Existence of radial solution for a quasilinear equation with singular nonlinearity

Kaushik Bal

Department of Mathematics and Statistics
Indian Institute of Technology
Kanpur-208016, India

Abstract
We prove that the equation

$$-\Delta_p u = \lambda \left( \frac{1}{u^\delta} + u^q + f(u) \right) \quad \text{in} \quad B_R(0)$$

$$u = 0 \quad \text{on} \quad \partial B_R(0), \quad u > 0 \quad \text{in} \quad B_R(0)$$

admits a weak radially symmetric solution for $\lambda > 0$ sufficiently small, $0 < \delta < 1$ and $p - 1 < q < p^* - 1$. We achieve this by combining a blow-up argument and a Liouville type theorem to obtain a priori estimates for the regularized problem. Using a variant of a theorem due to Rabinowitz we derive the solution for the regularized problem and then pass to the limit.

Keywords: Quasilinear Elliptic Equation, Boundary Singularity, a priori estimate, Bifurcation theory

2000 MSC: 35J20, 35J65, 35J70

1. Introduction

Consider the following quasilinear singular elliptic equation:

$$-\Delta_p u = \lambda \left( \frac{1}{u^\delta} + u^q + f(u) \right) \quad \text{in} \quad B_R(0)$$

$$u = 0 \quad \text{on} \quad \partial B_R(0), \quad u > 0 \quad \text{in} \quad B_R(0)$$

(1.1)

where $B_R(0) \subset \mathbb{R}^N$ is a ball of radius $R$ with center at 0 and $\delta > 0$.

Here, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$.

We assume that the pair $(p, q)$ satisfies

$$(p, q) \in \{ 1 < p < N : p - 1 < q < p^* - 1 \}$$

(1.2)
where, $p^* = \frac{pN}{N-p}$ is the critical Sobolev exponent.

We also assume that $f : \mathbb{R}^+ \to \mathbb{R}$ is a $C^1$ function satisfying the following assumptions (H):

- $f(t) + c_0 t^q \geq 0$ for all $t \geq 0$ and for some $0 < c_0 < 1$.
- $f(0) = 0$ and $\lim_{t \to \infty} \frac{f(t)}{t^q} = 0$.

Problem (2.4) arises in many branches of applied mathematics. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla u) - \nabla \cdot (g(u)\nabla u)$$

(1.3)

has been used to model the dynamics of thin films of viscous fluids, where $u(x,t)$ is the height of air/liquid interface. The zero set $\{u = 0\}$ is the liquid/solid interface and is known as rupture set. The coefficient $f(u)$ denotes the surface tension effects and typical chose to be $f(u) = u^3$. The coefficient of the second order term can be viewed as the Van der Waals interactions $g(u) = u^m$ where $m < 0$. For more physical motivation see [22] and the reference therein.

A huge literature is available for Singular elliptic problems of this type. Starting with the pioneering work of Crandall et al [12], who considered the problem

$$\begin{cases} 
-\Delta_p u = \frac{\lambda \kappa(x)}{u^\delta} + \mu(x)u^q + f(u) \quad &\text{in } \Omega, \\
 u = 0 \quad &\text{on } \partial \Omega, \\
 u > 0 \quad &\text{in } \Omega,
\end{cases}$$

(P)$_\chi$

For $p = 2$, $\kappa \geq 0$ is bounded, $f, \mu = 0$ they showed that for any $\lambda > 0$ there exists a unique classical solution in $C^2(\Omega) \cap C(\bar{\Omega})$ and $u \in C^1(\bar{\Omega})$ for $0 < \delta < 1$. For $\lambda$ small enough and $f = 0$, Coclit and Palmieri [11] showed the existence of solution for $0 < \lambda < \Sigma$, where $\Sigma = \{\inf \lambda > 0 : \text{the equation has no weak solutions}\}$. Assuming $0 < \delta < 1$, Yijing et al [26] applied the variational methods to show the existence of at least two solutions for $1 < q < 2^* - 1$, $N \geq 3$. The critical case was almost simultaneously settled by Haitao [29] using Perron’s Method and by Hirano et al [19] using Nehari Manifold technique. In Adimurthi-Giacomoni, existence of at least two weak solutions were shown using variational technique for the critical problem of Moser-Trudinger type for $0 < \delta < 3$, $N = 2$. For the subcritical problem with $p = 2$, $f = 0$, Bal-Giacomoni [8] proved an a priori estimate and a symmetry result using moving plane technique. Giacomoni-Saoudi [15] proved the existence of two weak solutions using variational technique for $0 < \delta < 3$. Coming to the case $p \neq 2$, In a beautiful paper Ambrosetti et
al. showed that there exists two distinct positive solutions to \((P_\chi)\) using uniform a priori estimate and global bifurcation theory for \(\delta < 0, p - 1 < q \leq p^* - 1\). A similar problem \((P_\chi)\) for \(f = 0\) was also considered by Giacomoni-Sreenadh [17] in a ball, i.e. in the radial symmetry case and they proved the existence using a shooting method. Another important contribution is the study of \((P_\chi)\) was Giacomoni et al [16] where using the variational method with \(f = 0\) they have been able to show the existence of two weak solutions for the case \(0 < \delta < 1\) and \(p - 1 < q \leq p^* - 1\). Other significant contributions can be found in Arcaya-Merida [4], Boccardo-Orsina [10] to name a few.

