ASYMPTOTICS OF STOCHASTIC BURGERS EQUATION WITH JUMPS

SHULAN HU¹, RAN WANG²∗

Abstract. For one-dimensional stochastic Burgers equation driven by Brownian motion and Poisson process, we study the \( \psi \)-uniformly exponential ergodicity with \( \psi(x) = 1 + \|x\| \), the moderate deviation principle and the large deviation principle for the occupation measures.

1. Introduction

As is well-known, Burgers equation was first studied to understand the turbulent fluid flow, see Burgers (1974). Since then the Burgers equation perturbed by different random noises have been considered, see monographs Da Prato and Zabczyk (2014), Peszat and Zabczyk (2007) and recent articles Dong and Xu (2007), Wu and Xie (2012), Dong et al. (2014) and references therein.

The ergodicity of the stochastic Burgers equation driven by Brownian motion and Poisson process was proved in Dong (2008) in the sense that the system converges to a unique invariant measure under the weak topology, but the convergence speed is not addressed. In this paper, we prove that the system converges to the invariant measure exponentially faster under a topology stronger than total variation by constructing a Lyapunov function in the same way as in Dong et al. (2019). The moderate deviation principle (MDP) for the occupation measure is also obtained.

The large deviation principle (LDP) for the occupation measure is one of the strongest ergodicity results for the long time behavior of Markov processes. It has been one of the classical research topics in probability since the pioneering work of Donsker and Varadhan (1975-1984). It gives an estimate on the probability that the occupation measures are deviated from the invariant measure, refer to Deuschel and Stroock (1989) for an introduction to large deviation theory of Markov processes. Wu (2001) gave the hyper-exponential recurrence criterion of the LDP of occupation measures for strong Feller and irreducible Markov processes. Based on this criterion, the large deviations of the occupation measures for the stochastic Burgers equation and stochastic Navier-Stokes equation driven by Brownian motion are proved in Gourcy (2007a) and Gourcy (2007b). There are some other papers about the applications of Wu’s criterion, see Jakšić et al. (2015), Nersesyan (2018) for some dissipative SPDEs. Wang et al. (2019) proposed a framework for verifying the hyper-exponential recurrence condition, which contains a family of strong dissipative SPDEs. In that framework, the strong dissipation produces to a stronger-norm moment estimate for the system after a fixed time uniformly over the initial values, which implies the hyper-exponential

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recurrence condition. See Wang and Xu (2018) for an application to stochastic reaction-diffusion equation driven by the subordinate Brownian motion.

However, the framework in Wang et al. (2019) is no longer available for the stochastic Burgers equation, which does not have the strong dissipation. In this paper, we check the hyper-exponential recurrence condition by using an exponential martingale argument. Due to the present of the jumps, the proof here is more complicated than that for the Brownian motion case in Gourcy (2007a).

The paper is organized as follows. The framework is given in Section 2. Section 3 is devoted to proving the $\psi$-uniformly exponential ergodicity and the moderate deviation principle. In Section 4, we prove the large deviation principle.

2. THE FRAMEWORK

Let $\mathbb{H} := L^2(0, 1)$ with the Dirichlet boundary condition and with vanishing mean values. Then $\mathbb{H}$ is a real separable Hilbert space with inner product

$$\langle x, y \rangle := \int_0^1 x(\xi)y(\xi)d\xi, \quad \forall x, y \in \mathbb{H}.$$  

Denote $\|x\|_\mathbb{H} := (\langle x, x \rangle_\mathbb{H})^{\frac{1}{2}}$. Let $\Delta x = x''$ be the second order differential operator on $\mathbb{H}$. Then $-\Delta$ is a positive self-adjoint operator on $\mathbb{H}$. Let $\alpha_k = \pi^2k^2$ and $\varepsilon_k(\xi) := \sqrt{2}\sin(k\pi\xi)$, for any $k \in \mathbb{N}^* = \{1, 2, \cdots \}$. Then $\{e_k\}_{k \in \mathbb{N}^*}$ forms an orthogonal basis of $\mathbb{H}$ and $-\Delta e_k = \alpha_k e_k$ for any $k \in \mathbb{N}^*$.

Let $\mathbb{V}$ be the domain of the fractional operator $(-\Delta)^\frac{1}{2}$, i.e.,

$$\mathbb{V} := \left\{ \sum_{k \in \mathbb{N}^*} \alpha_k^\frac{1}{2} a_k e_k; (a_k)_{k \in \mathbb{N}} \subset \mathbb{R} \text{ with } \sum_{k \in \mathbb{N}^*} a_k^2 < +\infty \right\},$$

with the inner product

$$\langle x, y \rangle_\mathbb{V} := \sum_{k \in \mathbb{N}^*} \alpha_k \langle x, e_k \rangle_{\mathbb{H}} \cdot \langle y, e_k \rangle_{\mathbb{H}},$$

