Desargues theorem, its configurations, 
and the solution to a long-standing 
enumeration problem

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Abstract

We solve a long-standing problem by enumerating the number of
non-degenerate Desargues configurations. We extend the result to the
more difficult case involving Desargues blockline structures in Sec-
tion 8.

A transparent proof of Desargues theorem in the plane and in space
is presented as a by-product of our methods.

Keywords: Desargues theorem, Desargues configuration, 5-compressor, pro-
jective spaces, polarity, finite field

1 Introduction, Background.

The celebrated theorems of Pappus and Desargues are the fundamental
building blocks in the axiomatic development of incidence and projective
geometry. Hilbert’s work showed that a projective plane π over a division
ring F is equivalent to one in which Desargues theorem holds, which is in

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turn equivalent to the assumption that \( \pi \) is embedded in a 3-dimensional projective space.

The Hessenberg theorem shows that the ancient theorem of Pappus implies the Desargues theorem. From this it follows that a projective plane over a commutative division ring \( F \), i.e., a field \( F \), is equivalent to one in which the Pappus theorem holds. Thus, since all finite division rings are fields, we have that Pappus and Desargues are equivalent for finite projective planes. For a discussion of non-Desarguesian planes see Lorimer [12]. The Desargues configuration still plays a fundamental role when studying the collineation of such planes.

In [15] page 145, the author, referring also to Baker [1] writes as follows: “After an argument that runs well over one hundred pages, Baker shows that beneath the statement of Desargues’ theorem, another far more interesting geometric structure lies concealed. This structure is nowadays called the Desargues configuration.”

As an aside, the Desargues configuration makes its appearance also in many areas of combinatorics and graph theory. In the paper on Colouring Problems by W.T. Tutte (see [17]), the author discusses connections between graph theory and the geometrical approach to the Four Colour Problem first explored by O. Veblen [18] in 1912.

In a 3-dimensional projective space over the binary field a tangential 2-block is a set \( S \) of points with the following two properties:

(a) Each line in space contains at least one point of \( S \);

(b) At each point \( P \) of \( S \) there exists a tangent line to \( S \) at \( P \), i.e., a line meeting \( S \) only in \( P \).

A remarkable fact is that, apart from a plane in the space, i.e., a Fano plane, the only other tangential 2-block is a spatial (i.e., a non-planar) Desargues configuration.

Generalizing the definition of tangential 2-block to projective spaces of arbitrary dimension, Tutte announces in [17] his famous tangential 2-block conjecture to the effect that the only tangential 2-blocks existing in projective space over the binary field correspond to the Fano plane, the Desargues configuration and the 5-dimensional Petersen block. The conjecture, as a special case, implies the Four Colour Theorem for graphs. The conjecture has been verified for several dimensions but remains unsolved as of this writing.

Enumerating the number of Desargues configurations in the plane is an old problem dating back to 1970-71 and the work in [14]. There the authors restrict attention to the special case where the characteristic of the underlying field is 2 or 3 and no line of the plane contains as many as 4 points of the 10 points of the configuration. This is tantamount to saying that no point of the configuration is self-conjugate. Later on we show in
Theorem 4.4 that no line contains more than 4 points of the configuration. For further discussion on conjugacy we refer to our paper [2].

Even in the special case discussed in [14] the proof is lengthy and complicated. In this paper, we produce a complete solution. We show in Section 9 how the standard approach doesn’t work given the many possibilities that must be considered and the impact of the characteristic of the field. A crucial factor here is that the type of configuration depends on the number of self-conjugate points, which in turn depends on the characteristic of the field. We also enumerate the 3-dimensional Desargues configurations, providing two proofs of the result.

2 Projective planes and spaces, Desargues configurations.

For basic definitions, we refer to Coxeter [6,7], Pedoe [13], Hirschfeld [10], Hartshorne [9], Veblen and Young [19], or Todd [16]. For some interesting related work, we mention articles by Conway and Ryba [4,5], Crannell and Douglas [8] and Lord’s book [11]. A recent paper of the authors [2] relates to the material in Section 8.

Much of the paper can be visualized in projective space over the reals. Here is a quick overview from the synthetic or algebraic point of view. The Euclidean plane can be embedded in, or extended to the real projective plane \( \text{PG}(2, \mathbb{R}) \) as follows. We adjoin an “infinite point” (or “slope point”) to each of a given parallel class of lines. Different parallel classes have different slope points. We decree that the newly created infinite points lie on an “infinite line” or “line at infinity”. The projective plane now has the property that two distinct points lie on a unique line and two distinct lines meet in a unique point.

There is also the more algebraic approach as follows. Start with a 3-dimensional vector space \( V \) over the reals. Define points to be all the 1-dimensional subspaces of \( V \) and the lines to be all the 2-dimensional subspaces of \( V \). The resulting structure is isomorphic to \( \text{PG}(2, \mathbb{R}) \) as described above as shown in Hartshorne [9].

Similarly, Euclidean 3-space can be embedded in the projective space \( \text{PG}(3, \mathbb{R}) \) by adjoining a “plane at infinity”. The algebraic model then comes from a 4-dimensional real vector space.

Initially we work in \( \Sigma = \text{PG}(n, F) \) the \( n \)-dimensional projective space based on the \((n + 1)\)–dimensional vector space over the skew field \( F \). The celebrated theorem of Desargues is as follows.

**Theorem 2.1.** Let \( ABC \) and \( A'B'C' \) be two triangles in \( \Sigma \) such that the lines \( AA' \), \( BB' \) and \( CC' \) pass through a point \( V \) of \( \Sigma \). Then the 3 intersection points \( P, Q, R \) of the lines \( AB \) with \( A'B' \), \( BC \) with \( B'C' \) and \( CA \) with \( C'A' \) lie on a line \( v \).
Figure 1: A Desargues configuration. (a) if we choose the point $V$ for the vertex, then $ABC$ and $A'B'C'$ are in perspective from $V$ with Desargues axis $v$ containing $P, Q, R$. (b) if we choose the point $C$ for the vertex, then $QRC'$ and $BAV$ are in perspective from $C$ with Desargues axis containing $A', B', P$. The block $A'B'P$ is the polar of $C$.

