On a method of introducing free-infinitely divisible probability measures

Zbigniew J. Jurek (University of Wroclaw)

December 14, 2014

Abstract. Random integral mappings \( I_{h,r}^{a,b} \) give isomorphisms between the sub-semigroups of the classical \((\text{ID}, \ast)\) and the free-infinite divisible \((\text{ID}, \boxplus)\) probability measures. This allows us to introduce new examples of such measures and their corresponding characteristic functionals.

Mathematics Subject Classifications (2010): Primary 60E07, 60H05, 60B11; Secondary 44A05, 60H05, 60B10.

Key words and phrases: Classical infinite divisibility; free infinite divisibility Lévy-Khintchine formula; Nevanlinna-Pick formula; characteristic (Fourier) functional; Voiculescu transform.

Abbreviated title: Free-infinitely divisible measures

In this paper we prove Fourier type characterizations for new classes of limiting distributions (subsets of infinitely divisible laws) in the free probability theory. One of them (Proposition 1) is the free analog of the class \( \mathcal{U} \). The class \( \mathcal{U} \) was defined as the class of weak limits for sequences of the form:

\[
U_{r_n}(X_1) + U_{r_n}(X_2) + ... + U_{r_n}(X_n) + x_n \Rightarrow \mu, \tag{\ast}
\]

where random variables \((X_n)\) are stochastically independent, the above summands are infinitesimal, \((x_n)\) are real numbers and the non-linear shrinking deformations \( U_r, r > 0 \), are defined as follows;

\[
U_r(0) := 0, \quad U_r(x) := \max\{|x| - r, 0\} \frac{x}{|x|}, \quad \text{for} \quad x \neq 0.
\]

*Research funded by Narodowe Centrum Nauki (NCN), grant no Dec2011/01/B/ST1/01257.*
Probability measures $\mu$ in (*) are called $s$-selfdecomposable ("$s$", because of the shrinking operators $U_r$.) They were introduced for measures on Hilbert spaces in Jurek (1977) and in Jurek (1981). Later on studied in Jurek (1984) and (1985). More recently, in Bradley and Jurek (2014), the Gaussian limit in (*) was proved in a case when the independence was replaced by some strong mixing conditions. On other hand, in Arzimendi and Hasebe (2014), Section 5, measures in class $\mathcal{U}$ have studied via the unimodality property of their Lévy spectral measures.

Replacing in (*) the $U'_r s$ by the linear dilations $T_r (x) = rx$ we get the Lévy class $L$ of so called selfdecomposable distributions. In particular, we obtain stable distributions, when $X'_i$’s are also identically distributed.

For purposes in this paper we need the descriptions of the classes $L$ and $\mathcal{U}$ in terms the random integral representations like the ones in (4); cf. Jurek and Vervaat (1983) and Theorem 2.1 in Jurek (1984), respectively.

0. The isomorphism. Traditionally, let $\phi_\mu$ denotes the Fourier transform (the characteristic function) of a probability measure $\mu$ and let $V_\nu$ denotes the Voiculescu transform of a probability measure $\nu$ (the definition is given in the subsection 1.2. below). Then for a class $\mathcal{C}$ of classical $\ast$ - infinitely divisible probability measures we define its free $\boxplus$ -infinitely divisible counterpart $\mathcal{C}$ as follows:

$$\mathcal{C} = \{ \nu : V_\nu (it) = it^2 \int_0^\infty \log \phi_\mu (-v) e^{-tv} dv, \ t > 0; \ \text{for some } \mu \in \mathcal{C} \} \ (1)$$

Conversely, for a class $\mathcal{C}$ of $\boxplus$ -infinitely divisible measures we define its classical $\ast$ - infinitely divisible counterpart $\mathcal{C}$ as follows:

$$\mathcal{C} = \{ \mu : it^2 \int_0^\infty \log \phi_\mu (-v) e^{-tv} dv = V_\nu (t), \ t > 0; \ \text{for some } \nu \in \mathcal{C} \} \ (2)$$

It is notably that above, and later on, we consider $V_\nu$ (and Cauchy transforms) only on the imaginary axis. Still, it is sufficient to perform the explicit inverse procedures; cf. Section 1.3. below.

We illustrate the relation between classes $\mathcal{C}$ and $\mathcal{C}$ (in fact, an isomorphism) via examples and will prove among others that :

$v$ is $\boxplus$ $s$-selfdecomposable if and only if for $t > 0$

$$V_\nu (it) = a^2 + \frac{\sigma^2}{3} \frac{1}{it} + \int_{\mathbb{R} \setminus \{0\}} \left( \frac{(it)^2 \frac{\log (it - x) - \log (it))}{x} - it \frac{1}{2} x_1 B(x) \right) M(dx)$$

for some constants $a \in \mathbb{R}, \sigma^2 \geq 0$ (variance) and a Lévy spectral measure $M$. The parameters $a, \sigma^2$ and the measure $M$ correspond to $\mu = [a, \sigma^2, M]$ in (1) and (2); for other details cf. Proposition 1.
The method (idea) of inserting the same characteristics (parameters) into different integral kernels can be traced to Jurek and Vervaat (1983), p. 254, where it was used to describe the class $L$ of selfdecomposable distributions. Namely, $[a, R, M]_L$ (the triple in the Lévy-Khintchine formula) was identified with $[a, R, M]$. For other such examples cf. Jurek (2011), an invited talk at 10th Vilnius Conference. Similarly, Bercovici and Pata (1999) introduced a bijection between the semigroups of classical and free infinite divisible probability measures. In this paper the identification is done on the level of Lévy exponents $\log \phi$ (cummulants) and Voiculescu transforms $V_\nu$.

