, 12)$ drop fast enough such that the integrals are convergent and the contributions from the semi-circle at infinity can be neglected. The high-energy behavior of the amplitudes $F_i$ is deduced from the high-energy behavior of the VCS helicity amplitudes of Eq. (2). After some algebra, we obtain the following Regge limit for $\nu \to \infty$, at fixed $t$ and $Q^2$:

$$ F_1, F_5 \sim Q^2 \nu^{\alpha_P(t)-1}, \quad \nu^{\alpha_M(t)}, \quad (5) $$

$$ F_7 \sim \nu^{\alpha_P(t)-2}, \quad \nu^{\alpha_M(t)-1}, \quad (6) $$

$$ F_2, F_3, (F_5 + 4F_{11}), \quad (6) $$

$$ F_6, F_8, F_9, F_{10}, F_{12} \sim \nu^{\alpha_P(t)-2}, \quad \nu^{\alpha_M(t)-2}, \quad (7) $$

$$ F_4 \sim \nu^{\alpha_P(t)-4}, \quad \nu^{\alpha_M(t)-3}. \quad (8) $$

In Eqs. (5)-(8), we have separately indicated the high energy behavior originating from the “pomeron” (with Regge trajectory $\alpha_p(t)$, and $\alpha_P(0) \approx 1.08$) and from $t$-channel meson-exchange contributions (with Regge trajectory $\alpha_M(t) \lesssim 0.5$). It then follows from Eqs. (5)-(8) for two amplitudes, $F_1$ and $F_5$, that an unsubtracted dispersion integral as in Eq. (4) does not exist, whereas the other 10 amplitudes on the lhs of Eqs. (5)-(8) can be evaluated through unsubtracted dispersion integrals. This situation is similar as for RCS, where 2 of the 6 invariant amplitudes cannot be evaluated by unsubtracted dispersion relations either [13].

To construct the VCS amplitudes $F_1$ and $F_5$ in an unsubtracted dispersion framework, one could proceed in an analogous way as has been proposed by L’vov [13] in the case of RCS. The unsubtracted dispersion integrals for $F_1$ and $F_5$ are evaluated along the real $\nu$-axis in a finite range $-\nu_{max} \leq \nu \leq +\nu_{max}$ (with $\nu_{max} \approx 1.5$ GeV).

The integral along a semi-circle of finite radius $\nu_{max}$ in the complex $\nu$-plane is described by the asymptotic contribution $F_1^{as}$, which is parametrized by $t$-channel poles (e.g. for $Q^2 = 0, F_1^{as}$ corresponds to $\sigma$-exchange, and $F_5^{as}$ corresponds to $\pi^0$-exchange). Since the parametrization of the asymptotic parts amounts to some phenomenology, we will limit ourselves in the present work to those 10 amplitudes $F_i \ (i \neq 1, 5)$ for which the integrals in Eq. (4) converge, and which do not involve such asymptotic contributions. A full study of VCS observables within a dispersion formalism - requiring, of course, all 12 amplitudes $F_i$, and a parametrization of the two asymptotic contributions - will be considered in a future work.

We next consider the non-Born VCS tensor at low energy ($|q'| \to 0$) but at arbitrary three-momentum $|q|$ of the virtual photon. In this limit, it has been shown [2] that the non-Born term can be parametrized by 6 generalized polarizabilities (GP’s), which are functions of $|q|$ and which are denoted by $P^{(\rho',L',\rho,L)}S(|q|)$. In this notation, $\rho$ ($\rho'$) refers to the electric (2), magnetic (1) or longitudinal (0) nature of the initial (final) photon, $L$ ($L'=1$) represents the angular momentum of the initial (final) photon, and $S$ differentiates between the spin-flip ($S=1$) and non-spin-flip ($S=0$) character of the transition at the nucleon side. A convenient choice for the 6 GP’s has been proposed in [2],

$$ P^{(01,01)}(|q|), \quad P^{(11,11)}(|q|), \quad (9) $$

$$ P^{(01,01)}(|q'|), \quad P^{(11,11)}(|q'|), \quad (10) $$

$$ P^{(01,12)}(|q|), \quad P^{(11,02)}(|q'|), \quad (11) $$

which reduces to the following expressions in the real photon limit ($|q'| = 0$) [3] :

