CONVEXITY, CRITICAL POINTS, AND CONNECTIVITY RADIUS

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Abstract. We study the level sets of the distance function from a boundary point of a convex set in Euclidean space. We provide a lower bound for the range of connectivity of the level sets, in terms of the critical points of the distance function in the sense of Grove–Shiohama–Gromov–Cheeger.

1. Introduction

Critical point theory for non-smooth functions such as the distance function from a point in a Riemannian manifold was developed by Grove and Shiohama [3], Gromov [2], and Cheeger [1] (see also [4]). We will exploit this notion of criticality to derive an optimal lower bound for the connectivity radius at a boundary point of a convex set in Euclidean space in terms of the least critical distance. The connectedness of the level sets is a question posed at MO. We apply the techniques along the lines of Grove–Shiohama–Gromov–Cheeger, to develop a kind of Morse theory to provide an optimal answer. Here the answer is optimal in the sense that it is easy to give nontrivial cases where the bound is optimal.

2. Connectivity radius and critical distance

Let $K \subseteq \mathbb{R}^n$ be a closed convex set of dimension at least 2. Consider a boundary point $O \in \partial K$. Let $S_r(O) \subseteq \mathbb{R}^n$ be the sphere of radius $r > 0$ centered at $O$.

Definition 2.1. The connectivity radius $CR(O)$ of $O \in \partial K$ is the supremum of all $\epsilon$ such that for all $r < \epsilon$ the intersection $S_r(O) \cap K$ is connected.

Definition 2.2. A point $P \in \partial K \setminus \{O\}$ is $O$-critical, or critical for short, if one of the following equivalent conditions is satisfied:

1. $\langle O - P, X - P \rangle \geq 0$ for all $X \in K$;
(2) the affine hyperplane through $P$ orthogonal to $OP$ is a supporting hyperplane for $K$.

**Example 2.3.** If $K \subseteq \mathbb{R}^2$ is a disk and $O \in \partial K$, then the only $O$-critical point is the antipodal point of $O$.

**Definition 2.4.** The least critical distance $LCD(O)$ is the infimum of $|OP|$ where the infimum is taken over all $O$-critical points $P \in \partial K$.

### 3. The results

**Theorem 3.1.** For each $O \in \partial K$ we have $CR(O) \geq LCD(O)$.

**Example 3.2.** Let $K \subseteq \mathbb{R}^2$ be an acute-angled triangle. If $O$ a vertex of $K$, then $CR(O)$ is the length of the altitude from $O$ to the opposite side, and the foot of the altitude is an $O$-critical point. If $O$ contained in an open side of $K$, then $CR(O)$ is the smaller of the two distances from $O$ to the remaining two sides, and the foot of each perpendicular is a critical point.

The lower bound provided by the theorem is nontrivial due to the following lemma.

**Lemma 3.3.** Let $K \subseteq \mathbb{R}^n$ be a convex set, and let $O \in \partial K$. Then $LCD(O) > 0$.

**Proof.** If $P_1$ and $P_2$ are critical points with angle $\angle P_1OP_2 \leq \frac{\pi}{3}$ then by the Pythagorean Theorem we have $\frac{|OP_1|}{|OP_2|} \leq 2$.

Suppose $(P_i)$ is a sequence of critical points tending to $O$. Passing to a subsequence if necessary, we can assume that $\frac{|OP_i|}{|OP_{i+1}|} > 2$ for each $i$. Then $\angle P_iOP_j > \frac{\pi}{3}$ whenever $i < j$. But the sphere of directions at $O$ can only contain finitely many directions such that all pairwise angles are greater than $\frac{\pi}{3}$. The contradiction shows that the critical points must be bounded away from $O$. \(\square\)

We first illustrate the theorem by considering the case $n = 2$.

**Lemma 3.4.** Let $K \subseteq \mathbb{R}^2$ be compact and convex. Let $O \in \partial K$ and suppose $S_r(O) \cap K$ has more than one connected component. Then there is an $O$-critical point $C$ with $|OC| \leq r$.

