Carleman estimates for the one-dimensional heat equation with non-smooth coefficients and applications

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Abstract. We study the observability and some of its consequences for one-dimensional heat equations $\partial_t \pm \partial_x (c \partial_x)$ with non-smooth coefficients $c$. The observability, for a linear equation, is obtained by a Carleman-type estimate. In a first step, we derive global Carleman estimates in the case of a piecewise $C^1$-coefficient. In a second step, we treat the case of a coefficient with bounded variations by approximating $c$ by piecewise regular coefficients, $c_\varepsilon$, and passing to the limit in the Carleman estimates associated to the operators defined with $c_\varepsilon$. This kind of observability inequality yields controllability results for a semi-linear equation as well as a stability result for the identification of the diffusion coefficient.

1. Introduction

The question of controllability of partial differential systems with discontinuous coefficients and its dual counterpart, observability, are not fully solved yet. Recently, a result of controllability for a semilinear heat equation with a discontinuous coefficient was proven in [7] by means of a Carleman observability estimate. Roughly speaking, as in the case of hyperbolic systems (see e.g. [14, page 357]), the authors of [7] proved their controllability result in the case where the control is supported in the region where the diffusion coefficient is the ‘lowest’. In both cases, however, the approximate controllability holds without any restriction on the ‘monotonicity’ of the coefficients. It is then natural to question whether or not an observability estimate holds in the case of non-smooth coefficients and arbitrary observation location.

In the one-dimensional case, the controllability result for linear parabolic equations was proven for coefficients with bounded variations in [9]. The proof relies on Russell’s method [15]. However, the question of the existence of a Carleman-type observability estimate had remained open.

We consider the elliptic operator $A$ formally defined by $-\partial_x (c \partial_x)$ on $L^2(\Omega)$ in the one-dimensional bounded domain $\Omega = (0,1)$, with $0 < c_{\text{min}} \leq c \leq c_{\text{max}}$, and $c$ is assumed to be in $L^\infty(\Omega)$. The domain of $A$ is $\mathcal{D}(A) = \{ u \in H^2_0(\Omega); c \partial_x u \in H^1(\Omega) \}$. Let $T > 0$. We shall use the following notations $Q = (0,T) \times \Omega$, $\Gamma = \{ 0,1 \}$, and $\Sigma = (0,T) \times \Gamma$.

Here, we show that we can achieve global Carleman estimates for the operators $\partial_t \pm A$ in $Q$, with a boundary ‘observation’ or an interior ‘observation’, in the case of a piecewise $C^1$ coefficient $c$, in a first part, and in the case of a coefficient $c$ with bounded variations ($BV$), in a second part.
Carleman estimates for parabolic equations with smooth coefficients were proven in [10]. The proof is based on the construction of suitable weight functions $\beta$ whose gradient is non-zero in the complement of the observation region. In the non-smooth case, in [7], to obtain the observability, the authors have to add the assumptions on the ‘monotonicity’ of the coefficients mentioned above. In both cases, the weight function $\beta$ was chosen in the domain of the operator $\nabla \cdot (c \nabla)$. In a first part, in the case of a piecewise $C^1$ coefficient, we do not impose this constraint, which, with the jump of the derivative of $\beta$ as a new parameter, enables us to control the interface terms in the derivation of the Carleman estimate and therefore allows us to relax the ‘monotonicity’ condition on the coefficient. Note however that the results of [7] are for the multidimensional heat equation. The relaxation of the ‘monotonicity’ condition in the $n-$dimensional case, $n \geq 2$, remains, to our knowledge, open.

In the case of a $BV$ coefficient, in the second part, the Carleman estimates derived for the operators $\partial_t \pm \partial_x (c \partial_x y)$ are obtained through a limiting process from the Carleman estimates associated for $\partial_t \pm \partial_x (c, \partial_x)$, for $c_\varepsilon$ piecewise regular converging to $c$ in $L^\infty(\Omega)$. The main issue in this limiting process is to keep both the weight functions and constants in the Carleman estimate under control. Note that the approximation of the $BV$ coefficient $c$ by some piecewise regular coefficient $c_\varepsilon$ is closely related to numerical methods.

