Quantum Hall Phase Diagram of Second Landau-level Half-filled Bilayers: Abelian versus Non-Abelian States

Michael R. Peterson and S. Das Sarma
Condensed Matter Theory Center, Department of Physics,
University of Maryland, College Park, MD 20742
(Dated: August 24, 2009)

The quantum Hall phase diagram of the half-filled bilayer system in the second Landau level is studied as a function of tunneling and layer separation using exact diagonalization. We make the striking prediction that bilayer structures would manifest two distinct branches of incompressible fractional quantum Hall effect (FQHE) corresponding to the Abelian 331 state (at moderate to low tunneling and large layer separation) and the non-Abelian Pfaffian state (at large tunneling and small layer separation). The observation of these two FQHE branches and the quantum phase transition between them will be compelling evidence supporting the existence of the non-Abelian Pfaffian state in the second Landau level.

PACS numbers: 73.43.-f, 71.10.Pm

The fractional quantum Hall effect (FQHE) in the second Landau level (SLL) at filling factor \( \nu = 5/2 \) [1] presents an amazing confluence of ideas from condensed matter physics, conformal field theory, topology, and quantum computation. In particular, the fundamental nature of the experimentally observed 5/2 FQHE, whether an exotic spin-polarized non-Abelian incompressible paired state or a more common Abelian incompressible paired state, has remained an enigma for more than 20 years. Although theoretical and numerical work indicates that the 5/2 FQHE belongs to a non-Abelian Pfaffian universality class [2], there is scant experimental evidence supporting this conclusion [3]. Since the question of non-Abelian-ness (or not) of the experimental 5/2 state is of profound importance beyond quantum Hall physics [2], it is useful to contemplate novel situations where the nature of the 5/2 state will manifest itself in a dramatic, but hitherto unexplored, manner. In this Letter, we make a specific suggestion for such a possibility by showing theoretically that a bilayer system may provide important new insight into the nature of the 5/2 FQHE by allowing for a novel quantum phase transition between an Abelian and non-Abelian 5/2 FQHE as system parameters are varied. The system we have in mind could either be a true bilayer double-quantum-well structure or a single wide-quantum-well which manifests effective bilayer behavior. We show that tuning the inter-layer tunneling and/or the layer separation would lead to a transition between the Abelian and the non-Abelian SLL 5/2 FQHE, which should be easily observable experimentally in standard FQHE transport experiments.

We consider the FQHE in spin-polarized bilayer quantum wells with non-zero tunneling at total filling factor \( \nu = 5/4 + 5/4 = 5/2 \), i.e., the half filled SLL with each layer having one quarter SLL filling. The best known variational \( N \) electron wavefunctions describing the physics of this system are the paired Abelian Halperin 331 (331) state [4]

\[
\Psi_{331} = \prod_{i<j}^{N/2} (z_i^R - z_j^R)^{3/2} \prod_{i<j}^{N/2} (z_i^L - z_j^L)^{3/2} \prod_{i,j}^{N/2} (z_i^R - z_j^L) \tag{1}
\]

which is known [8] to be a good description of two-component FQHE states at half-filling and the non-Abelian Moore-Read Pfaffian state (Pf) [6]

\[
\Psi_{\text{Pf}} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j}^{N} (z_i - z_j)^2 \tag{2}
\]

which is a good description of one-component FQHE states [2] at half filling \( z = x - iy \) is the electron position and the superscript \( R(L) \) indicates the right(left) layer.

The interesting possibility, which we validate through numerical calculations, is that by tuning the inter-layer tunneling, one can go from an incompressible FQH state at 5/2 described by a two-component 331 bilayer state in the weak-tunneling regime to a one-component Pf state in the strong-tunneling regime. Although a two-component 331 bilayer FQH state has been observed [2 8] at \( \nu = 1/2 \) in the lowest LL (LLL), the corresponding one-component strong-tunneling state turns out to be the non-FQHE compressible composite fermion sea in the LLL. In the SLL, we establish the possibility of observing the novel Abelian-to-non-Abelian quantum phase transition (with no compressible state in between) in the bilayer \( \nu = 5/2 \) situation. Our predictions can be experimentally tested in bilayer structures or in thick single-layer structures (where the self-consistent field from the electrons produces effective two-component behavior).

