Solution of three basic problems of elasticity for half-plane

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Abstract. Solution of elastic problem for half-plane is of the highest concern in almost every field of engineering. However, there are no relations binding the values of all components of the stress-strain state at the half-plane boundary. This paper shows that arbitrary formulation of the boundary conditions considerably simplifies integral equations and transfers them to the class of Fredholm’s first kind equations. Considering symmetry, the problem for half-plane with mathematical cut is reduced to the problem on half-plane boundary. Regardless of the method, any solution is reduced to ill-posed problem.

1. Introduction
There is an increasing trend in the number of poorly conditioned problems in rock mechanics, with many of them considered in the half-plane. Integral equations that require the boundary conditions formulation are introduced to solve specific problems for a half-plane. These are usually chosen in the simplest forms such as perfect sliding at the contact boundary, the adhesion with absolutely rigid body when the boundary values of the displacement components are equal to zero on the final intercept, etc. (Figure 1). Naturally, it is impossible to account for all the possibilities. Introducing the concept of concentrated force to the elastic problem, which is viewed as an incoherent and awkward attempt to assign the analytical solution to the concentrated force [1] is in principle ill-defined problem. However, it is convenient to use such ill-conditioned solutions in for various fields of knowledge (because of the analytical nature of the solution) in studying different regularities and patterns that cannot in turn be considered well-conditioned.

Figure 1. Possible formulations of the boundary conditions in different intercepts of the \( x \)-axis: \( u, v \)—displacement components, \( \sigma_y, \tau \)—stress components, \( F \)—concentrated force.
2. Two-dimensional problem solution
The general case of a singular integral equations system has the form [2]

\[ f(t_0) + 2\mu g(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{f(t) + 2\mu g(t)}{t - t_0} dt, \]

\[ \kappa f(t_0) - 2\mu g(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\kappa f(t) - 2\mu g(t)}{t - t_0} dt = \frac{1}{\pi i} \int_{\Gamma} \left[ f(t) + 2\mu g(t) \right] \frac{\bar{T} - t_0}{t - t_0}, \]

where \( \kappa = 3 - 4w; \)
\( \mu = E[2(1 + w)]^{-1}; \)
\( E — Young's \) modulus; \( w — \) Poisson ratio,

\[ f(t) = i \int_0 (X_n + iY_n) ds = f_1 + if_2 \]  

(2)

\( X_n, Y_n \) are components of the forces in the direction of the x and y axes; \( g = u + iv; \) \( u, v \) are components of the displacements in the direction of axes x and y; \( i \) is the imaginary unit; line over the function denotes the complex conjugate value; \( \Gamma \) is the boundary of the considered region; \( t_0 \) is the point affix of \( \Gamma \).

3. Solution for the half-plane boundary problem
Let’s assume the sought connection based on (1), (2), when \( \Gamma \) is the boundary of the half-plane that coincides with the x-axis. Given that

\[ d \frac{\bar{T} - t_0}{t - t_0} = 0 \]

we rewrite the system (1) for the half-plane boundary, as:

\[ f_1 + i f_2 + 2\mu (u + iv) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f_1 + i f_2 + 2\mu (u + iv)}{t - x} dt, \]

\[ \kappa (f_1 - i f_2) - 2\mu (u - iv) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\kappa (f_1 - i f_2) - 2\mu (u - iv)}{t - x} dt, \]

whence the real part is inferred to be

\[ f_1 + 2\mu u = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f_2 + 2\mu v}{t - x} dt, \]

\[ \kappa f_1 - 2\mu u = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\kappa f_2 - 2\mu v}{t - x} dt, \]

(3)

Accordingly, the imaginary

\[ f_2 + 2\mu v = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f_1 + 2\mu u}{t - x} dt, \]

\[ \kappa f_2 + 2\mu v = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\kappa f_1 - 2\mu u}{t - x} dt, \]

(4)

Exclude from (3) the term with \( f_2 \), and from (4) the term with \( f_1 \)

\[ 2\kappa f_1 = 2\mu (\kappa - 1) u + \frac{\kappa + 1}{\pi} \int_{-\infty}^{\infty} \frac{2\mu v}{t - x} dt, \]

\[ 2\kappa f_2 = 2\mu (\kappa - 1) v - \frac{\kappa + 1}{\pi} \int_{-\infty}^{\infty} \frac{2\mu u}{t - x} dt, \]

(5)

Now, contrariwise, excluding respectively \( v \) and \( u \) from (3), (4) we obtain
After differentiating (5) by $x$, i.e. passing on directly to the stress components in the left part, the equation can be written:

$$
\frac{\kappa - 1}{\kappa} \sigma_{xx} + \frac{\kappa + 1}{\kappa} \mu \nu' = \frac{\kappa + 1}{\kappa} \mu \sigma_{yy} - \frac{\kappa - 1}{\kappa} \nu' = + \frac{\kappa + 1}{\kappa} \mu \sigma_{yy} - \frac{\kappa - 1}{\kappa} \nu' = \int_{-\infty}^{\infty} \frac{u'}{t-x} dt. \quad (7)
$$

Similarly, after differentiating (6), we obtain

$$
\frac{\kappa - 1}{\kappa} \sigma_{yy} + \frac{\kappa + 1}{\kappa} \mu \sigma' = \frac{\kappa + 1}{\kappa} \mu \nu' = \frac{\kappa - 1}{\kappa} \nu' = \int_{-\infty}^{\infty} \frac{u'}{t-x} dt. \quad (8)
$$

As a result, system (7) determines the values of normal and shear stresses using the derivatives of displacements, whereas system (8)—in the opposite way. In other words, systems (7), (8) represent solutions of the three main elastic problems for the half-plane boundary in quadratures.

