Generalized Estimating Equation for the Student-t Distributions

Atin Gayen
Discipline of Mathematics
Indian Institute of Technology Indore
Indore, Madhya Pradesh 453552, India
Email: atinfordst@gmail.com

M. Ashok Kumar
Discipline of Mathematics
Indian Institute of Technology Indore
Indore, Madhya Pradesh 453552, India
Email: ashokm@iiti.ac.in

Abstract—In [12], it was shown that a generalized maximum likelihood estimation problem on a (canonical) $\alpha$-power-law model ($M^{(\alpha)}$-family) can be solved by solving a system of linear equations. This was due to an orthogonality relationship between the $M^{(\alpha)}$-family and a linear family with respect to the relative $\alpha$-entropy (or the $I_\alpha$-divergence). Relative $\alpha$-entropy is a generalization of the usual relative entropy (or the Kullback-Leibler divergence), $M^{(\alpha)}$-family is a generalization of the usual exponential family. In this paper, we first generalize the $M^{(\alpha)}$-family including the multivariate, continuous case and show that the Student-t distributions fall in this family. We then extend the above stated result of [12] to the general $M^{(\alpha)}$-family. Finally we apply this result to the Student-t distribution and find generalized estimators for its parameters.

I. INTRODUCTION AND PRELIMINARIES

The exponential families of probability distributions are important in statistics as many important probability distributions like Binomial, Poisson, Gaussian and so on fall in this class. Let $X = (X_1, \ldots, X_d)$ be a $d$-dimensional random vector that follows a probability distribution $p_\theta, \theta \in \Theta$, where $\Theta$ is an open subset of $\mathbb{R}^k$. Suppose also that $X_1, \ldots, X_d$ are jointly continuous (or jointly discrete). The family of probability distributions $\mathcal{E} = \{p_\theta : \theta \in \Theta\}$ is said to belong to a $k$-parameter exponential family if it can be represented in the following form [3] Eq. (7.7.5).

$$p_\theta(x) = \begin{cases} \exp[q(x) + Z(\theta) + w(\theta)^T f(x)] & \text{if } x \in \mathbb{S} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\mathbb{S}$ denotes the support of $p_\theta$ (that is, $p_\theta(x) > 0$ for $x \in \mathbb{S}$). Here $w := (w_1, \ldots, w_s)^T$ and $f := (f_1, \ldots, f_s)^T$ such that for all $i = 1, \ldots, s$, $w_i : \Theta \rightarrow \mathbb{R}$ and $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ are some functions, and $q : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative function. Further, $Z : \Theta \rightarrow \mathbb{R}$ is a function which makes $p_\theta$ a probability distribution and all $w_i(\theta)$ and $Z(\theta)$ are assumed to be differentiable on $\Theta$. The exponential family can be thought of as projections of the Kullback-Leibler (KL) divergence on a set of probability distributions determined by some linear constraints, called linear family [4, 5]. $I_\alpha$-divergence is a generalization of the KL-divergence and is defined as follows. For probability distributions $p$ and $q$ on $\mathbb{R}^d$,

$$I_\alpha(p, q) := \frac{1}{1-\alpha} \log \int p(x)q(x)^{\alpha-1}dx - \frac{1}{\alpha} \log \int p(x)^\alpha dx + \log \int q(x)^\alpha dx,$$

[15, 16, 13, 3] (also known as relative $\alpha$-entropy [11], [12], logarithmic density power divergence [14], projective power divergence [12], $\gamma$-divergence [8, 3]). Here $\alpha > 0, \alpha \neq 1$, called the order of the divergence. Notice that, $I_\alpha$ coincides with the KL-divergence as $\alpha \rightarrow 1$ [12, 3]. In this sense $I_\alpha$-divergence can be regarded as a generalization of KL-divergence. $I_\alpha$-divergence also arises in information theory as redundancy in the mismatched case of guessing (for $\alpha < 1$) [15], source coding [12], and in encoding of tasks [2]. An analogous fact to the projections of KL-divergence on linear families yield an exponential family, the projections of $I_\alpha$-divergence on linear families yield an $\alpha$-power-law family, $M^{(\alpha)}$. A general $M^{(\alpha)}$-family can be defined, including the continuous and multivariate case, as follows.

Definition 1: The family of probability distributions $\{p_\theta : \theta \in \Theta\}$ is said to belong to a $k$-parameter $M^{(\alpha)}$ family if it can be written as

$$p_\theta(x) = \begin{cases} Z(\theta)^{-1} [q(x)^{\alpha-1} + w(\theta)^T f(x)]^{\frac{1}{\alpha-1}} & \text{if } x \in \mathbb{S} \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where $[r]^+ := \max\{r, 0\}$, for any $r \in \mathbb{R}$ and the functions $w, f, q$ and $Z$ are as defined in [11]. If the number of $w_i$’s is equal to that of $\theta_i$’s and each $w_i(\theta) = \theta_i$, such a family is called canonical $M^{(\alpha)}$-family [12] Def. 8. This canonical form of the family arises as a projection of the $I_\alpha$-divergence on a linear family of probability distributions [11]. Many well-known distributions such as Wigner semi-circle distribution, Wigner parabolic distribution and more interestingly, the Student-t distributions fall in the class $M^{(\alpha)}$.

