GENERATORS OF DETAILED BALANCE QUANTUM MARKOV SEMIGROUPS

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Abstract

For a quantum Markov semigroup $T$ on the algebra $B(H)$ with a faithful invariant state $\rho$, we can define an adjoint $\tilde{T}$ with respect to the scalar product determined by $\rho$. In this paper, we solve the open problems of characterising adjoints $\tilde{T}$ that are also a quantum Markov semigroup and satisfy the detailed balance condition in terms of the operators $H, L_k$ in the Gorini Kossakovski Sudarshan Lindblad representation $L(x) = i[H, x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k)$ of the generator of $T$. We study the adjoint semigroup with respect to both scalar products $\langle a, b \rangle = \text{tr}(\rho a^* b)$ and $\langle a, b \rangle = \text{tr}(\rho^{1/2} a^* \rho^{1/2} b)$.

Keywords: Quantum detailed balance, quantum Markov semigroup, Lindblad representation

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1 Introduction

The principle of detailed balance is at the basis of equilibrium physics. The notion of detailed balance for open quantum systems (Alicki[2], Frigerio and Gorini[8], Kossakovski, Frigerio, Gorini and Verri[10], Alicki and Lendi[3]) when the evolution is described by a uniformly continuous Quantum Markov Semigroup (QMS) $T$ with a faithful normal invariant state $\rho$, is formulated as a property of the generator $L$. 



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Indeed, for a system with associated separable Hilbert space $\mathfrak{h}$, this can be written in the Gorini Kossakowski Sudarshan Lindblad (GKSL) form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2 L_k^* x L_k + x L_k^* L_k) \tag{1}$$

where $H$ and $L_k$ are operators on $\mathfrak{h}$ that can always be chosen satisfying $\text{tr}(\rho L_k) = 0$ and the natural summability and minimality conditions (see Theorem 30). The state $\rho$ defines a scalar product $(x, y) = \text{tr}(\rho x^* y)$ on the algebra $\mathcal{B}(\mathfrak{h})$ of operators on $\mathfrak{h}$, and $T$ admits a dual semigroup with respect to this scalar product if there exists another uniformly continuous QMS $\tilde{\rho}$ defined on $\mathfrak{h}$ such that $\text{tr}(\rho T_t(x)y) = \text{tr}(\rho x T_t(y))$. The QMS $T$ satisfies the quantum detailed balance condition if the effective Hamiltonian $H$ commutes with $\rho$ and $\tilde{\rho} = \mathcal{L} - 2i[H, \cdot]$ i.e. the dissipative part $\mathcal{L}_0 = \mathcal{L} - i[H, \cdot]$ of $\mathcal{L}$ is self-adjoint.

This generalizes the notion of detailed balance (reversibility) for a classical Markov semigroup which is called reversible when it is self-adjoint in the $L^2$ space of an invariant measure. It is worth noticing, however, that, in the commutative case, the adjoint (dual) of a Markov semigroup is always a Markov semigroup.

The dual of a QMS $T = (T_t)_{t \geq 0}$ with respect to the state $\rho$ may not be a QMS because the adjoint $\tilde{T}_t$ of the map $T_t$ may not be positive or, even more, may not be a $^*$-map i.e. $\tilde{T}_t(a)^* \neq \tilde{T}_t(a^*)$. It is known (see e.g. Ref.[10] Prop. 2.1) that $\tilde{T}_t$ is a completely positive map if it commutes with the modular group $(\sigma_t)_{t \in \mathbb{R}}$ associated with $\rho$. More recently, Majewski and Streater[11] (Thm.6 p.7985) showed that the $\tilde{T}_t$ are (completely) positive whenever they are $^*$-maps.

The structure of the generator (1) of a detailed balance QMS was studied in Ref.[2] and Ref.[10] under the additional assumption that it is a normal operator, i.e. $\mathcal{L}$ and $\tilde{\mathcal{L}}$ commute.

In this paper, we solve the open problems of characterising in terms of the operators $H, L_k$ in (1) dual semigroups $\tilde{T}$ that are QMSs and when they satisfy the detailed balance condition without additional assumptions on the generator $\mathcal{L}$. The main results of this paper, Theorems 26 and 30, describe the structure of generators of QMS whose dual is still a QMS and, among them, the structure of those satisfying the detailed balance condition.

The dual semigroup is a QMS if and only if the maps $T_t$ commute with the modular automorphism (Theorem 8). When this happens we can find particular GKSL representations of $\mathcal{L}$ as in (1), that we call privileged, with $H$ commuting with $\rho$ and the $L_k, L_k^*$’s eigenvalues of the modular automorphism (Definition 20) i.e. $\rho L_k \rho^{-1} = \lambda_k L_k$. Moreover, the generator $\tilde{\mathcal{L}}$ of the dual semigroup admits a privileged GKSL representation with $\tilde{H} = -H - c$ (c is a real constant) and $\tilde{L}_k = \lambda_k^{-1/2} L_k^*$ (Theorem 26).

Finally the quantum detailed balance condition $\mathcal{L} - \tilde{\mathcal{L}} = 2i[H, \cdot]$ (for some self-adjoint operator $K$) holds if and only if $H = K + c$ and there exists a unitary matrix $(u_{k\ell})_{k\ell}$ such that $\lambda_k^{-1/2} L_k^* = \sum_{\ell} u_{k\ell} L_{\ell}$ (Theorem 30).
There are other choices of the scalar product on $B(h)$ induced by $\rho$; we can define $\langle a, b \rangle_s = \text{tr}(\rho^{1-s}a^*\rho^s b)$ for any $s \in [0, 1]$. The most studied case is the previous one with $s = 0$. The case $s = 1/2$, sometimes called symmetric, however, is also interesting (see Goldstein and Lindsay[9]). Indeed, as wrote Accardi and Mohari[1] (p.409), “it is worth characterizing the class of Markov semigroup such that $T_t = \tilde{T}_t$” in full generality also for the dual semigroup with respect to the “symmetric” scalar product $\langle a, b \rangle = \text{tr}(\rho^{1/2}a^*\rho^{1/2} b)$ (Petz’s duality). Note that the “symmetric” dual semigroup is always a QMS.

In Section 4 we solve this problem and the more general problem of characterising QMSs satisfying the “symmetric” detailed balance condition $\mathcal{L} - \mathcal{L}' = 2i[K, \cdot]$ ($\mathcal{L}'$ is the generator of the symmetric dual QMS). We show that the “symmetric” detailed balance is weaker than usual detailed balance (Proposition 33) and establish the relationships among the $L_k$’s, the operator $G = -2^{-1}\sum_k L_k^*L_k - iH$ and $\rho$ of symmetric detailed balance $\mathcal{L}$ (Theorem 10). Examples 38 and 41 show that, in the “symmetric” case, the effective Hamiltonian $H$ may not commute with $\rho$.

The paper is organised as follows. In Section 2 we outline the detailed balance condition for classical Markov semigroups. Then we explore several possible definitions of the dual semigroup in Section 3 and study the generators of QMS whose dual is still a QMS in Section 4. In Section 5 we characterise generators of quantum detailed balance QMS. The special case of QMS on $2 \times 2$ matrices is analysed in Section 6; in this case it turns out that, if the dual semigroup is a QMS, then it satisfies the quantum detailed balance condition. Further examples of detailed balance QMSs, also with unbounded generators can be found in the literature and in Ref.[7]. Finally, in Section 7 we study the symmetric detailed balance condition.

2 Classical detailed balance

Let $(E, E, \mu)$ be a measure space with $\mu$ $\sigma$-finite and let $T = (T_t)_{t \geq 0}$ be a weakly* continuous Markov semigroup of bounded positive linear maps on $L^\infty(E, E, \mu)$. $T$ is the dual semigroup of a strongly continuous contraction semigroup on the predual space $L^1(E, E, \mu)$ denoted $T_*$. Suppose that $T$ admits a $T$-invariant probability density $\pi$ (a norm one, non-negative, function in $L^1(E, E, \mu)$ such that $T_{*t}\pi = 0$ for $t \geq 0$) vanishing only on an element of $E$ of measure 0. Then, it is well-known that the sesquilinear form

$$(f, g) = \int_E \bar{f}g\pi \, d\mu$$

defines a scalar product on $L^\infty(E, E, \mu)$, that we denote by $\langle \cdot, \cdot \rangle_\pi$, and putting

$$\tilde{T}_t(g) = \pi^{-1}T_{*t}(\pi g)$$

for each $t \geq 0$ one defines the adjoint of the operator $T_t$ with respect to this scalar product. Indeed, $\pi g$ belongs to $L^1(E, E, \mu)$ and the $T_*$-invariance of $\pi$ yields

$$\|\tilde{T}_t(g)\| \leq \pi^{-1}T_{*t}(\pi)\|g\|_\infty = \|g\|_\infty,$$
so that $\tilde{T}_t$ is a well defined bounded operator on $L^\infty(E, \mathcal{E}, \mu)$. Moreover, we have
\[
\left \langle \tilde{T}_t(g), f \right \rangle_\pi = \int_E \tilde{T}_t(g)f\pi d\mu = \int_E \pi^{-1}T_{*t}(\pi g)f\pi d\mu = \int_E (\pi g)T_t(f) d\mu(x) = \langle g, T_t(f) \rangle_\pi
\]
for every $f, g \in L^\infty(E, \mu)$. Clearly the maps $\tilde{T}_t$ are also positive, thus $\tilde{T}_t = (\tilde{T}_t)_{t \geq 0}$ is a weakly* continuous semigroup of bounded positive maps on $L^\infty(E, \mathcal{E}, \mu)$.

Finally, the semigroup $\tilde{T}_t$ is Markov since $\tilde{T}_t(1_E) = \pi^{-1}T_{*t}(\pi 1_E) = 1_E$.

**Definition 1** We say that $T_t$ satisfies classical detailed balance if every operator $T_t$ is selfadjoint with respect to $\langle \cdot, \cdot \rangle_\pi$, i.e. $\tilde{T}_t = T_t$.

Therefore, $T_t$ satisfies classical detailed balance if and only if
\[
T_t(f) = \pi^{-1}T_{*t}(\pi f).
\]

**Remark 2** Detailed balance is equivalent to reversibility of classical Markov chains. Indeed, when $E = \{1, \ldots, d\}$ is a finite (for simplicity) set, endowed with the discrete $\sigma$-algebra $\mathcal{E}$ and the counting measure $\mu$, with a Markov semigroup $(T_t)_{t \geq 0}$ we can associate the transition rate matrix $(q_{jk})_{1 \leq j,k \leq d}$ defined by
\[
q_{jk} = \lim_{t \to 0} t^{-1} (T_t 1_{\{k\}} - 1_{\{k\}})(j)
\]
$(1_{\{k\}})$ denotes the indicator function of the set $\{k\}$. Denoting $(\tilde{q}_{jk})_{1 \leq j,k \leq d}$ the transition rate matrix associated with the Markov semigroup $(\tilde{T}_t)_{t \geq 0}$, it follows immediately from the definitions that (3) is equivalent to the classical condition $\pi_j q_{jk} = \pi_k q_{kj}$ for all $j, k \in E$ called reversibility.

The same condition also arises in discrete time Markov chains.

### 3 The quantum dual semigroup

The definition of detail balance involves the dual semigroup with respect to the scalar product determined by the invariant state. When studying the non-commutative analogue two fundamental differences with the classical commutative case arise: 1) there are several possible dualities, 2) the dual semigroup might not be positive. In this section we analyze these problems.

Let $\mathfrak{h}$ be a complex separable Hilbert space and let $T$ be a uniformly continuous QMS on $\mathcal{B}(\mathfrak{h})$ generated by a bounded linear operator $L$. A faithful invariant state $\rho$ for $T$ can be written in the form
\[
\rho = \sum_{k \geq 1} \rho_k |e_k\rangle\langle e_k|,
\]
where $\rho_k > 0$ for every $k$, $\sum_{k \geq 1} \rho_k = 1$ and $(e_k)_{k \geq 1}$ is an orthonormal basis of $\mathfrak{h}$. Therefore $\rho$ is invertible but, when $\dim \mathfrak{h} = \aleph_0$, its inverse $\rho^{-1} = \sum_{k \geq 1} \rho_k^{-1} |e_k\rangle\langle e_k|$ is a positive operator with dense domain $\rho(\mathfrak{h})$. 


Definition 3 Let $s \in [0, 1]$ fixed. We say that $\mathcal{T}$ admits the $s$-dual semigroup with respect to $\rho$ if there exists a uniformly continuous semigroup $\widetilde{\mathcal{T}} = \{\widetilde{T}_t\}$ on $\mathcal{B}(h)$ such that

$$\text{tr}(\rho^{1-s} \widetilde{T}_t(a) \rho^s b) = \text{tr}(\rho^{1-s} a \rho^s \mathcal{T}_t(b))$$

(5)

for all $a, b \in \mathcal{B}(h)$, $t \geq 0$.

