REALIZATIONS OF INNER AUTOMORPHISMS
OF ORDER FOUR AND FIXED POINTS SUBGROUPS
BY THEM ON THE CONNECTED COMPACT
EXCEPTIONAL LIE GROUP $E_8$, PART III

By
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Abstract. The compact simply connected Riemannian 4-symmetric spaces were classified by J.A. Jiménez according to type of the Lie algebras. As homogeneous manifolds, these spaces are of the form $G/H$, where $G$ is a connected compact simple Lie group with an automorphism $\tilde{\gamma}$ of order four on $G$ and $H$ is a fixed points subgroup $G^\gamma$ of $G$. According to the classification by J.A. Jiménez, there exist seven simply connected Riemannian 4-symmetric spaces $G/H$ in the case where $G$ is of type $E_8$. In the present article, we give the explicit form of automorphisms $\tilde{\omega}_4, \tilde{\kappa}_4$ and $\tilde{e}_4$ of order four on $E_8$ induced by the $C$-linear transformations $\omega_4, \kappa_4$ and $e_4$ of the 248-dimensional $C$-vector space $e_8^C$, respectively. Further, we determine the structure of these fixed points subgroups $(E_8)^{\omega_4}, (E_8)^{\kappa_4}$ and $(E_8)^{e_4}$ of $E_8$. These amount to the global realizations of three spaces among seven Riemannian 4-symmetric spaces $G/H$ above corresponding to the Lie algebras $\mathfrak{h} = \mathfrak{su}(2) \oplus i\mathfrak{R} \oplus e_6, i\mathfrak{R} \oplus \mathfrak{so}(14)$ and $\mathfrak{h} = \mathfrak{su}(2) \oplus i\mathfrak{R} \oplus \mathfrak{so}(12)$, where $\mathfrak{h} = \text{Lie}(H)$. With this article, the all realizations of inner automorphisms of order four and fixed points subgroups by them have been completed in $E_8$.

1. Introduction

Let $G$ be a Lie group and $H$ a compact subgroup of $G$. A homogeneous space $G/H$ with $G$-invariant Riemannian metric $g$ is called a Riemannian 4-symmetric space if there exists an automorphism $\tilde{\gamma}$ of order four on $G$ such that $(G^\gamma)_0 \subset H \subset G^\gamma$, where $G^\gamma$ and $(G^\gamma)_0$ are the fixed points subgroup of $G$ by $\tilde{\gamma}$ and its identity component, respectively.

Now, for the exceptional compact Lie group of type $E_8$, as in Table below, there exist seven cases of the compact simply connected Riemannian 4-symmetric spaces which were classified by J.A. Jiménez as mentioned in abstract (2). Accordingly, our interest is to realize the groupification for the classification as Lie algebra.

Our results of groupification corresponding to the Lie algebra $\mathfrak{h}$ in Table are given as follows.

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In [3] and [4], the author has already realized the groupification for Case 1 and Cases 2, 3, 4 in Table, respectively. In the present article, we state the realizations of the group $H$ for Cases 5, 6 and Case 7.

Finally, the author would like to say that the feature of this article is to give elementary proofs of the isomorphism of groups by using the homomorphism theorem, and the all global realizations have been completed in $E_8$.

This article is a continuation of [4], hence we start from Section 7. The readers refer to [3] for preliminary results and also to [3],[4],[5] or [7] for notations. Note that we change the numbering of Case 5 and Case 6 in [3] to the numbering of Case 3 and Case 4 in the present article, respectively.

7. Case 5. The automorphism $\tilde{\omega}_4$ of order four and the group $(E_8)^{\omega_4}$

In this section (also in Section 8 and Section 9), we use the 248-dimensional $C$-vector space $e_8^C$ used in Case 1 ([3]) and the simply connected compact exceptional Lie group of type $E_8$ constructed by T. Imai and I. Yokota ([1]).

We define a $C$-linear transformation $\omega_4$ of $e_8^C$ by

$$\omega_4(\Phi, P, Q, r, s, t) = (i\Phi t^{-1}, -iP, -iQ, r, s, t),$$

where $\iota$ on the right hand side is the $C$-linear transformation of $\Psi^C$ defined in [7] Definition of Subsection 4.10 (p.131)] (the space $\Psi^C$, called the Freudenthal $C$-vector space, is defined in [3] Preliminaries (p.94))). Note that $\omega_4$ is the composition transformation of $\iota, \nu \in E_7 \subset E_8$, where $\nu$ is the $C$-linear transformation of $e_8^C$ defined in [7] Definition of Subsection 5.7(p.174)]. Moreover since $\iota, \nu$ are expressed as elements of $E_8$ by

$$\iota = \exp\left(\frac{2\pi i}{4}\ad(\Phi(0, 0, 0, 3), 0, 0, 0, 0, 0)\right),$$

$$\nu = \exp\left(\frac{2\pi i}{4}\ad(\Phi(0, 0, 0, 6), 0, 0, 0, 0, 0)\right),$$

| Case | $\mathfrak{h}$ | $\tilde{\gamma}$ | $H = G^\gamma$ |
|------|----------------|-----------------|----------------|
| 1    | $\mathfrak{so}(6) \oplus \mathfrak{so}(10)$ | $\bar{\sigma}_4'$ | $(\text{Spin}(6) \times \text{Spin}(10))/\mathbb{Z}_4$ |
| 2    | $i \mathfrak{R} \oplus \mathfrak{su}(8)$ | $\bar{\omega}_4$ | $(U(1) \times SU(8))/\mathbb{Z}_{24}$ |
| 3    | $i \mathfrak{R} \oplus \mathfrak{e}_7$ | $\bar{\nu}_4$ | $(U(1) \times E_7)/\mathbb{Z}_2$ |
| 4    | $\mathfrak{su}(2) \oplus \mathfrak{su}(8)$ | $\bar{\mu}_4$ | $(SU(2) \times SU(8))/\mathbb{Z}_4$ |
| 5    | $\mathfrak{su}(2) \oplus i \mathfrak{R} \oplus \mathfrak{e}_6$ | $\bar{\omega}_4$ | $(SU(2) \times U(1) \times E_6)/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ |
| 6    | $i \mathfrak{R} \oplus \mathfrak{so}(14)$ | $\bar{\kappa}_4$ | $(U(1) \times \text{Spin}(14))/\mathbb{Z}_4$ |
| 7    | $\mathfrak{su}(2) \oplus i \mathfrak{R} \oplus \mathfrak{so}(12)$ | $\bar{\delta}_4$ | $(SU(2) \times U(1) \times \text{Spin}(12))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ |
isomorphism \( P/r.p.c/o.p.c/o.p.c/s.p.c/t.p.c/i.p.c/o.p.c/n.p \)

where the mapping 
\[ \omega_4 = \exp \left( \frac{2\pi i}{4} \text{ad}(\Phi(0,0,0,9),0,0,0,0) \right) \]

Hence it follows from above that \( \omega_4 \in E_8 \) and \( (\omega_4)^4 = 1, (\omega_4)^2 = \nu \), so that \( \omega_4 \) induces the inner automorphism \( \tilde{\omega}_4 \) of order four on \( E_8 \): \( \tilde{\omega}_4(\alpha) = \omega_4 \alpha \omega_4^{-1}, \alpha \in E_8 \).

Now, we will study the subgroup \( (E_8)^{\omega_4} \) of \( E_8 \):
\[ (E_8)^{\omega_4} = \left\{ \alpha \in E_8 \left| \omega_4 \alpha = \alpha \omega_4 \right. \right\} \]

The aim of this section is to determine the structure of the group \( (E_8)^{\omega_4} \). Before that, we prove the lemma and propositions needed later.

**Lemma 7.1.** The Lie algebra \( (e_8)^{\omega_4} \) of the group \( (E_8)^{\omega_4} \) is given by
\[ (e_8)^{\omega_4} = \left\{ \text{ad}(R) \in \text{Der}(e_8) \left| \omega_4 \text{ad}(R) = \text{ad}(R) \omega_4, R \in e_8 \right. \right\} \]
\[ \cong \{ R \in e_8 \mid \omega_4 R = R \} \]
\[ = \{ R = (\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in (e_7)^i \cong u(1) \oplus e_6, r \in iR, s \in C \}, \]
where \( (e_7)^i \) is the Lie algebra of the group \( (E_7)^i \) := \( \{ \alpha \in E_7 \mid \alpha = \alpha t \} \).

In particular, we have \( \dim ((e_8)^{\omega_4}) = (1 + 78) + 1 + 2 = 82 \).

**Proof.** By doing straightforward computation, we can obtain the required results. The isomorphism \( (e_7)^i \cong u(1) \oplus e_6 \) as Lie algebras follows from [7, Theorem 4.10.2]. \( \square \)

**Proposition 7.2.** The group \( (E_8)^{\omega_4} \) contains the group \( (E_7)^i \) which is isomorphic to the group \( (U(1) \times E_6)/Z_3, Z_3 = \{(1, 1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\} \): \( (E_8)^{\omega_4} \supset (E_7)^i \cong (U(1) \times E_6)/Z_3 \), where \( \omega = (-1/2) + (\sqrt{3}/2)i \).

**Proof.** Immediately, we have \( (e_7)^i \subset (e_8)^{\omega_4} \) from Lemma [7.1]. Moreover since both of the groups \( (E_8)^{\omega_4} \) and \( (E_7)^i \) are connected, we have \( (E_7)^i \subset (E_8)^{\omega_4} \). However, we will prove it directly. Let \( \alpha \in (E_7)^i \). Note that \( -1 \in z(E_7) \) (the center of \( E_7 \)), it follows that
\[ \omega_4 \alpha(\Phi, P, Q, r, s, t) = \omega_4(\alpha \Phi \alpha^{-1}, \alpha P, \alpha Q, r, s, t) \]
\[ = (\alpha \Phi \alpha^{-1} t^{-1}, -\alpha P, -\alpha Q, r, s, t) \]
\[ = (\alpha (\Phi t^{-1}) \alpha^{-1}, \alpha (-t P), \alpha (-t Q), r, s, t) \]
\[ = \alpha \omega_4(\Phi, P, Q, r, s, t), (\Phi, P, Q, r, s, t) \in e_8^C, \]
that is, \( \omega_4 \alpha = \alpha \omega_4 \). Hence we have \( \alpha \in (E_8)^{\omega_4} \), so that the first half is proved.

As for the proof of the second half, we define a mapping \( \varphi_i : U(1) \times E_6 \rightarrow (E_7)^i \) by
\[ \varphi_i(\theta, \beta) = \phi(\theta)\beta, \]
where the mapping \( \phi : U(1) = \{ \theta \in C \mid (\tau \theta) \theta = 1 \} \rightarrow E_7 \) is defined by \( \phi(\theta)(X, Y, \xi, \eta) = (\theta^{-1} X, \theta Y, \theta^3 \xi, \theta^{-3} \eta), (X, Y, \xi, \eta) \in \mathbb{P}^C \).

