Combinatorial bases of standard modules of twisted affine Lie algebras in types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$, rectangular highest weights

Marijana Butorac$^a$ and Slaven Kožić$^b$

$^a$Faculty of Mathematics, University of Rijeka, Rijeka, Croatia; $^b$Department of Mathematics, Faculty of Science, University of Zagreb, Zagreb, Croatia

ABSTRACT

We consider the standard modules of rectangular highest weights of affine Lie algebras in types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$. By using vertex algebraic techniques we construct the combinatorial bases for standard modules and their principal subspaces and parafermionic spaces. Finally, we compute the corresponding character formulae and, as an application, we obtain two new families of combinatorial identities.

1. Introduction

The connection between Rogers–Ramanujan-type sums and characters of parafermionic spaces was originally established in the work of J. Lepowsky and M. Primc [23] by constructing the vertex operator bases of parafermionic spaces for the affine Lie algebra $A_1^{(1)}$. Later on, such combinatorial bases for the parafermionic spaces of standard modules of rectangular highest weights for all untwisted affine Lie algebras were found by G. Georgiev [15] (type $A_l^{(1)}$) and by M. Primc and the authors [5] (types $B_l^{(1)}, C_l^{(1)}$). These bases were used to prove the generalized Rogers–Ramanujan-type sums obtained by A. Kuniba, T. Nakanishi and J. Suzuki [21] and D. Gepner [16]. Generalizing this approach, M. Okado and R. Takenaka [29, 32] found bases for the parafermionic spaces of standard modules of highest weights $k\Lambda_0$ for twisted affine Lie algebras. Moreover, using these bases they proved character formulae conjectured by G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Z. Tsuboi [19].

In this paper, we construct the combinatorial bases for the parafermionic spaces of standard modules of rectangular highest weights, i.e. of weights of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j,$$

where $k_0, k_j \in \mathbb{Z}_{\geq 0}$ and $k = k_0 + k_j > 0$, (1.1)

for twisted affine Lie algebras of types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$, where $j = 1$ (resp. $j = l$) in type $A_{2l-1}^{(2)}$ (resp. $D_{l+1}^{(2)}$). As with [5, 15, 29], the construction consists of several steps, as follows.

First, we obtain the quasi-particle bases of the principal subspaces $W_{L^\hat{\nu}(\Lambda)}$ of standard modules $L^\hat{\nu}(\Lambda)$ for the highest weight $\Lambda$ as in (1.1). As with the corresponding bases in the case $\Lambda = k\Lambda_0$, found by the first author and C. Sadowski [6], they are expressed in terms of monomials of twisted quasi-particles applied on the highest weight vector of $L^\hat{\nu}(\Lambda)$, such that the quasi-particle energies satisfy certain difference and initial conditions. However, in this case, the quasi-particle monomials satisfy different...
initial conditions, which we determine by employing results of J. Lepowsky [22] and B. Bakalov and V. Kac [1] on the lattice construction of level one twisted modules. Despite an apparent analogy between the cases $\Lambda = k\Lambda_0$ and $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$, the latter requires a certain new ingredient in the proof of linear independence. More specifically, we use the so-called simple current operator from Li’s papers [27] and [28], which is a certain linear bijection between level one twisted modules from [22]. In addition, we also construct quasi-particle bases for the principal subspaces of generalized Verma modules.

Next, we use the quasi-particle bases for principal subspaces to construct the combinatorial bases for standard modules of rectangular highest weights in types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$, thus generalizing the aforementioned results of M. Okado and R. Takanaka [29]. As before, the proof of linear independence relies on Li’s simple current operators. Finally, we employ the parafermionic projection to obtain the combinatorial bases of parafermionic spaces for all rectangular highest weights in types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$.

At the end, we use all these bases to compute the character formulae of the corresponding standard modules and their principal subspaces and parafermionic spaces. In particular, we obtain the parafermionic character formulae which coincide with those of D. Gepner [17]. Furthermore, by comparing the quasi-particle bases for the principal subspaces of the generalized Verma modules with their Poincaré–Birkhoff–Witt bases, we discover two new families of combinatorial identities which correspond to types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$. The identities of such form often appear in different areas of mathematics; see, e.g., the papers by J. Fulman [13] and J. Hua [18].

2. Preliminaries

2.1. Level one standard modules

Let $\mathfrak{g}$ be a complex simple Lie algebra of type $A_{2l-1}$ or $D_{l+1}$ with the Cartan subalgebra $\mathfrak{h}$ and the root system $R$, equipped with the standard nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Using this form one can identify $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via $\langle \alpha, h \rangle = \alpha(h)$ for $\alpha \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We assume that the form is rescaled so that $\langle \alpha, \alpha \rangle = 2$ when $\alpha$ is a simple root. The simple roots $\alpha_1, \ldots, \alpha_{rkg}$, where $rkg$ stands for the rank of $\mathfrak{g}$, generate the root lattice $Q$ and the fundamental weights $\lambda_1, \ldots, \lambda_{rkg}$ generate the weight lattice $P$ of $\mathfrak{g}$. Let $\nu$ be the Dynkin diagram automorphism of $Q$ of order 2 as in Figure 1.

Let us recall the construction of standard modules of twisted affine Lie algebra $\widehat{\mathfrak{g}}[\nu]$ associated to $\mathfrak{g}$ and $\nu$ from [22] (see also [1, 7, 8, 11, 12, 27]). The central extension $\widehat{Q}$ of the root lattice $Q$ by the cyclic group $\langle -1 \rangle = \{ \pm 1 \}$,

$$
1 \rightarrow \langle -1 \rangle \rightarrow \widehat{Q} \rightarrow Q \rightarrow 1
$$

is determined by the commutator map $C_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$, i.e. by $aba^{-1}b^{-1} = C_0(\alpha, \beta)$ for $a, b \in \widehat{Q}$. We fix a section $e : Q \rightarrow \widehat{Q}$ normalized so that $e_0 = 1$ and $e_\alpha = \alpha$ for all $\alpha \in Q$ and

$$
e_\alpha e_\beta = \varepsilon C_0(\alpha, \beta) e_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in Q.
$$

![Finite Dynkin diagrams and the action of the automorphism $\nu$.](image-url)
Here \( \epsilon_{C_0}(\cdot, \cdot) \) denotes the normalized 2-cocycle \( \epsilon_{C_0} : Q \times Q \rightarrow \langle -1 \rangle \) defined as in [30].

Let \( \hat{\nu} \) be the lifting of \( \nu \) to the involutive automorphism of \( Q \) such that
\[
\hat{v}a = v\hat{a} \quad \text{for all} \quad a \in \hat{Q} \quad \text{and} \quad \hat{v}a = a \quad \text{if} \quad v\hat{a} = \hat{a}.
\]

By the same symbol we denote the extension of the automorphism \( \hat{\nu} \) to the lattice vertex operator algebra \( V_Q \). Moreover, for any \( k \) we write \( \hat{\nu} = \hat{\nu}^{\otimes k} \) for the extension of \( \hat{\nu} \) to the vertex operator algebra \( V_Q^{\otimes k} \).

Again, we have \( \hat{\nu}^2 = 1 \) (cf. [1, 22]).

For any integer \( m \) set
\[
h(m) = \{ x \in h \mid v(x) = (-1)^{m}x \}.
\]

This gives us the decomposition of the Cartan subalgebra
\[
h = h(0) \oplus h(1), \quad \text{where} \quad h(0) = \bigoplus_{i=1}^{l} \mathbb{C}(\alpha_i + v\alpha_i) \quad \text{and} \quad h(1) = \bigoplus_{i=1}^{l} \mathbb{C}(\alpha_i - v\alpha_i).
\]

For any \( \alpha \in h \) and \( m \in \mathbb{Z} \) we write \( \alpha(m) \) for the projection of \( \alpha \) onto \( h(m) \).

Let
\[
\hat{h}[v]_{\frac{1}{2}\mathbb{Z}} = \hat{h}[v]^+ \oplus \hat{h}[v]^- \oplus \mathbb{C}c
\]

Consider the Heisenberg subalgebra
\[
\hat{h}[v]_{\frac{1}{2}\mathbb{Z}} = \hat{h}[v]^+ \oplus \hat{h}[v]^- \oplus \mathbb{C}c
\]
of the twisted affine Lie algebra
\[
\hat{h}[v] = \bigcup_{m \in \mathbb{Z}} h(m) \otimes t^{m/2} \otimes \mathbb{C}c \oplus \mathbb{C}d = h(0) \otimes \mathbb{C}(t, t^{-1}) \oplus h(1) \otimes t^{1/2} \mathbb{C}(t, t^{-1}) \oplus \mathbb{C}c \oplus \mathbb{C}d.
\]

Furthermore, let \( S[v] \) be the irreducible \( \hat{h}[v]_{\frac{1}{2}\mathbb{Z}} \)-module
\[
S[v] = U(\hat{h}[v]) \otimes_{U_r} \mathbb{C}, \quad \text{where} \quad U_+ = U(\prod_{m \in \frac{1}{2}\mathbb{Z} \geq 0} h(2m) \otimes t^{m} \otimes \mathbb{C}c \oplus \mathbb{C}d).
\]

Note that \( S[v] \) is linearly isomorphic to the space of symmetric algebra \( S(\hat{h}[v]^-) \).

Form the induced \( Q \)-module
\[
U_T = \mathbb{C}[\hat{Q}] \otimes_{\mathbb{C}[\hat{N}]} T \cong \mathbb{C}[Q/N],
\]
where \( \hat{N} \) is the subgroup of \( \hat{Q} \) obtained by pulling back \( N = h(1) \cap Q \), and \( T \) denotes the one dimensional \( \hat{N} \)-module \( \mathbb{C} \) with character \( \tau : \hat{N} \rightarrow \mathbb{C}^\times \). Note that \( U_T \) can be regarded as an \( \hat{h}[v] \)-module with
\[
d \cdot a = -\frac{1}{2}(\hat{a} + v\hat{a}, \hat{a})a.
\]

Moreover, the \( \hat{h}[v] \)-module \( U_T \) is isomorphic to \( \mathbb{C}[\pi_0 Q] \otimes T \), where \( \pi_0 \) denotes the projection from \( h \) onto \( h(0) \) (cf. [7, 22]).

