THE CHROMATIC BRAUER CATEGORY AND ITS LINEAR REPRESENTATIONS

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Abstract. The Brauer category is a symmetric strict monoidal category that arises as a categorification of the Brauer algebras in the context of Banagl’s framework of positive topological field theories (TFTs). We introduce the chromatic Brauer category as an enrichment of the Brauer category in which the morphisms are component-wise labeled. Linear representations of the (chromatic) Brauer category are symmetric strict monoidal functors into the category of real vector spaces and linear maps equipped with the Schauenburg tensor product. We study representation theory of the (chromatic) Brauer category, and classify all its faithful linear representations. As an application, we use indices of fold lines to construct a refinement of Banagl’s concrete positive TFT based on fold maps into the plane.

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1. Introduction

The Brauer algebras $D_m$ have first appeared in work of Brauer [6] on representation theory of the orthogonal group $O(n)$. In view of Schur-Weyl duality, they replace the role played by the group algebras of symmetric groups in representation theory of the general linear group. Generators of $D_m$ are the diagrams consisting of $2m$ vertices and $m$ edges, where the vertices are arranged in two parallel rows of $m$ vertices, and each vertex lies in the boundary of exactly one edge. Given a commutative ground ring $k$ with unit, $D_m$ is the $k$-algebra freely generated as $k$-module by those diagrams. Multiplication is induced by concatenation of diagrams, where each arising free loop component gives rise to an additional multiplication with the indeterminate $x$. A signed variant of Brauer algebras has been studied in [15]. Brauer algebras play an important role in knot theory, where, for instance, Birman-Murakami-Wenzl algebras [5, 14, 18], which are the quantized version of Brauer algebras, have been used to construct generalizations of the Jones polynomial.

We are concerned with a natural categorification $Br$ of Brauer’s algebras that has been constructed by Banagl [3, 4] in search of new topological invariants in the context of his framework of positive topological field theory (TFT). A similar categorification has been considered independently by Lehrer-Zhang [11] in a modern categorical approach to the invariant theory of the orthogonal and symplectic groups. Roughly speaking, morphisms in the so-called Brauer category $Br$ are represented by 1-dimensional unoriented tangles in a high-dimensional Euclidean space. In particular, generators and relations for the strict monoidal category $Br$ have been found in [4] (compare also [11]) by adapting the methods that are used by Turaev [17] for deriving a presentation for the category of tangle diagrams.

Let us discuss the main ideas behind Banagl’s notion of positive TFT, and the role of the Brauer category $Br$ and its representation theory in this context. By definition, the axioms for positive TFT [3] differ from Atiyah’s original axioms for TFT [1] in that they are formulated over semirings instead of rings. Recall that semirings are not required to have additive inverse (“negative”) elements. In computer science, semirings and related structures have been studied by Eilenberg [7] in the context of automata theory and formal languages. The essential advantage of positive TFTs over usual TFTs is that so-called Eilenberg completeness of certain semirings can be used to give a rigorous construction of positive TFTs of arbitrary dimension. This construction is implemented by Banagl in a process of quantization that requires so-called fields and an action functional as input. Inspiration comes from theoretical quantum physics, where the state sum is expressed by fields and an action functional via the Feynman path integral.

In [4], Banagl applies his framework to produce in arbitrary dimension an explicit positive TFT for smooth manifolds. The construction uses singularity theory of so-called fold maps, and the resulting state sum invariants can distinguish exotic smooth spheres from the standard sphere. Now, in this concrete setting, the role of fields is played by certain fold maps into the plane, and the action functional assigns to such fields morphisms in the Brauer category $Br$ by extracting the 1-dimensional patterns that arise from the singular locus of fold maps. However, as pointed out in Section 8 of [3], it is desirable to compose such a category-valued action functional with a symmetric strict monoidal functor from $Br$ to the category $\text{Vect}$ of real vector spaces and linear maps. Note that one requires the category $\text{Vect}$ to be
equipped with a symmetric strict monoidal structure, which is provided by using the Schauenburg tensor product \cite{10}. In this way, the Brauer category serves only as an intermediary structure, and the state sum of the resulting positive TFT will become accessible through linear algebra. Of course, the loss of information should be kept at a minimum during this linearization process, which is motivation for studying faithfulness of such linear representations \( Br \to \text{Vect} \). This knowledge is required when it comes to the explicit computation of state sum invariants (compare Section 6.3 and Section 10.5 in \cite{19} as well as Remark 9.5 in \cite{20}).

In this paper we determine not only the faithful representations of \( Br \), but also those of the chromatic Brauer category \( cBr \) which will be introduced in Section 3.1 as an enrichment of \( Br \) in which morphisms are component-wise labeled (“colored”) by elements of a countable index set. Hence, in contrast to the Brauer category, isomorphic objects of the chromatic Brauer category need not be equal. Our reason for considering \( cBr \) is that it can be used to construct a refinement of Banagl’s positive TFT based on fold maps in the following way (see Section 5). In analogy with the index of non-degenerate critical points in Morse theory, one can associate a (reduced) index to the singularities of a fold map. Those fold indices are intrinsically defined, locally constant along the singular set, and carry topological information about the source manifold. For fold maps from \( n \)-dimensional source manifolds into the plane, the set of possible fold indices is \( \{0, \ldots, [(n-1)/2]\} \). We will modify Banagl’s original construction by defining a \( cBr \)-valued action functional which additionally remembers indices of fold lines as labels from the set \( \mathbb{N} = \{0, 1, 2, \ldots\} \) of natural numbers.

Concerning linear representations of \( Br \), Banagl has shown in Proposition 2.22 of \cite{4} that there exist linear representations that are faithful on loops. This suffices for his purpose to show that state sum invariants of the positive TFT are able to detect exotic smooth structures on spheres. As a much more general result, we have the following

**Theorem 1.1** \((\cite{13}, \cite{19})\). Let \( Y : Br \to \text{Vect} \) be a symmetric strict monoidal functor from the Brauer category into the category of real vector spaces and linear maps (equipped with the Schauenburg tensor product). Then the vector space \( Y([1]) \) has finite dimension \( d \). Moreover the functor \( Y \) is faithful if and only if \( d \geq 2 \).

What is more, we will show Theorem 1.2 below. Since the Brauer category is naturally (monochromatically) embedded in the chromatic Brauer category, Theorem 1.1 is implied by Theorem 1.2 (To conclude this, one has to use that any linear representation of \( Br \) can be extended to one of \( cBr \) by means of our structure results Theorem 3.6 and the corresponding result for \( Br \).)

In preparation of the statement of our result on linear representations of \( cBr \), note that the objects of \( cBr \) that are mapped to the object \([1]\) of \( Br \) under the forgetful functor \( cBr \to Br \) are parametrized by the labels \( k \in \mathbb{N} \), say \( ([1], k) \).

**Theorem 1.2.** Let \( Y : cBr \to \text{Vect} \) be a symmetric strict monoidal functor from the chromatic Brauer category into the category of real vector spaces and linear maps (equipped with the Schauenburg tensor product). Then, for each \( k \in \mathbb{N} \), the vector space \( Y(([1], k)) \) is of finite dimension, say \( d_k \). Suppose that \( d_k > 0 \) for all \( k \in \mathbb{N} \). Then, the functor \( Y \) is faithful if and only if the sequence \( d_0, d_1, \ldots \) satisfies
for all \((l_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{Z}\) the implication

\[
\prod_{k=0}^{\infty} d_k^k = 1 \quad \Rightarrow \quad l_k = 0 \text{ for all } k \in \mathbb{N}.
\]

In particular, faithful linear representations of \(\mathbf{cBr}\) exist because one can take \(d\) to be the sequence of prime numbers, and then apply Theorem 3.6 to construct a strict monoidal functor \(Y: \mathbf{cBr} \to \mathbf{Vect}\) which realizes \(d_k\) for \(k \in \mathbb{N}\) as the dimension of the real vector space \(Y(\{1\}, \mathbb{F})\).

The paper is structured as follows. In Section 2 we recall fundamental facts about monoidal categories in general, and the Schauenburg tensor product in particular. The chromatic Brauer category is introduced in Section 3, where its linear representations are classified by Theorem 3.6. The proof of our main result Theorem 1.2 will be given in Section 4. Finally, in Section 5 we discuss our application to Banagl’s positive TFT based on fold maps.

Notation. Throughout the paper, the natural numbers will be meant to be the set \(\mathbb{N} = \{0, 1, 2, \ldots\}\) (including zero).

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2. Preliminaries on strict monoidal categories

In this section, we introduce the definitions and notational conventions that will be used throughout the paper. We refer to [9] for the basic definitions in this section.

2.1. Strict monoidal categories. A monoidal category \((\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)\) is a category \(\mathbf{C}\) equipped with a bifunctor \(\otimes: \mathbf{C} \times \mathbf{C} \to \mathbf{C}\), an object \(I \in \text{Ob}(\mathbf{C})\), called unit with respect to the tensor product \(\otimes\), and three isomorphisms

\[
\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad \lambda_X: I \otimes X \to X \quad \text{and} \quad \rho_Y: Y \otimes I \to Y,
\]

which are functorial in \(X, Y, Z \in \text{Ob}(\mathbf{C})\). Furthermore, \(\alpha, \lambda\) and \(\rho\) satisfy the coherence conditions given by the pentagon axiom and triangle axiom. These are

\[
\begin{align*}
(W \otimes X) \otimes (Y \otimes Z) &\xrightarrow{\alpha_{W \otimes X, Y \otimes Z}} (W \otimes (X \otimes Y)) \otimes Z \xrightarrow{\alpha_{W, X \otimes Y, Z}} W \otimes ((X \otimes Y) \otimes Z) \\
(W \otimes X) \otimes (Y \otimes Z) &\xrightarrow{\alpha_{W, X \otimes Y, Z}} W \otimes (X \otimes (Y \otimes Z))
\end{align*}
\]

and

\[
\begin{align*}
(Y \otimes I) \otimes X &\xrightarrow{\rho_Y \otimes 1_X} Y \otimes X \\
Y \otimes (I \otimes X) &\xrightarrow{1_Y \otimes \lambda_X} Y \otimes (I \otimes X)
\]
\]
for all $W, X, Y, Z \in \text{Ob}(\mathbf{C})$. Here, $\alpha$ is called \textit{associativity constraint}, and $\lambda$ and $\rho$ are called \textit{left} and \textit{right unit constraints}, respectively. A monoidal category $\mathbf{C}$ is called \textit{strict} if the associativity and unit constraints $\alpha, \lambda, \rho$ are given by identity morphisms of the category.

If $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a (strict) monoidal category, then we will call the data $(\otimes, I, \alpha, \lambda, \rho)$ a (strict) \textit{monoidal structure} on $\mathbf{C}$.

Let $\mathbf{C}$ and $\mathbf{D}$ be monoidal categories. A \textit{monoidal functor} is a functor $F : \mathbf{C} \to \mathbf{D}$ which respects the monoidal structure. To be more precise, it is a functor

$$(F, \xi, \xi_0) : (\mathbf{C}, \otimes, I, \alpha, \lambda, \rho) \to (\mathbf{D}, \otimes, I, \lambda, \rho),$$

with isomorphisms $\xi_{X,Y} : F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$ for all $X, Y \in \text{Ob}(\mathbf{C})$, functorial in $X, Y$, i. e. for morphisms $f : X \to X'$ and $g : Y \to Y'$ the diagram

\[
\begin{array}{ccc}
F(X) \otimes_{\mathbf{D}} F(Y) & \xrightarrow{\xi_{F(X),F(Y)}} & F(X \otimes_{\mathbf{C}} Y) \\
F(f) \otimes_{\mathbf{D}} F(f') & \downarrow & F(f \otimes_{\mathbf{C}} f') \\
F(X') \otimes_{\mathbf{D}} F(Y') & \xrightarrow{\xi_{F(X'),F(Y')}} & F(X' \otimes_{\mathbf{C}} Y')
\end{array}
\]

and an isomorphism $\xi_0 : I_{\mathbf{D}} \to F(I_{\mathbf{C}})$ such that the diagrams

\[
\begin{array}{c}
(F(X) \otimes_{\mathbf{D}} F(Y)) \otimes_{\mathbf{D}} F(Z) \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} F(X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{D}} F(Z) \xrightarrow{\xi_{X,Y,Z}} F((X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{C}} Z) \\
F(X) \otimes_{\mathbf{D}} (F(Y) \otimes_{\mathbf{D}} F(Z)) \xrightarrow{1_{F(X)} \otimes_{\mathbf{D}} \xi_{Y,Z}} F(X \otimes_{\mathbf{C}} (Y \otimes_{\mathbf{C}} Z)) \\
I_{\mathbf{D}} \otimes_{\mathbf{D}} F(X) \xrightarrow{\xi_0 \otimes_{\mathbf{D}} 1_{F(X)}} F(I_{\mathbf{C}}) \otimes_{\mathbf{D}} F(X) \\
F(X) \xrightarrow{\xi_{I_{\mathbf{C}},X}} F(I_{\mathbf{C}} \otimes_{\mathbf{C}} X) \\
\end{array}
\]

\[
\begin{array}{c}
F(X) \otimes_{\mathbf{D}} I_{\mathbf{D}} \xrightarrow{\xi_0} F(X) \otimes_{\mathbf{D}} F(I_{\mathbf{C}}) \\
F(X) \xrightarrow{\lambda_{F(X)}} F(\lambda_{X,C}) \\
\end{array}
\]

commute for all $X, Y, Z \in \text{Ob}(\mathbf{C})$. The monoidal functor $(F, \xi, \xi_0)$ is called \textit{strict} if the isomorphisms $\xi_0$ and $\xi$ are identity morphisms of $\mathbf{D}$.

Let $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A \textit{symmetric braiding} $b$ on $\mathbf{C}$ is a monoidal functorial isomorphism $b : \otimes \to \otimes \circ \tau$, where the map $\tau : \text{Ob}(\mathbf{C}) \times \text{Ob}(\mathbf{C}) \to \text{Ob}(\mathbf{C}) \times \text{Ob}(\mathbf{C})$ is given by $\tau(X, Y) = (Y, X)$ for any pair $(X, Y)$ of objects of the category $\mathbf{C}$, satisfying the \textit{hexagon axiom}, the \textit{unity coherence}, and
the inverse law which are given by the commutativity of the diagrams

\[
\begin{array}{c}
\begin{array}{c}
(X \otimes Y) \otimes Z \\
\alpha_{X,Y,Z}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{b_{X,Y} \otimes 1_Z} \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \otimes (Y \otimes Z) \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{b_{X,Y} \otimes Z} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(Y \otimes X) \otimes Z \\
\alpha_{Y,X,Z}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{1_Y \otimes b_{X,Z}} \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
Y \otimes (X \otimes Z) \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{b_{X,Y} \otimes Y} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(Y \otimes X) \otimes Z \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{b_{X,Y} \otimes Z} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(Y \otimes Z) \otimes X \\
\alpha_{Y,Z,X}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

respectively, for all \(X, Y, Z \in \text{Ob}(C)\).

A monoidal category \(C\) together with a symmetric braiding \(b\) is called symmetric monoidal category. For most of the time, we will work with strict monoidal categories. In that case, we will omit the associativity and unit constraints in the notation of a monoidal category, i.e. we will write \((C, \otimes, I)\) instead of \((C, \otimes, I, \alpha, \lambda, \rho)\).

Let \((C, \otimes, I_C, b_C)\) and \((D, \otimes, I_D, b_D)\) be symmetric strict monoidal categories. A monoidal functor \((F, \xi, \xi_0): C \rightarrow D\) is called symmetric if \(F\) is compatible with the symmetric structures, i.e. if \(F(b_{C,X,Y}) = \xi_{F(Y),F(X)} \circ b_{D,F(X),F(Y)} \circ \xi_{X,Y}^{-1}\) for all \(X, Y \in \text{Ob}(C)\).

Let \((C, \otimes, I, \alpha, \lambda, \rho, b)\) be a symmetric monoidal category. An object \(X \in \text{Ob}(C)\) is called dualizable if there exists a triple \((X^*, i_X, e_X)\) consisting of an object \(X^* \in \text{Ob}(C)\), called a dual of \(X\) and morphisms \(i_X: I \rightarrow X^* \otimes X\), and \(e_X: X \otimes X^* \rightarrow I\), called unit and counit respectively, such that the diagrams

\[
\begin{array}{c}
\begin{array}{c}
X \otimes I \\
\xrightarrow{1_X \otimes i_X} \\
\rho_X
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \otimes (X^* \otimes X) \\
\xrightarrow{\alpha_{X,X^*,X}^{-1}} \\
(X \otimes X^*) \otimes X
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\lambda_{X^*}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
I \otimes X \\
\xrightarrow{1_I \otimes 1_{X^*}} \\
I \otimes (X^* \otimes X)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\alpha_{X^*,X,X^*}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \otimes X^* \\
\xrightarrow{e_X \otimes 1_X} \\
X \otimes I
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\rho_{X^*}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X^* \\
\xrightarrow{i_X \otimes 1_{X^*}} \\
X^* \otimes (X \otimes X^*)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\lambda_{X^*}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X^* \otimes I \\
\xrightarrow{1_{X^*} \otimes e_X} \\
I \otimes X
\end{array}
\end{array}
\end{array}
\end{array}
\]

commute. A symmetric strict monoidal category \((C, \otimes, I, \beta)\) is called compact if every object is dualizable. Let \(i\) denote the family of unit morphisms and let
denote the family of counit morphisms. We then write \((C, \otimes, I, \beta, i, e)\) for a compact category.

