QUASI-PERFECT SCHEME-MAPS AND BOUNDEDNESS OF
THE TWISTED INVERSE IMAGE FUNCTOR

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To Phillip Griffith, on his 65th birthday

Abstract. For a map $f: X \to Y$ of quasi-compact quasi-separated schemes, we discuss quasi-perfection, i.e., the right adjoint $f^\times$ of $Rf_*$ respects small direct sums. This is equivalent to the existence of a functorial isomorphism $f^\times O_Y \cong Lf^*(\cdot) \implies f^\times (-)$; to quasi-properness (preservation by $Rf_*$ of pseudo-coherence, or just properness in the noetherian case) plus boundedness of $Lf^*$ (finite tor-dimensionality), or of the functor $f^\times$; and to some other conditions. We use a globalization, previously known only for divisorial schemes, of the local definition of pseudo-coherence of complexes, as well as a refinement of the known fact that the derived category of complexes with quasi-coherent homology is generated by a single perfect complex.

1. INTRODUCTION

This paper, inspired by [V, p. 396, Lemma 1 and Corollary 2], deals with matters raised there, but not yet fully treated in the literature.

Throughout, scheme will mean quasi-compact quasi-separated scheme (see [GD, §6.1, p. 290ff]), though weaker assumptions would sometimes suffice. Unless otherwise indicated, a map $f: X \to Y$ will be a scheme-morphism, necessarily quasi-compact and quasi-separated.

For a scheme $X$, $D(X)$ is the (unbounded) derived category of the category of (sheaves of) $O_X$-modules, and $D_{qc}(X)$ is the full subcategory whose objects are the $O_X$-complexes whose homology sheaves are all quasi-coherent. For any map $f: X \to Y$, the derived functor $Rf_*: D(X) \to D(Y)$ takes $D_{qc}(X)$ to $D_{qc}(Y)$ [Lp, Prop. (3.9.2)]. Grothendieck Duality theory asserts, to begin, that the restriction $Rf_*: D_{qc}(X) \to D_{qc}(Y)$ has a right adjoint $f^\times$, the “twisted inverse image functor” in our title.

1 Warning: for nonproper maps of noetherian schemes, the usual twisted inverse image $f'$ differs from $f^\times$, and is not covered by this paper. For that, see, e.g., [Lp, §4.9].
A proof for maps of separated schemes, suggested by Deligne’s appendix to [H], is described in [Lp] §4.1. This proof depends ultimately on the Special Adjoint Functor Theorem, applied to categories of sheaves. A more direct approach, via Brown Representability—which applies immediately to derived categories—is given in [N1]. Originally this too required separability, but now that assumption can be dropped because of [BB, p. 9, Thm. 3.3.1], which gives that $\mathcal{D}_{\text{qc}}(X)$ is compactly generated, and because $Rf_*$ commutes with $\mathcal{D}_{\text{qc}}$-coproducts (= direct sums) [Lp] (3.9.3.3).

The functor $f^!$ emerging from these proofs commutes with translation (=suspension) of complexes, and is bounded-below (way-out right in the sense of [H, p. 68]), i.e., there exists an integer $m$ such that for every $F \in \mathcal{D}_{\text{qc}}(Y)$ with $H^iF = 0$ for all $i$ less than some integer $n(F)$, it holds that $H^i f^! F = 0$ for all $i < n(F) - m$ (see [Lp] (4.1.8) and the remarks preceding it).

"Bounded-below" has a similar meaning for any functor between derived categories. Bounded-above is defined in an analogous way, with $>$ (resp. $+$) in place of $<$ (resp. $-$). A functor is bounded if it is bounded both above and below. Boundedness enables a potent form of induction in derived categories, expressed by the “way-out Lemmas” [H, p. 68, Prop. 7.1 and p. 73, Prop. 7.3].

For example, the left adjoint $Lf^*$ of $Rf_*$ is always bounded-above; and $Lf^*$ is bounded iff $f$ has finite tor-dimension (a.k.a. finite flat dimension), that is, there is an integer $d \geq 0$ such that for each $x \in X$ there exists an exact sequence of $\mathcal{O}_{Y,f(x)}$-modules

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathcal{O}_{X,x} \to 0$$

with $P_i$ flat over $\mathcal{O}_{Y,f(x)}$ ($0 \leq i \leq d$).

We will be concerned with the relation between boundedness of the right adjoint $f^!$ and the left adjoint $Lf^*$, especially in the context of quasi-perfection, a property of maps to be discussed at length now in §2.

**Definition 1.1.** We say a map $f : X \to Y$ is quasi-perfect if $f^!$ respects direct sums in $\mathcal{D}_{\text{qc}}$, i.e., for any small $\mathcal{D}_{\text{qc}}(Y)$-family $(E_\alpha)$ the natural map

$$\bigoplus \alpha f^! E_\alpha \rightarrow \sim f^! (\bigoplus \alpha E_\alpha).$$

As will be explained below, quasi-perfection is also characterized by the existence of a canonical isomorphism

$$f^! \mathcal{O}_Y \otimes^L Lf^* F \rightarrow \sim f^* F \quad (F \in \mathcal{D}_{\text{qc}}(Y)).$$

More characterizations are given in §2—for instance, via compatibility of $f^!$ with tor-independent base change (Theorem 2.7). That section also brings in the related condition on maps of being perfect, i.e., pseudo-coherent.

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2Subsequently, a slightly simpler proof was given in [BB] p. 14, 3.3.4. (In that proof one needs to replace “flabby” by “quasi-flabby,” see [K] 2.)
and of finite tor-dimension. (Pseudo-coherence will be reviewed in §2. It holds, for instance, for all finite-type maps of noetherian schemes; and then descent to the noetherian case yields that every flat, finitely presentable map is pseudo-coherent.) For example, for a proper map $f$ of noetherian schemes, $f$ is quasi-perfect $\iff$ $f$ is perfect $\iff$ $f^\times$ is bounded.

It is stated in [V, p. 396, Lemma 1] that any proper map $f$ of finite-dimensional noetherian schemes is quasi-perfect. In general, however, this fails even for closed immersions. But $f^\times$ does respect direct sums when the summands $E_\alpha$ are uniformly homologically bounded below, i.e., there exists an integer $n$ such that for all $\alpha$, $H^iE_\alpha \cong 0$ whenever $i < n$ [Lp, (4.7.6)(b)]. Consequently, if the functor $f^\times$ is bounded, then $f$ is quasi-perfect.

Our main results say more. But first, call a map $f : X \to Y$ quasi-proper if $Rf_*X$ takes pseudo-coherent $O_X$-complexes to pseudo-coherent $O_Y$-complexes. (Again, pseudo-coherence is explained in §2. In particular, if $X$ is noetherian then $E \in D(X)$ is pseudo-coherent iff the homology sheaves $H^\bullet(E)$ are all coherent, and vanish for $n \gg 0$.) Kiehl showed that every proper pseudo-coherent map is quasi-proper. Consequently, any flat, finitely presentable, proper map, being pseudocoherent, is quasi-proper (and perfect and quasi-perfect as well). Moreover, when $Y$ is noetherian, every finite-type separated quasi-proper $f : X \to Y$ is proper.

Here are the main results.

**Theorem 1.2.** For a map $f : X \to Y$, the following are equivalent:

(i) $f$ is quasi-perfect (resp. perfect).
(ii) $f$ is quasi-proper (resp. pseudo-coherent) and has finite tor-dimension.
(iii) $f$ is quasi-proper (resp. pseudo-coherent) and $f^\times$ is bounded.

Hence, by Kiehl’s theorem, every proper perfect map is quasi-perfect.

The implication (i) $\Rightarrow$ (iii) is worked out in §4. The proofs in §4 are based on Theorems 4.1 and 4.2, which are of independent interest.

Theorem 4.1 states that for a scheme $X$, any pseudo-coherent $O_X$-complex can be “arbitrarily-well approximated,” globally, by a perfect complex. (Local approximability is essentially the definition of pseudo-coherence. The global result was previously known only for divisorial schemes.)

This leads to quasi-proper maps being characterized as those $f$ such that $Rf_*X$ takes perfect complexes to pseudo-coherent ones. Since by Prop. 2.1 quasi-perfect maps are those $f$ such that $Rf_*X$ takes perfect complexes to perfect ones, it follows at once that quasi-perfect maps are quasi-proper.

Theorem 4.2 refines a theorem of Bondal and van den Bergh [BB, p. 9, Thm. 3.1.1] which states that the triangulated category $D_{qc}(X)$ is generated by a single perfect complex. With this in hand, one can prove Corollary 4.3.1 which says that for any quasi-perfect or perfect $f$ as above, $f^\times$ is bounded.

The implication (iii) $\Rightarrow$ (ii) results from Theorem 3.1, which says, for any $f : X \to Y$ as above, if $f^\times$ is bounded then $f$ has finite tor-dimension.
Finally, the implication (ii) \(\Rightarrow\) (i) holds by definition for the resp. case, and is proved for the other case in §2, Example 2.2(a).

Let us call a map \(f : X \to Y\) **locally embeddable** if every \(y \in Y\) has an open neighborhood \(V\) over which the induced map \(f^{-1}V \to V\) factors as \(f^{-1}V \xrightarrow{i} Z \xrightarrow{p} V\) where \(i\) is a closed immersion and \(p\) is smooth. (For instance, any quasi-projective \(f\) satisfies this condition.) Proposition 2.5 asserts that any quasi-proper locally embeddable map is pseudo-coherent. A similar proof shows that any quasi-perfect locally embeddable map is perfect. By 1.2, then, a locally embeddable map is quasi-perfect iff it is quasi-proper and perfect.

The equivalence of (i) and (ii) in Theorem 1.2 generalizes [V, p. 396, Cor. 2], in view of the following characterization (mentioned above) of quasi-perfection.

For a map \(f : X \to Y\), and for any \(E \in \mathcal{D}_{qc}(X), F \in \mathcal{D}_{qc}(Y)\), with \(\otimes := \otimes^L\), the derived tensor product, the “projection map”

\[
\pi : (Rf_\ast E) \otimes F \to Rf_\ast (E \otimes Lf^\ast F),
\]

defined to be adjoint to the natural composite map

\[
Lf^\ast ((Rf_\ast E) \otimes F) \xrightarrow{\sim} (Lf^\ast Rf_\ast E) \otimes Lf^\ast F \to E \otimes Lf^\ast F,
\]

is an isomorphism. (This is well-known under more restrictive hypotheses; for a proof in the stated generality, see [Lp, Prop. (3.9.4)].) There results a natural map

\[
\chi_F : f^\ast O_Y \otimes Lf^\ast F \to f^\ast F \quad (F \in \mathcal{D}_{qc}(Y)),
\]

adjoint to the natural composite map

\[
Rf_\ast (f^\ast O_Y \otimes Lf^\ast F) \xrightarrow{\sim} Rf_\ast f^\ast O_Y \otimes F \to O_Y \otimes F = F.
\]

It is clear (since \(\otimes\) and \(Lf^\ast\) both respect direct sums, see e.g., [Lp, 3.8.2]) that if \(\chi_F\) is an isomorphism for all \(F \in \mathcal{D}_{qc}(Y)\) then \(f\) is quasi-perfect; and Proposition 2.1 gives the converse.

