Baryon Electromagnetic Properties in Partially Quenched Heavy Hadron Chiral Perturbation Theory

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Abstract

The electromagnetic properties of baryons containing a heavy quark are calculated at next-to-leading order in partially quenched heavy hadron chiral perturbation theory. Calculations are performed for three light flavors in the isospin limit and additionally for two light non-degenerate flavors. We use partially-quenched charge matrices that are easy to implement on the lattice. The results presented are necessary for the light quark mass extrapolation and zero-momentum extrapolation of lattice QCD and partially quenched lattice QCD calculations of heavy hadron electromagnetic properties. Additionally relations between the sextet electromagnetic form factors and transition form factors are derived.

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I. INTRODUCTION

Calculating physical properties of hadrons from QCD is a formidable challenge. The strong interaction dynamics involved is highly non-perturbative and results in the confinement of quarks and gluons into color-neutral hadronic states. Electromagnetic probes have often been sought experimentally as a means to glimpse the charge and magnetism distributions of quarks within hadrons. Electromagnetic observables, such as the charge radii and electromagnetic moments, provide a physically intuitive glimpse at the structure of hadrons. For baryons containing $b$ or $c$ quarks, however, the experimental difficulties in measuring these observables are substantial.

The dynamics underlying singly heavy baryons involves a rich interplay of opposite limits of QCD: heavy-quark physics and light-quark physics. Lattice QCD provides a first principles method for the calculation of QCD observables. These numerical simulations can determine the electromagnetic properties of baryons containing a heavy quark and thereby provide a physical picture of these baryons in terms of their quark degrees of freedom. Calculation of singly heavy baryon observables (masses and weak transitions) have been pursued in quenched QCD (QQCD) [1, 2, 3, 4, 5, 6, 7, 8]. Until now [9], partially quenched QCD (PQQCD) calculations have been very limited for any baryons. The appearance of quenched and partially quenched approximations to lattice QCD arises from numerically calculating the fermionic determinant. In QQCD, the time-costly numerical computation is radically simplified by replacing the determinant by a constant, whereas in PQQCD the determinant is calculated, but using larger quark masses for the quarks not coupled to external sources. In PQQCD, sea quark contributions are thus retained; while in QQCD, the sea quarks have been effectively discarded. Due to restrictive computational time, the valence quark masses, too, cannot be set to their physical values and one must extrapolate down from the values used on the lattice.

To perform quark mass extrapolations, one must understand how QCD observables behave as the quark masses vary. This can be done using chiral perturbation theory ($\chi$PT), which is a model-independent effective theory for low-energy QCD. This effective theory is written in terms of the pseudo-Goldstone bosons appearing from chiral symmetry breaking [10, 11]. While these bosons are not massless, they remain light compared to the scale of chiral symmetry breaking and dominate low-energy hadronic interactions. The light quark masses appear as parameters of $\chi$PT and enable the determination of the quark mass dependence of QCD observables by matching onto the effective theory. Without external input (e.g. experimental data or lattice QCD data), the effective theory has little predictive power because symmetries constrain the types of operators in the Lagrangian, but not the values of their coefficients. While providing a guide to extrapolate lattice QCD data, $\chi$PT can also benefit in turn from the determination of its low-energy constants (LECs) from these data. Heavy quark symmetry furthermore can be combined with chiral symmetry to describe hadrons formed from light quarks and a single heavy quark [12, 13, 14].

For QQCD lattice simulations, quenched chiral perturbation theory (Q$\chi$PT) has been developed to aid in the extrapolation in valence quark masses [15, 16, 17, 18, 19]. There is, however, no general connection of QQCD observables to QCD because, for example, QQCD lacks an axial anomaly. Consequently Q$\chi$PT contains operators that do not have analogues in $\chi$PT. Furthermore, the LECs of operators that have analogues are numerically unrelated to their counterparts in $\chi$PT. The sickness of the quenched approximation can be treated by utilizing partially quenched lattice simulations. For such simulations, partially quenched
chiral perturbation theory PQ\(\chi\)PT has been constructed to perform the extrapolation in both sea and valence quark masses [20, 21, 22, 23, 24, 25]. Unlike QQCD, PQQCD retains an axial anomaly and the flavor singlet field can be integrated out. In further distinction to Q\(\chi\)PT, the LECs of \(\chi\)PT appear in PQQ\(\chi\)PT, and hence PQQCD lattice simulations can be used to determine \(\chi\)PT parameters. Much work has resulted from this possibility, e.g., in the baryonic sector see [26, 27, 28, 29, 30, 31, 32, 33, 34].

In this work we determine the electromagnetic properties of baryons containing a heavy quark at next-to leading order (NLO) in partially quenched heavy hadron \(\chi\)PT. We determine the electromagnetic properties in three-flavor and two-flavor partially quenched simulations and use the isospin limit in the former. Such expressions allow one to perform the required extrapolation in light-quark masses. Additionally the electromagnetic LECs that appear in heavy hadron \(\chi\)PT can be extracted at NLO from lattice data and we explore how the arbitrariness of the light-quark charge matrix can be exploited to determine the LECs. Furthermore we derive relations between the electromagnetic form factors and electromagnetic transition form factors of the sextet baryons. These relations follow from the decoupling of the heavy and light spin degrees of freedom.

The organization of the paper is as follows. First in Sec. II we review the inclusion of baryons containing a heavy quark into PQ\(\chi\)PT. Next, the charge radii and magnetic moments of the \(s_\ell = 0\) baryons are calculated in Sec. III. The charge radii, magnetic moments and electric quadrupole moments of the \(s_\ell = 1\) baryons are calculated in Sec. IV. The corresponding results for \(SU(4|2)\) with non-degenerate light quarks are presented in Appendix A. For reference, in Appendix B we give the electromagnetic properties of these baryons in \(SU(3)\) and \(SU(2)\) \(\chi\)PT. In Appendix C we use heavy quark symmetry to derive relations between the sextet baryon form factors; and in Appendix D we present results for and comment on the commonly used form of the partially-quenched charge matrix. The summary, Sec. V, highlights the goal of understanding the electromagnetic properties of the triplet and sextet baryons from lattice QCD, and contrasts the situation with that of baryons formed from three light quarks.

II. PQ\(\chi\)PT FOR HEAVY HADRONS

In this section, we present the formulation of PQ\(\chi\)PT for heavy hadrons in \(SU(6|3)\). We begin by reviewing the set up of PQQCD. Next we describe the pseudo-Goldstone mesons of PQ\(\chi\)PT, and finally include the heavy hadrons into this partially quenched theory.

A. PQQCD

The light-quark sector Lagrangian in PQQCD is given by

\[
\mathcal{L} = \sum_{j,k=1}^{9} \bar{q}_j (i\not{D} - m_q)_{jk} q_k.
\]  

This differs from the QCD Lagrangian by the addition of six extra quarks; three bosonic ghost quarks, \((\tilde{u}, \tilde{d}, \tilde{s})\), and three fermionic sea quarks, \((j, l, r)\), in addition to the light physical quarks \((u, d, s)\). The nine quark fields transform in the fundamental representation...
of the graded group $SU(6|3)$ \[35, 36\]. They appear in the nine-component vector
\[ q = (u, d, s, j, r, \bar{u}, \bar{d}, \bar{s})^T. \] (2)

These quark fields obey graded equal-time commutation relations
\[ q_\alpha^i(x) q_\beta^j(y) - (-1)^{\eta_\alpha \eta_\beta} q_\beta^j(y) q_\alpha^i(x) = \delta^{\alpha \beta} \delta^{ij} (x - y), \] (3)
where $\alpha, \beta$ and $i, j$ are spin and flavor indices, respectively. The graded equal-time commutation relations which vanish can be written similarly. The different statistics of the PQQCD quark fields are reflecting in grading factors $\eta_k$ employed above, where
\[ \eta_k = \begin{cases} 1 & \text{for } k = 1, 2, 3, 4, 5, 6 \\ 0 & \text{for } k = 7, 8, 9 \end{cases}. \] (4)

In the isospin limit $m_u = m_d$, the quark mass matrix of $SU(6|3)$ is given by
\[ m_q = \text{diag}(m_u, m_u, m_s, m_j, m_j, m_r, m_u, m_u, m_s). \] (5)

Notice that each valence quark mass is degenerate with the corresponding ghost quark’s mass. This equality maintains an exact cancellation in the path integral between their respective determinants. Putting only valence quarks in the external states, the contributions to observables from disconnected quark loops come from the sea sector of PQQCD. Thus one can separate valence and sea contributions as their names suggest, and independently vary the masses of the valence and sea sectors.

The light quark electric charge matrix $Q$ is not uniquely defined in PQQCD \[37\]. By imposing the charge matrix $Q$ to be supertraceless in $SU(6|3)$, no new operators involving the singlet component are introduced. This can be accomplished with the choice
\[ Q = \text{diag}(q_u, q_d, q_s, q_j, q_l, q_r, q_u, q_d, q_s), \] (6)
where to maintain supertracelessness $q_j + q_l + q_r = 0$. Notice that when the sea quark masses are made degenerate with the valence quark masses, QCD is only recovered for the specific choice $q_u = q_j = \frac{2}{3}$, and $q_d = q_s = q_l = q_t = q_r + \frac{1}{3}$. As we shall detail below, one can use unphysical charges for both valence and sea quarks as a means to determine the LECs. The results and problems for the commonly used partially-quenched form of $Q$ proposed in \[26\] are presented in Appendix \[D\].

**B. Mesons of PQ\(\chi\)PT**

For massless light quarks, the theory described by the Lagrangian in Eq. (1) has the symmetry $SU(6|3)_L \otimes SU(6|3)_R \otimes U(1)_V$, that is assumed to be spontaneously broken to $SU(6|3)_V \otimes U(1)_V$ in analogy with QCD. One can build an effective low-energy theory of PQQCD by perturbing about the physical vacuum state. This theory is PQ\(\chi\)PT, and the dynamics of the pseudo-Goldstone mesons appearing from chiral symmetry breaking are described at leading order in the chiral expansion by the Lagrangian
\[ \mathcal{L} = \frac{f^2}{8} \text{str} \left( D^\mu \Sigma^\dagger D_\mu \Sigma \right) + \lambda \text{str} \left( m_q \Sigma^\dagger + m_q^\dagger \Sigma \right), \] (7)
where the field

$$\Sigma = \exp \left( \frac{2i\Phi}{f} \right) = \xi^2,$$

and the meson fields appear in the $SU(6|3)$ matrix

$$\Phi = \begin{pmatrix} M & \chi^\dagger \\ \chi & \bar{M} \end{pmatrix}.$$  

The $M$ and $\bar{M}$ matrices contain bosonic mesons (with quantum numbers of $q\bar{q}$ pairs and $\bar{q}\bar{q}$ pairs, respectively), while the $\chi$ and $\chi^\dagger$ matrices contain fermionic mesons (with quantum numbers of $q\bar{q}$ pairs and $\bar{q}q$ pairs, respectively). The upper $3 \times 3$ block of the matrix $M$ contains the familiar pions, kaons, and eta, while the remaining components consist of mesons formed from one or two sea quarks, see e.g. [26]. The operation $\text{str}()$ in Eq. (7) is a graded flavor trace. The gauge covariant derivative is defined by $D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu [Q, \Sigma]$, where $A_\mu$ is the photon field. To leading order, one finds that mesons with quark content $qq'$ are canonically normalized when their masses are given by

$$m^2_{qq'} = \frac{4\lambda_f^2}{f^2} (m_q + m_{q'}).$$

The flavor singlet field is $\Phi_0 = \text{str}(\Phi)/\sqrt{6}$, and because PQQCD has a strong axial anomaly $U(1)_A$, the mass of the singlet field has been taken to be on the order of the chiral symmetry breaking scale, and thus $\Phi_0$ has been integrated out of Eq. (7). The resulting $\eta$ two-point correlation functions, however, deviate from their familiar form in $\chi$PT. In calculating the electromagnetic properties of heavy hadrons, the results do not explicitly depend on the form of the flavor-neutral propagator.

