A Unified Approach to Solvable Models of Dilaton Gravity in Two-Dimensions Based on Symmetry

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Abstract

A large class of solvable models of dilaton gravity in two space-time dimensions, capable of describing black hole geometry, are analyzed in a unified way as non-linear sigma models possessing a special symmetry. This symmetry, which can be neatly formulated in the target-space-covariant manner, allows one to decompose the non-linearly interacting dilaton-gravity system into a free field and a field satisfying the Liouville equation with in general non-vanishing cosmological term. In this formulation, all the existent models are shown to fall into the category with vanishing cosmological constant. General analysis of the space-time structure induced by a matter shock wave is performed and new models, with and without the cosmological term, are discussed.

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1 Introduction

Everyone would agree that quantum physics of black holes is one of the most fascinating and challenging subjects in theoretical physics. It is the arena in which the two fundamental principles of modern physics, namely quantum mechanics and general relativity, are deeply interwoven to produce effects which defy our conventional wisdom. The ultimate fate of a black hole emitting Hawking radiation, the possible loss of quantum coherence across the horizon, and the statistical meaning of the black hole entropy are among such conundrums. What makes these questions difficult to answer is, among other things, that they require us to treat the matter and the gravitational degrees of freedom on the same footing, including their quantization. Unfortunately, the present state of the art is not advanced enough to do this in a realistic four dimensional setting and we need to resort to some simplified models in low dimensions.

Even in low dimensions, the task is far from trivial. Although interesting in many respects, non-linear sigma model description of string theory with a black hole in the target space [1, 2] is not capable of describing the all important back reaction. Recent attempts using suitable matrix models [3] should in principle be free of such limitations but there the space-time interpretation presents difficulty.

Along a more conventional field theoretical line, an attractive string-inspired dilaton gravity model was presented by Callan, Giddings, Harvey and Strominger (CGHS) [4] and has aroused a lot of interest as it contains black hole solutions and is classically exactly solvable [5]-[11]. With this model, the authors showed that one can partly incorporate the effects of back reaction in the limit of large number $N$ of matter fields in the form of anomaly-induced effective action and demonstrated the existence of Hawking radiation. The problem was, however, that this large $N$ approximation broke down for small black hole mass and hence the fate of the black hole could not be traced to the end. Furthermore, since the dilaton-gravity sector was not really quantized, the question of the possible loss of quantum coherence could not be addressed. It was clear that some solvable extensions or modifications were called for.

Since then, a number of such solvable models of CGHS type have been devised and analyzed from various points of view [12], [13], [14], [15]-[17], [18], [19]-[24]. In addition to being exactly solvable at the semiclassical level, these models are in principle amenable to full quantum treatment albeit with certain caveat about the range of the fields and the
choice of the functional measure. Indeed for the simplest of these models, exact quantization has been performed and some physical consequences have been extracted \[15\]-\[17], \[18\]. Although satisfactory answers to the aforementioned questions concerning the fate of the black hole etc. are yet to be sought for, these developments have demonstrated the usefulness of such solvable models of dilaton gravity. (For pedagogical reviews of the CGHS model and the subsequent developments, see \[25\], \[26\], \[27\].)

The main objective of the present article is to characterize and clarify the structure of a large class of such solvable models from a unified point of view. As has been emphasized in the literature \[23\], \[12\], there is an immense freedom in the possible form of the action for dilaton gravity models in two dimensions due to the dimensionless nature of the dilaton field. Namely, when one takes into account the effects of functional measure and renormalization, the general form of the action can take that of a non-linear sigma model formally similar to the string theory in curved space. Of course in order to be regarded as models of dilaton gravity, they must satisfy such requirements as two-dimensional general covariance and conformal invariance, but this still allows for infinite possibilities for the choice of the form of the “target space metric”, the coupling to the background scalar curvature etc.. This freedom is also reflected in the variety of solvable models proposed.

What we wish to do is to capture the characteristic feature common to all such solvable models and thereby treat them in a systematic way.

A conspicuous feature of the solvable models so far proposed is that by some field transformations one can reduce the models down to those of massless free fields. These transformations are however generally quite complicated and require certain amount of ingenuity to be discovered. What we shall show in this paper is that behind the existence of such transformations and hence the solvability of the models is in fact a powerful symmetry, which is an extensive generalization of the one noted in the specific model of Russo, Susskind and Thorlacius \[14\]. This symmetry, which can be neatly formulated in the target-space covariant manner in the general non-linear sigma model formulation mentioned above, guarantees the existence of at least one free field, which we call $F(x)$, in the dilaton-gravity sector, and further dictates that the field orthogonal to $F(x)$, which will be called $V(x)$, satisfies a Liouville equation with in general non-vanishing cosmological constant. It turned out that all the solvable models so far proposed fall into this scheme with vanishing cosmological constant. This means that not only do we have a unified framework to discuss existent models but also are able to treat a new class of solvable
models with a genuine Liouville type field. Indeed, after deriving some formulas to study the space-time structure induced by a matter shock wave in a general manner, we shall analyze new models, with and without the cosmological term.

Let us briefly indicate the organization of the rest of this article: In Sec.2, after specifying the non-linear sigma model we deal with, we describe the symmetry which allows us to decompose the system into a free field and a Liouville field. Condition of conformal invariance further determines the structure of the background charge terms and the system is completely solved in terms of two free fields. In Sec.3, we apply this formalism to a general class of dilaton gravity models. The condition of solvability following from the aforementioned symmetry takes the form of a differential equation relating various functions specifying the model and it is solved to give an explicit relation among them. From this formula we immediately see that existing models are realized as special cases of our general treatment. We then go on in Sec.4 to investigate the space-time structure of the models induced by a matter shock wave. By studying the singularity structure and the asymptotic form of the scalar curvature, we are lead to analyze a number of new models which exhibit a variety of physical behavior. Finally, Sec.5 is devoted to discussions concerning full quantization and on possible implication of our model as a model of string theory.

2 Sigma Model with a Symmetry

2.1 The Action

As was recalled in the introduction, due to the ambiguities in the choice of the functional measure and to the uncertainties in the renormalization counter terms, a systematic study of dilaton gravity models must deal with an action of a non-linear sigma model type. In this paper, for definiteness we shall focus on such an action of the form

$$S = \frac{1}{\gamma^2} \int d^2 x \sqrt{-\hat{g}} \left[ \hat{g}^{\mu\nu} \partial_\mu X^i \partial_\nu X^j G_{ij}(X) \partial_\mu X^j + Q(X) \hat{R} + \Lambda e^{V(X)} \right],$$  \hspace{1cm} (2.1)

where \(\gamma\) and \(\Lambda\) are constants and the “target space coordinates” \(X^i(x), (i = 1, 2)\) are to be identified later with the dilaton field \(\phi(x)\) and the conformal factor \(\rho(x)\) in a certain conformal gauge. We shall take the signature of the world-sheet metric to be \((-+)\).

Let us first study the properties of this action for the flat (world-sheet) reference metric
\[ \hat{g}_{\mu\nu} = \eta_{\mu\nu}, \]  

Namely

\[ S_{\text{flat}} = \frac{1}{\gamma^2} \int d^2 x \left[ \partial_{\mu} X^i G_{ij} \partial^\mu X^j + \Lambda e^V \right]. \] (2.2)

Under a general variation \( \delta X^i \), the Lagrangian changes by

\[ \gamma^2 \delta \mathcal{L} = 2 \partial^\mu \left( \partial_{\mu} X^i G_{ij} \delta X^j \right) - 2 \left( \Box X^k + \Gamma^k_{ij} \partial_{\mu} X^i \partial^\mu X^j \right) G_{kl} \delta X^l + \Lambda V_{,k} \delta X^k e^V, \] (2.3)

where \( \Gamma^k_{ij} \) is the usual Christoffel symbol constructed from the target space metric \( G_{ij} \). \( \Box \) is the world-sheet Laplacian, and we have used the abbreviation for the target space derivative \( V_{,k} \equiv \partial V / \partial X^k \). From this expression, we can immediately read off the equations of motion

\[ \Box X^k = -\Gamma^k_{ij} \partial_{\mu} X^i \partial^\mu X^j + \frac{1}{2} \Lambda V_{,k} e^V. \] (2.4)

### 2.2 Symmetry of the Model

What will be of utmost importance is the observation that the system possesses a symmetry if the variation \( \delta X^k \) satisfies the following two conditions:

\[ (i) \quad V_{,k} \delta X^k = 0, \] (2.5)

\[ (ii) \quad \nabla_i \delta X_j = \partial_i \delta X_j - \Gamma^k_{ij} \delta X_k = 0, \] (2.6)

where \( \delta X_j \equiv G_{jl} \delta X^l \). Indeed, for such variations, the last term in (2.3) vanishes, while the second term can be rewritten as

\[
-2 \left( \Box X^k \delta X_k + \Gamma^k_{ij} \partial_{\mu} X^i \partial^\mu X^j \delta X_k \right) \\
= -2 \left( \Box X^k \delta X_k + \partial_i \delta X_j \partial_{\mu} X^i \partial^\mu X^j \right) \\
= -2 \partial^\mu \left( \partial_{\mu} X^k \delta X_k \right),
\] (2.7)

which precisely cancels the first term. Thus we find \( \delta \mathcal{L} = 0 \) and we have a symmetry.

