Remarks on Liouville type theorems for the 3D steady axially symmetric Navier-Stokes equations

Wendong Wang

Dalian University of Technology, China
&University of Oxford, UK

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Abstract

In this note, we investigate the 3D steady axially symmetric Navier-Stokes equations, and obtained Liouville type theorems if the velocity or the vorticity satisfies some a priori decay assumptions.

Keywords: Liouville type theorem, Navier-Stokes equations, axially symmetric Navier-Stokes equations

1 Introduction

An interesting question about Liouville type theorem of the 3D stationary Navier-Stokes equations in $\mathbb{R}^3$ is as follows: whether the solution of

\[
\begin{aligned}
- \Delta u + u \cdot \nabla u &= -\nabla p, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

satisfying the vanishing property at infinity

\[
\lim_{|x| \to \infty} u(x) = 0,
\]

and the bounded Dirichlet energy

\[
D(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx < \infty
\]
implies \( u \equiv 0 \) is still an open problem, which is related to J. Leray (see also P12. Galdi [7]).

Many conditional criteria have been obtained for this issue. For example, Galdi proved the above Liouville type theorem by assuming \( u \in L^2_2(R^3) \) in [7]. Chae in [2] showed the condition \( \Delta u \in L^4(R^3) \) is sufficient for the vanishing property of \( u \). Also, Chae-Wolf gave an improvement of logarithmic form for Galdi’s result in [4] by assuming that \( \int_{R^3} |u|^8 \{ \ln(2 + \frac{1}{|u|}) \}^{-1} dx < \infty \). Seregin obtained the conditional criterion \( u \in BMO^{-1}(R^3) \) in [12]. Moreover, Kozonoa-Terasawab-Wakasugib proved \( u \equiv 0 \) if the vorticity \( w = o(|x|^{-\frac{\mu}{2}}) \) or \( \|u\|_{L^2_2(R^3)} \leq \delta D(u)^{1/3} \) for a small constant \( \delta \) in [10]. It is shown that all the above norms \( u \in L^2_2(R^3) \), the log form of \( u \in L^2_2(R^3) \) or \( u \in L^2_2(R^3) \) can be replaced by the norms in the annular domain \( B_R \setminus B_{R/2} \) in [16] by Seregin and the author, where the following energy description was stated:

\[
\int_{B_{R/2}} |\nabla u|^2 dx \leq CR^{-2} \left( \int_{B_R \setminus B_{R/2}} |u|^2 dx \right) + C(q) R^{2-q} \|u\|_{L^{q, \infty}(B_R \setminus B_{R/2})}^3
\]

where \( B_R = B_R(0) \) is a ball centered at 0 and \( q > 3 \). Note that the conditions (2) and (3) are not used in [16] as in [4]. More references, we refer to [3, 13, 14] and the references therein.

Moreover, the problem is not solved, even for the case of axially symmetric Navier-Stokes equations, to the best of the author’s knowledge. Motivated by the result Seregin in [14], where he proved that the condition \( |u| \lesssim \frac{1}{|x'|^\mu} \) with \( x' = (x_1, x_2) \) and \( \mu \approx 0.77 \) implies \( u \equiv 0 \), we are aimed to improve the decay assumption. At first, let us introduce the axially symmetric Navier-Stokes equations. Let \( u(x) = u_r(t, r, z)e_r + u_\theta(t, r, z)e_\theta + u_z(t, r, z)e_z \), where

\[
e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right) = (\cos \theta, \sin \theta, 0),
\]

\[
e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right) = (-\sin \theta, \cos \theta, 0),
\]

\[
e_z = (0, 0, 1)
\]

and (1) becomes

\[
\begin{cases}
    b \cdot \nabla u_r - \Delta_0 u_r + \frac{u_r}{r^2} - \frac{u_\theta^2}{r} + \partial_r p = 0, \\
    b \cdot \nabla u_\theta - \Delta_0 u_\theta + \frac{u_\theta}{r^2} + \frac{u_r u_\theta}{r} = 0, \\
    b \cdot \nabla u_z - \Delta_0 u_z + \partial_z p = 0, \\
    \partial_r (ru_r) + \partial_z (ru_z) = 0,
\end{cases}
\]

where

\[
b = u_r e_r + u_z e_z, \quad \Delta_0 = \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz}.
\]
The vorticity is represented as
\[
w = w_r e_r + w_\theta e_\theta + w_z e_z = (\partial_z u_\theta)e_r + (\partial_z u_r - \partial_r u_z)e_\theta + \frac{\partial_r (ru_\theta)}{r} e_z.
\]

There are also many developments on the Liouville type theorems of a xi-symmetric case. For example, Liouville type theorem was proved by assuming no swirl (i.e. \(u_\theta = 0\)), see Koch-Nadirashvili-Seregin-Sverak [9] or Korobkov-Pileckas-Russo[11]. The condition \(ru_\theta \in L^q\) with some \(q \geq 1\) or \(b \in L^3\) is enough, see Chae-Weng in [5]. Specially, for the axially symmetric case, the decay of the velocity or the vorticity can be obtained: Choe-Jin [6], Weng [17] proved that
\[
|u_r(r, z)| + |u_z(r, z)| + |u_\theta(r, z)| \lesssim \sqrt{\ln r},
\]
\[
|w_\theta(r, z)| \lesssim r^{-\left(\frac{q}{q+1}\right)},
\]
\[
|w_r(r, z)| + |w_z(r, z)| \lesssim r^{-\left(\frac{q}{q+1}\right)}
\]
Recently, Carrillo-Pan-Zhang in [1] gave an alternative method for the decay of \(u\) and an improvement for the decay bound of the vorticity
\[
|w_\theta(r, z)| \lesssim r^{-\frac{2}{3}}(\ln r)^{\frac{2}{3}},
\]
\[
|w_r(r, z)| + |w_z(r, z)| \lesssim r^{-\frac{2}{3}}(\ln r)^{\frac{1}{3}}
\]
by using Brezis-Gallouet inequality.

