SUPERSATURATED IDEALS

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Dedicated to the memory of Ken Kunen

Abstract. An ideal $I$ on a set $X$ is supersaturated iff $\text{add}(I) \geq \omega_2$ and for every family $\mathcal{F}$ of $I$-positive sets with $|\mathcal{F}| < \text{add}(I)$, there exists a countable set that meets every set in $\mathcal{F}$. We show that many well-known ccc forcings preserve supersaturation. We also show that the existence of supersaturated ideals is independent of ZFC plus “There exists an $\omega_1$-saturated $\sigma$-ideal”.

1. Introduction

Saturation properties of ideals are ubiquitous in modern set theory and there is a considerable body of work (for example, see [3, 5, 6, 7]) on the study of a large number of such properties. Throughout this paper, by an ideal $I$ on $X$, we mean an ideal $I$ on $X$ that contains every finite subset of $X$. Supersaturation is a strengthening of $\omega_1$-saturation defined as follows.

Definition 1.1. Suppose $I$ is an ideal on $X$ and $\lambda$ is a cardinal. We say that $I$ is $\lambda$-supersaturated iff $\text{add}(I) \geq \lambda^+$ and for every $A \subseteq I^+$, if $|A| < \text{add}(I)$, then there exists $W \in [X]^{<\lambda}$ such that for every $A \in A$, $A \cap W \neq \emptyset$. $I$ is supersaturated iff it is $\omega_1$-supersaturated.

Suppose $I$ is a supersaturated ideal on $X$. Since $\text{add}(I) \geq \omega_2$, it follows that $I^+$ cannot have an uncountable subfamily of pairwise disjoint sets because no countable set can meet all of them. So $I$ is $\omega_1$-saturated. Let $\mu = \text{add}(I)$. Ulam showed that either $\mu$ is a measurable cardinal or $\mu$ is a weakly inaccessible cardinal $\leq c$. Solovay showed that $\mu$ admits a normal $\omega_1$-saturated ideal $\mathcal{J}$ and $\mu$ is a measurable cardinal in the inner model $L[\mathcal{J}]$. For proofs of these facts, see [7].

Though closely related to some of the works of Fremlin, supersaturated ideals were formally introduced in [4] where it was shown that if $\kappa \leq c$ admits a normal supersaturated ideal then the order dimension of the Turing degrees is at least $\kappa$. An earlier motivation for investigating these ideals comes from the following question of Fremlin – See Problem EG(h) in [1].

Question 1.2 (Fremlin). Suppose $\kappa$ is real valued measurable and $m : \mathcal{P}(\kappa) \to [0,1]$ is a witnessing normal measure. Let $\mathcal{F}$ be a family of subsets of $\kappa$ such that $|\mathcal{F}| < \kappa$ and for every $A \in \mathcal{F}$, $m(A) > 0$. Must there exist a countable $N \subseteq \kappa$ such that for every $A \in \mathcal{F}$, $N \cap A \neq \emptyset$?

So Question 1.2 is asking if the null ideal of every normal witnessing measure on a real valued measurable cardinal must be supersaturated. One of the standard ways of obtaining $\omega_1$-saturated ideals on cardinals below the continuum is to start

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with a measurable cardinal $\kappa$ and a witnessing normal prime ideal $\mathcal{I}$ on $\kappa$, and force with a ccc forcing $\mathbb{P}$ that adds $\geq \kappa$ reals. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^\mathbb{P}$. Then $\mathcal{J}$ is always an $\omega_1$-saturated normal ideal on $\kappa \leq \omega_1$. But whether or not $\mathcal{J}$ is supersaturated will depend on the choice of $\mathbb{P}$. This motivates the notion of supersaturation preserving forcings (Definition 2.1). In Section 2, we show that a large class of ccc forcings for adding new reals are supersaturation preserving. In particular, the following holds.

**Theorem 1.3.** Let $\text{Random}_\lambda$ denote the forcing for adding $\lambda$ random reals.

1. Every $\sigma$-linked forcing is supersaturation preserving.
2. $\text{Random}_\lambda$ is supersaturation preserving for every $\lambda$.

The question of whether every $\omega_1$-saturated ideal must be supersaturated was raised in [4]. Our main result shows that this is independent.

**Theorem 1.4.** Each of the following is consistent.

1. There is an $\omega_1$-saturated ideal on a cardinal below the continuum and there are no supersaturated ideals.
2. There is an $\omega_1$-saturated ideal on a cardinal below the continuum and every $\omega_1$-saturated ideal is supersaturated.

**Notation:** Let $\mathcal{I}$ be an ideal on $X$. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. $\text{add}(\mathcal{I})$ denotes the least cardinality of a subfamily of $\mathcal{I}$ whose union is in $\mathcal{I}^+$. For $A \subseteq X$, define $\mathcal{I} \upharpoonright A = \{ Y \subseteq X : Y \cap A \in \mathcal{I} \}$. Suppose $V \subseteq W$ are transitive models of set theory, $X, \mathcal{I} \in V$ and $V \models \text{``$\mathcal{I}$ is an ideal on $X$''}$. Recall that the ideal generated by $\mathcal{I}$ in $W$ is $\mathcal{J} = \{ A \in W : (\exists B \in \mathcal{I}) (A \subseteq B) \}$.

For a set of ordinals $X$, $\text{otp}(X)$ denotes the order type of $X$. An ordinal $\delta$ is indecomposable iff for every $X \subseteq \delta$, either $\text{otp}(X) = \delta$ or $\text{otp}(\delta \setminus X) = \delta$. If $\mathbb{P}, \mathbb{Q}$ are forcing notions, we write $\mathbb{P} \leq \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ is also a maximal antichain in $\mathbb{Q}$. $\text{Cohen}_\lambda$ denotes the forcing for adding $\lambda$ Cohen reals. $\text{Random}_\lambda$ is the measure algebra on $2^\lambda$ equipped with the usual product measure denoted by $\mu_\lambda$. If $\lambda$ is clear from the context, then we drop it and just write $\mu$.

2. CCC FORCINGS AND SUPERSATURATION

**Definition 2.1.** A forcing $\mathbb{P}$ is $\kappa$-ssp (ssp = supersaturation preserving) iff for every normal supersaturated ideal $\mathcal{I}$ on $\kappa$, $V^\mathbb{P} \models \text{``the ideal generated by $\mathcal{I}$ is supersaturated''}$. $\mathbb{P}$ is ssp iff it is $\kappa$-ssp for every $\kappa$.

In [4], the following forcings were shown to be $\kappa$-ssp for every $\kappa$.

(a) $\text{Cohen}_\lambda$ for any $\lambda$.

(b) Any finite support iteration of ccc forcings of size $< \kappa$.

It was also shown that $\text{Random}_\lambda$ is $\kappa$-ssp for any measurable $\kappa$. The next theorem improves this to all $\kappa$.

**Theorem 2.2.** $\text{Random}_\lambda$ is $\kappa$-ssp for every $\kappa$ and $\lambda$.

**Proof.** Fix a normal supersaturated ideal $\mathcal{I}$ on $\kappa$. Put $\mathbb{B} = \text{Random}_\lambda$ and let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^\mathbb{B}$. Suppose $\theta < \kappa$ and $\models_{\mathbb{B}} \langle \hat{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets. It suffices to find $B \in [\kappa]^\omega_0$ such that $\models_{\mathbb{B}} (\forall i < \theta) (\hat{A}_i \cap B \neq \emptyset)$.

