A non-renormalizable neutrino mass model with $S_3 \otimes Z_2$ symmetry

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(Dated: October 26, 2021)

The lepton sector is studied within a flavored non-renormalizable model where the $S_3 \otimes Z_2$ flavor symmetry drives the Yukawa couplings. In this framework, the effective neutrino mass, that comes from the type II see-saw mechanism, as well as the charged lepton mass matrices are hierarchical and these have (under a benchmark in the charged sector) a kind of Fritzsch textures that accommodate the mixing angles in good agreement with the last experimental data. The model favors the normal hierarchy, this also predicts consistent values for the CP-violating phase and the $|m_{ee}|$ effective Majorana neutrino mass rate. Along with this, the branching ratio for the lepton flavor violation process, $\mu \rightarrow e\gamma$, is well below the current bound.

I. INTRODUCTION

In spite of the fact that Standard Model (SM) works almost perfectly, it fails to explain the neutrino experimental data, dark matter, baryon asymmetry of the universe and so forth [1]. Speaking about the mixings, the lepton sector exhibits a peculiar pattern which is totally different to the quark sector where the mixing matrix is almost diagonal and this puzzle remains unsolved.

In this line of thought, hierarchical quark mass matrices as the nearest neighbor interaction (NNI) textures [2–5] and those that possess the generalized Fritzsch textures [6], fit quite well the CKM matrix [7, 8]. In the lepton sector, according to the experimental data, the PMNS matrix [9, 10] has large values in its entries which can be understood by the presence of a symmetry behind the neutrino mass matrix. Currently, we can find in the literature elegant proposals (and their respective breaking) as the $\mu \leftrightarrow \tau$ symmetry [11–18], $\mu \leftrightarrow \tau$ reflection symmetry [19–26], Tri-Bimaximal [27, 32], Cobimaximal mixing matrices [33, 43]. Moreover, hierarchical mass matrices as the Fritzsch [44] and the generalized Fritzsch textures [6] also accommodate quite well the PMNS mixing matrix.

From the model building point of view, the flavor symmetries [45–48] have been useful to get desirable textures in the fermion mass matrices, and therefore, the well known mixing patterns. For example, the $S_3$ non-abelian group that has been explored exhaustively in different frameworks [49, 50]. In the mentioned literature there are few models
where the Fritzsch textures have been implemented. Hence, the main purpose that we pursue is to realize those textures by means the $S_3$ flavor symmetry, however, we obtain a modified Fritzsch textures which are different to previous studies.

Due to the last neutrino oscillations data seem to favor the normal hierarchy \[100\], in this paper, we construct a non-renormalizable lepton model in the type II see-saw scenario where the $S_3 \otimes Z_2$ flavor symmetry drives the Yukawa couplings. We stress that the scalar sector of the mentioned model keeps intact so that flavons are included to generate the mixings. In this work, the effective neutrino as well as the charged lepton mass matrices are hierarchical and these have (under a benchmark in the charged sector) a kind of Fritzsch textures that accommodate the mixing angles in good agreement with the last experimental data. The model predicts consistent values for the CP-violating phase and the $|m_{ee}|$ effective Majorana neutrino mass rate. Along with this, the branching ratio for the lepton flavor violation process, $\mu \rightarrow e\gamma$, is well below the current bound.

The plan of the paper is as follows: the framework, the matter content of the model and the fermion mass matrices are described in detail in section II; in the section III, the PMNS mixing matrix is obtained and relevant features are remarked. An analytical study is carried out on the mixing angles to find the parameter space that accommodates the observables, this together with a numerical study in section IV. In section V, we give some model predictions and relevant conclusions are shown in section VI.

II. THE FRAMEWORK

The current framework is a scalar extension of the SM so that the usual matter content under the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is considered. Explicitly, the fields are

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (3, 2, 1), \quad d_R \sim (3, 1, -2) \quad u_R \sim (3, 1, 4)$$

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \sim (1, 2, -1), \quad e_R \sim (1, 1, -2). \quad (1)$$

Additionally, in the scalar sector we have the following fields

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \sim (1, 2, 1), \quad \Delta = \begin{pmatrix} \Delta^+ \\ \Delta^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Delta^+ \\ \Delta^0 \end{pmatrix} \sim (1, 3, 2). \quad (2)$$

Having introduced the matter content, the Yukawa mass term is given by

$$\mathcal{L} = \mathcal{L}_{SM} - \frac{1}{2} Y^\nu \bar{L}(i\sigma_2)\Delta (L)^c - V(H, \Delta) + h.c. \quad (3)$$

Although the quark and scalar fields have been mentioned, the quark mixings and the scalar potential analysis will leave out in this work. We have to point out that scalar potential analysis is crucial to get a viable model but the full study is a working progress.

Speaking about the flavor symmetry, we will use the $S_3$ due to the three dimensional real representation can be decomposed as: $3_S = 2 \oplus 1_S$ or $3_A = 2 \oplus 1_A$. This structure seems to work quite well for obtaining hierarchical mass matrices. Along with this, the $Z_2$ discrete symmetry can be used to forbid some Yukawa couplings, in our work this is needed to prohibit the renormalizable terms.
A. The model

As we already commented, in the present model, the scalar sector contains one Higgs doublet \((H)\) and one triplet \((\Delta)\) so that some flavons will be added to the matter content in order to generate the mass textures that provide the mixings. Then, the matter fields transform in a non trivial way. Hence, the assignation under the \(S_3 \otimes Z_2\) is shown in the following table.

| Matter | \(L_1\) | \(L_3\) | \(e_{1R}\) | \(e_{3R}\) | \(\phi_1\) | \(\phi_3\) | \(\Delta\) | \(H\) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(S_3\) | 2      | 1\(_S\) | 2      | 1\(_S\) | 2      | 1\(_S\) | 1\(_S\) | 1\(_S\) |
| \(Z_2\) | 1      | 1      | 1      | -1     | -1     | -1     | -1     | -1     |