With such Carleman estimates at hand, following the (fixed-point) method of [1, 8] (and generalized in [6]), we treat the problem of the null controllability for classes of semilinear parabolic equations of the form

$$
\begin{cases}
\partial_t y - \partial_x (c \partial_x y) + G(y, \partial_x y) = 0 & \text{in } Q,
\end{cases}
\begin{align*}
y(t, 0) &= v(t), & y(t, 1) &= 0, & y(0, x) &= y_0(x) & \text{in } \Omega,
\end{align*}
$$

where $v$ is the control, $G : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz and $G(0,0) = 0$ (further assumptions on the nonlinear function $G$ or on the initial condition will be introduced below). Controllability results of this type can be motivated by problems from biology for example. We also provide a stability result for the inverse problem of the identification of the diffusion coefficient in the case of piecewise $C^1$ coefficients.

The first part (Section 2) on the derivation of Carleman estimates in the case of a piecewise $C^1$ coefficient is a joint work of the three authors [3, 2]. The second part (Section 3) on the derivation of Carleman estimates in the case of a $BV$ coefficient is a work of the third author [13].

2. Global Carleman estimates for a piecewise regular coefficient
We consider a piecewise $C^1$ diffusion coefficient with a finite number of singularities. We shall thus here assume that $0 = a_0 < a_1 < a_2 < \ldots < a_n = 1$ and $c_{[a_i, a_{i+1}]} \in C^1([a_i, a_{i+1}])$, $i = 0, \ldots, n - 1$. Let $j \in \{0, \ldots, n - 1\}$ be fixed in the sequel and $\mathcal{O}_0 \subset \mathcal{O} \subset (a_j, a_{j+1})$ be a non-empty open set. Let $T > 0$. We shall use the following notations $S = \{a_1, \ldots, a_{n-1}\}$, $\Omega^j = \Omega \setminus \{a_1, \ldots, a_{n-1}\}$, $Q = (0, T) \times \Omega$, and $Q^j = (0, T) \times \Omega^j$.

We first introduce a particular type of weight functions, which are constructed using the following lemma.

Lemma 2.1. There exists a function $\beta \in C(\Omega)$ such that $\beta_{[a_i, a_{i+1}]} \in C^2([a_i, a_{i+1}])$, $i = 0, \ldots, n - 1$, satisfying $\beta > 0$ in $\Omega$, $\beta = 0$ on $\{0, 1\}$, $(\tilde{\beta}_{[a_i, a_{i+1}]}')' \neq 0$ in $[a_j, a_{j+1}] \setminus \mathcal{O}_0$, $(\tilde{\beta}_{[a_i, a_{i+1}]}')' \neq 0$, $i \in \{0, \ldots, n - 1\}$, $i \neq j$, and the function $\beta$ satisfies the following trace properties: for some $\alpha > 0$, $(A_i u, u) \geq \alpha|u|^2$, $u \in \mathbb{R}^2$, with the matrices $A_i$, defined by

$$
A_i = \left( \begin{array}{ccc}
[a_i^2, a_i a_{i+1}], & \beta'(a_i^2) [a_i^2, a_i] & \beta'(a_i^2) [a_i^2, a_i] + [c^2(\beta')^3]a_i \\
\beta'(a_i^2) [a_i^2, a_i] & [a_i^2, a_i] & \beta'(a_i^2) [a_i^2, a_i] + [c^2(\beta')^3]a_i \\
\beta'(a_i^2) [a_i^2, a_i] & \beta'(a_i^2) [a_i^2, a_i] + [c^2(\beta')^3]a_i & [a_i^2, a_i] \\
\end{array} \right), \quad i = 1, \ldots, n - 1.
$$


where \([\rho]_x = \rho(x^+) - \rho(x^-)\) for \(x \in (0, 1)\).

Choosing a function \(\tilde{\beta}\), as in the previous lemma, we introduce \(\beta = \tilde{\beta} + K\) with \(K = m\|\tilde{\beta}\|_\infty\) and \(m > 1\). For \(\lambda > 0\) and \(t \in (0, T)\), we define the following weight functions

\[
\varphi(t, x) = \frac{e^{\lambda\beta(t)}}{t(T-t)}, \quad \eta(t, x) = \frac{e^{\lambda\tilde{\beta}} - e^{\lambda\beta(t)}}{t(T-t)},
\]

(2.1)

with \(\tilde{\beta} = 2m\|\tilde{\beta}\|_\infty\) (see [7]).