In this paper we show the explicit relation between $V_w(z) = 1/z$ and $\phi_{N(0,1)}(t) = \exp(-t^2/2)$, where $w$ is the Wigner semicircle distribution and $N(0,1)$ is the standard Gaussian measure.

1. The classical $\ast$- and the $\boxplus$- free infinite divisibility.

1.1. A probability measure $\mu$ is $\ast$-infinitely divisible (ID, $\ast$) if for each natural $n \geq 2$ there exists probability measure $\mu_n$ such that $\mu_n^n = \mu$. Equivalently, its characteristic function $\phi_\mu$ (Fourier transform) admits the following form (Lévy-Khinchine formula)

$$\log \phi_\mu(t) = ita - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{itx} - 1 - itx1_{|x|\leq 1}\right)M(dx), \quad t \in \mathbb{R}, \quad (3)$$

and the triplet $a \in \mathbb{R}, \sigma^2 \geq 0$ (covariance) and a positive Borel measure $M$ are uniquely determined by $\mu$; in short we write $\mu = [a, \sigma^2, M]$. A sigma-finite measure $M$ in (3) is finite on all open complements of zero and integrates $|x|^2$ in every finite neighborhood of zero. It is called the Lévy spectral measure of $\mu$.

In recent year has been considerable interest in studying random integral representations of infinitely divisible probability measures or their Lévy measures $M$; cf. Jurek (2012) and references therein. Namely, for a continuous $h$ and a monotone right continuous $r$, on an interval $(a, b)$, one defines

$$I_{h,r}^{[a,b]}(\nu) := \mathcal{L}\left( \int_{[a,b]} h(s) dY_\nu(r(s)) \right), \quad (4)$$

where $\mathcal{L}(X)$ denotes the probability distribution of $X$ and $Y_\nu$ is a cagdlag Lévy process such that $\mathcal{L}(Y_\nu(1)) = \nu$.

In terms of characteristic functions (4) means that

$$\left(I_{h,r}^{[a,b]}(\nu)\right)(t) = \exp \int_{[a,b]} \log \phi_\nu(\pm h(s) t)(\pm dr(t), \quad t \in \mathbb{R}, \quad (5)$$

3
where the minus sign is for decreasing $r$ and plus for increasing $r$; cf. Jurek and Vervaat (1983), Lemma 1.1 or Jurek (2007) (in the proof of Theorem 1) or Jurek (2012). Moreover, $(I_{(a,b)}^{h,r}(\nu))$ denotes here the characteristic function of the probability measure $I_{(a,b)}^{h,r}(\nu)$. In (5), we may write $\phi_{\nu}(-w) = \phi_{\nu^-}(w)$, $w \in \mathbb{R}$, where $\nu^-$ is the reflected measure, that is, $\nu^-(B) := \nu(-B)$ for all Borel sets $B$.

For the purposes below we consider the following specific random integral mapping:

$$(ID, *) \ni \mu \rightarrow K(\mu) \equiv I_{(a,b)}^{s,1-e^{-s}}(\mu) = \mathcal{L}\left(\int_0^\infty s\, dY_\mu(1 - e^{-s})\right) \in \mathcal{E}, \quad (6)$$

from Jurek (2007), formula (17). There, it was done for any real separable Hilbert space. (In Barndorff-Nielsen and Thorbjørnsen (2006), and in other works, the mapping (6) was denoted by the letter $\Upsilon$ and (originaly) was defined on the family of Lévy measures on a real line).

The mapping $K$ is an isomorphism between convolution semigroup $ID$ and $\mathcal{E}$ (range of the mapping $K$). Moreover, if $\mu = [a, \sigma^2, M]$ then from (3) we get

$$\phi_{K(\mu)}(t) = \exp\{ita - \sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{1-itx} - 1 - it1_{\{|x|\leq 1\}}(x)M(dx)\right)\}; \quad (7)$$
cf. Jurek (2006), Corollary 5. For a general theory of the calculus on random integral mappings of the form (2) cf. Jurek (2012).

1.2. D. Voiculescu and others studying so called free-probability introduced new binary operations on probability measures and termed them accordingly free-convolutions; cf. Bercovici-Voiculescu (1993) and references therein. To recall the definition $\boxplus$ convolution we need some auxiliary notions.

For a measure $\nu$, its Cauchy transform is given as follows

$$G_\nu(z) := \int_\mathbb{R} \frac{1}{z-x} \nu(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (8)$$

Furthermore, having $G_\nu$ we define $F_\nu(z) := 1/G_\nu(z)$ and then the Voiculescu transform as

$$V_\nu(z) := F_\nu^{-1}(z) - z, \quad z \in \Gamma_{n,M} := \{x + iy \in \mathbb{C}^+: |x| < \eta y, y > M\}, \quad (9)$$

where a such region (called Stolz angle) exists and the inverse function is well defined on it; cf. Bercovici and Voiculescu (1993), Proposition 5.4 and Corollary 5.5.
The functional $\nu \to V_\nu(z)$ is an analogue of the classical Fourier transform $\nu \to \phi_\nu(t), t \in \mathbb{R}$. The fundamental fact is that, for two measures $\nu_1$ and $\nu_2$ one has

$$V_{\nu_1}(z) + V_{\nu_2}(z) = V_{\nu_1 \boxplus \nu_2}(z),$$

for a uniquely determined probability measure, denoted as $\nu_1 \boxplus \nu_2$. This property allowed to introduce the notion of $\boxplus$ free-infinite divisibility. For this new $\boxplus$ infinite divisibility we have the following analog of the Lévy-Khintchine formula (3):

**Theorem 1.** A measure $\nu$ is $\boxplus$ infinitely divisible if and only if

$$V_\nu(z) = b + \int_{\mathbb{R}} \frac{1 + sz}{z - s} \rho(ds), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

for some uniquely determined real constant $b$ and a finite Borel measure $\rho$.

