Classical and Quantum Anisotropic Wormholes in Pure General Relativity

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Abstract

In the homogeneous and isotropic Friedmann-Robertson-Walker minisuperspace model, it is known that there are no Euclidean wormhole solutions in the pure gravity system. Here it is demonstrated explicitly that in Taub cosmology, which is one of the simplest anisotropic cosmology models, wormhole solutions do exist in pure general relativity in both classical and quantum contexts.

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Some time ago, the “Euclidean wormhole physics” stimulated an enormous excitement in the theoretical physics society due to the possible, provocative effects it may have on low-energy physics. For instance, the wormhole configuration may result in the generic loss of quantum coherence [2,3] or provide a mechanism for fixing the observable value of the cosmological constant to zero [1,2]. As for the quantitative aspect of this wormhole physics, the construction of explicit wormhole instanton solutions had been actively attempted [4-6]. To our disappointment, however, in the simplest minisuperspace model, homogeneous and isotropic wormhole solutions had been found only in the presence of somewhat exotic matter fields [8] but not in the pure gravity system. In this letter, we shall demonstrate that wormholes with the simplest anisotropic structure do exist in pure general relativity in both classical and quantum contexts. Classically, wormholes are Euclidean metrics which are solutions to the Euclidean classical field equations representing spacetimes consisting of two asymptotically Euclidean regions joined by a narrow tube. In the quantum sense, on the other hand, and particularly in the context of canonical quantum cosmology, “quantum wormholes” [8] may be identified with a state represented by a solution to the Wheeler-DeWitt (WD) equation satisfying a certain boundary condition [7] describing the typical wormhole configuration. Now consider the system of pure Einstein gravity with “negative” cosmological constant, i.e., the anti-de Sitter spacetime in 4-dim. Its Einstein-Hilbert action and the associated Einstein field equation are given respectively by

$$S = \frac{1}{16\pi G} \left[ \int_M d^4x \sqrt{g} (R - 2\Lambda) + 2 \int_{\partial M} d^3x \sqrt{h} (K - K_0) \right],$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

(1)

where we introduced the Gibbons-Hawking surface term [9] on the boundary $\partial M$ and $\Lambda = -|\Lambda| < 0$. We stress that the addition of negative cosmological constant here is not the necessary condition for the existence of wormhole solutions as we shall see in a moment. Then, in order to look for “anisotropic” (both classical and quantum) wormhole solutions, we choose to work in “Taub cosmology model” [10] (which is one of the simplest homogeneous, anisotropic cosmology models known) described by the metric
\[ ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}(e^{2\beta(t)})_{ij}\sigma^i \otimes \sigma^j \]  

(2)

where the Euclidean signature metric can be obtained by replacing \( t \) by the Euclidean time \( \tau = it \) and thus \(-N^2(t)dt^2\) by \(N^2(\tau)d\tau^2\). \( N(t) \) and \( a(t) = e^{\alpha(t)} \) denote the lapse function and the scale factor respectively and \( \beta_{ij}(t) = \text{diag}(\beta, \beta, -2\beta) \) represents the degree of anisotropy. Thus this Taub model can be thought of as a special case of more general Bianchi type-IX model of anisotropic cosmology. Namely in the Bianchi type-IX model, the anisotropy measure \( \beta_{ij} \) has two independent components \( \beta_+ \) and \( \beta_- \). And the special case in which one sets \( \beta_- = 0 \) keeping \( \beta_+ \equiv \beta \neq 0 \) amounts to the reduction of the model to the Taub model.

