GENERALIZED HILBERT COEFFICIENTS AND NORTHCOTT’S INEQUALITY

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ABSTRACT. Let $R$ be a Cohen-Macaulay local ring of dimension $d$ with infinite residue field. Let $I$ be an $R$-ideal that has analytic spread $\ell(I) = d$, $G_d$ condition and the Artin-Nagata property $AN_{d-2}$. We provide a formula relating the length $\lambda(I_{n+1}/J^n)$ to the difference $P_I(n) - H_I(n)$, where $J$ is a general minimal reduction of $I$, $P_I(n)$ and $H_I(n)$ are the generalized Hilbert-Samuel polynomial and the generalized Hilbert-Samuel function in the sense of C. Polini and Y. Xie. We then use it to establish formulas to compute the higher generalized Hilbert coefficients of $I$. As an application, we extend Northcott’s inequality to non $m$-primary ideals. When equality holds in the generalized Northcott’s inequality, the ideal $I$ enjoys nice properties. Indeed, in this case, we prove that the reduction number of $I$ is at most one and the associated graded ring of $I$ is Cohen-Macaulay. We also recover results of G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie on the positivity of the generalized first Hilbert coefficient $j_1(I)$. Our work extends that of S. Huckaba, C. Huneke and A. Ooishi to ideals that are not necessarily $m$-primary.

1. INTRODUCTION

Multiplicities and Hilbert functions play important role in commutative algebra and algebraic geometry. It is well-known that multiplicities are widely used to study intersection theory and singularity theory. Besides that, multiplicities and Hilbert functions reflect various algebraic and geometric properties of an ideal $I$ in a Noetherian local ring $R$. In particular, they provide useful information on the arithmetical properties, like the depth, of the associated graded ring $G$, where $G = \text{gr}_I(R) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ is an algebraic construction whose projective scheme represents the exceptional fiber of the blowup of a variety along a subvariety.

The classical multiplicities and Hilbert functions (i.e., the Hilbert multiplicity and the Hilbert function) are only defined for ideals that are primary to the maximal ideal $m$ of $R$. In order to study properties associated to non $m$-primary ideals, one has to define generalized multiplicities and generalized Hilbert functions. One of the generalizations of multiplicities of ideals is called the $j$-multiplicity. It was introduced by R. Achilles and M. Manaresi in 1993 to study improper intersections of two varieties [1]. In 1999, H. Flenner, L. O’Carroll and W. Vogel defined the generalized Hilbert function using the 0th local cohomology functor [6, Definition 6.1.5]. In 2003, C. Ciupercă introduced the generalized Hilbert coefficients via a different approach – the bigraded ring $\text{gr}_m(G)$ [3]. Recently, C. Polini and Y. Xie re-conciliated both approaches and defined the concepts of the generalized Hilbert polynomial and the generalized Hilbert coefficients following the approach of H. Flenner, L. O’Carroll and W. Vogel [20]. One of the fundamental properties

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proved by C. Polini and Y. Xie illustrates the behavior of the generalized Hilbert function under a hyperplane section \[20\]. Indeed, they proved that the first \(d-1\) generalized Hilbert coefficients \(j_0(I), \ldots, j_{d-2}(I)\), where \(d = \dim R\), are preserved under a general hyperplane section. This nice property allows us to study the generalized Hilbert coefficients by reducing to the lower dimensional case.

The generalized Hilbert coefficients are important invariants of the ideal \(I\). It is well-known that the normalized leading coefficient \(j_0(I)\) (i.e., the \(j\)-multiplicity of \(I\)) can be computed using general elements (by [11] Theorem 3.8 and [26] Corollary 2.5). This number was used to prove the refined Bezout’s theorem [6], to detect integral dependence of non \(m\)-primary ideals (extension of the fundamental theorem of Rees) [5], and to study the depth of the associated graded ring of an arbitrary ideal (see for instance, [11] and [10]). In 1987, C. Huneke provided a formula relating the Chern number of normalization of ideals [4]. Therefore it is very important to establish properties such as positivity for the higher generalized Hilbert coefficients.

In the case of \(m\)-primary ideals, there are a number of formulas to compute the Hilbert coefficients (see for instance, [11] and [10]). In 1987, C. Huneke provided a formula relating the length \(\lambda(I^{n+1}/I^n)\) to the difference \(P_I(n) - H_I(n)\), where \(I\) is an \(m\)-primary ideal in a 2-dimensional Cohen-Macaulay local ring, \(J\) is a minimal reduction of \(I\), \(P_I(n)\) and \(H_I(n)\) are respectively the usual Hilbert-Samuel polynomial and the usual Hilbert-Samuel function of \(I\) [11]. This formula was extended later by S. Huckaba to Cohen-Macaulay local rings of arbitrary dimension \(d\) [10]. S. Huckaba then established some formulas to compute the usual Hilbert coefficients of \(I\), and proved conditions in terms of \(e_1(I)\) for the associated graded ring to be almost Cohen-Macaulay [10].

If \(I\) is an \(m\)-primary ideal in a Cohen-Macaulay local ring, the positivity of \(e_1(I)\) can be observed from the well-known Northcott’s inequality

\[
e_1(I) \geq e_0(I) - \lambda(R/I) = \lambda(I/J),
\]

where \(J\) is a minimal reduction of \(I\). By this inequality, one has that \(e_1(I) = 0\) if and only if \(I\) is a complete intersection. Furthermore, the ideal \(I\) enjoys nice properties when equality holds in the above inequality. Indeed, it was shown that \(e_1(I) = \lambda(I/J)\) if and only if the reduction number of \(I\) is at most 1, and in this case, the associated graded ring \(G\) is Cohen-Macaulay (see [11] and [18]).

This paper generalizes the above results to ideals that are not necessarily \(m\)-primary. In Section 2, we fix the notation and recall some basic concepts and facts that will be used throughout the paper. For an ideal \(I\) in a Noetherian local ring that has maximal analytic spread \(\ell(I) = d = \dim R\) and \(G_d\) condition, we establish a formula to compute \(e_1(T)\), where \(T\) is an 1-dimensional reduction of \(I\) (see Section 2 for the definition of \(T\)). We then give a condition in terms of \(e_1(T)\) for the associated
graded ring of $I$ to be almost Cohen-Macaulay. This result generalizes [10, Theorem 3.1]. In Section 3, we provide a generalized version of [10, Theorem 2.4] relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where $I$ is an ideal in a $d$-dimensional Cohen-Macaulay local ring that satisfies $\ell(I) = d$, $G_d$ condition and $AN_d^{r-2}$. $J$ is a general minimal reduction of $I$. $P_I(n)$ and $H_I(n)$ are respectively the generalized Hilbert-Samuel polynomial and the generalized Hilbert-Samuel function of $I$. As an application, we establish some formulas to compute the higher generalized Hilbert coefficients. In the last section, we apply our formula to prove a generalized version of Northcott’s inequality, and recover the work of G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie on the positivity of the generalized first Hilbert coefficient $j_1(I)$. At the same time, we prove that, if equality holds in the generalized Northcott’s inequality, the reduction number of $I$ is at most one and the associated graded ring of $I$ is Cohen-Macaulay, which generalizes the classical results of [11] and [18].

