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Approximate solution fuzzy pantograph equation by using homotopy perturbation method

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Abstract. In this paper, Homotopy Perturbation Method (HPM) is modified and formulated to find the approximate solution for its employment to solve (FDDEs) involving a fuzzy pantograph equation. The solution that can be obtained by using HPM is in the form of infinite series that converge to the actual solution of the FDDE and this is one of the benefits of this method. In addition, it can be used for solving high order fuzzy delay differential equations directly without reduction to a first order system. Moreover, the accuracy of HPM can be detected without needing the exact solution. The HPM is studied for fuzzy initial value problems involving pantograph equation. Using the properties of fuzzy set theory, we reformulate the standard approximate method of HPM and obtain the approximate solutions. The effectiveness of the proposed method is demonstrated for third order fuzzy pantograph equation.

1. Introduction

Several dynamical real life phenomena may be formulated as a mathematical model where they are generally formulated as systems of ordinary or partial differential equations. Delay differential equations (DDEs) are differential equations in which the derivative of the unknown function at a certain time is given in terms of the value of the function at previous times. For example, actuators, sensors and field networks, which are involved in feedback loops, may exhibit delays. Time-delay systems are also used to model several different mechanisms in the dynamics of epidemics [1]. Fuzzy differential equations (FDEs) have been known as a useful tool to model a dynamical system when there is insufficient information about its behavior. FDE appears when the modeling of these problems is imperfect and its nature is under uncertainty or vagueness. FDEs are mathematical models that are suitable for modelling dynamical systems of such nature and they have been applied in modelling population [2] as well as modelling problems in mathematical physics [3] and medicine [4]. Fuzzy delay differential equations have been solved approximately using Adomian Decomposition Method in [5,6].

HPM has been introduced in 1999 [7] and since then the method has been applied to problems in mathematical and physical sciences [8,9]. This method, which is considered as a powerful method, has three important advantages. The first advantage is that the solution obtained from this method is given as a rapidly convergent series whose components are easily computed. The second advantage is that it can be used to solve all types of linear and nonlinear equations. The third advantage is that it is capable of not only reducing the size of computations, but also increasing the accuracy of the approximate solutions. In this paper, our aim is to formulate HPM from crisp domain into fuzzy domain in order to solve nonlinear $n^\text{th}$ order pantograph equation directly. To the best of our knowledge, HPM has never been used to solve the $n^\text{th}$ order pantograph equation and this paper demonstrates the first attempt.
2. Fuzzy Delay Differential Equations

Consider the \(n\)th order FDDE:

\[
\frac{d^n}{dt^n} \tilde{y}(t) = \tilde{f}(t, \tilde{y}(t), \sum_{i=2}^{n-1} \tilde{y}^{(i-1)}(t), \tilde{y}(t), \tilde{g}(t)) \quad t \in [t_0, T], \ t \in [t_0 - \alpha, t_0], r \in [0, 1]
\]

\[
\tilde{y}(t_0) = [a_0], r, \quad \frac{d}{dt} \tilde{y}(t_0) = [a_1], r, \quad \frac{d^2}{dt^2} \tilde{y}(t_0) = [a_2], r, \ldots, \quad \frac{d^{n-1}}{dt^{n-1}} \tilde{y}(t_0) = [a_n], r
\]

where \(\tilde{y}(t)\) and \(\tilde{y}(t)\) are the fuzzy function [10] of the crisp variable \(t\). \(\tilde{f}\) is a fuzzy function of the crisp variable \(t\), the fuzzy variable \(\tilde{y}\) and fuzzy inhomogeneous term \(\tilde{g}(t)\). In Eq. (1) \(\frac{d^n}{dt^n} \tilde{y}(t)\) and \(\sum_{i=2}^{n-1} \tilde{y}^{(i-1)}(t)\) be \(n\)th order fuzzy H-derivatives [1, 12] of \(\tilde{y}(t)\) at \(t \in [t_0, T]\) for \(n \geq 2\). The time delay \(\alpha\) is a known positive rational number with the fuzzy numbers \([a_j]\) for \(j = 0, 1, 2, \ldots, n\) numbers. In order to defuzzify Eq. (1), we denote the fuzzy function \(\tilde{y}\) by \(\bar{y} = [y, \bar{y}]\) for \(t \in [t_0, T]\) and \(r \in [0, 1]\). It means that the \(r\)-level set of \(\tilde{y}(t)\) can be defined as:

\[
[\bar{y}(t)]_r = \left[\begin{array}{c}
y(t) \\bar{y}(t) \\
y(t) \\bar{y}(t)
\end{array}\right], \quad [\bar{y}(t)]_r = \left[\begin{array}{c}
y(t) \\bar{y}(t) \\
y(t) \\bar{y}(t)
\end{array}\right], \quad \bar{y}(n)(t) = \left[\begin{array}{c}
y(n)(t) \\bar{y}(n)(t) \\
y(n)(t) \\bar{y}(n)(t)
\end{array}\right], \quad \bar{y}(n+1)(t) = \left[\begin{array}{c}
y(n+1)(t) \\bar{y}(n+1)(t) \\
y(n+1)(t) \\bar{y}(n+1)(t)
\end{array}\right]
\]

In Eq. (1) we have \(\bar{y}(n)(t) = \tilde{f}(t, \tilde{y}(t), \sum_{i=2}^{n-1} \tilde{y}^{(i-1)}(t), \tilde{y}(t), \tilde{g}(t))\) then we assume that \(\bar{y}(t), \sum_{i=2}^{n-1} \tilde{y}^{(i-1)}(t), \tilde{y}(t), \tilde{g}(t) = \bar{U}(t)\), such that

\[
\begin{aligned}
\{\bar{U}(t; r) &= [U(t; r), \bar{U}(t; r)] \\
[F(t, \bar{U})]_r &= [F(t, \bar{U})]
\end{aligned}
\]

By using Zadeh extension principles [14], we have

\[
\tilde{f}(t, \tilde{y}(t)) = \left[\begin{array}{c}
f(t, \bar{U}(t; r)) \\
f(t, \bar{U}(t; r))
\end{array}\right], \quad \text{such that}
\]

\[
\begin{aligned}
f(t, \bar{U}(t; r)) = \bar{F}(t, \bar{U}(t; r), \bar{U}(t; r)) = \bar{F}(t, \bar{U}(t; r)) \\
\tilde{f}(t, \tilde{y}(t)) = \bar{G}(t, \bar{U}(t; r), \bar{U}(t; r)) = \bar{G}(t, \bar{U}(t; r))
\end{aligned}
\]

Thus we have the following FDDEs

\[
\begin{aligned}
\bar{y}(n; t) &= \bar{F}(t, \bar{U}(t; r)) \\
\bar{y}(t_0; r) &= \bar{g}(t_0; r) = \bar{a}_0; r, \quad \bar{y}(t_0; r) = \bar{a}_0; r \ldots \bar{y}^{(n-1)}(t_0; r) = \bar{a}_n; r
\end{aligned}
\]

where the membership function of \(\bar{F}(t, \bar{U}(t; r))\) and \(\bar{G}(t, \bar{U}(t; r))\) can be defined as

\[
\bar{F}(t, \bar{U}(t; r)) = \min \{\bar{y}(n)(t, \bar{U}(t; r))\} ; \bar{U}(t; r) \in [\bar{U}(t; r)]_r,
\]
\[ G(t, \bar{U}(t; r)) = \max \{ \hat{y}^{(m)}(t, \bar{\mu}(r)); \mu \in [\bar{U}(t; r)]_r \}. \]

3. Analysis of Fuzzy HPM

The general structure of HPM for solving crisp DDEs of the form (1) has been given \[15,16\]. According to Section 2 and Section 3, we write Eqs. (5-6) in the following forms:

\[ \mathcal{L}_n y(t; r) = \mathcal{F} \left( t, \bar{U}(t; r) \right), \quad t \in [t_0, T] \quad (7) \]

\[ y(t_0; r) = a_0(r), y'(t_0; r) = a_0(r), \ldots, y^{(n-1)}(t_0; r) = a_0(r) \quad (8) \]

\[ \mathcal{L}_n \bar{y}(t; r) = \mathcal{G} \left( t, \bar{U}(t; r) \right), \quad t \in [t_0, T] \]

\[ \bar{y}(t_0; r) = \bar{a}_0(r), \bar{y}'(t_0; r) = \bar{a}_0(r), \ldots, \bar{y}^{(n-1)}(t_0; r) = \bar{a}_0(r) \]

where \( \mathcal{L}_n = [\mathcal{L}_n, \mathcal{L}_n] \) are the linear operators with \( \mathcal{L}_n = \frac{d^{(n)}}{dt^{(n)}} \) and \( \mathcal{F}, \mathcal{G} \) are nonlinear operators.