1 Analogous to the canonical exponential family.
As KL-divergence is closely related to the maximum likelihood estimation (MLE), the $\mathcal{J}_\alpha$-divergence is closely related to a robustified version of MLE. Indeed, if $X_1, \ldots, X_n$ is an independent and identically distributed (i.i.d.) sample drawn according to some $p_\theta$ of a parametric model, $\Pi = \{p_\theta : \theta \in \Theta\}$ and $\mathcal{S}$ is the common support of $\Pi$ (that is, support of members of $\Pi$ does not depend on $\theta$), to find the MLE of $\theta$, one needs to solve the so-called score equation or estimating equation for $\theta$, given by

$$\frac{1}{n} \sum_{j=1}^{n} s(X_j; \theta) = 0. \quad (3)$$

Here $s(\cdot; \theta) := \nabla \log p_\theta(\cdot)$, called the score function and $\nabla$ stands for gradient with respect to $\theta$. If there is contamination in the observed sample, one modifies the score equation by replacing the usual average of the score functions $s(X_j; \theta)$ in (3) by some weighted average that down-weights the effect of the outliers. Motivated by the works of Field and Smith [2] and Windham [17], the following estimating equation was proposed by Jones et al. [10]:

$$\frac{1}{n} \sum_{j=1}^{n} \frac{p_\theta(X_j)^{\alpha-1} s(X_j; \theta)}{\int p_\theta(x)^{\alpha-1} dx} = \frac{\int p_\theta(x)^{\alpha} s(x; \theta) dx}{\int p_\theta(x)^{\alpha} dx}, \quad (4)$$

where $\alpha > 1$. The above equation was proposed based on the following intuition. If a sample point $x$ is not compatible to the true distribution $p_\theta$, then $p_\theta(x)^{\alpha-1}$ would be smaller and thus down-weights the effect of $x$ in the average of the score functions. The equation is obtained by equating the normalized empirical weighted average to its hypothetical one (c.f. [11]). Observe that, (4) does not make sense in terms of robustness for $\alpha < 1$. However, it is a valid estimation problem even for $\alpha < 1$ as minimization of $\mathcal{J}_\alpha$-divergence for $\alpha < 1$ corresponds to certain estimation problems in information theory, such as guessing [10], source coding (see [12] Sec. II C) and encoding of tasks [2].

Csiszár and Shields showed that if $\Pi$ is a canonical exponential family with finite support $\mathcal{S}$, the MLE (if exists and unique) is a solution to a system of linear equations [5] Th. 3.3]. This was due to an orthogonality relationship between the exponential family and a linear family with respect to the relative entropy. By exploiting the geometry between $\mathcal{J}_\alpha$-divergence, $\mathcal{M}^{(\alpha)}$-family and linear family, analogously, Kumar and Sundaresan showed that if $\Pi$ is a canonical $\mathcal{M}^{(\alpha)}$-family with finite support $\mathcal{S}$, then the solution of (4) (if exists and unique) is same as solution of a system of linear equations [12] Th. 18 and Th. 21]. In this paper, we solve this problem for the general $\mathcal{M}^{(\alpha)}$-family by directly solving the estimating equation. We show that, under some regularity assumptions, the result continues to hold for general $\mathcal{M}^{(\alpha)}$-family as well. We then apply this result to find estimators for the student-t distributions. We assume the following regularity conditions unless stated otherwise.

(a) All the integrals are well-defined over $\mathcal{S}$ with respect to the Lebesgue measure on $\mathbb{R}^d$ in the continuous case and with respect to the counting measure in discrete case.

(b) For probability distributions $p_\eta, p_{\eta'} \in \Pi$, if $\theta \neq \eta$, then $p_\eta \neq p_{\eta'}$ on a set of positive measure.

(c) The support $\mathcal{S}$ does not depend on $\theta$. Integration with respect to $x$ and differentiation with respect to $\theta$ can be interchanged.

II. ESTIMATION ON $\mathcal{M}^{(\alpha)}$ FAMILY

In this section, we solve the estimation problems (MLE or the generalized MLE) associated with the $\mathcal{J}_\alpha$-divergence by solving the respective estimating equations.

**Theorem 2.** The following are true.

(i) The MLE on an exponential family as defined in (1) must satisfy

$$\partial_r [w(\theta)]^T \mathbb{E}_\theta [f(X)] = \partial_r [w(\theta)]^T \tilde{f}, \quad r = 1, \ldots, k. \quad (5)$$

(ii) The Jones et. al. estimator on an $\mathcal{M}^{(\alpha)}$ family as in Definition 1 must satisfy

$$\frac{\partial_r [w(\theta)]^T \mathbb{E}_\theta [f(X)]}{\mathbb{E}_\theta [q(\theta)^{\alpha-1} + w(\theta)^T f(X)]} = \frac{\partial_r [w(\theta)]^T \tilde{f}}{q^{\alpha-1} + w(\theta)^T \tilde{f}}, \quad (6)$$

$$r = 1, \ldots, k.$$  

Here $\partial_r$ denotes the partial derivative with respect to $\theta_r$, $\tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_k)^T$, $\tilde{f}_i = \frac{1}{n} \sum_{j=1}^{n} f_i(X_j)$ for $i = 1, \ldots, s$, and $q^{\alpha-1} := \frac{1}{n} \sum_{j=1}^{n} q(X_j)^{\alpha-1}$.