Let \( V_Q^T \) be the irreducible \( \hat{\nu} \)-twisted module for \( V_Q \) given by
\[
V_Q^T = S[v] \otimes U_T.
\]

For each \( e_\alpha \in \hat{Q} \), we consider \( \hat{\nu} \)-twisted vertex operator
\[
Y^\hat{\nu}(\iota(e_\alpha), z) = 2^{-\frac{[\alpha]}{2}} E^\pm(\alpha, z) E^\pm(-\alpha, z) e_\alpha z^{-[\alpha] + \frac{[\alpha(0), \alpha(0)]}{2} - \frac{a(0)}{2}},
\]
where
\[
E^\pm(\alpha, z) = \exp \left( \sum_{m \in \frac{1}{2}\mathbb{Z} \geq 0} \frac{-[\alpha(2m)]}{m} z^{-m} \right).
\]
The component operators $x_{\alpha}(m)$ are defined by
\[ Y^{\tilde{\nu}}(t(\varepsilon_{\alpha}), z) = \sum_{m \in \frac{1}{2} \mathbb{Z}} x_{\alpha}(m) z^{-m - \frac{\langle \alpha, \nu \rangle}{2}}. \]
By [22], they satisfy
\[ x_{\alpha}(m) = (-1)^{2m} x_{\alpha}(m) \quad \text{for all } m \in \frac{1}{2} \mathbb{Z}. \] (2.2)
We shall use the symbol $\tilde{\nu}$ to denote the lifting of the corresponding automorphism on $\mathfrak{h}$ to an automorphism of $\mathfrak{g}$, which is given by
\[ \tilde{\nu} x_{\alpha} = \begin{cases} \epsilon_{C_{0}}(\alpha, \alpha) x_{\alpha}, & \text{if } \mathfrak{g} \text{ is of type } A_{2l-1}, \\ x_{\alpha}, & \text{if } \mathfrak{g} \text{ is of type } D_{l+1}. \end{cases} \]
For any $m \in \mathbb{Z}$ set
\[ \mathfrak{g}(m) = \{ x \in \mathfrak{g} \mid \tilde{\nu}(x) = (-1)^{m} x \}. \]
We shall denote by $x_{(m)}$ the projection of the element $x \in \mathfrak{g}$ onto $\mathfrak{g}(m)$. Let $\tilde{\mathfrak{g}}[\tilde{\nu}]$ be the $\tilde{\nu}$-twisted affine Lie algebra associated with $\mathfrak{g}$ and $\tilde{\nu}$,
\[ \tilde{\mathfrak{g}}[\tilde{\nu}] = \tilde{\mathfrak{g}}[\tilde{\nu}] \oplus \mathbb{C} d, \quad \text{where } \tilde{\mathfrak{g}}[\tilde{\nu}] = \bigsqcup_{m \in \frac{1}{2} \mathbb{Z}} \mathfrak{g}(2m) \otimes t^{m} \oplus \mathbb{C} c. \]
Its Lie brackets are given by
\[ [x \otimes t^{m}, y \otimes t^{n}] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n,0} c, \quad [c, \tilde{\mathfrak{g}}[\tilde{\nu}]] = 0, \quad [d, x \otimes t^{m}] = nx \otimes t^{m}, \]
for $x \in \mathfrak{g}(2m)$, $y \in \mathfrak{g}(2n)$ and $m, n \in \frac{1}{2} \mathbb{Z}$. The Lie algebra $\tilde{\mathfrak{g}}[\tilde{\nu}]$ is isomorphic to the twisted affine Lie algebra of type $A^{(2)}_{2l-1}$ or $D^{(2)}_{l+1}$ with respect to the choice of $Q$ and $\nu$ (cf. [20]).

The representation of $\mathfrak{h}[\nu]$ on $V_{Q}^{\tilde{\nu}}$ uniquely extends to the representation of $\tilde{\mathfrak{g}}[\tilde{\nu}]$ by
\[ (x_{\alpha})_{(2m)} \otimes t^{m} \mapsto x^{\tilde{\nu}}_{\alpha}(m) \quad \text{for } m \in \frac{1}{2} \mathbb{Z}, \alpha \in Q. \]
Moreover, we have $V_{Q}^{\tilde{\nu}} \cong \tilde{L}^{\tilde{\nu}}(\Lambda_{0})$, where $\tilde{L}^{\tilde{\nu}}(\Lambda_{0})$ is the level one standard $\tilde{\mathfrak{g}}[\tilde{\nu}]$-module of the highest weight $\Lambda_{0}$ with the highest weight vector $v_{\Lambda_{0}} = v_{L^{\tilde{\nu}}(\Lambda_{0})} = 1 \otimes 1$ (cf. [22]). Let
\[ \nu = \pi \nu_{0} \lambda_{j}, \] (2.3)
where, as in (1.1), we set $j = 1$ (resp. $j = l$) for $\mathfrak{g}$ of type $A_{2l-1}$ (resp. $D_{l+1}$). The element $\nu \in \mathfrak{h}(0)$ satisfies $\langle \nu, \alpha \rangle \in \frac{1}{2} Q$ for all $\alpha \in Q$. Following [22] (see also [7]) we obtain the level one standard $\tilde{\mathfrak{g}}[\tilde{\nu}]$-module $\tilde{L}^{\tilde{\nu}}(\Lambda_{j})$ of the highest weight $\Lambda_{j}$ by defining a new representation of $\tilde{\mathfrak{g}}[\nu]$ on $V_{Q}^{\tilde{\nu}}$ as follows. The elements of $\mathfrak{h}(0)$ act on $U_{T}$ by
\[ \alpha \nu \cdot b \otimes t = \left( \alpha, \nu + \frac{1}{2} \right) b \otimes t \]
for $b \in \tilde{Q}, t \in T, \alpha \in \mathfrak{h}(0)$, the operators $e^{a \nu}$ act by
\[ z^{a \nu} \cdot b \otimes t = z^{\left( a, \nu + \frac{1}{2} \right)} b \otimes t \]
and the Heisenberg algebra $\tilde{\mathfrak{h}}[\nu]_{\frac{1}{2} \mathbb{Z}}$ acts trivially. A $\nu$-shifted $\tilde{\nu}$-twisted vertex operator $Y^{\tilde{\nu} \nu}(t(\varepsilon_{\alpha}), z)$ is defined by
\[ Y^{\tilde{\nu} \nu}(t(\varepsilon_{\alpha}), z) = Y^{\tilde{\nu}}(t(\varepsilon_{\alpha}), z) z^{\left( \alpha_{(0)}, \nu \right)} = 2^{-\frac{\langle a, \alpha \rangle}{2}} E^{\nu_{0}(-\alpha, z)} E^{\nu_{+}(-\alpha, z)} e_{\alpha} z^{\nu_{0}} + \frac{\langle a, \alpha \rangle}{z^{\nu_{0}}} \frac{\langle a, \alpha \rangle}{z^{\nu_{0}}} \] (cf. [7]). The component operators $x^{\tilde{\nu} \nu}_{\alpha}(m)$, $m \in \frac{1}{2} \mathbb{Z}$, are given by
\[ Y^{\tilde{\nu} \nu}(t(\varepsilon_{\alpha}), z) = \sum_{m \in \frac{1}{2} \mathbb{Z}} x^{\tilde{\nu} \nu}_{\alpha}(m) z^{-m} \]
and they satisfy

\[ x_{\alpha}^{\gamma}(m) = x_{\alpha}^{\gamma}(m + \langle \alpha(0), \gamma \rangle). \]  

(2.4)

Let \( \tilde{\nu}_{\gamma} \) be the automorphism of Lie algebra \( g \) such that

\[ \tilde{\nu}_{\gamma} h = vh \quad \text{for all} \quad h \in h \quad \text{and} \quad \tilde{\nu}_{\gamma} x_{\alpha} = (-1)^{2\langle \alpha, \gamma \rangle} x_{\alpha} \quad \text{for all} \quad \alpha \in Q. \]

Then \( \tilde{\nu}_{\gamma}^2 = 1 \) and the \( \tilde{\nu}_{\gamma} \)-twisted affine Lie algebra \( \tilde{g}[\tilde{\nu}_{\gamma}] \) associated to \( g \) coincides with \( \tilde{g} \tilde{V} \). Furthermore, from [22, Thm. 10.1] (see also [7, Thm. 4.2]) follows that the new representation of \( \tilde{g}[\nu] \) on \( V_{Q_{\nu}}^{\gamma} \) uniquely extends to the representation of \( \tilde{g} \tilde{V} \) by

\[ (x_{\alpha})_{2m} \otimes t^m \mapsto x_{\alpha}^{\gamma}(m) \quad \text{for all} \quad m \in \frac{1}{2} \mathbb{Z}, \quad \alpha \in Q. \]

It can be easily checked that we have the isomorphism of \( \tilde{g}[\nu] \)-modules \( V_{Q_{\nu}}^{\gamma} \cong \tilde{V}(\Lambda_{j}) \). We denote the highest weight vector of \( \tilde{V}(\Lambda_{j}) \) by \( v_{\Lambda_{j}} \). By [27, Prop. 5.4], \( V_{Q_{\nu}}^{\gamma} \) is a \( \nu_{\gamma} \)-twisted \( V_{Q} \)-module, where \( \nu_{\gamma} = e^{-2\pi \sqrt{-1} \nu_{\gamma}(0)} \) is an automorphism of \( V_{Q} \).

### 2.2. Higher level standard modules

Let \( k \in \mathbb{N} \) be fixed. The tensor product \( (V_{Q_{\nu}}^{\gamma})^{\otimes k} \) of \( k \) copies of \( \nu_{\gamma} \)-twisted \( V_{Q} \)-modules \( V_{Q_{\nu}}^{\gamma} \) possesses a structure of \( \nu_{\gamma} \)-twisted module for the vertex operator algebra \( V_{Q_{\nu}}^{\gamma} \) with vertex operators

\[ Y_{\nu_{\gamma}}^{\gamma}(v_1 \otimes \cdots \otimes v_k, z) = Y_{\nu_{\gamma}}^{\gamma}(v_1, z) \otimes \cdots \otimes Y_{\nu_{\gamma}}^{\gamma}(v_k, z). \]

The operators \( x_{\alpha}^{\gamma}(m) \) on \( (V_{Q_{\nu}}^{\gamma})^{\otimes k} \) are given by

\[ Y_{\nu_{\gamma}}^{\gamma}(x_{\alpha}^{\gamma}(-1) \cdot (1 \otimes \cdots \otimes 1), z) = \sum_{m \in \frac{1}{k} \mathbb{Z}} x_{\alpha}^{\gamma}(m) z^{-m-1}. \]

This construction gives the diagonal action of the sublattice \( kQ \subset Q \) on \( (V_{Q_{\nu}}^{\gamma})^{\otimes k} \):

\[ k\alpha \mapsto \rho(k\alpha) = e^{k \alpha} \otimes \cdots \otimes e^{k \alpha}, \quad \alpha \in Q. \]  

(2.5)

To simplify the notation, in the rest of the paper we will omit the symbol \( \gamma \) and write \( x_{\nu}(z) \) for \( x_{\nu}(z)^{\gamma} \) as well, as it will be clear from the context whether \( \gamma \) is 0 or \( \pi_{0} \lambda_{j} \). We shall need the following relations on the standard \( \tilde{g}[\nu] \)-module \( \tilde{V}(\Lambda) \), which are consequence of the adjoint action of \( e_{\alpha} \) on \( \tilde{g} \) (cf. [20], see also [29, 31]):

\[ e_{\alpha} d e_{\alpha}^{-1} = d + \alpha - \frac{\langle \alpha(0), \alpha(0) \rangle}{2} c, \]
\[ e_{\alpha} e_{\alpha}^{-1} = c, \]
\[ e_{\alpha} h_{(0)} e_{\alpha}^{-1} = h_{(0)} - \langle \alpha(0), h_{(0)} \rangle c, \quad h \in h, \]
\[ e_{\alpha} h_{(s)}(s) e_{\alpha}^{-1} = h_{(s)}(s), \quad s \neq 0, \]
\[ e_{\alpha} x_{\beta}(s) e_{\alpha}^{-1} = x_{\beta}(s - \langle \beta(0), \alpha(0) \rangle). \]  