2. Formulas for $e_1(I)$.

In this paper, we always assume that $(R, m, k)$ is a Noetherian local ring of dimension $d$ with maximal ideal $m$ and infinite residue field $k$ (we can enlarge the residue field to be infinite by replacing $R$ by $R(z) = R[z]_{mR[z]}$, where $z$ is a variable over $R$). Let $I$ be an $R$-ideal. We recall the concept of the generalized Hilbert-Samuel function of $I$. Let $G = \text{gr}_I(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ be the associated graded ring of $I$. As the homogeneous components of $G$ may not have finite length, one considers the $G$-submodule of elements supported on $m$: $W := \{ \xi \in G \mid \exists t > 0 \text{ such that } \xi \cdot m^t = 0 \} = H^0_m(G) = \bigoplus_{n=0}^{\infty} H^0_m(I^n/I^{n+1})$. Since $W$ is a finite graded module over $\text{gr}_I(R) \otimes_R R/m^t$ for some $t \geq 0$, its Hilbert-Samuel function $H_W(n) := \sum_{i=0}^{n} \lambda(I_m(I^i/I^{i+1}))$ is well defined. The generalized Hilbert-Samuel function of $I$ is defined to be: $H_I(n) := H_W(n)$ for every $n \geq 0$.

The definition of generalized Hilbert-Samuel function was introduced by H. Flenner, L. O’Carroll and W. Vogel in 1999 [6, Definition 6.1.5], and studied later by C. Polini and Y. Xie [19] as well as G. Colomé Nin, C. Polini, B. Ulrich and Y. Xie [4]. Since $\dim_G W \leq \dim R = d$, $H_I(n)$ is eventually a polynomial of degree at most $d$

$$P_I(n) = \sum_{i=0}^{d} (-1)^i j_i(I) \binom{n + d - i}{d - i}.$$

C. Polini and Y. Xie [19] defined $P_I(n)$ to be the generalized Hilbert-Samuel polynomial of $I$ and $j_i(I), 0 \leq i \leq d$, the generalized Hilbert coefficients of $I$. The normalized leading coefficient $j_0(I)$ is called the $j$-multiplicity of $I$ (see [1], [17], or [19]).

Recall that the Krull dimension of the special fiber ring $G/mG$ is called the analytic spread of $I$ and is denoted by $\ell(I)$. In general, $\dim_G W \leq \ell(I) \leq d$ and equalities hold if and only if $\ell(I) = d$. Therefore $j_0(I) \neq 0$ if and only if $\ell(I) = d$ [17].

If $I$ is $m$-primary, each homogeneous component of $G$ has finite length, thus $W = G$ and the generalized Hilbert-Samuel function coincides with the usual Hilbert-Samuel function; in particular, the generalized Hilbert coefficients $j_i(I), 0 \leq i \leq d$, coincide with the usual Hilbert coefficients $e_i(I)$.
The definition of generalized Hilbert coefficients is different from the one given by C. Ciupercă where he used the bigraded ring \( \operatorname{gr}_m(G) \) \cite{ciuperc}. Polini and Xie re-conciliated both approaches and proved that the generalized Hilbert coefficients \( j_0(I), \ldots, j_{d-2}(I) \) are preserved under a general hyperplane section \cite{polini-xie2}.

In this paper, we are going to use the tool of general elements to study the generalized Hilbert-Samuel function. We now recall this notion. Let \( I = (a_1, \ldots, a_t) \) and write \( x_i = \sum_{j=1}^t \lambda_{ij}a_j \) for \( 1 \leq i \leq s \) and \( (\lambda_{ij}) \in R^t \). The elements \( x_1, \ldots, x_s \) form a sequence of general elements in \( I \) (equivalently \( x_1, \ldots, x_s \) are general in \( I \)) if there exists a Zariski dense open subset \( U \) of \( k^t \) such that the image \( (\lambda_{ij}) \in U \. When \( s = 1 \), \( x = x_1 \) is said to be general in \( I \).

Recall an ideal \( J \subseteq I \) is called a reduction of \( I \) if \( J^r = I^{r+1} \) for some non negative integer \( r \). The least such \( r \) is denoted by \( r_j(I) \). A reduction is minimal if it is minimal with respect to inclusion. The reduction number \( r(I) \) of \( I \) is defined as \( \min \{ r_j(I) \mid J \) is a minimal reduction of \( I \} \). Since \( R \) has infinite residue field, the minimal number of generators \( \mu(J) \) of any minimal reduction \( J \) of \( I \) equals the analytic spread \( \ell(I) \). Furthermore, general \( \ell(I) \) elements in \( I \) form a minimal reduction \( J \) whose \( r_j(I) \) coincides with the reduction number \( r(I) \) (see \cite[2.2]{polini-xie2} or \cite[8.6.6]{huneke}). One says that \( J \) is a general minimal reduction of \( I \) if it is generated by \( \ell(I) \) general elements in \( I \).

The ideal \( I \) is said to satisfy \( G_{s+1} \) condition if for every \( p \in V(I) \) with \( htp = i \leq s \), the ideal \( I_p \) is generated by \( i \) elements, i.e., \( I_p = (x_1, \ldots, x_i)_p \) for some \( x_1, \ldots, x_i \) in \( I \).

From now on, we will assume \( I \) has \( \ell(I) = d \) and \( G_d \) condition. Let \( J = (x_1, \ldots, x_d) \), where \( x_1, \ldots, x_d \) are general elements in \( I \), i.e., \( J \) is a general minimal reduction of \( I \). Set \( J_t = (x_1, \ldots, x_t) \), \( 0 \leq t \leq d-1 \), \( \overline{R} = R/J_{d-1} : I^t \), where \( J_{d-1} : I^t = \{ a \in R \mid \exists t > 0 \text{ such that } a : I^t \subseteq J_{d-1} \} \), and use \( \overline{\cdot} \) to denote images in the quotient ring \( \overline{R} \). Then \( \overline{R} \) is an 1-dimensional Cohen-Macaulay local ring and \( \overline{T} \) is \( m \)-primary. Hence the generalized Hilbert-Samuel function \( H_T(n) \) and the generalized Hilbert-Samuel polynomial \( P_T(n) \) are the usual Hilbert-Samuel function and the usual Hilbert-Samuel polynomial of \( \overline{T} \), respectively. Note \( H_T(I) \) and hence \( P_T(n) \) do not depend on choices of general elements \( x_1, \ldots, x_{d-1} \) in \( I \) (see \cite{polini-xie2}), and \( P_T(n) = e_0(\overline{T})(n+1) - e_1(\overline{T}) \), where \( e_0(\overline{T}) = \lambda(\overline{R}/(\overline{x_d})) = j_0(I) \). If \( R \) is Cohen-Macaulay and \( I \) is \( m \)-primary, then \( e_1(\overline{T}) = e_1(I) \) (see for instance \cite[Proposition 1.2]{polini-xie3}). But they are in general not the same.

We will show later in Theorem \ref{thm:main} that \( e_1(\overline{T}) \) (like \( e_1(I) \), see \cite[Theorem 3.1]{polini-xie1}) characterizes the depth of the associated graded ring \( G \). For depth\( (G) \), we mean the depth of the local ring \( G_M \), where \( M := m/I \oplus I^2/I^3 \oplus \ldots \) denotes the maximal homogeneous ideal of \( G \). Since \( \text{depth}(G) \leq \dim G = \dim R = d \), \( G \) is said to be Cohen-Macaulay if \( \text{depth}(G) = d \) and almost Cohen-Macaulay if \( \text{depth}(G) = d - 1 \). The condition \( \text{depth}(G) \geq d - 1 \) is a useful one, especially when one considers questions about the behavior of \( I^n \). It reduces greatly the computation of the generalized Hilbert coefficients (see Corollary \ref{cor:depth} in Section 3).

