Invariant Measures and the Soliton Resolution Conjecture

SOURAV CHATTERJEE
Courant Institute

Abstract
The soliton resolution conjecture for the focusing nonlinear Schrödinger equation (NLS) is the vaguely worded claim that a global solution of the NLS, for generic initial data, will eventually resolve into a radiation component that disperses like a linear solution, plus a localized component that behaves like a soliton or multisoliton solution. Considered to be one of the fundamental open problems in the area of nonlinear dispersive equations, this conjecture has eluded a proof or even a precise formulation to date.

This paper proves a "statistical version" of this conjecture at mass-subcritical nonlinearity, in the following sense: The uniform probability distribution on the set of all functions with a given mass and energy, if such a thing existed, would be a natural invariant measure for the NLS flow and would reflect the long-term behavior for "generic initial data" with that mass and energy. Unfortunately, such a probability measure does not exist. We circumvent this problem by constructing a sequence of discrete measures that, in principle, approximate this fictitious probability distribution as the grid size goes to 0. We then show that a continuum limit of this sequence of probability measures does exist in a certain sense, and in agreement with the soliton resolution conjecture, the limit measure concentrates on the unique ground state soliton. Combining this with results from ergodic theory, we present a tentative formulation and proof of the soliton resolution conjecture in the discrete setting.

The above results, following in the footsteps of a program of studying the long-term behavior of nonlinear dispersive equations through their natural invariant measures initiated by Lebowitz, Rose, and Speer and carried forward by Bourgain, McKean, Tzvetkov, Oh, and others, are proved using a combination of techniques from large deviations, PDEs, harmonic analysis, and bare-hands probability theory. It is valid in any dimension. © 2014 Wiley Periodicals, Inc.

Contents

1. Introduction 1738
2. Main Result 1743
3. Is This a Proof of the Soliton Resolution Conjecture on a Large Discrete Torus with Small Grid Size? 1746
4. Microcanonical Invariant Measure for the Discrete NLS 1748
5. Main Ideas in the Proof 1751
6. Summary of Notation 1752

Communications on Pure and Applied Mathematics, Vol. LXVII, 1737–1842 (2014) © 2014 Wiley Periodicals, Inc.
1 Introduction

1.1 Probabilistic Motivation

Suppose that we are asked to choose a function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) uniformly at random from the set of all \( v : \mathbb{R}^d \rightarrow \mathbb{C} \) satisfying \( M(v) = m \) for some given constant \( m \), where

\[
M(v) := \int_{\mathbb{R}^d} |v(x)|^2 \, dx.
\]

While this question does not make sense mathematically, the only reasonable answer that one can give is that \( f \) must be equal to 0 almost everywhere. Paradoxically, this \( f \) does not satisfy \( M(f) = m \). The paradox is resolved if we view this question as the limit of a sequence of discrete questions: First approximate \( \mathbb{R}^d \) by a large box \([-L, L]^d\); then discretize this box by splitting it as a union of many small cubes; finally, choose a function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) uniformly from the set of all functions \( v \) that are piecewise constant in these small cubes and 0 outside the box \([-L, L]^d\) and satisfy \( M(v) = m \). This is a probabilistically sensible question; the resulting \( f \) approaches 0 in the \( L^\infty \)-norm as the box size goes to infinity.
Now suppose that we add one more constraint, namely, that \( f \) should satisfy
\[
H(f) = E,
\]
where \( H \) is the functional
\[
H(v) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v(x)|^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^d} |v(x)|^{p+1} \, dx,
\]
and \( p > 1 \) and \( E \in \mathbb{R} \) are given constants. The motivation for adding this second constraint comes from the study of microcanonical invariant measures of nonlinear Schrödinger equations (more on this later). One problem that arises immediately is that if \( v \) satisfies \( M(v) = m \) and \( H(v) = E \), so does the function \( u(x) := \alpha_0 v(x + x_0) \) for any \( x_0 \in \mathbb{R}^d \) and \( \alpha_0 \in \mathbb{S}^1 \), where \( \mathbb{S}^1 \) is the unit circle in \( \mathbb{C} \). Thus, it is reasonable to first quotient the function space by the equivalence relation \( \sim \), where \( u \sim v \) means that \( u \) and \( v \) are related in the above manner.