Cf. Bercovici and Voiculescu (1993). The integral formula (10), in complex analysis, is called the Nevanlinna-Pick formula.

**Remark 1.** Note that $V_\nu : \mathbb{C}^+ \to \mathbb{C}^-$ is an analytic function and for the mappings $T_c x := c x, x \in \mathbb{R}, (c > 0)$ and the image measure $T_c \nu$ we have $V_{T_c \nu}(z) = c V_\nu(c^{-1}z)$, for $z$ in appropriate Stolz angle; cf. Bercovici-Voiculescu (1993) or Barndorff-Nielsen (2006), Lemma 4.20. This is in contrast to characteristic functions of measures where we have

$$\phi_{T_c \mu}(t) = \phi_\mu(ct), \quad \text{for all } t \in \mathbb{R}.$$ 

1.3. For some analogies and comparison below, let us recall from Jurek (2006) that the restricted versions of $G_\nu$ and $V_\nu$ are just those functions considered only on the imaginary axis. Then we have that

$$\frac{1}{it} G_\nu \left(\frac{1}{it}\right) = \int_{\mathbb{R}} \frac{1}{1 - itx} \nu(dx) = \phi_{e \cdot \eta}(t), \quad t \neq 0, \quad \text{and} \quad \lim_{t \to 0} \frac{1}{it} G_\nu \left(\frac{1}{it}\right) = 1$$

where $e \cdot \eta$ means the probability distribution of a product of stochastically independent rv’s: the standard exponential $e$ and the variable $\eta$ with probability distribution $\nu$.

The identity (11) means that we can retrieve a measure $\nu$ from the characteristic function $\phi_{e \cdot \eta}$; cf. Jurek (2006), the proof of Theorem 1, on p.189 and Examples on pp. 195-198. This is in sharp contrast with the classical Stieltjes inversion formula where one needs to know $G_\nu$ in strips of complex plane; see it, for instance, in Nica and Speicher (2006), p. 31.

In fact, we have even more straightforward relation. Namely,

$$\int_0^\infty \phi_\nu(s)e^{-ts}ds = i G_\nu(it), \quad \text{for } t > 0,$$

where $e \cdot \eta$ means the probability distribution of a product of stochastically independent rv’s: the standard exponential $e$ and the variable $\eta$ with probability distribution $\nu$.
cf. Jankowski and Jurek (2012), Proposition 1. Thus, restricted Cauchy transforms are just Laplace transforms of characteristic functions.

In the spirit of (11) and (12), instead of (10), let us introduce the restricted Voiculescu transform as

\[ V_\nu(it) \equiv k_{b,\rho}(it) := b + \int_{\mathbb{R}} \frac{1 + i t s}{it - s} \rho(ds), \quad t \neq 0. \tag{13} \]

From the inversion formula in Theorem 1 in Jankowski and Jurek (2012) and from (13), we have that

\[ b = \Re k_{b,\rho}(i); \quad \rho(\mathbb{R}) := -\Im k_{b,\rho}(i); \quad \text{and for the measure } \rho \text{ we have} \]

\[ \int_{0}^{\infty} \phi_\rho(r)e^{-wr}dr = \frac{i k_{b,\rho}(-iw) - i \Re k_{b,\rho}(i) - w \Im k_{b,\rho}(i)}{w^2 - 1}, \quad w > 0. \tag{14} \]

(See there also the comment (a paradigm) in the first paragraph in the introduction on p. 298.)

In order to have (13) in a form more explicitly related to (7), let us define the new triple: shift \( a \), the variance \( \sigma^2 \) and the Lévy spectral measure \( M \), as follows:

\[ \sigma^2 := \rho(\{0\}); \quad \rho(dx) := \frac{x^2}{1 + x^2}M(dx), \quad \text{on } \mathbb{R} \setminus \{0\}; \]

\[ a := b + \int_{\mathbb{R}} s \left( \frac{1}{1+|x|^2} - \frac{1}{1 + s^2} \right) \rho(ds); \tag{15} \]

Then from (13), with some calculations, we get

\[ it k_{b,\nu}\left(\frac{1}{it}\right) = ia t - \sigma^2 t^2 + \int_{\mathbb{R}} \left( \frac{1}{1 - s t} - 1 - its 1_{|s| \leq 1} \right) M(ds), \quad \text{for } t < 0; \tag{16} \]

For computational details cf. Barndorff-Nielsen and Thorbjørnsen (2006), Proposition 4.16, p. 105

1.4. The functions \( G_\nu(z) \) and \( V_\nu(z) \) are analytic in some complex domains and thus are uniquely determined by their values on imaginary axis (more generally, on subsets with limiting points in their domains).

