Einstein-Podolski-Rosen Experiment from Noncommutative Quantum Gravity

Michael Heller
Vatican Observatory, V-12000 Vatican City State*

Wiesław Sasin
Institute of Mathematics, Warsaw University of Technology
Plac Politechniki 1, 00-661 Warsaw, Poland

August 26, 2018

Abstract

It is shown that the Einstein–Podolski–Rosen type experiments are the natural consequence of the groupoid approach to noncommutative unification of general relativity and quantum mechanics. The geometry of this model is determined by the noncommutative algebra $\mathcal{A} = C^\infty_c(G, \mathbb{C})$ of complex valued, compactly supported, functions (with convolution as multiplication) on the groupoid $G = E \times \Gamma$. In the model considered in the present paper $E$ is the total space of the frame bundle over space-time and $\Gamma$ is the Lorentz group. The correlations of the EPR type should be regarded as remnants of the totally non-local physics below the Planck threshold which is modelled by a noncommutative geometry.

*Correspondence address: ul. Powstańców Warszawy 13/94, 33-110 Tarnów, Poland. E-mail: mheller@wsd.tarnow.pl
1 Introduction

One of the greatest challenges of contemporary physics is to explain the non-local effects of quantum mechanics theoretically predicted (in the form of a gedanken experiment) by Einstein, Podolski and Rosen (EPR, for short) \[1\] and experimentally verified by Aspect et al. \[2, 3, 4\] (for a comprehensive review see \[5\]). Although non-local effects of this type logically follow from the postulates of quantum mechanics, it seems strange and against our ”realistic common sense” to accept that two particles separated in space could be so strongly correlated (provided they once interacted with each other) that they seem to “know” about each other irrespectively of the distance separating them. In spite of long lasting discussions, so far no satisfactory explanation of this effect has been offered. In the present paper we shall argue that effects of the EPR type are remnants of the totally non-local physics of the fundamental level (below the Planck threshold). We substantiate our argument by explaining the EPR experiment in terms of a quantum gravity model, based on a noncommutative geometry, proposed by us in \[6\] (see also \[7\]), although the explanation itself does not depend on particulars of the model.

The main physical idea underlying our model is that below the Planck threshold (we shall speak also on the ”fundamental level”) there is no space-time but only a kind of pregeometry which is modeled by a suitable noncommutative space, and that space-time emerges only in the transition process to the classical gravity regime. Accordingly, we start our construction not from a space-time manifold \(M\), but rather from a groupoid \(G = E \times \Gamma\) where \(E\) is a certain abstract space and \(\Gamma\) a suitable group of ”fundamental symmetries”.

In the present paper, for the sake of simplicity, we shall assume that \(E\) is the total space of the frame bundle over space-time \(M\) and \(\Gamma = SO(3, 1)\). We define, in terms of this geometry, the noncommutative algebra \(A = C^\infty(G, \mathbb{C})\) of smooth, compactly supported, complex-valued functions on the groupoid \(G\) with the usual addition and convolution as multiplication. We develop a noncommutative differential geometry basing on this algebra, and define a noncommutative version of Einstein’s equation (in the operator form). The algebra \(A\) can be completed to become a \(C^*\)-algebra, and this subalgebra of \(A\) which satisfies the generalized Einstein’s equation is called *Einstein* \(C^*\)-algebra, denoted by \(\mathcal{E}\) (for details see \[6\]). And now quantization is performed in the standard algebraic way. Since the explanation of the EPR type experiments depends on the noncommutative structure of the groupoid \(G\) rather
than on details of our field equations and the quantization procedure, we shall not review them here; the reader interested in the particulars of our model should consult the original paper [6].

It can be shown that the subalgebra $\mathcal{A}_{proj}$ (elements of $\mathcal{A}_{proj}$ are called projectible) of functions which are constant on suitable equivalence classes of fibres $\pi_E^{-1}(p)$, $\pi_E$ being the projection $G = E \times \Gamma \to \Gamma$ and $p \in E$, is isomorphic to the algebra $C^\infty(M)$ of smooth functions on $M$. Consequently, by making the restriction of $\mathcal{A}$ to $\mathcal{A}_{proj}$ we recover the ordinary space-time geometry and the standard general relativity. In our model, to simplify calculations, we have assumed that the noncommutative differential geometry is determined by the submodule $V$ of the module $\text{Der}\mathcal{A}$ of all derivations of $\mathcal{A}$, and that $V$ has the structure adapted to the product structure of the groupoid $G = E \times \Gamma$, i.e., $V = V_E \oplus V_\Gamma$, where $V_E$ and $V_\Gamma$ are ”parts” parallel to $E$ and $\Gamma$, correspondingly. It can be seen that in our model the geometry “parallel” to $E$ is responsible for generally relativistic effects, and that “parallel” to $\Gamma$ for quantum effects. In general, ”mixed terms” should appear, and then one would obtain stronger interaction between general relativity and quantum physics. This remains to be elaborated in the future.

