TOPOLOGICAL BIFURCATIONS
OF MINIMAL INVARIANT SETS
FOR SET-VALUED DYNAMICAL SYSTEMS

JEROEN S.W. LAMB, MARTIN RASMUSSEN AND CHRISTIAN S. RODRIGUES

Abstract. We discuss the dependence of set-valued dynamical systems on parameters. Under mild assumptions which are naturally satisfied for random dynamical systems with bounded noise and control systems, we establish the fact that topological bifurcations of minimal invariant sets are discontinuous with respect to the Hausdorff metric, taking the form of lower semi-continuous explosions and instantaneous appearances. We also characterise these transitions by properties of Morse-like decompositions.

1. Introduction

Dynamical systems usually refer to time evolutions of states, where each initial condition leads to a unique state of the system in the future. Set-valued dynamical systems allow a multi-valued future, motivated, for instance, by impreciseness or uncertainty. In particular, set-valued dynamical systems naturally arise in the context of random and control systems.

The main motivation for the work in this paper is the study of random dynamical systems represented by a mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with a bounded noise of size $\varepsilon > 0$,

$$x_{n+1} = f(x_n) + \xi_n,$$

where the sequence $(\xi_n)_{n \in \mathbb{N}}$ is a uniformly distributed random variable with values in $B_\varepsilon(0) := \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$. The collective behavior of all future trajectories is then represented by a set-valued mapping $F : \mathcal{K}(\mathbb{R}^d) \rightarrow \mathcal{K}(\mathbb{R}^d)$, defined by

$$F(M) := B_\varepsilon(f(M)) \text{ for all } M \in \mathcal{K}(\mathbb{R}^d),$$

where $\mathcal{K}(\mathbb{R}^d)$ is the set of all nonempty compact subsets of $\mathbb{R}^d$.

Under the natural assumption that the probability distribution on $B_\varepsilon(0)$ has a non-vanishing Lebesgue density, it turns out that the supports of stationary measures of the random dynamical system are minimal invariant sets of the set-valued mapping $F$ [Ara00, ZH07]. A minimal invariant set is a compact set $M \subset \mathbb{R}^d$ that is invariant (i.e. $F(M) = M$) and contains no proper invariant subset.

2010 Mathematics Subject Classification. 37G35, 37H20, 37C70, 49K21 (primary), 37B25, 34A60 (secondary).

The first author was supported by a FAPESP-Brazil Visiting Professorship (2009-18338-2). The first and second author gratefully acknowledge partial support by EU IRSES project DynEur-Braz and the warm hospitality of IMECC UNICAMP during the development of this paper. The second author was supported by an EPSRC Career Acceleration Fellowship and a Marie Curie Intra European Fellowship of the European Community.

We are grateful to Janosch Rieger for many useful comments on a preliminary version of this paper.
In this paper, we are mainly interested in topological bifurcations of minimal invariant sets, while considering a parameterized family of set-valued mappings \((F_\lambda)_{\lambda \in \Lambda}\), where \(\Lambda\) is a metric space. These bifurcations involve discontinuous changes as well as disappearances of minimal invariant sets under variation of \(\lambda\).

**Definition 1.1** (Topological bifurcation of minimal invariant sets). Let \((F_\lambda)_{\lambda \in \Lambda}\) be a continuously parameterized family of set-valued mappings on \(\mathbb{R}^d\) such that \(F_\lambda(x)\) contains a ball for all \(\lambda \in \Lambda\) and \(x \in \mathbb{R}^d\), and let \(M_\lambda\) denote the union of minimal invariant sets of \(F_\lambda\), \(\lambda \in \Lambda\). We say that \(F_\lambda\) admits a topological bifurcation of minimal invariant sets at \(\lambda = \lambda_*\) if for any neighbourhood \(V\) of \(\lambda_*\), there does not exist a family of homeomorphisms \((h_\lambda)_{\lambda \in V}\), \(h_\lambda: \mathbb{R}^d \to \mathbb{R}^d\), depending continuously on \(\lambda\), with the property that \(h_\lambda(M_\lambda) = M_{\lambda_*}\) for all \(\lambda \in V\).

The main result concerns the necessity of discontinuous changes of minimal invariant sets at bifurcation points with two possible local scenarios.

**Theorem 1.2.** Suppose that the family \((F_\lambda)_{\lambda \in \Lambda}\) admits a bifurcation at \(\lambda = \lambda_*\). Then a minimal invariant set changes discontinuously at \(\lambda = \lambda_*\) in one of the following ways:

(i) it explodes lower semi-continuously at \(\lambda_*\), or
(ii) it disappears instantaneously at \(\lambda_*\).

A more technical formulation of this result with the precise assumptions can be found in Theorem 5.1. In fact, the setting of set-valued dynamical systems in this paper is slightly more general than presented above and includes also continuous-time systems.

Another focus of this paper lies in extending Morse decomposition theory to study bifurcation problems in our context. Recently, Morse decompositions have been discussed for set-valued dynamical systems \([BBS, Li07, McG92]\), and we generalize certain fundamental results for attractors and repellers to complementary invariant sets. The second main result of this paper (Theorem 6.1) asserts that at a bifurcation point, these complementary invariant sets must touch.

In the context of the presented motivation above, we note that the study of random dynamical systems with bounded noise can be separated into a topological part (involving the mapping \(F\)) and the evolution of measures. In contrast, the topological part is redundant in the case of unbounded noise (modelled for instance by Brownian motions), where there is only one minimal invariant set, given by the whole space and supporting a unique stationary measure.