Note that equations (6) and (7) belong to the type of Fredholm’s equations of second kind it is necessary to follow the formulated boundary conditions for a specific problem. In the case of an unsuccessful formulation of the boundary conditions the systems (or one of the equations of systems (6), (7)) pass into the category of Fredholm’s equations of the first kind, which requires regularization or modification of the boundary conditions.

The last component of the stress state remains undefined at the boundary of the half-plane and is found from the ratio [2]

$$
(\kappa + 1) \varphi(x) = f(x) + 2 \mu g(x),
$$

which we rewrite as

$$
\sigma_{xx} + \sigma_{yy} = 4 \text{Re}(\varphi'(x)) = \frac{4}{\kappa + 1} (\sigma_{yy} + 2 \mu u').
$$

Substituting the value $u'$ from (8) in the last equality, we finally obtain

$$
\sigma_{xx} = \sigma_{yy} + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\tau_{xy}}{t-x} dt. \quad (9)
$$

Thus, all known solutions for the half-plane boundary are easily obtained from (7), (8), and (9). For example, if $\tau_{xy} = 0$ everywhere on the half-plane boundary, then, as follows from (8), (9)

$$
u' = \frac{\kappa - 1}{4 \mu} \sigma_{yy}, \quad \nu' = \frac{\kappa + 1}{4 \mu} \int_{-\infty}^{\infty} \frac{\sigma_{yy}}{t-x} dt, \quad \sigma_{yy}(x) = \sigma_{yy}(x),$$

which coincides with the Westergaard solution [2]. For passing into the region, the boundary values of the Kolosov—Muskhelishvili potentials have the form

$$
(\kappa + 1) \varphi(t) = f(t) + 2 \mu g(t),
$$

$$
(\kappa + 1) \varphi(t) = \kappa f(t) - 2 \mu g(t) - t \varphi'(t),
$$

then
4. Solution for the crack

Let’s stop at obtaining the solution from (7), (8) and (9) for a plane with a section of length $2a$ which is wedged by constant forces $\sigma_y = -\sigma_0$, $\tau = 0$. Due to the symmetry, we consider a mixed problem for a half-plane, where we formulate the boundary conditions in the following form (Figure 2)

$$\sigma_y = -\sigma_0, \quad |x| \leq a; \quad \tau_{xy} = 0, \quad |x| < \infty, \quad \nu(x) = 0, \quad |x| \geq a.$$ 

![Figure 2](image)

Figure 2. Formulation of boundary conditions in the section problem.

As it can be seen, from (8), (7)

$$u'(x) = \frac{\kappa - 1}{4\mu} \begin{cases} -\sigma_0, & |x| \leq a, \\ \sigma_y, & |x| > a, \end{cases}$$

$$\sigma_y = -\frac{\kappa - 1}{\kappa} \mu u' + \frac{\kappa + 1}{\kappa} \frac{\mu}{\pi} \int_{-a}^{a} \frac{v'}{t-x} dt$$

or for $|x| \leq a$

$$\frac{\mu}{\pi} \int_{-a}^{a} \frac{v'}{t-x} dt = -\frac{\kappa + 1}{4} \sigma_0.$$ 

Following [4], the last equation is reduced to

$$\mu v' = \frac{\kappa + 1}{\kappa} \sigma_0 \frac{x}{\sqrt{a^2 - x^2}}, \quad |x| \leq a,$$

whence for $|x| > a$

$$\sigma_y = \frac{\mu}{\pi} \int_{-a}^{a} \frac{v'}{t-x} dt = \frac{\kappa + 1}{4} \sigma_0 \left[ \frac{x}{\sqrt{a^2 - x^2}} - 1 \right].$$

(11)

The latter solution (11) is in agreement with the one obtained in [2] using the conformal mapping of an ellipse onto a mathematics section, and is incorrect due to the violation of conformality at two points. The cause of the incorrect solution (11) remains to be found. The former equation (10)
eliminates the question, since it assumes continuity of the stresses $\sigma_y$, which is essential in the elastic problem.

5. Conclusions
The solution to the three main problems in the elasticity theory for the half-plane boundary found in quadratures has been obtained for the first time. Arbitrarily formulated boundary conditions were shown to be capable to reduce the problem to an ill-conditioned one. Therefore, we shouldn’t overlook the thin line whose violation turns the correct problem into an ill-defined, which is critical when the solution utilizes numerical methods that are too rough for a thin dividing line.

References
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