DERIVATIVES OF EIGENVALUES AND JORDAN FRAMES

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Abstract. Every element in a Euclidean Jordan algebra has a spectral decomposition. This spectral decomposition is a generalization of the spectral decompositions of a matrix. In the context of Euclidean Jordan algebras, this is written using eigenvalues and the so-called Jordan frame. In this paper we deduce the derivative of eigenvalues in the context of Euclidean Jordan algebras. We also deduce the derivative of the elements of a Jordan frame associated to the spectral decomposition.

1. Introduction. Euclidean Jordan algebras have been used to deal with optimization problems involving symmetric cones (cf. [4, 5, 7, 6, 8, 9, 10, 11, 15, 16, 12, 13, 19, 18, 20, 21, 22]). The optimization problems involving symmetric cones are tractable ones. These problems are known by linear optimization over symmetric cones or symmetric optimization. The connection between Euclidean Jordan algebras and symmetric cones follows from the fact that any symmetric cone can be written as a cone of squares of some Euclidean Jordan algebra (cf. [3]). In the process of developing algorithms for symmetric optimization many properties concerning eigenvalues have been proved. We refer the reader to references given before.

Every element $x$ in a Euclidean Jordan algebra has a spectral decomposition. This is written using eigenvalues and idempotents, which are functions of $x$. This paper generalizes the derivatives of eigenvalues of matrices, presented in [14]. Here, we deduce the derivative of eigenvalues and the derivative of a Jordan frame, with respect to $x$. As a motivation, these derivatives might help in dealing with optimization problems in the framework of Euclidean Jordan algebras. More precisely, interior-point methods have been using the so-called barrier functions ([17]), where these functions, for symmetric optimization ([20]), have often the form

$$
\Psi(x) = \sum_{i=1}^{r} \psi(\lambda_i(x))c_i,
$$

where $\sum_{i=1}^{r} \lambda_i(x)c_i$ is the spectral decomposition of $x$ (see details below) and $\psi$ is a real valued function, called kernel function, see [2]. As these methods need
the computation of the gradient and hessian of the barrier function, the formulas deduced in this paper may play a role here.

The manuscript is organized as follows. In Section 2 we define the Euclidean Jordan algebras and present the principal concepts used here: the quadratic representation, the spectral decomposition and the Peirce decomposition. For more details about Euclidean Jordan algebras we refer to [3]. In Section 3 we deduce the derivatives of the eigenvalues and the derivatives of a Jordan frames.

2. Topics in Euclidean Jordan algebras. A (real) Jordan algebra is a pair \((V, \circ)\) where \(V\) is a finite-dimensional vector space over \(\mathbb{R}\) and “\(\circ\)” is a bilinear map \((x, y) \mapsto x \circ y\) (Jordan product) from \(V \times V\) into \(V\) such that

- \(x \circ (y \circ z) = (x \circ y) \circ z\), where \(x^2 = x \circ x\).

For an element \(x\) in \(V\) let \(L(x)\) be the linear map from \(V\) to \(V\) defined by \(L(x)y = x \circ y\).

For each \(x\) in \(V\) we define

\[ P(x) = 2L(x)^2 - L(x^2). \]

The map \(P\) is called the quadratic representation of \(V\).

From now on \(V\) denotes an Euclidean Jordan algebra with rank \(r\) \((\text{rank}(V) = r)\) and identity element \(e\).

Since \(V\) is a finite-dimensional vector space, for each \(x \in V\), there exists a positive integer \(k\) such that the set \(\{e, x, x^2, \ldots, x^k\}\) is linearly dependent. This implies the existence of a polynomial \(p \neq 0\) such that \(p(x) = 0\). We define the minimal polynomial of \(x \in V\) as polynomial \(p\) of minimal degree such that \(p(x) = 0\).

We define the degree of \(x\), denoted as \(\text{deg}(x)\), as the degree of the minimal polynomial of \(x\). Obviously \(\text{deg}(x) \leq \dim(V)\), where \(\dim(V)\) denotes the dimension of the vector space \(V\) over \(\mathbb{R}\).

We define the rank of \(V\) as

\[ \text{rank}(V) := \max\{\text{deg}(x) : x \in V\}. \]

An element \(x \in V\) is called regular if \(\text{deg}(x) = \text{rank}(V)\).

