NEW EXPPLICIT BALANCED SCHEMES FOR SDES WITH LOCALLY LIPSCHITZ COEFFICIENTS

ZHONGQIANG ZHANG †

Abstract. We introduce a class of explicit balanced schemes for stochastic differential equations with coefficients of superlinearly growth satisfying a global monotone condition. The first scheme is a balanced Euler scheme and is of order half in the mean-square sense whereas it is of order one under additive noise. The second scheme is a balanced Milstein scheme, which is of order one in the mean-square sense.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and \(\mathcal{F}_t^w\) be an increasing family of \(\sigma\)-subalgebras of \(\mathcal{F}\) induced by \(w(t)\) for 0 \(\leq t \leq T\), where \((w(t), \mathcal{F}_t^w) = ((w_1(t), \ldots, w_m(t))^\top, \mathcal{F}_t^w)\) is an \(m\)-dimensional standard Wiener process. We consider the system of Ito stochastic differential equations (SDE):

\[
dX = a(t, X)dt + \sum_{r=1}^m \sigma_r(t, X)dw_r(t), \quad t \in (t_0, T], \quad X(t_0) = X_0,
\]

where \(X, a, \sigma_r\) are \(d\)-dimensional column-vectors and \(X_0\) is independent of \(w\). We suppose that any solution \(X_{t_0, X_0}(t)\) of (1.1) is well-defined on \([t_0, T]\).

We consider numerical methods for (1.1) when the coefficients \(a(t, x)\) and \(\sigma_r\) satisfy no globally Lipschitz conditions and propose the following two explicit balanced schemes:

\[
X_{k+1} = X_k + \sin(a(t_k, X_k)h) + \sin(\sum_{r=1}^m \sigma_r(t_k, X_k)\xi_{r k} \sqrt{h}),
\]

which is of half-order mean-square convergence in general and is of first-order mean-square convergence for additive noise and

\[
X_{k+1} = X_k + \sin(a(t_k, X_k)h) + \sin(\sum_{r=1}^m \sigma_r(t_k, X_k)\xi_{r k} \sqrt{h}) + \sin \left( \sum_{i, r=1}^m \Lambda_i \sigma_r(t, X_k) I_{i, r, t_k} \right),
\]

where \(I_{i, r, t_k} = \int_{t_k}^{t_{k+1}} \int_{t_k}^s dw_i \, dw_r\). The scheme (1.3) is of first-order mean-square convergence. For commutative noises, i.e. \(\Lambda_i \sigma_r = \Lambda_r \sigma_i\) \((\Lambda_i = \sigma_i^\top \frac{\partial}{\partial x})\), we can use only increments of Brownian motions of double Ito integral in (1.3) since \(I_{i, r, t_k} + I_{r, i, t_k} = (\xi_{ik} \xi_{rk} - \delta_{ik})h/2\). In this case, we have a simplified version of (1.3)

\[
X_{k+1} = X_k + \sin(a(t_k, X_k)h) + \sin(\sum_{r=1}^m \sigma_r(t_k, X_k)\xi_{r k} \sqrt{h})
\]

\[
+ \sin \left( \frac{1}{2} \sum_{i, r=1}^m \Lambda_i \sigma_r(t, X_k)(\xi_{ik} \xi_{rk} - \delta_{ik})h \right).
\]

Date: February 18, 2014.

2000 Mathematics Subject Classification. Primary 60H35; secondary 65C30, 60H10.

Key words and phrases. nonglobally Lipschitz coefficients, balanced methods, explicit schemes, mean-square convergence, high-order schemes.
These two schemes are balanced explicit schemes and can be seen as a different type of tamed schemes, see e.g. [4, 6, 7, 8, 16, 17, 18, 19, 20, 21], where the coefficients are approximated by the function of the form $f(x)/(1 + h^\alpha |f(x)|)$ ($0 \leq \alpha \leq 1$) to control their superlinear growth. The difference here is that we use the sine function, which is motivated by obtaining higher order mean-square convergence for half-order schemes under additive noise and higher order mean-square convergence schemes. We note that for no globally Lipschitz coefficients, no high-order schemes have been proposed, i.e. all the schemes proposed are half-order, see e.g. [4, 8, 16, 19, 18].

For numerical methods for SDEs with non-globally Lipschitz coefficients, there are other type of explicit schemes, which uses stopping time techniques without any approximation of the coefficient functions, see e.g. [10, 15]. Also, some implicit schemes have been proposed, see e.g. [2, 11, 19].

Let us briefly review balanced explicit schemes for SDEs with non-globally Lipschitz coefficients. While if the coefficient $a(t, x)$ violates the Lipschitz condition and only satisfies one-sided Lipschitz condition (or monotone condition) and grows superlinearly, many existing explicit numerical schemes for SDEs with Lipschitz coefficient (see e.g. [9, 12, 14]) are not stable and thus not convergent any more. For example, the Euler scheme is not convergent in the moments and mean-square sense as the moments of its solutions is not bounded from above, see e.g. [6, 15]. Some explicit schemes (tamed schemes, one type of balanced schemes [13]) have been proposed for SDEs under such conditions, see e.g. tamed Euler schemes [14, 17], tamed Milstein scheme [20]. Compared with the classical Euler scheme and Milstein scheme, these schemes have an approximate drift term $a(t_k, X_k)/(1 + h^\alpha |a(t_k, X_k)|)$ instead of the drift terms $a(t_k, X_k)$ to control the growth of the drift, where $\alpha = 1$ in [20] and $\alpha = 1/2$ in [17].

When the coefficients, both $a(t, X)$ and $\sigma_r(t, X)$, violate the Lipschitz conditions, the aforementioned tamed schemes also fails to converge in the mean-square sense. In this case, Ref. [5] proposed a “fully tamed” Euler scheme

$$X_{k+1} = X_k + \frac{a(t_k, X_k)h + \sum_{r=1}^{m} \sigma_r(t_k, X_k)\xi_r\sqrt{h}}{\max(1, h |a(t_k, X_k)| h + \sum_{r=1}^{m} |\sigma_r(t_k, X_k)| \xi_r \sqrt{h})}. \quad (1.5)$$

This scheme was proved to converge without convergence order in [5]. However, it is shown that the scheme becomes oscillatory at certain values after the term $h |a(t_k, X_k)| h + \sum_{r=1}^{m} |\sigma_r(t_k, X_k)| \xi_r \sqrt{h}$ is larger than one. There have been several version of tamed schemes proposed for SDEs under such conditions on coefficients.

