On explosion of the chaotic attractor

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There are presented examples of the rather sudden and violent explosion of the strange attractor of a one-dimensional driven damped anharmonic oscillator induced by a relatively small change of the amplitude of the strongly nonperturbative periodic driving force. A phenomenologic characterization of the explosion of the strange attractor has been given in terms of the behavior of the average maximal Lyapunov exponent \( \lambda \) and that of the fractal dimension \( D_q \) for \( q = -4 \). It is shown that the building up of the exploding strange attractor is accompanied by a nearly linear increase of the maximal average Lyapunov exponent \( \lambda \). A sudden jump of the fractal dimension \( D_{-4} \) is detected when the explosion starts off from an attractor consisting of disjoint bunches separated by an empty phase-space region.

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I. INTRODUCTION

Various cases of the one-dimensional periodically driven damped anharmonic oscillator like the Duffing and the van der Pol oscillators are well-known and thoroughly investigated dissipative systems of relatively simple kinematics in which deterministic chaos appears for particular values of the parameters \([1,14]\). In the present paper we would like to concentrate our attention to the explosion of the strange attractor under which we mean the following phenomenon. Supposing that for a particular value of some control parameter the strange attractor consists of highly populated disjoint bunches separated by an empty or underpopulated phase-space region and the latter gets filled in gradually with the continuous change of the control parameter. Our concept of the explosion of the strange attractor is here more qualitative and general than that introduced originally in \([14]\). There a similar phenomenon has been reported for particular values of the parameters of the damped, purely quartic oscillator (called here oscillator O2) under the simultaneous exertion of a periodic and a constant driving forces, the latter serving the control parameter. The relatively weak driving forces enabled the author to give a detailed dynamical explanation of the explosion of the strange attractor. Namely, it arises when the unstable and stable regions of a hyperbolic fixed point happen to touch tangentially with the increase of the control parameter. It has been found that the explosion of the strange attractor is interrupted time-to-time by windows of regular motion when the control parameter increases.

Here we show that apart from its dynamical origin a similar phenomenon occurs for particular parameter sets of the periodically driven damped quartic oscillator with a nonvanishing linear component of the restoring force, but with a vanishing constant external force (called oscillator O1 below). Now the amplitude of the periodic driving force controls the explosion of the strange attractor and it has extremely large values that the driving is far away from being a perturbation, as opposed to the case of explosion for oscillator O2 in \([14]\). For oscillator O1 we present cases when the highly populated bunches of the unexplored strange attractor are separated by an empty phase-space region that is filled in rather suddenly when the control parameter is increased by a relatively small amount, much less than this happens in the case of the explosion of the strange attractor presented in \([14]\). Our term empty phase-space region needs some explanation. In the cases of exploding strange attractors belonging to the oscillator O1 we have established that the highly populated bunches of the yet unexplored attractors are separated by a phase-space region avoided by the trajectory points. For numerical samplings with similar statistics only a severely underpopulated region of the disjoint bunches has been found for the unexplored strange attractor for oscillator O2 given in \([14]\). Therefore, we shall say that in the examples found by us for oscillator O1 the explosion starts from an unexplored strange attractor characterized by an empty phase-space region separating the highly populated bunches. Moreover, in the examples that we are going to present no windows of periodic motion interrupt the explosion. We shall show that in these examples the explosion is more sudden and violent according to its quantitative characteristics as compared to the example discussed in \([14]\).

The qualitatively similar features of the explosion of the strange attractor found by us for the cases of oscillator O1 may or may not be the consequence of dynamics similar to that for oscillator O2. In our examples a detailed dynamical analysis of the explosion seems to be hardly available because of the strongly nonperturbative nature of the driving as opposed to the case discussed in \([14]\). Here our purpose is to show that there exist rather phenomenologic static global characteristics of the exploding strange attractor which are sensitive to the gradual change of the control parameter. We are showing that such characteristics are the average maximal Lyapunov exponent, as well as the generalized dimension \( D_q \) for some asymptotically large negative \( q \) value, (at \( q = -4 \) in our work). For a comparison, we analyse the explosion of the strange attractor for both types of oscillators O1 and O2.
Our paper is constructed as follows. In Sect. II we describe the externally driven, damped, anharmonic oscillators under consideration and the numerics used for the solution of their equations of motion. Also the qualitative features of the explosion of the strange attractor are discussed for the various cases. Sect. III presents the details of the determination of the maximal Lyapunov exponents averaged over various initial conditions and its tendency of monotonic increase during the building up of the exploding strange attractor is shown. In Sect. IV the details of the numerical determination of the fractal dimensions $D_q$ are given and a sudden jump of the fractal dimension $D_q$ with negative quotient $q = -4$ is shown for cases when the explosion starts off from an attractor of disjoint bunches separated by an empty phase-space region. A comparison of the various cases of the explosions of the strange attractor is given on the scale of the relative change of the control parameter. Finally, the conclusions are drawn in Sect. V.

II. EXPLoding STRange ATTRACTORS FOR DUFFING OSCILLATORS

Both oscillators O1 and O2 investigated by us are those of Duffing-type with a single-well potential. In terms of dimensionless parameters the equations of motion of oscillator O1 are

$$
\dot{x} = v,
\dot{v} = -\gamma v - \delta^2(x + x^3) + I \cos(2\pi t) \tag{1}
$$

with the damping factor $\gamma$, the amplitude $I$ of the periodic external driving force and the parameter $\delta^2$ of the anharmonic potential. Here the initial conditions $x(0) = v(0) = 1.0$ at $t = 0$ were chosen to get an overall picture on the position and extension of the bounded phase-space region in which the trajectory runs. Chaotic behaviour has been found by us for two sets of the parameters: (A) $\gamma = 0.325$, $\delta = 3.25$ and the control parameter $I \in [787.50, 788.75]$ increased with the steps $\Delta I \approx 0.015$, and (B) $\gamma = 0.65$, $\delta = 6.50$ and $I \in [1924.0, 1940.0]$ varied with the steps $\Delta I = 1.0$. No windows of regular motion occurred in these intervals. The equations of motion of oscillator O2 (see [14]) are given as

$$
\dot{x} = v,
\dot{v} = -\gamma v - x^3 + I_0 + I \cos(2\pi t) \tag{2}
$$

with the amplitude $I_0$ of the constant driving force, and the other parameters having the same meaning as for oscillator O1. We have chosen parameter set (C) used in [14] with the fixed values $\gamma = 0.05$ and $I = 0.16$ and the control parameter $I_0$ varying in the interval $[0.026, 0.055]$ with the discrete steps $\Delta I_0 \approx 0.001$. The initial conditions $x(0) = v(0) = 0.1$ at $t = 0$ were chosen in the close neighbourhood of the chaotic attractor found in [14].

