Absolute Summability Factor $|N, p_n|_k$ of Improper Integrals

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Abstract— In this paper, we defined the summability for integrals and established a theorem on absolute Nörlund summability $|N, p_n|_k$ factors of improper integral under sufficient conditions. Some auxiliary results (well-known) have also been deduced from the main results under suitable conditions.

Keyword - Absolute Summability, Nörlund summability, Improper Integrals, Inequalities for Integrals.

I. INTRODUCTION

1. Summability factor concerning infinite series: Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums, $s_n = \sum_{n=0}^{n} a_n$ and $\sigma_n$ be the $n^{th}$ Cesàro means of the series, i.e.,

$$\sigma_n = \frac{1}{n} \sum_{k=0}^{\infty} s_k$$

(1)

The series $\sum_{n=0}^{\infty} a_n$ is said to be $|C, 1|_k$, $k \geq 1$ summable [5], if

$$\lim_{n \to \infty} \sigma_n = s,$$

(2)

and

$$\sum_{n=1}^{\infty} p^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty$$

(3)

2. Summability factor concerning improper integrals: Let $f$ be a real valued continuous function of $t$ in the interval $[0, \infty)$ and $s(x) = \int_{0}^{x} f(t)dt$. Then, $\int_{0}^{\infty} f(t)dt$ is said to be summable $|C, 1|$, if

$$\int_{0}^{\infty} |\sigma'(x)| \, dx < \infty.$$  

(4)

and $\int_{0}^{\infty} f(t)dt$ is said to be summable $|C, 1|_k$, $k \geq 1$, if

$$\int_{0}^{\infty} x^{k-1} |\sigma'(x)|^k \, dx < \infty,$$

(5)

where $\sigma(x)$ is Cesàro mean of $s(x)$ and given by

$$\sigma(x) = \frac{1}{x} \int_{0}^{x} (x-t) f(t) dt,$$

(6)

or

$$\sigma(x) = \frac{1}{x} \int_{0}^{x} s(t) dt.$$  

(7)

The Kronecker identity: $s(x) - \sigma(x) = \nu(x)$, where

$$\nu(x) = \frac{1}{x} \int_{0}^{x} tf(t) dt.$$  

(8)
The condition (5) can be written as
\[ \int_{0}^{\infty} \left| \frac{1}{x} \right| v(x) \, dx < \infty. \]  

(9)

Considering the \((N, P_n)\) and \((K, 1, \alpha)\) summability, Parashar [9] obtained the minimum set of conditions for an infinite series to be \((K, 1, \alpha)\) summable. In 1986, Bor [1] found the relationship between two summability techniques \(|C, 1|\) and \(|N, p_n|\) and in [2], he used the \(|N, p_n|\) for generalization of a theorem based on minimal set of sufficient conditions for infinite series. In 2016, Sonker and Munjal [10] determined a theorem on generalized absolute Cesàro summability with the sufficient conditions for infinite series and in [11], they used the concept of triangle matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. In 2017, Sonker and Munjal [12] found the approximation of the function \(f \in \text{Lip}(\alpha, p)\) using infinite matrices of Cesàro submethod and in [13], they obtained boundness conditions of absolute summability factors.

In this way by using the advanced summability method, we can improve the quality of the filters.

Borwein [3] extended many results on ordinary and absolute summability methods of integral. Çanak [4] and Totur [14] worked on the concept of Cesàro summability with a very interesting result for integrals. In the same direction, we extended the results of Mazhar [7] with the help of some new generalized conditions and absolute Nörlund summability \(|N, p_n|\) factor for integrals.

II. KNOWN RESULTS

In [6], Kishore has proved the following theorem concerning \(|C, 1|\) and \(|N, p_n|\) summability methods.

**Theorem 1:** Let \(p_0 > 0, p_n \geq 0\) and \(p_n\) be a non-increasing sequence. If \(\sum a_n\) is summable \(|C, 1|\), then the series \(\sum a_n P_n (n + 1)^{-1}\) is summable \(|N, p_n|\).

By concerning absolute Cesàro summability \(|C, 1|\) factors and a positive monotonic non-decreasing function \(\gamma(x)\), Özgen [8] obtained the following results for integrals.

