INVERSE SCATTERING PROBLEM FOR A SPECIAL CLASS OF CANONICAL SYSTEMS AND NON-LINEAR FOURIER INTEGRAL. PART I: ASYMPTOTICS OF EIGENFUNCTIONS

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Abstract. An original approach to the inverse scattering for Jacobi matrices was recently suggested in [17]. The authors considered quite sophisticated spectral sets (including Cantor sets of positive Lebesgue measure), however they did not take into account the mass point spectrum. This paper follows similar lines for the continuous setting with an absolutely continuous spectrum on the half-axis and a pure point spectrum on the negative half-axis satisfying the Blaschke condition. This leads us to the solution of the inverse scattering problem for a class of canonical systems that generalizes the case of Sturm-Liouville (Schrödinger) operator.

1. Faddeev-Marchenko space in Szegő/Blaschke setting

One of the important aspects of the spectral theory of differential operators is the scattering theory [13, 14] and, in particular, the inverse scattering [11]. An original approach to the inverse scattering was recently suggested in [17]. The paper focused on classical Jacobi matrices and connections between the scattering and properties of a special Hilbert transform.

In this paper, we carry out the plan of [17] in the continuous situation. Compared with [17], a completely new feature is that the scattering data incorporate the pure point spectrum with infinitely many mass points. Of course, this is a natural and important step in the developing the theory. The discussion leads us to the solution of the inverse scattering problem for a class of canonical systems that include the Sturm-Liouville (Schrödinger) equations. At present, though, we are unable to characterize the scattering data corresponding to the last important special case.

This part of the work is mainly devoted to the asymptotic behavior of certain reproducing kernels (the generalized eigenfunctions). It is organized as follows. Section 1 contains definitions, some general facts and formulations of results on asymptotics. The asymptotic properties of reproducing kernels from certain model spaces are studied in Section 2. Special operator nodes arising from our construction are discussed in Sections 3 and 4. One of the nodes generates a canonical system we are interested in. Its properties and connections to the de Branges spaces of entire functions are also in Section 4. The Sturm-Liouville (Schrödinger) equations are considered in Section 5. An example is given in the first appendix (Section 6). The second appendix (Section 7) relates the whole construction to the matrix $A_2$ Hunt-Muckenhoupt-Wheeden condition.

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We define the $L^2$-norm on the real axis as
\[ ||f||^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |f(\lambda)|^2 d\lambda, \] (1.1)
so that the reproducing kernel of the $H^2$ subspace is of the form
\[ k(\lambda, \lambda_0) = \frac{i}{\lambda - \lambda_0}. \]
The section “Inverse scattering problem on the real axis” in [11, Chap. 3, Sect. 5] begins with a Sturm-Liouville operator
\[ -y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}, \] (1.2)
with the potential $q$ satisfying the a priori condition
\[ \int_{\mathbb{R}} (1 + |x|)|q(x)| dx < \infty. \] (1.3)
To such an operator one associates so called scattering data
\[ \{s_+, \nu_+\}, \] (1.4)
where $s_+$ is a contractive function on the real axis, $|s_+(\lambda)| \leq 1$, $\lambda \in \mathbb{R}$, possessing certain properties and $\nu_+$ is a discrete measure, in fact, supported on a finite number of points $\Lambda = \{\lambda_k\}$ of the imaginary axis, $\frac{1}{\lambda_k} > 0$.
We proceed in the opposite direction starting from the scattering data $\{s_+, \nu_+\}$ and going to the potential $q$. The key point of the construction is that we assume that the scattering data (1.4) satisfy only very natural (and minimal) conditions from the point of view of the function theory. Namely, we suppose that:
- a symmetric on the real axis function $s_+$, $s_+(\lambda) = s_+(-\lambda)$, satisfies the Szegő condition
\[ \int_{\mathbb{R}} \frac{\log(1 - |s_+(\lambda)|^2)}{1 + \lambda^2} d\lambda > -\infty, \] (1.5)
- the support $\Lambda$ of a discrete measure $\nu_+ = \sum_k \nu_+(\lambda_k) \delta_{\lambda_k}$ satisfies the Blaschke condition
\[ \sum_{(\lambda_k/i) \leq 1} \frac{\lambda_k}{i} < \infty, \quad \sum_{(\lambda_k/i) > 1} i \frac{\lambda_k}{\lambda_k} < \infty. \] (1.6)
Let us point out that we did not even assume that the measure $\nu_+$ is finite.

Our plan is to show that already in this case one can associate a certain differential operator of the second order to the given spectral data and then one can prove several specification theorems.

**Definition 1.1.** An element $f$ of the space $L^2_{\{s_+, \nu_+\}}$ is a function on $\mathbb{R} \cup \Lambda$ such that
\[ ||f||^2_{\{s_+, \nu_+\}} = \sum_{\lambda_k \in \Lambda} |f(\lambda_k)|^2 \nu_+(\lambda_k) \]
\[ + \frac{1}{4\pi} \int_{\mathbb{R}} \frac{f(\lambda) - f(-\lambda)}{s_+(\lambda)} \left[ \frac{1}{s_+(\lambda)} \frac{s_+(\lambda)}{1} \right] \left[ -f(-\lambda) \right] d\lambda \] (1.7)
is finite.
Using (1.5) and (1.6) we define the outer in the upper half-plane function $s_e$ as
\[ |s_e(\lambda)|^2 + |s_+(\lambda)|^2 = 1 \quad \text{a.e. on } \mathbb{R}, \] (1.8)
and the Blaschke product

\[ B(\lambda) = \prod_k b_{ik}(l), \quad (1.9) \]

where \( b_{ik}(l) = \frac{\lambda - \lambda_k}{\lambda - \lambda_k} \) if \( l_k/i \leq 1 \) and \( b_{ik}(l) = -\frac{\lambda - \lambda_k}{\lambda - \lambda_k} \) if \( l_k/i > 1 \). We also put

\[ S(\lambda) = \begin{bmatrix} s_- & s \\ s & s_+ \end{bmatrix}(\lambda), \quad \lambda \in \mathbb{R}, \quad (1.10) \]

where

\[ s := \frac{s_+}{B} \quad \text{and} \quad s_- := -\frac{s}{s_+}. \quad (1.11) \]

The matrix function \( S \) possesses two fundamental properties: \( S^*(-\bar{\lambda}) = S(\lambda) \), and it is unitary-valued. The third property is analyticity of the entry \( s \), which has analytic continuation to the upper half-plane as a function of bounded characteristic with a specific nature, that is, it is a ratio of an outer function and a Blaschke product.

The measure \( \nu_- \) is defined through \( s_- \) and \( \nu_+ \) by

\[ \frac{1}{\nu_+(\lambda_k)} \frac{1}{\nu_-(\lambda_k)} = \left| \left( \frac{1}{s} \right) (\lambda_k) \right|^2. \quad (1.12) \]

A reason for these and the following definitions will be clarified in a moment.

Set

\[ \begin{bmatrix} sf^+ \\ sf^- \end{bmatrix}(\lambda) = \begin{bmatrix} s & 0 \\ s_+ & 1 \end{bmatrix}(\lambda) \begin{bmatrix} f^+(\lambda) \\ -f^+(-\bar{\lambda}) \end{bmatrix} = \begin{bmatrix} 1 & s_- \\ 0 & s \end{bmatrix}(\lambda) \begin{bmatrix} -f^-(-\bar{\lambda}) \\ f^-(-\lambda) \end{bmatrix} \quad (1.13) \]

for \( \lambda \in \mathbb{R} \) and

\[ f^-(\lambda_k) = -i \left( \frac{1}{s} \right)'(\lambda_k) \nu_+(\lambda_k) f^+(\lambda_k) \quad (1.14) \]

for \( \lambda_k \in \Lambda \). It is evident that in this way we define a unitary map from \( L^2_{(s_+,\nu_+)} \) to \( L^2_{(s_-,\nu_-)} \). In fact, due to (1.13)

\[ \frac{1}{4\pi} \int_{\mathbb{R}} \left[ f(\lambda) \quad -f(-\lambda) \right] \begin{bmatrix} 1 \\ s_+(-\lambda) \end{bmatrix} \begin{bmatrix} f(\lambda) \\ -f(-\lambda) \end{bmatrix} d\lambda = \frac{||sf^+||^2 + ||sf^-||^2}{2} \quad (1.15) \]

where we have the standard \( L^2 \)-norm on \( \mathbb{R} \) in the RHS of the equality. The key point is that relations (1.13), (1.13) not only define a duality between these two spaces but, what is more important, a duality between corresponding Hardy subspaces.

Actually we give two versions of definitions of Hardy subspaces (in general, they are not equivalent, see an example in Section 4). By the first one, \( H^2_{(\nu_-,\nu_+)} \) is basically the closure of \( H^\infty \) with respect to the given norm (17). More precisely, let \( B_{N_1,N_2} = \prod_{k=N_1+1}^{N_2} b_{ik} \) and \( \mathcal{B} = \{ B_{N,\infty} \} \). Saying it differently, \( B' \in \mathcal{B} \) if and only if \( B' \) is a divisor of \( B \) such that \( B/B' \) is a finite Blaschke product. Then

\[ f = B_{N,\infty} g, \quad g \in H^\infty, \quad (1.16) \]

belongs to \( L^2_{(s_-,\nu_-)} \). By \( H^2_{(\nu_-,\nu_+)} \) we denote the closure in \( L^2_{(s_-,\nu_-)} \) of functions of the form (1.16). Let us point out that every element \( f \) of \( H^2_{(\nu_-,\nu_+)} \) is such that \( s_\cdot f \) belongs to the standard \( H^2 \), see (1.16). Therefore, in fact, \( f(\lambda) \) has an analytic
continuation from the real axis to the upper half-plane. Moreover, the value \( f(l) \) obtained by this continuation, and \( f(\lambda_k) \) which is defined for all \( \lambda_k \in \Lambda \), since \( f \) is a function from \( L^2_{\{s, \nu\}} \), still perfectly coincide.

The second space also consists of functions from \( L^2_{\{s, \nu\}} \) having an analytic continuation to the upper half-plane.

**Definition 1.2.** A function \( f \in L^2_{\{s, \nu\}} \) belongs to \( \hat{H}^2_{\{s, \nu\}} \) if \( g(\lambda) := (s_e f)(\lambda) \), \( \lambda \in \mathbb{R} \), belongs to the standard \( H^2 \) and

\[
\langle f(\lambda) = \left( \frac{g}{s_e} \right)(\lambda), \lambda \in \Lambda \rangle
\]

where in the RHS \( g \) and \( s_e \) are defined by their analytic continuation to the upper half-plane.

It turns out that spaces \( H^2_{\{s, \nu\}} \) and \( \hat{H}^2_{\{s, \nu\}} \) are dual in a certain sense.

**Theorem 1.3.** Let \( f^+ \in L^2_{\{s, \nu\}} \cap H^2_{\{s, \nu\}} \) and let \( f^- \in L^2_{\{s, \nu\}} \) be defined by (1.11), (1.14). Then \( f^- \in \hat{H}^2_{\{s, \nu\}} \). In short, we write

\[
(\hat{H}^2_{\{s, \nu\}})^+ = L^2_{\{s, \nu\}} \cap H^2_{\{s, \nu\}}. \tag{1.17}
\]

**Proof.** We notice that \( f^+ \in L^2_{\{s, \nu\}} \) implies

\[
(s f^-)(\lambda) = s_+ (\lambda) f^+(\lambda) - f^+(-\bar{\lambda}) \in L^2, \lambda \in \mathbb{R}.
\]

Since

\[
\langle f^+, Bh \rangle_{\{s, \nu\}} = \langle s_+(\lambda) f^+(\lambda) - f^+(-\bar{\lambda}), -B(-\bar{\lambda}) h(-\bar{\lambda}) \rangle, \quad h \in H^2,
\]

it follows from \( f^+ \in L^2_{\{s, \nu\}} \cap H^2_{\{s, \nu\}} \) that

\[
(s e f^-)(\lambda) = g(\lambda) := B(\lambda) s_+(\lambda) f^+(\lambda) - f^+(-\bar{\lambda}) \in H^2.
\]

Now we calculate the scalar product

\[
\langle f^+, \frac{iB(\lambda)}{\lambda - \lambda_k} \rangle_{\{s, \nu\}} = f^+(\lambda_k) i B(\lambda_k) \nu_+(\lambda_k) + \langle s e f^-, \frac{i}{\lambda - \lambda_k} \rangle
\]

\[
= f^+(\lambda_k) i B(\lambda_k) \nu_+(\lambda_k) + g(\lambda_k) = 0.
\]

Therefore, by (1.14) we get

\[
f^-(\lambda_k) = \left( \frac{g}{s_e} \right)(\lambda_k), \lambda_k \in \Lambda.
\]

\( \square \)

Both \( H^2_{\{s, \nu\}} \) and \( \hat{H}^2_{\{s, \nu\}} \) are spaces of analytic in the upper half-plane functions, so they have reproducing kernels. For \( \mu \in \mathbb{C}^+ \), we denote them by

\[
k_{\{s, \nu\}}(l, \mu) = k_{\{s, \nu\}}(\cdot, \mu), \quad \hat{k}_{\{s, \nu\}}(\lambda, \mu) = \hat{k}_{\{s, \nu\}}(\cdot, \mu).
\]

Recall also that

\[
k(l, \mu) = k(\cdot, \mu) = \frac{i}{l - \bar{\mu}}
\]

is the reproducing kernel of the standard Hardy space \( H^2 \). The first step is to prove asymptotics for the families \( \{ e^{itx} k_{\{s, \nu\}}(\cdot, \cdot, l_0) \}_{x \in \mathbb{R}} \) and \( \{ e^{itx} \hat{k}_{\{s, \nu\}}(\cdot, \cdot, l_0) \}_{x \in \mathbb{R}} \) with \( l_0 \in \mathbb{C}^+ \).
Theorem 1.4. The following relations hold true:

i) on \( \mathbb{R} \),
\[
\begin{align*}
  s \left( e^{ilx} k_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) \right) &= se^{ilx} k(\cdot, l_0) + o(1), \quad (1.18) \\
  s \left( e^{ilx} \tilde{k}_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) \right) &= se^{ilx} k(\cdot, l_0) + o(1)
\end{align*}
\]
as \( x \to +\infty \). Moreover,
\[
\begin{align*}
  s(-\bar{l}_0) s \left( e^{ilx} k_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) \right) &= e^{ilx} k(\cdot, l_0) + s(-e^{-ilx} k(\cdot, -l_0) + o(1), \quad (1.19) \\
  s(-\bar{l}_0) s \left( e^{ilx} \tilde{k}_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) \right) &= e^{ilx} k(\cdot, l_0) + s(-e^{-ilx} k(\cdot, -l_0) + o(1)
\end{align*}
\]
as \( x \to -\infty \) (of course, everything is in \( L^2 \)-sense).

ii) on \( \Lambda \),
\[
e^{ilx} k_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) = o(1), \quad e^{ilx} \tilde{k}_{(s,e^{2ilx},\nu_x,e^{2ilx})}(\cdot, l_0) = o(1) \quad (1.20)
in \( L^2_{\nu_x} \)-sense as \( x \to +\infty \). Furthermore, for a \( l_k \in \Lambda \)
\[
\begin{align*}
  \lim_{x \to +\infty} e^{-2ilx} k_{(s,e^{2ilx},\nu_x,e^{2ilx})}(l_k, l_k) &= \frac{1}{\nu_+(l_k)}, \quad (1.21) \\
  \lim_{x \to -\infty} e^{-2ilx} \tilde{k}_{(s,e^{2ilx},\nu_x,e^{2ilx})}(l_k, l_k) &= \frac{1}{\nu_-(l_k)}
\end{align*}
\]

It goes without saying that relations (1.18)–(1.20) correspond to scattering “from +\( \infty \) to \(- \infty \)”; compare these formulas to (0.8), (0.25) from [17]. Scattering in the inverse direction (“from \(- \infty \) to ++\( \infty \)”) is described similarly. We give the formulas for the family \( \{e^{ilx} k_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(\cdot, l_0)\} \) only; asymptotics for \( \{e^{ilx} \tilde{k}_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(\cdot, l_0)\} \) are the same.

