Fair Resource Sharing with Externailties*

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Abstract

We study a fair resource sharing problem, where a set of resources are to be shared among a set of agents. Each agent demands one resource and each resource can serve a limited number of agents. An agent cares about what resource they get as well as the externalities imposed by their mates, whom they share the same resource with. Apparently, the strong notion of envy-freeness, where no agent envies another for their resource or mates, cannot always be achieved and we show that even to decide the existence of such a strongly envy-free assignment is an intractable problem. Thus, a more interesting question is whether (and in what situations) a relaxed notion of envy-freeness, the Pareto envy-freeness, can be achieved: an agent \(i\) envies another agent \(j\) only when \(i\) envies both the resource and the mates of \(j\). In particular, we are interested in a dorm assignment problem, where students are to be assigned to dorms with the same capacity and they have dichotomous preference over their dorm-mates. We show that when the capacity of the dorms is 2, a Pareto envy-free assignment always exists and we present a polynomial-time algorithm to compute such an assignment; nevertheless, the result fails to hold immediately when the capacities increase to 3, in which case even Pareto envy-freeness cannot be guaranteed. In addition to the existential results, we also investigate the implications of envy-freeness on proportionality in our model and show that envy-freeness in general implies approximations of proportionality.

1 Introduction

It is the back-to-school season, freshmen are starting their college life which they have long expected. Amongst the long list of things they are waiting to discover, they are all concerned about where they are going to live and who they are going to stay with in the forthcoming years. This gives the accommodation administrator a hard time: whenever she proposes a dorm assignment, some student will come to tell her that they prefer some other dorm that they are not assigned to (due to layouts of the dorms, their locations, floors, surrounding environments, etc.), or prefer to stay with students in another dorm (due to the subjects they study, their hobbies, lifestyles, political views, etc.). In this paper, we formalize the problem faced by the accommodation administrator as a fair resource sharing problem. In this problem, a set of heterogeneous resources needs to be fairly shared among a set of agents who demands one resource each. The agents have preferences over both the

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resources and their mates, i.e., the other agents with whom they share the same resources. Each resource can be assigned to multiple agents, subject to its capacity, which is the maximum number of agents it can serve simultaneously. The aim is to find an envy-free (EF) assignment of the resources to the agents, such that no agent would prefer to exchange their resource with any other agent. Besides dorm assignment, the fair resource sharing problem also arises in other scenarios, such as project assignment, where each agent has a preference over the projects to work on, as well as a preference over their collaborators in the assigned project; or group activity selection problem, where agents have preferences over the activities as well as the other agents.

1.1 Main Results

We make the following contributions. We first study a strong EF notion, where no agent envies another for the resource she received or her mates who share this resource with her. We show that this strong EF notion does not ensure the existence of an EF assignment; even to decide the existence of an EF assignment is NP-hard, and the hardness result holds even when there are only two resources and all the agents have dichotomous preferences over the other agents. Indeed, envy-free assignments barely exist under the above strong notion even in very simple settings, so we are interested in a relaxed notion called Pareto envy-freeness (PEF). Informally speaking, an assignment is called PEF if for any two agents $i$ and $j$, either $i$ does not envy $j$ for the resource $j$ receives, or for the mates of $j$ that share the same resource with $j$. With this relaxed notion, we show additional negative and positive results. Specifically, a PEF assignment, still, may not exist and it is NP-hard to decide the existence; nevertheless, when we focus on a special class of the resource sharing problem where the resources have the same capacity and agents have dichotomous preference over their mates, we find that a PEF assignment always exists if the resource capacity is 2 and such an assignment can be found in polynomial time. The proof of this result is non-trivial and relies on the Gallai-Edmonds Theorem [34] and the Hall’s Theorem [27]. We also show that 2 is a tight upper bound of the resource capacity that guarantees the existence of a PEF assignment; when the capacities increase to 3, we present a counterexample that admits no PEF assignment.

Towards the end of this paper, we also investigate the implications of EF on proportionality (PROP). Unlike the classic fair resource allocation domain, in our model, an EF assignment may not be PROP. Nevertheless, we show that every EF assignment is a 2-approximation of proportionality if the capacity of each resource is at least 2, and the bound of the approximation ratio is tight. In addition, PEF also implies PROP under a Pareto optimality-style notion.

1.2 Related Work

The primary consideration of our work, that of the envy-freeness, is related to the substantial literature of fair division problems. The major settings of fair division problems include: (i) the division of, without loss of generality, one divisible good, e.g., a cake, that can be thought of as a segment that can be cut through at any points, which is hence also known as the cake-cutting problem [37, 20, 12, 7]; (ii) the allocation of a set of indivisible goods [33, 13, 14], each can only be assigned to some agent as a whole; and (iii) the inverse problems of allocating divisible or indivisible goods, i.e., items that are unwanted such as chores [8, 6, 28]. Envy-freeness and proportionality are the main considerations in the studies of these problems, where the latter asks for an allocation in which every agent obtains $1/n$ of the total value of the goods according to their own valuation ($n$ being the number of agents). Although an envy-free allocation and a proportional allocation normally always exist in the cake-cutting setting, they may not exist when the items are indivisible. Thus, relaxed notions are proposed. Typically, envy-freeness up to one item, or EF-1, is proposed.
as a relaxation of envy-freeness \[33\], which can often be guaranteed with indivisible items and is hence a pervasive notion when it comes to allocation of indivisible items. Similarly, the maximin share fairness (MMS-fairness) is proposed \[13\], which requires each agent to obtain at least the maximum value they can get by partitioning the items into \(n\) bundles and taking the least valuable bundle; constant approximations of the MMS-fairness can be achieved with indivisible items.

Our model differs from canonical models of fair division problems in that the items are to be shared, instead of being partitioned, among the agents and each agent only gets one item. However, we adopt envy-freeness as our key consideration and adapt the notion to our setting by considering the Pareto envy-freeness. A similar relaxed notion of envy-freeness was also considered by Chan et al. \[16\] and a followup work Huzhang et al. \[29\]. They showed that when the capacities of the dorms increase to 2, although individual envy-freeness cannot be guaranteed, a direct application of the result in \[36\] ensures room envy-freeness which treats every pair of dorm-mates as a whole. That is, by moving all the agents in a dorm to another dorm, the agents’ total utility cannot be increased. This is different from our objective, which is concerned with the utility of each individual agent, rather than that of a group.

The dorm assignment problem we study relates our work to the stable roommate problem \[25, 31\], where \(2m\) students are to be assigned to \(m\) dorms in a way such that no pair of students want to swap their positions in the assignment, so that the assignment is considered to be stable. (There are also other stability notions, such as exchange stability \[15\] and popular matchings \[19\].) These papers only consider the agents’ preferences over their mates. In other words, the dorms are assumed to be identical to every agent. Though we can generalize this stability notion by incorporating also the agents’ values to the dorms, a stable dorm assignment can be far from being EF. Arguable, in many scenarios, such as the dorm assignment problem, EF is a more desired notion: it is more important to ensure that everyone feels that they are treated equally, than to find an assignment in which no one can hope for a better resource only because they are occupied by someone who refuse to exchange. More restrictively, the fair house assignment problem initiated by \[30, 35\] (and more recently, \[26, 9, 2\]), studied a special case of our model where the capacities of the resources are one (and the external values between the agents are zero).

Similar problems have also been studied in various market settings. In a market, resources have (different) prices (e.g., rent) that needs to be shared between the agents that they are assigned to. Hence, the utility of each agent is the net value they obtain, i.e., the difference between the value they obtain from the resource and the price they pay. As shown by Shapley and Shubik \[36\], if each resource has capacity 1, then there is an assignment along with a price profile, such that the matching between the agents and resource is envy-free. In our model, however, we consider only scenarios where monetary transfer is not possible. The counterpart of this setting is related to the fair rent division problem \[3, 4, 1, 24\], which studies fair ways to assign rooms to agents and divide the rent among them. With monetary transfers, an envy-free solution is always feasible; thus one research interest is to find the “best” envy-free solutions \[3, 24\]. Apart from allowing monetary transfer, these papers also differ from our work in that they did not consider the agents’ preferences over their dorm-mates. Indeed, when these external preferences are considered, the setting with monetary transfer is an interesting parallel direction for future work.

Our work is also related to models from cooperative game theory, such as hedonic games \[11, 22\] and group activity selection games \[19, 18, 21\]. In hedonic games, the agents form coalitions and their utilities are decided solely by the members in the coalition, without any resource in the model. In group activity selection games, the agents’ utilities may also depend on the type of activities they take, which are similar to the resources in our model. Despite this similarity, models of group activity selection games usually employ different modeling assumptions, e.g., every agent always has the choice of deviating to singleton activities (i.e., by being alone), which is not feasible in our
model. The main objective of study group activity selection games is often to incentivize agents to form stable groups or to participate in (non-singleton) activities while our focus is EF. To the best of our knowledge, no result for these models implies ours in this paper.