[Note that for \( p(z) := i \Im z \) and \( q(z) := z \) we have that \( p(it) = q(it) \) although they are different. Of course, \( p \) is not an analytic function! ]

1.5. Because of (6), (7) and (16), here is the explicit relation (an isomorphism) between the free-⊗ and the classical - ∗ infinite divisibility:
Theorem 2. A probability measure $\nu$ is $\boxplus$-infinitely divisible if and only if there exist a unique $\ast$-infinitely divisible probability measure $\mu$ such that

\[
(it) \, V_\nu((it)^{-1}) = \log \left( I_{(0,\infty)}^{s,1-e^{-s}}(\mu) \right)(t) = \\
\log \left( \mathcal{L} \left( \int_0^\infty s \, dY_\mu(1-e^{-s}) \right) \right)(t) = \int_0^\infty \log \phi_\mu(ts) e^{-s} ds, \quad \text{for } t < 0, \quad (17)
\]

where $(Y_\mu(u), u \geq 0)$ is a Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu$.

Equivalently, we have that for $\boxplus$-infinitely divisible $\nu$ its Voiculescu transform $V_\nu$ is of the form

\[
V_\nu(it) = it \int_0^\infty \log \phi_\mu(-ts) e^{-s} ds = it^2 \int_0^\infty \log \phi_\mu(-v) e^{-tv} dv, \quad t > 0; \quad (18)
\]

for an uniquely determined $\ast$-infinitely divisible measure $\mu$.

This is a rephrased version of Corollary 6 in Jurek (2007).

Statements (17) and (18) are equivalent as one can be deducted from the other. Moreover, they provide easy way of computing examples (classes) of free $\boxplus$-infinitely divisible measures from their counterparts in $(ID, \ast)$; cf. Propositions 1-4 below.

Also note that in both cases (17) and (18) we have Laplace transform of functions $\log \phi_\mu(-t)$ of $\ast$-infinitely divisible measures $\mu$.

Remark 2. From the first line in (17) we see that measures from $(ID, \boxplus)$ can be identified with measures from the semigroup $\mathcal{E} = I_{(0,\infty)}^{s,1-e^{-s}}(ID)$.

2. Examples of explicit relations between free $\boxplus$- and classical $\ast$- infinite divisible probability measures.

In the first three subsections, for a given class $\mathcal{C}$ classical $\ast$-infinitely divisible measures we identify its counterpart $\mathcal{C}$ of free $\boxplus$-infinitely divisible companions.

2.1. For the free $\boxplus$ analog of $s$-selfdecomposable distributions we have

**Proposition 1.** A probability distribution $\mu$ is free $\boxplus$ $s$-selfdecomposable, in symbols, $\mu \in (\mathcal{U}, \boxplus)$, if and only if there exist a unique $\mu = [a, \sigma^2, M] \in (ID, \ast)$ such that its Voiculescu transforms have representations

\[
(a) \quad it \, V_\mu \left( \frac{1}{it} \right) = \log \left( I_{(0,\infty)}^{r_{\mu}(v),(\mu^{-})}(\mu) \right)(t), \quad \text{with } r_{\mu}(v) := e^{-v} - v \Gamma(0, v); \quad (19)
\]

\[
(b) \, V_\mu(it) = \frac{a}{2} + \frac{\sigma^2}{3} \, it + \int_{\mathbb{R} \setminus \{0\}} \left( \frac{(it)^2 [\log(it - x) - \log(it)]}{x} - it^{-1} e^{-it} x \right) M(dx)
\]
for $t > 0$.

(c) For $z \in \mathbb{C}^+$ we have

$$V_u(z) = \frac{a}{2} + \frac{\sigma^2}{3} + \int_{\mathbb{R}\backslash\{0\}} \left( z^2 \frac{\log(z - x) - \log(z)}{x} - z - \frac{1}{2} x 1_B(x) \right) M(dx).$$

Above $\Gamma(0; x) := \int_{x}^{\infty} \frac{e^{-s}}{s} ds$, $x > 0$, is the incomplete Euler gamma function.

Proof. Recall that $\lambda$ is classical $\ast\ast$-selfdecomposable, i.e., $\lambda \in (\mathcal{U}, \ast)$ if and only if $\lambda = I_{[0,1]}^{\ast\ast} (\mu)$ for some $\mu \in ID$; cf. Jurek (1984), Theorem 2.1 for different characterizations of this class or Jurek (1985). Thus for $w \neq 0$, using (3), we get

$$\log \phi_{I_{[0,1]}^{\ast\ast}}(w) = \log (I_{[0,1]}^{\ast\ast} (\mu))^{\hat{(t)}} = \int_{0}^{1} \log \phi_{I_{[0,1]}^{\ast\ast}}(tu) du
= \frac{i}{2} aw - \frac{1}{6} \sigma^2 w^2 + \int_{\mathbb{R}\backslash\{0\}} \left( e^{i tu} - 1 - i \frac{w u}{i u} \right) M(du).$$

(20)

[The formula (20) is as in Corollary 7.1 in Jurek (1984). However, it was obtained there by using Choquet’s Theorem on extreme points in a subset of all Lévy spectral measures.]