The non-holonomic basis \( \{\sigma^i\} \) form a basis of 1-forms on a spacelike 3-sphere \( S^3 \) satisfying the SU(2) Maurer-Cartan structure equation

\[ d\sigma^i = \frac{1}{2} \epsilon^{ijk} \sigma^j \wedge \sigma^k \]  

(3)

and can be represented in terms of 3-Euler angles \( 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \psi \leq 4\pi \) parametrizing \( S^3 \) as

\[
\begin{align*}
\sigma^1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\sigma^2 &= \sin \psi d\theta - \cos \psi \sin \theta d\phi, \\
\sigma^3 &= d\psi + \cos \theta d\phi.
\end{align*}
\]  

(4)

The two independent metric functions, \( a(t) = e^{\alpha(t)} \) and \( \beta(t) \) thus can be regarded as two “minisuperspace variables” \( \gamma^A = (a, \beta) \) and in terms of them the gravity action (dropping the Gibbons-Hawking surface term) can be written as

\[ S = \frac{1}{2} \int dt Na^3 \left[ \frac{1}{N^2} \{-\frac{\dot{a}}{a}^2 + \dot{\beta}^2\} - \left\{ \frac{1}{a^2}V(\beta) + \frac{4}{3}\Lambda \right\} \right] \]  

(5)

where we defined the “curvature potential” \( V(\beta) \) as

\[ V(\beta) \equiv \frac{1}{3}(e^{-8\beta} - 4e^{-2\beta}) \]  

(6)

and we redefined the action by multiplying it by an overall constant, \( S \to (3\pi/4G)S \) and then rescaled the lapse function as \( N \to 2N \). For later use, we also provide, in the Lorentzian
signature, the expressions for the curvature scalar, $R$ of the 4-dim. space and the trace of the extrinsic curvature, $K$ of a 3-dim. spacelike hypersurface:

$$R = \frac{6}{N^2}(\frac{\ddot{a}}{a}) + (\frac{\dot{a}}{a})^2 + \dot{\beta}^2 - (\frac{\dot{N}}{N})(\frac{\dot{a}}{a}) - \frac{3}{2a^2}V(\beta),$$

$$K = -\frac{3}{N}(\frac{\dot{a}}{a}).$$  \(7\)

Next, upon identifying the momenta $\pi_A = (P_a, P_\beta)$ conjugate to the minisuperspace variables $\gamma^A = (a, \beta)$ as $P_a = \partial S/\partial \dot{a} = -a\dot{a}/N$ and $P_\beta = \partial S/\partial \dot{\beta} = a^3\dot{\beta}/N$, one can go over to the Hamiltonian of the system via the Legendre transformation as

$$S = \int dt L_{ADM} = \int dt(P_a\dot{a} + P_\beta\dot{\beta} - H_{ADM})$$

with

$$H_{ADM} = NH_0 + N_iH_i, \quad N_i = 0,$$

$$H_0 = -\frac{\delta S}{\delta N} = \frac{1}{2}a^{-3}[-a^2P_a^2 + P_\beta^2 + a^4\{V(\beta) + \frac{4}{3}\Lambda a^2}\]$$

which (i.e., $H_0$) vanishes identically due to the time-reparametrization invariance in general relativity. This is called the classical Hamiltonian constraint, $H_0 = 0$ and, upon Dirac quantization, it turns into the Wheeler-DeWitt equation to which we shall come back later on. In order to find classical wormhole solutions, which are Euclidean objects, from this point on, we need to work in Euclidean signature. The Euclidean gravity action can be obtained, via the analytic continuation, as

$$I = -iS = \frac{1}{2} \int d\tau Na^3[\frac{1}{N^2}\{-(\frac{d'}{a})^2 + \beta'^2\} + \frac{1}{a^2}V(\beta) + \frac{4}{3}\Lambda a^2}]$$  \(9\)

where now the “prime” denotes the derivative with respect to the Euclidean time $\tau$ while earlier the “overdot” denoted that with respect to the Lorentzian time $t$. By varying this Euclidean action with respect to the lapse $N(\tau)$, once again one gets the Hamiltonian constraint

$$\frac{1}{N^2}(\frac{da}{d\tau})^2 - a^2(\frac{d\beta}{d\tau})^2 + [V(\beta) + \frac{4}{3}\Lambda a^2] = 0$$  \(10\)

where $(d\beta/d\tau) = -i(d\beta/dt) = -i(N/a^3)P_\beta$ using the Wick rotation rule, $\tau = it$ while varying it with respect to minisuperspace variables $a, \beta$, one gets, respectively,
\[
\frac{1}{N^2 a} \left( \frac{d^2 a}{d\tau^2} \right)^2 + \frac{2}{N^2} \left( \frac{d\beta}{d\tau} \right)^2 + \frac{4}{3} \Lambda = 0, 
\]
(11)
\[
\frac{1}{N^2} \left( \frac{d^2 \beta}{d\tau^2} \right) + \frac{3}{N^2 a} \left( \frac{d a}{d\tau} \right) \left( \frac{d\beta}{d\tau} \right) - \frac{1}{2a^2} \left( \frac{dV}{d\beta} \right) = 0. 
\]
(12)

