Instanton homology and the Alexander polynomial

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Abstract. We prove that the instanton knot homology $KHI(K)$ as defined by Kronheimer-Mrowka recovers the Alexander polynomial for knots $K$ in the 3-sphere.

1. Introduction

In a recent paper [8], Kronheimer and Mrowka revisit an instanton knot homology first defined and studied by A. Floer [4]. For simplicity we assume the case of knots $K$ in the 3-sphere $S^3$. Kronheimer and Mrowka refined the instanton knot homology theory by introducing a bigrading on the groups. The bigraded groups are denoted by $KHI(K)$. They conjecture that $KHI(K)$ is isomorphic to other knot homologies, such as the one defined by Ozsvath and Szabo. A small step in this direction is to show that $KHI(K)$ recovers the Alexander polynomial. In this note we prove this. In addition we state two simple corollaries of the result.

1.1. Instanton homology for knots. We review the version of instanton Floer homology for an oriented knot $K$ in the 3-sphere $S^3$, as considered in [8]. It is defined to be the instanton homology of a certain closure $K^T$ of the knot complement $S^3 - \nu K^c$ by $(T - D^o) \times S^1$. Here $T$ is a 2-torus and $D$ is a small disk in $T$. Let $\Sigma_K$ be a Seifert surface for $K$, oriented so that the boundary, with the induced orientation, matches the orientation of $K$ on $\partial \nu K$. The meridians for $K$ on $\partial \nu K$ and the fibers $\{pt\} \times S^1$ on the boundary of $(T - D^o) \times S^1$. An orientation of the 3-manifold is required in the definition of instanton homology. $K^T$ is given the orientation induced from $S^3 - \nu K^c$, and $S^3$ is given the standard orientation on $\mathbb{R}^3$, regarding it as $\mathbb{R}^3 \cup \{\infty\}$.

To define instanton homology in the non-homology 3-sphere case we also need to specify a complex line bundle $\omega \to K^T$ whose Chern class has a non-zero mod 2 evaluation on at least one integral homology class in $K^T$. Let $\alpha$ be a homologically non-trivial oriented simple closed curve in $T$. Choose $\omega_0$ to be the complex line with Chern class Poincaré dual to $\alpha$, thinking of $\alpha$ as living on $(T - D^o) \times \{1\}$ in $K^T$. $I_* (K^T, \omega_0)$ denotes the instanton homology for $(K^T, \omega_0)$ with complex coefficients.
1.2. Generalized eigenspace decomposition. There is only a relative mod 8 grading on \( I_s(K^T)_{\omega_0} \), but there is an absolute mod 2 grading due to Froyshov \([5]\) (details below). We shall always assume the canonical mod 2 grading, and any mod 8 grading used will be assumed to be consistent with the mod 2 grading. This makes the groups

\[
\widetilde{I}_0(K^T)_{\omega_0} = \bigoplus I_{2i}(K^T)_{\omega_0}, \quad \widetilde{I}_1(K^T)_{\omega_0} = \bigoplus I_{2i-1}(K^T)_{\omega_0}
\]

well-defined.

Let \( x_0 \) be a point in \( K^T \). The action of the \( \mu \)-map evaluated at \( x_0 \), \( \mu(x_0) \), sends \( I_s(K^T)_{\omega_0} \) to \( I_s-4(K^T)_{\omega_0} \). By \([5]\) this determines a splitting of \( \widetilde{I}_s(K^T)_{\omega_0} \) into \( \pm 2 \)-eigenspaces, in their normalization. Then by definition

\[
KHI_s(K) = \widetilde{I}_s^+(K^T)_{\omega_0} = +2\text{-eigenspace}.
\]

Let \( \hat{\Sigma} \) denote the surface \((\Sigma_K \cap (S^3 - \nu K^c)) \cup (T - D^o) \times \{1\}, \) with the induced orientation from \( \Sigma_K \). The action of the \( \mu \)-map evaluated on \( \hat{\Sigma}, \mu(\hat{\Sigma}) \), sends \( I_s(K^T)_{\omega_0} \) to \( I_s-2(K^T)_{\omega_0} \). The actions of \( \mu(\hat{\Sigma}) \) and \( \mu(x_0) \) commute; \( \widetilde{I}_s^+(K^T)_{\omega_0} \) is preserved under the action of \( \mu(\hat{\Sigma}) \). In \([5]\) it is shown (with the normalization used there) that the eigenvalues of \( \mu(\hat{\Sigma}) \) are the even integers \( n \) satisfying the bound \( |n| \leq 2g - 2 \), \( g \) being the genus of \( \hat{\Sigma} \). Then there is a decomposition of \( KHI_s(K) \) by the generalized eigenspaces of \( \mu(\hat{\Sigma}) \), which in the notation of \([5]\) is written

\[
(1.1) \quad KHI_s(K) = \bigoplus_{i = -g}^{g} KHI_s(K, i).
\]

Define the finite Laurent polynomial

\[
P_K(t) = P_{\omega_0}(K^T, \hat{\Sigma})(t) = \chi_{-g} t^{-g} + \cdots + \chi_0 + \chi_1 t + \cdots + \chi_g t^g,
\]

where \( \chi_i \) is the Euler characteristic of \( KHI_s(K, i) \).

