Brezis–Van Schaftingen–Yung formulae in ball Banach function spaces with applications to fractional Sobolev and Gagliardo–Nirenberg inequalities

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Abstract
Let $X$ be a ball Banach function space on $\mathbb{R}^n$. In this article, under some mild assumptions about both $X$ and the boundedness of the Hardy–Littlewood maximal operator on the associate space of the convexification of $X$, the authors prove that, for any locally integrable function $f$ with $\| | \nabla f | \|_X < \infty$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \right\|^{\frac{n}{q} + 1} \right\|_X \sim \| | \nabla f | \|_X$$

with the positive equivalence constants independent of $f$, where the index $q \in (0, \infty)$ is related to $X$ and $|\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \right\|^{\frac{n}{q} + 1}$ is the Lebesgue measure of the set under consideration. In particular, when $X := L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, the above formulae hold true for any given $q \in (0, \infty)$ with $n \left( \frac{1}{p} - \frac{1}{q} \right) < 1$, which when $q = p$ are exactly the recent surprising formulae of H. Brezis, J. Van Schaftingen, and P.-L. Yung, and which in other cases are new. This generalization has a wide range of applications and, particularly, enables the authors to establish new fractional Sobolev and new Gagliardo–Nirenberg inequalities in various function spaces, including Morrey spaces, mixed-norm...
Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces, Orlicz spaces, Orlicz-slice (generalized amalgam) spaces, and weak Morrey spaces, and, even in all these special cases, the obtained results are new. The proofs of these results strongly depend on the Poincaré inequality, the extrapolation, the exact operator norm on $X'$ of the Hardy–Littlewood maximal operator, and the exquisite geometry of $\mathbb{R}^n$.

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1 Introduction

It is well known that, for any given $s \in (0, 1)$ and $p \in [1, \infty)$, the homogeneous fractional Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ having the following finite Gagliardo semi-norm

$$
\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right]^\frac{1}{p}
$$

These spaces play a key role in harmonic analysis and partial differential equations (see, for instance, [10, 19, 20, 46, 51, 73, 74, 83]).

A well-known drawback of the Gagliardo semi-norm in (1.1) is that one can not recover the homogeneous Sobolev semi-norm $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ when $s = 1$, in which case the integral in (1.1) is infinite unless $f$ is constant (see [8, 11]). Here and thereafter, for any differentiable function $f$ on $\mathbb{R}^n$, $\nabla f$ denotes the gradient of $f$, namely

$$
\nabla f := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right),
$$

and, for any given $p \in [1, \infty)$, the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ is defined to be the set of all the locally integrable functions $f$ on $\mathbb{R}^n$ having the following finite semi-norm

$$
\|f\|_{\dot{W}^{1,p}(\mathbb{R}^n)} := \|\nabla f\|_{L^p(\mathbb{R}^n)}.
$$

An important approach to recover $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ out of Gagliardo semi-norms is due to Bourgain et al. [9] who in particular proved that, for any given $p \in [1, \infty)$ and any $f \in \dot{W}^{1,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$
\lim_{s \in (0,1), s \to 1} \left(1 - s\right) \|f\|_{\dot{W}^{1,p}(\mathbb{R}^n)}^p = C_{(p,n)} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p,
$$

where $C_{(p,n)}$ is a positive constant depending only on both $p$ and $n$. Very recently, Brezis et al. [17] discovered an alternative way to repair this defect by replacing the $L^p$ norm in (1.1) with the weak $L^p$ quasi-norm, namely $\|f\|_{L^{p,\infty}(\mathbb{R}^n \times \mathbb{R}^n)}$. For any given $p \in [1, \infty)$, Brezis et al. in [17] proved that there exist positive constants $C_1$ and $C_2$ such that, for any $f \in C^\infty_c(\mathbb{R}^n)$ (the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support),
\[ C_1 \| \nabla f \|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{f(x) - f(y)}{|x - y|^\frac{n}{p} + 1} \right\|_{L^{p, \infty}(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_2 \| \nabla f \|_{L^p(\mathbb{R}^n)}, \]  

where

\[ \left\| \frac{f(x) - f(y)}{|x - y|^\frac{n}{p} + 1} \right\|_{L^{p, \infty}(\mathbb{R}^n \times \mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \lambda \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(x) - f(y)|}{|x - y|^\frac{n}{p} + 1} > \lambda \right\} \frac{1}{p}, \]  

here and thereafter, for any given \( m \in \mathbb{N} \) and any Lebesgue measurable set \( E \subset \mathbb{R}^m \), the symbol \(|E|\) denotes its Lebesgue measure. Later, Brezis et al. in [15] showed that (1.2) holds true for any \( f \in \dot{W}^{1,p}(\mathbb{R}^n) \). The equivalence (1.2) in particular allows Brezis et al. in [17] to derive some surprising alternative estimates of fractional Sobolev and Gagliardo–Nirenberg inequalities in some exceptional cases involving \( \dot{W}^{1,1}(\mathbb{R}^n) \), where the anticipated fractional Sobolev and Gagliardo–Nirenberg inequalities fail; see also [12, 13] for more studies on the Gagliardo–Nirenberg inequality. For later discussions, we use the Fubini theorem to write the weak \( L^p \)-norm in (1.3) as

\[ \left\| \frac{f(x) - f(y)}{|x - y|^\frac{n}{p} + 1} \right\|_{L^{p, \infty}(\mathbb{R}^n \times \mathbb{R}^n)} = \sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, p)}(x, y) \, dy \right]^{\frac{1}{p}}, \]

where, for any \( \lambda \in (0, \infty) \),

\[ E_f(\lambda, p) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^\frac{n}{p} + 1 \right\}. \]

Consequently, the estimate (1.2) takes the following version: for any \( f \in \dot{W}^{1,p}(\mathbb{R}^n) \),

\[ \sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, p)}(x, y) \, dy \right]^{\frac{1}{p}} \sim \| \nabla f \|_{L^p(\mathbb{R}^n)} \]  

with the positive equivalence constants independent of \( f \). More related works can be found in [14, 16, 18, 31, 35–37, 44].

Let us also give a few comments on the proof of (1.2) in [17]. The proof of the lower bound is relatively simpler. Indeed, a substantially sharper lower bound was obtained in [17], using both a method of rotation and the Taylor remainder theorem. On the other hand, as was pointed out in [17], the stated upper bound for any given \( p \in (1, \infty) \) can be easily deduced from the following Lusin–Lipschitz inequality in [7]: for any differentiable function \( f \) and any \( x, y \in \mathbb{R}^n \),

\[ |f(x) - f(y)| \lesssim |x - y| \left[ M(|\nabla f|)(x) + M(|\nabla f|)(y) \right], \]

where the implicit positive constant is independent of \( x, y \), and \( f \). Here and thereafter, the Hardy–Littlewood maximal operator \( M \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) (the set of all locally integrable functions on \( \mathbb{R}^n \)) and \( x \in \mathbb{R}^n \),
\[ M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \]  
(1.6)

where the supremum is taken over all the balls \( B \subset \mathbb{R}^n \) containing \( x \). Thus, the hard core of the proof of the upper bound in (1.2) is the case \( p = 1 \). The proof in [17], which actually works for the full range \( p \in [1, \infty) \), uses the Vitali covering lemma in one variable and then a method of rotation. Thus, the rotation invariance of the space \( L^p(\mathbb{R}^n) \) seems to play an indispensable role in the proof of (1.2) in [17].

The main purpose of this article is to give an essential extension of (1.4). Such extensions are fairly nontrivial because our setting typically involves function spaces that are neither rotation invariance nor translation invariance. Somewhat surprisingly, even when returning to the standard Lebesgue space \( L^p(\mathbb{R}^n) \), we have the following new estimate (see Theorem 5.15 below): for any given \( p \in [1, \infty) \) and \( q \in (0, \infty) \) with \( n(\frac{1}{p} - \frac{1}{q}) < 1 \), and for any \( f \in \dot{W}^{1,p}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \, dx \right\}^\frac{1}{p} \sim \| \nabla f \|_{L^p(\mathbb{R}^n)},
\]  
(1.7)

where the positive equivalence constants are independent of \( f \). In the case of \( p = q \), (1.7) is exactly the surprising estimate (1.2) in [15, 17].

Our main result extends the results (1.2) in [15, 17] to a wide class of function spaces on \( \mathbb{R}^n \), including Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces, Orlicz spaces, Orlicz-slice spaces, and weak Morrey spaces (see, respectively, Sects. 5.1 through 5.7 below for their histories and definitions). We treat these spaces in a uniform manner in the setting of ball quasi-Banach function spaces recently introduced by Sawano et al. [93]. Ball quasi-Banach function spaces are quasi-Banach spaces of measurable functions on \( \mathbb{R}^n \) in which the quasi-norm is related to the Lebesgue measure on \( \mathbb{R}^n \) in an appropriate way (see Definition 2.1 below). These function spaces play an important role in many branches of analysis. They are less restrictive than the classical Banach function spaces introduced in the book [6, Chapter 1]. For more studies on ball quasi-Banach function spaces, we refer the reader to [21, 89, 93, 93, 103–105, 107] for the Hardy spaces associated with ball quasi-Banach function spaces, to [54, 101, 108] for the boundedness of operators on ball quasi-Banach function spaces, and to [58, 62, 99, 102] for the applications of ball quasi-Banach function spaces.

To be precise, in this article, our aim is to establish the following analogue of (1.4) for the quasi-norm \( \| \cdot \|_X \) of a given ball quasi-Banach function space \( X \) under some mild assumptions about both \( X \) and the boundedness of the Hardy–Littlewood maximal operator on the associate space of the convexification of \( X \) (see Theorems 4.5 and 4.10 below): for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^\frac{1}{q} \right\|_X \sim \| \nabla f \|_X,  
\]  
(1.8)

where the index \( q \in (0, \infty) \) is related to \( X \) under consideration and the positive equivalence constants are independent of \( f \). In particular, when returning to the special case of \( X := L^p(\mathbb{R}^n) \) with \( p \in [1, \infty) \), we obtain the formula (1.7), which when \( p = q \) is just (1.2) obtained in [15, 17], and which even when \( q \in (0, \infty) \) satisfying \( n(\frac{1}{p} - \frac{1}{q}) < 1 \) seems new. Moreover, we prove that, when \( q \in [1, \infty) \), the condition \( n(\frac{1}{p} - \frac{1}{q}) < 1 \) is sharp in some sense [see Remark 3.6(iii) below for the details]. Similarly to the case of \( X := L^p(\mathbb{R}^n) \) in [15, 17], which actually works for the full range \( p \in [1, \infty) \), uses the Vitali covering lemma in one variable and then a method of rotation. Thus, the rotation invariance of the space \( L^p(\mathbb{R}^n) \) seems to play an indispensable role in the proof of (1.2) in [17].
[15, 17], (1.8) also allows us to extend the fractional Sobolev and the Gagliardo–Nirenberg inequalities to the setting of ball quasi-Banach function spaces (see Corollaries 4.13 and 4.15 below).

The formula (1.8) gives an equivalence between the Sobolev semi-norm and the quantity involving the difference of the function under consideration. It is quite remarkable that such an equivalence holds true for a ball Banach function space \( X \). Indeed, finding an appropriate way to characterize the smoothness of functions via their finite differences is a notoriously difficult problem in approximation theory, even for some simple weighted Lebesgue space in one dimension (see [69, 72] and the references therein). A major difficulty comes from the fact that difference operators \( \Delta_h f := f(\cdot + h) - f(\cdot) \) for any \( h \in \mathbb{R}^n \setminus \{0\} \) are no longer bounded on general weighted \( L^p \) spaces. It turns out that, via using the extrapolation in [28] and the exact operator norm on the associate space of \( X \) of the Hardy–Littlewood maximal operator, the estimate (1.8) in \( X \) follows from the following estimates in weighted Lebesgue spaces with Muckenhoupt weights [see Definition 3.4 below for the precise definition of \( \omega \)-condition on the weights \( \omega \)].

**Theorem 1.1** Let \( p \in [1, \infty) \) and \( \omega \in A_1(\mathbb{R}^n) \). Then, for any \( f \in W^{1, p}_\omega(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, p)}(x, y) \, dy \, \omega(x) \, dx \sim \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]

(1.9)

where the positive equivalence constants are independent of \( f \).

Indeed, we prove an improved version of Theorem 1.1 (see Theorem 3.5 below). The \( A_p(\mathbb{R}^n) \)-condition on the weights \( \omega \) in Theorem 1.1 is necessary in some sense in the case of \( n = 1 \) (see Theorem 3.7 below). Note that, unlike the integral \( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \, dx \, dy \) in the estimate (1.2), the integral \( \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} \cdots \, dy] \omega(x) \, dx \) in (1.9) is not symmetric with respect to both \( x \) and \( y \), which causes additional technical difficulties in the proof of the upper bound in (1.9). Indeed, the proof of the upper estimate in (1.9) is fairly nontrivial. On one hand, the proof of (1.2) in the unweighted case in the article [15, 17] is based on a method of rotation and seems to be inapplicable in the weighted case here. On the other hand, using the Lusin-Lipschitz inequality (1.5) would give the stated upper bound in (1.9) for any given \( p \in (1, \infty) \), which is not enough for our purpose because it excludes the endpoint case \( p = 1 \). Recall that the hard core of both the main results in [15, 17] and also Theorem 1.1 is the endpoint case \( p = 1 \) [see Remark 3.6(ii) for more details]. Instead of the Vitali covering lemma, we use several adjacent systems of dyadic cubes in \( \mathbb{R}^n \) and hence the exquisite geometry of \( \mathbb{R}^n \) (see, for instance, [71, Sect. 2.2]) to overcome these obstacles.

The remainder of this article is organized as follows.

In Sect. 2, we first introduce the homogeneous ball Banach Sobolev space \( \dot{W}^{1, X}(\mathbb{R}^n) \) which extends the concept of the homogeneous Sobolev space \( \dot{W}^{1, p}(\mathbb{R}^n) \) to the ball Banach function space. Motivated by [45], we show that the set of all infinitely differentiable functions whose gradients have compact support is dense in \( \dot{W}^{1, X}(\mathbb{R}^n) \) (see Theorem 2.6 below), which plays a key role in the proofs of both Theorems 1.1 and 4.5. To prove Theorem 2.6, we first establish the Poincaré inequality on the homogeneous ball Banach Sobolev space \( \dot{W}^{1, X}(\mathbb{R}^n) \) (see Proposition 2.8 below).

Section 3 is devoted to the proof of Theorem 1.1 which characterizes the Sobolev semi-norm in weighted Lebesgue spaces. To show Theorem 1.1, we prove a more general result (see Theorem 3.5 below), which plays an essential role in the proof of Theorem 4.5. First,
we establish the lower estimate of Theorem 3.5 in ball quasi-Banach function spaces (see Theorem 3.18 below), which is a part of Theorem 4.5. In the proof of the upper estimate of Theorem 3.5, as already pointed out, since the integral in the estimate (3.1) of Theorem 3.5 is not symmetric, we use several adjacent systems of dyadic cubes in \( \mathbb{R}^n \) (see Lemma 3.10 below) and hence the exquisite geometry of \( \mathbb{R}^n \) to overcome this obstacle. Finally, we prove Theorem 3.7 which shows that the \( A_p(\mathbb{R}^n) \)-condition on the weight \( \omega \) in Theorem 1.1 is necessary in some sense in the case of \( n = 1 \).

In Sect. 4, we generalize (1.2) to ball Banach function spaces under some mild assumptions. However, the calculations in [17] rely on the following three crucial properties of \( L^p(\mathbb{R}^n) \), which are not available for ball Banach function spaces: the rotation invariance, the translation invariance, and the explicit expression of the norm. Borrowing some ideas from the proof of the extrapolation theorem in [28], using Theorem 3.5 and the exact operator norm on the associate space of \( X \) of the Hardy–Littlewood maximal operator, we establish the characterization of the Sobolev semi-norm in ball Banach function spaces (see both Theorems 4.13 and 4.15 below). As applications, we also establish alternative fractional Sobolev and alternative Gagliardo–Nirenberg inequalities in ball Banach function spaces (see Corollaries 4.13 and 4.15 below).

In Sect. 5, we apply all these results obtained in Sect. 4, respectively, to \( X := M^\alpha_q(\mathbb{R}^n) \) (the Morrey space), \( X := L^{p,(\cdot)}(\mathbb{R}^n) \) (the variable Lebesgue space), \( X := L^p(\mathbb{R}^n) \) (the mixed-norm Lebesgue space), \( X := L^p_\omega(\mathbb{R}^n) \) (the weighted Lebesgue space), \( X := L^\Phi(\mathbb{R}^n) \) (the Orlicz space), \( X := (E^p_q)_t(\mathbb{R}^n) \) (the Orlicz-slice space or the generalized amalgam space), or \( X := M^\alpha_{r,\infty}(\mathbb{R}^n) \) (the weak Morrey space), all these results are completely new. Due to the generality and the flexibility, more applications of the results of this article are predictable.

Finally, we make some conventions on notation. Let \( \mathbb{N} := \{1, 2, \ldots \} \) and \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \).

We always denote by \( C \) a positive constant which is independent of the main parameters, but it may vary from line to line. We also use \( C_{(\alpha, \beta, \ldots)} \) to denote a positive constant depending on the indicated parameters \( \alpha, \beta, \ldots \). The symbol \( f \lesssim g \) means that \( f \leq Cg \). If \( f \lesssim g \) and \( g \lesssim f \), we then write \( f \sim g \). If \( f \leq Cg \) and \( g = h \) or \( g \leq h \), we then write \( f \lesssim g = h \) or \( f \lesssim g \leq h \), rather than \( f \lesssim g \sim h \) or \( f \lesssim g \lesssim h \). We use \( 0 \) to denote the origin of \( \mathbb{R}^n \). If \( E \) is a subset of \( \mathbb{R}^n \), we denote by \( 1_E \) its characteristic function and, for any measurable set \( E \subset \mathbb{R}^n \) with \( |E| < \infty \) and for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), let

\[
\int_E f(x) \, dx := \frac{1}{|E|} \int_E f(x) \, dx =: f_E.
\]

For any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), let \( B(x, r) := \{ y \in \mathbb{R}^n : |x - y| < r \} \) and

\[
\mathbb{B} := \{ B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty) \}. \tag{1.10}
\]

For any \( \alpha \in (0, \infty) \) and any ball \( B := B(x_B, r_B) \) in \( \mathbb{R}^n \), with \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), let \( \alpha B := B(x_B, \alpha r_B) \). Also, for any \( q \in [1, \infty) \), we denote by \( q' \) its conjugate exponent, that is, \( 1/q + 1/q' = 1 \). Finally, when we prove a theorem or the like, we always use the same symbols in the wanted proved theorem or the like.

## 2 Density in \( \dot{W}^{1,X}(\mathbb{R}^n) \)

In this section, we first extend the concept of homogeneous Sobolev spaces to the ball Banach Sobolev space \( \dot{W}^{1,X}(\mathbb{R}^n) \). Moreover, we establish the Poincaré inequality on the homogeneous ball Banach Sobolev space \( \dot{W}^{1,X}(\mathbb{R}^n) \) and then show that the set of all infinitely
differentiable functions whose gradients have compact supports is dense in $\dot{W}^{1, X}(\mathbb{R}^n)$ (Theorems 2.8 and 2.6 below).

First, we recall some preliminaries on ball quasi-Banach function spaces introduced in [93]. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$.

**Definition 2.1** A quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$, equipped with a quasi-norm $\| \cdot \|_X$ which makes sense for all functions in $\mathcal{M}(\mathbb{R}^n)$, is called a **ball quasi-Banach function space** if it satisfies that

(i) For any $f \in \mathcal{M}(\mathbb{R}^n)$, $\| f \|_X = 0$ implies that $f = 0$ almost everywhere;
(ii) For any $f, g \in \mathcal{M}(\mathbb{R}^n)$, $|g| \leq |f|$ almost everywhere implies that $\| g \|_X \leq \| f \|_X$;
(iii) For any $\{f_m\}_{m \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$ and $f \in \mathcal{M}(\mathbb{R}^n)$, $0 \leq f_m \uparrow f$ almost everywhere as $m \to \infty$ implies that $\| f_m \|_X \uparrow \| f \|_X$ as $m \to \infty$;
(iv) $B \in \mathbb{B}$ implies that $1_B \in X$, where $\mathbb{B}$ is the same as in (1.10).

Moreover, a ball quasi-Banach function space $X$ is called a **ball Banach function space** if the norm of $X$ satisfies the triangle inequality: for any $f, g \in X$,

$$\| f + g \|_X \leq \| f \|_X + \| g \|_X,$$

and that, for any $B \in \mathbb{B}$, there exists a positive constant $C(B)$, depending on $B$, such that, for any $f \in X$,

$$\int_B |f(x)| \, dx \leq C(B) \| f \|_X.$$

**Remark 2.2** (i) Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$. By [103, Remark 2.5(i)] (see also [104, Remark 2.6(i)]), we conclude that, for any $f \in \mathcal{M}(\mathbb{R}^n)$, $\| f \|_X = 0$ if and only if $f = 0$ almost everywhere.

(ii) As was mentioned in [103, Remark 2.5(ii)] (see also [104, Remark 2.6(ii)]), we obtain an equivalent formulation of Definition 2.1 via replacing any ball $B$ by any bounded measurable set $E$ therein.

(iii) In Definition 2.1, if we replace any ball $B$ by any measurable set $E$ with $|E| < \infty$, we obtain the definition of (quasi-)Banach function spaces originally introduced in [6, Definitions 1.1 and 1.3]. Thus, a (quasi-)Banach function space is always a ball (quasi-)Banach function space. But, the reverse is not true.

(iv) By [32, Theorem 2], we conclude that both (ii) and (iii) of Definition 2.1 imply that any ball quasi-Banach function space is complete.

We recall the definition of ball Banach function spaces with absolutely continuous norm; see [6, Definition 3.1] and [101, Definition 3.2].

**Definition 2.3** A ball Banach function space $X$ is said to have an **absolutely continuous norm** if, for any $f \in X$ and any sequence of measurable sets, $\{E_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$, satisfying that $1_{E_j} \to 0$ almost everywhere as $j \to \infty$, $\| f 1_{E_j} \|_X \to 0$ as $j \to \infty$.

We extend the concept of homogeneous Sobolev spaces to ball Banach function spaces.

**Definition 2.4** Let $X$ be a ball Banach function space. The **homogeneous ball Banach Sobolev space** $\dot{W}^{1, X}(\mathbb{R}^n)$ is defined to be the set of all the distributions $f$ on $\mathbb{R}^n$ such that $|\nabla f| \in X$ equipped with the quasi-norm

$$\| f \|_{\dot{W}^{1, X}(\mathbb{R}^n)} := \| |\nabla f| \|_X,$$

where $\nabla f := (\partial_1 f, \ldots, \partial_n f)$ denotes the distributional gradient of $f$. Springer
Recall that, for any given bounded open set \( U \subset \mathbb{R}^n \), the Sobolev space \( W^{1,1}(U) \) is defined to be the set of all the integrable functions \( f \) on \( U \) such that
\[
\| f \|_{W^{1,1}(U)} := \| f \|_{L^1(U)} + \| \nabla f \|_{L^1(U)} < \infty.
\]
Moreover, we denote by \( W^{1,1}_{\text{loc}}(\mathbb{R}^n) \) the set of all the locally integrable functions \( f \) on \( \mathbb{R}^n \) satisfying that \( f \in W^{1,1}(U) \) for any bounded open set \( U \subset \mathbb{R}^n \). The following conclusion shows that, for any \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \), \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \).

**Proposition 2.5** Let \( X \) be a ball Banach function space. Then
\[
\dot{W}^{1,X}(\mathbb{R}^n) \subset W^1_{\text{loc}}(\mathbb{R}^n).
\]

**Proof** Let \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \). Then \( |\nabla f| \in X \). From this and the definition of \( X \), it follows that \( |\nabla f| \in L^1_{\text{loc}}(\mathbb{R}^n) \). This, together with [73, Sect. 1.1.2], implies that \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). This finishes the proof of Proposition 2.5. \( \square \)

In what follows, for any given \( r \in (0, \infty) \), we use \( L^r_{\text{loc}}(\mathbb{R}^n) \) to denote the set of all locally \( r \)-order integrable functions on \( \mathbb{R}^n \). For any given \( r \in (0, \infty) \), the centered ball average operator \( B_r \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),
\[
B_r(f)(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.
\] (2.1)

In addition, we denote by the symbol \( C_c(\mathbb{R}^n) \) [resp., \( C^\infty(\mathbb{R}^n) \)] the set of all continuous functions with compact support [resp., all infinitely differentiable functions] on \( \mathbb{R}^n \). The following conclusion is the main result of this section.

**Theorem 2.6** Let \( X \) be a ball Banach function space. Assume that the centered ball average operators \( \{B_r\}_{r \in (0,\infty)} \) are uniformly bounded on \( X \) and that \( X \) has an absolutely continuous norm. Then, for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \), there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \) with \( |\nabla f_k| \in C_c(\mathbb{R}^n) \) for any \( k \in \mathbb{N} \) such that
\[
\lim_{k \to \infty} \| f - f_k \|_{\dot{W}^{1,X}(\mathbb{R}^n)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \| (f - f_k)1_{B(0,R)} \|_X = 0
\]
for any \( R \in (0, \infty) \).

**Remark 2.7** (i) Let \( X \) be the same as in Theorem 2.6. Here and thereafter, let \( C^\infty(\mathbb{R}^n) := C_c(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \). We point out that, when \( n \in [2, \infty) \cap \mathbb{N} \), a slight modification of the proof of Theorem 2.6 shows that \( C^\infty(\mathbb{R}^n) \) is dense in \( \dot{W}^{1,X}(\mathbb{R}^n) \), which when \( X := L^p(\mathbb{R}^n) \) with \( p \in [1, \infty) \) is a part of [45, Theorem 4].

(ii) Let \( X \) be a ball Banach function space. Assume that \( X \) has an absolutely continuous norm and the Hardy–Littlewood maximal operator \( M \) is bounded on \( X \). In this case, Theorem 2.6 is included in [80, Theorem 1.2]. However, when \( X := L^1(\mathbb{R}^n) \), the Hardy–Littlewood maximal operator \( M \) is not bounded on \( L^1(\mathbb{R}^n) \) and hence, in this case, [80, Theorem 1.2] is inapplicable. However, when \( X := L^1(\mathbb{R}^n) \), since the centered ball average operators \( \{B_r\}_{r \in (0,\infty)} \) are obviously uniformly bounded on \( L^1(\mathbb{R}^n) \), Theorem 2.6 still holds true in this case. Thus, when \( X := L^1(\mathbb{R}^n) \), compared with the Hardy–Littlewood maximal operator, the centered ball average operators have obvious advantage.

We now establish the following Poincaré inequality on the homogeneous ball Banach Sobolev space \( \dot{W}^{1,X}(\mathbb{R}^n) \), which plays a key role in the proof of Theorem 2.6.
Proposition 2.8 Let \( x_0 \in \mathbb{R}^n \), \( R \in (0, \infty) \), and \( \Omega := B(x_0, R) \) when \( n \in [1, \infty) \cap \mathbb{N} \) or \( \Omega := \{ x \in \mathbb{R}^n : R < |x| < 2R \} \) when \( n \in [2, \infty) \cap \mathbb{N} \). Assume that \( X \) is a ball Banach function space and the centered ball average operators \( \{ B_r \}_{r \in (0, \infty)} \) are uniformly bounded on \( X \). Then there exists a positive constant \( C \), independent of \( R \), such that, for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \),

\[
\|(f - f_{\Omega})I_{\Omega}\|_X \leq CR\|\nabla f\|_{1,\Omega}\|_X.
\] (2.2)

Remark 2.9 Let \( X \) be the same as in Proposition 2.8 and \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \). Then, for any ball \( B \subset \mathbb{R}^n \), \( f1_B \in X \). Indeed, by Proposition 2.5, we find that \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), which, together with (2.2), further implies that, for any ball \( B \subset \mathbb{R}^n \),

\[
\|f1_B\|_X \leq \|(f - f_{\Omega})1_B\|_X + \|f_{\Omega}1_B\|_X \\
\quad \leq rB\|f\|_{\dot{W}^{1,X}(\mathbb{R}^n)} + \int_B |f(y)| dy\|1_B\|_X < \infty.
\]

Thus, \( f1_B \in X \), which completes the proof of the above claim.

To prove Proposition 2.8, we need several lemmas. Let \( n \in [2, \infty) \cap \mathbb{N} \). Recall that, for any given measurable function \( g \) on \( \mathbb{R}^n \), the Riesz potential \( I_1(g) \) is defined by setting, for any \( x \in \mathbb{R}^n \),

\[
I_1(g)(x) := \int_{\mathbb{R}^n} \frac{g(y)}{|x - y|^{n-1}} dy.
\]

We have the following conclusion.

Lemma 2.10 Let \( X \) be a ball Banach function space and the centered ball average operators \( \{ B_r \}_{r \in (0, \infty)} \) the same as in (2.1). Assume that \( \{ B_r \}_{r \in (0, \infty)} \) are uniformly bounded on \( X \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( n \in [2, \infty) \cap \mathbb{N} \). Then there exists a positive constant \( C \), independent of \( \Omega \), such that, for any \( g \in X \),

\[
\|I_1(|g|1_{\Omega})1_{\Omega}\|_X \leq C\text{diam}(\Omega)\|g1_{\Omega}\|_X,
\]

where \( \text{diam}(\Omega) := \sup_{x, y \in \Omega} |x - y| \).