Note, here, that the Hamiltonian constraint in eq.(10) coincides with the \( \tau \tau \)-component of the Einstein equation, \( R_{\tau\tau} - \frac{1}{2} g_{\tau\tau} R + \Lambda g_{\tau\tau} = 0 \) and to arrive at the expression for the eq.(11), we made use of the Hamiltonian constraint in eq.(10). Needless to say, constructing a general solution of the coupled equations (10),(11) and (12) is a highly non-trivial job. However, by restricting our interest to some particular class of wormhole solutions with specific character, we can have some insight into the existence of classical wormhole solutions. Namely, suppose we particularly look for wormhole solutions with very small/large anisotropy,

\[
V(\beta) = \frac{1}{3} (e^{-8\beta} - 4e^{-2\beta}) \rightarrow k = -1(\beta \rightarrow 0), \quad 0(\beta \rightarrow \infty), 
\]
(13)
\[
d\frac{V}{d\beta} = \frac{8}{3} (-e^{-8\beta} + e^{-2\beta}) \rightarrow 0(\beta \rightarrow 0 \text{ or } \beta \rightarrow \infty) 
\]
(14)

and with “constant” momentum associated with the anisotropy change,

\[
P_\beta = i\frac{a^3}{N} \beta' = \text{const.} 
\]
(15)

Imposing these conditions for the character of the solution, the eqs.(10), (11) and (12) above reduce to, respectively,

\[
\frac{1}{N^2} \left( \frac{d a}{d\tau} \right)^2 + \frac{P_\beta^2}{a^4} + [k + \frac{4}{3} \Lambda a^2] = 0, 
\]
(16)
\[
\frac{1}{N^2 a} \left( \frac{d^2 a}{d\tau^2} \right) - \frac{2}{a^6} P_\beta^2 + \frac{4}{3} \Lambda = 0, 
\]
(17)
\[
\frac{1}{N^2} \frac{d}{d\tau} \left( \frac{N}{a^3} \right) + \frac{3}{N a^4} \left( \frac{d a}{d\tau} \right) = 0. 
\]
(18)

Next, we now fix the gauge associated with the time-reparametrization invariance as \( N(\tau) = 1 \). Incidentally, then, the eq.(18) above is automatically satisfied and we are left just with the eq.(16) which is an ODE for the scale factor \( a(\tau) \) alone. (Eqs.(16) and (17), upon fixing the gauge \( N(\tau) = 1 \), become the same since one can get the latter by differentiating the former with respect to the Euclidean time \( \tau \).) Thus what remains is to solve the eq.(16), i.e.,
\[
\left( \frac{da}{d\tau} \right)^2 + \left[ k + \frac{4}{3} \Lambda a^2 \right] = -\frac{P_{\beta}^2}{a^4}
\] (19)

for \( a(\tau) \) and then plug it into
\[
\beta(\tau) = -iP_{\beta} \int_{-\infty}^{\tau} \frac{d\tau'}{a^3(\tau')}
\] (20)

to obtain the behavior of anisotropy. Here, we set the “initial” time to be \( \tau \to -\infty \) when yet neither the wormholes nor the baby universes were born and no anisotropy yet arises, i.e., \( \beta(-\infty) = 0 \). At this point, it seems worth noting that after fixing the gauge \( N(\tau) = 1 \), it becomes clear that the condition given in eq.(15), i.e., the constant momentum associated with the change in anisotropy, \( P_{\beta} = \text{const.} \) corresponds to the condition for wormhole solutions whose anisotropy changing rate is inversely proportional to the scale (i.e., the size) of the wormhole, \( (d\beta/d\tau) = (-i)P_{\beta}/a^3 \), which seems quite a reasonable expectation. We now attempt to solve the eq.(19) for exact wormhole solutions.