The crucial point is that the geometry as determined by the noncommutative algebra $\mathcal{A}$ is non-local, i.e., there are no maximal ideals in $\mathcal{A}$ which could determine points and their neighborhoods in the corresponding space, and consequently neither space points nor time instants can be defined in terms of $\mathcal{A}$. Physical states of a quantum gravitational system are identified with states on the algebra $\mathcal{A}$, i.e., with the set of positive linear functionals (normed to unity) on $\mathcal{A}$, and pure states in the mathematical sense are identified with pure states in the physical sense. Let $a \in \mathcal{A}$ be a quantum gravitational observable, i.e., a projectible and Hermitian element of $\mathcal{A}$ ($a$ must be projectible to leave traces in the macroscopic world), and $\varphi$ a state on $\mathcal{A}$. Then $\varphi(a)$ is the expectation value of the observable $a$ when the system is in the state $\varphi$. The fact that $a$ is an element of a ”non-local” (noncommutative) algebra $\mathcal{A}$ implies that when $a$ is projected to the space-time $M$ it becomes a real-valued (since $a$ is Hermitian) function on $M$, and the results of a measurement corresponding to $a$ are values of this function. Consequently, one should expect correlations between various measurement results even if they are performed at distant points of space-time $M$. We shall see that this is indeed the case.

The organization of our material is the following. In Section 2, we consider
the eigenvalue equation for quantum gravitational observables. In Section 3, we show that correlations of the EPR type between distant events in space-time are consequences of non-local (noncommutative) physics of the quantum gravitational regime, and in Section 4 we present details of the EPR experiment in terms of the noncommutative approach. Section 5 contains concluding remarks.

2 Measurement on Quantum Gravitational System

Let \( \varphi : \mathcal{A} \to \mathbb{C} \) be a state on the algebra \( \mathcal{A} \), i.e., \( \varphi(1) = 1 \) and \( \varphi(aa^*) \geq 0 \) for every \( a \in \mathcal{A} \). It can be easily seen that \( \varphi|_{\mathcal{A}_{proj}} : \mathcal{A}_{proj} \to \mathbb{C} \) is a state on the subalgebra \( \mathcal{A}_{proj} \).

Let now \( a \in \mathcal{A}_{proj} \) be Hermitian, then there exists a function \( \bar{a} \in C^\infty(M) \) with \( \bar{a} \circ pr = a \), where \( pr : G \to M \) is the projection, and the state \( \bar{\varphi} : C^\infty(M) \to \mathbb{R} \) on the algebra \( C^\infty(M) \), such that \( \varphi(a) = \bar{\varphi}(\bar{a}) \). Since the algebras \( \mathcal{A}_{proj} \) and \( C^\infty(M) \) are isomorphic, the spaces of states of these algebras are isomorphic as well.

To make a contact with the standard formulation of quantum mechanics we represent the noncommutative algebra \( \mathcal{A} \) in a Hilbert space by defining, for each \( p \in E \), the representation

\[ \pi_p : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \]

where \( \mathcal{B}(\mathcal{H}) \) is the algebra of operators on the Hilbert space \( \mathcal{H} = L^2(G_p) \) of square integrable functions on the fibre \( G_p = \pi_E^{-1}(p) \), \( \pi_E : G \to E \) being the natural projection, with the help of the formula

\[ \pi_p(a)\psi = \pi_p(a) * \psi \]

or more explicitly

\[ (\pi_p(a)\psi)(\gamma) = \int_{G_p} a(\gamma_1)\psi(\gamma_1^{-1}\gamma), \]