Then the mapping \( \varphi_i \) induces the required isomorphism (see [7, Theorem 4.10.2] in detail). \( \square \)
Let the special unitary group $SU(2) = \{ A \in M(2, C) \mid (\tau A)A = E, \det A = 1 \}$ and we define an embedding $\phi_v : SU(2) \rightarrow E_8$ by

$$\phi_v \left( \begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \tau a & b & 0 & 0 \\ 0 & -\tau b & a & 0 & 0 \\ 0 & 0 & 0 & (\tau a)a - (\tau b)b & (\tau a)(\tau b) \ ab \\ 0 & 0 & 0 & -2(\tau a)b & (\tau a)^2 - b^2 \\ 0 & 0 & 0 & 2a(\tau b) & (\tau b)^2 \ a^2 \end{pmatrix}.$$  

**Proposition 7.3.** [7, Theorem 5.7.4] The group $(E_8)^{\omega_4}$ contains a subgroup

$$\phi_v(SU(2)) = \{ \phi_v(A) \in E_8 \mid A \in SU(2) \}$$

which is isomorphic to the group $SU(2) = \{ A \in M(2, C) \mid (\tau A)A = E, \det A = 1 \}$.

**Proof.** For $A = \begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix} := \exp \left( \begin{pmatrix} -iv & -\tau \varrho \\ \varrho & iv \end{pmatrix} \right) \in SU(2)$, where $\begin{pmatrix} -iv & -\tau \varrho \\ \varrho & iv \end{pmatrix} \in \mathfrak{su}(2)$, we have $\phi_v(A) = \exp(\text{ad}(0,0,0,iv, \varrho, -\tau \varrho)) \in (E_8)^{\omega_4}$ (Lemma 7.1). \hfill $\Box$

Note that there are some errata in definition of $\varphi_3(A)$ of [7, Theorem 5.7.4].

Here we need the following result in the proof of Theorem 7.5 below.

**Theorem 7.4.** ([7, Theorem 5.7.6]) The group $(E_8)^{\nu}$ is isomorphic to the group $(SU(2) \times E_7)/\mathbb{Z}_2, \mathbb{Z}_2 = \{(E, 1), (-E, -1)\}$: $(E_8)^{\nu} \equiv (SU(2) \times E_7)/\mathbb{Z}_2$.

**Proof.** We define a mapping $\varphi : SU(2) \times E_7 \rightarrow (E_8)^{\nu}$ by

$$\varphi(A, \delta) = \phi_v(A)\delta.$$

Then the mapping $\varphi$ induces the required isomorphism. \hfill $\Box$

Now, we determine the structure of the group $(E_8)^{\omega_4}$.

**Theorem 7.5.** The group $(E_8)^{\omega_4}$ is isomorphic to the group $(SU(2) \times U(1) \times E_6)/(\mathbb{Z}_2 \times \mathbb{Z}_3), \mathbb{Z}_2 = \{(E, 1, 1), (-E, -1, 1)\}, \mathbb{Z}_3 = \{(E, 1, 1), (E, \omega, \phi(\omega^2)), (E, \omega^2, \phi(\omega))\}$:

$$(E_8)^{\omega_4} \equiv (SU(2) \times U(1) \times E_6)/(\mathbb{Z}_2 \times \mathbb{Z}_3).$$

**Proof.** We define a mapping $\varphi_{\omega_4} : SU(2) \times U(1) \times E_6 \rightarrow (E_8)^{\omega_4}$ by

$$\varphi_{\omega_4}(A, \theta, \beta) = \phi_v(A)\varphi(\theta, \beta)(= \varphi(A, \varphi(\theta, \beta))),$$

where $\varphi$ is defined in the proof of Theorem 7.4.

First, it is clear that $\varphi_{\omega_4}$ is well-defined from Propositions 7.2, 7.3 and since the mapping $\varphi_{\omega_4}$ is the restriction of the mapping $\varphi : SU(2) \times E_7 \rightarrow (E_8)^{\nu}$, $\varphi_{\omega_4}$ is a homomorphism.

Next, we will prove that $\varphi_{\omega_4}$ is surjective. Let $\alpha \in (E_8)^{\omega_4}$. From $(\omega_4)^2 = v$, we easily see $(E_8)^{\omega_4} \subset (E_8)^{\nu}$. Hence there exist $A \in SU(2)$ and $\delta \in E_7$ such that $\alpha = \varphi(A, \delta)$ (Theorem 7.4). Moreover, from the condition $\omega_4 \alpha \omega_4^{-1} = \alpha$, that is, $\omega_4 \varphi(A, \delta) \omega_4^{-1} = \varphi(A, \delta)$, we have $\varphi(A, i\delta \omega_4^{-1}) = \varphi(A, \delta)$. Indeed, note that the formula $\omega_4 \delta \omega_4^{-1} = i\delta \omega_4^{-1}$...
follows from $\delta \in E_7$, so that it follows from $\omega_4\phi_v(A)\omega_4^{-1} = \phi_v(A)$ (Proposition 7.3) that

$$\omega_4\varphi(A, \delta)\omega_4^{-1} = \omega_4(\phi_v(A)\delta)\omega_4^{-1} = (\omega_4\phi_v(A)\omega_4^{-1})(\omega_4\delta\omega_4^{-1}) = \phi_v(A)(\omega_4\delta\omega_4^{-1}) = \varphi(A, \omega_4)(\omega_4\delta\omega_4^{-1}),$$

that is, $\varphi(A, \omega_4\delta\omega_4^{-1}) = \varphi(A, \delta)$.

Thus, since $\text{Ker}\varphi = \{(E, 1), (-E, -1)\}$ (Theorem 7.4), we have the following

$$\begin{cases} A = A \\ \omega_4\delta\omega_4^{-1} = \delta \end{cases} \quad \text{or} \quad \begin{cases} A = -A \\ \omega_4\delta\omega_4^{-1} = -\delta. \end{cases}$$

In the latter case, this case is impossible because of $A \neq O$, where $O$ is the zero matrix. In the former case, $\delta \in (E_7)^4$ follows from the second condition, so that there exist $\theta \in U(1)$ and $\beta \in E_6$ such that $\delta = \varphi_1(\theta, \beta)$ (Proposition 7.2). Hence there exist $A \in SU(2)$, $\theta \in U(1)$ and $\beta \in E_6$ such that $\alpha = \varphi(A, \varphi_1(\theta, \beta)) = \varphi_\omega(A, \theta, \beta)$. The proof of surjective is completed.

Finally, we will determine $\text{Ker}\varphi_{\omega_4}$. From the definition of kernel, we have

$$\text{Ker}\varphi_{\omega_4} = \{(A, \theta, \beta) \in SU(2) \times U(1) \times E_6 \mid \varphi_{\omega_4}(A, \theta, \beta) = 1\}$$

$$= \{(A, \theta, \beta) \in SU(2) \times U(1) \times E_6 \mid \varphi(A, \varphi_1(\theta, \beta)) = 1\}.$$  

Here, the mapping $\varphi_{\omega_4}$ is the restriction of the mapping $\varphi$ and together with $\text{Ker}\varphi = \{(E, 1), (-E, -1)\}$ (Theorem 7.4), we will find the elements $(A, \theta, \beta) \in SU(2) \times U(1) \times E_6$ satisfying the following

$$\begin{cases} A = E \\ \varphi_1(\theta, \beta) = 1 \end{cases} \quad \text{or} \quad \begin{cases} A = -E \\ \varphi_1(\theta, \beta) = -1. \end{cases}$$

In the former case, from $\text{Ker}\varphi_1 = \{(1, 1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\}$ (Proposition 7.2), we have the following

$$\begin{cases} A = E \\ \theta = 1 \\ \beta = 1, \end{cases} \quad \begin{cases} A = E \\ \theta = \omega \\ \beta = \phi(\omega^2) \end{cases} \quad \text{or} \quad \begin{cases} A = E \\ \theta = \omega^2 \\ \beta = \phi(\omega). \end{cases}$$

In the latter case, the second condition can be rewritten as $\phi(-\theta)\beta = 1$ from $-1 = \phi(-1)$. Hence, as in the former case, we have the following

$$\begin{cases} A = -E \\ \theta = -1 \\ \beta = 1, \end{cases} \quad \begin{cases} A = -E \\ \theta = -\omega \\ \beta = \phi(\omega^2) \end{cases} \quad \text{or} \quad \begin{cases} A = -E \\ \theta = -\omega^2 \\ \beta = \phi(\omega). \end{cases}$$
Indeed, let
\[ \varphi_{\omega_4} = \{ (E, 1, 1), (E, \omega, \phi(\omega^2)), (E, \omega^2, \phi(\omega)) \} \]
\[ = \{(E, 1, 1), (-E, -1, 1)\} \times \{(E, 1, 1), (E, \omega, \phi(\omega^2)), (E, \omega^2, \phi(\omega))\} \]
\[ \cong \mathbb{Z}_2 \times \mathbb{Z}_3. \]

Therefore we have the required isomorphism
\[ (E_8)^{\omega_4} \cong (SU(2) \times U(1) \times E_6)/(\mathbb{Z}_2 \times \mathbb{Z}_3) \]

\[ \square \]

8. Case 6. The automorphism \( \tilde{\kappa}_4 \) of order four and the group \( (E_8)^{\kappa_4} \)

In this section, our aim is to prove main theorem: \( (E_8)^{\kappa_4} \cong (U(1) \times Spin(14))/\mathbb{Z}_4 \).

In [5, Theorem 5.10], the author has proved that the group \( (E_8^C)^{\kappa_3} \) is isomorphic to the group \( (C^* \times Spin(14, C))/\mathbb{Z}_4 \) using the automorphism \( \tilde{\kappa}_3 \) of order three on \( E_8^C \):
\[ (E_8^C)^{\kappa_3} \cong (C^* \times Spin(14, C))/\mathbb{Z}_4, \]
where the group \( (E_8^C)^{\kappa_3} \) is defined as \( (E_8^C)_0 \) in [5, Theorem 5.10] and \( \tilde{\kappa}_3, \kappa_3 \) are defined later. As in the proof of [5, Theorem 5.10], using the automorphism \( \tilde{\kappa}_4 \) of order four on \( E_8^C \), we will first prove that the group \( (E_8^C)^{\kappa_4} \) is isomorphic to the group \( (C^* \times Spin(14, C))/\mathbb{Z}_4 \) in Subsection 8.1:
\[ (E_8^C)^{\kappa_4} \cong (C^* \times Spin(14, C))/\mathbb{Z}_4, \]
and using the result above, the structure of the group \( (E_8)^{\kappa_3} \) will be determined in Subsection 8.2. We often will use the results obtained and definitions in [5, Section 5.3], then note that we change several signs used in [5, Section 5.3]. For example, \( \kappa_3 \) in [5, Section 5.3 (p.37)] is changed to \( \nu_3 \) in this section.