C. Baryons containing heavy quarks in PQ$\chi$PT

Heavy quark effective theory (HQET) and PQ$\chi$PT can be combined to describe the interactions of heavy hadrons and pseudo-Goldstone mesons. Let $m_Q$ denote the mass of the heavy quark, where $Q = c$ or $b$. In the limit $m_Q \to \infty$, baryons containing a heavy quark and two light quarks (and indeed all heavy hadrons) are classified by the spin of their light degrees of freedom, $s_\ell$, because the heavy quark’s spin decouples from the system. The inclusion of baryons containing a heavy quark into $\chi$PT was carried out in [14, 38, 39] and their magnetic moments were calculated to one-loop order in [40, 41]. The quenched chiral theory for baryons with a heavy quark was written down in [42] and the partially quenched theory was investigated in [29, 43].

Consider first the baryons with $s_\ell = 0$. To include these spin-$\frac{1}{2}$ baryons into PQ$\chi$PT, we use the method of interpolating fields to classify their representations of $SU(6|3)$ [19, 26, 27, 29, 44]. The $s_\ell = 0$ baryons are described by the field

$$T_{ij} \sim Q^{\gamma,c} \left( q_i^{\alpha,a} q_j^{\beta,b} + q_i^{\beta,b} q_j^{\alpha,a} \right) \varepsilon_{abc}(C\gamma_5)_{\alpha\beta},$$

1 As we largely work to leading order in $1/m_Q$, the label $Q$ is omitted from all baryon tensors.
where \(i\) and \(j\) are flavor indices; \(a\), \(b\), and \(c\) are color indices; and \(\alpha\), \(\beta\), and \(\gamma\) are spin indices. The tensor \(T\) has the symmetry property
\[
T_{ij} = (-)^{\eta_i \eta_j} T_{ji}
\]
and forms a 39-dimensional representation of \(SU(6|3)\). These states are conveniently classified under the quark sectors acted upon, thus we use the super-algebra terminology of \([26, 45]\). Under \(SU(3)_{\text{val}} \otimes SU(3)_{\text{sea}} \otimes SU(3)_{\text{ghost}}\), the ground floor, level A transforms as a \((3, 1, 1)\) and contains the familiar \(s_\ell = 0\) QCD baryon tensor \(T_{ij}\), i.e. \(T_{ij} = T_{ji}\), when the indices are restricted to the range \(1 - 3\). With our conventions, we have
\[
T_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \Lambda_Q & \Xi^{+\frac{1}{2}}_Q \\
-\Lambda_Q & 0 & \Xi^{-\frac{1}{2}}_Q \\
-\Xi^{+\frac{1}{2}}_Q & -\Xi^{-\frac{1}{2}}_Q & 0
\end{pmatrix}_{ij},
\]
and the superscript labels the 3-projection of isospin. The first floor of level A contains nine baryons that transform as a \((3, 1, 3)\), while the ground floor of level B contains nine baryons that transform as a \((3, 3, 1)\). The remaining floors and levels are not necessary for our calculation.

Next we consider the \(s_\ell = 1\) baryons. These spin \(\frac{1}{2}\) and \(\frac{3}{2}\) baryons are degenerate in the heavy quark limit and can be described by one field \(S^\mu_{ij}\). The interpolating field for these baryons is
\[
S^\mu_{ij} \sim Q^{\gamma,c} \left(q_i^{\alpha,a} q_j^{\beta,b} - q_i^{\beta,b} q_j^{\alpha,a}\right) \varepsilon_{abc} (C^\mu)_{\alpha\beta},
\]
where the tensor satisfies the symmetry
\[
S^\mu_{ij} = (-)^{1+\eta_i \eta_j} S^\mu_{ji},
\]
and makes up a 42-dimensional representation of \(SU(6|3)\). Under \(SU(3)_{\text{val}} \otimes SU(3)_{\text{sea}} \otimes SU(3)_{\text{ghost}}\), the ground floor, level A transforms as a \((6, 1, 1)\) and contains the familiar \(s_\ell = 1\) QCD baryon tensor \(S^\mu_{ij}\), i.e. \(S^\mu_{ij} = S^\mu_{ij}\), when the indices are restricted to the range \(1 - 3\). The QCD flavor tensor is
\[
S^\mu_{ij} = \frac{1}{\sqrt{3}} (v^\mu + \gamma^\mu) \gamma_5 B_{ij} + B^*_\mu_{ij},
\]
with
\[
B_{ij} = \begin{pmatrix}
\Sigma^{+1}_Q & \frac{1}{\sqrt{2}} \Sigma^0_Q & \frac{1}{\sqrt{2}} \Xi^{+\frac{1}{2}}_Q \\
\frac{1}{\sqrt{2}} \Sigma^0_Q & \Sigma^{-1}_Q & \frac{1}{\sqrt{2}} \Xi^{-\frac{3}{2}}_Q \\
\frac{1}{\sqrt{2}} \Xi^{+\frac{1}{2}}_Q & \frac{1}{\sqrt{2}} \Xi^{-\frac{3}{2}}_Q & \Omega_Q
\end{pmatrix}_{ij},
\]
and similarly for the \(B^*_\mu_{ij}\). Above \(v^\mu\) is the velocity vector of the heavy hadron, and we suppress velocity labels on all heavy hadron fields throughout. Again the superscript on these states labels the 3-projection of isospin. The first floor of level A contains nine baryons that transform as a \((3, 1, 3)\), while the ground floor of level B contains nine baryons that transform as a \((3, 3, 1)\). The remaining floors and levels are not necessary for our calculation.
The free Lagrangian for the $\mathcal{T}$ and $S^\mu$ fields is given by

$$\mathcal{L} = -i \left( \mathcal{S}^\mu v \cdot D S_\mu \right) + \Delta \left( \mathcal{S}^\mu S_\mu \right) + \lambda_1 \left( S^\mu M_+ S_\mu \right) + \lambda_2 \left( S^\mu S_\mu \right) \text{str} M_+ + i \left( \mathcal{T}^\mu v \cdot D \mathcal{T} \right) + \lambda_3 \left( \mathcal{T} M_+ \mathcal{T} \right) + \lambda_4 \left( \mathcal{T} \mathcal{T} \right) \text{str} M_+.$$  \hspace{1cm} (18)

We have employed $()$-notation for flavor contractions that are invariant under chiral transformations. The relevant contractions can be found in [29]. The mass of the $\mathcal{T}$ field has been absorbed into the static phase of the heavy hadron fields. Thus the leading-order mass splitting $\Delta$ appears as the mass of the $S^\mu$. This splitting remains finite as $\{m_Q \to \infty, m_q \to 0\}$ and cannot be removed by field redefinitions due to the interaction of the $\mathcal{T}$ and $S^\mu$ fields. We treat $\Delta \sim m_\pi$ in our power counting. The Lagrangian contains the chiral-covariant derivative $D^\mu$, the action of which is identical on $\mathcal{T}$ and $S^\mu$ fields and has the form

$$(D^\mu T)_{ij} = \partial^\mu T_{ij} + V^\mu_{ii} T_{ij} + (-)^{n_i(n_j + n_{j'})} V^\mu_{jj'} T_{ij'}.$$  \hspace{1cm} (19)

The vector and axial-vector meson fields are defined by

$$V^\mu = \frac{1}{2} \left( \xi \partial^\mu \xi^\dagger + \xi^\dagger \partial^\mu \xi \right), \quad A^\mu = \frac{i}{2} \left( \xi \partial^\mu \xi^\dagger - \xi^\dagger \partial^\mu \xi \right).$$  \hspace{1cm} (20)

The mass operator has the usual definition

$$M_+ = \frac{1}{2} \left( \xi m_q \xi + \xi^\dagger m_q \xi^\dagger \right).$$  \hspace{1cm} (21)

Adding electromagnetic interactions to the covariant derivative can be accomplished by the replacement $V^\mu_{ij} \to V^\mu_{ij} + ieA^\mu(Q_Q \delta_{ij}/2 + Q_{ij})$ in Eq. (20), where $Q_Q$ is the heavy quark charge and the factor of one-half normalizes the action of the Kronecker delta. Gauging of the axial-vector fields is not necessary to the order we work.

The Lagrangian that describes the interactions of the $\mathcal{T}$ and $S^\mu$ fields with the pseudo-Goldstone modes is given at leading order by the Lagrangian

$$\mathcal{L} = ig_2 \left( \mathcal{S}^\mu v^\nu A^\rho S^\sigma \right) \varepsilon_{\mu\nu\rho\sigma} + \sqrt{2} g_3 \left[ \left( \mathcal{T} A^\mu S_\mu \right) + \left( \mathcal{S}^\mu A_\mu \mathcal{T} \right) \right].$$  \hspace{1cm} (22)

The LECs appearing above, the $\lambda_j$, $g_2$, and $g_3$, all have the same numerical values as those used in $SU(3)$ heavy hadron $\chi$PT. This is described in Appendix [13].

III. ELECTROMAGNETIC PROPERTIES OF THE $s_\ell = 0$ BARYONS

In this section, we calculate the charge radii of the $s_\ell = 0$ baryons in PQ$\chi$PT to leading order in the heavy quark expansion. Additionally the magnetic moments are calculated to next-to-leading order in the heavy quark expansion in order to ascertain the leading light-quark mass dependence. Baryon matrix elements of the electromagnetic current $J^\mu$ can be parametrized in terms of two form factors $F_1$ and $F_2$. In the heavy hadron formalism, such a decomposition is

$$\langle T(p')|J^\mu|T(p)\rangle = \bar{u}(p') \left[ v^\mu F_1(q^2) + \frac{i\sigma^\mu \sigma^\nu}{2M_T} F_2(q^2) \right] u(p),$$  \hspace{1cm} (23)

with $q = p' - p$ as the momentum transfer. The Dirac form factor $F_1(q^2)$ is normalized at zero momentum transfer to the baryon charge $Q_T$ in units of the electron charge $e$, namely
\( F_1(0) = Q_T \). The baryon charge is a sum of the heavy quark charge \( Q_Q \), and the total charge of the light degrees of freedom \( Q_T \), i.e. \( Q_T = Q_Q + Q_T \). Keep in mind that the light-quark electric charge matrix leaves the valence quarks with arbitrary charges, hence \( Q_T \) and \( Q_T \) need not have their physical values. Ignoring the Dirac contribution, the magnetic moment is given by

\[
\mu = F_2(0),
\]

and the Dirac charge radius by

\[
\langle r^2 \rangle = 6 \frac{d}{dq^2} F_1(q^2) \bigg|_{q^2=0}.
\]

These electromagnetic observables receive two types of contributions: short range contributions, which arise from local operators in the effective theory; and long range contributions, which arise from loop graphs involving the pseudo-Goldstone modes. The local operators characterizing the short distance electromagnetic interactions of heavy quark baryons come in two forms. First there are electromagnetic operators corresponding to the current interacting with the heavy quark. These operators can be found by matching the HQET operators onto the chiral theory \([46, 47]\) and thus have fixed coefficients. Secondly there are operators which characterize the short distance interaction of the current with the light degrees of freedom. Chiral symmetry dictates only the form of these operators, leaving them with undetermined LECs.

