Cloaking by anomalous localized resonance for linear elasticity on a coated structure

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Abstract

We investigate anomalous localized resonance on the circular coated structure and cloaking related to it in the context of elasto-static systems. The structure consists of the circular core with constant Lamé parameters and the circular shell of negative Lamé parameters proportional to those of the core. We show that the eigenvalues of the Neumann-Poincaré operator corresponding to the structure converges to certain non-zero numbers determined by Lamé parameters and derive precise asymptotics of the convergence. We then show with estimates that cloaking by anomalous localized resonance takes place if and only if the dipole type source lies inside critical radii determined by the radii of the core and the shell.

1 Introduction

If a dielectric material is coated by a meta-material with a negative dielectric constant, then a source outside the structure may cause an anomalous localized resonance by which the source may be cloaked. This phenomenon was first discovered in [9, 10] when the core and shell are concentric disks and the source is single or multiple polarizable dipoles. It was shown in [9] that there is a critical radius determined by radii of the core and the shell such that if the dipole source is located inside the radius, then cloaking by anomalous localized resonance (CALR) takes place, and if the dipole source is located outside the radius, then CALR does not take place.

In a recent work [1], quantitatively precise analysis of CALR is presented on circular coated structure with arbitrary sources. Among the findings of the paper is that CALR is a spectral phenomenon occurring at the limit point of eigenvalues of the Neumann-Poincaré (NP) operator defined on concentric disks. In fact, by Plemelj’s symmetrization principle, the NP operator can be realized as a symmetric operator [6]. Since it is compact on concentric circles, the NP operator has real eigenvalues converging to 0 (exponentially fast), and there CALR takes place. On a single disk 0 is the only eigenvalue of the NP operator (other than 1/2 of multiplicity 1). So CALR does not take place. But by coating

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†This work is supported by A3 Foresight Program among China (NSF), Japan (JSPS), and Korea (NRF 2014K2A2A6000567). Work of HK is supported by NRF 2016R1A2B4011304

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the disk with another disk, the boundary becomes two circles and the NP eigenvalues are perturbed and converges to $0$, and hence CALR does take place on the coated structure. In fact, CALR may take place on a single inclusion (not a coated inclusion). For example, the NP eigenvalues on ellipses are not $0$ and converge to $0$, CALR takes place on ellipses \[4\]. There is yet another approach to CALR using the variational method, for which we refer to \[7\].

Since CALR is a spectral phenomenon at the limit point of eigenvalues of the NP operator, it is interesting to look into CALR in the context other than electro-statics where the notion of the NP operator is defined. In this regard the system of elasto-statics is particularly interesting since the elastic NP operator is not compact even on smooth boundaries. However, it is proved in the recent paper \[3\] that elastic NP eigenvalues on smooth boundaries of two-dimensional domains, whose Lamé parameters are $(\lambda, \mu)$, accumulate at either $k_0$ or $-k_0$

\[k_0 = \frac{\mu}{2(\lambda + 2\mu)}. \tag{1.1}\]

Using this spectral property of the elastic NP operator CALR for the linear isotropic elasticity is considered in \[3\] when the background is the usual isotropic elastic material with the pair of Lamé constants $(\lambda, \mu)$ and the inclusion has

\[(\tilde{\lambda}, \tilde{\mu}) = (c + i\delta)(\lambda, \mu) \tag{1.2}\]

as its Lamé constants, where $c$ is a negative constant and $\delta$ is a small positive constant representing dissipation. The asymptotics of NP eigenvalues are derived and it is proved that CALR occurs on ellipses if the constant $c$ in (1.2) satisfies

\[z(c) := \frac{c + 1}{2(1 - c)} = \pm k_0. \tag{1.3}\]

We emphasize that the condition (1.3) can be satisfied only when $c$ is negative since $0 < k_0 < 1/2$. The elastic NP eigenvalues on two dimensional disks are

\[
\frac{1}{2}, -\frac{\lambda}{2(2\mu + \lambda)}, \pm k_0, \tag{1.4}
\]

where the first two have multiplicity one, while the last two are of infinite multiplicities, and CALR does not occur since $\pm k_0$ are the elastic NP eigenvalues (not limit points). One may expect that like the electro-static case if we coat the disk by a concentric circle, then the NP eigenvalues in (1.4) are perturbed from $\pm k_0$ (while converging to them) and CALR may occur.

The purpose of this paper is to confirm it. To do so, we suppose that the disk $D_i$ is included in another concentric disk $D_e$ where the Lamé parameters are given by

\[
(\lambda^\delta, \mu^\delta) = \begin{cases} 
(\lambda, \mu), & \text{in } \mathbb{R}^2 \setminus D_e, \\
(c + i\delta)(\lambda, \mu), & \text{in } D_e \setminus D_i, \\
(\lambda, \mu), & \text{in } D_i.
\end{cases} \tag{1.5}
\]

Here $c$ is a negative constant and $\delta$ is a parameter which will be sent to $0$. We show that the elastic NP eigenvalues on the boundary of $D_e \setminus D_i$ deviate from $\pm k_0$, but accumulate at $\pm k_0$. We actually show that the NP operator (with respect to the inner product on $H^{-1/2}$, space defined by the single layer potential) can be realized as a series of block matrices,
and then using perturbation theory of eigenvalues we derive asymptotic formula of the convergence. We then show that CALR occurs if $c$ satisfies (1.3). It is worth emphasizing that the critical radii (for cloaking) when $z(c) = k_0$ and when $z(c) = -k_0$ are different, which is due to the different behavior of NP eigenvalues near $k_0$ and $-k_0$.

A few comments are in order before completing Introduction. The major emphasis of this paper is on the spectral property of the elastic NP operator on two circles. Once we obtain the asymptotic behavior of eigenvalues, the rest of arguments are parallel to those in [1] even though they are more involved since a system of equations are being dealt with. Existence of elastic meta-materials satisfying (1.2) is out of the scope of our expertise. In this regard, we mention that the method of this paper works only when (1.2) is satisfied. If, for example, only one of the parameters is negative, the method may not work.

While this work is in progress, we were informed by Hongyu Liu that their work [8] was completed. There it is proved CALR occurs on the annulus structure in elasticity when the source function $f$ (see (2.8)) is supported in a circle. We emphasize that the method of this paper is completely different from that of [8]: their method uses the variational one similar to that of [7] while our method uses spectral properties of the NP operator. The method of this paper reveals more clearly the spectral nature of anomalous localized resonance.

This paper is organized as follows. Section 2 is to formulate the problem using layer potentials and define the elastic NP operator. Section 3 is to derive asymptotics of the NP eigenvalues near $\pm k_0$. Occurrence of CALR is proved in section 4. Appendix is to provide proofs of two technical identities.

2 Layer potential formulation of the problem

2.1 Layer potentials for the Lamé system

Let us first recall definitions of the layer potential and the NP operator related to the Lamé system (see, for example, [2]). With the pair of Lamé constants $(\lambda, \mu)$ satisfying $\mu > 0$ and $\lambda + \mu > 0$, the isotropic elasticity tensor $C = (C_{ijkl})_{i,j,k,l=1}^2$ is defined by

$$C_{ijkl} := \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

Then the corresponding Lamé system of elasticity equations is defined to be $\mathcal{L}_{\lambda,\mu} := \nabla \cdot \hat{\nabla}$ where $\hat{\nabla}$ is the symmetric gradient, namely,

$$\hat{\nabla} u := \frac{1}{2} (\nabla u + \nabla u^T) \quad (T\text{ for transpose}).$$

Let $\Omega$ be a connected bounded domain with the Lipschitz boundary in $\mathbb{R}^2$. The single layer potential of the density function $\varphi$ on $\partial \Omega$ associated with the Lamé operator $\mathcal{L}_{\lambda,\mu}$ is defined by

$$S_{\partial \Omega}[\varphi](x) := \int_{\partial \Omega} \Gamma(x - y) \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2,$$

where $d\sigma$ is the arc length of $\partial \Omega$ and $\Gamma(x) = (\Gamma_{ij}(x))_{i,j=1}^2$ is the Kelvin matrix of the fundamental solution to the Lamé system in $\mathbb{R}^2$, namely,

$$\Gamma_{ij}(x) = \frac{\alpha_1}{2\pi} \delta_{ij} \ln |x| - \frac{\alpha_2}{2\pi} \frac{x_i x_j}{|x|^2}, \quad (2.1)$$
with
\[ \alpha_1 = \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right). \]

The (elasto-static) NP operator on \( \partial \Omega \) is defined by
\[ K^*_\partial \Omega [\varphi](x) := \text{p.v.} \int_{\partial \Omega} \partial_{\nu_x} \Gamma(x - y)\varphi(y) d\sigma(y) \quad \text{a.e.} \ x \in \partial \Omega. \]

Here p.v. stands for the Cauchy principal value, the conormal derivative on \( \partial \Omega \) associated with \( \nabla \cdot C \nabla \) is defined to be
\[ \partial_{\nu} u := (C \nabla u)n = \lambda (\nabla \cdot u) n + 2\mu (\nabla u)n \quad \text{on} \ \partial \Omega, \quad (2.2) \]
where \( n \) is the outward unit normal to \( \partial \Omega \), and \( \partial_{\nu_x} \Gamma(x - y) \) is defined by
\[ \partial_{\nu_x} \Gamma(x - y)b = \partial_{\nu_x} (\Gamma(x - y)b) \quad (2.3) \]
for any constant vector \( b \in \mathbb{R}^2 \).