**Proof.** Let $B_r(O)$ be the closed disk of radius $r$. The hypothesis of the lemma implies that the curve $\partial K \cap B_r(O)$ also has more than one connected component. Let $\gamma \subseteq \partial K \cap B_r(O)$ be a connected component not containing the point $O$. If $\gamma$ is a single point then it is $O$-critical. Otherwise, let $A, B \in \gamma$ be the endpoints of the curve $\gamma$. Clearly $|AO| = |BO| = r$. If $\gamma \subseteq S_r(O)$ then each interior point of $\gamma$ is $O$-critical,
proving the bound in this case. Thus we may assume that some points of $\gamma$ lie in the interior of $B_r(O)$. Let $C \in \gamma$ be the point at least distance $|CO|$ from $O$. By convexity of $K$ and first variation, the line through $C$ orthogonal to $OC$ is a supporting line for $K$. Thus $C$ is an $O$-critical point and $|OC| < r$ in this case. □

At every non-critical point, there is a tangent vector with positive (outward) $O$-radial component and pointing toward the interior of $K$.

A standard partition of unity argument using the convexity of the tangent cone of $K$ at boundary points yields the following lemma.

**Lemma 3.5.** Let $K \subseteq \mathbb{R}^n$ be convex. Let $L = S_r(O) \cap K$ be a level set not containing any critical points. Then there exists a smooth vector field along $L$ with constant positive radial component and pointing toward the interior of $K$.

We will exploit Lemma 3.5 to prove our main theorem.

**Proof of Theorem 3.1.** The case when $K$ is 2-dimensional was treated in Lemma 3.4. We now treat the general case $n \geq 2$. Let $K_r = S_r(O) \cap K$.

Suppose $K_r$ is not connected. We will show that $r > \text{LCD}(O)$.

Consider a pair of distinct connected components $X, Y$ of $K_r$. We identify $O$ with the origin and choose rays $\mathbb{R}^+ x, \mathbb{R}^+ y$ meeting $X$ and $Y$ respectively. Let $\beta \leq r$ be the infimum of radii $s \leq r$ such that the connected components of the points $x_s = \mathbb{R}^+ x \cap K_s$ and $y_s = \mathbb{R}^+ y \cap K_s$ are still distinct. Since dim $K \geq 2$, a path in $K$ connecting the two rays can be chosen to avoid the point $O$ and then pulled in radially to the level containing a point of the path nearest $O$. Hence $\beta > 0$.

Let us show that points $x_\beta$ and $y_\beta$ are in the same connected component of $K_\beta$. Suppose otherwise. Then there are disjoint open sets $U, V \subseteq \mathbb{R}^n$ such that $x_\beta \in U$, $y_\beta \in V$, and $K_\beta \subseteq U \cup V$. Let $\epsilon_n = \frac{1}{n}$. Since $K$ is star-shaped at $O$ and closed, the intersections $K_{\beta - \epsilon_n} \cap \hat{U}$ and $K_{\beta - \epsilon_n} \cap V$ are still non-empty for sufficiently large $n$. By definition of $\beta$, the points $x_{\beta - \epsilon_n}$ and $y_{\beta - \epsilon_n}$ are in the same connected component of $K_{\beta - \epsilon_n}$. Therefore there exists a point $z_n \in K_{\beta - \epsilon_n}$ such that $z_n \notin U \cup V$. Passing to a subsequence if necessary we can assume that $(z_n)$ converges. Let $z = \lim_{n \to \infty} z_n$. By compactness of $K$, we have $z \in K_\beta$. On the other hand by construction $z \notin U \cup V$. This contradicts the fact that $K_\beta \subseteq U \cup V$. The contradiction proves that $K_\beta$ is connected.

If the connected level set $K_\beta$ did not contain any $O$-critical point, we could use the flow generated by the vector field of Lemma 3.5 to push
it out into a level $K_{r+\epsilon}$ for $\epsilon > 0$, contradicting the minimality of $\beta$. Hence $K_{\beta}$ must contain a critical point, and thus $r > \beta \geq \text{LCD}(O)$, proving the theorem. \hfill \box

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