2. **Quasi-Perfect Maps**

For surveying quasi-perfection in more detail, starting with Proposition 2.1 we need some preliminaries.

First, a brief review of the notion of pseudo-coherence of complexes. (Details can be found in the primary source [IL, Exposé III], or, perhaps more accessibly, in [L1, pp. 283ff, §2]; a summary appears in [Lp, §4.3].) The idea is built up from that of **strictly perfect** \(O_X\)-complex, i.e., bounded complex of finite-rank free \(O_X\)-modules.

For \(n \in \mathbb{Z}\), a map \(\xi : P \to E\) in \(K(X)\), the homotopy category of \(O_X\)-complexes, (resp. in \(D(X)\)), is said to be an \(n\)-**quasi-isomorphism** (resp. \(n\)-**isomorphism**) if the following two equivalent conditions hold:

1. The homology map \(H^j(\xi) : H^j(P) \to H^j(E)\) is bijective for all \(j > n\) and surjective for \(j = n\).
(2) For any $K(X)$- (resp. $D(X)$-)triangle

\[
\begin{array}{ccc}
P & \xrightarrow{\xi} & E \\
& & \rightarrow Q \\
& & \rightarrow P[1],
\end{array}
\]

it holds that $H^j(Q) = 0$ for all $j \geq n$.

Then $E$ is said to be $n$-pseudo-coherent if $X$ has an open covering $(U_\alpha)$ such that for each $\alpha$ there exists a strictly perfect $O_{U_\alpha}$-complex $P_\alpha$ and an $n$-quasi-isomorphism (or equivalently, an $n$-isomorphism) $P_\alpha \rightarrow E|_{U_\alpha}$, see [I, p. 98, Définition 2.3]; and $E$ is pseudo-coherent if $E$ is $n$-pseudo-coherent for every $n$. If $O_X$ is coherent, this means simply that $F$ has coherent homology sheaves, vanishing in all sufficiently large degrees [ibid., p. 116, top]. When $X$ is noetherian and finite-dimensional, it means that $F$ is globally $D$-isomorphic to a bounded-above complex of coherent $O_X$-modules [ibid., p. 168, Cor. 2.2.2.1].

A complex $E \in D(X)$ ($X$ a scheme) is said to be perfect if it is locally $D$-isomorphic to a strictly perfect $O_X$-complex. More precisely, $E$ is said to have perfect amplitude in $[a, b]$ ($a \leq b \in \mathbb{Z}$) if locally on $X$, $E$ is $D$-isomorphic to a bounded complex of finite-rank free $O_X$-modules vanishing in all degrees $< a$ or $> b$. Thus $E$ is perfect iff it has perfect amplitude in some interval $[a, b]$.

By [I, p. 134, 5.8], $E$ has perfect amplitude in $[a, b]$ iff $E$ is $(a - 1)$-pseudo-coherent and has tor-amplitude in $[a, b]$ (i.e., is globally $D$-isomorphic to a flat complex vanishing in all degrees $< a$ and $> b$). So $E$ is perfect iff it is pseudo-coherent and has finite tor-dimension (the latter meaning that it is $D$-isomorphic to a bounded flat complex).

A map $f : X \rightarrow Y$ is pseudo-coherent if every $x \in X$ has an open neighborhood $U$ such that the restriction $f|_U$ factors as $U \xrightarrow{i} Z \xrightarrow{p} Y$, where $i$ is a closed immersion such that $i_*O_U$ is pseudo-coherent on $Z$, and $p$ is smooth [I, p. 228, Défn. 1.2]. Pseudo-coherent maps are finitely presentable. Compositions of pseudo-coherent maps are pseudo-coherent [I, p. 236, Cor. 1.14].

A map is perfect if it is pseudo-coherent and has finite tor-dimension [I, p. 250, Défn. 4.1]. Any smooth map is perfect, any regular immersion (= closed immersion corresponding to a quasi-coherent ideal generated locally by a regular sequence) is perfect, and compositions of perfect maps are perfect [I, p. 253, Cor. 4.5.1(a)].

For noetherian $Y$, any finite-type $f : X \rightarrow Y$ is pseudo-coherent. Pseudo-coherence (resp. perfection) of maps survives tor-independent base change [I, p. 233, Cor. 1.10; p. 257, Cor. 4.7.2]. Hence, by descent to the noetherian case [EGA] IV, (11.2.7)], every flat finitely-presentable map is perfect.

A map $f : X \rightarrow Y$ is quasi-proper if $Rf_*O_X$-complexes to pseudo-coherent $O_Y$-complexes.

Kiehl’s Finiteness Theorem [Kl, p. 315, Thm. 2.9′] (first proved by Illusie for projective maps [I, p. 236, Thm. 2.2]) generalizes preservation of coherence by higher direct images under proper maps of noetherian schemes. It states that every proper pseudo-coherent map is quasi-proper.
This theorem (or its special case [I] p. 240, Cor. 2.5), plus [Lp] Ex. (4.3.9])
implies that if $Y$ is noetherian then a finite-type separated $f : X \to Y$ is quasi-
proper iff it is proper.

For details in the proof of the following Proposition, and for some subse-
quent considerations, recall that an object $C$ in a triangulated category $T$ is
compact if for every small $T$-family $(E_\alpha)$ the natural map is an isomorphism
\[
\oplus \Hom(C, E_\alpha) \sim \to \Hom(C, \oplus E_\alpha).
\]

For any scheme $X$, the compact objects of $D_{qc}(X)$ are just the perfect
complexes, of which one is a generator [BB, p. 9, Thm. 3.1.1].

**Proposition 2.1.** For a map $f : X \to Y$, the following are equivalent:

(i) $f$ is quasi-perfect (Definition 1.1).

(ii) The functor $Rf_*$ takes perfect complexes to perfect complexes.

(iii) $f$ is proper.

(iv) For all $F \in D_{qc}(Y)$, the map in (1.3) is an isomorphism
\[
\chi^*_F : f^*O_Y \otimes_{\mathcal{O}_X} = Lf^*F \sim \to f^*F.
\]

**Proof.** (i) $\iff$ (ii) $\iff$ (ii): [N1] p. 224, Thm. 5.1.

(i) $\implies$ (iii): [N1] p. 223, Thm. 4.1.

(iii) $\implies$ (i): simple.

(i) $\implies$ (iv) $\implies$ (i): See [N1] p. 226, Thm. 5.4.

To be precise, the results in [N1] are proved for separated schemes; but with
the remark preceding Prop. 2.1, one readily verifies that the proofs survive
without any separability requirement. \qed

**Examples 2.2.** (a) Any quasi-proper map $f$ of finite tor-dimension—in par-
ticular, by Kiehl’s theorem, any proper perfect map—is quasi-perfect. Indeed,
$Rf_*$ preserves pseudo-coherence, and by [I] p. 250, 3.7.2 (a consequence of the
projection isomorphism mentioned near the end of the above Introduction),
$Rf_*$ preserves finite tor-dimensionality of complexes; so Prop. 2.1(ii) holds.

(b) Let $f : X \to Y$ be a map with $X$ divisorial—i.e., $X$ has an ample family
$(\mathcal{L}_i)_{i \in I}$ of invertible $O_X$-modules [I] p. 171, Défn. 2.2.5]. Then [N1] p. 212,
Example 1.11 and p. 224, Theorem 5.1] show that $f$ is quasi-perfect $\iff$ for
each $i \in I$, the $O_Y$-complex $Rf_*(\mathcal{L}_i^{\otimes -n})$ is perfect for all $n_i \gg 0$.

(c) Let $f$ be quasi-projective and let $\mathcal{L}$ be an $f$-ample invertible sheaf. Then
$f$ is quasi-perfect $\iff$ the $O_Y$-complex $Rf_*(\mathcal{L}^{\otimes -n})$ is perfect for all $n \gg 0$.

Indeed, condition (ii) in Prop 2.1 together with the compatibility of $Rf_*$
and open base change, implies that quasi-perfection is a property of $f$ which
can be checked locally on $Y$, and the same holds for perfection of $Rf_*(\mathcal{L}^{\otimes -n})$;
so we may assume $Y$ affine, and apply (b).
(d) For a finite map $f: X \to Y$ the following are equivalent:

(i) $f$ is quasi-perfect.

(ii) $f$ is perfect.

(iii) The complex $f_*\mathcal{O}_X \cong Rf_*\mathcal{O}_X$ is perfect.

This follows quickly from (a) and from Proposition 2.1(ii).

A tor-independent square is a fiber square of maps

$$
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow{s} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
$$

(2.3)

(that is, the natural map is an isomorphism $X' \cong X \times_Y Y'$) such that for all $x \in X$, $y' \in Y'$ and $y \in Y$ with $f(x) = u(y') = y$, and all $i > 0$,

$$
\text{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y}) = 0.
$$

The following stability properties will be useful.

**Proposition 2.4.** For any tor-independent square (2.3),

(i) If the functor $f^*$ is bounded then so is $g^*$.

(ii) If $f$ is quasi-perfect then so is $g$.

(iii) If $f$ is quasi-proper then so is $g$.

**Proof.** (i) and (ii) are proved in [Lp, (4.7.3.1)]; and (iii) is treated in Prop. 4.4 below (a slight change in whose proof gives another proof of (ii)).

Since perfection (resp. pseudo-coherence) is a local property of complexes, and $Rf_*$ is compatible with open base change on $Y$, we deduce:

**Corollary 2.4.1.** Let $f: X \to Y$ be a map, and let $(Y_i)_{i \in I}$ be an open cover of $Y$. Then $f$ is quasi-perfect (resp. quasi-proper) $\iff$ for all $i$, the same is true of the induced map $f^{-1}Y_i \to Y_i$.

**Proposition 2.5.** Let $f: X \to Y$ be a locally embeddable map, i.e., every $y \in Y$ has an open neighborhood $V$ over which the induced map $f^{-1}V \to V$ factors as $f^{-1}V \xrightarrow{i} Z \xrightarrow{p} V$ where $i$ is a closed immersion and $p$ is smooth. (For instance, any quasi-projective $f$ satisfies this condition [EGA II, (5.3.3)].)

(i) If $f$ is quasi-proper then $f$ is pseudo-coherent.

(ii) If $f$ is quasi-perfect then $f$ is perfect.

