Interaction of Vortices with an External Field

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Abstract

The interaction of a magnetic flux vortex with weak external fields is considered in the framework of the Abelian Higgs model. The approach is based on the calculation of the zero-mode excitation probability in the external field. The excitation of the field configuration is found perturbatively. As an example we consider the effect of interaction with an external current. The linear in the scalar field perturbation is also considered.

Introduction

As is known, the Abelian Higgs model in 2+1 dimensions possesses a finite-energy static topological nontrivial solution of the vortex type [1]. In the nonrelativistic limit the model has the form of the Ginzburg-Landau theory, which phenomenologically describes the magnetic flux vortices in the type II superconductor [2]. On the other hand, the extrapolation of the vortex solution of the Abelian Higgs model to 3+1 dimensions gives $U(1)$ string configuration, which could be produced as a topological defect at the early Universe [3]. Very recently a new class of closed vortex ring solutions was discovered in 3+1 dimensional $SU(2)$ Yang-Mills-Higgs theory [4]. Thus the strings or vortices have played many interesting roles in the interplay between high energy physics, condensed matter and cosmology.

The dynamical properties of vortices in the Abelian Higgs model and some of its modifications have been considered from different points of view. The much studied question seems to be a problem of interaction between strings.
The general proof of the existence of multivortex configurations was constructed in [5, 6] and a method for obtaining asymptotic multisoliton solutions in gauge theories was given in [7]. There is a nice description of vortices dynamics based on the moduli space technique [8, 9]. In this scheme low energy soliton dynamics is approximated by geodesic motion on the space of the collective coordinates of static multivortices configuration with respect to the metric induced by functional of the kinetic energy. Since no exact static multivortices solutions are known, some numerical calculations have been performed. The numerical investigation of the interaction between well-separated vortices was carried out by Shellard and Ruback [10]. An analytical study of the interaction between vortices was done recently in [11].

The general result of these works is that there are no long-range forces between static vortices. There is a difference from BPS multimonopole configuration where the repulsive and attractive long-range forces between monopoles exactly compensate, BPS monopoles experience no net interaction. That makes the problem of the string dynamics in an external field more complicated comparing to the case of monopoles. Actually the interaction of vortices with external fields was not much studied.

There is a regular perturbation scheme used to describe the motion of solitons under external force [12]. This approach is based on the calculation of the probability of the excitation of corresponding translations zero modes in an external field. In our previous publication [12] we discussed the application to the problem of motion of 1+1 dimensional kink in $\phi^4$ model and to the case of the interaction between the 't Hooft-Polyakov monopole and an external weak field.

In the present note we would like to apply this formalism to the problem of interaction between 2+1 dimensional vortex solution in the Abelian Higgs model and an external field.

\section{Current-current interaction}

The Lagrangian of the Abelian Higgs model in (2+1) dimensional space-time is given (in the units $c = 1$, $\hbar = 1$):

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^*(D^\mu \phi) + \frac{\lambda}{4} \left(|\phi|^2 - v^2\right)^2 ,$$

where a $U(1)$ real gauge potential $A_\mu = (A_0, A_k)$, $k = 1, 2$, is coupled to a charged complex scalar field $\phi = \phi_1 + i\phi_2$. Here $e$ is the gauge coupling constant, $\lambda$ is the scalar
field self-coupling constant, \( v \) is the vacuum expectation value of the modulus of the scalar field and

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu;
\]

\[
D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi, \quad \text{or} \quad D_\mu \phi^a = \partial_\mu \phi^a - e \varepsilon_{ab} A_\mu \phi^b, \quad a = 1, 2
\]

are the field strength tensor and the covariant derivative correspondingly. Hereafter, the space indices are \( m, n, k \ldots = 1, 2 \) and the real components of the complex Higgs field are labeled by the indices \( a, b, c \ldots = 1, 2 \).

This model is a relativistic analog of the Landau-Ginzburg theory of superconductivity. The finiteness of the energy determines the asymptotic form of the fields. There are nontrivial static vortex solutions of this model depending on the polar coordinates \( r \) and \( \varphi \) \cite{1, 2} which are described by the Nielsen-Olesen ansatz:

\[
A_0 = A_r = 0; \quad A_\varphi = \frac{n}{er} (1 - K(r)); \quad \phi = v H(r) e^{in\varphi}, \quad \text{where} \quad n \in \mathbb{Z}.
\]

Here the boundary conditions on the structure functions \( K(r) \), \( H(r) \) are:

\[
K(r) \underset{r \to \infty}{\to} 0, \quad H(r) \underset{r \to \infty}{\to} 1;
\]

\[
K(r) \underset{r \to 0}{\to} 1, \quad H(r) \underset{r \to 0}{\to} 0
\]

and the only nonzero component of the field strength tensor is \( F_{xy} = -(n/er)K'(r) = B_z \).

