The Probabilistic Profitable Tour Problem
under a specific graph structure

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April 18, 2022

Abstract

Among the most important variants of the traveling salesman problem (TSP) are those relaxing the constraint that every locus should necessarily get visited, rather taking into account a revenue (prize) for visiting customers. In the Profitable Tour Problem (PTP), we seek for a tour visiting a subset of customers while maximizing net gain (profit) as difference between total revenue collected from visited customers and incurred traveling costs. The metric TSP can be modeled as a PTP with large revenues. As such, PTP is well-known to be NP-hard and also APX-hardness follows. Nevertheless, PTP is solvable in polynomial time on particular graph structures like lines, trees and circles. Following recent emphasis on robust optimization, and motivated by current flourishing of retail delivery services, we study the Probabilistic Profitable Tour Problem (PPTP), the generalization of PTP where customers will show up with a known probability, in their respective loci, only after the tour has been planned. Here, the selection of customers has to be made a priori, before knowing if a customer will actually submit his request or will not. While the tour has to be designed without this knowledge, revenues will only be collected from customers who will require the service. The objective is to maximize the expected net gain obtained by visiting only the customers that show up. We provide a polynomial time algorithm computing and characterizing the space of optimal solutions for the special case of the PPTP where customers are distributed on a line.

Keywords: Traveling salesman problem with profits; probabilistic profitable tour problem; polynomial time complexity.

1 Introduction

Several variants of the traveling salesman problem, taking into account revenues (profits) for visiting customers, have been studied in the literature: with single or multiple vehicles, with and without time windows, with precedence (Hanafi et al. [5]) or other side constraints (see the surveys by Feillet et al. [3] and, more recently, by Gunawan et al [4]).

In the PTP, each customer with a known location is associated with a positive revenue (prize) and the problem looks for a tour originating at a depot and visiting a subset of the customers so to maximize the difference between collected revenues and total traveling costs (Feillet et al. [3]).

In the literature, PTP is sometimes formulated as a minimization problem trying to find a tour that minimizes the sum of cost and missed revenue for not visiting customers. The two formulations complement
each others and are equivalent as optimization problems (Johnson et al. [7]). A great interest has been
devoted also to the capacitated variant of the PTP, since it can be seen as a special case of the elementary
shortest path problem with resource constraints (see Jepsen et al. [6] where the authors introduce a branch-
and-cut and new valid inequalities for the problem). Recently, Angelelli et al. [2] study the deterministic
PTP and other variants of the Traveling Salesman problem with profits considering special structures of
the underlying graph (line, cycle, star rooted at the depot, tree rooted at the depot). The authors provide
computational complexity and approximation results for all studied problems generalizing them to the case
with positive service times associated with customers. As far as the profitable tour problem on a line (a
path) is concerned, the authors show that it is solvable in linear time under both service settings.

Frequently, in many real application contexts, the customers will require a service with a known proba-
bility. In such a case, the decision maker has to decide a strategy to construct an a priori solution, i.e. he/she
has to select which customers to visit before knowing who among them will submit a request. Only those
customers, among the ones selected in advance, who will submit a request, will also be served. The objective
of the problem is to select a subset of customers so to maximize the expected profit measured as difference
between revenue and traveled distance. We call this variant of the PTP, the Probabilistic Profitable Tour
Problem (PPTP).

In contrast to the deterministic variants, a very few contributions can be found on probabilistic TSP with
profits. Angelelli et al. [1] provide a linear integer stochastic formulation of the Orienteering Problem and
develop both a branch-and-cut approach and different matheuristic methods. To the best of our knowledge,
no contributions can be found on the PTP under uncertainty.

In this paper, we will analyze the probabilistic PTP where customers are distributed on a line. Starting
from main results provided in Angelelli et al. [2], our theoretical question is if the probabilistic PTP on a
line remains solvable in polynomial time or becomes NP-hard. Our main contribution is the complexity
of probabilistic PTP on a line and the characterization of its optimal solutions space. In particular, we
show that the problem can be solved in $O(n^3)$ time, where $n$ is the number of customers. Although the
main contribution of our work remains theoretical, one can figure out real contexts where the problem is
likely to find application: the road network topology is typical of a mountain valley where customers are
all located on a main road, whereas the a priori optimization is typical of a 2-stage decision process where
some decisions are made at the first stage (selection of the customers for whom service is guaranteed) and a
recourse action takes place at the second stage (maximize the expected value of the total collected revenues
minus the traveling costs related to the subset of a priori selected customers that actually showed up by
requiring the service).

