ADIABATIC LIMITS AND THE SPECTRUM OF THE
LAPLACIAN ON FOLIATED MANIFOLDS

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ABSTRACT. We present some recent results on the behavior of the spectrum of the differential form Laplacian on a Riemannian foliated manifold when the metric on the ambient manifold is blown up in directions normal to the leaves (in the adiabatic limit).

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INTRODUCTION

Let \((M, F)\) be a closed foliated manifold, \(\dim M = n, \dim F = p, p + q = n\), endowed with a Riemannian metric \(g\). Then we have a decomposition of the tangent bundle to \(M\) into a direct sum \(TM = F \oplus H\), where \(F = TF\) is the tangent bundle to \(F\) and \(H = F^\perp\) is the orthogonal complement of \(F\), and the corresponding decomposition of the metric: \(g = g_F + g_H\). Define a one-parameter family \(g_h\) of Riemannian metrics on \(M\) by

\[
g_h = g_F + h^{-2}g_H, \quad h > 0.
\]

By the adiabatic limit, we will mean the asymptotic behavior of Riemannian manifolds \((M, g_h)\) as \(h \rightarrow 0\).

In this form, the notion of the adiabatic limit was introduced by Witten in the study of the global anomaly. He considered a family of Dirac operators acting along the fibers of a Riemannian fiber bundle over the

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circle and gave an argument relating the holonomy of the determinant line bundle of this family to the adiabatic limit of the eta invariant of the Dirac operator on the total space. Witten’s result was proved rigorously in [6], [7] and [10], and extended to general Riemannian bundles in [5] and [13]. This study gave rise to the development of adiabatic limit technique for analyzing the behavior of certain spectral invariants under degeneration that has many applications in the local index theory (see, for instance, [8]).

New properties of adiabatic limits were discovered by Mazzeo and Melrose [30]. They showed that in the case of a fibration, a Taylor series analysis of so-called small eigenvalues in the adiabatic limit and the corresponding eigenforms leads directly to a spectral sequence, which is isomorphic to the Leray spectral sequence. This result was used in [13], and further developed in [16], where the very general setting of any pair of complementary distributions is considered. Nevertheless, the most interesting results of [16] are only proved for foliations satisfying very restrictive conditions. The ideas from [30] and [16] were also applied in the case of the contact-adiabatic (or sub-Riemannian) limit in [18, 36].

In this paper, we will discuss extensions of the results mentioned above to the adiabatic limits on foliated manifolds. For any $h > 0$, we will consider the Laplace operator $\Delta_h$ on differential forms defined by the metric $g_h$. It is a self-adjoint, elliptic, differential operator with the positive, scalar principal symbol in the Hilbert space $L^2(M, \Lambda T^*M, g_h)$ of square integrable differential forms on $M$, endowed with the inner product induced by $g_h$, which has discrete spectrum. Denote by

$$0 \leq \lambda_0(h) \leq \lambda_1(h) \leq \lambda_2(h) \leq \cdots$$

the spectrum of $\Delta_h$, taking multiplicities into account.

In Section 1 we discuss the asymptotic behavior as $h \to 0$ of the trace of $f(\Delta_h)$:

$$\text{tr} f(\Delta_h) = \sum_{i=0}^{+\infty} f(\lambda_i(h)),$$

for any sufficiently nice function $f$, say, for $f \in S(\mathbb{R})$. The results given in this Section should be viewed as a very first step in extending the adiabatic limit technique to analyze the behavior of spectral invariants to the case of foliations.

In Section 2 we study “branches” of eigenvalues $\lambda_i(h)$ that are convergent to zero as $h \to 0$ (the “small” eigenvalues) and the corresponding eigenspaces and discuss the differentiable spectral sequence of the foliation, which is a direct generalization of the Leray spectral sequence, and its Hodge theoretic description.

We will consider two basic classes of foliations — Riemannian foliations and one-dimensional foliations defined by the orbits of invariant flows on Riemannian Heisenberg manifolds.
We remark that the adiabatic limit is, up to scaling, an example of collapsing (in general, without bounded curvature) in the sense of [9]. For a discussion of the behavior of the spectrum of the differential form Laplacian on a compact Riemannian manifold under collapse, we refer, for instance, to [3, 11, 17, 22, 27] and references therein.

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1. ADIABATIC LIMITS AND EIGENVALUE DISTRIBUTION

Let $(M, F)$ be a closed foliated manifold, endowed with a Riemannian metric $g$. In this Section, we will discuss the asymptotic behavior of the trace of $f(\Delta_h)$ in the adiabatic limit.

1.1. Riemannian foliations. For Riemannian foliations, the problem was studied in [25]. Recall (see, for instance, [3, 11, 17, 22, 27]) that a foliation $F$ is called Riemannian, if there exists a Riemannian metric $g$ on $M$ such that the induced metric $g_\tau$ on the normal bundle $\tau = TM/F$ is holonomy invariant, or, equivalently, in any foliated chart $\phi : U \to \mathbb{R}^p \times I^q$ with local coordinates $(x, y)$, the restriction $g_H$ of $g$ to $H = F^\perp$ is written in the form

$$g_H = \sum_{\alpha, \beta = 1}^q g_{\alpha\beta}(y) \theta^\alpha \theta^\beta,$$

where $\theta^\alpha \in H^*$ is the 1-form, corresponding to the form $dy^\alpha$ under the isomorphism $H^* \cong T^*\mathbb{R}^q$, and $g_{\alpha\beta}(y)$ depend only on the transverse variables $y \in \mathbb{R}^q$. Such a Riemannian metric is called bundle-like.

It turns out that the adiabatic spectral limit on a Riemannian foliation can be considered as a semiclassical spectral problem for a Schrödinger operator on the leaf space $M/F$, and the resulting asymptotic formula for the trace of $f(\Delta_h)$ can be written in the form of the semiclassical Weyl formula for a compact Riemannian manifold, if we replace the classical objects, entering to this formula by their noncommutative analogues. This observation provides a very natural interpretation of the asymptotic formula for the trace of $f(\Delta_h)$ (see Theorem 1.1 below).

First, we transfer the operators $\Delta_h$ to the fixed Hilbert space $L^2\Omega = L^2(M, \Lambda T^*M, g)$, using an isomorphism $\Theta_h$ from $L^2(M, \Lambda T^*M, g_h)$ to $L^2\Omega$ as follows. With respect to a bigrading on $\Lambda T^*M$ given by

$$\Lambda^k T^* M = \bigoplus_{i=0}^k \Lambda^{i,k-i} T^* M, \quad \Lambda^i T^* M = \Lambda^i H^* \otimes \Lambda^i F^*,$$

we have, for $u \in L^2(M, \Lambda i^j T^* M, g_h)$,

$$\Theta_h u = h^j u.$$

The operator $\Delta_h$ in $L^2(M, \Lambda T^* M, g_h)$ corresponds under the isometry $\Theta_h$ to the operator $L_h = \Theta_h \Delta_h \Theta_h^{-1}$ in $L^2\Omega$.
With respect to the above bigrading of $\Lambda T^*M$, the de Rham differential $d$ can be written as
\[ d = d_F + d_H + \theta, \]
where

1. $d_F = d_{0,1} : C^\infty(M, \Lambda^i \wedge T^*M) \to C^\infty(M, \Lambda^{i+j} T^*M)$ is the tangential de Rham differential, which is a first order tangentially elliptic operator, independent of the choice of $g$;
2. $d_H = d_{1,0} : C^\infty(M, \Lambda^i \wedge T^*M) \to C^\infty(M, \Lambda^{i,j} T^*M)$ is the transversal de Rham differential, which is a first order transversally elliptic operator;
3. $\theta = d_{2,-1} : C^\infty(M, \Lambda^i \wedge T^*M) \to C^\infty(M, \Lambda^{i+j} T^*M)$ is a zeroth order differential operator.

