A novel approach to MADM problems using Fermatean fuzzy Hamacher prioritized aggregation operators

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Abstract
A generalized form of union and intersection on FFS can be formulated from a generalized t-norm (TN) and t-conorm (TCN). Hamacher operations such as Hamacher product and Hamacher sum are good alternatives to produce such product and sum. The Hamacher operations can generate more flexible and more accurate results in decision-making process due to the working parameter involved in these operations. The intuitionistic fuzzy set, briefly as IFS and its extension involving Pythagorean fuzzy set (PFS) and Fermatean fuzzy set (FFS), are all effective tools to express uncertain and incomplete cognitive information with membership, nonmembership and hesitancy degrees. The Fermatean fuzzy set (FF-set) carries out uncertain and imprecise information smartly in exercising decision making than IFS and PFS. By adjusting the prioritization of attributes in FF environment, in this course of this article, we first device new operations on FF information using prioritized attributes and by employing HTN and HTCN, we discuss the basic operations. Induced by the Hamacher operations and FF-set, we propose FF Hamacher arithmetic and also geometric aggregation operators (AOs). In the first section, we introduce the concepts of an FF Hamacher prioritized AO and FF Hamacher prioritized weighted AO. In the second part, we develop FF Hamacher prioritized geometric operator (GO) and FF Hamacher prioritized weighted GO. We study essential properties and a few special cases of our newly proposed operators. Then, we make use of these proposed operators in developing tools which are key factors in solving the FF multi-attribute decision-making situations with prioritization. The university selection phenomena are considered as a direct application for analysis and to demonstrate the practicality and efficacy of our proposed model. The working parameter considered in these AOs is analyzed in different existing and proposed AOs. Further, comparison analysis is conducted for the authenticity of proposed & existing operators.

Keywords MADM · FF Hamacher prioritized average (FFHPA) operator · FF Hamacher prioritized weighted average (FFHPWA) operator · FF Hamacher prioritized geometric (FFHPG) operator · FF Hamacher prioritized weighted geometric (FFHPWG) operator

1 Introduction
Pythagorean fuzzy sets (PFSs) Xu (2007); Xu and Yager (2006), an augmentation of intuitionistic fuzzy sets (IFSs), tremendously have attracted many potential researchers in recent times. Yager Xu and Yager (2008) was the first to develop a useful decision-making technique based on Pythagorean fuzzy information for use in MCDM scenarios involving Pythagorean fuzzy information. In Yager and Abbasov (2013), Yager and Abbasov treated the Pythagorean membership grades (PMGs) and they found it identical with PFSs. Their work also showed the relation between the PMGs and the complex numbers. PFNs were used by Reformat and Yager Xu and Yager (2008) to create a framework for dealing with collaboration-based recommendation. Gou et al. Gou...
et al. (2016) studied Pythagorean fuzzy mappings and investigated fundamental properties called derivability, continuity and differentiability. Zeng et al. Zeng et al. (2018) introduced an aggregation procedure involving PFS and applied its notion in solving MADM. Zhang Zhang (2016) proposed an approach to MCDM problems in terms of the idea of similarity measure for Pythagorean fuzzy sets. PFSs have been successfully introduced in different research areas; in particular, Garg used PFSs in investment decision process (see Garg, Garg (2016); Peng and Yang, Peng and Yang (2015)), utilized PFSs in the candidate selection procedure for Asian Infrastructure Investment Bank (Ren et al., Ren et al. (2016)) and the service excellence of national airlines (Zhang and Xu, Zhang and Xu (2014)). Senapati and Yager Senapati and Yager (2019a, b) introduced the notion of a Fermatean fuzzy set (FFS). They offered various numerical examples of FFS to help people understand the concept. It is also important to mention that the class of this type of fuzzy set has more ability to capture the uncertainties as compared to IFSs and PFSs, and is qualified to handle higher degree of vagueness. MADM has been extensively used in many areas of sciences, for example, (Xu & Xia; Xu & Chen; Xu Xu and Xia (2011); Xu and Chen (2011); Xu (2011)) introduced the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator.

The fuzzy information aggregation operators are necessarily appealing and significant research topics and are given profound regard among the researchers. Various forms of generalizations of T-norms and T-conorms exist in text, such as Archimedean T-norms and T-conorms, Hamacher T-norms and T-conorms, algebraic T-norms and T-conorms, Einstein T-norms and T-conorms, Frank T-norms and T-conorms and Dombi T-norms and T-conorms. Liu Liu (2014) used Hamacher aggregation operators in interval-valued intuitionistic fuzzy numbers (IVIFNs) and discussed MAGDM techniques. Zhao and Wei Zhou et al. (2014) initiated Einstein hybrid aggregation operators for IFNs and applied it to multi-attribute decision-making method. Wei et al. Chen (2012) studied multi-attribute decision-making problems by proposing the bipolar fuzzy Hamacher arithmetic and geometric aggregation operators and discussed the basic properties of these proposed operators. Hamacher T-norm and T-conorm, which are the generalization of algebraic and Einstein T-conorm and T-norm Beliakov et al. (2007), are more universal and adoptable. The value of research on aggregation operators based on Hamacher operation and their application to MADM problems is significant. Xiao Xiao (2014) gave induced interval-valued intuitionistic fuzzy Hamacher ordered weighted geometric (IIIVIFHOWG) operator. Li Li (2014) studied interval-valued intuitionistic fuzzy sets with various operations, known as the Hamacher sum and the Hamacher product, and introduced the interval-valued intuitionistic fuzzy Hamacher correlated averaging (IVIFHCA) operator. Tan et al. Tan et al. (2015) developed hesitant fuzzy Hamacher aggregation operators for multi-attribute decision-making.

Senapati and Yager Chen (2014a) introduced four new types of weighted aggregation operators for FFS, namely, Fermatean fuzzy weighted average (FFWA) operator, Fermatean fuzzy weighted geometric (FFWG) operator, Fermatean fuzzy weighted power average (FFWPA) operator and Fermatean fuzzy weighted power geometric (FFWPG) operator. Recently, in a paper Aydemir and Gunduz (2020), Aydemir & Gunduz discussed TOPSIS method in terms of Dombi aggregation operators based on FF-sets and gave a complete overview of FF-sets in the framework of Dombi operations. In [43], the authors have extended FF-set into Hamacher operations and investigated the basic properties of FF-sets in Hamacher operations. Some practical examples of real-world scenarios were discussed as practical examples for the validation of the theory. Keeping in view the work on FF-set, we intend to extend the work of [43] and propose a series of new aggregation operators in Fermatean fuzzy environment based upon the Hamacher operations with the prioritization of attributes. The organization and novel contributions of this article are mentioned as:

- The FF-set has more potential than the traditional IFS and PFS for decision makers to study the uncertain situation in the real-world problems.
- The flexibility parameter involved in Hamacher operations has the ability to produce more accurate results in a decision process.
- The proposed model can be applied in situations where the traditional models of IFS and PFS are failed.
- The prioritization factor of attributes makes the proposed operators more advanced in modern decision process.

The remaining parts of the paper are organized in the following lines.

The second section, in brief, recalls basic knowledge of the IFS, PFSs and FFSs and the elementary operational laws of FFSs. In Sect. 3, we develop Fermatean fuzzy Hamacher prioritized average (FFHPA) operator, and Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator, Fermatean fuzzy Hamacher prioritized geometric (FFHPG) operator, and Fermatean fuzzy Hamacher prioritized weighted geometric (FFHPWG) operator. In Sect. 4, we make the use of these operators to develop tools that are handy in solving the Fermatean fuzzy multi-attribute decision-making problems. A case study example of the university selection committee is analyzed in Sect. 5, and some comparisons of proposed operators are studied. The comparison of proposed and existing operators is stud-
ied, and some future directions are given at the end of the paper.

2 Preliminaries and basic results

In this section, we sum up requisite knowledge associated with IFS, PFS and FFS along with corresponding operations and related properties. We will consider more familiarized ideas, which are useful in the sequential analysis.

