ON THE POSSIBILITY OF $Z_N$ EXOTIC SUPERSYMMETRY IN
TWO DIMENSIONAL CONFORMAL FIELD THEORY

F. Ravanini

Service de Physique Théorique, C.E.A. - Saclay

Orme des Merisiers, F-91190 Gif-sur-Yvette, France

and

I.N.F.N. - Sez. di Bologna, Italy

Abstract

We investigate the possibility to construct extended parafermionic con-
formal algebras whose generating current has spin $1 + \frac{1}{K}$, generalizing the
superconformal (spin 3/2) and the Fateev Zamolodchikov (spin 4/3) algebras.
Models invariant under such algebras would possess $Z_K$ exotic supersymme-
tries satisfying (supercharge)$^K = (\text{momentum})$. However, we show that for
$K = 4$ this new algebra allows only for models at $c = 1$, for $K = 5$ it is a
trivial rephrasing of the ordinary $Z_5$ parafermionic model, for $K = 6, 7$ (and,
requiring unitarity, for all larger $K$) such algebras do not exist. Implications
of this result for existence of exotic supersymmetry in two dimensional field
theory are discussed.

Saclay preprint SPhT/91-121
August 1991

Submitted for publication to Int.J.Mod.Phys. A

* Laboratoire de la Direction des Sciences de la Matière du Commissariat à l’Energie Atomique
1 Introduction

Conformal Field Theory (CFT) in two dimensions (2D) \[1, 2\] can give information on the general structure of the space of all 2D Quantum Field Theories (QFT). Indeed, each reasonable QFT must possess ultraviolet (UV) as well as infrared (IR) fixed points for which the Callan-Symanzik $\beta$ function is zero, thus showing scale invariance. Following Polyakov \[3\] it is conceivable that all scale invariant QFT also possess conformal invariance. Hence the UV and IR limits of any 2D QFT must be described by suitable CFT’s.

Moreover, if a QFT has some particular symmetry preserved all along the renormalization flow, this symmetry should exhibit itself at the UV and IR points too. For example consider an $N = 1$ supersymmetric theory. Its action is invariant under transformations by a spin 1/2 charge $Q$ (the so called supercharge) such that $Q^2 = P$, where $P$ is the (conserved) total momentum. Corresponding to this charge $Q$ there is a conserved current of spin 3/2 that in the UV limit becomes the well known $G(z)$ current enlarging the conformal symmetry to an $N = 1$ superconformal one. Thus the UV limit of such a theory must be a superconformal model, minimal or not. In a certain sense, we could say that existence of a superconformal algebra guarantees the existence of reasonable UV limits for 2D $N = 1$ supersymmetric QFT’s.

One can ask if such a structure can be generalized to models having, say, $Z_3$ graded supersymmetry, i.e. models whose action is invariant under transformations $Q, Q^\dagger$ such that $Q^3 = Q^\dagger^3 = P$. If so, the UV limit of these $Z_3$ supersymmetric models is described by appropriate “$Z_3$ exotic” superconformal models, invariant under an algebra that generalizes the superconformal one to the case of $Z_3$ gradation. As $P$ has dimension 1, $Q$ and $Q^\dagger$ must have dimension 1/3, and the corresponding conserved current must be of dimension 4/3. Such an algebra extending the Virasoro algebra by means of a couple of conjugated currents $A(z), A^\dagger(z)$ of spin (and dimension) 4/3 has been investigated by Fateev and Zamolodchikov \[4\].
These $Z_2$ (Susy) and $Z_3$ (spin 4/3) algebras, that we shall call in the following $SZ_2$ and $SZ_3$ respectively, show some well known common structure. First of all, both allow for a series of unitary minimal models, accumulating to $c = 3/2$ and to $c = 2$ respectively [4, 5]. For $c$ larger than these values there is still a continuum of non-minimal models. The set of all $SZ_2 \otimes SZ_2$ invariant models, minimal or not, is the set of all UV or IR fixed point of 2D QFT invariant under $N = 1$ supersymmetry; the same set for $SZ_3$ represents all fixed points of QFT invariant under $Z_3$ exotic supersymmetry. Each minimal model in the two series can be perturbed by some relevant scalar operator contained in its Kac table. The least relevant operator (that with conformal dimension closer to 1) leads to two different behaviours: for negative values of the perturbing coupling constant the model is massive and integrable, and its scattering matrix is known [6]; for positive coupling constant the perturbation defines a massless flow that has been shown, at least by perturbative arguments, to have a non trivial IR limit also belonging to the same series [7, 8]. The $Z_2$ or $Z_3$ supersymmetry is preserved along the flow, i.e. one can define non-local charges $Q$ in the perturbed model, such that $Q^2 = P$ or $Q^3 = P$. For negative coupling constant the scattering matrix of the massive model is known [6] and is indeed invariant under such a charge $Q$. For positive coupling the models are know to flow to usual $Z_2$ (Ising) and $Z_3$ (Potts) models respectively. As these IR limits do not have $Q$ invariance any more, one concludes [7, 9] that along the flow there is spontaneous supersymmetry breaking. In the $Z_2$ case, it has been shown [7] that the resulting goldstino is a field that in the IR limit evolves in the spin 1/2 fermion of the Ising model. Similarly, in the $Z_3$ case, Zamolodchikov [9] has argued that the goldstini fields corresponding to the broken $A(z), A^i(z)$ currents should become, in the IR limit, the couple of 3-state Potts parafermions of spin 2/3. Notice that in
both cases the spin of the broken current and that of the resulting goldstino sum up to 2 (any better understanding of this fact should be welcome). It has also been observed that in both cases the scattering matrix of the negative coupling massive model coincides with the Boltzmann weights of the corresponding $Z_2$ Ising or $Z_3$ Potts models [9].