It’s a natural question: whether there exist the sharp constants \(\mu_1, \mu_2\) such that
\[
|(u_r(r, z), u_z(r, z), u_\theta(r, z))| \lesssim \frac{1}{r^{\mu_1}} \text{ or } |(w_r(r, z), w_z(r, z), w_\theta(r, z))| \lesssim \frac{1}{r^{\mu_2}}
\]
implies that \(u \equiv 0\) for the axially symmetric case?

With the help of energy estimates in [16] we can improve the result in [14] to \(\mu > \frac{2}{3}\), which is almost a equivalent form of \(u \in L^2_t L^\frac{q}{q+3} \rightarrow \infty\).

**Theorem 1.1.** Suppose that \(u\) is axially symmetric smooth solution of the equation (4) and for some \(\mu > \frac{2}{3}\),
\[
|u| \leq \frac{C}{(1 + r)^\mu}.
\]
Then \(u \equiv 0\).

Note that \(\Gamma = ru_\theta\) satisfies the special structure
\[
b \cdot \nabla \Gamma - \Delta_0 \Gamma + \frac{2}{r} \partial_r \Gamma = 0
\]
and Maximum principle can be applied, thus the condition \(u_\theta = o(\frac{1}{r})\) as \(|x| \rightarrow \infty\) implies \(u\) is trivial. However, it’s still known that whether \(u_\theta = o(\frac{1}{r})\) can be replaced by \(u_\theta = O(\frac{1}{r})\). But we show that the condition \(|b| = O(\frac{1}{r})\) or \(b \in BMO^{-1}(R^3)\) is sufficient, which improved the assumption \(b \in L^3(R^3)\) in [5].
Here we say a function $f \in BMO^{-1}(R^3)$ if there exists a vector-value function $d \in R^3$ and $d_j \in BMO(R^3)$ such that $f = \text{div} \, d = d_{j,j}$. It’s well-known that for the BMO space, we have

$$\Gamma(s) = \sup_{x_0 \in R^3, R > 0} \left( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |d - d_{x_0,R}|^s dx \right)^{\frac{1}{s}} < \infty.$$ 

for any $s \in [1, \infty)$.

In details, we obtained the following result.

**Theorem 1.2.** Suppose that $u$ is axially symmetric smooth solution of the equation (4) satisfying (2) and (3). Then $u \equiv 0$ if one of the following conditions is satisfied

(i) $b = (u_r, u_z) \in BMO^{-1}(R^3)$;

(ii) $|b| \leq \frac{C}{r}$

For the decay of the vorticity, we also state the following corresponding result.

**Theorem 1.3.** Suppose that $u$ is axially symmetric smooth solution of the equation (4) satisfying (2) and (3). Moreover,

$$|w_r, w_\theta, w_z| \leq \frac{C}{r^\beta}, \quad \beta > \frac{5}{3}.$$

Then $u \equiv 0$.

**Remark 1.** This conclusion generalized the result of [10] to the axially symmetric case, where the condition $|w| = o(|x|^{-\frac{5}{3}})$ was put.

Throughout this article, $C$ denotes a constant, which may be different from line to line.

## 2 Proof of Theorem 1.1

Recall a Caccioppoli inequality in [16], which is stated as follows.

**Proposition 2.1.** Let $(u, p)$ be the smooth solution of (1). Then for $0 < \delta \leq 1$ and $\frac{6(3-\delta)}{6-\delta} < q < 3$, we have

$$\int_{B_{R/2}} |\nabla u|^2 dx \leq \frac{C}{R^2} \left( \int_{B_R \setminus B_{R/2}} |u|^2 dx \right) + C(\delta) \left( \|u\|_{L^{\frac{3-\delta}{\delta}}(B_R \setminus B_{R/2})} R^2 \right)^\frac{\delta}{2} \left( \frac{\delta}{3-\delta} \right)^{\frac{q}{2}} \frac{R}{2}.$$
Proof of Theorem 1.1. Let $C_R$ denote the cylindrical region $\{x; |x'| \leq R, |z| \leq R\}$, then it’s easy to check that

$$B_R \subset C_R \subset B_{\sqrt{3}R}.$$  

Hence, it follows from Proposition 2.1 that

$$\int_{C_{\sqrt{3}R}} |\nabla u|^2 dx \leq \frac{C}{R^2} \left( \int_{C_R \setminus C_{\sqrt{3}R}} |u|^2 dx \right) + C(\delta) \left( \|u\|^3_{L^q,\infty(C_R \setminus C_{\sqrt{3}R})} R^{2-\frac{9-3\delta}{3} - \frac{\delta}{2}} \right)^{\frac{2}{2-\delta}}$$

\begin{equation}
\leq C\|u\|^2_{L^q(C_R)} R^{1-\frac{6}{q}} + C(\delta, q) \left( \|u\|^3_{L^q(C_R)} R^{2-\frac{9-3\delta}{3} - \frac{\delta}{2}} \right)^{\frac{2}{2-\delta}} \tag{5}
\end{equation}

for $q > 2$, where we used the property of Lorentz space

$$\|u\|_{L^q,\infty(\Omega)} \leq C(q, \ell) \|u\|_{L^{q,\ell}(\Omega)}$$

(for example, see Proposition 1.4.10 in [8]).

For $\mu q > 2$, we have

$$\|u\|_{L^q(C_R)} \leq C \left( R \int_0^R (1 + r)^{1-\mu q} dr \right)^{\frac{1}{q}} \leq C(\mu, q) R^{\frac{1}{q}}$$

Then the terms of the right hand side of (5) is controlled by

$$\int_{C_{\sqrt{2}R}} |\nabla u|^2 dx \leq C(\mu, q) R^{1-\frac{4}{q}} + C(\delta, \mu, q) \left( R^{2-\frac{4}{2} - \frac{6-2\delta}{q}} \right)^{\frac{2}{2-\delta}} \tag{6}$$

Claim that: for fixed $\mu > \frac{2}{3}$, there exist constants $\delta \in (0, 1)$ and $q$ such that

$$\max\{6 - 3\delta, 2\mu\} < q < 3, \quad \text{and} \quad 2 - \frac{\delta}{2} - \frac{6 - 2\delta}{q} < 0 \tag{7}$$

hence letting $R \to \infty$, by (6) we have

$$\int_{R^3} |\nabla u|^2 dx = 0,$$

which implies $u \equiv 0$.