The HQET part of the short distance dynamics comes from the anomalous magnetic moment and charge radius of the heavy quark. In PQ\(\chi\)PT the magnetic moment operator is matched in the \( s_\ell = 0 \) baryon sector onto the term \([46, 47]\)

\[
\mathcal{L} = -\frac{e \mu_Q}{4m_Q} \left( \overline{T} \sigma_{\mu\nu} T \right) F^{\mu\nu},
\]

where the heavy quark magnetic moment \( \mu_Q \) consists of the Dirac piece plus the anomalous part, i.e. \( \mu_Q = Q_Q + \frac{2\alpha_s(m_Q)}{3\pi} + \ldots \). The charge radius contribution from the photon coupling to the heavy quark arises from the electromagnetic Darwin term \([48, 49]\) that appears in the HQET Lagrangian as

\[
\mathcal{L} = -\frac{e Q_Q}{8m_Q^2} \overline{T} v_\mu D_\nu F^{\mu\nu} Q.
\]

In the \( s_\ell = 0 \) baryon sector, this HQET operator matches onto the PQ\(\chi\)PT term

\[
\mathcal{L} = -\frac{e Q_Q}{8m_Q^2} \left( \overline{T} T \right) v_\mu \partial_\nu F^{\mu\nu},
\]

which contributes to the baryon charge radii.

Now we assess short-distance contributions to electromagnetic observables from the light degrees of freedom. In PQ\(\chi\)PT the magnetic moment contribution from the light quarks is suppressed by \( \Lambda_{QCD}/m_Q \) because the light degrees of freedom have \( s_\ell = 0 \). Thus the leading magnetic moment operator has the form

\[
\mathcal{L} = -\frac{e \mu_T \Lambda_{QCD}}{4\Lambda_\chi m_Q} \left( \overline{T} Q \sigma_{\mu\nu} T \right) F^{\mu\nu}.
\]

\(^2\) We use \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \) for the electromagnetic field-strength tensor.
FIG. 1: Loop diagrams contributing to the charge radii of the $s_\ell = 0$ baryons. Mesons are denoted by a dashed line, flavor neutrals (hairpins) by a crossed dashed line, and the photon by a wiggly line. A thin (thick) solid line denotes an $s_\ell = 0$ ($s_\ell = 1$) baryon. The diagrams in the first row contribute to the Dirac form factor. The remaining diagrams with a photon have no $q^2$-dependence. These, along with the wavefunction renormalization diagrams in the last two rows, ensure non-renormalization of the electric charge.

The local contributions in PQ$\chi$PT to the charge radii of the $s_\ell = 0$ baryons arise from the dimension-six operator that is contained in the Lagrangian

$$\mathcal{L} = -\frac{ec_T}{\Lambda^2_\chi} (\overline{T Q T}) v_\mu \partial_\nu F^{\mu\nu}.$$  \hfill (30)

Although the quark charges are arbitrary, the LECs $\mu_T$ and $c_T$ of these two $SU(6|3)$ PQ$\chi$PT operators have the same numerical values as the LECs in three flavor $\chi$PT. We make this connection in Appendix B.

The long distance contributions to these electromagnetic observables arise from the one-loop diagrams depicted in Fig. 1. The diagrams involve the pseudo-Goldstone mesons and the vertices are generated from the leading-order interaction Lagrangian Eq. (22). These loops, however, do not contribute to the magnetic moments of the $s_\ell = 0$ baryons. The leading light-quark mass dependence of the magnetic moments enters from heavy quark symmetry breaking terms. We shall address these terms below. By contrast, the charge radii receive contributions from the one-loop diagrams depicted in the figure. These diagrams are divergent and the resultant scale dependence is absorbed by a counterterm of the form in Eq. (30). In this calculation and throughout this work, we use dimensional regularization with a modified minimal subtraction scheme, where we have only subtracted
terms proportional to
\[ \frac{1}{\varepsilon} - \gamma_E + 1 + \log 4\pi. \] (31)

To compactly write out the \( q^2 \)-dependence of the form factors, we define the abbreviation
\[ P_\phi = \sqrt{1 - x(1 - x) \frac{q^2}{m_\phi^2}}. \] (32)

Combining the local and loop contributions, for the Dirac form factor we find
\[ F_1(q^2) = Q_T + \left( Q_Q m_Q^2 + Q_T \frac{c_T}{A^2} \right) q^2 \]
\[ + \frac{1}{8\pi^2 f^2} \sum_\phi \alpha_\phi^T \int_0^1 dx \left[ -\frac{q^2}{6} \log \frac{m_\phi^2}{\mu^2} + 2m_\phi^2 P_\phi \log P_\phi \right] \]
\[ + \frac{3g_3^2}{(4\pi f)^2} \sum_\phi \alpha_\phi^T \int_0^1 dx \left[ J(m_\phi P_\phi, \Delta, \mu) - J(m_\phi, \Delta, \mu) - \frac{2}{9} q^2 \right. \]
\[ \left. - \frac{2}{3} x(1 - x) q^2 G(m_\phi P_\phi, \Delta, \mu) \right]. \] (33)

The non-analytic functions arising from loop integrals have been abbreviated in the above expression by
\[ G(m, \delta, \mu) = \log \frac{m^2}{\mu^2} - \frac{\delta}{\sqrt{\delta^2 - m^2 + i\varepsilon}} \]
\[ \log \frac{\delta - \sqrt{\delta^2 - m^2 + i\varepsilon}}{\delta + \sqrt{\delta^2 - m^2 + i\varepsilon}} \] (34)

and
\[ J(m, \delta, \mu) = (m^2 - 2\delta^2) \log \frac{m^2}{\mu^2} + 2\delta \sqrt{\delta^2 - m^2} \log \frac{\delta - \sqrt{\delta^2 - m^2 + i\varepsilon}}{\delta + \sqrt{\delta^2 - m^2 + i\varepsilon}} \] (35)

The sums in Eq. (33) are over loop mesons \( \phi \) of mass \( m_\phi \). The coefficients \( \alpha_\phi^T \) appear in Table II and depend on the particular \( s_\ell = 0 \) baryon state \( T \). Notice the charges \( q_j \) and \( q_l \) do not explicitly appear in these coefficients. In the isospin limit, they always enter in the combination \( q_j + q_l \) which is the same as \(-q_r\) by supertracelessness of \( Q \).

From the table, we see that with efficacious choices for the charges of valence and sea quarks contributions from certain loop mesons can be eliminated thereby simplifying the chiral extrapolation. For the \( \Lambda_Q \) baryon, we can eliminate all but one loop meson; while for the \( \Xi_Q \) baryons, we can reduce the competition from four loop mesons down to two. Alternately there is a choice of sea charges which decreases computation time. One can preserve the supertracelessness of the charge matrix with the choice \( q_j = q_l = q_r = 0 \). For this choice, the lattice practitioner does not calculate closed quark loops with current insertion. Furthermore either of the valence charges can be taken to zero, and the LECS can still be extracted from the resulting electromagnetic properties. Once the LECSs are known, physical predictions can be made.

For the magnetic moments of the \( s_\ell = 0 \) baryons, the leading light-quark mass dependence arises at next-to-leading order in the heavy quark expansion. Terms contributing to the magnetic moments are due to the mass-splitting of the \( B \) and \( B^* \) baryons [40], as well as
TABLE I: The coefficients $\alpha^T_\phi$ in $SU(6|3)$ PQ$\chi$PT. Coefficients are listed for the $s_\ell = 0$ baryon states $T$, and are grouped into contributions from loop mesons with mass $m_\phi$.

|   |   |   |   |
|---|---|---|---|
| $\Lambda_Q$ | ju | ru | js | rs |
| $\Xi^+ \Lambda_Q$ | $q_u + q_\ell + q_r$ | $\frac{1}{2}(q_u + q_d) - q_r$ | 0 | 0 |
| $\Xi^- \Lambda_Q$ | $q_u + \frac{1}{2}q_r$ | $\frac{1}{2}(q_u - q_r)$ | $q_s + \frac{1}{2}q_r$ | $\frac{1}{2}(q_s - q_r)$ |

FIG. 2: Loop diagrams contributing to the leading light-quark mass dependence of the $s_\ell = 0$ baryons’ magnetic moments. Mesons are denoted by a dashed line, the photon by a wiggly line, and a thin (thick) solid line denotes an $s_\ell = 0$ ($s_\ell = 1$) baryon. Heavy quark symmetry breaking vertices are depicted by squares and arise from terms in Eq. (36).

Aside from these loops, we have additionally contributions from the local operators in Eqs. (26) and (29). Thus at next-to-leading order in the heavy quark expansion, we find the Pauli form factors of the $s_\ell = 0$ baryons to be

$$F_2(q^2) = \frac{M_T}{m_Q} \mu_Q + \mu_T \frac{\Lambda_{QCD} M_T}{\Lambda \chi m_Q} Q_T + \lambda g_3 \frac{\Lambda_{QCD} M_T}{2 \pi^2 f^2 m_Q} \sum_\phi \alpha_\phi^T \int_0^1 dx \mathcal{F}(m_\phi P_\phi, \Delta, \mu)$$

The new non-analytic function $\mathcal{F}(m, \delta, \mu)$ entering above is given by

$$\mathcal{F}(m, \delta, \mu) = -\delta \log \frac{m^2}{\mu^2} + \sqrt{\delta^2 - m^2} \log \frac{\delta - \sqrt{\delta^2 - m^2 + i\varepsilon}}{\delta + \sqrt{\delta^2 - m^2 + i\varepsilon}}.$$

The coefficients $\alpha_\phi^T$ are the same as for $F_1(q^2)$ appearing in Table I and hence the charges can be adjusted in the same manner to simplify the chiral extrapolation or to decrease the lattice simulation time.
IV. ELECTROMAGNETIC PROPERTIES OF THE $s_t = 1$ BARYONS

In this section, we calculate the charge radii, magnetic moments and electric quadrupole moments of the $s_t = 1$ baryons in PQXPT. Similar to the above decomposition for the $s_t = 0$ baryons, the current matrix elements of the spin-$\frac{3}{2}$ $B$ baryons have a decomposition

$$
\langle B(p')|J^{\mu}|B(p)\rangle = \overline{u}(p') \left[ v^{\mu} F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m_B} F_2(q^2) \right] u(p),
$$

where $q = p' - p$ as the momentum transfer. The spin-$\frac{3}{2}$ matrix elements of the electromagnetic current $J^{\mu}$ for the $B^*$ baryons can be parametrized as

$$
\langle B^*_\mu(p')|J^{\rho}|B^*_\nu(p)\rangle = -\overline{u}_\mu(p') \mathcal{O}^{\rho\mu\nu} u_\nu(p),
$$

where $u_\mu(p)$ is a Rarita-Schwinger spin vector for an on-shell heavy baryon. This spin vector satisfies the constraints $v^{\rho} u_\rho(p) = \gamma^{\rho} u_\rho(p) = 0$. The tensor $\mathcal{O}^{\rho\mu\nu}$ can be written in terms of four independent form factors

$$
\mathcal{O}^{\rho\mu\nu} = g^{\rho\mu} \left[ v^{\rho} F_1^*(q^2) + \frac{i\sigma^{\rho\nu} q_\nu}{2m_B} F_2^*(q^2) \right] + \frac{q^\rho q^\nu}{(2m_B)^2} \left[ v^{\rho} G_1^*(q^2) + \frac{i\sigma^{\rho\nu} q_\nu}{2m_B} G_2^*(q^2) \right].
$$

To avoid notational confusion within this section, we have appended a superscript * to denote the $B^*$ baryon form factors. Extraction of these form factors for the $B^*$ baryons requires a non-trivial identity for on-shell Rarita-Schwinger spinors \cite{50}. For our purposes, the identity takes the form

$$
\overline{u}_\alpha(p') (q^\alpha g^{\mu\beta} - q^\beta g^{\mu\alpha}) u_\beta(p) = \overline{u}_\alpha(p') \left( -\frac{q^2}{2m_B} g^{\alpha\beta} v^\mu + ig^{\alpha\beta} \sigma^{\mu\nu} q_\nu + \frac{1}{m_B} q^\alpha q^\beta v^\mu \right) u_\beta(p).
$$

The conversion of the above covariant vertex form factors to multipole form factors is detailed in \cite{50}.