We refer to Figure [1](a). Note that the two triangles either lie in a plane or are contained in a 3-dimensional space so we may assume that $n = 3$. The 2 triangles are said to be in perspective with center $V$. For reasons
explained below and in Section 3, the line \( v \) is called the Desargues axis or polar of \( V \).

The two triangles yield a Desargues Configuration \( D \) with 10 points and 10 triples of collinear points called blocks, as follows.

Points of \( D \): \( V, A, B, C, A', B', C', P, Q, R \)

Blocks of \( D \): \( \{VAA'\}, \{VBB'\}, \{VCC'\}, \{APB\}, \{A'PB'\}, \{ARC\}, \{A'RC'\}, \{BQC\}, \{B'QC'\}, \{PQR\} \).

A line of \( \Sigma \) containing a block of \( D \) is called a blockline. Each point of \( D \) is contained in exactly 3 blocks of \( D \) and each block of \( D \) contains exactly 3 points of \( D \).

Throughout this paper, we assume that \( D \) is non-degenerate, i.e., that \( D \) has 10 distinct points and 10 distinct blocklines.

The configuration \( D \) is an example of a \((3, 3)\)-configuration in the terminology of Pedoe [13, p. 25], or a \(10_3\)-configuration in the terminology of Coxeter [6, p. 26].

As with \( V \), any of the 10 points of \( D \) can be regarded as the vertex of perspective of exactly two triangles formed from the points and blocklines of \( D \). For example, if we choose the point \( C \) for vertex then, using the 3 blocklines on \( C \), we get the 2 triangles in perspective from \( C \), namely, the triangles \( QRC \) and \( BAV \). The Desargues axis for the vertex \( C \) is the blockline containing the block \( \{A'B'P\} \). See Figure 1(b).

### 3 Planar Desargues configurations and 5-compressors.

Here we work with planar Desargues configurations \( D \) where the 2 triangles \( ABC \) and \( A'B'C' \) lie in the same plane lying in \( \Sigma = PG(3, F) \), with \( F \) any field.

**Definition 3.1.** A 5-compressor or a 5-point or a simplex \( S \) in \( \Sigma \) is a set \( S \) of 5 points with no 4 coplanar.

In what follows, the symbol \(<P_1, P_2, P_3>\) represents the plane containing the triangle \( P_1P_2P_3 \).

We now develop the important connection between 5-compressors and Desargues configurations.

**Theorem 3.2.** Let \( S = \{P_1, P_2, P_3, P_4, P_5\} \) be a 5-compressor, i.e., a set of 5 points in \( \Sigma = PG(3, F) \) with no 4 of its points coplanar. Let \( \pi \) be any plane of \( \Sigma \) containing no point of \( S \). The points \((ij)\), where \((ij)\) is the point of intersection of the line \( P_iP_j \) with \( \pi \), yield 10 distinct points in \( \pi \). The plane containing \( P_i, P_j, P_k \) intersects \( \pi \) in a line containing the block \([ijk]\) which denotes the set of 3 collinear points \((ij), (jk) \) and \((ik)\). The 10 points \((ij)\) and the 10 blocks such as \([ijk]\) form a Desargues \((3, 3)\)-configuration.
in \( \pi \). Conversely, any Desargues configuration arises from a 5-compressor in \( \Sigma \) using the above construction.

**Proof.** The assumption on \( S \) guarantees that no 3 points \( P_1, P_2, P_3 \) of \( S \) are collinear. If the line \( P_i P_j \) were to also meet \( \pi \) in the point \( P_uP_v \cap \pi \), the sets \( \{u,v\} \) and \( \{i,j\} \) being disjoint would imply that \( \{P_i, P_j, P_u, P_v\} \) is a set of 4 points lying in a plane. This contradicts the definition of \( S \). Thus the 10 lines \( P_i P_j \) meet \( \pi \) in \( \binom{5}{2} = 10 \) distinct points.

A plane containing \( \{P_i, P_j, P_k\} \), denoted by \( \langle P_i, P_j, P_k \rangle \), meets \( \pi \) in 3 distinct collinear points. Suppose that 2 planes \( \pi_1 = \{P_i, P_j, P_k\} \) and \( \pi_2 = \{P_u, P_v, P_w\} \) meet \( \pi \) in a line \( l \). Since no line contains more than 4 points of the configuration, it follows that \( P_u \), say, must equal one of \( P_i, P_j, P_k \), say \( P_i \). Then \( \pi_1 \) and \( \pi_2 \) are both equal to the plane \( \pi_3 \) containing \( P_u = P_i \) and \( l \). Therefore \( \pi_3 \) contains either 4 or 5 points of \( S \), which is impossible.

In this way we obtain \( \binom{5}{2} = 10 \) points \( (ij) \) and \( \binom{5}{3} = 10 \) blocks \( [ijk] \) corresponding to the lines \( P_iP_j \) and the planes \( \langle P_i, P_j, P_k \rangle \).

Points \( (ab) \) and \( (cd) \) lie in one of the 10 blocks if and only if the pairs \( (ab) \) and \( (cd) \) share a symbol. Two of the 10 blocks \( [abc] \) and \( [uvw] \) intersect in one of the 10 points \( (ij) \) if and only if the 2 triples share a common pair. In this way we see that the 5-compressor \( S \) yields a \((3,3)\)-Desargues configuration \( D \) in \( \pi \) with 10 points and 10 lines.

We will prove a strong form of the converse in Theorem 5.1.

In light of Theorem 3.2, we can think of 5-compressors as giving us data compression — high above the Desargues configuration, “in the clouds” — in that they store the information of Desargues configurations.

We refer to Figure 2 for an illustration of a 5-compressor, as described in Theorem 3.2. All of the lines \( P_iP_j \) and \( P_2P_3 \) are shown, \( i \neq 1, j \neq 2 \). The Desargues configuration \( D \) is shown in more detail in Figure 3.