From (17) and the first line in (20) we get

$$it V_m \left( \frac{1}{it} \right) = \int_{0}^{\infty} \log \phi_{I_{[0,1]}^{\ast\ast}}(t s) e^{-s} ds = \int_{0}^{\infty} \log \phi_{I_{[0,1]}^{\ast\ast}}(t s) e^{-s} duds, \quad (v := su)
= \int_{0}^{\infty} \int_{0}^{v} \log \phi_{I_{[0,1]}^{\ast\ast}}(tv) \frac{e^{-s}}{s} dv ds = \int_{0}^{\infty} \log \phi_{I_{[0,1]}^{\ast\ast}}(tv) \Gamma(0, v) dv. \quad (21)$$

Let us define the (decreasing) time change $r_u(v)$ for $v > 0$ as follows

$$r_u(v) := \int_{v}^{\infty} \Gamma(0, w) dw = \int_{v}^{\infty} \int_{w}^{\infty} \frac{e^{-s}}{s} ds dw
= \int_{v}^{\infty} \int_{v}^{s} \frac{e^{-s}}{s} ds dw = \int_{v}^{\infty} \frac{e^{-s}}{s} (s - v) ds = e^{-v} - v \Gamma(0, v).$$

Then taking into account (5) and putting $r_u$ into (21) to get

$$it V_m \left( \frac{1}{it} \right) = \int_{0}^{\infty} \log \phi_{I_{[0,1]}^{\ast\ast}}(-tv) (1) dr_u(v) = \log (I_{[0,\infty]}^{\ast\ast} (\mu^-))^{\hat{(t)}}$$

and this completes the proof of the part (a).
For part (b), using (17), (3) and the first line in (20), after interchanging the order of integration, we get for \( t < 0 \),

\[
\begin{align*}
\int V_m(\frac{1}{it}) &= \int_0^\infty \log \phi_\lambda(t/s)e^{-s}ds = \int_0^1 \int_0^\infty \log \phi_\mu(tus)e^{-s}dsdu \\
&= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}} \int_0^1 \left[ \int_0^\infty (e^{itux} - 1 - itux1_B(x))e^{-s}ds \right] duM(dx) \\
&= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}} \left[ \int_0^1 \frac{itux}{1-itux}du - \frac{1}{2}itux1_B(x) \right] M(dx) \\
&= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}} \left[ -\frac{\log(1-itx)}{itx} - 1 - \frac{1}{2}itx1_B(x) \right] M(dx). \quad (22)
\end{align*}
\]

Substituting \(-1/t\) for \( t \) in (22) we arrive at

\[
V(it) = \frac{a}{2} + \frac{\sigma^2}{3} \frac{1}{it} + \int_{\mathbb{R}\setminus\{0\}} \left( (it)^2 \frac{\log(it-x) - \log(it)}{x} - it - \frac{1}{2}x1_B(x) \right) M(dx),
\]

which gives (b). Part (c) is an analytic extension of (b) and this completes a proof of Proposition 1.

[Equality (b) can be also obtained by putting second line in (20) into (18).]

**Remark 3.**

(i) There is the connection between the class \((\mathcal{U}, \boxplus)\) and the \(L^r,\mathcal{U}(0,\infty)\) \((ID)\) via (a) in Proposition 1.

(ii) \(L^{r,\mathcal{U}}(0,\infty) \oslash L^{s,\mathcal{U}}(0,1) = L^{r,s}(0,\infty)\) with \(r_u(v) := e^{-v} - v\Gamma(0,v)\);

Let \(h_1 \otimes h_2\) denotes the tensor product of functions and let \(\rho_1 \times \rho_2\) denotes the product of measures. Then we have

**Corollary 1.** For \(h_1(t) := t1_{(0,1)}(t), \rho_1(dx) := 1_{(0,1)}(x)dx\) and \(h_2(s) := s1_{(0,\infty)}(s), \rho_2(dy) := e^{-y}dy\) we have that

\[
(h_1 \otimes h_2)(\rho_1 \times \rho_2)(dw) = 1_{(0,\infty)}(w)\Gamma(0,w)dw
\]

This is in fact the calculation performed in (21) starting with the first double integral.

**2.2.** For free \(\boxplus\)-selfdecomposability we have the following:

**Proposition 2.** A probability distribution \(\mathcal{S}\) is \(\boxplus\)-selfdecomposable, in symbols, \(\mathcal{S} \in (\mathcal{L}, \boxplus)\), if and only if there exist a unique \(\mu = [a, \sigma^2, M] \in (ID_{\log}, \boxplus)\) such that

\[
(a) \ \int V_{\mathcal{S}}(\frac{1}{it}) = \log \left( L^{r,\mathcal{U}}_{(0,\infty)}(\mu^-) \right) (t), \text{ for } t < 0; \quad \Gamma(0,w) := \int_w^\infty \frac{e^{-s}}{s} ds \quad (23)
\]
Equivalently,

\[ (b) \quad V_s(it) = a + \frac{\sigma^2}{2it} + \int_{\mathbb{R}\setminus\{0\}} [it \ln \frac{it}{it-x} - x 1_{\{|x| \leq 1\}}] M(dx), \quad \text{for } t > 0. \quad (24) \]

That is, for \( z \in \mathbb{C}^+ \),

\[ (c) \quad V_s(z) = a + \frac{\sigma^2}{2z} + \int_{\mathbb{R}\setminus\{0\}} [z \ln \frac{z}{z-x} - x 1_{\{|x| \leq 1\}}] M(dx). \]

**Proof.** Recall that \( \rho \in (L, *) \), in other words, \( \rho \) is \(*\) selfdecomposable if and only if \( \rho = I e^{-s, s}(0, \infty) (\mu) \) for some \( \mu \in ID_{\log} \); cf. Jurek-Vervaat (1983) or Jurek-Mason (1993), Chapter 3 (with operator \( Q = I \)). Hence

\[
\log \phi_\rho(w) = \log (I e^{-s, s}(\mu)) (w) = \int_0^\infty \log \phi_\mu(we^{-u}) du
\]

\[
= iaw - \frac{1}{4} \sigma^2 w^2 + \int_{\mathbb{R}\setminus\{0\}} \left( \int_{\{0,1\}} e^{iwrx} - \frac{1}{r} dr - iwx 1_{\{|x| \leq 1\}} \right) M(dx). \quad (25)
\]

[The characterization in (25) of selfdecomposable distributions was first obtained by K. Urbanik (by the method of extreme points) and then by Jurek and Vervaat (1983), formula (4.5) on p. 255.]