\( a \in \mathcal{A}, \gamma = \gamma_1 \circ \gamma_2, \gamma, \gamma_1, \gamma_2 \in G_p, \psi \in L^2(G_p), \) and the integration is with respect to the Haar measure. This representation is called the Connes representation (see [8, p.102], [6]).
Now, let us suppose that $a$ is an observable, i.e., $a \in \mathcal{A}_{proj}$, and we perform a measurement of the quantity corresponding to this observable. The eigenvalue equation for $a$ is

$$\int_{G_p} a(\gamma_1) \psi(\gamma_1^{-1}\gamma) = r_p \psi(\gamma)$$

where the eigenvalue $r_p$ is the expected result of the measurement when the system is in the state $\psi$. Here we must additionally assume that the "wave function" $\psi$ is constant on fibres of $G$ to guarantee for equation (1) to have its usual meaning in the non-quantum gravity limit. If this is the case, equation (1) can be written in the form

$$\psi(\gamma_1^{-1}\gamma) \int_{G_p} a(\gamma_1) = r_p \psi(\gamma)$$

and consequently

$$r_p = \int_{G_p} a(\gamma_1).$$

Let us notice that the measurement result is a measure in the mathematical sense.

It is obvious that if we define the "total phase space" of our quantum gravitational system

$$L^2(G) := \bigoplus_{p \in E} L^2(G_p)$$

and the operator

$$\pi(a) := (\pi_p(a))_{p \in E}$$

acting on $L^2(G)$ then the eigenvalue equation becomes

$$\pi(a)\psi = r\psi$$

where $r : M \to \mathbb{R}$ is a function on space-time $M$ given by

$$r(x) = \int_{G_p} a(\gamma_1)$$

where $x$ is a point in $M$ to which the frame $p$ is attached. Let us notice that the function $r$ is equal to the function $\bar{a} : M \to \mathbb{R}$ (see the beginning of the present Section). Let us now consider a composed quantum system
the state of which is described by the single vector in the Hilbert space, and let us perform a measurement on its parts when they are at a great distance from each other. Formula (2) asserts that in such a case the results of the measurement are not independent but are the values of the same function defined on space-time. This can be regarded as a "shadow" of a non-local character of the observable $a$ projected down to space-time $M$.

3 EPR Non-Localcy

So far we were mainly concerned with what happens when we project the algebra $A$ onto the "horizontal component" $E$ of the groupoid $G$. This, of course, gives us the transition to the classical space-time geometry (general relativity). In the present Section, we shall be interested in projecting $A$ onto the "vertical component" $\Gamma$ of $G$. This gives us quantum effects of our model.

Let us consider functions projectible to $\Gamma$. We define

$$A_\Gamma := \{ f \circ pr_\Gamma : f \in C^\infty_c(\Gamma, \mathbb{C}) \} \subset A.$$ 

The reasoning similar to that in the beginning of the present section shows that if $s \in A_\Gamma$ and $\psi : A_\Gamma \to \mathbb{C}$ is a state on $A_\Gamma$ then $\psi(s) = \psi(s^\prime)$, where $s = s' \circ pr_\Gamma$, $pr_\Gamma : G \to \Gamma$ is the projection, and $\psi : C^\infty_c(\Gamma, \mathbb{C}) \to \overline{\mathbb{C}}$ is a state on $C^\infty_c(\Gamma, \mathbb{C})$.

Let now $\Phi$ be a state on $C^\infty_c(\Gamma, \mathbb{C})$. We say that the state $\varphi : A \to \mathbb{C}$ is $\Gamma$-invariant associated to $\Phi$ on $A$ if

$$\varphi(s) = \begin{cases} \Phi(s), & \text{if } s \in A_\Gamma, \\ 0, & \text{if } s \not\in A_\Gamma. \end{cases}$$

Since all fibres of $G_p$, $p \in E$, of $G$ are isomorphic, the number $\varphi(s) = \Phi(s^\prime)$, for $s \in A_\Gamma$, is the same in each fibre $G_p$. If additionally $s$ is a Hermitian element of $A$, and if a measurement performed at a certain point of space-time $M$ gives the number $\varphi(s)$ as its result, then this result is immediately "known" at all other fibres $G_p$, $p \in E$, of $G$, and consequently at all other points of space-time $x = \pi_M(p) \in M$, where $\pi_M : E \to M$ is the canonical projection.