**Terminology 3.3.** We refer to the configuration \( D \) with its 10 points and 10 blocks as above as the section of the 5-compressor (5-point) \( S \) by \( \pi \), and we say that \( S \) sections \( D \).

This is consistent with the terminology in [19] p. 34.

**Theorem 3.4.** Let \( D \) be any (non-degenerate) Desargues configuration in the plane \( \pi = PG(2, F) \), with \( F \) any field. Let \( V \) be any point of \( D \). Let \( P_1, P_2 \) be two fixed points in space not in \( \pi \) such that \( P_1, P_2 \) and \( V \) are collinear. Then there exists two 5-compressors, \( S_1 = \{P_1, P_2, P_3, P_4, P_5\} \) and \( S_2 = \{P_1, P_2, Q_3, Q_4, Q_5\} \) which section to \( D \), where \( S_1, S_2 \) contain no point of \( \pi \).

**Proof.** Let \( D \) be labelled as indicated in Figure 1(a). Let \( P_3 \) denote \( P_1A \cap P_2A' \), i.e., \( P_3 \) is the point of intersection of the lines \( P_1A \) and \( P_2A' \) joining \( P_1 \).
Figure 2: A 5-compressor $S = \{P_1, P_2, P_3, P_4, P_5\}$ and the Desargues configuration $D$ in $\pi$ that $S$ sections. The Desargues configuration $D$ is shown in more detail in Figure 3.

We refer to Figures 3 and 4. In a similar way we can calculate that all of the 10 blocks in $D$ are of the form $[ijk]$. (Using $S_2$ we get a different labelling obtained by interchanging $ABC$ with $A'B'C'$. This completes the proof of Theorem 3.4.)
Figure 3: Alternative labels for the Desargues configuration in Figure 1(a). Here, \( V = (12) \), \( A = (13) \), \( A' = (23) \), \( B = (14) \), \( B' = (24) \), \( C = (15) \), \( C' = (25) \), \( P = (34) \), \( Q = (45) \), and \( R = (35) \). For this labeling, the points of block \([123]\) are on blockline \( AA' \), \([124]\) is on \( BB' \), \([125]\) is on \( CC' \), \([134]\) is on \( AB \), \([234]\) is on \( A'B' \), \([135]\) is on \( AC \), \([235]\) is on \( A'C' \), \([145]\) is on \( BC \), \([245]\) is on \( B'C' \), and \([345]\) is on \( PQ \).

4 Desargues theorem in the plane and polarities.

Using the notation of Section 2 we now use Theorem 3.4 to give a new transparent proof of the Desargues theorem in the plane. Unlike several standard proofs we do not explicitly rely on the Desargues theorem in space. Instead we use the fact that two planes intersect in a line.
Figure 4: The 5-compressors $S_1$ in (a), and $S_2$ in (b), section to $D$ for Theorem 3.4. Alternatives labels for the points and blocks of $D$ corresponding to $S_1$ are shown in Figure 3.

4.1. Proof of Theorem 2.1, the theorem of Desargues in the plane.
**Proof.** Construct the 5-compressor $S_1$ sectioning the given Desargues configuration in the plane $\pi$ as in Theorem 3.4. The intersections of the pairs of corresponding sides are $P, Q, R$ which are collinear since the 3 points lie on the intersection of two planes, namely the plane $\pi$ and the plane $\langle P_3, P_4, P_5 \rangle$. ■

**Remark 4.2.** The diagram in Figure 4 and Figure 2 is precisely the same as that in the standard proof of the theorem. See Coxeter [6].

Theorem 3.4 provides us with a convenient notation for the 10 points $(ij)$ and 10 blocks $[ijk]$ of a Desargues configuration $D$ as in Figure 3. The mapping

$$
\phi : (ij) \rightarrow \text{blockline containing } [uvw]
$$

and

$$
\phi : \text{blockline containing } [uvw] \rightarrow (ij),
$$

where $i, j, u, v, w$ are distinct, has period 2, maps points to blocklines, blocklines to points and preserves incidences in $D$. Thus $\phi$ is a polarity of $D$, mapping a point to its polar blockline, and a blockline to its pole. If a point $P$ lies on its polar blockline $l_P$, we say that this point $P$ is self-polar, or that $P$ is a self-conjugate (SC) point. Following the notation of [2], we think of $P$ as an “accidental” extra point on the blockline containing its polar block. Dually, if a line $l$ contains its pole, we say that $l$ is self-polar or self-conjugate. For example, in Figure 3 the point $(35)$ is an SC point on the blockline containing $(12)$.

As pointed out in Section 2, each of the 10 points $(ij)$ of $D$ serves as the vertex of perspective for two triangles in $D$. The polar of $(12)$ is the corresponding Desargues axis which is the blockline containing the block $[345]$.

Next, let $l$ be a blockline containing the block $[123]$ consisting of the points $(12), (13), (23)$. Let $l$ contain an extra point, $(ij)$. Suppose that $i$ (or $j$) is equal to 1 or 2 or 3. For example, let $i = 1, j = 5$. Then $l$ contains $(12), (13), (23), (15)$. Since $l$ contains $(12)$ and $(15)$, it contains $(25)$. Then the 5 points $V,A,A',C,C'$ lie on $l$ so that two of the blocklines are equal. This contradicts the fact that $D$ is non-degenerate.

We summarize as follows.

**Theorem 4.3.** A blockline containing the block $[ijk]$ can contain at most one additional point of $D$, namely the point $(uv)$, where $u,v,i,j,k$ are 5 different symbols. In this case, the point $(uv)$ lies on the Desargues axis $[ijk]$ of the two triangles in $D$ that are in perspective from $(uv)$. Thus $(uv)$ lies on its polar line, i.e., it is self-polar (=self-conjugate). For a detailed discussion of the polarity, see [2] and Section 8.

It is perhaps worth restating Theorem 4.3 since the result is so useful.
Figure 5: The triangles (13)(14)(15) and (23)(24)(25) are in perspective from (12). The point (35) is self-conjugate; it is on the blockline containing [124].