2.2. The Schauenburg tensor product. The usual tensor product \(\otimes\) in the category \(\text{Vect}\) of real vector spaces and linear maps determines a monoidal structure on \(\text{Vect}\) (with unit object \(I = \mathbb{R}\)) which is certainly not strict. When studying linear representations of strict monoidal categories, it is appropriate to endow \(\text{Vect}\) with a strict monoidal structure. As described in Theorem XI.5.3 in [9, p. 291], there is a general method for turning any given monoidal category \(C\) into a monoidally equivalent strict monoidal category \(C^{\text{str}}\). However, this procedure changes the category even on the object level, which is probably not convenient for studying properties of linear representations. Banagl employs an explicit strict monoidal construction is that the monoidal equivalence \(\pi\) for \(B\) and \(V,W\) all \(\otimes\) and \(\beta\) the identity on \(V\) and \(W\). Given elements \(u \in U, v \in V\) and \(w \in W\) the identity \((u \otimes v) \otimes w = u \otimes (v \otimes w)\) holds because of \((U \otimes V) \otimes W = U \otimes (V \otimes W)\).

There is a standard symmetric braiding \(\beta_{V,W}: V \otimes W \to W \otimes V\) on \(\text{Vect}\) for all \(V, W \in \text{Ob}(\text{Vect})\) (with respect to the standard tensor product \(\otimes\)). By defining \(b_{V,W} = \xi_{W,V} \circ \beta_{V,W} \circ \xi_{V,W}^{-1}\) we obtain a symmetric braiding with respect to \(\otimes\). All in all, the following proposition holds.

**Proposition 2.1.** The data \((\text{Vect}, \otimes, \mathbb{R}, b)\) define a symmetric strict monoidal category.

For the rest of this paper, we will use the Schauenburg tensor product on \(\text{Vect}\); thus, we will from now on write \(\otimes\) instead of \(\circ\).

**Remark 2.2.** If the vector spaces \(V\) and \(W\) are equipped with ordered bases \(\{v_i\}_i\) and \(\{w_j\}_j\), respectively, then we let the tensor product \(V \otimes W\) be equipped with the lexicographically ordered basis \(\{v_i \otimes w_j\}_{i,j}\). Given two morphisms \(A: V \to V'\) and \(B: W \to W'\) in \(\text{Vect}\), the tensor product \(A \otimes B: V \otimes W \to V' \otimes W'\) is defined. Suppose that each of the vector spaces \(V, W, V', W'\) is equipped with an ordered basis. If \(V \otimes W\) and \(V' \otimes W'\) are finite-dimensional, then it is well-known that the matrix representation of \(A \otimes B\) is given by the Kronecker product of the matrix representations of \(A\) and \(B\) with respect to the fixed bases.

Schauenburg’s result applies more generally to categories of structured sets. Essentially, a category \(C\) is called a category of structured sets (see Definition 4.1 in [16]) if there is a faithful functor from \(C\) into the category of sets, such that for any object in \(C\) and any bijection of the underlying set into another set there is a unique object in \(C\) and a unique morphism in \(C\) realizing the bijection. As pointed out by Schauenburg, all categories of algebraic structures are categories of structured sets by means of the forgetful functor to the category of sets and transfer of the algebraic structures. In particular, this applies to the (non-strict) monoidal category of real vector spaces with the standard monoidal structure \((\text{Vect}, \otimes, \mathbb{R}, \alpha, \lambda, \rho)\).

**Theorem 2.3** (cf. [16], Theorem 4.3). Let \(C\) be a category of structured sets. Then for each monoidal structure \((\otimes, I, \alpha, \lambda, \rho)\) on \(C\), there exists a strict monoidal
structure \((\otimes, I)\) with the same unit object \(I\) such that
\[
(Id_C, \xi, 1_I): (C, \otimes, I) \to (C, \otimes, I, \alpha, \lambda, \rho)
\]
is a monoidal category equivalence.

3. The chromatic Brauer category and its linear representations

3.1. The chromatic Brauer category. Recall that the Brauer category \(\text{Br}\) is considered in Section 10 of [3] and Section 2.3 of [4], where Banagl defines it as a natural categorification of Brauer algebras. Roughly speaking, isomorphism classes of finite sets serve as objects, and morphisms are isometry classes of unoriented tangles in Euclidean 4-space. We shall next introduce the chromatic Brauer category \(c\text{Br}\) as a certain enrichment of the Brauer category. Namely, we equip the components of objects and morphisms with colorings using a countable number of colors. The structure of our discussion will closely follow the above references to facilitate comparisons. Note that both categories \(\text{Br}\) and \(c\text{Br}\) have the structures of strict monoidal categories that are compact and symmetric. However, in contrast to the Brauer category, isomorphic objects of the chromatic Brauer category will not necessarily be equal.

Let us formally introduce the category \(c\text{Br}\). Given \(m = 1, 2, \ldots\), let \([m]\) denote the set \([1, \ldots, m]\). Let \([0]\) denote the empty set. We will also use the notation \(M[m] = \{1, \ldots, m\} \subset \mathbb{R}^1\) if we want to consider the set \([m]\) as a 0-submanifold of \(\mathbb{R}^1\). The objects of \(c\text{Br}\) are pairs of the form \([(m), c]\), where \(c\) is some map \(c: [m] \to \mathbb{N}\). For \(m = 0\), \(c = c_{\emptyset}\) denotes the unique map \(\emptyset \to \mathbb{N}\). Morphisms \([(m), c] \to [(m'), c']\) in \(c\text{Br}\) are represented by pairs \((W, \Omega)\), where

- \(W\) is a 1-cobordism from \([m]\) to \([m']\) together with a smooth embedding \(W \hookrightarrow [0, 1] \times \mathbb{R}^3\), where the boundary satisfies \(\partial W = W \cap \{0\} \times \mathbb{R}^3\) and
  \[
  \partial W \cap \{0\} \times \mathbb{R}^3 = \{0\} \times M[m] \times \{0\} \times \{0\},
  \partial W \cap \{1\} \times \mathbb{R}^3 = \{1\} \times M[m'] \times \{0\} \times \{0\},
  \]
such that near the boundary, \(W\) is embedded as the product
  \[
  [0, \varepsilon] \times M[m] \times \{0\} \times \{0\} \cup [1 - \varepsilon, 1] \times M[m'] \times \{0\} \times \{0\}
  \]
  for some \(\varepsilon > 0\), and
- \(\Omega\) is a locally constant map \(\Omega: W \to \mathbb{N}\) such that \(\Omega|_{[m]} = c\) and \(\Omega|_{[m']} = c'\).

Given \([(m), c]\) and \([(m'), c']\), two such pairs \((W_0, \Omega_0)\) and \((W_1, \Omega_1)\) determine the same morphism in \(c\text{Br}\) if there is a diffeomorphism \(\alpha: W_0 \to W_1\) of cobordisms such that the embedding of \(W_0\) is smoothly isotopic to the composition of \(\alpha\) with the embedding of \(W_1\), by an isotopy that is the identity near \(\{0, 1\} \times \mathbb{R}^3\), and \(\Omega_0 = \Omega_1 \circ \alpha\).

The composition of two morphisms \(\varphi: [(m), c] \to [(m'), c']\) and \(\psi: [(m'), c'] \to [(m''), c'']\) is defined by composing representatives of \(\varphi\) and \(\psi\) followed by rescaling. That is, if \(\varphi\) is represented by the pair \((W', \Omega')\) and \(\psi\) by the pair \((W'', \Omega'')\), then \(\psi \circ \varphi\) is represented by the pair \((W', \Omega)\), where \(W = W' \cup_{[m']} W''\) and \(\Omega|_{W'} = \Omega', \Omega|_{W''} = \Omega''\). The smooth embedding \(W \hookrightarrow [0, 1] \times \mathbb{R}^3\) is given by translating the embedding \(W'' \hookrightarrow [0, 1] \times \mathbb{R}^3\) to \([1, 2] \times \mathbb{R}^3\), gluing the embeddings \(W' \hookrightarrow [0, 1] \times \mathbb{R}^3\) and \(W'' \hookrightarrow [1, 2] \times \mathbb{R}^3\) at \(M[m']\), and then reparametrizing the interval \([0, 2]\) to \([0, 1]\). The identity morphism \(1_{([0], \emptyset)}: ([0], \emptyset) \to ([0], \emptyset)\) is represented by the
empty 1-cobordism $\emptyset$ together with the unique map $\emptyset \to \mathbb{N}$. For $m > 0$, the identity morphism $1_{([m], c)}$ on $([m], c)$ is represented by the product cobordism $[0, 1] \times M[m] \times \{0\} \times \{0\}$ and the map $\Omega = c \circ \text{proj}[m]$. This completes the definition of the category $\text{cBr}$. Note that $\text{Hom}_{\text{cBr}}(([m], c), ([m'], c'))$ is empty if and only if there is a positive integer $k \in \mathbb{N}$ for which the number $|c^{-1}(k)| + |c'^{-1}(k)|$ is odd. Notationally, for $k \in \mathbb{N}$ we denote by $\xi : [m] \to \mathbb{N}$, $\xi(i) = k$ for all $i \in [m]$.

For later reference, we record the following.

**Lemma 3.1.** Let $\iota : ([m], c) \to ([m'], c')$ be an isomorphism in $\text{cBr}$. Then, $m = m'$, and $\iota$ is uniquely determined by the object $([m], c')$ and a permutation $\sigma$, of the set $\{1, \ldots, m\}$.

**Proof.** If $\iota$ is represented by $(W, \Omega)$ and an embedding $W \hookrightarrow [0, 1] \times \mathbb{R}^3$, then every component of $W$ is diffeomorphic to $[0, 1]$ with one endpoint being mapped to $\{0\} \times \mathbb{R}^3$ and the other to $\{1\} \times \mathbb{R}^3$. Hence, we have $m = m'$, and $W$ induces a permutation $\sigma$, of the set $\{1, \ldots, m\}$. Hence, $\iota$ is uniquely determined by the object $([m], c)$ and the underlying permutation $\sigma$. In particular, observe that $c' = c \circ \sigma^{-1}$.

We equip $\text{cBr}$ with the structure $(\text{cBr}, \otimes, I)$ of a strict monoidal category as follows. The tensor product $\otimes : \text{cBr} \times \text{cBr} \to \text{cBr}$ is defined by “stacking” of objects and morphisms. More precisely, given objects $([m], c)$ and $([m'], c')$, we let $([m], c) \otimes ([m'], c') = ([m + m'], c''')$, where $c'''(k) = c(k)$ whenever $k \in [m]$ and $c'''(k) = c'(k - m)$ whenever $k \in [m + m'] \setminus [m]$. Given two morphisms $\varphi : ([m], c) \to ([m'], c')$ and $\psi : ([n], b) \to ([n'], b')$ represented by pairs $(W_{\varphi}, \Omega_{\varphi})$ and $(W_{\psi}, \Omega_{\psi})$, respectively, the tensor product $\varphi \otimes \psi$ is represented by the pair $(W_{\varphi} \sqcup W_{\psi}, \Omega_{\varphi} \sqcup \Omega_{\psi})$. More precisely, the embedding $W_{\varphi} \sqcup W_{\psi} \hookrightarrow [0, 1] \times \mathbb{R}^3$ is an extension of $W_{\varphi} \hookrightarrow [0, 1] \times \mathbb{R}^3$ by an embedding of $W_{\psi}$, which is obtained by first applying the translation $(x, y, z, t) \mapsto (x, y + m, z, t)$ to the original embedding $W' \hookrightarrow [0, 1] \times \mathbb{R}^3$, then modifying the embedding near the $[n']$-endpoints of $W_{\psi}$ appropriately to connect to the correct points in $\{1\} \times M[m' + n'] \times \{0\} \times \{0\}$, and finally making the resulting embedding disjoint to the embedding of $W_{\psi}$ by means of a small isotopy. Let the unit object $I$ be $([0], c_{\emptyset})$. Then it is easy to check that $(\text{cBr}, \otimes, I)$ defines a strict monoidal category (compare Section 2.1). Note that, in contrast to the original Brauer category, the tensor product of $\text{cBr}$ is not even commutative on objects.

Next we equip $(\text{cBr}, \otimes, I)$ with a compact, symmetric structure $(\text{cBr}, \otimes, I, b, i, e)$ by introducing families of morphisms called braiding $b$, unit $i$ and counit $e$. Given two objects $([m], c)$ and $([m'], c')$ in $\text{cBr}$ we define the braiding

$$b_{([m], c), ([m'], c')} : ([m], c) \otimes ([m'], c') \to ([m'], c') \otimes ([m], c)$$
to be the unique isomorphism whose underlying cobordism is represented by the loop-free diagram

\[
\begin{array}{c}
m + m' \\
\vdots \\
m + 1 \\
m \\
\vdots \\
1 \\
m' + m \\
\vdots \\
m' + 1 \\
m' \\
\vdots \\
1 \\
\end{array}
\]

Given an object \([m], c\) in \(c\text{Br}\) we define the unit \(i_{([m], c)} : ([0], c_{\emptyset}) \to ([m], c)\) to be the unique morphism whose underlying cobordism is represented by the loop-free diagram

\[
\begin{array}{c}
2m \\
\vdots \\
m + 1 \\
m \\
\vdots \\
1 \\
\end{array}
\]

Given an object \([m], c\) in \(c\text{Br}\) we define the counit \(e_{([m], c)} : ([m], c) \otimes ([m], c) \to ([0], c_{\emptyset})\) to be the unique morphism whose underlying cobordism is represented by the loop-free diagram

\[
\begin{array}{c}
2m \\
\vdots \\
m + 1 \\
m \\
\vdots \\
1 \\
\end{array}
\]

Then it is easy to check that \((c\text{Br}, \otimes, I, b, i, e)\) defines a compact, symmetric, strict monoidal category. Hereafter we write \(i_{(k)}\), \(e_{(k)}\) and \(b_{(k,l)}\), where \(k, l \in \mathbb{N}\) are called the elementary morphisms of \(c\text{Br}\).

For every \(k \in \mathbb{N}\), there is precisely one endomorphism \(\lambda_{(k)} : ([0], c_{\emptyset}) \to ([0], c_{\emptyset})\), such that \(\lambda_{(k)}\) is represented by \((S^1, \Omega_k)\), where \(S^1\) is the circle and \(\Omega_k \equiv k\). We will call this endomorphism \(\lambda_{(k)}\) the \(k\)-loop. The endomorphisms in \(c\text{Br}\) of the identity object \(I\) are then given by \(\text{End}_{c\text{Br}}(I) = \{\lambda_{(0)}^m \otimes \cdots \otimes \lambda_{(m)}^m \mid \exists m \in \mathbb{N} : n_0, \ldots, n_m \in \mathbb{N}\}\), where \(\lambda_{(k)}^n\) denotes the \(n\)-fold tensor product \(\lambda_{(k)} \otimes \cdots \otimes \lambda_{(k)}\) and \(\lambda_{(0)}^0 = 1_{\emptyset}\). The \(k\)-loop endomorphism \(\lambda_{(k)}\) can be factored as \(\lambda_{(k)} = e_{(k)} \circ i_{(k)}\). Every such \(k\)-loop commutes with every other morphism \(\varphi \in \text{Mor}(c\text{Br})\), that is \(\lambda_{(k)} \otimes \varphi = \varphi \otimes \lambda_{(k)}\). In particular, any isomorphism in \(c\text{Br}\) can never contain a \(k\)-loop. Loops are persistent, that is, if \(\varphi : ([m], c) \to ([m'], c')\) has \(\nu\) \(k\)-loops and \(\psi : ([m'], c') \to ([m''], c'')\) has \(\mu\) \(k\)-loops, then \(\psi \circ \varphi\) has at least \(\nu + \mu\) \(k\)-loops. Also,
loops are cancellative: if $\lambda_{(k)} \otimes \varphi = \lambda_{(k)} \otimes \psi$, then $\varphi = \psi$.

The usual Brauer category $\text{Br}$ introduced in \cite{3} Section 10 can be obtained by $\text{cBr}$ by forgetting about the coloring maps $c$ and $\Omega$ as a data of objects and morphisms. Naturally, we find the forgetful functor $\mathfrak{f}$ by word $w$\). Let

**3.2. Presentation of the chromatic Brauer category.** Next we will discuss a presentation of the chromatic Brauer category $\text{cBr}$. For that purpose we will recall the notion of presenting monoidal categories from Section 2, page 14, of \cite{4}. Let $(C, \otimes, I)$ be a strict monoidal category and $G$ be a collection of morphisms of $C$. Interpreting $G$ as an alphabet, we may form words as follows: For all $g \in G$ and $X \in \text{Ob}(C)$, $[g]$ and $[1_X]$ are words. If $w_1$ and $w_2$ are words, then the string $(w_1 \otimes w_2)$ is a word and the string $(w_2 \circ w_1)$ whenever $\text{cod} w_1 = \text{dom} w_2$. Every word $w$ determines a morphism $|w|$ of $C$ by the rules

$$|[g]| = g, \quad |[1_X]| = 1_X, \quad |w_1 \otimes w_2| = |w_1| \otimes |w_2|, \quad |w_2 \circ w_1| = |w_2| \circ |w_1|.$$ 

Two words are called freely equivalent (we write $\sim$), if they can be obtained from each other by a finite sequence of subword substitutions implementing associativity for $\circ$ and $\otimes$, identity cancellation for $\circ$ and $\otimes$ and compatibility between $\circ$ and $\otimes$. Note that if $w_1$ and $w_2$ are freely equivalent, then $|w_1| = |w_2|$.

**Lemma 3.2** (\cite{9}, Lemma XII.1.2.(c)). Any word in $G$ is freely equivalent to a word of the form $[1_X]$ for some object $X$ or to a word of the form

$$([1_{X_1}] \otimes [f_1] \otimes [1_{Y_1}]) \circ ([1_{X_2}] \otimes [f_2] \otimes [1_{Y_2}]) \circ \cdots \circ ([1_{X_k}] \otimes [f_k] \otimes [1_{Y_k}])$$

with morphisms $f_1, \ldots, f_k \in G_{\text{and objects}} X_1, \ldots, X_k, Y_1, \ldots, Y_k \in \text{Ob}(C)$.