At the end of the preface of this section, since the content of this section is related to the subalgebra \( g_0 \) of the simple graded Lie algebras \( g \) of second kind
\[ g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2, \quad [g_k, g_l] \subset g_{k+l}, \]
we will review about that. In a simple graded Lie algebra \( g \) of second kind, we know that there exists \( Z \in g_0 \), called a characteristic element, such that each \( g_k \) is the \( k \)-eigenspace of \( \text{ad} Z : g \to g \), so that
\[ g_0 = \{ X \in g \mid (\text{ad} Z)X = 0 \}. \]

Here, set \( \tilde{\gamma}_3 := \exp \left( \frac{2\pi i}{3} \text{ad} Z \right) \) as an inner automorphism of order three on \( g \), then we have
\[ g_0 = \{ X \in g \mid \tilde{\gamma}_3 X = X \} =: (g)^{\gamma_3}. \]

Indeed, let \( X \in g_0 \). Then it follows from \( (\text{ad} Z)X = 0 \) that
\[ \tilde{\gamma}_3 X = \left( \exp \left( \frac{2\pi i}{3} \text{ad} Z \right) \right) X = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{2\pi i}{3} \text{ad} Z \right)^n X = X. \]
Hence we have $X \in (g)^{\gamma_3}$. Conversely, let $X \in (g)^{\gamma_3}$. Then we easily see $X \in g_k$ for some $k \in \{-2, -1, 0, 1, 2\}$, so that $(\text{ad} Z)X = kX$ holds. Hence it follows from above that

$$X = \tilde{\gamma}_3 X = \left(\exp\left(\frac{2\pi i}{3} \text{ad} Z\right)\right) X = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2\pi i}{3} \text{ad} Z\right)^n\right) X$$

that $\exp\left(\frac{2\pi i}{3} k\right) = 1$. Thus we have $k = 3n, n \in \mathbb{Z}$, however since $|k| \leq 2$, we see $k = 0$, that is, $X \in g_0$. With above, $g_0 = (g)^{\gamma_3}$ is proved. This result will be useful later.

### 8.1. The group $(E_8^C)^{\kappa_4}$

We define $C$-linear transformations $\kappa_3$ and $\kappa_4$ of $e_8^C$ by

$$\kappa_3(\Phi, P, Q, r, s, t) = (v_3 \Phi v_3^{-1}, \omega^2 v_3 P, \omega v_3 Q, r, \omega s, \omega^2 t),$$

$$\kappa_4(\Phi, P, Q, r, s, t) = (v_4 \Phi v_4^{-1}, -iv_4 P, iv_4 Q, r, -s, -t),$$

where $\omega = (-1/2) + (\sqrt{3}/2)i$ and $v_3, v_4 \in E_7$ on the right hand side are defined by

$$v_3(X, Y, \xi, \eta) = \left(\begin{array}{ccc} \omega^2 \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \omega \xi_2 & \omega x_1 \\ x_2 & \omega x_1 & \omega \xi_3 \end{array}\right),$$

$$v_4(X, Y, \xi, \eta) = \left(\begin{array}{ccc} -i \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & i \xi_2 & i x_1 \\ x_2 & i x_1 & i \xi_3 \end{array}\right).$$

respectively. Then $\kappa_3, \kappa_4$ can be expressed by

$$\kappa_3 = \exp\left(\frac{2\pi i}{3} \text{ad} \kappa\right), \quad \kappa_4 = \exp\left(\frac{2\pi i}{4} \text{ad} \kappa\right),$$

respectively, where $\kappa := (\Phi(-2E_1 \lor E_1, 0, 0, -1), 0, 0, -1, 0, 0) \in e_8^C, E_1 \lor E_1 := (1/3)(2E_1 - E_2 - E_3)^{-1} \in e_6^C$. Besides, $v_3, v_4$ can be also expressed by

$$v_3 = \exp\left(\frac{2\pi i}{3} \Phi(-2E_1 \lor E_1, 0, 0, -1)\right), \quad v_4 = \exp\left(\frac{2\pi i}{4} \Phi(-2E_1 \lor E_1, 0, 0, -1)\right),$$

respectively. Hence it follows from above that $\kappa_3, \kappa_4 \in E_8^C \subset E_8^C$ and $(\kappa_3)^3 = (\kappa_4)^4 = 1$, so that $\kappa_3$ induces the inner automorphism $\tilde{\kappa}_3$ of order three on $E_8$: $\tilde{\kappa}_3(\alpha) = \kappa_3 \alpha \kappa_3^{-1}, \alpha \in E_8$ and $\kappa_4$ induces the inner automorphism $\tilde{\kappa}_4$ of order four on $E_8$: $\tilde{\kappa}_4(\alpha) = \kappa_4 \alpha \kappa_4^{-1}, \alpha \in E_8$, so are on $E_8^C$.

Now, we will study the subgroup $(E_8^C)^{\kappa_4}$ of $E_8^C$:

$$(E_8^C)^{\kappa_4} = \{\alpha \in E_8^C \mid \kappa_4 \alpha = \alpha \kappa_4\}.$$
LEMMA 8.1. The Lie algebra $(\mathfrak{e}_8^C)^{k_4}$ of the group $(E_8^C)^{k_4}$ coincides with the Lie algebra $(\mathfrak{e}_8^C)^{k_3}$ of the group $(E_8^C)^{k_3} = (\mathfrak{e}_8^C)^{k_3}$.

PROOF. By an argument similar to the proof of $g_0 = (\mathfrak{g})^{\gamma_3}$ in the beginning of this section, $g_0 = (\mathfrak{g})^{\gamma_4} := \{X \in \mathfrak{g} \mid \tilde{\gamma}_4X = X\}$ is proved, where $\tilde{\gamma}_4 := \exp\left(\frac{2\pi i}{4}\text{ad } Z\right)$. Hence, by replacing $\mathfrak{g}$ and $\gamma_3, \gamma_4$ with $\mathfrak{e}_8^C$ and $\kappa_3, \kappa_4$, respectively, we obtain $(\mathfrak{e}_8^C)^{k_4} = g_0 = (\mathfrak{e}_8^C)^{k_3}$, where the Lie algebra $g_0$ above is the same one obtained in [7] Theorem 5.7.. □

Now, we determine the structure of the group $(E_8^C)^{k_4}$.

THEOREM 8.2. The group $(E_8^C)^{k_4}$ is isomorphic to the group $(C^* \times \text{Spin}(14, C))/\mathbb{Z}_4$, $\mathbb{Z}_4 = \{(1, 1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}$: $(E_8^C)^{k_4} \cong (C^* \times \text{Spin}(14, C))/\mathbb{Z}_4$.

PROOF. First, we will prove that the group $(E_8^C)^{k_4}$ coincides with the group $(E_8^C)^{k_3}$. Since $E_8^C$ is the simply connected complex Lie group type $E_8$, both of the groups $(E_8^C)^{k_4}$ and $(E_8^C)^{k_3}$ are connected ([3] Preliminaries Lemma 2.2) in [6], moreover together with $(\mathfrak{e}_8^C)^{k_4} = (\mathfrak{e}_8^C)^{k_3}$ (Lemma 8.1), we have $(E_8^C)^{k_4} = (E_8^C)^{k_3}$.

Here, we define a mapping $\varphi : C^* \times \text{Spin}(14, C) \to (E_8^C)^{k_4}$ by the same mapping defined in the proof of [5] Theorem 5.10) as follows:

$$\varphi(a, \beta) = \phi(a)\beta,$$

where $\phi$ is defined in [3] Subsection 5.3 (p.45)] and $\text{Spin}(14, C)$ is constructed in [5] Proposition 5.8.7).

Therefore, from $(E_8^C)^{k_4} = (E_8^C)^{k_3}$, we have the required isomorphism

$$(E_8^C)^{k_4} \cong (C^* \times \text{Spin}(14, C))/\mathbb{Z}_4.$$ 

□

8.2. The group $(E_8)^{k_4}$

In this subsection, as for the construction of $\text{Spin}(14, C)$, we will give a briefly explain based on [5] Section 5.3], and note that several signs in [5] Section 5.3] are changed as mentioned in the beginning of this section. After that, we will construct the group $\text{Spin}(14)$ in $E_8$ and determine the structure of the group $(E_8)^{k_4}$.

Besides, as for the embedding sequence of the exceptional compact Lie groups: $\text{Spin}(8) \subset F_4 \subset E_6 \subset E_7 \subset E_8$, see [7] Theorems 2.7.1, 3.7.1, 4.7.2, 5.7.3 ] in detail.

We define 14-dimensional $C$-vector subspaces $(\mathfrak{e}_8^C)_2, (\mathfrak{e}_8^C)_{-2}$ of $\mathfrak{e}_8^C$ by

$$(\mathfrak{e}_8^C)_2 = \left\{R \in \mathfrak{e}_8^C \mid (\text{ad } R) = 2R\right\}$$

$$= \left\{R = (\Phi, 0, Q, 0, 0, t) \mid \begin{array}{l} \Phi = \Phi(0, 0, \xi_1E_1, 0), \xi_1 \in C, \\ Q = (\xi_2E_2 + \xi_3E_3 + F_1(x_1), \eta_1E_1, 0, \eta), \\ \xi_k, \eta_1, \eta \in C, x_1 \in C^C, \\ t \in C \end{array} \right\},$$
We denote the composition mapping

\[
\begin{aligned}
&\Phi = \Phi(0, v_1 E_1, 0, 0), v_1 \in C, \\
&\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0,
\end{aligned}
\]

\[
\begin{aligned}
P = (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0), \\
\xi_1, \eta_k, \xi \in C, y_1 \in \mathbb{C}^C.
\end{aligned}
\]

where the \(\xi_1\) restricts the mapping

\[
\begin{aligned}
(\mathbb{E}_8^C)_{-2} = \{ R \in \mathbb{E}_8^C \mid (\text{ad}k)R = -2R \}
\end{aligned}
\]

\[
\begin{aligned}
R = (\Phi, P, 0, 0, s, 0) \\
\xi_1, \eta_k, \xi \in C, y_1 \in \mathbb{C}^C,
\end{aligned}
\]

respectively, where \(\kappa = (\Phi(-2E_1 \vee E_1, 0, 0, -1), 0, 0, -1, 0, 0) \in \mathbb{E}_8^C\) is used in previous subsection.

