A note on small data soliton selection for nonlinear Schrödinger equations with potential

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Abstract

In this note, we give an alternative proof of the theorem on soliton selection for small energy solutions of nonlinear Schrödinger equations (NLS) studied in [3, 4]. As in [4] we use the notion of Refined Profile but unlike in [4] we do not modify the modulation coordinates and we do not search for Darboux coordinates.

1 Introduction

In this note we give an alternative and, in some respect, simplified proof of the selection of small energy standing waves for the nonlinear Schrödinger equation (NLS)

\[ i\partial_t u = Hu + g(|u|^2)u, \quad (t, x) \in \mathbb{R}^{1+3}, \]  

where \( H := -\Delta + V \) is a Schrödinger operator with \( V \in S(\mathbb{R}^3, \mathbb{R}) \) (Schwartz function) and \( g \in C^\infty(\mathbb{R}, \mathbb{R}) \) satisfies \( g(0) = 0 \) and the growth condition:

\[ \forall n \in \mathbb{N} \cup \{0\}, \exists C_n > 0, |g^{(n)}(s)| \leq C_n \langle s \rangle^{2-n} \text{ where } \langle s \rangle := (1 + |s|^2)^{1/2}. \]  

(1.2)

We consider the Cauchy problem of NLS (1.1) with the initial condition \( u(0) = u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \). It is well known that the NLS (1.1) is locally well-posed (LWP) in \( H^1 := H^1(\mathbb{R}^3, \mathbb{C}) \), see e.g. [2, 7]. It is easy also to conclude, by mass and energy conservation, that for small initial data \( u_0 \in H^1 \) the corresponding solution is globally defined.

The aim of this paper is to revisit the study of asymptotic behavior of small (in \( H^1 \)) solutions when the Schrödinger operator \( H \) have several simple eigenvalues. In such situation, it have been proved that solutions decouple into a soliton and dispersive wave [11, 13, 3]. More recently in [4] we have introduced the notion of Refined Profile, which simplifies significantly the proof of the result in [3]. In this note we exploit the notion of Refined Profile of [4], but we give an alternative proof of the result in [4] which does not exploit directly the hamiltonian structure of the NLS. In this sense, in this paper we are closer in spirit to Soffer and Weinstein [11] and Tsai and Yau [13], but our proof is at the same time simpler and with stronger results.

To state our main result precisely, we introduce some notation and several assumptions. The following two assumptions for the Schrödinger operator \( H \) hold for generic \( V \).

Assumption 1.1. 0 is neither an eigenvalue nor a resonance of \( H \).

Assumption 1.2. There exists \( N \geq 2 \) s.t.

\[ \sigma_d(H) = \{\omega_j | j = 1, \cdots, N\}, \text{ with } \omega_1 < \cdots < \omega_N < 0, \]  

and
where $\sigma_d(H)$ is the set of discrete spectrum of $H$. Moreover, we assume all $\omega_j$ are simple and
\[
\forall \mathbf{m} \in \mathbb{Z}^N \setminus \{0\}, \quad \mathbf{m} \cdot \omega \neq 0, \tag{1.3}
\]
where $\omega := (\omega_1, \cdots, \omega_N)$. We set $\phi_j$ to be the eigenfunction of $H$ associated to the eigenvalue $\omega_j$ satisfying $\|\phi_j\|_{L^2} = 1$. We also set $\phi = (\phi_1, \cdots, \phi_N)$.

Remark 1.3. The cases $N = 0, 1$ are easier and are not treated it in this paper. Unfortunately, Assumption (1.2) excludes radial potentials $V(r)$, for $r = |x|$, where in general we should expect eigenvalues with multiplicity higher than one.

As it is well known, $\phi_j$'s are smooth and decays exponentially. For $s \geq 0, \gamma \geq 0$, we set
\[
H^s_\gamma := \{u \in H^s | \|u\|_{H^s_\gamma} := \|\cosh(\gamma x)u\|_{H^s} < \infty\}.
\]
The following is well known.

**Proposition 1.4.** There exists $\gamma_0 > 0$ s.t. for all $1 \leq j \leq N$, we have $\phi_j \in \cap_{s \geq 0} H^s_{\gamma_0}$.

Using $\gamma_0 > 0$, we set
\[
\Sigma^s := H^s_{\gamma_0} \text{ if } s \geq 0, \quad \Sigma^s := (H^s_{\gamma_0})^* \text{ if } s < 0, \quad \Sigma^0 := (\Sigma^0)^* \text{ and } \Sigma^\infty := \cap_{s \geq 0} \Sigma^s.
\]
We will not consider any topology in $\Sigma^\infty$ and we will only consider it as a set.

In order to introduce the notion of refined profile, we need the following combinatorial set up, which is exactly the same as in [4].

We start the following standard basis of $\mathbb{R}^N$, which we view as "non–resonant" indices:
\[
\text{NR}_0 := \{e_j | j = 1, \cdots, N\}, \quad e_j := (\delta_{ij}, \cdots, \delta_{Nj}) \in \mathbb{Z}^N, \quad \delta_{ij} \text{ the Kronecker delta}. \tag{1.4}
\]

More generally, the sets of resonant and non–resonant indices $R, \text{NR}$, are
\[
R := \{\mathbf{m} \in \mathbb{Z}^N | \sum \mathbf{m} = 1, \quad \omega \cdot \mathbf{m} > 0\}, \quad \text{NR} := \{\mathbf{m} \in \mathbb{Z}^N | \sum \mathbf{m} = 1, \quad \omega \cdot \mathbf{m} < 0\}, \tag{1.5}
\]
where $\sum \mathbf{m} := \sum_{j=1}^N m_j$ for $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$.

From Assumption 1.2 it is clear that $\{\mathbf{m} \in \mathbb{Z}^N | \sum \mathbf{m} = 1\} = R \cup \text{NR}$ and $\text{NR}_0 \subset \text{NR}$. For $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define
\[
|\mathbf{m}| := (|m_1|, \cdots, |m_N|) \in \mathbb{Z}^N, \quad ||\mathbf{m}|| := \sum |\mathbf{m}| = \sum_{j=1}^N |m_j|, \tag{1.6}
\]
and introduce partial orders $\leq$ and $\prec$ by
\[
\mathbf{m} \preceq \mathbf{n} \iff \forall j \in \{1, \cdots, N\}, \quad m_j \leq n_j, \quad \text{and} \quad \mathbf{m} \prec \mathbf{n} \iff \text{def} \quad \mathbf{m} \preceq \mathbf{n} \text{ and } \mathbf{m} \neq \mathbf{n}, \tag{1.7}
\]
where $\mathbf{n} = (n_1, \cdots, n_N)$. We define the minimal resonant indices by
\[
R_{\min} := \{\mathbf{m} \in R | \exists \mathbf{n} \in R \text{ s.t. } n_j \prec |\mathbf{m}|\}. \tag{1.8}
\]
We also consider $\text{NR}_1$ formed by the nonresonant indices not larger than resonant indices:
\[
\text{NR}_1 := \{\mathbf{m} \in \text{NR} | \forall \mathbf{n} \in R_{\min}, \quad |\mathbf{n}| \neq |\mathbf{m}|\}. \tag{1.9}
\]
Both $R_{\min}$ and $NR_1$ are finite sets, see [4] for the elementary proof.