Theorem \ref{thm:main} is achieved from a formula computing \( e_1(\overline{T}) \) (see Lemma \ref{lem:formula} in the following). Since we do not have the finite length on \( R/I \), to compare the length \( \lambda(I^{n+1}/JI^n) \) with \( \lambda(\overline{T}^{n+1}/\overline{J}I^n) \), where \( J \) is a general minimal reduction of \( I \), we need the following lemma.
Lemma 2.1. Let $D \subseteq B \subseteq A$ and $D \subseteq C \subseteq A$ be finite modules over $R$ such that $A/B$ and $C/D$ have finite lengths (while the lengths of $B/D$ and $A/C$ are not necessarily finite). Then
\[ \lambda(A/B) + \lambda(B \cap C/D) = \lambda(C/D) + \lambda(A/B + C). \]

Proof. By the exact sequences
\[ 0 \to B \cap C/D \to B/D \to B/C \to 0, \]
\[ 0 \to B + C/C \to A/C \to A/B + C \to 0, \]
\[ 0 \to C/D \to A/D \to A/C \to 0, \]
\[ 0 \to B/D \to A/D \to A/B \to 0, \]
we have the following long exact sequences
\[ 0 \to B \cap C/D \to H^0_m(B/D) \to H^0_m(B + C/C) \to 0 \to H^1_m(B/D) \to H^1_m(B + C/C) \to 0, \]
\[ 0 \to H^0_m(B + C/C) \to H^0_m(A/C) \to A/B + C \to H^1_m(B + C/C) \to H^1_m(A/C) \to 0, \]
\[ 0 \to C/D \to H^0_m(A/D) \to H^0_m(A/C) \to 0 \to H^1_m(A/D) \to H^1_m(A/C) \to 0, \]
\[ 0 \to H^0_m(B/D) \to H^0_m(A/D) \to A/B \to H^1_m(B/D) \to H^1_m(A/D) \to 0, \]
and the commutative diagram
\[
\begin{array}{ccc}
0 & \to & \text{Im}(\Delta_2) \\
\downarrow{id} & & \downarrow{\tilde{\pi}_2} \\
0 & \to & \text{Ker}(\tilde{i}_1 \circ \tilde{\pi}_1) \\
\end{array}
\]
\[
\begin{array}{ccc}
& \to & \text{Im}(\Delta_2) \\
& \downarrow{id} & \downarrow{\tilde{\pi}_2} \\
& \to & \text{Ker}(\tilde{i}_1 \circ \tilde{\pi}_1) \\
\end{array}
\]
with exact rows and isomorphic vertical maps $id$ and $\tilde{\pi}_2$, hence $\text{Im}(\Delta_2) \cong \text{Ker}(\tilde{i}_1 \circ \tilde{\pi}_1)$. Since $\text{Ker}(\tilde{i}_1 \circ \tilde{\pi}_1) \cong \text{Ker}(\tilde{i}_1) = \text{Im}(\Delta_1)$, we have $\text{Im}(\Delta_2) \cong \text{Im}(\Delta_1)$. Now
\[
\lambda(A/B) + \lambda(B \cap C/D) = \lambda(\text{Im}(\Delta_2)) + \lambda(\text{Im}(\Delta_2)) - \lambda(H^0_m(A/D)) + \lambda(B \cap C/D) = \lambda(\text{Im}(\Delta_1)) + \lambda(H^0_m(A/D)) - \lambda(H^0_m(B/D)) + \lambda(B \cap C/D) = \lambda(A/B + C) + \lambda(H^0_m(B + C/C)) - \lambda(H^0_m(A/C)) + \lambda(H^0_m(A/D)) - \lambda(H^0_m(B/D)) + \lambda(B \cap C/D) = \lambda(A/B + C) + \lambda(H^0_m(B/D)) - \lambda(B \cap C/D) + \lambda(C/D) - \lambda(H^0_m(B/D)) + \lambda(B \cap C/D) = \lambda(C/D) + \lambda(A/B + C).
\]

\hfill \square

Proposition 2.2. Let $I$ be an $R$-ideal with $\ell(I) = d$ and $G_d$ condition. For general elements $x_1, \ldots, x_d$ in $I$, set $J = (x_1, \ldots, x_d)$, $J_{d-1} = (x_1, \ldots, x_{d-1})$, and $\mathcal{R} = R/J_{d-1} : I^n$ as above. Then for every $n \geq 0$, one has
\[ (a) \quad \lambda(I^{n+1}/J^n) - \lambda((J_{d-1} : I^n) \cap I^{n+1}/(J_{d-1} : I^n) \cap J^n) = \Delta[P_I(n) - H_I(n)]. \]
\[ (b) \quad \sum_{n=0}^{\infty} [\lambda(I^{n+1}/J^n) - \lambda((J_{d-1} : I^n) \cap I^{n+1}/(J_{d-1} : I^n) \cap J^n)] = e_1(\mathcal{R}). \]
Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal which satisfies Theorem 2.3.

Theorem 2.3. Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal which satisfies $\ell(I) = d$, $G_d$ condition and $(d - 2)$-weakly residually $(S_2)$. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of $I$, the following two statements are equivalent:

(a) $\sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) = e_1(\overline{I})$.

(b) For every $n \geq 0$, $J_{d-1}I^{n+1} = J_{d-1}I^n$, where $J_{d-1} = (x_1, \ldots, x_{d-1})$ defined as before.

Furthermore, if $I$ satisfies $AN_{d-2}$, then (a) or (b) is equivalent to that $\text{depth}(G) \geq d - 1$. 

Proof. (a) For every $n \geq 0$, we have

$$I^{n+1} + J_{d-1} : I^n \iff I^{n+1}$$

$$JI^n + J_{d-1} : I^n \iff I^n$$

with $I^{n+1} + J_{d-1} : I^n/JI^n + J_{d-1} : I^n$ and $I^{n+1}/JI^n$ all having finite lengths. By Lemma 2.1

$$\lambda(I^{n+1}/JI^n) = \lambda(I^{n+1} + J_{d-1} : I^n/JI^n + J_{d-1} : I^n) + \lambda([JI^n + J_{d-1} : I^n] \cap I^{n+1}/JI^n).$$

Since $[JI^n + J_{d-1} : I^n] \cap I^{n+1}/JI^n \cong (J_{d-1} : I^n) \cap I^{n+1}/(J_{d-1} : I^n) \cap JI^n$, we have

$$\lambda(I^{n+1}/JI^n) - \lambda([J_{d-1} : I^n] \cap I^{n+1}/(J_{d-1} : I^n) \cap JI^n) = \lambda(JI^{n+1}/JI^n) = \Delta(P_\ell(n) - H_\ell(n)],$$

where the latter equality follows from [10, Theorem 2.4]. Now (b) follows by (a) and [10 Corollary 2.10].