The remainder of this paper is organized as follows. In Section 2 we describe the resource sharing model and a special setting of the model called the dorm sharing model. Then, we formalize our EF and PEF notions in Section 3 and discuss the existence of an assignment under this notions, as well as the related computational complexity results. In Section 4 we study PEF assignments in dorm sharing models and present a key result — a polynomial-time algorithm to compute a PEF assignment when the dorm capacity is 2. In Section 5 we further investigate the implications of EF on PROP and present several results about approximations of PROP implied by EF. We conclude in Section 6 where we discuss several directions for future work.

2 Resource Sharing Model

There is a set $M$ of resources (e.g., dorms) that needs to be assigned to a set $N$ of agents. Let $m$ and $n$ be the sizes of $M$ and $N$, respectively, i.e., $|N| = n$ and $|M| = m$. Each agent $i \in N$ has a value $v_{ij} \geq 0$ for each resource $j \in M$ and demands exactly one resource. Besides the values for the resources, each agent also receives a value from their mates, i.e., the set of agents with whom they share the same resource. To distinguish between these two types of values, we will also refer to an agent’s value for a resource as an internal value and their value for another agent as an external value or an externality. We consider additive values throughout this paper: agent $i$ receives an external value $e_{ij} \in \mathbb{R}$ if she shares her resource with agent $j \in N$; the total external value she receives is defined to be $e_i(A) = \sum_{j \in A} e_{ij}$. We assume that $e_{ij} \geq 0$ for all $i, j \in N$, and $e_{ii} = 0$ (i.e., an agent has no external value for themselves).

In a feasible assignment, each agent is assigned to one resource, while each resource can serve multiple agents. We further set a capacity $c_j \geq 1$ for each resource $j \in M$, which is the maximum number of agents $j$ can serve. Without loss of generality, we assume that the supply meets the demand in our model, so it holds that $n = \sum_{j \in M} c_j$. With the above definitions, an instance of the resource sharing problem is then given by a tuple $\mathcal{I} = (N, M, \mathbf{v}, \mathbf{e}, \mathbf{c})$, where $\mathbf{c} = (c_j)_{j \in M}$, $\mathbf{v} = (v_{ij})_{i \in N, j \in M}$, and $\mathbf{e} = (e_{ij})_{i, j \in N}$.

We write an assignment as $X = (X_1, \ldots, X_m)$, where each $X_j \subseteq N$ denotes the subset of agents that are assigned to resource $j$. For each agent $i \in N$, we denote by $r_i(X)$ the resource that is assigned to $i$ in $X$, and by $S_i(X)$ the set of agents sharing the same resource with $i$ in $X$. With slight abuse of notation, the internal and external values each agent $i$ obtains from assignment $X$ are denoted as $v_i(X)$ and $e_i(X)$, respectively; we have

$$v_i(X) = v_{ir_i(X)}$$

and

$$e_i(X) = \sum_{\ell \in S_i(X)} e_{i\ell}.$$  

**Dorm Assignment Problem.** We are particularly interested in a special setting of the above model, which we term the dorm sharing model as it can be seen as a simple model for a dorm assignment task. In a dorm sharing model, the following conditions hold:

1. All resources (i.e., dorms) have the same capacity $c \geq 1$ (so $n = c \cdot m$);
2. Every agent has a dichotomous preference over the other agents, i.e., their external values are binary: \( e_{ij} \in \{0, 1\} \) for all \( i, j \in N \);

3. The external values are symmetric, i.e. \( e_{ij} = e_{ji} \) for all \( i, j \in N \). When \( e_{ij} = e_{ji} = 1 \), we say that agents \( i \) and \( j \) are friends of each other.

Thus in the dorm sharing model, the externalities among the agents can be described as an undirected graph \( G = (N, E) \), where each node represents an agent and there is an edge \( e = \{i, j\} \in E \) between two agents \( i \) and \( j \) if and only if they are friends of each other. We refer such a graph as an externality graph.

### 3 Envy-free Notions

Our goal is to find an envy-free assignment, in which no agent would envy another for what they have been assigned, considering both the internal and external utilities. In our model, this means that no agent would prefer to exchange their assigned resource with another agent. It is then necessary to specify how the agents would assess the benefit of such an exchange of positions. Arguably, the most straightforward way is to consider the sum of the internal and the external values. We define the (total) utility of an agent \( i \) in an assignment \( X \) as

\[
u_i(X) = v_i(X) + e_i(X).
\]

The envy-free notion defined below expects that no agent envies another agent for what they get at their position.

**Definition 3.1** (Envy-free assignment). For any assignment \( X \), let \( X^{i \leftrightarrow j} \) be the assignment resulting from switching agents \( i \) and \( j \) in \( X \), i.e., \( r_i(X) = r_j(X^{i \leftrightarrow j}), r_j(X) = r_i(X^{i \leftrightarrow j}) \), and \( r_\ell(X) = r_\ell(X^{i \leftrightarrow j}) \) for any \( \ell \notin \{i, j\} \). An assignment \( X \) is envy-free (EF) if \( u_i(X) \geq u_i(X^{i \leftrightarrow j}) \) for every pair of agents \( i, j \in N \).

Unfortunately, an EF assignment may not exist in a resource sharing instance, even in the special case of dorm sharing. For example, when all agents have zero external values and they all prefer dorm 1 to any other dorms, any agent who is not assigned to dorm 1 will envy those who get it. Moreover, even to decide the existence of an EF assignment appears to be computationally hard as we show in Proposition 3.2. Because of these negative results, we consider a relaxed EF notion, which we term the Pareto EF (PEF). The PEF notion treats an agent’s utility as a two-dimensional vector which has the internal and external utilities as its components; a utility vector is considered to be not worse than another vector if it is not dominated by that vector in both dimensions (Definition 3.3).

**Proposition 3.2.** To decide whether a given instance admits an EF assignment is NP-complete, even when the model is canonical and when there are only two resources of value 0 for every agent.

**Proof.** The problem is obviously in NP, as an EF assignment serves as a witness of a yes-instance; the EF of this assignment can be verified in polynomial time. For the hardness, we show a reduction from the classic NP-complete problem, the CLIQUE problem. An instance of CLIQUE is given by an undirected graph \( G = (V, E) \) and an integer \( k > 0 \). It is a yes-instance if and only if there exists a clique of size \( k \) on \( G \), i.e., a subset \( X \subseteq V \), such that \( \{i, j\} \in E \) for every pair of distinct vertices \( i, j \in X \). Without loss of generality, we can assume that \( |V|/2 < k \leq |V| \): when \( k \leq |V|/2 \), we can always modify an instance to one that satisfies this assumption by adding \( |V| \) dummy
vertices to the graph, that form a clique and are disconnected to any vertex in $V$; the new instance $\langle \tilde{G} = (\tilde{V}, \tilde{E}), \tilde{k} = k + |V| \rangle$ is such that $\tilde{k} \geq \frac{|V|}{2}$ and it is a yes-instance if and only if the original instance $\langle G, k \rangle$ is a yes-instance.

For ease of presentation, we first prove this result for a model where not every agent has value 0 to the resources, and then show how it can be extend to the situation when all agents have value 0 to the resources.

Given an instance $\langle G, k \rangle$ of CLIQUE, we construct an instance of our problem as follows. Let the set of agents be $N = N_V \cup N'_V \cup N_1$, where:

- $N_1$ contains a set of $4k - 2|V|$ agents who have value 1 to resource 1, and value 0 to resource 2 and to every other agent. Thus, for an assignment to be EF, these agents have to be assigned to resource 1.
- $N_V = \{a_i : i \in V\}$ contains $|V|$ agents, each corresponding to a vertex on $G$. We say that two agents $a_i, a_j \in N_V$ are friends if and only if $\{i, j\} \in E$. Each agent $a_i \in N_V$ has value $2k - d_i - 1$ to resource 1 and external value 1 to each of their friends, where $d_i$ denotes the degree of $i$ on $G$ (i.e., number of friends in $N_V$). All their other values (including external values) are 0.
- $N'_V = \{a'_i : i \in V\}$ contains $|V|$ agents, each corresponding to a vertex on $G$. Each $a'_i \in N'_V$ has external value 1 to the corresponding agent $a_i \in N_V$. All their other values (including external values) are 0. Thus, for an assignment to be EF, each $a'_i$ has to be assigned to the same resource $a_i$ is assigned to; we should always allocate them as a pair.

There are $4k$ agents in total, so we set $c_1 = c_2 = 2k$. We next argue that the CLIQUE instance is a yes-instance if and only if the above instance admits an EF assignment.