$$ P^{(01,01)}(0) \sim \alpha, \quad P^{(11,11)}(0) \sim \beta, \quad (12) $$

$$ P^{(01,01)}(0) = 0, \quad P^{(11,11)}(0) = 0, \quad (13) $$

$$ P^{(01,12)}(0) \sim \gamma_3, \quad P^{(11,02)}(0) \sim \gamma_2 + \gamma_4, \quad (14) $$

where $\alpha$ ($\beta$) are the electric (magnetic) polarizabilities, and $\gamma_2, \gamma_3, \gamma_4$ are 3 of the 4 spin polarizabilities of RCS.

In terms of the VCS invariants, the limit $|q'| \to 0$ at finite $|q|$ corresponds to $\nu \to 0$ and $t \to -Q^2$ at finite $Q^2$. One can therefore express the GP’s in terms of

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1We have redefined 4 of the 12 invariant amplitudes of [1] by dividing them through $\nu$, such that all of them are even functions of $\nu$. This simplifies the formalism since only one type of dispersion integrals needs to be considered then.

2The pomeron contributes to the sum of helicity conserving amplitudes $T_1, T_2, T_3, T_4$, i.e. gives the same amplitude for parallel or antiparallel orientation of the helicities.
the VCS amplitudes $F_i$ at the point $\nu = 0$, $t = -Q^2$ at finite $Q^2$, for which we introduce the shorthand: $\bar{F}_i(Q^2) = F_i^{NB}(Q^2, \nu = 0, t = -Q^2)$. The relations between the GP’s and the $\bar{F}_i(Q^2)$ can be found in \textsuperscript{3}.

From the high-energy behavior for the VCS invariant amplitudes, it follows that one can evaluate the $\bar{F}_i$ (for $i \neq 1, 5$) through the unsubtracted DR's

$$\bar{F}_i(Q^2) = \frac{2}{\pi} \int_{\nu_{thr}}^{+\infty} d\nu' \frac{\text{Im} \bar{F}_i(Q^2, \nu', t = -Q^2)}{\nu'} . \quad (15)$$

Unsubtracted DR's for the GP’s will therefore hold for those combinations of GP’s that do not depend upon the amplitudes $\bar{F}_1$ and $\bar{F}_5$. We note however that $\bar{F}_5$ can appear in the combination $\bar{F}_5 + 4 \bar{F}_{11}$, which has the high-energy behavior of Eq. (16) leading to a convergent integral. Among the 6 GP’s, we find the following 4 combinations that do not depend upon $\bar{F}_1$ and $\bar{F}_5$:

- $P^{(01,01)0} + \frac{1}{2} P^{(11,11)0} = -\frac{2}{\sqrt{3}} \left( \frac{E + M_N}{E} \right)^{1/2} M_N \tilde{q}_0 \times \left\{ \frac{q_0^2}{\tilde{q}_0} \bar{F}_2 + (2 \bar{F}_5 + \bar{F}_9) - \bar{F}_{12} \right\}$,
- $P^{(01,01)1} = \frac{1}{3\sqrt{2}} \left( \frac{E + M_N}{E} \right)^{1/2} \tilde{q}_0 \times \left\{ (\bar{F}_5 + \bar{F}_7 + 4 \bar{F}_{11}) + 4 M_N \bar{F}_{12} \right\}$,
- $P^{(01,12)1} = \frac{1}{\sqrt{2} \tilde{q}_0} P^{(11,11)1} = \frac{1}{3} \left( \frac{E + M_N}{E} \right)^{1/2} M_N \tilde{q}_0 \times \left\{ (\bar{F}_5 + \bar{F}_7 + 4 \bar{F}_{11}) + 4 M_N (2 \bar{F}_6 + \bar{F}_9) \right\}$,
- $P^{(01,12)1} + \frac{\sqrt{3}}{2} P^{(11,02)1} = \frac{1}{6} \left( \frac{E + M_N}{E} \right)^{1/2} \tilde{q}_0 \times \left\{ q_0 (\bar{F}_5 + \bar{F}_7 + 4 \bar{F}_{11}) + 8 M_N^2 (2 \bar{F}_6 + \bar{F}_9) \right\}$,

