Review of SU(2)-Calibrations

André Miemiec

Institut für Physik
Humboldt Universität
D-12489 Berlin, Germany
Newtonstr. 15
miemiec@physik.hu-berlin.de

Abstract
The purpose of this article is to provide a review of SU(2)-calibrations. The focus is on developing all techniques in full detail by studying selected examples. The supergravity point of view and the string theoretic one are explained.
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1 Introduction

In this notes we would like to give a self contained introduction into the subject of calibrated geometries [1] while setting the focus on the important example of $SU(2)$-calibrated geometries. Calibrated geometries arise in supergravity and string theory quite naturally in the investigation of classical solutions of the equations of motion (eom), which preserve some amount of supersymmetry $^\ast_1$ (Killing spinors) [2, 3, 4]. Typical examples are provided by the widely used brane solutions [5, 6, 7, 8]. Nowadays the application of calibrated geometries is one of the main computational tools in supergravity and string theory and the objective of current research. Therefore an understanding of the basic concepts is very useful [9, 10, 11]. In recent years the attempts to provide a full classification of supersymmetric solutions of all supergravities of physical interest have benefited a lot from the progress in understanding (generalised) calibrations [12, 13, 14, 15, 16]. Further work in this direction may be performed and belongs certainly to the most inspiring developments in supergravity and string theory nowadays. To set the scene we want to motivate the origin of calibrated geometries by giving a simple example. For this purpose we consider branes, which prove to be generic supersymmetric solutions of supergravities and are most important in applications. One can also look at them from a different point of view. Rather then considering them as solutions of a given supergravity one might regard them as dynamical objects moving in a fixed supergravity background, too. The action from which the dynamics of the brane in the background follows is called p-brane action [17, 18, 19].

Example 1. In a flat background and in the absence of other gauge fields the p-brane action governing the dynamics is simply the volume functional of the brane embedded into flat D-dimensional spacetime. The solutions of the corresponding equations of motion are minimal surfaces. If $f : M^{1,p} \rightarrow \mathbb{R}^{1,D-1}$ is the embedding map and for simplicity we assume it describes effectively the embedding of a 2-manifold $\mathcal{C}$ into $\mathbb{R}^4$, i.e.

$\mathbb{R}^{1,p-2} \times \mathcal{C} \rightarrow \mathbb{R}^{1,p-2} \times \mathbb{R}^4 \times \mathbb{R}^{D-p-3}$,

$(x^0, x^1, \ldots, x^{p-2}, \xi^1, \xi^2) \rightarrow (x^0, x^1, \ldots, x^{p-2}, \xi^1, \xi^2, X^1, X^2, \ldots, x^{D-1})$

with $X^1 = X^1(\xi^1, \xi^2)$ and $X^2 = X^2(\xi^1, \xi^2)$. We can consider $\mathbb{R}^4$ as the complex space $\mathbb{C}^2$ by introducing the following complex coordinates

$^\ast_1$For the case of 11d supergravity some important examples of brane solutions are discussed in [35].
\[
\begin{align*}
z^1 &= \xi^1 + iX^1 \\
z^2 &= \xi^2 + iX^2.
\end{align*}
\]

\(C^2\) is a Kähler manifold with Kähler form
\[
\omega = \frac{i}{2} \left( dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 \right). \tag{1}
\]

In addition one can define a second closed form via
\[
\Omega = dz^1 \wedge dz^2. \tag{2}
\]

The induced metric on \(C\) is computed in the standard way by computing all possible scalar products of the tangent vectors \(\vec{r}_1\) and \(\vec{r}_2\) at a point \((\xi_1, \xi_2)\) and we obtain:
\[
(f^*g)_{ij} = 
\begin{pmatrix}
1 + (\partial_1 X_1)^2 + (\partial_2 X_2)^2 & \partial_1 X_1 \partial_2 X_1 + \partial_1 X_2 \partial_2 X_2 \\
\partial_1 X_1 \partial_2 X_1 + \partial_1 X_2 \partial_2 X_2 & 1 + (\partial_2 X_1)^2 + (\partial_2 X_2)^2
\end{pmatrix}, \tag{3}
\]

The volume functional expressed in terms of the determinant of the induced metric \((f^*g)_{ij}\) on \(C\) reads (see appendix A)
\[
S = \int_{\mathbb{R}^{1,p-2} \times C^2} dp^{p+1} x \sqrt{|g|} = \text{Vol}(\mathbb{R}^{1,p-2}) \cdot \int_{\mathbb{C}^2} d^2 x \sqrt{|f^*g|}
= \text{Vol}(\mathbb{R}^{1,p-2}) \cdot \int_{\mathbb{C}^2} d^2 x \sqrt{|f^*\omega|^2 + |f^*\Re\Omega|^2 + |f^*\Im\Omega|^2}
\geq \text{Vol}(\mathbb{R}^{1,p-2}) \cdot \int_{\mathbb{C}^2} f^*\omega.
\]

The volume is bounded from below. If the bound is saturated, i.e. the equality sign holds, one obtains a peculiar geometry given by the equation
\[
f^*\omega = \sqrt{|g|} d\xi^1 \wedge d\xi^2, \tag{4}
\]
i.e. the pullback of the Kähler form \(\omega\) is the volume form on \(C\). This statement is a local one. According to Wirtingers theorem this is a property characteristic of complex submanifolds of \(\mathbb{C}^2\). So the surfaces calibrated with respect to the Kähler form \(\omega\) are just the complex submanifolds of \(\mathbb{C}^2\). Similar bounds but formally different geometries can be established by utilising one of the other three squares in \(|f^*g|^2\). This seems to be ambiguous. A complete resolution of this ambiguity will be given in this review.
Definition 1. (Preliminary) : A calibration consists of a closed differential form $\omega$, which provides a lower bound for the volume of a brane solution such that for all submanifolds $C$

$$\text{devol}_C \geq f^* \omega \bigg|_{TC},$$

when evaluated on any tangent plane. If a submanifold (brane) saturates the bound we call it calibrated. The saturation of the bound can be translated into constraints on the geometry of the submanifold (brane).

In the next sections we want to develop an understanding for the interplay of closed differential forms defining a calibration and geometrical constraints. In order to be as explicit as possible we limit ourself to a detailed study of the simplest but nevertheless nontrivial example of $SU(2)$-calibrated geometry. The reason for calling it a $SU(2)$-calibrated geometry will become clear later. The advantage of $SU(2)$-calibrations is, that everything can be computed easily and so most insight into the subject is possible. It can be seen as a role model for more sophisticated constructions of calibrated geometries [1]. There exists other reviews, which might be consulted for further reading [21, 22]. Here we choose the following strategy. As was pointed out in example [1] the study of the local geometry is enough in order to derive the constraints (differential equations) governing the geometry of the manifold. Therefore our plan is to start with a formal discussion of the local geometry of $SU(2)$-calibrated submanifolds by studying the geometry of two planes in $\mathbb{R}^4$ in section 2 first [23, 24, 25]. Having achieved a complete understanding of the geometries we want to investigate the whole topic from the perspective often taken in string theory in section 3 [26, 27, 28, 29, 30, 31]. Here we rederive the $SU(2)$-conditions from the string picture in the case of an D2-D4 bound state [32]. Finally we generalise the recipe to generate the differential equations to the case of $SU(d)$-calibrations in section 4 [33, 34].

2 Grassmanians of 2-planes

2.1 Grassmanian $G(2, 4)$

The Grassmanian of 2-planes in $\mathbb{R}^4$ is defined through:

$$G(2, 4) = \{ \xi \wedge \eta | \xi, \eta \in \Omega^1(\mathbb{R}^4) \} \subset \Lambda^2 \mathbb{R}^4,$$

i.e. as a subset of the six dimensional space $\Lambda^2 \mathbb{R}^4$. Its elements are 2-planes generated by two vectors dual to the two 1-forms $\xi$ and $\eta$.