**TABLE I. Matter content for the lepton sector.** \(I = 1, 2\)

As one can notice, due to the \(Z_2\) symmetry there are no-renormalizable Yukawa mass term, then, at the next leading order in the cutoff scale we have

\[
-\mathcal{L}_Y = \frac{y_1^v}{\Lambda} \left[ L_1 H (\phi_1 e_{2R} + \phi_2 e_{1R}) + L_2 H (\phi_1 e_{1R} - \phi_2 e_{2R}) \right] + \frac{y_3^v}{\Lambda} \left[ L_1 H \phi_3 e_{1R} + L_2 H \phi_3 e_{2R} \right] + \frac{y_5^v}{\Lambda} \left[ L_1 H \phi_1 + L_2 H \phi_2 \right] e_{3R}
\]

\[
+ \frac{y_2^v}{\Lambda} \left[ L_1 \phi_1 e_{1R} + H \phi_2 e_{2R} \right] + \frac{y_3^v}{\Lambda} \bar{L}_3 H \phi_3 e_{3R} + \frac{y_5^v}{\Lambda} \left[ \bar{L}_1 \Delta (\phi_1 L_2 + \phi_2 L_1) + \bar{L}_2 \Delta (\phi_1 L_1 - \phi_2 L_2) \right] 
\]

\[
+ \frac{y_2^v}{\Lambda} \left[ \bar{L}_1 \Delta \phi_3 L_1 + \bar{L}_2 \Delta \phi_3 L_2 \right] + \frac{y_3^v}{\Lambda} \left[ \bar{L}_1 \Delta \phi_3 + \bar{L}_2 \Delta \phi_2 \right] L_3 + \frac{y_5^v}{\Lambda} \bar{L}_3 \Delta \phi_3 L_3 + h.c. \quad (4)
\]

As result of this, the lepton mass matrices are given as

\[
M_e = \begin{pmatrix}
    a_e + b_e' & b_e & c_e \\
    b_e & a_e - b_e' & c_e' \\
    f_e & f_e' & g_e
\end{pmatrix}, \quad M_\nu = \begin{pmatrix}
    a_\nu + b_\nu' & b_\nu & c_\nu \\
    b_\nu & a_\nu - b_\nu' & c_\nu' \\
    c_\nu & c_\nu' & g_\nu
\end{pmatrix} \quad (5)
\]

with

\[
a_e = y_1^v \frac{\langle \phi_1 \rangle}{\Lambda}, \quad b_e' = y_1^v \frac{\langle \phi_2 \rangle}{\Lambda}, \quad b_e = y_1^v \frac{\langle \phi_1 \rangle}{\Lambda}, \quad c_e = y_3^v \frac{\langle \phi_1 \rangle}{\Lambda};
\]

\[
c_e' = y_3^v \frac{\langle \phi_3 \rangle}{\Lambda}, \quad f_e = y_4^v \frac{\langle \phi_1 \rangle}{\Lambda}, \quad f_e' = y_4^v \frac{\langle \phi_2 \rangle}{\Lambda}, \quad g_e = y_5^v \frac{\langle \phi_3 \rangle}{\Lambda};
\]

\[
a_\nu = y_2^v \Delta \frac{\langle \phi_3 \rangle}{\Lambda}, \quad b_\nu' = y_1^v \Delta \frac{\langle \phi_2 \rangle}{\Lambda}, \quad b_\nu = y_1^v \Delta \frac{\langle \phi_1 \rangle}{\Lambda};
\]

\[
c_\nu = y_3^v \Delta \frac{\langle \phi_3 \rangle}{\Lambda}, \quad c_\nu' = y_3^v \Delta \frac{\langle \phi_2 \rangle}{\Lambda}, \quad g_\nu = y_5^v \Delta \frac{\langle \phi_3 \rangle}{\Lambda}. \quad (6)
\]

Here, \(v\) and \(v_\Delta\) stand for the vacuum expectation values (vev’s) of the Higgs doublet and triplet, respectively. In order to reduce the free parameters in the lepton mass matrices, we assume the following vev’s pattern for the flavon doublet and singlet of \(S_3\), respectively: \(\langle \phi \rangle = v_\phi (1,0)\) and \(\langle \phi_3 \rangle = v_{\phi_3}\). At the same time, we set the magnitudes of the vev’s as follows: \(v_\phi \sim \lambda \Lambda\) and \(v_{\phi_3} \sim \lambda \Lambda\) where \(\lambda = 0.225\) is the Wolfenstein parameter. Before finishing this section, we would like to remark that the flavor symmetry is broken by the vev’s of the flavons and the cutoff \(\Lambda\) scale satisfies the hierarchy \(\Lambda \gg v \gg v_\Delta\). Therefore, the main role that the flavons play is to provide the mixings as was already commented.

\[\text{[1]}\] In fact, one might consider two different vev’s alignments: (a) \(\langle \phi \rangle = v_\phi (0,1)\) and \(\langle \phi_3 \rangle = v_{\phi_3}\) but this does not provide the right mixings; (b) \(\langle \phi \rangle = v_\phi (1,1)\) and \(\langle \phi_3 \rangle = v_{\phi_3}\), in this case, the free parameters increase.
III. PMNS MIXING MATRIX

Due to the alignment, the mass matrices read as

\[
M_e = \begin{pmatrix}
  a_e & b_e & c_e \\
  b_e & a_e & 0 \\
  f_e & 0 & g_e
\end{pmatrix}, \quad M_\nu = \begin{pmatrix}
  a_\nu & b_\nu & c_\nu \\
  b_\nu & a_\nu & 0 \\
  c_\nu & 0 & g_\nu
\end{pmatrix}.
\] (7)

As one can notice, if \(a_e(a_\nu)\) was zero, the charged lepton (neutrino) mass matrix would possess implicitly the NNI (Fritzsch) textures. In general, the charged lepton mass matrix has five complex free parameters, then, in order to reduce a little bit more the free parameters we will adopt the benchmark \(c_e \approx f_e\). As a result, the lepton mass matrices have the Fritzsch textures but the entry \(a_{(\nu,e)}\) will modify slightly those textures, as we will show next.