The equations of motion have been numerically integrated by fourth-order Runge-Kutta method. The usage of the time step $\Delta t = 0.001$ provided stable regular solutions, as well as stable distribution of the points on the strange attractor. The time evolution of the phase trajectories has been followed through $\sim 10^5 - 10^9$ time periods of the periodic driving force after the initial transient has been damped at around $t_{\text{damp}} \approx 10^3$ time periods. The Poincaré-sections were determined by projecting the points of the phase trajectory at each period of time, i.e., at times $t_n = t_0 + n(D\phi/2\pi)$ of the driving force onto the $(x, v)$ plane, were the first Poincaré section has always been taken at the time $t_0 > t_{\text{damp}}$. The phase shift was set to $D\phi = 0$ except of the determination of the fractal dimension $D_{-4}$ when an average over various choices of $D\phi$ has been taken (see below in Sect. IV in more detail). The final decision on the regular or chaotic nature of the trajectory was taken depending on the sign of the maximal Lyapunov exponent (see the discussion in Sect. III below).

Let us start with the description of the evolution of the strange attractor for oscillator O1 for parameter sets (A) and (B) illustrated in Figs. 1 and 2. Below we shall determine the maximal Lyapunov exponent and establish its definitively positive value in all of the cases shown in these figures. For oscillator O1 and parameter set (A) the explosion occurs with increasing control parameter $I$ as shown in Figs. 1. For $I = 787.91$ the chaotic attractor is initially concentrated in three disjoint bunches on the Poincaré map separated by an essentially empty phase-space region and with the increase of the control parameter from $I \approx 788.0$ to $788.4$ it bursts out filling in the whole phase-space region between the originally disjoint bunches. It was also observed that the strange attractor before the explosion for $I \in [787.50, 788.75]$ and just after it for $I \in [788.4, 788.6]$ remains qualitatively unaltered. The critical value is at $I_c \approx 788.0$ where the explosion of the attractor sets on and it continues to $I \approx 788.4$.

For oscillator O1 with parameter set (B) a similar explosion of the strange attractor occurs with increasing control parameter $I$ (see Fig. 2). For values $I \in [1926, 1929]$ the strange attractor consists of three disjoint bunches, for $I$ raising from 1929 to 1935 it bursts out, and for $I \in [1935, 1938]$ its extension and pictorial form are essentially kept. The critical value is at $I_c \approx 1929.5$ where the explosion sets on and it lasts until the value $I \approx 1935$ is reached.

Now let us have a closer look on the evolution of the strange attractor for the oscillator O2 with parameter set (C) used in [14]. In this case windows of regular and irregular motion alternate in the investigated parameter interval $I_0 \in [0.03, 0.045]$ which interrupt the explosion process of the strange attractor. The evolution of the attractor with the gradual increase of the control parameter $I_0$ has been mapped by the steps $\Delta I_0 = 0.001$ and more densely with the steps $\Delta I_0 = 0.00016$ in the intervals where the average maximal Lyapunov exponent $\lambda$ changes rather rapidly. Taking $I_0$ values from the reg-
ular windows, we have performed a thorough search for an additional strange attractor: we have started several trajectories with various initial conditions distributed in the phase-space region generally occupied by the strange attractor when it appears with certainty for close values of the control parameter, but we have always found only regular trajectories. Therefore it can be excluded that in these windows both a limit cycle and a strange attractor exist, as far as the investigated phase-space region is considered.

Taking the set of pictures on the strange attractor with these rather fine steps of the control parameter, we have found that there is a critical value $I_0 \approx 0.03476$, where the attractor can be considered unexploded, consisting of two bunches separated by a phase-space region of very low population, but not being empty as compared to the corresponding phase-space regions belonging to the unexploded strange attractors found by us for oscillator O1. This can be seen from the comparison of the top left picture in Fig. 3 to the top left pictures in Figs. 1 and 2, when one keeps in mind that all pictures of the Poincaré maps represent a statistical sampling of similar quality of the distribution of the trajectory points on the strange attractor. The serii of attractors in Figs. 3 and 4 show that both with increasing and decreasing values of the control parameter $I_0$ the attractor starts to build up with more and more points in the phase-space region separating the highly populated bunches. With increasing values of the control parameter $I_0 > I_0c$ the gradual building up of the attractor seems to continue until the value $I_0 \approx 0.038$ is reached where a wide regular window opens up (c.f. the negative average Lyapunov exponents in Fig. 8). Two isolated points of regular behavior has also been detected in the interval $I_0 \in [0.03476, 0.038]$ but those seem not to influence the explosion process. In the direction of decreasing $I_0 < I_0c$ the strange attractor starts to build up again, for $I_0 = 0.0346$ (the most left picture on the top of Fig. 4) there are more points populating the phase-space region between the bunches than for $I_0 = 0.03476$ (the most left picture on the top of Fig. 3), but then the set of pictures in Fig. 4 shows that for $I_0 = 0.03396$ the inner phase-space region becomes again almost empty, the explosion is suddenly interrupted, its result is almost erased and then a restart of the explosion leads to the gradual building up of the attractor till the control parameter falls off to the value $I_0 \approx 0.03316$. The sudden break up and restart of the explosion seems to appear as a result of the short regular window around
FIG. 2: Explosion of the chaotic attractor for oscillator O1 with parameter set \((B)\) and for the control parameter values \(I = 1929, 1930, 1933, 1935\) from the left to the right and from the top to the bottom, respectively.

\(I_0 \approx 0.034\) (see the points with negative average maximal Lyapunov exponents in Fig. 8). The explosion process seems then to continue until the upper edge of the wide regular window \(I_0 \in [0.030, 0.033]\) is reached. The evolution of the attractor found by us is in agreement with the observations reported in \([14]\) (see Figs. 3 (a) and (b) for \(I_0 = 0.030\) and 0.045, respectively, and Fig. 6 for \(I_0 = 0.035\) in \([14]\); \(B = I_0\) in the author’s notation). The case of explosion for \(I_0 > I_{0c} \approx 0.03476\) with increasing control parameter \(I_0\) has been thoroughly investigated in \([14]\) and also its dynamical explanation has been given. Namely, the explosion in that case arises when the stability and instability regions of a hyperbolic fixed point tangentially touch one another. The explosion of the attractor for control-parameter values \(I_0 < I_{0c}\) decreasing from the critical value, and its sudden interruption and restart are not mentioned in \([14]\).