Definition 2.1 [1,2] For a universe \( X \), intuitionistic fuzzy set (IFS) \( A \) is an expression of the form

\[
\tilde{A} = \{x, \overline{\alpha}(x), \overline{\beta}(x) : x \in X\}
\]

where \( \overline{\alpha}(x) \in [0, 1] \) is known as the “degree of membership of \( \tilde{A} \)”, and \( \overline{\beta}(x) \in [0, 1] \) is called the “degree of nonmembership of \( \tilde{A} \)”, and \( \overline{\alpha}(x), \overline{\beta}(x) \) satisfy the following condition: \( 0 \leq \overline{\alpha}(x) + \overline{\beta}(x) \leq 1 \), for all \( x \in X \). Apparently, when \( \overline{\beta}(x) = 1 - \overline{\alpha}(x) \), for all \( x \in X \), \( \tilde{A} \) turns to be a fuzzy set.

Definition 2.2 [4] A Pythagorean fuzzy set (PFS) \( P \) on the universal \( X \) is an object of the form

\[
P = \{x, \alpha_P(x), \beta_P(x) : x \in X\}
\]

where \( \alpha_P : X \longrightarrow [0, 1] \) is termed as the “degree of membership of \( P \)”, and \( \beta_P : X \longrightarrow [0, 1] \) is called the “degree of nonmembership of \( P \)”, and \( \alpha_P(x), \beta_P(x) \) satisfies the condition: \( 0 \leq (\alpha_P(x))^2 + (\beta_P(x))^2 \leq 1 \) for all \( x \in X \). For PFS, \( P \) and \( x \in X \), \( \pi(x) = \sqrt{1 - (\alpha_P(x))^2 - (\beta_P(x))^2} \) is the indeterminacy of \( x \) to \( P \).

Definition 2.3 Senapati and Yager (2019a) A Fermatean fuzzy set (FFS) defined on a nonempty set \( X \) is a structure of the form given as

\[
F = \{(x, \alpha_F(x), \beta_F(x)) : x \in X\}
\]

where \( \alpha_F : X \longrightarrow [0, 1] \), and \( \beta_F : X \longrightarrow [0, 1] \), respectively, are the degree of membership and non-membership of every element \( x \in X \) for the set \( F \). The condition \( 0 \leq (\alpha_F(x))^3 + (\beta_F(x))^3 \leq 1 \), holds for all \( x \in X \). For an FFS, \( F \) and \( x \in X \), the function \( \pi(x) = \sqrt[3]{1 - (\alpha_F(x))^3 - (\beta_F(x))^3} \) is the indeterminacy of \( x \) to \( F \). For simplicity, we use \( F = (\mu, \nu) \) for an FFS \( \{(x, \alpha_F(x), \beta_F(x)) : x \in X\} \) and call it a Fermatean fuzzy element (FFE) or Fermatean fuzzy number (FFN).

We shall notice that the Fermatean membership grades (FMGs) are greater than the Pythagorean membership grades (PMGs) and intuitionistic membership grades (IMGs), respectively.

Theorem 2.4 Senapati and Yager (2019a) The set of FMGs is larger than the set of PMGs and IMGs.

Definition 2.5 Senapati and Yager (2019a) Let \( F = (\mu, \nu) \), \( F_1 = (\mu_1, \nu_1) \), and \( F_2 = (\mu_2, \nu_2) \), be any three FFEs, and their set operations are defined as in the following:

(i) \( F_1 \cap F_2 = (\min \{\mu_1, \mu_2\}, \max \{\nu_1, \nu_2\}) \);
(ii) \( F_1 \cup F_2 = (\max \{\mu_1, \mu_2\}, \max \{\nu_1, \nu_2\}) \);
(iii) \( F_1 \subseteq F_2 \) if and only if \( \mu_1 \leq \mu_2 \), and \( \nu_1 \geq \nu_2 \);
(iv) \( F^c = (\nu, \mu) \).

Definition 2.6 [19]. Let \( F = (\mu, \nu) \), \( F_1 = (\mu_1, \nu_1) \), and \( F_2 = (\mu_2, \nu_2) \), be any three FFEs, and \( \xi > 0 \), then the following operations hold true.

(i) \( F_1 \oplus F_2 = \left(\sqrt[3]{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}, \nu_1 \nu_2\right) \);
(ii) \( F_1 \otimes F_2 = \left(\mu_1 \mu_2, \sqrt[3]{\nu_1^3 + \nu_2^3 - \nu_1 \nu_2^2}\right) \);
(iii) \( \xi (F) = \left(\sqrt[3]{1 - (1 - \mu_1^3)^\xi}, \nu_1^\xi\right) \);
(iv) \( (F)^\xi = \left(\mu_1^\xi, \sqrt[3]{1 - (1 - \nu_1^3)^\xi}\right) \).

Theorem 2.7 [19]. For three FFEs \( F = (\mu, \nu) \), \( F_1 = (\mu_1, \nu_1) \), and \( F_2 = (\mu_2, \nu_2) \), the following properties are valid:

(i) \( F_1 \oplus F_2 = F_2 \oplus F_1 \);
(ii) \( F_1 \otimes F_2 = F_2 \otimes F_1 \);
(iii) \( \xi (F_1 \oplus F_2) = \xi F_1 \oplus \xi F_2, \xi > 0 \);
(iv) \( (\xi_1 + \xi_2) F = \xi_1 F \oplus \xi_2 F, \xi_1, \xi_2 > 0 \);
(v) \( (F_1 \otimes F_2)^\xi = F_1^\xi \otimes F_2^\xi, \xi > 0 \);
(vi) \( F^{\xi_1} \oplus F^{\xi_2} = F^{\xi_1 + \xi_2}, \xi_1, \xi_2 > 0 \).

In order to grade FFEs, the rating function (briefly as rt) for FFEs is defined as:

Definition 2.8 [19] For FFE, \( F = (\mu, \nu) \), the rating function of \( F \) can be defined as follows:

\[
rt(F) = \frac{1 + \mu^3 - \nu^3}{2}.
\]
In particular \( rt(F) = \begin{cases} 1, & \text{if } F = (1, 0) \\ -1, & \text{if } F = (0, 1) \end{cases} \).

In the following, we examine a comparison relation for Fermatean fuzzy elements.

**Definition 2.9** [19] Let \( F_1 = (\mu_1, \nu_1) \) and \( F_2 = (\mu_2, \nu_2) \), be any two FFESs, and let \( rt(F_1) \) and \( rt(F_2) \) be the respective ratings of \( F_1 \) and \( F_2 \), then

(i) \( rt(F_1) > rt(F_2) \), then \( F_1 \) is superior than \( F_2 \) and denoted by \( F_1 > F_2 \);

(ii) \( rt(F_1) = rt(F_2) \), then \( F_1 \) and \( F_2 \) are equivalent, denoted by \( F_1 \sim F_2 \).

**Definition 2.10** [19] Let \( F = (\mu, \nu) \) be a FFE. The accuracy function of \( F \) can be defined as follows:

\[ Acc(F) = \mu^3 + \nu^3 \in [0, 1]. \]

Using the analogy of rating function (briefly as; \( rt \)) and accuracy function (briefly as; \( Acc \)), we give a complete criterion for the ranking of FFESs in the following.

**Definition 2.11** [19] Let \( F_1 = (\mu_1, \nu_1) \), and \( F_2 = (\mu_2, \nu_2) \), be any two FFESs, and let \( rt(F_1) \) and \( Acc(F_1) \) \((i = 1, 2)\) be the respective ratings and accuracies of \( F_1 \) and \( F_2 \), then

(I) \( rt(F_1) < rt(F_2) \implies F_1 < F_2 \);

(II) \( rt(F_1) > rt(F_2) \implies F_1 > F_2 \);

(III) If \( rt(F_1) = rt(F_2) \), then

(i) \( Acc(F_1) < Acc(F_2) \implies F_1 < F_2 \);

(ii) \( Acc(F_1) > Acc(F_2) \implies F_1 > F_2 \);

(iii) \( Acc(F_1) = Acc(F_2) \implies F_1 \sim F_2 \).

### 2.1 Hamacher operations

Hamacher proposed generalized form of T-norm as well as T-conorm, called Hamacher operations, which consists of Hamacher product and Hamacher sum. These are the respective blueprints of the well-known T-norm and T-conorm, mentioned in the definition below.

