DYNAMIC CHARACTERIZATIONS OF QUASI-ISOMETRY, AND APPLICATIONS TO COHOMOLOGY

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ABSTRACT. We build a bridge between geometric group theory and topological dynamical systems by establishing a dictionary between various notions of quasi-isometry and various notions of continuous orbit equivalence. As an application, we give conceptual explanations for previous results of Shalom and Sauer on quasi-isometry invariance of homological and cohomological dimensions and Shalom’s property $H_{FD}$. As another application, we produce many new quasi-isometry invariants which are given by group homology and cohomology for a class of coefficients including all induced and co-induced modules. This implies that for an arbitrary ring, being a duality group over that ring is a quasi-isometry invariant among all groups which have finite cohomological dimension over that ring. It also follows that vanishing of $\ell^2$-Betti numbers is a quasi-isometry invariant for all countable discrete groups.

1. INTRODUCTION

The philosophy of geometric group theory is to study groups not merely as algebraic objects but from a geometric point of view. There are two ways of developing a geometric perspective, by viewing groups themselves as geometric objects (for instance with the help of their Cayley graphs, which leads to the notion of quasi-isometry) or by studying groups by means of “nice” group actions on spaces which carry some topology or geometry. Once a geometric point of view is taken, an immediate question is: How much of the original algebraic structures is still visible from our new perspective? Or more precisely: Which algebraic invariants of groups are quasi-isometry invariants?

Our goals in this paper are twofold. First, we want to connect the two geometric perspectives mentioned above by giving dynamic characterizations of quasi-isometry (and of related notions). It turns out that for topological dynamical systems, the concept corresponding to quasi-isometry is given by (modified versions of) continuous orbit equivalence, as introduced in [14, 15]. The latter means that we can identify the orbit structure of our dynamical systems in a continuous way. The idea of developing dynamic characterizations of quasi-isometry goes back to Gromov’s notion of topological couplings and has been developed further in [28, 25]. Recently, independently from the author, a dynamic characterization of bilipschitz equivalence for finitely generated groups was obtained in [19], which is a special case of our result.

Secondly, we want to study the behaviour of algebraic invariants of groups under quasi-isometry. More precisely, we consider invariants of (co)homological nature. Using our dynamic characterizations of quasi-isometry, we give conceptual explanations of the results in [28, 25] on quasi-isometry invariance of homological and cohomological dimensions and Shalom’s property $H_{FD}$. Moreover, using a refined, more concrete version of our dynamic characterizations, we produce many new quasi-isometry invariants of (co)homological nature. We generalize the result in [10] that among groups $G$ satisfying the finiteness condition $\mathcal{F}_n$, the cohomology groups $H^n(G, RG)$ are quasi-isometry invariants for all commutative rings $R$ with unit. We show that for a class of coefficients (called res-invariant modules), including all induced and co-induced modules, group homology and cohomology are quasi-isometry invariants. In particular, we obtain that for an arbitrary commutative ring $R$ with unit, the property of being a duality group over $R$ (in the sense of [11]) is a quasi-isometry invariant among all groups which have finite cohomological dimension over $R$. This generalizes [10, Corollary 3]. We also obtain quasi-isometry invariance for reduced group homology and reduced group cohomology. As a consequence, we obtain that vanishing of $\ell^2$-Betti numbers is a quasi-isometry invariant among all finitely generated discrete groups. Actually, we show that this even applies to all countable discrete groups if we replace quasi-isometry by uniform equivalence (which we introduce below). This generalizes the corresponding result

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in [21] which only covered groups of type \( \mathcal{F}_m \). Independently from the author, it was recently shown in [27] that among all compactly generated unimodular locally compact second countable groups, vanishing of \( \ell^2 \)-Betti numbers is a quasi-isometry invariant.

Let us now formulate and explain our main results in more detail. At the same time, we fix some notations.

Throughout this paper, all our groups are countable and discrete. First, we recall the notion of uniform embedding, as explained in [28]. We need the following reformulation, which is clearly equivalent to the definition given in [28].

**Definition 1.1.** A map \( \varphi : G \to H \) between two groups \( G \) and \( H \) is called a uniform embedding if for every subset \( S \subseteq G \times G, \{s^{-1} : (s,t) \in S\} \) is finite if and only if \( \{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\} \) is finite.

Examples of uniform embeddings are subgroup embeddings and quasi-isometric embeddings.

**Definition 1.2.** Two maps \( \varphi, \phi : G \to H \) are called uniformly close if \( \{\varphi(x)\phi(x)^{-1} : x \in G\} \) is finite. We will write \( \varphi \sim \phi \) in that case.

**Definition 1.3.** A uniform embedding \( \varphi \) is called a uniform equivalence if it is uniformly invertible, i.e., there is a uniform embedding \( \psi : H \to G \) such that \( \psi \circ \varphi \sim \text{id}_G \) and \( \varphi \circ \psi \sim \text{id}_H \).

We say that two groups \( G \) and \( H \) are uniformly equivalent if there is a uniform equivalence \( G \to H \).

A bijective uniform embedding is called a uniform bijection.

For finitely generated groups, it is shown in [28] that uniform equivalences coincide with quasi-isometries, and thus, uniform bijections coincide with bilipschitz equivalences. We use the notions of uniform embeddings and uniform equivalences because they are more general and actually more natural in our context. Note that unlike in [28] [25], our definition of uniform embedding is right-invariant (i.e., we use \( st^{-1} \) and not \( s^{-1}t \)). This is because our groups usually act from the left when we consider dynamical systems.

To explain our dynamic characterizations of uniform embeddings and uniform equivalences, we introduce the following weaker notion of continuous orbit equivalence. Let \( G \acts X \) and \( H \acts Y \) be topological dynamical systems, where the groups act by homeomorphisms on locally compact Hausdorff spaces. A continuous orbit couple is a pair of continuous maps \( p : X \to Y \) and \( q : Y \to X \) which both preserve orbits in a continuous way, such that \( p \) and \( q \) are inverses up to orbits (i.e., \( q(p(x)) \) lies in the same \( G \)-orbit of \( x \) and similarly for \( p \circ q \)). “Preserving orbits in a continuous way” is made precise by continuous maps \( a : G \times X \to H \) such that \( p(g,x) = a(g,x).p(x) \) for all \( g \in G \) and \( x \in X \). If \( p \) and \( q \) are actual inverses (i.e., \( q \circ p = \text{id}_X \) and \( p \circ q = \text{id}_Y \)), then our systems are called continuously orbit equivalent.

Our first main result establishes the following dictionary: The existence of a uniform embedding \( G \to H \) corresponds to the existence of a continuous orbit couple for topologically free systems \( G \acts X \) and \( H \acts Y \), where \( X \) is compact. The existence of a uniform equivalence \( G \to H \) corresponds to the existence of continuously orbit equivalent topologically free systems \( G \acts X \) and \( H \acts Y \), where both \( X \) and \( Y \) are compact. We refer to Theorem 2.17 for precise statements.

It turns out that for compact \( X \), the existence of a continuous orbit couple for \( G \acts X \) and \( H \acts Y \) is equivalent to saying that \( G \acts X \) and \( H \acts Y \) are Kakutani equivalent. This means that there are clopen subspaces \( A \subseteq X \) and \( B \subseteq Y \) which are full with respect to the \( G \)- and \( H \)-actions such that the partial actions \( G \acts A \) and \( H \acts B \) are continuously orbit equivalent (in the sense of [15]). This implies that the transformation groupoids of \( G \acts X \) and \( H \acts Y \) are Morita equivalent. Building on this observation, we show that the results in [28] [25] on quasi-isometry invariance (or rather uniform equivalence invariance) of homological and cohomological dimensions and Shalom’s property \( H_{FD} \) are immediate consequences of Morita invariance of various notions of groupoid (co)homology. This gives a conceptual explanation for the results in [28] [25], and at the same time, our work isolates precise conditions on the dynamical systems which are needed to show quasi-isometry invariance.

The dynamic characterizations we described so far are abstract as the dynamical systems are not specified. It is striking that even such abstract characterizations suffice to derive the results in [28] [25]. However, to show quasi-isometry invariance (or rather uniform equivalence invariance) of group homology and cohomology with particular coefficients, we need more concrete versions of our dynamic characterizations. Inspired by [29], we first observe that in place of abstract dynamical systems, we may always take the canonical action \( G \acts \mathcal{B}G \) of
groups $G$ on their Stone-Čech compactifications $\beta G$. The appearance of $G \curvearrowright \beta G$ is not surprising because of its universal property. But now, our crucial observation is that we can go even further and consider the actions $G \curvearrowright G$ of groups acting on themselves by left multiplication. By doing so, it seems that we are losing all the information as any two actions $G \curvearrowright G$ and $H \curvearrowright H$ are continuously orbit equivalent as long as $G$ and $H$ have the same cardinality. The problem is that the spaces on which our groups act are no longer compact. However, we can replace compactness by asking for finiteness conditions on the maps $a$; which -- as in the definition of continuous orbit couples -- make precise that orbits are preserved in a continuous way: We require that for every $g \in G$, the map $a(g,\cdot)$ should have finite image. It is this finiteness condition which singles out "controlled" orbit equivalences which behave well in (co)homology. The point is that every uniform embedding $G \to H$ gives rise to a "controlled" orbit equivalence between $G \curvearrowright G$ and $H \curvearrowright H$. This change of perspective, putting the emphasis on this finiteness condition, already manifests itself in our reformulation of the notion of uniform embedding (Definition 1.1).

These ideas lead to the following results: Let $R$ be a commutative ring with unit and $W$ an $R$-module. The set $C(G,W)$ of functions $G \to W$ carries a natural $RG$-module structure. An $RG$-submodule $L \subseteq C(G,W)$ is called res-invariant if for every $f \in L$ and $A \subseteq G$, the restriction of $f$ to $A$ (viewed as a function on $G$ by extending it by $0$) still lies in $L$. Examples include all the modules appearing in the theorem (Corollary 4.27) below. It now turns out that a uniform equivalence $\phi : G \to H$ induces a one-to-one correspondence between res-invariant submodules of $C(G,W)$ and res-invariant submodules of $C(H,W)$, denoted by $L \mapsto \phi_*L$, together with isomorphisms $H_\ast(\phi) : H_\ast(G,L) \cong H_\ast(H,\phi_*L)$ for all $L$. Similarly, $\phi$ induces a one-to-one correspondence between res-invariant submodules of $C(H,W)$ and res-invariant submodules of $C(G,W)$, say $M \mapsto \phi^*M$, together with isomorphisms $H_\ast(\phi) : H_\ast(H,M) \cong H_\ast(G,\phi^*M)$ for all $M$. In particular, we obtain

**Theorem (Corollary 4.27).** Let $G$ and $H$ be countable discrete groups and $\phi : G \to H$ a uniform equivalence.

(i) For every commutative ring $R$ with unit and every $R$-module $W$, $\phi$ induces isomorphisms

(a) $H_\ast(G,C(G,W)) \cong H_\ast(H, C(H,W))$,

(b) $H^\ast(H,RH \otimes_R W) \cong H^\ast(G, RG \otimes_R W)$.

(ii) Let $R = \mathbb{R}$ or $R = \mathbb{C}$ and $W = R$.

1. For all $0 < p \leq \infty$, $\phi$ induces isomorphisms

\[ H_\ast(G, \ell^p(G,W)) \cong H_\ast(H, \ell^p(H,W)), \quad H^\ast(H, \ell^p(H,W)) \cong H^\ast(G, \ell^p(G,W)), \]

\[ \tilde{H}_\ast(G, \ell^p(G,W)) \cong \tilde{H}_\ast(H, \ell^p(H,W)), \quad \tilde{H}^\ast(H, \ell^p(H,W)) \cong \tilde{H}^\ast(G, \ell^p(G,W)), \]

2. $\phi$ induces isomorphisms

\[ H_\ast(G, c_0(G,W)) \cong H_\ast(H, c_0(H,W)), \quad H^\ast(H, c_0(H,W)) \cong H^\ast(G, c_0(G,W)), \]

\[ \tilde{H}_\ast(G, c_0(G,W)) \cong \tilde{H}_\ast(H, c_0(H,W)), \quad \tilde{H}^\ast(H, c_0(H,W)) \cong \tilde{H}^\ast(G, c_0(G,W)). \]

(iii) Let $G$ and $H$ be a finitely generated discrete groups. Then, for all $s \in \mathbb{R} \cup \{\infty\}$ and $1 \leq p \leq \infty$, $\phi$ induces isomorphisms

\[ H_\ast(G, H^{s,p}(G,W)) \cong H_\ast(H, H^{s,p}(H,W)), \quad H^\ast(H, H^{s,p}(H,W)) \cong H^\ast(G, H^{s,p}(G,W)), \]

\[ \tilde{H}_\ast(G, H^{s,p}(G,W)) \cong \tilde{H}_\ast(H, H^{s,p}(H,W)), \quad \tilde{H}^\ast(H, H^{s,p}(H,W)) \cong \tilde{H}^\ast(G, H^{s,p}(G,W)). \]

Here $c_0(G,W) = \{ f : G \to W : \lim_{x \to 0} |f(x)| = 0 \}$, $H^{s,p}(G,W) = \{ f : G \to W : f \cdot (1 + \ell)^s \in \ell^p(G,W) \}$, where $\ell$ is the word length on $G$, and $H^{s,p}(G,W) = \bigcap_{\ell \in \mathbb{R}} H^{s,p}(G,W)$. Actually, we show that every uniform embedding $\phi : G \to H$ induces a map $H_\ast(\phi) : H_\ast(G,L) \to H_\ast(H,\phi_*L)$ such that $H_\ast(\phi) = H_\ast(\phi)$ if $\phi \sim \phi$ and $H_\ast(\psi \circ \phi) = H_\ast(\psi) \circ H_\ast(\phi)$. It is then evident that uniform equivalences induce isomorphisms as they are precisely those uniform embeddings which are invertible modulo $\sim$. A similar remark applies to cohomology. Thus, not only these (co)homology groups are quasi-isometry invariants. By functoriality, the actions of the groups of quasi-isometries (modulo $\sim$) on these (co)homology groups are quasi-isometry invariants.

We obtain analogous results in the topological setting, i.e., for topological res-invariant modules and reduced (co)homology.

The results we mentioned above on quasi-invariance of being a duality group and of vanishing of $\ell^2$-Betti numbers are immediate consequences. We also obtain a new proof for quasi-isometry invariance of homological and cohomological dimensions (over arbitrary rings) as long as they are finite.
We refer to Theorem 4.22, Corollary 4.27 and §4 for precise statements and more details. §3 and §4 are independent from each other. Thus readers interested in this last set of results on quasi-isometry invariance of group (co)homology with res-invariant modules as coefficients may go directly from §2 to §4.

As far as our methods are concerned, we use groupoid techniques as in [28, 25, 20], but instead of working with abstract dynamical systems, we base our work on concrete dynamic characterizations of quasi-isometry. The difference between our work and [10] is that we do not work with descriptions of group (co)homology in terms of Eilenberg-MacLane spaces, as these descriptions require finiteness conditions (like $\mathcal{F}_n$ or $\mathcal{F}_\infty$) on our groups and have to be modified whenever we change coefficients. Instead, the finiteness condition leading to “controlled” orbit equivalences allows us to work with complexes coming from bar resolutions.

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2. DYNAMICAL CHARACTERIZATIONS OF QUASI-ISOMETRY

2.1. Preliminaries. The central notions of uniform embedding and uniform equivalence have been introduced in §1. We remark that, as in the case of quasi-isometries, it is easy to see that a uniform embedding $\varphi : G \to H$ is uniformly invertible if and only if $H$ can be covered by finitely many translates of $\varphi(G)$, i.e., there is a finite set $F \subseteq H$ such that $H = \bigcup_{h \in F} \varphi(hG)$. Moreover, the (set-theoretical) inverse of a uniform bijection is again a uniform bijection.

Note that in [28] and [25], uniform equivalences are called quasi-isometries and uniform bijections are called bilipschitz equivalences even for general countable groups which might not be finitely generated, i.e., where we might not have word metrics.

Remark 2.1. Note that unlike in [28], our definition of uniform embeddings is right-invariant, not left-invariant (i.e., we use $st^{-1}$ instead of $s^{-1}t$). For finitely generated groups, this amounts to considering right-invariant word lengths and word metrics. We do so because in the following, we will consider left actions of groups, in particular the action of a group by left multiplication on itself. Of course, this is merely a matter of convention.

The following concept, due to Gromov, builds a bridge between geometric group theory and topological dynamical systems.

Definition 2.2. For two groups $G$ and $H$, a $(G, H)$ topological coupling consists of a locally compact space $\Omega$ with commuting free and proper left $G$- and right $H$-actions which admit clopen $H$- and $G$-fundamental domains $X$ and $Y$.

Our $(G, H)$ topological coupling is called $G$-cocompact if $Y$ is compact, $H$-cocompact if $X$ is compact, and cocompact if it is both $G$- and $H$-cocompact. It is called topologically free (or free) if the combined action $G \times H \curvearrowright \Omega$ is topologically free (or free).

All our spaces are Hausdorff. Moreover, as we are only concerned with the topological setting, we sometimes simply call a $(G, H)$ topological coupling a $(G, H)$ coupling. We often write $G \curvearrowright \Omega_X \curvearrowright H$ to keep track of all the relevant data.

The following result goes back to ideas of Gromov and is proven in [28] and [25].

Theorem 2.3. Let $G$ and $H$ be countable discrete groups.

(i) There exists a uniform embedding $G \to H$ if and only if there exists a $H$-cocompact $(G, H)$ coupling.

(ii) There exists a uniform equivalence $G \to H$ if and only if there exists a cocompact $(G, H)$ coupling.

(iii) There is a uniform bijection $G \to H$ if and only if there is a cocompact $(G, H)$ coupling $G \curvearrowright \Omega_X \curvearrowright H$ with $X = Y$.

Proof. For (i), see [25, Theorem 2.2, (i) $\iff$ (ii)]. For (ii), see [25, Theorem 2.2, (iii) $\iff$ (iv)]. For (iii), see [28, Remark after Theorem 2.1.2].

Remark 2.4. The proofs in [25, 28] show that the underlying space $\Omega$ of the $(G, H)$ couplings can be chosen to be second countable and totally disconnected in the above statements.
Let us now isolate an idea from [19] which will be useful later on.

**Lemma 2.5.** If there exists a \((G, H)\) coupling \(G \bowtie \Omega \bowtie H\), then there exists a topologically free \((G, H)\) coupling \(G \bowtie \Omega' \bowtie H\). If \(G \bowtie \Omega \bowtie H\) is \(G\)-cocompact, \(H\)-cocompact or cocompact, \(G \bowtie \Omega' \bowtie H\) may be chosen with the same property. If \(\Omega\) is second countable and totally disconnected, we may choose \(\Omega'\) with the same property.

**Proof.** The idea of the proof appears in the proof of [19, Theorem 3.2]. Let \(G \times H \bowtie Z\) be a free action on the Cantor space \(Z\). It is easy to see that \(\Omega' = \Omega \times Z\) with diagonal \(G\) and \(H\)-actions is a \((G, H)\) coupling which is topologically free (even free). As \(Z\) is compact and totally disconnected, our additional claims follow. \(\square\)

### 2.2. Topological couplings and continuous orbit couples

We explain the connection between topological couplings and continuous orbit couples. First of all, a topological dynamical system \(G \bowtie X\) consists of a group \(G\) acting on a locally compact space \(X\) via homeomorphisms. We write \(g.x\) for the action.

**Definition 2.6.** Let \(G \bowtie X\) and \(H \bowtie Y\) be topological dynamical systems.

A continuous map \(p : X \to Y\) is called a continuous orbit map if there exists a continuous map \(a : G \times X \to H\) such that \(p(x) = a(g.x, x)\) for all \(g \in G\) and \(x \in X\).

A continuous orbit couple for \(G \bowtie X\) and \(H \bowtie Y\) consists of continuous orbit maps \(p : X \to Y\) and \(q : Y \to X\) such that there exist continuous maps \(g : X \to G\) and \(h : Y \to H\) such that \(q(p(x)) = g(x).x\) and \(p(q(y)) = h(y).y\) for all \(x \in X\) and \(y \in Y\).

**Definition 2.7.** A \((G, H)\) continuous orbit couple consists of topological dynamical systems \(G \bowtie X\) and \(H \bowtie Y\) and a continuous orbit couple for \(G \bowtie X\) and \(H \bowtie Y\). If \(G \bowtie X\) and \(H \bowtie Y\) are topologically free, then the \((G, H)\) continuous orbit couple is called topologically free. We call \(X\) the \(G\)-space and \(Y\) the \(H\)-space of our \((G, H)\) continuous orbit couple.

**Remark 2.8.** In this language, a continuous orbit equivalence for topological dynamical systems \(G \bowtie X\) and \(H \bowtie Y\) in the sense of [14] is the same as a continuous orbit couple for \(G \bowtie X\) and \(H \bowtie Y\) with \(g \equiv e\) and \(h \equiv e\), i.e., \(p = q^{-1}\).

**Definition 2.9.** A \((G, H)\) continuous orbit equivalence consists of topological dynamical systems \(G \bowtie X\) and \(H \bowtie Y\) and a continuous orbit equivalence for \(G \bowtie X\) and \(H \bowtie Y\).

**Theorem 2.10.** Let \(G\) and \(H\) be groups. There is a one-to-one correspondence between isomorphism classes of topologically free \((G, H)\) couplings and isomorphism classes of topologically free \((G, H)\) continuous orbit couples, with the following additional properties:

(i) A \((G, H)\) coupling \(G \bowtie \Omega \bowtie H\) corresponds to a \((G, H)\) continuous orbit couple with \(G\)-space homeomorphic to \(\tilde{X}\) and \(H\)-space homeomorphic to \(\tilde{Y}\).

(ii) A \((G, H)\) coupling \(G \bowtie \Omega \bowtie H\) with \(\tilde{X} = \tilde{Y}\) corresponds to a \((G, H)\) continuous orbit equivalence.

Here, the notions of isomorphisms are the obvious ones: Topological couplings \(G \bowtie \Omega_1 \bowtie H\) and \(G \bowtie \Omega_2 \bowtie H\) are isomorphic if there exists a \(G \times H\)-equivariant homeomorphism \(\Omega_1 \xrightarrow{\cong} \Omega_2\) sending \(\tilde{X}_1\) to \(\tilde{X}_2\) and \(\tilde{Y}_1\) to \(\tilde{Y}_2\). Continuous orbit couples \((p_i, q_i)\) for \(G \bowtie X_i\) and \(H \bowtie Y_i\), \(i = 1, 2\), are isomorphic if there exist \(G\)- and \(H\)-equivariant homeomorphisms \(X_1 \xrightarrow{\cong} X_2\) and \(Y_1 \xrightarrow{\cong} Y_2\) such that we obtain commutative diagrams

\[
\begin{array}{ccc}
X_1 & \xrightarrow{p_1} & Y_1 \\
\downarrow & & \downarrow \\
X_2 & \xrightarrow{p_2} & Y_2 \\
\end{array}
\quad\quad\quad
\begin{array}{ccc}
Y_1 & \xrightarrow{q_1} & X_1 \\
\downarrow & & \downarrow \\
Y_2 & \xrightarrow{q_2} & X_2 \\
\end{array}
\]

For the proof of Theorem 2.10 we now present explicit constructions of continuous orbit couples out of topological couplings and vice versa. The constructions are really the topological analogues of those in [9, § 3] (see also [28, 25]). In the following, we write \(g.x\) (\(g \in G, x \in \Omega\)) and \(xh\) (\(x \in \Omega, h \in H\)) for the left \(G\)- and right \(H\)-actions in topological couplings, and \(g.x, h.y\) for the actions \(G \bowtie X, H \bowtie Y\) from continuous orbit couples.

### 2.2.1. From topological couplings to continuous orbit couples

Let \(G \bowtie \Omega \bowtie H\) be a \((G, H)\) coupling. Set \(X := \tilde{X}\) and \(Y := \tilde{Y}\). Define a map \(p : X \to Y\) by requiring \(Gx \cap Y = \{p(x)\}\) for all \(x \in X\). The intersection \(Gx \cap Y\), taken in \(\Omega\), consists of exactly one point because \(\gamma\) is a \(G\)-fundamental domain. By construction, there is a map \(\gamma : X \to G\) such that \(p(x) = \gamma(x).x\). For \(g \in G\), \(\gamma\) takes the constant value \(g\) on \(X \cap g^{-1}Y\). As \(X \cap g^{-1}Y\) is clopen, because \(X\) and \(Y\) are, \(\gamma\) is continuous. \(p\) is continuous as it is so on \(X \cap g^{-1}Y\) for all \(g \in G\).
We now define a $G$-action, denoted by $G \times X \to X$, $(g, x) \mapsto g.x$, as follows: For every $g \in G$ and $x \in X$, there exists a unique $\alpha(g.x) \in H$ such that $g x = \alpha(g.x) x$. For fixed $g \in G$ and $h \in H$, we have $\alpha(g.x) = h$ for all $x \in X \cap g^{-1}X h$. As $X \cap g^{-1}X h$ is clopen because $X$ is, $\alpha : G \times X \to H$ is continuous. Set $g.x := g x \alpha(g.x)^{-1}$. It is easy to check that $\alpha$ satisfies the cocycle identity $\alpha(g_1 g_2.x) = \alpha(g_1, g_2.x) \alpha(g_2, x)$. Using this, it is easy to see that $G \times X \to X$, $(g, x) \mapsto g.x$ defines a (left) $G$-action on $X$ by homeomorphisms.