Theoretically, the system is parameterized using three variables: the width \( w \) of a single layer (both layers having the same width), the separation \( d \) between the two layers (by definition \( d > w \) since \( d \) is the distance between the center of the two layers), and the inter-layer
tunneling energy $\Delta$ (i.e., the symmetric-antisymmetric energy gap). The most natural theoretical approach is to use the symmetric-antisymmetric basis where $c_{mS} = (c_{mR} + c_{mL})/\sqrt{2}$ and $c_{mA} = (c_{mR} - c_{mL})/\sqrt{2}$ destroy an electron in the symmetric $(S)$ and antisymmetric $(A)$ superposition states, respectively, where $m$ is angular momentum and $c_{mL(R)}$ destroys an electron in the left(right) quantum well. $S(A)$ can be considered to be an effective pseudospin index for the bilayer system. In this basis, the Hamiltonian of spin-polarized interacting electrons in a bilayer quantum well system entirely confined to a single Landau level (all completely filled LLs are considered inert) with tunneling is given by

$$
H = \frac{1}{2} \sum \sum (m_1 \sigma_1, m_2 \sigma_2) [V |m_4 \sigma_4, m_3 \sigma_3| c_{m_1 \sigma_1} c_{m_2 \sigma_2} e^{i} c_{m_4 \sigma_4} e^{-i} c_{m_3 \sigma_3} - \frac{\Delta}{2} \sum (c_{mS} c_{mA} - c_{mA} c_{mS})].
$$

The intra-layer Coulomb potential energy between two electrons is $V = (e^2/\epsilon l)/\sqrt{r^2 + w^2}$ (we use the Zhang-Das Sarma potential to model the single layer quasi-2D interaction) and the inter-layer potential energy is $V = (e^2/\epsilon l)/\sqrt{r^2 + d^2}$ (both $d$ and $w$ are given in units of the magnetic length $l = \sqrt{\hbar c/\epsilon B}$ where $\epsilon$ is the electron charge and $B$ the magnetic field strength). In other words, we solve the bilayer interacting Hamiltonian in the $d$-$w$-$\Delta$ parameter space in the $S_z$ ($z$-component of the pseudospin operator) basis. All lengths and energies are given in units of the magnetic length from now on. We use the magnetic length $l$ and the corresponding Coulomb energy $e^2/\epsilon l$ ($\epsilon$ is the dielectric constant of the host semiconductor) as the unit of length and energy, respectively, throughout (including figures).

We exactly diagonalize $H$ for finite $N$, utilizing the spherical geometry where the electrons are confined to a spherical surface of radius $R = \sqrt{N_\phi / 2}$, $N_\phi / 2$ is the total magnetic flux piercing the surface ($N_\phi$ is an integer according to Dirac), and the filling factor in the partially occupied SLL is $\lim N_\rightarrow \infty N_\phi / N$. Incompressible ground states (those exhibiting the FQHE) are uniform states with total angular momentum $L = 0$ and a non-zero excitation gap. We are concerned with the Pf and 331 variational states for total $\nu = 5/4 + 5/4 = 5/2$ so $N_\phi = 2N - 3$. Throughout, we consider $(N, N_\phi) = (8, 13)$ and mention that due to the pseudospin component present in the bilayer problem the Hilbert space is much larger (more than $10^5$ states for $N = 8$ and over $7 \times 10^6$ for the next largest system at $N = 10$) than typical one-component FQHE systems. The $N = 6$ electron system is aliased with $2+3/7$ filled LLs leading to possible ambiguities. $N = 8$ should be adequate for a qualitative and semi-quantitative understanding of the physics.

We investigate this system with the usual probes used in theoretical FQHE studies by calculating: (i) wavefunction overlap between variational ansatz (Pf and 331) and the exact ground state—an overlap of unity(zero) indicating the physics is(is not) described by the ansatz; (ii) expectation value of the $z$-component of the pseudospin operator $S_z$ of the exact ground state—a value of zero($N/2$) indicating the ground state to be two-(one-)component; and (iii) energy gap (provided the ground state is a uniform state with $L = 0$)–a non-zero excitation gap indicating a possible FQHE state. In other words, we ask: (i) What is the physics (i.e., 331 or Pf)?; (ii) Is the system one- or two-component?; (iii) Will the system display FQHE (i.e., is the system compressible or incompressible)?

(i) What is the physics? The calculated wavefunction overlap between the exact ground state $\Psi_0$ of $H$ and the two appropriate candidate variational wavefunctions ($\Psi_{\text{Pf}}$ and $\Psi_{\text{331}}$) as a function of distance $d$ and tunneling energy $\Delta$ is shown in Fig. 1. First we focus on the situation with zero width $w = 0$ (Fig. 1b) and concentrate on the overlap with $\Psi_{\text{Pf}}$. In the limit of zero tunneling and zero $d$ the overlap with $\Psi_{\text{Pf}}$ is small—approximately 0.5 (unclear from the figure). However, only weak tunneling is required to produce a state with a sizeable overlap of approximately $\sim 0.96$ and as the tunneling is increased this overlap remains large and approximately constant. For $d \neq 0$, in the weak tunneling limit, the overlap with $\Psi_{\text{Pf}}$ decreases drastically. Adding moderate to strong tunneling we obtain a sizeable overlap, decreasing gently in the large $d$ limit. This shows that the strong-tunneling (one-component) regime is well described by the Pf state.