Proof of (7). First for fixed $\mu > \frac{2}{3}$, we choose $\delta_0 \in (0, 1)$ such that

$$\frac{2}{\mu} < 4 \frac{3 - \delta_0}{4 - \delta_0}$$
Since $0 < \delta_0 < 1$, we have

$$1 - \frac{\delta_0}{4} < 1 - \frac{\delta_0}{6},$$

and

$$\frac{6}{6 - \delta_0} > \frac{4}{4 - \delta_0}$$

so we take

$$q = \frac{1}{2} \left( \max \left\{ \frac{6}{6 - \delta_0}, \frac{2}{\mu} \right\} + \frac{4}{4 - \delta_0} \right)$$

Then we have

$$\max \left\{ \frac{6}{6 - \delta_0}, \frac{2}{\mu} \right\} < q < \frac{4}{4 - \delta_0} < 3,$$

which implies (7).

Hence the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

Let $\phi(x) = \phi(r, z) \in C_0^\infty(C_R)$ and $0 \leq \phi \leq 1$ satisfying

$$\phi(x) = \begin{cases} 1, & x \in C_{R/2}^c, \\ 0, & x \in C_R \end{cases}$$

and

$$|\nabla \phi| \leq \frac{C}{R}, \quad |\nabla^2 \phi| \leq \frac{C}{R^2}.$$ 

Without loss of generality, by Theorem X.5.1 in [7] we can assume that

$$\lim_{|x| \to \infty} |p| + |u| = 0.$$ 

Note that $\Delta p = -\partial_i \partial_j (u_i u_j)$, then using Calderón-Zygmund estimates and gradient estimates of harmonic function, we have

$$\int_{R^3} |p|^3 + |u|^6 dx < CD(u)^3,$$

and

$$\|\nabla p\|_{L^2(R^3)} < CD(u),$$
since \( \| \nabla u \|_{L^2(R^3)} \leq CD(u) \).

Multiplying \( \phi u \cdot \) on both sides of (1), integration by parts yields that

\[
\int_{C_R} \phi \left( |\nabla u_r|^2 + |\nabla u_\theta|^2 + |\nabla u_\varphi|^2 + \frac{u_r^2}{r^2} + \frac{u_\theta^2}{\theta^2} \right) \, dx
\leq \int_{C_R} \left( \frac{1}{2} |u|^2 + p \right) (u_r \partial_r + u_\varphi \partial_\varphi) \phi \, dx + C \| u \|_{L^6(C_R \cap C_{R/2})}^2 \leq I + C \| u \|_{L^6(C_R \cap C_{R/2})}^2.
\]

**Case (i).** Due to \( u_r, u_\varphi \in BMO^{-1}(R^3) \), we write

\[
u_r = \partial_j d_{1,j}, \quad u_\varphi = \partial_j d_{2,j}, \quad j = 1, 2, 3,
\]

where \( d_{1,j}, d_{2,j} \in BMO(R^3) \). Also, denote \( \bar{f} \) as the mean value of \( f \) on the domain \( C_R \).

Then we have

\[
I = \int_{C_R} \left( \frac{1}{2} |u|^2 + p \right) \left[ \partial_j (d_{1,j} - \bar{d}_{1,j}) \partial_r + \partial_j (d_{2,j} - \bar{d}_{2,j}) \partial_\varphi \right] \phi \, dx
\]

\[
- \int_{C_R} \partial_j \left( \frac{1}{2} |u|^2 + p \right) \left[ (d_{1,j} - \bar{d}_{1,j}) \partial_r \phi + (d_{2,j} - \bar{d}_{2,j}) \partial_\varphi \phi \right] \, dx
\]

\[
- \int_{C_R} \left( \frac{1}{2} |u|^2 + p \right) \left[ (d_{1,j} - \bar{d}_{1,j}) \partial_r (\partial_r \phi) + (d_{2,j} - \bar{d}_{2,j}) \partial_\varphi (\partial_\varphi \phi) \right] \, dx
\]

Recall that \( \phi(x) = \phi(r, \varphi) \) and

\[
\partial_j \partial_r \phi = \partial_\varphi \partial_j \phi, \quad \text{for} \ j = 1, 2, 3,
\]

\[
\partial_j \partial_\varphi \phi = \partial_\varphi \partial_j \phi, \quad \text{for} \ j = 3,
\]

\[
\partial_\varphi \partial_\varphi \phi = \cos \theta \partial_r^2 \phi, \quad \partial_\varphi \partial_r \phi = \sin \theta \partial_r^2 \phi,
\]

which and the property of BMO function yield that

\[
I \leq CR^{-1} \| \nabla(|u|^2) \| + \| \nabla p \|_{L^2(C_R \cap C_{R/2})} (\| d_{1,j} - \bar{d}_{1,j} \|_{L^3(C_R)} + \| d_{2,j} - \bar{d}_{2,j} \|_{L^3(C_R)})
\]

\[
+ CR^{-2} (\| u \|_{L^6(C_R \cap C_{R/2})} + \| p \|_{L^3(C_R \cap C_{R/2})}) (\| d_{1,j} - \bar{d}_{1,j} \|_{L^2(C_R)} + \| d_{2,j} - \bar{d}_{2,j} \|_{L^2(C_R)})
\]

\[
\leq C \| \nabla(|u|^2) \| + \| \nabla p \|_{L^2(C_R \cap C_{R/2})} + C (\| u \|_{L^6(C_R \cap C_{R/2})} + \| p \|_{L^3(C_R \cap C_{R/2})})
\]

\[
\to 0 \quad (\text{as} \ R \to \infty)
\]

Hence, the proof of case (i) is complete.

