More on Compression and Ranking

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Abstract

We study the role that honesty, selectivity, closure properties, relativization, and target spaces other than $\Sigma^*$ have in the recursion-theoretic study of compression and ranking.

1 Introduction

A recent paper by Hemaspaandra and Rubery [HR16] introduced and studied recursion-theoretic analogues of the complexity-theoretic notions of ranking and compression [GS91, HR90, GHK92].

Loosely put, a compression function for a set $A$ is a function over the domain $\Sigma^*$ (we will always take $\Sigma$ to be $\{0,1\}$) such that (a) $f(A) = \Sigma^*$ and (b) $(\forall a, b \in \Sigma^*: a \neq b)[f(a) \neq f(b)]$. That is, $f$ puts $A$ in one-to-one correspondence with $\Sigma^*$. This is sometimes described as providing a minimal perfect hash function for $A$; it is perfect since there are no collisions (among elements of $A$), and it is minimal since not a single element of the codomain is missed. Note that the above does not put any constraints on what strings the elements of $A$ are mapped to, or even about whether the compression function needs to be defined on such strings.

In the earlier paper, we studied compression for both the case of (total) recursive compression functions (which of course must be defined on all inputs in $\Sigma^*$), and for the case of partial-recursive compression functions (i.e., compression functions that on some or all elements of $A$ are allowed to be undefined). That paper proves many results focused mostly on getting an idea of which sets are or are not compressible under each of these two notions of compression.

The earlier paper also studies, also for both (total) recursive functions and partial recursive functions, the closely related notion of ranking. Ranking is basically compression with the additional constraint that among elements of $A$ it preserves ordering as it maps to $\Sigma^*$, i.e., the $i$’th element of $A$ must be mapped to the $i$’th string in $\Sigma^*$ (or to the integer $i - 1$ if we are viewing the codomain as the natural numbers, via the standard correspondence
that the $i$’th string in $\Sigma^*$ corresponds to the natural number $i - 1$). Again, the focus of that paper is determining which sets are or are not rankable with respect to (total) recursive ranking functions and/or partial-recursive ranking functions.

In the present paper, we study a number of different aspects of ranking and compression, still in the recursion-theoretic setting. In particular, we prove results that variously either display interactions between the notions of recursion-theoretic compressibility/rankability and other notions (such as selectivity and honesty) or that, in the case of Section 3, focus on compression with the target image of $A$ being allowed to be not necessarily $\Sigma^*$ but rather some set $B \subseteq \Sigma^*$.

2 Definitions

Since the preliminaries here—the notions of ranking and compression, and the types of functions with respect to which they are studied—are taken over from the earlier paper, and in order to stay precisely in sync with those notions, instead of including an in-house Definitions section, we instead treat the Definitions section of that earlier paper [HR16], which was Section 3 of that paper, as if it were included here. And we will draw freely, and without redefinition here, on the definitions/background/notions/notations/shorthands/etc. presented in that section. However, since the definitions/notations for the two most important concepts and the two most important function classes that we will speak of are so essential as to be important to have immediately at hand, we will here explicitly include them, taken from that paper (again, for all other definitions and notations, please see Section 3 of [HR16]).

$F_{REC}$ will denote the class of all total recursive functions from $\Sigma^*$ to $\Sigma^*$. $F_{PR}$ will denote the class of all partial recursive functions from $\Sigma^*$ to $\Sigma^*$.

**Definition 2.1 (Compressible sets [HR16]).**

1. Given a set $A \subseteq \Sigma^*$, we say that a (possibly partial) function $f$ is a compression function for $A$ exactly if

   (a) $\text{domain}(f) \supseteq A$,
   (b) $f(A) = \Sigma^*$, and
   (c) $(\forall a \in A)(\forall b \in A)[a \neq b \implies f(a) \neq f(b)]$.

2. Let $\mathcal{F}$ be any class of (possibly partial) functions mapping from $\Sigma^*$ to $\Sigma^*$. A set $A$ is $\mathcal{F}$-compressible exactly if $(\exists f \in \mathcal{F})[f$ is a compression function for $A]$.