III. MAXIMAL LYAPUNOV EXPONENT

A. On the numerical algorithm and its sensitivity

We have taken the decision on the regular or chaotic behaviour of the trajectories according to the sign of their maximal Lyapunov exponents for the determination of which we have developed a \(C^{++}\) code implementing Benettin’s method \([15–23]\). The time evolution of the minute vector \(\vec{\ell}(t) = (\xi(t), \eta(t))\) attached with its bottom to the reference trajectory at the time \(t = t_0 > t_d\) is then determined performing a sequential renormalization of its length to its original size after each time interval \(\tau\). The tip of the vector \(\vec{\ell}(t)\) points to the side-trajectory started at times \(t_k = t_0 + k\tau\) in the \((k + 1)\)-th sequence. The reference trajectory has been determined as described in the previous section, the side-trajectories have been determined by solving the linearized equations of motion using the same fourth-order Runge-Kutta algorithm and the same time step \(\Delta t = 0.001\). The linearized equations...
The equations of motion are given as

\[
\begin{align*}
\dot{\xi} &= \eta, \\
\dot{\eta} &= -\gamma \eta - \delta^2 [1 + 3x^2(t_k')]\xi
\end{align*}
\]  

(3)

for oscillator O1 and

\[
\begin{align*}
\dot{\xi} &= \eta, \\
\dot{\eta} &= -\gamma \eta - 3x^2(t_k')\xi
\end{align*}
\]  

(4)

for oscillator O2, where \(t_k' = t_k + t'\) with the time variable \(t' \in [0, \tau]\) started from zero at the beginning of each sequence and the dot stands here for differentiation with respect to \(t'\). At the end of any \((k + 1)\)-th sequence the vector \(\vec{\ell}(t_k + \tau - \epsilon)\) has been rescaled to its original size \(\ell_0 = |\vec{\ell}(t_0)|\) and the vector

\[
\vec{\ell}(t_{k+1}) = \frac{\ell_0}{|\vec{\ell}(t_k + \tau - \epsilon)|} \vec{\ell}(t_k + \tau - \epsilon)
\]  

(5)

(with \(\epsilon \to 0^+\)) obtained in this manner is used as initial condition for solving the linearized equations in the next sequence. Having determined the side-trajectories for the first \(n\) sequences, we obtained the quantities

\[
k_n(\ell_0, \tau) = \frac{1}{n\tau} \sum_{k=1}^{n} \ln \frac{|\vec{\ell}(t_k + \tau - \epsilon)|}{\ell_0}
\]  

(6)

which for large number \(n\) of the sequences converge to the maximal Lyapunov exponent [15–23],

\[
\lambda = \lim_{n \to \infty} k_n(\ell_0, \tau).
\]  

(7)

A detailed analysis of the numerical algorithm has been performed in order to determine its sensitivity to the number \(n\) of the time steps \(\tau\), the choice of the parameters \(\ell_0\) and \(\tau\), the orientation of the vector \(\vec{\ell}(t_0)\), as well as to the initial conditions. The parameters \(n \approx 15000\), \(\ell_0 = 10^{-9}\), \(\tau = 0.5\), \(\xi(0) = \ell_0\), \(\eta(0) = 0\) have been used for the determination of the maximal Lyapunov exponents. Fig. 5 illustrates the typical convergence of the \(k_n\) values. The maximal Lyapunov exponent has been determined as the mean of the last 20 \(k_n\) values in the sequence which show up a few per cent variance. A similar sensitivity has been obtained for varying the duration \(\tau\) of the time sequences taking the values \(\tau = 0.1, 0.2, 0.5, 0.7\), and for varying the length \(\ell_0\) of the initial minute vector.
The maximal Lyapunov exponents determined in the above described manner turned out to be much more sensitive to the initial conditions \((x(t_0), v(t_0))\) determining the point on the reference trajectory, where the minute vector \(\ell(t_0)\) has been attached to it. Therefore, we have determined maximal Lyapunov exponents averaged over several initial conditions. As described in Sect. III the reference trajectories have been followed up to \(t \approx 10^5\). For the various initial conditions such points of the corresponding Poincaré map, i.e., those of the attractor have been selected which belong to randomly chosen 5 to 10 integer \(t_0\) values in the time interval \([5 \cdot 10^4, 8 \cdot 10^4]\). Generally more initial conditions have been taken for values of the control parameter for which the chaotic attractor just explodes. It has been observed that the mean \(\bar{\lambda}\) of the maximal Lyapunov exponents belonging to the various initial conditions exhibit a variance of about 10 per
cent for any values of the control parameter involved in the investigation.

In order to test our numerical code for the determination of the maximal Lyapunov exponent we applied it to the Van der Pol oscillator investigated in [24] where the value $\lambda = 0.0985$ is given in comparison to our result $\lambda = 0.0992$ obtained for the same parameter values and initial condition. The discrepancy of 0.7 per cent is in agreement with the sensitivity of the algorithm to the various parameters.

B. Maximal Lyapunov exponents of the exploding chaotic attractors

The dependence of the average maximal Lyapunov exponent on the control parameter of the exploding strange attractor has been determined for oscillator O1 with parameter sets (A) and (B) and for oscillator O2 with parameter set (C). An approximately linear raise of the average maximal Lyapunov exponent $\bar{\lambda}$ during the explosion of the strange attractor with increasing values of the control parameter $I$ has been observed for oscillator O1 for both parameter sets (A) and (B), as shown in Figs. 6 and 7 respectively. Here the points represent maximal Lyapunov exponents averaged over 10 initial conditions in the region of the explosion. In these cases the explosion is not interrupted by windows of regular behaviour.

![FIG. 6: The average maximal Lyapunov exponent $\bar{\lambda}$ vs. the shifted control parameter $\delta I = I - 1920$ for oscillator O1 with parameter set (A). Also the scale $c = (I - I_c)/I_c$ with $I_c \approx 788.0$ is shown.](image)

A comparison of the graph in Fig. 6 with the plots in Fig. 1 reveals the following behaviour. For $I = 787.91$ (and similarly in the interval $I \in [787.4, \sim 787.9]$) the chaotic attractor consists of three disjoint bunches separated by an empty phase-space region, while the average maximal Lyapunov exponent remains almost constant. When the burst out of the attractor into the phase-space region between the highly populated bunches sets on at $I = I_c \approx 788.0$, an approximately linear raise of the average maximal Lyapunov exponent starts with increasing values of the control parameter $I$ and lasts in the whole interval $I \in [788.0, 788.3]$, where the explosion takes place. Finally, the $\lambda$ values saturate for $I \in [788.4, \sim 788.5]$, where the explosion stops. The average maximal Lyapunov exponent raised by a factor of $\sim 3$ during the explosion.