For $i < \theta$ and $\alpha < \kappa$, put $p_{i, \alpha} = \{ \alpha \in \hat{A}_i \}_\mathbb{B}$. So each $p_{i, \alpha}$ is a Baire subset of $2^\lambda$. Put $T_i = \{ \alpha < \kappa : p_{i, \alpha} \neq 0_\mathbb{B} \}$.
Claim 2.3. For each \( p \in B \setminus \{0_B\} \), \( \{\alpha \in T_i : p_i,\alpha \cap p \neq 0_B\} \in \mathcal{I}^+ \).

Proof. Put \( X_p = \{\alpha \in T_i : p_i,\alpha \cap p \neq 0_B\} \) and suppose \( X_p \in \mathcal{I} \). Since the empty condition forces that \( A_i \in \mathcal{F}^+ \), it follows that for every \( X \in \mathcal{I} \), \( \{p_i,\alpha : \alpha \in T_i \setminus X\} \) is predense in \( B \). But every condition in \( \{p_i,\alpha : \alpha \in T_i \setminus X_p\} \) is incompatible with \( p \) which is impossible. \( \square \)

For a finite partial function \( f \) from \( \lambda \) to 2, define \( \{f = \{x \in 2^\lambda : |\text{dom}(f)| = f \} \).

For a clopen \( K \subseteq 2^\lambda \), define \( \text{supp}(K) \) to be the smallest finite set \( S \subseteq \lambda \) such that \( (\forall x, y \in 2^\lambda)(x \mid S = y \mid S \iff x \in K = y \in K) \). If \( \text{supp}(K) = S \), then there is finite list \( \{f_{K,n} : n < n_*\} \) where \( f_{K,n} \)'s are pairwise distinct functions from \( S \) to 2 and \( K = \bigcup_{n < n_*} [f_{K,n}] \).

Definition 2.4. Suppose \( C \) is a family of clopen sets in \( 2^\lambda \). We say that \( C \) is a strong \( \Delta \)-system of width \( (n_*, N_*) \) iff \( n_*, N_* < \omega \) and the following hold.

(a) \( (\text{supp}(K) : K \in C) \) is a \( \Delta \)-system with root \( R \).
(b) For every \( K \in C \), \( |\text{supp}(K) \setminus R| = n_* \).
(c) For every \( K \in C \), \( K = \bigsqcup_{n < N_*} [f_{K,n}] \) where each \( f_{K,n} : \text{supp}(K) \rightarrow 2 \) and \( f_{K,n} \)'s are pairwise distinct.
(d) For every \( K_1, K_2 \in C \) and \( n < N_* \),
   (i) \( f_{K_1,n} \upharpoonright R = f_{K_2,n} \upharpoonright R \) and
   (ii) if for \( m \in \{1, 2\} \), \( \xi^* : j < |R| + n_* \) lists \( \text{supp}(K_m) \) in increasing order, then \( f_{K_1,j}(\xi^*_j) = f_{K_2,j}(\xi^*_j) \) for every \( j < |R| + n_* \).

Lemma 2.5. Suppose \( p \subseteq 2^\lambda \) is Baire and \( C \) is an infinite strong \( \Delta \)-system of clopen sets in \( 2^\lambda \) of width \( (n_*, N_*) \). Let \( \varepsilon > 0 \) and assume that for infinitely many \( K \in C \), \( \mu(p \cap K) \geq \varepsilon \). Then for all but finitely many \( K \in C \), \( \mu(p \cap K) \geq \varepsilon/2 \).

Proof. Let \( R \) be the root of \( (\text{supp}(K) : K \in C) \). For each \( K \in C \), fix \( \{f_{K,n} : n < N_*\} \) such that \( K = \bigsqcup_{n < N_*} [f_{K,n}] \). First suppose that \( p \) is clopen. Let \( C_p = \{K \in C : (\text{supp}(K) \setminus R) \cap \text{supp}(p) = \emptyset\} \). Then \( C \setminus C_p \) is finite and for each \( K \in C_p \),

\[
\mu(p \cap K) = \sum_{n < N_*} \mu(p \cap [f_{K,n}]) = 2^{-n_*} \sum_{n < N_*} \mu(p \cap [f_{K,n} \upharpoonright R])
\]

which does not depend on \( K \in C_p \). It follows that the result holds if \( p \) is clopen. The general case follows by applying the previous case to a clopen \( q \subseteq 2^\lambda \) satisfying \( \mu(p \Delta q) < \varepsilon/2 \). \( \square \)

For each \( \alpha \in T_i \), fix \( S_{i,\alpha} \subset |\lambda|^{\aleph_0} \) such that \( p_{i,\alpha} \) is supported in \( S_{i,\alpha} \). For every \( i < \theta, \alpha \in T_i \) and \( \varepsilon > 0 \) rational, choose a clopen set \( K_{i,\alpha,\varepsilon} \subseteq 2^\lambda \) with \( \text{supp}(K_{i,\alpha,\varepsilon}) \subseteq S_{i,\alpha} \) such that

\[
\frac{\mu(p_{i,\alpha} \Delta K_{i,\alpha,\varepsilon})}{\mu(K_{i,\alpha,\varepsilon})} < \varepsilon
\]

Claim 2.6. For each \( i < \theta \) and \( \varepsilon > 0 \) rational, we can find \( \mathcal{F}_{i,\varepsilon} \subseteq \mathcal{I}^+ \) and \( \{(n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y}) : Y \in \mathcal{F}_{i,\varepsilon} \} \) such that the following hold.

1. \( \mathcal{F}_{i,\varepsilon} \) is a countable family of pairwise disjoint sets and \( T_i \setminus \bigcup \mathcal{F}_{i,\varepsilon} \in \mathcal{I} \).
2. For each \( Y \in \mathcal{F}_{i,\varepsilon} \), \( \{K_{i,\alpha,\varepsilon} : \alpha \in Y\} \) is a strong \( \Delta \)-system of width \( (n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y}) \).
Proof. Fix $i < \theta$ and $\varepsilon > 0$ rational. To simplify notation, we write $K_\alpha$ instead of $K_{i,\alpha,\varepsilon}$. It suffices to show that for every $\mathcal{I}$-positive $X \subseteq T_i$, there exists $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and there exist $(n_\gamma, N_\gamma)$ such that $\{K_\alpha : \alpha \in Y\}$ is a strong $\Delta$-system of width $(n_\gamma, N_\gamma)$. Since then we can take $\mathcal{F}_{i,\varepsilon}$ to be a maximal disjoint family of such $Y$'s. That each $\mathcal{F}_{i,\varepsilon}$ is countable follows from the fact that $\mathcal{I}$ is $\omega_1$-saturated.

Fix a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $\alpha \in T_i \cap \gamma$, $\max(\supp(K_\alpha)) < \gamma$. Suppose $X \subseteq T_i \cap E$ and $X \in \mathcal{I}^+$. Since $\mathcal{I}$ is normal and the map $\alpha \mapsto \max(\supp(K_\alpha))$ is regressive on $X$, we can find $R \subseteq \kappa$ finite and $Y_1 \subseteq X$ such that $Y_1 \in \mathcal{I}^+$, $(\forall \alpha \in Y_1)(\supp(K_\alpha) \cap \alpha = R)$ and $\supp(K_\alpha) \setminus R = n_\ast$ does not depend on $\alpha \in Y_1$. It also follows that $(\supp(K_\alpha) : \alpha \in Y_1)$ forms a $\Delta$-system with root $R$. For each $\alpha \in Y_1$, let $K_\alpha = \bigcup_{n < N_\alpha}[f_{\alpha,n}]$ where each $f_{\alpha,n} : \supp(K_\alpha) \rightarrow 2$. Choose $Y_2 \subseteq Y_1$ such that $Y_2 \in \mathcal{I}^+$ and $N_\alpha = N_\ast$ does not depend on $\alpha \in Y_2$. Finally, choose $Y \subseteq Y_2$ such that $Y \in \mathcal{I}^+$ and $\{K_\alpha : \alpha \in Y\}$ is a strong $\Delta$-system of width $(n_\ast, N_\ast)$.