**Case (ii).** When \( |(u_r, u_\varphi)| \leq \frac{C}{r} \) for \( r > 0 \),

\[
I = \int_{C_R} \left( \frac{1}{2} |u|^2 + p \right) (u_r \partial_r + u_\varphi \partial_\varphi) \phi \, dx
\]
\[ \begin{array}{c}
\leq C \int_{C_R} \left( \frac{1}{2} |u|^2 + |p| \right) (\partial_r \ln(r) |\partial_r \phi| + \partial_r \ln(r) |\partial_z \phi|) \, dx.
\end{array} \]

Let \( g(r) = \ln(r) \) and \( \bar{g} \) be the mean value of \( g \) on \( \{ x' ; |x'| \leq R \} \). Then we have

\[ I \leq -C \int_{C_R} \partial_r \left( \frac{1}{2} |u|^2 + |p| \right) (g - \bar{g}) (|\partial_r \phi| + |\partial_z \phi|) \, dx \]

\[ -C \int_{C_R} \left( \frac{1}{2} |u|^2 + |p| \right) (g - \bar{g}) \partial_r (|\partial_r \phi| + |\partial_z \phi|) \, dx \]

\[ -C \int_{C_R} \left( \frac{1}{2} |u|^2 + |p| \right) (g - \bar{g}) \frac{1}{r} (|\partial_r \phi| + |\partial_z \phi|) \, dx \]

\[ = I_1 + I_2 + I_3 \]

Note that \( g \in BMO(R^2) \) (see, for example, Chapter IV [15]), and we have

\[ R^{-1} \left( \int_{C_R} |g - \bar{g}|^3 \, dx \right)^\frac{1}{3} \leq C \left( R^{-2} \int_{|x'| \leq R} |g - \bar{g}|^3 \, dx \right)^\frac{1}{3} \leq C \]

and

\[ R^{-2} \left( \int_{C_R} |g - \bar{g}|^\frac{4}{3} \, dx \right)^\frac{3}{4} \leq C, \quad R^{-3} \left( \int_{C_R} |g - \bar{g}|^{12} \, dx \right) \leq C \]

Hence as the arguments of (i), we have

\[ I_1 + I_2 \leq C \| \nabla (|u|^2) \| + \| \nabla p \|_{L^2(C_R \setminus C_{R/2})} + C(\| u \|_{L^6(C_R \setminus C_{R/2})}^2 + \| p \|_{L^3(C_R \setminus C_{R/2})}) \]

For the term of \( I_3 \), we get

\[ I_3 \leq CR^{-1} (\| u \|_{L^6(C_R \setminus C_{R/2})}^2 + \| p \|_{L^3(C_R \setminus C_{R/2})}) \| g - \bar{g} \|_{L^2(C_R)} \frac{1}{r} \| \tilde{2} \|_{L^2(C_R)} \]

\[ \leq CR^{-\frac{1}{2}} (\| u \|_{L^6(C_R \setminus C_{R/2})}^2 + \| p \|_{L^3(C_R \setminus C_{R/2})}) \| g - \bar{g} \|_{L^2(C_R)} \]

\[ \leq C (\| u \|_{L^6(C_R \setminus C_{R/2})}^2 + \| p \|_{L^3(C_R \setminus C_{R/2})}) \]

Hence, we can conclude that

\[ I \to 0 \quad (\text{as } R \to \infty) \]

The proof of Theorem 1.2 is complete.

4 Proof of Theorem 1.3

We are going to prove that
Proposition 4.1. Assume that the conditions of Theorem 1.3 hold. (1) Let \( w_\theta \leq Cr^{-\beta} \) with \( \beta > 1 \). Then we get for \( r > 1 \)

\[
|u_r(r, z)| + |u_z(r, z)| \leq C \begin{cases} 
(1 + r)^{\frac{3}{2} + \frac{\beta}{2}}, & \beta > 2, \\
(1 + r)^{1 - \beta}, & 1 < \beta < 2, \\
(1 + r)^{-1} \ln(r + 1), & \beta = 2.
\end{cases}
\]

(2) Let \( |w_r| + |w_z| \leq Cr^{-\beta} \) with \( \beta > 1 \). Then we get for \( r > 1 \)

\[
|u_\theta(r, z)| \leq C \begin{cases} 
(1 + r)^{\frac{3}{2} + \frac{\beta}{2}}, & \beta > 2, \\
(1 + r)^{1 - \beta}, & 1 < \beta < 2, \\
(1 + r)^{-1} \ln(r + 1), & \beta = 2.
\end{cases}
\]

Proof of Theorem 1.3. It follows from Proposition 4.1 and Theorem 1.1 directly.

Next we are aimed to prove Proposition 4.1. Firstly, we introduce a representation formula of \( u_r, u_z \) and \( u_\theta \) with the help of the vorticity. Since \( b = u_r e_r + u_z e_z \) and \( \nabla \times b = w_\theta e_\theta \), \( \nabla \times (u_\theta e_\theta) = w_re_r + w郅e_z \) by Biot-Savart law, we can get the integral representation of the velocity as follows (for example, see Lemma 2.2 for a local version by Choe-Jin [6], also see Lemma 3.10 by Weng [17]).