We now introduce the functions $\{G_m\}_{m \in R_{\min}} \subset \Sigma^\infty$ which are crucial in our analysis. For $m \in NR_1$, we inductively define $\tilde{\phi}_m(0)$ and $g_m(0)$ by

$$\tilde{\phi}_m(0) := \phi_j, \ g_m(0) = 0, \ j = 1, \cdots, N,$$

(1.10)

and, for $m \in NR_1 \setminus NR_0$, by

$$\tilde{\phi}_m(0) := -(H - m \cdot \omega)^{-1} g_m(0),$$

(1.11)

$$g_m(0) := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(m_1, \cdots, m_{2m+1}) \in A(m,m)} \tilde{\phi}_{m_1}(0) \cdots \tilde{\phi}_{m_{2m+1}}(0),$$

(1.12)

where

$$A(m, m) := \left\{ \{m_j\}_{j=1}^{2m+1} \in (NR_1)^{2m+1} \mid \sum_{j=0}^{m} m_{2j+1} - \sum_{j=1}^{m} m_{2j} = m, \ \sum_{j=0}^{2m+1} |m_j| = |m| \right\}$$

(1.13)

Remark 1.5. For each $m \geq 1$ and $m \in NR_1$, $A(m, m)$ is a finite set. Furthermore, for sufficiently large $m$, we have $A(m, m) = \emptyset$. Thus, even though we are expressing $g_m(0)$ in (1.12) by a series, the sum is finite.

For $m \in R_{\min}$, we define $G_m$ by

$$G_m := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(m_1, \cdots, m_{2m+1}) \in A(m,m)} \tilde{\phi}_{m_1}(0) \cdots \tilde{\phi}_{m_{2m+1}}(0).$$

(1.14)

Remark 1.6. $g_m(0)$ and $G_m$ are defined similarly. We are using a different notation to emphasize that $g_m(0)$ has $m \in NR_1$, while $G_m$ has $m \in R_{\min}$.

The following is the nonlinear Fermi Golden Rule (FGR) assumption essential in our analysis.

Assumption 1.7. For all $m \in R_{\min}$, we assume

$$\int_{|k|^2=m \omega} |\hat{G}_m(k)|^2 \ dS \neq 0,$$

(1.15)

where $\hat{G}_m$ is the distorted Fourier transform associated to $H$.

Remark 1.8. In the case $N = 2$ and $\omega_1 + 2(\omega_2 - \omega_1) > 0$, we have $G_m = g'(0)\phi_1\phi_2^2$, which corresponds to the condition in Tsai and Yau [14], based on the explicit formulas in Buslaev and Perelman [1] and Soffer and Weinstein [10]. These works are related to Sigal [9]. More general situations are considered in [3], where however the $G_m$ are obtained after a certain number of coordinate changes, so that the relation of the $G_m$ and the $\phi_j$’s is not discussed in [3] and is not easy to track.

In [4] it is proved that for a generic nonlinear function $g$ the condition (1.15) is a consequence of the following simpler one, which is similar to (11.6) in Sigal [9],

$$\int_{|k|^2=m \omega} |\hat{\phi}_m(k)|^2 \ dS \neq 0 \text{ for all } m \in R_{\min}$$

(1.16)

where $\phi_m := \prod_{j=1}^{N} \phi_j^{m_j}$. Specifically, in [4] the following is proved.
Proposition 1.9. Let $L = \sup \{ \| m \|^{-1/2} : m \in \mathbb{R}_{\text{min}} \}$ and suppose that the operator $H$ satisfies condition (1.16). Then there exists an open dense subset $\Omega$ of $\mathbb{R}^L$ s.t. if $(g'(0), \ldots, g^{(L)}(0)) \in \Omega$ such that Assumption 1.7 is true for (1.1).

For $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, $m = (m_1, \ldots, m_N) \in \mathbb{Z}^N$, we define

$$z^m := z_1^{(m_1)} \cdots z_N^{(m_N)} \in \mathbb{C}, \text{ where } z^{(m)} := \begin{cases} z^m & m \geq 0, \\ z^{-m} & m < 0, \end{cases}$$

(1.17)

$$|z|^k := (|z_1|^k, \ldots, |z_N|^k) \in \mathbb{R}^N, \|z\| := \sum_{j=1}^N |z_j| \in \mathbb{R}. \quad (1.18)$$

We will use the following notation for a ball in a Banach space $B$:

$$B_B(u, r) := \{ v \in B \mid \| v - u \|_B < r \}. \quad (1.19)$$

The Refined Profile is of the form $\phi(z) = z \cdot \phi + o(\|z\|)$ and is defined by the following proposition, proved in [4].

Proposition 1.10 (Refined Profile). For any $s \geq 0$, there exist $\delta_s > 0$ and $C_s > 0$ s.t. $\delta_s$ is nonincreasing w.r.t. $s \geq 0$ and there exist

$$\{ \psi_m \} \subset C^\infty(B_{\mathbb{R}^N}(0, \delta_s^2), (\Sigma^s)_{NR_1}), \quad \varphi(\cdot) \in C^\infty(B_{\mathbb{R}^N}(0, \delta_s^2), \mathbb{R}^N)$$

and $R \in C^\infty(B_{\mathbb{C}^N}(0, \delta_s), \Sigma^s)$,

s.t. $\varphi(0, \ldots, 0) = \omega$, $\psi_m(0) = 0$ for all $m \in NR_1$ and

$$\| R(z) \|_{\Sigma^s} \leq C_s \| z \|^2 \sum_{m \in \mathbb{R}_{\text{min}}} |z^m|, \quad (1.20)$$

where $B_X(a, r) := \{ u \in X \mid \| u - a \|_X < r \}$, and if we set

$$\phi(z) := z \cdot \phi + \sum_{m \in NR_1} z^m \psi_m(|z|^2) \quad \text{and} \quad z_j(t) = e^{-i\varphi_j(|z|^2) t} z_j, \quad (1.21)$$