Under a global monotone conditions and some polynomials growth conditions, Ref. [19] proposed the following balanced scheme (tamed scheme)

$$X_{k+1} = X_k + \frac{a(t_k, X_k)h + \sum_{r=1}^{m} \sigma_r(t_k, X_k)\xi_r\sqrt{h}}{1 + h |a(t_k, X_k)| h + \sum_{r=1}^{m} |\sigma_r(t_k, X_k)| h \beta}. \quad (1.6)$$

and proved a half-order convergence of this scheme. They showed that the scheme is still half-order for additive noise. Ref. [19] pointed that the scheme (1.6) is not applicable for some critical situations where the solution to (1.1) has only a finite number of moments. The author then proposed the following scheme

$$X_{k+1} = X_k + \frac{a(t_k, X_k)h + \sum_{r=1}^{m} \sigma_r(t_k, X_k)\xi_r\sqrt{h}}{1 + |a(t_k, X_k)| h^\beta + \sum_{r=1}^{m} |\sigma_r(t_k, X_k)| h^\beta}. \quad (1.7)$$

where the scheme was proved to converge in the mean-square sense with order half when $\beta = 1/2$. A general tamed scheme of this type (with drift and diffusion coefficients divided by some functional of coefficients plus one) was proposed in [18] for Lyapunov stability rather than simply $L^p$-stability. The author in [18] argued that tamed schemes should be adjusted towards different problems. Under general conditions, Refs. [4, 8]
proposed a tamed Euler scheme of a similar type for SDEs with exponential moments and proved stability and half-order convergence in the $L^p$ sense.

However, we note that under global monotone conditions and polynomial growth conditions, schemes (1.6) and (1.7) are only half-order schemes even for additive noise. We observe that for additive noise, classical schemes like the Euler scheme, under the Lipschitz conditions, are first-order. It seems that the use of the function $f(x)/(1 + h^a |f(x)|)$ prevents the lifting of order of tamed schemes for SDEs (e.g. (1.6) and (1.7)). We will show that the use of sine function allows us to control the growth of diffusion in our balanced schemes and to obtain higher-order schemes (up to first order). As we may see later, the use of sine function for diffusion is essential while the drift term can be still “tamed” as $a(t_k, X_k)/(1 + h |a(t_k, X_k)|)$ without losing order for our balanced schemes.

The advantage of our balanced schemes, (1.2) and (1.3), also lies in the following observation. In the critical situations, e.g. those in [16] for the 3/2 model of stochastic volatility, our schemes have almost the same number of bounded moments as the scheme (1.7), see Remark 3.3. Furthermore, we do not have any requirement on the growth condition for high dimensional SDEs ($d > 1$ or $m > 1$) while in this case we note that the coefficient is of at most third-order polynomial growth for the scheme (1.7) to converge with order half.

We will follow the recipe of the proofs in [19] to prove the convergence rate of our scheme under Assumption 2.1 given below. With this assumption given, we can apply the fundamental mean-square convergence theorem in [19] (see also Theorem 2.2 below) with only providing the boundedness of moments and local truncation error. The proof for our balanced Euler scheme will be given in details while the proof for our balanced Milstein scheme will be brief with only necessary details since the idea of the proof is very similar.

2. Preliminary

Throughout the paper, we use the letter $K$ to denote generic constants which are independent of $h$ (time step size) and $k$ (time steps).

Let $X_{t_0, x_0}(t) = X(t), t_0 \leq t \leq T$, be a solution of the system (1.1). We will assume the bounded moments of initial condition, global monotone condition and local Lipschitz condition as follows:

Assumption 2.1. (i) The initial condition is such that

$$E|X_0|^{2p} \leq K < \infty, \text{ for all } p \geq 1. \tag{2.1}$$

(ii) For a sufficiently large $p_0 \geq 1$ there is a constant $c_1 \geq 0$ such that for $t \in [t_0, T],

$$(x - y, a(t, x) - a(t, y)) + \frac{2p_0 - 1}{2} \sum_{r=1}^{m} |\sigma_r(t, x) - \sigma_r(t, y)|^2 \leq c_1 |x - y|^2, \; x, y \in \mathbb{R}^d. \tag{2.2}$$

(iii) There exist $c_2 \geq 0$ and $\kappa \geq 1$ such that for $t \in [t_0, T],

$$|a(t, x) - a(t, y)|^2 \leq c_2 (1 + |x|^{2\kappa - 2} + |y|^{2\kappa - 2})|x - y|^2, \; x, y \in \mathbb{R}^d. \tag{2.3}$$

Define $X_{t,x}(t + h)$ of (1.1) as

$$X_{t,x}(t + h) = x + \int_{t}^{t+h} a(\theta, X_{t,x}(\theta)) \, d\theta + \int_{t}^{t+h} \sum_{r=1}^{m} \sigma_r(\theta, X_{t,x}(\theta)) \, dw_r, \tag{2.4}$$

and introduce the one-step approximation $\tilde{X}_{t,x}(t + h), t_0 \leq t < t + h \leq T$, to the solution $X_{t,x}(t + h)$

$$\tilde{X}_{t,x}(t + h) = x + A(t, x, h; w_i(\theta) - w_i(t), i = 1, \ldots, m, \; t \leq \theta \leq t + h). \tag{2.5}$$
Using the one-step approximation (2.5), we recurrently construct the approximation \((X_k, \mathcal{F}_k), \ k = 0, \ldots, N, \ t_{k+1} - t_k = h_{k+1}, \ T_N = T\) with \(X_0 = X(t_0)\):

\[
X_{k+1} = \bar{X}_{t_k, t_{k+1}} = X_k + A(t_k, X_k, h_{k+1}; w_i(\theta) - w_i(t_k), \ i = 1, \ldots, m, \ t_k \leq \theta \leq t_{k+1}).
\]

For simplicity, we will consider a uniform time step size, i.e., \(h_k = h\) for all \(k\).