First, suppose that the CLIQUE instance is a yes-instance: there exists a clique $X \subseteq V$, $|X| = k$. Consider the following assignment. Assign all agents in $N_1$ to resource 1; assign each $a_i \in N_V$, $i \in X$, to resource 2 and the remaining agents in $N_V$ to resource 1; assign each $a'_i \in N'_V$ to the same resource $a_i$ is assigned to. The assignment satisfies the capacity constraint and it is EF:

- It is EF for every agent in $N_1$ and $N'_V$ by our observation above.
- Each $a_i \in N_V$, $i \in X$, gets external utility $k$ from $a'_i$ and their $k - 1$ friends on resource 2; they have $d_i - k + 1$ remaining friends on resource 1, so swapping them to resource 1 gives them utility at most $(2k - d_i - 1) + (d_i - k + 1) = k$. The assignment if EF for them.
- Each $a_i \in N_V$, $i \notin X$, gets external utility at least $d_i - k + 1$ from $a'_i$ and at least $d_i - k$ friends on resource 1, so their utility is at least $(2k - d_i - 1) + (d_i - k + 1) = k$. Swapping them to resource 2 gives them at most $k$ friends while they have value 0 to the resource, so the assignment if EF for them.

Conversely, suppose that there exists no clique of size $k$ on $G$, and for the sake of contradiction, there exists an EF assignment. Again, by our observation, all agents in $N_1$ has to be assigned to resource 1, and each pair $a_i$ and $a'_i$ has to be assigned to the same resource. Hence, $k$ pairs of $a_i$ and $a'_i$ are assigned to resource 2. When there exists no size-$k$ clique on $G$, some $a_i$ on resource 2 finds at most $k - 2$ friends on the same resource, obtaining utility at most $k - 1$. There are at least $d_i - k + 2$ friends of this agent on resource 1, so swapping them to resource 1 (with some agent in $N_1$) gives them utility at least $(2k - d_i - 1) + (d_i - k + 2) > k - 1$, which contradicts the assumption that the assignment is EF.
We have finished the proof for the setting where agents may have non-zero resource values. To extend the proof to the setting where every agent has value 0 to every resource, the idea is to convert non-zero resource values in our reduction to external values by adding extra agents. We briefly describe the approach below.

- We let \( N_1 \) now contain \( 4k + 2|V| \) agents who have external value 1 to each other; let \( N_2 \) be a new set of \( 4|V| \) agents who have external value 1 to each other. The idea is to ensure that all agents in \( N_1 \) (or \( N_2 \)) must be assigned to the same resource if we want the assignment to be EF. Now that we have \( 8|V| \) more agents, we set \( c_1 = c_2 = 2k + 4|V| \); this further ensures that \( N_1 \) and \( N_2 \) must be put on different resources as the capacity of one resource is not large enough to hold both groups simultaneously, so we end up having \( 2k \) free space on one resource and \( 2|V| - 2k \) on another, which is equivalent to our reduction above. It then only matters how we distribute the remaining agents in these free spaces.

- We let each agent \( a_i \in N_V \) have external value 1 with each of the first \( 2k - d_i - 1 \) agents in \( N_1 \), so given that all agents in \( N_1 \) should be on the same resource in an EF assignment, \( a_i \) will gain utility \( 2k - d_i - 1 \) by moving to this resource, which is equivalent to their value to resource 1 in our reduction above. Note that this does not change the observation that all agents in \( N_1 \) must be on the same resource, even though now they can also gain some (negligible) external utility from agents in \( N_V \).

This completes the proof.

**Definition 3.3** (Pareto envy-free assignment). An assignment \( X \) is *Pareto-envy-free* (PEF) if for every pair of agents \( i \) and \( j \) at least one of the following two conditions holds:

1. \( v_i(X) \geq v_i(X^{i \leftrightarrow j}) \); or
2. \( e_i(X) \geq e_i(X^{i \leftrightarrow j}) \).

By definition, EF implies PEF. Intuitively, the PEF notion assumes an agent to be happy if they find that no other agent gets a better resource or a better set of mates in the assignment. While comparing the internal and external values separately weakens the EF notion, it might be a more appropriate approach for some scenarios where these two types of values are not directly comparable. For example, the internal value may represent (the negative of) the monetary cost of a resource, while the external value may represent friendship, which cannot always be converted to monetary values. Likewise, the PEF notion also applies when the agents only have cardinal utilities for the resources and ordinal preferences over sets of mates, or vice versa.

Nevertheless, the relaxation of EF to PEF does not immediately free us from an “existential crisis”: a PEF assignment may not exist even in the dorm sharing model and even when there are only two dorms (see Example 3.4); similarly, even to decide the existence of a PEF assignment is computationally hard (see Proposition 3.6). Despite this string of negative results, our key finding in this paper is that a PEF assignment always exists in a dorm sharing instance when the dorms have capacity two, and such an assignment can be computed in polynomial time. We show these results in the next section.

**Example 3.4.** There are 2 dorms of capacity 5, and 10 agents \( N = \{1, \ldots, 10\} \).

- Each agent in \( \{1, \ldots, 7\} \) has value 1 for dorm 1 and value 0 for dorm 2;
- Each agent in \( \{8, 9, 10\} \) has value 0 for dorm 1 and value 1 for dorm 2.
All agents in \( \{1, \ldots, 5\} \) are friends with each other; and for each \( i \in \{1, \ldots, 5\} \), agent \( i \) and \( i + 5 \) are friends (see the externality graph in Figure 1). Proposition 3.5 shows that there is no PEF assignment for this instance.

Figure 1: The externality graph of Example 3.4

**Proposition 3.5.** There is no PEF assignments for the dorm sharing instance in Example 3.4.

**Proof.** Let \( N^* = \{1, \ldots, 5\} \). We consider all possible assignments of the agents in \( N^* \).

- **Case 1.** All agents in \( N^* \) are assigned to dorm 1. In this case, all the other agents in \( \{6, \ldots, 10\} \) are assigned to dorm 2. In this assignment, agent 6 does not share a dorm with her friends and is assigned to the worse dorm. Thus, agent 6 Pareto-envies every agent in \( N^* \setminus \{1\} \).

- **Case 2.** All agents in \( N^* \) are assigned to dorm 2. Similar to Case 1, agent 8 Pareto-envies every agent in \( N^* \setminus \{3\} \) in this assignment.

- **Case 3.** Four agents in \( N^* \) are assigned to dorm 1 and the other one is assigned to dorm 2. Suppose agent \( i \in N^* \) is the one assigned to dorm 2. Other than the four agents in \( N^* \), there is another agent \( j \notin N^* \) who is assigned to dorm 1. For agent \( i \), at least four of her friends are in a better dorm and in her own dorm there is at most one friend. Thus, agent \( i \) Pareto-envies \( j \).

- **Case 4.** Four agents in \( N^* \) are assigned to dorm 2 and the other one is assigned to dorm 1. Suppose agent \( i \in N^* \) is the one assigned to dorm 1. Other than the four agents in \( N^* \), there is another agent outside of \( N^* \) who is assigned to dorm 2, so at least two agents in \( \{8, 9, 10\} \) are assigned to dorm 1 and at least one of them, say \( j \), is not a friend of \( i \). In this case, \( j \) does not share a dorm with her friend (i.e., \( j - 5 \)) and is assigned to a worse dorm. Thus \( j \) Pareto-envies \( N^* \setminus \{i, j - 5\} \).

- **Case 5.** Three agents in \( N^* \) are assigned to dorm 1 and the other two are assigned to dorm 2. Suppose agent \( i \in N^* \) is assigned to dorm 2, which is the worse dorm for her. Agent \( i \) has at least three friends in dorm 1 and at least one friend in dorm 2. Thus, there is at least one agent \( j \) in dorm 1 who is not a friend of \( i \), and agent \( i \) Pareto-envies \( j \) in this assignment.
• Case 6. Three agents in \( N^* \) are assigned to dorm 2 and the other two are assigned to dorm 1. Assume that \( \{i, j, k\} \subset N^* \) are assigned to dorm 2 and \( \{l, h\} \subset N^* \) are assigned to dorm 1. Since each dorm has capacity 5, at least one agent in \( \{i + 5, j + 5, k + 5\} \) is not in dorm 2. Without loss of generality, assume this is agent \( i + 5 \). Then for agent \( i \), she shares a worse dorm with two of her friends \( j \) and \( k \) while three of her friends \( \{i + 5, l, h\} \) are in the better dorm. Thus, agent 1 Pareto-envies the other two agents in dorm 1.

Therefore, no assignment is PEF in this instance. \( \square \)

**Proposition 3.6.** To decide whether a given instance admits a Pareto EF assignment is NP-complete, even when there are only two resources with binary external valuation.\(^1\)

*Proof.* The problem is obviously in NP, with a Pareto EF assignment being a witness of a yes-instance, which can be verified in polynomial time. To show the hardness, our approach is similar to that in the proof of Proposition 3.2; we show a reduction from the CLIQUE problem.

