On analogues of the Arakawa-Kaneko zeta functions of Mordell-Tornheim type

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Abstract

In this paper, we construct certain analogues of the Arakawa-Kaneko zeta functions. We prove functional relations between these functions and the Mordell-Tornheim multiple zeta functions. Furthermore we give some formulas among Mordell-Tornheim multiple zeta values as their applications.

1 Introduction

Let \( \mathbb{Z} \) be the rational integer ring, \( \mathbb{N} \) the set of natural numbers, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mathbb{Q} \) the rational number field and \( \mathbb{C} \) the complex number field. We denote \( n \) repetitions of \( m \) by \( \{m\}_n \) for \( m, n \in \mathbb{N} \).

Arakawa and Kaneko \cite{1} introduced the “Arakawa-Kaneko zeta function” defined by

\[
\xi(k_1, k_2, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t} - 1} \ln_{k_1, k_2, \ldots, k_r} (1 - e^{-t}) \, dt
\]

for \( (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r \) and \( s \in \mathbb{C} \) with \( \Re(s) > 0 \), where \( \ln_{k_1, k_2, \ldots, k_r}(z) \) is the polylogarithm defined by

\[
\ln_{k_1, k_2, \ldots, k_r}(z) = \sum_{0 < m_1 < m_2 < \cdots < m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \quad (z \in \mathbb{C}, \, |z| < 1).
\]

When \( r = 1 \), \( \xi(k; s) \) is also denoted by \( \xi_k(s) \). They proved that for \( m \in \mathbb{N}_0 \), \( \xi(\{1\}_r^{-1}, k; m + 1) \) can be written in terms of multiple zeta values (MZVs) in \cite{1} Theorem 9.

On the other hand, Matsumoto defined the “Mordell-Tornheim \( r \)-ple zeta function” by

\[
\zeta_{MT,r}(s_1, s_2, \ldots, s_r; s_{r+1})
\]

\[
= \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r} (\sum_{j=1}^r m_j)^{s_{r+1}}} \quad (s_i \in \mathbb{C}, \, \Re(rs_i + s_{r+1}) > r),
\]

and proved that this function can be continued meromorphically to the whole \( \mathbb{C}^{r+1} \)-space in \cite{4} and \cite{5}. This zeta function in the double sum case was first
studied by Tornheim [8] for the values at positive integers in 1950s. He gave some evaluation formulas for $\zeta_{MT,2}(k_1, k_2; k_3)$ for $k_1, k_2, k_3 \in \mathbb{N}$. Mordell [6] independently proved that $\zeta_{MT,2}(k; k; \pi^{-3k}) \in \mathbb{Q}$ for all even $k \geq 2$. Tsumura [9, Theorem 4.5] and Nakamura [7, Theorem 1] showed certain functional relations among the Mordell-Tornheim double zeta functions and the Riemann zeta functions.

In this paper, for $k \in \mathbb{N}$, we first define the function

$$\xi_{MT}(k; s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \prod_{j=1}^{r} \text{Li}_{k_j}(1 - e^{-t}) \, dt \quad (s \in \mathbb{C}, \Re(s) > 0)$$

which can be regarded as an analogue of the Arakawa-Kaneko zeta function of Mordell-Tornheim type (see Definition 1). We construct functional relations between $\xi_{MT}(k; s)$ and the Mordell-Tornheim multiple zeta functions (see Theorem 8). For example,

$$\zeta(2)^2 \zeta(s) - 2\zeta(2)\xi_{MT}(2; s) + \xi_{MT}(2, 2; s) = \zeta_{MT,3}(2, 2, 0; s) + 2s \zeta_{MT,3}(2, 1, 0; s + 1) + s(s + 1) \zeta_{MT,3}(1, 1, 0; s + 2).$$

This can be proved by the method similar to the proof of [1, Theorem 8]. Secondly, we show certain relation formulas among Mordell-Tornheim multiple zeta values (see Corollary 9). For example,

$$\zeta(2)^2 \zeta(m + 1) - 2\zeta(2) \frac{1}{m!} \zeta_{MT,m+1}(2, \{1\}^m; 1) + \frac{1}{m!} \zeta_{MT,m+2}(2, 2, \{1\}^m; 1) = \zeta_{MT,3}(2, 2, 0; m + 1) + 2(m + 1) \zeta_{MT,3}(2, 1, 0; m + 2) + (m + 1)(m + 2) \zeta_{MT,3}(1, 1, 0; m + 3) \quad (m \in \mathbb{N}, \ m \geq 3),$$

and

$$\zeta_{MT,3}(2, 1, 1; 1) = 2\zeta(2)\zeta(3) - \zeta(5).$$

Lastly, we consider a generalization of main results (see Theorem 15).