The paper is organized as follows. In Section 2, the main properties of the problem and its optimal
solution are discussed for a given value of the unitary cost per traveled distance, whereas in Section 3 the
dependency on such a parameter is analyzed and the space of optimal solutions is characterized. Finally,
the solution algorithm is presented in Section 4 and its computational complexity discussed. Concluding
remarks are provided in Section 5.

## 2 Problem definition and main properties

Formally, the PPTP can be defined on a directed graph $G = (V, A)$ with $V = \{v_0\} \cup N$ where $v_0$ is the
depot, $N = \{v_1, \ldots, v_n\}$ is the set of potential customers and $A$ denotes the arc set. To each customer $v_i$, a
revenue (prize) $p_i > 0$ and a probability $\pi_i \in (0, 1]$ are assigned. The depot $v_0$ is located at the origin of the
semi-line in the point $x_0 = 0$. Each customer $v_i$ is positioned in $x_i > 0$. Each arc $(i, j)$ is weighted by a cost
$|x_j - x_i|c$ where $c \geq 0$ is a unitary cost per traveled distance. Without loss of generality, we assume that:
• $0 < x_i \leq x_{i+1}$ for $i = 1, \ldots, n - 1$;

• $x_i = x_{i+1} \Rightarrow \pi_i \leq \pi_{i+1}$ (if two customers are positioned at the same location, they are sorted in non-decreasing order of their probability)

• if two customers share the same location and have equal probability, the tie is broken randomly.

Let $S \subseteq N$ the set of customers selected a priori by the decision maker and let $X$ be the subset of customers that will submit a request. The set of customer that will be actually served by the vehicle is $S \cap X$. We indicate as $R(S)$ the corresponding expected revenue:

$$R(S) = \sum_{v_i \in S} \pi_i p_i. \quad (1)$$

Observe that the minimum length route to serve all customers in $S \cap X$ corresponds to reach the farthest customer while serving all the others on the way back to the depot. This means that in the case $v_i$ is the farthest customer in $S \cap X$, the total traveled distance is $2x_i$, that, for simplicity of notation, we denote as $l_i$. Thus, according to the assumption $0 < x_i \leq x_{i+1}$, the expected traveled distance can be computed as:

$$L(S) = \sum_{v_i \in S} \left[ l_i \pi_i \prod_{v_j \in S, j > i} (1 - \pi_j) \right]. \quad (2)$$

Given the cost $c \geq 0$ per unit of traveled length, the expected cost $C(S, c)$ and the expected profit $G(S, c)$ will be computed as follows:

$$C(S, c) = cL(S), \quad (3)$$

$$G(S, c) = R(S) - cL(S). \quad (4)$$

The PPTP on a line can thus be formulated as follows:

$$\text{PPTP}(N, c) = \max_{S \subseteq N} G(S, c) \quad (5)$$

We call a solution of (5) an optimal set. The optimal set may depend on $c$, but by now we assume $c$ as given.

Solving problem (5) may appear a difficult task as the number of potential options is exponential with respect to $n$ for every chosen $c$. However, in Section 4, we will show an approach that, looking at $\text{PPTP}(N, c)$ as a function of $c$, manages to build in polynomial time a family of optimal sets covering all possible values of $c \in [0, +\infty)$.

In this section, we start showing some basic properties of the problem for a fixed value of $c$ and conclude that, even though we may have several optimal sets for a given $c$, there is only one maximal optimal set and only one minimal optimal set, whereas any other optimal set is a subset of the former and a superset of the latter.

Given a non void subset of customers $S \subseteq N$, we refer to $v^S \in S$ as the customer with the highest index in $S$ (the farthest customer from the depot and with highest probability). Moreover, we indicate as $S'$ the set $S \setminus \{v^S\}$, whereas $l^S$, $p^S$ and $\pi^S$ are distance, score and probability of $v^S$.

**Proposition 1** Given a non void set of customers $S$, the following recursive formulas hold: (See proof in appendix)

$$R(S) = R(S') + \pi^S p^S \quad (6)$$

$$L(S) = (1 - \pi^S)L(S') + \pi^Sl^S \quad (7)$$

3
Proposition 4. Given any two subsets of customers $S_1$ and $S_2$ such that $S_1 \subseteq S_2$, then $L(S_1) \leq L(S_2)$. (See proof in appendix)

Proposition 3. Given any two subsets of customers $S_1$ and $S_2$, the following results hold: (See proof in appendix)

\[
R(S_1 \cup S_2) = R(S_1) + R(S_2) - R(S_1 \cap S_2),
\]

\[
L(S_1 \cup S_2) \leq L(S_1) + L(S_2) - L(S_1 \cap S_2).
\]

Proposition 4. Given any two subsets of customers $S_1$ and $S_2$, we have

\[
G(S_1, c) + G(S_2, c) \leq G(S_1 \cup S_2, c) + G(S_1 \cap S_2, c).
\]