In two dimensions, the condition \( (i) \) can be solved uniquely (up to a constant):

\[ \delta X^k = \frac{\epsilon^{kl}}{\sqrt{|G|}} V_l, \] (2.8)

where \( |G| \) is the determinant of \( G_{ij} \) and \( \epsilon^{kl} \) is the usual antisymmetric matrix with \( \epsilon^{12} = -\epsilon_{12} = 1 \). Then substitution into \( (ii) \) yields a condition for \( V(X) \):

\[ \nabla_i \nabla_j V = 0. \] (2.9)
This means that \( V_k \) is a target space Killing vector, and moreover, from \( 0 = [\nabla_i, \nabla_j] V_k = R_{ikij} V^l \), we deduce that the two-dimensional target space must be flat.

An important consequence of this symmetry is that it guarantees the existence of a free field. Let \( F(X) \) be a function of \( X^i \) and apply to it the world-sheet Laplacian. Using the equations of motion for \( X^i \) we get

\[
\square F(X) = \square X^k F_{,k} + \partial_\mu X^i \partial^\mu X^j F_{,ij} = \nabla_i \nabla_j F \partial_\mu X^i \partial^\mu X^j + \frac{1}{2} \Lambda F_{,k} V^l V^k \epsilon V^l.
\]

This vanishes if

\[
\nabla_i \nabla_j F = 0, \quad F_{,k} V^k = 0.
\]

The second condition is solved by

\[
F_{,k} = -\sqrt{-|G|} \epsilon_{kl} V^l,
\]

which automatically satisfies the first condition as well since \( \nabla_i \nabla_j V = 0 \). To assure that the function \( F(X) \) itself exists, we must check the integrability condition, \( \text{i.e.} \) if \( \epsilon^{kl} \nabla_k F_{,l} = 0 \) holds. After a simple calculation we get

\[
\epsilon^{kl} \nabla_k F_{,l} = -\sqrt{-|G|} \nabla_m V^m = 0,
\]

again due to \( \nabla_i \nabla_j V = 0 \). This establishes that \( F(X) \) defined by Eq.(2.14) is a free field.

In fact it is easy to show that \( F(X) \) is directly related to the Noether current associated with the symmetry. Recall that \( \delta \mathcal{L} = 0 \) identically under the symmetry variation. Thus the Noether current takes a simple form

\[
\begin{align*}
        j_\mu &= \partial_\mu X^i G_{ij} \delta X^j \\
        &= \partial_\mu X^i G_{ij} \epsilon^{jk} V^k.
    \end{align*}
\]

By making use of \( G_{ij} \epsilon^{jk} / \sqrt{-|G|} = \sqrt{-|G|} \epsilon_{ij} G^{jk} \), we find

\[
\begin{align*}
        j_\mu &= \partial_\mu X^i \sqrt{-|G|} \epsilon_{ij} V^j \\
        &= -\partial_\mu F.
    \end{align*}
\]
Therefore the current conservation and the free field equation for $F(X)$ are identical. This type of symmetry was previously noticed in a particular model of dilaton gravity \[14\]. We now have its generalization and will demonstrate its power as we develop our general treatment.

Let us now try to express the action (2.2) in terms of $F(x)$ and $V(x)$. To do this, compute $j_\mu j^\mu$ with $j_\mu$ given in (2.16). Using the identity $\epsilon^{ij}\epsilon_{mn} = -(\delta^i_m \delta^j_n - \delta^i_n \delta^j_m)$, we get

$$j_\mu j^\mu = - (\partial_\mu X^i G_{ij} \partial^\mu X^j) V^k V_k + \partial_\mu V \partial^\mu V.$$

(2.18)

When $\nabla_i \nabla_j V = 0$ holds, it is easy to see that the squared norm of the vector $V_k$

$$\Delta \equiv V^k V_k$$

(2.19)

is a constant. Thus, with $j_\mu = -\partial_\mu F$, we have the action in the form

$$S_{flat} = \frac{1}{\gamma^2} \int d^2x \left[ \partial_\mu X^i G_{ij} \partial^\mu X^j + \Lambda e^V \right]$$

$$= \frac{1}{\gamma^2} \int d^2x \left[ -\partial_\mu F \partial^\mu F + \partial_\mu V \partial^\mu V + \Lambda \Delta e^V \right].$$

(2.20)

We see that, as announced in the introduction, the system decomposes into a free field $F(x)$ and a field $V(x)$ orthogonal to it which satisfies the Liouville equation

$$\Box V = \frac{1}{2} \Lambda \Delta e^V.$$

(2.21)

Its general solution is well-known \[28\] and can be expressed in terms of a free field $\psi$ in the form

$$V = \psi - 2 \ln Y,$$

(2.22)

$$Y = 1 + \frac{\Lambda \Delta}{16} A(x^+) B(x^-),$$

(2.23)

where $A(x^+)$ and $B(x^-)$ are defined by

$$\partial_+ A(x^+) \partial_- B(x^-) = e^\psi.$$

(2.24)

\[1\]In this model, the variation and the current take the form $\delta \phi = \delta \rho = e^{2\phi}$ and $j^\mu = \partial^\mu (\phi - \rho)$ respectively in their notation.
2.3 Conditions for Conformal Invariance

In order for a sigma model to describe a world-sheet gravity theory, it is necessary that it has the conformal symmetry. This will impose conditions for the field \( Q(X) \) coupled to the background curvature \( \hat{R} \).

By varying (2.1) with respect to the reference metric \( \hat{g}^{\mu\nu} \), one obtains the energy-momentum tensor \( T^{DG}_{\mu\nu} \) (in a suitable normalization) for the dilaton-gravity sector. For flat metric \( \hat{g}^{\mu\nu} = \eta^{\mu\nu}, T^{DG}_{\mu\nu} \) and its trace take the form

\[
T^{DG}_{\mu\nu} = -\partial_\mu X^i G_{ij} \partial_\nu X^j + \frac{1}{2} \eta^{\mu\nu} \left( \partial_\alpha X^i G_{ij} \partial^\alpha X^j \right) - (\eta^{\mu\nu} \Box - \partial_\mu \partial_\nu) Q + \frac{1}{2} \eta^{\mu\nu} \Lambda e^V, \\
T^{DG}_{\mu\mu} = -\Box Q + \Lambda e^V = -\Box X^i Q_{,i} - \partial_\mu X^i \partial^\mu X^j Q_{,ij} + \Lambda e^V.
\] (2.25)

With the use of the equations of motion (2.4), the condition for the vanishing of \( T^{DG}_{\mu\mu} \) can be expressed as

\[
\frac{1}{2} \Lambda \left( Q^k V_{,k} - 2 \right) e^V = -\nabla_i \nabla_j Q \partial_\mu X^i \partial^\mu X^j.
\] (2.27)

Since we can change \( \partial_\mu X^i \) while keeping the values of \( X^i \) fixed, we must have

\[
Q^k V_{,k} = 2, \\
\nabla_i \nabla_j Q = 0.
\] (2.28)

The second condition tells us that \( Q_{,k} \) is a Killing vector with a constant norm and hence it must be expressible as a linear combination of two other (orthogonal) Killing vectors \( F_{,k} \) and \( V_{,k} \). Writing this relation in the form

\[
F_{,k} = c_V V_{,k} + c_Q Q_{,k},
\]

and contracting with \( V^{,k} \), we immediately find

\[
F_{,k} = c_V \left( V_{,k} - \frac{1}{2} \Delta Q_{,k} \right).
\] (2.30)

The constant \( c_V \) can be expressed in terms of \( \Delta \) and the squared norm \( Q_{,k} Q^{,k} \): First from (2.14) we find \( F_{,k} F^{,k} = -\Delta \). On the other hand, the same quantity can be computed from Eq. (2.30) above. Comparing them we get

\[
c_V = \frac{1}{\sqrt{1 - \frac{1}{4} \Delta Q_{,k} Q^{,k}}}. \\
\] (2.31)
The value of $Q,kQ^k$ will be specified for dilaton gravity models in the next section.