Let us consider the interaction of this configuration with an external field. Note that because both vector and scalar fields are coupled in the vortex configuration, there are a few different ways to introduce such an interaction. For example, one could add to the Lagrangian (1) an extra current-current perturbation term that is linear in external current \( J_m \) as well as in current of scalar field

\[
L^{(v)}_{int} = j_m J_m \equiv \frac{ie}{2} [\phi^*(D_m \phi) - (D_m \phi)^* \phi] J_m.
\]

As a result the classical field equations take the form

\[
\partial_m F_{mn} = \frac{ie}{2} [\phi^*(D_n \phi) - (D_n \phi)^* \phi] + \mathcal{F}^{(1)}_n;
\]

\[
D_m D_m \phi^a = \lambda (\phi^b \phi^b - v^2) \phi^a + \mathcal{F}^{(2)}_a
\]

where the last terms represents the external force acting on the configuration. They read
\[ F_n^{(1)} = e^2 |\phi|^2 J_n; \quad F_n^{(2)} = 2e \varepsilon_{ab} D_n \phi^b J_n \] (6)

The effect of such a perturbation is that there are corrections to the static vortex solution (3). In an analogy with the case of two-dimensional \( \lambda \phi^4 \) model [12] these corrections could be expanded in powers of external perturbation:

\[ A_m = (A_m)_0 + a_m + \ldots; \quad \phi^a = (\phi^a)_0 + \chi^a + \ldots \]

where \((A_m)_0, (\phi^a)_0\) correspond to the classical \( n\)-vortices solution given by the zeroth-order approximation (3) and the fluctuations of the vector and scalar fields on this background are of the same order as perturbation \( J_m \).

To the first order corrections, they can be found from the equations describing the fluctuations of vector and scalar fields:

\[
\begin{align*}
\left(-\frac{d^2}{dt^2} + \partial_m \partial_m - e^2 \phi^a \phi^a\right) a_n &= -2e\varepsilon_{ab} D_n \phi^a \chi^b + F_n^{(1)}; \\
\left(-\frac{d^2}{dt^2} + D_m D_m\right) \chi^a &= 2e\varepsilon_{ab} D_m \phi^b a_m + F_n^{(2)},
\end{align*}
\]

(7)

where we have decomposed the complex field via \( \chi = \chi_1 + i\chi_2 \), and the background field gauge \( \partial_m a_m = ie (\phi^* \chi - \chi^* \phi) \) is used.

Now one can apply the same approach that was already used in [12], i.e. write the expansion of the fields \( a_m(\mathbf{r}, t), \chi^a(\mathbf{r}, t) \) on the eigenmodes of the matrix \( D^2 \) of second functional derivatives of the action with respect to the fields \( A_m, \phi^a \):

\[
D^2 \begin{pmatrix} a_n \\ \chi^a \end{pmatrix} \equiv \begin{pmatrix} (\partial_m^2 - e^2 \phi^a \phi^a) a_n & 2e\varepsilon_{ab} D_n \phi^a \chi^b \\ -2e\varepsilon_{ab} D_m \phi^b a_m & D_m D_m \chi^a \end{pmatrix}.
\]

In matrix notation the equation of motion (7) can be rewritten in the form

\[
\left(-\frac{d^2}{dt^2} + D^2\right) f = \mathcal{F},
\]

(8)

where

\[
f = \begin{pmatrix} a_n \\ \chi^a \end{pmatrix}; \quad \mathcal{F} = \begin{pmatrix} F_n^{(1)} \\ F_n^{(2)} \end{pmatrix}.
\]
We seek for the solution of Eq. (8) in the form of an expansion

\[ f(r, t) = \sum_{i=0}^{\infty} C_i(t) \zeta_i(r) \]  

(9)
on the complete set of eigenfunctions \( \zeta_i(r) \) of the operator \( D^2 \). These eigenfunctions consist of a vector and a scalar component: \( \zeta_i(r) = \begin{pmatrix} \eta_{mi}(r) \\ \eta_i^a(r) \end{pmatrix} \) describing the fluctuations of the corresponding fields on the vortex background [5], [8]. Thus, there are indices of two kind: index \( i \) is the number of a mode and the indices \( m, a \) correspond to its spatial and ‘isotopic’ components.