**Definition 2.12** Hamacher (1978) Assume \( a_1, b_1, a_2, b_2 \in R \). Then, Hamacher T-norms (HT-norms) and Hamacher T-conorms (HT-conorms) are expressed as:

\[ T(a_1, b_1) = a_1 \otimes b_1 = \frac{a_1 b_1}{\lambda + (1 - \lambda)(a_1 + b_1 - a_1 b_1)}, \lambda > 0 \quad (1) \]

\[ T^*(a_2, b_2) = a_2 \oplus b_2 = \frac{a_2 + b_2 - a_2 b_2 - (1 - \lambda)a_2 b_2}{1 - (1 - \lambda)a_2 b_2}, \lambda > 0 \quad (2) \]

Particularly, for \( \lambda = 1 \), HT-norm and HT-conorm take the following forms:

\[ T(a_1, b_1) = a_1 \otimes b_1 = a_1 b_1 \quad (3) \]

\[ T^*(a_2, b_2) = a_2 \oplus b_2 = a_2 + b_2 - a_2 b_2 \quad (4) \]

and these are algebraically equivalent. When \( \lambda = 2 \),

\[ T(a_1, b_1) = a_1 \otimes b_1 = \frac{a_1 b_1}{1 + (1 - a_1)(1 - b_1)} \quad (5) \]

\[ T^*(a_2, b_2) = a_2 \oplus b_2 = \frac{a_2 + b_2}{1 + a_2 b_2} \quad (6) \]

Equations (5) and (6), respectively, are known as the Einstein T-norm and Einstein T-conorm.

### 3 FF Hamacher operators

In this section, utilizing the notion of HTN and HTCN, we explain Hamacher operations with respect to FFESs. We propose the Hamacher arithmetic AOs with FFESs. In this regard, the operation rules for FF Hamacher operation are recalled in the following definition.

**Definition 3.1** Let \( F_j = (\mu_j, \nu_j) \) \((j = 1, 2)\) be a array of FFESs, \( \succ > 0, k > 0 \), then, the fundamental Hamacher operations for FFESs are introduced as:

(i) \( F_1 \oplus F_2 = \left( \sqrt[\mu_1 + \mu_2 - (\mu_1)^2 - (\mu_2)^2}{1 - (\mu_1)^2 \mu_2^2}, \sqrt[\nu_1 + \nu_2 - (\nu_1)^2 - (\nu_2)^2]{1 - (\nu_1)^2 \nu_2^2} \right) \);

(ii) \( F_1 \otimes F_2 = \left( (\sqrt[\mu_1 \mu_2]{1 + (1 - \lambda)(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2 (\mu_2)^2)}{1 - (\mu_1)^2 \mu_2^2} \right), \left( (\sqrt[\nu_1 \nu_2]{1 + (1 - \lambda)(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2 (\nu_2)^2)}{1 - (\nu_1)^2 \nu_2^2} \right) \);

(iii) \( \kappa(F_1) = \left( (\sqrt[\mu_1 \mu_2]{1 + (1 - c)(\mu_1)^2 + (\mu_1)^2 - (\mu_1)^2 (\mu_2)^2)}{1 - (\mu_1)^2 \mu_2^2} \right), \left( (\sqrt[\nu_1 \nu_2]{1 + (1 - c)(\nu_1)^2 + (\nu_1)^2 - (\nu_1)^2 (\nu_2)^2)}{1 - (\nu_1)^2 \nu_2^2} \right) \);

(iv) \( (F_1)^k = \left( (\sqrt[\mu_1 \mu_2]{1 + (1 - c)(\mu_1)^2 + (\mu_1)^2 - (\mu_1)^2 (\mu_2)^2)}{1 - (\mu_1)^2 \mu_2^2} \right), \left( (\sqrt[\nu_1 \nu_2]{1 + (1 - c)(\nu_1)^2 + (\nu_1)^2 - (\nu_1)^2 (\nu_2)^2)}{1 - (\nu_1)^2 \nu_2^2} \right) \).
3.1 Fermatean fuzzy Hamacher prioritized aggregation operators

3.1.1 Fermatean fuzzy Hamacher prioritized arithmetic aggregation operators

Let us denote by $£$ the set of all nonempty FFNs, i.e.,
$$£ = \{(\omega, \sigma) | (\omega, \sigma) \in [0, 1]^2, \ 0 \leq \omega + \sigma \leq 1\}$$
with the partial order $\preceq$ defined by $(\omega_1, \sigma_1) \preceq (\omega_2, \sigma_2)$ if and only if $\omega_1 \leq \omega_2$ and $\sigma_1 \geq \sigma_2$. The top and bottom elements of $£$ are defined by $1_£ = (1, 0)$ and $0_£ = (0, 1)$. Then $£$ becomes a lattice with the partial order $\preceq$. If $F_1$ and $F_2$ are two FFNs, then $F_1 \preceq F_2$ implies $F_1 \leq F_2$.

The concept of prioritized average (PA) was first introduced by Yager [39] in 2008 and was defined as follows:

Definition 3.2 (Yager [39]) Assume that $C = \{C_1, C_2, ..., C_n\}$ is a collection of criteria with the prioritization among the criteria defined by the linear ordering as $C_1 > C_2 > ... > C_n$, where criteria $C_j$ have a higher priority than $C_k$ for $j < k$ and $n \in \mathbb{N}$. The real number $C_j(x) \in [0, 1]$ is the performance of any alternative $x$ under the criterion $C_j$ and

$$PA(C_j) = \frac{\sum_{j=1}^{n} \Phi_j C_j(x)}{\sum_{j=1}^{n} T_j}$$

where $\Phi_j = \frac{T_j}{\sum_{j=1}^{n} T_j}, T_j = \sum_{r=1}^{n} C_k(x) \ (j = 2, ..., n), T_1 = 1$.

Then PA is called the prioritized average (PA) operator.

The PA operators are used in the situations where the input arguments are exact values. By combining the Hamacher operations with prioritized inputs based on FS-sets, develop prioritized arithmetic aggregation operators with Fermatean fuzzy numbers based on Hamacher operators. Let $F_r = (\mu_r, \nu_r) \ (r = 1, ..., p)$ be a family of FFEs. We define Fermatean fuzzy Hamacher prioritized arithmetic aggregation operator as follows:

Definition 3.3 The Fermatean fuzzy Hamacher prioritized averaging (FFHPA) operator is a mapping FFHPA: $F^p \rightarrow F$ such that

$$FFHPA(F_1, F_2, ..., F_p) = \oplus_{r=1}^{p} \left( \frac{T_r}{\sum_{r=1}^{p} T_r} F_r \right)$$

$$= \frac{T_1}{\sum_{r=1}^{p} T_r} F_1 \oplus \frac{T_2}{\sum_{r=1}^{p} T_r} F_2 \oplus ... \oplus \frac{T_p}{\sum_{r=1}^{p} T_r} F_p \ (7)$$

where $T_r = \prod_{r=1}^{p} r t (F_r) \ (r = 2, 3, ..., p), T_1 = 1$ and $rt (F_r)$ represent the score value of $F_r \ (r = 1, 2, ..., p)$.

In the following theorem we use the mathematical induction and operational rule of FFEs to prove that the aggregation value of a family of FFEs by using $FFHPA$ operator is again an FFE.

Theorem 3.4 Let $F_r = (\mu_r, \nu_r) \ (r = 1, ..., p)$ be a family of FFEs, then the aggregation value of this family by the FFHPA operator is also a FFE, and

$$FFHPA(F_1, F_2, ..., F_p) = \oplus_{r=1}^{p} \left( \frac{T_r}{\sum_{r=1}^{p} T_r} F_r \right) \ (8)$$

$$= \frac{T_1}{\sum_{r=1}^{p} T_r} F_1 \oplus \frac{T_2}{\sum_{r=1}^{p} T_r} F_2 \oplus ... \oplus \frac{T_p}{\sum_{r=1}^{p} T_r} F_p \ (9)$$

where $Tr = \prod_{r=1}^{p} r t (F_r) \ (r = 2, 3, ..., p), T_1 = 1$ and $rt (F_r)$ represent score value of $F_r \ (r = 1, 2, ..., p)$.