When $s = 0$ we shall abbreviate the name of $\widetilde{T}$ speaking of dual semigroup.

We denote by $\mathcal{T}^*_{st}$ and $\widetilde{T}^*_{st}$ the predual maps of $\mathcal{T}_t$ and $\widetilde{T}_t$ respectively.

We remark that for every $s \in [0, 1]$ the sesquilinear form

$$\langle a, b \rangle_s := \text{tr}(\rho^{1-s} a^* \rho^s b)$$

defines a scalar product on $\mathcal{B}(h)$: indeed

$$\langle a, a \rangle_s = \text{tr}((\rho^{s/2} a \rho^{(1-s)/2})^* (\rho^{s/2} a \rho^{(1-s)/2})) \geq 0$$

and $\langle a, a \rangle_s = 0$ implies $\rho^{s/2} a \rho^{(1-s)/2} = 0$, i.e. $a = 0$ because $\rho$ is invertible.

If $\widetilde{T}_t$ is a $\ast$-map, then it is exactly the adjoint operator of $\mathcal{T}_t$ with respect to the scalar product $\langle \cdot, \cdot \rangle_s$.

In our framework, we will always suppose that $\mathcal{T}$ admits the $s$-dual semigroup.

Proposition 4 For each $t \geq 0$ and $a \in \mathcal{B}(h)$ we have

$$\rho^{1-s} \widetilde{T}_t(a) \rho^s = \mathcal{T}_{st}(\rho^{1-s} a \rho^s).$$

(6)

Moreover, the following properties hold:
1. $\widetilde{T}_t(1) = 1$;
2. $\widetilde{T}_{st}(\rho) = \rho$;
3. if $\widetilde{T}_t$ is positive, then it is also normal.

Proof. The identity (6) is easily checked starting from (5) and using that

$$\text{tr}(\rho^{1-s} a \rho^s \mathcal{T}_t(b)) = \text{tr}(\mathcal{T}_{st}(\rho^{1-s} a \rho^s) b).$$

Putting $a = 1$, we find then $\rho^{1-s} \widetilde{T}_t(1) \rho^s = \mathcal{T}_{st}(\rho) = \rho$ by the invariance of $\rho$; this implies $(\widetilde{T}_t(1) - 1) \rho^s = 0$, i.e. $\widetilde{T}_t(1) = 1$ for the density of $\rho(h)$ in $h$.

Taking $b = 1$ in (5) yields $\text{tr}(\widetilde{T}_t(a) \rho) = \text{tr}(a \rho)$ for all $a \in \mathcal{B}(h)$. This means in particular that the map $a \mapsto \text{tr}(\widetilde{T}_t(a) \rho)$ is weakly*-continuous, so $\rho$ belongs to the domain of $\mathcal{T}_{st}$ and $\widetilde{T}_{st}(\rho) = \rho$.

To prove property 3 it is enough to show that, for every increasing net $(x_\alpha)_\alpha$ of positive elements in $\mathcal{B}(h)$ with $\text{sup}_\alpha x_\alpha = x \in \mathcal{B}(h)$, we have

$$\lim_\alpha \langle u, \widetilde{T}_t(x_\alpha) u \rangle = \langle u, \widetilde{T}_t(x) u \rangle$$
for each \( u \) in a dense subspace of \( h \).

So, let \( u \in \rho(h) \); then \( u = \rho^{1-s}v = \rho^s w \) for some \( v, w \in h \). Therefore, equation 6 implies

\[
\lim_{\alpha} \langle u, \tilde{T}_t(x_{\alpha})u \rangle = \lim_{\alpha} \langle v, \rho^{1-s} \tilde{T}_t(x_{\alpha}) \rho^s w \rangle = \lim_{\alpha} \langle v, T_{st}(\rho^{1-s} x_{\alpha} \rho^s)w \rangle = \langle v, T_{st}(\rho^{1-s} x \rho^s)w \rangle = \langle v, \rho^{1-s} \tilde{T}_t(x \rho^s)w \rangle = \langle v, \rho^{1-s} \tilde{T}_t(x) \rho^s w \rangle = \langle u, \tilde{T}_t(x)u \rangle,
\]

since \( T_{st} \) is normal. (q.e.d.)

It is clear from (6) that

\[
\tilde{T}_t(a) = \rho^{-(1-s)}T_{st}(\rho^{1-s} \alpha \rho^s) \rho^{-s}
\]

on the dense subset \( \rho^s(h) = \rho(h) \) of \( h \), so that the 1/2-dual semigroup is completely positive and then it is a QMS thanks to Proposition 4.3. However, for \( s \neq 1/2 \), contrary to what happens in the commutative case, the maps \( \tilde{T}_t \) might not be positive. In this case \( \tilde{T} \) is not a QMS (see Example 25).

Remark 5 If \( h \) is finite-dimensional, then any uniformly continuous QMS \( T \) on \( B(h) \) admits the \( s \)-dual semigroup, since equation 7 defines a uniformly continuous semigroup of bounded operators on \( B(h) \) satisfying \( \text{tr}(\rho^{1-s} \tilde{T}_t(a) \rho^s b) = \text{tr}(\rho^{1-s} \alpha \rho^s T_t(b)) \).

The relationships between the generators \( L, \tilde{L}, L_s \) and \( \tilde{L}_s \), of \( T, \tilde{T}, T_s \) and \( \tilde{T}_s \) respectively are easily deduced.

Proposition 6 The semigroups \( T \) and \( \tilde{T} \) satisfy (5) if and only if, for all \( a, b \in B(h) \), we have

\[
\text{tr}(\rho^{1-s} \tilde{L}(a) \rho^s b) = \text{tr}(\rho^{1-s} \alpha \rho^s L(b)).
\]

(8)

In this case, the following identity holds

\[
\rho^{1-s} \tilde{L}(a) \rho^s = L_s(\rho^{1-s} \alpha \rho^s).
\]

(9)

Moreover, if \( \tilde{T} \) is a QMS, then

\[
\rho^s L(a) \rho^{1-s} = \tilde{L}_s(\rho^{1-s} \alpha \rho^s)
\]

(10)

Proof. The identity (8) clearly follows differentiating (5) at \( t = 0 \). Conversely, the identity (8), implies that, for all \( n \geq 0 \) we have

\[
\text{tr}(\rho^{1-s} \tilde{L}^n(a) \rho^s b) = \text{tr}(\rho^{1-s} \tilde{L}^{n-1}(a) \rho^s L(b)) = \ldots = \text{tr}(\rho^{1-s} \alpha \rho^s L^n(b)).
\]

Multiplying by \( t^n/n! \) and summing on \( n \), we obtain (8) because \( T_t = \sum_{n \geq 0} t^n L^n/n! \), and \( \tilde{T}_t = \sum_{n \geq 0} t^n \tilde{L}_s^n/n! \). Finally (9) and (10) follow from (8) by the same arguments leading to the identity (6) starting from (5). (q.e.d.)
We now characterise QMSs with s-dual for \( s = 0 \) which is still a QMS. To this end, we start recalling some basic ingredient of Tomita-Takesaki theory.

Let \( L^2(h) \) be the space of Hilbert-Schmidt operators on \( h \), with scalar product given by \( \langle x, y \rangle_{HS} = \text{tr}(x^*y) \). If we set \( \Omega = \rho^{1/2} \in L^2(h) \) and \( \pi_\rho(a) : L^2(h) \to L^2(h) \) the left multiplication by \( a \in B(h) \), then we obtain a representation of \( B(h) \) on \( L^2(h) \) such that \( \Omega \) is a cyclic and separating vector, and \( \text{tr}(\rho a) = \langle \Omega, \pi_\rho(a)\Omega \rangle_{HS} \) for every \( a \in B(h) \). Under these hypothesis, identifying \( B(h) \) with \( \pi_\rho(B(h)) \), the modular operator \( \Delta \) (see section 2.5.2 of Ref.\[6\]) is defined on the dense set \( B(h)\rho^{1/2} \) by

\[
\Delta a\rho^{1/2} = \rho a\rho^{-1}\rho^{1/2} = \rho a\rho^{-1/2},
\]

whereas a calculation shows that the modular group \( (\sigma_t)_{t \in \mathbb{R}} \) on \( B(h) \) is given by

\[
\sigma_t(a) = \rho^t a \rho^{-it}.
\]

We recall that an element \( a \) in \( B(h) \) is analytic for \( (\sigma_t) \) if there exists a strip

\[
I_\lambda = \{z \in \mathbb{C} \mid |\Im z| < \lambda \}
\]

and a function \( f : I_\lambda \to B(h) \) such that:

1. \( f(t) = \sigma_t(a) \) for all \( t \in \mathbb{R} \);
2. \( I_\lambda \ni z \rightarrow \text{tr}(\eta f(z)) \) is analytic for all \( \eta \in L^1(h) \) or, equivalently, \( I_\lambda \ni z \rightarrow \langle u, f(z)v \rangle \) is analytic for all \( u, v \in h \).

We denote by \( \mathcal{A} \) the set of all analytic elements for \( (\sigma_t) \).

It is a well known fact (Proposition 5 of \[11\]) that \( \mathcal{A} \rho^{1/2} \) is a core for \( \Delta \) and \( \sigma_z(a) = \rho^z a \rho^{-iz} \in B(h) \) for all \( a \in \mathcal{A} \) and \( z \in \mathbb{C} \).

In particular, the modular automorphism \( \sigma_{-i} \) on \( B(h) \) is defined by \( \sigma_{-i}(a) = \rho a\rho^{-1} \) for all \( a \in \mathcal{A} \) and it satisfies the following property

**Lemma 7** If \( \sigma_{-i}(a) = \alpha a \), then we have \( \sigma_{-i}(a^*) = \alpha^{-1} a^* \) and \( \alpha = \text{tr}(\rho a a^*)/\text{tr}(\rho a^*a) \).

In particular, every eigenvalue of \( \sigma_{-i} \) is strictly positive.

**Proof.** Let \( \sigma_{-i}(a) = \alpha a \); then \( \alpha \neq 0 \) for \( \sigma_{-i} \) is invertible, and

\[
\sigma_{-i}(a^*) = \rho a^* \rho^{-1} = (\sigma_{-i}^{-1}(a))^* = (\alpha^{-1} a^*)^* = \overline{\alpha^{-1}} a^*.
\]

But \( \text{tr}(\rho a a^*) = \text{tr}(\rho a \rho^{-1} \rho a^*) = \text{tr}(\rho a^*a) \), so that we obtain \( \alpha = \text{tr}(\rho a a^*)/\text{tr}(\rho a^*a) \) positive. Therefore, \( \sigma_{-i}(a^*) = \alpha^{-1} a^* \). \( \text{(q.e.d.)} \)

We say that a linear bounded operator \( X \) on \( B(h) \) commute with \( \sigma_z \) for some \( z \in \mathbb{C} \) if \( X(\sigma_z(a)) = \sigma_z(X(a)) \) for all \( a \in \mathcal{A} \).

We can now show the following characterisation of QMSs whose 0-dual is still a semigroup of positive linear maps, i.e. a QMS, adapting an argument from Majewski and Streater\[11\] (proof of Theorem 6).
Theorem 8 The following conditions are equivalent:

1. $\tilde{T}$ is a QMS;
2. any $T_t$ commutes with $\sigma_{-i}$;
3. $\mathcal{L}$ commutes with $\sigma_{-i}$.

If the above conditions hold, also the maps $T_r$, $T_{sr}$, $\tilde{T}_r$, $\tilde{T}_{sr}$ and the generators $\mathcal{L}$, $\mathcal{L}_s$, $\mathcal{L}_r$ commute with the homorphisms $\sigma_t$ for all $t, r \geq 0$.

Proof. (1) $\Rightarrow$ (3) If $\tilde{T}$ is a QMS, then, in particular, $\tilde{T}_r$ satisfies $\tilde{T}_r(a^*) = \tilde{T}_r(a)^*$ for all $a \in B(h)$; therefore, by (11) with $s = 0$ and the same formula taking the adjoint we have

$$\mathcal{L}(a) = \tilde{\mathcal{L}}_s(a\rho) \quad \rho \mathcal{L}(a) = \tilde{\mathcal{L}}_s(\rho a),$$

so that, replacing $a$ by $\rho a \rho^{-1}$,

$$\mathcal{L}(\rho a \rho^{-1}) = \tilde{\mathcal{L}}_s((\rho a \rho^{-1})\rho) \rho^{-1} = \rho \mathcal{L}_r(a) \rho^{-1}$$

for all $a \in A$. This means $\mathcal{L} \circ \sigma_{-i} = \sigma_{-i} \circ \mathcal{L}$ in the previous sense.

(3) $\Rightarrow$ (2) By induction we can show that $\mathcal{L}^n$ and $\sigma_{-i}$ commute for every $n \geq 0$; then, also $T_t$ commute with $\sigma_{-i}$, for $T_t = \exp(t\mathcal{L}) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mathcal{L}^n$.