We would like to study the structure of the manifold $G(2, 4)$. To that purpose we consider the action of the Hodge star operator on $\Lambda^2 \mathbb{R}^4$. The action of the Hodge star is simply defined by

$$* (dx^i \wedge dx^j) = \text{sgn} (i j k l) dx^k \wedge dx^l.$$
Since $*^2 = \mathbb{1}$ there is a split of the six dimensional space

$$\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$$

into three dimensional eigenspaces of $*$. The split is realized via the projections $P_{\pm} = \frac{1}{2} (1 \pm *)$.

Now we investigate the decomposition of $G(2,4)$ according to this split. Every element $x$ of $G(2,4)$ is of the form $x = \xi \wedge \eta$, where the vectors $\xi$ and $\eta$ can be chosen to be orthonormal. The following statement is true:

**Proposition 1.**

$$x \in G(2,4), \text{ iff } x \wedge x = 0.$$ 

**Proof.** Without proof. \(\square\)

**Proposition 2.**

$$G(2,4) = S^2_+ \times S^2_-$$

**Proof.** Each $x \in G(2,4)$ can be decomposed according to

$$x = P_+ x + P_- x$$

and we will see that this splitting defines a map of $G(2,4)$ into $S^2_+ \times S^2_-$. To that purpose we compute the norm of the three vector obtained via the projection of $x$ into the “$+$”-eigenspace of $*$. The following equation holds$^3$:

$$\|P_+ x\|^2 \text{ vol} = P_+ x \wedge * P_+ x = P_+ x \wedge P_+ x$$

$$= \frac{1}{2} (x + * x) \wedge \frac{1}{2} (x + * x)$$

$$= \frac{1}{2} x \wedge * x = \frac{1}{2} \|x\|^2 \text{ vol}.$$ 

$^3$This is due to the definition of the scalar product of two differential $p$-forms $\omega \wedge \eta = \langle \omega, \eta \rangle \text{ vol}$. Here vol denotes the volume form.
Setting $\|x\|^2 = 1$ merely reflects the fact that the area spanned by normed vectors $\xi$ and $\eta$ is equal to one. We obtain the result that the three “vector” $P_+ x$ is of norm $1/2$.

The general element of $P_+ G(2, 4)$ can be written as the three “vector”

$$P_+ x = a_{12} P_+ (dx^1 \wedge dx^2) + a_{13} P_+ (dx^1 \wedge dx^3) + a_{23} P_+ (dx^2 \wedge dx^3).$$

Using the following abbreviation $(a_{12}, a_{13}, a_{23}) \mapsto (a_1, a_2, a_3) = \vec{a}$ the following identity holds:

$$< P_+ x, P_+ y > \text{ vol} = \frac{1}{2} < \vec{a}, \vec{b} > \text{ vol.}$$

Comparing the previous result of the norm of $P_+ x$ with the latter identity one obtains

$$\|P_+ x\|^2 \text{ vol} = \frac{1}{2} \|\vec{a}\|^2 \text{ vol} = \frac{1}{2} \text{ vol},$$

i.e. the vector $\vec{a}$ associated with $P_+ x$ is of unit norm. A completely analogous result holds for the “−” eigenspace:

$$< P_- x, P_- y > \text{ vol} = \frac{1}{2} < \vec{a}, \vec{b} > \text{ vol}$$

$$\|P_- x\|^2 \text{ vol} = \frac{1}{2} \|\vec{a}\|^2 \text{ vol} = \frac{1}{2} \|x\|^2 \text{ vol.}$$

Basically the vector $\vec{a}$ associated with $P_- x$ is again of unit norm. In addition for two 2-forms belonging to the “+” and the “−” eigenspaces, respectively, the formula

$$\omega_+ \wedge \omega_- = 0$$

holds, i.e. the decomposition is an orthogonal one. We obtain the announced map:

$$\phi : G(2, 4) \rightarrow S_+^2 \times S_-^2,$$

$$x \mapsto P_+ x + P_- x$$

**Lemma 1.** The map $\phi : G(2, 4) \rightarrow S_+^2 \times S_-^2$ is bijective.
Proof.

Surjectivity: Choose \((a, b) \in S^2_+ \times S^2_\) and define a section \(s\) of \(G(2, 4)\) by

\[
s : S^2_+ \times S^2_- \longrightarrow G(2, 4) \\
(a, b) \mapsto x = \omega_a + \omega_b
\]

Then compute

\[
x \wedge x = \omega_a \wedge \omega_a + 2 \omega_a \wedge \omega_b + \omega_b \wedge \omega_b = 0,
\]

i.e. \(x \in G(2, 4)\), indeed.

Injectivity: Choose two \(x, x' \in G(2, 4)\) having the same images in \(S^2_+ \times S^2_-\). Each can be written as \(x = P_+ x + P_- x\) and \(x' = P_+ x' + P_- x'\), respectively. Since the images coincide, i.e.

\[
P_+(x) = P_+(x') \\
P_-(x) = P_-(x')
\]

or more detailed

\[
x - x' = - \ast (x - x') \\
x - x' = + \ast (x - x'),
\]

one concludes \(x = x'\) due to the orthogonality of the split, i.e. \(\Lambda^2_+ \mathbb{R}^4 \cap \Lambda^2_- \mathbb{R}^4 = \emptyset\).

Having understood the structure of the manifold \(G(2, 4)\), i.e. the parameter space of two planes in \(\mathbb{R}^4\), from the decomposition into two spheres sitting in the spaces of self- and antiselfdual two forms, respectively, we want now discuss the geometries arising from restricting the parameter space of two planes to submanifolds of \(G(2, 4)\).

2.2 Grassmanian of complex planes \(C(\mathbb{R}^4)\)

A complex structure is given by some \(J : \mathbb{R}^4 \rightarrow \mathbb{R}^4\) with \(J^2 = -\mathbb{1}\).

Each point \(x_J \in S^2_+\) defines trough

\[
< J \xi, \eta > = < x_J, \xi \wedge \eta >
\]
a complex structure and we can construct the map $J$ out of $x_J$:

$$x_J = \sum_{i<j} a_{ij} P_+ (dx^i \wedge dx^j)$$

$$< x_J, \xi \wedge \eta > \cdot \text{vol} = *x_J \wedge \xi \wedge \eta$$

Then it is a simple exercise to compute $J$ in terms of $x_J$ which is given by

$$J = \begin{pmatrix}
0 & -a_{12} & -a_{13} & -a_{23} \\
-a_{12} & 0 & -a_{23} & a_{13} \\
a_{13} & a_{23} & 0 & -a_{12} \\
a_{23} & -a_{13} & a_{12} & 0
\end{pmatrix}$$

and in fact squares to the $-1$. Fixing a complex structure is equivalent to considering the subset of complex 2-planes in $G(2,4)$ with respect to the complex structure chosen before. The corresponding submanifold is simply

$$\mathcal{C}(\mathbb{R}^4) = \text{point} \times S^2_\perp.$$ 

This is the situation we found in example 1.

**Remark 1.** In principle one could have played the game the other way around, i.e. fixing a point in $S^2_\perp$ instead. The difference is a change in the orientation. Since we usually stick to the orientation we have started with this possibility has to be excluded.