Similarly, we define a continuous map $q : Y \to X$ by requiring $X \cap \gamma H = \{q(y)\}$ for all $y \in Y$, and let $\eta : Y \to H$ be the continuous map satisfying $q(y) = \eta y(y)$. To define an $H$-action on $Y$, let $\beta(y, h) \in G$ be such that $y h \in \beta(y, h) Y$. Again, $\beta : Y \times H \to G$ is continuous. Set $h, y := \beta(y, h^{-1}) y h^{-1}$. It is easy to check that $\beta$ satisfies $\beta(y, h_1 h_2) = \beta(y, h_1) \beta(h_1^{-1} \cdot h_2)$. Using this, it is again easy to see that $H \times Y \to Y$, $(h, y) \mapsto h, y$ defines an $H$-action on $Y$ by homeomorphisms.

Let us check that $(p, q)$ is a $(G, H)$ continuous orbit couple. We need to identify $G g x \alpha(g.x)^{-1} \cap Y$ in order to determine $p(g.x) = p(g x \alpha(g.x)^{-1})$. We have

$$G g x \alpha(g.x)^{-1} \ni \beta(y(x), x, \alpha(g.x)^{-1})^{-1} y(x) x \alpha(g.x)^{-1} \in Y,$$

so $p(g.x) = \beta(y(x), x, \alpha(g.x)^{-1})^{-1} y(x) x \alpha(g.x)^{-1} = \alpha(g.x, y(x)). y(x) x = \alpha(g.x, p(x)).$ Similarly, in order to identify $q(h, y) = q(\beta(y, h^{-1}) y h^{-1})$, we need to determine $X \cap \beta(y, h^{-1}) y h^{-1} H$. As

$$X \ni \beta(y, h^{-1}) y h^{-1} \eta(y) \alpha(\beta(y, h^{-1})^{-1} \cdot y h^{-1}, \eta(y))^{-1} \in \beta(y, h^{-1}) y h^{-1} H,$$

we deduce $q(h, y) = \beta(y, h^{-1})^{-1} \eta(y) \alpha(\beta(y, h^{-1})^{-1} \cdot y h^{-1}, \eta(y))^{-1} = \beta(y, h^{-1})^{-1} \eta(y))^{-1} \cdot y h^{-1}, \eta(y)$. Finally, $q(p(x)) = q(y(x)) = y(x) x \alpha(g.x, x)^{-1} = y(x) x$ and $p(q(y)) = p(y h^{-1}) \eta(y))^{-1} \cdot y h^{-1}, \eta(y) = \eta(y)^{-1} \cdot y h^{-1}, \eta(y)$. All in all, we see that $p$ and $q$ give rise to a continuous orbit couple for $G \times X$ and $H \times Y$, with $g(x) = y(x)$ and $h(y) = \eta(y)^{-1}$.

Note that our coupling does not need to be topologically free for this construction. However, it is clear that $G \times \Omega \times H$ is topologically free (i.e., $G \times H \times \Omega$ is topologically free) if and only if $G \times H$ and $\Omega \times H$ are topologically free.

**Remark 2.11.** Our notation differs slightly from the one in [28] and [25]. Our $\alpha(g.x)$ is $\alpha(g^{-1}, x)^{-1}$ in [28] § 2.2, Equation (3) and [25] § 2.2, Equation (2.2)]. This is closely related to Remark 2.1.

**Remark 2.12.** The dynamical system $G \times X$ we constructed above can be canonically identified with $G \times \Omega \times H$. Similarly, our system $H \times Y$ can be identified with $G \times \Omega \times H$ in a canonical way.

### 2.2.2. From continuous orbit couples to topological couplings

Let $G \times X$ and $H \times Y$ be topologically free systems on locally compact spaces $X$ and $Y$. Assume that we are given a continuous orbit couple for $G \times X$ and $H \times Y$, and let $p, q, a, g$ and $h$ be as in Definition 2.6 and let $b : H \times Y \to G$ be a continuous map with $q(h, y) = b(h, y), q(y)$ for all $y \in H$ and $y \in Y$. Define commuting left $G$- and right $H$-actions on $X \times H$ by $g(x, h) = (g, x, a(g, x) h), (x, h) h' = (x, h')$. Furthermore, define commuting left $G$- and right $H$-actions on $G \times Y$ by $g'(g, y) = (g, g' y)$ and $(g, y) h = (g b(h^{-1}, y)^{-1}, h^{-1}, y)$.

A straightforward computation, using the cocycle identities ([14 Lemma 2.8]) for $a$ and $b$, shows that $\Theta : X \times H \to G \times Y$, $(x, h) \mapsto (g(x, h)^{-1} b(h^{-1}, p(x))^{-1} h, p(x))$ is a $G$- and $H$-equivariant homeomorphism with inverse $\Theta^{-1} : G \times Y \to X \times H$, $(g, y) \mapsto (g, q(y), a(g, q(y)) h(y))$. Thus, if we set $\Omega = X \times H$ as a $G \times H$-space and set $X = X \times \{e\}, Y = \Theta^{-1}(\{e\} \times Y)$, then this yields the desired topologically free $(G, H)$ coupling $G \times \Omega \times H$.

Note that topological freeness of $G \times X$ and $H \times Y$ ensures that $a$ and $b$ satisfy the cocycle identities (as in [14 Lemma 2.8]), which are needed in the preceding computations.

### 2.2.3. One-to-one correspondence

**Proof of Theorem 2.10.** It is straightforward to check that the constructions described in § 2.2.1 and § 2.2.2 are inverse to each other up to isomorphism. If we start with a topologically free $(G, H)$ coupling $G \times \tilde{\Omega} \times H$, construct a continuous orbit couple and then again a $(G, H)$ coupling, we end up with a $(G, H)$ coupling of the form $G \times \tilde{\Omega} \times H$ where $\tilde{\Omega} = \bar{X} \times H \cong G \times \bar{Y}$, $\bar{X} = X \times \{e\}$ and $\bar{Y} \cong \{e\} \times Y$. It is then obvious
that \( \tilde{\Omega} = \tilde{X} \times H \to \Omega \), \((x, h) \mapsto xh\) is an isomorphism of the couplings \( G \curvearrowright \tilde{\Omega}_X \times H \) and \( G \curvearrowright \Omega \times H \). Conversely, if we start with a continuous orbit couple for topologically free systems \( G \curvearrowright X \) and \( H \curvearrowright Y \), construct a \((G, H)\) coupling and then again a \((G, H)\) continuous orbit couple, we end up with a continuous orbit couple for \( G \curvearrowright \tilde{X} \) and \( H \curvearrowright \tilde{Y} \) where \( \tilde{X} = X \times \{e\} \) and \( \tilde{Y} \cong \{e\} \times Y \). The canonical isomorphisms \( X \cong X \times \{e\} \) and \( Y \cong \{e\} \times Y \) yield an isomorphism between the original \((G, H)\) continuous orbit couple and the one we obtained at the end.

Additional property (i) is clear from our constructions. For (ii), take \( \tilde{X} = \tilde{Y} \) in the construction of §2.2.2 Then it is clear that our maps \( p \) and \( q \) become the identity map on \( \tilde{X} = \tilde{Y}, \) that \( y \) becomes the constant function with value \( e \in G \) and \( \eta \) the constant function with value \( e \in H \). Hence it is obvious that our construction yields a \((G, H)\) continuous orbit equivalence (see also Remark 2.3).

\[ \square \]

**Remark 2.13.** The maps \( p, q \) constructed in §2.2.2 are open. Thus the maps \( p, q \) appearing in a continuous orbit couple (Definition 2.6) are automatically open. This is also easy to see directly from the definition.

### 2.3. Continuous orbit couples and Kakutani equivalence.

**Definition 2.14.** (Compare also [13 Definition 4.1] with Remark 2.15.) Topological dynamical systems \( G \curvearrowright X \) and \( H \curvearrowright Y \) are Kakutani equivalent if there exist clopen subsets \( A \subseteq X \) and \( B \subseteq Y \) such that \( G.A = X, H.B = Y \) and \( \{X \times G\} \) \( \cong \) \((Y \times H)\) as topological groupoids. Here \( \{X \times G\} = s^{-1}(A) \cap r^{-1}(A) \) and \( \{Y \times H\} = s^{-1}(B) \cap r^{-1}(B) \).

**Remark 2.15.** \((X \times G)\) \( \cong \) \((Y \times H)\) if \((X, G)\) is (isomorphic to) the transformation groupoid attached to the partial action \( G \curvearrowright A \) which is obtained by restricting \( G \curvearrowright X \) to \( A \). Similarly, \((Y \times H)\) \( \cong \) \((Y \times H)\) if \((Y, H)\) is (isomorphic to) the transformation groupoid attached to the partial action \( H \curvearrowright B \) which is obtained by restricting \( H \curvearrowright Y \) to \( B \). In view of this, two topologically free systems \( G \curvearrowright X \) and \( H \curvearrowright Y \) are Kakutani equivalent if and only if there exist clopen subsets \( A \subseteq X \) and \( B \subseteq Y \) with \( G.A = X, H.B = Y \) such that the partial actions \( G \curvearrowright A \) and \( H \curvearrowright B \) are continuously orbit equivalent in the sense of [15]. This follows from [15 Theorem 2.7].

The reader may find more about partial actions in [15 §2] and the relevant references in [15].

**Theorem 2.16.** Let \( G \curvearrowright X \) and \( H \curvearrowright Y \) be topologically free systems. There exists a continuous orbit couple for \( G \curvearrowright X \) and \( H \curvearrowright Y \) with \( p(X) \) closed if and only if \((G \curvearrowright X, H \curvearrowright Y)\) are Kakutani equivalent.

**Here** \( p : X \to Y \) is as in Definition 2.6. The assumption that \( p(X) \) is closed always holds if \( X \) is compact. This will be the case of interest later on.

**Proof.** By Remark 2.15 we have to show that there exists a continuous orbit couple for \( G \curvearrowright X \) and \( H \curvearrowright Y \) if and only if there exist clopen subspaces \( A \subseteq X \) and \( B \subseteq Y \) with \( G.A = X, H.B = Y \) such that the partial actions \( G \curvearrowright A \) and \( H \curvearrowright B \) are continuously orbit equivalent. For \( \Rightarrow \), suppose we are given a continuous orbit couple for \( G \curvearrowright X \) and \( H \curvearrowright Y \), and let \( p, q, a, b, g \) and \( h \) be as in Definition 2.6 and §2.2.2. For \( g \in G \), let \( U_g = \{ x \in X : g(x) = g \} \). Then \( U_g \) is clopen, and \( X = \bigsqcup_{g \in G} U_g \).

For every \( g \in G \), \( V_g := p(U_g) \) is clopen, and \( p : U_g \to V_g \) is a homeomorphism, whose inverse is given by \( V_g \to U_g, y \mapsto g^{-1}.q(y) \). Set \( B := p(X) \). By assumption, \( B \) is closed, hence clopen. We have \( B = \bigsqcup_{g \in G} V_g \).

As \( G \) is countable, this is a countable union. Hence by inductively choosing compact open subspaces \( B_g \) of \( V_g \), we can arrange that \( B \) is the disjoint union \( B = \bigsqcup_{g \in G} B_g \). Let \( A_g := U_g \cap p^{-1}(B_g) \) and \( A := \bigsqcup_{g \in G} A_g \). As every \( A_g \) is clopen, \( A = \bigsqcup_{g \in G} A_g \) in \( X = \bigsqcup_{g \in G} U_g \). Set \( \varphi := p|A = \bigsqcup_{g \in G} p|A_g \). By construction, \( \varphi \) is a homeomorphism with inverse \( \varphi^{-1} = \bigsqcup_{g \in G} (p|A_g)^{-1} = \bigsqcup_{g \in G} (g^{-1}.q)|_{B_g} \).

We have \( \varphi(g.x) = p(g.x) = a(g,x), p(x) \) for all \( x \in A \), \( g \in G \) with \( g.x \in A \). Moreover, take \( y \in B_g \) and \( h \in H \) with \( y \in B_g \). Then \( \varphi^{-1}(h.y) = g_2^{-1}.q(h.y) = g_2^{-1}b(h.y), q(y) = g_2^{-1}b(h.y)g_1, \varphi^{-1}(y) \). Define a map \( b' \) by setting \( b'(h.y) = g_2^{-1}b(h.y)_g_1 \) if \( y \in B_g \cap h^{-1}.B_g \). Then \( b' \) is continuous, and we have \( \varphi^{-1}(h.y) = b'(h.y), \varphi^{-1}(y) \) for all \( y \in B, h \in H \) with \( y \in B \). This shows that \( \varphi \) gives rise to a continuous orbit equivalence for \( G \curvearrowright A \) and \( H \curvearrowright B \). To see that \( \varphi \) is a homeomorphism, take \( x' \in X \) an \( x \in A \) such that \( p(x) = p(x') \). Then \( g(x).x = q(p(x)) = q(p(x')) = g(x').x' \), and therefore \( x' \in G.x \). To see \( H.B = Y \), take \( y \in Y \) arbitrary. Then \( p(q(y)) = h(y), y \) shows that \( y = h(y)^{-1}.p(q(y)) \) in \( H.B \). This shows \( \Rightarrow \).

For \( \Leftarrow \), suppose that \( G \curvearrowright X \) and \( H \curvearrowright Y \) are Kakutani equivalent, i.e., there are clopen subsets \( A \subseteq X \) and \( B \subseteq Y \) with \( G.A = X, H.B = Y \) and the partial actions \( G \curvearrowright A \) and \( H \curvearrowright B \) are continuously orbit equivalent via a homeomorphism \( \varphi : A \to B \). By definition of continuous orbit equivalence (see [15]), there exist continuous
Let $G$ be two countable discrete groups. Let $\varphi : G \to H$ be a uniform embedding. Consider the Stone-Čech compactification $\beta G$ of $G$. It is homeomorphic to the spectrum $\text{Spec}(\ell^\infty(G))$, and can be identified with the space of all ultrafilters on $G$. We will think of elements in $\beta G$ as ultrafilters on $G$. Given any subset $X \subseteq G$, we obviously have the identification $\{ \mathcal{F} \in \beta G : X \subseteq \mathcal{F} \} \cong \beta X$, $\mathcal{F} \mapsto \mathcal{F} \cap X := \{ F \cap X : F \in \mathcal{F} \}$.  

2.4. Dynamic characterizations of uniform embeddings, equivalences and bijections. Putting together our previous results, we obtain the following

**Theorem 2.17.** Let $G$ and $H$ be countable discrete groups.

- The following are equivalent:
  - There exists a uniform embedding $G \to H$.
  - There exist Kakutani equivalent topologically free $G \times \sim H$ and $H \times \sim Y$, with $X$ compact.
  - There is a continuous orbit couple for topologically free $G \sim X$ and $H \sim Y$, with $X$ compact.

- The following are equivalent:
  - There is a uniform equivalence $G \sim H$.
  - There are Kakutani equivalent topologically free $G \times \sim X$ and $H \times \sim Y$ on compact spaces $X, Y$.
  - There is a continuous orbit couple for topologically free $G \sim X$ and $H \sim Y$, with $X, Y$ compact.
  - There is a uniform bijection $G \to H$ if and only if there exist continuously orbit equivalent topologically free systems $G \sim X$ and $H \sim Y$ on compact spaces $X$ and $Y$.

In all these statements, the spaces $X$ and $Y$ can be chosen to be totally disconnected and second countable.

This is a generalization of [19] Theorem 3.2, where the authors independently prove the last item of our theorem in the special case of finitely generated groups.

**Remark 2.18.** The last observation in Theorem 2.17 says that we can always choose our spaces $X, Y$ to be totally disconnected. In that case, [19] Theorem 3.2 tells us that we can replace Kakutani equivalence in the theorem above by stable continuous orbit equivalence. Two topological dynamical systems $G \times \sim X$ and $H \times \sim Y$ are called stably continuously orbit equivalent if $Z \times G \sim Z \times X$ and $Z \times H \sim Z \times Y$ are continuously orbit equivalent. Here the integers $\mathbb{Z}$ act on themselves by translation.

2.5. Dynamic characterizations of uniform embeddings, equivalences and bijections in terms of actions on Stone-Čech compactifications. Inspired by [29], we characterize uniform embeddings, equivalences and bijections in terms of Kakutani equivalence (or stable continuous orbit equivalence) and continuous orbit equivalence of actions on Stone-Čech compactifications.

Let $G, H$ be two countable discrete groups. Let $\varphi : G \to H$ be a uniform embedding. Consider the Stone-Čech compactification $\beta G$ of $G$. It is homeomorphic to the spectrum $\text{Spec}(\ell^\infty(G))$, and can be identified with the space of all ultrafilters on $G$. We will think of elements in $\beta G$ as ultrafilters on $G$. Given any subset $X \subseteq G$, we obviously have the identification $\{ \mathcal{F} \in \beta G : X \subseteq \mathcal{F} \} \cong \beta X$, $\mathcal{F} \mapsto \mathcal{F} \cap X := \{ F \cap X : F \in \mathcal{F} \}$. 

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Lemma 2.20. \( \varphi \) induces a continuous orbit equivalence between \( G \rightleftharpoons \beta X \) and \( H \rightleftharpoons \beta Y \), in the sense of Definition 2.6.

Proof. For all \( g \in G \), we need to find a continuous map \( a : \{ g \} \times U_{g^{-1}} \rightarrow H \) with \( \beta \varphi(g, \mathcal{F}) = a(g, \mathcal{F}) \cdot \beta \varphi(\mathcal{F}) \).

Let \( F_{\mathcal{F}} = \{ \mathcal{F} : \mathcal{F} \in \beta X : g, \mathcal{F} \in \beta X \} \). Then \( \mathcal{F} \in \beta G : X \in \mathcal{F}, g^{-1}X \in \mathcal{F} \Rightarrow \beta(X \cap g^{-1}X) \). For \( x \in X \cap g^{-1}X \), define the ultrafilter \( \mathcal{F}_x \) by setting \( a(g, \mathcal{F}_x) = \beta \varphi(g, x) \). Then \( \mathcal{F}_x = \lim_{i \rightarrow i} a(g, \mathcal{F}_x) \cdot \beta \varphi(\mathcal{F}_x) = \beta \varphi(\mathcal{F}_x) \).

The following observation will be used several times.

Proposition 2.21. Let \( G \) and \( H \) be countable discrete groups.

(i) The following are equivalent:

- There is a uniform embedding \( G \rightarrow H \).
- There is an open, dense, \( H \)-invariant subspace \( \tilde{Y} \subseteq \beta H \) such that \( G \rightleftharpoons \beta G \) and \( H \rightleftharpoons \tilde{Y} \) are Kakutani equivalent.
- There is an open, dense, \( H \)-invariant subspace \( \tilde{Y} \subseteq \beta H \) such that \( G \rightleftharpoons \beta G \) and \( H \rightleftharpoons \tilde{Y} \) are stably Kakutani equivalent.

(ii) There is an open, dense, \( H \)-invariant subspace \( \tilde{Y} \subseteq \beta H \) such that \( G \rightleftharpoons \beta G \) and \( H \rightleftharpoons \tilde{Y} \) are stably Kakutani equivalent.

(iii) There is a uniform bijection \( G \rightarrow H \) if and only if \( \tilde{Y} = \beta H \) works in the statements in (i).

Proof. (i): Let \( \varphi : G \rightarrow H \) be a uniform embedding. Let \( Y \) and \( X \) be as in Lemma 2.20. As the restriction of \( \varphi \) to \( X \) is a bijection onto \( Y \), Proposition 2.19 yields that \( G \rightleftharpoons \beta X \) and \( H \rightleftharpoons \beta Y \) are continuously orbit equivalent.

As there is a finite subset \( F \subseteq X \) with \( G \rightleftharpoons \beta G \) and \( H \rightleftharpoons \beta H \), we have \( \beta G = G, \beta X \). Let \( \tilde{Y} := H, \beta Y \). Then \( G \rightleftharpoons \beta G \) and \( H \rightleftharpoons \tilde{Y} \) are Kakutani equivalent. \( \tilde{Y} \) is \( H \)-invariant by construction, and it is easy to see that \( \tilde{Y} \) is open and dense.

Now (i) follows from Theorem 2.17 and Remark 2.18.

(ii): A uniform embedding \( \varphi : G \rightarrow H \) is uniformly invertible if and only if there is a finite subset \( F \subseteq H \) such that \( \beta \varphi = G = H \). This happens if and only if in the proof of (i), we get \( \tilde{Y} = \beta H \).
Theorem 2.17. □

Remark 2.22. In combination with [29], Corollary 2.21 implies that nuclear Roe algebras have distinguished Cartan subalgebras. We will elaborate on this in ongoing joint work with Jean Renault [16].

Remark 2.23. Corollary 2.21 shows that quasi-isometry rigidity can be interpreted as a special case of continuous orbit equivalence rigidity (in the sense of [14]), applied to actions on Stone–Čech compactifications. This point towards an interesting connection between these two types of rigidity phenomena and would be worth exploring further.

3. Applications to (Co)homology I

We now show how the results in [28, 25] on quasi-isometry invariance of (co)homological dimensions and property $HF_{loc}$ follow from Morita invariance of groupoid (co)homology. Let us first define groupoid (co)homology. We do this in a concrete and elementary way which is good enough for our purposes. We refer to [6] for a more general and more conceptual approach, and for more information about groupoids. Let $\mathcal{G}$ be an étale locally compact groupoid with unit space $X = \mathcal{G}^{(0)}$, and $R$ a commutative ring with unit. A $\mathcal{G}$-sheaf of $R$-modules is a sheaf $\mathcal{A}$ of $R$-modules over $X$, i.e., we have a locally compact space $\mathcal{A}$ with an étale continuous surjection $\pi: \mathcal{A} \to X$ whose fibres are $R$-modules, together with the structure of a right $\mathcal{A}$-module of $\mathcal{G}$-sheaves of $d$-modules over $\mathcal{A}$, with compact support such that $\pi(f(\tilde{\gamma})) = r(\tilde{\gamma})$. Now we define a chain complex $\cdots \to \Gamma_{c}(\mathcal{G}((n), \mathcal{A})) \xrightarrow{d_{n-1}} \Gamma_{c}(\mathcal{G}((n-1), \mathcal{A})) \xrightarrow{d_{n}} \cdots \xrightarrow{d_{2}} \Gamma_{c}(\mathcal{G}, \mathcal{A}) \xrightarrow{d_{1}} \Gamma_{c}(X, \mathcal{A}) \to 0$, with $d_{1}(f)(x) = \sum_{\tilde{\gamma} \in \mathcal{G}} f(\tilde{\gamma}) - \sum_{\tilde{\gamma} \in \mathcal{G}} f(\tilde{\gamma})$ for $f \in \Gamma_{c}(\mathcal{G}, \mathcal{A})$, and for $n \geq 1$: $d_{n}(f) = \sum_{i=0}^{n}(-1)^{i}d_{n}^{(i)}(f)$

We then define the $n$-th homology group $H_{n}(\mathcal{G}, \mathcal{A}) := \ker(d_{n})/\text{im}(d_{n+1})$. In the case $R = \mathbb{Z}$ and where $\mathcal{A}$ is a constant sheaf with trivial $\mathcal{G}$-action, we recover [18, Definition 3.1].