Corollary 1.5. We have
\[
\begin{align*}
  s \left( e^{-ilx} k_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(\cdot, l_0) \right) &= se^{-ilx} k(\cdot, l_0) + o(1), \quad x \to -\infty, \\
  s(-\bar{l}_0) s \left( e^{-ilx} k_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(\cdot, l_0) \right) &= e^{-ilx} k(\cdot, l_0) + s(e^{ilx} k(\cdot, -l_0) + o(1), \quad x \to +\infty
\end{align*}
\]
in \( L^2 \)-sense on the real line. As for \( \Lambda \),
\[
e^{-ilx} k_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(\cdot, l_0) = o(1)
\]
in \( L^2_{\nu_x} \)-sense as \( x \to -\infty \). As before, for a \( l_k \in \Lambda \),
\[
\lim_{x \to +\infty} e^{2ilx} k_{(s,e^{-2ilx},\nu_x,e^{-2ilx})}(l_k, l_k) = \frac{1}{\nu_-(l_k)}.
\]

We set
\[
\tilde{k}(\cdot, l_0) = \begin{cases} k(\cdot, l_0), & l \in \mathbb{R}, \\
0, & l \in \Lambda.
\end{cases}
\]
Theorem 1.4 follows immediately from the following result.
The following relations hold true:

\[
\lim_{x \to +\infty} \left| e^{itx} k_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) - e^{it\hat{k}}(\cdot, l_0) \right| = 0, \quad (1.22)
\]

\[
\lim_{x \to +\infty} \left| e^{itx} \hat{k}_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) - e^{it\hat{k}}(\cdot, l_0) \right| = 0. \quad (1.23)
\]

The proof of this theorem is the main purpose of Section 2.

Remark 1.7. Relations (1.18)–(1.20) follow at once from Theorem 1.6.

Indeed, let us have a look at (1.22). Recalling that \( \hat{k}(\cdot, l_0) = 0 \) on \( \Lambda \), we see

\[
\left| e^{itx} k_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) - e^{it\hat{k}}(\cdot, l_0) \right| = \left| e^{itx} k_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) \right|^2_{s_+} + \left| e^{it\hat{k}}(\cdot, l_0) \right|^2 \to 0,
\]
as \( x \to +\infty \), so the first relation in (1.20) is proved. Then we notice that

\[
\left[ \begin{array}{c}
1 \\
s_+ \\
1
\end{array} \right] = \left[ \begin{array}{ccc}
|s|^2 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{array} \right] \left[ \begin{array}{c}
s_+ \\
1
\end{array} \right].
\]

This implies that the first summand on the right-hand side of the above equality is

\[
\left| \cdots \right|^2_{s_+} = \left| s \left( e^{itx} k_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) - e^{it\hat{k}}(\cdot, l_0) \right) \right|^2_{s_+} + \left| s \left( e^{it\hat{k}}(\cdot, l_0) \right) \right|^2_{s_-} - \left( e^{-itx} k_{\{s, -l_0\}} + s e^{it\hat{k}}(\cdot, l_0) \right)^2_{s_-}.
\]

The presence of the first term on the right-hand side shows that we are done with (1.18). To deal with \( e^{itx} k_{\{s_+, e^{2itx}, \nu e^{2itx}\}}(\cdot, l_0) \) in (1.20), we use Lemma 2.2 and its corollary saying

\[
\lim_{x \to +\infty} k_{\{s, e^{-2itx}(b_0 b_{-l_0})^{-1}, \nu e^{-2itx}(b_0 b_{-l_0})^{-1}\}}(-l_0, -l_0) = \frac{1}{2\text{Im} l_0 |s(l_0)|^2}
\]

(see also Lemma 2.8). Hence, we come to

\[
s(l_0) s \left( e^{-itx} k_{\{s, e^{-2itx}(b_0 b_{-l_0})^{-1}, \nu e^{-2itx}(b_0 b_{-l_0})^{-1}\}} \right) = e^{-itx} k_{\{s, -l_0\}} + s e^{it\hat{k}}(\cdot, l_0) + o(1)
\]
as \( x \to +\infty \). This is the second relation in (1.19) up to changes \( x \mapsto -x, l_0 \mapsto -l_0, s_- \mapsto s_+ \), \( s_+ \mapsto s_- \) and \( s_+ (b_0 b_{-l_0}) \mapsto s_- \).

Equalities (1.21) are proved in Corollary 2.6.

2. Asymptotics of reproducing kernels

2.1. Definitions and notation. In this subsection, we prove several propositions concerning special properties of the reproducing kernels introduced in Section 1.

For \( l_0 \in \mathbb{C}_+ \), let

\[
K_{\{s_+, \nu_+\}}(\cdot, l_0) = \frac{k_{\{s_+, \nu_+\}}(\cdot, l_0)}{\sqrt{k_{\{s_+, \nu_+\}}(l_0, l_0)}}, \quad \hat{K}_{\{s_+, \nu_+\}}(\cdot, l_0) = \frac{\hat{k}_{\{s_+, \nu_+\}}(\cdot, l_0)}{\sqrt{\hat{k}_{\{s_+, \nu_+\}}(l_0, l_0)}},
\]

be their normalized versions. It is also convenient to put

\[
K(l, l_0) = \frac{k(l, l_0)}{\|k(\cdot, l_0)\|} = \frac{i \left( 2\text{Im} l_0 \right)^{1/2}}{|l - l_0|}. 
\]
For a fixed $x \in \mathbb{R}$ we define $H^2_{s_+, \nu_+}(x)$ as the closure of the functions
\[ f(\lambda) = B_N, \infty(\lambda) g(\lambda) e^{i\lambda x}, \quad g \in H^\infty, \quad B_N, \infty \in \mathcal{B}. \] (2.1)

In particular, $H^2_{s_+, \nu_+} = H^2_{s_+, \nu_+}(0)$. In the similar way we define the set of spaces $\hat{H}^2_{s_+, \nu_+}(x)$, so that $\hat{H}^2_{s_+, \nu_+}$ is related to $x = 0$.

It is easy to see that
\[ H^2_{s_+}(x) = e^{i\lambda x} H^2_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}, \quad \hat{H}^2_{s_+}(x) = e^{i\lambda x} \hat{H}^2_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}, \]
and
\[ k_{s_+, \nu_+}(l, l; x) = e^{i\lambda (l - l_0)} k_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}(l, l_0), \]
\[ \hat{k}_{s_+, \nu_+}(l, l; x) = e^{i\lambda (l - l_0)} \hat{k}_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}(l, l_0) \]
are the reproducing kernels of these spaces, respectively. We also have their normalized versions
\[ K_{s_+, \nu_+}(. , l_0; x) = e^{i\lambda (l - \text{Re} l_0)} K_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}(., l_0), \]
\[ \hat{K}_{s_+, \nu_+}(. , l_0; x) = e^{i\lambda (l - \text{Re} l_0)} \hat{K}_{s, e^{2i\lambda x}, \nu_+ e^{2i\lambda x}}(., l_0). \]

This section is mainly devoted to the proof of asymptotic formulas for both types of kernels as $x \to +\infty$.

2.2. Some special properties of the reproducing kernels. The following lemma is trivial but probably the notations are slightly confusing. We believe that the diagram below will help to avoid misunderstanding: $\pm$-mappings $L^2_{s_+, \nu_+} \leftrightarrow L^2_{s_-, \nu_-}$, given by (1.13), (1.14), actually depend on the scattering data $\{s_\pm, \nu_\pm\}$, although we do not indicate this dependence explicitly in most cases.

Lemma 2.1. Let $w(\lambda)$ be an inner meromorphic function in the upper half-plane such that $w(\lambda_k) \neq 0$, $w(\lambda_k) \neq \infty$ for all $\lambda_k \in \Lambda$. Put $w_*(\lambda) := w(-\lambda)$. The following diagram is commutative
\[
\begin{array}{cccc}
L^2_{w, s_+, w, s_+} & \xrightarrow{w} & L^2_{s_+, \nu_+} \\
\downarrow w & & \downarrow w \\
L^2_{w^{-1}, w^{-1}, s, w^{-1}, s, \nu_-} & \xrightarrow{w^{-1}} & L^2_{s_-, \nu_-}
\end{array}
\] (2.2)

Here the horizontal arrows are related to the unitary multiplication operators and the vertical arrows are related to two different $\pm$-duality mappings.

Proof. Note that both $w$ and $w^{-1}$ are well defined on $\mathbb{R} \cup \Lambda$. Evidently, $wf \in L^2_{s_+, \nu_+}$ means that $f \in L^2_{s_+, \nu_+}$. Since $|w(\lambda)| = 1$, $\lambda \in \mathbb{R}$, we have that $\{w^{-1}w^{-1}w^{-1}s_-, w^{-1}w^{-1}w^{-1}w^{-1}s_-, w^{-1}w^{-1}w^{-1}w^{-1}w^{-1}s_-, w^{-1}w^{-1}w^{-1}s_-, \nu_-, \nu_-, \nu_-, \nu_-\}$ are minus–scattering data for $\{w, s_+, w, s_+, \nu_+, \nu_+, \nu_+, \nu_+\}$ if $\{s_-, \nu_-, \nu_-, \nu_-, \nu_-, \nu_-, \nu_-, \nu_-\}$ corresponds to $\{s_+, \nu_+, \nu_+, \nu_+\}$. In other words, the $s$-function remains the same for both sets of scattering data. Then we use definitions (1.13), (1.14). \hfill \square

Let $b_{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0 - \lambda}$. Note that $(b_{\lambda_0})_* = b_{-\lambda_0}$.

Lemma 2.2. We have
\[
(k_{s_+, \nu_+}(\lambda, \lambda_0))^- = \frac{1}{s(-\lambda_0) 2i \text{Im} \lambda_0} \frac{b_{\lambda_0}^{-1}(\lambda) \hat{k}_{b_{\lambda_0}^{-1}b_{-\lambda_0}^{-1}s_{\lambda_0}b_{\lambda_0}^{-1}b_{-\lambda_0}^{-1}s_{\lambda_0}^{-1}, \nu_-, \nu_-}(\lambda, -\lambda_0)}{k_{b_{\lambda_0}^{-1}b_{-\lambda_0}^{-1}s_{\lambda_0}b_{\lambda_0}^{-1}b_{-\lambda_0}^{-1}s_{\lambda_0}^{-1}, \nu_-, \nu_-}(-\lambda_0, -\lambda_0)}, \] (2.3)
and, consequently,
\[
 k_{\{s_+, \nu_+\}}(\lambda_0, \lambda_0) \hat{k}_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}}(-\bar{\lambda}_0, -\bar{\lambda}_0) = \frac{1}{s(-\bar{\lambda}_0)s(\lambda_0)} \frac{1}{(2\text{Im} \lambda_0)^2}.
\]

(2.4)

**Proof.** First we note that the following one-dimensional spaces coincide
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\}^* = \{b^{-1}_0 \hat{k}_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}}(\lambda, -\bar{\lambda}_0)\}.
\]

This follows immediately from Theorem 1.3, but we prefer to give a formal proof. Starting with the orthogonal decomposition
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\} = H^2_{\{s_+, \nu_+\}} \oplus b_{\lambda_0} H^2_{\{b_{\lambda_0} b_{-\lambda_0}, b_{\lambda_0} b_{-\lambda_0} \nu_+\}}
\]
we have
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\}^* = (H^2_{\{s_+, \nu_+\}})^* \ominus (b_{\lambda_0} H^2_{\{b_{\lambda_0} b_{-\lambda_0}, b_{\lambda_0} b_{-\lambda_0} \nu_+\}})^*;
\]
or, due to (2.3),
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\}^* = (H^2_{\{s_+, \nu_+\}})^* \ominus b^{-1}_{-\lambda_0} (H^2_{\{b_{\lambda_0} b_{-\lambda_0}, b_{\lambda_0} b_{-\lambda_0} \nu_+\}})^*.
\]

Now we use Theorem 1.3
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\}^* = (L^2_{\{s_-, \nu_-\}} \oplus \hat{H}^2_{\{s_-, \nu_-\}})
\]
\[
\ominus b^{-1}_{-\lambda_0} (L^2_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}} \ominus \hat{H}^2_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}})
\]
\[
= b^{-1}_{-\lambda_0} (\hat{H}^2_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}} \ominus b_{-\lambda_0} \hat{H}^2_{\{s_-, \nu_-\}}).
\]
Thus
\[
\{k_{\{s_+, \nu_+\}}(\lambda, \lambda_0)\}^* = C b^{-1}_{-\lambda_0} \hat{k}_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}}(\lambda, -\bar{\lambda}_0).
\]

(2.5)

The essential part of the lemma deals with the constant C. We calculate the scalar product
\[
\left\langle k_{\{s_+, \nu_+\}}(\lambda, \lambda_0), \frac{iB(\lambda)}{\lambda - \lambda_0}\right\rangle_{\{s_+, \nu_+\}}.
\]

On the one hand, since \(\frac{iB(\lambda)}{\lambda - \lambda_0}\) belongs to the intersection of \(L^2_{\{s_+, \nu_+\}}\) with \(H^2\), we can use the reproducing property of \(k_{\{s_+, \nu_+\}}\):
\[
\left\langle k_{\{s_+, \nu_+\}}(\lambda, \lambda_0), \frac{iB(\lambda)}{\lambda - \lambda_0}\right\rangle_{\{s_+, \nu_+\}} = \frac{B(\lambda_0)}{2\text{Im} \lambda_0} = \frac{B(-\bar{\lambda}_0)}{2\text{Im} \lambda_0}.
\]

(2.6)

On the other hand we can reduce the given scalar product to the scalar product in the standard \(H^2\). Since \(B(\lambda_0) = 0\), the \(\nu\)-component disappears and we get
\[
\frac{1}{2} \left(\begin{array}{l}1 \bar{s}_+ \\ s_+ \end{array}\right) \left(\begin{array}{l}1 \\ \bar{s}_+ \end{array}\right) \left(k_{\{s_+, \nu_+\}}(\lambda, \lambda_0), -k_{\{s_+, \nu_+\}}(-\bar{\lambda}, -\bar{\lambda}_0)\right), \left(\begin{array}{l}iB(\lambda) \\ iB(-\bar{\lambda}) \end{array}\right)\right)
\]
\[
= \left\langle s(\lambda)k_{\{s_+, \nu_+\}}(\lambda, \lambda_0), \frac{iB(\lambda)}{\lambda + \lambda_0}\right\rangle.
\]

Substituting here (2.3) and using \(s = s_e/B\) we come to
\[
C \left\langle s(\lambda)k_{\{b^{-1}_0, b^{-1}_0 s_-, b^{-1}_0 b^{-1}_0 \nu_-\}}(\lambda, -\bar{\lambda}_0), b_{-\lambda_0}(\lambda) \frac{i}{\lambda + \lambda_0}\right\rangle.
\]
we see $f$ is trivial.