**Theorem 4.4.** Any blockline contains at most four points of the Desargues configuration.

5 5-Compressors sectioning to $D$.

Let $V$ be a fixed point in $PG(2,F)$ and let $D$ be a (non-degenerate) Desargues configuration containing $V$. Let $P_1, P_2$ be two fixed points in space not lying in $\pi = PG(2,F)$ such that $P_1, P_2$ and $V$ are collinear.

The following result is a strengthening of Theorem 3.4. We use the notation in Figure 1(a).

**Theorem 5.1.** There exist exactly two 5-compressors $S_1, S_2$ with no point of either lying in $\pi$ which contain $P_1$ and $P_2$ and whose section is $D$.

**Proof.** From Theorem 3.4 there are at least two 5-compressors sectioning to $D$. Let the 5-compressor $S = \{P_1, P_2, P_3, P_4, P_5\}$ section to $D$. There are 3 blocks, namely $VAA', VBB'$ and $VCC'$ containing $V$. We can assume that the 3 planes $<P_1, P_2, P_3>$, $<P_1, P_2, P_4>$, $<P_1, P_2, P_5>$ intersect $\pi$ in
the 3 blocklines containing \(VAA', VBB', VCC'\) respectively. The blockline containing \(VAA'\) might well contain a fourth point \(X\) of \(D\) with \(X\) being the pole of the block \(VAA'\). However, neither of the triples \(\{VAX\}, \{VA'X\}\) are blocks of \(D\). It follows that we only have 2 possibilities, namely

(a) \(P_3\) is the point \(P_1A \cap P_2A'\)

(b) \(P_3\) is the point \(P_1A' \cap P_2A\).

Assume that case (a) holds. Examining the blockline containing \(\{V,B,B'\}\) we again have 2 possibilities

(a) \(P_4 = P_1B \cap P_2B'\)

(b) \(P_4 = P_1B' \cap P_2B\).

Assume case (b) here. We already have \(P_3 = P_1A \cap P_2A'\) so that \(P_1P_3A\) are collinear. We now assume that \(P_4 = P_1B' \cap P_2B\) so that \(P_1P_4B'\) are collinear. By hypothesis the points of intersection of the lines \(P_1P_3, P_1P_4\) and \(P_3P_4\) with \(\pi\) are the 3 points of a block which lies on the blockline formed by the intersection of \(<P_1,P_3,P_4>\) with \(\pi\). Now \(P_1P_3\) meets \(\pi\) in \(A\) and \(P_1P_3\) meets \(\pi\) in \(B'\). Thus \(A\) and \(B'\) must lie in one of the 10 blocks of \(D\) which is false. We conclude that either

\[P_1A \cap P_2A' = P_3\text{ and }P_1B \cap P_2B' = P_4\]

or

\[P_1A' \cap P_2A = P_3\text{ and }P_1B' \cap P_2B = P_4\]

By examining the block \(VCC'\) in a similar way we conclude that there are just 2 possibilities for the 5-compressor, namely \(S_1\) and \(S_2\) as in Theorem 3.4,

namely

\[S_1 = \{P_1, P_2, P_1A \cap P_2A', P_1B \cap P_2B', P_1C \cap P_2C'\}\]

or

\[S_2 = \{P_1, P_2, P_1A' \cap P_2A, P_1B' \cap P_2B, P_1C' \cap P_2C\}\].

\[\blacksquare\]

6 The number of planar Desargues configurations.

In this section \(F = GF(q)\) the finite field of order \(q\). [The easiest example is when \(q\) is a prime \(p\) and our field is formed from the integers modulo \(p\) with the usual addition and multiplication.] First we count the number of Desargues configurations \(D\) containing a given point \(V\) of the plane \(\pi\) contained in \(\Sigma = PG(3,q)\). We have already shown the following.

(a) \(D\) can be obtained as the section of a 5-compressor \(S\) in \(\Sigma\) with no point of \(S\) lying in \(\pi\).
(b) \( S = \{P_1, P_2, P_3, P_4, P_5\} \) where \( P_1, P_2 \) are distinct fixed points in space not in \( \pi \) such that the line \( P_1P_2 \) passes through the given point \( V \) of \( S \).

(c) There are exactly 2 such 5-compressors \( S_1, S_2 \) whose section is the Desargues configuration \( D \) in \( \pi \) containing \( V \).

Our task now is to calculate the total number, say \( \theta \), of 5-compressors \( S \) containing \( P_1, P_2 \) as in (b) above. It will then follow that there are exactly \( \frac{\theta}{2} \) Desargues configurations in \( \pi \) containing \( V \).

In what follows, \( \Lambda = AG(3, q) \) denotes the affine 3-dimensional space over the field \( F = GF(q) \) obtained from \( \Sigma \) by removing \( \pi \).

**Lemma 6.1.** Let \( P_1, P_2, P_3, P_4 \) be a set of 4 given points in \( \Lambda \) not contained in a plane of \( \Lambda \). Let \( P_5 \) be a point of \( \Lambda \) such that \( S = \{P_1, P_2, P_3, P_4, P_5\} \) is a 5-compressor in \( \Lambda \), i.e., a set of 5 points in \( \Lambda \) such that no four of its points lie in a plane of \( \Lambda \) (and thus, such that no 3 points lie on a line of \( \Lambda \)). Then the numbers of possibilities for \( P_5 \) is \((q - 2)(q^2 - 2q + 2)\).

**Proof.** We calculate \( u \), the number of points in the union \( U \) of the 4 planes \(<P_1, P_2, P_3>, <P_1, P_2, P_4>, <P_1, P_3, P_4>, <P_2, P_3, P_4>\). Recall that \(<P_i, P_j, P_k>\) denotes the unique plane of \( \Lambda \) containing the points \( P_i, P_j \) and \( P_k \).