**Theorem 2.2** ([19]). Suppose (i) Assumption 2.1 holds;

(ii) The one-step approximation \(\bar{X}_{t,x}(t + h)\) from (2.5) has the following orders of accuracy: for some \(p \geq 1\) there are \(\alpha \geq 1, \ h_0 > 0, \) and \(K > 0\) such that for arbitrary \(t_0 \leq t \leq T - h, \ x \in \mathbb{R}^d, \) and all \(0 < h \leq h_0\):

\[
|\mathbb{E}[X_{t,x}(t + h) - \bar{X}_{t,x}(t + h)]| \leq K(1 + |x|^{2\alpha})^{1/2}h^{q_1},
\]

\[
|\mathbb{E}[X_{t,x}(t + h) - \bar{X}_{t,x}(t + h)]^{2p}|^{1/(2p)} \leq K(1 + |x|^{2\alpha p})^{1/(2p)}h^{q_2}
\]

with

\[
q_2 \geq \frac{1}{2}, \ q_1 \geq q_2 + \frac{1}{2};
\]

(iii) The approximation \(X_k\) from (2.6) has bounded moments, i.e., for some \(p \geq 1\) there are \(\beta \geq 1, \ h_0 > 0, \) and \(K > 0\) such that for all \(0 < h \leq h_0\) and all \(k = 0, \ldots, N:\)

\[
\mathbb{E}|X_k|^{2p} < K(1 + \mathbb{E}|X_0|^{2p\beta}).
\]

Then for any \(N\) and \(k = 0, 1, \ldots, N\) the following inequality holds:

\[
|\mathbb{E}[X_{t_0, X_0}(t_k) - \bar{X}_{t_0, X_0}(t_k)]^{2p}|^{1/(2p)} \leq K(1 + \mathbb{E}|X_0|^{2\gamma p})^{1/(2p)}h^{q_2 - 1/2},
\]

where \(K > 0\) and \(\gamma \geq 1\) do not depend on \(h\) and \(k\), i.e., the order of accuracy of the method (2.6) is \(q = q_2 - 1/2\).

According to this theorem, we can obtain the convergence order of a one-step method by providing boundedness of moments and local truncation error of the one-step method. With this theorem, we will prove convergence orders of our balanced Euler and Milstein scheme in next two sections. The ideas of the proofs are similar to those in [19, Section 3].

In the proofs, we will frequently use the following facts

\[
|a(t, x)|^2 \leq K(1 + |x|^{2\kappa}), \quad \sum_{r=1}^m |\sigma_r(t, x)|^2 \leq K(1 + |x|^{\kappa+1}).
\]

which can be readily seen from (2.3) and (2.2). From the global monotone condition (2.2), we can readily obtain

\[
\mathbb{E}[|X_{t_0, X_0}(t)|^{2p}] < K(1 + \mathbb{E}|X_0|^{2p}), \quad 1 \leq p < p_0, \quad t \in (t_0, T).
\]

**3. The balanced Euler scheme**

In this section, we prove a half-order mean-square convergence of our balanced Euler scheme (1.2). For additive noise, we prove that (1.2) is a first-order scheme. By Theorem 2.2 we need to prove boundedness of moments and local truncation error, which are presented in the following two subsections.
3.1. Boundedness of moments of the solutions to (1.2). We will follow the recipe of the proof of moments boundedness in [19 Section 3], which uses a stopping time technique, see also, e.g. [3, 15].

Lemma 3.1. Suppose Assumption [2, 4] holds with sufficiently large $p_0$. For all natural $N$ and all $k = 0, \ldots, N$ the following inequality holds for moments of the scheme (1.2)

$$E|X_k|^{2p} \leq K(1 + E|X_0|^{2p\beta}), \quad 2 \leq 2p < \frac{2p_0}{G(\kappa)} - 1,$$

(3.1)

where the constants $\beta \geq 1$ and $K > 0$ are independent of $h$ and $k$ and $G(\kappa) = \max(2\kappa - 1, \chi_{p > 1}(\kappa - 1))$.

Proof. As t the case $\kappa = 1$ (i.e., when $a(t, x)$ is globally Lipschitz) is trivial, we will consider only the case $\kappa > 1$.

We will find some events

$$\tilde{\Omega}_{R,k} := \{\omega : |X_l| \leq R(h), \ l = 0, \ldots, k\},$$

(3.2)

where $hR^\kappa(h) < 1$ such that

$$E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|X_k|^{2p}] \leq K(1 + E|X_0|^{2p}).$$

(3.3)

For the compliments of $\tilde{\Omega}_{R,k}$, denoted by $\bar{\Omega}_{R,k}$, we will prove the boundedness of moments starting from the following observation for (1.2) that

$$|X_{k+1}| \leq |X_k| + 2 \leq |X_0| + 2(k + 1).$$

(3.4)

We first prove the lemma for integer $p \geq 1$. We have

$$E[\chi_{\tilde{\Omega}_{R,k+1}}(\omega)|X_{k+1}|^{2p}]$$

$$\leq E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|X_{k+1}|^{2p}] = E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|(X_{k+1} - X_k) + X_k|^{2p}]$$

$$\leq E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|X_k|^{2p}] + K\sum_{l=1}^{2p} E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|X_k|^{2p-l}|X_{k+1} - X_k|^l]$$

$$+ E[\chi_{\tilde{\Omega}_{R,k}}(\omega)|X_k|^{2p-2}A],$$

(3.5)

where $A = \chi_{\tilde{\Omega}_{R,k}}(\omega)E[2p(X_k, X_{k+1} - X_k) + p(2p - 1)|X_{k+1} - X_k|^2|F_{t_k}]$.