When \( p \) satisfies the “subcriticality” condition \( p < 1 + 4/d \), standard results from the theory of nonlinear Schrödinger equations imply that the set of functions \( v \) that minimize \( H(v) \) given \( M(v) = m \) form a unique equivalence class of the relation \( \sim \). This equivalence class is known as the “ground state soliton” of mass \( m \). The main result of this manuscript (Theorem 2.1) says that if we attempt to choose an equivalence class uniformly at random from all classes satisfying \( M(v) = m \) and \( H(v) = E \) by first discretizing the problem and then passing to the continuum limit, then we end up choosing this ground state soliton. As before, there is no paradox in the fact that the ground state soliton may not satisfy the constraint \( H(v) = E \). While the problem is quite simple for the single constraint \( M(v) = m \), the addition of the second constraint \( H(v) = E \) somehow renders it unreasonably difficult; indeed, nearly the entirety of this long manuscript is devoted to the proof of Theorem 2.1.

The above result is a small step towards understanding uniform probability distributions on manifolds in function spaces that are defined by a finite number of constraints. These distributions arise as “microcanonical” invariant measures for Hamiltonian flows on such manifolds. The conserved quantities for the flow give the constraints defining the manifold. A preliminary attempt with a simpler problem was made in \([14]\). All of this and how it connects to the behavior of nonlinear Schrödinger flows and ideas from statistical physics will be discussed in greater detail in the remainder of this section.

1.2 The Nonlinear Schrödinger Equation

A complex-valued function \( u \) of two variables \( x \) and \( t \), where \( x \in \mathbb{R}^d \) is the space variable and \( t \in \mathbb{R} \) is the time variable, is said to satisfy a \( d \)-dimensional nonlinear Schrödinger equation (NLS) if
\[
i \partial_t u = -\Delta u + \kappa |u|^{p-1} u,
\]
where \( \Delta \) is the Laplacian operator in \( \mathbb{R}^d \), \( p > 1 \) is the nonlinearity parameter, and \( \kappa \) is a parameter that is either \(+1\) or \(-1\). When \( \kappa = 1 \), the equation is called “defocusing,” and when \( \kappa = -1 \) it is called “focusing.”
The study of the NLS and other nonlinear dispersive equations is a large and growing area in the analysis of PDEs, with numerous open questions and conjectures. For a very readable general introduction, see Tao [65]. For a more specialized account of the state of affairs in the study of NLS, see the lecture notes of Raphaël [50]. The NLS arises in many areas of the pure and applied sciences, including Bose-Einstein condensation, Langmuir waves in plasmas, nonlinear optics, and a number of other fields [4,21,23,32,54,73,74].

The NLS is an infinite-dimensional Hamiltonian flow, with Hamiltonian given by

\[ H(v) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v(x)|^2 \, dx + \frac{\kappa}{p + 1} \int_{\mathbb{R}^d} |v(x)|^{p+1} \, dx. \]

(Note that in the focusing case, \( \kappa = -1 \), this is just the function \( H \) defined in (1.2).) Consequently, if \( u \) is a solution to (1.3), then \( H(u(t, \cdot)) \) is the same for all \( t \). Since \( H(v) \) is commonly called the energy of \( v \) in the context of Hamiltonian flows, the previous sentence can be restated as “The NLS flow conserves energy.” Another important conserved quantity is the mass \( M(v) \), defined in (1.1).