3. For each $\mathcal{F}$ as above, $\mathcal{F}$-compressible $= \{A \mid A$ is $\mathcal{F}$-compressible$\}$ and $\mathcal{F}$-compressible$' = \mathcal{F}$-compressible $\cup \{A \subseteq \Sigma^* \mid A$ is a finite set$\}$.

4. For each $\mathcal{F}$ as above and each $C \subseteq 2^{\Sigma^*}$, we say that $C$ is $\mathcal{F}$-compressible exactly if $(\forall A \in C)[\text{If A is an infinite set, then A is F-compressible}]$.

**Definition 2.2 (Rankable sets [HR16]).**
1. Given a set $A \subseteq \Sigma^*$, we say that a (possibly partial) function $f$ is a ranking function for $A$ exactly if

(a) $\text{domain}(f) \supseteq A$, and

(b) if $x \in A$, then $f(A) = \|A \leq x\|$. (That is, if $x$ is the $i$'th string in $A$, then $f(x)$ is the $i$'th string in $\Sigma^*$.)

2. Let $F$ be any class of (possibly partial) functions mapping from $\Sigma^*$ to $\Sigma^*$. A set $A$ is $F$-rankable exactly if $(\exists f \in F)[f$ is a ranking function for $A]$.

3. For each $F$ as above, $F$-rankable $= \{A \mid A$ is $F$-rankable$\}$.

4. For each $F$ as above and each $C \subseteq 2^{\Sigma^*}$, we say that $C$ is $F$-rankable exactly if $(\forall A \in C)[A$ is $F$-rankable$]$.

3 Compression onto $B$

Compression, both in the definition used in this paper (Definition 2.1) and in the complexity-theoretic notion that this paper is transferring to the recursion-theoretic setting [GHK92], is speaking of a map $f$ from a set $A$ to $\Sigma^*$ such that the map is (total on $A$, and) 1-to-1 from $A$ and satisfies $f(A) = \Sigma^*$.

It is natural to wonder about the same issue, except regarding maps not from $A$ to $\Sigma^*$ but from $A$ to some set $B \subseteq \Sigma^*$. We now define that notion, of which compression is the $B = \Sigma^*$ case. Note that in our definition of this notion, we do allow strings in $A$ to be mapped to $B$ or to $\overline{B}$ or even, for the case of $F_{PR}$ maps, to legally be undefined. In particular, this definition does not require that $f(\Sigma^*) = B$. As in the main body of the paper, here too $\Sigma = \{0, 1\}$.

Definition 3.1 (Compressible to $B$).

1. Given set $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$, we say that a (possibly partial) function $f$ is a compression function for $A$ to $B$ exactly if

(a) $\text{domain}(f) \supseteq A$.

(b) $f(A) = B$.

(c) $(\forall a \in A)(\forall b \in A)[a \neq b \implies f(a) \neq f(b)]$.

2. Let $F$ be any class of (possibly partial) functions mapping from $\Sigma^*$ to $\Sigma^*$. A set $A$ is $F$-compressible to $B$ exactly if $(\exists f \in F)[f$ is a compression function for $A$ to $B]$.

The classes $F$ of interest to us will again be $F_{REC}$ and $F_{PR}$.

As mentioned above, compression is simply the $B = \Sigma^*$ case of this more flexible definition.

Observation 3.2. $f$ is a compression function for $A$ if and only if $f$ is a compression function for $A$ to $\Sigma^*$. $A$ is $F$-compressible if and only if $A$ is $F$-compressible to $\Sigma^*$.
The natural first question to ask is whether compression to \( B \) is a new notion, or whether it in fact coincides with our existing notion, at least for sets \( B \) from common classes such as REC and RE. The following result shows that for REC and RE this new notion does coincide with our existing one.

**Theorem 3.3.**
1. \( (\forall \text{ infinite } B \in \text{REC})(\forall A)[A \text{ is } F_{\text{REC}}\text{-compressible to } B \iff A \text{ is } F_{\text{REC}}\text{-compressible (to } \Sigma^*)]. \)
2. \( (\forall \text{ infinite } B \in \text{RE})(\forall A)[A \text{ is } F_{\text{PR}}\text{-compressible to } B \iff A \text{ is } F_{\text{PR}}\text{-compressible (to } \Sigma^*)]. \)

**Proof.** Define \( \text{rank}_C(y) = \| \{ z \mid z \leq_{\text{lex}} y \land z \in C \} \| \).

We first prove part 1 and we begin with the “if” direction. Suppose \( A \) is \( F_{\text{REC}}\)-compressible to \( \Sigma^* \) via a recursive function, \( f \), and suppose \( B \) is recursive and infinite.