Since $\mathcal{I}$ is supersaturated, we can choose $B \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$, $\varepsilon > 0$ rational and $Y \in \mathcal{F}_{i,\varepsilon}$, we have $|B \cap Y| = \aleph_0$. It suffices to show that for each $i < \theta$, $\{p_\alpha : \alpha \in B\}$ is predense in $\mathbb{B}$.

Suppose not. Fix $i < \theta$ and $p \subseteq 2^\lambda$ Baire such that $\mu(p) > 0$ and for every $\alpha \in B$, $\mu(p_\alpha \cap p) = 0$. Let $X = \{\alpha \in T_i : \mu(p_\alpha \cap p) > 0\}$. By Claim 2.6, $X \in \mathcal{I}^+$. Using the argument in the proof of Claim 2.6, we can choose $\varepsilon > 0$ rational, $X_\ast \subseteq X$ and $n_\ast, N_\ast < \omega$ such that

- (a) $X_\ast \in \mathcal{I}^+$ and for each $\alpha \in X_\ast$, $\mu(p_\alpha \cap p) \geq 4\varepsilon$.
- (b) $\{K_{i,\alpha,\varepsilon} : \alpha \in X_\ast\}$ is a strong $\Delta$-system of width $(n_\ast, N_\ast)$.

Choose $Y \in \mathcal{F}_{i,\varepsilon}$ such that $Y \cap X_\ast \in \mathcal{I}^+$. Since $|Y \cap X_\ast| \geq \aleph_0$ and $|Y \cap B| = \aleph_0$, by Lemma 2.5, we can choose $\alpha \in Y \cap B$ such that $\mu(p \cap K_{i,\alpha,\varepsilon}) \geq 2\varepsilon$. But since $\mu(p_\alpha \Delta K_{i,\alpha,\varepsilon}) \leq \varepsilon \mu(K_{i,\alpha,\varepsilon}) \leq \varepsilon$, it follows that $\mu(p \cap p_\alpha) \geq \varepsilon > 0$: Contradiction. This completes the proof of Theorem 2.2.

**Theorem 2.7.** Every $\sigma$-linked forcing is $\kappa$-ssp for every $\kappa$.

**Proof.** Let $\mathcal{I}$ be a normal supersaturated ideal on $\kappa$. Suppose $\mathbb{P}$ is a $\sigma$-linked forcing and $\mathcal{J}$ is the ideal generated by $\mathcal{I}$ in $V^\mathbb{P}$. Fix $\theta < \kappa$ and WLOG, assume that the trivial condition forces that $(\dot{A}_i : i < \theta)$ is a sequence of $\mathcal{J}$-positive sets. It suffices to construct $X \in [\kappa]^{\aleph_0}$ such that $\Vdash \theta(\forall i < \theta)(X \cap \dot{A}_i \neq \emptyset)$.

Since $\mathbb{P}$ is $\sigma$-linked, we can write $\mathbb{P} = \bigcup \{L_n : n < \omega\}$ where each $L_n \subseteq \mathbb{P}$ has pairwise compatible members. For each $i < \theta$ and $n < \omega$, define

$$B_{i,n} = \{\alpha < \kappa : (\exists p \in L_n)(p \Vdash \alpha \in \dot{A}_i)\}$$

**Claim 2.8.** $W_i = \bigcup \{L_n : n < \omega, B_{i,n} \in \mathcal{I}^+\}$ is dense in $\mathbb{P}$.

**Proof.** Suppose not and fix $p \in \mathbb{P}$ such that no extension of $p$ lies in $W_i$. Put $C = \{\alpha < \kappa : (\exists q \leq p)(q \Vdash \alpha \in \dot{A}_i)\}$. Since no extension of $p$ lies in $W_i$, it follows that $C \subseteq \bigcup \{B_{i,n} : n < \omega, B_{i,n} \in \mathcal{I}\}$ and hence $C \in \mathcal{I}$. It now follows that $p \Vdash \dot{A}_i \in \mathcal{J}$ which is impossible. □

Since $\mathcal{I}$ is supersaturated, we can find a countable $X \subseteq \kappa$ such that for every $i < \theta$ and $n < \omega$, if $B_{i,n} \in \mathcal{I}^+$, then $X \cap B_{i,n} \neq \emptyset$. We claim that $\Vdash (\forall i < \theta)(X \cap \dot{A}_i \neq \emptyset)$. □
Suppose not and fix \( p \in \mathbb{P} \) and \( i < \theta \) such that \( p \Vdash X \cap \check{A}_i = \emptyset \). Using Claim 2.8, choose \( n < \omega \) and \( p' \leq p \) such that \( p' \in L_n \) and \( B_{i,n} \in \mathbb{I}^+ \). Choose \( \alpha \in B_{i,n} \cap X \) and \( q \in L_n \) such that \( q \Vdash \alpha \in \check{A}_i \). Since \( L_n \) is linked, we can find a common extension \( r \in \mathbb{P} \) of \( p', q \). But \( r \Vdash \alpha \in X \cap \check{A}_i \); Contradiction. \( \square \)

**Corollary 2.9.** Each of the following forcings is ssp: Cohen, random, Amoeba, Hechler, Eventually different real forcing.

We do not know if we can improve Theorem 2.7 to the class of \( \sigma \)-finite-cc forcings. For example, one can ask the following.

**Question 2.10.** Suppose \( \mathbb{B} \) is a boolean algebra and \( m : \mathbb{B} \to [0,1] \) is a strictly positive finitely additive measure on \( \mathbb{B} \). Must \( \mathbb{B} \) be supersaturation preserving?

The next two facts are well known.

**Fact 2.11.** Suppose \( \mathbb{P} \) is a separative \( \sigma \)-linked forcing. Then \( |\mathbb{P}| \leq \mathfrak{c} \).

**Fact 2.12.** Let \( \langle (P_\xi, Q_\xi) : \xi < \lambda \rangle \) be a finite support iteration with limit \( P_\lambda \) where for every \( \xi < \lambda \), \( V^{P_\xi} \models \theta \), Assume \( \lambda < \mathfrak{c}^+ \). Then \( P_\lambda \) is also \( \sigma \)-linked.

**Theorem 2.13.** Let \( \mathcal{I} \) be a normal supersaturated ideal on \( \kappa \) and let \( \lambda \leq \kappa^+ \). Suppose \( \langle (P_\xi, Q_\xi) : \xi < \lambda \rangle \) is a finite support iteration with limit \( P_\lambda \) where for every \( \xi < \lambda \), \( V^{P_\xi} \models \theta \). Let \( J \) be the ideal generated by \( \mathcal{I} \) in \( V^{P_\lambda} \). Then \( J \) is supersaturated.