2 Preliminaries

We first construct a Mordell-Tornheim type analogue of $\xi(k_1, k_2, \ldots, k_r; s)$ and continue it analytically to an entire function. We define $\{C_{m,MT}^{(k)}\}$ by

$$\prod_{j=1}^{r} \text{Li}_{k_j}(1 - e^{-t}) = \sum_{m=0}^{\infty} C_{m,MT}^{(k)} \frac{t^m}{m!}$$

for $k = (k_1, k_2, \ldots, k_r) \in \mathbb{Z}^r$. These are generalizations of poly-Bernoulli numbers $\{C_m^{(k)}\}$ defined by

$$\text{Li}_k(1 - e^{-t}) = \sum_{m=0}^{\infty} C_m^{(k)} \frac{t^m}{m!}$$

for $k \in \mathbb{Z}$ (see [1]). Since $\text{Li}_k(1 - e^{-t}) = O(t)$ ($t \to 0$) and $\text{Li}_k(1 - e^{-t}) = O(t)$ ($t \to \infty$) for all $k \in \mathbb{N}$, we can define the following function.
Definition 1. For $k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - r$, let

$$\xi_{MT}(k; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t-1} \prod_{j=1}^r \text{Li}_{k_j}(1-e^{-t}) \, dt,$$

(4)

where $\Gamma(s)$ is the gamma function.

The integral on the right-hand side of (4) converges absolutely uniformly in the region $\Re(s) > 1 - r$. When $r = 1$, $\xi_{MT}(k; s) = \xi_k(s)$ holds for $k = (k) \in \mathbb{N}$.

Theorem 2. For $k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r$, the function $\xi_{MT}(k; s)$ can be continued analytically to an entire function, and satisfies

$$\xi_{MT}(k; -m) = (-1)^m C_{m,MT}^k \quad (m \in \mathbb{N}_0).$$

(5)

Proof. Let

$$A(k; s) = \int_{C_{\varepsilon}} \frac{t^{s-1}}{e^t-1} \prod_{j=1}^r \text{Li}_{k_j}(1-e^{-t}) \, dt$$

$$= \left( e^{2\pi \sqrt{-1} s} - 1 \right) \int_{\varepsilon}^\infty \frac{t^{s-1}}{e^t-1} \prod_{j=1}^r \text{Li}_{k_j}(1-e^{-t}) \, dt$$

$$+ \int_{C_{\varepsilon}} \frac{t^{s-1}}{e^t-1} \prod_{j=1}^r \text{Li}_{k_j}(1-e^{-t}) \, dt \quad (s \in \mathbb{C}),$$

where $C$ is the contour which is the path consisting of the real axis (top side), a circle $C_{\varepsilon}$ around the origin of radius $\varepsilon$ (sufficiently small), and the positive real axis (bottom side). Since the integrand has no singularity on $C$ and the contour integral converges absolutely for all $s \in \mathbb{C}$, we can see that $A(k; s)$ is entire. Suppose $\Re(s) > 1 - r$, then the second integral tends to 0 as $\varepsilon \to 0$. Therefore we have

$$\xi_{MT}(k; s) = \frac{1}{\left( e^{2\pi \sqrt{-1} s} - 1 \right) \Gamma(s)} A(k; s).$$

Since $\xi_{MT}(k; s)$ is holomorphic for $\Re(s) > 1 - r$, this function has no singularity at any positive integer. Therefore this gives the analytic continuation of $\xi_{MT}(k; s)$ to an entire function. Let $s = -m$ for $m \in \mathbb{N}_0$. Using (4), we have

$$\xi_{MT}(k; -m) = \frac{(-1)^m m!}{2\pi \sqrt{-1}} A(k; -m)$$

$$= \frac{(-1)^m m!}{2\pi \sqrt{-1}} \int_{C_r} t^{-m-1} \sum_{n=0}^\infty C_{n,MT}^k \frac{t^n}{n!} \, dt$$

$$= (-1)^m C_{m,MT}^k.$$

This completes the proof.