**Proposition 1 ([3]).** Let \((V, \circ)\) be a Jordan algebra with rank \(r\). The set of regular elements is open and dense in \(V\). There exist polynomials \(a_1, a_2, \ldots, a_r \in \mathbb{R}[X], \)

\(i = 1, \ldots, r\) such that \(a_i(x) \in \mathbb{R}\) and the minimal polynomial of every regular element \(x \in V\) in the variable \(\lambda\) is given by

\[ f(\lambda; x) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \cdots + (-1)^ra_r(x). \]

The polynomials \(a_1, \ldots, a_r\) are unique and \(a_j\) is homogeneous of degree \(j\), for \(1 \leq j \leq r\).

The polynomial \(f(\lambda; x)\) in the above proposition is called the characteristic polynomial of \(x\).

**Definition 2.1.** The coefficient \(a_1(x)\) is called the trace of \(x\), denoted as \(\text{tr}(x)\). The coefficient \(a_r(x)\) is called the determinant of \(x\), denoted as \(\text{det}(x)\). The roots of the characteristic polynomial are called the eigenvalues of \(x\).

A Jordan algebra \((V, \circ)\) endowed with an inner product, denoted \(\langle , \rangle\), such that

\[ \langle x \circ y, z \rangle = \langle x, y \circ z \rangle, \]
for all \( x, y, z \in V \) is said to be Euclidean. We use the inner product
\[
\langle x, y \rangle = \operatorname{tr} (x \circ y).
\]

An element \( c \in V \) is said to be idempotent if \( c^2 = c \). Two idempotents \( c \) and \( d \) are said to be orthogonal if \( c \circ d = 0 \). Since
\[
\langle c, d \rangle = \langle c^2, d \rangle = \langle c, c \circ d \rangle = \langle c, 0 \rangle = 0,
\]
optential idempotents are orthogonal with respect to the inner product. One says that \( c_1, \ldots, c_k \), with \( k \leq r \), is a complete system of orthogonal idempotents if
\[
c_i^2 = c_i, \quad \forall i
\]
\[
c_i \circ c_j = 0 \text{ if } i \neq j,
\]
\[
c_1 + c_2 + \cdots + c_k = e.
\]

The following theorem gives a spectral decomposition for any element \( x \in V \).

**Theorem 2.2 ([3]). Spectral theorem, type I.** For \( x \) in \( V \) there exists unique real numbers \( \lambda_1, \ldots, \lambda_k \), all distinct, and a unique complete system of orthogonal idempotents such that
\[
x = \lambda_1 c_1 + \cdots + \lambda_k c_k.
\]

For each \( j = 1, \ldots, k \), we have \( c_j \in \mathbb{R}[x] \). The numbers \( \lambda_j \) are the eigenvalues of \( x \) and \( \sum_{j=1}^k \lambda_j c_j \) is called the spectral decomposition of \( x \).

We say that an idempotent is primitive if it is non-zero and cannot be written as the sum of two (necessarily orthogonal) non-zero idempotents. We say that \( c_1, \ldots, c_m \) is a complete system of orthogonal primitive idempotents, or a Jordan frame, if each \( c_j \) is a primitive idempotent and if
\[
c_i \circ c_j = 0, \quad i \neq j
\]
\[
\sum_{j=1}^m c_j = e.
\]

**Theorem 2.3 ([3]). Spectral theorem, type II.** Suppose \( V \) has rank \( r \). Then for every \( x \) in \( V \) there exists a Jordan frame \( c_1, \ldots, c_r \) and real numbers \( \lambda_1, \ldots, \lambda_r \) such that
\[
x = \sum_{j=1}^r \lambda_j c_j.
\]
The numbers \( \lambda_j \) (with their multiplicities) are uniquely determined by \( x \). Furthermore,
\[
\det(x) = \prod_{j=1}^r \lambda_j, \quad \operatorname{tr}(x) = \sum_{j=1}^r \lambda_j.
\]

The decomposition of type I can be obtained from the decomposition of type II as follows. Let \( x = \sum_{i=1}^r \lambda_i c_i \) be the spectral decomposition of type II of \( x \). Let us define the integers \( s, q_1, \ldots, q_s \) such that \( q_s = r \) and
\[
\lambda_1(x) = \cdots = \lambda_{q_1}(x) > \lambda_{q_1+1}(x) = \cdots = \lambda_{q_2}(x) > \cdots > \lambda_{q_s}(x).
\]
Define \( J_i = \{q_{i-1} + 1, \ldots, q_i\} \) (with \( q_0 = 0 \)) and put \( e_i = \sum_{j \in J_i} c_j \). Then \( e_1, \ldots, e_s \) is a complete system of idempotents and
\[
x = \sum_{i=1}^s \lambda_{q_i}(x) e_i \quad (1)
\]
is the spectral decomposition of type I.
Let \( c \) be an idempotent. We define the space
\[
V(c, \lambda) = \{ x \in V : x \circ c = \lambda x \}.
\]
Let \( c_1, \ldots, c_r \) be a Jordan frame. We consider the following subspaces of \( V \),
\[
V_{ii} = V(c_i, 1) = \mathbb{R}c_i,
\]
\[
V_{ij} = V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}).
\]