According to HPM \[7\] multiplying both sides of Eqs. (7-8) by inverse operators \[17\] \( \check{\mathcal{L}}^{-1} = \int_0^t \int_0^t \cdots \int_0^t \, d\tau_1 d\tau_2 \ldots d\tau_n \) to obtain the following equations

\[ \check{y}(t; r) = y_0(t; r) + \check{\mathcal{L}}^{-1} \mathcal{F} \left( t, \bar{U}(t; r) \right) \quad (9) \]

\[ \check{\bar{y}}(t; r) = \bar{y}_0(t; r) + \check{\mathcal{L}}^{-1} \mathcal{G} \left( t, \bar{U}(t; r) \right) \quad (10) \]

where \( y_0(t; r) \), and \( \bar{y}_0(t; r) \) are initials guessing that satisfy the initial conditions of Eqs. (9-10) and can be defining as follows:

\[ y_0(t; r) = a_1(r) + a_2(r)t + \frac{a_3(r)}{2!} t^2 + \cdots + \frac{a_n(r)}{(n-1)!} t^{(n-1)} \quad (11) \]

\[ \bar{y}_0(t; r) = \bar{a}_1(r) + \bar{a}_2(r)t + \frac{\bar{a}_3(r)}{2!} t^2 + \cdots + \frac{\bar{a}_n(r)}{(n-1)!} t^{(n-1)} \quad (12) \]

In HPM, the proper choice of the initial guess and the auxiliary linear operator will guarantee the convergence of the HPM solution series \[17\]. According to the HPM \[8\], we construct a homotopy form into Eqs. (9) and (10) which satisfies the following relation

\[ \check{\mathcal{H}}(t, p; r) = (1 - p) \mathcal{L}_n \left[ \check{y}(t; r) - y_0(t; r) \right] + p \left[ \mathcal{L}_n y_0(t; r) - \mathcal{F} \left( t, \bar{U}(t; r) \right) \right] = 0 \quad (13) \]

\[ \check{\mathcal{H}}(t, p; r) = (1 - p) \check{\mathcal{L}}_n \left[ \check{\bar{y}}(t; r) - \bar{y}_0(t; r) \right] + p \left[ \check{\mathcal{L}}_n \bar{y}_0(t; r) - \mathcal{G} \left( t, \bar{U}(t; r) \right) \right] = 0 \quad (14) \]

where \( p \in [0, 1] \) is an embedding parameter such that

\[ \begin{cases} \check{\mathcal{H}}(t, 0; r) = \mathcal{L}_n \left[ \check{y}(t; r) - y_0(t; r) \right] = 0 \\ \check{\mathcal{H}}(t, 1; r) = \mathcal{L}_n y_0(t; r) - \mathcal{F} \left( t, \bar{U}(t; r) \right) = 0 \end{cases} \quad (15) \]

In addition, for the upper bound we have

\[ \begin{cases} \check{\mathcal{H}}(t, 0; r) = \check{\mathcal{L}}_n \left[ \check{\bar{y}}(t; r) - \bar{y}_0(t; r) \right] = 0 \\ \check{\mathcal{H}}(t, 1; r) = \check{\mathcal{L}}_n \bar{y}_0(t; r) - \mathcal{G} \left( t, \bar{U}(t; r) \right) = 0 \end{cases} \quad (16) \]