We now recall some residual intersection properties. Let $J_i = (x_1, \ldots, x_i)$, where $x_1, \ldots, x_i$ are elements in $I$. Define $J_i : I = \{a \in R \mid a \cdot I \subseteq J_i\}$. One says that $J_i : I$ is an i-residual intersection of $I$ if $I_p = (x_1, \ldots, x_i)_p$ for every $p \in \text{Spec}(R)$ with $\text{dim}R_p = i - 1$. An i-residual intersection $J_i : I$ is called a geometric i-residual intersection of $I$ if, in addition, $J_p = (x_1, \ldots, x_i)_p$ for every $p \in V(I)$ with $\text{dim}R_p \leq i$. It was shown that if $I$ satisfies $G_\ell$ condition, then for general elements $x_1, \ldots, x_s$ in $I$ and each $0 \leq i < s$, the ideal $J_i : I$ is a geometric i-residual intersection of $I$, and $J_s : I$ is an s-residual intersection of $I$ (see [23] or [19, Lemma 3.1]).

Assume $R$ is Cohen-Macaulay. The ideal $I$ is s-weakly residually $(S_2)$ (respectively, has the weak Artin-Nagata property $AN^-_d$) if for every $0 \leq i \leq s$ and every geometric i-residual intersection $J_i : I$ of $I$ the quotient ring $R/J_i : I$ satisfies Serre’s condition $(S_2)$ (respectively, is Cohen-Macaulay).

The notion of residual intersections was introduced by Artin and Nagata [2] as a generalization of the concept of linkage to the case where the two “linked” ideals do not necessarily have the same height. The issue on the Cohen-Macaulayness of residual intersections has been addressed in a series of results (for instance, [12], [8], [4] and [23]), which require either depth conditions on all of the Koszul homology modules of $I$ such as the “strong Cohen-Macaulayness” or weaker “sliding depth condition”, or depth conditions on sufficiently many powers of $I$.

The following theorem generalizes [10, Theorem 2.1] to ideals that are not necessarily m-primary. Notice if $R$ is Cohen-Macaulay and $I$ is m-primary, then $e_1(I) = e_1(\overline{I})$ (21 Proposition 1.2), and $I$ automatically satisfies $\ell(I) = d$, $G_d$ condition, $(d - 2)$-weakly residually $(S_2)$ as well as the weak Artin-Nagata property $AN^-_{d-2}$.

Theorem 2.3. Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal which satisfies $\ell(I) = d$, $G_d$ condition and $(d - 2)$-weakly residually $(S_2)$. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of $I$, the following two statements are equivalent:

(a) $\sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) = e_1(\overline{I})$.

(b) For every $n \geq 0$, $J_{d-1}I^{n+1} = J_{d-1}I^n$, where $J_{d-1} = (x_1, \ldots, x_{d-1})$ defined as before.
**Proof.** First if $d = 1$, then $\overline{R} = R/0 : I^\infty = R/0 : I$ is an 1-dimensional Cohen-Macaulay local ring and $\overline{I}$ is $\overline{m}$-primary. By $(0 : I) \cap I = 0$ and [10] Theorem 3.1, one has

$$e_1(\overline{I}) = \sum_{n=0}^{\infty} \lambda(\overline{I}^{n+1} / \overline{I} J^{n}) = \sum_{n=0}^{\infty} \lambda(\overline{I}^{n+1} / \overline{I}^n).$$

Assume $d \geq 2$. Since $I$ satisfies $\ell(I) = d$ and $G_d$ condition, one has that $J : I$ is a geometrically $i$-residual intersection of $I$, where $J_i = (x_1, \ldots, x_i), 0 \leq i \leq d - 1$ [19]. Furthermore, since $I$ is $(d - 2)$-weakly residually $(S_2)$, for each $0 \leq i \leq d - 1$, one has that $R/J_i : I$ has no embedded associated prime ideals and thus $J_i : I = J_i : x_{i+1}$. Note that $\text{depth}(R/J_i : I) = \text{depth}(R/J_i : I) \geq 2$. We will show $\text{depth}(R/J_i) \geq 1$ by induction on $i$. The case $i = 0$ (i.e., $J_0 = (0)$) is clear. Assume $1 \leq i \leq d - 1$ and $\text{depth}(R/J_{i-1}) \geq 1$, then by the exact sequence

$$0 \to R/J_{i-1} : x_i \to R/J_{i-1} \to R/J_i \to 0,$$

one has that $\text{depth}(R/J_i) \geq \min\{\text{depth}(R/J_{i-1} : x_i) - 1, \text{depth}(R/J_{i-1})\} \geq 1$. We claim that $(I_{d-1} : I) \cap I = I_{d-1}$. Indeed, since $I_{d-1} \subseteq (I_{d-1} : I) \cap I$, we just need to show that $(x_1, \ldots, x_{d-1})_p = (I_{d-1} : I)_p \cap I_p$ for every $p \in \text{Ass}(R/J_{d-1})$, which follows by the fact that for every $p \in \text{Ass}(R/J_{d-1})$, since $\text{height}p \leq d - 1$, then either $I_p = R_p$ or $I_p = (J_{d-1})_p$.

Now for $n \geq 0$, $(J_{d-1} : I) \cap I^{n+1} = J_{d-1} \cap I^{n+1}$ and $(J_{d-1} : I) \cap I^n = J_{d-1} \cap I^n$. Therefore if (b) is true, then for $n \geq 0$,

$$\lambda[J_{d-1} : I] \cap I^{n+1} / (J_{d-1} : I) \cap I^n] = \lambda[J_{d-1} \cap I^{n+1} / J_{d-1} \cap I^n]$$

$$= \lambda[J_{d-1} \cap I^n / J_{d-1} \cap I^n] = 0.$$

And (a) follows by Proposition 2.2 (b) and the fact that $R/J_{d-1} : I$ is Cohen-Macaulay and thus $J_{d-1} : I^n = J_{d-1} : I$.

Assume (a). By Proposition 2.2 (b), for every $n \geq 0$, $\lambda[J_{d-1} : I] \cap I^{n+1} / (J_{d-1} : I) \cap I^n] = 0$. Hence

$$J_{d-1} \cap I^{n+1} = (J_{d-1} : I) \cap I^{n+1} = (J_{d-1} : I) \cap I^n = J_{d-1} \cap I^n.$$

We use induction on $n$ to prove that for every $n \geq 0$, $J_{d-1} \cap I^{n+1} = J_{d-1} I^n$. This is clear if $n = 0$. Assume $n \geq 1$ and $J_{d-1} \cap I^n = J_{d-1} I^{n-1}$. Then (b) follows by the following equalities:

$$J_{d-1} \cap I^{n+1} = J_{d-1} \cap I^n$$

$$= J_{d-1} \cap (J_{d-1} I^n + x_d I^n)$$

$$= J_{d-1} I^n + J_{d-1} \cap x_d I^n$$

$$= J_{d-1} I^n + x_d [J_{d-1} \cap I]$$

$$= J_{d-1} I^n + x_d J_{d-1} I^{n-1}$$

$$= J_{d-1} I^n.$$

Finally assume $I$ satisfies $AN_{d-2}$, we will show that (b) is equivalent to that $\text{depth}(G) \geq d - 1$.