then, setting $z(t) = (z_1(t), \ldots, z_n(t))$, the function $u(t) := \phi(z(t))$ satisfies

$$i \partial_t u = Hu + g(|u|^2) u - \sum_{m \in \mathbb{R}_{\text{min}}} z(t)^m G_m - R(z(t)), \quad (1.22)$$

where $\{ G_m \}_{\mathbb{R}_{\text{min}}} \subset (\Sigma^s)^{\mathbb{R}_{\text{min}}}$ is given in (1.14). Finally, writing $\psi_m = \psi_m^{(s_1)}$, $\varphi = \varphi^{(s_2)}$ and $R = R^{(s_2)}$, for $s_1 < s_2$ we have $\psi_m^{(s_1)}(|\cdot|^2) = \psi_m^{(s_2)}(|\cdot|^2)$, $\varphi^{(s_1)}(|\cdot|^2) = \varphi^{(s_2)}(|\cdot|^2)$ and $R^{(s_1)} = R^{(s_2)}$ in $B_{\mathbb{R}^N}(0, \delta_{s_2})$.

We give several formulae related to the refined profile. Let $X$ be a Banach space and $F \in C^1(B_{\mathbb{C}^N}(0, \delta), X)$ for some $\delta > 0$. For $z \in B_{\mathbb{C}^N}(0, \delta)$ and $w \in \mathbb{C}^N$, we set

$$D_z F(z) w := \left. \frac{d}{dc} \right|_{c=0} F(z + cw).$$
For \( z(t) \) given by the 2nd equation of (1.21), that is \( z_j(t) = e^{-i\omega_j(|z|^2)t}z_j \), we have
\[
  i\partial_t z = \nabla(|z|^2)z, \quad \text{where } \nabla(|z|^2)z := (\nabla_1(|z|^2)z_1, \ldots, \nabla_N(|z|^2)z_N).
\]

Thus, \( i\partial_t \phi(z(t)) = iD_\nabla \phi(z(t))(-i\nabla(|z(t)|^2)z(t)) \) and we have the following formula identically satisfied by \( \phi(z) \),
\[
  iD_\nabla \phi(z)(-i\nabla(|z|^2)z) = H\phi(z) + g(|\phi(z)|^2)\phi(z) - \sum_{m \in R_{\min}} z^m G_m - R(z). \tag{1.23}
\]

Furthermore, differentiating (1.23) w.r.t. \( z \) in any given direction \( \bar{z} \in \mathbb{C}^N \), we obtain
\[
  H[z]D_\nabla \phi(z)\bar{z} = iD_\nabla^2 \phi(z)(-i\nabla(|z|^2)z, \bar{z}) + iD_\nabla \phi(z) \left( D_\nabla(-i\nabla(|z|^2)z)\bar{z} \right) + \sum_{m \in R_{\min}} D_\nabla(z^m)\bar{z}G_m + D_\nabla R(z)\bar{z}, \tag{1.24}
\]

where the operator \( H[z] \) is defined by
\[
  H[z]f := Hf + g(|\phi(z)|^2)\bar{f} + 2g'(|\phi(z)|^2)\text{Re} \left( \bar{\phi(z)} \cdot f \right) \phi(z) \tag{1.25}
\]
and is selfadjoint for the inner product \( \langle u, v \rangle = \text{Re} \int_{\mathbb{R}^3} u \bar{v} \, dx \).

The refined profile \( \phi(z) \) contains as a special case the small standing waves bifurcating from the eigenvalues, when they are simple.

**Corollary 1.11.** Let \( s > 0 \) and \( j \in \{1, \ldots, N\} \). Then, \( \phi(z(t)e_j) \) solves (1.1) if \( z \in B_C(0, \delta_s) \) and \( z(t) = e^{-i\omega_j(|z e_j|^2)t}z \).

**Proof.** Since \( (ze_j)^m = 0 \) for \( m \in R_{\min} \), we see that from (1.20) and (1.22) the remainder terms \( \sum_{m \in R_{\min}} z(t)^m G_m + R(z(t)) \) are 0 in (1.22). Therefore, we have the conclusion. \( \square \)

**Remark 1.12.** If the eigenvalues of \( H \) are not simple the above does not hold anymore in general. See Gustafson-Phan [6].

We call solitons, or standing waves, the functions
\[
  \phi_j(z) := \phi(ze_j) \quad \text{for } z \in B_C(0, \delta_s). \tag{1.26}
\]

The main result, which have first proved in [3] is the following.

**Theorem 1.13.** Under the Assumptions 1.1, 1.2 and 1.7, there exist \( \delta_0 > 0 \) and \( C > 0 \) s.t. for all \( u_0 \in H^1 \) with \( \epsilon_0 := ||u_0||_{H^1} < \delta_0 \), there exists \( j \in \{1, \ldots, N\} \), \( z \in C^1(\mathbb{R}, \mathbb{C}) \), \( \eta_+ \in H^1 \) and \( \rho_+ \geq 0 \) s.t.
\[
  \lim_{t \to \infty} ||u(t) - \phi_j(z(t)) - e^{it\Delta} \eta_+||_{H^1} = 0, \tag{1.27}
\]
with \( C^{-1} \epsilon_0^2 \leq \rho_+^2 + ||\eta_+||_{H^1}^2 \leq C\epsilon_0^2 \) and
\[
  \lim_{t \to +\infty} |z(t)| = \rho_+. \tag{1.28}
\]
When written in the Modulation parameters, the NLS appears like a complicated system where some discrete modes are coupled to radiation. The discrete modes tend to produce complicated patterns, similar to the ones of a linear system with eigenvalues. However, asymptotically in time the nonlinear interaction is responsible of spilling of energy into radiation which disperses at space infinity and to the selection of a unique nonlinear standing wave. Theorem 1.13 is the same of the main theorem in [4] and is very similar to the main theorem in [3]. The proofs here and in [4] are much simpler than in [3] or in earlier papers containing early partial results, like [11, 13].

In [4], in order to detect the nonlinear redistribution of the energy it was necessary to make full use of the hamiltonian structure of our NLS, by first introducing Darboux coordinates and by then considering a normal forms argument. The discovery of the notion of Refined Profile made in [8] and its further development in [4] allows to forgo the normal forms argument because an almost optimal system of coordinates is provided automatically by the Refined Profile. In [4] we introduced Darboux coordinates in a way much simpler than in [3]. Undoubtedly, Darboux coordinates are quite natural for a Hamiltonian system and in [4] they contribute to simplify the system. In the present note however, we provide a different proof which, except for the information that mass and energy are constant, thus guaranteeing the global existence of our small $H^1$ solutions, does not make explicit use of the hamiltonian structure of the equations.