Let us also introduce cohomology. Let $\mathcal{G}$, $R$, and $\mathcal{A}$ be as above, and let $\Gamma(\mathcal{G}((n), \mathcal{A}))$ be the $R$-module of continuous functions $f: \mathcal{G}((n)) \to \mathcal{A}$ with $\pi(f(\tilde{\gamma})) = r(\tilde{\gamma})$. We define a cochain complex $0 \to \Gamma(X, \mathcal{A}) \xrightarrow{d^{0}} \Gamma(\mathcal{G}, \mathcal{A}) \xrightarrow{d^{1}} \cdots \xrightarrow{d^{n-1}} \Gamma(\mathcal{G}((n), \mathcal{A})) \xrightarrow{d^{n}} \cdots \xrightarrow{d^{n}} \Gamma(\mathcal{G}((n-1), \mathcal{A}))$ with $d^{0}(f)(\gamma) = \gamma, f(s(\gamma)) - f(r(\gamma))$, and for $n \geq 1$: $d^{n}(f) = \sum_{i=0}^{n+1}(-1)^{i}d^{n}((f))$, where

$d^{n}((f))(\gamma_{0}, \ldots, \gamma_{n}) = \gamma_{0}, f(\gamma_{1}, \ldots, \gamma_{n})$;

$d^{n+1}((f))(\gamma_{0}, \ldots, \gamma_{n+1}) = f(\gamma_{0}, \ldots, \gamma_{n-1}, \gamma_{n+1})$ for $1 \leq i \leq n$;

$d^{n+1}((f))(\gamma_{0}, \ldots, \gamma_{n}) = f(\gamma_{0}, \ldots, \gamma_{n-1}, \gamma_{n})$.  

We set $H^{n}(\mathcal{G}, \mathcal{A}) := \ker(d^{n})/\text{im}(d^{n-1})$. 

(iii): If $\varphi: G \to H$ is a uniform bijection, then we can take $X = G, Y = H$ in the above proof of (i) and obtain that $G \cong H$ and $H \cong H$ are continuously orbit equivalent. The reverse implication "⇐" in (ii) is proven in Theorem 2.17.
Now let \( G \times X \) be a topological dynamical system. For notational purposes, and to keep the conventions in the literature, let us pass to the right action \( X \times G \), \( x,g \mapsto g^{-1}x \), and consider the corresponding transformation groupoid \( X \times G \) with source and range maps given by \( s(x,g) = xg \), \( r(x,g) = x \). We note that the transformation groupoid \( G \times X \) attached to the original action, as in [14,15], is isomorphic to \( X \times G \) via \( G \times X \to X \times G \), \( (g,x) \mapsto (gx,g) \). It is easy to see that a \((X \times G)\)-sheaf of \( R\)-modules is nothing else but a sheaf \( \mathcal{A} \) of \( R\)-modules over \( X \times G \), \( \pi : \mathcal{A} \to X \), together with a left \( G \)-action on \( \mathcal{A} \) via homeomorphisms (denoted by \( G \times \mathcal{A} \to \mathcal{A} \), \( (g,a) \mapsto g.a \)) such that \( \pi \) becomes \( G \)-equivariant, and \( \mathcal{A}_x \to \mathcal{A}_{g,x} , a \mapsto g.a \) is an isomorphism of \( R\)-modules. We call these \( G\)-sheaves of \( R\)-modules over \( X \).

3.1. Isomorphisms in homology and cohomology. Our goal is to prove Theorem 3.1. Let \( G \times X \) and \( H \times Y \) be topologically free systems, where \( G \) and \( H \) are countable discrete groups. Suppose that \( G \times X \) and \( H \times Y \) are Kakunui equivalent. Then there is an equivalence of categories between \( G\)-sheaves of \( R\)-modules over \( X \times H\)-sheaves of \( R\)-modules over \( Y \), denoted by \( \mathcal{F}_X \to \mathcal{F}_Y \) on the level of objects, such that \( H_n(\Gamma(X,\mathcal{F}_X)) \cong H_n(H,\Gamma(\mathcal{F}_Y)) \) and \( H^*(\Gamma(X,\mathcal{F}_X)) \cong H^*(H,\Gamma(\mathcal{F}_Y)) \).

Here \( \Gamma \) stands for continuous sections and \( \Gamma_c \) for those with compact support.

First we need to identify group (co)homology with groupoid (co)homology.

Lemma 3.2. Let \( G \times X \) be a topological dynamical system and \( \mathcal{A} \) a \( G\)-sheaf of \( R\)-modules over \( X \). Then

\[
H_n(\Gamma_c(X,\mathcal{A})) \cong H_n(\Gamma(X,\mathcal{A})) \cong H^*(\Gamma(X,\mathcal{A})).
\]

Proof. For homology, we identify our above chain complex with the bar cocomplex \( C_n \) defining \( H_n(G,\Gamma_c(X,\mathcal{A})) \) (see [4, III.1]). \( C_n \) is given by \( \Gamma_c(X,\mathcal{A}) \otimes_R R[G^n] \). There is an obvious identification of

\[
(X \times G)^{(n)} = \{(x_1,g_1), \ldots, (x_n,g_n) : x_i, g_i = x_{i+1} \text{ for all } 1 \leq i \leq n-1 \}
\]

with \( \{x_1, x_2, \ldots, x_n \} : x \in X, g_1, g_2, \ldots, g_n \in G \}. Every section in \( \Gamma_c((X \times G)^{(n)}, \mathcal{A}) \) is of the form \( (X \times G)^{(n)} \to \mathcal{A}, (x_1,g_1, \ldots, g_n) \mapsto a_{g_1, \ldots, g_n}(x) \), where \( a_{g_1, \ldots, g_n} \in \Gamma_c(X,\mathcal{A}) \) and \( a_{g_1, \ldots, g_n} \equiv 0 \) for almost all \( (g_1, \ldots, g_n) \in G^n \).

Hence we have an identification \( C_n \cong \Gamma_c((X \times G)^{(n)}, \mathcal{A}) \) which sends \( a \otimes (g_1, \ldots, g_n) \in C_n \) to the section

\[
(x_1,g'_1, \ldots, g'_n) \mapsto \begin{cases} a(x) & \text{if } g'_1 = g_1, \ldots, g'_n = g_n \\ 0 & \text{else} \end{cases} \quad \text{in } \Gamma_c((X \times G)^{(n)}, \mathcal{A}).
\]

It is easy to check that these identifications are compatible with the differentials.

We need a version of Morita invariance of groupoid (co)homology, for which we insert a short proof (see [5] for a more conceptual treatment). Let \( G \times X \) be a topological dynamical system, where \( G \) is a countable discrete group. Let \( F \subseteq X \) be a clopen subspace which is \( G\)-full, i.e., \( G.F = X \). Write \( \mathcal{G} := X \times G \), and let \( \mathcal{G} | F \) be the subgroupoid \( \mathcal{G} | F := r^{-1}(F) \cap s^{-1}(F) \). Let \( \mathcal{A}\mathcal{F} \) be a \( G\)-sheaf of \( R\)-modules over \( X \), or equivalently a \( \mathcal{A}\mathcal{F}\) of \( R\)-modules, and \( \mathcal{A}\mathcal{F} | F \) the restriction of \( \mathcal{A}\mathcal{F} \) to \( \mathcal{G} | F \). Then the inclusion \( \iota_n : \mathcal{A}\mathcal{F} | F \to \mathcal{A}\mathcal{F} \) induces homomorphisms

\[
i_n : \Gamma_c((\mathcal{A}\mathcal{F} | F)^{(n)}, \mathcal{A}\mathcal{F} | F) \to \Gamma_c(\mathcal{A}\mathcal{F}^{(n)}, \mathcal{A}\mathcal{F} | F), \quad f \mapsto (\iota^n)_*(f), \text{ where } (\iota^n)_*(f)(\bar{\xi}) = \sum_{\xi \in (\mathcal{A}\mathcal{F} | F)^{(n)}} f(\bar{\xi}) ;
\]

\[
i^* : \Gamma((\mathcal{A}\mathcal{F}^{(n)}, \mathcal{A}\mathcal{F} | F) \to \Gamma((\mathcal{A}\mathcal{F} | F)^{(n)}, \mathcal{A}\mathcal{F} | F), \quad f \mapsto (\iota^n)^*(f) := f \circ \iota^n.
\]

Lemma 3.3. The homomorphisms \( \iota_n \) induce isomorphisms \( H_n(\iota_n) : H_n(\mathcal{A}\mathcal{F} | F, \mathcal{A}\mathcal{F} | F) \cong H_n(\mathcal{A}\mathcal{F}, \mathcal{A}\mathcal{F}) \) for all \( n \).

The homomorphisms \( \iota^n \) induce isomorphisms \( H^n(\iota^n) : H^n(\mathcal{A}\mathcal{F}, \mathcal{A}\mathcal{F} | F) \cong H^n(\mathcal{A}\mathcal{F} | F, \mathcal{A}\mathcal{F} | F) \) for all \( n \).
Proof. We have $X = G.F = \bigcup_{g \in G} g.F$. As $G$ is countable, we can inductively construct clopen subspaces $F_i \subseteq F$ and $g_i \in G$ for $i = 1, 2, \ldots$, with $F_1 = F$, $g_1 = e$ (the identity in $G$) such that we have a disjoint union $X = \bigcup_{i=1}^\infty g_i.F_i$. Define $\theta : X \to \mathcal{F}$ by setting $\theta(x) = (x, g_i)$ for $x \in g_i.F_i$. Let $\rho : \mathcal{F} \to \mathcal{F}/F, \gamma \mapsto \theta(r(\gamma)^{-1} \gamma \theta(s(\gamma)))$. For $(x, g) \in \mathcal{F}$ with $x \in g_i.F_i$ and $g^{-1}x \in g_j.F_j$, we have $\theta(x, g) = (g_i^{-1}x, g_i^{-1}gg_j)$. It is easy to see that $\rho$ is a groupoid homomorphism. Moreover, we have $\rho \circ i = \id$ on $\mathcal{F}/F$. Let $p^n : \mathcal{F}^{(n)} \to (\mathcal{F}/F)^{(n)}$, $(\gamma_1, \ldots, \gamma_n) \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n))$.

Let us first treat homology. Define homomorphisms $p_n : \Gamma_c((\mathcal{F})^{(n)}, \mathcal{A}) \to \Gamma_c((\mathcal{F}/F)^{(n)}, \mathcal{A}/F)$ by setting

$$p_0(f)(y) = \sum_{\rho(x) = y} \theta(x)^{-1} f(x), \text{ and } p_n(f)(\bar{\gamma}) = \sum_{\rho^n(\bar{\gamma}) = \bar{\gamma}} \theta(r(\gamma_1)^{-1} f(\gamma_1), \ldots, f(r(\gamma_n)^{-1} f(\gamma_n))$$

for $n \geq 1$.

$p_*$ is a chain map since $\rho$ is a groupoid homomorphism. We have $p_n \circ i = \id$ on $\Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$ for all $n$. We now show that $i_n p_n$ is homotopic to the identity on $\Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$. To do so, construct homomorphisms $k_n : \Gamma_c((\mathcal{F})^{(n)}, \mathcal{A}) \to \Gamma_c((\mathcal{F}^{(n+1)}, \mathcal{A})$ for all $n \geq 0$ by setting $k_n = \sum_{h=1}^{n+1} (-1)^h 1_k^{(h)}$, $k_n^h = (k_n^h)^*$, where $k_n^h : \mathcal{F}^{(n)} \to \mathcal{F}^{(n+1)}$ for $n \geq 0$ are given by $k_1^1 = \theta$ and, for $n \geq 1$,

$$k_n^h(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_n-1, \theta(r(\gamma_n)), \rho(\gamma_2), \ldots, \rho(\gamma_n))$$

for $1 \leq h \leq n$.

It is straightforward to check that $d_1 k_0 = i_0 p_0 - \id$ on $\Gamma_c(X, \mathcal{A})$ and $d_{n+1} k_n + k_{n-1} d_n = i_n p_n - \id$ on $\Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$ for all $n \geq 1$. Thus $i_n p_n \sim_h \id$.

Hence $H_n(i_* \gamma) \cong H_n(\mathcal{F}, \mathcal{A}/F)$ with inverse $H_n(p_*)$, for all $n$.

Now we turn to cohomology. Define homomorphisms $p^* : \Gamma_c((\mathcal{F}/F)^{(n)}, \mathcal{A}/F) \to \Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$ by setting

$$p^0(f)(x) = \theta(x) f(\rho(x)), \text{ and } p^n(f)(\gamma_1, \ldots, \gamma_n) = \theta(r(\gamma_1)) f(p^n(\gamma_1, \ldots, \gamma_n))$$

for $n \geq 1$.

$p^*$ is a cochain map since $\rho$ is a groupoid homomorphism. We have $i^* p^* = \id$ on $\Gamma_c((\mathcal{F}/F)^{(n)}, \mathcal{A}/F)$. We now show that $p^* p^* = \id$ on $\Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$. To do so, construct homomorphisms $k^* : \Gamma_c((\mathcal{F})^{(n)}, \mathcal{A}) \to \Gamma_c((\mathcal{F}^{(n-1)}, \mathcal{A})$ for all $n \geq 1$ by setting $k^* = \sum_{h=1}^{n} (-1)^h 1_k^{(h)}$, $k^*_h = (k^*_h)^*$, where $k^*_h : \mathcal{F}^{(n)} \to \mathcal{F}^{(n-1)}$ for $n \geq 1$ are given by $k_1^1 = \theta$ and, for $n \geq 2$,

$$k_n^h(\gamma_1, \ldots, \gamma_n-1) = (\gamma_1, \ldots, \gamma_n-1, \theta(s(\gamma_n)), \rho(\gamma_2), \ldots, \rho(\gamma_n-1))$$

for $1 \leq h \leq n-1$.

It is straightforward to check that $d_1^* d^0 = p^0 d^0 - \id$ on $\Gamma_c(X, \mathcal{A})$ and $d_{n+1}^* k_n^* + k_{n-1}^* d_n^* = p^n d^n - \id$ on $\Gamma_c((\mathcal{F})^{(n)}, \mathcal{A})$ for all $n \geq 1$. Thus $p^* i_* \sim_h \id$.

Hence $H^n(i^*)$ is an isomorphism $H^n(\mathcal{F}, \mathcal{A}/F) \xrightarrow{\cong} H^n(\mathcal{F}/F, \mathcal{A}/F)$ with inverse $H^n(p^*)$, for all $n$.

Proof of Theorem 3.2 By assumption, there are clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $X = G.A$, $Y = H.B$ and an isomorphism of topological groupoids $\chi : (X \times G)A \cong (Y \times H)B$. Let $t_A : (X \times G)A \to X \times G$ and $t_B : (Y \times H)B \to Y \times H$ be the canonical inclusions. As $A$ is $G$-full and $B$ is $H$-full, $t_A$ and $t_B$ induce equivalences of categories of sheaves. So we obtain an equivalence of categories between $G$-sheaves of $R$-modules over $X$ and $H$-sheaves of $R$-modules over $Y$, denoted by $\mathcal{A}_X \to \mathcal{A}_Y$ on the level of objects, such that $\mathcal{A}_Y$ is uniquely determined by $\chi^*(\mathcal{A}_Y|B) = \mathcal{A}_X|A$.

By Lemmas 3.2 and 3.3

$$H_n(\Gamma_c(X, \mathcal{A})) \cong H_n(X \times G, \mathcal{A}_X) \cong H_n((X \times G)|A, \mathcal{A}_X|A)$$

$$\cong H_n((Y \times H)|B, \mathcal{A}_Y|B) \cong H_n(Y \times H, \mathcal{A}_Y) \cong H_n(H, \Gamma_c(Y, \mathcal{A}_Y))$$

and $H^n(\Gamma_c(X, \mathcal{A})) \cong H^n(X \times G, \mathcal{A}_X) \cong H^n((X \times G)|A, \mathcal{A}_X|A)$

$$\cong H^n((Y \times H)|B, \mathcal{A}_Y|B) \cong H^n(Y \times H, \mathcal{A}_Y) \cong H^n(H, \Gamma(Y, \mathcal{A}_Y)|A, \mathcal{A}_X)$$. 

For every topological dynamical system $G \acts X$, we have $\sup \{n : H_n(G, \Gamma_c(X, \mathcal{A})) \not\cong \{0\}\} \leq \hd(G)$ and $\sup \{n : H^n(G, \Gamma_c(X, \mathcal{A})) \not\cong \{0\}\} \leq \cd(G)$ by the definitions of homological and cohomological dimensions. Here the suprema are taken over all $G$-sheaves $\mathcal{A}$ of $R$-modules over $X$. 

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Definition 3.4. A \((G,H)\) continuous orbit couple is called \(H,\emptyset G\)-full if \(\sup \{ n : H_n(G, \mathcal{F}(X, \mathcal{A})) \not\subseteq \{ 0 \} \} = \text{hd}_G(G)\) holds for its topological dynamical system \(G \curvearrowright X\). It is called \(H^*G\)-full if its topological dynamical system \(G \curvearrowright X\) satisfies \(\sup \{ n : H^n(G, \mathcal{F}(X, \mathcal{A})) \not\subseteq \{ 0 \} \} = \text{cd}_G(G)\).

The following is an immediate consequence of Theorem 3.1.

Corollary 3.5. If there exists an \(H,\emptyset G\)-full topologically free \((G,H)\) continuous orbit couple, then \(\text{hd}_G(G) \leq \text{hd}_H(H)\). If there exists an \(H^*G\)-full topologically free \((G,H)\) continuous orbit couple, then \(\text{cd}_G(G) \leq \text{cd}_H(H)\).

Remark 3.6. Together with Theorem 3.17 Corollary 3.5 can be viewed as an explanation and generalization of the results in \([28, 25]\) concerning quasi-isometry invariance of homological and cohomological dimension. In our terminology, the conditions from \([28, 25]\) that the topological dynamical system \(G \curvearrowright X\) of a \((G,H)\) continuous orbit couple admits a \(G\)-invariant probability measure and \(\mathbb{Q} \subseteq R\) ensure that the \((G,H)\) continuous orbit couple is \(H,\emptyset G\)-full and \(H^*G\)-full (see \([28, \S\ 3.3]\) and \([25, \S\ 4]\)). Existence of a \(G\)-invariant probability measure is guaranteed if \(G\) is amenable and the \(G\)-space of our continuous orbit couple is compact. Moreover, again in our terminology, it is shown in \([25, \S\ 4]\) that a \((G,H)\) continuous orbit couple with compact \(G\)-space is \(H,\emptyset G\)-full if \(\text{hd}_G(G) < \infty\) and \(H^*G\)-full if \(\text{cd}_G(G) < \infty\). Once we know this, \([28, \text{Theorem 1.5}]\) and \([25, \text{Theorem 1.2}]\) are immediate consequences of Theorems 3.17 and Corollary 3.5.

3.2. Isomorphisms in reduced cohomology. Let \(\mathcal{G}\) be an étale locally compact groupoid and \(\mathcal{L} = (\mu, \mathcal{H}, L)\) a (unitary) representation of \(\mathcal{G}\) as in \([24, \text{Chapter II, Definition 1.6}]\). Here \(\mu\) is a quasi-invariant measure on \(\mathcal{L}^{(0)}, \mathcal{H}\) a Hilbert bundle over \((\mathcal{L}^{(0)}, \mu)\), and \(L\) a representation of \(\mathcal{G}\), i.e., for each \(\gamma \in \mathcal{G}\), \(L(\gamma)\) is a unitary \(\mathcal{H}(\gamma) \hookrightarrow \mathcal{H}(\gamma)\), and the conditions in \([24, \text{Chapter II, Definition 1.6}]\) are satisfied (\(\sigma\) in \([24, \text{Chapter II, Definition 1.6}]\) is the trivial cocycle in our case). Let \(D\) be the modular function attached to \(\mu\), as in \([24, \text{Chapter I, Definition 3.4}]\). In particular, we are interested in the case \(\mathcal{G} = X \times G\) of a transformation groupoid attached to a topological dynamical system \(G \curvearrowright X\) on a compact space \(X\). A representation \(\mathcal{L}\) of \(X \times G\) gives rise – through its integrated form – to a \(*\)-representation of \((X, \times G)\), which in turn corresponds in a one-to-one way to a covariant representation \((\pi_2, \sigma_2)\) of \((G \curvearrowright X)\) or rather of \((X, \times G)\).

Now let \(\mathcal{G} = X \times G\) be as above and \(\mathcal{L}\) a representation of \(\mathcal{G}\). We define cohomology groups \(H^n(\mathcal{G}, \mathcal{L})\) and reduced cohomology groups \(\tilde{H}^n(\mathcal{G}, \mathcal{L})\) with coefficients in \(\mathcal{L}\). Let us write \(\mathcal{L} = (\mu, \mathcal{H}, L)\). Let \(\mathcal{G}^{(n)} = \{ (\gamma_1, \ldots, \gamma_n) \in \mathcal{G}^{(n)} : s(\gamma_i) = r(\gamma_{i+1}) \text{ for } 1 \leq i \leq n - 1 \},\) and set \(r(\gamma_1, \ldots, \gamma_n) = r(\gamma_n)\). We will write \(\gamma\) for elements in \(\mathcal{G}^{(n)}\). Let \(\Gamma(\mathcal{G}^{(n)}, \mathcal{H})\) be the set of all Borel functions \(f : \mathcal{G}^{(n)} \to \mathcal{H}\) with \(f(\gamma) \in \mathcal{H}(\gamma)\), such that for every compact subset \(K \subseteq \mathcal{G}^{(n)}\), \(\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in K} \| f(\gamma) \| d\mu(x) < \infty\), divided by the equivalence relation saying that \(f_1 \sim f_2\) if for every compact subset \(K \subseteq \mathcal{G}^{(n)}\), \(\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in K \cap r(\gamma) = x} \| f_1(\gamma) - f_2(\gamma) \|^2 d\mu(x) = 0\). The topology on \(\Gamma(\mathcal{G}^{(n)}, \mathcal{H})\) is given by the following notion of convergence: A net \((f_i)\) converges to an element \(f\) in \(\Gamma(\mathcal{G}^{(n)}, \mathcal{H})\) if for every compact subset \(K \subseteq \mathcal{G}^{(0)}\), \(\lim_{i \to \infty} \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in K} \| f_i(\gamma) - f(\gamma) \|^2 d\mu(x) = 0\). We define a cochain complex \(0 \to \Gamma(\mathcal{G}^{(0)}, \mathcal{H}) \xrightarrow{d^0} \Gamma(\mathcal{G}^{(1)}, \mathcal{H}) \xrightarrow{d^1} \ldots\) with \(d^0(f)(\gamma) = D^{-\frac{1}{2}}(\gamma)(L(\gamma)f(s(\gamma)) - f(r(\gamma)))\), and for \(n \geq 1\) \(d^n = \sum_{i=0}^{n+1} (-1)^i d^i\), where
\[
\begin{align*}
    d^0_{(0)}(f)(\gamma_0, \ldots, \gamma_n) &= D^{-\frac{1}{2}}(\gamma_0)L(\gamma_0)f(\gamma_1, \ldots, \gamma_n); \\
    d^0_{(1)}(f)(\gamma_0, \ldots, \gamma_n) &= f(\gamma_0, \ldots, \gamma_{i-1}, \gamma_i, \ldots, \gamma_n) & \text{for } 1 \leq i \leq n; \\
    d^0_{(n+1)}(f)(\gamma_0, \ldots, \gamma_n) &= f(\gamma_0, \ldots, \gamma_{n-1}).
\end{align*}
\]
It is easy to check that \(d^n \circ d^{n-1} = 0\) for all \(n \geq 1\). Thus \(\text{im}(d^{n-1}) \subseteq \ker(d^n)\). Since all the \(d^n\) are continuous, we also have \(\overline{\text{im}(d^{n-1})} \subseteq \ker(d^n)\). We set \(H^n(\mathcal{G}, \mathcal{L}) := \ker(d^n)/\text{im}(d^{n-1})\) and \(\tilde{H}^n(\mathcal{G}, \mathcal{L}) := \ker(d^n)/\overline{\text{im}(d^{n-1})}\).

Our goal is to prove the following

**Theorem 3.7.** Suppose there is a continuous orbit couple for topological dynamical systems \(G \curvearrowright X\) and \(H \curvearrowright Y\) on compact spaces \(X\) and \(Y\). Then there is an one-to-one correspondence between representations of \(X \times G\) and \(Y \times H\), denoted by \(\Sigma \leftrightarrow \mathfrak{M}\), with \(H^\bullet(G, \sigma_2) \cong H^\bullet(H, \sigma_2|\mathfrak{M})\) and \(H^\bullet(G, \sigma_2) \cong H^\bullet(H, \sigma_2|\mathfrak{M})\).
For the definition of reduced cohomology $\tilde{H}^*$, we refer to [12, Chapitre III].

We first need to identify groupoid cohomology with group cohomology.

**Lemma 3.8.** We have $H^*(X \times G, \mathcal{L}) \cong H^*(G, \sigma_\mathcal{L})$ and $\tilde{H}^*(X \times G, \mathcal{L}) \cong \tilde{H}^*(G, \sigma_\mathcal{L})$.

