Setting of the Dupin cyclide by three straight lines and sphere

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Abstract. Over the past few years, interest in Dupin cyclides has re-emerged: they can be used both in mechanical engineering and in construction, for covering the spans of civil buildings and in temple architecture; the properties of the Dupin cyclide can be widely used in computer graphics. The article considers a method for setting of Dupin cyclides using three lines and a sphere, which greatly the possibilities for the construction of these surfaces, especially when it is necessary to cut compartments of a certain size from them and to dock them.

Keywords: Dupin cyclide, cyclic surface, surface design, modeling.

1. Introduction
Over the past six to seven years, interest in the surface of the Dupin cyclide [1] has grown quite strongly. From one work to another [2-7], new information on cyclides appears. This can be explained both by the new “discovery” [8] of this surface for a wide circle of specialists, and by the fact that such well-known surfaces as rotating cone, rotating cylinder and torus are special cases of this surface, which is remarkable in all respects [8].

The use of the surface of the Dupin cyclide is a fairly wide range – this is the use in mechanical engineering [4; 5; 7; 8], and in the creation of pipelines of different diameters [3; 8], the use in the design and construction of thin shells structures [9], in the construction of conics in computer graphics [10; 11], in the design of domes in temple architecture [12], and much more.

2. Formulation of the problem
Until now, all the construction of the compartments of Dupin cyclides was carried out individually for each task without applying theoretical knowledge of geometric modeling, which is now applied in any direction of design and production in the form of engineering geometry, including computer graphics. And there are more and more needs for the development of computer-aided design over time. The proposed research will help to formalize this process and involve computer technology in the development of various design options for the compartments of Dupin cyclides and their docking. Based on the foregoing, there is a practical need for developing of new methods for setting Dupin cyclides for the surface design in mechanics and in construction.

3. Theory
3.1. Setting of the Dupin cyclide
Statement. Three pairwise intersecting straight lines l, t, a, belonging to the plane Σ, and a sphere Ω of radius R centered at the point O of intersection of straight lines l, and t define ∞ o Dupin cyclide.

Let’s prove this statement.
Fig. 1 shows three pairwise intersecting straight lines \( l \), \( t \), and \( a \), as well as a sphere \( \Omega \) of radius \( R = \) \(| O_1; N | = | O; M | \) intersecting the straight lines \( l \) and \( t \) at four points: \( M, 1 \) and \( N, 2 \), respectively. The straight lines \( l \), \( t \), and \( a \) lie in the plane \( \Sigma \). Let the plane \( \Sigma \) be the frontal plane of projections. Note that the designations in the figures do not correspond to those adopted in descriptive geometry and, nevertheless, the constructions are understandable.

Let's draw the plane \( \perp \Sigma \) through the points \( M \) and \( N \), which intersects the plane \( \Sigma \) along the straight line \( d \) (Fig. 2). The point \( j \) will actually mean a straight line perpendicular to the plane \( \Sigma \), and will be the axis of rotation of the plane \( \Gamma \). The plane \( \Gamma \), in turn, intersects the sphere \( \Omega \) along the circumference \( m \) (Fig. 2) and it is the generatrix of the surface of the Dupin cyclide. When the plane \( \Gamma \) rotates around the axis \( j \), point \( M \) moves along a circumference with a center \( O_1 \) of radius \( O_1M \); point \( N \) will move in a circumference with the center \( O_2 \) of radius \( O_2N \). Obviously, the diameter of the generatrix of the circumference \( m \) will change, and when the plane \( \Gamma \) passes through points \( A \) and \( B \), its radius will be zero. Point \( O \) will move (in this example) along the ellipse [8]. The circumferences intersecting at points \( A \) and \( B \) create an outline of the Dupin cyclide [5; 13].

Next, calculate the number of parameters involved in the construction of the Dupin cyclide.

The plane \( \Sigma \), in which the straight lines are constructed, fixes 3 parameters.

Three lines in the plane \( \Sigma \), fix 6 parameters (each 2).

Finally, a sphere having a center at the point \( O \) of intersection of the straight lines \( l \) and \( t \) and therefore not occupying any other parameters than the radius, fixes 1 parameter. Thus, the total number of parameters is 10.

Therefore, it is necessary to proceed from the sum of these parameters, the number of which should not be less than ten.

It can also be counted in another way.

Setting of one of the straight lines captures 4 parameters [14]; the second straight line fixes 3 parameters, since one parameter goes to the intersection with the first straight line; the third straight line fixes only 2 parameters, since 2 of the 4 parameters are spent on the intersection with the two previous straight lines; finally, the tenth parameter goes to the radius of the sphere.