![FIG. 7: The average maximal Lyapunov exponents $\bar{\lambda}$ vs. the shifted control parameter $\delta I = I - 1920$ for oscillator O1 with parameter set (B). Also the scale $c = (I - I_c)/I_c$ with $I_c = 1930$ is shown.](image)

A similar approximately linear increase of the average maximal Lyapunov exponent $\bar{\lambda}$ with increasing control parameter $I$ can be recognized in Fig. 4. The comparison of Fig. 4 with the series of plots in Fig. 2 reveals again that the linear raise of $\bar{\lambda}$ accompanies the explosion of the strange attractor. For $I \in [1926, \sim 1929]$, when the yet unexploded strange attractor consists of three disjoint bunches, the average maximal Lyapunov exponent does not change significantly. In the interval $I \in [\sim 1930, \sim 1935]$ where the strange attractor explodes there occurs an almost linear raise of $\bar{\lambda}$ again. Finally, the $\lambda$ values saturate for $I \in [\sim 1935, 1940]$ where the explosion of the attractor stops. The average maximal Lyapunov exponent raised by a factor of $\sim 2.5$ in this case.

For the oscillator O2 with the parameter set (C) we recovered the same windows of the regular behaviour breaking time-to-time the explosion of the chaotic attractor with increasing control parameter $I_0$ observed in [14] previously. This was realized by making a point-to-point comparison with Fig. 7 in [14] showing the exponent-like quantities determined there. Even the few points where the author could not uniquely decide whether the motion is regular or chaotic reappeared in our calculation showing up both positive and negative $\lambda$ values for the various
initial conditions. These points generally are such where the system is rather close to an edge-of-chaos system. Our results on the maximal Lyapunov exponents $\lambda$ averaged over the various initial conditions are shown in Fig. 8. Here the trajectory started at $x(0) = v(0) = 0.1$ has been used to obtain the chaotic attractor on which additional initial conditions for 10 reference trajectories have been chosen, in order to evaluate the average exponent.

Here we identified the critical value with $I_{0c} \approx 0.03476$ at which we have found the smallest positive value of the average maximal Lyapunov exponent $\lambda_c \approx 0.0243$. In between the wider regular windows $I_0 \in [0.030, 0.033]$ on the left and $I_0 \in [0.038, 0.043]$ on the right the average exponent $\lambda$ shows up a rapid increase from its minimal value when $I_0$ moves away from its critical value in both directions and a kind of saturation occurs close to the edges of the wide regular windows. Such a matter-of-fact reflects the explosion of the strange attractor for $I_0$ values tuned away from its critical value as shown in Figs. 8 and 11 and discussed in Sect. 11. The overall trend of raising is approximately linear in both directions, again accompanied by the total increase of $\lambda$ with the factor of $\sim 2.0$ and $\sim 2.5$ for decreasing and increasing $I_0$ values, respectively. Under the present accuracy of our calculations such an overall trend seems not to be disturbed by the rather short regular windows breaking the chaotic behaviour in the interval $I_0 \in [0.33, 0.38]$. Neither the structural change of the strange attractor (mentioned in connection with the plots in Fig. 11) breaks the trend of variation of the average maximal Lyapunov exponent when the control parameter takes the values $I_0 = 0.03460$, $0.03400$, and $0.03396$ and swaps a rather short regular window in the close neighborhood of $I_0 = 0.03428$.

IV. GENERALIZED DIMENSIONS

A. Determination of the generalized dimensions

The generalized dimensions $D_q$ characterize the multifractal structure of the strange attractor. We have chosen the sandbox method which provides the generalized dimensions via the slope of the logarithm of the so-called generalized correlation sums $C_q$ vs. the logarithm of the box size $R$. Let $P_i (i = 1, 2, \ldots, N)$ be a series of points on the multifractal under consideration given via the vectors $\vec{r}_i$ in a $d$-dimensional Euclidean space. Hyperspheres of radii $R$ are centered at each of the points $P_i (i = 1, 2, \ldots, N)$ and the relative frequencies

$$w_i(R) = \frac{1}{N-1} \sum_{j=1, j\neq i}^N \Theta(R - |\vec{r}_i - \vec{r}_j|)$$

(8)

of finding in those another point $P_j (j \neq i)$ of the series are determined. For $q \neq 1$ the average of the powers $w_i^{q-1}(R)$ are taken over the series of points in order to determine the generalized correlation sums

$$C_q(R) = \frac{1}{N} \sum_{i=1}^N w_i^{q-1}(R).$$

(9)

Let us remind that for $q > 1$ integers and $N \gg 1$ the sum $C_q(R)$ represents the average probability to find a number of $q$ points in a hypersphere of radius $R$. The generalized dimensions are read off from the asymptotic scaling $C_q(R) \sim R^{(q-1)D_q}$ for $R \to 0$,

$$D_q \sim \frac{1}{q-1} \frac{\ln C_q(R)}{\ln(R/R_{\max})}, \quad q \neq 1$$

(10)

where $R_{\max}$ is the maximal distance of the points in the series. The dimension $D_1$, the information dimension is obtained as the limit $D_1 = \lim_{q \to 0} D_{1+q}$ that yields

$$D_1 \sim \frac{(1/N) \sum_{i=1}^N \ln w_i(R)}{\ln(R/R_{\max})}. $$

(11)

It is well-known that $D_2$ and $D_0$ are the correlation and the Hausdorff dimensions, respectively. The latter should be smaller than the dimension $d_{ph.sp.}$ of the phase-space ($D_0 < d_{ph.sp.} = 3$ in our case) because of the not space-filling fractal structure of the strange attractor. Furthermore, the generalized dimensions are monotonically decreasing with increasing value of the parameter $q$, i.e., it holds the inequality $D_q \geq D_{q'}$ for $q < q'$. The sum $C_q(R)$ is dominated by the regions of the strange attractor with large and small occupation probabilities, respectively, for $q \geq 0$ and $q < 0$. Therefore one expects that the generalized dimensions $D_q$ for negative parameter values $q < 0$ get large when the strange attractor exhibits extended regions of low occupation probability. This is the feature we shall use to characterize the explosion of the strange attractor.
We have used Grassberger and Procaccia’s correlation sum approach \cite{26,28,32} when the time series of a single variable measured on the attractor is embedded first into a $d$-dimensional vector space and then the correlation sum is evaluated from the series of vectors in the embedding space. For a proper choice of the dimension $d$ of the embedding space the embedded trajectories will have the same geometric and dynamical properties as the true trajectory has in the phase space. We constructed a number $N$ of $d$-dimensional vectors of the embedding space from the time series of the coordinate variable $x_n = x(t_n)$ with $n = 1, 2, \ldots, N$ ‘measured’ on the Poincaré map of the attractor. The series of vectors

$$\vec{r}_j = (x_j, x_{j+1}, \ldots, x_{j+d-1}) \quad (12)$$

($j = 0, 1, \ldots, N - 1$) have been constructed where we have chosen $t_0 \approx 10^5$ and the time interval between the sampled values as well as the lag time between the successive vectors has been set equal to the time period of the periodic driving force. The trajectories started from the point $(x(t_0), v(t_0))$ of the strange attractor have been determined numerically as described in Sect. II. The embedding space has been endowed by the Euclidean distance. The various parameters of the embedding algorithm were settled on the strange attractor for oscillator O1 with parameter set $(A)$ for the value $I = 788.22$ of the control parameter, i.e., for a strange attractor in the ‘midway’ of being exploded.