Initial research on bifurcations in random dynamical systems with unbounded noise started in the 1980s, mainly by Ludwig Arnold and co-workers \([Arn98, Bax94, JKP02]\). Two types of bifurcation have been distinguished so far: the *phenomenological bifurcation* (P-bifurcation), concerning qualitative changes in stationary densities, and the *dynamical bifurcation* (D-bifurcation), concerning the sign change of a Lyapunov exponent, cf. also \([Ash99]\). To a large extent, however, bifurcations in random dynamical systems remain unexplored.

In modelling, bounded noise is often approximated by unbounded noise with highly localized densities in order to enable the use of stochastic analysis. In this approximation, topological tools to identify bifurcations are inaccessible, leaving the manifestation of a topological bifurcation as a cumbersome quantitative and qualitative change of properties of invariant measures.
Our work contributes to the abstract theory of set-valued dynamical systems dating back to the 1960s. Early contributions were motivated mainly by control theory [Rox65, Klo78], and later developments include stability and attractor theory [Ara00, GK01, Gr¨u02, KMR11, McG92, Rox97], Morse decompositions [BBS, Li07, McG92] and ergodic theory [Art00].

Our results build upon initial piloting studies concerning bifurcations in random dynamical systems with bounded noise [BHY, CGK08, CHK10, HY06, HY10, ZH07, ZH08] and control systems [CK03, CMKS08, CW09, Gay04, Gay05]. In particular, Theorem 1.2 unifies and generalises observations in [BHY, HY06, ZH07] to higher dimensions and non-invertible (set-valued) systems, while the bifurcation analysis in terms of Morse-like decompositions is a novel perspective.

We finally remark that set-valued dynamical systems appear in the literature also as closed relations, general dynamical systems, dispersive systems or families of semi-groups.

2. Set-valued dynamical systems

Throughout this paper, we consider the phase space of a set-valued dynamical system to be a compact metric space \((X, d)\). We restrict to the setting of a compact phase space throughout the paper, although some of our results extend naturally to noncompact complete phase spaces.

Let \(B_\varepsilon(x_0) = \{ x \in X : d(x, x_0) < \varepsilon \}\) denote the \(\varepsilon\)-neighbourhood of a point \(x_0 \in X\). For arbitrary nonempty sets \(A, B \subset X\) and \(x \in X\), let \(\text{dist}(x, A) := \inf \{ d(x, y) : y \in A \}\) be the distance of \(x\) to \(A\) and \(\text{dist}(A, B) := \sup \{ \text{dist}(x, B) : x \in A \}\) be the Hausdorff semi-distance of \(A\) and \(B\). The Hausdorff distance of \(A\) and \(B\) is then defined by \(h(A, B) := \max \{ \text{dist}(A, B), \text{dist}(B, A) \}\).

The set of all nonempty compact subsets of \(X\) will be denoted by \(\mathcal{K}(X)\). Equipped with the Hausdorff distance \(h\), \(\mathcal{K}(X)\) is also a metric space \((\mathcal{K}(X), h)\). It is well-known that if \(X\) is complete or compact, then \(\mathcal{K}(X)\) is also complete or compact, respectively. Define for a sequence \((M_n)_{n \in \mathbb{N}}\) of bounded subsets of \(X\),

\[
\limsup_{n \to \infty} M_n := \left\{ x \in X : \liminf_{n \to \infty} \text{dist}(x, M_n) = 0 \right\}
\]

and

\[
\liminf_{n \to \infty} M_n := \left\{ x \in X : \limsup_{n \to \infty} \text{dist}(x, M_n) = 0 \right\}
\]

(see [AF90, Definition 1.1.1]).

In this paper, a set-valued dynamical system is understood as a mapping \(\Phi : T \times X \to \mathcal{K}(X)\) with time set \(T = \mathbb{N}_0\) (discrete) or \(T = \mathbb{R}_+^\circ\) (continuous), which fulfills \(\Phi(1, X) = X\) and the following properties:

(H1) \(\Phi\) is continuous.

(H2) \(\Phi(0, \xi) = \{\xi\}\) for all \(\xi \in X\).

(H3) \(\Phi(t + \tau, \xi) = \Phi(t, \Phi(\tau, \xi))\) for all \(t, \tau \geq 0\) and \(\xi \in X\).

Note that in (H2), the extension \(\Phi(t, M) := \bigcup_{x \in M} \Phi(t, x)\) for \(M \subset X\) was used.

There is a one-to-one correspondence between discrete set-valued dynamical systems and continuous mappings \(f : X \to \mathcal{K}(X)\). On the other hand, set-valued dynamical systems arise in the context of differential inclusions, which canonically generalize ordinary differential equations to multi-valued vector fields [ACS4, Dei92]. Note that the \(\varepsilon\)-perturbation of a discrete mapping as discussed in the Introduction yields a set-valued dynamical system with continuous dependence on \(x\).
Associated to every set-valued dynamical system is a so-called dual system.

**Definition 2.1** (Dual of a set-valued dynamical system). Let \( \Phi : \mathbb{T} \times X \to \mathcal{K}(X) \) be a set-valued dynamical system. Then the *dual* of \( \Phi \) is defined by \( \Phi^* : \mathbb{T} \times X \to \mathcal{K}(X) \), where

\[
\Phi^*(t, \xi) := \{ x \in X : \xi \in \Phi(t, x) \} \quad \text{for all} \quad (t, \xi) \in \mathbb{T} \times X .
\]

Note that in case of an invertible (single-valued) dynamical system, \( \Phi^* \) coincides with the system under time reversal.