**Proof.** By induction on \( \lambda \). First suppose \( \lambda < \mathfrak{c} \). If \( \lambda < \kappa^+ \), then by Fact 2.12, \( P_\lambda \) is \( \sigma \)-linked and the claim holds by Theorem 2.7. So assume \( \lambda = \kappa^+ \) and fix any \( P_\lambda \)-generic filter \( G_\lambda \) over \( V \). Let \( \langle A_i : i < \theta \rangle \) be a sequence of \( J \)-positive sets in \( V[G_{\lambda}] \) where \( \theta < \kappa \). Since \( P_\lambda \) is a finite support iteration of ccc forcings, there exists \( \eta < \lambda = \kappa^+ \) such that \( \langle A_i : i < \theta \rangle \in V[G_{\eta}] \) where \( G_{\eta} = P_\eta \cap G_\lambda \). Note that each \( A_i \) is \( J_{\eta} \)-positive where \( J_{\eta} \) is the ideal generated by \( \mathcal{I} \) in \( V[G_{\eta}] \). By inductive hypothesis, there is a countable set that meets \( A_i \) for every \( i < \theta \). Hence \( J \) is supersaturated.

Next assume \( \kappa > \mathfrak{c} \). Then \( \kappa \) is measurable and \( \mathcal{I} \) is a normal prime ideal on \( \kappa \). First suppose \( \lambda \leq \kappa \). By Fact 2.11, \( |P_\xi| \leq |\xi \cdot \mathfrak{c}| = \kappa \) for every \( \xi < \kappa \). Hence by Theorem 4.9 in \([4]\), it follows that \( J \) is supersaturated. Next suppose \( \kappa < \lambda \leq \kappa^+ \). Note that \( V^{P_\lambda} \models \mathfrak{c} \geq \kappa \) since Cohen reals are added at each stage of cofinality \( \omega \). So we can work in \( V^{P_\lambda} \) and repeat the argument for the case \( \kappa \leq \mathfrak{c} \). \( \square \)

It is now natural to ask the following.

**Question 2.14** ([3]). Suppose \( \kappa \) is measurable. Is every ccc forcing \( \kappa \)-ssp?

In Section [4] we’ll show that the answer is negative. We end this section with the following weaker positive result.

**Theorem 2.15.** Suppose \( \kappa \) is measurable and \( \mathcal{I} \) is a normal prime ideal on \( \kappa \). Let \( \mathbb{B} \) be a ccc complete boolean algebra. Then \( V^\mathbb{B} \models \text{"the ideal generated by \( \mathcal{I} \) is } \omega_2 \text{-supersaturated."} \)

**Proof.** It suffices to show that the following holds in \( V^\mathbb{B} \): For every \( A \subseteq J^+ \), if \( |A| < \kappa \), then there exists \( X \in [\kappa]^{\mathfrak{c}} \) such that \( X \) meets every member of \( A \).
Suppose $\theta < \kappa$ and $\forces \{ A_i : i < \theta \} \subseteq J^+$. Choose $Y \subseteq \kappa$ of $\mathcal{I}$-measure one such that for every $i < \theta$ and $\alpha \in Y$, $p_{i,\alpha} = [\alpha \in A_i] > 0\mathbb{B}$. Using the inaccessibility of $\kappa$, the following claim is easy to check.

**Claim 2.16.** There exists $(\mathbb{B}_\alpha : \alpha < \kappa)$ such that the following hold.

(i) $\mathbb{B}_\alpha \triangleleft \mathbb{B}$ and $|\mathbb{B}_\alpha| < \kappa$.

(ii) $\mathbb{B}_\alpha$'s are increasing and continuous at $\alpha$ when $\text{cf}(\alpha) > \aleph_0$.

(iii) $\{ p_{i,\beta} : \beta < \alpha, i < \theta \} \subseteq \mathbb{B}_\alpha$.

Let $\pi_\alpha : \mathbb{B} \to \mathbb{B}_\alpha$ be a projection map witnessing $\mathbb{B}_\alpha \triangleleft \mathbb{B}$. Choose $f : \kappa \to \kappa$ such that for every $i < \theta$ and $\alpha < \kappa$, we have $\alpha < f(\alpha)$ and $p_{i,\alpha} \in \mathbb{B}_{f(\alpha)}$. Choose $Y_1 \subseteq Y$ of measure one and $\alpha_* < \kappa$ such that for every $i < \theta$, $\pi_\alpha(p_{i,\alpha}) = p_{i,\alpha} \in \mathbb{B}_{\alpha_*}$ does not depend on $\alpha \in Y_1$ and $\text{range}(f \lceil \alpha) \subseteq \alpha$ for every $\alpha \in Y_1$. Note that $p_{i,\alpha} = 1_\mathbb{B}$ since $\forces \forall A_i \in J^+$. Let $X \subseteq Y \setminus \alpha_*$ be such that $\text{otp}(X) = \omega_1$ and for every $\alpha < \beta$ in $X$, $f(\alpha) < f(\beta)$.

**Claim 2.17.** For every $i < \theta$, $\{ p_{i,\alpha} : \alpha \in X \}$ is predense in $\mathbb{B}$.

**Proof.** Let $\text{sup}(X) = \gamma_*$. Then $\text{cf}(\gamma_*) = \aleph_1$ and hence $\mathbb{B}_{\gamma_*} = \bigcup \{ \mathbb{B}_\gamma : \gamma \in X \}$. Fix $i < \theta$. Given $p \in \mathbb{B}$, choose $\gamma \in X$ such that $\pi_{\gamma_*}(p) \in \mathbb{B}_\gamma$. Now since

$$\mathbb{B} = \mathbb{B}_\gamma \ast \mathbb{B}_{\gamma_+} / \mathbb{B}_\gamma \ast \mathbb{B} / \mathbb{B}_\gamma,$$

we can decompose $p = (\pi_{\gamma_*}(p), 1, x)$ and $p_{i,\gamma} = (1, y, 1)$. Hence $p, p_{i,\gamma}$ are compatible. $\square$

It follows that $\mathcal{J}$ is $\omega_2$-supersaturated. $\square$

3. Consistently, there are $\omega_1$-saturated ideals on $\mathfrak{c}$ and all of them are supersaturated

The aim of this section is to show that it is consistent that every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Theorem 3.1.** It is consistent that there is a normal supersaturated ideal on $\mathfrak{c}$ and every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Lemma 3.2.** Suppose that every $\sigma$-ideal $\mathcal{I}$ satisfying (i)-(iv) below is supersaturated.

(i) $\mathcal{I}$ is a uniform ideal on $\lambda$,

(ii) $\mu \subseteq \lambda$,

(iii) for every $X \in \mathcal{I}^+$, $\text{add}(\mathcal{I} \upharpoonright X) = \mu$ and

(iv) $\mathcal{I}$ is $\omega_1$-saturated.

Then every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Proof.** Suppose $\mathcal{J}$ is an $\omega_1$-saturated $\sigma$-ideal on $X$. Note that for every $A \in \mathcal{J}^+$, there exists $B \subseteq A$ such that $(\ast)_B$ holds where

$$(\ast)_B \text{ says the following: } B \in \mathcal{J}^+, [B]^{<|B|} \subseteq \mathcal{J} \text{ and for every } C \subseteq B, \text{ if } C \in \mathcal{J}^+, \text{ then } \text{add}(\mathcal{J} \upharpoonright C) = \text{add}(\mathcal{J} \upharpoonright B).$$

Since $\mathcal{J}$ is $\omega_1$-saturated, we can find a countable partition $\mathcal{F}$ of $X$ such that for each $B \in \mathcal{F}$, $(\ast)_B$ holds. Now by assumption, each $\mathcal{J} \upharpoonright B$ is supersaturated. Hence $\mathcal{J}$ is also supersaturated. $\square$
Lemma 3.3. Suppose $\mathbb{P}$ is a ccc forcing, $\kappa > \mathfrak{c}$ and $V^\mathbb{P} \models \mathcal{J}$ is a $\kappa$-complete $\omega_1$-saturated uniform ideal on $\lambda$. Let $\mathcal{I} = \{X \subseteq \kappa : 1_p \models X \in \mathcal{J}\}$. Then there is a countable partition $\mathcal{F}$ of $\lambda$ such that for every $A \in \mathcal{F}$, $\mathcal{I} \models A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a $\kappa$-complete prime ideal on $\lambda$.