From (17) and the first line in (25), for \( t < 0 \), we have

\[
\begin{align*}
\int_0^\infty \log \phi_\rho(ts)e^{-s} ds &= \int_0^\infty \int_0^\infty \log \phi_\mu(tse^{-u})e^{-s} duds (w := se^{-u}) \\
&= \int_0^\infty \log \phi_\mu(tw) \frac{1}{w} dw e^{-s} ds = \int_0^\infty \log \phi_\mu(tw) \frac{1}{w} \int_w^\infty e^{-s} ds dw \\
&= \int_0^\infty \log \phi_\mu(tw) \frac{e^{-w}}{w} dw = \int_0^\infty \log \phi_\mu(-tw)(-1)(\Gamma(0, w))' dw \\
&= \log (I_{(0,\infty)}(\mu^-))(t), \quad (26)
\end{align*}
\]

(see the equality (5), which gives (a).
Or equivalently, using the second line in (25) and (3) we get

\[ itV_s \left( \frac{1}{it} \right) = \int_0^\infty \left[ iats - \frac{1}{4} \sigma^2(ts)^2 \right] e^{-s} ds + \int_{\mathbb{R}\{0\}} \left( \int_{(0,1)} e^{itsx} \frac{1}{r} dr - itsx 1_{|x| \leq 1}(x) \right) M(dx) \]

\[ = iat - \frac{1}{2} \sigma^2 t^2 + \int_{\mathbb{R}\{0\}} \left( \int_{(0,1)} \left( \frac{1}{1 - itx} - 1 \right) \frac{1}{r} dr - itx 1_{|x| \leq 1}(x) \right) M(dx) \]

\[ = iat - \frac{1}{2} \sigma^2 t^2 + \int_{\mathbb{R}\{0\}} \left( - \ln(1 + itx) - itx 1_{|x| \leq 1}(x) \right) M(dx). \]

Now substituting for \( s = -1/t > 0 \) we get

\[ \frac{1}{is} V_s(is) = \frac{-ia}{s} - \frac{\sigma^2}{2s^2} + \int_{\mathbb{R}\{0\}} \left( - \ln(1 + ix/s) + ix/s 1_{|x| \leq 1}(x) \right) M(dx), \]

and hence the equality (b).

Part (c) is an analytic continuation of (b) and this concludes a proof of Proposition 2.

In terms of tensor product we have

**Corollary 2.** For \( h_1(t) := t, \rho_1(dt) := e^{-t}, 0 < t < \infty \) and \( h_2(s) := e^{-s}; \rho_2(ds) := ds, 0 < s < \infty \) then

\[ (h_1 \otimes h_2)(\rho_1 \times \rho_2)(du) = \frac{e^{-u}}{u} du \]

This follows from (26) starting from the double integral and the calculations that follows. See also Jurek (2012).

**Remark 4.** (a) From (26) we get \( I_{(0,\infty)}^{1-t} \circ I_{(0,\infty)}^{e-t}(\mu) = I_{(0,\infty)}^{w, \Gamma(w)}(\mu), \) for \( \mu \in ID_{\log}. \) This is Thorin class \( T \) and thus \( T \) can be identified with the class of free-selfdecomposable measures. (Compare Remarks 2 and 3(i).)

(b) For the standard exponential measure \( e \) and standard Poisson measure \( e(\delta_1) \) we have

\[ I_{(0,\infty)}^{w, \Gamma(w)} e(\delta_1)) = e. \] (27)

To see (b) note that

\[ \log \phi_e(v) = \int_0^\infty (e^{ivx} - 1) \frac{e^{-x}}{x} dx, \quad \text{for} \; v \in \mathbb{R}, \]
therefore for $e(\delta_1)$ we conclude
\[
\log \left( I_{(0,\infty)}^{(w, \Gamma(w))} (e(\delta_1) ) \right) (t) = \int_0^\infty \log \phi_{e(\delta_1)} (tw) \frac{e^{-w}}{w} dw = \int_0^\infty (e^{itw} - 1) \frac{e^{-w}}{w} dw = \log \phi_e (t),
\]
which proves that (unexpected ?) relation (27) between Poisson (discrete) and exponential (continuous) distributions.

2.3. For free $\boxplus$-stable distributions we have:

**Proposition 3.** A measure $\nu$ is non-Gaussian free-$\boxplus$ stable if and only if for $t > 0$ its Voiculescu transform $V_\nu$ should be such that it is EITHER

\[
V_\nu (it) = a - C \frac{\Gamma(2-p)}{1-p} \Gamma(p+1) \cos \frac{\pi p}{2} \left[ p^{-1}(it)^{1-p} (i - \beta \tan \left( \frac{\pi p}{2} \right) \right], \quad (28)
\]

where $a \in \mathbb{R}, C > 0, 0 < p < 1$ or $1 < p < 2$ and $|\beta| \leq 1$,

OR $p = 1$ and

\[
V_\nu (it) = a - C \beta (1 - \gamma) + \frac{C}{2} \left( 2 \beta \log (it) - i \pi (1 + \beta) \right) \quad (29)
\]

where $1 - \gamma = \int_0^\infty w \log w e^{-w} dw$ (Euler constant $\gamma \sim 0.577$).