Given an instance \( \langle G = (V, E), k \rangle \) of CLIQUE, without loss of generality, \( |V|/2 < k \leq |V| \), we construct the following instance of our problem. Let the set of agents be \( N = N_V \cup N_1 \), where:

- Every agent has value 1 to resource 1, and value 0 to resource 2, so in a Pareto EF assignment, each agent assigned to resource 2 must not be able to obtain a higher external utility by swapping their position with an agent on resource 1.

- \( N_1 \) contains a set of \( 2k \) agents who have external value 1 to every other agent in \( N_1 \) and external value 0 to each agent in \( N_V \) unless defined to be 1 below.

- \( N_V = \{a_i : i \in V\} \) contains \( |V| \) agents, each corresponding to a vertex on \( G \). Each agent \( a_i \in N_V \) has external value 1 to each \( a_j \in N_V \) such that \( \{i, j\} \in E \), as well as the first \( 2k - d_i - 1 \) agents in \( N_1 \), where \( d_i \) denotes the degree of \( i \) on \( G \). All other external values are 0.

Let the resource capacities be \( c_1 = k + |V| \) and \( c_2 = k \).

Observe that by the above valuations, each agent \( a \in N_1 \) has to be assigned to resource 1 in a Pareto EF assignment: if they are assigned to resource 2, their external utility is at most \( k - 1 \) while they can get external utility at least \( k \) from their remaining friends on resource 1. Thus, only agents in \( N_V \) can be assigned to resource 2.

It can be verified that if the CLIQUE instance is a yes-instance, assigning the \( k \) agents corresponding to the clique on \( G \) gives a Pareto EF assignment. Conversely, if the CLIQUE instance is a no-instance, no matter which \( k \) agents we assign to resource 2, some of them gets external utility at most \( k - 2 \); however, they have \( 2k - d_i - 1 \) friends in \( N_1 \) and at least \( d_i - (k - 2) \) remaining friends in \( N_V \) who are all on resource 1, so swapping their position with a non-friend on resource 1 would give them external utility \( 2k - d_i - 1 + d_i - (k - 2) = k + 1 > k - 2 \), which implies that no assignment can be Pareto EF. \( \square \)

4 Pareto-Envyfree Dorm Assignment

In this section, we demonstrate the existence of a PEF assignment in a dorm sharing instance with dorm capacity 2 by presenting an (efficient) algorithm to compute one such assignment. We first

\(^1\)We remark that we are only able to show the hardness result for the situation where resources have different capacities. The computational complexity when resources are restricted to have the same capacity is an interesting open problem.
introduce some additional notation and several useful results. Let $G = (V, E)$ be an arbitrary externality graph. A matching $\mathcal{M}$ in $G$ is a set of edges without common nodes; it is called:

- a *maximum matching*, if no other matching contains more edges;
- a *perfect matching*, if the edges in it cover all the $|V|$ nodes of $G$; and
- a *nearly perfect matching*, if the edges in it cover $|V| - 1$ nodes of $G$.

For any set of nodes $A \subseteq V$, denote by $G \setminus A$ the induced subgraph of $G$ after the nodes in $A$ and the associated edges are removed from $G$. The subgraph $G \setminus A$ may be composed of one or more disjoint components (i.e., connected subgraphs). A component is called *even* (or odd) if it contains an even (or odd) number of nodes. We will use the concept of Tutte set defined as follows.

**Definition 4.1** (Tutte set). Given a graph $G = (V, E)$, a set $A \subseteq V$ of nodes is called a Tutte set if every maximum matching $\mathcal{M}$ of $G$ can be decomposed as

$$\mathcal{M} = \mathcal{M}_D \cup \mathcal{M}_B \cup \mathcal{M}_{A,D},$$

where the set $\mathcal{M}_D$ contains a nearly perfect matching in each (odd) component of $G \setminus A$; the set $\mathcal{M}_B$ contains a perfect matching in each (even) component of $G \setminus A$; and $\mathcal{M}_{A,D}$ is a matching that matches every node in $A$ to a node in some odd component of $G \setminus A$.

The decomposition in the above definition is called Gallai-Edmonds decomposition (see figure:tte). Note that all the components in this decomposition are disjoint from each other in the induced graph $G \setminus A$. A nice property of Tutte set is that one can be computed in polynomial time, according to the following result.

**Lemma 4.2** (Gallai-Edmonds structure theorem [34, 17]). Given a graph $G = (V, E)$, a Tutte set $A$ on $G$ can be constructed in $O(n^3)$ time.

We will also make use of the Hall’s theorem presented below.

![Figure 2: An illustration of Gallai-Edmonds decomposition. The solid lines form a maximum matching and dotted lines are edges not included in this matching. Here $G^\text{odd}_{A^-}$ contains two odd components, which contain 1 and 3 nodes, respectively, and $G^\text{even}_{A^-}$ contains two even components, which contain 2 and 4 nodes, respectively.](image)
Lemma 4.3 (Hall’s theorem [27]). For any bipartite graph $G = (L, R; E)$ with node sets $L$ and $R$, and edge set $E$ such that $|L| \leq |R|$, there exists a matching with size $|L|$ if and only if for every subset $S \subseteq L$, it holds that $|S| \leq |N(S)|$, where

$$N(S) = \{i \in R \mid \{i, j\} \in E \text{ for some } j \in S\}$$

denotes the neighbourhood of $S$, i.e., the set of all nodes in $R$ adjacent to some element of $S$.

When the dorm capacity is 2, PEF essentially requires that at least one of the following two situations holds for every agent $i$: (1) agent $i$ shares an arbitrary dorm with one of her friends; or (2) agent $i$ (weakly) prefers the dorm she gets to the dorms her friends get (otherwise, $i$ would envy the agent who shares a dorm with her friend). Thus, one possible approach to find a PEF assignment is to compute a maximum matching in the externality graph. If all the agents are covered by this matching, then we pair them up according to this matching and assign each pair to an arbitrary dorm, by which the first situation will hold for all the agents. However, if this maximum matching does not cover all agents, we need to make sure that every unpaired agent gets a better dorm than all their friends do. To this end, we make use of the Gallai-Edmonds decomposition. The idea is to leave dorms that are “bad” for the unpaired agents to the paired agents. We present Algorithm 1 which computes a PEF assignment for any given instance with dorm capacity 2. The correctness of this algorithm is shown via Lemmas 4.4 and 4.5.

Lemma 4.4. Algorithm 1 always terminates with an assignment of $N$ to $M$.

Proof. It suffices to show that the while-loop at Step 4 always terminates. We first argue that the following inequalities hold throughout Step 4:

$$|S| \leq |N(S)|, \quad \text{for all } S \subseteq A,$$

where $N(S)$ denotes the neighborhood of $S$ on the bipartite graph $G^*$.

Indeed, since $A$ is a Tutte set and $M$ is a maximum matching, according to Definition 4.1, $M$ matches each node in $A$ to a node in $L$. Thus, (1) holds before the while-loop is executed. We will argue that:

(i) If (1) holds at the beginning of some round, at least one of the conditions that define Cases 1–4 must be true;

(ii) Moreover, no matter which case is selected, the algorithm will proceed as described, and (1) will hold at the end of that round as long as it holds at the beginning of it.

By induction, this will imply that the while-loop will continue as long as $A \cup L \neq \emptyset$. Since at least one pair of agents is assigned to a dorm in each of the four cases, it follows that the while-loop will indeed end with $A \cup L = \emptyset$, whereby every agent is assigned to some dorm.

Indeed, given that (1) holds, at least one of the conditions defining Cases 2–3 must be true, so (i) is obvious and it remains to show (ii). We consider the case selected by the algorithm.

- If it is Case 1, since no agent in $A$ is a neighbor of $i$ or $i'$, the neighborhood $N(S)$ of each subset $S \subseteq A$ will not change after $i$ and $i'$ are removed from $G'$, and (1) will still hold.

- If it is Case 2, first, we need to argue that we can find a perfect matching as described in the algorithm. Indeed, given (1), by the Hall’s theorem (Lemma 4.3), there exists a matching of size $|S|$, which is a perfect matching between $S$ and $N(S)$ given that $|S| = |N(S)|$ in this case.
Algorithm 1: Find a PEF assignment when dorms have capacity 2.

**Input**: A dorm assignment instance $I = (N, M, v, e, c)$ with $c_i = 2$ for all $i \in N$.

**Output**: An assignment of agents in $N$ to dorms in $M$.

0. Initialization:

Let $G = (N, E)$ be the externality graph of $I$;

Let $\tilde{M}$ denote the set of unassigned dorms throughout (initially, $\tilde{M} = M$).

1. Compute a Tutte set $A \subseteq N$ of $G$, and a maximum matching $M$.

2. for each even component $X$ in $G \setminus A$ do
   
   Since $A$ is a Tutte set, $M$ contains a perfect matching for $X$. Assign each matched pair in $X$ to an arbitrary dorm in $\tilde{M}$.