**Proof.** By Corollary 2.4.1, quasi-properness (resp. quasi-perfection) of $f$ is a property local over $Y$; and since they are compatible with tor-independent base change, the same is true of pseudo-coherence and perfection. So we may as well assume that $X = f^{-1}V$. Then it suffices to show that the complex $i_*\mathcal{O}_X$ is pseudo-coherent when $f$ is quasi-proper, (resp., by [H] p. 252, Prop. 4.4], that $i_*\mathcal{O}_X$ is perfect when $f$ is quasi-perfect).
But \( i \) factors as \( X \xrightarrow{\gamma} X \times_Y Z \xrightarrow{\varphi} Z \) with \( \gamma \) the graph of \( i \) and \( g \) the projection. The map \( \gamma \) is a local complete intersection \([EGA\ IV, (17.12.3)]\), so the complex \( \gamma_! \mathcal{O}_X \) is perfect. Also, \( g \) arises from \( f \) by flat base change, so by Proposition 2.4, \( g \) is quasi-proper (resp. quasi-perfect). Hence \( i_* \mathcal{O}_X = R_i_* \mathcal{O}_X = Rg_* \gamma_* \mathcal{O}_X \) is indeed pseudo-coherent (resp. perfect). \( \square \)

(2.6). For any tor-independent square (2.3), the map

\[(2.6.1) \quad \theta(E): Lu^* Rf_* E \to Rg_* Lo^* E \quad (E \in D_{qc}(X))\]

adjoint to the natural composition

\[Rf_* E \to Rf_* Rv_* Lo^* E \cong Ru_* Rg_* Lo^* E\]

(equivalently, to \( Lg^* Lu^* Rf_* E \cong Lv^* Lf^* Rf_* E \to Lo^* E \)) is an isomorphism, so that one has a base-change map

\[(2.6.2) \quad \beta(F): Lv^* f^! F \to g^! Lu^* F \quad (F \in D_{qc}(Y))\]

adjoint to the natural composition

\[Rg_* Lo^* f^! F \xrightarrow{\theta^{-1}} Lw^* Rf_* f^! F \to Lu^* F.\]

The fundamental independent base-change theorem states that:

Let there be given a tor-independent square (2.3) and an \( F \in D_{qc}(Y) \). If \( f \) is quasi-proper, \( u \) has finite tor-dimension, and \( H^n F = 0 \) for all \( n < 0 \), then \( \beta(F) \) is an isomorphism.

This theorem is well-known, at least under more restrictive hypotheses. For a treatment in full generality, see \([Lp, \S\S 4.4–4.6]\).

One consequence, in view of Proposition 2.4(i), is:

**Corollary 2.6.3.** Let \( f: X \to Y \) be a quasi-proper map and let \( (Y_i)_{i \in I} \) be an open cover of \( Y \). Then \( f^! \) is bounded \( \iff \) for all \( i \), the same is true of the induced map \( f^{-1} Y_i \to Y_i \).

For quasi-perfect \( f \), a stronger base-change theorem holds—which, together with boundedness of \( f^! \) (Corollary 4.3.1), characterizes quasi-perfection:

**Theorem 2.7** (\([Lp, Thm. 4.7.4]\)). Let

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow s & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

be a tor-independent square, with \( f \) quasi-perfect. Then for all \( F \in D_{qc}(Y) \) the base-change map of (2.6.2) is an isomorphism

\[\beta(F): v^* f^! F \xrightarrow{\sim} g^! u^* F.\]

The same holds, with no assumption on \( f \), whenever \( u \) is finite and perfect.
Conversely, the following conditions on a map \( f: X \to Y \) are equivalent; and if \( f^\times \) is bounded above, they imply that \( f \) is quasi-perfect:

(i) For any flat affine universally bicontinuous map \( u: Y' \to Y \), the base-change map associated to the (tor-independent) square
\[
\begin{array}{ccc}
Y' \times_Y X & \xrightarrow{u} & X \\
\downarrow g & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
\]
is an isomorphism
\[\beta(O_Y): v^*f^\times O_Y \xrightarrow{\sim} g^*u^*O_Y.\]

(ii) The map in \([13]\) is an isomorphism
\[\chi_F: f^\times O_Y \otimes Lf^*F \xrightarrow{\sim} f^\times F\]
whenever \( F \) is a flat quasi-coherent \( O_Y \)-module.

Keeping in mind Corollary 4.3.1 below \(( f \) quasi-perfect \( \Rightarrow f^\times \) bounded), we can deduce:

**Corollary 2.7.1.** A map \( f: X \to Y \) is quasi-perfect \( \iff f^\times \) is bounded and the following two conditions hold:

(i) If \( u: Y' \to Y \) is an open immersion, and if \( v: Y' \times_Y X \to X \) and \( g: Y' \times_Y Y \to Y \) are the projection maps, then the base-change map is an isomorphism
\[\beta(O_Y): v^*f^\times O_Y \xrightarrow{\sim} g^*u^*O_Y.\]

Equivalently (see \([Lp, \S 4.6, \text{subsection V}]\)), for all \( E \in D_{qc}(X) \) the natural composite map
\[\mathbf{R}f_\ast \mathbf{R}\text{Hom}_X^\bullet(E, f^\times O_Y) \to \mathbf{R}\text{Hom}_Y^\bullet(\mathbf{R}f_\ast E, \mathbf{R}f_\ast f^\times O_Y) \to \mathbf{R}\text{Hom}_Y^\bullet(\mathbf{R}f_\ast E, O_Y)\]
is an isomorphism.

(ii) If \((F_\alpha)\) is a filtered direct system of flat quasi-coherent \( O_Y \)-modules, then for all \( n \in \mathbb{Z} \) the natural map is an isomorphism
\[\lim_{\alpha} H^n(f^\times F_\alpha) \xrightarrow{\sim} H^n(f^\times \lim_{\alpha} F_\alpha).\]

*Remarks.* 1. Conditions (i) and (ii) in Theorem 2.7 are connected via the flat, affine, and universally bicontinuous natural map \( \text{Spec}(S_{\leq 1}(F)) \to Y \), where \( S_{\leq 1}(F) \) is the \( O_Y \)-algebra \( O_Y \oplus F \) with \( F^2 = 0 \).

2. The idea behind the proof of Corollary 2.7.1 is to use Lazard’s theorem that over a commutative ring \( A \) any flat module is a \( \lim \) of finite-rank free \( A \)-modules \([GD, \text{p. 163, (6.6.24)]}\), to show that (i) and (ii) imply condition (ii) in Theorem 2.7.

3*Universally bicontinuous* means that for any \( Y'' \to Y \) the resulting projection map \( Y' \times_Y Y'' \to Y'' \) is a homeomorphism onto its image \([GD, \text{p. 249, Défn. (3.8.1)]}\).
3. Boundedness of \( f^\times \) implies finite tor-dimension

**Theorem 3.1.** Let \( f : X \to Y \) be a map. If \( f^\times \) is bounded then \( f \) has finite tor-dimension.

The proof uses the following two Lemmas.

An \( \mathcal{O}_X \)-complex \( E \) is \( a \)-locally projective \((a \in \mathbb{Z})\) if there is a \( b \geq a \) and an affine open covering \( \{U_i := \text{Spec}(A_i)\}_{i \in I} \) of \( X \) such that for each \( i \in I \), the restriction \( E|_{U_i} \) is \( D \)-isomorphic to a quasi-coherent direct summand of a complex \( F \) of free \( \mathcal{O}_{U_i} \)-modules, with \( F \) vanishing in all degrees outside \([a, b]\).

Every complex with perfect amplitude in \([a, b]\) \((\S 2)\) is \( a \)-locally projective.

**Lemma 3.2.** For any scheme \( X \), there is an integer \( s > 0 \) such that for all \( a \in \mathbb{Z} \) and \( a \)-locally projective \( E \in D(X) \), if \( G \in D_{qc}(X) \) and \( H^jG = 0 \) for all \( j > a - s \) then \( \text{Hom}_{D(X)}(E, G) = 0 \).

**Lemma 3.3.** Let \( f : X \to Y \) be a perfect map, of tor-dim \( d < \infty \). Then there exists an integer \( t > 0 \) such that for any \( a \)-locally projective \( E \in D_{qc}(X) \), \( R^j f_* E \in D_{qc}(Y) \) is \((a - d - t)\)-locally projective.

These Lemmas are proved below.

**Proof of Theorem 3.1**

Part (i) of Proposition 2.4 gives an immediate reduction to the case where \( Y \) is affine, say \( Y = \text{Spec}(A) \). We need to show in this case that for any open immersion \( \iota: U \hookrightarrow X \) with \( U \) affine the \( \mathcal{O}_Y \)-module \( f_{\iota*} \mathcal{O}_U \) has finite tor-dimension.

Since \( U \) is affine, there are natural isomorphisms

\[
  f_{\iota*} \mathcal{O}_U = (f \iota)_* \mathcal{O}_U \xrightarrow{\sim} R(f \iota)_* \mathcal{O}_U \xrightarrow{\sim} R f_{\iota*} R f_* \mathcal{O}_U.
\]

So for any \( G \in D_{qc}(Y) \) there are natural isomorphisms

\[
\text{Hom}_{D(Y)}(f_{\iota*} \mathcal{O}_U, G) \cong \text{Hom}_{D(Y)}(R f_{\iota*} R f_* \mathcal{O}_U, G) \cong \text{Hom}_{D(X)}(R f_* \mathcal{O}_U, f^\times G).
\]

Lemma 3.3 provides an integer \( t \) such that if \( U \) is any quasi-compact open subscheme of \( X \), with inclusion \( \iota: U \subset X \), then \( R f_{\iota*} \mathcal{O}_U \) is \((-t)\)-locally projective. By Lemma 3.2 and the boundedness of \( f^\times \), it follows, for \( U \) affine, \( G \) a quasi-coherent \( \mathcal{O}_Y \)-module, and some \( j \gg 0 \) not depending on \( G \), that

\[
\text{Ext}^j(f_{\iota*} \mathcal{O}_U, G) = \text{Hom}_{D(Y)}(f_{\iota*} \mathcal{O}_U, G[j]) \cong \text{Hom}_{D(X)}(R f_{\iota*} \mathcal{O}_U, f^\times G[j]) = 0.
\]

The natural equivalences \( D(A) \xrightarrow{\sim} D(Y_{qc}) \xrightarrow{\sim} D_{qc}(Y) \) (where \( Y_{qc} \) is the category of quasi-coherent \( \mathcal{O}_Y \)-modules—see [BN, p. 30, Cor. 5.5]) show then that \( f_{\iota*} \mathcal{O}_U \) has a resolution by the sheafification of a bounded projective \( A \)-complex, and thus has finite tor-dimension, as desired. \( \square \)
Proof of Lemma 3.2

Let us call an open $U \subset X$ $E$-good if $U$ is affine, say $U = \text{Spec}(A)$, and if there is a $b \geq a$ such that the restriction $E|_U$ is $D$-isomorphic to the sheafification of a projective $A$-complex $E$ vanishing in all degrees outside $[a, b]$.

Clearly, every quasi-compact open subset of $X$ is a finite union of $E$-good open subsets. Hence, as in the proof of [BB] p. 13, Prop. 3.3.1, it will suffice to show that Lemma 3.2 holds for $X$ if $X$ itself is $E$-good, or if $X = X_1 \cup X_2$ with $X_1$ and $X_2$ quasi-compact open subsets such that Lemma 3.2 holds for $X_1$, $X_2$ and $X_1 \cap X_2$ (which is also quasi-compact, since $X$ is quasi-separated).

Suppose first that $X = \text{Spec}(A)$ is $E$-good. Let $E$ be as in the definition of $E$-good, and let $G \in D_{qc}(X)$ be such that $H^jG = 0$ for all $j > a - 1$. The natural equivalence of categories $D(X_{qc}) \cong D_{qc}(X)$ (where $X_{qc}$ is the category of quasi-coherent $O_X$-modules) allows us to assume $G$ quasi-coherent, so that $G$ is the sheafification of an $A$-complex $G$; and further, after applying the well-known truncation functor (see e.g., [Lp], §1.10) we can assume that $G$ vanishes in all degrees $> a - 1$.