As in crisp case, this is called deformation. From Eqs. (15) and (16) we can represent each \( y \) in these equations as follows:

\[ \check{y}(t; r) = \sum_{k=0}^{\infty} p^k \check{y}_k(t; r), \check{\bar{y}}(t; r) = \sum_{k=0}^{\infty} p^k \check{\bar{y}}_k(t; r), \ldots, \check{y}^{(n)}(t; r) = \sum_{k=0}^{\infty} p^k \check{y}_k^{(n)}(t; r) \quad (17) \]
Now substituting the above equations into Eq. (15) and (16) and collecting the y terms of the same powers of p, we have:

\[ \begin{align*}
p^1: & \quad L_n \left[ y_1(t;r) + y_0(t;r) \right] - F(t, y_0(t;r), \sum_{i=2}^{n-1} y_0^{(i-1)}(t;r), y_0(t;r)) - \gamma(t;r) = 0 \\
& \quad y_1(t_0;r) = 0, y_1'(t_0;r) = 0, \ldots, y_1^{(n-1)}(t_0;r) = 0 \\
p^2: & \quad L_n \left[ y_2(t;r) - F(t, y_1(t;r), \sum_{i=2}^{n-1} y_1^{(i-1)}(t;r), y_1(t;r)) - \gamma(t;r) = 0 \\
& \quad y_2(t_0;r) = 0, y_2'(t_0;r) = 0, \ldots, y_2^{(n-1)}(t_0;r) = 0 \\
\ldots & \quad \\
p^{n+1}: & \quad L_n \left[ y_{n+1}(t;r) - F(t, y_n(t;r), \sum_{i=2}^{n-1} y_n^{(i-1)}(t;r), y_n(t;r)) - \gamma(t;r) = 0 \\
& \quad y_{n+1}(t_0;r) = 0, y_{n+1}'(t_0;r) = 0, \ldots, y_{n+1}^{(n-1)}(t_0;r) = 0 \\
\end{align*} \]

Similarly for the upper bound

\[ \begin{align*}
p^1: & \quad \bar{L}_n \left[ \bar{y}_1(t;r) + \bar{y}_0(t;r) \right] - G(t, \bar{y}_0(t;r), \sum_{i=2}^{n-1} \bar{y}_0^{(i-1)}(t;r), \bar{y}_0(t;r)) - \bar{\gamma}(t;r) = 0 \\
& \quad \bar{y}_1(t_0;r) = 0, \bar{y}_1'(t_0;r) = 0, \ldots, \bar{y}_1^{(n-1)}(t_0;r) = 0 \\
p^2: & \quad \bar{L}_n \left[ \bar{y}_2(t;r) - G(t, \bar{y}_1(t;r), \sum_{i=2}^{n-1} \bar{y}_1^{(i-1)}(t;r), \bar{y}_1(t;r)) = 0 \\
& \quad \bar{y}_2(t_0;r) = 0, \bar{y}_2'(t_0;r) = 0, \ldots, \bar{y}_2^{(n-1)}(t_0;r) = 0 \\
\ldots & \quad \\
p^{n+1}: & \quad \bar{L}_n \left[ \bar{y}_{n+1}(t;r) - G(t, \bar{y}_n(t;r), \sum_{i=2}^{n-1} \bar{y}_n^{(i-1)}(t;r), \bar{y}_n(t;r)) = 0 \\
& \quad \bar{y}_{n+1}(t_0;r) = 0, \bar{y}_{n+1}'(t_0;r) = 0, \ldots, \bar{y}_{n+1}^{(n-1)}(t_0;r) = 0 \\
\end{align*} \]

for all \( r \in [0,1] \). Then the approximate solution is given by setting \( p=1 \) as follows:

\[ \tilde{y}(t;r) = S_m(t;r) = \sum_{i=0}^{m-1} \tilde{U}_i(t;r) \quad (18) \]

Now the exact solution of Eq. (1), therefore, can be obtained by setting \( p = 1 \) as follows:

\[ \tilde{Y}(t;r) = \lim_{p \to 1} \tilde{y}(t;r) = \lim_{p \to 1} \left[ \sum_{i=0}^{\infty} p^i \sum_{i=0}^{m-1} \tilde{U}_i(t;r) \right] = \sum_{i=0}^{\infty} \tilde{U}_i(t;r). \quad (19) \]