In this paper we address the problem to generalize these examples to $Z_K$ graded exotic superconformal algebras, $SZ_K$ for short. If such algebras exist, they can be the base for the construction of $SZ_K \otimes SZ_K$ invariant models. Then one can address the problem of perturbing these models by their least relevant operator thus getting examples of massive and massless non-conformal $Z_K$ exotic supersymmetric models. In particular, the picture valid for $Z_2$ and $Z_3$ should naively suggest that suitably perturbing the $SZ_K$ model with lowest central charge, one could get a case of spontaneously broken $Z_K$ exotic supersymmetry, whose goldstino, in the IR limit, could describe the usual $Z_K$ parafermion of the $Z_K$ Ising model. Boltzmann weights for the $Z_K$ Ising models are known [10]. Assuming that they can be as well used as scattering matrices of some 2D QFT, Bernard and Pasquier [11] have shown that they are indeed invariant under transformations $Q$ such that $Q^K = P$. Thus, they are the natural candidates of an eventual massive model obtained by suitably deforming the lowest $c$ $SZ_K$ model.

If conversely the $SZ_K$ models do not exist, namely because the $SZ_K$ algebras are inconsistent for some $K$, then no UV limit can be defined for a $Z_K$ exotic supersymmetric theory, and, as a reasonable QFT must have an UV limit, we can conclude that no $Z_K$ exotic supersymmetry exists for that $K$ at all, and the Boltzmann weights of [11] cannot be used as scattering matrices of a QFT.

The interest of searching possible $SZ_K$ algebras is even more general: in appendix A of [12], Fateev and Zamolodchikov describe the most general $Z_K$ symmetric parafermionic algebra, of which the usual $Z_K$ parafermions are a particular case. All $Z_K$-invariant conformal models should have some realization of this general $Z_K$ algebra somewhere hidden in their operator product expansion (OPE) algebra.
Thus, knowledge of possible associative \( Z_K \) symmetric algebras (and of their representations) should help in the classification of all conformal models having \( Z_K \) symmetry. \( SZ_K \) algebras explored in the present paper are one among the many possibilities described in [12].

2 The \( Z_K \)-superconformal algebras

We begin our investigation by giving the general form of the \( SZ_K \) algebras we are interested in. We proceed by direct generalization of the known \( Z_2 \) and \( Z_3 \) cases.

The \( SZ_2 \) algebra has the simple \( Z_2 \)-graded fusion rules \( \psi \psi = 1 \), and the \( SZ_3 \) one is described by \( Z_3 \)-graded fusion rules \( \psi \psi = \psi^\dagger, \psi \psi^\dagger = 1 \). Requirement of \( Z_K \)-gradation of the \( SZ_K \) algebra means considering a set of \( Z_K \) symmetric fusion rules for the currents:

\[
\psi_i \psi_j = \psi_{i+j} , \quad (\psi_0 = 1, \psi_i^\dagger = \psi_{K-i})
\]  

(1)

Here and in the following \( i, j, k, ... \) indices are always to be taken modulo \( K \) and we introduce the notation \( \hat{i} = K - i \). We are interested in \( N = 1 \) supersymmetry, i.e. we require that for a given conformal dimension \( \Delta_k \) there can be only one couple of currents \( \psi_k \) and \( \psi_k^\dagger \). Furthermore, we require that no currents of spin one are present as secondaries in the family of the identity, otherwise they would form the Kac-Moody algebra of a continuous internal symmetry, while we are interested in the case where no symmetry additional to the \( Z_K \)-susy is postulated. We also require that the only current of dimension two appearing in the identity family is the stress-energy tensor. Currents of higher spin \( 3, 4, 5, ... \) are allowed, which means that the identity family is not necessarily that of pure Virasoro algebra, it can as well contain currents generating some W-algebra. All these requirements fix the form of the operator product expansion (OPE) algebra to be the following \((z_{12} = z_1 - z_2\)
and $j \neq i$

$$\psi_i(z_1)\psi_j(z_2) = C_{ij}z_{12}^{\alpha_{ij}} \left[ \psi_{i+j}(z_2) + z_{12}^{\alpha_{ij} + 2\Delta_{i+j}} \partial_{z_2} \psi_{i+j}(z_2) + O(z_{12}^3) \right]$$

$$\psi_i(z_1)\psi_i^+(z_2) = z_{12}^{-2\Delta_i} \left[ 1 + z_{12}^2 \frac{2\Delta_i}{c} T(z_2) + O(z_{12}^3) \right]$$

$$T(z_1)\psi_i(z_2) = \frac{\Delta_i \psi_i(z_2)}{z_{12}^2} + \frac{\partial_{z_2} \psi_i(z_2)}{z_{12}} + O(1)$$

$$T(z_1)T(z_2) = \frac{c/2}{z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial_{z_2} T(z_2)}{z_{12}} + O(1)$$

(2)

where we introduced the useful notation $\alpha_{ij} = \Delta_{i+j} - \Delta_i - \Delta_j$. The last two equations show that Virasoro algebra with central charge $c$ is a subalgebra of $SZ_K$ algebra and that $\psi_i$'s are Virasoro primary fields of left conformal dimension $\Delta_i$. As they are to be conserved currents, their right conformal dimension $\bar{\Delta}_i$ must be zero. Hence the spin of the current $\psi_i$, as well as its full conformal dimension, is given by $\Delta_i$. Of course there will be a “right” algebra of currents $\bar{\psi}_i(\bar{z})$ and $\bar{T}(\bar{z})$ pertaining the right chiral part of the models and commuting with the $\{\psi_i, T\}$ algebra, so that the models will be invariant under $SZ_K \otimes \bar{SZ}_K$ symmetry. All the considerations in the following will be done for the left algebra and apply as well to the right one.