1.3. Statement of results.

Theorem 1.1. For any knot \( K \) in the 3-sphere, \( P_K(t) \) is exactly the symmetrized and normalized Alexander polynomial \( \Delta_K(t) \) of \( K \).

This proves Conjecture 7.26 of \([5]\). The symmetrized and normalized Alexander polynomial \( \Delta_K(t) \) of \( K \) is the unique representative that satisfies \( \Delta_K(t^{-1}) = \Delta_K(t) \) and \( \Delta_K(1) = 1 \). The proof of the theorem involves a straightforward application of Floer’s surgery exact triangle.

Instead of the closure of the knot complement by \((T - D^o) \times S^1 \) we can consider the standard closure by \( D^2 \times S^1 \) given by 0-surgery on \( K \), which we denote by \( K^D \). We consider the instanton homology of \( K^D \) but this time choose the complex line \( \omega' \) with Chern class Poincaré dual to an oriented meridian of \( K \), regarding the meridian as a curve in \( K^D \).

Corollary 1.2. For an oriented \( K \), let \( Q_K(t) \) be the finite Laurent polynomial defined analogously to \( P_K(t) \) but with \( K^D \) and the complex line \( \omega' \) instead. Then

\[
Q_K(t) = \frac{\Delta_K(t) - 1}{t - 2 + t^{-1}}
\]

where \( \Delta_K(t) \) is as above. In particular \( Q_K(1) = \frac{1}{2} \Delta_K'(1) \).
For example the unknot has \( Q_K(t) = 0 \), the trefoil knot has \( Q_K(t) = 1 \) and the figure-8 knot has \( Q_K(t) = -1 \). We mention that for \( I_\ast(K^D)_{\omega'} \), we do not actually know that \( \mu(x_0) \) splits \( I_\ast(K^D)_{\omega'} \) into \( \pm 2 \)-eigenspaces. However this is not needed, and we can simply use the generalized \( +2 \)-eigenspace in place of the \( +2 \)-eigenspace.

**Corollary 1.3.** If the symmetrized and normalized Alexander polynomial \( \Delta_K(t) \) of a knot \( K \) in \( S^3 \) is non-trivial, i.e. \( \Delta_K(t) \neq 1 \), then \( \pm 1 \)-surgery on \( K \) never yields a simply connected 3-manifold.

**Remark 1.4.** This corollary is a special case of the property P conjecture. (Property P is proven in [2] and [3], independently of Perelman’s proof of the Poincaré conjecture. It uses a result in [2] that states that consideration of \( \pm 1 \)-surgery is sufficient.)

**Proof.** If \( \Delta_K(t) \neq 1 \), then \( Q_K(t) \neq 0 \) and the instanton homology groups \( I_\ast(K^D)_{\omega'} \) are non-trivial. By Floer’s surgery exact triangle the instanton homology for the \( \nu \)-surgery on \( K \) is also non-trivial. Thus there must be at least one non-trivial representation of \( \pi_1(K) \) into \( SU(2) \). \( \square \)

2. **Preliminaries**

### 2.1. Mod 2 grading

We briefly review the canonical mod 2 grading since we will be using it throughout this note (see [5] and also [3], Sect. 6.5). Let \([g] \in I_\ast(Y)_{\omega} \). Suppose that \( Y = \partial X \) as oriented manifolds and that \( E \to X \) is a \( U(2) \)-bundle with connection \( A \) that extends \( g \) on \( Y \).

Let \( \hat{X} \) be the cylindrical-end manifold obtained by joining the semi-infinite tube \( Y \times [0, \infty) \) to the boundary. Likewise, extend \( E \) to \( \hat{E} \) and also extend \( A \) to \( \hat{A} \) by the pullback of \( g \) over the cylindrical end. We let \( \text{Ind} E \) be the index of the anti-self-dual operator on \( \hat{X} \) coupled to \( \hat{A} \). We also have indices \( \text{Ind}^\pm X \) of the anti-self-dual operator on forms on \( \hat{X} \), on the positive/negative \( \delta \)-weighted spaces where the weight is non-zero and sufficiently small in absolute value [3, Sect. 3.3.1].

