An abstract theorem on the existence of hylomorphic solitons

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Abstract

In this paper we prove an abstract theorem which can be used to study the existence of solitons for various dynamical systems described by partial differential equations. We also give an idea of how the abstract theorem can be applied to prove the existence of solitons in some dynamical systems.

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1 Introduction

In some recent papers (1, 3, 4, 9, 6, 7) the existence of solitons has been proved using very similar techniques. In this paper we prove a general, abstract theorem which applies to most of the situations analyzed in the mentioned papers.

In section 2 we give an abstract definition of soliton. Section 3 is devoted to the main result, namely an existence result for hylomorphic solitons (Theorem 11). These solitons are minimizers, satisfying suitable stability properties, of a constrained functional. The proof of Theorem 11 is carried out in two steps: in the first step the research of the minimizers of a constrained functional is reduced to the study of the minimizers of a suitable free functional (Theorem 15). In the second step stability properties of these minimizers are proved (subsection 3.2).

In section 4 we give an idea (the complete proofs being contained in the quoted papers) of how the abstract theorem 11 can be used to prove the existence of solitons in some dynamical systems, namely for the nonlinear Schrödinger equation, the nonlinear wave equation, the nonlinear beam equation and the Klein-Gordon-Maxwell equation. In a forthcoming paper we shall use theorem 11 to study new situations.

2 An abstract definition of solitons

Solitons are particular states of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more fields which mathematically are represented by functions

\[ u : \mathbb{R}^N \rightarrow V \]  

(1)

where V is a vector space with norm \[ | \cdot |_V \] which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system \((X, \gamma)\) where \(X\) is the set of the states and \(\gamma : \mathbb{R} \times X \rightarrow X\) is the time evolution map. If \(u_0(x) \in X\), the evolution of the system will be described by the function

\[ u(t, x) := \gamma_t u_0(x). \]  

(2)

We assume that the states of \(X\) have "finite energy" so that they decay at \(\infty\) sufficiently fast. Roughly speaking, the solitons are "bump" solutions characterized by some form of stability.
To define them at this level of abstractness, we need to recall some well-known notions in the theory of dynamical systems.

**Definition 1** A set \( \Gamma \subset X \) is called invariant if \( \forall u \in \Gamma, \forall t \in \mathbb{R}, \gamma_t u \in \Gamma \).

**Definition 2** Let \((X, d)\) be a metric space and let \((X, \gamma)\) be a dynamical system. An invariant set \( \Gamma \subset X \) is called stable, if \( \forall \varepsilon > 0, \exists \delta > 0, \forall u \in X, \)

\[
d(u, \Gamma) \leq \delta,
\]
implies that

\[
\forall t \geq 0, d(\gamma_t u, \Gamma) \leq \varepsilon.
\]

For every \( z \in \mathbb{Z}^N \), and \( u \in X \), we set

\[
(g_z u)(x) = u(x - Az).
\]

(3)

where \( A \) is an invertible matrix; such a \( g_z \) will be called lattice transformation. We set

\[
G = \{ g_z | z \in \mathcal{G} \};
\]

(4)

where \( \mathcal{G} \) is a subgroup of \((\mathbb{Z}^N, +)\). \( G \) is a group of transformations acting on the space \( X \); actually it is a linear representation of the group \( \mathcal{G} \).

**Definition 3** A non-empty subset \( \Gamma \subset X \) is called \( G \)-invariant if

\[
\forall u \in \Gamma, \forall z \in \mathbb{Z}^N, g_z u \in \Gamma.
\]

**Definition 4** A closed \( G \)-invariant set \( \Gamma \subset X \) is called \( G \)-compact if for any sequence \( u_n(x) \) in \( \Gamma \) there is a sequence \( g_n \in G \), such that \( u_n(g_n x) \) has a converging subsequence.

Now we are ready to give the definition of soliton:

**Definition 5** A state \( u(x) \in X \) is called soliton if there is an invariant set \( \Gamma \) such that

- (i) \( \forall t, \gamma_t u(x) \in \Gamma \),
- (ii) \( \Gamma \) is stable,
- (iii) \( \Gamma \) is \( G \)-compact.

**Remark 6** The above definition needs some explanation. For simplicity, we assume that \( \Gamma \) is a manifold (actually, in many concrete models, this is the generic case). Then (iii) implies that \( \Gamma \) is finite dimensional. Since \( \Gamma \) is invariant, \( u_0 \in \Gamma \Rightarrow \gamma_t u_0 \in \Gamma \) for every time. Thus, since \( \Gamma \) is finite dimensional, the evolution of \( u_0 \) is described by a finite number of parameters. The dynamical system \((\Gamma, \gamma)\) behaves as a point in a finite dimensional phase space. By the stability of \( \Gamma \), a small perturbation of \( u_0 \) remains close to \( \Gamma \). However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite dimensional system with a small perturbation.
3 Existence of hylomorphic solitons

We now assume that the dynamical system \((X, \gamma)\) has \(l + 1\) integrals of motion. One of them will be called energy and it will be denoted by \(E\); the set of the other integrals can be considered as a functional

\[ C : X \to \mathbb{R}^l \]

and it will be called hylenic charge. At this level of abstractness, the name energy and hylenic charge are conventional, but \(E\) and \(C\) need to satisfy different assumptions see assumption (EC-3) below. In our applications to PDE, \(E\) will be the usual energy. The name hylenic charge has been introduced in [4], [1], [2] and here this name will denote just a set of other integrals.

The presence of \(E\) and \(C\) allows to give the following definition of hylomorphic soliton.