Let \( H^{1/2}(\partial \Omega)^2 \) be the usual \( L^2 \)-Sobolev space of order \( 1/2 \) and \( H^{-1/2}(\partial \Omega)^2 \) be its dual space with respect to \( L^2 \)-pairing \( \langle \cdot, \cdot \rangle \). Let \( \Psi \) be the subspace of \( H^{-1/2}(\partial \Omega)^2 \) spanned by
\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} y \\ -x \end{bmatrix}. \quad (2.4) \]

We denote by \( H_{\Psi}^{-1/2}(\partial \Omega) \) the annihilator of \( \Psi \), namely, the collection of all \( \varphi \in H^{-1/2}(\partial \Omega)^2 \) such that \( \langle \varphi, f \rangle = 0 \) for all \( f \in \Psi \).

The NP operator \( K^*_\partial \Omega \) is bounded on \( H^{-1/2}(\partial \Omega)^2 \), and maps \( H_{\Psi}^{-1/2}(\partial \Omega) \) into itself. If \( \lambda \notin [-1/2, 1/2] \), then \( \lambda I - K^*_\partial \Omega \) is invertible on \( H^{-1/2}(\partial \Omega)^2 \) (and on \( H_{\Psi}^{-1/2}(\partial \Omega) \)). The NP operator is related to the single layer potential through the following jump formula (see [2, 5]):
\[ \partial_{\nu} S_{\partial \Omega}[\varphi]|_\pm = \left( \pm \frac{1}{2} I + K^*_\partial \Omega \right)[\varphi] \quad \text{a.e.} \ \text{on} \ \partial \Omega, \quad (2.5) \]
where the subscripts \( \pm \) denote the limits (to \( \partial \Omega \)) from outside and inside of \( \partial \Omega \). Even if \( K^*_\partial \Omega \) is not self-adjoint with respect to the usual \( L^2 \)-inner product, it can be realized as a self-adjoint operator on \( H_{\Psi}^{-1/2}(\partial \Omega) \). In fact, it is proved in [3] (following the discovery of [6]) that the product \( \langle \cdot, \cdot \rangle_* \), defined by
\[ (\varphi, \psi)_* := -\langle \varphi, S_{\partial \Omega}[\psi] \rangle, \quad (2.6) \]
is actually an inner product on \( H_{\Psi}^{-1/2}(\partial \Omega) \) which yields a norm equivalent to the usual \( H^{-1/2} \)-norm. Then \( K^*_\partial \Omega \) is self-adjoint with respect to this new inner product thanks to the Plemelj's symmetrization principle:
\[ S_{\partial \Omega} K^*_\partial \Omega = K^*_\partial \Omega S_{\partial \Omega}, \]
where \( K_{\partial \Omega} \) is the \( L^2 \)-adjoint operator of \( K^*_\partial \Omega \). It is well-known (see, for example, [5]) that \( K^*_\partial \Omega \) is not a compact operator on \( H^{-1/2}(\partial \Omega) \) even if \( \partial \Omega \) is smooth. However, it is proved in [3] that the spectrum of \( K_{\partial \Omega} \) consists of pure point spectrum converging to \( \pm k_0 \), which is defined in (1.4).
2.2 The CALR problem on coated disks

To formulate the problem on coated disks, let \( D_i := \{|x| < r_i\} \) and \( D_e := \{|x| < r_e\} \), \( 0 < r_i < r_e \), and write \( \Gamma_i = \partial D_i \) and \( \Gamma_e = \partial D_e \). The distribution of Lamé parameters are given by \( [15] \), and the elasticity tensor \( C^\delta = (C_{ijkl}^\delta)_{i,j,k,l=1} \) is given by

\[
C_{ijkl}^\delta := \lambda^\delta \delta_{ij} \delta_{kl} + \mu^\delta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\]

We then consider the following elasticity equation:

\[
\nabla \cdot C^\delta \hat{\nabla} u^\delta = f \quad \text{in} \ \mathbb{R}^2
\]

with the decaying condition \( u^\delta(x) \to 0 \) as \( |x| \to \infty \), where \( u^\delta = (u_1^\delta, u_2^\delta)^T \) and the source \( f \) is compactly supported in \( \mathbb{R}^2 \setminus \overline{D^e} \) which satisfies the following conservation law:

\[
\int_{\mathbb{R}^2} f = 0.
\]

Let \( u^\delta \) be the solution to (2.8) and define

\[
E(u^\delta) := \int_{D_e \setminus D_i} \lambda |\nabla \cdot u^\delta|^2 + 2\mu |\hat{\nabla} u^\delta|^2.
\]

Here and afterwards, \( A : B \) for two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) denotes \( \sum_{i,j} a_{ij} b_{ij} \), and \( |A|^2 = A : A \). Then CALR is characterized by the following two conditions:

- the blow up of the dissipated energy on the annulus:
  \[
  \delta E^\delta := \delta E(u^\delta) \to \infty \quad \text{as} \ \delta \to 0,
  \]

- the boundedness of the solution \( u^\delta(x) \) far away from the structure; more precisely, there is a radius \( a > r_e \) such that \( |u^\delta(x)| < C \) for some \( C > 0 \) on \( |x| > a \) as \( \delta \to 0 \).

It is worth mentioning that \( E^\delta \) is the imaginary part of the total energy, namely,

\[
E^\delta = \Im \int_{\mathbb{R}^2} C^\delta \hat{\nabla} u^\delta : \hat{\nabla} u^\delta.
\]

2.3 Layer potential formulation of the CALR problem

We now formulate the CALR problem (2.8) using layer potentials and define the NP operator for the problem. The formulation here is parallel to that in [1] for the electrostatic case. So we omit proofs.

Let us fix notation first: let \( \mathcal{H}^* = H^{-1/2}(\Gamma_i)^2 \times H^{-1/2}(\Gamma_e)^2 \) and \( \mathcal{H}_{\psi^*} = H_{\psi}^{-1/2}(\Gamma_i)^2 \times H_{\psi}^{-1/2}(\Gamma_e)^2 \). Let \( F \) be the Newtonian potential of \( f \):

\[
F(x) := \int_{\mathbb{R}^2} \Gamma(x - y)f(y)dy, \quad x \in \mathbb{R}^2.
\]

Note that \( F \) satisfies \( \mathcal{L}_{\lambda,\mu}F = f \) in \( \mathbb{R}^2 \) and \( F(x) \to 0 \) as \( |x| \to \infty \) since \( f \) satisfies (2.9).

Let \( S_{\Gamma_i} \) and \( S_{\Gamma_e} \) be the single layer potentials on \( \Gamma_i \) and \( \Gamma_e \), respectively, with respect to the Lamé parameters \((\lambda, \mu)\). Then we seek the solution \( u^\delta \) of (2.8) in the following form:

\[
u^\delta(x) = F(x) + S_{\Gamma_i}[\varphi^\delta_i](x) + S_{\Gamma_e}[\varphi^\delta_e](x)
\]

(2.13)
for some \((\varphi_i^\delta, \varphi_e^\delta) \in \mathcal{H}_\psi\). We emphasize that the solution can be represented as \([2.13]\) because the Lamé parameters of the shell is of the form \((c + i\delta)(\lambda, \mu)\). If they are not of the form, then the single layer potential with different Lamé parameter should be used, and arguments of this paper may not be applied.