A significant amount of information is known about the defocusing NLS; in particular, it is known that in many situations, solutions of the defocusing equation disperse like solutions of the linear Schrödinger equation (see [65, p. 154]). Here dispersion means that while \( M(u(t, \cdot)) \) remains conserved, for every compact set \( K \subseteq \mathbb{R}^d \),

\[ \lim_{t \to \infty} \int_K |u(x, t)|^2 \, dx = 0. \]

In the focusing case, however, dispersion may not occur. This is demonstrated quite simply by a special class of solutions called “solitons” or “standing waves.” These are solutions of the form \( u(x, t) = v(x) e^{i \omega t} \), where \( \omega \) is a positive constant and the function \( v \) is a solution of the soliton equation

\[ -\omega v = -\Delta v - |v|^{p-1} v. \]

Often the function \( v(x) \) is also called a soliton. Such functions are known to be smooth and exponentially rapidly decreasing (see, e.g., [13, sec. 8.1]), and if one makes the further assumption that \( v \) is nonnegative and spherically symmetric, then there is a unique solution to (1.4) for each \( \omega > 0 \) [3,16,63]; we refer to this \( v \) as the “ground state.” There also exist radial solutions that change sign; see [2]. Such solutions are called “excited states.”

The focusing equation is said to have mass-subcritical nonlinearity if the nonlinearity parameter \( p \) satisfies the subcriticality condition

\[ 1 < p < 1 + \frac{4}{d}. \]

Mass-subcritical nonlinearity has important consequences. For instance, if \( p < 1 + 4/d \), then it is easy to show that all solutions with initial data in \( H^1 \) are global.
and bounded in $H^1$ (see [50, sec. 1.1]). Another important feature of this regime is that for any $m > 0$,

$$E_{\text{min}}(m) := \inf_{v : M(v) = m} H(v) \in (-\infty, 0),$$

and the infimum is achieved at the ground state soliton with mass $m$. In fact, it is simple to prove by a scaling argument that when $p < 1 + 4/d$, the function $E_{\text{min}}$ has the form

$$E_{\text{min}}(m) = m^\alpha E_{\text{min}}(1),$$

where $E_{\text{min}}(1) \in (-\infty, 0)$ and $\alpha > 1$ is a constant that is explicitly determined by $p$ and $d$ (see [50, sec. 1.4]). The infimum is achieved uniquely: any energy-minimizing function $v$ must be of the form

$$v(x) = Q_{\lambda(m)}(x - x_0)e^{i\gamma_0},$$

where $x_0 \in \mathbb{R}^d$ and $\gamma_0 \in \mathbb{R}$, and $Q_{\lambda(m)}$ is the unique ground state soliton with mass $m$. The ground state soliton $Q_{\lambda(m)}$ has the following explicit form: Let $Q$ be the unique positive and radially symmetric solution of the equation

$$-Q = -\Delta Q - |Q|^{p-1}Q.$$

For each $\lambda > 0$, let

$$Q_{\lambda}(x) := \lambda^{2/(p-1)}Q(\lambda x).$$

Then for each $m > 0$, there is a unique $\lambda(m) > 0$ such that $Q_{\lambda(m)}$ is the ground state soliton of mass $m$. For all of the above claims about ground state solitons in the mass-subcritical regime, see [50, secs. 1.2–1.3]. The uniqueness of the ground state is a deep result. See [65, app. B] for details.

When $p \geq 1 + 4/d$, much less is known; it is currently an area of active research (see [28,30] for recent developments and pointers to the literature).