## 2 The proof

We start from constructing the modulation coordinate. First, we have the following.

**Lemma 2.1.** There exist $\delta > 0$ and $\mathbf{z} \in C^\infty (\mathbb{B}_{\Sigma^1}(0, \delta), \mathbb{C}^N)$ s.t.

$$\forall \tilde{z} \in \mathbb{C}^N, \ (i(u - \phi(z(u)))) , D_z \phi(z(u)) \tilde{z} = 0.$$  

**Proof.** Standard.

We set

$$\eta(u) := u - \phi(z(u)). \quad (2.1)$$

In the following we write $\eta = \eta(u)$ and $\mathbf{z} = \mathbf{z}(u)$. Substituting $u = \phi(\mathbf{z}) + \eta$ to (1.1) and using (1.23), we have

$$i \partial_t \eta + i D_z \phi(\mathbf{z}) \left( \partial_z \mathbf{z} + i \nabla(|\mathbf{z}|^2) \mathbf{z} \right) = H[\mathbf{z}] \eta + \sum_{m \in \mathbb{R}_{\min}} \mathbf{z}^m G_m + \mathcal{R}(\mathbf{z}) + F(\mathbf{z}, \eta), \quad (2.2)$$

where

$$F(\mathbf{z}, \eta) = g(\phi(\mathbf{z}) + \eta^2) \phi(\mathbf{z}) + \eta - g(|\phi(\mathbf{z})|^2) \phi(\mathbf{z}) - g(|\phi(\mathbf{z})|^2) \eta - 2g'(|\phi(\mathbf{z})|^2) \text{Re} \left( \overline{\phi(\mathbf{z})} \eta \right) \phi(\mathbf{z}).$$

Given an interval $I \subseteq \mathbb{R}$ we set

$$\text{Stz}(I) := L^\infty_t H^j(I) \cap L^2_t W^{j-6}(I), \quad \text{Stz}^*(I) := L^1_t H^j(I) + L^2_t W^{j-6/5}(I), \quad j = 0, 1, \quad (2.3)$$

where $H^0 = L^2$ and $W^{0,p} = L^p$ and use Yajima’s [15] Strichartz inequalities, for $t_0 \in \mathbb{T},$

$$\|e^{-itH} P_x v \|_{\text{Stz}(I)} \lesssim \|v\|_{H^j}, \quad \| \int_{t_0}^t e^{-i(t-s)H} P_x f(s) ds \|_{\text{Stz}^*(I)} \lesssim \|f\|_{\text{Stz}^*(I)}, \quad j = 0, 1. \quad (2.4)$$

Under the assumptions of Theorem 1.13 we have $\|u\|_{L^\infty H^1(\mathbb{R})} \lesssim \epsilon_0$ from energy and mass conservation. Since $\|u\|_{H^1} \sim ||z|| + ||\eta||_{H^1}$, we conclude

$$||z||_{L^\infty_t (\mathbb{R})} + ||\eta||_{L^\infty_t H^1(\mathbb{R})} \lesssim \epsilon_0.$$
Theorem 2.2 (Main Estimates). There exist $\delta_0 > 0$ and $C_0 > 0$ s.t. if $\epsilon_0 = \|u_0\|_{H^1} < \delta_0$, we have

$$
\|\eta\|_{\text{Str}^1(I)} + \sum_{m \in \mathcal{R}_{\min}} \|z^m\|_{L^2(I)} + \|\partial_t z + i \nabla (|z|^2) z\|_{L^2(I)} \leq C \epsilon_0,
$$

(2.5)

for $I = [0, \infty)$ and $C = C_0$. Notice that (2.5), the equation (2.2) satisfied by $\eta$, estimate (1.20) for $\mathcal{R}(z)$ and Lemma 2.4 below for $F(z, \eta)$, allow to prove in a standard and elementary fashion that $\eta(t)$ scatters as $t \to +\infty$, i.e. there exists $\eta_+ \in H^1$ such that $\|\eta(t) - e^{it\Delta} \eta_+\|_{H^1} \to +\infty$. From (2.5) we have $\|\eta_+\|_{H^1} \leq C \epsilon_0$.

Using mass conservation we have

$$
\|\phi(z(t))\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \langle \phi(z(t)), e^{it\Delta} \eta_+ \rangle - 2 \langle \phi(z(t)), \eta(t) - e^{it\Delta} \eta_+ \rangle - \|\eta(t)\|_{L^2}^2
$$

\[t \to +\infty \Rightarrow \lim_{t \to +\infty} \|z(t)\|_{L^2}^2 = \rho_+^2 \leq C \epsilon_0.

So, by $\|\phi(z(t))\|_{L^2}^2 = \|z(t)\|_{L^2}^2 + o(\|z(t)\|_{L^2}^2)$, we get $\lim_{t \to +\infty} \|z(t)\|_{L^2}^2 = \rho_+^2$ for some $0 \leq \rho_+ \leq 2C \epsilon_0$.

The fact that $z^m \in L^2(\mathbb{R}_+)$ and, as it is easy to see, $\partial_t (z^m) \in L^\infty(\mathbb{R}_+) \cap C^0([0, \infty))$, imply $z^m \to +\infty$ for any $m \in \mathcal{R}_{\min}$. This implies $z_k \to +\infty$ for all $k$ except at most for one, yielding the selection of one coordinate $j$ in the statement of Theorem 1.13. The proof that Theorem 2.2 implies Theorem 1.13 is like in [3].

By completely routine arguments discussed in [3], (2.5) for $I = [0, \infty)$ is a consequence of the following Proposition.

Proposition 2.3. There exists a constant $c_0 > 0$ s.t. for any $C_0 > c_0$ there is a value $\delta_0 = \delta_0(C_0)$ s.t. if (2.5) holds for $I = [0, T]$ for some $T > 0$, for $C = C_0$ and for $u_0 \in B_{H^1}(0, \delta_0)$, then in fact for $I = [0, T]$ the inequalities (2.5) holds for $C = C_0/2$.

In the remainder of the paper we prove Proposition 2.3.

2.1 Estimate of the continuous variable $\eta$

In the following, we set $\epsilon_0 = \|u_0\|_{H^1}$. Further, when we use $\lesssim$, the implicit constant will not depend on $C_0$. We start from the estimate of the remainder term $F$.