### 2.3 Grassmanian of Lagrangian planes $\mathcal{L}(\mathbb{R}^4)$

There exists other geometric structures. A very popular one and closely related to Kähler forms are symplectic geometries. In flat space this geometry is again defined via the Kähler form. We define the Grassmanian of Lagrangian 2-planes as the set on which the Kähler form restricts to zero, i.e.

$$\omega_J(\xi, \eta) = < J\xi, \eta > = < x_J, \xi \wedge \eta > = 0.$$ \hspace{1cm} (6)

Therefore Lagrangian planes are somehow opposite to complex planes.
Choose $\xi \land \eta = P_+ (\xi \land \eta) + P_- (\xi \land \eta) = \omega_+ + \omega_-$. 

$$\omega_J (\xi, \eta) = \langle x_J , \xi \land \eta \rangle$$
$$= \langle x_J , \omega_+ + \omega_- \rangle$$
$$= \langle x_J , \omega_+ \rangle$$
$$= \frac{1}{2} \langle \vec{a} , \vec{b} \rangle \equiv 0 \text{ Hessische Normalform !}$$

with $\vec{a}$ a representative for $x_J$ and $\vec{b}$ a representative of $\omega_+$. Therefore all planes parametrised by $S^2_+ \subset S^2_+$ orthogonal to $\vec{a}$, i.e. the complex structure, satisfy the defining constraint (6).

$$\mathcal{L}(\mathbb{R}^4) = S^1 \times S^2_-$$

The $S^1$ part is the intersection of $S^2_+ \subset S^2_+$ with the plane orthogonal to $x_J \in S^2_+$. 

Taking into account that $\mathcal{L}(\mathbb{R}^4) = U(2)/SO(2)$ as a homogeneous space one can reinterpret the topology of $\mathcal{L}(\mathbb{R}^4)$ in terms of $U(2)$ as indicated below:

\[
\begin{array}{ccc}
U(2) & \downarrow & \leftarrow \\
\uparrow & & \leftarrow \\
U(1) \times SU(2) & \leftarrow & \text{Hopf} \\
\downarrow & & \\
S^1_+ & \leftarrow & S^2_-
\end{array}
\]

Let us concentrate on the interpretation of $S^1_+$ as the phase of $U(2)$, i.e. we want to show, that fixing the point in $S^1_+$ corresponds to singling out all elements in $U(2)$ with the same phase. 

To that purpose we choose a special complex structure, say:

$$x_J = (1, 0, 0)$$

which is

$$J = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.$$  

This choice of $x_J$ corresponds to the Kähler form $\omega$ in \ref{2}. Using $J$ a real vector can be made
into a complex one by the map

\[
(a_1, a_2, a_3, a_4)^T \mapsto \begin{pmatrix} a_1 + i \cdot a_2 \\ a_3 + i \cdot a_4 \end{pmatrix}.
\]

(7)

Each element of \(U(2)\) can be represented by a \(2 \times 2\)-matrix with the common additional restrictions.

\[
U = \begin{pmatrix} a_1 + i \cdot a_2 & b_1 + i \cdot b_2 \\ a_3 + i \cdot a_4 & b_3 + i \cdot b_4 \end{pmatrix}
\]

The projection of the corresponding plane \(\eta = a \wedge b\) into the “+”-eigenspace and parametrising the projection by the coordinates of \(S^1_+\) yields to

\[
P_+ (a \wedge b) = \begin{cases} 0 & \text{P}_+(dx^1 \wedge dx^2) \\ \cos \alpha & \text{P}_+(dx^1 \wedge dx^3) \\ \sin \alpha & \text{P}_+(dx^2 \wedge dx^3) \end{cases}
\]

In Fig. 1 we also show the sphere \(S^2_\pm\) spanned by the three directions \(\omega, \Re \Omega\) and \(\Im \Omega\), which follow from the differential forms defined in eq. (1) and eq. (2), respectively. But according to (7) the complex coordinates are now \(z^1 = x^1 + x^2\) and \(z^2 = x^3 + x^4\).

One can compute the determinant of \(U(2)\) which yields

\[
\det U = \begin{cases} \cos \alpha & \text{Im} \\Omega \\ \sin \alpha & \text{Re} \\Omega \end{cases} \cos \alpha + i \begin{cases} \cos \alpha & \text{Im} \\Omega \\ \sin \alpha & \text{Re} \\Omega \end{cases} \begin{cases} \cos \alpha & \text{Im} \\Omega \\ \sin \alpha & \text{Re} \\Omega \end{cases} = e^{i \alpha}.
\]

Fig. 1: G(2,4)
2.4 Grassmanian of Special Lagrangian planes $\mathcal{SL}(\mathbb{R}^4)$

Combining the results from subsection 2.2 and subsection 2.3 it becomes very simple to analyse the Grassmanian of Special Lagrangian planes. The Special Lagrangian planes are defined as those Lagrangian planes which share the same phase. But then obviously

$$\mathcal{SL}(\mathbb{R}^4) = \text{point} \times S^2$$

$$\downarrow$$

$$U(2) \quad \downarrow \quad U(1) \times SU(2)$$

Phase $\swarrow$ Hopf

$$S^1_+ \quad S^2_-$$

On the other side this point can be seen as defining a certain complex structure $\tilde{J}$, so that

$$\mathcal{SL}_J(\mathbb{R}^4) = \mathcal{C}_J(\mathbb{R}^4).$$

It is convenient to select the point with $\alpha = 0$, i.e. $\tilde{J} = \Re \Omega$ and the Special Lagrangian planes with respect to the pair $(\omega, \Omega)$ can be understood as complex planes with respect to the Kähler structure $\tilde{J} = \Re \Omega$.

The detailed investigation we finished here made it clear what is meant by the notion of a $SU(2)$-calibration. The group $SU(2) \subset SO(4)$ can be understood as arising from the reduction of structure group of a general Riemannian 2-manifold embedded in a 4-dimensional space in order to be consistent with the geometric structures we considered (complex or special Lagrangian). In the next section we would like to understand the origin of such a geometrical constraint from the presence of supersymmetry.

3 Linear Algebra of D4-D2 System with Flux

We consider again an example consisting of a system of a D2 and a D4-brane with additional magnetic flux on the D4-brane. The BPS-equations (8) and (9) are derived from the quantisation of open strings [26]. Equation (8) contains the modification of the BPS-equations of a single D4-brane due to presence of the magnetic B-field which is taken into account by acting with the rotation matrix $\tilde{R}$ asymmetrically on the right ($\varepsilon$) and left ($\tilde{\varepsilon}$) moving Killing spinors. Its angles $\tilde{\phi}_{ij} = \tan F_{ij}$ corresponds to the flux on the D4-brane. The second equation (9) describes a general rotation $R$ of the D2-brane inside the D4-brane**:

**Up to now this is not an eigenvalue equation, since it relates only $\varepsilon$ to $\tilde{\varepsilon}$.**
\[ \Gamma_{01234} \stackrel{\rightarrow}{R} \varepsilon = (\stackrel{\rightarrow}{R})^{-1} \varepsilon \quad (8) \]
\[ \Gamma_{012} \stackrel{\rightarrow}{R} \varepsilon = \stackrel{\rightarrow}{R} \varepsilon \quad (9) \]

Here the matrices of symmetric and asymmetric rotation are

\[ \stackrel{\rightarrow}{R} = e^{\varphi_{12} \Sigma_{12}} \cdots e^{\varphi_{34} \Sigma_{34}} \quad (10) \]
\[ \stackrel{\rightarrow}{\tilde{R}} = e^{\tilde{\varphi}_{12} \Sigma_{12}} \cdots e^{\tilde{\varphi}_{34} \Sigma_{34}} \quad (11) \]

with \( \Sigma_{ij} = \Gamma_{ij}/2 \) the generators of \( \text{Spin}(4) \). The arrow indicates the order of the exponentials. The ordering in the above expressions is due to ascending tuples of numbers \( (12) \leq (ij) \leq (34) \). The opposite ordering is denoted by

\[ \stackrel{\leftarrow}{R} = e^{\varphi_{34} \Sigma_{34}} \cdots e^{\varphi_{12} \Sigma_{12}} \]

and for instance the following relation holds:

\[ (\stackrel{\rightarrow}{R})^{-1} = \stackrel{\leftarrow}{R}^{-1} \]