Next we consider the overlap between $\Psi_0$ and $\Psi_{\text{331}}$. In the zero tunneling limit, as a function of $d$, we find the overlap starts small, increases to a moderate maximum of $\approx 0.80$ at $d \sim 1$ before achieving an essentially constant value of $\approx 0.56$. For $d > 4$ the overlap remains relatively constant and slowly decreases as the tunneling is increased (in fact, there is a slight increase in the overlap to $\sim 0.6$ for a region of positive $d > 4$ and $0.1 \lesssim \Delta \lesssim 0.15$). Thus, the weak-tunneling (two-component) regime is well described by $\Psi_{\text{331}}$.

In Fig. 1d, we consider the finite layer width situation (i.e., $w \neq 0$) which may enhance the overlap with paired FQHE states in the SLL. Indeed, if we consider zero tunneling and $w > 0$ (and necessarily $d > w$) we find that
the overlap between \( \Psi_0 \) and \( \Psi_{331} \) can be increased significantly to \( \sim 0.94 \) by using \( w = 1.2 \) and \( d = 1.4 \). There is a region of large overlap for \( 1.1 < d/w < 2.5 \) (provided \( w \lesssim 2 \)). The overlap with \( \Psi_{331} \) increases for finite \( w \) compared with the \( w = 0 \) case, while in the strong tunneling limit, the overlap decreases with increasing \( w \). The overlap with \( \Psi_{Pf} \) is also affected; namely, in the strong tunneling regime the overlap increases for \( d \neq 0 \) and the amount of tunneling required to increase the overlap decreases.

(ii) Is the system one- or two-component? Our conclusion so far, based on overlap calculations, is completely consistent with the calculated expectation value of the \( z \)-component of the pseudospin operator, \( \langle S_z \rangle \) which is essentially the order parameter describing the one-component to two-component transition, i.e., \( \langle S_z \rangle \approx N/2 \) describes a one-component phase whereas \( \langle S_z \rangle \approx 0 \) describes a two-component phase. Fig. 2 shows the calculated \( \langle S_z \rangle \) as a function of \( d \) and \( \Delta \) with \( w = 0 \) (the \( w = 1.2 \) result is almost identical). For \( d = 0 \) and \( \Delta > 0.1 \) the system is essentially one-component as evidenced by \( \langle S_z \rangle \sim N/2 \) and it is expected that the Pf would provide a good physical description as it indeed does (cf. Fig. 1).

In the region where \( d \neq 0 \) and \( \Delta \) is small, the system is essentially two-component as evidenced by \( \langle S_z \rangle \sim 0 \) and one would expect that the 331 state would provide a good physical description as it indeed does.

(iii) Will the system display the FQHE? Wavefunction overlap and pseudo-spin are only two properties that elucidate the physics. Another property is the energy gap above the \( L = 0 \) ground state in the excitation spectra; a crucial characteristic that determines whether the system would display the FQHE at all. In Fig. 3(b) we show the energy gap, defined as the difference between the first excited and ground state energies at constant \( N_0 \), as a function of \( d \) and \( \Delta \) for the SLL system for \( w = 0 \). In the \( d = 0 \) and \( \Delta = 0 \) limit the system is gapless, however, a gap suddenly appears when the SU(2) symmetry is broken explicitly for \( d \neq 0 \). Most interestingly, the gap (other than the SU(2) symmetric point) is always positive throughout the parameter space. Thus, if one starts in a region described by \( \Psi_{331} \), then one can smoothly move into a region described by \( \Psi_{Pf} \) without the gap closing. The phase transition between two topologically distinct FQHE states, the Abelian 331 and non-Abelian Pf state, is continuous at \( w = 0 \) with no compressible state in between. This is an unexpected interesting feature of our results.