Set $\delta(I) = d - g$, where $\text{ht} I = g$. We use the induction on $\delta$. If $\delta = 0$, the assertion follows because
(b) is equivalent to that $x_1^t, \ldots, x_{d-1}^t$ form a regular sequence on $G$ (see [25 Proposition 2.6]), and the latter is equivalent to that depth$(G) \geq d - 1$. Thus we may assume $\delta(I) \geq 1$ and the theorem holds for smaller values of $\delta(I)$. In particular, $d \geq g + 1$. Since $x_1^t, \ldots, x_n^t$ form a regular sequence on $G$, we may factor out $x_1, \ldots, x_t$ to assume $g = 0$. Now $d = \delta(I) \geq 1$. Set $S = R/0 : I$. Then $S$ is Cohen-Macaulay since $I$ satisfies $AN_{d-2}$. Note $\dim S = \dim R = d$, grade$(IS) \geq 1$, $IS$ still satisfies $G_d$ condition, $AN_{d-2}$. $\ell(IS) = \ell(I) = d$ (see for instance [19]). Since $I \cap (0 : I) = 0$, there is a graded exact sequence

$$0 \to 0 : I \to G \to \gr_{IS}(S) \to 0.$$  

Since depth$(0 : I) \geq d$, one has that depth$(G) \geq d - 1 \Leftrightarrow$ depth$(\gr_{IS}(S)) \geq d - 1$. We claim that (b) is equivalent to $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$. Indeed, if (b) holds, then clearly $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$. On the other hand, if $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$, then

$$J_{d-1} \cap I^{n+1} \subseteq J_{d-1}I^n + (0 : I) \cap I^{n+1} = J_{d-1}I^n,$$

again by $I \cap 0 : I = 0$. We are done by induction hypothesis since $\delta(IS) = d - \text{grade}(IS) < d = \delta(I)$.

3. Formulas for $j_i(I), 1 \leq i \leq d$.

In this section we will provide a formula relating the length $\lambda(I^{n+1}/J^m)$ to the difference $P_t(n) - H_t(n)$, where $I$ is an ideal with $\ell(I) = d$, $G_d$ condition and $AN_{d-2}$. $J$ is a general minimal reduction of $I$, $P_t(n)$ and $H_t(n)$ are the generalized Hilbert-Samuel polynomial and the generalized Hilbert-Samuel function of $I$. This formula generalizes [10] Theorem 2.4

**Theorem 3.1.** Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal with $\ell(I) = d$, $G_d$ condition and $AN_{d-2}$. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of $I$, one has that for all $n \geq 0$,

$$\lambda(I^{n+1}/J^m) + \omega_n(J, I) = \Delta^d[P_t(n) - H_t(n)],$$

where $\omega_0(J, I) = \lambda(R/J_{d-1} : I + I) - \lambda([H_0^nJ]/R/I)$, and for $n \geq 1$,

$$\omega_n(J, I) = \Delta^d \left[ \lambda(\bar{K}_{n-1}^0) \right] + \Delta^d \left[ \lambda(\bar{K}_{n-1}^1) \right] + \cdots + \Delta^d \left[ \lambda(\bar{K}_{n-1}^{d-2}) \right]$$

$$+ \Delta^d \left[ \lambda(\bar{L}_{m-1}^0) - \lambda(L_{m-1}^0) + \lambda(N_{m-1}^0) \right]$$

$$+ \Delta^d \left[ \lambda(\bar{L}_{m-1}^1) - \lambda(L_{m-1}^1) + \lambda(N_{m-1}^1) \right] + \cdots$$

$$+ \Delta^d \left[ \lambda(\bar{L}_{m-1}^{d-2}) - \lambda(L_{m-1}^{d-2}) + \lambda(N_{m-1}^{d-2}) \right]$$

$$- \lambda([J_1 : I] \cap J^{n+1}/(J_1 : I) \cap J^n) - \lambda([J_2 : I] \cap J^{n+1} + (J_1 : I)/(J_2 : I) \cap J^n + (J_1 : I)]$$

$$- \cdots - \lambda([J_{d-1} : I] \cap J^{n+1} + (J_{d-2} : I)/(J_{d-1} : I) \cap J^n + (J_{d-2} : I)]$$

$$- (-1)^n \binom{d - 1}{n} \beta,$$

$(d - 1)_n := 0$ if $n \geq d$, and for $0 \leq i \leq d - 2$,

$$\bar{K}_{n-1}^i = I^{n+1} : x_1/J_i : I + I^n,$$
$$L_n^i = (J_i : I) \cap I^n / \{(J_i : I) \cap I^n : m^n\},$$  
$$N_n^i = (J_{i+1} : I) \cap I^n + (J_i : I) \cap I^{n+1} + x_{i+1} I^{n-1},$$

Proof. Recall for each $0 \leq i \leq d-1$, $J_i : I$ is a geometric $i$-residual intersection of $I$, where $J_i = (x_1, \ldots, x_i)$. Set $R^i = R/J_i : I$ and $G^i = \text{gr}_{IR^i}(R^i)$. Then $[G^i]_0 = R/(0 : I + I)$ and $[G^i]_n = [G]_n$ for every $n \geq 1$. Hence

$$\Delta[H_I(0)] = \lambda(H^0_m(R/I))$$
$$= \lambda(H^0_m(R/0 : I + I)) + [\lambda(H^0_m(R/I)) - \lambda(H^0_m(R/0 : I + I))]$$
$$= \Delta[H_{IR^0}(0)] + \beta,$$

with $\beta$ defined above, and $\Delta[H_I(n)] = \Delta[H_{IR^0}(n)]$ for $n \geq 1$. Therefore we have that for $n \geq 0$,

$$\Delta^d[H_I(n)] = \Delta^d[H_{IR^0}(n)] + (-1)^n \binom{d-1}{n} \beta,$$

with the binomial coefficient $\binom{d-1}{n} = 0$ if $n > d-1$.

We use induction on $d$ to prove the theorem. First assume $d = 1$. If $n = 0$, one has

$$\lambda(I/J) + \omega_0(J, I)$$
$$= \lambda(1R^0/IR^0) + \lambda(R/0 : I + I) - \lambda[H^0_m(R/I)]$$
$$= \Delta[P_{IR^0}(0) - H_{IR^0}(0)] + \lambda(R/0 : I + I) - \lambda[H^0_m(R/I)]$$
$$= \Delta[P_{IR^0}(0)] - \lambda[H^0_m(R/I)]$$
$$= \Delta[P_I(0) - H_I(0)].$$

where the second equality follows from [10] Theorem 2.4 since $R^0$ is an 1-dimensional Cohen-Macaulay local ring and $IR^0$ is $mR^0$-primary, and the third equality follows from $\Delta[P_{IR^0}(n)] = \Delta[P_I(n)] = j_0(I)$ for every $n \geq 0$.