**Definition 7** A soliton \(u_0 \in X\) is hylomorphic if \(\Gamma\) (as in Def. 5) has the following structure

\[ \Gamma = \Gamma(e_0, c_0) = \{ u \in X \mid E(u) = e_0, C(u) = c_0 \} \]

where

\[ e_0 = \min \{ E(u) \mid C(u) = c_0 \} \]  \hspace{1cm} (5)

for some \(c_0 \in \mathbb{R}^l\).

Notice that, by (5), we have that a hylomorphic soliton \(u_0\) satisfies the following nonlinear eigenvalue problem:

\[ E'(u_0) = \sum_{j=1}^{l} \lambda_j C'_j(u_0) \]

where we have set \(C(u) = (C_1(u), ..., C_l(u))\).

Clearly, for a given \(c_0\), the minimum of \(E\) might not exist; moreover, even if the minimum exists, it is possible that \(\Gamma\) does not satisfy (ii) or (iii) of def. [5].

In order to prove an existence result for hylomorphic solitons, we impose some assumptions to \(E\) and \(C\); to do this we need some other definitions:

**Definition 8** We say that a functional \(F\) on \(X\) has the splitting property if given a sequence \(u_n = u + w_n \in X\) such that \(w_n\) converges weakly to 0, we have that

\[ F(u_n) = F(u) + F(w_n) + o(1) \]  \hspace{1cm} (6)
Remark 9 Every quadratic form satisfies the splitting property; in fact, in this case, we have that
\[ F(u) := \langle Lu, u \rangle \]
for some continuous selfadjoint operator \( L \); then, given a sequence \( u_n = u + w_n \) with \( w_n \to 0 \) weakly, we have that
\[
F(u_n) = \langle Lu, u \rangle + \langle Lw_n, w_n \rangle + 2 \langle Lu, w_n \rangle \\
= F(u) + F(w_n) + o(1)
\]

Now we can formulate the properties we require:

- (EC-1) there are \( l+1 \) prime integrals \( E(u) \) and \( C(u) = (C_1(u), ..., C_l(u)) \) of the dynamical system \((X, \gamma)\) such that
  \[ E(0) = 0, \ C(0) = 0; \ E'(0) = 0; \ C'(0) = 0 \]
- (EC-2) \( E(u) \) and \( C(u) \) are \( G \)-invariant.
- (EC-3) (coercivity assumption) suppose that there exists \( a \geq 0 \), and \( s \geq 1 \) such that
  - (i) \( E(u) + a |C(u)|^s \geq 0 \);
  - (ii) if \( \|u\| \to \infty \), then \( E(u) + a |C(u)|^s \to \infty \)
  - (iii) for any sequence \( u_n \) in \( X \) such that \( E(u_n) + a |C(u_n)|^s \to 0 \), we have that \( u_n \to 0 \)
- (EC-4) \( E \) and \( |C| \) satisfy the splitting property.

Definition 10 A norm (or a seminorm) \( \| \cdot \|^\# \) is called an auxiliary norm (or seminorm) on \( X \) if it satisfies the following property:

- given any sequence \( \|u_n\| \) bounded in \( X \) such that \( \|u_n\|^\# \geq \delta > 0 \), we can extract a subsequence \( u_{n_k} \) and we can take a sequence \( g_k \in G \) such that \( g_k u_{n_k} \) is weakly convergent to some \( \tilde{u} \neq 0 \).

The notion of auxiliary norm is related to a result obtained by Lieb in \cite{[12]}.
In our abstract scheme, this norm allows to define the following number:
\[
A_0 := \liminf_{\|u\|^\# \to 0} \frac{E(u)}{|C(u)|^s}.
\]  
(7)

Now we can state the main result:
Theorem 11  Assume that the dynamical system \((X, \gamma)\) satisfies (EC-1),..., (EC-4). Moreover assume that

\[
\inf_{u \in X} \frac{E(u)}{|C(u)|} < \Lambda_0.
\]

Then, the dynamical system \((X, \gamma)\) admits a continuous family \(u_\delta \ (\delta \in (0, \bar{\delta}) \ , \ \bar{\delta} > 0)\) of independent, hylomorphic solitons (two solitons \(u_{\delta_1}, u_{\delta_2}\) are called independent if \(u_{\delta_1} \neq g u_{\delta_2}\) for every \(g \in G\)).

We will prove Theorem 11 in two steps:

- first step: existence of a set of minimizers \(\Gamma\) as in def. 7
- second step: proof of the stability of \(\Gamma\)

3.1 An existence result for constrained minimizers

In this section, we will be concerned with the first step. We need some definitions. These definitions are related to a couple \((X, G)\) where \(G\) is a group acting on \(X\). In our applications \(G\) will be the group \(G\), but these results hold also in an abstract framework.

Definition 12  Let \(G\) be a group acting on \(X\). A sequence \(u_n\) in \(X\) is called \(G\)-compact if we can extract a subsequence \(u_{n_k}\) such that there exists a sequence \(g_k \in G\) such that \(g_k u_{n_k}\) is convergent.

Definition 13  A functional \(J\) on \(X\) is called \(G\)-invariant if

\[
\forall g \in G, \ \forall u \in X, \ J(gu) = J(u).
\]

Definition 14  A functional \(J\) on \(X\) is called \(G\)-compact if any minimizing sequence \(u_n\) is \(G\)-compact.

Notice that a \(G\)-compact functional has a \(G\)-compact set of minimizers (see def. 9).