### 3.1 Choosing a gauge

The most general flux is given by the antisymmetric tensor below:

\[ F = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix} \quad (12) \]

By choosing proper coordinates (for details see appendix B) it can be put into the form

\[ \tilde{F} = \begin{pmatrix} 0 & \tilde{F}_{12} & 0 & 0 \\ -\tilde{F}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{F}_{34} \\ 0 & 0 & -\tilde{F}_{34} & 0 \end{pmatrix} \quad (13) \]

Here
\[ F_{12} = \frac{1}{\sqrt{2}} \sqrt{\sum_{i<j} F_{ij}^2 + \delta} \]  
(14)

\[ F_{34} = \frac{1}{\sqrt{2}} \sqrt{\sum_{i<j} F_{ij}^2 - \delta} \]  
(15)

with

\[ \delta = \left[ \left( F_{13} + F_{24} \right)^2 + \left( F_{12} - F_{34} \right)^2 + \left( F_{14} - F_{23} \right)^2 \right] \cdot \left[ \left( F_{13} - F_{24} \right)^2 + \left( F_{12} + F_{34} \right)^2 + \left( F_{14} + F_{23} \right)^2 \right]. \]

The both factors in \( \delta \) are related to the selfdual and antiselfdual parts of \( F \).

Choosing this gauge (see appendix C) eq. (11) simplifies since the corresponding matrix is now

\[ \hat{R} = e^{\hat{\varphi}_{12}\Sigma_{12}} \cdot e^{\hat{\varphi}_{34}\Sigma_{34}}. \]  
(16)

### 3.1.1 The symmetric form of the BPS equations

For practical purposes it is convenient to write the rotation matrix \( \hat{R} \) in the form

\[ \hat{R} = e^{\hat{\varphi}_{12}\Sigma_{12}} \cdot e^{\hat{\varphi}_{13}\Sigma_{13}} \cdot \ldots \cdot e^{\hat{\varphi}_{24}\Sigma_{24}} \cdot e^{\hat{\varphi}_{34}\Sigma_{34}}, \]  
(17)

with \( \hat{R} \) containing all angles with the exception of \( \varphi_{12} \) and \( \varphi_{34} \). Now the BPS-equations (8) and (9) can be rewritten as:

\[ \hat{R} \Gamma_{01234} \hat{R} \varepsilon = \varepsilon \]
\[ (R)^{-1} \Gamma_{012} R \varepsilon = \varepsilon \]

Intertwining the rotation matrices with the other gamma matrices ends up with:

\[ \Gamma_{01234} \hat{R} \hat{R} \varepsilon = \varepsilon \]
\[ \Gamma_{012} e^{-\varphi_{34}\Sigma_{34}} R \hat{R} \varepsilon = \varepsilon \]

Comparing the two left hand sides of the last set of equations and skipping the redundant \( \Gamma_{012} \),
the expression reads:

\[ \Gamma_{34} \leftarrow R e^{-\varphi_{34} \Sigma_{34}} R \rightarrow (\tilde{R} \varepsilon) = (\tilde{R} \varepsilon) \]

Multiplying both sides with \( e^{\varphi_{34} \Sigma_{34}} \) and reordering the last expressions slightly yields

\[ \Gamma_{34} \leftarrow R \rightarrow (\varepsilon \Sigma_{34} \rightarrow \tilde{R} \varepsilon) = (\varepsilon \Sigma_{34} \tilde{R} \varepsilon) \]

Thus the problem is reduced to determine the +1 eigenvalues of

\[ \Gamma_{34} \leftarrow \tilde{R} \varepsilon = + \eta \quad (18) \]

3.1.2 The structure of \( Spin(4) \)

The point to note is that \( \tilde{R} \rightarrow \tilde{R} \rightarrow \tilde{R} \rightarrow \tilde{R} \) represents the general element of the group \( Spin(4) \). So we can use any representation of the general group element we want only guided by practical needs. Combining this freedom with the following standard properties of \( Spin(4) \) eq. (18) can be solved completely.

For this purpose we use the group isomorphism \( Spin(4) \equiv SU(2) \times SU(2) \). The splitting reflects the splitting of \( \Lambda^2(\mathbb{R}^4) \) with respect to the Hodge star operation and the isomorphism mentioned above is constructed by introducing the following selfdual and antiselfdual combinations of generators

\[
\begin{align*}
T_1 &= \frac{\Sigma_{12} + \Sigma_{34}}{2} & R_1 &= \frac{\Sigma_{12} - \Sigma_{34}}{2} \\
T_2 &= \frac{\Sigma_{13} - \Sigma_{24}}{2} & R_2 &= \frac{\Sigma_{13} + \Sigma_{24}}{2} \\
T_3 &= \frac{\Sigma_{23} + \Sigma_{14}}{2} & R_3 &= \frac{\Sigma_{23} - \Sigma_{14}}{2}
\end{align*}
\]

whose algebra is

\[
[T_i, T_j] = \epsilon_{ijk} T_k \quad [R_i, R_j] = \epsilon_{ijk} R_k \quad [T_i, R_j] = 0.
\]

Since in eq. (18) the generators \( \Sigma_{12} \) and \( \Sigma_{34} \) appear only in \( \tilde{R} \) and the other only in \( R \) we use the freedom to change the representation of the general element of \( Spin(4) \) by choosing an ordering in which generators combining under (anti)selfduality follow each other. Then the general element
of $\text{Spin}(4)$ reads

$$
\begin{align*}
\mathcal{R} \mathcal{R} & = e^{\alpha_3 T_3} e^{\alpha_2 T_2} e^{\alpha_1 T_1} \cdot e^{\beta_3 R_3} e^{\beta_2 R_2} e^{\beta_1 R_1} = U_T(\alpha) \cdot U_R(\beta) \\
\mathcal{R} \mathcal{R} & = e^{\beta_1 R_1} e^{\beta_2 R_2} e^{\beta_3 R_3} \cdot e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3} = U_R(\beta) \cdot U_T(\alpha)
\end{align*}
$$

with $\alpha$ or $\beta$ the selfdual or antiselfdual combinations of the $\varphi_{ij}$ and $\tilde{\varphi}_{ij}$ respectively**. Finally the product of the last two expressions becomes

$$
\begin{align*}
\mathcal{R} \mathcal{R} \mathcal{R} \mathcal{R} & = U_R(\beta) \cdot U_T(\alpha) \cdot U_T(\alpha) \cdot U_R(\beta) \\
& = \left( U_R(\beta) \cdot U_R(\beta) \right) \cdot \left( U_T(\alpha) \cdot U_T(\alpha) \right)
\end{align*}
$$

(19)

(20)

3.1.3 Transformation into standard form:

Here we note only some formulas which transform the representation of the Lie algebra in a form where the direct product is more obvious. The main purpose is to study the presentation of $\Gamma_{34}$ in this new basis, so that the problem of finding eigenvalues of eq. (18) can be reduced to the two independent $SU(2)$ factors.

