TOPOLOGICAL MODAL LOGICS WITH DIFFERENCE MODALITY

KUDINOV ANDREY

Abstract. We consider propositional modal logic with two modal operators □ and [≠]. In topological semantics □ is interpreted as an interior operator and [≠] as difference. We show that some important topological properties are expressible in this language. In addition, we present a few logics and proofs of f.m.p. and of completeness theorems.

1. Introduction.

This paper deals with the topological semantics of modal logic. The study of topological semantics of modal logic was started in 1944 by McKinsey and Tarski [12]. Recently, this topic has been attracting more attention partly due to applications in AI (cf. [2] and [8]). Reading the modal box as an interior operator one can easily show that logic of all topological spaces is S4. In addition, McKinsey and Tarski proved that S4 is also the complete logic of the reals, Cantor space and indeed of any metric separable space without isolated points (for a new proof of this fact see [1]). Therefore, all these spaces are modally equivalent, hence many natural properties of topological spaces such as connectedness, density-it-itself and T1 are undefinable. For more information on spatial logics and spatial reasoning see [1, 9, 15, 16].

There are two ways of enriching the definability of a language: to change semantics or to extend the language. According to the first way, Esakia in [6] and Shehtman in [16] considered the derivational logic (more recent paper on this [3]). According to the other way, we can add the universal modality. In this new language we can express connectedness (cf. [13]).

In this paper, however, we add difference modality (or modality of inequality) [≠], interpreted as “true everywhere except here”. Difference modality was suggested to use by several people independently (in [10] for one). More deeply this modality and its interpretation in Kripke frames were studied in [14]. It has been shown that difference modality increase greatly the expressive power of a language (cf. [11, 14]). The expressive power of this language in topological spaces has been studied by Gabelia in [9], the author presented axioms that defines T1 and T0 spaces. Being added to the topological modal logic, the difference modality allows us to express topological properties that were unreachable before. The topological properties mentioned in the end of the first paragraph became definable. The universal modality is expressible as well in the following way: ∀A = [≠]A ∧ A.

Here we also introduce three logics: S4D, S4DS, and S4DT1S. We prove their f.m.p. and following completeness theorems: S4D is complete with respect to all
topological spaces (Theorem 6.6), $S4DS$ is complete with respect to all dense-in-itself topological spaces (Theorem 6.8), and $S4DT_1S$ is complete with respect to any zero-dimensional dense-in-itself metric space (Theorem 6.11).

2. Definitions and basic notions.

Let us introduce some notations the reader will meet in this paper. Assume that $B$ is a set, $R, R' \subseteq B \times B$ are relations on $B$, then

$$R \mid_A = R \cap (A \times A), \text{ for any } A \subseteq B;$$

$$Id_B = \{(x, x) | x \in B\};$$

$$R^+ = R \cup Id_B;$$

$$R \circ R' = \{(x, z) | \exists y (xRy \& yR'z)\};$$

$$R^1 = R, R^n = R^{n-1} \circ R;$$

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

In this paper, we study propositional modal logics with two modal operators, $\Box$ and $[\neq]$. A formula is defined as follows:

$$\phi := p | \bot | \phi \rightarrow \phi | \Box \phi | [\neq] \phi.$$

The standard classic logic operators ($\lor, \land, \neg, \top, \equiv$) are expressed in terms of $\rightarrow$ and $\bot$. The dual modal operators $\Diamond, [\neq]$ are defined in the usual way as $\Diamond A = \neg \Box \neg A$, $[\neq] A = \neg [\neq] \neg A$ respectively. $[\forall] A$ stands for $[\neq] A \land A$.

**Definition 2.1.** A bimodal logic (or a logic, for short) is a set of modal formulas closed under Substitution ($A(p_i)$), Modus Ponens ($A, A \rightarrow B, B$) and two Generalization rules ($\Box A, [\neq] A$); containing all classic tautologies and the following axioms

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),$$

$$[\neq](p \rightarrow q) \rightarrow ([\neq]p \rightarrow [\neq]q).$$

$K_2$ denotes the minimal bimodal logic.

Let $L$ be a logic and let $\Gamma$ be a set of formulas, then $L + \Gamma$ denotes the minimal logic containing $L$ and $\Gamma$. If $\Gamma = \{A\}$, then we write $L + A$ rather then $L + \{A\}$.

In this paper, however, we consider a few additional axioms:

$$\Box A \rightarrow [\neq] p \rightarrow [\neq] [\neq] p$$

$$\Box [\neq] p \rightarrow [\neq] [\neq] p$$

$$\Box [\neq] p \rightarrow [\neq] [\neq] p$$

$$\Box [\neq] p \rightarrow [\neq] [\neq] p$$

The first two axioms are for $[\neq]$ and they are from the paper by de Rijke [14]. These axioms correspond to some basic properties of inequality: symmetry and pseudo-transitivity respectively.