Now, we will prove the following existence result:

Theorem 15  Assume that \(E\) and \(C\) satisfy (EC-1),..., (EC-4) and \(G\). Then there are \(\delta > 0\) and a family of values of the charge \(c_\delta\), \(\delta \in (0, \bar{\delta})\), such that, for all \(\delta \in (0, \bar{\delta})\), the minimum

\[
e_\delta = \min \{ E(u) \ | \ C(u) = c_\delta \}
\]

(9)
exists and the set of minimizers $\Gamma_\delta$ is $G$-compact. Moreover, $\Gamma_\delta$ can be characterized also as the set of minimizers of the $G$-compact functional

$$J_\delta(u) = \Lambda(u) + \delta \Phi(u)$$

where

$$\Lambda(u) = \frac{E(u)}{|C(u)|}$$
$$\Phi(u) = E(u) + 2a |C(u)|^s.$$  

**Proof.** Now take the functional

$$J_\delta(u) = \Lambda(u) + \delta \Phi(u)$$

where $\delta \in (0, \bar{\delta})$ is so small that

$$\inf_{u \in X} J_\delta(u) < \Lambda_0.$$  

(10)

Such $\bar{\delta} > 0$ does exist by virtue of assumption (8).

It is easy to show that

$$J_\delta(u) \geq \frac{\delta}{2} \Phi(u) - M$$  

(11)

where $M$ is a suitable constant; in fact by (EC-3)(i) we have that

$$E(u) \geq -a |C(u)|^s$$  

(12)

and hence

$$\frac{E(u)}{|C(u)|} \geq -a |C(u)|^{s-1}$$  

(13)

Then, by (12) and (13)

$$J_\delta(u) = \frac{E(u)}{|C(u)|} + \delta \Phi(u) \geq -a |C(u)|^{s-1} + \frac{\delta}{2} [E(u) + 2a |C(u)|^s] + \frac{\delta}{2} \Phi(u)$$

$$\geq -a |C(u)|^{s-1} + \frac{\delta}{2} [-a |C(u)|^s + 2a |C(u)|^s] + \frac{\delta}{2} \Phi(u)$$

$$= -a |C(u)|^{s-1} + \frac{a \delta}{2} |C(u)|^s + \frac{\delta}{2} \Phi(u) \geq \frac{\delta}{2} \Phi(u) - M$$

where

$$M = -\min_{t \geq 0} \left( \frac{\delta t^s - t^{s-1}}{2} \right).$$

Next, we will prove that $J_\delta$ is $G$-compact. Let $u_n$ be a minimizing sequence of $J_\delta$. We have to prove that there exists a sequence $g_k \in G$ and a subsequence $u_{n_k}$ such that $u_k := g_k u_{n_k}$ is convergent.
By (10), there exists $\eta > 0$ such that, for $n$ sufficiently large,

$$\frac{E(u_n)}{C(u_n)} + \delta \Phi(u_n) < \Lambda_0 - \eta.$$  

So we have that for $n$ sufficiently large

$$\frac{E(u_n)}{C(u_n)} < \Lambda_0 - \eta$$

and hence, by the definition of $\Lambda_0$, we have that

$$\|u_n\|_g \geq b$$

for some $b > 0$. By (EC-3)(ii) and (11), we have that $\|u_n\|$ is bounded. Then, by Def[10] we can extract a subsequence $u_{n_k}$ and we can take a sequence $g_k \in G$ such that $u_k := g_k u_{n_k}$ is weakly convergent to some

$$\bar{u} \neq 0.$$  

(15)

We can write

$$u_n = \bar{u} + w_n$$

with $w_n \to 0$ weakly. We want to prove that $w_n \to 0$ strongly. First of all we will show that

$$\lim \Phi(\bar{u} + w_n) \geq \Phi(\bar{u}) + \lim \Phi(w_n).$$

In fact, using (EC-4) we have that

$$\lim \Phi(\bar{u} + w_n) = \lim (E(\bar{u} + w_n) + 2a |C(\bar{u} + w_n)|^s)$$

$$= E(\bar{u}) + \lim E(w_n) + 2a \lim (|C(\bar{u})| + |C(w_n)|)^s$$

$$\geq E(\bar{u}) + \lim E(w_n) + 2a \lim (|C(\bar{u})|^s + |C(w_n)|)^s$$

$$= E(\bar{u}) + 2a |C(\bar{u})|^s + \lim E(w_n) + 2a \lim |C(w_n)|^s$$

$$= \Phi(\bar{u}) + \lim \Phi(w_n).$$

(16)

Next we show that

$$C(\bar{u} + w_n)$$

does not converge to 0.

(17)

Arguing by contradiction assume that $C(\bar{u} + w_n)$ converges to 0. Then, since $\bar{u} + w_n$ is a minimizing sequence for $J_\delta$, also $E(\bar{u} + w_n)$ and $\Phi(\bar{u} + w_n)$ converge to 0. Then, by (EC-3)(iii), we get

$$\bar{u} + w_n \to 0$$

in $X$.