Also note the following useful fact which is used in the proof of Theorem 3.4 below.

**Lemma 3.3.** Let $(C, \otimes, I)$ be a strict monoidal category, then the group $(\text{End}_C(I), \circ)$ is abelian.

**Proof.** Let $\varphi, \psi \in \text{End}_C(I)$ and write $1$ for the map $1_I$, then the statement follows from free equivalence.

$$\varphi \circ \psi = (1 \otimes \varphi) \circ (\psi \otimes 1) = (1 \circ \psi) \otimes (\varphi \circ 1) = \psi \otimes \varphi$$

$$= (\psi \circ 1) \otimes (1 \circ \varphi) = (\psi \otimes 1) \circ (1 \otimes \varphi) = \psi \circ \varphi.$$ 

Let $\mathcal{F}(G)$ denote the class of free equivalence classes of words in $G$. Then the realization $|\cdot|$ is still well defined on $\mathcal{F}(G)$. Let $R$ be the collection of pairs $(w_1, w_2)$ of words in $G$ such that $|w_1| = |w_2|$. For two elements $x, y \in \mathcal{F}(G)$ we define $x \sim_R y$, and say that $x, y$ are $R$-equivalent, if and only if one can obtain some representative of $y$ from some representative of $x$ by a finite sequence of subword substitutions, where an allowable substitution consists of replacing a subword $w_1$
by \(w_2\) for \((w_1, w_2) \in R\). We say that \((C, \otimes, I)\) is generated by the generators \(G\) and the relations \(R\), if

- any morphism in \(C\) can be obtained as \(|w|\) for some word \(w\) in \(G\), and
- for any \(x, y \in F(G)\), we have \(x \sim_R y\) if and only if \(|x| = |y|\) in \(C\).

The structure of morphisms in \(cBr\) is then elucidated by the following Theorem 3.4.

Note that for the Brauer category \(Br\) the structure is obtained by forgetting the colorings of generators and relations via \(?\), see [11], Theorem 2.6. We will denote the set of generators of \(Br\) by \(G_0\) and the set of relations of \(Br\) by \(R_0\).

**Theorem 3.4 (Presentation of \(cBr\)).** The compact symmetric strict monoidal category \(cBr\) is generated by the elementary morphisms \(G\) and the relations \(R\), if any morphism in \(C\) can be obtained as \(|w|\) for some word \(w\) in \(G\), and for any \(x, y \in F(p, q)\), we have \(x \sim_R y\) if and only if \(|x| = |y|\) in \(C\).

The compact symmetric strict monoidal category \(cBr\) is generated by the elementary morphisms \(G\) and the following relations \(R\), where \(1_{(k)}\) denotes the identity on \(D\) for \(k \in \mathbb{N}\):

(C1) Zig-Zag (straightening):

\[
(e_{(k)} \otimes 1_{(k)}) \circ (1_{(k)} \otimes i_{(k)}) = 1_{(k)},
\]

(C2) Sliding:

\[
(e_{(k)} \otimes 1_{(l)}) \circ (1_{(k)} \otimes b_{(l,k)}) = (1_{(l)} \otimes e_{(k)}) \circ (b_{(k,l)} \otimes 1_{(k)}),
\]

(C3) Reidemeister 1 (de-looping):

\[
b_{(k,k)} \circ i_{(k)} = i_{(k)},
\]

(C4) Reidemeister 2 (double crossing):

\[
b_{(l,k)} \circ b_{(k,l)} = 1_{(k)} \otimes 1_{(l)},
\]

(C5) Reidemeister 3 (braiding relation, a.k.a. Yang-Baxter equation):

\[
(1_{(l)} \otimes b_{(j,k)}) \circ (b_{(j,l)} \otimes 1_{(k)}) \circ (1_{(j)} \otimes b_{(k,l)})
= (b_{(k,l)} \otimes 1_{(j)}) \circ (1_{(k)} \otimes b_{(j,l)}) \circ (b_{(j,k)} \otimes 1_{(l)}),
\]

(In the above diagrams, differently structured lines correspond to independent labels.)

**Proof.** Let \(\varphi\) be a morphism in \(cBr\) and let \((W, \Omega)\) be a representative of \(\varphi\). Then \(?(\varphi)\) is a morphism in \(Br\), therefore it can be expressed as a word in \(i_1, e_1\) and \(b_{1,1}\) (i.e. having the form of the expression (3.1)). Then \(W\) is a gluing of cobordisms
representing $e_1$, $i_1$ and $b_{1,1}$ in $\text{Br}$. Finally, we extend those cobordisms to pairs in $\text{cBr}$ by restriction of $\Omega$.

Now, let $\varphi: ([m], c) \to ([m'], c')$ be a morphism in $\text{cBr}$ and let $x$ and $y$ be free equivalence classes in $\mathcal{F}(G)$ having $\varphi$ as their realizations. Let $x_0$ and $y_0$ be the induced free equivalence classes in $\mathcal{F}(G_0)$ ($G_0 = \{i_1, e_1, b_{1,1}\} \subset \text{Mor}(\text{Br})$) having $\varphi_0 = \varphi(\varphi)$ as their realizations. Therefore, $x_0 \sim_{R_0} y_0$. On the other hand, $\varphi_0$ can be written as $\varphi_0 = \lambda^\otimes \otimes \varphi_0^\otimes$ where $\varphi_0^\otimes$ is loop-free. $\varphi_0^\otimes$ can be obtained as $|w_0|$ for some word $w_0$ in $G_0$ of the form (3.1). Thus, $\varphi_0$ can be obtained by the expression $w = [(1_{[m']} \otimes e_1) \circ (1_{[m]} \otimes i_1)]^w \circ w_0$ of the form (3.1). We choose a word $w_x$ of the form (3.1) representing $x_0$. There exists a word $w'_x$ in $G$ of the form (3.1) representing $x$ such that $w_x$ is the induced free equivalence class in $\mathcal{F}(G_0)$. Indeed, the choice colorings is made as in the first half of the proof. Since $|w_x| = |w|$ one can obtain $w$ from $w_x$ by a finite sequence of subword substitutions by relations of $R_0$. In each of these steps we can easily extend the colorings to the next word. All in all, we obtain a lifting $w''$ of $w$ in $\text{cBr}$ such that $w'$ is obtained from $w''$ by a finite sequence of subword substitutions using relations from $R$. Analogously, there exists $w'_r$ and a lifting $w'''$ of $w$ such that $w'''$ is obtained from $w''$ by a finite sequence of subword substitutions using relations from $R$. It remains to show that $w'''$ is obtained from $w'$ in this way. Note that both of them colorings of $w$, the induced colorings of $w_0$ must be identical because on components with non-empty boundary the coloring is determined by the coloring of the boundary points. Therefore it suffices to show that loops of different colors commute, but this is precisely the statement of Lemma 3.3 for $C = \text{cBr}$. $$\square$$

3.3. Linear representations of the chromatic Brauer category. For the purpose of constructing strict monoidal functors $Y: \text{cBr} \to \text{Vect}$, where $\text{Vect}$ is equipped with the Schauenburg tensor product, we recall the notion of a duality structure on a finite dimensional real vector space $V$ from Definition 2.5 in [4]. Namely, a duality structure on $V$ is a pair $(i, e)$ whose components are a symmetric copairing $i: \mathbb{R} \to V \otimes V$ and a symmetric pairing $e: V \otimes V \to \mathbb{R}$, also called unit and counit, respectively, satisfying the zig-zag equation $(e \otimes 1_V) \circ (1_V \otimes i) = 1_V$. This also means that $V$ is dualizable with dual $V$, hence $V$ is self-dual.

Now let $d$ be the dimension of $V$ and let $\{v_1, ..., v_d\}$ be a basis of $V$. Then the set of all duality structures on $V$ stands in an $1$-$1$ correspondence to the symmetric and invertible $(d \times d)$-matrices $\text{Sym}(d, \mathbb{R}) \cap \text{Gl}(d, \mathbb{R})$. Indeed, let $e_{jk} = e(v_j \otimes v_k)$ and write $X := \text{Mat}(e) = (e_{jk})_{j,k=1}^d$, then $X$ is symmetric due to the symmetry of $e$, and $-$ by the zig-zag equation $-X$ is invertible with inverse $X^{-1} = \text{Mat}(i) = (i_{jk})_{j,k=1}^d$, where $i_{jk}$ are given by $i(1) = \sum_{j,k=1}^d i_{jk} v_j \otimes v_k$. Conversely, let $X \in \text{Sym}(d, \mathbb{R}) \cap \text{Gl}(d, \mathbb{R})$ be symmetric and invertible. Then the matrices $\text{vec}(X^{-1})$ and $\text{vec}(X)^T$ define a symmetric copairing and pairing such that the zig-zag equation is satisfied, where $(\cdot)^T$ denotes the transposition of a matrix and $\text{vec}(\cdot)$ denotes the vectorization of a matrix formed by stacking the columns of the matrix into a single column vector.

The trace of the duality structure $(i, e)$ on $V$ is defined by $\text{Tr}(i, e) = e \circ i$. By the above description of duality structures we easily compute $\text{Tr}(i, e)(1) = d = \text{dim}(V)$.

If $Y: \text{cBr} \to \text{Vect}$ is a linear representation of $\text{cBr}$, then $Y(([1], k)) = V_k$ is for all $k \in \mathbb{N}$ a finite dimensional vector space (cf. Proposition 2.7 in [4]) since
the pair \((Y(\iota(k)), Y(\epsilon(k)))\) forms a duality structure on \(V_k\). Note that \(Y(\lambda(k)) = Y(\epsilon(k)) \circ Y(\iota(k)) = \delta_k\).

Since \(Y\) is required to be symmetric, we also have \(Y(b_{k,l}, i) = b_{k,l}, \eta\), where \(b_{k,l}, \eta \cdot V_k \otimes V_l \rightarrow V_l \otimes V_k\) is the braiding isomorphism induced by \(v_k \otimes v_l \mapsto v_l \otimes v_k\).

Given a strict monoidal category \(\mathbf{C}\) which is presented by generators and relations, the following result provides a construction of strict monoidal functors on \(\mathbf{C}\).

**Proposition 3.5** ([9], Proposition XII.1.4). Let \((\mathbf{C}, \otimes, \mathbf{I})\) and \((\mathbf{D}, \otimes, \mathbf{I})\) be strict monoidal categories. Suppose that \(\mathbf{C}\) is generated by morphisms \(G\) and relations \(R\). Let \(F_0 : \text{Ob}(\mathbf{C}) \rightarrow \text{Ob}(\mathbf{D})\) be a map such that \(F_0(\mathbf{I}) = \mathbf{I}\) and \(F_0(X \otimes Y) = F_0(X) \otimes F_0(Y)\) for all \(X, Y \in \text{Ob}(\mathbf{C})\). Let \(F_1 : G \rightarrow \text{Mor}(\mathbf{D})\) be a map such that \(\text{dom}(F_1(g)) = F_0(\text{dom}(g))\) and \(\text{cod}(F_1(g)) = F_0(\text{cod}(g))\). Suppose that any pair \((w_1, w_2) \in R\) yields equal morphisms in \(\mathbf{D}\) after replacing any symbol \(g \in G\) of \(w_1\) and \(w_2\) by \(F_1(g)\) and any symbol \(1_X\) by \(1_{F_0(X)}\). Then there exists a unique strict monoidal symmetric functor \(F : \mathbf{C} \rightarrow \mathbf{D}\) such that \(F(X) = F_0(X)\) for all \(X \in \text{Ob}(\mathbf{C})\) and \(F(g) = F_1(g)\) for all \(g \in G\).

Now, using Theorem 3.4, we may construct strict monoidal functors \(Y : \mathbf{cBr} \rightarrow \text{Vect}\) by choosing duality structures.

**Theorem 3.6** (Linear representations of \(\mathbf{cBr}\)). Let \(V_k\) be a finite dimensional real vector space and let the pair \((\iota(k), \epsilon(k))\) be a duality structure on \(V_k\) for all \(k \in \mathbb{N}\). Then there exists a unique strict monoidal symmetric functor \(Y : (\mathbf{cBr}, \otimes, \mathbf{I}, \iota) \rightarrow (\text{Vect}, \otimes, \mathbb{R}, \iota)\), which satisfies \(Y([1], \mathbf{k}) = V_k\) and preserves duality, i.e. \(Y(\iota(k)) = \iota(k)\) and \(Y(\epsilon(k)) = \epsilon(k)\) for all \(k \in \mathbb{N}\).

**Proof.** Set \(Y_0 : \text{Ob}(\mathbf{cBr}) \rightarrow \text{Ob}(\text{Vect})\) by fixing the images of \([1], \mathbf{k}\) via \(Y_0([1], \mathbf{k}) = V_k\) for all \(k \in \mathbb{N}\) and extend \(Y_0\) to all of \(\text{Ob}(\mathbf{cBr})\) while respecting the strict monoidal structure, in particular we have \(Y_0(\mathbf{I}) = \mathbb{R}\).

By Theorem 3.4, \(\mathbf{cBr}\) is generated by the elementary morphisms \(\epsilon(k), \iota(k)\) and \(b_{k,l}, i\). For the relation \((\mathbf{C}1)\) to be satisfied as an expression in \(\text{Vect}\), the image of \((\iota(k), \epsilon(k))\) under \(Y\) need to be a duality structure on \(V_k\), therefore we define \(Y_1(\iota(k)) = \iota(k)\) and \(Y_1(\epsilon(k)) = \epsilon(k)\). For \(Y\) becoming symmetric, we define \(Y_1(b_{k,l}, i) = b_{V_k, V_l}\).

To apply Proposition 3.5, we need to verify that the relations \((\mathbf{C}1)-(\mathbf{C}5)\) are valid in \(\text{Vect}\) (as described in Proposition 3.5). The zig-zag relation \((\mathbf{C}1)\) is satisfied by the definition of the duality structure. Fix a basis \(\{v_{1}^{(k)}, ..., v_{d_{k}}^{(k)}\}\) of \(V_k\) and a basis \(\{v_{1}^{(l)}, ..., v_{d_{l}}^{(l)}\}\) of \(V_l\), where \(d_k\) and \(d_l\) denote the dimension of the vector space \(V_k\) and the vector space \(V_l\), respectively. Then \((\mathbf{C}2)\) follows by the following computation.

\[
(e(k) \otimes 1_{V_l})(1_{V_k} \otimes b_{V_k, V_l})(v_{\eta}^{(k)} \otimes v_{\mu}^{(l)} \otimes v_{\nu}^{(l)})) = (e(k) \otimes 1_{V_l})(v_{\eta}^{(k)} \otimes v_{\mu}^{(k)} \otimes v_{\nu}^{(l)}))
\]

\[
= e(k)(v_{\eta}^{(k)} \otimes v_{\mu}^{(k)})v_{\nu}^{(l)}
\]

\[
= (1_{V_l} \otimes e(k))(v_{\eta}^{(k)} \otimes v_{\mu}^{(k)} \otimes v_{\nu}^{(l)}
\]

\[
= (1_{V_l} \otimes e(k))(b_{V_k, V_l} v_{\eta}^{(k)} \otimes v_{\mu}^{(l)} \otimes v_{\nu}^{(l)}))
\]

for all \(\eta, \nu \in \{1, ..., d_k\}\) and \(\mu \in \{1, ..., d_l\}\). By definition of \(i\) being a symmetric copairing, \((\mathbf{C}3)\) is automatically satisfied in \(\text{Vect}\). Also, the transposition \(b\) clearly
satisfies \((C4)\). Furthermore, \(b\) is a well-known solution of the Yang-Baxter equation so that \((C5)\) is satisfied as well. \(\Box\)

4. Proof of Theorem 1.2

The proof of Theorem 1.2 presented here is based on the proof of Theorem 1.1 given in Chapter 5 of [13]. It is divided into two parts as follows. The first part culminates in Corollary 4.3, which states that the given symmetric strict monoidal functor \(Y: \mathbf{cBr} \to \mathbf{Vect}\) is faithful if and only if it is faithful on loops (a notion that will be defined below). As it turns out, Corollary 4.3 is a consequence of Proposition 4.2, which takes place solely in the chromatic Brauer category. Secondly, we prove Theorem 4.4 which classifies all symmetric strict monoidal functors \(Y\) which are faithful on loops. We will also give a second proof of Theorem 1.1 in Section 4.1.

We introduce some operations \(\varphi^{op}, \circ\varphi\) and \(\varphi^{o}\) on morphisms \(\varphi: ([m], c) \to ([m'], c')\) in \(\mathbf{cBr}\). If the morphism \(\varphi\) is represented by the pair \((W, \Omega)\) (together with an embedding \(i: W \to [0, 1] \times \mathbb{R}^3\)), then there is a morphism \(\varphi^{op}: ([m'], c') \to ([m], c)\) which can also be represented by \((W, \Omega)\) but where the embedding is given by \(op \circ i\), where \(op: (t, x, y, z) \mapsto (1 - t, x, y, z)\). Note that for isomorphisms \(\alpha: ([m], c) \to ([m], c')\) the identity \(\alpha^{op} = \alpha^{-1}\) holds by means of Lemma 3.1.

Indeed, for the underlying permutation \(\sigma_\alpha, \sigma_\alpha^{op}\) of \(\alpha, \alpha^{op}\), respectively, we have \(\sigma_\alpha^{-1} = \sigma_\alpha^{op}\). Also note the validity of the equations \(i^{op}_{(k)} = e_{(k)}\) and \(e^{op}_{(k)} = i_{(k)}\).