We define a \(C\)-linear mapping \(\zeta : \mathbb{E}_8^C \rightarrow \mathbb{E}_8^C\) by

\[
\begin{aligned}
\zeta(\Phi, P, Q, r, s, t) = (\xi_1 \Phi \xi_1^{-1}, i \xi_1 Q, i \xi_1 P, -r, t, s),
\end{aligned}
\]

where the \(C\)-linear transformation \(\xi_1\) of \(\Psi^C\) on the right hand side is defined by

\[
\begin{aligned}
\zeta_1(X, Y, \xi, \eta) &= \begin{pmatrix}
    i\eta & \chi_3 & \chi_2 \\
    \chi_3 & i\eta_3 & -i\eta_1 \\
    \chi_2 & -i\chi_1 & i\eta
\end{pmatrix}, \\
&\begin{pmatrix}
    i\xi & \eta_3 & \chi_2 \\
    \eta_3 & i\eta_3 & -i\eta_1 \\
    \eta_2 & -i\eta_1 & i\eta
\end{pmatrix}, \eta_1, i\xi_1.
\end{aligned}
\]

In particular, the restriction of the mapping \(\zeta\) to \((\mathbb{E}_8^C)^{-2}\) induces a mapping \((\mathbb{E}_8^C)^{-2} \rightarrow (\mathbb{E}_8^C)_2\), so the explicit form of its restriction mapping is given by

\[
\begin{aligned}
\zeta(\Phi(0, v_1 E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0), 0, 0, s, t) = (\Phi(0, 0, v_1 E_1, 0, 0, 0, Q, 0, 0, -q_1).
\end{aligned}
\]

Note that the restriction of the mapping \(\zeta\) to \((\mathbb{E}_8^C)_2\) is also denoted by the same sign.

Moreover, we define a \(C\)-linear mapping \(\delta : (\mathbb{E}_8^C)_2 \rightarrow (\mathbb{E}_8^C)_2\)

\[
\begin{aligned}
\delta(\Phi(0, 0, v_1 E_1, 0, 0, Q, 0, 0, t) = (\Phi(0, 0, -tE_1, 0, 0, 0, -q_1).
\end{aligned}
\]

We denote the composition mapping \(\delta \zeta : (\mathbb{E}_8^C)^{-2} \rightarrow (\mathbb{E}_8^C)_2\) of \(\zeta\) and \(\delta\) by \(\zeta_\delta\), then the explicit form of the mapping \(\zeta_\delta\) is given by

\[
\begin{aligned}
\zeta_\delta(\Phi(0, v_1 E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0), 0, 0, s, t) = (\Phi(0, 0, -sE_1, 0, 0, 0, -q_1).
\end{aligned}
\]

in addition, the explicit form of the inverse mapping \(\zeta_\delta^{-1} : (\mathbb{E}_8^C)_2 \rightarrow (\mathbb{E}_8^C)^{-2}\) is given by

\[
\begin{aligned}
\zeta_\delta^{-1}(\Phi(0, 0, v_1 E_1, 0, 0, (\xi_2 E_2 + \xi_3 E_3 + F_1(x_1), \eta_1 E_1, 0, \eta), 0, 0, t) = (\Phi(0, -tE_1, 0, 0, (\eta E_1, -xi_2 E_2 - \xi_3 E_3 + F_1(x_1), -\xi_1 E_1, 0, 0, 0, -q_1).
\end{aligned}
\]

Now, we define a subgroup \((G_{14})^C\) of \(\mathbb{E}_8^C\) by

\[
(G_{14})^C := \{ \beta \in \mathbb{E}_8^C \mid (\text{ad}k)\beta = \beta(\text{ad}k), \zeta_\delta \beta R = \beta \zeta_\delta R, R \in (\mathbb{E}_8^C)^{-2} \}.
\]

Then, from [5], Proposition 5.8.7, the group \((G_{14})^C\) is isomorphic to the group \(Spin(14, C)\) as the universal covering group of \(SO(14, C)\):

\[
(G_{14})^C \cong Spin(14, C).
\]
In particular, note that \((G_{14})^C, (e_8^C)_{-2}\) above are denoted by \(G_{14}, (V^C)^{14}\) in [5] Proposition 5.8.7, respectively.

We prove the following lemma needed below and later.

**Lemma 8.3.** The C-linear transformation \(\tau \lambda_\omega\) satisfies the formula \((\text{ad} \kappa) \tau \lambda_\omega = -\tau \lambda_\omega\) \((\text{ad} \kappa)\) and commutes with the C-linear transformation \(\kappa_4; (\tau \lambda_\omega) \kappa_4 = \kappa_4 (\tau \lambda_\omega)\), where \(\tau \lambda_\omega\) is a composition transformation of \(\tau\) and \(\lambda_\omega\) defined in [3] Preliminaries (p.96).

**Proof.** First, we denote \(\Phi(-2E_1 \lor E_1, 0, 0, -1)\) by \(\Phi_\kappa\) only in this lemma: \(\Phi_\kappa := \Phi(-2E_1 \lor E_1, 0, 0, -1)\). Then, note that \(\tau \lambda \Phi_\kappa = -\Phi_\kappa \tau \lambda\), it follows that

\[
(\text{ad} \kappa) \tau \lambda_\omega (\Phi, P, Q, r, s, t) = \text{ad} \kappa (\tau \lambda \Phi \lambda^{-1} \tau, \tau \lambda Q, -\tau \lambda P, -\tau r, -\tau t, -\tau s)
\]

\[
= ([\Phi_\kappa, \tau \lambda \Phi \lambda^{-1} \tau], \Phi_\kappa (\tau \lambda Q) - \tau \lambda Q, \Phi_\kappa (\tau \lambda P) - \tau \lambda P, 0, 2 \tau t, -2 \tau s)
\]

\[
=- (\tau \lambda ([\Phi_\kappa, \Phi] \lambda^{-1} \tau), \tau \lambda (\Phi_\kappa Q + Q), -\tau \lambda (\Phi_\kappa P - P), 0, -2(\tau t), 2(\tau s))
\]

\[
=- \tau \lambda_\omega (\text{ad} \kappa)(\Phi, P, Q, r, s, t), (\Phi, P, Q, r, s, t) \in e_8^C,
\]

that is, \((\text{ad} \kappa) \tau \lambda_\omega = -\tau \lambda_\omega (\text{ad} \kappa).\) The first half is proved.

The second half is easily proved by doing straightforward computation under the definitions of \(\tau \lambda_\omega\) and \(\kappa_4\).

Here, in order to prove the proposition below, we use the following lemma.

**Lemma 8.4.** (1) For \(R \in (e_8^C)_{-2}\), the formula \((\tau \lambda_\omega) \zeta_\delta R = \zeta_\delta^{-1} (\tau \lambda_\omega) R\) holds.

(2) For \(\beta \in \text{Spin}(14, C), \beta\) satisfies the formula \(\zeta_\delta^{-1} \beta R' = \beta \zeta_\delta^{-1} R', R' \in (e_8^C)_2\).

**Proof.** (1) Under the definition of \(\tau \lambda_\omega\) and the mappings \(\zeta_\delta, \zeta_\delta^{-1}\) mentioned above, we do straightforward computation of both sides:

\[
(\tau \lambda_\omega) \zeta_\delta R
\]

\[
= (\tau \lambda_\omega) (\Phi(0, v_1 E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0), 0, 0, s, 0)
\]

\[
= \tau \lambda_\omega (\Phi(0, 0, 0, sE_1, 0), 0, (-\eta_3 E_2 - \eta_2 E_3 + F_1(y_1), -\xi E_1, 0, -\xi), 0, 0, -v_1)
\]

\[
= (\tau \lambda \Phi(0, 0, -sE_1, 0) \lambda^{-1} \tau, \tau \lambda (-\eta_3 E_2 - \eta_2 E_3 + F_1(y_1), -\xi E_1, 0, -\xi), 0, 0, \tau v_1, 0)
\]

\[
= (\Phi(0, \tau s E_1, 0), 0, (-\tau \xi E_1, \tau \eta_3 E_2 + \tau \eta_2 E_3 - F_1(\tau y_1), -\tau \xi), 0, 0, \tau v_1, 0),
\]

\[
\zeta_\delta^{-1} (\tau \lambda_\omega) R
\]

\[
= \zeta_\delta^{-1} (\tau \lambda \Phi(0, 0, 0, 0) \lambda^{-1} \tau, 0, -\tau \lambda (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0), 0, 0, -\tau s)
\]

\[
= \zeta_\delta^{-1} (\Phi(0, 0, -\tau v_1 E_1, 0), 0, (-\tau \eta_2 E_2 - \tau \eta_3 E_3 - F_1(\tau y_1), \tau \xi E_1, 0, \tau \xi), 0, 0, -\tau s)
\]

\[
= (\Phi(0, \tau s E_1, 0, 0), (-\tau \xi E_1, \tau \eta_3 E_2 + \tau \eta_2 E_3 - F_1(\tau y_1), -\tau \xi), 0, 0, \tau v_1, 0).
\]

With above, the required formula is proved.

(2) Let \(\text{Spin}(14, C)\) as the group \((G_{14})^C\). In the formula \(\zeta_\delta R = \beta \zeta_\delta R, R \in (e_8^C)_{-2}\), the required formula is proved by setting \(\zeta_\delta R = R', R' \in (e_8^C)_2\).

\(\square\)
Proposition 8.5. The $C$-linear transformation $\tau \lambda_\omega$ induces the involutive inner automorphism of the group $\text{Spin}(14, C)$: $\tau \lambda_\omega(\beta) = (\tau \lambda_\omega)\beta(\lambda_\omega \tau), \beta \in \text{Spin}(14, C)$.

Proof. Let $\text{Spin}(14, C)$ as the group $(G_{14})^{C}$. We define a mapping $g : (G_{14})^{C} \rightarrow (G_{14})^{C}$ by

$$g(\beta) = (\tau \lambda_\omega)\beta(\lambda_\omega \tau).$$

We will prove $g(\beta) \in (G_{14})^{C}$. First, it follows from the first half of Lemma 8.3 that

$$(\text{ad}_\kappa)g(\beta) = (\text{ad}_\kappa)(\tau \lambda_\omega)\beta(\lambda_\omega \tau)$$

$$= -(\tau \lambda_\omega)(\text{ad}_\kappa)\beta(\lambda_\omega \tau)$$

$$= -(\tau \lambda_\omega)\beta(\text{ad}_\kappa)(\lambda_\omega \tau)$$

$$= -(\tau \lambda_\omega)\beta(-\lambda_\omega \tau(\text{ad}_\kappa))$$

$$= (\tau \lambda_\omega)\beta(\lambda_\omega \tau)(\text{ad}_\kappa)$$

$$= g(\beta)(\text{ad}_\kappa),$$

that is, $(\text{ad}_\kappa)g(\beta) = g(\beta)(\text{ad}_\kappa)$.

Next, as for $\zeta_\delta g(\beta) = g(\beta)\zeta_\delta R, R \in (e_{8}^{C})_{-2}$, it follows from Lemma 8.4 (1), (2) that

$$\zeta_\delta g(\beta) = \zeta_\delta(\tau \lambda_\omega\beta\lambda_\omega \tau)R \quad ((\tau \lambda_\omega\beta\lambda_\omega \tau)R \in (e_{8}^{C})_{-2})$$

$$= (\tau \lambda_\omega)\zeta_\delta^{-1}(\tau \lambda_\omega)(\tau \lambda_\omega\beta\lambda_\omega \tau R)$$

$$= \tau \lambda_\omega\zeta_\delta^{-1}\beta\lambda_\omega \tau R \quad (\lambda_\omega \tau R \in (e_{8})_{2})$$

$$= \tau \lambda_\omega\beta\zeta_\delta^{-1}\lambda_\omega \tau R$$

$$= \tau \lambda_\omega\beta\tau \lambda_\omega \zeta_\delta R$$

$$= (\tau \lambda_\omega\beta\lambda_\omega \tau)\zeta_\delta R$$

$$= g(\beta)\zeta_\delta R,$$

that is, $\zeta_\delta g(\beta) = g(\beta)\zeta_\delta R, R \in (e_{8}^{C})_{-2}$.

