ON THE DEPENDENCE OF THE COMPONENT COUNTING PROCESS OF A DISCRETE UNIFORM RANDOM VARIABLE

Joseph Squillace
University of California, Irvine
Department of Mathematics
440V Rowland Hall
Irvine, CA 92697-3875, USA

Abstract. We are concerned with the general problem of proving the existence of joint distributions of two discrete random variables M and N subject to infinitely many constraints of the form \( P(M = i, N = j) = 0 \). In particular, the variable M has a countably infinite range and the other variable N is uniformly distributed with finite range. The constraints placed on the joint distribution will require, for some j’s in the range of N, \( p(i, j) = 0 \) for infinitely many values of i in the range of M. To prove the existence of such a joint distribution, we provide a technique that furnishes the existence of an \( \infty \times n \) matrix consisting of non-negative real numbers whose row and column sums are known, with zeros in infinitely many pre-specified locations.

Given \( n \in \mathbb{N} \), consider an assembly, multiset, or selection \( A_n \) among elements of \([n] = \{1, 2, \ldots, n\}\), and consider a uniformly distributed random variable \( N(n) \) on \( A_n \). For each \( i \leq n \), denote by \( C_i(n) \) the number of components of \( N(n) \) of size \( i \) so that \( \sum_{i \leq n} iC_i(n) = n \). In each of these combinatorial structures, there exists infinitely many processes \( (Z_i(x))_{i \in \mathbb{N}} \) indexed by a real parameter \( x \), consisting of non-negative independent variables \( (Z_i(x))_i \) such that \( \mathcal{L}(C_1(n), \ldots, C_n(n)) = \mathcal{L}(Z_1(x), \ldots, Z_n(x)) \sum_{i \leq n} iZ_i(x) = n \).

Let \( M(n, x) \) denote a random variable whose components are given by \( (Z_i(x))_{i \leq n} \). We introduce the notion of pivot mass, which, when combined with the theory of transportation polytopes and some results from topology, provides a technique that furnishes couplings of \( M(n, x) \) and \( N(n) \) with desired properties. For each of these combinatorial structures, we prove that there exists real numbers \( x(n) \) for which we can couple \( M(n, x) \) and \( N(n) \) with \( \sum_{i \leq n} (C_i(n) - Z_i(x))^+ \leq 1 \) when \( x > x(n) \). We are providing a partial answer to the question “how much dependence is there in the process \((C_i(n))_{i \leq n}\)’?”

1. Introduction

Our results regard the component counting process of a discrete uniform random variable in a combinatorial structure, and these results are provided by establishing the existence of couplings of random variables.

Definition. Let X and Y be random variables defined on the probability space \((\Omega_X, \mathcal{F}_X, P_X)\) and \((\Omega_Y, \mathcal{F}_Y, P_Y)\). A coupling of X and Y is a probability space \((\Omega, \mathcal{F}, P)\) in which there exists random variables \(X'\) and \(Y'\) such that X has the same distribution as \(X'\) and Y has the same distribution as \(Y'\).

For each of the random variables X considered in this paper, X and \(X'\) will share the same range. Thus, for each coupling of X and Y, the definition implies that there exists a joint probability mass function \(p(x, y) := P(X' = x, Y' = y)\) whose marginal distributions satisfy

\[
\sum_{y:p(x,y) > 0} p(x, y) = P(X = x),
\]

\[
\sum_{x:p(x,y) > 0} p(x, y) = P(Y = y).
\]

Equivalently, \(P_X(X' = x) = P_X(X = x)\) and \(P_Y(Y' = y) = P_Y(Y = y)\) for all x in the range of X and all y in the range of Y. In particular, we provide couplings, with some constraints, of a uniform random variable N, necessarily consisting of a dependent component process, with another random variable M having the following properties:

1\textsuperscript{In} our discrete setting, we can consider probability spaces \((S, F, P)\) of the following form: (i) \(S\) is a nonempty set that is at most countable; (ii) \(F\) is the power set of \(S\); and (iii) the probability measure \(P\) is defined as \(P(E) = \sum_{e \in E} P(e)\) for all \(E \in F\), where \(p\) is a probability mass function—i.e., \(p : S \to [0, 1]\) with \(\sum_{s \in S} p(s) = 1\).

2\textsuperscript{When} describing a particular coupling of X and Y, we often write X and Y instead of \(X'\) and \(Y'\), respectively.
(1) $M$ has infinite range.
(2) $M$ and $N$ have the same number of components.
(3) The components of $M$ are independent and nonnegative.

The constraints imposed on our couplings are motivated by the following conjecture, proposed by Richard Arratia in §2.2 of [1], which we now describe. Consider a uniformly distributed variable $N(n) \in [n]$ with prime factorization

$$N(n) = \prod_{p \leq n} p^{C_p(n)}.$$ 

It can be shown that the prime power process $(C_p(n))_{p \leq n}$ converges in distribution to a process $(Z_p)_{p \leq n}$ of independent variables where $Z_p$ is a geometric random variable of parameter $\frac{1}{p}$ for each prime $p \leq n$. Defining

$$M(n) = \prod_{p \leq n} p^{Z_p},$$

we state Arratia’s conjecture.

**Conjecture.** For all $n \geq 1$, it is possible to construct $N(n)$ uniformly distributed from 1 to $n$, $M(n)$ and a prime $P(n)$ such that

$$\text{always} \quad N(n) \text{ divides } M(n) P(n).$$

Equivalently, the conjecture states that there exists a coupling of $M(n)$ and $N(n)$ such that we always have

$$\sum_{p \leq n} (C_p(n) - Z_p) \geq 1.$$ 

This is also equivalent to the existence of a joint probability mass function $p(\cdot, \cdot)$ with marginal distributions $M(n)$ and $N(n)$ such that $p(\cdot, \cdot) = 0$ when

$$\sum_{p \leq n} (C_p(n) - Z_p) < 1.$$ 

We impose an analogous constraint on the couplings provided in this paper, but now we will point out some differences between these couplings and the coupling conjectured by Arratia. First, we drop the requirement that $N(n) \in [n]$; rather, from now on we let $N(n)$ denote a uniform variable in a combinatorial structure over $[n]$ (these structures are defined in §1.1). Instead of prime factorizations, we consider a component counting process $(C_i(n))_{1 \leq i \leq n}$ (here $i$ is any positive integer less than or equal $n$) of $N(n)$ which satisfies

$$\sum_{1 \leq i \leq n} i C_i(n) = n - \text{the latter equation is not always true for the prime power process } (C_p(n))_{p \leq n} \text{ of a uniformly distributed variable over } [n].$$

In Arratia’s conjecture, there exists a natural candidate for $M(n)$ since the prime power process $(C_p(n))_{p \leq n}$ converges in distribution to the process $(Z_p)_{p \leq n}$ described above. However, in each of the examples considered in this paper, all of the components $C_i(n), 1 \leq i \leq n$, diverge to $\infty$ as $n \to \infty$; so we do not find a comparable process by taking a limit of $C_i(n)$ with respect to $n$. Rather, we take advantage of the fact that in either an assembly, multiset, or selection, there exists infinitely many processes $(Z_i(x))_{1 \leq i \leq n}$, indexed by some positive real parameter $x$, which furnish candidates for a random variable $M(n, x)$ to be compared with a uniform random variable $N(n)$ (see equation (1) below).

The combinatorial structures listed in §1.1 provide the frameworks in which we obtain our couplings. Theorem 1, the main result of this paper, is stated in §1.2. In §2, we describe how our constraints force a significant proportion of the entries of a prospective joint mass distribution of our variables to be 0. In §3, we introduce the notion of pivot mass, which depends on the constraints placed on the desired joint distribution. Some properties of the pivot mass are proved in §4. In §5, we apply results on the pivot mass, a result of Brualdi, and some theorems from topology to prove the existence of our couplings.