The spins $\Delta_k$ can not take arbitrary values. As explained in Appendix A of [12], or equivalently using the techniques of [13, 14], it is possible to show that the most general value for $\Delta_k$ compatible with the fusion rules (1) is given by

$$\Delta_k = \frac{pk(K-k)}{K} + M_k$$

(3)

where $M_k \in \mathbb{Z}$, $M_k = M_k$, $M_0 = 0$ and $p$ can be integer if $K$ is odd and integer or half-integer if $K$ is even. Furthermore, in the case of $SZ_K$ algebra, we must require that one of the currents $\psi_k$, say $\psi_1$, has spin $\Delta_1 = 1 + 1/K$, in order to be able to define a conserved charge $Q$ such that $Q^K = P$. This fixes $M_1 = M_1 = 2$ and $p = -1$, hence the formula for $\Delta_k$ we shall assume in the following is

$$\Delta_k = M_k - \frac{k(K-k)}{K}, \quad M_0 = 0 \quad M_1 = 2 \quad M_k = M_k$$

(4)

where the integers $M_k$ have to be constrained by the allowed behaviours of correlation functions near their singularities (see below).
The structure constants $C_{ij}$ are chosen such that all non-zero two point functions are normalized to 1 ($C_{\bar{i} \bar{i}} = 1$), and enjoy full symmetry, i.e. defining

$$C_{ij} = Q_{i,j}^{i+j} = Q_{i,j,N-i-j}$$

(5)

full symmetry of the symbols $Q_{i,j,k}$ must be required. This restricts the number of independent structure constants. Moreover charge conjugation symmetry implies $C_{ij} = C_{\bar{i} \bar{j}}$.

### 3 Associativity

The most important requirement on $SZ_K$ algebras is their associativity or, equivalently, duality of the 4-point correlation functions of fields $\psi_i$. As fields $\psi_i(z)$ do not depend on $\bar{z}$, their 4-point functions will have dependence on $z_1, ..., z_4$ only, and not on $\bar{z}_1, ..., \bar{z}_4$. Invariance under the projective group $\text{SL}(2,\mathbb{C})$ implies that one can choose 3 of the 4 positions in any 4-point function as 0,1 and $\infty$ so that it depends essentially only on one projective invariant variable $x$, the so called anharmonic ratio

$$\langle 0|\psi_i(\infty)\psi_k(1)\psi_i(x)\psi_j(0)|0\rangle = G_{ij}^{kl}(x)$$

(6)

$Z_K$ invariance forces this correlation function to be 0 if $i + j + k + l \neq 0 \mod K$.

The requirement of duality on the 4-point function can be put as conditions on $G_{ij}^{kl}(x)$

$$G_{ij}^{kl}(x) = G_{ik}^{jl}(1-x) = e^{\pi i \alpha_{ik} x^{-2\Delta_i}} G_{il}^{kj}(1/x)$$

(7)

The phase in the last equality comes from the braiding of the (semilocal) fields $\psi_i$ and $\psi_k$ necessary to bring $x$ close to $\infty$. Due to the mutual semilocality of fields \cite{12}, $G_{ij}^{kl}$ is not in general a single valued function. Its expansion in blocks must reproduce its monodromy properties, which in turn can be read from the OPE’s \cite{2}. Obviously, blocks for the right chiral part of the correlation function are trivially equal to 1. Moreover, in each channel there is only one possible exchanged family and therefore
only one block:

\[ G_{ij}(x) = C_{ij} C_{kl} F_{kl}(x) \]
\[ G_{ik}(1-x) = C_{ik} C_{ji} F_{ik}(1-x) \]
\[ G_{il}(1/x) = C_{il} C_{jk} F_{il}(1/x) \]

The behaviour of the blocks at \( x = 0, 1, \infty \) can be easily inferred from the OPE’s

\[ \mathcal{F}_{ij}(x) \sim x^{\alpha_{ij}} \sum_{n=0}^{\infty} C_n x^n \]
\[ \mathcal{F}_{ik}(1-x) \sim (1-x)^{\alpha_{ik}} \sum_{n=0}^{\infty} d_n (1-x)^n \]
\[ \mathcal{F}_{il}(1/x) \sim \left( \frac{1}{x} \right)^{\alpha_{il}} \sum_{n=0}^{\infty} h_n x^{-n} \]

The series expansions are convergent in a neighborhood of 0, 1, \( \infty \) respectively. Some of the coefficients \( c_n, d_n, h_n \) can be computed from the information contained in the OPE’s (8). In particular \( c_0, d_0, h_0 \) are always guaranteed to be equal to 1. Moreover, if \( i \neq j \)

\[ \mathcal{F}_{ij}(x) = x^{\alpha_{ij}} \left( 1 + \frac{(\alpha_{ij} + 2\Delta_i)(\alpha_{ij} + 2\Delta_j)}{2\Delta_{i+j}} x + O(x^2) \right) \]

while if \( i = j \)

\[ \mathcal{F}_{ii}(x) = x^{-2\Delta_i} \left( 1 + \frac{2\Delta_i \Delta_k}{c} x^2 + O(x^3) \right) \]

The behaviour for the blocks in eq. (9) reproduces the correct monodromy properties of the multivalued correlation function. The blocks are (up to the branch singularity in the leading factor) locally holomorphic functions of \( x \). They can be analytically continued in the whole plane, excluding the branch points. Closure under analytic continuation requires \[ [13, 15] \]

\[ \alpha_{ij} + \alpha_{ik} + \alpha_{il} - 2\Delta_i = -R \]

where \( R \) is a non-negative integer. This in turn fixes the general form of blocks to be

\[ \mathcal{F}_{ij}(x) = x^{\alpha_{ij}} (1-x)^{\alpha_{ik}} P(x) \]
\[ \mathcal{F}_{ik}(1-x) = (1-x)^{\alpha_{ik}} x^{\alpha_{ij}} Q(1-x) \]
\[ \mathcal{F}_{il}(1/x) = x^{-\alpha_{il}} (1-x^{-1})^{\alpha_{ik}} T(1/x) \]
\[ = e^{-\pi i \alpha_{ik} x^{2\Delta_i} x^{\alpha_{ij}} (1-x)^{\alpha_{ik}} x^R T(1/x)} \]
where $P(x) = \sum_{n=0}^{R} P_n x^n$, $Q(x) = \sum_{n=0}^{R} Q_n x^n$ and $T(x) = \sum_{n=0}^{R} T_n x^n$ are polynomials of degree $R$ in $x$. In the last equality eq. (12) has been used. To avoid heavy notation we dropped indices $i, j, k, l$ from the polynomials, but it must be bared in mind that they, as well as the integer $R$, are specific of the particular correlation function $G_{ij}^{kl}(x)$. Normalization of the blocks implies $P_0 = Q_0 = T_0 = 1$.