Hence we have $g(\beta) \in (G_{14})^{C}$. The proof of this proposition is completed. \qed

From Proposition 8.5, we can define a subgroup $(\text{Spin}(14, C))^{\tau \lambda_\omega}$ of $\text{Spin}(14, C)$:

$$(\text{Spin}(14, C))^{\tau \lambda_\omega} = \{ \beta \in \text{Spin}(14, C) \mid (\tau \lambda_\omega)\beta = \beta(\tau \lambda_\omega) \}.$$

We prove the following lemma needed in the proof of theorem below.

Lemma 8.6. The Lie algebra $(\text{spin}(14, C))^{\tau \lambda_\omega}$ of the group $(\text{Spin}(14, C))^{\tau \lambda_\omega}$ is given by

$$(\text{spin}(14, C))^{\tau \lambda_\omega} = \{ R \in (g_{14})^{C} \mid (\tau \lambda_\omega)R = R \}.$$
Then it follows from

\[
\begin{cases}
\Phi = \Phi(D + \bar{A}(d_1)) + i(t_1E_1 + t_2E_2 + t_3E_3 + F_1(t_1))^{-}, A, -\tau A, \nu, \\
D \in s\theta(8), d_1, t_1 \in C, \\
\tau_k \in R, \tau_1 + \tau_2 + \tau_3 = 0, \\
A = a_2E_2 + a_3E_3 + F_1(a_1) \in \mathbb{H}^C, \nu \in iR, \\
P = (p_2E_2 + p_3E_3 + F_1(p_1), \rho_1E_1, 0, \rho) \in \Psi^C, \\
r \in iR, \\
\end{cases}
\]

where the Lie algebra \((g_{14})^C\) is defined as \(g_{14}\) in \([5, \text{Lemma 5.8.3}].\)

In particular, we have \(\dim((spin(14, C)^{\tau A_\omega}) = (28 + 16 + 20 + 1) + 24 + 1 - 1 = 91.\)

**Proof.** Let \(R = (\Phi(\phi, A, B, \nu), P, Q, r, 0, 0) \in (g_{14})^C,\)

\[
\phi = D + \bar{A}(d_1) + (t_1E_1 + t_2E_2 + t_3E_3 + F_1(t_1))^{-}, D \in s\theta(8),
\]

\[
d_1, t_1 \in C, \tau_k \in C, \tau_1 + \tau_2 + \tau_3 = 0,
\]

\[
A = a_2E_2 + a_3E_3 + F_1(a_1) \in \mathbb{H}^C, a_k \in C, a_1 \in \mathbb{C}^C,
\]

\[
B = b_2E_2 + b_3E_3 + F_1(b_1) \in \mathbb{H}^C, b_k \in C, b_1 \in \mathbb{C}^C,
\]

\[
\nu \in C,
\]

\[
P = (p_2E_2 + p_3E_3 + F_1(p_1), \rho_1E_1, 0, \rho) \in \Psi^C, \rho_k, \rho \in C, p_1 \in \mathbb{C}^C,
\]

\[
Q = (\zeta_1E_1, \zeta_2E_2 + \zeta_3E_3 + F_1(z_1), \zeta_1E_1, \zeta, 0) \in \Psi^C, \zeta_k, \zeta \in C, z_1 \in \mathbb{C}^C,
\]

\[
r \in C,
\]

\[
\tau_1 + (2/3)\nu + 2r = 0.
\]

Then it follows from

\[
(\tau A_\omega)R = (\tau A_\omega)(\Phi(\phi, A, B, \nu), P, Q, r, 0, 0,)
\]

\[
= (\tau \lambda \Phi(\phi, A, B, \nu)\lambda^{-1}\tau, \tau \lambda Q, -\tau \lambda P, -\tau r, 0, 0)
\]

that

\[
\tau \lambda \Phi(\phi, A, B, \nu)\lambda^{-1}\tau = \Phi(\phi, A, B, \nu), \quad \tau \lambda Q = P, \quad -\tau \lambda P = Q, \quad -\tau r = r.
\]

Hence, using the formula \(\tau \lambda \Phi(\phi, A, B, \nu)\lambda^{-1}\tau = \Phi(-\tau^{t}\phi, -\tau B, -\tau A, -\tau \nu),\) the required result is obtained. \(\square\)

From Lemma [8.6], we have

\[
(spin(14, C)^{\tau A_\omega} \subseteq e_8 = \left\{ R = (\Phi, P, -\tau \lambda P, r, s, -\tau s) \mid \Phi \in e_7, P \in \Psi^C, r \in iR, s \in C \right\}.
\]

Now, we construct the spinor group \(Spin(14)\) in \(E_8.\)

**Theorem 8.7.** The group \((Spin(14, C))^{\tau A_\omega}\) is isomorphic to the group \(Spin(14) : (Spin(14, C))^{\tau A_\omega} \cong Spin(14).\)
Proof. First, since the group $\text{Spin}(14, C)$ is the simply connected Lie group, the group $(\text{Spin}(14, C))^{\tau_4 \omega}$ is connected ([3] Preliminaries Lemma 2.2) in ([6]). Hence, since both of the groups $(\text{Spin}(14, C))^{\tau_4 \omega}$ and $E_8$ are connected and $(\text{spin}(14, C))^{\tau_4 \omega} \subset e_8$, we confirm $(\text{Spin}(14, C))^{\tau_4 \omega} \subset E_8$.

We define a 14-dimensional $R$-vector space $V^{14}$ by
\[
V^{14} = \{ R \in (e_8^C)_{-2} \mid (\tau \lambda_\omega) \zeta_\delta R = -R \} = \left\{ R = (\Phi(\Phi, \nu E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\tau v, 0) \right\}
\]
with the norm
\[
(R, R)_\zeta = \frac{1}{30} B_8(\zeta_\delta R, R) = 4(\tau \nu)\nu + (\tau \eta)\eta + y\bar{y} + (\tau \xi)\xi,
\]
where $B_8$ is the Killing form of $e_8^C$ (as for the Killing form $B_8$, see [7] Theorem 5.3.2 in detail). Obviously, the group $(\text{Spin}(14, C))^{\tau_4 \omega}$ acts on $V^{14}$.

Here, let the orthogonal group
\[
O(14) = O(V^{14}) = \{ \alpha \in \text{Iso}_R(V^{14}) \mid (\alpha R, \alpha R) = (R, R) \zeta \}.
\]
We consider the restriction $\alpha \left|_{V^{14}} \right.$ of $\alpha \in (\text{Spin}(14, C))^{\tau_4 \omega}$ to $V^{14}$, then we see $\alpha \left|_{V^{14}} \in O(14) = O(V^{14})$. Indeed, the group $(\text{Spin}(14, C))^{\tau_4 \omega}$ acts on $V^{14}$, so that we have $\alpha \left|_{V^{14}} \in \text{Iso}_R(V^{14})$. Moreover, for $R \in V^{14}$, it follows that
\[
(\alpha \left|_{V^{14}} R, \alpha \left|_{V^{14}} R \right)_\zeta = \frac{1}{30} B_8(\zeta_\delta \alpha R, \alpha R) = \frac{1}{30} B_8(\zeta_\delta R, R) = (R, R) \zeta.
\]
Hence we can define a homomorphism $\pi : (\text{Spin}(14, C))^{\tau_4 \omega} \to O(14) = O(V^{14})$ by $\pi(\alpha) = \alpha \left|_{V^{14}}$.

Moreover, since the mapping $\pi$ is continuous and the group $(\text{Spin}(14, C))^{\tau_4 \omega}$ is connected, $\pi$ induces a homomorphism $\pi : (\text{Spin}(14, C))^{\tau_4 \omega} \to SO(14) = SO(V^{14})$.

We will determine $\text{Ker} \pi$. First, from the definition of kernel, we have
\[
\text{Ker} \pi = \{ \beta \in (\text{Spin}(14, C))^{\tau_4 \omega} \mid \pi(\beta) = 1 \}
= \{ \beta \in (\text{Spin}(14, C))^{\tau_4 \omega} \mid \beta \left|_{V^{14}} = 1 \} (\subset E_8).\]
Then let $\beta \in \text{Ker}\, \pi$. Since it follows from $(\Phi(0, -E_1, 0, 0), 0, 0, 0, 1, 0), i(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \in V^{14}$ that

$$\beta(\Phi(0, -E_1, 0, 0), 0, 0, 0, 1, 0) = (\Phi(0, -E_1, 0, 0), 0, 0, 0, 1, 0),$$
$$\beta i(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0) = i(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0),$$

note that $\beta \in \text{Iso}_c(e^8)$, we have $\beta(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0)$. Here, if $\beta(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0)$, we have $\beta(0, 0, 0, 0, 0, 1) = (0, 0, 0, 0, 0, 1)$. Indeed, it follows from $(\tau, \lambda_\omega)\beta = \beta(\tau, \lambda_\omega)$ that

$$(0, 0, 0, 0, 0, 1) = \tau\lambda_\omega(0, 0, 0, 0, 0, 1) = -\tau\lambda_\omega(0, 0, 0, 0, 1, 0)$$
$$= -\beta(0, 0, 0, 0, 1) = -\beta(0, 0, 0, 0, 0, 1) = \beta(0, 0, 0, 0, 0, 1),$$

that is, $\beta(0, 0, 0, 0, 0, 1) = (0, 0, 0, 0, 0, 1)$. Hence we have $\beta \in E_7 \subset (E_8)_{(0,0,0,0,0,1)}$ (Theorem 5.7.3)).

Moreover, since it follows from $(0, (E_1, 0, 1, 0), 0, 0, 0, 0), i(0, (-E_1, 0, 1, 0), 0, 0, 0, 0) \in V^{14}$ that

$$\beta(0, (E_1, 0, 1, 0), 0, 0, 0, 0) = (0, (E_1, 0, 1, 0), 0, 0, 0, 0)$$
$$\beta i(0, (-E_1, 0, 1, 0), 0, 0, 0, 0) = i(0, (-E_1, 0, 1, 0), 0, 0, 0, 0),$$

again note that $\beta \in \text{Iso}_c(e^8)$, we have $\beta(0, (0, 0, 1, 0), 0, 0, 0, 0) = (0, (0, 0, 1, 0), 0, 0, 0, 0)$ and $\beta(0, (E_1, 0, 0, 0), 0, 0, 0, 0) = (0, (E_1, 0, 0, 0), 0, 0, 0, 0)$, that is, $\beta(0, 0, 1, 0) = (0, 0, 1, 0)$ and $\beta E_1 = E_1$ in $\mathfrak{g}^C$. Hence we have $\beta \in E_6 \subset (E_7)_{(0,0,0,1)}$ (Theorem 4.7.2)) and $\beta$ satisfies $\beta E_1 = E_1$.