Proof. From Proposition 3 we get

\[
G(S_1 \cup S_2, c) = R(S_1 \cup S_2) - cL(S_1 \cup S_2)
\]

\[
\geq [R(S_1) + R(S_2) - R(S_1 \cap S_2)] - c[L(S_1) + L(S_2) - L(S_1 \cap S_2)]
\]

\[
\geq [R(S_1) - cL(S_1)] + [R(S_2) - cL(S_2)] - [R(S_1 \cap S_2) - cL(S_1 \cap S_2)]
\]

\[
\geq G(S_1, c) + G(S_2, c) - G(S_1 \cap S_2, c).
\]

Proposition 5. Given any two subsets of customers $S_1$ and $S_2$ such that $G(S_1, c) = G(S_2, c)$ for a given unitary cost $c$, then

\[
G(S_1, c) \leq \max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c)).
\]

Proof. By contradiction, if $\max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c)) < G(S_1, c)$ then we get $G(S_1 \cup S_2, c) + G(S_1 \cap S_2, c) < G(S_1, c) + G(S_2, c)$ in contrast with Proposition 4.

Proposition 6. If two subset of customers $S_1$ and $S_2$ are optimal sets for a given unitary cost $c$, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are also optimal sets.

Proof. From Proposition 5 we know that $G(S_1, c) \leq \max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c))$, but for the optimality of $S_1$ and $S_2$ we have $G(S_1, c) = \max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c))$, which proves that at least one between $S_1 \cap S_2$ and $S_1 \cup S_2$ is optimal. We now show that both are optimal.

From Proposition 4 we know that

\[
G(S_1, c) + G(S_2, c) \leq \max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c)) + \min(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c)),
\]

but from equality $G(S_1, c) = \max(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c))$ it follows that

\[
G(S_2, c) \leq \min(G(S_1 \cup S_2, c), G(S_1 \cap S_2, c))
\]

and thus, from the optimality of $S_2$, the optimality of both $S_1 \cap S_2$ and $S_1 \cup S_2$ follows.

Proposition 7. For each unitary cost $c$, there exists a unique minimal optimal set and a unique maximal optimal set. Moreover, any optimal set is a subset of the maximal one and a superset of the minimal one.
Proof. By contradiction, let $S_1$ and $S_2$ be two distinct maximal optimal sets, we know from Proposition 6 that their proper superset $S_1 \cup S_2$ is optimal, thus neither $S_1$ or $S_2$ can be maximal optimal sets. Moreover, if we had an optimal set $S$ which is not a subset of the maximal one $\bar{S}$, then $S \cup \bar{S}$ would be optimal and a proper superset of $\bar{S}$, in contrast with maximality of $\bar{S}$.

We use the same argument to show that the minimal optimal set is unique and any optimal set is a superset of the minimal one.

3 Dependency from the unitary cost

In this section, we study the dependency of $\text{PPTP}(N, c)$ from the unitary cost $c$ and characterize the property of optimal sets accordingly. We end this section with a hint on how to build a description of $\text{PPTP}(N, c)$ in terms of values and optimal sets.

In the following, we call characteristic function of a set of customers $S$ its expected revenue $G(S, c) = R(S) - cL(S)$ seen as a function of the unitary cost $c$. Note that the characteristic function of a set $S$ can be graphically represented in the Cartesian plane $G(S, c)$ as a line with non positive slope. In particular, the slope is negative for each set $S \neq \emptyset$ and null for $S = \emptyset$; in such a case $G(\emptyset, c) = 0$ for all $c$.

Proposition 8 $\text{PPTP}(N, c)$ as a function of $c$ holds the following characteristics:

1. $\text{PPTP}(N, 0) = R(N) = \sum_{i \in N} p_i \pi_i$

2. There exists a finite positive value $\tilde{c}$ such that
   \[
   \begin{cases} 
   \text{PPTP}(N, c) > 0 & \text{if } c < \tilde{c} \\
   \text{PPTP}(N, c) = 0 & \text{otherwise}
   \end{cases}
   \]

3. In the interval $[0, \tilde{c}]$ function $\text{PPTP}(N, c)$ is strictly decreasing, piece-wise linear and convex. In $[0, \infty)$ function $\text{PPTP}(N, c)$ is non-increasing piece-wise linear and convex.