Example 3.5 Let $F_1 = (0.8, 0.5), F_2 = (0.8, 0.7), F_3 = (0.7, 0.8)$ and $F_4 = (0.6, 0.7)$ be four FFEs, $\frac{T_1}{\sum_{r=1}^{4} T_r} = 0.3180, \frac{T_2}{\sum_{r=1}^{4} T_r} = 0.3098, \frac{T_3}{\sum_{r=1}^{4} T_r} = 0.1287, \frac{T_4}{\sum_{r=1}^{4} T_r} = 0.0561$. Suppose $\lambda = 3$, then

$$FFHPA(F_1, F_2, F_3, F_4) = \oplus_{r=1}^{4} \left( \frac{T_r}{\sum_{r=1}^{4} T_r} F_r \right) = \left( \prod_{r=1}^{4} r t (F_r) \right) \left[ \frac{T_1}{\sum_{r=1}^{4} T_r} (F_1), \frac{T_2}{\sum_{r=1}^{4} T_r} (F_2), \frac{T_3}{\sum_{r=1}^{4} T_r} (F_3), \frac{T_4}{\sum_{r=1}^{4} T_r} (F_4) \right] = (0.7254, 0.7424).$$

The working parameter in FFHPA operator has two special types which are discussed in the following:

(1) If $\lambda = 1$, then FFHPA is equivalent to the Fermatean fuzzy prioritized average (FFPA) operator:

$$FFPA(F_1, F_2, ..., F_p) = \oplus_{r=1}^{p} \left( \frac{T_r}{\sum_{r=1}^{p} T_r} F_r \right) = \left( \prod_{r=1}^{p} (1 - (\mu_r)^3) \sum_{r=1}^{p} \frac{T_r}{\sum_{r=1}^{p} T_r} \right) \left[ \prod_{r=1}^{p} (\nu_r) \sum_{r=1}^{p} \frac{T_r}{\sum_{r=1}^{p} T_r} \right]. \ (10)$$

(2) If $\lambda = 2$, then FFHPA becomes the Fermatean fuzzy Einstein prioritized average (FFEPA) operator:

$$FFEPA(F_1, F_2, ..., F_p) = \oplus_{r=1}^{p} \left( \frac{T_r}{\sum_{r=1}^{p} T_r} F_r \right) = \left( \prod_{r=1}^{p} (1 - (\mu_r)^3) \sum_{r=1}^{p} \frac{T_r}{\sum_{r=1}^{p} T_r} \right) \left[ \prod_{r=1}^{p} (\nu_r) \sum_{r=1}^{p} \frac{T_r}{\sum_{r=1}^{p} T_r} \right]. \ (11)$$
If we consider the weights \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_p)^T \) of FF-set \( F_r (r = 1, 2, ..., p) \) such that \( \Phi_r > 0 \) and \( \sum_{r=1}^{p} \Phi_r = 1 \). Then we define FF Hamacher prioritized weighted average (FFHPWA) operator as follows:

**Definition 3.6** Let \( F_r = (\mu_r, v_r) (r = 1, ..., p) \) be a family of FFEs. A Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator of a dimension \( p \) with the corresponding weighting vector \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_p)^T \) such that \( \Phi_r > 0 \), and \( \sum_{r=1}^{p} \Phi_r = 1 \). Then

\[
FFHPWA_\Phi (F_1, F_2, ..., F_p) = \bigoplus_{r=1}^{p} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{p} \Phi_r T_r} F_r \right)
\]

where \( T_r = \prod_{k=1}^{p} r \) \( F_r \) (\( r = 2, 3, ..., p \), \( T_1 = 1 \)) and \( r \) \( F_r \) represent the rating value of \( F_r \) (\( r = 1, 2, ..., p \)).

As a consequence of Definition 3.6, we have the theorem stated below.

**Theorem 3.7** A Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator of a dimension \( p \) returns a \( F \)-set and

\[
FFHPWA_\Phi (F_1, F_2, ..., F_p) = \bigoplus_{r=1}^{p} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{p} \Phi_r T_r} F_r \right)
\]

where \( T_r = \prod_{k=1}^{p} r \) \( F_r \) (\( r = 2, 3, ..., p \), \( T_1 = 1 \)) and \( r \) \( F_r \) represent the rating value of \( F_r \) (\( r = 1, 2, ..., p \)) and \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_p)^T \) is the weight vector such that \( \Phi_r > 0 \), and \( \sum_{r=1}^{p} \Phi_r = 1 \).

In the following we discuss two special cases of FFHPWA operator for the working parameter \( \lambda \).

(1) For \( \lambda = 1 \), the \( FFHPWA \) reduces to Fermatean fuzzy prioritized weighted average (FPWA) operator:

\[
FFPW_\Phi (F_1, F_2, ..., F_p) = \bigoplus_{r=1}^{p} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{p} \Phi_r T_r} F_r \right)
\]

(2) For \( \lambda = 2 \), FFHPWA reduces to Fermatean fuzzy Einstein prioritized weighted average (FFEPWA) operator:

\[
FFEPWA_\Phi (F_1, F_2, ..., F_p) = \bigoplus_{r=1}^{p} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{p} \Phi_r T_r} F_r \right)
\]

Example 3.8 Let \( F_1 = (0.6, 0.4), F_2 = (0.7, 0.6), F_3 = (0.8, 0.5), \) and \( F_4 = (0.5, 0.8) \) be four FFEs and \( \Phi = (0.2, 0.1, 0.3, 0.4)^T \) be the weighting vector. Then

\[
\frac{\Phi_1 T_1}{\sum_{r=1}^{4} \Phi_r T_r} = 0.4747, \quad \frac{\Phi_2 T_2}{\sum_{r=1}^{4} \Phi_r T_r} = 0.1336, \quad \frac{\Phi_3 T_3}{\sum_{r=1}^{4} \Phi_r T_r} = 0.2781, \quad \frac{\Phi_4 T_4}{\sum_{r=1}^{4} \Phi_r T_r} = 0.1134
\]

and the aggregation value of FFEs for \( (\lambda = 3) \) and by using definition of FFHPWA operator we get

\[
CFHPWA_\Phi (F_1, F_2, F_3, F_4) = \bigoplus_{r=1}^{4} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{4} \Phi_r T_r} F_r \right)
\]

and

\[
= (0.6752, 0.4902).
\]

4 Fermatean fuzzy Hamacher prioritized geometric aggregation operators

Let \( F_r = (\mu_r, v_r) (r = 1, ..., p) \) be a family of FFEs. We define Fermatean fuzzy Hamacher prioritized geometric aggregation operator as follows:

**Definition 4.1** The Fermatean fuzzy Hamacher prioritized averaging (FFHPA) operator is a mapping FFHPA: \( F^p \rightarrow \) such that

\[
FFHPA_\Phi (F_1, F_2, ..., F_p) = \bigoplus_{r=1}^{p} \left( \frac{\Phi_r T_r}{\sum_{r=1}^{p} \Phi_r T_r} F_r \right)
\]

where \( T_r = \prod_{k=1}^{p} r \) \( F_r \) (\( r = 2, 3, ..., p \)), \( T_1 = 1 \) and \( r \) \( F_r \) represent score value of \( F_r \) (\( r = 1, 2, ..., p \)).
In the following theorem we use the mathematical induction and operational rule of FFEs to prove that the aggregation value of a family of FFEs by using \( F F H P G \) operator is again an FFE.

**Theorem 4.2** Let \( F_r = (\mu_r, v_r) \) \((r = 1, ..., p)\) be a family of FFEs, then the aggregation value of this family by the FFHPG operator is also an FFE, and

\[
FFHPG (F_1, F_2, ..., F_p) = \otimes_{r=1}^p (F_r)^{\gamma_r \over \sum_{r=1}^p \gamma_r}
\]

Further, we offer two special cases of FFHPG operator:

1. For \( \lambda = 1 \), FFHPG becomes the Fermatean fuzzy prioritized geometric (FFPG) operator:

\[
FFPG (F_1, F_2, ..., F_p) = \otimes_{r=1}^p (F_r)^{\gamma_r \over \sum_{r=1}^p \gamma_r}, \quad \lambda = 1,
\]

(2) For \( \lambda = 2 \), FFHPG reduces to Fermatean fuzzy Einstein prioritized geometric (FFEKG) operator:

\[
FFEKG (F_1, F_2, ..., F_p) = \otimes_{r=1}^p (F_r)^{\gamma_r \over \sum_{r=1}^p \gamma_r}, \quad \lambda = 2.
\]

In the following we introduce FFHPWG operator.