Since $\xi_{r_k}$ are independent of $F_{t_k}$ and the Gaussian density function is symmetric, we obtain

$$\chi_{\tilde{\Omega}_{R,k}}E[\sum_{r=1}^{m}\sigma_r(t_k, X_k)\xi_{r_k}\sqrt{h}|F_{t_k}|] = 0,$$

(3.6)

and

$$\chi_{\tilde{\Omega}_{R,k}}E[\sum_{r=1}^{m}|\sigma_r(t_k, X_k)|^2|F_{t_k}|] = \chi_{\tilde{\Omega}_{R,k}}\sum_{r=1}^{m}|\sigma_r(t_k, X_k)|^2.$$  

(3.7)

Similarly, we have, also by the asymmetry of the sine function,

$$\chi_{\tilde{\Omega}_{R,k}}E[\sin(\sum_{r=1}^{m}\sigma_r(t_k, X_k)\xi_{r_k}\sqrt{h})|F_{t_k}|] = 0.$$  

(3.8)
Then the conditional expectation in (3.5) becomes

\[ A = 2p\chi_{R,k}E[(X_k, \sin(a(t_k, X_k)) h + \frac{2p-1}{2} |X_{k+1} - X_k|^2 |F_{t_k}]) \]

\[ = 2p\chi_{R,k}E[(X_k, a(t_k, X_k)) h + \frac{2p-1}{2}(|\sin(a(t_k, X_k)) h|^2 + |\sin(\sum_{r=1}^{m} \sigma_r(t_k, X_k) \xi_{r,k} \sqrt{h})|^2 |F_{t_k}]) \]

\[ + 2p\chi_{R,k}E[(X_k, a(t_k, X_k) h - \sin(a(t_k, X_k)) h)|F_{t_k}] \]

\[ \leq 2p\chi_{R,k} hE[(X_k, a(t_k, X_k) + \frac{2p-1}{2} \sum_{r=1}^{m} |\sigma_r(t_k, X_k)|^2 |F_{t_k}] + p(2p-1)\chi_{\tilde{R},k} |a(t_k, X_k)|^2 h^2 \]

\[ + 2p\chi_{\tilde{R},k} |X_k| |a(t_k, X_k)|^2 h^2, \]

where we have used the fact that \(|\sin(y)| \leq |y|\) and

\[ |y - \sin(y)| \leq 2|y| |\sin(\theta y/2)|^2, \quad 0 \leq \theta \leq 1. \quad (3.9) \]

Using the global monotone condition (2.2) and the growth condition (2.11), we obtain

\[ A \leq K \chi_{\tilde{R},k} (h + |X_k|^2 h + |X_k|^{2\kappa - 1} h^2). \quad (3.10) \]

Now consider the second term in (3.5):

\[ \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-1} |X_{k+1} - X_k|^l] \]

\[ \leq K \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-1} |h^l a(t_k, X_k)|^l + h^{l/2} \sum_{r=1}^{m} |\sigma_r(t_k, X_k)|^l |\xi_{r,k}|^l] \]

\[ \leq K \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-1} h^{l/2} \left[ 1 + |X_k|^{(\kappa+1)/2} \right], \]

where we used the growth condition (2.11) and the fact that \(\chi_{\tilde{R},k}(\omega)\) and \(X_k\) are independent of \(\xi_{r,k}\). Then by (3.5), (3.10), and (3.11), we have

\[ \mathbb{E}[\chi_{\tilde{R},k+1}(\omega) |X_{k+1}|^{2p}] \]

\[ \leq \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p} + K h \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-2} \left[ 1 + |X_k|^2 h |X_k|^{2\kappa+1} \right] \]

\[ + K \sum_{l=3}^{2p} \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-l} h^{l/2} \left[ 1 + |X_k|^{(\kappa+1)/2} \right] \]

\[ \leq \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p}] + K h \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p}] + K \sum_{l=2}^{2p} \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p-l} h^{l/2} \]

\[ + K h^2 \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p+2\kappa-1}] + K h \sum_{l=3}^{2p} \mathbb{E}[\chi_{\tilde{R},k}(\omega) |X_k|^{2p+l(\kappa-1)/2} h^{l/2-1}. \]

If we choose

\[ R = R(h) = h^{-1/G(\kappa)}, \quad \text{where} \quad G(\kappa) = \max(2\kappa - 1, \chi_{p>13}(\kappa - 1)), \quad (3.13) \]

we get, for \(l = 3, \ldots, 2p, \)

\[ \chi_{\tilde{R},k}(\omega) |X_k|^{2p+2\kappa-1} h \leq \chi_{\tilde{R}(h),k}(\omega) |X_k|^{2p}, \]

\[ \chi_{\tilde{R}(h),k}(\omega) |X_k|^{2p+l(\kappa-1)/2} h^{l/2-1} \leq \chi_{\tilde{R}(h),k}(\omega) |X_k|^{2p}. \]
Thus we have for (3.5),
\[
\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_{k+1}|^{2p}]
\leq \mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2p} + Kh\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2(2p-1)}]h^l + \sum_{l=0}^{p}\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2(p(l-1))}h^l]
\leq \mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2p} + Kh\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2p}] + Kh,
\]
where in the last line we have used Young’s inequality. From here, we get (3.3) by Gronwall’s inequality.

It remains to estimate \(\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2p}]\). We recall that, see [19] Section 3,
\[
\chi_{\hat{\Omega}_{R}} = \sum_{l=0}^{k} \chi_{\hat{\Omega}_{R,l-1}}\chi_{|X_l|>R},
\]
where we put \(\chi_{\hat{\Omega}_{R,1-1}} = 1\). Then, using (3.4), (3.3), and Hölder’s and Markov’s inequalities, we obtain
\[
\mathbb{E}[\chi_{\hat{\Omega}_{R(h),k}}(\omega)|X_k|^{2p}] \leq \left(\mathbb{E}[||X_0| + 2k^{2p'p'}] \right)^{1/p'} \sum_{l=0}^{k} \frac{\mathbb{E}[\chi_{\hat{\Omega}_{R(h),l-1}}|X_l|^{q(2p+1)G(\kappa)}]}{R(h)^{2p+1}G(\kappa)} \leq K\left(\mathbb{E}[||X_0| + 2k^{2p'p'}] \right)^{1/p'} \left(\mathbb{E}[|X_0|^{q(2p+1)G(\kappa)}] \right)^{1/q'} \leq K\left(1 + \mathbb{E}[|X_0|^{2pp'+q(2p+1)G(\kappa)}] \right),
\]
where Jensen’s inequality, (3.1) holds for non-integer \(p\) as well. □