The other notable feature is the large energy gap in the parameter region where the system crosses over from a Pf to a 331 state. However, we stress that the region with the largest gap is the one with \( d = 0 \) and finite \( \Delta \) described by \( \Psi_{Pf} \). It makes intuitive sense that the largest gap would be when \( d = 0 \) since any form of finite thickness trivially lowers the energy by softening the Coulomb interaction, but this is in stark contrast to the results (not shown) in the LLL where the \( d \approx 0 \) region has a lower gap than the “crossover” region which has the largest gap. The reason for this quantitative difference between the LLL and SLL regarding the location of the optimal gap in the parameter space, i.e. at the crossover...
regime for the LLL and at \( d = 0 \) for the SLL, is a matter of quantitative details and arises from the different behaviors of the Coulomb pseudopotentials in the two different Landau levels. We note from investigating the energy gap that at \( \nu = 5/2 \) the Pf one-component FQHE state would be more robust, i.e., would survive to higher temperatures and larger disorder strength, than the 331 two-component FQHE state where the gap, in that parameter region, is lower in part because the Coulomb interaction has been softened by the finite distance required to stabilize it. This may be one simple way to experimentally distinguish the two states in the bilayer system.

Finally, we present the quantum phase diagram for the bilayer system at \( \nu = 5/2 \) in Fig. 3. This phase diagram is created by identifying the 331 and Pf phases as the regions in the parameter space where the overlap with \( \Psi_0 \) is larger, cf. Fig.1. In general, the one-component 331 phase (the upper left region) has a weaker SLL FQHE (\( \nu = 5/2 \)) than that of the two-component Pf (the lower right region). This should be contrasted with the corresponding LLL case where the 331 phase is the only observed bilayer \( \nu = 1/2 \) FQHE state \[3, 8, 11\] and the Pf state loses out to the 331 state \[12\]. Figure 3, which is the most important prediction of our work, shows that in the SLL bilayer system, both the 331 and Pf states should be visible, and in a realistic finite thickness system (Fig. 3b), the FQHE gap would become very small as one goes from the 331 to the Pf regime. In fact, it is conceivable that in the finite \( w \) system (Fig. 3b) the quantum phase transition between the 331 and the Pf phase can only occur by going through an intermediate compressible phase. The experimental observation of two strong FQHE regimes, like those shown in Fig. 3b, with the FQHE gap being largest along the two ridges in the phase diagram as \( d \) and \( \Delta \) are varied should be a spectacular verification of our theory and strong evidence for the existence of a Pf FQHE in the SLL.

To summarize, we predict the existence of both the two-component 331 Abelian (at intermediate to large \( d \)) and the one-component Pfaffian non-Abelian (at small \( d \) and intermediate to large \( \Delta \)) \( \nu = 5/2 \) SLL FQHE phase in bilayer structures. The observation of these two topologically distinct phases, one (Pf) stabilized by large inter-layer tunneling and the other (331) stabilized by large inter-layer separation would be a spectacular verification of the theoretical expectation that bilayer structures allow quantum phase transitions between topologically trivial and non-trivial paired even denominator incompressible FQHE states. The direct experimental observation of our predicted “two distinct branches” of SLL 5/2 FQHE in bilayer structures, as shown in Fig. 3b, will be compelling evidence for the existence of the \( \nu = 5/2 \) non-Abelian Pf state.

We acknowledge support from Microsoft Project Q. MRP thanks Kentaro Nomura for a helpful discussion.

[1] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. 59, 1776 (1987).
[2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Rev. Mod. Phys. 80, 1083 (2008) and references therein. See, also, M. Storni, R. H. Morf, and S. Das Sarma, arXiv:0812.2691 (unpublished), for the current theoretical status of this issue.
[3] C. R. Dean, B. A. Piot, P. Hayden, S. D. Sarma, G. Gervais, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 100, 146803 (2008).
[4] B. I. Halperin, Helv. Phys. Acta 56, 783 (1983); Surf. Sci. 305, 1 (1994).
[5] See articles by J. P. Eisenstein and S. M. Girvin and A. H. MacDonald in Perspectives in Quantum Hall Effects, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997).
[6] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[7] J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, K. W.
West, and S. He, Phys. Rev. Lett. 68, 1383 (1992).
[8] Y. W. Suen, M. B. Santos, and M. Shayegan, Phys. Rev. Lett. 69, 3551 (1992).
[9] F. C. Zhang and S. Das Sarma, Phys. Rev. B 33, 2903 (1986).
[10] M. R. Peterson, T. Jolicoeur, and S. D. Sarma, Phys. Rev. Lett. 101, 016807 (2008); Phys. Rev. B 78, 155308 (2008).
[11] D. R. Luhman, W. Pan, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Physical Review Letters 101, 266804 (2008).
[12] S. He, S. Das Sarma, and X. C. Xie, Phys. Rev. B 47, 4394 (1993); Z. Papić, G. Möller, M. V. Milovanović, N. Regnault, and M. O. Goerbig, Phys. Rev. B 79, 245325 (2009); K. Nomura and D. Yoshioka, J. Phys. Soc. Jpn. 73, 2612 (2004); M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. B 46, 9586 (1992).