(2) $\Rightarrow$ (1) Let us define a contraction semigroup $(\tilde{T}_r)_{r \geq 0}$ on $L^2(h)$ by

$$\tilde{T}_r(a\rho^{1/2}) = T_r(a)\rho^{1/2}$$

for all $a \in B(h)$ and $r \geq 0$. Indeed, since $T_r(a^*)T_r(a) \leq T_r(a^*a)$ by the 2-positivity of $T_r$, we have

$$\|\tilde{T}_r(a\rho^{1/2})\|_{HS}^2 = \text{tr}\left(\rho^{1/2}T_r(a^*)T_r(a)\rho^{1/2}\right) \leq \text{tr}\left(\rho^{1/2}T_r(a^*a)\rho^{1/2}\right) = \|a\rho^{1/2}\|_{HS}^2$$

by the invariance of $\rho$ and the semigroup property follows from a straightforward algebraic computation. Condition (2) implies then

$$\tilde{T}_r(\Delta a\rho^{1/2}) = \tilde{T}_r(\rho a\rho^{-1}\rho^{1/2}) = T_r(\rho a\rho^{-1})\rho^{1/2} = \rho T_r(a)\rho^{-1}\rho^{1/2}$$

for all $a \in A$, i.e. any map $\tilde{T}_r$ commutes with $\Delta$ (a $\rho^{1/2}$ is a core for $\Delta$). Therefore, by spectral calculus, $\tilde{T}_r$ also commutes with $\Delta^it$ for all $t \in \mathbb{R}$. It follows that $T_r$ commutes with $\sigma_t$ for all $t \geq 0$. Thus

$$\text{tr}(\sigma_t(T_{sr}(b))a) = \text{tr}(T_{sr}(b)\sigma_{-t}(a)) = \text{tr}(bT_{sr}(\sigma_{-t}(a))) = \text{tr}(\sigma_t(b)\tilde{T}_r(a) \rho^{-i} T_r(\rho^{i} b \rho^{-i}))$$

for all $a, b$, i.e. also $T_{sr}$ commutes with $\sigma_t$. Then, for all $r \geq 0$ and $t \in \mathbb{R}$, we get

$$\rho^{it}T_{sr}(b)\rho^{-it} = \sigma_t(T_{sr}(b)) = T_{sr}(\sigma_t(b)) = T_{sr}(\rho^{it} b \rho^{-it}). \quad (11)$$
We want to show that this equation holds for \( b = \rho^{1/2}a\rho^{1/2} \) and for certain complex \( t \). Since the maps
\[
z \rightarrow \rho^{iz} = e^{iz\ln \rho}, \quad z \rightarrow \rho^{-iz} = e^{-iz\ln \rho}
\]
are analytic on \( \Im z \leq 0 \) and \( \Im z \geq 0 \) respectively, and the operator
\[
\rho^{i(t+is)}\rho^{1/2}a\rho^{1/2} = \rho^{it}\rho^{-s+1/2}a\rho^{1/2+s}\rho^{-it}
\]
is trace class into the strip \( 1/2 \leq s \leq 1/2 \), both sides of equation (11) have an analytic continuation into this strip, so that \( \rho^{iz}\mathcal{T}_s(b)\rho^{-iz} = \mathcal{T}_s(\rho^{iz}b\rho^{-iz}) \) holds for all complex \( z \) with \( |\Im z| \leq 1/2 \) and \( b = \rho^{1/2}a\rho^{1/2} \). Taking \( z = -i/2 \), we get
\[
\rho^{1/2}\mathcal{T}_s(\rho^{1/2}a\rho^{1/2})\rho^{-1/2} = \mathcal{T}_s(\rho^{1/2}\rho^{1/2}a\rho^{1/2}\rho^{-1/2}) = \mathcal{T}_s(\rho a) = \rho \tilde{\mathcal{T}}_s(a).
\]
Hence
\[
\tilde{\mathcal{T}}_s(a) = \rho^{-1/2}\mathcal{T}_s(\rho^{1/2}a\rho^{1/2})\rho^{-1/2},
\]
therefore any operator \( \tilde{\mathcal{T}}_s \) is completely positive and \( \tilde{\mathcal{T}} \) is a QMS.

The above arguments also prove the claimed commutation of semigroups, their generators and the homomorphisms \( \sigma_t \).

\( \text{(q.e.d.)} \)

4 The generator of the dual semigroup

Suppose now that the dual semigroup \( \tilde{\mathcal{T}} \) (for \( s = 0 \)) is a QMS with generator \( \tilde{\mathcal{L}} \). In this section we find the relationship between the operators \( H, L_k \) and \( \tilde{G}, \tilde{L}_k \) which appear in the Lindblad representation of \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \).

To this end, we start recalling the following result from Parthasarathy[12] (Th. 30.16) on the representation of the generator of a uniformly continuous QMS in a special form of GKLS (Gorini, Kossakowski, Sudarshan, Lindblad) type.

**Theorem 9** Let \( \mathcal{L} \) be the generator of a uniformly continuous QMS on \( \mathcal{B}(\mathfrak{h}) \) and let \( \rho \) be any normal state on \( \mathcal{B}(\mathfrak{h}) \). Then there exist a bounded selfadjoint operator \( H \) and a sequence \( (L_k)_{k \geq 1} \) of elements in \( \mathcal{B}(\mathfrak{h}) \) such that:

1. \( \text{tr}(\rho L_k) = 0 \) for each \( k \geq 1 \),
2. \( \sum_{k \geq 1} L_k^*L_k \) is strongly convergent,
3. if \( \sum_{k \geq 0} |c_k|^2 < \infty \) and \( c_0 + \sum_{k \geq 1} c_k L_k = 0 \) for scalars \( (c_k)_{k \geq 0} \) then \( c_k = 0 \) for every \( k \geq 0 \),
4. \( \mathcal{L}(a) = i[H,a] - \frac{1}{2} \sum_{k \geq 1} (L_k^*L_k a - 2L_k^*aL_k + aL_k^*L_k) \) for all \( a \in \mathcal{B}(\mathfrak{h}) \).

Moreover, if \( H', (L'_k)_{k \geq 1} \) is another family of bounded operators in \( \mathcal{B}(\mathfrak{h}) \) with \( H' \) selfadjoint, then it satisfies conditions (1)-(4) if and only if the lengths of sequences \( (L_k)_{k \geq 1}, (L'_k)_{k \geq 1} \) are equal and
\[
H' = H + \alpha, \quad L'_k = \sum_j u_{kj} L_j
\]
for some scalar \( \alpha \) and a unitary matrix \( (u_{kj})_{kj} \).
In our framework \( \rho \) will always be a faithful normal \( \mathcal{T} \)-invariant state.

We now introduce a terminology in order to distinguish GKSL representations with properties (1) and (3) in Theorem 9 from standard GKSL representations.

**Definition 10** We call special GKSL representation with respect to a state \( \rho \) by means of the operators \( H, L_k \) any representation of \( \mathcal{L} \) satisfying conditions (1), . . . , (4) of Theorem 9.

**Remark 11** Condition 3 of Theorem 9 means that \( \{1, L_1, L_2, \ldots\} \) is a set of linearly independent elements of \( \mathcal{B}(\mathfrak{h}) \). If \( \dim \mathfrak{h} = d \), then the length of \( (L_k)_{k \geq 1} \) in a special GKSL representation of \( \mathcal{L} \) is at most \( d^2 - 1 \).

We recall that we can also write \( \mathcal{L} \) as

\[
\mathcal{L}(a) = G^*a + aG + \sum_k L_k^* aL_k,
\]

where \( G \) is the bounded operator on \( \mathfrak{h} \) defined by

\[
G = -iH - \frac{1}{2} \sum_k L_k^* L_k.
\]

**Remark 12** The last statement of Theorem 9 implies that, in a special GKSL representation of \( \mathcal{L} \), the above operator \( G \) is unique up to a purely imaginary multiple of the identity operator. Indeed the operator \( G' \) defined as in (13) replacing \( H, L_k \) by \( H', L'_k \) satisfies

\[
G' = -iH - i\alpha - \frac{1}{2} \sum_{k,j,m} \bar{u}_{kj} u_{km} L_j^* L_m
\]

\[
= -iH - i\alpha - \frac{1}{2} \sum_{j,m} \left( \sum_k \bar{u}_{kj} u_{km} \right) L_j^* L_m
\]

\[
= -iH - i\alpha - \frac{1}{2} \sum_j L_j^* L_j = G - i\alpha
\]

because the matrix \( (u_{kj})_{kj} \) is unitary.

Let \( k \) be a Hilbert space with Hilbertian dimension equal to the length of the sequence \( (L_k)_k \) and let \( (f_k)_k \) be an orthonormal basis of \( k \). Defining a linear bounded operator \( L : \mathfrak{h} \to \mathfrak{h} \otimes k \) by \( Lu = \sum_k L_k u \otimes f_k \), Theorem 9 takes the following form (Theorem 30.12 Ref.[12])

**Theorem 13** If \( \mathcal{L} \) is the generator of a uniformly continuous QMS on \( \mathcal{B}(\mathfrak{h}) \), then there exist an Hilbert space \( k \), a bounded linear operator \( L : \mathfrak{h} \to \mathfrak{h} \otimes k \) and a bounded selfadjoint operator \( H \) in \( \mathfrak{h} \) satisfying the following:
1. \( \mathcal{L}(x) = i[H, x] - \frac{1}{2} (L^*Lx - 2L^*(x \otimes 1_k)L + xL^*L) \) for all \( x \in \mathcal{B}(h) \);
2. the set \( \{ (x \otimes 1_k)Lu : x \in \mathcal{B}(h), u \in h \} \) is total in \( h \otimes k \).

**Proof.** Letting \( Lu = \sum_k L_ku \otimes f_k \), where \((f_k)\) is an orthonormal basis of \( k \) and the \( L_k \) are as in Theorem 9, a simple calculation shows that condition (1) is fulfilled.

Suppose that there exists a non-zero vector \( \xi \in \{ (x \otimes 1_k)Lu : x \in \mathcal{B}(h), u \in h \}^\perp \); then \( \xi = \sum_k v_k \otimes f_k \) with \( v_k \in h \) and

\[
0 = \langle \xi, (x \otimes 1_k)Lu \rangle = \sum_k \langle v_k, xL_ku \rangle = \sum_k \langle L_k^*x^*v_k, u \rangle
\]

for all \( x \in \mathcal{B}(h) \), \( u \in h \). Hence, \( \sum_k L_k^*x^*v_k = 0 \). Since \( \xi \neq 0 \), we can suppose \( \|v_1\| = 1 \); then, putting \( p = |v_1\rangle\langle v_1| \) and \( x = py^* \), \( y \in \mathcal{B}(h) \), we get

\[
0 = L_1^*yv_1 + \sum_{k \geq 2} \langle v_k, v_{L_k^*y} \rangle L_k^*v_1 = \left( L_1^* + \sum_{k \geq 2} \langle v_1, v_k \rangle L_k^* \right)yv_1.
\]

(14)

Since \( y \in \mathcal{B}(h) \) is arbitrary, equation (14) contradicts the linear independence of the \( L_k \)'s. Therefore the set in (2) must be total. (q.e.d.)

We now study the generator \( \mathcal{L} \) of QMS \( T \) whose dual \( \tilde{T} \) is a QMS. As a first step we find an explicit form for the operator \( G \) defined by (13).

**Proposition 14** If \( \mathcal{L}(u) = G^*u + aG + \sum_j L_j^*aL_j \) is a special GKSL representation of \( \mathcal{L} \) and \( \rho \) is the \( T \)-invariant state (4) then

\[
G^*u = \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|)e_k - \text{tr}(\rho G)u
\]

(15)

\[
Gv = \sum_{k \geq 1} \rho_k \mathcal{L}_s(|v\rangle\langle e_k|)e_k - \text{tr}(\rho G^*)v
\]

(16)

for every \( u, v \in h \).

**Proof.** Since \( \mathcal{L}(|u\rangle\langle v|) = |G^*u\rangle\langle v| + |u\rangle\langle Gv| + \sum_j |L_j^*u\rangle\langle L_j^*v| \), letting \( v = e_k \) we have \( G^*u = |G^*u\rangle\langle e_k|e_k \) and

\[
G^*u = \mathcal{L}(|u\rangle\langle e_k|)e_k - \sum_j \langle e_k, L_j e_k \rangle L_j^*u - \langle e_k, Ge_k \rangle u.
\]

Multiplying both sides by \( \rho_k \) and summing on \( k \), we find then

\[
G^*u = \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|)e_k - \sum_{k \geq 1} \rho_k \langle e_k, L_j e_k \rangle L_j^*u - \sum_{k \geq 1} \rho_k \langle e_k, Ge_k \rangle u
\]

= \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|)e_k - \sum_j \text{tr}(\rho L_j)L_j^*u - \text{tr}(\rho G)u
\]

and (15) follows since \( \text{tr}(\rho L_j) = 0 \). Computing the adjoint of \( G \) we find immediately (16).