To see that \( \Phi^* \) is well-defined, i.e. \( \Phi^*(t, \xi) \in \mathcal{K}(X) \), consider for given \((t, \xi) \in \mathbb{T} \times X \) a sequence \((x_n)_{n \in \mathbb{N}} \) in \( \Phi^*(t, \xi) \) converging to \( x \in X \). This means that \( \xi \in \Phi(t, x_n) \) for all \( n \in \mathbb{N} \), and hence, \( \xi \in \lim_{n \to \infty} \Phi(t, x_n) = \Phi(t, x) \) by continuity of \( \Phi \). Thus, \( x \in \Phi^*(t, \xi) \), which proves that this set belongs to \( \mathcal{K}(X) \). Note that \( \Phi(1, X) = X \) implies that the images of \( \Phi^* \) are non-empty.

The dual \( \Phi^* \) was introduced already in [McG92] without formalising its properties. The following proposition shows that indeed \( \Phi^* \) fulfills the initial value condition (H2) and the group property (H3), but it can be shown that \( \Phi^* \) can be discontinuous.

**Proposition 2.2.** The dual system \( \Phi^* \) fulfills (H2)–(H3).

*Proof.* (H2) One has \( \Phi^*(0, \xi) = \{ x \in X : \xi \in \Phi(0, x) \} = \{ x \in X : \xi \in \{ x \} \} = \{ \xi \} \) for all \( \xi \in X \).

(H3) It follows that

\[
\Phi^*(t + \tau, \xi) = \{ x \in X : \xi \in \Phi(t + \tau, x) \} = \{ x \in X : \xi \in \Phi(\tau, \Phi(t, x)) \} = \{ x \in X : \exists y \in \Phi(t, x) : \xi \in \Phi(\tau, y) \}
\]

\[
= \{ x \in X : \exists y \in \Phi(t, x) : \exists y \in \Phi^*(\tau, y) \}
\]

\[
= \{ x \in X : \exists y \in \Phi^*(\tau, y) : x \in \Phi^*(t, y) \}
\]

\[
= \Phi^*(t, \Phi^*(\tau, \xi)) .
\]

This finishes the proof of this proposition. \( \square \)

3. **Minimal invariant sets**

In the following, the focus lies on the determination and bifurcation of so-called *minimal invariant sets* of a set-valued dynamical system \( \Phi \). Given a set-valued dynamical system \( \Phi : \mathbb{T} \times X \to \mathcal{K}(X) \), a nonempty and compact set \( M \subset X \) is called *\( \Phi \)-invariant* if

\[
\Phi(t, M) = M \quad \text{for all} \quad t \geq 0 .
\]

A \( \Phi \)-invariant set is called *minimal* if it does not contain a proper \( \Phi \)-invariant set.

Minimal \( \Phi \)-invariant sets are pairwise disjoint, and under the assumption that there exists an \( \varepsilon > 0 \) and \( T > 0 \) such that \( \Phi(T, x) \) contains at least an \( \varepsilon \)-ball for all \( x \in X \), there are only finitely many of such sets, since \( X \) is compact.

Minimal \( \Phi \)-invariant sets are important, because they are exactly the supports of stationary measures of a random dynamical system, whenever \( \Phi \) describes the topological part of the random system (see [HY06, HY10, BHY] in the continuous case of random differential equations, and [ZH07] for random maps). Moreover, in case \( \Phi \) describes a control system, minimal \( \Phi \)-invariant sets coincide with invariant control sets (see the monograph [CK00]).
Proposition 3.1. Let \( \Phi : \mathbb{T} \times X \to K(X) \) be a set-valued dynamical system and let \( M \subset X \) be compact with \( \Phi(t,M) \subset M \) for all \( t \geq 0 \), and suppose that no proper subset of \( M \) fulfills this property. Then \( M \) is \( \Phi \)-invariant.

Proof. Standard arguments lead to the fact that the \( \omega \)-limit set
\[
\bigcap_{t \geq 0} \bigcup_{s \geq t} \Phi(s,M) = \bigcap_{t \geq 0} \Phi(t,M)
\]
is a nonempty and compact \( \Phi \)-invariant set \([AF90]\). Since \( \bigcap_{t \geq 0} \Phi(t,M) \subset M \), it follows that this set coincides with \( M \).

The existence of minimal \( \Phi \)-invariant sets follows from Zorn’s Lemma.

Proposition 3.2 (Existence of minimal invariant sets). Let \( \Phi : \mathbb{T} \times X \to K(X) \) be a set-valued dynamical system and \( M \subset X \) be compact such that \( \Phi(t,M) \subset M \) for all \( t \geq 0 \). Then there exists at least one subset of \( M \) which is minimal \( \Phi \)-invariant.

Proof. Consider the collection \( C := \{ A \subset K(M) : \Phi(t,A) \subset A \text{ for all } t \geq 0 \} \). \( C \) is partially ordered with respect to set inclusion, and let \( C' \) be a totally ordered subset of \( C \). It is obvious that \( \bigcap_{A \in C} A \) is nonempty, compact and lies in \( C \). Thus, Zorn’s Lemma implies that there exists at least one minimal element in \( C \) which is a minimal \( \Phi \)-invariant set.