**Remark 3.2.** Consider the balanced scheme (1.6) in [19]. With the growth condition (2.11), we can estimate the conditional expectation in (3.3) by
\[
\chi_{\hat{\Omega}_{R,k}}(\omega)\mathbb{E}[\{(2pX_k, X_{k+1} - X_k) + p(2p-1)|X_{k+1} - X_k|^2\} |\mathcal{F}_{k}] \\
\leq \chi_{\hat{\Omega}_{R,k}} h(1 + |X_k|^2 + |X_k|^{2\kappa} h + |X_k|^{2\kappa+1} h + |X_k|^{3/(2\kappa+1)} h^{1/2}).
\]
We then can set \(R(h) = h^{-1/(3\kappa-1)}\). By the same argument for the event \(\hat{\Lambda}_{R(h),k}\), we have
\[
\mathbb{E}[|X_k|^{2p}] \leq K\left(1 + \mathbb{E}[|X_0|^{2pp'+q(2p+1)(3\kappa-1)}] \right), \quad 2p \leq \frac{2p_0}{3\kappa-1} - 1. \tag{3.15}
\]

**Remark 3.3.** For different balanced schemes, there are different requirements on the coefficients as different schemes usually have different number of bounded moments. Consider the 3/2-stochastic volatility model.
\[
dX = \lambda X(\theta - X) dt + \mu |X|^{3/2} dW, \quad X = X_0, \tag{3.16}
\]
where \(\lambda, \theta, \mu, X_0 > 0\). It can be readily checked that \(1 \leq p_0 \leq \lambda/\mu^2 + 1/2\).

For the scheme (1.2), a bounded second moment \((p = 1)\) requires \(2 < \frac{2p_0}{3\kappa-1} - 1\) and thus for \(\kappa = 2\), it requires \(2p_0 \geq 6\kappa - 3 = 9\).

For the scheme (1.6), a bounded second moment \((p = 1)\) requires \(2 < 2p_0/(3\kappa-1) - 1\), and thus \(2p_0 > 9\kappa - 3 = 15\) for \(\kappa = 2\), according to (3.15).

For the scheme (1.7), a bounded second moment \((p = 1)\) requires \(2 \leq \frac{2p_0}{3\kappa} \) and thus \(2p_0 \geq 4\kappa = 8\) for \(\kappa = 2\), see [16].

It is clear that the scheme (1.7) can deal with less restricted coefficients for the 3/2-stochastic volatility model. However, we note that the scheme (1.7) requires that \(\kappa \leq 3\) for high dimensional SDEs or one dimensional SDEs with several noises while the schemes (1.6) and (1.2) have no this limitation.
3.2. One-step error. The next lemma gives estimates for the one-step error of the balanced Euler scheme (1.2).

Lemma 3.4. Assume that (2.12) holds. Assume that the coefficients \( a(t, x) \) and \( \sigma_r(t, x) \) have continuous first-order partial derivatives in \( t \) and that these derivatives and the coefficients satisfy inequalities of the form (2.11). Then the scheme (1.2) satisfies the inequalities (2.6) and (2.7) with \( q_1 = 3/2 \) and \( q_2 = 1 \), respectively.

Moreover, consider additive noise, i.e., \( \sigma_r(t, x) = \sigma_r(t) \). If the coefficient \( a(t, x) \) also has continuous first-order and second-order derivatives in \( x \) and their derivatives satisfy the polynomial growth condition of the form (2.11), then we have \( q_1 = 2 \) and \( q_2 = 3/2 \).

The proof of this lemma is given below. According to Theorem 2.2, the following proposition can be readily deduced from Lemmas 3.1 and 3.4.

Proposition 3.5. Under the assumptions of Lemmas 3.1 and 3.4, the balanced Euler scheme (1.2) has a mean-square convergence order half, i.e., for it the inequality (2.10) holds with \( q = 1/2 \).

For additive noise, we have that the scheme (1.2) is of first order convergence, i.e. \( q = 1 \).

We need the following lemma for the proof.

Lemma 3.6 ([19]). Let a function \( \phi(t, x) \) have continuous first-order partial derivative in \( t \) and that the derivative and the function satisfy inequalities of the form (2.3). For \( l \geq 1 \) and \( s \geq t \), we have

\[
|E[\phi(s, \tilde{X}_{t,x}(s)) - \phi(t, x)]| \leq K(1 + |x|^{2l-1})[(s - t)^{l/2} + (s - t)^l].
\] (3.17)

The proof of Lemma 3.6 can be found in [19] Appendix C. Now we prove Lemma 3.4, the order of accuracy for one-step error of the balanced Euler scheme (1.2).

Proof. Now consider the one-step approximation of the SDE (1.1), which corresponds to the balanced method (1.2):

\[
X = x + \sin(a(t, x)h) + \sin\left(\sum_{r=1}^{m} \sigma_r(t, x)\xi_r \sqrt{h}\right)
\] (3.18)

and the one-step approximation corresponding to the explicit Euler scheme:

\[
\tilde{X} = x + a(t, x)h + \sum_{r=1}^{m} \sigma_r(t, x)\xi_r \sqrt{h}.
\] (3.19)

Step 1. We start with analysis of the one-step error of the Euler scheme:

\[\hat{\rho}(t, x) := X_{t,x}(t + h) - \tilde{X}.\]

By Lemma 3.6, we have

\[
|E[\hat{\rho}(t, x)]| = \left| E \int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x))ds \right|
\leq E \int_t^{t+h} |a(s, X_{t,x}(s)) - a(t, x)|ds \leq Kh^{3/2}(1 + |x|^{2q-1}).
\] (3.20)
NEW BALANCED EXPLICIT SCHEMES FOR NONLINEAR SDES

(3.21)

Also we have

\[
\mathbb{E}[|\tilde{\rho}|^{2p}(t,x)] \leq K \mathbb{E}\left[ \int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x)) ds \right]^{2p}
\]

\[
+ K \sum_{r=1}^{q} \mathbb{E}\left[ \int_t^{t+h} (\sigma_r(s, X_{t,x}(s)) - \sigma_r(t, x)) dw_r(s) \right]^{2p}.
\]