In addition, since $(0, (0, E_2 - E_3, 0, 0), 0, 0, 0, 0), (0, (0, i(E_2 + E_3), 0, 0), 0, 0, 0, 0) \in V^{14}$, $\beta$ satisfies $\beta(E_2 - E_3) = E_2 - E_3$ and $\beta(E_2 + E_3) = E_2 + E_3$ in $\mathfrak{g}^C$ by an argument similar to above, together with $\beta E_1 = E_1$ above, we have $\beta E_i = E_i, i = 1, 2, 3$, so we have $\beta \in F_4 \subset (E_6)_E$ (Theorem 3.7.1)), and moreover we see $\beta \in \text{Spin}(8) \subset (F_4)_{E_1,E_2,E_3}$ (Theorem 2.7.1)). Here, we denote $\beta \in \text{Spin}(8)$ by $\beta = (\beta_1, \beta_2, \beta_3) \in SO(8) \times SO(8) \times SO(8)$, then since $\beta_1$ satisfies the condition $\beta_1 y = y$ for all $y \in \mathfrak{c}$, we have $\beta_1 = 1$. Hence, from the Principle of triality on $SO(8)$, we have the following

$$\beta = (1, 1, 1) = 1 \quad \text{or} \quad \beta = (1, -1, -1) =: \sigma.$$

Hence we have $\text{Ker}\, \pi \subset \{1, \sigma\}$ and vice versa, so that $\text{Ker}\, \pi = \{1, \sigma\} \equiv \mathbb{Z}_2$.

Finally, since the group $SO(14)$ is connected and $\text{Ker}\, \pi$ is discrete, together with $\dim((\text{spin}(14, C))^{\tau_4\omega}) = 91 = \dim(\text{so}(14))$ (Lemma 8.6), $\pi$ is surjective. Thus we have the isomorphism $(\text{Spin}(14, C))^{\tau_4\omega}/\mathbb{Z}_2 \cong SO(14)$.

Therefore the group $(\text{Spin}(14, C))^{\tau_4\omega}$ is isomorphic to the group $\text{Spin}(14)$ as the universal covering group of $SO(14)$:

$$(\text{Spin}(14, C))^{\tau_4\omega} \cong \text{Spin}(14).$$
Let the mapping \( \phi : C^* \to E_8^C \) defined in [5] Subsection 5.3 (p.45)]. Then we prove the following lemma needed in the proof of theorem below.

**Lemma 8.8.** For \( a \in C^* \), the action of \( \phi(a) \) on \( E_8^C \) is given by

\[
\phi(a)(\Phi, P, Q, r, s, t) = (\psi(a)\Phi\psi(a)^{-1}, a\psi(a)P, a^{-1}\psi(a)Q, r, a^2s, a^{-2}t),
\]

where the action to \( \mathfrak{g}^C \) of \( \psi(a) \in E_7^C \) on right hand side is defined by

\[
\psi(a)(X, Y, \xi, \eta) = \left( \begin{array}{ccc} a\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & a^{-1}\xi_2 & a^{-1}x_1 \\ x_2 & a^{-1}x_1 & a^{-1}\xi_3 \end{array} \right), \left( \begin{array}{ccc} a^{-1}\eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & a\eta_2 & ay_1 \\ y_2 & ay_1 & a\eta_3 \end{array} \right), a\xi, a^{-1}\eta).
\]

Moreover, we have the following formula

\[
(\tau\lambda_\omega)\phi(a)(\lambda_\omega\tau) = \phi((\tau a)^{-1}).
\]

**Proof.** As for the first half, its proof is proved in [5] Lemma 5.9]. As for the second half, we first have \( (\tau\lambda)\psi(a)(\lambda^{-1}\tau) = \psi((\tau a)^{-1}) \). Indeed, it follows that

\[
(\tau\lambda)\psi(a)(\lambda^{-1}\tau)(X, Y, \xi, \eta) = (\tau\lambda\psi(a)(-\tau Y, \tau X, -\tau\eta, \tau\xi)
\]

\[
= (\tau\lambda)\left( \begin{array}{ccc} -a\tau\eta_1 & -\tau y_3 & -\tau\bar{y}_2 \\ -\tau y_3 & -a^{-1}\tau\eta_2 & -a^{-1}\tau y_1 \\ -\tau y_2 & -a^{-1}\tau\bar{y}_1 & -a^{-1}\tau\eta_3 \end{array} \right), \left( \begin{array}{ccc} a^{-1}\tau\xi_1 & \tau x_3 & \tau\bar{x}_2 \\ \tau x_3 & a\tau\xi_2 & a\tau x_1 \\ \tau x_2 & a\tau\bar{x}_1 & a\tau\xi_3 \end{array} \right), -a\tau\eta, a^{-1}\tau\xi)
\]

\[
= \left( \begin{array}{ccc} (\tau\eta)\xi_1 & \tau x_3 & \tau\bar{x}_2 \\ \tau x_3 & (\tau\xi)\xi_2 & (\tau\eta)x_1 \\ \tau x_2 & (\tau\xi)x_1 & (\tau\xi)\xi_3 \end{array} \right), \left( \begin{array}{ccc} (\tau a\eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & (\tau a\eta_2 & (\tau a^{-1})y_1 \\ y_2 & (\tau a^{-1})\eta_1 & (\tau a^{-1})\eta_3 \end{array} \right), (\tau a^{-1})\xi, (\tau a)\eta
\]

\[
= \psi((\tau a)^{-1}).
\]

Hence, using \( \tau\lambda_\omega(\Phi, P, Q, r, s, t) = (\tau\lambda P^{-1}\tau, \tau\lambda Q, -\tau r P, -\tau r, -\tau t, -\tau s) \) and the formula \( (\tau\lambda)\psi(a)(\lambda^{-1}\tau) = \psi((\tau a)^{-1}) \) shown above, we can obtain the formula \( (\tau\lambda_\omega)\phi(a)(\lambda_\omega\tau) = \phi((\tau a)^{-1}) \) by doing straightforward computation. \( \square \)

Now, we determine the structure of the group \( (E_8)^{\xi_4} \).

**Theorem 8.9.** The group \( (E_8)^{\xi_4} \) is isomorphic to the group \( (U(1) \times Spin(14))/Z_4, Z_4 = \{(1, 1), (-1, -1), (i, \phi(-i)), (-i, \phi(i))\} : (E_8)^{\xi_4} \cong (U(1) \times Spin(14))/Z_4. \)

**Proof.** Let \( U(1) := \{a \in C \mid (\tau a)a = 1\}(\subset C^*) \) and \( Spin(14)(\subset Spin(14, C)) \) as the group \( Spin(14, C)^{\tau\lambda_\omega} \) (Proposition [5.7]). Then we can define a mapping \( \varphi_{\xi_4} : U(1) \times Spin(14) \to (E_8)^{\xi_4} \) by the restriction of the mapping \( \varphi : C^* \times Spin(14, C) \to (E_8^C)^{\xi_4} : \varphi_{\xi_4}(a, \beta) = \varphi(a, \beta) = \varphi(a, \beta) \) (Theorem 8.2).

First, we will prove that \( \varphi_{\xi_4} \) is well-defined. It is clear \( \phi(a) \in (E_8^C)^{\xi_4} \), so that we have \( \phi(a) \in (E_8)^{\xi_4} \). Indeed, since \( a \) satisfies the condition \( (\tau a)a = 1 \), we have \( (\tau\lambda_\omega)\phi(a)(\lambda_\omega\tau) = \phi(a) \) by Lemma 8.8, that is, \( \phi(a) \in ((E_8^C)^{\xi_4})^{\tau\lambda_\omega} \) and moreover it
follows from \((\tau \lambda_\omega)_{\kappa_4} = \kappa_4 (\tau \lambda_\omega)\) (Lemma 8.3) and \((E_8)_{\tau \lambda_\omega} = E_8\) (Preliminaries (p.96)) that
\[
\phi(a) \in ((E_8)_{\tau \lambda_\omega})_{\kappa_4} = ((E_8)_{\tau \lambda_\omega})_{\kappa_4} = (E_8)^{\kappa_4}.
\]

By an argument similar to above, it follows from Lemma 8.3 that
\[
\beta \in Spin(14) = (Spin(14, C))_{\tau \lambda_\omega} \subset ((E_8)_{\tau \lambda_\omega})_{\kappa_4} = ((E_8)_{\tau \lambda_\omega})_{\kappa_4} = (E_8)^{\kappa_4},
\]
that is, \(\beta \in (E_8)^{\kappa_4}\). Hence \(\varphi_{\kappa_4}\) is well-defined. Subsequently, we will prove that \(\varphi_{\kappa_4}\) is a homomorphism, however since \(\varphi_{\kappa_4}\) is the restriction of the mapping \(\varphi\), it is clear.

Next, we will prove that \(\varphi_{\kappa_4}\) is surjective. Let \(\alpha \in (E_8)^{\kappa_4} \subset (E_8)_{\tau \lambda_\omega}\), there exist \(a \in C^+\) and \(\beta \in Spin(14, C)\) such that \(\alpha = \varphi(a, \beta)\) (Theorem 8.2). Moreover, from the condition \((\tau \lambda_\omega)_{\alpha(\lambda_\omega \tau)} = \alpha\), that is, \((\tau \lambda_\omega)\varphi(a, \beta)(\lambda_\omega \tau) = \varphi(a, \beta)\), we have \(\varphi((\tau a)^{-1}, (\tau \lambda_\omega)\beta(\lambda_\omega \tau)) = \varphi(a, \beta)\). Indeed, it follows from Lemma 8.3 that
\[
(\tau \lambda_\omega)\varphi(a, \beta)(\lambda_\omega \tau) = (\tau \lambda_\omega)\phi(a)\beta(\lambda_\omega \tau)
= (\tau \lambda_\omega)\phi(a)(\lambda_\omega \tau)(\tau \lambda_\omega)\beta(\lambda_\omega \tau)
= \phi((\tau a)^{-1})(\tau \lambda_\omega)\beta(\lambda_\omega \tau)
= \varphi((\tau a)^{-1}, (\tau \lambda_\omega)\beta(\lambda_\omega \tau))
\]
that \(\varphi((\tau a)^{-1}, (\tau \lambda_\omega)\beta(\lambda_\omega \tau)) = \varphi(a, \beta)\).