One can show that
\[ d_h = \Theta_h d \Theta_h^{-1} = d_F + h d_H + h^2 \theta, \]
and the adjoint of $d_h$ in $L^2 \Omega$ is
\[ \delta_h = \Theta_h d \Theta_h^{-1} = \delta_F + h \delta_H + h^2 \theta^*. \]

Therefore, one has
\[ L_h = d_h \delta_h + \delta_h d_h = \Delta_F + h^2 \Delta_H + h^4 \Delta_\theta + h K_1 + h^2 K_2 + h^3 K_3, \]
where $\Delta_F = d_F d_F^* + d_F^* d_F$ is the tangential Laplacian, $\Delta_H = d_H d_H^* + d_H^* d_H$ is the transverse Laplacian, $\Delta_\theta = \theta \theta^* + \theta^* \theta$ and $K_1 = d_F \theta^* + \theta^* d_F + \delta_F \theta + \theta \delta_F$ are of zeroth order, and $K_2 = d_F \delta_H + \delta_H d_F + \delta_F d_H + K_3 = d_H \theta^* + \theta^* d_H + \delta_H \theta + \theta \delta_H$ are first order differential operators.

Suppose that $\mathcal{F}$ is a Riemannian foliation and $g$ is a bundle-like metric. The key observation is that, in this case, the transverse principal symbol of the operator $\delta_H$ is holonomy invariant, and, therefore, the first order differential operator $K_1$ is a leafwise differential operator. Using this fact, one can show that the leading term in the asymptotic expansion of the trace of $f(\Delta_h)$ or, equivalently, of the trace of $f(L_h)$ as $h \to 0$ coincides with the leading term in the asymptotic expansion of the trace of $f(\bar{L}_h)$ as $h \to 0$, where
\[ \bar{L}_h = \Delta_F + h^2 \Delta_H. \]

More precisely, we have the following estimates (with some $C_1, C_2 > 0$):
\[ |\text{tr} f(L_h)| < C_1 h^{-q}, \quad |\text{tr} f(L_h) - \text{tr} f(\bar{L}_h)| < C_2 h^{1-q}, \quad 0 < h \leq 1, \]
where we recall that $q$ denotes the codimension of $\mathcal{F}$.

We observe that the operator $L_h$ has the form of a Schrödinger operator on the leaf space $M/\mathcal{F}$, where $\Delta_H$ plays the role of the Laplace operator, and $\Delta_F$ the role of the operator-valued potential on $M/\mathcal{F}$.
Recall that, for a Schrödinger operator $H_h$ on a compact Riemannian manifold $X$, $\dim X = n$, with a matrix-valued potential $V \in C^\infty(X, \mathcal{L}(E))$, where $E$ is a finite-dimensional Euclidean space and $V(x)^* = V(x)$:

$$H_h = -\hbar^2 \Delta + V(x), \quad x \in X,$$

the corresponding asymptotic formula (the semiclassical Weyl formula) has the following form:

$$\text{tr} f(H_h) = (2\pi)^{-n} \hbar^{-n} \int_{T^*X} \text{Tr} f(p(x, \xi)) \, dx \, d\xi + o(\hbar^{-n}), \quad \hbar \to 0^+,$$

where $p \in C^\infty(T^*X, \mathcal{L}(E))$ is the principal $\hbar$-symbol of $H_h$:

$$p(x, \xi) = |\xi|^2 + V(x), \quad (x, \xi) \in T^*X.$$

Now we demonstrate how the asymptotic formula for the trace of $f(\Delta_h)$ in the adiabatic limit can be written in a similar form, using noncommutative geometry. (For the basic information on noncommutative geometry of foliations, we refer the reader to \cite{26} and references therein.)

Let $G$ be the holonomy groupoid of $\mathcal{F}$. Let us briefly recall its definition. Denote by $\sim_h$ the equivalence relation on the set of piecewise smooth leafwise paths $\gamma : [0, 1] \to M$, setting $\gamma_1 \sim_h \gamma_2$ if $\gamma_1$ and $\gamma_2$ have the same initial and final points and the same holonomy maps. The holonomy groupoid $G$ is the set of $\sim_h$ equivalence classes of leafwise paths. $G$ is equipped with the source and the range maps $s, r : G \to M$ defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. Recall also that, for any $x \in M$, the set $G^x = \{ \gamma \in G : r(\gamma) = x \}$ is the covering of the leaf $L_x$ through the point $x$, associated with the holonomy group of the leaf. We will identify any $x \in M$ with the element of $G$ given by the constant path $\gamma(t) = x, t \in [0, 1]$.

Let $\lambda_L$ denote the Riemannian volume form on a leaf $L$ given by the induced metric, and $\lambda^x, x \in M$, denote the lift of $\lambda_L$ via the holonomy covering map $s : G^x \to L_x$.

Denote by $\pi : N^*\mathcal{F} \to M$ the conormal bundle to $\mathcal{F}$ and by $\mathcal{F}_N$ the linearized foliation in $N^*\mathcal{F}$ (cf., for instance, \cite{32} \cite{26}). Recall that, for any $\gamma \in G$, $s(\gamma) = x, r(\gamma) = y$, the codifferential of the corresponding holonomy map defines a linear map $dh^\gamma_N : N^*_y \mathcal{F} \to N^*_x \mathcal{F}$. Then the leaf of the foliation $\mathcal{F}_N$ through $\nu \in N^*\mathcal{F}$ is the set of all $dh^\gamma_N(\nu) \in N^*\mathcal{F}$, where $\gamma \in G, r(\gamma) = \pi(\nu)$.

The holonomy groupoid $G_{\mathcal{F}_N}$ of the linearized foliation $\mathcal{F}_N$ can be described as the set of all $(\gamma, \nu) \in G \times N^*\mathcal{F}$ such that $r(\gamma) = \pi(\nu)$. The source map $s_N : G_{\mathcal{F}_N} \to N^*\mathcal{F}$ and the range map $r_N : G_{\mathcal{F}_N} \to N^*\mathcal{F}$ are defined as $s_N(\gamma, \nu) = dh^\gamma_N(\nu)$ and $r_N(\gamma, \nu) = \nu$. We have a map $\pi_G : G_{\mathcal{F}_N} \to G$ given by $\pi_G(\gamma, \nu) = \gamma$. Denote by $\mathcal{L}(\pi^*\Lambda T^*M)$ the vector bundle on $G_{\mathcal{F}_N}$, whose fiber at a point $(\gamma, \nu) \in G_{\mathcal{F}_N}$ is the space of linear maps

$$(\pi^*\Lambda T^*M)_{s_N(\gamma, \nu)} \to (\pi^*\Lambda T^*M)_{r_N(\gamma, \nu)}.$$
There is a standard way (due to Connes [12]) to introduce the structure of involutive algebra on the space $C_c^\infty(G_{F_N}, \mathcal{L}(\pi^*\Lambda^*M))$ of smooth, compactly supported sections of $\mathcal{L}(\pi^*\Lambda^*M)$. For any $\nu \in N^*F$, this algebra has a natural representation $R_\nu$ in the Hilbert space $L^2(G_{F_N}, s_N^*(\pi^*\Lambda^*M))$ that determines its embedding to the $\ast$-algebra of all bounded operators in $L^2(G_{F_N}, s_N^*(\pi^*\Lambda^*M))$. Taking the closure of the image of this embedding, we get a $\ast$-algebra $C^*(N^*F, F_N, \pi^*\Lambda^*M)$, called the twisted foliation $\ast$-algebra. The leaf space $N^*F/F_N$ can be informally considered as the cotangent bundle to $M/F$, and the algebra $C^*(N^*F, F_N, \pi^*\Lambda^*M)$ can be viewed as a noncommutative analogue of the algebra of continuous vector-valued differential forms on this singular space.

Let $g_N \in C^\infty(N^*F)$ be the fiberwise Riemannian metric on $N^*F$ induced by the metric on $M$. The principal $h$-symbol of $\Delta_h$ is a tangentially elliptic operator in $C^\infty(N^*F, \pi^*\Lambda^*M)$ given by $$\sigma_h(\Delta_h) = \Delta_{F_N} + g_N,$$

where $\Delta_{F_N}$ is the lift of the tangential Laplacian $\Delta_F$ to a tangentially elliptic (relative to $F_N$) operator in $C^\infty(N^*F, \pi^*\Lambda^*M)$, and $g_N$ denotes the multiplication operator in $C^\infty(N^*F, \pi^*\Lambda^*M)$ by the function $g_N$. (Observe that $g_N$ coincides with the transversal principal symbol of $\Delta_H$.) Consider $\sigma_h(\Delta_h)$ as a family of elliptic operators along the leaves of the foliation $F_N$ and lift these operators to the holonomy coverings of the leaves. For any $\nu \in N^*F$, we get a formally self-adjoint uniformly elliptic operator $\sigma_h(\Delta_h)_\nu$ in $C^\infty(G_{F_N}, s_N^*(\pi^*\Lambda^*M))$, which essentially self-adjoint in the Hilbert space $L^2(G_{F_N}, s_N^*(\pi^*\Lambda^*M))$. For any $f \in S(\mathbb{R})$, the family $\{f(\sigma_h(\Delta_h)_\nu), \nu \in N^*F\}$ defines an element $f(\sigma_h(\Delta_h))$ of the $\ast$-algebra $C^*(N^*F, F_N, \pi^*\Lambda^*M)$.