**Proof.** Let $\mathcal{L} = (\mu, \mathcal{H}, \Lambda)$. The underlying Hilbert space of $\sigma_\mathcal{L}$ is the space of $L^2$-sections $\Gamma(X, \mathcal{H})$. We have an obvious identification $(X \times G)^{(n)} \cong \{(x, g_1, \ldots, g_n) : x \in X, g_i \in G\}$. Under this identification, every element in $\Gamma((X \times G)^{(n)}, \mathcal{H})$ is of the form $(x, g_1, \ldots, g_n) \mapsto \xi_{g_1, \ldots, g_n}(x)$, where $\xi_{g_1, \ldots, g_n} \in \Gamma(X, \mathcal{H})$, and a collection $(\xi_{g_1, \ldots, g_n})_{g_1, \ldots, g_n}$ of elements in $\Gamma(X, \mathcal{H})$ determines a unique element in $\Gamma((X \times G)^{(n)}, \mathcal{H})$. In other words, we have a canonical isomorphism

$$
\Gamma((X \times G)^{(n)}, \mathcal{H}) \xrightarrow{\sim} C(G^n, \Gamma(X, \mathcal{H}))
$$

sending $(x, g_1, \ldots, g_n) \mapsto \xi_{g_1, \ldots, g_n}(x)$ to $(g_1, \ldots, g_n) \mapsto \xi_{g_1, \ldots, g_n}$. Note that we view $G$ as a discrete space, so that every map $G^n \to \Gamma(X, \mathcal{H})$ is continuous. The isomorphism in (2) is a homeomorphism with respect to the topology on $\Gamma((X \times G)^{(n)}, \mathcal{H})$ defined at the beginning of §3.2, and the topology of pointwise convergence on $C(G^n, \Gamma(X, \mathcal{H}))$. Finally, it is easy to check that the linear homeomorphisms in (2) taken together for all $n \geq 0$ give rise to an identification of the cochain complexes used in the definition of $H^n(X \times G, \mathcal{L})$ (or $\tilde{H}^n(X \times G, \mathcal{L})$) and $H^n(G, \sigma_\mathcal{L})$ (or $\tilde{H}^n(G, \sigma_\mathcal{L})$).

We also need a version of Morita invariance for our groupoid cohomology. Let $\mathcal{G} = X \times G$ be as above and $\mathcal{L}$ a representation of $\mathcal{G}$. Suppose that $F \subseteq X$ is a clopen subspace with $X = GF$. Let $\mathcal{L}|F$ be the representation of $\mathcal{G}|F$ given by restriction of $\mathcal{L}$. Then the inclusion $i : \mathcal{G}|F \to \mathcal{G}$ induces continuous homomorphisms $p^i : \Gamma((\mathcal{G}|F)^{(n)}, \mathcal{H}|F) \to \Gamma((\mathcal{G})^{(n)}, \mathcal{H})$ given by $p^i(f)(\gamma_1, \ldots, \gamma_n) = f(\gamma_1, \ldots, \gamma_n)$. Here we write $\mathcal{H}|F$ for the Hilbert bundle $\mathcal{H}$ restricted to $F$. It is easy to see that $i^*$ is a cochain map.

**Lemma 3.9.** For all $n$, $i^*$ induces isomorphisms

$$
H^n(i^*) : H^n(\mathcal{G}|F, \mathcal{L}|F) \xrightarrow{\sim} H^n(\mathcal{G}, \mathcal{L})
$$

and $\tilde{H}^n(i^*) : \tilde{H}^n(\mathcal{G}|F, \mathcal{L}|F) \xrightarrow{\sim} \tilde{H}^n(\mathcal{G}, \mathcal{L}).$

**Proof.** Construct continuous maps $\theta : X \to \mathcal{G}$ and $\rho : \mathcal{G} \to \mathcal{G}|F$ as in the proof of Lemma 3.3. Define homomorphisms $p^n : \Gamma((\mathcal{G}|F)^{(n)}, \mathcal{H}|F) \to \Gamma((\mathcal{G})^{(n)}, \mathcal{H})$ by setting $p^n(f)(x) = D^{-\frac{1}{2}}(\theta(r(x)))(L(\theta(x))f(s(\theta(x))))$ and

$$
p^n(f)(\gamma_1, \ldots, \gamma_n) = D^{-\frac{1}{2}}(\theta(r(\gamma_i)))(L(\theta(r(\gamma_i))))f(s(\theta(\gamma_i))) for n \geq 1.
$$

As $\rho$ is a groupoid homomorphism, $p^i$ is a cochain map. In addition, as $\theta$ and $\rho$ are continuous, $p^i$ is continuous for every $n$. Moreover, as $\rho \circ i = \text{id}$ on $\mathcal{G}|F$, we have $i^*p^n = \text{id}$ on $\Gamma((\mathcal{G}|F)^{(n)}, \mathcal{H}|F)$ for all $n$. Let us now show that $p^i$ is continuously homotopic to the identity on $\Gamma((\mathcal{G})^{(n)}, \mathcal{H})$. To do so, construct continuous homomorphisms $k^n : \Gamma((\mathcal{G})^{(n)}, \mathcal{H}) \to \Gamma((\mathcal{G})^{(n-1)}, \mathcal{H})$ for all $n \geq 1$ by setting $k^n = \sum_{k=1}^n (-1)^{n-k}k^n_{(h);}(\mathcal{K}(h), k)^{\mathcal{N}(h)}$, where $k^n_{(1)} = \theta$ and

$$
k^n_{(h)}(\gamma_1, \ldots, \gamma_{n-1}) = \gamma_1, \ldots, \gamma_{n-1}, \theta(r(\gamma_1)), \rho(\gamma_1), \ldots, \rho(\gamma_{n-1}) for 1 \leq h \leq n - 1;
$$

$$
k^n_{(n)}(\gamma_1, \ldots, \gamma_{n-1}) = (\gamma_1, \ldots, \gamma_{n-1}, \theta(s(\gamma_{n-1}))).
$$

It is straightforward to check $k^1d^0 = p^0d^0 - \text{id}$ on $\Gamma((\mathcal{G})^{(0)}, \mathcal{H})$ and $d^{n-1}k^n + k^{n+1}d^n = p^n d^n - \text{id}$ on $\Gamma((\mathcal{G})^{(n)}, \mathcal{H})$ for all $n \geq 1$.

This shows that for all $n$, $i^*$ induces an isomorphism $H^n(i^*) : H^n(\mathcal{G}|F, \mathcal{L}|F) \xrightarrow{\sim} H^n(\mathcal{G}, \mathcal{L})$ with inverse $H^n(p^i)$, and also an isomorphism $\tilde{H}^n(i^*) : \tilde{H}^n(\mathcal{G}|F, \mathcal{L}|F) \xrightarrow{\sim} \tilde{H}^n(\mathcal{G}, \mathcal{L})$ with inverse $\tilde{H}^n(p^i)$.

**Proof of Theorem 3.7**. If there is a continuous orbit couple for topological dynamical systems $G \subset X$ and $H \circlearrowright Y$ on compact spaces $X$ and $Y$, then by Theorem 2.16 $G \subset X$ and $H \circlearrowright Y$ are Kakutani equivalent. So there exist clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $G/A = X$, $H/B = Y$, together with an isomorphism of topological groupoids $\chi : (X \times G)|A \cong (Y \times H)|B$.

As $A$ is $G$-full and $B$ is $H$-full, we get one-to-one correspondences $\mathcal{L} \leftrightarrow \mathcal{L}|A$ and $\mathfrak{M} \leftrightarrow \mathfrak{M}|B$ between representations of $X \times G$ and $(X \times G)|A$, and between representations of $Y \times H$ and $(Y \times H)|B$, respectively. Thus we obtain a one-to-one correspondence between representations of $X \times G$ and $Y \times H$, denoted by $\mathcal{L} \leftrightarrow \mathfrak{M}$, where $\mathfrak{M}$ is uniquely determined by $\chi^{-1}(\mathfrak{M}|B) = \mathcal{L}|A$.

Therefore, we obtain by Lemma 3.8 and Proposition 3.9

$$
H^*(G, \sigma_\mathcal{L}) \cong H^*((X \times G)|A, \mathcal{L}|A) \cong H^*((Y \times H)|B, \mathfrak{M}|B) \cong H^*(Y \times H, \mathfrak{M}) \cong H^*(H, \sigma_\mathfrak{M}).
$$
The proof for reduced cohomology is completely analogous. □

Remark 3.10. If the topological dynamical system $G ∼ X$ is on a second countable space $X$, then every $*$-representation of $C_r^*(X ∗ G)$ on a Hilbert space is the integrated form of a representation of $X ∗ G$. Actually, $*$-representations of $C_r^*(X ∗ G)$ and representations of $X ∗ G$ are in one-to-one correspondence (see [24, Chapter II, Theorem 1.21 and Corollary 1.23]). Thus we obtain a reformulation of Theorem [3.7]. Suppose there is a continuous orbit couple for topological dynamical systems $G ∼ X$ and $H ∼ Y$ on second countable compact spaces $X$ and $Y$. By Theorem 2.16, $G ∼ X$ and $H ∼ Y$ are Kakutani equivalent, so there exist clopen subspaces $A ⊆ X$ and $B ⊆ Y$ with $G.A = X, H.B = Y$, together with an isomorphism of topological groupoids $\chi: (X ∗ G)|A \xrightarrow{\cong} (Y ∗ H)|B$. Let $\Phi: C^*(X ∗ G)|A \xrightarrow{\cong} C^*(Y ∗ H)|B$ be the corresponding isomorphism of groupoid $C^*$-algebras. Then the one-to-one correspondence $\mathcal{L} \leftrightarrow \mathcal{M}$ from Theorem 3.7 translates to a one-to-one correspondence $(\pi, \sigma) \mapsto (\rho, \tau)$ between covariant representations of $G ∼ X$ and $H ∼ Y$, where $(\rho, \tau)$ is uniquely determined (up to unitary equivalence) by the requirement that $(\rho \times \tau)[C^*((Y ∗ H)|B)] \circ \Phi = \pi \times \sigma|C^*((X ∗ G)|A)$. Here we view $C^*((X ∗ G)|A)$ as full corners in $C(Y) ∗ H$ and $C(X) ∗ G$. We write $(\rho, \tau) = \text{Ind}_{\Phi^{-1}}(\pi, \sigma)$ and $(\pi, \sigma) = \text{Ind}_\Phi(\rho, \tau)$.

Corollary 3.11. Let $G ∼ X$ and $H ∼ Y$ be topological dynamical systems on second countable compact spaces $X$ and $Y$, and assume that there is a continuous orbit couple for $G ∼ X$ and $H ∼ Y$. Let $(\pi, \sigma) \mapsto (\rho, \tau)$ be as in Remark 3.10. Then we have $H^*(G, \sigma) \cong H^*(H, \tau)$ and $\hat{\text{R}}^*(G, \sigma) \cong \hat{\text{R}}^*(H, \tau)$.

Remark 3.12. Theorem 3.7 and Corollary 3.11 have natural analogues in homology, i.e., for $H_*$ and $\hat{H}_*$.

3.3. Quasi-isometry invariance of property $H_{FD}$. As a consequence of Theorem 3.7, we discuss quasi-isometry invariance of Shalom’s property $H_{FD}$ from [28]. In this section (§3.3), we assume that our spaces are second countable. Let us start with the following

Lemma 3.13. Let $G ∼ \hat{\mathcal{X}} \hat{\otimes} H$ be a topological groupoid, let $\alpha$ and $\beta$ be as in §2.2.1, let $G ∼ \hat{\mathcal{Y}}, H ∼ \hat{\mathcal{X}}$ be the actions given by $g.x = g\alpha(g,x)^{-1}, h.y = \beta(y,h^{-1})^{-1}xh^{-1}$, and let $\hat{\mathcal{X}} ∗ G, \hat{\mathcal{Y}} ∗ H$ be the corresponding transformation groupoids. Then

$$\hat{\mathcal{X}} ∗ G \to (\mathcal{X} ∗ (G ∗ H))|\hat{\mathcal{X}}, (x,g) \mapsto (x,g,\alpha(g^{-1},x)^{-1})$$

$$\hat{\mathcal{Y}} ∗ H \to (\mathcal{X} ∗ (G ∗ H))|\hat{\mathcal{Y}}, (y,h) \mapsto (y,\beta(y,h),h)$$

are isomorphisms of topological groupoids.

Proof. As $r(x,g) = x = r(x,g,\alpha(g^{-1},x)^{-1}), s(x,g) = x, g = g^{-1}.x = g^{-1}x\alpha(g^{-1},x)^{-1} = s(x,g,\alpha(g^{-1},x)^{-1})$ and

$$(x,g,\alpha(g^{-1},x)^{-1})(g^{-1}x\alpha(g^{-1},x)^{-1}, g, \alpha(g^{-1},g^{-1}x\alpha(g^{-1},x)^{-1})) = (x,gg^\alpha(g^{-1},x)^{-1})$$

$$(x,gg^\alpha(g^{-1},x)^{-1}, \alpha(g^{-1},g^{-1}x\alpha(g^{-1},x)^{-1})) = (x,gg^\alpha((gg^\alpha(g^{-1},x)^{-1}))^{-1})$$

$$(x,g,\alpha(g^{-1},x)^{-1})\alpha(g^{-1},g^{-1}x\alpha(g^{-1},x)^{-1})^{-1}) = (x,g\alpha(g^{-1},x)^{-1},h)$$

$$(x,gg^\alpha(g^{-1},x)^{-1}, \alpha(g^{-1},g^{-1}x\alpha(g^{-1},x)^{-1})) = (x,gg^\alpha((gg^\alpha(g^{-1},x)^{-1}))^{-1})$$

Then, the second statement is analogous. □

Given a topologically free $(G,H)$ continuous orbit couple which corresponds to the $(G,H)$ coupling $G ∼ \hat{\mathcal{X}} \hat{\otimes} H$ with compact $X$ and $Y$, the proof of Theorem 2.16 provides a concrete way to construct Kakutani equivalent dynamical systems $G ∼ X$ and $H ∼ Y$ together with clopen subspaces $A ⊆ X$ and $B ⊆ Y$ such that $\mathcal{X} ∗ G)|A \cong (Y ∗ H)|B$. We need the following

Lemma 3.14. We can modify our $(G,H)$ continuous orbit couple above, without changing its topological dynamical system $G ∼ X$, so that the described process yields a topological coupling and subspaces $A$ with $A = B$ as subspaces of $\mathcal{X}$. □

Proof. In the proof of Theorem 2.16, we had constructed $A$ and $B$ as disjoint unions $A = \bigsqcup A_g$ and $B = \bigsqcup B_g$. Following the construction of the continuous orbit couple out of our topological coupling in §2.2.1, we see that these subspaces $A_g$ and $B_g$ were related by $gA_x = B_x$ in $\mathcal{X}$. Set $Y' := (Y \setminus B) \sqcup A$. Then $X$ and $Y'$ are still fundamental domains for the $H$- and $G$-actions on $\mathcal{X}$. So we obtain a new topologically free $(G,H)$ coupling $G ∼ \hat{\mathcal{Y}} \hat{\otimes} H$. The construction in §2.2.1 yields a continuous orbit couple with new continuous orbit map $p' : X \to Y'$ satisfying $p'(X) = A$. Hence our construction in the proof of Theorem 2.16 gives us the subspaces $A ⊆ X$ and $A ⊆ Y'$ implementing the Kakutani equivalence between $G ∼ X$ and $H ∼ Y'$. □
Let $G \curvearrowright Y \Omega X \curvearrowright H$ and $G \curvearrowright X, H \curvearrowright Y$ be as above, with a clopen subspace $A \subseteq X \cap Y$ such that $G.A = X, H.A = Y$ and $(X \times G)|A \cong (Y \times H)|A$. Let $\Phi : C^*((X \times G)|A) \xrightarrow{\cong} C^*((Y \times H)|A)$ be the induced C*-isomorphism. Lemma 5.13 yields an isomorphism of C*-algebras $C(X) \rtimes G \cong 1_X(C_0(\Omega) \rtimes (G \times H))|X$, and $1_X$ is a full projection. Therefore, $C(X) \rtimes G$ is Morita equivalent to $C_0(\Omega) \rtimes (G \times H)$, and a $(C(X) \rtimes G-C_0(\Omega) \rtimes (G \times H))$-imprimitivity bimodule is given by $\mathcal{B} = 1_X(C_0(\Omega) \rtimes (G \times H))$ (with respect to the identification $C(X) \rtimes G \cong 1_X(C_0(\Omega) \rtimes (G \times H))|X$ provided by Lemma 5.13). We obtain (up to unitary equivalence) bijections between representations of $C(X) \rtimes G$ and representations of $C_0(\Omega) \rtimes (G \times H)$ and also between covariant representations of $G \curvearrowright X$ and $G \rtimes H \curvearrowright \Omega$. We denote both of them by $\text{Ind}_X$. Also, let $\mathcal{B}$ be the $(C(X) \rtimes G-C_0(\Omega) \rtimes (G \times H))$-imprimitivity bimodule given by $1_Y(C_0(\Omega) \rtimes (G \times H))$ with respect to the identification $C(Y) \rtimes H \cong 1_Y(C_0(\Omega) \rtimes (G \times H))|Y$ provided by Lemma 5.13. We define $\text{Ind}_Y$ similarly as $\text{Ind}_X$. Now we have two ways to go from covariant representations of $G \curvearrowright X$ to covariant representations of $H \curvearrowright Y$: $\text{Ind}_{\Phi^{-1}}$ introduced in Remark 3.10 and $\text{Ind}_{\Phi^{-1}}$ introduced in Remark 5.10. It turns out that they coincide.

Proposition 3.15. In the situation described above, $\text{Ind}_{\Phi^{-1}} \text{Ind}_X(\pi, \sigma)$ is unitarily equivalent to $\text{Ind}_{\Phi^{-1}}(\pi, \sigma)$ for every covariant representation $(\pi, \sigma)$ of $G \times X$.

Proof. Let $\text{Ind}_{\Phi^{-1}}(\pi, \sigma) = (\rho, \tau)$ and let $\text{Ind}_{\Phi^{-1}} \text{Ind}_X(\pi, \sigma) = (\rho', \tau')$. Let $i^X : C((X \times G)|A) \to C(X) \times G$ and $i^Y : C((Y \times H)|A) \to C(Y) \rtimes H$ be the canonical embeddings. Also, let $i_X : C(X) \rtimes G \to C_0(\Omega) \rtimes (G \times H)$ and $i_Y : C(Y) \rtimes H \to C_0(\Omega) \rtimes (G \times H)$ be the embeddings obtained with the help of Lemma 5.13. Then $(\rho, \tau)$ is uniquely determined by $(\pi \times \sigma)i^X \circ \Phi^{-1} \sim_u (\rho \times \tau)\circ i^Y$. We want to show that $\rho \times \tau$ has the same property. $(\rho', \tau')$ is uniquely determined by the existence of a representation $\Pi$ of $C_0(\Omega) \rtimes (G \times H)$ with $\Pi \circ i_X \sim_u \pi \times \sigma$ and $\Pi \circ i_Y \sim_u \rho' \times \tau'$. Hence $(\rho' \times \tau')\circ i^Y \sim_u (\pi \times \sigma)\circ i^X \circ \Phi^{-1}$.

Our claim follows. \hfill \Box

Let $G \curvearrowright X \Omega Y \curvearrowright H$ and $G \curvearrowright X, H \curvearrowright Y$ be as above. Let $A \subseteq X \cap Y$ be a clopen subspace with $G.A = X, H.A = Y$ and $(X \times G)|A \cong (Y \times H)|A$. Let $\Phi : C^*((X \times G)|A) \xrightarrow{\cong} C^*((Y \times H)|A)$ be the induced C*-isomorphism. Let $\Pi = (\Pi^X, \Pi^H)$ be a covariant representation of $G \times X$ on the Hilbert space $\mathcal{H}$. Let $\sigma$ be a unitary representation of $G$ on $\mathcal{H}_\sigma$. It is clear that $(1 \otimes \Pi^X, \sigma \otimes \Pi^X)$ is a covariant representation of $G \times X$ on $\mathcal{H}_\sigma \otimes \mathcal{H}$. Let $\text{Ind}_{\Phi^{-1}}(\sigma, \Pi)$ be the unitary representation of $H$ which is part of the covariant representation $\text{Ind}_{\Phi^{-1}}(1 \otimes \Pi^X, \sigma \otimes \Pi^X)$. Moreover, let $\tau$ be a unitary representation of $H$ on $\mathcal{H}_\tau$. Let $\Theta = (\Theta^Y, \Theta^H) = \text{Ind}_{\Phi^{-1}}(\Pi^Y, \Pi^H).$ Denote by $\text{Ind}_D(\Theta, \tau)$ the unitary representation of $G$ which is part of the covariant representation $\text{Ind}_{\Phi}(\Theta^Y \otimes 1, \Theta^H \otimes 1, \sigma \otimes \text{Ind}_D(\Theta, \tau))$.

Lemma 3.16. $(1 \otimes \Pi^Y \otimes 1, \sigma \otimes \text{Ind}_D(\Theta, \tau)) = \text{Ind}_D(1 \otimes \Theta^Y \otimes 1, 1, \text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)$.

Proof. We have to show that

$$(1 \otimes \Pi^Y \otimes 1) \circ (\sigma \otimes \text{Ind}_D(\Theta, \tau))|C((X \times G)|A) = (1 \otimes \Theta^Y \otimes 1) \times (\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)|C((Y \times H)|B) \circ \Phi.$$ Fix $g \in G$ and $h \in H$. Let $f$ be the characteristic function of a compact subset of $(X \times \{g\}) \cap (X \times G)|A$ whose image under $\chi$ lies in $(Y \times \{h\}) \cap (Y \times H)|B$. It suffices to consider such $f$ as they span a dense subset in $C^*((X \times G)|A)$. We have

$$(1 \otimes \Theta^Y \otimes 1) \times (\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)(\Phi(f)) = ((1 \otimes \Theta^Y) \times (\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)(\Phi(f))) \otimes \tau(h)$$

$$= ((1 \otimes \Pi^Y) \times (\sigma \otimes \Pi^H)(f)) \otimes \tau(h) = \sigma(g) \otimes \Pi(f) \otimes \tau(h) = \sigma(g) \otimes (\Theta(\Phi(f)) \otimes \tau(h))$$

$$= \sigma(g) \otimes ((\Pi^Y \otimes 1) \times \text{Ind}_D(\Theta, \tau)(f)) = (1 \otimes \Pi^Y \otimes 1) \times (\sigma \otimes \text{Ind}_D(\Theta, \tau)(f)).$$ \hfill \Box
Let $\Lambda$ be a representation of $C(X) \rtimes G$, and set $\tilde{\Lambda} := \text{Ind}_X \Lambda$. Let

$$\mathcal{H}_{\Lambda,c} := \left\{ \eta \in \mathcal{H}_{\Lambda}; \eta = \tilde{\Lambda}(1_K)\eta \text{ for some compact } K \subseteq \Omega \right\},$$

and let $\mathcal{L}$ be the complex vector space of linear maps $\mathcal{H}_{\Lambda,c} \to \mathbb{C}$ which are bounded whenever restricted to a subspace of the form $\tilde{\Lambda}(1_K)\mathcal{H}_{\Lambda}$, with $K \subseteq \Omega$ compact. Moreover, let $\Lambda^G$ be the unitary representation of $G$ on $\mathcal{H}_{\Lambda}$ induced by $\Lambda$, and denote by $\Lambda^G$ and $\Lambda^H$ the unitary representations of $G$ and $H$ on $\mathcal{H}_{\Lambda}$ induced by $\Lambda$. As $\mathcal{H}_{\Lambda,c}$ is obviously invariant under the $G$- and $H$-actions, we obtain by restriction $G$- and $H$-actions on $\mathcal{H}_{\Lambda,c}$. Finally, by dualizing, we obtain $G$- and $H$-actions on $\mathcal{L}$.

**Lemma 3.17.** There is a $G$-equivariant linear isomorphism $\mathcal{H}_{\Lambda} \xrightarrow{\phi} \mathcal{L}^H$.

**Proof.** Up to unitary equivalence, we have $\mathcal{H}_{\Lambda} = \tilde{\Lambda}(1_X) \mathcal{H}_{\Lambda}$, and $\Lambda^G$ is given by the composite

$$G \to C(X) \rtimes G \xrightarrow{\phi} \tilde{1}_G C_0(\Omega) \rtimes (G \times H) \tilde{1}_G \xrightarrow{\tilde{\Lambda}} \mathcal{L} \left(\tilde{\Lambda}(1_X) \mathcal{H}_{\Lambda}\right),$$

where the first map is given by $G \to C(X) \rtimes G$, $g \mapsto u_g$.