Lemma 2.4. Under the assumption of Proposition 2.3, we have

$$
\|F(z, \eta)\|_{\text{Str}^1(I)} \lesssim C \epsilon_0^3.
$$

(2.6)

Proof. By (1.2), we have the pointwise bound

$$
|F(z, \eta)| + \|\nabla F(z, \eta)\| \lesssim (1 + |\eta|^2) (|\phi(z)| + \|\nabla \phi(z)\| + |\eta|) |\eta| + \|\nabla \eta\|.
$$

(2.7)

Using this, we obtain the conclusion by Hölder and Sobolev estimates. \qed

We set

$$
\mathcal{H}_c[z] := \{ v \in L^2 \; | \; \forall \mathbf{z} \in \mathbb{C}^N, \; \langle iv, D_v \phi(z) \mathbf{z} \rangle = 0 \}.
$$

(2.8)

Notice that for $u \in H^1$, $\eta(u) \in \mathcal{H}_c[z(u)] \cap H^1$. Following Gustafson, Nakanishi and Tsai [5], we can construct an inverse of $P_c$ on $\mathcal{H}_c[z]$. 7
Lemma 2.5. There exists $\delta > 0$ s.t. there exists $\{a_{jA}\}_{j=1,\ldots,N,A=R,I} \in C^\infty(\mathcal{B}_{CN}(0,\delta), \Sigma^1)$ s.t.

$$\|a_{jA}(z)\|_{\Sigma^1} \lesssim \|z\|^2, \ j = 1, \ldots, N, \ A = R, I$$

(2.9)

and

$$R[z] := \text{Id} - \sum_{j=1}^N (\langle \cdot, a_{jR}(z) \rangle \phi_j + \langle \cdot, a_{jI}(z) \rangle i\phi_j),$$

(2.10)

satisfies $R[z]P_c|_{\mathcal{H}_L[z]} = \text{Id}|_{\mathcal{H}_L[z]}$, $P_cR[z]|_{P_cL^2} = \text{Id}|_{P_cL^2}$.

Proof. A proof is in [3].

We set $\tilde{\eta} = P_c\eta$. By Lemma 2.5, we have $\eta = R[z]\tilde{\eta}$ and $\|\eta\|_{\text{Str}} \sim \|\tilde{\eta}\|_{\text{Str}}$. Applying $P_c$ to (2.2), we have

$$i\partial_t \tilde{\eta} = H\tilde{\eta} - iP_cD_z\phi(z) (\partial_t z + i\varpi(|z|^2)z) + \sum_{m \in \mathcal{R}_{\min}} z^m P_cG_m$$

(2.11)

$$+ P_c R(z) + P_cF(z, \eta) + P_c(H[z] - H)\eta.$$

Lemma 2.6. Under the assumption of Proposition 2.3, we have

$$\|\eta\|_{\text{Str}^2(L)} \lesssim \epsilon_0 + C(C_0)\epsilon_0^3 + \sum_{m \in \mathcal{R}_{\min}} \|z^m\|_{L^2(L)}.$$  

(2.12)

Proof. Obviously, from $\|\eta\|_{\text{Str}} \sim \|\tilde{\eta}\|_{\text{Str}}$, it is enough to bound the latter. By Strichartz estimates (2.4) and Lemma 2.4 we easily obtain

$$\|\tilde{\eta}\|_{\text{Str}^2(L)} \lesssim \epsilon_0 + C(C_0)\epsilon_0^3 + \|P_c D_z\phi(z) (\partial_t z + i\varpi(|z|^2)z)\|_{L^2(L)} + \sum_{m \in \mathcal{R}_{\min}} \|z^m\|_{L^2(L)}.$$  

Using the fact that $\|P_c D_z\phi(z)\|_{\Sigma^1} = O(\|z\|^2)$, we obtain (2.12).

We set $Z(z) := -\sum_{m \in \mathcal{R}_{\min}} z^m R_{\pm}(m \cdot \omega)P_cG_m$ and $\xi := \tilde{\eta} + Z$, where $R_{\pm}(\lambda) := (H - \lambda - i0)^{-1}$. Using the identity

$$(D_xz^m) (i\omega z) = im \cdot \omega z$$

(2.13)

with $\omega z := (\omega_1 z_1, \ldots, \omega_N z_N)$, we see that $Z$ satisfies

$$-i\partial_t Z(z) + HZ(z) = \sum_{m \in \mathcal{R}_{\min}} z^m P_cG_m + R_Z(z),$$

(2.14)

where

$$R_Z(z) = i \sum_{m \in \mathcal{R}_{\min}} D_x (z^m) \left[ (\partial_t z + i\varpi(|z|^2)z) + (i\omega z - i\varpi(|z|^2)z) \right] R_{\pm}(m \cdot \omega)P_cG_m.$$  

Substituting $\tilde{\eta} = \xi - Z(z)$ into (2.11), we obtain

$$i\partial_t \xi = H\xi - iP_cD_x\phi(z) (\partial_t z + i\varpi(|z|^2)z) + P_c R(z) + P_cF(z, \eta) + P_c(H[z] - H)\eta + R_Z(z).$$

(2.15)
Lemma 2.7. Under the assumption of Proposition 2.3, we have
\[ \|\xi\|_{L^2S^0-\{t\}} \lesssim \epsilon_0 + C_\theta \epsilon_0^3. \]