While minimal \( \Phi \)-invariant sets always exists, they are typically non-unique. Uniqueness directly follows for set-valued dynamical systems that are contractions in the Hausdorff metric. Such contractions can be identified by the following lemma, whose proof will be omitted.

Lemma 3.3. Consider the set-valued dynamical system \( \Phi : \mathbb{N}_0 \times K(X) \to K(X) \), defined by \( \Phi(1,x) := U(f(x)) \) for all \( x \in X \), where \( f : X \to X \) is a contraction on the compact metric space \((X,d)\), i.e. one has
\[
d(f(x),f(y)) \leq Ld(x,y) \quad \text{for all } x,y \in X
\]
with some Lipschitz constant \( L < 1 \), and \( U : X \to K(X) \) is a function such that \( U(x) \) is a neighbourhood of \( x \) for all \( x \in X \). Assume that \( U \) is globally Lipschitz continuous (but not necessarily a contraction) with Lipschitz constant \( M > 0 \) such that \( ML < 1 \). The mapping \( \Phi(1,\cdot) \) then is a contraction in \((K(X),h)\). The unique fixed point of \( \Phi(1,\cdot) \) is the unique minimal \( \Phi \)-invariant set, which is also globally attractive.

The above lemma applies in particular to the motivating example presented in the Introduction. In this case, \( U(x) := \overline{B}_\varepsilon(x) \) with Lipschitz constant 1. Hence, if \( f \) is a contraction, then the set-valued mapping \( F \) has a globally attractive unique minimal invariant set.

4. Generalisation of attractor-repeller decomposition

The purpose of this section is to provide generalisations of attractor-repeller decompositions which were introduced in [MW06, Li07] for the study of Morse decompositions of set-valued dynamical systems. These generalisations are necessary for our purpose, because we deal with invariant sets rather than attractors, and they will be applied in Section 6 in the context of bifurcation theory.
Fundamental for the definition of Morse decompositions are domains of attraction (of attractors), because associated repellers are then identified as complements of these sets. For a given $\Phi$-invariant set $M$, the domain of attraction is defined by

$$\mathcal{A}(M) = \left\{ x \in X : \lim_{t \to \infty} \text{dist} \left( \Phi(t, x), M \right) = 0 \right\}.$$

If $M$ is an attractor, that is a $\Phi$-invariant set such that there exists an $\eta > 0$ with

$$\lim_{t \to \infty} \text{dist} \left( \Phi(t, B_\eta(M)), M \right) = 0,$$

then the complementary set $X \setminus \mathcal{A}(M)$ is a $\Phi^*$-invariant set, which has the interpretation of a repeller, because all points outside of this set converge to the attractor in forward time. It is worth to note that this repeller is not necessarily $\Phi$-invariant (which is a difference from the classical Morse decomposition theory).

For a $\Phi$-invariant set $M$ which is not an attractor, the complementary set $X \setminus \mathcal{A}(M)$ is not necessarily $\Phi^*$-invariant, but this property can be attained when $\mathcal{A}(M)$ is replaced by a slightly smaller set.

**Proposition 4.1.** Let $\Phi : T \times X \to \mathcal{K}(X)$ be a set-valued dynamical system, and let $M \subset X$ be $\Phi$-invariant such that $\mathcal{A}(M) \neq X$, i.e. $M$ is not globally attractive. Then the complement of the set

$$\mathcal{A}_-(M) := \mathcal{A}(M) \setminus \left\{ x \in \mathcal{A}(M) : \text{there exist } t \geq 0 \text{ with } \Phi(t, x) \cap \partial \mathcal{A}(M) \neq \emptyset, \right\},$$

or for all $\gamma > 0$, one has

$$\lim_{t \to \infty} \sup \left( \text{dist} \left( \Phi(t, B_\gamma(x)), M \right) > 0 \right)$$

i.e. the set $M^* := X \setminus \mathcal{A}_-(M)$, is $\Phi^*$-invariant.

The set $M^*$ is called the dual of $M$. Under the additional assumption that $M$ is an attractor in Proposition [4.1] i.e. $\mathcal{A}(M)$ is a neighbourhood of $M$, the pair $(M, M^*)$ is an attractor-repeller pair as discussed in [MW06]. This pair can be extended to obtain Morse decompositions, see [Li07].

Before proving this proposition, we will derive an alternative characterization of the set $\mathcal{A}_-(M)$.

**Lemma 4.2.** Let $\Phi : T \times X \to \mathcal{K}(X)$ be a set-valued dynamical system and $M \subset X$ be $\Phi$-invariant. Then the set $\mathcal{A}_-(M)$ admits the representation

$$\mathcal{A}_-(M) = \left\{ x \in X : \text{for all } T \geq 0, \text{ there exists a neighbourhood } V \text{ of } \Phi(T, x) \right. \left. \text{with } \lim_{t \to \infty} \text{dist} \left( \Phi(t, V), M \right) = 0 \right\}.$$

**Proof.** First, note that compact subsets $K$ of $\mathcal{A}_-(M)$ are attracted by $M$, i.e. we have $\lim_{t \to \infty} \text{dist} \left( \Phi(t, K), M \right) = 0$. We have to show two set inclusions.

($\subset$) Let $x \in \mathcal{A}_-(M)$ and $T > 0$. Since $\Phi(T, x)$ lies in the interior of $\mathcal{A}(M)$, there exists a compact neighbourhood $V$ of $\Phi(T, x)$ that is contained in $\mathcal{A}(M)$. This proves that $\lim_{t \to \infty} \text{dist} \left( \Phi(t, V), M \right) = 0$, and hence, $x$ is contained in the right hand side.