The next two axioms are axioms for $S4$. These axioms have well-known correspondence to the properties of topological interior operator: $IY \subseteq Y$ and $IY \subseteq IIY$.

\[1\] In this paper relation $R$ is pseudo-transitive iff $R^+$ is transitive. In some papers this property calls weakly transitive (cf. [10, 11])
respectively (where \( Y \) is an arbitrary set). We denote the interior and the closure operators by \( I \) and \( C \) respectively.

Axiom \((D\Box)\) is needed to connect \( \Box \) and \([\neq]\) and to make sure that \([\forall]\) is the universal modality.

The meaning of the next two axioms will be explained later.

In this paper we study the following three logics:

\[
S4D = K_2 + \{B_D, 4\Box_D, D\Box, T\Box, 4\Box\}, \\
S4DS = S4D + DS, \\
S4DT_1 S = S4DS + AT_1.
\]

3. Topological models.

Definition 3.1. A topological model is a pair \((X, \theta)\), where \( X \) is a topological space and \( \theta \) is a function assigning to each proposition letter \( p \) a subset \( \theta(p) \) of \( X \). The function \( \theta \) is called a valuation.

Definition 3.2. The truth of a formula at a point of a topological model is defined by induction:

(i) \( X, \theta, x \models p \) iff \( x \in \theta(p) \)

(ii) \( X, \theta, x \models \bot \) iff \( X, \theta, x \models \phi \) or \( X, \theta, x \models \psi \)

(iii) \( X, \theta, x \models \Box \phi \) iff there is a neighborhood \( U \) of \( x \) such that for any \( y \in U \) \( X, \theta, y \models \phi \)

(iv) \( X, \theta, x \models [\neq] \phi \) iff \( X, \theta, y \models \phi \) for any \( y \neq x \)

If \( U \) is a subset of \( X \), then \( X, \theta, U \models A \) denotes that \( X, \theta, x \models A \) for any \( x \in U \). A formula \( A \) is called valid in a topological space \( X \) (notation: \( X \models A \)), if it is true at any point under any valuation. Also in notation \( X, \theta, x \models A \) we will omit the space and/or the valuation, if it is clear what space and/or valuation we consider.

Definition 3.3. The D-logic of a class of topological spaces \( T \) (in notation \( L_D(T) \)) is the set of all formulas that are valid in all topological spaces from \( T \).

Let us describe the classes of topological spaces axiomatized by \((AT_1),(DS)\).

Definition 3.4. A \( T_1 \)-space is a topological space such that all its one-element subsets are closed.

As we mentioned in introduction there is an axiom that defines \( T_1 \) spaces in [9], but it has a little bit different form. And due to the next lemma they are equivalent on topological spaces.

Lemma 3.5. Let \( X \) be a topological space then \( X \models AT_1 \) iff \( X \) is a \( T_1 \)-space.

Proof. \((\Rightarrow)\) Ad absurdum. Suppose there exists \( x \in X \) such that \( \{x\} \) is not closed. Hence \( X - \{x\} \neq I(X - \{x\}) \). Let \( U = X - \{x\} \). There exists

\[
y \in U - IU
\]

We take a valuation \( \theta \) in \( X \) such that \( \theta(p) = U \). Then \( x \models [\neq]p \) but \( x \models AT_1 \); hence \( x \models [\neq]\Box p \). Since \( y \neq x \), we have \( y \models \Box p \), which means that \( y \) together with some its neighborhood is in \( U \). This contradicts to (3.1).
(⇐) Assume that $\mathcal{X}$ is a $T_1$-space. Let $\mathcal{X}, \theta, x \models \lnot p$ then $\theta(p) \supseteq \mathcal{X} - \{x\}$. We need to prove that $x \models \lnot \Box p$. It means that for all $y \in \mathcal{X} - \{x\}$ $y \models \Box p$. Take any $y \in \mathcal{X} - \{x\}$. Since $\mathcal{X} - \{x\}$ is open, there exists an open $U \ni y$ and $U \subseteq \mathcal{X} - \{x\}$. So $U \models p$, then $y \models \Box p$, hence $x \models \lnot \Box p$.

Definition 3.6. Let $\mathcal{X}$ be a topological space. A point $x \in \mathcal{X}$ is called isolated, if $\{x\}$ is open. $\mathcal{X}$ is called dense-in-itself, if it has no isolated points.

Lemma 3.7. Let $\mathcal{X}$ be a topological space then $\mathcal{X} \models DS$ iff $\mathcal{X}$ is dense-in-itself.

Proof. ($\Rightarrow$) Ad absurdum. Assume that $\mathcal{X}$ is not dense-in-itself and $x \in \mathcal{X}$ is isolated.

Let us take a valuation $\theta$ in $\mathcal{X}$ such that $\theta(p) = \mathcal{X} - \{x\}$; then $x \models \lnot p$. Since $\{x\}$ is open and $x \models \lnot p$, it follows that $x \models \Box \lnot p$ or equivalently, $x \models \lnot \Box p$. This contradicts to the axiom $(DS)$.