Secondly, we show a relation between the Mordell-Tornheim multiple zeta values and $\xi_{MT}(k; m + 1)$ for $m \in \mathbb{N}_0$. For this aim, we consider the following function and give a lemma.
Definition 3. For \( k = (k_1, k_2, \ldots, k_{r+1}) \in \mathbb{N}^r \times \mathbb{N}_0 \) and \( z \in \mathbb{C} \) with \( |z| < 1 \), let
\[
\mathcal{L}_k(z) = \sum_{m_1, m_2, \ldots, m_r=1}^{\infty} \frac{z^{\sum_{j=1}^{r} m_j} m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r} (\sum_{j=1}^{r} m_j)^{k_{r+1}}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}.
\]

Under the above condition, the sum on the right-hand side of (6) converges absolutely uniformly. We note that \( \mathcal{L}_k(z) = L_{k_1+k_2}(z) \) holds for \( r = 1 \) and \( k = (k_1, k_2) \). By direct calculation, we have

Lemma 4. For \( k = (k_1, k_2, \ldots, k_r, k_{r+1}) \in \mathbb{N}^{r+1} \) and \( z \in \mathbb{C} \) with \( |z| < 1 \),
\[
\frac{d}{dz} \mathcal{L}_k(z) = \begin{cases} 
\frac{1}{z} \mathcal{L}_k(z) & (k_{r+1} \geq 2), \\
\frac{1}{z} \prod_{j=1}^{r} L_{k_j}(z) & (k_{r+1} = 1),
\end{cases}
\]
where \( k^{(r+1)} = (k_1, k_2, \ldots, k_r, k_{r+1} - 1) \).

Proposition 5. For \( k_1 = (k_1, k_2, \ldots, k_r, 0) \in \mathbb{N}^r \times \mathbb{N}_0 \), \( k = (k_1, k_2, \ldots, k_r) \) and \( m \in \mathbb{N}_0 \),
\[
\xi_{MT}(k; m + 1) = \frac{1}{m!} \zeta_{MT, m+r}(k_1, k_2, \ldots, k_r, \{1\}^m; 1).
\]

We can recover [3, Corollary 4.2 and Theorem 4.4] as follows.

Corollary 6. For \( m \in \mathbb{N}_0 \),
\[
\zeta_{MT, m+1}(\{1\}^m+1; 1) = (m + 1)! \zeta(m + 2).
\]

Proof. By \( \xi_1(s) = s \zeta(s + 1) \) and Proposition 5, we obtain the assertion.

3 Main results

In this section, we give main results. We first prepare the following lemma which is necessary to show the first and second main results.

Lemma 7. For \( s_j \in \mathbb{C} \) with \( \Re(s_j) > 0 \) (\( 2 \leq j \leq r \)) and \( \Re(s_{r+1}) > r \),
\[
\zeta_{MT, r}(0, s_2, \ldots, s_r; s_{r+1}) = 
\frac{1}{\prod_{j=2}^{r+1} \Gamma(s_j)} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\prod_{j=2}^{r+1} e^{s_j-1}}{(e^{s_{r+1}} - 1) \prod_{j=2}^{r+1} (e^{t_{r+1}} - 1) dt_2 \cdots dt_r dt_{r+1}}.
\]
Proof. Using the well-known relation
\[
m^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-mt} \, dt \quad (m \in \mathbb{N}, \ s \in \mathbb{C}, \ \Re(s) > 0)
\]
for \( s_j \in \mathbb{C} \) with \( \Re(s_j) > 0 \) (2 \leq j \leq r) and \( \Re(s_{r+1}) > r \), we have
\[
\prod_{j=2}^{r+1} \Gamma(s_j) \times \zeta_{MT,r}(0, s_2, \ldots, s_r; s_{r+1})
\]
\[
= \sum_{m_1, m_2, \ldots, m_{r+1}} \prod_{j=2}^{r+1} \left( \int_0^\infty t_j^{m_j-1} e^{-m_j t_j} \, dt_j \right)
\times \left( \int_0^\infty t_{r+1}^{m_{r+1}-1} e^{-(\sum_{j=1}^{r} m_j) t_{r+1}} \, dt_{r+1} \right)
\]
\[
= \sum_{m_1, m_2, \ldots, m_{r+1}} \int_0^\infty \cdots \int_0^\infty dt_2 dt_3 \cdots dt_{r+1}
\times \left( \prod_{j=2}^{r+1} t_j^{m_j-1} \right) \left( e^{-t_{r+1}} \right) \prod_{j=2}^{r+1} e^{-m_j (t_j + t_{r+1})}
= \int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=2}^{r+1} t_j^{m_j-1}}{(e^{t_{r+1}} - 1) \prod_{j=2}^{r+1} (e^{t_j + t_{r+1}} - 1)} dt_2 dt_3 \cdots dt_{r+1}.
\]
Changing the order of summation and integration is justified by absolutely convergence. Therefore we complete the proof.