Proof. Let’s analyze each point separately:

1. For $c = 0$ we get $\text{PPTP}(N, 0) = \max_{S \in P(N)} (R(S) - 0 \cdot L(S)) = \max_{S \in P(N)} (R(S)) = R(N)$.

2. For each non void set $S \subseteq N$ we get $G(S, c) \leq 0$ for $c \geq c_S = R(S)/L(S)$. Let us indicate $\tilde{c} = \max_{S \subseteq N} \{c_S\}$. Then we get that for each $c < \tilde{c}$, inequality $\text{PPTP}(N, c) > 0$ holds since there exists at least one set $S \subseteq N$ for which $G(S, c) > 0$; when $c \geq \tilde{c}$ then we get $G(S, c) \leq 0$ for each set $S \subseteq N$ and thus the void set corresponds to the optimal solution since $G(\emptyset, c) = 0$.

3. For $c \in [0, \tilde{c}]$ function $\text{PPTP}(N, c) = \max_{S \in P(N)} G(S, c)$ is the envelop of a finite set of linear functions strictly decreasing. For $c \geq \tilde{c}$, function $\text{PPTP}(N, c)$ becomes constant. In the interval $[0, \infty)$, it only loses monotonicity.

The function $\text{PPTP}(N, c)$ is thus described by a finite number of linear pieces. Each linear piece is the characteristic function of an optimal set within the corresponding range. To determine the actual shape of $\text{PPTP}(N, c)$ can, in principle, be hard as it is the outcome of a number, exponential in $n$, of lines. However, we will show that this task can be accomplished in polynomial time $O(n^3)$. Let us start with some properties of function $\text{PPTP}(N, c)$. 

5
Proposition 9 If for some $\bar{c}$ there are two distinct optimal sets, then the maximal optimal set is optimal on $(\bar{c} - \varepsilon, \bar{c})$ and the minimal optimal set is optimal on $[\bar{c}, \bar{c} + \varepsilon)$ for some $\varepsilon > 0$. In particular, $\bar{c}$ is a corner point of $PPTP(N,c)$.

Proof. Let us consider the minimal and maximal optimal sets in $\bar{c}$, and call them $A$ and $B$, respectively. By Proposition 7 we know that they are unique distinct and that $A \subset B$. It is easy to see that $R(A) < R(B)$ and $L(A) < L(B)$ (Proposition 2) so that, being $G(A, \bar{c}) = G(B, \bar{c})$ it must necessarily be $G(A, c) < G(B, c)$ for $c < \bar{c}$ and viceversa $G(A, c) > G(B, c)$ for $c > \bar{c}$. Easy to see that any other intermediate optimal set in $\bar{c}$ is dominated by the maximal optimal set for $c < \bar{c}$ and by the minimal optimal set for $c > \bar{c}$.

Furthermore, if for a fixed $\varepsilon > 0$ there were some distinct optimal sets in interval $(\bar{c} - \varepsilon, \bar{c} + \varepsilon)$ other than $A$ and $B$, then we can repeatedly halve the value of $\varepsilon$; the process must come to an end as we have only a finite number of potential optimal sets.

Point $\bar{c}$ is a corner point because function $PPTP(N,c)$ takes different slopes around $\bar{c}$. 

Proposition 10 If $(c_1, c_2)$ is an interval such that $PPTP(N,c)$ is linear (no corner points in the interval), then there is only one optimal set for all $c \in (c_1, c_2)$.

Proof. If two distinct set of customers are optimal for some $c \in (c_1, c_2)$ then $c$ is a corner point.

Proposition 11 There are at most $n$ corner points in function $PPTP(N,c)$.

Proof. According to Proposition 9, at each corner point the optimal set loose some customers, and since we have $n$ customers, we can have $n$ corner points at most.

Resuming, we showed that:

1. In the corner points of function $PPTP(N,c)$, we have two optimal solutions defined by the linear pieces belonging to the envelop, of which one is subset of the other. More precisely, the piece belonging to higher values of $c$ is characterized by the minimal optimal set which is a subset of the maximal optimal set characterized by the piece associated with lower values of $c$ (Proposition 9);

2. In each corner point $\tilde{c}$ we may have other optimal sets, but all of them are dominated by the maximal optimal set for $c < \tilde{c}$ and by the minimal optimal set for $c > \tilde{c}$; each of these intermediate sets is a subset of the maximal one and a superset of the minimal one (Proposition 9);

3. Each linear piece of function $PPTP(N,c)$ is characterized by one and only one optimal maximal solution (Proposition 10);

4. given two consecutive corner points $c_1$ and $c_2$, and the line segment representing $PPTP(N,c)$ for $c \in [c_1, c_2]$, the corresponding optimal set is the minimal optimal set for $c = c_1$, the only optimal set for $c \in (c_1, c_2)$ and the maximal optimal set in $c = c_2$ (Propositions 9 and 10);