Hence, since \(\text{Ker} \varphi = \{(1, 1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}\), we have the following

\[
\begin{align*}
i (\tau a)^{-1} &= a \\
(\tau \lambda_\omega)\beta(\lambda_\omega \tau) &= \beta,
\end{align*}
\]

\[
\begin{align*}
ii (\tau a)^{-1} &= -a \\
(\tau \lambda_\omega)\beta(\lambda_\omega \tau) &= \phi(-1)\beta,
\end{align*}
\]

\[
\begin{align*}
iii (\tau a)^{-1} &= ia \\
(\tau \lambda_\omega)\beta(\lambda_\omega \tau) &= \phi(-i)\beta,
\end{align*}
\]

\[
\begin{align*}
iv (\tau a)^{-1} &= -ia \\
(\tau \lambda_\omega)\beta(\lambda_\omega \tau) &= \phi(i)\beta.
\end{align*}
\]

Case (i). From \((\tau a)^{-1} = a\), we have \(a \in U(1) = \{a \in C^+ \mid (\tau a)a = 1\}\). From \((\tau \lambda_\omega)\beta(\lambda_\omega \tau) = \beta\), we have \(\beta \in Spin(14)\) (Theorem 8.7). Hence there exist \(a \in U(1)\) and \(\beta \in Spin(14)\) such that \(\alpha = \varphi(a, \beta) = \varphi_{\kappa_4}(a, \beta)\).

Case (ii). From \((\tau a)^{-1} = -a\), we have \((\tau a)a = -1\). However, this case is impossible because of \((\tau a)a > 0\).

Case (iii). From \((\tau a)^{-1} = ia\), we have \((\tau a)a = i\). As in Case (ii), this case is also impossible.

Case (iv). From \((\tau a)^{-1} = -ia\), we have \((\tau a)a = -i\). As in Case (ii), this case is also impossible.

With above, the proof of surjective is completed.

Finally, we will determine \(\text{Ker} \varphi_{\kappa_4}\). Since \(\varphi_{\kappa_4}\) is the restriction of the mapping \(\varphi\), we have \(\text{Ker} \varphi_{\kappa_4} = \text{Ker} \varphi\), that is, \(\text{Ker} \varphi_{\kappa_4} = \{(1, 1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\} \cong \mathbb{Z}_4\).

Therefore we have the required isomorphism
\[
(E_8)^{\kappa_4} \cong (U(1) \times Spin(14))/\mathbb{Z}_4.
\]
9. Case 7. The automorphism \( \tilde{e}_4 \) of order four and the group \( (E_8)^{\epsilon_4} \)

We define a \( C \)-linear transformations \( \epsilon_4 \) of \( e_8^C \) by

\[
\epsilon_4(\Phi, P, Q, r, s, t) = (\nu_4 \Phi \nu_4^{-1}, -\nu_4 P, -\nu_4 Q, r, s, t),
\]

where \( \nu_4 \in E_7 \) on the right hand side is the same one as that defined in previous section. Note that \( \epsilon_4 \) is the composition mapping of \( \nu_4, \nu \in E_7 \subset E_8 \), moreover since \( \nu_4, \nu \) are expressed as elements of \( E_8 \) by

\[
\nu_4 = \exp \left( \frac{2\pi i}{4} \mathrm{ad}(\Phi(-2E_1 \lor E_1, 0, 0, 0, 0)) \right),
\]

\[
\nu = \exp \left( \frac{2\pi i}{4} \mathrm{ad}(\Phi(0, 0, 0, 0, 0)) \right),
\]

respectively and together with \( \{\Phi(-2E_1 \lor E_1, 0, 0, 0, 0), \Phi(0, 0, 0, 0, 0)\} = 0 \), we have

\[
\epsilon_4 = \exp \left( \frac{2\pi i}{4} \mathrm{ad}(\Phi(-2E_1 \lor E_1, 0, 0, 0, 0)) \right).
\]

Hence it follows from above that \( \epsilon_4 \in E_8 \) and \( \epsilon_4^4 = 1 \), so that \( \epsilon_4 \) induces the inner automorphism \( \tilde{e}_4 \) of order four on \( E_8 \): \( \tilde{e}_4(\alpha) = \epsilon_4 \alpha \epsilon_4^{-1}, \alpha \in E_8 \).

Now, we will study the subgroup \( (E_8)^{\epsilon_4} \) of \( E_8 \):

\[
(E_8)^{\epsilon_4} = \{ \alpha \in E_8 \mid \epsilon_4 \alpha = \alpha \epsilon_4 \}.
\]

The aim of this section is to determine the structure of the group \( (E_8)^{\epsilon_4} \). Before that, we make some preparations. First, in order to prove the proposition below, we use the following proposition and theorem.

**Proposition 9.1.** The group \( (E_7)^{\nu_4} \) contains a group

\[
\phi_{U(1)}(U(1)) = \{ \phi_{U(1)}(\theta) \mid \theta \in U(1) \}
\]

which is isomorphic to the group \( U(1) = \{ \theta \in C \mid (\tau \theta) = 1 \} \), where \( \phi_{U(1)} \) is the restriction of the mapping \( \varphi \), defined in [7] Theorem 4.11.13. For \( \theta \in U(1) \), the mapping \( \phi_{U(1)}(\theta) : \Psi^C \to \Psi^C \) is given by

\[
\phi_{U(1)}(\theta)(X, Y, \xi, \eta) = \varphi_{\tau}(\theta, 0)(\xi_1, x_3, \xi_2, x_1, \eta_1, y_3, \eta_2, y_1, \xi, \eta) = \varphi_{\tau}(\theta, \tau \theta)(\xi_1, x_3, \xi_2, x_1, \eta_1, y_3, \eta_2, y_1, \xi, \eta).
\]
Proof. From the definition of \( \phi_{(1)} \), it is clear \( \phi_{(1)} \in E_7 \). Moreover, since \( \nu_4 \) is expressed by \( \phi_{(1)}(-i) : \nu_4 = \phi_{(1)}(-i) \), it is also clear \( \nu_4 \phi_{(1)}(\theta) = \phi_{(1)}(\theta) \nu_4 \). Hence we have \( \phi_{(1)}(\theta) \in (E_7)^{\nu_4} \). \( \Box \)

Theorem 9.2. ([7] Theorem 4.11.15]) The group \((E_7)^{\sigma}\) is isomorphic to the group \((SU(2) \times \text{Spin}(12))/\mathbb{Z}_2, \mathbb{Z}_2 = \{(1, 1), (-1, -\sigma)\} : (E_7)^{\sigma} \cong (SU(2) \times \text{Spin}(12))/\mathbb{Z}_2\).

Proof. We define a mapping \( \varphi : SU(2) \times \text{Spin}(12) \to (E_7)^{\sigma} \) by
\[
\varphi(A, \beta) = \varphi_2(A) \beta,
\]
where \( \varphi_2 \) is defined in [7] Theorem 4.11.13].

Then the mapping \( \varphi \) induces the required isomorphism. As for the \( R \)-linear transformation \( \sigma \) of \( \mathbb{B}^C \), see [7] Subsection 4.11(p.133) in detail. \( \Box \)

Proposition 9.3. The group \((E_8)^{\nu_4}\) contains the group \((E_7)^{\nu_4}\) which is isomorphic to the group \((U(1) \times \text{Spin}(12))/\mathbb{Z}_2, \mathbb{Z}_2 = \{(1, 1), (-1, -\sigma)\} : (E_8)^{\nu_4} \supset (E_7)^{\nu_4} \cong (U(1) \times \text{Spin}(12))/\mathbb{Z}_2\).

Proof. Let \( \alpha \in (E_7)^{\nu_4} \). Note that \(-1 \in \mathfrak{z}(E_7) \) (the center of \( E_7 \)), we have
\[
epsilon_4 \alpha(\Phi, P, Q, r, s, t) = \varepsilon_4(\alpha \Phi \alpha^{-1}, \alpha P, \alpha Q, r, s, t)
\]
\[= (\nu_4 \alpha \Phi \alpha^{-1}, -\nu_4 \alpha P, -\nu_4 \alpha Q, r, s, t)
\]
\[= (\alpha (\nu_4 \Phi \nu_4^{-1}) \alpha^{-1}, \alpha (-\nu_4 P), \alpha (-\nu_4 Q), r, s, t)
\]
\[= \alpha \varepsilon_4(\Phi, P, Q, r, s, t), \quad (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C,
\]
that is, \( \varepsilon_4 \alpha = \alpha \varepsilon_4 \). Hence we have \( \alpha \in (E_8)^{\nu_4} \), so the first half is proved.

Next, we will move the proof of the second half. Let \( U(1) = \{ \theta \in C \mid (\tau \theta) \theta = 1 \} \) and \( \text{Spin}(12) \) be constructed in [7] Theorem 4.11.11]. We define a mapping \( \varphi_{\nu_4} : U(1) \times \text{Spin}(12) \to (E_7)^{\nu_4} \) by
\[
\varphi_{\nu_4}(\theta, \beta) = \phi_{(1)}(\theta) \beta.
\]
Note that \( \varphi_{\nu_4} \) is the restriction of the mapping \( \varphi : SU(2) \times \text{Spin}(12) \to (E_7)^{\sigma} \) defined in the proof of Theorem 9.2.

First, we will prove that \( \varphi_{\nu_4} \) is well-defined. From Proposition 9.1, we have \( \phi_{(1)}(\theta) \in (E_7)^{\nu_4} \). Since \( \varphi_2(A) \) and \( \beta \in \text{Spin}(12) \) are commutative, we have \( \nu_4 \beta = \beta \nu_4 \) because of \( \nu_4 = \phi_{(1)}(-i) = \varphi_2(\text{diag}(-i, i)) \), that is, \( \beta \in (E_7)^{\nu_4} \). Hence \( \varphi_{\nu_4} \) is well-defined. Subsequently, we will prove that \( \varphi_{\nu_4} \) is a homomorphism. However, since the mapping \( \varphi_{\nu_4} \) is the restriction of the mapping \( \varphi \), it is clear.

Next, we will prove that \( \varphi_{\nu_4} \) is surjective. Let \( \alpha \in (E_7)^{\nu_4} \). Then it follows from
\[(\nu_4)^2 = (\phi_{(1)}(-i))^2 = \phi_{(1)}(-1) = -\sigma \text{ that } \alpha \in (E_7)^{-\sigma} = (E_7)^{\sigma} \]. Hence there exist \( A \in SU(2) \) and \( \beta \in \text{Spin}(12) \) such that \( \alpha = \varphi(A, \beta) \) (Theorem 9.2). Moreover from the condition \( \nu_4 \alpha = \alpha \nu_4^{-1} = \alpha, \) that is, \( \nu_4 \varphi(A, \beta) \nu_4^{-1} = \varphi(A, \beta), \) using \( \nu_4 = \phi_{(1)}(-i) \) we have \( \varphi(\frac{a}{c} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \nu_4 \beta \nu_4^{-1}) = \varphi(\frac{a}{c} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta) \) as \( A := \frac{a}{c} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).
Thus, since $\ker \varphi = \{(E, 1), (-E, -1)\}$, we have the following
\[
\begin{align*}
\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} & \quad \text{or} \quad \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & d \end{pmatrix} \\
\nu_4 \beta \nu_4^{-1} = \beta & \quad \text{or} \quad \nu_4 \beta \nu_4^{-1} = -\sigma \beta.
\end{align*}
\]
In the latter case, from $\nu_4 \beta = \beta \nu_4$, we have $\beta = -\sigma \beta$. Hence the latter case is impossible. Indeed, if there exists $\beta \in \Spin(12)$ such that $\beta = -\sigma \beta$. Then apply $\beta^{-1}$ on both side of $\beta = -\sigma \beta$, we have $1 = -\sigma$. This is contradiction.