**Proof.**

$f \in H$.

Recall that Lemma 2.3.

We say few more words about spaces $\mathcal{H}$.

$x - \lambda_0$ that the

Thus (2.3) is proved. Comparing the norms of these vectors and taking into account

As a consequence of the above lemma, we have

Indeed, using diagram (2.2), we get for $x \geq 0$

by (2.3). So, the latter function is in $\mathcal{H}_x^2$. We always have

For discrete measures $\nu_\pm$, let $\nu_{\pm,N}$ be their truncations

We say few more words about spaces $H_2$, $\mathcal{H}_x^2$ and $H_2^x$, $\mathcal{H}_x^2$. Recall that $H_2^x \subset \mathcal{H}_x^2$.

**Lemma 2.3.**

i) Let $||s_\pm||_\infty < 1$ (or, what is the same, $\inf \mathbb{R} |s_\pm| > 0$). Then

$H_2^x \subset \mathcal{H}_x^2$.

and, consequently, $K^x_{s_+} (\cdot, l_0) = \hat{K}_x^x (\cdot, l_0)$.

ii) We always have

The equality above takes place if and only if $K^x_{s_+} (\cdot, l_0) = \hat{K}_x^x (\cdot, l_0).

iii) Obviously,

$H_2^x \subset H_2^x$, $\mathcal{H}_x^2 \subset \mathcal{H}_x^2$,

and

$K^x_{s_+} (l_0, l_0) \leq K^x_{s_+,N} (l_0, l_0),$

$\hat{K}_x^x (l_0, l_0) \leq \hat{K}_x^x (l_0, l_0).

As before, the inequalities become equalities if and only if the corresponding reproducing kernels coincide.

**Proof.** To prove i), we only have to show the inverse inclusion. Suppose that

$f \in \mathcal{H}_x^2$. By Definition 1.12, $f \in L_2^{s_+ (\cdot, \nu_+)}$ and $s_e f \in H^2$. Since $s_e, 1/s_e \in H^\infty$, we see $f \in H^2$ and hence $f \in \mathcal{H}_x^2$. The claim about the reproducing kernels is trivial.
The inequality in ii) of course follows from inclusion $H^2_{\{s_+,\nu_+\}} \subset \hat{H}^2_{\{s_+,\nu_+\}}$. Consider a system $\{f_n\}_{n \in \mathbb{Z}^+} f_n = b_{l_0}' \hat{K}_{\{s_+(b_0 \nu_0^{-1} \nu_0^{-1} b_0^{-1}\}(-l_0, l_0)$, This is an orthonormal basis in $\hat{H}^2_{\{s_+,\nu_+\}}$. We have $K_{\{s_+,\nu_+\}}(., l_0) \in H^2_{\{s_+,\nu_+\}}$ and $\|K_{\{s_+,\nu_+\}}(., l_0)\|_{\{s_+,\nu_+\}} = 1$. So

$$K_{\{s_+,\nu_+\}}(., l_0) = \sum_n a_n f_n$$

and $a_0 = K_{\{s_+,\nu_+\}}(l_0, l_0)/\hat{K}_{\{s_+,\nu_+\}}(l_0, l_0)$. Obviously, $|a_0|^2 \leq 1$ and claim ii) is proved.

Let us have a look at iii). The first inclusion follows from the fact that for $f \in H^2(\mathbb{C}^+)$

$$\|B_{N, \infty} f\|_{\{s_+,\nu_+\}} \leq \|B_{N, \infty} f\|_{\{s_+,\nu_+\}}.$$ 

The second one follows from Definition 2.2 of $\hat{H}^2_{\{s_+,\nu_+\}}$. The inequalities for the reproducing kernels are corollaries of these inclusions; to prove them just argue as in ii).

In particular, we have

$$K_{\{s_+e^{-2i\xi}, \nu_+e^{-2i\xi}\}}(l_0, l_0) K_{\{s_-e^{2i\xi} (b_0 \nu_0^{-1} b_0^{-1})^{-1}, \nu_-e^{2i\xi}(b_0 \nu_0^{-1} b_0^{-1})^{-1}\}}(-l_0, -l_0)$$

$$= \frac{1}{|s(l_0)/(2\text{Im} l_0)|}.$$ 

under assumptions i) of the above lemma.

We denote by $P_{\{s_+,\nu_+\}}$ the orthogonal projector from $L^2_{\{s_+,\nu_+\}}$ on $H^2_{\{s_+,\nu_+\}}$. Furthermore, $P_{\{s_+,\nu_+\}}$ and $\hat{P}_{\{s_+,\nu_+\}}$ are orthogonal projectors on $H^2_{\{s_+,\nu_+\}}(x)$ and $\hat{H}^2_{\{s_+,\nu_+\}}(x)$, correspondingly.

**Lemma 2.4.** We have for any $f \in L^2_{\{s_+,\nu_+\}}$:

i) $\lim_{x \to -\infty} P_{\{s_+,\nu_+\}} f = f$, $\lim_{x \to +\infty} \hat{P}_{\{s_+,\nu_+\}} f = f$ \hspace{1cm} (2.7)

ii) $\lim_{x \to -\infty} P_{\{s_+,\nu_+\}} f = 0$, $\lim_{x \to +\infty} \hat{P}_{\{s_+,\nu_+\}} f = 0$ \hspace{1cm} (2.8)

Symbolically, we may say that

i) $\lim_{x \to -\infty} e^{ilx_1} H^2_{\{s_+,\nu_+\}} = L^2_{\{s_+,\nu_+\}}$, $\lim_{x \to -\infty} e^{ilx_2} \hat{H}^2_{\{s_+,\nu_+\}} = L^2_{\{s_+,\nu_+\}}$

ii) $\lim_{x \to +\infty} e^{ilx_1} H^2_{\{s_+,\nu_+\}} = \{0\}$, $\lim_{x \to +\infty} e^{ilx_2} \hat{H}^2_{\{s_+,\nu_+\}} = \{0\}$

**Proof.** We prove the first equality in (2.7); the argument for the second equality is likewise. Relations in (2.4) drop by duality, since

$$L^2_{\{s_+,\nu_+\}} = e^{ilx} H^2_{\{s_+,\nu_+\}} \oplus (e^{-ilx} \hat{H}^2_{\{s_+,\nu_+\}})^*.$$ 

Obviously, $e^{ilx_2} H^2_{\{s_+e^{2i\xi_2}, \nu_+e^{2i\xi_2}\}} \subset e^{ilx_1} H^2_{\{s_+e^{2i\xi_1}, \nu_+e^{2i\xi_1}\}}$ for $x_1 \leq x_2$ and so $k_{\{s_+,\nu_+\}}(l_0, l_0)$ is decreasing with respect to $x \in \mathbb{R}$. We have to prove that

$$\lim_{x \to +\infty} e^{-2ilx} k_{\{s_+,\nu_+\}}(l_0, l_0) = 0,$$

which is trivial since the second factor tends to $k(l_0, l_0)$ by Lemma 2.4. \square
Corollary 2.5. We have
\[
\lim_{x \to -\infty} e^{-2\text{Im} l_k \tau} k_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}}(l_k, l_k) = \frac{1}{\nu_+(l_k)}.
\]

Proof. Let us consider \( g_k = (1/\nu_+(l_k)) \delta_{l_k} \). Recall that \( k_{\{s, \nu\}}(., l_0; x) = e^{i\tau(l_0 \to l_k)} k_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}}(., l_0) \). Hence, we obtain for a \( f \in H^2_\{s, \nu\}(x) \)
\[
(f, g_k)_{\{s, \nu\}} = \frac{f(l_k)}{\nu_+(l_k)} \nu_+(l_k) = f(l_k).
\]
On the other hand,
\[
f(l_k) = (f, k_{\{s, \nu\}}(., l_k; x))_{\{s, \nu\}} = (f, P_{x, \{s, \nu\}} g_k)_{\{s, \nu\}}.
\]
By Lemma 2.4
\[
\lim_{x \to -\infty} ||k_{\{s, \nu\}}(., l_k; x)||^2_{\{s, \nu\}} = ||g_k||^2_{\{s, \nu\}}
\]
which becomes the claim of the corollary if we write the norms explicitly. \( \square \)

2.3. Proof of Theorem 1.6

Lemma 2.6. We have
\[
k_{\{s, \nu\}}(., l_0) = \lim_{\varepsilon \to 0^+} (\varepsilon + I + H_{\{s, \nu\}})^{-1} k(., l_0),
\]
where \( H_{\{s, \nu\}} \) is the Hankel operator coming from the metric \( l_0 \) and the limit is understood in \( L^2_{\{s, \nu\}} \)-sense.

The argument follows [17], Lemma 1.2, and is omitted.

Lemma 2.7. Let \( ||s||_\infty < 1 \) and \( \nu_+ \) be a measure with a finite support. Then
\[
\lim_{x \to +\infty} \frac{K_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}}(l_0, l_0)}{K(l_0, l_0)} = 1.
\]

Proof. We see that
\[
|k_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}}(l_0, l_0) - k(l_0, l_0)|^2
\]
\[
= |k_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}}(., l_0) - k(., l_0)|^2
\]
\[
= \left| \left( (I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})(I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})^{-1} k(., l_0) - k(., l_0) \right) \right|^2
\]
\[
\leq C \left( (H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}} k(., l_0), (I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})^{-1} k(., l_0)) \right)^2
\]
\[
+ (H_{\nu, e^{2i\tau x}} k(., l_0), (I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})^{-1} k(., l_0))^2
\]
The bound for the first term is easy
\[
| \ldots | \leq \frac{1}{1 - ||s||_\infty} ||P_{s, e^{-2i\tau x}} k(-l_0) ||_2 |k(., l_0)|_2 \to 0
\]
as \( x \to +\infty \) by the \( L^2 \)-Fourier theorem. Since \( F = (I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})^{-1} k(., l_0) \) \( \in H^2 \) satisfies \( ||F||_2 \leq C \), we get \( |F(l_k)| \leq C/\sqrt{\text{Im} l_k} \) and
\[
||H_{\nu, e^{2i\tau x}} k(., l_0), (I + H_{\{s, e^{2i\tau x}, \nu, e^{2i\tau x}\}})^{-1} k(., l_0)||
\]
Proof. We start with the proof of the first equality. Taking the square root of both sides
we have

\[ \sqrt{\nu_n} e^{-2i\Omega_n x} (l_0, l_0) = 1, \] (2.9)

\[ \sqrt{\nu_n} e^{-2i\Omega_n x} (l_0, l_0) = 1. \] (2.10)

Proof. We start with the proof of the first equality. Taking the square root of both sides of (2.4), we see

\[ K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0) \]

\[ = \frac{1}{2\text{Im} l_0 |s(l_0)|} K_{s_{\pm}, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}, \nu_n, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}} (-l_0, -l_0) \]

\[ \geq 2\text{Im} l_0 |s(l_0)| K_{s_{\pm}, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}, \nu_n, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}} (-l_0, -l_0) \]

\[ = \frac{|B(l_0)|}{|B_{1,N}(l_0)|} K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0). \] (2.11)

Then we continue as

\[ K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0) \]

\[ \geq |B_{N,\infty}(l_0)| (K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0))^{1/2} \]

\[ \geq |B_{N,\infty}(l_0)| ((\varepsilon + 1 + H_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}}) - 1) k((l_0, l_0))^{1/2} (l_0) \]

\[ = \frac{1}{\sqrt{1 + \varepsilon}} |B_{N,\infty}(l_0)| K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0). \] (2.12)

Let \( s_N = s_{N}/B_{1,N}. \) We have

\[ K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0) \leq K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0) \]

\[ = \frac{1}{2\text{Im} l_0 |s_N(l_0)|} K_{s_{\pm}, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}, \nu_n, e^{-2i\Omega_n} (b_0 b_{-l_0})^{-1}} (-l_0, -l_0) \]

\[ \leq \frac{1}{2\text{Im} l_0 |s_N(l_0)|} K_{s_{\pm}, e^{2i\Omega_n}, \nu_n, e^{2i\Omega_n}} (l_0, l_0). \] (2.13)

by ii). Lemma 2.3 We set \( s_{-\varepsilon} = s_{-\varepsilon}/(1 + \varepsilon), \nu_{-N,\varepsilon} = \nu_{-N}/(1 + \varepsilon); \) the functions \( s_{N,\varepsilon}, s_{+\varepsilon} \) are defined by unitarity of the scattering matrix, and \( \nu_{N,\varepsilon} \) is defined by \( \nu_{N,\varepsilon} \) through relations (1.12). Notice that the support of \( \nu_{N,\varepsilon} \) is the same as the support of \( \nu_n \) (and equals \( \{l_k\}_{k=1,N} \)). Since \( K \) and \( K \)-kernels are the same for pairs \( \{s_{-\varepsilon}, \nu_{-N,\varepsilon}\} \) and \( \{s_{+\varepsilon}, \nu_{N,\varepsilon}\} \) by ii), Lemma 2.3 we continue as

\[ \frac{1}{\sqrt{1 + \varepsilon}} \frac{|s_N(l_0)|}{|s_N(l_0)|} K_{s_{+}, e^{2i\Omega_n}, \nu_{N,\varepsilon}, e^{2i\Omega_n}} (l_0, l_0). \]
That is,

\[
\frac{|B_{N,\infty}(l_0)|}{\sqrt{1+\varepsilon}} K\left(\frac{\nu}{1+i\varepsilon}, \frac{N-\varepsilon}{1+i\varepsilon} \right)(l_0, l_0) \leq K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0) \\
\leq \frac{\sqrt{1+\varepsilon} |s_N(c_0)|}{|s_N(l_0)|} K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0).
\]

The quantities \(K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0)\) and \(K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0)\) tend to \(K(l_0, l_0)\) as \(x \to +\infty\) by Lemma 2.7. Remaining factors in the left- and right-hand side parts of the inequality go to 1 with \(\varepsilon \to +0, N \to +\infty\). Hence, for any \(\varepsilon' > 0\) we can choose appropriate \(\varepsilon, N\) to have

\[
1 - \varepsilon' \leq \liminf_{x \to +\infty} \frac{K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0)}{K(l_0, l_0)} \leq \limsup_{x \to +\infty} \frac{K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(l_0, l_0)}{K(l_0, l_0)} \leq 1 + \varepsilon',
\]

and (2.9) is proved.

The proof of (2.10) is almost identical. First of all, to keep the notation we used to, we prove

\[
\lim_{x \to +\infty} \frac{K\left(s_+, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0)}{K(-\bar{r}_0, -\bar{r}_0)} = 1. \tag{2.14}
\]

instead of (2.10). This is obviously the same thing up to changes \(-\bar{r}_0 \to r_0\) and \(s_- \to s_+\). The second modification is that we estimate the value of a \(K\)-kernel by the values of \(K\)-kernels (and not vice versa as we have just done to prove (2.9)).