Furthermore, we can express \(\varphi^{op}: ([m'], c') \to ([m], c)\) in terms of \(\varphi: ([m], c) \to ([m], c')\) via the formulas

\[
\varphi^{op} = (e_{([m], c')} \otimes 1_{([m], c)}) \circ (1_{([m'], c')} \otimes \varphi \otimes 1_{([m], c)}), \quad \text{or} \\
\varphi^{op} = (1_{([m], c)} \otimes e_{([m'], c)}) \circ (1_{([m], c)} \otimes \varphi \otimes 1_{([m], c')}) \circ (i_{([m], c)} \otimes 1_{([m'], c')}).
\]

**Example 4.1.** We illustrate (4.1) by a concrete example. Let \(\varphi: ([4], c) \to ([4], c')\) and \(\varphi^{op}: ([4], c') \to ([4], c)\) be represented by the diagrams

- \[
\begin{array}{ccccccc}
4 & & & & & & 4 \\
\hline
3 & \text{----} & & \text{----} & & 3 \\
2 & & & & & & 2 \\
1 & & & & & & 1 \\
\end{array}
\]

and

- \[
\begin{array}{ccccccc}
4 & & & & & & 4 \\
\hline
3 & \text{----} & & \text{----} & & 3 \\
2 & & & & & & 2 \\
1 & & & & & & 1 \\
\end{array}
\]

respectively. Then one can check (by comparing) that \(\varphi^{op}\) is also represented by Figure 1.

Furthermore, we define \(\circ\varphi: ([m], c) \otimes ([m'], c') \to ([0], c_{(0)})\) and \(\varphi^{o}: ([0], c_{(0)}) \to ([m], c) \otimes ([m'], c')\) by \(\circ\varphi := e_{([m], c)} \circ (1_{([m], c)} \otimes \varphi^{op})\) and \(\varphi^{o} := (1_{([m], c)} \otimes \varphi) \circ i_{([m], c)}\).

A morphism \(\varphi: ([m], c) \to ([m'], c')\) can be written for some suitable \((l_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^\infty \mathbb{N}\) in the form

\[
\varphi = \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{(l_k)} \right) \otimes (\beta \circ \varphi_0 \circ \alpha),
\]
where $\alpha: ([m], c) \rightarrow ([m], c_0)$ and $\beta: ([m'], c_0') \rightarrow ([m'], c')$ are isomorphisms in $\mathbf{cBr}$, and $\varphi_0: ([m], c_0) \rightarrow ([m'], c_0')$ is a loop-free morphism in $\mathbf{cBr}$ that can be written for some suitable $(p_k)_{k\in\mathbb{N}}, (q_k)_{k\in\mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N}$ in the form

$\varphi_0 = \bigotimes_{k=0}^{\infty} 1([c_0^{-1}(k)]-2p_k), k \otimes c_e^{p_k} (k) \otimes s^{-q_k} (k)$.

(In particular, note that the maps $c_0: [m] \rightarrow \mathbb{N}$ and $c_0': [m'] \rightarrow \mathbb{N}$ are monotonous, and satisfy $|c_0^{-1}(k)| - 2p_k = |c_0'^{-1}(k)| - 2q_k$ for all $k \in \mathbb{N}$.)

**Proposition 4.2.** Let $\varphi, \psi: ([m], c) \rightarrow ([m'], c')$ be loop-free morphisms in $\mathbf{cBr}$. Then $\varphi = \psi$ if and only if

$$\alpha \circ \varphi \circ \psi = \bigotimes_{k=0}^{\infty} \lambda^{\frac{1}{2}([c_0^{-1}(k)]+|c_0'^{-1}(k)|)}.$$  

**Proof.** Let $\varphi = \psi$, then

$$\alpha \circ \varphi \circ \psi = e_{([m], c)} \circ (1\{|m\}, c) \otimes (\varphi^{op}) \circ (1\{|m\}, c) \otimes i_{([m], c)}$$

$$= e_{([m], c)} \circ (1\{|m\}, c) \otimes (\varphi^{op} \circ \varphi) \circ i_{([m], c)}.$$

Let us compute the expression $\varphi^{op} \circ \varphi$ seperately by using the normal form of $\varphi$ as described above,

$$\varphi^{op} \circ \varphi = \alpha^{-1} \circ \left( \bigotimes_{k=0}^{\infty} 1([c_0^{-1}(k)]-2p_k), k \otimes (e_{(k)}^{p_k} \otimes e_{(k)}^{q_k}) \right) \circ \beta^{-1} \circ$$

$$\circ \beta \circ \left( \bigotimes_{k=0}^{\infty} 1([c_0^{-1}(k)]-2p_k), k \otimes (e_{(k)}^{p_k} \otimes (e_{(k)}^{p_k} \otimes e_{(k)}^{q_k})) \right) \circ \alpha$$

$$= \alpha^{-1} \circ \left( \bigotimes_{k=0}^{\infty} 1([c_0^{-1}(k)]-2p_k), k \otimes (i_{(k)} \circ e_{(k)}^{p_k} \otimes (e_{(k)} \circ i_{(k)}^{p_k} \otimes e_{(k)}^{q_k})) \right) \circ \alpha$$

$$= \left[ \alpha^{-1} \circ \left( \bigotimes_{k=0}^{\infty} 1([c_0^{-1}(k)]-2p_k), k \otimes (i_{(k)} \circ e_{(k)}^{p_k} \otimes (e_{(k)} \circ i_{(k)}^{p_k} \otimes e_{(k)}^{q_k})) \right) \circ \alpha \right] \otimes \bigotimes_{k=0}^{\infty} \lambda^{q_k} (k).$$

**Figure 1.**
Now, for the isomorphism $\alpha$ we have the relations valid in $\text{cBr}$ given by
\[
e_{([m],c)} \circ (1_{([m],c)} \otimes \alpha^{-1}) = e_{([m],c)} \circ (\alpha \otimes 1_{([m],c)}),\] and
\[(1_{([m],c)} \otimes \alpha) \circ i_{([m],c)} = (\alpha^{-1} \otimes 1_{([m],c)}) \circ i_{([m],c)}.
\]
(Indeed, if we write $\alpha$ as a product of adjacent transpositions $\alpha = T_1 \circ \cdots \circ T_N$, then we can shift the $T_i$ along $e_{([m],\bullet)}$ (resp. $i_{([m],\bullet)}$) from $T_i \otimes 1$ to $1 \otimes T_i$, but in the reverse order. Note that during this procedure, the coloring $\bullet$ is changing after each step, after the last shift of $T_N$ it has become $c_{0\alpha}$.) This leads to
\[
a \circ \phi \circ \phi^0 = 
= \left[ e_{([m],c_{0\alpha})} \circ (1_{([m],c_{0\alpha})} \otimes \left( \bigotimes_{k=0}^{\infty} 1_{([c^{-1}(k)]-2p_k]} \otimes \left( i_{(k)} \circ e_{(k)} \right)^{\otimes p_k} \right) \right) \circ i_{([m],c_{0\alpha})} \right] \otimes 
= \left[ e_{([2 \sum p_k],\tilde{c})} \circ (1_{([2 \sum p_k],\tilde{c})} \otimes \left( \bigotimes_{k=0}^{\infty} \left( i_{(k)} \circ e_{(k)} \right)^{\otimes p_k} \right) \right) \circ i_{([2 \sum p_k],\tilde{c})} \right] \otimes 
= \left[ \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes (q_k-2p_k+|c^{-1}(k)|)} \right)^{\text{op}} \circ \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes p_k} \right)^{\text{op}} \otimes \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes (q_k-2p_k+|c^{-1}(k)|)} \right) \right],
\]
where $\tilde{c} : [2 \sum p_k] \to \mathbb{N}$ is monotone and satisfies $|c^{-1}(k)| = 2p_k$.

Conversely, if the morphisms $\phi, \psi : ([m], c) \to ([m'], c')$ satisfy (12), it suffices to show that $?\phi = ?\psi$ in $\text{Br}$. After applying $?^m$ to (12) we obtain $a \circ \phi_0 \circ \psi^0 = \lambda^{\otimes \frac{1}{2}(m+m')}$, where $\phi_0 = ?(\phi)$, $\psi_0 = ?(\psi)$ and $a = ? \circ \alpha$. Therefore, it suffices to show that for every component $C$ of a representative $W(a \phi_0)$ of $a \phi_0$ there exists a component $\mathcal{C}$ of a representative $W(\phi_0^0)$ of $\phi_0^0$ such that $C$ and $\mathcal{C}$ have the same endpoints in $[m+m']$. Let $P$ be a point in $[m+m']$ and let $P_x$ and $P_y$ denote the other endpoint of the connected component $C$ and $\mathcal{C}$, respectively, containing $P$. Note first, that the number $\frac{1}{2}(m+m')$ is the maximal number of loops $\lambda$ which can be contained in $a \phi_0 \circ \psi_0^0$, since $W(\psi_0^0)$ and $W(a \phi_0)$ each consist of $\frac{1}{2}(m+m')$ distinguished connected components. This means that for every component $C$ of $W(a \phi_0)$ there is a component $\mathcal{C}$ of $W(\phi_0^0)$ such that $C$ and $\mathcal{C}$ close up to $S^1$. In other words, if $P \in C$ and $P \in \mathcal{C}$, then $P_x = P_y$.

The symmetric strict monoidal functor $Y : \text{cBr} \to \text{Vect}$ is called faithful on loops if for any two morphisms $\phi, \psi : ([m], c) \to ([m'], c')$ in $\text{cBr}$ the condition $Y(\phi) = Y(\psi)$ implies that there are a sequence $(\ell_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N}$ and loop-free morphisms $\phi_0$ and $\psi_0$ such that $\phi = \otimes k \lambda_{\ell_k}^{\otimes k} \otimes \phi_0$ and $\psi = \otimes k \lambda_{\ell_k}^{\otimes k} \otimes \psi_0$.

Recall from the discussion in Section 3.2 that $Y([[1], \ell]) = V_\ell$ is for all $k \in \mathbb{N}$ a finite dimensional vector space, whose dimension will be denoted by $d_\ell$. Furthermore, recall that $Y(\lambda_{\ell}) = d_\ell$. 

As an immediate consequence of Proposition 4.2, we obtain the following corollary.

**Corollary 4.3.** $Y : \text{cBr} \to \text{Vect}$ is faithful on loops if and only if $Y$ is faithful.

**Proof.** What we need to show is that $Y$ being faithful is implied by $Y$ being faithful on loops. Let $\varphi_0, \psi_0 : ([m], c) \to ([m], c')$ be loop-free such that $Y(\varphi_0) = Y(\psi_0)$, then $Y(\varphi_0^0) = Y(\psi_0^0)$. Therefore, we have

$$Y(\varphi_0) = Y(\varphi_0^0) = Y(\psi_0)$$

for all loops. Now, under the assumption that $\varphi$ is faithful on loops, Proposition 4.2 implies that $\varphi_0 = \psi_0$. Hence, $Y$ is also faithful on loop-free morphisms. Now let $\varphi, \psi : ([m], c) \to ([m'], c')$ be morphisms (possibly containing loops) such that $Y(\varphi) = Y(\psi)$, and rewrite them as $\varphi = \left( \bigotimes_k \lambda_{(k)}^{\otimes l_k} \right) \otimes \varphi_0$ and $\psi = \left( \bigotimes_k \lambda_{(k)}^{\otimes l_k} \right) \otimes \psi_0$, for a sequences $(l_k)_{k \in \mathbb{N}} \in \bigoplus_k \mathbb{N}$ and loop-free morphisms $\varphi_0, \psi_0 : ([m], c) \to ([m'], c')$. Since $Y$ is faithful on loops, we have in particular $Y(\lambda_{(k)}) = d_k > 0$ for all $k \in \mathbb{N}$. We obtain

$$Y(\varphi_0) = \frac{1}{\prod_k d_k} Y(\varphi) = \frac{1}{\prod_k d_k} Y(\psi) = Y(\psi_0),$$

i.e. $\varphi_0 = \psi_0$ and therefore $\varphi = \psi$. Hence, $Y$ is faithful.

If $Y$ is faithful on loops, it is clear that the dimension $d_k$ of $V_k$ needs to satisfy for all $(l_k)_{k \in \mathbb{N}} \in \bigoplus_k \mathbb{N}$ the implication (1.1), namely

$$\prod_{k=0}^{\infty} d_k = 1 \quad \Rightarrow \quad l_k = 0 \text{ for all } k \in \mathbb{N}.$$

**Theorem 4.4.** Suppose that $d_k > 0$ for all $k \in \mathbb{N}$, and that the implication (1.1) holds for all sequences $(l_k)_{k \in \mathbb{N}} \in \bigoplus_k \mathbb{N}$. Then, the functor $Y : \text{cBr} \to \text{Vect}$ is faithful on loops.

**Proof.** Let $\varphi_0 : ([m], c) \to ([m], c')$ be a loop-free morphisms presented in its normal form $\varphi_0 = \beta \circ \varphi_0 \circ \alpha$, with $\varphi_0 = \bigotimes_k 1([l-1(k)] \otimes d_k \otimes \lambda_{(k)}^{\otimes l_k} \otimes \lambda_{(k)}^{\otimes q_k})$. We will compute $\text{Tr}(Y(\varphi_0^0 \circ \varphi_0))$. Recall that the trace is invariant under cyclic permutation, and the trace of the tensor product of two matrices is the product of their traces. We will also use the identity $\text{Tr}(\lambda_{(k)}^p) = d_k$. Indeed, this follows from the invariance of the trace under cyclic permutation and $Y(\lambda_{(k)}) = e(\lambda_{(k)}) = d_k$. Thus, we obtain

$$\text{Tr}(Y(\varphi_0^0 \circ \varphi_0)) = \text{Tr}(Y(\alpha^{-1}) \circ Y(\varphi_0^0) \circ Y(\beta^{-1}) \circ Y(\beta) \circ Y(\varphi_0) \circ Y(\alpha)) = \text{Tr}(Y(\varphi_0^0) \circ Y(\varphi_0))$$

$$= \text{Tr} \left( Y \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes l_k} \otimes \bigotimes_{k=0}^{\infty} 1([l-1(k)] \otimes d_k \otimes (\lambda_{(k)}^p) \otimes (\lambda_{(k)}^{\otimes q_k})) \right) \right)$$

$$= \prod_{k=0}^{\infty} d_k \cdot \text{Tr}(1_{V_k})^{[l-1(k)]-2p_k} \cdot \text{Tr}(\lambda_{(k)}^p)^{p_k}$$

$$= \prod_{k=0}^{\infty} d_k^{[l-1(k)]+q_k-p_k} = \prod_{k=0}^{\infty} d_k^{[l-1(k)]+|c^{-1}(k)|+|c^{-1}(k)|}. $$
Note two things: The number $\text{Tr}(Y(\varphi_0^\text{op} \circ \varphi_0))$ does not vanish in any case and it only depends on the domain $([m], c)$ and codomain $([m], c')$ of the morphism $\varphi_0$.

Now, let $\varphi, \psi: ([m], c) \rightarrow ([m'], c')$ such that $Y(\varphi) = Y(\psi)$. Then there are sequences $(\mu_k)_{k \in \mathbb{N}}, (\nu_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N}$ and loop-free morphisms $\varphi_0, \psi_0: ([m], c) \rightarrow ([m'], c')$ such that $\varphi = \bigotimes_{k=0}^{\infty} \chi(\mu_k) \otimes \varphi_0$ and $\psi = \bigotimes_{k=0}^{\infty} \chi(\nu_k) \otimes \psi_0$. Then, $Y(\varphi) = Y(\psi)$ implies that $Y(\varphi^\text{op}) = Y(\psi^\text{op})$ by (1.1) and $Y$ being a strict monoidal functor. Consequently,

$$\prod_{k=0}^{\infty} d_{2k}^{\mu_k} \cdot \text{Tr}(Y(\varphi_0 \circ \varphi_0^\text{op})) = \text{Tr}(Y(\varphi \circ \varphi^\text{op}))$$

$$= \text{Tr}(Y(\psi \circ \psi^\text{op})) = \prod_{k=0}^{\infty} d_{2k}^{\nu_k} \cdot \text{Tr}(Y(\psi_0 \circ \psi_0^\text{op})),$$

which is equivalent to $\prod_k (\mu_k - \nu_k) = 1$. By implication (1.1), $\mu_k = \nu_k$ for all $k \in \mathbb{N}$. Hence, $Y$ is faithful on loops. □

This completes the proof of Theorem 1.2

4.1. **An alternative proof of Theorem 4.4**. Based on Section 3.2 in [19], we sketch an alternative proof of Theorem 4.4 which does not make use of a trace argument, but exploits the Kronecker product (see Remark 2.2) instead.

Firstly, we study the behavior of $Y$ on isomorphisms.

**Lemma 4.5.** If $\iota: ([m], c) \rightarrow ([m], c')$ is an isomorphism in $\mathbf{cBr}$, then

$$Y(\iota)(w_1 \otimes \cdots \otimes w_m) = w_{\iota^{-1}(1)} \otimes \cdots \otimes w_{\iota^{-1}(m)} \text{ for all } w_i \in V_{\iota(i)}, 1 \leq i \leq m,$$

where $\iota$ denotes the underlying permutation $[m] \rightarrow [m]$ (see Lemma 3.7).