No two of these 4 distinct planes are parallel so any two of them intersect in an affine line. Any 3 of the planes meet in a unique point in space which is one of the \( P_i \). No point lies on all 4 planes. Then, by inclusion-exclusion we get that \( u = |U| = 4q^2 - \binom{4}{2}q + \binom{4}{3}1 = 4q^2 - 6q + 4 \). Therefore the number of possibilities for \( P_5 \) is equal to \( q^3 - [4q^2 - 6q + 4] = (q - 2)(q^2 - 2q + 2) \).

**Theorem 6.2.** The number of 5-compressors \( S = \{P_1, P_2, P_3, P_4, P_5\} \) containing 2 given points \( P_1, P_2 \) in the affine space \( \Lambda = AG(3, q) \) is

\[
\theta = \frac{(q^3 - q)(q^3 - q^2)(q - 2)(q^2 - 2q + 2)}{6}.
\]

**Proof.** The point \( P_3 \) can be any point in \( \Lambda \) not on the line \( P_1P_2 \). So the number of choices for \( P_3 \) is \( q^3 - q \). \( P_4 \) can be any point in \( \Lambda \) not in the plane \(<P_1, P_2, P_3>\). So there are \((q^3 - q^2)\) choices for \( P_4 \). From Lemma 6.1 the number of choices for \( P_5 \) having chosen \( P_1, P_2, P_3, P_4 \) is \((q - 2)(q^2 - 2q + 2)\). The product \((q^3 - q)(q^3 - q^2)(q - 2)(q^2 - 2q + 2)\) counts ordered triples \( \{P_3, P_4, P_5\} \). Permuting these 3 points yields the same (unordered) set of three points. Since we are counting sets, not ordered sets, we divide this product by \(3! = 6\) to obtain the result. \( \blacksquare \)
Theorem 6.3. The total number of Desargues configurations in $PG(2,q)$ is

$$q^3(q^3-1)(q^2-1)(q-2)(q^2-2q+2)/120.$$ 

Proof. Let $x$ denote the total number of Desargues configurations $D$ in $\pi = PG(2,q)$. We count $|J|$ the number of incidences in $J = \{(V,D) \mid \text{point } V \text{ is incident with Desargues configuration } D\}$. 

For each point $V$ there are, from Theorem 6.2, exactly $\frac{q}{2}$ configurations $D$ containing $V$. Thus $|J| = (q^2 + q + 1)\frac{q}{2}$, since $\pi$ has $q^2 + q + 1$ points. Also $|J| = 10x$ since $D$ contains exactly 10 points of $\pi$. Thus $x = (\frac{q^2 + q + 1}{20})q$. Using the fact that $q^2 + q + 1 = q^3 - q^2 - q + 1$, and simplifying, the result follows. 

7 Three-dimensional Desargues configurations.

We will work here in $\Omega = PG(4,q)$, where $\Sigma = PG(3,q)$ is a given 3-dimensional subspace of $\Omega$. As in Section 3, a set $S = \{P_1, P_2, P_3, P_4, P_5\}$ of 5 points in $\Omega$ with no 4 being coplanar is called a 5-compressor. The following lemma is analogous to Theorem 3.2 in 3-dimensions.

Lemma 7.1. Suppose no point of the 5-compressor $S$ lies in $\Sigma$. Then the 10 lines $P_iP_j$ joining $P_i$ to $P_j$ meet $\Sigma$ in 10 distinct points $(ij)$. Each of the 10 planes $<P_i, P_j, P_k>$ sections $\Sigma$ in a block $[ijk]$ of points $(ij), (ik), (jk)$ which are collinear on a line called a blockline. The resulting configuration of 10 distinct points and 10 distinct blocks is a Desargues configuration $D$ in $\Sigma$.

As above we say that $S$ sections $D$ or $S$ sections to $D$. $D$ is either “planar”, i.e., lies in a plane in $\Sigma$, or $D$ is “non-planar”, so that the points of $D$ span a 3-dimensional space $\Sigma$. The details are as follows.

Lemma 7.2. (a) If $D$ is spatial, i.e., non-planar, then $S$ spans the space $\Omega = PG(4,q)$.

(b) If $S$ spans $\Omega$ then $D$ is spatial.

Proof. The lines $P_iP_2, P_iP_3, P_iP_4, P_iP_5$ meet $\Sigma$ in the 4 points $(12), (13), (14)$ and $(15)$. The points $(13), (14), (15)$ are the vertices of one of the two triangles in $D$ that are in perspective from $(12)$. Since $D$ is non-planar these 4 points, namely $(12), (13), (14), (15)$ span $\Sigma$. $P_i$ is not in $\Sigma$. The line joining $P_i$ to $P_i$ contains $(1i), i = 2, 3, 4, 5$. Thus $S$ spans $\Omega = <P_i, \Sigma>$. This proves (a).
To prove (b), we suppose by way of contradiction that the points (12), (13), (14) and (15) lie in a plane \( \pi \). Now \( P_1 \) is not in \( \Sigma \). Consider the 3-dimensional space \(<P_1, \pi>\). It contains \( P_1 \) and the points \((1i), i = 2, 3, 4, 5\). Thus it contains \( S = \{P_1, P_2, P_3, P_4, P_5\} \). Then \( S \) is contained in a 3-dimensional space, a contradiction to the assumption that \( S \) spans \( \Omega \). ■

**Notation 7.3.** A 5-compressor \( S \) in \( \Omega = PG(4, q) \) that spans \( \Omega \) is termed a 5-arc in \( \Omega \). \( \Gamma = AG(4, q) \) is the affine space obtained by removing \( \Sigma \) from \( \Omega \).

If \( S \) is a 5-arc then, as for a 5-compressor, no 4 points of \( S \) are coplanar. But, in addition, we demand that the 5 points of \( S \) are not contained in a 3-dimensional subspace of \( \Omega \).

Using this notation and a similar method of proof to that of Theorem 3.4, we have the following result.