and with the norm $\|x\|_\mathbb{V} := \langle x, x \rangle_\mathbb{V}^{\frac{1}{2}}$. Clearly, $\mathbb{V}$ is densely and compactly embedded in $\mathbb{H}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a completed filtered probability space, and $N(dt, du)$ the Poisson measure with finite intensity measure $n(du)$ on a given measurable space $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$. Then

$$\widetilde{N}(dt, du) := N(dt, du) - n(du)dt$$

is the compensated martingale measure. Let $W$ be the cylindrical Wiener process, which is independent with $N(dt, du)$, e.g., $W := \sum_{k \in \mathbb{N}^*} W^k e_k$, where $\{W^k\}_{k \in \mathbb{N}^*}$ are a sequence of independent standard one-dimensional Brownian motions independent with $N(dt, du)$.

Consider the following stochastic Burgers equation in the Hilbert space $\mathbb{H}$:

$$\begin{align*}
\{ &dX_t = \Delta X_t dt + B(X_t)dt + QdW_t + \int_{\mathbb{U}} f(X_{t-}, u)\widetilde{N}(dt, du), \\
&X(0) = x \in \mathbb{H}\}
\end{align*}$$

(2.1)

Here $B(x) := B(x, x)$ is a bilinear operator, which is defined by $B(x, y) := xy'$ for $x \in \mathbb{H}, y \in \mathbb{V}$, and $Q \in \mathcal{L}(\mathbb{H})$ (the space of all Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$) is given by

$$Qx = \sum_{k \in \mathbb{N}^*} \beta_k \langle x, e_k \rangle e_k, \quad x \in \mathbb{H},$$
with $\|Q\|_{HS} := \sqrt{\text{tr}(Q^*Q)} = \sqrt{\sum_{k \in \mathbb{N}^*} |\beta_k|^2} < \infty$.

Assume that the coefficient $f$ satisfies the following conditions:

(H.1) $f(\cdot, \cdot) : \mathbb{H} \times \mathbb{U} \to \mathbb{H}$ is measurable;
(H.2) $\int_\mathbb{U} \|f(0, u)\|_H^2 n(du) < \infty$;
(H.3) $\int_\mathbb{U} \|f(x, u) - f(y, u)\|_H^2 n(du) \leq K\|x - y\|_H^2, \forall x, y \in \mathbb{H}$;
(H.4) $f(\cdot, u) \in C_b^1((0, T); \mathbb{H}), \forall u \in \mathbb{U}$.

Let $\mathbb{D}([0, +\infty); \mathbb{H})$ be the space of all càdlàg functions from $[0, +\infty)$ to $\mathbb{H}$ equipped with the Skorokhod topology. Denote by $S(t) = e^{\Delta t}$.

**Definition 2.1.** The process $X = \{X_t\}_{t \geq 0}$ is called a mild solution of (2.1), if for any $x \in \mathbb{H}$, $X \in \mathbb{D}([0, +\infty); \mathbb{H})$ satisfying that for any $t > 0$,

$$\int_0^t \left[\|X(s)\|_H^2 + \|B(X(s))\|_H^2\right] ds < \infty,$$

and

$$X_t = S(t)x + \int_0^t S(t-s)B(X_s)ds + \int_0^t S(t-s)QdW_s + \int_0^t \int_\mathbb{U} S(t-s)f(X_{s^-}, u)\tilde{N}(ds, du), \quad \mathbb{P} - \text{a.s.}$$

For all $\varphi \in \mathcal{B}_b(\mathbb{H})$ (the space of all bounded measurable functions on $\mathbb{H}$), define

$$P_t \varphi(x) := \mathbb{E}[\varphi(X_t)] \text{ for all } t \geq 0, x \in \mathbb{H},$$

where $\mathbb{E}_x$ denotes the expectation with respect to (w.r.t. for short) the law of stochastic process $X$ with initial value $X_0 = x$. For any $t > 0$, $P_t$ is said to be strong Feller if $P_t \varphi \in C_b(\mathbb{H})$ for any $\varphi \in \mathcal{B}_b(\mathbb{H})$, where $C_b(\mathbb{H})$ is the space of all bounded continuous functions on $\mathbb{H}$. $P_t$ is irreducible in $\mathbb{H}$ if $P_t 1_O(x) > 0$ for any $x \in \mathbb{H}$ and any non-empty open subset $O$ of $\mathbb{H}$.

Recall the following properties about the solution to Eq.(2.1).

**Theorem 2.2 (Dong and Xu (2007), Dong (2008)).** Under (H.1)-(H.4), the following statements hold:

(i) For every $x \in \mathbb{H}$ and $\omega \in \Omega$ a.s., Eq.(2.1) admits a unique mild solution $X = \{X_t\}_{t \geq 0} \in \mathbb{D}([0, +\infty); \mathbb{H}) \cap L^2((0, +\infty); \mathcal{V})$, which is a Markov process.

(ii) $X = \{X_t\}_{t \geq 0}$ is strong Feller and irreducible in $\mathbb{H}$, and it admits a unique invariant probability measure $\mu$.