The foliation $F_N$ has a natural transverse symplectic structure, which can be described as follows. Consider a foliated chart $\varphi : U \subset M \to I^p \times I^q$ on $M$ with coordinates $(x, y) \in I^p \times I^q$ ($I$ is the open interval $(0, 1)$) such that the restriction of $F$ to $U$ is given by the sets $y = \text{const}$. One has the corresponding coordinate chart in $T^*M$ with coordinates denoted by $(x, y, \xi, \eta) \in I^p \times I^q \times \mathbb{R}^p \times \mathbb{R}^q$. In these coordinates, the restriction of the conormal bundle $N^*F$ to $U$ is given by the equation $\xi = 0$. So we have a coordinate chart $\varphi_n : U_1 \subset N^*F \to I^p \times I^q \times \mathbb{R}^q$ on $N^*F$ with the coordinates $(x, y, \eta) \in I^p \times I^q \times \mathbb{R}^q$. Indeed, the coordinate chart $\varphi_n$ is a foliated coordinate chart for $F_N$, and the restriction of $F_N$ to $U_1$ is given by the level sets $y = \text{const}, \eta = \text{const}$. The transverse symplectic structure for $F_N$ is given by the transverse two-form $\sum_j dy_j \wedge d\eta_j$.

The corresponding canonical transverse Liouville measure $dy \, d\eta$ is holonomy invariant and, by noncommutative integration theory [12], defines the trace $\text{tr}_{F_N}$ on the $\ast$-algebra $C^*(N^*F, F_N, \pi^*\Lambda^*M)$. Combining the Riemannian volume forms $\lambda_I$ and the transverse Liouville measure, we get a volume form $d\nu$ on $N^*F$. For any $k \in C_c^\infty(G_{F_N}, \mathcal{L}(\pi^*\Lambda^*M))$, its trace is
given by the formula
\[ \text{tr}_{\mathcal{F}}(k) = \int_{N^*_{\mathcal{F}}} k(\nu) d\nu. \]
The trace \( \text{tr}_{\mathcal{F}} \) is a noncommutative analogue of the integral over the leaf space \( N^*_{\mathcal{F}}/\mathcal{F}_N \) with respect to the transverse Liouville measure. One can show that the value of this trace on \( f(\sigma_h(\Delta_h)) \) is finite.

Replacing in the formula (3) the integration over \( T^*X \) and the matrix trace \( \text{Tr} \) by the trace \( \text{tr}_{\mathcal{F}_N} \) and the principal \( h \)-symbol \( p \) by \( \sigma_h(\Delta_h) \), we obtain the correct formula for \( \text{tr} f(\Delta_h) \) in the adiabatic limit.

**Theorem 1.1** ([25]). For any \( f \in S(\mathbb{R}) \), the asymptotic formula holds:
\[ \text{tr} f(\Delta_h) = (2\pi)^{-q} h^{-q} \text{tr}_{\mathcal{F}_N} f(\sigma_h(\Delta_h)) + o(h^{-q}), \quad h \to 0. \]

The formula (4) can be rewritten in terms of the spectral data of leafwise Laplace operators. We will formulate the corresponding result for the spectrum distribution function
\[ N_h(\lambda) = \sharp \{ i : \lambda_i(h) \leq \lambda \}. \]

Restricting the tangential Laplace operator \( \Delta_{\mathcal{F}} \) to the leaves of the foliation \( \mathcal{F} \) and lifting the restrictions to the holonomy coverings of leaves, we get the the Laplacian \( \Delta = \Delta_{\mathcal{F}} \) acting in \( C_c^\infty(\mathbb{G}, s^*\Lambda T^*M) \). Using the assumption that \( \mathcal{F} \) is Riemannian, it can be checked that, for any \( x \in M \), \( \Delta_{\mathcal{F}} \) is formally self-adjoint in \( L^2(\mathbb{G}, s^*\Lambda T^*M) \), that, in turn, implies its essential self-adjointness in this Hilbert space (with initial domain \( C_c^\infty(\mathbb{G}, s^*\Lambda T^*M) \)). For each \( \lambda \in \mathbb{R} \), let \( E_x(\lambda) \) be the spectral projection of \( \Delta_{\mathcal{F}} \), corresponding to the semi-axis \( (-\infty, \lambda] \). The Schwartz kernels of the operators \( E_x(\lambda) \) define a leafwise smooth section \( e_\lambda \) of the bundle \( \mathcal{L}(\Lambda T^*M) \) over \( G \).

We introduce the spectrum distribution function \( N_{\mathcal{F}}(\lambda) \) of the operator \( \Delta_{\mathcal{F}} \) by the formula
\[ N_{\mathcal{F}}(\lambda) = \int_M \text{Tr} e_{\lambda}(x) \, dx, \quad \lambda \in \mathbb{R}, \]
where \( dx \) denotes the Riemannian volume form on \( M \). By [24], for any \( \lambda \in \mathbb{R} \), the function \( \text{Tr} e_{\lambda} \) is a bounded measurable function on \( M \), therefore, the spectrum distribution function \( N_{\mathcal{F}}(\lambda) \) is well-defined and takes finite values.

As above, one can show that the family \( \{ E_x(\lambda) : x \in M \} \) defines an element \( E(\lambda) \) of the twisted von Neumann foliation algebra \( W^*(G, \Lambda T^*M) \), the holonomy invariant transverse Riemannian volume form for \( \mathcal{F} \) defines a trace \( \text{tr}_{\mathcal{F}} \) on \( W^*(G, \Lambda T^*M) \), and the right hand side of the last formula can be interpreted as the value of this trace on \( E(\lambda) \).

**Theorem 1.2** ([25]). Let \( (M, \mathcal{F}) \) be a Riemannian foliation, equipped with a bundle-like Riemannian metric \( g \). Then the asymptotic formula for \( N_h(\lambda) \) has the following form:
\[ N_h(\lambda) = h^{-q} (4\pi)^{-q/2} \frac{1}{\Gamma((q/2) + 1)} \int_{-\infty}^{\lambda} (\lambda - \tau)^{q/2} d\tau N_{\mathcal{F}}(\tau) + o(h^{-q}), \quad h \to 0. \]
1.2. A linear foliation on the 2-torus. In this Section, we consider the simplest example of the situation studied in the previous Section, namely, the example of a linear foliation on the 2-torus. So consider the two-dimensional torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) with the coordinates \((x, y) \in \mathbb{R}^2\), taken modulo integer translations, and the Euclidean metric \( g \) on \( T^2 \):

\[
g = dx^2 + dy^2.
\]

Let \( \tilde{X} \) be the vector field on \( \mathbb{R}^2 \) given by

\[
\tilde{X} = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y},
\]

where \( \alpha \in \mathbb{R} \). Since \( \tilde{X} \) is translation invariant, it determines a vector field \( X \) on \( T^2 \). The orbits of \( X \) define a one-dimensional foliation \( \mathcal{F} \) on \( T^2 \). The leaves of \( \mathcal{F} \) are the images of the parallel lines \( \tilde{L}_{(x_0, y_0)} = \{(x_0 + t, y_0 + t\alpha) : t \in \mathbb{R}\} \), parameterized by \((x_0, y_0) \in \mathbb{R}^2\), under the projection \( \mathbb{R}^2 \to T^2 \).

In the case when \( \alpha \) is rational, all leaves of \( \mathcal{F} \) are closed and are circles, and \( \mathcal{F} \) is given by the fibers of a fibration of \( T^2 \) over \( S^1 \). In the case when \( \alpha \) is irrational, all leaves of \( \mathcal{F} \) are everywhere dense in \( T^2 \).

