Generalised Poisson-Dirichlet Distributions and the Negative Binomial Point Process

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Abstract

When $S = (S_t)_{t \geq 0}$ is an $\alpha$-stable subordinator, the sequence of ordered jumps of $S$, up till time 1, omitting the $r$ largest of them, and taken as proportions of their sum $(r)S_t$, defines a 2-parameter distribution on the infinite dimensional simplex, $\nabla_\infty$, which we call the PD$^{(r)}_\alpha$ distribution. When $r = 0$ it reduces to the PD$\alpha$ distribution introduced by Kingman in 1975. We observe a serendipitous connection between PD$^{(r)}_\alpha$ and the negative binomial point process of Gregoire (1984), which we exploit to analyse in detail a size-biased version of PD$^{(r)}_\alpha$. As a consequence we derive a stick-breaking representation for the process and a useful form for its distribution. This program produces a large new class of distributions available for a variety of modelling purposes.

Keywords: generalised Poisson-Dirichlet laws; negative binomial point process; trimmed $\alpha$-stable subordinator; stick-breaking representation; size-biased permutation

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1 Introduction

Developments related to the Poisson-Dirichlet distribution and its generalisations have had an enormous impact in recent times, stimulating as well as synthesising a host of theoretical results connected in particular to the excursion theory of stochastic processes and to random partitions, and opening up a wealth of applications areas, especially for example in Bayesian statistics and population genetics. We refer to Bertoin (1996) and Feng (2010) for up-to-date accounts of various aspects.

To motivate the ideas that concern us here, start with a stable subordinator $(S_t)_{t \geq 0}$ of index $\alpha \in (0, 1)$ on $\mathbb{R}^+$ having jump process $(\Delta S_t := S_t - S_{t-})_{t > 0}$, and order the jumps up till 1 as $\Delta S_1^{(1)} \geq \Delta S_1^{(2)} \geq \cdots$. The random sequence $(\Delta S_1^{(i)}/S_1)_{i \geq 1}$

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specifies a distribution on the infinite dimensional simplex $\nabla_\infty$ which we will refer to as a PD$_\alpha$ distribution. It was introduced by Kingman [1975] and subsequently gave rise to a large body of research. Of special interest to us are papers by Perman et al. [1992] and Pitman & Yor [1992, 1997] (hereafter, referred to as PPY (1992) and PY (1992; 1997)). They contain in particular formulae for the distribution of the size-biased vector associated with PD$_\alpha$.

The PD$_\alpha$ distribution arises by considering the ordered jumps $(\Delta S_{1}^{(i)})_{i\geq 1}$ and their relation to the sum, $S_{1}$. As a natural generalisation, delete the $r$ largest jumps ($r \in \mathbb{N}$) up till time 1 and consider the distribution of the remaining jumps $(\Delta S_{1}^{(i)})_{i\geq r+1}$ taken as proportions of their sum, $(r)S_{1}$, the latter being $S_{1}$ with the $r$ largest terms removed. Again we obtain a distribution on $\nabla_\infty$, now with an extra parameter, $r$. When $r = 0$ (no trimming) this is a PD$_\alpha$ distribution, while for $r = 1, 2, \ldots$, it defines a 2-parameter distribution on $\nabla_\infty$ which we call the PD$_{(r)}$ distribution.

Laplace transforms of the stable ratios take a reasonably explicit form, and reveal a close connection with the negative binomial point process of Gregoire (1984). This suggests some rewarding new lines of enquiry, and we proceed to define a size-biased version of PD$_{(r)}$ and use the point process representation to derive a corresponding stick-breaking representation.

This program produces a large new class of distributions available for a variety of modelling purposes. We illustrate its relevance by reference to two applied situations, occurring in finance and linguistics which are analysed in papers by Sosnovskiy (2015) and Goldwater et al. (2011). We discuss these further in Section 3.

2 Jumps of a Normalised Stable Subordinator

We consider a driftless stable subordinator, that is, a real valued Lévy process $(S_{t})_{t\geq 0}$, with $S_{0} \equiv 0$, on a filtered probability space $(\Omega, (\mathcal{F}_{t})_{t\geq 0}, P)$, with canonical triplet $(\gamma, 0, \Lambda)$; thus, having Lévy measure

$$\Lambda(dx) = c_0 x^{-\alpha-1} dx 1_{\{x>0\}}, \text{ for some } c > 0 \text{ and } 0 < \alpha < 1,$$

(2.1)

with tail measure

$$\overline{\Lambda}(x) = cx^{-\alpha}, \quad x > 0,$$

and Laplace transform

$$E e^{-\lambda S_{t}} = e^{-t \Psi(\lambda)}, \quad \text{where} \quad \Psi(\lambda) = \int_{(0,1)} (1 - e^{-\lambda x}) \Lambda(dx), \quad \lambda > 0.$$

(2.2)

Write $(\Delta S_{t} := S_{t} - S_{t-})_{t>0}$, with $\Delta S_{0} = 0$, for the jump process of $S$, and $\Delta S_{t}^{(1)} \geq \Delta S_{t}^{(2)} \geq \cdots$ for the ordered jumps at time $t > 0$. Since $\Lambda\{(0, \infty)\} = \infty$ there are infinitely many jumps, a.s. (almost surely), in any finite time interval, and since $\Lambda$ is diffuse, the ordered jumps $(\Delta S_{t}^{(i)})_{i=1,2,\ldots}$ are uniquely defined, a.s. Our objective

\footnote{Throughout, let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.}
is to study the “trimmed” process, by which we mean $S_t$ minus its large jumps, at a given time $t$:

$$(^rS)_{t} := S_{t} - \sum_{i=1}^{r} \Delta S_{t}^{(i)}, ~ r \in \mathbb{N}, ~ t > 0$$

(2.3)

(and we set $(0)S_t \equiv S_t$).

In the next subsection we define the PD$_{\alpha}^r$ distribution as that of the sequence of ratios $(\Delta S_{1}^{(i)}/(^rS)_{1})_{i \geq r+1}$.

### 2.1 Generalised Poisson-Dirichlet Distributions

Fix $r \in \mathbb{N}_0$ and define

$$V_{n}^{(r)} := \frac{\Delta S_{1}^{(r+n)}}{(^rS)_{1}}, ~ n \in \mathbb{N}.$$ 

Since, for $m > n$,

$$\sum_{n<j \leq m} V_{j}^{(r)} \leq \frac{(r+n)S_{1}}{(^rS)_{1}} \to 0, \text{ a.s., as } n \to \infty,$$

the series $\sum_{n} V_{n}^{(r)}$ converges a.s. for each $r \in \mathbb{N}$, and clearly, $\sum_{n \geq 1} V_{n}^{(r)} = 1$.