The transmission conditions to be satisfied by \(u^\delta\) on \(\Gamma_i\) and \(\Gamma_e\) are

\[
\begin{cases}
(c + i\delta) \partial_v u^\delta|_+ = \partial_v u^\delta|_- & \text{on } \Gamma_i, \\
\partial_v u^\delta|_+ = (c + i\delta) \partial_v u^\delta|_- & \text{on } \Gamma_e.
\end{cases}
\]

So \((\varphi_i^\delta, \varphi_e^\delta)\) is the solution to the following system of integral equations:

\[
\begin{cases}
(c + i\delta) \partial_v S_{\Gamma_i}[\varphi_i^\delta] + \partial_v S_{\Gamma_e}[\varphi_e^\delta] = (c - 1 + i\delta) \partial_v S_{\Gamma_e}[\varphi_e^\delta] & \text{on } \Gamma_i, \\
(1 - c - i\delta) \partial_v S_{\Gamma_i}[\varphi_i^\delta] + \partial_v S_{\Gamma_e}[\varphi_e^\delta] = (c - 1 + i\delta) \partial_v S_{\Gamma_e}[\varphi_e^\delta] & \text{on } \Gamma_e,
\end{cases}
\]

where \(\partial_v\) and \(\partial_v\) are the conormal derivatives on \(\Gamma_i\) and \(\Gamma_e\), respectively. Using the jump formula \([2.5]\), the above integral equations are written as

\[
\begin{bmatrix}
-z_\delta I + K_{\Gamma_i}^* & \partial_v S_{\Gamma_e} \\
\partial_v S_{\Gamma_i} & z_\delta I + K_{\Gamma_e}^*
\end{bmatrix}
\begin{bmatrix}
\varphi_i^\delta \\
\varphi_e^\delta
\end{bmatrix}
= -\begin{bmatrix}
\partial_v F \\
\partial_v F
\end{bmatrix},
\]

(2.14)

where

\[
z_\delta = \frac{1 + c + i\delta}{2(1 - c) - 2i\delta}.
\]

(2.15)

We emphasize that \(\partial_v F \in H_{\psi}^{-1/2}(\Gamma_i)\) and \(\partial_v F \in H_{\psi}^{-1/2}(\Gamma_e)\) since \(\mathcal{L}_{\lambda, \mu} F = 0\) in \(D_e\).

Following section 2 in \([\text{I}]\) we define \(\mathbb{K}^* : \mathcal{H}_* \to \mathcal{H}_*\) as

\[
\mathbb{K}^* := \begin{bmatrix}
-K_{\Gamma_i}^* & -\partial_v S_{\Gamma_e} \\
\partial_v S_{\Gamma_i} & K_{\Gamma_e}^*
\end{bmatrix}.
\]

(2.16)

Then the equation \((2.14)\) can be rewritten as

\[
(z_\delta I + \mathbb{K}^*) [\Phi^\delta] = \mathbf{G},
\]

(2.17)

where

\[
\Phi^\delta := \begin{bmatrix}
\varphi_i^\delta \\
\varphi_e^\delta
\end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix}
\partial_v F \\
\partial_v F
\end{bmatrix}.
\]

The operator \(\mathbb{K}^*\) is the NP operator associated with two circles.

Next, we define \(\mathcal{S}\) on \(\mathcal{H}_*\) as

\[
\mathcal{S} := \begin{bmatrix}
S_{\Gamma_i} & S_{\Gamma_e} \\
S_{\Gamma_i} & S_{\Gamma_e}
\end{bmatrix}.
\]

(2.18)

We emphasize that the \(S_{\Gamma_e}\) in the first row is defined to be \(S_{\Gamma_e}[\varphi]|_{\Gamma_i}\) and \(S_{\Gamma_i}\) in the second row is \(S_{\Gamma_i}[\varphi]|_{\Gamma_e}\). Define, for \(\Phi, \Psi \in \mathcal{H}_*\),

\[
\langle \Phi, \Psi \rangle_* := -\langle \Phi, \mathcal{S}[\Psi] \rangle,
\]

(2.19)

where \(\langle \cdot, \cdot \rangle\) denotes the pairing of \(L^2(\Gamma_i)^2 \times L^2(\Gamma_e)^2\). Following the same arguments in subsection 3.3 of \([\text{I}]\), one can show that \(\langle \cdot, \cdot \rangle_*\) is actually an inner product on \(\mathcal{H}_\psi\), and
\( K^* \) is a self-adjoint operator with respect to this inner product. Let \( \| \cdot \|_\ast \) be the norm induced by the new inner product. One can show in the same way as in \[3\] that this norm is equivalent to the usual \( H^{-1/2} \)-norm.

Let \( u_\delta \) be the solution to (2.8) given in the form (2.13). Using Green’s formula, namely,
\[
\int_{\partial \Omega} u \cdot \partial_\nu v d\sigma = \int_\Omega u \cdot \mathcal{L}_{\lambda, \mu} v + \int_\Omega \lambda (\nabla \cdot u) (\nabla \cdot v) + 2\mu \nabla u : \nabla v,
\]
one can see that
\[
\int_{D_e \setminus D_i} \lambda |\nabla \cdot (S_{\Gamma_i}[\varphi^\delta] + S_{\Gamma_e}[\varphi^\delta])|^2 + 2\mu |\nabla (S_{\Gamma_i}[\varphi^\delta] + S_{\Gamma_e}[\varphi^\delta])|^2 = \| \Phi^\delta \|_\ast^2.
\]
So, we obtain the following lemma.

**Lemma 2.1.** There are constants \( C_1 \) and \( C_2 \) such that
\[
C_1 (\| \Phi^\delta \|_\ast^2 - 1) \leq E(u_\delta) \leq C_2 (\| \Phi^\delta \|_\ast^2 + 1). \tag{2.20}
\]

### 3 Asymptotics of NP eigenvalues

In this section we represent the NP operator as block matrices acting on finite dimensional subspaces, and derive asymptotics of the NP eigenvalues. To do so, we first recall some computations in \[3\].

#### 3.1 Computations on a disk

Let \( \Gamma \) be the circle of radius \( r_0 \). For an integer \( m \) let
\[
\varphi_m = \begin{bmatrix} \cos m\omega \\ \sin m\omega \end{bmatrix} \quad \text{and} \quad \bar{\varphi}_m = \begin{bmatrix} -\sin m\omega \\ \cos m\omega \end{bmatrix}.
\]

The following computations are from \[3\]. For \( x = (r \cos \omega, r \sin \omega) \), if \( m = 1 \), then
\[
-S_{\Gamma}[\varphi_1](x) = \begin{cases} \frac{\alpha_1 - \alpha_2}{2} r \varphi_1(\omega), & r < r_0, \\ \frac{\alpha_1 - \alpha_2}{2} r_0^2 \varphi_1(\omega), & r > r_0. \end{cases}
\]

If \( m \geq 2 \), then
\[
-S_{\Gamma}[\varphi_m](x) = \begin{cases} \frac{\alpha_1}{2m} r_0^{m-1} \varphi_m(\omega) + \frac{\alpha_2}{2} \left( \frac{r_0^2}{r} - r \right) r_0^{m-1} \varphi_{m-2}(\omega), & r < r_0, \\ \frac{\alpha_1}{2m} r_0^{m+1} \varphi_m(\omega), & r > r_0. \end{cases}
\]

If \( m \leq -1 \), then
\[
-S_{\Gamma}[\varphi_m](x) = \begin{cases} \frac{\alpha_1}{2m} r_0^{m+1} \varphi_m(\omega), & r < r_0, \\ \frac{\alpha_1}{2m} r_0^{m+1} \varphi_m(\omega) + \frac{\alpha_2}{2} \left( r - \frac{r_0^2}{r} \right) r_0^{m+1} \varphi_{m-2}, & r > r_0. \end{cases}
\]
It is also shown that if \( m \geq 2 \), then
\[
- S_r [\tilde{\varphi}_m](x) = \begin{cases} 
\frac{\alpha_1}{2m} r^{m-r} \tilde{\varphi}_m(\omega) - \frac{\alpha_2}{2} \left( \frac{r^2}{r} - r \right) \frac{r^{m-1}}{r_0^{m-1}} \tilde{\varphi}_{m+2}(\omega), & r < r_0, \\
\frac{\alpha_1}{2m} \frac{r_0^{m+1}}{r} \tilde{\varphi}_m(\omega), & r > r_0.
\end{cases}
\] (3.5)