We define $L : \mathcal{H}_{\Lambda} \to \mathcal{L}$ by setting $L(\xi)(\eta) = \sum_{h \in H} \langle \tilde{\Lambda}^H(h)\xi, \eta \rangle$. Here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}_{\Lambda}$, and our convention is that it is linear in the second component. Note that in the definition of $L(\xi)(\eta)$, the sum is always finite since $\eta$ lies in $\mathcal{H}_{\Lambda,c}$. It is clear that $L$ is linear. Moreover, we have

$$L(\xi)(\tilde{\Lambda}^H(h')\eta) = \sum_h \langle \tilde{\Lambda}^H(h)\xi, \tilde{\Lambda}^H(h')\eta \rangle = \sum_h \langle \tilde{\Lambda}^H((h')^{-1}h)\xi, \eta \rangle = L(\xi)(\eta).$$

Therefore, the image of $L$ lies in $\mathcal{L}^H$, and we obtain a linear map $\mathcal{H}_{\Lambda} \to \mathcal{L}^H$. We claim that the inverse is given by $R : \mathcal{L}^H \to \mathcal{H}_{\Lambda} \cong \mathcal{H}_{\Lambda}$, where the first map is given by restriction, $l \mapsto l|_{\tilde{\Lambda}(1_X)\mathcal{H}_{\Lambda}}$, and the second map is the canonical isomorphism, identifying $\xi \in \mathcal{H}_{\Lambda}$ with the element $\langle \xi, \cdot \rangle \in \mathcal{H}_{\Lambda}^*$. Note that $l|_{\tilde{\Lambda}(1_X)\mathcal{H}_{\Lambda}}$ is bounded because of our definition of $\mathcal{H}_{\Lambda,c}$. Let us show that $R$ is the inverse of $L$. For $l \in \mathcal{L}^H$, we have

$$L(R(l))(\eta) = \sum_h \langle \tilde{\Lambda}^H(h)R(l), \eta \rangle = \sum_h \langle R(l), \tilde{\Lambda}^H(h^{-1})\eta \rangle = \sum_h l(\tilde{\Lambda}(1_X)\tilde{\Lambda}^H(h^{-1})\eta) = \sum_h l(\tilde{\Lambda}(1_X)\eta) = l(\xi).$$

For $\xi \in \mathcal{H}_{\Lambda} = \tilde{\Lambda}(1_X)\mathcal{H}_{\Lambda}$, we have $R(L(\xi)) = \xi$ since

$$L(\xi)(\tilde{\Lambda}(1_X)\eta) = \sum_h \langle \tilde{\Lambda}^H(h)\xi, \tilde{\Lambda}(1_X)\eta \rangle = \sum_h \langle \tilde{\Lambda}^H(h)\tilde{\Lambda}(1_X)\eta, \xi \rangle = \langle \xi, \eta \rangle.$$

because $\tilde{\Lambda}(1_X)\eta = \xi$ if $h = e$ and $\tilde{\Lambda}(1_X)\eta = 0$ if $h \neq e$.

Finally, let us show that $L$ is $G$-equivariant:

$$L(\Lambda^G(g)\xi)(\eta) = \sum_h \langle \tilde{\Lambda}^H(h)(\Lambda^G(g)\xi), \eta \rangle = \sum_h \sum_j \langle \tilde{\Lambda}^H(h)\tilde{\Lambda}(1_X j^{-1} g)\Lambda^G(g)^j \xi, \eta \rangle = \sum_{h,j} \langle \tilde{\Lambda}^H(h)\Lambda^H(j^{-1})\tilde{\Lambda}(1_X j^{-1} g)\Lambda^G(g)^j \xi, \eta \rangle = \sum_{h,j} \langle \tilde{\Lambda}^H(h)\tilde{\Lambda}(1_X g)\Lambda^G(g)^j \xi, \eta \rangle = \sum_h \langle \Lambda^G(g)\tilde{\Lambda}^H(h)\tilde{\Lambda}(1_X)\xi, \eta \rangle = \sum_h \langle \tilde{\Lambda}^H(h)\xi, \Lambda^G(g)^{-1} \eta \rangle = L(\xi)(\Lambda^G(g)^{-1} \eta).$$

□

**Corollary 3.18.** We have $\{\Lambda^G$-invariant vectors$\} = \mathcal{H}^G \cong \mathcal{L}^{G \times H}$.

**Theorem 3.19.** There exists a one-to-one correspondence between $(\text{Ind}_{\phi^{-1}}(\sigma, \Pi) \otimes \tau)$-invariant vectors and $(\sigma \otimes \text{Ind}_{\phi}(\Theta, \tau))$-invariant vectors.

**Proof.** Obviously, $(1 \otimes \Pi^\delta \otimes 1, \sigma \otimes \text{Ind}_{\phi}(\Theta, \tau))$ is a covariant representation of $G \rtimes X$. Let $\Lambda := (1 \otimes \Pi^\delta \otimes 1) \rtimes (\sigma \otimes \text{Ind}_{\phi}(\Theta, \tau))$. Set $\Lambda := \text{Ind}_X \Lambda$, and define $\mathcal{L}$ as in Lemma 3.17. Then Corollary 3.18 yields a one-to-one correspondence between $(\sigma \otimes \text{Ind}_{\phi}(\Theta, \tau))$-invariant vectors and $\mathcal{L}^{G \times H}$.

Let $\text{Ind}_{\phi^{-1}} \Lambda$ be the representation of $C(Y) \rtimes H$ corresponding to $\text{Ind}_{\phi^{-1}}(1 \otimes \Pi^\delta \otimes 1, \sigma \otimes \text{Ind}_{\phi}(\Theta, \tau))$. By Proposition 3.15, $\text{Ind}_{\phi^{-1}} \Lambda \sim_u \Lambda$. Hence, together with Lemma 3.16, Corollary 3.18 yields a one-to-one correspondence between $(\text{Ind}_{\phi^{-1}}(\sigma, \Pi) \otimes \tau)$-invariant vectors and $\mathcal{L}^{G \times H}$. 

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All in all, we obtain
\[
\{(\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)\text{-invariant vectors}\} \overset{1:1}{\longleftrightarrow} \mathcal{L}^{G \times H} \overset{1:1}{\longleftrightarrow} \{((\sigma \otimes \text{Ind}_{\Phi}(\Theta, \tau))\text{-invariant vectors}\}.
\]

**Corollary 3.20.** \(\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau\) has an invariant vector if and only if \(\sigma \otimes \text{Ind}_{\Phi}(\Theta, \tau)\) has an invariant vector.

We now come to Shalom’s property \(H_{FD}\). Recall that a group \(G\) has \(H_{FD}\) if for every unitary representation \(\sigma\) of \(G\), \(\hat{H}^1(G, \sigma) \not\cong \{0\}\) implies that \(\sigma\) contains a finite dimensional subrepresentation.

**Definition 3.21.** A \((G, H)\) continuous orbit couple is called \(H^1\)-faithful if its \(G\)- and \(H\)-spaces are second countable compact, and its topological dynamical system \(G \curvearrowright X\) has the property that for every unitary representation \(\sigma\) of \(G\) with \(\hat{H}^1(G, \sigma) \not\cong \{0\}\), there exists a covariant representation \((\Pi^X, \Pi^G)\) of \(G \curvearrowright X\) such that \(\hat{H}^1(G, (\sigma \otimes \Pi^G) \not\cong \{0\}\).

**Theorem 3.22.** Let \(G, H\) be countable discrete groups. Suppose there exists an \(\hat{H}^1\)-faithful topologically free \((G, H)\) continuous orbit couple. If \(H\) has property \(H_{FD}\), then \(G\) has property \(H_{FD}\).

For the proof, let us recall the following observation which is explained in [28, § 3.1]:

**Lemma 3.23.** A unitary representation \(\sigma\) of a countable discrete group \(G\) contains a finite dimensional subrepresentation if and only if \(\sigma\) is a unitary representation \(\sigma'\) of \(G\) such that \(\sigma \otimes \sigma'\) has an invariant vector.

**Proof of Theorem 3.22.** By Lemma [3.14], we may assume that our \(\hat{H}^1\)-faithful topologically free \((G, H)\) continuous orbit couple corresponds to a topologically free \((G, H)\) coupling \(G \curvearrowright Y \times H\) with second countable compact spaces \(X\) and \(Y\), which leads to topological dynamical systems \(G \curvearrowright X\) and \(H \curvearrowright Y\) together with a clopen subspace \(A \subseteq X \cap Y\) with \(G.A = X, H.A = Y\) and \((X \times G)|A \cong (Y \times H)|A\). Now let \(\sigma\) be a unitary representation of \(G\) with \(\hat{H}^1(G, \sigma) \not\cong \{0\}\). By \(\hat{H}^1\)-faithfulness, there exists a covariant representation \((\Pi^X, \Pi^G)\) of \(G \curvearrowright X\) with \(\hat{H}^1(G, \Pi^G) \not\cong \{0\}\). By Corollary [3.11], \(\hat{H}^1(H, \text{Ind}_{\Phi^{-1}}(\sigma, \Pi)) \cong \hat{H}^1(G, (\sigma \otimes \Pi^G))\), so that \(\hat{H}^1(H, \text{Ind}_{\Phi^{-1}}(\sigma, \Pi)) \not\cong \{0\}\). As \(H\) has property \(H_{FD}\), \(\text{Ind}_{\Phi^{-1}}(\sigma, \Pi)\) must have a finite dimensional subrepresentation. Thus Lemma [3.23] implies that there is a \(\sigma\) is a unitary representation \(\tau\) of \(H\) such that \(\text{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau\) has an invariant vector. By Corollary [3.20], \(\sigma \otimes \text{Ind}_{\Phi}(\Theta, \tau)\) must have an invariant vector. Again by Lemma [3.23], this implies that \(\sigma\) has a finite dimensional subrepresentation. Hence \(G\) has property \(H_{FD}\).

**Remark 3.24.** A \((G, H)\) continuous orbit couple with second countable compact \(G\)- and \(H\)-spaces is \(\hat{H}^1\)-faithful if its topological dynamical system \(G \curvearrowright X\) admits a \(G\)-invariant probability measure. To see this, let \(\mu\) be such a measure. Let \((\Pi^X, \Pi^G)\) be the canonical covariant representation of \(G \curvearrowright X\) on \(L^2(\mu)\). Then \(\Pi^G\) contains the trivial representation, so that \(\sigma \otimes \Pi^G\) contains \(\sigma\). This shows \(\hat{H}^1\)-faithfulness. In particular, this is the case when \(G\) is amenable. Therefore, Theorem [2.17] and Theorem [3.22] imply [28, Theorem 4.3.3]. The case of amenable groups is not the only situation where invariant probability measures exist. It follows easily from [7] and Theorem [2.17] that for residually finite groups \(G\) and \(H\) with coercively equivalent box spaces, there exists a \((G, H)\) continuous orbit couple with second countable compact \(G\)- and \(H\)-spaces such that its topological dynamical system \(G \curvearrowright X\) admits a \(G\)-invariant probability measure.

4. Applications to (co)homology II

We now produce many new quasi-isometry invariants of (co)homological nature.

4.1 Uniform embeddings and res-invariant modules. Let \(G\) be a group, \(R\) a commutative ring with unit and \(W\) an \(R\)-module. Let \(C(G, W)\) be the set of functions from \(G\) to \(W\). The \(G\)-action on itself by left multiplication induces a canonical left \(RG\)-module structure on \(C(G, W)\). Explicitly, given \(g \in G\) and \(f \in C(G, W)\), \(g.f\) is the element in \(C(G, W)\) given by \((g.f)(x) = f(g^{-1}.x)\) for all \(x \in G\). We are interested in the following class of \(RG\)-submodules of \(C(G, W)\). Given a subset \(A\) of \(G\), let \(1_A\) be its indicator function, i.e., \(1_A \in C(G, R)\) is given by \(1_A(x) = 1\) if \(x \in A\) and \(1_A(x) = 0\) if \(x \notin A\). Here 1 is the unit of \(R\). Given \(f \in C(G, W)\) and \(A \subseteq G\), we form the pointwise product \(1_A \cdot f \in C(G, W)\). This is nothing else but the restriction of \(f\) to \(A\), extended by 0 outside of \(A\) to give a function \(G \to W\).

**Definition 4.1.** An \(RG\)-submodule \(L \subseteq C(G, W)\) is called res-invariant if \(1_A \cdot f\) lies in \(L\) for all \(f \in L\) and \(A \subseteq G\).
Examples 4.2. For arbitrary $R$ and $W$, $\{f \in C(G,W) : \text{supp}(f) \text{ is finite}\} \cong RG \otimes_R W$ and $C(G,W)$ are res-invariant.

If $R = \mathbb{R}$ or $R = \mathbb{C}$, $W = R$, then for all $0 < p \leq \infty$, $\{f \in C(G,W) : \sum_{x \in G} |f(x)|^p < \infty\} = \ell^p(G,W)$ is res-invariant. Also, $\{f \in C(G,W) : \lim_{x \to e} f(x) = 0\} = c_0(G,W)$ is res-invariant.

Let $G$ be a finitely generated discrete group and $\ell$ the right-invariant word length coming from a finite symmetric set of generators. Let $R = \mathbb{R}$ or $R = \mathbb{C}$ and $W = R$. As in [13], we define for $s \in R$ and $1 \leq p \leq \infty$ the Sobolev space $H^{s,p}(G,W) := \{f : G \to W : f : (1 + \ell)^s \in \ell^p(G,W)\}$, and $H^{\infty,p}(G,W) := \bigcap_{p \in \mathbb{R}} H^{s,p}(G,W)$. All these Sobolev spaces are res-invariant.

In the last examples ($\ell^p, c_0$ and $H^{s,p}$), we can also replace $W$ by any normed space over $R$.

We are also interested in the following topological setting: Let $R$ be a topological field and $W$ an $R$-module.

**Definition 4.3.** A topological res-invariant $RG$-submodule $L$ of $C(G,W)$ is a res-invariant $RG$-submodule of $C(G,W)$ together with the structure of a topological $R$-vector space on $L$ such that

(3) for every $A \subseteq G$, $L \to L$, $f \mapsto 1_A \cdot f$ is continuous,

(4) for every $g \in G$, $L \to L$, $f \mapsto g \cdot f$ is continuous.

When we consider topological res-invariant modules, $R$ will always be a topological field, though we might not mention this explicitly. For instance, in $\mathbb{R}$, $\ell^p(G,W)$ and $c_0(G,W)$ are topological res-invariant modules.

Also, $H^{s,p}(G,W)$ becomes a topological res-invariant module with respect to the topology induced by the norm $\|f\|_{s,p} = \|f \cdot (1 + \ell)^s\|_{\ell^p(G,W)}$ for $s \in R$, and with respect to the projective limit topology for $s = \infty$.

In the following, we explain how uniform embeddings interact with res-invariant modules. Recall that all our groups are countable and discrete, and that a map $\varphi : G \to H$ between groups $G$ and $H$ is a uniform embedding if for every $S \subseteq G \times G$, $\{st^{-1} : (s,t) \in S\}$ is finite if and only if $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\}$ is finite (Definition [11]).

**Remark 4.4.** Let $\varphi : G \to H$ be a uniform embedding. Given $g \in G$, let $S = \{(g^{-1}x,x) : x \in G\}$. Then $\{st^{-1} : (s,t) \in S\} = \{g^{-1}\}$ is finite, so that $\varphi(\{g^{-1}x\}) = \varphi(x) \in G$. In other words, we can find a finite decomposition $G = \bigsqcup_{i \in I} X_i$, where $I$ is a finite index set, and a finite subset $\{h_i : i \in I\} \subseteq H$ such that $\varphi(g^{-1}X_i) = h_i^{-1}\varphi(x)$ for all $x \in X_i$ and $i \in I$.

**Lemma 4.5.** Let $\varphi : G \to H$ be a uniform embedding, and let $Y := \varphi(G)$. Then we can find $X \subseteq G$ such that the restriction of $\varphi$ to $X$, $X \to Y$, $x \mapsto \varphi(x)$, is a bijection. In addition, we can find a finite decomposition $G = \bigsqcup_{i \in I} X_i$, $g(i) \in G$ for $1 \leq i \leq I$, and $h(i) \in H$ for $1 \leq i \leq I$, such that $X_i = g(i)^{-1}X(i)$ for some $X(i) \subseteq X$ and $\varphi(x) = h(i)\varphi(g(i)x)$ for all $x \in X_i$ and $1 \leq i \leq I$. We can always arrange $g(1) = e$ (the identity in $G$), $h(1) = e$ (the identity in $H$), and $X_1 = X(1) = X$.

**Proof.** By Lemma [20] we can find $X$ such that the restriction of $\varphi$ to $X$ is bijective onto its image and that there are finitely many $g(i) \in G$, $1 \leq i \leq I$, such that $G = \bigsqcup_{i = 1}^I g(i)^{-1}X(i)$, where we can certainly arrange $g(1) = e$. Now define recursively $X_1 := X$ and $X(i) = X \setminus g(i)^{-1}X \cup \ldots \cup g(i-1)^{-1}X_{i-1}$. Then $G = \bigsqcup_{i = 1}^I g(i)^{-1}X(i)$. Using Remark [21] we can further decompose each $X(i)$ to guarantee that there exist $h(i) \in H$ for $1 \leq i \leq I$ such that $\varphi(x) = h(i)\varphi(g(i)x)$ for all $x \in g(i)^{-1}X(i)$ and $1 \leq i \leq I$. Setting $X_i := g(i)^{-1}X(i)$, we are done. □

Recall that two maps $\varphi, \varphi : G \to H$ are uniformly close (written $\varphi \sim \varphi$) if $\{\varphi(x)\varphi(x)^{-1} : x \in G\}$ is finite.

**Remark 4.6.** If $\varphi, \varphi : G \to H$ are uniformly close, then there is a finite decomposition $G = \bigsqcup_{i \in I} X_i$, where $I$ is a finite index set, and a finite subset $\{h_i : i \in I\} \subseteq H$ such that we have $\varphi(x) = h_i\varphi(x)$ for all $x \in X_i$ and $i \in I$.

Let $\varphi : G \to H$ be a uniform embedding. Note that for every $y \in H$, $\varphi^{-1}\{\{y\}\}$ is finite. To see this, let $S = \varphi^{-1}\{\{y\}\} \times \varphi^{-1}\{\{y\}\}$. Then $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\} = \{e\}$, so that $\{st^{-1} : (s,t) \in S\}$ must be finite. Hence $\varphi^{-1}\{\{y\}\}$ is finite. Therefore, given $f \in C(G,W)$, we may define $\varphi_*(f) \in C(H,W)$ by setting $\varphi_*(f)(y) = \sum_{x \in G} f(x)$. Moreover, given $f \in C(H,W)$, define $\varphi^*(f) = f \circ \varphi \in C(G,M)$.

**Definition 4.7.** Given a res-invariant $RG$-submodule $L$ of $C(G,W)$, let $\varphi_*L$ be the smallest res-invariant $RH$-submodule of $C(H,W)$ containing $\{\varphi_*(f) : f \in L\}$. Given a res-invariant $RH$-submodule $M$ of $C(G,W)$, let $\varphi^*M$ be the smallest res-invariant $RG$-submodule of $C(H,W)$ containing $\{\varphi^*(f) : f \in M\}$.

**Lemma 4.8.** We have

(5) $\varphi_*L = \langle \{h \cdot \varphi_*(f) : h \in H, f \in L\}\rangle_R$

(6) $\varphi^*M = \langle \{1_A \cdot \varphi^*(f) : f \in M, A \subseteq G\}\rangle_R$. 

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Lemma 4.9. Proof. We obviously have “⊇” in (5). To show “⊆”, it suffices to show that the right-hand side is res-invariant as it is obviously an RH-submodule. Given $B \subseteq H$, we have for all $h \in H$ and $f \in L$ that

$$1_B \cdot (h \cdot \varphi_s(f)) = h \cdot (1_{h^{-1}B} \cdot \varphi_s(f)) = h \cdot \varphi_s \left( 1_{h^{-1}(h^{-1}B)} \cdot f \right),$$

which lies in the right-hand side as $L$ is res-invariant.

From (6), we again have “⊇” by construction. As the right-hand side is res-invariant, it suffices to show that it is an RG-submodule in order to prove “⊆”. Given $g \in G$, by Remark 4.6 we can find a finite decomposition $G = \bigsqcup_{i \in I} X_i$ and a finite subset $\{ h_i : i \in I \} \subseteq H$ such that $\varphi(g^{-1}x) = h_i^{-1} \varphi(x)$ for all $x \in X_i$ and $i \in I$. Then, for all $A \subseteq G$, $g \cdot (1_A \cdot \varphi^*(f)) = 1_{A \cdot g^{-1}} \cdot (g \cdot \varphi^*(f)) = \sum_{i \in I} 1_{X_i} \cdot 1_{g^{-1}A} \cdot (\varphi^*(h_i \cdot f))$ lies in the right-hand side of (6) as $M$ is an RH-submodule. \hfill \Box

Note that in general, $\varphi_s L$ is not equal to $\{ \varphi_s(f) : f \in L \}$, and $\varphi^* M$ is not equal to $\{ \varphi^*(f) : f \in M \}$.

**Lemma 4.9.** (i) If $\varphi, \phi : G \to H$ are uniform embeddings with $\varphi \sim \phi$, then $\varphi \circ \phi \circ \psi = \phi \circ \varphi \circ \psi$ for all $L$ and $M$.

(ii) If $\psi : G \to H$ and $\psi : H \to K$ are uniform embeddings, then $\psi \bullet (\varphi \circ \phi) = (\psi \circ \phi) \circ \psi$ for all $L$ and $N$.

Proof. (i) Let us show $\varphi \circ \phi \circ \psi = \psi \circ \varphi \circ \psi$. By symmetry, we have $\phi \circ \varphi \circ \psi = \psi \circ \varphi \circ \psi$. Let $I$, $\{ X_i : i \in I \}$ and $\{ h_i : i \in I \}$ be as above. We have that

$$\varphi^* (f) = \sum_{i \in I} 1_{X_i} \cdot \varphi^*(f) = \sum_{i \in I} 1_{X_i} \cdot \varphi^*(h_i \cdot f) \in \varphi^* M$$

for all $f \in M$. Hence $\varphi^* M \subseteq \varphi^* M$. By symmetry, we have $\varphi^* M \subseteq \varphi^* M$.

(ii) Let us show $\psi \circ (\varphi \circ \phi) = (\psi \circ \phi) \circ \psi$. Obviously, “⊇” holds as $\psi \bullet (\varphi \circ \phi) \subseteq \psi \bullet (\psi \circ \phi)$ for all $f \in L$. Let us show “⊆”. By (5), it suffices to show that $\psi \bullet (\varphi \circ \phi) \subseteq (\psi \circ \phi) \circ \psi$ for all $h \in H$ and $f \in L$. By Remark 4.6, we can find a finite decomposition $H = \bigsqcup_{i \in I} Y_i$ and a finite subset $\{ k_i : i \in I \} \subseteq K$ such that $\psi(h^{-1}y) = k_i^{-1} \psi(y)$ for all $y \in Y_i$ and $i \in I$. Then

$$\psi \bullet (\varphi \circ \phi) = \sum_{i \in I} 1_{X_i} \cdot (\psi \circ \phi)(f) = \sum_{i \in I} 1_{X_i} \cdot (\psi \circ \phi)(h_i \cdot f) \subseteq (\psi \circ \phi) \circ \psi$$

lies in $(\psi \circ \phi) \circ \psi$ for all $f \in L$, as $L$ is res-invariant. This shows “⊆”.

Let us show $\psi \circ (\psi \circ \phi) \subseteq (\psi \circ \phi) \circ \psi$. “⊇” holds as $\psi \circ (\psi \circ \phi) \subseteq \psi \circ (\psi \circ \phi)$. By (6), it suffices to prove that $\psi \circ (\psi \circ \phi) \subseteq (\psi \circ \phi) \circ \psi$ for all $f \in N$. Let us prove “⊆”. By (5), it suffices to prove that $\psi \circ (\psi \circ \phi) \subseteq (\psi \circ \phi) \circ \psi$ for all $f \in N$. We have

$$\psi \circ (\psi \circ \phi) = \psi \circ (\psi \circ \phi) \circ \psi = \psi \circ (\psi \circ \phi) \circ \psi$$

which lies in $(\psi \circ \phi) \circ \psi$ as the latter is res-invariant. This shows “⊆”. \hfill \Box

Our next goal is to define a suitable topology on $\varphi_s L$ in case $L$ is a topological res-invariant RG-submodule of $C(G, W)$. We start with some preparations. Let $\varphi : G \to H$ be a uniform embedding, and set $Y := \varphi(G)$. Lemma 2.20 gives us a start with some preparations. Let $\varphi : G \to H$ be a uniform embedding, and set $Y := \varphi(G)$.