**Proof.** The permutation $\iota$ of $\{1, \ldots, m\}$ can be written as the composition of adjacent transpositions. Hence, the isomorphism $\iota$ can be written as the composition of isomorphisms in $\mathbf{cBr}$ of the form $\delta_u: ([m], a) \rightarrow ([m], a'), u \in \{1, \ldots, m-1\}$, where

$$\delta_u := 1_{(a(1))} \otimes \cdots \otimes 1_{(a(u-1))} \otimes b_{(a(u), a(u+1))} \otimes 1_{(a(u+2))} \otimes \cdots \otimes 1_{(a(m))}.$$

Note that the permutation $\sigma_{\delta_u}$ of $\{1, \ldots, m\}$ is the transposition that interchanges $u$ with $u+1$. That is, $\sigma_{\delta_u}(u) = u+1, \sigma_{\delta_u}(u+1) = u$ and $\sigma_{\delta_u}(j) = j$ for all $j \in \{1, \ldots, m\} \setminus \{u, u+1\}$. It suffices to show the following statements:

(i) **The claim holds for the isomorphisms $\delta_1, \ldots, \delta_{m-1}: ([m], a) \rightarrow ([m], a')$.**

In fact, let $w_i \in V_{\delta(i)}, 1 \leq i \leq m$. Since $Y$ is symmetric, we have

$$Y(b_{(a(u), a(u+1))}) = b_{(a(u), a(u+1))},$$

where $b_{(a(u), a(u+1))}: V_{a(u)} \otimes V_{a(u+1)} \rightarrow V_{a(u+1)} \otimes V_{a(u)}$ is the braiding automorphism induced by $v \otimes w \rightarrow w \otimes v$.

Therefore,

$$Y(\delta_u)(w_1 \otimes \cdots \otimes w_m) = w_{\delta_u^{-1}(1)} \otimes \cdots \otimes w_{\delta_u^{-1}(m)};$$

(ii) **If the claim holds for two isomorphisms**

$$\alpha: ([m], a) \rightarrow ([m], a') \text{ and } \beta: ([m], a') \rightarrow ([m], a''),$$

then...
in cBr, then it also holds for their composition \( \beta \circ \alpha : ([m], a) \to ([m], a'') \).

In fact, using \( \sigma_{\beta \circ \alpha} = \sigma_\beta \circ \sigma_\alpha \), we obtain for all \( w_i \in V_{\alpha(i)}, \ 1 \leq i \leq m \),

\[
Y(\beta \circ \alpha)(w_1 \otimes \cdots \otimes w_m) = Y(\beta)(Y(\alpha)(w_1 \otimes \cdots \otimes w_m)) \\
= Y(\beta)(w_{\sigma^{-1}_\alpha(1)} \otimes \cdots \otimes w_{\sigma^{-1}_\alpha(m)}) \\
= w_{\sigma^{-1}_\alpha(\sigma_\beta^{-1}(1))} \otimes \cdots \otimes w_{\sigma^{-1}_\alpha(\sigma_\beta^{-1}(m))} \\
= w_{\sigma^{-1}_{\beta \circ \alpha}(1)} \otimes \cdots \otimes w_{\sigma^{-1}_{\beta \circ \alpha}(m)}. 
\]

\[\square\]

An immediate consequence of Lemma 4.5 is that \( Y \) is faithful on isomorphisms in cBr.

**Corollary 4.6.** Suppose that \( d_k \geq 2 \) for all \( k \in \mathbb{N} \). If \( \iota_1, \iota_2 : ([m], c) \to ([m], c') \) are two isomorphisms in cBr such that \( Y(\iota_1) = Y(\iota_2) \), then \( \iota_1 = \iota_2 \).

**Proof.** For every \( k \in \mathbb{N} \) we fix an ordered basis \( \{ v_k, \ldots, v_k' \} \) of the real vector space \( V_k = Y([1], \mathbb{H}) \). Thus, for every \( s \in \{1, \ldots, m\} \) we may consider the element

\[
v^s := v_1^{(1)} \otimes \cdots \otimes v_1^{(s-1)} \otimes v_2^{(s)} \otimes v_2^{(s+1)} \otimes \cdots \otimes v_1^{(m)} \in \bigotimes_{i=1}^m V_{\iota(i)}. \]

Note that if \( \iota : ([m], c) \to ([m], c') \) is an isomorphism, then we have \( Y(\iota)(v^s) = v^{\iota(s)} \) in \( \bigotimes_{i=1}^m V_{\iota(i)} \) by Lemma 4.5. (In fact, if \( w_s := v_2^{(s)} \) and \( w_r := v_1^{(r)} \) for \( r \neq s \), then \( w_{\iota^{-1}(i)} = v_2^{(s)} \) for \( i = \iota(s) \) and \( w_{\iota^{-1}(i)} = v_1^{(\iota^{-1}(i))} \) for \( i \neq \iota(s) \).

Hence, \( Y(\iota)(v^s) = Y(\iota)(w_1 \otimes \cdots \otimes w_m) = w_{\iota^{-1}(1)} \otimes \cdots \otimes w_{\iota^{-1}(m)} = v^{\iota(s)} \).

Let us assume that \( \iota_1 \neq \iota_2 \). Then, there exists \( t \in \{1, \ldots, m\} \) such that \( \sigma_{\iota_1}(t) \neq \sigma_{\iota_2}(t) \). However, we have \( v^{\sigma_{\iota_1}(t)} = Y(\iota_1)(v^t) = Y(\iota_2)(v^t) = v^{\sigma_{\iota_2}(t)} \) in \( \bigotimes_{i=1}^m V_{\iota(i)} \).

This is a contradiction to the linear independence of \( v^{\sigma_{\iota_1}(t)} \) and \( v^{\sigma_{\iota_2}(t)} \) for \( \sigma_{\iota_1}(t) \neq \sigma_{\iota_2}(t) \). Hence, \( \iota_1 = \iota_2 \). \[\square\]

The next crucial step is to compute the preimage under \( Y \) of scalar square matrices.

**Proposition 4.7.** Suppose that \( d_k \geq 2 \) for all \( k \in \mathbb{N} \). If \( \varphi \in \text{Hom}_{cBr}(([m], c), ([m], c)) \) satisfies \( Y(\varphi) = \mu \cdot 1 \otimes \bigotimes_{i=1}^m V_{\iota(i)} \) for some \( \mu \in \mathbb{R} \), then there exists \( (l_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^\infty \mathbb{N} \) such that \( \mu = \prod_{k=0}^\infty d_k^{l_k} \) and \( \varphi = \left( \bigotimes_{k=0}^\infty \alpha_{l_k} \right) \otimes 1_{([m], c)} \).

**Proof.** The given morphism \( \varphi : ([m], c) \to ([m], c) \) can be written for some suitable \( (l_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^\infty \mathbb{N} \) in the form

\[
\varphi = \left( \bigotimes_{k=0}^\infty \alpha_{l_k} \right) \otimes (\beta \circ \varphi_0 \circ \alpha),
\]

where \( \alpha : ([m], c) \to ([m], c') \) and \( \beta : ([m], c') \to ([m], c) \) are isomorphisms in cBr, and \( \varphi_0 : ([m], c') \to ([m], c'') \) is a loop-free morphism in cBr that can be written for some suitable \( (p_k)_{k \in \mathbb{N}}, (q_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^\infty \mathbb{N} \) in the form

\[
\varphi_0 = \left( \bigotimes_{k=0}^\infty 1_{([c^{-1}(k)], [-2p_k], \mathbb{H})} \otimes e_{\iota(k)}^{p_k} \right) \otimes \left( \bigotimes_{k=0}^\infty 1_{([k], [q_k])} \otimes i_{l_k}^{q_k} \right). 
\]
(In particular, note that the maps \( c', \sigma' : [m] \to \mathbb{N} \) are monotonous, and satisfy \(|c'^{-1}(k)| - 2p_k = |\sigma'^{-1}(k)| - 2q_k \) for all \( k \in \mathbb{N} \).

Applying \( Y \) and using that \( Y(\lambda_{(k)}) = d_k \) for all \( k \in \mathbb{N} \), we obtain

\[
Y(\varphi) = \prod_{k=0}^{\infty} d_k^{k} \cdot (Y(\beta) \circ Y(\varphi_0) \circ Y(\alpha)).
\]

Using the assumption \( Y(\varphi) = \mu \cdot 1_{\otimes_{i=1}^{m} V_{c(i)}} \) and setting \( \gamma := \beta^{-1} \circ \alpha^{-1} : ([m], c') \to ([m], c'') \), we have

\[
(4.3) \quad \frac{\mu}{\prod_{k=0}^{\infty} d_k^{k}} \cdot Y(\gamma) = Y(\varphi_0).
\]

For every \( k \in \mathbb{N} \) we fix an ordered basis \( \{ v_1^k, \ldots, v_{d_k}^k \} \) of the real vector space \( V_k = Y([1], \mathbb{A}) \). For any object \( ([m], c) \) in \( \text{cBr} \) we assume that

\[
Y(([m], c)) = Y \left( \bigotimes_{i=1}^{m} ([1], c_{(i)}) \right) = \bigotimes_{i=1}^{m} Y(([1], c_{(i)})) = \bigotimes_{i=1}^{m} V_{c(i)}
\]

is equipped with the lexicographically ordered basis

\[
\{ v_{r_1}^{c_{(1)}}, \ldots, v_{r_m}^{c_{(m)}} \}_{r_1, \ldots, r_m}.
\]

Then, by Remark 2.3, the matrix representation of \( Y(\varphi_0) \) is the Kronecker product of the matrix representations of \( Y(1_{([c^{-1}(k)]-2p_k]} \mathbb{A}) \) and \( Y(\varepsilon_{(k)}) \otimes_{\mathbb{A}} \), \( Y(\varepsilon_{(k)}) = Y(1_{([e^{-1}(k)]-2q_k]} \mathbb{A}) \) and \( Y(\varepsilon_{(k)}) = Y(e_{(k)}) \otimes_{\mathbb{A}} \), \( e_{(k)} \in \mathbb{N} \). Note that the matrix representation of \( 1_{V_k} \) is the identity matrix of size \( d_k \times d_k \). Moreover, the matrix representation of \( Y(\varepsilon_{(k)}): V_k \otimes V_k \to \mathbb{R} \) is a \((1 \times d_k^2)\)-matrix which contains at least two nonzero entries because \( d_k \geq 2 \), and the \((d_k \times d_k)\)-matrix \( Y(\varepsilon_{(k)})(v_i^k \otimes v_j^k) \) is invertible according to the discussion in Section 4.3. Similarly, the matrix representation of \( Y(\varepsilon_{(k)}): \mathbb{R} \to V_k \otimes V_k \) is a \((d_k^2 \times 1)\)-matrix which contains at least two nonzero entries. Therefore, if there exists \( k \in \mathbb{N} \) such that \( p_k > 0 \) or \( q_k > 0 \), then the matrix representation of \( Y(\varphi_0) \) has a column or a row which contains at least two nonzero entries. On the other hand, it follows from Lemma 4.3 that \( Y(\gamma) \) is represented by a permutation matrix because \( \gamma : ([m], c') \to ([m], c'') \) is an isomorphism. In particular, every column and every row of \( Y(\gamma) \) has exactly one nonzero entry, which yields a contradiction in Equation (4.3). Consequently, \( p_k = 0 \) and \( q_k = 0 \) for all \( k \in \mathbb{N} \). Hence, we have \( c' = c'' \), and \( \varphi_0 = \bigotimes_{k=0}^{\infty} 1_{([c^{-1}(k)]-2p_k]} \mathbb{A} = 1_{([m], c')} \). Then it follows from Equation (4.3) that \( \mu = \prod_{k=0}^{\infty} d_k^{k} \) because \( Y(\gamma) \) is a permutation matrix and \( Y(\varphi_0) \) is an identity matrix. Moreover, we can apply Corollary 4.6 to the isomorphisms \( \gamma, \varphi_0 : ([m], c') \to ([m], c') \) to obtain \( \gamma = \varphi_0 \). Finally, we conclude that

\[
\varphi = \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes_{k=0}^{\infty}} \right) \otimes (\beta \circ \varphi_0 \circ \alpha), \quad \psi = \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes_{k=0}^{\infty}} \right) \otimes (\beta' \circ \psi_0 \circ \alpha'),
\]

Finally, we give the proof of Theorem 4.4.

**Proof of Theorem 4.4** Assume that \( Y(\varphi) = Y(\psi) \) for two given morphisms \( \varphi, \psi : ([m], b) \to ([m], c) \) in \( \text{cBr} \). For some suitable \( (\mu_k)_{k \in \mathbb{N}}, (\nu_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N} \) we can write

\[
\varphi = \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes_{k=0}^{\infty}} \right) \otimes (\beta \circ \varphi_0 \circ \alpha), \quad \psi = \left( \bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes_{k=0}^{\infty}} \right) \otimes (\beta' \circ \psi_0 \circ \alpha'),
\]
where $\alpha, \alpha': ([m], c) \to ([m], b_0)$ and $\beta, \beta': ([n], c_0) \to ([m], c')$ are isomorphisms in $\text{cBr}$, and $\varphi_0, \psi_0: ([m], b_0) \to ([n], c_0)$ are loop-free morphisms in $\text{cBr}$ that can be written for some suitable $(p_k)_{k \in \mathbb{N}}, (p'_k)_{k \in \mathbb{N}}, (q_k)_{k \in \mathbb{N}}, (q'_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N}$ in the form

$$\varphi_0 = \bigotimes_{k=0}^{\infty} 1([b_0^{-1}(k)]-2p_k) \otimes e_k^{\otimes p_k} \otimes i_k^{\otimes q_k},$$

$$\psi_0 = \bigotimes_{k=0}^{\infty} 1([b_0^{-1}(k)]-2p'_k) \otimes e_k^{\otimes p'_k} \otimes i_k^{\otimes q'_k}.$$  

(In particular, note that the maps $b_0: [m] \to \mathbb{N}$ and $c_0: [n] \to \mathbb{N}$ are monotonically increasing, and satisfy $|b_0^{-1}(k)| - 2p_k = |c_0^{-1}(k)| - 2q_k$ and $|b_0^{-1}(k)| - 2p'_k = |c_0^{-1}(k)| - 2q'_k$ for all $k \in \mathbb{N}$.)

We have to show that $\varphi$ and $\psi$ have the same number of $k$-loops for each $k \in \mathbb{N}$, that is, $(\mu_k)_{k \in \mathbb{N}} = (\nu_k)_{k \in \mathbb{N}}$. In the following, we will only show $\nu_k \leq \mu_k$. Then, $\nu_k = \mu_k$ follows by symmetry.

We will use Proposition 4.7 to reduce the assumption $Y(\varphi) = Y(\psi)$ to equation (4.4) below, which is a statement in the chromatic Brauer category.

Fix $k \in \mathbb{N}$. Define $a_k = 1([b_0^{-1}(k)]-2p_k) \otimes e_k^{\otimes p_k}$ and $b_k = 1([b_0^{-1}(k)]-2p'_k) \otimes e_k^{\otimes p'_k}$. Then, an easy calculation shows that

$$b(k) \circ \left(1([b_0^{-1}(k)]-2p_k) \otimes e_k^{\otimes p_k} \otimes i_k^{\otimes q_k}\right) \circ a(k) = \lambda_k^{\otimes (p_k+q_k)} \otimes 1([b_0^{-1}(k)]-2p_k) \otimes i_k^{\otimes q_k}.$$ 

Thus, setting $a := \alpha^{-1} \circ \left(\bigotimes_{k=0}^{\infty} a_k\right)$ and $b := \left(\bigotimes_{k=0}^{\infty} b_k\right) \circ \beta^{-1}$, we obtain

$$b \circ \varphi \circ a = \left(\bigotimes_{k=0}^{\infty} \lambda_k^{(p_k+q_k)}\right) \otimes \left(\bigotimes_{k=0}^{\infty} 1([b_0^{-1}(k)]-2p_k) \otimes i_k^{\otimes q_k}\right).$$

Applying the monoidal functor $Y$ to the previous equation and using $Y(\varphi) = Y(\psi)$, we obtain

$$Y(b \circ \psi \circ a) = Y(b \circ \varphi \circ a) = \prod_{k=0}^{\infty} \lambda_k^{(p_k+q_k)} \cdot \left(\bigotimes_{k=0}^{\infty} 1([b_0^{-1}(k)]-2p_k) \otimes i_k^{\otimes q_k}\right).$$

Note that the assumptions on the sequence $d_0, d_1, \ldots$ imply that $d_k \geq 2$ for all $k \in \mathbb{N}$. Hence, it follows from Proposition 4.7 that there exists $(l_k)_{k \in \mathbb{N}} \in \bigoplus_{k=0}^{\infty} \mathbb{N}$ such that $\prod_{k=0}^{\infty} d_k^{l_k+2p_k+q_k} = \prod_{k=0}^{\infty} d_k^{l_k}$ and

$$b \circ \psi \circ a = \left(\bigotimes_{k=0}^{\infty} \lambda_k^{l_k}\right) \otimes \left(\bigotimes_{k=0}^{\infty} 1([b_0^{-1}(k)]-2p_k) \otimes i_k^{\otimes q_k}\right).$$

Note that the implication (4.4) implies that $l_k = \mu_k + p_k + q_k$ for all $k \in \mathbb{N}$. It suffices to show that for each $k \in \mathbb{N}$, $\nu_k + p_k + q_k$ is an upper bound for the number of $k$-loops contained in the composition $b \circ \psi \circ a$. (Indeed, then it follows from equation (4.4) that $\nu_k + p_k + q_k \leq \mu_k + p_k + q_k$. Thus, $\nu_k \leq \mu_k$.)

Setting $\psi'_0 := \beta^{-1} \circ \beta' \circ \psi_0 \circ \alpha' \circ \alpha^{-1}$, $\tilde{a} = \bigotimes_{k=0}^{\infty} a_k$ and $\tilde{b} = \bigotimes_{k=0}^{\infty} b_k$, we have

$$b \circ \psi \circ a = \left(\bigotimes_{k=0}^{\infty} \lambda_k^{l_k}\right) \otimes \left(\tilde{b} \circ \psi'_0 \circ \tilde{a}\right).$$

Fix $k \in \mathbb{N}$. It suffices to show that the number of loops of label $k$ in $\tilde{b} \circ \psi'_0 \circ \tilde{a}$ is $\leq p_k + q_k$. 