(2.6)–(2.9)

For any positive integer \( r \) consider the twisted vertex operators

\[ x_{\alpha_{j}}^{\gamma}(z) = Y_{\gamma}(x_{\alpha_{j}}^{\gamma}(-1)^{j} \mathbf{i}, z) = \sum_{m \in \frac{1}{r} \mathbb{Z}} x_{\alpha}^{\gamma}(m) z^{-m-r} = (Y_{\gamma}(x_{\alpha_{j}}^{\gamma}(-1)^{j} \mathbf{i}, z))^{r}, \]  

(2.11)

associated with the vector \( x_{\alpha_{j}}^{\gamma}(-1)^{j} \mathbf{i} \in L(k\Lambda_{0}) \) (cf. [27]). They satisfy the commutator formula for twisted vertex operators,

\[ \left[ x_{\alpha}^{\gamma}(z_{1}), x_{\beta}^{\gamma}(z_{2}) \right] = \sum_{s \geq 0} \frac{(-1)^{s}}{2s!} \left( \frac{d}{dz_{1}} \right)^{s} z_{1}^{-1} \sum_{q \in \mathbb{Z}/2\mathbb{Z}} \delta \left( -1 \frac{q}{2} \frac{z_{1}}{z_{2}} \right) Y_{\gamma}(v^{\beta} x_{\alpha}(s) x_{\beta}(-1)^{j} \mathbf{i}, z_{2}). \]
For any $h \in \mathfrak{h}$ and $m \in \frac{1}{2}\mathbb{Z}$ we have
\[
[h_{(2m)}(z), x_{ra}^\vee(z)] = r(h_{(2m)}, \alpha(-2m)) z^m x_{ra}^\vee(z).
\]

For any dominant integral weight $\lambda \in (\mathfrak{h}(0))^*$ let $U_\lambda$ be the finite-dimensional irreducible $\mathfrak{g}(0)$-module of highest weight $\lambda$. The generalized Verma module $N^\vee(\Lambda)$ associated with $U_\lambda$ is defined by $N^\vee(\Lambda) = U(\mathfrak{g}[\hat{\nu}]) \otimes_{U(\mathfrak{g}[\nu])} U_\lambda$, where the action of the Lie algebra
\[
\mathfrak{g}[\hat{\nu}]^\geq = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} (\mathfrak{g}(2n) \otimes t^n) \otimes \mathbb{C}c
\]
on $U_\lambda$ is given by
\[
\mathfrak{g}(2n) \otimes t^n \cdot u = 0 \quad \text{for all} \quad n > 0 \quad \text{and} \quad c \cdot u = ku \quad \text{for all} \quad u \in U_\lambda.
\]
Recall that the Poincaré–Birkhoff–Witt theorem for the universal enveloping algebra gives the vector space isomorphism
\[
N^\vee(\Lambda) \cong U(\mathfrak{g}[\hat{\nu}]^0) \otimes_U U_\lambda, \quad \text{where} \quad \mathfrak{g}[\hat{\nu}]^0 = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} (\mathfrak{g}(2n) \otimes t^n).
\]

Let $V$ denote the generalized Verma module $N^\vee(\Lambda)$ or its unique simple quotient $L^\vee(\Lambda)$. Then there exists a linear map
\[
Y^\vee : g(-1) \otimes 1 \rightarrow (\text{End} V)[[z^\frac{1}{2}, z^{-\frac{1}{2}}]],
\]
\[
Y^\vee(x(-1) \otimes 1, z) = x^\vee(z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} x^\vee(n) z^{-n-1},
\]
which defines a structure of $\hat{\nu}$-twisted $N(k\Lambda_0)$-module over $V$; see, e.g., [27]. Its coefficients $x^\vee(n)$ satisfy the identity in (2.2) as well.

### 2.3. Principal subspaces

Consider the nilpotent Lie subalgebra

\[
n = \bigoplus_{\alpha \in R_+} \mathbb{C}x_\alpha
\]
of $\mathfrak{g}$ generated by all root vectors $x_\alpha$ which correspond to positive roots $\alpha \in R_+$. Let

\[
\mathfrak{n}(\hat{\nu}) = \mathfrak{n}(\hat{\nu}) \oplus \mathbb{C}c, \quad \text{where} \quad \mathfrak{n}(\hat{\nu}) = \bigsqcup_{m \in \frac{1}{2}\mathbb{Z}, \alpha \in R_+} \mathbb{C}x_{\alpha(2m)} \otimes t^m,
\]
be the corresponding subalgebra of $\mathfrak{g}[\hat{\nu}]$. Denote by $\nu_\Lambda$ the highest weight vector of $V = N^\vee(\Lambda), L^\vee(\Lambda)$. Following [10], define the principal subspace of $V = N^\vee(\Lambda), L^\vee(\Lambda)$ by

\[
W^T_V = U(\mathfrak{n}(\hat{\nu})) \nu_\Lambda.
\]

The principal subspace $W^T_V$ coincides with

\[
U(\mathfrak{n}_{\alpha_1}(\hat{\nu})) \cdots U(\mathfrak{n}_{\alpha_l}(\hat{\nu})) \nu_\Lambda, \quad \text{where} \quad \mathfrak{n}_{\alpha_i}(\hat{\nu}) = \bigsqcup_{m \in \frac{1}{2}\mathbb{Z}} \mathbb{C}x_{\alpha_i(2m)} \otimes t^m,
\]
(see [6, Lemma 3.1] and [14, Lemma 3.1]). In addition, as with the untwisted case in [4, Section 5], for $V = N^\vee(\Lambda)$ we have the isomorphism of $\mathfrak{n}(\hat{\nu})$-modules

\[
W^T_{N^\vee(\Lambda)} \cong U(\mathfrak{n}(\hat{\nu})_{<0}), \quad \text{where} \quad \mathfrak{n}(\hat{\nu})_{<0} = \bigsqcup_{m \in \frac{1}{2}\mathbb{Z}, \alpha \in R_+} \mathbb{C}x_{\alpha(2m)} \otimes t^m. \quad (2.13)
\]
3. Quasi-particle bases of principal subspaces

From now on, we consider the rectangular highest weights $\Lambda$, as defined by (1.1). We use the realization of the standard $\tilde{\mathfrak{g}}[\tilde{\nu}]$-module $L_{\tilde{\nu}}^\nu(\Lambda) \cong U(\tilde{\mathfrak{g}}[\tilde{\nu}]) \cdot v_\Lambda$ as a submodule of $(V_Q^T)^{\otimes k}$ with the highest weight vector

$$v_\Lambda = v_\Lambda j_k \otimes \cdots \otimes v_\Lambda j_1,$$

where $j_s = \begin{cases} 0 & \text{for } 1 \leq s \leq k_0, \\ j & \text{for } k_0 + 1 \leq s \leq k. \end{cases}$ (3.1)

3.1. Twisted quasi-particles

For positive integers $r$ and $i = 1, \ldots, l$ and $m \in \frac{1}{2} \mathbb{Z}$ we define a quasi-particle of charge $r$, color $i$ and energy $-m$ as the coefficient $x^\nu_{\alpha_i} (m)$ of the twisted vertex operator $x^\nu_{\alpha_i} (z)$, as given by (2.11) (cf. [6, 14]). Our goal is to construct the so-called quasi-particle bases of the principal subspaces, i.e. the bases consisting of some monomials of quasi-particles applied on the highest weight vector. Charges of quasi-particles which appear in the basis of the principal subspaces $W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T$ will be less than or equal to $k$ since we have the integrability relations

$$x^\nu_{\alpha_i} (z) = 0$$

on the standard $\mathfrak{g}[\nu]$-modules $L_{\nu}^\nu(\Lambda)$ of level $k$ (cf. [27]). On the other hand, the elements of basis for $W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T$ will contain quasi-particle of all charges.

By (2.11), it follows that in the case of $W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T$ all quasi-particles satisfy

$$x^\nu_{\alpha_i} (z) v_\Lambda \in \left\{ z^{-\frac{r}{2} \sum_{j=1}^{l} \delta_{j,i}} W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T \left[ [z^{1/2}] \right], \quad \text{when } \nu \alpha_i \neq \alpha_i, \right. \left. W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T \left[ [z] \right], \quad \text{when } \nu \alpha_i = \alpha_i, \right\}$$

while in the case of $W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T$ we have

$$x^\nu_{\alpha_i} (z) v_\Lambda \in \left\{ z^{-\frac{r}{2} + \frac{1}{2} \sum_{j=1}^{l} \delta_{j,i}} W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T \left[ [z^{1/2}] \right], \quad \text{when } \nu \alpha_i \neq \alpha_i, \right. \left. W_{L_{\tilde{\nu}}^\nu(\Lambda)}^T \left[ [z] \right], \quad \text{when } \nu \alpha_i = \alpha_i, \right\}$$

The quasi-particle bases of principal subspaces will be expressed in the form

$$b v_\Lambda = b^\nu (\alpha_1) \cdots b^\nu (\alpha_l) v_\Lambda$$

$$= \underbrace{x_{n_{1,i}^{(1)}, \alpha_i} (m_{1,i}^{(1)}) \cdots x_{n_{1,i}^{(2)}, \alpha_i} (m_{1,i}^{(2)}) \cdots x_{n_{1,i}^{(t)}, \alpha_i} (m_{1,i}^{(t)})}_b^\nu (\alpha_i) \underbrace{\cdots x_{n_{1,l}^{(1)}, \alpha_i} (m_{1,l}^{(1)}) \cdots x_{n_{1,l}^{(2)}, \alpha_i} (m_{1,l}^{(2)}) \cdots x_{n_{1,l}^{(t)}, \alpha_i} (m_{1,l}^{(t)})}_b^\nu (\alpha_l) v_\Lambda,$$ (3.5)

where for $i = 1, \ldots, l$ and $t \in \mathbb{N}$ we have

$$0 \leq n_{i,j}^{(1)} \leq \cdots \leq n_{i,j} \leq n_{1,i}, \quad r^{(1)}_i \geq r^{(2)}_i \geq \cdots \geq r^{(t)}_i \geq 0, \quad m_{j,i}^{(1)} \leq \cdots \leq m_{1,j}.$$