Define the mod 2 grading of \([g] \) to be

\[
\nu([g]) = \text{Ind} E - 3\text{Ind}^\pm X \text{ mod } 2.
\]

Index calculations (see for instance [3], Sect. 3.3.1) show that \( \text{Ind}^\pm X = b_1(X) - b_2^\pm(X) \). (Here we assume that \( Y \) is connected.) \( b_2^+ \) is the dimension of a maximal positive definite subspace for the (possibly degenerate) intersection form on the image of \( H_2(X) \) in \( H_2(X, \partial X) \).

Let us now suppose that \( W \) is a cobordism between \( Y \) and \( Y' \) that induces a map \( I_W : I_\ast(Y)_{\omega} \to I_{\ast+k}(Y')_{\omega'} \). We wish to determine the value of \( k \text{ mod } 2 \), the mod 2 degree of \( I_W \).

**Lemma 2.1.** The degree \( k \) of the map \( I_W \) above satisfies

\[
k = 3(b_1(W) - b_1(Y) + b_0(Y') - b_0(W) - b_2^+(W)) \text{ mod } 2.
\]

**Remark 2.2.** If \( Y \) or \( Y' \) is disconnected the lemma is still valid. We need however to interpret the instanton homology of a disjoint union \( I_\ast(Y_0 \cup Y_1)_{\omega'} \) as the tensor product \( I_\ast(Y_0)_{\omega} \otimes I_\ast(Y_1)_{\omega'} \). In particular the grading satisfies \( \nu([g] \otimes [g']) = \nu([g]) + \nu([g']) \text{ mod } 2 \).
Proof. Let \([\varrho] \in I_\vartriangle(Y)_{\omega}\) and \(I_W([\varrho]) = [\varrho'] \in I_{*+k}(Y')_{\omega'}\). Let \(E_W\) be the \(U(2)\)-bundle with connection that limits to \(\varrho\) and \(\varrho'\) at the ends. Then by assumption \(\text{Ind}E_W = 0\). Let \(Y = \partial X\) and \(E\) be as above. The additivity property of the index [3 Sect. 3.3] tells us that
\[
\text{Ind}(E \cup E_W) = \text{Ind}E + \text{Ind}E_W,
\]
\[
\text{Ind}^-(X \cup W) = \text{Ind}^+X + \text{Ind}^-W.
\]
On the other hand, according to [3 Prop. 3.10],
\[
\text{Ind}^+X - \text{Ind}^-X = -(b_0(\partial X) + b_1(\partial X))
\]
and, by [3 Prop. 3.15] (with additional terms added for a non-connected boundary),
\[
\text{Ind}^-W = b_1(W) - (b_0(W) - b_0(\partial W)) - b_2^+(W).
\]
Since the difference \(\nu[\varrho] - \nu[\varrho']\) is given by
\[
\text{Ind}(E \cup E_W) - \text{Ind}E - 3(\text{Ind}^-(X \cup W) - \text{Ind}^-X) \mod 2,
\]
the result follows. \(\square\)

2.2. Instanton invariants for 2-component links. We will need to introduce versions of instanton homology associated to an oriented 2-component link \(L\) in the 3-sphere. The definitions are parallel to those for knots. They are defined as the instanton homology of certain closures of the link complement. Let \(C = [0, 1] \times S^1\). Let \(D_1\) and \(D_2\) be two small disjoint disks in \(T\) and set \(T_2 = T - D_1^* - D_2^*\). Let \(\Sigma_L\) be an oriented (connected) Seifert surface with induced orientation on the boundary equal to the orientation on \(L\). Set
\[
L^C = (S^3 - \nu L^\circ) \cup (C \times S^1),
\]
\[
L^{T_2} = (S^3 - \nu L^\circ) \cup (T_2 \times S^1).
\]
Again the union is taken along the boundaries such that the oriented boundary of \(\Sigma_L \cap (S^3 - \nu L^\circ)\) matches up with the oriented boundary of \(C \times \{1\}\) or \(T_2 \times \{1\}\). Additionally the meridians of the link should match up with the fibers \(\{pt\} \times S^1\) on the boundaries. In the case of \(L^C\) the genus of \(\hat{\Sigma}\), the extension of \(\Sigma_L\), is one greater than that of \(\Sigma_L\), but in the case of \(L^{T_2}\) it is two greater.