Consequently, the distribution of $(V_{n}^{(r)})_{n \in \mathbb{N}}$ when $r \in \mathbb{N}$ defines a new family of distributions on $\nabla_\infty$ derived from the subordinator $S$.

**Definition 2.1.** Let $(S_{t}, 0 \leq t \leq 1)$ be a driftless stable subordinator with index $\alpha \in (0, 1)$ and take $r \in \mathbb{N}_0$. Then the distribution of the sequence

$$(V_{n}^{(r)})_{n \in \mathbb{N}} = (V_{1}^{(r)}, V_{2}^{(r)}, \ldots) = \left(\frac{\Delta S_{1}^{(r+1)}}{(^rS)_{1}}, \frac{\Delta S_{1}^{(r+2)}}{(^rS)_{1}}, \ldots\right)$$

(2.4)

we call a PD$_{\alpha}^r$ distribution. When $r = 0$, PD$_{\alpha}$ is recovered.

**Remark 2.1.** PD$_{\alpha}^r$ is obtained from the deletion of the $r$ largest jumps of $S_1$, followed by renormalisation, rather than from the deletion of the first $r$ size-biased picks from $(\Delta S_{1}^{(i)})$, as considered in Pitman (2003) and PY (1997, Prop. 34, 35). This results in a different dependence structure in PD$_{\alpha}^r$ for the stick-breaking representation than in the PY situations (cf. Theorem 2.1 in the next section).

Results related to those of PY (1997) concerning deletion of excursion intervals of certain Bessel bridges are in PPY (1992, Sect.3); see also James (2013, 2015).

**Remark 2.2.** Similar to (2.4), any distribution on $\nabla_\infty$ with a subordinator representation can be generalised by removing the $r$ largest jumps up till time $t > 0$ from the subordinator. For example: (i) the usual Poisson-Dirichlet distribution, denoted as PD(0, $\theta$) in PY (1997), can be generalised by trimming a Gamma subordinator up till time $\theta > 0$;

(ii) the two parameter PD($\alpha, \theta$) distribution in PY (1997) can be extended by trimming a generalised Gamma subordinator up till a random time mixed with a Gamma($\theta/\alpha, 1$) distribution (see PY (1997, Prop. 21)).
We do not pursue these generalisations here, going on instead to explore a connection with the negative binomial process (in the next subsection), but we conclude this subsection with a formula for the Laplace transform of the ratio \((r)S_1/\Delta S_1^{(r)}\):

\[
E\left(e^{-\lambda(r)S_1/\Delta S_1^{(r)}}\right) = 1 \left(1 + \Psi(\lambda)\right)^{-r}, \quad r \in \mathbb{N},
\]

where \(W = (W_v)_{v \geq 0}\) is a driftless subordinator with measure \(\Lambda(dx)1_{(0,1)}\), and \(\Gamma_r\) denotes a Gamma\((r,1)\) random variable independent of \(W\). Recall that \(\Psi\) is defined in (2.2). Formula (2.5) appears in PY (1997, Prop. 11), as well as in Kevei & Mason (2014).

Later we will need the density function of a Gamma random variable with parameter \(r\):

\[
P(\Gamma_r \in dx) = \frac{x^{r-1}e^{-x}}{\Gamma(r)}1\{x>0\},
\]

and the density of a Beta random variable \(B_{a,b}\) with parameters \(a, b > 0\):

\[
f_B(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1 - x)^{b-1}1\{0 < x < 1\},
\]

where \(\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx, r > 0\), is the Gamma function.

### 2.2 The Negative Binomial Point Process and the Distribution of \(PD^{(r)}_\alpha\)

In this subsection we connect the previous results with the negative binomial point process introduced by Gregoire (1984). As previously, \(S\) is the driftless stable subordinator with index \(\alpha \in (0, 1)\). It turns out that to generate the Laplace transform in (2.5), we have to construct a point process from ratios of stable jumps, rather than from the jumps themselves. Thus, for \(r \in \mathbb{N}\), and with \(\delta_x\) denoting a point mass at \(x \in \mathbb{R}\), define a random point measure on the Borel sets of \((0,1)\) by

\[
\mathbb{B}^{(r)} = \sum_{i \geq 1} \delta_{J_r(i)}, \quad \text{where} \quad J_r(i) = \frac{\Delta S_{1}^{(r+i)}}{\Delta S_1^{(r)}}, \quad i = 1, 2, \ldots .
\]

Let \((\mathbb{M}, \mathcal{M})\) be the space of all point measures\(^2\) on \((0,1)\) with its usual Borel \(\sigma\)-algebra and let \(\mathcal{F}^+\) be the set of nonnegative measurable functions on \((0,1)\). A random measure \(\xi\) on \((\mathbb{M}, \mathcal{M})\) has Laplace functional defined as

\[
\Phi(f) = E(e^{-\xi(f)}) = \int_{M \in \mathbb{M}} e^{-\int_{(0,1)} f(x) M(dx)} P(\xi \in dM), \quad f \in \mathcal{F}^+.
\]

Given a measure \(\Pi\) on \((0,\infty)\), locally finite at infinity, Gregoire (1984) defines the point process \(\mathbb{BN}^{(r)}(\Pi)\) on \((\mathbb{M}, \mathcal{M})\) in terms of its Laplace functional as

\[
\Phi(f) = \left(1 + \int_{(0,\infty)} (1 - e^{-f(x)}) \Pi(dx) \right)^{-r}, \quad f \in \mathcal{F}^+.
\]

\(^2\)We generally follow the exposition in Resnick (1987, Chap. 3) for the following setup.
Recall (2.1) and let \( \tilde{\Lambda}(dx) := \alpha x^{-\alpha-1} dx \mathbf{1}_{\{0 < x < 1\}} \) be the normalised \( \Lambda(dx) \) restricted to \((0, 1)\). For \( r \in \mathbb{N} \), denote the law of \( B\mathcal{N}(r, \tilde{\Lambda}) \) by \( P_r \), so that

\[
P_r(dM) = P(B\mathcal{N}(r, \tilde{\Lambda}) \in dM), \quad M \in \mathcal{M}.
\]

Let the family of Palm distributions of \( P_r \) be \( P_r^{(x)} \), \( x \in (0, 1) \). The correspondence between (2.8) and the righthand side of (2.5) suggests the following proposition.

**Proposition 2.1.** Let \( B^{(r)} \) be defined as in (2.7). Then

(i) \( B^{(r)} \) is a negative binomial point process with distribution \( P_r \) such that

(ii) \( E(B^{(r)}(A)) = r \tilde{\Lambda}(A) \) for any Borel set \( A \subset (0, 1) \).