Here $r_i^{(p)}$ for $1 \leq p \leq t$ represent the parts of the partition $\mathcal{R}_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(t)})$ of some fixed positive number $r_i$. The partition $\mathcal{R}_i$ is conjugate to the partition $\mathcal{R}_i' = (n_{i,j}^{(1)}, \ldots, n_{i,j})$. We now recall terminology from [6, 14]. For every color $i = 1, \ldots, l$, the number $r_i$ is said to be a color-type, a partition $\mathcal{R}_i$ a charge-type, a partition $\mathcal{R}_i$ a dual-charge-type and a finite sequence of energies $\mathcal{E}_i = (m_{i,j}^{(1)}, \ldots, m_{i,j})$ an energy-type of the monomial $b_i = b^\nu (\alpha_i)$. Generalizing this notions to the quasi-particle monomial $b = b^\nu (\alpha_1) \cdots b^\nu (\alpha_l)$ we define its color-type $\mathcal{C} = (r_1, \ldots, r_l)$, charge-type $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l)$, dual-charge-type $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l)$ and energy-type $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_l)$. Moreover, its total charge $\text{chg} b$ and total energy $en b$ are defined by

$$\text{chg} b = \sum_{i=1}^{l} r_i \quad \text{and} \quad \text{en} b = -m_{i,j}^{(1)} - \cdots - m_{i,j}^{(q)} - \cdots - m_{1,j}.$$
Let $M_{\text{QP}}$ denote the set of all quasi-particle monomials as in (3.5). We introduce the linear order “$<$” on the set of all monomials from $M_{\text{QP}}$ of fixed degree and $h_{(0)}$-weight as follows. Let $b$ and $\overline{b}$ be quasi-particle monomials of color-types $C$ and $\overline{C}$, charge-types $R'$ and $\overline{R}'$ and energy-types $\mathcal{E}$ and $\overline{\mathcal{E}}$, respectively. Then we write $bv_{\Lambda} < \overline{bv}_{\Lambda}$ if one of the following conditions holds:

1. $\text{chg } b > \text{chg } \overline{b}$
2. $\text{chg } b = \text{chg } \overline{b}$ and $C < \overline{C}$
3. $C = \overline{C}$ and $en b < en \overline{b}$
4. $C = \overline{C}$, $en b = en \overline{b}$ and $R' < \overline{R}'$
5. $R' = \overline{R}'$ and $\mathcal{E} < \overline{\mathcal{E}}$,

where for integer sequences we write $(x_p, \ldots, x_1) < (y_r, \ldots, y_1)$ if there exists $s$ such that

$$x_1 = y_1, \ldots, x_{s-1} = y_{s-1} \quad \text{and} \quad s = p + 1 < r \quad \text{or} \quad x_s < y_s. \quad (3.6)$$

### 3.2. Quasi-particle bases of principal subspaces

By (3.3) and (3.4), the vectors of the form (3.5), such that for $i = 1, \ldots, l$ their quasi-particle energies satisfy

$$m_{p,i} \leq -\mu_i n_{p,i} - \sum_{s=1}^{n_{p,i}} \mu_i \delta_{i,j} \quad \text{with} \quad 1 \leq p \leq r_i^{(1)} \quad \text{and} \quad \mu_i := \frac{1}{2} \langle \alpha_i (0), \alpha_i (0) \rangle,$$

span entire principal subspace. We shall now strengthen the constraints above by using relations among quasi-particles. First, by employing the commutator formula for twisted vertex operators on $N^\hat{V} (\Lambda)$ we generalize [6, Lemma 4.2] as follows.

**Lemma 3.1.** Let $n_1, n_2$ be positive integers and $\alpha, \beta \in \mathbb{R}_+$ positive simple roots.

a) If $\langle \alpha, \beta \rangle = -1 = \langle \nu \alpha, \beta \rangle$, we have

$$(z_1 - z_2)^{\min[n_1,n_2]} x_{\alpha}^{\hat{V}} (z_1) x_{\beta}^{\hat{V}} (z_2) = (z_1 - z_2)^{\min[n_1,n_2]} x_{\alpha}^{\hat{V}} (z_2) x_{\alpha}^{\hat{V}} (z_1).$$

b) If $\langle \alpha, \beta \rangle = -1$ and $\langle \nu \alpha, \beta \rangle = 0$, we have

$$(z_1^{1/2} - z_2^{1/2})^{\min[n_1,n_2]} x_{\alpha}^{\hat{V}} (z_1) x_{\beta}^{\hat{V}} (z_2) = (z_1^{1/2} - z_2^{1/2})^{\min[n_1,n_2]} x_{\beta}^{\hat{V}} (z_2) x_{\alpha}^{\hat{V}} (z_1).$$

The interaction between quasi-particles of the same color [6, Lemma 4.1] can be summarized as follows.

**Lemma 3.2.** Let $i = 1, \ldots, l$ and $v \in W_{\hat{V}}^\Lambda$ for $V = N^\hat{V} (\Lambda), L^\hat{V} (\Lambda)$. For any charges $n_1, n_2$ such that $n_1 \geq n_2$ and integer $M = m_1 + m_2$ the $2\mu_i n_2$ vectors

$$x_{n_1 \alpha_i}^{\hat{V}} (m_1) x_{n_2 \alpha_i}^{\hat{V}} (m_2) v, \quad x_{n_1 \alpha_i}^{\hat{V}} (m_1 - 1) x_{n_2 \alpha_i}^{\hat{V}} (m_2 + 1) v, \ldots$$

$$\ldots, x_{n_1 \alpha_i}^{\hat{V}} (m_1 - 2 \mu_i n_2 + 1) x_{n_2 \alpha_i}^{\hat{V}} (m_2 + 2 \mu_i n_2 - 1) v$$

can be expressed as a finite linear combination of the monomial vectors

$$x_{n_1 \alpha_i}^{\hat{V}} (s_1) x_{n_2 \alpha_i}^{\hat{V}} (s_2) v \quad \text{such that} \quad s_1 + s_2 = M \quad \text{and} \quad s_1 \leq m_1 - 2 \mu_i n_2 \text{ or } s_2 < m_2$$

and monomial vectors which contain a quasi-particle of color $i$ and charge $n_1 + 1$. Moreover, if $n_1 = n_2$, the monomial vectors

$$x_{n_1 \alpha_i}^{\hat{V}} (m_1) x_{n_1 \alpha_i}^{\hat{V}} (m_2) v \quad \text{such that} \quad m_1 + m_2 = M \quad \text{and} \quad m_2 - 2 \mu_i n_1 < m_1 \leq m_2$$

can be expressed as a finite linear combination of monomial vectors

$$x_{n_1 \alpha_i}^{\hat{V}} (s_1) x_{n_1 \alpha_i}^{\hat{V}} (s_2) v \quad \text{such that} \quad s_1 + s_2 = M \quad \text{and} \quad s_1 \leq s_2 - 2 \mu_i n_1$$

and monomial vectors which contain a quasi-particle of color $i$ and charge $n_1 + 1$. 


Denote by $B_{W^\nu(\Lambda)}$ the set of all quasi-particle monomials of the form (3.5) such that their energies $m_{p,i}$ satisfy the following difference conditions for all $i = 1, \ldots, l$:

$$\begin{align*}
m_{p,i} &\leq (1 - 2p)\mu_i n_{p,i} - \langle \alpha_{i(0)}^{(0)}, \alpha_{i-1}^{(0)} \rangle \sum_{q=1}^{q_{i-1}} \min \{ n_{q,i-1}, n_{p,i} \} - \sum_{i=1}^{n_p} \mu_i \delta_{i,j}, \\
m_{p+1,i} &\leq m_{p,i} - 2\mu_i n_{p,i} \quad \text{if} \quad n_{p+1,i} = n_{p,i},
\end{align*}$$

(3.7) (3.8)

where we set $r_0^{(1)} := 0$ and $j_s$ are given by (3.1). As for the standard module, we denote by $B_{W^\nu(\Lambda)}$ the subset of $B_{W^\nu(\Lambda)}$ such that its monomials contain only quasi-particles of charge less than or equal to $k$; recall (3.2). For $V = N^\nu(\Lambda)$, $L^\nu(\Lambda)$ we have

**Theorem 3.3.** For any highest weight $\Lambda$ as in (1.1) the set $B_{W^\nu} = \{ bv_\Lambda \mid b \in B_{W^\nu} \}$ forms a basis of the principal space $W^T_{V^\nu}$.

The proof that $B_{W^\nu}$ spans the principal subspace goes in parallel with [4, Section 5.5] and [14, Sect. 5]. Furthermore, the linear independence of $B_{W^\nu(\Lambda)}$ can be verified by arguing as in [2, Section 3]. As for the set $B_{W^\nu(\Lambda)}$ in comparison with [6] (where the analogous basis for the highest weights of the form $k\Lambda_0$ is established), the proof of linear independence in the rectangular case requires a certain new ingredient, the map $[\pi_0\lambda_j]$ which connects modules $L^\nu(\Lambda_0)$ and $L^\nu(\Lambda_j)$. In Section 3.3 below, we recall the Georgiev projection $\pi_\mathcal{R}$ and the aforementioned map $[\pi_0\lambda_j]$. Finally, in Section 3.4 we employ them to finalize the proof.

### 3.3. Georgiev’s projection and the simple current map

Let $\mathcal{R}$ be the dual-charge-type of the quasi-particle monomial from (3.5). Consider Georgiev’s projection $\pi_\mathcal{R}$ of the principal subspace $W^T_{L^\nu(\Lambda)}$ on the tensor product space

$$W^T_{L^\nu(\Lambda)}(r_1^{(k)}, \ldots, r_s^{(k)}) \otimes \cdots \otimes W^T_{L^\nu(\Lambda)}(r_1^{(l)}, \ldots, r_s^{(l)}) \subset (W^T_{L^\nu(\Lambda)})^\otimes k_j \otimes (W^T_{L^\nu(\Lambda_0)})^\otimes k_0,$$

where $W^T_{L^\nu(\Lambda)}(r_1^{(k)}, \ldots, r_s^{(k)})$, $s = 1, \ldots, k$, denote the $h(0)$-weight subspaces of the level one principal subspace $W^T_{L^\nu(\Lambda_0)}$, consisting of vectors of charges $r_1^{(s)}, \ldots, r_s^{(s)}$; see [6, 14] for more details. We will also use the symbol $\pi_\mathcal{R}$ to denote the generalization of the projection to the space of formal power series with coefficients in $(W^T_{L^\nu(\Lambda)})^\otimes k_j \otimes (W^T_{L^\nu(\Lambda_0)})^\otimes k_0$. As in [6, Section 5.1], one can use the integrability relations (3.2) at the level $k = 1$, $x^\nu_{2\alpha_i}(z)^2 = 0$ for $i = 1, \ldots, l$, to express the projection of the monomial (3.5) as the coefficient of the product of the corresponding vertex operators applied on the highest weight vector (3.1).