**Definition 4.4** Let \( F_r = (\mu_r, v_r) \) \((r = 1, 2, ..., p)\) be a family of FFEs. The FFHPWG is a mapping from \( F^p \) to \( F \) such that

\[
FFHPWG (F_1, F_2, ..., F_p) = \otimes_{r=1}^p (F_r)^{\gamma_r \over \sum_{r=1}^p \gamma_r}, \quad \Phi = \begin{pmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_p \end{pmatrix},
\]

where \( \Phi_r = \prod_{r=1}^{p-1} r (F_r) \) \((r = 2, 3, ..., p)\), \( T_1 = 1 \) and \( r (F_r) \) represent score value of \( F_r \) \((r = 1, 2, ..., p)\).

**Example 4.3** Let \( F_1 = (0.8, 0.5) \), \( F_2 = (0.8, 0.7) \), \( F_3 = (0.7, 0.8) \) and \( F_4 = (0.6, 0.7) \) be four FFEs, \( T_1 = 0.3180 \), \( T_2 = 0.3098 \), \( T_3 = 0.1287 \), \( T_4 = 0.0561 \). Suppose \( \lambda = 3 \), then

\[
FFHPA (F_1, F_2, F_3, F_4) = \otimes_{r=1}^4 (F_r)^{\gamma_r \over \sum_{r=1}^4 \gamma_r}
\]

In the following we introduce FFHPWG operator.

**Definition 4.4** Let \( F_r = (\mu_r, v_r) \) \((r = 1, 2, ..., p)\) be a family of FFEs. The FFHPWG is a mapping from \( F^p \) to \( F \) such that

\[
FFHPWG (F_1, F_2, ..., F_p) = \otimes_{r=1}^p (F_r)^{\gamma_r \over \sum_{r=1}^p \gamma_r}, \quad \Phi = \begin{pmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_p \end{pmatrix},
\]

where \( \gamma_r = \prod_{r=1}^{p-1} r (F_r) \) \((r = 2, 3, ..., p)\), \( T_1 = 1 \) and \( r (F_r) \) represent score value of \( F_r \) and \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_p)^T \) is the weighting vector of \( F_r \) \((r = 1, break2, ..., p)\) such that \( \Phi_r > 0 \) and \( \sum_{r=1}^p \Phi_r = 1 \).

Using FFHPWG operator, and the operational rules of Definition 3.1, we can prove the following submsequent theorem easily.

**Theorem 4.5** Let \( F_r = (\mu_r, v_r) \) \((r = 1, 2, ..., p)\) be a family of FFEs. The FFHPWG operator is a mapping \( F^p \rightarrow F \) such that
\[ F F H P W G_{\Phi} (F_1, F_2, ..., F_p) = \bigotimes_{r=1}^{p} (F_r)_{\sum_{r=1}^{p} \phi_r T_r} \]
\[ = \left( \frac{1}{\sqrt[p]{\prod_{r=1}^{p} (1 - (v_r)^3)^{\sum_{r=1}^{p} \phi_r T_r}}} \right) \quad (17) \]

where \( T_r = \prod_{k=1}^{r-1} r_t (F_r) \) (\( r = 2, 3, ..., p \)), \( T_1 = 1 \) and \( r_t (F_r) \) represent score value of \( F_r \) (\( r = 1, 2, ..., p \)) and \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_p)^T \) is the weight vector of \( F_r \) (\( r = 1, 2, ..., p \)) such that \( \Phi_r > 0 \) and \( \sum_{r=1}^{p} \Phi_r = 1 \).

We investigate two special cases of FFHPWG operator in the following:

(1) For \( \lambda = 1 \), FFHPWG minimizes to Fermatean fuzzy prioritized weighted geometric (FFPWG) operator:

\[ F F P W G_{\Phi} (F_1, F_2, ..., F_p) = \bigotimes_{r=1}^{p} (F_r)_{\sum_{r=1}^{p} \phi_r T_r} \]
\[ = \left( \prod_{r=1}^{p} (\mu_r)_{\sum_{r=1}^{p} \phi_r T_r} \right) \right) \]
\[ \left( 1 - \prod_{r=1}^{p} (1 - \phi_r)^{3\sum_{r=1}^{p} \phi_r T_r} \right) \right) \quad (18) \]

(2) For \( \lambda = 2 \), FFHPWG minimizes to Fermatean fuzzy Einstein prioritized weighted geometric (FFEFPWG) operator:

\[ F F E F P W G_{\Phi} (F_1, F_2, ..., F_r) = \bigotimes_{r=1}^{p} (F_r)_{\sum_{r=1}^{p} \phi_r T_r} \]
\[ = \left( \prod_{r=1}^{p} (\mu_r)_{\sum_{r=1}^{p} \phi_r T_r} \right) \quad (19) \]

\[ \left( 1 - \prod_{r=1}^{p} (1 - \phi_r)^{3\sum_{r=1}^{p} \phi_r T_r} \right) \]

5 Fuzzy modeling of MADM: the case of Fermatean fuzzy information

We shall apply FF-Dombi prioritized AOs constructed in the previous sections to solve a MADM problem with FF information. Denote a discrete set of alternatives by \( A = \{ A_1, A_2, ..., A_m \} \), we also denote by \( \mathbb{G} = \{ G_1, G_2, ..., G_r \} \), the set of attributes, we assume that there is a prioritization among these attributes and let the prioritization be a linear ordering \( G_1 \succ G_2 \succ ... \succ G_r \), indicating that the attribute \( G_\xi \) has a higher priority than \( G_\zeta \) if \( \xi < \zeta \). Let \( \mathbb{\varnothing} = \{ \varnothing_1, \varnothing_2, ..., \varnothing_r \} \) be the weight vector for the attributes \( G_{\xi} \) (\( \xi = 1, 2, 3, ..., r \)) such that \( \varnothing_\xi > 0 \) and \( \sum_{\xi=1}^{r} \varnothing_\xi = 1 \).

Suppose that \( M = (F_{g\xi})_{m \times n} = (\mu_{g\xi}, \delta_{g\xi})_{m \times n} \) is the FF decision matrix, where \( \mu_{g\xi} \) represents the degree of membership, that the alternative \( A_\xi \in \mathbb{A} \) satisfies the alternative \( A_\xi \in \mathbb{A} \) does not satisfy the attribute \( G_{g\xi} \) considered by the decision makers such that \( \mu_{g\xi} + \delta_{g\xi} \leq 1 \) and \( \mu_{g\xi}, \delta_{g\xi} \subset [0, 1] \), (\( \xi = 1, 2, ..., m \)) and (\( \xi = 1, 2, ..., n \)) the DMs proposed for the attributes \( G_{g\xi} \). To follow the above discussion we utilize the methods developed in previous section and design an algorithm to solve multiple attribute decision-making problem based on FF-environment.

\[ T_{g\xi} = \prod_{k=1}^{\xi-1} \text{Score} (F_{g\xi}) \quad (\xi = 1, 2, ..., m; \ k = 2, 3, ..., m) \quad (20) \]

Algorithm

Step 1. Calculate the value of \( T_{g\xi} \) (\( \xi = 1, 2, ..., m; \ k = 2, 3, ..., m \))

\[ T_{g\xi} = \prod_{k=1}^{\xi-1} \text{Score} (F_{g\xi}) \quad (\xi = 1, 2, ..., m; \ k = 2, 3, ..., m) \]

5 Fuzzy modeling of MADM: the case of Fermatean fuzzy information

We shall apply FF-Dombi prioritized AOs constructed in the previous sections to solve a MADM problem with FF information. Denote a discrete set of alternatives by \( A = \{ A_1, A_2, ..., A_m \} \), we also denote by \( \mathbb{G} = \{ G_1, G_2, ..., G_r \} \), the set of attributes, we assume that there is a prioritization among these attributes and let the prioritization be a linear ordering \( G_1 \succ G_2 \succ ... \succ G_r \), indicating that the attribute \( G_\xi \) has a higher priority than \( G_\zeta \) if \( \xi < \zeta \). Let \( \mathbb{\varnothing} = \{ \varnothing_1, \varnothing_2, ..., \varnothing_r \} \) be the weight vector for the attributes \( G_{\xi} \) (\( \xi = 1, 2, 3, ..., r \)) such that \( \varnothing_\xi > 0 \) and \( \sum_{\xi=1}^{r} \varnothing_\xi = 1 \).