The concatenation method applied in §5 provides a technique that allows one to prove the existence of an $\infty \times n$ matrix $A = [a_{ij}]$ of nonnegative numbers with known row sum vector $R = (r_1, r_2, \ldots)$, known column sum vector $S = (s_1, \ldots, s_n)$, and constraints of the form $a_{ij} = 0$ for infinitely many pairs $(i, j)$.
concatenation method takes an $m \times n$ matrix $A$, a $k \times n$ matrix $B$ and forms an $(m+k) \times n$ matrix of the form

$$\begin{bmatrix} A \\ B \end{bmatrix}.$$  

Our concatenation method will give us a sequence of finite matrices, each having a finite row sum vector of the form $(r_1, r_2, \ldots, r_m)$, with $m$ strictly increasing for each successive matrix, zeros in the correct positions, and a column sum vector $((1-\varepsilon)s_1, \ldots, (1-\varepsilon)s_n)$, with $\varepsilon$ strictly decreasing for each successive matrix. A theorem of Brualdi and ideas introduced in Sections 3 and 4 of this paper make such a construction possible. The fact that the limit of this concatenation process exists will be deduced from topological results. If we let $r_i = P(M = i)$ and $s_j = P(N = j)$, then the $\infty \times n$ matrix corresponds to a desired coupling of $M$ and $N$.

1.1. Three Major Combinatorial Structures. All couplings constructed in this paper involve a discrete uniform random variable in any one of the following three combinatorial classes. An assembly $A_n$ is an example of a combinatorial structure in which the set $[n]$ is partitioned into blocks and for each block of size $i$ one of $m_i$ possible structures is chosen. An example of an assembly is the collection of set partitions of $[n]$, in which case $m_i = 1$ for $i \leq n$ (since the order of the elements in a particular block is irrelevant -- i.e., once $i$ numbers $n_1, \ldots, n_i \in [n]$ are chosen and placed in a box of size $i$, there is a unique block consisting of these $i$ elements). Moreover, for set partitions of $[n]$, we have $\#A_n = B_n$, the $n$th Bell number. Another example of an assembly is the set $S_n$ of permutations of $[n]$, in which case $m_i = (i-1)!$ (since there are $(i-1)!$ distinct cycles of length $i$ among $i$ chosen numbers $n_1, \ldots, n_i \in [n]$) for $i \leq n$. Further, for permutations of $[n]$, we have $\#A_n = n!$. A multiset $A_n$ is a pair $([n], m)$, where $m: A \to \mathbb{N}$ is a function that gives the multiplicity $m(a)$ of each element $a \in [n]$. Equivalently (see Meta-example 2.2 of §2.2 of [2]), the integer $n$ is partitioned into parts, and for each part of size $i$, one of the $m_i$ objects of weight $i$ is chosen. In the example of integer partitions of a positive integer $n$, we have $m_i = 1$ (for each part of size $i$, we have only $m_i = 1$ choice for the size of $i$) for $1 \leq i \leq n$. When $A_n$ is the set of integer partitions of $n$, we have $\#A_n = p(n)$, where $p$ is the integer partition function. Selections are similar to multisets, but now we require all parts to be distinct. An example of a selection is the set of all integer partitions of a positive integer $n$ with distinct parts. In the case of integer partitions with distinct parts, we have $\#A_n = q(n)$, where $q$ is the integer partition function with distinct parts. To simplify the notation, let’s define $n := \#A_n$ for each of these structures.

These three structures are characterized by the following generating relations between $k_n$ and $m_i$. Assemblies are characterized by

$$\sum_{n \geq 0} \frac{(k_n) z^n}{n!} = \exp \left( \sum_{i \geq 1} \frac{m_i z^i}{i!} \right),$$

multisets are characterized by

$$\sum_{n \geq 0} (k_n) z^n = \prod_{i \geq 1} (1 - z^i)^{-m_i},$$

and selections are characterized by

$$\sum_{n \geq 0} (k_n) z^n = \prod_{i \geq 1} \left(1 + z^i\right)^{m_i}$$

(§2.2 of [2]). Revisiting the example of an assembly in which $A_n$ denotes the set of all set partitions of $[n]$ (so that $m_i = 1$ for $1 \leq i \leq n$), it is known that the $n$th Bell number $B_n$ satisfies the generating equation

$$\sum_{n \geq 0} \frac{B_n}{n!} z^n = \exp (e^z - 1),$$

and the right hand side may be expressed as $\exp \left( \sum_{i \geq 1} \frac{z^i}{i!} \right)$.

1.2. Couplings of Random Variables. In each of the assembly, multiset, and selection settings, our methods of arriving at our desired couplings are similar. We start by considering $N(n) \sim \text{Unif}(A_n)$. For $i \leq n$, if we denote by $C_i(n)$ the number components of $N(n)$ of size $i$, then $0 \leq C_i(n) \leq n$ and $\sum_{i=1}^n iC_i(n) = n$. In particular, the variables $C_i(n), i \leq n$, are independent and their distributions are determined by the uniform variable $N(n)$. The process $(C_i(n))_{1 \leq i \leq n} = (C_1(n), \ldots, C_n(n))$ is called the component counting process of $N(n)$. 


Example 1. In the example $A_n = S_n$, the term $C_i (n)$ is the number of cycles of $N (n)$ of length $i$, and $(C_1 (n), \ldots , C_n (n))$ is often referred to as the cycle type of $N (n)$. In the example for which $A_n$ is the collection of set partitions of $[n]$, $C_i (n)$ is the number of blocks of $N (n)$ of size $i$. In the example for which $A_n$ is the set of integer partitions of $n$, $C_i (n)$ is the number of $i$’s in the integer partition $N (n)$ of $n$.

In each of these combinatorial settings, there exists an infinite family $\{(Z_i (x))_{i \in \mathbb{N}}\}_x$, parametrized by positive values of $x$ (specifically, $x > 0$ for assemblies, $x \in (0, 1)$ for multisets, and $x \in (0, \infty)$ for selections), of infinite sequences $(Z_i (x))_{i \in \mathbb{N}}$ of nonnegative integer-valued independent random variables $Z_i (x)$ for which

\[
\mathcal{L} (C_1 (n), \ldots , C_n (n)) = \mathcal{L} \left( Z_1 (x), \ldots , Z_n (x) \middle| \sum_{i \leq n} i Z_i (x) = n \right)
\]

(S2.3 of [2]). Equation (1) states that the probability that the vector $(C_1 (n), \ldots , C_n (n))$ belongs to some region $\Gamma \in \mathbb{R}^n$ (where $\Gamma$ is an element of the $n$-fold direct product $\prod_{i \leq n} \mathcal{B} (\mathbb{R})$ of the Borel $\sigma$-algebra on $\mathbb{R}$) is the same as the conditional probability that $(Z_1 (x), \ldots , Z_n (x))$ belongs to $\Gamma$ if we condition on the event $\{\sum_{i \leq n} i Z_i (x) = n\}$. For a fixed $x$, we consider another random variable $M (n, x)$ whose component counting process is given by $(Z_i (x))_{i \leq n}$, so the distribution of $M (n, x)$ is determined by the independent process $(Z_i (x))_{i \leq n}$. The main result of this paper is the following theorem.

Theorem 1. Let $n \in \mathbb{N}$ and suppose $A_n$ denotes an assembly, multiset, or a selection among elements of $[n]$. Given $N (n) \sim \text{Unif} (A_n)$ with component counting process $(C_i (n))_{i \leq n}$, there exist processes $(Z_i (x))_{i \in \mathbb{N}}$, for some $x > 0$, of non-negative independent random variables satisfying (1) for which we can couple $M (n, x)$ and $N (n)$ with

\[
\sum_{i \leq n} (C_i (n) - Z_i (x))^+ \leq 1.
\]

In particular, when $A_n$ denotes an assembly or selection, the couplings can be obtained for sufficiently large $x$ – depending on $n$; when $A_n$ denotes a multiset, the couplings can be obtained when $x$ is sufficiently close to $1$ – depending on $n$.