By Lemma 3.6 we get for the first term in (3.21):

\[
\mathbb{E}\left[ \int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x)) ds \right]^{2p} \leq K h^{2p-1} \int_t^{t+h} \mathbb{E}[|a(s, X_{t,x}(s)) - a(t, x)|^{2p}] \, ds
\]

\[
\leq K h^{2p}(1 + |x|^{4p\kappa - 2p}).
\]

Using the inequality for powers of Ito integrals from [1, pp. 26] and Lemma 3.6 we obtain

\[
\mathbb{E}\left[ \int_t^{t+h} \left( \sigma_r(s, X_{t,x}(s)) - \sigma_r(t, x) \right) dw_r(s) \right]^{2p}
\]

\[
\leq K h^{p-1} \int_t^{t+h} \mathbb{E}[|\sigma_r(s, X_{t,x}(s)) - \sigma_r(t, x)|^{2p}] \, ds \leq K h^{2p}(1 + |x|^{4p\kappa - 2p}).
\]

It follows from (3.21) and (3.23) that

\[
\mathbb{E}[|\tilde{\rho}|^{2p}(t,x)] \leq K h^{2p}(1 + |x|^{4p\kappa - 2p}).
\]

Step 2. Now we compare the one-step approximations (3.18) of the balanced scheme (3.21) and (3.19) of the Euler scheme:

\[
X = x + \sin(a(t,x)h) + \sin(\sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h}) = \hat{X} - \rho(t,x),
\]

where

\[
\rho(t,x) = a(t,x) - \sin(a(t,x)) + \sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h} - \sin(\sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h})
\]

By the symmetry of Gaussian density function and the asymmetry of sine function, we have

\[
\mathbb{E}[\sin(\sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h})] = 0,
\]

and then by the inequality (3.9) and the fact that \[ |\sin(y)| \leq |y|, \] we have

\[
|\mathbb{E}[\rho(t,x)]| = |\mathbb{E}[\rho(t,x)h - \sin(a(t,x)h)]| \leq |a(t,x)h| 2 |\sin(\theta a(t,x)h/2)|^2
\]

\[
\leq |a(t,x)h|^2 \leq K h^2(1 + |x|^2\kappa),
\]

whence, from (3.25) and (3.21), we obtain that (3.18) satisfies (2.6) with \( q_1 = 3/2 \).

From the inequality (3.9) and the fact that \[ |\sin(y)| \leq |y|, \] we can readily obtain

\[
\mathbb{E}[|\tilde{\rho}|^{2p}(t,x)] \leq K \mathbb{E}[|a(t,x)h - \sin(a(t,x)h)|^{2p}]
\]

\[
+ K \mathbb{E}\left\[ \sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h} - \sin(\sum_{r=1}^{m} \sigma_r(t,x)\xi_r \sqrt{h}) \right\]^{2p}
\]

\[
\leq K |a(t,x)h|^{3p} + K h^{3p}\mathbb{E}\left[ \sum_{r=1}^{m} |\sigma_r(t,x)\xi_r| \right]^{6p} \leq K h^{3p}(1 + |x|^{3p(\kappa+1)}),
\]

which together with (3.25) and (3.23) implies that (3.18) satisfies (2.7) with \( q_2 = 1. \)
Lemma 4.1. Suppose Assumption 2.1 holds with sufficiently large $p_0$. Assume the following polynomial growth for $\Lambda_i\sigma_i(t, x)$:

$$\sum_{i, r=1}^{m} |\Lambda_i\sigma_i(t, x)|^2 \leq K(1 + |x|^{2\kappa'}), \quad \kappa \geq 0.$$ (4.1)

For all natural $N$ and all $k = 0, \ldots, N$ the following inequality holds for moments of the scheme (1.2)

$$\mathbb{E}|X_k|^{2p} \leq K(1 + \mathbb{E}|X_0|^{2p_0}), \quad 2 \leq 2p < \frac{2p_0}{G(\kappa)} - 1,$$ (4.2)

where the constants $\beta \geq 1$ and $K > 0$ are independent of $h$ and $k$ and

$$G(\kappa) = \max(2\kappa - 1, 2\kappa' - 2, \chi_{\beta>1}3(\kappa - 1)).$$
Proof. The idea of the proof is similar to that for Lemma 3.1. We thus present part of the proof with necessary details.

Consider the third term in the right-hand side of (3.5). It is essential to provide proper upper bound for $A = \chi_{\mathcal{R},k} (|2p(X_k, X_{k+1} - X_k) + p(2p - 1)|X_{k+1} - X_k|^2)$. Similar to the proof of upper bound for $A$ in Lemma 3.1 we have

$$
A = 2p\chi_{\mathcal{R},k} E[(X_k, \sin(t_k, X_k)h) + \frac{2p - 1}{2} |X_{k+1} - X_k|^2 |\mathcal{F}_{t_k}]
$$

$$
= 2p\chi_{\mathcal{R},k} E[(X_k, \sin(\sigma(t_k, X_k)I_{i,r,t_k}))^2 |\mathcal{F}_{t_k}]
$$

$$
+ p(2p - 1)\chi_{\mathcal{R},k} E\left[\sin\left(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k}\right)\right]^2 |\mathcal{F}_{t_k}]
$$

$$
+ p(2p - 1)\chi_{\mathcal{R},k} E[2(\sin(\sum_{i,r=1}^{m} \sigma_r(t_k, X_k)\xi_{kr}\sqrt{h})), \sin(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k})] |\mathcal{F}_{t_k}]
$$

$$
+ p(2p - 1)\chi_{\mathcal{R},k} E[2(\sin(a(t_k, X_k)h), \sin(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k})] |\mathcal{F}_{t_k}]
$$

$$
\leq \chi_{\mathcal{R},k} (K|X_k|^2 + Kh^2 |X_k|^{2\alpha} + Kh^2 |X_k|^{2\alpha + 1})
$$

$$
+ p(2p - 1)\chi_{\mathcal{R},k} E\left[\sin\left(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k}\right)\right]^2 |\mathcal{F}_{t_k}]
$$

$$
+ p(2p - 1)\chi_{\mathcal{R},k} E[2(\sin(\sum_{i,r=1}^{m} \sigma_r(t_k, X_k)\xi_{kr}\sqrt{h})), \sin(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k})] |\mathcal{F}_{t_k}]
$$