As in the case of knots we can take the instanton homology groups \(I_\omega^+(L^C)_{\omega}\) and \(I_\omega^+(L^{T_2})_{\omega}\) as the basis for the definition of instanton link homology. We shall define our choices \(\omega = \omega_1, \omega_2\) below. Using the action of \(\mu(\hat{\Sigma})\) to give a decomposition in terms of \(\tilde{I}_\omega^+(L^C)_{\omega_1}\) and \(\tilde{I}_\omega^+(L^{T_2})_{\omega_2}\), we can again define the finite Laurent polynomials
\[
P_{\omega_1}(L^C, \hat{\Sigma})(t) \quad \text{and} \quad P_{\omega_2}(L^{T_2}, \hat{\Sigma})(t).
\]
Let \(\alpha_0\) be an oriented simple arc in \(S^3 - \nu L^\circ\) connecting the two boundary components. Let \(\alpha_1\) be an oriented simple arc in \(C \times S^1\) connecting the two boundary components such that the boundary of \(\alpha_1\) matches up with the boundary of \(\alpha_0\) and the union is an oriented simple closed curve \(\alpha'\). Then in the case of \(L^C\) choose \(\omega = \omega_1\) where \(\omega_1\) has Chern class Poincaré dual to \(\alpha'\).

Think of \(T_2\) as the connected sum of surfaces \(C \sharp T\). Let \(\alpha''\) be an oriented simple closed curve of the form \(\alpha_0 \cup \alpha_1\) where \(\alpha_1\) is the arc as before but living in the \(C\) factor of \(C \sharp T\). Let \(\alpha\) be a homologically non-trivial simple closed curve in the \(T\) factor of the connected sum. We think of \(\alpha\) as living on \(T_2 \times \{1\}\). Then for \(L^{T_2}\) choose \(\omega = \omega_2\) where \(\omega_2\) has Chern class Poincaré dual to \(\alpha'' + \alpha\).
2.3. The excision principle and instanton homology for $L^T_2$. With our choices for $\omega$, $P_{\omega_2}(L^T_2, \tilde{\Sigma})(t)$ is actually determined by $P_{\omega_1}(L^C, \tilde{\Sigma})(t)$. This is a consequence of Floer’s excision principle.

**Lemma 2.3.** Let $\Sigma_2$ be the surface of genus 2. The finite Laurent polynomial is such that

$$P_{\omega_2}(L^T_2, \tilde{\Sigma})(t) = P_{\omega_1}(L^C, \tilde{\Sigma})(t) \cdot P_{\omega_3}(\Sigma_2 \times S^1, \Sigma_2)(t),$$

where $P_{\omega_3}(\Sigma_2 \times S^1, \Sigma_2)(t)$ is the finite Laurent polynomial for $\Sigma_2 \times S^1$ derived from the instanton homology of $\Sigma_2 \times S^1$ and defined in an analogous manner. The complex line $\omega_3$ has Chern class the Poincaré dual of the curve $\alpha$ above.

**Proof.** Let $F_1$ and $F_2$ be the two tori that form the boundary of $(S^3 - \nu L^C) \subset L^T_2$. Since $\omega_2$ evaluates non-trivially mod 2 on $F_i$ we may apply Floer’s excision principle (see [3, Theorem 7.7]) to conclude that there is an isomorphism

$$\tilde{I}^+_s(L^T_2)_{\omega_2} \cong \tilde{I}^+_s(\Sigma_2 \times S^1)_{\omega_3} \otimes \tilde{I}^+_s(L^C)_{\omega_1}.$$

However it is not immediately evident that this isomorphism preserves the mod 2 grading; we shall prove this below but only in this specific situation.

In the above isomorphism the action of $\mu(\tilde{\Sigma})$ on $\tilde{I}^+_s(L^T_2)_{\omega_2}$ corresponds to the action of $\mu(\tilde{\Sigma}) \otimes 1 + 1 \otimes \mu(\Sigma_2)$ on the tensor product space. It follows that the generalized $\lambda$-eigenspaces $W_\lambda$ in $\tilde{I}^+_s(L^T_2)_{\omega_2}$ obey a relation of the form

$$W_\lambda \cong \bigoplus_{\lambda = \lambda_0 + \lambda_1} U_{\lambda_0} \otimes V_{\lambda_1},$$

where $U_{\lambda_0}$ and $V_{\lambda_1}$ are corresponding generalized eigenspaces in $\tilde{I}^+_s(\Sigma_2 \times S^1)_{\omega_3}$ and $\tilde{I}^+_s(L^C)_{\omega_1}$ respectively. The lemma follows easily, assuming the mod 2 grading claim.