(18)

From (18) and since $w_n \to 0$ weakly in $X$, we have that $\bar{u} = 0$, contradicting (15). Then, by (17),

$$C(\bar{u} + w_n) = C(\bar{u}) + C(w_n) + o(1) \geq const. > 0$$

(19)
up to a subsequence. Now we set
\[
j_\delta = \lim J_\delta(u'_n); \quad e_\delta = E(\bar{u}); \quad c_\delta = |C(\bar{u})| \\
e_1 = \lim E(w_n); \quad c_1 = \lim |C(w_n)|.
\]

By (16) and (19), we have that
\[
j_\delta = \lim \left[ \frac{E(u'_n)}{|C(u'_n)|} + \delta \Phi(u'_n) \right] \\
= \lim \frac{E(\bar{u}) + E(w_n) + o(1)}{|C(\bar{u}) + C(w_n) + o(1)|} + \delta \lim \Phi(\bar{u} + w_n) \\
\geq \frac{e_\delta + e_1}{c_\delta + c_1} + \delta \lim \Phi(w_n) + \delta \Phi(\bar{u}). \quad (20)
\]

Now we want to prove that
\[
\frac{e_1}{c_1} \geq \frac{e_\delta}{c_\delta} \quad (21)
\]

We argue indirectly and we suppose that
\[
\frac{e_\delta}{c_\delta} > \frac{e_1}{c_1} \quad (22)
\]

By the above inequality it follows that
\[
\frac{e_\delta + e_1}{c_\delta + c_1} = \frac{\frac{e_\delta}{c_\delta} c_\delta + \frac{e_1}{c_1} c_1}{c_\delta + c_1} > \frac{\frac{e_1}{c_1} c_\delta + \frac{e_\delta}{c_\delta} c_1}{c_\delta + c_1} = \frac{e_1}{c_1} \quad (23)
\]
and hence
\[
j_\delta \geq \frac{e_\delta + e_1}{c_\delta + c_1} + \delta \lim \Phi(w_n) + \delta \Phi(\bar{u}) \\
\geq \frac{e_1}{c_1} + \delta \lim \Phi(w_n) + \delta \Phi(\bar{u}) \\
= \lim J_\delta(w_n) + \delta \Phi(\bar{u}) \geq j_\delta + \delta \Phi(\bar{u}) > j_\delta.
\]

So (22) cannot occur and then we have (21). In this case, arguing as in (23)
\[
\frac{e_\delta + e_1}{c_\delta + c_1} \geq \frac{e_\delta}{c_\delta} \quad (24)
\]
ans so, using (20) and the above inequality,
\[
j_\delta \geq \frac{e_\delta + e_1}{c_\delta + c_1} + \delta \Phi(\bar{u}) + \delta \lim \Phi(w_n) \\
\geq \frac{e_\delta}{c_\delta} + \delta \Phi(\bar{u}) + \delta \lim \Phi(w_n) \\
= J_\delta(\bar{u}) + \delta \lim \Phi(w_n) \geq j_\delta + \delta \lim \Phi(w_n).
\]

Then
\[
\delta \lim \Phi(w_n) \leq 0
\]
and by (EC-3)(iii) \( w_n \to 0 \) and hence \( u_n' \to \bar{u} \) strongly. Thus \( J_\delta \) is \( G \)-compact and the set of minimizer \( \Gamma_\delta \) is not empty. Clearly, if \( u \in \Gamma_\delta \), it turns out that \( u \) minimises also the functional
\[
\frac{E(u)}{c_\delta} + \delta [E(u) + ac_\delta^4] = \left( \frac{1}{c_\delta} + \delta \right) E(u) + \delta ac_\delta^4
\]
on the set \( \{ u \in X \mid C(u) = c_\delta \} \) and hence it minimizes \( E(u) \) on this set.
\[\square\]

In the next section we consider the second step, namely we prove that \( \Gamma_\delta \) is stable and then we complete the proof of Theorem 11.

### 3.2 A stability result

We need the (well known) Liapunov theorem in following form:

**Theorem 16** Let \( \Gamma \) be an invariant set and assume that there exists a differentiable function \( V \) (called a Liapunov function) such that

- (a) \( V(u) \geq 0 \) and \( V(u) = 0 \Leftrightarrow u \in \Gamma \)
- (b) \( \partial_t V(\gamma_t(u)) \leq 0 \)
- (c) \( V(u_n) \to 0 \Leftrightarrow d(u_n, \Gamma) \to 0 \).

Then \( \Gamma \) is stable.

**Proof.** For completeness, we give a proof of this well known result. Arguing by contradiction, assume that \( \Gamma \), satisfying the assumptions of Th. 16, is not stable. Then there exists \( \varepsilon > 0 \) and sequences \( u_n \in X \) and \( t_n > 0 \) such that

\[
d(u_n, \Gamma) \to 0 \quad \text{and} \quad d(\gamma_{t_n}(u_n), \Gamma) > \varepsilon.
\]

Then we have
\[
d(u_n, \Gamma) \to 0 \implies V(u_n) \to 0 \implies V(\gamma_{t_n}(u_n)) \to 0 \implies d(\gamma_{t_n}(u_n), \Gamma) \to 0
\]
where the first and the third implications are consequence of property (c). The second implication follows from property (b). Clearly, this fact contradicts (25).
\[\square\]

The following Theorem holds:

**Theorem 17** Assume (EC-1) and (EC-2). For \( u \in X \) and \( e_0, c_0 \in \mathbb{R} \), we set
\[
V(u) = (E(u) - e_0)^2 + (C(u) - c_0)^2.
\]

If \( V \) is \( G \)-compact and
\[
\Gamma = \{ u \in X : E(u) = e_0, \ C(u) = c_0 \} \neq \emptyset,
\]
then every \( u \in \Gamma \) is a soliton.
Proof: We have to prove that $\Gamma$ in (27) satisfies (i),(ii) and (iii) of Def. 5. The property (iii), namely the fact that $\Gamma$ is $G$-compact, is a trivial consequence of the fact that $\Gamma$ is the set of minimizers of a $G$-compact functional $V$. The invariance property (i) is clearly satisfied since $E$ and $C$ are constants of the motion. It remains to prove (ii), namely that $\Gamma$ is stable. To this end we shall use Th. 16. So we need to show that $V(u)$ satisfies (a), (b) and (c). Statements (a) and (b) are trivial. Now we prove (c). First we show the implication $\Rightarrow$. Let $u_n$ be a sequence such that $V(u_n) \to 0$. By contradiction we assume that $d(u_n, \Gamma) \to 0$, namely that there is a subsequence $u_{n_k}$ such that

$$d(u_{n_k}', \Gamma) \geq a > 0.$$  

(28)

Since $V(u_n) \to 0$ also $V(u_{n_k}') \to 0$, and, since $V$ is $G$ compact, there exists a sequence $g_n$ in $G$ such that, for a subsequence $u_{n_k}'$, we have $g_n u_{n_k}' \to u_0$. Then

$$d(u_{n_k}', \Gamma) = d(g_n u_{n_k}'', \Gamma) \leq d(g_n u_{n_k}'', u_0) \to 0$$

and this contradicts (28).