Proof Let \( D := 2\text{diam}(\Omega) \). Then, for any \( x \in \Omega \),

\[
I_1(|g|1_{\Omega})(x) = \int_{\Omega} \frac{|g(y)|}{|x - y|^{n-1}} dy = \int_{B(x, D)} \frac{|g(y)|1_{\Omega}(y)}{|x - y|^{n-1}} dy \\
\quad \leq \sum_{k=1}^{\infty} (2^{-k}D)^{-n} \int_{2^{-k}D \leq |x - y| < 2^{-k+1}D} |g(y)|1_{\Omega}(y) |dy| \\
\quad \leq D \sum_{k=1}^{\infty} 2^{-k} \int_{B(x, 2^{-k+1}D)} |g(y)|1_{\Omega}(y) |dy| \\
\quad = D \sum_{k=1}^{\infty} 2^{-k} B_{2^{-k+1}D}(g1_{\Omega})(x).
\]

From this and the assumptions that both \( X \) is a ball Banach function space and the operators \( \{ B_r \}_{r \in (0, \infty)} \) are uniformly bounded on \( X \), it follows that
∥I_1(g)1_Ω∥_X \lesssim \text{diam} (\Omega) \sup_{g \in L^1(\mathbb{R}^n): \|g\|_X = 1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty.

This finishes the proof of Lemma 2.10.

The following concept of the associate space of ball Banach function spaces can be found in [93, Definition 2.2]; see also, for instance, [6, Chapter 1, Definitions 2.1 and 2.3] for the corresponding one of Banach function spaces.

**Definition 2.11** For any ball Banach function space X, the associate space (also called the Köthe dual) X' is defined by setting

\[
X' := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup_{g \in X: \|g\|_X = 1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\},
\]

(2.3)

where \(\|\cdot\|_{X'}\) is called the associate norm of \(\|\cdot\|_X\).

**Remark 2.12** By [93, Proposition 2.3], we find that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

The following Hölder inequality is a simple corollary of both Definition 2.1(i) and (2.3) (see [6, Theorem 2.4]).

**Lemma 2.13** Let X be a ball Banach function space and X' its associate space. If f \in X and g \in X', then fg is integrable and

\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
\]

**Lemma 2.14** Assume that X is a ball Banach function space and the centered ball average operators \(\{B_r\}_{r \in (0, \infty)}\) are uniformly bounded on X. Then, for any ball B \subset \mathbb{R}^n,

\[
\|1_B\|_X \|1_B\|_{X'} \sim |B|
\]

with the positive equivalence constants depending only on X.

**Proof** Let B be a ball in \(\mathbb{R}^n\). Notice that, for any g \in L^1_{loc}(\mathbb{R}^n) and x \in \mathbb{R}^n,

\[
\frac{1}{|B|} \int_B |g(y)| \, dy 1_B(x) \leq \frac{2^n}{|B(x, 2r_B)|} \int_{B(x, 2r_B)} |g(y)| \, dy 1_B(x) = 2^n B_{2r_B}(g)(x) 1_B(x),
\]

where \(r_B\) denotes the radius of B. From this, the definition of X', and the assumption that the centered ball average operators \(\{B_r\}_{r \in (0, \infty)}\) are uniformly bounded on X, it follows that

\[
\frac{\|1_B\|_X \|1_B\|_{X'}}{|B|} \leq \sup_{\|g\|_X = 1} \frac{1}{|B|} \int_B |g(y)| \, dy 1_B \|_X
\]

\[
\leq \sup_{\|g\|_X = 1} 2^n \|B_{2r_B}(g) 1_B\|_X \lesssim 1.
\]
On the other hand, by Lemma 2.13, we obtain
\[ |B| = \int_{\mathbb{R}^n} 1_B(x) \, dx \leq \|1_B\|_X \|1_B\|_{X'} . \]

This finishes the proof of Lemma 2.14. \(\square\)

Now, we show Proposition 2.8.

**Proof of Proposition 2.8** Let \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \). By Proposition 2.5, we find that \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \). Next, we show the present proposition by considering the following two cases on \( n \).

Case 1) \( n = 1 \). In this case, \( \Omega = (x_0 - R, x_0 + R) \) for some \( x_0 \in \mathbb{R} \) and \( R \in (0, \infty) \). We first show that, for almost every \( x \in \Omega \),
\[ |f(x) - f_\Omega| \leq \int_\Omega |f'(y)| \, dy. \tag{2.4} \]

Let \( \eta \in C_c^\infty(\mathbb{R}) \) satisfy \( \text{supp}(\eta) \subset (-1, 1) \) and \( \int_\mathbb{R} \eta(x) \, dx = 1 \). For any \( k \in \mathbb{N} \), let \( \eta_k(\cdot) := k \eta(k \cdot) \) and \( f_k(\cdot) := (f \ast \eta_k)(\cdot) \). Using both (i) and (vi) of [39, Theorem 4.1], we find that \( f_k \in C^\infty(\mathbb{R}) \) for any \( k \in \mathbb{N} \),
\[ \lim_{k \to \infty} \|f - f_k\|_{L^1(\Omega)} = 0, \quad \text{and} \quad \lim_{k \to \infty} \|f' - f'_k\|_{L^1(\Omega)} = 0. \tag{2.5} \]

By the Riesz theorem, we may assume that \( \lim_{k \to \infty} f_k(x) = f(x) \) for almost every \( x \in \Omega \). Since \( f_k \in C^\infty(\mathbb{R}) \) for any \( k \in \mathbb{N} \), it follows that, for any \( k \in \mathbb{N} \) and \( x, y \in \Omega \),
\[ |f_k(x) - f_k(y)| = \int_y^x f'_k(t) \, dt \leq \int_\Omega |f'_k(t)| \, dt \]
and hence
\[ \left| f_k(x) - \frac{1}{|\Omega|} \int_\Omega f_k(y) \, dy \right| \leq \frac{1}{|\Omega|} \int_\Omega |f_k(x) - f_k(y)| \, dy \leq \int_\Omega |f'_k(t)| \, dt. \]

Using this and (2.5), and letting \( k \to \infty \), we conclude that (2.4) holds true.

By (2.4) and Lemmas 2.13 and 2.14, we obtain
\[ \|f - f_\Omega\|_X \leq \|f_\Omega\|_X \int_\Omega |f'(y)| \, dy \leq \|1_\Omega\|_X \|x\|_X \|f'1_\Omega\|_X \]
\[ \lesssim R \|f'1_\Omega\|_X. \]

This proves (2.2) in this case.

Case 2) \( n \in [2, \infty) \cap \mathbb{N} \). In this case, from [33, Lemma 8.2.1(b)], we infer that there exists a ball \( B \subset \Omega \) and positive constants \( c_1, c_2 \) such that, for almost every \( x \in \Omega \),
\[ |f(x) - f_B| \leq c_1 I_1(\|\nabla f\|_1) \tag{2.6} \]
and
\[ |B| \leq |\Omega| \leq c_2 |B|, \tag{2.7} \]
where the positive constants \( c_1 \) and \( c_2 \) depend only on \( n \). Using (2.6) and Lemma 2.10, we further obtain
\[ \|(f - f_B)1_\Omega\|_X \lesssim I_1(\|\nabla f\|_1)1_\Omega\|_X \lesssim R \|\nabla f\|_1. \tag{2.8} \]
On the other hand, by [33, Theorem 8.2.4(b)], we have the following classical Poincaré inequality
\[ \|f - f_\Omega\|_{L^1(\Omega)} \lesssim R \|\nabla f\|_{L^1(\Omega)}, \]
where the implicit positive constant depends only on \( n \). From this, Lemmas 2.13 and 2.14, and (2.7), it follows that
\[ \|(f_B - f_\Omega)1_\Omega\|_X \]

\[ \leq \frac{\|1_\Omega\|_X}{|B|} \int_B |f(x) - f_\Omega| \, dx \lesssim \frac{\|1_\Omega\|_X}{|\Omega|} \int_\Omega |f(x)| \, dx \]

\[ \lesssim R \frac{\|1_\Omega\|_X}{|\Omega|} \int_\Omega |\nabla f(x)| \, dx \leq R \frac{\|1_\Omega\|_X \|1_\Omega\|_{X'}}{|\Omega|} \|\nabla f\|_X \]

\[ \lesssim R \frac{\|B(0, 2R)\|_X \|B(0, 2R)\|_{X'}}{|B(0, 2R)|} \|\nabla f\|_X \lesssim R \|\nabla f\|_X. \]

This, together with both (2.8) and the assumption that \( X \) is a ball Banach function space, further implies that (2.2) holds true. This finishes the proof of Proposition 2.8.

To prove Theorem 2.6, we need the following conclusion.

**Proposition 2.15** Let \( X \) be a ball Banach function space. Assume that the centered ball average operators \( \{B_r\}_{r \in (0, \infty)} \) are uniformly bounded on \( X \) and that \( X \) has an absolutely continuous norm. Then, for any \( f \in W^{1, \infty}(\mathbb{R}^n) \), there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap \tilde{W}^{1, \infty}(\mathbb{R}^n) \) such that
\[ \lim_{k \to \infty} \|f - f_k\|_{W^{1, \infty}(\mathbb{R}^n)} = 0 \]
and
\[ \lim_{k \to \infty} \|(f - f_k)1_{B(0, R)}\|_X = 0 \]
for any \( R \in (0, \infty) \).

**Proof** Let \( f \in \tilde{W}^{1, X}(\mathbb{R}^n) \). It is well known that there exists a function \( \eta \in C^\infty_c(\mathbb{R}^n) \) such that \( \text{supp}(\eta) \subset B(0, 1) \) and \( \int_{\mathbb{R}^n} \eta(x) \, dx = 1 \). For any \( k \in \mathbb{N} \), let \( \eta_k(\cdot) := k^n \eta(k \cdot) \) and \( f_k(\cdot) := (f * \eta_k)(\cdot) \). Next, we show that
\[ \lim_{k \to \infty} \|f - f_k\|_{\tilde{W}^{1, X}(\mathbb{R}^n)} = 0. \] (2.9)

First of all, by Proposition 2.5, we conclude that \( f \in W^{1, 1}_{\text{loc}}(\mathbb{R}^n) \). Using this and both (i) and (v) of [39, Theorem 4.1], we conclude that, for any \( k \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \), \( f_k \in C^\infty(\mathbb{R}^n) \) and
\[ \frac{\partial f_k}{\partial x_i} = \frac{\partial f}{\partial x_i} * \eta_k. \] (2.10)

Let \( i \in \{1, \ldots, n\} \). Since \( X \) has an absolutely continuous norm, from [99, Proposition 3.8], it follows that \( C_c(\mathbb{R}^n) \) is dense in \( X \). Thus, for any \( \epsilon \in (0, \infty) \), there exists a function \( g \in C_c(\mathbb{R}^n) \) such that
\[ \left\| \frac{\partial f}{\partial x_i} - g \right\|_X \leq \epsilon. \] (2.11)

By this, (2.10), and the assumption that \( X \) is a ball Banach function space, we have
\[ \left\| \frac{\partial f}{\partial x_i} - \frac{\partial f_k}{\partial x_i} \right\|_X \leq \left\| \frac{\partial f}{\partial x_i} - g \right\|_X + \|g - g * \eta_k\|_X. \]
\[
+ \left\| g \ast \eta - \frac{\partial f}{\partial x_i} \ast \eta \right\|_X.
\] (2.12)

Assume that \( \text{supp} \,(g) \subset B(0, N) \) with some \( N \in (0, \infty) \). Then we have
\[
\| g - g \ast \eta \|_X \leq k^n \int_{\mathbb{R}^n} |g(\cdot) - g(y)| |\eta(k[\cdot - y])| \, dy \|_X
\leq \int_{B(0, k^{-1})} |g(\cdot) - g(y)| \, dy \|_X
\leq \sup_{|x - y| \leq k^{-1}} |g(x) - g(y)| \|\mathbf{1}_{B(0,N+1)}\|_X \rightarrow 0
\] (2.13)
as \( k \rightarrow \infty \). On the other hand, using (2.11) and the assumption that the operators \( \{B_r\}_{r \in (0, \infty)} \) are uniformly bounded on \( X \), we conclude that, for any \( k \in \mathbb{N} \),
\[
\left\| g \ast \eta - \frac{\partial f}{\partial x_i} \ast \eta \right\|_X
\leq k^n \int_{\mathbb{R}^n} |g(\cdot - y) - \frac{\partial f}{\partial x_i}(\cdot - y)| \eta(ky) \, dy \|_X
= \int_{\mathbb{R}^n} g(\cdot - y/k) - \frac{\partial f}{\partial x_i}(\cdot - y/k) \eta(y) \, dy \|_X
\leq \int_{B(0, k^{-1})} g(y) - \frac{\partial f}{\partial x_i}(y) \, dy \|_X
= B_{k^{-1}} \left( g - \frac{\partial f}{\partial x_i} \right) \|_X \lesssim \left\| g - \frac{\partial f}{\partial x_i} \right\|_X \lesssim \epsilon.
\]

From this, (2.11), (2.13), and (2.12), it follows that, for any \( \epsilon \in (0, \infty) \),
\[
\limsup_{k \rightarrow \infty} \left\| \frac{\partial f}{\partial x_i} - \frac{\partial f_k}{\partial x_i} \right\|_X \lesssim \epsilon.
\]
Let \( \epsilon \rightarrow 0 \). Then we have, for any \( i \in \{1, \ldots, n\} \),
\[
\lim_{k \rightarrow \infty} \left\| \frac{\partial f}{\partial x_i} - \frac{\partial f_k}{\partial x_i} \right\|_X = 0.
\]
This implies that (2.9) holds true.

Finally, we show that
\[
\lim_{k \rightarrow \infty} \| (f - f_k) \mathbf{1}_{B(0,R)} \|_X = 0 \quad (2.14)
\]
for any \( R \in (0, \infty) \). By [39, Theorem 4.1(iii)], we find that, for any \( R \in (0, \infty) \),
\[
\lim_{k \rightarrow \infty} \int_{B(0,R)} |f(y) - f_k(y)| \, dy = 0.
\]
Using this, the assumption that \( X \) is a ball Banach function space, Proposition 2.8, Remark 2.9, and (2.9), we obtain, for any \( R \in (0, \infty) \),
\[
\| (f - f_k) \mathbf{1}_{B(0,R)} \|_X \lesssim R \| \nabla f - \nabla f_k \|_X
+ \frac{1}{|B(0, R)|} \int_{B(0,R)} |f(y) - f_k(y)| \, dy \|\mathbf{1}_{B(0,R)}\|_X
\rightarrow 0
\]
Now, we show Theorem 2.6.

**Proof of Theorem 2.6** Let \( f \in \tilde{W}^{1, X}(\mathbb{R}^n) \). By Proposition 2.15, we can assume that \( f \in C^\infty(\mathbb{R}^n) \cap \tilde{W}^{1, X}(\mathbb{R}^n) \). Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \) be such that \( 0 \leq \varphi \leq 1 \), \( \text{supp}(\varphi) \subset B(0, 2) \), and \( \varphi \equiv 1 \) in \( B(0, 1) \). For any given \( k \in \mathbb{N} \) and any \( x \in \mathbb{R}^n \), let \( \eta_k(x) := \varphi(x/k) \). Now, we prove the present theorem by considering the following two cases on \( n \).

**Case 1)** \( n = 1 \). In this case, let \( f_k(x) := \int_0^x f'(t) \eta_k(t) \, dt + f(0) \) for any \( x \in \mathbb{R} \) and any given \( k \in \mathbb{N} \). By the assumption that \( X \) has an absolutely continuous norm, we have

\[
\lim_{k \to \infty} \left\| f'_k - f' \right\|_X = \lim_{k \to \infty} \left\| f'_k - f'_k \right\|_X = 0.
\]

Let \( R \in (0, \infty) \). Notice that, for any \( k \in \mathbb{N} \cap [R, \infty) \) and \( x \in B(0, R) \),

\[
f_k(x) = \int_0^x f'(t) \, dt + f(0) = f(x)
\]

and hence

\[
\lim_{k \to \infty} \left\| (f - f_k) \mathbf{1}_{B(0, R)} \right\|_X = 0.
\]

This finishes the proof of the present theorem in this case.

**Case 2)** \( n \in [2, \infty) \cap \mathbb{N} \). In this case, for any \( k \in \mathbb{N} \), let \( \Omega_k := \{ x \in \mathbb{R}^n : k < |x| < 2k \} \). By Proposition 2.8, we obtain, for any \( k \in \mathbb{N} \),

\[
\left\| (f - f_{\Omega_k}) \mathbf{1}_{\Omega_k} \right\|_X \lesssim k \left\| \nabla f \mathbf{1}_{\Omega_k} \right\|_X. \tag{2.15}
\]

For any \( k \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), let \( f_k(x) := (f(x) - f_{\Omega_k}) \eta_k(x) + f_{\Omega_k} \). Now, we show that \( f_k \to f \) in \( \tilde{W}^{1, X}(\mathbb{R}^n) \). Notice that, for any \( j \in \{1, \ldots, n\} \),

\[
\frac{\partial(f_k - f)}{\partial x_j} = \frac{\partial f}{\partial x_j}(\eta_k - 1) + (f - f_{\Omega_k}) \frac{\partial \eta_k}{\partial x_j}. \tag{2.16}
\]

Since \( X \) has an absolutely continuous norm, we deduce that

\[
\lim_{k \to \infty} \left\| \frac{\partial f}{\partial x_j}(\eta_k - 1) \right\|_X = 0. \tag{2.17}
\]

Observe that, for any \( k \in \mathbb{N} \) and \( x \in \mathbb{R}^n \),

\[
\left| \frac{\partial \eta_k}{\partial x_j}(x) \right| = \frac{1}{k} \left| \frac{\partial \varphi}{\partial x_j}(\frac{x}{k}) \right| \leq \frac{1}{k} \left\| \nabla \varphi \right\|_{L^\infty(\mathbb{R}^n)}.
\]

From this, (2.15), and the assumption that \( X \) has an absolutely continuous norm, it follows that

\[
\left\| (f - f_{\Omega_k}) \frac{\partial \eta_k}{\partial x_j} \right\|_X = \left\| (f - f_{\Omega_k}) \frac{\partial \eta_k}{\partial x_j} \mathbf{1}_{\Omega_k} \right\|_X \lesssim \left\| \nabla f \mathbf{1}_{\Omega_k} \right\|_X \leq \left\| \nabla f \mathbf{1}_{B(0, k)} \right\|_X \to 0
\]

as \( k \to \infty \). This, together with both (2.17) and (2.16), further implies that, for any \( j \in \{1, \ldots, n\} \),

\[
\lim_{k \to \infty} \left\| \frac{\partial(f_k - f)}{\partial x_j} \right\|_X = 0
\]

This completes the proof of (2.14) and hence finishes the proof of Theorem 2.15. □
and hence
\[ \lim_{k \to \infty} \| f - f_k \|_{\dot{W}^1, X(\mathbb{R}^n)} = 0. \]
Let \( R \in (0, \infty) \). Observe that, for any \( k \in \mathbb{N} \cap [R, \infty) \) and \( x \in B(0, R) \),
\[ f_k(x) = f(x) - f_{\Omega_k} + f_{\Omega_k} = f(x) \]
and hence
\[ \lim_{k \to \infty} \| (f - f_k)1_B(0, R) \|_X = 0. \]
This then finishes the proof of Theorem 2.6. \( \square \)

The following definition of the convexification of a ball Banach function space can be found in [93, Definition 2.6].

**Definition 2.16** Assume that \( X \) is a ball quasi-Banach function space and \( p \in (0, \infty) \). The \( p \)-convexification \( X^p \) of \( X \) is defined by setting \( X^p := \{ f \in \mathcal{M} (\mathbb{R}^n) : |f|^p \in X \} \) equipped with the quasi-norm \( \| f \|_{X^p} := \| |f|^p \|^{1/p}_X \) for any \( f \in X^p \).

The following lemma gives a sufficient condition for the uniform boundedness of centered ball average operators \( \{B_r\}_{r \in (0, \infty)} \) on \( X \), which is just [30, Lemma 3.11].

**Lemma 2.17** Let \( X \) be a ball Banach function space and \( p \in [1, \infty) \). Assume that \( X^{1/p} \) is a ball Banach function space and the Hardy–Littlewood maximal operator \( \mathcal{M} \) is bounded on \( (X^{1/p})' \). Then the centered ball average operators \( \{B_r\}_{r \in (0, \infty)} \) are uniformly bounded on \( X \).

Using Lemma 2.17 and Theorem 2.6, we immediately obtain the following conclusion.

**Corollary 2.18** Let \( X \) be a ball Banach function space and \( p \in [1, \infty) \). Assume that \( X^{1/p} \) is a ball Banach function space, the Hardy–Littlewood maximal operator \( \mathcal{M} \) is bounded on \( (X^{1/p})' \), and \( X \) has an absolutely continuous norm. Then, for any \( f \in \dot{W}^{1, X}(\mathbb{R}^n) \), there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \) with \( |\nabla f_k| \in C_c(\mathbb{R}^n) \) for any \( k \in \mathbb{N} \) such that
\[ \lim_{k \to \infty} \| f - f_k \|_{\dot{W}^1, X(\mathbb{R}^n)} = 0 \] and
\[ \lim_{k \to \infty} \| (f - f_k)1_B(0, R) \|_X = 0 \]
for any \( R \in (0, \infty) \).

## 3 Estimates in weighted Lebesgue spaces

In this section, we establish the characterization of the Sobolev semi-norm in the weighted Lebesgue space (see Theorem 3.5 below), which is just Theorem 1.1 when \( p = q \). It should be pointed out that Theorem 3.5 plays a vital role in the proof of Theorem 4.5 below. Moreover, we show that the \( A_p(\mathbb{R}^n) \) condition in Theorem 1.1 is sharp in some sense (see Theorem 3.7 below).

We first recall the concept of Muckenhoupt weights \( A_p(\mathbb{R}^n) \) (see, for instance, [43]).

**Definition 3.1** An \( A_p(\mathbb{R}^n) \)-weight \( \omega \), with \( p \in [1, \infty) \), is a nonnegative locally integrable function on \( \mathbb{R}^n \) satisfying that, when \( p \in (1, \infty) \),
\[ [\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left\{ \frac{1}{|Q|} \int_Q [\omega(x)]^{1/p} \, dx \right\}^{p-1} < \infty, \]
and 

\[ [\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(x) \, dx \left[ \|\omega^{-1}\|_{L^\infty(Q)} \right] < \infty, \]

where the suprema are taken over all cubes \( Q \subset \mathbb{R}^n \). Moreover, let \( A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n) \).

**Definition 3.2** Let \( p \in [0, \infty) \) and \( \omega \in A_\infty(\mathbb{R}^n) \). The weighted Lebesgue space \( L^p_\omega(\mathbb{R}^n) \) is defined to be the set of all the measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[ \|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right]^\frac{1}{p} < \infty. \]

The following lemma is a part of [43, Proposition 7.1.5].

**Lemma 3.3** Let \( p \in [1, \infty) \) and \( \omega \in A_p(\mathbb{R}^n) \). Then the following statements hold true.

(i) For any \( \lambda \in (1, \infty) \) and any cube \( Q \subset \mathbb{R}^n \), one has \( \omega(\lambda Q) \leq [\omega]_{A_p(\mathbb{R}^n)} \lambda^n \omega(Q) \);

(ii) \( [\omega]_{A_p(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \sup_{f_1 \in L^p_\omega(\mathbb{R}^n)} \frac{\left[ \frac{1}{|Q|} \int_Q |f(x)| \, dx \right]^p}{\omega(Q) \int_Q |f(x)|^p \omega(x) \, dx} \)

where the first supremum is taken over all cubes \( Q \subset \mathbb{R}^n \).

**Definition 3.4** Let \( p \in [1, \infty) \) and \( \omega \in A_\infty(\mathbb{R}^n) \). The homogeneous weighted Sobolev space \( \dot{W}^1_p(\mathbb{R}^n) \) is defined to be the set of all the distributions \( f \) on \( \mathbb{R}^n \) whose distributional gradients \( \nabla f := (\partial_1 f, \ldots, \partial_n f) \) satisfy \( |\nabla f| \in L^p_\omega(\mathbb{R}^n) \) and, moreover, for any \( f \in \dot{W}^1_p(\mathbb{R}^n) \), let

\[ \|f\|_{\dot{W}^1_p(\mathbb{R}^n)} := \|\nabla f\|_{L^p_\omega(\mathbb{R}^n)}. \]

The following conclusion is the main result of this section.

**Theorem 3.5** Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). Assume that \( \omega \in A_1(\mathbb{R}^n) \). Then there exist positive constants \( C_1 \) and \( C_{([\omega]_{A_1(\mathbb{R}^n)})} \) such that, for any \( f \in \dot{W}^1_p(\mathbb{R}^n) \),

\[ \left[ \frac{K(q, n)}{n} \right]^\frac{1}{q} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \leq \sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right] \omega(x) \, dx \leq C_1 C_{([\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx, \]

where, for any \( \lambda \in (0, \infty) \), \( q \in (0, \infty) \), and any measurable function \( f \),

\[ E_f(\lambda, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^{\frac{n}{q} + 1} \right\}. \]
the positive constant $C_1$ depends only on $p, q,$ and $n$, the positive constant $C_{(\omega) A_1(\mathbb{R}^n)}$ increases as $[\omega]_{A_1(\mathbb{R}^n)}$ increases, and $C_{(\cdot)}$ is continuous on $(0, \infty)$. Moreover, for any $f \in \dot{W}^{-1,p}(\mathbb{R}^n)$,

$$
\lim_{\lambda \to \infty} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx = \left[ \frac{K(q, n)}{n} \right]^{\frac{p}{q}} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
$$

where

$$K(q, n) := \int_{[q_{n-1}}|\xi| \cdot e|^q \, d\sigma(\xi)$$

(3.3)

with $e$ being some unit vector in $\mathbb{R}^n$.

**Remark 3.6**

(i) As a consequence of Theorem 3.5, we obtain Theorem 1.1.

(ii) For any given $p \in (1, \infty)$, the upper estimate of (3.1) can be easily deduced from both the Lusin–Lipschitz inequality (1.5) and the boundedness of the Hardy–Littlewood maximal operator on $L^p(\mathbb{R}^n)$. Based on this, we find that, since the Hardy–Littlewood maximal operator is not bounded on $L^1(\mathbb{R}^n)$, the hard core of the proof of (3.1) is to include the case $p = 1$.

(iii) Let $p, q \in [1, \infty]$ satisfy $n \max(0, \frac{1}{p} - \frac{1}{q}) < s < 1$. By [57, Theorem 1.3], we conclude that, if $f \in \dot{L}^\min[p,q]_{\loc}(\mathbb{R}^n)$, then $f \in F^s_{p,q}(\mathbb{R}^n)$ if and only if

$$I := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^q}{|\cdot - y|^{n+q}} \, dy \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{q}} < \infty$$

and, moreover, in this case,

$$I \sim \|f\|_{F^s_{p,q}(\mathbb{R}^n)}$$

(3.4)

with the positive equivalence constants independent of $f$, where $F^s_{p,q}(\mathbb{R}^n)$ denotes the classical Triebel–Lizorkin space (see [100, Sect. 2.3] for the precise definition). Using (3.4), we find that, for any given $p \in [1, \infty]$ and $s \in (0, 1)$, $F^s_{p,q}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ with equivalent norms. Also, by [57, Theorem 1.5], we conclude that, under the assumption that $p, q \in [1, \infty)$, (3.4) is valid only if $n \max(0, \frac{1}{p} - \frac{1}{q}) \leq s < 1$. Moreover, using both (i) and (ii) of Theorem 4.3 below, we find that, when $s = 1$, $p \in [1, \infty)$, and $q \in [1, p]$,

$$\left\| \int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^q}{|\cdot - y|^{n+q}} \, dy \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{q}} = \infty$$

unless $f$ is a constant. Thus, the Gagliardo quasi-semi-norm in (3.4) cannot recover the Triebel–Lizorkin quasi-semi-norm $\| \cdot \|_{F^1_{p,q}(\mathbb{R}^n)}$ when $s = 1$ and, in this sense, the assumption $n(\frac{1}{p} - \frac{1}{q}) < 1$ in Theorem 3.5 seems to be sharp. Replacing the strong type quasi-norm in (3.4) by the weak type quasi-norm, and using Theorem 3.5 with $\omega = 1$, we conclude that, for any $f \in \dot{W}^{-1,p}(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} + \sup_{\lambda \in (0, \infty)} \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right]^{\frac{p}{q}} \, dx \right\}^{\frac{1}{p}} \sim \|f\|_{F^1_{p,q}(\mathbb{R}^n)}$$
with the positive equivalence constants independent of \( f \). This indicates that Theorem 3.5 is a perfect replacement of (3.4) in the critical case \( s = 1 \).

(iv) Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n \left( \frac{1}{p} - \frac{1}{q} \right) < 1 \). Let \( \dot{W}^1_{L^p, q}(\mathbb{R}^n) \) be the weak-type space \( \dot{W}^1_{X, q}(\mathbb{R}^n) \) in Definition 4.1 below with \( X := L^p_0(\mathbb{R}^n) \). Assume that \( q_1, q_2 \in (0, \infty) \) satisfy \( n \left( \frac{1}{p} - \frac{1}{q_1} \right) < 1 \) and \( n \left( \frac{1}{p} - \frac{1}{q_2} \right) < 1 \). From Theorem 3.5, it follows that

\[
\dot{W}^1_{L^p_0(\mathbb{R}^n), q_1}(\mathbb{R}^n) \cap \dot{W}^1_{\omega, p}(\mathbb{R}^n) = \dot{W}^1_{L^p_0(\mathbb{R}^n), q_2}(\mathbb{R}^n) \cap \dot{W}^1_{\omega, p}(\mathbb{R}^n)
\]

with equivalent quasi-norms. Thus, when \( q \in (0, \infty) \) satisfies \( n \left( \frac{1}{p} - \frac{1}{q} \right) < 1 \), the space \( \dot{W}^1_{L^p_0(\mathbb{R}^n), q}(\mathbb{R}^n) \cap \dot{W}^1_{\omega, p}(\mathbb{R}^n) \) is independent of \( q \).

(v) For the purpose of our applications later, we only consider \( A_1(\mathbb{R}^n) \)-weights here. However, our proof actually works equally well for more general \( A_p(\mathbb{R}^n) \)-weights. For instance, a slight modification of the proofs in this section shows that (3.1) holds true for any given \( 1 \leq p = q < \infty \) and \( \omega \in A_{\min[p, 1 + \frac{p}{n}]}(\mathbb{R}^n) \).