We introduce

\[
\mathcal{R} = \left\{ q \in \mathcal{C}(Q, \mathbb{R}); q_{|_{[0, T] \times [a_i, a_{i+1}]}} \in \mathcal{C}^2([0, T] \times [a_i, a_{i+1}]), \ i = 0, \ldots, n-1, q_{|_{\Gamma}} = 0, \text{ and } q \text{ satisfies } (2.2), \text{ for all } t \in (0, T) \right\},
\]

with

\[
q(a_i^-) = q(a_i^+), \quad c(a_i^-)\partial_x q(a_i^+) = c(a_i^+\partial_x q(a_i^+), \ i = 1, \ldots, n-1. \quad (2.2)
\]

**Theorem 2.2.** There exist \(\lambda_1 = \lambda_1(\Omega, \mathcal{O}) > 0\), \(s_1 = (T + T^2)\hat{s}_1 > 0\) and a positive constant \(C = C(\Omega, \mathcal{O})\) so that the following estimate holds

\[
s^{-1} \int_Q e^{-2s\eta} \varphi^{-1} \left( |\partial_x q|^2 + |\partial_x(c\partial_x q)|^2 \right) dx dt + s\lambda^2 \int_Q e^{-2s\eta} \varphi \left| \partial_x q \right|^2 dx dt
\]

\[
+ s^3\lambda^4 \int_Q e^{-2s\eta} \varphi^3 |q|^2 dx dt
\]

\[
+ s\lambda \sum_{i=1}^{n-1} \int_0^T \varphi(t, a_i) e^{-2s\eta(t, a_i)} |\partial_x q(t, a_i^-)|^2 dt + s^3\lambda^3 \sum_{i=1}^{n-1} \int_0^T \varphi^3(t, a_i) e^{-2s\eta(t, a_i)} |q(t, a_i)|^2 dt
\]

\[
\leq C \left[ s^3\lambda^4 \int_{(0, T) \times \mathcal{O}} e^{-2s\eta} \varphi^3 |q|^2 dx dt + \int_Q e^{-2s\eta} \left| \partial_t q \pm \partial_x(c\partial_x q) \right|^2 dx dt \right],
\]

for \(s \geq s_1\), \(\lambda \geq \lambda_1\) and for all \(q \in \mathcal{R}\).

**Proof.** Arguing as in [10, 7], we set \(\psi = e^{-s\eta} q\). We obtain, in the derivation of the Carleman estimate, integral terms over \(Q'\) and some time integrals over \((0, T)\) with trace terms at \(a_i\), \(i = 1, \ldots, n-1\). In fact, the leading order terms for these time integrals at \(a_i\) (w.r.t. to the parameters \(s\) and \(\lambda\)) are given by

\[
\mu_i := s\lambda \int_0^T \varphi(t, a_i) \left[ \beta' c \partial_x \psi |^2(t, \cdot) \right]_{a_i} dt + s^3\lambda^3 \left[ c^2(\beta')^3 \right]_{a_i} \int_0^T \varphi^3(t, a_i) \left| \psi(t, a_i) \right|^2 dt,
\]

with \(i = 1, \ldots, n-1\). We obtain

\[
\mu_i = s\lambda \int_0^T \varphi(t, a_i) \left( A_i u(t, a_i), u(t, a_i) \right) dt,
\]

with \(u(t, a_i) = (c(a_i^-)\partial_x \psi(t, a_i^-), s\lambda \varphi(t, a_i) \psi(t, a_i))^T\), with the \(2 \times 2\) matrix \(A_i\) as in Lemma 2.1. This term is thus positive and can ‘absorb’ the remaining time integrals at \(a_i\) if we choose the parameters \(s\) and \(\lambda\) to be sufficiently large. The rest of the proof can be adapted from [10, 7].
Remark 2.3. In a similar fashion, we can obtain Carleman estimates with a ‘side observation’, for instance at $x=0$, for the operators $\partial_t \pm \partial_x (c \partial_x)$, i.e.,

\[
\begin{align*}
\int_Q e^{-2s\eta} \phi^{-1} (|\partial_t q|^2 + |\partial_x (c \partial_x q)|^2) \, dxdt \
+ s^3 \lambda^4 \int_Q e^{-2s\eta} \phi^3 |q|^2 \, dxdt
\end{align*}
\]