3. Let $L = \emptyset$.

   for each odd component $X$ in $G \setminus A$ do
   
   Since $A$ is a Tutte set, $M$ contains a nearly perfect matching for $X$. Let $i$ be the unmatched agent in $X$ and add it into $L$. Assign each matched pair in $X$ to one of the $|X| - 1$ least preferred dorms of agent $i$ in $\tilde{M}$.

4. Let $G^* = (A, L; E^*)$ be the bipartite graph between $A$ and $L$, such that $(a, l) \in E^*$ if and only if $a \in A$, $l \in L$, and $(a, l) \in E$. For every $S \subseteq A$, let $\mathcal{N}(S)$ denote the neighborhood of $S$ on $G^*$.

   while $A \cup L \neq \emptyset$ do
   
   if (Case 1) there exists a pair of agents $i, i' \in L$, such that $\{i, i'\} \cap e = \emptyset$ for all $e \in E^*$ then
   
   Assign $i$ and $i'$ to an arbitrary dorm in $\tilde{M}$.

   else if (Case 2) $|S| = |\mathcal{N}(S)|$ for some (nonempty) $S \subseteq A$ then
   
   Find a perfect matching $\mathcal{M}'$ between $S$ and $\mathcal{N}(S)$. Assign each matched pair in $\mathcal{M}'$ to an arbitrary dorm in $\tilde{M}$.

   else if (Case 3) $|S| = |\mathcal{N}(S)| - 1$ for some $S \subseteq A$ then
   
   Find a nearly perfect matching $\mathcal{M}'$ between $S$ and $\mathcal{N}(S)$, in which all nodes in $S$ are covered. Let $i \in \mathcal{N}(S)$ be the unmatched agent in $\mathcal{M}'$, and assign each matched pair in $\mathcal{M}'$ to one of the $|S|$ least preferred dorms of agent $i$ in $\tilde{M}$.

   else if (Case 4) $|S| \leq |\mathcal{N}(S)| - 2$ for all (nonempty) $S \subseteq A$ then
   
   Find a pair of agents $i, i' \in L$ who have the same most preferred dorm in $\tilde{M}$ and assign them to this most preferred dorm.

   Remove all the assigned agents from $A$ and $L$, and remove their adjacent edges from $G^*$.
To see that (I) will still hold at the end of this round, suppose for the sake of contradiction that it is violated after agents in \( S \) and \( \mathcal{N}(S) \) (and the adjacent edges) are removed from \( G^* \). In other words, we have \(|Q| > |\mathcal{N}(Q) \setminus \mathcal{N}(S)|\) for some \( Q \subseteq A \setminus S \) (where \( \mathcal{N}(Q) \) denotes the neighborhood of \( Q \) before the removal of \( \mathcal{N}(S) \)). It follows that

\[
|Q \cup S| = |Q| + |S| > |\mathcal{N}(Q) \setminus \mathcal{N}(S)| + |\mathcal{N}(S)|
= |\mathcal{N}(Q) \cup \mathcal{N}(S)|
= |\mathcal{N}(Q \cup S)|.
\]

Since \( Q \cup S \) is a subset of \( A \), this means that (I) does not hold even before \( S \) and \( \mathcal{N}(S) \) are removed from \( G^* \), which contradicts our assumption.

- If it is Case 3, then similarly to Case 2, by the Hall’s theorem and the assumption that (I) holds at the beginning of this round, there exists a matching of size \(|S|\) between \( S \) and \( \mathcal{N}(S) \). Suppose for the sake of contradiction that (I) breaks after the removal of the assigned agents (in this case, these are agents in \( S \) and \( \mathcal{N}(S) \setminus \{i\} \)). We have

\[
|Q| > |\mathcal{N}(Q) \setminus (\mathcal{N}(S) \setminus \{i\})|
\]

for some \( Q \subseteq A \setminus S \). Since we also have \(|S| = |\mathcal{N}(S)| - 1\) in Case 3, it follows that

\[
|Q \cup S| = |Q| + |S| > |\mathcal{N}(Q) \setminus (\mathcal{N}(S) \setminus \{i\})| + |\mathcal{N}(S)| - 1
= |\mathcal{N}(Q) \setminus (\mathcal{N}(S) \setminus \{i\})| + |\mathcal{N}(S) \setminus \{i\}|
= |\mathcal{N}(Q) \cup (\mathcal{N}(S) \setminus \{i\})|
\geq |\mathcal{N}(Q \cup S)| - 1.
\]

Since sizes of sets are integers, this means that \(|Q \cup S| \geq |\mathcal{N}(Q \cup S)|\). However, the fact that the algorithm selected Case 3, instead of Case 2, means that the condition defining Case 2 does not hold; namely, we have \(|X| < |\mathcal{N}(X)|\) for all subsets \( X \subseteq A \). This contradicts the above inequality since \( Q \cup S \) is a subset of \( A \).

- If it is Case 4, we need to argue first that we can indeed find a pair of agents who have the same most preferred dorm. By the condition defining Case 4, we have \(|A| \leq |\mathcal{N}(A)| - 2 \leq |L| - 2\), which means that

\[
|L| \geq \frac{|A| + |L|}{2} + 1 = |\mathcal{M}| + 1.
\]

Thus, there is one more agents in \( L \) than the number of unassigned dorms and by the pigeonhole principle there exist two agents who prefer the same unassigned dorm the most. The removal of these two agents reduces \(|\mathcal{N}(S)|\) by at most 2 for all \( S \subseteq A \), so given that in this case we have \(|S| \leq |\mathcal{N}(S)| - 2\) for all \( S \subseteq A \) at the beginning of the round, (I) will still hold after the removal of \( i \) and \( i' \).

This completes the proof.

\[ \square \]

**Lemma 4.5.** The assignment Algorithm \( \Pi \) generates is PEF.

**Proof.** Apparently, the assignment is PEF for all the agents assigned as a matched pair, each of whom shares their dorm with a friend in the assignment. Observe that those who are not assigned as a pair only appear in \( L \), so it suffices to show that the assignment is PEF for all the agents in \( L \), who are assigned only in Step 4 of the algorithm. Moreover, since each agent \( i \in L \) comes
from a unique component in $G \setminus A$, they would only Pareto-envy agents in $A$ or agents in the same component with them. They way the other agents in each odd component are assigned in Step 3 ensures that these agents do not get a better dorm than $i$ does, so $i$ will not Pareto-envy them. Thus, in what follows we only need to argue that $i$ does not Pareto-envy any agent in $A$.

Consider an arbitrary agent $i^* \in A$, and suppose for the sake of contradiction that some agent $\ell \in L$ Pareto-envies $i^*$. Observe that $i^*$ is assigned a dorm in some round where the algorithm proceeds with Case 2 or 3.

- If it is Case 2, then apparently, $\ell$ cannot be assigned in the same round when $i^*$ is assigned, as all those assigned in Case 2 share dorms with their friends. Now if $\ell$ has been assigned in some previous round, then either $\ell$ is not connected to $A$ (i.e., Case 1), or $\ell$ gets a dorm which she prefers to the dorm agent $i^*$ gets (i.e., Case 4), so $\ell$ does not Pareto-envy $i^*$. Thus, $\ell$ must be assigned in a subsequent round, which however means that $\ell$ is not a neighbor of $i^*$ (all the remaining neighbors of $i^*$ will be assigned in the same round with $i^*$), so she does not Pareto-envy $i^*$, either.

- If it is Case 3, we further consider two possibilities. First, if $\ell$ is not the unmatched agent $i \in \mathcal{N}(A')$ that is selected to assign the other agents, then the same argument above applies, and $\ell$ does not Pareto-envy $i^*$. On the other hand, if $\ell$ is indeed the agent selected to assign the other agents, since her least preferred dorms are assigned in that round, she will only get a better dorm in a subsequent round; consequently, she will not Pareto-envy $i^*$, either.

Thus, no agent $\ell \in L$ will Pareto-envy any agent $i^* \in A$. The assignment is PEF and this completes the proof.

In fact, Algorithm 1 can be implemented in polynomial time. The key to this implementation is to find an efficient way to determine whether the conditions defining Cases 2 and 3 are true or not, and to compute a subset $S$ satisfying these conditions. We demonstrate how this can be done and summarize our results as the key theorem below.

**Theorem 4.6.** Given any dorm assignment instance with capacity 2 for all dorms, a PEF assignment always exists and can be computed in polynomial time.

**Proof.** Given Lemmas 4.4 and 4.5, we show that there is a way to implement Algorithm 1 in polynomial time to complete this proof. In what follows, we will use the same notation in Algorithm 1. Thus, $G^*$ is the bipartite graph constructed in Step 4 of Algorithm 1 and for each $S \subseteq A$, we denote the neighborhood of $S$ on $G^*$ by $\mathcal{N}(S)$.