Now we prove the other implication $\Leftarrow$. Let $u_n$ be a sequence such that $d(u_n, \Gamma) \to 0$, then there exists $v_n \in \Gamma$ s.t.

$$d(u_n, \Gamma) \geq d(u_n, v_n) - \frac{1}{n}.$$  

(29)

Since $V$ is $G$-compact, also $\Gamma$ is $G$-compact; so, for a suitable sequence $g_n$, we have $g_n v_n \to \bar{w} \in \Gamma$. We get the conclusion if we show that $V(u_n) \to 0$. We have by (29), that $d(u_n, v_n) \to 0$ and hence $d(g_n u_n, g_n v_n) \to 0$ and so, since $g_n v_n \to \bar{w}$, we have $g_n u_n \to \bar{w} \in \Gamma$. Therefore, by the continuity of $V$ and since $\bar{w} \in \Gamma$, we have $V(g_n u_n) \to V(\bar{w}) = 0$ and we can conclude that $V(u_n) \to 0$.

$\square$

In the cases in which we are interested, $X$ is an infinite dimensional manifold; then if you choose generic $e_0$ and $c_0$, $V$ is not $G$-compact since the set $\Gamma = \{u \in X : E(u) = e_0, C(u) = c_0\}$ has codimension 2. However, Th. 15 allows to determine $e_0$ and $c_0$ in such a way that $V$ is $G$-compact and hence to prove the existence of solitons by using Theorem 17.

Proof of Th. 11 In order to prove Th. 11 we will use Th. 17 with $e_0 = e_\delta$ and $c_0 = c_\delta$ where $e_\delta$ and $c_\delta$ are given by Th. 15.

We set

$$V(u) = (E(u) - e_\delta)^2 + (C(u) - c_\delta)^2.$$  

(30)

We show that $V$ is $G$-compact: let $w_n$ be a minimizing sequence for $V$, then $V(w_n) \to 0$ and consequently $E(w_n) \to e_\delta$ and $C(w_n) \to c_\delta$. Now, since

$$\min J_\delta = \frac{e_\delta}{c_\delta} = \delta [e_\delta + a c_\delta]^p,$$

we have that $w_n$ is a minimizing sequence also for $J_\delta$. Then, since $J_\delta$ is $G$-compact, we get

$$w_n \text{ is } G\text{-compact}.$$  

(31)
So we conclude that $V$ is $G$-compact and hence the conclusion follows by using Theorem 17.

\[ \square \]

4 Some applications

In this section we show how the abstract theorem can be used to prove existence of solitons in some dynamical systems, namely for the nonlinear Schrödinger equation, the nonlinear wave equation, the nonlinear beam equation and the Klein-Gordon-Maxwell equation. We do not carry out detailed proofs since they are contained in (1, 3, 4, 9, 6, 7).

4.1 The nonlinear Schroedinger equation

Let us consider the nonlinear Schrödinger equation (NSE):

\[ i\partial_t \psi = -\frac{1}{2}\Delta \psi + \frac{1}{2}W'(\psi) \tag{NSE} \]

The solutions of this equations are critical points of the functional

\[ S = \int \left[ \text{Re}(i\partial_t \psi \bar{\psi}) - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \right] dx \ dt \]

Noether’s theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. 11 or 4). They are derived by a continuity equation.

Now we describe the first integrals which will be relevant for this paper, namely the energy, the “hylenic charge” and the momentum.

**Energy.** The energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; it has the following form

\[ E(\psi) = \int \left[ \frac{1}{2m} |\nabla \psi|^2 + W(\psi) \right] dx. \tag{32} \]

**Hylenic charge.** Following 3 the hylenic charge, is defined as the quantity which is preserved by the invariance of the Lagrangian with respect to the action

\[ \psi \mapsto e^{i\theta} \psi. \]

Thus, in this case, the charge is nothing else but the $L^2$ norm, namely:

\[ C(\psi) = \int |\psi|^2 dx \]

The phase space is given by

\[ X = H^1(\mathbb{R}^N, \mathbb{C}) \]
and the generic point in $X$ will be denoted by

$$u = \psi$$

The norm of $X$ is given by

$$\|u\| = \left( \int \left( |\nabla \psi| + |\psi|^2 \right) dx \right)^{\frac{1}{2}}$$

and the auxiliary seminorm is given by

$$\|u\|_z = \sup_{z \in \mathbb{R}^N} \left( \int_{B_1(z)} |\psi|^2 dx \right)^{\frac{1}{2}}$$

Applying theorem 11 we get the following existence result (see e.g. [10], [3]):

**Theorem 18** Assume that

$$W(s) = \frac{1}{2} as^2 + N(s)$$

and that

$$|N'(s)| \leq c_1 |s|^{q-1} + c_2 |s|^{p-1} \text{ for some } 2 < q < p < 2^*.$$  \hspace{1cm} (33)

$$N(s) \geq -c_1 s^2 - c_2 |s|^\gamma \text{ for some } c_1, c_2 \geq 0, \ \gamma < 2 + \frac{4}{N} \text{ for some } s \geq 0.$$  \hspace{1cm} (34)

Then NSE has a continuous family of hylomorphic solitons.