Solving the equation (2.30) for $Q$, we finally obtain the expression for the full action (2.1) in terms of $F$ and $V$:

$$S = \frac{1}{\gamma^2} \int d^2x \sqrt{-\hat{g}} \left[ -\hat{g}^{\mu\nu} \partial_\mu F \partial_\nu F + \hat{g}^{\mu\nu} \partial_\mu V \partial_\nu V + \Lambda \Delta e^V \right] + \left( 2V - \frac{2}{c_V} F \right) \hat{R}.$$  \hfill (2.32)

The energy-momentum tensor following from this action is

$$T^{DG}_{\pm\pm} = T^F_{\pm\pm} + T^V_{\pm\pm},$$

$$T^F_{\pm\pm} = \frac{1}{\Delta} \left( (\partial_\pm F)^2 - \frac{2}{c_V} \partial^2_\pm F \right),$$

$$T^V_{\pm\pm} = -\frac{1}{\Delta} \left( (\partial_\pm V)^2 - 2\partial^2_\pm V \right) = -\frac{1}{\Delta} \left( (\partial_\pm \psi)^2 - 2\partial^2_\pm \psi \right).$$  \hfill (2.33)

These expressions appear to be singular when the $\Delta \to 0$, i.e. when $V$ becomes a free field. In fact from (2.30) one sees that $F$ and $V$ become degenerate in this limit. This however is not a problem. From (2.31) we see that $c_V = 1 + O(\Delta)$ in that limit and hence $F - \psi$ will be of $O(\Delta)$, as is easily seen from (2.30) and the structure of $V$ in (2.23). With this in mind, we define, for general $\Delta$, a new free field $\chi$ by the equation

$$F = \frac{1}{c_V} \psi + \frac{c_V \Delta}{2} \chi.$$  \hfill (2.34)

Then after a simple calculation we get

$$T^{DG}_{\pm\pm} = \partial_\pm \psi \partial_\pm \chi + \frac{c^2_V \Delta}{4} \left( (\partial_\pm \chi)^2 - \partial^2_\pm \chi \right) - \frac{1}{4} Q_k Q^k \left( (\partial_\pm \psi)^2 - 2\partial^2_\pm \psi \right).$$  \hfill (2.35)

This expression is completely regular as $\Delta \to 0$ and the system will be described by a pair of free fields $\psi$ and $\chi$.

3 General Analysis of Solvable Models of Dilaton Gravity

3.1 The Model

Having formulated a sigma model with a special symmetry, we now apply it to dilaton gravity models in a unified manner. A large class of such models can be represented by the action of the form

$$S = \frac{1}{\gamma^2} \int d^2x \sqrt{-g} \left[ g^{\mu\nu} K(\phi) \partial_\mu \phi \partial_\nu \phi + q(\phi) R + \Lambda e^{\psi(\phi)} \right],$$  \hfill (3.1)
where $\phi$ is the dilaton field. In order to be able to deal with possibilities of various
conformal frames, we shall first make a conformal transformation of the metric by a factor
e$^{2\omega(\phi)}$ and then take a conformal gauge. This amounts to the combined transformation

$$g_{\mu\nu} = e^{2(\rho + \omega(\phi))} \hat{g}_{\mu\nu}.$$  

(3.2)

Then the curvature scalar takes the form

$$R = e^{-2(\rho + \omega)} \left( \hat{R} - 2 \hat{g}^{\mu\nu} \hat{D}_\mu \hat{D}_\nu (\rho + \omega) \right).$$  

(3.3)

We now add a conformal anomaly term. Various possibilities exist depending on the
choice of the measure, and again to cover a general situation we take it to be of the form\footnote{The form assumed here is natural from the point of view of world-sheet general covariance. However, if one wishes, it can be reproduced by the replacement $K \rightarrow K + \kappa \Omega_{,\phi}^2$ and $q \rightarrow q + \kappa \Omega$. Furthermore, replacements of this type can produce a variety of forms for the anomaly term.}

$$S_{\text{anom}} = \frac{\kappa}{\gamma^2} \int d^2x \sqrt{-\hat{g}} \left( \hat{g}^{\mu\nu} \partial_\mu (\rho + \Omega(\phi)) \partial_\nu (\rho + \Omega(\phi)) 

+ (\rho + \Omega(\phi)) \hat{R} \right),$$  

(3.4)

allowing an arbitrary function $\Omega(\phi)$. The dependence of $\kappa$ on the number $N$ of matter
fields, to be explicitly incorporated later, may differ depending again on the choice of the
measure \cite{22, 14}, but here it need not be specified. With this term added, the resultant
action can be identified with a sigma model discussed in the previous section with

$$(X^1, X^2) = (\phi, \rho),$$  

(3.5)

$$Q = q + \kappa (\rho + \Omega),$$  

(3.6)

$$V = 2\rho + v + 2\omega,$$  

(3.7)

$$G_{ij} = \begin{pmatrix} K(\phi) + 2q \omega_{,\phi} + \kappa (\Omega_{,\phi})^2 & Q_{,\phi} \\
Q_{,\phi} & \kappa \end{pmatrix}.$$  

(3.8)

It is useful to note that the contravariant vector $Q^k$ has a particularly simple form, namely
$Q^k = (0, 1)$, due to the fact that $Q_{,k}$ is precisely the same as the second column vector of
$G_{ij}$. Using this, one can easily check that the conformal invariance conditions $Q^k V_k = 2$
and $\nabla_i \nabla_j Q = 0$ are satisfied for any choice of the functions above. This is understood as
follows: Since $S_{\text{anom}}$ has the form maintaining the world-sheet covariance, the total action
$S + S_{\text{anom}}$ also has this covariance. Then the traceless condition becomes identical with
the invariance with respect to the variation of the conformal factor. Thus \((T^{DG})_\mu = 0\) is satisfied automatically by the equation of motion.

Also, inner product of any target-space vector with \(Q^k\) is easy to compute. In particular, we note

\[ Q_{,k} Q^k = Q_{,\rho} = \kappa. \]  

(3.9)

3.2 Condition for Solvability

In the previous section, we have seen that when \(\nabla_i \nabla_j V = 0\) is satisfied a powerful symmetry exists and it leads to the solvability of the model. We now analyze this condition in detail.

From the structure of \(G_{ij}\) it is easily to check that the only non-vanishing components of the Christoffel symbols are

\[ \Gamma^\phi_{\phi\phi} = \frac{\partial_\phi |G|}{2 |G|}, \]  

(3.10)

\[ \Gamma^\rho_{\phi\phi} = \frac{1}{2 |G|} (2 G_{\phi\phi} G_{\rho,\phi} - G_{\phi\rho} G_{\phi,\phi}), \]  

(3.11)

and that all the components of \(\nabla_i \nabla_j V = 0\), except \(\nabla^2 \phi V = 0\), are automatically satisfied.