The determination of the asymptotic scaling region of the correlation sums is the basic ingredient of Grassberger and Procaccia’s method. The double-logarithmic plots in $C_q$ vs. $R$ have been taken by the steps $\Delta \ln R \approx 0.3$. Asymptotic scaling generally occurs in the interval $R_l \leq R \leq R_u$ where $R_u \approx R_{\text{max}}$ and $R_l > R_{\text{min}}$ with the estimated lower bound $R_{\text{min}}$. When data resulted from numerical computations are used - like in our case - $R_{\text{min}}$ can be estimated in terms of the bit resolution $b$ by which the data are represented in the computer as $R_{\text{min}} \approx 2^{-(b-2)}R_{\text{max}} \sim 10^{-4}R_{\text{max}}$ for $b = 15$, i.e., for double precision computations \cite{28}. According to our numerical experience the lengths of the scaling intervals are different for the various strange attractors and the various choices of the parameter $q$ and had to be determined in each particular case separately. The relatively small number of embedded vectors used (c.f. the discussion below) resulted in a statistical noise causing almost constant tails in the double-logarithmic plots in $C_q$ vs. $\ln R$ and the actual value of $R_l$ exceeded generally the estimated lower bound $R_{\text{min}}$ with its rather typical values $R_l/R_{\text{max}} \sim 10^{-3} - 10^{-2}$. Although we were able to identify the scaling regions in each of the cases, there occurred some ambiguity as to their boundaries $R_l$ and $R_u$. The statistical errors of the generalized dimensions $D_q$ presented by us below include the error of the fit of a straight line to the log-log plot in the scaling region and the generally even larger error from the somewhat ambiguous choice of the scaling region. The latter error has been estimated through the variance of $D_q$ values obtained for various, slightly different choices of the scaling region.

| $q$ | \(D_q \pm \Delta D_q\) |
|-----|----------------------|
| $d = 3$ | 3.5 ± 0.5 | 2.2 ± 0.5 | 2.8 ± 0.5 | 3.8 ± 0.5 |
| $d = 4$ | 2.8 ± 0.4 | 2.1 ± 0.4 | 2.7 ± 0.4 | 3.8 ± 0.4 |
| $d = 5$ | 1.9 ± 0.3 | 2.1 ± 0.3 | 2.2 ± 0.2 | 2.3 ± 0.3 |
| $d = 6$ | 1.1 ± 0.1 | 1.1 ± 0.1 | 1.05 ± 0.08 | 1.2 ± 0.1 |

| $d = 4$ | 1.03 ± 0.04 | 1.05 ± 0.04 | 0.99 ± 0.03 | 1.08 ± 0.04 |

TABLE I: Dependence of the generalized dimensions $D_q$ on the embedding dimension $d$, the errors include those of the fit of a straight line to the curve $\ln C_q$ vs. $\ln(R/R_{\text{max}})$ and the ambiguity of the scaling interval.

Recording a number $N = 5 \cdot 10^4$ of vectors we investigated the dependence of the various generalized dimensions $D_q$ for $q = -4, -3, \ldots, 3, 4$ on the dimension $d$ of the embedding space in the range $d \in [3, 6]$. It was established that the $D_q$ values for $q > 0$ are relatively stable although slightly raising with the increase of the embedding dimension (see Table I), while for $q < 0$ they blow up with increasing $d$. For our analysis we have chosen $d = 5$ for the dimension of the embedding space. This is in accordance with the observation \cite{20} that for a dissipative system the choice of $d$ at about twice the fractal dimension of the attractor can be sufficient to mimic the dynamics on the attractor. For positive parameter values $q > 0$ one would expect the saturation of the $D_q$ values with the further increase of the dimension $d$, but then in our case the relatively small number $N$ of the vectors would result in a very low probability to find another vector in the neighbourhood of radius $R_{\text{min}}$ of any given vector \cite{20}. In order to get at least 2 vectors in average in a neighborhood of radius $R_{\text{min}}$ we would need the number $N \approx 2 \cdot 2^{(b-2)d} \approx 10^{20}$ of vectors for $b = 15$ and $d = 5$. For $N \approx 5 \cdot 10^4$, $b = 15$ and $d = 5$ we get $N(R_{\text{max}}/R_{\text{min}})^{d-d} \approx 1/2$ points in a neighborhood of radius $10^{-1}R_{\text{max}}$. Therefore the statistics we have is rather poor, but a significant increase of the number $N$ of the vectors with several orders of magnitude is not also available. The poor statistics explains why the scaling intervals $R_l \leq R \leq R_u$ were found by us in some cases rather short and their endpoints somewhat ambiguous.

Also the dependence of the algorithm on the number $N$ of the vectors involved in the computation has been investigated. It has been established that the correlation dimension $D_2$ remains stable within the estimated errors for $N = 2.5 \cdot 10^4$, $5 \cdot 10^4$, $7.5 \cdot 10^4$, and $10^5$. The generalized dimensions $D_q$ for $-4 \leq q \leq 4$ have been computed for $N = 2.5 \cdot 10^4$ and $5 \cdot 10^4$. Table I shows that for $q \geq 0$ the $D_q$ values are identical within the errors in both cases, while for $q < 0$ the increase of the factor of 2 of the number of vectors leads to a slight fall of the corresponding $D_q$ values but still within the errors. Because the computer time for the evaluation of the correlation sums runs with $N^2$, we have chosen $N = 2.5 \cdot 10^4$ for the
systematic investigations.

The numerically evaluated values of the generalized dimensions are rather sensitive to the phase of the periodic driving force at which the Poincaré sections are taken. We have performed calculations for phase shifts \( \Delta \phi = 0, \pi, 3\pi/2 \). Typical values for the oscillator \( O1 \) with parameter set \( (A) \) for \( I = 788.22, d = 5 \) and \( N = 2.5 \cdot 10^4 \) are shown in Table I. As a rule, the changes of the generalized dimensions \( D_q \) with the phase shift \( \Delta \phi \) exceed the estimated errors \( \Delta D_q \). As to the investigated exploding attractors of oscillator \( O1 \), it has been observed that a phase shift of \( \Delta \phi = \pi \) corresponds to an almost rigid rotation of the Poincaré map by the angle \( \pi \), although the phase shifts with intermediate values \( 0 < \Delta \phi < \pi \) involve the distortion of the strange attractor, as well. This explains that the generalized dimensions of the system determined from Poincaré sections taken with a phase difference \( \pi \) are identical within their errors, but those belonging to Poincaré sections taken with the phase shift \( \pi/2 \) deviate much more than their statistical errors. For the discussed exploding attractor of oscillator \( O2 \), the phase shift seems always to cause a combination of some rigid rotation and distortion of the strange attractor. Due to this state of affairs we have determined the weighted averages \( \bar{D}_q \) of the generalized dimensions belonging to the Poincaré sections taken with phase shifts \( 0, \pi/2, \pi, \) and \( 3\pi/2 \) and these average values have been used for the characterization of the strange attractors.