The Dirac form factors $F_1(q^2)$ and $F_1^*(q^2)$ for the $B$ and $B^*$ baryons are normalized at zero momentum transfer to the baryon charge $Q_B$ in units of the electron charge $e$, namely $F_1(0) = F_1^*(0) = Q_B$. The baryon charge is a sum of the heavy quark charge $Q_Q$, and the total charge of the light degrees of freedom $Q_S$, i.e. $Q_B = Q_Q + Q_S$. Keep in mind that the light-quark electric charge matrix leaves the valence quarks with arbitrary charges, hence $Q_S$ and $Q_S$ need not have their physical values. The Dirac charge radii for have the form $\langle r^2 \rangle = 6F_1^*(0)$, for the $B$ baryons, and $\langle r^2 \rangle = 6F_1^{*'}(0)$, for the $B^*$ baryons. The magnetic moments of the $B$ and $B^*$ baryons are defined through their Pauli form factors: $\mu = F_2(0)$ and $\mu^* = F_2^*(0)$, respectively. The electric quadrupole moments $Q$ of the $B^*$ baryons are given by

$$
Q = -\frac{1}{2} G_2^*(0).
$$

To the order we work in the heavy quark expansion, the form factor $G_2^*(q^2) = 0$ because the light degrees of freedom have $s_t = 1$. Hence the magnetic octupole moments vanish; see Appendix C for further discussion. The relations between the electromagnetic form factors of the $B$ and $B^*$ baryons are also discussed in Appendix C.

The electromagnetic observables of the $B$ and $B^*$ baryons receive short and long range contributions in the chiral effective theory. The short range contributions stem from both
the electromagnetic interaction of the heavy quark and the electromagnetic interaction of the light degrees of freedom. As above, the HQET part arises from the magnetic moment of the heavy quark as well as the electromagnetic Darwin term. These contributions are encoded in the effective theory through the heavy quark magnetic moment operator

\[ L = \frac{e\mu_Q}{4m_Q} (\mathcal{S}^a \sigma_{\mu\nu} S_a) F^{\mu\nu}, \]  

(44)

where \( \mu_Q = Q_Q + \frac{2\alpha_s(m_Q)}{3\pi} + \ldots \), and the heavy quark charge radius operator

\[ L = \frac{eQ_Q}{8m_Q^2} (\mathcal{S}_a S_a) v_\mu \partial_\nu F^{\mu\nu}. \]  

(45)

The electric quadrupole moments of the \( B^* \) baryons cannot receive any contribution from the electromagnetic interactions of the heavy quark.

In PQ\( \chi \)PT the leading contribution from the light degrees of freedom to the magnetic moments of the \( B \) and \( B^* \) baryons is contained in the Lagrangian

\[ \mathcal{L} = \frac{ie\mu_S}{\Lambda_\chi} (\mathcal{S}_\mu \mathcal{Q}_S) F^{\mu\nu}. \]  

(46)

To be complete to the order we work, the LEC \( \mu_S \) must be treated as a linear function of \( \Delta \), i.e. \( \mu_S = \mu_S^{(0)} + \frac{\Delta}{\Lambda_\chi} \mu_S^{(1)} \). Moreover we absorb the finite contributions from loop integrals into the parameter \( \mu_S^{(1)} \). Determination of this LEC requires the ability to vary \( \Delta \) and for this reason we treat the \( \Delta \) dependence only implicitly. The magnetic moment coefficient \( \mu_S \) is the only LEC that must be treated in this fashion to the order we are working. The charge radii receive local contributions from the dimension-six operator contained in the term

\[ \mathcal{L} = \frac{ecS}{\Lambda^2_\chi} (\mathcal{S}^a \mathcal{Q}_S a) v_\mu \partial_\nu F^{\mu\nu}, \]  

(47)

and finally the electric quadrupole moments arise from

\[ \mathcal{L} = -\frac{eQ_S}{\Lambda^2_\chi} (\mathcal{S}^{\{\mu} \mathcal{Q}_S S^{\nu\}}) v_\alpha \partial_\mu F^{\alpha\nu}. \]  

(48)

The brackets \( \{ \ldots \} \) produce the symmetric traceless part of Lorentz tensors, i.e. \( \mathcal{O}^{(\mu\nu)} = \mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu} - \frac{1}{2} g^{\mu\nu} \mathcal{O}_a \). The magnetic octupole moment operator enters at \( \mathcal{O}(1/m_Q\Lambda_\chi^3) \) which is beyond the order we work. Although the quark electric charges are arbitrary, the electromagnetic LECs for the \( s_t = 1 \) baryons, \( \mu_S, c_S \) and \( Q_S \), have the same numerical values as in \( SU(3) \chiPT \), see Appendix B.

The long distance contributions to the electromagnetic observables of the \( B \) and \( B^* \) baryons arise from the one-loop diagrams depicted in Fig. IV. These diagrams involve the pseudo-Goldstone mesons which enter from the vertices generated by the leading-order interaction Lagrangian Eq. (22). Calculation of these diagrams along with the tree-level
FIG. 3: Loop diagrams contributing to the electromagnetic moments and charge radii of the \( s_\ell = 1 \) baryons. Mesons are denoted by a dashed line, flavor neutrals (hairpins) by a crossed dashed line, and the photon by a wiggly line. A thick (thin) solid line denotes an \( s_\ell = 1 \) (\( s_\ell = 0 \)) baryon. The diagrams in the first row contribute to the electromagnetic moments and charge radii. The remaining diagrams with a photon have no \( q^2 \)-dependence. These, along with the wavefunction renormalization diagrams in the last two rows, ensure non-renormalization of the electric charge.

TABLE II: The coefficients \( \alpha_\phi^{B} \) in \( SU(6|3) \) PQ\( \chi \)PT. Coefficients are listed for the \( s_\ell = 1 \) baryon states \( B \), and are grouped into contributions from loop mesons with mass \( m_\phi \).

| \( B \)     | \( j_u \) | \( r_u \) | \( j_s \) | \( r_s \) |
|------------|-----------|-----------|-----------|-----------|
| \( \Sigma_{Q}^{+1} \) | \( 2q_u + q_r \) | \( q_u - q_r \) | 0         | 0         |
| \( \Sigma_{Q}^{0} \)  | \( q_u + q_d + q_r \) | \( \frac{1}{2}(q_u + q_d) - q_r \) | 0         | 0         |
| \( \Sigma_{Q}^{-1} \)  | \( 2q_d + q_r \) | \( q_d - q_r \) | 0         | 0         |
| \( \Xi_{Q}^{+1/2} \)   | \( q_u + \frac{1}{2}q_r \) | \( \frac{1}{2}(q_u - q_r) \) | \( q_s + \frac{1}{2}q_r \) | \( \frac{1}{2}(q_s - q_r) \) |
| \( \Xi_{Q}^{-1/2} \)   | \( q_d + \frac{1}{2}q_r \) | \( \frac{1}{2}(q_d - q_r) \) | \( q_s + \frac{1}{2}q_r \) | \( \frac{1}{2}(q_s - q_r) \) |
| \( \Omega_{Q} \)       | 0         | 0         | 2q_s + q_r | q_s - q_r |
contributions for the $B$ baryons yields

\[
F_1(q^2) = Q_B + \left( Q_Q \frac{1}{8m_Q^2} + Q_S c_s \frac{1}{\Lambda^2_x} - Q_S \frac{\mu_s}{2\Lambda_x M_B} - Q_S \frac{Q_S}{2\Lambda^2_x} \right) q^2
\]

\[
+ \frac{1}{8\pi^2 f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left[ -\frac{q^2}{6} \log \frac{m^2_{\phi}}{\mu^2} + 2m^2_{\phi} P^2_{\phi} \log P_{\phi} \right]
\]

\[
+ \frac{g^2_{\phi}}{(4\pi f)^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left\{ 2 \left[ m^2_{\phi} - \frac{5}{3} x(1-x)q^2 \right] \log P_{\phi} - \frac{5}{18} q^2 \left[ \log \frac{m^2_{\phi}}{\mu^2} + 1 \right] \right\}
\]

\[
+ \frac{g^2_{\phi}}{(4\pi f)^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left[ J(m_{\phi} P_{\phi}, -\Delta, \mu) - J(m_{\phi}, -\Delta, \mu) + \frac{1}{9} q^2
\]

\[- \frac{2}{3} x(1-x)q^2 G(m_{\phi} P_{\phi}, -\Delta, \mu) \right] \]

(49)

for the Dirac form factor and

\[
F_2(q^2) = \frac{M_B}{m_Q} \mu_Q + \mu_S \frac{4M_B}{3\Lambda_x} Q_S + \frac{g^2_{\phi} M_B}{24\pi f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx m_{\phi} P_{\phi}
\]

\[
+ \frac{g^2_{\phi}}{12\pi^2 f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx F(m_{\phi} P_{\phi}, -\Delta, \mu) \]

(50)

for the Pauli form factor. The function $F(m, \delta, \mu)$ appearing above is given in Eq. (38). In the above expressions, the sums run over loop mesons $\phi$ with mass $m_{\phi}$. The coefficients $\alpha^B_{\phi}$ are listed in Table II and depend on the particular baryon state $B$. Notice the charges $q_j$ and $q_l$ do not explicitly appear in these coefficients. In the isospin limit, they always enter in the combination $q_j + q_l$ which is the same as $-q_r$ by supertracelessness of $Q$.

Carrying out the calculation for the $B^*$ baryons, we obtain

\[
F_1^*(q^2) = Q_B + \left( Q_Q \frac{1}{8m^2_Q} + Q_S c_s \frac{1}{\Lambda^2_x} - Q_S \frac{\mu_s}{2\Lambda_x M_B} - Q_S \frac{Q_S}{2\Lambda^2_x} \right) q^2
\]

\[
+ \frac{1}{8\pi^2 f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left[ -\frac{q^2}{6} \log \frac{m^2_{\phi}}{\mu^2} + 2m^2_{\phi} P^2_{\phi} \log P_{\phi} \right]
\]

\[
+ \frac{g^2_{\phi}}{(4\pi f)^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left\{ 2 \left[ m^2_{\phi} - 2x(1-x)q^2 \right] \log P_{\phi} - \frac{1}{3} q^2 \left[ \log \frac{m^2_{\phi}}{\mu^2} + 1 \right] \right\}
\]

\[
+ \frac{g^2_{\phi}}{(4\pi f)^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx \left[ J(m_{\phi} P_{\phi}, -\Delta, \mu) - J(m_{\phi}, -\Delta, \mu) \right] \]

(51)

for the Dirac form factor,

\[
F_2^*(q^2) = \frac{M_B}{m_Q} \mu_Q + \mu_S \frac{2M_B}{\Lambda_x} Q_S + \frac{g^2_{\phi} M_B}{16\pi f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx m_{\phi} P_{\phi}
\]

\[
+ \frac{g^2_{\phi}}{8\pi^2 f^2} \sum_{\phi} \alpha^B_{\phi} \int_0^1 dx F(m_{\phi} P_{\phi}, -\Delta, \mu) \]

(52)
for the Pauli form factor, and finally

\begin{align}
G_1^*(q^2) &= 4Q_S \frac{M_B}{\Lambda_x} \left( \mu_S + 2Q_S \frac{M_B}{\Lambda_x} \right) \\
&\quad + \frac{g_2^2 M_B^2}{4\pi^2 f^2} \sum_\phi \alpha_B^\phi \int_0^1 dx \left\{ 2x(1-x) \log P_\phi + \frac{1}{6} \left[ \log \frac{m_\phi^2}{\mu^2} + 1 \right] \right\} \\
&\quad - \frac{g_2^2 M_B^2}{2\pi^2 f^2} \sum_\phi \alpha_B^\phi \int_0^1 dx \left[ x(1-x) \mathcal{G}(m_\phi P_\phi, -\Delta, \mu) + \frac{1}{6} \right].
\end{align}

(53)

The coefficients $\alpha_B^\phi$ appearing in the expressions for the $B^*$ baryon form factors are listed in Table II and are identical for baryon states $B$ and $B^*$. Taking the $m_Q \to \infty$ limit, the relations between the $B$ and $B^*$ electromagnetic form factors (derived in Appendix C) are satisfied by our one-loop PQ$\chi$PT results.