This non-trivial condition reads

\[ \nabla^2 \phi V = \partial_\phi V_{,\phi} - \frac{1}{2} V_{,\phi} \frac{\partial_\phi |G|}{|G|} \]

\[ - \frac{1}{|G|} (2 G_{\phi\phi} G_{\rho,\phi} - G_{\phi\rho} G_{\phi,\phi}) = 0. \]  

(3.12)

After a straightforward but somewhat tedious calculation, this becomes

\[ |G| \nabla^2 \phi V = q_3 \partial_\phi \left( \frac{K}{q_3} - \frac{v_3}{q_3} \right) \]

\[ \quad + \kappa W_{,\phi} (K + 2 q_3 u_{,\phi}) - \frac{\kappa}{2} W_{,\phi} (K + 2 q_3 u_{,\phi})_{,\phi} \]

\[ = 0, \]  

(3.13)

\[ = 0, \]  

(3.14)

where

\[ u \equiv \omega - \Omega, \]  

(3.15)

\[ W \equiv v + 2 u. \]  

(3.16)
It is convenient to call the expression in the parenthesis of the first line a function \( a(\phi) \). In other words, we parametrize the function \( K(\phi) \) by

\[
K = aq^2ador + q,\phi v,\phi .
\] (3.17)

Substitute this back into the above equation and regard all the quantities as functions of \( q \) instead of \( \phi \). Then the equation simplifies to

\[
a_q \left( 1 - \frac{\kappa}{2} W_q \right) + \kappa W_{qq} \left( a + \frac{1}{2} W_q \right) = 0 .
\] (3.18)

There are two types of solutions to this equation.

Type I: When the quantity in the first parenthesis does not vanish, one can convert this into a differential equation for the function \( a \) with respect to \( y \equiv W_q \). The general solution can then be readily obtained:

\[
\kappa a = 1 - \kappa W_q - c(\kappa W_q - 2)^2
\]

\[
= (1 - 4c)(1 - \kappa W_q) - c\kappa^2 W_q^2 ,
\] (3.19) (3.20)

where \( c \) is an arbitrary integration constant. The second line suggests that \( c = 1/4 \) is a rather special value. The significance of this value is understood when we compute the quantity \( \Delta = V^k V_k \), which gives the cosmological constant for the Liouville equation satisfied by \( V \). With the parametrization (3.17), \( \Delta \) can be expressed as

\[
\Delta = \frac{4a + \kappa W_q^2}{\kappa(a + W_q) - 1} .
\] (3.21)

Substituting (3.20) into this, we get

\[
\Delta = \frac{4c - 1}{\kappa c} .
\] (3.22)

Thus we see that \( c = 1/4 \) is precisely the value at which the field \( V \) becomes a free field. As we shall see, all the existent models fall into this category.

Type II: In the special case where \( \kappa W_q = 2 \), the equation (3.18) is satisfied with an arbitrary \( a(\phi) \) since \( W_{qq} = 0 \). The general formula (3.21) gives in this case

\[
\Delta = \frac{4}{\kappa} .
\] (3.23)

Thus models of this category are ill-defined without the anomaly term and we shall not consider them further.
3.3 Further Analysis of Type I Solution

As was demonstrated in Sec. 2, the defining equation for $F$, namely

$$F_k = -\sqrt{-G} \epsilon_{kl} V^l,$$

is integrable. For the type I solution, integration can indeed be carried out explicitly. It is most convenient to cast the result in the form

$$F = \frac{1}{2\sqrt{c}} (V - 2c\Delta r), \quad (3.24)$$

where

$$r \equiv q - \frac{\kappa}{2} W = q - \frac{\kappa}{2} (v + 2\omega - 2\Omega). \quad (3.25)$$

This quantity $r(\phi)$ will be useful in analyzing the space-time structure as it can be readily expressed in terms of the free fields $\psi$ and $\chi$. Substituting (2.34) and (2.23) into (3.24) we get

$$r = -\chi - \frac{1}{c\Delta} \ln \left(1 + \frac{\Lambda \Delta}{16} AB\right), \quad (3.26)$$

where we have used

$$c_v = 2\sqrt{c} = \left(1 - \frac{\kappa \Delta}{4}\right)^{-1/2}. \quad (3.27)$$

The latter equation is obtained by noting that Eq. (3.24) can also be written in the form $F = c_v (V - \frac{1}{2} \Delta Q)$ as in (2.30) and by using Eqs. (3.6), (3.7) and (3.22). It conforms to the general formula (2.31) derived towards the end of Sec. 2.

Now let us examine the type I solution (3.20) more closely. Using $K = q^2 \phi (a + v_q)$ and $u = \omega - \Omega$, it can be rewritten as a relation between various functions appearing in the action:

$$c(\kappa v_{\phi} + 2(\kappa u_{\phi} - q_{\phi}))^2 = q_{\phi}^2 - 2\kappa q_{\phi} u_{\phi} - \kappa K. \quad (3.28)$$

Thus we can solve for one of the functions if the others are known. For instance, if we regard the potential function $v(\phi)$ as unknown, we can easily solve for it to get

$$v = \frac{2}{\kappa} (q - \kappa u)$$

$$\pm \frac{1}{\kappa \sqrt{c}} \int d\phi \sqrt{q_{\phi}^2 - \kappa K - 2\kappa q_{\phi} u_{\phi}}. \quad (3.29)$$

We shall shortly describe an application of this formula.
3.4 Existing Models

We now show how the various solvable models so far proposed are naturally described in our unified scheme. It turns out that they can be classified into two categories depending on whether $W_{qq}$ in Eq. (3.18) vanishes or not.

First, the models with non-vanishing $W_{qq}$ can all be regarded as special cases of the one analyzed by de Alwis [12], which is specified by

$$\omega(\phi) = \Omega(\phi) = 0, \quad (3.30)$$
$$K(\phi) = 4e^{-2\phi}(1 + h(\phi)), \quad (3.31)$$
$$Q,\phi = q,\phi = -2e^{-2\phi}(1 + \bar{h}(\phi)), \quad (3.32)$$

where $h(\phi)$ and $\bar{h}(\phi)$ describe possible deviations from the CGHS model. Integrating the last equation, we get

$$q = e^{-2\phi} - 2 \int^{\phi}\! e^{-2\phi} \bar{h} \, d\phi. \quad (3.33)$$

Substituting these functions into (3.29) with a choice of the + sign immediately yields the potential function

$$v(\phi) = \frac{2q}{\kappa} + \frac{2}{\kappa\sqrt{c}} \int^{\phi}\! e^{-2\phi}\sqrt{(1 + \bar{h})^2 - \kappa e^{2\phi}(1 + h)} \, d\phi$$

For the special value $c = 1/4$ discussed previously, this is precisely the form obtained in [12] by use of complicated field transformations. (Our sign convention for $\kappa$ is opposite to de Alwis’s.)

For certain choices of $h$ and $\bar{h}$, the integrals can be explicitly performed. For the model of Bilal and Callan [13], $h = \bar{h} = 0$ and one gets

$$q = e^{-2\phi}, \quad (3.34)$$
$$v = \frac{2}{1 + Z} + \ln \frac{1 + Z}{1 - Z} + \text{const.}, \quad (3.35)$$
$$Z \equiv \sqrt{1 - \kappa q^{-1}}. \quad (3.36)$$

As for the RST model [14], $h = 0, \bar{h} = (\kappa/4) \exp(2\phi)$ and one has

$$q = e^{-2\phi} - \frac{\kappa}{2} \phi, \quad (3.37)$$
$$v = -2\phi. \quad (3.38)$$
On the other hand, for the model treated in [15]-[17], [18], \( W_{qq} \) vanishes and hence one has an additional relation

\[
W = v + 2(\omega - \Omega) = 2bq + \text{const.},
\]

where \( b \) is a constant. The model is then specified by

\[
\begin{align*}
\Omega &= \kappa = 0, \\
q &= e^{-2\phi}, \quad K = 4q, \\
v &= \ln q, \\
\omega &= q - \frac{1}{2}\ln q \quad (b = 1).
\end{align*}
\]

(It can be easily checked that \( \Delta = 0 \) (i.e. \( c = 1/4 \)) and the general formula (3.29) is still valid with the limiting procedure \( \kappa \to 0 \).)

In the discussions above, which evidently demonstrate the generality of our scheme, the value of \( c \) turned out to be invariably equal to 1/4 for all the models treated. There is in fact a simple way to understand it. Suppose we take a restrictive point of view that the potential function \( e^V \) in the action (2.20) be a marginal deformation as in the procedure of David, Distler and Kawai [29]. Then it should be a \((1, 1)\) operator and one can easily show that this requirement leads to the value \( c = 1/4 \).