Suppose that \( M = (F_{g\xi})_{m \times n} = (\mu_{g\xi}, \delta_{g\xi})_{m \times n} \) is the FF decision matrix, where \( \mu_{g\xi} \) represents the degree of membership, that the alternative \( A_\xi \in \mathbb{A} \) satisfies the alternative \( A_\xi \in \mathbb{A} \) does not satisfy the attribute \( G_{g\xi} \) considered by the decision makers such that \( \mu_{g\xi} + \delta_{g\xi} \leq 1 \) and \( \mu_{g\xi}, \delta_{g\xi} \subset [0, 1] \), (\( \xi = 1, 2, ..., m \)) and (\( \xi = 1, 2, ..., n \)) the DMs proposed for the attributes \( G_{g\xi} \). To follow the above discussion we utilize the methods developed in previous section and design an algorithm to solve multiple attribute decision-making problem based on FF-environment.

\[ T_{g\xi} = \prod_{k=1}^{\xi-1} \text{Score} (F_{g\xi}) \quad (\xi = 1, 2, ..., m; \ k = 2, 3, ..., m) \quad (20) \]

Algorithm

Step 1. Calculate the value of \( T_{g\xi} \) (\( \xi = 1, 2, ..., m; \ k = 2, 3, ..., m \)) using the formula as follows by

\[ T_{g\xi} = \prod_{k=1}^{\xi-1} \text{Score} (F_{g\xi}) \quad (\xi = 1, 2, ..., m; \ k = 2, 3, ..., m) \quad (20) \]

and

\[ T_{g1} = 1, (\xi = 1, 2, ..., m). \quad (21) \]
Step 2. Apply the operator FFHPWA on the decision matrix $M$ where

$$F_q = FFHPWA(F_{q1}, F_{q2}, ..., F_{qr})$$

$$= \sum_{t=1}^{r} \left( \frac{\sum_{x=1}^{s} \sum_{y=1}^{u} F_{txy} F_{qy}}{\sum_{x=1}^{s} \sum_{y=1}^{u} F_{txy} G_{xy}} \right) F_{r}$$

or apply FFHPWG operator

$$F_q = FFHPWG(F_{q1}, F_{q2}, ..., F_{qr})$$

$$= \sum_{t=1}^{r} \left( \frac{\sum_{x=1}^{s} \sum_{y=1}^{u} F_{txy} F_{qy}}{\sum_{x=1}^{s} \sum_{y=1}^{u} F_{txy} G_{xy}} \right) F_{r}$$

(22)

(23)

to get the aggregated values of $F_q$ ($\ell = 1, 2, ..., m$) of the alternatives $A_q$.

Step 3. Calculate the values of the score function $Score(F_q)$ ($\ell = 1, 2, ..., m$) of all the aggregated FFNs $F_q$ ($\ell = 1, 2, ..., m$) obtained in Step 2. If the value of score functions $Score(F_q)$ and $Score(F_{q\ell})$ is not different, then apply the accuracy function $acc(F_q)$ and $acc(F_{q\ell})$ for the ranking order of alternatives $A_q$ ($\ell = 1, 2, ..., m$).

Step 5. End

6 Experimental example and comparative discussion

6.1 Example description

We discuss the selection process of teaching staff of our university. To promote the education system of Abdul Wali Khan University, the Department of Mathematics wants to recruit overseas outstanding educationists. After some important meetings in the department, an expert team is selected to complete the process of selection of outstanding teachers. The panel of experts consists of university vice chancellor (VC), dean of physical and numerical sciences (P&NS) and human resource development officer. This team of expert will analyze a set of five candidates $A_q$ ($\ell = 1, 2, 3, 4, 5$) following the four attributes $G_1$: qualification, $G_2$: teaching ability, $G_3$: research expertise and $G_4$: quality research publications. University VC has absolute priority in decision-making, dean of P&NS comes next. Further they will be strict in their principle of combine ability and will not influence by any political integrity. The prioritization criteria are defined as $G_1 > G_2 > G_3 > G_4$, where the symbol $>$ is used to represent prefer than relation. The team will use FFNs in the evaluation of candidates $A_q$ ($\ell = 1, 2, 3, 4, 5$).

The attribute weight vector in the selection process is $\phi = (0.2, 0.2, 0.3, 0.3)^T$, and the decision matrix for this model is $P = (F_{q\ell})_{4 \times 5}$ which is represented in Table 1, where $F_{q\ell}$ are FFNs.

In order to select the most desirable candidate $A_q$ ($\ell = 1, 2, 3, 4, 5$), we apply FFHPWA and FFHPWG operators in the following steps of algorithm

6.1.1 FFHPWA operator

Step 1. Apply Eqs. (7) and (8) to compute the values of $T_{q\ell}$ ($\ell = 1, 2, ..., m$; $\xi = 1, 2, ..., r$) which are given in the following matrix

$T_{q\ell} = \begin{bmatrix} 1.000 & 0.4240 & 0.3188 & 0.1594 & 0.1048 \\ 1.000 & 0.6480 & 0.4143 & 0.1945 & 0.0596 \\ 1.000 & 0.7555 & 0.4600 & 0.2649 & 0.2003 \\ 1.000 & 0.7425 & 0.4076 & 0.2720 & 0.1187 \end{bmatrix}$

Step 2. For $\lambda = 1$, we apply FFHPWA operator to aggregate the overall preference values of FFNs $F_q$ of the candidates $A_q$ ($\ell = 1, 2, 3, 4, 5$), and we get $F_1 = (0.8389, 0.1640), F_2 = (0.7027, 0.3188), F_3 = (0.5097, 0.5226), F_4 = (0.5654, 0.8165), F_5 = (0.2163, 0.9519)$.

Step 3. Calculate the score value using score($F_q$) function of the FFNs $F_q$ ($\ell = 1, 2, 3, 4, 5$), and we get: score($F_1$) = 0.7929, score($F_2$) = 0.6572, score($F_3$) = 0.4948, score($F_4$) = 0.3182, score($F_5$) = 0.0737.

Table 1 Fermatean fuzzy decision matrix

|   | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| $G_1$ | (4.6) | (8.2) | (5.5) | (7.3) | (4.8) |
| $G_2$ | (8.6) | (7.4) | (4.5) | (5.8) | (5.6) |
| $G_3$ | (8.1) | (7.5) | (6.4) | (9.6) | (3.9) |
| $G_4$ | (8.3) | (5.3) | (7.2) | (6.7) | (3.7) |

Step 4. Rank all the candidates $A_q$ ($\ell = 1, 2, 3, 4, 5$), according to the values of score function score($F_q$) ($\ell = 1, 2, 3, 4, 5$), of the candidates as $A_1 > A_2 > A_3 > A_4 > A_5$. 
Methods & $S(\mathcal{F}_1)$ & $S(\mathcal{F}_2)$ & $S(\mathcal{F}_3)$ & $S(\mathcal{F}_4)$ & $S(\mathcal{F}_5)$ & Ranking order
---
$FFWA$ & $-1.305$ & $-1.175$ & $-0.429$ & $-1.477$ & $-0.498$ & $A_3 > A_5 > A_2 > A_1 > A_4$
$FFWG$ & $-1.271$ & $-1.176$ & $-0.406$ & $-1.351$ & $-1.181$ & $A_3 > A_2 > A_5 > A_1 > A_4$
$FFWPA$ & $-0.799$ & $-0.879$ & $-0.376$ & $-1.237$ & $-0.736$ & $A_3 > A_5 > A_1 > A_2 > A_4$
$FFWPG$ & $-0.533$ & $-0.705$ & $-0.338$ & $-1.080$ & $-0.586$ & $A_3 > A_1 > A_5 > A_2 > A_4$
$FFHWA$ & $0.0996$ & $0.2004$ & $0.0110$ & $0.1646$ & $0.2061$ & $A_5 > A_2 > A_4 > A_1 > A_3$
$FFHWG$ & $-0.0472$ & $0.0904$ & $-0.0008$ & $0.1497$ & $0.1427$ & $A_4 > A_5 > A_2 > A_3 > A_1$
$FFHOWA$ & $0.1506$ & $0.1959$ & $0.0026$ & $0.1609$ & $0.2170$ & $A_5 > A_2 > A_4 > A_1 > A_3$
$FFHOWG$ & $0.0261$ & $0.0796$ & $-0.0066$ & $0.1435$ & $0.1742$ & $A_5 > A_2 > A_4 > A_1 > A_3$
$FFHPA$ & $0.7929$ & $0.6573$ & $0.4949$ & $0.3184$ & $0.1063$ & $A_1 > A_2 > A_1 > A_4 > A_5$
$FFHPG$ & $0.4513$ & $0.5977$ & $0.6142$ & $0.7557$ & $0.6905$ & $A_1 > A_2 > A_1 > A_4 > A_5$
$FFHPWA$ & $0.7929$ & $0.6572$ & $0.4948$ & $0.3182$ & $0.0737$ & $A_1 > A_2 > A_1 > A_4 > A_5$
$FFHPWG$ & $0.4513$ & $0.5978$ & $0.6144$ & $0.7558$ & $0.6909$ & $A_4 > A_5 > A_3 > A_2 > A_1$

### Step 5

$A_1$ is selected as the best candidate.