**Theorem 3.7** Let \( p \in [1, \infty) \) and \( \omega \) be a non-negative function on \( \mathbb{R} \). Assume that there exists a positive constant \( C_1 \) such that, for any \( f \in C^1(\mathbb{R}) \) satisfying that \( f' \) has compact support,

\[
\sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^2} 1_{E_f(\lambda, p)}(x, y)\omega(x) \, dx \, dy \leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx,
\]

where \( E_f(\lambda, p) \) is the same as in (3.2) for any \( \lambda \in (0, \infty) \). Then \( \omega \in A_p(\mathbb{R}) \).

**Remark 3.8** In one dimension, Theorem 3.7 implies that the \( A_1(\mathbb{R}) \) condition in Theorem 1.1 with \( p = 1 \) is sharp.

The proof of Theorem 3.5 is given in Sects. 3.1 and 3.2. In Sect. 3.3, we prove Theorem 3.7.

### 3.1 Proof of Theorem 3.5: upper estimate

We begin with recalling some conclusions about Muckenhoupt weights \( A_p(\mathbb{R}^n) \). The following lemma is a part of [43, Theorem 7.1.9].

**Lemma 3.9** Let \( p \in (1, \infty) \) and \( \omega \in A_p(\mathbb{R}^n) \). Then there exists a positive constant \( C \), independent of \( \omega \), such that, for any \( f \in L^p_0(\mathbb{R}^n) \),

\[
\| M(f) \|_{L^p_0(\mathbb{R}^n)} \leq C [\omega]_{A_p(\mathbb{R}^n)}^{\frac{1}{p-1}} \| f \|_{L^p_0(\mathbb{R}^n)},
\]

where \( M \) is the same as in (1.6).

For the proof of the upper bound in (3.1), we need to use several adjacent systems of dyadic cubes, which can be found, for instance, in [71, Sect. 2.2].

**Lemma 3.10** For any \( \alpha \in \{0, \frac{1}{3}, \frac{2}{3} \}^n \), let

\[
D^\alpha := \{ 2^j(k + [1, 0]^n + (-1)^j \alpha) : j \in \mathbb{Z}, k \in \mathbb{Z}^n \}.
\]

Then

(i) For any \( Q, Q' \in D^\alpha \) with the same \( \alpha \in \{0, \frac{1}{3}, \frac{2}{3} \}^n \), \( Q \cap Q' \in \{\emptyset, Q, Q'\} \);
(ii) For any ball \( B \subset \mathbb{R}^n \), there exists an \( \alpha \in [0, \frac{1}{3}, \frac{3}{2}]^n \) and a \( Q \in \mathcal{D}^\alpha \) such that \( B \subset Q \subset CB \), where the positive constant \( C \) depends only on \( n \).

Applying Proposition 2.5 with \( X := L^p_w(\mathbb{R}^n) \), we obtain the following corollary.

**Corollary 3.11** Let \( p \in [1, \infty) \) and \( \omega \in A_p(\mathbb{R}^n) \). Then
\[
\dot{W}^{1,p}_w(\mathbb{R}^n) \subset W^{1,1}_{\text{loc}}(\mathbb{R}^n).
\]

We also need the following conclusion.

**Lemma 3.12** Let \( p \in [1, \infty) \), \( \omega \in A_p(\mathbb{R}^n) \), and \( f \in \dot{W}^{1,p}_w(\mathbb{R}^n) \). Then there exists an \( A \subset \mathbb{R}^n \) with \( |A| = 0 \) and a positive constant \( C_n \), depending only on \( n \), such that, for any \( x \in \mathbb{R}^n \setminus A \), \( r \in (0, \infty) \), and any ball \( B_1 \subset B := B(x, r) \subset 3B_1 \),
\[
|f(x) - f_{B_1}| \leq C_n r \sum_{j=0}^{\infty} 2^{-j} \int_{2^{-j}B} |\nabla f(y)| \, dy.
\]  

**Proof** By Corollary 3.11, we find that \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \). This, together with the classical Poincaré inequality (see, for instance, [33, Theorem 8.2.4(b)]), implies that, for any ball \( B_0 \subset \mathbb{R}^n \),
\[
\int_{B_0} |f(y) - f_{B_0}| \, dy \leq Cr_{B_0} \int_{B_0} |\nabla f(y)| \, dy,
\]  

where the positive constant \( C \) depends only on \( n \). By the Lebesgue differentiation theorem, we find that there exists an \( A \subset \mathbb{R}^n \) with \( |A| = 0 \) such that, for any \( x \in \mathbb{R}^n \setminus A \),
\[
\lim_{j \to \infty} \frac{1}{|2^{-j}B|} \int_{2^{-j}B} |f(x) - f(y)| \, dy = 0
\]
and hence
\[
f(x) = \lim_{j \to \infty} \frac{1}{|2^{-j}B|} \int_{2^{-j}B} f(y) \, dy = \lim_{j \to \infty} f_{2^{-j}B},
\]  

where \( B := B(x, r) \) with \( r \in (0, \infty) \). This, together with (3.6), implies that
\[
|f(x) - f_B| = \lim_{j \to \infty} |f_{2^{-j}B} - f_B| \leq \sum_{j=0}^{\infty} |f_{2^{-j-1}B} - f_{2^{-j}B}|
\]
\[
\leq \sum_{j=0}^{\infty} \frac{1}{|2^{-j-1}B|} \int_{2^{-j-1}B} |f(y) - f_{2^{-j}B}| \, dy
\]
\[
\lesssim r \sum_{j=0}^{\infty} 2^{-j} \int_{2^{-j}B} |\nabla f(y)| \, dy.
\]  

On the other hand, using (3.6) again, we have
\[
|f_B - f_{B_1}| \leq \frac{1}{|B_1|} \int_{B_1} |f(y) - f_B| \, dy \lesssim \frac{1}{|B|} \int_{B} |f(y) - f_B| \, dy
\]
\[
\lesssim r \int_{B} |\nabla f(y)| \, dy.
\]  

From this and (3.7), we deduce that (3.5) holds true. This finishes the proof of Lemma 3.12. \( \square \)
We next establish the upper estimate of Theorem 3.5 in the following separate theorem.

**Theorem 3.13** Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n\left(\frac{1}{p} - \frac{1}{q}\right) < 1 \). Assume that \( \omega \in A_1(\mathbb{R}^n) \). Then there exist positive constants \( C_1 \) and \( C_{(\omega)} \) such that, for any \( f \in \dot{W}^{1,p}_{\omega}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \left[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} 1_{E_f}(\lambda, q)(x, y) \, dy \right)^{\frac{p}{q}} \omega(x) \, dx \right]^{\frac{q}{p}} \leq C_1 C_{(\omega)} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]

(3.8)

where \( E_f(\lambda, q) \) for any \( \lambda \in (0, \infty) \) is the same as in (3.2), the positive constant \( C_1 \) is independent of \( \omega \), the positive constant \( C_{(\omega)} \) increases as \( [\omega]_{A_1(\mathbb{R}^n)} \) increases, and \( C(\cdot) \) is continuous on \((0, \infty)\).

Let \( p, q, \omega, \) and \( f \) be the same as in Theorem 3.13. For any \( x, y \in \mathbb{R}^n \), let \( B_{x,y} := B(\frac{x+y}{2}, |x - y|) \),

\[
E_f^{(1)}(1, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f_{B_{x,y}}| \geq 2^{1-q} |x - y|^{\frac{n}{q} + 1} \right\} \setminus (A \times \mathbb{R}^n),
\]

(3.9)

and

\[
E_f^{(2)}(1, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y) - f_{B_{x,y}}| \geq 2^{1-q} |x - y|^{\frac{n}{q} + 1} \right\} \setminus (\mathbb{R}^n \times A),
\]

(3.10)

where \( A \) is the same as in Lemma 3.12. We need the following several lemmas.

**Lemma 3.14** Let \( p \in [1, \infty) \), \( q \in (0, \infty) \), and \( \omega \in A_1(\mathbb{R}^n) \). Then there exists a positive constant \( C \) such that, for any \( f \in C_c^{\infty}(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f^{(1)}(1, q)}(x, y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \leq C \omega_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]

(3.11)

where \( E_f^{(1)}(1, q) \) is the same as in (3.9) and the positive constant \( C \) is independent of \( \omega \).

**Proof** Indeed, to show this lemma, in the proof below, we only need to consider the case \( \epsilon = \frac{1}{2} \). However, some of the estimates below with an arbitrarily given \( \epsilon \in (0, 1) \) are needed in the proof of Lemma 3.15 below and hence we give the details here.

By Lemma 3.12 with both \( B := B(x, 2|x - y|) \) and \( B_1 := B(\frac{x+y}{2}, |x - y|) =: B_{x,y} \), we find that there exists a positive constant \( c_1 \), depending only on \( n \), such that, for any \( x, y \in \mathbb{R}^n \),

\[
|f(x) - f_{B_{x,y}}| \leq c_1 |x - y| \sum_{j=0}^{\infty} 2^{-j} \int_{B(x, 2^{-j+1}|x - y|)} |\nabla f(z)| \, dz.
\]

Using the pigeonhole principle, we conclude that, for any \((x, y) \in E_f^{(1)}(1, q)\), there exists a \( j, x, y \in \mathbb{Z}_+ \), depending only on both \( x \) and \( y \), such that

\[
\int_{B(x, 2^{-j}x,y+1|x - y|)} |\nabla f(z)| \, dz > c_2 2^{j,x,y(1-\epsilon)} |x - y|^{n/q},
\]

(3.12)
Applying Lemma 3.10 to the ball $B_{j,x,y} := B(x, 2^{-j,x,y} |x - y|)$.

For any $A$, we prove (3.11) by considering the following two cases on dyadic cubes and the fact that the dyadic cubes in $A$ contained in a dyadic cube in $A$, it follows that $(3.13)$ with the positive constant $C$ depends only on $n$. This implies that, for any $(x, y) \in E_f^{(1)}(1, q)$, there exists $j, x, y \in \mathbb{Z}_+^n$ and a positive constant $c_3$, depending only on $n$, $q$, and $\varepsilon$, such that

\[
\int_{Q_{j,x,y}} |\nabla f(z)| \, dz > c_3 2^{2j,x,y(1-\varepsilon)} |2^{j,x,y} Q_{j,x,y}|^{\frac{q}{n}}.
\]  

(3.13)

For any $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ and $j \in \mathbb{Z}_+$, we denote by the symbol $\mathcal{A}_{\alpha}^j$ the collection of all the dyadic cubes $Q \in \mathcal{D}^n$ which satisfies (3.13) with $Q_{j,x,y}$ replaced by $Q$, where $\mathcal{D}^n$ is the same as in Lemma 3.10. From the definition of $\mathcal{A}_{\alpha}^j$, Lemma 3.3(ii) with $\omega$ regarded as an $A_p(\mathbb{R}^n)$ weight, and the fact that $[\omega]_{A_p(\mathbb{R}^n)} \leq [\omega]_{A_1(\mathbb{R}^n)}$, we deduce that, for any $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, $j \in \mathbb{Z}_+$, and $Q \in \mathcal{A}_{\alpha}^j$,

\[
\omega(Q)|2^j Q|^\frac{n}{q} \leq c_3^{-p} [\omega]_{A_1(\mathbb{R}^n)} 2^{-j(1-\varepsilon)p} \int_Q |\nabla f(z)|^p \omega(z) \, dz
\]

\[
\leq c_3^{-p} [\omega]_{A_1(\mathbb{R}^n)} 2^{-j(1-\varepsilon)p} \|f\|^p_{L^p(\mathbb{R}^n)},
\]  

(3.14)

which further implies that, for any given $j \in \mathbb{Z}_+$,

\[
\sup_{Q \in \mathcal{A}_{\alpha}^j} l(Q) < \infty
\]  

with $l(Q)$ for any $Q \in \mathcal{A}_{\alpha}^j$ being the edge length of $Q$. Thus, every cube $Q \in \mathcal{A}_{\alpha}^j$ is contained in a dyadic cube in $\mathcal{A}_{\alpha}^j$ that is maximal with respect to the set inclusion. For any $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, we denote by the symbol $\mathcal{A}_{\alpha,\text{max}}^j$ the collection of all the dyadic cubes in $\mathcal{A}_{\alpha}^j$ that are maximal with respect to the set inclusion. Clearly, the maximal dyadic cubes in $\mathcal{A}_{\alpha,\text{max}}^j$ are pairwise disjoint. For any $(x, y) \in E_f^{(1)}(1, q)$, since $(x, y) \in B_{j,x,y} \times 2^{j,x,y} B_{j,x,y}$, it follows that $(x, y) \in Q_{j,x,y} \times 2^{j,x,y} Q_{j,x,y}$ for some $Q_{j,x,y} \in \mathcal{A}_{\alpha,\text{max}}^{j,x,y}$. Thus, we have

\[
E_f^{(1)}(1, q) \subset \bigcup_{j=0}^{\infty} \bigcup_{\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n} \bigcup_{Q \in \mathcal{A}_{\alpha,\text{max}}^j} (Q \times 2^j Q),
\]

which implies that, for any $x \in \mathbb{R}^n$,

\[
\int_{\mathbb{R}^n} 1_{E_f^{(1)}(1, q)}(x, y) \, dy \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n} \sum_{Q \in \mathcal{A}_{\alpha,\text{max}}^j} 1_Q(x)|2^j Q|.
\]  

(3.15)

Now, we prove (3.11) by considering the following two cases on $q$.

Case 1) $q \in (0, p)$. In this case, by (3.15), the Minkowski inequality on $L^\frac{p}{q}(\mathbb{R}^n)$, (3.14), and the fact that the dyadic cubes in $\mathcal{A}_{\alpha,\text{max}}^j$ are pairwise disjoint, we obtain

\[
I := \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f^{(1)}(1, q)}(x, y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \right\}^{\frac{q}{p}}
\]

\[\geq \sup_{Q \in \mathcal{A}_{\alpha,\text{max}}^j} l(Q)^\frac{q}{p}.
\]
conclude that

This gives the desired estimate (3.11) for any given $q \in (0, p]$.

Case 2) $q \in (p, \infty)$. In this case, recall that, for any $r \in (0, 1]$ and $\{a_j\}_{j \in \mathbb{Z}^+} \subset (0, \infty)$,

$$\left( \sum_{j \in \mathbb{Z}^+} a_j \right)^r \leq \sum_{j \in \mathbb{Z}^+} a_j^r;$$

(3.16)

By this, (3.15), (3.14), and the fact that the dyadic cubes in $\mathcal{R}_{d, \max}^j$ are pairwise disjoint, we conclude that

$$\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f^{(1)}(1, q)}(x, y) \right]^{\frac{p}{q}} \omega(x) dx$$

$$\leq \sum_{j=0}^\infty \sum_{\alpha \in \{0, \frac{1}{2}, \frac{2}{3}\}^n} \omega(Q) |2^j Q|^{\frac{p}{q}}$$

$$\leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^\infty \sum_{\alpha \in \{0, \frac{1}{2}, \frac{2}{3}\}^n} \omega(Q) |2^j Q|^{\frac{p}{q}}$$

$$\leq \sum_{j=0}^\infty \sum_{\alpha \in \{0, \frac{1}{2}, \frac{2}{3}\}^n} \sum_{Q \in \mathcal{R}_{d, \max}^j} 2^{-j(1-\epsilon)p} \int_{Q} |\nabla f(z)|^p \omega(z) dz$$

$$\leq [\omega]_{A_1(\mathbb{R}^n)} 3^n \sum_{j=0}^\infty \sum_{\alpha \in \{0, \frac{1}{2}, \frac{2}{3}\}^n} 2^{-j(1-\epsilon)p} \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) dz.$$
Lemma 3.15 \textit{Let } p \in [1, \infty) \text{ and } q \in (0, \infty) \text{ satisfy } n \left( \frac{1}{p} - \frac{1}{q} \right) < 1. \text{ Let } \omega \in A_1(\mathbb{R}^n). \text{ Then there exist positive constants } C \text{ and } C_{(\omega)} A_1(\mathbb{R}^n) \text{ such that, for any } f \in C_c^\infty(\mathbb{R}^n),

\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1 E_f^{(2)}(1, q)(x, y) \, dy \right]^{\frac{n}{q}} \omega(x) \, dx

\leq C C_{(\omega)} A_1(\mathbb{R}^n) \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,

(3.17)

where } E_f^{(2)}(1, q) \text{ is the same as in (3.10), the positive constant } C \text{ is independent of } \omega, \text{ the positive constant } C_{(\omega)} \text{ increases as } [\omega] A_1(\mathbb{R}^n) \text{ increases, and } C(\cdot) \text{ is continuous on } (0, \infty). \text{ }

Before we prove Lemma 3.15, we need the following lemma, which can be found in [28, p.18].

Lemma 3.16 \textit{Let } p \in [1, \infty) \text{ and } \omega \in A_p(\mathbb{R}^n). \text{ For any } g \in L_p^0(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n, \text{ let }

Rg(x) := \sum_{k=0}^\infty \frac{\mathcal{M}^k g(x)}{2^k \|\mathcal{M}\|^k_{L_p^0(\mathbb{R}^n) \to L_p^0(\mathbb{R}^n)}},

\text{where, for any } k \in \mathbb{N}, \mathcal{M}^k := \mathcal{M} \circ \cdots \circ \mathcal{M} \text{ is the } k \text{ iterations of the Hardy–Littlewood maximal operator and } \mathcal{M}^0 g(x) := |g(x)|. \text{ Then, for any } g \in L_p^0(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n,

\begin{enumerate}
\item \|g(x)\| \leq Rg(x);
\item Rg \in A_1(\mathbb{R}^n) \text{ and } [Rg] A_1(\mathbb{R}^n) \leq 2\|\mathcal{M}\|_{L_p^0(\mathbb{R}^n) \to L_p^0(\mathbb{R}^n)}, \text{ where } \|\mathcal{M}\|_{L_p^0(\mathbb{R}^n) \to L_p^0(\mathbb{R}^n)} \text{ denotes the operator norm of } \mathcal{M} \text{ mapping } L_p^0(\mathbb{R}^n) \text{ to } L_p^0(\mathbb{R}^n);
\item \|Rg\|_{L_p^0(\mathbb{R}^n)} \leq 2\|g\|_{L_p^0(\mathbb{R}^n)}.
\end{enumerate}

The following lemma is just [108, Lemma 2.6].

Lemma 3.17 \textit{Let } X \text{ be a ball Banach function space. Then } X \text{ coincides with its second associate space } X'' \text{. In other words, a function } f \text{ belongs to } X \text{ if and only if it belongs to } X'' \text{ and, in that case, }

\|f\|_X = \|f\|_{X''}.

\textbf{Proof of Lemma 3.15} \textit{Let } \varepsilon \in (0, 1) \text{ be a sufficiently small absolute constant. By an argument similar to that used in the proof of Lemma 3.14, we know that there exist positive constants } C_1 \text{ and } C_2, \text{ depending only on } n, q, \text{ and } \varepsilon, \text{ such that, for any } j \in \mathbb{Z}_+, \alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n, \text{ and } Q \in \mathcal{A}_{\alpha, \max}^j,

C_1 2^{j(1 - \varepsilon)} |2^j Q|^{\frac{1}{p}} < \int_Q |\nabla f(z)| \, dz,

|2^j Q|^{\frac{p}{q}} \leq C_2 [\omega] A_1(\mathbb{R}^n) 2^{-j(1 - \varepsilon)p} \frac{1}{\omega(Q)} \int_Q |\nabla f(z)|^p \omega(z) \, dz,

(3.18)

and

E_f^{(2)}(1, q) \subset \bigcup_{j=0}^\infty \bigcup_{\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n} \bigcup_{Q \in \mathcal{A}_{\alpha, \max}^j} \left(2^j Q \times Q\right),

(3.19)
where $\mathcal{A}_{x,\text{max}}^j$ is the same as in the proof of Lemma 3.14. Using (3.19), we conclude that, for any $x \in \mathbb{R}^n$,

$$
\int_{\mathbb{R}^n} 1_{E_f}^{(2)}(1, q)(x, y) \, dy \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} |Q| 1_{2j/Q}(x).
$$

(3.20)

Now, we prove (3.17) by considering the following two cases on $q$.

Case 1) $q \in [p, \infty)$ and $n\left(\frac{1}{p} - \frac{1}{q} \right) < 1$. In this case, by (3.20), (3.16), (3.18), and Lemma 3.3(i), we find that

$$
\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} 1_{E_f}^{(2)}(1, q)(x, y) \, dy \right|^{\frac{p}{q}} \omega(x) \, dx
\leq \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} |Q| 1_{2j/Q}(x) \left| \int_{\mathbb{R}^n} 1_{2j/Q}(x) \right|^{\frac{p}{q}} \omega(x) \, dx
\leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} |Q|^{\frac{p}{q}} \omega(2^j Q)
= \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} |Q|^{\frac{p}{q}} \omega(2^j Q)
\leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} 2^j Q |Q|^{\frac{p}{q}} 2^j n(1 - \frac{1}{p} - \frac{1}{q}) \omega(Q)
\leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} 2^j p n(1 - \frac{1}{p} - \frac{1}{q}) \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz
\leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, \frac{3}{2} \}^n} \sum_{Q \in \mathcal{A}_{x,\text{max}}^j} 2^j p n(1 - \frac{1}{p} - \frac{1}{q}) \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz
\leq [\omega]_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz,
$$

(3.21)

where, in the fourth step, we took an $\varepsilon \in (0, 1)$ sufficiently small so that $n\left(\frac{1}{p} - \frac{1}{q} \right) < 1 - \varepsilon$. This can be done because $n\left(\frac{1}{p} - \frac{1}{q} \right) < 1 - \varepsilon$. This finishes the proof of (3.17) in this case.

Case 2) $q \in (0, p)$. In this case, let $r := \frac{p}{q}$, $r' := \frac{r}{r-1}$, and $\mu(x) := [\omega(x)]^{1-r'}$ for any $x \in \mathbb{R}^n$. Since $\omega \in A_1(\mathbb{R}^n) \subset A_r(\mathbb{R}^n)$, it follows that $\mu \in A_{r'}(\mathbb{R}^n)$ and

$$
[\omega^{1-r'}]_{A_{r'}(\mathbb{R}^n)} = [\omega]_{A_r(\mathbb{R}^n)} \leq [\omega]_{A_1(\mathbb{R}^n)}
$$

(3.22)

(see, for instance, both (4) and (6) of [43, Proposition 7.1.5]). It is known that $[L_\omega^r(\mathbb{R}^n)]' = L_\omega^{r'}(\mathbb{R}^n)$, where $[L_\omega^r(\mathbb{R}^n)]'$ denotes the associate space of $L_\omega^r(\mathbb{R}^n)$ in Definition 2.11 (see [33, Theorem 2.7.4]). From this, Lemma 3.17 with $X := L_\omega^r(\mathbb{R}^n)$, Definition 2.11 with $X := L_\mu^r(\mathbb{R}^n)$, and Lemma 3.16(i), we deduce that

$$
\left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f}^{(2)}(1, q)(x, y) \, dy \right]^{\frac{r}{r'}} \omega(x) \, dx \right\}^{\frac{1}{r'}}
$$
\[ \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(\cdot,y) \, dy \right\|_{L_{\mu}(\mathbb{R}^n)} = \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(\cdot,y) \, dy \right\|_{L_{\mu}'(\mathbb{R}^n)}' \]
\[ = \sup_{\|g\|_{L_{\mu}'(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(x,y) \, dy \right] g(x) \, dx \]
\[ \leq \sup_{\|g\|_{L_{\mu}'(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(x,y) \, dy \right] Rg(x) \, dx \]
\[ \lesssim \sup_{\|g\|_{L_{\mu}'(\mathbb{R}^n)} = 1} \left[ Rg \right]_{A_1(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |\nabla f(x)|^q Rg(x) \, dx, \quad (3.23) \]
where, in the last step, we used (3.21) with \( p := q \) and \( \omega := Rg \). On the other hand, by the Hölder inequality, both (ii) and (iii) of Lemma 3.16, Lemma 3.9, and (3.22), we find that, for any \( g \in L_{\mu}'(\mathbb{R}^n) \) with \( \|g\|_{L_{\mu}'(\mathbb{R}^n)} = 1 \),
\[ \left[ Rg \right]_{A_1(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |\nabla f(x)|^q Rg(x) \, dx \]
\[ \lesssim \left[ Rg \right]_{A_1(\mathbb{R}^n)}^2 \|g\|_{L_{\mu}'(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{p}} \]
\[ \lesssim \|M\|_{L_{\mu}'(\mathbb{R}^n) \rightarrow L_{\mu}'(\mathbb{R}^n)}^{\frac{2}{pq-1}} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{p}} \]
\[ \lesssim [\mu]_{A_r(\mathbb{R}^n)}^{\frac{2}{pq-1}} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{p}} \]
\[ \leq [\omega]_{A_1(\mathbb{R}^n)}^2 \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{p}}. \quad (3.24) \]
This, combined with (3.23), implies that
\[ \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(x,y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \right\}^{\frac{1}{p}} \]
\[ \lesssim [\omega]_{A_1(\mathbb{R}^n)}^2 \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{p}}. \quad (3.24) \]
This finishes the proof of (3.17) in this case.

Let
\[ C_{(\omega)_{A_1(\mathbb{R}^n)}} := \begin{cases} [\omega]_{A_1(\mathbb{R}^n)}^2 & \text{if } q \in [p, \infty) \text{ and } n \left( \frac{1}{p} - \frac{1}{q} \right) < 1, \\ [\omega]_{A_1(\mathbb{R}^n)}^{2p/q} & \text{if } q \in (0, p). \end{cases} \]
It is easy to show that \( C_{(\omega)_{A_1(\mathbb{R}^n)}} \) increases as \( [\omega]_{A_1(\mathbb{R}^n)} \) increases. From both (3.21) and (3.24), we deduce that
\[ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E_f^{(2)}(1,q)}(x,y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \]
\[ \lesssim C_{(\omega)_{A_1(\mathbb{R}^n)}} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx. \]
This finishes the proof of Lemma 3.15. □

Finally, we show Theorem 3.13.

**Proof of Theorem 3.13** Without loss of generality, by both Corollary 2.18 with $X := L^p_{\omega}(\mathbb{R}^n)$ and a density argument [see, for instance, the proof of (4.22) below], we may assume that $f \in C_0^\infty(\mathbb{R}^n)$ and also that $\lambda = 1$ because, otherwise, we can replace $f$ by $f/\lambda$ for any $\lambda \in (0, \infty)$. Let $A$ be the same as in Lemma 3.12. Since $|A| = 0$, it follows that, to show the present theorem, it suffices to prove (3.8) with $E_f(1, q)$ replaced by $E_f(1, q) \setminus ((A \times \mathbb{R}^n) \cup (\mathbb{R}^n \times A))$. On the other hand, it is easy to show that

$$E_f(1, q) \setminus ((A \times \mathbb{R}^n) \cup (\mathbb{R}^n \times A)) \subset E_f^{(1)}(1, q) \cup E_f^{(2)}(1, q),$$

where $E_f^{(1)}(1, q)$ and $E_f^{(2)}(1, q)$ are the same, respectively, as in (3.9) and (3.10). Thus, it suffices to prove the corresponding upper estimates for both the sets $E_f^{(1)}(1, q)$ and $E_f^{(2)}(1, q)$, which are done, respectively, in Lemmas 3.14 and 3.15. This finishes the proof of Theorem 3.13. □

### 3.2 Proof of Theorem 3.5: lower estimate

In this subsection, we give a generalization of the lower estimate of Theorem 3.5 on ball quasi-Banach function spaces, which plays an essential role in the proof of Theorem 4.5 below. In what follows, we use the symbol $C^2(\mathbb{R}^n)$ to denote the set of all twice continuously differentiable functions on $\mathbb{R}^n$.