where $E = \sqrt{|q^2| + M_N^2}$ denotes the initial proton c.m. energy, and $q_0 = M_N - E$ the virtual photon c.m. energy in the limit $|q^2| = 0$. Unfortunately, the 4 combinations of GP’s of Eqs. (16)-(19) can at present not yet be compared with the data. In particular, the only unpolarized experiment \textsuperscript{4} measured two structure functions which cannot be evaluated in an unsubtracted DR formalism, as they contain in addition to $P^{(01,01)0} + 1/2 P^{(11,11)0}$ of Eq. (16), which is proportional to $\alpha + \beta$ at $Q^2 = 0$, also the generalization of $\alpha - \beta$.

The 4 combinations of GP’s on the lhs of Eqs. (16)-(19) can then be evaluated by unsubtracted DR’s, from the dispersion integrals of Eq. (19) for the $\bar{F}_i(Q^2)$. To this end, the imaginary parts $\text{Im} \bar{F}_i$ in Eq. (15) have to be calculated by use of unitarity. For the VCS helicity amplitudes of Eq. (1) (denoted for short by $T_{ji}$), the unitarity equation reads:

$$2 \text{Im} \bar{T}_{ji} = \sum_X (2\pi)^4 \delta^4(P_X - P_i) T_{Xj}^1 T_{Xi} , \quad (20)$$

where the sum runs over all possible intermediate states $X$ that can be formed. In our present calculation, we saturate the dispersion integrals of Eq. (15) by the dominant contribution of the $\pi N$ intermediate states. For the pion photoproduction and electroproduction helicity amplitudes in the range $Q^2 \leq 0.5 \text{ GeV}^2$, we use the phenomenological analysis of MAID \textsuperscript{16}, which contains both resonant and non-resonant pion production mechanisms.

In Fig. 1, we show the imaginary parts of the 12 VCS reduced helicity amplitudes $\tau_i$ as calculated with $\pi N$ intermediate states. These reduced amplitudes are obtained from the full VCS helicity amplitudes of Eq. (1) by dividing out a common angular factor:

$$T_{\lambda's'; \lambda s} = (\cos \theta/2)^{|\Lambda + \Lambda'|} (\sin \theta/2)^{|\Lambda' - \Lambda'|} \tau_i . \quad (21)$$

In Eq. (21), the total helicities in the initial and final states are denoted by $\Lambda = \lambda - s$ and $\Lambda' = \lambda' - s'$ respectively, and the correspondence between the 12 $\tau_i$ and the
helicity labels is given in Fig. 4. The imaginary parts were calculated in two different ways, first through the helicity amplitudes as expressed in Eq. (24). The sum over the final states denoted by $X$ contains a phase-space integral for the $\pi N$ intermediate state, which is then performed numerically. In a second calculation, the helicity amplitudes are first decomposed into a multipole series, and the unitarity equation is then implemented for the $\pi N$ multipoles. It was found that the partial wave expansion up to orbital angular momentum $l \leq 3$, is already in very good agreement with the numerical integration, thus providing a valuable cross-check on the numerical calculation.

In Fig. 2, we show the results for the 4 combinations of GP’s of Eqs. (16)-(19) in the DR formalism, and compare them to the results of the $O(p^3)$ HBChPT \[3\] (dashed curves) and the linear $\sigma$-model \[1\] (dashed-dotted curves).

In conclusion, we have presented a DR formalism for VCS and given, for the first time, a dispersive result for 4 of the GP’s of the proton. These evaluations could be used to check the convergence of the chiral expansions in ChPT calculations. It also provides a new tool to analyze VCS experiments at higher energies (above pion threshold) where there is an increased sensitivity to the GP’s of the proton. In the future we will study further details of such an analysis, including possible parametrizations of the 2 non-convergent dispersion integrals.

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