($\supset$) Let $x \in \mathcal{A}_-(M)$ such that for all $T \geq 0$, there exists a neighbourhood $V$ of $\Phi(T, x)$ with $\lim_{t \to \infty} \text{dist} \left( \Phi(t, V), M \right) = 0$. This implies that for all $T \geq 0$, one has $\Phi(t, x) \cap \partial \mathcal{A}(M) = 0$, which finishes the proof of this lemma.

The set $\mathcal{A}_-(M)$ thus describes all trajectories in the domain of attraction that are attracted also under perturbation.
Proof of Proposition 4.1. It will be shown that $\Phi^*(t, M^*) = M^*$ for all $t \geq 0$.

($) Assume that there exist $t > 0$ and $x \in \Phi^*(t, M^*) \setminus M^* = \Phi^*(t, M^*) \cap A_-(M)$. This implies that $\Phi(t, x) \cap M^* \neq \emptyset$ and $x \in A_-(M)$, which contradicts the fact that $A_-(M)$ fulfills $\Phi(t, A_-(M)) \subset A_-(M)$ for all $t \geq 0$.

($) Assume that there exist $t > 0$ and $x \in M^* \setminus \Phi^*(t, M^*)$. This means that $\Phi(t, x) \cap M^* = \emptyset$, and hence, $\Phi(t, x) \subset A_-(M)$. We will show that this implies that $x \in A_-(M)$, which is a contradiction. Let $T \geq 0$, and consider first the case that $T \leq t$. The fact that $A_-(M)$ is open and $\Phi(t, x) \subset A_-(M)$ is compact implies that there exists a $\gamma > 0$ such that $B_\gamma(\Phi(t, x)) \subset A_-(M)$. Moreover, the continuity of $\Phi$ and the relation $\Phi(t-T, \Phi(T, x)) = \Phi(t, x)$ yield the existence of $\delta > 0$ such that $\Phi(t-T, B_\delta(\Phi(T, x))) \subset B_\gamma(\Phi(t, x)) \subset A_-(M)$. Since compact subsets of $A_-(M)$ are attracted to $M$, the assertion follows. Consider now the case $T > t$. Since $A_-(M)$ is invariant and $\Phi(t, x) \subset A_-(M)$, $\Phi(T, x)$ is a compact subset of $A_-(M)$. $A_-(M)$ is open, so there exists a compact neighbourhood of $\Phi(T, x)$ which is attracted by $M$. This finishes the proof of this proposition.

5. Dependence of minimal invariant sets on parameters

The main goal of this section is to describe how minimal invariant sets depend on parameters. We consider a family $(\Phi_\lambda)_{\lambda \in A}$ of set-valued dynamical systems $\Phi_\lambda : T \times X \to K(X)$, where $(A, d_\lambda)$ is a metric space.

We assume now the conditions (H4) and (H5) that are naturally fulfilled for set-valued dynamical systems generated by mappings $f_\lambda : X \to X$, depending continuously on a real parameter $\lambda$ and perturbed by a closed $\varepsilon$-ball as in (1.1). The first condition addresses uniform continuity in $x \in X$.

(H4) $(\lambda, t) \mapsto \Phi_\lambda(t, x)$ is continuous in $(\lambda, t) \in A \times T$ uniformly in $x$.

As in Theorem 4.1 we exclude single-valued dynamical systems and assume

(H5) $\Phi_\lambda(t, x)$ contains a ball of positive radius for all $(t, x) \in T \times X$ with $t > 0$

and $\lambda \in A$, and moreover, there exist $T > 0$ and $\varepsilon > 0$ such that $\Phi_\lambda(T, x)$ contains a ball of size $\varepsilon$ for all $x \in X$.

The union of all minimal $\Phi_\lambda$-invariant sets in $X$ will be denoted by $M_\lambda$. The following theorem describes how $M_\lambda$ depends on the parameter.

Theorem 5.1 (Dependence of minimal invariant sets on parameters). Given a family of set-valued dynamical systems $(\Phi_\lambda)_{\lambda \in A}$ satisfying (H1)–(H5), and let $M_{\lambda_\infty} \subset M_{\lambda_k}$ be a minimal $\Phi_{\lambda_k}$-invariant set for some $\lambda_\infty \in A$. Then for each sequence $(\lambda_n)_{n \in \mathbb{N}}$ converging to $\lambda_\infty$, there exist a subsequence $(\lambda_{nk})_{k \in \mathbb{N}}$ and a $\delta > 0$ such that exactly one of the following statements holds.

(i) Lower semi-continuous dependence:

$$M_{\lambda_\infty} \subset \liminf_{k \to \infty} \left( M_{\lambda_{nk}} \cap B_\delta(M_{\lambda_\infty}) \right).$$

(ii) Instantaneous appearance:

$$\emptyset = \limsup_{k \to \infty} \left( M_{\lambda_{nk}} \cap B_\delta(M_{\lambda_\infty}) \right).$$

Proof. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence with $\lambda_n \to \lambda_\infty$ as $n \to \infty$. Define the sequence $(c_n)_{n \in \mathbb{N}}$ by

$$c_n := \begin{cases} 1 \quad : \quad M_{\lambda_n} \cap M_{\lambda_\infty} \neq \emptyset \\ 2 \quad : \quad M_{\lambda_n} \cap M_{\lambda_\infty} = \emptyset \quad \text{for all } n \in \mathbb{N}, \end{cases}$$
and choose $\delta > 0$ such that $B_{\delta}(M_{\lambda_{\infty}}) \cap M_{\lambda_{\infty}} = M_{\lambda_{\infty}}$. Since $\{1, 2\}$ is finite, there exists a constant subsequence $c_{n_k} k \in \mathbb{N}$.