Based on the results obtained, we can conclude: no matter how you count the number of parameters involved, there should always be only ten.
Figure 3 shows how the Dupin cyclide can be obtained from the same configuration shown in Fig. 1. The fact is that the intersection of a given circumference with straight lines $l$ and $t$ gives four points: $M$, $N$, $I$ and 2. If we draw the plane $\Gamma$ through the points $M$ and $N$ or $I$ and 2, we get Dupin cyclides similar to those shown in Fig. 2, and the generating set of circumferences $m$ is the inner family for the cyclide itself. If we draw the plane $\Gamma$ through the points 2 and $M$ or $I$ and $N$ (Fig. 3), then all the generating set of circumferences $m$ for the Dupin cyclide will be external (Fig. 4). Moreover, on the line $a$ there is point $i$, which marks the position of the outer axis of the cyclide. This axis in this case is also perpendicular to the plane $\Sigma$ containing these lines. The radii of the outline spheres $O_1M$ and $O_22$.

Thus, the principle of setting of the Dupin cyclide with three straight lines and a sphere can be considered proven. Moreover, in [4; 5; 8] it is shown that the Dupin cyclide is uniquely defined by its outlines, which we see in Fig. 2 and 4.

When setting three pairwise intersecting straight lines and a sphere, we get four Dupin cyclides: two with an internal tangent line to the family of circumferences, one of the family of which is the given sphere, and the centers of which move along ellipses, and two with the external tangent line to the family of circumferences whose centres move along the hyperbola for this example. As a result, we have $\infty \circ$ Dupin cyclide.

### 3.2. Variants of setting of Dupin cyclides

Consider the variants of setting of the Dupin cyclide, based on those shown in Fig. 2 and 4.

Fig. 5-7 show three options. The distance between the centers of the circumferences is indicated as $A$. 

![Figure 5](image1.png)

![Figure 6](image2.png)

![Figure 7](image3.png)
As a result, we get:
1. \( A + R1 < R2 \) – there are no conical points (Fig. 5).
2. \( A + R1 = R2 \) – we have one conical point (Fig. 6, point Q).
3. \( A + R1 > R2 \) – we get two conical points (Fig. 7, points S and T).

All conical points are located in the plane \( \Sigma \) defined by the straight lines \( l \), \( t \) and \( a \). Moreover, as we will see later, the conical points will be located on the axis \( i \).

Fig. 2 and 4 show variants of Dupin cyclides with the passage of the axis \( a \) past a given sphere.

Let's consider other options. Let the straight line \( a \) intersect the given sphere \( \Omega \) (Fig. 8). In this case, the point \( N \) moves along the outer circumference of radius \( ON \), the point \( M \) moves along the inner circumference of radius \( OM \), and as a result, we get the Dupin cyclide with two conical points \( G \) and \( Q \) located on the axis \( j \). If the straight line \( a \) intersects a given sphere \( \Omega \) at the intersection of its straight line \( l \) or \( t \) (Fig. 9), then the circle \( m \), rotating around the axis \( j \) (the point \( M \) remains in place, only the point \( N \) moves along the outer contour circumference), we obtain the cyclide with one conical point on the axis \( j \). It should be noted that the designations in Fig. 8 and 9 (one orthogonal projection is made on the plane \( \Sigma \), and the second – on the plane perpendicular to it) differ from those adopted in descriptive geometry. This, according to the author, should not cause difficulties in reading the draft.

**Figure 8.**

**Figure 9.**

### 3.3. The determination of the axes of Dupin cyclides

The axes of the Dupin cyclide, which are the intersection lines of bundles of planes containing families of circumferences, will be described. These axes are intersecting mutually perpendicular straight lines \( i \) and \( j \) \[8\]. The setting of the Dupin cyclide shown in Fig. 1, automatically gives the position of the axis \( j \). There is a need to determine the second axis – \( i \). Let's find it.

Consider the variants of setting the Dupin cyclide shown in Fig. 2 and 4.
Figure 10 shows a well-known configuration called hyperbolic involution. If we apply it to the Dupin cyclide shown in fig. 2, then we find the point \( T \) belonging to the second axis \( i \) of the cyclide (Fig. 10). Indeed, with two tangent planes to the Dupin cyclide [8], as a result of their intersection we obtain the axis \( T \), which certainly passes through the point \( T \) since the configuration \( MNT \) (Fig. 11) can be represented as a rotating cone in contact with this sphere \( \Omega \), and since the tangent plane to the cyclide must also touch the given sphere, and the point of tangency belongs to the circle \( m \), this plane must also touch the constructed rotating cone, and therefore pass through its vertex \( T \).

Since the axis \( i \) passes through the point \( T \) perpendicular to the axis of symmetry \( a \) of the Dupin cyclide, then, having determined one point on the axis \( i \), we also determine the position of the entire axis (Fig. 11). Obviously, the conical points \( A \) and \( B \) will be located on the axis \( i \).

