Wormhole and time machine are very interesting objects in general relativity. However, they need exotic matters which are impossible in classical level to support them. But if we introduce the quantum effects of gravity into the stress-energy tensor, these peculiar objects can be constructed self-consistently. Fortunately, loop quantum cosmology (LQC) has the potential to serve as a bridge connecting the classical theory and quantum gravity. Therefore it provides a simple way for the study of quantum effect in the semiclassical case. As is well known, loop quantum cosmology is very successful to deal with the behavior of early universe. In the early stage, if taken the quantum effect into consideration, inflation is natural because of the violation of every kinds of local energy conditions. Similar to the inflationary universe, the violation of the averaged null energy condition is the necessary condition for the traversable wormholes. In this paper, we investigate the averaged null energy condition in LQC in the framework of effective Hamiltonian, and find out that LQC do violate the averaged null energy condition in the massless scalar field coupled model.

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I. INTRODUCTION

The wormhole and the time machine are attractive objects in general relativity and have been ever used as allurements to attract young students to research general relativity [1]. And they are always active research fields in general relativity [2]. The deducing stress-energy tensor components of these exotic spacetime through Einstein’s equation must violate all the known pointwise energy condition, which is forbidden in classical general relativity. In contrast, the energy condition violation can be easily met in quantum field because of quantum fluctuation [3,4,5]. For example, the Casimir vacuum for the electromagnetic field between two perfectly conducting plates has a negative local energy density [6]; squeezed states of light can entail negative energy densities [7]. But these quantum effects are always confined to an extremely thin band [8]. On the other hand, the topological censorship theorem proved by Friedman, Schleich, and Witt [9] implies that, the existence of macroscopic traversable wormholes requires the violation of the averaged null energy condition (ANEC). ANEC can be stated as

$$\int_{\gamma} T_{\mu\nu} k^\mu k^\nu dl \geq 0,$$

where the integral is along any complete, achronal null geodesic $\gamma$, $k^\mu$ denotes the geodesic tangent, and $l$ is an affine parameter.

If we take the effect of quantum gravity into consideration, the quantum effect will enter stress-energy tensor. Then every kind of energy condition violation is easily satisfied with respect to effective stress-energy tensor which is reduced from metric through Einstein’s equation. Based on semi-classical gravitational analysis, many self-consistent wormhole solutions have been found [10,11,12,13,14,15]. As a quantum gravitational theory, loop quantum gravity (LQG) [10,17,18,19] is a non-perturbative and background independent quantization of gravity. The application of the techniques of LQG to the cosmological sector is known as loop quantum cosmology (LQC). Some of the main features of LQG such as discreteness of spatial geometry are inherited in LQC. LQC has the potential to serve as a bridge connecting the classical theory and quantum gravity. Therefore it provides a simple way for the study of quantum effect in the semiclassical case. It exhibits an overwhelming strength in solving the fundamental problems of the universe. A major success of LQC is that it resolves the problem of classical singularities both in an isotropic model [20] and in a less symmetric homogeneous model [21], a result that depending crucially on the discreteness of the theory. It has been shown that non-perturbative modification of the matter Hamiltonian leads to a generic phase of inflation [21,22,23]. Furthermore, in this successful quantum gravitational theory, another point worthy of being paid some attention to is energy condition violation [31]. We do find kinds of energy condition violation. For example, in loop quantum cosmology non-perturbative modification to a scalar matter field at short scales induces a violation of the strong energy condition [27] and results in inflation [24,25,26]. Given so many local energy condition violated by loop quantum cosmology and motivated by the topological censorship theorem, we investigate that whether the averaged null energy condition is violated also in LQC. The difference between the ANEC and other energy conditions such as strong energy condition mentioned above is that the ANEC is an integral along any complete null-like geodesic, not be confined to the neighborhood of a certain point of the space-time. For a system without symmetry, it is a very complicated
issue, making it almost impossible to calculate. But in the context of LQC, taking the cosmological principle into consideration, it will give us an exact result. The cosmological principle states that the Universe appears the same in every direction form every point in space. This allows us to get a precise result and this result can provide a hint for researching the wormholes in LQG and testing the validity of LQG. In this paper, we assume that all the quantum effect in LQC can be described approximately by effective loop quantum cosmology. Through our calculation, we find that LQC do violate the averaged null energy condition in the massless scalar field coupled model.