To establish the mod 2 grading, consider the surgery cobordism $W$ between $L^C \cup (\Sigma_2 \times S^1)$ and $L^T_2$ in the proof of the excision principle. Let $H = T \times [-1, 1] \times [-1, 1]$. The boundary is $T \times \{0\} \times [-1, 1] \cup \{1\} \times [-1, 1] \times \{-1, 1\}$. We regard $T \times \{0\} \times [-1, 1]$ as the ‘core’ of $H$. Then $W$ is obtained from $(L^C \cup (\Sigma_2 \times S^1)) \times [0, 1]$ by identifying $\nu F_1 \cup \nu F_2$ in $(L^C \cup (\Sigma_2 \times S^1)) \times \{1\}$ with $T \times [0, 1]$ in $H$. Clearly $W$ deformation retracts onto $W_0$, the union of $L^C \cup (\Sigma_2 \times S^1)$ and the core of $H$, where $F_1$ and $F_2$ are identified with $\partial (T \times \{0\} \times [-1, 1])$.

In $L^C$ let $a$ be an oriented simple closed curve corresponding to $\{0\} \times S^1 \subset C \times S^1$. Let $b$ be an oriented simple closed curve corresponding to $\{pt\} \times S^1 \subset C \times S^1$. Then thinking of $a$ and $b$ as homology classes, we have $a = 0$ in $H_1(L^C)$ and that $b$ is a generator of $H_1(L^C)$. On the other hand, by the identification via the core of $H$, $a$ is a generator of $H_1(\Sigma_2 \times \{pt\})$ and $b$ corresponds to a fiber $\{pt\} \times S^1$.

Since $H_1(L^C) \cong \mathbb{Z} \oplus \mathbb{Z}$ we have that the rank of $H_1(\Sigma_2 \times S^1)$ is 7. Then $H_1(W) = H_1(W_0)$ is obtained from $H_1(\Sigma_2 \times S^1) \oplus H_1(L^C)$ by introducing the two relations above. Thus $H_1(W_0)$ has rank 5. It is easily seen that $b_2^+(W) = 0$; thus by Lemma 2.1 the dimension shift is

$$k = b_1(W) - b_1(Y) + b_0(Y') - b_0(W) - b_2^+(W) = 5 - (2 + 5) + 1 - 1 - 0 \equiv 0 \mod 2.$$

This completes the proof. \qed
2.4. Some calculations. Lemma 2.3 above necessitates evaluation of $P_{\omega_4}(\Sigma_2 \times S^1, \Sigma_2)(t)$. For later use we will also need the evaluation of $P_{\omega_4}(T \times S^1)(t)$ where $\omega_4$ has Chern class dual to any homologically non-trivial oriented simple closed curve in $T$.

Lemma 2.4. $P_{\omega_4}(T \times S^1)(t) = 1$ for either orientation on $T \times S^1$.

Proof. After applying a diffeomorphism, the line bundle $\omega_4$ can be assumed to have Chern class dual to the fiber $\{pt\} \times S^1$. It is well-known (see for instance [1] Prop. 1.14) that the $SO(3)$-bundle with 2nd Steifel-Whitney class $w_2 \equiv c_1(\omega_4) \mod 2$ carries a unique flat connection, up to gauge equivalence. In terms of $U(2)$-bundles this gives two flat connections $\eta_0$ and $\eta_1$ on the adjoint bundle. The instanton homology is such that $I_*(T \times S^1)_{\omega_4} \cong \mathbb{C} \oplus \mathbb{C}$, where the dimensions differ by 4. The rest of the lemma follows easily once we can show that the generators lie in even dimensions.

Let $X = T \times D^2$ so that $Y = \partial X$. Let $E_{\eta_i} \to X$ be a $U(2)$-bundle carrying connection $A$ that extends $\eta_i$. Then $\text{Ind} E_{\eta_1} = \text{Ind} E_{\eta_0} + 4 \mod 8$ since the Floer dimensions differ by 4. We wish to evaluate $\text{Ind} E_{\eta_0}$. Let $\varphi: T \times S^1 \to T \times S^1$ be the orientation reversing diffeomorphism that reverses the $S^1$ factor; denote by $\tilde{\varphi}$ a lift to the bundle level. Then $\tilde{\varphi}^*(\eta_0)$ is equivalent in instanton homology to either $\eta_0$ or $\eta_1$. Now form the double of $X$, identifying the boundaries via $\varphi$, and let $E = E_{\eta_0} \cup E_{\tilde{\varphi}^*(\eta_0)}$ be the corresponding bundle over the double. There are two ways of forming $E$. We choose the one that has $c_1$ dual to the surface $\{y_0\} \times D^2 \cup \{y_0\} \times D^2$ in the double. Then

$$\text{Ind} E = 2(4c_2 - c_1^2) - 3(1 - b_1 + b_2^+) \equiv 0 \mod 8.$$  

On the other hand, by the additivity of the index we have

$$\text{Ind} E_{\eta_0} + \text{Ind} E_{\tilde{\varphi}^*(\eta_0)} = \text{Ind} E_{\eta_0} + \text{Ind} E_{\eta_0} + 4n \equiv 0 \mod 8.$$  