Denote by $[\pi_0\lambda_j]$ the identity map on the vector space $V^\nu_Q$ endowed with two different structures of level one standard $\tilde{\mathfrak{g}}[\tilde{\mathfrak{h}}]$-modules, $L^\nu(\Lambda_0)$ and $L^\nu(\Lambda_j)$ (cf. [9, 27, 28]). By the construction of these modules follows that, for the suitably normalized highest weight vector $v_\Lambda$, we have for any root $\alpha$, $m \in \frac{1}{2}\mathbb{Z}$ and $i = 1, \ldots, l$ the identities

$$\begin{align*}
[\pi_0\lambda_j] v_\Lambda &= v_\Lambda, \\
x^\nu_{\alpha_i}(m) [\pi_0\lambda_j] &= [\pi_0\lambda_j] x^\nu_{\alpha_i}(m + \langle \lambda_j^{(0)}, \alpha^{(0)} \rangle), \\
[\pi_0\lambda_j] \circ e_{\alpha_i} &= e_{\alpha_i} \circ [\pi_0\lambda_j].
\end{align*}$$

(3.9) (3.10) (3.11)
Suppose $b$ is the quasi-particle monomial from (3.5). The equalities (3.9) and (3.10) imply that

$$\pi_R b(\pi_0 \lambda_j) v_{\Lambda_0} \otimes k_j \otimes v_{\Lambda_0}^{\otimes k_0}$$

coincides with the coefficient $\left( \pi_0 \lambda_j \right) \otimes k_j \otimes 1 \otimes k_0 \left( \pi_R b^+ v_{\Lambda_0}^k \right)$ of the variables

$$-m_{r_i^{(1)}, j} - n_{r_i^{(1)}, j} \cdots -m_{r_i^{(k_0+1)}, j} + 1, j - n_{r_i^{(k_0+1)}, j} + 1, j \cdots$$

$$-m_{r_i^{(k_0+1)}, j} - n_{r_i^{(k_0+1)}, j} + \lambda_{i(0)} \alpha_{i(0)} \left( n_{r_i^{(k_0+1)}, j} - k_0 \right) \cdots$$

$$\cdots z_{r_i^{(k_0+1)}, j} \cdots z_{r_i^{(1)}, j} \cdots$$

in the image of the (generalized) Georgiev projection

$$\pi_R \left( x_{r_i^{(1)}, j} a_i (z_{r_i^{(1)}, j}) \cdots x_{n_{1,1}, a_1} (z_{1,1}) \left( \pi_0 \lambda_j \right) v_{\Lambda_0} \otimes k_j \otimes v_{\Lambda_0}^{\otimes k_0} \right).$$

Here we denote by $b^+$ the quasi-particle monomial

$$b^+ = b_{a_1}^{+1} \cdots b_{a_l}^{+l}$$

such that its factors $b_{a_i}^{+i}$, $i = 1, \ldots, l$, are given by

$$b_{a_i}^{+i} = x_{r_i^{(1)}, j} a_i \left( m_{r_i^{(1)}, j} \right) \cdots x_{r_i^{(k_0+1)}, j} a_i \left( m_{r_i^{(k_0+1)}, j} \right) + \lambda_{i(0)} \alpha_{i(0)} \left( n_{r_i^{(k_0+1)}, j} - k_0 \right) \cdots$$

$$\cdots x_{n_{1,1}, a_1} a_1 \left( m_{1,1} + \lambda_{1(0)} \alpha_{1(0)} \right) \left( n_{1,1} - k_0 \right).$$

Finally, it remains to observe that by (3.7) and (3.8) the energies of $b^+ v_{k\Lambda_0}$ satisfy the initial and difference conditions for the quasi-particle basis of principal subspace $W_{L^F(k\Lambda_0)}$, as given by [6, Theorem 5.1].

### 3.4. Proof of linear independence

We are now ready to complete the proof of Theorem 3.3. Suppose that the quasi-particle monomial $b$ of charge-type $R'$ and dual-charge-type $\bar{R}$ is the smallest monomial, with respect to the linear ordering from Section 3.1, in the linear combination (indexed by some finite set $A$),

$$\sum_{a \in A} c_a b_a v_\Lambda = 0. \quad (3.14)$$

We can assume that all quasi-particle monomials $b_a \in B_{W_{L^F}(\Lambda)}$ are of the same color-type and that the scalars $c_a \in \mathbb{C}$ are nonzero. Let us apply the projection $\pi_\mathcal{R}$ to (3.14). By its definition, all nonzero summands which appear in

$$\sum_{a \in A} c_a \pi_\mathcal{R} b_a v_\Lambda = 0$$

contain monomials $b_a$ of the same charge-type $R'$. Using (3.9) we find

$$0 = \sum_{a \in A} c_a \pi_\mathcal{R} b_a \left( \pi_0 \lambda_j \right) v_{\Lambda_0} \otimes k_j \otimes v_{\Lambda_0}^{\otimes k_0} = \left( \pi_0 \lambda_j \right) \otimes k_j \otimes 1 \otimes k_0 \sum_{a \in A} c_a \pi_\mathcal{R} b_a^+ v_{\Lambda_0}^k$$

with $b_a^+$ as in (3.12) and (3.13). Dropping the invertible operator $\left( \pi_0 \lambda_j \right) \otimes k_j \otimes 1 \otimes k_0$, we get

$$\sum_{a \in A} c_a \pi_\mathcal{R} b_a^+ v_{\Lambda_0}^k = 0.$$

Thus, using the quasi-particle basis for the weights of the form $k\Lambda_0$, as given by [6, Theorem 5.1], we obtain $c_a = 0$, so that the assertion of Theorem 3.3 follows.
4. Combinatorial bases of standard modules and parafermionic spaces

In this section, we use the quasi-particle bases for principal subspaces to construct combinatorial bases of standard modules and their parafermionic spaces. Throughout the section, \( \Lambda \) denotes the rectangular highest weight; recall (1.1).

4.1. Standard modules

Let \( B_{U(\widehat{g}[v]^-)} \) be the Poincaré–Birkhoff–Witt basis of the irreducible \( \widehat{h}[v] \frac{1}{2} \widehat{Z} \)-module \( U(\widehat{h}[v]^-) \) of level \( k \) which consists of all elements \( h = h_{a_1} \cdots h_{a_l} \) such that for \( i = 1, \ldots, l \) we have

\[
h_{a_i} = (\alpha_i(2i-1))^{r_{i,j}} \cdots (\alpha_i(2i-1))^{r_{1,j}},
\]

where \( t_i \in \mathbb{Z}_{\geq 0} \) and \( r_{p,i}, s_{p,i} \in \mathbb{N} \) are such that \( s_{1,i} \leq \cdots \leq s_{t_i,i} \) for all \( p = 1, \ldots, t_i \).

We shall now generalize the definition of the linear order from Section 3.1 to the set

\[
\left\{ e_{\mu} h b v_{\Lambda} \mid \mu \in Q, h \in B_{U(\widehat{h}[v]^-)}, b \in M'_{QP} \right\},
\]

where \( M'_{QP} \) denotes the subset of \( M_{QP} \) which consists of all vectors of the form (3.5) such that their quasi-particle charges are less than or equal to \( k - 1 \). Let \( e_{\mu} h b v_{\lambda} \) and \( e_{\nu} h b v_{\mu} \) be any two elements of the set (4.2) of the same degree and \( h(0) \)-weight. We shall write \( e_{\mu} h b v_{\lambda} < e_{\nu} h b v_{\mu} \) if we have \( b v_{\lambda} < b v_{\mu} \) (i.e., in other words, if one of the conditions (1)–(5) holds) or if the energy types of \( b \) and \( b' \) coincide and we have \( h < h' \). The order on \( B_{U(\widehat{h}[v]^-)} \) is defined as follows. For any two elements \( h = h_{a_1} \cdots h_{a_l} \) and \( h' = h_{a_1} \cdots h_{a_i} \) of the basis \( B_{U(\widehat{h}[v]^-)} \) we write \( h < h' \) if one of the conditions holds:

1. \( (r_{1,i}, \ldots, r_{1,1}) < (\tilde{r}_{1,i}, \ldots, \tilde{r}_{1,1}) \)
2. \( (r_{1,i}, \ldots, r_{1,1}) = (\tilde{r}_{1,i}, \ldots, \tilde{r}_{1,1}) \) and \( (s_{1,i}, \ldots, s_{1,1}) < (\tilde{s}_{1,i}, \ldots, \tilde{s}_{1,1}) \).

The integer sequences in (1) and (2) have the same meaning as in (4.1) and we compare them as in (3.6).

In order to construct the quasi-particle basis of standard module \( L^\widehat{\Lambda} (\Lambda) \) we need the canonical isomorphism of \( d \)-graded vector spaces given by Lepowsky-Wilson in [25, 26],

\[
U(\widehat{h}[v]^-) \otimes L^\widehat{\Lambda} (\Lambda) \xrightarrow{\cong} L^\widehat{\Lambda} (\Lambda)
\]

where \( L^\widehat{\Lambda} (\Lambda) \) denotes the vacuum subspace of standard module \( L^\widehat{\Lambda} (\Lambda) \), i.e.

\[
L^\widehat{\Lambda} (\Lambda) \cong \{ v \in L^\widehat{\Lambda} (\Lambda) \mid \widehat{h}[v] \cdot v = 0 \}.
\]

Moreover, the construction relies the vertex operator formula

\[
\frac{1}{p!} (2z^\alpha(x)(z))^p = \frac{1}{q!} \epsilon_{C_2}(\alpha, -\alpha)^{-q} E^{-}(\alpha, z)(2z^\alpha(x)(z))^q E^{+}(-\alpha, z)e_{\mu_\alpha} z^{\mu_\alpha + \alpha(0)},
\]

where \( p, q \geq 0 \) are such that \( k = p + q \) and \( \mu_\alpha = \langle \alpha(0), \alpha(0) \rangle / 2 \) (cf. [29]). Formula (4.5) implies that the quasi-particle of color \( i \) and charge \( k \) acts on \( L^\widehat{\Lambda} (\Lambda) \) as an operator

\[
x_{k,\alpha_i}^\widehat{\Lambda}(m) = e_{\alpha_i}^k h \quad \text{for some} \quad h \in U(\widehat{h}[v]).
\]

Consequently, we do not need quasi-particles of the highest charge \( k \) to form the basis of the standard module. Let \( Q(0) = \{ \alpha(0) \mid \alpha \in Q \} \). We have

**Theorem 4.1.** For any highest weight \( \Lambda \) as in (1.1) the set

\[
B_{L^\widehat{\Lambda} (\Lambda)} = \left\{ e_{\mu} h b v_{\Lambda} \mid \mu \in Q(0), h \in B_{U(\widehat{h}[v]^-)}, b \in B_{\widetilde{L}^\widehat{\Lambda}(\Lambda)} \cap M'_{QP} \right\}
\]

forms a basis of the standard module \( L^\widehat{\Lambda} (\Lambda) \).
The proof that $B_{\hat{L}^\vee(\Lambda)}$ spans the standard module $\hat{L}^\vee(\Lambda)$ relies on the quasi-particle relations (4.5) and Lemmas 3.1 and 3.2. It can be carried out by arguing as in [5, Lemma 2.3], so we omit its details.