If \( m \leq -1 \), then
\[
- S_r [\tilde{\varphi}_m](x) = \begin{cases} 
\frac{\alpha_1}{2|m|} r^{|m|} \tilde{\varphi}_m(\omega), & r < r_0, \\
\frac{\alpha_1}{2|m|} \frac{r^{m+1}}{r^{|m|}} \tilde{\varphi}_m(\omega) - \frac{\alpha_2}{2} \left( r - \frac{r_0^2}{r} \right) \frac{r_0^{|m|+1}}{r^{|m|+1}} \tilde{\varphi}_{m+2}(\omega), & r > r_0.
\end{cases}
\] (3.6)

As consequences of these computations, eigenvalues of \( K^+_1 \) are obtained in [9]:
\[
K^+_1[\varphi_m] = -k_0 \varphi_m \ (m < 0), \quad K^+_1[\tilde{\varphi}_m] = -k_0 \tilde{\varphi}_m \ (m < 0),
\] (3.7)
and
\[
K^+_1[\varphi_1] = -\frac{\lambda}{2(2\mu + \lambda)} \varphi_1, \quad K^+_1[\tilde{\varphi}_m] = k_0 \tilde{\varphi}_m \ (m \geq 2).
\] (3.8)

Even though it is irrelevant to this work, we mention for the completeness’s sake that
\[
K^+_1[\tilde{\varphi}_1] = \frac{1}{2} \tilde{\varphi}_1.
\]

3.2 Block matrix structure of the inner product

For \( n = 0, 1, 2, \ldots \), we define
\[
\Phi_{n,1} := \begin{bmatrix} \varphi_{n+1} \\ 0 \end{bmatrix}, \quad \Phi_{n,2} := \begin{bmatrix} \varphi_{-n+1} \\ 0 \end{bmatrix}, \quad \Phi_{n,3} := \begin{bmatrix} 0 \\ \varphi_{n+1} \end{bmatrix}, \quad \Phi_{n,4} := \begin{bmatrix} 0 \\ \varphi_{-n+1} \end{bmatrix}.
\] (3.9)

Here \( 0 \) denotes two-dimensional zero vector, and so \( \Phi_{n,j} \) are four-dimensional vector-valued functions. We then define finite dimensional subspaces \( \mathcal{V}_n \) of \( \mathcal{H}^* \) by
\[
\mathcal{V}_n := \text{span} \{ \Phi_{n,1}, \Phi_{n,2}, \Phi_{n,3}, \Phi_{n,4} \}, \quad n \geq 0.
\] (3.10)

It is worth mentioning that \( \mathcal{V}_0 \) is of two dimensions and spanned by \( \Phi_{0,1} \) and \( \Phi_{0,3} \).

Similarly, we define
\[
\tilde{\Phi}_{n,1} := \begin{bmatrix} \tilde{\varphi}_{n+1} \\ 0 \end{bmatrix}, \quad \tilde{\Phi}_{n,2} := \begin{bmatrix} \tilde{\varphi}_{-n+1} \\ 0 \end{bmatrix}, \quad \tilde{\Phi}_{n,3} := \begin{bmatrix} 0 \\ \tilde{\varphi}_{n+1} \end{bmatrix}, \quad \tilde{\Phi}_{n,4} := \begin{bmatrix} 0 \\ \tilde{\varphi}_{-n+1} \end{bmatrix},
\] (3.11)
and define
\[
\tilde{\mathcal{V}}_n = \text{span} \{ \tilde{\Phi}_{n,1}, \tilde{\Phi}_{n,2}, \tilde{\Phi}_{n,3}, \tilde{\Phi}_{n,4} \}, \quad n \geq 0.
\] (3.12)

Using (3.2)-(3.6) in the previous subsection, one can see that
\[
S(\mathcal{V}_n) \subset \mathcal{V}_n \quad \text{and} \quad S(\tilde{\mathcal{V}}_n) \subset \tilde{\mathcal{V}}_n \quad \text{for each } n.
\] (3.13)

For example, according to the definition (2.18), \( S[\Phi_{n,1}] \) is given by
\[
S[\Phi_{n,1}] = \begin{bmatrix} S_{r_1}[\varphi_{n+1}] \big| r_1 \\ S_{r_1}[\varphi_{n+1}] \big| r_2 \end{bmatrix}.
\]
So, using (3.2)–(3.4), one can see that \( S[C] \) and \( \rho \) straight-forward, but tedious. Set

\[
\Phi_{n,i} := \frac{1}{n} \sum_{j=1}^{n} a_{ij} \Phi_{n,j}, \quad i = 1, \ldots, 4.
\]

In fact, they are given as follows. We do not include details of derivations since they are straightforward, but tedious. Set \( \rho := r_i/r_e \).

(i) On \( \mathcal{V}_n, n \geq 2, \)

\[
-\mathbb{S} = \begin{bmatrix}
\frac{\alpha_1}{2(n+1)} r_i & 0 & \frac{\alpha_1}{2(n+1)} r_i \rho^n & 0 \\
0 & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} (r_i^2 - r_e^2) \rho^n & 0 \\
\frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_e^2 \rho^n & \frac{\alpha_1}{2(n+1)} r_i \rho^n \\
0 & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_i & 0 \\
\end{bmatrix}.
\] (3.14)

(ii) On \( \mathcal{\bar{V}}_n, n \geq 2, \)

\[
-\mathbb{S} = \begin{bmatrix}
\frac{\alpha_1}{2(n+1)} r_i & 0 & \frac{\alpha_1}{2(n+1)} r_i \rho^n & 0 \\
0 & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} (r_i^2 - r_e^2) \rho^n & 0 \\
\frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_e^2 \rho^n & \frac{\alpha_1}{2(n+1)} r_i \rho^n \\
0 & \frac{\alpha_1}{2(n+1)} r_i & \frac{\alpha_1}{2(n+1)} r_i & 0 \\
\end{bmatrix}.
\] (3.15)

We mention that only the cases when \( n \geq 2 \) are presented above. It is because the cases of large \( n \) matter for our analysis.

As a consequence we obtain the following lemma.

**Lemma 3.1.** (i) It holds that \( \mathcal{V}_m \perp \mathcal{V}_n \) and \( \mathcal{\bar{V}}_m \perp \mathcal{\bar{V}}_n \) if \( m \neq n \), and \( \mathcal{V}_m \perp \mathcal{\bar{V}}_n \) for all \( m \) and \( n \), with respect to the inner product \( \langle \cdot, \cdot \rangle_* \).

(ii) For \( j = 1, \ldots, 4, \)

\[
\langle \Phi_{n,j}, \Phi_{n,j} \rangle_* \approx \frac{1}{n}, \quad \langle \Phi_{n,j}, \bar{\Phi}_{n,j} \rangle_* \approx \frac{1}{n}.
\] (3.16)

(iii) If \( i \neq j \), then

\[
|\langle \Phi_{n,i}, \Phi_{n,j} \rangle_*| \lesssim \rho^n, \quad |\langle \Phi_{n,i}, \bar{\Phi}_{n,j} \rangle_*| \lesssim \rho^n.
\] (3.17)

In the above and afterwards \( \langle \Phi_{n,j}, \Phi_{n,j} \rangle_* \approx \frac{1}{n} \) mean that there are constants \( C_1 \) and \( C_2 \) independent of \( n \) such that

\[
\frac{C_1}{n} \leq \langle \Phi_{n,j}, \Phi_{n,j} \rangle_* \leq \frac{C_2}{n},
\]

and \( |\langle \Phi_{n,i}, \Phi_{n,j} \rangle_*| \lesssim \rho^n \) means that there is a constant \( C \) independent of \( n \) such that

\[
|\langle \Phi_{n,i}, \Phi_{n,j} \rangle_*| \leq C \rho^n.
\]

**Proof of Lemma 3.1.** The item (i) is an immediate consequence of (3.13). For (ii) and (iii), we note that \( \langle \Phi_{n,i}, \Phi_{n,j} \rangle_{L^2} = 0 \) if \( i \neq j \), and \( \langle \Phi_{n,j}, \Phi_{n,j} \rangle_{L^2} = 2\pi r_i \) if \( j = 1, 2, \) and \( 2\pi r_e \) if \( j = 3, 4 \). We then observe that the diagonal elements of the matrices in (3.14) and (3.15) are of order \( n^{-1} \), and off-diagonal elements are of order \( \rho^n \). So, (3.16) and (3.17) follow. \( \square \)
3.3 Block matrix structure of the NP operator

We now represent the NP operator $K^*$ on $\mathcal{V}_n$ and $\tilde{\mathcal{V}}_n$. According to the definition (2.16), $K^*[\Phi_{n,1}]$, for examples, is given by

$$K^*[\Phi_{n,1}] = \begin{bmatrix} -K_{\Gamma_1}^* [\varphi_{n+1}] |_{\Gamma_i} \\ \partial_{\nu} S_{\Gamma_1} [\varphi_{n+1}] |_{\Gamma_i} \end{bmatrix}.$$ 

So, using (3.3) and (3.7), one can compute $K^*[\Phi_{n,1}]$. In this way, one can show that

$$K^*(\mathcal{V}_n) \subset \mathcal{V}_n \quad \text{and} \quad K^*(\tilde{\mathcal{V}}_n) \subset \tilde{\mathcal{V}}_n \quad \text{for each} \quad n.$$ (3.18)

We also obtain the following lemma for the block matrix representation of the NP operator.