(i) wormhole solution with very small anisotropy (\( \beta \to 0 \))

This case amounts to choosing \( k = -1 \), and then the eq.(19) admits an exact solution in the absence of the cosmological constant. Namely \( (da/d\tau)^2 - 1 = -P_{\beta}^2/a^4 \) yields, upon integration,
\[
\tau = \int_0^\tau d\tau' = \int_{a(0)}^{a(\tau)} \frac{a^2 da}{\sqrt{(a^2 + P_{\beta})(a^2 - P_{\beta})}}
\] (21)

\[
= \sqrt{\frac{P_{\beta}}{2}} F[\arccos(\sqrt{\frac{P_{\beta}}{a}}), \frac{1}{\sqrt{2}}] - \sqrt{2P_{\beta}} E[\arccos(\sqrt{\frac{P_{\beta}}{a}}), \frac{1}{\sqrt{2}}] + \frac{1}{a} \sqrt{a^4 - P_{\beta}^2}
\]

where we chose that the wormhole throat, i.e., the minimum value of \( a(\tau) \) occurs for \( \tau = 0 \), namely \( (da/d\tau)|_{\tau=0} = 0 \) and \( a(\tau = 0) = \sqrt{P_{\beta}} \) and \( F \) and \( E \) are the elliptic integrals of the 1st and 2nd kind respectively. This solution can be identified with a classical wormhole configuration since it displays the asymptotic behavior \( a^2(\tau) \to \tau^2 \) as \( \tau \to \pm\infty \).

Now that we have found the exact wormhole solution in closed form. Then next, we discuss precisely what topology changing process this wormhole instanton solution describes. To this end, we note ;

For \( \tau \to -\infty \),
\[ a^2(\tau) \to \tau^2 \to \infty, \quad \left( \frac{da}{d\tau} \right)_{\tau \to -\infty} \to 1, \]
\[ \beta(-\infty) = 0, \quad \left( \frac{d\beta}{d\tau} \right)_{\tau \to -\infty} = -i P_\beta \frac{1}{a^3(-\infty)} \to 0 \]

thus the curvature scalar there goes like \( R(\tau \to -\infty) \to 0 \), namely \( \tau \to -\infty \) spacelike hypersurface is flat.

For \( \tau = 0 \),
\[ a(0) = \sqrt{P_\beta}, \quad \left( \frac{da}{d\tau} \right)_{\tau = 0} = 0, \]
\[ \beta(0) = -i P_\beta \int_{-\infty}^{0} \frac{d\tau}{a^3(\tau)} = \text{const.}, \quad \left( \frac{d\beta}{d\tau} \right)_{\tau = 0} = -i P_\beta \frac{1}{a^3(0)} = -i \frac{1}{\sqrt{P_\beta}} \]

thus the trace of the extrinsic curvature there is \( K(\tau = 0) = 3 \left( \frac{a^2}{a} \right)_{|\tau = 0} = 0 \), namely this \( \tau = 0 \) spacelike hypersurface is a surface of vanishing extrinsic curvature and corresponds to the wormhole throat. And this surface has small, constant anisotropy and constant anisotropy changing rate. To conclude, this wormhole solution with very small anisotropy is an instanton configuration representing the topology change from the initial \( (\tau \to -\infty) \) state with topology \( R^3 \) to the final \( (\tau = 0) \) state with the topology \( R^3 \oplus S^3 \), i.e., flat space with an additional closed baby universe.

(ii) wormhole solution with very large anisotropy \((\beta \to \infty)\)

This case amounts to choosing \( k = 0 \) and the eq.(19) admits an exact solution even in the presence of the cosmological constant. Namely \((da/d\tau)^2 - 4|\Lambda|a^2/3 = -P_\beta^2/a^4\), upon integration,
\[ \tau = \int_{0}^{\tau} d\tau' = \int_{a(0)}^{a(\tau)} \left( \frac{1}{\kappa} \right) \frac{a^2 da}{\sqrt{a^6 - \tau^2}}, \]

yields,
\[ a(\tau) = \left[ \frac{P_\beta}{\kappa} \cosh(3\kappa \tau) \right]^{1/3} \quad (22) \]

where again we chose the wormhole neck to occur for \( \tau = 0 \) and \( a(\tau = 0) = (\frac{3}{4|\Lambda|} P_\beta^2)^{1/6} \) and we defined \( \kappa^2 \equiv \frac{4}{3}|\Lambda| \) and \( r \equiv P_\beta / \kappa = P_\beta \sqrt{\frac{3}{4|\Lambda|}} \).