**Theorem 3.18** Let $X$ be a ball quasi-Banach function space and $q \in (0, \infty)$. Then, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$,

$$\liminf_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (\cdot, y) \in E_f(\lambda, q)\}}(y) \, dy \right\|_{X^{\frac{1}{q}}} \geq \frac{K(q, n)}{n} \| \nabla f \|_X^q,$$  \hspace{1cm} (3.25)

where $E_f(\lambda, q)$ for any $\lambda \in (0, \infty)$ is the same as in (3.2), and $K(q, n)$ is the same as in (3.3). Moreover, if $X$ is a ball Banach function space, then, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$,

$$\lim_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (\cdot, y) \in E_f(\lambda, q)\}}(y) \, dy \right\|_{X^{\frac{1}{q}}} = \frac{K(q, n)}{n} \| \nabla f \|_X^q.$$  \hspace{1cm} (3.26)

**Proof** Let $q \in (0, \infty)$, $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$, and $M \in (0, \infty)$ be such that $\text{supp}(|\nabla f|) \subset B(0, M)$. Let $K := B(0, M + 1)$. In order to show (3.25), we first prove (3.25). For any $\lambda \in (0, \infty)$, $x \in \mathbb{R}^n$, and $\xi \in \mathbb{S}^{n-1}$, let

$$F_f(x, \xi, \lambda, q) := \left\{ t \in (0, \infty) : \frac{|f(x + t\xi) - f(x)|^q}{t^q} > \lambda^q t^n \right\}.$$

Then, by the Fubini Theorem, we have, for any $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (x, y) \in E_f(\lambda, q)\}}(y) \, dy = \int_{\mathbb{S}^{n-1}} \int_0^\infty F_f(x, \xi, \lambda, q)(t) t^{n-1} \, dt \, d\sigma(\xi).$$  \hspace{1cm} (3.27)
Since $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$, it follows that there exist constants $L_1 \in (\|\nabla f\|_{L^\infty(\mathbb{R}^n)}, \infty)$ and $L_2 \in (0, \infty)$ such that, for any $t \in (0, \infty)$, $x \in \mathbb{R}^n$, and $\xi \in \mathbb{S}^{n-1}$,

$$|f(x + t\xi) - f(x)| \leq L_1 t$$  \hspace{1cm} (3.28)

and

$$|f(x + t\xi) - f(x) - t\nabla f(x) \cdot \xi| \leq L_2 t^2.$$  \hspace{1cm} (3.29)

Let $\lambda \in (L_1, \infty)$. By (3.28), we conclude that, for any $t \in (0, \infty)$, $x \in \mathbb{R}^n$, and $\xi \in \mathbb{S}^{n-1}$,

$$\frac{|f(x + t\xi) - f(x)|^q}{t^q} \leq L_1^q$$  \hspace{1cm} (3.30)

and hence

$$F_f(x, \xi, \lambda, q) = \left\{ t \in (0, (L_1/\lambda)^{\frac{q}{n}}) : \frac{|f(x + t\xi) - f(x)|^q}{t^q} > \lambda^q t^n \right\}.$$  \hspace{1cm} (3.31)

From (3.29), we deduce that, for any $t \in (0, \infty)$, $x \in \mathbb{R}^n$, and $\xi \in \mathbb{S}^{n-1}$,

$$|\xi \cdot \nabla f(x)| - tL_2 \leq \frac{|f(x + t\xi) - f(x)|}{t} \leq |\xi \cdot \nabla f(x)| + tL_2,$$

which, combined with both (3.30) and (3.31), further implies that, for any $t \in (0, (L_1/\lambda)^{\frac{q}{n}})$,

$$A_f^-(x, \xi, \lambda, q) \leq \frac{|f(x + t\xi) - f(x)|^q}{t^q} \leq A_f^+(x, \xi, \lambda, q),$$

where

$$A_f^-(x, \xi, \lambda, q) := \left\lceil \max \left\{ |\xi \cdot \nabla f(x)| - (L_1/\lambda)^{\frac{q}{n}}L_2, 0 \right\} \right\rceil^q$$

and

$$A_f^+(x, \xi, \lambda, q) := \min \left\lceil \left\lceil |\xi \cdot \nabla f(x)| + (L_1/\lambda)^{\frac{q}{n}}L_2 \right\rceil^q, L_1^q \right\rceil.$$  \hspace{1cm} (3.32)

Using this, we conclude that, for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{S}^{n-1}$,

$$F_f^-(x, \xi, \lambda, q) \subset F_f(x, \xi, \lambda, q) \subset F_f^+(x, \xi, \lambda, q),$$

where

$$F_f^\pm(x, \xi, \lambda, q) := \{ t \in (0, \infty) : A_f^\pm(x, \xi, \lambda, q) > \lambda^q t^n \}. $$

We now show that, for any $\lambda \in (L_1, \infty)$, $\xi \in \mathbb{S}^{n-1}$, and $x \in K^C$,

$$F_f(x, \xi, \lambda, q) = \emptyset,$$

and hence, for any $x \in \mathbb{R}^n$,

$$\int_{\mathbb{S}^{n-1}} \int_0^\infty 1_{F_f^-(x, \xi, \lambda, q)}(t) t^{n-1} dt d\sigma(\xi) \leq \int_{\mathbb{S}^{n-1}} \int_0^\infty 1_{F_f^+(x, \xi, \lambda, q)}(t) t^{n-1} dt d\sigma(\xi) 1_K(x).$$  \hspace{1cm} (3.33)

Indeed, by (3.31), for any $\lambda \in (L_1, \infty)$, we have $F_f(x, \xi, \lambda, q) \subset (0, 1)$. Thus, for any $\lambda \in (L_1, \infty)$, $x \in K^C$, $\xi \in \mathbb{S}^{n-1}$, and $t \in F_f(x, \xi, \lambda, q) \subset (0, 1)$, we obtain $x + t\xi \in \mathbb{R}^n$.\[\square\]
which completes the proof of (3.33). By (3.27), (3.32), and Definition 2.1(ii) together with the assumption that $X$ is a ball quasi-Banach function space, we have

$$
\lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (\cdot, y) \in E_f(\lambda, q)\}}(y) \, dy \right\|_{X^q}^{\frac{1}{q}} \\
\geq \lambda^q \left\| \int_{S^{n-1}} \int_0^\infty \frac{1}{\lambda} F_f(\cdot, \lambda, q)(t) \, dt \, d\sigma(\xi) \right\|_{X^q}^{\frac{1}{q}} \\
= \frac{1}{n} \left\| \int_{S^{n-1}} A_f(\cdot, \lambda, q) \, d\sigma(\xi) \right\|_{X^q}^{\frac{1}{q}}. \tag{3.34}
$$

Notice that the function $\lambda \mapsto A_f(x, \xi, \lambda)$ is increasing on $(0, \infty)$ and

$$
\lim_{\lambda \to \infty} A_f(x, \xi, \lambda) = |\xi \cdot \nabla f(x)|^q,
$$

and that $K(q, n)$ is independent of $e$. Using this and Definition 2.1(iii), and letting $\lambda \to \infty$ in (3.34), we then conclude

$$
\lim_{\lambda \to \infty} \inf \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (\cdot, y) \in E_f(\lambda, q)\}}(y) \, dy \right\|_{X^q}^{\frac{1}{q}} \\
\geq \frac{1}{n} \left\| \int_{S^{n-1}} |\xi \cdot \nabla f|^q \, d\sigma(\xi) \right\|_{X^q}^{\frac{1}{q}} \\
= \frac{1}{n} \left\| \int_{S^{n-1}} |\xi \cdot \nabla f|^q \, d\sigma(\xi) |\nabla f|^q \right\|_{X^q}^{\frac{1}{q}} = \frac{K(q, n)}{n} \| |\nabla f|\|^q_X. \tag{3.35}
$$

This finishes the proof of (3.25).

Next, we prove that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_2(\mathbb{R}^n)$,

$$
\lim_{\lambda \to \infty} \sup \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n: (\cdot, y) \in E_f(\lambda, q)\}}(y) \, dy \right\|_{X^q}^{\frac{1}{q}} \leq \frac{K(q, n)}{n} \| |\nabla f|\|^q_X. \tag{3.36}
$$

To this end, recall that, for any $\theta \in (0, 1)$, there exists a positive constant $C(\theta)$ such that, for any $a, b \in (0, \infty)$,

$$
(a + b)^q \leq (1 + \theta)a^q + C(\theta)b^q \tag{3.37}
$$

(see, for instance, [11, p.699]). Obviously, we have, for any $x \in \mathbb{R}^n$ and $\xi \in S^{n-1}$,

$$
|\xi \cdot \nabla f(x)| \leq \| |\nabla f|\|_{L^\infty(\mathbb{R}^n)} < L_1.
$$

From this and (3.37), we deduce that, for any $\lambda \in (L_1, \infty)$ sufficiently large,

$$
A_f(\xi, \lambda, q) = |||\xi \cdot \nabla f(x)| + (L_1/\lambda)^q L_2 \|^q \\
\leq (1 + \theta)|||\xi \cdot \nabla f(x)|^q + C(\theta)(L_1/\lambda)^q L_2^q, \\
$$

where $C(\theta)$ is a positive constant depending only on $\theta$. This, together with (3.27), (3.32), (3.33), and (3.37) with $q$ replaced by $\frac{1}{q}$, implies that, for any $\lambda \in (L_1, \infty)$ sufficiently large,
Applying Corollary 2.18 with $\eta \equiv 0$, we obtain the following corollary; we omit the details here.

**Corollary 3.19** Let $p \in [1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then, for any $f \in \dot{W}^{1,p}_\omega(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ with $|\nabla f_k| \in C_\omega(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ such that

$$
\lim_{k \to \infty} \| f - f_k \|_{\dot{W}^{1,p}_\omega(\mathbb{R}^n)} = \lim_{k \to \infty} \| \nabla f - \nabla f_k \|_{L^p_\omega(\mathbb{R}^n)} = 0.
$$

**Remark 3.20** We should point that $C_\omega^\infty(\mathbb{R}^n)$ may not be dense in $\dot{W}^{1,p}_\omega(\mathbb{R}^n)$. Indeed, when $n = 1$ and $\omega \equiv 1$, Hajłasz and Kalamajska in [45, Theorem 4] proved that $C_\omega^\infty(\mathbb{R})$ is not dense in $\dot{W}^{1,p}(\mathbb{R})$.
As a consequence of Theorem 3.18 with $X$ replaced by $L^p_{\omega}(\mathbb{R}^n)$, we have the following conclusion.

**Corollary 3.21** Let $p \in [1, \infty)$, $q \in (0, \infty)$ with $n\left(\frac{1}{p} - \frac{1}{q}\right) < 1$, and $\omega \in A_1(\mathbb{R}^n)$. Then, for any $f \in \dot{W}^{1,p}_{\omega}(\mathbb{R}^n)$,

\[
\lim_{\lambda \to \infty} \lambda^p \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx = \left[ \frac{K(q, n)}{n} \right]^\frac{p}{q} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx.
\]

**Proof** Using Theorem 3.18 with $X$ replaced by $L^p_{\omega}(\mathbb{R}^n)$, we find that (3.38) holds true for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$. Next, we show that (3.38) holds true for any $f \in \dot{W}^{1,p}_{\omega}(\mathbb{R}^n)$.

Let $f \in \dot{W}^{1,p}_{\omega}(\mathbb{R}^n)$ and $C(q, n) := \left[ \frac{K(q, n)}{n} \right]^\frac{1}{q}$. From Corollary 3.19, we infer that there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ with $|\nabla f_k| \in C_c(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ such that

\[
\lim_{k \to \infty} \| |\nabla f - \nabla f_k| \|_{L^p_{\omega}(\mathbb{R}^n)} = 0.
\]

By the Hölder inequality, (3.37) with $q$ replaced by $\frac{1}{q}$, Theorem 3.18 with $f$ replaced by $f_k$, and Theorem 3.13 with $f$ replaced by $f - f_k$, we obtain

\[
\| |\nabla f| \|_{L^p_{\omega}(\mathbb{R}^n)} - \| |\nabla f - \nabla f_k| \|_{L^p_{\omega}(\mathbb{R}^n)} \leq\| |\nabla f - \nabla f_k| \|_{L^p_{\omega}(\mathbb{R}^n)} \leq\]

\[
\leq C^{-1}_{(q, n)} \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f_k (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx \right\} \leq C^{-1}_{(q, n)} \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f - f_k (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx \right\}
\]

\[
\leq C^{-1}_{(q, n)} C(\theta) \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f - f_k (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx \right\} \leq \left( C_1 C_{(\omega)_{A_1(\mathbb{R}^n)}} \right)^{1/p} C^{-1}_{(q, n)} C(\theta) \delta^{-1} \| |\nabla f_k| - |\nabla f| \|_{L^p_{\omega}(\mathbb{R}^n)}
\]

\[
+ C^{-1}_{(q, n)} \left( 1 + \theta \right) \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx \right\} \leq \left( C_1 C_{(\omega)_{A_1(\mathbb{R}^n)}} \right)^{1/p} C^{-1}_{(q, n)} \frac{1}{1 - \delta} \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx \right\},
\]

where $C_1$ is the same as in Theorem 3.13 and $C(\theta)$ a positive constant depending on $\theta$. Let $k \to \infty, \theta \to 0$, and $\delta \to 0$. Then we obtain

\[
\liminf_{\lambda \to \infty} \lambda^p \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E f (\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx
\]
\[
\geq \left[ \frac{K(q, n)}{n} \right]^\frac{p}{q} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx. \tag{3.39}
\]

Similarly, we also have
\[
\limsup_{\lambda \to \infty} \lambda^p \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right]^\frac{p}{q} \omega(x) \, dx
\leq \left[ \frac{K(q, n)}{n} \right]^\frac{p}{q} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]
which, together with (3.39), further implies that (3.38) holds true. This then finishes the proof of Corollary 3.21.

As a consequence of Corollary 3.21, we obtain the lower estimate of Theorem 3.5. Now, we can complete the proof of Theorem 3.5.

**Proof of Theorem 3.5** As a consequence of Theorem 3.13 and Corollary 3.21, we immediately obtain the desired conclusions of the present theorem, which then completes the proof of Theorem 3.5.

\[\square\]

**3.3 Proof of Theorem 3.7**

**Proof of Theorem 3.7** Assume that there exists a positive constant \(C_1\) such that, for any \(f \in C^1(\mathbb{R})\) satisfying that \(f'\) has compact support,
\[
\sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^2} 1_{E_f(\lambda, p)}(x, y) \omega(x) \, dx \, dy \leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx, \tag{3.40}
\]
where, for any \(\lambda \in (0, \infty), p \in [1, \infty), \) and any measurable function \(f, E_f(\lambda, p)\) is the same as in (3.2) and the constant \(C_1\) is independent of \(f\). We now show that \(\omega \in A_p(\mathbb{R})\) with \(p \in [1, \infty)\). Observe that the inequality (3.40) is both dilation and translation invariance; that is, for any \(\delta \in (0, \infty)\) and \(x_0 \in \mathbb{R}, \) both the weights \(\omega(\delta x)\) and \(\omega(x - x_0)\) satisfy (3.40) with the same constant \(C_1\). This, combined with Lemma 3.3(ii), further implies that, to prove \(\omega \in A_p(\mathbb{R}), \) it suffices to show that there exists a positive constant \(C, \) depending only on \(C_1, \) such that, for any nonnegative function \(g \in L^1_{\text{loc}}(\mathbb{R}), \)
\[
\left[ \int_{-1}^1 g(x) \, dx \right]^p \leq \frac{C}{\omega([-1, 1])} \int_{-1}^1 [g(x)]^p \omega(x) \, dx. \tag{3.41}
\]

To show (3.41), we first prove that, for any \(0 \leq g \in C^\infty(\mathbb{R}), \)
\[
\left[ \int_{-1}^1 g(x) \, dx \right]^p \leq \frac{C_1 6^p + 1}{4\omega(I_0)} \int_{-4}^4 [g(x)]^p \omega(x) \, dx, \tag{3.42}
\]
where \(I_0 := [-3, -1] \cup [1, 3]. \) Let \(\eta \in C^\infty(\mathbb{R})\) be such that \(\eta(x) \in [0, 1]\) for any \(x \in \mathbb{R}, \) \(\eta(x) = 1\) for any \(x \in [-3, 3], \) and \(\eta(x) = 0\) for any \(x \in \mathbb{R}\) with \(|x| \in [4, \infty). \) For any \(x \in \mathbb{R}, \) let
\[
f(x) := \int_{-\infty}^x g(t) \eta(t) \, dt.
\]
Clearly, \( f \in C^\infty(\mathbb{R}) \) and \( \text{supp}(f') \subset [-4, 4] \). Let \( \lambda := 6^{-1-\frac{1}{p}} \int_{-1}^{1} g(t) \, dt \). Then, for any \( x \in [-3, -1] \) and \( y \in [1, 3] \), we have
\[
|f(y) - f(x)| = \int_{x}^{y} g(t) \, dt \geq \int_{-1}^{1} g(t) \, dt = 6^{1+\frac{1}{p}} \lambda \geq \lambda|x - y|^{\frac{1}{p}+1}.
\]
This, together with the symmetry further implies that
\[
([1, 3] \times [-3, -1]) \cup ([-3, -1] \times [1, 3]) \subset E_f(\lambda, p).
\]
Thus, using this and (3.40), we conclude that
\[
4\lambda^p \int_{I_0} \omega(x) \, dx \leq \lambda^p \int_{\mathbb{R}^2} \omega(x)1_{E_f(\lambda, p)}(x, y) \, dx \, dy
\]
\[
\leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx \leq C_1 \int_{-4}^{4} [g(x)]^p \omega(x) \, dx.
\]
This proves (3.42).

Second, we show that, for any nonnegative locally integrable function \( g \),
\[
\left[ \int_{-1}^{1} g(x) \, dx \right]^p \leq C_1 \frac{6^{p+1}}{\omega(I_0)} \int_{-1}^{1} [g(x)]^p \omega(x) \, dx.
\]
(3.43)

Without loss of generality, we may assume that \( g \) is bounded because, otherwise, one may replace \( g \) by \( \min\{g, n\} \) for any \( n \in \mathbb{N} \), and then apply the monotone convergence theorem to obtain the desired inequality. Let \( \varphi \in C^\infty(\mathbb{R}) \) be such that \( \varphi(t) \geq 0 \) for any \( t \in \mathbb{R} \), \( \varphi(t) = 0 \) for any \( t \in \mathbb{R} \) with \( |t| \geq 1 \), and \( \int_{\mathbb{R}} \varphi(t) \, dt = 1 \). For any \( \varepsilon \in (0, \infty) \) and \( t \in \mathbb{R} \), let \( \varphi_\varepsilon(t) := \varepsilon^{-1} \varphi(t/\varepsilon) \) and
\[
g_\varepsilon(t) := (g1_{[-1,1]}) \ast \varphi_\varepsilon(t) = \int_{-1}^{1} g(u) \varphi_\varepsilon(t-u) \, du.
\]
Then \( 0 \leq g_\varepsilon \in C_c^\infty(\mathbb{R}) \) and, using (3.42), we obtain
\[
\left[ \int_{-1}^{1} g_\varepsilon(x) \, dx \right]^p \leq \frac{1}{\omega(I_0)} \int_{-4}^{4} [g_\varepsilon(x)]^p \omega(x) \, dx.
\]
(3.44)

Since, for almost every \( t \in \mathbb{R} \),
\[
\lim_{\varepsilon \to 0} g_\varepsilon(t) = g(t)1_{[-1,1]}(t)
\]
and
\[
\sup_{\varepsilon \in (0, \infty)} \|g_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \sup_{\varepsilon \in (0, \infty)} \|\varphi_\varepsilon\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} < \infty,
\]
from these, (3.44), and the Lebesgue dominated convergence theorem, we deduce (3.43).

Finally, we prove (3.41). Let \( g := 1 \) and \( I := [-1, 1] \). By (3.43), we find that \( \frac{\omega(I)}{\omega(I_0)} \geq c_1 \), where \( c_1 := \frac{2}{C_13^{p+1}} \). Thus,
\[
\omega(2I) \leq \omega(I_0) + \omega(I) \leq (1 + 1/c_1)\omega(I).
\]
Since (3.40) is both dilation and translation invariance for the weight \( \omega \), it follows that the inequality \( \omega(2I) \leq (1 + 1/c_1)\omega(I) \) holds true for any compact interval \( I \subset \mathbb{R} \). By this, we find that
\[
\omega([-1, 1]) \leq \omega([-4, 0]) + \omega([0, 4])
\]
This, combined with (3.43), further implies that (3.41) holds true. This then finishes the proof of Theorem 3.7. \[ \square \]

4 Estimates in ball Banach function spaces

In this section, we establish the Brezis–Van Schaftingen–Yung formulae in ball Banach function spaces (see Theorem 4.5 below). As applications, we also obtain some fractional Sobolev and Gagliardo–Nirenberg type inequalities in ball Banach function spaces.

We begin with introducing the following concepts of homogeneous (weak) $X$-based Triebel–Lizorkin-type spaces.

**Definition 4.1** Let $q \in (0, \infty)$, $s \in [0, \infty)$, and $X$ be a ball quasi-Banach function space.

(i) The homogeneous $X$-based Triebel–Lizorkin-type space $\dot{F}^{s}_{X,q}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{F}^{s}_{X,q}(\mathbb{R}^n)} := \left\| \left[ \int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^q}{|\cdot - y|^{n+qs}} \, dy \right]^{\frac{1}{q}} \right\|_X < \infty.$$  

(ii) The homogeneous weak $X$-based Triebel–Lizorkin-type space $\dot{W}^{s}_{X,q}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{W}^{s}_{X,q}(\mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \lambda \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda q)}(\cdot, y) \, dy \right]^{\frac{1}{q}} \right\|_X < \infty,$$

where, for any $\lambda \in (0, \infty)$,

$$E_f(\lambda, q, s) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{s}{q}} \right\}. \quad (4.1)$$

**Remark 4.2** The space $\dot{F}^{s}_{X,q}(\mathbb{R}^n)$ is different from the Triebel–Lizorkin type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ introduced in [106], although both include the classical Triebel–Lizorkin space as their special cases. Moreover, the space $\dot{F}^{s}_{X,q}(\mathbb{R}^n)$ is defined by differences and the Triebel–Lizorkin type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ in [106] is defined by the Littlewood–Paley $g$ function. Both spaces cannot cover each other.

Similarly to Brezis et al. [17, (1.2)] (see also [8, 11]), we have the following conclusions on the “drawback” of $\dot{F}^{s}_{X,q}(\mathbb{R}^n)$.

**Theorem 4.3** Let $X$ be a ball quasi-Banach function space and $s, q \in (0, \infty)$. Assume that $X^{1/q}$ is a ball Banach function space. If $s \min\{1, q\} \in [1, \infty)$ and $f \in \dot{F}^{s}_{X,q}(\mathbb{R}^n)$, then $f$ is a constant function.

**Proof** Let $s, q \in (0, \infty)$ and $f \in \dot{F}^{s}_{X,q}(\mathbb{R}^n)$. By Lemma 3.17 and Definition 2.11, we have

$$\infty > \left\| \int_{\mathbb{R}^n} \frac{|f(\cdot) - f(y)|^q}{|\cdot - y|^{n+qs}} \, dy \right\|_{X^{1/q}} \leq (1 + 1/c_1) \omega([-3, -1]) + \omega([1, 3]) = (1 + 1/c_1) \omega(I_0).$$
By this, we obtain, for any \( j \) and any \( s \),

\[
\sup_{\|g\|_{L^q(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \left| \frac{f(x) - f(y)}{|x-y|^{n+sq}} - g(x) \right| dy dx = \sup_{\|g\|_{L^q(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \left| \frac{f(x) - f(x-h)}{|h|^{n+sq}} - g(x) \right| dh dx. \tag{4.2}
\]

For any \( N \in (0, \infty) \), let \( g := \mathbf{1}_{B(0,N)} / \| \mathbf{1}_{B(0,N)} \|_{L^q(\mathbb{R}^n)} \). Using (4.2), we conclude that, for any \( N \in (0, \infty) \),

\[
\int_{|x| < N} \int_{|h| < r} \frac{|f(x) - f(x-h)|^q}{|h|^{n+sq}} dh dx < \infty.
\]

From this, we deduce that, for any \( N \in (0, \infty) \) and \( r \in (0, N) \),

\[
\int_{|x| < N} \int_{|h| < r} \frac{|f(x) - f(x-h)|^q}{|h|^{n+sq}} dh dx \\
> \sum_{j=0}^{\infty} 2^{jn+sq} r^{-n+sq} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \frac{|f(x) - f(x-h)|^q}{|h|^{n+sq}} dh dx.
\tag{4.3}
\]

We prove the present theorem by considering the following two cases on both \( s \) and \( q \).

Case 1) \( q \in [1, \infty) \) and \( s \in [1, \infty) \). Recall the discrete Hölder inequality that, for any \( m \in \mathbb{Z}_+ \) and \( \{a_j\}_{j=0}^m \subset [0, \infty) \),

\[
\left( \sum_{j=0}^m a_j \right)^q \leq (m+1)^{q-1} \left( \sum_{j=0}^m a_j^q \right). \tag{4.4}
\]

By this, we obtain, for any \( j \in \mathbb{Z}_+ \),

\[
\int_{2^{-j}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x-h)|^q dx dh \\
= 2^{jn} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \int_{|x| < N-r} |f(x) - f(x-2^j h)|^q dx dh \\
= 2^{jn} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \int_{|x| < N-r} |\sum_{i=0}^{2^j-1} [f(x - ih) - f(x - (i+1)h)]|^q dx dh \\
\leq 2^{jn+jq-j} \sum_{i=0}^{2^j-1} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \int_{|x| < N-r} |f(x - ih) - f(x - (i+1)h)|^q dx dh \\
\leq 2^{jn+jq-j} \sum_{i=0}^{2^j-1} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \int_{|x| < N-r+2^{-j}r} |f(x) - f(x-h)|^q dx dh \\
\lesssim 2^{jn+jq} \int_{2^{-(j+1)r} \leq |h| < 2^{-j}r} \int_{|x| < N} |f(x) - f(x-h)|^q dx dh, \tag{4.5}
\]
which, combined with (4.3), further implies that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
\sum_{j=0}^{\infty} 2^{j(s-1)q} \int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x-h)|^q \, dx \, dh < \infty.
\]
From this and $(s-1)q \in [0, \infty)$, we deduce that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
\int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x-h)|^q \, dx \, dh = 0.
\]
Using this and letting $N \to \infty$, we then obtain, for any $r \in (0, \infty)$,
\[
\int_{2^{-1}r \leq |h| < r} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0.
\]
By this, we further conclude that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0,
\]
which implies that $f$ is a constant function on $\mathbb{R}^n$. This finishes the proof of the present theorem in this case.

Case 2) $q \in (0, 1)$ and $sq \in [1, \infty)$. By an argument similar to that used in the proof of (4.5) with (4.4) replaced by (3.16), we find that, for any $j \in \mathbb{Z}_+$,
\[
\int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x-h)|^q \, dx \, dh \\
\leq 2^{jn+j} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N} |f(x) - f(x-h)|^q \, dx \, dh,
\]
which, combined with (4.3), implies that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
\sum_{j=0}^{\infty} 2^{j(sq-1)} \int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x-h)|^q \, dx \, dh < \infty.
\]
From this and $sq - 1 \in [0, \infty)$, we deduce that, for any $r \in (0, \infty)$,
\[
\int_{2^{-1}r \leq |h| < r} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0.
\]
By this and the arbitrariness of $r$, we conclude that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0,
\]
which further implies that $f$ is a constant function on $\mathbb{R}^n$. This then finishes the proof of Theorem 4.3.

\[\square\]

**Remark 4.4** In Theorem 4.3, if $X := L^p(\mathbb{R}^n)$, $0 < q \leq p < \infty$, and $s \min\{1, q\} \in [1, \infty)$, then the conclusions of Theorem 4.3 hold true with $X$ replaced by $L^p(\mathbb{R}^n)$, which, when $p = q \in [1, \infty)$ and $s = 1$, coincide with [17, (1.2)] (see also [8, 11]).

One of the main targets in this section is to prove the equivalence (1.8) in a ball Banach function space $X$ under some mild assumptions on both $X$ and $p$. Theorem 4.3 justifies the use of the semi-norm $\|f\|_{W^{1,q}_X(\mathbb{R}^n)}$ instead of $\|f\|_{\bar{W}^{1,q}_X(\mathbb{R}^n)}$ in the equivalence (1.8) as follows.
Theorem 4.5  Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). Assume that \( X \) is a ball Banach function space, \( X^{1/p} \) a ball Banach function space, and \( \mathcal{M} \) in (1.6) bounded on its associate space \((X^{1/p})'\).

(i) Then there exist positive constants \( C_2 \) and \( C(\|\mathcal{M}\|_{(X^{1/p})' \rightarrow (X^{1/p})'}) \) such that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
\left[ \frac{K(q, n)}{n} \right]^\frac{1}{q} \| |\nabla f| \|_X \leq \| f \|_{\mathcal{W}^{1,q}_{X,q}(\mathbb{R}^n)} \leq C_2 C(\|\mathcal{M}\|_{(X^{1/p})' \rightarrow (X^{1/p})'}) \| |\nabla f| \|_X, \tag{4.6}
\]

where \( K(q, n) \) is the same as in (3.3), the positive constant \( C_2 \) depends only on \( p, q, \) and \( n \), and the positive constant \( C(\|\mathcal{M}\|_{(X^{1/p})' \rightarrow (X^{1/p})'}) \) depends only on \( \|\mathcal{M}\|_{(X^{1/p})' \rightarrow (X^{1/p})'} \), \( p \), and \( q \), increases as \( \|\mathcal{M}\|_{(X^{1/p})' \rightarrow (X^{1/p})'} \) increases, and \( C(\cdot) \) is continuous on \((0, \infty)\). Moreover, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
\lim_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X^{1/q}} = \frac{K(q, n)}{n} \| |\nabla f| \|_X, \tag{4.7}
\]

where \( E_f(\lambda, q) \) for any \( \lambda \in (0, \infty) \) is the same as in (3.2).

(ii) Assume that \( X \) has an absolutely continuous norm. Then both (4.6) and (4.7) hold true for any \( f \in \mathcal{W}^{1,1}(\mathbb{R}^n) \).

Recall that the space \( \mathcal{M}(\mathbb{R}^n) \) is the set of all measurable functions on \( \mathbb{R}^n \). To prove Theorem 4.5, we need the following conclusion whose proof is a slight modification of [28, p. 18] via replacing \( L^0_p(\mathbb{R}^n) \) in [28, p. 18] by \( X \).

**Lemma 4.6** Let \( X \subset \mathcal{M}(\mathbb{R}^n) \) be a linear normed space, equipped with a norm \( \| \cdot \|_X \) which makes sense for all functions in \( \mathcal{M}(\mathbb{R}^n) \). Assume that the Hardy–Littlewood maximal operator \( \mathcal{M} \) is bounded on \( X \). For any \( g \in X \) and \( x \in \mathbb{R}^n \), let

\[
R_X g(x) := \sum_{k=0}^{\infty} \frac{\mathcal{M}^k g(x)}{2^k \|\mathcal{M}\|^k_{X \rightarrow X}},
\]

where, for any \( k \in \mathbb{Z}_+ \), \( \mathcal{M}^k \) is the same as in Lemma 3.16. Then, for any \( g \in X \) and \( x \in \mathbb{R}^n \),

(i) \( |g(x)| \leq R_X g(x) \);

(ii) \( R_X g \in A_1(\mathbb{R}^n) \) and \([R_X g]_{A_1(\mathbb{R}^n)} \leq 2 \|\mathcal{M}\|_{X \rightarrow X} \), where \( \|\mathcal{M}\|_{X \rightarrow X} \) denotes the operator norm of \( \mathcal{M} \) mapping \( X \) to \( X \);

(iii) \( \|R_X g\|_X \leq 2\|g\|_X \).