**Theorem 2.4 ([3]).** The vector space \( V \) has the following orthogonal direct sum decomposition:
\[
V = \bigoplus_{i \leq j} V_{ij}.
\]
If we denote by \( P_{ij} \) the orthogonal projection onto \( V_{ij} \), then
\[
P_{ii} = P(c_i),
\]
\[
P_{ij} = 4L(c_i)L(c_j).
\]
In view of the Theorem 2.4 we can decompose
\[
x = \sum_{i=1}^{r} x_{ii} + \sum_{i<j} x_{ij},
\]
with \( x_{ij} \in V_{ij} \) and \( i \leq j \) or equivalently,
\[
x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij}
\]
with \( x_i \in \mathbb{R}, i = 1, \ldots, r \) and \( x_{ij} \in V_{ij} \), \( i < j \). We call it the Peirce decomposition of \( x \) with respect to the Jordan frame \( c_1, \ldots, c_r \).

**Example 2.1.** Denoting \((x_0; x_1; \ldots; x_n) \in \mathbb{R}^{n+1}\) as \((x_0; \bar{x})\), with \( \bar{x} = (x_1; \ldots; x_n) \), and defining the product as
\[
x \circ y = (x^T y; x_0 \bar{y} + \bar{x}_0 \bar{x}),
\]
then \((\mathbb{R}^{n+1}, \circ)\) is a (real) Jordan algebra. This Jordan algebra endowed with inner product \( \langle x, y \rangle = \text{tr}(x \circ y) = 2x^T y, x, y \in \mathbb{R}^{n+1} \). Moreover, any \( x \in \mathbb{R}^{n+1} \), with \( \bar{x} \neq 0 \), has the spectral decomposition
\[
x = \lambda_1(x)c_1(x) + \lambda_2(x)c_2(x),
\]
where
\[
\lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \end{bmatrix} = \begin{bmatrix} x_0 - \|\bar{x}\| \\ x_0 + \|\bar{x}\| \end{bmatrix}
\]
and
\[
c_1(x) = \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix},
c_2(x) = \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix},
\]
where \( \|\bar{x}\| \) is the Euclidean norm in \( \mathbb{R}^n \). When \( \bar{x} = 0 \) the spectral decomposition is trivial.
3. Derivatives of eigenvalues. In this section we obtain derivatives of eigenvalues. To obtain these formulas we followed the ideas in [14], where similar formulas for matrices were deduced.

We denoted \( D_x f(x) \) as the derivative of \( f \) at \( x \), where \( f : V \to V \) is a function whose domain has a non-empty interior and \( x \) is a point of the interior of \( f \)'s domain. The second derivative is denoted as \( D^2_x f(x) \). For simplicity of notation and in case \( V = \mathbb{R} \) we will sometimes use \( f'(t) \) and \( f''(t) \) as the first and second derivative of \( f \) with respect to \( t \), respectively, for \( t \in \mathbb{R} \).

Let \( b \in \mathbb{R}^r \) and we assume that the coordinates of \( b \) are non-increasing order.

We can write

\[
\begin{align*}
b_1 &= \cdots = b_{k_1} > b_{k_1+1} = \cdots = b_{k_2} > b_{k_2+1} \cdots b_{k_d}, \quad (k_d = r).
\end{align*}
\]

Thus \( d \) is the number of distinct values in the coordinates of the vector \( b \). We define the corresponding partition \( \{I_1, \ldots, I_r\} \) of the index set \( \{1, \ldots, n\} \) such that

\[
I_1 = \{1, 2, \ldots, k_1\}, I_2 = \{k_1 + 1, \ldots, k_2\}, \ldots, I_d = \{k_{d-1} + 1, \ldots, k_d\},
\]

and we call these sets blocks of \( b \).

Let \( \lambda(x) \) be the vector of eigenvalues of \( x \in V \) in non-increasing order. We say that an eigenvalue of \( x \) is simple if the correspondent block has size 1.

