Phase diagram of two dimensional electron gas in a perpendicular magnetic field around Landau level filling factors $\nu = 1$ and 3.

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The measured melting curve $T_m(\nu)$ between the crystal and liquid phases is analyzed using thermodynamics to extract the change of magnetization $\Delta M$ as a function of the Landau level filling factor $\nu$, near $\nu = 1$. An explanation of $\Delta M(\nu)$ is proposed in terms of Skyrmions. Near $\nu = 3$, a Wigner crystal is the most probable solid phase, experiments excluding Skyrmions.

In previous works\textsuperscript{1,2,3} the authors have proposed the possible existence of a Hall crystal and/or Skyrmions at very low temperatures near the Landau filling factor $\nu = 1$. The recent NMR relaxation experiments of Gervais et al.\textsuperscript{4} strengthen the earlier claim of Barrett et al.\textsuperscript{5} that Skyrmions exist around $T \sim 40$ mK near $\nu = 1$, and a possible transition from a liquid to a lattice of Skyrmions could occur there. We point out here the thermodynamical result\textsuperscript{6} for the slope of the melting curve $T_m(\nu)$:

$$\frac{\partial}{\partial \nu} T_m(\nu) = \frac{B \Delta M}{\nu \Delta S},$$

where $\Delta M$ and $\Delta S$ represent magnetization and entropy changes respectively at the transition temperature $T_m$ of Skyrmion melting and $B$ and $\nu$ are the magnetic field and the filling factor, respectively. The three estimated values of the solid-liquid transition temperature $T_m$, according to the measurements of Gervais et al., are illustrated by the black points in Fig. 1. The continuous curve in the figure is determined by a fit to the measured points after also adding two physical conditions: a) The transition temperature in the limit $\nu - 1$ should vanish because the density of the Skyrmion gas tends to zero, and b) To assume, in a first approximation, a Skyrmion-anti-Skyrmion symmetry around $\nu = 1$. The even polynomial of the filling factor deviation $X = \nu - 1$, employed for the fit, included up to the sixth powers of $X$. It should be noticed that the curves for the liquid to solid phase of the Skyrmions estimated from the specific heat measurement of Bayot et al.\textsuperscript{8} do not show the particle and antiparticle symmetry. However, taking into account that the vanishing of the critical temperature $T_m$ for $\nu = 1$ should be valid in any case (See, eg. Ref. \textsuperscript{7}), the qualitative argument below will not be affected by this simplifying assumption in the first stage of the analysis. The qualitative behavior shown in Fig. 1 (apart from the above mentioned asymmetry) is also shown by the measurements in Ref. \textsuperscript{8}. It will be seen that the analysis given here will supply a possible mechanism for the lack of symmetry around $\nu = 1$.

The behavior of $T_m$ in Fig.1 can be studied by considering eqn. (1). The derivative of $T_m$ is depicted in Fig 2. Firstly, it may be noted that for $X < 0$ (the region in which the experimental points of Gervais et al were measured)
there exists a negative value of $X = X_c$ for which $\Delta M$ should be zero. This is so, because we expect physically that $S_{\text{solid}} < S_{\text{liquid}}$. Defining now $\Delta S = S_{\text{solid}} - S_{\text{liquid}} = S_s - S_l$ (and also $\Delta M = M_s - M_l$) it follows that $\Delta S$ is non-vanishing and negative. Then, the magnetization of the liquid at $X = X_c$, is identical to that of the solid at the zero of $T_m$ for $X = X_c$. Over $X_c$ up to $X = 0$ the quantity $\Delta M$ should become positive, since from thermodynamics $\Delta M/\Delta S < 0$, hence $\Delta M$ and $\Delta S$ must have the opposite sign for $X > X_c$. Similarly, for the region $X < X_c$, the magnetization change should acquire negative values.

The filling factor values for which the internal energies of the crystal and the liquid coincide can also be estimated by considering the relation

$$\frac{\partial}{\partial \nu} T_m(\nu) = -\frac{T_m(\nu)}{\nu},$$

(2)

which should be obeyed at those points. This equation has the following solutions: $X = -0.104$ and $X = 0$, lying in the interval defined by $X = -0.16$ (corresponding to the lowest measured filling factor) and $X = 0$. It is of interest that the equal energy filling factors are very close to the values for which the magnetization of the crystal and the liquid are also coinciding. The equality of the internal energy at $X = 0$, that is at $\nu = 1$, is an expected result since the Skyrmion density vanishes at $\nu = 1$.