As with the $s_\ell = 0$ baryons, Table III shows that the chiral extrapolations can be simplified with efficacious choices for the charges of valence and sea quarks. For the $\Sigma_Q$ and $\Omega_Q$ baryons, we can eliminate all but one loop meson; while for the $\Xi_Q^0$ baryons, we can reduce the contributions from four loop mesons down to two. In $SU(6|3)$ one can again decrease the computation time by choosing the sea quarks to have vanishing electric charges as this preserves the supertracelessness of the charge matrix. Thus all closed quark loops with a current insertion vanish. Furthermore for the $\Sigma_Q^0$, and $\Xi_Q^0$ baryons, either of the contributing valence charges can be taken to zero, and the LECs can still be extracted from the lattice data. For the $\Sigma_Q^{s=1}$, $\Sigma_Q^{-1}$, and $\Omega_Q$ baryons, this choice is not possible since these states have two valence quarks of the same flavor. Nonetheless, LECs extracted from calculations involving one baryon can be used to test PQ$\chi$PT predictions for the remaining baryons.

V. SUMMARY

Above we have calculated the electromagnetic properties of baryons containing a heavy quark. For baryons with light degrees of freedom $s_\ell = 0$, we obtained the charge radii at NLO in the chiral expansion. The magnetic moments of these baryons vanish at leading order in the heavy-quark expansion and thus to obtain the leading light-quark mass dependence we worked to NLO in the heavy quark expansion. For the $s_\ell = 1$ baryons, we obtained the charge, magnetic moment and electric quadrupole moment form factors at NLO in the chiral expansion. Expressions for these quantities are derived in PQ$\chi$PT for three light flavors in the main text, and two light flavors in Appendix A. Additionally the corresponding $\chi$PT results for three and two light flavors are presented in Appendix B. Knowledge of the quark mass dependence of these observables is essential to extrapolate lattice QCD data to the physical light quark masses. Additionally the effective field theory predicts the momentum transfer dependence of the electromagnetic form factors near zero recoil. This information should be utilized to perform the zero-momentum extrapolation of lattice data in order to extract static electromagnetic properties. Furthermore there are relations between the various sextet baryon electromagnetic form factors that follow from the decoupling of the heavy and light spins; these are found in Appendix C.

There are a number of interesting points that arise when considering the electromagnetic properties of baryons containing a heavy quark as compared to baryons containing only light quarks. Firstly the choice of the heavy quark lattice action allows for more freedom in
separating contributions to baryon properties. For the octet and decuplet baryons formed from three light quarks, in order to have a consistent power counting the average inverse baryon mass $1/M$ becomes an expansion parameter $[51, 52]$. On the lattice, however, there is no direct control over this expansion, whereas the choice of the HQET action leads to control over the $1/m_Q$ expansion. This in turn can provide information about the chiral regime. Consider the $s_\ell = 0$ baryons. The form factors $F_1(q^2)$ and $F_2(q^2)$ enter at different orders in the heavy-quark expansion. Thus if the light-quark mass dependence from $\chi$PT can be trusted, then increasing the heavy quark mass should diminish the variation of $F_2(q^2)$ and isolate the leading loop contributions to $F_1(q^2)$. If this behavior is not observed, one is certainly not in the chiral regime.

A further contrast to the baryons with only light quarks, is the case of the $\Lambda_Q$ and $\Sigma_Q^0$ baryons. In $SU(2)$ flavor, the loop contributions to the electromagnetic properties of these baryons vanish due to cancellations between $\pi^+$ and $\pi^-$ loops. Such complete cancellations do not occur for the nucleons and deltas. From the point of view of chiral physics, this is perhaps uninteresting as there is no quark mass dependence. This situation is true at NLO in the heavy quark expansion as well, because the flavor structure of the heavy quark symmetry breaking operators is identical to the LO operators. If there is observed quark mass variation of the electromagnetic properties of the $\Lambda_Q$ and $\Sigma_Q^0$, then one is not in the chiral regime. Even though there is no quark-mass dependence in QCD, the couplings $g_2$ and $g_3$ can still be extracted from considering the $\Lambda_Q$ and $\Sigma_Q^0$ in PQ$\chi$PT, because the arbitrary quark electric charges can be chosen so that loop diagrams do contribute.

A feature of three-flavor electromagnetism, is the tracelessness of the charge matrix. This carries over into the partially quenched theory as supertracelessness, and enables the LECs to be extracted from lattice data in which the sea quarks have vanishing charges. This reduces the computation time, as closed quark loops with current insertion vanish. We have accordingly used a charge matrix which clearly separates valence and sea contributions; see Appendix D. Similar electroweak operators should be utilized for other observables.

Lattice QCD calculations will enable an understanding of the electromagnetic properties of hadrons in terms of quark and gluon degrees of freedom. These observables paint an intuitive picture of the low-energy electromagnetic structure of hadrons. Foreseeable lattice calculations of these properties, however, will require extrapolation in the light quark masses. The expressions derived in this paper for heavy hadrons in $\chi$PT and PQ$\chi$PT allow such extrapolations, as well as the zero-momentum extrapolation. The PQ$\chi$PT expressions are not complicated by new non-analytic functions arising from the flavor-neutral propagator. Moreover a study of the electromagnetic properties of baryons containing a heavy quark allows the exploration of our ability to extrapolate lattice QCD on two fronts: the light-quark regime and the heavy-quark regime. Control over the latter scale is not a feature for nucleons and deltas treated as heavy baryons. Lastly the exploitation of arbitrary electroweak quark charges allows for calculational simplifications to extract physical low-energy constants.

---

3 There is the case of the spin-$3/2$ $\Sigma^{*0}$ baryon in SU(3) for which the pion and kaon loops individually sum to zero. The local electromagnetic operators, however, are all proportional to the baryon charge, thus to one-loop order all electromagnetic observables of the $\Sigma^{*0}$ vanish.
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APPENDIX A: ELECTROMAGNETIC PROPERTIES IN SU(4|2)

Heavy hadron chiral perturbation theory in the baryon sector for SU(4|2) parallels that of SU(6|3) in Sec. III. The partially quenched heavy hadron Lagrangian has been written down in [29]. In this appendix, we briefly review the setup of SU(4|2) PQQCD and PQχPT. We then present the calculation of heavy hadron electromagnetic properties for this graded flavor group.

1. PQχPT for SU(4|2)

In SU(4|2) PQQCD, the light-quark sector Lagrangian appears as

\[ \mathcal{L} = \sum_{j,k=1}^{6} \bar{q}_j (i \gamma - m_q)_{jk} q_k, \]  

(A1)

and differs from the usual QCD Lagrangian by the inclusion of four extra quarks; two bosonic ghost quarks, (\(\tilde{u}, \tilde{d}\)), and two fermionic sea quarks, (\(j, l\)), in addition to the light physical quarks (\(u, d\)). The six quark fields transform in the fundamental representation of \(SU(4|2)\). They appear in the six-component vector

\[ q = (u, d, j, l, \tilde{u}, \tilde{d})^T. \]  

(A2)

The quark fields obey the graded equal-time commutation relations in Eq. (3) but with the grading factor \(\eta_k\) now defined by

\[ \eta_k = \begin{cases} 
1 & \text{for } k = 1, 2, 3, 4 \\
0 & \text{for } k = 5, 6 
\end{cases}. \]  

(A3)

The \(SU(4|2)\) mass matrix with non-degenerate quarks is given by

\[ m_q = \text{diag}(m_u, m_d, m_j, m_l, m_{\tilde{u}}, m_{\tilde{d}}), \]  

(A4)

where the ghost quarks remain degenerate with their valence partners. Defining ghost and sea quark charges is constrained only be the restriction that QCD be recovered in the limit of appropriately degenerate quark masses. The general form of the charge matrix we choose is

\[ \mathcal{Q} = \text{diag}(q_u, q_d, q_j, q_l, q_{\tilde{u}}, q_{\tilde{d}}), \]  

(A5)

which is not supertraceless. In the limit \(m_j \to m_u\) and \(m_l \to m_d\), QCD with two light flavors is recovered only for the values \(q_u = q_j = \frac{2}{3}\) and \(q_{\tilde{u}} = q_{\tilde{d}} = -\frac{1}{3}\). The results for and problems with the commonly used partially-quenched charge matrix are presented in Appendix D.
For massless light quarks, the Lagrangian in Eq. (A1) has a graded symmetry $SU(4|2)_L \otimes SU(4|2)_R \otimes U(1)_V$, which is assumed to be spontaneously broken to $SU(4|2)_V \otimes U(1)_V$. The Lagrangian describing the pseudo-Goldstone mesons of this theory is identical in form to Eq. (7). The meson fields appearing in Eq. (9) are replaced by an $SU(4|2)_V \otimes U(1)_V$ matrix $\Phi$. The flavor singlet field that appears in $SU(4|2)_L$ is defined to be $\Phi_0 = \text{str}(\Phi)/\sqrt{2}$. As before, the mass of the singlet field $m_0$ can be taken to be on the order of the chiral symmetry breaking scale.

In this limit, the two-point correlation functions deviate from their familiar form in $\chi$PT, and their explicit forms are not needed for expressing our results.

To include baryons with one heavy quark into the theory, we use the same interpolating fields, Eqs. (11) and (14). The $s_\ell = 0$ baryons are then described by the field $T_{ij}$ which forms a 17-dimensional representation of $SU(4|2)$. The baryon tensor of QCD $T_{ij}$ is contained as $T_{ij} = T_{ij}$, when the indices are restricted to the range 1–2. In our conventions, we have

$$T_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \Lambda Q \\ -\Lambda Q & 0 \end{pmatrix}_{ij}. \tag{A6}$$

The $s_\ell = 1$ baryons are described by $S_{ij}^\mu$ which makes up a 19-dimensional representation of $SU(4|2)$. The baryon tensor of QCD $S_{ij}^\mu$ is embedded as $S_{ij}^\mu = S_{ij}^\mu$, when the indices are restricted to the range 1–2. Here the QCD flavor tensor $S_{ij}^\mu$ is given in terms of $B_{ij}$ and $B^*_{ij}$ as in Eq. (16), but with

$$B_{ij} = \begin{pmatrix} \Sigma^0_Q & \frac{1}{\sqrt{2}} \Sigma^0_Q \\ \frac{1}{\sqrt{2}} \Sigma^0_Q & \Sigma^{-1}_Q \end{pmatrix}_{ij}, \tag{A7}$$

and similarly for $B^*_{ij}$. The superscript on these states labels the 3-projection of isospin. The remaining states relevant to our calculation have been classified in [29].