This can be transparently seen if we consider the problem in the Hilbert space by using the Connes representation of the algebra $A$. Let $a \in A_\Gamma$, and
let us consider the following Connes representations

\[ \pi_p(a)(\xi_p) = a_p \ast \xi_p, \]  \hspace{1cm} (3)

and

\[ \pi_q(a)(\xi_q) = a_q \ast \xi_q, \]  \hspace{1cm} (4)

where \( \xi_p \in L^2(G_p), \xi_q \in L^2(G_q), p, q \in E, p \neq q. \) Since \( G_p \) and \( G_q \) are isomorphic, we can choose \( \xi_p \) and \( \xi_q \) to be isomorphic with each other, which implies that \( \pi_p(a) \) and \( \pi_q(a) \) are isomorphic as well. We have the following important

Lemma. If \( a \in A_\Gamma \) then its image under the Connes representation \( \pi_p \) does not depend of the choice of \( p \in E \) (up to isomorphism).

Since \( p \in E \) projects down to the space-time point \( \pi_M(p) \in M, \pi_M : E \to M \), the above result should be interpreted as stating that all points of \( M \) “know” what happens in the fiber \( G_g, g \in \Gamma \). This, together with the fact that vectors \( \xi_p \), upon which the observable \( \pi_p(a) \), \( a \in A_\Gamma \) acts, also do not depend of \( p \), in principle, explains the EPR type experiments. However, let us go deeper into details.

4 EPR Experiment in Terms of Noncommutative Geometry

In this section we consider a group \( \Gamma \) such that \( \Gamma_0 = SU(2) \) is its compact subgroup. We look for an element \( s \in A_\Gamma \) such that

\[ \pi_p(s) : L^2(\Gamma_0) \to L^2(\Gamma_0). \]

Of course, \( C^2 \subset L^2(\Gamma_0) \). We define two linearly independent functions on the group \( \Gamma_0 \), for instance the constant function

\[ 1 : \Gamma_0 \to C, \]

and

\[ \text{det} : \Gamma_0 \to C, \]

which span the linear space \( C^2 \), i.e., \( C^2 = \langle 1, \text{det} \rangle_C \). Let \( \hat{S}_z = \pi_p(s)|_{C^2} \) be the usual z-component spin operator. We have

\[ \pi_p(s)\psi = \hat{S}_z\psi, \]
for $\psi \in \mathbb{C}^2$ or, by using the Connes representation and the fact that $\hat{S}_z\psi = \pm \frac{\hbar}{2}\psi$, 
\[
\int_{\Gamma_0} s_p(\gamma_1)\psi(\gamma_1^{-1}\gamma) = \pm \frac{\hbar}{2}\psi.
\]
Since $s_p$ = const, one obtains 
\[
\int_{\Gamma_0} \psi(\gamma_1^{-1}\gamma) \sim \psi(\gamma).
\]
One of the solutions of this equation is $\psi = 1_{\Gamma_0}$. Therefore 
\[
\frac{\hbar}{2} = \pm \int_{\Gamma_0} s_p(\gamma_1).
\]
Hence 
\[
(s_p)_1 = + \frac{\hbar}{2} \frac{1}{\text{vol}\Gamma_0},
\]
\[
(s_p)_2 = - \frac{\hbar}{2} \frac{1}{\text{vol}\Gamma_0},
\]
and consequently 
\[
\pi_p((s_p)_1)\psi = + \frac{\hbar}{2} \psi \text{ for } \psi \in \mathbb{C}^+, 
\]
\[
\pi_p((s_p)_2)\psi = - \frac{\hbar}{2} \psi \text{ for } \psi \in \mathbb{C}^-,
\]
where $\mathbb{C}^+ := \mathbb{C} \times \{0\}$, and $\mathbb{C}^- := \{0\} \times \mathbb{C}$. To summarize these results we can define 
\[
\hat{S}_z\psi = \pi_p(s_1, s_2)\psi := \begin{cases} (s_1)_p * \psi & \text{if } \psi \in \mathbb{C}^+, \\ (s_2)_p * \psi & \text{if } \psi \in \mathbb{C}^-.
\end{cases}
\]
Now, the analysis of the “EPR paradox” proceeds in the same way as in the standard textbooks on quantum mechanics (see for instance [4], pp. 179-181). An observer $A$, situated at $\pi_M(p) = x_A \in M$, measures the z-spin component of the one of the electrons\footnote{Let us notice that when $A$ measures the spin of the electron, he simultaneously determines the position of the electron (at least roughly), i.e., the position $x_A$ at which he himself is situated (spin and position operators commute).} i.e., he applies the operator $\hat{S}_z \otimes 1_{\mathbb{C}^2}$ to the vector $\xi = \frac{1}{\sqrt{2}}(\psi \otimes \varphi - \varphi \otimes \psi)$ where $\psi \in \mathbb{C}^+$ and $\varphi \in \mathbb{C}^-$. Let us suppose that the result of the measurement is $\frac{\hbar}{2}$. This means that the state
vector $\xi = \frac{1}{\sqrt{2}}(\psi \otimes \varphi - \varphi \otimes \psi) \in C^2 \otimes C^2 \subset L^2(G_r) \otimes L^2(G_r)$, $r \in E$, has collapsed to $\xi_0 = \frac{1}{\sqrt{2}}(\psi \otimes \varphi)$, and that immediately after the measurement the system is in the state $\xi_0$ which is the same (up to isomorphism) for all fibres $G_r$ whatever $r \in E$, and consequently it does not depend of the point in space-time to which $r$ is attached (see formulae (3) and (4) which are obviously valid also for tensor products). In particular, the vector $\xi_0$ is the same for the fibres $G_p$ and $G_q$ where $p$ is such that $\pi_M(p) = x_A$ and $q$ is such that $\pi_M(q) = x_B$ ($x_A \neq x_B$). It is now obvious that if an observer $B$, situated a $x_B$ measures the z-spin component of the second electron, i.e., if he applies the operator $1 |_{C^2} \otimes \hat{S}_z$ to the vector $\xi_0$, he will obtain the value $-\frac{h}{2}$ as the result of his measurement.