### 4 Space-time Structure of the Models

In the preceding sections, we have systematically analyzed a general class of solvable dilaton gravity models and shown that such systems can be invariably described by a set of free fields \( \psi \) and \( \chi \). We now add free massless matter fields to the system and study the space-time structure thereby induced.

#### 4.1 Energy-Momentum Tensor Constraints

We shall consider the situation where left-going matter fields are sent in from the past null infinity, which is described by an energy-momentum tensor \( T^f_{++} \). Further, to make the analysis simple, we take the \( \psi = 0 \) gauge. Then, recalling the form of \( T^{DG}_{++} \) given in Eq.(2.35), with \( c_V = 2\sqrt{c} \), the energy-momentum constraint becomes

\[
\partial_+^2 \chi - c\Delta(\partial_+\chi)^2 = T^f_{++}.
\]

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Depending on the behavior of the model, it is sometimes necessary to add a contribution of a boundary function [4] if one wishes to realize asymptotic flatness in the region which is in the causal past to the injection of the matter. To make the analysis transparent, we will first focus on the cases where such a function is not necessary and will briefly discuss the remaining cases later.

Now when the cosmological constant $\Delta$ is non-vanishing, Eq.(4.1) is non-linear as well as inhomogeneous and it is difficult to solve it for general $T^f_{++}$. However, for a shock-wave configuration of the form

$$T^f_{++} = T_0 \delta(x^+ - x^+_0), \quad (4.2)$$

$$T_0 > 0, \quad x^+_0 > 0, \quad (4.3)$$

the exact solution can be obtained.

We shall now solve it with the boundary condition $\chi = 0$ for $x^+ < x^+_0$. (More general choice is possible, but it does not appear to give qualitatively different results. ) Let us write $x = x^+ - x^+_0$ and set $\partial_x \chi(x) = \theta(x)w(x)$. Then the equation becomes

$$\delta(x)w(0) + \theta(x)w'(x) - c\Delta w(x)^2\theta(x) = T_0 \delta(x) . \quad (4.4)$$

Thus $w(x)$ must satisfy

$$w(0) = T_0 , \quad (4.5)$$

$$w' - c\Delta w^2 = 0 . \quad (4.6)$$

The solution is

$$w(x) = \frac{T_0}{1 - c\Delta T_0 x} . \quad (4.7)$$

Integrating once again, we obtain the desired solution

$$\chi(x^+ - x^+_0) = \frac{-1}{c\Delta} \ln Z , \quad (4.8)$$

where

$$Z \equiv 1 - c\Delta T_0 \theta(x^+ - x^+_0)(x^+ - x^+_0) . \quad (4.9)$$
4.2 Formula for the Curvature Scalar

We are now ready to examine the behavior of the scalar curvature. The original metric can be written in the form

\[ g_{\mu\nu} = e^{2(\rho+\omega)} \eta_{\mu\nu} = e^{V-v} \eta_{\mu\nu}. \]  

(4.10)

Using the Liouville equation of motion for \( V \) and its solution given in (2.22) – (2.24), the curvature scalar in the \( \psi = 0 \) gauge can be written as

\[ R = -e^{v-V} \Box (V-v) = e^v \left(-\frac{1}{2} \Lambda \Delta + Y^2 \Box v \right), \]  

(4.11)

where

\[ Y = 1 + \frac{\Lambda \Delta}{16} x^+ x^- . \]  

(4.12)

In order for the field \( V \) to be real, \( Y \) must be positive. Rather than enumerating all possibilities, we shall take \( \Delta \leq 0 \) so that the quadrant \( 0 < x^+ < \infty, -\infty < x^- < 0 \), which is considered for the existing models, is included in the allowed region.

Let us recall the quantity \( r(\phi) \) given in (3.26). Since \( \chi \) and \( Y \) are already known, we immediately find the explicit form of \( r(x^+, x^-) \):

\[ r = \frac{1}{c\Delta} \ln \left( \frac{Z}{Y} \right), \]  

(4.13)

where \( Z \) and \( Y \) are as in (4.9) and (4.12). It is important to note that the behavior of \( r \) is completely fixed independently of the choice of various functions \( q(\phi), K(\phi), v(\phi), \omega(\phi) \) and \( \Omega(\phi) \). Therefore in analyzing \( R \) it is convenient to regard \( v \) as a function of \( r \).

Before deriving a general formula for \( R \), it is instructive to look at the behavior of \( r(x^+, x^-) \) for several regions, which we shall use shortly. First, as \( x^+ \to 0 \), \( r \) tends to a \( \Delta \)-independent expression

\[ r \to \frac{-\Lambda}{16c} x^+ x^- . \]  

(4.14)

On the other hand, for the asymptotic region of large \( x^+ \), the behavior of \( r \) depends crucially on whether \( \Delta \) vanishes or not. For models with \( \Delta = 0 \), we have

\[ r \to -\frac{1}{4} x^+ (\Lambda x^- + 4 T_0), \]  

(4.15)
whereas for non-vanishing $\Delta$, $r$ tends to a finite function of $x^-$ given by

$$ r \rightarrow \frac{1}{c\Delta} \ln \frac{16cT_0}{-\Lambda x^-} + O(1/x^+) . \quad (4.16) $$

Another important region is the one where $r$ vanishes, which occurs for $Z = Y$. There are two such regions. One occurs after the shock-wave has passed, namely for $x^+ > x^+_0$, along the hyperbolic line

$$ x^+(\Lambda x^- + 16cT_0) = 16cT_0 x^+_0 , \quad (4.17) $$

which is independent of $\Delta$. We shall see that for many models, a black hole singularity is located along this line. The other region of vanishing $r$ is along the asymptotic line $x^+x^- \sim 0$ before the shock-wave has passed.

Let us now go back to the evaluation of $R$. Regarding $v = v(r)$, Eq.(4.11) becomes

$$ -R = e^v \left[ \frac{1}{2} \Lambda \Delta + 4Y^2 (v_r \partial_+ \partial_- r + v_{rr} \partial_+ r \partial_- r) \right] . \quad (4.18) $$

Derivatives of $r$ are easily computed and we obtain a general formula:

$$ -R = \Lambda e^v \left\{ \frac{1}{2} \Delta - \frac{1}{4c} v_r + \frac{1}{(8c^2)^2} \left[ x^+(\Lambda x^- + 16cT_0 \theta(x^+ - x^+_0)) \right] + c\Delta \lambda T_0 x^+_0 \theta(x^+ - x^+_0) x^+ x^- v_{rr} \right\} . \quad (4.19) $$

In this form one can readily take the limit $\Delta \rightarrow 0$, in which the expressions for $r$ and $R$ simplify considerably to

$$ r = -T_0 \theta(x^+ - x^+_0)(x^+ - x^+_0) - \frac{\Lambda}{4} x^+ x^- , \quad (4.20) $$

$$ R = \Lambda e^v \left( v_r + (r - T_0 x^+_0 \theta(x^+ - x^+_0)) v_{rr} \right) . \quad (4.21) $$

### 4.3 New Models

We now make use of the formula for $R$ and explore models with reasonable physical behaviour. Because of the generality of our scheme, large possibilities exist and we shall have to limit our investigation to only a certain class of models satisfying a number of simplifying features. First, we assume $\omega = \Omega = 0$ as in the original CGHS model and concentrate on the cases of $\kappa \geq 0$. Next it is natural to suppose that the potential function $v(\phi)$ is independent of the number of matter fields and hence on $\kappa$. $q(\phi)$, on the other hand, may or may not depend on $\kappa$ but we shall consider the $\kappa$-independent cases. This
means that we can regard \( v \) as a \( \kappa \)-independent function of \( q \) and we have the relation
\[
r = q - \left( \frac{\kappa}{2} \right) v(q)
\] from (3.24). It would be most convenient if we can solve this relation to get \( v(r) \), but it is in general difficult. Thus we treat \( v \) as \( v(q) \) and substitute in the formula for \( R \) the relations
\[
v_r = \frac{v_q}{1 - \frac{\kappa}{2} v_q}, \quad v_{rr} = \frac{v_{qq}}{(1 - \frac{\kappa}{2} v_q)^3}
\] (4.22)

Our analysis will be such that the specific form of \( q(\phi) \) will not be needed.