$$
P = \left( \begin{array}{cccc}
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2}
\end{array} \right) \in SU(4)
$$

then

**$\alpha_1 = \tilde{\varphi}_{12} + \tilde{\varphi}_{34}, \alpha_2 = \varphi_{13} - \varphi_{24}, \alpha_3 = \varphi_{23} + \varphi_{14}$ and $\beta_1 = \tilde{\varphi}_{12} - \tilde{\varphi}_{34}, \ldots$
\[ \tilde{T}_1 = P_1^{-1} T_1 P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & -
frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \tilde{T}_2 = P_1^{-1} T_2 P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & -
frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \tilde{T}_3 = P_1^{-1} T_3 P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -
\frac{i}{2} & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \tilde{R}_1 = P_1^{-1} R_1 P_1 = \begin{pmatrix} \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -
\frac{i}{2} \end{pmatrix} \]

\[ \tilde{R}_2 = P_1^{-1} R_2 P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \end{pmatrix} \]

\[ \tilde{R}_3 = P_1^{-1} R_3 P_1 = \begin{pmatrix} 0 & 0 & 0 & -
\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -
\frac{i}{2} & 0 & 0 & 0 \end{pmatrix} \]

The matrix \( \Gamma_{34} \) is transformed into

\[ \tilde{\Gamma}_{34} = P^{-1} \Gamma_{34} P = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \] (21)

### 3.2 Final step: Solution

Now, by putting all steps together, we can discuss the existence of +1 eigenvalues for each of the two \( SU(2) \)s separately. With the standard form of the last section and the decomposition of eq. 19 equation 18 reads:

\[ U_{\hat{F}}(\alpha) \cdot U_{\hat{F}}(\alpha) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \] (22)

\[ U_{\hat{R}}(\beta) \cdot U_{\hat{R}}(\beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \] (23)
It is enough to study one of the both equations. The other problem is completely identical. So we concentrate on the equation (22). If one compares the generators \( \tilde{T}_i \) with the generators \( t_i \) given in appendix D one finds the following map \( \tilde{T}_i = t_i \) and it is straightforward to write out the equation (22) explicitly.

The BPS equations are obtained as follows. When we split eq. (18) into the two independent conditions eq. (22) and eq. (23) the condition of finding eigenvectors is translated into an identity on the level of matrices. This is due to the fact that the eigenvalues of \( SU(2) \) come in pairs \( e^{i\lambda}, e^{-i\lambda} \) and so the existence of one +1-eigenvalue implies the existence of two +1-eigenvalues.

There are three sets of commuting rotations,

\[
\begin{align*}
\text{a)} \quad [\Gamma_{12}, \Gamma_{34}] & = 0, \\
\text{b)} \quad [\Gamma_{13}, \Gamma_{24}] & = 0, \\
\text{c)} \quad [\Gamma_{14}, \Gamma_{23}] & = 0,
\end{align*}
\]

those simultaneous in 12 and 34, those in 23 and 14 and those in 24 and 13 directions. By decomposing \( \tilde{R}_1^2 \tilde{R}_2^2 \) into these three components we then get the conditions to preserve any supersymmetry:

\[
\begin{align*}
\text{a)} \quad 0 & = F_{12} + \frac{1}{F_{34}}, \\
\text{b)} \quad 0 & = \varphi_{23} - \varphi_{14} + \arctan(F_{23}) - \arctan(F_{14}), \\
\text{c)} \quad 0 & = \varphi_{24} + \varphi_{13} + \arctan(F_{24}) + \arctan(F_{13}).
\end{align*}
\]

Each line of (24) states a condition that is capable to define a flat supersymmetric D-brane bound state by being satisfied globally, whereas they may only be patched together locally. The three rotations, symmetric or asymmetric, corresponds to three different relative \( U(1) \) rotation of the two branes. Only together they generate the most general local \( SU(2) \) deformation of the globally flat cycle. One may simplify the conditions by choosing coordinates where one of the \( U(1) \) rotations is absorbed, such that e.g. \( f^* \mathfrak{Im} \Omega = 0 \). The choice of relative signs in (24) is arbitrary and stems from a particular choice of complex structure. It relates the equation with minus signs to the symplectic structure \( f^* \omega \) and the one without to \( f^* \mathfrak{Im} \Omega \).
We now follow the usual procedure to replace the flat intersecting branes by a smooth curve $X_1(x_3, x_4), X_2(x_3, x_4)$, which means replacing the global angles $\phi_{ij}$ by local quantities according to

$$\tan(\phi_{ij}) = \partial_j X_i .$$

Then we find

$$f^* \omega = - \frac{F_{23} - F_{14}}{1 + F_{23} F_{14}},$$

$$f^* \Im \Omega = - \frac{F_{24} + F_{13}}{1 - F_{24} F_{13}}. $$

These are the conditions that the deformed cycle preserves any supersymmetry in the presence of the 2-form flux on the D4-brane. Note, that in the case $F_{23} = F_{14} = (\ast F)_{23}$ and $F_{24} = -F_{13} = (\ast F)_{24}$ the standard conditions of a special Lagrangian calibration are recovered.

Then, the field strength and the cycle are separately supersymmetric. In (26) the deviation of the flux from being self-dual or anti-self-dual is compensated by the deviation of the cycle from being special Lagrangian.

In this section we have seen how the supersymmetry constraints eq. (8) and eq. (9) of a D2-D4 bound state with magnetic flux leads to the geometrical conditions of $SU(2)$-calibrations discussed in section 2 before.

In the next section we want to discuss the derivation of the calibration conditions for the case of $SU(d)$-calibrations systematically.
4 The \( SU(d) \)-Cycle Equations

A \( d \)-dimensional ‘curve’ \( \Sigma^{(d)} \), embedded into \( 2d \)-dimensional flat space \( \mathbb{R}^{2d} \) with coordinates \( x^i \) \((i = 1, \ldots, 2d)\) can be described at least locally by the zero locus of \( d \) real functions \( f^m(x^1, \ldots, x^{2d}) \):

\[
\Sigma^{(d)} = \mathcal{V}(f^1, \ldots, f^d) = \{(x_1, \ldots, x_{2d}) \mid f^m(x^1, \ldots, x^{2d}) = 0, \ m = 1, \ldots, d\}.
\]

If one wants to deal with a so called supersymmetric \( d \)-cycle, the choice of the functions \( f^m \) is highly constrained. To study these restrictions we first introduce \( d \) real coordinates \( \xi^i \) \((i = 1, \ldots, d)\) which parametrise the curve \( \Sigma^{(d)} \). Furthermore we consider complex coordinates \( u^i = x^{2i-1} + ix^{2i} \), of \( \mathbb{C}^d \). Then the \( d \)-cycle can be characterised by making the complex \( u^i \) to be functions of the real coordinates \( \xi^i \), i.e. by the following embedding map \( i \) from \( \Sigma^{(d)} \) into \( \mathbb{C}^d \):

\[
i : \Sigma^{(d)} \longrightarrow \mathbb{C}^d: \quad \xi^i \rightarrow u^i(\xi^i), \quad i = 1, \ldots, d.
\]  

Fig. 3: The intersecting configuration

The intersection configuration (for the case \( d = 3 \)) is depicted in figure 3.