We again discuss exactly what type of topology changing process this wormhole instanton
solution describes. Thus we note:

For \( \tau \to -\infty \),

\[
a^3(\tau) \to e^{-3\kappa \tau} \to \infty, \quad \left( \frac{da}{d\tau} \right)_{|\tau \to -\infty} \to -\kappa \to \infty,
\]

\[
\beta(-\infty) = 0, \quad \left( \frac{d\beta}{d\tau} \right)_{|\tau \to -\infty} = -iP_\beta \frac{1}{a^2(-\infty)} \to 0\]

thus the curvature scalar there goes like \( R(\tau \to -\infty) \to 12\kappa^2 = 16|\Lambda| \), namely \( \tau \to -\infty \) spacelike hypersurface is the 3-dim. space of constant curvature with large radius, i.e., \( S^3_\infty \).

For \( \tau = 0 \),

\[
a(0) = \left( \frac{3}{4|\Lambda|} P_\beta^2 \right)^{1/6}, \quad \left( \frac{da}{d\tau} \right)_{|\tau = 0} = 0,
\]

\[
\beta(0) = -iP_\beta \int_{-\infty}^{0} \frac{d\tau}{a^2(\tau)} = \text{const.}, \quad \left( \frac{d\beta}{d\tau} \right)_{|\tau = 0} = -iP_\beta \frac{1}{a^4(0)} = -i\sqrt{\frac{4|\Lambda|}{3}}
\]

thus the trace of the extrinsic curvature there is again \( K(\tau = 0) = 3(\frac{d^2}{da^2})_{|\tau = 0} = 0 \), namely this \( \tau = 0 \) spacelike hypersurface is a surface of vanishing extrinsic curvature and hence is the wormhole throat. And this surface has large, constant anisotropy and constant anisotropy changing rate. Consequently, this wormhole solution with very large anisotropy is an instanton configuration describing the topology change from the initial \( \tau \to -\infty \) state with topology \( S^3_\infty \) to the final \( \tau = 0 \) state with the topology \( S^3_\infty \oplus S^3 \), i.e., large 3-sphere with an additional closed baby universe.

Having constructed wormhole instanton solutions, we now turn to their contribution to the topology-changing tunnelling amplitude. Namely, we evaluate the wormhole instanton action, \( I(\text{instanton}) \), then the quantity, \( \exp[-I(\text{instanton})] \), would represent the semi-classical approximation to the topology-changing tunnelling amplitude. Thus by substituting the \((\tau \tau \text{-component of})\) Einstein equation in (10) satisfied by the wormhole solution into the Euclidean action in eq.(1) or (9), one gets

\[
I(\text{instanton}) = \int_{a(-\infty)}^{a(0)} \frac{[V(\beta) - \frac{4}{3}|\Lambda|a^2]a^3da}{\sqrt{\frac{4}{3}|\Lambda|a^6 - V(\beta)a^4 - P_\beta^2}}
\]

(23)

where we used \( (d\beta/d\tau) = -i(N/a^3)P_\beta \), \( d\tau = a^2da/N\sqrt{\frac{4}{3}|\Lambda|a^6 - V(\beta)a^4 - P_\beta^2} \) and the Gibbons-Hawking surface term vanishes on each boundary, one at \( \tau = -\infty \) and the other
at $\tau = 0$ as we have noted earlier. Note, first, that above expression for $I(\text{instanton})$ is independent of the lapse function $N$ and hence the “physically observable” topology-changing tunnelling amplitude $\exp \left[-I(\text{instanton})\right]$ possesses manifest gauge-invariance associated with the time reparametrization invariance as it should. Unfortunately, it does not seem possible to carry out the integral in eq.(23) to obtain the precise value of the wormhole instanton action. Still, however, we can evaluate $I(\text{instanton})$ for wormhole solution with very small/large anisotropy we constructed earlier. Firstly, for the wormhole with very small anisotropy given in eq.(21),

$$I(\text{instanton}) = -\int_{\lambda_1}^{a(0)} \frac{a^3 da}{\sqrt{a^4 - P_\beta^2}} = \frac{1}{2} \sqrt{\lambda_1^4 - P_\beta^2}$$