The dual versions of [Lp] (2.3.4) and (2.3.8)(v)], and the equivalences $D(A) \cong D(X_{qc})$, $D(X_{qc}) \cong D_{qc}(X)$, yield natural isomorphisms, with $K(A)$ the homotopy category of $A$-complexes:

$$\text{Hom}_{K(A)}(E, G) \cong \text{Hom}_{D(A)}(E, G) \cong \text{Hom}_{D(X_{qc})}(E, G) \cong \text{Hom}_{D(X)}(E, G).$$

So since $E$ vanishes in all degrees $< a$ and $G$ vanishes in all degrees $> a - 1$, therefore $\text{Hom}_{D(X)}(E, G) = 0$, proving Lemma 3.2 in this case.

Suppose next that $X = X_1 \cup X_2$ as above. Let $s > 0$ be such that Lemma 3.2 holds with this $s$ for all three of $X_1$, $X_2$, and $X_1 \cap X_2$. Let $G \in D_{qc}(X)$ satisfy $H^jG = 0$ for all $j > a - (s + 1)$. Let $i: X_1 \hookrightarrow X$, $j: X_2 \hookrightarrow X$, and $k: X_1 \cap X_2 \hookrightarrow X$ be the inclusion maps. One gets the natural triangle

$$G \longrightarrow R_i_*i^!G \oplus R_j_*j^!G \longrightarrow R_k_*k^!G \longrightarrow G[1],$$

by applying the usual exact sequence, holding for any flasque $O_X$-module $F$,

$$0 \to F \to i_*i^*F \oplus j_*j^*F \to k_*k^*F \to 0$$

to an injective $q$-injective resolution of $E^+$. There results an exact sequence

$$\text{Hom}(E, Rk_*k^!G[-1]) \to \text{Hom}(E, G) \to \text{Hom}(E, Ri_*i^!G) \oplus \text{Hom}(E, Rj_*j^!G).$$

Adjointness of $Rk_*$ and $Lk^* = k^*$ gives that

$$\text{Hom}_{D(X)}(E, Rk_*k^!G[-1]) \cong \text{Hom}_{D(X_1 \cap X_2)}(k^*E, k^!G[-1]).$$

and Lemma 3.2 makes these groups vanish. Similarly, $\text{Hom}(E, Rj_*j^!G) = 0$ and $\text{Hom}(E, Ri_*i^!G) = 0$. Hence $\text{Hom}(E, G) = 0.$

\[\text{Another term for “q-injective” is “K-injective”—see [Lp] (2.3.2.3), (2.3.5).}\]
Proof of Lemma \[3.3\]

The question is local on $Y$, so we may assume $Y$ affine, say $Y = \text{Spec}(B)$. Arguing as in the preceding proof, suppose first that $X$ is $E$-good. We begin with the case $E = \mathcal{O}_X$. Then for some $t > 0$, $f$ factors as

$$X \xrightarrow{\iota} Y_t := \text{Spec}(B[T_1, T_2, \ldots, T_t]) \xrightarrow{\pi} \text{Spec}(B),$$

where $T_1, \ldots, T_t$ are independent indeterminates, $\iota$ is a closed immersion, and $\pi$ is the natural map. By [I, p. 252, Prop. 4.4(ii) and p. 174, Prop. 2.2.9(b)], the sheaf $\iota_* \mathcal{O}_X$ is a bounded quasi-coherent complex $G$ of direct summands of finite-rank free $\mathcal{O}_{Y_t}$-modules, vanishing in all degrees $<-d - t$. Hence $\mathcal{R}f_* \mathcal{O}_X \cong \pi_* \iota_* \mathcal{O}_X \cong \pi_* G$ is $(-d - t)_a$-locally projective.

Since $\mathcal{R}f_*$ commutes with direct sums in $\mathsf{D}_{qc}$ (because $\mathcal{R}f_*$ has a right adjoint, or more directly, by [Lp, 3.9.3.3]), it follows that for any free $\mathcal{O}_X$-module $E$, $\mathcal{R}f_* E$ is $(-d - t)_a$-locally projective. Finally, to show that for any $a_\omega$-locally projective $E$, $\mathcal{R}f_* E$ is $a_\omega$-locally projective, one reduces easily to where $E$ is a bounded free complex, and then argues by induction on the number of degrees in which $E$ is nonvanishing, using the following observation:

$$(\ast): \text{ In a } \mathsf{D}(X)\text{-triangle } N[-1] \xrightarrow{\delta} L \rightarrow M \xrightarrow{\rho} N, \text{ if } N \text{ is } a_\omega\text{-locally projective then } M \text{ is } a_\omega\text{-locally projective iff so is } L.$$  

(To see this, one may suppose that $X$ is affine, say $X = \text{Spec}(A)$. If $N$ and $L$ are $a_\omega$-locally projective, then one may assume they are sheafifications of bounded projective $A$-complexes vanishing in all degrees $< a$, so that by the dual versions of [Lp, (2.3.4) and (2.3.8)(v)], $\delta$ comes from a $\mathsf{K}(X)$-morphism $\delta_0: N[-1] \rightarrow L$; and $M$ is isomorphic to the mapping cone of $\delta_0$, an $a_\omega$-projective complex. Similarly, if $N$ and $M$ are sheafifications of bounded projective $A$-complexes vanishing in all degrees $< a$, then $\rho$ comes from a $\mathsf{K}(X)$-morphism $\rho_0: M \rightarrow N$, and since $L[1]$ is isomorphic to the mapping cone of $\rho_0$, therefore $L$ is $a_\omega$-locally projective.)

Suppose next that $X = X_1 \cup X_2$ with $X_1$ and $X_2$ quasi-compact open subsets for which there exists a $t > 0$ such that Lemma 3.3 holds with this $t$ for all three of $X_1$, $X_2$, and $X_1 \cap X_2$. As in the proof of Lemma \[3.2\] there is a $\mathsf{D}(Y)$-triangle

$$\mathcal{R}f_* E \rightarrow \mathcal{R}(f i)_* (E|_{X_1}) \oplus \mathcal{R}(f j)_* (E|_{X_2}) \rightarrow \mathcal{R}(f k)_* (E|_{X_1 \cap X_2}) \rightarrow \mathcal{R}f_* E[1]$$

in which the two vertices other than $\mathcal{R}f_* E$ are $(a - d - t)_a$-projective, whence, by $(\ast)$, so is $\mathcal{R}f_* E$.

As before, this completes the proof of Lemma \[3.3\] and so of Theorem \[3.1\].

4. APPROXIMATION BY PERFECT COMPLEXES.

Terminology remains as in \S 2.

The main results in this section are the following two theorems.
Theorem 4.1. For any scheme $X$, there exists a positive integer $B = B(X)$ such that for any $E \in \mathbf{D}_{qc}(X)$ and integer $m$, if $E$ is $(m-B)$-pseudo-coherent then there exists in $\mathbf{D}_{qc}(X)$ an $m$-isomorphism $P \to E$ with $P$ perfect.

Theorem 4.2. Let $X$ be a scheme. Then $\mathbf{D}_{qc}(X)$ has a perfect generator, i.e., there is a perfect $\mathcal{O}_X$-complex $S$ such that for each $E \neq 0$ in $\mathbf{D}_{qc}(X)$ there is an $n \in \mathbb{Z}$ and a nonzero $\mathbf{D}_{qc}(X)$-morphism $S[n] \to E$.

Moreover, for each such $S$ there is an integer $A = A(S)$ such that for all $E \in \mathbf{D}_{qc}(X)$ and $j \in \mathbb{Z}$ with $H^j(E) \neq 0$,

\[ \text{Hom}(S[n], E) \neq 0 \quad \text{for some} \quad n \leq A - j. \]

Theorem 4.1 may be compared to [I, p. 173, 2.2.8(b)]. The first statement in Theorem 4.2 comes from [BB, p. 9, Thm. 3.1.1].

Proofs are given in section 5 below.

Corollary 4.3.1. If a map $f$ is either perfect or quasi-perfect, then the functor $f^\times$ is bounded.

Proof. As mentioned in the Introduction, $f^\times$ commutes with translation of complexes, and $f^\times$ is bounded below. So to show that $f^\times$ is bounded, it is enough to find a $j_0$ such that for every $m \in \mathbb{Z}$ and $F \in \mathbf{D}_{qc}(Y)$ with $H^i(F) = 0$ for all $i > m$, it holds that $H^j f^\times F = 0$ for all $j \geq m + j_0$.

Suppose $H^j(f^\times F) \neq 0$. With $S$ and $A$ as in Theorem 4.2 there exists $k \leq A$ and a nonzero $\mathbf{D}(X)$-morphism $S \to f^\times F[j - k]$, the latter corresponding under adjunction to a nonzero morphism $\lambda: Rf_* S \to F[j - k]$.

For some $a$, $Rf_* S$ is $a$-locally projective—when $f$ is perfect, that results from Lemma 3.3 and when $f$ is quasi-perfect, it’s because $Rf_* S$ is perfect. It follows from Lemma 3.3 that there is an integer $s = s(Y)$ such that $\lambda$ cannot exist if $j \geq m + A - a + s$. With $j_0 := A - a + s$, we must have then that $H^j(f^\times F) = 0$ for all $j \geq m + j_0$; and so $f^\times$ is indeed bounded.

Corollary 4.3.2. For a map $f: X \to Y$, the following are equivalent.

(i) $f$ is quasi-proper.

(ii) For any perfect $\mathcal{O}_X$-complex $P$, $Rf_* P$ is pseudo-coherent.

(iii) If $S$ is as in Theorem 4.2 then $Rf_* S$ is pseudo-coherent.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The first implication is clear (since perfect complexes are pseudo-coherent); and the second is trivial.

(iii) $\Rightarrow$ (ii). Let $\mathcal{R}$ be the smallest triangulated subcategory of $\mathbf{D}_{qc}(X)$ containing $S$, and let $\mathcal{R}'$ be the full subcategory of $\mathbf{D}_{qc}(X)$ whose objects are all the direct summands of objects in $\mathcal{R}$. The subcategory $\mathcal{R}' \subset \mathbf{D}_{qc}(X)$ is triangulated, and closed under formation of direct summands [N2, p. 99, 2.1.39].

The full subcategory $\mathcal{R}^c$ of $\mathcal{R}$ whose objects are the compact ones in $\mathcal{R}$ is triangulated, whence every object in $\mathcal{R}'$—and in $\mathcal{R}'$—is compact. Consequently, [N1, p. 222, Lemma 3.2] shows that the smallest full subcategory of $\mathbf{D}_{qc}(X)$ which contains $\mathcal{R}$ and is closed with respect to coproducts is $\mathbf{D}_{qc}(X)$ itself.
Hence, by [N1, p. 214, Thm. 2.1.3], every perfect complex lies in \( \hat{R} \). (Alternatively, see [N2, p. 285, Prop. 8.4.1 and p. 140, Lemma 4.4.5].)