The duality requirement (7) can then be simply translated in conditions on these polynomials

\begin{align*}
    \text{st-duality: } & \Rightarrow C_{ij} C_{kl}^{\ast} P(x) = C_{ik} C_{jl}^{\ast} Q(1-x) \\
    \text{su-duality: } & \Rightarrow C_{ij} C_{kl}^{\ast} P(x) = C_{il} C_{kj}^{\ast} x^R T(1/x)
\end{align*}

These equations can take a particularly simple form when they are considered for some special cases. Here and in the following we use the notation $G_{ij}^{kl} = \langle i j k l \rangle$. Remembering that $C_{i,i} = 1$, we first consider the correlation functions $\langle i \hat{i} j k \rangle$.

The constraints

\begin{equation}
    \frac{2\Delta_i \Delta_k}{c} = P_2 - \frac{1}{2}(\alpha_{ik}^2 + \alpha_{ik})
\end{equation}

and

\begin{align*}
    P_1 &= \alpha_{ik} \\
    Q_1 &= \frac{(\alpha_{ik} + 2\Delta_i)(\alpha_{ik} + 2\Delta_k)}{2\Delta_{i+k}} + \alpha_{ij} \\
    T_1 &= \frac{(\alpha_{ik} + 2\Delta_i)(\alpha_{ik} + 2\Delta_k)}{2\Delta_{i-k}} + \alpha_{ik}
\end{align*}

obtained comparing the blocks with the expansions (10,11) can be conveniently used, together with the duality constraints

\begin{align*}
    |C_{ik}|^2 = P(1) &= \sum_{n=0}^{R} P_n, \quad Q_n = \frac{1}{|C_{ik}|^2} \sum_{p=n}^{R} \binom{n}{p} P_p \\
    |C_{i,j}^{\ast}|^2 = P_n &= \frac{1}{|C_{i,j}^{\ast}|^2} P_{R-n}
\end{align*}

(17) (18)

to fix as much as possible the parameters. In particular for $R = 0$ it should be $\alpha_{ik} = 0$ and then $c = \infty$. Hence if at least one of the correlation functions $\langle i \hat{i} i \hat{i} \rangle$ has $R = 0$ the whole algebra is inconsistent.
In the case $i = k$, the correlation function $\langle i \, i \, i \, i \rangle$ has even more constraints. Indeed, here $su$-duality brings again to the same block, hence the polynomial $T(x)$ is equal to $P(x)$, and we have the condition

$$P(x) = x^R P(1/x) \Rightarrow P_{R-n} = P_n$$

(19)

It is easy to convince oneself that if $R \leq 3$ the polynomial is completely fixed. For $R = 1$ the only way to avoid inconsistency is to have $\alpha_{ik} = 1$. In the $R = 2, 3$ cases eq.(15) can be conveniently used to fix $c$. For higher values of $R$ (13) can be used to express the coefficient $P_2$ in terms of $c$.

Another case of strongly constrained correlation function is that of four equal fields $\langle i \, i \, i \, i \rangle$. In this case the blocks in all the three channels coincide and there is only one polynomial whose coefficients are constrained by the equations $P(x) = x^R P(1/x)$ and $P(x) = P(1-x)$. They have no solution for $R$ odd, for $R$ even the coefficients $P_n$ are fully determined if $R < 6$.

A last remark is in order about conformal dimensions: we have seen in the previous section that they are determined up to integers $M_k$. Now, conformal dimensions enter in the determination of $R$, eq.(12). The fundamental requirement that for all 4-point functions $R$ must be a non negative integer can then be translated into a set of inequalities to be satisfied by the integers $M_k$, thus strongly selecting, as we shall see in the next section, the possible choices of conformal dimensions.

The strategy to study a given algebra will then be the following:

1. identify the parameters in the algebra. These are $c$ and the independent structure constants. Moreover, spin of fields will eventually depend on a set of integers $M_k$ as in eq.(4).

2. list all the non-zero 4-point functions and compute $R$ for each of them. Imposing $R \geq 0$ the integers $M_k$ can be selected to a few possible choices.

3. For each choice consider the functions $\langle i \, i \, k \, \bar{k} \rangle$. If at least one of them has $R = 0$, discard the choice. Else, identify the $\langle i \, i \, k \, \bar{k} \rangle$ functions with lower values
of $R$ and, after having constrained as much as possible the coefficients of the polynomials using duality, try to fix $c$ and/or some $C_{ij}$ through eqs. (15, 17, 18). If different correlation functions provide inconsistent values of $c$ or $C_{ij}$, discard the choice.

4. Otherwise use the values so obtained to fix as much as possible the other correlation functions and check all the possible duality constraints. If somewhere some unconsistency appears, discard the choice. If instead the choice passes all the checks it defines a consistent associative algebra. If some parameter is left free, the algebra allows for an infinity of associative realizations, one for each value of the parameter.

5. redo steps 3 and 4 for all the choices selected by step 2.

Next section will illustrate this procedure on some simple examples.