The one-parameter family \( g_h \) of Riemannian metrics on \( T^2 \) defined by (1) is given by

\[
g_h = \frac{1 + h^{-2} \alpha^2}{1 + \alpha^2} dx^2 + 2\alpha \frac{1 - h^{-2}}{1 + \alpha^2} dx dy + \frac{\alpha^2 + h^{-2}}{1 + \alpha^2} dy^2.
\]

The Laplace operator (on functions) defined by \( g_h \) has the form

\[
\Delta_h = \Delta_F + h^2 \Delta_H,
\]

where

\[
\Delta_F = -\frac{1}{1 + \alpha^2} \left( \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} \right)^2,
\]

\[
\Delta_H = -\frac{h^2}{1 + \alpha^2} \left( -\alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2
\]

are the tangential and the transverse Laplace operators respectively.

The operator \( \Delta_h \) has a complete orthogonal system of eigenfunctions

\[
u_{kl}(x, y) = e^{2\pi i(kx + ly)}, \quad (x, y) \in T^2,
\]

with the corresponding eigenvalues

\[
\lambda_{kl}(h) = (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (k + \alpha l)^2 + \frac{h^2}{1 + \alpha^2} (-\alpha k + l)^2 \right), \quad (k, l) \in \mathbb{Z}^2.
\]

The eigenvalue distribution function of \( \Delta_h \) has the form

\[
N_h = \# \{(k, l) \in \mathbb{Z}^2 : (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (k + \alpha l)^2 + \frac{h^2}{1 + \alpha^2} (-\alpha k + l)^2 \right) < \lambda \}.
\]

Thus we come to the following problem of number theory:

**Problem 1.3.** Find the asymptotic for \( h \to 0 \) of the number of integer points in the ellipse

\[
\{(\xi, \eta) \in \mathbb{R}^2 : (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (\xi + \alpha \eta)^2 + \frac{h^2}{1 + \alpha^2} (-\alpha \xi + \eta)^2 \right) < \lambda \}.
\]
In the case when $\alpha$ is rational, this problem can be easily solved by elementary methods of analysis. In the case when $\alpha$ is irrational, such an elementary solution seems to be unknown, and, in order to solve the problem, the connection of this problem with the spectral theory of the Laplace operator and with adiabatic limits plays an important role.

**Theorem 1.4 ([40]).** The following asymptotic formula for the spectrum distribution function $N_h(\lambda)$ of the operator $\Delta_h$ for a fixed $\lambda \in \mathbb{R}$ holds:

1. For $\alpha \not\in \mathbb{Q}$,
   \[
   N_h(\lambda) = \frac{1}{4\pi} h^{-1} \lambda + o(h^{-1}), \quad h \to 0.
   \]
2. For $\alpha \in \mathbb{Q}$ of the form $\alpha = \frac{p}{q}$, where $p$ and $q$ are coprime,
   \[
   N_h(\lambda) = h^{-1} \sum_{k \in \mathbb{Z}} \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} + o(h^{-1}), \quad h \to 0.
   \]

**Remark 1.** The asymptotic formulas (6) and (7) of Theorem 1.4 look quite different. Nevertheless, it can be shown that

\[
\lim_{p \to +\infty \atop q \to +\infty} \sum_{k \in \mathbb{Z}} \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} = \frac{1}{4\pi} \lambda.
\]

Indeed, one can write

\[
\sum_k \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} = \sum_k g(\xi_k) \Delta \xi_k,
\]

where

\[
g(\xi) = \frac{1}{2\pi^2} (\lambda - \xi^2)^{1/2}, \quad \xi_k = \frac{2\pi}{\sqrt{p^2 + q^2}} k.
\]

This immediately implies that

\[
\lim_{p \to +\infty \atop q \to +\infty} \sum_{k \in \mathbb{Z}} \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} = \frac{1}{2\pi^2} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} (\lambda - \xi^2)^{1/2} d\xi = \frac{1}{4\pi} \lambda.
\]

We now show how to derive the asymptotic formulae of Theorem 1.4 from Theorem 1.2 (see [40] for more details).

**Case 1:** $\alpha \not\in \mathbb{Q}$. In this case $G = \mathbb{T}^2 \times \mathbb{R}$. The source and the range maps $s, r : G \to \mathbb{T}^2$ are defined for any $\gamma = (x, y, t) \in G$ by $s(\gamma) = (x - t, y - \alpha t)$.
and \( r(\gamma) = (x, y) \). For any \((x, y) \in \mathbb{T}^2\), the set \( G^{(x,y)} \) coincides with the leaf \( L_{(x,y)} \) through \((x, y)\) and is diffeomorphic to \( \mathbb{R} \):

\[ L_{(x,y)} = \{(x - t, y - \alpha t) : t \in \mathbb{R} \}. \]

The Riemannian volume form \( \lambda^{(x,y)} \) on \( L_{(x,y)} \) equals \( \sqrt{1 + \alpha^2} \). Finally, the restriction of the operator \( \Delta_F \) to each leaf \( L_{(x,y)} \) coincides with the operator

\[ A = -\frac{1}{1 + \alpha^2} d^2, \]

acting in the space \( L^2(\mathbb{R}, \sqrt{1 + \alpha^2}) \).

Using the Fourier transform, it is easy to compute the Schwartz kernel \( E_\lambda(t, t_1) \) of the spectral projection \( \chi_\lambda(A) \) of the operator \( A \), corresponding to the semi-axis \((-\infty, \lambda] \) (relative to the volume form \( \sqrt{1 + \alpha^2} \)):

\[ E_\lambda(t, t_1) = \frac{1}{2\pi \sqrt{1 + \alpha^2}} \int_{\mathbb{R}} e^{i(t - t_1)\xi} \chi_\lambda \left( \frac{\xi^2}{1 + \alpha^2} \right) d\xi. \]

Then, for any \( \gamma = (x, y, t) \in G = \mathbb{T}^2 \times \mathbb{R} \), we have

\[ e_\lambda(\gamma) = E_\lambda(0, t) = \frac{1}{2\pi \sqrt{1 + \alpha^2}} \int_{\mathbb{R}} e^{-it\xi} \chi_\lambda \left( \frac{\xi^2}{1 + \alpha^2} \right) d\xi. \]

The restriction of \( e_\lambda \) to \( \mathbb{T}^2 \) is given by

\[ e_\lambda(x, y) = E_\lambda(0, 0) = \frac{1}{2\pi \sqrt{1 + \alpha^2}} \int_{\mathbb{R}} \chi_\lambda \left( \frac{\xi^2}{1 + \alpha^2} \right) d\xi = \frac{1}{\pi} \sqrt{\lambda}, \lambda > 0. \]

We get that the spectrum distribution function \( N_F(\lambda) \) of the operator \( \Delta_F \) has the form:

\[ N_F(\lambda) = \int_{\mathbb{T}^2} e_\lambda(x, y) dxdy = \frac{1}{\pi} \sqrt{\lambda}, \lambda > 0. \]

By Theorem 1.2, we obtain

\[ N_h(\lambda) = h^{-1} \frac{1}{\pi} \int_{-\infty}^\lambda (\lambda - \tau)^{1/2} d\tau N_F(\tau) + o(h^{-1}) \]
\[ = \frac{1}{4\pi} h^{-1} \lambda + o(h^{-1}), \quad h \to 0. \]

**Case 2:** \( \alpha \in \mathbb{Q} \) of the form \( \alpha = \frac{p}{q} \), where \( p \) and \( q \) are coprime. In this case, the holonomy groupoid is \( \mathbb{T}^2 \times (\mathbb{R}/q\mathbb{Z}) \). The leaf \( L_{(x,y)} \) through any \((x, y)\) is the circle \( \{ (x + t, y + \alpha t) : t \in \mathbb{R}/q\mathbb{Z} \} \) of length \( l = \sqrt{p^2 + q^2} \). The restriction of the operator \( \Delta_F \) to each \( L_{(x,y)} \) coincides with the operator

\[ A = -\frac{1}{1 + \alpha^2} d^2, \]

acting in the space \( L^2(\mathbb{R}/q\mathbb{Z}, \sqrt{1 + \alpha^2}) \).
Using the Fourier transform, it is easy to see that the kernel of the spectral projection $\chi_\lambda(A)$ in $L^2(\mathbb{R}/\mathbb{q}\mathbb{Z}, \sqrt{1+\alpha^2} dt)$ is given by the formula

$$E_\lambda(t, t_1) = \frac{1}{\sqrt{p^2 + q^2}} \sum_{k \in \mathbb{Z}} e^{\frac{2\pi i}{\sqrt{p^2 + q^2}} k(t-t_1)}.$$