**Proof:** 1. Consider an i.i.d. sample $X_1, \ldots, X_n$ drawn according to some $p_\theta$, where $X_i = (X_{1i}, \ldots, X_{di})^T$, $\forall i = 1, \ldots, n$.

(i) If $p_\theta$ is in an exponential family as in (1), we have for $r = 1, \ldots, k$, 

$$\partial_r [\log p_\theta(x)] = \partial_r [Z(\theta)] + \partial_r [w(\theta)]^T f(x).$$

$$\mathbb{E}_\theta [\partial_r \log p_\theta(X)] = 0, \text{ by the regularity condition (c), thus we have}$$

$$\partial_r [Z(\theta)] + \partial_r [w(\theta)]^T \mathbb{E}_\theta [f(X)] = 0. \quad (7)$$

Using (7) in the estimating equation (3), we have (5).

(ii) The estimating equation (4) can be re-written as

$$\frac{\frac{1}{n} \sum_{i=1}^{n} p_\theta(X_i)^{\alpha-2} \nabla p_\theta(X_i)}{\frac{1}{n} \sum_{i=1}^{n} p_\theta(X_i)^{\alpha-1}} = \frac{\int p_\theta(x)^{\alpha-1} \nabla p_\theta(x) dx}{\int p_\theta(x)^{\alpha} dx}. \quad (8)$$

Since $p_\theta \in \mathcal{M}^{(\alpha)}$, from (2) we have

$$p_\theta(x)^{\alpha-2} \partial_r \partial_r [p_\theta(x)] = Z(\theta)^{-1-\alpha} \left\{ Z(\theta)^{-1} \partial_r [Z(\theta)] [q(x)]^{\alpha-1} + w(\theta)^T f(x) \right\} - \frac{1}{\alpha-1} \partial_r [w(\theta)]^T f(x),$$

for $r = 1, \ldots, k$. Using this in (8) we get,

$$Z(\theta)^{-1} \partial_r [Z(\theta)] = \frac{\frac{1}{n} \sum_{i=1}^{n} \partial_r [w(\theta)]^T f(X_i)}{\frac{1}{n} \sum_{i=1}^{n} f(X_i)}.$$

$$= Z(\theta)^{-1} \partial_r [Z(\theta)] - \frac{1}{\alpha-1} \partial_r [w(\theta)]^T \int p_\theta(x)f(x) dx.$$
which implies (6).

In the following theorem we show that for a regular $E$ or $M^{(a)}$-family, Theorem 2 can be improved further.

**Definition 3:** If, for an exponential family $E$ (respectively, $M^{(a)}$-family), the following conditions are true, then such a family is said to be a $k$-parameter regular exponential family (respectively, $k$-parameter regular $M^{(a)}$-family).

(i) Number of $w_i$'s and number of $\theta_i$'s are equal ($s = k$),
(ii) $w_1(\cdot), \ldots, w_k(\cdot)$ are functionally independent on $\Theta$,
(iii) $f_1(\cdot), \ldots, f_k(\cdot)$ are functionally independent on $\mathbb{S}$.

**Theorem 4:** If the families in Theorem 2 are further regular, then
(i) the MLE on exponential family must satisfy
\[ E_\theta[f(X)] = \bar{f}, \]  
(9)
(ii) the Jones et al. estimator on an $M^{(a)}$-family must satisfy
\[ \frac{E_\theta[f(X)]}{E_\bar{\theta}[q(X)^{a-1}] - \bar{f}} = \frac{\bar{f}}{q^{a-1}}. \]  
(10)

**Proof:** Since a $k$-parameter regular family, 1, $w_1(\cdot), \ldots, w_k(\cdot)$ are functionally independent on $\Theta$. This implies the vectors \( \{\partial_i[w_i(\theta)]\}_{i=1,\ldots,k} \), \( i = 1, \ldots, k \), are linearly independent for every $\theta \in \Theta$, because if
\[ \sum_{i=1}^k c_i(\partial_i[w_i(\theta)]), \ldots, \partial_k[w_i(\theta)] = 0, \]
for some scalars $c_1$, $\ldots$, $c_k$, then
\[ c_1w_1(\theta) + \cdots + c_kw_k(\theta) = m, \]
where $m$ is a constant. Then functional independence of 1, $w_1(\cdot), \ldots, w_k(\cdot)$ implies that $m = c_1 = \cdots = c_k = 0$.

(i) (5) can be re-written as
\[ \partial_r[w(\theta)]^T E_\theta[f(X)] = 0. \]  
(11)

For any $\theta \in \Theta$, this is a system of $k$-homogeneous equations in $k$ unknowns where the coefficient matrix is given by
\[ D := (\partial_i[w_j(\theta)])_{k \times k}. \]

As all the columns of $D$ are linearly independent, the determinant of $D$ is non-zero. Hence (11) implies $E_\theta[f_i(X)] = \bar{f}_i = 0$, for $i = 1, \ldots, k$. By the regularity assumption (b), the estimator must satisfy (10).