4 \textbf{SZ}_K \textbf{ algebras for } K \leq 7

To illustrate the general theory of the previous section, let us study some particular case in detail. We shall discover that even for low values of $K$ some interesting surprises arise.

\textbf{SZ}_2 \textbf{ algebra} - In the case $K = 2$, i.e. usual $N = 1$ superconformal algebra there is only one fermionic field $\psi_1$ of spin 3/2. No non trivial structure constant appear and the only parameter in the algebra is $c$. There is only one non-trivial 4-point function $\langle 1 \ 1 \ 1 \ 1 \rangle$ with $R = 6$. Crossing symmetry fixes it up to a free parameter:

$$\langle 1 \ 1 \ 1 \ 1 \rangle = x^{-3}(1 - x)^{-3}(1 - 3x + P x^2 + \frac{9 - 2P}{3} x^3 + P x^4 - 3x^5 + x^6) \quad (20)$$

that can be re-expressed in terms of $c$ as $P = (15c - 3)/2c$. No other restriction can be found on the algebra. Therefore we conclude that \textbf{SZ}_2 is associative for any value of $c$. Unitarity \cite{1} will then restrict to $c = \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right)$ or $c \geq 3/2$. 

10
**SZ\(_3\) algebra** - The case \(K = 3\), namely the spin 4/3 algebra of Fateev and Zamolodchikov [4], has two free parameters: \(c\) and the structure constant \(C_{1,1} = \lambda\). On the other hand, there is only one non trivial 4-point function \(\langle 1 \hat{1} 1 \hat{1} \rangle\), for which \(R = 4\). Crossing symmetry then determines the 4-point function up to a free parameter \(P\)

\[
\langle 1 \hat{1} 1 \hat{1} \rangle = x^{-8/3}(1-x)^{-4/3}(1 - \frac{4}{3}x + Px^2 - \frac{4}{3}x^3 + x^4)
\]  

(21)

Both \(c\) and \(\lambda\) can be computed in terms of \(P\) by use of eqs.(15,17). Eliminating \(P\) in the result we reobtain the well known relation computed by Fateev and Zamolodchikov 9

\[c|\lambda|^2 = 4(8 - c).\]

Hence the **SZ\(_3\)** algebra has only one free parameter, that can be chosen to be \(c\). Unitarity [16] will fix it to \(c = 2 \left(1 - \frac{12}{m(m+4)}\right)\) or \(c \geq 2\).

**SZ\(_4\) algebra** - The first new case \(K = 4\) has parafermions \(\psi_1, \psi_2\) and \(\psi_3 = \psi_1^\dagger\) of dimensions \(\Delta_1 = \Delta_3 = 5/4\) and \(\Delta_2 = M_2 - 1\) respectively. The algebra has two parameters: \(c\) and the structure constant \(C_{1,1} = \lambda\). There are four non trivial 4-point functions. The requirement that for each of them \(R \geq 0\) amounts to a set of inequalities in \(M_2\):

- from \(\langle 1 \hat{1} 1 \hat{1} \rangle \Rightarrow M_2 \leq 6\)
- from \(\langle 2 \ 2 \ 2 \ 2 \rangle \Rightarrow M_2 \geq 1\)
- from \(\langle 1 \hat{1} 2 \ 2 \rangle \Rightarrow M_2 \geq 1\)
- from \(\langle 1 \ 1 \ 1 \ 1 \rangle \Rightarrow M_2 \leq 2\)

We see that these, together with the fact that \(\Delta_2\) must be different from zero to avoid doubling of the vacuum, imply \(M_2 = 2\) and therefore \(\Delta_2 = 1\). The correlation function \(\langle 2 \ 2 \ 2 \ 2 \rangle\) with \(R = 4\) is completely fixed by crossing symmetry

\[
\langle 2 \ 2 \ 2 \ 2 \rangle = x^{-2}(1-x)^{-2}(1 - 2x + 3x^2 - 2x^3 + x^4)
\]  

(22)

and can be conveniently used to fix \(c = 1\) by use of eq.(13). The other correlation function \(\langle 1 \hat{1} 1 \hat{1} \rangle\) has \(R = 4\). Duality should fix it up to a parameter depending on...
c. The value \( c = 1 \) fixes this parameter, and the correlation function reads

\[
\langle 1 \hat{1} 1 \hat{1} \rangle = x^{-5/2}(1 - x)^{-3/2}(1 - \frac{3}{2}x + \frac{7}{2}x^2 - \frac{3}{2}x^3 + x^4)
\]  

Eq. (23) then allows to compute the structure constant (up to an inessential phase that we fix to 1) as \( \lambda = \sqrt{5/2} \). One can check that the remaining correlation functions are compatible with this fixing of the parameters.

The surprising result here is that \( c \) is no more free. \( \text{SZ}_4 \) does not allow for a series of minimal models at different \( c \) plus continuum, but rather for a series of models at \( c = 1 \), that can be indeed easily identified with points on the gaussian and orbifold lines for suitable values of the compactification radius. In a certain sense, the series of models present in \( \text{SZ}_2 \) and \( \text{SZ}_3 \) cases are here “squeezed” to the \( c = 1 \) lines. The models can still “flow” from one another, but the perturbing operator is now the limiting case of least relevant operators, namely the marginal operator that allows to move along the \( c = 1 \) lines. All the \( \text{SZ}_4 \) invariant models are connected by this marginal operator to one of the two modular invariant solutions of \( Z_4 \) parafermion (the diagonal one lying on the orbifold line). One could then still speculate about \( \text{SZ}_4 \) spontaneous symmetry breaking with goldstini given by \( Z_4 \) parafermions. This picture is however somewhat delicate here: the current \( \psi_1 \) of spin \( 5/4 \) seems to break nicely to give a paragoldstino of spin \( 3/4 \) (again \( 5/4 + 3/4 = 2 \)), but the other current of spin 1 cannot break spontaneously due to Coleman theorem, and indeed we find it again in the \( Z_4 \) usual parafermion. As the stress-energy tensor can be related to this current via \( U(1) \) Sugawara construction, this implies also that conformal symmetry cannot be broken along this “flow” and the central charge should not change, as indeed is the case. All these strange features indicate that \( K = 4 \) is somewhat a “limiting” case for \( Z_K \) exotic superconformal algebras.