For the parabolic counterpart few results are available. See for example Badra et al [6], [7], [8] among others.

We denote \(B_R(0) = \Omega\) until otherwise mentioned and \(C_c^\infty(\Omega)\) denotes the space of all \(C^\infty\) functions \(\phi : \Omega \to \mathbb{R}\) with compact support.

Define the cone \(\mathbf{C}\) as

\[ \mathbf{C} = \{ u \in C^1_0(\bar{\Omega}) \text{ s.t. } u(x) \geq C \text{dist}(x, \partial\Omega) \} \]

for some \(C > 0\) and \(C^1_0(\bar{\Omega}) = \{ u \in C^1(\bar{\Omega}) \mid u(x) = 0 \text{ for all } x \in \partial\Omega \}\) equipped with the norm \(\| u \|_{C^1} = \| u \|_\infty + \| \nabla u \|_\infty\).

By the weak solution of \((1.1)\) we mean \(u \in \mathbf{C}\) such that \(u\) satisfies:

\[ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \lambda \int_\Omega \left( \frac{1}{u^p} + u^q + f(u) \right) \phi \, dx, \quad \forall \phi \in C^\infty_c(\Omega) \]

**Remark 1.1.** Note that one need the assumptions \(0 < \delta < 1\) to make sense of the definition of weak solution, since in the cone \(\mathbf{C}\) we have

\[
\int_\Omega \frac{\phi(x)}{u^\delta(x)} \, dx \leq \int_\Omega \frac{\phi(x)}{d(x)} \, dx \leq \int_\Omega \left( \frac{\phi^2(x)}{d^2(x)} \right)^\frac{1}{2} \left( \frac{d^2(x)}{u^2(x)} \right)^\frac{1}{2} \, dx \\
\leq \int_\Omega C'(N) \left( \| \nabla u \|^2 \right)^\frac{1}{2} \left( d^2(1-\delta) \right)^\frac{1}{2} \, dx \quad \text{(Hardy’s Inequality)} \\
< \infty \quad \text{for } 0 < \delta < 1.
\]

and some \(C'(N)\) is constant dependent on \(N\), \(\phi \in C^\infty_c(\Omega)\) with \(d(x) = \text{dist}(x, \partial\Omega)\).
2. Main Result

We will show that there exists a radially symmetric weak solution to \((1.1)\). Precisely, we have the following result:

**Theorem 2.1.** Assume that \( f \) satisfying the hypothesis \((H)\) and \( 0 < \delta < 1 \) then for \( \lambda > 0 \) sufficiently small, there exists a radially symmetric weak solution \( u \in C \) to \((1.1)\).

Before we proceed with the proof of Theorem \((2.1)\) we point out the major difficulties one encounters while handling such equations:

1. The nonlinearity \( \frac{1}{u} \) is singular hence blows up near boundary subject to the Dirichlet Boundary conditions.
2. Another vital problem to find a radially symmetric solution to any equation involving the operator \( \Delta_p \) is its degeneracy for \( p > 2 \) and singular for \( 1 < p < 2 \), which prevents one to deduce the symmetry results available for the laplacian.

Now we lay down our strategy to give the existence result.

To overcome the above mentioned difficulties we start by regularising \((1.1)\) as follows:

For every \( \epsilon > 0 \) consider the equation:

\[
-\Delta_p u = \lambda \left( \frac{1}{(u + \epsilon)\delta} + u^q + f(u) \right) \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega, \quad u > 0 \quad \text{in } \Omega
\]

We will show the existence of a weak solution to \((2.1)\) by using a degree theory argument as in Azizieh-Clement [5] by proving the existence of continuum of solution to a slightly modified problem as follows:

For every \( \epsilon > 0 \) and \( \mu \geq 0 \) we consider the equation:

\[
-\Delta_p u = \lambda \left( \frac{1}{(u + \epsilon)\delta} + u^q + f(u) \right) + \mu \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega, \quad u > 0 \quad \text{in } \Omega
\]

Consequently the existence of positive nontrivial solution to \((2.1)\) for \( \mu = 0 \) will be reached by means of a theorem due to Rabinowitz.