5. The function $PPTP(N,c)$ contains at most $n$ corner points and $n + 1$ linear pieces (Proposition 11);

6. The function $PPTP(N,c)$ is defined for all $c > 0$ by at most $n + 1$ optimal sets.

Next, we show that optimal sets of function $PPTP(N,c)$, (and the corresponding corner points) can be computed in polynomial time with respect to the size $n$ of the instance.
By defining $N^{(k)}$ as the set of the first $k$ customers closest to the depot for $k = 0, 1, \ldots, n$, we can also write:

$$PPTP(N^{(k)}, c) = \max_{S \in P(N^{(k)})} G(S, c), \quad k = 0, ..., n, \quad (14)$$

where in the particular case $k = 0$ we get an empty optimal set with $PPTP(N^{(0)}, c) = 0$ for all $c \geq 0$

Our idea is to iteratively build function $PPTP(N^{(k+1)}, c)$ from $PPTP(N^{(k)}, c)$, starting with $k = 0$ up to $k = n - 1$.

Let us indicate with $S^{(k)}$ the family of optimal sets defining function $PPTP(N^{(k)}, c)$; We also indicate with $S^{(k)}_c \in S^{(k)}$ the optimal set for a given unit cost $c$, that is

$$PPTP(N^{(k)}, c) = \max_{S \in S^{(k)}} (G(S, c)) = G(S^{(k)}_c, c). \quad (15)$$

Finally, we indicate with $E^{(k)} = \{S \cup \{v_{k+1}\} \mid S \in S^{(k)}\}$ the family of extended sets obtained from sets in $S^{(k)}$ by adding a new customer $v_{k+1}$. We recall that according to Proposition 11 family $S^{(k)}$ contains at most $k + 1$ elements.

Next, we show that the family of optimal sets $S^{(k+1)}$ for $PPTP(N^{(k+1)}, c)$ can be extracted from $S^{(k)} \cup E^{(k)}$ which is the fundamental property the iterative step of the building process lays on.

**Proposition 12** The family of optimal sets $S^{(k+1)}$ for function $PPTP(N^{(k+1)}, c)$ is a subset of $S^{(k)} \cup E^{(k)}$ with cardinality at most $(k + 1) + 1$.

**Proof.** Let $S_1 \subseteq N^{(k+1)}$ be a set of customers. We show that $G(S_1, c) \leq \max_{S \in S^{(k)} \cup E^{(k)}} (G(S, c))$ for all $c > 0$. We proceed by cases.

a) If $v_{k+1} \notin S_1$, then by construction we have

$$G(S_1, c) \leq PTPP(N^{(k)}, c) = \max_{S \in S^{(k)}} (G(S, c)) \leq \max_{S \in S^{(k)} \cup E^{(k)}} (G(S, c)).$$
b) If $v_{k+1} \in S_1$, then let us define $S'_1 = S_1 \setminus \{v_{k+1}\} \subseteq N^{(k)}$. By using (3), we get:

$$G(S_1, c) = G(S'_1, (1 - \pi_{k+1})c + \pi_{k+1}(p_{k+1} - c_l + 1))
\leq \max_{S \in S^{(k)}} (G(S, (1 - \pi_{k+1})c) + \pi_{k+1}(p_{k+1} - c_l + 1),$$

now let us indicate with $\tilde{S}_c \in S^{(k)}$ the optimal set of problem $\max_{S \in S^{(k)}} (G(S, (1 - \pi_{k+1})c))$ for a given $c$ and observe that $\tilde{S}_c \cup \{v_{k+1}\} \in E^{(k)}$; Thus, we have

$$G(S_1, c) \leq G(\tilde{S}_c, (1 - \pi_{k+1})c + \pi_{k+1}(p_{k+1} - c_l + 1)) = G(\tilde{S}_c \cup \{v_{k+1}\}, c) \leq \max_{S \in E^{(k)}} (G(S, c)) \leq \max_{S \in S^{(k)} \cup E^{(k)}} (G(S, c)).$$

Thus, $S^{(k)} \cup E^{(k)}$ is enough to determine function $PPTP(N^{(k+1)}, c)$. It is worth noticing that, according to Proposition 11, the family of customer sets $S^{(k)} \cup E^{(k)}$ can be reduced to have a cardinality no larger than $k + 2$. $\blacksquare$

## 4 Solution algorithm and computational complexity

This section is devoted to the presentation of the procedure to compute function $PPTP(N, c)$. Such a function is described by a sequence of maximal optimal sets that change according to $c$ value. The corner points depend on the sequence of such solutions.