\( \text{SZ}_5 \) algebra - In this case there are four parafermions of spin \( \Delta_1 = \Delta_4 = 6/5, \Delta_2 = \Delta_3 = M_2 - 6/5 \) respectively. The algebra has 3 parameters: \( c, \lambda = C_{1,1} \) and \( \mu = C_{1,2} \). The requirement that \( R \geq 0 \) for all 4-point functions translates again into a set of inequalities for \( M_2 \) that can be simultaneously satisfied only if \( M_2 = 2, 3 \).
In the first case \( M_2 = 2 \) we have \( \Delta_1 = 6/5, \Delta_2 = 4/5 \) and we reobtain the usual \( Z_5 \) parafermionic model of Fateev and Zamolodchikov [12], merely with the name of the two parafermions reversed. The other case is \( \Delta_1 = 6/5, \Delta_2 = 9/5 \). Using the correlation function \( \langle \hat{1} \hat{1} \hat{1} \hat{1} \rangle \), with \( R = 3 \), we can fix \( c = -6 \) and \( \lambda = \sqrt{4/5} \). The \( \langle \hat{1} \hat{2} \hat{2} \rangle \) correlation function, with \( R = 3 \), reads

\[
\langle \hat{1} \hat{2} \hat{2} \rangle = x^{-12/5}(1-x)^{-6/5} \left(1 - \frac{6}{5}x - \frac{3}{5}x^2 + \frac{4}{5}x^3\right)
\]

where we have fixed the coefficient of \( x^2 \) in the polynomial inserting \( c = -6 \) in eq.(15), and that of \( x^3 \) by imposing su-duality, eq.(18). su-duality then requires that the sum of coefficients in the polynomial \( P(x) \) is equal to \( |\mu|^2 \). But this sum is zero, thus implying \( \mu = 0 \) in contradiction with the fusion rules. This argument rules out the \( M_2 = 3 \) case.

\( \text{SZ}_5 \) can then only coincide with the \( Z_5 \) parafermionic algebra with \( c = 8/7, \lambda = \sqrt{8/5}, \mu = \sqrt{9/5} \). This is a very particular case: it is easy to check that no other \( Z_K \) parafermionic model contains a current of spin \( 1+1/K \). See also appendix A for a more general result on this point.

**\( \text{SZ}_{6,7} \) algebras** - For \( K = 6 \) we have 5 parafermions of spin \( \Delta_1 = \Delta_5 = 7/6, \Delta_2 = \Delta_4 = M_2 - 4/3, \Delta_3 = M_3 - 3/2 \). The parameters in the algebra are \( c, \lambda = C_{1,1}, \mu = C_{1,2} \) and \( \rho = C_{2,2} \). The requirement \( R \geq 0 \) for all 4-point functions restricts \( M_2 \) and \( M_3 \) to the following chioces:

- **A.** \( M_2 = M_3 = 2 \)  \( \Delta_1 = 7/6, \Delta_2 = 2/3, \Delta_3 = 1/2 \)
- **B.** \( M_2 = 2, M_3 = 3 \)  \( \Delta_1 = 7/6, \Delta_2 = 2/3, \Delta_3 = 3/2 \)
- **C.** \( M_2 = M_3 = 3 \)  \( \Delta_1 = 7/6, \Delta_2 = 5/3, \Delta_3 = 3/2 \)

Case A is immediately ruled out because \( \{1, \psi_2, \psi_4\} \) form in this case a \( Z_3 \) parafermion subalgebra with \( c = 4/5 \) while \( \{1, \psi_3\} \) form a \( Z_2 \) algebra with \( c = 1/2 \), incompatible with the former. Case B also is nonsense, as \( \{1, \psi_2, \psi_4\} \) still forms a \( Z_3 \) subalgebra with \( c = 4/5 \), but we know that at \( c = 4/5 \) there is no room for Virasoro primary fields of dimensions \( 7/6 \) or \( 3/2 \). We are left with case C, where
the correlation function $\langle 1 \hat{1} 1 \hat{1} \rangle$ (with $R = 3$) fixes $c = -49/10$. Besides the fact that this automatically excludes possibility to build up any unitary model, the $\text{SZ}_6$ algebra of case $C$ is inconsistent even for this negative value of $c$. Indeed it can be checked that the correlation functions $\langle 1 \hat{1} 2 \hat{2} \rangle$ and $\langle 1 \hat{1} 3 \hat{3} \rangle$ give two incompatible (and negative!) values of $|\mu|^2$.

Thus there is no possibility to realize an associative $\text{SZ}_6$ algebra. The same happens for $K = 7$, where the only possibility $\Delta_1 = 8/7, \Delta_2 = 11/7, \Delta_3 = 9/7$ gives a negative value of $c$ and is ruled out by arguments similar to those of $K = 6$. For higher values of $K$ the analysis is in principle still possible, but the number of correlation functions to analyze increases and computations can become cumbersome. To get more general results we have to turn to a closer analysis of some particular correlation function and restrict more the problem by some new input.

5 General unitary case - Proof of unconsistency for $K > 5$

If we are content to explore the possibility to have unitary $Z_K$ exotic superconformal theories, then the additional requirements $c > 0$ and $\Delta_i > 0$ help to get a general answer. This can be given in form of a

**Theorem** - With the constraints $c > 0$ and $\Delta_i > 0, i = 1, 2, 3, 4$, there is no associative $\text{SZ}_K$ algebra for $K > 5$.