(iii) The Laplace functional of the probability measure \( P_r^{(x)} \) on \((\mathfrak{M}, \mathcal{M})\) satisfies

\[
\Phi_{P_r^{(x)}}(f) = \Phi_{\delta_x}(f) \Phi_{P_{r+1}}(f), \quad f \in \mathcal{F}^+.
\]

**Remark 2.3** (Interpretation of \( P_r^{(x)} \)). We can think of the Palm distribution \( P_r^{(x)} \) as the conditional distribution of \( B\mathcal{N}(r, \tilde{\Lambda}) \) given \( B\mathcal{N}(r, \tilde{\Lambda}) \{ \{x\} \} > 0 \). From (2.9), we can interpret \( P_r^{(x)} \) in the following way. Let \( \xi \) be distributed as \( B\mathcal{N}(r+1, \tilde{\Lambda}) \). Then \( P_r^{(x)} \) is the distribution of \( \xi + \delta_x \).

**Proof of Proposition 2.1.** (i): Conditional on \( \{ \Delta S_t^{(r)} = v \} \), \( v > 0 \), the truncated point process \( \{ \Delta S_{t+j}^{(r)} \}, j \in \mathbb{N} \) is a Poisson point process with intensity measure \( \Lambda(dx) \mathbf{1}_{\{x < v\}} \). The following distributional equivalence can be deduced, for example, from Lemma 1.1 of Buchmann et al. (2016):

\[
(\Delta S_t^{(i)})_{1 \leq i \leq r} \equiv \mathcal{D} (\tilde{\Lambda}^{-1}(\Gamma_i/t))_{1 \leq i \leq r}, \quad t > 0, \quad r \in \mathbb{N},
\]

where the \( \Gamma_i \) are Gamma\((i, 1)\) random variables and \( \tilde{\Lambda}^{-1}(x) = c^{1/\alpha}x^{-1/\alpha} \) is the inverse function of \( \Lambda \). Using (2.10) we can write the Laplace functional of \( B^{(r)} \) as

\[
E(e^{-B^{(r)}(f)}) = \int_{v>0} \exp \left( - \int_0^v (1 - e^{-f(x/v)}) \Lambda(dx) \right) P(\tilde{\Lambda}^{-1}(\Gamma_r) \in dv)
\]

\[
= \int_{v>0} \exp \left( - \int_0^1 (1 - e^{-f(x)}) \Lambda(dx) \right) P(\tilde{\Lambda}^{-1}(\Gamma_r) \in dv)
\]

\[
= \int_{v>0} \exp \left( - cv^{-\alpha} \int_0^1 (1 - e^{-f(x)}) x^{-1-\alpha} dx \right) P(\Gamma_r \in d(cv^{-\alpha}))
\]

for each \( f \in \mathcal{F}^+ \). By change of variable in (2.11) with \( y = cv^{-\alpha} \), we have

\[
E(e^{-B^{(r)}(f)}) = \int_0^\infty \exp \left( - y \int_0^1 (1 - e^{-f(x)}) \tilde{\Lambda}(dx) \right) P(\Gamma_r \in dy)
\]

\[
= \int_0^\infty \exp \left( - y \int_0^1 (1 - e^{-f(x)}) \tilde{\Lambda}(dx) \right) y^{r-1} e^{-y} \Gamma(r) dy
\]

\[
= \left( 1 + \int_0^1 (1 - e^{-f(x)}) \tilde{\Lambda}(dx) \right)^{-r}.
\]

Comparing (2.12) with (2.8) proves Part (i). Parts (ii) and (iii) follow from Part (i) by Propositions 3.3 and 4.3 in Gregoir (1984).
Remark 2.4. (i) The sum of the points in $B^{(r)}$ is $(r)S_1/\Delta S_1^{(r)}$, hence the connection with (2.5).

(ii) A variety of formulae relating to the Poisson-Dirichlet distributions have been derived over the years, including an iterative formula for the joint density of the first $n$ terms of $PD_\alpha$ (Perman (1993, Thm. 2)). Such formulae, while explicit, are “rather intractable” (PY (1992, p.329)), and simpler structures can be revealed for the corresponding size-biased permutation; see, e.g., PPY (1992, Thm. 1.2) (attributed to Perman (1990)), which allows for a “stick-breaking” representation of $PD_\alpha$ in terms of independent beta rvs. See also Pitman & Tran (2015) for the setting of a finite sequence of i.i.d. random variables. This motivates us to consider the size-biased permutation of $PD_\alpha^{(r)}$ and to investigate a stick-breaking-like representation in the $r$-trimmed case through the random point measure $B^{(r)}$.

Pitman (1995) proved that the PD$(\alpha, \theta)$ of PY (1997) is the largest class of distributions with a stick-breaking representation in terms of independent beta rvs; inevitably, then, our enlarged class $PD_\alpha^{(r)}$ requires a dependent stick-breaking representation. James (2013) derives another class of distributions, PG$(\alpha, \zeta)$, by mixing generalised Gamma subordinators, which also has a dependent stick-breaking representation.) The dependence structure will become clear in the main Theorem 2.1 of this section which gives a formula for the density of the size-biased version of the sequence $(V_n^{(r)})$ in (2.4). The remaining calculations in this section lead up to Theorem 2.1.

Henceforth fix $r \in \mathbb{N}$. Write

$$\mathfrak{J}_r := \{J_r(1), J_r(2), J_r(3), \ldots\} \quad (2.13)$$

for the points of $B^{(r)}$, with sum

$$(r)T := \sum_{i \geq 1} J_r(i) = \frac{(r)S_1}{\Delta S_1^{(r)}}. \quad (2.14)$$

Define the size-biased random permutation of $\mathfrak{J}_r$, denoted by $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \ldots)$, in the following way. Conditional on $\mathfrak{J}_r$, the first term $\tilde{J}_1$ takes values among the members of $\mathfrak{J}_r$ with probabilities

$$P(\tilde{J}_1 = J_r(i) \mid \mathfrak{J}_r) = \frac{J_r(i)}{\sum_{\ell \geq 1} J_r(\ell)} = \frac{\Delta S_1^{(r+i)}}{(r)S_1}, \quad i = 1, 2, \ldots$$

Conditional on $\mathfrak{J}_r$ and $\tilde{J}_1, \ldots, \tilde{J}_n$, for each $n = 1, 2, \ldots$, the $(n+1)^{st}$ term $\tilde{J}_{n+1}$ takes values among $\{J_r(j), j = 1, 2, \ldots; J_r(j) \neq \tilde{J}_i, 1 \leq i \leq n\}$, with probabilities

$$P(\tilde{J}_{n+1} = J_r(j) \mid \tilde{J}_1, \ldots, \tilde{J}_n, \mathfrak{J}_r) = \frac{\Delta S_1^{(r+j)}1\{\Delta S_1^{(r+j)} \neq \tilde{J}_i \cdot \Delta S_1^{(r)}; 1 \leq i \leq n\}}{(r)S_r - \Delta S_1^{(r)} \cdot (\sum_{i=1}^n \tilde{J}_i)}.$$