We can see that the assumptions of Th. 11 but (EC-3) are easy to verify. In order to verify (EC-3) we need the Nash inequality in the following form

$$\|\psi\|_{L^p}^p \leq b_p \|\psi\|_{L^2}^r \|\nabla \psi\|_{L^q}^q$$  \hspace{1cm} (35)

where

$$p < 2 + \frac{4}{N},$$

$$q = pN \left( \frac{1}{2} - \frac{1}{p} \right)$$

$$r = p - q.$$

then, for $s = p/2$ and $a$ sufficiently large, we have that

$$E(u) + a |C(u)|^s \geq \frac{1}{2} \|\nabla \psi\|_{L^2}^2 + \int W(|\psi|) + a \|\psi\|_{L^p}^p \geq (by \ (34))$$

$$\geq \frac{1}{2} \|\nabla \psi\|_{L^2}^2 - c_3 \|\psi\|_{L^p}^p + a \|\psi\|_{L^2}^2 \geq (by \ (35)) \geq 0$$

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4.2 The nonlinear wave equation

The nonlinear wave equation (NWE) has the following structure

\[ \Box \psi + W'(\psi) = 0 \]  (NWE)

where \( \psi : \mathbb{R}^4 \to \mathbb{C} \) and where, with some abuse of notation we have set

\[ W'(\psi) = F'(|\psi|) \frac{\psi}{|\psi|} \]

for some smooth function \( F : [0, \infty) \to \mathbb{R} \). The NWE has a variational structure, namely it is the Euler-Lagrange equation with respect to the functional

\[ S = \frac{1}{2} \int \int (|\partial_t \psi|^2 - |\nabla \psi|^2) \, dx \, dt - \int \int W(\psi) \, dx \, dt \]

The energy and the hylenic charge take the following form

\[ E = \frac{1}{2} \int \int (|\partial_t \psi|^2 + |\nabla \psi|^2) \, dx + \int W(\psi) \, dx \]

\[ C = \text{Im} \int \psi^* \overline{\psi} \, dx. \]

In order to describe the phase space \( X \), we rewrite NWE as an Hamiltonian system:

\[
\begin{align*}
\partial_t \psi &= \phi \\
\partial_t \phi &= \Delta \psi - W'(\psi);
\end{align*}
\]

so we can see that the phase space is given by

\[ X = H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C}) \]

and the generic point in \( X \) will be denoted by

\[ u = \begin{bmatrix} \psi \\ \phi \end{bmatrix} \]

The norm of \( X \) is given by

\[ \|u\| = \left( \int (|\psi|^2 + |\nabla \psi|^2 + |\phi|^2) \, dx \right)^{\frac{1}{2}} \]

the auxiliary seminorm is given by

\[ \|u\|_s = \sup_{z \in \mathbb{R}^N} \left( \int_{B_1(z)} |\psi|^2 \, dx \right)^{\frac{1}{2}} \]

The following existence theorem holds (see [1])
Theorem 19 Assume that

\[ W(s) = \frac{1}{2} m^2 s^2 + N(s) \]

and that

- (W-i) **Positivity** \( W(s) \geq 0 \)
- (W-ii) **Nondegeneracy** \( W''(0) = m^2 > 0 \)
- (W-iii) **Hylomorphy** \( \exists s_0 : N(s_0) < 0 \)
- (W-iii) **Growth** there is a constant \( c > 0 \) such that
  \[ N'(s) \geq -c_1 s - c_2 s^{p-1}, \quad 2 < p < 2^* \]

Then NWE has a continuous family of hylomorphic solitons.

### 4.3 The nonlinear beam equation

Let us consider the nonlinear beam equation (NBE)

\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + W'(u) = 0. \]  

(NBE)

where \( u : \mathbb{R}^2 \to \mathbb{R}, \ W : \mathbb{R} \to \mathbb{R} \).

Equation (NBE), with a suitable choice of \( W(s) \), has been proposed as model for a suspension bridge (see [15], [13], [14], [16]).

NBE has a variational structure, namely it is the Euler-Lagrange equation with respect to the functional

\[ S = \frac{1}{2} \int \int (u_t^2 - u_{xx}^2) \, dx \, dt - \int \int W(u) \, dx \, dt. \]

The energy and the momentum take the following form

\[ E = \frac{1}{2} \int \int (u_t^2 + u_{xx}^2) \, dx \, dx + \int W(u) \, dx \]

\[ C = -\int u_t u_x \, dx. \]

In this case the momentum will play the role of hylenic charge. Equation (NBE), can be rewritten as an Hamiltonian system as follows:

\[ \begin{align*}
    \partial_t u &= v \\
    \partial_t v &= -\partial^4_x u - W'(u)
\end{align*} \]
The phase space is given by
\[ X = H^2(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C}) \]
and the generic point in \( X \) will be denoted by
\[ u = \begin{bmatrix} u \\ v \end{bmatrix}. \]

The norm of \( X \) is given by
\[ \|u\| = \left( \int (v^2 + u_{xx}^2 + u^2) \, dx \right)^{\frac{1}{2}}. \]

The auxiliary seminorm is given by
\[ \|u\|^\# = \|u\|_{L^{\infty}}. \]

The energy and the momentum (which in this case plays the role of hylenic charge), as functionals defined on \( X \), take the following form
\[ E(u) = \frac{1}{2} \int \left( v^2 + u_{xx}^2 \right) \, dx + \int W(u) \, dx \]
\[ C(u) = -\int vu_x \, dx. \]

By using Theorem 11 the following existence result can be proved (9)

**Theorem 20** Assume that

- *(W-i) (Positivity)* \( W(s) > 0 \) for \( s \neq 0 \) and \( \exists \delta > 0 \) such that \( |s| \geq 1 \Rightarrow W(s) \geq \delta \)
- *(W-ii) (Nondegeneracy)* \( W''(0) > 0 \)
- *(W-iii) (Hylomorphy)* \( \exists M > 0, \exists \alpha \in [0, 2), \forall s \geq 0, W(s) \leq M |s|^\alpha \)

Then NWE has a continuous family of hylomorphic solitons.