4.3.1 Models with \( \Delta = 0 \)

Let us begin with models with \( \Delta = 0 \). First we look at the behavior of \( R \) for the asymptotic region of large \( x^+ \), where \( r \to +\infty \). To motivate our search for a reasonable \( v(q) \), recall the CGHS model, where \( q = e^{-2\phi} \) is a measure of the inverse coupling strength. It would then be natural to require that \( r \to +\infty \) corresponds, at least for sufficiently large \( |x^-| \), to the weak coupling region with large positive \( q \). Then we wish to have \( r \sim q^a \) with positive \( a \) in that region. This prompted us to investigate the following two cases:

\[
(i) \quad v(q) = -\alpha q^{1+\beta}, \quad (\alpha > 0, \beta > -1), \quad r = q + (\kappa \alpha/2) q^{1+\beta}
\]

\[
(ii) \quad v(q) = -\alpha \ln q, \quad r = q + (\kappa \alpha/2) \ln q
\]

(4.23) (4.24)

Although we are focusing only on a particular asymptotic region, these forms for \( v(q) \) are rather generic and useful for the investigation of other regions as well.

Let us begin with the type (i) potential. As we wish to allow for non-integral value of \( \beta \) and in fact for just such a case we shall find interesting space-time structure, the range of \( q \) is restricted to \( q \geq 0 \) in order for \( v(q) \) to be real. This implies that \( r \) should also be non-negative. Recalling the expression of \( r \) given in (4.20) we shall therefore concentrate on (a part of) the quadrant \( 0 < x^+ < \infty, -\infty < x^- < 0 \). The exact form of \( R \) is given by

\[
R_{(i)} = -\alpha \Lambda e^{-\alpha q^{1+\beta}} \left[ \frac{(1+\beta)q^{\beta}}{1 + \frac{\kappa \alpha}{2} (1+\beta)q^{\beta}} + \frac{\beta(1+\beta)}{(1 + \frac{\kappa \alpha}{2} (1+\beta)q^{\beta})^3} \left( q^{\beta} \left( 1 + \frac{\kappa \alpha}{2} q^{\beta} \right) - T_0 x^- x^{0+} q^{\beta-1} \theta(x^{-} - x^{0+}) \right) \right].
\]

(4.25)

From this expression, we can read off the behavior of \( R \). First, \( R_{(i)} \) vanishes in the limit \( x^+ \to \infty \) for positive \( \alpha \) because large \( r \) corresponds to large \( q \). Next we study the behavior
as \( x^+ \to 0 \) and also along the hyperbolic line (4.17). As was pointed out previously, \( r \) tends to vanish in both of these regions. For \( x^+ \to 0 \), \( q \) also vanishes and this leads to the vanishing of \( R \) for any positive \( \beta \) since the term proportional to the \( \theta \)-function describing the shock wave is absent in this region. In contrast, for \( x^+ > x^+_0 \), a curvature singularity may develop along \( r = 0 \) line. Indeed for a fractional value \( 1 > \beta > 0 \), we get a singularity of strength \( R \sim q^{\beta-1} \) precisely due to the shock wave and by drawing a Penrose diagram one can easily identify this as a black hole singularity accompanied by an event horizon along \( x^- = -16cT_0/\Lambda \).

As for the type (ii), the exact form of \( R \) becomes

\[
R_{(ii)} = \Lambda\frac{\alpha q^{-\alpha}}{q + \frac{\kappa \alpha}{2}} \left[ -1 + \frac{q}{(q + \frac{\alpha}{2})^2} \left( q + \frac{\kappa \alpha}{2} \ln q - T_0 x^+ \theta(x^+ - x^+_0) \right) \right].
\] (4.26)

In this case, the behavior of \( R \) is very different depending on whether \( \kappa \) vanishes or not. For \( \kappa = 0 \), \( R_{(ii)} \) becomes proportional to the \( \theta \)-function. Then, noting that \( r = q \) in this case, we see that \( R_{(ii)} \) vanishes identically for \( x^+ < x^+_0 \). For \( x^+ > x^+_0 \), the scalar curvature tends to vanish as \( x^+ \to \infty \) for \( \alpha > -2 \), and a curvature singularity due to the shock wave forms along the \( r = 0 \) line. Note that \( v(q) \) is well-defined only in the quadrant \( 0 < x^+ < \infty, -\infty < x^- < 0 \).

Next, let us consider the case with non-zero \( \kappa \). For positive \( \alpha \), \( r \) goes like \( q \) for large \( r \) and by simple power counting we see that \( R_{(ii)} \to 0 \). When \( \alpha < 0 \), large \( r \) corresponds both to large and small \( q \). For small \( q \), again by power counting we ascertain the vanishing of \( R_{(ii)} \). For large \( q \), the asymptotic behavior is \( R_{(ii)} \sim q^{-(2+\alpha)} \) and it vanishes for \( \alpha > -2 \). Thus we find that these models do have asymptotically flat regions for large \( x^+ \) provided \( \alpha > -2 \).

As for the behavior as \( x^+ \to 0 \) and along the hyperbolic line \( r = 0 \), the situation again depends very much on the value of \( \alpha \). First for \( \alpha < 0 \), \( r(q) \) has a minimum \( r_{\min} \) at \( q = -\kappa \alpha/2 \), and just at this point the denominators of \( v,rr \) and \( v,rrr \) vanish. Therefore a curvature singularity develops along the line \( r = r_{\min} \) before \( x^+ \to 0 \) or \( r = 0 \) is reached.

On the other hand, for positive \( \alpha \), \( r = 0 \) corresponds to a non-vanishing value \( q = q_c \), defined as the solution of the equation \( q_c + (\kappa \alpha/2) \ln q_c = 0 \). It is easy to see that \( 0 < q_c < 1 \) and it vanishes as \( \kappa \to 0 \). Then for both of the above regions \( R \) becomes constant and
takes the form

\[ R \rightarrow -\frac{\alpha \Lambda}{q_c^2 (q_c + \frac{\kappa \alpha}{2})} \delta_{\kappa,0} \quad (x^+ \to 0), \]  
\[
R \rightarrow -\frac{\alpha \Lambda}{q_c^2 (q_c + \frac{\kappa \alpha}{2})} \left(1 - \delta_{\kappa,0} + \frac{T_0 x^+ q_c}{(q_c + \frac{\alpha \kappa}{2})^2}\right) \quad (x^+ > x^+_0, r \to 0). \]  

(4.27)

(4.28)

This means that as \( x^+ \to 0 \) the space-time is anti de Sitter instead of flat and at the same time the curvature singularity along \( r = 0 \) disappears. Although this appears interesting, there is actually a problem: For \( \kappa \neq 0 \), \( v(q) \) becomes well-defined even in the region for negative \( r \), and as \( r \to -\infty \) the curvature \( R \) is seen to diverge.

Combining the results obtained above, we can construct an interesting new model. It is obtained by smoothly joining the type (i) potential for the region \( x^+ < x^+_0 \) and the type (ii) potential for the remaining region \( x^+ \gtrsim x^+_0 \). An example is given by

\[ v(q) = -\alpha q^{1+\beta} + q^2 \ln q, \quad (\alpha > 0, \ 0 < \beta < 1) \]  

(4.29)

The allowed region, where \( v(q) \) is well-defined, is again seen to be the quadrant \( 0 < x^+ < \infty, -\infty < x^- < 0 \).

The model so obtained has the following attractive properties: (a) a black hole singularity forms along the space-like hyperbolic line \((4.17)\), (b) the space-time is flat for both \( x^+ \to \infty \) (outside the horizon) and \( x^+ \to 0 \), and (c) Hawking radiation exists.