As for the linear independence, its proof employs the Georgiev-type projection

$$\pi_{\mathcal{R}_{\alpha_1}} : \left( L(\Lambda) \right)_{r_1} \rightarrow L(\Lambda_{\alpha_1})_{r_1(1)} \otimes \cdots \otimes L(\Lambda_{\alpha_1})_{r_1(k)},$$

with respect to the decomposition

$$\hat{L}^\vee(\Lambda) = \bigoplus_{r \in \mathbb{Z}} \hat{L}^\vee(\Lambda)_r \subset \hat{L}^\vee(\Lambda_{\alpha_1}) \otimes \cdots \otimes \hat{L}^\vee(\Lambda_{\alpha_1}),$$

where

$$\hat{L}^\vee(\Lambda)_r = \bigoplus_{r_2, \ldots, r_1 \in \mathbb{Z}} \hat{L}^\vee(\Lambda)_{k \Lambda | b_{(0)} + r_1 \alpha_1 + \cdots + r_2 \alpha_2 + r_1 \alpha_1}.$$

The projection $\pi_{\mathcal{R}_{\alpha_1}}$, as well as its generalization to the space of formal power series with coefficients in (4.6) (which we again denote by the same symbol), are uniquely determined by the dual-charge type $\mathcal{R}_{\alpha_1} = (r_1(1), r_1(2), \ldots, r_1(k))$ and the color-type $r_1 = \sum_{t=1}^{k} r_{1(t)}$ with respect to the color $i = 1$. Consider the projection of the vector

$$e_\mu (a_{(2)}, a_{(1)}) r_{1(1)} \ldots (a_{1(2)}, a_{1(1)}) r_{1(1)} \times x^\vee_{r_{1(1)}, \alpha_1} (m_{r_{1(1)}, 1}) \ldots x^\vee_{r_{1(k)}, \alpha_1} (m_{r_{1(k)}, 1}) v_{\Lambda},$$

such that its submonomial with respect to color $i = 1$,

$$b^\vee_{\alpha_1} = x^\vee_{r_{1(1)}, \alpha_1} (m_{r_{1(1)}, 1}) \ldots x^\vee_{r_{1(k)}, \alpha_1} (m_{r_{1(k)}, 1}),$$

is of dual-charge-type $\mathcal{R}_{\alpha_1}$. It coincides with the corresponding coefficient in

$$\pi_{\mathcal{R}_{\alpha_1}} e_\mu (a_{i(2)}, a_{i(1)}) r_{1(1)} \ldots (a_{1(2)}, a_{1(1)}) r_{1(1)} \times x^\vee_{r_{1(1)}, \alpha_1} (z_{r_{1(1)}, 1}) \ldots x^\vee_{r_{1(k)}, \alpha_1} (z_{r_{1(k)}, 1}) v_{\Lambda},$$

where $a_{i(-)} (z) = \sum_{m=0}^{\infty} a_{i(2m)} (m) z^{-m-1}$ for $i = 1, \ldots, l$. Suppose that there exists a finite linear combination

$$\sum c_{\mu, h, b} e_{\mu} h b v_{\Lambda} = 0 \quad (4.7)$$

of vectors $e_{\mu} h b v_{\Lambda} \in B_{\hat{L}^\vee(\Lambda)}$ of the same degree and $b_{(0)}$-weight such that all coefficients $c_{\mu, h, b} \in \mathbb{C}$ are nonzero. Choose any monomial vector in (4.7) of the form

$$e_{\mu} h b v_{\Lambda} \quad \text{with} \quad \text{chg} b(a_{1}) = r_1 \quad (4.8)$$

and the maximal charge-type $\mathcal{R}'_{\alpha_1}$. As before, in (4.8) we write $b(a_{1})$ for the submonomial of $b$ in color $i = 1$. Its dual-charge-type $\mathcal{R}_{\alpha_1} = (r_{1(1)}, \ldots, r_{1(p)})$ in color $i = 1$, where $p < k$ and $r_1 = r_{1(1)} + \cdots + r_{1(p)}$, uniquely determines the Georgiev-type projection $\pi_{\mathcal{R}_{\alpha_1}}$ (with $r_{1(t)} = 0$ for $t > p$). The projection $\pi_{\mathcal{R}_{\alpha_1}}$ ensures that only nonzero summands in

$$\sum c_{\mu, h, b} \pi_{\mathcal{R}_{\alpha_1}} e_{\mu} h b v_{\Lambda} = 0 \quad (4.9)$$

are those of the form as in (4.8). Indeed, from

$$e_{\alpha_1} (v_{\Lambda_{jk}} \otimes \cdots \otimes v_{\Lambda_{j_1}}) = e_{\alpha_1} v_{\Lambda_{jk}} \otimes \cdots \otimes e_{\alpha_1} v_{\Lambda_{j_1}},$$
and
\[ e_\mu h b v_\Lambda \in \bigcap_{r_{k-1} \in \mathbb{Z}} \bigcap_{r_k > 0} L^\Lambda (\Lambda_{j_1})_{r_1} \otimes \cdots \otimes L^\Lambda (\Lambda_{j_l})_{r_k}, \]
follows that \( \pi_{\mathcal{R}_{a_1}} \) annihilates all vectors \( e_\mu h b v_\Lambda \) in (4.7) such that \( \text{chg } b(\alpha_1) < r_1 \).

By using the identities (3.9)–(3.11) and arguing as in Section 3.4 we write (4.9) as
\[ \sum \epsilon_{\mu,h,b} \pi_{\mathcal{R}_{a_1}} e_\mu h b^+ v_{k_\Lambda 0} = 0, \]  
(4.10)
where the monomials \( b^+ \) are found as in (3.12) and (3.13). Hence we can now proceed with the iterated use of the constant terms of twisted \( \Delta \)-maps from [27] and the bijective map \( e_{a_1} \) as in [6, Section 5.4], until we reduce (4.10) to a linear combination of the vectors of the form \( \pi_{\mathcal{R}_{a_1}} e_\mu h b v_{k_\Lambda 0} \) which satisfy the initial and difference conditions and \( \text{chg } b(\alpha_1) = 0 \); see [6, Theorem 5.1] for more details. Next, we apply the same procedure for the remaining colors \( i = 2, \ldots, l \). Finally, we find that all coefficients \( \epsilon_{\mu,h,b} \) in (4.7) are zero, thus verifying the linear independence of the set \( B_{L^\Lambda(\Lambda)} \).

### 4.2. Vacuum spaces

Following J. Lepowsky and R. Wilson [24], for a quasi-particle monomial of charge-type \( \mathcal{R}' = (n_{i_1}, \ldots, n_{i_l}, 1) \) we define \( Z \)-operators
\[ Z_{\mathcal{R}'}(z_{i_1}, \ldots, z_{i_1}, 1) = E^-(\alpha_1, z_{i_1})^{n_{i_1}/k} \cdots E^-(\alpha_1, z_{1,1})^{n_{1,1}/k} x_{\mathcal{R}'}(z_{i_1}, \ldots, z_{1,1}) \times E^+(\alpha_i, z_{i_1})^{n_{i_1}/k} \cdots E^+(\alpha_1, z_{1,1})^{n_{1,1}/k}, \]
where
\[ x_{\mathcal{R}'}(z_{i_1}, \ldots, z_{1,1}) = x_{\mathcal{R}'}(z_{i_1}) \cdots x_{\mathcal{R}'}(z_{1,1}). \]

We shall write the above formal Laurent series as
\[ Z_{\mathcal{R}'}(z_{i_1}, \ldots, z_{1,1}) = \sum_{m_{i_1}, \ldots, m_{1,1} \in \mathbb{Z}} Z_{\mathcal{R}'}(m_{i_1}, \ldots, m_{1,1}) z_{1,1}^{-m_{i_1}} \cdots z_{1,1}^{-m_1}. \]

Its coefficients act on the vacuum space, i.e. we have
\[ Z_{\mathcal{R}'}(m_{i_1}, \ldots, m_{1,1}) : L^\Lambda(\Lambda) \hat{g}[v]^+ \rightarrow L^\Lambda(\Lambda) \hat{g}[v]^+. \]  
(4.11)

By (2.9) we also have the action of the Weyl group translations \( e_\alpha \) on the vacuum space,
\[ e_\alpha : L^\Lambda(\Lambda) \hat{g}[v]^+ \rightarrow L^\Lambda(\Lambda) \hat{g}[v]^+. \]

Consider the projection
\[ \pi_{\hat{g}[v]^+} : L^\Lambda(\Lambda) \rightarrow L^\Lambda(\Lambda) \hat{g}[v]^+. \]  
(4.12)
defined by the direct sum decomposition
\[ L^\Lambda(\Lambda) = L^\Lambda(\Lambda) \hat{g}[v]^+ \oplus \hat{g}[v] - U(\hat{g}[v]) \cdot L^\Lambda(\Lambda) \hat{g}[v]^+. \]

By (4.11) it follows that the action of projection (4.12) satisfies
\[ \pi_{\hat{g}[v]^+} : x_{\mathcal{R}'}(z_{i_1}, \ldots, z_{1,1}) v_\Lambda \mapsto Z_{\mathcal{R}'}(z_{i_1}, \ldots, z_{1,1}) v_\Lambda. \]

By employing the isomorphism (4.3) along with the projection (4.12), one easily obtains the following consequence of Theorem 4.1:
Theorem 4.2. For any highest weight $\Lambda$ as in (1.1) the set of vectors
\[ e_{\mu} \mathcal{Z}_{R'}(m_{i_1}^{(1)}, \ldots, m_{1,1})v_{\Lambda}, \]
where $\mu \in Q(0)$ and the charge-type $R'$ and the energy-type $(m_{i_1}^{(1)}, \ldots, m_{1,1})$ satisfy the initial and difference conditions for $B_{W,\mathcal{L}_{\psi}(\Lambda)}$, forms a basis of the vacuum space $L^{\mu}(\Lambda)\hat{h}[v]^+$. 