Let \( f'(x) \) output the element of \( B \) whose rank within \( B \) is \( \text{rank}_{\Sigma^*}(f(x)) \). Then \( f' \) is a recursive function and \( A \) is clearly \( F_{\text{REC}}\)-compressible to \( B \) by \( f' \).

For the “only if” direction, let \( B \) be recursive and infinite and suppose \( A \) is \( F_{\text{REC}}\)-compressible to \( B \) by a recursive function, \( f \).

Let \( f'(x) = \epsilon \) if \( f(x) \notin B \). Otherwise, let \( f'(x) \) be the string whose rank in \( \Sigma^* \) is \( \text{rank}_B(f(x)) \). Then \( f' \) is recursive, and compresses \( A \) to \( \Sigma^* \).

Let us turn to part 2 of the theorem. Again, we begin with the “if” direction. Let \( B \) be an infinite r.e. set, and let enumerating Turing machine \( E \) enumerate without repetitions the elements of \( B \). Suppose \( A \) is \( F_{\text{PR}}\)-compressible to \( \Sigma^* \) via a partial recursive function, \( f \). Then on input \( x \), \( f' \) does the following:

1. Attempt to compute \( f(x) \). This may run forever, namely in the case that \( x \notin \text{domain}(f) \).
2. If \( f(x) \) outputs a value, simulate \( E \) until it enumerates \( \text{rank}_{\Sigma^*}(f(x)) \) strings.
3. Output the \( \text{rank}_{\Sigma^*}(f(x)) \)'th string output by \( E \).

\( f' \) is partial recursive and \( A \) is \( F_{\text{PR}}\)-compressible to \( B \) via \( f' \).

Similarly, for the “only if” direction, let \( B \) be infinite and r.e. and let enumerating Turing machine \( E \) enumerate \( B \). Suppose \( A \) is \( F_{\text{PR}}\)-compressible to \( B \) via a partial recursive function, \( f \). On input \( x \), \( f' \) computes as follows:

1. Attempt to compute \( f(x) \).
2. If \( f(x) \) outputs a value, then run \( E \) until it outputs \( f(x) \). This may run forever, namely if \( f(x) \notin B \).
3. Suppose \( f(x) \) is the \( l \)'th string output by \( E \). Then output the \( l \)'th string in \( \Sigma^* \).

\( f' \) is partial recursive, and \( A \) is \( F_{\text{PR}}\)-compressible to \( \Sigma^* \) by \( f' \). \( \square \)
Theorem 3.3 covers the two most natural “pairings”: recursive sets $B$ with $F_{REC}$ compression, and r.e. sets $B$ with $F_{PR}$ compression. What about the two “nonmatching” cases among those notions? We note that one and a half of them hold, and conjecture that the remaining half fails.

Corollary 3.4. 1. $(\forall$ infinite $B \in REC)(\forall A)[A$ is $F_{PR}$-compressible to $B \iff A$ is $F_{PR}$-compressible (to $\Sigma^*$)].

2. $(\forall$ infinite $B \in RE)(\forall A)[A$ is $F_{REC}$-compressible (to $\Sigma^*$) $\implies A$ is $F_{REC}$-compressible to $B$. $(Indeed, even implied is compressibility to $B$ via an $F_{REC}$-compressor function that additionally satisfies $f(\Sigma^*) = B$.)

Proof. The first part follows immediately from Theorem 3.3 part 2. The second part follows as a corollary to the proof of Theorem 3.3 part 2. In particular, that proof’s “$\iff$” direction proves this, since it is clear that if $f$ is a recursive function (as it is in this part of the present theorem), the $f'$ defined there is a recursive function.