(ii) (6) can be re-written as
\[ \partial_r[w(\theta)]^T \left[ \frac{E_\theta[f(X)]}{E_\bar{\theta}[q(X)^{a-1} + w(\theta)f(X)]} - \frac{\bar{f}}{q^{a-1} + w(\theta)f(X)} \right] = 0. \]

As the family is regular, the above equation becomes
\[ \frac{E_\theta[f(X)]}{E_\bar{\theta}[q(X)^{a-1} + w(\theta)f(X)]} = q^{a-1} + w(\theta)f(X), \]
by a similar argument as in (i). Now,
\[ q^{a-1} + w(\theta)f(X) = q^{a-1} + \frac{q^{a-1} + w(\theta)f(X)}{E_\theta[q(X)^{a-1} + w(\theta)f(X)]}w(\theta)^T E_\theta[f(X)] \]

Thus,
\[ \left\{ q^{a-1} + w(\theta)^T f \right\} \{ E_\theta[q(X)^{a-1} + w(\theta)^T f(X)] \} = q^{a-1} E_\theta[q(X)^{a-1} + w(\theta)^T f(X)] + \left[ q^{a-1} + w(\theta)^T f \right] w(\theta)^T E_\theta[f(X)], \]

that is,
\[ \{ q^{a-1} + w(\theta)^T f \} E_\theta[q(X)^{a-1}] = q^{a-1} E_\theta[q(X)^{a-1}] + w(\theta)^T f(X). \]

This implies
\[ \frac{q^{a-1}}{E_\theta[q(X)^{a-1}]} = \frac{q^{a-1} + w(\theta)^T f}{E_\theta[q(X)^{a-1}] + w(\theta)^T f(X)}. \]

Thus the estimator must satisfy (10).

**Remark 1:** Observe that, Theorem 4 (ii) essentially extends the result known for a canonical $M^{(a)}$-family with finite support [12] Th. 18 and Th. 21.

III. GENERALIZED MAXIMUM LIKELIHOOD ESTIMATION ON STUDENT-T DISTRIBUTIONS

In this section, we first show that the Student-t distributions form an $M^{(a)}$-family and then we apply Theorem 2 to find the estimator for their parameters.

The $d$-dimensional Student-t distribution with mean $\mu := (\mu_1, \ldots, \mu_d)^T$ and a positive-definite covariance matrix $\Sigma := (\sigma_{ij})_{d \times d}$ is given by
\[ p_\theta(x) = N_{\theta,\alpha}[1 + b_\alpha(x - \mu)^T \Sigma^{-1}(x - \mu)]^{\frac{1}{\nu}}, \]  
(12)

where $d/(d+2) < \alpha$, $\alpha \neq 1$ and $b_\alpha = [1 - \alpha]/[2\alpha - d(1 - \alpha)]$. Here $\nu := [b_\alpha + 2(1 - \alpha)^2]/[(1 - \alpha)^2]$ is the degrees of freedom. Let $\Sigma^{-1} := (\sigma_{ij})$, the inverse of $\Sigma$. The normalizing constant $N_{\theta,\alpha}$ is given by
\[ N_{\theta,\alpha} := \begin{cases} \frac{b^{d/2}\Gamma((1/2)\alpha)}{\Gamma(1/2)\Gamma((1/2)\alpha-1)} \sqrt{\det(\Sigma)}^{1/2} & \text{if } \alpha < 1 \\ \frac{[-b_\alpha]^{d/2}\Gamma((1/2)\alpha-1)}{\Gamma(1/2)\Gamma((1/2)\alpha-1)+[d/2]} \sqrt{\det(\Sigma)}^{1/2} & \text{if } \alpha > 1. \end{cases} \]

The support of this distribution is given by
\[ \mathbb{S} = \begin{cases} \mathbb{R}^d & \text{if } \alpha < 1 \\ \{ x : (x - \mu)^T \Sigma^{-1}(x - \mu) \geq -1/b_\alpha \} & \text{if } \alpha > 1. \end{cases} \]

We use the following notations. For a matrix $A = (a_{ij})_{d \times d}$,
\[ \text{Tr}(A) := \text{Trace of } A = \sum_i a_{ii}, \quad A^{-T} := (A^{-1})^T, \]
\[ \text{Vec}(A) := (a_{11}, \ldots, a_{1d}, a_{21}, \ldots, a_{2d}, \ldots, a_{d1}, \ldots, a_{dd})^T, \]
that is, if $A$ is a matrix of order $d$, then $\text{Vec}(A)$ is a $d^2$-dimensional column vector whose $(i-1)d+j$-th entry is $a_{ij}$, for all $i, j = 1, \ldots, d$. For $x \in \mathbb{S}$, we can re-write (12) as
\[
p_0(x) = N_{\theta, \alpha} \left[ 1 + b_0 \{ x^T \Sigma^{-1} x - 2 \mu^T \Sigma^{-1} x + \mu^T \Sigma^{-1} \mu \} \right] = N_{\theta, \alpha} \left[ 1 + b_0 \left\{ \text{Tr}(\Sigma^{-1} xx^T) - 2(\Sigma^{-1})^T x + \mu^T \Sigma^{-1} \mu \right\} \right] = N_{\theta, \alpha} \left[ 1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \text{Vec}(xx^T) - 2(\Sigma^{-1})^T x + \mu^T \Sigma^{-1} \mu \} \right].
\]