Then the sums of the remaining points in the point process, after removing points by size-biased sampling, are

$$(r)T_1 := (r)T - \tilde{J}_1, \quad \text{and for each } n > 1, \quad (r)T_n := (r)T_{n-1} - \tilde{J}_n. \quad (2.15)$$
The successive residual fractions are
\[ (r)U_1 := \frac{(r)T_1}{(r)T} = 1 - \frac{\tilde{J}_1}{(r)T}, \] (2.16)
and for each \( n > 1, \)
\[ (r)U_n := \frac{(r)T_n}{(r)T_{n-1}} = 1 - \frac{\tilde{J}_n}{(r)T_{n-1}}. \] (2.17)

For \( M \) a point measure in \( \mathfrak{M} \), let \( T(M) = \sum_{x \in M} x \) be the sum of the magnitudes of the points in \( M \). For each \( r \in \mathbb{N} \) let the density of \( T(\mathbb{B}^{(r)}) \) with \( \mathbb{B}^{(r)} \) distributed as \( \mathcal{B}\Lambda(r, \tilde{\Lambda}) \) be
\[ g_r(t) := \frac{P(T(\mathbb{B}^{(r)}) \in dt)}{dt} = \frac{P((r)T \in dt)}{dt}, \quad t > 0. \] (2.18)

By (2.14), \( (r)T = (r)S_1/\Delta S_1^{(r)} \), so by (2.5), \( g_r \) satisfies
\[ \int_0^\infty e^{-\lambda x} g_r(x) dx = \left( 1 + \int_0^1 (1 - e^{-\lambda x})\tilde{\Lambda}(dx) \right)^{-r}, \] (2.19)
for \( \lambda > 0, \ r \in \mathbb{N} \). Alternatively expressed, \( g_r \) is the density of \( \tilde{W}_{T_{r+1}} \), where \( (\tilde{W}_v)_{v \geq 0} \) is the driftless subordinator with \( \text{Levy measure} \ \tilde{\Lambda}(dx) \).

The next lemma derives important properties of \( \mathbb{B}^{(r)} \). It will be apparent that our proofs owe much to the methods of PPY (1992), PY (1992) and Fitzsimmons, Pitman and Yor (1993, Sect. 5). See Lemma 2.2 of PPY (1992). In a remark at the end of this section we discuss briefly the differences as well as similarities between our approaches.

**Proposition 2.2.** For \( r \in \mathbb{N} \) let \( \mathbb{B}^{(r)} \) be a negative binomial point process. Let \( \mathbb{B}^{(r)}_1 = \mathbb{B}^{(r)} - \tilde{J}_1 \) be the remaining process after removing the first size-biased pick.

(i) Then for \( 0 < x < 1, \ M \in \mathfrak{M}, \)
\[ P(\tilde{J}_1 \in dx, \mathbb{B}^{(r)}_1 \in dM) = \frac{x}{T(M) + x} r\tilde{\Lambda}(dx) P(\mathbb{B}^{(r+1)} \in dM). \] (2.20)

(ii) For \( 0 < x < 1, \ M \in \mathfrak{M}, \ t > 0, \) we have
\[ P(\tilde{J}_1 \in dx, \mathbb{B}^{(r)}_1 \in dM, (r)T_1 \in dt) = \frac{x}{t + x} r\tilde{\Lambda}(dx) P(\mathbb{B}^{(r+1)} \in dM, (r)T \in dt). \] (2.21)

(iii) For \( 0 < x < 1, \ t > 0, \) we have
\[ P(\tilde{J}_1 \in dx, (r)T \in dt) = \frac{x}{t + x} r\tilde{\Lambda}(dx) P((r+1)T \in dt). \] (2.22)

**Proof of Proposition 2.2:** (i) The definition of the size-biased picks implies
\[ P(\tilde{J}_1 \in dx \mid \mathbb{B}^{(r)} = M) = \frac{x}{T(M)} M(dx), \quad 0 < x < 1, \ M \in \mathfrak{M} \setminus \{\phi\}. \] (2.23)
Recall that $P_r(dM) = P(B^{(r)} \in dM)$. We use the following property of Palm distributions (see for instance Daley and Vere-Jones (1988, Sect. 12.1)):

$$r\tilde{\Lambda}(dx)P_r^{(x)}(dM) = M(dx)P_r(dM) = M(dx)P(B^{(r)}(x) \in dM)$$  \hspace{1cm} (2.24)

(noting that the first moment measure of $B^{(r)}$ is $r\tilde{\Lambda}(dx)$, by Prop. 2.1 (ii)). Write $P_{r+i}(dM) = P(B^{(r+i)} \in dM)$ for $i \in \mathbb{N}_0$ and $M \in \mathbb{M}$. Then, from (2.23) and (2.24),

$$P(\bar{J}_1 \in dx, B^{(r)} \in dM) = \frac{x}{T(M)}M(dx)P_r(dM) = \frac{x}{T(M)}r\tilde{\Lambda}(dx)P_{r+1}(dM_1).$$  \hspace{1cm} (2.25)

By (2.9), $P_r^{(x)}$ is the distribution of $\delta_x + \xi$ where $\xi$ is distributed as $BN(r + 1, \tilde{\Lambda})$. For each $x \in (0, 1)$, let $B^{(r,x-)} = B^{(r)} - \delta_x$. Changing variable to $M_1 = M - \delta_x$ in (2.25) gives

$$P(\bar{J}_1 \in dx, B^{(r,x-)} \in dM_1) = \frac{x}{T(M_1) + x}r\tilde{\Lambda}(dx)P_{r+1}(dM_1).$$  \hspace{1cm} (2.26)

Then noting that, jointly, $P(\bar{J}_1 \in dx, B^{(r)}_1 \in dM_1) = P(\bar{J}_1 \in dx, B^{(r,x-)} \in dM_1)$, we have proved (2.20).