Since the righthand side of the equation \((2.2)\) is locally Lipchitz in \([0, \infty)\) and positive in \((0, \infty)\) for each \( \epsilon > 0 \) we can use the result by Damascelli-Sciunzi [14] which is as follows:
Theorem 2.2. Let $f$ be locally Lipchitz continuous function in $[0, \infty)$ and positive in $(0, \infty)$ and $u \in C^1(\Omega)$ be a weak solution of the equation

$$-\Delta_p u = f(u) \text{ in } \Omega$$
$$u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega$$

Then $u$ is radially symmetric and $\frac{\partial u}{\partial r} < 0$, where $\frac{\partial u}{\partial r}$ is the derivative in radial direction.

to derive the following result:

Theorem 2.3. Let $f$ satisfies $(H)$ and $u_{\lambda, \epsilon} \in C^1(\Omega)$ is a weak solution of equation (2.2), then $u_{\lambda, \epsilon}$ is radial in $\Omega$ and moreover $\frac{\partial u_{\lambda, \epsilon}}{\partial r} < 0$ where $\frac{\partial u}{\partial r}$ is the derivative in radial direction.

Since we will be using degree theory we need the nonexistence of solution for large $\mu$. To that aim we have the following nonexistence result for the equation (2.2):

Theorem 2.4. Let $q > p - 1$ and $\mu$ large enough with $f$ satisfying $(H)$ then the equation

$$-\Delta_p u = \lambda \left( \frac{1}{(u + \epsilon)^{\delta}} + u^q + f(u) \right) + \mu \text{ in } \Omega$$
$$u \in C^1(\Omega) \quad u > 0 \text{ in } \Omega$$

does not admit a solution, where $\lambda_1$ is the first eigenvalue of $-\Delta_p$ under homogeneous Dirichlet boundary condition.

Remark 2.5. Theorem 2.4 holds for any domain $U$ which is smooth and bounded in $\mathbb{R}^n$.

Before we proceed with the proof of Theorem 2.4 we need the following lemma which is an easy consequence of Picone’s Identity due to Allegretto-Huang [1]. See also Ruiz [25]

Lemma 2.6. Let $u$ be a positive solution of the problem:

$$-\Delta_p u = h(x) \text{ in } \Omega$$
$$u(x) = 0 \text{ on } \partial \Omega$$

Then,

$$\int_{\Omega} h(x) \phi_1^{p} \leq \lambda_1 \int_{\Omega} \phi_1^m$$
where \( h(x) \) is a continuous positive function and \( \lambda_1 \) is the first eigenvalue of the p-laplacian with Dirichlet boundary condition and \( \phi_1 \) is the associated eigenfunction.

Observe that \( \frac{\phi_1^p}{p-1} \) belongs to \( W^{1,p}_0(\Omega) \) since \( u \) is positive in \( \Omega \) and nonzero outward derivative on the boundary due to the Strong Maximum Principle due to Vazquez [28].

**Proof.** Since \( u_\varepsilon \in C^1(\overline{\Omega}) \) is a solution of (2.2) we have,

\[
\lambda_1 \int_\Omega \frac{\phi_1^p}{p-1} \geq \int_\Omega \left[ \frac{\lambda}{(u_\varepsilon + \varepsilon)^q} + \lambda u_\varepsilon^q + \lambda f(u_\varepsilon) + \mu \right] \frac{\phi_1^p}{u_\varepsilon^{p-1}} \, dx
\]

\[
\geq \lambda(1 - c_0) \int_\Omega u_\varepsilon^q \frac{\phi_1^p}{u_\varepsilon^{p-1}} + \mu \int_\Omega \frac{\phi_1^p}{u_\varepsilon^{p-1}}
\]

Now define,

\[
k = \min \left\{ \frac{\mu + \lambda(1 - c_0)t^q}{tp^{p-1}} : t \geq 0 \right\}
\]

Hence we have,

\[
\lambda_1 \int_\Omega \frac{\phi_1^p}{p-1} \geq k \int_\Omega \phi_1^p
\]

Thus, \( k \leq \lambda_1 \). \( \square \)

Now we deduce some monotonicity for the equation (2.2), the proof of which is inspired by Proposition 1.1 of Giacomoni-Saoudi [15] and an algebraic inequality due to Lindqvist [21] which is provided below:

**Lemma 2.7.** Let \( u, v \in W^{1,p}_0(\Omega) \) then we have:

\[
\int_\Omega (|\nabla w|^{p-2}\nabla w - |\nabla v|^{p-2}\nabla v) \nabla (w - v) \, dx \geq C_1 \int_\Omega |\nabla (w - v)|^p \, dx
\]

for \( p \geq 2, C_1 > 0 \) being a constant and

\[
\int_\Omega (|\nabla w|^{p-2}\nabla w - |\nabla v|^{p-2}\nabla v) \nabla (w - v) \, dx \geq \frac{C_2 \left( \int_\Omega |\nabla (w - v)|^p \, dx \right)^{\frac{2}{p}}}{\left( \left( \int_\Omega |\nabla w|^p \, dx \right)^{\frac{1}{p}} + \left( \int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} \right)^{2-p}}
\]

for \( 1 < p < 2, C_2 > 0 \) being a constant.
We also state a strong comparison principle due to Cuesta-Takac \[13\].