**Proposition 2** ([1]). Let \( 1 \leq k \leq r, \; x = \sum_{i=1}^{r} \lambda_i(x) c_i \in V \) and \( \lambda(x) \in \mathbb{R}^r \). Denote \( S_p(x) = \sum_{i=1}^{p} \lambda_i(x) \), with \( p \leq r \). For any \( 1 \leq j \leq d \), \( S_{k_j}(x) \) is differentiable with respect to \( x \) and \( D_x S_{k_j}(x) = \sum_{i=1}^{k_j} c_i \).

**Proposition 3.** Under the assumptions of Proposition 2, we have

\[
D_x \left( \sum_{i \in I_\ell} \lambda_i(x) \right) = \sum_{i \in I_\ell} c_i,
\]

for \( \ell = 1, \ldots, d \).

**Proof.** By Proposition 2 we have

\[
D_x \left( \sum_{i \in I_{k_j+1}} \lambda_i(x) \right) = D_x S_{k_j+1}(x) - D_x S_{k_j}(x) = \sum_{i \in I_{k_j+1}} c_i,
\]

which proves the proposition.

In fact, we can say that the sum of equal eigenvalues of \( x \), i.e. \( \sum_{i \in I_\ell} \lambda_i(x) \), is differentiable at \( x \). Indeed, as we show below by example, if an eigenvalue is not simple then it may not be differentiable.

**Example 3.1.** Let \( V \) be the Euclidean Jordan algebra defined in Example 2.1. We have already seen that for \( x \in V \),

\[
\lambda_1(x) = x_0 - \|\bar{x}\| \quad \text{and} \quad \lambda_2(x) = x_0 + \|\bar{x}\|.
\]

Obviously

\[
\lambda_1(x) = \lambda_2(x)
\]
holds if and only if \( \| \bar{x} \| = 0 \), if and only if \( \bar{x} = 0 \). For this case, we compute the directional derivative of \( \lambda_2(x) \) at \( x \) following the direction \( u = (u_0; \bar{u}) \):

\[
D_u \lambda_2(x) = \lim_{t \to 0} \frac{\lambda_2(x + tu) - \lambda_2(x)}{t} = \lim_{t \to 0} \frac{x_0 + t u_0 + \| \bar{x} + t \bar{u} \| - x_0}{t} = \lim_{t \to 0} \frac{t u_0 + \| t \bar{u} \|}{t} = \lim_{t \to 0} u_0 + \frac{|t|\| \bar{u} \|}{t}.
\]

Since we have

\[
\lim_{t \to 0^-} \frac{|t|\| \bar{u} \|}{t} = -\| \bar{u} \|
\]

and

\[
\lim_{t \to 0^+} \frac{|t|\| \bar{u} \|}{t} = \| \bar{u} \|
\]

we conclude that \( \lambda_2(x) \) is not differentiable at \( x \) if \( \bar{u} \neq 0 \).

**Corollary 1.** If \( \lambda_i(x) \), with \( 1 \leq i \leq r \), is a simple eigenvalue then

\[
D_x \lambda_i(x) = c_i, \quad i = 1, \ldots, r.
\]

**Proof.** This follows from Proposition 3, with \( |I_\ell| = 1 \).

From now on, we will assume that \( x \) depends linearly in parameter \( t \) as follows:

\[
x(t) = x_0 + tu, \quad x_0, u \in V, \ t \in \mathbb{R},
\]

but we may write \( x \) instead of \( x(t) \) for simplicity of notation. Let

\[
x(t) = \sum_{i=1}^{r} \lambda_i(x(t))c_i
\]

be the spectral decomposition of \( x(t) \) and

\[
u = \sum_{i=1}^{r} u_i c_i + \sum_{i<j} u_{ij}
\]

the Peirce decomposition of \( u \) with respect to the Jordan frame \( c_1, \ldots, c_r \) (see (2)).

**Proposition 4.** Let \( \lambda_i(x(t)) \) be a simple eigenvalue of \( x(t) \). Then

\[
D_t \lambda_i(x(t)) = u_i.
\]

**Proof.** Applying the chain rule we have \( D_x \lambda_i(x(t)) = \langle D_x \lambda_i(x(t)), x'(t) \rangle \). Therefore by Corollary 1,

\[
D_t \lambda_i(x(t)) = \langle c_i, u \rangle = u_i.
\]

The proposition is proved.

**Proposition 5.** Under the assumptions of Proposition 2, we have

\[
D_t \left( \sum_{\ell \in I_\ell} \lambda_i(x(t)) \right) = \sum_{\ell \in I_\ell} u_i,
\]

for \( \ell = 1, \ldots, d \).