Lemma 4.2. Like the vorticity at the point \( (r \cos \theta, r \sin \theta, z) \) denoted by \( (w_r, w_\theta, w_z)(r, z) \), we write the vorticity at the point \( (\rho \cos \phi, \rho \sin \phi, k) \) as \( (w_\rho, w_\phi, w_k)(\rho, k) \). Then we have

\[
u_r(r, z) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma_1(r, \rho, z - k) w_\phi(\rho, k) \rho d\rho dk,
\]

\[
u_z(r, z) = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma_2(r, \rho, z - k) w_\phi(\rho, k) \rho d\rho dk
\]

\[
u_\theta(r, z) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma_3(r, \rho, z - k) w_k(\rho, k) \rho d\rho dk
\]

\[ -\int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma_1(r, \rho, z - k) w_\rho(\rho, k) \rho d\rho dk\]

where

\[
\Gamma_1(r, \rho, z - k) = \frac{1}{4\pi} \int_{0}^{2\pi} \frac{z - k}{[r^2 + \rho^2 - 2r\rho \cos \phi + (z - k)^2]^{\frac{3}{2}}} \cos \phi d\phi,
\]

\[
\Gamma_2(r, \rho, z - k) = -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{\rho - r \cos \phi}{[r^2 + \rho^2 - 2r\rho \cos \phi + (z - k)^2]^{\frac{3}{2}}} d\phi,
\]

\[
\Gamma_3(r, \rho, z - k) = -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{\rho - r \cos \phi}{[r^2 + \rho^2 - 2r\rho \cos \phi + (z - k)^2]^{\frac{3}{2}}} \cos \phi d\phi.
\]
Secondly, we give the bounds of estimate of $\Gamma_2, \Gamma_3$ and $\Gamma_1$, which will be used in the proof. This is similar to that in [6], where $\rho \approx r$ was assumed. Here we consider all $\rho > 0$ and large $r > 0$. In details, we have the following estimates.

**Lemma 4.3 (Estimate of $\Gamma_2, \Gamma_3$ and $\Gamma_1$).**

\[
|\Gamma_2(r, \rho, z - k)| + |\Gamma_3(r, \rho, z - k)| \leq \frac{C}{(\max\{\rho, r\})^0[(r - \rho)^2 + (z - k)^2]^{\frac{2-\alpha}{4}}}, \tag{11}
\]

for $r > 1$ and $0 \leq \alpha \leq 1$;

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{C|z - k|}{(\max\{\rho, r\})^0[(r - \rho)^2 + (z - k)^2]^{\frac{3-\alpha}{4}}}, \tag{12}
\]

where $r > 1$, $0 \leq \alpha \leq 1$ for $\frac{r}{4} \leq \rho \leq 4r$, and $0 \leq \alpha \leq 3$ for $\rho < \frac{r}{4}$ or $\rho \geq 4r$.

Thirdly, we assume Lemma 4.3 holds and complete the proof of Proposition 4.1 and Lemma 4.3 is proved later.

**Proof of Proposition 4.1:** At first, we estimate the term of $u_r(r, z)$. Let

\[
I = u_r(r, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_1 w_\phi \rho d\rho dk
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{r^{\gamma/8}} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty} \int_{r^{\gamma/8}}^{r^{\gamma/4}} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty} \int_{r^{\gamma/4}}^{r^{\gamma/2}} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty} \int_{r^{\gamma/2}}^{r^{\gamma/2} + \beta} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty} \int_{r^{\gamma/2} + \beta}^{r} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty} \int_{r}^{\infty} \Gamma_1 w_\phi \rho d\rho dk
\]

where $0 \leq \gamma, \delta \leq 1$, to be decided.

For the term $I_1$, by (12) and $\|w_\phi\|_{L^2(\mathbb{R}^3)} \leq CD(u) < \infty$ we get

\[
I_1 \leq C \left( \int_{-\infty}^{\infty} \int_{0}^{r^{\gamma/8}} |\Gamma_1(r, \rho, z - k)|^2 \rho d\rho dk \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \int_{-\infty}^{\infty} \int_{0}^{r^{\gamma/8}} \frac{|z - k|^2}{r^{2\alpha} + (z - k)^2} d\rho dk \right)^{\frac{1}{2}}
\]

\[
\leq Cr^{-\frac{\gamma}{2}} \left( \int_{-\infty}^{\infty} \int_{0}^{r^{\gamma/8}} \frac{r^{-2}|z - k|^2}{[1 + r^{-2}(z - k)^2]^{3-\alpha} r^{-\gamma} d\rho dk} \right)^{\frac{1}{2}} \leq Cr^{-\frac{\gamma}{2} + \gamma}
\]

where $0 \leq \alpha < \frac{3}{7}$. For the term $I_2$, using $r > 1$, (12) and $w_\theta \leq Cr^{-\beta}$

\[
I_2 \leq C \int_{r^{\gamma/8}}^{r^{\gamma/4}} \int_{0}^{r^{\gamma/8}} \Gamma_1 \rho^{1-\beta} d\rho dk
\]
\[ C \left( \int_{-\infty}^{\infty} \int_{r^{\gamma/8}}^{r^{\gamma/4}} \frac{|z-k|}{r^\alpha (r^2 + (z-k)^2)^{\frac{\alpha}{2}}} \rho^{1-\beta} \, d\rho \, dk \right) \leq C \left\{ \begin{array}{ll} r^{-1+\gamma(2-\beta)} & (\beta > 2) \\ r^{-1} \ln r & (\beta = 2) \\ r^{1-\beta} & (1 < \beta < 2) \end{array} \right. \]

where \(0 \leq \alpha < 1\).

Moreover, for the term \(I_3\), by (12) and \(w_\theta \leq C r^{-\beta}\)

\[ I_3 \leq C \int_{-\infty}^{\infty} \int_{r^{\delta/2}}^{r^{\delta/4}} \Gamma_1 \rho^{1-\beta} \, d\rho \, dk \leq C \left( \int_{-\infty}^{\infty} \int_{r^{\delta/2}}^{r^{\delta/4}} \frac{|z-k|}{r^\alpha [(r-\rho)^2 + (z-k)^2]^{\frac{3-\alpha}{2}}} \rho^{1-\beta} \, d\rho \, dk \right) \leq C r^{-\alpha-\delta+\alpha\delta} \left( \int_{-\infty}^{\infty} \int_{r^{\delta/2}}^{r^{\delta/4}} \frac{r^{-\delta} |z-k|}{\left[ \frac{1}{4} + r^{-2\delta (z-k)^2} \right]^{\frac{3-\alpha}{2}}} r^{-\delta} \, d\rho \, dk \rho^{1-\beta} \, d\rho \right) \leq C \left\{ \begin{array}{ll} r^{2-\beta-\alpha-\delta+\alpha\delta} & (\beta < 2 \text{ or } \beta > 2) \\ r^{-\alpha-\delta+\alpha\delta} \ln r & (\beta = 2) \end{array} \right. \]

where \(0 \leq \alpha < 1\).