If $c_{n_k} \equiv 2$, assume to the contrary that for all $k \in \mathbb{N}$, there exist $m \geq k$ and $a_k \in M_{\lambda_{n_k}} \cap B_{1/k}(M_{\lambda_{\infty}})$, the sequence $(a_k)_{k \in \mathbb{N}}$ has a convergent subsequence with limit $a_{\infty} \in M_{\lambda_{\infty}}$. Now $\Phi_{\lambda_{n_k}}(T,a_{\infty}) \subset M_{\lambda_{\infty}}$, and the continuity of $\Phi$ implies that there exists a $\gamma > 0$ such that $\Phi_{\lambda_{n_k}}(T,\varpi) \subset B_{\gamma/4}(\Phi_{\lambda_{n_k}}(T,a_{\infty}))$ for all $x \in B_\gamma(a_{\infty})$. (H4) then implies the existence of $N > 0$ such that for all $m > N$, we have $\Phi_{\lambda_{n_k}}(T,x) \subset B_{\gamma/4}(\Phi_{\lambda_{n_k}}(T,x)) \subset B_{\gamma/2}(\Phi_{\lambda_{n_k}}(T,a_{\infty})) \subset B_{\gamma/2}(M_{\lambda_{\infty}})$ for all $m > N$ and $x \in B_\gamma(a_{\infty})$. Since $\Phi_{\lambda_{n_k}}(T,x)$ contains an $\varepsilon$-ball and is within the $\varepsilon/2$-neighbourhood of $M_{\lambda_{\infty}}$, one gets $\Phi_{\lambda_{n_k}}(T,x) \cap M_{\lambda_{\infty}} \neq \emptyset$. This is a contradiction to the definition of the sequence $c_{n_k}$ and this proves that there exists $\delta \in (0,\bar{\delta})$ with $M_{\lambda_{n_k}} \cap B_{\delta}(M_{\lambda_{\infty}}) = \emptyset$ whenever $\tfrac{1}{k} < \delta$. Hence, (ii) holds.

If $c_{n_k} \equiv 1$, define $\delta := \bar{\delta}$. Choose minimal $\Phi_{\lambda_{n_k}}$-invariant sets $M_{\lambda_{n_k}} \subset M_{\lambda_{n_k}}$ such that $M_{\lambda_{n_k}} \cap M_{\lambda_{\infty}} \neq \emptyset$ for $k \in \mathbb{N}$. Since $\Phi_{\lambda_{n_k}}(T,M_{\lambda_{n_k}} \cap M_{\lambda_{\infty}}) \subset M_{\lambda_{\infty}}$, (H4) implies that there exists a $k_0 \in \mathbb{N}$ such that

$$(5.1) \quad \Phi_{\lambda_{n_k}}(T,M_{\lambda_{n_k}} \cap M_{\lambda_{\infty}}) \subset B_{\varepsilon/4}(M_{\lambda_{\infty}}) \cap M_{\lambda_{n_k}} \quad \text{for all } k \geq k_0.$$ 

Let $B_{\varepsilon/4}(d_1), \ldots, B_{\varepsilon/4}(d_r)$ with $d_1, \ldots, d_r \in M_{\lambda_{\infty}}$ be finitely many $\varepsilon/4$-balls covering the compact set $B_{\varepsilon/4}(M_{\lambda_{\infty}})$. Because of (H5) and (5.1), each of the sets $M_{\lambda_{n_k}}$ contains (at least) one of the points $d_1, \ldots, d_r$. We can thus put the sets $M_{\lambda_{n_k}}$ into $r$ different systems of sets $C_i$ such that $\bigcap_{M \in C_i} M \supset \{d_i\}$ for $i \in \{1, \ldots, r\}$. We show now that the asserted limit relation in (i) holds when restricting to a subsequence corresponding to each category, from which the assertion follows, since there are thus only finitely many categories. For simplicity, assume now that there is only one category. It will be shown now that $\liminf_{k \to \infty} M_{\lambda_{n_k}}$ cannot be left in forward time for $\lambda = \lambda_{\infty}$, i.e. fulfills the conditions of Proposition 5.1. Since this set is nonempty and intersects $M_{\lambda_{\infty}}$, minimality of $M_{\lambda_{\infty}}$ then implies the assertion. Assume to the contrary that there exists an $\bar{x} \in \liminf_{k \to \infty} M_{\lambda_{n_k}}$ such that $\Phi_{\lambda_{n_k}}(\tau,\bar{x}) \setminus \liminf_{k \to \infty} M_{\lambda_{n_k}} \neq \emptyset$ for some $\tau > 0$, i.e. there exists a $\xi \in \Phi_{\lambda_{n_k}}(\tau,\bar{x})$ such that $\xi \notin \liminf_{k \to \infty} M_{\lambda_{n_k}}$. (H4) implies the existence of a sequences $(x_{n_k})_{k \in \mathbb{N}}$ (converging to $\bar{x}$) and $(y_{n_k})_{k \in \mathbb{N}}$ (converging to $\xi$) such that $x_{n_k} \in M_{\lambda_{n_k}}$ and $y_{n_k} \in \Phi_{\lambda_{n_k}}(\tau, x_{n_k}) \subset M_{\lambda_{n_k}}$. Hence, $\xi \in \liminf_{k \to \infty} M_{\lambda_{n_k}}$, which is a contradiction and finishes the proof of this theorem.