(24)

while secondly, for the wormhole with very large anisotropy given in eq.(22),

$$I(\text{instanton}) = -\kappa \int_{\lambda_2}^{a(0)} \frac{a^5 da}{\sqrt{a^6 - r^2}} = \frac{\kappa}{3} \sqrt{\lambda_2^6 - r^2} = \frac{2\sqrt{3}}{9} \sqrt{|\Lambda|} \lambda_2^6 - \frac{3P_\beta^2}{4}$$

(25)

where $\kappa$ and $r$ are as defined earlier and $\lambda_1$ and $\lambda_2$ denote large cut-offs for asymptotic wormhole sizes $a(-\infty)$. In particular, the expression for the wormhole instanton action in eq.(25) indicates that the tunnelling amplitude for the topology change driven by the wormhole solution in eq.(22) gets maximized for the smallest possible value of the cosmological constant, $|\Lambda| = \frac{3P_\beta^2}{4\lambda_2^6}$.

We now turn to the discussion of anisotropic wormholes in the quantum regime. The formulation we shall employ for quantum treatment of wormholes [8] can be summarized as follows: we construct and study a minisuperspace model (again based on the Taub cosmology) of canonical quantum cosmology in which the main objective is to solve the Wheeler-DeWitt equation to find the universe wave function. Then to see if there is an excitation that can be interpreted as a “quantum wormhole”, we look for a particular solution to the WD equation with “wormhole boundary condition” that allows one to identify the universe wave function as representing an excitation corresponding to a wormhole state. And as a proposal for such wormhole boundary condition, we shall employ the one advocated by Hawking and Page [7]. According to them, wormhole boundary conditions can generally be classified into two
categories; one for the “ground state” and the other for the “excited states” of quantum wormholes. Firstly, on the boundary condition for the ground state. When $\sqrt{\hbar} \to 0$, the universe wave function should be regular to represent a non-singular 4-metric. And when $\sqrt{\hbar} \to \infty$, the universe wave function should be damped, say, exponentially to represent an asymptotically Euclidean 4-metric, namely to represent the fact that there are no gravitational excitations asymptotically. Next, on the boundary condition for the excited states. When $\sqrt{\hbar} \to 0$, again the universe wave function should be regular but it may oscillates for small-$\sqrt{\hbar}$. And when $\sqrt{\hbar} \to \infty$, the universe wave function should be damped. Thus now by applying the “Dirac quantization procedure” to general relativity, which is one of the most well-known constraint systems, the classical Hamiltonian constraint given in eq.(8), turns into its quantum version, namely the WD equation

$$\frac{1}{2}[a^2 \frac{\partial^2}{\partial a^2} + (p + 1)a \frac{\partial}{\partial a} - \frac{\partial^2}{\partial \beta^2} + a^4 \{V(\beta) + \frac{4}{3} \Lambda a^2\}]\Psi[a, \beta] = 0$$

(26)

where the suffix “$p$” represents the well-known ambiguity in “operator-ordering” and in passing from the classical Hamiltonian constraint to this quantum WD equation, we substituted the conjugate momenta $P_a = -\dot{a}/N, P_\beta = a^3 \dot{\beta}/N$ by quantum momentum operators $\hat{P}_a = -i\partial/\partial a, \hat{P}_\beta = -i\partial/\partial \beta$. And of course $\Psi[a, \beta]$ is the (physical) universe wave function. Again, we do not intend to challenge the general solution to this WD equation. Instead, we will be content with the universe wave function for quantum wormholes which have specific characters we considered earlier in the classical treatment. Namely, wormholes with very small/large anisotropy and with constant momentum associated with the anisotropy change described by conditions given in eqs.(13) (14) and (15). In this particular case, since $V(\beta)$ is replaced by a constant $k$ (which is either $-1$ or $0$), the WD equation above admits a separation of variable. Namely by setting $\Psi[a, \beta] = A(a)B(\beta)$, the WD equation decomposes into two sectors

$$\frac{d^2}{da^2} + \frac{(p + 1)}{a} \frac{d}{da} + a^2(k + \frac{4}{3} \Lambda a^2) + \frac{P_\beta^2}{a^2}A(a) = 0,$$