(⇐) Assume that $\mathcal{X}$ is dense-in-itself and $(\mathcal{X}, \theta), x \models \lnot p$; then there are two cases:

(i) $\theta(p) = \mathcal{X}$, in this case it is obvious that $(\mathcal{X}, \theta), x \models \Diamond p$;
(ii) $\theta(p) = \mathcal{X} - \{x\}$, then $(\mathcal{X}, \theta), x \models \Diamond p$ since $x \in C(\mathcal{X} - \{x\}) = \mathcal{X}$.

4. Kripke frames and models.

Kripke frames and models are well-known basic notions of modal logic (cf. [4] and [5]).

Definition 4.1. A Kripke frame is a tuple $F = (W, R_1, \ldots, R_n)$ such that

(i) $W$ is a non-empty set,
(ii) $R_i$ for $i = 1 \ldots n$ are binary relations on $W$.

In this paper however, we consider Kripke frames with one or two relations only. The first is denoted as $R$ and the second (if it is present) — as $R_D$.

Definition 4.2. A Kripke model is a pair $\mathcal{M} = (F, \theta)$, where $F$ is a frame and $\theta$ is a valuation (a function from the set of all proposition letters to the set of all subsets of $W$).

$\mathcal{M}, x \models A$ denotes that formula $A$ is true in model $\mathcal{M}$ at point $x$; $\mathcal{M} \models A$ denotes that $A$ is true at all points of model $\mathcal{M}$; $F \models A$ denotes that $(F, \theta), x \models A$ for all valuations $\theta$ and all points $x \in W$; $F, x \models A$ denotes that $(F, \theta), x \models A$ for all valuations $\theta$. For a subset $U \subseteq W$ $\mathcal{M}, U \models A$ denotes that for any $x \in U$ $(\mathcal{M}, x \models A)$.

Definition 4.3. The logic of a class of frames $\mathcal{F}$ (in notation $L(\mathcal{F})$) is the set of all formulas that are valid in all frames from $\mathcal{F}$. For a single frame $F$, $L(F)$ stands for $L(\{F\})$.

Definition 4.4. A frame $F$ is called a $\Lambda$-frame for a modal logic $\Lambda$, if $\Lambda \subseteq L(F)$.

Definition 4.5. A p-morphism from a Kripke frame $F = (W, R, R_D)$ onto a Kripke frame $F' = (W', R', R'_D)$ is a map $f : W \rightarrow W'$ satisfying the following conditions:

1. $f$ is surjective;
2. $\forall x \forall y (xRy \Rightarrow f(x)R'f(y))$ and the same for $R_D$ and $R'_D$;
3. $\forall x \forall z (f(x)R'z \Rightarrow \exists y (xRy \land f(y) = z))$ and the same for $R_D$ and $R'_D$.
In notation: \( f : F \rightarrow F' \).

**Definition 4.6.** By cone \( F^x \) we will understand the frame

\[
(W^x, R_{W^x}, R_D_{|W^x}),
\]

where \( W^x = (R \cup R_D)^+ (x) \). If for some \( x \in F = F^x \) then \( F \) called rooted.

The following two lemmas are well-known (cf. [4] and [5]).

**Lemma 4.7.** Let \( F = (W, \ldots) \) be a Kripke frame, then

\[
L(F) = \bigcap \{ L(F^x) \mid x \in W \}.
\]

**Lemma 4.8.** (\( p \)-morphism Lemma) \( f : F \rightarrow F' \) implies \( L(F) \subseteq L(F') \).

In this paper we consider only \( S4D \)-frames. The axioms \( B_D, 4\_D, D\Box, T\Box, 4\Box \) put constraints on relations \( R \) and \( R_D \). So from now on we assume that all Kripke frames satisfy the following conditions:

- \( R \) is reflexive (axiom \( T\Box \)) and transitive (\( 4\Box \)),
- \( R_D \) is symmetric (\( B_D \))
- \( R_D \) is pseudo-transitive (\( 4\_D \)),
- \( R \subseteq R_D \cup Id_W \) (\( D\Box \)).

Note that we can further assume that \( R_D \cup Id = W \times W \), because according to Lemma [4.7] we can consider only generated subframes.

Now let us see what formulas \( AT_1 \) and \( DS \) mean in a Kripke frame.

Let \( F = (W, R, R_D) \) be a \( S4D \)-frame, then \( Top(F) = Top(W, R) \) denotes the topological space on the set \( W \) with the topology \( \{ R(V) \mid V \subseteq W \} \). For formulas with the difference modality the validity in \( F \) and \( Top(F) \) may not be equivalent. This is because \( R_D \) could be not the real inequality relation.

**Definition 4.9.** Let \( R \) be a transitive reflexive relation on \( W \). Then \( x \in W \) is called \( R \)-minimal (respectively \( R \)-maximal), if for any \( y, yRx \) (respectively \( xRy \)) implies \( x = y \).

**Definition 4.10.** Let \( F = (W, R, R_D) \) be an \( S4D \)-frame; we say that \( F \) is a \( T_1 \)-frame (or has the \( T_1 \)-property), if all \( R_D \)-irreflexive points are \( R \)-minimal.