Similarly, for \(I_5\) we have

\[ I_5 \leq C \left\{ \begin{array}{ll} r^{2-\beta-\alpha-\delta+\alpha\delta} & (\beta < 2 \text{ or } \beta > 2) \\ r^{-\alpha-\delta+\alpha\delta} \ln r & (\beta = 2) \end{array} \right. \]

where \(0 \leq \alpha < 1\).

Furthermore, for \(0 \leq \alpha < 1\) by (12) and \(w_\theta \leq C r^{-\beta}\) we have

\[ I_4 \leq C \int_{-\infty}^{\infty} \int_{r^{\delta/2}}^{r^{\delta/4}} \Gamma_1 \rho^{1-\beta} \, d\rho \, dk \leq C \left( \int_{-\infty}^{\infty} \int_{r^{\delta/2}}^{r^{\delta/4}} \frac{|z-k|}{r^\alpha [(r-\rho)^2 + (z-k)^2]^{\frac{3-\alpha}{2}}} \rho^{1-\beta} \, d\rho \, dk \right) \leq C \left( \int_{r^{\delta/2}}^{r^{\delta/4}} r^{-\alpha} (r-\rho)^{-1+\alpha} \rho^{1-\beta} \, d\rho \right) \leq C r^{1-\beta-\alpha} \left( \int_{r^{\delta/2}}^{r^{\delta/4}} (r-\rho)^{-1+\alpha} \, d\rho \right) \leq C r^{1-\beta-\alpha+\alpha\delta} \ (\beta > 1) \]

Finally, (12) and \(w_\theta \leq C r^{-\beta}\) yield that

\[ I_6 \leq C \int_{-\infty}^{\infty} \int_{4r}^{\infty} \Gamma_1 \rho^{1-\beta} \, d\rho \, dk \]
\[ I \leq C \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|z - k|}{\rho^\alpha \left[ \rho^2 + (z - k)^2 \right]^\frac{1+\beta}{2}} \rho^{1-\beta} d\rho dk \right) \]
\[ \leq Cr^{1-\beta} \quad (\beta > 1) \]

Hence, concluding the estimates of \( I_1, \cdots, I_6 \), we have the following arguments.

**Case a.** \( \beta > 2 \). At this time, we have

\[ I \leq C \left[ r^{-\frac{3}{2}+\gamma} + r^{-1+\gamma(2-\beta)} + r^{2-\beta-\alpha-\delta+\delta\alpha} + r^{1-\beta-\alpha+\delta\alpha} + r^{1-\beta} \right] \]

where \( 0 \leq \alpha < 1 \) and \( 0 \leq \gamma, \delta \leq 1 \).

First, we choose \( \gamma = \frac{1}{2(\beta-1)} \) such that \( -\frac{3}{2} + \gamma = -1 + \gamma(2-\beta) \). Furthermore, we take \( \alpha \uparrow 1, \delta \uparrow 1 \) such that

\[ (1 - \delta)(1 - \alpha) \leq \beta - \frac{5}{2} + \frac{1}{2(\beta - 1)} \]

which implies

\[ -1 + \gamma(2 - \beta) \geq 2 - \beta - \alpha - \delta + \delta\alpha \]

Moreover, note that

\[ 2 - \beta - \alpha - \delta + \delta\alpha \geq 1 - \beta \geq 1 - \beta - \alpha + \delta\alpha \]

Then, we get for \( r > 1 \)

\[ |u_r(r, z)| \leq Cr^{-\frac{3}{2}+\frac{1}{2(\beta-1)}}. \]

**Case b.** \( \beta < 2 \). At this time, we have

\[ I \leq C \left[ r^{-\frac{3}{2}+\gamma} + r^{2-\beta-\alpha-\delta+\delta\alpha} + r^{1-\beta-\alpha+\delta\alpha} + r^{1-\beta} \right] \]

where \( 0 \leq \alpha < 1 \) and \( 0 \leq \gamma, \delta \leq 1 \). We choose \( \gamma = 0 \) and \( \delta = 1 \), then we get

\[ |u_r(r, z)| \leq Cr^{1-\beta}. \]

**Case c.** \( \beta = 2 \). At this time, we have

\[ I \leq C \left[ r^{-\frac{3}{2}+\gamma} + r^{-1} \ln r + r^{-\alpha-\delta+\delta\alpha} \ln r + r^{1-\beta-\alpha+\delta\alpha} + r^{1-\beta} \right] \]

where \( 0 \leq \alpha < 1 \) and \( 0 \leq \gamma, \delta \leq 1 \). We choose \( \gamma = 0 \) and \( \delta = 1 \), then we get

\[ |u_r(r, z)| \leq Cr^{-1} \ln r. \]

Hence we complete the estimate of \( u_r(r, z) \).
Note that the bound of $\Gamma_1$ used as above is similar to the estimates of $\Gamma_2$ and $\Gamma_3$. Hence similar arguments hold for $u_z$ and $u_\theta$. The proof of Proposition 4.1 is complete.