For any $\gamma = (x, y, t) \in G = \mathbb{T}^2 \times (\mathbb{R}/\mathbb{q}\mathbb{Z})$, we have

$$e_\lambda(\gamma) = E_\lambda(0, t) = \frac{1}{\sqrt{p^2 + q^2}} \sum_{k \in \mathbb{Z}} e^{-\frac{2\pi i}{\sqrt{p^2 + q^2}} k t}.$$

We get that the spectrum distribution function $N_F(\lambda)$ of $\Delta_F$ is of the form:

$$N_F(\lambda) = \int_{\mathbb{T}^2} e_\lambda(x, y) dxdy = \frac{1}{\sqrt{p^2 + q^2}} \#\{k \in \mathbb{Z} : |k| < \sqrt{\frac{\lambda}{2\pi}} \sqrt{p^2 + q^2}\}.$$

By Theorem 1.2 we obtain for $h \to 0$

$$N_h(\lambda) = h^{-1} \frac{1}{\pi} \int_{-\infty}^{\lambda} (\lambda - \tau)^{1/2} d\tau N_F(\tau) + o(h^{-1})$$

$$= h^{-1} \frac{1}{\pi \sqrt{p^2 + q^2}} \sum_{|k|<\frac{\lambda}{2\pi} \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} + o(h^{-1}).$$

1.3. **Riemannian Heisenberg manifolds.** In this Section we consider the adiabatic limits associated with one-dimensional foliations given by the orbits of invariant flows on Riemannian Heisenberg manifolds. These foliations are examples of non-Riemannian foliations.

Recall that the real three-dimensional Heisenberg group $H$ is the Lie subgroup of $\text{GL}(3, \mathbb{R})$ consisting of all matrices of the form

$$\gamma(x, y, z) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

Its Lie algebra $\mathfrak{h}$ is the Lie subalgebra of $\text{gl}(3, \mathbb{R})$ consisting of all matrices of the form

$$X(x, y, z) = \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

A Riemannian Heisenberg manifold $M$ is defined to be a pair $(\Gamma \backslash H, g)$, where $\Gamma = \{\gamma(x, y, z) : x, y, z \in \mathbb{Z}\}$ is a uniform discrete subgroup of $H$ and $g$ is a Riemannian metric on $\Gamma \backslash H$ whose lift to $H$ is left $H$-invariant.

It is easy to see that $g$ is uniquely determined by the value of its lift to $H$ at the identity $\gamma(0, 0, 0)$, that is, by a symmetric positive definite $3 \times 3$-matrix.
In the following, we will assume that the metric $g$ corresponds to a $3 \times 3$-matrix of the form

$$
\begin{pmatrix}
    h_{11} & h_{12} & 0 \\
    h_{12} & h_{22} & 0 \\
    0 & 0 & g_{33}
\end{pmatrix}.
$$

(8)

The lift of $g$ to $H$ is given by the formula

$$
g(\gamma(x, y, z)) = h_{11} dx^2 + 2h_{12} dx dy + h_{22} dy^2 + g_{33}(dz - x dy)^2,
$$

$(x, y, z) \in \mathbb{R}^3$.

The corresponding Laplace operator has the form

$$
\Delta = - \left\{ \frac{1}{h_{11}h_{22} - h_{12}^2} \left[ h_{22} \frac{\partial^2}{\partial x^2} - h_{12} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) \right) + \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) \frac{\partial}{\partial x} \right] + h_{11} \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right)^2 \right\} + \frac{1}{g_{33}} \frac{\partial^2}{\partial z^2}.
$$

Theorem 1.5 ([19]). The spectrum of the Laplace operator $\Delta$ on functions on $M$ (with multiplicities) has the form

$$
\text{spec } \Delta = \Sigma_1 \cup \Sigma_2,
$$

where

$$
\Sigma_1 = \{ \lambda(a, b) = 4\pi^2 \frac{h_{22}a^2 - 2h_{12}ab + h_{11}b^2}{h_{11}h_{22} - h_{12}^2} : a, b \in \mathbb{Z} \},
$$

$$
\Sigma_2 = \{ \mu(c, k) = \frac{4\pi^2 c^2}{g_{33}} + \frac{2\pi c(2k + 1)}{\sqrt{h_{11}h_{22} - h_{12}^2}} \text{ with mult. } 2c : c \in \mathbb{Z}^+, \; k \in \mathbb{Z}^+ \cup \{0\} \}.
$$

Remark 2. As shown in [19], for an arbitrary left $H$-invariant metric $g$ on $H$, there exists a left $H$-invariant metric $g_1$, which corresponds to a $3 \times 3$-matrix of the form (8), such that Riemannian Heisenberg manifolds $(\Gamma\backslash H, g)$ and $(\Gamma\backslash H, g_1)$ are isometric. Therefore, Theorem 1.5 provides a solution of the problem of calculation of the spectrum of the Laplace operator on functions for an arbitrary Riemannian Heisenberg manifold.

Now we assume that the metric $g$ on $M$ corresponds to a $3 \times 3$-matrix of the form

$$
\begin{pmatrix}
    h_{11} & 0 & 0 \\
    0 & h_{22} & 0 \\
    0 & 0 & g_{33}
\end{pmatrix}.
$$

In this case, one can write down explicitly all the eigenfunctions of the corresponding Laplace operator on functions. This fact plays an important role in the proof of the following theorem.
Theorem 1.6 ([11]). The spectrum of the Laplace operator $\Delta$ on differential one forms on $M$ (with multiplicities) has the form
\[
\text{spec } \Delta = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,
\]
where
\[
\Sigma_1 = \{ \lambda_{\pm}(a, b) = 4\pi^2 \left( \frac{a^2}{h_{11}} + \frac{b^2}{h_{22}} \right)
+ \frac{g_{33}}{h_{11}h_{22}} \pm \sqrt{\frac{g_{33}^2}{h_{11}h_{22}} + 16\pi^2 \frac{g_{33}}{h_{11}h_{22}} \left( \frac{a^2}{h_{11}} + \frac{b^2}{h_{22}} \right)^2}
\right) \quad \text{with mult. } 2 : a, b \in \mathbb{Z},
\]
\[
\Sigma_2 = \{ \mu(c, k) = \frac{4\pi^2 c^2}{g_{33}} + \frac{2\pi c(2k + 1)}{\sqrt{h_{11}h_{22}}} \quad \text{with mult. } 2c :
\]
\[
c \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \cup \{0\},
\]
\[
\Sigma_3 = \{ \mu_{\pm}(a, k) = \frac{4\pi^2 c^2}{g_{33}} + \frac{2\pi c(2k + 1)}{\sqrt{h_{11}h_{22}}}
+ \frac{g_{33}}{h_{11}h_{22}} \pm \sqrt{\left( \frac{4\pi c}{\sqrt{h_{11}h_{22}}} + \frac{g_{33}}{h_{11}h_{22}} \right)^2 + 8k \frac{2\pi c g_{33}}{\sqrt{h_{11}h_{22}}}}
\right) \quad \text{with mult. } 2c :
\]
\[
c \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \cup \{0\}.
\]
We refer the reader to [11] for a similar calculation of the spectrum of the Dirac operator on Riemannian Heisenberg manifolds.