The above theorem asserts that discontinuous changes in minimal invariant sets occur either as explosions or as instantaneous appearances. We are led to address the question if a continuous merging process of two minimal invariant sets is possible (note that this is not ruled out by (i) of Theorem 5.1). The following proposition shows that the answer to this question is negative.

**Proposition 5.2.** Let $(\Phi_{\lambda})_{\lambda \in \Lambda}$ be a family of set-valued dynamical systems fulfilling (H1)–(H5), and let $M^1_{\lambda}$ and $M^2_{\lambda}$ be two different minimal $\Phi_{\lambda}$-invariant sets. Then for all $\lambda^* \in \Lambda$, one has

$$\liminf_{\lambda \to \lambda^*} \inf_{(x,y) \in M^1_{\lambda} \times M^2_{\lambda}} d(x,y) > 0,$$

i.e. the sets $M^1_{\lambda}$ and $M^2_{\lambda}$ cannot collide under variation of $\lambda$. 
Proof. Suppose the contrary, which means that there exist an $x^* \in X$ and a sequence $\lambda_n \to \lambda^*$ as $n \to \infty$ with
\[
\lim_{n \to \infty} \text{dist}(x^*, M^1_{\lambda_n}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \text{dist}(x^*, M^2_{\lambda_n}) = 0.
\]
Due to (H4) and (H5), for $t > 0$, the set $\Phi_{\lambda^*}(t, x^*)$ intersects the interior of both $M^1_{\lambda_n}$ and $M^2_{\lambda_n}$ when $n$ is large enough. This, however, contradicts the fact that $M^1_{\lambda_n}$ and $M^2_{\lambda_n}$ are $\Phi$-invariant and finishes the proof of this proposition. 

6. A NECESSARY CONDITION FOR BIFURCATION

Consider a family $(\Phi_\lambda)_{\lambda \in \Lambda}$ of set-valued dynamical systems $\Phi_\lambda : \mathbb{T} \times X \to \mathcal{K}(X)$, where $(\Lambda, d_\Lambda)$ is a metric space, and suppose that (H1)–(H5) hold.

Recall the definition of a topological bifurcation (Definition 1.1) and the fact that $\mathcal{M}_\Lambda$ denotes the union of all minimal $\Phi_\lambda$-invariant sets. As a direct consequence of Theorem 5.1 and Proposition 5.2, a topological bifurcation of $\mathcal{M}_\Lambda$ is characterised by a minimal $\Phi_{\lambda_\infty}$-invariant set $M_{\lambda_\infty}$, a sequence $\lambda_n \to \lambda_\infty$ as $n \to \infty$ and $\delta > 0$ such that
\[
(6.1) \quad M_{\lambda_\infty} \subseteq \liminf_{n \to \infty} (\mathcal{M}_{\lambda_n} \cap B_\delta(M_{\lambda_\infty})) \quad \text{or} \quad \emptyset = \limsup_{n \to \infty} (\mathcal{M}_{\lambda_n} \cap B_\delta(M_{\lambda_\infty}))
\]

The following theorem provides a necessary condition for a topological bifurcation of minimal invariant sets involving the dual $M^*_{\lambda_\infty}$ of $M_{\lambda_\infty}$ as introduced in Section 4.

**Theorem 6.1** (Necessary condition for bifurcation). Let $(\Phi_\lambda)_{\lambda \in \Lambda}$ be a family of set-valued dynamical systems fulfilling (H1)–(H5), and assume that $(\Phi_\lambda)_{\lambda \in \Lambda}$ admits a topological bifurcation such that (6.1) holds for a minimal invariant set $M_{\lambda_\infty}$. Then $M^*_{\lambda_\infty}$ has nonempty intersection with $M_{\lambda_\infty}$.

**Proof.** Consider the sequence $\lambda_n \to \lambda_\infty$ as defined before the statement of the theorem. Assume to the contrary that there exists a $\gamma > 0$ such that $B_\gamma(M_{\lambda_\infty}) \subset \mathcal{A}_\perp(M_{\lambda_\infty})$. Then for each $\delta > 0$, there exists a compact absorbing set $B$ such that $M_{\lambda_\infty} \subset B \subset B_\delta(M_{\lambda_\infty})$, that is, $\Phi_{\lambda_\infty}(t, B) \subset \text{int } B$ for $t > 0$. Due to continuous dependence on $\lambda$, there exists an $n_0 \in \mathbb{N}$ such that $\Phi_{\lambda_n}(t, B) \subset \text{int } B$ for all $n \geq n_0$ and $t > 0$. This means that there exists a minimal $\Phi_{\lambda_n}$-invariant set in $B$ for all $n \geq n_0$. Note that $n_0$ depends on $\delta$, and in the limit $\delta \to 0$, this minimal invariant set converges to $M_{\lambda_\infty}$, because of Theorem 5.1. Hence, there is no bifurcation, which shows that $X \setminus \mathcal{A}_\perp(M_{\lambda_\infty}) \cap M_{\lambda_\infty} \neq \emptyset$. 