**Lemma 4.11.** Let \( F = (W, R, R_D) \) be \( S4D \)-frame. Then \( F \models AT_1 \) iff \( F \) is a \( T_1 \)-frame.

**Proof.** \((\Rightarrow)\) Suppose \( F \models AT_1 \) and there exists an \( R \)-non-minimal and \( R_D \)-irreflexive point in \( F \). To be more specific, let \( x \) and \( y \) be two different points such that \( \neg xR_D x \) and \( yR_x \). Take a valuation \( \theta \) such that \( \theta(p) = W - \{ x \} \). Then \( x \models [\neq]p \) and \( x \models \neg p \), thus \( y \models \Diamond \neg p \). Since \( x \neq y \) and \( yR_x \), we have \( xR_D y \), \( x \models \Diamond \neg p \). Hence \( x \models \neg [\neq] \Diamond p \). This contradicts \( x \models AT_1 \).

\((\Leftarrow)\) Assume that \( F \) is a \( T_1 \)-frame and for some valuation for \( F \) we have \( x \models [\neq]p \).

Let us show that \( x \models [\neq] \Diamond p \). As we mentioned above, generated subframes preserve validity, so we can assume that \( F = F^x \) hence \( R_D(x) \cup \{ x \} = W \). There are two possibilities:

1) \( xR_D x \). Then \( y \models p \) for any \( y \in W \), hence for all \( y \in W \) we have \( y \models \Box p \); so \( x \models [\neq] \Box p \).

2) \( \neg xR_D x \). Then \( y \models p \) for every \( y \neq x \). By assumption, \( y \neq x \), \( yRz \) implies \( z \neq x \), hence \( z \models p \). So for any \( y \neq x \) \( y \models \Box p \); hence \( x \models [\neq] \Box p \). \( \square \)
Definition 4.12. Let $F = (W, R, R_D)$ be an S4D-frame; we say that $F$ is a $DS$-frame, if every $R_D$-irreflexive point has an R-successor (called just a successor further on).

Lemma 4.13. Let $F = (W, R, R_D)$ be an S4D-frame. Then $F \models DS$ iff $F$ is a $DS$-frame.

Proof. ($\Rightarrow$) Suppose $F \models DS$ and there exists an $R_D$-irreflexive point $x$ without successors. We take a valuation $\theta$ such that $\theta(p) = W - \{x\}$; then $x \models [\neg]p$ but $x \not\models \lozenge p$. This contradicts $F \models DS$.

($\Leftarrow$) Suppose that every $R_D$-irreflexive point in $F$ has a successor. Let us prove that for any $x \in W$ $x \models DS$. Suppose $(F, \theta), x \models [\neg]p$, then there are two cases: (i) $x$ is $R_D$-reflexive; then $\theta(p) = W$, and so $x \models \lozenge p$ since $R$ is reflexive; (ii) $x$ is $R_D$-irreflexive, then $\theta(p) \supseteq W - \{x\}$, and by our assumption, there exists $y \neq x$ such that $xRy$; hence $y \models p$ and $x \not\models \lozenge p$. □

5. Kripke completeness and finite model property.

All our axioms are Sahlqvist formulas. So we easily obtain Kripke completeness for logics S4D, S4DS, S4DT1S.

Following the common way of proving f.m.p. we use filtration (cf. [5] or [12]).

Definition 5.1. Let $M = (F, \theta)$ be a Kripke model, where $F = (W, R, R_D)$ is a Kripke frame and $\Psi$ is a set of formulas closed under subformulas. Let $\approx_{\Psi}$ be the equivalence relation on the elements of $W$ defined as follows:

$w \approx_{\Psi} v$ iff for all $\phi$ in $\Psi$: $(M, w \models \psi$ iff $M, v \models \phi$).

By $[w]$ we denote the equivalence class of $w$. Suppose $M' = (F', \theta')$ and $F' = (W', R', R_D')$ such that

(1) $W' = W_{\Psi} = \{[w] \mid w \in W\}$.
(2) If $wRv$ then $[w]R'[v]$ (and similarly for $R_D$),
(3) If $[w]R'[v]$ then for all $\Box \phi \in \Psi$; $M, w \models \Box \phi$ only if $M, v \models \phi$ (and similarly for $R_D$ and $[\neg \phi]$).
(4) $\theta'(p) = \{[w] \mid M, w \models p\}$, for all atomic symbols $p$ in $\Psi$.

Then $M'$ is called a filtration of $M$ through $\Psi$.

Lemma 5.2. (Filtration Lemma) Let $M'$ be a filtration of $M$ through $\Psi$, then for any $x \in M_1$ and for any $\psi \in \Psi$

$M, w \models \psi \iff M', [w] \models \psi$.

Lemma 5.3. Let $F_1$ be an S4D-frame, $M_1 = (F_1, \theta_1)$ a model, $\Psi$ a finite set of formulas closed under subformulas. Then there exists a filtration $M_2$ of $M_1$ through $\Psi$, such that $M_2 = (F_2, \theta_2)$ and $F_2$ is an S4D-frame.