#### 6.1.2 FFHPWG operator

If we utilize the FFHPWG operator, then the procedure is similar as above.

**Step 1.** For $\lambda = 1$, we apply FFHPWG operator to aggregate the overall preference values of FFNs $\mathcal{F}_\ell$ of the candidates $A_\ell$ ($\ell = 1, 2, 3, 4, 5$), and we get:

$\mathcal{F}_1 = (0.4848, 0.5956)$, $\mathcal{F}_2 = (0.6377, 0.3993)$, $\mathcal{F}_3 = (0.6580, 0.3826)$, $\mathcal{F}_4 = (0.8594, 0.4974)$, $\mathcal{F}_5 = (0.8094, 0.5293)$.

**Step 2.** Calculate the score value using score($\mathcal{F}_\ell$) function of the FFNs $\mathcal{F}_\ell$ ($\ell = 1, 2, 3, 4, 5$), and we get:

score($\mathcal{F}_1$) = 0.4513, score($\mathcal{F}_2$) = 0.5978, score($\mathcal{F}_3$) = 0.6144, score($\mathcal{F}_4$) = 0.7558, score($\mathcal{F}_5$) = 0.6909.

**Step 3.** Rank all the candidates $A_\ell$ ($\ell = 1, 2, 3, 4, 5$), according to the values of score function, score($\mathcal{F}_\ell$) ($\ell = 1, 2, 3, 4, 5$), of the candidates as $A_4 > A_5 > A_3 > A_2 > A_1$.

**Step 4.** $A_4$ is selected as the best candidate for the post.

From the above discussion, we observe that the overall ranking orders of all the candidates are different by utilizing the two operators, in FFHPWA operator $A_1$ is the most suitable candidate for the post, while by applying FFHPWG operator, the most desirable candidate is $A_4$.

#### 6.2 Comparison of proposed and existing operators

From Table 2, we observe that the ranking order of alternative by using different methods is different. But the best alternatives of the same type of aggregation operator are same, e.g., the best alternative in FFWA, FFWG, FFWPA and FFWPG operators is $A_3$. The best alternatives for the methods FFHWA operator are $A_3$ and for FFHWG operator is $A_4$. This means that the FFHWA and FFHWG operators have a rare fluctuation in ranking order. FFHOWA and FFHOWG operators have same ranking for best alternative, i.e., $A_5$. The other methods FFHPA, FFHPG, FFHPWA and FFHPWG operators have similar fluctuations in the ranking order as in case of FFHWA and FFHWG operators.

The graphical views of these ranking orders are shown in Fig. 1.

From Fig. 1, we observe that the ranking orders of all the alternatives using several existing and proposed operators are different. We observe that the graphs of existing operators (FFWA (green line), FFWG (brown line), FFWPA (yellow line) and FFWG (aquamarine line)) are monotonically increasing & decreasing between the alternatives $A_1$ to $A_5$, and we cannot observe stability in these operators. On the other hand, the graphs of proposed operators (FFHWA (purple line), FFHWG (red line), FFHOWA (teal line), and FFHWG (black line)) are more stable and their graphs have very rare fluctuations. Therefore, the proposed operators seem to be more stable.
Table 3  Ranking order in proposed operators

| Alternatives | FFHPA  | FFHPG  |
|--------------|--------|--------|
| A1           | 0.7929 | 0.4513 |
| A2           | 0.6573 | 0.5977 |
| A3           | 0.4949 | 0.6142 |
| A4           | 0.3184 | 0.7557 |
| A5           | 0.1063 | 0.6905 |
| Ranking      | A1 ≻ A2 ≻ A3 ≻ A4 ≻ A5 | A4 ≻ A5 ≻ A3 ≻ A2 ≻ A1 |

Alternatives FFHPWA FFHPWG
A1 0.7929 0.4513
A2 0.6572 0.5978
A3 0.4948 0.6144
A4 0.3184 0.7558
A5 0.0737 0.6909
Ranking A1 ≻ A2 ≻ A3 ≻ A4 ≻ A5 A4 ≻ A5 ≻ A3 ≻ A2 ≻ A1

6.2.1 Effect of prioritizations of attributes

In Table 3, the score values and their ranking orders of alternatives in FFHPA, FFHPG, FFHPWA and FFHPWG operators are given.

From Table 3, we observe that the ranking order of alternatives in FFHPA and FFHPWA operators has very rare fluctuations. Similarly, the ranking orders in FFHPG and FFHPWG operator have very small changes. It is concluded that the weights of alternatives in prioritized aggregation operators of FF-sets have a very low effect on the order of alternatives based on Hamacher operations.

From Fig. 2, it is clear that the ranking order of alternatives in (FFHPA) (green line) and (FFHPWA) (dotted line) is approximately parallel, while the ranking order in (FFHPG) (yellow line) and (FFHPWG) (dotted line) is apparently parallel. This means that the weights in Fermatean fuzzy Hamacher (prioritized) weighted averaging and (prioritized) weighted geometric operators have very small effect on the ranking order of alternatives. It is also observed that the ranking order is monotonically increasing in FFHPA and FFHPWA operators while monotonically decreasing in FFHPG and FFHPWG operators.

To compare our proposed operators with the results obtained by applying Pythagorean fuzzy aggregation operators with and without prioritization of weights of attributes, we consider the methods proposed in [19] and [29].

In [19], Pythagorean fuzzy Hamacher weighted averaging and geometric aggregation operators have been applied; if we consider the attribute weight of alternatives as $\Phi = (0.1, 0.2, 0.3, 0.4)^T$, then the score values of alternative are given in following Table 4.

On the other hand, if we consider the prioritization of attributes and use PFHPWA and PFHPWG operators, then the score values and their corresponding ranking orders are shown in Table 5.

Ranking order of alternatives using PFHPWA and PFHPWG operators and proposed operators with weighted prioritization is shown in Tables 6 and 7.

In the following figures, we compare our proposed operators with the existing operators of Pythagorean fuzzy Hamacher averaging and geometric operators with prioritization of attributes.

From Fig. 3, the ranking order of alternatives using PFHPWA operator (green line) is smoothly decreasing, while the ranking of alternatives using FFHPWA operator (brown line) is strictly decreasing.