**Lemma 2.8.** Let \(\Omega \subset \mathbb{R}^n\) be a smooth, bounded domain. If \(f, g \in L^\infty(\Omega)\) satisfy \(0 \leq f \leq g\) with \(f \neq g\) in \(\Omega\). Assume that \(u, v \in W^{1,p}_0(\Omega)\) are weak solution of the equations

\[
-\Delta_p u = f \quad \text{in} \quad \Omega \\
-\Delta_p v = g \quad \text{in} \quad \Omega
\]

Then \(0 \leq u \leq v\) in \(\Omega\) and \(\frac{\partial u}{\partial \gamma} < \frac{\partial v}{\partial \gamma} \leq 0\) on \(\partial \Omega\), where \(\gamma\) denotes the outward unit normal to \(\partial \Omega\).

With the above lemmas in our hand we state our result on the monotonicity of the solution to (2.2).

**Theorem 2.9.** For \(\lambda > 0\) sufficiently small then we have \(u_{\lambda, \epsilon} \geq u_{\lambda, \epsilon'}\) for any \(\epsilon' \geq \epsilon > 0\) and \(u_{\lambda, \epsilon}\) being the solution of equation (2.2).

In particular one has: \(u_{\lambda, \epsilon}(x) \geq C \text{dist}(x, \partial \Omega)\) for any \(\epsilon > 0\) and some constant \(C > 0\).

**Proof.** Let us consider the problem:

\[
-\Delta_p u = \frac{\lambda}{(u + \epsilon)^\delta} \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial \Omega, \quad u > 0 \quad \text{in} \quad \Omega
\]

Using a minimisation argument as in Theorem 1.1 of Badra et al \[8\] with the elliptic regularity of Lieberman \[20\] we get \(\bar{u} \in C^{1,\alpha}(\Omega)\) for every \(\epsilon > 0\) solving (2.3). It is also easy to see that \(\eta \phi_1\) is a subsolution of (2.3), where \(\eta > 0\) is sufficiently small, depending on \(\lambda\) but independent of \(\epsilon\) and \(\phi_1\) is defined in Lemma 2.6. Therefore, any solution \(\bar{u}\) of (2.3) must satisfy \(\bar{u} \geq \eta \phi\) hence, \(\bar{u}(x) \geq C \text{dist}(x, \partial \Omega)\) for some \(C > 0\). Note that for \(\lambda > 0\) small enough and for every \(\epsilon > 0\) we have

\[g_\epsilon(t) = \lambda \left( \frac{1}{(t + \epsilon)^\delta} + t^\eta + f(t) \right) + \mu\]

is nonincreasing for \(t \in [0, |\bar{u}|_\infty]\).

We assert that \(u_{\lambda, \epsilon} \geq u_{\lambda, \epsilon'}\) for \(\epsilon' \geq \epsilon > 0\).

Indeed we have,

\[-\Delta_p u_{\lambda, \epsilon'} + \Delta_p u_{\lambda, \epsilon} = \lambda(g(u_{\lambda, \epsilon}) - g(u_{\lambda, \epsilon'}))\] (2.4)
Multiplying (2.4) by \((u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+\) and then integrating over \(\Omega\) we get using Lemma 2.7,

\[
C_1 \int_{\Omega} |\nabla (u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+|^p \, dx \leq \int_{\Omega} (g(u_{\lambda,\epsilon}) - g(u_{\lambda,\epsilon'}))(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+ \, dx \leq 0
\]

for \(p \geq 2\), \(C_1 > 0\) and,

\[
C_2 \left( \int_{\Omega} |\nabla (u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})|^p \, dx \right)^{\frac{2}{p}} \leq \int_{\Omega} (g(u_{\lambda,\epsilon}) - g(u_{\lambda,\epsilon'}))(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+ \, dx \leq 0
\]

for \(1 < p < 2\) and \(C_2 > 0\).

Combining the above inequalities we get,

\[
u_{\lambda,\epsilon} \geq u_{\lambda,\epsilon'}\text{ for } \epsilon' \geq \epsilon > 0
\]

for any solution \(u_{\lambda,\epsilon}\) of (2.2). Therefore, \(u_{\lambda,\epsilon}(x) > C \text{dist}(x, \partial \Omega)\), where \(C\) is a constant independent of \(\epsilon\).

3. A Priori Estimate

In this section we apply Theorem 2.9 and the blowup technique due to Gidas et al [18] to get an uniform estimate on the solution of the regularised problem (2.2). But before that we need to show that there cannot be a concentration of maxima of family of solutions approaching the boundary. To show that we first start with a geometric lemma due to Madsen-Tornehave [23].