Proof. By \( \| \cdot \|_{L^2S^0} \lesssim \| \cdot \|_{\text{Str}^\alpha} \) and Strichartz estimates (2.4), we have
\[ \|\xi\|_{L^2S^0} \leq \|\bar{\eta}(0)\|_{L^2} + \| e^{itH} Z(\xi(0)) \|_{L^2S^0} + \| \int_0^t e^{-i(t-s)H} R_Z(\xi(u)) \| ds \|_{L^2S^0-} \]
\[ + \| iP_e D_\xi \phi(z) (\tilde{z} + i\varphi(|z|^2) z) - R(z) - F(z, \eta) - (H[z] - H) \|_{\text{Str}^\alpha} \]
where \( z(t) = z(u(t)) \). One can bound the contribution of the 2nd line of (2.16) by \( \lesssim C(C_\theta) \epsilon_0^3 \) using, as in Lemma 2.6, \( \| P_e D_\xi \phi(z) \|_{S^1} = O(\|\xi\|^2) \) and \( \| D_z (z^m) (i\omega z - i\varphi(|z|^2) z) \| = O(\|\xi\|^2) |z^m| \) by (2.13). Similarly, the first term in the r.h.s. of (2.16) can be bounded by \( \lesssim \epsilon_0 \). For the 2nd and 3rd term in the r.h.s. of (2.16), we use the estimate
\[ \| e^{-itH} R_+ (m \cdot \omega) P_e f \|_{S^0-} \lesssim \langle t \rangle^{-3/2} \| f \|_{S^0}. \]
Using (2.17), we have
\[ \| e^{-itH} Z(\xi(0)) \|_{L^2S^0-\{t\}} \lesssim \sum_{m \in \mathbb{R}_{\min}} \|z(0)^m\| \langle t \rangle^{-3/2} \|G_m \|_{S^0} \lesssim \epsilon_0, \]
and
\[ \| \int_0^t e^{-i(t-s)H} R_Z(\xi(u(s)) \| ds \|_{L^2S^0-\{t\}} \]
\[ \leq \sum_{m \in \mathbb{R}_{\min}} \| \int_0^t |D_z (z^m(s)) (\partial_t z(s) + i \varphi(|z|^2) z(s))| e^{-i(t-s)H} R_+ (m \cdot \omega) P_e G_m \| ds \|_{L^2(s)} \]
\[ \lesssim \sum_{m \in \mathbb{R}_{\min}} |c_\theta^2| \int_0^t \langle \partial_t z(s) + i \varphi(|z|^2(s)) z(s) \rangle + \|z(s)^m\| \langle t-s \rangle^{-3/2} \|L^2(s) \rangle \lesssim C(C_\theta) \epsilon_0^3, \]
where we have used \( \partial_t z + i\varphi = \partial_t \tilde{z} + i\varphi(|z|^2) z - i(\varphi(|z|^2) z - \omega z) \) and (2.13) in the 2nd inequality and Young’s convolution inequality in the 3rd inequality. Therefore, we have the conclusion. \( \square \)

2.2 Estimate of discrete variables
We next estimate the quantities \( \|\partial_t z + i\varphi(|z|^2) z\|_{L^2} \) and \( \sum_{m \in \mathbb{R}_{\min}} \|z^m\|_{L^2} \). To do so, we first compute the inner product \( \langle\langle \xi, D_z \phi(z) \rangle z \rangle_{\mathbb{C}^N} \) for any given \( z \in \mathbb{C}^N \). First, notice that by \( \eta \in \mathcal{H}_c[z] \) we obtain the orthogonality relation
\[ \langle i \partial_t \eta, D_z \phi(z) \rangle = - \langle \eta, D_z^2 \phi(z) (\partial_t z, \tilde{z}) \rangle. \]
Second, applying the inner product \( \langle \eta, \cdot \rangle \) to equation (1.24), we have
\[ \langle H[z] \eta, D_z \phi(z) \rangle = \langle \eta, D_z^2 \phi(z) (\varphi(|z|^2) z, \tilde{z}) \rangle + \sum_{m \in \mathbb{R}_{\min}} \langle \eta, (D_z (z^m) \tilde{z}) G_m \rangle + \langle \eta, D_z R(z) \rangle. \]
where we exploited the selfadjointness of $H[z]$ and the orthogonality in Lemma 2.1. Thus, applying $\langle \cdot, D_\phi(z)\tilde{z}\rangle$ to equation (2.2) for $\eta$ and using the last two equalities, we obtain
\[
\langle iD_\phi(z)(\partial_t z + i\varpi(|z|^2)z), D_\phi(z)\tilde{z}\rangle = \langle i\eta, D_\phi(z)(\partial_t z + i\varpi(|z|^2)z) \rangle + \langle \eta, D_\phi(z)\tilde{z}\rangle \\
+ \sum_{m \in R_{\min}} \langle \eta, (D_\phi(z^m)\tilde{z}) G_m \rangle + \sum_{m \in R_{\min}} \langle z^m G_m, D_\phi(z)\tilde{z}\rangle \\
+ \langle R(z), D_\phi(z)\tilde{z}\rangle + \langle F(z, \eta), D_\phi(z)\tilde{z}\rangle.
\] (2.18)

Using $\tilde{z} = e_j, ie_j$ we have the following.

**Lemma 2.8.** Under the assumption of Proposition 2.3, we have
\[
\partial_t z_j + i\varpi(|z|^2)z_j = -i \sum_{m \in R_{\min}} z^m (G_m, \phi_j) + r_j(z, \eta),
\] (2.19)

where $r_j(z, \eta)$ satisfies
\[
\|r_j(z, \eta)\|_{L^2(I)} \lesssim C_0 e_0^3.
\]

In particular, we have
\[
\|\partial_t z + i\varpi(|z|^2)z\|_{L^2(I)} \lesssim \|z^m\|_{L^2(I)} + C_0 e_0^3.
\] (2.20)

**Proof.** First since $D_\phi(0)\tilde{z} = \tilde{z} \cdot \phi$, we have
\[
\langle iD_\phi(z)(\partial_t z + i\varpi(|z|^2)z), D_\phi(z)\tilde{z}\rangle = \sum_{j=1}^{N} \text{Re}(i(\partial_t z_j + i\varpi(|z|^2)z_j)\tilde{z}_j) + r(z, \tilde{z}),
\] (2.21)

where
\[
r(z, \tilde{z}) = \langle i(D_\phi(z) - D_\phi(0))(\partial_t z + i\varpi(|z|^2)z), D_\phi(z)\tilde{z}\rangle \\
+ \langle iD_\phi(0)(\partial_t z + i\varpi(|z|^2)z), (D_\phi(z) - D_\phi(0))\tilde{z}\rangle.
\] (2.22)

Since $\|D_\phi(z) - D_\phi(0)\|_{L^2} \lesssim |z|^2 \lesssim e_0^2$, by the assumptions of Proposition 2.3 we have
\[
\|r(z, \tilde{z})\|_{L^2(I)} \lesssim C_0 e_0^3 \quad \text{for all } \tilde{z} = e_1, ie_1, \cdots, e_N, ie_N.
\] (2.23)

Setting
\[
\tilde{r}(z, \tilde{z}, \eta) := \langle i\eta, D_\phi^2(z)\left(\partial_t z + i\varpi(|z|^2)z, \tilde{z}\right) + \langle \eta, D_\phi(z)R(z)\tilde{z}\rangle + \sum_{m \in R_{\min}} \langle \eta, (D_\phi(z^m)\tilde{z}) G_m \rangle \\
+ \sum_{m \in R_{\min}} \langle z^m G_m, (D_\phi(z) - D_\phi(0))\tilde{z}\rangle + \langle R(z), D_\phi(z)\tilde{z}\rangle + \langle F(z, \eta), D_\phi(z)\tilde{z}\rangle,
\] (2.24)

by the assumptions of Proposition 2.3 we have
\[
\|\tilde{r}(z, \tilde{z}, \eta)\|_{L^2(I)} \lesssim C_0 e_0^3 \quad \text{for all } \tilde{z} = e_1, ie_1, \cdots, e_N, ie_N.
\] (2.25)