**Proof.** This follows from Proposition 3.
Lemma 3.1. Let $c(t) \in V$ be an idempotent dependent on a real parameter $t$. If $c(t)$ is differentiable with respect to $t$ then

$$c(t) \circ c'(t) = \frac{1}{2}c'(t).$$

Proof. Since $c(t)^2 = c(t)$, we have $2c(t) \circ c'(t) = c'(t)$. The result follows.

Lemma 3.2. Let $c_i(t)$ and $c_j(t)$ be two orthogonal idempotents. If $c_i(t)$ and $c_j(t)$ are differentiable with respect to $t$ then

$$c_i'(t) \circ c_j(t) + c_i(t) \circ c_j'(t) = 0.$$

Proof. The proof follows from $c_i(t) \circ c_j(t) = 0$.

Let $x = \sum_{i=1}^r \lambda_i(x)c_i$ be a spectral decomposition of $x$. If all the eigenvalues are simple, by Theorem 2.2 the Jordan frame $c_1, \ldots, c_r$ is unique with respect to $x$ and every $c_i$ is a polynomial in $x$. Hence, every $c_i$ is differentiable with respect to $x$. The following theorem gives a formula for $D_i c_i(x(t))$.

Theorem 3.3. Let $x(t) = x_0 + tu$ and $x(t) = \sum_{i=1}^r \lambda_i(x(t))c_i$ be its spectral decomposition. Suppose that the eigenvalues of $x(t)$ are simple. Then

$$D_i c_i(x(t)) = 4 \sum_{j \neq i} \frac{(c_i \circ u) \circ c_j}{\lambda_i - \lambda_j} = \sum_{j \neq i} \frac{u_{ij}}{\lambda_i - \lambda_j}.$$  

Proof. In this proof we use $c_j$ or $c_j(t)$ instead of $c_j(x(t))$. Clearly $x \circ c_i = \lambda_i c_i$. Differentiating this expression to $t$ we get

$$x'(t) \circ c_i + x \circ c_i'(t) = \lambda_i'(t)c_i + \lambda_i c_i'(t),$$

which is equivalent to

$$\lambda_i'(t)c_i = u \circ c_i + (x - \lambda_i e) \circ c_i'(t).$$

This implies that

$$0 = c_j \circ (u \circ c_i) + c_j \circ ((x - \lambda_i e) \circ c_i'(t))$$

for $j \neq i$. From Lemma 3.2, we get

$$(c_i'(t) \circ c_j(t)) \circ x + (c_i(t) \circ c_j'(t)) \circ x = 0,$$

for $i \neq j$, which, commutating $x$ with $c_i$, is equivalent to

$$(c_i'(t) \circ x) \circ c_j + (x \circ c_i'(t)) \circ c_i = 0, \quad i \neq j.$$  

On the other hand, we have $(x \circ c_j) \circ c_i = (x \circ c_j) \circ c_j$. Differentiating this expression with respect to $t$ we get

$$(u \circ c_j) \circ c_i + (x \circ c_j') \circ c_i + (x \circ c_j) \circ c_j' = (u \circ c_i) \circ c_j + (x \circ c_i') \circ c_j + (x \circ c_i) \circ c_j',$$

where we used $c_j'$ instead of $c_j'(t)$. Rearranging the terms in the last expression we get

$$(x \circ c_j'(t)) \circ c_i + (x \circ c_j) \circ c_i'(t) = (x \circ c_j'(t)) \circ c_j + (x \circ c_i) \circ c_j'.$$

From expressions (5) and (6) we obtain

$$-(x \circ c_i'(t)) \circ c_j + (x \circ c_j) \circ c_i'(t) = (x \circ c_i'(t)) \circ c_j + (x \circ c_i) \circ c_j'.$$
From here it follows
\[
2(x \circ c'_i(t)) \circ c_j = (x \circ c_j) \circ c'_i(t) - (x \circ c_i) \circ c'_j(t)
\]
\[
= \lambda_j c_j \circ c'_i(t) - \lambda_i c_i \circ c'_j(t)
\]
\[
= \lambda_j c_j \circ c'_i(t) + \lambda_i c'_i(t) \circ c_j \quad \text{(Lemma 3.2)}
\]
\[
= (\lambda_j + \lambda_i) c'_i(t) \circ c_j
\]

Using equation (4), we get
\[
0 = c_j \circ (u \circ c_i) + \frac{1}{2} (\lambda_j + \lambda_i) c'_i(t) \circ c_j - (c_j \circ \lambda_i e) \circ c'_i(t) \quad \text{for } j \neq i,
\]
hence
\[
(\lambda_i - \lambda_j) c'_i(t) \circ c_j = 2c_j \circ (u \circ c_i) \quad \text{for } j \neq i
\]
which is equivalent to
\[
c'_i(t) \circ c_j = 2\frac{c_j \circ (u \circ c_i)}{\lambda_i - \lambda_j} \quad \text{for } j \neq i.
\]