This paper is organized as follows. First, we briefly review the framework of effective loop quantum cosmology in Sec. II and introduce an exactly solvable model containing a massless scalar field in Sec. III. Then in Sec. IV, we investigate the averaged null energy condition in this exactly solvable model. At last in Sec. V we give some discussion and implication of our results. In this paper we adopt $c = \hbar = G = 1$.

II. FRAMEWORK OF EFFECTIVE LQC

In order to deduce the effective stress-energy tensor conveniently, let us begin with the classical Hamiltonian description of the universe. For simplicity, we only consider flat universe in this work. Under the assumption of the cosmological principle, the metric of spacetime is described by FRW metric

$$ds^2 = -dt^2 + a^2 (dx^2 + dy^2 + dz^2),$$

where $a$ is the scale factor of the universe, only depending on $t$ due to the homogeneity of our universe. The classical Hamiltonian for this system is given by

$$H_{cl} = -\frac{3}{8\pi\gamma^2\hbar c^2} + H_M(p, \phi). \quad (2)$$

Here we have adopted the variables in the form of loop quantum gravity. The phase space is spanned by coordinates $c = \gamma \dot{a}$, being the gravitational gauge connection, and $p = a^2$, being the densitized triad. Note that the Gaussian constraint in loop formalism implies that changing $p$ to $-p$ will lead to the same physical results, which can be seen obviously from the relationship $p = a^2$. In this paper we fix this gauge freedom by $p > 0$. $\gamma$ is the Barbero-Immirzi parameter. $H_M$ denotes the Hamiltonian of matter part and $\phi$ means matter field. The dynamical equations together with Hamiltonian constraint $H_{cl} = 0$ which corresponds to Einstein’s equations reduce to the following Friedmann and Raychaudhuri equations

$$H^2 = \frac{8\pi}{3} \rho, \quad (3)$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi}{3} (\rho + 3P). \quad (4)$$

Here the energy density $\rho$ and pressure $P$ of matter are defined as

$$\rho = a^{-3}H_M, \quad P = -\frac{1}{3}a^{-2}\frac{\partial H_M}{\partial a}, \quad (5)$$

which is consistent with the form of stress-energy tensor for ideal fluid

$$T_{\mu\nu} = \rho U_\mu U_\nu + P (g_{\mu\nu} + U_\mu U_\nu)$$

$$= \rho (dt)_\mu (dt)_\nu + a^2 P \left[(dx)_\mu (dx)_\nu + (dy)_\mu (dy)_\nu + (dz)_\mu (dz)_\nu\right], \quad (6)$$

where $U_\mu = (1, 0, 0, 0)$ is the comoving observer of matter field which corresponds to the comoving observer of the universe naturally.

Correspondingly, the effective Hamiltonian in LQC is given by

$$H_{eff} = -\frac{3}{8\pi\gamma^2\hbar^2} \sqrt{\rho} \sin^2 (\bar{\mu}c) + H_M(p, \phi). \quad (7)$$

The variable $\bar{\mu}$ corresponds to the dimensionless length of the edge of the elementary loop and is given by

$$\bar{\mu} = \xi p^{-1/2}, \quad (8)$$

where $\xi$ is a constant $\xi > 0$ and depends on the particular scheme in the holonomy corrections. In this paper we take $\bar{\mu}$-scheme, which gives

$$\xi^2 = 2\sqrt{3\pi\gamma l_p^2}, \quad (9)$$

where $l_p$ is Planck length.

Similarly, the canonical equations and the Hamiltonian constraint give out modified Friedmann and Raychaudhuri equations

$$H^2 = \frac{8\pi}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right), \quad (10)$$

$$\dot{H} + H^2 = -\frac{4\pi}{3} \left[\rho \left(1 - \frac{\rho}{\rho_c}\right) + 3P \left(1 - \frac{2\rho}{\rho_c}\right) - \frac{3\rho^2}{\rho_c}\right], \quad (11)$$

where $\rho_c = \frac{3}{8\pi\gamma^2\hbar^2}$ is the critical energy density coming from the quantum effect in loop quantum cosmology. These modified equations give us the effective energy density and the effective pressure reduced by Einstein’s equation as

$$\rho_{eff} = \rho \left(1 - \frac{\rho}{\rho_c}\right), \quad (12)$$

$$P_{eff} = P \left(1 - \frac{2\rho}{\rho_c}\right) - \frac{\rho^2}{\rho_c}. \quad (13)$$