Even in the mass-subcritical case, little is known about the long-term behavior of solutions. One particularly important conjecture, sometimes called the “soliton resolution conjecture” (see Tao [65, p. 154]), claims (vaguely) that as $t \to \infty$, the solution $u(\cdot, t)$ would look more and more like a soliton, or a union of a finite number of receding solitons. The claim may not hold for all initial conditions, but is expected to hold for “most” (i.e., generic) initial data. In the critical and supercritical regimes, the conjecture is still supposed to be true, but with the additional imposition that the solution does not blow up. The conjecture is based mainly on numerical simulations, although there has been a limited amount of progress towards a proof (see [43,59,64,66,67] and references therein). The only case where one can give a heuristic treatment is when $d = 1$ and $p = 3$, where the NLS is completely integrable (see [44,58,75]). The soliton resolution conjecture has been investigated for other dispersive systems, with partial results [19,20,41,56,57]. For significant recent progress on the soliton resolution conjecture for the energy-critical wave equation and a far more extensive survey of the literature around the conjecture, see [18].
1.3 Invariant Measures for the NLS

One approach to understanding the long-term behavior of global solutions is through the study of invariant Gibbs measures. Roughly the idea is as follows. Since the NLS is a Hamiltonian flow, one might expect by Liouville’s theorem that Lebesgue measure on the space of all functions of suitable regularity, if such a thing existed, would be an invariant measure for the flow (see, e.g., [1, p. 68] for a statement of Liouville’s theorem in the finite-dimensional setting). Since the flow preserves energy, this would imply that Gibbs measures that have density proportional to

\[ \exp(-\kappa \beta H(v)) \]

with respect to this fictitious Lebesgue measure (where \( \beta \) is arbitrary) would also be invariant for the flow. One way to make this rigorous is to first restrict the system to the unit torus \( \mathbb{T}^d \) and then consider Gibbs measures that have density proportional to

\[ \exp\left( \kappa \beta \int_{\mathbb{T}^d} |v(x)|^{p+1} \, dx \right) \]

with respect to the free-field Gaussian measure (see [34]) on the appropriate space of distributions on \( \mathbb{T}^d \). This is the pioneering idea of Lebowitz, Rose, and Speer [35]. For such a thing to make sense in \( d \geq 2 \), one has to interpret the integral in the Wick-ordered sense.

These Gibbs measures exist for the defocusing case (\( \kappa = 1 \)) for all \( p \) in \( d = 1 \) (without Wick ordering) and for \( p \leq 5 \) in \( d = 2 \), and \( p \leq 3 \) in \( d = 3 \) [25]. Furthermore, despite the fact that this measure is supported on rough functions, Bourgain showed that it is invariant under the dynamics given by (1.3) for \( d \leq 2 \) [6]. This means that the dynamics can be defined (after Wick ordering modification in \( d = 2 \)) on a set of full measure with respect to this Gibbs measure.

The focusing case (\( \kappa = -1 \)) is more delicate. Since \( H \) is unbounded from below, it is obvious that the Gibbs measure cannot exist without some restrictions on its domain. It was shown in [35] that in \( d = 1 \), the Gibbs measure exists for \( p = 3 \) when restricted to \( L^2 \) balls and that it exists for \( p = 5 \) with the additional condition of small \( \beta \). The development was continued by Bourgain [5], McKean [37], McKean and Vaninsky in [38–40], and Zhidkov [76]. In \( d = 2 \), Jaffe showed that the measure exists for \( p = 2 \) for real \( v \) when restricted to \( L^2 \) balls and after Wick ordering (see [34]), while Brydges and Slade [9] showed that this does not work when \( p = 3 \).

Invariant measures coupled with Bourgain’s development [5–8] of the so-called \( X^{s,b} \) spaces (“Bourgain spaces”) for constructing global solutions has led to important developments in this field. Recently striking advances have been made by Tzvetkov and coauthors [10–12, 68–71], Oh and coauthors [17, 45–49], and others (e.g., [42]) who use invariant measures and Bourgain’s method to construct global
solutions of the NLS and other nonlinear dispersive equations with random initial data.

Qualitative features of the infinite volume limit of Gibbs measures were studied by Brydges and Slade \cite{9} in $d = 2$ and Rider \cite{51, 52} in $d = 1$. Invariant Gibbs measures for the cubic discrete nonlinear Schrödinger equation (DNLS) in $d \geq 3$ were studied in \cite{15}.