2. The Joint Mass Distribution of $(M (n, x), N (n))$

For some fixed value of $x$, if we are to successively construct a joint probability mass function $p (\cdot , \cdot)$ for $M (n, x)$ and $N (n)$ for which inequality (2) holds, we must ensure that $\mathbb{P} (\mathcal{L} (M (n, x) = \cdot , N (n) = \cdot) = 0$ when

\[
\sum_{i \leq n} (C_i (n) - Z_i (x))^+ > 1.
\]

We can index the joint distribution by using the range of $N (n)$ and the range of $M (n, x)$ for the column labels and row labels, respectively. In particular, we can label the columns with the range of $(C_i (n))_{i \leq n}$ in lexicographic order. Since we have infinitely many row labels, for each $m \in \mathbb{Z}_{\geq 0}$, we apply the lexicographic ordering on all elements $(m_1, \ldots , m_n) \in (\mathbb{Z}_{\geq 0})^n$ with $\sum_{i \leq n} m_i = m$, starting with $m = 0$ (we start with $m = 0$ since the $Z_i$’s are non-negative). With respect to this ordering, we will often enumerate the columns by $1, 2, \ldots , k_n$ and the rows by $1, 2, \ldots$. 

The following example shows that it is possible for several elements of $A_n$ to have the same component process (hence the same column label). Note that in the setting of Arratia’s conjecture, it is impossible for two columns to have the same label – the uniqueness of prime factorization in $\mathbb{N}$ ensures that each $(C_p (n))_{p \leq n}$ uniquely determines $N (n)$.

Example 2. Fix $n = 3$ and consider the assembly $A_3 = S_3$ of permutations of $\{1, 2, 3\}$. The elements of $S_3$ are $1, (1 \ 2), (1 \ 3), (2 \ 3), (1 \ 2 \ 3), (1 \ 3 \ 2)$, and their respective component counts are $(3, 0, 0), (1, 1, 0), (1, 1, 0)$, $(3, 0, 0), (1, 1, 0), (1, 1, 0)$.

\[3\text{For fixed } x, \text{since the variables } Z_i (x), i \leq n, \text{are independent, it is not always true that } \sum_{i \leq n} i Z_i (x) = n. \text{Therefore, the variable } M (n, x) \text{does not always correspond to an element of } A_n.\]

\[4\text{To simplify the notation, we will sometimes (Figure 2, Theorem 2, and §4) replace } Z_i (x) \text{with } Z_i, \text{replace } C_i (n) \text{with } C_i, \text{replace } N (n) \text{with } N, \text{and replace } M (n, x) \text{with } M. \text{Note that the term } Z_i, \text{in which we drop the } x, \text{is usually reserved for the case } n = \infty, \text{which is not considered in this paper.}\]
\begin{align*}
(1, 1, 0), (0, 0, 1), (0, 0, 1). & \\
\text{For any } n \in \mathbb{N}, \text{ Cauchy proved that there are}
\prod_{a_i \geq 0, 1 \leq i \leq n} n!	ext{ permutations in } S_n
\text{ with cycle type } (i_1, \ldots, i_n), \text{ so this gives the number of elements in } S_n
\text{ with the component counting process } (C_i(n))_{i \leq n} = (i_1, \ldots, i_n).
\end{align*}

For our purposes, when we have multiple columns with the same component counting process, we enumerate these columns in any order. The reason that we do not combine these into one column with larger probability mass is due to the fact we are coupling \( M(n, x) \) and \( N(n) \) instead of coupling the two processes \( (C_i(n))_{i \leq n} \) and \( (Z_i(x))_{i \leq n} \) i.e., two columns with the same label correspond to different values of \( N(n) \).

For the interested reader, equations (2.2), (2.3), and (2.4) in \S 2.2 of [2] give the number of columns with a given column label \((a_1, \ldots, a_n)\) for each of our combinatorial structures.

In each of these three settings, there are additional constraints on any joint probability mass function of \( M(n, x) \) and \( N(n) \) since the marginal distributions are known:

- The sum along column \( N(n) = j, 1 \leq j \leq k_n \), is

\[
\sum_{a_i \geq 0, 1 \leq i \leq n} \mathbb{P}\left( N(n) = j, (Z_i(x))_{i \leq n} = (a_i)_{i \leq n} \right) = \mathbb{P}(N(n) = j) = \frac{1}{k_n}.
\]

- The sum along the row \( M(n, x) = m, m \in \mathbb{N} \), labeled \((Z_i(x))_{i \leq n} = (m_i)_{i \leq n}\) is

\[
\sum_{a_i \geq 0, 1 \leq i \leq n} \mathbb{P}\left( (C_i(n))_{i \leq n} = (a_i)_{i \leq n}, (Z_i(x))_{i \leq n} = (m_i)_{i \leq n} \right) = \prod_{i \leq n} \mathbb{P}(Z_i(x) = m_i),
\]

where the latest equation is due to the independence of the process \((Z_i(x))_{i \leq n}\).

3. Pivot Mass

One way to obtain a coupling of \( M(n, x) \) and \( N(n) \) (without regarding inequality (2)) is to couple them with

\[
\mathbb{P}\left( M(n, x) = a, N(n) = b \right) = \mathbb{P}(M(n, x) = a) \mathbb{P}(N(n) = b),
\]

and this is known as the independent coupling. This will ensure that for each row label \( i \) and each column label \( j \) we have

\[
\mathbb{P}\left( M(n, x) = i, N(n) = j \right) = \frac{\mathbb{P}(M(n, x) = i)}{k_n}.
\]

This is not a coupling which satisfies \( \sum_{i \leq n} (C_i(n) - Z_i(x))^+ \leq 1 \) since all row labels \( \mathbb{P}(M(n, x) = i), i \in \mathbb{N} \), will be positive in this paper. In order to modify this joint mass table to obtain a coupling satisfying (2), one would have to remove the probability mass in the joint mass table of the independent coupling corresponding to pairs \((i, j)\) of row and column labels for which (2) is violated. Then one would have to distribute that probability mass to other locations in the table while still preserving both the row and column sums. If we start by using the joint mass table corresponding to the independent coupling, then the function introduced in the following definition is defined on \( A_n \) and gives the proportion of the column mass \( 1/k_n \) in column \( N(n) \) that lies in rows labeled \((Z_i(x))_{i \leq n}\) satisfying \( \sum_{i \leq n} (C_i(n) - Z_i(x))^+ > 1 \).

**Definition.** We call the pair \((i, j), \) corresponding to the \( i \)th row label \((Z_i(x))_{i \leq n}\) and the \( j \)th column label \((C_i(n))_{i \leq n}, \) a **pivot** if \( \sum_{i \leq n} (C_i(n) - Z_i(x))^+ > 1 \). Denote the set of all pivots by \( P \). The **pivot mass** in column \( N(n) = j \) is defined as

\[
\mathbb{P}M(j) := \sum_{i(i,j) \in P} \mathbb{P}(M(n, x) = i).
\]

Given a subset \( L(n) \) of column labels of \([n]\), the pivot mass in \( L(n) \) is defined as

\[
\mathbb{P}M(L(n)) := \sum_{i(i,j) \in P \wedge j \in L(n)} \mathbb{P}(M(n, x) = i).
\]
Theorem 2 gives a formula for $\mathcal{PM}(j)$. Due to the role of the parameter $x$, it is not necessary to derive a formula for $\mathcal{PM}(L(n))$ to prove Theorem 1; the fact that $\mathcal{PM}(L(n)) \leq \mathcal{PM}(j)$ for any $j \in L(n)$ will be sufficient.

**Figure 1.** If $(i_0, j_0)$ is a pivot, then our desired joint distribution table should have a 0 in the $(i_0, j_0)$ entry.

| $M(n, x)$ | $N(n)$ | Row sum |
|-----------|--------|---------|
| 1         | 1      | $\mathbb{P}(M(n, x) = 1)$ |
| 2         | $\vdots$ | $\vdots$ |
| $i_0$     | $j_0$  | $\mathbb{P}(M(n, x) = i_0)$ |
| $\vdots$  | $\vdots$ | $\vdots$ |
| $i$       | $\mathbb{P}(M(n, x) = i, N(n) = j)$ |
| $\vdots$  | $\vdots$ | $\vdots$ |

**Column sum**

| 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
|-----|-----|-----|-----|-----|

**Example 3.** Revisiting the example $A_3 = S_3$, let us illustrate some key features of a desired joint mass distribution of $(M(3, x), N(3))$.