The mixing matrices that take place in the PMNS matrix are obtained as follows: \(M_e\) and \(M_\nu\) are diagonalized respectively by \(U_{e(L,R)}\) and \(U_\nu\) such that \(U_{e(L,R)}^\dagger M_e U_{e(R)} = \hat{M}_e\) and \(U_\nu^\dagger M_\nu U_\nu^* = \hat{M}_\nu\) with \(\hat{M}_{(e,\nu)} = \text{Diag.}(m_{(e,1)}, m_{(\mu,2)}, m_{(\tau,3)})\) being the physical lepton masses. Then, we make the following rotation \(U_{e(L,R)} = S_{12} u_{e(L,R)}\) and \(U_\nu = S_{12} u_\nu\) so that one obtains \(u_{e(L)}^\dagger m_e u_{e(R)} = \hat{M}_e\) and \(u_\nu^\dagger m_\nu u_\nu^* = \hat{M}_\nu\) where \(m_{(e,\nu)}\) and \(S_{12}\) are given respectively as

\[
m_\ell = \begin{pmatrix}
  a_\ell & b_\ell & 0 \\
  b_\ell & a_\ell & c_\ell \\
  0 & c_\ell & g_\ell
\end{pmatrix}, \quad S_{12} = \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix},
\] (9)

where \(\ell = \nu, e\).

We can observe that both mass matrices can be written as

\[
m_\ell = a_\ell 1_{3 \times 3} + \begin{pmatrix}
  0 & b_\ell & 0 \\
  b_\ell & 0 & c_\ell \\
  0 & c_\ell & g_\ell - a_\ell
\end{pmatrix}.
\] (10)

As one can realize, the second mass matrix has the Fritzsch texture but the there is a shift due to the \(a_\ell\) parameter. Consequently, we expect a deviation to the Fritzsch prediction on the mixings. Let us diagonalize the mass matrix, \(m_\ell\), where the CP violating phases are factorized as \(m_\ell = P_\ell \tilde{m}_\ell P_\ell\). Explicitly, we obtain

\[
P_\ell = \begin{pmatrix}
  e^{i\eta_1} & 0 & 0 \\
  0 & e^{i\eta_2} & 0 \\
  0 & 0 & e^{i\eta_3}
\end{pmatrix}, \quad \tilde{m}_\ell = \begin{pmatrix}
  |a_\ell| & |b_\ell| & 0 \\
  |b_\ell| & |a_\ell| & |c_\ell| \\
  0 & |c_\ell| & |g_\ell|
\end{pmatrix}
\] (11)

with the following condition on the CP phases

\[
\eta_1 = \frac{\text{arg}(a_\ell)}{2}, \quad \eta_2 = \frac{\text{arg}(a_\ell)}{2}, \quad \eta_3 = \frac{\text{arg}(g_\ell)}{2}, \quad \eta_{\ell_1} + \eta_{\ell_2} = \text{arg}(b_\ell), \quad \eta_{\ell_2} + \eta_{\ell_3} = \text{arg}(c_\ell).
\] (12)

As is well known, the Fritzsch textures are given by

\[
M = \begin{pmatrix}
  0 & A & 0 \\
  A^* & 0 & B \\
  0 & B^* & C
\end{pmatrix}.
\] (8)
As a result of factorizing the CP violating phases, we have that \( \mathbf{u}_{eL} = \mathbf{P}_e \mathbf{O}_e \), \( \mathbf{u}_{eR} = \mathbf{P}_e^\dagger \mathbf{O}_e \) and \( \mathbf{u}_\nu = \mathbf{P}_\nu \mathbf{O}_\nu \). Let us obtain the orthogonal matrix that diagonalizes the real symmetric mass matrix, \( \mathbf{m}_e \).

Here, we will consider two cases: the normal and inverted hierarchy in the neutrino masses.

1. Normal Hierarchy (NH)

For this case, the diagonalization procedure is valid for charged lepton and the active neutrinos. Then considering the mass matrix, \( \mathbf{m}_e \), we can fix three free parameters in terms of the physical masses and unfixed one free parameter, \( |a_\ell| \). This is,

\[
\begin{align*}
|g_\ell| &= m_{\ell_1} - |m_{\ell_2}| + m_{\ell_1} - 2|a_\ell| \\
|b_\ell| &= \sqrt{\frac{(m_{\ell_3} - |a_\ell|)(|m_{\ell_2}| + |a_\ell|)(m_{\ell_1} - |a_\ell|)}{m_{\ell_3} - |m_{\ell_2}| + m_{\ell_1} - 3|a_\ell|}} \\
|c_\ell| &= \sqrt{\frac{(m_{\ell_3} + m_{\ell_2} - 2|a_\ell|)(|m_{\ell_2}| - |m_{\ell_1}| - 2|a_\ell|)(|m_{\ell_2}| - m_{\ell_1} + 2|a_\ell|)}{m_{\ell_3} - |m_{\ell_2}| + m_{\ell_1} - 3|a_\ell|}}
\end{align*}
\]

(13)

where we have taken \( m_{\ell_2} = -|m_{\ell_2}| \) in order to get real parameters. In addition, there is a constraint for the unfixed free parameter \( m_{\ell_3} > |m_{\ell_2}| > m_{\ell_1} > |a_\ell| > 0 \). After a lengthy task, we obtain the orthogonal real matrix

\[
\mathbf{O}_\ell = \begin{pmatrix}
\sqrt{(m_{\ell_2} + a_\ell)(1 - a_\ell)\mathcal{M}_2} & -\sqrt{(m_{\ell_2} + a_\ell)(1 - a_\ell)\mathcal{M}_1} & \sqrt{(m_{\ell_2} + a_\ell)(m_{\ell_1} - a_\ell)\mathcal{M}_1} \\
\sqrt{(m_{\ell_2} + a_\ell)\mathcal{M}_2} & \sqrt{(m_{\ell_2} - a_\ell)\mathcal{M}_1} & \sqrt{(m_{\ell_2} - a_\ell)\mathcal{M}_1} \\
\sqrt{(m_{\ell_2} - a_\ell)\mathcal{M}_1} & \sqrt{(m_{\ell_2} + a_\ell)\mathcal{M}_1} & \sqrt{(1 - a_\ell)\mathcal{M}_1}
\end{pmatrix}
\]

