The surface of a lattice polytope

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Abstract

My main results are simple formulas for the surface area of \(d\)-dimensional lattice polytopes using Ehrhart theory.

1 Introduction

Throughout the paper a lattice polytope \(P \subseteq \mathbb{R}^d\) is a polytope whose vertices have integral coordinates.

Let \(S \subseteq \mathbb{R}^d\) be a subset of the Euclidean space \(\mathbb{R}^d\). Let \(G(S)\) denote the lattice point enumerator of the set \(S\), the number of lattice (integral) points in \(S\), i.e., \(G(S) = |(S \cap \mathbb{Z}^d)|\).

Let \(P\) denote an arbitrary \(d\)-dimensional lattice polytope. In the following we denote by

\[ \nu P := \{ \nu \mathbf{x} : \mathbf{x} \in P \} \]

the dilatate of \(P\) by the integer factor \(\nu \geq 0\).

In 1962 E. Ehrhart proved (see e.g. [1, Chapter 3, Chapter 5], [6]) the following Theorem:

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Theorem 1.1 Let \( \mathcal{P} \) be a convex \( d \)-dimensional lattice polytope in the Euclidean space \( \mathbb{R}^d \). Then there exists a unique polynomial (the Ehrhart polynomial)

\[
E_{\mathcal{P}}(x) := \sum_{i=0}^{d} e_i(\mathcal{P}) x^i \in \mathbb{Q}[x],
\]

which has the following properties:

1. For all integers \( \nu \geq 0 \),

\[
E_{\mathcal{P}}(\nu) = |(\nu \mathcal{P}) \cap \mathbb{Z}^d|.
\]

2. The leading coefficient \( e_d(\mathcal{P}) \) of \( E_{\mathcal{P}}(x) \) is \( \text{vol}(\mathcal{P}) \), the volume of \( \mathcal{P} \).

3. If \( \text{int}(\mathcal{P}) \) denotes the interior of \( \mathcal{P} \), then the reciprocity law states that for all integers \( \nu > 0 \),

\[
E_{\mathcal{P}}(-\nu) = (-1)^d |(\nu \cdot \text{int}(\mathcal{P})) \cap \mathbb{Z}^d|.
\]

4. The second leading coefficient \( e_{d-1}(\mathcal{P}) \) of \( E_{\mathcal{P}}(x) \) is the half of the lattice surface area of \( \mathcal{P} \):

\[
e_{d-1}(\mathcal{P}) = \frac{1}{2} \sum_{F \text{ facet of } \mathcal{P}} \frac{\text{vol}_{d-1}(F)}{\text{det}(\text{aff } F \cap \mathbb{Z}^d)}.
\]

Here \( \text{vol}_{d-1}(\cdot) \) denotes the \( (d-1) \)-dimensional volume and \( \text{det}(\text{aff } F \cap \mathbb{Z}^d) \) denotes the determinant of the \( (d-1) \)-dimensional sublattice contained in the affine hull of \( F \).

5. The constant coefficient \( e_0(\mathcal{P}) \) of \( E_{\mathcal{P}}(x) \) is 1.

Let \( \mathcal{P} \) be a convex \( d \)-dimensional lattice polytope, which contains the origin of the lattice in its interior. We say that \( \mathcal{P} \) is reflexive if the dual polytope \( \mathcal{P}^* \) is a lattice polytope, where the dual polytope of \( \mathcal{P} \) is defined as

\[
\mathcal{P}^* := \{ y \in \mathbb{R}^d : \langle x, y \rangle \geq -1 \text{ for all } x \in \mathcal{P} \}.
\]

A. M. Kasprzyk proved in \cite[Proposition 3.9.2]{Kasprzyk} the following equivalent characterization of reflexive Fano polytopes:

Proposition 1.2 Let \( \mathcal{P} \) be a \( d \)-dimensional Fano polytope. Then \( \mathcal{P} \) is reflexive iff

\[
\text{vol}(\mathcal{P}) = \frac{\text{surf} \mathcal{P}}{d}.
\]
In 1899 G. A. Pick published his famous formula in [11]. Using this formula we can compute easily the area of a lattice polygon. Pick showed that the following expression gives the area of a simple lattice polygon $Q$:

$$\text{Area}(Q) = I + \frac{B}{2} - 1,$$

where $B$ is the number of lattice points on the boundary of $Q$ and $I$ is the number of lattice points in the interior of $Q$.

This formula can be derived easily from Ehrhart Theorem 1.1 (see e.g. [4, Chapter 4]).

My main results are similar simple formulas for the surface area of 3–dimensional and 4–dimensional lattice polytopes using Ehrhart theory.