**Figure 2.** A desired coupling of $M(3, x)$ and $N(3)$ should have a zero at any location $(Z_i(x))_{i \leq 3}, (C_i(3))_{i \leq 3}$ satisfying $\sum_{i \leq 3} (C_i(3) - Z_i(x))^+ > 1$.

| $M$   | $(0, 0, 1)$ | $(0, 0, 1)$ | $(1, 1, 0)$ | $(1, 1, 0)$ | $(3, 0, 0)$ | Row sum |
|-------|-------------|-------------|-------------|-------------|-------------|---------|
| $(0, 0, 0)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (0, 0, 0))$ |
| $(0, 0, 1)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (0, 0, 1))$ |
| $(0, 1, 0)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (0, 1, 0))$ |
| $(1, 0, 0)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (1, 0, 0))$ |
| $(1, 1, 0)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (1, 1, 0))$ |
| $(0, 0, 2)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (0, 0, 2))$ |
| $(0, 1, 2)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (0, 1, 2))$ |
| $(1, 0, 2)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (1, 0, 2))$ |
| $(1, 1, 2)$ |             | 0           | 0           | 0           | 0           | $\mathbb{P}(Z_i(1)_{i \leq 3} = (1, 1, 2))$ |

| Column sum | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
|------------|-----|-----|-----|-----|-----|
| $\mathcal{PM}(N)$ | 0   | 0   | $\mathbb{P}(Z_1 = Z_2 = 0)$ | $\mathbb{P}(Z_1 = Z_2 = 0)$ | $\mathbb{P}(Z_1 = Z_2 = 0)$ |

Each column with a pivot contains infinitely many pivots. In Figure 2, column $(3, 0, 0)$ has a pivot in any row of the form $(a, b, c)$ with $a \in \{0, 1\}, b, c \geq 0$. Columns labeled $(1, 1, 0)$ have a pivot in any row of the form $(0, 0, l)$ for any $l \in \mathbb{Z}_{\geq 0}$. Moreover, note that the independence of the process $(Z_i(x))_{i \leq 3}$ allows us to distribute $\mathbb{P}$ through the parentheses in the row sums and pivot mass expressions. The actual value of the row sum and pivot masses depends on the choice of the process $(Z_i(x))_{i \leq 3}$. In Section 4, we mention several choices for such processes $(Z_i(x))_{i \leq 3}$ which will satisfy equation (1).
The following theorem plays a key role in the proof of Theorem 1.\footnote{When Theorem 2 is applied in §4, additional indicator functions will be included to remind us that $\P(Z_i(x) \leq k) = 0$ if $k < 0$.} For convenience, in the following theorem and its proof, we simplify the notation by writing $C_i(n) = C_i$, $Z_i(x) = Z_i$, $M(n, x) = M$ and $N(n) = N$. Moreover, the notion of pivot mass introduced in this section may be generalized; in a particular setting, one should define pivot mass based on the constraints required of their desired coupling. It is both a combinatorial and probabilistic object since it is a sum of probability masses indexed by the counting constraint \footnote{\label{fn:pivot}}.

\textbf{Theorem 2.} (Pivot Mass Formula for 1 Column) Consider a fixed column label $N \in A_n$ and denote its component counting process by $(C_i)_{i \leq n}$. Its pivot mass is

$$
\P M(N) = 1 - \sum_{j \leq n} \left( 1_{\{C_j > 0\}} (1 - \P(Z_j \leq C_j - 2)) \prod_{i \neq j, i \leq n} (1 - \P(Z_i \leq C_i - 1)) \right)^{1_{\{C_j > 0\}}} + \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \prod_{i \leq n} (1 - \P(Z_i \leq C_i - 1)).
$$

\textit{Proof.} Given $1 \leq j \leq n$, let $\vec{e}_j$ denote the row vector of length $n$ whose $j$th entry is 1 and whose other entries are 0. Given two vectors $(a_i)_{i \leq n}, (b_i)_{i \leq n}$ in $\mathbb{R}^n$, we write $(a_i)_{i \leq n} \preceq (b_i)_{i \leq n}$ if $a_i \leq b_i$ for each $i \leq n$. Since $\sum_{k=1}^\infty \P(M(n, x) = k) = 1$, we have

$$
P M(N) = 1 - \sum_{k: (k, N) \notin P} \P(M = k).
$$

We have the event equality

$$
\{(M, N) \notin P\} = \{\exists j \leq n : (Z_i)_{i \leq n} \succeq (C_i)_{i \leq n} - \vec{e}_j \cdot 1_{\{C_j > 0\}}\}
$$

since the pair $(M, N)$ is a not pivot if and only if $Z_j \geq C_j$ for all $i$ except possibly one value $j$ with $Z_j = C_j - 1$. Since each $Z_i, 1 \leq i \leq n$ is nonnegative, we can only have $Z_j = C_j - 1$ when $C_j > 0$. Note that if $Z_i \geq C_i$ for all $i$, then any $j$ satisfies $(Z_i)_{i \leq n} \succeq (C_i)_{i \leq n} - \vec{e}_j \cdot 1_{\{C_j > 0\}}$. On the other hand, if there exists a value $j$ for which $Z_j = C_j - 1$ and $Z_i \geq C_i$ for all $i \neq j$, then $(Z_i)_{i \leq n} \succeq (C_i)_{i \leq n} - \vec{e}_j \cdot 1_{\{C_j > 0\}}$. Therefore, the right hand side of equation \footnote{\label{fn:pivot}} is

$$
1 - \sum_{k: (k, N) \notin P} \P(M = k) = 1 - \P\left(\exists j \leq n : (Z_i)_{i \leq n} \succeq (C_i)_{i \leq n} - \vec{e}_j \cdot 1_{\{C_j > 0\}}\right).
$$

We rewrite the probability $\P\left(\exists j \leq n : (Z_i)_{i \leq n} \succeq (C_i)_{i \leq n} - \vec{e}_j \cdot 1_{\{C_j > 0\}}\right)$ by applying an inclusion-exclusion argument. Corresponding to any $j \leq n$ with $C_j > 0, Z_j \geq C_j - 1$, and $Z_i \geq C_i$ for $i \neq j$, we add the term $\P(Z_j \geq C_j - 1, \text{and} \, Z_i \geq C_i \text{for all } i \neq j)$. As a result, we have added those elements with $Z_i \geq C_i$ for all $i$ a total of $\sum_{i=1}^n 1_{\{C_i > 0\}}$ many times. Therefore, we compensate by subtracting the term $(\sum_{i=1}^n 1_{\{C_i > 0\}} - 1) \P\left((Z_i)_{i \leq n} \geq (C_i)_{i \leq n}\right)$. Further, applying independence of the process $(Z_i)_{i \leq n}$, we have

$$
\P(Z_j \geq C_j - 1 \text{ and } Z_i \geq C_i \text{ for all } i \neq j) = \P(Z_j \geq C_j - 1) \P(Z_i \geq C_i \text{ for all } i \neq j) = \P(Z_j \geq C_j - 1) \prod_{i \neq j, i \leq n} \P(Z_i \geq C_i)
$$

and

$$
\P\left((Z_i)_{i \leq n} \succeq (C_i)_{i \leq n}\right) = \prod_{i \leq n} \P(Z_i \geq C_i).
$$
Thus, the right hand side of equation (4) becomes

\[
1 - \sum_{j \leq n} \left(1_{\{C_j > 0\}} \mathbb{P}(Z_j \geq C_j - 1) \prod_{i \neq j, i \leq n} \mathbb{P}(Z_i \geq C_i)\right) + \left(\sum_{i \leq n} 1_{\{C_i > 0\}} - 1\right) \prod_{i \leq n} \mathbb{P}(Z_i \geq C_i).
\]