Define the occupation measure $\mathcal{L}_t$ by

$$\mathcal{L}_t(\Gamma) := \frac{1}{t} \int_0^t \delta_{X_s}(\Gamma) ds,$$

where $\Gamma$ is a Borel measurable set in $\mathbb{H}$, $\delta$ is the Dirac measure. Then $\mathcal{L}_t$ is in $\mathcal{M}_1(\mathbb{H})$, the space of probability measures on $\mathbb{H}$. On $\mathcal{M}_1(\mathbb{H})$, let $\sigma(\mathcal{M}_1(\mathbb{H}), \mathcal{B}_b(\mathbb{H}))$ be the $\tau$-topology of convergence against measurable and bounded functions, which is much stronger than the usual weak convergence topology $\sigma(\mathcal{M}_1(\mathbb{H}), C_b(\mathbb{H}))$.
3. $\psi$-UNIFORM EXPONENTIALLY ERGODICITY AND MODERATE DEVIATION PRINCIPLE

Let $\mathcal{M}_b(\mathbb{H})$ be the space of signed $\sigma$-additive measures of bounded variation on $\mathbb{H}$ equipped with the Borel $\sigma$-field $\mathcal{B}(\mathbb{H})$. On $\mathcal{M}_b(\mathbb{H})$, we consider the $\tau$-topology $\sigma(\mathcal{M}_b(\mathbb{H}), \mathcal{B}_b(\mathbb{H}))$.

Given a measurable function $\psi : \mathbb{H} \to \mathbb{R}^+$, define

$$B_\psi := \{g : \mathbb{H} \to \mathbb{R}; |g(x)| \leq \psi(x)\}.$$  

For a function $b(t) : \mathbb{R}_+ \to (0, +\infty)$, define

$$M_t := \frac{1}{b(t)\sqrt{t}} \int_0^t (\delta_{X_s} - \mu) ds.$$  

where $b(t)$ satisfies

$$\lim_{t \to \infty} b(t) = +\infty, \quad \lim_{t \to \infty} \frac{b(t)}{\sqrt{t}} = 0.$$  

Let $P_\nu$ be the probability measure of the system $X$ with initial measure $\nu$.

(H.5) Assume that there exists a constant $M > 0$ satisfying that

$$\sup_{x \in \mathbb{H}} \int_U \|f(x,u)\|^2_n du < +\infty.$$  

Theorem 3.1. Assume (H.1)-(H.5) hold. Then the following statements hold for $\psi(x) = 1 + \|x\|_H$.

1. The invariant measure $\mu$ satisfies that $\mu(\psi) < \infty$ and the Markov semigroup $\{P_t\}_{t \geq 0}$ is $\psi$-uniformly exponentially ergodic, i.e., there exist some constants $C, \gamma > 0$ satisfying that for

$$\sup_{g \in B_\psi} |P_t g(x) - \mu(g)| \leq Ce^{-\gamma t}\psi(x), \quad x \in \mathbb{H}, \ t \geq 0.$$  

2. For any initial measure $\nu$ verifying $\nu(\psi) < +\infty$, the measure $P_\nu(M_t \in \cdot)$ satisfies the large deviation principle w.r.t. the $\tau$-topology with speed $b^2(t)$ and the rate function

$$I(\nu) := \sup \left\{ \int \varphi d\nu - \frac{1}{2}\sigma^2(\varphi); \varphi \in B_\psi(\mathbb{H}) \right\}, \quad \forall \nu \in \mathcal{M}_b(\mathbb{H}),$$  

where

$$\sigma^2(\varphi) := \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_\mu \left( \int_0^t (\varphi(X_s) - \mu(\varphi)) ds \right)^2$$  

exists in $\mathbb{R}$ for every $\varphi \in B_\psi$. More precisely, the following three properties hold:

(a1) for any $r \geq 0$, $\{\beta \in \mathcal{M}_b(\mathbb{H}); I(\beta) \leq r\}$ is compact in $(\mathcal{M}_b(\mathbb{H}), \tau)$;

(a2) (the upper bound) for any closed set $\mathcal{E}$ in $(\mathcal{M}_b(\mathbb{H}), \tau)$,

$$\limsup_{t \to \infty} \frac{1}{b^2(t)} \log \mathbb{P}_\beta(M_t \in \mathcal{E}) \leq -\inf_{\beta \in \mathcal{E}} I(\beta);$$  

(a3) (the lower bound) for any open set $\mathcal{D}$ in $(\mathcal{M}_b(\mathbb{H}), \tau)$,

$$\liminf_{t \to \infty} \frac{1}{b^2(t)} \log \mathbb{P}_\beta(M_t \in \mathcal{D}) \geq -\inf_{\beta \in \mathcal{D}} I(\beta).$$  

Recall that a measurable function $h : \mathbb{H} \to \mathbb{R}$ belongs to the extended domain $D_e(\mathfrak{L})$ of the generator $\mathfrak{L}$ of $\{P_t\}_{t \geq 0}$, if there is a measurable function $g : \mathbb{H} \to \mathbb{R}$ satisfying that for all $t > 0$, $\int_0^t |g|(X_s) ds < +\infty$, $\mathbb{P}_x$-a.s., and

$$h(X_t) - h(X_0) - \int_0^t g(X_s) ds,$$

is a càdlàg $\mathbb{P}_x$-local martingale for all $x \in \mathbb{H}$. In that case, we write $g := \mathfrak{L}h$.

