Hardy’s theorem for the $q$-Bessel Fourier transform

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Abstract

In this paper we give a $q$-analogue of the Hardy’s theorem for the $q$-Bessel Fourier transform. The celebrated theorem asserts that if a function $f$ and its Fourier transform $\hat{f}$ satisfying $|f(x)| \leq c.e^{-\frac{1}{4}x^2}$ and $|\hat{f}(x)| \leq c.e^{-\frac{1}{4}x^2}$ for all $x \in \mathbb{R}$ then $f(x) = \text{const}.e^{-\frac{1}{4}x^2}$.

1 The $q$-Fourier Bessel transform

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [2]. We put

\[ \mathbb{R}_q^{+} = \{q^n, \quad n \in \mathbb{Z}\}, \]

and for complex $a$

\[ (a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad n = 1 \ldots \infty. \]

Jackson’s $q$-integral (see [3]) in the interval $[0, \infty]$ is defined by

\[ \int_0^\infty f(x)d_qx = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n). \]
We introduce the following functional spaces $L_{q,1,\nu}$ of even functions $f$ defined on $\mathbb{R}^+_q$ such that
\[
\|f\|_{q,1,\nu} = \left[ \int_0^\infty |f(x)| x^{2\nu+1} d_q x \right] < \infty.
\]
The normalized Hahn-Exton q-Bessel function of order $\nu > -1$ (see [5]) is defined by
\[
j_\nu(z, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}(n-\frac{1}{2})}{(q, q)_n (q^{\nu+1}, q)_n} z^n.
\]
It’s an entire analytic function in $z$.

**Lemma 1** For every $p \in \mathbb{N}$, there exist $\sigma_p > 0$ for which
\[
|z|^{2p} |j_\nu(z, q^2)| < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}.
\]

**Proof.** In fact
\[
|z|^{2p} |j_\nu(z, q^2)| \leq \frac{1}{(q^2, q^2)_{\infty} (q^{2\nu+2}, q^2)_{\infty}} \sum_{n=0}^{\infty} q^{n(n-1)} |z|^{2n+2p} \leq \frac{q^{p(p+1)}}{(q^2, q^2)_{\infty} (q^{2\nu+2}, q^2)_{\infty}} \sum_{n=p}^{\infty} q^{n(n-2p-1)} |z|^{2n}.
\]
Now using the Stirling’s formula
\[
n! \sim \sqrt{2\pi n} \frac{n^n}{e^n},
\]
we see that there exist an entire $n_0 \geq p$ such that
\[
q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0,
\]
which implies
\[
\sum_{n=n_0}^{\infty} q^{n(n-2p-1)} |z|^{2n} < \sum_{n=n_0}^{\infty} \frac{1}{(2n)!} |z|^{2n} < e^{|z|}.
\]
Finally there exist $\sigma_p > 0$ such that
\[
\frac{|z|^{2p} |j_\nu(z, q^2)|}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}.
\]
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This finish the proof. ■

The q-Bessel Fourier transform \( \mathcal{F}_{q,\nu} \) introduced in [1,4] as follow

\[
\mathcal{F}_{q,\nu} f(x) = c_{q,\nu} \int_0^\infty f(t) j_\nu(xt, q^2) t^{2\nu+1} dq t.
\]

where

\[
c_{q,\nu} = \frac{1}{1 - q^{(q^2 \nu^2 + 2, q^2)}}.
\]

The following theorem was proved in [1]

**Theorem 1** Given \( f \in \mathcal{L}_{q,1,\nu} \) then we have

\[
\mathcal{F}_{q,\nu}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}^+.
\]

**Proof.** See [1] p 3. ■

\section{Hardy’s theorem}

The following Lemma from complex analysis is crucial for the proof of our main theorem.

**Lemma 2** Let \( h \) be an entire function on \( \mathbb{C} \) such that

\[
|h(z)| \leq C e^{a|z|^2}, \quad z \in \mathbb{C},
\]

\[
|h(x)| \leq C e^{-ax^2}, \quad x \in \mathbb{R},
\]

for some positive constants \( a \) and \( C \). Then \( h(z) = \text{Const}. e^{-ax^2} \).

**Proof.** See [6] p 4. ■

Now we are in a position to state and prove the q-analogue of the Hardy’s theorem

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Theorem 2 Suppose \( f \in L_{q,1,\nu} \) satisfying the following estimates
\[
|f(x)| \leq Ce^{-\frac{x}{2}}, \quad \forall x \in \mathbb{R}_q^+,
\]
\[
|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\frac{x}{2}}, \quad \forall x \in \mathbb{R},
\]
where \( C \) is a positive constant. Then there exist \( A \in \mathbb{R} \) such that
\[
f(z) = A.c_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}z^2}\right)(z), \quad \forall z \in \mathbb{C}.
\]