Comparing (13) with (2), we have Student-t distributions as a $(d^2 + d)$-parameter $M(\alpha)$-family, with
\[
\theta = (\mu, \Sigma^{-1}), \quad Z(\theta)^{-1} = N_{\theta, \alpha}, \quad q(x) = 1,
\]
\[
w_1(\theta) = b_0 (\mu^T \Sigma^{-1} \mu), \quad f_1(x) = 1,
\]
\[
w_2(\theta) = -2b_0 (\Sigma^{-1})^T x, \quad f_2(x) = x,
\]
\[
w_3(\theta) = b_0 \text{Vec}(\Sigma^{-1})^T, \quad f_3(x) = \text{Vec}(xx^T).
\]

Since for $\alpha < 1$, the support $\mathbb{S}$ does not depend upon the parameters, we can apply Theorem 2 (ii) to estimate the parameters $\mu$ and $\Sigma$. Let us first calculate the derivative of each $w_i(\theta)$ with respect to each parameter.
\[
\partial_\mu [w_1(\theta)] = 2b_0 (\Sigma^{-1})^T \mu,
\]
\[
\partial_\mu [w_2(\theta)] = -2b_0 (\Sigma^{-1})^T,
\]
\[
\partial_\mu [w_3(\theta)] = 0,
\]

where $O_{d \times d^2}$ is the zero matrix of order $d \times d^2$. For $i, j = 1, \ldots, d$
\[
\partial_{u_{ij}} [w_1(\theta)] = 2b_0 \mu_i \mu_j,
\]
\[
\partial_{u_{ij}} [w_2(\theta)] = u_{ij},
\]
\[
\partial_{u_{ij}} [w_3(\theta)] = u_{ij},
\]

where $u_{ij}$ is a $d$-dimensional row vector whose entries are zero except the $i$-th and $j$-th entries which are $(-2b_0 \mu_i \mu_j)$ and $(-2b_0 \mu_j)$, respectively. Similarly, $v_{ij}$ is a $d^2$-dimensional row vector whose entries are zero except the $[(i-1)d+j]$-th and $[(j-1)d+i]$-th which are equal to $b_0$.

Consider now an i.i.d. sample $X_1, \ldots, X_n$ according to a $p_\theta$ of the form (12), where $X_i = (X_{i1}, \ldots, X_{id})^T$, $i = 1, \ldots, n$. Define
\[
\bar{X} = (\bar{X}_1, \ldots, \bar{X}_d)^T, \quad \bar{X}_i = \frac{1}{n} \sum_{i=1}^{n} X_{id},
\]

for $i = 1, \ldots, d$, and $\bar{X}X^T$ is the matrix of order $d \times d$ whose $(i, j)$-th entry is $\frac{1}{n} \sum_{i=1}^{n} X_{id}X_{jd}$. Let us denote
\[
Y := 1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \text{Vec}(\bar{X}X^T) - 2(\Sigma^{-1})^T \bar{X} + \mu^T \Sigma^{-1} \mu \}.
\]

Using (6), we have the following estimating equations for the Student-t distributions.
\[
-2b_0 \Sigma^{-1} \bar{E}[X_i] + 2b_0 \Sigma^{-1} \mu = -2b_0 \Sigma^{-1} \bar{X} + 2b_0 \Sigma^{-1} \mu, \quad 2b_0 \mu_i \mu_j + u_{ij} \bar{E}[X_i + v_{ij} \bar{E}[\text{Vec}(\bar{X}X^T)]] = 2b_0 \mu_i \mu_j + u_{ij} \bar{X} + v_{ij} \text{Vec}(\bar{X}X^T),
\]
for $i, j = 1, \ldots, d$. Since $\bar{E}[X_i] = \mu$ and $\bar{E}[X_iX_j] = \sigma_{ij} + \mu_i \mu_j$, the above system reduces to
\[
\mu = \bar{X}, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X_{il}X_{jl} = \mu_i \mu_j + \bar{X}_{ij}^T \sigma_{ij}.
\]