In addition, the inset of Fig. 3 of Gervais et al, as these workers stress, suggests that there is a critical filling factor in the range $\nu \approx 0.80 - 0.83$ where a quantum phase transition exists. Following the treatment of reference March, Suzuki and Parrinello, it is then relevant to ask what is the order parameter of this transition. We propose the low frequency shear modulus, say $G$, of the Skyrmion crystal, as a candidate. If the quantum phase transition remains first order, in analogy with (1) for the melting curve, then it follows from this equation that, since the Third Law of Thermodynamics requires $\Delta S > 0$, that the melting curve has a positively infinite slope at the critical filling factor near $0.8$. This assumes the unlikely event that $\Delta M > 0$ such that $\Delta M/\Delta S$ remains finite does not occur. Also, such a first order transition would imply that the low frequency shear modulus of the Skyrmion crystal goes discontinuously to zero at the quantum phase transition.

![FIG. 2: The derivative of the estimated curve for $T_m(\nu)$ showing the various change of signs of $\Delta M/\Delta S$ as indicated by Eq. (1).](image)

We next advance an explanation for the form of the estimated solid-liquid phase transition curve of an assembly of Skyrmions. The picture to be exposed incorporates concepts coming from the ‘anyon’ type of description of the QHE state near $\nu = 16.10.11$. We expect that the qualitative explanation given can help to make contact between ‘anyon’ descriptions and the transition between liquid and crystal Skyrmion phases.

Let us consider the region $X < 0$ near $\nu = 1$ and first assume that $X$ is close to zero. That is, the density of Skyrmion excitations $N$ is low and naturally they will tend to crystallize. This situation is illustrated at the left hand side in Fig. 3, which shows a crystalline arrangement of Skyrmions. For $\nu < 1$ the Skyrmions should really be of the anti-Skyrmion kind having a positive net charge. This charge distribution is indicated in the figure by the plus sign. The curly arrows lying inside each Skyrmion picture signal the flow of diamagnetic Hall currents, giving rise to the magnetic moment of the assumed almost isolated Skyrmion. This magnetic moment is depicted as the white vertical arrow over each Skyrmion picture. Since, the temperature is non-vanishing and is also close to the critical value $T_m$, the excitations will also have a kinetic motion. This should produce a mean increment of the magnetic moment per particle. It is clear that this increment will be always a diamagnetic one: that is, opposite to the external magnetic field $B$. This contribution is shown as a black arrow below each of the particle symbols. In the crystal phase (left hand side) it seems that the crystal potential could drastically restrain the realization of large circular motion of the center of mass of each Skyrmion at thermal velocities, around the crystal position. Then, the
diamagnetic orbital moment addition to each Skyrmion $m_{S,o-s}$, can be expected to be reduced in absolute value by the crystal potential.

The liquid phase state of the Skyrmion gas is illustrated in the right hand side of Fig. 3. The same conventions for the symbols are used and the temperature is also assumed to be equal to $T_m$ but slightly above the phase transition. In this case, since the gas is very diluted in the considered situation, the Skyrmion proper magnetic moments can be expected to be coincident in value with the ones in the crystal phase. The situation seems to be different for the orbital component of the magnetic moment created by the thermal fluctuations. In this case, the constraining action of the crystal is not present and the Skyrmions have the chance of orbiting with similar thermal velocities but in greater circles with appreciably more degrees of freedom than their solid counterparts. Thus, an increase in magnitude of the orbital magnetic moment contribution $m_{S,o-l}$ with respect to its solid counterpart $m_{S,o-s}$, should be expected in the liquid phase.

![Diagram](image)

**FIG. 3:** Diagrams representing the Skyrmion crystal (left hand side) and the Skyrmion liquid (right hand side). The white vertical arrows represent the individual magnetic moments of each particle due to its internal (Hall) motion. The black vertical ones indicate the contribution to the magnetic moments coming from the thermally induced orbital motion of the center of mass. The less constrained orbital movement in the liquid phase can be expected to increase the thermally induced orbital components. This is illustrated by the longer black arrows in the liquid diagram.

Therefore, in the region $X < 0$ down to $X_c$ it is to be expected that

$$M_s - M_l = N (m_S + m_{S,o-s}) - N(m_S + m_{S,o-l}),$$

$$= N (-m_{S,o-l} + m_{S,o-s}),$$

$$\Delta M = M_s - M_l > 0.$$ 

This is the same conclusion that the estimated phase diagram predicts, according to eqn. (1).

As the density of anti-Skyrmions $N$ grows as $X$ diminishes approaching to $X_c$ in the liquid phase, the scattering among the Skyrmions becomes more intense. This effect can be expected to inhibit the realization of large circular motions of the center of mass of orbiting anti-Skyrmions, leading to a decrease of the magnitude of $m_{S,o-l}$ down to equalize it to $m_{S,o-s}$ at $X = X_c$. Further, as the density increases even more, the scattering plus overlapping effects of the Skyrmions in the liquid phase makes natural a further reduction of the net magnetization of the liquid with respect to the value in the more organized solid arrangement, allowing steady currents and magnetic moments. This situation is in accord with the property $\Delta M < 0$ following from the estimated critical curve for $X < X_c < 0$.