The introduction of these effective quantities makes the reduced stress-energy tensor in the form of the ideal fluid again.
III. AN EXACTLY SOLVABLE MODEL

Now following closely [30], we consider a universe containing a massless scalar field. Then the Hamiltonian for the matter part in equation (17) can be written as

$$ H_M(p, \phi) = \frac{1}{2} p_\phi^2, $$

where $p_\phi$ is the conjugate momentum for the scalar field $\phi$. The complete equations of motion for the universe containing a massless scalar field are

$$ \begin{cases} \dot{c} = -\frac{\kappa}{\pi} \left( \sqrt{p} \left[ \frac{\sin(\mu c)}{\mu} \right]^2 \right) - \frac{\kappa}{\pi} \frac{p_\phi^2}{p^{3/2}}, \\ \dot{p} = \frac{2}{5} \frac{\kappa}{\mu} \sin(\mu c) \cos(\mu c), \end{cases} \tag{14} $$

and

$$ \begin{cases} \dot{\phi} = p^{-3/2} p_\phi, \\ \dot{p}_\phi = 0, \end{cases} \tag{15} $$

where $\kappa = 8\pi$. In addition, the Hamiltonian constraint $H_{\text{eff}} = 0$ becomes

$$ \frac{3}{8\pi^2 \mu^2} \sqrt{p} \sin^2(\mu c) = \frac{1}{2} \frac{p_\phi^2}{p^{3/2}}. \tag{16} $$

Combining equations (14) and (16) we obtain

$$ \left( \frac{dp}{dt} \right)^2 = \Omega_1 p^{-1} - \Omega_3 p^{-4}, \tag{17} $$

with $\Omega_1 = \frac{2}{3} \kappa p_\phi^2$ and $\Omega_3 = \frac{1}{5} \kappa^2 \gamma^2 \xi^2 p_\phi^4$. Note that from equation (15), $p_\phi$ is a constant which characterizes the scalar field in the system. To solve equation (17) we introduce a new dependent variable $u$ in the form

$$ u = p^3. \tag{18} $$

With use of the variable $u$ the equation (17) becomes

$$ \left( \frac{du}{dt} \right)^2 = 9\Omega_1 u - 9\Omega_3, \tag{19} $$

and has a solution

$$ u = \frac{\Omega_3}{\Omega_1} + \frac{9}{4} \Omega_1 t^2 - \frac{9}{2} \Omega_1 C_1 t + \frac{9}{4} \Omega_1 C_1^2, \tag{20} $$

where $C_1$ is a constant of integration. We can choose $C_1 = 0$ through coordinate freedom. Then the solution for $p$ is

$$ p = \left[ \frac{\Omega_3}{\Omega_1} + \frac{9}{4} \Omega_1 t^2 \right]^{1/3}. \tag{21} $$

IV. THE AVERAGED NULL ENERGY CONDITION IN LQC

Based on the above discussion, we come to calculate the averaged null energy condition in the context of LQC. Because of the homogeneity of the universe described by the FRW metric, the null geodesic curves pass through different spacial points are the same. So, to investigate the ANEC, we need only consider one of the null geodesic lines pass any point in space. Due to the isotropy of FRW metric, the null geodesic curves passing through the same point in different directions are also the same. Therefore, our problem reduces to test along any null geodesic line. Specifically, we consider this null geodesic line generated by vectors

$$ \left( \frac{\partial}{\partial t} \right)^\mu + \frac{1}{a} \left( \frac{\partial}{\partial x} \right)^\mu. $$

According to definition $(\frac{\partial}{\partial t})^\mu \nabla^\nu (\frac{\partial}{\partial t})^\nu = 0$ we can reparameterize it with affine parameter $l$ to get

$$ k^\mu = \left( \frac{\partial}{\partial l} \right)^\mu = \frac{1}{a} \left( \frac{\partial}{\partial t} \right)^\mu + \frac{1}{a^2} \left( \frac{\partial}{\partial x} \right)^\mu. $$