It follows that $\text{Ind} E_{\eta_0} \equiv 0 \mod 2$. Clearly $b_1(X) - b_2^+(X) = 2$, and so $\nu[\eta_0] \equiv 0 \mod 2$. □

Lemma 2.5. $P_{\omega_4}(\Sigma_2 \times S^1, \Sigma)(t) = t^{-1} - 2 + t$ for either orientation on $\Sigma_2 \times S^1$.

Proof. According to [1] Prop. 1.15 we have, in increasing order of dimension,

$$I_*(\Sigma_2 \times S^1)_{\omega_4} = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \{0\}.$$  

We shall fix this grading, beginning with zero for the first group and reading from left to right. We assume for the moment that this is consistent with the mod 2 grading. We shall prove this below by showing that the group $\mathbb{C}^2$ in dimension 1 above must be an odd graded group.

The action of $\mu(\Sigma_2)$ has eigenvalues $-2, 0$ or $+2$ on $I_+^*(\Sigma_2 \times S^1)_{\omega_4}$. On the odd dimensions $\mu(\Sigma_2)$ necessarily shifts each summand of the odd groups $\mathbb{C}^2 \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}$ to the next. Thus $\mu(\Sigma_2)$ is zero on $I_+^*(\Sigma_2 \times S^1)_{\omega_4}$ and this makes up its entire generalized eigenspace decomposition. On the other hand, the $\pm 2$-eigenvalues for $\mu(\Sigma_2)$ are simple on $I_*^*(\Sigma_2 \times S^1)$ [5 Prop. 7.4], so only the $\pm 2$-eigenvalues on $I_0^*(\Sigma_2 \times S^1)_{\omega_4}$ have non-trivial eigenspaces, each of dimension 1. This completely decomposes $I_+^*(\Sigma_2 \times S^1)_{\omega_4}$ and the result follows, modulo the claim regarding the mod 2 grading.

Let $\rho$ be the (perturbed) flat connection that is a generator $I_1(\Sigma_2 \times S^1)_{\omega_3}$. Let $\varphi: \Sigma \times S^1 \to \Sigma_2 \times S^1$ be the orientation reversing diffeomorphism that reverses dimensions differ by 4.
the $S^1$ factor. Let $X$ be such that $\partial X = Y$ and let $E_\varphi \to X$ be the bundle with connection that extends $\varphi$. Let $\tilde{\varphi}$ be a lift to a bundle map. It must be the case then that $\tilde{\varphi}^*(\varphi)$ is a generator in dimension 1 or 5, so by the same reasoning as in the preceding lemma, $\text{Ind}_{E_\varphi^*(\varphi)} \equiv \text{Ind}_{E_\varphi} \mod 4$. Repeating the doubling argument, we find that

$$\text{(2.1)} \quad 2\text{Ind}_{E_\varphi} \equiv 2(4c_2 - c_1^2) - 3(1 - b_1 + b_2^+) \mod 4,$$

where the right-hand terms refer to the doubled manifold (and bundle). We now need to construct an appropriate $X$. This is done as follows.

Let $H$ be a handlebody whose boundary is $\Sigma_2$. Let $a_1, b_1, a_2, b_2$ be the standard oriented simple closed curves in $\Sigma_2$ that represent a basis for $H_1$ with the property that $a_1$ and $a_2$ bound disks $D_1$ and $D_2$ respectively in $H$. Without loss of generality we may assume that $\omega_3$ has Chern class Poincaré dual to $a_1 \times \{\text{pt}\}$ in $\Sigma_2 \times S^1$. We now choose $X = H \times S^1$. The bundles $E_\varphi$ and $E_{\tilde{\varphi}^*(\varphi)}$ extend over $X$. For the bundle $E_\varphi \cup E_{\tilde{\varphi}^*(\varphi)}$ over the doubled manifold, we may choose $c_1$ Poincaré dual to $D_1 \cup D_1$. Thus $c_1^2 = 0$. The second homology of the doubled manifold (over $\mathbb{Z}$) is generated by $S_i = D_i \cup D_i$ and $b_i \times S^1$ ($i = 1, 2$). The intersection form can then be written as two copies of

$$\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.$$

This gives $b_2^+(X \cup X) = 2$. It is seen that $b_1(X \cup X) = 3$. Therefore the left-hand side of (2.1) evaluates to 0 mod 4 so that $\text{Ind}_{E_\varphi} \equiv 0 \mod 2$.