**Lemma 7.4.** Let \( D \) be a spatial Desargues configuration in \( \Sigma = PG(3, q) \) and let \( V \) be a point of \( D \). Let \( P_1, P_2 \) be fixed points in \( \Omega = PG(4, q) \) not in \( \Sigma \) such that the line \( P_1P_2 \) meets \( \Sigma \) in \( V \). Then

(a) there exists 5-compressors \( S_1, S_2 \) in \( \Gamma = AG(4, q) \) such that \( S_1, S_2 \) contain \( P_1, P_2 \) and section \( \Sigma \) in \( D \).

(b) Each of \( S_1, S_2 \) is a 5-arc in \( \Gamma \).

(c) Any 5-arc in \( \Gamma \) containing \( P_1, P_2 \) sections \( \Sigma \) in a spatial Desargues configuration containing \( V \).

The following analogue of Theorem 5.1 is proved in a similar way.

**Theorem 7.5.** Let \( D \) be a spatial Desargues configuration in \( \Sigma = PG(3, q) \) where \( \Sigma \) is contained in \( \Omega = PG(4, q) \). Let \( V \) be a point of \( D \). Let \( P_1, P_2 \) be two points in the affine space \( \Lambda = AG(4, q) \) obtained by removing \( \Sigma \) from \( \Omega \) such that the line \( P_1P_2 \) meets \( \Sigma \) in \( V \). Then there are exactly two 5-compressors \( S_1, S_2 \) in \( \Omega \) containing \( P_1 \) and \( P_2 \) which section to \( D \). Each of \( S_1, S_2 \) is a 5-arc of \( \Omega \) which is contained in the affine space \( \Gamma = AG(4, q) \).

**Theorem 7.6.** The number of 5-arcs \( S \) of \( \Omega \) contained in \( \Gamma \) and containing \( P_1, P_2 \) is

\[
\theta = \frac{(q^4 - q)(q^4 - q^2)(q^4 - q^3)}{6}.
\]

**Proof.** There are \( q^4 - q \) possibilities for \( P_3 \), followed by \( q^4 - q^2 \) possibilities for \( P_4 \) and \( q^4 - q^3 \) possibilities for \( P_5 \). In total we have \( (q^4 - q)(q^4 - q^2)(q^4 - q^3) \) ordered possibilities for \( S \). Thus to find \( \theta \) we divide by \( 3! = 6 \) since this product counts ordered triples \( P_3P_4P_5 \). ■
Theorem 7.7. The total number of spatial Desargues configurations is
\[ \frac{(q^3 + q^2 + q + 1)(q^4 - q)(q^4 - q^2)(q^4 - q^3)}{120}. \]

Proof. Let \( J \) denote the set of incidences \((V, D)\), where \( V \) is a point of \( \Sigma = PG(3, q) \) and \( D \) is a spatial Desargues configuration containing \( V \). If there are \( x \) spatial Desargues configurations in \( PG(3, q) \) we get that \( |J| = 10x \).

On the other hand, from Theorem 7.6 the number of spatial Desargues configurations containing \( V \) is \( \theta^2 \). Since there are \( q^3 + q^2 + q + 1 \) possibilities for \( V \) in \( \Sigma \) we get that \( |J| = (q^3 + q^2 + q + 1)^2 \). Thus
\[ x = \frac{(q^3 + q^2 + q + 1)(q^4 - q)(q^4 - q^2)(q^4 - q^3)}{120}. \]

We now sketch another proof of Theorem 7.7.

Alternative proof of Theorem 7.7

Proof. Our goal is to show that the number of spatial Desargues configurations containing \( V \) is \( \frac{q^2}{2} \).

To this end let \( VAA', VBB' \) and \( VCC' \) be 3 blocks of a spatial Desargues configuration. There are \( q^2 + q + 1 \) choices for the line \( VA \) and \( q^2 + q \) choices for the line \( VB \). Now the line \( VC \) cannot lie in the plane formed by the lines \( VA \) and \( VB \) since the configuration is spatial. Thus there are \( q^2 \) choices for the line \( VC \). All told we have \( (q^2 + q + 1)(q^2 + q)q^2 \) choices for the (unordered) set of 3 lines.

There are \( \binom{3}{2} \) choices for each of the sets \( AA', BB', CC' \). There are then 8 choices for the triangles \( ABC \) and \( A'B'C' \). However, interchanging \( A \) with \( A' \), \( B \) with \( B' \), \( C \) with \( C' \) will yield the same configuration so we have effectively 4 choices for the triangles. Thus the number of spatial configurations containing \( V \) is
\[ \frac{(q^2 + q + 1)(q^2 + q)q^2(\frac{3}{2})(\frac{3}{2})(\frac{3}{2})}{6} \cdot 4 \]
which is equal to \( \frac{q^2}{2} \), proving the result.
In this section, we address a historical ambiguity as to the definition of a Desargues configuration by both a Desargues configuration and a Desargues blockline structure.

We are motivated by the following question: Using the notation of Figure 3, given a Desargues configuration \( D \) in \( PG(2, F) \) with points: \((12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\) and blocks: \{ \((12)(13)(23)\) \}, \{ \((12)(14)(24)\) \}, \{ \((12)(15)(25)\) \}, \{ \((13)(14)(34)\) \}, \{ \((23)(24)(34)\) \}, \{ \((13)(15)(35)\) \}, \{ \((23)(25)(35)\) \}, \{ \((14)(15)(45)\) \}, \{ \((24)(25)(45)\) \}, \{ \((34)(35)(45)\) \}, could there be a second Desargues configuration that has the same points and the same blocklines as \( D \), but with different blocks than \( D \)? A Desargues configuration’s blocks each have exactly three points by definition. But, some blocklines could “accidentally” contain a fourth point of the Desargues configuration. This happens precisely when a point lies on its polar line.

Suppose a new block is formed by replacing one of the points of an existing block with the fourth accidental point. The question is whether it is possible to find a Desargues configuration containing the new block which has the same points and blocklines as the original Desargues configuration?

For example, in Figure 5, the triangles \((13)(14)(15)\) and \((23)(24)(25)\) are in perspective from the point \((12)\). Notice that the point \((35)\) is self-conjugate as it is accidentally on the blockline that contains \([124]\).