Proof of Theorem 3.1. According to (Down et al., 1995, Theorem 5.2c) and (Wu, 2001, Theorem 2.4), it is sufficient to prove that there exist some continuous function $1 \leq \psi \in D_e(\mathfrak{L})$, compact subset $K \subset \mathbb{H}$ and constants $\varepsilon, C > 0$ such that

$$\frac{\mathfrak{L}\psi}{\psi} \geq \varepsilon 1_{K^c} - C 1_K.$$

(3.7)

Here, we construct the Lyapunov function $\psi$ in the same way as in Dong et al. (2019). Since $1 + \|x\|_H$ is comparable with $(1 + \|x\|_H^2)^{\frac{1}{2}}$, we will take

$$\psi(x) = (1 + \|x\|_H^2)^{\frac{1}{2}}$$

instead of $1 + \|x\|_H$. First observe that

$$\nabla \psi(x) = \frac{x}{(1 + \|x\|_H^2)^{\frac{1}{2}}}$$

and

$$\text{Hess } \psi(x) = -\frac{x \times x}{(1 + \|x\|_H^2)^{\frac{3}{2}}} + \frac{I_H}{(1 + \|x\|_H^2)^{\frac{1}{2}}}.$$

(3.10)

here $I_H$ stands for the identity operator. Then, we have

$$\sup_{x \in \mathbb{H}} \|\text{Hess } \psi(x)\| \leq 1, \quad \sup_{x \in \mathbb{H}} \|\nabla \psi(x)\| \leq 1,$$

(3.11)

here $\|\text{Hess } \psi(x)\|$ and $\|\nabla \psi(x)\|$ denote their operator norms. Moreover, we have

$$\langle \Delta x, \nabla \psi(x) \rangle = \frac{\langle \Delta x, x \rangle}{(1 + \|x\|_H^2)^{\frac{1}{2}}} = -\frac{\|x\|^2_H}{(1 + \|x\|_H^2)^{\frac{1}{2}}}, \quad \forall x \in \mathbb{V},$$

(3.12)

and

$$\langle B(x), \nabla \psi(x) \rangle = \frac{\langle B(x), x \rangle}{(1 + \|x\|_H^2)^{\frac{1}{2}}} = 0, \quad \forall x \in \mathbb{V}.$$

(3.13)

By Taylor’s expansion, for any $x \in \mathbb{H}, u \in \mathbb{U}$, there exists constant $\theta \in (0, 1)$ satisfying that

$$\psi(x + f(x, u)) - \psi(x) - \langle \nabla \psi(x), f(x, u) \rangle$$

$$= \frac{1}{2} \langle \text{Hess } \psi(x + \theta f(x, u)), f(x, u), f(x, u) \rangle.$$

(3.14)

By Itô’s formula, we have

$$d\psi(X_t) = \langle \Delta X_t, \nabla \psi(X_t) \rangle dt + \langle B(X_t), \nabla \psi(X_t) \rangle dt$$

$$+ \langle \nabla \psi(X_t), QdW_t \rangle + \frac{1}{2} \text{tr}(Q^* \text{Hess } \psi(X_t) Q) dt.$$
\[
+ \int_U (\psi(X_{t-} + f(X_{t-}, u)) - \psi(X_{t-})) \tilde{N}(dt, du)
\]
(3.15) \[
+ \int_U (\psi(X_{t-} + f(X_{t-}, u)) - \psi(X_{t-}) - \langle \nabla \psi(X_{t-}), f(X_{t-}, u) \rangle) n(du) dt.
\]

Then, by (H.5), (3.9)-(3.14), we know that
\[
\mathcal{L}\psi(x) = \langle \Delta x, \nabla \psi(x) \rangle + \langle B(x), \nabla \psi(x) \rangle + \frac{1}{2} \text{tr}(Q^* \text{Hess} \psi(x) Q)
\]
\[
+ \int_U (\psi(x + f(x, u)) - \psi(x) - \langle \nabla \psi(x), f(x, u) \rangle) n(du)
\]
\[
\leq - \frac{\|x\|^2_V}{(1 + \|x\|^2_H)^{2/3}} + \frac{1}{2} \|Q\|^2_{\text{HS}} + \frac{1}{2} \int_U \|f(x, u)\|^2_{\|x\|_H} n(du)
\]
\[
\leq - \frac{1 + \|x\|^2_V}{(1 + \|x\|^2_H)^{2/3}} + \frac{1}{2} \|Q\|^2_{\text{HS}} + \frac{M}{2}
\]
(3.16)
\[
\leq - (1 + \|x\|^2_{\|x\|_V}^{2/3} + c_1,
\]
where in the last inequality the Poincaré inequality \(\|x\|_V \geq \pi \|x\|_H\) is used, \(c_1 := 1 + \frac{1}{2} (\|Q\|^2_{\text{HS}} + M).