Let us propose below a qualitative explanation for the Skyrmion-anti-Skyrmion observed asymmetry in the heat capacity measurements in Ref. 8. For this purpose we notice that the orbital magnetic moment generated by the thermal motion is not changing its sign when the filling factor passes across the $\nu = 1$ value. In addition it can be also underlined that the Skyrmion has a rotating internal motion. These circumstances are illustrated in Fig. 4. Therefore, such rotational movements inside the crystal should generate shear forces tending to distort it in some measure, as illustrated in the left hand side of Fig. 4. But, after taking into account that for $\nu > 1$, both, the internal Skyrmion rotational motion and the thermal one coincide in sense, and that, on the contrary, for $\nu < 1$ these motions are opposed, it follows that the shear forces distorting the Skyrmion crystal should be higher for $\nu > 1$. But, the crystal melting is strongly determined by the resistance to shear deformations, and then, the $\nu > 1$ crystal should be in a state closer to an instability under shear deformations. Thus, it seems natural to expect that the melting in the $\nu < 1$ region will require higher temperatures to be attained, as experimentally observed. 8
Let us remark that in contrast to the behavior near $\nu = 1$, where experiment strongly points to the ground-state being a Skyrmion crystal, there is accord between experimental groups that a Skyrmion crystal does not form the ground state of the two dimensional electron assembly near $\nu = 3$, where a Wigner crystal seems the most likely candidate from existing experiments. Therefore, the whole situation around integral filling factors seems to reinforce the view advanced in the literature that the main ground state at exactly integer filling factors shows an electromagnetic Chern-Simons (equivalent to a Hall one) response that either:

a) Transforms the set of impurities, when they exist in dirty samples, in a kind of ”charge reservoir”, which accepts or releases electrons above or below $\nu = 1$, respectively.

b) Or, in rough terms, generates ”artificial impurities” localizing charges, as Skyrmions or anti-Skyrmion excitations near $\nu = 1$ and ’electron’ or ’hole’ like Wigner crystals near $\nu = 3$, again when $\nu$ is above or below the corresponding filling factor, respectively.

The coexistence of all these crystal or liquid structures for magnetic fields well inside plateaus is a supporting experimental fact for this view, as stated in. The demonstration of the validity of this property for realistic planar samples is under consideration. In Ref. it was only shown for the simpler case of a superlattice of planar samples.

To conclude, near $\nu = 1$ the measured phase diagram given by Gervais et al has been interpreted using the thermodynamic relation and making the plausible assumption that $S_2 < S_1$. The measured melting curve requires various change of signs of the magnetization difference $\Delta M$ in the region $\nu = 0.8 - 1.2$. A qualitative interpretation of the nature of these changes is advanced by considering the internal structure of the Skyrmions and their thermal motions. The observed asymmetry of the melting into a Skyrmion liquid phase above and below $\nu = 1$ is proposed to be produced by the diamagnetic moment created by the thermal motion, which reduces or increases the magnitude of the internal magnetic moments of the Skyrmions and anti-Skyrmions, respectively. Therefore, an explanation of the observed properties of the Skyrme excitations near $\nu = 1$ is proposed and based on an electromagnetic and 'anyon' description. However, further work, both experiment and theory, is required on this potentially important matter.

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1 A. Cabo and F. Claro, Phys. Rev. B 70, 235320 (2004).
2 F. Claro, A. Cabo and N. H. March, *Phys. Stat. Sol. B* **242**, 1817 (2005).
3 A. Cabo, F. Claro and N. H. March, arXiv:cond-mat/0309166 (2003).
4 G. Gervais, H. L. Stormer, D. C. Tsui, P. L. Kuhns, W. G. Moulton, A. P. Reyes, L. N. Pfeiffer, K. W. Baldwin and K. W. West, *Phys. Rev. Lett.* **94**, 196803 (2005).
5 S. E. Barrett, G. Dabbagh, L. N. Pfeiffer, K. W. West and R. Tycko, *Phys. Rev. Lett.* **74**, 5112 (1995).
6 M. J. Lea, N. H. March and W. Sung, *J. Phys. Cond. Matter* **4**, 5263 (1992), *J. Phys. Cond. Matter* **3**, 6810 (1991).
7 L. Brey, H. A. Fertig, R. Côte and A. H. MacDonald, *Phys. Rev. Lett.* **75**, 2562 (1995).
8 V. Bayot, E. Grivei and S. Melinte, M. B. Santos and M. Shayegan, *Phys. Rev. Lett.* **76**, 4584 (1996).
9 N. H. March, M. Suzuki and M. Parrinello, *Phys. Rev. B* **19**, 2027 (1979).
10 A. Cabo and D. Martinez-Pedrera, *Phys. Rev. B* **67**, 245310 (2003).
11 S. L. Sondi, A. Karlhede, S. Kivelson and E. H. Rezayi, *Phys. Rev. B* **47**, 16419 (1993).