So, as in (2.12), we have

\[
K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0) \geq K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0) \\
\geq \frac{1}{\sqrt{1+\varepsilon}} K\left(\frac{\nu}{1+i\varepsilon}, \frac{N-\varepsilon}{1+i\varepsilon} \right)(-\bar{r}_0, -\bar{r}_0) \\
= \frac{|B_{N,\infty}(l_0)|}{\sqrt{1+\varepsilon}} K\left(\frac{\nu}{1+i\varepsilon}, \frac{N-\varepsilon}{1+i\varepsilon} \right)(-\bar{r}_0, -\bar{r}_0).
\]

The first inequality in the above estimate is (ii), Lemma 2.3 and the last one repeats computation (2.11). Similarly to (2.13), we get

\[
K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0) \leq K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0) \\
= \frac{1}{2\text{Im} l_0 |s_N(l_0)|} K\left(s_+, e^{-2i\varepsilon z (b_0 b_{-l_0})^{-1}}, \frac{N-\varepsilon}{1+i\varepsilon} e^{-2i\varepsilon (b_0 b_{-l_0})^{-1}} \right)(l_0, l_0) \\
\leq \frac{1}{2\text{Im} l_0 |s_N(l_0)|} K\left(\frac{\nu}{1+i\varepsilon}, e^{-2i\varepsilon (b_0 b_{-l_0})^{-1}} \right)(l_0, l_0) \\
= \frac{|B_{N,\infty}(l_0)|}{|s_N(l_0)|} K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0) \\
= \frac{|B_{N,\infty}(l_0)|}{|s_N(l_0)|} K\left(s_-, e^{2i\varepsilon z}, \nu e^{2i\varepsilon z} \right)(-\bar{r}_0, -\bar{r}_0).
\]
Above, the pair \( \{s_{-\varepsilon}, \nu_{-N, \varepsilon}\} \) comes from \( \{\frac{s_{-\varepsilon}}{1 + \varepsilon}, \frac{\nu_{-N, \varepsilon}}{1 + \varepsilon}\} \) as explained after (2.13). Hence,

\[
\frac{|B_{N, \infty}(l_0)|}{\sqrt{1 + \varepsilon}} K_{\{s_{-\varepsilon}, \nu_{-N, \varepsilon} e^{2ilx}\}}(-l_0, -l_0) \leq K_{\{s_{-\varepsilon} e^{2ilx}, \nu_{-N, \varepsilon} e^{2ilx}\}}(-l_0, -l_0).
\]

Repeating the argument from the first part of the proof, we see that for every \( \varepsilon' > 0 \)

\[
1 - \varepsilon' \leq \liminf_{x \to +\infty} \frac{\hat{K}_{\{s_{-\varepsilon} e^{2ilx}, \nu_{-N, \varepsilon} e^{2ilx}\}}(-l_0, -l_0)}{K(-l_0, -l_0)} \leq \limsup_{x \to +\infty} \frac{\hat{K}_{\{s_{-\varepsilon} e^{2ilx}, \nu_{-N, \varepsilon} e^{2ilx}\}}(-l_0, -l_0)}{K(-l_0, -l_0)} \leq 1 + \varepsilon',
\]

and relation (2.14) is proved.

Proof of Theorem 1.8. At present, the claim of the theorem is an easy consequence of Lemma 2.8. For an arbitrary \( N \), we have

\[
|e^{ilx}K_{\{s_{+}, e^{2ilx}, \nu_{+} e^{2ilx}\}}(l, l_0)| - e^{ilx}\tilde{K}(l, l_0)|_{\{s_{+}, \nu_{+}\}} \leq |e^{ilx}K_{\{s_{+} e^{2ilx}, \nu_{+} e^{2ilx}\}}(l, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|_{\{s_{+}, \nu_{+}\}} + |e^{ilx}\tilde{K}(\cdot, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|_{\{s_{+}, \nu_{+}\}}.
\]

The claim will then follow if we prove

\[
\limsup_{x \to +\infty} |e^{ilx}\tilde{K}(\cdot, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|_{\{s_{+}, \nu_{+}\}} \leq C_1 |1 - B_{N, \infty}(l_0)| \quad (2.15)
\]

\[
\limsup_{x \to +\infty} |e^{ilx}K_{\{s_{+} e^{2ilx}, \nu_{+} e^{2ilx}\}}(l, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|_{\{s_{+}, \nu_{+}\}} \leq C_2 |1 - B_{N, \infty}(l_0)| \quad (2.16)
\]

with some constants \( C_1, C_2 \). The computation for (2.15) is easy and elementary

\[
|e^{ilx}\tilde{K}(\cdot, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|^2_{\{s_{+}, \nu_{+}\}} \leq |e^{ilx}(1 - B_{N, \infty}) K(\cdot, l_0)|^2_{\{s_{+}, \nu_{+}\}} + |e^{ilx}B_{N, \infty} K(\cdot, l_0)|^2_{\{s_{+}, \nu_{+}\}}
\]

The second term above obviously goes to 0 as \( x \to +\infty \); for the first one we have

\[
|e^{ilx}(1 - B_{N, \infty}) K(\cdot, l_0)|^2_{s_{+}} \leq 2|e^{ilx}(1 - B_{N, \infty}) K(\cdot, l_0)|^2_{s_{+}} \leq 4|1 - B_{N, \infty}(l_0)|.
\]

We pass to (2.16) now. Once again, for an arbitrary \( N \),

\[
|e^{ilx}K_{\{s_{+} e^{2ilx}, \nu_{+} e^{2ilx}\}}(l, l_0) - B_{N, \infty} e^{ilx} K(\cdot, l_0)|^2_{\{s_{+}, \nu_{+}\}} \leq |e^{ilx}(1 - B_{N, \infty}) K(\cdot, l_0)|^2_{\{s_{+}, \nu_{+}\}}.
\]

By Lemma 2.8 we get for the second term

\[
\Re(\ldots) = \frac{B_{N, \infty}(l_0) K(l_0, l_0)}{K_{\{s_{+} e^{2ilx}, \nu_{+} e^{2ilx}\}}(l_0, l_0)} \to B_{N, \infty}(l_0)
\]

as \( x \to +\infty \). The third term is

\[
(\ldots) = |B_{N, \infty} K(\cdot, l_0)|^2_{s_{+} e^{2ilx}} + |B_{N, \infty} K(\cdot, l_0)|^2_{\nu_{+} e^{2ilx}}
\]
\[ \leq \|K(\cdot, l_0)\|^2_2 + \|P_+ [s_+ e^{2i\xi x} B_{N, \infty} K(\cdot, l_0)] (-\bar{\ell})\|_2 \|B_{N, \infty} K(\cdot, l_0)\|_2 \]

\[ + \|B_{N, \infty} K(\cdot, l_0)\|^2_2 e^{2i\xi x} \to 1, \]

since \( \|K(\cdot, l_0)\|^2_2 = 1 \) and the rest tends to 0 with \( x \to +\infty \) (for the second term, this is Fourier \( L^2 \)-theorem). So, summing up

\[ \limsup_{x \to +\infty} \|e^{ix} K_{\{s_+ e^{2i\xi x}, \nu_+ e^{2i\xi x}\}} (\cdot, l_0) - B_{N, \infty} e^{ix} K(\cdot, l_0)\|_{s, \nu_+} \leq 2 \text{Re} (1 - B_{N, \infty} (l_0)), \]

and \( (2.16) \) is proved.

The proof of (1.23) is likewise, we just have to use (2.10) instead of (2.3). \( \square \)

3. UNITARY NODE, I

Consider the multiplication operator by \( \tilde{v} \), \( v = \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_0} \), acting in

\[ L^2_{\{s_+, \nu_+\}} = (\hat{H}^2_{\{s_-, \nu_-\}})^\perp \oplus H^2_{\{s_+, \nu_+\}}. \]  \( (3.1) \)

**Lemma 3.1.** The multiplication operator by \( \tilde{v} \) acts as a unitary operator from

\[ \{ \hat{k}^+_\{s_-, \nu_-\} (\lambda, \lambda_0) \} \oplus H^2_{\{s_+, \nu_+\}}(x) \]

\[ \text{to} \]

\[ \{ \hat{k}^+_\{s_-, \nu_-\} (\lambda, -\bar{\lambda}_0) \} \oplus H^2_{\{s_+, \nu_+\}}(x). \]  \( (3.3) \)

**Proof.** It is obvious that the multiplication by \( \tilde{v} = \frac{b_{-\bar{\lambda}_0}}{b_{\lambda_0}} \) acts from

\[ \{ f \in \hat{H}^2_{\{s_-, \nu_-\}} : f(\lambda_0) = 0 \} = b_{\lambda_0} \hat{H}^2_{\{b_{\lambda_0} b_{-\bar{\lambda}_0} s_- b_{\lambda_0} b_{-\bar{\lambda}_0} \nu_-\}} \]

to

\[ \{ f \in \hat{H}^2_{\{s_-, \nu_-\}} : f(-\bar{\lambda}_0) = 0 \} = b_{-\bar{\lambda}_0} \hat{H}^2_{\{b_{\lambda_0} b_{-\bar{\lambda}_0} s_- b_{\lambda_0} b_{-\bar{\lambda}_0} \nu_-\}}. \]

Therefore it acts in their orthogonal complements \( (3.2), (3.3) \). \( \square \)

We now recall the definition of the characteristic function of a unitary node and its functional model. An extensive discussion of the subject and its application to interpolation problems can be found in [8, 10, 11].

Let \( K, E_1, E_2 \) be Hilbert spaces and \( U \) be a unitary operator acting from \( K \oplus E_1 \) to \( K \oplus E_2 \). We assume that \( E_1 \) and \( E_2 \) are finite-dimensional (\( \dim E_1 = \dim E_2 = 1 \) in this section, and \( \dim E_1 = \dim E_2 = 2 \) in Section 4). The characteristic function is defined by

\[ \Theta(\zeta) := P_{E_2} U (I_{K \oplus E_1} - \zeta P_K U)^{-1} |E_1|. \]  \( (3.4) \)

It is a holomorphic in the unit disk \( \{ \zeta : |\zeta| < 1 \} \) contractive-valued operator function. We make a specific assumption that \( \Theta(\zeta) \) has an analytic continuation in the exterior of the unite disk through a certain arc \( (a, b) \subset \mathbb{T} \) by the symmetry principle

\[ \Theta(\zeta) = \Theta^*(\frac{1}{\zeta})^{-1}. \]

For \( f \in K \) define

\[ F(\zeta) := P_{E_2} U (I - \zeta P_K U)^{-1} f. \]  \( (3.5) \)

This \( E_2 \)-valued holomorphic function belongs to the functional space \( K_\Theta \) with the following properties.

- \( F(\zeta) \in H^2(E_2) \) and it has analytic continuation through the arc \( (a, b) \).
- \( F_*(\zeta) := \Theta^*(\zeta) F \left( \frac{1}{\zeta} \right) \in H^2(E_1) \).
For almost every \( \zeta \in \mathbb{T} \) the vector \( \begin{bmatrix} F_x \\ F_y \end{bmatrix} \) belongs to the image of the operator \( \begin{bmatrix} I & \Theta^* \\ \Theta & I \end{bmatrix} \), and therefore the scalar product
\[
\left\langle \begin{bmatrix} I & \Theta^* \\ \Theta & I \end{bmatrix}^{-1} \begin{bmatrix} F_x \\ F_y \end{bmatrix}, \begin{bmatrix} F_x \\ F_y \end{bmatrix} \right\rangle \quad \text{is well-defined and does not depend on the choice of a preimage (the first term in the above scalar product). Moreover,}
\]
\[
\int_{\mathbb{T}} \left\langle \begin{bmatrix} I & \Theta^* \\ \Theta & I \end{bmatrix}^{-1} \begin{bmatrix} F_x \\ F_y \end{bmatrix}, \begin{bmatrix} F_x \\ F_y \end{bmatrix} \right\rangle dm < \infty. \quad (3.6)
\]
The integral in (3.6) represents the square of the norm of \( F \) in \( K_0 \).

Note that \( P_K U | K \) becomes a certain “standard” operator in the model space
\[
f \mapsto F(\zeta) \quad \implies \quad P_K U f \mapsto \frac{F(\zeta) - F(0)}{\zeta}, \quad (3.7)
\]
see (3.5).

The following simple identity is a convenient tool in the forthcoming calculation.

**Lemma 3.2.** For a unitary operator \( U : K \oplus E_1 \to K \oplus E_2 \)
\[
U^* P_{E_2} U (I - \zeta P_K U)^{-1} = I + (\zeta - U^*) P_K U (I - \zeta P_K U)^{-1}. \quad (3.8)
\]

**Proof.** Since \( I_{K \oplus E_2} = P_K + P_{E_2} \) and \( U \) is unitary we have
\[
U^* P_{E_2} U = (I - \zeta P_K U) + (\zeta - U^*) P_K U.
\]
Then we multiply this identity by \( (I - \zeta P_K U)^{-1} \).

**Theorem 3.3.** Let \( e_1, e_2 \) be the normalized vectors in the one-dimensional spaces (3.2) and (3.3),
\[
e_1(\lambda) = \frac{\hat{k}^{+}_{(s+,\nu_+)}(\lambda, \lambda_0)}{\sqrt{\hat{k}^{-}_{(s-,\nu_-)}(\lambda_0, \lambda) \hat{k}^{-}_{(s-,\nu_-)}(\lambda_0, \lambda_0)}}, \quad e_2(\lambda) = \frac{\hat{k}^{+}_{(s-,\nu_-)}(\lambda, -\bar{\lambda}_0)}{\sqrt{\hat{k}^{-}_{(s-,\nu_-)}(\lambda_0, \lambda) \hat{k}^{-}_{(s-,\nu_-)}(\lambda_0, \lambda_0)}}. \quad (3.9)
\]
Then the reproducing kernel of \( H^2_{(s+,\nu_+)} \) is of the form
\[
k^{(s+,\nu_+)}(\lambda, \mu) = \frac{(ve_2)(\lambda)(ve_2)(\mu) - e_1(\lambda)e_2(\mu)}{1 - v(\lambda)v(\mu)}. \quad (3.10)
\]

**Proof.** First, we are going to find the characteristic function of the multiplication operator by \( \bar{v} \) with respect to decompositions (3.2) and (3.3) and the corresponding functional representation of this node.

By (6.3) we fixed “bases” in the one-dimensional spaces. So, instead of the operator we get a scalar function \( \theta(\zeta) \):
\[
\Theta(\zeta)e_1 := P_{E_2} U (I - \zeta P_K U)^{-1} e_1 = e_2(\zeta). \quad (3.11)
\]
We substitute (3.11) in (3.8)
\[
v(\lambda)e_2(\lambda)\theta(\zeta) = e_1(\lambda) + (\zeta - v(\lambda))(P_K U (I - \zeta P_K U)^{-1} e_1)(\lambda). \quad (3.12)
\]
Recall an important property of \( \hat{k}^{+}_{(s-,\nu_-)}(\lambda, \lambda_0) \): it has analytic continuation in the upper half-plane with the only pole at \(-\lambda_0\) (see Lemma 2.2). Therefore all terms
in (3.12) are analytic in \( \lambda \) and we can choose \( \lambda \) satisfying \( v(\lambda) = \zeta \). Then we obtain the characteristic function in terms of the reproducing kernels

\[
\theta(v(\lambda)) = \frac{e_1(\lambda)}{v(\lambda)e_2(\lambda)}.
\]  

(3.13)

Similarly for \( f \in K = H^2_{\{s+,v_+\}} \) we define the scalar function \( F(\zeta) \) by

\[
P_{E_2}U(I - \zeta P_K U)^{-1}f = e_2F(\zeta).
\]

(3.14)

Using again (3.8) we get

\[
v(\lambda)e_2(\lambda)F(\zeta) = f(\lambda) + (\zeta - v(\lambda))(P_K U(I - \zeta P_K U)^{-1}f)(\lambda).
\]

Therefore,

\[
F(v(\lambda)) = \frac{f(\lambda)}{v(\lambda)e_2(\lambda)}.
\]

(3.15)

Now we are in a position to get (3.10). Indeed, by (3.14) and (3.15) we proved that the vector

\[
P_K(I - v(\mu)U*P_K)^{-1}U^*e_2v(\mu)e_2(\mu)
\]

is the reproducing kernel of \( K = H^2_{\{s+,v_+\}} \) with respect to \( \mu \), \( |v(\mu)| < 1 \). Using the Darboux identity

\[
P_{E_2}U(I - \zeta P_K U)^{-1}P_K(I - \bar{\zeta}0U^*P_K)^{-1}U^*|E_2 = \frac{I - \Theta(z)\Theta^*(\bar{\zeta}0)}{1 - \zeta0}
\]

(in this setting this is a simple and pleasant exercise) we obtain

\[
k\{s+,\nu_+\}(\lambda, \mu) = v(\lambda)e_2(\lambda)\frac{I - \theta(v(\lambda))\theta(v(\mu))}{1 - v(\lambda)v(\mu)}v(\mu)e_2(\mu)
\]

for \( |v(\lambda)| < 1 \), \( |v(\mu)| < 1 \). By analyticity and (3.13) we have that relation (3.10) holds for all \( l, \mu \in \mathbb{C}_+ \).