**Proof** Let \( g \in X \). It is easy to show that (i) holds true.

Since \( \mathcal{M} \) is sub-linear, it follows that, for any \( x \in \mathbb{R}^n \),

\[
\mathcal{M}(R_X g)(x) \leq \sum_{k=0}^{\infty} \frac{\mathcal{M}^{k+1} g(x)}{2^k \|\mathcal{M}\|^k_{X \rightarrow X}} \leq 2 \|\mathcal{M}\|_{X \rightarrow X} R_X g(x).
\]

Thus, \( R_X g \in A_1(\mathbb{R}^n) \) and \([R_X g]_{A_1(\mathbb{R}^n)} \leq 2 \|\mathcal{M}\|_{X \rightarrow X} \). This proves (ii).

Finally, by the triangle inequality of \( \| \cdot \|_X \) and the assumption that the Hardy–Littlewood maximal operator \( \mathcal{M} \) is bounded on \( X \), we have

\[
\|R_X g\|_X \leq \sum_{k=0}^{\infty} \frac{\|\mathcal{M}^k g\|_X}{2^k \|\mathcal{M}\|^k_{X \rightarrow X}} = 2\|g\|_X.
\]

\( \square \) Springer
This finishes the proof of (ii) and hence Lemma 4.6.

As a consequence of Lemma 4.6, we have the following conclusion.

**Lemma 4.7** Let $p \in [1, \infty)$, $X$ be a ball quasi-Banach function space, and $Y \subset \mathcal{M}(\mathbb{R}^n)$ be a linear normed space, equipped with a norm $\| \cdot \|_Y$ which makes sense for all functions in $\mathcal{M}(\mathbb{R}^n)$. Assume that, for any $f \in X^{1/p}$,

$$\| f \|_{X^{1/p}} = \sup_{\| h \|_Y = 1} \left| \int_{\mathbb{R}^n} f(x) h(x) \, dx \right|$$

(4.8)

and that the Hardy–Littlewood maximal operator $M$ is bounded on $Y$. Then, for any $f \in X$,

$$\| f \|_X \leq \sup_{\| g \|_Y = 1} \left[ \int_{\mathbb{R}^n} |f(x)|^p R_Y g(x) \, dx \right]^{1/p} \leq 2^{1/p} \| f \|_X.$$  

(4.9)

**Proof** By Lemma 4.6, we find that, for any $f \in X$,

$$\| f \|^p_X = \| f^p \|_{X^{1/p}} = \sup_{\| g \|_Y = 1} \left[ \int_{\mathbb{R}^n} [f(x)]^p g(x) \, dx \right]$$

$$\leq \sup_{\| g \|_Y = 1} \int_{\mathbb{R}^n} |f(x)|^p R_Y g(x) \, dx$$

$$\leq \sup_{\| g \|_Y = 1} \| f^p \|_{X^{1/p}} \| R_Y g \|_Y \leq 2 \| f \|^p_X.$$

This finishes the proof of Lemma 4.7. □

**Remark 4.8** Let $p \in [1, \infty)$ and $X$ be a ball Banach function space. Assume that $X^{1/p}$ is a ball Banach function space and that the Hardy–Littlewood maximal operator $M$ is bounded on $(X^{1/p})'$, where $(X^{1/p})'$ denotes the associate space of $X^{1/p}$. By Lemma 3.17, we find that (4.8) holds true with $Y := (X^{1/p})'$. Thus, by Lemma 4.7, we conclude that (4.9) holds true with $Y := (X^{1/p})'$.

**Proof of Theorem 4.5** We first prove that, for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_X^{1/q} \leq C_2 C(\| M \|_{(X^{1/p})' \to (X^{1/p})'}) \| \nabla f \|_X,$$

(4.10)

where the positive constant $C_2$ depends only on $p, q$, and $n$, the positive constant $C(\| M \|_{(X^{1/p})' \to (X^{1/p})'})$ depending only on $\| M \|_{(X^{1/p})' \to (X^{1/p})'}$, $p$, and $q$, increases as $\| M \|_{(X^{1/p})' \to (X^{1/p})'}$ increases, and $C(\cdot)$ is continuous on $(0, \infty)$.

Let $Y := X^{1/p}$. Then both $Y$ and $Y'$ are ball Banach function spaces. By Lemma 4.7 and Remark 4.8, we have, for any $\lambda \in (0, \infty)$,

$$\left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_X^{1/q} \leq \sup_{\| g \|_{Y'} \leq 1} \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(x, y) \, dy \right]^{p/q} R_{Y'} g(x) \, dx \right\}^{1/p}.$$  

(4.11)
Let $g \in Y'$ with $\|g\|_{Y'} \leq 1$. Using both Lemma 4.6(ii) and Theorem 3.5 with $\omega$ replaced by $R_{Y'}g$, we find that there exist positive constants $C$ and $C((R_{Y'}g)_{A_{1}(\mathbb{R}^n)})$ such that, for any $\lambda \in (0, \infty)$,

$$
\lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \right]^{\frac{p}{q}} R_{Y'}g(x) \, dx \right\}^{\frac{1}{p}} \leq CC((R_{Y'}g)_{A_{1}(\mathbb{R}^n)}) \left[ \int_{\mathbb{R}^n} |\nabla f(x)|^p R_{Y'}g(x) \, dx \right]^{\frac{1}{p}} \leq CC(2\|M\|_{Y' \to Y'}) \left[ \int_{\mathbb{R}^n} |\nabla f(x)|^p R_{Y'}g(x) \, dx \right]^{\frac{1}{p}},
$$

where $E_f(\lambda, q)$ is the same as in (3.2), the positive constant $C(2\|M\|_{Y' \to Y'})$ increases as $\|M\|_{Y' \to Y'}$ increases, $C(\cdot)$ is continuous on $(0, \infty)$, and the positive constant $C$ depends only on $p, q,$ and $n$. From this, (4.11), Lemma 4.7, and Remark 4.8 again, we deduce that, for any $\lambda \in (0, \infty)$,

$$
\lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{1}{q}} \right\} \|\nabla f\|_X \leq CC(2\|M\|_{Y' \to Y'}) \sup_{\|g\|_{Y'} \leq 1} \left[ \int_{\mathbb{R}^n} |\nabla f(x)|^p R_{Y'}g(x) \, dx \right]^{\frac{1}{p}} \leq CC(2\|M\|_{Y' \to Y'}) 2^{\frac{1}{p}} \|\nabla f\|_X.
$$

This proves (4.10).

By both (3.25) and (4.10), we conclude that both (4.6) and (4.7) hold true for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$. This proves (i).

Next, we prove (ii). Assume that $X$ has an absolutely continuous norm. By (4.10), we find that, to show (ii), it suffices to prove that (4.7) holds true for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$. We first show that, for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$,

$$
\liminf_{\lambda \to \infty} \lambda^q \left\{ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right\}^{\frac{1}{q}} \|\nabla f\|_X \geq \frac{K(q, n)}{n} \|\nabla f\|_X. \tag{4.12}
$$

Let $f \in \dot{W}^{1,X}(\mathbb{R}^n)$. Using Corollary 2.18, we find that there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ with $|\nabla f_k| \in C_c(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ such that

$$
\lim_{k \to \infty} \|\nabla f - \nabla f_k\|_X = 0. \tag{4.13}
$$

By the assumption that $X$ is a ball Banach function space, we have

$$
\|\nabla f\|_X \leq \|\nabla f_k\|_X + \|\nabla f - \nabla f_k\|_X. \tag{4.14}
$$

Let $C(q, n) := \left\lfloor \frac{K(q, n)}{n} \right\rfloor^{\frac{1}{q}}$. From (3.26), (3.37) with $q$ replaced by $\frac{1}{q}$, the assumption that $X$ is a ball Banach function space, and (4.10), we deduce that, for any $\eta, \delta \in (0, 1)$,

$$
\|\nabla f_k\|_X^q = C^{-1}_{(q, n)} \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right\}^{\frac{1}{q}} \|\nabla f - \nabla f_k\|_X \leq C^{-1}_{(q, n)} \liminf_{\lambda \to \infty} \lambda \left\{ \int_{\mathbb{R}^n} 1_{E_f(1-(1-\delta)\lambda, q)}(\cdot, y) \, dy \right\}^{\frac{1}{q}} \|\nabla f - \nabla f_k\|_X.
$$
\[ + \mathbf{1}_{E_f(\delta \lambda, q)}(\cdot, y) \, dy \rceil \frac{1}{q} \leq C_\eta C_{(q, n)}^{-1} \liminf_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f_k-f}((1-\delta)\lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ + (1 + \eta)C_{(q, n)}^{-1} \liminf_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\delta \lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ \leq C_\eta (1 - \delta)^{-1} C_{(q, n)}^{-1} \sup_{\lambda \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f_k-f}(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ + (1 + \eta)\delta^{-1}C_{(q, n)}^{-1} \liminf_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ \leq c_0C_\eta (1 - \delta)^{-1} C_{(q, n)}^{-1} \| \|f - \nabla f\|\|_{X} \]
\[ + (1 + \eta)\delta^{-1}C_{(q, n)}^{-1} \liminf_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \],

where \( c_0 := C_2C_{(\|M\|_{(X^1)\rightarrow(X^1)/\eta}^\infty}} \) and \( C_\eta \) is a positive constant depending only on \( \eta \). Using this and (4.14), and letting \( k \to \infty \), we conclude that

\[ \| \|f - \nabla f\|\|_{X} \leq (1 + \eta)\delta C_{(q, n)}^{-1} \liminf_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \].

Let \( \eta \to 0 \) and \( \delta \to 1 \). We then prove (4.12).

Finally, we show that

\[ \limsup_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \leq \frac{K(q, n)}{n} \| \|f - \nabla f\|\|_{X}^q. \] (4.15)

By (3.26), we find that (4.15) holds true for any \( f \in C^2(\mathbb{R}^n) \) with \( \|f\| \in C_\eta(\mathbb{R}^n) \). Using (4.15) with \( f := f_k \), (4.10) with \( f := f - f_k \), (3.37), and the assumption that \( X \) is a ball Banach function space, we have, for any \( k \in \mathbb{N} \),

\[ \limsup_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \leq C_\eta \limsup_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f-f}(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ + (1 + \theta) \limsup_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\delta \lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ \leq C_\eta (1 - \delta)^{-1} \sup_{\lambda \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f-f}(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \]
\[ + (1 + \theta)\delta^{-1} \limsup_{\lambda \to \infty} \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q)}(\cdot, y) \, dy \right\|_{X} \].
We first prove (i). From Theorem 3.18, it follows that, for any $p$ and $m$, when $X \in \text{any } m$, $f$ for any $\delta \to 1$, and using (4.13), we conclude that

$$\lim \sup_{\lambda \to \infty} \lambda \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{1}{q}} \leq C_{(q,n)} \| \nabla f \|_X.$$ 

Thus, (4.15) holds true, which, combined with (4.12), further implies that (4.7) holds true for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$. This finishes the proof of Theorem 4.5.

**Remark 4.9**

(i) In Theorem 4.5(ii), if $X := L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$ and if $q \in (0, \infty)$ satisfies $n(\frac{1}{p} - \frac{1}{q}) < 1$, then the conclusions of Theorem 4.5(ii) hold true with $X$ replaced by $L^p(\mathbb{R}^n)$, which, when $p = q \in [1, \infty)$, is just (1.2) in [15, 17].

(ii) Let $p \in [1, \infty)$ and $X$ be the same as in Theorem 4.5(ii). Assume that $q_1, q_2 \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q_1}) < 1$ and $n(\frac{1}{p} - \frac{1}{q_2}) < 1$. From Theorem 4.5(ii), it follows that

$$\dot{W}^{1,q}_X(\mathbb{R}^n) \cap \dot{W}^{1,X}(\mathbb{R}^n) = \dot{W}^{1, q_1}_X(\mathbb{R}^n) \cap \dot{W}^{1, X}(\mathbb{R}^n)$$

with equivalent quasi-norms. Thus, when $q \in (0, \infty)$ satisfies $n(\frac{1}{p} - \frac{1}{q}) < 1$, the space $\dot{W}^{1,q}_X(\mathbb{R}^n) \cap \dot{W}^{1,X}(\mathbb{R}^n)$ is independent of $q$.

When $p = 1$ and the Hardy–Littlewood maximal operator $M$ in (1.6) is not known to be bounded on its associate space $X'$, Theorem 4.5 does not work anymore in this case; instead of this, we have the following conclusion.

**Theorem 4.10**

Let $X$ be a ball Banach function space and $q \in (0, \infty)$ with $n(1 - \frac{1}{q}) < 1$. Assume that there exists a sequence $\{\theta_m\}_{m \in \mathbb{N}} \subset (0, 1)$ such that $\lim_{m \to \infty} \theta_m = 1$ and, for any $m \in \mathbb{N}$, $X^{1/\theta_m}$ is a ball Banach function space, $M$ in (1.6) bounded on its associate space $(X^{1/\theta_m})'$, and

$$\lim_{m \to \infty} \| M \|_{(X^{1/\theta_m})' \to (X^{1/\theta_m})'} < \infty. \tag{4.16}$$

(i) Then, for any $f \in C^2(\mathbb{R}^n)$ with $| \nabla f | \in C_\subset(\mathbb{R}^n)$,

$$\| f \|_{\dot{W}^{1,q}}(\mathbb{R}^n) \sim \| \nabla f \|_X,$$

where the positive equivalence constants are independent of $f$. Moreover, for any $f \in C^2(\mathbb{R}^n)$ with $| \nabla f | \in C_\subset(\mathbb{R}^n)$,

$$\lim_{\lambda \to \infty} \lambda^\theta \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{1}{q}} = \frac{K(q,n)}{n} \| \nabla f \|_X^q \tag{4.18}$$

where $E_f(\lambda, q)$ for any $\lambda \in (0, \infty)$ is the same as in (3.2).

(ii) Assume that the centered ball average operators $B_{r, f}$ are uniformly bounded on $X$ and $X$ has an absolutely continuous norm. Then, for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$, both (4.17) and (4.18) hold true.

**Proof**

We first prove (i). From Theorem 3.18, it follows that, for any $f \in C^2(\mathbb{R}^n)$ with $| \nabla f | \in C_\subset(\mathbb{R}^n)$, (4.18) holds true. Thus, to show (4.17), it remains to prove that, for any $f \in C^2(\mathbb{R}^n)$ with $| \nabla f | \in C_\subset(\mathbb{R}^n)$,

$$\| f \|_{\dot{W}^{1,q}_X(\mathbb{R}^n)} \lesssim \| \nabla f \|_X. \tag{4.19}$$
To this end, it suffices to show that, for any \( f \in C^2(\mathbb{R}^n) \) with \(|\nabla f| \in C_c(\mathbb{R}^n)\) and for any \( \lambda \in (0, \infty) \),

\[
\lambda \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_{\frac{1}{q}} \lesssim \|\nabla f\|_X, \tag{4.20}
\]

where \( E_f(\lambda, q) \) for any \( \lambda \in (0, \infty) \) is the same as in (3.2). Let \( f \in C^2(\mathbb{R}^n) \) with \(|\nabla f| \in C_c(\mathbb{R}^n)\). Let \( \{q_m\}_{m \in \mathbb{N}} \) be the same as in the present theorem. By Theorem 4.5 and the facts that, for any \( m \in \mathbb{N} \), \( X^{1/q_m} \) is a ball Banach function space and that \( M \) in (1.6) is bounded on its associate space \(((X^{1/q_m})')\)' , we conclude that, for any \( m \in \mathbb{N} \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_{\frac{1}{q}} \lesssim C(\|M\|_{(X^{1/q_m})' \to (X^{1/q_m})'}) \|\nabla f\|_X, \quad \tag{4.21}
\]

which further implies that, for any \( \lambda \in (0, \infty) \),

\[
\lambda^{1/q_m} \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_{\frac{1}{q}} \lesssim C^{\frac{1}{q_m}} \|\nabla f\|_X. \tag{4.21}
\]

From this, (4.21), Definition 2.1(iii), (4.16), and the fact that \( C(\cdot) \) is continuous on \((0, \infty)\), together with the assumption that \( X \) is a quasi-Banach space, we deduce that, for any \( \lambda \in (0, \infty) \),

\[
\left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X = \lim_{m \to \infty} \inf_{j \geq m} \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_{\left\| f \right\|_{q_m}} = \lim_{m \to \infty} \inf_{j \geq m} \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \left\| \nabla f \right\|_{X} \lesssim \lim_{m \to \infty} \lambda^{-\frac{1}{q_m}} C^{\frac{1}{q_m}} \|\nabla f\|_X \leq \lambda^{-1} C(\lim_{m \to \infty} \|M\|_{(X^{1/q_m})' \to (X^{1/q_m})'}) \lim_{m \to \infty} \left\| \nabla f \right\|_{X} \lesssim \lambda^{-1} \lim_{m \to \infty} \left\| \nabla f \right\|_{X}
\]

which implies that (4.20) and hence (4.19) hold true. This finishes the proof of (i).
Next, we prove (ii). We first show that, for any \( f \in \hat{W}^{1, X}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \frac{1}{\| \lambda \|_X} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{1}{q}} \right\|_X \lesssim \| \nabla f \|_X. \tag{4.22}
\]

By Theorem 2.6, we conclude that there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \) with \( \{\| \nabla f_k \|_{k \in \mathbb{N}} \subset C_c(\mathbb{R}^n) \) such that

\[
\lim_{k \to \infty} \| \nabla f - \nabla f_k \|_X = 0 \quad \text{and} \quad \lim_{k \to \infty} \| (f - f_k)1_{B(0, R)} \|_X = 0 \tag{4.23}
\]

for any \( R \in (0, \infty) \). For any \( N \in \mathbb{N} \), let

\[
E_N := \left\{ (x, y) \in B(0, N) \times B(0, N) : |x - y| \geq \frac{1}{N} \right\}.
\]

It is easy to show that, for any \( k \in \mathbb{N}, \lambda \in (0, \infty) \), and \( \delta \in (0, 1) \),

\[
E_f(\lambda, q) \subset E_{f_k}((1 - \delta)\lambda, q) \cup E_k^{(1)}(2^{-1}\delta\lambda, q) \cup E_k^{(2)}(2^{-1}\delta\lambda, q),
\]

where

\[
E_k^{(1)}(2^{-1}\delta\lambda, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f_k(x)| > \frac{\delta\lambda}{2} |x - y|^{1 + \frac{n}{2}} \right\}
\]

and

\[
E_k^{(2)}(2^{-1}\delta\lambda, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y) - f_k(y)| > \frac{\delta\lambda}{2} |x - y|^{1 + \frac{n}{2}} \right\}.
\]

Using this and applying (3.37) with \( q \) replaced by \( \frac{1}{q} \), we obtain, for any \( \lambda \in (0, \infty), N \in \mathbb{N}, k \in \mathbb{N}, \) and \( \delta, \theta \in (0, 1) \),

\[
\begin{align*}
&\lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q) \cap E_N(\cdot, y)} \, dy \right\|_X^{\frac{1}{q}} \\
&\leq \lambda(1 + \theta) \left\| \int_{\mathbb{R}^n} 1_{E_{f_k}((1 - \delta)\lambda, q) \cap E_N(\cdot, y)} \, dy \right\|_X^{\frac{1}{q}} \\
&\quad + \lambda C(\theta) \left\| \int_{\mathbb{R}^n} \left[ 1_{E_k^{(1)}(2^{-1}\delta\lambda, q) \cap E_N(\cdot, y)} + 1_{E_k^{(2)}(2^{-1}\delta\lambda, q) \cap E_N(\cdot, y)} \right] \, dy \right\|_X^{\frac{1}{q}} \\
&=: I + II, \tag{4.24}
\end{align*}
\]

where \( C(\theta) \) is a positive constant depending only on \( \theta \). Applying (4.19) with \( f := f_k \), we have, for any \( \lambda \in (0, \infty), k \in \mathbb{N}, \) and \( \delta, \theta \in (0, 1) \),

\[
I \leq \lambda(1 + \theta) \left\| \int_{\mathbb{R}^n} 1_{E_{f_k}((1 - \delta)\lambda, q)}(\cdot, y) \, dy \right\|_X^{\frac{1}{q}} \\
\lesssim (1 - \delta)^{-1}(1 + \theta) \| \nabla f_k \|_X \\
\leq (1 - \delta)^{-1}(1 + \theta) \| \nabla f \|_X + \| \nabla f - \nabla f_k \|_X. \tag{4.25}
\]

Next, we estimate II. It is obvious that, for any \( x \in \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} 1_{E_k^{(1)}(2^{-1}\delta\lambda, q) \cap E_N(x, y)} \, dy
\]
\[
\begin{align*}
&\leq (2\lambda^{-1}\delta^{-1})^q \int_{\mathbb{R}^n} \frac{|f(x) - f_k(x)|^q}{|x-y|^{q+n}} 1_{E_N}(x,y) \, dy \\
&\leq (2\lambda^{-1}\delta^{-1})^q N^{q+N} |B(0, N)| |f(x) - f_k(x)|^q 1_{B(0,N)}(x). 
\end{align*}
\] (4.26)

On the other hand, by Lemma 2.13, we find that, for any \(x \in \mathbb{R}^n\),
\[
\begin{align*}
\int_{\mathbb{R}^n} 1_{E_k^{(1)}(2^{-1}\delta^\lambda, q)} \cap E_N(x,y) \, dy \\
&\leq \frac{2}{\delta^\lambda} \int_{\mathbb{R}^n} \frac{|f(y) - f_k(y)|}{|x-y|^{1+\frac{n}{q}}} 1_{E_N}(x,y) \, dy \\
&\leq \frac{2N^{1+\frac{n}{q}}}{\delta^\lambda} \int_{B(0,N)} |f(y) - f_k(y)| \, dy 1_{B(0,N)}(x) \\
&\leq \frac{2N^{1+\frac{n}{q}}}{\delta^\lambda} \|f - f_k\|_X \|1_{B(0,N)}\|_X \|1_{B(0,N)}\|_X.
\end{align*}
\]

This, combined with (4.26), further implies that, for any \(\lambda \in (0, \infty), \, N \in \mathbb{N}, k \in \mathbb{N}, \) and \(\delta, \theta \in (0, 1)\),
\[
\begin{align*}
\|f - f_k\|_X &\leq \lambda C(\theta) 2^{\frac{1}{\theta}} \left[ \left\| \left( \int_{\mathbb{R}^n} 1_{E_k^{(1)}(2^{-1}\delta^\lambda, q)} \cap E_N(\cdot, y) \, dy \right)^{\frac{1}{\theta}} \right\|_X \right. \\
&\quad + \lambda C(\theta) 2^{\frac{1}{\theta}} \left[ \left\| \left( \int_{\mathbb{R}^n} 1_{E_k^{(2)}(2^{-1}\delta^\lambda, q)} \cap E_N(\cdot, y) \, dy \right)^{\frac{1}{\theta}} \right\|_X \right. \\
&\quad \leq C(\theta) 2^{\frac{1}{\theta}} \frac{2^{\lambda^{-1}}N^{1+\frac{n}{q}}}{\delta^\lambda} |B(0, N)|^{\frac{1}{\theta}} \|f - f_k\|_X \\
&\quad + \lambda C(\theta) 2^{\frac{1}{\theta}} \left( \frac{2N^{1+\frac{n}{q}}}{\delta^\lambda} \right)^{\frac{1}{\theta}} \|f - f_k\|_X \|1_{B(0,N)}\|_X \|1_{B(0,N)}\|_X.
\end{align*}
\]

Using this, (4.25), (4.24), and (4.23), and letting \(k \to \infty\), we conclude that, for any \(\lambda \in (0, \infty), \, N \in \mathbb{N}, \) and \(\delta, \theta \in (0, 1)\),
\[
\lambda \left\| \left( \int_{\mathbb{R}^n} 1_{E_f(\delta, q)} \cap E_N(\cdot, y) \, dy \right)^{\frac{1}{\theta}} \right\|_X \lesssim (1 - \delta)^{-1} (1 + \theta) \|\nabla f\|_X.
\]

Let \(\delta \to 0, \theta \to 0, \) and \(N \to \infty\). Then we find that (4.22) holds true.

By (4.22) and a density argument similar to that used in the proof of (4.12), we conclude that (4.18) holds true for any \(f \in \dot{W}^{1,n}(\mathbb{R}^n)\). This finishes the proof of (ii) and hence Theorem 4.10.

\begin{remark}
(i) In Theorem 4.10(ii), if \(X := L^p(\mathbb{R}^n)\) with \(p \in [1, \infty)\) and if \(q \in (0, \infty)\) satisfies \(n(1 - \frac{1}{p}) < 1\), then the conclusions of Theorem 4.10(ii) hold true with \(X\) replaced by \(L^p(\mathbb{R}^n)\). Moreover, when \(q := 1\) and \(X := L^1(\mathbb{R}^n)\), (4.17) is just (1.2) with \(p = 1\) in [15, 17].

(ii) Theorem 4.10 is used to solve the endpoint case of concrete examples of ball Banach function spaces in Sect. 5. For example, when \(X := L^{\gamma^*}(\mathbb{R}^n)\) with \(\gamma^* = 1\) (see Sect. 5.3 below for the precise definitions of both \(L^{\gamma^*}(\mathbb{R}^n)\) and \(\tilde{\gamma}_{-}\)), since it is still unclear whether or not the Hardy–Littlewood maximal operator \(M\) is bounded on \([L^{\gamma^*}(\mathbb{R}^n)]^\prime\), it follows that Theorem 4.5 does not work anymore in this case, while Theorem 4.10 is applicable in this case.
\end{remark}
(iii) We should point out that we do not need the assumption (4.16) in the proof of the lower estimate of (4.17).

In both Theorems 4.10 and 4.5, we assume that $X$ is a ball Banach function space. In the case that $X$ is a ball quasi-Banach function space, we have the following conclusion, the proof of which is similar to that of Theorem 4.5; we omit the details here.

**Theorem 4.12** Let $p \in [1, \infty)$, $q \in (0, \infty)$ with $n(\frac{1}{p} - \frac{1}{q}) < 1$, $X$ be a ball quasi-Banach function space, and $Y \subset \mathcal{M}(\mathbb{R}^n)$ be a linear normed space, equipped with a norm $\| \cdot \|_Y$ which makes sense for all functions in $\mathcal{M}(\mathbb{R}^n)$. Assume that, for any $f \in X^{1/p}$,

$$\| f \|_{X^{1/p}} = \sup_{\| h \|_Y = 1} \left| \int_{\mathbb{R}^n} f(x)h(x) \, dx \right|$$

and the Hardy–Littlewood maximal operator $M$ is bounded on $Y$. Then there exists a positive constant $C$ such that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$,

$$\left[ \frac{K(q, n)}{n} \right]^{\frac{1}{q}} \| |\nabla f| \|_X \leq \sup_{\lambda \in (0, \infty)} \lambda \left\{ x \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot \gamma \cdot |x - y|^{\frac{n}{q} + 1} \right\}^{\frac{1}{q}} \leq C \| |\nabla f| \|_X,$$

where $K(q, n)$ is the same as in (3.3).

As a consequence of Theorem 4.10, we obtain the following fractional Sobolev type inequality on ball Banach function spaces.

**Corollary 4.13** Let $q_1 \in [1, \infty)$, $\theta \in (0, 1)$, and $q \in [1, q_1]$ satisfy $\frac{1}{q} = \frac{1-\theta}{q_1} + \theta$. Let $X$ be a ball Banach function space. Assume that there exists a sequence $\{\alpha_m\} \in (0, 1)$ such that, for any $m \in \mathbb{N}$, $\lim_{m \to \infty} \alpha_m = 1$, $X^{1/\alpha_m}$ is a ball Banach function space, $\mathcal{M}$ in (1.6) bounded on its associate space $(X^{1/\alpha_m})'$, and

$$\lim_{m \to \infty} \| M \|_{(X^{1/\alpha_m})' \to (X^{1/\alpha_m})'} < \infty. \quad (4.28)$$

(i) Let $q_1 \in [1, \infty)$. Then there exists a positive constant $C$ such that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$,

$$\| f \|_{W^{\theta}_{q_1, q}(\mathbb{R}^n)} \leq C \| f \|_{X^{q_1}} \| |\nabla f| \|_{X}^{\theta}. \quad (4.29)$$

Assume further that $X$ has an absolutely continuous norm, $X^{q_1}$ is a ball Banach function space, and the centered ball average operators $\{B_r\}_{r \in (0, \infty)}$ are uniformly bounded on $X$. Then (4.29) holds true for any $f \in \dot{W}^{1, X}(\mathbb{R}^n)$.

(ii) Let $q_1 = \infty$. Then there exists a positive constant $C$ such that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$,

$$\| f \|_{W^{\theta}_{q_1, q}(\mathbb{R}^n)} \leq C \| f \|_{L^{\infty}(\mathbb{R}^n)} \| |\nabla f| \|_{X}^{\theta}. \quad (4.30)$$

Assume further that the centered ball average operators $\{B_r\}_{r \in (0, \infty)}$ are uniformly bounded on $X$ and that $X$ has an absolutely continuous norm. Then (4.30) holds true for any $f \in \dot{W}^{1, X}(\mathbb{R}^n)$.

**Proof** We first prove (ii). Let $f \in \dot{W}^{1, X}(\mathbb{R}^n)$. If $\| f \|_{L^{\infty}(\mathbb{R}^n)} = 0$ or $\| f \|_{L^{\infty}(\mathbb{R}^n)} = \infty$, then (4.30) holds true automatically. Now, we assume that $\| f \|_{L^{\infty}(\mathbb{R}^n)} \in (0, \infty)$. Let $q_1 := \infty.
Then \( q\theta = 1 \). For any \( \lambda, r, s \in (0, \infty) \), let \( E_f(\lambda, r, s) \) be the same as in (4.1). Since, for any \( \lambda \in (0, \infty) \),
\[
E_f \left( \lambda, \frac{1}{\theta}, \theta \right) \subset E_f \left( \frac{\lambda^{1/\theta}}{[2\|f\|_{L_\infty(\mathbb{R}^n)}]^{(1-\theta)/\theta}}, 1, 1 \right),
\]
from Definition 2.1(ii), we deduce that
\[
\parallel f \parallel_{W^{q,q}_{\theta,\theta}(\mathbb{R}^n)} = \sup_{\lambda \in (0, \infty)} \lambda^{1/\theta} \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, \theta)}(\cdot, y) \, dy \right\|_X 
\leq \sup_{\lambda \in (0, \infty)} \lambda^{1/\theta} \left\| \int_{\mathbb{R}^n} 1_{E_f(2\|f\|_{L_\infty(\mathbb{R}^n)}^{(1-\theta)/\theta}, 1, 1)}(\cdot, y) \, dy \right\|_X 
= [2\|f\|_{L_\infty(\mathbb{R}^n)}]^{1-\theta/\theta} \sup_{\lambda \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X. \tag{4.31}
\]

If \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \), then, applying Theorem 4.10(i) with \( q := 1 \), we find that
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X \lesssim \| \nabla f \|_X. \tag{4.32}
\]
This, together with (4.31), further implies that (4.30) holds true for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \).