Now by applying the partial derivative \( \partial_{\xi^k} \) to the defining equations \( f^m \) of the \( d \)-cycle, we get the following relations:

\[
\sum_{n=1}^{d} (f^m_{u^n} u^k_{\xi^k} + f^m_{\bar{u}^n} \bar{u}^k_{\bar{\xi}^k}) = 0
\]  

(28)

(here \( f^m_{u^n} = \frac{\partial f^m}{\partial u^n}, \ u^k_{\xi^k} = \frac{\partial u^k}{\partial \xi^k} \)). These can be grouped into the following matrix expressions

\[
\begin{pmatrix}
  f_{u^1}^1 & f_{u^2}^1 & \cdots & f_{u^d}^1 \\
  f_{u^1}^2 & f_{u^2}^2 & \cdots & f_{u^d}^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{u^1}^d & f_{u^2}^d & \cdots & f_{u^d}^d
\end{pmatrix}
\begin{pmatrix}
  u_{\xi^k}^1 \\
  u_{\xi^k}^2 \\
  \vdots \\
  u_{\xi^k}^d
\end{pmatrix}
= (-1)^d
\begin{pmatrix}
  f_{\bar{u}^1}^1 & f_{\bar{u}^2}^1 & \cdots & f_{\bar{u}^d}^1 \\
  f_{\bar{u}^1}^2 & f_{\bar{u}^2}^2 & \cdots & f_{\bar{u}^d}^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{\bar{u}^1}^d & f_{\bar{u}^2}^d & \cdots & f_{\bar{u}^d}^d
\end{pmatrix}
\begin{pmatrix}
  \bar{u}_{\bar{\xi}^k}^1 \\
  \bar{u}_{\bar{\xi}^k}^2 \\
  \vdots \\
  \bar{u}_{\bar{\xi}^k}^d
\end{pmatrix}
\]

We will denote the left matrix by \( M \) and the right matrix as \( \bar{M} \), henceforth. Note, the sign in
front of $\bar{M}$ depends on the dimension $d$ of the cycle. With the help of these matrices we can express the bared derivatives by the unbared ones in the following way:

$$\partial_k \bar{U} = (-1)^d \bar{M}^{-1} M \cdot \partial_k U = N \cdot \partial_k U.$$ (29)

By definition $N$ shares the properties:

1. $N^{-1} = (-1)^d M^{-1} \bar{M} = \bar{N}$
2. $|\det N| = 1$

Remembering the $d$-cycle should be supersymmetric we can ask for restrictions of the matrix $N$ following from this condition. It is well known that the notion of supersymmetric cycles [2] coincides with the notion of special Lagrangian submanifolds [1] which can be rephrased in terms of the embedding map $i: \Sigma^{(d)} \rightarrow \mathbb{C}^d$ and the two conditions:

$$i^* \Im \Omega = 0 \quad \text{volume minimizing}$$

$$i^* \omega = 0 \quad \text{Lagrangian submanifold}$$ (30)

With $\Omega = du^1 \wedge \ldots \wedge du^d$ and $\omega = \frac{1}{2} \sum_i du^i \wedge d\bar{u}^i$. The requirement of minimal volume reads

$$0 = i^* \Im \Omega = \Im (du^1(\xi_1, \ldots, \xi_d) \wedge \ldots \wedge du^d(\xi_1, \ldots, \xi_d))$$

$$= \Im (\epsilon_{i_1 \ldots i_d} \bar{u}_{\xi_1}^{i_1} \bar{u}_{\xi_2}^{i_2} \ldots \bar{u}_{\xi_d}^{i_d}) \, d\xi^1 \wedge \ldots \wedge d\xi^d$$

$$\Rightarrow 0 = \frac{1}{2^d} (\epsilon_{i_1 \ldots i_d} \bar{u}_{\xi_1}^{i_1} \bar{u}_{\xi_2}^{i_2} \ldots \bar{u}_{\xi_d}^{i_d} - \epsilon_{i_1 \ldots i_d} \bar{u}_{\xi_1}^{\bar{i}_1} \bar{u}_{\xi_2}^{\bar{i}_2} \ldots \bar{u}_{\xi_d}^{\bar{i}_d})$$

$$= \frac{1}{2^d} (\epsilon_{i_1 \ldots i_d} \bar{u}_{\xi_1}^{\bar{i}_1} \bar{u}_{\xi_2}^{\bar{i}_2} \ldots \bar{u}_{\xi_d}^{\bar{i}_d} - \epsilon_{i_1 \ldots i_d} \bar{N}_{\xi_1}^{i_1} \ldots \bar{N}_{\xi_d}^{i_d} \bar{u}_{\xi_1}^{\bar{i}_1} \bar{u}_{\xi_2}^{\bar{i}_2} \ldots \bar{u}_{\xi_d}^{\bar{i}_d})$$

$$= \frac{1}{2^d} (\epsilon_{i_1 \ldots i_d} - \epsilon_{i_1 \ldots i_d} \bar{N}_{\xi_1}^{i_1} \ldots \bar{N}_{\xi_d}^{i_d} \bar{u}_{\xi_1}^{\bar{i}_1} \bar{u}_{\xi_2}^{\bar{i}_2} \ldots \bar{u}_{\xi_d}^{\bar{i}_d})$$

$$= \frac{1}{2^d} (1 - \det N) \cdot \partial (u^1, \ldots, u^d) \over \partial (\xi_1, \ldots, \xi_d)$$

which yields

$$\det N|_{\forall (f_1, \ldots, f_n)} = 1 \quad \text{or for short} \quad \det N \equiv 1.$$ 

For the calculation of the det-equation the following relation is useful.

$$\det N \equiv 1 \iff \det M - (-1)^d \det \bar{M} \equiv 0$$
Now we turn to the second equation. With the canonical Kähler (symplectic) form $\omega$, the pull back operation results in

$$0 = i^* \omega = \frac{1}{2i} \sum_{i} du^i (\xi_1 \ldots \xi_d) \wedge d\bar{u}^i (\xi_1 \ldots \xi_d)$$

$$= \frac{1}{2i} \sum_i \left( \sum_k u^i_{\xi_k} d\xi_k \right) \wedge \left( \sum_l \bar{u}^i_{\xi_l} d\xi_l \right)$$

$$= \frac{1}{2i} \sum_{k<l} \sum_i \left[ u^i_{\xi_k} \bar{u}^i_{\xi_l} - u^i_{\xi_l} \bar{u}^i_{\xi_k} \right] d\xi_k \wedge d\xi_l$$

which is satisfied if we set $N \equiv N^T$. However, as it stands, this requirement is sufficient, only.

Now we intend to give a proof that the condition is necessary, too.

To proof $N \equiv N^T$ we remember some facts from symplectic geometry especially various ways of characterising Lagrangian planes in symplectic vector spaces. The utility of this investigation rests on the simple observation that our conditions on the $d$-cycle to be a special Lagrangian submanifolds are in fact conditions on its tangent bundle, i.e. Lagrangian planes locally.

To begin with, we consider a complex vector space $\mathbb{C}^d$ furnished with a Hermitian structure

$$< x, y > = \sum_i x_i \bar{y}_i = g(x, y) + i \sigma(x, y)$$

which splits into an Euclidean metric $g$ and a symplectic form $\sigma$. One can check that $\sigma$ coincides
with
\[
\omega = \frac{1}{2i} \sum_i du^i \wedge d\bar{u}^i.
\]
given before. Therefore we identify both objects. The two-form \(\omega\) is non degenerated, antisymmetric and bilinear. With help of \(\omega\) we can define the notion of symplectic orthogonality.