(q.e.d.)
Proposition 15 Let $\overline{T}$ be the $s$-dual of a QMS $T$ with generator $\overline{L}$. If $G$ and $\tilde{G}$ are the operators \[16\] in some special GKSL representations of $L$ and $\overline{L}$ then
\[
\tilde{G}\rho^s = \rho^s G^* + \left(\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*)\right) \rho^s.
\] (17)

Moreover, we have $\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) = ic_\rho$ for some $c_\rho \in \mathbb{R}$.

Proof. The identities \[16\] and \[49\] yield
\[
\tilde{G}\rho^s = \sum_{k \geq 1} \tilde{L}_s(\rho^s |v\rangle\langle \rho^s e_k |) \rho^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v
\]
\[
= \sum_{k \geq 1} \tilde{L}_s(\rho^s |v\rangle\langle \rho^s e_k |) \rho^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v
\]
\[
= \sum_{k \geq 1} \rho^s L(\rho^s |v\rangle\langle e_k |) \rho^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v
\]
\[
= \rho^s G^* v + \left(\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*)\right) \rho^s v.
\]

Therefore, we obtain \[17\].

Right multiplying equation \[17\] by $\rho^1 - s$ we have $\tilde{G}\rho^s = \rho^s G^* \rho^1 - s + \left(\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*)\right) \rho$, so that taking the trace,
\[
\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) = \text{tr}(\tilde{G}\rho) - \text{tr}(\rho^s G^* \rho^1 - s)
\]
\[
= \text{tr}(\tilde{G}\rho) - \text{tr}(G^* \rho) = -\left(\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*)\right);
\]

this proves that $\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) = ic_\rho$ for some real constant $c_\rho$. (q.e.d.)

Proposition 16 Let $\overline{T}$ be the $0$-dual of a QMS $T$ and let
\[
L(a) = G^* a + \sum_j L_j^* a L_j + a G, \quad \overline{L}(a) = \tilde{G}^* a + \sum_j \tilde{L}_j^* a \tilde{L}_j + a \tilde{G}
\]
be special GKSL representations of $L$ and $\overline{L}$. Then:
1. $\tilde{G} = G^* + ic$ with $c \in \mathbb{R}$,
2. both $G$ and $\tilde{G}$ commute with $\rho$,
3. $\sum_k L_k^* L_k$, $\sum_k \tilde{L}_k^* \tilde{L}_k$, $H$ and $\overline{H}$ commute with $\rho$.

Proof. (1) It follows by Proposition 15 for $s = 0$ and Theorem 9, Remark 12. Indeed, in any special GKSL representations of $L$ and $\overline{L}$, $G$ and $\tilde{G}$ are unique up to a purely imaginary multiple of the identity operator.
(2) Let \( G \) and \( \tilde{G} \) be the operators (16) and in the given special GKSL representations of \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \). Since \( \mathcal{L},(\rho a) = \rho \mathcal{L}(a) \) holds for \( \tilde{T} \) is a QMS, we have

\[
\tilde{G}\rho v = \sum_{k=1}^{n} \rho_k \tilde{\mathcal{L}}_k(\rho|v\rangle\langle pe_k|)e_k - \text{tr}(\rho \tilde{G}^*)\rho
\]

\[
= \sum_{k=1}^{n} \rho_k \rho \mathcal{L}(|v\rangle\langle e_k|)e_k - \text{tr}(\rho \tilde{G}^*)\rho v
\]

\[
= \rho \left( (G^*v + \text{tr}(\rho G)v) - \text{tr}(\rho \tilde{G}^*)\rho v \right)
\]

for every \( v \in \mathfrak{h} \), that is \( \tilde{G}\rho = \rho G^* + ic\rho \). But \( G \) and \( \tilde{G} \) are the operators (16), therefore by (1) we have also \( \tilde{G}\rho = G^* \rho + ic\rho \), and so \( G^* \rho = \rho G^* \). This, together with Remark 12 clearly implies (2).

(3) Follows from (2) by decomposing \( G \) and \( \tilde{G} \) into their self-adjoint and anti self-adjoint parts. (q.e.d.)

We now study the properties of the \( L_k \) when \( \tilde{T} \) is a QMS.

**Lemma 17** With the notations of Theorem 13, if \( \mathfrak{h} \) is finite-dimensional then the equation

\[
\sum_k L^*_k a L_k = \sum_k \rho^{-1} L^*_k \rho a \rho^{-1} L_k \rho \quad \forall a \in \mathcal{B}(\mathfrak{h})
\]

(18)

implies \( \rho L_k \rho^{-1} = \lambda_k L_k \) and \( \rho L^*_k \rho^{-1} = \lambda_k^{-1} L^*_k \) for some positive \( \lambda_k \).

**Proof.** Define two linear maps \( X_1, X_2 \) on \( \mathfrak{h} \otimes k \) by

\[
X_1(x \otimes 1_k)Lu = (x \otimes 1_k)(\rho \otimes 1_k)Lu, \rho^{-1} u
\]

\[
X_2(x \otimes 1_k)Lu = (x \otimes 1_k)(\rho^{-1} \otimes 1_k)Lu u
\]

for all \( x \in \mathcal{B}(\mathfrak{h}) \) and \( u \in \mathfrak{h} \).

We postpone to Lemma 18 the proof that \( X_1 \) and \( X_2 \) are well defined on the total (Theorem 13) set \( \{(x \otimes 1_k)Lu \mid x \in \mathcal{B}(\mathfrak{h}), u \in \mathfrak{h}\} \) in \( \mathfrak{h} \otimes k \). We can now extend \( X_1 \) and \( X_2 \) to a bounded operator on \( \mathfrak{h} \otimes k \). Moreover, (21) implies

\[
\langle X_1(x \otimes 1_k)Lu, X_2(y \otimes 1_k)Lv \rangle = \sum_k \langle u, \rho^{-1} L^*_k \rho x^* y \rho^{-1} L_k \rho v \rangle
\]

\[
= \sum_k \langle u, L^*_k x^* y L_k v \rangle
\]

\[
= \langle (x \otimes 1_k)Lu, (y \otimes 1_k)Lv \rangle
\]

for all \( x, y \in \mathcal{B}(\mathfrak{h}) \) and \( u \in \mathfrak{h} \). As a consequence we have \( X_1^* X_2 = 1_{\mathfrak{h} \otimes k} \).

By the definition of \( X_1, X_2 \) we have also \( X_j(y \otimes 1_k) = (y \otimes 1_k)X_j \) for \( j = 1, 2 \). Therefore \( X_j \) can be written in the form \( 1_{\mathfrak{h}} \otimes Y_j \) for some invertible bounded map \( Y_j \) on \( k \) satisfying \( Y_1^* Y_2 = 1_k \).
The definition of $X_1, X_2$ implies then

\[(1_l \otimes Y_1)L = (\rho \otimes 1_l)\rho^{-1}, \quad (1_l \otimes Y_1^*)^{-1} L = (\rho^{-1} \otimes 1_l)\rho, \quad (19)\]

right multiplying by $\rho$ and left multiplying by $(\rho \otimes 1_l)$ the first and the second identity we find

\[(1_l \otimes Y_1)L\rho = (\rho \otimes 1_l)L, \quad (1_l \otimes Y_1^*)^{-1} \rho L = L\rho.\]

Writing the second as $(\rho \otimes 1_l)L = (1_l \otimes Y_1^*)L\rho$ we obtain

\[(1_l \otimes Y_1)L\rho = (\rho \otimes 1_l)L = (1_l \otimes Y_1^*)L\rho.\]

Since $\rho$ is faithful, it follows that $(1_l \otimes Y_1)L = (1_l \otimes Y_1^*)L$ (and also $(1_l \otimes Y_1)(x \otimes 1_l) = (1_l \otimes Y_1^*) (x \otimes 1_l)\rho$ for all $x$) proving that $Y_1$ is self-adjoint.

Therefore, there exist non-zero $\lambda_k \in \mathbb{R}$ and a unitary operator $U$ on $k$ such that

\[Y_1 = U^*DU\]

with $D = \text{diag}(\lambda_1, \lambda_2, \ldots)$. The identities (19) yield then

\[(1_l \otimes DU)L = (1_l \otimes U)(\rho \otimes 1_l)L = (\rho \otimes 1_l)(1_l \otimes U)L\rho^{-1}\]

\[(1_l \otimes D^{-1}U)L = (1_l \otimes U)(\rho^{-1} \otimes 1_l)L = (\rho^{-1} \otimes 1_l)(1_l \otimes U)L\rho\]

Thus, putting $L' = UL$, or, more precisely $L'_k = \sum u_{k\ell}L_\ell$ for all $k$, we have

\[\rho L'_k\rho^{-1} = \lambda_k L'_k \quad \text{and} \quad \rho^{-1} L'_k\rho = \lambda^{-1}_k L'_k\]

for every $k$. To conclude the proof it suffices to recall that $\lambda_k > 0$ by Lemma 7 since the above identities mean that $\lambda_k$ is an eigenvalue of $\sigma_i$. (q.e.d.)

We now check that the maps $X_1, X_2$ introduced in the proof of Lemma 17 are well defined.

**Lemma 18** With the notations of Lemma 17 if $\mathfrak{h}$ is finite-dimensional and equation (17) holds, then

\[\sum_{j=1}^{m} (x_j \otimes 1_l) Lu_j = 0 \quad (20)\]

for $x_1, \ldots, x_m \in \mathcal{B}(\mathfrak{h})$, $u_1, \ldots, u_m \in \mathfrak{h}$ implies:

1. $\sum_{j=1}^{m} (x_j \otimes 1_l)(\rho \otimes 1_l)L\rho^{-1}u_j = 0$;
2. $\sum_{j=1}^{m} (x_j \otimes 1_l)(\rho^{-1} \otimes 1_l)L\rho u_j = 0$.

**Proof.** Suppose that (20) holds. Taking the adjoint of (21) we find

\[\sum_k \rho L_k^* \rho^{-1} a \rho L_k \rho^{-1} = \sum_k L_k^* a L_k\]

for every $a \in \mathcal{B}(\mathfrak{h})$. (q.e.d.)
for every $a \in \mathcal{B}(h)$ and compute

$$\langle (y \rho^{-1} \otimes 1_k)L \rho v, \sum_{j=1}^{m} (x_j \otimes 1_k)(\rho \otimes 1_k)L \rho^{-1} u_j \rangle$$

$$= \sum_{j=1}^{m} \sum_{k} \langle v, L^*_k y^* x_j L_k u_j \rangle = \langle (y \otimes 1_k)L v, \sum_{j=1}^{m} (x_j \otimes 1_k)L u_j \rangle = 0$$

for all $y \in \mathcal{B}(h)$ and $v \in h$. But the set $S = \{(y \rho^{-1} \otimes 1_k)L \rho v \mid y \in \mathcal{B}(h), v \in h\}$ is total in $h \otimes k$, because $\{(y \otimes 1_k)L v \mid y \in \mathcal{B}(h), v \in h\}$ is total (Theorem 13) and the maps $y \mapsto y \rho^{-1}, v \mapsto \rho v$ are bijective. This proves (1). The proof of (2) is similar and we omit it.

(q.e.d.)

**Proposition 19** Suppose that $\mathcal{L}$ and $\sigma_{-i}$ commute. Then there exists a special GKS representation of $\mathcal{L}$ in which, for all $k$, we have

$$\rho L_k = \lambda_k L_k \rho, \quad \rho L^*_k = \lambda^{-1}_k L^*_k \rho, \quad \lambda_k > 0.$$  

**Proof.** Define $p_n := \sum_{k \leq n} |e_k \rangle \langle e_k|$ for $n \geq 0$ and consider $u, v \in h, a \in \mathcal{B}(h)$; since the map

$$z \mapsto \langle u, \rho^i z p_n a p_n \rho^{-i z} \rangle = \sum_{k, h \leq n} \rho^i_k \rho^{-i z} \langle u, e_k \rangle \langle e_k, a e_h \rangle \langle e_h, v \rangle$$

is analytic on $\mathbb{C}$, then $b := p_n a p_n$ belongs to $a$ for all $n \geq 0$. As a consequence, since $\mathcal{L}$ and $\sigma_{-i}$ commute, we have $\mathcal{L}(b) = \rho^{-1} \mathcal{L}(\rho b \rho^{-1}) \rho$, so that

$$i[H, b] - \frac{1}{2} \sum_{k} (L^*_k L_k b - 2L^*_k b L_k + b L^*_k L_k) = \rho^{-1} [H, \rho b \rho^{-1}] \rho$$

$$- \frac{1}{2} \sum_{k} (\rho^{-1} L^*_k L_k \rho b - 2 \rho^{-1} L^*_k \rho b \rho^{-1} L_k \rho + b \rho^{-1} L^*_k L_k \rho).$$

Both $H$ and $\sum_{k} L^*_k L_k$ commute with $\rho$ by Proposition 16 (3). We have then

$$\sum_{k} L^*_k p_n a p_n L_k = \sum_{k} \rho^{-1} L^*_k \rho p_n a p_n \rho^{-1} L_k \rho,$$

and so

$$\sum_{k} \rho^{-1} L^*_k \rho p_n a p_n L_k \rho = \sum_{k} \rho^{-1} L^*_k \rho p_n a p_n \rho^{-1} L_k \rho.$$  

for all $a \in \mathcal{B}(h), n \geq 0$, right and left multiplying (21) by $p_n$. Remembering that $p_n \rho^{-1} = \rho^{-1} p_n$ on $\rho(h)$ and setting $L_{(n)} := p_n L_k p_n, \rho_{(n)} := \rho p_n$, the above equality gives

$$\sum_{k} \rho^{-1}_{(n)} L^*_{(n)} \rho_{(n)} b \rho^{-1}_{(n)} L_{(n)} \rho_{(n)} = \sum_{k} L^*_{(n)} b L_{(n)}.$$  

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for all $b \in \mathcal{B}(p_n(h))$.