Conjecture 3.5. It does not hold that $(\forall$ infinite $B \in RE)(\forall A)[A$ is $F_{REC}$-compressible to $B \implies A$ is $F_{REC}$-compressible (to $\Sigma^*$)].

We further conjecture that for the class coRE, both directions fail, for both the $F_{REC}$ and the $F_{PR}$ cases.

Another interesting question is how recursive compressibility to $B$ is, or is not, linked to recursive isomorphism. We show that recursive isomorphism of sets implies mutual compressibility to each other. We conjecture that the other direction does not hold.

Theorem 3.6. If $A \equiv_{iso} B$, then $A$ is $F_{REC}$-compressible to $B$ and $B$ is $F_{REC}$-compressible to $A$.

Proof. $A$ is $F_{REC}$-compressible to $B$ by simply letting our $F_{REC}$-compression function be the isomorphism function. And since each recursive isomorphism has a recursive inverse, $B$ is $F_{REC}$-compressible to $A$ by simply letting our $F_{REC}$-compression function be the inverse of the isomorphism function. $\square$

Conjecture 3.7. There exists sets $A$ and $B$ such that $A$ is $F_{REC}$-compressible to $B$, $B$ is $F_{REC}$-compressible to $A$, yet $A \not\equiv_{iso} B$.

4 Closure under Variations

How robust are the recursively compressible and the recursively rankable sets? Do sets lose these properties under the subtraction, or addition, or (better yet) symmetric difference with finite sets? Or even with sufficiently nice infinite sets? The following result addresses that.
Theorem 4.1.  \[1\] If \(A\) is an \(F_{\text{REC}}\)-rankable set, and \(B_1 \subseteq A\) is a recursive set, and \(B_2 \subseteq \overline{A}\) is a recursive set, then \(A \triangle (B_1 \cup B_2)\) (equivalently, \((A - B_1) \cup B_2)\) is an \(F_{\text{REC}}\)-rankable set.

2. If \(A\) is an \(F_{\text{REC}}\)-compressible set, and \(B_1 \subseteq A\) is a recursive set, and \(A - B_1\) is infinite, then \(A - B_1\) is an \(F_{\text{REC}}\)-compressible set.

3. If \(A\) is an \(F_{\text{REC}}\)-compressible set, and \(B_2 \subseteq \overline{A}\) is a recursive set, then \(A \cup B_2\) is an \(F_{\text{REC}}\)-compressible set.

Proof. Let us define \(\text{lexshift}(x,n)\), for \(x \in \Sigma^*\) and \(n\) an integer, as follows. If \(n \in \{0,1,2,\ldots\}\), then \(\text{lexshift}(x,n)\) is the string \(n\) spots in lexicographical order after \(x\) within \(\Sigma^*\). And if \(n \in \{-1,-2,-3,\ldots\}\) then \(\text{lexshift}(x,n)\) is \(\epsilon\) if \(x\) is one of the lexicographically first \(-1 \times n\) strings in \(\Sigma^*\), and otherwise is the string \(n\) spots in lexicographical order before \(x\) within \(\Sigma^*\). For example, \(\text{lexshift}(\epsilon,3) = 00\), \(\text{lexshift}(\epsilon,-3) = \epsilon\), and \(\text{lexshift}(1010,-3) = 111\).

For the first part of this theorem, the ranking part, let \(f\) be an \(F_{\text{REC}}\)-ranking function for \(A\), and let \(f_1(x) = \|B_1^x\|\) and let \(f_2(x) = \|B_2^x\|\). Our \(F_{\text{REC}}\) ranking function for \(A \triangle (B_1 \cup B_2)\) is \(f'(x) = \text{lexshift}(f(x), f_2(x) - f_1(x))\). This directly accounts for the additions and deletions done by \(B_1\) and \(B_2\).

For the second part of this theorem, let \(f\) be an \(F_{\text{REC}}\)-compression function for \(A\), and let \(f_1\) be as above. Our \(F_{\text{REC}}\) compression function for \(A - B_1\) is \(f'(x) = f(\text{lexshift}(x, -1 \times f_1(x)))\). This directly accounts for the deletions done by \(B_1\). The “infinite” clause in the statement is simply because if \(A - B_1\) is finite, \(f'(A - B_1)\) will be finite and so of course will not cover \(\Sigma^*\).