**Definition 2.** The orthogonal complement of a vector subspace \(E \in \mathbb{C}^d\) is defined by
\[
E^\perp = \{ x \in \mathbb{C}^d \mid \omega(x, E) = 0 \}
\]
In the special case that \(E = E^\perp\) we call \(E\) a Lagrangian plane. Obviously on a Lagrangian plane the symplectic form restricts to zero. So we recognise the content of the constraint \(i^*\omega = 0\). It simply states that all tangent spaces to the supersymmetric cycle are Lagrangian planes embedded in the tangent space of the embedding space. Here we collect some facts:

1. \(Sp(E)\) operates transitively on Lagrangian planes
2. Since \(U(d)\) preserves the Hermitian form, it is contained in \(Sp(E)\).
3. By \(L(\mathbb{C}^d)\) we denote the Grassmannian of Lagrangian planes
4. \(\lambda \in L(\mathbb{C}^d)\) is characterised by choosing an orthonormal basis \((a_1, \ldots, a_n)\) with respect to the Euclidean metric \(g\). But then it is orthonormal with respect to the Hermitian form, too:
\[
< a_i, a_j > = g(a_i, a_j) + i \omega(a_i, a_j) = 1 = \delta_{ij},
\]
i.e. the matrix \(a = (a_1, \ldots, a_n)\) is unitary. The other direction works, too. Hence
\[
\lambda \in L(\mathbb{C}^d) \iff \exists a \in U(d), \lambda = a(\mathbb{R}^d)
\]

---

**Fig. 4:** The operation of \(a\) on Lagrangian planes

21
5. Obviously each Lagrangian plane will be stabilised by any element in $O(n)$, i.e. we can regard the Grassmannian of Lagrangian planes as the quotient space

$$L(\mathbb{C}^d) = \frac{U(d)}{O(d)}$$

How can we define a projection from $U(d)$ onto $L(\mathbb{C}^d)$? We observe that two elements $a$ and $a'$ determine the same Lagrangian plane, iff

$$\lambda = a(\mathbb{R}^d) = a'(\mathbb{R}^d) \iff a\bar{a}^{-1} = a'a'^{-1},$$

which is constant on the $O(d)$-orbits of the fibration. Now we can identify $L(\mathbb{C}^d)$ with the image of the projection map

$$\begin{align*}
\pi : U(d) &\to L(\mathbb{C}^d) \\
a &\mapsto \lambda = a\bar{a}^{-1}
\end{align*}$$

By abuse of language we denote the matrix representative $a\bar{a}^{-1}$ of the Lagrangian plane $\lambda = a(\mathbb{R}^n)$ by $\lambda$ again. But how can we associate the geometrical object with this artificial matrix representative? The connection between the matrix $\lambda$ on the one side and the concrete Lagrangian plane $\lambda$ on the other side is given through the central equation

$$x \in \lambda \iff x = \lambda \bar{x}$$

In the last formula we recognise the familiar equation (29). But now we know, that we can represent $\lambda$ as $\lambda = a\bar{a}^{-1}$ and this yields straightforward

$$\begin{align*}
\lambda^+ &\equiv \bar{a}^{-1}a^+ = a^{-1}a^T a^{-1} = \bar{a}a^{-1} = \bar{\lambda} \\
\Rightarrow \lambda^T &= \lambda
\end{align*}$$

But then we can finally conclude by identifying $\lambda = N^{-1}$ and performing some mild manipulations that

$$N \equiv N^T$$

In summary, all what we have done so far can be formulated in a short but important proposition which is the starting point for all further computations:
**Proposition:** A \(d\)-cycle, represented as an intersection of \(d\) real valued functions is supersymmetric, iff \(N \equiv N^T\) and \(\det N \equiv 1\).

It will turn out to be very useful to reformulate the last proposition \(N \equiv N^T\) in a different, but equivalent way. Namely, it is not difficult to show that the requirement \(N \equiv N^T\) is equivalent to the condition that the matrix \(MM^+\) should be real modulo \(I(\mathbb{V})\). To prepare this reformulation we remark that by the split of the coordinates of \(\mathbb{R}^{2d}\) into the coordinates of \(\mathbb{C}^d\) they inherit an intrinsic meaning as the spatial and momentum variables of symplectic geometry. This is given by

\[
u^i = q^i + ip^i,
\]

i.e. the real part of \(u^i\) gets the meaning of a spatial coordinate whereas the \(p^i\) is a momentum variable. Then we are free to define the convenient Poisson brackets of phase-space functions \(\{f, g\}\). This is done in the standard way as

\[
\{f, g\} = \sum_{i=1}^d \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i} \right) = \sum_{i=1}^d (f_{2i-1}g_{2i} - f_{2i}g_{2i-1}),
\]

where \(f_{2i-1} = \frac{\partial f}{\partial q^i} = \frac{\partial f}{\partial x^{2i-1}}\) and \(f_{2i} = \frac{\partial f}{\partial p^i} = \frac{\partial f}{\partial x^{2i}}\). Then the matrix \(MM^+\) reads

\[
(MM^+)^{mn} = \langle \nabla f^m, \nabla f^n \rangle \pm i \cdot \{f^m, f^n\}.
\]

So \(MM^+\) is a real matrix modulo \(I(\mathbb{V})\), i.e. \(N \equiv N^T\), if all Poisson brackets among the defining functions \(f^m\) and \(f^n\) vanish:

\[
\{f^m, f^n\} \equiv 0.
\]

So we get a more suitable set of equations for concrete calculations.

**Corollary:** A \(d\)-cycle, represented as an intersection of \(d\) real valued functions is supersymmetric, iff \(\{f^i, f^j\} \equiv 0\) and \(\det N \equiv 1\)

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A Determinant of a two dimensional metric

The determinant of the induced metric given in eq. (3) is

\[
|f^*g| = \left[ 1 + \left( \frac{\partial X_1}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_1} \right)^2 \right] \left[ 1 + \left( \frac{\partial X_1}{\partial \xi_2} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_2} \right)^2 \right] - \left[ \frac{\partial X_1}{\partial \xi_1} \frac{\partial X_1}{\partial \xi_2} + \frac{\partial X_2}{\partial \xi_1} \frac{\partial X_2}{\partial \xi_2} \right]^2
\]

\[
= 1 + \left( \frac{\partial X_1}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_1}{\partial \xi_2} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_2} \right)^2
\]

\[
+ \left[ \left( \frac{\partial X_1}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_1} \right)^2 \right] \left[ \left( \frac{\partial X_1}{\partial \xi_2} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_2} \right)^2 \right] - \left[ \frac{\partial X_1}{\partial \xi_1} \frac{\partial X_1}{\partial \xi_2} + \frac{\partial X_2}{\partial \xi_1} \frac{\partial X_2}{\partial \xi_2} \right]^2
\]

\[
= 1 + \left( \frac{\partial X_1}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_1} \right)^2 + \left( \frac{\partial X_1}{\partial \xi_2} \right)^2 + \left( \frac{\partial X_2}{\partial \xi_2} \right)^2
+ \left( \frac{\partial X_1}{\partial \xi_1} \frac{\partial X_2}{\partial \xi_2} - \frac{\partial X_2}{\partial \xi_1} \frac{\partial X_1}{\partial \xi_2} \right)^2
\]

\[
= \left[ f^*\omega \right]^2 + \left[ f^*\Im \Omega \right]^2 + \left[ f^*\Re \Omega \right]^2
\]  

(35)

B Decomposition of the adjoint representation of \(\text{Spin}(4)\)

The change from eq. (12) to the normal form of eq. (13) can be described quite explicitly by using the isomorphism of \(\text{Spin}(4)\) to \(SU(2) \times SU(2)\) constructed in subsection 3.1.2 and the relation to the corresponding \(SO\)-groups as shown in the diagram below:

\[
\text{Spin}(4) \quad \text{SU}(2) \times \text{SU}(2) \\
\downarrow \pi \quad \downarrow \pi
\]

\[
\text{SO}(4) \quad \text{SO}(3) \times \text{SO}(3)
\]  

(36)