In the proof of this statement, we make use of three conformal dimensions

$$\Delta_1 = 1 + \frac{1}{K}, \quad \Delta_2 = M_2 - 2 + \frac{4}{K}, \quad \Delta_3 = M_3 - 3 + \frac{9}{K} \quad (25)$$

We already know from previous section that $\text{SZ}_K$ algebras exist for $K \leq 5$ and are absent for $K = 6, 7$. Here we shall consider values of $K \geq 6$. Requirement $\Delta_i > 0$ bounds $M_i$ to

$$M_2 \geq 2, \quad M_3 \geq \begin{cases} 2 & \text{for } K \leq 8 \\ 3 & \text{for } K \geq 9 \end{cases} \quad (26)$$
Let us first consider the correlation function \( \langle 1 \hat{1} 1 \hat{1} \rangle \), for which \( R = 4\Delta_1 - \Delta_2 = 6 - M_2 \geq 0 \) requires \( M_2 \leq 6 \). The case \( M_2 = 6 \) leads to \( R = 0 \) and, as explained in section 3, is inconsistent, the same happens for \( M_2 = 5, R = 1 \) as \( \alpha_{11} = M_2 - 4 + \frac{2}{K} \neq 1 \). The case \( M_3, R = 3 \) always gives, through eq. (15) negative values of \( c \), hence it is discarded too. We are left with two possibilities:

A. \( M_2 = 2, R = 4 \) \( \Rightarrow P(x) = 1 - 2K_1 x + P_2 x^2 - 2K_1 x^3 + x^4 \)

B. \( M_2 = 4, R = 2 \) \( \Rightarrow P(x) = 1 - \frac{2}{K} x + x^2 \Rightarrow c = \frac{2(K+1)}{K-2} \)

Case A - To go on we have to resort to another correlation function, namely to \( \langle 1 \hat{1} 2 \hat{2} \rangle \), that, for \( M_2 = 2 \), has \( R = 4 - M_3 \). The requirement \( R > 0 \) (\( R = 0 \) again is inconsistent) then restrict \( M_3 \leq 3 \). Thus for \( K \geq 9, M_3 = 3, R = 1 \) is the only possible value. In this case \( P(x) = 1 + 4K_1 x \) and \( |C_{11}|^2 = P_1 = \frac{4 - K}{K} \) is negative for all \( K > 4 \). This rules out this case. There is still the possibility of \( M_3 = 2 \) for \( K = 6, 7, 8 \). The cases \( K = 6, 7 \) are ruled out by the results of the previous section. \( K = 8 \) is the only case where we have also to consider \( \Delta_4 = M_4 - 2 \). \( \Delta_4 > 0 \) implies \( M_4 > 2 \) while the correlation function \( \langle 2 2 2 2 \rangle \) requires \( M_4 \leq 2 \). Also this case is ruled out.

Case B - The function \( \langle 1 \hat{1} 1 \hat{1} \rangle \) previously considered is completely fixed in this \( R = 2 \) case and yields

\[
|C_{11}|^2 = 2 + \frac{2}{K} \quad (27)
\]

In this case the correlation function \( \langle 1 \hat{1} 2 \hat{2} \rangle \) has \( R = 8 - M_3 \), hence it must be \( M_3 < 8 \). Here it is convenient to resort to the function \( \langle 1 1 1 \hat{3} \rangle \), having \( R = M_3 - 3M_2 + 6 = M_3 - 6 \), that requires \( M_3 \geq 6 \). Hence \( M_3 = 6, 7 \). Consider again the function \( \langle 1 \hat{1} 2 \hat{2} \rangle \). For \( M_3 = 7, R = 1 \) it is possible to compute \( |C_{11}|^2 = 1 + \frac{4}{K} \), in contradiction with \((27)\). For \( M_3 = 6 \) the value of \( |C_{11}|^2 \) in \((27)\) helps to fix the coefficient \( P_2 \) from which \( c \) can be evaluated back. We obtain \( c = \frac{4(K+1)(K+2)}{K^2 - 4K + 8} \), in contradiction with \((27)\).

All the possible cases are then ruled out by simply considering a set of few correlation functions for some of the fields in the algebra. We believe that this
result, surely valid for unitary theories, is in fact absolutely general: there are no $Z_K$ exotic (in the sense of $Q^K = P$) superconformal algebras for $K > 5$.

6 Conclusions and implications

The main result of this paper is the impossibility to construct $Z_K$ exotic $N = 1$ superconformal algebras for $K > 5$. For $K = 5$ the result is trivial, for $K = 4$ is $c = 1$ theory, and finally we are “seriously” left only with the already known cases $K = 2, 3$, i.e. ordinary $N = 1$ superconformal algebra and the spin 4/3 algebra. What is established is that for $K > 5$ it is not possible to realize at the conformal point, an algebra of currents such that it can define a charge $Q$ satisfying $Q^K = P$. This does not mean that “more exotic” supersymmetric algebras can not be constructed in two dimensions: for example, Fateev [18] has recently shown that in some parafermionic models it is possible to consider a charge $Q$ such that $Q^K = P_s$, where $P_s$ is an appropriate higher spin local conserved charge. These models are in connection with the parafermionic algebras introduced in the appendix A of [12], and further studied, from the unitarity point of view, in [16]. In fact, $SZ_3$ is also a particular case of these algebras.

As $Q^K = P$ can not be realized at criticality, it is presumably impossible also in perturbations of CFT’s. It then becomes problematic to identify the Boltzmann weights of [11] with the scattering matrices of a 2D QFT. It can happen that such a QFT does not exist, but also that it exists and its UV limit is quite tricky. Also, the spontaneous symmetry breakdown and the goldstino problem needs more understanding. An intriguing observation in this connection is that the goldstino picture breaks down at $K = 4$. Palla [19] has studied perturbations of $Z_K$ parafermionic models that can convert some of the parafermionic currents into conserved quantities. Now, this is possible exactly starting from $K = 4$. Many indications point to the fact that the behaviour of parafermionic theories should be quite different for $K < 4$ and $K > 4$, with $K = 4$ as a limiting case. If this is related or not to more
fundamental issues like Galois theory is to be understood.