(14)

with

\[
\begin{align*}
\mathcal{M}_1 &= 1 + \hat{m}_{\ell_1} - 2\hat{a}_\ell, \quad \mathcal{M}_2 = 1 - \hat{m}_{\ell_2} - 2\hat{a}_\ell, \quad \mathcal{M}_3 = \hat{m}_{\ell_3} - \hat{m}_{\ell_1} + 2\hat{a}_\ell, \\
\mathcal{D}_1 &= (1 - \hat{m}_{\ell_1})(\hat{m}_{\ell_2} + \hat{m}_{\ell_1})D, \quad \mathcal{D}_2 = (1 + \hat{m}_{\ell_2})(\hat{m}_{\ell_2} + \hat{m}_{\ell_1})D, \quad \mathcal{D}_3 = (1 + \hat{m}_{\ell_2})(1 - \hat{m}_{\ell_1})D,
\end{align*}
\]

(15)

where \( \hat{m}_{\ell_2} = |m_{\ell_2}|/m_{\ell_1} \), \( \hat{m}_{\ell_1} = m_{\ell_1}/m_{\ell_3} \) and \( \hat{a}_\ell = |a_\ell|/m_{\ell_1} \). As we observed, for simplicity, the mixing matrix elements have been normalized by the heaviest mass. Therefore, the constraint is replaced by \( 1 > \hat{m}_{\ell_2} > \hat{m}_{\ell_1} > \hat{a}_\ell > 0 \).

2. Inverted Hierarchy (IH)

For this ordering, we obtain the fixed free parameters

\[
\begin{align*}
|d_\nu| &= m_2 - |m_1| + m_3 - 2|a_\nu| \\
|b_\nu| &= \sqrt{\frac{(m_4 - |a_\nu|)(|m_1| + |a_\nu|)(m_2 - |a_\nu|)}{m_2 - |m_1| + m_3 - 3|a_\nu|}} \\
|c_\nu| &= \sqrt{\frac{(|m_1| - m_3 + 2|a_\nu|)(m_2 + m_3 - 2|a_\nu|)(m_2 - |m_1| - 2|a_\nu|)}{m_2 - |m_1| + m_3 - 3|a_\nu|}}
\end{align*}
\]

(16)

where we have taken \( m_1 = -|m_1| \) for getting the real parameters. Therefore, the orthogonal real matrix is given by

\[
\mathbf{O}_\nu = \begin{pmatrix}
-\sqrt{\frac{(1 - \hat{a}_\nu)(\hat{m}_3 - \hat{a}_\nu)\mathcal{N}_2}{\mathcal{D}_2}} & \sqrt{\frac{\hat{m}_1 + \hat{a}_\nu)(\hat{m}_3 - \hat{a}_\nu)\mathcal{N}_1}{\mathcal{D}_2}} & \sqrt{\frac{(1 - \hat{a}_\nu)(\hat{m}_1 + \hat{a}_\nu)\mathcal{N}_3}{\mathcal{D}_2}} \\
\sqrt{\frac{\hat{m}_1 + \hat{a}_\nu)\mathcal{N}_1\mathcal{D}_2}{\mathcal{D}_2}} & \sqrt{\frac{(1 - \hat{a}_\nu)\mathcal{N}_1\mathcal{D}_2}{\mathcal{D}_2}} & \sqrt{\frac{(1 - \hat{a}_\nu)(\hat{m}_1 + \hat{a}_\nu)\mathcal{N}_3}{\mathcal{D}_2}} \\
\sqrt{\frac{\hat{m}_1 + \hat{a}_\nu)\mathcal{N}_3\mathcal{D}_2}{\mathcal{D}_2}} & \sqrt{\frac{(1 - \hat{a}_\nu)\mathcal{N}_2\mathcal{D}_2}{\mathcal{D}_2}} & -\sqrt{\frac{(\hat{m}_3 - \hat{a}_\nu)\mathcal{N}_1\mathcal{D}_2}{\mathcal{D}_2}}
\end{pmatrix}
\]

(17)
where

\[ N_1 = \tilde{m}_1 - \tilde{m}_3 + 2\tilde{a}_\nu, \quad N_2 = 1 + \tilde{m}_3 - 2\tilde{a}_\nu, \quad N_3 = 1 - \tilde{m}_1 - 2\tilde{a}_\nu; \]
\[ D_\nu = 1 - \tilde{m}_1 + \tilde{m}_3 - 3\tilde{a}_\nu; \]
\[ D_{\nu_1} = (1 + \tilde{m}_1)(\tilde{m}_1 + \tilde{m}_3)D_\nu, \quad D_{\nu_2} = (1 + \tilde{m}_1)(1 - \tilde{m}_3)D_\nu, \quad D_{\nu_3} = (1 - \tilde{m}_3)(\tilde{m}_1 + \tilde{m}_3)D_\nu, \]

(18)

where \( \tilde{m}_1 = |m_1|/m_2, \tilde{m}_3 = m_3/m_2 \) and \( \tilde{a}_\nu = |a_\nu|/m_2 \). In this parametrization, there is a constraint among the neutrino masses and the free parameter \( \tilde{a}_\nu \), this is \( 1 > \tilde{m}_1 > \tilde{m}_3 > \tilde{a}_\nu > 0 \).