Using the fact that \(\mathbb{P}(Z_i \geq a) = 1 - \mathbb{P}((Z_i \leq a - 1))\), expression (5) becomes

\[
1 - \sum_{j \leq n} \left(1_{\{C_j > 0\}} (1 - \mathbb{P}(Z_j \leq C_j - 2)) \prod_{i \neq j, i \leq n} (1 - \mathbb{P}(Z_i \leq C_i - 1))\right) + \left(\sum_{i \leq n} 1_{\{C_i > 0\}} - 1\right) \prod_{i \leq n} (1 - \mathbb{P}(Z_i \leq C_i - 1)).
\]

The following result shows that only columns with label \((C_i(n))_{i \leq n} = \overrightarrow{c}_n\) have zero pivot mass. In this paper, we will only apply the \((\Leftarrow)\) part of the statement.\footnote{Note that \((\Rightarrow)\) implies that each column label other than \(\overrightarrow{C}_n\) has pivots. Using equations (2.2) – (2.4) in §2.2 of [2] (which give the number of columns with label \(\overrightarrow{c}_n\) in each of these combinatorial settings) we can always determine the number of columns that contain pivots.}

**Theorem 3.** For any nonempty collection \(L(n)\) of column labels, \(\mathcal{PM}(L(n)) = 0\) if and only if a column with label \((C_i(n))_{i \leq n} = \overrightarrow{c}_n\) belongs to \(L(n)\).

**Proof.** \((\Leftarrow)\) Given any row label \((Z_i(n, x))_{i \leq n}\), the vector \((C_i(n))_{i \leq n} = \overrightarrow{c}_n\) satisfies

\[
\sum_{i \leq n} (C_i(n) - Z_i(x))^+ = (C_n(n) - Z_n(x))^+ = (1 - Z_n(x))^+ \leq 1.
\]

Thus, \(\mathcal{PM}(\overrightarrow{c}_n) = 0\). Therefore, given \(\overrightarrow{c}_n \in L(n)\), we have

\[
\mathcal{PM}(L(n)) \leq \mathcal{PM}(\overrightarrow{c}_n) = 0.
\]

\((\Rightarrow)\) Now suppose \(\overrightarrow{c}_n \notin L(n)\). Recall that any column label \((C_i(n))_{i \leq n}\) satisfies \(\sum_{i \leq n} i C_i(n) = n\). Since \(\overrightarrow{c}_n\) is the only column label with \(\sum_{i \leq n} C_i(n) = 1\), this gives us one of two cases for each column label in \(L(n)\). Either (a) there exists some \(j\) with \(C_j(n) \geq 2\) or (b) there exists distinct \(j, k\) with \(C_j(n) \geq 1, C_k(n) \geq 1\). In case (a), using any row label \((Z_i(x))_{i \leq n}\) with \(Z_j(x) = 0\), we have

\[
\sum_{i \leq n} (C_i(n) - Z_i(x))^+ \geq C_j(n) - Z_j(x) \geq 2.
\]

In case (b), we can take any \((Z_i(x))_{i \leq n}\) with \(Z_j(x) = Z_k(x) = 0\) to ensure that

\[
\sum_{i \leq n} (C_i(n) - Z_i(x))^+ \geq (C_j(n) - Z_j(x)) + (C_k(n) - Z_k(x)) \geq 2.
\]

Since we have just showed that each column label other than \(\overrightarrow{c}_n\) has a pivot, we use the fact that each of these columns has a pivot in the first row (labeled \((Z_i(x))_{i \leq n} = (0, 0, \ldots, 0)\)). Note that \(\mathbb{P}(M(n, x) = i) > 0\)
for all distributions in this paper (see §4), so we have
\[
\mathcal{PM} (L (n)) \geq \mathbb{P} (Z_i (x) = 0, \forall i \leq n) = \mathbb{P} (M (n, x) = 1) > 0.
\]
\[\square\]

4. Pivot Mass can be made Arbitrarily Small for Assemblies, Multisets, and Selections

The following condition on \(\mathcal{PM}\) will be verified for our three combinatorial structures:

\[
\forall n \in \mathbb{N} \forall \varepsilon > 0 \exists x: \text{equation } (1) \text{ holds and } \mathcal{PM} (\cdot) < \varepsilon.
\]

4.1. Assemblies. In the assembly setting, we can take \(Z_i (x) \sim \text{Po} \left( \frac{m_i x^i}{n} \right)\) for any \(x > 0\) to obtain equation (1) (§2.3 of [2]). Recall that the CDF of a random variable \(Z \sim \text{Po} (\lambda)\) is given by \(\mathbb{P} (Z \leq k) = \frac{\Gamma (\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor!}\) for \(k \in \mathbb{Z}_{\geq 0}\), where \(\Gamma (a, b)\) is the upper incomplete gamma function – i.e., \(\Gamma (a, b) = \int_b^\infty t^{a-1} e^{-t} dt\).

**Lemma 1.** For a fixed \(a > 0\), we have \(\lim_{b \to \infty} \Gamma (a, b) = 0\).

**Proof.** Since \(\Gamma (a, 0) = \Gamma (a)\) is convergent for \(a > 0\), we have
\[
\Gamma (a, b) = \Gamma (a) - \int_0^b t^{a-1} e^{-t} dt \\
\rightarrow \Gamma (a) - \Gamma (a) \text{ as } b \to \infty \\
= 0.
\]
\[\square\]

We can use Lemma 1 and take \(x \to \infty\) to obtain
\[
\Gamma \left( C_i \frac{m_i x^i}{n} \right) \left( C_i - 1 \right)! \to 0 \text{ when } C_i > 0
\]
and
\[
\Gamma \left( C_j - 1, \frac{m_j x^j}{n^j} \right) \left( C_j - 2 \right)! \to 0 \text{ when } C_j > 1
\]

Therefore, Theorem 2 implies that \(\mathcal{PM} (N (n))\) equals
\[
1 - \sum_{j \leq n} \left( \begin{array}{c} 1 \{ C_j > 0 \} \left( 1 - 1 \{ C_j > 1 \} \right) \frac{\Gamma \left( C_j - 1, \frac{m_j x^j}{n^j} \right)}{(C_j - 2)!} \prod_{i \neq j, i \leq n} \left( 1 - 1 \{ C_i > 0 \} \frac{\Gamma \left( C_i, \frac{m_i x^i}{n} \right)}{(C_i - 1)!} \right) \right) \\
+ \left( \sum_{i \leq n} 1 \{ C_i > 0 \} - 1 \right) \prod_{i \leq n} \left( 1 - 1 \{ C_i > 0 \} \frac{\Gamma \left( C_i, \frac{m_i x^i}{n} \right)}{(C_i - 1)!} \right).
\]
If we let \( x \to \infty \), we can apply (7) and (8) to deduce that

\[
\mathcal{P}(N(n)) \rightarrow 1 - \sum_{j \leq n} \left( 1_{\{C_j > 0\}} \prod_{i \neq j, i \leq n} (1 - 0) \right) \\
+ \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \prod_{i \leq n} (1 - 0) \\
= 1 - \sum_{j \leq n} 1_{\{C_j > 0\}} + \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \\
= 0.
\]

This verifies condition (6) for assemblies.

4.2. Multisets. In the multiset setting, we can take \( \mathcal{Z}_i(x) \sim \text{NB}(m_i, x_i) \), for any \( x \in (0, 1) \), to obtain equation (1) (§2.3 of [2]). Recall that the CDF of \( Z \sim \text{NB}(r, p) \) is given by \( \mathbb{P}(Z \leq k) = 1 - I_p(k + 1, r) \), where \( I_p \) is the regularized incomplete beta function. That is, \( I_p(x; a, b) = \frac{B(x; a, b)}{B(a, b)} \), where \( B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} \, dt \), defined for \( \text{Re}(a) > 0 \) and \( \text{Re}(b) > 0 \), is the beta function and \( B(x; a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} \, dt \) is the incomplete beta function.