**Corollary 3.4.** The following Wronskian-type identity is satisfied for the reproducing kernels

\[
\begin{vmatrix}
(se_2^*)(\mu) \\
e_2(\mu)
\end{vmatrix}
\begin{vmatrix}
(se_1^*)(\mu) \\
e_1(\mu)
\end{vmatrix}
= \frac{1}{i} \left( \log|v(\mu)| \right)', \text{ Im } \mu > 0.
\]

(3.16)

**Proof.** To be brief, we write \( k\{s+,\nu_+\}(\ldots) \) instead of \( (k\{s+,\nu_+\}(\ldots))^− \). So we multiply \( k\{s+,\nu_+\}(\lambda, -\bar{\mu}) \) by \( b_\mu(\lambda) \) and calculate the resulting function of \( \lambda \) at \( \lambda = \mu \). By (2.30) we get

\[
\{b_\mu(\lambda)k\{s+,\nu_+\}(\lambda, -\bar{\mu})\}_{\lambda=\mu} = \frac{1}{s(\mu)2\text{Im } \mu}
\]

(3.17)

Now we make the same calculation using representation (3.14). Since

\[
k\{s+,\nu_+\}(\lambda, -\bar{\mu}) = \frac{v(\mu)}{v(\lambda) - v(\mu)} \frac{v(\lambda)e_2(\lambda)}{e_1(\bar{-\mu})} \frac{e_1(\lambda)}{v(-\bar{\mu})e_2(-\bar{\mu})},
\]

we get in combination with (3.14)

\[
-\frac{v'(\mu)}{v(\mu)s(\mu)} = \begin{vmatrix}
v(\mu)e_2(\mu) & e_1(\bar{-\mu}) \\
e_1(\bar{-\mu}) & v(-\bar{\mu})e_2(-\bar{\mu})
\end{vmatrix}.
\]

By the symmetry \( k\{s-,\nu_-\}(\lambda, \lambda_0) = \bar{k}\{s-,\nu_-\}(\lambda, -\lambda_0) \), we have \( e_2(-\bar{\mu}) = e_1(\mu) \). Thus (3.10) is proved. \( \Box \)
Corollary 3.5. Let \( \mu \in \mathbb{R}^+ \) and as before \( \Re \lambda_0 > 0 \), then
\[
|e_2(\mu)|^2 - |e_1(\mu)|^2 = \frac{1}{4} (\log v(\mu))'.
\] (3.18)

Proof. All terms in \( \text{3.16} \) have boundary values. Recall that on the real axis \((se_{1,2})'(\mu) = (s - e_{1,2})(\mu) - e_{1,2}(-\bar{\mu})\). Then use again the symmetry of the reproducing kernel. \( \square \)

We finish this section with a translation of the relation \( \|f\|_{\{s_+,\nu_+\}}^2 = \) a unitary map from \( H^2_{\{s_+,\nu_+\}} \) to \( K_\theta \) to the following proposition.

Theorem 3.6. Let
\[
s^\theta_\pm(\lambda) := \frac{e_2(-\lambda)}{e_2(\lambda)}, \quad \lambda \in \mathbb{R}^+,
\] (3.19)
extended by the symmetry \( s^\theta_+(\lambda) = s^\theta_+(\lambda) \) to the whole \( \mathbb{R} \). Let \( \nu_+^\theta \) be a positive measure on the imaginary half-axis
\[
d\nu_+^\theta(\lambda) := \frac{|d\nu(\lambda)|}{2\pi |e_2(\lambda)|^2}, \quad \lambda \in i\mathbb{R}^+.
\] (3.20)
Then
\[
||f||_{\{s_+,\nu_+\}}^2 = \int_{i\mathbb{R}^+} |f(\lambda)|^2 d\nu_+^\theta(\lambda)
\]
\[
+ \frac{1}{4\pi} \int_{\mathbb{R}} \left[ \frac{f(\lambda)}{s_+^\theta(\lambda)} - f(-\lambda) \right] \left[ \frac{1}{s_+^\theta(\lambda)} \frac{s_+^\theta(\lambda)}{1} f(\lambda) - f(-\lambda) \right] d\lambda
\] (3.21)
for all \( f \in H^2_{\{s_+,\nu_+\}} \). In other words
\( \text{id} : H^2_{\{s_+,\nu_+\}} \to H^2_{\{s^\theta_+,\nu_+^\theta\}} \)
is an isometry.

Proof. We use definition of the scalar product in \( K_\theta \), relations \( \text{3.13}, \text{3.15} \), and \( \text{3.18} \). \( \square \)

4. Unitary node, II: A canonical system

In this section we associate a canonical system (see \( \text{3.15} \)) with the given chain \( \{H^2_{\{s_+,\nu_+\}}(x)\}_{x \in \mathbb{R}} \) of subspaces of \( L^2_{\{s_+,\nu_+\}} \).

4.1. Characteristic function of a unitary node and transfer matrix. Definitions. This time we consider the unitary multiplication operator by \( \bar{v}, \ v = \frac{\lambda^2 - \lambda_0^2}{\lambda^2 - 3\lambda_0^2} \), with respect to the decomposition
\[
L^2_{\{s_+,\nu_+\}} = (\bar{H}^2_{\{s_-,\nu_+\}})^+ \oplus K_{\{s_+,\nu_+\}}(x) \oplus H^2_{\{s_+,\nu_+\}}(x).
\] (4.1)
Actually this is definition of the space \( K_{\{s_+,\nu_+\}}(x) \).

The following lemma is similar to Lemma \( \text{3.1} \).
Lemma 4.1. The multiplication operator by \( \bar{v} \) acts from
\[
\{k^+_{s,v} (\lambda, \lambda_0)\} \oplus K_{s,v} (x) \oplus \{k_{s,v} (\lambda, \lambda_0; x)\}
\]
(4.2)
to
\[
\{k^+_{s,v} (\lambda, -\bar{\lambda}_0)\} \oplus K_{s,v} (x) \oplus \{k_{s,v} (\lambda, -\bar{\lambda}_0; x)\}.
\]
(4.3)

We define normalized vectors that form orthonormal basises in \( E_1 \) and \( E_2 \)
\[
e^{(1)}_1 (\lambda) = \frac{k^+_{s,v} (\lambda, \lambda_0)}{||k^+_{s,v} (\lambda, \lambda_0)||}, \quad e^{(1)}_2 (\lambda) = \frac{k_{s,v} (\lambda, \lambda_0; x)}{||k_{s,v} (\lambda, \lambda_0; x)||},
\]
and
\[
e^{(2)}_1 (\lambda) = \frac{k^+_{s,v} (\lambda, -\bar{\lambda}_0)}{||k^+_{s,v} (\lambda, -\bar{\lambda}_0)||}, \quad e^{(2)}_2 (\lambda) = \frac{k_{s,v} (\lambda, -\bar{\lambda}_0; x)}{||k_{s,v} (\lambda, -\bar{\lambda}_0; x)||}.
\]
(4.4)
(4.5)

We point out that the vectors \( e^{(i)}_r (\lambda), i = 1, 2, \) depend also on \( x \) and \( e^{(i)}_r (\lambda), i = 1, 2, \) do not.

Generally for an operator \( A : H_1 \oplus H_2 \to \tilde{H}_1 \oplus \tilde{H}_2 \) its Potapov-Ginzburg transform \( \tilde{A} : H_1 \oplus H_2 \to H_1 \oplus H_2 \) is defined by
\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \tilde{A}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}, \quad \text{where} \quad \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = A
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

In terms of the block decomposition of \( A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \) we have
\[
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & -I
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
I & -A_{12} \\
0 & -A_{22}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
x_2
\end{bmatrix}.
\]

Therefore,
\[
\tilde{A} = \begin{bmatrix}
I & -A_{12} \\
0 & -A_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & -I
\end{bmatrix} = \begin{bmatrix}
A_{11} - A_{12}A^{-1}_{22}A_{21} & A_{12}A^{-1}_{22} \\
-A^{-1}_{22}A_{21} & A^{-1}_{22}
\end{bmatrix}.
\]
(4.6)

The transformation is well-defined if \( A_{22} \) is invertible. Note, that if \( A \) is unitary,
\[
||y_1||^2 + ||y_2||^2 = ||x_1||^2 + ||x_2||^2,
\]
then \( \tilde{A} \) preserves the indefinite metric
\[
||y_1||^2 - ||y_2||^2 = ||x_1||^2 - ||x_2||^2.
\]

For the unitary node given by the multiplication operator by \( \bar{v} \) and decompositions (4.2), (4.3):
\[
U : (K \oplus \{e^{(1)}_1\}) \oplus \{e^{(2)}_2\} \to (K \oplus \{e^{(2)}_2\}) \oplus \{e^{(1)}_1\}
\]
(4.7)
we define the j-unitary node
\[
\tilde{U} : (K \oplus \{e^{(1)}_1\}) \oplus \{e^{(2)}_2\} \to (K \oplus \{e^{(2)}_2\}) \oplus \{e^{(1)}_1\}
\]
(4.8)
by (4.4), separating in this way \( x \)-depending "channels".

The characteristic operator-valued function for the node (4.7) is
\[
\Theta(\zeta) := P_{E_2}U(I - \zeta P_K U)^{-1}|E_1,
\]
and its matrix with respect to the chosen basises is
\[
\Theta(\zeta) \begin{bmatrix}
e^{(1)}_1 (\lambda) & e^{(1)}_2 (\lambda) \\
e^{(2)}_1 (\lambda) & e^{(2)}_2 (\lambda)
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = \begin{bmatrix}
e^{(2)}_1 (\lambda) & e^{(2)}_2 (\lambda)
\end{bmatrix}
\begin{bmatrix}
\theta(\zeta) \\
\epsilon_2
\end{bmatrix},
\]
(4.10)
where
\[ \theta(\zeta) = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} (\zeta). \]

Respectively, its functional representation is of the form
\[ P_{E_{\delta}} U (I - \zeta P_K U)^{-1} f = \begin{bmatrix} e_2^{(2)}(\lambda) & e_1^{(2)}(\lambda) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} (\zeta), \] (4.11)

for \( f \in K_{\{\lambda,\nu\}}(x) \).

The transfer matrix is actually the characteristic matrix function of the node (4.8). Having (4.9), (4.10), we rewrite (4.7) in the block form as
\[ U \begin{bmatrix} \zeta k(\zeta) \\ \theta \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} (\zeta) \begin{bmatrix} \theta \\ c_1 \\ c_2 \end{bmatrix}. \] (4.12)

Consequently, we get for \( \tilde{U} \):
\[ \tilde{U} \begin{bmatrix} \zeta k(\zeta) \\ \theta \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} (\zeta) \begin{bmatrix} \theta \\ c_1 \\ c_2 \end{bmatrix}. \] (4.13)

Therefore the transfer matrix \( \mathfrak{A}(\zeta) \) of the \( j \)-node is related to \( \theta(\zeta) \) by
\[ \mathfrak{A}(\zeta) \begin{bmatrix} 1 \\ \theta_{21}(\zeta) & 0 \\ 0 & \theta_{22}(\zeta) \end{bmatrix} = \begin{bmatrix} \theta_{11}(\zeta) & \theta_{12}(\zeta) \\ 0 & 1 \end{bmatrix}. \]

Thus
\[ \mathfrak{A}(\zeta) = \begin{bmatrix} \theta_{11}(\zeta) & \theta_{12}(\zeta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \theta_{21}(\zeta) & 0 \\ 0 & \theta_{22}(\zeta) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\theta_{12}(\zeta) \\ 0 & -\theta_{22}(\zeta) \end{bmatrix}^{-1} \begin{bmatrix} \theta_{11}(\zeta) \\ 0 \end{bmatrix} -1. \] (4.14)

4.2. Calculating \( \theta \) and \( \mathfrak{A} \). We are following the same lines as in Section 3. Let us substitute (4.10) into (3.8)
\[ v(\lambda) \begin{bmatrix} e_2^{(2)}(\lambda) & e_1^{(2)}(\lambda) \end{bmatrix} \theta(\zeta) = \begin{bmatrix} e_1^{(1)}(\lambda) & e_2^{(1)}(\lambda) \end{bmatrix} \]
\[ \quad + (\zeta - v(\lambda)) \left( P_K U (I - \zeta P_K U)^{-1} \begin{bmatrix} e_1^{(1)} \\ e_2^{(1)} \end{bmatrix} \right)(\lambda). \]

All terms here are analytic in \( \lambda \) and we can choose \( \lambda \in \mathbb{C}_+ \) with the property \( v(\lambda) = \zeta \). Then we get
\[ v(\lambda) \begin{bmatrix} e_2^{(2)}(\lambda) & e_1^{(2)}(\lambda) \end{bmatrix} \theta(v(\lambda)) = \begin{bmatrix} e_1^{(1)}(\lambda) & e_2^{(1)}(\lambda) \end{bmatrix}. \] (4.15)

Similarly, by (4.11)
\[ v(\lambda) \begin{bmatrix} e_2^{(2)}(\lambda) & e_1^{(2)}(\lambda) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} (v(\lambda)) = f(\lambda). \] (4.16)

It is tempting to make the change of variable \( \lambda \rightarrow -\bar{\lambda} \), \( \lambda \in \mathbb{R} \), in (4.10), (4.16) and to write
\[ v(\lambda) \begin{bmatrix} (e_2^{(2)})^{-}(\lambda) & (e_1^{(2)})^{-}(\lambda) \end{bmatrix} \theta(v(\lambda)) = \begin{bmatrix} (e_1^{(1)})^{-}(\lambda) & (e_2^{(1)})^{-}(\lambda) \end{bmatrix}, \] (4.17)
and
\[ v(\lambda) \begin{bmatrix} (e_2^{(2)})^{-}(\lambda) & (e_1^{(2)})^{-}(\lambda) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} (v(\lambda)) = f^{-}(\lambda). \] (4.18)
However, to succeed with this plan, we need to prove that \( \theta(\zeta) \) has an analytic continuation in \( \mathbb{C} \setminus \mathbb{D} \) etc. That is why we prefer to consider a dual node given by the diagram

\[
\begin{array}{c}
K \oplus E_1 \\
\uparrow \quad \uparrow \quad \uparrow \\
K^- \oplus E_1' \\
K \oplus E_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
K^- \oplus E_2'
\end{array}
\]

The characteristic matrix-valued function remains the same since we choose basis in \( E_{1,2}^- \) as the image of the basis in \( E_{1,2} \). Then we obtain \( (4.17) \) and \( (4.18) \) simply repeating the arguments from \( (4.15) \) and \( (4.16) \). Hence

\[
v(\lambda) \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^-
\end{bmatrix} (\lambda) \begin{bmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22}
\end{bmatrix} (v(\lambda)) = \begin{bmatrix}
(e_1^{(1)})^- & (e_2^{(1)})^- \\
\frac{1}{s(\lambda)} & \frac{1}{v(\lambda)}
\end{bmatrix} (\lambda),
\]

and

\[
v(\lambda) \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^- \\
e_2^{(2)} & e_1^{(2)}
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} (v(\lambda)) = \begin{bmatrix}
f^- \\
f
\end{bmatrix} (\lambda).
\]

**Lemma 4.2.** Both \( \det \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^- \\
e_2^{(2)} & e_1^{(2)}
\end{bmatrix} (\lambda) \) and \( \theta_{22}(\lambda) \) do not vanish identically.