The substitution of expansion (9) into Eq.(7) results in the following system of equations for coefficients \( C_i(t) \):

\[ \sum_{i=0}^{\infty} \left( \ddot{C}_i + \Omega_i^2 C_i \right) \eta_{mi}(r) - 2e\varepsilon_{ab} \chi^b D_m \phi^a = F_m^{(1)}(r); \]

\[ \sum_{i=0}^{\infty} \left( \ddot{C}_i + \omega_i^2 C_i \right) \eta_i^a(r) - 2e\varepsilon_{ab} a_mD_m \phi^b = F^{(2)}_a(r), \]  

(10)

This is a system describing two sets of coupled forced oscillators.

It is known that among all fluctuations on the vortex background there are modes with zero energy (\( \Omega_0 = \omega_0 = 0 \)) [5]. Such modes are collective translation coordinates of the multivortices configuration. All the other solutions of the system (10) correspond to the oscillations on the classical background. Unfortunately neither analitical solution for the vortex structure functions nor the eigenfunctions \( \eta_{mi}(r), \eta_i^a(r) \) are known. Nevertheless, some information about the vortex dynamic can be obtained from the system (10).

Following the approach by Manton [13], one can find the acceleration of the vortex under perturbation \( J_m \) if the excitations of the zero modes are treated as a nontrivial time dependent translation of the n-vortex configuration (3).

However, the structure of the Lagrangian (1) suggests [14] that the normalizable zero modes are not only translations of the topologically non-trivial configurations but

\[ \zeta^{(k)}(r) \equiv \zeta_0(r) = \begin{pmatrix} \eta_m(r)^{(k)} \\ \eta^a(r)^{(k)} \end{pmatrix} \]

where

\[ \eta_n(r)^{(k)} = F_{kn} = \partial_k A_n - \partial_n A_k; \quad \eta^a(r)^{(k)} = D_k \phi^a = \partial_k \phi^a - e\varepsilon_{ab} \phi^b A_k. \]  

(11)
Here the index $k$ corresponds to the translation in the direction $\hat{r}_k$.

Similarly to the case of 3+1 dimensional Georgi-Glashow model, these normalizable zero modes coincide with the pure translational quasi-zero modes of the vector and scalar fields $\bar{\eta}_n(r) = \partial_k A_n$; $\bar{\eta}^a(r) = \partial_k \phi^a$ up to a gauge transformation with a special choice of the parameter which is just the gauge potential $A_k$ itself. These modes are normalized in such a way that makes $C_0$ in expansion (9) equal to the displacement of the vortex in the direction $\hat{r}_k$:

$$A_n(r + \delta r) \approx A_n(r) + \partial_k A_n(r) \delta x_k = A_n(r) + C_0(t) \bar{\eta}_n(r)^{(k)};$$

$$\phi^a(r + \delta r) \approx \phi^a(r) + \partial_k \phi^a(r) \delta x_k = \phi^a(r) + C_0(t) \bar{\eta}^a(r)^{(k)}$$

(12)

Now we can project the Eq.(10) onto the zero modes (11) which yield the equation:

$$\tilde{C}_0 \int d^2 x \left[ (\eta^a(r))^2 + (\eta_m(r))^2 \right] - 2 e \int d^2 x \varepsilon_{ab} \left\{ D_m \phi^a \chi^b \eta_m + a_m D_m \phi^b \eta^a \right\}$$

$$= \int d^2 x F_m(r) \eta_m(r)^{(k)} + \int d^2 x F^a(r) \eta^a(r)^{(k)}.$$  

(13)

The second term on the left-hand side of this equation describes the transitions between the vortex zero modes $\eta^a, \eta_m$ and all the other fluctuations $\eta^a_i, \eta_{mi}$ on the vortex background. A substitution of the expansion (9) gives

$$-2 e \int d^2 x \varepsilon_{ab} \left\{ D_m \phi^a \chi^b \eta_m + a_m D_m \phi^b \eta^a \right\}$$

$$= -2 e \int d^2 x \varepsilon_{ab} \sum_{i=0}^{\infty} C_i(t) \left\{ \eta^{a(m)} \eta^b \eta_m + \eta^a \eta^{b(m)} \eta_m \right\}$$

(14)

The first term in the sum (14) is equal to zero: the vortex configuration behave as a particle-like object and there is no effect of mutual transitions between the collective coordinates of scalar and vector fields. All the other terms ($i \neq 0$) describe the effect of a bremsstrahlung of both vector and scalar massive fields from a vortex accelerated by an external force. Such effects of radiation are suppressed as $\sim \exp \left\{ -m_{s(v)} r \right\}$ and, if scalar (vector) fields fluctuation are very heavy, they can be considered as an additional small perturbation of the second order. But if $m_{s(v)}$ are small, the contribution of this transition term (14) could be of the same order as the probability of the zero mode excitation. We will not consider here this situation and neglect the contribution from the term (14).