(2.4)

For the sake of presentation, we choose here to derive a Carleman estimate with a boundary imposed on $\gamma$ choosing the weight function

\[
\tilde{\gamma}_c \in C[0, 1), \quad \tilde{\gamma}_c(0) = 0, \quad \tilde{\gamma}_c(t) \geq 0, \quad t \in [0, 1),
\]

\[
\tilde{\gamma}_c(t) \leq \nu, \quad t \in [0, 1),
\]

properties given by Lemma 2.1.

Lemma 3.1. There exist a function $\tilde{\beta} \in C(\Omega)$ such that $\tilde{\beta}^i_j(\eta, z) = 0$, $(\tilde{\beta}^i_j(\eta, z)^*)'$, satisfying $\tilde{\beta} > 0$ in $\Omega$, $\tilde{\beta}(1) = 0$, and the function $\tilde{\beta}$ satisfies the following trace properties: for some $\alpha > 0$, $(A_i u, u) \geq \alpha|u|^2$, $u \in \mathbb{R}^n$ with the matrices $A_i$, defined as in Lemma 2.1.

Remark 2.5. Note that an inequality, of the form of (2.3), with these pointwise terms on the l.h.s. of the Carleman estimates can still be obtained in the case of a smooth coefficient by simply choosing the weight function $\beta$ to have a jump condition for its derivative and satisfying the properties given by Lemma 2.1.

3. Carleman estimates in the case of a $BV$ coefficient

For the sake of presentation, we choose here to derive a Carleman estimate with a boundary observation, similar to that obtained in (2.4). The proof in the case of an inner observation is similar but requires work on both sides of the observation region (see Remark 3.5).

We consider a diffusion coefficient $c \in BV(\Omega)$, with $0 < c_{\min} \leq c \leq c_{\max}$. We denote the total variation of $c$ on $(0, 1)$ by $\theta := \int_0^1 \theta (c) \, dt$. Let $\varepsilon > 0$. There exists a piecewise-constant function $c_{\varepsilon}$ such that $\|c - c_{\varepsilon}\|_{L^\infty(\Omega)} \leq \varepsilon$, $c_{\varepsilon}(x) \leq \theta$ (see e.g. [5]).

We denote by $a_1, \ldots, a_k$ the points of discontinuity of $c_{\varepsilon}$. We then have $\sum_{i=1}^k |c_{\varepsilon}(a_i^+) - c_{\varepsilon}(a_i^-)| \leq \theta$. Let $Y_i = c_{\varepsilon}(a_i^+)/c_{\varepsilon}(a_i^-)$ and $X_i$, $i = 1, \ldots, k$, be defined by $X_i = f(Y_i)$, with $f(s) = s$ if $s \geq 1$ and $f(s) = 2 - s$ if $s < 1$. We define the piecewise-constant function $\gamma_{\varepsilon}$ as $\gamma_{\varepsilon}(x) := \gamma_{\varepsilon}(1) \prod_{x < a_j} X_j$, for $x \notin \{a_1, \ldots, a_k\}$, for some fixed $\gamma_{\varepsilon} < 0$. We set the function $\tilde{\theta}_{\varepsilon}(x) := \int_{\varepsilon}^x \gamma_{\varepsilon}(y) \, dy$, which satisfies the properties listed in Lemma 2.4 by the jump conditions imposed on $\gamma_{\varepsilon} = \tilde{\theta}_{\varepsilon}$ at $a_1, \ldots, a_k$. Concerning the total variation of the functions $\gamma_{\varepsilon}$ we have the following lemma.

Lemma 3.1. There exist $K > 0$ and $\varepsilon_0 > 0$ that solely depend on the diffusion coefficient $c \in BV(\Omega)$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, $V_0^1(\gamma_{\varepsilon}) \leq K |\gamma_{\varepsilon}|(1)$.