Hence, we end up having the PMNS mixing matrix \( V^i = U^{i}_{\nu L}U^{i}_{\nu R} = O^{i}_{\nu}P_{\nu}O^{i}_{\nu} \) with \( i = NH, IH \). In addition, \( P_{\nu} = P^{i}_{\nu}P_{\nu} \equiv \text{Diag}(1,1,e^{i\eta_\nu}) \) with \( \eta_\nu = \eta_{\beta\delta} - \eta_\tau \). Thus, we can compare our expression with the standard parametrization of the PMNS mixing matrix such that the reactor, atmospheric and solar angles are well determined by

\[ \sin \theta_{13} = |(V^i)_{13}| = |(O_{\nu})_{11}(O^{i}_{\nu})_{13} + (O_{\nu})_{21}(O^{i}_{\nu})_{23} + (O_{\nu})_{31}(O^{i}_{\nu})_{33}e^{i\eta_\nu}|, \]
\[ \sin \theta_{23} = \sqrt{1 - \sin^2 \theta_{13}} = |(O_{\nu})_{12}(O^{i}_{\nu})_{13} + (O_{\nu})_{22}(O^{i}_{\nu})_{23} + (O_{\nu})_{32}(O^{i}_{\nu})_{33}e^{i\eta_\nu}|, \]
\[ \sin \theta_{12} = \sqrt{1 - \sin^2 \theta_{13}} = |(O_{\nu})_{11}(O^{i}_{\nu})_{12} + (O_{\nu})_{21}(O^{i}_{\nu})_{22} + (O_{\nu})_{31}(O^{i}_{\nu})_{32}e^{i\eta_\nu}|. \]

(19)

In the PMNS matrix there are three free parameters namely: \( |a_e|, |a_\nu| \) and one CP violating phase \( \eta_\nu \), in Eq. [19]. In fact, due to of lacking information on the absolute neutrino masses, the lightest one may be considered as an extra free parameter.

On the other hand, we would like to point out a little comment on the Majorana phases for each hierarchy. We have considered the CP parities for the complex neutrino masses which means that these can be either 0 or \( \pi \). Thus, for the normal and inverted ordering we have \( (m_3, m_2, m_1) = (+, -, +) \) and \( (m_3, m_2, m_1) = (+, +, -) \), respectively. Those CP parities values ensure that the fixed parameters given in Eq. [13] and Eq. [16] are reals.

**IV. RESULTS**

**A. Analytical study**

In order to try of figuring out the allowed region for free parameters, let us make a brief analytical study on the mixing angles formulas. To do so, we have to keep in mind that for the normal and inverted hierarchy, two neutrino masses can be fixed in terms of the squared mass scales and the lightest neutrino mass. This is,

\[ m_3 = \sqrt{\Delta m^2_{31} + m_1^2}, \quad |m_2| = \sqrt{\Delta m^2_{21} + m_1^2}, \quad \text{Normal Hierarchy} \]
\[ m_2 = \sqrt{\Delta m^2_{31} + \Delta m^2_{21} + m_3^2}, \quad |m_1| = \sqrt{\Delta m^2_{31} + m_3^2}, \quad \text{Inverted Hierarchy} \]

(20)

In addition, the experimental data, that will be used in this analytical and numerical study, is given in the table [II].

In the current analysis, central values will be used for the normalized masses and there is a hierarchy among those, this is, \( \tilde{m}_\mu > \tilde{m}_e/\tilde{m}_\mu > \tilde{m}_e, \tilde{m}_2 > \tilde{m}_1/\tilde{m}_2 > \tilde{m}_1 \) (for normal ordering) and \( \tilde{m}_1 > \tilde{m}_3/\tilde{m}_1 \geq \tilde{m}_3 \) (for inverted ordering); actually, for the last hierarchy we have \( m_2 \approx m_1(1 + \Delta m^2_{31}/2m_1^2) \), then \( \tilde{m}_3 \approx \tilde{m}_3/\tilde{m}_1 \). Consequently, we get the following values \( \tilde{m}_e \approx 2.9 \times 10^{-4}, \tilde{m}_e/\tilde{m}_\mu \approx 4.8 \times 10^{-3} \) and \( \tilde{m}_\mu \approx 5.9 \times 10^{-2} \). At the same time, for the neutrinos one obtains

- Normal Hierarchy

\[ \tilde{m}_1 \approx 2 \times 10^{-2}, \quad \frac{\tilde{m}_1}{\tilde{m}_2} \approx 0.115, \quad \tilde{m}_2 \approx 0.173. \]

(21)
with $m_1 \approx 0.001$ for the lightest mass.

- Inverted Hierarchy

\[
\begin{align*}
\tilde{m}_3 &\approx 0.195; & \tilde{m}_2 &\approx 0.198, & \tilde{m}_1 &\approx 1
\end{align*}
\]

with $m_3 \approx 0.01$.

Notice that particular values for the lightest neutrino mass have been considered for the normal and inverted hierarchy. Thus, we will obtain approximately the matrices $O_e$ and $O_\nu$ for the normal and inverted ordering, then the mixing angles must be calculated in analytical way for different scenarios.

**Normal Hierarchy:** ($1 > m_{\ell_2} > m_{\ell_1} > \tilde{a}_{\ell} > 0$).

- Case I: $\tilde{a}_{\ell} \approx 0$. In this limit, the Fritzsch textures are recovered and the orthogonal matrix is given by

\[
O_\ell \approx \begin{pmatrix}
\sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})}{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})+m_{\ell_2}m_{\ell_1}+m_{\ell_2}m_{\ell_1}}}

\sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})}{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})+m_{\ell_2}m_{\ell_1}+m_{\ell_2}m_{\ell_1}}}

\sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}(1-m_{\ell_2})}{m_{\ell_2}(1-m_{\ell_2})+m_{\ell_2}+m_{\ell_2}}}
& \sqrt{\frac{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})}{m_{\ell_2}m_{\ell_1}(m_{\ell_2}-m_{\ell_1})+m_{\ell_2}m_{\ell_1}+m_{\ell_2}m_{\ell_1}}}
\end{pmatrix}
\]

- Case II: $\tilde{a}_{\ell} \approx \tilde{m}_{\ell_1}$.

\[
O_\ell \approx \begin{pmatrix}
1
& 0
& 0

0
& \sqrt{\frac{m_{\ell_1}}{1+m_{\ell_1}}}
& \sqrt{\frac{m_{\ell_2}+m_{\ell_1}}{1+m_{\ell_2}}}

0
& \sqrt{\frac{m_{\ell_1}+m_{\ell_1}}{1+m_{\ell_1}}}
& \sqrt{\frac{m_{\ell_2}}{1+m_{\ell_2}}}
\end{pmatrix}
\]