### 4.4 The Klein-Gordon-Maxwell equations

Now let us consider the Klein-Gordon-Maxwell equations:

\[ D_{\varphi}^2 \psi - D_{A}^2 \psi + W'(\psi) = 0 \]  \hspace{1cm} (36)
\[ \nabla \cdot (\partial_t A + \nabla \varphi) = q \Re (iD_{\varphi} \overline{\psi}) \]  \hspace{1cm} (37)
\[ \nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \varphi) = q \Re (iD_A \overline{\psi}) \]  \hspace{1cm} (38)
Here $q$ denotes a positive parameter which, in some physical models, represents the unit electric charge, $\nabla \times$ and $\nabla$ denote respectively the curl and the gradient operators;

$$A = (A_1, A_2, A_3) \in \mathbb{R}^3 \text{ and } \varphi \in \mathbb{R}$$

are the gauge potentials;

$$D_\varphi \psi = (\partial_t + iq\varphi) \psi$$

is the covariant derivative with respect to the $t$ variable, and

$$D_A \psi = (\nabla - iqA) \psi$$

is the covariant derivative with respect to the $x$ variable (see for example [5] and [17]).

Equations (36), ..., (38) are invariant with respect to the gauge transformations

$$\psi \rightarrow e^{iq\chi} \psi$$  \hspace{1cm} (39)

$$\varphi \rightarrow \varphi - \partial_t \chi$$  \hspace{1cm} (40)

$$A \rightarrow A + \nabla \chi$$  \hspace{1cm} (41)

where $\chi \in C^2(\mathbb{R}^4)$.

If we make the following change of variables:

$$E = - (\partial_t A + \nabla \varphi)$$  \hspace{1cm} (42)

$$H = \nabla \times A$$  \hspace{1cm} (43)

$$\rho = -q \text{Re} \left( iD_\varphi \psi \overline{\psi} \right)$$  \hspace{1cm} (44)

$$j = q \text{Re} \left( iD_A \psi \overline{\psi} \right).$$  \hspace{1cm} (45)

you get the following equations:

$$\nabla \cdot E = \rho, \quad \text{Gauss equation}$$

$$\nabla \times H - \frac{\partial E}{\partial t} = j, \quad \text{Ampère equation}$$

$$\nabla \times E + \frac{\partial H}{\partial t} = 0, \quad \text{Faraday equation}$$

$$\nabla \cdot H = 0, \quad \text{No-monopole equation}$$

We write $\psi$ in polar form

$$\psi(x, t) = u(x, t) e^{iS(x, t)}, \quad u \geq 0, \quad S \in \mathbb{R}/2\pi\mathbb{Z}. \hspace{1cm} (46)$$

Equation (36) can be split in the two following ones

$$\Box u + W'(u) + \left[ |\nabla S - qA|^2 - (\partial_t S + q\varphi)^2 \right] u = 0 \hspace{1cm} (47)$$
\[ \frac{\partial}{\partial t} \left[ (\partial_t S + q\phi) u^2 \right] - \nabla \cdot \left[ (\nabla S - qA) u^2 \right] = 0. \quad (48) \]

and, using the variables \( j \) and \( \rho \), these equations can be written as follows:

\[ \Box u + W'(u) + \frac{j^2 - \rho^2}{q^2 u^3} = 0 \]

\[ \partial_t \rho + \nabla \cdot j = 0. \]

Thus, our equations, in the gauge invariant variables, become:

\[ \Box u + W'(u) + \frac{j^2 - \rho^2}{q^2 u^3} = 0, \quad \text{Matter equation} \]

\[ \nabla \cdot E = \rho, \quad \text{Gauss equation} \]

\[ \nabla \times H - \frac{\partial E}{\partial t} = j, \quad \text{Ampère equation} \]

\[ \nabla \times E + \frac{\partial H}{\partial t} = 0, \quad \text{Faraday equation} \]

\[ \nabla \cdot H = 0, \quad \text{No-monopole equation} \]

Peculiar difficulties with the NKGM equations:

- (j) the Lagrangian density

\[ L(\psi, \partial_t \psi, A, \partial_t A, \phi) = \frac{1}{2} \left( |D_\phi \psi|^2 - |D_A \psi|^2 \right) + W(\psi) + \frac{1}{2} \left( |\partial_t A + \nabla \phi|^2 - |\nabla \times A|^2 \right). \]

is incomplete, namely it does not depend explicitly on \( \partial_\phi \phi \)

- (ji) the energy

\[ E(\mathbf{u}) = \frac{1}{2} \int \left( |\partial_t \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + \frac{\rho^2 + j^2}{q^2 u^2} + \mathbf{E}^2 + \mathbf{H}^2 \right) dx + \int W(\mathbf{u}) dx \]

does not have the "usual" form

\[ \{\text{energy}\} = \{\text{positive quadratic form}\} + \{\text{higher order terms}\}. \]