One may be cautious about the ability of our procedure to yield the exact values of \( D_q \)'s basically due to the poor statistical sampling of the attractor, i.e., due to the rather small number \( N \) of the vectors in the embedding space. Nevertheless, the obtained values of the generalized dimensions enable one for making comparisons of the strange attractors of a given oscillator when the control parameter gradually changes. The \( q \)-dependence of the generalized dimensions for any of the investigated particular systems follows the theoretically expected monotonically falling off tendency with increasing parameter \( q \). As a rule the \( D_0 \) values satisfy the inequality \( 2 < D_0 < 3 \) which means that the strange attractor does not fill the 3-dimensional phase space. Although the \( D_q \) values for \( q < 0 \) may be far away of their exact values, but for all attractors they are determined by the same algorithm and in that manner their changes should be characteristic for the alteration of the fractal structure of the exploding strange attractor.

### Table I

| \( q \) | \( D_q \) ± \( \Delta D_q \) |
|-------|-----------------------------|
| \( -4 \) | 3.2 ± 0.5 2.8 ± 0.5 |
| \( -2 \) | 3.0 ± 0.5 2.7 ± 0.4 |
| \( 0 \)  | 2.2 ± 0.3 2.2 ± 0.2 |
| \( 2 \)  | 1.06 ± 0.11 1.05 ± 0.08 |
| \( 4 \)  | 0.91 ± 0.04 0.99 ± 0.03 |

### Table II

| \( q \) | \( D_q \) ± \( \Delta D_q \) |
|-------|-----------------------------|
| \( -4 \) | 3.2 ± 0.5 2.8 ± 0.5 |
| \( -2 \) | 3.0 ± 0.5 2.7 ± 0.4 |
| \( 0 \)  | 2.2 ± 0.3 2.2 ± 0.2 |
| \( 2 \)  | 1.06 ± 0.11 1.05 ± 0.08 |
| \( 4 \)  | 0.91 ± 0.04 0.99 ± 0.03 |

### Table III

| \( q \) | \( D_q \) ± \( \Delta D_q \) |
|-------|-----------------------------|
| \( -4 \) | 3.2 ± 0.5 2.8 ± 0.5 |
| \( -2 \) | 3.0 ± 0.5 2.7 ± 0.4 |
| \( 0 \)  | 2.2 ± 0.3 2.2 ± 0.2 |
| \( 2 \)  | 1.06 ± 0.11 1.05 ± 0.08 |
| \( 4 \)  | 0.91 ± 0.04 0.99 ± 0.03 |

**B. Explosion of the strange attractor and the generalized dimensions**

Before going into the detailed discussion of the results let us emphasize two features of the generalized dimensions which are of particular importance in characterizing the explosion of the chaotic attractor. It is well-known that the correlation sums for positive and negative parameters \( q \) are dominated by the contributions of the densely and rarely occupied regions of the attractor, respectively. Therefore, one expects that dimensions \( D_q \) with negative \( q \) values for which \( D_q \) approaches already the limiting value \( D_{-\infty} \) are rather sensitive to the explosion of the attractor. When the strange attractor has not yet been exploded and consists of several bunches of points separated by an empty phase-space region one should get much smaller values for the dimensions \( D_{-\infty} \) with \( |q| \gg 1 \) than the values obtained after the sampled points of the trajectory start to occupy the region separating the highly populated bunches of the strange attractor. Therefore one expects a sudden jump in the values of \( D_{-\infty} \) for \( |q| \gg 1 \). Moreover, the \( x \) coordinates of the points on the Poincaré map of the unexploded attractor represent a data set with gaps, i.e., these data belong to disjoint intervals. Consequently, the correlation sums are expected to show up two different scaling regions one for small and one for large separations of the point-pairs \( 26 \). For small \( R_l < R < R_b \) the pairs of points contributing to the sum \( C_{-\infty}(R) \) belong to the same bunch of the attractor, while for large \( R_b < R < R_a \) each point of the pairs belong to different bunches. Here \( R_b \) and \( R_a \) are, respectively, the characteristic size of a single bunch and that of the separation distance of the various bunches. After the explosion has been started, the gap in the data disappears and one obtains a single scaling region. Typical scalings of \( \ln C_{-\infty} \) vs. \( \ln (R/R_{\max}) \) are shown in Fig. 4 for oscillator \( O1 \) with parameter set \( (A) \). Just before the explosion of the attractor there are two scaling regions: a long one for small point separations and a rather short
one for large separations, but only a single scaling region appears when the explosion process has already been set on work. When the explosion process proceeds the originally empty phase-space region between the bunches becomes more and more occupied by trajectory points and, consequently, the two scaling regions merge into a single one. Similar behaviour has been observed for oscillator O1 with parameter set (B). For oscillator O2 with parameter set (C) no double scaling has been observed for values $J_0 \in [0.033, 0.038]$, i.e., in between the left and right wide regular windows. This is a consequence of not having an empty phase space region between the highly populated bunches of the strange attractor, i.e., that of not having a gap in the data neither for $I_{\text{uc}} = 0.03476$ (where the maximal Lyapunov exponent take its smallest value) nor for $J_0 = 0.03400$ (where a sudden break and restart of the explosion has been observed). For cases when two scaling regions occurred for oscillator O1, we have determined the $D_{-4}$ value from the much longer scaling region for small separations. Thus, for the exploding attractor the $D_{-4}$ values have been enhanced via the contributions of the separations $R$ of the order of the size of the underpopulated region, as compared to the unexploded attractor for which the contributions of separations not exceeding the size of the bunches dominate $\ln c$ and yield a suppressed $D_{-4}$ value. We have to mention that the second scaling regions for large separations are generally rather short and not available for a reliable determination of another scaling dimension due to the low statistics (due to the relatively small number of embedded vectors) in our computations. Nevertheless, its scaling exponent can be estimated generally a few times larger than the dimension $D_{-4}$ determined from the much longer scaling regions for small separations.