Proof. Let $M' = (W_{\Psi}, R', R_D', \theta')$ be the minimal filtration of $M$. The minimal filtration is well-known (cf. [5] or [12]). Briefly, $[x]R'[y]$ iff there exist $x' \in [x]$ and $y' \in [y]$ such that $x'R_1y'$, and the same for $R_D'$ and $R_D$.

Let $R_2$ be the transitive closure of $R'$:

$R_2 = R'^* = \bigcup_{n \geq 1} R'^n$;
and let $R_{D2}$ be the pseudo-transitive closure of $R'_D$:

$$R_{D2} = R'_D \cup (Id - R'_D).$$

Note that the only difference between the pseudo-transitive and the transitive closure is that the irreflexive points remain irreflexive.

One can easily see that the reflexivity of $R'$ is inherited by $R_2$, and the reflexivity of $R'$ follows from the reflexivity of $R_1$. In the same way the symmetry of $R_{D1}$ implies the symmetry of $R_{D2}$. The transitivity of $R_2$ and the pseudo-transitivity of $R_{D2}$ are provided by construction. Next, we can easily show that $R' \subseteq R'_D \cup Id$; hence $R_2 \subseteq R_{D2} \cup Id$ holds.

To complete the proof, we have to show that the relations $R_2$ and $R_{D2}$ satisfy the definition of filtration. Since Filtration Lemma for the minimal filtration and its transitive closure in transitive logics are well-known (cf. [3]), we will only check $R_{D2}$.

1. For arbitrary $w, v \in W_1$ assume $vR_D w$, let us prove that $[v]R_{D2}[w]$. If $[v] \neq [w]$, the proof is the same as for the transitive closure. So assume $[w] = [v]$; then $[v]R'_D[w]$, and so $[v]R_{D2}[w]$.
2. Assume $[v]R_{D2}[w]$, let us prove that for all $[\varphi] \in \Psi; M_1, [v] \models [\varphi]$ only if $M_1, w \models \varphi$. If $[v] \neq [w]$, then the proof is the same as for the transitive closure. If $[w] = [v]$, then from $[v]R_{D2}[v]$ follows $[v]R'_D[v]$. But $R'_D$ was already filtration.

So we obtain a filtration that reduces $M_1$ to a finite model over an $S4D$-frame.

**Theorem 5.4.** Let $L$ be one of the logics: $S4D$, $S4DS$, $S4DT_1S$. Then $L$ has the finite model property.

**Proof.** Assume that $A$ is a formula such that $A$ is not in $L$. Hence, $A$ is refuted in some generated submodel $M_1 = (W_1, R_1, R_{D1}, \theta)$ of the canonical model of logic $L$. Note that since $M_1$ is a generated submodel, $R_{D1} \cup Id_{W_1}$ is the universal relation.

Let $\Psi$ be the set of all subformulas of formula $A$. By Lemma 5.3 there exists model $M_2 = (F_2, \theta_2)$ such that $F_2$ is a $S4D$-frame and $M_2$ is a filtration of $M_1$ through $\Psi$.

Since $M_2$ is a filtration, $A$ is refuted in $M_2$. So it remains to prove (if needed) the $T_1$–property and the $DS$–property for $F_2$.

Let us prove that axiom $AT_1$ is valid in frame $F_2$. By Lemma 4.11 it is sufficient to prove that for any $\eta$ such that $\neg \eta R_{D2} \eta$ there does not exist $\psi$ such that $\psi \neq \eta$ and $\psi R_2 \eta$.

Assume the contrary, i.e. there exists a point $\psi \neq \eta$ such that $\psi R_2 \eta$. Then consider their inverse images: $[x] = \eta \& [y] = \psi$. By construction of $R_2$ we obtain

$$y \approx_\psi y_0 R_1 z_0 \approx_\psi y_1 \ldots y_k R_1 z_k \approx_\psi x$$

Since $y \neq_\psi x$, we can take maximal $l$ such that $y_l \neq_\psi z_l$. By transitivity of $\approx_\psi$ we conclude that $z_l \approx_\psi x$ and $[z_l] = [x] = \eta$. Assume that $z_l \neq x$, since $R_{D1} \cup Id$ is the universal relation $z_l R_{D1} x$, hence $[z_l] R_{D2} [x]$; which contradicts $\neg \eta R_{D2} \eta$.

So $y_l \neq x \& y_l R_1 x (= z_l)$, at the same time reflexivity is preserved under filtration; hence $\neg x R_{D1} x$. So we came to a contradiction, because the generated subframe $F_1 = (W_1, R_1, R_{D1})$ of the canonical model has the $T_1$–property.
Now let \([x]\) be an \(R_{D_2}\)-irreflexive point. Hence, \(x\) is also an \(R_{D_1}\)-irreflexive point, so for some \(y\), \(xR_1y\) and \(xR_{D_1}y\), then \([x]R_2[y]\) and \([x]R_{D_2}[y]\), so \([y] \neq [x]\); hence \([x]\) is not maximal. \(\square\)

6. Topological completeness.

Let us define analogue of \(p\)-morphism for maps from topological space onto finite \(S4D\)-frame.