Acknowledgements - I am grateful to M.Bauer, D.Bernard, V.Pasquier and J.B.Zuber for many useful discussions. I thank the Service de Physique Théorique of C.E.A. - Saclay for the kind hospitality. The Theory Group of I.N.F.N. - Bologna and the Director of Sez. di Bologna of I.N.F.N. are acknowledged for the financial support allowing me to spend this year in Saclay.

Appendix

It is interesting to ask if there are $SZ_P$ algebras hidden in the usual $Z_K$ parafermionic models, even for $P \neq K$, generalizing the curious phenomenon observed for $Z_5$. We shall show here that no such case is possible for $P > 5$, in agreement with the results of the main part of the paper, and that the only other cases can always be traced back to the well known $SZ_2$ or $SZ_3$ algebras.

First of all let us prove that for $P > 5$ there is no parafermion of spin $\frac{P+1}{P}$ contained in any $Z_K$ parafermionic algebra, for all $K$. To do that, we have to consider the equation

$$\frac{P + 1}{P} = \frac{k(K - k)}{K}$$

where the expression on the r.h.s., for $k = 1, ..., K-1$ gives the most general spin for a parafermion in $Z_K$. The case $k = 1$ is clearly impossible, for all $K$ and all $P$ as it equates a l.h.s. greater than 1 with a r.h.s. less than 1. So consider $k \geq 2$. Solving (28) for $K$ one gets

$$K = k + 1 + \frac{k + P + 1}{P}$$

As $K$ must be an integer, we have to require $k + P + 1 \geq Pk - P - 1$, i.e.

$$k \leq \frac{2P + 1}{P - 1}$$
For $P > 5$ this implies $k \leq 2$, hence the only possibility is $k = 2$. Substituting in (29) we get

$$K = 4 + \frac{4}{P - 1}$$  \hspace{1cm} (31)

which can never be integer if $P > 5$. This proves that no $SZ_P$ is contained in a usual parafermionic algebra for $P > 5$.

For $P \leq 5$ use of (29) and (30) allows to list all the possible occurrences.

- For $P = 5$ we have $K = 5$, $k = 2, 3$. This is the result noticed in the paper that $SZ_5$ coincides with the $Z_5$ parafermion.

- For $P = 4$ no solution appears.

- For $P = 3$ we have $K = 6$, $k = 2, 4$, thus showing that a realization of $SZ_3$ is contained in the $Z_6$ parafermionic algebra. This is not surprising as the $Z_6$ model is known to belong to the unitary minimal series of spin $4/3$ algebra, namely for $m = 4$.

- For $P = 2$ there are two solutions: $K = 6$, $k = 3$ says that the $Z_6$ model is also supersymmetric, (it belongs indeed also to the superconformal minimal series for $m = 6$) while $K = 8$, $k = 2, 6$ shows two fields of spin $3/2$ for the $Z_8$ model.

- For $P = 1$ the solution $K = 8$, $k = 4$ completes the result for $P = 2$: the two spin $3/2$ curents are associated to a current of spin 2, thus $\{T, \psi_2, \psi_4, \psi_6\}$ form in this case two copies of the $N = 1$ superconformal algebra: the $Z_8$ model is doubly $N = 1$ supersymmetric. Another solution appears for $P = 1$ when $K = 9$, $k = 3, 6$.

This exhausts all possible realizations of $SZ_P$ algebras in $Z_K$ parafermionic models.

References
[1] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl. Phys. B241 (1984) 333

[2] For comprehensive reviews see lectures by J.Cardy, P.Ginsparg and J.B.Zuber, in Fields, Strings and Critical Phenomena, Les Houches, Ed. E.Brézin and J.Zinn-Justin, (Elsevier Science Publishers, 1989)

[3] A.M.Polyakov, Sov. Phys. JETP Lett. 12 (1970) 381

[4] A.B.Zamolodchikov and V.A.Fateev, Teor. Math. Phys. 71 (1987) 163

[5] D.Friedan, Z.Qiu and S.Shenker, Phys. Lett. B151 (1985) 37

[6] C.Ahn, D.Bernard and A.LeClair, Nucl. Phys. B346 (1990) 409

[7] D.Kastor, E.Martinec and S.Shenker, Nucl. Phys. B316 (1989) 590

[8] Č.Crnković, G.Sotkov and M.Stanishov, Phys. Lett. B226 (1989) 297

[9] A.B.Zamolodchikov, Landau Institute preprint, 1989

[10] V.A.Fateev and A.B.Zamolodchikov, Phys. Lett. A92 (1982) 37

[11] D.Bernard and V.Pasquier, Int. J. Mod. Phys. B4 (1990) 913

[12] V.A.Fateev and A.B.Zamolodchikov, Sov. Phys. JETP 62 (1985) 215

[13] P.Christe and F.Ravanini, Phys. Lett. B217 (1989) 252

[14] C.Vafa, Phys. Lett. B206 (1988) 421

[15] P.Christe and R.Flume, Phys. Lett. 188B (1987) 219; Nucl. Phys. B282 (1987) 466;
P.Christe, PhD thesis, BONN-IR-86-32 (Dec. 1986)

[16] P.Goddard and A.Schwimmer, Phys. Lett. B206 (1988) 62

[17] S.Coleman, Comm. Math. Phys. 31 (1973) 259
[18] V.A. Fateev, LPHT preprint 90-52

[19] L. Palla, Phys. Lett. B253 (1991) 342