Proof of Lemma 4.3. The remaining part is devoted to proving Lemma 4.3, which is similar to that of [6], where the case $\frac{r}{4} < \rho < 4r$ is discussed. Here we consider all the value $\rho > 0$ and sketch the proof. First, for $k > 0$ and $\beta \geq 1$ we find

\[
I = \int_0^{\frac{\pi}{2}} \frac{d\phi}{(\sqrt{1 + k \sin^2 \phi})^\beta} \leq \begin{cases} 
C(\delta) \min\{1, k^{-\frac{\delta}{2}}\}, & \beta = 1 \\
C(\beta) \min\{1, k^{-\frac{1}{2}}\}, & \beta > 1 
\end{cases}
\]  
(13)

for any $0 \leq \delta < 1$. Obviously, $k \leq C$ holds, and next we assume that $k$ is large enough. Then for $0 < \ell < 1$

\[
I \leq \ell + \int_0^{\frac{\pi}{2}} \frac{d\phi}{(k \sin^2 \phi)^{\beta/2}}
\]

Due to $\phi \leq 2 \sin \phi$ for $\phi \in (0, \frac{\pi}{2})$, we have

\[
I \leq \ell + 2k^{-\beta/2}(\ln(\frac{\pi}{2}) - \ln \ell), \quad \beta = 1,
\]

and

\[
I \leq \ell + 2k^{-\beta/2}\frac{(\frac{\pi}{2})^{1-\beta} - \ell^{1-\beta}}{1-\beta}, \quad \beta > 1,
\]

which yield the required bound (13) by choosing a suitable $\ell$.

Obviously, from the formulas of $\Gamma_2, \Gamma_3$ and $\Gamma_1$, we have

\[
|\Gamma_i(r, \rho, z - k)| \leq \frac{\rho + r}{[(r - \rho)^2 + (z - k)^2]^{\frac{3}{2}}}, \quad i = 2, 3;
\]

(14)

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{|z - k|}{[(r - \rho)^2 + (z - k)^2]^{\frac{3}{2}}}
\]

(15)

for all $\rho > 0$ and $r > 0$.

Next we go on estimating $\Gamma_2, \Gamma_3$, and $\Gamma_1$ carefully, respectively.

Step I. Noting the periodic and even property and variable transform for $\phi$, we also have

\[
\Gamma_2 = -\int_0^{2\pi} \frac{d\phi}{4\pi \left[r^2 + \rho^2 - 2r \rho \cos \phi + (z - k)^2\right]^{\frac{3}{2}}} \\
= -\int_0^{\frac{\pi}{2}} \frac{d\phi}{\pi \left[r^2 + \rho^2 - 2r \rho \cos 2\phi + (z - k)^2\right]^{\frac{3}{2}}}
\]

and

\[
\Gamma_2 = -\int_0^{\frac{\pi}{2}} \frac{d\phi}{2\pi \rho \left[(r - \rho)^2 + 4r \rho \sin^2 \phi + (z - k)^2\right]^{\frac{3}{2}}}
\]
\[
\leq C \frac{1}{\rho \sqrt{(r - \rho)^2 + (z - k)^2}} \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 + K \sin^2 \phi}} - \frac{1}{2\pi \rho [(r - \rho)^2 + (z - k)^2]^{3/2}} \int_{0}^{\pi/2} \frac{\rho^2 - r^2}{(\sqrt{1 + K \sin^2 \phi})^3} d\phi = I_1 + I_2
\]

where

\[
K = \frac{4r\rho}{(r - \rho)^2 + (z - k)^2}
\]

When \( K \leq 1 \), that is \( 4r\rho \leq (r - \rho)^2 + (z - k)^2 \), we have \( (r - \rho)^2 + (z - k)^2 \geq \frac{1}{2}r^2 \) for \( \rho \leq \frac{r}{2} \) and \( (r - \rho)^2 + (z - k)^2 \geq 2r^2 \) for \( \frac{r}{2} \leq \rho \leq 4r \). Moreover, for \( \rho \geq 4r \) we have

\[
(r - \rho)^2 + (z - k)^2 \geq \left( \frac{3}{4}\rho \right)^2 \geq \left( \frac{3}{5}(\rho + r) \right)^2 \geq \frac{9}{25}(\rho + r)^2
\]

Hence for \( K \leq 1 \) we have

\[
\Gamma_2 \leq C \frac{1}{\rho \sqrt{(r - \rho)^2 + (z - k)^2}}
\]

When \( K > 1 \), by (13) we have

\[
\Gamma_2 \leq C(\delta) \frac{1}{\rho \sqrt{(r - \rho)^2 + (z - k)^2}} \left[ \left( \frac{(r - \rho)^2 + (z - k)^2}{4r\rho} \right)^{\frac{3}{2}} + \frac{|\rho^2 - r^2|}{(r - \rho)^2 + (z - k)^2} \left( \frac{(r - \rho)^2 + (z - k)^2}{4r\rho} \right)^{\frac{3}{2}} \right]
\]

where \( 0 \leq \delta < 1 \).

**Case a.** For \( r > 1 \) and \( \rho \leq \frac{r}{4} \) or \( \rho > 4r \), by (14) we know the estimate (11) holds.

**Case b.** For \( r > 1 \) and \( \frac{r}{4} \leq \rho \leq 4r \) with \( K \leq 1 \), by (14) and (16) we know the estimate (11) holds.

**Case c.** For \( r > 1 \) and \( \frac{r}{4} \leq \rho \leq 4r \) with \( K >> 1 \), by (14) and (17) we know the estimate (11) holds by noting that \( (r - \rho)^2 + (z - k)^2 \leq 16r^2 \) and

\[
\frac{|\rho^2 - r^2|}{(r - \rho)^2 + (z - k)^2} \left( \frac{(r - \rho)^2 + (z - k)^2}{4r\rho} \right)^{\frac{3}{2}} \leq \frac{\rho + r}{\sqrt{4r\rho}} \leq 5.
\]

Hence the proof of \( \Gamma_2 \) is complete.