There are four points of \( D \) on the blockline through \([124]\). The given Desargues configuration has a block \{ \((12), (14), (24)\) \} from that line. Could another Desargues configuration with the same points and the same blocklines have a block containing \((35)\) and two of \((12), (14), (24)\)? We will show that the answer is no.

We introduce some terminology. Associated with any Desargues configuration \( D \) is an incidence structure \( D_{BL} \) of points and subsets of the points. The points of \( D_{BL} \) are the points of \( D \). The subsets of \( D_{BL} \) are the points of \( D \) that lie on a blockline, i.e., a line of the plane containing a block. (Such a subset has either three or four points.) The resulting incidence structure is called a Desargues blockline structure.

Note: ostensibly, a better-sounding name for this might be a Desargues blockline configuration. However, as we have defined it in Section 2, a “Desargues blockline configuration” would not actually be a configuration as the incidence structure of points and subsets of points of a configuration must have a constant number of points per subset. In this situation, there will
often be three points in a subset, but there might be a fourth accidental point in a subset.

For example, if $D$ is the Desargues configuration represented by Figure 5, the blocks of $D$ are

$$\begin{align*}
\{(12)(13)(23)\}, \{(12)(14)(24)\}, \{(12)(15)(25)\}, \\
\{(13)(14)(34)\}, \{(23)(24)(34)\}, \\
\{(13)(15)(35)\}, \{(23)(25)(35)\}, \\
\{(14)(15)(45)\}, \{(24)(25)(45)\}, \\
\{(34)(35)(45)\}.
\end{align*}$$

Each block has three points, as required by the definition of a Desargues configuration. If we focus on the block $[124]$ which contains $(12)$, $(14)$, $(24)$, in $D$, the corresponding subset in $D_{BL}$ contains four points, namely $(12)$, $(14)$, $(24)$ and the accidental point $(35)$.

**Observation 8.1.** If a Desargues configuration has no self-conjugate points, then the corresponding Desargues blockline structure is the same as the given Desargues configuration.

The main result of this section is the following.

**Theorem 8.2.** Let $\pi = PG(2, F)$, where $F$ has characteristic different from 2, 3. Let $ABC$ and $A'B'C'$ be two triangles in perspective from $V$ that give rise to the Desargues configuration $D$. Assume that two other triangles in perspective from $V$ give rise to the Desargues configuration $D'$. Suppose that $D$ and $D'$ have the same points and the same blocklines. Then $D$ and $D'$ have the same blocks as well.

**Proof.** Consider a Desargues configuration $D$. Since $F$ does not have characteristic 3, $D$ can have at most three self-conjugate points [2, Theorem 3.20]. We first show that there $D$ must have at least one point that is not self-conjugate and does not lie on any self-conjugate line. (This is not necessarily the case if the field has characteristic 3!)

To see this, there are four possibilities for the number of self-conjugate points. We use coordinates as in Figure 3. If there are no self-conjugate points, choose $V = (12)$ for the desired point. If there is one self-conjugate point, label it $(13)$, say, and then choose $V = (12)$ as the desired point. If there are two self-conjugate points, they must be on a block by [2, Lemma 3.10]. So, the labels for the two points must share a symbol. Without loss of generality, they are $(13), (14)$. In this case, choose $V = (12)$ as the desired point. The remaining case is when there are three self-conjugate points. Recall that each pair of them must be on a block. So, each pair of them must share a symbol (in their labels). But the three points are not on a block since the characteristic is not 2 [2, Theorem 3.21]. Without loss of generality, they are $(13), (14), (15)$. Again, we choose $V = (12)$ as the desired point.
Therefore, let $V = (12)$ be a point that is not self-conjugate and that does not lie on any self-conjugate line. There are exactly three blocklines of $D$ through $V$. Now we consider $D'$, which has the same points and the same blocklines as $D$. As discussed in Section 2, $D'$ must have a pair of triangles in perspective from $V$. There are only three lines through $V$ and they only have two points in addition to $V$ on each of the three lines. One blockline through $V$ contains the block $[123]$, another contains $[124]$, and the final one contains $[125]$.

There are four possibilities for the possible pair of triangles in $D'$:

(i) $$(13)(14)(25), (23)(24)(15),$$

(ii) $$(13)(24)(15), (23)(14)(25),$$

(iii) $$(13)(24)(25), (23)(14)(15),$$

(iv) $$(13)(14)(15), (23)(24)(25).$$

In case (i), there is a blockline in $D'$ through $(13)(25)$. That means that $(13)$ or $(25)$ is self-conjugate (in $D$). However, $(25)$ cannot be self-conjugate in $D$. For this would imply that $(25)(13)(14)(34)$ lie on a blockline in $D$ and thus on a blockline of $D'$. But the points $(25)(14)(13)$ are postulated to form a triangle in $D'$. Therefore $(13)$ is self-conjugate (in $D$). Similarly, $(14)$ is self-conjugate (in $D$). Also, there is a blockline through $(23)(15)$. Therefore, $(23)$ must be self-conjugate (in $D$). But, $(14)$ and $(23)$ cannot both be self-conjugate (in $D$) because they do not lie on a block (in $D$). So, case (i) is not possible.

In case (ii), there is a blockline through $(13)(24)$. We know that $(24)$ is not self-conjugate (in $D$) because $(13)(24)(15)$ is a triangle in $D'$. Therefore $(13)$ is self-conjugate (in $D$). Similarly, $(15)$ is self-conjugate (in $D$). Also, there is a blockline through $(14)(23)$. We know that $(14)$ is not self-conjugate (in $D$) because $(23)(14)(25)$ is a triangle in $D'$. Therefore $(23)$ is self-conjugate (in $D$). But, $(23)$ and $(15)$ cannot both be self-conjugate (in $D$) because they do not lie on a block (in $D$). So, case (ii) is not possible.