Proof. It is clear that $\mathcal{I}$ is a $\kappa$-complete uniform ideal on $\lambda$. Suppose $\mathcal{F} \subseteq \mathcal{I}^+$ is an uncountable family of pairwise disjoint sets. For each $A \in \mathcal{F}$, choose $p_A \in \mathbb{P}$ such that $p_A \models A \notin \mathcal{J}$. Since $\mathbb{P}$ is ccc, some $p \in \mathbb{P}$ forces uncountably many $p_A$’s into the $\mathbb{P}$-generic filter. But this contradicts the fact that $\mathcal{J}$ is $\omega_1$-saturated in $V^\mathbb{P}$. So $\mathcal{I}$ is $\omega_1$-saturated. Since $\mathcal{I}$ is $\kappa$-complete and $\kappa > \mathfrak{c}$, $\mathcal{I}$ is nowhere atomless. Hence there is a countable partition $\mathcal{F}$ of $\lambda$ such that for every $A \in \mathcal{F}$, $\mathcal{I} \models A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a $\kappa$-complete prime ideal on $\lambda$. \[\square\]

Lemma 3.4. Suppose $\kappa$ is an inaccessible cardinal and $\mathcal{U}$ is a $\kappa$-complete uniform ultrafilter on $\lambda$. Let $\mathbb{P} = \text{Cohen}_\kappa$. Let $\mathcal{J}$ be the ideal generated by the dual ideal of $\mathcal{U}$ in $V^\mathbb{P}$. Then for each $A \subseteq \mathcal{J}^+$, if $|A| < \kappa$, then there exists a countable set that meets every member of $A$.

Proof. We identify conditions $p \in \mathbb{P}$ as members of the Baire algebra on $2^\kappa$ which is the $\sigma$-algebra generated by clopen subsets of $2^\kappa$. Note that for every Baire $p \subseteq 2^\kappa$, there is a countable $S \subseteq \kappa$ such that for every $x, y \in 2^\kappa$ satisfying $x \upharpoonright S = y \upharpoonright S$, we have $x \in p$ iff $y \in p$. We call such an $S$, a support of $p$. The ordering on $\text{Cohen}_\kappa$ is defined by $p \leq q$ iff $p \setminus q$ is meager in $2^\kappa$. Recall that if $p \subseteq 2^\kappa$ is Baire and $S \in [\kappa]^\omega$ is a support of $p$ then there is a countable family $\mathcal{P}$ of clopen subsets of $2^\kappa$ each supported in $S$ such that the symmetric difference of $p$ and $\bigcup \mathcal{P}$ is meager. So $p$ is completely determined by the family $\mathcal{P}$.

It is clear that $\mathcal{J}$ is a $\kappa$-complete uniform ideal on $\lambda$. Suppose $\theta < \kappa$ and $\langle \dot{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets in $V^\mathbb{P}$. WLOG, assume that the trivial condition forces this. For $i < \theta$ and $\alpha < \lambda$, let $p_{i,\alpha} = [[\alpha \in \dot{A}_i]]_p$. Note that for each $i < \theta$, and $Z \in \mathcal{U}$, $\{p_{i,\alpha} : \alpha \in Z\}$ is predense in $\mathbb{P}$ since otherwise some condition will force $\dot{A}_i \in \mathcal{J}$. Since $\mathcal{U}$ is $\kappa$-complete, we can choose $X \in \mathcal{U}$ such that for every $i < \theta$ and $\alpha \in X$, $p_{i,\alpha} > 0_p$. Let $S_{i,\alpha} \in [\kappa]^\omega$ be a support of $p_{i,\alpha}$. Since $\kappa$ is inaccessible, we can choose $Y \subseteq X$ such that $Y \in \mathcal{U}$ and for each $i < \theta$, the following hold.

(a) For every $\alpha, \beta \in Y$, $(S_{i,\alpha}, 2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (S_{i,\beta}, 2^{S_{i,\beta}}, p_{i,\beta})$. Put $\text{otp}(S_{i,\alpha}) = \gamma_i$. Let $h_{i,\alpha} : \gamma_i \rightarrow S_{i,\alpha}$ be the order isomorphism and define $H_{i,\alpha} : 2^{\gamma_i} \rightarrow 2^{S_{i,\alpha}}$ by $H_{i,\alpha}(x) = x \circ h_{i,\alpha}^{-1}$. Choose $p_i \subseteq 2^{\gamma_i}$ such that $H_{i,\alpha}[p_i] = p_{i,\alpha}$.

(b) For each $\gamma < \gamma_i$, either $|\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1$ or for every $Z \in \mathcal{U}$, $|\{h_{i,\alpha}(\gamma) : \alpha \in Z \cap Y\}| \geq \kappa$. Put $\Gamma_i = \{\gamma < \gamma_i : |\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1\}$ and $h_{i,\alpha}^{\Gamma_i} = R_i$.

Define $B_{i,\alpha} = \{x \in 2^{R_i} : y \upharpoonright (S_{i,\alpha} \setminus R_i) : y \in p_{i,\alpha} \wedge y \upharpoonright R_i = x\}$ is meager}. Then $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y$ and $B_i$ is meager in $2^{R_i}$. Since otherwise $\{p_{i,\alpha} : \alpha \in Y\}$ will not be predense in $\mathbb{P}$.

Using (b), choose $B \in [Y]^\omega$ such that for every $i < \theta$ and $\alpha \neq \beta$ in $B$, $S_{i,\alpha} \cap S_{i,\beta} = R_i$. It follows now that for every $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in $\mathbb{P}$. Hence $\models (\forall i < \theta)(B \cap \dot{A}_i \neq \emptyset)$. \[\square\]
Proof of Theorem 3.1 Let $V \models \kappa = \omega_1$ and $\kappa$ is the least measurable cardinal. Let $\mathbb{P} = \text{Cohen}_\kappa$. We already know that there is a normal supersaturated ideal on $\kappa = \kappa$ in $V^\mathbb{P}$. Let us check that, $V^\mathbb{P} \models \text{“Every } \omega_1\text{-saturated } \sigma\text{-ideal is supersaturated”}$. By Lemma 3.2, it suffices to consider ideals $\mathcal{J}$ that satisfy the following for some $\omega_1 \leq \mu \leq \kappa$.

(i) $\mathcal{J}$ is a uniform ideal on $\lambda$,
(ii) for every $X \in \mathcal{J}^+$, $\text{add}(\mathcal{J} \upharpoonright X) = \mu$ and
(iii) $\mathcal{J}$ is $\omega_1$-saturated.