If $n \geq 1$, since $\omega_0(J, I) = 0$, one has

$$\lambda(I^{n+1}/J^n) + \omega_n(J, I) = \lambda(I^{n+1}R^0/IR^nR^0)$$
$$= \Delta[P_{IR^0}(n) - H_{IR^0}(n)] = \Delta[P_I(n) - H_I(n)].$$

Now assume $d \geq 2$ and the assertion holds for $d - 1$. By the proof of Proposition 2.2,

$$\lambda(I^{n+1}/J^n) - \lambda((J_{d-1} : I) \cap I^{n+1} / (J_{d-1} : I) \cap J^n) = \lambda(I^{n+1}R^{d-1}/IR^nR^{d-1})$$
$$= \Delta[P_{IR^{d-1}}(n) - H_{IR^{d-1}}(n)] = \Delta^d[P_I(n)] - \Delta[H_{IR^{d-1}}(n)],$$

by the fact that $\Delta[P_{IR^{d-1}}(n)] = \Delta^d[P_I(n)] = j_0(I)$. If $n = 0$, one has

$$\lambda((J_{d-1} : I) \cap I / (J_{d-1} : I) \cap J) = \lambda(J_{d-1} / J_{d-1}) = 0,$$
and therefore
\[ \lambda(I/J) + \omega_0(J, I) \]
\[ = \Delta^d[P_I(0)] - \Delta[H_{IR^0}(0)] + \lambda(R/J_{d-1} : I + I) - \lambda[H_{R^0}(R/I)] \]
\[ = \Delta^d[P_I(0)] - \lambda(R/J_{d-1} : I + I) + \lambda(R/J_{d-1} : I + I) - \lambda[H_{R^0}(R/I)] \]
\[ = \Delta^d[P_I(0)] - \lambda[H_{R^0}(R/I)] \]
\[ = \Delta^d[P_I(0) - H_I(0)]. \]

Let \( n \geq 1 \). We have the following exact sequences for every \( n \geq 1 \):
\[
0 \to K_{n-1}^0 \to H_{m}^0((G^0)_{m-1}) \xrightarrow{x_t} H_{m}^0((G^0)_{n}) \to H_{m}^0((G^0)_{n-1}) \to 0,
\]
\[
0 \to L_n^0 \to H_{m}^0((G^0)_{n})/x_t H_{m}^0((G^0)_{n-1}) \to H_{m}^0([G^1]_{n}) \to N_n^0 \to 0,
\]
where
\[
K_{n-1}^0 = \left((0 : I) \cap I^p + I^{n+1} : p_{n-1}x_1\right) \cap \left(\left((0 : I) \cap I^{n-1} + I^p : p_{n-1} \cdot \text{m}^n\right) / \left((0 : I) \cap I^{n-1} + I^p\right)\right),
\]
\[
L_n^0 = \left((0 : I) \cap I^p + I^{n+1} : (J_1 : I) \cap \text{m}^n/\left(\left((0 : I) \cap I^p + (J_1 : I) \cap I^{n+1} + x_1\left((0 : I) \cap I^{n-1} + I^p : p_{n-1} \cdot \text{m}^n\right)\right)\right)\right),
\]
\[
N_n^0 = \left((J_1 : I) \cap I^p + I^{n+1} : p \cdot \text{m}^n/\left(\left((J_1 : I) \cap I^n + ((0 : I) \cap I^p + I^{n+1}) : p \cdot \text{m}^n\right)\right)\right).
\]
Note \(((0 : I) \cap I^p + I^{n+1}) : p_{n-1}x_1/((0 : I) \cap I^{n-1} + I^p)\) has finite length because \( G \) is Cohen-Macaulay on the punctured spectrum by [15, Theorem 3.1]. Hence
\[
K_{n-1}^0 = \left((0 : I) \cap I^p + I^{n+1} : p_{n-1}x_1/((0 : I) \cap I^{n-1} + I^p)\right).
\]

Therefore
\[
\Delta^d[H_{IR^0}(n)] = \Delta^d-2[\lambda[H_{m}^0((G^0)_{n})] - \lambda[H_{m}^0((G^0)_{n-1})]]
\]
\[
= \Delta^d-2[\lambda[H_{m}^0((G^0)_{n})/x_t H_{m}^0((G^0)_{n-1})] - \lambda(K_{n-1}^0)]
\]
\[
= \Delta^d-2[\lambda[H_{m}^0((G^1)_{n})] + \Delta^d-2[\lambda(L_n^0) - \Delta^d-2[\lambda(N_n^0)] - \Delta^d-2[\lambda(K_{n-1}^0)]
\]
\[
= \Delta^d-1[H_{IR^0}(n)] + \Delta^d-2[\lambda(L_n^0) - \lambda(N_n^0) - \lambda(K_{n-1}^0)].
\]

By Lemma 2.3, the induction hypothesis, and the above equality,
\[
\lambda(I^{n+1}/J^p) = \lambda(I^{n+1}R^0/J^pR^0)
\]
\[
= \lambda(I^{n+1}R^1/J^pR^1) + \lambda((J_1 : I) \cap I^{n+1} + (J_1 : I) \cap J^p)
\]
\[
= \Delta_{n-1}[P_{IR^0}(n) - H_{IR^0}(n)] - \omega_0(JR^1, IR^1) + \lambda((J_1 : I) \cap I^{n+1} + (J_1 : I) \cap J^p)
\]
\[
= \Delta^d[P_{IR^0}(n)] - \Delta^d[H_{IR^0}(n)] + \Delta^d-2[\lambda(L_n^0) - \lambda(N_n^0) - \lambda(K_{n-1}^0)]
\]
\[
- \omega_0(JR^1, IR^1) + \lambda((J_1 : I) \cap I^{n+1} + (J_1 : I) \cap J^p)
\]
\[
= \Delta^d[P_{IR^0}(n) - H_{IR^0}(n)]
\]
\[
- \omega_0(JR^1, IR^1) + \Delta^d-2[\lambda(K_{n-1}^0)] + \Delta^d-2[-\lambda(L_n^0) + \lambda(N_n^0) - \lambda((J_1 : I) \cap I^{n+1} + (J_1 : I) \cap J^p)].
\]

We claim that for every \( n \geq 1 \),
\[
\lambda(K_{n-1}^0) = \Delta[\lambda(\tilde{K}_{n-1}^0)] + \lambda(\tilde{L}_n^0),
\]
In particular, if $d$ is odd, then by equation (2), we have
\begin{align*}
\lambda(K_{n-1}^0) = \lambda(I^{n+1} : x_1/(0 : I) \cap I^{n-1} + I^n),
\lambda(L_n) = \lambda[(x_1) \cap I^n/(x_1) \cap I^{n+1} + x_1I^{n-1}],
\end{align*}

since $(0 : I) \cap I^n = 0$ for $n \geq 1$. This follows by the following equalities:
\begin{align*}
\lambda(K_{n-1}^0) &= \lambda[I^{n+1} : x_1/(0 : I) \cap I^{n-1} + I^n] \\
&= \lambda[(I^{n+1} : x_1) + 0 : I/I^n + 0 : I] \\
&= \lambda[I^{n+1} : x_1/I^n + 0 : I] - \lambda[I^{n+1} : x_1/(I^{n+1} : x_1) + 0 : I] \\
&= \Delta[\lambda(K_{n-1}^0)] + \lambda[I^n : x_1/(0 : I) + I^{n-1}] - \lambda[I^{n+1} : x_1 + I^{n-1}/0 : I + I^{n-1}] \\
&= \Delta[\lambda(K_{n-1}^0)] + \lambda[I^n : x_1/(I^{n+1} : x_1) + I^{n-1}] \\
&= \Delta[\lambda(K_{n-1}^0)] + \lambda[(x_1) \cap I^n/(x_1) \cap I^{n+1} + x_1I^{n-1}] \\
&= \Delta[\lambda(K_{n-1}^0)] + \lambda(L_n).
\end{align*}

Now
\begin{align*}
\omega_n(JR^1, IR^1) + \Delta^{d-2}[\lambda(K_{n-1}^0)] +\Delta^{d-2}[-\lambda(L_n^0) + \lambda(N_n^0)] - \lambda[(J_1 : I) \cap I^{n+1}/(J_1 : I) \cap I^n]
&= \omega_n(JR^1, IR^1) + \Delta^{d-1}[\lambda(K_{n-1}^0)] + \Delta^{d-2}[\lambda(L_n^0) - \lambda(L_n^0) + \lambda(N_n^0)] - \lambda[(J_1 : I) \cap I^{n+1}/(J_1 : I) \cap I^n]
&= \omega_n(JR^0, IR^0).
\end{align*}

Therefore by equation (2), we have
\begin{align*}
\lambda(I^{n+1}/JJ^n) &= \Delta^d[P_I(n) - H_I(n)] - \omega_n(JR^0, IR^0) \\
&= \Delta^d[P_I(n) - H_I(n)] - \omega_n(JR^0, IR^0) - (-1)^n \binom{d-1}{n} \beta = \Delta^d[P_I(n) - H_I(n)] - \omega_n(J, I).
\end{align*}

The following lemma is inspired by [10] Proposition 2.9.