The free Lagrangian for the $\mathcal{T}$ and $\mathcal{S}^\mu$ fields is the same as Eq. (18) and the Lagrangian that describes the interactions of these fields with the pseudo-Goldstone modes is given by Eq. (22). The LECs appearing in the Lagrangian, $\lambda_1$, $\lambda_2$, $g_2$, and $g_3$, all have the same numerical values as those used in $SU(2)$ heavy hadron $\chi$PT, see Appendix [3]. These are of course numerically different than those in $SU(3)$. The parameters $\lambda_3^{(PQ)}$ and $\lambda_4^{(PQ)}$ are different and can be related to the unquenched coefficient $\lambda_4$ by matching, namely $\lambda_4 = \frac{1}{2} \lambda_3^{(PQ)} + \lambda_4^{(PQ)}$.

2. Baryon electromagnetic properties

Calculation of the baryon electromagnetic properties in $SU(4|2)$ is similar to that in $SU(6|3)$ because the Lagrangian has the same structure. There is, however, one crucial difference because the charge matrix is not supertraceless and hence there are new operators contributing at tree level to each of the electromagnetic properties.

For the Dirac form factor of the $s_\ell = 0$ baryons, there is an additional operator that contributes at tree level, namely

$$\mathcal{L} = -\frac{\bar{e}_\mathcal{T}}{\Lambda_\chi^2} (\mathcal{T}^\mu \mathcal{T}_\mu) \mathcal{Q}. \tag{A8}$$
TABLE III: The coefficients $\alpha_T^\phi$ for the $\Lambda_Q$ in $SU(4|2)$ PQ$\chi$PT. Coefficients are grouped into contributions from loop mesons with mass $m_\phi$.

| $\Lambda_Q$ | $\frac{1}{2}(q_u - q_j)$ | $\frac{1}{2}(q_u - q_l)$ | $\frac{1}{2}(q_d - q_j)$ | $\frac{1}{2}(q_d - q_l)$ |
|-------------|---------------------------|---------------------------|---------------------------|---------------------------|

The contribution of this operator to the Dirac form factor of the $\Lambda_Q$ baryon is

$$\delta F_1(q^2) = \frac{q_{jl} \bar{c}_T \Lambda^2}{\Lambda^2 m_Q} q^2,$$

where $q_{jl} = q_j + q_l$ is the supertrace of the charge matrix. The remainder of the form factor is given in Eq. (33), however the coefficients $\alpha_T^\phi$ are now listed for the $\Lambda_Q$ baryon for $SU(4|2)$ in Table III. For the magnetic moment of the $\Lambda_Q$, we have the additional operator

$$\mathcal{L} = -\frac{e\bar{T}_T \Lambda_{QCD}}{4\Lambda^2 m_Q} (\bar{T}_\mu \sigma_{\mu\nu} T) F_{\mu\nu} \text{str} Q,$$

which leads to a contribution to the Pauli form factor

$$\delta F_2(q^2) = q_{jl} \bar{\mu}_T \frac{\Lambda_{QCD} M_T}{\Lambda^2 m_Q},$$

and the remainder of the Pauli form factor is given in Eq. (37) with the coefficients $\alpha_T^\phi$ for $SU(4|2)$.

From Table III we see that the charges of the valence and sea quarks can be chosen so that there are no loop contributions at this order. This dramatically simplifies the partially quenched chiral extrapolation and can be anticipated from the $SU(2)$ result, see Appendix B. One can choose $Q$ to be supertraceless to sort out $c_T$ from $\bar{c}_T$, and $\mu_T$ from $\bar{\mu}_T$. The specific choice $q_j = q_l = 0$ is possible, but the result is not sensitive to all the LECs. Thus in $SU(4|2)$ quark disconnected contributions with current insertion must be dealt with to ascertain all LECs and hence make physical predictions.

Considering next the $s_\ell = 1$ baryon electromagnetic properties, we must add similar types of operators involving the supertrace of the charge matrix. These operators are

$$\mathcal{L} = \frac{ie\bar{S}_{\mu} \bar{S}_{\nu} \Lambda}{\Lambda^2} F^{\mu\nu} \text{str} Q + \frac{e\bar{c}_S}{\Lambda^2} (\bar{S}^\alpha S_\alpha) v_\mu \partial_\nu F^{\mu\nu} \text{str} Q - \frac{e Q_S}{\Lambda^2} (\bar{S}^{\mu\nu}) v_\mu \partial_\nu F^{\mu\nu} \text{str} Q.
$$

(A12)

For the spin-$\frac{1}{2}$ $B$ baryons, the contributions to the form factors from the above terms are

$$\delta F_1(q^2) = \left( \frac{\bar{c}_S}{\Lambda^2} + \frac{Q_S}{6\Lambda^2} \right) q_{jl} q^2$$

$$\delta F_2(q^2) = \frac{4M_B}{3\Lambda^2} q_{jl} \bar{\mu}_S.
$$

(A13)

The remaining parts of the Dirac and Pauli form factors are the same as in Eqs. (49) and (50), respectively. The coefficients $\alpha^B_\phi$ for $SU(4|2)$ are listed in Table IV for the $\Sigma_Q$ baryons.
TABLE IV: The coefficients $\alpha_{B}^{\phi}$ in $SU(4|2)$ PQ$\chi$PT. Coefficients are listed for the $s_{\ell} = 1$ baryon states $B$, and are grouped into contributions from loop mesons with mass $m_{\phi}$.

| $\Sigma_{Q}^{\pm,0,\pm1}$ | $j_{u}$ | $l_{u}$ | $j_{d}$ | $l_{d}$ |
|--------------------------|--------|--------|--------|--------|
| $\Sigma_{Q}^{+}$        | $q_{u} - q_{j}$ | $q_{u} - q_{l}$ | $0$    | $0$    |
| $\Sigma_{Q}^{-}$        | $\frac{1}{2}(q_{u} - q_{j})$ | $\frac{1}{2}(q_{u} - q_{l})$ | $\frac{1}{2}(q_{d} - q_{j})$ | $\frac{1}{2}(q_{d} - q_{l})$ |
| $\Sigma_{Q}^{0}$        | $0$    | $0$    | $q_{d} - q_{j}$ | $q_{d} - q_{l}$ |

Lastly, for the spin-$\frac{3}{2}$ $B^{*}$ baryons, the contributions to the electromagnetic form factors from the above operators are

$$\delta F_{1}^{*}(q^{2}) = \left(\tau_{S} - \frac{\mu_{S}}{2\Lambda_{\chi}M_{B}} - \frac{\Omega_{S}}{2\Lambda_{\chi}^{2}}\right) q_{jl} q^{2}$$
$$\delta F_{2}^{*}(q^{2}) = \frac{2M_{B}}{\Lambda_{\chi}} q_{jl} \mu_{S}$$
$$\delta G_{1}^{*}(q^{2}) = \frac{4M_{B}}{\Lambda_{\chi}} \left(\mu_{S} + 2\Omega_{S} \frac{M_{B}}{\Lambda_{\chi}}\right) q_{jl},$$

with the remaining parts of these three form factors given in Eqs. (A11), (A52) and (A53) with the coefficients $\alpha_{B}^{\phi}$ listed in Table IV.

From Table IV we see that for each $\Sigma_{Q}$ baryon the charges of the valence and sea quarks can be chosen so that there are no loop contributions at this order. Consequently there is a simple partially-quenched chiral extrapolation. Compared with the $SU(2)$ result, see Appendix B, only the $\Sigma_{Q}^{0}$ has this simplification. One can choose $Q$ to be supertraceless to sort out $c_{S}$ from $\tau_{S}$, etc. The specific choice $q_{j} = q_{l} = 0$ is possible, but again does not lead to results that depend on all LECs. Quark disconnected contributions with current insertion must be calculated in $SU(4|2)$ to ascertain the relevant LECs and make physical predictions.

APPENDIX B: ELECTROMAGNETIC PROPERTIES IN SU(3) AND SU(2)

For completeness, we include calculations of the baryon electromagnetic properties in $\chi$PT. The magnetic moments were calculated to one-loop order in $[40, 41]$; the remaining properties have not been calculated before. To compare with the main text, we consider the case of $SU(3)$ flavor in the isospin limit and to compare with the results of Appendix A we consider non-degenerate $SU(2)$. For these calculations, we retain the tensors $T_{ij}$ and $S_{ij}^{\mu}$ in $\chi$PT. For the case of $SU(3)$ flavor the tensors are given in Eqs. (13) and (17), respectively; while for $SU(2)$ flavor, they are given by Eqs. (A6) and (A7), respectively.

The free Lagrangian for the $T_{ij}$ and $S_{ij}^{\mu}$ fields in $SU(3)$ and $SU(2)$ both have the form

$$\mathcal{L} = -i \left(\bar{S}_{\mu}^{\nu} \cdot DS_{\mu}\right) + \Delta \left(\bar{S}_{\mu}^{\mu} S_{\mu}\right) + \lambda_{1} \left(\bar{S}_{\mu}^{\mu} M_{+} S_{\mu}\right) + \lambda_{2} \left(\bar{S}_{\mu}^{\mu} S_{\mu}\right) \text{tr}M_{+}$$
$$+ i \left(\bar{T}_{\nu} \cdot DT\right) + \lambda_{3} \left(\bar{T}_{\nu} M_{+} T\right) + \lambda_{4} \left(\bar{T} \cdot T\right) \text{tr}M_{+},$$

with the exception that $\lambda_{3} = 0$ in the two-flavor theory to avoid redundancy [43]. Similarly the interaction Lagrangian for $T_{ij}$ and $S_{ij}^{\mu}$ has the same form for both flavor groups, namely

$$\mathcal{L} = ig_{2} \left(\bar{S}_{\mu}^{\nu} \cdot A_{\nu} S_{\mu}\right) \varepsilon_{\mu\nu\rho\sigma} + \sqrt{2} g_{3} \left[\left(\bar{T} A_{\mu} S_{\mu}\right) + \left(\bar{S}_{\mu} A_{\mu} T\right)\right]$$

(21)
TABLE V: The coefficients $\alpha^T_\phi$ in $SU(3)$ $\chi$PT. Coefficients are listed for the $s_\ell = 0$ baryon states $B$, and are grouped into contributions from loop mesons with mass $m_\phi$.

| | $\pi$ | $K$ |
|---|---|---|
| $\Lambda_Q$ | 0 | $\frac{1}{2}$ |
| $\Xi^0_Q$ | $\frac{1}{2}$ | 0 |
| $\Xi^{-}_Q$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

The factor $\sqrt{2}$ is chosen so that $g_3$ agrees with the literature, e.g. \cite{53}. The numerical values of the $\lambda_j$, $g_2$, and $g_3$ are of course different for the $SU(3)$ and $SU(2)$ theories.