Let us see how they come about. First note that for large \( q \) the potential is dominated by the log term, while for small \( q \) the power term prevails and in both regions \( r \sim q \) holds. Thus from the previous analysis the properties (a) and (b) are guaranteed. To see that Hawking radiation exists in this model, recall the form of the metric. In the \( \psi = 0 \) gauge with \( \Delta = 0 \), the form given in \((4.10)\) reduces to \( g_{\mu\nu} = e^{-v} h_{\mu\nu} \). For \( x^+ \to 0 \), \( v(q) \) vanishes and hence \( x^\pm \) are the manifestly flat coordinates. On the other hand, for \( x^+ \to \infty \), we have \( q \sim r \sim -\frac{1}{4} x^+(\Lambda x^- + 4T_0) \) and the line element becomes

\[ ds^2 \quad = \quad \frac{4 dx^+ dx^-}{x^+(\Lambda x^- + 4T_0)}. \]  

(4.30)

Thus to get manifestly flat coordinates we must make a familiar exponential type conformal transformation, and by standard arguments this leads to Hawking radiation.
4.3.2 Models with a boundary function

Let us now make a brief discussion on the possibility of adding a boundary function \( \kappa t_{++}(x^+) \) to \( T_{++}^f \) in (4.1) in order to realize a flat region for \( x^+ < x^+_0 \) when the anomaly term is included. Originally [1], possible existence of a boundary function was inferred from the covariant conservation law \( \nabla_\mu T^\mu_\nu = 0 \) together with the anomaly equation \( g^{\mu\nu}T_{\mu\nu} \propto \kappa R \) [30]. For us, a slightly different viewpoint will be more convenient. Quantum mechanically, \( T_{\mu\nu} \) in these equations must be understood as the expectation value, \( \langle \hat{T}_{\mu\nu} \rangle_g \), of the operator \( \hat{T}_{\mu\nu} \) in a curved space-time described by a metric \( g_{\mu\nu} \). More precisely, as the form of the anomaly equation dictates, \( \hat{T}_{\mu\nu} \) must be defined such that for a flat region described by manifestly flat metric \( g_{\mu\nu} = \eta_{\mu\nu} \) (in a coordinate system we call \( \sigma^\mu \)) \( \langle \hat{T}_{\mu\nu}(\sigma) \rangle_\eta \) vanishes. If one wishes to use a different coordinate system, call it \( x^\mu \), related to \( \sigma^\mu \) by a conformal transformation, one must make a re-normal-ordering and this can give rise to a boundary function. Since \( R \) still vanishes in \( x^\mu \), one can turn the argument around and use \( R = 0 \) (in an appropriate region) as the equation for determining the boundary function. In what follows, we shall carry out this procedure explicitly.

For simplicity, we shall limit ourselves to \( \Delta = 0 \) case with \( \omega = \Omega = 0 \) only. In this case, the curvature, the energy-momentum constraint for \( x^+ < x^+_0 \) and the expression for \( \chi \) take the following forms:

\[
R = e^v \Box v, \quad (4.31)
\]
\[
\partial_+^2 \chi = \kappa t_{++}, \quad (4.32)
\]
\[
\chi = -\frac{\Lambda}{4}x^+x^- - r = -\frac{\Lambda}{4}x^+x^- - q + \frac{\kappa}{2}v, \quad (4.33)
\]

where the last equation is obtained from (3.25) and (3.26). Suppose we want \( R \) to vanish identically in the above region. Then \( \Box v = 0 \) must hold and we may write \( v = v^+(x^+) + v^-(x^-) \). Putting this into the expression for \( \chi \) and applying \( \partial_+^2 \), we get

\[
\partial_+^2 \chi = -\partial_+^2 q + \frac{\kappa}{2}\partial_+^2 v^+. \quad (4.34)
\]

On the other hand, from \( \Box \chi = 0 \) it follows that \( \partial_+\partial_- q = -\frac{\Lambda}{4} \) and we get

\[
q = -\frac{\Lambda}{4}x^+x^- + \kappa p(x), \quad (4.35)
\]
\[
p(x) = (p^+(x^+) + p^-(x^-)), \quad (4.36)
\]
where $p^\pm$ are arbitrary functions of the specified variables. From these equations, we get the expression for the boundary function $t_{++}$ as

$$\kappa t_{++} = \partial_+^2 \chi = -\kappa \partial_+^2 p^+ + \frac{\kappa}{2} \partial_+^2 v^+.$$  \hfill(4.37)

With the shock wave, $\chi$ is then solved as

$$\chi = \kappa \left( -p(x) + \frac{1}{2} v(x) \right) + T_0(x^+ - x^+_{0}) \theta(x^+ - x^+_{0}),$$

where $p^-(x^-)$ and $v^-(x^-)$ are arbitrary functions.

When $v$ is specified as a function of $q$, as we have been assuming, the form of $v(q)$ satisfying $\Box v = 0$ is actually severely restricted because of the special form of $q$ given in (4.35). In such a case, the condition $\Box v = 0$ can be written as

$$0 = \partial_+ \partial_- v = \partial_+ \partial_- q v_q + \partial_+ q \partial_- q v_{qq}$$
$$= -\frac{\Lambda}{4} v_q + v_{qq} \left( \frac{\Lambda}{4} x^- - \partial_+ p^+ \right) \left( \frac{\Lambda}{4} x^+ - \partial_- p^- \right).$$  \hfill(4.39)

From this the following expression must be a function only of $q$:

$$\frac{\Lambda}{4} v_q = \left( \frac{\Lambda}{4} x^- - \kappa \partial_+ p^+ \right) \left( \frac{\Lambda}{4} x^+ - \kappa \partial_- p^- \right).$$  \hfill(4.40)

It is not difficult to prove that it can happen only for the following two cases:

$$(i) \quad q + c = -\frac{\Lambda}{4} (x^+ + a)(x^- + b),$$
$$v = -\alpha \ln(q + c) + \beta,$$
$$\kappa p^+ = -\frac{\Lambda}{4} bx^+ + \text{const.},$$
$$v^+ = -\alpha \ln(x^+ + a) + \text{const.},$$

$$(ii) \quad q + c = \frac{\Lambda}{4} (ax^+ - \frac{1}{2a} x^- + b)^2,$$
$$v = \alpha \sqrt{q + c} + \beta,$$
$$\kappa p^+ = \frac{\Lambda}{4} \left( a^2(x^+)^2 + 2abx^+ \right) + \text{const.},$$
$$v^+ = \alpha \sqrt{\Lambda/4ax^+} + \text{const.},$$

where $a, b, c, \alpha$ and $\beta$ are constants.

The form of $r$ when a left-going matter shock wave is sent in from the past null infinity can then be readily obtained from (4.38) and (4.33) for the above two cases:

$$r_{(i)} = -\frac{\Lambda}{4} x^+ x^- + \frac{\kappa}{2} \alpha \ln \left( -\frac{\Lambda}{4} x^+ x^- \right) - T_0(x^+ - x^+_{0}) \theta(x^+ - x^+_{0}),$$  \hfill(4.43)
\[ r_{(ii)} = \sqrt{\frac{\Lambda}{4}} (ax^+ - \frac{1}{2a} x^- + b) \left\{ \frac{\Lambda}{4} (ax^+ - \frac{1}{2a} x^- + b) - \frac{\kappa \alpha}{2} \right\} \]
\[ -T_0 (x^+ - x^+_0) \theta(x^+ - x^+_0) - c - \frac{\kappa}{2} \beta. \] (4.44)

Here we have set \( a, b, c, \beta = 0 \) in \( R_{(i)} \) for simplicity.

Now from Eqs. (4.31) and (4.22) the exact form of \( R \) for these cases is obtained in the same way as in the previous subsection:

\[ R_{(i)} = -\alpha \Lambda \frac{q^{-\alpha}}{q + \kappa \alpha/2} \left[ 1 - \frac{q}{(q + \kappa \alpha/2)^2} \times \left\{ \frac{\Lambda}{4} x^+ x^- + \kappa \alpha - \frac{(\kappa \alpha)^2}{\Lambda x^+ x^-} - T_0 \theta(x^+ - x^+_0) \left( x^+ - \frac{2\kappa \alpha}{\Lambda x^-} \right) \right\} \right], \] (4.45)

\[ R_{(ii)} = \frac{\alpha}{2} \sqrt{\frac{\Lambda}{q + c - \kappa \alpha/4}} \left[ 1 - \frac{1}{(\sqrt{q + c - \kappa \alpha/4})^2} \left( \sqrt{\frac{\Lambda}{4}} \left( ax^+ - \frac{x^-}{2a} + b \right) - \frac{\kappa \alpha}{4} \right) \right] \times \left\{ \left( \sqrt{\frac{\Lambda}{4}} (ax^+ - \frac{x^-}{2a} + b) - \frac{\kappa \alpha}{4} \right) - \frac{T_0}{a \sqrt{\Lambda}} \theta(x^+ - x^+_0) \right\} \right]. \] (4.46)

We can easily check that indeed \( R_{(i),(ii)} = 0 \) holds for \( x^+ < x^+_0 \) as planned.