But $\rho_{(n)}$ is faithful on the finite-dimensional Hilbert space $p_n(h)$ and \{(x \otimes 1_h)L_{(n)}u \mid x \in \mathcal{B}(p_n(h)), \ u \in p_n(h)\} is total in $p_n(h) \otimes k$, therefore Lemma 17 assures that

$$\rho_{(n)}L_{(n)k}\rho_{(n)}^{-1} = \lambda_{k,n}L_{(n)k} \quad \rho_{(n)}L^*_{(n)k}\rho_{(n)}^{-1} = \lambda_{k,n}^{-1}L^*_{(n)k}$$

for some $\lambda_{k,n} > 0$, i.e.

$$\rho_{p_k}L_{p_k} = \lambda_{k,n}^2 p_k L_{p_k} \quad \rho_{p_k}L^*_{p_k} = \lambda_{k,n}^{-2}L^*_{p_k}$$

(22)

Since $(p_n)_n$ is an increasing sequence of projections, this implies $\lambda_{k,n} = \lambda_k$ for $n \gg 0$, and then, letting $n \to \infty$ in equation (22), we obtain

$$\rho k = \lambda_k L_k \rho, \quad \rho L^*_k = \lambda_k^{-1}L^*_k \rho,$$

for $(p_n)_n$ converges to $1$ in the strong operator topology. (q.e.d.)

**Definition 20** Let $\mathcal{L}$ be the generator of a QMS and let $\rho$ be a faithful normal state. A special GKSL representation of $\mathcal{L}$ with respect to $\rho$ by means of operators $H, L_k$ is called privileged if their operators $L_k$ satisfy $\rho L_k = \lambda_k L_k \rho$ and $\rho L^*_k = \lambda_k^{-1}L^*_k \rho$ for some $\lambda_k > 0$ and $H$ commutes with $\rho$.

**Remark 21** The operator $\sum_k L_k^* L_k$ (the self-adjoint part of $G$) in a privileged GKSL representation clearly commutes with $\rho$. Moreover, the constants $\lambda_k$ are determined by the eigenvalues of $\rho$. Indeed, writing $\rho$ as in (4), the identity $\rho L_k = \lambda_k L_k \rho$ yields

$$\rho \langle e_j, L_k e_m \rangle = \langle e_j, \rho L_k e_m \rangle = \lambda_k \langle e_j, L_k \rho e_m \rangle = \lambda_k \rho \langle e_j, L_k e_m \rangle.$$

Therefore $\lambda_k = \rho \rho^{-1}$ for all $j, m$ such that $\langle e_j, L_k e_m \rangle \neq 0$. In particular, if we write $\rho = \lambda e^{-H S}$ for some bounded selfadjoint operator $H_S = \sum \varepsilon_j |e_j \rangle \langle e_j|$ on $h$, we find $\lambda_k = \lambda e^{\varepsilon_m - \varepsilon_j}$.

**Proposition 22** Given two privileged GKSL of $\mathcal{L}$ with respect to the same state $\rho$ by means of operators $H, L_k$ and $H', L'_k$, with $D = \text{diag}(\lambda_1, \lambda_2, \ldots)$ and $D' = \text{diag}(\lambda'_1, \lambda'_2, \ldots)$, there exists a unitary operator $V$ on $k$ and $\alpha \in \mathbb{R}$ such that

$$H' = H + \alpha, \quad L' = (1_h \otimes V)L, \quad D' = VDV'^*.$$

**Proof.** By Theorem 9 there exist $\alpha \in \mathbb{R}$ and a unitary $V$ on $k$ such that $H' = H + \alpha$ and $L' = (1_h \otimes V)L$. Since both families $H, L_k$ and $H', L'_k$ give privileged GKSL representations with respect to the same state $\rho$, we have

$$(\rho \otimes 1_k)L = (1_h \otimes D)L\rho, \quad (\rho \otimes 1_k)L' = (1_h \otimes D'L')L\rho.$$

Left multiplying the first identity by $(1_h \otimes V)$ and replacing $L'$ by $VL$ in the second we find

$$(\rho \otimes 1_k)(1_h \otimes V)L = (1_h \otimes VD)L\rho, \quad (\rho \otimes 1_k)(1_h \otimes V)L = (1_h \otimes D'V)L\rho.$$

It follows that $VD = D'V$, i.e. $D' = VDV^*$. (q.e.d.)
Remark 23 The identity $D' = VDV^*$ means that $V$ is a change of coordinates that transforms $D$ into another diagonal matrix; in particular, if $D = \text{diag}(\lambda_1, \lambda_2, \ldots)$ and $D' = \text{diag}(\lambda'_1, \lambda'_2, \ldots)$, we have

$$\lambda'_i(f_i, Vf_j) = \lambda_j(f_i, Vf_j).$$

Since $V$ is a unitary operator this implies that, when the $\lambda_k$ are all different, for every $i$ there exists a unique $j$ such that $\langle f_i, Vf_j \rangle \neq 0$ and for every $j$ there exists a unique $i$ such that $\langle f_i, Vf_j \rangle \neq 0$. Thus

$$Vf_j = e^{i\theta_{\sigma(j)} f_{\sigma(j)}^j}$$

and $L'_k = e^{i\theta_{\sigma(k)} L_{\sigma(k)}}$ with $\theta_{\sigma(j)} \in \mathbb{R}$ and $\sigma$ a permutation. Therefore, when the $\lambda_k$ are all different, privileged GKSL representations of $L$ exist, if and only there exists a privileged GKSL representation of $L$ with respect to $\rho$.

The results of this section are summarized by the following

Theorem 24 The 0-dual semigroup $\tilde{T}$ of a QMS $T$ generated by $\mathcal{L}$ with faithful normal invariant state $\rho$ is a QMS if and only there exists a privileged GKSL representation of $\mathcal{L}$ with respect to $\rho$.

Proof. If $\tilde{T}$ is a QMS, then $\mathcal{L}$ commutes with $\sigma_i$ by Theorem 8 and so there exists a special GKSL representation of $\mathcal{L}$ by Propositions 19 and 16. The converse is trivial. (q.e.d.)

We now exhibit an example of semigroup whose dual is not a QMS.

Example 25 Consider the semigroup $T$ on $M_2(\mathbb{C})$ generated by

$$\mathcal{L}(a) = i \frac{\Omega}{2} [\sigma_1, a] - \mu^2 (\sigma^+ \sigma^- a - 2 \sigma^+ a \sigma^- + a \sigma^+ \sigma^-),$$

where $\mu > 0$, $\Omega \in \mathbb{R}$, $\Omega \neq 0$ and $\sigma_k$ are the Pauli matrices and $\sigma^\pm = \sigma_1 \pm i \sigma_2$ are the raising and lowering operator.

A straightforward computation shows the state

$$\rho = \frac{1}{2} \left( 1 + \frac{2 \mu^2 \Omega}{2\Omega^2 + \mu^4} \sigma_2 - \frac{\mu^4}{2\Omega^2 + \mu^4} \sigma_3 \right) = \frac{1}{2\Omega^2 + \mu^4} \begin{pmatrix} \Omega^2 & -i \mu^2 \Omega \\ i \mu^2 \Omega & \Omega^2 + \mu^4 \end{pmatrix}$$

is invariant and faithful. The generator $\mathcal{L}$ can be written in a special GKSL form (with respect to the invariant state $\rho$) with

$$L_1 = \mu \sigma^- - \frac{\mu}{2} \text{tr}(\rho \sigma^-) 1 = \mu \sigma^- + i \frac{\mu^3 \Omega}{2(2\Omega^2 + \mu^4)} , \quad H = \left( \frac{\Omega}{2} + \frac{\mu^3 \Omega}{2\Omega^2 + \mu^4} \right) \sigma_1.$$

The dual semigroup $\tilde{T}$ of $T$ is not a QMS because $H$ does not commute with $\rho$. 

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We now establish the relationship between the privileged GKSL representations of a generator \( \mathcal{L} \) and its 0-dual \( \tilde{\mathcal{L}} \).

**Theorem 26** If \( \tilde{\mathcal{T}} \) is a QMS, then, for every privileged GKSL representation of \( \mathcal{L} \), by means of operators \( H, L_k \), there exists a privileged GKSL representation of \( \tilde{\mathcal{L}} \), by means of operators \( \tilde{H}, \tilde{L}_k \) such that:

1. \( \tilde{H} = -H - \alpha \) for some \( \alpha \in \mathbb{R} \);
2. \( \tilde{L}_k = \lambda_k^{-1/2} L_k^* \) for some \( \lambda_k > 0 \).

**Proof.** Consider a privileged GKSL representation of \( \mathcal{L} \)

\[
\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \geq 1} \left( L_k^* L_k a - 2L_k^* a L_k + a L_k L_k^* \right),
\]

with \( H \rho = \rho H \) and \( \rho L_k \rho^{-1} = \lambda_k L_k, \rho L_k^* \rho^{-1} = \lambda_k^{-1} L_k^* \) for some \( \lambda_k > 0 \).

Since \( \rho \tilde{\mathcal{L}}(a) = \mathcal{L}_* (\rho a) \), we have

\[
\rho \tilde{\mathcal{L}}(a) = -i[H, a] - \frac{1}{2} \sum_k (\rho a L_k^* L_k - 2L_k \rho a L_k^* + L_k^* L_k \rho a) = -i\rho [H, a] - \frac{1}{2} \sum_k (\rho a L_k^* L_k - 2\lambda_k^{-1} \rho L_k a L_k^* + L_k^* L_k \rho a).
\]

But \( \rho \) is \( \mathcal{T} \)-invariant and commutes with \( H \) thus \( \sum_k \rho L_k^* L_k = \sum_k L_k^* L_k \rho = \sum_k L_k \rho L_k^* = \sum_k \lambda_k^{-1} \rho L_k L_k^* \). It follows that \( \sum_k L_k^* L_k = \sum_k \lambda_k^{-1} L_k L_k^* \) and

\[
\rho \tilde{\mathcal{L}}(a) = \rho \left( -i[H, a] - \frac{1}{2} \sum_k (a \lambda_k^{-1} L_k L_k^* - 2\lambda_k^{-1} L_k a L_k^* + \lambda_k^{-1} L_k L_k^* a) \right).
\]

Therefore, putting \( \tilde{H} = -H - \alpha \) \( (\alpha \in \mathbb{R}) \) and \( \tilde{L}_k = \lambda_k^{-1/2} L_k^* \), we find a GKSL representation of \( \tilde{\mathcal{L}} \).

Since \( [\tilde{H}, \rho] = 0, \text{tr}(\rho \tilde{L}_k) = 0 \) for every \( k \) and \( \{ 1, \tilde{L}_k \mid k \geq 1 \} \) is clearly a set of linearly independent elements, we found a special GKSL representation of \( \tilde{\mathcal{L}} \) by means of the operators \( \tilde{H}, \tilde{L}_k \). Moreover, we have

\[
\rho \tilde{L}_k = \lambda_k^{-1/2} \rho L_k^* = \lambda_k^{-1/2} \lambda_k^{-1} L_k^* \rho = \lambda_k^{-1} \tilde{L}_k \rho
\]

and, in the same way \( \rho \tilde{L}_k^* \rho^{-1} = \lambda_k^{-1} \tilde{L}_k^* \). Therefore we found a privileged GKSL representation of \( \tilde{\mathcal{L}} \) by means of the operators \( \tilde{H}, \tilde{L}_k \).