By Helly’s theorem [12, 5], up to a subsequence, the functions $\gamma_{\varepsilon}$ converge everywhere to a function $\gamma$ as $\varepsilon$ goes to 0. Moreover, this function satisfies $V_0^1(\gamma) \leq K |\gamma(1)|$. By the dominated convergence theorem, the associated functions $\tilde{\beta}_{\varepsilon}$ converge everywhere to the continuous function $\tilde{\beta}(x) := \int_0^x \gamma(y) \, dy$.

Let $a$ be any point of discontinuity of $c_{\varepsilon}$. As above, we set $Y = \frac{c_{\varepsilon}(a^+)}{c_{\varepsilon}(a^-)}$ and define the matrix $A$ as in Lemma 2.1. Then we have the following crucial lemma.
Lemma 3.2. The eigenvalues $\nu_1, \nu_2$ of the matrix $A$ satisfy $\nu_i \geq C|Y - 1|$, $i = 1, 2$, with $C$ uniform w.r.t. $\varepsilon$ and the considered singularity of $c_\varepsilon$.

With $\tilde{\beta}_\varepsilon$ and $\tilde{\beta}$, we define weight functions according to (2.1). With a detailed inspection of the proof of the Carleman estimate in the case of a piecewise $\mathcal{C}^1$ coefficient (Section 2 and [2]), the previous lemma yields

Proposition 3.3. The constant $C$ on the r.h.s. of the Carleman estimate (2.3) for $\partial_t \pm \partial_x (c_\varepsilon \partial_x)$ and the constants $s_1$ and $\lambda_1$ can be chosen uniformly w.r.t. $\varepsilon$ for $0 < \varepsilon \leq \varepsilon_0$, with $\varepsilon_0$ sufficiently small.

The previous results show that both the weight functions and the constants in the Carleman estimate remain ‘well behaved’, as $\varepsilon$ goes to zero. We consider $q$ and $q_\varepsilon$ (weak) solutions to

$$
\begin{aligned}
& \partial_t q \pm \partial_x (c\partial_x q) = f \\
& q = 0 \\
& q(T, x) = q_0(x) \quad (\text{resp. } q(0, x) = q_0(x))
\end{aligned}
$$

in $Q$, $q = 0$ on $\Sigma$, $q(T, x) = q_0(x)$ in $\Omega$, with $\partial_x (c\partial_x q_0) = \mu$ and $\partial_x (c_\varepsilon \partial_x q_0) = \mu$, with $\mu$ and $f$ sufficiently regular. Since $\partial_t q_\varepsilon, \partial_x q_\varepsilon$ and $\partial_x (c\partial_x q_\varepsilon)$ converge to $\partial_t q, \partial_x q$ and $\partial_x (c\partial_x q)$ in $L^2$ norm, the previous results show that the Carleman estimate holds for $q_\varepsilon$ and the operators $\partial_t \pm \partial_x (c \partial_x)$ yield a similar estimate for $q$ and the operators $\partial_t \pm \partial_x (c \partial_x)$ with the same constants, as $\varepsilon$ goes to zero. We can now relax the assumptions on $\mu$ and $f$. We have thus obtained the following theorem.

Theorem 3.4. Let $c \in BV(\Omega)$ with $0 < c_{\min} \leq c \leq c_{\max}$. There exists $\lambda_1 > 0, s_1 = (T + T^2)\tilde{s}_1 > 0$ and $C > 0$ so that the Carleman estimate (2.4) holds for $s \geq s_1, \lambda \geq \lambda_1$ and for all $q$ (weak) solution to $\partial_t q \pm \partial_x (c\partial_x q) = f$ in $Q, q = 0$ on $\Sigma, q(T, x) = q_0(x)$ (resp. $q(0, x) = q_0(x)$) in $\Omega, with q_0 \in L^2(\Omega)$ and $f \in L^2(Q).

In the case where $\partial \in \Omega$, with $\partial$ a non-empty open set and $c$ is of class $\mathcal{C}^1$ in $\partial$, we obtain Carleman estimate (2.3), for the operators $\partial_t \pm \partial_x (c \partial_x)$, with an interior ‘observation’ on $(0, T) \times \partial$.