**Lemma 4.10.** Let $\bar{Y} \subset Y$ and $\bar{X} = X \cap \varphi^{-1}(\bar{Y})$. Then $1_{\bar{X}} \cdot L \to 1_{\bar{Y}} \cdot (\varphi_L)$ is an isomorphism of $R$-modules.
Proof. Our map is definitely $R$-linear, so it suffices to prove that it is bijective. Injectivity holds as we can recover $f$ from $\varphi_*(f)$ using

$$\varphi^*(\varphi_*(f))(\bar{x}) = \varphi_*(f)(\varphi(\bar{x})) = \sum_{x \in G, \varphi(x) = \varphi(\bar{x})} f(x) = f(\bar{x})$$

for $f \in 1_R \cdot L$ and $\bar{x} \in \bar{X}$. For surjectivity, (5) implies that it suffices to show that for all $h \in H$ and $f \in L$, $1_R \cdot (h \cdot \varphi_*(f))$ lies in the image of our map. Fix $h \in H$. Consider $S = \{(s,t) \in \bar{G} \times \bar{X}: \varphi(s) = h^{-1} \varphi(t)\}$. By construction, $\{(s) \varphi(t)^{-1}: (s,t) \in S\} = \{h^{-1}\}$, so $\varphi$ is a uniform embedding, this implies that $F := \{s^{-1}: (s,t) \in S\}$ is finite. Enumerate $F$, say $F = \{g_1, \ldots, g_n\}$ (where the $g_i$ are pairwise distinct). Define recursively $X_1 := \{\bar{x} \in \bar{X}: \varphi(g_1 \bar{x}) = h^{-1} \varphi(\bar{x})\}$, and $X_i := \{\bar{x} \in \bar{X}: \varphi(g_i \bar{x}) = h^{-1} \varphi(\bar{x})\} \setminus \{g_j \cdot x \in X_{i-1} \}$. Then for all $\bar{x} \in \bar{X}$, we have

$$\{x \in G: \varphi(x) = h^{-1} \varphi(\bar{x})\} = \{g_i \bar{x}: \bar{x} \in X_i, 1 \leq i \leq n\}.$$  

(8) "$\supseteq"$ is clear, and to show "$\subseteq"$, take $x \in G$ with $\varphi(x) = h^{-1} \varphi(\bar{x})$. Then there is a smallest index $j$ with $1 \leq j \leq n$ such that $x = g_j \bar{x}$. Then $\bar{x}$ lies in $X_j$, because if not, then $\bar{x} = g_j^{-1} g_i x$ for some $x_i \in X_i$ with $i < j$. But then $x = g_i x_i$, and thus $h^{-1} \varphi(\bar{x}) = \varphi(x) = \varphi(g_i x_i) = h^{-1} \varphi(x_i)$, which implies $\bar{x} = x_i$ as $\varphi$ is injective on $X_i$. Hence $x = g_i \bar{x}$, contradicting minimality of $j$. Hence (8) holds. We now claim

$$1_R \cdot (h \cdot \varphi_*(f)) = \varphi_*(\sum_{i=1}^n 1_{X_i} \cdot (g_i^{-1} \cdot f)) \quad \text{for all } f \in L.$$  

Here is the reason: If $y \notin \bar{Y}$, then for every $1 \leq i \leq n$ and $x \in X_i$, $\varphi(x)$ lies in $\varphi(X_i) = \varphi(\bar{X}) = \bar{Y}$, so that

$$\varphi_*(\sum_{i=1}^n 1_{X_i} \cdot (g_i^{-1} \cdot f)) (y) = \sum_{x \in G, \varphi(x) = y} \sum_{i=1}^n (1_{X_i} \cdot (g_i^{-1} \cdot f))(x) = 0 = (1_R \cdot (h \cdot \varphi_*(f))) (y).$$  

If $y \in \bar{Y}$, then for $\bar{x} \in \bar{X}$ with $\varphi(\bar{x}) = y$, we have

$$\varphi_*(\sum_{i=1}^n 1_{X_i} \cdot (g_i^{-1} \cdot f)) (y) = \sum_{x \in G, \varphi(x) = y} \sum_{i=1}^n 1_{X_i}(x) \cdot f(g_i x) = \sum_{i=1}^n 1_{X_i}(\bar{x}) \cdot f(g_i \bar{x}) = f(x) = \varphi_*(f)(h^{-1} \cdot y) = (h \cdot \varphi_*(f))(y) = (1_R \cdot (h \cdot \varphi_*(f)))(y).$$

This shows that $1_R \cdot (h \cdot \varphi_*(f))$ lies in the image of our map. \hfill \Box

For $j \in \mathbb{Z}$, $j \geq 1$, set $X_j := X \cap \varphi^{-1}(Y_j)$. Obviously, for all $j \geq 1$, we have $1_{h_j Y_j} \cdot (\varphi, L) = h_j \cdot (1_{Y_j} \cdot (\varphi, L))$. Thus $1_{h_j Y_j} \cdot L \to h_j \cdot 1_{Y_j} \cdot (\varphi, L), f \mapsto h_j \cdot \varphi_*(f)$ is an isomorphism. For $J \in \mathbb{Z}, J \geq 1$, define

$$\Phi' : \bigoplus_{j=1}^J 1_{X_j} \cdot L \to \varphi_*(L), (f_j) \mapsto \sum_{j=1}^J h_j \cdot \varphi_*(f_j).$$

Definition 4.11. Let $L$ be a topological res-invariant RG-submodule of $C(G,W)$. Let $\tau$ be the finest topology on $\varphi, L$ such that for all $J \in \mathbb{Z}, J \geq 1$, $\Phi'$ is continuous. Here $1_{X_j} \cdot L$ is given the subspace topology from $L$, and $\bigoplus_{j=1}^J 1_{X_j} \cdot L$ is given the product topology.

Lemma 4.12. $\tau$ is the finest topology on $\varphi, L$ satisfying the following properties:

(T1) $\varphi(L, \tau)$ is a topological res-invariant RH-submodule of $C(H,W)$.

(T2) $L \to (\varphi, L, \tau), f \mapsto \varphi_*(f)$ is continuous.

Proof. Let us first show that $\tau$ satisfies (T1) and (T2). $(\varphi, L, \tau)$ is a topological $R$-vector space as for every $J$, $\bigoplus_{j=1}^J 1_{X_j} \cdot L$ is one and $\Phi'$ is $R$-linear.

Given $B \subseteq H$, let us show that $\varphi(B, \tau) : (\varphi, V, \tau) \to (\varphi, V, \tau), f \mapsto 1_B \cdot f$ is continuous. By definition of $\tau$, it suffices to show that the composition $\text{res}_B \circ \Phi'$ is continuous for all $J$. For every $J$, we have a commutative
Hence we have a commutative diagram
\[
\begin{array}{ccc}
(\varphi_L, \tau) & \xrightarrow{\text{res}_\beta} & (\varphi_L, \tau) \\
\Phi^j & \downarrow & \downarrow \Phi^j \\
\bigoplus_{j=1}^{J} 1_{X_j} \cdot L & \rightarrow & \bigoplus_{j=1}^{J} 1_{X_j} \cdot L
\end{array}
\]
where the lower horizontal map is given by \( \bigoplus_{j=1}^{J} 1_{X_j} \cdot L \rightarrow \bigoplus_{j=1}^{J} 1_{X_j} \cdot L, (f_j) \mapsto \left( 1_{\varphi^{-1} (\partial_i^{-1} \beta) \cdot f_j} \right)_j \). It is continuous as \( L \) satisfies (3). \( \Phi^j \) is continuous by construction of \( \tau \). Thus \( \text{res}_\beta \circ \Phi^j \) is continuous for all \( J \).

Let us show that for every \( h \in H \), \( (\varphi_L, \tau) \rightarrow (\varphi_L, \tau), f \mapsto h \cdot f \) is continuous. By construction of \( \tau \), it suffices to show that for every \( J \geq 1, \bigoplus_{j=1}^{J} 1_{X_j} \cdot L \rightarrow (\varphi_L, \tau), (f_j) \mapsto \sum_{j=1}^{J} h_j \cdot \varphi(f_j) \) is continuous. As addition is continuous, it suffices to show that for every \( j \), \( h_j \cdot \varphi(f) = \sum_{j=1}^{J} h_j \cdot (1_{X_j} \cdot \varphi(f_j)) \). Set \( \hat{h}_j := h_j^{-1} \cdot h_h \).

By (9), there exists \( n_j \geq 1 \), \( X_j \subseteq X_j \) and \( f_{ij} \in G \) for \( 1 \leq i \leq n_j \) such that for all \( f \in 1_{\hat{X}_j} \cdot L \), we have
\[
1_{X_j} \cdot (\hat{h}_j \cdot \varphi(f)) = \varphi \left( \sum_{j=1}^{n_j} 1_{X_{ij}} \cdot (g_{ij}^{-1} \cdot f) \right) \cdot \varphi \left( \sum_{j=1}^{J} 1_{X_j} \cdot (g_{ij}^{-1} \cdot f) \right).
\]
Hence we obtain a commutative diagram
\[
\begin{array}{ccc}
1_{\hat{X}} \cdot L & \xrightarrow{h \cdot \varphi, (\cdot)} & (\varphi_L, \tau) \\
\bigoplus_{j=1}^{J} 1_{X_j} \cdot L & \downarrow \Phi^j & \downarrow \Phi^j \\
1_{X_j} \cdot L & \rightarrow & 1_{X_j} \cdot L
\end{array}
\]
where the left vertical arrow is given by \( 1_{\hat{X}} \cdot L \rightarrow \bigoplus_{j=1}^{J} 1_{X_j} \cdot L, f \mapsto \left( \sum_{j=1}^{n_j} 1_{X_{ij}} \cdot (g_{ij}^{-1} \cdot f) \right) \). It is continuous as \( L \) satisfies (3) and (4). \( \Phi^j \) is continuous by construction of \( \tau \). We conclude that the upper horizontal map is continuous as well.

This shows that \( \tau \) satisfies (T1).

Let us show that \( \tau \) satisfies (T2). By Lemma 3.5 there are a finite decomposition \( G = \bigsqcup_{i=1}^{I} X_i \) and finite subsets \( \{g(i) : 1 \leq i \leq I\} \subseteq G \), \( \{h(i) : 1 \leq i \leq I\} \subseteq H \) such that \( X_i = g(i)^{-1} X(i) \) for \( X(i) \subseteq X \) and \( \varphi(x) = h(i) \varphi(g(i)x) \) for all \( x \in X_i \) and \( 1 \leq i \leq I \). We have \( L = \bigoplus_{i=1}^{I} 1_{X_i} \cdot L \) as topological \( R \)-vector spaces. Thus, in order to show that \( L \rightarrow \varphi_L, f \mapsto \varphi(f) \) is continuous, it suffices to show that \( \varphi_L(\cdot) : 1_{X_i} \cdot L \rightarrow \varphi_L, f \mapsto \varphi(f) \) is continuous for all \( i \). For all \( f \in 1_{X_i} \cdot L \) and \( y \in H \), we have
\[
\varphi_L(f)(y) = \sum_{x \in X_i} f(x) = \sum_{x \in X_i} (g(i)f)(g(i)x) = \sum_{x \in X(i)} (g(i)f)(x) = h(i) \varphi(g(i)f)(y).
\]
Hence we have a commutative diagram
\[
\begin{array}{ccc}
1_{X_i} \cdot L & \xrightarrow{\varphi, (\cdot)} & (\varphi_L, \tau) \\
\downarrow & & \downarrow \\
1_{X} \cdot L & \rightarrow & 1_{X} \cdot L
\end{array}
\]
where the left vertical arrow is given by \( 1_{X_i} \cdot L \rightarrow 1_{X} \cdot L, f \mapsto g(i)f \) and the diagonal arrow is given by \( 1_{X} \cdot L \rightarrow (\varphi_L, \tau), f \mapsto h(i) \cdot \varphi_L(f) = h(i) \cdot \Phi^j(f) \). The left vertical arrow is continuous because \( L \) satisfies (4), and the diagonal arrow is continuous by construction of \( \tau \) and because we have already shown that \( \varphi_L \) satisfies (4). Hence the upper horizontal map is continuous as well. This shows that \( \tau \) satisfies (T2).

Finally, let \( \check{\tau} \) be a topology on \( \varphi_L \) satisfying (T1) and (T2). We want to show \( \check{\tau} \subseteq \tau \), i.e., \( \text{id} : (\varphi_L, \tau) \rightarrow (\varphi_L, \check{\tau}) \) is continuous. By construction of \( \tau \), it is enough to show that \( \Phi^j : \bigoplus_{j=1}^{J} 1_{X_j} \cdot L \rightarrow (\varphi_L, \tau), (f_j) \mapsto \sum_{j=1}^{J} h_j \cdot \varphi(f_j) \) is continuous for all \( J \). But this is immediately clear since \( \check{\tau} \) satisfies (T1) and (T2). \[ \square \]
Now let us define a suitable topology on \( \varphi^*M \) in case \( M \) is a topological res-invariant RH-submodule of \( C(H,W) \). Again, some preparations are necessary. Let \( \varphi : G \to H \) be a uniform embedding and \( M \) a res-invariant RH-submodule of \( C(H,W) \).

**Lemma 4.13.** Let \( \bar{X} \subseteq G \) be such that the restriction of \( \varphi \) to \( \bar{X} \) is injective. Let \( \bar{Y} := \varphi(\bar{X}) \). Then \( 1_{\bar{Y}} \cdot M \to 1_{\bar{X}} \cdot (\varphi^*M), f \mapsto 1_{\bar{X}} \cdot \varphi^*(f) \) is an isomorphism of R-modules.

**Proof.** For every \( f \in 1_{\bar{Y}} \cdot M \) and \( y \in H \), we have

\[
\varphi_\ast(1_{\bar{X}} \cdot \varphi^*(f))(y) = \sum_{x \in \bar{X}} \varphi^*(f)(x) = \sum_{x \in \bar{X}} (f)(\varphi(x)) = f(y).
\]

Hence \( \varphi_\ast(1_{\bar{X}} \cdot \varphi^*(f)) = f \), and our map is injective. To show surjectivity, it suffices by (6) to show that for every \( f \in M \) and \( A \subseteq G \), \( 1_{\bar{X}} \cdot (1_A \cdot \varphi^*(f)) \) lies in the image of our map. This follows from \( 1_{\bar{X}} \cdot (1_A \cdot \varphi^*(f)) = 1_{A \cap \bar{X}} \cdot \varphi^*(1_{(\varphi(A \cap \bar{X}))} \cdot f) \). \( \square \)

Now let \( Y = \varphi(G) \). Lemma 4.12 gives us \( X \subseteq G \) such that the restriction of \( \varphi \) to \( X \) is a bijection \( X \stackrel{\sim}{\to} Y, x \mapsto \varphi(x) \). By Lemma 4.5, we can find a finite decomposition \( G = \bigsqcup_{i=1}^f X_i \) and finite subsets \( \{g(i) : 1 \leq i \leq I\} \subseteq G \), \( \{h(i) : 1 \leq i \leq I\} \subseteq H \) such that \( X_i = g(i)^{-1}X(i) \) for some \( X(i) \subseteq X \) and \( \varphi(x) = h(i) \varphi(g(i)x) \) for all \( x \in X_i \) and \( 1 \leq i \leq I \). Let \( Y_i := \varphi(X_i) \) and \( \Phi : \bigsqcup_{i=1}^f 1_{Y_i} \cdot M \to \varphi^*M, (f_i) \mapsto \sum_{i=1}^f 1_{X_i} \cdot \varphi^*(f_i) \). As we obviously have \( \varphi^*M = \bigoplus_{i=1}^f 1_{X_i} \cdot (\varphi^*M), \Phi \) is surjective. And by Lemma 4.13, \( \Phi \) is injective. Thus \( \Phi \) is an isomorphism of R-modules.

**Definition 4.14.** Let \( M \) be a topological res-invariant RH-submodule of \( C(H,W) \). Define the topology \( \tau \) on \( \varphi^*M \) so that \( \Phi \) becomes a homeomorphism. Here \( 1_{Y_i} \cdot M \) is given the subspace topology from \( M \), and \( \bigoplus_{i=1}^f 1_{Y_i} \cdot M \) is given the product topology.

**Lemma 4.15.** \( \tau \) is the finest topology on \( \varphi^*M \) satisfying the following properties:

(\( T^1 \)) \( (\varphi^*M, \tau) \) is a topological res-invariant RG-submodule of \( C(G,W) \).

(\( T^2 \)) \( M \to (\varphi^*M, \tau), f \mapsto \varphi^*(f) \) is continuous.

**Proof.** Let us first show that \( \tau \) satisfies (\( T^1 \)) and (\( T^2 \)). \( (\varphi^*M, \tau) \) is obviously a topological R-vector space as \( \bigoplus_{i=1}^f 1_{Y_i} \cdot M \) is one. Let us show that for every \( A \subseteq G \), \( \text{res}_A : (\varphi^*M, \tau) \to (\varphi^*M, \tau), f \mapsto 1_A \cdot f \) is continuous.

We have a commutative diagram

\[
\begin{array}{c}
(\varphi^*M, \tau) \\
\Phi \downarrow \cong \downarrow \Phi \\
\bigoplus_{i=1}^f 1_{Y_i} \cdot M = \bigoplus_{i=1}^f 1_{Y_i} \cdot M
\end{array}
\]

where the lower horizontal map is given by \( \bigoplus_{i=1}^f 1_{Y_i} \cdot M \to \bigoplus_{i=1}^f 1_{Y_i} \cdot M, (f_i) \mapsto (1_{\varphi(X(i) \cap A)} \cdot f_i) \). It is continuous as \( M \) satisfies (3). As \( \Phi \) is a homeomorphism by construction, \( \text{res}_A \) is continuous.

Let us show that for every \( g \in G, g \cdot \downarrow : (\varphi^*M, \tau) \to (\varphi^*M, \tau), f \mapsto g \cdot f \) is continuous. By construction of \( \tau \), it suffices to show that \( (\cdot \downarrow) \circ \Phi \) is continuous. This amounts to saying that for all \( i, 1_{Y_i} \cdot M \to (\varphi^*M, \tau), f \mapsto g \cdot (1_{X_i} \cdot \varphi^*(f)) \) is continuous. Fix \( i \) and set \( \bar{Y} := Y_i, \bar{X} := X_i \). By Remark 4.4, we can find a finite decomposition \( G = \bigsqcup_{j \in J} X_j \) and a finite subset \( \{h_j : j \in J\} \subseteq H \) such that \( \varphi(g^{-1}x) = h_j \varphi(x) \) for all \( x \in X_j \) and \( j \in J \). Thus, for every \( f \in 1_{\bar{Y}} \cdot M \), we have

\[
g \cdot (1_{\bar{X}} \cdot \varphi^*(f)) = 1_{\bar{X}} \cdot (g \cdot \varphi^*(f)) = \sum_{j \in J} 1_{X_j \cap g \bar{X}} \cdot \varphi^*(h_j \cdot f) = \sum_{i=1}^f \sum_{j \in J} 1_{X_j \cap g \bar{X} \cap X_i} \cdot \varphi^*(h_j \cdot f) = \sum_{i=1}^f 1_{X_i} \cdot \varphi^*(\sum_{j \in J} (1_{\varphi(X_j \cap g \bar{X} \cap X_i)} \cdot (h_j \cdot f)))
\]

So we have a commutative diagram

\[
\begin{array}{c}
1_{\bar{Y}} \cdot M \\
\Phi \downarrow \cong \downarrow \Phi \\
\bigoplus_{i=1}^f 1_{Y_i} \cdot M
\end{array}
\]
where the upper horizontal arrow is given by \( 1_Y \cdot M \to (\varphi^* M, \tau), f \mapsto g \cdot (1_Y \cdot \varphi^* (f)) \), and the left vertical arrow is given by \( 1_Y \cdot M \to \bigoplus_{i=1}^I 1_{Y_i} \cdot M, f \mapsto \left( \sum_{j \in J} 1_{\varphi(X_i \cap \tau \cap X_i)} \cdot (h_j \cdot f) \right)_i \). The last map is continuous as \( M \) satisfies (3) and (4).

This shows that \( \tau \) satisfies (T1).

Let us show that \( \varphi^* (\sqcup) : M \to (\varphi^* M, \tau), f \mapsto \varphi^* (f) \) is continuous. We have a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi^* (\sqcup)} & (\varphi^* M, \tau) \\
\bigoplus_{i=1}^I 1_{Y_i} \cdot M & \xrightarrow{\Phi} & \\
\end{array}
\]

where the left vertical arrow is given by \( M \to \bigoplus_{i=1}^I 1_{Y_i} \cdot M, f \mapsto (1_{Y_i} \cdot f)_i \). This is continuous as \( M \) satisfies (3).

Hence \( \varphi^* (\sqcup) \) is continuous, and \( \tau \) satisfies (T2).

Finally, let \( \tilde{\tau} \) be another topology on \( \varphi^* M \) satisfying (T1) and (T2). We want to show that \( \tilde{\tau} \subseteq \tau \), i.e., \( \tilde{\tau} \subseteq \tau \), i.e., \( \text{id} : (\varphi^* M, \tau) \to (\varphi^* M, \tilde{\tau}) \) is continuous. By construction of \( \tau \), it suffices to show that \( \Phi : \bigoplus_{i=1}^I 1_{Y_i} \cdot M \to (\varphi^* M, \tilde{\tau}), (f)_j \mapsto \sum_{j \in J} 1_{X_j} \cdot \varphi^* (f_j) \) is continuous. But this is clear since \( \tilde{\tau} \) satisfies (T1) and (T2).

We have the following topological analogue of Lemma 4.9.

**Lemma 4.16.** (i) If \( \varphi, \psi : G \to H \) are uniform embeddings with \( \varphi \circ \phi \), then \( \varphi, L = \varphi, L \) and \( \varphi^* M = \varphi^* M \) as topological spaces, for all topological res-invariant RG-submodules \( L \) of \( C(G, W) \) and all topological res-invariant RH-submodules \( M \) of \( C(H, W) \).

(ii) If \( \varphi : G \to H \) and \( \psi : H \to K \) are uniform embeddings, then \( \psi_* (\varphi, L) = (\psi \circ \varphi, L) \) and \( \varphi^* (\psi^* N) = (\varphi \circ \psi)^* N \) for all topological res-invariant RG-submodules \( L \) of \( C(G, W) \) and all topological res-invariant RK-submodules \( N \) of \( C(K, W) \).

**Proof.** (i) By Lemma 4.5, we can find a finite decomposition \( G = \bigsqcup_{i \in I} X_i \) and a finite subset \( \{ h_i : i \in I \} \subseteq H \) such that \( \varphi(x) = h_i \varphi(x) \) for all \( x \in X_i \) and \( i \in I \). Let \( \tau \) be the topology on \( \varphi, L \) and \( \tilde{\tau} \) the topology on \( \varphi, L \). We want to prove that \( \tau \subseteq \tilde{\tau} \), i.e., \( \text{id} : (\varphi, L, \tau) \to (\varphi, L, \tilde{\tau}) \) is continuous. Let \( \Phi^J \) be the maps arising in the definition of \( \tilde{\tau} \). By construction, it suffices to show that \( \Phi^J : \bigoplus_{i=1}^I 1_{X_i} \cdot L \to (\varphi, L, \tau), (f)_j \mapsto \sum_{j \in J} 1_{X_j} \cdot \varphi (f_j) \) is continuous for all \( J \). Thus it suffices to show that for every \( j, \sum_{j \in J} 1_{X_j} \cdot \varphi (f_j) \) is continuous. We have for all \( f \in 1_{X_j} : h_j \cdot \varphi (f) = \sum_{j \in J} 1_{X_j} \cdot \varphi (f_j) = \sum_{j \in J} h_j \cdot \varphi (f_j) \), and this shows continuity as \( \tau \) satisfies (T1) and (T2). So indeed, \( \tau \subseteq \tilde{\tau} \), and by symmetry, \( \tilde{\tau} \subseteq \tau \).