**Lemma 3.2.** The NP-operator $K^*$ admits the following matrix representation:

(i) On $\mathcal{V}_n$, $n \geq 2$,

$$K^* = J + M_n,$$

where

$$J = k_0 \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$M_n = \rho^n \begin{bmatrix} 0 & 0 & 0 & \mu \alpha_2 \\ 0 & 0 & -(n-1) \mu \alpha_2 (\rho^2 - 1) & 0 \\ \mu \alpha_1 \rho^2 & -n \mu \alpha_2 (\rho^2 - 1) & 0 & 0 \\ 0 & 0 & 0 & \mu \alpha_1 \rho^2 \end{bmatrix}.$$ 

(ii) On $\tilde{\mathcal{V}}_n$, $n \geq 2$:

$$K^* = J + \tilde{M}_n,$$

where

$$\tilde{M}_n = \rho^n \begin{bmatrix} 0 & 0 & 0 & \mu \alpha_2 \\ 0 & 0 & (n-1) \mu \alpha_2 (1 - \rho^2) & 0 \\ \mu \alpha_1 \rho^2 & (n+1) \mu \alpha_2 (\rho^2 - 1) & 0 & 0 \\ 0 & 0 & 0 & \mu \alpha_1 \rho^2 \end{bmatrix}.$$ 

3.4 Asymptotics of NP eigenvalues

It turns out that the exact expression of the eigenvalues of $K^*$ on $\mathcal{V}_n$, or of the matrix $J + M_n$ in (3.19), is extremely lengthy and complicated. However, for analysis of this paper, it is enough to have their asymptotic behavior as $n \to \infty$, which we obtain in this subsection using the perturbation theory.

To investigate the asymptotic behavior of the eigenvalues, it is more convenient to express $K^* = J + M_n$ on $\mathcal{V}_n$ as follows:

$$K^* = P_n + \rho^n Q,$$
where

\[
P_n = \begin{bmatrix}
-k_0 & 0 & 0 & 0 \\
0 & k_0 & -n\rho^n\mu\alpha_2(1 - \rho^{-2}) & 0 \\
0 & -n\rho^n\mu\alpha_2(\rho^2 - 1) & k_0 & 0 \\
0 & 0 & 0 & -k_0 \\
\end{bmatrix},
\]

and

\[
Q = \begin{bmatrix}
0 & 0 & \mu\alpha_2 & 0 \\
0 & 0 & \mu\alpha_2(1 - \rho^{-2}) & \mu\alpha_1\rho^{-2} \\
\mu\alpha_1\rho^2 & -\mu\alpha_2(\rho^2 - 1) & 0 & 0 \\
0 & \mu\alpha_2 & 0 & 0 \\
\end{bmatrix}.
\]

It is worth emphasizing that \(Q\) is independent of \(n\).

It is easy to find exact eigenvalues and eigenfunctions of \(P_n\). In fact, they are given as follows:

\[
\begin{align*}
\lambda^0_{n,1} &= -k_0, & E^0_{n,1} &= [1, 0, 0, 0]^T, \\
\lambda^0_{n,2} &= +k_0 - n\rho^n\mu\alpha_2(\rho - \rho^{-1}), & E^0_{n,2} &= [0, 1, \rho, 0]^T, \\
\lambda^0_{n,3} &= +k_0 + n\rho^n\mu\alpha_2(\rho - \rho^{-1}), & E^0_{n,3} &= [0, -1, \rho, 0]^T, \\
\lambda^0_{n,4} &= -k_0, & E^0_{n,4} &= [0, 0, 0, 1]^T.
\end{align*}
\]

Here and afterwards, \(E^0_{n,j}\), written as a vector in \(\mathbb{R}^4\) actually represents a four dimensional vector-valued function. For example, \(E^0_{n,2}\) in \((3.21)\) represents \(\Phi_{n,2} + \rho\Phi_{n,3}\).

Note that, if \(n\) is large, the matrix \(\rho^2Q\) becomes a small perturbation matrix. So we can derive asymptotic formula for eigenvalues using standard arguments of the eigenvalue perturbation theory (see, for example, section XII of \([11]\)). We only mention that since eigenvalues \(\lambda^0_3\) and \(\lambda^0_4\) are simple, we apply non-degenerate perturbation theory. For the cases of \(\lambda^0_0 = \lambda^0_4\), we apply the degenerate perturbation theory.

Let \(\lambda_{n,j}\) and \(E_{n,j}\) \((n \geq 1, j = 1, 2, 3, 4)\) be the eigenvalues and eigenvectors of \(\mathbb{K}^*\), respectively. Then we have

\[
\begin{align*}
\lambda_{n,1} &= -k_0 - \rho^{2n}\frac{\mu^2\alpha_1\alpha_2\rho^4}{k_0(1 + \rho^2)} + O(\rho^{3n}), & E_{n,1} &= E^0_{n,1} + O(n\rho^n), \\
\lambda_{n,2} &= +k_0 - n\rho^n\mu\alpha_2(\rho - \rho^{-1}) + O(\rho^n), & E_{n,2} &= E^0_{n,2} + O(\rho^n), \\
\lambda_{n,3} &= +k_0 + n\rho^n\mu\alpha_2(\rho - \rho^{-1}) + O(\rho^n), & E_{n,3} &= E^0_{n,3} + O(\rho^n), \\
\lambda_{n,4} &= -k_0 - \rho^{2n}\frac{\mu^2\alpha_1\alpha_2\rho^2}{k_0(1 + \rho^2)} + O(\rho^{3n}), & E_{n,4} &= E^0_{n,4} + O(n\rho^n).
\end{align*}
\]

Similarly, we can also obtain the asymptotic behavior of eigenvalues of \(\mathbb{K}^*\) on \(\tilde{\mathcal{F}}_n\) as follows

\[
\begin{align*}
\tilde{\lambda}_{n,1} &= -k_0 - \rho^{2n}\frac{\mu^2\alpha_1\alpha_2\rho^4}{k_0(1 + \rho^2)} + O(\rho^{3n}), \\
\tilde{\lambda}_{n,2} &= +k_0 + n\rho^n\mu\alpha_2(\rho - \rho^{-1}) + O(\rho^n), \\
\tilde{\lambda}_{n,3} &= +k_0 - n\rho^n\mu\alpha_2(\rho - \rho^{-1}) + O(\rho^n), \\
\tilde{\lambda}_{n,4} &= -k_0 - \rho^{2n}\frac{\mu^2\alpha_1\alpha_2\rho^2}{k_0(1 + \rho^2)} + O(\rho^{3n}).
\end{align*}
\]
and the corresponding eigenvectors are
\[
\begin{align*}
\tilde{E}_{n,1} &= [1,0,0,0]^T + O(n\rho^n), \\
\tilde{E}_{n,2} &= [0,1,\rho,0]^T + O(\rho^n), \\
\tilde{E}_{n,3} &= [0,-1,\rho,0]^T + O(\rho^n), \\
\tilde{E}_{n,4} &= [0,0,0,1]^T + O(n\rho^n).
\end{align*}
\] (3.32)

We emphasize that all eigenvalues in (3.24)–(3.31) converge to either \(k_0\) or \(-k_0\).

In view of (3.16), (3.17), (3.24)–(3.27), and (3.32)–(3.35), we obtain the following lemma.