**Lemma 3.2.** Let $I$ be an $R$-ideal. Then
\begin{align*}
\sum_{n=1}^{\infty} \binom{n}{i-1} \Delta^d[P_I(n) - H_I(n)] = j_i(I) \text{ for } 1 \leq i \leq d.
\end{align*}

By Theorem 3.1 and the above lemma, we obtain formulas to compute the generalized Hilbert coefficients.

**Corollary 3.3.** Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal with $\ell(I) = d$, $G_d$ condition and $AN(d-2)$ condition and $G_d$ condition and $AN_d$. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of $I$, one has
\begin{align*}
\sum_{n=1}^{\infty} \binom{n}{i-1} [\lambda(I^{n+1}/JJ^n) + \omega_n(J, I)] = j_i(I) \text{ for } 1 \leq i \leq d.
\end{align*}

In particular, if $d = 1$,
\begin{align*}
\sum_{n=0}^{\infty} \lambda(I^{n+1}/JJ^n) + \lambda(R/0 : I + I) - \lambda[H_0^0(R/I)],
\end{align*}
and if $d \geq 2$,
\[
j_1(I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/J^{I^n}) + \lambda(R/J_{d-1} : I + I) - \lambda[H_m^0(R/I + I)] \\
+ \Delta^{d-2}[\lambda(L_0^n) - \lambda(N_0^n)] + \ldots + \Delta[\lambda(L_0^{d-3}) - \lambda(N_0^{d-3})] \\
+ \sum_{n=1}^{\infty} [\lambda(L_n^{d-2}) - \lambda(L_{n-2}^{d-2}) + \lambda(N_n^{d-2})] \\
- \sum_{n=1}^{\infty} \lambda((J_1 : I) \cap I^{n+1}/(J_1 : I) \cap J^{I^n}) - \ldots \\
- \sum_{n=1}^{\infty} \lambda[(J_{d-1} : I) \cap I^{n+1} + J_{d-2} : I/(J_{d-1} : I) \cap J^{I^n} + J_{d-2} : I].
\]

**Proof.** If $d = 1$, by Theorem 3.1 one has $\omega_0(J, I) = \lambda(R/0 : I + I) - \lambda[H_m^0(R/I)]$ and $\omega_n(J, I) = 0$ for $n \geq 1$. Hence
\[
j_1(I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/J^{I^n}) + \omega_n(J, I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/J^{I^n}) + \lambda(R/0 : I + I) - \lambda[H_m^0(R/I)].
\]
Assume $d \geq 2$. Then
\[
j_1(I) = \sum_{n=0}^{\infty} [\lambda(I^{n+1}/J^{I^n}) + \omega_n(J, I)] \\
= \sum_{n=0}^{\infty} \lambda(I^{n+1}/J^{I^n}) + \lambda(R/J_{d-1} : I + I) - \lambda[H_m^0(R/I)] \\
+ \Delta^{d-2}[\lambda(L_0^n) - \lambda(N_0^n)] + \ldots + \Delta[\lambda(L_0^{d-3}) - \lambda(N_0^{d-3})] \\
+ \sum_{n=1}^{\infty} [\lambda(L_n^{d-2}) - \lambda(L_{n-2}^{d-2}) + \lambda(N_n^{d-2})] \\
- \sum_{n=1}^{\infty} \lambda((J_1 : I) \cap I^{n+1}/(J_1 : I) \cap J^{I^n}) - \ldots \\
- \sum_{n=1}^{\infty} \lambda[(J_{d-1} : I) \cap I^{n+1} + J_{d-2} : I/(J_{d-1} : I) \cap J^{I^n} + J_{d-2} : I] \\
- \beta \left[ \sum_{n=0}^{d-1} (-1)^n \binom{d-1}{n} \right] + \beta,
\]
which is equal to the desired result since $\sum_{n=0}^{d-1} (-1)^n \binom{d-1}{n} = 0$ and
\[
\beta = \lambda(H_m^0(R/I)) - \lambda(H_m^0(R/0 : I + I)).
\]
where \( H = 0 \) if \( d = 1 \), or \( H = 0 : I \) if \( d \geq 2 \).

## 4. Generalized Northcott’s inequality

As an application of Corollary 3.3, we obtain the following generalized Northcott’s inequality.

**Theorem 4.1.** Assume \( R \) is Cohen-Macaulay. Let \( I \) be an \( R \)-ideal with \( \ell(I) = d \), \( G_d \) condition and weakly \((d - 2)\)-residually \((S_2)\). Then for a general minimal reduction \( J = (x_1, \ldots, x_d) \) of \( I \), one has the following generalized Northcott’s inequality:

\[
j_1(I) \geq \lambda(I/J) + \lambda(R/J_{d-1} : I) + (J_{d-2} : I + I) : m^\infty.
\]

**Proof.** Set \( S = R/J_{d-2} : I \), where \( J_{d-2} = (x_1, \ldots, x_d) \). Then \( j_1(I) = j_1(IS) \), \( IS \) satisfies \( \ell(IS) = 2 = \dim S \), \( G_2 \) condition and \( AN_0^S \) (see [20] and [19]). By Corollary 3.3 we have

\[
j_1(I) = j_1(IS).
\]

\[
= \sum_{n=0}^{\infty} \lambda(I^{n+1}S/ Jt^nS) - \sum_{n=1}^{\infty} \lambda((x_{d-1}S : IS) \cap I^{n+1}S/ (x_{d-1}S : IS) \cap Jt^nS)
\]

\[
+ \lambda(S/x_{d-1}S : IS + IS) - \lambda(H^0_m(S/IS))
\]

\[
+ \sum_{n=1}^{\infty} [\lambda((x_{d-1}S) \cap I^nS/(x_{d-1}S) \cap I^{n+1}S + x_{d-1}I^{n+1}S]
\]

\[
- \lambda(I^{n+1}S \cap(x_{d-1}S : IS) \cap I^{n+1}S + x_{d-1}(I^nS : p IS \, m^\infty))]
\]

\[
+ \sum_{n=1}^{\infty} \lambda((x_{d-1}S : IS) \cap I^nS + I^{n+1}S : p IS \, m^\infty/(x_{d-1}S : IS) \cap I^nS + I^{n+1}S : p IS \, m^\infty)]
\]

\[
\geq \lambda(I/J) + \lambda(R/J_{d-1} : I + (J_{d-2} : I + I) : m^\infty).
\]