Furthermore,

\[
\theta_{22}(\lambda) \det \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^- \\
e_2^{(2)} & e_1^{(2)}
\end{bmatrix} (\lambda) = -\frac{i}{s(\lambda)} \left( \frac{1}{v(\lambda)} \right)^{\prime}.
\]

In particular, the characteristic matrix-valued function \( \theta(\zeta) \) and the map \( K_{\{s_+\nu_+\}}(x) \rightarrow K_\theta \) are well-defined by \( (4.20), (4.21) \) in terms of the reproducing kernels.

**Proof.** This follows from an obvious consequence of \( (4.21) \)

\[
v(\lambda) \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^- \\
e_2^{(2)} & e_1^{(2)}
\end{bmatrix} (\lambda) \begin{bmatrix}
1 & -\theta_{12} \\
0 & -\theta_{22}
\end{bmatrix} (v(\lambda)) = \begin{bmatrix}
v(e_2^{(2)})^- & -v(e_2^{(1)})^- \\
v e_2^{(2)} & -e_2^{(1)}
\end{bmatrix} (\lambda),
\]

and \( (4.10) \) that says

\[
\det \begin{bmatrix}
v(e_2^{(2)})^- & -v(e_2^{(1)})^- \\
v e_2^{(2)} & -e_2^{(1)}
\end{bmatrix} (\lambda) = i \frac{v'(\lambda)}{s(\lambda)}.
\]

\( \square \)

By \( (4.12) \), we have for the transfer matrix

\[
\Omega_x(\lambda^2) = \begin{bmatrix}
1 & -\theta_{12} & \theta_{11} & 0 \\
0 & -\theta_{22} & \theta_{21} & -1
\end{bmatrix} (v(\lambda)).
\]

The map from \( K_\theta \) to the corresponding de Branges space \( \mathcal{H}(\Omega) \) \( \text{[Sect. 28]} \) is of the form

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} (\lambda^2) = \begin{bmatrix}
1 & -\theta_{12} & \theta_{11} & 0 \\
0 & -\theta_{22} & \theta_{21} & -1
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} (v(\lambda)).
\]

Combining \( (4.20) \) with

\[
v(\lambda) \begin{bmatrix}
(e_2^{(2)})^- & (e_1^{(2)})^- \\
e_2^{(2)} & e_1^{(2)}
\end{bmatrix} (\lambda) \begin{bmatrix}
\theta_{11} & 0 \\
\theta_{21} & -1
\end{bmatrix} (v(\lambda)) = \begin{bmatrix}
(e_1^{(1)})^- & -v(e_1^{(2)})^- \\
\frac{1}{s(\lambda)} & \frac{1}{v(\lambda)}
\end{bmatrix} (\lambda),
\]

\( (4.26) \)
we have
\[ \begin{bmatrix} v(e_2^{(2)})^{-} & -(e_2^{(1)})^{-} \\ v e_2^{(2)} & -e_2^{(1)} \end{bmatrix} (\lambda) \mathcal{A}_x(\lambda^2) = \begin{bmatrix} (e_1^{(1)})^{-} & -v(e_1^{(2)})^{-} \\ e_1^{(1)} & -v e_1^{(2)} \end{bmatrix} (\lambda), \] (4.27)
and
\[ \begin{bmatrix} v(e_2^{(2)})^{-} & -(e_2^{(1)})^{-} \\ v e_2^{(2)} & -e_2^{(1)} \end{bmatrix} (\lambda) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} (\lambda^2) = \begin{bmatrix} f^{-} \\ f \end{bmatrix} (\lambda). \] (4.28)

Note that the condition \( I - \theta \theta^* \geq 0 \) is the same as
\[ \begin{bmatrix} 1 & -\theta_{12} \\ 0 & -\theta_{22} \end{bmatrix} \begin{bmatrix} 1 & -\theta_{12} \\ 0 & -\theta_{22} \end{bmatrix}^* \begin{bmatrix} \theta_{11} & 0 \\ 0 & -\theta_{21} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{11} & 0 \\ 0 & 1 \end{bmatrix}^* \geq 0. \]
That is, the transformation maps contractive matrices into \( j \)-contractions,
\[ j - \mathcal{A} j \mathcal{A}^* \geq 0, \quad j := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \]
although this was clear from the definition of \( j \)-node.

4.3. de Branges’ Theorem.

**Theorem 4.3.** For every \( x > 0 \),
\[ \mathcal{A}_x(\lambda^2) = \frac{-i\lambda}{v'(\lambda)} \begin{bmatrix} e_2^{(1)} \\ v e_2^{(2)} \end{bmatrix} (\lambda) \begin{bmatrix} (e_1^{(1)})^{-} & -v(e_1^{(2)})^{-} \\ e_1^{(1)} & -v e_1^{(2)} \end{bmatrix} (\lambda), \] (4.29)
is an entire matrix-valued function of \( \lambda^2 \) and
\[ \mathcal{H}(\mathcal{A}_x) = \left\{ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} : (\lambda^2) = \frac{-i\lambda}{v'(\lambda)} \begin{bmatrix} e_2^{(1)} \\ v e_2^{(2)} \end{bmatrix} (\lambda) \begin{bmatrix} (e_1^{(1)})^{-} & -v(e_1^{(2)})^{-} \\ e_1^{(1)} & -v e_1^{(2)} \end{bmatrix} (\lambda) \begin{bmatrix} f^{-} \\ f \end{bmatrix} (\lambda), \ f \in K_{(s_+, \nu_+)}(x) \right\}, \] (4.30)
is the de Branges space of entire functions [3, Sect. 28].

The proof is omitted.

We point out that for all \( x \) the \( x \)-depending matrix in the RHS of (4.29) meets the following normalization condition
\[ (v e_2^{(2)})(\lambda_0) = 0, \quad e_2^{(1)}(\lambda_0) > 0. \] (4.31)

As the result we get a family of \( 2 \times 2 \) \( j \)-contractive matrix-valued functions with a certain normalization at \( \lambda_0 \). The family is monotonic in \( x \), and every matrix is an entire function in \( \lambda^2 \) of the zero mean type (concerning the corollary of the last condition see [3, Theorem 39]). According to de Branges’ Theorem [3, Sect. 36, 37], Theorem 37, such a family can be included in the chain
\[ j \frac{d}{dt} \mathcal{A}(\lambda^2, t) = \left\{ i\lambda^2 \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} (t) + \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} (t) \right\} \mathcal{A}(\lambda^2, t), \ |\beta| = \alpha, \] (4.32)
such that \( \mathcal{A}_x(\lambda^2) = \mathcal{A}(\lambda^2, t_x) \), where \( x = x(t) \) is a monotonic function. Here we choose \( \lambda_0^2 = i \) as the normalization point.
4.4. Parameters of the system in terms of reproducing kernels.

**Theorem 4.4.** For the system
\[
\frac{\beta}{\alpha} = -\frac{s(\lambda_0)}{s(-\lambda_0)} \frac{d\hat{k}(\lambda_0, -\bar{\lambda}_0)}{d\hat{k}(-\lambda_0, -\bar{\lambda}_0)}.
\] (4.33)

**Proof.** Set
\[
E = \begin{bmatrix}
-e_2^{(1)} & v(e_2^{(1)}) \\
-v(e_2^{(2)}) & -v(k_{\lambda_0})
\end{bmatrix}
= \frac{1}{\sqrt{k(\lambda_0, \lambda_0)}} \begin{bmatrix}
-k_{\lambda_0} & k_{\bar{\lambda}_0} \\
-k\lambda_0 & v(k_{\lambda_0})
\end{bmatrix}.
\] (4.34)

We note that due to (4.29)
\[
j\dot{AA}^{-1} = j\dot{EE}^{-1}.
\] In particular, for \(\lambda_0 = i\)
\[
j\dot{E}(\lambda_0)E^{-1}(\lambda_0) = -\begin{bmatrix}
\alpha & 2\beta \\
0 & \alpha
\end{bmatrix}.
\] (4.35)

On the other hand, using (2.3), (2.4), we have
\[
E(\lambda_0) = \begin{bmatrix}
\tau & a\tau \\
0 & \tau^{-1}
\end{bmatrix} \begin{bmatrix}
-1 & 0 \\
0 & C
\end{bmatrix}
\] (4.36)
with a constant \(C\) and
\[
\tau := \sqrt{k(\lambda_0, \lambda_0)}, \quad a := s^2(\lambda_0)(2\text{Im} \lambda_0)^2\hat{k}(\lambda_0, -\bar{\lambda}_0).
\] (4.37)

In these notations,
\[
j\dot{E}(\lambda_0)E^{-1}(\lambda_0) = \begin{bmatrix}
\dot{\tau} & a\dot{\tau}^2 \\
0 & \dot{\tau}
\end{bmatrix}.
\]
Comparing this with (4.33) we get \(\frac{\partial}{\partial \tau} = -\frac{da}{d(\tau - 2)}\). Since
\[
\tau^{-2} = s(\lambda_0)s(-\bar{\lambda}_0)(2\text{Im} \lambda_0)^2\hat{k}(\lambda_0, -\bar{\lambda}_0),
\]
by (4.36) we have (4.33). \(\square\)

5. de Branges system and Sturm-Liouville equation

In this section, we rewrite the results of the previous sections for a particular case of the Sturm-Liouville equation. Let
\[
Ly = -y'' + qy
\] (5.1)
be a self-adjoint operator acting on \(L^2(\mathbb{R})\), and let
\[
u = \frac{L - \lambda^2_0}{L - \bar{\lambda}_0}, \quad \text{Re} \lambda_0 > 0, \text{Im} \lambda_0 > 0,
\] (5.2)
be its Cayley transform.

**Lemma 5.1.** Let \(e^\pm(x, \lambda) \in L^2(\mathbb{R}_\pm), ||e^\pm(x, \lambda)|| = 1\), be such that
\[
-\frac{d^2}{dx^2}e^\pm(x, \lambda) + q(x)e^\pm(x, \lambda) = \lambda^2 e^\pm(x, \lambda), \quad x \in \mathbb{R}_\pm, \text{Im} \lambda > 0, \text{Re} \lambda \neq 0.
\] (5.3)
Then for all \(f_+ \in L^2(\mathbb{R}_+)\)
\[
u f_+ = Ce^-(x, \lambda_0)(f_+, e^+(x, -\bar{\lambda}_0)) + g_+,
\] (5.4)
where \(C = C(u)\) and \(g_+ \in L^2(\mathbb{R}_+)\).
Proof. Let
\[(uf_+)(x) = \begin{cases} g_-(x), & x \in \mathbb{R}_- \\ g_+(x), & x \in \mathbb{R}_+ \end{cases},\]
or, what is the same,
\[\left(\frac{\lambda_0^2 - \bar{\lambda}_0^2}{L - \lambda_0^2}f_+\right)(x) = \begin{cases} g_-(x), & x \in \mathbb{R}_- \\ g_+(x) - f_+(x), & x \in \mathbb{R}_+ \end{cases}.\]
Therefore,
\[-g''_+ + qg_+ = \lambda_0^2 g_+, \quad x \in \mathbb{R}_-,
\]
and \(g_- \in L^2(\mathbb{R}_-)\). That is,
\[g_-(x) = C(f_+)e^-(x, \lambda_0).\]
Thus we get
\[uf_+ = C(f_+)e^-(x, \lambda_0) + g_+.
\]
For \(C(f_+)\) we have
\[C(f_+) = (uf_+, e^-(x, \lambda_0)) = (f_+, u^*e^-(x, \lambda_0)).\]
Now we are looking at
\[u^*e^-(x, \lambda_0) = \left( I + \frac{\lambda_0^2 - \bar{\lambda}_0^2}{L - \lambda_0^2} \right) e^-(x, \lambda_0) = \begin{cases} h_-(x), & x \in \mathbb{R}_- \\ h_+(x), & x \in \mathbb{R}_+ \end{cases},\]
or
\[\left(\lambda_0^2 - \bar{\lambda}_0^2\right)e_-(x, \lambda_0) = \begin{cases} -\tilde{h}''_+ + q\tilde{h}_+ - \bar{\lambda}_0^2\tilde{h}_-, & x \in \mathbb{R}_- \\ -\tilde{h}''_+ + q\tilde{h}_+ - \lambda_0^2\tilde{h}_+, & x \in \mathbb{R}_+ \end{cases},\]
where \(\tilde{h} = h - e^-(x, \lambda_0)\). This implies
\[\tilde{h}(-0) = \tilde{h}(+0), \quad \tilde{h}'(-0) = \tilde{h}'(+0).
\]
Notice that the equality \(C_2 = 0\) contradicts the linear independence of \(e^-(x, \lambda_0)\) and \(e^-(x, -\lambda_0)\).
Hence,
\[u^*e^-(x, \lambda_0) = \tilde{h} + e^-(x, \lambda_0) = \begin{cases} C_1e^-(x, -\lambda_0), & x \in \mathbb{R}_- \\ C_2e^+(x, -\bar{\lambda}_0), & x \in \mathbb{R}_+ \end{cases},\]
and (5.4) is proved with
\[C(f_+) = (f_+, C_2e^+(x, -\bar{\lambda}_0)).\]
\[\square
\]
Corollary 5.2. The operator \(u\) acts from \(L^2(\mathbb{R}_+) \oplus \{e^-(x, -\lambda_0)\}\) to \(L^2(\mathbb{R}_+) \oplus \{e^-(x, \lambda_0)\}\).
Proof. Similarly to (5.4),
\[ u_f = Ce^+(x, \lambda_0)\langle f_-, e^-(x, -\bar{\lambda}_0) \rangle + g_- \]  
(5.6)
That is, \( u_f = g_- \in L^2(\mathbb{R}_- \cup \mathbb{R}_+^n) \) if \( f_- \perp e^-(x, -\bar{\lambda}_0) \). Moreover, \( (u_f) \) is orthogonal to \( e^-(x, \lambda_0) \) by (5.3) in this case. \qed

**Corollary 5.3.** For every \( x_0 > 0 \), the operator \( u \) acts from
\[ (L^2[0, x_0] \oplus \{ e^-(x, -\bar{\lambda}_0) \}) \oplus \{ e^{x_0}_+(x, -\bar{\lambda}_0) \} \]  
(5.7)
to
\[ (L^2[0, x_0] \oplus \{ e^{x_0}_+(x, \lambda_0) \}) \oplus \{ e^-(x, \lambda_0) \} \]  
(5.8)
where \( e^{x_0}_+(x, \lambda) \in L^2[x_0, \infty) \) is the normalized solution of (5.3).