An idea that is gaining traction in physics circles in recent years is that of considering microcanonical ensembles (see, e.g., \cite{54, 55} and references therein). The general idea—which has already been discussed at the beginning of this section—is to consider an abstract manifold of functions satisfying certain constraints (usually two) and then trying to understand the characteristics of a function picked uniformly at random from this manifold. Often the physicists alternately characterize the uniform distribution as the “maximum entropy” distribution. For example, in the context of the NLS, one looks at the “uniform distribution” on the space of all functions with a given mass and energy. The relevance of this to the long-term behavior of NLS flows is heuristically justified through Liouville’s theorem; we have more on this in the next section. It is in an attempt to understand these physical heuristics that I got interested in this line of research (and I thank Persi Diaconis—who heard about it from Julien Barré—for communicating these problems to me a few years ago). In an early paper \cite{14}, I tried to understand the behavior of functions chosen uniformly from all functions satisfying $\int |u(x)|^2 dx = m$ and $\int |u(x)|^{p+1} dx = -E$, completely ignoring the gradient term in the Hamiltonian. Already in this simplified situation one can prove interesting phase transitions and localization phenomena. In a later paper with Kay Kirkpatrick \cite{15}, the gradient term was added to the analysis, but the nonlinearity parameter was taken to be so large that the gradient term became practically unimportant. In both \cite{14} and \cite{15}, the settings were discrete and too crude to allow passage to a continuum limit. The purpose of the current manuscript is to undertake the more serious task of analyzing regimes where the gradient term actually matters and a continuum limit can be taken.

2 Main Result

Assume that $\kappa = -1$ for the rest of this manuscript. Given $m > 0$ and $E > E_{\min}(m)$ (where $E_{\min}(m)$ is the minimum energy for mass $m$, as defined in \cite{1.5}), let

\begin{equation}
S(E, m) := \{ v \in H^1(\mathbb{R}^d) : M(v) = m \text{ and } H(v) = E \}
\end{equation}

be the set of all $H^1$-functions of mass $m$ and energy $E$. Since the NLS flow \cite{1.3} preserves mass and energy, the same heuristic via Liouville’s theorem that led to \cite{1.9} would imply that a “uniform distribution” on $S(E, m)$, if such a thing existed, would be an invariant measure for the flow. In physics parlance, these measures would be the “microcanonical ensembles” corresponding to the “canonical ensembles” given by \cite{1.9}.
If the soliton resolution conjecture is indeed true and an invariant measure like the microcanonical ensemble suggested above indeed exists and describes the long-term behavior of the typical NLS flow with a given mass and energy, then it should put all its mass on soliton or multisoliton functions. This may seem like a contradiction since such functions may not have energy $E$ required for membership in $S(E, m)$. However, there is no actual contradiction since $S(E, m)$ is not compact under any reasonable metric.

Our goal is to go ahead and try to give a meaning to the abstract nonsense outlined above. To give a meaning to the notion of a uniform probability distribution on the set of all functions with a given mass and energy, we restrict ourselves first to a finite region of space, and then to a discretization of it. Instead of $\mathbb{R}^d$, therefore, our space would be the discrete grid $V_n = \{0, 1, \ldots, n-1\}^d = (\mathbb{Z}/n\mathbb{Z})^d$.

We imagine this set embedded in $\mathbb{R}^d$ as $hV_n$, where $h > 0$ is a parameter representing the grid size. Note that $hV_n$ is a discrete approximation of the box $[0, nh]^d$. We would eventually want to send $h$ to 0 and $nh$ to $\infty$.