Let $\alpha \in \mathbb{R}$. Consider the left-invariant vector field on $H$ associated with
\[
X(1, \alpha, 0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}.
\]
Since $X(1, \alpha, 0)$ is a left-invariant vector field, it determines a vector field on $M = \Gamma \setminus H$. The orbits of this vector field define a one-dimensional foliation $\mathcal{F}$ on $M$. The leaf through a point $\Gamma \gamma(x, y, z) \in M$ is described as
\[
L_{\Gamma \gamma(x, y, z)} = \{ \Gamma \gamma(x + t, y + \alpha t, z + \alpha tx + \frac{\alpha t^2}{2}) \in \Gamma \setminus H : t \in \mathbb{R} \}.
\]
We assume that $g$ corresponds to the identity $3 \times 3$-matrix. Consider the adiabatic limit associated with the Riemannian Heisenberg manifold $(\Gamma \setminus H, g)$ and the one-dimensional foliation $\mathcal{F}$. The Riemannian metric $g_h$ on $\Gamma \setminus H$ defined by ([11]) corresponds to the matrix
\[
\frac{1 + h^{-2}}{1 + \alpha^2} \begin{bmatrix} 0 & \frac{\alpha}{h^2} & \frac{\alpha}{h^2} \\ \frac{\alpha}{h^2} & 0 & \frac{\alpha}{h^2} \\ \frac{\alpha}{h^2} & \frac{\alpha}{h^2} & 0 \end{bmatrix}, \quad h > 0.
\]
The corresponding Laplacian (on functions) on the group $H$ has the form:

$$
\Delta_h = -\frac{1}{1 + \alpha^2} \left[ \left( \frac{\partial}{\partial x} + \alpha \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) \right)^2 + h^2 \left( -\alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right)^2 \right] - h^2 \frac{\partial^2}{\partial z^2}.
$$

Using an explicit computation of the heat kernel on the Heisenberg group, one can show the following asymptotic formula.

**Theorem 1.7** ([39]). For any $t > 0$, we have as $h \to 0$

$$
\text{tr} e^{-t\Delta_h} = \frac{\hbar^{-2}}{4\pi} \int_{-\infty}^{+\infty} \frac{\eta}{\sinh(t\eta)} e^{-t\eta^2} d\eta + o(h^{-2}).
$$

**Remark 3.** The formula (9) looks quite different from what we have in the case of a Riemannian foliation. For instance, if $\mathcal{F}$ is a one-dimensional Riemannian foliation on a three-dimensional closed Riemannian manifold $M$ given by the orbits of a non-singular isometric flow such that the set of closed orbits has measure zero, then, by Theorem 1.1 (or, equivalently, by Theorem 1.2), the asymptotic formula for the trace of the heat operator $e^{-t\Delta_h}$ in the adiabatic limit has the following form: for any $t > 0$,

$$
\text{tr} e^{-t\Delta_h} = \frac{\hbar^{-2}}{4\pi t} \int_{-\infty}^{+\infty} e^{-t\eta^2} d\eta + o(h^{-2})
$$

$$
= \frac{\hbar^{-2}}{4\sqrt{\pi}t^3} + o(h^{-2}), \quad h \to 0.
$$

So, in comparison with the case of Riemannian foliations, the formula (9) contains an additional factor $\frac{\eta}{\sinh(t\eta)}$ related with the distortion of the transverse part of the Riemannian metric along the orbits of the flow.

**Remark 4.** It would be quite interesting to write the formula (9) in a form similar to the formula (4).

2. **Adiabatic limits and differentiable spectral sequence**

In this Section, we will discuss the problem of “small eigenvalues” of the Laplace operator in the adiabatic limit and its relation with the differentiable spectral sequence of the foliation. We will start with some background information on the differentiable spectral sequence.

2.1. **Preliminaries on the differentiable spectral sequence.** As usual, let $\mathcal{F}$ be a codimension $q$ foliation on a closed manifold $M$. The differentiable spectral sequence $(E_k, d_k)$ of $\mathcal{F}$ is a direct generalization of the differentiable version of the Leray spectral sequence for fibrations, which converges to the de Rham cohomology of $M$. 
Denote by $\Omega$ the space of smooth differential forms and by $\Omega'$ the space of smooth differential $r$-forms on $M$. Similar to the bundle case, the differentiable spectral sequence $(E_k, d_k)$ of $F$ is defined by the decreasing filtration by differential subspaces

$$\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_q \supset \Omega_{q+1} = 0,$$

where the space $\Omega^r_k$ of $r$-forms of filtration degree $\geq k$ consists of all $\omega \in \Omega^r$ such that $i_X \omega = 0$ for all $X = X_1 \land \cdots \land X_{r-k+1}$, where the $X_i$ are vector fields tangent to the leaves. Roughly speaking, $\omega$ in $\Omega^r_k$ iff it is of degree $\geq k$ transversely to the leaves.

Recall that the induced spectral sequence $(E_k, d_k)$ is defined in the following standard way (see, for instance, [28]):

$$Z^{u,v}_k = \Omega^{u+v}_u \cap d^{-1} (\Omega^{u+v+1}_{u+k}) \ , \quad Z^{u,v}_\infty = \Omega^{u+v}_u \cap \ker d \ ,$$

$$B^{u,v}_k = \Omega^{u+v}_u \cap d (\Omega^{u+v-1}_{u-k}) \ , \quad B^{u,v}_\infty = \Omega^{u+v}_u \cap \text{Im } d \ ,$$

$$E^{u,v}_k = \frac{Z^{u,v}_k}{Z^{u+1,v-1}_{k-1} + B^{u,v}_{k-1}} \ , \quad E^{u,v}_\infty = \frac{Z^{u,v}_\infty}{Z^{u+1,v-1}_\infty + B^{u,v}_\infty} .$$

We assume $B_{-1}^{u,v} = 0$, so $E^{u,v}_0 = \Omega^{u+v}_u / \Omega^{u+v}_{u+1}$. Each homomorphism $d_k : E^{u,v}_k \to E^{u+k,v+k+1}_k$ is canonically induced by $d$.

The terms $E^{0,*}_1$ and $E^{*,0}_2$ are respectively called leafwise cohomology and basic cohomology, and $E^{*,0}_2$ is isomorphic to the transverse cohomology [20] (also called Haefliger cohomology).

The $C^\infty$ topology of $\Omega$ induces a topological vector space structure on each term $E_k$ such that $d_k$ is continuous. A subtle problem here is that $E_k$ may not be Hausdorff [20]. So it makes sense to consider the subcomplex given by the closure of the trivial subspace, $0_k \subset E_k$, as well as the quotient complex $\tilde{E}_k = E_k / 0_k$, whose differential operator will be also denoted by $d_k$.

2.2. Riemannian foliations. For a Riemannian foliation $F$, each term $E_k$ of the differentiable spectral sequence $(E_k, d_k)$ is Hausdorff if $k \geq 2$, and $H(0_1) = 0$. So $E_k \cong \tilde{E}_k$ for $k \geq 2$. The proof of this result given in [29] uses the structure theorem for Riemannian foliations due to Molino [31, 32] to reduce the problem to transitive foliations, and, for transitive foliations, it uses a construction of a parametrix for the de Rham complex given by Sarkaria [37]. Moreover, it turns out that, for $k \geq 2$, the terms $E_k$ are homotopy invariants of Riemannian foliations [3]. (This result generalizes a previous work showing the topological invariance of the basic cohomology [15].)

Now return to adiabatic limits. So let $g$ be a Riemannian metric on $M$ and $g_h$ be the one-parameter family of metrics defined by $h$. Denote by $\Delta^*_h$ the Laplace operator on differential $r$-forms on $M$ defined by $g_h$, and by

$$0 \leq \lambda^*_0(h) \leq \lambda^*_1(h) \leq \lambda^*_2(h) \leq \cdots$$
its spectrum (with multiplicities). It is well known that the eigenvalues of the Laplacian on differential forms vary continuously under continuous perturbations of the metric, and thus the “branches” of eigenvalues $\lambda^r_i(h)$ depend continuously on $h > 0$. In this Section, we shall only consider the “branches” $\lambda^r_i(h)$ that are convergent to zero as $h \to 0$; roughly speaking, the “small” eigenvalues. The asymptotics as $h \to 0$ of these metric invariants are related to the differential invariant $\hat{E}^r_1$ and the homotopy invariants $E^r_k$, $k \geq 2$, as follows.

Theorem 2.1 (II). With the above notation, for Riemannian foliations on closed Riemannian manifolds we have

$$\dim \hat{E}^r_1 = \sharp \{ i \mid \lambda^r_i(h) = O \left( h^2 \right) \quad \text{as} \quad h \to 0 \},$$
$$\dim E^r_k = \sharp \{ i \mid \lambda^r_i(h) = O \left( h^{2k} \right) \quad \text{as} \quad h \to 0 \}, \quad k \geq 2.$$

We refer to [23] for a particular form of this theorem in the case of Riemannian flows.

As a part of the proof of Theorem 2.1 and also because of its own interest, the asymptotics of eigenforms of $\Delta^r_h$ corresponding to “small” eigenvalues were also studied. This study was begun in [30] for the case of Riemannian bundles, and continued in [16] for general complementary distributions.