Let \(\mathcal{K} := \{x \in \mathbb{H}; \|x\|_V \leq 2c_1\}\). Then \(\mathcal{K}\) is a compact set in \(\mathbb{H}\). For any \(x \in \mathcal{K}\), we have
\[
(1 + \|x\|^2_{\|x\|_V}^{2/3} - c_1)
\]
(3.17)
\[
(1 + \|x\|^2_H)^{2/3}
\]
for any \(x \notin \mathcal{K}\), we have
\[
(1 + \|x\|^2_{\|x\|_V}^{2/3} - c_1) \geq (1 + \|x\|^2_{\|x\|_V}^{2/3} - \frac{\|x\|_V}{2}) \geq \frac{1}{2}.
\]
(3.18)
Putting (3.16)-(3.18) together, we obtain that
\[
\frac{\mathcal{L}\psi(x)}{\psi(x)} \geq (1 + \|x\|^2_{\|x\|_V}^{2/3} - c_1) \geq \frac{1}{2} \|x\|_V \geq c_1 1_{\mathcal{K}},
\]
which implies (3.7). The proof is complete. \(\square\)

4. LARGE DEVIATION PRINCIPLE

(H.6) Assume that there exists a constant \(a_0 > 0\) satisfying that
\[
\sup_{x \in \mathbb{H}} \int_U \|f(x, u)\|^2_{\|x\|_H} \exp \left( a_0 \|f(x, u)\|_{\|x\|_H} \right) n(du) < +\infty.
\]
(4.1)
For any \(\lambda_0 > 0, L > 0\), let
\[
\mathcal{M}_{\lambda_0, L} := \left\{ \nu \in \mathcal{M}_1(\mathbb{H}); \int e^{\lambda_0 \|x\|_H^2} \nu(dx) \leq L \right\}.
\]
(4.2)

Theorem 4.1. Assume (H.1)-(H.4) and (H.6) hold. Then the family \(\mathbb{P}_\nu(\mathcal{L}_t \in \cdot)\) as \(t \to +\infty\) satisfies the LDP with respect to the \(\tau\)-topology, with the speed \(t\) and the rate function \(J\), uniformly for any initial measure \(\nu\) in \(\mathcal{M}_{\lambda_0, L}\). More precisely, the following three properties hold:
(a1) for any $a \geq 0$, \(\{\beta \in \mathcal{M}_1(\mathbb{H}) ; J(\beta) \leq a\} \) is compact in \((\mathcal{M}_1(\mathbb{H}), \tau)\); 
(a2) (the upper bound) for any closed set \(\mathcal{E}\) in \((\mathcal{M}_1(\mathbb{E}), \tau)\),
\[
\limsup_{t \to \infty} \frac{1}{t} \log \sup_{\nu \in \mathcal{M}_{A_0, t}} \mathbb{P}_\nu(\mathcal{L}_t \in \mathcal{E}) \leq -\inf_{\beta \in \mathcal{E}} J(\beta);
\]
(a3) (the lower bound) for any open set \(\mathcal{D}\) in \((\mathcal{M}_1(\mathbb{E}), \tau)\),
\[
\liminf_{t \to \infty} \frac{1}{t} \log \inf_{\nu \in \mathcal{M}_{A_0, t}} \mathbb{P}_\nu(\mathcal{L}_t \in \mathcal{D}) \geq -\inf_{\beta \in \mathcal{D}} J(\beta).
\]

Remark 4.2. Assumptions (H.1)-(H.4) are standard conditions for the existence and uniqueness of the solution for Eq. (2.1), see Dong and Xu (2007). Condition (H.6) ((H.5) resp.) guarantees for the exponential (square resp.) integrability of the solution. The similar conditions are often used in the study of the large deviation theory for small Poisson noise perturbations of SPDEs, e.g., see (Röckner and Zhang, 2007, Section 4), (Budhiraja et al., 2013, Condition 3.1) and (Yang, Zhai and Zhang, 2015, Condition 3.1). Inspiring by (Budhiraja et al., 2013, Section 4.1), we give the following example of the Poisson random measure \(N\) and \(f\) satisfying (H.1)-(H.4) and (H.6):

Let \(\{N(t)\}_{t \geq 0}\) be a Poisson process with the rate 1, \(\{A^j\}_{j \in \mathbb{N}}\) independent and identically distributed random variables, with a common distribution function \(F\), which are also independent of \(\{N(t)\}_{t \geq 0}\). Then
\[
N([0, t] \otimes B) = \sum_{j=1}^{N(t)} 1_B(A^j), \quad t \geq 0, B \in \mathcal{B}(\mathbb{R}_+),
\]
is a Poisson random measure on the space \(\mathbb{R}_+ \times \mathbb{R}_+\). The intensity measure of \(\{N(t)\}_{t \geq 0}\) is given by
\[
\nu(A \otimes B) = \rho(A) \cdot F(B), \quad A, B \in \mathcal{B}(\mathbb{R}_+).
\]
Here \(\rho(\cdot)\) denotes the Lebesgue measure.