Consider each step of Algorithm 1. In Step 1, a Tutte set and a maximum matching can be computed in polynomial time by Lemma 4.2. Following that, Steps 2 and 3 trivially run in polynomial time. Thus, it suffices to argue that Step 4 can be implemented in polynomial time.

Indeed, in the while-loop at Step 4, if we have determined which case to proceed with, the subsequent procedure for each case can be implemented efficiently. Specifically, for Cases 1, the assignment procedure is trivial. For Cases 2 and 3, given $S$, to find a perfect or nearly perfect matching it suffices to compute a maximum matching, which can be done in polynomial time as we already know. For Case 4, to find the pair $\{i, i'\}$, we can enumerate all the $O(n^2)$ agent pairs, and the subsequent assignment procedure is trivial, too. Therefore, we only need to show that we can efficiently determine whether the conditions defining Cases 1–3 are true or not. When none of them hold, the condition defining Case 4 must be true as we have argued in the proof of Lemma 4.4.

To check if the condition defining Case 1 is true, we can simply enumerate all the agent pairs in $L$, which takes time $O(n^2)$. To check if the condition defining Case 2 is true, we enumerate every
\( \ell \in L \) and apply the following procedure, which attempts to generate a (nonempty) set \( S \subseteq A \) such that \( |S| = |\mathcal{N}(S)| \).

- We first compute a maximum matching \( \mathcal{M} \) on \( G^* \). Since (1) holds as we have shown in the proof of Lemma 4.4, by the Hall’s theorem, \( \mathcal{M} \) covers every agent in \( A \). For every agent \( i \in A \cup L \), we let
  \[
  \mathcal{M}(i) = \{ j \in A \cup L : j \text{ is matched to } i \text{ in } \mathcal{M} \}.
  \]

Note that \( \mathcal{M}(i) \) is either a singleton or an empty set.

- Let \( S = \mathcal{M}(\ell) \).

- If \( |S| = |\mathcal{N}(S)| \), then we are done with a desired subset \( S \), and we terminate this procedure.

- Otherwise, for each \( i \in \mathcal{N}(S) \), we add the agent in \( \mathcal{M}(i) \) into \( S \) if it is not in \( S \) yet, and repeat this step, until no new agent can be added into \( S \) in this way (i.e., when \( \mathcal{M}(i) \subseteq S \) for all \( i \in \mathcal{N}(S) \)).

Apparently, the above procedure finishes in polynomial time. We show next that, if there exists some nonempty \( S \subseteq A \) with \( |S| = |\mathcal{N}(S)| \), then the above procedure will successfully generate such an \( S \) for some \( \ell \in L \).

Since \( S \) is nonempty and \( |S| = |\mathcal{N}(S)| \), we have \( \mathcal{N}(S) \neq \emptyset \). Hence, for some \( \ell \in L \), we have \( \ell \in \mathcal{N}(S) \). The following claim then implies that the agent matched with \( \ell \) in \( \mathcal{M} \) must also be in \( S \). Indeed, \( S \) is initialized to be \( \mathcal{M}(\ell) \) in the above procedure, and the next step gradually expands \( S \) by adding into it necessary elements implied by the following claim.

**Claim 1.** If \( |S| = |\mathcal{N}(S)| \), then it holds that \( \mathcal{M}(i) \subseteq S \) for every \( i \in \mathcal{N}(S) \).

Note that to prove the above claim, it suffices to show that \( |S| = |Q| \), where \( Q = \{ i \in \mathcal{N}(S) : \mathcal{M}(i) \subseteq S \text{ and } \mathcal{M}(i) \neq \emptyset \} \). Indeed, since every \( i \in Q \) is matched in \( \mathcal{M} \) with a unique agent in \( S \), we have \( |S| \geq |Q| \). On the other hand, since \( \mathcal{M} \) covers all agents in \( S \), every \( i \in S \) is matched with a unique agent in \( \mathcal{N}(S) \), which must also be in \( Q \), so we also have \( |S| \leq |Q| \). Hence, \( |S| = |Q| \). Immediately, this also implies the following claim, which we will use to design a similar procedure to check if the condition defining Case 3 is true.

**Claim 2.** If \( |S| = |\mathcal{N}(S)| - 1 \), then there exists \( i^* \in \mathcal{N}(S) \) such that \( \mathcal{M}(i^*) \cap S = \emptyset \), but \( \mathcal{M}(i) \subseteq S \) for every \( i \in \mathcal{N}(S) \) \( \setminus \{ i^* \} \).

To check the condition defining Case 3, we enumerate every agent pairs \( \{ \ell, \ell^* \} \subseteq L \) (such that \( \ell \neq \ell^* \) ) and apply the following procedure, which attempts to generate a set \( S \subseteq A \) such that \( |S| = |\mathcal{N}(S)| - 1 \).

- Compute a maximum matching \( \mathcal{M} \) on \( G^* \).

- Let \( S' = \{ \mathcal{M}(\ell) \} \).

- Let \( S = S' \cup \{ \ell^* \} \). If \( |S| = |\mathcal{N}(S)| - 1 \), then we are done with a desired subset \( S \), and we terminate this procedure. Otherwise, for all \( i \in \mathcal{N}(S') \), we add \( \mathcal{M}(i) \) into \( S' \) if it is not in \( S' \) yet, and repeat this step, until no new agent can be added into \( S' \) in this way.

If there exists a satisfying \( S \), then at some point during the enumeration of agent pairs, we will pick a pair \( \{ \ell, \ell^* \} \) such that \( \{ \ell, \ell^* \} \subseteq \mathcal{N}(S), \mathcal{M}(\ell^*) \notin S \), and \( \mathcal{M}(\ell) \in S \). Thus, by Claim 1 the above procedure will correctly generate \( S \).

In summary, Algorithm [c] generates a PEF assignment for any given instance with capacity 2 for all dorms, and it can be implemented in polynomial time. This completes the proof. \( \square \)
Impossibility for $c \geq 3$. Unfortunately, if the capacity of the dorms increases to 3, a PEF assignment may not exist. We demonstrate this via the following example.

**Example 4.7.** There are 3 dorms and 9 agents. The agents’ external values are defined by the graph in Figure 3, namely, for each $i \in \{1, 3, 5, 7\}$, agents $i$ and $i + 1$ are friends with each other. Every agent has value $j$ for each dorm $j \in \{1, 2, 3\}$.

Suppose for the sake of contradiction that there is a PEF assignment. Then for any agent $i \neq 9$ who is assigned to dorm 3, the friend of agent $i$ has to be assigned to dorm 3 as well; otherwise, this friend will Pareto-envy the other agents in dorm 3. Since the capacity of dorm 3 is 3, after assigning agent $i$ and her friend in this dorm, we can only assign agent 9 to fill up the dorm as every other agent has a friend. Thus, the other three pairs of agents are assigned to dorms 1 and 2, which means that at least one pair of friends must be assigned to two different dorms. As a result, for this pair of friends, the one in dorm 1 will Pareto-envy the other agents in dorm 2 because they share a better dorm with her friend. Therefore, no PEF assignment exists for this instance.

5 Implications on Proportionality

We have investigated the EF notion in the previous sections. In this section, we shift our focus to another extensively studied notion in the literature of fair division, proportionality (PROP), and study the implication of EF on this notion. In the classic resource allocation setting (where agents are assigned disjoint sets of resources), a proportional assignment is one in which every agent gets their fair share of at least $1/n$ of the total value of the resources to be assigned (according to their own valuation). This notion does not directly apply to our model because the agents also have external values for each other in addition to their values for the resources. Nevertheless, in the same spirit, we can define the agent’s “average” external value, which is the product of their average value for the other agents, i.e., $\frac{1}{n-1} \sum_{\ell \in N} e_{i\ell}$ (note that $e_{ii} = 0$), and the average number of agents sharing the same resource with this agent, i.e., $\frac{1}{m} \sum_{j \in M} c_j - 1 = \frac{n}{m} - 1$. With this in hand, we defined the PROP notion for our model below.

**Definition 5.1** (PROP assignment). An assignment $X$ is proportional (PROP) if $u_i(X) \geq PROP_i$ for every agent $i \in N$, where

$$PROP_i = \frac{1}{m} \sum_{j \in M} v_{ij} + \left( \frac{n}{m} - 1 \right) \cdot \left( \frac{1}{n-1} \sum_{\ell \in N} e_{i\ell} \right).$$

5.1 The Relationship between EF and PROP

It is not hard to see that a PROP assignment does not have to be EF. This is true even in the special setting where the agents have no external values and each resource has capacity one. As a simple example, when there are three agents, if agent 1 has values $v_{11} = 1$, $v_{12} = 0$, and $v_{13} = 2$ for the resources while agents 2 and 3 have value 1 for every resource, assigning each resource $i$ to agent
$i$ is a PROP assignment but it is not EF for agent 1. On the other hand, it is well known that in the classic resource allocation setting, envy-freeness implies proportionality. But unfortunately this is not the case for our model; the following example shows that an EF assignment may not provide any guarantee for proportionality if the capacity of some resource is one.