In the former case. From the first condition $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SU(2)$, we have
\[
A = \begin{pmatrix} a & 0 \\ 0 & \tau a \end{pmatrix}, \quad (\tau a) a = 1, \text{ that is, } a \in U(1), \text{ and it is trivial that } \beta \in \Spin(12). \text{ Thus there exist } \theta \in U(1) \text{ and } \beta \in \Spin(12) \text{ such that } \alpha = \varphi(\text{diag}(\theta, \tau \theta), \beta) = \varphi_{\nu_4}(\theta, \beta).
\]
With above, the proof of surjective is completed.

Finally, we will determine $\ker \varphi_{\nu_4}$. Since $\varphi_{\nu_4}$ is restriction of the mapping $\varphi$, we have $\ker \varphi_{\nu_4} = \ker \varphi$, that is, $\ker \varphi_{\nu_4} = \{(1, 1), (-1, -\sigma)\} \cong \mathbb{Z}_2$.

Therefore we have the required isomorphism
\[
(E_7)^{\nu_4} \cong (U(1) \times \Spin(12))/\mathbb{Z}_2
\]

Lemma 9.4. (1) The Lie algebra $(e_8)^\nu$ of the group $(E_8)^\nu$ is given by
\[
(e_8)^\nu = \{\text{ad}(R) \in \text{Der}(e_8) \mid \nu \text{ad}(R) = \text{ad}(R) \nu\}
\]
\[
\cong \{R \in e_8 \mid \nu R = R\} = \{R = (\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in e_7, r \in iR, s \in C\}.
\]
In particular, we have $\dim((e_8)^\nu) = 133 + 1 + 2 = 136$.

(2) The Lie algebra $(e_8)^{\nu_4}$ of the group $(E_8)^{\nu_4}$ is given by
\[
(e_8)^{\nu_4} = \{\text{ad}(R) \in \text{Der}(e_8) \mid \epsilon_4 \text{ad}(R) = \text{ad}(R) \epsilon_4\}
\]
\[
\cong \{R \in e_8 \mid \epsilon_4 R = R\} = \{R = (\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in (e_7)^{\nu_4} \cong \mathfrak{u}(1) \oplus \mathfrak{so}(12), r \in iR, s \in C\}.
\]
In particular, we have $\dim((e_8)^{\nu_4}) = (1 + 66) + 1 + 2 = 70$.

Proof. By doing straightforward computation, we can prove this lemma. The Lie-isomorphism $(e_7)^{\nu_4} \cong \mathfrak{u}(1) \oplus \mathfrak{so}(12)$ follows from the group isomorphism $(E_7)^{\nu_4} \cong (U(1) \times \Spin(12))/\mathbb{Z}_2$ (Proposition 9.3). □

Proposition 9.5. The group $(E_8)^{\nu_4}$ contains a subgroup
\[
\phi_\nu(SU(2)) = \{\phi_\nu(A) \in E_8 \mid A \in SU(2)\}
\]
which is isomorphic to the group $SU(2) = \{A \in M(2, C) \mid (\tau^t A) A = E, \det A = 1\}$, where $\phi_\nu$ is defined in Proposition 7.3.
Proof. For $A = \begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix} := \exp \left( \begin{pmatrix} -iv & -\tau \varrho \\ \varrho & iv \end{pmatrix} \right) \in SU(2)$, where $\begin{pmatrix} -iv & -\tau \varrho \\ \varrho & iv \end{pmatrix} \in \mathfrak{su}(2)$, we have $\phi_v(A) = \exp(ad(0,0,iv,\varrho,-\tau \varrho)) \in (E_8)^{e_4}$ (Lemma 9.4(2)).

Now, we determine the structure of the group $(E_8)^{e_4}$.

Theorem 9.6. The group $(E_8)^{e_4}$ is isomorphic to the group $(SU(2) \times U(1) \times Spin(12))/(Z_2 \times Z_2)$, where $Z_2 = \{(E,1,1), (E,-1,-\sigma)\}, Z_2 = \{(E,1,1), (-E,1,-1)\}$: $(E_8)^{e_4} \cong (SU(2) \times U(1) \times Spin(12))/(Z_2 \times Z_2)$.

Proof. We define a mapping $\varphi_{e_4} : SU(2) \times U(1) \times Spin(12) \to (E_8)^{e_4}$ by

$$\varphi_{e_4}(A, \theta, \beta) = \phi_v(A)\varphi_{e_4}(\theta, \beta)(= \varphi(A, \varphi_{e_4}(\theta, \beta)))$$

Note that this mapping is the restriction of the mapping $\varphi : SU(2) \times E_7 \to (E_8)^{v_4}$ defined in the proof of Theorem 7.4.

First, we will prove that $\varphi_{e_4}$ is well-defined. However, from Propositions 9.3, 9.5, it is clear that $\varphi_{e_4}$ is well-defined. Subsequently, we will prove that $\varphi_{e_4}$ is a homomorphism. Since the mapping $\varphi_{e_4}$ is the restriction of the mapping $\varphi$ and $\varphi_{e_4}$ is a homomorphism (Proposition 9.3), $\varphi_{e_4}$ is a homomorphism.

Next, we will prove that $\varphi_{e_4}$ is surjective. Let $\alpha \in (E_8)^{e_4}$. Then, since the group $E_8$ is the simply connected Lie group, both of the groups $(E_8)^{e_4}$ and $(E_8)^{v_4}$ are connected ([3 Preliminaries Lemma 2.2]) in [6], together with $(e_8)^{e_4} \subset (e_8)^{v_4}$ (Lemma 9.4 (1), (2)), we have $\alpha \in (E_8)^{e_4} \subset (E_8)^{v_4}$. Hence there exist $A \in SU(2)$ and $\delta \in E_7$ such that $\alpha = \varphi(A, \delta)$ (Theorem 7.4). Moreover, from the condition $e_4\alpha e_4^{-1} = \alpha$, that is, $e_4\varphi(A, \delta)e_4^{-1} = \varphi(A, \delta)$, we have $\varphi(A, v_4\delta v_4^{-1}) = \varphi(A, \delta)$. Indeed, from $e_4(0,0,iv,\varrho,-\tau \varrho) = (0,0,iv,\varrho,-\tau \varrho)$ and $\phi_v(A) = \exp(ad(0,0,iv,\varrho,-\tau \varrho))$, we have $e_4\phi_v(A)e_4^{-1} = \phi_v(A)$ by a computation similar to that in the proof of Theorem 7.5. In addition, $e_4\delta e_4^{-1} = v_4\delta v_4^{-1}$ follows from $\delta \in E_7$. Hence it follows from

$$e_4\varphi(A, \delta)e_4^{-1} = e_4(\phi_v(A)\delta)e_4^{-1} = (e_4\phi_v(A)e_4^{-1})(e_4\delta e_4^{-1}) = \phi_v(A)(v_4\delta v_4^{-1}) = \varphi(A, v_4\delta v_4^{-1})$$

that $\varphi(A, v_4\delta v_4^{-1}) = \varphi(A, \delta)$.

Thus, since $\text{Ker} \varphi = \{(E,1), (-E,1)\}$, we have the following

$$\begin{cases} A = A \\ v_4\delta v_4^{-1} = \delta \end{cases} \quad \text{or} \quad \begin{cases} A = -A \\ v_4\delta v_4^{-1} = -\delta. \end{cases}$$

In the latter case, this case is impossible because of $A \neq O$, where $O$ is the zero matrix. In the former case, $\delta \in (E_7)^{v_4}$ follows from the second condition, so that there exist $\theta \in U(1)$ and $\beta \in Spin(12)$ such that $\delta = \varphi_{e_4}(\theta, \beta)$ (Proposition 9.3). Hence there
exist $A \in SU(2), \theta \in U(1)$ and $\beta \in Spin(12)$ such that $\alpha = \varphi(A, \theta, \varphi_4(\theta, \beta)) = \varphi_{e_8}(A, \theta, \beta)$. The proof of surjective is completed.

Finally, we will determine $\text{Ker}\varphi_{e_8}$. From the definition of kernel, we have

$$\text{Ker}\varphi_{e_8} = \{(A, \theta, \beta) \in SU(2) \times U(1) \times Spin(12) \mid \varphi_4(A, \theta, \beta) = 1\}$$

$$= \{(A, \theta, \beta) \in SU(2) \times U(1) \times Spin(12) \mid \varphi(A, \varphi_4(\theta, \beta)) = 1\}.$$

Here, the mapping $\varphi_4$ is the restriction of the mapping $\varphi$ and together with $\text{Ker}\varphi = \{(E, 1), (-E, -1)\}$ (Theorem 7.3), we will find the elements $(A, \theta, \beta) \in SU(2) \times U(1) \times Spin(12)$ satisfying the following

$$\left\{ \begin{array}{l} A = E \\
\varphi_4(\theta, \beta) = 1 \\
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} A = -E \\
\varphi_4(\theta, \beta) = -1. \\
\end{array} \right.$$

In the former case, from $\text{Ker}\varphi_{e_8} = \{(1, 1), (-1, -\sigma)\}$ (Proposition 9.3), we have the following

$$\left\{ \begin{array}{l} A = E \\
\theta = 1 \\
\beta = 1 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} A = E \\
\theta = -1 \\
\beta = -\sigma. \end{array} \right.$$

In the latter case, the second condition can be rewritten as $\varphi_4(\theta, -\beta) = 1$ from $-1 \in z(E_7)$, moreover $-\beta \in Spin(12)$. Hence, as in the former case, we have the following

$$\left\{ \begin{array}{l} A = -E \\
\theta = 1 \\
\beta = -1 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} A = -E \\
\theta = -1 \\
\beta = \sigma. \end{array} \right.$$

Hence we can obtain

$$\text{Ker}\varphi_{e_8} = \{(E, 1, 1), (E, -1, -\sigma), (-E, 1, -1), (-E, -1, \sigma)\}$$

$$\text{Ker}\varphi_{e_8} = \{(E, 1, 1), (E, -1, -\sigma)\} \times \{(E, 1, 1), (-E, 1, -1)\}$$

$$\approx Z_2 \times Z_2.$$

Therefore we have the required isomorphism

$$(E_8)^{e_8} \cong (SU(2) \times U(1) \times Spin(12))/(Z_2 \times Z_2).$$

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