**Inverted Hierarchy:** ($1 > m_1 > m_3 > \tilde{a}_\nu > 0$).

- Case I: $\tilde{a}_\nu \approx 0$.

\[
O_\nu \approx \begin{pmatrix}
\sqrt{\frac{m_1(m_1+m_3)}{m_1(m_1+m_3)+m_1+m_3}}
& \sqrt{\frac{m_1m_3(1-m_3)}{m_1m_3(1-m_3)+m_1+m_3}}
& \sqrt{\frac{m_1(1-m_1)}{m_1(1-m_1)+m_1+m_3}}

\sqrt{\frac{m_1(m_1+m_3)}{m_1(m_1+m_3)+m_1+m_3}}
& \sqrt{\frac{m_1m_3(1-m_3)}{m_1m_3(1-m_3)+m_1+m_3}}
& \sqrt{\frac{m_1(1-m_1)}{m_1(1-m_1)+m_1+m_3}}

\sqrt{\frac{m_1(m_1+m_3)}{m_1(m_1+m_3)+m_1+m_3}}
& \sqrt{\frac{m_1m_3(1-m_3)}{m_1m_3(1-m_3)+m_1+m_3}}
& \sqrt{\frac{m_1(1-m_1)}{m_1(1-m_1)+m_1+m_3}}
\end{pmatrix}
\]
Case II: \( \tilde{a}_\nu \approx \tilde{m}_3 \).

\[
O_\nu \approx \begin{pmatrix}
0 & 0 & 1 \\
\sqrt{\frac{1-\tilde{m}_3}{1+\tilde{m}_3}} & \sqrt{\frac{\tilde{m}_1+\tilde{m}_3}{1+\tilde{m}_3}} & 0 \\
-\sqrt{\frac{\tilde{m}_1+\tilde{m}_3}{1+\tilde{m}_3}} & \sqrt{\frac{1-\tilde{m}_3}{1+\tilde{m}_3}} & 0
\end{pmatrix}.
\]

(26)

Having obtained the above approximated matrices, then we can obtain the mixing angles for different scenarios and some combinations:

1. Normal hierarchy

- Scenario A: If \( O_e \) and \( O_\nu \) were like Eqn. (23), then the mixing angles would be

\[
\sin \theta_{13} \approx |\tilde{m}_2\sqrt{\tilde{m}_1 \left( 1 - \frac{\tilde{m}_1}{\tilde{m}_2} \right)} + \sqrt{\tilde{m}_e^2 \tilde{m}_2 - \tilde{m}_e \sqrt{1 - \tilde{m}_2} e^{i\eta_\nu}}|;
\]

\[
\sin \theta_{23} \approx |\frac{-\sqrt{\tilde{m}_e^2 \tilde{m}_2 - \tilde{m}_e \sqrt{1 - \tilde{m}_2} e^{i\eta_\nu}}}{\sqrt{1 - \sin^2 \theta_{13}}}|;
\]

\[
\sin \theta_{12} \approx |\frac{\sqrt{\tilde{m}_1^2 (1 - \frac{\tilde{m}_1}{\tilde{m}_2})}}{\sqrt{1 - \sin^2 \theta_{13}}}|.
\]

where the notable hierarchy in the charged lepton has been taken into account. In the above expressions, the reactor, atmospheric and solar angles are controlled by the ratio \( \sqrt{\tilde{m}_e^2/\tilde{m}_\mu} \approx 0.069, \sqrt{\tilde{m}_2} \approx 0.41 \) and \( \sqrt{\tilde{m}_1/\tilde{m}_2} \approx 0.34 \), respectively. In order to enhance the angle values, the phase \( \eta_\nu \) must be near to \( \pi \). In this way, we have that \( \sin \theta_{13} \approx 0.06, \sin \theta_{23} \approx 0.6 \) and \( \sin \theta_{12} \approx 0.25 \). As result of this, the reactor and solar angle are not in the allowed experimental region with the neutrino masses values given in Eq. (21).

- Scenario B. If \( O_e \) and \( O_\nu \) were like Eqn. (24), then one would get

\[
\sin \theta_{13} \approx 0;
\]

\[
\sin \theta_{23} \approx |\frac{1 - \tilde{m}_e}{1 + \tilde{m}_\mu} \sqrt{\tilde{m}_2 + \tilde{m}_1} - \sqrt{\tilde{m}_\mu + \tilde{m}_e} \sqrt{\tilde{m}_2 (1 - \tilde{m}_2)} e^{i\eta_\nu}|;
\]

\[
\sin \theta_{12} \approx 0.
\]

(28)

So that this case is completely discarded.

- Scenario C: If \( O_\nu \) and \( O_e \) were like Eqs. (23) and (24) respectively, then the mixing angles would be

\[
\sin \theta_{13} \approx |\sqrt{\frac{\tilde{m}_1^2 (1 - \frac{\tilde{m}_1}{\tilde{m}_2})}}{\sqrt{1 - \sin^2 \theta_{13}}}|;
\]

\[
\sin \theta_{23} \approx |\frac{\sqrt{\tilde{m}_e^2 \tilde{m}_2 - \tilde{m}_e \sqrt{1 - \tilde{m}_2} e^{i\eta_\nu}}}{\sqrt{1 - \sin^2 \theta_{13}}}|;
\]

\[
\sin \theta_{12} \approx |\sqrt{\frac{\tilde{m}_1 (1 - \frac{\tilde{m}_1}{\tilde{m}_2})}{\sqrt{1 - \sin^2 \theta_{13}}}}|.
\]

(29)