**Proof.** For classical $\ast$-stable measures from Meerscheart and Scheffler (2001), Theorem 7.3.5, p. 265 we have that

$\mu$ is non-Gaussian $\ast$--stable iff and only if there exist $C > 0, a \in \mathbb{R}, 0 < p < 1, 1 < p < 2, -1 \leq \beta \leq 1$ such that for each $t \in \mathbb{R}$

\[
\log \phi_\mu (t) = ita - C \frac{\Gamma(2-p)}{1-p} \cos \left( \frac{\pi p}{2} \right) |t|^p \left( 1 - i \beta \text{sign}(t) \tan \left( \frac{\pi p}{2} \right) \right); \quad (30)
\]

and for $p = 1$ we have

\[
\log \phi_\mu (t) = ita - C \frac{\pi}{2} |t| \left( 1 + i \beta \frac{2}{\pi} \text{sign}(t) \log |t| \right), \quad t \in \mathbb{R}, \quad (31)
\]

where $\beta := 2\theta - 1$ is the skewness parameter; $0 \leq \theta \leq 1$ is the probability of the positive tail of Lévy measure $M$ of $\mu$, that is, for $r > 0$, we have $M(x > r) = \theta Cr^{-p}$. And $1 - \theta$ is the probability of the negative tail of $M$.

In order to get (28) one needs insert (30) into first equality in (18) and perform some easy calculations. Similarly, putting (31) into (18) and using the identity $\log i = i\pi/2$ one gets equality (29), which completes a proof of Proposition 3.
Remark 5. (i) In some papers and books often there is a small but essential error. Namely, in (30), there is $\beta$ instead of $(-\beta)$; cf. P. Hall (1981).
(ii) Note that the expressions in square brackets in (28) and (29) are identical, up to the sign, with those in Proposition 5.12 in Bercovici-Pata (1999). Also compare Biane’s formulas for free-stable distributions in the Appendix there.

2.4. For a finite Borel measure $m$, let $e(m) := e^{-m(\mathbb{R})} \sum_{k=0}^{\infty} \frac{m^k}{k!}$ denotes the $*$-compound Poison probability measures.

Proposition 4. A probability measure $\nu$ is $\boxplus$-compound Poison probability measure if and only if

$$V_{\nu}(it) = it \int_{\mathbb{R}} \frac{x}{it-x} m(dx), \text{ for } t > 0,$$

for some finite Borel measure $m$ on the real line. Moreover, $\nu$ is free-infinitely divisible if and only if

$$V_{\nu}(z) = b + c^2 \frac{1}{z} + \int_{\mathbb{R}\setminus\{0\}} \frac{1}{z-x} m(dx) + z \int_{\mathbb{R}} \frac{x}{z-x} m(dx), \quad z \in \mathbb{C} \setminus \{0\}$$

for some $b,c \in \mathbb{R}$ and finite Borel measure $m$ on $\mathbb{R}$.

Proof. Since $\log \phi_e(m)(t) = \int_{\mathbb{R}} (e^{itx} - 1)m(dx), \ t \in \mathbb{R}$, therefore by (17)

$$V_{\nu}(it) = it \int_{0}^{\infty} \log \phi_e(m)(-t^{-1}s)e^{-s} ds$$

$$= it \int_{\mathbb{R}} \int_{0}^{\infty} (e^{-it^{-1}sx} - 1)e^{-s} ds \ m(dx) = it \int_{\mathbb{R}} \frac{x}{it-x} m(dx), \text{ for } t > 0,$$

which completes a proof of (32). The remaining part is a consequence of above and Theorem 1. [Also see pp. 203-206 in Nica-Speicher (2006) for the discussion of free compound Poisson distributions]

2.5. In this subsection, for a given three examples of $\boxplus$-infinitely divisible measures we identify their classical $*$- infinitely divisible companions.

Example 1. The probability measure $w$ such that $V_{w}(z) = \frac{1}{z}, z \neq 0$, is called free-Gaussian measure. Why such a term?

Note that from Theorem 2, we get $(it)V_{w}(\frac{1}{it}) = -t^2$. On the other hand, taking standard normal distribution $N(0, 1)$ for the measure $\mu$ we get

$$\int_{0}^{\infty} \log \phi_{\mu}(ts)e^{-*} ds = -t^2 \int_{0}^{\infty} s^2/2e^{-*} ds = -t^2 = \frac{1}{it} V_{w}(\frac{1}{it})$$
So, it is right to call $w$ an analogue of free Gaussian distribution.

More importantly, $w$ is a weak limit of free-analog of CLT and $w$ is the standard Wigner’s semicircle law with the density $\frac{1}{\pi \sqrt{4 - x^2}} 1_{[-2, 2]}(x)$, mean value zero and variance 1. [Using the inversion formula in (14) for $V_m$ we get $b = 0$ and $\rho = \delta_0$ in (10).]