By making use of these expressions in the region where \( v(q) \) is well-defined, it is straightforward to find the behavior of \( R \) for \( x^+ > x^+_0 \). Although there are interesting features in each case, we omit the details for brevity and comment only on the type (i) case with \( \alpha < 0 \).

First for \( \kappa = 0 \), a curvature singularity develops along the line \( r = 0 \) and \( R \) tends to vanish as \( x^+ \to \infty \) for \( \alpha > -2 \). On the other hand, when \( \kappa(> 0) \) is turned on, a curvature singularity now forms along \( r = -(\kappa \alpha/2)(1 - \ln(-\kappa \alpha/2)) \), which is the minimum of \( r(q) \).

In the limit \( x^+ \to \infty \), \( R \) vanishes for \( 0 > \alpha > -1 \) and diverges for \( \alpha \leq -1 \). Especially, at the critical value \( \alpha = -1 \), \( R \) diverges like \( \ln x^+ \) and the space-time is essentially the same as that in [14]. Although we have only dealt with a few models, this illustrates how one can incorporate possible boundary functions and widen the possibilities for new models.

### 4.3.3 Models with \( \Delta \neq 0 \)

When we allow \( \Delta \neq 0 \), it turns out to be quite difficult to find a model with asymptotically flat regions. (This is partly due to the fact that \( r \) stays finite even for \( x^+ \to \infty \), as already noted in (4.16).) Nevertheless, if we relax this condition and look for models which are at least asymptotically finite and at the same time exhibit a shock-wave-induced black hole
singularity, we find that, for example, the power type potential $v(q) = -\alpha q^{1+\beta}$ examined for $\Delta = 0$ is still viable for the same range of $\alpha$ and $\beta$, namely for $\alpha > 0$, $0 < \beta < 1$, provided that $\Delta < 0$. Recall that $q^{1+\beta}$ is well-defined only in the quadrant $0 < x^+ < \infty$, $-\infty < x^- < 0$. In this case, as $x^+$ goes to zero, $R$ tends to a positive constant:

$$R \rightarrow -\frac{1}{2}\Lambda\Delta. \quad (4.47)$$

On the other hand, for $x^+ \rightarrow \infty$, it goes to a finite function of $x^-$. Especially in the limit $-x^- \rightarrow \infty$,

$$R \rightarrow \frac{\alpha\beta(\beta + 1)\Lambda}{4c^2\Delta q^{1-\beta}(1 + (\kappa\alpha/2)(1 + \beta)q^\beta)^3} \left(\frac{-\Lambda x^-}{16cT_0}\right)^{1-2/(1-4c)}, \quad (4.48)$$

$$q^{1+\beta} = \frac{2}{\alpha(1 - 4c)} \ln \left(\frac{-\Lambda x^-}{16cT_0}\right), \quad (4.49)$$

where we have used (3.22). Since $\Delta < 0$ means $0 < c < 1/4$, this expression tends to vanish as $x^- \rightarrow -\infty$ and we have a flat space at the spatial infinity. As for the black hole singularity, it can easily be seen to occur along the $r = 0$ ($q = 0$) hyperbolic line in just the same way as for the $\Delta = 0$ case previously discussed. Admittedly the space-time described by this model is somewhat unusual. It is an open question whether one can find models with more conventional flat asymptotic behavior by considering different $v(q)$ and/or by incorporating suitable boundary functions.

## 5 Discussions

By observing that solvability of various models of dilaton gravity in two-dimensions can be understood as due to a powerful symmetry, we have formulated in a systematic and unified way how one can analyze a rather general class of such models from the point of view of non-linear sigma model. This formulation then allowed us to explore a number of new models and their space-time behaviors under the influence of the matter shock wave. Although these represent only a small subset of models which can be treated by our method, they serve to illustrate the general strategy and the power of our formulation.

In this article, we have been using the word “solvable” in a broad sense; it means that the model, including the effect of the measure as anomaly term, can be reduced to a collection of free fields and in general a field satisfying the Liouville equation. As was stated in the introduction, to be able to attack the challenging questions listed there, it
would be desirable to have a model which is quantum mechanically fully solvable, for any number of matter fields.

Let us briefly discuss what that would require in our formulation. First, in this respect, models with a genuine Liouville field are not likely to be useful since quantum Liouville dynamics is notoriously difficult, especially in the operator formalism. (For its problems, see, for example, [31]).

Now restricting to $\Delta = 0$ case, the immediate question is how we can perform the full quantization. In a special model with $\kappa = 0$ treated in [15], it was shown that with an appropriate measure the transformation from the original fields to the free fields $\psi$ and $\chi$ is a quantum canonical transformation and hence it was justified to use the usual free field quantization. It is not difficult to show that this argument can be extended to a generalization of that model with non-vanishing $\kappa$. It is constructed in such a way that the potential term becomes perturbatively a conformally $(1, 1)$ operator. Specifically the model is specified by (compare with Eqs.(3.39)$\sim$(3.42))

$$q = e^{-2\phi}, \quad v = \ln q, \quad \omega = \frac{1}{2}(2bq - \ln q),$$

$$\Omega = 0, \quad a(\phi) = -\kappa b^2, \quad c = \frac{1}{4}. \quad (5.1)$$

As in the model of [15], it is possible to construct all the physical states explicitly by DDF operators [32] and to further perform the analysis similar to the one presented in [16, 17].

For a more general class of models treated in this article, similar statement has not been demonstrated. As we shall discuss shortly, validity of the canonicity of transformation has an important bearing on the question of the target space general covariance in the string theory interpretation of the models.

Assuming for the moment that free-field quantization is justified, let us proceed to examine what other problems we must overcome. In that case, since the energy-momentum tensor has the simple form

$$T^{DG}_{\pm\pm} = \partial_+ \psi \partial_{\pm} \chi - \partial_\pm^2 \chi - \frac{\kappa}{4} \left( (\partial_{\pm} \psi)^2 - 2\partial_\pm^2 \psi \right), \quad (5.3)$$

the system can be shown to possess quantum conformal invariance. Thus, using the techniques of conformal field theory, it should be possible to find all the physical states and so on. Difficulty appears when one starts expressing various quantities in terms of
the free fields. The quantity directly expressible in terms of $\psi$ and $\chi$ is $r$, which for $\Delta = 0$ takes the form

$$r = -\chi - \frac{\Lambda}{4}AB. \quad (5.4)$$

On the other hand, quantities of interest may not be explicitly given as functions of $r$. For example, the conformal factor for the metric $e^{\psi - v}$ involves the potential function $v$. Since it is related to $r$ by $r = q - (\kappa/2)v$, if we specify $v$ as a function of $q$, as we did in the previous section, it is in general not possible to get $v(r)$ in a closed form. A way to circumvent this difficulty is, obviously, to specify the model by giving $v(r)$ directly but then $v(\phi)$ would in general be $\kappa$-dependent. Although we did not adopt this point of view in this article, it may be worth pursuing in the fully quantum treatment.

Finally let us discuss possible implications of our work when the non-linear sigma model is interpreted as a model in string theory. Since the special symmetry we have imposed can be characterized completely in a target space covariant manner, naively all the models with equal $\Delta = V_k V^k$ and $Q_k Q^k$, which are target space invariant, are expected to describe the same string theory in flat background and hence they should have identical spectrum. Indeed the energy-momentum tensor expressed in terms of the free fields $\psi$ and $\chi$ depends only on these two parameters. But of course the validity of this argument crucially hinges on the nature of the functional measure. Only if we can find a general coordinate invariant measure for $X^k$ such that it reduces to a simple measure for the free fields $\psi$ and $\chi$ through a canonical transformation, can we realize the general coordinate invariance in string theory. This is quite similar to the line of thought pursued some years ago in [33] in a more general setting. Since the understanding of the emergence of target space general covariance in string theory is an unsolved problem, it would be interesting if one can demonstrate it for our simple models.

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