Since the tangents \( MT \) and \( NT \) to the sphere \( \Omega \) are also tangent to the circumferences of the cyclide, they are perpendicular to the straight lines \( O1M \) and \( O2N \), respectively. From here, the following configuration is valid for the setting of the Dupin cyclide shown in Fig. 12. Here also 10 parameters are involved: three parameters on the plane in which the constructions are carried out, six on the three straight lines \( l \), \( t \), \( a \) and one – the length of the segments \( OM = ON \) located on different straight lines \( l \) and \( t \).

Fig. 13, which is an addition to Fig. 4, shows similar constructions of the vertex \( T \), through which the axis \( j \) passes, perpendicular to the axis \( i \).

Now consider the settings shown in fig. 14 and 15.

Fig. 14a shows the setting of the Dupin cyclide with a sphere, axis \( a \), and point \( T \) belonging to the
axis $i$. The number of parameters involved: $\Sigma$ plane – 3 parameters; sphere $\Omega$ – 3 parameters; straight line $a$ – 2 parameters; point $T$ – 2 parameters. Total – 10 parameters. This means that the Dupin cyclide is uniquely set.

Fig. 14b shows the construction of the axes $j$ and $i$. Through the point $T$, the straight lines $TM$ and $TN$ are tangent to the sphere $\Omega$ in the plane $\Sigma$. Points $O$, $M$, and $N$ will give the position of the centers of the circumferences $O1$ and $O2$. Thus, the cyclide will be fully displayed with its circumferences. And what sets the configuration shown in Fig. 10, when referring to the Dupin cyclide?

The sphere and points $T$ and $j$ are given (Fig. 15). These are three parameters of the plane $\Sigma$; three sphere parameters; two parameters of one of the points, for example, $T$ (or $j$) and one parameter for the second point, since the straight line $TC$ for the point $T$ (or $jN$ for the point $j$) will be known from the configuration in Fig. 10. One parameter remains free. The result is a one-parameter multitude ($\propto 1$) of Dupin cyclides. Any additional parameter cuts out one unique from $\propto 1$ Dupin cyclides, since there are points $M$ and $N$.

3.4. Special cases of setting of Dupin cyclides
Let $l$ and $t$ coincide identically (Fig. 16-18). Here we can have several variants of setting of Dupin cyclides.

1. We have our own center of the sphere $\Omega$ (Fig. 16) and its real radius $R$. The straight line $a$ (the axis of the Dupin cyclide) intersects $l \equiv t$ outside this sphere. The tangent lines to the sphere $\Omega$ at the points $M$ and $N$ intersect in the plane $\Sigma$ at the improper point $i \infty$. Circumferences will be concentric, with a common center $j \equiv O1 \equiv O2$. As a result, we obtain a torus as a special case of the Dupin cyclide.

2. The straight line $a$ intersects the straight line $l \equiv t$ inside the given sphere. As a result, we get a torus with two conical points.

3. The straight line $a$ intersects the straight line $l \equiv t$ at the points $M$ or $N$ of a given sphere. As a result, we get a torus with one conic point.

4. The straight line $a$ intersects the straight $l \equiv t$ at a point located between the center of the sphere and the points $M$ or $N$. As a result, we obtain a torus with two conical points.

5. The straight line $a$ intersects the straight $l \equiv t$ inside the given sphere, and more specifically, at the point $O$. Obviously, we get the sphere (Fig. 17).

6. If the straight line $a$ is parallel to the straight lines $l \equiv t$ (Fig. 18), then in this case their intersection point $j$ is improper ($j \infty$), and the result will be a rotating cylinder.
Finally, consider the configuration shown in fig. 1, but in the case when the straight line $a$ passes through the intersection of the straight lines $l$ and $t$ – the point $O$. (Fig. 19).

We are dealing with a special case. Here, both lines coincide, and as a result we obtain a given sphere $\Omega$ of radius $OM = ON$.

Thus, it can be stated that the sphere is a special case of the Dupin cyclide along with a rotating cone, a rotating cylinder, and a torus. And the sphere is obtained only when the axis $a$ of the cyclides (the third given straight line) passes through its center.

4. Discussion of results
The results were presented in the form of a report at the annual All-Russian Scientific Conference with International Participation “Problems of Engineering Geometry”, held at the Russian Technological University (MIREA) on October 21st, 2019, and received a positive assessment.

5. Conclusion
It was proved that three pairwise intersecting straight lines and a sphere with a center at one of the intersection points set a finite number of Dupin cyclides ($\infty$1 cyclides). It was found that the sphere along with the rotating cone and rotating cylinder and with the torus is a special case of the Dupin cyclide.
The proposed configurations can be convenient in the design of both Dupin cyclides and its compartments, as well as their docking in computer-aided design, mechanical engineering, and construction.

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