Note that the second and third parts of this theorem are weakened analogues for compressible sets of the first part. The proof of the third part, however, is slightly different in character than that of the second part, due to the finite and infinite cases receiving separate treatments.

In particular, if \(B_2\) is finite, our \(F_{\text{REC}}\) compression function for \(A \cup B_2\) is \(f'(x) = \text{lexshift}(f(x), \|B_2\|)\) for \(x \notin B_2\) and \(f'(x) = \|B_2^x\|\) for \(x \in B_2\).

On the other hand, if \(B_2\) is infinite, let \(g\) be an \(F_{\text{REC}}\) compression function for \(B_2\), e.g., \(g\) can be taken to be (recall that \(B_2\) is recursive) defined by \(g(x)\) being the \(\max(\text{rank}_B(x),1)\)'st string in \(\Sigma^*\); so this in fact is even a ranking function for \(B_2\). We define \(f'(x)\) as follows. (Recall that for us \(\Sigma\) is always fixed as being \(\{0,1\}\).) For any string, \(z \in \Sigma^*\) other than \(\epsilon\), \(\text{pred}(z)\) denotes the immediate lexicographical predecessor of \(z\) within \(\Sigma^*\), e.g., \(\text{pred}(1100) = 1011\). If \(x \notin B_2\) then \(f'(x) = 1f(x)\) (i.e., \(f(x)\) prefixed with a one). If \(x \notin B_2\) and \(g(x) = \epsilon\) then \(f'(x) = \epsilon\). And, finally, if \(x \notin B_2\) and \(g(x) \neq \epsilon\) then \(f'(x) = 0\text{pred}(g(x))\). (The shift-by-one treatment of the \(x \notin B\) case is because we must ensure that \(\epsilon\) is mapped to by some string in \(A \cup B_2\).)

\(\square\)

Corollary 4.2.  \[1\] The class of \(F_{\text{REC}}\)-rankable sets is closed under symmetric difference with finite sets (and thus also under removing and adding finite sets).

2. The class of \(F_{\text{REC}}\)-compressible sets is closed addition and subtraction of finite sets.
5 Closures and Boolean Operations

Section 4 establishes some closure properties of recursively compressible and rankable sets. However, there are limits to how far one can go. For example, the \( F_{REC} \)-rankable sets and the \( F_{REC} \)-compressible sets are not closed under intersection.

**Theorem 5.1.** There exist infinite \( F_{REC} \)-rankable sets \( A \) and \( B \) such that \( A \cap B \) is an infinite set yet is not \( F_{PR} \)-compressible.

**Proof.** Let our alphabet (relative to which the class of RE sets is defined) be \( \Sigma = \{0, 1\} \). The join is usually defined as: \( L_1 \oplus L_2 = \{0x \mid x \in L_1\} \cup \{1x \mid x \in L_2\} \). But in this proof, we’ll use a version, also used in [HR16], that puts the mark in the rightmost rather than leftmost bit; this will support the fact that the theorem is even making a claim about ranking. In particular, let \( \hat{\oplus} \) be defined by \( A \hat{\oplus} B = \{x0 \mid x \in A\} \cup \{x1 \mid x \in B\} \).

Let \( L \) denote a set that is not \( F_{PR} \)-compressible. Such sets are known to exist (indeed, they even exist within \( \Delta^0_2 \)) [HR16]. Consider \( A = L \hat{\oplus} \emptyset \) and \( B = \Sigma^* \hat{\oplus} \emptyset \). Both \( A \) and \( B \) are infinite. \( B \) is clearly an \( F_{REC} \)-rankable set. \( A \) also is \( F_{REC} \)-rankable, namely via the function that maps \( \epsilon \) to \( \epsilon \) and that for each \( y \) maps \( y0 \) and \( y1 \) to \( y \).

\( A \cap B = L \hat{\oplus} \emptyset \) is an infinite set. Yet it cannot be \( F_{PR} \)-compressible, since a compression function \( g \) for \( L \hat{\oplus} \emptyset \) can be restricted to give a compression function \( f \) for \( L \) by taking \( f(x) = g(x0) \).