Using Lemma 7, we have the first main result as follows.

**Theorem 8.** For \( r \in \mathbb{N} \) and \( s \in \mathbb{C}, \)
\[
\sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j (2)^{r-1-j} \zeta_{MT}([2]^j; s)
= \sum_{j=0}^{r-1} \binom{r-1}{j} (s) \zeta_{MT,r}([2]^{r-1-j}, \{1\}^j, 0; s + j).
\]

Proof. We first assume \( r \geq 2 \). For \( s \in \mathbb{C} \) with \( \Re(s) > 0 \), let
\[
J_{MT,r}(s) = \int_0^\infty \cdots \int_0^\infty dt_1 dt_2 \cdots dt_r \frac{t_{r+1}^{r-1} - 1}{e^{t_r} - 1} \prod_{j=1}^{r-1} \frac{t_j + t_r}{e^{t_j + t_r} - 1}.
\]
Using
\[
\frac{\partial}{\partial \gamma_j} \text{Li}_2(1 - e^{-t_j - t_r}) = \frac{t_j + t_r}{e^{t_j + t_r} - 1} \quad (1 \leq j \leq r - 1),
\]
we have
\[
J_{MT,r}(s) = \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} (2)^{r-1-j} \zeta_{MT}([2]^j; s).
\]
On the other hand, by Lemma 7 and
\[ \zeta_{MT,r}(i, s_i, j, s_j, \ldots; s_r+1) = \zeta_{MT,r}(j, s_j, i, s_i, \ldots; s_r+1) \]
for \(1 \leq i \leq j \leq r\), we have
\[ J_{MT,r}(s) = \sum_{j=0}^{r-1} \binom{r-1}{j} \Gamma(s+j) \zeta_{MT,r}(\{2\}^{r-1-j}, \{1\}^j, 0; s+j). \]
By the analytic continuation, we obtain the desired identity in the case \(r \geq 2\). When \(r = 1\), it holds trivially. Therefore we complete the proof.

By Theorem 8 and Proposition 5, we immediately obtain the second main result as follows.

**Corollary 9.** For \(r \in \mathbb{N}\) and \(m \in \mathbb{N}_0\),
\[ \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{(-1)^j \zeta(2)^{r-1-j}}{m!} \zeta_{MT,j+m}(\{2\}^j, \{1\}^m; 1) = \sum_{j=0}^{r-1} \binom{r-1}{j} (m+1) \zeta_{MT,r}(\{2\}^{r-1-j}, \{1\}^j, 0; m + 1 + j). \]

Next, in order to evaluate \(\zeta_{MT,2k+1}(2, \{1\}^{2k}; 1)\), we quote \([2, (75)]\):
\[ \zeta(a, b) = \frac{1}{2} \left\{ (-1)^b \binom{M}{a} - 1 \right\} \zeta(M) + (1 + (-1)^b) \zeta(a) \zeta(b) \]
\[ + (-1)^{b+1} \sum_{k=1}^{(M-3)/2} \left\{ \binom{2k}{a-1} + \binom{2k}{b-1} \right\} \zeta(2k+1) \zeta(M-2k-1), \]
where \(a, b \in \mathbb{N}\) with \(a, b \geq 2\) and \(M = a + b \equiv 1 \pmod{2}\).