Since the pseudo-coherent complexes in \( D_{\mathsf{qc}}(Y) \) are the objects of a triangulated subcategory closed under formation of direct summands [Lp, p. 99, b) and p. 105, 2.12], therefore the complexes \( Q \in D_{\mathsf{qc}}(X) \) such that \( Rf_*Q \) is pseudo-coherent are the objects of a triangulated subcategory closed under formation of direct summands. Thus if \( S \) is such a \( Q \) then every complex in \( \hat{R} \)—and so every perfect complex—is such a \( Q \).

(ii) \( \Rightarrow \) (i). Let \( E \) be a pseudo-coherent \( \mathcal{O}_X \)-complex, let \( m \in \mathbb{Z} \), and let

\[
P \xrightarrow{\alpha} E \xrightarrow{} Q \xrightarrow{} P[1]
\]

be a triangle with \( \alpha \) an \( m \)-isomorphism as in Theorem 4.1. Thus \( H^k(Q) = 0 \) for all \( k \geq m \). As \( Rf_* \) is bounded above [Lp, (3.9.2)], there is an integer \( t \) depending only on \( f \) such that \( H^k(Rf_*Q) = 0 \) for all \( k \geq m + t \), that is, \( Rf_*\alpha \) is an \( (m + t) \)-quasi-isomorphism. So if \( Rf_*P \) is pseudo-coherent then \( Rf_*E \) is \( (m + t) \)-pseudo-coherent; and since \( m \) is arbitrary, therefore \( Rf_*E \) is pseudo-coherent. \( \square \)

From [Lp, 4.3.2(ii)] we get:

**Corollary 4.3.3.** Every quasi-perfect map is quasi-proper.

Next, we deduce “stability” of quasi-properness.

**Proposition 4.4.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

be a tor-independent square. If \( f \) is quasi-proper then so is \( g \).

**Proof.** Since pseudo-coherence is a local property, it suffices to prove the Proposition when \( Y' \) is affine and \( u(Y') \) is contained in an affine subset of \( Y \). So we can assume that \( u = u'u'' \) where \( u' \) is an open immersion and \( u'' \) is an affine map. It follows that it suffices to prove the Proposition (a) when \( u \)—hence \( v \)—is an open immersion and (b) when \( u \)—hence \( v \)—is an affine map (see [GI, p. 358, (9.1.16)(iii), (9.1.17)].)

In either of these two cases, it holds that

\((*)\) if \( S \) is as in Theorem 4.1 then \( Lv^*S \) is a generator of \( D_{\mathsf{qc}}(X') \).

Indeed, in case \( v \) is an open immersion and \( 0 \neq E \in D_{\mathsf{qc}}(X) \) (since \( E \cong v^*Rv_*E \)) and the same holds in case \( v \) is affine, by [Lp, (3.10.2.2)]. So in either case, for some \( n \),

\[
0 \neq \operatorname{Hom}_{D_{\mathsf{qc}}(X)}(S[n], Rv_*E) \cong \operatorname{Hom}_{D_{\mathsf{qc}}(X')}(Lv^*S[n], E),
\]

proving (*).
It is easy to see that the complex $L^v S$ is perfect. So by Corollary 4.3.2 to prove the Proposition for $u$ as in (*) it suffices to show that $Rg_*L^v S$ is pseudo-coherent. But by [Lp] (3.10.3), $Rg_*L^v S \cong L^u Rf_* S$; and since $Rf_* S$ is pseudo-coherent, therefore, by [I] p. 111, 2.16.1, so is $L^u Rf_* S$.

5. Proofs of Theorems 4.1 and 4.2

Heavy use will be made of the following technical notion.

**Definition 5.1.** Let $\mathcal{T}$ be a triangulated category, and let $S \subset \mathcal{T}$ be a class of objects. Let $m \leq n$ be integers. The full subcategory $S[m, n] \subset \mathcal{T}$ is the smallest among (= intersection of) all full subcategories $S \subset \mathcal{T}$ such that:

(i) $0$ is contained in $S$.
(ii) If $E \in S$, then $E[-\ell] \in S$ for all integers $\ell$ in the interval $[m, n]$.
(iii) For any $\mathcal{T}$-triangle $E \rightarrow F \rightarrow G \rightarrow E[1]$,
if $E$ and $G$ are in $S$ then so is $F$.

**Remark 5.2.** One checks that $S[m, n] = \left( \bigcup_{\ell=m}^{n} S[-\ell] \right)[0, 0]$.

**Remark 5.3.** Defn. 5.1 expands to allow $m = -\infty$ or $n = \infty$. For example, $S[m, \infty) := \bigcup_{n=m}^{\infty} S[m, n]$. Furthermore, $S(\infty, \infty) := \bigcup_{m \leq n} S[m, n]$, being closed under translation (see 5.3.1), is the smallest triangulated subcategory of $\mathcal{T}$ containing $S$ [N2, p. 60, Defn. 1.5.1].

**Remark 5.4.** The following are easy observations.

(i) If $E \in S[m, n]$ and $j \in \mathbb{Z}$ then $E[-j] \in S[m + j, n + j]$.
Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $S \subset \mathcal{T}$ whose objects are those $E \in S[m, n]$ such that $E[-j] \in S[m + j, n + j]$.
One deduces that, with $S[m, n]_o$ the class of objects in $S[m, n]$,

$$(S[m, n]_o)[m', n'] = S[m + m', n + n'].$$

(ii) If every object of $S$ is compact, then so is every object of $S[m, n]$.
Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $S \subset \mathcal{T}$ whose objects are those $E \in S[m, n]$ which are compact.

(iii) Let $A$ be an abelian category, and $H : \mathcal{T} \rightarrow A$ a cohomological functor, see [N2] p. 32, 1.1.9. If for every object $F \in S$ we have $H(F[-i]) = 0$ for all $i$ in some interval $[a, b]$, then for all $E \in S[m, n]$,

$$H(E[-j]) = 0 \text{ for all } j \in [a - m, b - n].$$

Indeed, (i), (ii) and (iii) in 5.1 hold for the full subcategory $S \subset \mathcal{T}$ whose objects are those $E \in S[m, n]$ which satisfy 5.11.
(iv) Let $\phi: \mathcal{T} \to \mathcal{T}'$ be a triangle-preserving additive functor [Lp] §1.5. Then

$$\phi(S[m, n]) \subset \{ \phi S \}[m, n].$$

Indeed, (i), (ii) and (iii) hold for the full subcategory $S \subset \mathcal{T}$ whose objects are those $E \in S[m, n]$ such that $\phi E \in \{ \phi S \}[m, n]$.

(v) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be morphisms inside $\mathcal{T}$-triangles

$$\begin{align*}
E &\longrightarrow A \longrightarrow B \longrightarrow E[1], \\
F &\longrightarrow A \longrightarrow B \longrightarrow E[1], \\
G &\longrightarrow B \longrightarrow C \longrightarrow G[1].
\end{align*}$$

If $E$ and $G$ are in $S[m, n]$ then so is $F$.

Indeed, the octahedral axiom [N2] p. 60, 1.4.7 produces a triangle

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1].$$

Example 5.5. Remark 5.4(iii) will be used thus. Let $G$ be an object of $\mathcal{T}$, and $H$ the cohomological functor $H(\_):=\text{Hom}(\_ , G)$, see [N2] p. 33, 1.1.11. Then for $a=m$ and $b=n$ the assertion becomes:

If for every object $F \in S$ we have $\text{Hom}(F[-i], G)=0$ for all $i \in [m,n]$, then $\text{Hom}(E,G)=0$ for all $E \in S[m,n]$.

A key role in the proofs will be played by Koszul complexes.

Example 5.6. Let $R$ be a commutative ring, $(f_1, f_2, \ldots, f_r)$ a sequence in $R$, and $(n_1, n_2, \ldots, n_r)$ a sequence of positive integers. The associated Koszul complex (see, e.g., [EGA] III, (1.1.1)) is

$$K_\ast(f_1^{n_1}, \ldots, f_r^{n_r}) := \bigoplus_{i=1}^r K_\ast(f_i^{n_i}),$$

where $K_\ast(f_i^{n_i})$ is $R \xrightarrow{f_i^{n_i}} R$ in degrees $-1$ and $0$, and $0$ elsewhere.

For $r=0$, set $K_\ast(\phi) := R$. For all $r \geq 0$, $K_\ast(f_1^{n_1}, \ldots, f_r^{n_r})$ is a complex with perfect amplitude in $[-r, 0]$, and homology killed by each $f_i^{n_i}$.

For any complex $E$, and $f \in R$, $K_\ast(f \otimes E)$ is the mapping cone of the endomorphism “multiplication by $f$” of $E$. Thus for $1 \leq i < r$, $K_\ast(f_i^{n_i}, \ldots, f_r^{n_r})$ is the mapping cone of the endomorphism “multiplication by $f_i^{n_i}$” of the complex $K_\ast(f_i^{n_i+1}, f_i^{n_i+2}, \ldots, f_r^{n_r})$. It follows that

$$K_\ast(f_1^{n_1}, f_2^{n_2}, \ldots, f_r^{n_r}) \in \{ K_\ast(f_1, f_2, \ldots, f_r) \} [0, 0].$$

This is shown by a straightforward induction, based on application of 5.4(v) to the following three natural triangles (where $\top$ signifies “omit,”):

$$K_\ast(f_1^{n_1}, \ldots, f_i^{n_i}, f_{i+1}, \ldots, f_r) \longrightarrow$$

$$K_\ast(f_1^{n_1}, \ldots, f_{i+1}^{n_i}, f_{i+2}, \ldots, f_r)[1] \xrightarrow{f_i^{n_i}} K_\ast(f_1^{n_1}, \ldots, f_{i+1}^{n_i}, f_{i+2}, \ldots, f_r)[1] \xrightarrow{+}$$
The proofs of Theorems 4.1 and 4.2 will involve induction on the number of affine open subschemes needed to cover $X$. One needs to begin with some results on affine schemes.

In the situation of Example 5.6, denote the sequence $(f^n_1, \ldots, f^n_r)$ by $F^n$, omitting the superscript “$n$” when $n = 1$. Let $C_\bullet(f^n)$ be the cokernel of that map of complexes $R[-1] \to K_\bullet(f^n)[-1]$ which is the identity map of $R$ in degree 1. The complex $C_\bullet(f^n)$ has perfect amplitude in $[1 - r, 0]$; and there is a natural homotopy triangle

$$C_\bullet(f^n) \longrightarrow R \longrightarrow K_\bullet(f^n) \longrightarrow C_\bullet(f^n)[1].$$

There is a map of complexes $K_\bullet(f^{n+m}) \to K_\bullet(f^n)$ ($f \in R$; $m, n > 0$) depicted by

$$\begin{align*}
R &\longrightarrow R \\
\quad f^{n+m} &\quad \quad f^n \\
\quad R &\quad \quad f^m \quad \quad R
\end{align*}$$

Tensoring such maps gives a map $K_\bullet(f^{n+m}) \to K_\bullet(f^n)$, and hence a map $C_\bullet(f^{n+m}) \to C_\bullet(f^n)$. For any $R$-complex $E$, we have then the Čech complex

$$\check{C}^\bullet(f, E) := \lim_{n \to \infty} \text{Hom}_R^\bullet(C_\bullet(f^n), E).$$

Let $U$ be the complement of the closed subscheme $\text{Spec}(R/fR) \subset \text{Spec}(R)$, with inclusion $i: U \hookrightarrow X$. From [EGA III, §1.3] it follows readily that, with $E^\sim$ the quasi-coherent complex corresponding to $E$, $R_\bullet i^* E^\sim$ is naturally $D$-isomorphic to the sheafified Čech complex $\check{C}^\bullet(f, E) := C^\bullet(f, E)^\sim$.