Remark 3.5. For the case of an interior ‘observation’, the proof is similar to the one presented here. The function $\tilde{\beta}_\varepsilon$ is constructed on both sides of $\partial$, through its derivative, $\tilde{\beta}_\varepsilon$, and the regularity of $c$ in $\partial$ allows to smoothly connect the two parts of the function, with some convergence in $\mathcal{C}^2(\partial)$ as $\varepsilon$ goes to zero. As above, we then pass to the limit in a Carleman estimate (2.3) for $\partial_t \pm \partial_x (c \partial_x), with c \varepsilon piecewise $\mathcal{C}^1$, with an interior ‘observation’.

4. A Carleman estimate for the heat equation with a r.h.s. in $L^2(0, T, H^{-1}(\Omega))$

Following [11], we obtain a Carleman estimate for $\partial_t q \pm \partial_x (c\partial_x q) = f$ if $f \in H^{-1}$. We set

$$
\mathcal{N}_\pm = \{ q \in \mathcal{C}([0, T], H^1_0(\Omega)); q(t) \in D(A) \forall t \in [0, T], q(t) = 0 \text{ on } \Sigma, q(T, x) = q_0(x) \}.
$$

Theorem 4.1. Let $\partial \in \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{\min} \leq c \leq c_{\max}$ and $c$ of class $\mathcal{C}^1$ in $\partial$. There exists $\lambda_2 = \lambda_2(\partial, c) > 0$, $s_2 = (T + T^2)s_2(\partial, c) > 0$ and a positive constant $C = C(\partial, c)$ so that the following estimate holds, for $s \geq s_2, \lambda \geq \lambda_2$ and for all $q \in \mathcal{N}_\pm$,

$$
s\lambda^2 \int_Q e^{-2s\eta} |\partial_x q|^2 \ dx \ dt + s^3 \lambda^4 \int_Q e^{-2s\eta} \varphi^3 |q|^2 \ dx \ dt
$$

(4.1)
The solution

Lemma 5.1. The Carleman estimate (4.1) proven in the previous section allows to give observability estimates that yield results of controllability to the trajectories for classes of semilinear heat equations. We let \( \omega \subseteq \Omega \) be a non-empty open set and \( c \in BV(\Omega) \) with \( 0 < c_{\min} \leq c \leq c_{\max} \) and \( c \) of class \( C^1 \) in some non-empty open subset \( \mathcal{O} \) of \( \omega \). We let \( a \) and \( b \) be in \( L^\infty(\mathcal{Q}) \) and \( q \in L^2(\Omega) \). From Carleman estimate (4.1) we obtain the following lemma.

Lemma 5.1. The solution \( q \) to \(-\partial_t q - \partial_x(c\partial_x q) + aq - \partial_x(bq) = 0 \) in \( Q \), \( q = 0 \) on \( \Sigma \), and \( q(T) = q_T \) satisfies \( \|q(0)\|^2_{L^2(\mathcal{Q})} \leq c^{CH} \left( \int_{[0,T] \times \omega} |q| \, dxdt \right)^2 \), where \( H = H(T, \|a\|_{\infty}, \|b\|_{\infty}) = 1 + \frac{T}{2} + T + (T + T^{1/2}) \|a\|_{\infty} + \|a\|_{\infty}^{3/2} + (1 + T) \|b\|_{\infty}^2 \).

Since the coefficient \( c \) is \( C^1 \) in some open subset \( \mathcal{O} \) of \( \omega \), the proof of [6, Theorem 2.5, Lemma 2.5] can be adapted. See also [7, Proposition 4.2, Lemma 4.3]. Such an observability estimate yields the null controllability of the semilinear parabolic system (1.1), as well as the following system

\[
\begin{align*}
\partial_t y - \partial_x(c\partial_x y) + G(y, \partial_x y) &= 1_\omega v \quad \text{in} \ Q, \\
y(t, \cdot) &= 0 \quad \text{on} \ \Sigma, \\
y(0, x) &= y_0(x) \quad \text{in} \ \Omega,
\end{align*}
\]

(5.1)

where \( G : \mathbb{R}^2 \rightarrow \mathbb{R} \) is locally Lipschitz and \( G(0, 0) = 0 \), which implies that \( G(x, y) = xg(x, y) + yg(x, y) \), with \( g \) and \( G \) in \( L^\infty_{loc}(\mathbb{R}) \).