(ii) $T_1 = T(B^{(r)}_1)$ is a deterministic transformation of $B^{(r)}_1$, so for each $y > 0$,

$$P(\bar{J}_1 \in dx, B^{(r)}_1 \in dM, T(B^{(r)}_1) \leq y) = P(\bar{J}_1 \in dx, B^{(r)}_1 \in dM, M \in Q_y) = \mathbf{1}_{\{M \in Q_y\}}P(\bar{J}_1 \in dx, B^{(r)}_1 \in dM),$$

where $Q_y := \{M \in \mathbb{M} : T(M) \leq y\}$. By (2.20) the last expression equals

$$\frac{x}{T(M) + x}r\tilde{\Lambda}(dx)P(B^{(r+1)} \in dM, T(B^{(r+1)}) \leq y),$$

from which (2.21) follows.

(iii) Integrating $M$ out of (2.21) and recalling (2.18) gives (2.22) via

$$P(\bar{J}_1 \in dx, T(B^{(r)}_1) \in dt) = \frac{x}{t + x}r\tilde{\Lambda}(dx)P(T(B^{(r+1)}) \in dt).$$

This completes the proof of Proposition 2.2.

We can now compute the joint density of the size-biased points of $BN(r, \tilde{\Lambda})$. Write the ascending factorial as $r^{(n)} = r(r + 1) \cdots (r + n - 1)$, $n \in \mathbb{N}$.

**Proposition 2.3.** Fix $r, n \in \mathbb{N}$. Given $x_i \in (0, 1), 1 \leq i \leq n, x_i \neq x_j$ for $i \neq j$, and $t > \sum_{i=1}^n x_i$, we have (interpreting $\sum_0^n = 0$)

$$P(\bar{J}_1 \in dx_1, \ldots, \bar{J}_n \in dx_n, (r)T \in dt) = r^{(n)}a^n \prod_{i=1}^n \frac{x_i^{-\alpha}dx_i}{t - \sum_{j=1}^{i-1} x_j}P((r+n)T \in d\left(t - \sum_{i=1}^n x_i\right)).$$  \hspace{1cm} (2.27)
Proof of Proposition 2.3: Given \(x_1, \ldots, x_n \in (0, 1), x_i \neq x_j\) for \(i \neq j\), and \(M \in \mathcal{M}\), write \(M_{i+1} = M_i - \delta_{x_{i+1}}\), with \(M_0 = M\) and \(i = 0, \ldots, n - 1\). We consider only the first two size-biased picks with \(x_1 \neq x_2\). The extension to general \(n\) is similar. Letting \(M_1 := M - \delta_{x_1}\), we compute

\[
P(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, T(\mathbb{B}^{(r)}) \in dt) = \int_{M \in \mathcal{M}} P(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}^{(r)} \in dM, T(\mathbb{B}^{(r)}) \in dt) = \int_{M_1 \in \mathcal{M}_1} P(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}^{(r,x_1-)} \in dM_1, T(\mathbb{B}^{(r,x_1-)}) \in d(t - x_1)) = \int_{M_1 \in \mathcal{M}_1} P(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2, \mathbb{B}^{(r)}_1 \in dM_1, T(\mathbb{B}^{(r)}_1) \in d(t - x_1)). \tag{2.28}
\]

The probability on the RHS of (2.28) can be replaced by

\[
P(\tilde{J}_2 \in dx_2 | \tilde{J}_1 = x_1, \mathbb{B}^{(r)}_1 = M_1, (r)T_1 = t - x_1) = \int_{M \in \mathcal{M}} P(\tilde{J}_1 \in dx_1, \mathbb{B}^{(r)}_1 \in dM_1, (r)T_1 \in d(t - x_1)) = \int_{M_1 \in \mathcal{M}_1} P(\tilde{J}_1 \in dx_1, \mathbb{B}^{(r)}_1 \in dM_1, (r)T_1 \in d(t - x_1)) = \frac{x_2}{t - x_1} M_1(dx_2)
\]

\[
\times \frac{x_1}{t} r \tilde{\Lambda}(dx_1) P(\mathbb{B}^{(r+1)} \in dM_1, T(\mathbb{B}^{(r+1)}) \in d(t - x_1)). \tag{2.29}
\]

The first equality in (2.29) comes from (2.21) and the fact that \(\tilde{J}_2\) is conditionally independent of \(\tilde{J}_1\) given \(\mathbb{B}^{(r)}_1\). In the last equality of (2.29), we used the definition of size-biased picks, as in (2.23). Using (2.29), the RHS of (2.28) equals

\[
\frac{x_2}{t - x_1} \int_{M_1 \in \mathcal{M}_1} M_1(dx_2) P(\mathbb{B}^{(r+1)} \in dM_1, T(\mathbb{B}^{(r+1)}) \in d(t - x_1)). \tag{2.30}
\]

When \(\xi\) is a point process with distribution \(P^{(r)}_{r+1}\), for each \(M \in \mathcal{M}\) and \(t > 0\), we abbreviate \(P(\xi \in dM, T(\xi) \in dt)\) to \(P^{(r)}_{r+1}(dM, dt)\). Recalling that \((r)T_1 = T(\mathbb{B}^{(r)}_1)\) is a deterministic function of \(\mathbb{B}^{(r)}_1\), then by (2.24),

\[
M_1(dx_2) P_{r+1}(dM_1, d(t - x_1)) = (r + 1) \tilde{\Lambda}(dx_2) P^{(x_2)}_{r+1}(dM_1, d(t - x_1)).
\]
Substituting this in (2.30), we obtain from (2.28) that
\[
\begin{align*}
P(\tilde{J}_1 \in dx_1, \tilde{J}_2 \in dx_2; T(B^{(r)}) & \in dt) \\
&= \frac{x_2}{t-x_1} \frac{x_1}{t} r \tilde{\Lambda}(dx_1) \int_{M_1 \in M_1} (r+1) \tilde{\Lambda}(dx_2) P_{r+1}^{(x_2)}(dM_1, d(t-x_1)) \\
&= \frac{x_2}{t-x_1} \frac{x_1}{t} r \tilde{\Lambda}(dx_1) (r+1) \tilde{\Lambda}(dx_2) \\
&\quad \times \int_{M_2 \in M_2} P(B^{(r+2)} \in dM_2, T(B^{(r+2)}) \in d(t-x_1-x_2)) \\
&= \prod_{i=1}^{2} \frac{(r+i-1)x_i \tilde{\Lambda}(dx_i)}{t - \sum_{j=1}^{i-1} x_j} P^{(r+2)}(T \in d(t-x_1-x_2)).
\end{align*}
\]

Here the second equality is obtained by changing variable to \( M_2 = M_1 - \delta_{x_2} \) as in (2.26), with \( M_2 := M_1 - \delta_{x_2} \). In the last equality, we note that \( P(B^{(r+2)} \in M_2) = 1 \), as \( B^{(r+2)} \) has a diffuse mean measure, hence
\[
P(B^{(r+2)}(\{x_1\}) > 0) = P(B^{(r+2)}(\{x_2\}) > 0) = 0.
\]

This proves (2.27) when \( n = 2 \). By a similar argument, we can show that for each \( n \in \mathbb{N} \), \( x_i \in (0,1) \), \( t > \sum_{i=1}^{n} x_i \),
\[
P(\tilde{J}_1 \in dx_1, \ldots, \tilde{J}_n \in dx_n; T(B^{(r)}) \in dt) \\
= \prod_{i=1}^{n} \frac{(r+i-1)x_i \tilde{\Lambda}(dx_i)}{t - \sum_{j=1}^{i-1} x_j} P^{(r+n)}(T \in d\left(t - \sum_{i=1}^{n} x_i\right)),
\]
and this is the same as (2.27). \( \square \)

Next we use (2.27) to derive the joint densities of the size-biased quantities in (2.15)–(2.17). Write \( \Theta(x) = \alpha x^{-\alpha} 1_{\{0 < x < 1\}} \) and recall \( (r)T_0 \equiv (r)T \).