If the centered ball average operators \( \{B_r\}_{r \in (0, \infty)} \) are uniformly bounded on \( X \) and \( X \) has an absolutely continuous norm, then, by Theorem 4.10(ii) with \( q := 1 \), we conclude that (4.32) holds true for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \). This, combined with (4.31), further implies that (4.30) holds true for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \). This finishes the proof of (ii) of the present corollary.

Next, we prove (i). Assume \( q_1 \in [1, \infty) \). We first show that (4.29) holds true for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \). Let \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \) and \( A \in (0, \infty) \) be a constant which is specified later. Since, for any \( x, y \in \mathbb{R}^n \) with \( x \neq y \),
\[
\frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{q} + \theta}} = \left[ \frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{q} + 1}} \right]^{1-\theta} \left[ \frac{|f(x) - f(y)|}{|x - y|^{n+1}} \right]^{\theta},
\]
we deduce that, for any \( \lambda \in (0, \infty) \),
\[
E_f(\lambda, q, \theta) \subset \left[ E_f \left( A^{-\theta} \lambda, q_1, 0 \right) \cup E_f \left( A^{1-\theta} \lambda, 1, 1 \right) \right].
\]
This, together with the fact that \( \| \cdot \|_X \) is a quasi-norm, further implies that, for any \( \lambda \in (0, \infty) \),
\[
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, \theta)}(\cdot, y) \, dy \right\|_X^{\frac{1}{q}} \lesssim \left\| \int_{\mathbb{R}^n} 1_{E_f(A^{-\theta} \lambda, q_1, 0)}(\cdot, y) \, dy \right\|_X^{\frac{1}{q}} + \left\| \int_{\mathbb{R}^n} 1_{E_f(A^{1-\theta} \lambda, 1, 1)}(\cdot, y) \, dy \right\|_X^{\frac{1}{q}} 
\leq \left( \frac{A^\theta G}{\lambda} \right)^{\frac{q_1}{q}} + \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}}, \tag{4.33}
\]
where

\[
G := \sup_{\lambda \in (0, \infty)} \lambda \left\| \frac{1}{\sqrt[\frac{q}{2}]{\lambda}} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_X
\]

and

\[
H := \sup_{\lambda \in (0, \infty)} \lambda \left\| \frac{1}{\sqrt[\frac{q}{2}]{\lambda}} \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X.
\]

Choose an \( A \in (0, \infty) \) such that

\[
\left( \frac{A^\theta G}{\lambda} \right)^{\frac{q_1}{q}} = \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}}.
\]

This, combined with (4.33), then implies that

\[
\left\| \frac{1}{\sqrt[\frac{q}{2}]{\lambda}} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, \theta)}(\cdot, y) \, dy \right\|_X^{\frac{1}{q}} \lesssim \left( \frac{A^\theta G}{\lambda} \right)^{\frac{q_1}{q}} + \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}} \sim \left( \frac{A^\theta G}{\lambda} \right)^{\frac{q_1}{q}} \sim \lambda^{-1} G^{1-\theta} H^\theta.
\]

Using this and applying Theorem 4.10(i) with \( q := 1 \), we conclude that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{q_1}{2}} \right\} \right\|_{X^{q_1}} \lesssim G^{1-\theta} \|\nabla f\|_{X^q}.
\]

Next, we prove that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
G \lesssim \|f\|_{X^{q_1}}.
\]

To this end, by Lemma 3.17 and Lemma 4.6(i), we have, for any \( \lambda \in (0, \infty) \) and \( m \in \mathbb{N} \),

\[
\left\| \frac{1}{\sqrt[\frac{q}{2}]{\lambda}} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_{X^{1/\alpha m}} = \sup_{\|g\|_{X^{1/\alpha m}} = 1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(x, y) \, dy \right] g(x) \, dx \leq \sup_{\|g\|_{X^{1/\alpha m}} = 1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(x, y) \, dy \right] R_{X^{1/\alpha m}} g(x) \, dx.
\]

For any \( x, y \in \mathbb{R}^n \), let

\[
D_f(\lambda, q_1)_1 := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x)| > \lambda |x - y|^{\frac{n}{q_1}} / 2 \right\}
\]

and

\[
D_f(\lambda, q_1)_2 := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y)| > \lambda |x - y|^{\frac{n}{q_1}} / 2 \right\}.
\]
Observe that, for any $\lambda \in (0, \infty)$, $E_f(\lambda, q_1, 0) \subset D_f(\lambda, q_1)_1 \cup D_f(\lambda, q_1)_2$. From this, both (ii) and (iii) of Lemma 4.6, and the definition of $A_1(\mathbb{R}^n)$, we deduce that, for any $\lambda \in (0, \infty)$, $m \in \mathbb{N}$, and $g \in (X^{1/\alpha_m})'$ with $\|g\|_{(X^{1/\alpha_m})'} = 1$,

$$
\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(x, y) \, dy \right] R_{(X^{1/\alpha_m})'} g(x) \, dx \\
\leq \int_{\mathbb{R}^n} \left[ \int_{y \in \mathbb{R}^n \cap (x, y) \in D_f(\lambda, q_1)_1} \, dy \right] R_{(X^{1/\alpha_m})'} g(x) \, dx \\
+ \int_{\mathbb{R}^n} \left[ \int_{x \in \mathbb{R}^n \cap (x, y) \in D_f(\lambda, q_1)_2} R_{(X^{1/\alpha_m})'} g(x) \, dx \right] \, dy \\
\lesssim \lambda^{-q_1} \int_{\mathbb{R}^n} |f(x)|^{q_1} R_{(X^{1/\alpha_m})'} g(x) \, dx \\
+ [R_{(X^{1/\alpha_m})'}]_{A_1(\mathbb{R}^n)} \lambda^{-q_1} \int_{\mathbb{R}^n} |f(y)|^{q_1} R_{(X^{1/\alpha_m})'} g(y) \, dy \\
\lesssim \left[ 1 + \|M\|_{(X^{1/\alpha_m})' \to (X^{1/\alpha_m})'} \right] \lambda^{-q_1} \|f\|_{X^{1/\alpha_m}} \|R_{(X^{1/\alpha_m})'} g\|_{(X^{1/\alpha_m})'} \\
\lesssim \left[ 1 + \|M\|_{(X^{1/\alpha_m})' \to (X^{1/\alpha_m})'} \right] \lambda^{-q_1} \|f\|_{X^{1/\alpha_m}}. 
$$

By this and (4.37), we find that there exists a positive constant $C$ such that, for any $m \in \mathbb{N}$ and $f \in C^2_c(\mathbb{R}^n)$,

$$
\lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_{X^{1/\alpha_m}}^{1/q_1} \leq C \left[ 1 + \|M\|_{(X^{1/\alpha_m})' \to (X^{1/\alpha_m})'} \right] \|f\|_{X^{1/\alpha_m}}. 
$$

Using this, Definition 2.1(iii), (4.38), and (4.16), together with a quasi-Banach space $X$, we conclude that, for any $\lambda \in (0, \infty)$,

$$
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_X \\
= \left\| \lim_{m \to \infty} \inf_{j \geq m} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right]^{1/\alpha_j} \right\|_X \\
= \left\| \lim_{m \to \infty} \inf_{j \geq m} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right]^{1/\alpha_j} \right\|_X \\
\leq \lim_{m \to \infty} \inf \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right]^{1/\alpha_m} \right\|_X \\
\leq \lim_{m \to \infty} \left[ 1 + \|M\|_{(X^{1/\alpha_m})' \to (X^{1/\alpha_m})'} \right]^{q_1/\alpha_m} \lambda^{-q_1/\alpha_m} \left\| f \right\|_{X^{q_1/\alpha_m}} \\
\lesssim \lambda^{-q_1} \limsup_{m \to \infty} \left\| f \right\|_{X^{q_1/\alpha_m}} \\
= \lambda^{-q_1} \limsup_{m \to \infty} \left\| \frac{f}{\|f\|_{L^\infty(\mathbb{R}^n)}} \right\|_{X^{q_1/\alpha_m}} \\
\leq \lambda^{-q_1} \limsup_{m \to \infty} \left\| \frac{f}{\|f\|_{L^\infty(\mathbb{R}^n)}} \right\|^{q_1}_{X^{q_1/\alpha_m}} = \lambda^{-q_1} \left\| f \right\|_{X^{q_1}}.
$$

This implies that (4.36) holds true. From both (4.36) and (4.35), it follows that (4.29) holds true for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_c(\mathbb{R}^n)$. 
Assume that $X$ has an absolutely continuous norm, $X^{q_1}$ is a ball Banach function space, and the centered ball average operators $\{B_r\}_{r \in (0, \infty)}$ are uniformly bounded on $X$. It remains to show that (4.29) holds true for any $f \in \dot{W}^{1, X}(\mathbb{R}^n)$ under these additional assumptions.

Indeed, by both (4.34) and Theorem 4.10(ii) with $q := 1$, we find that (4.35) holds true for any $f \in \dot{W}^{1, X}(\mathbb{R}^n)$. This implies that, to prove (4.29) for any $f \in \dot{W}^{1, X}(\mathbb{R}^n)$, it suffices to show (4.36) for any $f \in \dot{W}^{1, X}(\mathbb{R}^n) \cap X^{q_1}$. Let $f \in \dot{W}^{1, X}(\mathbb{R}^n) \cap X^{q_1}$. By the assumption that the centered ball average operators $\{B_r\}_{r \in (0, \infty)}$ are uniformly bounded on $X$, it is easy to prove that the centered ball average operators $\{B_r\}_{r \in (0, \infty)}$ are also uniformly bounded on $X^{q_1}$. Moreover, the assumption that $X$ has an absolutely continuous norm obviously implies that $X^{q_1}$ also has an absolutely continuous norm. From these and the assumption that $X^{q_1}$ is a ball Banach function space, it follows that $C_c^\infty(\mathbb{R}^n)$ is dense in $X^{q_1}$ (see, for instance, [30, Corollary 3.10]). Thus, there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that

$$
\lim_{k \to \infty} \|f - f_k\|_{X^{q_1}} = 0.
$$

(4.39)

For any $N \in \mathbb{N}$, let

$$
E_N := \{(x, y) \in B(0, N) \times B(0, N) : |x - y| \geq N^{-1}\}.
$$

Notice that, for any $k \in \mathbb{N}$ and $\lambda \in (0, \infty)$,

$$
E_f(\lambda, q_1, 0) \subset \left[ E_{f_k}(3^{-1}\lambda, q_1, 0) \cup E_k^{(1)}(3^{-1}\lambda, q_1, 0) \cup E_k^{(2)}(3^{-1}\lambda, q_1, 0) \right],
$$

where

$$
E_k^{(1)}(3^{-1}\lambda, q_1, 0) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f_k(x)| > 3^{-1}\lambda|x - y|^\frac{n}{q_1}\}
$$

and

$$
E_k^{(2)}(3^{-1}\lambda, q_1, 0) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y) - f_k(y)| > 3^{-1}\lambda|x - y|^\frac{n}{q_1}\}.
$$

This, combined with the assumption that $X$ is a ball Banach function space and the fact that $\frac{1}{q_1} \in (0, 1]$, further implies that, for any $\lambda \in (0, \infty)$ and $k, N \in \mathbb{N}$,

$$
\lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda, q_1, 0) \cap E_N}(\cdot, y) \, dy \right\|_{X^{q_1}}^\frac{1}{q_1}
\leq \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f_k}(3^{-1}\lambda, q_1, 0) \cap E_N}(\cdot, y) \, dy \right\|_{X^{q_1}}^\frac{1}{q_1}
+ \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_k^{(1)}(3^{-1}\lambda, q_1, 0) \cap E_N}(\cdot, y) \, dy \right\|_{X^{q_1}}^\frac{1}{q_1}
+ \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_k^{(2)}(3^{-1}\lambda, q_1, 0) \cap E_N}(\cdot, y) \, dy \right\|_{X^{q_1}}^\frac{1}{q_1}
=: I + II + III.
$$

(4.40)

Applying (4.36) with $f := f_k$ and using the assumption that $X^{q_1}$ is a ball Banach function space, we find that, for any $\lambda \in (0, \infty)$ and $k, N \in \mathbb{N}$,

$$
I \leq \lambda \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_{f_k}(3^{-1}\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_{X^{q_1}}^\frac{1}{q_1} \lesssim \|f_k\|_{X^{q_1}} \leq \|f_k - f\|_{X^{q_1}} + \|f\|_{X^{q_1}}.
$$

(4.41)
Observe that, for any \( \lambda \in (0, \infty) \), \( k, N \in \mathbb{N} \), and \( x \in \mathbb{R}^n \),

\[
\lambda^{q_1} \int_{\mathbb{R}^n} 1_{E^k} (3^{-1} \lambda, q_1, 0) \cap E_N (x, y) \, dy \\
\leq 3^{q_1} \int_{\mathbb{R}^n} \frac{|f(x) - f_k(x)|^{q_1}}{|x - y|^n} 1_{E_N} (x, y) \, dy \\
\leq 3^{q_1} N^n |B(0, N)| |f(x) - f_k(x)|^{q_1}.
\]

Thus, for any \( \lambda \in (0, \infty) \) and \( k, N \in \mathbb{N} \),

\[
\Pi \leq 3 [N^n |B(0, N)|]^{\frac{1}{q_1}} \| f - f_k \|_{X^{q_1}}.
\]  \hspace{1cm} (4.42)

By Lemma \ref{lem:2.13}, we find that, for any \( \lambda \in (0, \infty) \), \( k, N \in \mathbb{N} \), and \( x \in \mathbb{R}^n \),

\[
\lambda^{q_1} \int_{\mathbb{R}^n} 1_{E^k} (3^{-1} \lambda, q_1, 0) \cap E_N (x, y) \, dy \\
\leq 3^{q_1} \int_{\mathbb{R}^n} \frac{|f(y) - f_k(y)|^{q_1}}{|x - y|^n} 1_{E_N} (x, y) \, dy \\
\leq 3^{q_1} N^n \int_{B(0, N)} |f(y) - f_k(y)|^{q_1} \, dy 1_{B(0, N)} (x) \\
\leq 3^{q_1} N^n \| f - f_k \|_{X^{q_1}}^{q_1} \| 1_{B(0, N)} \|_{X'} 1_{B(0, N)} (x)
\]

and hence

\[
\Pi \leq 3 [N^n \| 1_{B(0, N)} \|_X \| 1_{B(0, N)} \|_{X'}]^{\frac{1}{q_1}} \| f - f_k \|_{X^{q_1}}.
\]

From this, (4.42), (4.41), and (4.40), it follows that, for any \( \lambda \in (0, \infty) \) and \( k, N \in \mathbb{N} \),

\[
\lambda \left\| \int_{\mathbb{R}^n} 1_{E_f (\lambda, q_1, 0) \cap E_N (\cdot, y)} \, dy \right\|_{X}^{\frac{1}{q_1}} \\
\lesssim \left\{ 1 + [N^n |B(0, N)|]^{\frac{1}{q_1}} + [N^n \| 1_{B(0, N)} \|_X \| 1_{B(0, N)} \|_{X'}]^{\frac{1}{q_1}} \right\} \| f - f_k \|_{X^{q_1}} \\
+ \| f \|_{X^{q_1}}.
\]

Using this and (4.39), and letting \( k \to \infty \) and \( N \to \infty \), we find that (4.36) holds true for any \( f \in X^{q_1} \), which completes the proof of (ii) and hence Corollary \ref{cor:4.13}. \( \square \)

**Remark 4.14** \( (i) \) Let \( X := L^p (\mathbb{R}^n) \) with \( p \in [1, \infty) \). Then Corollary \ref{cor:4.13} holds true with \( X \) replaced by \( L^p (\mathbb{R}^n) \). In particular, if \( q_1 := \infty, q \in [1, \infty], \theta := 1/q, X := L^1 (\mathbb{R}^n), \) and \( f \in C_c^\infty (\mathbb{R}^n) \), then, in this case, Corollary \ref{cor:4.13}(ii) is just \cite[Corollary 5.1]{17}; furthermore, in this case, if \( n = 1 \), by Corollary \ref{cor:4.13}(ii), we find that, for any \( f \in C_c^\infty (\mathbb{R}) \),

\[
\left\| \begin{array}{c} |f(x) - f(y)| \\ |x - y|^{\frac{2}{q}} \end{array} \right\|_{L^{q, \infty} (\mathbb{R} \times \mathbb{R})} \lesssim \left\| f \right\|_{L^\infty (\mathbb{R})} \left\| f' \right\|^\theta_{L^1 (\mathbb{R})} \lesssim \left\| f' \right\|_{L^1 (\mathbb{R})},
\]

which is just \cite[Corollary 4.1]{17}.

\( (ii) \) Let \( q_1 := \infty \). In this case, when we replace the weak type norm \( \| \cdot \|_{W^\omega_{X^{q_1}, q} (\mathbb{R}^n)} \) in (4.30) by the strong type norm \( \| \cdot \|_{X^{q_1}_q (\mathbb{R}^n)} \), (4.30) may not hold true (see, for instance, \cite[(5.3)]{17}). In this sense, (4.30) with \( q_1 = \infty \) seems to be sharp.
(iii) Let \( q_1 \in [1, \infty) \). We should point out that, if the weak type norm \( \| \cdot \|_{W^{\theta}_{Xq, q}(\mathbb{R}^n)} \) in (4.29) is replaced by the strong type norm \( \| \cdot \|_{\dot{W}^{\theta}_{Xq, q}(\mathbb{R}^n)} \), it is still unclear whether or not Corollary 4.13(i) still holds true.

(iv) Let \( X \) be the same as in Theorem 4.5(ii). Then both (4.29) and (4.30) hold true for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \), which can be proved by a slight modification of the proof of Corollary 4.13 with Theorem 4.10 replaced by Theorem 4.5(ii); we omit the details.

(v) Let \( X \) be the same as in Theorem 4.12. Then both (4.29) and (4.30) hold true for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_\varepsilon(\mathbb{R}^n) \), which can be proved by a slight modification of the proof of Corollary 4.13 with Theorem 4.10 replaced by Theorem 4.12; we omit the details.

Similarly, using Theorem 4.5, we obtain the following fractional Gagliardo–Nirenberg type inequality on ball Banach function spaces.

**Corollary 4.15** Let \( s_1 \in [0, 1) \), \( q_1 \in (1, \infty) \), and \( \theta \in (0, 1) \). Let \( s \in (s_1, 1) \) and \( q \in (1, q_1) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1}{q_1} - \theta \). Let \( X \) be a ball Banach function space. Assume that there exists a sequence \( \{\alpha_m\}_{m \in \mathbb{N}} \subset (0, 1) \) such that, for any \( m \in \mathbb{N} \), \( \lim_{m \to \infty} \alpha_m = 1 \), \( X^{1/\alpha_m} \) is a ball Banach function space, \( \mathcal{M}(1.6) \) bounded on its associate space \( (X^{1/\alpha_m})' \), and (4.28) holds true. Then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_\varepsilon(\mathbb{R}^n) \),

\[
\| f \|_{W^{\theta}_{Xq, q}(\mathbb{R}^n)} \leq C \| f \|_{\dot{W}^{\theta}_{Xq_1, q_1}(\mathbb{R}^n)}^{1/q_1} \| \nabla f \|_X^q \leq C \| f \|_{\dot{W}^{\theta}_{Xq_1, q_1}(\mathbb{R}^n)}^{1/q_1} \| \nabla f \|_X^q \cdot (4.43)
\]

Assume further that the centered ball average operators \( \{B_f\}_{f \in C(0,\infty)} \) are uniformly bounded on \( X \) and \( X \) has an absolutely continuous norm. Then (4.43) holds true for any \( f \in \dot{W}^{1,X}(\mathbb{R}^n) \).

**Proof** Let \( f \in \dot{F}^{1}_{Xq_1, q_1}(\mathbb{R}^n) \cap \dot{W}^{1}_{X, 1}(\mathbb{R}^n) \). For any \( \lambda, r, s \in (0, \infty) \) and \( x \in \mathbb{R}^n \), let \( E_f(\lambda, r, s) \) be the same as in (4.1). Let

\[
G_1 := \sup_{\lambda \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, s_1)}(\cdot, y) \, dy \right\|_{X^{q_1}}^{1/q_1} = \| f \|_{\dot{W}^{1}_{X^{q_1}, q_1}(\mathbb{R}^n)}
\]

and

\[
H_1 := \sup_{\lambda \in (0, \infty)} \lambda \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X = \| f \|_{\dot{W}^{1}_{X, 1}(\mathbb{R}^n)}.
\]

By an argument similar to that used in the proof of (4.34) with \( E_f(\lambda, q, \theta) \) and \( E_f(\lambda_1, q_1, 0) \) replaced, respectively, by \( E_f(\lambda, q, s) \) and \( E_f(\lambda, q_1, s_1) \), we conclude that

\[
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, s)}(\cdot, y) \, dy \right\|_X \lesssim \frac{1}{\lambda^q} G_1^{q(1-\theta)} H_1^q.
\]

From this and the fact that \( \| f \|_{\dot{W}^{\theta}_{Xq_1, q_1}(\mathbb{R}^n)} \leq \| f \|_{\dot{F}^{1}_{Xq_1, q_1}(\mathbb{R}^n)} \), we deduce that

\[
\| f \|_{\dot{W}^{\theta}_{Xq, q}(\mathbb{R}^n)} \leq \sup_{\lambda \in (0, \infty)} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, s)}(\cdot, y) \, dy \right\|_X \lesssim G_1^{q(1-\theta)} H_1^q = \| f \|_{\dot{F}^{1}_{Xq_1, q_1}(\mathbb{R}^n)} \| f \|_{\dot{W}^{\theta}_{X, 1}(\mathbb{R}^n)} \lesssim \| f \|_{\dot{W}^{\theta}_{Xq_1, q_1}(\mathbb{R}^n)} \| f \|_{\dot{W}^{\theta}_{X, 1}(\mathbb{R}^n)} \cdot (4.44)
\]
If \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \), then, applying Theorem 4.10(i) with \( q := 1 \), we find that
\[
\| f \|_{\dot{W}^{1,1}_{\chi}(\mathbb{R}^n)} \lesssim \| |\nabla f| \|_{X}.
\] (4.45)

This, together with (4.44), further implies that (4.43) holds true for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \).

If the centered ball average operators \( \{B_r\}_{r \in (0, \infty)} \) are uniformly bounded on \( X \) and \( X \) has an absolutely continuous norm, then, by Theorem 4.10(ii) with \( q := 1 \), we find that (4.45) holds true for any \( f \in \dot{W}^{1,1}_{\chi}(\mathbb{R}^n) \). This, combined with (4.44), further implies that (4.43) holds true for any \( f \in \dot{W}^{1,1}_{\chi}(\mathbb{R}^n) \). This finishes the proof of Corollary 4.15.

**Remark 4.16**
(i) In Corollary 4.15, if \( X := L^p(\mathbb{R}^n) \) with \( p \in [1, \infty) \), then the conclusions of Corollary 4.15 hold true with \( X \) replaced by \( L^p(\mathbb{R}^n) \). Moreover, when \( X := L^1(\mathbb{R}^n) \) and \( f \in C^\infty_c(\mathbb{R}^n) \), Corollary 4.15 in this case is just [17, Corollary 5.2].

(ii) Let \( X \) be the same as in Theorem 4.5(ii). Then, similarly to Remark 4.14(iv), we find that (4.43) holds true for any \( f \in \dot{W}^{1,1}_{\chi}(\mathbb{R}^n) \), which can be proved by a slight modification of the proof of Corollary 4.15 with Theorem 4.10 replaced by Theorem 4.5(ii); we omit the details.

(iii) Let \( X \) be the same as in Theorem 4.12. Then, similarly to Remark 4.14(v), we find that (4.43) holds true for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \), which can be proved by a slight modification of the proof of Corollary 4.15 with Theorem 4.10 replaced by Theorem 4.12; we omit the details.

### 5 Applications

In this section, we apply Theorems 4.5 and 4.10 and Corollaries 4.13 and 4.15, respectively, to six concrete examples of ball Banach function spaces, namely Morrey spaces (see Sect. 5.1 below), mixed-norm Lebesgue spaces (see Sect. 5.2 below), variable Lebesgue spaces (see Sect. 5.3 below), weighted Lebesgue spaces (see Sect. 5.4 below), Orlicz spaces (see Sect. 5.5 below), Orlicz-slice spaces (see Sect. 5.6 below), and weak Morrey spaces (see Sect. 5.7 below). These examples explicitly indicate both the university and the practicability of the results of this article and more their applications to other (new) function spaces are obviously possible.

#### 5.1 Morrey spaces

For any \( 0 < r \leq \alpha < \infty \), the Morrey space \( M_r^\alpha(\mathbb{R}^n) \) is defined to be the set of all the measurable functions \( f \) on \( \mathbb{R}^n \) with the finite semi-norm
\[
\| f \|_{M_r^\alpha(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} |B|^{1/\alpha - 1/r} \| f \|_{L^r(B)}.
\]

These spaces were introduced in 1938 by Morrey [77] in order to study the regularity of solutions to partial differential equations. They have important applications in the theory of elliptic partial differential equations, potential theory, and harmonic analysis (see, for instance, [1, 23, 47–50, 63, 90, 91, 98]). As was indicated in [93, p. 86], the Morrey space \( M_r^\alpha(\mathbb{R}^n) \) for any \( 1 \leq r \leq \alpha < \infty \) is a ball Banach function space, but is not a Banach function space in the terminology of Bennett and Sharpley [6]. Using both Theorems 4.5(i) and 4.10(i), we obtain the following conclusions.
**Theorem 5.1** Let \( 1 \leq r \leq \alpha < \infty \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{r} - \frac{1}{q}) < 1 \). Then, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
\sup_{B \in \mathcal{B}} \lambda |B|^{\frac{1}{q} - \frac{1}{r}} \left[ \int_B \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^\frac{n}{q} + 1 \right\} \frac{1}{y} \, dx \right]^{1/r} \sim \| |\nabla f| \|_{M^q_p(\mathbb{R}^n)},
\]

where the positive equivalence constants are independent of \( f \). Moreover, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),

\[
\lim_{\lambda \to \infty} \sup_{B \in \mathcal{B}} \lambda |B|^{\frac{1}{q} - \frac{1}{r}} \left[ \int_B \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^\frac{n}{q} + 1 \right\} \frac{1}{y} \, dx \right]^{1/r} = \left[ \frac{K(q, n)}{n} \right]^{\frac{1}{q}} \| |\nabla f| \|_{M^q_p(\mathbb{R}^n)}.
\]

**Proof** We prove (5.1) by considering the following two cases on \( r \).

Case 1) \( r \in (1, \infty) \). In this case, since \( n(\frac{1}{r} - \frac{1}{q}) < 1 \), it follows that there exists a \( p \in [1, r) \) satisfying \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). It is known that, for any \( 1 < r \leq \alpha < \infty \), the associate space \( X' \) of the Morrey space \( X := M^\alpha_p(\mathbb{R}^n) \) is a block space, on which the Hardy–Littlewood maximal operator \( M \) is bounded (see, for instance, [95, Theorem 4.1], [22, Theorem 3.1], and [52, Lemma 5.7]). By this and [93, p. 86], we find that \( X^{1/p} = M^\alpha_p(\mathbb{R}^n) \) is a ball Banach function space and \( M \) bounded on its associate space \( (X^{1/p})' \).

Thus, all the assumptions of Theorem 4.5(i) are satisfied for both \( X := M^\alpha_p(\mathbb{R}^n) \) with \( 1 < r \leq \alpha < \infty \) and \( p \in [1, r) \).

By Theorem 4.5(i) with \( X \) replaced by \( M^\alpha_p(\mathbb{R}^n) \), we find that both (5.1) and (5.2) hold true. This finishes the proof of the theorem in this case.

Case 2) \( r = 1 \). In this case, we then have \( n(1 - \frac{1}{q}) < 1 \). By the proof of [22, Theorem 3.1] (see also [94]), we find that, when \( 1 \leq r \leq \alpha < \infty \) and \( \theta \in (0, 1) \), for any \( f \in (M^\alpha_{r/\theta}(\mathbb{R}^n))' \),

\[
\| Mf \|_{(M^\alpha_{r/\theta}(\mathbb{R}^n))'} \leq \frac{r}{\theta} \| f \|_{(M^\alpha_{r/\theta}(\mathbb{R}^n))'},
\]

where the implicit positive constant depends only on \( n \). Thus, all the assumptions of Theorem 4.10(i) are satisfied for \( X := M^\alpha_1(\mathbb{R}^n) \). In this case, by Theorem 4.10(i) with \( X := M^\alpha_1(\mathbb{R}^n) \), we find that both (5.1) and (5.2) hold true. This then finishes the proof of this case and hence Theorem 5.1. \( \square \)

**Remark 5.2**

(i) Let \( r \in [1, \infty) \) and \( q = \alpha = r \). In this case, Theorem 5.1 is just [17, Theorem 1.1].

(ii) Let \( 1 \leq r \leq \alpha < \infty \). Since the Morrey space does not have an absolutely continuous norm, it is still unclear whether or not (5.1) holds true for any \( f \in W^{1,X}(\mathbb{R}^n) \) with \( X := M^\alpha_{r, \infty}(\mathbb{R}^n) \).

Using Corollaries 4.13 and 4.15, we obtain the following conclusions.
Corollary 5.3 Let $1 \leq r \leq \alpha < \infty$, $q_1 \in [1, \infty)$, and $\theta \in (0, 1)$. Let $q \in [1, q_1]$ satisfy $\frac{1}{q} = \frac{1-\theta}{q_1} + \theta$.