FIG. 9: Logarithm of the correlations sum $-\ln c_{-4}/5$ vs. $\ln(R/R_{\text{max}})$ for oscillator O1 with parameter set (A) for the strange attractor just before the explosion for $I = 787.92$ (top) and just after it has been started for $I = 788.01$ (bottom) with phase shift $\Delta \phi = 0$.

FIG. 10: The generalized dimension $D_{-4}$ vs. the shifted control parameter $\delta I$ (as given in Figs. 6 and 7, respectively) for the strange attractor belonging to oscillator O1 with parameter sets (A) (top) and (B) (bottom) for various phase shifts: $\Delta \phi = 0$ (circle), $\pi/2$ (square), $\pi$ (triangle), and $3\pi/2$ (diamond). Also the scale $c = (I - I_c)/I_c$ is shown. The $D_{-4}$ values are normalized to their average value in the ‘asymptotic tail’ for $I < I_c$.

Now let us discuss the behaviour of the fractal dimension $D_{-4}$ characterizing the exploding strange attractors for gradually varying control parameter. This is shown in Fig. 11 for oscillator O1 with parameter sets (A) and (B) for various choices of the phase shift $\Delta \phi$ of the pe-
periodic driving force for which the Poincaré sections were taken. Making the comparison more straightforward, the values \(D_{-4}(I, \Delta \phi)\) for each \(\Delta \phi\) are normalized to their average values taken in interval \(I < I_c\), i.e., for the unexploded attractor. It has been observed that for both discussed cases of the explosion the phase shift with \(\pi\) produces an almost rigid rotation of the Poincaré map of the strange attractor with the angle \(\pi\). This results in the agreement of the \(D_{-4}\) values within the error bars for \(\Delta \phi = 0\) and \(\pi\) and for \(\Delta \phi = \pi/2\) and \(3\pi/2\) that is illustrated by a few points for parameter set \((A)\) in Fig. 10. In order to reduce the CPU time the detailed analysis has been performed on the base of \(D_{-4}\) values obtained for the choices \(\Delta \phi = 0\) and \(\pi/2\). In Fig. 10 one can see that there occurs a significant jump of the dimension \(D_{-4}\) at \(c = 0\) when the explosion starts, i.e., when the approximately linear rise of the average maximal Lyapunov exponent begins in Figs. 6 and 7. Furthermore, it is also seen in Fig. 10 that the height of the jump is rather sensitive to the choice of the phase shift \(\Delta \phi\). Nevertheless, taking the weighted average \(\bar{D}_{-4}\) of the \(D_{-4}\) values over the various phase shifts, the jump, i.e., the effect of the start of the explosion remains still significant in both cases. One would need much better statistics and smaller error bars, i.e., orders of magnitude more vectors in the embedding space in order to decide whether the singular behaviour of \(D_{-4}(c)\) is the jump at \(c = 0\) of an approximately steplike function or that of a peaked function with a wider tail for \(c > 0\). Anyhow, the effect, i.e., the jump of the average dimension \(\bar{D}_{-4}\) at the start of the explosion is significant even for the accuracy of our method. One should also notice that the jump is of the factor of \(\sim 3\) for the case \((A)\) and \(\sim 6\) for case \((B)\), i.e., it becomes more expressed when the explosion of the strange attractor occurs for larger amplitude of the periodic driving force. The interval of almost linear increase of the maximal Lyapunov exponent is of the width \(\Delta c \approx 0.0004\) and 0.002 for cases \((A)\) and \((B)\), respectively, i.e., the process of the explosion accomplishes during a rather small relative change of the control parameter. The interval in which the singular behaviour of \(\bar{D}_{-4}\) occurs is even much shorter, \((\Delta c)_{\text{sing}} \approx 0.0001\) for both cases. Even if a few trajectory points burst out into the originally empty phase-space region, the value of \(D_{-4}\) jumps suddenly.

For the strange attractor belonging to oscillator \(O2\) with parameter set \((C)\) we have also determined the dependence of the dimensions \(D_{-4}\) on the control parameter \(I_0\) for various choices of the phase shift of the periodic external force (see Fig. 12). Here the Poincaré maps belonging to various choices of the phase shift cannot be obtained from each other by a rigid rotation. This is reflected by the variation of the \(D_{-4}\) values with the phase shift \(\Delta \phi\). Let us restrict our discussion to the interval \(I_0 \in [0.033, 0.038]\) surrounded by the wide regular windows from both sides. The \(D_{-4}(I_0)\) values are more or less peaked at \(I_{0c} \approx 0.03476\) for which we have found the smallest positive value of the average maximal Lyapunov exponent, although that peak is almost within the estimated errors except the single point for the phase shift \(\Delta \phi = 3\pi/2\). A thorough looking through the numerically evaluated scalings of \(\ln C_{-4}\) vs. \(\ln (R/R_{\text{max}})\) shows that in all cases only a single scaling interval exists. The average \(\bar{D}_{-4}\) values remain constant within the estimated errors in the interval \(I_0 \in [0.033, 0.038]\), as shown in Fig. 13. This may be a hint to suggest that the effect if it is present at all is so weak here that it cannot be seen by the accuracy of our computations. Let us remind the reader that we have established for oscillator \(O1\) that a decreasing amplitude of the control parameter causes a smaller jump of \(\bar{D}_{-4}\) when the explosion starts. For the strange attractor belonging to oscillator \(O2\) under discussion the control parameter has extremely small values. This together with the fact that the highly populated bunches of the strange attractor are not separated for \(I_{0c}\) with a really empty phase-space region can result in washing out the singularity in the control-parameter dependence of \(\bar{D}_{-4}\) within our computational accuracy.
For the strange attractor with parameter set (C) for oscillator O2 the widths of the intervals in which the average maximal Lyapunov exponent increases linearly are $\Delta c \approx 0.02$ and $\Delta c \approx 0.05$, respectively, to the left and to the right of the critical value $c = 0$. This means that the explosion in this case is much less sudden than in the above discussed cases for oscillator O1. The overall increase of the maximal Lyapunov exponent is a factor of $\sim 3$ and $\sim 2$ on the right-hand and left-hand sides of $I_{0c}$, respectively. This is quite similar to the factors of $\sim 2$ and $\sim 2.5$ of overall increase during the linear raise for the strange attractors with parameter sets (A) and (B), respectively, for oscillator O1. Afterall one has to conclude that the explosion of the strange attractor in the cases found by us for oscillator O1 is much more rapid and violent with the change of the control parameter, then the explosion for oscillator O2 with parameter set (C).