**Theorem 5.4.** The transfer matrix of the unitary node \( [\lambda^2_1, \lambda^2_2] \) is of the form
\[ \mathfrak{A}_{x_0}(\lambda^2) = \begin{bmatrix} e^+_{x_0}(x_0, \lambda_0) & -e^+_{x_0}(x_0, -\bar{\lambda}_0) \\ e^+_{x_0}(x_0, \lambda_0) & -e^+_{x_0}(x_0, -\bar{\lambda}_0) \end{bmatrix}^{-1} \mathfrak{B}_{x_0}(\lambda^2) \begin{bmatrix} -e^-(0, -\bar{\lambda}_0) & e^-(0, \lambda_0) \\ -e^-(0, -\bar{\lambda}_0) & e^-(0, \lambda_0) \end{bmatrix}, \]  
(5.9)
where
\[ \mathfrak{B}_x(\lambda^2) = \begin{bmatrix} e(x, \lambda) & s(x, \lambda) \\ \dot{e}(x, \lambda) & \dot{s}(x, \lambda) \end{bmatrix} \]  
(5.10)
is the standard transfer matrix for equation (5.7).

**Proof.** In the block form we have
\[ u \begin{bmatrix} \zeta k_\xi \\ c_1 \\ d_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} k_\xi \\ d_1 \\ c_1 \\ d_2 \end{bmatrix}, \]  
(5.12)
with
\[ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \mathfrak{A}(\zeta) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \]  
(5.13)
In other words,
\[ \left( I + \frac{\lambda^2_0 - \bar{\lambda}_0^2}{L - \lambda^2_0} \right) \{ \zeta k_\xi + c_1 e^-(x, -\bar{\lambda}_0) + d_2 e^{x_0}_+(x, -\bar{\lambda}_0) \} = k_\xi + d_1 e^{x_0}_+(x, \lambda_0) + c_2 e^-(x, \lambda_0), \]  
(5.14)
or
\[ \frac{\lambda^2_0 - \bar{\lambda}_0^2}{L - \lambda^2_0} \{ \zeta k_\xi + c_1 e^-(x, -\bar{\lambda}_0) + d_2 e^{x_0}_+(x, -\bar{\lambda}_0) \} = (1 - \zeta) k_\xi + d_1 e^{x_0}_+(x, \lambda_0) + c_2 e^-(x, \lambda_0) - c_1 e^-(x, -\bar{\lambda}_0) - d_2 e^{x_0}_+(x, -\bar{\lambda}_0). \]  
(5.15)
This means that the RHS of (5.14) has the second derivative and we have on the interval \([0, x_0]\)
\[ -k''_\xi + q k_\xi = \lambda^2 k_\xi, \]  
(5.15)
for the spectral parameter
\[ \lambda^2 = \lambda^2_0 + \frac{\zeta}{1 - \zeta} (\lambda^2_0 - \bar{\lambda}_0^2). \]
Above, \( \zeta = \frac{\lambda^2 - \lambda_0^2}{\lambda^2 - \bar{\lambda}_0^2} \).

Let

\[
(1 - \zeta)k_\zeta = Ac(x, \lambda) + Bs(x, \lambda).
\]

Then the continuity at \( x = 0 \) implies

\[
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
e^{-}(0, -\bar{\lambda}_0) & \bar{e}^{-}(0, \lambda_0) \\
-\bar{e}^{-}(0, -\bar{\lambda}_0) & e^{-}(0, \lambda_0)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix},
\]

and by the continuity at \( x = x_0 \),

\[
\begin{bmatrix}
e^\pm(x_0, \lambda_0) - e^\pm(x_0, -\bar{\lambda}_0) \\
\bar{e}^\pm(x_0, \lambda_0) - \bar{e}^\pm(x_0, -\bar{\lambda}_0)
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} = \begin{bmatrix}
e(x_0, \lambda) & s(x_0, \lambda) \\
\bar{e}(x_0, \lambda) & \bar{s}(x_0, \lambda)
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}.
\]

The theorem is proved. \( \square \)

We now compute the parameters of the related canonical system under the chosen normalization.

We start observing that, up to the initial matrix \( \mathfrak{A}_0 \), the transfer matrix has the same normalization as the transfer matrix \( \mathfrak{A}_0 \) (or \( \mathfrak{A}_0 \)) in Section 4.

**Corollary 5.5.** The transfer matrix of unitary node \( \mathfrak{A}_0 \) (5.7), \( \mathfrak{A}_0 \) is of the form

\[
\mathfrak{A}_x(\lambda^2) = \mathfrak{A}_x(\lambda^2)\mathfrak{A}_0,
\]

where

\[
\mathfrak{A}_x(\lambda^2) = \left[\begin{array}{cc} e^+(x_0, \lambda_0) & e^+(x_0, -\bar{\lambda}_0) \\
\bar{e}^+(x_0, \lambda_0) & \bar{e}^+(x_0, -\bar{\lambda}_0) \end{array}\right]^{-1} \mathfrak{B}_{x_0}(\lambda^2) \left[\begin{array}{cc} e^0(0, \lambda_0) & e^0(0, -\bar{\lambda}_0) \\
\bar{e}^0(0, \lambda_0) & \bar{e}^0(0, -\bar{\lambda}_0) \end{array}\right].
\]

Therefore, \( \mathfrak{A}_x(\lambda^2) \) meets the normalization:

\[
(\mathfrak{A}_x(\lambda_0^2))_{11} > 0, \quad (\mathfrak{A}_x(\lambda_0^2))_{21} = 0
\]

for all \( x > 0 \).

We use the same notation as in \( \mathfrak{A}_0 \).

**Theorem 5.6.** Let \( m_+(\lambda) \) be the Weyl function of operator \( \mathfrak{B}_\lambda \) and let

\[
\mathfrak{A}_x(\lambda_0^2) = \left[\begin{array}{cc} \tau & \alpha \tau \\
0 & \tau^{-1} \end{array}\right].
\]

Then

\[
\frac{d}{d(\tau^{-2})} \mathfrak{A}_x(\lambda_0^2) = \frac{c(x, \lambda_0) m_+(\lambda_0) + s(x, \lambda_0)}{c(x, \lambda_0) m_+(\lambda_0) + s(x, \lambda_0)}.
\]

**Proof.** First of all,

\[
\left[\begin{array}{c} e^\pm(x_0, \lambda) \\
\bar{e}^\pm(x_0, \lambda) \end{array}\right] = \mathfrak{B}_x \left[\begin{array}{c} m_+(\lambda) \\
1 \end{array}\right] \rho(x), \quad x \geq x_0,
\]

where \( \rho(x_0) \) should be found from the condition

\[
\int_{x_0}^{\infty} |e^\pm(x, \lambda)|^2 dx = 1.
\]

Using

\[
\frac{d}{dx}(\mathfrak{B}_x(\lambda^2)J \mathfrak{B}_x(\lambda^2)) = -(\lambda^2 - \bar{\lambda}^2) \mathfrak{B}_x(\lambda^2) \left[\begin{array}{cc} 1 & 0 \\
0 & \mathfrak{B}_x(\lambda^2) \end{array}\right],
\]

(5.21)
where $\tau$.

Therefore, we deduce from (5.18) that

$$
\rho^2(x_0) = \frac{\lambda^2 - \bar{\lambda}^2}{m_+(\lambda) - \bar{m}_+(\lambda)}.
$$

That is,

$$
\rho^2(x_0) = \frac{\lambda^2 - \bar{\lambda}^2}{m_+(\lambda) - \bar{m}_+(\lambda)}.
$$

In particular

$$
\rho^2(0) = -\frac{\lambda^2 - \bar{\lambda}^2}{m_+(\lambda) - \bar{m}_+(\lambda)}.
$$

Since for $x \geq x_0$

$$
\mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} \rho^2(0, \lambda_0) \\ \rho^2_0(0, \lambda_0) \end{array} \right] = \mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \rho(0) = \left[ \begin{array}{c} \rho^+_0(x, \lambda_0) \\ \rho^+_0(x, \lambda_0) \end{array} \right] = \left[ \begin{array}{c} \rho(0) \\ \rho(0) \end{array} \right],
$$

we get for the first column of the matrix $\mathcal{A}_x(\lambda_0^2)$

$$
\left[ \begin{array}{c} \rho^+_0(x, \lambda_0) \\ \rho^+_0(x, \lambda_0) \end{array} \right]^{-1} = \left[ \begin{array}{c} \rho(0) \\ \rho(0) \end{array} \right].
$$

Therefore, we deduce from (5.18) that $\tau = \frac{\rho(0)}{\rho(x_0)}$ and, recalling (5.23), (5.24), we come to

$$
\tau^{-2} = -\frac{m_+(\lambda_0) - \bar{m}_+(\lambda_0)}{m_+(\lambda_0) - \bar{m}_+(\lambda_0)}.
$$

To compute $a\tau$, we proceed as

$$
a\tau \Delta = -\left[ \begin{array}{c} \rho^+_0(x, -\lambda_0) \\ \rho^+_0(x, -\lambda_0) \end{array} \right] \mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \rho(0)

= -\left[ \begin{array}{c} \rho^+_0(x, -\lambda_0) \\ \rho^+_0(x, -\lambda_0) \end{array} \right] J \mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \rho(0)

= -\rho(x_0) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \rho(0),
$$

where

$$
\Delta = \det \left[ \begin{array}{cc} \rho^+_0(x, \lambda_0) & -\rho^+_0(x, -\lambda_0) \\ \rho^+_0(x, \lambda_0) & -\rho^+_0(x, -\lambda_0) \end{array} \right]

= \left[ \begin{array}{c} \rho^+_0(x, -\lambda_0) \\ \rho^+_0(x, -\lambda_0) \end{array} \right] J \left[ \begin{array}{c} \rho^+_0(x, \lambda_0) \\ \rho^+_0(x, \lambda_0) \end{array} \right]

= \rho(x_0) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \mathcal{B}_x(\lambda_0^2) \left[ \begin{array}{c} m_+(\lambda_0) \\ 1 \end{array} \right] \rho(x_0).
$$
Combining (5.21) and (5.23), we obtain

\[
a = \frac{[m_+(\lambda_0)]}{[m_+(\lambda_0)]} B_2^* (\lambda_0^2) J \frac{m_+(\lambda_0)}{1}. \tag{5.29}
\]

Using (5.21) and the Wronskian identity for \( B_2 (\lambda) \) we get (5.19) from (5.20) and (5.21) by a direct computation.  

6. Appendix 1. An example

In this section we give an example which shows that the class of canonical systems discussed in Section 4 is larger than the class of Sturm-Liouville equations from Section 5. We will see that generally

\[
H^2_{\{s_+, \nu_+\}} \neq \hat{H}^2_{\{s_+, \nu_+\}}, \tag{6.1}
\]

although always \( H^2_{\{s_+, \nu_+\}} \subset \hat{H}^2_{\{s_+, \nu_+\}} \) and we will also discuss some other interesting phenomena.

Throughout this section we set \( s_+ = \frac{1}{\lambda + \tau} \) and \( \nu_+ = 0 \).

First, we prove \( \text{(5.1)} \). Since

\[
\langle (s_+ f)(\lambda), -f(-\bar{\lambda}) \rangle = 0
\]

for all \( f \in H^2 \), we get \( ||f||_{\{s_+, \nu_+\}} = ||f|| \). Therefore, in this case \( H^2_{\{s_+, \nu_+\}} \) coincides with the standard \( H^2 \).

On the other hand, we have \( s(\lambda) = \frac{1}{\lambda + \tau} \), so \( s \cdot 1 \in H^2 \). Let us check that \( 1 \in L^2_{\{s_+, \nu_+\}} \). This follows from the identity

\[
[1 \quad -1] \left[ \begin{array}{c}
\frac{1}{\lambda + \tau} \\
\frac{1}{1}
\end{array} \right] = \frac{2}{|\lambda + i\tau|^2}. \tag{6.2}
\]

Hence, by the definition of \( \hat{H}^2_{\{s_+, \nu_+\}} \) a constant function belongs to this space, but of course \( 1 \notin H^2 \) and \( \text{(6.1)} \) is proved.

The above conclusion can be sharpened. Using

\[
\frac{1}{2} \left[ \begin{array}{c}
s_+(\lambda) \\
1
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{s_+(\lambda)} \\
1
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{\lambda + \tau} \\
\frac{1}{1}
\end{array} \right] = \langle s_+(\lambda) - 1, -f(-\bar{\lambda}) \rangle = 0, \tag{6.3}
\]

for all \( f \in H^2 \), we get that 1 is orthogonal to \( H^2_{\{s_+, \nu_+\}} \subset \hat{H}^2_{\{s_+, \nu_+\}} \). Actually we have the following orthogonal decomposition

\[
\hat{H}^2_{\{s_+, \nu_+\}} = \{1\} \oplus H^2_{\{s_+, \nu_+\}} = \{1\} \oplus H^2. \tag{6.4}
\]

This implies that the reproducing kernel of \( \hat{H}^2_{\{s_+, \nu_+\}} \) is

\[
\hat{k}(\lambda, \lambda_0) = k_{\{s_+, \nu_+\}}(\lambda, \lambda_0) = \frac{1}{||1||^2_{\{s_+, \nu_+\}}} \left( \frac{1}{\lambda - \lambda_0} + \frac{i}{\lambda - \lambda_0} \right), \tag{6.5}
\]

and, by \( \text{(5.2)} \), \( ||1||^2_{\{s_+, \nu_+\}} = \frac{1}{2} \).

Now we show that the property \( H^2_{\{s_+, \nu_+\}}(x) \neq \hat{H}^2_{\{s_+, \nu_+\}}(x) \) is not \( x \)-invariant. Namely, \( H^2_{\{s_+, \nu_+\}}(x) = \hat{H}^2_{\{s_+, \nu_+\}}(x) \) for \( x > 0 \) despite \( \text{(6.1)} \) for \( x = 0 \).
Lemma 6.1. Let \( s_+ = \frac{\lambda}{\lambda + i}, \nu_+ = 0 \). Then \( H^2_{\{s_+,\nu_+\}}(x) = \hat{H}^2_{\{s_+,\nu_+\}}(x) \) for all \( x > 0 \).

Proof. Notice that
\[
\hat{H}^2_{\{s_+,\nu_+\}}(x) = e^{i\lambda x} \hat{H}^2_{\{s_+,e^{2i\lambda x},\nu_+,e^{2i\lambda x}\}},
\]
and \( H^2_{\{s_+,\nu_+\}}(x) = e^{i\lambda x} H^2 \). So we have to show that \( \hat{H}^2_{\{s_+,e^{2i\lambda x},\nu_+,e^{2i\lambda x}\}} = H^2 \).

By definition \( f \in \hat{H}^2_{\{s_+,e^{2i\lambda x},\nu_+,e^{2i\lambda x}\}} \) means that
\[
\frac{i}{\lambda + i} f(\lambda) \in H^2, \\
e^{2i\lambda x} \frac{\lambda}{\lambda + i} f(\lambda) - f(-\lambda) \in L^2.
\]
(6.6)

We have to prove that \( f \in H^2 \).