**Definition 6.1.** Let \(\mathfrak{X}\) be a topological space and let \(F = (W, R, R_D)\) be a finite Kripke frame. A function \(f : \mathfrak{X} \to F\) is called a \(cd\)-\(p\)-morphism, if it is surjective and satisfies the following two conditions

\begin{align}
(6.1) & \quad Cf^{-1}(w) = f^{-1}(R^{-1}(w)), \\
(6.2) & \quad R_D^{-1}(f^{-1}(w)) = f^{-1}(R_D^{-1}(w)),
\end{align}

where \(R_D = \{\neq\}\) in \(\mathfrak{X}\) (in particular \(R_D^{-1}\{\{x\}\} = \mathfrak{X} - \{x\}\)). In notation \(f : \mathfrak{X} \overset{cd}{\longrightarrow} F\).

Note that since \(f\) is surjective, (6.2) is equivalent to the following: if \(w\) is \(R_D\)-irreflexive then \(f^{-1}(w)\) is one-element.

**Lemma 6.2.** If \(F\) is a finite Kripke frame, \(\mathfrak{X}\) is a topological space and \(f : \mathfrak{X} \overset{cd}{\longrightarrow} F\) then \(L_D(\mathfrak{X}) \subseteq L(F)\).

**Proof.** Note that \(f\) is \(cd - p\)-morphism and \(C\) distributes over finite unions. So for \(U \subseteq W\) we have

\begin{align}
(6.3) & \quad f^{-1}(R^{-1}(U)) = f^{-1}( \bigcup_{w \in U} R^{-1}(w)) = \bigcup_{w \in U} f^{-1}(R^{-1}(w)) = \bigcup_{w \in U} Cf^{-1}(w) = Cf^{-1}(U).
\end{align}

In other terms, \(f\) is an interior map between topological spaces \(\mathfrak{X}\) and \(\text{Top}(F)\).

Similarly

\begin{align}
(6.4) & \quad f^{-1}(R_D^{-1}(U)) = f^{-1}( \bigcup_{w \in U} R_D^{-1}(w)) = \bigcup_{w \in U} f^{-1}(R_D^{-1}(w)) \overset{\text{induction}}{=} \bigcup_{w \in U} R_D^{-1}f^{-1}(w) = R_D^{-1}f^{-1}(U).
\end{align}

Now let \(\theta\) be an arbitrary valuation on the frame \(F\). Take a valuation \(\Theta\) on \(\mathfrak{X}\) such that \(\Theta(p) = f^{-1}(\theta(p))\). Then a standard inductive argument shows that for any formula \(\phi\)

\begin{align}
(6.5) & \quad \Theta(\phi) = f^{-1}(\theta(\phi)),
\end{align}

where \(\theta(\phi) = \{v | (F, \theta), v \models \phi\}\) and \(\Theta(\phi) = \{x | (\mathfrak{X}, \Theta), x \models \phi\}\).

For this proof we rewrite all formulas using \(\Diamond\) and \(\langle\neq\rangle\) (rather then \(\Box\) or \([\neq]\)).

There are only two nontrivial cases:

i) \(\phi \equiv \Diamond \psi\). Then

\(f^{-1}(\theta(\Diamond \psi)) = f^{-1}(R^{-1}(\theta(\psi))) = Cf^{-1}(\theta(\psi)) = C \Theta(\psi) = \Theta(\Diamond \psi)\).

\(\square\)\(C\) is not distributes over infinite unions so finiteness of \(F\) is essential.
ii) $\phi \equiv (\neq) \psi$. Then
\[
f^{-1}(\theta((\neq) \psi)) = f^{-1}(R_D^{-1}(\theta(\psi))) \overset{\text{def.}}{=} R_D^{-1}f^{-1}(\theta(\psi)) \overset{\text{induction}}{=} R_D^{-1}\Theta(\psi) = \Theta((\neq) \psi).
\]
Now if $\phi \notin \mathbf{L}(F)$, there exists a valuation $\theta$ such that $\theta(\phi) \notin W$. By (6.5) $\Theta(\phi) = f^{-1}(\theta(\phi))$, and so $\Theta(\phi) \notin \mathbf{X}$ since $f$ is subjective. Thus $\phi \notin \mathbf{L}(\mathbf{X})$. \qed

The following proposition uses ideas from [14, 6]

**Proposition 6.3.** Let $F = (W, R, R_D)$ be a S4D-Kripke frame, $R_D \cup \text{Id}_W = W \times W$. There exists S4D-Kripke frame $F' = (W', R', R_D')$, such that $F' \rightarrow F$ and $x'R'D'y'$ iff $x' \neq y'$.

**Proof.** Let us put $W^0 = \{x \in W | x \not\in D\}$ and $W^x = W - W^0$. Then
\[
W' = W^x \cup W^0 \times \{0, 1\}
\]
Let us define the function $f : F' \rightarrow F$ such that
\[
f(x') = \begin{cases} x, & \text{if } x' = x \in W^x; \\ x, & \text{if } x' = (x, i); \end{cases}
\]
and the relation $R'$:
\[
x'R'y' \iff f(x')Rf(y')
\]
Let us prove that $f$ is a p-morphism.