Assume that there exists \(a_0 > 0\) such that
\[
\int_0^\infty u^2 e^{a_0 u} F(du) < \infty.
\]
For any function \(G \in C^1_b(\mathbb{H})\), the function
\[
f(x, u) := G(x)u, \quad x \in \mathbb{H}, u \in \mathbb{R}_+
\]
satisfies all the conditions required in Theorem 4.1.

Remark 4.3. The rate function \(J\) can be expressed by the entropy of Donsker-Varadhan, see Donsker and Varadhan (1975-1984), (Deuschel and Stroock, 1989, Chapter V) or (Wu, 2001, Section 2.2). Under the Feller assumption:
\[
P_t(C_b(\mathbb{H})) \subset C_b(\mathbb{H}), \quad \forall t \geq 0,
\]
we know that (for instance see Lemma B.7 in Wu (2000))
\[
J(\nu) = \sup \left\{ -\int \frac{\xi \varphi}{\varphi} d\nu ; 1 \leq \varphi \in D_\mathcal{L}(\mathcal{L}) \right\}, \quad \nu \in \mathcal{M}_1(\mathbb{H}).
\]

(4.3)
Remark 4.4. For every $\varphi : \mathbb{H} \to \mathbb{R}$ measurable and bounded, as $\nu \mapsto \int_{\mathbb{H}} \varphi d\nu$ is continuous w.r.t. the $\tau$-topology, then by the contraction principle (Deuschel and Stroock (1989)),

$$\mathbb{P}_\nu \left( \frac{1}{t} \int_0^t \varphi(X_s) ds \in \cdot \right)$$

satisfies the LDP on $\mathbb{R}$ uniformly over $\nu$ in $\mathcal{M}_{\lambda_0, L}$, with the rate function given by

$$J^\varphi(r) := \inf \left\{ J(\beta) < +\infty ; \beta \in \mathcal{M}_1(\mathbb{H}) \text{ and } \int \varphi d\beta = r \right\}, \forall r \in \mathbb{R}.$$ 

The proof of Theorem 4.1. By Theorem 2.2, we know that $P_t$ is strong Feller and irreducible in $\mathbb{H}$ for any $t > 0$. According to (Wu, 2001, Theorem 2.1), to prove Theorem 4.1, it is sufficient to prove that for any $\lambda > 0$, there exists a compact set $\mathcal{K}$ in $\mathbb{H}$

$$\sup_{\nu \in \mathcal{M}_{\lambda_0, L}} \mathbb{E}_\nu \left[ e^{\lambda \tau_{\mathcal{K}}} \right] < \infty, \quad \text{and} \quad \sup_{x \in \mathcal{K}} \mathbb{E}_x \left[ e^{\lambda \tau_{\mathcal{K}}^{(1)}} \right] < \infty,$$

where

$$\tau_{\mathcal{K}} := \inf \{ t \geq 0 ; X_t \in \mathcal{K} \}, \quad \tau_{\mathcal{K}}^{(1)} := \inf \{ t \geq 1 ; X_t \in \mathcal{K} \}.$$ 

The basic ingredient for the proof of (4.4) is to show the exponential decay of the tails of the stopping times $\tau_{\mathcal{K}}$ and $\tau_{\mathcal{K}}^{(1)}$ for a suitable choice of compact set $\mathcal{K} \subset \mathbb{H}$. It can be proved by using arguments in (Gourcy, 2007a, Lemma 6.1) or (Wang et al., 2019, Lemma 3.8), combining with the critical exponential estimate in Proposition 4.6 below.

The proof is complete. \hfill \Box

The following result is similar to Lemma 4.1 in Röckner and Zhang (2007).