Note, that the first integral on the left-hand side of the Eq.(13) gives a very simple result. Because of the virial theorem (see e.g, [15]), the kinetic energy of a vortex is
equal to the potential energy and we obtain
\[
\int d^2x \left[ (\eta_a^a(r))^2 + (\eta_m(r))^2 \right] = \int d^2x \left( F_{kn}^2 + (D_k\phi)^2 \right) \\
= \int d^2x \left\{ \frac{1}{2} \left( F_{kn}^2 + (D_k\phi)^2 \right) + V[\phi] \right\} = M \tag{15}
\]
that is simple the energy of static vortex per unit length, or its mass. Note that the same relation between the normalization factors of monopole zero modes and the monopole mass holds in 't Hooft-Polyakov model [12].

Now we turn to the right-hand side of the Eq.(13). Suppose that the external constant current \( J_m \) is directed along the \( x \) axis: \( J_m = (J, 0) \). Taking into account the definition (6) of external force acting on a vortex and substitute the ansatz (3), one could find
\[
I_1 = \int d^2x F_m^{(1)}(r) \eta_m(r)^{(k)} = e^2 J_m \int dxdy \, \phi^a \phi^a F_{km} \\
I_2 = \int d^2x F_m^{(2)}(r) \eta_a(r)^{(k)} = -2e J_m \int dxdy \, \varepsilon_{ab} D_m \phi^a D_k \phi^b. \tag{16}
\]
Thus the non-trivial result gives the projection onto the zero mode component \( \zeta^{(y)}(r) \):
\[
I_1 = e^2 J \int dxdy \, \phi^a \phi^a F_{xy} = 2\pi ev^2 J_n \int dr H^2(r) \frac{dK(r)}{dr}; \\
I_2 = -2e J \int dxdy \, \varepsilon_{ab} D_x \phi^a D_y \phi^b = -4\pi ev^2 J_n \int dr H(r) \frac{dH(r)}{dr} \\
= -2\pi ev^2 J_n, \tag{17}
\]
i.e. the only zero modes orthogonal to the external force are excited

Obviously, the result depends upon the relation between the masses of scalar \( (m_s^2 = 2\lambda v^2) \) and vector \( (m_v^2 = e^2 v^2) \) particles. For example, in the London limit \( m_s \gg m_v \), and on the distance ranges at \( m_s^{-1} \ll r \ll m_v^{-1} \) one could neglect the core structure of scalar field, i.e. suppose that \( H \sim 1 \) everywhere. Then we have for the second integral in (16)
\[
2\pi ev^2 J_n \int dr H^2(r) \frac{dK(r)}{dr} \sim 2\pi ev^2 J_n \int dr \frac{dK(r)}{dr} = 2\pi ev^2 J_n.
\]

\footnote{It is interesting to compare this conclusion with another problem of the interaction between well separated vortices discussed in [16], where the intervortex forces lead to the well known effect of \( \pi/2 \)-scattering of two vortices by head-on collision [8].}
Thus, taking into account the definition of the magnetic flux \( \Phi_n = \int d^2x F_{xy} = \frac{2\pi n}{e} \), we finally obtain

\[
M \ddot{C}_0 = -4\pi ev^2 Jn = -2v^2 e^2 J\Phi_n = -2m^2 v J\Phi_n \tag{18}
\]

and the acceleration of a vortex along the direction orthogonal to the external force is

\[
W = \frac{2Jm^2 v}{M} \Phi_n \tag{19}
\]

## 2 A linear perturbation of the scalar field

Alongside with the Lagrangian of interaction (4) there is another possibility to introduce linear on scalar field interaction between the vortex scalar field and and external perturbation:

\[
L_{\text{int}}^a = \frac{\nu}{2} \left( \phi e^{-im\varphi} + \phi^* e^{im\varphi} \right) = \nu \varepsilon_{ab} \phi^a n^b = \nu v^2 H(r) \cos(n - m) \varphi, \tag{20}
\]

where \( \varepsilon \ll 1 \) is a perturbation parameter and \( n^a = (-\sin m\varphi, \cos m\varphi) \) is a unit vector. Obviously this is a perturbation of the vortex configuration connected with the external scalar field of another vortex of topological charge \( m \). The case \( m = 0 \) corresponds to the topologically trivial external constant scalar field coupled with only one component of the vortex scalar field \( \phi^2 = vH(r) \cos \varphi \).