1. Obviously $f$ is surjective.
2. Assume that $x'R'y'$ then by definition of $R'$ $f(x')Rf(y')$. Assume that $x'R_D'y'$ (or $x' \neq y'$), $f(x') = x$ and $f(y') = y$. If $x \neq y$ then $x \not\in D$ and $y' = (x, 1)$ (or vice versa); using (6.6) we conclude that $x \not\in D$.
3. Assume that $f(x')Ry$. If $y \in W^0$ then $y' = (y, 0)$ or $y' = y$ otherwise. Easy to see that $f(y') = y$ and $x'R'y'$. Assume that $f(x')R_Dy$. Case when $f(x') \neq y$ is obvious so let $f(x') = y$. It means that $y \in W^0$ and $x' = (y, 1)$. So we put $y' = (y, (i + 1) \mod 2)$ and this will do. \qed

**Corollary 6.4.** Let $\mathcal{C}$ be the class S4D-frames of the form $F = (W, R, \neq)$ then S4D is complete with respect to $\mathcal{C}$.

It is easy to show that for any S4D-frame $F = (W, R, \neq)$
\[
\text{Top}(F) \overset{cd}{\rightarrow} F
\]
but we can prove a stronger statement:

**Lemma 6.5.** Let $(F, \theta)$ be a Kripke model then for any formula $A$ and $x \in W$
\[
F, \theta, x \models A \iff \text{Top}(F), \theta, x \models A
\]

**Proof.** By induction on the complexity of $A$. The only case that is not trivial or classical is when $A = [\neq]B$.
\[
F, \theta, x \models [\neq]B \iff \forall y (y \neq x \Rightarrow F, \theta, y \models B)
\]
but by induction it holds iff
\[
\forall y (y \neq x \Rightarrow \text{Top}(F), \theta, y \models B) \iff \text{Top}(F), \theta, x \models [\neq]B
\]
\qed
Theorem 6.6. S4D is the D-logic of all topological spaces.

Proof. Let A be a formula that is not in S4D. Then by Corollary 6.4 there exists a Kripke frame F = (W, R, ≠) such that F ↣ A. By Lemma 6.5 we obtain Top(F) ↣ A. □

Proposition 6.7. Let F = (W, R, ≠) be a DS-frame, then Top(F) is a dense-in-itself topological space.

Proof. In Top(F) the least open neighborhood of point x is R(x). Since F is a DS-frame, R(x) − {x} ≠ ∅; hence Top(F) is dense-in-itself. □

Theorem 6.8. S4DS is logic of all dense-in-itself topological spaces.

Proof. From Theorem 5.4 we know that S4DS is complete with respect to all finite DS-frames. Now we can apply Proposition 6.7 and Lemma 6.5. □

If a logic contains the axiom (AT₁) then we cannot use the above methods. Indeed if F = (W, R, ≠) and F |= AT₁, then R = Id_W. The logic of such frames will be the logic of isolated points. So we need to find more sophisticated ways.

Recall a few definitions.

Definition 6.9. A non-empty topological space X is called zero-dimensional if clopen sets constitute its open base.

Definition 6.10. A pair (X, ρ) called metric space if X is a set and ρ is a function from X × X onto R, such that ρ(x, y) ≥ 0, ρ(x, y) = 0 iff x = y, ρ(y, x) = ρ(x, y), and ρ(x, y) + ρ(y, z) ≥ ρ(x, z).

On metric space can be defined natural topology based on open balls: \{ y | ρ(x, y) < r \}.

Theorem 6.11. S4DT₁S is complete with respect to any zero-dimensional dense-in-itself metric space.

Proof. Let X be a zero-dimensional dense-in-itself metric space and ρ is the distance in it; O(x, r) denotes the open ball \{ y ∈ X | ρ(x, y) < r \}.

We know from Theorem 5.4 that S4DT₁S is complete with respect to all finite DS-T₁-frames. If we prove that for an arbitrary finite DS-T₁-frame F = (W, R, R_D),

\[ X \xrightarrow{cd} F \]

then we prove the theorem.

We use induction on the size of F. Consider three cases.

Case I. W = R(w₀), R_D = W × W for some w₀. Since S4 is complete with respect to X(cf. [1]) and F⁻ = (W, R) is S4-frame, then there exists a continuous function f : X → Top(F⁻). It is easy to check that f : X \xrightarrow{cd} F.

Case II. W = R(w₀), R_D = W × W − (w₀, w₀). Since W is finite let us enumerate all points in W starting with w₀: W = \{ w₀, w₁, w₂, ..., w_n \}. Any generated subframe F^w_i for i > 0 satisfies to case I.