In Fig. 4, the ranking order of alternatives in PFHPWG operator (green line) is smoothly decreasing, while the ranking order of alternatives in FFHPWG operator (brown line) is smoothly increasing.
| Alternatives | PFHWA | PFHWG |
|-------------|-------|-------|
| A1          | 0.2323| 0.0071|
| A2          | 0.0161| -0.2471|
| A3          | 0.3704| 0.1563|
| A4          | 0.2438| 0.0108|
| A5          | 0.0062| -0.3125|

**Table 5** Score values of alternatives using PFHPWA and PFHPWG operator

| Alternatives | PFHPWA | PFHPWG |
|-------------|--------|--------|
| A1          | 0.6522 | 0.5496 |
| A2          | 0.4816 | 0.3618 |
| A3          | 0.6228 | 0.4888 |
| A4          | 0.5964 | 0.4918 |
| A5          | 0.4648 | 0.3568 |

**Table 6** Score values of alternatives using PFHPWA and FFHPWG operator

| Alternatives | PFHPWA | FFHPWA |
|-------------|--------|--------|
| A1          | 0.6522 | 0.7929 |
| A2          | 0.4816 | 0.6572 |
| A3          | 0.6228 | 0.4948 |
| A4          | 0.5964 | 0.3184 |
| A5          | 0.4648 | 0.0737 |

| Alternatives | PFHPWG | FFHPWG |
|-------------|--------|--------|
| A1          | 0.5496 | 0.4513 |
| A2          | 0.3618 | 0.5978 |
| A3          | 0.4888 | 0.6144 |
| A4          | 0.4918 | 0.7558 |
| A5          | 0.3568 | 0.6909 |

**7 Concluding remarks**

In this paper, an attempt is made to present several types of aggregation operators based on Hamacher operations for use in FFS decision-making process. Previously, the FFSs environment described different aggregation operators of Hamacher operations without prioritization of attributes. We presented here arithmetic and geometric operations to initiate some Fermatean fuzzy Hamacher prioritized aggregation operators from the rationale of Hamacher operations as Fermatean fuzzy Hamacher prioritized average (FFHPA) operator, Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator, Fermatean fuzzy Hamacher prioritized geometric (FFHPG) operator and Fermatean fuzzy Hamacher prioritized weighted geometric (FFHPWG) operator. Several new aspects of these recommended operators are considered. As a fact check, we have applied these operators to look into strategies remedying MADM situations. Eventually, a genuine example for the selection process of teaching staff of our university is considered to develop a
strategy and usefulness about the presented method. The new operators are compared with Pythagorean fuzzy Hamacher aggregation operators which gave the reliability of these operators. In future, we will consider risk theory and other areas under uncertain conditions for the proposed Fermatean fuzzy sets with prioritized attributes structure.

Author Contributions Khan and Khan discussed and formulated the measures. Jan and Afridi wrote the paper together.

Declarations

Conflict of interest The authors declare no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animal performed by any of the authors.

References

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Atanassov KT (1989) More on intuitionistic fuzzy sets. Fuzzy Sets Syst 33:37–46

Aydemir SB, Gunduz SY (2020) Fermatean fuzzy TOPSIT method with dombi aggregation operators and its application in multi-criteria decision making. J Intell Fuzzy Syst. https://doi.org/10.3233/JIFS-191763

Beliakov G, Pradera A, Calvo T (2007) Aggregation functions: a guide for practitioners. Springer, Heidelberg

Chen T-Y (2012) Nonlinear assignment-based methods for interval-valued intuitionistic fuzzy multi-criteria decision-making analysis with incomplete preference information. Int J Inf Tech Decision Making 11:821–855

Chen T-Y (2014) The extended linear assignment methods for multiple criteria decision-making based on interval-valued intuitionistic fuzzy sets. Appl Math Model 38:2101–2117

Chen T-Y (2014) Interval-valued intuitionistic fuzzy QUALIFLEX method with a likelihood-based comparison approach for multiple criteria decision analysis. Inf Sci 261:149–169

Chen T-Y (2018) An interval-valued Pythagorean fuzzy compromise approach with correlation-based closeness indices for multiple-criteria decision analysis of bridge construction methods. Complexity. https://doi.org/10.1155/2018/6463039

Chen T-Y (2018) A novel PROMETHEE-based outranking approach for multiple criteria decision analysis with Pythagorean fuzzy information. IEEE Access 6:54495–54506

Chen TY (2018) An interval-valued Pythagorean fuzzy outranking method with a closeness-based assignment model for multiple criteria decision making. Int J Intell Syst 33:126–168

Garg H (2016) A new generalized pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int J Intell Syst 31:886–920

Garg H, Shahrzadi G, Akram M (2020) Decision-making analysis based on Fermatean fuzzy Yager aggregation operators with application in COVID-19 testing facility. Math Problems Eng. https://doi.org/10.1155/2020/7279027

Gou X, Xu Z, Ren P (2016) The properties of continuous Pythagorean fuzzy information. Int J Intell Syst 31:401–424

Hadi A, Khan W, Khan A (2021) A novel approach to MADM problems using Fermatean fuzzy Hamacher aggregation operators. Int J Intell Syst. https://doi.org/10.1002/int.22324

Hamachar H (1978) Uber logische verknupfungenn unssharfer Aus sagen und deren Zugenhorige Bewertungsfunktione Trappl, Klir, Riccardi (Eds), Progr Cybern Syst Res, 3: 276–288

Li W (2014) Approaches to decision making with interval-valued intuitionistic fuzzy information and their application to enterprise financial performance assessment. J Intell Fuzzy Syst 27(1):1–8

Liu PD (2014) Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. IEEE Trans Fuzzy Syst 22(1):83–97

Parvathi R (2005) Theory of Operators on Intuitionistic Fuzzy Sets of Second Type and their Applications to Image Processing. Alagappa Univ, Karaikudi, India ((Ph.D. dissertation). Dept. Math)

Parvathi R, Vassilev P, Atanassov KT (2012) A note on the bijective correspondence between intuitionistic fuzzy sets and intuitionistic fuzzy sets of pth type. In: New Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations. SRI PAS IBS PAN, Warsaw, pp. 143–147

Peng X, Yang Y (2015) Some results for Pythagorean fuzzy sets. Int J Intell Syst 30:1133–1160

Reformat MZ, Yager RR (2014) Suggesting recommendations using Pythagorean fuzzy sets illustrated using netflix movie data. Information processing and management of uncertainty in knowledge-based systems. Springer, Cham, pp 546–556

Ren P, Xu Z, Gou X (2016) Pythagorean fuzzy TODIM approach to multi-criteria decision making. Appl Soft Comput 42:246–259
Senapati T, Yager RR (2019) Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods. Eng Appl Artif Intell 85:112–121
Senapati T, Yager RR (2019a) Fermatean fuzzy sets. J Ambient Intell Human Comput 11:663–674
Senapati T, Yager RR (2019b) Some new operations over Fermatean fuzzy numbers and application of Fermatean fuzzy WPM in multi criteria decision making. Informatica 30(2):391–412
Tan CQ, Yi WT, Chen XH (2015) Hesitant fuzzy Hamacher aggregation operators for multicriteria decision making. Appl Soft Comput 26:325–349
Wang J-C, Chen T-Y (2015) Likelihood-based assignment methods for multiple criteria decision analysis based on interval-valued intuitionistic fuzzy sets. Fuzzy Optim Decision Making 14:425–457
Wei G, Alsaadi FE, Hayat T, Alsaedi A (2018) Bipolar Fuzzy Hamacher aggregation Operators in multi Attribute Decision Making. Int J Fuzzy Syst. https://doi.org/10.1007/s40815-017-0338-6
Wu SJ, Wei GW (2017) Pythagorean fuzzy Hamacher aggregation operators and their application to multi attribute decision making. Int J Knowl-based Intell Eng Syst 21:189–201
Wu S-J, Wei G-W (2017) Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Int J Knowl-based Intell Eng Syst 21:189–201
Xiao S (2014) Induced interval-valued intuitionistic fuzzy Hamacher ordered weighted geometric operator and their application to multiple attribute decision making. J Intell Fuzzy Syst 27(1):527–534
Xu ZS (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15(6):1179–1187
Xu ZS (2011) Approaches to multi attribute group decision making based on intuitionistic fuzzy power aggregation operators. Knowl Based Syst 24(6):749–760
Xu ZS, Chen Q (2011) A multi-criteria decision making procedure based on intuitionistic fuzzy bonferroni means. J Syst Sci Syst Eng 20(2):217–228
Xu ZS, Xia MM (2011) Induced generalized intuitionistic fuzzy operators. Knowl Based Syst 24(2):197–209
Xu ZS, Yager RR (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 35:417–433
Xu ZS, Yager RR (2008) Dynamic intuitionistic fuzzy multi-attribute decision making. Int J Approx Reason 48(1):246–262
Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 28:436–452
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–356
Zeng S, Mu Z, Balezentis T (2018) A novel aggregation method for Pythagorean fuzzy multi attribute group decision making. Int J Intell Syst 33(3):573–585
Zhang X (2016) A novel approach based on similarity measure for Pythagorean fuzzy multi criteria group decision making. Int J Intell Syst 31:593–611
Zhang X, Xu Z (2014) Extension of TOPSIS to multi criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29:1061–1078
Zhou LY, Zhao XF, Wei GW (2014) Hesitant fuzzy hamacher aggregation operators and their application to multi attribute decision making. J Intell Fuzzy Syst 26(6):2689–2699

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