Lemma 3.1. (Tubular Neighbourhood) Let \(M(\subset \mathbb{R}^N)\) be a \(C^2\) compact submanifold of dimension \(N - 1\). Then there exists an open set \(V \subset \mathbb{R}^N\) and an extension to the \(\text{Id}_M\) to a continuous map \(r : V \rightarrow M\) such that

1. For \(x \in V\) and \(y \in M\), \(|x - r(x)| \leq |x - y|\) with equality iff \(y = r(x)\).
2. For every \(x_0 \in M\) the fiber \(r^{-1}(x_0)\) consists of \(\{x \in \mathbb{R}^N : x = x_0 + t \gamma(x_0)\} \) with \(|t| < \rho\) for some \(\rho > 0\).
We call $V = V_p$ the open tubular neighbourhood of $M$ of radius $\rho$.
Moreover $M = \partial \Omega$ with $\Omega$ convex and bounded then

$$\{x \in \mathbb{R}^N : x = y + t\gamma(y), \ 0 < t < 2\rho, \ y \in \partial \Omega \} \subset \Omega$$

where $\gamma(y)$ denotes the inward unit normal to $\partial \Omega$ at $y$.

**Lemma 3.2.** For each $\epsilon > 0$ and for every positive solution $u_{\lambda, \epsilon}$ to (2.2) there exists a global maximum $y_{\epsilon} \in \Omega$ of $u_{\lambda, \epsilon}$ such that $\text{dist}(y_{\epsilon}, \partial \Omega) \geq \rho$, where $\rho$ is defined in Lemma 3.1 and independent of $\epsilon$.

**Proof.** Assume by contradiction that there exists $\epsilon_n$ s.t $u_{\lambda, \epsilon_n}$ always attains its maximum in the set $A = \{x : \text{dist}(x, \partial \Omega) < \rho\}$.
Choose $p \in A$ s.t $u_{\lambda, \epsilon_n} = u_{\lambda, \epsilon_n}(p)$.
One can find an $x_0 \in \partial \Omega$ such that $p$ belongs to the normal line through $\partial \Omega$ at $x_0$ with $\text{dist}(p, x_0) < \rho - \gamma$ for some $0 < \gamma < \rho$. Let the normal line be $P$ which also intersects the $B(0, R - \rho)$ at $y$.
Clearly, $u_{\lambda, \epsilon_n}(y) < u_{\lambda, \epsilon_n}(p)$, but that contradicts the fact that $u_{\lambda, \epsilon_n}$ is decreasing in the radial direction. Hence the result.

We will also need a nonexistence result due to Ni-Serrin [24] which will allow us to deduce a contradiction while applying the blow-up technique.

**Lemma 3.3.** Consider the problem

$$\begin{align*}
-\Delta_p u &= u^q \text{ in } \mathbb{R}^N \\
u &> 0; \ u \in C^1(\mathbb{R}^N) \\
\|u\|_{\infty} &= 1
\end{align*}$$

if $p - 1 < q < \frac{Np}{N - p} - 1$ and $1 < p < N$ then the problem (3.1) does not admit a nontrivial radial solution.

With the above lemmas at our disposal we proceed with the main result of this section:

**Theorem 3.4.** Assume that $f$ satisfies (H), $p - 1 < q < \frac{Np}{N - p} - 1$ and $1 < p < N$. Then for $\lambda$ sufficiently small there exists $C > 0$ such that $\|u_{\lambda, \epsilon}\| < C$ for any $C^1$ solution of (2.2) where $C$ is independent of $\epsilon$ and depends only on $\Omega$.

Henceforth we will denote $u_{\lambda, \epsilon}$ by $u_\epsilon$. 

9
Proof. From Lemma 3.2 we have the existence of a minima \( p_\epsilon \) of \( u_\epsilon \) which lies on the set \( \{ x : \text{dist}(x, \partial \Omega) \geq \rho \} \) for each \( \epsilon > 0 \) i.e., \( \max u_\epsilon = u_\epsilon (p_\epsilon) \) and \( p_\epsilon \in \{ x : \text{dist}(x, \partial \Omega) \geq \rho \} \) and \( \rho \) is defined as in Lemma 3.1.

We assert that \( \exists C > 0 \) s.t \( ||u_\epsilon||_\infty < C \) for every positive solution \( u_\epsilon \) of (2.2).

Let on the contrary we have a subsequence \( u_n \) of \( u_\epsilon \) s.t \( H_n = ||u_n||_\infty = u_n(p_n) \to \infty \) and \( \text{dist}(p_n, \partial \Omega) \geq \rho \).

Define, \( v_n(x) = H_n^{-1} u_n(y) \) where \( y = M_n x + p_n \) and \( M_n > 0 \) will be defined later. The function \( v_n \) is well defined in \( B(0, \frac{\rho}{2M_n}) \) and \( v_n(0) = ||v_n||_\infty = 1 \).