We will see that the effect of such an interaction term depends from the topology of external configuration. Indeed, if both vortices have the same magnetic flux (i.e., if \( m = n \)), the only effect of the additional term (20) is a small increasement of the configuration mass. But if, for example \( n - m = 1 \), this term lifts degeneration of vacuum as it takes place in the case of 2D \( \lambda \phi^4 \) model or ‘t Hooft-Polyakov monopole [12].

The difference from the above discussed situation is that the interaction term (4) now only affects the scalar component of the vortex configuration. Indeed, the field equations are still given by Eq.(13), but the external force acting on the vortex is now

\[
F_m^{(1)} = 0; \quad F_{2(2)}^a = \varepsilon v \varepsilon_{ab} n^b \tag{21}
\]

Projection of this formula onto zero modes along \( x \)-axis gives

\[
\int d^2x \mathcal{F}^a(r) \eta^a(r)^{(\kappa)} = \varepsilon v \int d^2x \varepsilon_{ab} D_x \phi^a n^b \tag{22}
\]
\[
\begin{align*}
\ &= \varepsilon v^2 \int d^2 x \left\{ H' \cos \varphi \cos(n - m)\varphi + \frac{n}{r} H K \sin \varphi \sin(n - m)\varphi \right\} \\
\ &= \varepsilon v^2 \pi \left[ (\delta_{m,n+1} + \delta_{m,n-1}) \int dr r H' + (\delta_{m,n-1} - \delta_{m,n+1}) \int dr n HK \right],
\end{align*}
\]
i.e. the vortices move under an external force described by (20) only if \( m = n \pm 1 \). On the same way one can see that the projection of (21) onto zero modes along \( y \)-axis is trivial:
\[
\begin{align*}
\ &= \varepsilon v \int d^2 x \varepsilon_{ab} D_y \phi^a n^b \\
\ &= \varepsilon v^2 \int d^2 x \left\{ H' \sin \varphi \cos(n - m)\varphi - \frac{n}{r} H K \cos \varphi \sin(n - m)\varphi \right\} = 0 \quad (23)
\end{align*}
\]
for any values of \( m, n \).

In order to estimate the integrals let us consider a limiting case. In the Bogomol’nyi limit the first order equations on the shape functions are simply [17]
\[
H' = \frac{n}{r} H K; \quad 2n \frac{r}{r} K' = m^2 (H^2 - 1)
\]
Thus the force acting on the vortex in this limit goes to
\[
\mathcal{F}_x \longrightarrow 2\pi \varepsilon v^2 \delta_{m,n-1} \int dr H K; \quad \mathcal{F}_y = 0. \quad (24)
\]
Thus, the effect of interaction is nontrivial only if \( m = n - 1 \). This is just the situation when the perturbation term, being considered as a correction to the Higgs potential, lifts the degeneration of the vacuum\(^2\). For example we can consider a simple case when a vortex of a unitary magnetic flux interacts with a trivial external homogenious perturbation. Note, that then we again have the situation when the force is orthogonal to the external perturbation.

Further aproximation is to neglect the core structure of the vortex, i.e. suppose, as it was done at the end of the previous section, that everywhere \( H \sim 1, K \approx \delta(0) \). Then the force acts on the configuration in the London limit and can be approximated as
\[
\mathcal{F}_x \longrightarrow \pi n \varepsilon v^2 = \frac{\varepsilon m_n^2}{e} \Phi_n; \quad \mathcal{F}_y = 0,
\]
\(^2\)This corresponds to the so-called thin-wall approximation of the problem of spontaneous vacuum decay [18].
that gives the acceleration of the vortex

\[ W = \frac{\varepsilon m^2_n}{M e} \Phi_n. \]

This result can be compared with the final formula from the previous section (19). In both situations the force is orthogonal to the perturbation and the acceleration is proportional to the magnetic flux of the vortex and \( m^2_n \).

**Conclusions**

This work is based on a paper with V. Kiselev [12]. We have found a regular perturbation scheme to describe the motion of vortices in the Abelian Higgs model under external force. The key point of our approach is to treat the excitation of the zero modes of the vortex as a nontrivial time-dependent translation of the whole configuration. The amplitude of this excitation can be calculated from the field equation. We have considered two different kinds of external perturbation connected with current-current interaction and with an external scalar field. In both cases the force acting on the vortex is orthogonal to the external perturbation.

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