Since $V^\mathbb{P} \models \kappa = \kappa$, we can assume that $\mu \leq \kappa$. Otherwise there is a countable partition $\mathcal{E}$ of $\lambda$ into $\mathcal{J}$-positive sets such that for each $X \in \mathcal{E}$, $\mathcal{J} \upharpoonright X$ is a $\mu$-complete prime ideal and it easily follows that $\mathcal{J}$ is supersaturated.

Towards a contradiction, suppose $\mu < \kappa$. Working in $V^\mathbb{P}$, define an ideal $\mathcal{K}$ on $\mu$ as follows. Since $\text{add}(\mathcal{J}) = \mu$, we can choose a family $\{A_i : i < \mu\} \subseteq \mathcal{J}$ of pairwise disjoint sets such that $\bigcup_{i<\mu} A_i \in \mathcal{J}^+$. Define

$$\mathcal{K} = \{\Gamma \subseteq \mu : \bigcup\{A_i : i \in \Gamma\} \in \mathcal{J}\}$$

It is easy to see that $\mathcal{K}$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. For simplicity, assume that $1_\mathbb{P} \models \mathcal{K}$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. Coming back to $V$, define $\mathcal{K}' = \{X \subseteq \mu : 1_\mathbb{P} \models X \in \mathcal{K}\}$. It is clear that $V \models \mathcal{K}'$ is a $\mu$-additive ideal on $\mu$. We claim that $V \models \mathcal{K}'$ is $\omega_1$-saturated. Suppose not and fix $\langle (A_\xi, p_\xi) : \xi < \omega_1\rangle$ such that $A_\xi$’s are pairwise disjoint subsets of $\mu$ and for every $\xi < \omega_1$, $p_\xi \models A_\xi \notin \mathcal{K}$. Since $\mathbb{P}$ is ccc, we can find some $p_\kappa \in \mathbb{P}$ that forces uncountable many $p_\xi$’s into the generic $G_\mathbb{P}$. But this means that $p_\kappa \models \check{K}$ is not $\omega_1$-saturated which is impossible. So $V \models \mathcal{K}'$ is $\omega_1$-saturated. So $\mu$ is weakly inaccessible in $V$. Since $V \models \mu > \omega_1 = \kappa$, it follows that $\mu$ must be measurable in $V$. But $\kappa$ is the least measurable cardinal in $V$. Hence $\mu \geq \kappa$: Contradiction.

So we must have $\mu = \kappa$. Let $\mathcal{I} = \{Y \subseteq \lambda : 1_\mathbb{P} \models X \in \mathcal{J}\}$. By Lemma 3.3, there is a countable partition $\mathcal{F}$ of $\lambda$ such that for each $X \in \mathcal{F}$, $\mathcal{I} \upharpoonright X$ is a $\kappa$-complete prime ideal on $\lambda$. For each $X \in \mathcal{F}$, let $\mathcal{I}_X$ be the ideal generated by $\mathcal{I} \upharpoonright X$ in $V^\mathbb{P}$.

By Lemma 3.4, for every $\mathcal{A} \subseteq \mathcal{I}^+_\lambda$, if $|\mathcal{A}| < \kappa$, then there is a countable set that meets every member of $\mathcal{A}$. Since $\mathcal{I}_\lambda \subseteq \mathcal{J} \upharpoonright A$ and $\text{add}(\mathcal{J} \upharpoonright A) = \kappa$, it follows that $\mathcal{J} \upharpoonright A$ is supersaturated for each $A \in \mathcal{F}$. Since $\mathcal{F}$ is a countable partition of $\lambda$, it follows that $\mathcal{J}$ is also supersaturated. \hfill \Box

4. Killing supersaturated ideals

Definition 4.1. Suppose $\delta < \omega_1$ is indecomposable and $\kappa$ is an infinite cardinal. Let $Q^\kappa_\delta$ consist of all countable partial maps from $\kappa$ to $2$ such that

1. $\text{otp}(\text{dom}(p)) < \delta$ and
2. $\{\xi \in \text{dom}(p) : p(\xi) = 1\}$ is finite.

For $p, q \in Q^\kappa_\delta$ define $p \leq q$ iff $q \subseteq p$. Let $\mathbb{P}_\kappa$ be the finite support product of $\{Q^\kappa_\delta : \delta < \omega_1, \delta \text{ indecomposable}\}$.

Lemma 4.2. Let $\mathbb{P}_\kappa$ be as in Definition 4.1.

1. $\mathbb{P}_\kappa$ is ccc.
2. If $\kappa \geq \omega_1$, then $\mathbb{P}_\kappa$ is not $\sigma$-finite-cc.
Proof. (1) Towards a contradiction, suppose $A = \{p_i : i < \omega_1\}$ is an uncountable antichain in $P_\kappa$. Put $D_i = \text{dom}(p_i)$. By passing to an uncountable subset of $A$, we can assume that $D_i$'s form a $\Delta$-system with root $D$. For each $\delta \in D$ and $i < \omega_1$, put $s_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 1\}$ and $X_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 0\}$. Note that $\text{otp}(X_{i,\delta}) < \delta$. Choose $B \subseteq [A]^\omega$ such that for each $\delta \in D$, $\langle s_{i,\delta} : i \in B \rangle$ is a $\Delta$-system with root $s_\delta$ and for every $i < j$ in $B$, $s_{j,\delta} \cap X_{i,\delta} = \emptyset$.

Choose $j \in B$ and $\delta \in D$ such that letting $C = \{i \in B \cap j : p_i(\delta) \perp_{Q_\delta} p_j(\delta)\}$, every transversal of $\{s_{i,\delta} \setminus s_\delta : i \in C\}$ has order type $\geq \delta$. Now observe that $X_{j,\delta}$ has to meet $s_{i,\delta} \setminus s_\delta$ for every $i \in C$. Hence $\text{otp}(X_{j,\delta}) \geq \delta$: Contradiction.

(2) It is enough to show that $Q = Q_\kappa^{\omega_1}$ is not $\sigma$-finite-cc. Towards a contradiction, suppose $Q = \bigcup_{n < \omega} W_n$ where no $W_n$ has an infinite antichain. Choose $\langle A_n : n < \omega \rangle$ as follows.

(a) $A_0 \subseteq W_0$ is a maximal antichain of conditions $p$ such that $\text{max} (\text{dom}(p)) = \gamma_p$ exists and $p(\gamma_p) = 1$. Define $\gamma_0 = \text{max} (\{\gamma_p : p \in A_0\})$.

(b) $A_{n+1} \subseteq W_{n+1}$ is a maximal antichain of conditions $p \in W_{n+1}$ such that $\text{max} (\text{dom}(p)) = \gamma_p$ exists, $\gamma_p > \gamma_n$ and $p(\gamma_p) = 1$. If $A_{n+1} \neq \emptyset$, define $\gamma_{n+1} = \text{max} (\{\gamma_p : p \in A_{n+1}\})$. Otherwise, $\gamma_{n+1} = \gamma_n$.