From the preceding considerations it is also straightforward to see that $b_1(X) = 3$ and $b_2^+(X) = 0$. So

$$\nu[\varphi] = \text{Ind}_{E_\varphi} - 3(b_1(X) - b_2^+(X)) \equiv 1 \mod 2,$$

and the lemma follows. \hfill \Box

2.5. **Skein relations.** Let $J$ denote an oriented knot or link in the 3-sphere. Let $J_+, J_-$ and $J_0$ denote in the usual way the knots or links that in a projection differ in a neighborhood of a single crossing. We use the conventions for $J_+$ and $J_-$ in [6]. If $J_\pm$ is modeled on the $x$- and $y$-axes in the plane (the strands agreeing with the standard orientation of the axes), then in $J_+$ the $x$-axis strand goes over the $y$-axis strand.

If $J_\pm$ are knots, then $J_0$ is a 2-component link. If $J_\pm$ are 2-component links and the crossing is between different components of $J_\pm$, then $J_0$ is a knot. We have a version of the following well-known theorem that applies only to knots and 2-component links.

**Theorem 2.6.** Let $J$ denote either an oriented knot or an oriented 2-component link. Let $\Delta_J(t)$ be a finite Laurent polynomial in powers of $t^{1/2}$ that is an oriented isotopy invariant of $J$. Assume that (1) $\Delta_J(t) = 1$ for the unknot, (2) $\Delta_J(t) = 0$ for split 2-component links, and (3) the following skein rule holds:

$$\Delta_{J_+}(t) - \Delta_{J_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{J_0}(t),$$

where if $J$ is a 2-component link, then the crossing change is between different strands. Then for knots $K$, $\Delta_K(t)$ is exactly the symmetrized and normalized Alexander polynomial of $K$; i.e. it satisfies $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

**Proof.** See [6] Proof of Theorem 1.5]. \hfill \Box
3. Proofs

3.1. Theorem 1.1. The strategy is to show that for oriented knots $K$, $P_K(t) = P_{\omega_0}(K^T, \Sigma)(t)$ satisfies the conditions of Theorem 2.6 upon applying Floer’s exact triangle (II) will be our general reference). However this will require, for oriented 2-component links $L$, a slight alteration of $P_{\omega_1}(L^C, \hat{\Sigma})(t)$ before the skein relation can be satisfied.

We let

$$\hat{P}_J(t) = \begin{cases} P_{\omega_0}(J^T, \hat{\Sigma})(t) & \text{if } J \text{ is a knot}, \\ -(t^{1/2} - t^{-1/2})P_{\omega_1}(J^C, \hat{\Sigma})(t) & \text{if } J \text{ is a link}. \end{cases}$$

According to Floer, for oriented knots $K$ there is an exact sequence of the form

$$\rightarrow \tilde{I}_i(K^T_0)_{\omega_0} \xrightarrow{a} \tilde{I}_i(K^T_0)_{\omega_0} \xrightarrow{b} \tilde{I}_i(K^T_0)_{\omega_0} \xrightarrow{c} \tilde{I}_{i-1}(K^T_0)_{\omega_0} \rightarrow$$

where the maps are induced by the various surgery cobordisms, essentially adding 2-handles to the product cobordism along a knot. The maps $a$ and $b$ are of degree zero and $c$ is of degree $-1$, by applying Lemma 2.1 to the surgery cobordisms $W_a$, $W_b$, and $W_c$ respectively. $W_a$ and $W_b$ are obtained by adding two handles along homologically trivial knots with $-1$-framing. Therefore $b_1$ is the same as the ends of the cobordism. Clearly, for $W_a$ and $W_b$, $b_2^+ = 0$. By Lemma 2.1 this gives a dimension shift of $0$ mod $2$. On the other hand, $W_c$ is obtained by adding a 2-handle along a knot which is a generator for 1st homology, so $b_1$ drops by 1 in $W_c$, but $b^+_2 = 0$. This results in a dimension shift of $-1$ mod $2$.

Since all the surgery cobordisms are connected, the action of $\mu(x_0)$ in each group commutes with $a$, $b$ and $c$. A similar statement is true for the action of $\mu(\Sigma)$ on each group, because all the various oriented surfaces $\Sigma$ are homologous in the surgery cobordisms.