Therefore, since $D_\phi(0)ik e_j = ik \phi_j$ $(k = 0, 1)$, we have
\[
-\text{Im}(\partial_t z_j + i\varpi(|z|^2)z_j) = \sum_{m \in R_{\min}} \langle z^m G_m, \phi_j \rangle - r(z, e_j) + \tilde{r}(z, e_j, \eta),
\]
\[
\text{Re}(\partial_t z_j + i\varpi(|z|^2)z_j) = \sum_{m \in R_{\min}} \langle z^m G_m, i\phi_j \rangle - r(z, ie_j) + \tilde{r}(z, ie_j, \eta).
\]
Therefore, from (2.23) and (2.25), we have the conclusion with
\[ r_j(z, \eta) = -r(z, \i e_j) + i \eta r(z, e_j) - i \bar{r}(z, e_j, \eta). \]

Using the stationary Refined Profile equation (1.23), the last line of (2.28) can be written as
\[ \langle \sum_{m \in R_{\min}} z_m \| z_m \|_{L^2} \rangle = e(2.32). \]

Having estimated \( \eta \) and \( \partial_t z + i \varpi(|z|^2) z \) in terms of \( \sum_{m \in R_{\min}} \| z_m \|_{L^2} \), we need to estimate the latter quantity. Here we use the Fermi Golden Rule.

**Lemma 2.9.** Under the assumption of Proposition 2.3, we have
\[ \sum_{m \in R_{\min}} \| z_m \|_{L^2} \leq \epsilon_0 + (C_0 \epsilon_0) \epsilon_0. \]  

**Proof.** We substitute \( \bar{z} = i \varpi(|z|^2) z \) into (2.18) and we make various simplifications. The first, by \( \langle f, 0 \rangle = 0 \), is
\[ \langle i D_{z} \phi(z) \partial_{t} z + i \varpi(|z|^2) z, D_{x} \phi(z) \rangle = \langle i D_{x} \phi(z) \partial_{t} z, D_{x} \phi(z) \rangle \varpi(|z|^2) z. \]  

Next,
\[ \langle \sum_{m \in R_{\min}} z_m G_m + R(z), D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle = \langle \sum_{m \in R_{\min}} \sum_{m \in R_{\min}} \partial_{t} |z|^2 z, D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle \]
\[ - \langle \sum_{m \in R_{\min}} \sum_{m \in R_{\min}} \partial_{t} |z|^2 z, D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle \]  

The 1st term of the r.h.s. of (2.28) can be written as
\[ \langle \sum_{m \in R_{\min}} \sum_{m \in R_{\min}} \partial_{t} |z|^2 z, D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle \]  

where
\[ R_1(z) = \langle \sum_{m \in R_{\min}} \sum_{m \in R_{\min}} \partial_{t} |z|^2 z, D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle \]
\[ + \langle \partial_{t} z, D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle, \]
satisfies
\[ \int_{0}^{T} |R_1(z(t))| dt \leq C_0^2 \epsilon_0^4. \]  

Using the stationary Refined Profile equation (1.23), the last line of (2.28) can be written as
\[ - \langle H \phi(z) + g(|\phi(z)|^2) \phi(z), D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle \]
\[ + \langle D_{x} \phi(z)(i \varpi(|z|^2) z, i D_{x} \phi(z)) \rangle \varpi(|z|^2) z \rangle \]  

Notice that we have
\[ \langle H \phi(z) + g(|\phi(z)|^2) \phi(z), D_{x} \phi(z) \rangle \varpi(|z|^2) z \rangle = \frac{d}{dt} E(\phi(z)), \]  

\[ \text{(2.32)} \]
and the right hand side of (2.27), i.e. the left hand side of (2.18), cancels with the 2nd line of (2.31). Therefore, from (2.18) with $z = i \varpi (|z|^2) z$, (2.27), (2.28), (2.29), (2.31) and (2.32), we have

$$
\frac{d}{dt} E(\phi(z)) - \sum_{m \in R_{\min}} m \cdot \omega \langle \phi, i z^m G_m \rangle = \sum_{m \in R_{\min}} \langle z^m G_m, D_z \phi(0) \left( \partial_t z + i \varpi (|z|^2) z \right) \rangle + R_2(z, \eta),
$$

(2.33)

where

$$
R_2(z, \eta) = R_1(z) + \langle \eta, D_z^2 \phi(z) \left( \partial_t z + i \varpi (|z|^2) z, i \varpi (|z|^2) z \right) \rangle + \langle \eta, D_z \varphi(z) \rangle + \langle \varphi(z), D_z \varphi(z) \rangle,
$$

(2.34)

satisfies

$$
\int_0^T |R_2(z(t), \eta(t))| dt \lesssim (C_0^2 \epsilon_0^2 + C_0^2 \epsilon_0^2) \epsilon_0^2.
$$

(2.35)

By Lemma 2.8 and $D_z \phi(0) z = z \cdot \phi$, the 1st term of right hand side of (2.33) can be written as

$$
\sum_{m \in R_{\min}} \langle z^m G_m, \phi \cdot \left( \partial_t z + i \varpi (|z|^2) z \right) \rangle = \sum_{m \in R_{\min}} \sum_{j=1}^N \langle z^m G_m, \phi_j \left( -iz^n g_{n,j} + r_j(z, \eta) \right) \rangle
$$

(2.36)

$$
= \sum_{m \in R_{\min}, \neq n} \sum_{j=1}^N \text{Re} \left( iz^m \overline{m} \right) g_{m,j} g_{n,j} + \sum_{m \in R_{\min}} \sum_{j=1}^N \langle z^m G_m, r_j(z, \eta) \phi_j \rangle,
$$

where we have set $g_{m,j} := \langle G_m, \phi_j \rangle$ and used the fact that $\langle z^m G_m, -iz^m \phi_j \rangle = 0$ due to $G_m$ and $\phi_j$ being $R$ valued. Now, for $m \neq n$, we have

$$
\partial_t (z^n \overline{m}) = \frac{1}{i(\overline{m} - n) \cdot \omega} \partial_t (z^n \overline{m}) + r_{n,m}(z),
$$