1. $SU(3)$

In contrast to the partially quenched theories considered above, the light-quark mass matrix in the isospin limit of $SU(3)$ flavor is given by

$$ m_q = \text{diag}(m_u, m_u, m_s), \quad (B3) $$

and the light-quark charge matrix is

$$ Q = \text{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right). \quad (B4) $$

The local operators contributing to the electromagnetic properties of the $s_\ell = 0$ baryons are contained in the Lagrangian

$$ \mathcal{L} = -\frac{e^{\mu Q}}{4m_Q} (\overline{T} T \sigma^{\mu \nu} T) F_{\mu \nu} - \frac{e Q_\mu}{8m_Q^2} (\overline{T} T) v_{\mu} \partial_{\nu} F^{\mu \nu}, \quad (B5) $$

for the heavy quark contributions and

$$ \mathcal{L} = -\frac{e^{\mu T} \Lambda_{QCD}}{4 \Lambda^2 \chi m_Q} (\overline{T} Q \sigma^{\mu \nu} T) F_{\mu \nu} - \frac{e c_T}{\Lambda^2 \chi} (\overline{T} Q T) v_{\mu} \partial_{\nu} F^{\mu \nu}, \quad (B6) $$

for the brown muck. The LECs $\mu_T$ and $c_T$ in $SU(6|3)$ have the same numerical values as in $SU(3)$ $\chi$PT. The heavy quark symmetry breaking operators have the form

$$ \mathcal{L} = \lambda \frac{\Lambda_{QCD}^2}{m_Q} (\overline{S}^a i \sigma^{\mu \nu} S^a) - \sqrt{2} \lambda g \frac{\Lambda_{QCD}}{m_Q} [(\overline{T} A^{\mu} i \sigma_{\mu \nu} S^a) + (\overline{T} S^a i \sigma_{\mu \nu} A^{\mu} T)] \quad (B7) $$

for the $SU(3)$ flavor group. We trivially find the LECs above, $\lambda$ and $\lambda g$, have the same values as in $SU(6|3)$. The Dirac and Pauli form factors can now be determined in $SU(3)$ and they have a form identical to Eqs. (33) and (37). The coefficients $\alpha^T_\phi$ are listed for $SU(3)$ in Table V.

The local operators contributing to the electromagnetic properties of the $s_\ell = 1$ baryons are contained in the Lagrangian

$$ \mathcal{L} = \frac{e^{\mu Q}}{4m_Q} (\overline{S}^a \sigma_{\mu \nu} S^a) F_{\mu \nu} + \frac{e Q_\mu}{8m_Q^2} (\overline{S}^a S^a) v_{\mu} \partial_{\nu} F^{\mu \nu}, \quad (B8) $$
TABLE VI: The coefficients $\alpha_B^\varphi$ in $SU(3)$ $\chi$PT. Coefficients are listed for the $s_\ell = 1$ baryon states $B$, and are grouped into contributions from loop mesons with mass $m_\varphi$.

| $\Sigma^{\pm}_Q$ | $\pi$ | $K$ |
|------------------|------|-----|
| $\Sigma^0_Q$     | 0    | $\frac{1}{2}$ |
| $\Sigma^{-1}_Q$  | $-1$ | 0   |
| $\Xi^{+}_{Q}$    | $\frac{1}{2}$ | 0   |
| $\Xi^{-1}_{Q}$   | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $\Omega_Q$       | 0    | $-1$ |

for the heavy quark contributions and

$$\mathcal{L} = \frac{i e \mu_S}{\Lambda_\chi} \left( \vec{S} \mu Q S_\nu \right) F_{\mu\nu} + \frac{e c_S}{\Lambda_\chi^2} \left( \vec{S}^\alpha Q S_\alpha \right) v_\mu \partial_\nu F_{\mu\nu} - \frac{e Q_\varphi}{\Lambda_\chi^2} \left( \vec{S}^{(\mu} Q S^{\nu)} \right) v_\alpha \partial_\mu F^{\alpha}_{\nu}, \quad (B9)$$

for the brown muck. The LECs $\mu_S, c_S$, and $Q_\varphi$ in $SU(6|3)$ have the same numerical values as those in $SU(3)$ $\chi$PT. The three form factors $F_1(q^2)$, and $F_2(q^2)$ can now be determined in $SU(3)$ for the $B$ baryons and they have a form identical to Eqs. (49) and (50). The form factors $F_1^*(q^2), F_2^*(q^2)$ and $G_1^*(q^2)$ for the $B^*$ baryons have the same form as Eqs. (51), (52) and (53). For each of the form factors for the $B$ and $B^*$ baryons, the coefficients $\alpha_B^\varphi$ are listed for $SU(3)$ in Table VI.

2. $SU(2)$

The light-quark mass matrix in $SU(2)$ flavor is given by

$$m_q = \text{diag}(m_u, m_d), \quad (B10)$$

and the light-quark charge matrix is

$$Q = \text{diag} \left( \frac{2}{3}, -\frac{1}{3} \right). \quad (B11)$$

The operators contributing to baryon electromagnetic properties have the same form as in $SU(3)$. One must keep in mind, however, that the LECs appearing in Eqs. (B16), (B17), and (B19) have different numerical values in $SU(2)$. These values are the same as in $SU(4|2)$. Additionally there are local electromagnetic operators in $SU(2)$ that involve the trace of the charge matrix. These operators have the form

$$\mathcal{L} = -\frac{e c_T}{\Lambda_\chi^2} \left( \mathcal{T} T \right) v_\mu \partial_\nu F^\mu_{\nu \tau} \text{tr} Q - \frac{e (T^\mu)^{\alpha}\omega_{\alpha}}{4\Lambda_\chi} \left( T_{\sigma\mu\nu} T \right) F^\mu_{\nu \tau} \text{tr} Q$$

$$+ \frac{i e \mu_S}{\Lambda_\chi} \left( \vec{S}_\mu S_\nu \right) F^\mu_{\nu \tau} \text{tr} Q + \frac{e c_S}{\Lambda_\chi^2} \left( S^\alpha S_\alpha \right) v_\mu \partial_\nu F^\mu_{\nu \tau} \text{tr} Q - \frac{e Q_\varphi}{\Lambda_\chi^2} \left( \vec{S}^{(\mu} S^{\nu)} \right) v_\alpha \partial_\mu F^{\alpha}_{\nu \tau} \text{tr} Q.$$  

(B12)
The LECs appearing in $SU(4|2)$ are all numerically equal to those in the above Lagrangian for $SU(2)$. Including contributions from these operators, the electromagnetic form factors of baryons in $SU(2)$ have exactly the same form as those in $SU(4|2)$ in Appendix A with $q_{jl} = \frac{1}{3}$. For the $\Lambda_Q$ baryon, however, the are no net loop contributions since $\alpha_{\pi Q} = 0$. For the $\Sigma_Q$ baryons, the value of the coefficient $\alpha$ is given by the isospin: $\Sigma^c_{\ell} = I$.

**APPENDIX C: FORM FACTOR RELATIONS IN THE HEAVY QUARK LIMIT**

In this Appendix, we derive relations between the electromagnetic form factors and transition form factors of the $B$ and $B^*$ baryons. These relations follow from the decoupling of the spins of heavy and light degrees of freedom in the heavy quark limit. We begin for simplicity with the case of the $s_\ell = 0$ baryons.

The electromagnetic current $J^\mu$ can be written in terms of light quark and heavy quark contributions,

$$J^\mu = \sum_i Q_i \bar{q}_i \gamma^\mu q_i + \sum_i \bar{Q}_i v^\mu Q_i, \quad (C1)$$

where the first sum is over the light flavors while the second is over the heavy flavors. Written this way, our arguments generalize to partially quenched theories. Notice that only the heavy quark charge operator survives the heavy quark limit, thus the action of $\mu$ between heavy quark states is proportional to unity. Consider the current matrix elements of the $s_\ell = 0$ baryons. As the spins of the heavy and light degrees of freedom are decoupled, the baryon states are direct products of $s_\ell = 0$ states and heavy quark states. The action of the current operator is always unity in the heavy quark subspace, thus we have

$$\langle \bar{T}(p') | J^\mu | T(p) \rangle = \langle \bar{Q}(1) | q^\mu \rangle.$$

Comparing the above form with the current matrix element decomposition in Eq. (23), we find $F_1(q^2) \propto f(q^2)$ and $F_2(q^2) = 0$.

Now we consider the current matrix elements of the sextet baryons. In the heavy quark limit, the $B$ and $B^*$ baryons are degenerate and are described by the $s_\ell = 1$ field $S^\mu$ defined in Eq. (10). Schematically current matrix elements of the $S$ field can be written as

$$\langle \bar{S}_\mu(p') | J^\rho | S_\nu(p) \rangle = \left( \langle s_\ell = 1, p' | \otimes | Q \rangle \right) J^\mu \left( | Q \rangle \otimes | s_\ell = 1, p \rangle \right), \quad (C3)$$

because of the decoupling of heavy and light spins. As the action of $J^\rho$ in the heavy quark sector is unity, current matrix elements of the $S$ field can be parametrized by

$$\langle \bar{S}_\mu(p') | J^\rho | S_\nu(p) \rangle = -\bar{\epsilon}_\mu(p') \mathcal{O}^{\mu\rho} \epsilon_\nu(p), \quad (C4)$$

where $\epsilon_\mu(p)$ is the polarization spinor of the $S$-field that is proportional to the direct product: $u(\nu) \otimes \epsilon_\mu(p)$, where $u(\nu)$ is the heavy quark spinor and $\epsilon_\mu(p)$ is the polarization vector of the light degrees of freedom. The polarization spinor satisfies the constraints $p \cdot \epsilon(p) = \not{v}(p) = v \cdot \epsilon(p) = 0$. The tensor $\mathcal{O}^{\mu\rho}$ has the decomposition

$$\mathcal{O}^{\mu\rho} = g^{\mu\nu} v^\rho f_1(q^2) + \frac{1}{2M_B} (g^{\mu\rho} g^{\nu\mu} - g^{\nu\mu} g^{\mu\rho}) f_2(q^2) + \frac{q^\rho q^\nu}{(2M_B)^2} v^\rho g_1(q^2). \quad (C5)$$
Notice that there are only three form factors for the $S$-field because the light degrees of freedom have spin one. The above current matrix element encodes the electromagnetic form factors of both the $B$ and $B^*$ baryons, as well as their electromagnetic transition form factors.

Comparing the current matrix element of the $B^*$ fields in Eq. (41) with that of Eq. (C4) allows us to determine

\[
F_1^*(q^2) = f_1(q^2),
\]

\[
F_2^*(q^2) = f_2(q^2),
\]

\[
G_1^*(q^2) = g_1(q^2), \quad \text{and}
\]

\[
G_2^*(q^2) = 0,
\]

in the heavy quark limit. Notice that the magnetic octupole form factor $G_2^*(q^2)$ vanishes by heavy quark symmetry. Carrying out the procedure for the current matrix element of the $B$ fields in Eq. (39), we find

\[
F_1(q^2) = f_1(q^2) + \frac{q^2}{12M_B^2}g_1(q^2), \quad \text{and}
\]

\[
F_2(q^2) = \frac{2}{3}f_2(q^2),
\]

in the heavy quark limit. Thus we have relations between the $B$ and $B^*$ electromagnetic form factors, namely

\[
F_1(q^2) - F_1^*(q^2) = \frac{q^2}{12M_B^2}G_1^*(q^2),
\]

and

\[
F_2(q^2) - \frac{2}{3}F_2^*(q^2) = 0.
\]

These in turn yield relations between the static electromagnetic properties of the $B$ and $B^*$ baryons

\[
<r^{*2}> - <r^2> = \frac{Q}{M_B},
\]

and

\[
\mu = \frac{2}{3}\mu^*.
\]

These expressions are satisfied in the heavy quark limit by our one-loop PQ\(\chi\)PT and \(\chi\)PT results.