Lemma 4.5. For any $g \in C^2_b(\mathbb{H})$,

$$M^g_t := \exp \left( g(X_t) - g(X_0) - \int_0^t h(X_s) ds \right)$$

is an $\mathcal{F}_t$-local martingale, where

$$h(x) = \langle \Delta x, \nabla g(x) \rangle + \langle B(x), \nabla g(x) \rangle + \frac{1}{2} \| Q^* \nabla g(x) \|^2_H + \frac{1}{2} \text{tr}(Q^* \text{Hess } g(x) Q)$$

$$+ \int_{\mathcal{U}} \left( \exp \left[ g(x + f(x, u)) - g(x) \right] - 1 - \langle \nabla g(x), f(x, u) \rangle \right) n(du).$$

Proof. We follow the argument in (Röckner and Zhang, 2007, Lemma 4.1). Applying Itô’s formula first to $\exp(g(X_t))$ and then to $\exp \left( g(X_t) - g(X_0) - \int_0^t h(X_s) ds \right)$ proves the lemma. \hfill \Box

Proposition 4.6. Assume that (H.1)-(H.4) and (H.6) hold. For any $\lambda \in (0, a_0]$, $\theta \in (0, 1)$, there exist constants $c_1(\theta), c_2(\theta), c_3(\lambda)$ such that for any $T > 0$,

$$\mathbb{E}_x \left[ \exp \left( \theta \lambda \int_0^T \| X_t \|_{\mathcal{V}} dt \right) \right] \leq c_1(\theta) + c_2(\theta) e^{c_3(\lambda) T} e^{\lambda \| x \|_H}.$$ 

Remark 4.7. For the stochastic Burgers equation driven by Brownian motion (i.e., $f \equiv 0$ in (2.1)), the Lemma 5.2 of Gourcy (2007a) tells us that for any $\lambda \in (0, \pi^2\|Q\|/2]$, where $\|Q\|$ is the norm of $Q$ as an operator in $\mathbb{H}$,

\begin{equation}
\mathbb{E}_x \left[ \exp \left( \lambda \int_0^T \|X_t\|_V^2 dt \right) \right] \leq e^{\lambda t_0} e^{\lambda \|x\|_{\mathbb{H}}^2}.
\end{equation}

This is difficult to prove in the jump case. Here, we replace $\int_0^T \|X_t\|_V^2 dt$ by $\int_0^T \|X_t\|_V dt$ and add an extra parameter $\theta \in (0, 1)$ in (4.7), which is also enough to show the exponential decay of the tails of the stopping times $\tau_{\mathcal{K}}$ and $\tau_{\mathcal{K}}^{(1)}$ for a suitable choice of $\mathcal{K}$.

The proof of Proposition 4.6. For any $\lambda \in (0, a_0]$, let

\[ Z_\lambda := \int_0^T \frac{\lambda^2 \|X_t\|_V^2}{(1 + \lambda^2 \|X_t\|_{\mathbb{H}}^2)^{\frac{3}{2}}} dt. \]

Since $\|x\|_V \geq \pi \|x\|_{\mathbb{H}}$, we have

\begin{equation}
\frac{\lambda^2 \|X_t\|_V^2}{(1 + \lambda^2 \|X_t\|_{\mathbb{H}}^2)^{\frac{3}{2}}} \geq \left( 1 + \lambda^2 \|X_t\|_V^2 \right)^{\frac{1}{2}} - 1.
\end{equation}

Thus, to prove (4.7), it is enough to prove that for any $\lambda \in (0, a_0]$, there exist constants $c_1(\theta), c_2(\theta), c_3(\lambda)$ such that

\begin{equation}
\mathbb{E}_x \left[ \exp \left( \theta Z_\lambda \right) \right] \leq c_1(\theta) + c_2(\theta) e^{c_3(\lambda) T} e^{\lambda \|x\|_{\mathbb{H}}}. \tag{4.10}
\end{equation}

Let

\[ \psi_\lambda(x) := (1 + \lambda^2 \|x\|_{\mathbb{H}}^2)^{\frac{1}{2}}, \]

a generalization of $\psi$ given in (3.8). Then $\psi_\lambda$ has the similar estimates (3.9)-(3.13) with $\psi$ up to some constants.

Let $G(x) := e^{\psi_\lambda(x)}$. Note that

\[ \text{Hess } G(x) = G(x) \text{Hess } \psi_\lambda(x) + G(x) \nabla \psi_\lambda(x) \times \psi_\lambda(x). \]

It implies that

\begin{equation}
\|\text{Hess } G(x)\|_{\mathbb{H}} \leq \lambda^2 G(x), \ \forall x \in \mathbb{V}. \tag{4.11}
\end{equation}

By Taylor’s expansion, there exist constants $\theta_1 \in (0, 1), \theta_2 \in (0, \theta_1)$ satisfying that

\[
\begin{align*}
&\left| \exp \left[ \psi_\lambda(x + f(x, u)) - \psi_\lambda(x) \right] - 1 - \langle \nabla \psi_\lambda(x), f(x, u) \rangle \right| \\
&= \left| e^{-\psi_\lambda(x)} \left[ G(x + f(x, u)) - G(x) - \langle \nabla G(x), f(x, u) \rangle \right] \right| \\
&= \left| \frac{1}{2} e^{-\psi_\lambda(x)} \left\langle \text{Hess } G(x + \theta_1 f(x, u)) f(x, u), f(x, u) \right\rangle \right| \\
&\leq \frac{\lambda^2}{2} \exp \left( \psi_\lambda(x + \theta_1 f(x, u)) - \psi_\lambda(x) \right) \|f(x, u)\|_{\mathbb{H}}^2 \\
&\leq \frac{\lambda^2}{2} \exp \left( \|\nabla \psi_\lambda(x + \theta_2 f(x, u)) - \psi_\lambda(x)\|_{\mathbb{H}} \right) \|f(x, u)\|_{\mathbb{H}}^2 \\
&\leq \frac{\lambda^2}{2} \exp \left( \lambda \|f(x, u)\|_{\mathbb{H}} \right) \|f(x, u)\|_{\mathbb{H}}^2. \tag{4.12}
\end{align*}
\]
Applying Lemma 4.5 with the above choice of $\psi_\lambda$, we know that