As one can notice, the reactor angle is tiny in comparison to the scenario A, the atmospheric and solar angle are handled by the \( \sqrt{\tilde{m}_2^2} \approx 0.41 \) and \( \sqrt{\tilde{m}_1/\tilde{m}_2} \approx 0.34 \); the atmospheric angle value can be increased by allowing that the phase \( \eta_\nu \) must be \( \pi \). Therefore, we obtain \( \sin \theta_{13} \approx 0.016, \sin \theta_{23} \approx 0.58 \) and \( \sin \theta_{12} \approx 0.32 \).
• Scenario D: If $\mathbf{O}_\nu$ and $\mathbf{O}_e$ were like Eq. (24) and Eq. (23) respectively, then one would obtain

$$
\sin \theta_{13} \approx \left| \sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}} \sqrt{\frac{\tilde{m}_2 + \tilde{m}_1}{1 + \tilde{m}_2}} - \sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}} \sqrt{\frac{1 - \tilde{m}_1}{1 + \tilde{m}_2}} e^{i\eta_\nu} \right|; \\
\sin \theta_{23} \approx \left| \sqrt{\frac{\tilde{m}_2 + \tilde{m}_1}{1 + \tilde{m}_2}} - \sqrt{\frac{\tilde{m}_\mu}{1 + \tilde{m}_2}} e^{i\eta_\nu} \right| \sqrt{1 - \sin^2 \theta_{13}}; \\
\sin \theta_{12} \approx \left| \sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}} \sqrt{\frac{1 - \tilde{m}_1}{1 + \tilde{m}_2}} - \sqrt{\frac{\tilde{m}_\mu}{1 + \tilde{m}_2}} \sqrt{\tilde{m}_2 + \tilde{m}_1} e^{i\eta_\nu} \right| \sqrt{1 - \sin^2 \theta_{13}} \right|
$$

(30)

In this scenario, the reactor angle is smaller (larger) than scenario A (C); the solar angle is smaller than the scenarios A and C so that this case is ruled out.

2. Inverted hierarchy

• Scenario E: If the charged lepton and the neutrino mixing matrices were like Eq. (23) and Eq. (25), then the observables would be

$$
\sin \theta_{13} \approx \sqrt{\tilde{m}_e}; \\
\sin \theta_{23} \approx \sqrt{\tilde{m}_\mu}; \\
\sin \theta_{12} \approx \frac{1}{\sqrt{2}} \left( 1 + \sqrt{\frac{\tilde{m}_e}{\tilde{m}_\mu}} \right).
$$

(31)

This scenario is discarded since that the reactor and atmospheric angles are tiny.

• Scenario F: If the charged lepton and the neutrino mixing matrices were like Eq. (24) and Eq. (25), then the observable would be

$$
\sin \theta_{13} \approx 0; \\
\sin \theta_{23} \approx \sqrt{\frac{\tilde{m}_\mu + \tilde{m}_e}{1 + \tilde{m}_e}}; \\
\sin \theta_{12} \approx \frac{1}{\sqrt{2}}.
$$

(32)

Analogously to the previous case, this scenario is ruled out by the predictions on the reactor and atmospheric angles which come out being small.

• Scenario G: $\mathbf{O}_\nu$ and $\mathbf{O}_e$ given by Eqs. (26) and (23), respectively, then

$$
\sin \theta_{13} \approx 1; \\
\sin \theta_{23} > 1; \\
\sin \theta_{12} > 1.
$$

(33)

• Scenario H: $\mathbf{O}_\nu$ and $\mathbf{O}_e$ given by Eqs. (26) and (24), respectively, then

$$
\sin \theta_{13} \approx 1; \\
\sin \theta_{23} > 1; \\
\sin \theta_{12} > 1.
$$

(34)
The last two scenarios are completely ruled out due to the reactor angle is close to 1.

As consequence of this analytical study, speaking roughly there are two scenarios (A and C) which seem to provide allowed values for the observables. Let us add that one would expect changes in the mentioned scenarios when the lightest neutrino mass varies in its allowed region.

### B. Numerical study

The numerical analysis consists of scattered plots to constrain the allowed region for each free parameters. Then, we will be working with the following expressions

\[
\begin{align*}
\sin^2 \theta_{13} &= \sin^2 \theta_{13} (|a_e|, |a_\nu|, \eta_\nu, m_j) \\
\sin^2 \theta_{23} &= \sin^2 \theta_{23} (|a_e|, |a_\nu|, \eta_\nu, m_j) \\
\sin^2 \theta_{12} &= \sin^2 \theta_{12} (|a_e|, |a_\nu|, \eta_\nu, m_j)
\end{align*}
\]

(35)

where \(m_j\) with \(j = 1,3\) stands for the lightest neutrino mass for normal and inverted hierarchy, respectively.

In the scattered plots, we will vary the free parameters in such a way those satisfy their respective constraints. For the lightest neutrino mass, in the normal (inverted) case, we have \(1 > \tilde{m}_2 > \tilde{m}_1 > \tilde{a}_\nu > 0 \) (\(1 > \tilde{m}_1 > \tilde{m}_3 > \tilde{a}_\nu > 0\)); along with this, for each hierarchy, the lightest mass varies in the region \(0 - 0.9\) eV, the effective phase \(2\pi \geq \eta_\nu \geq 0\) and the charged lepton parameter \(1 > \tilde{m}_\mu > \tilde{m}_e > \tilde{a}_e > 0\). Then, we demand that our theoretical expressions satisfy the experimental bounds up to \(3\sigma\), this allows us to scan the allowed regions for the free parameters that fit quite well the experimental results. Finally, as a model prediction, the \(\delta_{CP}\) CP-violating phase and the effective Majorana neutrino mass are fitted.

\[\text{FIG. 1. From left to right: the reactor, solar, atmospheric angles and CP phase versus the lightest neutrino mass. The thick line stands for 3 \sigma of C. L.}\]

In the Fig. [1], we observe that there is a region \((0.01 - 0.014\) eV) for the lightest neutrino mass where the observables are in great according to the experimental results.
FIG. 2. From left to right: the reactor, solar, atmospheric angles and CP phase versus the $|a_\nu|$ parameter. The thick line stands for 3 $\sigma$ of C. L.

According to the Fig. (2), the $a_\nu$ ($\tilde{a}_\nu$) prefers small values for fitting the mixing angles. This means the Fritzsch textures are favored but a small deviation is necessary to accommodate the observables up to 3 $\sigma$.

FIG. 3. From left to right: the reactor, solar, atmospheric angles and CP phase versus the $|b_e|$ parameter. The thick line stands for 3 $\sigma$ of C. L.

