A refined model for piezoelectric composite beams

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Abstract. This work presents and compares few simple one-dimensional models for the piezoelectric actuation and detection of beams. The 1D nature, which allows an easy embedding of the model in the classical Euler-Bernoulli beam equations, is obtained by adopting simplifying assumptions along directions of the cross-sectional plane. By changing such assumptions, different models can be built. Their validity is discussed and compared with results of FEM simulations for varying geometries. We show that commonly adopted models fail in a series of practical cases and propose a new model capable of accurately describing wide beams.

1. Introduction
Piezoelectricity is widely used in MEMS devices, such as resonators [1, 2], sensors [3] and energy harvesters [4], as well as in macroscopic harvesters [5]. In several applications, a bimorph beam with the structure shown in figure 1 is used, and for this reason compact descriptions of the electromechanical behavior of a piezoelectric beam have been developed in the past [6, 7]. Such models typically take the form of simple one-dimensional problems, taking advantage of the classic Euler-Bernoulli and Timoshenko beam equations. The reduction of the full 3D problem to a 1D model is obtained by adopting simplifying assumptions on the stress and strain distributions in the directions of the beam cross-sectional plane. These assumptions may be justified for specific geometries, or if one or more piezoelectric coefficients apart from $e_{31}$ are assumed to be negligible, which is rarely the case in actual piezoelectric materials. In actual materials, other coefficients may play a significant role and cannot be neglected.

In this work we review some of the assumptions proposed in literature and investigate their range of validity by comparing them with FEM results. We show that, in the case of wide beams, established assumptions are not justified and a different model, taking more reasonable boundary conditions into account, accurately describe this case.

2. Model
We present here a complete model for the electromechanical behavior of piezoelectrically actuated beams. We consider the cantilever geometry of figure 1, with a mechanical actuation in the form of a vertical force $T_0$ applied at the cantilever tip. Problems of this form are frequency encountered in piezoe actuators or energy harvesters.

The generic constitutive equations for the piezoelectric material (orange in figure 1) and beam structural material (grey in the same figure) can be written as:

$$T_f^B = C_f^B S_f$$ (1)
Figure 1. Schematic structure of the piezoelectric device. The area shaded in red is clamped.

\[ T_i^p = C_{ij}^p S_j - e_{ij} E_j \]  

(2)

where \( T_i \) and \( S_j \) are the stress and strain vectors in contracted (engineering) notation. \( C_{ij} \) and \( e_{ij} \) are the stiffness and piezoelectric stress matrices, respectively, and \( E_j \) is the electric field. Superscripts \( B \) and \( p \) stand for beam and piezo, respectively. The equation for the electric displacement \( D_i \) in the piezoelectric material is:

\[ D_i = \epsilon_{ij} E_j + e_{ij} S_j \]  

(3)

where \( \epsilon_{ij} \) is the electrical permittivity matrix at zero strain.

These general three-dimensional equations can be (and have been) simplified to obtain a one-dimensional model for the system of figure 1. We will show that the choice of the simplifying assumptions, however, is crucial for the accuracy of the model. Different hypotheses have been proposed in the literature. Garcia and coworkers [6] assume no displacement in the structure either along \( y \) or \( z \), i.e., \( S_{yy} = S_{zz} = 0 \). We call this approach Full Zero Displacement or FZD. On the other hand, Inman and coworkers [7] assume the dual hypothesis of Full Zero Force (FZF), i.e., the normal stresses along \( y \) and \( z \) are both zero (\( T_{yy} = T_{zz} = 0 \)). These two choices correspond to the two limiting cases of a continuum, and as such neither is always justified.

Since beams are typically slender (i.e., \( L \gg t_p + t_B \)), the assumption of zero force along \( z \) (\( T_{zz} = 0 \)) is reasonable. Along the \( y \) direction however, the situation is complicated by the fact that the two materials are clamped at their interface. This fact may play a significant role in blocking deformations along \( y \) if the in-plane dimensions \( L, w \) are comparable, as suggested by FEM simulations by the authors.

For these reasons, in this work we propose an alternative, yet still very simple, hybrid assumption, i.e. that the beam displacement is zero only along the \( y \) direction, and retain the assumption of zero vertical stress (\( S_{yy} = 0, T_{zz} = 0 \)). We call this model In-Plane Zero Displacement or IPZD. The observations above suggest that FZF should be valid at small \( w/L \), and IPZD at large \( w/L \).

Reduction of equations (1)-(3) to the one-dimensional case, given one of the three cases, is straightforward and was already carried in the aforementioned papers. Equations (1)-(3) can be solved for \( T_{xx} \) and \( D_z \), giving:
Table 1. Equivalent longitudinal stiffness and piezoelectric coupling coefficients for the three proposed models. $Y$ is the Young’s modulus, $\nu$ the Poisson’s ratio and $d$ the piezo strain coefficient.

| equivalent parameter | FZD         | FZF         | IPZD         |
|----------------------|-------------|-------------|--------------|
| $K_*$                | $C_{11}^*$  | $Y_x^*$     | $\frac{Y_x^*}{(1 - \nu_{xy}^*)^2}$ |
| $P_c$                | $e_{31}$    | $d_{31}Y_x = e_{31}(1 - \nu_{xy}) - e_{33}\nu_{xz}$ | $\frac{d_{31}Y_x}{1 - \nu_{xy}} = \left[ e_{31}(1 - \nu_{xy}) - e_{33}\nu_{xz} \right]$ |

where $K_B$ and $K_p$ are equivalent axial stiffness constants for the structural and piezoelectric layers, respectively, $P_c$ is a piezoelectric coupling coefficient, and $\varepsilon_p$ is an equivalent dielectric permittivity constant. These constants depend on the assumptions along $y$ and $z$ directions, with only $\varepsilon_p$ having a negligible variation which allows its substitution with $\varepsilon_{zz}$. The values of each equivalent constant for the three models are reported in Table 1. For the derivation of these constants, hexagonal anisotropy, which is typical of piezoelectric materials such as PZT and AlN, has been assumed for the piezoelectric material. It is worth mentioning that, if $e_{33}$ and $e_{32}$ are negligible, the three models give the same $P_c$.