\( \text{(q.e.d.)} \)

## 5 Quantum detailed balance

In this section we characterise the generator of a uniformly continuous QMS satisfying the quantum detailed balance condition.
Definition 27 A QMS $\mathcal{T}$ on $\mathcal{B}(h)$ satisfies the quantum $s$-detailed balance condition (s-DB) with respect to a normal faithful invariant state $\rho$, if its generator $\mathcal{L}$ and the generator $\tilde{\mathcal{L}}$ of the $s$-dual semigroup $\tilde{T}$ satisfy

$$\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i[K, a]$$

with a bounded self-adjoint operator $K$ on $h$ for all $a \in \mathcal{B}(h)$.

This definition generalises the concept of classical detailed balance discussed in Section 2. Indeed, a classical Markov semigroup $T$ satisfies the classical detailed balance condition if and only if $T = \tilde{T}$, i.e. the generators $A$ and $\tilde{A}$ coincide.

Lemma 28 If $\mathcal{T}$ satisfies the quantum $s$-detailed balance condition then $\tilde{T}$ is a QMS and the self-adjoint operator $K$ in (23) commutes with $\rho$.

Proof. The identity (23) implies that $\tilde{\mathcal{L}}$ is conditionally completely positive. Therefore $\tilde{T}$ is a QMS. Moreover, recalling that $\rho$ is an invariant state for both $T$ and $\tilde{T}$ by Proposition 4, for any $a \in \mathcal{B}(h)$, we have then

$$0 = \text{tr} \left( \rho (\mathcal{L}(a) - \tilde{\mathcal{L}}(a)) \right) = 2i \text{tr}(\rho [K, a]) = 2i \text{tr}(\rho [K, a]),$$

i.e. $[K, \rho] = 0$. This completes the proof. (q.e.d.)

Notice $[K, \rho] = 0$ and equation (23) imply that the linear operator $\mathcal{L}' = \mathcal{L} - i[K, \cdot]$ is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_0$ on $\mathcal{B}(h)$.

Throughout this section we consider the duality with $s = 0$.

Proposition 29 Given a special GKSL representation of the generator $\mathcal{L}$ of a QMS $\mathcal{T}$ by means of operators $H, L_k$. Define

$$\mathcal{L}_0(a) = -\frac{1}{2} \sum_k (L_k^*L_k a - 2L_k^*aL_k + aL_k^*L_k).$$

The QMS $\mathcal{T}$ satisfies the quantum 0-detailed balance condition if and only if $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$ with $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ and $[H, \rho] = 0$.

Proof. Clearly, if $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$ with $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ and $[H, \rho] = 0$, the QMS $\mathcal{T}$ satisfies the 0-DB. Indeed, if $\mathcal{L}_0$ is self-adjoint and $H$ commutes with $\rho$, we have $\tilde{\mathcal{L}} = \mathcal{L}_0 - i[H, \cdot]$. Therefore $\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i[H, a]$.

Conversely, if $\mathcal{T}$ satisfies the 0-DB condition can find a privileged GKSL of $\mathcal{L}$ by means of operators $K, M_k$ by Theorem 24. Note that $K$ commutes with $\rho$ because it is the Hamiltonian in a privileged GKSL representation. On the other hand, the Hamiltonian $K$ in a special GKSL representation is unique up to a scalar multiple of the identity by Theorem 9, therefore we can take $H = K$ and we know that: 1) $H$ commutes with $\rho$, 2) the operators $L_k$ and $M_k$ define the same map $\mathcal{L}_0$. 

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It follows that $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$ and then $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 - i[H, \cdot]$. Moreover, $\mathcal{T}$ satisfies the 0-DB condition so that $\mathcal{L} = \tilde{\mathcal{L}}$. It follows that $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$. (q.e.d.)

We can now characterise generators $\mathcal{L}$ of QMS satisfying the 0-DB condition.

**Theorem 30** A QMS $\mathcal{T}$ satisfies the 0-detailed balance condition $\mathcal{L} - \tilde{\mathcal{L}} = 2i[K, \cdot]$ if and only if there exists a privileged GKSL representation of $\mathcal{L}$, by means of operators $H, L_k$, such that:

1. $H = K + c$ for some $c \in \mathbb{R}$,
2. $L_k^{1/2} = \sum_j u_{kj} L_j$ for some $\lambda_k > 0$ and some unitary operator $(u_{kj})_{kj}$ on $k$.

In particular both $H$ and $\sum_k L_k^* L_k$ commute with $\rho$.

**Proof.** If $\mathcal{T}$ satisfies the 0-DB condition its generator $\mathcal{L}$ and the generator $\tilde{\mathcal{L}}$ of the dual QMS satisfy $\mathcal{L}(a) - i[K, a] = \tilde{\mathcal{L}}(a) + i[K, a]$. Let $H, L_k$ be the operators in a privileged GKSL representation of $\mathcal{L}$. By Theorem 26, the operators $H = H + c$ and $L_k = L_k^{1/2} L_k^*$ give us a privileged GKSL representation of $\mathcal{L}$.

It follows that the operators $H - K, L_k$ and $-H + K - c, L_k^{1/2} L_k^*$ arise in a special GKSL representation of $\mathcal{L}(\cdot) - i[H, \cdot]$. Therefore, by Theorem 9, $H - K = -H + K - c'$ leading us to (1) and there exists a unitary operator $(u_{kj})_{kj}$ on $k$ such that (2) holds.

Conversely if conditions (1) and (2) hold, writing $\mathcal{L}(a) = \mathcal{L}_0(a) + i[H, a]$, a straightforward computation shows that $\text{tr}(\rho L(a)b) = \text{tr}(\rho aL(b))$ with $\tilde{\mathcal{L}}(a) = \mathcal{L}_0(a) - i[H, a]$. We have then $\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i[H, a]$ and the 0-DB condition holds with $K = H$.

(q.e.d.)

**Remark 31** The proof also shows that we can replace “there exists a privileged GKSL ...” by “for every privileged GKSL ...” in Theorem 26.

We conclude this section by showing an example of a QMS $\mathcal{T}$ whose $s$-dual semigroup $\mathcal{T}$ is still a QMS but does not satisfy the $s$-detailed balance condition.

**Example 32** We consider $h = \ell^2(\mathbb{Z}_n, \mathbb{C})$, $n \geq 3$, with the orthonormal basis $(e_j)_{j=1, \ldots, n}$, and define

$$\mathcal{L}(a) = S^* a S - a,$$

where $S$ is the unitary shift operator on $\ell^2(\mathbb{Z}_n)$, $Se_j = e_{j+1}$ (sum modulo $n$).

The QMS $\mathcal{T}$ generated by $\mathcal{L}$ admits $\rho = n^{-1} I$ as a faithful invariant state because $\mathcal{L}_s(I) = SS^* - I = 0$. A straightforward computation shows that the dual semigroup $\mathcal{T}$ is the QMS generated by the linear map $\tilde{\mathcal{L}}$ defined by $\tilde{\mathcal{L}}(a) = SaS^* - a$.

We now check that $\mathcal{T}$ does not satisfy the 0-detailed balance condition.

Letting $H = 0$ and $L_1 = S$ we find a privileged GKSL representation of $\mathcal{L}$. Suppose that $\mathcal{T}$ satisfies the 0-detailed balance condition. Then, by Theorem 26...
$\mathcal{L} = \tilde{\mathcal{L}}$ because $K$ is a multiple of the identity operator. This identity, however, is not true since

$$\mathcal{L}(|e_2\rangle\langle e_2|) - \tilde{\mathcal{L}}(|e_2\rangle\langle e_2|) = |e_1\rangle\langle e_1| - |e_3\rangle\langle e_3| \neq 0.$$ 

Note that the condition $n \geq 3$ is necessary. Indeed, for $n = 2$, we can easily check that $\mathcal{L} = \tilde{\mathcal{L}}$ and the $s$-detailed balance condition holds for all $s \in [0, 1]$.

### 6 Quantum Markov semigroups on $M_2(\mathbb{C})$

In this section we study in detail the case $h = \mathbb{C}^2$ and $\mathcal{B}(h) = M_2(\mathbb{C})$. We establish the general form of the generator of a QMS $\mathcal{T}$ whose 0-dual $\mathcal{T}$ is a QMS and show that, in this case, $\mathcal{T}$ satisfies the 0-detailed balance condition.

This can be viewed as a non-commutative counterpart of a well-known fact: any 2-state classical Markov chain satisfies the classical detailed balance condition.

We consider, as usual, the basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ of $M_2(\mathbb{C})$, where

$$\sigma_0 = 1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. Any state on $M_2(\mathbb{C})$ has the form

$$\frac{1}{2} (\sigma_0 + u_1 \sigma_1 + u_2 \sigma_2 + u_3 \sigma_3)$$

for some vector $(u_1, u_2, u_3)$, in the unit ball of $\mathbb{R}^3$. This state is faithful if the vector $(u_1, u_2, u_3)$ belongs to the interior of the unit ball, i.e. $u_1^2 + u_2^2 + u_3^2 < 1$. After a suitable change of coordinates then we can write a faithful state as

$$\rho = \begin{pmatrix} \nu & 0 \\ 0 & 1 - \nu \end{pmatrix} = \frac{1}{2} (\sigma_0 + (2\nu - 1)\sigma_3)$$

for some $0 < \nu < 1$.

We can now characterise special GKSL representations of the generator $\mathcal{L}$ of a QMS on $M_2(\mathbb{C})$ in the following way

**Lemma 33** If $L_k = \sum_{j=0}^3 z_{kj} \sigma_j$ with $z_{kj} \in \mathbb{C}$, $k \in J \subseteq \mathbb{N}$, then

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \in J} (L_k^* L_k a - 2 L_k^* a L_k + a L_k^* L_k)$$

is a special GKSL representation of $\mathcal{L}$ with respect to $\rho$ if and only if

1. $L_k = -(2\nu - 1)z_k \mathbb{1} + \sum_{j=1}^3 z_{kj} \sigma_j$ for all $k \in J$,

2. $\text{card}(J) \leq 3$ and $\{z_k : k \in J\}$ (with $z_k = (z_{k1}, z_{k2}, z_{k3})$) is a set of linearly independent vectors in $\mathbb{C}^3$.
Proof. A simple calculation shows that \( \text{tr}(\rho L_k) = 2(z_{k0} + (2\nu - 1)z_{k3}) \) thus, the condition \( \text{tr}(\rho L_k) = 0 \) is equivalent to \( z_{k0} = -(2\nu - 1)z_{k3} \).

Finally, \( \{ 1, L_k : k \in J \} \) is a set of linearly independent elements in \( M_2(\mathbb{C}) \) if and only if the vectors of coefficients w.r.t. the Pauli matrices
\[
\{(1, 0, 0, 0), (- (2\nu - 1)z_{k3}, z_{k1}, z_{k2}, z_{k3}) : k \in J \}
\]
are linearly independent in \( \mathbb{C}^4 \); this is clearly equivalent to have \( \text{card}(J) \leq 3 \) and \( \{z_k : k \in J\} \) linearly independent on \( \mathbb{C}^3 \), \( z_k := (z_{k1}, z_{k2}, z_{k3}) \). (q.e.d.)

Theorem 34 Suppose \( \nu \neq 1/2 \) (i.e. \( \rho \neq 1/2 \)) and \( \rho \) invariant for \( T \). Then \( \mathcal{T} \) is a QMS if and only if the special Lindblad representation of \( L \) has the form
\[
L(a) = i[H, a] - \frac{|\eta|^2}{2} (L^2 a - 2LaL + aL^2) - \frac{|\lambda|^2}{2} (\sigma^- \sigma^+ - 2\sigma^- a \sigma^+ + a \sigma^- \sigma^+) - \frac{|\mu|^2}{2} (\sigma^+ \sigma^- - 2\sigma^+ a \sigma^- + a \sigma^+ \sigma^-)
\]
where
\[
H = v_0 \sigma_0 + v_3 \sigma_3 = (v_0 + v_3) \sigma_+ \sigma^- + (v_0 - v_3) \sigma^- \sigma^+,
\]
\[
L = -(2\nu - 1) \mathbb{1} + \sigma_3 = (1 - \nu) \sigma^+ - \nu \sigma^-,
\]
\( \sigma^+ = \sigma_1 + i\sigma_2, \sigma^- = \sigma_1 - i\sigma_2, v_0, v_1 \in \mathbb{R} \) and \( \lambda, \mu, \eta \in \mathbb{C} \) satisfy
\[
|\lambda|^2/|\mu|^2 = \nu/(1 - \nu).
\]

Proof. Consider a special GKSL representation
\[
L(a) = i[H, a] - \frac{1}{2} \sum_{k \in J} (L_k^* L_k a - 2L_k^* a L_k + aL_k^* L_k)
\]
of \( L \) with respect to \( \rho \), where \( J \subseteq \{ 1, 2, 3 \} \), \( H = \sum_{j=0}^3 v_j \sigma_j \) and
\[
L_k = -(2\nu - 1) \mathbb{1} + \sum_{j=1}^3 z_{kj} \sigma_j = \begin{pmatrix}
2(1 - \nu)z_{k3} & (z_{k1} - iz_{k2}) \\
(z_{k1} + iz_{k2}) & -2\nu z_{k3}
\end{pmatrix},
\]
\( \{ z_k : k \in J \} \) linearly independent (Lemma 33).