(27)

$$\frac{d^2 B}{d\beta^2} + P_\beta^2 B = 0$$

(28)
where we chose the constant involved in the separation of variable to be the conjugate momentum \( P_\beta \) (which is a constant, too) since, as is manifest in eq.(28), it is the eigenvalue of the quantum momentum operator \( \hat{P}_\beta = -i\partial/\partial\beta \) which has to be the classical momentum \( P_\beta \). Obviously, then, the solution of \( \beta \)-sector is \( B(\beta) = (\text{const.})e^{iP_\beta \beta} \). Next, in order to have some insight into the behavior of the solution of \( a \)-sector, we view the \( a \)-sector of the WD equation given in eq.(27) as a Schrödinger-type equation with zero total energy. Then the “potential” energy can be identified with

\[
U(a) = -a^2(k - \frac{4}{3}|Aa^2| - \frac{P_\beta^2}{a^2})
\]

which involves the term \((-P_\beta^2/a^2)\) representing an “abyss” in the small-\( a \) region. Since the total energy is zero, the emergence of this potential abyss reveals the fact that (part of) the universe wave function, \( A(a) \) should be a highly oscillating function of \( a \) in the small-\( a \) region. And this enormous oscillatory behavior for small scale factor \( a \) appears to signal the existence of “quantum wormholes” in the small-\( a \) region of the minisuperspace [8] and hence seems consistent with the existence of classical wormhole solutions we studied earlier. Thus in order to confirm this belief of ours, we now solve the \( a \)-sector of WD equation in eq.(27) explicitly.

(i) Quantum wormhole with very small anisotropy \((\beta \to 0)\)

This case amounts to choosing \( k = -1 \) and then an exact solution to the \( a \)-sector of WD equation in (27) is available in the absence of the cosmological constant.

\[
\left[ \frac{d^2}{da^2} + \frac{(p + 1)}{a} \frac{d}{da} - a^2 + \frac{P_\beta^2}{a^2} \right]A(a) = 0,
\]

\[
A(a) = a^{-\frac{1}{2}} Z_{i\frac{1}{4}\sqrt{4P_\beta^2 - p^2}}(\frac{i}{2}a^2), \quad (-2\beta < p < 2\beta)
\]

where \( Z_\nu(x) \) denotes a Bessel function of order \( \nu = i\frac{1}{4}\sqrt{4P_\beta^2 - p^2} \). For now \( Z_\nu(x) \) could be the Bessel function of the 1st kind or the 2nd kind (i.e., the Neumann function) or their linear combinations, i.e., the Hankel functions. However, imposing the appropriate “wormhole boundary condition” of Hawking and Page, it should be, for small 3-geometry (i.e., \( a \to 0 \)), the Bessel function of the 1st kind since \( A(a) = a^{-\frac{1}{2}} J_{i\frac{1}{4}\sqrt{4P_\beta^2 - p^2}}(\frac{i}{2}a^2) \to \)
\[ a^{-\frac{1}{2}p}a^{2\nu} = a^{-\frac{1}{2}p} \exp [i2\nu |ln a] \] (where we used \( J_\nu(x) \rightarrow x^\nu/2^\nu \nu! \) for \( x \rightarrow 0 \)) with \(-2P_\beta < p \leq 0\) for the regularity at \( a = 0 \), which indeed possesses highly oscillatory behavior for \( a \rightarrow 0 \). Next, for large 3-geometry (i.e., \( a \rightarrow \infty \)), it should be the Hankel function since