Here we formulate the results obtained in [1] for the case of Riemannian foliations. Recall that $\Theta_h$ is an isomorphism of Hilbert spaces, which moves our setting to the fixed Hilbert space $L^2 \Omega$ (see (2)). The “rescaled Laplacian” $L_h = \Theta_h \Delta_h \Theta_h^{-1}$ has the same spectrum as $\Delta_h$, and eigenspaces of $\Delta_h$ are transformed into eigenspaces of $L_h$ by $\Theta_h$. It turns out that eigenspaces of $L_h$ corresponding to “small” eigenvalues are convergent as $h \to 0$ when the metric $g$ is bundle-like, and the limit is given by a nested sequence of bigraded subspaces,

$$\Omega \supset H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_\infty.$$

The definition of $H_1, H_2$ was given in [2] as a Hodge theoretic approach to $(E_1, d_1)$ and $(E_2, d_2)$, which is based on the study of leafwise heat flow. The space $H_1$ is defined as the space of smooth leafwise harmonic forms:

$$H_1 = \{ \omega \in \Omega : \Delta_F \omega = 0 \}.$$

As shown in [2], the orthogonal projection in $L^2 \Omega$ on the kernel of $\Delta_F$ in $L^2 \Omega$ restricts to smooth differential forms, yielding an operator $\Pi : \Omega \to H_1$. We define the operator $d_1$ on $H_1$ as $d_1 = \Pi d_H$. The adjoint of $d_1$ in $H_1$ equals $\delta_1 = \Pi \delta_H$. Finally, we take $\Delta_1 = d_1 \delta_1 + \delta_1 d_1$ on $H_1$ and put

$$H_2 = \ker \Delta_1.$$

The other spaces $H_k$ are defined in [1] as an extension of this Hodge theoretic approach to the whole spectral sequence $(E_k, d_k)$. In particular,

$$H_1 \cong \hat{E}_1, \quad H_k \cong E_k, \quad k = 2, 3, \ldots, \infty,$$
as bigraded topological vector spaces. Thus this sequence stabilizes (that is, \( \mathcal{H}_k = \mathcal{H}_\infty \) for \( k \) large enough) because the differentiable spectral sequence is convergent in a finite number of steps. The convergence of eigenforms corresponding to “small” eigenvalues is precisely stated in the following result, where \( L^2\mathcal{H}_1 \) denotes the closure of \( \mathcal{H}_1 \) in \( L^2\Omega \).

**Theorem 2.2.** For any Riemannian foliation on a closed manifold with a bundle-like metric, let \( \omega_i \) be a sequence in \( \Omega^r \) such that \( \|\omega_i\| = 1 \) and

\[
\langle L_{h_i}\omega_i, \omega_i \rangle \in o\left(h_i^{2(k-1)}\right)
\]

for some fixed integer \( k \geq 1 \) and some sequence \( h_i \to 0 \). Then some subsequence of the \( \omega_i \) is strongly convergent, and its limit is in \( L^2\mathcal{H}_1^r \) if \( k = 1 \), and in \( \mathcal{H}_k^r \) if \( k \geq 2 \).

To simplify notation let \( m^r_i = \dim \tilde{E}^r_1 \), and let \( m^r_k = \dim E^r_k \) for each \( k = 2, 3, \ldots, \infty \). Thus Theorem 2.1 establishes \( \lambda_i^r(h) = O\left(h^{2k}\right) \) for \( i \leq m^r_k \), yielding \( \lambda_i^r(h) \equiv 0 \) for \( i \) large enough. For every \( h > 0 \), consider the nested sequence of graded subspaces

\[
\Omega \supset \mathcal{H}_1(h) \supset \mathcal{H}_2(h) \supset \mathcal{H}_3(h) \supset \cdots \supset \mathcal{H}_\infty(h),
\]

where \( \mathcal{H}_k^r(h) \) is the space generated by the eigenforms of \( \Delta_h \) corresponding to eigenvalues \( \lambda_i^r(h) \) with \( i \leq m^r_k \); in particular, we have \( \mathcal{H}_k(h) = \mathcal{H}_\infty(h) = \ker \Delta_h \) for \( k \) large enough. Set also \( \mathcal{H}_k(0) = \mathcal{H}_k \). We have \( \dim \mathcal{H}_k^r(h) = m^r_k \) for all \( h > 0 \), so the following result is a sharpening of Theorem 2.1.

**Corollary 2.3.** For any Riemannian foliation on a closed manifold with a bundle-like metric and \( k = 2, 3, \ldots, \infty \), the assignment \( h \mapsto \mathcal{H}_k^r(h) \) defines a continuous map from \( [0, \infty) \) to the space of finite dimensional linear subspaces of \( L^2\Omega^r \) for all \( r \geq 0 \). If \( \dim \tilde{E}^r_1 < \infty \), then this also holds for \( k = 1 \).

By the standard perturbation theory, the map \( h \mapsto \mathcal{H}_k^r(h) \) is, clearly, \( C^\infty \) on \( (0, \infty) \) for any Riemannian foliation on a closed manifold with a bundle-like metric, \( k = 2, 3, \ldots, \infty \) and \( r \geq 0 \). As shown in \( [30] \), this map is \( C^\infty \) up to \( h = 0 \), if the foliation is given by the fibers of a Riemannian fibration. In the next section, we will see an example of a Riemannian foliation and a bundle-like metric such that the map \( h \mapsto \mathcal{H}_k^r(h) \) is not \( C^\infty \) at \( h = 0 \).

### 2.3. A linear foliation on the 2-torus.

In this Section, we consider the simplest example of the situation studied in the previous section, namely — the example of a linear foliation on the 2-torus. So, as in Section 1.2 consider the two-dimensional torus \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \) with the coordinates \( (x, y) \), the one-dimensional foliation \( \mathcal{F} \) defined by the orbits of the vector field

\[
\tilde{X} = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y},
\]

where \( \alpha \in \mathbb{R} \), and the Euclidean metric \( g = dx^2 + dy^2 \) on \( \mathbb{T}^2 \). The eigenvalues of the corresponding Laplace operator \( \Delta_h \) (counted with
multiplicities) are described as follows:

\[ \text{spec } \Delta_h^0 = \text{spec } \Delta_h^2 = \{ \lambda_{kl}(h) : (k, l) \in \mathbb{Z}^2 \}, \]

\[ \text{spec } \Delta_h^1 = \{ \lambda_{k_1l_1}(h) + \lambda_{k_2l_2}(h) : (k, l) \in \mathbb{Z}^2, (k_2, l_2) \in \mathbb{Z}^2 \}, \]

where \( \lambda_{kl}(h) \) are given by (15). So, for \( \alpha \notin \mathbb{Q} \), small eigenvalues appear only if \((k, l) = (0, 0)\) and \((k_1, l_1) = (k_2, l_2) = (0, 0)\) and have the form

\[ \lambda_0^0(h) = \lambda_0^2(h) = 0, \quad \lambda_1^0(h) = \lambda_1^1(h) = 0. \]

For \( \alpha \in \mathbb{Q} \) of the form \( \alpha = \frac{p}{q} \), where \( p \) and \( q \) are coprime, small eigenvalues appear only if \((k, l) = t(p, q), t \in \mathbb{Z}, \) and \((k_1, l_1) = t_1(p, q), (k_2, l_2) = t_2(p, q), t_1, t_2 \in \mathbb{Z} \). So there are infinitely many different branches of eigenvalues \( \lambda_h \) with \( \lambda_h = O(h^2) \) as \( h \to 0 \), and all the branches of eigenvalues \( \lambda_h \) with \( \lambda_h = O(h^4) \) as \( h \to 0 \) are given by (10).

Now let us turn to the differential spectral sequence. By a straightforward computation, one can show that

\[ E_2^{u, v} = E_\infty^{u, v} = \mathbb{R}, \quad u = 0, 1, \quad v = 0, 1, \]

that agrees with the above description of small eigenvalues.