### 4.3. Parafermionic spaces

We now extend the definition of the parafermionic space of highest weight $k\Lambda_0$ from [29] (see also [5, 15] for the untwisted case) to any rectangular highest weight $(1.1)$. The parafermionic space is defined as the quotient
\[ L^{\mu}(\Lambda)_{kQ}^\hat{h}[v]^+ = L^{\mu}(\Lambda)\hat{h}[v]^+ / \text{span} \left\{ (\rho(\alpha) - 1) \cdot v \mid \alpha \in Q, v \in L^{\mu}(\Lambda)\hat{h}[v]^+ \right\}, \tag{4.13} \]
where the map $\rho$ is given by (2.5). Denote by $\pi_{kQ}^{\hat{h}[v]^+}$ the canonical projection
\[ \pi_{kQ}^{\hat{h}[v]^+} : L^{\mu}(\Lambda)\hat{h}[v]^+ \rightarrow L^{\mu}(\Lambda)_{kQ}^\hat{h}[v]^+. \]
We have
\[ L^{\mu}(\Lambda)_{kQ}^\hat{h}[v]^+ \cong \bigoplus_{\mu \in \Lambda + Q/kQ} L^{\mu}(\Lambda)\hat{h}[v]^+. \]
As in [29], we assign to quasi-particle monomials of charge-type $R' = (n_{i_1}^{(1)}, \ldots, n_{1,1})$ the $\Psi$-operators
\[ \Psi_{R'}(z_{i_1}^{(1)}, \ldots, z_{1,1}) = Z_{R'}(z_{i_1}^{(1)}, \ldots, z_{1,1}) z_{i_1}^{(1)} \cdots z_{1,1}^{-n_{i_1}^{(1)} \cdot a_{i_0}(0)/k} \cdots z_{1,1}^{-n_{1,1} \cdot a_{1}(0)/k} / \epsilon_{a_1} \cdots \epsilon_{a_1}, \]
where $\epsilon_{a_\alpha} := \epsilon_{C_0}(\alpha, \cdot)$ is the 2-cocycle on the weight lattice $P$. In parallel with [29, Lemma 12], we have
\[ \Psi_{n_1 \beta_1, \ldots, n_1 \beta_1}(z_1, \ldots, z_1) = \prod_{1 \leq p < q \leq r} \left( \left( z_s^{1/2} - z_p^{1/2} \right)^{n_{p} \epsilon_{p} \beta_{p} / k} \left( z_s^{1/2} + z_p^{1/2} \right)^{\left( n_{p} \epsilon_{p} \beta_{p} / k \right)} \right) \Psi_{n_1 \beta_1}(z_1) \cdots \Psi_{n_1 \beta_1}(z_1). \tag{4.14} \]
Since the $\Psi$-operators commute both with the action of the Heisenberg subalgebra $\mathcal{H}^\dagger \mathbb{Z}$ and with the action of $\rho(kQ)$, their coefficients $\psi_{R'}(m_{i_1}^{(1)}, \ldots, m_{1,1})$, given by
\[ \psi_{R'}(z_{i_1}^{(1)}, \ldots, z_{1,1}) = \sum_{m_{i_1}^{(1)}, \ldots, m_{1,1}} \psi_{R'}(m_{i_1}^{(1)}, \ldots, m_{1,1}) z_{i_1}^{(1)} \cdots z_{1,1}^{-m_{i_1}^{(1)} \cdot n_{i_1}^{(1)} / k} \cdots z_{1,1}^{-m_{1,1} \cdot n_{1,1} / k}, \]
are induced operators on the quotient (4.13). The summation on the right-hand side goes over all sequences $(m_{i_1}^{(1)}, \ldots, m_{1,1})$ such that $m_{p,i} \in \frac{n_{p,j}^{(1)} \cdot a_{i_0}(0) / k + \mu_i \mathbb{Z}}{k}$ for all $1 \leq p \leq i_1$ when $\psi_{R'}(m_{i_1}^{(1)}, \ldots, m_{1,1})$ acts on the $\mu$-weight subspace of $L^{\mu}(\Lambda)\hat{h}[v]^+$. The following relations, which connect the coefficients of $\Psi$-operators and $\mathcal{Z}$-operators, follow from [29, Lemma 10] (see also [5, Lemma 3.2]):
\[ \mathcal{Z}_{R'}(m_{i_1}^{(1)}, \ldots, m_{1,1}) \bigg|_{L^{\mu}(\Lambda)\hat{h}[v]^+} = \prod_{i_1} \prod_{p=1}^{j_{i_1}^{(1)}} \left( \epsilon_{C_0}(1 / k n_{p,i}^{(1)} \cdot a_{i_0}(0) / k, \mu) \right)^{-1} \times \psi_{R'}(m_{i_1}^{(1)} + 1 / k n_{i_1}^{(1)} \cdot a_{i_0}(0) / k, \ldots, m_{1,1} + 1 / k n_{1,1} \cdot a_{1}(0) / k, \mu) \bigg|_{L^{\mu}(\Lambda)\hat{h}[v]^+}. \]
Finally, Theorem 4.2 implies...
Theorem 5.1. For any highest weight \( \Lambda \) as in (1.1) the set of vectors
\[
\pi_{kQ}^+ \mathcal{Z}_\mathcal{R}'(m_{i_1}^{(i)}, \ldots, m_{i_1}) v_\Lambda
\]
\[
= \psi_{\mathcal{R}'}(m_{i_1}^{(i)} + j, \alpha_{(0)}, \Lambda), \ldots, m_{i_1} + \frac{1}{k} n_{i_1}(\alpha_{(0)}, \Lambda)) v_\Lambda,
\]
where the charge-type \( \mathcal{R}' \) and the energy-type \( (m_{i_1}^{(i)}, \ldots, m_{i_1}) \) satisfy the initial and difference conditions for \( B_{W_L^\nu(\Lambda)} \), forms a basis of the parafermionic space \( L^\nu(\Lambda)_{kQ}^+ \).

5. Character formulae and combinatorial identities

In this section, we use our main results, Theorems 3.3, 4.1, 4.2, and 4.3 to compute the character formulae of the corresponding bases. Furthermore, as an application, in the case of the principal subspace of the generalized Verma module we obtain two new families of combinatorial identities. Throughout the entire section, \( \Lambda \) denotes an arbitrary rectangular highest weight, i.e. the weight of the form (1.1).

5.1. Principal subspaces

Let \( V = N^\nu(\Lambda), L^\nu(\Lambda) \). We define the character of the principal subspace \( W_T^T \) by
\[
\text{ch } W_T^T = \sum_{m,r_1,\ldots,r_l \geq 0} \dim W_T^{V(m,r_1,\ldots,r_l)} q^m y_1^{r_1} \cdots y_l^{r_l},
\]
where \( W_T^{V(m,r_1,\ldots,r_l)} \) is the weight subspace of the Cartan subalgebra \( h_{(0)} \oplus \mathbb{C}c \oplus \mathbb{C}d \) of weight \(-m\delta + r_1\alpha_1 + \cdots + r_l\alpha_l\).

For the dual-charge-type of quasi-particle monomial as in (3.5) let \( p_i^{(s)} \) be the number of quasi-particles of color \( i \) and charge \( s = 1, \ldots, t \) which appear in the monomial. Furthermore, we organize the numbers \( p_i^{(s)} \) into the sequence \( P = (P_1, \ldots, P_l) \) such that \( P_i = (p_1^{(1)}, \ldots, p_i^{(t)}) \). Also, we write
\[
(a; q)_r = \prod_{i=1}^r (1 - a q^{i-1}) \quad \text{for } r \geq 0 \quad \text{and} \quad (a; q)_\infty = \prod_{i \geq 1} (1 - a q^{i-1}).
\]
In addition, to express the character formulae we shall need the following notation
\[
\frac{1}{(q^{\mu_i}; q^{\mu_i^*})_r} = \sum_{s \geq 0} p_r(s) q^{\mu_i^*},
\]
where \( p_r(s) \) is the number of partitions of \( s \) with most \( r \) parts. By arguing as in [14, Sect.5] and using the difference conditions (3.7) and (3.8), which determine the quasi-particle basis of the principal subspace of standard module via Theorem 3.3, we find

Theorem 5.1. For the affine Lie algebras \( A_{2l-1}^{(2)} \) and \( D_{2l+1}^{(2)} \) we have
\[
\text{ch } W_T^T_{\mathcal{L}^\nu(\Lambda)} = \sum_{P} q^{l \sum_{s=1}^l \sum_{m,n=1}^k (\alpha_{(0)}, \alpha_{(0)} \min (m,n)|p_r^{(m)} p_r^{(n)} + \tilde{p}_j \prod_{i=1}^l y_i^{n_i}},
\]
where \( \tilde{p}_j = \sum_{s=k_0+1}^k (s - k_0) \mu_j p_i^{(s)} \) and \( n_i = \sum_{s=1}^k s \mu_i^{(s)} \). The sum goes over all finite sequences \( P = (P_1, \ldots, P_l) \) of \( lk \) nonnegative integers.

The character formula for the principal subspace of the generalized Verma module is obtained analogously:
Theorem 5.2. For the affine Lie algebras $A^{(2)}_{2l-1}$ and $D^{(2)}_{l+1}$ we have

$$
ch W^T_{\mathfrak{N}^\Lambda}(\Lambda) = \sum_{\mathcal{P}} q^{\frac{1}{2} \sum_{i=1}^l \sum_{n=1}^\infty \langle \alpha_i, r(0) \rangle \min(m, n) p_i^{(m)} p_r^{(n)}} \prod_{i=1}^l \prod_{s=1}^\infty (q^{\mu_i}; q^{\mu_i})_s^{(s)} \prod_{i=1}^l y_i^{n_i},
$$

where $n_i = \sum_{s=1}^l s p_i^{(s)}$, and the sum goes over all finite sequences $\mathcal{P} = (P_1, \ldots, P_l)$ of nonnegative integers with finite support.

Due to the vector space isomorphism (2.13), we can compute the character of $W^T_{\mathfrak{N}^\Lambda}(\Lambda)$ from the Poincaré–Birkhoff–Witt basis of $U(\widehat{\mathfrak{n}}_{+}^{<0})$ as well, thus getting

$$
ch W^T_{\mathfrak{N}^\Lambda}(\Lambda) = \frac{1}{\prod_{\alpha \in R_+} (\alpha; q^{\langle \alpha, \alpha(0) \rangle/2})_\infty}.
$$

On the right-hand side we used the notation

$$(\alpha; q)_\infty = (q^{a_1}; q^{a_2})_{\infty},$$

for any positive root $\alpha \in R_+$ such that $\alpha(0) = a_1 \alpha_1(0) + \cdots + a_l \alpha_l(0)$. Theorem 5.2 and the character formula (5.1) imply the following generalization of the identities from [3]:

Corollary 5.3. For the affine Lie algebras $A^{(2)}_{2l-1}$ and $D^{(2)}_{l+1}$ we have

$$
\frac{1}{\prod_{\alpha \in R_+} (\alpha; q^{\langle \alpha, \alpha(0) \rangle/2})_\infty} = \sum_{\mathcal{P}} q^{\frac{1}{2} \sum_{i=1}^l \sum_{n=1}^\infty \langle \alpha_i, r(0) \rangle \min(m, n) p_i^{(m)} p_r^{(n)}} \prod_{i=1}^l \prod_{s=1}^\infty (q^{\mu_i}; q^{\mu_i})_s^{(s)} \prod_{i=1}^l y_i^{n_i},
$$

where $n_i = \sum_{s=1}^l s p_i^{(s)}$, and the sum goes over all finite sequences $\mathcal{P} = (P_1, \ldots, P_l)$ of nonnegative integers with finite support.

5.2. Standard modules

Let $\mathcal{R} = (r_1^{(1)}, \ldots, r_l^{(k-1)})$ be the dual-charge-type of a quasi-particle monomial $b \in B_L^{\mathfrak{N}^\Lambda}(\Lambda)$. Consider the sequences $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k-1)})$ given by

$$
\mathcal{P}_i = (r_i^{(1)} - r_i^{(2)}, r_i^{(2)} - r_i^{(3)}, \ldots, r_i^{(k-2)} - r_i^{(k-1)}, r_i^{(k-1)}) \quad \text{for} \quad i = 1, \ldots, l.
$$

Note that the number $p_i^{(m)}$ equals the number of quasi-particles of color $i$ and charge $m$ in the quasi-particle monomial $b$. We define the character of $L^\mathfrak{N}(\Lambda)$ by

$$
ch L^\mathfrak{N}(\Lambda) = \sum_{m, r_1, \ldots, r_l \geq 0} \dim L^\mathfrak{N}(\Lambda)_{(m, r_1, \ldots, r_l)} q^m y_1^{r_1} \cdots y_l^{r_l},
$$

where $L^\mathfrak{N}(\Lambda)_{(m, r_1, \ldots, r_l)}$ is the weight subspace of weight $-m\delta + r_1\alpha_1 + \cdots + r_l\alpha_l$, with respect to the Cartan subalgebra $\mathfrak{h}(0) \oplus \mathbb{C}c \oplus \mathbb{C}d$. In a similar way, we can define the character of the vacuum space $L^\mathfrak{N}(\Lambda)_{\mathfrak{h}[v]^+}$. By the isomorphism in (4.3) we have

$$
ch L^\mathfrak{N}(\Lambda) = \frac{1}{\prod_{i=1}^l (q^{\mu_i}; q^{\mu_i})_\infty} ch \tilde{L}^\mathfrak{N}(\Lambda)_{\mathfrak{h}[v]^+}.
$$

Hence Theorems 4.1 and 4.2 imply
Theorem 5.4. For the affine Lie algebras $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$ we have

$$
\text{ch} \hat{L}^\mu(\Lambda) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{s \in \mathbb{Q}} q^{\frac{1}{\lambda_1}s^2 \lambda_1 + \Lambda + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)),
$$

where the first sum goes over all finite sequences $s = (s_1, \ldots, s_3)$ of $(k-1)$ nonnegative integers, $\hat{p}_j = \sum_{s=s_0+1}^{k-1} (s - s_0) \mu_j p_j(\alpha_1)$ and $h_{\lambda_1}$ are the fundamental coweights of $\mathfrak{g}(0)$.