Now let \( \tau \) be the topology on \( \varphi^* M \) and \( \tilde{\tau} \) the topology on \( \varphi^* M \). We want to show that \( \tilde{\tau} \subseteq \tau \), i.e., \( \text{id} : (\varphi^* M, \tilde{\tau}) \to (\varphi^* M, \tau) \) is continuous. Let \( \Phi^J \) be as in the definition of \( \tilde{\tau} \). By construction, it suffices to show that \( \Phi^J : \bigoplus_{i=1}^I 1_{Y_i} \cdot M \to (\varphi^* M, \tilde{\tau}), (f)_j \mapsto \sum_{j \in J} 1_{X_j} \cdot \varphi^* (f_j) \) is continuous. Thus it is enough to fix \( i \) and show that for \( \tilde{Y} := Y_i, X_i := X_i \) that \( 1_{Y_i} \cdot M \to (\varphi^* M, \tilde{\tau}), f \mapsto 1_{Y_i} \cdot \varphi^* (f) \) is continuous. We have for all \( f \in 1_{Y_i} : 1_{Y_i} \cdot \varphi^* (f) = \sum_{j \in J} 1_{X_j \cap \tilde{Y}} \cdot \varphi^* (f_j) = \sum_{j \in J} 1_{X_j \cap \tilde{Y}} \cdot \varphi^* (h_j) \), and this shows continuity as \( \tau \) satisfies (T1) and (T2). Hence \( \tilde{\tau} \subseteq \tau \), and by symmetry, \( \tau \subseteq \tilde{\tau} \).

(ii) Let \( \tau \) be the topology on \((\psi \circ \varphi)_* L = \psi_* \circ \varphi_* : L \to (\psi_* (\varphi_* L), \tilde{\tau}) \) is continuous. Hence, by maximality of \( \tau \), we must have \( \tilde{\tau} \subseteq \tau \). It remains to show that \( \tau \subseteq \tilde{\tau} \), i.e., \( \text{id} : (\varphi_* (\psi_* L, \tilde{\tau}) \to (\varphi_* (\psi_* L, \tau) \to (\varphi_* (\psi_* L, \tau) \) is continuous. Let \( \Phi^J : \bigoplus_{i=1}^I 1_{Y_i} \cdot (\varphi_* L, \tilde{\tau}) \to (\varphi_* (\psi_* L, \tau), (f)_j \mapsto \sum_{j \in J} 1_{Y_j} \cdot \varphi_* (f_j) \) be as in the construction of \( \tilde{\tau} \). By definition of \( \tilde{\tau} \), it suffices to prove that for every fixed \( j \) and \( \tilde{Y} := Y_j, k := k_j, 1_{Y_j} \cdot (\varphi_* L, \tilde{\tau}), f \mapsto k \cdot \varphi_* (f) \) is continuous. Hence it suffices to show that \( \Phi^J : \bigoplus_{i=1}^I 1_{Y_i} \cdot (\varphi_* L, \tilde{\tau}), f \mapsto \sum_{j \in J} 1_{Y_j} \cdot \varphi_* (f_j) \) be as in the construction of \( \tilde{\tau} \). By definition of \( \tilde{\tau} \), it suffices to show that for fixed \( i \) and \( \tilde{X} := X_i, h := h_i, 1_{X_i} \cdot \tilde{X} : (\varphi_* (\psi_* L, \tau), f \mapsto (1_{X_i} \cdot (h \cdot \varphi_* (f))) \) is continuous. By Remark 4.4, there is a finite decomposition \( Y = \bigsqcup_{i \in I} Y_i \) and a finite subset \( \{ k_i : i \in I \} \subseteq K \) such that \( \psi (h^{-1} \cdot y) = k_i^{-1} \psi (y) \) for all \( y \in Y_i \) and \( i \in I \). Hence

\[
k \cdot \varphi_* (1_{Y_i} \cdot (h \cdot \varphi_* (f))) = \sum_{i \in I} k_i \cdot \varphi_* (1_{X_i \cap \tilde{Y}} \cdot (1_{X_i \cap \tilde{Y}} \cdot \varphi_* (f))) = \sum_{i \in I} k_i \cdot \varphi_* (1_{X_i \cap \tilde{Y}} \cdot (1_{X_i \cap \tilde{Y}} \cdot \varphi_* (f)))
\]

and this is a continuous function of \( f \). Hence \( \tilde{\tau} \subseteq \tau \).

Finally, let \( \tau \) be the topology on \( (\varphi \circ \psi)^* N \) and \( \tilde{\tau} \) the topology on \( \varphi^* (\psi^* N) \). \( (\varphi \circ \psi)^* = \varphi^* \circ \psi^* : N \to (\varphi^* (\psi^* N), \tilde{\tau}) \) is continuous. Hence by maximality of \( \tau \), we have \( \tilde{\tau} \subseteq \tau \). It suffices to prove \( \tau \subseteq \tilde{\tau} \), i.e., \( \text{id} :
Let $G$ be a group, $R$ a commutative ring with unit, $L$ an $RG$-module. We write $g.f$ for the action of $g$ on $f \in L$. We recall the chain and cochain complexes coming from the bar resolution (see [4, Chapter III, § 1]): Let $(C_*(L), \partial_*)$ be the chain complex

\[ \ldots \to \partial_3 \circ C_3(L) \to \partial_2 \circ C_2(L) \to \partial_1 \circ C_1(L) \to C_0(L) \to 0, \]

with $C_0(L) = L$ and $C_n(L) = C_f(G^n, L) \cong R[G^n] \otimes_R L$, where $C_f$ stands for maps with finite support, and \( \partial_n = \sum_{i=0}^{n-1} (-1)^i \partial_n^{(i)} \), where

\[
\partial_n^{(0)}(f)(g_1, \ldots, g_{n-1}) = \sum_{g_0 \in G} g_0^{-1} f(g_0, g_1, \ldots, g_{n-1}),
\]

\[
\partial_n^{(i)}(f)(g_1, \ldots, g_{n-1}) = \sum_{g, \bar{g} \in G, \bar{g} = g_1 \cdot \cdots \cdot g_i} f(g, g_1, \ldots, g_{i-1}, \bar{g}, g_{i+1}, \ldots, g_{n-1}) \text{ for } 1 \leq i \leq n - 1,
\]

\[
\partial_n^{(n)}(f)(g_1, \ldots, g_{n-1}) = \sum_{g \in G} f(g_1, \ldots, g_{n-1}, g).
\]

Let $(C^*(L), \partial^*)$ be the cochain complex

\[ C^0(L) \xrightarrow{\partial^0} C^1(L) \xrightarrow{\partial^1} C^2(L) \xrightarrow{\partial^2} \ldots \]

where $C^0(L) = L$, $C^n(L) = C^*(G^n, L)$ for $n \geq 1$, and $\partial^n = \sum_{i=0}^{n} (-1)^i \partial^n_{(i)}$, with:

\[
\partial^n_{(0)}(f)(g_0, \ldots, g_n) = g_0 f(g_1, \ldots, g_n),
\]

\[
\partial^n_{(i)}(f)(g_0, \ldots, g_n) = f(g_0, \ldots, g_{i-1}, g_i, \ldots, g_n) \text{ for } 1 \leq i \leq n,
\]

\[
\partial^n_{(n)}(f)(g_0, \ldots, g_n) = f(g_0, \ldots, g_{n-1}).
\]

Now let $W$ be an $R$-module and $L \subseteq C(G, W)$ be an $RG$-submodule. Consider the transformation groupoid $\mathcal{G} := G \times G$ attached to the left multiplication action of $G$ on $G$. By definition, $\mathcal{G} = \{(x, g) : x \in G, g \in G\}$, and the range and source maps are given by $r(x, g) = x$, $s(x, g) = g^{-1}x$, whereas the multiplication is given by $(x, g_1)(g_2^{-1}x, g_2) = (x, g_1 g_2)$. Define $\sigma : \mathcal{G} \to G$, $(x, g) \mapsto g$. Let $\mathcal{G}^{(0)} = G$, and for $n \geq 1$, set

\[ \mathcal{G}^{(n)} := \{(\gamma_1, \ldots, \gamma_n) \in \mathcal{G}^n : s(\gamma_i) = r(\gamma_{i+1}) \text{ for all } 1 \leq i \leq n - 1\}, \]

and define, for $n \geq 1$, $\sigma : \mathcal{G}^{(n)} \to G^n$ as the restriction of $\sigma^n : \mathcal{G}^n \to G^n$ to $\mathcal{G}^{(n)}$.

Note that

\[ \mathcal{G}^{(n)} := \{(x_1, g_1), \ldots, (x_n, g_n) \in \mathcal{G}^n : g_i^{-1} x_i = x_{i+1} \text{ for all } 1 \leq i \leq n - 1\}, \]

so that we have a bijection

\[ \mathcal{G}^{(n)} \overset{\cong}{\to} G \times G^n, \quad ((x_1, g_1), \ldots, (x_n, g_n)) \mapsto (x_1, g_1, \ldots, g_n). \]

This is because for $2 \leq i \leq n$, $x_i$ is determined by the equation $x_i = g_{i-1}^{-1} \cdots g_1^{-1}x_1$. We will often use this identification of $\mathcal{G}^{(n)}$ with $G \times G^n$ without explicitly mentioning it.

Now, given $f \in C(\mathcal{G}^{(n)}, W)$ and $\bar{g} \in G^n$, we view $f|_{\sigma^{-1}(\bar{g})}$ as the map in $C(G, W)$ given by $x \mapsto f(x, \bar{g})$. Set $\text{supp}(f) := \{\bar{g} \in G^n : f|_{\sigma^{-1}(\bar{g})} \neq 0\}$. 

\[ \text{supp}(f) := \{\bar{g} \in G^n : f|_{\sigma^{-1}(\bar{g})} \neq 0\}. \]
Let us define a chain complex \((D_n(L), d_n)\) as follows: For \(n = 0, 1, 2, \ldots\), set
\[
D_n(L) := \left\{ f \in C(\mathcal{G}^{(n)}), W): \text{supp}(f) \text{ is finite, } f|_{\sigma^{-1}(\bar{g})} \in L \text{ for all } \bar{g} \in G^n \right\}.
\]
Moreover, for all \(n \geq 1\), define maps \(d_n : D_n(L) \to D_{n-1}(L)\) by setting \(d_n = \sum_{i=0}^{n} (-1)^i d^{(i)}_n\) with \(d^{(i)}_n = (\delta^{(i)}_n)_*\), where \(\delta^{(i)}_0 = s, \delta^{(i)}_1 = r\), and for \(n \geq 2,\)
\[
\delta^{(i)}_n(\gamma_1, \ldots, \gamma_n) = (\gamma_2, \ldots, \gamma_n),
\]
\[
\delta^{(i)}_n(\gamma_1, \ldots, \gamma_n) = (\gamma_i, \ldots, \gamma_{i+1}, \ldots, \gamma_n) \text{ for } 1 \leq i \leq n-1,
\]
\[
\delta^{(n)}_n(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_{n-1}).
\]
Here, we use the same notation as in §3.1 i.e., \((\delta^{(i)}_n)_*(f)(\bar{\eta}) = \sum_{\delta^{(i)}_n(\bar{\eta}) = \bar{\eta}} f(\bar{\eta})\).

Let us define a cochain complex \((D^*(L), d^*)\) by setting, for all \(n = 0, 1, 2, \ldots,\)
\[
D^*_n(L) := \left\{ f \in C(\mathcal{G}^{(n)}), W): \text{supp}(f) \text{ is finite, } f|_{\sigma^{-1}(\bar{g})} \in L \text{ for all } \bar{g} \in G^n \right\}.
\]
Moreover, for all \(n\), define maps \(d^n : D^*_n(L) \to D^*_{n+1}(L)\) by setting \(d^n = \sum_{i=0}^{n+1} (-1)^i d^n_i\) with \(d^n_i = (\delta^n_i)^*\) (as in §3.1) \((\delta^n_i)^*(f) = f \circ \delta^n_i\), where \(\delta^n_0 = s, \delta^n_1 = r\), and for \(n \geq 1,\)
\[
\delta^n_0(\gamma_0, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_n),
\]
\[
\delta^n_1(\gamma_0, \ldots, \gamma_n) = (\gamma_0, \ldots, \gamma-1, \gamma_n, \ldots), \text{ for } 1 \leq i \leq n,
\]
\[
\delta^n_{n+1}(f)(\gamma_0, \ldots, \gamma_n) = (\gamma_0, \ldots, \gamma_{n-1}).
\]

We are also interested in the topological setting, where we assume that \(R\) is a topological field, \(L \subseteq C(G, W)\) a \(RG\)-submodule together with the structure of a topological \(R\)-vector space such that the \(G\)-action \(G \curvearrowright L\) is by homeomorphisms. Equip the above chain and cochain complexes \(C_*(L)\) and \(C^*(L)\) with the topologies of pointwise convergence. We also equip \(D_*(L)\) and \(D^*(L)\) with the topologies of pointwise convergence, i.e., \(f_i \in C(\mathcal{G}^{(n)}), W\) converges to \(f \in C(\mathcal{G}^{(n)}), W\) if and only if \(\lim_i f_i|_{\sigma^{-1}(\bar{g})} = f|_{\sigma^{-1}(\bar{g})}\) in \(L\) for all \(\bar{g} \in G^n\).

The following is now immediate:

**Lemma 4.17.** (i) We have isomorphisms \(\chi_*\) of chain complexes and \(\chi^*\) of cochain complexes given by \(\chi_n : C_n(L) \to D_n(L), \chi_n(f)(x, \bar{g}) = f(\bar{g})(x)\) and \(\chi^n : C^n(L) \to D^n(L), \chi^n(f)(x, \bar{g}) = f(\bar{g})(x)\).

(ii) In the topological setting, \(\chi_*\) and \(\chi^*\) from (i) are topological isomorphisms.

By definition of group (co)homology, we have \(H_n(G, L) = H_n(C_*(L))\) and \(H^n(G, L) = H^n(C^*(L))\). By definition of reduced group (co)homology, we have \(\hat{H}_n(G, L) = \hat{H}_n(C_*(L))\) and \(\hat{H}^n(G, L) = \hat{H}^n(C^*(L))\) in the topological setting (recall that \(\hat{H}_n(C_*(L)) = \ker(\partial_n)/\text{im}(\partial_{n+1})\) and \(\hat{H}^n(C^*(L)) = \ker(\partial^n)/\text{im}(\partial^{n+1})\) by definition). Hence we obtain

**Corollary 4.18.** (i) \(\chi_*\) and \(\chi^*\) from Lemma 4.17 induce isomorphisms \(\hat{H}_n(\chi)_* : H_n(G, L) \xrightarrow{\cong} H_n(D_*(L))\) and \(\hat{H}^n(\chi)^* : H^n(G, L) \xrightarrow{\cong} H^n(D^*(L))\) for all \(n\).

(ii) In the topological setting, \(\chi_*\) and \(\chi^*\) from Lemma 4.17 induce isomorphisms \(\hat{H}_n(\chi)_* : \hat{H}_n(G, L) \xrightarrow{\cong} \hat{H}_n(D_*(L))\) and \(\hat{H}^n(\chi)^* : \hat{H}^n(G, L) \xrightarrow{\cong} \hat{H}^n(D^*(L))\) for all \(n\).

In this groupoid picture of group (co)homology, let us now explain how uniform embeddings induce chain and cochain maps. Let \(\varphi : G \to H\) be a uniform embedding. Let \(\mathcal{G} = G \times G\) and \(\mathcal{H} = H \times H\). Define \(\varphi^1 : \mathcal{G} \to \mathcal{H}, (x, y) \mapsto (\varphi(x), \varphi(y) \varphi^{-1}(x))\). It is easy to see that \(\varphi^1\) is a groupoid homomorphism. This means that if \(\gamma_1\) and \(\gamma_2\) are composable, then so are \(\varphi^1(\gamma_1)\) and \(\varphi^1(\gamma_2)\), and we have \(\varphi^1(\gamma_1 \gamma_2) = \varphi^1(\gamma_1) \varphi^1(\gamma_2)\). For all \(n \geq 1\), define \(\varphi^n : \mathcal{G}^{(n)} \to \mathcal{H}^{(n)}, (\gamma_1, \ldots, \gamma_n) \mapsto (\varphi^1(\gamma_1), \ldots, \varphi^1(\gamma_n))\).

Now let \(L\) be a res-invariant \(RG\)-submodule of \(C(G, W)\). For \(f \in D_n(L)\), consider \((\varphi^n)_*(f)(\bar{\eta}) = \sum_{\varphi^n(\bar{\eta}) = \bar{\eta}} f(\bar{\eta})\).
Lemma 4.19. (i) For all $n$, $D_n(\phi) : D_n(L) \to D_n(\phi,L)$, $f \mapsto (\phi^n)_*(f)$ is well-defined and gives rise to a chain map $D_* : D_*(L) \to D_*(\phi,L)$. If $\psi : H \to K$ is another uniform embedding, then we have
\begin{equation}
D_*(\psi \circ \phi) = D_*(\psi) \circ D_*(\phi).
\end{equation}

(ii) If $L$ is a topological res-invariant RG-submodule of $C(G,W)$, then for all $n$, $D_n(\phi)$ is continuous.

Note that for (11) to make sense, we implicitly use (ii) in Lemma 4.9.

Proof. (i) To show that $D_n(\phi)$ is well-defined, we have to show that $(\phi^n)_*(f) \in D_n(\phi,L)$ for all $f \in D_n(L)$. It suffices to treat the case that $\text{supp}(f) = \{g\}$ for a single $g = (g_1, \ldots, g_n) \in G^n$, as a general element in $D_n(L)$ is a finite sum of such $f$. Let us first show that $(\phi^n)_*(f)$ has finite support. For $1 \leq i \leq n$, let $N = \{x, g_i^{-1}x : x \in G\}$. Obviously, $\{st^{-1} : (s,t) \in S\}$ is finite. As $\phi$ is a uniform embedding, it follows that $\{\phi(s)\phi(t)^{-1} : (s,t) \in S\}$ is finite. Hence
\begin{equation}
F := \{\phi(x)\phi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n\}
\end{equation}
is finite.

It is clear that $\text{supp}((\phi^n)_*(f)) \subseteq F^n$.

Now let us show that for every $\tilde{h} = (h_1, \ldots, h_n) \in H^n$, $(\phi^n)_*(f)|_{\sigma^{-1}(\tilde{h})}$ lies in $\phi_*L$. Define
\begin{equation}
A := \{x \in G : \phi(g_i^{-1} \cdots g_1^{-1}x)\phi(g_i^{-1} \cdots g_1^{-1}x)^{-1} = h_i \text{ for all } 1 \leq i \leq n\}.
\end{equation}
Then $\phi^n(x, \tilde{g}) \in \sigma^{-1}(\tilde{h})$ if and only if $x \in A$. Hence
\begin{equation}
(\phi^n)_*(f)(y, \tilde{h}) = \sum_{x \in A} f(x, \tilde{g}) = \sum_{x \in A} f(x, \tilde{g}) = \phi_* \left(1_A \cdot \left(f|_{\sigma^{-1}(\tilde{g})}\right)\right)(y),
\end{equation}
so that
\begin{equation}
(\phi^n)_*(f)|_{\sigma^{-1}(\tilde{h})} = \phi_* \left(1_A \cdot \left(f|_{\sigma^{-1}(\tilde{g})}\right)\right).
\end{equation}

As $f|_{\sigma^{-1}(\tilde{g})}$ lies in $L$, $L$ is res-invariant and $\phi_* f \in \phi_* L$ for all $f \in L$, this shows that $(\phi^n)_*(f)|_{\sigma^{-1}(\tilde{h})} \in \phi_* L$. Hence $D_n(\phi)$ is well-defined for all $n$. $(D_n(\phi))_n$ is a chain map because $\phi^n$ is a groupoid homomorphism for all $n$. (11) holds because we have $(\phi^n)_* \circ (\phi^n)_* = ((\psi \circ \phi)^n)_*$, for all $n$.

(ii) (13) shows continuity of $D_n(\phi)$ for all $n$ as the right-hand side depends continuously on $f$. This is because $L$ satisfies (3) and the topology on $\phi_*L$ satisfies (T2).

Now let $M$ be a res-invariant RH-submodule of $C(H,W)$. For $f \in D^n(M)$, consider $(\phi^n)^*(f) = f \circ \phi^n$.

Lemma 4.20. (i) For all $n$, $D^n(\phi) : D^n(M) \to D^n(\phi^* M)$, $f \mapsto (\phi^n)^*(f)$ is well-defined and gives rise to a cochain map $D^* : D^*(M) \to D^*(\phi^* M)$. If $\psi : H \to K$ is another uniform embedding, we have
\begin{equation}
D^*(\psi \circ \phi) = D^*(\psi) \circ D^*(\phi).
\end{equation}

(ii) If $M$ is a topological res-invariant RH-submodule of $C(H,W)$, then $D^n(\phi)$ is continuous for all $n$.

For (14) to make sense, we implicitly use (ii) in Lemma 4.9.

Proof. (i) To show that $D^n(\phi)$ is well-defined, we have to show that for all $f \in D^n(M)$, $(\phi^n)^*(f) \in D^n(\phi^* M)$, i.e., $(\phi^n)^*(f)|_{\sigma^{-1}(\tilde{g})} \in D^n(M)$ for all $\tilde{g} = (g_1, \ldots, g_n) \in G^n$. $F = \{\phi(x)\phi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n\}$ is finite by (12). We also know that $\phi^n(x, \tilde{g}) \in \sigma^{-1}(F^n)$ for all $x \in G$. For $\tilde{h} = (h_1, \ldots, h_n) \in F^n$, let
\begin{equation}
A_{\tilde{h}} := \{x \in G : \phi(g_i^{-1} \cdots g_1^{-1}x)\phi(g_i^{-1} \cdots g_1^{-1}x)^{-1} = h_i \text{ for all } 1 \leq i \leq n\}.
\end{equation}
Then $G = \bigsqcup_{\tilde{h} \in F^n} A_{\tilde{h}}$, and for $x \in A_{\tilde{h}}$, we have $\phi^n(x, \tilde{g}) = (\phi^n, \tilde{h})$. Hence
\begin{equation}
(\phi^n)^*(f)|_{\sigma^{-1}(\tilde{g})}(x) = \phi^n(x, \tilde{g}) = \sum_{h \in F^n} 1_{A_{\tilde{h}}} \cdot \left(f|_{\sigma^{-1}(\tilde{h})}\right)(\phi^n(x)),
\end{equation}
and thus
\begin{equation}
(\phi^n)^*(f)|_{\sigma^{-1}(\tilde{g})} = \sum_{h \in F^n} 1_{A_{\tilde{h}}} \cdot \phi^n \left(f|_{\sigma^{-1}(\tilde{h})}\right).
\end{equation}
As \( f_{\sigma^{-1}(\vec{h})} \in M, \varphi^*(\vec{f}) \in \varphi^* M \) for all \( \vec{f} \in M \) and \( \varphi^* M \) is res-invariant, this shows that \( (\varphi^n)^*(f)|_{\sigma^{-1}(\vec{g})} \in \varphi^* M \)

Hence \( D^n(\varphi) \) is well-defined for all \( n \). \( (D^n(\varphi))_a \) is a cochain map because \( \varphi^n \) is a groupoid homomorphism for all \( n \). \( 12 \) holds because we have \( (\varphi^n)^* \circ (\varphi^n)^* = ((\psi \circ \varphi)^n)^* \) for all \( n \).

(ii) \( 15 \) shows that \( D^n(\varphi) \) is continuous for all \( n \) as the right-hand side depends continuously on \( f \). This is because the topology on \( \varphi^* M \) satisfies \((T^1)\) and \((T^2)\).

Our next goal is to show that uniformly close uniform embeddings induce the same chain and cochain maps up to homotopy. Let \( \varphi : G \to H \) be two uniform embeddings with \( \varphi \sim \varphi \). Let \( L \) be a res-invariant RG-submodule of \( C(G, W) \) and \( M \) a res-invariant RH-submodule of \( C(H, W) \). Let \( \mathcal{G} = G \times G \) and \( \mathcal{H} = H \times H \).

Define \( \theta : G \to \mathcal{H}, x \mapsto (\varphi(x), \varphi(x)\varphi(x)^{-1}) \). For \( n \geq 0 \) and \( 1 \leq h \leq n+1 \), let \( \kappa^n_{\theta} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)} \) be given by \( \kappa^n_{\theta}(1) = \theta \), and for \( n \geq 1 \),

\[
\kappa^n_{\theta}(\gamma_1, \ldots, \gamma_n) = (\varphi^1(\gamma_1), \ldots, \varphi^1(\gamma_{n-1}), \theta(r(\gamma_n)), \varphi^1(\gamma_n), \ldots, \varphi^1(\gamma_n)) \quad \text{for} \quad 1 \leq h \leq n,
\]

\[
\kappa^{n+1}_{\theta}(\gamma_1, \ldots, \gamma_n) = (\varphi^1(\gamma_1), \ldots, \varphi^1(\gamma_n), \theta(s(\gamma_n))).
\]

Moreover, for \( n \geq 1 \) and \( 1 \leq h \leq n \), let \( \kappa^n_{\theta} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)} \) be given by \( \kappa^n_{\theta}(1) = \theta \), and for \( n \geq 2 \),

\[
\kappa^n_{\theta}(\gamma_1, \ldots, \gamma_{n-1}) = (\varphi^1(\gamma_1), \ldots, \varphi^1(\gamma_{n-1}), \theta(r(\gamma_n)), \varphi^1(\gamma_n), \ldots, \varphi^1(\gamma_n)) \quad \text{for} \quad 1 \leq n \leq n-1,
\]

\[
\kappa^n_{\theta}(\gamma_1, \ldots, \gamma_{n-1}) = (\varphi^1(\gamma_1), \ldots, \varphi^1(\gamma_n), \theta(s(\gamma_n))).
\]

**Lemma 4.21.** (i) \( \kappa^n_{\theta} = (\kappa^n_{\theta})^* : D_n(L) \to D_{n+1}(\varphi,L) = D_{n+1}(\varphi,L) \) is well-defined for all \( n \) and \( h \). \( k_n := \sum_{h=1}^{n+1}(-1)^{h+1} \kappa^n_{\theta}(h) \) gives a chain homotopy \( D_n(\varphi) \sim_h D_n(\varphi) \).