Proof. We claim that \( \mathcal{F}_{q,\nu}f \) is an analytic function and there exist \( C' > 0 \) such that
\[
|\mathcal{F}_{q,\nu}f(z)| \leq C'e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.
\]
We have
\[
|\mathcal{F}_{q,\nu}f(z)| \leq c_{q,\nu}\int_0^\infty |f(x)||j_\nu(zx,q^2)|x^{2\nu+1}d_qx.
\]
From the Lemma 1, if \( |z| > 1 \) then there exist \( \sigma_1 > 0 \) such that
\[
x^{2\nu+1}|j_\nu(zx,q^2)| = \frac{1}{|z|^{2\nu+1}}|j_\nu(zx,q^2)| < \frac{\sigma_1}{1 + |z|^2} e^{\sigma_1|x|}, \quad \forall x \in \mathbb{R}_q^+.
\]
Then we obtain
\[
|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_1 c_{q,\nu}\left[\int_0^\infty \frac{e^{-\frac{1}{2}(x-|z|)^2}}{1 + |z|^2}d_qx\right] e^{\frac{1}{2}|z|^2} < C\sigma_1 c_{q,\nu}\left[\int_0^\infty \frac{1}{1 + x^2}d_qx\right] e^{\frac{1}{2}|z|^2}.
\]
Now, if \( |z| \leq 1 \) then there exist \( \sigma_2 > 0 \) such that
\[
x^{2\nu+1}|j_\nu(zx,q^2)| \leq \sigma_2 e^x, \quad \forall x \in \mathbb{R}_q^+.
\]
Therefore
\[
|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_2 c_{q,\nu}\left[\int_0^\infty e^{-\frac{1}{2}x^2}d_qx\right] \leq C\sigma_2 c_{q,\nu}\left[\int_0^\infty e^{-\frac{1}{2}x^2}d_qx\right] e^{\frac{1}{2}|z|^2}.
\]
Which leads to the estimate (1). Using Lemma 2, we obtain
\[
\mathcal{F}_{q,\nu}f(z) = \text{const}.e^{-\frac{1}{2}z^2}, \quad \forall z \in \mathbb{C},
\]
and by theorem 1, we conclude that
\[
f(z) = \text{const} \cdot \mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}t^2}\right)(z), \quad \forall z \in \mathbb{C}.
\]
This finish the proof. \( \blacksquare \)
Corollary 1 Suppose \( f \in L_{q,1} \) satisfying the following estimates
\[
|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}^+_q,
\]
\[
|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R},
\]
where \( C, p, \sigma \) are a positive constant and \( p \sigma = \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}^+_q \) such that \( a^2 p = \frac{1}{2} \). Then there exist \( A \in \mathbb{R} \) such that
\[
f(z) = A c_{q,\nu} \mathcal{F}_{q,\nu} \left( e^{-\sigma t^2} \right) (z), \quad \forall z \in \mathbb{C}.
\]

Proof. Let \( a \in \mathbb{R}^+_q \), and put
\[
f_a(x) = f(ax),
\]
then
\[
\mathcal{F}_{q,\nu} f_a(x) = \frac{1}{a^{2\nu + 2}} \mathcal{F}_{q,\nu} f(x/a).
\]
In the end, applying Theorem 2 to the function \( f_a \).

Remark 1 Using the \( q \)-Central limit Theorem (see [1]) we give a probability interpretation of the function \( c_{q,\nu} \mathcal{F}_{q,\nu} \left( e^{-\sigma t^2} \right) \). In fact if \( (\xi_n)_{n\geq 0} \) be a sequence of positive probability measures of \( \mathbb{R}^+_q \), satisfying

\[
\lim_{n \to \infty} n\sigma_n = \frac{(q^2, q^2)_1(q^{2\nu+2}, q^2)_1}{q^2}, \quad \text{where} \quad \sigma_n = \int_0^\infty t^2 t^{2\nu+1} d_q \xi_n(t),
\]
and
\[
\lim_{n \to \infty} n\bar{\sigma}_n = 0, \quad \text{where} \quad \bar{\sigma}_n = \int_0^\infty \frac{t^4}{1 + t^2} t^{2\nu+1} d_q \xi_n(t),
\]
then the \( n \)th \( q \)-convolution product \( \xi_n^{*n} \) converge strongly toward a measure \( \xi \) defined by
\[
d_q \xi(x) = c_{q,\nu} \mathcal{F}_{q,\nu} \left( e^{-\sigma t^2} \right) (x) d_q x.
\]
Corollary 2 Suppose \( f \in \mathcal{L}_{q,1,\nu} \) satisfying the following estimates
\[
|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,
\]
\[
|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},
\]
where \( C, p, \sigma \) are a positive constant and \( p\sigma > \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}_q^+ \) such that \( a^2p = \frac{1}{2} \). Then \( f \equiv 0 \).

**Proof.** In fact there exist \( \sigma' < \sigma \) such that \( p\sigma' = \frac{1}{4} \). Then the function \( f \) satisfying the estimates of Corollary 1, if we replacing \( \sigma \) by \( \sigma' \). Which implies
\[
|\mathcal{F}_{q,\nu} f(x)| = \text{const.} e^{-\sigma'x^2}, \quad \forall x \in \mathbb{R}.
\]
On the other hand, \( f \) satisfying the estimates of Corollary 2, then
\[
|\text{const.} e^{-\sigma'x^2}| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}.
\]
This implies \( \mathcal{F}_{q,\nu} f \equiv 0 \), and by Theorem 1 we conclude that \( f \equiv 0 \).

**Remark 2** Hardy’s theorem asserts the impossibility of a function and its \( q \)-Fourier Bessel transform to be simultaneously "very rapidly decreasing". Hardy’s theorem can also be viewed as a sort of "Qualitative uncertainty principles". One such example can be the fact that a function and its \( q \)-Bessel Fourier transform cannot both have compact support.

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