Theorem 5.2. Let \( T > 0 \). Let \( c \in BV(\Omega) \) with \( 0 < c_{\min} \leq c \leq c_{\max} \).

(i) Local null controllability: There exists \( \varepsilon > 0 \) such that for all \( y_0 \in L^2(\Omega) \) with \( \|y_0\|_{L^2(\mathcal{O})} \leq \varepsilon \), there exists a control \( v \in C^0([0, T]) \) such that the solution to (1.1) satisfies \( y(T) = 0 \).

(ii) Global null controllability: Let \( G \) satisfy in addition

\[
\begin{align*}
\frac{|g(x, y)|}{|x - y|} \rightarrow 0 \quad \text{as} \quad |x - y| \rightarrow \infty \quad \text{and} \quad \frac{|G(x, y)|}{|x - y|} \rightarrow 0 \quad \text{as} \quad |x - y| \rightarrow \infty.
\end{align*}
\]

For all \( y_0 \in L^2(\Omega) \), there exists \( v \in C^0([0, T]) \) such that the solution to (1.1) satisfies \( y(T) = 0 \).

We have similar results for system (5.1) with a non-empty open subset \( \omega \subseteq \Omega \), and \( v \in L^\infty((0, T) \times \omega) \), assuming that \( c \) of class \( C^1 \) in some non-empty open subset \( \mathcal{O} \) of \( \omega \).

The proof is based on a fixed-point argument and is along the same lines as that in [6] and originates from [1, 8]. Note that as usual, \( y(T) = y^*(T) \) can replace \( y(T) = 0 \) in the previous statements, where \( y^* \) is any trajectory defined in \([0, T]\) of system (1.1), corresponding to some initial data \( y_0^* \in L^2(\Omega) \) and any \( v^* \) in \( C^0([0, T]) \) ([resp. \( L^\infty((0, T) \times \omega) \) in the case of an interior control]). For the local controllability result, one has to assume \( \|y_0 - y_0^*\|_{L^2(\Omega)} \leq \varepsilon \), with \( \varepsilon \) sufficiently small.
6. Stability for a discontinuous diffusion coefficient

In [4], the authors establish a uniqueness result for the discontinuous diffusion coefficient \( c \) as well as a stability inequality. This inequality estimates the discrepancy in the coefficients \( c \) and \( \tilde{c} \) of two materials (with the same geometry) with an upper bound given by some Sobolev norms of the difference between the solutions \( y \) and \( \tilde{y} \) to

\[
\begin{align*}
\begin{cases}
\partial_t \tilde{y} - \partial_x (\tilde{c} \partial_x \tilde{y}) = 0 & \text{in } Q, \\
y(t, x) = h(t, x) & \text{on } \Sigma, \\
\tilde{y}(0, x) = \tilde{y}_0(x) & \text{in } \Omega,
\end{cases}
\begin{cases}
\partial_t y - \partial_x (c \partial_x y) = 0 & \text{in } Q, \\
y(t, x) = h(t, x) & \text{on } \Sigma, \\
y(0, x) = y_0(x) & \text{in } \Omega.
\end{cases}
\end{align*}
\]

(6.1)

Set \( u = y - \tilde{y} \) and \( q = \partial_t u \). Then \( q \) is solution to the following problem

\[
\begin{cases}
\partial_t q - \partial_x (c \partial_x q) = \partial_x ((c - \tilde{c}) \partial_x \tilde{y}) & \text{in } Q', \\
q = 0 & \text{on } \Sigma, \\
\text{transmission conditions (6.2)} & \text{on } S \times [0, T],
\end{cases}
\]

with

\[
q(x^-) = q(x^+), \quad (c \partial_x q)(x^-) = (\tilde{c} \partial_x \tilde{q})(x^+) + q(x, t),
\]

(6.2)

where \( x \in S = \{a_1, \ldots, a_{n-1}\} \), the set of singularities for both \( c \) and \( \tilde{c} \), and

\[
g(x, t) = ((c - \tilde{c}) \partial_x \tilde{y})(x^+) - ((c - \tilde{c}) \partial_x \tilde{y})(x^-).
\]

If the solutions \( y \) and \( \tilde{y} \) to (6.1) satisfy some (regularity) conditions (that can be achieved with some choices of boundary conditions \( h \) and initial conditions \( y_0 \) and \( \tilde{y}_0 \) in \( L^2(\Omega) \) — see [4] for details) we have the following stability result.