2 The main results

Let $\mathcal{P}$ be a convex $d$–dimensional lattice polytope in $\mathbb{R}^d$. Denote by $\text{surf}(\mathcal{P})$ the lattice surface area of $\mathcal{P}$:

$$\text{surf}(\mathcal{P}) := \sum_{F \text{ facet of } \mathcal{P}} \frac{\text{vol}_{d-1}(F)}{\det(\text{aff } F \cap \mathbb{Z}^d)}.$$

Let $i(\mathcal{P})$ and $b(\mathcal{P})$ denote the numbers $|\text{int}(\mathcal{P}) \cap \mathbb{Z}^d|$ and $|\partial(\mathcal{P}) \cap \mathbb{Z}^d|$, respectively.

Here $\partial(\mathcal{P})$ denotes the boundary of the polytope $\mathcal{P}$.

**Theorem 2.1** Let $\mathcal{P}$ be a convex $d$–dimensional lattice polytope in $\mathbb{R}^d$.

Suppose that $d$ is an odd number. Let $t := \frac{d-1}{2}$. Then define the matrix

$$A(\mathcal{P}, d) := \begin{pmatrix} b(\mathcal{P}) - 2 & 1^{d-3} & \cdots & 1^2 \\ b(2\mathcal{P}) - 2 & 2^{d-3} & \cdots & 2^2 \\ \vdots & \vdots & \ddots & \vdots \\ b(t\mathcal{P}) - 2 & t^{d-3} & \vdots & t^2 \end{pmatrix}$$

and

$$D(\mathcal{P}, d) := \begin{pmatrix} 1^{d-1} & 1^{d-3} & \cdots & 1^2 \\ \vdots & \vdots & \ddots & \vdots \\ t^{d-1} & t^{d-3} & \cdots & t^2 \end{pmatrix}$$
Then
\[
surf(P) = \frac{\det(A(P, d))}{\det(D(P, d))}
\] (3)

Suppose that \(d\) is an even number.
Let \(t := \frac{d}{2}\). Then define the matrix
\[
B(P, d) := \begin{pmatrix}
b(P) & 1^{d-3} & \cdots & 1 \\
b(2P) & 2^{d-3} & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
b(tP) & t^{d-3} & \cdots & t
\end{pmatrix}
\]
and
\[
D(P, d) := \begin{pmatrix}
1^{d-1} & 1^{d-3} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
t^{d-1} & t^{d-3} & \cdots & t
\end{pmatrix}
\]

Then
\[
surf(P) = \frac{\det(B(P, d))}{\det(D(P, d))}
\] (4)

**Proof.** Let
\[
E_P(x) := \sum_{i=0}^{d} e_i(P)x^i \in \mathbb{Q}[x],
\]
denote the Ehrhart polynomial of the polytope \(P\).

First suppose that \(d\) is an odd number. Let \(0 \leq k \leq \frac{d-1}{2}\). Then using Theorem 1.1
\[
i(kP) + b(kP) = L_P(k) = e_d(P)k^d + e_{d-1}(P)k^{d-1} + \ldots + 1
\] (5)

and
\[-i(kP) = -L_P(-k) = -e_d(P)k^d + e_{d-1}(P)k^{d-1} - \ldots + 1
\] (6)

Suming (5) and (6) we get that
\[
b(kP) = 2e_{d-1}(P)k^{d-1} + 2e_{d-3}(P)k^{d-3} + \ldots + 2
\]
i.e.,
\[
b(kP) - 2 = 2e_{d-1}(P)k^{d-1} + 2e_{d-3}(P)k^{d-3} + \ldots e_2(P)k^2
\]

for each $0 \leq k \leq \frac{d-1}{2}$. Solving this linear equation system using Cramer’s rule we get that
\[
e_{d-1}(P) = \frac{\det(A(P, d))}{2\det(D(P, d))}.
\]
But using Theorem 1.1 (4) we get that
\[
surf(P) = \frac{e_{d-1}(P)}{2},
\]
and we get our result for odd $d$.

Suppose that $d$ is an even number. Let $t := \frac{d}{2}$. Let $0 \leq k \leq \frac{d}{2}$. Using Theorem 1.1
\[
i(kP) + b(kP) = L_P(k) = e_d(P)k^d + e_{d-1}(P)k^{d-1} + \ldots + 1
\]
and
\[
i(kP) = L_P(-k) = e_d(P)k^d - e_{d-1}(P)k^{d-1} + \ldots + 1
\]
Subtracting (8) from (7) we get that
\[
b(kP) = 2e_{d-1}(P)k^{d-1} + 2e_{d-3}(P)k^{d-1} + \ldots + 2e_1(P)k
\]
for each $0 \leq k \leq \frac{d}{2}$. We can again solve this linear equation system using Cramer’s rule, hence
\[
e_{d-1}(P) = \frac{\det(B(P, d))}{2\det(D(P, d))}.
\]
Theorem 1.1 (4) implies that
\[
surf(P) = \frac{e_{d-1}(P)}{2},
\]
and we get our result from (10) and (11).

Examples.

If $d = 3$, then $surf(P) = b(P) - 2$.