Put $A = \bigcup_{n < \omega} A_n$ and $\gamma = \sup (\{\gamma_n : n < \omega\})$. Fix $\gamma_* \in (\gamma, \omega_1)$. Let $p_\gamma$ be defined by $\text{dom}(p_\gamma) = \{\gamma_p : p \in A\} \cup \{\gamma_*\}$ and for every $\xi \in \text{dom}(p_\gamma)$, $p(\xi) = 1$ iff $\xi = \gamma_*$. Note that $\text{otp}(\text{dom}(p_\gamma)) \leq \omega + 1 < \omega^2$ and hence $p_\gamma \in Q$. Choose $p_* \in Q$. But now $A_n \cup \{p_*\} \subseteq W_n$ is an antichain which contradicts the maximality of $A_n$. 

\[\square\]

Theorem 4.3. Suppose $\omega_1 \leq \kappa \leq \lambda$, $I$ is an $\omega_1$-saturated uniform ideal on $\lambda$ and $\text{add}(I) = \kappa$. Let $\mathbb{P}_\kappa$ be as in Definition 4.1. Let $\mathcal{J}$ be the ideal generated by $I$ in $V^{\mathbb{P}_\kappa}$. Then there exists $A \subseteq \mathcal{J}^+$ such that $|A| = \omega_1$ and there is no countable set that meets every member of $A$. Hence $V^{\mathbb{P}_\kappa} \models \mathcal{J}$ is an $\omega_1$-saturated $\kappa$-complete uniform ideal on $\lambda$ which is not supersaturated.

Proof. As $\mathbb{P}_\kappa$ is ccc, it is easy to see that in $V^{\mathbb{P}_\kappa}$, $\mathcal{J}$ is an $\omega_1$-saturated $\kappa$-complete uniform ideal on $\lambda$. So it suffices to show that in $V^{\mathbb{P}_\kappa}$, there exists $A \subseteq \mathcal{J}^+$ such that $|A| = \omega_1$ and there is no countable set that meets every member of $A$.

Since $\text{add}(I) = \kappa$, we can fix $Y \in \mathcal{I}^+$ and a partition $Y = \bigcup_{\alpha < \kappa} W_\alpha$ such that for each $\alpha \in [\kappa]^{< \kappa}$, $\bigcup_{\alpha \in \mathcal{I}} W_\alpha \in \mathcal{I}$. Let $G$ be $\mathbb{P}_\kappa$-generic over $V$. Let $G_\delta = \{p(\delta) : p \in G\}$. So $G_\delta$ is $Q_\delta$-generic over $V$. Define $\dot{A}_\delta \in V^{\mathbb{P}_\kappa} \cap \mathcal{P} (\lambda)$ by

$$\gamma \in \dot{A}_\delta \iff (\exists p \in G) (p(\delta)(\alpha) = 1 \land \gamma \in W_\alpha)$$

Suppose $Y \in \mathcal{I}$ and $p \in \mathbb{P}_\kappa$ with $\delta \in \text{dom}(p)$. Choose $\alpha < \kappa$ such that $W_\alpha \setminus Y \neq \emptyset$ and $\alpha \notin \text{dom}(p(\delta))$. Let $q \leq p$ be such that $q(\delta)(\alpha) = 1$. Then $q \Vdash_{\mathbb{P}_\kappa} A_\delta \setminus Y \neq \emptyset$. Hence $\Vdash_{\mathbb{P}_\kappa} A_\delta \in \mathcal{J}^+$.

Towards a contradiction suppose that in $V^{\mathbb{P}_\kappa}$, there is a countable $X \subseteq \lambda$ that meets each $\dot{A}_\delta$. Since $\mathbb{P}$ satisfies ccc, we can assume that $X \subseteq V$. Fix $p \in \mathbb{P}_\kappa$ such that $p \Vdash (\forall \delta)(X \cap \dot{A}_\delta \neq \emptyset)$. Put $W = \{\alpha < \kappa : W_\alpha \cap X \neq \emptyset\}$. So $W \subseteq \kappa$ is countable. Choose $\delta \in \omega_1 \setminus \text{dom}(p)$ indecomposable such that $\delta > \text{otp}(W)$. Define
fact 4.6. \( J \) is nowhere prime iff every \( J \)-positive set can be partitioned into two \( J \)-positive subsets.

Lemma 4.7. Suppose \( J \) is a nowhere prime supersaturated ideal on \( X \) and \( \mu = \text{add}(J) \). Then \( \mu \leq \kappa \) and there exists a \( \mu \)-additive supersaturated ideal on \( \mu \).

Proof. Towards a contradiction, suppose \( \mu > \kappa \). Construct a tree \( \{ A_\sigma : \sigma \in 2^{<\omega_1} \} \) of subsets of \( X \) as follows.

(i) \( A_\emptyset = X \).

(ii) If \( A_\sigma \in J^+ \), then \( \{ A_{\sigma 0}, A_{\sigma 1} \} \) is a partition of \( A_\sigma \) into two \( J \)-positive sets. This is possible since \( J \) is nowhere prime.

(iii) If \( A_\sigma \in J \), then \( A_{\sigma 0} = A_{\sigma 1} = A_\sigma \).

(iv) If \( \alpha < \omega_1 \) is limit and \( \sigma \in 2^\alpha \), then \( A_\sigma = \bigcap \{ A_{\sigma |\beta} : \beta < \alpha \} \).

Put \( F = \{ A_\sigma : \sigma \in 2^{<\omega_1} \text{ and } A_\sigma \in J \} \). We claim that \( X = \bigcup F \). Suppose not and fix \( y \in X \setminus \bigcup F \). Now observe that \( \{ A_{\sigma k} : \sigma \in 2^{<\omega_1} \land k < 2 \land y \in (A_\sigma \setminus A_{\sigma k}) \} \) is an uncountable family of pairwise disjoint \( J \)-positive sets which contradicts the fact that \( J \) is \( \omega_1 \)-saturated. So \( X = \bigcup F \). But since \( |F| \leq |2^{<\omega_1}| = \kappa \), this contradicts the fact that \( \text{add}(J) = \mu > \kappa \). Hence \( \mu \leq \kappa \).

Since \( \text{add}(J) = \mu \), there are \( Y \in J^+ \) and a partition \( Y = \bigcup_{\alpha < \mu} W_\alpha \) such that for every \( \Gamma \in [\mu]^{<\mu} \), \( \bigcup_{\alpha \in \Gamma} W_\alpha \in J \). Define

\[
K = \{ \Gamma \subseteq \mu : \bigcup_{\alpha \in \Gamma} W_\alpha \in J \}
\]

Then \( K \) is a \( \mu \)-additive \( \omega_1 \)-saturated ideal on \( \mu \). So \( \mu \) is weakly inaccessible. We claim that \( K \) must also be supersaturated. To see this, suppose \( A \subseteq K^+ \) and \( |A| < \mu \). For each \( A \in A \), define \( Y_A = \bigcup_{\alpha \in A} W_\alpha \). Then \( \{ Y_A : A \in A \} \subseteq J^+ \).
Since $\mathcal{J}$ is supersaturated, we can choose a countable $T \subseteq Y$ that meets $Y_A$ for every $A \in \mathcal{A}$. Let $B = \{ \alpha < \mu : T \cap W_\alpha \neq \emptyset \}$. Then $B \subseteq \mu$ is countable (as $W_\alpha$’s are pairwise disjoint) and it meets every $A \in \mathcal{A}$. Hence $\mathcal{K}$ is a $\mu$-additive supersaturated ideal on $\mu$. □

**Proof of Theorem 4.5.** Clause (a) is easy to check. Let us prove Clause (b). Suppose $\mathcal{J}$ is a supersaturated ideal on $X$. Put $\mu = \text{add}(\mathcal{J})$. We claim that it suffices to show that $V^\mathcal{S} \models \mu > \mathfrak{c}$. First note that, by Lemma 4.7, this would imply that for every $Y \in \mathcal{J}^+$, there exists $\mathcal{J}$-positive $Z \subseteq Y$ such that $\mathcal{J} \upharpoonright Z$ is a prime ideal. Hence by $\omega_1$-saturation of $\mathcal{J}$, we can find a countable partition of $X$ into $\mathcal{J}$-positive sets such that the restriction of $\mathcal{J}$ to each one of them is a prime ideal.