In the following, we speak of \( F_{REC} \)-compressible', since the fact that \( F_{REC} \)-compressible is not closed under intersection would follow simply from the fact that \( 1(0+1)^* \) and \( 0(0+1)^* \) are both \( F_{REC} \)-compressible yet their intersection is finite and finite sets can never be \( F_{REC} \)-compressible. But our statement below about \( F_{REC} \)-compressible' is a “real” claim, in that it is not exploiting the no-finite-sets feature of the \( F_{REC} \)-compressible sets.

**Corollary 5.2.** None of \( F_{REC} \)-rankable, \( F_{REC} \)-compressible', \( F_{PR} \)-rankable, and \( F_{PR} \)-compressible' are closed under intersection.

For union, we have the following similar claim.

**Theorem 5.3.** There exist infinite \( F_{REC} \)-rankable sets \( A \) and \( B \) such that \( A \cup B \) is not \( F_{REC} \)-compressible.

**Proof.** The proof is essentially the same as that of Theorem 5.1, except changing the choice of \( B \) to \( B = \emptyset \hat{\oplus} \Sigma^* \), replacing \( L \) with the RE-complete set \( K = \{x \mid x \in L(M_x)\} \), and replacing the final paragraph of the proof with the following paragraph:

Yet \( A \cup B = K \hat{\oplus} \Sigma^* \), which is an infinite RE set that clearly is not recursive. Since it is known that every infinite \( F_{REC} \)-compressible RE set is recursive [HR16], we conclude that \( A \cup B \) is an infinite set that is not \( F_{REC} \)-compressible.
Corollary 5.4. None of $F_{REC}$-rankable, $F_{REC}$-compressible, and $F_{REC}$-compressible' are closed under union.

By a similar approach, we can show that the $F_{REC}$-rankable sets are not closed under complementation.

Theorem 5.5. There exists a $F_{REC}$-rankable set $A$ such that $\overline{A}$ is not $F_{PR}$-rankable.

Proof. As above, $K = \{x \mid x \in L(M_x)\}$. Let $A$ be the set $\{x00 \mid x \in K\} \cup \{x10 \mid x \in \overline{K}\}$. $A$ is clearly $F_{REC}$-rankable. Yet note that

$$\overline{A} = \{\epsilon, 0, 1\} \cup \{x00 \mid x \in \overline{K}\} \cup \{x01 \mid x \in \Sigma^*\} \cup \{x10 \mid x \in K\} \cup \{x11 \mid x \in \Sigma^*\}.$$ 

If $A$ were $F_{PR}$-rankable with ranking function $f$, then we have $x \in K \iff f(x11) - f(x01) = 2$, contradicting the undecidability of $K$. □

Corollary 5.6. Neither $F_{REC}$-rankable nor $F_{PR}$-rankable is closed under complementation.

6 Relativization

The results of [HR16] all relativize in a straightforward manner. In this section, we include a few examples. This justifies our limitation to $F_{REC}$ and $F_{PR}$: By relativization, we get analogous results about more powerful function classes, such as $F_{\Delta_i}$.

Theorem 6.1. For each $i \geq 1$, $\Delta_i = \Sigma_i \cap F_{\Delta_i}$-compressible'.

Proof. Relativization of [HR16, Theorem 17]. □

Since, for $i \geq 1$ $F_{\Delta_i} \supseteq F_{REC}$, we get the following easy corollary.

Corollary 6.2. For each $i \geq 1$, $\Sigma_i \cap F_{REC}$-compressible' $\subseteq \Delta_i$.

Theorem 6.3. For each $i \geq 1$, $\Pi_i \cap F_{\Delta_i}$-rankable $= \Pi_i \cap F^{\Sigma_i}_{PR}$.

Proof. Relativization of [HR16, Theorem 10]. □

\[1\] $F_{\Delta_{i+1}}$ will denote the class of total functions computed by Turing machines given access to a $\Sigma_i$ oracle. Equivalently, $F_{\Delta_{i+1}} = (F_{REC})^{\Sigma_i}$. Note that $F_{REC} = F_{\Delta_1}$. The class of partial functions computed by Turing machines given access to a $\Sigma_i$ oracle will be denoted $(F_{PR})^{\Sigma_i}$ or simply as $F^\Sigma_{PR}$. 