The algorithm iteratively constructs $PPTP(N^{(k+1)}, c)$ by using $PPTP(N^{(k)}, c)$. Starting point is the function $PPTP(N^{(1)}, c)$. It is important to notice that, at each iteration, the description of $PPTP(N^{(k)}, c)$ contains at most $k + 1$ solutions. We use $D^{(k)}$ to indicate the description of function $PPTP(N^{(k)}, c)$. In particular, we indicate as $D^{(k)}, S$ and $D^{(k)}, S_i$ the list of maximal optimal solutions and the $i$-th solution in such a list, respectively. Finally, $D^{(k)}, C_i^{\text{min}}$ and $D^{(k)}, C_i^{\text{max}}$ are the minimum and maximum values of $c$ for which $D^{(k)}, S_i$ is optimal for the problem on $N^{(k)}$.

It follows that:

- $D^{(k)}, S_1 = N^{(k)}$ (see Proposition 8 point 1)
- $D^{(k)}, S_{|D^{(k)}, S|} = \emptyset$ (see Proposition 8 point 2)
- $D^{(k)}, C_i^{\text{min}} = \begin{cases} 0 & \text{for } i = 1 \\ c \text{ such that } G(D^{(k)}, S_i, c) = G(D^{(k)}, S_{i-1}, c) & \text{for } i = 2, \ldots, |D^{(k)}| \end{cases}$
- $D^{(k)}, C_i^{\text{max}} = \begin{cases} c \text{ such that } G(D^{(k)}, S_i, c) = G(D^{(k)}, S_{i+1}, c) & \text{for } i = 1, \ldots, |D^{(k)}| - 1 \\ +\infty & \text{for } i = |D^{(k)}| \end{cases}$

Function JointSortFilter executes the main task of creating the upper envelope of the solutions in $D^{(k)}$ and $E^{(k)}$. More precisely, it consider a set $O(k)$ of solutions: the ones coming from $D^{(k)}$ are already sorted and define a base of the envelope for $c \in [0, +\infty)$; the ones coming from $E^{(k)}$ are added one at a time to the existing envelope by modifying it accordingly. All solutions of $E^{(k)}$ that do not modify the envelope are discarded along with the ones of the existing envelope that are dominated by the insertion of a new solution.

**Theorem 13** Algorithm PPTPLINE$(N)$ provides the description of function POP$(N, c)$ in a computational time $O(n^3)$. 

8
Algorithm 1 PPTPLINE(\(N\))

1: Set \(h = |N|\) and \(D^{(0)} . S = [\emptyset] , D^{(0)} . C_{\text{min}} = 0 , D^{(0)} . C_{\text{max}} = \infty\)
2: for \(k = 0 \) to \(h - 1 \) do
3: \(E^{(k)} \leftarrow \{D^{(k)} . S \cup \{v_{k+1}\} \mid i = 1 , ..., |D^{(k)} . S|\}\)
4: \(D^{(k+1)} \leftarrow \text{JointSortFilter}(D^{(k)}, E^{(k)})\)
5: end for
6: return \((D^{(n)})\)

Proof. Initially, the algorithm provides the description of \(POP(N^{(1)}, c)\). By construction there exists exactly two optimal maximal solutions: the one that includes the unique customer in \(N^{(1)}\) and the void solution. The moving from the (minimal) description \(D^{(k)}\) of \(PPTP(N^{(k)}, c)\) at the beginning of each iteration, to the (minimal) description \(D^{(k+1)}\) of \(PPTP(N^{(k+1)}, c)\) at the end of the iteration is guaranteed by Proposition 12.

As far as computational complexity is concerned, we observe that, since \(|D^{(k)} . S|\) is \(O(k)\), each one of the \(O(k)\) solutions in \(E^{(k)}\) have to be compared with the \(O(k)\) solutions of the current envelope, which means a computational complexity of \(O(k^2)\). Since the operation has to be repeated for \(k = 1 , ..., n - 1 \) times, the complexity \(O(n^3)\) immediately follows.

5 Conclusions

In this paper, we analyze the Probabilistic Orienteering Problem for the special case where customers are located on a line. A straightforward algorithm is devised that allows to determine the upper envelope of the function describing the problem. The algorithm takes a polynomial time to find the optimal solution. As future work, the extension of the problem to more general cases will be taken into account.

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Appendix

Proof of Proposition 1

- Proof of formula (6)

\[ R(S) = \sum_{v_i \in S} \pi_i p_i = \left( \sum_{v_i \in S'} \pi_i p_i \right) + \pi^S p^S = R(S') + \pi^S p^S; \]

- Proof of formula (7)

\[ L(S) = \sum_{v_i \in S} \left[ l_i \pi_i \cdot \prod_{v_j \in S, j > i} (1 - \pi_j) \right] \]
\[ = \sum_{v_i \in S'} \left[ l_i \pi_i \cdot \prod_{v_j \in S', j > i} (1 - \pi_j) \right] + l^S \pi^S \cdot 1 \]
\[ = (1 - \pi^S) \cdot \sum_{v_i \in S'} \left[ l_i \pi_i \cdot \prod_{v_j \in S', j > i} (1 - \pi_j) \right] + l^S \pi^S \cdot 1 \]
\[ = (1 - \pi^S) \cdot L(S') + l^S \pi^S; \]