### 5 Concluding Remarks

To conclude our analysis it seems suitable to make the following remarks.

It should be emphasized that our scheme for noncommutative quantum gravity does not “explain” quantum mechanical postulates. In the very construction of our scheme it has been assumed that the known postulates which, in the standard formulation of quantum mechanics are valid for the algebra of observables, can be extended to a more general noncommutative algebra. However, the very fact that these postulates are valid in the conceptual framework of noncommutative geometry gives them a new flavour. For instance, since in the noncommutative regime there is no time in the usual sense, the sharp distinction between the continuous unitary evolution and the non-continuous process of measurement (“collapse of the wave function”) disappears. This distinction becomes manifest only when time emerges (see [10]) in changing from the noncommutative regime to the usual space-time geometry.

What our approach does explain is the fact that some quantum effects are strongly correlated even if they occur at great distances from each other. These effects are “projections” from the fundamental level at which all concepts have purely global meanings.

This explanation does not depend on “details” of our model, such as some particulars of the construction of noncommutative differential geometry, the concrete form of generalized Einstein’s equation, or the dynamical equation for quantum gravity. However, it does depend on (or even more, it is deeply
rooted in) the noncommutative character of the algebra $\mathcal{A} = C_c^\infty(G, \mathbb{C})$ and the product structure of the groupoid $G = E \times \Gamma$.

References

[1] A. Einstein, B. Podolski and N. Rosen, ”Can quantum description of physical reality be considered complete?” Phys. Rev. 47, 777–780 (1935).

[2] A. Aspect, P. Grangier and G. Roger, ”Experimental tests of realistic local theories via Bell’s theorem”, Phys. Rev. Lett., 47, 460–463 (1981).

[3] A. Aspect, P. Grangier and G. Roger, ”Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A new violation of Bells inequalities”, Phys. Rev. Lett., 49, 91–94 (1982).

[4] A. Aspect, J. Dalibard and G. Roger, ”Experimental tests of Bell inequalities using time-varying analyzers”, Phys. Rev. Lett., 49, 1804–1807 (1982).

[5] M.L.G. Redhead, Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics, Clarendon Press, Oxford, 1987.

[6] M. Heller, W. Sasin and D. Lambert, ”Groupoid approach to noncommutative quantization of gravity, J. Math. Phys., 38, 5840–5853 (1997).

[7] M. Heller, and W. Sasin, ”Towards noncommutative quantization of gravity, gr-qc/9712009.

[8] A. Connes, Noncommutative Geometry, Academic Press, San Diego-New York, 1994.

[9] C. J. Isham, Lectures on Quantum Theory, Imperial College Press, London, 1995.

[10] M. Heller and W. Sasin, “Emergence of Time”, gr-qc/9711051.