The mass and energy of a function $v : V_n \to \mathbb{C}$ at grid size $h$ and box size $n$ are defined in analogy with (1.1) and (1.2) as

$$M_{h,n}(v) := h^d \sum_{x \in V_n} |v(x)|^2$$

and

$$H_{h,n}(v) := \frac{h^d}{2} \sum_{x,y \sim V_n \atop x \sim y} \left| \frac{v(x) - v(y)}{h} \right|^2 - \frac{h^d}{p+1} \sum_{x \in V_n} |v(x)|^{p+1},$$

where $x \sim y$ means that $x$ and $y$ are neighbors in $V_n$. For simplicity, we endow $V_n$ with the graph structure of a discrete torus, i.e., identifying $n$ with 0. Let $\mathbb{C}V_n$ denote the set of all functions from $V_n$ into $\mathbb{C}$. Take any $\epsilon > 0$, $m > 0$, and $E \in \mathbb{R}$, and define the set

$$S_{\epsilon,h,n}(E, m) := \{v \in \mathbb{C}V_n : |M_{h,n}(v) - m| \leq \epsilon, |H_{h,n}(v) - E| \leq \epsilon\}.$$

Clearly, $S_{\epsilon,h,n}(E, m)$ is a finite volume subset of the finite-dimensional space $\mathbb{C}V_n$. This set is a “manageable” version of the set $S(E, m)$ defined in (2.1). Indeed, as $\epsilon \to 0$, $h \to 0$, and $nh \to \infty$, the set $S_{\epsilon,h,n}(E, m)$ may be imagined as tending to the limit set $S(E, m)$.

We have chosen $\epsilon > 0$ to ensure that the volume is nonzero whenever the set is nonempty. In this situation, the uniform probability distribution on this set is well-defined. This uniform probability distribution, besides being an approximation to our abstract object of interest, has also a concrete interpretation as a natural invariant measure for an appropriate discrete NLS evolution on $V_n$, to be discussed in the next section.
Fix $E$ and $m$ such that $E > E_{\text{min}}(m)$. Given $\epsilon$, $h$, and $n$, let $\mu_{\epsilon,h,n}$ be the uniform probability distribution on $S_{\epsilon,h,n}(E,m)$. Let $f_{\epsilon,h,n}$ be a random function on $V_n$ with law $\mu_{\epsilon,h,n}$. Our main result, stated below, is that when the nonlinearity is mass-subcritical, the random function $f_{\epsilon,h,n}$ converges in a certain sense to the unique ground state soliton $Q_{\lambda(m)}$ of mass $m$ defined in Section 1 as $(\epsilon,h,nh)$ goes to $(0,0,\infty)$ in a certain manner.

To define the notion of convergence, we first need a way of comparing functions on $V_n$ with functions on $\mathbb{Z}^d$ and $\mathbb{R}^d$. Given $v : V_n \to \mathbb{C}$, first define its extension $v^e$ to $\mathbb{Z}^d$ by simply defining $v^e$ as equal to $v$ on $V_n$ and $0$ outside. Next, given a function $w : \mathbb{Z}^d \to \mathbb{C}$, define its “continuum image at grid size $h$” as the function $\tilde{w} : \mathbb{R}^d \to \mathbb{C}$, defined as follows. Given $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, let $x = (x_1, \ldots, x_d)$ be the unique point in $\mathbb{Z}^d$ such that for each $i$,

$$x_i \leq y_i / h < x_i + 1,$$

and let $\tilde{w}(y) := w(x)$. (In other words, $x_i = \lfloor y_i / h \rfloor$.) Lastly, given $v : V_n \to \mathbb{C}$, define its continuum image $\tilde{v} : \mathbb{R}^d \to \mathbb{C}$ at grid size $h$ as the function $\tilde{v}^e$, that is, the continuum image of the extended function $v^e$.