Take an arbitrary point x₀ and clopen sets Y₀, Y₁, ... such that

\{ x₀ \} ⊂ ... ⊂ Y_n ⊂ ... ⊂ Y₁ ⊂ Y₀ = X

and

\[ Y_n ⊆ O(x₀, \frac{1}{n}) \]
for every $n > 0$. We can do it because $X$ is zero-dimensional (cf. [13])

Since $Y_n \subseteq O(x_0, \frac{1}{n})$, it follows that

$$\bigcap_n Y_n = \{x_0\}$$

and further we obtain

$$X - \{x_0\} = X - \bigcap_n Y_n = \bigcup_n X_n,$$

where $X_n = Y_n - Y_{n+1}$. The sets $X_n$ are open, metric, dense-in-itself and zero-dimensional.

For any open neighborhood $U$ of $x_0$ there exists $n$ such that $O(x_0, \frac{1}{n}) \subseteq U$, it follows that $Y_n \subseteq U$, hence for all $i \geq n$, $X_i \subseteq U$.

So, by induction, for any $j > 0$ there exists

$$f_j : X_j \xrightarrow{cd} F^{w_k}, \text{ where } (k - 1) \equiv j \pmod{n}$$

Now consider

$$f(x) = \begin{cases} w_0, & \text{if } x = x_0; \\ f_j(x), & \text{if } x \in X_j. \end{cases}$$

Let us prove that $f : X \xrightarrow{cd} F$.

First, we note that $f$ is surjective.

Second, we check (6.1). Assume that $y \in X_j$ then:

$$y \in C f^{-1}(w) \Rightarrow y \in C f_j^{-1}(w) = f_j^{-1}(R^{-1}(w)) \subseteq f^{-1}(R^{-1}(w));$$

and the other way around:

$$y \in f^{-1}(R^{-1}(w)) \Rightarrow f_j^{-1}(R^{-1}(w)) = y \in C f_j^{-1}(w) \subseteq C f^{-1}(w).$$

Now assume that $y = x_0$. For any $w \in W$, $w_0 \in R^{-1}(w)$; hence

$$x_0 \in f^{-1}(R^{-1}(w)).$$

On the other hand, for some $i$, $w = w_i$ and for any open neighborhood of $x_0$, there exists $m$ such that $(i - 1) \equiv m \pmod{n}$ and $U \supseteq X_m \supseteq f_m^{-1}(w_i)$. In other words, $x_0$ is a limit point for $f^{-1}(w_i)$, hence $x_0 \in C f^{-1}(w_i)$.

Third, we check (6.2). Since $f^{-1}(w_0)$ is a one-element set, (6.2) holds.

Case III. Everything else. Let us take all $R$-minimal $R$-clusters of $F$ and from each one of them we choose an arbitrary point. So we get the following set: $\{v_1, v_2, \ldots, v_k\}$. Standard unravelling arguments show that

$$F' = F^{v_1} \sqcup F^{v_2} \sqcup \ldots \sqcup F^{v_k} \Rightarrow F$$

So we need to show that $X \xrightarrow{cd} F'$.

Since $F$ is a $\mathbf{S4DT}_1\mathbf{S}$-frame, each $F^{v_i}$ satisfies case I or case II.

Since $X$ is zero-dimensional, we can present $X$ as disjunctive union of clopen subsets:

$$X = X_1 \sqcup \ldots \sqcup X_{k-1} \sqcup X_k.$$

By induction we have

$$f_1 : X_1 \xrightarrow{cd} F^{v_1},$$

$$\ldots$$

$$f_{k-1} : X_{k-1} \xrightarrow{cd} F^{v_{k-1}},$$

$$f_k : X_k \xrightarrow{cd} F' = F^{v_1} \sqcup F^{v_2} \sqcup \ldots \sqcup F^{v_k} \Rightarrow F.$$
It is easy to show that \( f = f_1 \sqcup \ldots \sqcup f_k \) (if \( x \in X_i \) then \( f(x) = f_i(x) \)) is a cd-p-morphism. □

The immediate and obvious corollary of this theorem is that \( \mathbf{S4DT}_1 \mathbf{S} \) is complete with respect to all dense-in-itself \( T_1 \) spaces.

7. Conclusions and open problems.

The language with difference modality shows much more expressive power then basic topological language, and even more then basic language with universal modality. We can express density-in-itself, \( T_1 \), and connectedness in it. Moreover

\[
(AE_1) \quad [\neq] p \land \neg p \land \square (p \rightarrow \square q \lor \square \neg q) \rightarrow \square (p \rightarrow q) \lor \square (p \rightarrow \neg q)
\]

differs \( \mathbb{R} \) from \( \mathbb{R}^2 \) (cf. [7]). It was proved that logic \( \mathbf{S4DT}_1 \mathbf{S} + (AE_1) + \text{“connectedness”} \) is complete with respect to \( \mathbb{R}^n \), \( n \geq 2 \) (the full proof is to be published). We still do not know the D-logic of \( \mathbb{R} \) and whether \( \mathbf{S4D} + (AT_1) \) is complete with respect to all \( T_1 \) spaces.

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This paper was published at Advances in Modal Logic, Volume 6, 2006

E-mail address: kudinov--at--iitp--dot--ru