In case (iii), there is a blockline through $(13)(24)$ and one through $(13)(23)$. Therefore, $(23)$ and $(24)$ are self-conjugate (in $D$). Also, there is a blockline through $(23)(14)$ and one through $(23)(15)$. Therefore $(14)$ and $(15)$ are self-conjugate (in $D$). But, $(23)$ and $(14)$ cannot both be self-conjugate (in $D$). So, case (iii) is not possible.

That leaves case (iv), in which case $D = D'$. ■

**Corollary 8.3.** Let $\pi = PG(2,F)$, where $F$ has characteristic different from 2, 3. If $D$ is a Desargues configuration that gives rise to the Desargues blockline structure $D_{BL}$, then no other Desargues configuration gives rise to the blockline structure $D_{BL}$.
It can be shown by similar synthetic methods that the restriction in Theorem 8.2 on the characteristic of the field is not really necessary. Corollary 8.3 then provides a significant strengthening to our calculation of the total number of Desargues configurations containing a point in Section 6.

**Corollary 8.4.** Let \( \pi = PG(2, F) \), where \( F \) is a finite field. Then the total number of distinct Desargues blockline structures is the same as the total number of Desargues configurations.

Added in proof. Using similar methods we can show that Corollary 8.4 holds in all cases regardless of the characteristic.

9 A possibly easier approach to the enumeration of the number of planar Desargues configurations through a given point that was given in Section 6.

In this section, we consider the possibility of an easier calculation of the number of planar Desargues configurations through a given point that was given in Section 6.

We have seen that the answer is \( \theta/2 \), where

\[
\theta = \frac{(q^3 - q)(q^3 - q^2)(q - 2)(q^2 - 2q + 2)}{6}
\]

as in Theorem 6.2 and Theorem 6.3. Is there a quicker way to calculate this by just working in the plane?

The following calculation seems plausible at first glance, but it is incorrect.

**ALTERNATE (INCORRECT) CALCULATION:**

Fix a point \( V \) in the plane \( PG(2, q) \). Consider a prospective Desargues configuration \( D \) containing that point. We wish to choose the points of the triangles \( ABC, A'B'C' \) that are in perspective from \( V \), with \( V = AA' \cap BB' \cap CC' \).

We first choose the three lines through \( V \) that will be \( AA', BB' \) and \( CC' \). There are \( (q+1)^3 \) choices for those lines.

On one of those three lines we choose two points unequal to \( V \) to be the set \( \{A, A'\} \). This gives us \( \binom{q+1}{2} \) choices. On the second line we choose the set \( \{B, B'\} \) giving \( \binom{q+1}{2} \) choices. Then any point of \( \{A, A'\} \) can be paired with any one of \( \{B, B'\} \) - there are two choices here - to yield the sides \( AB, A'B' \).

Finally, we need to choose \( C \) and \( C' \) on the remaining line. Let us choose \( C \) first. We ensure that \( C \) is not \( V \) and \( C \) is not on the line \( AB \). That gives us \( q + 1 - 2 = q - 1 \) choices.
To choose $C'$, we need to make sure that $C' \neq V, C$ and that $C'$ is not on the line $A'B'$. That gives us $q + 1 - 3 = q - 2$ choices.

This yields a total of

$$\binom{q + 1}{3} \binom{q}{2} \binom{q}{2} 2(q - 1)(q - 2).$$

However, this is not the answer obtained in Section 6.

What went wrong? We did not pay attention to the possibility that a point might be a self-conjugate point! For example, when we chose $C'$, we did not consider the possibility that $C$ might be self-conjugate. If it is self-conjugate, then $C$ would equal $CC' \cap A'B'$. So, it was over-simplistic to just subtract three in our calculation for the number of choices for $C'$. Furthermore, what if $B$ happened to be a self-conjugate point? Note that all of $A, B, C$ might perhaps be self-conjugate or not. To make things even more interesting, characteristic 3 fields play a special role. In other words the characteristic of the field has to be taken into account using the naive approach above! It is possible to have four self-conjugate points if and only if the field has characteristic 3. This was shown in [2, Theorem 3.20].

10 Concluding remarks.

Desargues theorem in the plane can easily be shown using coordinates as in Lord ([11]). The synthetic proofs seem to all use a construction which is in effect a compressor or a 5-point. Baker ([1]) uses the Desargues theorem in three dimensions to finish the proof. The Veblen-Young proof ([19]) is a very slightly longer version of our proof here. But the important continuation here, using the pairs-triples notation is the connection with polarities developed in [2, 3] and used in the enumeration. One possible method is to count the configurations having 0 self-polar points or exactly 1, 2, 3 or 4 self-polar points. From the Puystjens-Thas ([14]) paper we have seen how difficult this is. The fact that 4 self-conjugate points only occur in characteristic 3 shows that for general fields of order $q$ the method is doomed to failure. Fortunately the necessary information for enumeration is stored in compressed form in the 5-compressor and we can count those.

In [14] the authors concentrate on the cases when the underlying field has characteristic 2 or 3. They also demand that, in our terminology, each blockline must contain exactly 3 points of the Desargues configuration.

In the general case it is possible that a blockline contains as many as 4 points of the configuration, consisting of a block and an “accidental” extra point.

An easy example in which there are four points of a Desargues configuration on a blockline is as follows. Let $T$ be a translation with centre $V$ and axis $l_\infty$, where $l_\infty$ is the line at infinity and $V$ is on $l_\infty$. Let $ABC$
denote an affine triangle and let $A'B'C'$ be its image under $T$. These two triangles are in perspective from $V$. The Desargues axis is $l_\infty$ and it contains 4 points of the resulting configuration consisting of the 3 intersections of corresponding sides together with the “accidental extra point” $V$. See Figure 6. What is happening here is that the point $V$ is self-polar or self-conjugate (see Section 4). A detailed discussion of self-conjugate points in a Desargues configuration is provided in [2].

The case of characteristic 3 actually plays a very special role. In [2] we show that there can be at most three self-conjugate points in a Desargues configuration unless the characteristic is 3, in which case there can exist as many as four self-conjugate points.

The number of configurations in [14] is smaller than the number in Theorem 6.3 stemming from the fact that the authors there are working with a restricted class of configurations.

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