Using these we have
\[
\bar{E}[Y] = \bar{E}[1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \text{Vec}(XX^T) - 2(\Sigma^{-1})^T \bar{X} + \mu^T \Sigma^{-1} \mu \}] = 1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \{ \text{Vec}(\bar{X}) + \text{Vec}(\mu \mu^T) \} - 2(\Sigma^{-1})^T \mu + \mu^T \Sigma^{-1} \mu \}
\]
\[
= 1 + b_0 \{ \text{Tr}(\Sigma^{-1} \bar{X}) + \text{Tr}(\Sigma^{-1} \mu \mu^T) - \mu^T \Sigma^{-1} \mu \} = 1 + b_0 \{ d + \text{Tr}(\mu^T \Sigma^{-1} \mu) - \text{Tr}(\mu^T \Sigma^{-1} \mu) \}
\]
\[
= 1 + d \cdot b_0 \bar{E}[Y].
\]

and
\[
Y = 1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \text{Vec}(\bar{X}X^T) - 2(\Sigma^{-1})^T \bar{X} + \mu^T \Sigma^{-1} \mu \} = 1 + b_0 \{ \text{Vec}(\Sigma^{-1})^T \text{Vec}(\bar{X}X^T) \bar{X} + \mu \text{Vec}(\Sigma^{-1})^T \mu \}
\]
\[
+ \text{Vec}(\Sigma^{-1})^T \{ d + \text{Tr}(\Sigma^{-1} \mu \mu^T) - \mu^T \Sigma^{-1} \mu \} = 1 + b_0 \{ \bar{X}X^T \bar{X} + d \text{Tr}(\Sigma^{-1} \mu \mu^T) - \mu^T \Sigma^{-1} \mu \}
\]
\[
= 1 + d \cdot b_0 \bar{E}[Y].
\]

Equations (14) and (15) together imply $Y = \bar{E}[Y]$. Thus the estimating equations for Student-t distributions become
\[
\mu = \bar{X}, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X_{il}X_{jl} = \mu_i \mu_j + \sigma_{ij}.
\]

Hence the estimators for $\mu$ and $\Sigma$ are
\[
\hat{\mu} = \bar{X}, \quad \text{and} \quad \hat{\sigma}_{ij} = \frac{1}{n} \sum_{i=1}^{n} (X_{il} - \hat{\mu}_i)(X_{jl} - \hat{\mu}_j).
\]

These are summarized in the following theorem.

**Theorem 5:** Let $\alpha < 1$. The estimator for the mean and covariance parameters of a Student-t distribution in (12) by the estimating equation (4) of Jones et al., are
\[
\hat{\mu} = \bar{X}, \quad \hat{\sigma}_{ij} = \frac{1}{n} \sum_{i=1}^{n} (X_{il} - \hat{\mu}_i)(X_{jl} - \hat{\mu}_j).
\]

**Remark 2:** It can be shown that as $\alpha \to 1$, the Student-t distributions coincide with a normal distribution with mean $\mu$ and covariance matrix $\Sigma$. Also, for $\alpha = 1$, the estimating equation (4) is actually the usual score equation (3) of MLE. Thus there is a continuity on the $\alpha$-estimation for $\alpha$ in $[0, 1]$. The condition that the support $\mathbb{S}$ is independent of the parameters is necessary for Theorem 2. The uniform distribution in $(0, \theta)$, where $\theta$ is the unknown parameter, can be expressed as an exponential family. But the support $\mathbb{S}$ of this family depends on the parameter $\theta$. Also the MLE for $\theta$ cannot be obtained by simply solving the estimating equation (5). Here
we present such an example for the Jones et al. estimation 

Let us consider, for simplicity, the Student-t distributions with \( \alpha = 2 \) and variance \( \sigma = 1 \). Then the pdf is given by

\[
p_{\mu}(x) = N_2 \left[ 1 - \frac{(x - \mu)^2}{5} \right],
\]

where \( N_2 = \Gamma(5/2)/\sqrt{\pi} \Gamma(2) = 3/4\sqrt{\pi} \) and the support is given by

\[
S = \{ x : \mu - \sqrt{5} \leq x \leq \mu + \sqrt{5} \},
\]

which depends on the unknown parameter \( \mu \). Thus we cannot use Theorem 2(ii) directly to estimate \( \mu \). However, solving the estimating equation (4) is same as maximizing the following generalized likelihood function for \( p_\theta \),

\[
L^{(\alpha)}(\theta) := \frac{1}{\alpha - 1} \log \left[ \frac{1}{n} \sum_{j=1}^{n} p_{\theta}(X_j)^{\alpha - 1} \right] - \log \left[ \int p_{\theta}(x)^{\alpha} \, dx \right].
\]

Suppose that \( X_1, \ldots, X_n \) is an i.i.d. sample drawn according to \( p_\mu \) in \((18)\). Then

\[
L^{(2)}(\mu \mid X_1, \ldots, X_n) = 2 \log \left[ \frac{1}{n} \sum_{i=1}^{n} p_\mu(X_i) 1(\mu - \sqrt{5} \leq X_i \leq \mu + \sqrt{5}) \right]
\]

\[
- \log \left( \mathbb{E}_\mu \left[ N_2 \left( 1 - \frac{(X - \mu)^2}{5} \right) \right] \right)
\]

\[
= 2 \log \left[ \frac{1}{n} \sum_{i=1}^{n} p_\mu(X_i) 1(X_i - \sqrt{5} \leq \mu \leq X_i + \sqrt{5}) \right]
- \log \frac{4N_2}{5},
\]

where \( 1(\cdot) \) denotes the indicator function.