In [9, Proposition 10.3.2] A. M. Kasprzyk proved this formula from Pick’s Theorem.
If \( d = 4 \), then
\[
\text{surf}(\mathcal{P}) = \frac{b(2\mathcal{P}) - 2b(\mathcal{P})}{6}.
\]
If \( d = 5 \), then
\[
\text{surf}(\mathcal{P}) = \frac{b(2\mathcal{P}) - 4b(\mathcal{P}) - 6}{12}.
\]

**Remark.** A. M. Kasprzyk called my attention for the following consequence of Theorem 2.1.

**Corollary 2.2** Let \( \mathcal{P} \) be a convex \( d \)-dimensional Fano lattice polytope in \( \mathbb{R}^d \).

Suppose that \( d \) is an odd number. Let \( t := \frac{d-1}{2} \). Then define the matrix

\[
\mathbf{A}(\mathcal{P}, d) := \begin{pmatrix}
b(P) - 2 & 1^{d-3} & \cdots & 1^2 \\
b(2P) - 2 & 2^{d-3} & \cdots & 2^2 \\
\vdots & \vdots & \ddots & \vdots \\
b(tP) - 2 & t^{d-3} & \cdots & t^2
\end{pmatrix}
\]

and

\[
\mathbf{D}(\mathcal{P}, d) := \begin{pmatrix}
1^{d-1} & 1^{d-3} & \cdots & 1^2 \\
\vdots & \vdots & \ddots & \vdots \\
t^{d-1} & t^{d-3} & \cdots & t^2
\end{pmatrix}
\]

Then
\[
\text{vol}(\mathcal{P}) = \frac{\det(\mathbf{A}(\mathcal{P}, d))}{d \cdot \det(\mathbf{D}(\mathcal{P}, d))} \quad (12)
\]

Suppose that \( d \) is an even number.

Let \( t := \frac{d}{2} \). Then define the matrix

\[
\mathbf{B}(\mathcal{P}, d) := \begin{pmatrix}
b(P) & 1^{d-3} & \cdots & 1 \\
b(2P) & 2^{d-3} & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
b(tP) & t^{d-3} & \cdots & t
\end{pmatrix}
\]

and

\[
\mathbf{D}(\mathcal{P}, d) := \begin{pmatrix}
1^{d-1} & 1^{d-3} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
t^{d-1} & t^{d-3} & \cdots & t
\end{pmatrix}
\]
Then \( \mathcal{P} \) is a reflexive polytope iff

\[
\text{vol}(\mathcal{P}) = \frac{\det(B(\mathcal{P}, d))}{d \cdot \det(D(\mathcal{P}, d))},
\]

(13)

**Proof.**

Corollary 2.2 is the obvious consequence of Theorem 2.1 and Proposition 1.2.

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**References**

[1] M. Beck, S. Robins, Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra, Springer, 2007

[2] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, R.P. Stanley, Coefficients and roots of Ehrhart polynomials. *Integer points in polyhedra—geometry, number theory, algebra, optimization*, 15–36, *Contemp. Math.*, 374, Amer. Math. Soc., Providence, RI, 2005.

[3] H. S. M. Coxeter, Introduction to geometry. Reprint of the 1969 edition. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989

[4] D. A. Cox, Lectures on Toric Varieties, lecture notes available online on \link{http://www3.amherst.edu/~dacox}{http://www3.amherst.edu/~dacox}

[5] D. DeTemple, J. M. Robertson, The equivalence of Euler’s and Pick’s theorems. *Math. Teacher* 67 (1974), no. 3, 222–226.

[6] E. Ehrhart, Sur les polyèdres rationnels homothétiques à \( n \) dimensions. *C. R. Acad. Sci. Paris* 254 (1962) 616–618

[7] R. W. Gaskell, M. S. Klamkin, P. Watson, Triangulations and Pick’s theorem. *Math. Mag.* 49 (1976), no. 1, 35–37.
[8] C. Haase, I. V. Melnikov, The reflexive dimension of a lattice polytope. Ann. Comb. 10 (2006), no. 2, 211–217

[9] A. M. Kasprzyk, Toric Fano Varieties and Convex Polytopes, PhD Thesis, 2006, available online on [http://magma.maths.usyd.edu.au/users/kasprzyk/research/pdf/Thesis.pdf](http://magma.maths.usyd.edu.au/users/kasprzyk/research/pdf/Thesis.pdf)

[10] A. C. F. Liu, Lattice points and Pick’s theorem. Math. Mag. 52 (1979), no. 4, 232–235.

[11] G. Pick, Geometrisches zur Zahlenlehre, Sitzungber Lotos (Prague) 19 (1899), 311–319

[12] Rosenholtz, Ira; Calculating Surface Areas from a Blueprint. Math. Mag. 52 (1979), no. 4, 252–256.

[13] P. R. Scott, The fascination of the elementary. Amer. Math. Monthly 94 (1987), no. 8, 759–768