$$M^{\psi_\lambda}(t) := \exp \left( \psi_\lambda(X_t) - \psi_\lambda(x) - \int_0^t h(X_s)ds \right)$$

is an $\mathcal{F}_t$-local martingale, where

$$h(x) := \langle \Delta x, \nabla \psi_\lambda(x) \rangle + \langle B(x), \nabla \psi_\lambda(x) \rangle + \frac{1}{2} \|Q^* \nabla \psi_\lambda(x)\|_{\mathcal{H}}^2 + \frac{1}{2} \text{tr}(Q^* \text{Hess} \psi_\lambda(x)Q)$$

$$+ \int_U \left( \exp \left[ \psi_\lambda(x + f(x, u)) - \psi_\lambda(x) \right] - 1 - \langle \nabla \psi_\lambda(x), f(x, u) \rangle \right) n(du)$$

(4.13) \[ \leq - \frac{\lambda^2 \|x\|_V^2}{(1 + \lambda^2 \|x\|_V^2)^\frac{3}{2}} + \lambda^2 \|Q\|_{\mathcal{H}}^2 + \frac{\lambda^2}{2} \int_U \exp(\lambda \|f(x, u)\|_{\mathcal{H}}) \|f(x, u)\|_{\mathcal{H}}^2 n(du). \]

By (H.6), we know that for any fixed $\lambda \in (0, a_0]$,

(4.14) \[ M_\lambda := \sup_{x \in \mathbb{R}} \int_U \|f(x, u)\|_{\mathcal{H}}^2 \exp(\lambda \|f(x, u)\|_{\mathcal{H}}) n(du) < \infty. \]

Hence, by (4.13) and (4.14), we have that for any $r \geq 0$,

$$\mathbb{P}(Z_\lambda > r) \leq \mathbb{P} \left( \psi_\lambda(X_T) + \int_0^T \frac{\lambda^2 \|X_s\|_V^2}{(1 + \lambda^2 \|X_s\|_V^2)^\frac{3}{2}} ds > r \right)$$

$$= \mathbb{P} \left( \psi_\lambda(X_T) - \psi_\lambda(x) - \int_0^T h(X_s)ds + \psi_\lambda(x) + \int_0^T h(X_s)ds \right.$$

$$+ \int_0^T \frac{\lambda^2 \|X_s\|_V^2}{(1 + \lambda^2 \|X_s\|_V^2)^\frac{3}{2}} ds > r \bigg)$$

$$\leq \mathbb{P} \left( \psi_\lambda(X_T) - \psi_\lambda(x) - \int_0^T h(X_s)ds > r - \psi_\lambda(x) - T\lambda^2 \left( \frac{M_\lambda}{2} + \|Q\|_{\mathcal{H}}^2 \right) \right)$$

$$\leq \mathbb{E} \left[ M_T^{\psi_\lambda} \right] \exp \left( -r + \psi_\lambda(x) + T\lambda^2 \left( \frac{M_\lambda}{2} + \|Q\|_{\mathcal{H}}^2 \right) \right)$$

(4.15) \[ \leq \exp \left( -r + \psi_\lambda(x) + T\lambda^2 \left( \frac{M_\lambda}{2} + \|Q\|_{\mathcal{H}}^2 \right) \right), \]

where in the last inequality we have used the fact that $M_T^{\psi_\lambda}$ is a non-negative local martingale.

For any $\theta \in (0, 1)$ and $\lambda \leq a_0$, by (4.15), we have

$$\mathbb{E} \left[ \exp(\theta Z_\lambda) \right] \leq \theta + \theta \int_0^\infty e^{\theta r} \mathbb{P}(Z_\lambda > r) \, dr$$

$$\leq \theta + \theta \int_0^\infty e^{\theta r} \exp \left( -r + \psi_\lambda(x) + T\lambda^2 \left( \frac{M_\lambda}{2} + \|Q\|_{\mathcal{H}}^2 \right) \right) \, dr$$

(4.16) \[ = \theta + \exp \left( \psi_\lambda(x) + T\lambda^2 \left( \frac{M_\lambda}{2} + \|Q\|_{\mathcal{H}}^2 \right) \right) \frac{\theta}{1 - \theta}. \]

This implies (4.7). The proof is complete. \qed
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