**Example 5.2.** Let there be 2 resources and 4 agents. Let the capacities of the two resources be $c_1 = 1$ and $c_2 = 3$. Suppose that agent 1 has value 0 for both resources, and external value 0 for agents 2 and 3, and external value $T \gg 0$ for agent 4. In addition, all the other three agents have value 0 for both resources and external value 0 for every agent, so any assignment will be EF for these three agents.

It follows that assigning agent 4 to resource 1 and agents 1, 2, 3 to resource 2 is an EF assignment. (Note that the utility of agent 1 will still be 0 if we exchange this agent with agent 4.) However, agent 1 gets utility 0 from this assignment, whereas PROP$_1 = T/3 \gg 0$ by (2). In other words, the utility agent 1 gets in this EF assignment does not even provide any approximation guarantee to proportionality.

Nevertheless, if all resources have capacities at least 2, we are able to show that every EF assignment $X$ is approximately PROP for a multiplicative factor of \(1 - \frac{1}{\min_{j \in M} c_j} \geq \frac{1}{2}\), i.e., \(u_i(X) \geq \alpha \cdot \text{PROP}_i\) for every agent $i \in N$. We will refer to an assignment $X$ with $u_i(X) \geq \alpha \cdot \text{PROP}_i$ for all $i \in N$ as an $\alpha$-approximate PROP assignment.

**Proposition 5.3.** When $c_j \geq 2$ for all $j \in M$, any EF assignment is also a \(\left(1 - \frac{1}{c_{\min}}\right)\)-approximate PROP assignment, where $c_{\min} = \min_{j \in M} c_j$.

**Proof.** Suppose that $X$ is an EF assignment. Consider an arbitrary agent $i$ and let $r \in M$ be the resource assigned to $i$. Thus,

\[
u_i(X) = v_{ir} + \sum_{j \in X_r} e_{ij}.
\]

(3)

Since $X$ is EF, for any resource $j \in M \setminus \{r\}$ and any agent $i' \in X_j$, we have

\[
u_i(X) \geq \nu_i\left(X^{i+i'}\right) = v_{ij} + \sum_{\ell \in X_j \setminus \{i'\}} e_{i\ell}.
\]

(4)

Summing up (4) for all $i' \in X_j$ and dividing both sides of the inequality by $c_j$ gives

\[
u_i(X) \geq v_{ij} + \left(1 - \frac{1}{c_j}\right) \sum_{\ell \in X_j} e_{i\ell} \geq v_{ij} + \left(1 - \frac{1}{c_{\min}}\right) \sum_{\ell \in X_j} e_{i\ell}.
\]

(5)

Now summing up (3) and (5) for all resource $j \neq r$ and dividing both sides of the inequality by $m$ gives

\[
u_i(X) \geq \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{m} \left(1 - \frac{1}{c_{\min}}\right) \sum_{j \in M} \sum_{\ell \in X_j} e_{i\ell}.
\]
Therefore, by definition of PROP assignment, we have

\[ \text{PROP}_i = \frac{1}{m} \sum_{j \in M} v_{ij} + \left( \frac{n}{m} - 1 \right) \cdot \left( \frac{1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right) \]

\[ \leq \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{m} \sum_{\ell \in N} e_{i\ell} \]

\[ \leq \frac{c_{\min}}{c_{\min} - 1} \cdot \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{m} \sum_{\ell \in N} e_{i\ell} \]

\[ \leq \frac{c_{\min}}{c_{\min} - 1} \cdot u_i(X), \]

where the first inequality holds because \( n = \sum_{j \in M} c_j \geq 2m \) given that \( c_j \geq 2 \) for all \( j \). In other words, \( u_i(X) \geq \left( 1 - \frac{1}{c_{\min}} \right) \text{PROP}_i \) and this holds for all \( i \in N \), so \( X \) is a \( \left( 1 - \frac{1}{c_{\min}} \right) \)-approximate PROP assignment.

The lower bound in Proposition 5.3 is essentially tight because of the following example.

**Example 5.4.** Let \( c \geq 2 \) be an arbitrary integer. Suppose that there are \( m \) resources, which need to be assigned to \( n = (c - 1)m^2 + cm - c + 1 \) agents. The capacity of the first resource is \( c_1 = (c - 1)m^2 + 1 \) and the capacity of any other resource \( j > 1 \) is \( c_j = c \). Note that \( \sum_{j=1}^{m} c_j = (c - 1)m^2 + 1 + c(m - 1) = n \). Let \( X = (X_1, \ldots, X_m) \) be an allocation where agent 1 is assigned to resource 1, i.e., \( 1 \in X_1 \). Let the external values of agent 1 be \( e_{1\ell} = 1 \) if \( \ell \in X_1 \), and \( e_{1\ell} = m^2 \) otherwise. Assume that agent 1 has value 0 for all the resources.

Thus,

\[ u_1(X) = \sum_{\ell \in X_1} e_{1\ell} = (c - 1)m^2, \]

and exchanging agent 1 with any agent assigned to a resource \( j > 1 \) does not increase the utility of agent 1, so \( X \) is EF. Let us compute PROP_i. By construction of this instance, we have

\[ \frac{n}{m} - 1 = \frac{(c - 1)m^2 + 1 + c(m - 1)}{m} - 1 = \frac{(c - 1)m^2 + (c - 1)m - c + 1}{m}, \]

and

\[ \sum_{\ell \in \mathcal{N}} e_{1\ell} = (c - 1)m^2 + cm^2 \cdot (m - 1) = (cm - 1)m^2. \]

Hence,

\[ \text{PROP}_i = \left( \frac{n}{m} - 1 \right) \cdot \left( \frac{1}{n - 1} \sum_{\ell \in \mathcal{N}} e_{i\ell} \right) \]

\[ = \left( \frac{(c - 1)m^2 + (c - 1)m - c + 1 \cdot (cm - 1)}{(c - 1)m^2 + cm - c} \right) \cdot \text{PROP}_i. \]

When \( m/c \) is sufficiently large, for any constant \( \delta > 0 \), we have

\[ u_1(X) = \frac{((c - 1)m^2 + cm - c) \cdot (c - 1)m^2}{((c - 1)m^2 + (c - 1)m - c + 1) \cdot (cm - 1) \cdot m} \cdot \text{PROP}_i \]

\[ < \left( 1 - \frac{1}{c} + \delta \right) \cdot \text{PROP}_i. \]

The assignment \( X \) cannot be a \( \left( 1 - \frac{1}{c} + \delta \right) \)-approximate PROP assignment. \( \square \)
When all the resources have the same capacity \( c \geq 2 \) (e.g., in the dorm sharing model), we are able to improve the bound of the approximation ratio in Proposition 5.3 to \( 1 - \frac{1}{n} \) according to the following result. Note that \( 1 - \frac{1}{n} \geq 1 - \frac{m}{n} = 1 - \frac{1}{c_{\min}} \).

**Proposition 5.5.** When \( c_j = c \geq 2 \) for all resources \( j \in M \), any EF assignment is also a \((1 - \frac{1}{n})\)-approximate PROP assignment.

**Proof.** The proof follows the same idea of Proposition 5.3 so we only show the differences below. Let \( X \) be an arbitrary envy-free allocation, and \( i \) be an arbitrary agent. Similar to 5.3, we have

\[
u_i(X) \geq v_{ij} + \sum_{\ell \in X_j} e_{i\ell} - \frac{1}{c} \sum_{\ell \in X_j} e_{i\ell} \geq v_{ij} + \frac{c - 1}{c} \sum_{\ell \in X_j} e_{i\ell}.
\]

Summing up the above inequalities over all \( j \in M \) and dividing the inequality by \( m \) gives

\[
u_i(X) \geq \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{n} \sum_{j \in M} \sum_{\ell \in X_j} e_{i\ell} = \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{n} \sum_{\ell \in N} e_{i\ell}
\]

\[
= \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{c - 1}{n} \sum_{\ell \in N} e_{i\ell}
\]

\[
\geq \frac{n - 1}{n} \left( \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{c - 1}{n} \sum_{\ell \in N} e_{i\ell} \right) = \frac{n - 1}{n} \cdot \text{PROP}_i.
\]

Hence, assignment \( X \) is \((1 - \frac{1}{n})\)-approximate PROP.

Again, the above bound is essentially tight because of the following example.

**Example 5.6.** Suppose that there are \( m \) resources, which need to be assigned to \( n = c \cdot m \) agents, where \( c \geq 2 \) is the capacity of every resource. Let \( X = (X_1, \cdots, X_m) \) be an allocation where agent 1 is assigned to resource 1, i.e., \( 1 \in X_i \). Let the value of agent 1 for resource 1 be \( v_{11} = c - 1 \) and for every other resource \( j \geq 1 \) be \( v_{ij} = 0 \). Let the external value of agent 1 be \( e_{i\ell} = 0 \) for every \( \ell \in X_1 \), and \( e_{i\ell} = 1 \) for every other agent \( \ell \notin X_1 \).