8
7 Compressibility, Honesty, and Selectivity

If we restrict our attention to honest functions, we can get some very clean results. There is a little subtlety here, since there are many non-equivalent definitions of honesty. We use the following:

A (possibly partial) function \( f \) is honest on \( B \) if there is a recursive function \( g : \mathbb{N} \to \mathbb{N} \) such that for any \( x \in \text{domain}(f) \cap B \), \( g(|f(x)|) \geq |x| \). If \( f \) is honest on \( \Sigma^* \), we say \( f \) is honest.

This gives two potential ways to define honest compressibility. For a given set \( A \), we can require the compression function to be honest on \( \Sigma^* \), or only on \( A \). We call the former notion honestly-FPR-compressible, and the latter honestly-on-A-FPR-compressible. The following theorem asserts that these two notions are equivalent for FPR and FREC functions.

**Theorem 7.1.**
1. For each set \( A \), \( A \) is honestly-FPR-compressible if and only if \( A \) is honestly-on-A-FPR-compressible.
2. For each set \( A \), \( A \) is honestly-FREC-compressible if and only if \( A \) is honestly-on-A-FREC-compressible.

**Proof.** The “only if” direction is trivial, since every honest on \( \Sigma^* \) function is honest on \( A \).

For the “if” direction, let \( f \) be an honest on \( A \) compression function for \( A \), and let \( g \) be a recursive honesty-bound function for \( f \). Define \( f' \) as follows. \( \text{domain}(f') = \text{domain}(f) \). Over the domain of \( f \), if \( g(|f(x)|) \geq |x| \), then \( f'(x) = f(x) \). If \( g(|f(x)|) < |x| \), let \( f'(x) = x \). Since \( f \) was honest on \( A \), for any \( x \in A \), \( f'(x) = f(x) \). Thus \( f' \) is still a compression function for \( A \). The recursive function \( g'(n) = \max(g(n), n) \) satisfies, for all \( x \in \text{domain}(f') \), \( g'(|f'(x)|) \geq |x| \), and thus proves that \( f' \) is honest (on \( \Sigma^* \)).

Note that when \( f \) is recursive, so is \( f' \), giving us the second part of the theorem.

The following proof uses F-selectivity, which was very rarely useful. A set \( A \) is F-selective if there is a function \( f \in F \) of two arguments such that the following hold:

1. For any \( x, y \in \Sigma^* \), either \( f(x, y) = x \) or \( f(x, y) = y \).
2. If \( x \in A \) or \( y \in A \), \( f(x, y) \in A \).

Intuitively, \( f \) selects the “more likely” of its two inputs. When \( x, y \not\in A \), or \( x, y \in A \), \( f \) can choose either input. It’s only restricted when one input is in \( A \), and the other is not. Both FREC-selectivity and honestly-FREC-compressibility are fairly strong claims. Only the infinite recursive sets satisfy both. Let INFINITE denote the class of all infinite sets (over the alphabet \( \Sigma^* \), and recall that for us \( \Sigma \) is always \{0, 1\}).

**Theorem 7.2.** \( \text{FREC-selective} \cap \text{honestly-FREC-compressible} = \text{REC} \cap \text{INFINITE} \).

**Proof.** Every infinite recursive set is easily FREC-selective and honestly-FREC-compressible giving the \( \supseteq \) inclusion.
For the \( \subseteq \) inclusion, let \( A \) be honestly-F\(_{\text{REC}}\)-compressible by \( f \), with honesty bound \( g \), and let \( h \) be a F\(_{\text{REC}}\) selector function for \( A \). Then, for any \( z \), by the definition of compressibility and honesty:

\[
\| \{ w \mid f(w) = z \land |w| \leq g(|z|) \} \cap A \| = 1
\]

So define the finite set \( Q_z = \{ w \mid f(w) = z \land |w| \leq g(|z|) \} \). We know this set contains exactly one element of \( A \), and this will allow us to decide \( A \).