**Lemma 3.3.** We have
\[
\begin{align*}
(|E_{n,j}, E_{n,j}|_s &\approx n^{-1}, \\
|\tilde{E}_{n,j}, \tilde{E}_{n,j}|_s &\approx n^{-1} \quad \text{for } 1 \leq j \leq 4, \\
(|E_{n,j}, \Phi_{n,j}|_s &\approx n^{-1}, \\
|\tilde{E}_{n,j}, \tilde{E}_{n,j}|_s &\approx n^{-1} \quad \text{for } j = 1, 4, \\
|E_{n,j}, \Phi_{n,k}|_s &\lesssim \rho^n, \\
|\tilde{E}_{n,j}, \tilde{E}_{n,k}|_s &\lesssim \rho^n \quad \text{for } j = 1, 4, k \neq j, \\
|E_{n,j}, \Phi_{n,k}|_s &\approx n^{-1}, \\
|\tilde{E}_{n,j}, \tilde{E}_{n,k}|_s &\approx n^{-1} \quad \text{for } j = 2, 3, k = 2, 3, \\
|E_{n,j}, \Phi_{n,k}|_s &\lesssim \rho^n, \\
|\tilde{E}_{n,j}, \tilde{E}_{n,k}|_s &\lesssim \rho^n \quad \text{for } j = 2, 3, k = 1, 4.
\end{align*}
\] (3.36)

**4 CALR**

**4.1 Estimates of the solution to the integral equation**

We now look into the integral equation (2.17). We first observe that since \(G \in \mathcal{H}_0^s\), \(G\) is orthogonal to \(\widetilde{\mathcal{V}}_0\) in particular. So \(G\) is uniquely represented as

\[
G = \sum_{n=0}^{\infty} G_n + \sum_{n=1}^{\infty} \tilde{G}_n, \quad G_n \in \mathcal{V}_n, \quad \tilde{G}_n \in \widetilde{\mathcal{V}}_n.
\]

It may be helpful to mention that \(G_1\) has no component of \(\Phi_{1,2}\) and \(\Phi_{1,4}\) since \(G\) is perpendicular to constant vectors. The solution \(\Phi^\delta\) to (2.17) is given by

\[
\Phi^\delta = \sum_{n=0}^{\infty} \Phi^\delta_n + \sum_{n=1}^{\infty} \tilde{\Phi}^\delta_n,
\] (4.1)

where \(\Phi^\delta_n \in \mathcal{V}_n\) and \(\tilde{\Phi}^\delta_n \in \widetilde{\mathcal{V}}_n\) are, respectively, the solutions to the finite dimensional equations

\[
(z_\delta I + \mathbb{K}^*)\Phi^\delta_n = G_n \quad \text{on } \mathcal{V}_n, \quad (z_\delta I + \mathbb{K}^*)\tilde{\Phi}^\delta_n = \tilde{G}_n \quad \text{on } \widetilde{\mathcal{V}}_n.
\] (4.2)

Let us consider the first equation in the above. Since \(E_{n,j}, 1 \leq j \leq 4\), is an orthogonal basis for \(\mathcal{V}_n\), we have

\[
\Phi^\delta_n = \sum_{j=1}^{4} \frac{(G_n, E_{n,j})_s}{(E_{n,j}, E_{n,j})_s (z_\delta + \lambda_{n,j})} E_{n,j},
\]

and hence

\[
\|\Phi^\delta_n\|^2 = \sum_{j=1}^{4} \frac{|(G_n, E_{n,j})_s|^2}{|E_{n,j}|^2 (z_\delta + \lambda_{n,j})^2}.
\] (4.3)
Note that \( z_\delta \to z(c) \) as \( \delta \to 0 \) where \( z(c) \) is defined in (1.3). On the other hand, \( \lambda_{n,j} \) approaches to either \( k_0 \) or \( -k_0 \) as \( n \to \infty \). So, if \( z(c) \neq \pm k_0 \), then \( |z_\delta + \lambda_{n,j}| \geq C \) for some constant \( C \) for all sufficiently large \( n \) if \( \delta \) is small. So the norm given in (4.3) does not blow up. So, we assume that \( z(c) \) is either \( k_0 \) or \( -k_0 \).

Suppose that \( z(c) = k_0 \). If \( j = 2, 3 \), then we infer from (3.25) and (3.26) that \( |z_\delta + \lambda_{n,j}| \geq C \) for some constant \( C \) independent of \( n \). So, we have

\[
\sum_{j=2,3} \frac{|(G_n, E_{n,j})|^2}{\|E_{n,j}\|^2_\|z_\delta + \lambda_{n,j}\|^2} \lesssim \|G_n\|^2_\|. \tag{4.4}
\]

If \( j = 1, 4 \), then we infer from (3.24) and (3.27) that

\[
|z_\delta + \lambda_{n,j}|^2 \approx \delta^2 + \rho^{4n}, \tag{4.5}
\]

and hence

\[
\sum_{j=1,4} \frac{|(G_n, E_{n,j})|^2}{\|E_{n,j}\|^2_\|z_\delta + \lambda_{n,j}\|^2} \approx \sum_{j=1,4} \frac{|(G_n, E_{n,j})|^2}{\|E_{n,j}\|^2_\|\delta^2 + \rho^{4n}\|.} \tag{4.6}
\]

Since \( G_n \in \mathcal{Y}_n \) and \( \tilde{G}_n \in \mathcal{Y}_n \), we have

\[
G_n = \sum_{k=1}^4 g_{n,k} \Phi_{n,k}, \quad \tilde{G}_n = \sum_{k=1}^4 \tilde{g}_{n,k} \Phi_{n,k}, \tag{4.7}
\]

for some constants \( g_{n,k} \) and \( \tilde{g}_{n,k} \). Since \( (G_n, E_{n,1})_* = \sum_{k=1}^4 g_{n,k} (\Phi_{n,k}, E_{n,1})_* \), one can see from Lemma 3.3 that

\[
\frac{|g_{n,1}|^2}{n^2} - \rho^{2n} \sum_{k \neq 1} |g_{n,k}|^2 \lesssim |(G_n, E_{n,1})_*|^2 \lesssim \frac{|g_{n,1}|^2}{n^2} + \rho^{2n} \sum_{k \neq 1} |g_{n,k}|^2. \tag{4.8}
\]

Likewise, we have

\[
\frac{|g_{n,4}|^2}{n^2} - \rho^{2n} \sum_{k \neq 4} |g_{n,k}|^2 \lesssim |(G_n, E_{n,4})_*|^2 \lesssim \frac{|g_{n,4}|^2}{n^2} + \rho^{2n} \sum_{k \neq 4} |g_{n,k}|^2. \tag{4.9}
\]

It then follows from Lemma 3.3 and (4.6)–(4.9) that

\[
\frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) - n \rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}} \lesssim \sum_{j=1,4} \frac{|(G_n, E_{n,j})_*|^2}{\|E_{n,j}\|^2_\|z_\delta + \lambda_{n,j}\|^2} \lesssim \frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) + n \rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}}. \tag{4.10}
\]

So, we obtain from (4.3), (4.4), and (4.10) that

\[
\frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) - n \rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}} - \|G_n\|^2_\| \lesssim \|\Phi_{n,j}\|^2_\| \lesssim \frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) + n \rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}} + \|G_n\|^2_\|. \tag{4.11}
\]
One can estimate $\|\Phi_\delta^f\|^2_*$ in a similar way to obtain
\[
\frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) - n\rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}} - \|\tilde{G}_n\|^2_* \lesssim \|\Phi_\delta^f\|^2_* \lesssim \frac{n^{-1}(|g_{n,1}|^2 + |g_{n,4}|^2) + n\rho^{2n}(|g_{n,2}|^2 + |g_{n,3}|^2)}{\delta^2 + \rho^{4n}} + \|\tilde{G}_n\|^2_*.
\]  
(4.12)

Let, for ease of notation,
\[
I_n := |g_{n,1}|^2 + |g_{n,4}|^2 + |g_{n,1}|^2 + |g_{n,4}|^2
\]  
(4.13)
and
\[
II_n := |g_{n,2}|^2 + |g_{n,3}|^2 + |g_{n,2}|^2 + |g_{n,3}|^2
\]  
(4.14)
Since
\[
\sum_{n=0}^\infty \|G_n\|^2_* + \sum_{n=1}^\infty \|\tilde{G}_n\|^2_* = \|G\|^2_*,
\]
we infer by summing (4.11) and (4.12) over $n$ that there are constants $C_1$ and $C_2$ independent of $\delta$ such that
\[
C_1 \left( \sum_{n=1}^\infty \frac{n^{-1}I_n - n\rho^{2n}II_n}{\delta^2 + \rho^{4n}} - 1 \right) \leq \|\Phi_\delta^f\|^2_* \leq C_2 \left( \sum_{n=1}^\infty \frac{n^{-1}I_n + n\rho^{2n}II_n}{\delta^2 + \rho^{4n}} + 1 \right).
\]  
(4.15)