We must find \( v_j \) and \( z_{kj} \) such that:
1. \( [H, \rho] = 0 \);
2. \( \rho L_k \rho^{-1} = \lambda_k L_k \) and \( \rho L_k^* \rho^{-1} = \lambda_k L_k^* \) for some \( \lambda_k > 0 \);
3. \( \rho \) is \( T \)-invariant.
(1) Clearly $H$ commutes with $\rho$ if and only if $v_1 = v_2 = 0$, i.e.

$$H = v_0 \mathbb{1} + v_3 \sigma_3 = (v_0 + v_3)\sigma^+\sigma^- + (v_0 - v_3)\sigma^-\sigma^+.$$

(2) Fix $k \in \mathcal{J}$. One can easily check that

$$\rho L_k \rho^{-1} = \left( \frac{2(1 - \nu)z_{k3}}{1 - \nu} (z_{k1} + iz_{k2}) \right),$$

and, since $\nu \neq 1/2$, the identity $\rho L_k \rho^{-1} = \lambda_k L_k$ holds if and only if either

$$\begin{cases}
\lambda_k = 1 \\
z_{k1} - iz_{k2} = 0
\end{cases}$$

or

$$\begin{cases}
z_{k3} = 0 \\
(1 - \nu - \lambda_k) (z_{k1} - iz_{k2}) = 0
\end{cases}$$

or

$$\begin{cases}
z_{k3} = 0 \\
(1 - \nu - \lambda_k) (z_{k1} + iz_{k2}) = 0
\end{cases}$$

In the first case, we get $L_k = z_{k3} (-2(\nu - 1) \mathbb{1} + \sigma_3) = z_{k3} ((1 - \nu)\sigma^+ - \nu \sigma^-)$; since $\{L_k : k \in \mathcal{J}\}$ is a set of linearly independent elements in $M_2(\mathbb{C})$, this means that there exists an unique $k_0 \in \mathcal{J}$ such that $\lambda_{k_0} = 1$. We can suppose $k_0 = 3$.

Therefore, for $k = 1, 2$, conditions (27) are equivalent to

$$\begin{cases}
z_{k3} = 0 \\
\frac{\nu}{1 - \nu} = \lambda_k \\
z_{k1} + iz_{k2} = 0
\end{cases}$$

or

$$\begin{cases}
z_{k1} - iz_{k2} = 0 \\
z_{k3} = 0
\end{cases}$$

or

$$\begin{cases}
z_{k3} = 0 \\
\frac{\nu}{1 - \nu} = \lambda_k
\end{cases}$$

that is

$$L_k = \begin{pmatrix} 0 & -iz_{k2} \\ 0 & 0 \end{pmatrix} = -iz_{k2} \sigma^+ \text{ and } \lambda_k = \frac{\nu}{1 - \nu},$$

or

$$L_k = \begin{pmatrix} 0 & iz_{k2} \\ 0 & 0 \end{pmatrix} = iz_{k2} \sigma^- \text{ and } \lambda_k = \frac{1 - \nu}{\nu},$$

so that we have $L_1 = -iz_{12} \sigma^+ = \lambda_1 \sigma^+$ and $L_2 = iz_{22} \sigma^- = \mu \sigma^-$, with $\lambda_1 = \nu/(1 - \nu)$ and $\lambda_2 = \nu/(1 - \nu)$.

Moreover, with this choice of $L_1, L_2$ and $L_3$, the equalities $\rho L_k \rho^{-1} = \lambda_k^{-1} L_k$ are automatically satisfied.

(3) Since $H, L_3$ and $\rho$ commute, $\rho$ is $T$-invariant if and only if

$$0 = \frac{1}{2} \sum_{k=1}^{2} (L_k L_k \rho - 2L_k \rho L_k^* + \rho L_k^* L_k) = \frac{1}{2} \sum_{k=1}^{2} (L_k L_k \rho - 2L_k (\rho L_k^* \rho^{-1}) \rho + (\rho L_k^* \rho^{-1}) (\rho L_k \rho^{-1}) \rho) = \sum_{k=1}^{2} (L_k^* L_k \rho - \lambda_k^{-1} L_k L_k^*),$$

that is

$$\frac{|\lambda|^2}{|\mu|^2} = \frac{|z_{12}|^2}{|z_{22}|^2} = \frac{\nu}{1 - \nu}.$$
This concludes the proof.

(q.e.d.)

**Theorem 35** Suppose $\nu \neq 1/2$.

If $\tilde{T}$ is a QMS, then $T$ satisfies detailed balance.

**Proof.** By Theorem 34 there exists a privileged GKSL representation of $L$ with

\[
\begin{align*}
L_1 &= \eta L \\
L_2 &= \lambda \sigma^+ \\
L_3 &= \mu \sigma^-
\end{align*}
\]

and

\[
\frac{|\lambda|^2}{|\mu|^2} = \frac{\nu}{1-\nu}.
\]

Therefore,

\[
\left( \begin{array}{c}
\sqrt{\frac{1}{\lambda} L_1} \\
\sqrt{\frac{1}{\lambda} L_2} \\
\sqrt{\frac{1}{\lambda} L_3}
\end{array} \right)
= 
\left( \begin{array}{ccc}
\eta L & 0 & 0 \\
0 & 0 & \sqrt{\frac{1-\nu}{\nu}} \\
0 & \sqrt{\frac{1-\nu}{\nu}} & \lambda \\
0 & \sqrt{\frac{1-\nu}{\nu}} & \lambda
\end{array} \right)
\left( \begin{array}{c}
L_1 \\
L_2 \\
L_3
\end{array} \right)
\]

and

\[
\left( \begin{array}{cccc}
\eta L & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{1-\nu}{\nu}} & \lambda \\
0 & \sqrt{\frac{1-\nu}{\nu}} & \lambda & 0 \\
0 & \sqrt{\frac{1-\nu}{\nu}} & \lambda & 0
\end{array} \right)
\]

is unitary thanks to (28). (q.e.d.)

7 The symmetric dual semigroup and detailed balance condition

We now study the $s$-dual semigroup and the quantum $s$-detailed balance condition for $s = 1/2$. In this case we call $T'$ the symmetric dual semigroup of $T$ and call symmetric detailed balance condition the 1/2-detailed balance condition.

By Proposition 4, the symmetric dual semigroup of $T$ is defined by

\[
\rho^{1/2} T'_t(a) \rho^{1/2} = T_{st}(\rho^{1/2} a \rho^{1/2}),
\]

so that

\[
T'_t(a) \supset \rho^{-1/2} T_t(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2}
\]

for all $a \in \mathcal{B}(\mathcal{H})$. The name symmetric is then justified by the left-right symmetry of multiplication by $\rho^{1/2}$ and $\rho^{-1/2}$.

Equation (30) ensures that any map $T'_t$ is completely positive, contrary to the case $s = 0$ (Example 25). Therefore the symmetric dual semigroup $T'$ is always a QMS with generator given by (Proposition 6)

\[
\rho^{1/2} \mathcal{L}(a) \rho^{1/2} = \mathcal{L}_s(\rho^{1/2} a \rho^{1/2}).
\]

The relationship between dual semigroups $\tilde{T}$ and $T'$ is described by the following...
Theorem 36  The 0-dual $\tilde{T}$ and the symmetric dual $T'$ of a QMS $T$ coincide if and only if each map $T_t$ commutes with $\sigma_{-i}$.

Proof. If $\tilde{T} = T'$, then $\tilde{T}$ is a QMS; hence, by Theorem 8 $T$ commutes with the modular automorphism $\sigma_{-i}$.

On the other hand, we showed in the proof of Theorem 8 that the commutation between $T$ and $\sigma_{-i}$ implies $\tilde{T}_t(a) = \rho^{-1/2}T_s(\rho^{1/2}a\rho^{1/2})\rho^{-1/2}$ for all $a \in B(h)$, $t \geq 0$, and then $\tilde{T} = T'$.

We now establish the relationship between the generator $L$ of a QMS and the generator $L'$ of the symmetric dual semigroup.

Theorem 37  For all special GKSL representation $L(a) = G^*a + \sum_k L^*_k a L_k + a G$ of $L$ there exists a special GKSL representation of $L'$ by means of operators $G', L'_k$ such that:

1. $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ for some $c \in \mathbb{R}$,
2. $L'_k\rho^{1/2} = \rho^{1/2}L^*_k$

Proof. Since $T'$ is a uniformly continuous QMS, its generator $L'$ admits a special GKSL representation, $L'(a) = G'^*a + \sum_k L'^*_k a L'_k + a G'$. Moreover, by Proposition 15 we have $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$, $c \in \mathbb{R}$, and so the relation $\rho^{1/2}L'(a)\rho^{1/2} = L'_s(\rho^{1/2}a\rho^{1/2})$ implies

$$\sum_k \rho^{1/2}L^*_k a L_k \rho^{1/2} = \sum_k L_k \rho^{1/2}a \rho^{1/2} L_k.$$  (31)

Define

$$X(x \otimes 1_k) L' \rho^{1/2} u = (x \otimes 1_k)(\rho^{1/2} \otimes 1_k) L^* u$$

for all $x \in B(h)$ and $u \in h$, where $L : h \to h \otimes k$, $Lu = \sum_k L_k u \otimes f_k$, $L' : h \to h \otimes k'$, $L'u = \sum_k L'_k u \otimes f'_k$, $(f_k)_k$ and $(f'_k)_k$ orthonormal basis of $k$ and $k'$ respectively. Thus, by (31),

$$(X(x \otimes 1_k) L' \rho^{1/2} u, X(y \otimes 1_k) L' \rho^{1/2} v) = \sum_k \langle u, \rho^{1/2}L^*_k x^* y L_k \rho^{1/2} v \rangle$$

$$= \langle (x \otimes 1_k) L' \rho^{1/2} u, (y \otimes 1_k) L' \rho^{1/2} v \rangle$$

for all $x, y \in B(h)$ and $u, v \in h$, i.e. $X$ preserves the scalar product. Therefore, since the set $\{x \otimes 1_k) L' \rho^{1/2} u \mid x \in B(h), u \in h\}$ is total in $h \otimes k'$ (for $\rho^{1/2} \in h$ is dense in $h$ and Theorem 13 holds), we can extend $X$ to an unitary operator from $h \otimes k'$ to $h \otimes k$. As a consequence we have $X^* X = \mathbf{1}_{h \otimes k'}$.

Moreover, since $X(y \otimes 1_k) = (y \otimes 1_k) X$ for all $y \in B(h)$, we can conclude that $X = \mathbf{1}_h \otimes Y$ for some unitary map $Y : k' \to k'$.

The definition of $X$ implies then

$$(\rho^{1/2} \otimes 1_k) L^* = XL' \rho^{1/2} = (\mathbf{1}_h \otimes Y)L' \rho^{1/2}.$$
This means that, by substituting $L'$ by $(1 \otimes Y)L'$, or more precisely $L'_k$ by $\sum_l u_{kl} L'_l$ for all $k$, we have $\rho^{1/2} L'_k = L'_k \rho^{1/2}$.

Since $\text{tr}(\rho L'_k) = \text{tr}(\rho L'_k) = 0$ and, from $L'(1) = 0$ (Proposition 4), $G^* + G' = \sum_k L'_k L'_k$, the properties of a special GKSL representation follow. (q.e.d.)

Contrary to what happens in the case $s = 0$, the operators $G, G'$ may not commute with $\rho$, as the following example shows.

**Example 38** Fix a faithful state $\rho = (1 + (2 \nu - 1) \sigma_3)/2$ on $M_2(\mathbb{C})$ with $\nu \in [0, 1[$, $\nu \neq 1/2$ and consider the semigroup on $M_2(\mathbb{C})$ generated by

$$L(a) = i[H,a] - \frac{1}{2} (L^*La - 2L^*aL + aL^*L)$$

with $H = \Omega \sigma_1$, $L = (1 - 2\nu) 1 + ir \sigma_1 + s \sigma_2 + \sigma_3$ and $\Omega, r, s \in \mathbb{R}$, $\Omega \neq 0$. Clearly $L$ is represented in a special GKSL form with respect to the faithful state $\rho$ and $H$ does not commute with $\rho$.

We now show that $\rho$ is an invariant state for the QMS generated by $L$ for a special choice of the constants $\Omega, r, s$ and so we find the desired example.