Hence we have,

\[
\nabla v_n(x) = H_n^{-1} M_n \nabla u_n(y) \\
\Delta_p v_n(x) = H_n^{1-p} M_n^{p} \Delta_p u_n(y)
\]

Now choosing \( M_n^p = H_n^{p-1-q} \) we have, \( \Delta_p v_n(x) = H_n^{-q} \Delta_p u_n(y) \)

Therefore,

\[
-\Delta_p v_n(x) = \lambda H_n^{-q} \left( \frac{1}{(u_n + \frac{1}{n})^\delta} + u_n^q + f(u_n) \right) \\
= \lambda H_n^{-q} \left( \frac{1}{(H_n v_n + \frac{1}{n})^\delta} + H_n^q v_n^q + f(H_n v_n) \right) \\
= \Theta(\lambda, x, v_n)
\]

Note since \( q > p - 1 \) and \( M_n \to 0 \) we have by Theorem 2.9,

\[
(H_n v_n + \frac{1}{n})^{-\delta} \leq \frac{C}{\rho^\delta} \tag{3.2}
\]

since, \( \text{dist}(y, \partial \Omega) > \frac{\rho}{2} \).

Again, since \( f \) satisfies (H) we have,

\[
v_n^q + H^{-q} f(H_n v_n) \leq v_n^q + \chi v_n^q \leq C \tag{3.3}
\]
as \( n \to \infty \) and \( C \) dependent of \( \Omega \) and independent of \( n \).

Combining (3.2) and (3.3) we have,

\[
\Theta(\lambda, x, v_n) \leq C
\]

for some constant \( C > 0 \) independent of \( n \).
Define, 
\[ z_n(x) = \begin{cases} v_n(x), & |x| < \frac{p}{2M_n} \\ 0 & \text{otherwise} \end{cases} \]

Using regularity result of Tolksdorf \[27\] we get \( z_n \in C^{1,\alpha} \) in \( B(0, R) \) for \( 2R \leq \frac{p}{2M_n} \) and \( ||z_n||_{C^{1,\alpha}} < C \) in \( B(0, R) \) for \( C \) independent of \( n \). Therefore \( z_n \) converges in the \( C^1 \) norm to \( z_0 \) and \( z_0(0) = 1 \). So one can pass to the limit to obtain that \( z_0 \) is a nontrivial radial solution (being the uniform limit of radial functions) of

\[ -\Delta z_0 = z_0^q \text{ in } \mathbb{R}^n \]
\[ z_0(0) = 1 \]

for \( p - 1 < q < \frac{Np}{N-p} - 1 \), which is a contradiction to Lemma 3.1.

4. Existence of solution to equation (2.1)

We are now in a position to provide the existence theorem for the regularised equation (2.1). This proofs and statements of this section closely follows that of Azizieh-Clement \[5\]. We start with a lemma below due to Azizieh-Clement \[5\]

**Lemma 4.1. (Azizieh and Clément)** Let \((E, ||.||)\) be a real Banach space. Let \( G : \mathbb{R}^+ \times E \rightarrow E \) be a continuous and maps bounded subsets on relatively compact subsets. Suppose moreover \( G \) satisfies

1. \( G(0, 0) = 0 \)
2. there exists \( R > 0 \) such that:
   (a) \( u \in E, ||u|| \leq R \) and \( u = G(0, u) \) implies \( u = 0 \).
   (b) \( \deg(Id - G(0, ), B(0, R), 0) = 1 \).

Let \( K \) denotes the set of all solutions of the problem \( P \)

\[ u = G(t, u) \]

in \( \mathbb{R}^+ \times E \). Let \( C \) denotes the closed connected subset of \( P \) to which \((0,0)\) belongs. If

\[ C \cap \{0\} \times E = \{(0,0)\}, \]

Then \( C \) is unbounded in \( \mathbb{R}^+ \times E \).

Using the above lemma we have the following result:
**Theorem 4.2.** For $0 < \delta < 1$ and $(p, q)$ satisfies \((1.2)\) with $f$ satisfying \((H)\). Then there exists at least one radially solution $u_{\lambda, \epsilon}$ to the problem \((2.1)\) in $C^1_0(\tilde{\Omega})$ for each $\epsilon > 0$.

We will now modify our problem \((2.2)\) in such a way that we can apply Lemma 4.1.

Define, $[W_0^{1,p}(\Omega)]^+ = \{u \in W_0^{1,p}(\Omega) : u \geq 0\}$ for $1 < p < \infty$.

Then for $\epsilon > 0$ and $\lambda \geq 0$ we consider the operator $A : [W_0^{1,p}(\Omega)]^+ \to W^{-1, \frac{p}{p-1}}(\Omega)$ such that

$$A(u) = -\Delta_p u - \frac{\lambda}{(u + \epsilon)^q}$$

We have the following lemma concerning the operator $A$:

**Lemma 4.3.** Let $\Omega$ be bounded set of class $C^{1,\alpha}$ for some $\alpha > 0$ and $g \in L^\infty(\Omega)$. Then the problem

$$Au = g \text{ in } \Omega$$

$$u \in W_0^{1,p}(\Omega), \ p > 1$$

has a unique solution $u \in C^1_0(\tilde{\Omega})$. Moreover if we define the operator $K : L^\infty(\Omega) \to C^1_0(\tilde{\Omega}) : g \to u$ where $u$ is the unique solution of \((4.1)\), then $K$ is continuous, compact and order-preserving.