In the charged lepton sector, the $a_e$ ($\tilde{a}_e$) parameter region is close to the electron mass as can be seen in Fig. (3), this is, $a_e \approx m_e$, so that the observables are well accommodated in the scenario C. Let us focus in the $\eta_\nu$ phase which lies in a region around $\pi$ value, the full region is shown in the Fig. (4).
FIG. 4. From left to right: the reactor, solar, atmospheric angles and CP phase versus the effective phase, $\eta_\nu$, parameter. The thick line stands for 3 $\sigma$ of C. L.

To summarize, a set of free parameters has been found in which the reactor, solar and the atmospheric angles can accommodate quite well but this latter lies in the allowed low region (3 $\sigma$). In addition, the model predicts large values for the Dirac CP-violating phase which is close to the up region according to the experimental data.

V. MODEL PREDICTIONS

A. Effective Majorana neutrino mass rate

Going back to the comment about CP parities for the complex neutrino masses, we want to perform the effective Majorana mass of the electron neutrino, which is defined by

$$|m_{ee}| = |m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2|,$$

(36)

where $m_i$ and $V_{ei}$ ($i = 1, 2, 3$) are the complex neutrino masses and PMNS matrix elements. As it is well known, the lowest upper bound on $|m_{ee}| < 0.22$ eV was provided by GERDA phase-I data [102] and this value has been significantly reduced by GERDA phase-II data [103].

In the previous section, we found a set of values for the free parameters (see Fig. 1-4) which fit the mixing angles. As a result, those values were used to find the regions for the effective Majorana mass of the electron neutrino, as shown in the Fig. (5). For this observable, two scattered plots have been only shown since that parameters $m_1$ and $a_\nu$ are more restrictive for the allowed region.
FIG. 5. From left to right: $|m_{ee}|$ versus $m_1$ and $|a_\nu|$ parameters, respectively. These scattered plots correspond to the normal ordering where the CP parities for the complex neutrino masses are $(m_3, m_2, m_1) = (+, -, +)$.

B. Lepton violation process: $\mu \rightarrow e\gamma$

In this section, we have calculated the branching ratio for the lepton flavor violation process $\mu \rightarrow e\gamma$ [104, 105] that is mediated by the doubly $(\Delta^{++})$ and singly $(\Delta^+)$ charged scalars that come from the Higgs triplet (see Eq.(2)). The branching ratio [104] is given by

$$
BR(\mu \rightarrow e\gamma) \approx 4.5 \times 10^{-3} \left( \frac{1}{\sqrt{2} v_{\Delta^+}} \right)^4 \left| \left( V^* M^\dagger \hat{M} \nu V^T \right)_{e\mu} \right|^2 \left( \frac{200 GeV}{m_{\Delta^{++}}} \right)^4
$$

(37)

where $m_{\Delta^+} = m_{\Delta^{++}} \equiv m_\Delta$ has been assumed in the previous result. Besides, $V$ stands for the PMNS mixing matrix.

The branching ratio depends on the PMNS mixing parameters, the single and doubly charged scalars; along with this, the vev of the Higgs triplet takes place. In here, we use the following regions $80 GeV < m_\Delta$ and $v_\Delta < 5 GeV$ [106]; the PMNS mixing parameters have been already constrained in the previous section, to be more explicit, we use the following regions: $0.01 eV < m_1 < 0.014 eV$, $0.35 MeV < |a_e| < m_e$, $0.004 eV < |a_\nu| < 0.006 eV$ and $\pi < \eta_\nu < 6\pi/5$.

FIG. 6. From left to right: $BR(\mu \rightarrow e\gamma)$ versus the $v_\Delta$ and $m_\Delta$ parameter. The thick line stands for $3 \sigma$ of C. L.

In the Fig. [6], the predicted region is shown for the branching ratio as function of the vev of the Higgs triplet and the mass of the singly and doubly charged scalars. Our model predicted a region, $BR(\mu \rightarrow e\gamma) \approx 10^{-40}$, that is too much below of the experimental bound $BR(\mu \rightarrow e\gamma) \approx 4.2 \times 10^{-13}$. 
VI. CONCLUSIONS

We have built an economical non-renormalizable lepton model for getting the mixings where the type II see-saw mechanism is responsible to explain tiny neutrino masses. Under a particular benchmark, in the charged lepton sector, the mass matrices have the Fritzsch textures with a shift parameter which makes different to the previous studies. Our main finding is: a set of values for the relevant parameters was found to be consistent (up to 3 $\sigma$) with the last experimental data on lepton observables for the normal neutrino mass ordering.

To finish, we would like to add that the $S_3 \otimes Z_2$ symmetry is an excellent candidate to be the flavor symmetry at low energy. However, one has to look for the best framework where the flavor symmetry solve the majority of open questions on the flavor problem and related issues. In this direction, the quark mixings and the scalar potential analysis will be included to have a complete study but this is a working progress.

ACKNOWLEDGEMENTS

García-Aguilar appreciates the facilities given by the IPN through the SIP project number 20211170. JCGI thanks Valentina A. and A. Emiliano Gómez Nabor for sharing great moments and experiences during this long time. This work was partially supported by Project 20211423 and PAPIIT IN109321.

Appendix A: $S_3$ flavour symmetry

The non-Abelian group $S_3$ is the permutation group of three objects \[15\] and this has three irreducible representations: two 1-dimensional, $1_S$ and $1_A$, and one 2-dimensional representation, $2$. We list the multiplication rules among them:

\begin{align*}
1_S \otimes 1_S &= 1_S, & 1_S \otimes 1_S &= 1_S, & 1_S \otimes 1_A &= 1_A \\
1_A \otimes 1_A &= 1_S, & 1_S \otimes 2 &= 2, & 1_A \otimes 2 &= 2 \\
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} a_1 b_1 + a_2 b_2 \\ (a_1 b_2 - a_2 b_1) \end{pmatrix} 1_A + \begin{pmatrix} a_1 b_1 + a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix} 2.
\end{align*}

(A1)

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