(i) If $q_1 \in [1, \infty)$, then there exists a positive constant $C$ such that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_2(\mathbb{R}^n)$,

\[
\sup_{\lambda \in (0, \infty)} \frac{1}{|B|} \left( \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^\frac{n}{q_1} + \theta \right\} \right)^r dx \leq C \| f \|_{M^q_{1,q_1}^{\alpha}} \| \nabla f \|_{M^q_{1,q_1}^{\alpha}}^{\theta}.
\]

(ii) If $q_1 = \infty$, then there exists a positive constant $C$ such that, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_2(\mathbb{R}^n)$,

\[
\sup_{\lambda \in (0, \infty)} \frac{1}{|B|} \left( \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^\frac{n}{q_1} + \theta \right\} \right)^r \theta dx \leq C \| f \|_{M^q_{1,q_1}^{\alpha}} \| \nabla f \|_{M^q_{1,q_1}^{\alpha}}^{\theta}.
\]

Corollary 5.4 Let $1 \leq r \leq \alpha < \infty$, $s_1 \in (0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, r)$ satisfy $s = (1 - \theta)s_1 + \theta$ and $\frac{1}{q} = \frac{1-\theta}{q_1} + \theta$. Then, for any $f \in C^2(\mathbb{R}^n)$ with $|\nabla f| \in C_2(\mathbb{R}^n)$, (4.43) holds true with $X$ replaced by $M^q_{s,q_1}^{\alpha}$.

Remark 5.5 (i) We point out that the Gagliardo–Nirenberg type inequality in the Sobolev–Morrey space related to the Riesz potential was established by Sawano et al. in [96].

The Gagliardo–Nirenberg type inequalities in the Sobolev–Morrey spaces, as given in Corollaries 5.3 and 5.4, appear new.

(ii) Let $1 \leq r < \alpha < \infty$. Since the Morrey space does not have an absolutely continuous norm, it is still unclear whether or not Corollaries 5.3 and 5.4 hold true for any $f \in \dot{W}^{1,X}(\mathbb{R}^n)$ with $X := M^q_{s,q_1}^{\alpha}$.

5.2 Mixed-norm Lebesgue spaces

For a given vector $r := (r_1, \ldots, r_n) \in (0, \infty]^n$, the mixed-norm Lebesgue space $L^r(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ with the finite quasi-norm

\[
\| f \|_{L^r(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \cdots \left[ \int_{\mathbb{R}} \cdots \left[ \int_{\mathbb{R}} |f(x_1, \ldots, x_n)| \right]^{\frac{1}{r_1}} dx_1 \cdots dx_n \right]^{\frac{1}{r_n}} \right\}.
\]

where the usual modifications are made when $r_i = \infty$ for some $i \in \{1, \ldots, n\}$. In the remainder of this subsection, let $r_- := \min\{r_1, \ldots, r_n\}$. The study of mixed-norm Lebesgue spaces can be traced back to Hörmander [56] and Benedek and Panzone [5]. For more studies on mixed-norm Lebesgue spaces, we refer the reader to [24, 59–61] for the Hardy space associated with mixed-norm Lebesgue spaces, to [25, 41, 42] for the Triebel and the Besov spaces associated with mixed-norm Lebesgue spaces, to [24, 26] for the (anisotropic) mixed-norm Lebesgue space, and to [85, 86] for the mixed Morrey space.

From the definition of $L^r(\mathbb{R}^n)$, we easily deduce that $L^r(\mathbb{R}^n)$, where $r \in (0, \infty)^n$, is a ball quasi-Banach function space. But, $L^r(\mathbb{R}^n)$ may not be a quasi-Banach function space (see, for instance, [108, Remark 7.20]). When $X := L^r(\mathbb{R}^n)$, we denote $\dot{W}^{1,X}(\mathbb{R}^n)$ simply by $\dot{W}^{1,r}(\mathbb{R}^n)$. Using Theorem 4.5(ii), we obtain the following conclusions.
Theorem 5.6 Let \( r := (r_1, \ldots, r_n) \in (1, \infty)^n \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{r_\cdot} - \frac{1}{q}) < 1 \). Then, for any \( f \in \dot{W}^{1,r}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q+1} \right\} \right\|_{L^r(\mathbb{R}^n)} \sim \| \nabla f \|_{L^r(\mathbb{R}^n)},
\]

where the positive equivalence constants are independent of \( f \). Moreover, for any \( f \in \dot{W}^{1,r}(\mathbb{R}^n) \),

\[
\lim_{\lambda \to \infty} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q+1} \right\} \right\|_{L^r(\mathbb{R}^n)} = \left[ \frac{K(q, n)}{q} \right]^{\frac{1}{q}} \| \nabla f \|_{L^r(\mathbb{R}^n)}. \tag{5.3}
\]

**Proof** Since \( n(\frac{1}{r_\cdot} - \frac{1}{q}) < 1 \), it follows that there exists a \( p \in [1, r_\cdot) \) satisfying \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). Using [40, Lemma 4.1], we find that \( L^r(\mathbb{R}^n) \) has an absolutely continuous norm. By the definition of \( L^r(\mathbb{R}^n) \), we have

\[
\left[ L^r(\mathbb{R}^n) \right]^{\frac{1}{r}} = L^{\frac{r}{q}}(\mathbb{R}^n)
\]

and hence \( \left[ L^r(\mathbb{R}^n) \right]^{\frac{1}{r}} \) is a ball Banach function space (see, for instance, [59, Remark 2.8(iii)]). This, together with [5, Theorems 1 and 2], further implies that

\[
\left( \left[ L^r(\mathbb{R}^n) \right]^{\frac{1}{r}} \right)' = L^{\left( \frac{r}{q} \right)'}(\mathbb{R}^n),
\]

where \((\frac{r}{q})' := (r_1', \ldots, r_n')\) with \( \frac{p}{r_i} + \frac{1}{r_i'} = 1 \) for any \( i \in \{1, \ldots, n\} \). Moreover, from [59, Lemma 3.5], we infer that the Hardy–Littlewood maximal operator \( M \) is bounded on \( \left[ L^r(\mathbb{R}^n) \right]^{\frac{1}{r}} \). Thus, all the assumptions of Theorem 4.5(ii) are satisfied for \( X := L^r(\mathbb{R}^n) \). By Theorem 4.5(ii) with \( X := L^r(\mathbb{R}^n) \), we conclude that both (5.3) and (5.4) hold true for any \( f \in \dot{W}^{1,r}(\mathbb{R}^n) \). This finishes the proof of Theorem 5.6. \( \square \)

Using Corollaries 4.13 and 4.15 and Remarks 4.14(iv) and 4.16(ii), we obtain the following conclusions, the proofs of which are similar to that of Theorem 5.6; we omit the details here.

**Corollary 5.7** Let \( r := (r_1, \ldots, r_n) \in (1, \infty)^n \), \( q_1 \in [1, \infty] \), and \( \theta \in (0, 1) \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \).

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in \dot{W}^{1,r}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q+1} \right\} \right\|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^{1,\theta}r(\mathbb{R}^n)} \| \nabla f \|_{L^r(\mathbb{R}^n)}^\theta.
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in \dot{W}^{1,r}(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q+1} \right\} \right\|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^{1,\infty}r(\mathbb{R}^n)} \| \nabla f \|_{L^r(\mathbb{R}^n)}^\theta.
\]
Corollary 5.8 Let \( r := (r_1, \ldots, r_n) \in (1, \infty)^n, s_1 \in (0, 1), q_1 \in (1, \infty), \) and \( \theta \in (0, 1). \) Let \( s \in (s_1, 1) \) and \( q \in (1, r_-) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1 - \theta}{q_1} + \theta. \) Then, for any \( f \in \tilde{W}^1 \cdot r(\mathbb{R}^n), \) (4.43) holds true with \( X \) replaced by \( L^r(\mathbb{R}^n). \)

Remark 5.9 (i) To the best of our knowledge, the Gagliardo–Nirenberg type inequalities of Corollaries 5.7 and 5.8 on the mixed-norm Sobolev space are completely new. (ii) In both Theorem 5.6 and Corollaries 5.7 and 5.8, the reason why the case \( r_- = 1 \) was excluded is that it is still unclear whether or not (4.16) holds true with \( L^r(\mathbb{R}^n) \) in this case.

5.3 Variable Lebesgue spaces

Let \( r : \mathbb{R}^n \rightarrow (0, \infty) \) be a nonnegative measurable function. Let
\[
\tilde{r}_- := \text{ess inf}_{x \in \mathbb{R}^n} r(x) \quad \text{and} \quad \tilde{r}_+ := \text{ess sup}_{x \in \mathbb{R}^n} r(x).
\]

A function \( r : \mathbb{R}^n \rightarrow (0, \infty) \) is said to be globally log-Hölder continuous if there exists an \( r_\infty \in \mathbb{R} \) and a positive constant \( C \) such that, for any \( x, y \in \mathbb{R}^n, \)
\[
|r(x) - r(y)| \leq \frac{C}{\log(e + 1/|x - y|)} \quad \text{and} \quad |r(x) - r_\infty| \leq \frac{C}{\log(e + |x|)}.
\]

The variable Lebesgue space \( L^{r(\cdot)}(\mathbb{R}^n) \) associated with the function \( r : \mathbb{R}^n \rightarrow (0, \infty) \) is defined to be the set of all the measurable functions \( f \) on \( \mathbb{R}^n \) with the finite quasi-norm
\[
\|f\|_{L^{r(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[ \frac{|f(x)|}{\lambda} \right]^{r(x)} \, dx \leq 1 \right\}.
\]

It is known that \( L^{r(\cdot)}(\mathbb{R}^n) \) is a ball quasi-Banach function space (see, for instance, [93, Sect. 7.8]). In particular, if \( 1 < \tilde{r}_- \leq \tilde{r}_+ < \infty, \) then \( (L^{r(\cdot)}(\mathbb{R}^n), \| \cdot \|_{L^{r(\cdot)}(\mathbb{R}^n)}) \) is a Banach function space in the terminology of Bennett and Sharpley [6] and hence also a ball Banach function space. For related results on variable Lebesgue spaces, we refer the reader to [27, 29, 34, 70, 78, 81, 82]. We first show that the average operators \( \{B_s\}_{s \in (0, \infty)} \) are uniformly bounded on \( L^{r(\cdot)}(\mathbb{R}^n). \)

Lemma 5.10 Let \( r : \mathbb{R}^n \rightarrow (0, \infty) \) be globally log-Hölder continuous and \( 1 \leq \tilde{r}_- \leq \tilde{r}_+ < \infty. \) Then there exists a positive constant \( C \) such that, for any \( s \in (0, \infty) \) and \( f \in L^{r(\cdot)}(\mathbb{R}^n), \)
\[
\|B_s(f)\|_{L^{r(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{r(\cdot)}(\mathbb{R}^n)}.
\]

Proof Let \( m \in \mathbb{N} \) satisfy that \( \int_{\mathbb{R}^n} (e + |y|)^{-m} \, dy \leq 1. \) Since \( r(\cdot) \) is globally log-Hölder continuous and \( 1 \leq \tilde{r}_- \leq \tilde{r}_+ < \infty, \) it follows that \( \frac{1}{r(x)} \) is also globally log-Hölder continuous (see, for instance, [27, Proposition 2.3(5)]). From this and [33, Theorem 4.2.4], we deduce that there exists a \( \beta \in (0, 1) \), depending only on both \( m \) and \( r(\cdot), \) such that, for any \( s \in (0, \infty), \)
\[
\int_{\mathbb{R}^n} (e + |y|)^{-m} \, dy \leq 1, \quad \text{and} \quad x \in \mathbb{R}^n,
\]
\[
|\beta B_s(f)(x)|^{r(x)} = \left[ \beta \int_{B(x, s)} |f(y)| \, dy \right]^{r(x)} \leq \int_{B(x, s)} |f(y)|^{r(y)} \, dy + 2^{-1} (e + |x|)^{-m}
\]
On the other hand, observe that, for any $r \in [1, \infty)$ and (5.6), further implies that, for any $s \in (0, \infty)$ and $f \in L^r(\mathbb{R}^n)$ with $\|f\|_{L^r(\mathbb{R}^n)} \leq 1$,

$$\int_{\mathbb{R}^n} |2^{-1} \beta B_s(f)(x)|^{r(x)} \, dx$$

and hence $\|B_s(f)\|_{L^r(\mathbb{R}^n)} \leq 2\beta^{-1}$. By this and a scaling argument, we conclude that (5.5) holds true with $C := 2\beta^{-1}$. This finishes the proof of Lemma 5.10. □

When $X := L_{\tilde{r}}(\mathbb{R}^n)$, we denote $\hat{W}^{1, X}(\mathbb{R}^n)$ simply by $\hat{W}^{1, r(\cdot)}(\mathbb{R}^n)$. Using both Theorems 4.5(ii) and 4.10(ii), we obtain the following conclusions.

**Theorem 5.11** Let $r : \mathbb{R}^n \to (0, \infty)$ be globally log-Hölder continuous. Assume that $1 \leq \tilde{r}_- \leq \tilde{r}_+ < \infty$ and $q \in (0, \infty)$ satisfy $n(\frac{1}{r_+} - \frac{1}{q}) < 1$. Then, for any $f \in \hat{W}^{1, r(\cdot)}(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{n}{q} + 1} \right\} \right\|_{L^r(\mathbb{R}^n)}$$

$$= \| |\nabla f| \|_{L^r(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of $f$. Moreover, for any $f \in \hat{W}^{1, r(\cdot)}(\mathbb{R}^n)$,

$$\lim_{\lambda \to \infty} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{n}{q} + 1} \right\} \right\|_{L^r(\mathbb{R}^n)}$$

$$= \left[ \frac{K(q, n)}{n} \right]^{\frac{1}{q}} \| |\nabla f| \|_{L^r(\mathbb{R}^n)}. \tag{5.8}$$

**Proof** We prove the present theorem by considering the following two cases on both $\tilde{r}_1$ and $\tilde{r}_+$. Case 1) $1 < \tilde{r}_- \leq \tilde{r}_+ < \infty$. In this case, since $n(\frac{1}{r_+} - \frac{1}{q}) < 1$, it follows that there exists a $p \in (1, \tilde{r}_-)$ such that $n(\frac{1}{p} - \frac{1}{q}) < 1$. By [27, p.73], we conclude that $L^{r(\cdot)}(\mathbb{R}^n)$ has an absolutely continuous norm. From the definition of $L^{r(\cdot)}(\mathbb{R}^n)$, we infer that

$$\left[ L^{r(\cdot)}(\mathbb{R}^n) \right]^\frac{1}{p} = L^{\frac{r(\cdot)}{p}}(\mathbb{R}^n).$$

Using this and the dual theorem on the variable Lebesgue space (see, for instance, [27, Theorem 2.80]), we have

$$\left( \left[ L^{r(\cdot)}(\mathbb{R}^n) \right]^\frac{1}{p} \right)' = L^{\frac{r(\cdot)}{p}}(\mathbb{R}^n).$$
where
\[
\frac{1}{r(x)} + \frac{1}{\rho(x)} = 1
\]
for any \( x \in \mathbb{R}^n \). This, together with [27, Theorem 2.80], further implies that the Hardy–Littlewood maximal operator \( \mathcal{M} \) is bounded on \( (L^r(\mathbb{R}^n))^{1/2} \). Thus, all the assumptions of Theorem 4.5(ii) are satisfied for \( X := L^r(\mathbb{R}^n) \) and hence, by Theorem 4.5(ii), we find that both (5.7) and (5.8) hold true for any \( f \in W^{1,r}(\mathbb{R}^n) \).

Corollary 5.12
Let \( r \in \mathbb{R}^n \) be globally log-Hölder continuous. Let \( n \in \mathbb{R}^n \) for any \( x \in \mathbb{R}^n \), where the implicit positive constant depends only on both \( n \) and \( r(\cdot) \). Thus, (4.16) holds true for any \( f \in L^r(\mathbb{R}^n) \). From this and Lemma 5.10, we deduce that all the assumptions of Theorem 4.10(ii) are satisfied for \( X := L^r(\mathbb{R}^n) \). Applying Theorem 4.10(ii) with \( X := L^r(\mathbb{R}^n) \), we conclude that both (5.7) and (5.8) hold true for any \( f \in W^{1,r}(\mathbb{R}^n) \). This finishes the proof of Theorem 5.11.

Using Corollaries 4.13 and 4.15, we obtain the following conclusions, the proofs of which are similar to that of Theorem 5.11; we omit the details here.

Corollary 5.12
Let \( r : \mathbb{R}^n \to (0, \infty) \) be globally log-Hölder continuous. Let \( 1 \leq r_0 \leq r_+ < \infty \), \( q_1 \in [1, \infty] \), and \( \theta \in (0, 1) \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1}{q_1} + \theta \).

(i) If \( q_1 \in (1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in W^{1,r}(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f| - f(y) > \lambda \right\} \right\|_{L^1(\mathbb{R}^n)}^{1/q} \leq C \| f \|_{L^{1,q}(\mathbb{R}^n)} \| \nabla f \|_{L^r(\mathbb{R}^n)}^{\theta}.
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in W^{1,r}(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f| - f(y) > \lambda \right\} \right\|_{L^1(\mathbb{R}^n)}^{1/q} \leq C \| f \|_{L^{\infty}(\mathbb{R}^n)} \| \nabla f \|_{L^r(\mathbb{R}^n)}^{\theta}.
\]

Corollary 5.13
Let \( r : \mathbb{R}^n \to (0, \infty) \) be globally log-Hölder continuous. Let \( 1 \leq r_0 \leq r_+ < \infty \), \( s_1 \in (0, 1) \), \( q_1 \in (1, \infty) \), and \( \theta \in (0, 1) \). Let \( s \in (s_1, 1) \) and \( q \in (1, r_-) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1 - \theta}{q_1} + \theta \). Then, for any \( f \in W^{1,r}(\mathbb{R}^n) \), (4.43) holds true with \( X \) replaced by \( L^{r}(\mathbb{R}^n) \).

Remark 5.14
We point out that a different Gagliardo–Nirenberg type inequality was established in the variable Sobolev spaces related to the Riesz potential in [68, 76]. However, Corollaries 5.12 and 5.13 appear new.
5.4 Weighted Lebesgue spaces

Recall that, for any given $1 \leq r \leq \infty$ and any given weight $\omega$ on $\mathbb{R}^n$, $L^r_{\omega}(\mathbb{R}^n)$ denotes the weighted Lebesgue space with respect to the measure $\omega(x)\,dx$ on $\mathbb{R}^n$ (see Definition 3.2) and, when $X := L^r_{\omega}(\mathbb{R}^n)$, $\dot{W}^{1,r}(\mathbb{R}^n)$ is denoted simply by $\dot{W}^{1,r}_{\omega}(\mathbb{R}^n)$ (see, Definition 3.4). It is worth pointing out that a weighted Lebesgue space with an $A_{\infty}(\mathbb{R}^n)$-weight may not be a Banach function space; see [93, Sect. 7.1]. Using Theorem 4.5(ii), we obtain the following conclusions.

**Theorem 5.15** Let $1 \leq p \leq r < \infty$, $\omega \in A_{r/p}(\mathbb{R}^n)$, and $q \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q}) < 1$. Then, for any $f \in \dot{W}^{1,r}_{\omega}(\mathbb{R}^n)$,

$$
\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left| \{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^{\frac{q}{r} + 1} \} \right|^\frac{r}{q} \omega(x) \, dx \right]^{1/r} \sim \| \nabla f \|_{L^p_{\omega}(\mathbb{R}^n)},
$$

(5.9)

where the positive equivalence constants are independent of $f$. Moreover, for any $f \in \dot{W}^{1,r}_{\omega}(\mathbb{R}^n)$,

$$
\lim_{\lambda \to \infty} \lambda \left[ \int_{\mathbb{R}^n} \left| \{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^{\frac{q}{r} + 1} \} \right|^\frac{r}{q} \omega(x) \, dx \right]^{1/r} = \left[ \frac{K(q,n)}{n} \right] \frac{1}{q} \| \nabla f \|_{L^p_{\omega}(\mathbb{R}^n)}.
$$

(5.10)

**Proof** By [88, Theorem 1.34], we find that $L^r_{\omega}(\mathbb{R}^n)$ has an absolutely continuous norm. Using the definition of $L^r_{\omega}(\mathbb{R}^n)$, we know that

$$
\left[ L^r_{\omega}(\mathbb{R}^n) \right]^{\frac{1}{r}} = L^r_{\omega}(\mathbb{R}^n).
$$

(5.11)

We consider two cases based on the size of $p$. If $p \in [1, r)$, then, from both (5.11) and [97, Lemma 4.2], we infer that

$$
\left[ L^r_{\omega}(\mathbb{R}^n) \right]^{\frac{1}{r}} = L^{(r/p)'}_{\omega^{1-(r/p)}}(\mathbb{R}^n).
$$

(5.12)

By both the assumption that $\omega \in A_{r/p}(\mathbb{R}^n)$ and [43, Proposition 7.1.5(4)], we conclude that

$$
\omega^{1-(r/p)'} \in A_{(r/p)'}(\mathbb{R}^n).
$$

Using this, (5.12), and Lemma 3.9, we find that the Hardy–Littlewood maximal operator $M$ is bounded on $([L^r_{\omega}(\mathbb{R}^n)]^{\frac{1}{r}})'$. Thus, in this case, all the assumptions of Theorem 4.5(ii) are satisfied for $X := L^r_{\omega}(\mathbb{R}^n)$.

If $p = r$, applying the conclusion in [62, p.9] and using the assumption that $\omega \in A_{r/p}(\mathbb{R}^n)$, we obtain

$$
\left( \left[ L^r_{\omega}(\mathbb{R}^n) \right]^{\frac{1}{r}} \right)' = L^\infty_{\omega^{-1}}(\mathbb{R}^n).
$$

This, combined with both [2, Theorem 3.1(b)] and [62, p.9], further implies that $M$ is bounded on $([L^r_{\omega}(\mathbb{R}^n)]^{1/r})'$. Thus, in this case, all the assumptions of Theorem 4.5(ii) are satisfied for $X := L^r_{\omega}(\mathbb{R}^n)$.

By Theorem 4.5(ii) with $X := L^r_{\omega}(\mathbb{R}^n)$, we conclude that both (5.10) and (5.9) hold true. This then finishes the proof of Theorem 5.15. \qed
Corollary 5.16 Assume that $p \in [1, \infty)$, $\omega \in A_p(\mathbb{R}^n)$, $q_1 \in [1, \infty]$, and $\theta \in (0, 1)$. Let $q \in [1, q_1]$ satisfy $1/q = 1/q_1 + \theta$.

(i) If $q_1 \in (1, \infty)$, then there exists a positive constant $C$ such that, for any $f \in \dot{W}^1_{\omega} \dot{p}^1(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left| y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{n+\theta} \right|^p \omega(x) \, dx \right]^{1/p} \leq C \| f \|_{L^{\omega q_1}(\mathbb{R}^n)} \| \nabla f \|_{L^{\omega q_1}(\mathbb{R}^n)}^{\theta}.$$

(ii) If $q_1 = \infty$, then there exists a positive constant $C$ such that, for any $f \in \dot{W}^1_{\omega} \dot{p}^1(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left| y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{n+\theta} \right|^p \omega(x) \, dx \right]^{1/p} \leq C \| f \|_{L^{\infty}(\mathbb{R}^n)} \| \nabla f \|_{L^{\infty}(\mathbb{R}^n)}^{\theta}.$$

Corollary 5.17 Let $s_1 \in (0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, q_1)$ satisfy $s = (1 - \theta)s_1 + \theta$ and $1/q = 1/q_1 + \theta$. Assume that $\omega \in A_p(\mathbb{R}^n)$. Then, for any $f \in \dot{W}^1_{\omega} \dot{p}^1(\mathbb{R}^n)$, (4.43) holds true with $X$ replaced by $L^{\omega q_1}(\mathbb{R}^n)$.

Remark 5.18 We point out that the Gagliardo–Nirenberg type inequality in the weighted Sobolev space related to the Riesz potential was obtained in [38, 84]. However, to the best of our knowledge, the Gagliardo–Nirenberg type inequalities of both Corollaries 5.16 and 5.17 on the weighted Sobolev space are completely new.

5.5 Orlicz spaces

First, we describe briefly some necessary concepts and facts on Orlicz spaces. A non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t \to \infty} \Phi(t) = \infty$. An Orlicz function $\Phi$ is said to be of lower (resp., upper) type $r$ for some $r \in \mathbb{R}$ if there exists a positive constant $C_r$ (resp., $C_{r'}$) such that, for any $t \in [0, \infty)$ and $s \in (0, 1)$ (resp., $s \in [1, \infty)$),

$$\Phi(st) \leq C_r s^r \Phi(t).$$

In the remainder of this subsection, we always assume that $\Phi : [0, \infty) \to [0, \infty)$ is an Orlicz function with positive lower type $r^-$ and positive upper type $r^+$. The Orlicz norm $\| f \|_{L^{\Phi}(\mathbb{R}^n)}$ of a measurable function $f$ on $\mathbb{R}^n$ is then defined by setting

$$\| f \|_{L^{\Phi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.$$

Accordingly, the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ with finite norm $\| f \|_{L^{\Phi}(\mathbb{R}^n)}$. It is easy to prove that the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is a quasi-Banach function space (see [93, Sect. 7.6]). For related results on Orlicz spaces, we refer the reader to [32, 79, 87]. We first prove the following lemma.
Lemma 5.19 Let $\Phi$ be an Orlicz function with positive lower type $r^-_\Phi \in [1, \infty)$ and positive upper type $r^+_\Phi$. Then there exists a positive constant $C$ such that, for any $r \in (0, \infty)$ and any $g \in L^\Phi(\mathbb{R}^n)$,

$$\|B_r(g)\|_{L^\Phi(\mathbb{R}^n)} \leq C \|g\|_{L^\Phi(\mathbb{R}^n)}, \quad (5.13)$$

where $B_r$ is the same as in (2.1).

Proof Let $r \in (0, \infty)$ and $g \in L^\Phi(\mathbb{R}^n)$. Since $\Phi$ is of lower type $r^-_\Phi \in [1, \infty)$, it follows that, for any $0 < t_1 < t_2 < \infty$,

$$\Phi(t_1) \leq C_{(r^-_\Phi)} \left( \frac{t_1}{t_2} \right)^{r^-_\Phi} \Phi(t_2) \leq C_{(r^-_\Phi)} \frac{t_1}{t_2} \Phi(t_2)$$

and hence

$$\frac{\Phi(t_1)}{t_1} \leq C_{(r^-_\Phi)} \frac{\Phi(t_2)}{t_2}. \quad (5.14)$$

From this and [67, Lemma 1.1.1], we deduce that, for any $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\Phi \left( \frac{B_r(g)(x)}{\lambda} \right) \lesssim \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi \left( \frac{|g(y)|}{\lambda} \right) dy,$$

where the implicit positive constant depends only on $\Phi$. This, together with the Tonelli theorem, further implies that, for any $r \in (0, \infty)$ and $\lambda \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \Phi \left( \frac{B_r(g)(x)}{\lambda} \right) dx \lesssim \int_{\mathbb{R}^n} \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi \left( \frac{|g(y)|}{\lambda} \right) dy \, dx$$

$$= \int_{\mathbb{R}^n} \Phi \left( \frac{|g(y)|}{\lambda} \right) \, dx$$

and hence

$$\|B_r(g)\|_{L^\Phi(\mathbb{R}^n)} \lesssim \|g\|_{L^\Phi(\mathbb{R}^n)}. \quad (5.15)$$

This finishes the proof of Lemma 5.19. \hfill \Box

When $X := L^\Phi(\mathbb{R}^n)$, we denote $\dot{W}^{1,X}(\mathbb{R}^n)$ simply by $\dot{W}^{1,\Phi}(\mathbb{R}^n)$. Using both Theorems 4.5(ii) and 4.10(ii), we obtain the following conclusions.

Theorem 5.20 Let $\Phi$ be an Orlicz function with positive lower type $r^-_\Phi$ and positive upper type $r^+_\Phi$. Let $1 \leq r^-_\Phi \leq r^+_\Phi < \infty$ and $q \in (0, \infty)$ satisfy $n \left( \frac{1}{r^-_\Phi} - \frac{1}{q} \right) < 1$. Then, for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^\frac{n}{q} \right\} \right\|_{L^\Phi(\mathbb{R}^n)} \sim \|\nabla f\|_{L^\Phi(\mathbb{R}^n)}, \quad (5.14)$$

where the positive equivalence constants are independent of $f$. Moreover, for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$,

$$\lim_{\lambda \to \infty} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^\frac{n}{q} \right\} \right\|_{L^\Phi(\mathbb{R}^n)} = \left[ \frac{K(q, n)}{n} \right]^{\frac{1}{q}} \|\nabla f\|_{L^\Phi(\mathbb{R}^n)}. \quad (5.15)$$
Proof We prove the present theorem by considering the following two cases on both $r_\Phi^-$ and $r_\Phi^+$.

Case 1) $1 < r_\Phi^- \leq r_\Phi^+ < \infty$. In this case, since $n\left(\frac{1}{r_\Phi^-} - \frac{1}{q}\right) < 1$, it follows that there exists a $p \in [1, r_\Phi^-)$ such that $n\left(\frac{1}{p} - \frac{1}{q}\right) < 1$. Let $\Phi_p(t) := \Phi(t\frac{1}{p})$ for any $t \in [0, \infty)$. By the definition of $L^\Phi(\mathbb{R}^n)$, we have

$$\left[L^\Phi(\mathbb{R}^n)\right]^{\frac{1}{p}} = L^{\Phi_p}(\mathbb{R}^n).$$

Moreover, using the proof of [109, Lemma 4.5], [67, Theorem 1.2.1], and the dual theorem of $L^\Phi(\mathbb{R}^n)$ (see, for instance, [87, Theorem 13]), we further conclude that, if $1 < r_\Phi^- \leq r_\Phi^+ < \infty$ and $p \in [1, r_\Phi^-)$, then $L^\Phi(\mathbb{R}^n)$ has an absolutely continuous norm and $\mathcal{M}$ is bounded on $([L^\Phi(\mathbb{R}^n)]^{\frac{1}{p}})'$. Thus, all the assumptions of Theorem 4.5(ii) are satisfied for $X := L^\Phi(\mathbb{R}^n)$ and hence, by Theorem 4.5(ii) with $X := L^\Phi(\mathbb{R}^n)$, we find that both (5.14) and (5.15) hold true.