- Proof of formula (8)

\[ C(S, c) = cL(S) \]
\[ = c(1 - \pi^S) L(S') + \pi^S l^S c \]
\[ = C(S', c(1 - \pi^S)) + \pi^S l^S c; \]

- Proof of formula (9)

\[ G(S, c) = R(S) - C(S, c) \]
\[ = R(S') + \pi^S p^S - (C(S', c(1 - \pi^S)) + \pi^S l^S c) \]
\[ = [R(S') - (1 - \pi^S) c L(S')] + \pi^S (p^S - l^S c) \]
\[ = G(S', (1 - \pi^S)c) + \pi^S (p^S - l^S c). \]

Proof of Proposition 2

Let \( S_1 \) and \( S_2 \) be two set of customers such that \( S_1 \subseteq S_2 \). We show that \( L(S_1) \leq L(S_2) \) by induction on cardinality of \( S_2 \) using equality 7.

**Base cases.** If \( |S_2| = 0 \), both \( S_1 \) and \( S_2 \) are empty and we get \( L(S_1) = 0 \leq L(S_2) = 0 \). If \( |S_2| = 1 \), then either \( S_1 = \emptyset \) or \( S_1 = S_2 \); and we get \( L(S_1) = 0 \leq L(S_2) \); and \( L(S_1) = L(S_2) \), respectively.

**Induction hypothesis.** Let us assume that the property holds for all \( |S_2| \leq n \) and show that it hold also for \( |S_2| = n + 1 \).
**Induction step.** Let $n > 1$ be the cardinality of $S_2$. Then either $v^{S_2} \in S_1$ or $v^{S_2} \notin S_1$.

If $v^{S_2} \in S_1$, we have $v^{S_2} = v^{S_1} = v^S$, and $L(S_1) = (1 - \pi^S \cdot L(S'_1) + l^S \pi^S \leq L(S'_2) + l^S \pi^S = L(S_2)$; the last inequality comes from induction hypothesis because $|S'_2| = |S_2| - 1 = n$.

If $v^{S_2} \notin S_1$, we have $S_1 \subseteq S_2$ and $L(S_1) \leq L(S'_2) \leq L(S'_2) + l^{S_2} \pi^{S_2} = L(S_2)$; the first inequality comes from induction hypothesis because $|S'_2| = |S_2| - 1 = n$. 

**Proof of Proposition 3.**

- **Proof of formula (10)** Equality follows straightforwardly from the following decompositions:

$$R(S_1) = \sum_{v_i \in S_1 \setminus S_2} \pi_i \pi_j + \sum_{v_i \in S_1 \cap S_2} \pi_i \pi_j,$$

$$R(S_2) = \sum_{v_i \in S_2 \setminus S_1} \pi_i \pi_j + \sum_{v_i \in S_1 \cap S_2} \pi_i \pi_j,$$

$$R(S_1 \cup S_2) = \sum_{v_i \in S_1 \setminus S_2} \pi_i \pi_j + \sum_{v_i \in S_2 \setminus S_1} \pi_i \pi_j + \sum_{v_i \in S_1 \cap S_2} \pi_i \pi_j,$$

$$R(S_1 \cap S_2) = \sum_{v_i \in S_1 \cap S_2} \pi_i \pi_j;$$

- **Proof of formula (11)** We first discuss two special cases.

a) When $S_2 \subseteq S_1$ (or $S_1 \subseteq S_2$) the property is trivially true since $S_1 \cup S_2 = S_1$ and $S_1 \cap S_2 = S_2$ (or $S_1 \cup S_2 = S_2$ and $S_1 \cap S_2 = S_1$) and equality boils down to an identity.

b) When $S_1 \cap S_2 = \emptyset$ the property boils down to the inequality $L(S_1 \cup S_2) \leq L(S_1) + L(S_2)$ which comes from the comparison of the following expressions:

$$L(S_1) = \sum_{v_i \in S_1} \pi_i \prod_{j \in S_1, j > i} (1 - \pi_j),$$

$$L(S_2) = \sum_{v_i \in S_2} \pi_i \prod_{j \in S_2, j > i} (1 - \pi_j),$$

$$L(S_1 \cup S_2) = \sum_{v_i \in S_1} \pi_i \prod_{j \in S_1 \cup S_2, j > i} (1 - \pi_j) + \sum_{v_i \in S_2} \pi_i \prod_{j \in S_1 \cup S_2, j > i} (1 - \pi_j),$$

where inequalities

$$\sum_{v_i \in S_1} \pi_i \prod_{j \in S_1 \cup S_2, j > i} (1 - \pi_j) \leq \sum_{v_i \in S_1} \pi_i \prod_{j \in S_1, j > i} (1 - \pi_j),$$

$$\sum_{v_i \in S_2} \pi_i \prod_{j \in S_1 \cup S_2, j > i} (1 - \pi_j) \leq \sum_{v_i \in S_2} \pi_i \prod_{j \in S_2, j > i} (1 - \pi_j)$$

hold because each term in the left summations has more factor less than 1.