The maximizer of \( L^{(2)}(\mu) \) is same as the maximizer of

\[
\ell^{(2)}(\mu) := \sum_{i=1}^{n} p_\mu(X_i) 1(X_i - \sqrt{5} \leq \mu \leq X_i + \sqrt{5}).
\]

Without loss of generality, let us assume that \( X_1 < X_2 < \cdots < X_n \). It is clear from \((21)\) that one needs to have the knowledge of the entire sample to decide the maximizer of \( \ell^{(2)}(\mu) \).

Let us first suppose that \( (X_n - X_1) \leq 2\sqrt{5} \). (22)

Then one can choose a \( \mu \) in \( [X_1 - \sqrt{5}, X_n + \sqrt{5}] \) such that \( p_\mu(X_i) > 0 \) for some \( i \in \{1, \ldots, n\} \). Thus, in this case, we have

\[
\ell^{(2)}(\mu) = \begin{cases} 
\sum_{i=1}^{n} p_\mu(X_i), & \text{for } \mu \in [X_1 - \sqrt{5}, X_2 - \sqrt{5}] \\
\sum_{i=1}^{n} p_\mu(X_i), & \text{for } \mu \in [X_2 - \sqrt{5}, X_3 - \sqrt{5}] \\
\vdots & \\
\sum_{i=1}^{n} p_\mu(X_i), & \text{for } \mu \in [X_{n-1} - \sqrt{5}, X_n - \sqrt{5}] \\
\sum_{i=1}^{n} p_\mu(X_i), & \text{for } \mu \in [X_n - \sqrt{5}, X_1 + \sqrt{5}] \\
\sum_{i=2}^{n} p_\mu(X_i), & \text{for } \mu \in [X_1 + \sqrt{5}, X_2 + \sqrt{5}] \\
\vdots & \\
\sum_{i=n-1}^{n} p_\mu(X_i), & \text{for } \mu \in [X_{n-2} + \sqrt{5}, X_{n-1} + \sqrt{5}] \\
p_\mu(X_n), & \text{for } \mu \in [X_{n-1} + \sqrt{5}, X_n + \sqrt{5}] \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( \bar{X}^{(k)} := \frac{1}{k} \sum_{i=1}^{k} X_i \) and \( \bar{X}_{(k)}^{(k)} := \frac{1}{n-k} \sum_{i=k+1}^{n} X_i \) for \( k = 1, \ldots, n-1 \).

The maximizer of \( \ell^{(2)}(\mu) \) on \([X_k - \sqrt{5}, X_{k+1} - \sqrt{5}]\) is

\[
\mu^{(k)} := \text{median} \{ X_k - \sqrt{5}, \bar{X}^{(k)}, X_{k+1} + \sqrt{5} \},
\]

that on \([X_n - \sqrt{5}, X_1 + \sqrt{5}]\) is

\[
\mu^{(n)} := \text{median} \{ X_n - \sqrt{5}, X_1 + \sqrt{5} \},
\]

and on \([X_k + \sqrt{5}, X_{k+1} + \sqrt{5}]\) is

\[
\mu^{(k)} := \text{median} \{ X_k + \sqrt{5}, \bar{X}^{(k)}, X_{k+1} + \sqrt{5} \},
\]

for \( k = 1, \ldots, n - 1 \). Let

\[
\mathcal{M} := \{ \mu^{(k)}, \mu^{(n)}, \mu^{(k)} : k = 1, \ldots, n-1 \}.
\]

Then the estimator of \( \mu \) is

\[
\hat{\mu} := \arg\max_{\mu \in \mathcal{M}} \ell^{(2)}(\mu).
\]

Thus it is clear that \( \hat{\mu} \) is not necessarily \( \bar{X} \). For illustration, let us suppose that the observed sample is 4.6, 4.7, 6.0, 7.0, 8.2, 8.6, 8.7, 8.8, 8.9, and 9.0. Then \( X_n - X_1 = 9 - 4.6 = 4.4 < 2\sqrt{5} \), \( \mu^{(10)} = 6.84 \) and

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\mu^{(1)} & \mu^{(2)} & \mu^{(4)} & \mu^{(6)} & \mu^{(7)} & \mu^{(8)} & \mu^{(9)} \\
2.46 & 3.76 & 4.76 & 5.57 & 6.1 & 6.66 & 6.76 \\
\hline
\mu^{(1)} & \mu^{(2)} & \mu^{(4)} & \mu^{(6)} & \mu^{(7)} & \mu^{(8)} & \mu^{(9)} \\
6.94 & 8.13 & 8.46 & 9.24 & 10.44 & 10.84 & 11.04 & 11.14 \\
\hline
\end{array}
\]

The respective maximum values of \( \ell^{(2)}(\mu) \) are given by 3.7 \( N_2 \) and

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
0.08 N_2 & 1.68 N_2 & 2.69 N_2 & 3.21 N_2 & 3.11 N_2 & 3.67 N_2 \\
3.15 N_2 & 3.33 N_2 & 3.52 N_2 & 4.02 N_2 & 6.37 N_2 & 6.42 N_2 \\
5.57 N_2 & 2.33 N_2 & 0.82 N_2 & 0.5 N_2 & 0.25 N_2 & 0.84 N_2 \\
\hline
\end{array}
\]

Thus, the maximum value of \( \ell^{(2)}(\mu) \) is 6.42 \( N_2 \) and the maximizer is \( \mu^{(3)} = 8.46 \). Hence \( \hat{\mu} = 8.46 \), which is not equal to \( \bar{X} = 7.45 \).