V. SUMMARY

We have presented examples on a very rapid and violent explosion of the strange attractor of a one-dimensional externally driven damped anharmonic oscillator when the control parameter of the explosion process, the amplitude of the strongly nonperturbative periodic driving force gradually increases by a relatively small amount. As compared to its use in [14], the term ‘explosion’ of the strange attractor is used by us in a rather phenomenologic and more general sense, disregarding of the dynamical origin of the explosion. It is shown that the explosion process can be reliably characterized by the dependence on the control parameter of such phenomenologic characteristics as the average maximal Lyapunov exponent $\lambda$ and the average generalized dimension $D_{\bar{4}}$. The former has been determined by Benettin’s method, the latter by means of the combination of the embedding technique and the sandbox method. For comparison the exploding strange attractor discussed in [14] has also been analysed in the same manner.

It has been shown that the explosion of the strange attractor is accompanied by an approximately linear increase of the average maximal Lyapunov exponent $\lambda$ in the cases presented by us as well as in the case given in [14]. This reflects the increasing chaoticity of the strange attractor when it gradually builds up during the explosion process. The overall increment of the maximal Lyapunov exponent is of the factor of cca. 2 to 3 in the various cases. In the cases presented by us the explosion is accomplished rather rapidly, after $\leq 0.3$ per cent of the relative change of the control parameter. As opposed to this, in the case discussed in [14] the explosion accomplishes much slowly, it needs 1 to 5 per cents of relative change of the control parameter.

In the examples presented by us also a rather sudden jump of the generalized dimensions $D_q$ with negative parameter $q$, in particular that of the average dimension $D_{\bar{4}}$ occurs when the explosion sets on, while we have not seen such a singularity in the case given in [14] within our computational accuracy. This disagreement may be explained, on the one hand, by the fact that in our examples the explosion starts from an attractor consisting of disjoint bunches which are separated by an empty phase-space region, while the latter is only underpopulated but not empty just before the strange attractor discussed in [14] bursts out and therefore one expects a much smaller effect in the latter case. On the other hand, our computations have been performed by using a relatively small number of 25000 embedded vectors, so that the weaker effect may not exceed the numerical errors. The empty phase-space region induces a gap in the string of the one-
dimensional data for the determination of the generalized correlation sum $C_{-4}(R)$ and that results in the dominance of the short-distance scaling of $\ln C_{-4}(R)$ for the yet unexplored strange attractor. When a few trajectory points start to occupy the originally empty phase-space region the contributions of inter-bunch distances become overemphasized in the generalized correlation sum $C_{-4}$ because of the much lower occupation probabilities in the phase-space region between the bunches than in the bunches themselves and this results in a sudden jump of the average generalized dimension $\bar{D}_{-4}$. The larger is the control parameter, the amplitude of the periodic driving force, the greater is the factor by which the dimension $\bar{D}_{-4}$ increases. The smallness of the driving force in the case discussed in [14] may be an additional source of such a weak effect of the explosion on $\bar{D}_{-4}$ that does not reveals itself under the accuracy of our approach.

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[1] K. Tomita, *Periodically forced nonlinear oscillators* in Chaos, ed. by A.V. Holden (Princeton Univ. Press, Princeton, 1986) pp. 211-236.
[2] Y. Ueda, *Some problems in the theory of nonlinear oscillations* Doctoral dissertation, Kyoto Univ., 1965. pp. 72.
[3] C. Hayashi, Y. Ueda, N. Akamatsu, H. Itakura, Trans. Inst. Elec. Commun. Eng. 53-A, 150-158 (1970).
[4] Y. Ueda, N. Akamatsu, and C. Hayashi, Trans. Inst. Elec. Commun. Eng. 56-A, 218-225 (1973).
[5] Y. Ueda, J. Stat. Phys. 20, 181-196 (1979).
[6] J. Testa, J. Pérez, and C. Jeffries, Phys. Rev. Lett. 48, 714 (1982).
[7] S. Novak and R.G. Frehlich, Phys. Rev. A 26, 3660-3663 (1982).
[8] R.G. Frehlich and S. Novak, Int. J. Non-Linear Mechanics 20, 123-134 (1985).
[9] Y. Ueda, Int. J. Non-Linear Mechanics 20, 481-491 (1985).
[10] Y. Ueda, Chaos, Solitons and Fractals 1, 199-231 (1991).
[11] M. Lakshmanan, *Bifurcations, Chaos, Controlling and Synchronization of Certain Nonlinear Oscillators*, arXiv:chaos-dyn/9709031 pp.31.
[12] A. Venkatesan, S. Parthasarathy and M. Lakshmanan, Chaos, Solitons and Fractals 18, 891-898 (2003).
[13] C. Bonatto, J.A.C. Gallas and Y. Ueda, Phys. Rev. E 77, 026217 (2008).
[14] Y. Ueda, *Explosion of strange attractors exhibited by Duffing’s equation*, Ann. NY Acad. Sci. 357 (1980) 422-434.
[15] B.V. Chirikov, CERN Trans. N° 71-40 (1971); B.V. Chirikov, F.M. Izrailev, V.A. Tayurski, comput. Physics Commun. 5, 11 (1973).
[16] M. Casartelli, E. Diana, L. Galgani, A. Scotti, Phys. Rev. A13, 1921 (1976).
[17] G. Benettin, L. Galgani, J.-M. Strelcyn, Phys. Rev. A14, 2338 (1976).
[18] G. Contopoulos, L. Galgani, A. Giorgilli, Phys. Rev. A18, 1183 (1978).
[19] A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, Physica 16D, 285 (1985).
[20] V.I. Oseledec, Trans. Mosc. Math. Soc. 19, 197 (1968).
[21] I. Shimada, T. Nagashima, Prog. Theor. Phys. 61, 1605 (1979).
[22] G. Benettin, L. Galgani, A. Giorgilli, J.-M. Strelcyn, Meccanica 15, 9 (1980).
[23] G. Benettin, L. Galgani, A. Giorgilli, J.-M. Strelcyn, Meccanica 15, 21 (1980).
[24] K. Ramasubramanian, M. S. Sriram, Physica D139, 72 (2000).
[25] H.G. Schuster, W. Just, *Deterministic Chaos*, (WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2005).
[26] R.C. Hilborn, *Chaos and Nonlinear Dynamics*, (Oxford Univ. Press, Oxford, 2000).
[27] C. Beck, F. Schögl, *Thermodynamics of chaotic systems*, (Cambridge Univ. Press, New York, 1993).
[28] P. Grassberger and I. Procaccia, Phys. Rev. Lett. 50, 346 (1983).
[29] P. Grassberger and I. Procaccia, Physica 9D, 189 (1983).
[30] P. Grassberger, Phys. Lett. 97A, 227 (1983).
[31] A. Ben-Mizrachi, I. Procaccia, and P. Grassberger, Phys. Rev. A29, 975 (1984).
[32] J.D. Farmer, E. Ott, and J.A. Yorke, Physica D7, 153 (1983).