By the symmetry of the Gaussian density function and the asymmetry of the sine function, we have

$$
\chi_{\mathcal{R},k} E[2(\sin(\sum_{i,r=1}^{m} \sigma_r(t_k, X_k)\xi_{kr}\sqrt{h})), \sin(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k})] |\mathcal{F}_{t_k}] = 0. \quad (4.3)
$$

From the fact that $|\sin(y)| \leq |y|$ and the inequality (4.3), we can readily obtain

$$
\chi_{\mathcal{R},k} E\left[\sin\left(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k}\right)\right]^2 |\mathcal{F}_{t_k}] \leq K\chi_{\mathcal{R},k} \left(\sum_{i,r=1}^{m} |\Lambda_i \sigma_r(t, X_k)|^2 \mathbb{E}[I_{i,r,t_k}^2 |\mathcal{F}_{t_k}]\right), \quad (4.4)
$$

$$
\chi_{\mathcal{R},k} E[2(\sin(a(t_k, X_k)h), \sin(\sum_{i,r=1}^{m} \Lambda_i \sigma_r(t, X_k)I_{i,r,t_k})] |\mathcal{F}_{t_k}]
$$

$$
\leq K\chi_{\mathcal{R},k} |a(t_k, X_k)| \sum_{i,r=1}^{m} |\Lambda_i \sigma_r(t, X_k)| \mathbb{E}[I_{i,r,t_k} |\mathcal{F}_{t_k}], \quad (4.5)
$$
Proposition 4.3. Under the assumptions of Lemmas 4.1 and 4.2, the balanced Milstein scheme (1.3) has a mean-square convergence order one, i.e., for it the inequality (2.10) holds with $q = 1$. 
Proof. Step 1. We start with analysis of the one-step error of the Milstein scheme:

$$\tilde{\rho}(t, x) := X_{t,x}(t+h) - \hat{X}.$$

By Itô’s formula, we can obtain,

$$|\mathbb{E}[\tilde{\rho}(t, x)]| = \left| \mathbb{E}\left[\int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x))ds + \sum_{i,r=1}^{m} \int_t^{t+h} \int_t^s [\Lambda_i\sigma_r(\theta, X_{t,x}(\theta)) - \Lambda_i\sigma_r(t, x)]\,dw_i\,dw_r \right] \right|$$

$$\leq \left| \mathbb{E}\left[\int_t^{t+h} \int_t^s \partial_t a(\theta, X_{t,x}(\theta)) + La(\theta, X_{t,x}(\theta))\,d\theta\,ds \right] \right|$$

$$+ \left| \mathbb{E}\left[\sum_{r=1}^{m} \int_t^{t+h} \int_t^s \partial_t(\Lambda_i\sigma_r(\theta, X_{t,x}(\theta))) + L\Lambda_i\sigma_r(\theta, X_{t,x}(\theta))\,d\theta\,dw_r(\theta)\,dw_r(s) \right] \right|$$

$$\leq Kh^2(1 + |x|^\nu'),$$

where we have used Hölder’s inequality and the growth condition (4.10).

For the mean-square one-step error, we have

$$\mathbb{E}[|\tilde{\rho}|^{2p}(t, x)] \leq K\mathbb{E}\left[\int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x))ds \right]^{2p}$$

$$+ K \sum_{r=1}^{m} \mathbb{E}\left[\int_t^{t+h} \sigma_r(s, X_{t,x}(s)) - \sigma_r(t, x)\,dw_r(s) - \sum_{i,r=1}^{m} \Lambda_i\sigma_r(t, x)I_{i,r,t} \right]^{2p}.$$

By Itô’s formula, using the inequality for powers of Itô integrals from [1] pp. 26, we obtain

$$\mathbb{E}\left[\int_t^{t+h} (\sigma_r(s, X_{t,x}(s)) - \sigma_r(t, x))\,dw_r(s) - \sum_{i,r=1}^{m} \Lambda_i\sigma_r(t, x)\int_t^{t+h} s\,dw_i\,dw_r \right]^{2p}$$

$$\leq Kh^{p-1} \int_t^{t+h} E\left[\sum_{i,r=1}^{m} \int_t^s [\Lambda_i\sigma_r(\theta, X_{t,x}(\theta)) - \Lambda_i\sigma_r(t, x)]\,dw_i(\theta) \right]^{2p} ds$$

$$+ Kh^{p-1} \int_t^{t+h} E\left[\int_t^s \partial_t\sigma_r(\theta, X_{t,x}(\theta)) + L\sigma_r(\theta, X_{t,x}(\theta))\,d\theta \right]^{2p} ds$$

$$\leq Kh^{p-1} \int_t^{t+h} (t-s)^{p-1} \sum_{i,r=1}^{m} \int_t^s E[|\Lambda_i\sigma_r(\theta, X_{t,x}(\theta)) - \Lambda_i\sigma_r(t, x)|^{2p}]\,d\theta ds$$

$$+ Kh^{p-1} \int_t^{t+h} E\left[\int_t^s \partial_t\sigma_r(\theta, X_{t,x}(\theta)) + L\sigma_r(\theta, X_{t,x}(\theta))\,d\theta \right]^{2p} ds$$

which can be further estimated by, using Itô’s formula for $\Lambda_i\sigma_r$ and Hölder’s inequality,

$$\leq Kh^{p-1} \int_t^{t+h} (t-s)^{2p}(1 + |x|^{2p\nu'' - 2p}) ds + Kh^{3p}(1 + |x|^{2p\nu'' - 2p})$$

$$\leq Kh^{3p}(1 + |x|^{2p\nu'' - 2p}),$$

where we have used the growth condition (4.11). Similarly, we have

$$\mathbb{E}\left[\int_t^{t+h} (a(s, X_{t,x}(s)) - a(t, x))ds \right]^{2p} \leq Kh^{2p-1} \int_t^{t+h} \mathbb{E}[|a(s, X_{t,x}(s)) - a(t, x)|^{2p}] ds$$

$$\leq Kh^{3p}(1 + |x|^{2p\nu'' - 2p}).$$
By (4.13) and (4.14), we obtain
\[ E[|\rho|^2p(t, x)] \leq Kh^{3p}(1 + |x|^{2p\alpha'' - 2p}). \tag{4.15} \]