Let \( g(\lambda) = \frac{1}{\lambda^2} f(\lambda) \). Conditions (6.6) can be easily transformed into
\[
\lambda\{e^{2i\lambda x} g(\lambda) + g(-\bar{\lambda})\} \in L^2
\]
with \( g \in H^2 \). Let \( G \) denote the Fourier transform of \( g \). Obviously, \( G \in L^2(\mathbb{R}_+) \) since \( g \in H^2 \). In these terms we have
\[
\{G(2x + t) + G(-t)\}' \in L^2.
\]
(6.7)

Since the supports of the functions \( G(2x + t) \) and \( G(-t) \) do not intersect, we get from (6.7) that \( G'(t) \in L^2 \). Therefore \( \lambda g(\lambda) \in L^2 \), and, consequently, \( f \) belongs to \( L^2 \) and, in fact, to \( H^2 \). \(\square\)

Corollary 6.2. Let \( s_+ = \frac{\lambda}{\lambda + i}, \nu_+ = 0 \). Then \( H^2_{\{s_+,\nu_+\}}(x) = \hat{H}^2_{\{s_+,\nu_+\}}(x) \) for all \( x < 0 \).

Proof. We only have to mention that \( s_- = s_+ \) in our case and to use Theorem 6.3. \(\square\)

Corollary 6.3. Let \( s_+ = \frac{\lambda}{\lambda + i}, \nu_+ = 0 \). Then
\[
\lim_{x \to -0} H^2_{\{s_+,\nu_+\}}(x) = \hat{H}^2_{\{s_+,\nu_+\}}(0).
\]
(6.8)

Proof. Obviously, \( \lim_{x \to x_0+0} H^2_{\{s_+,\nu_+\}}(x) = H^2_{\{s_+,\nu_+\}}(x_0) \) for \( x_0 \geq 0 \). Therefore by \( s_- = s_+ \) and the duality stated in Theorem 6.3,
\[
\lim_{x \to x_0-0} \hat{H}^2_{\{s_+,\nu_+\}}(x) = \hat{H}^2_{\{s_+,\nu_+\}}(x_0)
\]
for \( x_0 \leq 0 \). Finally, we use Corollary 6.2
\[
\lim_{x \to -0} H^2_{\{s_+,\nu_+\}}(x) = \lim_{x \to -0} \hat{H}^2_{\{s_+,\nu_+\}}(x) = \hat{H}^2_{\{s_+,\nu_+\}}(0).
\]
\(\square\)

This means, in particular, that the canonical system related to the given scattering data is not a Sturm-Liouville equation.

Indeed, let \( \mathcal{A}(x_0,x_1; \lambda^2) \), \( x_0 < x_1 \), be the transfer matrix (4.24) (or (4.29)). Recalling (4.27), we write
\[
\begin{bmatrix}
ve_2^{(2)}(x_1) & v_2^{(1)}(x_1)
\end{bmatrix} =
\begin{bmatrix}
\mathcal{C}_1^{(1)}(x_0) & ve_1^{(2)}(x_0)
\mathcal{C}_1^{(2)}(x_0)
\end{bmatrix} \mathcal{A}(x_0,x_1; \lambda^2).
We also introduce $\mathfrak{B}(x_0, x_1; \lambda^2)$ by
\[
\begin{bmatrix}
ve^{(2)}_2(x_1) & e^{(1)}_2(x_1)
\end{bmatrix} = \begin{bmatrix}
ve^{(2)}_2(x_0) & e^{(1)}_2(x_0)
\end{bmatrix} \mathfrak{B}(x_0, x_1; \lambda^2),
\]
so that we have the chain rule
\[\mathfrak{A}(x_0, x_2; \lambda^2) = \mathfrak{A}(x_0, x_1; \lambda^2) \mathfrak{B}(x_1, x_2; \lambda^2).\]

Fix $x_0 < 0$, put $x_2 = 0$, and let
\[\mathfrak{B}(\lambda^2) = \lim_{x_1 \to 0} \mathfrak{B}(x_1, 0; \lambda^2).\]

Using (6.9) and (6.8) we have
\[
\begin{bmatrix}
ve^{(2)}_2(x_1) & e^{(1)}_2(x_1)
\end{bmatrix} = \begin{bmatrix}
ve^{(2)}_2(x_1) & \hat{e}^{(1)}_2(x_1)
\end{bmatrix} \mathfrak{B}(\lambda^2),
\]
where (see (4.4), (4.5))
\[e^{(1)}_2(l) = \hat{k}(s, \nu_+)(\lambda, \lambda_0; x) \frac{e^{(2)}_2(x_1)}{||\hat{k}(s, \nu_+)(\lambda, \lambda_0; x)||},
\]
\[e^{(2)}_2(l) = \hat{k}(s, \nu_+)(\lambda, -\lambda_0; x) \frac{e^{(2)}_2(l)}{||\hat{k}(s, \nu_+)(\lambda, -\lambda_0; x)||}.
\]
Thus $\mathfrak{B}(\lambda^2)$ is a non-trivial divisor of $\mathfrak{A}(x_0, x; \lambda^2)$ for $x \geq 0$. The chain $\mathfrak{A}(x_0, x; \lambda^2)$ is not even continuous in $x$,
\[\lim_{x \to 0} \mathfrak{A}(x_0, x; \lambda^2) \neq \mathfrak{A}(x_0, 0; \lambda^2).
\]

Of course, we can get an explicit formula for $\mathfrak{B}(\lambda^2)$. Since $\mathfrak{B}$ depends on $\lambda^2$, (6.10) implies
\[
\begin{bmatrix}
e^{(2)}_2(\lambda) & e^{(1)}_2(\lambda, \lambda_0) \\
e^{(2)}_2(-\lambda) & e^{(1)}_2(-\lambda)
\end{bmatrix} \begin{bmatrix}v(\lambda) & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}e^{(2)}_2(\lambda) & \hat{e}^{(1)}_2(\lambda) \\
e^{(2)}_2(-\lambda) & e^{(1)}_2(-\lambda)
\end{bmatrix} \begin{bmatrix}v(\lambda) & 0 \\
0 & 1
\end{bmatrix} \mathfrak{B}(\lambda^2),
\]
or, with the help of (6.5),
\[
\sqrt{\frac{k(\lambda_0, \lambda_0)}{k(\lambda, \lambda_0)}} \begin{bmatrix}
k(\lambda, -\lambda_0) & k(\lambda, \lambda_0) \\
k(\lambda, -\lambda_0) & k(\lambda, \lambda_0)
\end{bmatrix} \begin{bmatrix}v(\lambda) & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}\hat{k}(\lambda, -\lambda_0) & \hat{k}(\lambda, \lambda_0) \\
\hat{k}(\lambda, -\lambda_0) & \hat{k}(\lambda, \lambda_0)
\end{bmatrix} \begin{bmatrix}v(\lambda) & 0 \\
0 & 1
\end{bmatrix} \mathfrak{B}(\lambda^2)
\]
\[
= \left\{ 2 \begin{bmatrix}1 & 1 \\
1 & 1
\end{bmatrix} + \begin{bmatrix}k(\lambda, -\lambda_0) & k(\lambda, \lambda_0) \\
\hat{k}(\lambda, -\lambda_0) & \hat{k}(\lambda, \lambda_0)
\end{bmatrix} \begin{bmatrix}v(\lambda) & 0 \\
0 & 1
\end{bmatrix} \mathfrak{B}(\lambda^2).
\]
Thus, directly,
\[
\sqrt{1 + 4 \text{Im} \lambda_0} I = \left\{ \frac{i}{\text{Re} \lambda_0} \begin{bmatrix} \lambda^2 - \lambda_0^2 & \lambda^2 - \lambda_0^2 \\
-\lambda^2 + \lambda_0^2 & -\lambda^2 + \lambda_0^2
\end{bmatrix} + I \right\} \mathfrak{B}(\lambda^2).
\]
Note that the determinant of the matrix in curly brackets is $1 + 4 \text{Im} \lambda_0$, so $\mathfrak{B}(\lambda^2)$ is indeed an entire function of $\lambda^2$ (a linear polynomial),
\[
\mathfrak{B}(\lambda^2) = \frac{1}{\sqrt{1 + 4 \text{Im} \lambda_0}} \left\{ I + \frac{i}{\text{Re} \lambda_0} \begin{bmatrix} -\lambda^2 + \lambda_0^2 & -\lambda^2 + \lambda_0^2 \\
\lambda^2 - \lambda_0^2 & \lambda^2 - \lambda_0^2
\end{bmatrix} \right\}.
\]
7. Appendix 2. On a certain sufficient condition

7.1. An extension of $A_2$ in the presence of the mass points.

**Theorem 7.1.** Let $E = [-2, 2]$ and $X = \{x_k\}$ be a set of points on $\mathbb{R} \setminus E$ that satisfies the Blaschke condition in the domain $\mathbb{C} \setminus E$. Let $\Sigma$ be an $n \times n$ matrix-measure supported on $E \cup X$ which is absolutely continuous on $E$,

$$d\Sigma(x) = W(x) \, dx, \quad (7.1)$$

moreover, $W(x)^{-1}$ exists for almost all $x \in E$, and $\Sigma(x_k) = \Sigma_k$. Define

$$(\Omega f)(x) = \lim_{\epsilon \to +0} \int_{E \cup X} \frac{d\Sigma(t) f(t)}{t - (x + i\epsilon)}, \quad x \in E, \quad (7.2)$$

for smooth vector–functions $f(t)$. Then there exists $Q > 0$ such that

$$\int_E (\Omega f)^*(x) W(x) (\Omega f)(x) \, dx \leq Q \int_{E \cup X} f^*(t) d\Sigma(t) f(t) \quad (7.3)$$

for all such $f$’s if and only if $W$ belongs to matrix $A_2$, and we have the following Carleson type inequality for any vector function $f \in L^2(W^{-1})$:

$$\sum_{x - k \in X} \langle \Sigma_k f(x_k), f(x_k) \rangle \leq Q \int_E \langle W^{-1} f, f \rangle \, dx. \quad (7.4)$$

Here $f(x_k) := \int_E \frac{f(t) dt}{x_k - t}$.

7.2. On a certain sufficient condition. The following lemmas are related to attempts to rewrite the $A_2$ condition for the spectral density directly in terms of the scattering function.

**Lemma 7.2.** Let

$$W = \begin{bmatrix} 1 & \bar{s}_+ \\ s_+ & 1 \end{bmatrix}. \quad (7.6)$$

The following conditions are equivalent

$$\left\langle W^{-1} P_+ W \begin{bmatrix} f(t) \\ \bar{f}(t) \end{bmatrix}, P_+ W \begin{bmatrix} f(t) \\ \bar{f}(t) \end{bmatrix} \right\rangle \leq Q \left\langle W \begin{bmatrix} f(t) \\ \bar{f}(t) \end{bmatrix}, \begin{bmatrix} f(t) \\ \bar{f}(t) \end{bmatrix} \right\rangle \quad (7.5)$$

for all $f \in L^2_{s_+} \otimes H^2_{s_+}$ and

$$||f^-||^2 \leq Q ||f^-||^2_{s_+}. \quad (7.6)$$

for $f^-(t) \in \hat{H}^2_{s_-}.$

**Lemma 7.3.** If (7.6) holds, then $\hat{H}^2_{s_-} = H^2_{s_-}$, moreover the norm in $\hat{H}^2_{s_-}$ is equivalent to the standard $H^2$-norm.

**Lemma 7.4.** $\hat{H}^2_{s_-} = H^2_{s_-}$ implies $\hat{H}^2_{s_+} = H^2_{s_+}$.

**Proof.**

$$\hat{H}^2_{s_+} = (L^2_{s_-} \otimes H^2_{s_-})^+ = (L^2_{s_-} \otimes \hat{H}^2_{s_-})^+ = L^2_{s_+} \otimes (\hat{H}^2_{s_-})^+ = H^2_{s_+}. \quad \square$$
Nevertheless, we cannot guarantee that the norm in $H^2_{s+}$ is equivalent to the $H^2$-norm. Thus in addition to (7.1) we have to impose the condition
\[ \langle f, f \rangle \leq Q \left\langle W \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix}, f(t) \right\rangle \tag{7.7} \]
for all $f \in H^2$ (this is exactly the condition on equivalence of the norms). Obviously, the last inequality is the same as
\[ \langle WP^+ \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix}, P^+ \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix} \rangle \leq Q \left\langle W \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix}, \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix} \right\rangle, \quad f \in H^2. \tag{7.8} \]
Thus we get

**Theorem 7.5.** The combination of the following two conditions
\[ ((W^{-\frac{1}{2}}P^+W^{-\frac{1}{2}})F, (W^{-\frac{1}{2}}P^+W^{-\frac{1}{2}})F) \leq Q(F, F) \tag{7.9} \]
with $F = W^\frac{1}{2} \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix}$ and $f \in L^2_{s+} \subseteq H^2_{s+}$, and
\[ ((W^\frac{1}{2}P^+W^{-\frac{1}{2}})F, (W^\frac{1}{2}P^+W^{-\frac{1}{2}})F) \leq Q(F, F) \tag{7.10} \]
with $F = W^\frac{1}{2} \begin{bmatrix} f(t) \\ \bar{f}(\bar{t}) \end{bmatrix}$ and $f \in H^2$ is equivalent to the first (or the second) condition from [17], Theorem 3.1.

**Proof.** Relation (7.9) is a slight modification of (7.5) and (7.10) of (7.8), respectively. Observe that if (7.10) holds with $f \in H^2$, then every $f \in H^2_{s+}$ belongs to $H^2$. Therefore, in fact, (7.9) may have a perfect sense for $f \in H^2_{s+}$. \hfill \Box

In any case, $W \in A_2$ is a sufficient condition for (7.9), (7.10). Let us transform this matrix condition into a scalar one.

**Lemma 7.6.** $W$ is in $A_2$ if and only if
\[ \sup_{I} \frac{1}{|I|} \int_I \left| s_+ - \langle s_+ \rangle_I \right|^2 + (1 - |\langle s_+ \rangle_I|^2) \frac{|1 - |s_+|^2|}{1 - |s_+|^2} \, dm < \infty, \tag{7.11} \]
where for an arc $I \subset \mathbb{T}$ we put
\[ \langle s_+ \rangle_I := \frac{1}{|I|} \int_I s_+ \, dm. \tag{7.12} \]

**Proof.** By definition we have that there exists $Q > 0$ such that
\[ \langle W^{-1} \rangle_I \leq Q \langle W \rangle_I^{-1} \tag{7.13} \]
for all $I \subset \mathbb{T}$. Note that
\[ \langle W \rangle_I = \begin{bmatrix} 1 & \langle s_+ \rangle_I \\ \langle s_+ \rangle_I & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \langle s_+ \rangle_I & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 - |\langle s_+ \rangle_I|^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \langle s_+ \rangle_I \\ 0 & 1 \end{bmatrix}. \]
Therefore (7.13) is equivalent to
\[ \begin{bmatrix} 1 & 0 \\ \sqrt{1 - |\langle s_+ \rangle_I|^2} & 1 \end{bmatrix} \langle W^{-1} \rangle_I \begin{bmatrix} 1 & 0 \\ \langle s_+ \rangle_I & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{1 - |\langle s_+ \rangle_I|^2} \end{bmatrix} \leq Q. \]
Since the matrix in the RHS is positive, its boundedness is equivalent to the boundedness of its trace. The last condition with a small effort gives (7.11) and vice versa. \hfill \Box
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