For each $q \in [1, \infty]$ define a pseudometric $\bar{L}^q$ on the set of measurable complex-valued functions on $\mathbb{R}^d$ as

$$(2.5) \quad \bar{L}^q(u,v) := \inf_{x_0 \in \mathbb{R}^d} \inf_{\alpha_0 \in S^1} \| u(\cdot) - \alpha_0 v(\cdot + x_0) \|_q,$$

where $S^1$ is the unit circle in the complex plane and $\| \cdot \|_q$ denotes the usual $L^q$-norm of a complex-valued function on $\mathbb{R}^d$ with respect to Lebesgue measure. (Note that $L^\infty$ is the essential supremum norm and not the supremum norm.) This is a pseudometric since $\bar{L}^q(u,v)$ may be $0$ even if $u$ and $v$ are not equal, but $v$ is of the form $v(x) = \alpha_0 u(x + x_0)$ for some $x_0 \in \mathbb{R}^d$ and $\alpha_0 \in S^1$. It is necessary to work with pseudometrics since the law of $f_{\epsilon,h,n}$ is invariant under translations and multiplication by scalars of unit modulus.

**Theorem 2.1.** Suppose that $1 < p < 1 + 4/d$. Fix $E$ and $m$ such that $E > E_{\text{min}}(m)$, where $E_{\text{min}}$ is defined in (1.5). Let $f_{\epsilon,h,n}$ be a uniform random choice from the set $S_{\epsilon,h,n}(E,m)$ defined in (2.4), and let $\tilde{f}_{\epsilon,h,n}$ denote its continuum image at grid size $h$, as defined above. Let $\bar{L}^q$ be the pseudometric defined above. Then for any $\delta > 0$ and any $q \in (2, \infty]$,

$$\lim_{h \to 0} \lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbb{P}(\bar{L}^q(\tilde{f}_{\epsilon,h,n}, Q_{\lambda(m)}) > \delta) = 0,$$

where $Q_{\lambda(m)}$ is the unique ground soliton of mass $m$ defined in Section 1. Moreover, there is a sequence $(\epsilon_k, h_k, n_k)$ with $\epsilon_k \to 0$, $h_k \to 0$, and $n_k h_k \to \infty$ as $k \to \infty$ such that for any fixed $\delta > 0$ and $q \in (2, \infty]$,

$$\lim_{k \to \infty} \mathbb{P}(\bar{L}^q(\tilde{f}_{\epsilon_k,h_k,n_k}, Q_{\lambda(m)}) > \delta) = 0.$$
Note that the energy of \( f_{\epsilon,h,n} \) is converging to \( E \), whereas the energy of \( Q_{\lambda(m)} \) is \( E_{\min}(m) \), which is strictly less than \( E \). There is no contradiction here, since the metrics of convergence are not strong enough to ensure convergence of the Hamiltonian. Nor should they be, since the difference \( E - E_{\min}(m) \) denotes the amount of energy that has “escaped” to infinity when the NLS has flowed for a long (infinite) time.

Let me emphasize here that Theorem 2.1 does not say that a certain Gibbs measure concentrates on its lowest energy state in the infinite volume limit. In fact, the soliton \( Q_{\lambda(m)} \) is not in the support of the measure \( \mu_{\epsilon,h,n} \) at all, even in the limit. What Theorem 2.1 says is more subtle: In the infinite volume continuum limit, a typical function with mass \( m \) and energy \( E \) decomposes into an “invisible” or “radiating” part that is small in \( L^\infty \)-norm but contains a significant amount of energy due to microscopic fluctuations, and a “visible part” that is close to the soliton \( Q_{\lambda(m)} \) in the \( L^\infty \)-distance.

It may seem strange that while a certain amount of energy escapes to infinity, there is no escape of mass. Again there is no contradiction, since functions of arbitrarily small \( L^2 \)-mass on \( \mathbb{R}^d \) can hold arbitrarily large amounts of energy by being very wiggly.

Theorem 2.1 does not model the full dynamics of NLS; in particular, it does not model stable multisoliton solutions consisting of two or more receding solitons that do not collapse into a single ground state. This is possibly because the effect of recession “outruns” the thermodynamic convergence to equilibrium in the infinite volume setting, whereas multisoliton solutions eventually merge into a single soliton on the finite discrete torus considered in Theorem 2.1.

Nevertheless, in the case of a finite discrete torus, Theorem 2.1 may be used to prove a version of the soliton resolution conjecture. This is the topic of the next section.