We choose 1-manifolds $W_0 \subset [0, 1] \times \mathbb{R}^3$, $W' \subset [1, 2] \times \mathbb{R}^3$ and $W_1 \subset [2, 3] \times \mathbb{R}^3$ which represent (up to translations along the first coordinate) the Brauer morphisms $\tilde{a}$, $\psi_0'$ and $\tilde{b}$, respectively. Then, $\tilde{b} \circ \psi_0' \circ \tilde{a}$ is represented (after reparametrization of the first coordinate) by the union $W := W_0 \cup W' \cup W_1 \subset [0, 3] \times \mathbb{R}^3$.

For a 1-manifold $X \subset [s, t] \times \mathbb{R}^3$ which represents some morphism in $\text{cBr}$, let $X \{e_{(k)}\}$ (respectively, $X \{i_{(k)}\}$) be the set of label $k$ components of $X$ whose endpoints are both contained in $[s] \times \mathbb{R}^3$ (respectively, in $[t] \times \mathbb{R}^3$). Moreover, denote by $X \{1_{(k)}\}$ the set of label $k$ components of $X$ which have one endpoint in $[s] \times \mathbb{R}^3$ and the other one in $[t] \times \mathbb{R}^3$. Finally, let $X \{\lambda_{(k)}\}$ be the set of closed label $k$ components of $X$.

Note that the number of $k$-loops in $\tilde{b} \circ \psi_0' \circ \tilde{a}$ is given by the cardinality of $W \{\lambda_{(k)}\}$, and we have to show that this number is $\leq p_k + q_k$. By definition of $\tilde{a}$ and $\tilde{b}$, $|W_0\{i_{(k)}\}| = p_k$ and $|W_1\{e_{(k)}\}| = q_k$. Hence, it suffices to construct an injective map $W\{\lambda_{(k)}\} \to W_0\{i_{(k)}\} \cup W_1\{e_{(k)}\}$.

Let $L \in W\{\lambda_{(k)}\}$ be a closed label $k$ component of $W$. The intersections $L \cap W_0$, $L \cap W'$ and $L \cap W_1$ can be written as disjoint union of components of $W_0$, $W'$ and $W_1$ respectively. It follows from $L \cap \{(0) \times \mathbb{R}^3\} = \emptyset$, $W_0\{e_{(k)}\} = \emptyset$ and $W_0\{\lambda_{(k)}\} = \emptyset$ that $L \cap W_0$ is a disjoint union of elements of $W_0\{i_{(k)}\}$. Analogously, it follows from $L \cap \{(3) \times \mathbb{R}^3\} = \emptyset$, $W_1\{i_{(k)}\} = \emptyset$ and $W_1\{\lambda_{(k)}\} = \emptyset$ that $L \cap W_1$ is a disjoint union of elements of $W_1\{e_{(k)}\}$. Moreover, $L$ has nonempty intersection with $W_0 \cup W_1$. (In fact, $\psi_0'$ is a loop-free Brauer morphism, being the composition of the loop-free Brauer morphism $\psi_0$ and Brauer isomorphisms. Therefore, $W'$ does not contain any closed components, so in particular $W\{\lambda\} = \emptyset$. Hence, $L$ cannot be entirely contained in $W'$. Thus, $L \cap W_0 \cup W_1 \neq \emptyset$. ) Hence, we can pick an element of $W_0\{i_{(k)}\} \cup W_1\{e_{(k)}\}$ which is contained in $L$. This defines a map $W\{\lambda\} \to W_0\{i_{(k)}\} \cup W_1\{e_{(k)}\}$. By construction, this map is injective. (Indeed, assume that $L, L' \in W\{\lambda_{(k)}\}$ are mapped to the same element $C \in W_0\{i_{(k)}\} \cup W_1\{e_{(k)}\}$. Then, $\emptyset \neq C \subset L \cap L'$ implies $L = L'$.)

5. Positive TFTs, fold maps, and exotic Kervaire spheres

Using the Brauer category $\text{Br}$ and singularity theory of fold maps, Banagl [3, 4] has constructed a high-dimensional positive TFT which is defined on smooth cobordisms. He also showed that the state sum of the theory can distinguish exotic smooth structures on spheres from the standard smooth structure. The construction is sketched in Section 10 of [3] as an application of the general framework of positive TFTs, and has been worked out in full detail in [4]. In the present section, we construct a refinement of Banagl’s theory in which we replace the Brauer category $\text{Br}$ by its chromatic enrichment $\text{cBr}$. The power of our state sum invariant is illustrated by Theorem 5.7, where we show that the associated aggregate invariant can detect exotic Kervaire spheres in infinitely many dimensions.

The present section is structured as follows. In Section 5.1 we outline the general features of the framework of positive TFTs, and explain the process of quantization. Section 5.2 provides the concrete definitions of fold fields and the $\text{cBr}$-valued action functional while pointing out the changes that arise from using the chromatic Brauer category instead of $\text{Br}$. Quantization is discussed in Section 5.3 where we carefully indicate the necessary modifications in the algebraic process of profinite idempotent
5.1. General framework. In [8] Banagl presents an individual approach to the construction of certain TFTs in arbitrary dimension. The idea is to modify Atiyah’s original axioms [1] by formulating them over semirings instead of rings. Compared to a ring, a semiring is not required to have additive inverses, i.e. “negative” elements. Banagl introduces the notion of positive TFTs, and shows that any system of so-called fields and action functionals gives rise to a positive TFTs by means of a process called quantization. This framework is inspired by quantization from theoretical physics. In order to avoid set theoretic difficulties that may arise in the definition of the Feynman path integral, Banagl employs the concept of complete semirings due to Eilenberg [7]. The reason is that a complete semiring has a summation law that allows to sum families of elements indexed by arbitrary index sets. Positive TFTs can motivate the construction of new invariants for smooth manifolds like the aggregate invariant of homotopy spheres (see Section 10 in [4]).

In the following, we outline Banagl’s construction [8] of a n-dimensional positive TFT from given systems of fields and action functionals. Following Section 5 in [3], a system $\mathcal{F}$ of fields assigns to every closed $(n-1)$-manifold $M$ and to every $n$-cobordism $W$ sets $\mathcal{F}(M)$ and $\mathcal{F}(W)$ of fields on $M$ and $W$, respectively. Fields on a cobordism can be restricted to subcobordisms and to codimension 1 submanifolds. Apart from desirable behavior with respect to the action of homeomorphisms and disjoint union, fields are especially required to glue under the gluing of cobordisms. The axioms for a system $T$ of action functionals (or action exponentials) with values in a strict monoidal category $C$ are inspired by the exponential of the action that appears in the integrand of the Feynman path integral. To every $n$-cobordism $W$ one associates a map $T_W: \mathcal{F}(W) \to \text{Mor}(C)$ in such a way that disjoint union of cobordisms is reflected by tensor product of morphisms in $C$, and gluing of cobordisms corresponds to composition of morphisms. More precisely, it is required that $T_W(f) = T_U(f|U) \otimes T_V(f|V)$ for fields $f$ on the disjoint union $W = W' \sqcup W''$ of cobordisms $W'$ and $W''$, and $T_W(f) = T_U(f|U) \circ T_V(f|V)$ for fields $f$ on the gluing $W = U \cup_N V$ along $N$ of cobordisms $U$ from $M$ to $N$ and $V$ from $N$ to $P$. Furthermore, the action functional is invariant under the action of homeomorphisms.

In Section 5.2 we will specifically take $C = c\text{Br}$. Next, we describe the process of quantization (see Section 6 in [3]). For this purpose, we fix a system $\mathcal{F}$ of fields, a $C$-valued system $T$ of action functionals, and a complete semiring $S$. Following Section 4 in [3], one first constructs a complete additive monoid $Q$ from the semiring $S$ and the strict monoidal category $C$. The elements of $Q$ are just maps $\text{Mor}(C) \to S$. Then, one exploits the completeness of $S$ to define two different multiplications on $Q$. As a result, one obtains a pair $(Q^c, Q^m)$ of generally non-commutative complete semirings. Multiplication in $Q^c$ is based on the composition of morphisms in $C$, whereas multiplication in $Q^m$ exploits the monoidal structure of $C$. As explained in Section 6 of [3], one assigns to every $n$-cobordism $W$ the composition $T_W: \mathcal{F}(W) \to Q$ of $T_W: \mathcal{F}(W) \to \text{Mor}(C)$ with the map $\text{Mor}(C) \to Q$ that assigns to every morphism in $C$ its characteristic function. Then, the state sum $Z_W: \mathcal{F}(\partial W) \to Q$ is defined on a boundary condition.
f ∈ F(∂W) as
\[ Z_W(f) = \sum_{F ∈ \mathcal{F}(W,f)} T_W(F) ∈ Q, \]
where the sum ranges over all fields F on W that extend f, i.e., F|∂W = f. Note that Z_W is well-defined due to the completeness of Q. In analogy with the quantum Hilbert state from physics, the state module Z(M) of a closed n-manifold M consists of all maps (“states”) \( \mathcal{F}(M) \to Q \) that satisfy a certain constraint equation. It can be shown that Z_W satisfies the constraint equation and is thus an element of the state module Z(∂W). Furthermore, the state modules and state sums thus defined can be shown to satisfy Banagl’s axioms of a positive TFT, including the essential gluing axiom. For a topologically meaningful choice of fields and action functionals the state sum Z_W is an invariant of n-cobordisms W that is interesting for further investigation.

5.2. Fold fields and cBr-valued actions. Fix an integer n ≥ 2. In this section we specify the fields and actions that determine our modification of the n-dimensional positive TFT constructed in [4]. All manifolds considered (with or without boundary) will be smooth, that is, differentiable of class \( C^∞ \).

5.2.1. Cobordisms. We recall the terminology concerning manifolds and cobordisms from Section 7.1 of [4].

From now on, we always use the terminology M, N, P etc. for closed \((n−1)\)-dimensional manifolds. Fix an integer \( D ≥ 2n + 1 \). We will always assume that any M is smoothly embedded in \( \mathbb{R}^D \), and that every connected component of M is contained in a hyperplane of the form \( \{k\} × \mathbb{R}^{D−1} \) for some \( k ∈ \{0,1,2,...\} \).

**Definition 5.1.** A cobordism from M to N is a compact n-dimensional smoothly embedded manifold \( W ⊂ [0,1] × \mathbb{R}^D \) with the following properties:

1. the boundary of W is \( ∂W = M ∪ N \), where \( M ⊂ \mathbb{R}^D = \{0\} × \mathbb{R}^D \) is the ingoing boundary and \( N ⊂ \mathbb{R}^D = \{1\} × \mathbb{R}^D \) is the outgoing boundary,
2. the interior of W satisfies \( W\setminus ∂W ⊂ (0,1) × \mathbb{R}^D \),
3. there exists \( 0 < ε < \frac{1}{2} \) such that \( W ∩ [0,ε] × \mathbb{R}^D = [0,ε] × M \) and \( W ∩ [1−ε,1] × \mathbb{R}^D = [1−ε,1] × N \) are product embeddings (any such ε is referred to as a cylinder scale), and
4. every connected component of W is contained in a set of the form \( [0,1] × \{k\} × \mathbb{R}^{D−1} \) for some \( k ∈ \{0,1,2,...\} \).

The advantage of working with embedded cobordisms \( W ⊂ [0,1] × \mathbb{R}^D \) is that they are naturally equipped with time functions \( ω: W → [0,1] \) induced by projection to the first coordinate. For every regular value \( t ∈ [0,1] \) of the time function \( ω: W → [0,1] \) the preimage \( ω^{-1}(t) \) is a smoothly embedded codimension 1 submanifold of W.

5.2.2. System of fold fields. Our theory will use exactly the same definition of fold fields on n-cobordisms that is employed in the original construction. Thus, in this section we will outline the content of Section 7.2 of [4]. We also use the same sets of fields on closed \((n−1)\)-manifolds although their definition relies on our modified action functional (see the end of Section 5.2.3).

The construction of fold fields on an n-dimensional cobordism W is based on the notion of fold maps from W into the plane \( \mathbb{R}^2 ≅ \mathbb{C} \). By definition, a fold map of an
n-manifold $X$ without boundary into the plane is a smooth map $F: X \to \mathbb{R}^2$ such that for every point $x \in X$ there exist coordinate charts centered at $x$ and $F(x)$ in which $F$ takes one of the following two normal forms:

$$(t, x_1, \ldots, x_{n-1}) \mapsto \begin{cases} (t, x_1) & \text{(regular point of $F$)}, \\ (t, -x_1^2 - \cdots - x_{i-1}^2 + x_{i+1}^2 + \cdots + x_{n-1}^2) & \text{(fold point of $F$)}. \end{cases}$$

Let $S(F)$ denote the set of fold points of a fold map $F: X \to \mathbb{R}^2$. It can be shown that $S(F) \subset X$ is a smoothly embedded 1-dimensional submanifold that is closed as a subset, and that $F$ restricts to an immersion $S(F) \to \mathbb{R}^2$. In analogy with the Morse index of non-degenerate critical points, there is the following notion of an (absolute) index for fold points.

**Proposition 5.2.** To any fold map $F: X \to \mathbb{R}^2$ one can associate a well-defined locally constant map

$$\iota_F: S(F) \to \mathbb{N}, \quad \iota_F(x) = \min\{i, n-1-i\},$$

where $i \in \{0, \ldots, n-1\}$ is the number of minus signs that appear in the local normal form of fold points above.

Let $W$ be an $n$-dimensional cobordism from $M$ to $N$. A smooth map $F: W \to \mathbb{R}^2$ is called fold map if $F$ has for some $\varepsilon > 0$ an extension to a fold map $\tilde{F}: ((-\varepsilon, 0] \times M) \cup_M W \cup_N ([1, 1+\varepsilon] \times N) \to \mathbb{R}^2$.

Given a fold map $F: W \to \mathbb{R}^2$, the intersection $S(\tilde{F}) \cap W$ does not depend on the choice of the fold map extension $\tilde{F}$, and will in the following be denoted by $S(F)$. The independence of the choice of $\tilde{F}$ can be shown by using the characterization of fold maps by means of transversality in jet spaces as discussed in Section 3 of [3]. Furthermore, for an open subset $U \subset \partial W$ we write $S(F) \cap \partial U$ if $S(\tilde{F}) \cap \partial U$ for some (and hence, any) fold map extension $\tilde{F}$ of $F$. If $S(F) \cap \partial W$, then $S(F) \subset W$ is a 1-dimensional smoothly embedded compact submanifold with boundary $\partial S(F) = S(F) \cap \partial W$. In this case, we write $\iota_F: S(F) \to \mathbb{N}$ for the restriction of $\iota_F: S(\tilde{F}) \to \mathbb{N}$ to $S(F)$ for some (and hence, any) fold map extension $\tilde{F}$ of $F$.

Let $\omega: W \to [0,1]$ denote the time function associated to $W$ (see Section 5.2.1).

**Definition 5.3.** Given a fold map $F: W \to \mathbb{C}$, we set

$$\mathfrak{n}(F) = \{t \in [0,1] : t \text{ is a regular value of } \omega, \text{ and } S(F) \cap \omega^{-1}(t) \subset [0,1] \}. $$

**Definition 5.4.** A fold map $F: W \to \mathbb{C}$ has generic imaginary parts over $t \in [0,1]$ if the restriction $\text{Im} \circ F|: S(F) \cap \omega^{-1}(t) \to \mathbb{R}$ is injective. Let

$$\text{GenIm}(F) = \{t \in [0,1] : F \text{ has generic imaginary parts over } t \} \subset [0,1].$$

For $k \in \{0,1,2,\ldots\}$ let $F(k)$ denote the restriction of a fold map $F: W \to \mathbb{C}$ to the part of $W$ that lies in $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$ (see Definition 5.1 (4)):

$$F(k) = F|: W \cap ([0,1] \times \{k\} \times \mathbb{R}^{D-1}) \to \mathbb{C}.$$  

Fields on $W$ are fold maps $F: W \to \mathbb{C}$ with certain properties concerning the subsets $\mathfrak{n}(F(k))$ and $\text{GenIm}(F(k))$ of $[0,1]$.

**Definition 5.5.** A fold field on $W$ is a fold map $F: W \to \mathbb{C}$ so that for all $k \in \{0,1,2,\ldots\}$ the following conditions hold:

1. $0,1 \in \mathfrak{n}(F(k)) \cap \text{GenIm}(F(k))$, and
(2) $\text{GenIm}(F(k))$ is residual in $[0, 1]$.

Condition (1) is exploited in the construction of the $\mathbf{Br}$-valued action functional $S$ in Section 7.3 of [4] (as well as in our modified construction in Section 5.2.3). Condition (2) is crucial for the proof of the indispensable gluing theorem (see Section 7.7 in [4]).

Let $\mathcal{F}(W)$ denote the set of all fold fields on $W$. If $W = \emptyset$, then one puts $\mathcal{F}(W) = \{\ast\}$ (set with a single element). Fields on closed $(n - 1)$-dimensional manifolds will be introduced at the end of Section 5.2.3 which completes the definition of the system $\mathcal{F}$ of fields.

5.2.3. System of $\mathbf{cBr}$-valued action functionals. In Section 7.3 in [4], Banagl uses singularity theory of fold maps into the plane to construct a system $S$ of $\mathbf{Br}$-valued action functionals. Namely, for every $n$-cobordism $W$ there is a function $S: \mathcal{F}(W) \to \text{Mor}(\mathbf{Br})$ assigning to every fold field on $W$ a morphism in $\mathbf{Br}$ that encodes the combinatorial information of the 1-dimensional singular set of the fold map. In the present section, we modify the original construction by replacing the Brauer category $\mathbf{Br}$ by its chromatic enrichment $\mathbf{cBr}$. The idea is to capture not only the singular patterns provided by the singular sets of fold fields, but also to remember the indices of fold lines by using labels from the set $\mathbb{N}$. Hence, we construct a system $\mathfrak{S}$ of $\mathbf{cBr}$-valued action functionals which is a lift of $S$ under the forgetful map $\text{Mor}(\mathbf{cBr}) \to \text{Mor}(\mathbf{Br})$. During the process of quantization in Section 5.2.3 we will linearize the system $S$ of action functionals by means of a faithful linear representation $Y: \mathbf{cBr} \to \mathbf{Vect}$ (see Theorem 1.2).