So towards a contradiction, assume $V^\mathcal{S} \models \mu \leq \mathfrak{c}$. Fix $Y \in \mathcal{J}^+$ such that for every $\mathcal{J}$-positive $Z \subseteq Y$, $\text{add}(\mathcal{J} \upharpoonright Z) = \mu$. Since $\mu \leq \mathfrak{c}$, it follows that $\mathcal{J} \upharpoonright Y$ is a nowhere prime supersaturated ideal. Using Lemma 4.7 again, we can get a $\mu$-additive supersaturated ideal $\mathcal{K}$ on $\mu$. Let us assume that the trivial condition in $\mathcal{S}$ forces all of this about $\mathcal{K}$.

Since $V^\mathcal{S} \models "\mu \leq \mathfrak{c} = \kappa^+"$ and $\mu$ is weakly inaccessible”, we must have $\mu \leq \kappa$. We consider two cases.

Case $\mu < \kappa$: In $V$, define $\mathcal{I}' = \{ X \subseteq \mu : 1_\mathcal{S} \Vdash X \in \mathcal{K} \}$. Since $\mathcal{S}$ is ccc, $V \models \mathcal{I}'$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. As $V \models \mu > \omega_1 = \mathfrak{c}$, $\mu$ is measurable in $V$. Since $\kappa$ is the least measurable cardinal in $V$, $\mu \geq \kappa$: Contradiction.

Case $\mu = \kappa$: In $V$, define $\mathcal{I}' = \{ X \subseteq \kappa : 1_\mathcal{S} \Vdash X \in \mathcal{K} \}$. Since $V \models \kappa > \mathfrak{c} = \omega_1$, we must have $V \models \mathcal{I}'$ is a $\kappa$-additive prime ideal on $\kappa$. Let $\mathcal{K}'$ be the ideal generated by $\mathcal{I}'$ in $V^\mathcal{S}$. Then $V^\mathcal{S} \models \mathcal{K}' \subseteq \mathcal{K}$ are $\omega_1$-saturated $\kappa$-additive ideals on $\kappa$. Using Fact 4.6, fix $B \in \mathcal{K}^+$ such that $\mathcal{K}' \upharpoonright B = \mathcal{K} \upharpoonright B$.

Choose $\gamma < \kappa^+$ such that $B \in V^{\mathcal{S}^\gamma}$. Let $\mathcal{K}''$ be the ideal generated by $\mathcal{I}'$ in $V^{\mathcal{S}^\gamma}$. By Theorem 4.3 it follows that in $V^{\mathcal{S}^{\gamma+1}}$, the ideal generated by $\mathcal{K}'' \upharpoonright B$ is not supersaturated. Now observe that $\mathcal{K} \upharpoonright B = \mathcal{K}' \upharpoonright B$ is the ideal generated by $\mathcal{K}'' \upharpoonright B$ in $V^\mathcal{S}$. It follows that $\mathcal{K}$ is not a supersaturated ideal: Contradiction. □

Using some results about separating families and supersaturated ideals from [2, 4], we can also get the following.

**Theorem 4.8.** Suppose $\kappa$ is a measurable cardinal with a witnessing normal prime ideal $\mathcal{I}$. Let $\mathbb{P}_\kappa$ be the forcing in Definition 4.4. Then the following hold in $V^{\mathbb{P}_\kappa}$.

(a) $\mathfrak{c} = \kappa$ and the ideal generated by $\mathcal{I}$ is a normal $\omega_1$-saturated ideal on $\kappa$.
(b) There is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| = \omega_1$ and for every countable $X \subseteq \kappa$ and $\alpha \in \kappa \setminus X$, there exists $S \in \mathcal{F}$ such that $\alpha \in S$ and $S \cap X = \emptyset$.
(c) The order dimension of Turing degrees is $\omega_1$.
(d) There are no nowhere prime supersaturated ideals.

Proof. (a) Since $\mathbb{Q}^\kappa_\kappa$ adds $\kappa$ Cohen reals, $\mathfrak{c} \geq \kappa$. The other inequality follows by a name counting argument using the facts that $\mathbb{P}_\kappa$ is a ccc forcing, $|\mathbb{P}_\kappa| = \kappa$ and $\kappa^{\omega} = \kappa$. That the ideal generated by $\mathcal{I}$ is a normal $\omega_1$-saturated ideal on $\kappa$ follows from the fact that $\mathbb{P}_\kappa$ is ccc.
(b) For each indecomposable \( \delta < \omega_1 \), define

\[
S_\delta = \{ \alpha < \kappa : (\exists p \in G_{P_\kappa})(\delta \in \text{dom}(p) \land p(\delta)(\alpha) = 1) \}
\]

Let \( \mathcal{F} = \{ S_\delta : \delta < \omega_1 \text{ is indecomposable} \} \). Suppose \( X \subseteq \kappa \) is countable and \( \alpha \in \kappa \setminus X \). We'll find an \( S_\delta \in \mathcal{F} \) such that \( \alpha \in S_\delta \) and \( X \cap S_\delta = \emptyset \). Since \( P_\kappa \) is ccc, we can find a countable \( Y \in V \) such that \( X \subseteq Y \subseteq \kappa \setminus \{ \alpha \} \). Now an easy density argument shows that the set

\[
D_{\alpha,Y} = \{ p \in P_\kappa : (\exists \delta \in \text{dom}(p))[p(\delta)(\alpha) = 1 \land (\forall \beta \in Y)(p(\delta)(\beta) = 0)] \}
\]

is dense in \( P_\kappa \). So we can choose \( p \in D_{\alpha,Y} \cap G_{P_\kappa} \). Let \( \delta \) witness that \( p \in D_{\alpha,Y} \). Then it is clear that \( \alpha \in S_\delta \) and \( X \cap S_\delta = \emptyset \).

(c) This follows from Theorem 3.9 in [2] and part (b) above.

(d) Suppose not. Then by Lemma 4.7, we can find some \( \mu \leq \kappa = \omega_1 \) and a \( \mu \)-additive supersaturated ideal on \( \mu \). Define \( \mathcal{E} = \{ S \cap \mu : S \in \mathcal{F} \} \). Then \( |\mathcal{E}| = \omega_1 \) and for every countable \( X \subseteq \mu \) and \( \alpha \in \mu \setminus X \), there exists \( S \in \mathcal{E} \) such that \( \alpha \in S \) and \( S \cap X = \emptyset \). Now applying Lemma 4.2 in [4] gives us a contradiction. \( \square \)

We conclude with the following questions.

1. Suppose \( \mathcal{I}, \mathcal{J} \) are normal ideals on \( \kappa \), \( \mathcal{I} \) is supersaturated and \( P(\kappa)/\mathcal{I} \) is isomorphic to \( P(\kappa)/\mathcal{J} \). Must \( \mathcal{J} \) be supersaturated?
2. Suppose \( \kappa \) is regular uncountable, \( \mathcal{I} \) is a \( \kappa \)-complete normal ideal on \( \kappa \) and \( P(\kappa)/\mathcal{I} \) is a Cohen algebra. Must \( \mathcal{I} \) be supersaturated?
3. Do \( \sigma \)-finite/bounded-cc forcings preserve supersaturation? What about Boolean algebras that admit a strictly positive finitely additive measure?

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