**Example 2.** The probability measure $c$ with $V_c(z) = -i$ is called free-Cauchy distribution. Why such a term?

From Theorem 2, $(it)V_c(\frac{1}{it}) = t$. On the other hand, taking the standard Cauchy distribution (with the probability density $\frac{1}{\pi (1 + x^2)}$) for the measure $\mu$ we get

$$\int_0^\infty \log \phi_\mu(ts)e^{-s}ds = -|t| \int_0^\infty se^{-s}ds = t = (it)V_c(\frac{1}{it}), \quad \text{for } t < 0,$$

so it justifies the term free-Cauchy measure. In fact, we have that

**Remark 6.** The measure $c$ is the standard Cauchy distribution. To see that we use the inversion procedure from (14). Thus $b = 0, c(\mathbb{R}) = 1$ and

$$\int_0^\infty \phi_c(r)e^{-wr}dr = \frac{1}{w + 1}; \quad \text{i. e., } \phi_c(r) = e^{-r}, \quad \text{for } r > 0.$$

Consequently, $\phi_c(r) = e^{-|r|}$, for $r \in \mathbb{R}$ and hence $c$ is the standard Cauchy probability measure.

**Example 3.** The probability measure $m$ such that $V_m(z) = \frac{z}{z-1}$ is called free-Poisson distribution. Why?

From Theorem 2, $(it)V_m(\frac{1}{it}) = \frac{it}{1-it} = \frac{1}{1-it} - 1$. On the other hand, if $\mu = \delta(\delta_1)$ is the standard Poisson distribution then $\log \phi_\mu(t) = e^{it} - 1$, and by Theorem 2,

$$\int_0^\infty \log \phi_\mu(ts)e^{-s}ds = \int_0^\infty (e^{its} - 1)e^{-s}ds = \frac{1}{1-it} - 1 = (it)V_m(\frac{1}{it}).$$

In fact, $m$ has the probability density $\frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{(0, 4]}(x)$ (so called Marchenko-Pastur law); cf. Bozejko and Hasebe (2014).

**Remark 7.** Putting $m := \delta_1$ in Proposition 4 we retrieve the Example 3.

**Acknowledgments.** Author would like thank Professor Takahiro Hasebe (Japan) for his help with references to free probability theory.
References

O. Arizmendi and T. Hasebe (2014), Classical Scale mixtures of Boolean stable laws, preprint.

O.E. Barndorff-Nielsen and S. Thorbjørnsen (2006), Classical and Free Infinite divisibility and Lévy Processes; Lect. Notes Math. 1866, pp. 33-159.

H. Bercovici and V. Pata (1999), Stable laws and domains of attraction in free probability, Ann. Math. 149, pp. 1023-1060.

H. Bercovici and D. Voiculescu (1993), Free convolution of measures with unbounded support, Indiana Univ. Math. J. 42, pp. 733-773.

M. Bozejko and T. Hasebe (2014), On free infinite divisibility for classical Meixner distributions, preprint.

R. C. Bradley and Z. J. Jurek (2014), On central limit theorem for shrunken weakly dependent random variables, to appear in Houston J. Math., arXiv:1410.0214 [math.PR]

P. Hall (1981), A comedy of errors: the canonical form for a stable characteristic function, Bull. London Math. Soc. 13 no 1, pp. 23-27.

L. Jankowski and Z. J. Jurek (2012), Remarks on restricted Nevalinna transforms, Demonstratio Math. vol. XLV, no2 , pp. 297-307.

Z. J. Jurek (1977), Limit distributions for sums of shrunken random variables. In: Second Vilnius Conference on Probability Theory and Mathematical Statistics. Abstract of Communications 3, pp. 95-96.

Z. J. Jurek (1981). Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. 185, PWN Warszawa.

Z. J. Jurek (1984), S-selfdecomposable probability measures as probability distributions of some random integrals. Limit Theorems in Probability and Statistics (Veszprém, 1982), Vol. I, II, pp. 617-629. Coll. Math. Societatis János Bolyai, Vol. 36.

Z. J. Jurek (1985), Relations between the s- selfdecomposable and selfdecomposable measures, Ann. Probab. 13, pp. 592-608.

Z. J. Jurek (2006), Cauchy transforms of measures as some functionals of Fourier transforms, Probab. Math. Stat. vol. 26, Fasc. 1, pp. 187-200.

Z. J. Jurek (2007), Random integral representations for free-infinitely divisible and tempered stable distributions, Stat. & Probab. Letters, 77, no 4, pp. 417-425.
Z. J. Jurek (2011), The random integral representation conjecture: a quarter of a century later, *Lithuanian Math. Journal*, 51, no 3, pp. 362-369.

Z. J. Jurek (2012), Calculus on random integral mappings $I_{(a,b)}^{h,r}$ and their domains, arXiv:0817121 [math PR]

Z. J. Jurek and W. Vervaat (1983), An integral representation for self-decomposable Banach space valued random variables, *Z. Wahrsch. verw. Gebiete* 62, pp. 246-262.

M. M. Meerscheart and H.P. Scheffler (2001), *Limit Distributions for Sums of Independent Random Vectors*, Wiley Series in Probability and Statistics, John Wiley & Sons, Inc. New York.

A. Nica and R. Speicher (2006), *Lectures on the combinatorics of free probability*, London Mathematical Society, Lecture Note series 335, Cambridge University Press.

Zbigniew J. Jurek
Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-386 Wrocław
Poland

E-mail: zjjurek@math.uni.wroc.pl, www.math.uni.wroc.pl/~zjjurek