A long but straightforward computation shows that, if we choose $r, s$ satisfying

$$2\nu = (r - s)^2/(r^2 + s^2), \quad (32)$$

then we find $L_*(\rho) = ((-4\nu^2 + 4\nu + 1)r + (2\nu - 1)(s - \Omega)) \sigma_2$. It is now a simple exercise to show that for all fixed $\nu$ and $\Omega \neq 0$ there exist $r, s$ satisfying (32) and

$$s = \Omega + r (4\nu^2 - 4\nu - 1)/(2\nu - 1). \quad (33)$$

A little computation yields

$$s = \frac{\pm \Omega}{2 \sqrt{\nu (1 - \nu)}}, \quad r = \frac{\pm \Omega (1 - 2\nu)}{2 \sqrt{\nu (1 - \nu)} (1 \pm 2 \sqrt{\nu (1 - \nu)})} \quad (34)$$

($\pm$ are all $+$ or all $-$. With this choice of $r$ and $s$ the state $\rho$ is invariant.

The 0-detailed balance is stronger than the symmetric detailed balance (see also [5] Th. 6.6 p.296).

**Proposition 39** If $T$ satisfies the 0-detailed balance, then it also fulfills the symmetric detailed balance. Moreover, these conditions are equivalent if and only if the 0-dual $\tilde{T}$ is a QMS.

**Proof.** Suppose that $\tilde{T}$ is a QMS. As we showed in the proof of Theorem 38 $\tilde{T}_t(a) = \rho^{-1/2} T_t(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2}$. Then $\tilde{T} = T'$ by (30), as a consequence $\tilde{L} = L'$, i.e. $\tilde{L} - L = L - L'$, and both detailed balance conditions are equivalent.
On the other hand, if $T$ satisfies the 0-detailed balance, then $\overline{T}$ is a QMS. Therefore $\overline{T} = T'$ and $T$ also fulfills the symmetric detailed balance condition. (q.e.d.)

We end this section by finding the relationships between the operators $H, L_k$ in a special GKSL representation of the generator of a QMS satisfying symmetric detailed balance.

**Theorem 40** A QMS $T$ satisfies the symmetric detailed balance condition $\mathcal{L} = -i[K, \cdot]$ if and only if there exists a special GKSL representation of the generator $\mathcal{L}$ by means of operators $H, L_k$ such that, letting $2G = -\sum_k L_k^* L_k - 2iH$, we have:

1. $G\rho^{1/2} = \rho^{1/2} G^* - (2iK + ic)\rho^{1/2}$ for some $c \in \mathbb{R}$,
2. $\rho^{1/2} L_k^* = \sum_k u_{kj} L_j \rho^{1/2}$, for all $k$, for some unitary matrix $(u_{kj})_{kj}$.

**Proof.** Choose a special GKSL representation of $\mathcal{L}$ by means of operators $H, L_k$. Theorem [37] allows us to write the dual $\mathcal{L}'$ in a special GKSL representation by means of operators $H', L'_k$, with $H' = (G^* - G')/(2i)$,

$$G'\rho^{1/2} = \rho^{1/2} G^*, \quad L'_k \rho^{1/2} = \rho^{1/2} L_k^*.$$  \hspace{0.5cm} (35)

Suppose first that $T$ satisfies the symmetric detailed balance condition. Then $\mathcal{L} - i[K, \cdot] = \mathcal{L}' + i[K, \cdot]$ and $K$ commutes with $\rho$ by Lemma [28]. Comparing the special GKSL representations of $\mathcal{L} - i[K, \cdot]$ and $\mathcal{L}' + i[K, \cdot]$, by Theorem [9] and Remark [12] we find

$$G + iK = G' - iK + ic, \quad L_k^* = \sum_j u_{kj} L_j,$$

for some unitary matrix $(u_{kj})_{kj}$ and some $c \in \mathbb{R}$. This, together with (35) implies that conditions (1) and (2) hold.

Conversely, notice that the dual $\mathcal{L}'$ admits the special GKSL representation

$$\mathcal{L}'(a) = G'^* a + \sum_k L'_k^* a L'_k + aG'.$$

Therefore, if conditions (1) and (2) are satisfied, by (35), we have

$$G'\rho^{1/2} = \rho^{1/2} G^* = (G + 2iK)\rho^{1/2},$$

so that $G' = G + 2iK$ and then

$$G'^* a + aG' = (G^* - 2iK)a + a(G + 2iK) = G^* a + aG - 2i[K, a]$$

$$\sum_k L'_k^* a L'_k = \sum_{k,j,m} \bar{u}_{kj} L_j^* a u_{km} L_m = \sum_{j,m} \left( \sum_k \bar{u}_{kj} u_{km} \right) L_j^* a L_m = \sum_k L_k^* a L_k.$$  \hspace{0.5cm} (27)
It follows that \( L'(a) = L(a) - 2i[K, a] \) and the symmetric detailed balance condition holds.

(q.e.d.)

The Hamiltonian \( H \) in a special GKSL representation of the generator of a QMS satisfying the symmetric detailed balance condition does not need to commute with the invariant state \( \rho \) (as in the case of 0-detailed balance) as shows the following

**Example 41** Let \( \mathcal{L} \) be the generator described in Example 38 and let \( \mathcal{L}' \) be its symmetric dual. The linear map \( K = (\mathcal{L} + \mathcal{L}')/2 \) is clearly the generator of a QMS. Moreover, \( \rho \) is an invariant state for \( K \) because it is an invariant state for \( \mathcal{L} \) and \( \mathcal{L}' \) by Proposition 4. \( K \) satisfies the symmetric detailed balance condition by its definition.

The special GKSL representation of \( \mathcal{L} \) by means of operators \( H, L \) as in Example 38 yields a special GSKL representation of \( \mathcal{L}' \) choosing \( L' = \rho^{1/2}L^*\rho^{-1/2} \) and \( H' = (G^* - G')/(2i) \) with \( G' = \rho^{1/2}G^*\rho^{-1/2} \) and \( 2G = -L^*L - iH \). Putting

\[
M_1 = L/\sqrt{2}, \quad M_2 = \rho^{1/2}L^*\rho^{-1/2}/\sqrt{2},
\]

\[
F = (G + G')/2, \quad F_0 = (F + F^*)/2, \quad K = (F^* - F)/(2i)
\]

we have \( 2F_0 = M_1^*M_1 + M_2^*M_2 \) and a special GKSL representation of \( K \) by means of operators \( K, M_j \).

We now check that \( K \) does not commute with \( \rho \). To this end it suffices that \( K \) is not linearly independent of \( \sigma_j \) for \( j = 1 \) or \( j = 2 \), namely \( \text{tr}(\sigma_j K) \neq 0 \). But

\[
2\text{tr}(\sigma_j K) = 2\text{tr}(\sigma_j F) = 3\text{tr}(\sigma_j(G + G')) = 3\text{tr}(\sigma_j G) + 3\text{tr}(\rho^{-1/2}\sigma_j\rho^{1/2}G^*),
\]

where, defining \( s \) and \( r \) as in (34) with \( - \) signs and computing \( (2\nu - 1)s + r = (2\nu - 1)\Omega/(1 - 2\sqrt{\nu(1 - \nu)}) \) we have

\[
G = \frac{(1 - 2\nu)^2 + 1 + s^2 + r^2}{2} - i\Omega\sigma_1 + \frac{(2\nu - 1)\Omega}{1 - 2\sqrt{\nu(1 - \nu)}}\sigma_2 + ((2\nu - 1) - rs)\sigma_3.
\]

Another straightforward computation yields

\[
\rho^{1/2} = \frac{\kappa}{2} \left( 1 + \frac{2\nu - 1}{\kappa^2} \sigma_3 \right), \quad \rho^{-1/2} = \frac{1}{\kappa} \left( 1 - \kappa^{-4}(2\nu - 1)^2 \right) \left( 1 - \frac{2\nu - 1}{\kappa^2} \sigma_3 \right)
\]

where \( \kappa := \sqrt{1 + 2\sqrt{\nu(1 - \nu)}} = \sqrt{\nu} + \sqrt{1 - \nu} \) and

\[
\rho^{-1/2}\sigma_1\rho^{1/2} = \frac{1}{2\sqrt{\nu(1 - \nu)}} \left( \sigma_1 - i(2\nu - 1)\sigma_2 \right).
\]

It follows that \( 2\text{tr}(\sigma_1 K) = -2\Omega \). Therefore we find \( \text{tr}(\sigma_1 K) \neq 0 \) for all \( \nu \in ]0, 1[ \) with \( \nu \neq 1/2 \) and \( \Omega \neq 0 \).
8 Case $s \in (0, 1/2) \cup (1/2, 1)$

We conclude the discussion on the $s$-dual semigroup by considering $s \in (0, 1/2) \cup (1/2, 1)$. In this framework, we show that $\tilde{T}(s)$ is a QMS if and only if the 0-dual semigroup is a QMS and, in this case, they coincide. Therefore, it is enough to study the case $s = 0$.

**Proposition 42** The following facts are equivalent:

1. $\tilde{T}(s)$ is a QMS;
2. $\tilde{T}(0)$ is a QMS.

Moreover, if the above conditions hold, then $\tilde{T}(s) = \tilde{T}(0)$.

**Proof.** 1 $\Rightarrow$ 2. Since $\tilde{T}(s)$ and $T_{st}$ are $*$-maps, by the second formula (3) and the same formula taking the adjoint we have

$$\rho^s T_t(a) \rho^{1-s} = \tilde{T}_t(s)(\rho^s a \rho^{1-s})$$

and

$$\rho^{1-s} T_t(a) \rho^s = \tilde{T}_t(s)(\rho^{1-s} a \rho^s).$$

Therefore, given $a \in \mathcal{A}$, we get

$$\rho^s T_t(a) \rho^{1-s} = \tilde{T}_t(s)(\rho^s a \rho^{1-s}) = \rho^{1-s} T_t(a) \rho^s.$$ 

and then

$$T_t(a) = \rho^{1-2s} T_t(s)(\rho^{2s-1} a \rho^{1-2s}) \rho^{2s-1}$$

i.e. any $T_t$ commutes with $\sigma_{-i(2s-1)}$.

This means that the contraction semigroup $(\tilde{T}_t)$ defined on $L^2(\mathfrak{h})$ by

$$\tilde{T}_t(a \rho^{1/2}) = T_t(a) \rho^{1/2}$$

commutes with $\Delta^{1-2s}$ and then, by spectral calculus, it also commutes with $\rho$; it follows that $\tilde{T}_t$ commutes with the modular automorphism $\sigma_{-i}$ and so $\tilde{T}(0)$ is a QMS by Theorem 8.

2 $\Rightarrow$ 1. If $\tilde{T}(0)$ is a QMS, by Theorem 24 there exists a privileged GKSL representation of $\mathcal{L}$ by means of operators $H$ and $L_k$ such that

$$\Delta(L_k \rho^{1/2}) = \rho L_k \rho^{-1/2} = \lambda_k L_k \rho^{1/2} \quad \text{and} \quad \Delta(L_k^* \rho^{1/2}) = \rho L_k^* \rho^{-1/2} = \lambda_k^{-1} L_k^* \rho^{1/2}.$$ 

It follows by spectral calculus that

$$\rho^\alpha L_k \rho^{-\alpha} \rho^{1/2} = \Delta^\alpha(L_k \rho^{1/2}) = \lambda_k^\alpha L_k \rho^{1/2}$$

and

$$\rho^\alpha L_k^* \rho^{-\alpha} \rho^{1/2} = \Delta^\alpha(L_k^* \rho^{1/2}) = \lambda_k^{-\alpha} L_k^* \rho^{1/2}.$$ 

for all $\alpha \neq 0$. Therefore, since $H$ and $\rho$ commute, we have
\[
\tilde{\mathcal{L}}^{(s)}(a) = \rho^{s-1}L_s(\rho^{1-s}a\rho^s)\rho^{-s} = -i\rho^{s-1}[H, \rho^{1-s}a\rho^s]\rho^{-s} - \frac{1}{2} \sum_k \left( \rho^{s-1}L_k^*L_k\rho^{1-s}a - 2\rho^{s-1}L_k\rho^{1-s}a\rho^sL_k^*\rho^{-s} + a\rho^sL_k^*L_k\rho^{-s} \right) \\
= -i[H, a] - \frac{1}{2} \sum_k \left( L_k^*L_k a - 2\lambda_k^{-1}L_k a L_k^* + a L_k^*L_k \right)
\]
for all $a \in \mathfrak{a}$; since $-i[H, a] - \frac{1}{2} \sum_k \left( L_k^*L_k a - 2\lambda_k^{-1}L_k a L_k^* + a L_k^*L_k \right) = \tilde{\mathcal{L}}^{(0)}(a)$ by Theorem 26 and $\mathfrak{a}$ is $\sigma$-weakly dense in $\mathcal{B}(h)$, the above equality means $\tilde{\mathcal{L}}^{(s)} = \tilde{\mathcal{L}}^{(0)}$, so that $\mathcal{T}^{(s)}$ is a QMS and it coincides with $\mathcal{T}^{(0)}$. (q.e.d.)

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