(2.37)

where

$$
r_{n,m}(z) = - \frac{(m - n) \cdot \varpi (|z|^2) - \omega}{(m - n) \cdot \omega} z^n \overline{m}
$$

$$
+ \frac{1}{(m - n) \cdot \omega} D_z (z^n) \left( \partial_t z + i \varpi (|z|^2) z \right) \overline{m} + z^n D_z (z^n) \left( \partial_t z + i \varpi (|z|^2) z \right).
$$

Thus, since $(m - n) \cdot \omega \neq 0$ from Assumption 1.2, we have

$$
z^n \overline{m} = \frac{1}{(m - n) \cdot \omega} \partial_t (z^n \overline{m}) + r_{n,m}(z),
$$

(2.38)

Then, by the hypotheses of Proposition 2.3 we have

$$
\int_0^T |r_{n,m}(z)| dt \lesssim C_0^2 \epsilon_0^4.
$$

(2.39)

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Thus, we have
\[
\sum_{m,n \in \mathbb{R}_{\text{min}}} \sum_{j=1}^{N} \text{Re}(iz^{m}z^{n}) g_{m,j}g_{n,j} = \partial_{t} A_{1}(z) + R_{3}(z),
\]
where
\[
A_{1}(z) = \sum_{m,n \in \mathbb{R}_{\text{min}}} \sum_{j=1}^{N} \frac{1}{(n-m) \cdot \omega} \text{Re}(z^{m}z^{n}) g_{m,j}g_{n,j}, \quad \text{and}
\]
\[
R_{3}(z) = \sum_{m,n \in \mathbb{R}_{\text{min}}} \sum_{j=1}^{N} \text{Re}(ir_{n,m}(z)) g_{m,j}g_{n,j}.
\]
Thus,
\[
\sum_{m \in \mathbb{R}_{\text{min}}} \langle z^{m}G_{m}, \phi \cdot (\partial_{z} + iz^{2}) \rangle = \partial_{t} A_{1}(z) + R_{4}(z, \eta),
\]
where
\[
R_{4}(z, \eta) = R_{3}(z) + \sum_{m,n \in \mathbb{R}_{\text{min}}} \sum_{j=1}^{N} \langle z^{m}G_{m}, r_{j}(z, \eta) \phi_{j} \rangle.
\]
By (2.38) and Lemma 2.8, we have
\[
\int_{0}^{T} |R_{4}(z(t), \eta(t))| dt \lesssim C_{0}^{2} \epsilon_{0}^{4}.
\]
Substituting \( \eta = R[z] \xi - (R[z] - 1)Z(z) - Z(z) \) into the 2nd term of the l.h.s. of (2.33), we have
\[
\sum_{m \in \mathbb{R}_{\text{min}}} m \cdot \omega \langle \eta, iz^{m}G_{m} \rangle = - \sum_{m \in \mathbb{R}_{\text{min}}} m \cdot \omega z^{m} \langle R_{+}(m \cdot \omega)P_{c}G_{m}, iz^{m}G_{m} \rangle - \sum_{m \in \mathbb{R}_{\text{min}}} m \cdot \omega \langle R[z] \xi - (R[z] - 1)Z(z), iz^{m}G_{m} \rangle.
\]
By (2.37), the 2nd term of the r.h.s. of (2.39) can be written as
\[
- \sum_{m,n \in \mathbb{R}_{\text{min}}} m \cdot \omega \langle z^{n}R_{+}(n \cdot \omega)P_{c}G_{n}, iz^{m}G_{m} \rangle = \partial_{t} A_{2}(z) + R_{5}(z),
\]
where
\[
A_{2}(z) = - \text{Re} \sum_{m,n \in \mathbb{R}_{\text{min}}} \frac{m \cdot \omega}{i(m-n) \cdot \omega} z^{m}z^{n} \langle R_{+}(n \cdot \omega)P_{c}G_{n}, iz^{m}G_{m} \rangle,
\]
\[
R_{5}(z) = - \sum_{m,n \in \mathbb{R}_{\text{min}}} m \cdot \omega \langle r_{n,m}(z)R_{+}(n \cdot \omega)P_{c}G_{n}, iz^{m}G_{m} \rangle.
\]
The last term of r.h.s. of (2.39) can be written as

\[ \sum_{m \in \mathbb{R}\min} m \cdot \omega \left( |R[z|\xi, iz^mG_m| + R_6(z)\right), \]

with \( R_6(z) \) satisfying

\[ \int_0^T |R_6(z(t))| dt \lesssim C_0^2 \epsilon_0^4. \]

Therefore, we have

\[ \frac{d}{dt} (E(\phi(z)) - A_1(z) - A_2(z)) = - \sum_{m \in \mathbb{R}\min} m \cdot \omega |z|^2 \langle R_+ (m \cdot \omega) P_m, iP_m, G_m \rangle \]

\[ + \sum_{m \in \mathbb{R}\min} m \cdot \omega \langle R[z|\xi, iz^mG_m \rangle + R_7(z, \eta) \]

where \( R_7(z, \eta) = R_2(z) + R_4(z) + R_5 + R_6. \)

Now, by \( R_+ (\omega \cdot m) = P.V. \frac{1}{\pi} \frac{1}{\omega \cdot m} + i\pi \delta(H - \omega \cdot m) \) and formula (2.5) p. 156 [12] and Assumption 1.7, we have

\[ \langle iG_m, (H - \omega \cdot m - i0)^{-1}G_m \rangle = \frac{1}{16\pi \sqrt{\omega \cdot m}} \int_{|k|^2 = \omega \cdot m} |\hat{G}_m(k)| dS(k) \geq 1, \]

with \( \hat{G}_m(k) \) like in Assumption 1.7. Thus, we have

\[ \|z^m\|^2_{L^2(I)} \lesssim \epsilon_0^2 + \delta^{-1} \|\xi\|^2_{L^2\Sigma_{\eta}^{-}(I)} + \delta \|z^m\|^2_{L^2(I)} + C_0^2 \epsilon_0^4, \]

where we have used Schwartz inequality. Taking \( \delta \) so that the \( \|z^m\|^2_{L^2(I)} \lesssim \epsilon_0^2 + \delta^{-1} \|\xi\|^2_{L^2\Sigma_{\eta}^{-}(I)} + C_0^2 \epsilon_0^4 \)

and using \( \|\xi\|^2_{L^2\Sigma_{\eta}^{-}(I)} \lesssim \epsilon_0 \) by Lemma 2.7, we obtain (2.26).

\[ \square \]

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