These models can be used in the classical Euler-Bernoulli beam equation, which, along with the electrical equation of the piezoelectric capacitor, leads to the following ODE system in the frequency domain:

$$
\begin{cases}
\frac{d}{dx} \begin{bmatrix} U(x) \\ \Psi(x) \\ M(x) \\ T(x) \end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1/(KI_c) & 0 \\
-\omega^2(\rho A)c & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} U(x) \\ \Psi(x) \\ M(x) \\ T(x) \end{bmatrix} + \Gamma_p V \\
Q = C_0 V + \Gamma_p \left( \frac{d\omega(u)}{dx} - \frac{d\omega(v)}{dx} \right)
\end{cases}
$$

where $U(x)$, $\Psi(x)$, $M(x)$ and $T(x)$ are the displacement, rotation, bending moment and shear force along the beam, respectively. $V$, $Q$ and $C_0$ are the voltage across the piezo, the charge over the upper electrode and the static capacitance of the piezo itself. $(KI)_c$ and $(\rho A)_c$ are the flexural stiffness and the mass per unit length of the composite structure. $\omega$ is the angular frequency, while $\Gamma_p$ is the system piezoelectric coupling coefficient, with value:

$$
\Gamma_p = P_c\omega \left( \ell_B + \frac{\ell_p}{2} - z_N \right).
$$

In equation (8), $z_N$ is the position of the neutral axis. Once that the equivalent axial stiffnesses $K_B$ and $K_p$ are known, $(KI)_c$ and $z_N$ can be computed with composite beam theory [8].

3. Validation
To validate the model we consider both static and dynamic results and compare the outcomes of the three models with FEM simulations. Polysilicon is used as the structural material and PZT as the piezoelectric material.
Properties used in simulations and models are summarized in Table 2. The beam and piezo thicknesses are 10µm and 1µm, respectively. The length L is fixed at 500 µm, while the width is swept from 25µm to 5000µm. FEM simulations were performed with ANSYS, using element SOLID226 for the piezoelectric layer, and SOLID186 for the structural layer. Both a static analysis and a modal analysis were performed for each geometry.

### 3.1. Static behavior

We solved the ODE system (7) in static conditions ($\omega = 0$) with clamped boundary conditions at one end of the beam and a constant vertical force at the other end (figure 1). Depending on the electrical boundary condition, different expressions for the tip deflection are obtained. For the two limiting cases of short-circuit (sc) and open-circuit (oc) conditions at the piezoelectric actuator, we found:

$$U_{sc}(L) = \frac{L^3}{3(kI)_c} T_0$$

$$U_{oc}(L) = \frac{L^3}{3(kI)_c} \frac{f_p^2 L/4 + C_0(kI)_c}{f_p^2 L + C_0(kI)_c} T_0$$

(9)

The short circuit solution coincides with the Euler-Bernoulli theory for a cantilever with a force load at its tip, while the open circuit solution includes the stiffening effect introduced by piezoelectric coupling. Static results are reported in figure 2 as a function of the ratio $w/L$ for short circuit (dashed lines) and open circuit (solid lines) electrical boundary conditions. The simulated tip displacement decreases for increasing $w/L$, passing from the FZF approximation ($w/L \to 0$) to the IPZD approximation ($w/L \to \infty$). Limit values are very accurate in short circuit while presenting small errors in open circuit. Since $(kI)_c$ is the only parameter affecting the short circuit model (see equation (9)), we conclude that the error is due to inaccuracy on the $f_p$ model. As expected from the discussion presented in the previous section, the FZD model is inaccurate for both electrical boundary conditions.

![Figure 2](image.png)

**Figure 2.** Tip deflection as a function of $w/L$ ratio.
3.2. Dynamic behavior
We compute the resonance frequency by writing the eigenproblem relative to the system (7) and then numerically solving for the roots of the characteristic equation, whose only variable is the angular frequency $\omega$. A comparison between simulations and models is reported in figure 3. The behavior is very similar to the one observed in the static analysis, with the resonance frequency moving from the FZF limit to the IPZD limit as the $w/L$ ratio increases. Small errors at the limit values for the open circuit case are observed as well.

4. Conclusions
In this work we discussed three one-dimensional models describing the electromechanical behavior of piezoelectrically actuated beams. The models are based on different hypotheses on the stress/strain state along the two directions of the beam cross-sectional plane. The hypothesis of zero displacement, present in the literature, is disproved by FEM simulations. The assumption of zero stress fails in the case of wide structures. We show that a new hypothesis developed by the authors, that of in-plane zero displacement, correctly models this latter case.

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Figure 3. Resonance frequency as a function of $w/L$ ratio.
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