**Theorem 2:** Let \(\gamma(x)\) be a positive monotonic non-decreasing function such that
\[ \lambda(x) \gamma(x) = O(1) \quad \text{as} \quad x \to \infty, \]
\[ \int_{0}^{\infty} u |\lambda''(u)| \gamma(u) \, du = O(1), \]
\[ \int_{0}^{\infty} \frac{|v(u)|^k}{u} \, du = O(\gamma(x)) \quad \text{as} \quad x \to \infty, \]

then the integrals \(\int_{0}^{\infty} f(t) \, dt\) is said to be summable \(|C, 1|\), \(k \geq 1\).

III. MAIN RESULTS

In the present research article, we extended the result of Özgen [8] by using the \(|C, 1|\) summability and some other concepts. With the help of functions \(\phi(x)\) and \(\chi(x)\) Cesàro summability \(|C, 1|\), we established the following theorem.

**Theorem 3:** Let \(p(0) > 0, p(x) \geq 0\) and \(p(x)\) be a non-increasing function. Let \(\chi'(x)\) be a positive non-decreasing function and there be two functions \(\beta(x)\) and \(\epsilon(x)\) such that
\[ |\epsilon'(x)| \leq \beta(x), \]
\[ \beta(x) \to 0 \quad \text{as} \quad x \to \infty, \]
\[ \int_{0}^{\infty} u |\beta''(u)| \chi(u) \, du < \infty, \]
\[ |\epsilon(x)| \chi(x) = O(1), \]
\[ \int_{0}^{\infty} \frac{|v(u)|^k}{u} \, du = O(\chi(x)) \quad \text{as} \quad x \to \infty, \]

then the integrals \(\int_{0}^{\infty} f(t) \, dt\) is said to be summable \(|N, p_n|\) for \(k \geq 1\).
Note: The above theorem can be proved by using the concept of example that \( \int_0^\infty x | \beta'(x) | \chi(x) dx < \infty \) is weaker \( \int_0^\infty x | \epsilon''(x) | \chi(x) dx < \infty \), and hence the introduction of the function \( \{ \beta(x) \} \) is justified.

Proof: It may be possible to choose the function \( \beta(x) \) such that
\[
| \epsilon'(x) | \leq \beta(x),
\]
(18)

When \( \epsilon'(x) \) oscillates, \( \beta(x) \) may be chosen such that \( | \beta(x) | < | \epsilon''(x) | \). Hence, \( \beta'(x) < | \epsilon''(x) | \), so that
\[
\int_0^\infty x | \epsilon''(x) | \chi(x) dx < \infty
\]
is a weaker requirement than \( \int_0^\infty x | \epsilon'(x) | \chi(x) dx < \infty \).

IV. PROOF OF THEOREM

In order to prove the theorem, we need to consider only the special case in which \( N, p_u |_k \) is \( \{ C, 1 \} \), that is, we shall prove that \( \int_0^\infty f(t) dt \) is summable \( \{ C, 1 \} \). Our theorem will then follow by means of theorem 1. Let \( T(x) \) be the function of \( n^\alpha (C, 1) \) means of the integral \( \int_0^\infty f(t) dt \). The integral is \( \{ C, 1 \} \) summable, if
\[
\int_0^\infty x^{\alpha-1} | T'(x) |^4 dx = O(1) \quad \text{as} \quad x \to \infty,
\]
(19)

where \( T(x) \) is given by
\[
T(x) = \frac{1}{X} \int_0^X \int_0^x \epsilon(u)f(u)du dt
= \frac{1}{X} \int_0^x \epsilon(u)f(u)du \int_0^x dt
= \frac{1}{X} \int_0^x (x-u)\epsilon(u)f(u)du
= \int_0^x \left( 1 - \frac{u}{x} \right) \epsilon(u)f(u)du
\]
(20)

On differentiating both sides with respect to \( x \), we get
\[
T'(x) = \frac{1}{X^2} \int_0^x u\epsilon(u)f(u)du
= \frac{\epsilon(x)}{X^2} \int_0^x uf(u)du - \frac{1}{X^2} \int_0^x \epsilon'(u)f(t)du dt
= \frac{\epsilon(x)}{X} \int_0^x (x+u)\epsilon(u)f(u)du
= \frac{1}{X} \int_0^x u\epsilon(u)du + \int_0^x \epsilon'(u)f(t)du dt
= T_1(x) + T_2(x).
\]
(21)