On input \( x \), compute \( f(x) \) and \( Q_{f(x)} \). Then use the selector function to find the unique element \( y \in Q_{f(x)} \) such that for any \( z \in Q_{f(x)} \), \( h(y, z) = y \). Such a \( y \) exists and is unique because there is exactly one element of \( A \) in \( Q_{f(x)} \).

If \( x = y \), then \( x \in A \). Otherwise, \( x \not\in A \), so \( A \) is recursive. Since \( A \) was compressible, it is infinite as well.

In fact, honestly-F\(_{\text{REC}}\)-compressible is much stronger than F\(_{\text{REC}}\)-compressible. While all co-r.e. cylinders were F\(_{\text{REC}}\)-compressible, we have that no co-r.e. set is honestly-F\(_{\text{REC}}\)-compressible.

**Theorem 7.3.** honestly-F\(_{\text{REC}}\)-compressible \( \cap \text{coRE} = \text{REC} \cap \text{INFINITE} \).

*Proof.* Every infinite recursive set is easily coRE and honestly-F\(_{\text{REC}}\)-compressible giving the \( \supseteq \) inclusion.

For the \( \subseteq \) inclusion, let \( A \) be co-r.e. and honestly-F\(_{\text{REC}}\)-compressible by a compression function \( f \). Let \( M \) accept \( \overline{A} \) Define the sets \( Q_z \) from the proof of Theorem 7.2.

Then for any input \( x \), compute \( f(x) \) and \( Q_{f(x)} \). Then dovetail applying \( M \) to each element of \( Q_{f(x)} \) until only one remains. If the remaining element is \( x \), then \( x \in A \). Otherwise, \( x \not\in A \), so \( A \) is recursive and infinite.

This next group of theorems builds to a result that if \( A \) is nonrecursive, F\(_{\text{REC}}\)-selective and F\(_{\text{REC}}\)-compressible then \( \overline{A} \) has an infinite r.e. subset. Since F\(_{\text{REC}}\)-selectivity is such a strong assumption, this theorem is of limited use. However, the arguments used to show it may prove useful in the proof of other claims.

**Theorem 7.4.** If \( A \) is F\(_{\text{REC}}\)-compressible via \( f \) and \( f(\overline{A}) \) is finite, then \( A \) is recursive.

*Proof.* Using the definition of compressibility, \( L = \{ x \in A \mid f(x) \in f(\overline{A}) \} \) is finite. By the assumptions of the theorem, so is \( f(\overline{A}) \). But, \( x \in A \) if and only if \( x \in L \lor f(x) \not\in f(\overline{A}) \) Since both of these sets are finite, this condition is recursive, and so is \( A \).

Now we consider the case where \( f(\overline{A}) \) is infinite.

**Theorem 7.5.** If \( A \) is F\(_{\text{REC}}\)-compressible via \( f \) and \( f(\overline{A}) \) is infinite, then there is an infinite r.e. set \( B_A = \{(p_1,q_1),(p_2,q_2),...\} \) such that no string appears in more than one pair and each pair contains at least one element of \( \overline{A} \)^{\footnote{In the theorem and the proof (and similarly regarding the proof of the corollary) we should, to be formally correct, define and work with the one-dimensional set \( \{(a,b) \mid (a,b) \in B_A \} \), where \( \langle \cdot, \cdot \rangle \) is a nice, standard pairing function; but let us consider that implicit.}}
Proof. We describe a machine that enumerates the desired set.

Initialize $Q = \emptyset$. Begin running $f(\epsilon), f(0), f(1), \ldots$ in sequence. Since $f(A)$ is infinite, there will be two strings $x, y \not\in Q$ where $f(x) = f(y)$. Enumerate $(x, y)$, and add both $x$ and $y$ to $Q$. The enumerated set will have the desired properties.

Corollary 7.6. If $A$ is nonrecursive, $\text{F}_{\text{REC}}$-selective, and $\text{F}_{\text{REC}}$-compressible, then $\overline{A}$ has an infinite r.e. subset.

Proof. Create the set from Theorem 7.5 and apply the $\text{F}_{\text{REC}}$-selector to each pair. If the selector chooses $p_i$, then $q_i \in \overline{A}$, and vice versa. So we can enumerate an infinite r.e. subset of $\overline{A}$ by enumerating the elements not chosen by the selector.

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