**References**

[AC84] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Grundlehren der Mathematischen Wissenschaften, vol. 264, Springer, Berlin, 1984.

[AF90] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Systems and Control: Foundations and Applications, vol. 2, Birkhäuser, Boston, 1990.

[Ara00] V. Araújo, *Attractors and time averages for random maps*, Annales de l’Institut Henri Poincaré – Analyse Non Linéaire 17 (2000), no. 3, 307–369.

[Arn98] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, Heidelberg, New York, 1998.

[Art00] Z. Artstein, *Invariant measures of set-valued maps*, Journal of Mathematical Analysis and Applications 252 (2000), no. 2, 696–709.

[Ash99] P. Ashwin, *Minimal attractors and bifurcations of random dynamical systems*, The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences 455 (1999), no. 1987, 2615–2634.

[Bax94] P.H. Baxendale, *A stochastic Hopf bifurcation*, Probability Theory and Related Fields 99 (1994), no. 4, 581–616.
[BBS] C.J. Braga Barros and J.A. Souza, Attractors and chain recurrence for semigroup actions, to appear in: Journal of Dynamics and Differential Equations.

[BHY] R.T. Botts, A.J. Homburg, and T.R. Young, The hopf bifurcation with bounded noise, submitted.

[CGK08] F. Colonius, T. Gayer, and W. Kliemann, Near invariance for Markov diffusion systems, SIAM Journal on Applied Dynamical Systems 7 (2008), no. 1, 79–107.

[CHK10] F. Colonius, A.J. Homburg, and W. Kliemann, Near invariance and local transience for random diffeomorphisms, Journal of Difference Equations and Applications 16 (2010), no. 2–3, 127–141.

[CK00] F. Colonius and W. Kliemann, The Dynamics of Control, Birkhäuser, Boston, 2000.

[CK03] Limits of input-to-state stability, Systems & Control Letters 49 (2003), no. 2, 111–120.

[CMKS08] F. Colonius, A. Marquardt, E. Kreuzer, and W. Sichermann, A numerical study of capsizing: comparing control set analysis and Melnikov’s method, International Journal of Bifurcation and Chaos 18 (2008), no. 5, 1503–1514.

[CW09] F. Colonius and T. Wichtrey, Control systems with almost periodic excitations, SIAM Journal on Control and Optimization 48 (2009), no. 2, 1055–1079.

[Dei92] K. Deimling, Multivalued Differential Equations, de Gruyter Series in Nonlinear Analysis and Applications, vol. 1, Walter de Gruyter & Co., Berlin, 1992.

[Gay04] T. Gayer, Control sets and their boundaries under parameter variation, Journal of Differential Equations 201 (2004), no. 1, 177–200.

[Gay05] Controllability and invariance properties of time-periodic systems, International Journal of Bifurcation and Chaos 15 (2005), no. 4, 1361–1375.

[GK01] L. Grüne and P.E. Kloeden, Discretization, inflation and perturbation of attractors, Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems (B. Fiedler, ed.), Springer, Berlin, 2001, pp. 399–416.

[Grü02] L. Grüne, Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization, Springer Lecture Notes in Mathematics, vol. 1783, Springer, Berlin, Heidelberg, 2002.

[HY06] A.J. Homburg and T. Young, Hard bifurcations in dynamical systems with bounded random perturbations, Regular & Chaotic Dynamics 11 (2006), no. 2, 247–258.

[HY10] Bifurcations for random differential equations with bounded noise on surfaces, Topological Methods in Nonlinear Analysis 35 (2010), no. 1, 77–98.

[JKP02] R.A. Johnson, P.E. Kloeden, and R. Pavani, Two-step transition in nonautonomous bifurcations: An explanation, Stochastics and Dynamics 2 (2002), no. 1, 67–92.

[Klo78] P.E. Kloeden, General control systems, Mathematical Control Theory (Proc. Conf., Australian Nat. Univ., Canberra, 1977), Lecture Notes in Mathematics, vol. 680, Springer, Berlin, 1978, pp. 119–137.

[KMR11] Peter E. Kloeden and Pedro Marín-Rubio, Negatively invariant sets and entire trajectories of set-valued dynamical systems, Set-Valued and Variational Analysis 19 (2011), no. 1, 43–57.

[Li07] D. Li, Morse decompositions for general dynamical systems and differential inclusions with applications to control systems, SIAM Journal on Control and Optimization 46 (2007), no. 1, 35–60.

[McG92] R.P. McGehee, Attractors for closed relations on compact Hausdorff spaces, Indiana University Mathematics Journal 41 (1992), no. 4, 1165–1209.

[MW06] R.P. McGehee and T. Wiandt, Conley decomposition for closed relations, Journal of Difference Equations and Applications 12 (2006), no. 1, 1–47.

[Rox65] E.O. Roxin, On generalized dynamical systems defined by contingent equations, Journal of Differential Equations 1 (1965), 188–205.

[Rox97] Control Theory and Its Applications, Stability and Control: Theory, Methods and Applications, vol. 4, Gordon and Breach Science Publishers, Amsterdam, 1997.

[ZH07] H. Zmarrou and A.J. Homburg, Bifurcations of stationary measures of random diffeomorphisms, Ergodic Theory and Dynamical Systems 27 (2007), no. 5, 1651–1692.

[ZH08] Dynamics and bifurcations of random circle diffeomorphisms, Discrete and Continuous Dynamical Systems B 10 (2008), no. 2–3, 719–731.