In particular, if the homology of $E$ is $fR$-torsion (i.e., for all $i \in \mathbb{Z}$, each element of $H_i(E)$ is annihilated by a power of $fR$) — or equivalently, if $i^* E^\sim$ is exact — then $\check{C}^\bullet(f, E)$ is exact; and since the complex $C_\bullet(f^n)$ is bounded and projective, therefore

$$H^0 \text{Hom}_R^\bullet(C_\bullet(f^n), E) \cong H^0 \text{RHom}_R^\bullet(C_\bullet(f^n), E) \cong \text{Hom}_D(R)(C_\bullet(f^n), E),$$

whence

$$\lim_{n \to \infty} \text{Hom}_D(R)(C_\bullet(f^n), E) \cong H^0 \check{C}^\bullet(f, E) = 0.$$
show that for any \( \lambda \in \text{Hom}_{D(P)}(R, E) \) there is an \( n > 0 \) such that \( \lambda \) factors through a \( D(R) \)-morphism \( K_\bullet(f^n) \to E \).

If \( P \) is a bounded complex of finitely generated projective \( R \)-modules, then the homology of \( \text{Hom}_R^\bullet(P, E) \) is still \( fR \)-torsion (as one sees, e.g., by induction on the number of nonvanishing components of \( P \); and replacing \( E \) in what precedes by \( \text{Hom}_R^\bullet(P, E) \), one obtains, via Hom-\( \otimes \) adjunction, that for any \( \lambda \in \text{Hom}_{D(R)}(P, E) \) there is an \( n > 0 \) such that \( \lambda \) factors through a \( D(R) \)-morphism \( K_\bullet(f^n) \otimes P \to E \).

**Lemma 5.7.** Let \( E \) be an \( R \)-complex such that \( H^j(E) \) is \( fR \)-torsion for all \( j \geq -r \), \( P \) an \( R \)-complex with perfect amplitude in \([0, b]\) for some \( b \geq 0 \), and \( \lambda \in \text{Hom}_{D(R)}(P, E) \). Then there is an integer \( n > 0 \) and a homomorphism of \( R \)-complexes \( \lambda_n: K_\bullet(f^n) \otimes P \to E \) such that for all \( j \geq -r \), the homology map \( H^j(\lambda): H^j(P) \to H^j(E) \) factors as

\[
H^j(P) = H^j(R \otimes P) \xrightarrow{\text{natural}} H^j(K_\bullet(f^n) \otimes P) \xrightarrow{H^j(\lambda_n)} H^j(E).
\]

**Proof.** We may assume that \( P \) is a complex of finitely generated projective \( R \)-modules, vanishing in all degrees outside \([0, b]\), see [1, p.175, b)]. Let \( \tau_{\geq -r}E \) be the usual truncation, and \( \pi: E \to \tau_{\geq -r}E \) the natural map, which induces homology isomorphisms in all degrees \( \geq -r \) (see, e.g., [Ep, §1.10]). By the preceding remarks, \( \pi \lambda \) factors in \( D(R) \) as

\[
P = R \otimes P \xrightarrow{\text{natural}} K_\bullet(f^n) \otimes P \xrightarrow{\lambda_n} \tau_{\geq -r}E.
\]

Since \( K_\bullet(f^n) \otimes P \) is bounded and projective, we may assume that \( \lambda_n \) is a map of \( R \)-complexes. Then the \( R \)-homomorphism

\[
(\lambda_n)^{-r}: (K_\bullet(f^n) \otimes P)^{-r} = P^0 \to (\tau_{\geq -r}E)^{-r} = \text{coker}(E^{-r-1} \to E^{-r})
\]

lifts to a map \( P^0 \to E^{-r} \), giving a map \( \lambda_n \) with the desired properties. \( \square \)

**Corollary 5.7.1.** Set \( I := fR = (f_1, f_2, \ldots, f_r)R \). Let \( m \in \mathbb{Z} \) and let \( E \) be an \( R \)-complex such that \( H^i(E) \) is \( I \)-torsion for all \( i \geq m - r \).

(i) If \( E \) is \( m \)-pseudocoherent, and \( p \geq m \) is such that \( H^i(E) = 0 \) for all \( i > p \), then there exists in the homotopy category of \( R \)-complexes an \( m \)-quasi-isomorphism \( P \to E \) with \( P \in \{K_\bullet(f)\}[m, p] \).

(ii) For any \( i \geq m \) with \( H^i(E) \neq 0 \), there is a nonzero map \( K_\bullet(f)[-i] \to E \).
Proof. (i) By [1] p.103, 2.10(b)], $H^p(E)$ is a finitely generated $R$-module. So there is an $\ell > 0$ and a surjective homomorphism $R^\ell \to H^p(E)$, which lifts to $R^\ell \to \ker(E^p \to E^{p+1})$, and thus there is a homomorphism $R^\ell \to E[p]$, or equivalently, $\lambda: R^\ell[m \to p] \to E[m]$, giving rise, by Lemma 5.7 to an $R$-homomorphism $\lambda_n[-m]: P_1 := (K_\bullet(f^n) \otimes R^\ell[-p]) \to E$ such that $H^p(\lambda_n[-m])$ is surjective. By Example 5.6 and Remark 5.4(i), we have $K_\bullet(f^n)[-p] \in \{K_\bullet(f)\}[p, p]$. So we get a homotopy triangle $P_1 \longrightarrow E \longrightarrow P_1[1]$ with $P_1 \in \{K_\bullet(f)\}[p, p]$ and $H^i(Q_1) = 0$ for all $i \geq p$, giving (i) when $p = m$.

In any case, $Q_1$ is $m$-pseudocoherent [1] p.100, 2.6; and since all the homology of $P_1$ is $I$-torsion, the exact homology sequence of the preceding triangle shows that $H^i(Q_1)$ is $I$-torsion for all $i \geq m - r$. If $m < p$ then, using induction on $p - m$, one may assume that there is a homotopy triangle $P_2 \longrightarrow Q_1 \longrightarrow P_2[1]$ with $P_2 \in \{K_\bullet(f)\}[m, p - 1] \subset \{K_\bullet(f)\}[m, p]$ and $H^i(Q) = 0$ for all $i \geq m$. There exists then a homotopy triangle $P \longrightarrow E \longrightarrow P[1]$ which, by Remark 5.4(v), is as desired.

(ii) There is, by assumption, a nonzero map $R \to H^i(E)$, which lifts to a map $R \to \ker(E^i \to E^{i+1})$; and so there is a nonzero map $\lambda: R \to E[i]$ with $H^i(\lambda) \neq 0$. If $j \geq -r$ then $j + i \geq i \geq m - r \geq m - r$, so $H^j(E[i]) = H^j(E)$ is $I$-torsion, whence by Lemma 5.7 there is for some $n > 0$ a nonzero map $K_\bullet(f^n) \to E[i]$. By Example 5.6 $K_\bullet(f^n) \in K_\bullet(f)[0, 0]$; so by Example 5.5 there is a nonzero map $K_\bullet(f) \to E[i]$, proving (ii).

For dealing with the nonaffine situation, we need to set up some notation.

**Notation 5.8.** A scheme $X$ can be covered by finitely many open affine subsets, say $X = \bigcup_{k=1}^t U_k$, with $U_k = \text{Spec}(R_k)$. For $1 \leq k \leq t$, set

(i) $V_k := \bigcup_{i=k}^t U_i$.

(ii) $Y_k := X - V_{k+1} := X$ when $k = t$.

So we have a filtration by closed subschemes $Y_1 \subset Y_2 \subset \cdots \subset Y_t = X$.

Both $U_k$ and $V_{k+1}$ are quasi-compact open subsets of the (quasi-separated) scheme $X$, whence so is $U_k \cap V_{k+1}$. So there is a sequence $f_k = \{f_{k1}, f_{k2}, \ldots, f_{kr_k}\}$ in $R_k$ such that

$U_k \cap V_{k+1} = \bigcup_{i=1}^{r_k} \text{Spec}(R_k[1/f_{ki}])$.
Set

(iii) $I_k := f_k R_k$, (so that $U_k \cap V_{k+1}$ is the complement of the closed subscheme $\text{Spec}(R_k/I_k) \subset U_k$).

(1) (iv) $C_k := (K_\bullet(f_k) \oplus K_\bullet(f_k)[1]) \sim = (K_\bullet(0, f_{k1}, f_{k2}, \ldots, f_{kr_k})) \sim$

with $K_\bullet(*)$ the Koszul complex over $R_k$ associated to the sequence $(*)$, and $(-)^\sim$ the sheafification functor from $R_k$-modules to quasi-coherent $O_{U_k}$-modules—so that $C_k$ is a perfect $O_{U_k}$-complex.

(The reason for introducing this $\oplus$ will emerge shortly.)

We have the cartesian diagram of (open) inclusion maps

$$
\begin{array}{ccc}
U_k \cap V_{k+1} & \xrightarrow{\nu} & V_{k+1} \\
\downarrow{\lambda} & & \downarrow{\xi} \\
U_k & \xrightarrow{\mu} & V_k = U_k \cup V_{k+1}
\end{array}
$$

The restriction $\lambda^* C_k$ is homotopically trivial, whence, in $D(V_{k+1})$,

$$
\xi^* R\mu_* C_k \cong R\nu_* \lambda^* C_k = 0.
$$

Thus, the restrictions of $R\mu_* C_k$ to both $V_{k+1}$ and $U_k$ are perfect, and so $R\mu_* C_k$ is itself perfect.

For any $O_{V_k}$-complex $G$, the obvious triangle

$$
G \xrightarrow{0} G \xrightarrow{\cdot} G \oplus G[1] \xrightarrow{} G[1]
$$

shows that the complex $G \oplus G[1]$ vanishes in the Grothendieck group $\mathcal{K}_0(V_k)$. Taking $G := R\mu_* (K_\bullet(f_k)) \sim$, we deduce then from Thomason’s localization theorem [11, p. 338, 5.2.2(a)] that the perfect $O_{V_k}$-complex $R\mu_* C_k$ is $D(V_k)$-isomorphic to the restriction of a perfect $O_X$-complex.

(v) Let $S_k \in D_{qc}(X)$ be a perfect $O_X$-complex whose restriction to $V_k$ is $D(V_k)$-isomorphic to $R\mu_* C_k$.

(vi) Let $S_k$ be the finite set $\{S_1, S_2, \ldots, S_k\}$.

According to Lemma 3.2, there is for each $k$ an integer $N_k > 0$ such that, if $Q \in D_{qc}(X)$ satisfies $H^j(Q) = 0$ for all $\ell \geq -N_k$ then $\text{Hom}_{D(X)}(S_k, Q) = 0$.