\[
A(a) = a^{-\frac{1}{2}p} J_\nu^{(1)} \left( \frac{i\nu}{\sqrt{4P_\beta - p^2}} \left( \frac{i}{2} a^2 \right) \right) \rightarrow a^{-\frac{1}{2}(p+2)} e^{-\frac{1}{2}a^2} \] (where we used \( H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \rightarrow x^{-\frac{1}{2}} e^{ix} \) for \( x \rightarrow \infty \)), which indeed possesses rapidly damping behavior for \( a \rightarrow \infty \) as desired. Thus this solution \( \Psi[a, \beta] = A(a) e^{iP_\beta \beta} \) is a legitimate universe wave function of a quantum wormhole.

(ii) Quantum wormhole with very large anisotropy (\( \beta \rightarrow \infty \))

This case amounts to choosing \( k = 0 \) and an exact solution to eq.(27) is available even in the presence of the cosmological constant.

\[
\frac{d^2}{da^2} + \frac{(p + 1)}{a} \frac{d}{da} - \frac{4}{3} |\Lambda| a^4 + \frac{P_\beta^2}{a^2} |\Lambda| A(a) = 0, \tag{31}
\]

\[
A(a) = a^{-\frac{1}{2}p} Z \left[ i\sqrt{4P_\beta - p^2} \right] \left( \frac{2}{3} \sqrt{\frac{|\Lambda|}{3}} a^3 \right), \quad (-2P_\beta < p < 2P_\beta).
\]

Again, imposing the “wormhole boundary condition” of Hawking and Page, this general solution of the Bessel equation should be, for small 3-geometry (\( a \rightarrow 0 \)), the Bessel function of the 1st kind since \( A(a) = a^{-\frac{1}{2}p} J_\nu^{(1)} \left( \frac{i\nu}{\sqrt{4P_\beta - p^2}} \left( \frac{i}{2} \sqrt{\frac{|\Lambda|}{3}} a^3 \right) \right) \rightarrow a^{-\frac{1}{2}p} a^{3\nu} = a^{-\frac{1}{2}p} \exp [i3|\nu| |ln a|] \) with \( \nu = \frac{i1}{6} \sqrt{4P_\beta - p^2} \) now and \(-2P_\beta < p \leq 0\) for the regularity at \( a = 0 \), which exhibits highly oscillatory behavior for \( a \rightarrow 0 \). Next, for large 3-geometry (i.e., \( a \rightarrow \infty \)), it should be the Hankel function since \( A(a) = a^{-\frac{1}{2}p} H_\nu^{(1)} \left( \frac{i\nu}{\sqrt{4P_\beta - p^2}} \left( \frac{i}{2} \sqrt{\frac{|\Lambda|}{3}} a^3 \right) \right) \rightarrow a^{-\frac{1}{2}(p+3)} e^{-\frac{1}{2}\sqrt{\frac{|\Lambda|}{3}} a^3} \), which exhibits rapidly damping behavior for \( a \rightarrow \infty \) as desired. And this solution \( \Psi[a, \beta] = A(a) e^{iP_\beta \beta} \) can be identified with a legitimate quantum wormhole wave function as well. Finally, we note that the wormhole wave functions given above in (i) and (ii) can be identified with the ones representing “excited states” of quantum wormholes we stated earlier as they are regular at \( a = 0 \) and possess oscillatory behaviors for small-\( a \).

To summarize, in this letter, we demonstrated explicitly that Euclidean, anisotropic wormhole solutions do exist even in pure Einstein gravity system. This, among others, strongly supports our intuitive expectation that there should be generic quantum fluctuations in spacetime alone even in the complete absence of matter. Noticing that homogeneous,
isotropic wormhole solutions do not exist in pure general relativity, one may naturally sus-
pect that the anisotropy degree of freedom could play a decisive role in switching on the
wormhole configuration. This is indeed the case as one can see in this work that it is the
nonvanishing $P_\beta^\gamma$, i.e., the momentum or the energy associated with the anisotropy change,
that essentially renders the occurrence of both classical and quantum wormholes possible.
After all, wormholes seem to be one of the rare means which are under our control (to some
extent) to challenge the quantum gravity.

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