Thus, \( u_1(X) = v_{11} = c - 1 \), and exchanging agent 1 with any agent in a resource \( j > 1 \), will not increase the utility of agent 1; \( X \) is envy-free. Let us compute \( \text{PROP}_i \). By construction of the instance, we have

\[
\text{PROP}_i = \frac{1}{m} \sum_{j \in M} v_{ij} + \left( \frac{n}{m} - 1 \right) \cdot \left( \frac{1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right)
\]

\[
= \frac{c - 1}{m} + \left( c - 1 \right) \cdot \frac{c(m - 1)}{cm - 1} = u_1(X) \cdot \left( \frac{1}{m} + \frac{c(m - 1)}{cm - 1} \right)
\]

\[
= u_1(X) \cdot \frac{n^2 - c}{n^2 - n}.
\]

Hence, when \( n/c \) is sufficiently large, for any constant \( \delta > 0 \), we have

\[
u_1(X) = \frac{n^2 - n}{n^2 - c} \cdot \text{PROP}_i = \left( 1 - \frac{n - c}{n^2 - c} \right) \cdot \text{PROP}_i < \left( 1 - \frac{1}{n} + \delta \right) \cdot \text{PROP}_i.
\]

The assignment \( X \) cannot be a \((1 - \frac{1}{n} + \delta)\)-approximate PROP assignment. \( \square \)
5.2 The Relationship between PEF and PROP

If we modify the average measures of the internal and external values in (2), we are able to obtain Proposition 5.7 for two special settings of the dorm sharing model: essentially, PEF implies a property that resembles the Pareto frontier of the PROP notion, where for every agent at least one of their internal and the external values is PROP. In Proposition 5.7, the first condition means that the resource agent \( i \) gets is at least as valuable as half of the resources, which can be seen as proportionality defined on ordinal preferences: if we assign value \( k \) to the resource at the \( k \)-th position in the agent’s ordinal preference (with the most preferred resource at the first position), then this condition results in the agent getting at least \( 1/n \) of the total value. The second condition is a direct modification of the external value in (2) that rounds up the value to the nearest integer. We denote by \( \lfloor x \rfloor \) the nearest integer of a number \( x \), i.e., \( \lfloor x \rfloor = \lceil x \rceil \) if \( x \geq \lfloor x \rfloor + 1/2 \), and \( \lfloor x \rfloor = [x] \), otherwise.

**Proposition 5.7.** For any dorm sharing instance with \( c = 2 \) or \( m = 2 \), if an assignment \( X \) is PEF, then \( X \) satisfies at least one of the following conditions:

1. \( \left| \{ j \in M : v_{ir_j}(X) \geq v_{ij} \} \right| \geq \frac{1}{2} m \); or
2. \( e_i(X) \geq \left\lfloor \frac{c-1}{n-1} \sum_{\ell \in N} e_{i\ell} \right\rfloor \).

**Proof.** We first consider the case where \( c = 2 \) (and \( m \) is arbitrary). In this case \( m = n/2 \). Consider an arbitrary agent \( i \) and the following two possibilities with respect to the value \( \sum_{\ell \in N} e_{i\ell} \).

- If \( \sum_{\ell \in N} e_{i\ell} < m \), then we have
  \[
  \left\lfloor \frac{c-1}{n-1} \sum_{\ell \in N} e_{i\ell} \right\rfloor \leq \left\lfloor \frac{1}{n-1} \cdot \left( \frac{n}{2} - 1 \right) \right\rfloor = 0.
  \]
  Thus the second condition holds for agent \( i \).

- If \( m \leq \sum_{\ell \in N} e_{i\ell} \leq 2m - 1 \), then we have
  \[
  \left\lfloor \frac{c-1}{n-1} \sum_{\ell \in N} e_{i\ell} \right\rfloor = 1.
  \]
  Hence, as long as agent \( i \) shares a resource with a friend of hers, the second condition will hold. On the other hand, if agent \( i \) does not have a friend in \( X \), then given that \( \sum_{\ell \in N} e_{i\ell} \geq m \) and now the agents have binary external values, the friends of agent \( i \) would occupy at least \( \left\lfloor \frac{n}{2} \right\rfloor \) resources in \( X \). Since assignment \( X \) is PEF and agent \( i \) now envies every agent who shares a dorm with one of her friends, the resource agent \( i \) gets must have a higher value for agent \( i \) than all those occupied by her friends; hence, the first condition holds for agent \( i \).

Next we consider the case where \( m = 2 \) (and \( c \) is arbitrary). In this case \( X = (X_1, X_2) \) and each resource is shared among \( n/2 \) agents. Consider an arbitrary agent \( i \) and without loss of generality we can assume that \( i \in X_1 \). Suppose that condition 1 does not hold for agent \( i \); since now there are only two resources, this means that \( v_{i1} < v_{i2} \). We show that condition 2 must hold in this case if

---

2This condition is not a weaker condition than proportionality defined on cardinal utilities. It can happen that an agent gets at least \( 1/n \) of the total value of the resources but the resource this agent gets is not ranked in the top 50% most valuable resources.
$X$ is PEF for agent $i$. Indeed, if $X$ is PEF for agent $i$ while $v_{i1} < v_{i2}$, it must be that agent $i$ does not envy any other agent for their external values, i.e, $e_i(X) \geq e_i(X^{i \leftrightarrow \ell})$ for all $\ell \in N$. Consider the following situations.

- If all the $\frac{m}{2}$ agents in $X_2$ are friends of agent $i$, then all the other $\frac{m}{2} - 1$ agents in $X_1$ must also be friends of agent $i$ as otherwise, agent $i$ would envy every agent in $X_2$ for their external value. This implies that $\sum_{\ell \in N} e_{i\ell} = n - 1$ and $e_i(X) = \frac{m}{2} - 1$; consequently,

$$\left\lfloor \frac{c - 1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right\rfloor = \left\lfloor \frac{n - 1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right\rfloor = \frac{n}{2} - 1 = e_i(X),$$

so the second condition holds.

- If some agent $j \in X_2$ is not a friend of agent $i$, then agent $i$ must have as many friend in $X_1$ as in $X_2$ (otherwise, $e_i(X) < e_i(X^{i \leftrightarrow j})$). It follows that

$$e_i(X) \geq \left\lfloor \frac{1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right\rfloor \geq \left\lfloor \frac{c - 1}{n - 1} \sum_{\ell \in N} e_{i\ell} \right\rfloor,$$

so the second condition holds, too.

Since the choice of agent $i$ is arbitrary, this completes the proof. \qed

We cannot hope to derive the same result beyond the above special settings. The following example shows that Proposition 5.7 does not hold when $c = 3$ and $m = 3$.

**Example 5.8.** Suppose that the instance is a dorm sharing instance with $c = 3$ and $m = 3$ (so there are $n = c \cdot m = 9$ agents). Agents 1, 2, 3, and 4 are friends with each other, agents 5 and 6 are friends with each other, and agents 7, 8, and 9 are friends with each other (see the externality graph in Figure 4). Suppose all the agents have the same value over the resources: with value 1 for resource 1, value 2 for resource 2, and value 3 for resource 3.

The assignment $X = (X_1, X_2, X_3)$ with $X_1 = \{1, 2, 3\}$, $X_2 = \{4, 5, 6\}$ and $X_3 = \{7, 8, 9\}$ is PEF: other than agent 4, every agent either gets their favorite resource, or share a resource with a friend of theirs; agent 4 on the other hand gets a better resource than all of her friends. However, $X$ is not PPROP for agent 4 as she did not get her top 1 resource and has external value 0. \qed

![Figure 4: The externality graph of Example 5.8](image)
6 Conclusion and Future Directions

In this paper we study a resource sharing problem with externalities, in which we consider both the agents’ values for the resources and their external values for other agents. We studied EF assignments in this model, the existence of such assignments, and the computation. In general, an EF assignment may not exist. Only in a special setting where all resources have the same capacity and all agents have dichotomous preferences over other agents, an EF assignment is guaranteed to exists under the notion of Pareto EF and can be computed efficiently. The existence guarantee is invalidated even when we slightly generalize this setting. We also investigated the implication of EF on proportionality and showed that EF implies approximations of PROP.

There are several interesting future directions of this work. In our work, we focused on the case of unit-demand agents so a natural extension is the case when agents demand multiple resources. It would also be interesting to adopt other ways to soften the strong requirements of envy-freeness, such as envy-free up to one or more items. One may also consider other popular fairness notions in the literature such as the maximin share fairness and extend these notions to the setting with externalities.

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