Taking the sum over \(j \neq i\), it follows that
\[
\sum_{j \neq i} c_j \circ c'_i(t) = 2 \sum_{j \neq i} \frac{c_j \circ (u \circ c_i)}{\lambda_i - \lambda_j}.
\]

Note that \(\sum_{j \neq i} c_j = e - c_i\). Therefore, by Lemma 3.1,
\[
\sum_{j \neq i} c_j \circ c'_i(t) = (e - c_i) \circ c'_i(t) = c'_i(t) - c_i \circ c'_i(t) = \frac{1}{2} c'_i(t).
\]

We can now conclude that
\[
c'_i(t) = 4 \sum_{j \neq i} \frac{c_j \circ (u \circ c_i)}{\lambda_i - \lambda_j}.
\]

Since \(P_{ij} = 4L(c_i)L(c_j)\) (see Theorem 2.4) and \(P_{ij}u = u_{ij}\) the last equality of the theorem follows.

\[\square\]

Theorem 3.3 provides an easy way to compute the derivative of \(c_i(x(t))\) with respect to \(t\). See the example below.

**Example 3.2.** Let \((\mathbb{R}^{n+1}, o)\) be the Euclidean Jordan algebra with the Jordan product defined by
\[
x \circ y = (x^T y; x_0 y_i + y_0 \bar{x}),
\]
and denoting \((x_0; x_1; \ldots; x_n) \in \mathbb{R}^{n+1}\) as \((x_0; \bar{x})\) with \(\bar{x} = (x_1; \ldots; x_n)\). Recall from Example 2.1 that, the spectral decomposition of \(x \in \mathbb{R}^{n+1}\) is
\[
x = \lambda_1(x)c_1(x) + \lambda_2(x)c_2(x),
\]
where the eigenvalues are
\[
\lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \lambda_2(x) \end{bmatrix} = \begin{bmatrix} x_0 - \|\bar{x}\| \\ x_0 + \|\bar{x}\| \end{bmatrix}
\]
and the Jordan frame is
\[
c_1(x) = \frac{1}{2} \begin{bmatrix} \frac{1}{\|\bar{x}\|} \\ -\frac{\bar{x}}{\|\bar{x}\|^2} \end{bmatrix}, \quad c_2(x) = \frac{1}{2} \begin{bmatrix} \frac{1}{\|\bar{x}\|} \\ \frac{\bar{x}}{\|\bar{x}\|^2} \end{bmatrix}.
\]
Setting \( x = x_0 + tu \), we differentiate \( c_1(x(t)) \) with respect to \( t \). This gives

\[
c'_1(t) = \frac{1}{2} \left( 0; D_t \left( -\frac{\tilde{x}}{||\tilde{x}||} \right) \right) = \frac{1}{2} \left( 0; -\frac{\tilde{u}}{||\tilde{x}||} + \frac{\tilde{x}^T \bar{u}}{||\tilde{x}||^2} \right),
\]

(7)

On the other hand, if we apply directly Theorem 3.3, we get

\[
c'_1(t) = 4 c_2 \frac{(u \circ c_1)}{\lambda_1 - \lambda_2} = 4 c_2 \frac{1}{2} (u_0 - \bar{u}^T \tilde{x} \tilde{x} - u_0 \bar{x}) = 4 \frac{1}{2} (u_0 - \bar{u}^T \tilde{x} \tilde{x} - u_0 \bar{x} + (u_0 - \bar{u}^T \tilde{x} \tilde{x} + u_0 \bar{x}) = \frac{1}{2} (0; \bar{u} - \frac{\tilde{u}}{||\tilde{x}||} + \frac{\tilde{u}^T \bar{x}}{||\tilde{x}||}) = \frac{1}{2} (0; \bar{u} - \frac{\tilde{u}}{||\tilde{x}||} + \frac{\tilde{u}^T \bar{x}}{||\tilde{x}||^2} \tilde{x}),
\]

in agreement with (7).

The following corollary gives an explicit derivative of simple eigenvalues. If the eigenvalues of \( x \in V \) are simple then they are twice differentiable at \( x \). This follows from Corollary 1 and because in this case the Jordan frame is differentiable at \( x \).