\textbf{Proposition 2.4.} Fix \( r \in \mathbb{N} \).

(i) The joint density of \((r)T, (r)T_1, (r)T_2, \ldots, (r)T_n\) with respect to Lebesgue measure is, for \( t_0 > t_1 > \cdots > t_n > 0 \) and \( n \in \mathbb{N} \),
\[
f(t_0, t_1, \ldots, t_n) = r(n) g_{r+n}(t_n) \prod_{i=0}^{n-1} \frac{\Theta(t_i - t_{i+1})}{t_i}.
\] (2.31)

(ii) The sequence \((r)T, (r)T_1, (r)T_2, \ldots\) is a (non-homogeneous) Markov Chain with transition density, for \( t_n > t_{n+1} > 0 \) and \( n \in \mathbb{N}_0 \),
\[
P((r)T_{n+1} \in dt_{n+1} \mid (r)T_n = t_n) = (r+n) \frac{\Theta(t_n - t_{n+1}) g_{r+n+1}(t_{n+1})}{g_{r+n}(t_n)} dt_{n+1}.
\] (2.32)

(iii) The joint density of \((r)T_n, (r)U_1, (r)U_2, \ldots, (r)U_n\) is, for \( t_n > 0, 0 < u_i < 1, 1 \leq i \leq n, \) and \( n \in \mathbb{N} \),
\[
h(t_n, u_1, \ldots, u_n) = \frac{r(n) g_{r+n}(t_n) t_n^{-n\alpha}}{K_n} \\
\quad \times \prod_{i=1}^{n} \frac{\Gamma(i\alpha + 1 - \alpha)}{\Gamma(i\alpha) \Gamma(1 - \alpha)} u_i^{\alpha - 1} u_i^{-\alpha} 1_{\{t_n < \prod_{j=i}^{n} u_j / \bar{u}_i\}},
\] (2.33)
where \( \bar{u}_i = 1 - u_i \), and

\[
K_n = \frac{\prod_{i=0}^{n-1} \Gamma(1 + i\alpha)}{\alpha^n \Gamma(n - 1) \prod_{i=1}^{n} \Gamma(i\alpha)} = \frac{\Gamma(n + 1)}{(n - 1)! \Gamma(n\alpha + 1)}. \tag{2.34}
\]

**Remark 2.5.** A routine calculation shows that

\[
\frac{1}{\Gamma(n\alpha)} \int_0^\infty t^{n\alpha - 1} \left(1 + \int_0^\infty (1 - e^{-tx}) \alpha x^{-\alpha - 1} dx\right)^{r - n} dt = \frac{K_n}{\gamma(n)} \tag{2.35}
\]

Notice that the inner integration in (2.35) is over \( x \in [0, \infty) \), whereas that in (2.19) is over \( x \in (0, 1) \), and our \( \Theta(x) \) is restricted to \( (0, 1) \), whereas that of PPY (1992, Eq. 2.21) is not. This is a reflection of the truncation induced by eliminating the large points. Still, the \( K_n \) in (2.34) and (2.35) exactly equals the \( K_n \) in Eq. (2.21) of PPY (1992), when the stable scaling constant \( c \) in their Eq. (2.21) is set equal to \( \Gamma(1 - \alpha) \). In both notations, \( K_n = E(S_1^{-\alpha n}) \) (see also Eq. (30) of PY (1997)).

In general, we have the following relation:

\[
K_n = r^n \int_{u_1=0}^1 \cdots \int_{u_n=0}^1 \int_{t_n=0}^1 d(u_1, \ldots, u_n) t_n^{r-n} g_{r+n}(t_n) dt_n \]

\[
\times \prod_{i=1}^{n} f_{B_{a_i-\alpha}}(u_i) du_1 \cdots du_n, \tag{2.36}
\]

where \( d(u_1, \ldots, u_n) := \min_{1 \leq i \leq n} \prod_{j=1}^{n} u_j/\bar{u}_i \) for \( 0 < u_i < 1, 1 \leq i \leq n, n \in \mathbb{N} \), and \( f_{B_{a,b}} \) is the density of a Beta\((a, b)\) distribution as in (2.6).

**Proof of Proposition 2.4:** (i): By change of variable in (2.27), we have

\[
P(\{^rT\} \in dt_0, \{^rT_1\} \in dt_1, \{^rT_2\} \in dt_2, \ldots, \{^rT_n\} \in dt_n) = P(\{^rT\} \in dt_0, \tilde{J}_1 \in d(t_0 - t_1), \tilde{J}_2 \in d(t_1 - t_2), \ldots, \tilde{J}_n \in d(t_{n-1} - t_n))
\]

\[
= r^n \prod_{i=1}^{n-1} \frac{\Theta(t_i - t_{i+1})}{t_i} g_{r+n}(t_n) dt_0 dt_1 \cdots dt_n.
\]

This proves (2.31). Part (ii) follows immediately from Part (i):

\[
P(\{^rT_{n+1}\} \in dt_{n+1} | \{^rT\} = t_0, \{^rT_1\} = t_1, \{^rT_2\} = t_2, \ldots, \{^rT_n\} = t_n)
\]

\[
= (r + n) \frac{\Theta(t_n - t_{n+1})}{t_n} g_{r+n+1}(t_{n+1}) g_{r+n}(t_n) dt_{n+1},
\]

which does not depend on \( t_0, t_1, \ldots, t_{n-1} \). Thus (2.32) is established.