**Remark 10.** We note that \([3]\) also holds for \(a = 1\) providing we remove the term containing \(\zeta(1)\).

Combining \([3]\) and Corollary 9 in the case \(r = 2\), we have the third main result as follows.

**Proposition 11.** For \(k \in \mathbb{N}\),
\[ \zeta_{MT,2k+1}(2, \{1\}^{2k}; 1) = (2k)! \left\{ \zeta(2) \zeta(2k+1) - \frac{1}{2} (2k^2 + k - 2) \zeta(2k+3) \right\} \]
\[ + \sum_{n=1}^{k-1} (2k+1-2n) \zeta(2n+1) \zeta(2k+2-2n) \].

**Example 12.**
\[ \zeta_{MT,3}(2, 1, 1; 1) = 2 \zeta(2) \zeta(3) - \zeta(5), \]
\[ \zeta_{MT,5}(2, 1, 1, 1, 1; 1) = 4! \left\{ \zeta(2)^2 \zeta(3) + 3 \zeta(3) \zeta(4) - 4 \zeta(7) \right\}. \]
Theorem 15. Using (9) and (10), we can define the following function.

4 A generalization of the function $\xi_{\text{MT}}(k; s)$

In this section, we consider a certain generalization of the function $\xi_{\text{MT}}(k; s)$ and aim to generalize Theorem 8.

By the definition (8), for $k = (k_1, k_2, \ldots, k_r, k_{r+1}) \in \mathbb{N}^r \times \mathbb{N}_0$, we have

$$ (9) \quad \mathcal{L}_k(1 - e^{-t}) = \begin{cases} O(t^l) & \text{if } k_{r+1} = 0 \text{ and } l = 2\{j \mid k_j = 1\} \geq 1, \\ O(1) & \text{otherwise } \quad (t \to \infty) \end{cases} $$

and

$$ (10) \quad \mathcal{L}_k(1 - e^{-t}) = O(t^l) \quad (t \to 0). $$

Using (9) and (10), we can define the following function.

Definition 13. For $r_1, r_2, \ldots, r_g \in \mathbb{N}$, $k_i = (k_1^{(i)}, k_2^{(i)}, \ldots, k_r^{(i)}, k_{r+1}^{(i)}) \in \mathbb{N}^r \times \mathbb{N}_0$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - \sum_{i=1}^g r_i$, let

$$ (11) \quad \xi_{\text{MT},g}(k_1, k_2, \ldots, k_g; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \prod_{i=1}^g \mathcal{L}_{k_i}(1 - e^{-t}) dt. $$

The integral on the right-hand side of (11) converges absolutely uniformly in the region $\Re(s) > 1 - \sum_{i=1}^g r_i$. Further we note that

$$ \xi_{\text{MT},1}(k_1; s) = \xi_{\text{MT}}(k; s) $$

for $k_1 = (k_1, k_2, \ldots, k_r, 0)$, $k = (k_1, k_2, \ldots, k_r)$ and $s \in \mathbb{C}$. Therefore we can see that Definition 13 is a generalization of the function $\xi_{\text{MT}}(k; s)$. By the same method as in the proof of Theorem 8 we have

Theorem 14. For $g, r_1, r_2, \ldots, r_g \in \mathbb{N}$ and $k_i \in \mathbb{N}^r \times \mathbb{N}_0 \ (1 \leq i \leq g)$, the function $\xi_{\text{MT},g}(k_1, k_2, \ldots, k_g; s)$ can be continued analytically to an entire function.