**Theorem 6.1.** We assume that the diffusion coefficients \( c \) and \( \tilde{c} \) are piecewise constant with the same singularity locations. Then there exists a constant \( C \) such that

\[
|c - \tilde{c}|_{L^\infty(\Omega)}^2 \leq C \left| \partial_x \partial_t y - \partial_x \tilde{y} \right|_{L^2(0, T)}^2 + C \left| \Delta y(T', \cdot) - \Delta \tilde{y}(T', \cdot) \right|_{L^2(\Omega')}^2,
\]

(6.3)

where \( \Omega' \) is the open set \( \Omega \) with the singularities of \( c \) removed.

A Carleman estimate is the key ingredient in the proof of such a stability estimate. In [4], this Carleman estimate was proven in any dimension but with an additional ‘monotonicity’ assumption on the discontinuous diffusion coefficient. In the present case, we can establish such a Carleman estimate for a piecewise \( C^1 \) diffusion coefficient. Choosing the weight function as in Lemma 2.4, we have the following estimate.

**Theorem 6.2.** Let \( t_0 > 0 \), in \( (0, T) \) and \( g(\cdot, a_i) \in H^1(t_0, T), i = 1, \ldots, n - 1 \). There exist \( \lambda_1 > 1, s_1 = s_1(\lambda_1) > 0 \) and a positive constant \( C \) so that the following estimate holds

\[
|M_1(e^{-s_1q})|_{L^2(Q')}^2 + |M_2(e^{-s_1q})|_{L^2(Q')}^2 + s_1^2 \int_Q e^{-2s_1\varphi} |\partial_x q|^2 \, dx \, dt
\]

\[
+ s_1^3 \lambda^4 \int_Q e^{-2s_1\varphi^3} |q|^2 \, dx \, dt
\]

\[
\leq C \left[ s \lambda \int_{t_0}^T e^{-s_1\varphi} |\partial_x q|^2(t, 0) \, dt + \int_Q e^{-2s_1\varphi} |\partial_t q \pm \partial_x (c \partial_x q)|^2 \, dx \, dt 
\]

\[
+ s \lambda \int_{t_0}^T e^{-2s_1\varphi} |g|^2 \, d\sigma \, dt + \int_{t_0}^T \int_S e^{-2s_1\varphi^2} |g|^2 \, d\sigma \, dt + s^{-2} \int_{t_0}^T \int_S e^{-2s_1\varphi^2} |\partial_q g|^2 \, d\sigma \, dt \right],
\]
for \( s \geq s_1 \), \( \lambda \geq \lambda_1 \) and for all \( q \in \mathcal{R}_g \), with \( M_1 \) and \( M_2 \) given by

\[
M_1 \psi = \partial_x(c \partial_x \psi) + s^2 \lambda^2 \varphi^2 (\beta')^2 c \psi + s(\partial_t \eta) \psi, \quad M_2 \psi = \partial_t \psi - 2s \lambda \varphi c \beta' \partial_x \psi - 2s \lambda^2 \varphi c (\beta')^2 \psi,
\]

and \( \mathcal{R}_g \) is given by

\[
\mathcal{R}_g = \{ q \in H^1(t_0, T, H^1_0(\Omega)); q|_{(t_0,T) \times (a_i, a_{i+1})} \in L^2(t_0, T, H^2((a_i, a_{i+1}))), \quad i = 0, \ldots, n - 1, \quad q|_{\Sigma} = 0 \text{ and } q \text{ satisfies (6.2) a.e. w.r.t. } t \}.
\]

**Remark 6.3.** Observe that in Theorem 6.1 and Theorem 6.2, we need not assume that jumps for \( c \) are greater than some positive constants \( K \) at its points of discontinuities, as is done in [4]. This is due to the choice made on the weight function \( \tilde{\beta} \) in Lemma 2.4. This remark is to be connected to the proof of Theorem 1.3 in [4, estimate (1.16) and following arguments].

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