The case of \( E_1 \) is more interesting. First of all, it depends on whether \( \alpha \) is rational or not. For \( \alpha \in \mathbb{Q} \), \( \mathcal{F} \) is given by the fibers of a trivial fibration \( \mathbb{T}^2 \to S^1 \), and, therefore, for any \( u = 0, 1 \) and \( v = 0, 1 \), we have

\[ E_1^{u, v} = E_1^{u, v} = \Omega_u(S^1) \otimes H^v(S^1) = C^\infty(S^1). \]

For \( \alpha \notin \mathbb{Q} \), we have

\[ \hat{E}_1^{u, v} = \mathbb{R}, \quad u = 0, 1, \quad v = 0, 1. \]

The description of \( E_1 \) is more complicated and depends on the diophantine properties of \( \alpha \). Recall that \( \alpha \notin \mathbb{Q} \) is called diophantine, if there exist \( c > 0 \) and \( d > 1 \) such that, for any \( p \in \mathbb{Z} \setminus \{0\} \) and \( q \in \mathbb{Z} \setminus \{0\} \), we have

\[ |q\alpha - p| > \frac{c}{|q|^d}. \]

Otherwise, \( \alpha \) is called Liouville. It is easy to see that \( E_1^{1, 0} = E_1^{0, 0} \) and \( E_1^{1, 1} = E_1^{0, 1} \). As shown in [21] and [35], we have

- \( E_1^{0, 0} = \mathbb{R} \);
- \( E_1^{0, 1} = \mathbb{R} \) if \( \alpha \) is diophantine and \( E_1^{0, 1} \) is infinite dimensional if \( \alpha \) is Liouville.

So when \( \alpha \) is a Liouville’s number, \( \hat{E}_1^{0, 1} = \hat{E}_1^{1, 1} \neq 0 \). As a direct consequence of this fact and [1] Theorem D, we obtain that, when \( \alpha \) is a Liouville’s number, there exists a bundle-like metric on \( \mathbb{T}^2 \) such that the associated map \( h \mapsto \mathcal{H}_h \), which is continuous on \([0, \infty)\) by Corollary 2.3 and \( C^\infty \) on \((0, \infty)\), is not \( C^\infty \) at \( h = 0 \).
2.4. Riemannian Heisenberg manifolds. In this Section, we discuss similar problems for adiabatic limits associated with the Riemannian Heisenberg manifold $(\Gamma \backslash H, g)$ and the one-dimensional foliation $\mathcal{F}$ introduced in Section 1.3 (see [41]). We assume that $g$ corresponds to the identity matrix, and the one-dimensional foliation $\mathcal{F}$ is defined by the vector field $X(1,0,0)$.

The corresponding Riemannian metric $g_h$ on $\Gamma \backslash H$ defined by (1) is given by the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & h^{-2} & 0 \\
0 & 0 & h^{-2}
\end{pmatrix}, \quad h > 0.
\]

By Theorem 1.5, it follows that the spectrum of the Laplacian $\Delta_h$ on 0- and 3-forms on $M$ (with multiplicities) is described as
\[
\text{spec } \Delta^0_h = \text{spec } \Delta^3_h = \Sigma_{1,h} \cup \Sigma_{2,h},
\]
where
\[
\begin{align*}
\Sigma_{1,h} &= \{ \lambda_h(a,b) = 4\pi^2(a^2 + h^2b^2) : a, b \in \mathbb{Z} \}, \\
\Sigma_{2,h} &= \{ \mu_h(c,k) = 2\pi c(2k+1)h + 4\pi^2 c^2 h^2 \text{ with mult. } 2c, c \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \cup \{0\} \}.
\end{align*}
\]

First, note that, for any $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z} \setminus \{0\}$,
\[
\lambda_h(a,b) > 4\pi^2 h^2, \quad h > 0,
\]
and for any $c \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+ \cup \{0\}$
\[
\mu_h(c,k) > 4\pi^2 h^2, \quad h > 0.
\]

Therefore, for any $h > 0$,
\[
\lambda^0(h) = \lambda^3(h) = 0, \quad \lambda^0_1(h) = \lambda^3_1(h) > 4\pi^2 h^2.
\]

Next, we see that, for any $b \in \mathbb{Z} \setminus \{0\}$,
\[
\lambda_h(0,b) = 4\pi^2 b^2 h^2 = O(h^2), \quad h \to 0.
\]

Since we have infinitely many different branches of eigenvalues $\lambda_h$ with $\lambda_h = O(h^2)$ as $h \to 0$, we conclude that, for any $i > 0$,
\[
\lambda^0_i(h) = \lambda^3_i(h) = O(h^2), \quad h \to 0.
\]

By Theorem 1.6, the spectrum of the Laplace operator $\Delta_h$ on one and two forms on $M$ (with multiplicities) has the form
\[
\text{spec } \Delta^1_h = \text{spec } \Delta^2_h = \Sigma_{1,h} \cup \Sigma_{2,h} \cup \Sigma_{3,h},
\]
where
\[ \Sigma_{1, h} = \{ \lambda_h, \pm (a, b) = 4\pi^2 (a^2 + h^2 b^2) + \frac{1 \pm \sqrt{1 + 16\pi^2 (a^2 + h^2 b^2)}}{2} \text{ with mult. } 2 : a, b \in \mathbb{Z} \}, \]
\[ \Sigma_{2, h} = \{ \mu_h (c, k) = 4\pi^2 c^2 h^2 + 2\pi c (2k + 1)h \text{ with mult. } 2c : c \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \cup \{0\} \}, \]
\[ \Sigma_{3, h} = \{ \mu_h, \pm (c, k) = 4\pi^2 c^2 h^2 + 2\pi c (2k + 1)h + \frac{1 \pm \sqrt{(4\pi ch + 1)^2 + 16k\pi ch}}{2} \text{ with mult. } 2c : c \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \cup \{0\} \}. \]

Observe that, for any \( b \in \mathbb{Z} \setminus \{0\}, \)
\[ \lambda_{h, -}(0, b) = 4\pi^2 b^2 h^2 + \frac{1 - \sqrt{1 + 16\pi^2 b^2 h^2}}{2} = 16\pi^4 b^4 h^4 + O(h^4), \quad h \to 0, \]
and, for any \( \lambda \in \text{spec } \Delta_h \setminus \{0\}, \)
\[ \lambda > C h^4, \quad h > 0, \]
with some constant \( C > 0. \) Therefore, we have
\[ \lambda_0^0(h) = \lambda_0^1(h) = \lambda_1^1(h) = \lambda_1^2(h) = 0, \quad h > 0, \]
and, by the above argument, we obtain, for any \( i > 1 \)
\[ \lambda_i^1(h) = \lambda_i^2(h) = O(h^4), \quad h \to 0. \]

We now turn to the differentiable spectral sequence. By a straightforward computation, one can show that \( E_3 = E_\infty, \) all the terms \( \hat{E}_1^r \) are infinite-dimensional and, for the basic cohomology, we have
\[ E_2^{0, 0} = \mathbb{R}, \quad E_2^{1, 0} = \mathbb{R}, \quad E_2^{2, 0} = C^\infty(S^1). \]
So we get that, in this case, for \( r = 0 \) and \( r = 3, \)
\[ \dim \hat{E}_1^r = \# \left\{ i \mid \lambda_i^r(h) = O(h^2) \text{ as } h \to 0 \right\} = \infty, \]
\[ \dim E_k^r = \# \left\{ i \mid \lambda_i^r(h) = O(h^{2k}) \text{ as } h \to 0 \right\} = 1, \quad k \geq 2. \]
and, for \( r = 1 \) and \( r = 2 \)
\[ \dim \hat{E}_1^r = \# \left\{ i \mid \lambda_i^r(h) = O(h^2) \text{ as } h \to 0 \right\} = \infty, \]
\[ \dim E_k^r = \# \left\{ i \mid \lambda_i^r(h) = O(h^4) \text{ as } h \to 0 \right\} = \infty, \]
\[ \dim E_k^r = \# \left\{ i \mid \lambda_i^r(h) = O(h^{2k}) \text{ as } h \to 0 \right\} = 2, \quad k \geq 3. \]

A more precise information can be obtained from the consideration of the corresponding eigenspaces that will be discussed elsewhere.

**Remark 5.** It is quite possible that both the asymptotic formula of Theorem [17] and the investigation of small eigenvalues given in this Section can be extended to the differential form Laplace operator on an arbitrary
Riemannian Heisenberg manifold. Nevertheless, we believe that many essentially new features of adiabatic limits on Riemannian Heisenberg manifolds can be already seen in the particular cases, which were considered in this paper, and we don’t expect anything rather different in the general case.

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