(ii) \( k^n_{\theta}(1) = (\kappa^n_{\theta})^* : D^n(M) \to D_{n+1}(\varphi,M) = D_{n+1}(\varphi,M) \) is well-defined for all \( n \) and \( h \). \( k^n := \sum_{h=1}^{n+1}(-1)^{h+1} \kappa^n_{\theta}(h) \) gives a cochain homotopy \( D^n(\varphi) \sim_h D^n(\varphi) \).

**Proof.** (i) Let us show that \( \kappa^n_{\theta} \) is well-defined, i.e., \( (\kappa^n_{\theta})^* : D_n(L) \to D_{n+1}(\varphi,L) \) for all \( f \in D_n(L) \). We may assume \( \text{supp}(f) = \{ \vec{g} \} \) for a single \( \vec{g} = (g_1, \ldots, g_n) \in G^n \), as a general element in \( D_n(L) \) is a finite sum of such \( f \). We first show that \( \text{supp}((\kappa^n_{\theta})^*(f)) \) is finite. By \( 12 \) and because \( \varphi \sim \varphi \), we know that

\[
F := \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n \} \cup \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G \} \cup \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n \}
\]

is finite. As \( \kappa^n_{\theta}(x, \vec{g}) \) lies in \( \sigma^{-1}(F^{n+1}) \) for all \( x \in G \), we conclude that \( \text{supp}((\kappa^n_{\theta})^*(f)) \) is contained in \( F^{n+1} \), which is finite. Let us show that for every \( \vec{h} = (h_1, \ldots, h_n) \in H^{n+1} \), \( (\kappa^n_{\theta})^*(f)|_{\sigma^{-1}(\vec{h})} \) lies in \( \varphi,M \). Define

\[
A := \{ x \in G : (g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_i \text{ for all } 1 \leq i \leq h-1, \text{ and } \}
\]

\[
\varphi(g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_k, \text{ and } \}
\]

\[
\varphi(g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_{i+1} \text{ for all } h \leq i \leq n \}.
\]

Then \( \kappa^n_{\theta} \in \sigma^{-1}(\vec{h}) \) if and only if \( x \in A \). Hence \( (\kappa^n_{\theta})^*(f)|_{\sigma^{-1}(\vec{h})} = \varphi_{\sigma^{-1}(\vec{h})} \left( \text{1.} \cdot \left(f|_{\sigma^{-1}(\vec{h})} \right) \right) \). As \( f|_{\sigma^{-1}(\vec{h})} \) lies in \( L \), \( L \) is res-invariant, and \( \varphi(x) \in \varphi,L \) for all \( \vec{f} \in L \), we see that \( (\kappa^n_{\theta})^*(f)|_{\sigma^{-1}(\vec{h})} \in \varphi,L \). Hence \( \kappa^n_{\theta} \) is well-defined for all \( n \) and \( h \). A straightforward computation shows that \( k_n \) indeed gives us the desired chain homotopy.

(ii) Let us show that \( k^n_{\theta} \) is well-defined, i.e., \( \kappa^n_{\theta}^* : D^n(M) \to \varphi^*M \) for all \( \vec{g} = (g_1, \ldots, g_n) \in G^n \) and \( f \in D^n(M) \). As in the proof of (i), note that

\[
F := \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n-1 \}
\]

\[
\cup \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G \}
\]

\[
\cup \{ \varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G, 1 \leq i \leq n-1 \}
\]

is finite, and that \( \kappa^n_{\theta}(x, \vec{g}) \in \sigma^{-1}(F^n) \). For \( \vec{h} = (h_1, \ldots, h_n) \in F^n \), set

\[
A_{\vec{h}} := \{ x \in G : (g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_i \text{ for all } 1 \leq i \leq h-1, \text{ and } \}
\]

\[
\varphi(g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_k, \text{ and } \}
\]

\[
\varphi(g_{i-1} \cdots g_1 x)\varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_{i+1} \text{ for all } h \leq i \leq n-1 \}.
\]
Then $G = \bigsqcup_{h \in F_n} A_h$, and for $x \in A_h$, $\kappa^n_{(h)}(x, \bar{g}) = (\varphi(x), \bar{h})$. Hence

$$(\kappa^n_{(h)})^*(f)|_{\sigma^{-1}(\bar{g})}(x) = f(\kappa^n_{(h)}(x, \bar{g})) = \sum_{h \in F_n} 1_{A_h}(x) \cdot \left( f|_{\sigma^{-1}(\bar{h})} \right)(\varphi(x))$$

and thus $(\kappa^n_{(h)})^*(f)|_{\sigma^{-1}(\bar{g})} = \sum_{h \in F_n} 1_{A_h} \cdot \left( \varphi^* \left( f|_{\sigma^{-1}(\bar{h})} \right) \right)$. Since $f|_{\sigma^{-1}(\bar{h})} \in M$, $\varphi^*(\tilde{f}) \in \varphi^*M$ for all $\tilde{f} \in M$ and $\varphi^*M$ is res-invariant, this shows that $(\kappa^n_{(h)})^*(f)|_{\sigma^{-1}(\bar{g})} \in \varphi^*M$. Hence $\kappa^n_{(h)}$ is well-defined. It is straightforward to check that $k^n$ indeed gives us the desired cochain homotopy. \hfill \Box

**Theorem 4.22.** Let $\varphi : G \to H$ be a uniform embedding, $L$ a res-invariant RG-submodule of $C(G,W)$ and $M$ a res-invariant RH-submodule of $C(G,W)$.

1. $D_*(\varphi)$ induces homomorphisms $H_*(\varphi) : H_*(G,L) \to H_*(H,\varphi_*L)$. In case $L$ is a topological res-invariant RG-submodule of $C(G,W)$, $D_*(\varphi)$ also induces continuous homomorphisms $\bar{H}_*(\varphi) : \bar{H}_*(G,L) \to \bar{H}_*(H,\varphi_*L)$.

2. If $\psi : H \to K$ is another uniform embedding, then $H_*(\psi \circ \varphi) = H_*(\psi) \circ H_*(\varphi)$ and $\bar{H}_*(\psi \circ \varphi) = \bar{H}_*(\psi) \circ \bar{H}_*(\varphi)$.

In particular, for uniform equivalences, i.e., uniform embeddings which are invertible modulo $\sim$, we get

**Corollary 4.23.** If $\varphi : G \to H$ is a uniform equivalence, then we obtain isomorphisms

$$H_*(\varphi) : H_*(G,L) \xrightarrow{\cong} H_*(H,\varphi_*L), \quad H^*(\varphi) : H^*(H,M) \xrightarrow{\cong} H^*(G,\varphi^*M),$$

and, in the topological case,

$$\bar{H}_*(\varphi) : \bar{H}_*(G,L) \xrightarrow{\cong} \bar{H}_*(H,\varphi_*L), \quad \bar{H}^*(\varphi) : \bar{H}^*(H,M) \xrightarrow{\cong} \bar{H}^*(G,\varphi^*M).$$

**Remark 4.24.** Our constructions are functorial in $W$: Let $L_1 \subseteq C(G,W_1)$ and $L_2 \subseteq C(G,W_2)$ be res-invariant RG-submodules, and assume that an $R$-linear map $\varphi : W_1 \to W_2$ induces an $R$-linear map $\lambda : L_1 \to L_2$. Then we also obtain an induced map $\varphi_*\lambda : \varphi_*L_1 \to \varphi_*L_2$, and we get a commutative diagram

$$\begin{array}{ccc}
D_*(L_1) & \xrightarrow{D_*(\varphi)} & D_*(\varphi_*L_1) \\
\xrightarrow{D_*(\lambda)} & & \xrightarrow{D_*(\varphi, \lambda)} \\
D_*(L_2) & \xrightarrow{D_*(\varphi)} & D_*(\varphi_*L_2)
\end{array}$$

so that the following diagram commutes as well:

$$\begin{array}{ccc}
H_*(G,L_1) & \xrightarrow{H_*(\varphi)} & H_*(H,\varphi_*L_1) \\
\xrightarrow{H_*(\lambda)} & & \xrightarrow{H_*(\varphi, \lambda)} \\
H_*(G,L_2) & \xrightarrow{H_*(\varphi)} & H_*(H,\varphi_*L_2)
\end{array}$$

A similar statement applies to reduced homology in the topological setting, and to (reduced) cohomology.

**Remark 4.25.** We can rephrase our results in the language of categories as follows: Consider the category $\mathfrak{G}$ of countable discrete groups with uniform embeddings as morphisms. Given a group $G$, let $\mathfrak{M}_R(G)$ be the category of res-invariant submodules of $C(G,W)$, where $W$ runs through all $R$-modules, and morphisms are given by maps like $\lambda$ coming from $R$-linear maps on the level of $W$ as in Remark 4.24. Then we have constructed a covariant functor $F$ assigning to $G$ the category $\mathfrak{M}_R(G)$ and to a uniform embedding $\varphi : G \to H$ the functor $F_\varphi : \mathfrak{M}_R(G) \to \mathfrak{M}_R(H)$ given by $F_\varphi(L) = \varphi_*L$ and $F_\varphi(\lambda) = \varphi_*\lambda$. Moreover, let $\mathcal{C}_R$ be the category of chain complexes over $R$. Then every $G$ gives rise to a covariant functor $C_G : \mathfrak{M}_R(G) \to \mathcal{C}_R$ given by $C_G(L) = D_*(L)$ and $C_G(\lambda) = D_*(\lambda)$. We have now constructed, for every uniform embedding $\varphi : G \to H$
a natural transformation \( T_\psi : C_G \to C_H \circ F_\psi \), whose component at \( L \) is given by \( D_*(\psi) : D_*(L) \to D_*(\varphi_*L) \). And this assignment \( G \mapsto C_G, \varphi \mapsto T_\psi \) is functorial in the following sense: Given another uniform embedding \( \psi : H \to K \), we obtain a natural transformation \( T_\psi^\varphi : C_H \circ F_\psi \Rightarrow C_K \circ F_\psi \circ F_\varphi \) whose component at \( L \) is given by \( D_*(\psi) : D_*(\varphi_*L) \to D_*(\varphi_*(\varphi_*L)) \), and we have \( T_\psi^\varphi \circ T_\psi = T_\psi \circ T_\varphi \).

As we saw, the first functor \( F \) factorizes through the equivalence relation of being uniform close for morphisms, i.e., \( \varphi \sim \psi \) implies that \( F_\varphi = F_\psi \). The crucial point is that once we replace \( C_R \) by \( [\mathcal{C}]_R \), the category of chain complexes over \( R \) with morphisms given by homotopy classes of chain maps, then for the induced natural transformations \( [T]_\varphi \), we also have \([T]_\psi = [T]_\varphi\) if \( \varphi \sim \psi \).

It is now evident that every uniform equivalence \( \varphi \) gives rise to an equivalence of categories \( F_\varphi \) (which actually has an exact inverse) and a natural isomorphism \([T]_\varphi\).

A similar remark applies to the contravariant setting (replacing \( \varphi_* \) by \( \varphi^* \) and \( D_* \) by \( D^* \)).

**Remark 4.26.** Our constructions are compatible with completions, in the following sense: Let \( \varphi : G \to H \) be a uniform embedding, \( L \) a topological res-invariant \( RH \)-submodule of \( C(G, W) \) and \( M \) a topological res-invariant \( RH \)-submodule of \( C(H, W) \). Passing to completions, \( \varphi \) induces continuous homomorphisms
\[
H_*(\varphi) : H_*(G, \overline{L}) \to H_*(H, \overline{\varphi_*L}), \quad \bar{H}_*(\varphi) : \bar{H}_*(G, \overline{L}) \to \bar{H}_*(H, \overline{\varphi_*L}),
\]
\[
H^*(\varphi) : H^*(G, \overline{M}) \to H^*(H, \overline{\varphi_*M}), \quad \bar{H}^*(\varphi) : \bar{H}^*(G, \overline{M}) \to \bar{H}^*(H, \overline{\varphi_*M})
\]
with the same properties as in Theorem 4.22. In particular, if \( \varphi \) is a uniform equivalence, these homomorphisms are topological isomorphisms.

Moreover, if \( 0 \to L' \to L \to L/L' \to 0 \) is a short exact sequence of \( RG \)-modules, where \( L' \) and \( L \) are res-invariant \( RG \)-submodules of \( C(G, W') \) and \( C(G, W) \), and the map \( L' \to L \) is induced by an \( R \)-linear map \( W' \to W \) as above, then we obtain a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & D_*(L') & \to & D_*(L) & \to & D_*(L)/D_*(L') & \to & 0 \\
& & \downarrow{D_*(\varphi)} & & \downarrow{D_*(\varphi)} & & \downarrow{D_*(\varphi)^*} & \\
0 & \to & D_*(\varphi_*L') & \to & D_*(\varphi_*L) & \to & D_*(\varphi_*L)/D_*(\varphi_*L') & \to & 0
\end{array}
\]
with exact rows. It induces the following commutative diagram in homology
\[
\begin{array}{cccccc}
\cdots & \to & H_*(G, L') & \to & H_*(G, L) & \to & H_*(G, L/L') & \to & \cdots \\
& & \downarrow{H_*(\varphi)} & & \downarrow{H_*(\varphi)} & & \downarrow{H_*(\varphi)^*} & \\
\cdots & \to & H_*(H, \varphi_*L') & \to & H_*(H, \varphi_*L) & \to & H_*(H, \varphi_*L/L') & \to & \cdots
\end{array}
\]
whose rows are given by the long exact sequences in group homology.

Such a commutative diagram also exists for exact sequences of the forms \( 0 \to L' \to \overline{L} \to \overline{L}/L' \to 0 \) and \( 0 \to \overline{L} \to L \to \overline{L}/L' \to 0 \), where \( L' \) and \( L \) are res-invariant \( RG \)-submodules of \( C(G, W') \) and \( C(G, W) \), \( \overline{L} \) and \( \overline{L} \) are completions of topological res-invariant submodules, and the maps \( L' \to \overline{L} \) and \( L \to \overline{L} \) are induced by \( R \)-linear maps \( W' \to W \).

Similar statements are also valid for cohomology.

### 4.3. Consequences

Let us apply our results to the Examples in 4.2. (b) in (i) of Corollary 4.27 below generalizes the result in [10] that \( H^0(G, RG) \) is a quasi-isometry invariant for groups with property \( \mathcal{F}_n \) (i.e., there exist models for Eilenberg-MacLane spaces with finite \( n \)-skeleton). (1) in (ii) was known in special cases. For instance, in [23][2][17], reduced group cohomology in degree 1 (i.e., \( \bar{H}^1 \)) with \( \ell^p \) coefficients has been identified with \( L^p \)-cohomology, as studied in [11][21]. Since \( L^p \)-cohomology is known to be a quasi-isometry invariant, this gives the special case of (ii) (1) where \( p \in [1, \infty[ \) and our groups are finitely generated. Also, the case \( p = \infty \) in (ii) (1) was known since \( H_* (G, \ell^p G) \) can be identified with uniformly finite homology (see [3]).

**Corollary 4.27.** Let \( G \) and \( H \) be countable discrete groups and \( \varphi : G \to H \) a uniform equivalence.

(i) For every commutative ring \( R \) with unit and every \( R \)-module \( W \), \( \varphi \) induces isomorphisms
\[
H_*(G, C(G, W)) \cong H_*(H, C(H, W)),
\]
\[
H^*(H, RH \otimes_R W) \cong H^*(G, RG \otimes_R W).
\]

(ii) Let \( R = \mathbb{R} \) or \( R = \mathbb{C} \) and \( W = R \).
Corollary 4.27 have the property that for every uniform equivalence
Remark 4.28.

isometries on these (co)homology groups are quasi-isometry invariants. To explain this, let
(and also topologically in the topological setting). For such
discrete group and
an isomorphism
invariant under uniform equivalence as long as they are finite.

We obtain an improvement since we can work over arbitrary rings
groups of type
As a consequence of Corollary 4.27 (i) (b), we obtain a generalization of the result in [10] which says that for
For
isometry invariant among groups of type
We also obtain a new proof of the result in [25] that homological and cohomological dimensions over
R are invariant under uniform equivalence as long as they are finite.

(1) For all $0 < p \leq \infty$, $\varphi$ induces isomorphisms

$$H_*(G, \ell^p(G,W)) \cong H_*(H, \ell^p(H,W)), \quad H^*(H, \ell^p(H,W)) \cong H^*(G, \ell^p(G,W)),$$

$$\tilde{H}_*(G, \ell^p(G,W)) \cong \tilde{H}_*(H, \ell^p(H,W)), \quad \tilde{H}^*(H, \ell^p(H,W)) \cong \tilde{H}^*(G, \ell^p(G,W)).$$

(2) $\varphi$ induces isomorphisms

$$H_*(G, c_0(G,W)) \cong H_*(H, c_0(H,W)), \quad H^*(H, c_0(H,W)) \cong H^*(G, c_0(G,W)),$$

$$\tilde{H}_*(G, c_0(G,W)) \cong \tilde{H}_*(H, c_0(H,W)), \quad \tilde{H}^*(H, c_0(H,W)) \cong \tilde{H}^*(G, c_0(G,W)).$$

(3) Let $G$ and $H$ be a finitely generated discrete groups. Then, for all $s \in \mathbb{R} \cup \{\infty\}$ and $1 \leq p \leq \infty$, $\varphi$
induces isomorphisms

$$H_*(G, H^{s,p}(G,W)) \cong H_*(H, H^{s,p}(H,W)), \quad H^*(H, H^{s,p}(H,W)) \cong H^*(G, H^{s,p}(G,W)),$$

$$\tilde{H}_*(G, H^{s,p}(G,W)) \cong \tilde{H}_*(H, H^{s,p}(H,W)), \quad \tilde{H}^*(H, H^{s,p}(H,W)) \cong \tilde{H}^*(G, H^{s,p}(G,W)).$$

Remark 4.28. The point is that all the examples $L(G) = RG, C(G,W), \ell^p(G,W), c_0(G,W)$ or $H^{s,p}(G,W)$ in
Corollary 4.27 have the property that for every uniform equivalence $\varphi : G \to H$, we have $\varphi_* L(G) = L(H)$
and also topologically in the topological setting). For such $L(G)$, by functoriality (see Theorem 4.24
and Remark 4.25), not only the (co)homology groups themselves, but even the actions of the groups of quasi-isometries on these (co)homology groups are quasi-isometry invariants. To explain this, let $G$ be a countable discrete group and $U(G)$ the group of uniform equivalences $G \to G$ modulo $\sim$. If $G$ is finitely generated, this is the usual group $QI(G)$ of quasi-isometries of $G$. Now assume that for every group $G$, $L(G)$ is a res-invariant module with the property that for all uniform equivalences $G \to H$, we have $\varphi_* L(G) = L(H)$. Then our results imply that if there is a uniform equivalence $G \to H$, we will find a group isomorphism $U(G) \cong U(H)$ and an isomorphism $H_*(G, L(G)) \cong H_*(H, L(H))$ which is equivariant for the $U(G)$- and $U(H)$-actions. In other words, the actions $U(G) \acts H_*(G, L(G))$ and $U(H) \acts H_*(H, L(H))$ are isomorphic. For $L = H^{s,p}$, we would have to replace $U$ by $QI$. Similar remarks apply to $\tilde{H}_*$, $\tilde{H}^*$.

As a consequence of Corollary 4.27(i) (b), we obtain a generalization of the result in [10] which says that for
groups of type $\mathcal{F}_\infty$, being a Poincaré duality group in the sense of [4] Chapter VIII, § 10] is a quasi-isometry invariant. We obtain an improvement since we can work over arbitrary rings $R$, where the $\mathcal{F}_\infty$ assumption becomes unnatural, and only need our groups to have finite cohomological dimension over $R$. We refer to [1] for the notion of duality group over general $R$.

Corollary 4.29. Let $R$ be a commutative ring with unit. Let $G$ and $H$ be finitely generated groups with finite cohomological dimension over $R$. If $G$ and $H$ are quasi-isometric, then $G$ is a duality group over $R$ if and only if $H$ is a duality group over $R$.

If $G$ and $H$ are finitely generated amenable groups and $\mathbb{Q} \subseteq R$, then $G$ is a duality group over $R$ if and only if $H$ is a duality group over $R$.

Proof. By [1] Theorem 5.5.1 and Remark 5.5.2], we know that a group $G$ is a duality group if and only if it has finite cohomological dimension, there is $n$ such that $H^n(G,A) \cong \{0\}$ for all $k \neq n$ and all induced $RG$-modules $A$, and $G$ is of type $FP_n$ over $R$. The second property is a quasi-isometry invariant by Corollary 4.27(ii) (b). The third property is a quasi-isometry invariant by [8] Theorem 9.61].

The special case follows from the first part of the corollary and [25] Theorem 1.2 (ii)].

For $p = 2$, Corollary 4.27(ii) (1) generalizes the result in [21] that vanishing of $\ell^2$-Betti numbers is a quasi-isometry invariant among groups of type $\mathcal{F}_\infty$ (see also the explanations in [27]). It was shown in [22] Proposition 3.8] that vanishing of the $n$-th $\ell^2$-Betti numbers is equivalent to vanishing of $H^n(G, \ell^2 G))$. Hence Corollary 4.27(ii) (1) implies the following

Corollary 4.30. Let $G$ and $H$ be countable discrete groups. Assume that there is a uniform equivalence $G \to H$. Then, for all $n$, the $n$-th $\ell^2$-Betti number of $G$ vanishes if and only if the $n$-th $\ell^2$-Betti number of $H$ vanishes.

As mentioned in [26], there is also a generalization of [21] in [20]. Moreover, independently from the author, Sauer and Schrödl obtained the result that even among all compactly generated unimodular locally compact second countable groups, vanishing of $\ell^2$-Betti numbers is a quasi-isometry invariant (see [27]). Note, however, that we do not need to assume that our groups are finitely generated.

We also obtain a new proof of the result in [25] that homological and cohomological dimensions over $R$ are invariant under uniform equivalence as long as they are finite.
Corollary 4.31. Let $R$ be a commutative ring with unit. Let $G$ and $H$ be countable discrete groups, and assume that there is a uniform equivalence $G \to H$.

If $G$, $H$ have finite homological dimensions over $R$, then their homological dimensions over $R$ coincide.

If $G$, $H$ have finite cohomological dimensions over $R$, then their cohomological dimensions over $R$ coincide.

Proof. Write $\text{hd}_R$ for homological dimension. Assume that $\text{hd}_R G = n < \infty$. Let $W$ be an $R$-module such that $H^n(H, G) \not\cong 0$. Define $W \to C(G, W), w \mapsto f_w$, where $f_w(x) = x^{-1}.w$. It is easy to see that this is an embedding of $R$-modules when we view $W$ as an $R$-module to construct $C(G, W)$ (i.e., we define the $R$-module structure by setting $(g.f)(x) = f(g^{-1}.x)$ for $f \in C(G, W)$). The long exact sequence in homology gives $0 \to H^n(H, G) \to H^n(H, G, C, W) \to \ldots$ because the $(n+1)$-th group homology of $G$ vanishes for all coefficients. Hence $H^n(H, G, C, W) \not\cong 0$. This shows that

$$\text{hd}_R G = \sup \{n: H_n(G, C, W) \not\cong 0\} \text{ for some R-module } W.$$ 

The same is true for $H$. Now our claim is a direct consequence of Corollary 4.27 (i) (a).

Write $\text{cd}_R$ for cohomological dimension. By [4, Proposition (2.3)], we know that if $\text{cd}_R G < \infty$, then

$$\text{cd}_R G = \sup \{n: H^n(G, RG \otimes_R W) \not\cong 0\} \text{ for some R-module } W.$$ 

The same is true for $H$. Now our claim is a direct consequence of Corollary 4.27 (i) (b). $\square$

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