After enlarging $N_k$ if necessary, we have also that $\text{Hom}_{D(X)}(Q, S_k) = 0$.

Set

(vii) $N := \text{max}\{N_1, N_2, \ldots, N_t, r_1, r_2, \ldots, r_t\} + 1$.

Next comes the key statement.

**Proposition 5.9.** With the preceding notation, let $m, k \in \mathbb{Z}$, $1 \leq k \leq t$, let $E \in D_{qc}(X)$ be such that $H^j(E)$ is supported in $Y_k$ for all $j \geq m - kN$, and set

$$
a_k = \binom{k + 1}{2} N \quad (1 \leq k \leq t).
$$
(i) If $E$ is $(m-(k-1)N)$-pseudo-coherent then there is an $m$-isomorphism $P \to E$ with $P \in S_k[m-a_k, \infty)$ (so that $P$ is perfect, see 5.4(ii)).

(ii) If $H^i(E) \neq 0$ for some $\ell \geq m$, then for some $i \geq m - a_k$ and some $j \in [1, k]$, there is a nonzero map $S_j[-i] \to E$.

Before proving this, let us see how to derive Theorems 4.1 and 4.2. Since $Y_t = X$, Theorem 4.1 with $B := (t - 1)N$ is contained in 5.9(ii).

Next, 5.9(ii) with $k = t$ shows that if $H^i(E) \neq 0$, then there exist integers $i \geq \ell - a_t$ and $j \in [1, t]$, and a non-zero map $S_j[-i] \to E$. This gives Theorem 4.2 for the specific choices

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_t, \quad A(S) := a_t = \left(\frac{t + 1}{2}\right)N.$$

The rest of Theorem 4.2 results from the following general fact, applied to $\mathcal{H} = \{ E \in D_{qc}(X) \mid H^i(E) \neq 0 \}$, $\mathcal{T} = D_{qc}(X)$ and $A = A(S) - \ell$.

**Proposition 5.10.** Let $\mathcal{T}$ be a triangulated category with coproducts. Let $\mathcal{H}$ be a collection of objects of $\mathcal{T}$. Suppose there exists a compact generator $S \in \mathcal{T}$ and an integer $A$ such that

$$E \in \mathcal{H} \implies \text{Hom}(S^n, E) \neq 0 \text{ for some } n \leq A.$$

Then every compact generator has a similar property: for each compact generator $S' \in \mathcal{T}$ there is an integer $A'$ such that

$$E \in \mathcal{H} \implies \text{Hom}(S'^n, E) \neq 0 \text{ for some } n \leq A'.$$

**Proof.** Let $\tilde{\mathcal{H}}$ be the full subcategory of $\mathcal{T}$ whose objects are all the direct summands of objects in $\{S'\}(-\infty, \infty) = \bigcup_{M \geq 0} \{S'\}[-M, M]$ (see Remark 5.3).

As in the proof of Corollary 4.2, (iii) $\implies$ (ii), one sees that $S \in \tilde{\mathcal{H}}$, i.e., there is an $S^* \in \tilde{\mathcal{H}}$ and an $M \geq 0$ such that $S \oplus S^* \in \{S'\}[-M, M]$.

Now if $E \in \mathcal{H}$ then, since $\text{Hom}(S[k], E) \neq 0$ for some $k \leq A$, and $S[k] \oplus S^*[k] \in \{S'\}[-M - k, M - k]$ (Remark 5.3(i)), therefore Example 5.5 gives $\text{Hom}(S'[n], E) \neq 0$ for some $n$ with $n \leq M + k \leq M + A$. \(\square\)

It remains to prove Proposition 5.9 which we do now by induction on $k$.

For $k = 1$, $a_1 = N$, so $H^i(E)$ is supported in $Y_1 = \text{Spec}(R_1/I_1)$ for all $j \geq m - r_1 - 1 \geq m - N$. As usual, when considering the restriction $E|_{U_1}$ we may assume it to be a quasi-coherent complex, then relate facts about it to facts about the corresponding complex $E$ of $R_1$-modules. For example, it holds that $H^k(E)$ is $I_1$-torsion for all $i \geq m - r_1 - 1$.

Thus, from Corollary 5.7(i), applied to $I_1 = (0, f_{11}, f_{22}, \cdots, f_{r_1})R_1$, it follows via 5.8(iv) that, if $E$ is $m$-pseudo-coherent then there exists an $m$-isomorphism $P \to E|_{U_1}$ with $P \in \{C_1\}[m, \infty)$. Likewise (and more easily), Corollary 5.7(ii) gives that if $H^i(E) \neq 0$ for some $i \geq m$—whence, $H^i(E)$ being supported in $Y_1 \subset U_1$, $H^i(E|_{U_1}) \neq 0$—then there is a nonzero map

$$C_1[-i] = (K_\bullet(f_1)[-i])^\sim \oplus (K_\bullet(f_1)[-i + 1])^\sim \to E|_{U_1}. $$
Lemma 5.11. Let $U$ and $V$ be open subsets of a scheme $X$, and let
\[
U \cap V \xrightarrow{\nu} V \\
\lambda \downarrow \qquad \downarrow \xi \\
U \xrightarrow{\mu} U \cup V
\]
be the natural diagram of inclusion maps. Let $C \in \mathcal{D}(U)$ satisfy $\lambda^* C = 0$. Let $E \in \mathcal{D}(U \cup V)$. Then:

(i) Every $\mathcal{D}(U)$-morphism $C \to \mu^* E$ extends uniquely to a $\mathcal{D}(U \cup V)$-morphism $R_{\mu_*} C \to E$.

(ii) If $C$ is perfect then so is $R_{\mu_*} C$.

(iii) If $S \subset \mathcal{D}(U)$ and $m \leq n \in \mathbb{Z}$ then $R_{\mu_*}(S[m,n]) \subset \{ R_{\mu_*} S \}[m,n]$.

Proof. (i) In view of the natural isomorphisms
\[
\text{Hom}_{\mathcal{D}(U)}(C, \mu^* E) \cong \text{Hom}_{\mathcal{D}(U)}(\mu^* R_{\mu_*} C, \mu^* E) \cong \text{Hom}_{\mathcal{D}(U \cup V)}(R_{\mu_*} C, R_{\mu_*} \mu^* E)
\]
we need only show that the natural map is an isomorphism
\[
R \text{Hom}^* (R_{\mu_*} C, E) \xrightarrow{\sim} R \text{Hom}^* (R_{\mu_*} C, R_{\mu_*} \mu^* E)
\]
(to which we can apply the homology functor $H^0$). Thus for any triangle
\[
G \longrightarrow E \xrightarrow{\text{natural}} R_{\mu_*} \mu^* E \longrightarrow G[1]
\]
we’d like to see that $R \text{Hom}^* (R_{\mu_*} C, G) = 0$. But $\mu^* G = 0 = \lambda^* C$, so that $\mu^* R \text{Hom}^* (R_{\mu_*} C, G) \cong R \text{Hom}^* (\mu^* R_{\mu_*} C, \mu^* G) = 0$, and $\xi^* R \text{Hom}^* (R_{\mu_*} C, G) \cong R \text{Hom}^* (\xi^* R_{\mu_*} C, \xi^* G) \cong R \text{Hom}^* (R_{\nu_*} \lambda^* C, \xi^* G) = 0$, whence the conclusion.

(ii) Since both $\xi^* R_{\mu_*} C \cong R_{\nu_*} \lambda^* C = 0$ and $\mu^* R_{\mu_*} C = C$ are perfect, therefore so is $R_{\mu_*} C$.

(iii) This is a special case of Remark 5.4(iv). \[\square\]

Lemma 5.12. For $k > 1$, suppose Proposition 5E(ii) holds with $k - 1$ in place of $k$. Then for any $E \in \mathcal{D}_{qc}(X)$ and $\mathcal{D}(U_k)$-morphism
\[
\psi: F \longrightarrow E|_{U_k} \quad (F \in \{ C_k \}[m, \infty)),
\]
there exists a $\mathcal{D}(X)$-morphism
\[
\tilde{\psi}: \tilde{F} \longrightarrow E \quad (\tilde{F} \in S_k[m - N - a_{k-1}, \infty))
\]
whose restriction $\tilde{\psi}|_{U_k}$ is isomorphic to $\psi$. \[\square\]
Before proving this Lemma, let us see how it is used to establish the induction step in the proof of Proposition 5.9. With reference to that Proposition, we show, for $k > 1$:

1. Assertion (i) for $k - 1$ implies assertion (i) for $k$.
2. Assertions (i) and (ii) for $k - 1$, together, imply assertion (ii) for $k$.

To prove (1), let $E \in D_{qc}(X)$ be $(m - (k - 1)N)$-pseudocoherent, with $H^j(E)$ supported in $Y_k$ for all $j \geq m - kN$. Since $m - (k - 1)N - r_k \geq m - kN$, therefore (after replacement of $K_\bullet(f_k)^\sim$ by $C_k$, see above) Corollary 5.7.1 provides a $D(U_k)$-triangle

\[
\begin{align*}
P_k \longrightarrow & E|_{U_k} \longrightarrow Q_k \longrightarrow P_k[1] \\
\end{align*}
\]

with $P_k \in \{C_k\}[m - (k - 1)N, \infty)$ and $H^j(Q_k) = 0$ for all $j \geq m - (k - 1)N$. By Lemma 5.12, the map $P_k \longrightarrow E|_{U_k}$ is isomorphic to the restriction of a $D(X)$-morphism $\psi': P' \rightarrow E$, with $P' \in S_k[m - (k - 1)N - a_{k - 1}, \infty)$, i.e., since

\[
a_{k - 1} + kN = \binom{k}{2}N + kN = \binom{k + 1}{2}N = a_k,
\]

with $P' \in S_k[m - a_k, \infty)$. Any $D_{qc}(X)$-triangle

\[
\begin{align*}
P' \longrightarrow & E \longrightarrow Q' \longrightarrow P'[1] \\
\end{align*}
\]

restricts on $U_k$ to one isomorphic to (5.12.1). So when $j \geq m - (k - 1)N$, then $H^j(Q')$ vanishes on $U_k$; furthermore, $H^j(E)$ is supported on $Y_k$, and since all the members of $S_k$ are exact outside $Y_k$ therefore so is $P'$ (argue as in Remark 5.4(i)–(iv)); and thus $H^j(Q')$ is supported in $(Y_k \setminus U_k) = Y_{k - 1}$.

Moreover, $Q'$ is $(m - (k - 2)N)$-pseudocoherent, since both $P'$ and $E$ are $\mathbb{I}$ p. 100, 2.6. So now the inductive assumption produces a triangle

\[
\begin{align*}
P'' \longrightarrow & Q' \longrightarrow Q \longrightarrow P''[1] \\
\end{align*}
\]

with $P'' \in S_{k - 1}[m - a_{k - 1}, \infty)$, and $H^j(Q) = 0$ whenever $j \geq m$.

There is then a triangle

\[
\begin{align*}
P \longrightarrow & E \longrightarrow Q \longrightarrow P[1], \\
\end{align*}
\]

and the assertion 5.9(i), for the integer $k$, results from Remark 5.4(v).