By the same method as in the proof of Theorem 8 we obtain

Theorem 15. For $N \in \mathbb{N}$, $r = (r_1, r_2, \ldots, r_{N-1}) \in \mathbb{N}^{N-1}$ and $s \in \mathbb{C}$,

$$ (12) \quad \sum_{n=0}^{N-1} (-1)^n \sum_{J \subseteq I_{N-1} \setminus \{j\}} \left\{ \prod_{j \in I_{N-1} \setminus J} \xi_{\text{MT},r_j}(\{1\}^{r_j}; 1) \right\} \xi_{\text{MT},n}(\{1_{r_j+1} \mid j \in J\}; s) $$

$$ = \sum_{n=0}^{\text{wt}(r)} \binom{s}{n} \sum_{i_1 + \cdots + i_{N-1} = n} \left\{ \prod_{j=1}^{N-1} \binom{r_j}{i_j} (r_j - i_j)! \right\} $$

$$ \times \xi_{\text{MT},N}(r_1 - i_1 + 1, r_2 - i_2 + 1, \ldots, r_{N-1} - i_{N-1} + 1, 0; s + n), $$

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where \(1_{r+1} = (\{1\}^{r+1}) \in \mathbb{N}^{r+1}, \xi_{MT,N}(\emptyset; s) = \zeta(s), I_{N-1} = \{1, 2, \ldots, N-1\}\) and \(\text{wt}(r) = \sum_{i=1}^{N-1} r_i\).

**Proof.** We first assume \(N \geq 2\) and define the function \(J_{MT,r}(s)\) by

\[
J_{MT,r}(s) = \int_0^\infty \cdots \int_0^\infty dt_1 dt_2 \cdots dt_N \\
\times \frac{t_N^{-1}}{e^{t_N} - 1} \prod_{j=1}^{N-1} \frac{(t_j + t_N)^{r_j}}{e^{t_j + t_N} - 1} \quad (s \in \mathbb{C}, \Re(s) > 0)
\]

for \(r = (r_1, r_2, \ldots, r_{N-1}) \in \mathbb{N}^{N-1}.\) It follows from Lemma 4 that

\[
\frac{\partial}{\partial t_j} \mathcal{L}_{1_{r+1}}(1 - e^{-t_j - t_N}) = \frac{(t_j + t_N)^{r_j}}{e^{t_j + t_N} - 1}.
\]

Therefore we have

\[
J_{MT,r}(s) = \sum_{n=0}^{N-1} \frac{(-1)^n}{\prod_{j=1}^{N-1} (i_{j1} + \cdots + i_{jN-1} = n)} \left( \prod_{j=1}^{N-1} \frac{(t_j)^{r_j}}{i_{j1}! \cdots i_{jN-1}!} \right)
\]

for \(\Re(s) > 1.\) On the other hand, by Lemma 7 we have

\[
J_{MT,r}(s) = \sum_{n=0}^{N-1} \Gamma(s + n) \sum_{i_1 \geq i_2 \geq \cdots \geq i_{N-1} \geq 0} \left( \prod_{j=1}^{N-1} \frac{(t_j)^{r_j}}{i_{j1}! \cdots i_{jN-1}!} \right)
\]

for \(\Re(s) > N.\) By the analytic continuation, we obtain (12) for all \(s \in \mathbb{C}\) when \(N \geq 2.\) When \(N = 1,\) (12) holds obviously. Therefore the proof is completed.

**Remark 16.** In particular, Theorem 15 in the case \((N, r) = (r, 1_{r-1})\) coincides with Theorem 8. Hence we can see that Theorem 15 is a generalization of Theorem 8.

We have not obtained the values of \(\xi_{MT,g}(k_1, k_2, \ldots, k_g; m+1)\) for \(m \in \mathbb{N}_0.\) But we have a certain lemma as follows.

**Lemma 17.** For \(g, r_1, r_2, \ldots, r_g \in \mathbb{N},\)

\[
\sum_{j=1}^{g} r_j! \xi_{MT,g-1}(1_{r_1+1}, \ldots, 1_{r_{j-1}+1}, 1_{r_{j+1}}, \ldots, 1_{r_g+1}; r_j + 1)
\]

\[
= \prod_{j=1}^{g} \xi_{MT,r_j}(\{1\}^{r_j}; 1).
\]
Remark 18. In particular, combining Corollary 6, Theorem 15 in the case $N = 2$ and Lemma 17 in the case $g = 2$, we have the Euler decomposition (cf. [1]).

$$\zeta(k+1)\zeta(r+1) = \sum_{m=0}^{k} \binom{r+m}{r} \zeta(k+1-m, r+1+m)$$

$$+ \sum_{n=0}^{r} \binom{k+n}{k} \zeta(r+1-n, k+1+n) \quad (r, k \in \mathbb{N}).$$

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