**Proof.** Clearly the operator $A$ is well-defined and continuous from $[W_0^{1,p}(\Omega)]^+$ to $W^{-1, \frac{p}{p-1}}(\Omega)$. Moreover $A$ is strictly monotone and coercive for $0 < \delta < 1$ (See Badra et al [8]).

Hence by Minty-Browder theorem there exists a unique solution $u \in [W_0^{1,p}(\Omega)]^+$ to the problem $Au = f$ for $f \in W^{-1, \frac{p}{p-1}}(\Omega)$.

From the $L^\infty$ estimates of Anane [3] and the $C^{1,\alpha}$ estimates of Lieberman [20] and Tolksdorf [27] we have $K$ is continuous and compact from $L^\infty(\Omega)$ to $C^1_0(\tilde{\Omega})$. The Strong Maximum Principle of Cuesta-Takac (Lemma 2.8) gives the order-preserving property of $K$.

With this we have the following lemma:

**Lemma 4.4.** Under the hypothesis of Theorem 4.2 and for each $\epsilon > 0$ there exists a real number $R > 0$ such that if $(u_\epsilon, \gamma) \in C^1_0(\tilde{\Omega}) \times [0, 1]$ is a solution of

$$u = K(\gamma g_\lambda(u))$$

$$u_\epsilon \neq 0$$

\(12\)
where \( g_{\lambda}(u) = \lambda(u^q + f(u)) \) then \( ||u_\epsilon||_{C^0_\beta(\bar{\Omega})} > R \)

**Proof.** Applying the exact same proof of Theorem 2.9 we have that any solution \( u_\epsilon \) of (4.2) satisfies \( u(x) \geq C \text{dist}(x, \partial \Omega) \). Hence we have, \( ||u_\epsilon||_{C^0_\beta(\bar{\Omega})} > R \), where \( R \) does not depend on \( \epsilon \).

Once we have the above lemma we can use the proof of Theorem 2.1 of Azizieh-Clement [5] to prove Theorem 4.2. We will provide it for completeness.

**Proof.** Define \( G : [0, \infty) \times C^1_0(\bar{\Omega}) \rightarrow C^1_0(\bar{\Omega}) \) such that \( G(\mu, u) = K(g_{\lambda}(u) + \mu) \).

Taking \( E = C^1_0(\bar{\Omega}) \) we have by Lemma 4.1 that \( G \) is compact and continuous and satisfies part 1(a) of Theorem 4.1.

Let us define \( h : [0, 1] \times B'(0, R) \rightarrow C^1_0(\bar{\Omega}) \) where \( B'(0, a) \) is the ball in \( C^1_0(\bar{\Omega}) \) of radius \( a \) and centered at 0 as

\[
h(\gamma, u) = K(\gamma g_{\lambda}(u)) \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad \forall \ u \in B'(0, R)
\]

By Lemma 4.1, \( h \) is compact continuous and satisfies \( h(1, \cdot) = G(0, \cdot) \) and \( h(0, \cdot) = 0 \).

Again by Lemma 4.2 we have \( u - h(\gamma, u) \neq 0 \) for all \( u \in \partial B'(0, R) \) and \( \gamma \in [0, 1] \).

So using Theorem 3.4 we get that all solution of (2.2) are bounded. Thus we have the existence of a \( u_\epsilon \in C^1_0(\bar{\Omega}) \) from Theorem 4.1. Moreover from Theorem 2.9 we get the \( u_\epsilon(x) \geq C \text{dist}(x, \partial \Omega) \), where \( C \) is independent of \( \epsilon \). Combining these results we have radially symmetric \( u_\epsilon \in C \) solving (2.1).

**Proof of Theorem 2.1**

We have from Theorem 3.4 that for any solution of (2.2), \( ||u_\epsilon||_{C^1_0(\bar{\Omega})} \leq C \) where \( C \) is independent of \( \epsilon \). Using the Boundary regularity theorem of Lieberman [20] on (2.2), we have \( u_\epsilon \in C^{1, \alpha}(\Omega) \) for some \( 0 < \alpha < 1 \) and \( ||u_\epsilon||_{C^{1, \alpha}(\Omega)} \leq C \) and \( C \) depends only on \( N, \Omega \) and \( p \) only.

Therefore Ascoli-Arzela theorem we have a subsequence \( u_{\epsilon_n} \) of \( u_\epsilon \) such that \( u_{\epsilon_n} \) converges uniformly to \( u \in C^1_0(\bar{\Omega}) \).

Note that since \( (u_\epsilon) \) is a sequence of radially symmetric function and the convergence of \( u_{\epsilon_n} \rightarrow u \) is uniform, we have \( u \) to be radially symmetric in \( \Omega \). Again since by Theorem 2.9 we have that \( u_\epsilon(x) \geq C \text{dist}(x, \partial \Omega) \) with \( C \)
independent of $\epsilon$ we have by Hardy Inequality we have,

$$\sup_{\epsilon_n > 0} \frac{\phi}{(u_{\epsilon_n} + \epsilon_n)^\delta} < \infty$$

for any $\phi \in C^\infty_c(\Omega)$.