Let $W$ be an $n$-cobordism from $M$ to $N$. We construct the function $\mathfrak{S}_n: \mathcal{F}(W) \to \text{Mor}(\mathbf{cBr})$ as follows. If $W$ is empty, then we set $\mathfrak{S}_n(\ast) = \text{id}_{(\{0\}, \mathcal{C})}$. Next suppose that $W$ is non-empty and entirely contained in a set of the form $[0, 1] \times \{k\} \times \mathbb{R}^{D-1}$, where $k \in \{0, 1, 2, \ldots\}$. Let $F (= F(k)) \in \mathcal{F}(W)$ be a field on $W$. By condition (1) for fold fields (see Definition 5.5), we have $0, 1 \in \partial(F)$, so that the intersections $S(F) \cap M$ and $S(F) \cap N$ are compact manifolds of dimension 0. Furthermore, since $F$ has generic imaginary parts over 0 and 1 (see Definition 5.4), the composition $\text{Im} \circ F: W \rightarrow \mathbb{R}$ restricts to injective maps on both $S(F) \cap M$ and $S(F) \cap N$. Let $m$ and $m'$ denote the number of points in $S(F) \cap M$ and $S(F) \cap N$, respectively. Then, we obtain orderings $S(F) \cap M = \{p_1, \ldots, p_m\}$ and $S(F) \cap N = \{q_1, \ldots, q_{m'}\}$ which are uniquely determined by requiring that $(\text{Im} \circ F)(p_i) < (\text{Im} \circ F)(p_j)$ if and only if $i < j$, and $(\text{Im} \circ F)(q_i) < (\text{Im} \circ F)(q_j)$ if and only if $i < j$. The resulting bijections $S(F) \cap M \cong M[m]$, $p_i \mapsto i$, and $S(F) \cap N \cong M[m']$, $q_i \mapsto i$, are exactly the same as those described in the original construction of $S$. We define maps $c: [m] \to \mathbb{N}$ and $c': [m'] \to \mathbb{N}$ by assigning to each point $x \in [m] = M[m] \cong S(F) \cap M$ and $x' \in [m'] = M[m'] \cong S(F) \cap N$ the index of the fold map $F$ at $x$ and $x'$, respectively (see Proposition 5.2). So far, we have constructed objects $([m], c)$ and $([m'], c')$ in $\mathbf{cBr}$. The desired morphism $\mathfrak{S}_n(F): ([m], c) \to ([m'], c')$ in $\mathbf{cBr}$ is now represented by the pair $(S(F), c')$, where the embedding $S(F) \subset [0, 1] \times \mathbb{R}^3$ is defined in exactly the same manner as described in the construction of $S$. That is, every component of $S(F)$ with non-empty boundary is embedded as a smooth arc that connects the corresponding points in $([0] \times M[m]) \cup ([1] \times M[m'])$. (For components of $S(F)$ without boundary one may choose an arbitrary embedding into $(0, 1) \times \mathbb{R}^3$.) Finally, for an arbitrary non-empty cobordism $W$, we define $\mathfrak{S}(F) = \bigotimes_{k=0}^\infty \mathfrak{S}_n(F(k))$. (Note that the tensor product is actually finite because $W$ is compact.) This completes...
our construction of a system \( S \) of \( \mathbf{cBr} \)-valued action functionals which lifts \( S \) under the forgetful map \( \text{Mor}(\mathbf{cBr}) \to \text{Mor}(\mathbf{Br}) \). Note that Lemma 7.12 in [4] remains valid when replacing \( S \) with \( \overline{S} \) in the formulation. That is, given a fold field \( F \) on \( W \) and some \( t \in (0, 1) \) such that \( t \in \text{GenIm}(F(k)) \) for all \( k \in \{0, 1, \ldots\} \), \( F \) restricts to fold fields \( F_{\leq t} \) on \( W \cap ([0, t] \times \mathbb{R}^D) \) and \( F_{> t} \) on \( W \cap ([1 - t, 1] \times \mathbb{R}^D) \), and we have \( \overline{S}(F) = \overline{S}(F_{\leq t}) \circ \overline{S}(F_{> t}) \) in \( \mathbf{cBr} \).

Finally, fields on a closed \((n - 1)\)-manifold \( M \) are defined to be certain fold fields on the cylinder \([0, 1] \times M \subset [0, 1] \times \mathbb{R}^D\), i.e., the trivial cobordism from \( M \) to \( M \). Namely, when \( M \) is non-empty, we put

\[
\mathcal{F}(M) = \{ f \in \mathcal{F}([0, 1] \times M); \overline{S}(f) = 1 \in \text{Mor}(\mathbf{cBr}) \},
\]

where \( 1 \) denotes some identity morphism in \( \mathbf{cBr} \). Note that, by Lemma 5.1, a fold field \( f \in \mathcal{F}([0, 1] \times M) \) satisfies \( \overline{S}(f) = 1 \) in \( \mathbf{cBr} \) if and only if \( S(f) = 1 \) in \( \mathbf{Br} \). If \( M = \emptyset \), then one puts \( \mathcal{F}(M) = \{ * \} \) (set with a single element). Hence, the set of fields on closed \((n - 1)\)-manifold remains unchanged when replacing \( S \) by \( \overline{S} \). In particular, Lemma 7.13 and Lemma 7.14 (additivity axiom) in [4] remain valid when replacing \( S \) with our modified \( \mathbf{cBr} \)-valued action functional \( \overline{S} \) in the formulation.

5.3. Quantization. As pointed out in Section 8 of [3], it can be advantageous to linearize the category-valued system of action functionals used for quantization. We construct a linearization \( \mathbf{T} \) of our system \( \overline{S} \) of \( \mathbf{cBr} \)-valued action functionals from Section 5.2.3 as follows. Fix once and for all a faithful symmetric strict monoidal functor \( Y: \mathbf{cBr} \to \mathbf{Vect} \) by means of Theorem 1.2. Then, a \( \mathbf{Vect} \)-valued action functional \( \mathbf{T} \) is defined by assigning to every \( n \)-cobordism \( W \) the composition

\[
\mathbf{T}_W: \mathcal{F}(W) \xrightarrow{\overline{S}} \text{Mor}(\mathbf{cBr}) \xrightarrow{Y} \text{Mor}(\mathbf{Vect}).
\]

In the present section, we quantize the system \( \mathcal{F} \) of fold fields of Section 5.2.2 and our system \( \overline{S} \) of \( \mathbf{Vect} \)-valued action functionals. For this purpose, we will first modify in Section 5.3.2 below the algebraic process of profinite idempotent completion (see Section 6 in [3]) to represent loops of different colors in \( \mathbf{cBr} \) by a countable family of loop parameters. Then, we proceed to define our positive TFT \( \overline{\mathbf{Z}} \). In Section 5.3.3 we will specify the state modules \( \overline{\mathbf{Z}}(M) \) of closed \((n - 1)\)-manifolds \( M \), and define the state sums of \( n \)-cobordisms \( W \) as certain elements \( \overline{\mathbf{Z}}_W \in \overline{\mathbf{Z}}(\partial W) \). First of all, Section 5.3.1 provides the necessary background on semirings.

5.3.1. Semirings and semimodules. We collect some basic material from the theory of semirings and semimodules that is needed for the process of quantization. A detailed background is provided in Section 2 of [3] and Section 4 of [4].

Recall that a (commutative) monoid is a triple \( M = (M, \ast, e) \), where \( M \) is a set equipped with a (commutative) associative binary operation \( \ast \) and two-sided identity element \( e \in M \), that is, \( e \ast m = m \ast e = m \) for all \( m \in M \). A semiring is a tuple \( S = (S, +, \cdot, 0, 1) \), where \( S \) is a set together with two binary operations \( + \) and \( \cdot \) and two elements \( 0, 1 \in S \) such that \((S, +, 0)\) is a commutative monoid, \((S, \cdot, 1)\) is a monoid, the multiplication \( \cdot \) distributes over the addition from either side, and \( 0 \) is absorbing, i.e., \( 0 \cdot s = 0 = s \cdot 0 \) for every \( s \in S \). The semiring \( S \) is called commutative if the monoid \((S, \cdot, 1)\) is commutative. A morphism of semirings sends \( 0 \) to \( 0 \), \( 1 \) to \( 1 \) and respects addition and multiplication. Fix a semiring \( S \). A (left)
S-semimodule is a commutative monoid \( M = (M, +, 0_M) \) together with a scalar multiplication \( S \times M \to M, (s, m) \mapsto sm \), such that for all \( r, s \in S \), \( m, n \in M \), we have \((rs)m = r(sm), r(m + n) = rm + rn, (r + s)m = rm + sm, 1m = m \), and \( r0_M = 0_M = 0m \). Given a morphism \( \varphi: S \to T \) of semirings, it is clear that \( T \) becomes a \( S \)-semimodule via \( st = \varphi(s)t \).

A monoid \( (M, \ast, e) \) is called idempotent if \( m \ast m = m \) for all elements \( m \in M \). The semiring \( (S, +, \cdot, 0, 1) \) is idempotent if \( (S, +, 0) \) is an idempotent monoid. A semimodule is called idempotent if its underlying additive monoid is idempotent.

Next, we discuss the important notion of Eilenberg-completeness [7, p. 125] for semirings and semimodules. A complete monoid is a commutative monoid that is complete as a semiring, then \( \hat{M} \) is continuous for all elements \( m \in \hat{M} \) and infinite distributivity holds, that is,

\[
\sum_{i \in \emptyset} m_i = 0, \quad \sum_{i \in \{1\}} m_i = m_1, \quad \sum_{i \in \{1,2\}} m_i = m_1 + m_2,
\]

and for every partition \( I = \bigcup_{j \in J} I_j \), we have

\[
\sum_{j \in J} \left( \sum_{i \in I_j} m_i \right) = \sum_{i \in I} m_i.
\]

A complete semiring is a semiring \( S \) for which \( (S, +, 0, \Sigma) \) is a complete monoid, and infinite distributivity holds, that is,

\[
\sum_{i \in I} s_i = \left( \sum_{i \in I} s_i \right), \quad \sum_{i \in I} s_i s = \left( \sum_{i \in I} s_i \right) s.
\]

A semimodule \( M \) over a commutative semiring \( S \) is called complete if its underlying additive monoid is equipped with a summation law that makes it complete as a commutative monoid, and infinite distributivity

\[
\sum_{i \in I} sm_i = s \left( \sum_{i \in I} sm_i \right)
\]

holds for every \( s \in S \) and every family \( (m_i)_{i \in I} \) in \( M \). If \( \varphi: S \to T \) is a morphism of semirings and \( T \) is complete as a semiring, then \( T \) can be easily seen to be complete as an \( S \)-semimodule.

We will also need an notion of continuity for idempotent complete semirings. Here, we only state the definition, and refer to the discussion preceding Proposition 4.2 in [8] for more details. Observe that any idempotent monoid \((M, \ast, e)\) admits a natural partial order \( \leq \) given by \( m \leq m' \) if and only if \( m + m' = m' \). An idempotent complete monoid \((M, +, 0, \Sigma)\) is continuous if for all families \((m_i)_{i \in I} \), \( m_i \in M \), and for all \( c \in M \), \( \sum_{i \in F} m_i \leq c \) for all finite \( F \subset I \) implies \( \sum_{i \in I} m_i \leq c \). An idempotent complete semiring (semimodule) is called continuous if its underlying additive monoid is continuous.

It is useful to note that the product \( \prod_{i \in I} M_i \) of a family \((M_i)_{i \in I}\) of continuous idempotent complete monoids is a continuous idempotent complete monoid.

**Example 5.6.** The minimal example of a semiring that is not a ring is given by the Boolean semiring \( \mathbb{B} \). This is the set \( \mathbb{B} = \{0, 1\} \) equipped with addition defined by \( 1 + 1 = 1 \) and multiplication given by \( 0 \cdot 0 = 0 \) (where 0 and 1 serve as identity
elements for addition and multiplication, respectively). Distributivity holds, but in \( \mathbb{B} \) there exists no additive inverse for 1. We leave it to the reader to check that the commutative semiring \( \mathbb{B} \) is idempotent, complete, and continuous.

5.3.2. Profinite idempotent completion. The algebraic process of profinite idempotent completion (see Section 6 in \([4]\)) adapts the general construction of a pair \((Q^c, Q^m)\) of complete semirings (see Section 4 in \([3]\)) to reflect the specific inner structure of morphism sets of the Brauer category, as we recall next. Note that the sets \( \text{Hom}_{\mathbf{Br}}([m], [m']) \) have the special property that they are naturally equipped with the action \( \tau^i \varphi = \varphi \otimes \lambda^{\otimes i} \) of the (multiplicatively written) monoid \( \mathbb{N} = \{ \tau^i; \ i \in \mathbb{N} \} \). Fix a linear representation \( U: \mathbf{Br} \to \mathbf{Vect} \) and write \( V = U([1]) \) and \( \hat{\lambda} = U(\lambda) \in \mathbb{R} \). Then, the subset \( H_{m,n} = U(\text{Hom}_{\mathbf{Br}}([m], [m'])) \) of the real vector space \( \text{Hom}_{\mathbf{Vect}}(V^\otimes m, V^\otimes m') \) inherits an action of the monoid \( \mathbb{N} \) via \( \tau^i f = \hat{\lambda}^i f \). Given a set \( A \), let \( FM(A) \) denote the free commutative monoid generated by \( A \). In particular, \( FM(H_{m,n}) \) has the structure of a \( \mathbb{N}[\tau] \)-semimodule by Lemma 4.1 in \([4]\). A \( \mathbb{N}[\tau] \)-semimodule is given by the algebraic tensor product \( Q(H_{m,n}) = FM(H_{m,n}) \otimes_{\mathbb{N}[\tau]} \mathbb{B}[[q]] \), where \( \mathbb{B} \) denotes the Boolean semiring, and \( \mathbb{B}[[q]] \) is the associated semiring of formal power series. It can be shown (see Lemma 6.7 in \([4]\) that \( Q(H_{m,n}) \) is isomorphic as a \( \mathbb{N}[\tau] \)-semimodule to a finite sum of copies of \( \mathbb{B}[[q]] \), so that its elements consist of a number of power series in the loop parameter \( q \). This can be derived more abstractly by using minimal shells of projectively finite subsets of a real vector space, see Definition 6.1 in \([4]\). Finally, the profinite idempotent completion of the set \( U(\text{Mor}(\mathbf{Br})) \) is the \( \mathbb{N}[\tau] \)-semimodule

\[
Q = Q(U) = \prod_{m, m' \in \mathbb{N}} Q(H_{m,n}).
\]

Provided that the underlying functor \( U \) is chosen to be faithful on loops, the additive monoid \( (Q, +, 0) \) can be promoted to idempotent complete semirings \( Q^c \) (the composition semiring, see Proposition 6.12 in \([3]\)) and \( Q^m \) (the monoidal semiring, see Proposition 6.14 in \([4]\)). It can be shown that the semirings \( Q^c \) and \( Q^m \) are both continuous (see Proposition 6.15 in \([4]\)). Continuity is exploited in \([4]\) to check several axioms of positive TFTs, namely the behavior of state sums under disjoint union (Proposition 7.22), and the gluing axiom (Theorem 7.26).

When using the chromatic Brauer category instead of \( \mathbf{Br} \), we need to replace the semimodule \( \mathbb{B}[[q]] \) over \( \mathbb{N}[\tau] \) by the semimodule \( \mathbb{B}[q] = \mathbb{B}[q_0, q_1, \ldots] \) over \( \mathbb{N}[\tau] = \mathbb{N}[\tau_0, \tau_1, \ldots] \). Here, the different parameters represent loops of different labels in \( \mathbf{cBr} \). Let us introduce the semirings \( \mathbb{N}[\tau] \) and \( \mathbb{B}[[q]] \) (compare Section 4 in \([4]\)). Set \( \mathbb{N} = \bigoplus_{i=0}^{\infty} \mathbb{N} \), which is a commutative monoid with respect to component wise addition and identity element \( 0 = (0, 0, \ldots) \). In general, a (formal) power series in a countable number of indeterminates \( q = q_0, q_1, \ldots \) and with coefficients in the Boolean semiring \( \mathbb{B} \) is a function \( a: \mathbb{N} \to \mathbb{B} \), written as a formal sum \( \sum_{\nu \in \mathbb{N}} a(\nu)q^\nu \), where \( q^\nu \) denotes the (finite) product \( \prod_{s=0}^{\infty} q_s^\nu_s \). The element \( a(\nu) \) is referred to as the coefficient of \( q^\nu \). Let \( \mathbb{B}[[q]] \) be the set of all power series over \( \mathbb{B} \) having a countable number of indeterminates \( q = q_0, q_1, \ldots \). We write 0 for the power series
a with \(a(\nu) = 0\) for all \(\nu\), and 1 for the power series \(a\) with \(a(0) = 1\) and \(a(\nu) = 0\) for all \(\nu \neq 0\). Define an addition on power series by \(a + b = c\), where \(c(\nu) = a(\nu) + b(\nu)\) for all \(\nu\). Define a multiplication on power series by the Cauchy product, that is, \(a \cdot b = c\) where \(c(\nu) = \sum_{\mu + \kappa = \nu} a(\mu)b(\kappa)\). Then, \((\mathbb{B}[[q]], +, \cdot, 0, 1)\) is a commutative idempotent semiring, the semiring of power series over \(\mathbb{B}\) in a countable number of indeterminates. In a similar way, using finite sums rather than power series, one defines the polynomial semiring \(\mathbb{N}[t]\) in a countable number of indeterminates \(t = \tau_0, \tau_1, \ldots\). Note that \(\mathbb{B}[[q]]\) is a \(\mathbb{N}[t]\)-semimodule via the semiring morphism \(\mathbb{N}[t] \rightarrow \mathbb{B}[[q]]\) that extends the unique semiring morphism \(\mathbb{N} \rightarrow \mathbb{B}\) by \(\tau_k \mapsto q_k\), \(k \in \mathbb{N}\). It can be shown that \(\mathbb{B}[[q]]\) is a complete semiring, and is hence complete as a \(\mathbb{N}[t]\)-semimodule. Furthermore, the idempotent complete semiring \(\mathbb{B}[[q]]\) can be shown to be continuous.