If $z(c) = -k_0$, then we obtain in a similar way that
\[
C_1 \left( \sum_{n=1}^\infty \frac{n^{-1}II_n - n\rho^{2n}I_n}{\delta^2 + n^2\rho^{2n}} - 1 \right) \leq \|\Phi_\delta^f\|^2_* \leq C_2 \left( \sum_{n=1}^\infty \frac{n^{-1}II_n + n\rho^{2n}I_n}{\delta^2 + n^2\rho^{2n}} + 1 \right).
\]  
(4.16)
It is worth emphasizing that the quantity $n((\delta^2 + n^2\rho^{2n})$ in the denominator is different from that in (4.15). This discrepancy is caused by the different asymptotic behaviors of eigenvalues near $k_0$ and $-k_0$ as shown in (3.21)–(3.31).

In summary, we obtain the following proposition from Lemma 2.1, (4.15) and (4.16).

**Proposition 4.1.** There are constants $C_1$ and $C_2$ independent of $\delta$ such that

(i) if $z(c) = k_0$, then
\[
C_1 \left( \sum_{n=1}^\infty \frac{n^{-1}I_n - n\rho^{2n}II_n}{\delta^2 + \rho^{4n}} - 1 \right) \leq E(u_\delta^f) \leq C_2 \left( \sum_{n=1}^\infty \frac{n^{-1}I_n + n\rho^{2n}II_n}{\delta^2 + \rho^{4n}} + 1 \right),
\]  
(4.17)

(ii) if $z(c) = -k_0$, then
\[
C_1 \left( \sum_{n=1}^\infty \frac{n^{-1}II_n - n\rho^{2n}I_n}{\delta^2 + n^2\rho^{2n}} - 1 \right) \leq E(u_\delta^f) \leq C_2 \left( \sum_{n=1}^\infty \frac{n^{-1}II_n + n\rho^{2n}I_n}{\delta^2 + n^2\rho^{2n}} + 1 \right).
\]  
(4.18)
4.2 Resonance by dipole sources

In this section we assume that the source function \( f \) in (2.8) is given by the dipole-type function, namely,

\[
f(x) = b^T \nabla_x \left( \begin{bmatrix} \delta_z(x) & 0 \\ 0 & \delta_x(x) \end{bmatrix} a \right),
\]

where \( z = (z_1, z_2)^T \) is the location of the dipole outside \( D_e \), and \( a = (a_1, a_2)^T, b = (b_1, b_2)^T \) are constant vectors. With this source function we obtain the following theorem using Proposition 4.1. In what follows, the notation \( A \sim B \) (\( A \) and \( B \) are two quantities depending on \( \delta \)) indicates that there are constants \( C_1 \) and \( C_2 \) independent of \( \delta \) such that \( C_1 \leq A/B \leq C_2 \).

**Theorem 4.2.** Let \( f \) be given by (4.19), and let \( \rho_z = r/e/|z| \). It holds that

(i) if \( z(c) = k_0 \), then

\[
E(u^\delta) \sim (\ln \delta)^3 \frac{\ln \rho_z}{\rho_z^2} \delta \to 0,
\]

(ii) if \( z(c) = -k_0 \), then

\[
E(u^\delta) \sim (\ln \delta)^{-1} \frac{2\ln \rho_z}{\rho_z} \delta \to 0.
\]

As an immediate consequence we obtain the following corollary.

**Corollary 4.3.** Suppose that \( f \) is given by (4.19).

(i) If \( z(c) = k_0 \), let \( r_* := r_e^2/r_i \). As \( \delta \to 0 \),

\[
\delta E(u^\delta) \to \begin{cases} 
\infty & \text{if } r_e < |z| \leq r_* , \\
0 & \text{if } |z| > r_* .
\end{cases}
\]

(ii) If \( z(c) = -k_0 \), let \( r_{**} := \sqrt{r_e^2/r_i} \). As \( \delta \to 0 \),

\[
\delta E(u^\delta) \to \begin{cases} 
\infty & \text{if } r_e < |z| < r_{**} , \\
0 & \text{if } |z| \geq r_{**} .
\end{cases}
\]

**Proof of Theorem 4.2.** Since \( f \) is assumed to be of the form (4.19), \( F \) defined by (2.12) is given by

\[
F(x) = b^T \nabla_x (\Gamma(x - z)a) = -b^T \nabla_z (\Gamma(x - z)a).
\]

Let \( g_{m,j}, \tilde{g}_{m,j}, j = 1, 2, 3, 4 \), be the coefficients of \( G = (\partial_{u_i} F, -\partial_{v_i} F)^T \) as defined by (4.7). It turns out that \( g_{n,j} \) and \( \tilde{g}_{n,j} \) satisfy

\[
g_{n,1, \tilde{g}_{n,1}} \approx \left( \frac{r_i}{|z|} \right)^n, \quad g_{n,2, \tilde{g}_{n,2}} \approx n^2 \left( \frac{r_i}{|z|} \right)^n,
\]

and

\[
g_{n,3, \tilde{g}_{n,3}} \approx \left( \frac{r_e}{|z|} \right)^n, \quad g_{n,4, \tilde{g}_{n,4}} \approx n^2 \left( \frac{r_e}{|z|} \right)^n.
\]

We include proofs of these estimates in Appendix A.
Suppose that $z(c) = k_0$. Then we infer from (4.13), (4.14), (4.25) and (4.26) that
\begin{equation}
I_n \approx n^4 \rho_z^{2n} \quad \text{and} \quad II_n \approx \rho_z^{2n}.
\end{equation}

It then follows from (4.17) that
\begin{equation}
E(u^\delta) \sim \sum_{n=1}^{\infty} \frac{n^3 \rho_z^{2n}}{\delta^2 + \rho_z^{4n}}.
\end{equation}

For small $\delta$, let $N$ be a positive integer such that
\begin{equation}
\rho_z^{2N} \leq \delta \leq \rho_z^{2(N-1)}.
\end{equation}

Note that $N \sim |\ln \delta|$. Then we have
\begin{equation}
E(u^\delta) \sim \sum_{n<N} + \sum_{n \geq N} \sum_{n<N} \frac{n^3 \rho_z^{2n}}{\rho_z^{4n}} + \sum_{n>N} \frac{1}{\delta^2 n^3 \rho_z^{2n}} =: S_1^\delta + S_2^\delta.
\end{equation}

Since
\begin{equation}
S_1^\delta \sim \int_1^N y^3 e^{2y \ln(\rho_z/\rho^2)} \, dy,
\end{equation}

we have
\begin{equation}
S_1^\delta \sim N^3 e^{2N \ln(\rho_z/\rho^2)} \sim |\ln \delta|^3 \delta \frac{\ln \rho}{\ln \rho - 2}.
\end{equation}

Likewise, since
\begin{equation}
S_2^\delta \sim \delta^{-2} \int_N^\infty y^3 e^{2y \ln(\rho_z)} \, dy,
\end{equation}

we have
\begin{equation}
S_2^\delta \sim \delta^{-2} N^3 e^{2N \ln(\rho_z)} \sim |\ln \delta|^3 \delta \frac{\ln \rho}{\ln \rho - 2}.
\end{equation}

This proves (4.21).

If $z(c) = -k_0$, we have
\begin{equation}
E(u^\delta) \sim \sum_{n=1}^{\infty} \frac{\rho_z^{2n}}{n(\delta^2 + \rho_z^{2n})}.
\end{equation}

In this case we choose $N$, instead of (4.28), so that
\begin{equation}
N \rho^N \leq \delta \leq (N-1) \rho^{N-1}.
\end{equation}

Then we have $\delta/N \sim e^{N \ln \rho}$ and $N \sim |\ln \delta|$. Then, in a way similar to above, one can show (4.21). \hfill \Box

### 4.3 Cloaking due to ALR

Recall that
\begin{equation}
u^\delta = F + S_{\Gamma} [\varphi^\delta] + S_{\Gamma_c} [\varphi^\delta].
\end{equation}

In this section, we show that
\begin{equation}|u^\delta(x)| \leq C, \quad |x| = r > r_0,
\end{equation}
for some \( r_0 > r_e \). Note that the above boundedness means that CALR happens, since we have already proved the energy \( E^\delta \) blows up when a point source located inside the critical radius.