(iii) To show (2.33), we first consider the case \( n = 2 \). Note that

\[
h(t_2, u_1, u_2) = f \left( \frac{t_2}{u_1 u_2}, \frac{t_2}{u_2}, t_2 \right) t_2^2 u_1^{-2} u_2^{-3};
\]

where \( f \) is defined in (2.31) and \( t_2^2 u_1^{-2} u_2^{-3} \) is the Jacobian from the change of variables. Expanding the expression in (2.31) with \( \Theta(x) = ax^{-\alpha}1_{\{0 < x < 1\}} \), we get
\[ h(t_2, u_1, u_2) \text{ equal to} \]
\[
r^{(2)} g_{r+2}(t_2) u_1^{-1} u_2^{-1} \Theta \left( \frac{t_2}{u_1 u_2} \right) \Theta \left( \frac{t_2}{u_2} \right) \\
= r^{(2)} g_{r+2}(t_2) t_2^{-2\alpha} \left( u_2^2 \bar{v}_2^{-\alpha} \right) \left( u_1 u_2 \right) \mathbf{1}_{\{t_2 u_2 / u_2 < 1\}} \mathbf{1}_{\{t_2 u_1 / (u_1 u_2) < 1\}} \\
= \frac{r^{(2)}}{K_2} g_{r+2}(t_2) t_2^{-2\alpha} \left[ \frac{\Gamma(1+\alpha)}{\Gamma(2\alpha) \Gamma(1-\alpha) u_2^{-1} \bar{v}_2^{-\alpha}} \right] \\
\times \left[ \frac{\Gamma(1)}{\Gamma(\alpha) \Gamma(1-\alpha) u_1^{\alpha-1} \bar{v}_1^{-\alpha}} \right] \mathbf{1}_{\{t_2 < u_2 / \bar{u}_2\}} \mathbf{1}_{\{t_2 < u_1 u_2 / \bar{u}_1\}},
\]

where
\[ K_2 = \frac{\prod_{i=0}^{1} \Gamma(1+i\alpha)}{\alpha^2 \Gamma^2(1-\alpha) \prod_{i=1}^{2} \Gamma(i\alpha)}. \]

This formula can be generalised to \(n \geq 2\) similarly, and (2.33) follows.

To complete this section our final theorem gives formulae for the distributions of the size-biased sequence constructed from \(\text{PD}^{(r)}_n\) as defined in (2.21), as well as for the residual fractions defined in (2.16)–(2.17).

**Theorem 2.1.** (i) For each \(r \in \mathbb{N}\) let \((V^{(r)}_n)_{n \in \mathbb{N}}\) have a \(\text{PD}^{(r)}_n\) distribution as defined in (2.21), with corresponding size-biased sequence \((\tilde{V}^{(r)}_n)\). Then for each \(n \in \mathbb{N}\) the joint density of \((\tilde{V}^{(r)}_1, \ldots, \tilde{V}^{(r)}_n, (r) T)\) with respect to Lebesgue measure is

\[
p_{(r)}(v_1, \ldots, v_n, t) = r^{(n)} \alpha^n \prod_{i=1}^{n} v_i^{-\alpha} 1_{\{v_i < 1\}} \times t^{-na} g_{r+n}(t \bar{v}_n), \quad (2.37)
\]

where \(t > 0, 0 < v_i < 1\) are such that \(\sum_{i=1}^{n} v_i < 1, \bar{v}_0 \equiv 1\), and, for each \(i \geq 1\),

\(\bar{v}_i = 1 - v_1 - \cdots - v_i\).

(ii) The joint distribution of \((r) T_n, \ldots, (r) U_n\) can be written as

\[
(r) T_n, (r) U_1, (r) U_2, \ldots, (r) U_n \overset{D}{=} (Y_d(U_1, \ldots, U_n), U_1, U_2, \ldots, U_n), \quad (2.38)
\]

where the \((U_i)\) are independent \(\text{Beta}(\alpha, 1-\alpha)\) rv's, independent of \(\mathbb{B}^{(r)}\), the function \(d(u_1, \ldots, u_n) := \min_{1 \leq i \leq n} \prod_{j=1}^{n} u_j / \bar{u}_i\) and, for each \(c > 0\), \(Y_c \overset{D}{=} (r+n)T^{-na} 1_{\{(r+n)T < c\}}\).

**Proof of Theorem 2.1.** (i) Identify the size-biased \(\tilde{V}^{(r)}_i\) with the points \(\tilde{J}_r(i)\) in (2.13) normalised by their sum \(T(\mathbb{B}^{(r)})\). Then change variable in (2.27) to \(v_i = x_i / t\) and substitute for \(\bar{\Lambda}\) to get

\[
P(\tilde{V}^{(r)}_1 \in dv_1, \ldots, \tilde{V}^{(r)}_n \in dv_n, T(\mathbb{B}^{(r)}) \in dt) \\
= P(\tilde{J}_r(1) \in tdv_1, \ldots, \tilde{J}_r(n) \in tdv_n, (r) T \in dt) \\
= r^{(n)} \alpha^n \prod_{i=1}^{n} t^{-\alpha} v_i^{-\alpha} dv_i \\
\times P\left( (r+n)T \in d\left( t - \sum_{i=1}^{n} x_i \right) \right) \quad \text{(by (2.27))}
\]

\[
= r^{(n)} \alpha^n t^{-na} \prod_{i=1}^{n} v_i^{-\alpha} dv_i P((r+n)T \in dt \bar{v}_n),
\]

\[ a = 1 + \alpha, \quad b = a + 1, \quad c = b + 1, \quad \Gamma(a) = \frac{1}{a}, \quad \Gamma(a + 1) = \frac{1}{a} \]
Recall that \( g_{r+n}(\cdot) \) is the density of \((r+n)T\) (see (2.38)), to complete the proof of (2.37).

(ii) Representation (2.38) is immediate from Proposition 2.4 and (2.36), with the vector on the RHS of (2.38) having the structure specified. \(\square\)

Remark 2.6. (i) There are some quite involved manipulations in obtaining the above formulae. As a check on the calculations, in the Appendix of [arxiv:1611.09980] we give direct verifications that (2.31) and (2.37) are probability densities (integrate to 1).

(ii) For a stick-breaking representation, solve (2.16) and (2.17) to get

\[
\tilde{V}_n^{(r)} = (1 - (r) U_n) \prod_{i=1}^{n-1} (r) U_i. \tag{2.39}
\]

The joint distribution of \((r) U_i)_{1 \leq i \leq n}\) can be computed from (2.38), in which we note that \(U_1, U_2, \ldots, U_n\) are individually independent but dependence overall is introduced via the connection with the \(Y\) term. In this respect the result is different from the \(\text{PD}_\alpha\) situation, as we would expect, but the distribution of \(\tilde{V}_n^{(r)}\) as given by (2.39) is sufficiently explicit to enable computations or simulations.

(iii) (2.37) generalises the corresponding version for \(\text{PD}_\alpha = \text{PD}(\alpha, 0)\) in PY (1997, Prop. 47).