Applying Minkowski’s inequality,
\[
| T_n |^4 = | T_1 + T_2 |^4 < 2^4 \left( | T_1 |^4 + | T_2 |^4 \right)
\]
(22)
Applying Hölder’s inequality, we have
\[
\int_0^t x^{k-1} \left| T_1(t) \right|^k dt = \int_0^t x^{k-1} \left| \frac{v(t)}{t} \right|^k dt \\
= \int_0^t \frac{1}{t} \left| v(t) \right|^k \left| x(t) \right| | x(t) | dt \\
\leq \int_0^t \frac{1}{t} \left| v(t) \right|^k dt \\
= \left| x(t) \right| \int_0^t \frac{1}{t} \left| v(t) \right|^k dt - \int_0^t \left| x(t) \right| dt \left| v(t) \right|^k du dt \\
= O(1) \left| x(t) \right| \chi(t) - \int_0^t \beta(t) \chi(t) dt \\
= O(1) - \int_0^t \left| \beta'(x) \right| dx \int_0^t \chi(u) du \\
\leq O(1) - \int_0^t \left| \beta'(u) \right| \chi(u) du \\
= O(1) \text{ as } x \to \infty. \tag{23}
\]

By virtue of the hypotheses of theorem 3,
\[
\int_0^t x^{k-1} \left| T_2(t) \right|^k dt = \int_0^t x^{k-1} \left| \frac{1}{t^{2k}} \int_0^t \left| u e'(u) v(u) \right| du \right|^k dt \\
\leq \int_0^t \frac{1}{t^{2k}} \left( \int_0^t \left| u e'(u) \right|^k \left| v(u) \right|^k du \right) \left( \frac{1}{t} \right)^{k-1} dt \\
= \int_0^t \left| u e'(u) \right|^k \left| v(u) \right|^k du \left( \frac{1}{t} \right)^{k-1} dt \\
= \int_0^t \left| u e'(u) \right| \left| v(u) \right|^k du \left( \frac{1}{t} \right)^{k-1} dt \\
= x \left| \beta(x) \right| \chi(x) - \int_0^t \left| \beta(u) \right| \chi(u) du - \int_0^t \left| \beta'(u) \right| \chi(u) du \\
\leq O(1) \text{ as } x \to \infty. \tag{24}
\]

On collecting (20)-(24), we have
\[
\int_0^t x^{k-1} \left| T'(t) \right|^k dt = O(1) \text{ as } t \to \infty, \tag{25}
\]

Hence proof of the theorem is complete.
V. COROLLARIES

**Corollary 1:** Let \( p(0) > 0, p(x) \geq 0 \) and \( p(x) \) be a non-increasing function. Let \( \chi(x) \) be a positive non-decreasing function such that

\[
\varepsilon(x)\chi(x) = O(1) \quad \text{as} \quad x \to \infty, \tag{26}
\]

\[
\int_0^\infty |\varepsilon'(u)| \chi(u)du = O(1), \tag{27}
\]

\[
\int_0^\infty \frac{|\varepsilon(u)|}{u}du = O(\chi(x)) \quad \text{as} \quad x \to \infty, \tag{28}
\]

then the integrals \( \int_0^\infty f(t)dt \) is said to be summable \( [N, p_n] \) for \( k \geq 1 \).

**Corollary 2:** Let \( p(0) > 0, p(x) \geq 0 \) and \( p(x) \) be a non-increasing function and \( \varepsilon(x) \) be a convex function such that \( \int \frac{\varepsilon(x)}{x} dx \) is convergent. If \( f \) is bounded on \( [R, \log n, 1] \) with index \( k \), then \( \int_0^\infty f(t)dt \) is summable \( [N, p_n] \) for \( k \geq 1 \).

**Corollary 3:** Let \( p(0) > 0, p(x) \geq 0 \) and \( p(x) \) be a non-increasing function and \( \varepsilon(x) \) be a convex function such that \( \int \frac{\varepsilon(x)}{x} dx \) is convergent. If \( f \) is bounded on \( [R, \log n, 1] \), then \( \int_0^\infty f(t)dt \) is summable \( [N, p_n] \).

**Note:** The above corollaries can be derived by taking the following assumptions in the main result,

(i) For corollary 1, we take \( |\varepsilon'(x)| = \beta(x) \).

(ii) For corollary 2, we take \( \chi(x) = \log(x) \) and \( \varepsilon(x) \) as a convex function.

(iii) For corollary 3, we take \( \chi(x) = \log(x), k = 1 \) and \( \varepsilon(x) \) as a convex function.

VI. CONCLUSION

The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Nörlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nutshell, the absolute summability methods are a motivation for the researchers, interested in studies of improper integrals.

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