As an illustration of the theorem, we now give some examples of character formulae.

Example 5.5. For the affine Lie algebra $A_{2}^{(2)}$ we have

$$
\text{ch} \hat{L}^\mu(\Lambda_1) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{\alpha \in \mathbb{Q}} q^{\alpha_1^2 \lambda_1 + \Lambda_1} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)),
$$

$$
\text{ch} \hat{L}^\mu(2\Lambda_1) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{(\lambda_1, \lambda_2, \lambda_3)} q^{\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - p_1(\lambda_1) + 2p_2(\lambda_2) + p_3(\lambda_3))} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)),
$$

$$
\text{ch} \hat{L}^\mu(\Lambda_0 + \Lambda_1) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{\alpha \in \mathbb{Q}} q^{\alpha_1^2 \lambda_1 + \Lambda_0 + \Lambda_1} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)),
$$

Example 5.6. For the affine Lie algebra $D_{3}^{(2)}$ we have

$$
\text{ch} \hat{L}^\mu(\Lambda_2) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{\alpha \in \mathbb{Q}} q^{\alpha_1^2 \lambda_1 + \Lambda_2} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \Lambda_2),
$$

$$
\text{ch} \hat{L}^\mu(2\Lambda_2) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{(\lambda_1, \lambda_2)} q^{\frac{1}{2}(2p_1(\lambda_1)^2 + 2p_2(\lambda_2)^2 + \lambda_1^2 - p_1(\lambda_1))} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)),
$$

$$
\text{ch} \hat{L}^\mu(\Lambda_0 + \Lambda_2) = \frac{1}{\prod_{l=1}^3(q^{\mu_1}; q^{\mu_2})} \sum_{\alpha \in \mathbb{Q}} q^{\alpha_1^2 \lambda_1 + \Lambda_0 + \Lambda_2} \prod_{l=1}^3 h_{\lambda_1}(\lambda_1 + \sum_{l=1}^3 \lambda_1 p_l(\alpha_1)).
5.3. Parafermionic spaces

As in [29], to define the character of the parafermionic space we use the parafermionic grading operator \( D \) on \( L_+^\Lambda (\Lambda) \) given by

\[
D = -d - D \hat{h}^{[v]} + \quad \text{and} \quad \left. D \hat{h}^{[v]} \right|_{L_+^\Lambda (\Lambda) \hat{h}^{[v]}} = \frac{\langle \mu_{(0)} \mu_{(0)} \rangle}{2k} - \langle \Lambda, \Lambda \rangle \frac{1}{2k}.
\] (5.3)

Note that by (5.3) the conformal energy of the basis vector

\[
\psi_{R'}(E) \nu_{\Lambda} = \psi_{\nu ((m_{1,1}^1 \cdots m_{l,1}^1) \cdots m_{1,1})} \nu_{\Lambda},
\] (5.4)

which corresponds to the quasi-particle monomial

\[
x_{n_{r_1}^1}^1 \alpha_1 (m_{r_1}^1 \cdots x_{n_{r_l}^l}^l \alpha_1 (m_{r_l}) \cdots x_{n_{r_1}^1}^1 \alpha_1 (m_{r_1}),
\]

is equal to

\[
- \sum_{i=1}^{l} \sum_{t=1}^{r_i} m_{u,i} \frac{k_j}{k} \sum_{t=1}^{k-1} t P^{(t)}_j
\]

\[
- \sum_{i=1}^{l} \sum_{t=1}^{r_i} \left( n_{u,j} \alpha_{t(0)} \sum_{s=1}^{u-1} n_{s,j} \alpha_{t(s)} + \sum_{p=1}^{i-1} \sum_{t=1}^{r_p} n_{s,p} \alpha_{t(0)} \right),
\]

where the second summand is due to the identity

\[
- \frac{1}{k} \sum_{t=1}^{k-1} t P^{(t)}_j \alpha_{t(0)} = - \frac{k_j}{k} \mu_j \sum_{t=1}^{k-1} t P^{(t)}_j.
\]

Define the character of the parafermionic space \( L_+^\Lambda (\Lambda) \) by

\[
\text{ch} \ L_+^\Lambda (\Lambda) \hat{h}^{[v]} = \sum_{m_{r_1}, \ldots, r_l} \dim \ L_+^\Lambda (\Lambda) \hat{h}^{[v]} (m_{r_1}, \ldots, r_l) q^m,
\]

where \( L_+^\Lambda (\Lambda) \hat{h}^{[v]} (m_{r_1}, \ldots, r_l) \) is the weight space spanned by monomial vectors (5.4) of conformal energy \(-m\) and color-type \((r_1, \ldots, r_l)\).

Recall the notation from (5.2) and introduce the following expressions:

\[
D \varphi(q) = \frac{1}{\prod_{l=1}^{l} \prod_{t=1}^{k-1} (q^{\mu_t}; q^{\mu_t})_{P_{l(t)}}}, \quad B \varphi(q) = q^{\mu_{l(t)}} \sum_{i=k_0+1}^{k_0} (l-k_0) P_{l(t)} \varphi(q)^{l(t)} q^{\mu_{l(t)}} \sum_{l=1}^{k_1} q^{l(t)}
\]

\[
G \varphi(q) = q^{\frac{1}{2} \sum_{i=1}^{l} \sum_{m,n=1}^{k_0+1} \langle \alpha_{t(0)} \alpha_{t(0)} \rangle (\min[m,n]-\frac{m+n}{2}) \langle \mu_{l(t)} \rangle \langle \mu_{l(t)} \rangle}.
\]

From the above considerations and Theorem 5.3 we get

**Theorem 5.7.** For the affine Lie algebras \( A_{2l-1}^{(2)} \) and \( D_{l+1}^{(2)} \) we have

\[
\text{ch} \ L_+^\Lambda (\Lambda) \hat{h}^{[v]} = \sum_{\mathcal{P}} D \varphi(q) G \varphi(q) B \varphi(q),
\] (5.5)

where the sum goes over all finite sequences \( \mathcal{P} = (P_l^{(1)}, \ldots, P_l^{(l-1)}) \) of \((k-1)\) nonnegative integers such that \( P_l = (P_l^{(1)}, \ldots, P_l^{(k-1)}) \).
It is worth noting that the character formula (5.5) coincides with [17, Eq. (14)]. At the end, let us give some examples of the parafermionic character formulae.

**Example 5.8.** For the affine Lie algebra $A_5^{(2)}$ we have

$$\text{ch} L^\mathcal{V}(2\Lambda_1) \hat{\mathcal{b}}[v]^+ = \sum_{P=(P_1,P_2,P_3)} \frac{q^{\frac{1}{2}(p_1^{(1)} - p_1^{(2)})^2 - p_2^{(1)} p_2^{(2)} + p_3^{(1)} p_3^{(2)}}}{\prod_{i=1}^3 (q^{\mu_i}; q^{\mu_i})^{p_3^{(1)}}} ,$$

$$\text{ch} L^\mathcal{V}(\Lambda_0 + \Lambda_1) \hat{\mathcal{b}}[v]^+ = \sum_{P=(P_1,P_2,P_3)} \frac{q^{\frac{1}{2}(p_1^{(1)} - p_1^{(2)})^2 - p_2^{(1)} p_2^{(2)} + p_3^{(1)} p_3^{(2)} + p_3^{(2)} - \frac{1}{2} p_3^{(1)}}}{\prod_{i=1}^3 (q^{\mu_i}; q^{\mu_i})^{p_3^{(1)}}} .$$

**Example 5.9.** For the affine Lie algebra $D_3^{(2)}$ we have

$$\text{ch} L^\mathcal{V}(2\Lambda_2) \hat{\mathcal{b}}[v]^+ = \sum_{P=(P_1,P_2)} \frac{q^{\frac{1}{2}(2p_1^{(1)} - 2p_1^{(2)} + p_2^{(1)} + p_2^{(2)})}}{\prod_{i=1}^2 (q^{\mu_i}; q^{\mu_i})^{p_2^{(1)}}} ,$$

$$\text{ch} L^\mathcal{V}(\Lambda_0 + \Lambda_2) \hat{\mathcal{b}}[v]^+ = \sum_{P=(P_1,P_2)} \frac{q^{\frac{1}{2}(2p_1^{(1)} - 2p_1^{(2)} + p_2^{(1)} + p_2^{(2)} - \frac{1}{2} p_2^{(1)})}}{\prod_{i=1}^2 (q^{\mu_i}; q^{\mu_i})^{p_2^{(1)}}} .$$

**Acknowledgments**

The authors are grateful to Mirko Primc for numerous helpful discussions. The authors would also like to thank Ole Warnaar for bringing to their attention some useful references on combinatorial identities.

**Data availability statement**

Data sharing not applicable to this article.

**Disclosure statement**

The authors report there are no competing interests to declare.

**Ethical statement**

Not applicable to this article.

**Funding**

This work has been supported in part by Croatian Science Foundation under the project UIP-2019-04-8488. The first author is partially supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

**References**

[1] Bakalov, B. N., Kac, V. G. (2004). Twisted modules over lattice vertex algebras. In Doebner, H.-D., Dobrev, V. K., eds. *Lie theory and its Applications in Physics V*. River Edge, NJ: World Scientific Publishing Co., Inc., pp. 3–26.
[29] Okado, M., Takenaka, R. (2022). Parafermionic bases of standard modules for twisted affine Lie algebras of type $A_{2l-1}^{(2)}, D_{l}^{(2)}, E_{6}^{(2)}$, and $D_{4}^{(3)}$. *Algebras Represent. Theory*. DOI: 10.1007/s10468-022-10145-2

[30] Penn, M., Sadowski, C. (2018). Vertex-algebraic structure of principal subspaces of the basic modules for twisted Affine Kac-Moody Lie algebras of type $A_{2n-1}^{(2)}, D_{n}^{(3)}, E_{6}^{(2)}$. *J. Algebra* 496:242–291. DOI: 10.1016/J.JALGEBRA.2017.10.022

[31] Primc, M. (1994). Vertex operator construction of standard modules for $A_{n}^{(1)}$. *Pacific J. Math*. 162:143–187. DOI: 10.2140/pjm.1994.162.143

[32] Takenaka, R. (2022). Vertex algebraic construction of modules for twisted affine Lie algebras of type $A_{2l}^{(2)}$. arXiv preprint 2205.05271.