For simplicity, we only consider \( S_{\Gamma_e} [\varphi^\delta_e] \). The potential \( S_{\Gamma_e} [\varphi^\delta_e] \) can be estimated in a similar way. Let us write the solutions of the equations \((4.2)\) using \( \Phi_{n,j} \) and \( \bar{\Phi}_{n,j} \).

\[
\Phi^\delta_n = \sum_{j=0}^{4} \frac{g_{n,j}}{z_\delta + \lambda_{n,j}} \Phi_{n,j}, \quad \bar{\Phi}^\delta_n = \sum_{j=1}^{4} \frac{\tilde{g}_{n,j}}{z_\delta + \lambda_{n,j}} \bar{\Phi}_{n,j},
\]

where \( g_{n,j} \) and \( \tilde{g}_{n,j} \) are given by \((4.7)\). Then, from \((4.1)\), we have

\[
\varphi^\delta_e = \sum_{n=0}^{\infty} \left( \frac{g_{n,3} \varphi_{n,3}}{z_\delta + \lambda_{n,3}} + \frac{\tilde{g}_{n,3} \tilde{\varphi}_{n,3}}{z_\delta + \lambda_{n,3}} \right) + \sum_{n=1}^{\infty} \left( \frac{g_{n,4} \varphi_{n,4} + \tilde{g}_{n,4} \tilde{\varphi}_{n,4}}{z_\delta + \lambda_{n,4}} \right).
\]

Hence, we have

\[
S_{\Gamma_e} [\varphi^\delta_e] = S_{\Gamma_e} [\varphi^\delta_{e,3}] + S_{\Gamma_e} [\varphi^\delta_{e,4}] = S_{\Gamma_e} [\varphi^\delta_{e,3}] + S_{\Gamma_e} [\varphi^\delta_{e,4}] + S_{\Gamma_e} [\bar{\varphi}^\delta_e]. \quad (4.31)
\]

Suppose \( z(c) = k_0 \). Let us estimate \( S_{\Gamma_e} [\varphi^\delta_{e,4}] \) in \((4.31)\). The other terms can be estimated in the same manner. From \((3.3)\), we have

\[
|S_{\Gamma_e} [\varphi^\delta_{e,4}](x)| \lesssim \sum_{n=1}^{\infty} \frac{|g_{n,4}| r^2}{\delta + \rho^{2n} r^{n-1}} \lesssim \sum_{n=1}^{\infty} n^2 \left( \frac{r^3}{r_c} \right)^n \left( \frac{1}{r |z|} \right)^{n-1}.
\]

Therefore, for \( |x| = r > r^3/r^2_i \), we have \( |S_{\Gamma_e} [\varphi^\delta_{e,4}](x)| < C \) for some \( C > 0 \).

Next, suppose \( z(c) = -k_0 \). For the same reason, we only estimate \( S_{\Gamma_e} [\varphi^\delta_{e,3}] \). From \((3.3)\), we have

\[
|S_{\Gamma_e} [\varphi^\delta_{e,3}](x)| \lesssim \sum_{n=1}^{\infty} \frac{|g_{n,3}| r^2}{\delta + \rho^{2n} r^{n-1}} \lesssim \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r^3}{r_i} \right)^n \left( \frac{1}{r |z|} \right)^{n-1}.
\]

Therefore, for \( |x| = r > r^2/r_i \), we have \( |S_{\Gamma_e} [\varphi^\delta_{e,3}](x)| < C \) for some \( C > 0 \).

Since \( \mathbf{F}(x) \to 0 \) as \( |x| \to \infty \), \((4.32)\) and \((4.33)\) together with Theorem \((4.2)\) yield the following theorem.

**Theorem 4.4.** Suppose that \( f \) be given by \((4.19)\).

(i) If \( z(c) = k_0 \), then CALR occurs with the critical radius \( r_* = r^2_e/r_i \); more precisely, \( E^\delta = \delta E(\mathbf{u}^\delta) \to \infty \) as \( \delta \to 0 \) if \( r_e < |z| \leq r_* \), and there is a constant \( C \) such that

\[
|\mathbf{u}^\delta(x)| < C \quad \text{for } |x| = r > r^2_e/r^2_i \quad \text{as } \delta \to 0.
\]

(ii) If \( z(c) = -k_0 \), then CALR occurs with the critical radius \( r_{**} = \sqrt{r^3_e/r_i} \); more precisely, \( E^\delta \to \infty \) as \( \delta \to 0 \) if \( r_e < |z| < r_{**} \), and there is a constant \( C \) such that

\[
|\mathbf{u}^\delta(x)| < C \quad \text{for } |x| = r > r^2_e/r_i \quad \text{as } \delta \to 0.
\]
A Proofs of (4.25) and (4.26)

Here we use the complex representation of the displacement vectors. So we identify \( a, b, z, x \) with complex numbers
\[
a = a_1 + ia_2, \quad b = b_1 + ib_2, \quad z = z_1 + iz_2 \quad \text{and} \quad x = x_1 + ix_2,
\]
respectively, where \( a, b, z \) are vectors appearing in the definition of the dipole source given in (4.19). Let
\[
\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}.
\]
We have
\[
2\mu \left[ (\Gamma(x - z)a)_1 + i (\Gamma(x - z)a)_2 \right] = \kappa \phi(x) - x \phi'(x) - \psi(x),
\]
where
\[
\phi(x) = \frac{a}{2\pi (\kappa + 1)} \ln(x - z),
\]
and
\[
\psi(x) = -\frac{\kappa x}{2\pi (\kappa + 1)} \ln(x - z) - \frac{a}{2\pi (\kappa + 1)} \left( \frac{x}{x - z} - \frac{a}{a} \right).
\]
We emphasize that since \( z \) lies outside \( D_e \), \( \ln(x - z) \) is well-defined as a function of \( x \in D_e \).

By a straightforward computation using the complex potential representation of \( \Gamma(x - z)a \), we can obtain
\[
\left. \left[ (\partial_{\nu} \Gamma(x - z)a)_1 + i (\partial_{\nu} \Gamma(x - z)a)_2 \right] \right|_{\Gamma_e} \tag{A.2}
\]
\[
= \sum_{m=0}^{\infty} \Re \left\{ G_m(z, \overline{z}) \right\} e^{i(m+1)\omega} + \Im \left\{ G_m(z, \overline{z}) \right\} ie^{i(m+1)\omega}
\]
\[
+ \sum_{m=2}^{\infty} \Re \left\{ H_m(z, \overline{z}) \right\} e^{-i(m-1)\omega} + \Im \left\{ H_m(z, \overline{z}) \right\} ie^{-i(m-1)\omega}, \tag{A.3}
\]
where
\[
G_m(z, \overline{z}) = \frac{1}{2\pi (\kappa + 1)} \left( \frac{a}{z} + \frac{\overline{a}}{\overline{z}} \right) \frac{r^m}{s^m},
\]
and
\[
H_m(z, \overline{z}) = \frac{1}{2\pi (\kappa + 1)} \left[ (m - 1) \frac{\overline{a}}{z} - \frac{\overline{z}}{r^2} \left( \kappa a + \frac{\overline{a} z}{s} (m - 1) \right) \right] \frac{r^m}{s^m}.
\]
Since
\[
\partial_{\nu} F|_{\Gamma_e} = -b^T \nabla_z \left( \partial_{\nu} \Gamma(x - z)a \right)_{|\Gamma_e},
\]
then we have
\[
g_{m,3} = -b^T \nabla_z \Re \left\{ G_m(z, \overline{z}) \right\}, \quad \overline{g}_{m,3} = -b^T \nabla_z \Im \left\{ G_m(z, \overline{z}) \right\},
\]
and
\[
g_{m,4} = -b^T \nabla_z \Re \left\{ H_m(z, \overline{z}) \right\}, \quad \overline{g}_{m,4} = -b^T \nabla_z \Im \left\{ H_m(z, \overline{z}) \right\}.
\]
Estimates in (4.26) follows from these.

Estimates in (4.25) can be proved similarly.
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