This follows by the following inequalities. First

\[
\sum_{n=0}^{\infty} \lambda(I^{n+1}S/ Jt^nS) - \sum_{n=1}^{\infty} \lambda((x_{d-1}S : IS) \cap I^{n+1}S/ (x_{d-1}S : IS) \cap Jt^nS)
\]

\[
= \lambda(IS/JS) + \sum_{n=1}^{\infty} [\lambda(I^{n+1}S/ Jt^nS) - \lambda((x_{d-1}S : IS) \cap I^{n+1}S/ (x_{d-1}S : IS) \cap Jt^nS)]
\]

\[
= \lambda(I/J) + \sum_{n=1}^{\infty} \lambda[I^{n+1}S/ Jt^nS + (x_{d-1}S : IS) \cap I^{n+1}S]
\]

\[
\geq \lambda(I/J),
\]

where the second equality follows by Lemma 2.1 and \( \lambda(IS/JS) = \lambda(I/(J_{d-1} : I) \cap I + J) = \lambda(I/J) \).

Next, because depth \((S/x_{d-1}S) \geq 1\) (see [23]), for every \( p \in \text{Ass}(S/x_{d-1}S) \), one has that \( p \) is not maximal and \( IS_p = (x_{d-1})S_p \). Hence \((x_{d-1}S : IS) \cap (JS : m^\infty)_p = x_{d-1}S_p \) for every \( p \in \text{Ass}(S/x_{d-1}S) \), which yields that \((x_{d-1}S : IS) \cap (JS : m^\infty) = x_{d-1}S \). Therefore

\[
(x_{d-1}S : IS + JS) \cap (JS : m^\infty) = JS + (x_{d-1}S : IS) \cap (JS : m^\infty) = JS.
\]
Since $\lambda(I/J) < \infty$ and $(x_{d-1}S : IS) \cap IS = (x_{d-1})S$, by Lemma 2.1 one has
\[
\lambda(S/x_{d-1}S : IS + IS) - \lambda(H_{I}^{d}(S/IS)) = \lambda(S/x_{d-1}S : IS + JS) - \lambda(H_{I}^{d}(S/JS)) = \lambda(S/x_{d-1}S : IS + JS : m^{\infty}) - \lambda[(x_{d-1}S : IS + JS) \cap (JS : m^{\infty})/JS] = \lambda(S/x_{d-1}S : IS + JS : m^{\infty}) = \lambda(R/J_{d-1} : I + (J_{d-2} : I + I) : m^{\infty}).
\]
Finally for $n \geq 1$,
\[
\lambda[(x_{d-1}S) \cap P^{n}S/(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S] - \lambda[P^{n+1}S : (x_{d-1}S) \cap P^{n}S m^{\infty}/(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S] = \lambda[(x_{d-1}S) \cap P^{n}S/(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S] - \lambda[P^{n+1}S : (x_{d-1}S) \cap P^{n}S m^{\infty}/(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S]
\geq 0,
\]
since there is a map
\[P^{n+1}S : (x_{d-1}S) \cap P^{n}S m^{\infty} \rightarrow (x_{d-1}S) \cap P^{n}S/(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S\]
with kernel
\[\left[P^{n+1}S : (x_{d-1}S) \cap P^{n}S m^{\infty}\right] \cap [(x_{d-1}S) \cap P^{n+1}S + x_{d-1}P^{n-1}S] = (x_{d-1}S) \cap P^{n+1}S + [P^{n+1}S : (x_{d-1}S) \cap P^{n}S m^{\infty}] \cap x_{d-1}P^{n-1}S = (x_{d-1}S) \cap P^{n+1}S + x_{d-1}(P^{n}S : P^{n-1}S m^{\infty}),\]
where the second equality holds because $\lambda(P^{n+1}S/x_{d-1}P^{n}S) < \infty$. \qed

The following theorem shows that the ideal $I$ enjoys nice properties when equality holds in the above inequality. It generalizes the classical result of [11] and [18].

**Theorem 4.2.** Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal with $\ell(I) = d$, $G_{d}$ condition, $AN_{d-2}$ and $\text{depth}(R/I) \geq \min\{1, \text{dim } R/I\}$. Then for a general minimal reduction $J = (x_{1}, \ldots, x_{d})$ of $I$, one has that $j_{1}(I) = \lambda(I/J) + \lambda[R/J_{d-1} : I + (J_{d-2} : I + I) : m^{\infty}]$ if and only if $r(I) \leq 1$. In this case, the associated graded ring $G$ is Cohen-Macaulay.

**Proof.** By the proof of Theorem 2.1 if $j_{1}(I) = \lambda(I/J) + \lambda[R/J_{d-1} : I + (J_{d-2} : I + I) : m^{\infty}]$ then for every $n \geq 1$, the length $\lambda[I^{n+1}S/JP^{n}S + (x_{d-1}S : IS) \cap P^{n+1}S] = 0$. Hence
\[I^{2} \subseteq JI + (J_{d-1} : I) \cap I^{2} = JI\]
since $(J_{d-1} : I) \cap I^{2} = J_{d-1}I$ by [19] Lemma 3.2. Now the desired result follows from [15] Theorem 3.1. \qed
Corollary 4.3. Assume $R$ is Cohen-Macaulay. Let $I$ be an $R$-ideal with $\ell(I) = d$, $G_d$ condition and weakly $(d-2)$ residually $(S_2)$. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of $I$, one has

(a) $j_1(I) \geq 0$.

(b) $j_1(I) = \lambda[R/J_{d-1} : I + (J_{d-2} : I + I) : m^\infty]$ implies that $I = J$ is a minimal reduction.

(c) Assume $R$ is excellent. Then $j_1(I) = \lambda(I/J)$ implies that $I$ is $m$-primary.

(d) Assume $R$ is excellent. Then $j_1(I) = 0$ if and only if $I$ is a complete intersection.

Proof. (a) and (b) are clear. Assume (c). Then $\lambda[R/J_{d-1} : I + (J_{d-2} : I + I) : m^\infty] = 0$, which implies $J_{d-1} : I + (J_{d-2} : I + I) : m^\infty = R$. Since $\ell(I) = d$, one has $J_{d-1} : I \neq R$. Hence $(J_{d-2} : I + I) : m^\infty = R$, i.e., $ht(J_{d-2} : I + I) = d$. Since $R$ is excellent by [4], $ht(J_{d-2} : I + I) = \max\{ht(I, d-1)\} = d$, which yields $htI = d$, i.e., $I$ is $m$-primary. The assertion (d) follows by (b) and (c).

We remark that (a) and (d) recover the work on the positivity of $j_1(I)$ by G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie [4].

We will finish the paper by an example from [4] which shows that if residual properties do not satisfy then the generalized Northcott’s inequality fails to hold.

Example 4.4. Let $R = k[[x,y]]/(x^3 - x^2y)$ and $J = (xy^t)$ for any $t \geq 0$. Notice that $R$ is an one-dimensional Cohen-Macaulay local ring and $\ell(J) = 1$. However, $J$ does not satisfy $G_1$. By Macaulay2 [7], one sees that $j_0(J) = t + 1$, $j_1(J) = 2 - t$, which is strictly less than 0 if $t > 2$.

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