Case 2) $1 = r_\Phi^- \leq r_\Phi^+ < \infty$. In this case, by the proof of [67, Theorem 1.2.1], we find that, when $1 = r_\Phi^- \leq r_\Phi^+ < \infty$ and $\theta \in (0, 1)$, for any $f \in ([L^\Phi(\mathbb{R}^n)]^{\frac{1}{p}})'$, 

$$\|Mf\|_{([L^\Phi(\mathbb{R}^n)]^{\frac{1}{p}})'} \lesssim (3C_{r_\Phi^+})^{-\frac{\theta}{\beta}} \|f\|_{([L^\Phi(\mathbb{R}^n)]^{\frac{1}{p}})'}$$

where the implicit positive constant depends only on $n$. Thus, (4.16) holds true for $X := L^\Phi(\mathbb{R}^n)$. From this and Lemma 5.19, we deduce that all the assumptions of Theorem 4.10(ii) are satisfied for $X := L^\Phi(\mathbb{R}^n)$. By Theorem 4.10(ii) for $X := L^\Phi(\mathbb{R}^n)$, we conclude that both (5.14) and (5.15) hold true for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$. This finishes the proof of Theorem 5.20. □

Using Corollaries 4.13 and 4.15, we obtain the following conclusions, the proofs of which are similar to that of Theorem 5.20; we omit the details here.

**Corollary 5.21** Assume that $\Phi$ is an Orlicz function with positive lower type $r_\Phi^-$ and positive upper type $r_\Phi^+$. Let $1 \leq r_\Phi^- \leq r_\Phi^+ < \infty$, $q_1 \in [1, \infty]$, and $\theta \in (0, 1)$. Let $q \in [1, q_1]$ satisfy $\frac{1}{q} = \frac{1-\theta}{q_1} + \theta$.

(i) If $q_1 \in [1, \infty)$, then there exists a positive constant $C$ such that, for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q_1 + \theta} \right\} \right\|_{L^\Phi(\mathbb{R}^n)}^{\frac{1}{q}} \lesssim C \|f\|_{L^{q_1,\Phi}(\mathbb{R}^n)} \|\nabla f\|_{L^{\Phi}(\mathbb{R}^n)}^{\theta}.$$

(ii) If $q_1 = \infty$, then there exists a positive constant $C$ such that, for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^\frac{q}{q_1 + \theta} \right\} \right\|_{L^\Phi(\mathbb{R}^n)}^{\frac{1}{q}} \lesssim C \|f\|_{L^{\infty,\Phi}(\mathbb{R}^n)} \|\nabla f\|_{L^{\Phi}(\mathbb{R}^n)}^{\theta}.$$

**Corollary 5.22** Let $\Phi$ be an Orlicz function with positive lower type $r_\Phi^-$ and positive upper type $r_\Phi^+$. Let $1 \leq r_\Phi^- \leq r_\Phi^+ < \infty$, $s_1 \in (0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, r_\Phi^-)$ satisfy $s = (1-\theta)s_1 + \theta$ and $\frac{1}{q} = \frac{1-\theta}{q_1} + \theta$. Then, for any $f \in \dot{W}^{1,\Phi}(\mathbb{R}^n)$, (4.43) holds true with $X$ replaced by $L^\Phi(\mathbb{R}^n)$.  

\[ \text{Springer} \]
Remark 5.23 We point out that the Gagliardo–Nirenberg type inequality in the Sobolev–Orlicz space related to the Riesz potential was obtained in [64, 65, 75]. However, to the best of our knowledge, the Gagliardo–Nirenberg type inequalities of both Corollaries 5.21 and 5.22 on the Sobolev–Orlicz space are completely new.

5.6 Orlicz-slice spaces

First, we give the definition of Orlicz-slice spaces and briefly describe some related facts. Throughout this subsection, we always assume that \( \Phi : [0, \infty) \to [0, \infty) \) is an Orlicz function with positive lower type \( r^- \) and positive upper type \( r^+ \). For any given \( t, r \in (0, \infty) \), the Orlicz-slice space \( (E_\Phi^r)_t(\mathbb{R}^n) \) is defined to be the set of all the measurable functions \( f \) on \( \mathbb{R}^n \) with the finite quasi-norm

\[
\| f \|_{(E_\Phi^r)_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[ \frac{\| f 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)}}{\| 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}}.
\]

The Orlicz-slice spaces were introduced in [109] as a generalization of both the slice space of Auscher and Mourgoglou [3, 4] and the Wiener amalgam space in [53, 55, 66]. According to both [109, Lemma 2.28] and [108, Remark 7.41(i)], the Orlicz-slice space \( (E_\Phi^r)_t(\mathbb{R}^n) \) is a ball Banach function space, but in general is not a Banach function space. We first prove the following lemma.

Lemma 5.24 Let \( t \in (0, \infty) \), \( r \in [1, \infty) \), and \( \Phi \) be an Orlicz function with positive lower type \( r^- \in [1, \infty) \) and positive upper type \( r^+ \). Then there exists a positive constant \( C \) such that, for any \( s \in (0, \infty) \) and \( g \in (E_\Phi^r)_t(\mathbb{R}^n) \),

\[
\| B_s(g) \|_{(E_\Phi^r)_t(\mathbb{R}^n)} \leq C \| g \|_{(E_\Phi^r)_t(\mathbb{R}^n)},
\]

where \( B_s \) is the same as in (2.1) with \( r \) replaced by \( s \).

Proof Let \( s \in (0, \infty) \) and \( g \in (E_\Phi^r)_t(\mathbb{R}^n) \). We consider the following two cases on both \( s \) and \( t \).

Case 1 \( s \in (0, t) \). In this case, by an argument similar to that used in the proof of (5.13), we conclude that, for any \( x \in \mathbb{R}^n \),

\[
\| B_s(g) 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)} \leq \| g 1_{B(x,2t)} \|_{L^\Phi(\mathbb{R}^n)}.
\]

From this and the proof of [109, Theorem 2.20], we infer that

\[
\| B_s(g) \|_{(E_\Phi^r)_t(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left[ \frac{\| B_s(g) 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)}}{\| 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}} \leq \left\{ \int_{\mathbb{R}^n} \left[ \frac{\| g 1_{B(x,2t)} \|_{L^\Phi(\mathbb{R}^n)}}{\| 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}} = \| g \|_{(E_\Phi^r)_t(\mathbb{R}^n)}.
\]

This proves (5.16) in this case.

Case 2 \( s \in [t, \infty) \). In this case, by the Fubini theorem, we have, for any \( x \in \mathbb{R}^n \),

\[
\| B_s(g) 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)} = \left\| \int_{B(s,t)} \int_{B(z,t)} |g(z)| \, d\xi \, dz 1_{B(x,t)} \right\|_{L^\Phi(\mathbb{R}^n)}.
\]

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We show the present theorem by considering the following two cases on both

\[ \int_{B(x,2s)} \int_{B(\xi,4t)} |g(z)| \, dz \, d\xi 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)} \]

\[ \int_{B(x,4s)} \int_{B(\xi,t)} |g(z)| \, dz \, d\xi 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)} \]

\[ = \int_{B(x,4s)} \int_{B(\xi,t)} |g(z)| \, dz \, d\xi 1_{B(x,t)} \|_{L^\Phi(\mathbb{R}^n)} . \]

Furthermore, by \[109, \text{Theorem 2.26 and Lemmas 4.4 and 4.5}\], we conclude that

\[ \text{Theorem 5.25} \]

Let \( t \in (0, \infty) \), \( r \in [1, \infty) \), and \( \Phi \) be an Orlicz function with positive lower type \( r^-_\Phi \) and positive upper type \( r^+_\Phi \). Let \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{\min\{r^-_\Phi, r\}} - \frac{1}{q}) < 1 \). Then, for any \( f \in \dot{W}^{1,\infty, \Phi}(\mathbb{R}^n) \),

\[ \sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda \cdot |x - y|^{\frac{n}{q} + 1} \right\} \right\|_{(E^{\Phi}_\lambda)(\mathbb{R}^n)} \]

\[ \sim \| \nabla f \|_{(E^{\Phi}_\lambda)(\mathbb{R}^n)} , \tag{5.17} \]

where the positive equivalence constants are independent of \( f \). Moreover, for any \( f \in \dot{W}^{1,\infty, \Phi}(\mathbb{R}^n) \),

\[ \sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda \cdot |x - y|^{\frac{n}{q} + 1} \right\} \right\|_{(E^{\Phi}_\lambda)(\mathbb{R}^n)} \]

\[ = \left[ \frac{K(q, n)}{n} \right]^{\frac{1}{q}} \| \nabla f \|_{(E^{\Phi}_\lambda)(\mathbb{R}^n)} . \tag{5.18} \]

**Proof** We show the present theorem by considering the following two cases on both \( r^-_\Phi \) and \( r^+_\Phi \).

Case 1) \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \). In this case, by both \[109, \text{Lemma 2.28}\] and \[108, \text{Remark 7.41(i)}\], we find that the Orlicz-slice space \( (E^{\Phi}_\lambda)(\mathbb{R}^n) \) is a ball Banach function space. Since \( n(\frac{1}{\min\{r^-_\Phi, r\}} - \frac{1}{q}) < 1 \), it follows that there exists a \( p \in (1, \min\{r^-_\Phi, r\}) \) such that \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). Using \[109, \text{Lemma 2.31}\], we find that

\[ \left( (E^{\Phi}_\lambda)(\mathbb{R}^n) \right)^{1/p} = (E^{\Phi/p}_\lambda)(\mathbb{R}^n) . \]

Furthermore, by \[109, \text{Theorem 2.26 and Lemmas 4.4 and 4.5}\], we conclude that \( (E^{\Phi}_\lambda)(\mathbb{R}^n) \)

has an absolutely continuous norm and that the Hardy–Littlewood maximal operator \( M \) is
bounded on \(((E^r_{\Phi})_t(\mathbb{R}^n))^{1/p}\)' . Thus, all the assumptions of Theorem 4.5(ii) are satisfied for 
\(X := (E^r_{\Phi})_t(\mathbb{R}^n)\) and hence, by Theorem 4.5(ii) with \(X := (E^r_{\Phi})_t(\mathbb{R}^n)\), we find that both 
(5.18) and (5.17) hold true for any \(f \in W^{1,(E^r_{\Phi})_t}(\mathbb{R}^n)\).

Case 2) \(1 = r^-_{\Phi} \leq r^+_{\Phi} < \infty\). In this case, by the proof of [109, Theorem 2.20], we conclude that, when 
\(1 = r^-_{\Phi} \leq r^+_{\Phi} < \infty, r \in (1, \infty), \) and \(\theta \in (0, 1)\), for any \(f \in ((E^{r^-_{\Phi}/\theta})_t(\mathbb{R}^n))'\),
\[\|Mf\|_{((E^{r^-_{\Phi}/\theta})_t(\mathbb{R}^n))'} \lesssim (3C_{r^+_{\Phi}})\frac{3r^+_{\Phi}}{\theta} + \frac{r}{\theta}\|f\|_{((E^{r^-_{\Phi}/\theta})_t(\mathbb{R}^n))'},\]
where the implicit positive constant depends only on \(n\). Thus, (4.16) holds true for \(X := (E^r_{\Phi})_t(\mathbb{R}^n)\). From this and Lemma 5.24, we deduce that all the assumptions of Theorem 
4.10(ii) are satisfied for \(X := (E^r_{\Phi})_t(\mathbb{R}^n)\). By Theorem 4.10(ii) with \(X := (E^r_{\Phi})_t(\mathbb{R}^n)\), we conclude that, for any \(f \in \hat{W}^{1,(E^r_{\Phi})_t}(\mathbb{R}^n)\), both (5.18) and (5.17) hold true. This finishes the proof of Theorem 5.25. \(\square\)

Using both Corollaries 4.13 and 4.15, we obtain the following conclusions, the proofs of which are similar to that of Theorem 5.25; we omit the details here.

**Corollary 5.26** Assume that \(t \in (0, \infty), r \in (1, \infty), \) and \(\Phi\) is an Orlicz function with positive lower type \(r^-_{\Phi}\) and positive upper type \(r^+_{\Phi}\). Let \(1 \leq r^-_{\Phi} \leq r^+_{\Phi} < \infty, q_1 \in [1, \infty], \) and \(\theta \in (0, 1)\). Let \(q \in [1, q_1]\) satisfy \(\frac{1}{q} = \frac{1-\theta}{q_1} + \theta\).

(i) If \(q_1 \in [1, \infty)\), then there exists a positive constant \(C\) such that, for any \(f \in \hat{W}^{1,(E^r_{\Phi})_t}(\mathbb{R}^n)\),
\[\sup_{\lambda \in (0, \infty)} \left(\int \left|y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^{\frac{3}{q}} + \theta\right|^\frac{1}{q}\right) \leq C\|f\|_{((E^r_{\Phi})_t(\mathbb{R}^n))}
\|\nabla f\|_{(E^r_{\Phi})_t(\mathbb{R}^n)}^{\theta}.\]

(ii) If \(q_1 = \infty\), then there exists a positive constant \(C\) such that, for any \(f \in \hat{W}^{1,(E^r_{\Phi})_t}(\mathbb{R}^n)\),
\[\sup_{\lambda \in (0, \infty)} \left(\int \left|y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda|x - y|^{\frac{3}{q}} + \theta\right|^\frac{1}{q}\right) \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \|\nabla f\|_{(E^r_{\Phi})_t(\mathbb{R}^n)}^{\theta}.\]

**Corollary 5.27** Let \(t \in (0, \infty), r \in (1, \infty), \) and \(\Phi\) be an Orlicz function with positive lower type \(r^-_{\Phi}\) and positive upper type \(r^+_{\Phi}\). Let \(1 \leq r^-_{\Phi} \leq r^+_{\Phi} < \infty, s_1 \in (0, 1), q_1 \in (1, \infty), \) and \(\theta \in (0, 1)\). Let \(s \in (s_1, 1)\) and \(q \in [1, \min\{r^-_{\Phi}, r\}]\) satisfy \(s = (1-\theta)s_1 + \theta\) and \(\frac{1}{q} = \frac{1-\theta}{q_1} + \theta\). Then, for any \(f \in \hat{W}^{1,(E^r_{\Phi})_t}(\mathbb{R}^n), (4.43)\) holds true with \(X\) replaced by \((E^r_{\Phi})_t(\mathbb{R}^n)\).

**Remark 5.28** To the best of our knowledge, the Gagliardo–Nirenberg type inequalities of both Corollaries 5.26 and 5.27 on the Sobolev–Orlicz-slice space are completely new.

### 5.7 Weak Morrey spaces

We first recall the weak Morrey space, which was introduced in [92]. For any \(0 < r \leq \infty\), the weak Morrey space \(M^r_{\infty}(\mathbb{R}^n)\) is defined to be the set of all the measurable functions \(f\) on \(\mathbb{R}^n\) with the finite semi-norm
\[\|f\|_{M^r_{\infty}(\mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \lambda\|f\|_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}} M^r_\mathbb{R}(\mathbb{R}^n).\]
It is easy to show that, for any $0 < r \leq \alpha < \infty$, $M_{r,\infty}^{\alpha}(\mathbb{R}^n)$ is a ball quasi-Banach function space, but in general is not a quasi-Banach function space. For any given $1 < r \leq \alpha < \infty$, it was proved in [92, Theorem 2.5] that the predual space of $M_{r,\infty}^{\alpha}(\mathbb{R}^n)$ is the following block space $\mathcal{H}_{r,1}^{\alpha}(\mathbb{R}^n)$. Recall that, for any given $p \in [1, \infty)$, the Lorentz space $L^{p,1}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ having the following finite semi-norm

$$
\|f\|_{L^{p,1}(\mathbb{R}^n)} := p \int_0^\infty |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} d\lambda.
$$

**Definition 5.29** Let $1 < r \leq \alpha < \infty$, $\frac{1}{r} + \frac{1}{\alpha} = 1$, and $\frac{1}{\alpha} + \frac{1}{\alpha} = 1$.

(i) A measurable function $b$ is called an $(\alpha', r', 1)$-block if there exists a cube $Q$ such that

$$
supp(b) \subset Q \quad \text{and} \quad \|b\|_{L^{r',1}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{r'} - \frac{1}{r}}.
$$

(ii) The block space $\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f \in L^{\alpha'}(\mathbb{R}^n)$ for which there exists a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a sequence $\{b_j\}_{j \in \mathbb{N}}$ of $(\alpha', r', 1)$-blocks such that

$$
f = \sum_{j \in \mathbb{N}} \lambda_j b_j
$$

in $L^{\alpha}(\mathbb{R}^n)$ and $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$. For any $f \in \mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)$, let

$$
\|f\|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} := \inf \sum_{j \in \mathbb{N}} |\lambda_j|,
$$

where the infimum is taken over all the possible decompositions of $f$ in (5.20).

We first show that, for any given $1 < r \leq \alpha < \infty$, the Hardy–Littlewood maximal operator $M$ is bounded on $\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)$.

**Proposition 5.30** Let $1 < r \leq \alpha < \infty$. Then there exists a positive constant $C$ such that, for any $f \in \mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)$,

$$
\|M(f)\|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)}.
$$

To show Proposition 5.30, we need the following lemma.

**Lemma 5.31** Let $p \in (1, \infty)$. Then there exists a positive constant $C_{(n)}$, depending only on $n$, such that, for any $f \in L^{p,1}(\mathbb{R}^n)$,

$$
\|M(f)\|_{L^{p,1}(\mathbb{R}^n)} \leq C_{(n)} \frac{2^{\frac{1}{p}}}{1 - 2^{\frac{1}{p} - 1}} \|f\|_{L^{p,1}(\mathbb{R}^n)}.
$$

**Proof** Let $f \in L^{p,1}(\mathbb{R}^n)$. For any $s \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$
f_{1, s}(x) := \begin{cases} f(x) & \text{if } |f(x)| > s, \\ 0 & \text{if } |f(x)| \leq s. \end{cases}
$$

and $f_{2, s}(x) := f(x) - f_{1, s}(x)$. It is obvious that, for any $s \in (0, \infty)$,

$$\{x \in \mathbb{R}^n : |M(f)(x)| > 2s\}$$

is bounded on $\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)$.
\begin{align*}
\leq |\{ x \in \mathbb{R}^n : |M(f_{1,s})(x)| > s \}| + |\{ x \in \mathbb{R}^n : |M(f_{2,s})(x)| > s \}|.
\end{align*}

For any \( s \in (0, \infty) \), since \( \|M(f_{2,s})\|_{L^\infty(\mathbb{R}^n)} \leq \|f_{2,s}\|_{L^\infty(\mathbb{R}^n)} \leq s \), it follows that
\begin{align*}
|\{ x \in \mathbb{R}^n : |M(f_{2,s})(x)| > s \}| = 0.
\end{align*}

This, combined with the fact that \( M \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{1,\infty}(\mathbb{R}^n) \), implies that
\begin{align*}
\|M(f)\|_{L^{p,1}(\mathbb{R}^n)} &= 2p \int_0^\infty |\{ x \in \mathbb{R}^n : |M(f)| > s \}| \frac{1}{p} \, ds \\
&\leq 2p \int_0^\infty |\{ x \in \mathbb{R}^n : |M(f_{1,s})| > s \}| \frac{1}{p} \, ds \\
&\leq p \int_0^\infty s^{-\frac{1}{p}} \left[ \int_{\{ x \in \mathbb{R}^n : |f(y)| > s \}} |f(y)| \, dy \right] \frac{1}{p} \, ds \\
&= p \int_0^\infty s^{-\frac{1}{p}} \left[ \sum_{k=0}^{\infty} \int_{\{ x \in \mathbb{R}^n : 2^k s \leq |f(x)| < 2^{k+1} s \}} |f(y)| \, dy \right] \frac{1}{p} \, ds \\
&\leq p \sum_{k=0}^{\infty} \int_0^\infty s^{-\frac{1}{p}} \left[ \int_{\{ x \in \mathbb{R}^n : 2^k s \leq |f(x)| < 2^{k+1} s \}} |f(y)| \, dy \right] \frac{1}{p} \, ds \\
&\leq p \sum_{k=0}^{\infty} 2^{(k+1)\frac{1}{p}} \left| \{ x \in \mathbb{R}^n : |f(x)| > 2^k s \} \right| \frac{1}{p} \, ds \\
&= \frac{2^{\frac{1}{p}}}{1 - 2^{\frac{1}{p} - 1}} \|f\|_{L^{p,1}(\mathbb{R}^n)}.
\end{align*}

This finishes the proof of Lemma 5.31. \qed

**Proof of Proposition 5.30** We first show that, for any \((\alpha', r', 1)\)-block \( b \) associated with cube \( Q := Q(x_0, l(Q)) \) for some \( x_0 \in \mathbb{R}^n \) and \( l(Q) \in (0, \infty) \),
\begin{equation}
\|M(b)\|_{H_{r',1}^\infty(\mathbb{R}^n)} \lesssim 1,
\end{equation}
where the implicit positive constant depends only on \( n, \alpha, \) and \( r \). Let \( U_0 := B(x_0, 2^{-1} \sqrt{n}l(Q)) \) and \( U_k := B(x_0, 2^{k-1} \sqrt{n}l(Q)) \setminus B(x_0, 2^{k-2} \sqrt{n}l(Q)) \) for any \( k \in \mathbb{N} \). Then, for any \( x \in \mathbb{R}^n \),
\begin{equation*}
M(b)(x) = \sum_{k=0}^{\infty} 1_{U_k}(x) M(b)(x) =: \sum_{k=0}^{\infty} m_k(x).
\end{equation*}

When \( k \in \{0, 1\} \), by Lemma 5.31 and (5.19), we find that
\begin{equation}
\|m_k\|_{L^{r',1}(\mathbb{R}^n)} \leq \|M(b)\|_{L^{r',1}(\mathbb{R}^n)} \lesssim \|b\|_{L^{r',1}(\mathbb{R}^n)} \lesssim |Q|^{\frac{1}{2} - \frac{1}{p}}.
\end{equation}

Let \( k \in \mathbb{N} \cap [2, \infty) \). By [43, Exercise 1.4.1(b)], we find that, for any \( g \in L^{r',1}(\mathbb{R}^n) \) and \( h \in L^{r,\infty}(\mathbb{R}^n) \),
\begin{equation*}
\|gh\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^{r',1}(\mathbb{R}^n)} \|h\|_{L^{r,\infty}(\mathbb{R}^n)}.
\end{equation*}

This implies that, for any \( x \in \mathbb{R}^n \),
\begin{equation*}
m_k(x) = 1_{U_k}(x) M(b)(x) \lesssim 1_{U_k}(x) \frac{1}{(2^k l(Q))^n} \int_Q b(x) \, dx
\end{equation*}
\[
\lesssim 1_u (x) \frac{1}{(2^k l(Q))^n} \|b\|_{L^r,1(\mathbb{R}^n)} \|1_Q\|_{L^{r,\infty}(\mathbb{R}^n)}
\]

and hence
\[
\| m_k \|_{L^r,1(\mathbb{R}^n)} \lesssim 2^{-kn} [l(Q)]^\frac{n}{2} - n \| 1_u (x) \|_{L^r,1(\mathbb{R}^n)}
\]

\[
\lesssim 2^{-\frac{kn}{2}} [2^k \sqrt{n} l(Q)]^\frac{n}{2} - \frac{n}{2} .
\] (5.23)

For any \( k \in \mathbb{N} \cap [2, \infty) \), let \( \sigma_0 := 1 =: \sigma_1, \sigma_k := 2^{-\frac{kn}{2}}, \tilde{m}_0 := \sigma_0^{-1} m_0, \tilde{m}_1 := \sigma_1^{-1} m_1 \), and \( \tilde{m}_k := \sigma_k^{-1} m_k \). By (5.22) and (5.23), we find that, for any \( k \in \mathbb{N} \), \( \tilde{m}_k \) is an \((\alpha', r', 1)\)-block associated with cube \( 2^k \sqrt{n} Q \) multiplied by a positive constant depending only on \( n, \alpha \), and \( r \). Moreover, from [92, Remark 2.4], we infer that
\[
M(b) = \sum_{k=0}^{\infty} \sigma_k \tilde{m}_k
\]
in \( L^\alpha(\mathbb{R}^n) \). Thus, \( M(b) \in \mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n) \) and
\[
\| M(b) \|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} \sigma_k \sim 1,
\]
where the implicit positive constant depends only on \( n, \alpha \), and \( r \). This proves (5.21).

Let \( f \in \mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n) \). Then there exists a sequence \( \{b_j\}_{j \in \mathbb{N}} \) of \((\alpha', r', 1)\)-blocks such that
\[
f = \sum_{j \in \mathbb{N}} \lambda_j b_j
\]
in \( L^\alpha(\mathbb{R}^n) \) and
\[
\| f \|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} \sim \sum_{j \in \mathbb{N}} |\lambda_j|.
\]
From this and (5.21), it follows that
\[
\| M(f) \|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{N}} |\lambda_j| \| M(b_j) \|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)} \lesssim \sum_{j \in \mathbb{N}} |\lambda_j| \sim \| f \|_{\mathcal{H}_{r',1}^{\alpha'}(\mathbb{R}^n)}.
\]
This finishes the proof of Proposition 5.30.

Using Theorems 4.12 and 4.12, we obtain the following conclusion.

**Theorem 5.32** Let \( 1 < r \leq \alpha < \infty \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{r} - \frac{1}{q}) < 1 \). Then, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^\frac{n+1}{q} \right\} \right\|_{M^{p,\infty}_{r,\infty}(\mathbb{R}^n)} \\
\sim \| \nabla f \|_{M^{p,\infty}_{r,\infty}(\mathbb{R}^n)},
\]
(5.24)
where the positive equivalence constants are independent of \( f \).

**Proof** Since \( n(\frac{1}{r} - \frac{1}{q}) < 1 \), it follows that there exists a \( p \in [1, r) \) satisfying \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). By [92, Theorem 2.5], we find that (4.27) holds true for both \( H \in \mathcal{H}(\alpha/p)' \) and \( q \in \mathcal{H}(\alpha/p)'1(\mathbb{R}^n) \). Moreover, by Proposition 5.30, we conclude that \( \mathcal{M} \) is bounded on \( Y \). Thus, all the assumptions of Theorem 4.12 are satisfied for \( X := M^{\alpha}_{r,\infty}(\mathbb{R}^n) \) and \( Y := \mathcal{H}(\alpha/p)'1(\mathbb{R}^n) \). By Theorem 4.12, we find that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \), (5.24) holds true. This finishes the proof of Theorem 5.32. \( \square \)

**Remark 5.33** (i) Let \( \alpha \in [1, \infty) \). In Theorem 5.32, the reason why the case \( r = 1 \) was excluded is that it is still unclear whether or not (4.16) holds true with \( X := M^{\alpha}_{1,\infty}(\mathbb{R}^n) \).

(ii) Let \( 1 < r \leq \alpha < \infty \). Since the weak Morrey space does not have an absolutely continuous norm, it is still unclear whether or not (5.24) holds true for any \( f \in \mathcal{W}^{1,X}(\mathbb{R}^n) \) with \( X := M^{\alpha}_{r,\infty}(\mathbb{R}^n) \).

Using Corollaries 4.13 and 4.15 and Remarks 4.14(v) and 4.16(iii), we obtain the following conclusions, the proofs of which are similar to that of Theorem 5.32; we omit the details here.

**Corollary 5.34** Let \( 1 < r \leq \alpha < \infty \), \( q_1 \in [1, \infty] \), and \( \theta \in (0, 1) \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \).

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^\frac{n+1}{q} \right\} \right\|_{M^{p,\infty}_{r,\infty}(\mathbb{R}^n)} \\
\leq C \| f \|_{M^{\frac{\alpha-\theta}{q_1},\infty}_{r,\infty}(\mathbb{R}^n)} \| \nabla f \|_{M^{\frac{\alpha}{q_1},\infty}_{r,\infty}(\mathbb{R}^n)}.\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^\frac{n+1}{q} \right\} \right\|_{M^{\frac{\alpha-\theta}{q_1},\infty}_{r,\infty}(\mathbb{R}^n)} \\
\leq C \| f \|_{L^\infty(\mathbb{R}^n)} \| \nabla f \|_{M^{\frac{\alpha}{q_1},\infty}_{r,\infty}(\mathbb{R}^n)}.\]

**Corollary 5.35** Let \( 1 < r \leq \alpha < \infty \), \( s_1 \in (0, 1) \), \( q_1 \in (1, \infty) \), and \( \theta \in (0, 1) \). Let \( s \in (s_1, 1) \) and \( q \in (1, r) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \). Then, for any \( f \in C^2(\mathbb{R}^n) \) with \( |\nabla f| \in C_c(\mathbb{R}^n) \), (4.43) holds true with \( X \) replaced by \( M^{\alpha}_{r,\infty}(\mathbb{R}^n) \).

**Remark 5.36** (i) To the best of our knowledge, the Gagliardo–Nirenberg type inequalities of both Corollaries 5.34 and 5.35 on the Sobolev–weak Morrey space are completely new.

(ii) Let \( 1 < r \leq \alpha < \infty \). Since the weak Morrey space does not have an absolutely continuous norm, it is still unclear whether or not Corollaries 5.34 and 5.35 hold true for any \( f \in \mathcal{W}^{1,X}(\mathbb{R}^n) \) with \( X := M^{\alpha}_{r,\infty}(\mathbb{R}^n) \).
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Declarations

Conflict of interest  All authors state no conflict of interest.

Informed consent  Informed consent has been obtained from all individuals included in this research work.

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