**Lemma 2.** Given \( a > 0 \), \( \lim_{x \to 1} I_x(a, b) = 1 \).

**Proof.** We have

\[\lim_{x \to 1} I_x(a, b) = \lim_{x \to 1} \frac{B(x; a, b)}{B(a, b)} \]

\[= \lim_{x \to 1} \frac{\int_0^x t^{a-1} (1 - t)^{b-1} \, dt}{\int_0^1 t^{a-1} (1 - t)^{b-1} \, dt} \]

\[= \frac{\int_0^1 t^{a-1} (1 - t)^{b-1} \, dt}{\int_0^1 t^{a-1} (1 - t)^{b-1} \, dt} = 1.\]

Using Theorem 2, \( \mathcal{P}(N(n)) \) equals

\[
1 - \left( \sum_{j \leq n} \left( 1_{\{C_j > 0\}} \left( 1 - 1_{\{C_j > 1\}} (1 - I_x \left( C_j - 1, m_j \right)) \prod_{i \neq j, i \leq n} (1 - 1_{\{C_i > 0\}} (1 - I_x \left( C_i, m_i \right)) \right) \right) \\
+ \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \prod_{i \leq n} (1 - 1_{\{C_i > 0\}} (1 - I_x \left( C_i, m_i \right)) .
\]
Taking $x \to 1$ and applying Lemma 2, we have

$$PM(N(n)) \to 1 - \left( \sum_{j \leq n} 1_{\{C_j > 0\}} (1 - 1_{\{C_j > 1\}} (1 - 1)) \prod_{i \neq j, i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1)) \right)$$

$$+ \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \prod_{i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1))$$

$$= 1 - \sum_{j \leq n} 1_{\{C_j > 0\}} (1 - 1_{\{C_j > 1\}}) \prod_{i \neq j, i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1)) \prod_{i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1))$$

which verifies condition (6) for multiset.

4.3. **Selections.** In the selection setting, we can take $Z_i(x) \sim \text{Bin} \left( m_i, \frac{x^i}{1 + x^i} \right)$ with $0 < x < \infty$ in order to obtain equation (1) (§2.3 of [2]). In our case, we are taking $p = \frac{x^i}{1 + x^i}$, so $p \to 1$ if and only if $x \to \infty$. Recall that the CDF of $Z \sim \text{Bin} \left( n, p \right)$ is given by $P(Z \leq k) = I_{1-p}(n-k, 1+k)$. Using Theorem 2, we can express $PM(N(n))$ as

$$1 - \sum_{j \leq n} 1_{\{C_j > 0\}} (1 - 1_{\{C_j > 1\}}) \prod_{i \neq j, i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1)) \prod_{i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1))$$

$$+ \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right) \prod_{i \leq n} (1 - 1_{\{C_i > 0\}} (1 - 1))$$

**Lemma 3.** We have $\lim_{p \to 1} I_{1-p}(n-k, 1+k) = 0$.

**Proof.**

$$\lim_{p \to 1} I_{1-p}(n-k, 1+k) = \lim_{p \to 1} \frac{B(1-p; n-k, 1+k)}{B(n-k, 1+k)}$$

$$= \lim_{p \to 1} \int_0^1 t^{n-k-1} (1-t)^k$$

$$= 0.$$  

Using Lemma 3, we see that

$$I_{1-p} = I_{1-\frac{x^i}{1 + x^i}} \to 0$$

if $x \to \infty$. Thus, we apply Theorem 2 and Lemma 3 while taking $x \to \infty$ to obtain

$$PM(N(n)) \to 1 - \sum_{j \leq n} 1_{\{C_j > 0\}} + \left( \sum_{i \leq n} 1_{\{C_i > 0\}} - 1 \right)$$

$$= 0,$$

which verifies condition (6) for selections.

5. **Using Pivot Mass to Provide Couplings**

In this section, we combine our results on pivot mass with ideas from the theory of transportation polytopes and topology to prove Theorem 1.
5.1. **Transportation Polytopes.**

Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be positive real vectors. Denote by $\mathcal{N}(R, S)$ the class of all nonnegative matrices with row sum vector $R$ and column sum vector $S$. The space $\mathcal{N}(R, S)$ is a convex polytope – a bounded subspace of $\mathbb{R}^{mn}$ resulting from the intersection of a finite number of closed half-spaces. It can be shown that

\[(10) \quad \mathcal{N}(R, S) \neq \emptyset \iff \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.\]

The space $\mathcal{N}(R, S)$ is called a **transportation polytope** since the elements of $\mathcal{N}(R, S)$ correspond to solutions of a transportation problem in which material from $m$ distinct sources is transported to one of $n$ destinations, where the supply at the $i$th source, $1 \leq i \leq m$, is $r_i$ and the demand at the $j$th destination, $1 \leq j \leq n$, is $s_j$.

Given an $m \times n$ matrix $A = [a_{i,j}]$, any sub-matrix of $A$ is specified by choosing a subset of the row index set $\{1, 2, \ldots, m\}$ and a subset of the column index set $\{1, 2, \ldots, n\}$ of $A$. Given $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$, define $A[I, J] := [a_{i,j} : i \in I, j \in J]$, where $J^c$ denotes the complement of the set $J$ relative to the set $[n]$. The following theorem appears in §8.1 of [1].

**Theorem 4.** Let $(r_1, r_2, \ldots, r_m)$ and $(s_1, s_2, \ldots, s_n)$ be nonnegative vectors with $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Let $W$ be any $m \times n$ $(0,1)$-matrix. There exists a matrix in $\mathcal{N}(R, S)$ with pattern equal to $W$ if and only if the following condition is satisfied:

For each $K$ with $\emptyset \subset K \subset \{1, 2, \ldots, m\}$ and each $L$ with $\emptyset \subset L \subset \{1, 2, \ldots, n\}$ such that $W[K, L] = O$,

\[(11) \quad \sum_{l \in L} s_l \geq \sum_{k \in K} r_k,
\]

with equality if and only if $W(K, L) = 0$.

Since the range of $N(n)$ is of size $k_n$, we replace $n$ in Theorem 4 with $k_n$. Define an $\infty \times k_n$ matrix $W = [w_{i,j}]$ by

\[w_{i,j} := \begin{cases} 0 & (i,j) \text{ is a pivot,} \\ 1 & \text{otherwise.} \end{cases}\]

Each time we apply Theorem 4, we will apply it to a finite sub-matrix of $W$ (specifically, the sub-matrices will have finitely many rows and $k_n$ many columns).

5.2. **From $\mathbb{R}^{m \times k_n}$ to $\mathbb{R}^\infty$: Topological Considerations.**

Condition [3] and Theorem 4 allow us to apply some results from topology in order to prove the existence of couplings of $M(n, x)$ and $N(n)$. The following Theorem 5 appears on page 65 of [3].

**Theorem 5.** Suppose that $Z$ is a nonempty complete metric space and $Y_\mu$ is a sequence of closed nested subsets of $Z$ whose diameters tend to 0, where $\text{diam}(Y_\mu) := \sup \{d(x,y) : x, y \in Y_\mu\}$. Then $\bigcap_{\mu=1}^{\infty} Y_\mu$ is a singleton.