**Corollary 2.** Let \( x(t) = \sum_{i=1}^r \lambda_i(x(t))c_i(x(t)) \). If the eigenvalues are simple then

\[
D^2_t \lambda_i(x(t)) = 4 \sum_{j \neq i} \frac{\text{tr} \left( (u \circ (c_j \circ u)) \circ c_i \right)}{\lambda_i - \lambda_j} = \sum_{j \neq i} \frac{\text{tr} (u^2_j)}{\lambda_i - \lambda_j}.
\]

**Proof.** From Proposition 4 we get

\[
D^2_t \lambda_i(x(t)) = D_t (u_i) = \langle c'_i(t), u \rangle.
\]

(8)

Now, the result follows from Theorem 3.3.

We prove similar properties for eigenvalues which are not simple. However, its proof requires special attention because, we cannot guarantee differentiability for the eigenvalues at \( x \). Aggregating all the idempotents in the same block, for \( \lambda(x) \in \mathbb{R}^k \), we have, by Theorem 2.2, that \( e_\ell = \sum_{i \in I_\ell} c_i \) with \( \ell = 1, \ldots, d \) is a unique complete system of idempotents and \( e_\ell \) is a polynomial in \( x \). Consequently \( e_\ell \), with \( \ell = 1, \ldots, d \) are differentiable with respect to \( x \). The Propositions 2 and 3 provide us the tools for the proof of the next result.

**Theorem 3.4.** Let \( x(t) = \sum_{i=1}^r \lambda_i(x(t))c_i(x(t)) \). If the eigenvalues are not simple and \( \lambda(x) \in \mathbb{R}^k \), then

\[
D_t \left( \sum_{i \in I_\ell} c_i(x(t)) \right) = 4 \sum_{i \in I_\ell} \sum_{j \notin I_\ell} \frac{(c_j \circ u) \circ c_i}{\lambda_i - \lambda_j} = \sum_{i \in I_\ell} \sum_{j \notin I_\ell} \frac{u_{ij}}{\lambda_i - \lambda_j},
\]

for \( \ell = 1, \ldots, d \).
Proof. Let $e_{\ell} = \sum_{i \in I_{\ell}} c_i$ and $e_m = \sum_{i \in I_m} c_i$, with $m \neq \ell$. Clearly

$$ (x \circ e_{\ell}) \circ e_m = 0. \tag{9} $$

Differentiating both sides of (9) to $t$, we get

$$ (x'(t) \circ e_{\ell}) \circ e_m + (x \circ e'_{\ell}(t)) \circ e_m + (x \circ e_{\ell}) \circ e'_m(t) = 0, $$

which is equivalent to

$$ (u \circ e_{\ell}) \circ e_m + (x \circ e'_{\ell}(t)) \circ e_m + \sum_{i \in I_{\ell}} \lambda_i c_i \circ e'_m(t) = 0. \tag{10} $$

From Lemma 3.2 we get that

$$ (e'_\ell(t) \circ e_m(t)) \circ x + (e_{\ell}(t) \circ e'_m(t)) \circ x = 0, $$

for $m \neq \ell$, which commutating $x$ with $e_{\ell}$ and $e_m$, is equivalent to

$$ (e'_\ell(t) \circ x) \circ e_m + (x \circ e'_m(t)) \circ e_{\ell} = 0, \quad m \neq \ell. \tag{11} $$

On the other hand, we have $(x \circ e_m) \circ e_{\ell} = (x \circ e_{\ell}) \circ e_m$. Differentiating this expression to $t$ we get

$$ (u \circ e_m) \circ e_{\ell} + (x \circ e'_m) \circ e_{\ell} + (x \circ e_m) \circ e'_{\ell} = (u \circ e_{\ell}) \circ e_m + (x \circ e'_{\ell}) \circ e_m + (x \circ e_{\ell}) \circ e'_m, $$

where we used $e'_{\ell}$ instead of $e'_\ell(t)$. Rearrange the last expression we get

$$ (x \circ e'_m(t)) \circ e_{\ell} + (x \circ e_m) \circ e'_{\ell}(t) = (x \circ e'_\ell(t)) \circ e_m + (x \circ e_{\ell}) \circ e'_m(t). \tag{12} $$

Using the identity (11) in the identity (12) we obtain

$$ -(x \circ e'_\ell(t)) \circ e_m + (x \circ e_m) \circ e'_{\ell}(t) = (x \circ e'_\ell(t)) \circ e_m + (x \circ e_{\ell}) \circ e'_m(t). $$