Therefore passing to the limit with the assumption that $f$ satisfies (H) we have $u \in C^1_0(\overline{\Omega})$ satisfies (1.1) and since $u$ is the limit of a non-increasing sequence $u_\epsilon$ we have, $u(x) \geq C \text{dist}(x, \partial \Omega)$. Thus we have the existence of radially symmetric $u \in C$ satisfying (1.1) in weak sense.

5. Acknowledgement

The following work was supported by DST Inspire Faculty Fellowship 2013-MA29.

References

[1] Walter Allegretto and Yin Xi Huang. A Picone’s identity for the $p$-Laplacian and applications. *Nonlinear Anal.*, 32(7):819–830, 1998.

[2] Antonio Ambrosetti, Jesus Garcia Azorero, and Ireneo Peral. Multiplicity results for some nonlinear elliptic equations. *J. Funct. Anal.*, 137(1):219–242, 1996.

[3] Aomar Anane. Etude des valeurs propres et de la résonance pour l’opérateur $p$-laplacien. Thèse de doctorat, Université Libre de Bruxelles, 1988.

[4] David Arcoya and Lourdes Moreno-Mérida. Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity. *Nonlinear Anal.*, 95:281–291, 2014.

[5] Céline Azizieh and Philippe Clément. A priori estimates and continuation methods for positive solutions of $p$-Laplace equations. *J. Differential Equations*, 179(1):213–245, 2002.

[6] Mehdi Badra, Kaushik Bal, and Jacques Giacomoni. Existence results to a quasilinear and singular parabolic equation. *Discrete Contin. Dyn. Syst.*, (Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. I):117–125, 2011.
[7] Mehdi Badra, Kaushik Bal, and Jacques Giacomoni. Some results about a quasilinear singular parabolic equation. *Differ. Equ. Appl.*, 3(4):609–627, 2011.

[8] Mehdi Badra, Kaushik Bal, and Jacques Giacomoni. A singular parabolic equation: Existence, stabilization. *J. Differential Equation*, 252:5042–5075, 2012.

[9] Kaushik Bal and Jacques Giacomoni. Symmetry and apriori estimate to a singular elliptic problem. *Zaragoza-Pau Conference on Applied Mathematics and Statistics, JACA 2010*, 11:1–11, 2011.

[10] Lucio Boccardo and Luigi Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, 2010.

[11] M. M. Coclite and G. Palmieri. On a singular nonlinear Dirichlet problem. *Comm. Partial Differential Equations*, 14(10):1315–1327, 1989.

[12] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, 1977.

[13] Mabel Cuesta and Peter Takáč. A strong comparison principle for positive solutions of degenerate elliptic equations. *Differential Integral Equations*, 13(4-6):721–746, 2000.

[14] Lucio Damascelli and Berardino Sciunzi. Regularity, monotonicity and symmetry of positive solutions of $m$-Laplace equations. *J. Differential Equations*, 206(2):483–515, 2004.

[15] J. Giacomoni and K. Saoudi. Multiplicity of positive solutions for a singular and critical problem. *Nonlinear Anal.*, 71(9):4060–4077, 2009.

[16] Jacques Giacomoni, Ian Schindler, and Peter Takáč. Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 6(1):117–158, 2007.

[17] Jacques Giacomoni and Konijeti Sreenadh. Multiplicity results for a singular and quasilinear equation. *Discrete Contin. Syst., Proceedings of the 6th AIMS International Conference, Suppl.*, pages 429–435, 2007.
[18] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8):883–901, 1981.

[19] Norimichi Hirano, Claudio Saccon, and Naoki Shioji. Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities. *Adv. Differential Equations*, 9(1-2):197–220, 2004.

[20] Gary Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[21] Peter Lindqvist. On the equation div ($\|\nabla u\|^{p-2}\nabla u$) + $\lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990.

[22] Li Ma and J. C. Wei. Properties of positive solutions to an elliptic equation with negative exponent. *J. Funct. Anal.*, 254(4):1058–1087, 2008.

[23] Ib Madsen and Jørgen Tornehave. *From calculus to cohomology*. Cambridge University Press, Cambridge, 1997. de Rham cohomology and characteristic classes.

[24] Wei-Ming Ni and James Serrin. Nonexistence theorems for quasilinear partial differential equations. In *Proceedings of the conference commemorating the 1st centennial of the Circolo Matematico di Palermo (Italian) (Palermo, 1984)*, number 8, pages 171–185, 1985.

[25] D Ruiz. A priori estimates and existence of positive solutions for strongly nonlinear problems. *Journal of Differential Equation*, 199:96–114, 2004.

[26] Yijing Sun, Shaoping Wu, and Yiming Long. Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. *J. Differential Equations*, 176(2):511–531, 2001.

[27] Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.

[28] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[29] Haitao Yang. Positive versus compact support solutions to a singular elliptic problem. *J. Math. Anal. Appl.*, 319(2):830–840, 2006.