Returning to the category \(\text{cBr}\), we observe that the \(k\)-loops \(\lambda_{(k)}\), \(k \in \mathbb{N}\), induce an action of the (multiplicatively written) commutative monoid \(\mathbb{N} = \{t^\nu; \nu \in \mathbb{N}\}\) on the morphism sets

\[
\text{Hom}_{\text{cBr}}\left(\left([m], c\right), \left([m'], c'\right)\right)
\]

via \((t^\nu, \varphi) \mapsto t^\nu \varphi = \left(\bigotimes_{k=0}^{\infty} \lambda_{(k)}^{\otimes q_k}\right) \otimes \varphi\). Using the fixed linear representation \(Y: \text{cBr} \rightarrow \text{Vect}\), we write \(V_k = Y(\left([1], k\right))\) and \(\hat{\lambda}_{(k)} = \lambda\left(k, k\right) \in \mathbb{R}\). Then, the subset

\[
H_{\left([m], c\right), \left([m'], c'\right)} = Y(\text{Hom}_{\text{cBr}}\left(\left([m], c\right), \left([m'], c'\right)\right))
\]

of the real vector space \(\text{Hom}_{\text{Vect}}(Y(\left([m], c\right)), Y(\left([m'], c'\right)))\)

inherits an action of the monoid \(\mathbb{N}\) via \(t^\nu f = \left(\prod_{k=0}^{\infty} \hat{\lambda}_{(k)}^{q_k}\right) \cdot f\). In analogy to Lemma 4.1 in [4], it follows that \(FM(H_{\left([m], c\right), \left([m'], c'\right)})\), the free commutative monoid generated by \(H_{\left([m], c\right), \left([m'], c'\right)}\), has the structure of a \(\mathbb{N}[t]\)-semimodule via

\[
\sum_{\nu \in \mathbb{N}} m_\nu t^\nu \cdot \sum_j \alpha_j f_j = \sum_{\nu, j} (m_\nu \alpha_j) (t^\nu f_j), \quad m_\nu, \alpha_j \in \mathbb{N}, f_j \in H_{\left([m], c\right), \left([m'], c'\right)}.
\]

Using the algebraic tensor product of semimodules over the commutative semiring \(\mathbb{N}[t]\) (compare Section 4 in [4]) we can now define a \(\mathbb{N}[t]\)-semimodule by

\[
\overline{Q}(H_{\left([m], c\right), \left([m'], c'\right)}) = FM(H_{\left([m], c\right), \left([m'], c'\right)}) \otimes \mathbb{N}[t] \mathbb{B}[[q]]).
\]

Let \(OP_{\left([m], c\right), \left([m'], c'\right)}\) denote the (finite) set of loop-free morphisms \(\left([m], c\right) \rightarrow \left([m'], c'\right)\) in \(\text{cBr}\). Since the linear representation \(Y: \text{cBr} \rightarrow \text{Vect}\) has been chosen to be faithful, it can be shown that \(\overline{Q}(H_{\left([m], c\right), \left([m'], c'\right)})\) is isomorphic in the category of \(\mathbb{N}[t]\)-semimodules to the product of copies of \(\mathbb{B}[[q]]\) indexed by the elements of \(OP_{\left([m], c\right), \left([m'], c'\right)}\). (In fact, in analogy to Lemma 6.6 in [4], one can show that every element in \(FM(H_{\left([m], c\right), \left([m'], c'\right)})\) can be uniquely written as

\[
\sum_{\sigma = 1}^{r} p_\sigma(t) Y(\varphi_\sigma)
\]

for suitable polynomials \(p_\sigma(t) \in \mathbb{N}[t]\), where \(\varphi_1, \ldots, \varphi_r\) is the list of elements of \(OP_{\left([m], c\right), \left([m'], c'\right)}\). Finally, the profinite idempotent completion of the set \(Y(Mor(\text{cBr}))\) is the \(\mathbb{N}[t]\)-semimodule

\[
\overline{Q} = \overline{Q}(Y) = \prod_{\left([m], c\right), \left([m'], c'\right)} \overline{Q}(H_{\left([m], c\right), \left([m'], c'\right)}).
\]
Similarly to the construction in Section 6 in \cite{4}, the additive monoid $(\overline{Q}, +,0)$ is complete, and can hence be advanced to idempotent complete semirings $\overline{Q}$ and $\overline{Q}^n$. Since $\mathbb{B}[a]$ is continuous, it follows that $\overline{Q}$ and $\overline{Q}^n$ are both continuous.

5.3.3. State modules and state sums. Let $\overline{Q}$ denote the profinite idempotent completion of the set $Y(\text{Mor(cBr)})$ associated to a fixed faithful linear representation $Y : \text{cBr} \rightarrow \text{Vect}$ as constructed in of Section 5.3.2. We proceed to define our smooth positive TFT $\overline{Z}$. The state module of a closed $(n-1)$-manifold is defined to be $\overline{Z}(M) = \{ z : \mathcal{F}(M) \rightarrow \overline{Q} \}$. By Proposition 3.1 in \cite{3}, $\overline{Z}(M)$ inherits the structure of a two-sided $\overline{Q}$-semialgebra and a two-sided $\overline{Q}^n$-semialgebra, and $\overline{Z}(M)$ is complete. Then, it follows from the corresponding properties of $\overline{Q}$ that $\overline{Z}(M)$ is idempotent and continuous. The construction of a contraction product

$$\langle \cdot , \cdot \rangle : (\overline{Z}(M) \otimes \overline{Z}(N)) \times (\overline{Z}(N) \otimes \overline{Z}(P)) \rightarrow \overline{Z}(M) \otimes \overline{Z}(P)$$

is analogous to the discussion in Section 7.4 of \cite{4}. Here, $\otimes$ denotes the complete tensor product of complete idempotent continuous semimodules (see Section 5 in \cite{4}) rather than the algebraic tensor product $\otimes$ of function semimodules discussed in \cite{2}.

Let $W^n$ be a cobordism from $M$ to $N$. The state sum will be defined as an element $\overline{Z}_W \in \overline{Z}(M) \otimes \overline{Z}(N)$. Fix a cylinder scale $\varepsilon_W$ for $W$. Given a boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, we define

$$\mathcal{F}(W ; f_M, f_N) = \{ F \in \mathcal{F}(W) \mid \exists \varepsilon(k), \varepsilon'(k) \in (0, \varepsilon_W) : \quad F\mid_{[0,\varepsilon(k)] \times M(k)} \approx f_M(k), \quad F\mid_{[1-\varepsilon'(k),1] \times N(k)} \approx f_N(k), \forall k \},$$

where the equivalence relation $\approx$ for fold fields on closed $(n-1)$-manifolds $X$ from Definition 7.18 in \cite{4} is used. Namely, two smooth maps $f : [a, b] \times X \rightarrow \mathbb{C}$ and $f' : [a', b'] \times X \rightarrow \mathbb{C}$ are equivalent, $f \approx f'$, if there exists a diffeomorphism $\xi : [a, b] \rightarrow [a', b']$ with $\xi(a) = a'$ such that $f(t,x) = f'(\xi(t),x)$ for all $(t, x) \in [a, b] \times X$. On $(f_M, f_N)$ the state sum $\overline{Z}_W$ is then defined as

$$\overline{Z}_W(f_M, f_N) = \sum_{F \in \mathcal{F}(f_M, f_N)} \mathbb{T}_W(F),$$

which is a well-defined element of the complete semiring $\overline{Q}$. Note that, when $\mathbb{T}(F) : [(m), c] \rightarrow [(m'), c']$ in cBr, the element $\mathbb{T}_W(F)$ is supposed to be identified with the element $\mathbb{T}_W(F) \otimes 1 \in \overline{Q}(H_{[(m), c], [(m'), c']}) \subset \overline{Q}$.

In close analogy with the further steps in \cite{4}, one can prove that our assignment $\overline{Z}$ is in fact a positive topological field theory. Namely, following Section 7.6 in \cite{4}, one checks the correct behavior of our state sum under disjoint union. Moreover, following Section 7.7 in \cite{4}, one proves the essential gluing formula $\overline{Z}_W = \langle \overline{Z}_W', \overline{Z}_W'' \rangle$ (see Theorem 7.26 in \cite{4}), where $W$ is the result of gluing a cobordism $W'$ from $M$ to $N$ with a cobordism $W''$ from $N$ to $P$ along $N$. Note that the preparatory results Proposition 7.23, Lemma 7.24, and Proposition 7.25 in \cite{4} need only be modified by replacing the Br-valued action functional $\$ with the cBr-valued action functional $\$ in the formulation. Diffeomorphism invariance $\varphi_*(\overline{Z}_W) = \overline{Z}_{W'}$ (see Theorem 9.16 in \cite{4}) of our state sum under diffeomorphisms $\varphi : \partial W \rightarrow \partial W'$ that can be extended to so-called time consistent diffeomorphisms $\varphi : W \rightarrow W'$ can be shown along the lines of Section 9 in \cite{4}. In particular, Lemma 9.12 and Lemma 9.14 in \cite{4} remain
valid when replacing \( S \) with \( \overline{S} \) in the formulation. The map \( \varphi_s : \mathbb{Z}(\partial W) \to \mathbb{Z}(\partial W') \)
can then be defined on a function \( z : \mathcal{F}(\partial W) \to \overline{Q} \) and a field \( g \in \mathcal{F}(\partial W') \) by
\[
\varphi_s(z)(g) = z(g \circ (\text{id}_{[0,1]} \times \varphi)) \in \overline{Q}.
\]

5.4. **The aggregate invariant and exotic Kervaire spheres.** Positive TFTs have been created with the intention to provide new topological invariants for high-dimensional manifolds (see [3]). In this section, we explain how our positive TFT \( \mathcal{Z} \) can be used to assign to any homotopy sphere \( M \) its aggregate invariant \( \overline{\mathfrak{A}}(M) \), an element of the complete semiring \( \overline{Q} \) from Section 5.3.2. The construction of \( \overline{\mathfrak{A}} \) is analogous to that of the aggregate invariant \( \mathfrak{A} \) studied Section 10 in [3]. While the invariant \( \mathfrak{A} \) is known to distinguish exotic spheres from the standard sphere (see Corollary 10.4 in [3]), we will indicate briefly that the invariant \( \overline{\mathfrak{A}} \) can distinguish exotic Kervaire spheres from other exotic spheres in infinitely many dimensions.

Fix a closed \( (n-1) \)-manifold \( M \) which is homeomorphic (but not necessarily diffeomorphic) to the sphere \( S^{n-1} \). Without loss of generality, we assume in the following that \( S^{n-1} \) and \( M \) are smoothly embedded in \( \{0\} \times \mathbb{R}^{D-1} \) (compare Section 5.2.1). From now on, we suppose that \( n - 1 \geq 5 \). Then, by classical Morse theory, \( M \) admits Morse functions with exactly two non-degenerate critical points, namely one minimum and one maximum. Given any diffeomorphism \( \xi : [0,1] \to [a,b] \) with \( \xi(0) = a \), and any Morse function \( f_M : M \to \mathbb{R} \) with exactly two non-degenerate critical points, we observe that the map
\[
\tilde{f}_M : [0,1] \times M \to \mathbb{R}^2, \quad \tilde{f}_M(t,x) = (\xi(t), f_M(x)),
\]
is a fold field on \( M \) (see Definition 5.5). Let \( C_2(M) \subset \mathcal{F}(M) \) denote the (non-empty) subset of all such maps \( \tilde{f}_M \). Fix an element \( \tilde{f}_S \in C_2(S^{n-1}) \) of the form \( \tilde{f}_S = \text{id}_{[0,1]} \times f_S \). Let us write \( \text{Cob}(S^{n-1},M) \) for the collection of all oriented cobordisms from \( S^{n-1} \) to \( M \) that are embedded in \( \{0\} \times \{0\} \times \mathbb{R}^{D-1} \) (compare property (4) of Definition 5.1). Since \( M \) is homeomorphic to \( S^{n-1} \), it can be shown that \( \text{Cob}(S^{n-1},M) \) is non-empty (see Lemma 10.1 in [3]). Now, for any cobordism \( W \in \text{Cob}(S^{n-1},M) \) and any fold field \( \tilde{f}_M \in C_2(M) \), the state sum \( \mathcal{Z}_W \in \mathcal{Z}(S^{n-1}) \otimes \mathcal{Z}(M) \) of Section 5.3.3 can be evaluated at \( (\tilde{f}_S, \tilde{f}_M) \in \mathcal{F}(S^{n-1}) \times \mathcal{F}(M) \) to yield an element \( \mathcal{Z}_W(\tilde{f}_S, \tilde{f}_M) \) in the complete semiring \( \overline{Q} \) from Section 5.3.2 that is associated to a faithful linear representation \( Y : \text{cBr} \to \text{Vect} \). Hence, summation in the complete semiring \( \overline{Q} \) yields a well-defined element
\[
\overline{\mathfrak{A}}(M) := \sum_{\tilde{f}_M \in C_2(M)} \sum_{W \in \text{Cob}(S^{n-1},M)} \mathcal{Z}_W(\tilde{f}_S, \tilde{f}_M) \in \overline{Q}.
\]

In conclusion, we outline an application to Kervaire spheres, which are a concrete family of homotopy spheres that can be obtained from a plumbing construction as follows (see [10] p. 162). The unique Kervaire sphere \( \Sigma_K^{n-1} \) of dimension \( n - 1 = 4r + 1 \) can be defined as the boundary of the parallelizable \( (4r+2) \)-manifold given by plumbing together two copies of the tangent disc bundle of \( S^{2r+1} \). According to the classification theorem of homotopy spheres (see [12] Theorem 6.1, pp. 123f)], as well as recent work of Hill-Hopkins-Ravenel [8] on the Kervaire invariant one problem, it is known that \( \Sigma_K^{n-1} \) is an exotic sphere, i.e., homeomorphic but not diffeomorphic to \( S^{n-1} \), except when \( n - 1 \in \{5, 13, 29, 61, 125\} \).

Note that, according to Remark 6.3 in [21], there are infinitely many dimensions of the form \( n - 1 \equiv 13 \mod 16 \) in which there exist exotic spheres that are not diffeomorphic to the Kervaire sphere \( \Sigma_K^{n-1} \). The following result shows that our
aggregate invariant $\Xi$ can distinguish exotic Kervaire spheres from other exotic spheres in infinitely many dimensions. We give a sketch of the proof by referring to the results of [19]. A detailed proof is beyond the scope of this paper, and will appear elsewhere.

**Theorem 5.7.** Suppose that $n - 1 \equiv 13 \pmod{16}$ and $n - 1 \geq 237$. Then, an exotic $(n - 1)$-sphere $\Sigma^{n-1}$ is diffeomorphic to the Kervaire sphere if and only if $\Xi(\Sigma^{n-1}) = \Xi(\Sigma^1_K)$.

**Sketch of proof.** Recall from Section 5.3.2 that elements of $\overline{Q}$ are families of power series in $\mathbb{B}[[q]]$ which are indexed by the loop-free morphisms of $cBr$. It follows from the construction of the state sum $Z_W$ (see Section 5.3.3) that non-trivial power series of the element $\Xi(M) \in \overline{Q}$ can only occur in the factor

$$\overline{Q}(H_{([2],[2]),([2],[2])}) = \mathbb{B}[[q]] \oplus \mathbb{B}[[q]] \oplus \mathbb{B}[[q]],$$

where the three copies of $\mathbb{B}[[q]]$ correspond to the three possible loop-free morphisms $([2],[2]) \to ([2],[2])$ in $cBr$, namely $1_{([2],[2])}, b_{([0],[0])},$ and $i_{([0],[0])}c_{([0],[0])}$. Let $\zeta(\Sigma^{n-1}) \in \mathbb{B}[[q]]$ denote the component of $\Xi(\Sigma^{n-1})$ that corresponds to the loop-free morphism $i_{([0],[0])}c_{([0],[0])}$. Then, for every $\nu \in \mathbb{N}$ the coefficient of $q^\nu$ in $\zeta(\Sigma^{n-1})$ is nonzero if and only if there exists a fold field $F \in \mathcal{F}(\mathcal{T}_S, \mathcal{T}_{\Sigma})$ such that $\Xi(F) = \left( \bigotimes_{k=0}^\infty \lambda_k \right) \otimes (i_{([0],[0])}c_{([0],[0])})$. We choose $\nu$ such that $\nu_j = 1$ for $j = n/2$ and $\nu_j = 0$ for $j \neq n/2$. Then, it follows from Corollary 10.1.4 and Theorem 3.4.9 in [19] that the coefficient of $q^{\nu}$ in $\zeta(\Sigma^{n-1})$ is 1 whenever $\Sigma^{n-1}$ is diffeomorphic to $\Sigma^1_K$. Conversely, if $\Sigma^{n-1}$ is not diffeomorphic to $\Sigma^1_K$, then Corollary 10.1.4 in [19] implies that the coefficient of $q^{\nu}$ in $\zeta(\Sigma^{n-1})$ is 0. \qed

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