To overcome the difficulty (j), we set

\[ M_L = \left\{ (\mathbf{Q}, \partial_t \mathbf{Q}) : \partial_t \phi + \nabla \cdot \mathbf{A} = 0, \nabla \cdot (\partial_t \mathbf{A} + \nabla \phi) = q \Re (iD_\phi \psi \bar{\psi}) \right\} \quad (49) \]

where

\[ \mathbf{Q} = (\psi, \mathbf{A}, \phi), \quad \partial_t \mathbf{Q} = (\partial_t \psi, \partial_t \mathbf{A}, \partial_t \phi) \]

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and we consider the modified Lagrangian

$$
\mathcal{L}(Q, \partial_t Q) = \frac{1}{2} \left[ |D_{D_t} \psi|^2 - |D_A \psi|^2 \right] - W(\psi) \\
+ \frac{1}{2} \left[ |\partial_t A|^2 - |\nabla A|^2 - (\partial_t \phi)^2 + |\nabla \phi|^2 \right].
$$

(50)

The dynamics induced by $\mathcal{L}$ is given by the following equations:

$$
D_{D_t}^2 \psi - D_{D_t}^2 A \psi + W'(\psi) = 0 \tag{51}
$$

$$
\Box A - q \text{Re} (iD_A \psi \bar{\psi}) = 0 \tag{52}
$$

$$
\Box \phi + q \text{Re} (iD_{D_t} \psi \bar{\psi}) = 0 \tag{53}
$$

The following result is well known (for a proof see e.g. [8])

**Theorem 21** The set $M_L$ is invariant for the dynamics induced by equations (51, 52, 53). Moreover, if the initial data are in $M_L$ and $(\psi, A, \phi)$ is a smooth solution of eq. (51, 52, 53) then it is also a solution of NKGM.

Now it is possible to define the conjugate variable of $Q$ via the Legendre transform. These conjugate variables will be denoted by $P = (\hat{\psi}, \hat{A}, \hat{\phi})$. We have:

$$
\hat{\psi} = \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} = \partial_t \psi + iq\phi \psi = D_{D_t} \psi \tag{54}
$$

$$
\hat{A} = \frac{\partial \mathcal{L}}{\partial (\partial_t A)} = \partial_t A \tag{55}
$$

$$
\hat{\phi} = \frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = -\partial_t \phi. \tag{56}
$$

Now, the Hamiltonian is well defined and takes the form:

$$
\mathcal{H}(Q, P) = \frac{1}{2} \int \left( |\psi|^2 + |D_A \psi|^2 + |\nabla \phi + \hat{A}|^2 + |\nabla \times A|^2 \right) + \int W(\psi) \tag{57}
$$

Now let us see how to overcome the difficulty (jj). Using the gauge independent variables, the energy takes the form

$$
E(u) = \frac{1}{2} \int \left[ v^2 + |\nabla u|^2 + \frac{\rho^2 + j^2}{q^2 u^2} + \mathbf{E}^2 + \mathbf{H}^2 \right] dx + \int W(u) dx.
$$

where $v = \partial_t u$.

We introduce new gauge invariant variables which eliminate this singularity:

$$
\theta = \frac{-\rho}{qu}; \quad \Theta = \frac{j}{qu}. \tag{58}
$$
Using these new variables the energy and the charge take the form:

\[ E(u) = \frac{1}{2} \int \left[ v^2 + |\nabla u|^2 + \theta^2 + \Theta^2 + E^2 + H^2 \right] + \int W(u). \] (59)

\[ C(u) = -q \int \theta u dx. \] (60)

Now we can define the functional framework which allows to apply Th. 11. We can take a norm defined by the quadratic part of the energy, namely

\[ \|u\| = \left( \int \left[ v^2 + |\nabla u|^2 + m^2 u^2 + \theta^2 + \Theta^2 + E^2 + H^2 \right] dx \right)^{\frac{1}{2}}. \] (61)

and the auxiliary seminorm:

\[ \|u\|_2 = \sup_{z \in \mathbb{R}^3} \left( \int_{B_1(z)} u^2 \, dx \right)^{\frac{1}{2}}. \]

So we get the following functional framework. We will denote by \( V \) the completion of \( C_0^\infty(\mathbb{R}^3, \mathbb{R}^{12}) \) with respect to the norm (61) so that

\[ u = (u, v, \theta, \Theta, E, H) \in V \cong H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3, \mathbb{R}^{11}) \]

Finally \( X \subset V \) will denote the closure of

\[ X_0 = \{ u \in C(\mathbb{R}^3, \mathbb{R}^{12}) \mid \nabla \cdot E = \rho, \, \nabla \cdot H = 0, \, E(u) < +\infty \} \]

with respect to the norm \( \|u\| \).

By using Theorem 11 the following existence result can be proved (7).

**Theorem 22** Assume that

\[ W(s) = \frac{1}{2} m^2 s^2 + N(s) \]

and that

- (W-i) (Positivity) \( W(s) \geq 0 \)
- (W-ii) (Nondegeneracy) \( m^2 > 0 \)
- (W-iii) (Hylomorphy) \( \exists \delta > 0 \) and \( \alpha \in (0, m) \) such that \( W(s) \leq \frac{1}{2} \alpha^2 s^2 \)
- (W-iii) (Growth condition) There are constants \( a, b > 0, 6 > p > 2 \) s.t.

\[ |N'(s)| \leq as^{p-1} + bs^{\frac{p}{2}-\frac{3}{2}}. \]

Then NKGM have a continuous family of hylomorphic solitons.
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