**Step II.** The term \( \Gamma_2 \) is similar and we omitted the details.

**Step III.** The term \( \Gamma_1 \) is estimated as follows.

\[
\Gamma_1(r, \rho, z - k) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{z - k}{[r^2 + \rho^2 - 2r\rho \cos \phi + (z - k)^2]^{3/2}} \cos \phi d\phi
\]
\[
\begin{align*}
= & \frac{1}{\pi} \int_{0}^{\pi/2} \frac{z - k}{[(r - \rho)^2 + 4r \rho \sin^2 \phi + (z - k)^2]^{1/2}} \cos 2\phi d\phi \\
\leq & \frac{C}{\sqrt{|z - k|}} \int_{0}^{\pi/2} \frac{1}{(\sqrt{1 + K \sin^2 \phi})^3} d\phi \\
\leq & I'
\end{align*}
\]

where

\[
K = \frac{4r \rho}{(r - \rho)^2 + (z - k)^2}
\]

When \(K \leq 1\), i.e. \(4r \rho \leq (r - \rho)^2 + (z - k)^2\), we have \((r - \rho)^2 + (z - k)^2 \geq \frac{1}{2}r^2\) for \(\rho \leq \frac{r}{2}\) and \((r - \rho)^2 + (z - k)^2 \geq 2r^2\) for \(\frac{r}{2} \leq \rho \leq 4r\). Moreover, for \(\rho \geq 4r\) we have

\[
(r - \rho)^2 + (z - k)^2 \geq \left(\frac{3}{4}\rho\right)^2
\]

Hence for \(K \leq 1\) we have

\[
(r - \rho)^2 + (z - k)^2 \geq \frac{1}{2}(\max\{r, \rho\})^2
\]

Using \((15)\), for \(K \leq 1\) we get

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{C|z - k|}{\sqrt{\rho}[(r - \rho)^2 + (z - k)^2]^{3/2}}, \tag{18}
\]

where \(0 \leq \alpha \leq 3\).

When \(K > 1\), i.e. \(4r \rho \geq (r - \rho)^2 + (z - k)^2\), which implies \(\rho \geq \frac{1}{2}r\), by \((13)\) we have

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{C|z - k|}{\sqrt{r}[(r - \rho)^2 + (z - k)^2]} \left(\frac{(r - \rho)^2 + (z - k)^2}{4r \rho}\right)^{1/2}
\]

Thus for \(\frac{1}{8}r \leq \rho \leq 4r\), we have

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{C|z - k|}{\sqrt{r} \max\{\rho, r\}^{\alpha\left[\frac{3}{2}(r - \rho)^2 + (z - k)^2\right]^{3/2}}}, \tag{19}
\]

where \(0 \leq \alpha \leq 1\). For \(\rho \geq 4r\), by \((15)\) we also derive that

\[
|\Gamma_1(r, \rho, z - k)| \leq \frac{C|z - k|}{\max\{\rho, r\}^{\alpha\left[\frac{3}{2}(r - \rho)^2 + (z - k)^2\right]^{3/2}}}, \tag{20}
\]

where \(0 \leq \alpha \leq 3\).

Concluding the estimates \((18)\), \((19)\) and \((20)\), we complete the proof of the inequality \((12)\).

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References

[1] B. Carrillo, X. Pan, Q. Zhang, Decay and vanishing of some axially symmetric D-solutions of the Navier-Stokes equations, arXiv:1801.07420 [math.AP].

[2] D. Chae, Liouville-type theorem for the forced Euler equations and the Navier-Stokes equations. Commun. Math. Phys., 326: 37-48 (2014).

[3] D. Chae, T. Yoneda, On the Liouville theorem for the stationary Navier-Stokes equations in a critical space, J. Math. Anal. Appl. 405 (2013), no. 2, 706-710.

[4] D. Chae, J. Wolf, On Liouville type theorems for the steady Navier-Stokes equations in $R^3$, arXiv:1604.07643.

[5] Chae, G., Weng, S., Liouville type theorems for the steady axially symmetric Navier-Stokes and Magnetohydrodynamic equations, Discrete and Continuous Dynamical Systems, Volume 36, Number 10, 2016, 5267-5285.

[6] H. Choe, B. Jin, Asymptotic properties of axis-symmetric D-solutions of the Navier-Stokes equations. J. Math. Fluid Mech. 11 (2009), no. 2, 208-232.

[7] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Second edition. Springer Monographs in Mathematics. Springer, New York, 2011. xiv+1018 pp.

[8] Grafakos, Loukas, Classical Fourier analysis, Graduate Texts in Mathematics, 249 (2nd ed.), Berlin, New York: Springer-Verlag, 2008. doi:10.1007/978-0-387-09432-8, ISBN 978-0-387-09431-1, MR 2445437.

[9] Koch, G., Nadirashvili, N., Seregin, G., Sverak, V., Liouville theorems for the Navier-Stokes equations and applications, Acta Mathematica, 203 (2009), 83-105.

[10] H. Kozono, Y. Terasawa, Y. Wakasugi, A remark on Liouville-type theorems for the stationary Navier-Stokes equations in three space dimensions, Journal of Functional Analysis, 272(2017), 804-818.

[11] M. Korobkov, K. Pileckas and R. Russo, The Liouville theorem for the steady-state Navier-Stokes problem for axially symmetric 3D solutions in absence of swirl, J. Math. Fluid Mech., 17 (2015), 287-293.

[12] G. Seregin, Liouville type theorem for stationary Navier-Stokes equations, Nonlinearity, 29 (2016), 2191-2195.
[13] G. Seregin, *A liouville type theorem for steady-state Navier-Stokes equations*, arXiv:1611.01563 and J. E. D. P. (2016), Expos no IX, 5 p.

[14] G. Seregin, *Remarks on Liouville type theorems for steady-state Navier-Stokes equations*, arXiv:1703.10822v1 and Algebra i Analiz, 2018, Vol. 30, no.2., 238-248.

[15] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, New Jersey, 1993.

[16] G. Seregin, W. Wang, *Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations*, arXiv:1805.02227v1

[17] S. Weng, *Decay properties of axially symmetric D-solutions to the steady Navier-Stokes equations*, J. Math. Fluid Mech. (2017). DOI 10.1007/s00021-016-0310-5