As for (2), let $E$ satisfy the hypotheses of 5.9(ii) for $k$. If $H^i(E|_{U_k}) = 0$ for all $i \geq m - (k - 1)N$ then $H^j(E)$ is supported in $Y_{k - 1}$ for all $j \geq m - (k - 1)N$, $H^j(E)$ is non-zero for some $j \geq m$, and $m - a_{k - 1} \geq m - a_k$; so in this case assertion (ii) for $k$ is already given by assertion (ii) for $k - 1$.

If, on the other hand, $H^i(E|_{U_k}) \neq 0$ for some $i \geq m - (k - 1)N$, then, since $m - (k - 1)N - r_k \geq m - kN$, Corollary 5.7.1 (suitably modified) provides a nonzero map $C_k[-i] \rightarrow E|_{U_k}$. By Remark 5.4(i),

\[
C_k[-i] \in \{C_k\}[i, \infty) \subset \{C_k\}[m - (k - 1)N, \infty),
\]
so by Lemma 5.12 there exists a nonzero \( D(X) \)-morphism \( \tilde{F} \to E \) with

\[
\tilde{F} \in S_k[m - (k - 1)N - N - a_{k-1}, \infty) = S_k[m - a_k, \infty).
\]

Hence, by Example 5.5.5.9(ii) holds for \( k \).

We come finally to the proof of Lemma 5.12.

Let \( S \subset D(U_k) \) be the full subcategory with objects those \( F \in \{ C_k \}[m, \infty) \) for which the Lemma holds. We need to verify the conditions in Definition 5.1, i.e., we need to show:

(a) \( C_k[-\ell] \in S \) for all \( \ell \geq m \); and

(b) for any \( D(U_k) \)-triangle

\[
F' \longrightarrow F \longrightarrow F'' \longrightarrow F'[1],
\]

if \( F', F'' \in S \) then \( F \in S \).

For (a), we first use Lemma 5.11 to extend \( \psi: C_k[-\ell] \to E|_{U_k} \) to a \( D(V_k) \)-morphism \( \phi: S_k[-\ell]|_{V_k} \to E|_{V_k} \). By Thomason’s localization theorem, as formulated in [N2, p. 214, 2.1.5] (and further elucidated in [ibid., p. 216, proof of Lemma 2.6]) there is then a \( D_{qc}(X) \)-diagram, with top row a triangle of perfect complexes:

\[
\begin{array}{c}
\overline{P} \longrightarrow \overline{F}_1 \longrightarrow S_k[-\ell] \longrightarrow \overline{P}[1] \\
\downarrow g \\
E
\end{array}
\]

and with \( \overline{P} \) exact on \( V_k \), so that \( f|_{V_k} \) is an isomorphism; and furthermore,

\[
\phi = (g|_{V_k}) \circ (f|_{V_k})^{-1}.
\]

Since \( S_k[-\ell] \in S_k[\ell, \infty] \) (see Remark 5.4(i)), we need only show that we can choose \( \overline{P} \in S_{k-1}[\ell - N - a_{k-1}, \infty) \), because then we’ll have

\[
\overline{F}_1 \in S_k[\ell - N - a_{k-1}, \infty) \subset S_k[m - N - a_{k-1}, \infty).
\]

The perfect complex \( \overline{P} \) is exact outside \( X - V_k = Y_{k-1} \), and we are assuming that 5.9(i) is true for \( k - 1 \). It follows that there exists a triangle

\[
P \longrightarrow \overline{P} \longrightarrow Q \longrightarrow P[1]
\]

with \( P \in S_{k-1}[\ell - N - a_{k-1}, \infty) \) and \( H^i(Q) = 0 \) for all \( i \geq \ell - N \). Since all the members of \( S_{k-1} \) are exact on \( V_k \), the same is true of \( P \) (argue as in Remark 5.4(i)-(iv)).

\[5\] where in the absence of separatedness, \( j_* \) should become \( Rj_* \).
Now [N2, p. 58, 1.4.6] produces an octahedron on $P \to \tilde{P} \to \tilde{F}_1$, where the rows and columns are triangles:

\[
\begin{array}{cccccc}
P & \rightarrow & F_1 & \rightarrow & S_k[-\ell] & \rightarrow & \tilde{P}[1] \\
\downarrow & \beta & \downarrow & \downarrow & \downarrow \\
Q & \rightarrow & F' & \rightarrow & S_k[-\ell] & \rightarrow & Q[1] \\
\downarrow & \downarrow & \downarrow & h \\
P[1] & \rightarrow & P[1] \\
\end{array}
\]

Since $H^i(Q[1 + \ell]) = 0$ for all $i \geq -N - 1$, the definition of $N$ (see Notation 5.8(vii)) forces the map $g: S_k[-\ell] \to Q[1]$ to vanish. The exact sequence

\[
\text{Hom}(S_k[-\ell], F') \xrightarrow{\text{via } f'} \text{Hom}(S_k[-\ell], S_k[-\ell]) \xrightarrow{\text{via } g = 0} \text{Hom}(S_k[-\ell], Q[1])
\]

shows there is a map $\iota: S_k[-\ell] \to F'$ with $f'\iota$ the identity map of $S_k[-\ell]$.

This gives rise to yet another octahedron, on $S_k[-\ell] \xrightarrow{\iota} F' \xrightarrow{h} P[1]$:

\[
\begin{array}{cccccc}
P & \rightarrow & \tilde{F} & \rightarrow & \tilde{F}_1 & \rightarrow & Q \rightarrow \tilde{F}[1] \\
\downarrow & \gamma & \downarrow & \downarrow & \downarrow \\
S_k[-\ell] & \rightarrow & F' & \rightarrow & Q & \rightarrow & S_k[-\ell + 1] \\
\downarrow & \downarrow & \downarrow & h \\
P[1] & \rightarrow & P[1] \\
\end{array}
\]

The first column is a triangle, with $\tilde{F} \in S_k[\ell - N - a_{k-1}, \infty)$, and $P|_{V_k}$ exact, so that $\alpha|_{V_k}$ is an isomorphism.

Moreover, if $\psi: \tilde{F} \to E$ is the composite $\tilde{F} \xrightarrow{\gamma} \tilde{F}_1 \xrightarrow{g} E$, then

\[
\alpha = f'\iota \alpha = f'\beta \gamma = f \gamma,
\]

so that on $V_k$,

\[
\phi \alpha = \phi f \gamma = g \gamma = \tilde{\psi},
\]

proving (a).
Proof of (b).

Let \( \psi: F \to E|_{U_k} \) be a \( D(U_k) \)-morphism. Since \( F' \in S \), there exists a complex \( \tilde{F}' \in S_k[m-N-a_{k-1}, \infty) \) and a \( D(X) \)-morphism \( \tilde{\psi}': \tilde{F}' \to E \) whose restriction to \( U_k \) is isomorphic to the composite \( F' \to F' \to E|_{U_k} \). There results a triangle

\[
\tilde{F}' \xrightarrow{\tilde{\psi}'} E \xrightarrow{\gamma'} E' \to \tilde{F}'[1] ,
\]

and hence a commutative \( D(U_k) \)-diagram (part of an octahedron):

\[
\begin{array}{cccc}
F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & F'[1] \\
\| & \downarrow{\psi} & \| & \downarrow{g} & \| \\
F'|_{U_k} & \longrightarrow & E|_{U_k} \xrightarrow{\gamma'|_{U_k}} E'|_{U_k} & \longrightarrow & F'[1] \\
\| & \downarrow{\chi} & & & \downarrow{h} & \\
G & = & G & \longrightarrow & \\
\end{array}
\]

Since \( F'' \in S \), there is an \( \tilde{F}'' \in S_k[m-N-a_{k-1}, \infty) \) and a \( D(X) \)-morphism \( \tilde{\psi}'': \tilde{F}'' \to E' \) whose restriction to \( U_k \) is isomorphic to \( g : F'' \to E'|_{U_k} \). So there is a triangle

\[
\tilde{F}'' \xrightarrow{\tilde{\psi}''} E' \xrightarrow{\gamma''} E'' \to \tilde{F}''[1] ;
\]

whose restriction to \( U_k \) is isomorphic to

\[
F'' \xrightarrow{\gamma'} E'|_{U_k} \xrightarrow{h} G \longrightarrow F''[1] .
\]

The restriction to \( U_k \) of the composite \( E \xrightarrow{\gamma'} E' \xrightarrow{\gamma''} E'' \) is isomorphic to the composite \( \chi : E|_{U_k} \xrightarrow{\gamma'|_{U_k}} E'|_{U_k} \xrightarrow{h} G \). Completing \( \gamma'' \gamma' \) to a triangle

\[
\tilde{F} \xrightarrow{\tilde{\psi}} E \xrightarrow{\gamma'' \gamma'} E'' \longrightarrow \tilde{F}[1]
\]

and restricting to \( U_k \), we obtain a triangle isomorphic to

\[
F \xrightarrow{\psi} E|_{U_k} \xrightarrow{\chi} G \longrightarrow \tilde{F}[1] .
\]

That \( \tilde{F} \in S_k[m-N-a_{k-1}, \infty) \) follows from Remark 5.4(v). □
References

[BB] A. Bondal, M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Moscow Math. J.* 3 (2003), 1–36.

[BN] M. Bökstedt, A. Neeman, Homotopy limits in triangulated categories, *Compositio Math.* 86 (1993), 203–234.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique II*, Publ. Math. IHES 8, 1961; *III*, Publ. Math. IHES 11, 1961; *IV*, Publ. Math. IHES 24, 1965.

[GD] ———, *Éléments de Géométrie Algébrique I*, Springer-Verlag, New York, 1971.

[H] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math., no. 20, Springer-Verlag, New York, 1966.

[I] L. Illusie, Généralités sur les conditions de finitude dans les catégories dérivées, etc., in *Théorie des Intersections et Théorème de Riemann-Roch (SGA 6)*, Lecture Notes in Math., no. 225, Springer-Verlag, New York, 1971, pp. 78–296.

[Kf] G. R. Kempf, Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves, *Rocky Mountain J. Math* 10 (1980), 637–645.

[Ki] R. Kiehl, Ein “Descente”-Lemma und Grothendiecks Projektionssatz für nicht-noethersche Schemata, *Math. Annalen* 198 (1972), 287–316.

[Lp] J. Lipman, *Notes on Derived Functors and Grothendieck Duality*, preprint, available at [http://www.math.purdue.edu/~lipman](http://www.math.purdue.edu/~lipman).

[N1] A. Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, *J. Amer. Math. Soc.* 9 (1996), 205–236.

[N2] ———, *Triangulated Categories*, Annals of Math. Studies, no. 148, Princeton University Press, Princeton, N.J., 2001.

[TT] R. W. Thomason, T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in *The Grothendieck Festschrift* Vol. III, Progr. Math., no. 88, Birkhäuser, Boston, 1990, pp. 247–435.

[V] J.-L. Verdier, Base change for twisted inverse image of coherent sheaves, in *Algebraic Geometry (Bombay, 1968)*, Oxford Univ. Press, London, 1969, pp. 393–408.

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