Therefore the above exact sequence respects the decomposition of each group into the $\pm 2$-eigenspaces of $\mu(x_0)$ and the generalized eigenspaces for $\mu(\Sigma)$ contained therein. Thus

$$P_{\omega_0}(K^T, \hat{\Sigma}) - P_{\omega_0}(K^T_0, \hat{\Sigma}) = -P_{\omega_1}(K^T_0, \hat{\Sigma}) = -(t^{-1} - 2 + t)P_{\omega_1}(K^C_0, \hat{\Sigma}),$$

where the last equality is obtained from Lemmas 2.3 and 2.6. It immediately follows that

$$\hat{P}_{K+}(t) - \hat{P}_{K-}(t) = (t^{1/2} - t^{-1/2})\hat{P}_{K_o}(t).$$

In the case of a 2-component oriented link $L$ where the crossing change is between different components of the link, we have the exact sequence

$$\rightarrow \tilde{I}_i(L^C_0)_{\omega_1} \xrightarrow{a} \tilde{I}_i(L^C_0)_{\omega_1} \xrightarrow{b} \tilde{I}_i(L^T_0)_{\omega_0} \xrightarrow{c} \tilde{I}_{i-1}(L^C_0)_{\omega_1} \rightarrow .$$

A similar argument gives

$$P_{\omega_1}(L^C_0, \hat{\Sigma}) - P_{\omega_1}(L^C_0, \hat{\Sigma}) = -P_{\omega_0}(L^T_0, \hat{\Sigma}),$$

and again relation (3.1) holds. For split 2-component links $L'$, the instanton homology must vanish because $\omega_1$ is non-trivial on the 2-sphere $S$ that separates the components of the link; there are no flat $SO(3)$-connections over $S$. Thus $0 = P_{\omega_1}(L')(t) = \hat{P}_L(t)$. For the unknot $U$, $U^T$ is clearly $T \times S^1$ so that $\hat{P}_U(t) = P_{\omega_0}(U^T)(t) = 1$ by Lemma 2.4 This completes the proof. \hfill \Box
3.2. Corollary [1]. In our definition of $\omega_0$ (defined for $K^T$) we could have also twisted it by the complex line that has Chern class Poincaré dual to an oriented fiber $\{pt\} \times S^1$ in the subset $(T - D^2) \times S^1$. We could have made corresponding changes to $\omega_1$ and $\omega_2$ (recall that these are defined for $L^C$ and $LT^2$ respectively), ensuring that these are compatible over the knot or link complement with each other. We will not need to change $\omega_3$ (defined for $\Sigma_2 \times S^1$) but will need a similar change to $\omega_4$ (defined for $T \times S^1$). Denote the changed complex lines by $\omega'_0$, $\omega'_1$, $\omega'_2$ and $\omega'_4$ respectively.

Lemma 2.3 remains unchanged with $\omega_1$ and $\omega_2$ replaced by $\omega'_1$ and $\omega'_2$ respectively. Excision does not require $\omega_3$ to be changed.

Lemma 2.4 remains the same with $\omega_4$ replaced by $\omega'_4$. There is a diffeomorphism that moves $\omega'_4$ back to $\omega_4$.

Then the proof of Theorem 1.1 goes through with $\omega_1$ replaced by $\omega'_1$. In particular the conclusion about $P_\ell(t)$, defined using $\omega'_1$, remains the same. Therefore we now assume the $\omega_i$’s are changed to $\omega'_i$’s.

Then, according to Floer, for oriented knots $K$ there is an exact sequence of the form

$$\rightarrow \tilde{I}_i(K^D)_{\omega} \xrightarrow{a} \tilde{I}_i(K^D)_{\omega'} \xrightarrow{b} \tilde{I}_i(K^C)_{\omega'_1} \xrightarrow{c} \tilde{I}_{i-1}(K^D)_{\omega'} \rightarrow .$$

Repeating the argument of the proof of Theorem 1.1 we have

$$Q_{K_+}(t) - Q_{K_-}(t) = -P_{\omega'_1}(K^C, \Sigma)(t),$$

and therefore

$$(t - 2 + t^{-1})(Q_{K_+}(t) - Q_{K_-}(t)) = (t^{1/2} - t^{-1/2}) \Delta_{K_0}(t).$$

Here we make use of the proof of Theorem 1.1 (with the new $\omega'_1$), which identifies $-(t^{1/2} - t^{-1/2})P(K^C, \Sigma)_{\omega'_1}(t)$ with $\Delta_{K_0}(t)$. Thus there is a universal constant $C$ such that

$$(t - 2 + t^{-1})Q_K(t) + C = \Delta_K(t),$$

where $\Delta_K(t)$ is the symmetrized and normalized Alexander polynomial. By considering the unknot, which has $Q_K(t) = 0$, we must have $C = 1$. The corollary follows immediately.

\[\square\]

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