4. Numerical Example

In this section, we employ HPM on a numerical example involving third order fuzzy pantograph equation [18]:

\[ \begin{align*}
y''''(t) &= t\tilde{y}''(t) + \tilde{y}'(t) - \tilde{y} \left( \frac{t}{2} \right) + t\cos(2t) + \cos \left( \frac{t}{2} \right), \quad 0 \leq t \leq 1 \\
y(0) &= [r, 2-r], \tilde{y}'(0) = [r, 3-2r], \tilde{y}''(0) = [r, 5-4r], \forall r \in [0,1] \\
\end{align*} \]

Define the linear operator of Eq. (20) \( \tilde{L}_3 \) with the inverse operator \( \tilde{L}_3^{-1} \). According to section (3) the initial approximation guesses of Eq. (20) is

\[ \tilde{y}_0(t;r) = [r, 2-r] + [r, 3-2r]t + \frac{[r, 5-4r]}{2!}t^2 \quad (21) \]

Since HPM obtain the solution in the form of infinite series we let

\[ f_1 = \cos(2t) = 1 - 2t^2 + \frac{2t^4}{3} - \frac{4t^6}{45} + \frac{2t^8}{315} - \frac{4t^{10}}{14175} \]
According to section 3 the HPM approximate series solution of Eq. (21) is given by

\[ f_2 = \cos \left( \frac{t}{2} \right) = 1 - \frac{t^2}{8} + \frac{t^4}{384} - \frac{t^6}{46080} + \frac{t^8}{10321920} - \frac{t^{10}}{3715891200} \]

Since Eq. (20) is without exact analytical solution, to show the accuracy of 10-order HPM solution \( \tilde{y}(t; r) \), we define the residual absolute error \([19,20]\) as follows

\[
[E]_r = \left| \sum_{i=0}^{10} \tilde{y}_i(t; r) - t \sum_{i=0}^{10} \tilde{y}_i'(t; r) - \sum_{i=0}^{10} \tilde{y}_i(t; r) + \sum_{i=0}^{10} \tilde{y}_i \left( \frac{t}{2}; r \right) - t \cos(2t) - \cos \left( \frac{t}{2} \right) \right|
\]

### Table 1. 10-order HPM of Eq. (20) at \( t = 1 \) for all \( r \in [0,1] \)

| \( r \) | \( y(1; r) \) | \( [E]_r \) | \( \bar{y}(1; r) \) | \( [E]_r \) |
|---|---|---|---|---|
| 0 | 0.2019228857139570 | 8.36629 \times 10^{-6} | 6.746485470231250 | 8.36484 \times 10^{-6} |
| 0.2 | 0.6268292839253067 | 8.36618 \times 10^{-6} | 5.862478631539141 | 8.36502 \times 10^{-6} |
| 0.4 | 1.051736821366561 | 8.36606 \times 10^{-6} | 4.9784729847033 | 8.36520 \times 10^{-6} |
| 0.6 | 1.4766420803480058 | 8.36595 \times 10^{-6} | 4.094466754154923 | 8.36537 \times 10^{-6} |
| 0.8 | 1.9015484785593553 | 8.36584 \times 10^{-6} | 3.210460815462813 | 8.36555 \times 10^{-6} |
| 1 | 2.3264548767707045 | 8.36572 \times 10^{-6} | 2.326454876770704 | 8.36572 \times 10^{-6} |

Figure 1. Accuracy of 10-order HPM solution of Eq. (20) \( \forall r \in [0,1] \) and \( t \in [0,1] \)
From Table 1 and Figures 1 and 2 one can see that the numerical results satisfies the convex symmetric triangular fuzzy number \([11, 12]\) for all \(0 \leq t \leq 1\) and \(0 \leq r \leq 1\).

5. Conclusions
In this research, an approximate-analytical method was applied to obtain an approximate solution \(n^{th}\) order FDDEs. A scheme-based HPM approximate solution of \(n^{th}\) order FDDEs has been formulated to obtain the approximate solution directly has been formulated to obtain the approximate solution directly without reduced into a first order system. From the numerical example, the accuracy of HPM can be determined even though these equations are without the exact analytical solution. A numerical example of third order linear FDDE shows the efficiency of the implemented approximate-analytical method. The results satisfied the fuzzy number properties of the triangular fuzzy number shape.

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