We conduct the proof by induction on the cardinality of $S = S_1 \cup S_2$ and make use of notation of equation (7) which we recall here for reader’s convenience

$$L(S) = (1 - \pi^S)L(S') + \pi^S l^S;$$

where we indicate as $v^S$ the farthest customer of a given set $S$ (with location $l^S$, profit $p^S$ and probability $\pi^S$), and define $S' = S \setminus \{v^S\}$. 

11
Base cases. If \(|S| \leq 2\) we have (w.l.o.g.) \(S_2 \subseteq S_1\) and/or \(S_1 \cap S_2 = \emptyset\). Thus, we fall in one of cases a,b discussed above.

Induction hypothesis. We assume that the property holds for \(|S| = |S_1 \cup S_2| \leq n\) and show that it holds also for \(|S| = n + 1\).

Induction step. Given \(|S_1 \cup S_2| = n + 1\), we assume w.l.o.g. that \(v^S \in S_1\). The following two cases may occur:

a) \(v^S \in S_2\) (thus \(v^S \in S_1 \cap S_2\)). From (7), we get:

\[
L(S_1 \cup S_2) = \pi^S l^S + (1 - \pi^S) L(S_1') \cup S_2')
\]

\[
L(S_1 \cap S_2) = \pi^S l^S + (1 - \pi^S) L(S_1' \cap S_2')
\]

\[
L(S_1) = \pi^S l^S + (1 - \pi^S) L(S_1')
\]

\[
L(S_2) = \pi^S l^S + (1 - \pi^S) L(S_2')
\]

and thus \(L(S_1 \cup S_2) + L(S_1 \cap S_2) = 2\pi^S l^S + (1 - \pi^S) [L(S_1') \cup S_2') + L(S_1' \cap S_2')]\), where \(|S_1' \cup S_2'| = n\).

Then, by induction hypothesis:

\[
L(S_1 \cup S_2) + L(S_1 \cap S_2) \leq 2\pi^S l^S + (1 - \pi^S) [L(S'_1) + L(S'_2)]
\]

\[
\leq (\pi^S l^S + (1 - \pi^S) L(S'_1)) + (\pi^S l^S + (1 - \pi^S) L(S'_2)) = L(S_1) + L(S_2).
\]

b) \(v^S \notin S_2\), thus

\(v^S \notin S_1 \cap S_2\) with \(S' = S'_1 \cup S_2\) and \(|S'| = n\).

From (7), we get \(L(S_1) = \pi^S l^S + (1 - \pi^S) L(S'_1)\), and then

\[
L(S'_1) = L(S_1) + \pi^S (L(S_1') - l^S).
\]

By using the induction hypothesis (\(|S'_1 \cup S_2| = n\)) we get

\[
L(S'_1 \cup S_2) + L(S'_1 \cap S_2) \leq L(S'_1) + L(S_2) = L(S_1) + \pi^S (L(S_1') - l^S) + L(S_2)
\]

Now, the final result is obtained by substituting back and recalling that \(L(S_1 \cap S_2) = L(S'_1 \cap S_2)\) (from \(v^S \notin S_2\)) and \(L(S'_1) \leq L(S'_1 \cup S_2)\) (from Proposition 2):

\[
L(S_1 \cup S_2) + L(S_1 \cap S_2) = \pi^S l^S + (1 - \pi^S) L(S'_1 \cup S_2) + L(S'_1 \cap S_2)
\]

\[
= \pi^S l^S - \pi^S L(S'_1 \cup S_2) + L(S'_1 \cup S_2) + L(S'_1 \cap S_2)
\]

\[
\leq \pi^S l^S - \pi^S L(S'_1 \cup S_2) + L(S'_1) + L(S_2)
\]

\[
= \pi^S (l^S - L(S'_1 \cup S_2) + L(S'_1) - l^S) + L(S_1) + L(S_2)
\]

\[
= \pi^S (L(S'_1) - L(S'_1 \cup S_2)) + L(S_1) + L(S_2)
\]

\[
\leq L(S_1) + L(S_2).
\]

The same considerations hold for the case \(v \in S_2 \setminus S_1\).