Step 2. Now we compare the one-step approximations (4.8) of the balanced scheme (1.3) and (4.9) of the Milstein scheme. Define
\[
\rho(t, x) = a(t, x)h - \sin(a(t, x)h) + \sum_{r=1}^{m} \sigma_{r}(t, x)\xi_{r}\sqrt{h} - \sin\left(\sum_{r=1}^{m} \sigma_{r}(t, x)\xi_{r}\sqrt{h}\right) \\
+ \sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t} - \sin\left(\sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t}\right). \tag{4.16}
\]

By the symmetry of Gaussian density function and the sine function, we have
\[
|E\rho(t, x)| \leq |E[a(t, x)h - \sin(a(t, x)h)]| \\
+ |E\left[ \sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t} - \sin\left(\sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t}\right) \right]|^{2} \\
\leq |a(t, x)h|^{2} + K \sum_{i, r=1}^{m} |\Lambda_{i}\sigma_{r}(t, x)|^{2} E[I_{i, r, t}^{2}] \\
\leq Kh^{2}(1 + |x|^{\max(2\alpha, 2\alpha')}), \tag{4.17}
\]
where we have used the inequality (3.3), the polynomial growth conditions (2.11) and (4.1), and \( E[I_{i, r, t}^{2}] \leq Kh^{2}. \) From here and (4.11), we have the one-step approximation of (1.3), (4.8), satisfies (2.6) with \( q_{1} = 2. \)

From the inequality (3.3) and \(|\sin(y)| \leq |y|\), we can readily obtain
\[
E[|\rho|^{2p}(t, x)] \leq K|E[a(t, x)h - \sin(a(t, x)h)]|^{2p} \\
+ K E\left[ \sum_{r=1}^{m} \sigma_{r}(t, x)\xi_{r}\sqrt{h} - \sin\left(\sum_{r=1}^{m} \sigma_{r}(t, x)\xi_{r}\sqrt{h}\right) \right]^{2p} \\
+ K E\left[ \sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t} - \sin\left(\sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t}\right) \right]^{2p} \\
\leq K |a(t, x)h|^{3p} + K E\left[ \sum_{r=1}^{m} \sigma_{r}(t, x)\xi_{r}\sqrt{h} \right]^{6p} + K E\left[ \sum_{i, r=1}^{m} \Lambda_{i}\sigma_{r}(t, x)I_{i, r, t} \right]^{3p} \\
\leq Kh^{3p}(|a(t, x)|^{3p} + \sum_{r=1}^{m} |\sigma_{r}(t, x)|^{6p} + \sum_{i, r=1}^{m} |\Lambda_{i}\sigma_{r}(t, x)|^{3p}). \tag{4.18}
\]

Thus by the polynomial growth conditions (2.11) and (4.1), we obtain
\[
E[|\rho|^{2p}(t, x)] \leq Kh^{3p}(1 + |x|^{\max(3p(\alpha+1), 3p\alpha')}),
\]
which together with (4.15) implies that the one-step approximation of (1.3), (4.8), satisfies (2.7) with \( q_{2} = 3/2. \) \( \square \)
References

[1] I. I. Gihman and A. V. Skorohod, Stochastic differential equations, Springer-Verlag, New York, 1972.
[2] D. J. Higham, X. Mao, and A. M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal., 40 (2002), pp. 1041–1063.
[3] M. Hutzenthaler and A. Jentzen, Numerical approximation of stochastic differential equations with non-globally Lipschitz continuous coefficients, ArXiv e-prints, (2012).
[4] M. Hutzenthaler and A. Jentzen, On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients, ArXiv, (2014).
[5] M. Hutzenthaler and A. Jentzen, Numerical approximation of stochastic differential equations with non-globally Lipschitz continuous coefficients, Mem. Amer. Math. Soc., (In press).
[6] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proc. R. Soc. A, (2011), pp. 1563–1576.
[7] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients, Ann. Appl. Probab., 22 (2012), pp. 1611–1641.
[8] M. Hutzenthaler, A. Jentzen, and X. Wang, Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations, ArXiv, (2013).
[9] P. E. Kloeden and E. Platen, Numerical solution of stochastic differential equations, Springer-Verlag, Berlin, 1992.
[10] W. Liu and X. Mao, Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations, Appl. Math. Comput., 223 (2013), pp. 389 – 400.
[11] X. Mao and L. Szpruch, Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Comput. Appl. Math., 238 (2013), pp. 14–28.
[12] G. N. Milstein, Numerical integration of stochastic differential equations, Kluwer Academic Publishers Group, Dordrecht, 1995.
[13] G. N. Milstein, E. Platen, and H. Schurz, Balanced implicit methods for stiff stochastic systems, SIAM J. Numer. Anal., 35 (1998), pp. 1010–1019.
[14] G. N. Milstein and M. V. Tretyakov, Stochastic numerics for mathematical physics, Springer-Verlag, Berlin, 2004.
[15] ———. Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients, SIAM J. Numer. Anal., 43 (2005), pp. 1139–1154.
[16] S. Sabanis, Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients, ArXiv, (2013).
[17] S. Sabanis, A note on tamed Euler approximations, Electron. Commun. Probab., 18 (2013), pp. no. 47, 1–10.
[18] L. Szpruch, V-stable tamed Euler schemes, ArXiv, (2013).
[19] M. V. Tretyakov and Z. Zhang, A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications, SIAM J. Numer. Anal., 51 (2013), pp. 3135–3162.
[20] X. Wang and S. Gan, The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Difference Equ. Appl., 19 (2013), pp. 466–490.
[21] X. Zong, F. Wu, and C. Huang, Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-Lipschitz continuous coefficients, Appl. Math. Comput., 228 (2014), pp. 240–250.

† EMAIL: ZHONGQIANG.ZHANG@BROWN.EDU