From here, it follows that

$$ 2(x \circ e'_\ell(t)) \circ e_m = (x \circ e_m) \circ e'_\ell(t) - (x \circ e_{\ell}) \circ e'_m(t) = \alpha_m e_m \circ e'_\ell(t) - \alpha_{\ell} e_{\ell} \circ e'_m(t) = \alpha_m e_m \circ e'_\ell(t) + \alpha_{\ell} e_m \circ e'_{\ell}(t) \quad \text{(Lemma 3.2)} = (\alpha_m + \alpha_{\ell}) e'_\ell(t) \circ e_m, $$

where we denoted $\alpha_m = \lambda_j$ for $j \in I_m$, because $\lambda_i = \lambda_j$ for all $i, j \in I_m$. Using the last expression in equation (10), we get

$$ 0 = (u \circ e_{\ell}) \circ e_m + \frac{1}{2}(\alpha_m + \alpha_{\ell}) e'_\ell(t) \circ e_m - \alpha_{\ell} e_m \circ e'_\ell(t) \quad \text{for } \ell \neq m. $$

Hence

$$ (\alpha_{\ell} - \alpha_m) e'_\ell(t) \circ e_m = 2e_m \circ (u \circ e_{\ell}) \quad \text{for } m \neq \ell $$

which is equivalent to

$$ e'_\ell(t) \circ e_m = \frac{2e_m \circ (u \circ e_{\ell})}{\alpha_{\ell} - \alpha_m} \quad \text{for } m \neq \ell. $$

Taking the sum over $m \neq \ell$, it follows that

$$ \sum_{m \neq \ell} e_m \circ e'_\ell(t) = 2 \sum_{m \neq \ell} \frac{e_m \circ (u \circ e_{\ell})}{\alpha_m - \alpha_{\ell}}. $$

Regard that $\sum_{m \neq \ell} e_m = e - e_{\ell}$. Therefore, by Lemma 3.1,

$$ \sum_{m \neq \ell} e_m \circ e'_\ell(t) = (e - e_{\ell}) \circ e'_\ell(t) = e'_\ell(t) - e_{\ell} \circ e'_\ell(t) = \frac{1}{2} e'_\ell(t). $$
We can now conclude that
\[ e'_\ell(t) = 4 \sum_{m \neq \ell} \frac{e_m \circ (u \circ e_\ell)}{\alpha_\ell - \alpha_m}. \]
Replacing \( e_m \) by \( \sum_{j \in I_m} c_j \) and \( e_\ell \) by \( \sum_{i \in I_\ell} c_i \) we obtain
\[ D_t \sum_{i \in I_\ell} c_i(t) = 4 \sum_{m \neq \ell} \sum_{i \in I_\ell} \frac{c_j \circ (u \circ c_i)}{\alpha_\ell - \alpha_m} \alpha_\ell - \alpha_m \]
\[ = 4 \sum_{m \neq \ell} \sum_{i \in I_\ell} \sum_{j \in I_m} c_j \circ (u \circ c_i) \lambda_i - \lambda_j \]
\[ = 4 \sum_{i \in I_\ell} \sum_{j \notin I_\ell} c_j \circ (u \circ c_i) \lambda_i - \lambda_j. \]
Since \( P_{ij} = 4L(c_i)L(c_j) \) (see Theorem 2.4) and \( P_{ij}u = u_{ij} \) the last equality of the theorem follows.

**Corollary 3.** Under the assumptions of the Theorem 3.4 we have
\[ D_t^2 \left( \sum_{i \in I_\ell} \lambda_i(x(t)) \right) = 4 \sum_{i \in I_\ell} \sum_{j \notin I_\ell} \frac{\text{tr} ((c_j \circ (u \circ c_i)) \circ u)}{\lambda_i - \lambda_j} = \sum_{i \in I_\ell} \sum_{j \notin I_\ell} \frac{\text{tr} (u^2_{ij})}{\lambda_i - \lambda_j}, \]
for \( \ell = 1, \ldots, d \).

**Proof.** From Theorem 5 we get
\[ D_t^2 \sum_{i \in I_\ell} \lambda_i(x(t)) = \langle D_t \left( \sum_{i \in I_\ell} c_i(t) \right), u \rangle, \]
for \( l = 1, \ldots, d \). Since
\[ \text{tr} ((c_j \circ (u \circ c_i)) \circ u) = \text{tr} (P_{ij}u \circ u) = \text{tr} (u^2_{ij}), \]
the result follows by Theorem 3.4.

In the properties presented in this section, i.e., for the derivatives of eigenvalues and Jordan frames, we had two versions, one supposing that the eigenvalues are simple and another supposing that they are not. These both versions appear naturally when we are deducing the formulas.

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