Similarly, if \((22)\) is true excluding one sample, say \( X_1 \), that is, if \( X_n - X_2 \leq 2\sqrt{5} \), but \( X_n - X_1 > 2\sqrt{5} \), then we can follow the same procedure with \( X_2, \ldots, X_n \) to find \( \hat{\mu} \). Thus, in

\footnote{This coincides with the usual log likelihood function for MLE as \( \alpha \to 1 \).}
general, if there are \( k \) samples such that (22) is true excluding these \( k \) samples, then we can proceed similarly with the rest \((n - k)\) samples to find \( \hat{\mu} \). Finally, when all the samples are more than \(2\sqrt{5} \) apart from each other, the intervals in (21) will be disjoint and in this case any sample point can be taken to be the estimator.

IV. SUMMARY

In this paper we extended the already known projection theorem of the \( J_\alpha \)-divergence to the canonical \( M^{(\alpha)} \)-family on the finite alphabet set of (22) to the more general multivariate, continuous \( M^{(\alpha)} \)-family and applied the result to find estimators for the Student-t distributions. In the case when \( \alpha < 1 \), we showed that the estimators are same as the maximum likelihood estimators of the Gaussian distribution, and can be obtained by solving the estimating equations (or projection equations). However, in the case when \( \alpha > 1 \), the estimators cannot be obtained by solving the estimating equations and one needs to obtain it by maximizing the generalized likelihood function on a case-by-case basis.

ACKNOWLEDGMENTS

Atin Gayen is supported by an INSPIRE fellowship, the Department of Science and Technology, Government of India.

REFERENCES

[1] Basu, A., Shioya, H., and Park, C. (2011). ”Statistical Inference: The Minimum Distance Approach," Chapman & Hall/ CRC Monographs on Statistics and Applied Probability 120.
[2] Bunte, C. and Lapidoth, A. (2014). “Encoding tasks and Rényi entropy,” IEEE Trans. Info. Theory, 60(9), pp. 5065-5076.
[3] Cichocki, A. and Amari, S. (2010). “Families of Alpha-Beta-and Gamma-Divergences: Flexible and Robust Measure of Similarities,” Entropy, 12, pp. 1532–1568.
[4] Csiszár, I. (1975). “I-divergence geometry of probability distributions and minimization problems,” Ann. Probab., 3, pp. 146–158.
[5] Csiszár, I. and Shields, P. C. (2004). “Information Theory and Statistics: A Tutorial,” Foundations and Trends in Communications and Information Theory, 1(4), pp. 417–528.
[6] Eguchi, S. and Kato, S. (2010). “Entropy and divergence associated with power function and the statistical application,” Entropy, 12, pp. 262–274.
[7] Field, C. and Smith, B. (1994). “Robust estimation: A weighted maximum likelihood approach,” International Statistical Review, 62(3), pp. 405–424.
[8] Fujisawa, H. and Eguchi, S. (2008). ” Robust parameter estimation with a small bias against heavy contamination”, J. Multivariate Anal., 99, pp. 2053–2081.
[9] Hogg, R. V., Craig, A. and McKean, J. W. (2013). Introduction to Mathematical Statistics, Pearson, Sixth Ed.
[10] Jones, M. C., Hjort, N. L., Harris, I. R., and Basu, A. (2001). “A comparison of related density based minimum divergence estimators,” Biometrika, 88(3), pp. 865–873.
[11] Kumar, M. A. and Sundaresan, R. (2015). “Minimization problems based on relative \( \alpha \)-entropy I: Forward Projection,” IEEE Trans. Info. Theory, 61(9), pp. 5063–5080.
[12] Kumar, M. A. and Sundaresan, R. (2015). “Minimization problems based on relative \( \alpha \)-entropy II: Reverse Projection,” IEEE Trans. Info. Theory, 61(9), pp. 5081–5095.
[13] Lutwak, E., Yang, D., and Zhang, G. (2005). “Cramér-Rao and moment-entropy inequalities for Rényi entropy and generalized Fisher information,” IEEE Trans. Info. Theory, 51(2), pp. 473–478.
[14] Maji, A., Ghosh, A., and Basu, A. (2016). “The logarithmic super divergence and asymptotic inference properties,” Advances in Statistical Analysis, 100, pp. 99–131.
[15] Sundaresan, R. (2002). “A measure of discrimination and its geometric properties,” in Proc. 2002 IEEE Int. Symp. Inf. Theory, Lausanne, Switzerland, pp. 264.
[16] Sundaresan, R. (2007). “Guessing under source uncertainty,” IEEE Trans. Info. Theory, 53(1), pp. 269–287.
[17] Windham, M. P. (1995). “Robustifying model fitting,” Journal of the Royal Statistical Society, Series B (Methodological), 57(3), pp. 599–609.