(iv) When we sample from a Poisson process, the various quantities in Proposition 2.4 are computed in PPY (1992, Theorem 2.1).

(v) Although motivated by the idea of trimming an integer number \(r\) of large jumps, our formulae once derived are valid for \(r > 0\), and available for modelling purposes in this generality.

To conclude this section we expand briefly on the differences as well as the similarities between the \(\text{PD}_\alpha\) and \(\text{PD}_\alpha^{(r)}\) approaches. In both cases, start with a stable (\(\alpha\)) subordinator \(S\) with ranked jumps \(\Delta S \geq \Delta S_1 \geq \cdots\). The sequence \((\Delta S_i / S_1)_{i \geq 1}\) then has a \(\text{PD}_\alpha\) distribution. We can think of these as the points from a Poisson point process with intensity measure \(\Lambda(dx) = \alpha x^{-\alpha-1}dx\), normalised by their sum. For \(\text{PD}_\alpha^{(r)}\), the analogous process is the negative binomial point process \(\mathcal{B}N(r, \Lambda)\) formed from ratios of jumps rather than from the jumps themselves, i.e.,

\[
\mathbb{B}^{(r)} = \sum_{i \geq 1} \delta_{J_r(i)}, \quad \text{with} \quad J_r(i) = \frac{\Delta S_{i+1}^{(r)}}{\Delta S_i^{(r)}}, \quad i \in \mathbb{N}.
\]

The normalised jumps on which a size-biased version is based are

\[
\frac{J_r(i)}{\sum_{j \geq 1} J_r(j)} = \frac{\Delta S_{i+1}^{(r)}}{\Delta S_i^{(r)}} / \frac{(r) S_1}{\Delta S_i^{(r)} S_1}, \quad i \in \mathbb{N}, \tag{2.40}
\]

and the sequence formed from these has a \(\text{PD}_\alpha^{(r)}\) distribution, as we define it, on the infinite simplex.
We may set \( r = 0 \) in (2.4) to have the distribution of \( (V_n^{(r)})_{n \in \mathbb{N}} \), that is, \( \text{PD}_\alpha^{(r)} \), reduce to that of \( (V_n)_{n \in \mathbb{N}} \), that is, \( \text{PD}_\alpha \). But we cannot take \( r = 0 \) in (2.40) with the idea that the size-biased distribution associated with \( \text{PD}_\alpha^{(r)} \) might then reduce to the one associated with \( \text{PD}_\alpha \). Our analysis proceeds via the process \( \mathbb{B}^{(r)} \), which is not defined for \( r = 0 \) (its points \( J_r(i) \) are not defined for \( r = 0 \)). Setting \( r = 0 \) in formulae such as (2.31), (2.33), (2.37), etc., which result from an analysis of \( \mathbb{B}^{(r)} \), is not permissible.

3 Discussion and Applications

We mention two applications papers which vividly illustrate the possibilities for useful and revealing application of our results. Sosnovskiy (2015) shows “capital distribution curves” (CDCs) for over 20 countries listed on the NASDAQ stock exchange. The CDC is a log plot of normalized stock capitalizations ranked in descending order, against their log-ranks. The curves display remarkable stability over periods of time and are very well fitted by a \( \text{PD}_\alpha \) distribution over much of their range. But a glance at Sosnovskiy’s Figure 3, for example, shows that an even better fit would result from discarding a small number of the largest stocks – suggesting a \( \text{PD}_\alpha^{(r)} \) distribution. We might indeed expect that a small number of very large stocks would show aberrant behaviour, compared to the majority.

A very similar situation occurs with the Zipf plots (log frequencies of words in the Penn Wall St. journal, against their log-ranks), in the paper of Goldwater et al. (2011). Half a dozen or so of the most frequent words appear as outliers, while the rest conform closely to a \( \text{PD}_\alpha \) fit (see their Figure 4).

In general, we can expect that our generalised \( \text{PD}_\alpha^{(r)} \) distribution could be used to robustify analyses and reveal interesting features in this kind of data.

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4 Appendix: (2.31) and (2.37) integrate to 1

As a check on the calculations, we give here a direct verification that (2.31) and (2.37) integrate to 1 in the case \( n = 1 \). An extension to larger \( n \) is straightforward.

Eq. (2.31) gives, for \( n = 1, \ r \in \mathbb{N}, \ t_0 > t_1 > 0 \),

\[
f(t_0, t_1) = r g_{r+1}(t_1) \frac{\Theta(t_0 - t_1)}{t_0} = r \alpha g_{r+1}(t_1) \frac{(t_0 - t_1)^{-\alpha} 1_{\{t_0-t_1<1\}}}{t_0}
\]

Notice that

\[
\int_{t_0=t_1}^{1+t_1} \frac{(t_0 - t_1)^{-\alpha}}{t_0} \ dt_0 = \int_0^1 \frac{t^{-\alpha}}{t + t_1} \ dt
\]

so

\[
\int_{t_1=0}^{\infty} \int_{t_0=t_1}^{1+t_1} f(t_0, t_1) \ dt_1 \ dt_0 = r \alpha \int_{t_1=0}^{\infty} g_{r+1}(t_1) \int_{t=0}^{1} \frac{t^{-\alpha}}{t + t_1} \ dt
\]
Introduce an integral over $\lambda$ and then substitute from (2.18) to write the last expression as

$$r\alpha \int_{t=0}^{1} t^{-\alpha} dt \int_{t_1=0}^{\infty} \frac{g_{r+1}(t_1)}{t + t_1} dt_1. \quad (4.1)$$

The last equation used (2.19). The final integral can be evaluated as

$$r \int_{\lambda=0}^{\infty} \frac{\alpha \int_{t=0}^{1} t^{-\alpha} e^{-\lambda t} dt \lambda}{\left(1 + \int_{0}^{1} (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{r+1}} = r \left(1 + \int_{0}^{1} (1 - e^{-\lambda x}) \tilde{\Lambda}(dx)\right)^{-r} \bigg|_{0}^{\infty} = 1.$$

Next we give a direct verification that (2.37) integrates to 1 in the case $n = 1$.

$$r \int_{v_1=0}^{1} \int_{x=0}^{1/v_1} \frac{v_1^{-\alpha}}{1 + \frac{x}{v_1}} \int_{t=0}^{1/v_1} t^{-\alpha} g_{r+1}(t(1 - v_1)) dt = r \int_{t=0}^{\infty} \frac{g_{r+1}(t) dt}{1 + ty} \int_{y=0}^{\infty} \frac{y^{\alpha-1}}{1 + y} dy$$

The last is the RHS of (4.1) which equals 1.

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