Then we can get the relation between $t$ and affine parameter $l$

$$ l = \frac{t}{a}. $$

For the considered universe containing a massless scalar field, the energy density and the pressure of matter can be expressed as

$$ \rho = \frac{1}{2} \dot{\phi}^2, \tag{22} $$

$$ P = \frac{1}{2} \dot{\phi}^2, \tag{23} $$

according to the definition (13). In the context of LQC, taking the quantum effects into consideration, the energy density and the pressure of matter reduce to the effective forms

$$ \rho_{\text{eff}} = \frac{1}{2} \dot{\phi}^2 \left( 1 - \frac{1}{2} \frac{\dot{\phi}^2}{\rho_c} \right), \tag{24} $$

$$ P_{\text{eff}} = \frac{1}{2} \dot{\phi}^2 \left( 1 - \frac{\dot{\phi}^2}{\rho_c} - \frac{1}{4} \frac{\dot{\phi}^4}{\rho_c}. \tag{25} $$

Since the stress-energy tensor reduced from the Einstein’s equations takes ideal fluid form in terms of the effective quantities

$$ T^{\text{eff}}_{\mu\nu} = \rho_{\text{eff}} (dt)^\mu (dt)^\nu + a^2 P_{\text{eff}} \times \left[ (dx)^\mu (dx)^\nu + (dy)^\mu (dy)^\nu + (dz)^\mu (dz)^\nu \right]. \tag{26} $$
the average null energy condition \[ (1) \] becomes
\[
\int_\gamma T^{\mu\nu}_{\text{eff}} k^\mu k^\nu dl = \int_{-\infty}^{\infty} \frac{1}{a} \left( \rho_{\text{eff}} + P_{\text{eff}} \right) dt
\]
\[
= \int_{-\infty}^{\infty} \frac{1}{a} \left( \phi^2 - \frac{\dot{\phi}^2}{\rho c} \right) dt
\]
\[
= p_0^2 \int_{-\infty}^{\infty} \left( p^{-7/2} - \rho^{-13/2} \frac{p_0^2}{\rho c} \right) dt.
\]

In the last line we have used equation (15) and the relationship between \( p \) and \( a \). Substitute the exact solution \[ (1) \] in above equation, we get
\[
\int_\gamma T^{\mu\nu}_{\text{eff}} k^\mu k^\nu dl = -\frac{\Gamma(5/6)\Gamma(2/3)}{7\rho c \Omega_1 (\Omega_2^{1/3})^{13/6}} \sqrt{\frac{\pi}{1111}} p_0^4,
\]
where \( \Gamma \) is the gamma function. From above result it is obvious that
\[
\int_\gamma T^{\mu\nu}_{\text{eff}} k^\mu k^\nu dl < 0.
\]

From the above discussion, we can clearly see that, in addition to violation of every kind of local energy condition, loop quantum cosmology also violates the averaged null energy condition. This averaged null energy condition violation results from quantum effect completely.

V. SUMMARY AND DISCUSSION

Given many kinds of local energy condition violation in loop quantum cosmology and motivated by the topological censorship theorem which rules out traversable wormholes in spacetime where the averaged null energy condition is satisfied, we investigate this kind of nonlocal energy condition in the context of LQC. Our analyses are based on a flat universe containing a massless scalar field. This model can be solved analytically. With the help of the analytical solution and taking advantage of the homogenous and isotropic symmetry of universe, we calculate the average of energy directly. Our result is interesting. Although the quantum correction is focused on the early universe around Planck scale, the correction is so strong that it makes the universe violate the null averaged energy condition. Mathematically we have written the modified Einstein equations in LQC in original Einstein equations form (refer to Eqs. (10)-(11)) but with effective stress-energy tensor. This form of equations makes the proof of the censorship theorem in \[ (9) \] valid also. So the ANEC (for the original stress-energy tensor instead of the effective one) argument can not forbid existence of wormhole once the Loop Quantum Gravity effects are taken into account. But we do not expect the existence of wormhole in LQC, because it is too symmetric to support wormholes \[ (22) \]. On the other hand, LQC adopts the essence of LQG, so our result can shed some light on the ANEC of LQG. And we hope this result can give some hints for looking for wormhole solutions in the LQG theory. These interesting objects will provide some gedanken-experiments to test our quantum gravity theory.

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Physically speaking, the Einstein equations in LQC are modified, while the stress-energy tensor $T_{\mu\nu}$ is unchanged. But mathematically, we can move the terms modified by LQC to the side of stress-energy tensor, and call them the quantum effect of stress-energy tensor. This viewpoint will bring some advantages and make some previous analysis result still valid. For an example, the proof of censorship theorem in [9] used the Einstein equations extensively. But it does not care what’s the form of matter. Instead, the geometric quantities are very important to the proof. If we just put the modified terms into the stress-energy tensor, the proof in that reference is still valid. In this paper we take this viewpoint.

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