The two \(SU(2)\) act via the adjoint representations (\(SO(3)\)) on the two 3-dimensional subspaces in the orthogonal decomposition of \(\Lambda^2(\mathbb{R}^4) = \Lambda^2_+ (\mathbb{R}^4) \oplus \Lambda^2_- (\mathbb{R}^4)\). Since \(F_{\mu\nu} \in \Lambda^2(\mathbb{R}^4)\) we can obtain the normal form of eq. (12) by computing the self-dual and anti-self-dual parts of \(F_{\mu\nu}\) due to

\[
F = \frac{1}{2} (1 + \ast) F + \frac{1}{2} (1 - \ast) F
\]

\[
(F_{12}, \ldots, F_{34}) \quad \mapsto \quad \left( [F_{12} + F_{34}] / 2, \ldots \right) \oplus \left( [F_{12} - F_{34}] / 2, \ldots \right)
\]

and determine the two rotations \(S_1 \times S_2 \in SO(3) \times SO(3)\), which rotate each of the two 3-vectors on the right hand side of the above decomposition to the standard position

\[
\left( [\tilde{F}_{12} + \tilde{F}_{34}] / 2, 0, 0 \right) \oplus \left( [\tilde{F}_{12} - \tilde{F}_{34}] / 2, 0, 0 \right)
\]

Since the rotations preserve the norm it is quite easy to work out from this requirement the form of the expressions in eq. (14) and eq. (15). Explicit matrices \(S_1\) and \(S_2\) can be written down.
using the formula (37) in appendix E. In the following we will need $S_1$ only, so we include the result for convenience. Note that the angles shown below are not unique due to the remaining rotation of a three vector around its own axis:

$$\theta_3 = -\arctan \frac{b}{a} \quad \theta_2 = \arctan \frac{c}{\sqrt{a^2 + b^2}} \quad \theta_1 = 0$$

Here $a$, $b$ and $c$ denote the selfdual components of $F_{\mu\nu}$.

C The invariant relation of flux and angles

Eq. (16) can be read as a contraction of the antisymmetric tensor $\tilde{\phi}_{ij} = \arctan \tilde{F}_{ij}$ with the antisymmetric gamma matrices $\Gamma_{ij}$.

In appendix B we constructed the matrix $S_1 \times S_2 \in SO(3) \times SO(3)$, which transforms a general $F$ into normal form $\tilde{F}$. This matrix is in the $6$ representation of $SO(4)$. Here we need the corresponding element in the $4$ representation of $SO(4)$, which generates the same action as $S_1 \times S_2$. Such an $S \in SO(4)$ acts like

$$\tilde{F}_{ij} = S^k_i S^l_j F_{kl} = (SFST)_{ij}$$

and can be constructed by going from the lower right to the lower left corner in diagram (36). Utilising this $S \in SO(4)$ the exponent in eq. (16) can be written as

$$\tilde{\phi}_{ij} \Gamma^{ij} = S^k_i (\arctan F)_{kl} S^l_j \Gamma^{ij} = (\arctan F)_{kl} S^k_i \Gamma^{ij} S^l_j = (\arctan F)_{kl} (ST)^k_i (ST)^l_j \Gamma^{ij}$$

But the action of $SO(4)$ on $\Gamma^{ij}$ in the last line can be seen as the adjoint action of $Spin(4)$ on its Lie algebra $\Sigma_{ij} = \Gamma_{ij}/2$. So we can rewrite the last transformation in terms of a $Spin(4)$ rotation and finally we obtain:

$$e^{(\arctan F)_{kl} (ST \Sigma S)^{kl}} = \pi^{-1}(ST) e^{(\arctan F)_{kl} \Sigma^{kl}} \pi^{-1}(ST^{-1})$$

D SU(2)

The generators of the Lie algebra $su(2)$ are

$$t_1 = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad t_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

The three one parameter subgroups generated by $t_1 \ldots t_3$ are

$$g_1 = \begin{pmatrix} \cos(\frac{\theta_1}{2}) & i \sin(\frac{\theta_1}{2}) \\ i \sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{pmatrix}$$

$$g_2 = \begin{pmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{pmatrix}$$

$$g_3 = \exp(-i \frac{\theta_3}{2}) \begin{pmatrix} 1 & 0 \\ 0 & \exp(i \frac{\theta_3}{2}) \end{pmatrix}$$
The product of each of the three gives
\[
g_1 \cdot g_2 \cdot g_3 = \left( \begin{array}{ccc} \cos(\theta) & -i \sin(\theta) & 0 \\ i \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) & 0 \end{array} \right) = \left( \begin{array}{ccc} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{array} \right) \]
\[
\cdot \left( \begin{array}{ccc} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{array} \right) \cdot \left( \begin{array}{ccc} x^1 \\ x^2 \\ x^3 \end{array} \right)
\]

E \quad SO(3)

The SO(3) action associated to the SU(2) of appendix B is defined by the action of SU(2) on the Lie algebra $su(2)$, i.e.
\[
\pi(U) \tilde{x} = U \tilde{x} U^{-1}
\]
with $\tilde{x} = x^i t_i$. The adjoint action on each generator can be computed easily and reads:
\[
\begin{align*}
\pi(g_1) t_1 & = \begin{pmatrix} \frac{i}{2} \sin \theta_1 & \frac{i}{2} \cos \theta_1 \\ -\frac{i}{2} \cos \theta_1 & \frac{i}{2} \sin \theta_1 \end{pmatrix} = t_1 \\
\pi(g_1) t_2 & = \begin{pmatrix} -\frac{i}{2} \sin \theta_1 & \frac{i}{2} \cos \theta_1 \\ \frac{i}{2} \cos \theta_1 & -\frac{i}{2} \sin \theta_1 \end{pmatrix} = \cos \theta_1 \cdot t_2 + \sin \theta_1 \cdot t_3 \\
\pi(g_1) t_3 & = \begin{pmatrix} -\frac{i}{2} \cos \theta_1 & -\frac{i}{2} \sin \theta_1 \\ \frac{i}{2} \sin \theta_1 & -\frac{i}{2} \cos \theta_1 \end{pmatrix} = -\sin \theta_1 \cdot t_2 + \cos \theta_1 \cdot t_3 \\
\pi(g_2) t_1 & = \begin{pmatrix} \frac{i}{2} \sin \theta_2 & \frac{i}{2} \cos \theta_2 \\ -\frac{i}{2} \cos \theta_2 & -\frac{i}{2} \sin \theta_2 \end{pmatrix} = -\sin \theta_2 \cdot t_3 + \cos \theta_2 \cdot t_1 \\
\pi(g_2) t_2 & = \begin{pmatrix} \frac{i}{2} \cos \theta_2 & -\frac{i}{2} \sin \theta_2 \\ \frac{i}{2} \sin \theta_2 & \frac{i}{2} \cos \theta_2 \end{pmatrix} = t_2 \\
\pi(g_2) t_3 & = \begin{pmatrix} -\frac{i}{2} \cos \theta_2 & -\frac{i}{2} \sin \theta_2 \\ \frac{i}{2} \sin \theta_2 & \frac{i}{2} \cos \theta_2 \end{pmatrix} = \cos \theta_2 \cdot t_3 + \sin \theta_2 \cdot t_1 \\
\pi(g_3) t_1 & = \begin{pmatrix} 0 & \frac{i}{2} e^{-i \theta_3} \\ -\frac{i}{2} e^{i \theta_3} & 0 \end{pmatrix} = \cos \theta_3 \cdot t_1 + \sin \theta_3 \cdot t_2 \\
\pi(g_3) t_2 & = \begin{pmatrix} 0 & -\frac{i}{2} e^{-i \theta_3} \\ \frac{i}{2} e^{i \theta_3} & 0 \end{pmatrix} = -\sin \theta_3 \cdot t_1 + \cos \theta_3 \cdot t_2 \\
\pi(g_3) t_3 & = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = t_3
\end{align*}
\]

The action rewritten with respect to the coordinates is then:
\[
\pi(g_1 \cdot g_2 \cdot g_3) \tilde{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (37)
\]
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