Lastly we consider the electromagnetic transition matrix element

\[
\langle B^*(p') | J^\rho | B(p) \rangle = i \bar{u}_\mu(p') \mathcal{O}^{\mu\rho\gamma_5} u(p),
\]

where the tensor $\mathcal{O}^{\mu\rho}$ is given by

\[
\mathcal{O}^{\mu\rho} = \left( g^{\mu\rho} - \frac{q^{\mu}q^{\rho}}{2M_B^2} \right) G_1(q^2) - \frac{q^{\mu}\gamma_5 q^{\rho}}{2M_B^2} G_2(q^2) + \frac{1}{2M_S^2}(q^2 q^{\mu\rho} - q^{\mu}q^{\rho}) G_3(q^2).
\]
TABLE VII: The coefficients \(\alpha_T^{\phi}\) in \(SU(6|3)\) PQ\(\chi\)PT for the charge matrix \(Q\) in Eq. (D1). Coefficients are listed for the \(s_f = 0\) baryon states \(T\), and are grouped into contributions from loop mesons with mass \(m_{\phi}\). The abbreviation \(q_{jl}\) is defined to be \(q_{jl} = q_j + q_l\).

\[
\begin{array}{|c|cccccc|}
\hline
\text{Baryon} & \pi & K & \eta_s & ju & ru & js & rs \\
\hline
\Lambda_{Q^{+\frac{1}{2}}} & -\frac{1}{3} + q_{jl} & \frac{1}{3} + q_r & 0 & \frac{1}{3} - q_{jl} & \frac{1}{6} - q_r & 0 & 0 \\
\Xi_{Q^{+\frac{1}{2}}} & -\frac{1}{6} + \frac{1}{2} q_{jl} & \frac{1}{2} (q_{jl} + q_r) & 0 & \frac{1}{2} - \frac{1}{2} q_{jl} & \frac{1}{3} - \frac{1}{2} q_r & -\frac{1}{3} - \frac{1}{2} q_{jl} & -\frac{1}{6} - \frac{1}{2} q_r \\
\Xi_{Q^{-\frac{1}{2}}} & -\frac{1}{6} + \frac{1}{2} q_{jl} & \frac{1}{2} (q_{jl} + q_r) & 0 & \frac{1}{2} - \frac{1}{2} q_{jl} & -\frac{1}{3} - \frac{1}{2} q_r & -\frac{1}{3} - \frac{1}{2} q_{jl} & -\frac{1}{6} - \frac{1}{2} q_r \\
\hline
\end{array}
\]

Matching this decomposition onto that in Eq. (C4) for the \(S\)-field yields

\[
G_1(q^2) = \frac{1}{\sqrt{3}} f_2(q^2),
\]
\[
G_2(q^2) = \frac{1}{\sqrt{3}} g_1(q^2), \quad \text{and}
\]
\[
G_3(q^2) = 0. \tag{C14}
\]

In particular these relations imply that the magnetic dipole transition moment is given by

\[
G_1(0) = \frac{1}{\sqrt{3}} \mu^*, \tag{C15}
\]

while the electric quadrupole transition moment is given by

\[
G_2(0) = -\frac{1}{2\sqrt{3}} Q, \tag{C16}
\]

and the Coulomb quadrupole moment \(G_3(0)\) vanishes.

**APPENDIX D: QUARK CHARGES**

In this Appendix, we provide results for the more commonly used choice of the charge matrix \(Q\). The choice used in the main text has advantages over this commonly used form as we explain below. Insertion of different quark charges results only in modification of the charge dependent factors in our above results. Thus the only factors that are altered for a different choice of \(Q\) are the baryon charge \(Q_T\) and \(Q_B\), the charge of the light degrees of freedom \(Q_T\) and \(Q_S\), and the coefficients \(\alpha_T^{\phi}\) and \(\alpha_B^{\phi}\) that depend on the charge of the loop meson \(\phi\).

The commonly used form of the charge matrix \(Q\) in \(SU(6|3)\) PQ\(\chi\)PT is \[26\]

\[
Q = \text{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, q_j, q_l, q_r, q_j, q_l, q_r\right). \tag{D1}
\]

When the sea and valence quark masses are made degenerate, one rather elegantly recovers QCD independent of the values of the charges \(q_j\), \(q_l\), and \(q_r\). Implementation of this choice for the charge matrix, however, is not the most economical in terms of computation time. This is because closed valence quark loops with a photon insertion are not canceled by
TABLE VIII: The coefficients $\alpha^B_\phi$ in $SU(6|3)$ PQXPT for the charge matrix $Q$ in Eq. (D1). Coefficients are listed for the $s_f = 1$ baryon states $B$, and are grouped into contributions from loop mesons with mass $m_\phi$. The abbreviation $q_{jl}$ is defined to be $q_{jl} = q_j + q_l$.

| $Q$ | $\alpha^B_\phi$ | $\eta_u$ | $\pi$ | $\eta_d$ | $j_u$ | $l_u$ | $j_d$ | $l_d$ |
|-----|----------------|-------|-----|-------|-----|-----|-----|-----|
| $\Sigma^+_Q$ | $-\frac{1}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $\frac{1}{3} - q_j$ | $\frac{2}{3} - q_l$ | $0$ | $0$ |
| $\Sigma^0_Q$ | $-\frac{1}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $\frac{1}{3} - q_j$ | $\frac{1}{3} - q_l$ | $0$ | $0$ |
| $\Sigma^{-1}_Q$ | $-\frac{1}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $\frac{1}{3} - q_j$ | $\frac{1}{3} - q_l$ | $0$ | $0$ |
| $\Xi^{+\frac{1}{2}}_Q$ | $-\frac{1}{6} + \frac{1}{3} q_jl + \frac{1}{3} q_r$ | $\frac{1}{6} + \frac{1}{3} q_j$ | $\frac{1}{3} - q_jl + \frac{1}{3} q_r$ | $\frac{1}{3} - q_jl + \frac{1}{3} q_r$ | $0$ | $0$ |
| $\Xi^{-\frac{1}{2}}_Q$ | $-\frac{1}{6} + \frac{1}{3} q_jl + \frac{1}{3} q_r$ | $\frac{1}{6} + \frac{1}{3} q_j$ | $\frac{1}{3} - q_jl + \frac{1}{3} q_r$ | $\frac{1}{3} - q_jl + \frac{1}{3} q_r$ | $0$ | $0$ |
| $\Omega_Q$ | $0$ | $\frac{1}{3} + q_jl$ | $\frac{1}{3} + q_r$ | $0$ | $0$ | $\frac{2}{3} - q_jl$ | $\frac{2}{3} - q_r$ |

TABLE IX: The coefficients $\alpha^T_\phi$ for the $\Lambda_Q$ in $SU(4|2)$ PQXPT for the charge matrix $Q$ in Eq. (D2). Coefficients are grouped into contributions from loop mesons with mass $m_\phi$. The abbreviation $q_{jl}$ is defined to be $q_j + q_l$.

| $\Lambda_Q$ | $\eta_u$ | $\pi$ | $\eta_d$ | $j_u$ | $l_u$ | $j_d$ | $l_d$ |
|-------------|-------|-----|-------|-----|-----|-----|-----|
| $-\frac{1}{3} + \frac{1}{2} q_jl + \frac{1}{2} q_r$ | $\frac{1}{6} + \frac{1}{3} q_jl + \frac{1}{3} q_r$ | $\frac{1}{3} - \frac{1}{2} q_jl + \frac{1}{3} q_r$ | $\frac{1}{3} - \frac{1}{2} q_jl + \frac{1}{3} q_r$ | $0$ | $0$ | $\frac{2}{3} - q_jl$ | $\frac{2}{3} - q_r$ |

The corresponding ghost quark loops, as their charges are not identical. Thus the lattice practitioner must calculate closed valence quark loops with photon insertion (using the effective charges $q_u = \frac{2}{3} - q_j$, $q_d = -\frac{1}{3} - q_l$, and $q_s = \frac{1}{3} - q_r$ to mimic the partial cancellation from ghost quark loops) in addition to closed sea quark loops and closed sea quark loops with photon insertion. Closed valence quark loops without photon insertion are of course canceled by closed ghost loops because their charges never enter. As valence quarks now appear in the closed loops with an operator insertion, the benefits of using lighter valence masses than sea masses cannot be maximally obtained. In the foreseeable future, the choice of $Q$ above is not ideal for performing partially quenched simulations.

In order that the valence and sea sectors be separated as their names suggest, one must have the ghost charges equal to their valence counterparts. With this choice, closed valence loops with operator insertion are always canceled by the corresponding ghost loop. Thus the only quark disconnected contributions arise from the sea sector and these quarks efficaciously can be given larger masses to reduce computation time. Thus a more practical choice of the $SU(6|3)$ charge matrix is that employed in the main text

$$Q = \text{diag}(q_u, q_d, q_s, q_j, q_l, q_r, q_u, q_d, q_s),$$

TABLE X: The coefficients $\alpha^B_\phi$ in $SU(4|2)$ PQXPT for the charge matrix $Q$ in Eq. (D2). Coefficients are listed for the $s_f = 1$ baryon states $B$, and are grouped into contributions from loop mesons with mass $m_\phi$. The abbreviation $q_{jl}$ is defined to be $q_{jl} = q_j + q_l$.

| $Q$ | $\alpha^B_\phi$ | $\eta_u$ | $\pi$ | $\eta_d$ | $j_u$ | $l_u$ | $j_d$ | $l_d$ |
|-----|----------------|-------|-----|-------|-----|-----|-----|-----|
| $\Sigma^+_Q$ | $-\frac{2}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $\frac{2}{3} - q_j$ | $\frac{2}{3} - q_l$ | $0$ | $0$ |
| $\Sigma^0_Q$ | $-\frac{2}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $\frac{2}{3} - q_j$ | $\frac{1}{3} - q_l$ | $0$ | $0$ |
| $\Sigma^{-1}_Q$ | $0$ | $\frac{2}{3} + q_j$ | $\frac{1}{3} + q_l$ | $0$ | $0$ | $\frac{1}{3} - q_j$ | $\frac{1}{3} - q_l$ |
where to maintain supertracelessness \( q_j + q_l + q_r = 0 \). Notice that when the sea quark masses are made degenerate with the valence quark masses, QCD is only recovered for the specific choice \( q_u = q_j = \frac{2}{3} \), and \( q_d = q_s = q_l = q_r = -\frac{1}{3} \). One can use unphysical charges for both valence and sea quarks to determine the electromagnetic LECs.

To contrast with the calculation in \( SU(6|3) \) in the main text, we calculate the baryon electromagnetic properties using the charge matrix in Eq. (D1). The results have the same functional form as in the main text. One must be careful, however, to use the appropriate values of the baryon charge, the charge of the light degrees of freedom, and replace the \( SU(6|3) \) coefficients \( \alpha_T^T \) and \( \alpha_B^B \) with those listed in Tables VII and VIII. Incomplete cancellations between valence and ghost sectors imply that valence-valence meson loops are not completely canceled by valence-ghost meson loops. Not surprisingly then these results have more complicated chiral extrapolation formula because more loop mesons contribute and one can try to simplify the formula by using clever choices for the unfixed charges.

As with the case of \( SU(6|3) \), the choice of charge matrix in [27] for \( SU(4|2) \)

\[
Q = \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, q_j, q_l, q_j, q_l \right) \quad (D2)
\]

is computationally intensive. A choice that more readily can be put to use on current lattices is

\[
Q = \text{diag}(q_u, q_d, q_j, q_l, q_u, q_d),
\]

which we used in Appendix A. This choice mandates that the only closed quark loops are those with sea quarks. Carrying out the calculation using \( Q \) in Eq. (D2), we find the modified \( SU(4|2) \) coefficients \( \alpha_T^T \) and \( \alpha_B^B \) listed in Tables IX and X respectively. Again there are more contributing loop mesons due to incomplete cancellations between the valence and ghost sectors of the theory.

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