Our goal is to apply Theorem 5 to the set $Z = \mathbb{R}^\infty := \prod_{i \in \mathbb{N}} \mathbb{R}$, which is a nonempty complete metric space with respect to uniform metric $D(x,y) := \sup \{d(x_i, y_i) : i \in \mathbb{N}\}$, where $d(a,b) := \min\{|a - b|, 1\}$ is the standard bounded metric on $\mathbb{R}$ (Theorem 43.5 of [3]). Our coupling matrix will be obtained via concatenation of finite matrices. Each of the matrices considered have $k_n$ many columns. By concatenation, we mean the that matrices

\[A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k_n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k_n} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,k_n} \\ \vdots & \ddots & \vdots \\ b_{q,1} & \cdots & b_{q,k_n} \end{bmatrix}\]

---

\[7\]We thank Ethan Bolker for suggesting that the theory of transportation polytopes may provide methods for constructing couplings with desired properties.

\[8\]Theorem 5 is a variant of Cantor’s intersection theorem. We thank Anthony Quas for suggesting the use of a nested intersection theorem [3].
will be combined into a new finite matrix

\[
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
a_{1,1} & \cdots & a_{1,k_n} \\
\vdots & & \vdots \\
a_{m,1} & \cdots & a_{m,k_n} \\
 b_{1,1} & \cdots & b_{1,k_n} \\
\vdots & & \vdots \\
b_{q,1} & \cdots & b_{q,k_n}
\end{bmatrix}.
\]

Since a matrix with \(m\) rows and \(k_n\) columns corresponds to a point in \(\mathbb{R}^{m \times k_n}\), we may take \(\mathbb{R}^\omega\) as the underlying space. E.g., if \(m = 1\), then our matrix has \(k_n\) entries, say \([a_{1,1}, \ldots, a_{1,k_n}]\), which corresponds to an infinite sequence in \(\mathbb{R}^\omega\) whose first \(k_n\) entries are given by this matrix, followed by countably many zeros. If \(m = 2\), then our matrix has \(2 \times k_n\) many entries, say

\[
\begin{bmatrix}
a_{1,1} & \cdots & a_{1,k_n} \\
a_{2,1} & \cdots & a_{2,k_n}
\end{bmatrix},
\]

and we can identify this matrix with the element \((a_{1,1}, \ldots, a_{1,k_n}, a_{2,1}, \ldots, a_{2,k_n}, 0, 0, \ldots) \in \mathbb{R}^\omega\).

The next step is to construct an infinite family of sets \((Y_\mu)_\mu\) which satisfy the hypotheses of Theorem 5. Each such \(Y_\mu\) comes from a transportation polytope in \(\mathbb{R}^{m \times k_n} \subset \mathbb{R}^\omega\), for some \(m \in \mathbb{N}\). Condition (6) and Theorem 4 will allow us to prove this polytope contains an element in which the specified pivot positions are 0. The singleton \(\bigcap_\mu Y_\mu\) will correspond to a desired coupling of \(M(n, x)\) and \(N(n)\).

Define \(r_i := \mathbb{P}(M(n, x) = i)\) and \(s_j := \frac{1}{k_n}\) for all \(i \in \mathbb{N}\) and \(1 \leq j \leq k_n\). Note that \(\sum_{i \in \mathbb{N}} r_i = \sum_{j=1}^{k_n} s_j = 1\) (so the analogue of the right-hand side of (10) holds when \(m = \infty\)) and \(r_i, s_j > 0\). The latter inequalities are a requirement of elements of \(\mathcal{N}(R, S)\) in the case \(m < \infty\). It is worth pointing out that, in the case \(m = \infty\), inequality (11) becomes

\[
P_M(L(n)) \leq 1 - \frac{\#L(n)}{k_n}.
\]

Based on Theorem 4, if we hope to provide a coupling with the same pattern as \(W\), one may expect (12) to hold. This is indeed the case; if \(\#L = k_n\), then \(P_M(L(n)) = 0\) by Theorem 3. Hence both sides of (12) are 0 in this case. Otherwise, if \(\#L(n) < k_n\), then the right hand side is positive and condition (6) implies that it is possible to construct variables \(M\) such that inequality (12) holds. However, we do not deduce that (12) proves the existence of desired couplings when \(m = \infty\) since we can only apply Theorem 4 in the finite case. As we will soon see, we can consider inequalities similar to (12), using the notion of pivot mass to restate inequality (11) for finite truncated sub-matrices of our desired coupling (and then we apply (6) to verify inequality (11) for our sub-matrices). It remains to show that space of \(\infty \times k_n\) matrices with prescribed row sum vector \((r_1, r_2, \ldots)\) and column sum vector \((s_1, \ldots, s_{k_n})\), and same pattern as \(W\) is nonempty.

The remaining part of this section is devoted to defining sets \(Y_\mu\) for which Theorem 5 applies and \(\bigcap_{\mu=1}^\infty Y_\mu\) corresponds to a desired coupling.

5.2.1. Concatenation of Finite Matrices. Choose \(\mu \in \mathbb{N}\) large enough to ensure that for each \(j \in L(n)\), there exists a row label \(i \leq \mu\) such that column \(j\) does not have a pivot in row \(i\). The reason for placing this constraint on \(\mu\) is that we want to ensure that we are including enough rows in our starting matrix is to ensure that we do not have a column that has pivots in each of the first \(\mu\) rows – such a column would necessarily have column sum 0, violating the key assumption that the set \(S\) for \(\mathcal{N}(R, S)\) consists of positive numbers. There are infinitely many sufficiently large row labels \(\mu\) since, for example, we can take \(\mu_1, l \in \mathbb{N}\), to be the label corresponding to row \((Z_l(n, x))_{i \leq n} = (nl, nl, \ldots, nl)\) – we will have \(\sum_{i \leq n} (C_i(n) - nl)^+ = 0\) since \(C_i(n) \leq n\). Define

\[
\varepsilon_\mu := \sum_{i=\mu+1}^\infty r_i.
\]
so that $\varepsilon_\mu \downarrow 0$ as $\mu \to \infty$. Define $R_\mu := \{r_1, \ldots, r_\mu\}$ and $S_\mu := \{(1 - \varepsilon_\mu) s_1, \ldots, (1 - \varepsilon_\mu) s_{k_n}\}$. Then $X_\mu := \mathcal{N}(R_\mu, S_\mu)$ is a nonempty collection of $\mu \times k_n$ matrices by (10) since
\[
\sum_{r \in R_\mu} r = \sum_{i=1}^\mu r_i = (1 - \varepsilon_\mu) \sum_{i=1}^{k_n} s_i = \sum_{s \in S_\mu} s.
\]

Given $L(n)$ and a collection $I$ of row labels, define the truncated pivot mass by
\[
\mathcal{P}(L(n); I) := \sum_{i \in I, (i,j) \in P} \mathbb{P}(M(n,x) = i),
\]

Using the truncated pivot mass, inequality (11) becomes
\[
(1 - \varepsilon_\mu) \left(1 - \frac{\# L(n)}{k_n}\right) \geq \mathcal{PM}(L(n); [\mu]),
\]

which we now verify. In the first case, $1 - \frac{\# L(n)}{k_n} = 0$, we have $e_n \in L(n)$. By Theorem 2, this implies $\mathcal{PM}(L(n); [\mu]) = 0$. In the second case, $1 - \frac{\# L(n)}{k_n} > 0$. Let $j \in L(n)$ and $0 < \varepsilon < (1 - \varepsilon_\mu) \left(1 - \frac{\# L(n)}{k_n}\right)$. By (6) it is possible to construct $M(n,x)$ such that $\mathcal{PM}(j) < \varepsilon$. Therefore,
\[
\mathcal{PM}(L(n); [\mu]) \leq \mathcal{PM}(L(n)) \leq \mathcal{PM}(j) < \varepsilon.
\]

Theorem 4 implies the existence of a matrix $x_\mu \in X_\mu$ in for which $x_\mu$ has row vector $R_\mu$, column vector $S_\mu$ and such that each pivot location has zero mass – $x_\mu$ is our starting point for the concatenation process.

**Figure 3.** The matrix $x_\mu$ is our first approximation to a desired coupling. If $(i,j) \in P$, then $x_\mu$ has a 0 in the $(i,j)$ entry.

\[
\begin{array}{ccccccc}
M(n,x) & N(n) & 1 & \cdots & \cdots & j & \cdots & k_n & \text{Row sum} \\
1 & & & & & & r_1 \\
\vdots & & & & & & \vdots \\
i & & & & & & 0 \\
\vdots & & & & & & \vdots \\
\mu & & & & & & r_\mu \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Column sum} & & & \\
1 - \varepsilon_\mu & \cdots & \cdots & 1 - \varepsilon_\mu \\
\frac{k_n}{x_n} & \cdots & \frac{k_n}{x_n} & \frac{k_n}{x_n} \\
\end{array}
\]

Note that $x_\mu \in \mathbb{R}^{\mu \times k_n}$, and $\mathbb{R}^{\mu \times k_n}$ can be viewed as a subset of the infinite Cartesian product $\mathbb{R}^\omega$. Extend $x_\mu$ to $Y_\mu$, the set of $\infty \times k_n$ matrices whose first $\mu$ rows are identical to the rows in $x_\mu$ and the remaining row $i$ sums are $r_i = \mathbb{P}(M(n,x) = i)$, for $i > \mu$. Such a set is nonempty since, for example, for rows $i \geq \mu + 1$, we can set the $(i,j)$ entry to $\frac{1}{k_n}$.
5.2.2. A Nested Family \( \{ Y_\mu \}_\mu \). Choose any row label \( \mu' > \mu \) for which there are no pivots in row \( \mu' \). We recursively construct \( x_{\mu'} \) and \( Y_{\mu'} \) as follows. For \( X_{\mu'} \), we have \( R_{\mu'} = (r_1, \ldots, r_{\mu'}) \) and \( S_{\mu'} = \left( \frac{1-\varepsilon_{\mu'}}{k_n}, \ldots, \frac{1-\varepsilon_{\mu'}}{k_n} \right) \).

Regarding the sub-matrix consisting of the first \( \mu \) rows, we can use our element \( x_{\mu} \) to ensure that the row sums are \( r_i \) for \( i \leq \mu \), the column sums along the first \( \mu \) rows are each \( \frac{1-\varepsilon_{\mu'}}{k_n} \), and each pivot has mass 0. The remaining \( (\mu' - \mu) \times k_n \) sub-matrix \( \gamma \), corresponding to rows \( i \in \{ \mu+1, \ldots, \mu' \} \), must have row \( i \) mass \( r_i \) and each column sums to

\[
\frac{1-\varepsilon_{\mu'}}{k_n} - \frac{1-\varepsilon_{\mu'}}{k_n} = \frac{\varepsilon_{\mu} - \varepsilon_{\mu'}}{k_n} = \sum_{i=\mu+1}^{\mu'} r_i.
\]

This equation shows that \( \sum_{r \in R_{\mu'} \setminus R_{\mu'}} r = \sum_{s \in S_{\mu'}} s \), so \( \gamma \) exists by (10). Further, if we want to ensure the existence of such an \( (\mu' - \mu) \times k_n \) sub-matrix such that the pivot positions have 0 mass, we apply Theorem 4. Inequality (11) becomes

\[
P(\mathcal{L}(n); \{ \mu+1, \ldots, \mu' \}) \leq \left( \sum_{i=\mu+1}^{\mu'} r_i \right) \left( 1 - \frac{\# \mathcal{L}(n)}{k_n} \right).
\]

This inequality is true. By a previous argument, when the right hand side is 0, so is the left hand side. Otherwise, the right hand side is positive, and the left hand side can be made arbitrarily small by using (10).

Therefore, such a \( (\mu' - \mu) \times k_n \) sub-matrix \( \gamma = \gamma_{\mu' - \mu} \) exists, and so does our second finite matrix \( x_{\mu'} \) whose first \( \mu \) rows are given by \( x_{\mu} \) and rows \( \mu+1, \ldots, \mu' \) are given by \( \gamma_{\mu' - \mu} \). I.e.,

\[
x_{\mu'} = \begin{bmatrix} x_{\mu} \\ \gamma_{\mu' - \mu} \end{bmatrix}.
\]

We see that \( x_{\mu'} \) is a \( \mu' \times k_n \) matrix with row \( i \) sum \( r_i \), for \( 1 \leq i \leq \mu' \), each column sum is \( \frac{1-\varepsilon_{\mu'}}{k_n} \), and all pivots have zero mass. As before, we extend \( x_{\mu'} \) to \( Y_{\mu'} \) — the set of all \( \infty \times k_n \) matrices whose first \( \mu' \) rows are \( x_{\mu'} \) and whose row \( i \) sums are \( r_i \) for \( i > \mu' \). Using a previous argument, it is possible to show \( Y_{\mu'} \) is nonempty. By construction, \( Y_{\mu'} \subset Y_\mu \). Continuing in this fashion gives a nested family \( \{ Y_\mu \}_\mu \).

5.2.3. \( \text{diam}(Y_\mu) \to 0 \). In our case, given \( x, y \in Y_\mu \), and viewing \( Y_\mu \) as a subset of \( \mathbb{R}^\omega \), we have

\[
D(x, y) = \sup \{ d(x_i, y_i) : i \in \mathbb{N} \} \leq r_{\mu+1}
\]

since the first row in which elements in the matrices \( x, y \in Y_\mu \) may differ is in row \( \mu + 1 \) (since the first \( \mu \) rows of \( x \), and \( y \) are determined by \( x_\mu \)). Therefore, \( \text{diam}(Y_\mu) \to 0 \) as \( \mu \to \infty \).

5.2.4. The Sets \( Y_\mu \) are Closed with respect to the Uniform Topology on \( \mathbb{R}^\omega \). Now we show that each of the sets \( Y_\mu \) are closed in \( \mathbb{R}^\omega \) with respect to the uniform metric \( D \). Suppose \( \{ y_\alpha \}_\alpha \) is a sequence of elements in \( Y_\mu \) that converges to some vector \( z \in \mathbb{R}^\omega \). For the purpose of arriving at a contradiction, suppose \( z \notin Y_\mu \).

The first \( \mu \) rows of each matrix in \( Y_\mu \) are given by \( x_\mu \). Therefore, there exists an \( i_0 \geq \mu + 1 \) for which the row \( i_0 \) sum of \( z \) is not \( r_{i_0} \). WLOG, suppose the row \( i_0 \) sum is \( r_{i_0} + \varepsilon \). This is absurd since there exists \( \alpha \) such that \( D(y_\alpha, z) < \frac{\varepsilon}{2k_n} \). As a result, the row \( i_0 \) sum of \( z \) and the row \( i_0 \) sum of \( y_\alpha \) differ by at most \( k_n \cdot \frac{\varepsilon}{2k_n} = \varepsilon/2 \). This contradiction proves that \( z \in Y_\mu \), so \( Y_\mu \) is closed.

Therefore, we have verified all of the conditions needed to apply Theorem 5. The singleton \( \bigcap_\mu Y_\mu \) corresponds to a desired coupling since row \( i \) sums to \( r_i \) for all \( i \in \mathbb{N} \), column \( j \) sums to \( \lim_{\mu \to \infty} (1-\varepsilon_\mu) \frac{1}{k_n} = \frac{1}{k_n} \), for all \( j \in [k_n] \), and each pivot has zero mass. Hence, the proof of Theorem 1 is complete.

Acknowledgment

I would like to thank my PhD advisors Michael Cranston and Nathan Kaplan for their feedback and support.
References

[1] Arratia, R. On the amount of dependence in the prime factorization of a uniform random integer. In B. Bollobás, editor, Contemporary Combinatorics, pages 29–91. Bolyai Society Mathematical Studies, Volume 10, 2002.

[2] Arratia, R., Barbour, A.D., and Tavare, S. Logarithmic Combinatorial Structures: A Probabilistic Approach. EMS Monographs in Mathematics. European Mathematical Society (EMS), 2003.

[3] Brualdi, R. Combinatorial Matrix Classes. Encyclopedia of Mathematics and its Applications, 108. Cambridge University Press, Cambridge, 2006.

[4] Fulkerson, D.R. Hitchcock transportation problem, RAND Corporation Report, p-890, 1956.

[5] Kolmogorov, A.N. & Fomin, S.V. Introductory Real Analysis, Translated & Edited by Richard A. Silverman. Dover Publications, Inc. New York, 1970.

[6] Munkres, J.R. Topology. Prentice Hall, Upper Saddle River, 2000.

[7] Quas, A. Reference Request for Couplings with Conditions, URL (version: 2017-10-27): https://mathoverflow.net/q/284525.