3 Is This a Proof of the Soliton Resolution Conjecture on a Large Discrete Torus with Small Grid Size?

A problem with the soliton resolution conjecture (SRC) is that its statement is not mathematically precise. The term “generic initial data” is particularly open to interpretation. For this reason, it may never be possible to completely settle the conjecture to everyone’s satisfaction. Even so, I will now make an attempt to prove a certain formulation of the conjecture on a large discrete torus. Whether this is actually a “correct” formulation may be a matter of contention.

Let all notation be as in Section 2. Fix a positive integer \( n \) and positive real numbers \( h \) and \( \epsilon \). As in Section 2, let \( V_n \) be the discrete torus \( \{0, 1, \ldots, n - 1\}^d \). The discrete Laplacian on the torus \( V_n \) with grid size \( h \) is defined as

\[
(3.1) \quad \Delta v(x) := \frac{1}{h^2} \sum_{y:y\sim x} (v(y) - v(x)),
\]
where \( y \sim x \) denotes the sum over all neighbors of \( x \) in \( V_n \), and \( v \) is any complex-valued function on \( V_n \).

The discrete focusing nonlinear Schrödinger equation (DNLS) on \( V_n \) with grid size \( h \) and nonlinearity parameter \( p \) is a family of coupled ODEs

\[
\frac{du}{dt} = -\Delta u - |u|^{p-1} u,
\]

where \( \Delta \) is the discrete Laplacian defined above and \( u(x, t) \) is a function on \( V_n \times \mathbb{R} \). The function \( u_0(x) = u(x, 0) \) is called the "initial data" for the flow. It is not difficult to show that the mass \( M_{h,n} \) defined in (2.2) and the energy \( E_{h,n} \) defined in (2.3) are conserved quantities for the DNLS flow.

The DNLS has been studied widely by physicists, but not so much by mathematicians, particularly in dimensions higher than 1. For a recent survey of the mix of rigorous and nonrigorous results that exist in the literature, see [29].

Let us formally christen the DNLS flow as \( T_t \). That is, let us denote the function \( u \) by \( T_t u \). It is easy to establish by Picard iterations and the conservation of mass that for any \( p > 1 \), \( T_t \) is a well-defined, one-to-one, and continuous map for all \( t \in \mathbb{R} \) and satisfies \( T_{t+s} = T_t T_s \); this is because in the discrete setting the right-hand side of (3.2) is a Lipschitz function of \( u \) (under any reasonable metric) where the Lipschitz constant may be bounded by a function of the mass of \( u \) and fixed quantities like \( h \) and \( n \). This is vastly simpler than the continuous case, where one has to use conservation of both mass and energy, together with the Gagliardo-Nirenberg inequality, to establish global well-posedness under the mass-subcriticality condition \( p < 1 + 4/d \). This was proved by Ginibre and Velo [24] (see also [27]).

Now fix some \( m > 0 \) and \( E > E_{\min}(m) \). Let \( S = S_{\varepsilon,h,n}(E,m) \) denote the set of all functions on \( V_n \) with mass \( [m - \varepsilon, m + \varepsilon] \) and energy \( [E - \varepsilon, E + \varepsilon] \) at grid size \( h \), defined in (2.4). By the Hamiltonian nature of the DNLS and the conservation of mass and energy, for any \( t \in \mathbb{R} \) the function \( T_t \) maps \( S \) onto itself, and the uniform probability distribution \( \mu \) on \( S \) is an invariant measure for \( T_t \).

Suppose that \( p < 1 + 4/d \). Given any probability measure \( \nu \) on \( S \), we will say that \( \nu \) satisfies the soliton resolution conjecture (SRC) with error \( \delta \) if

\[
\nu \left\{ f \in \mathbb{C}^{V_n} : \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}\{ L^\infty(\tilde{T_s f}, Q_{\lambda(m)}) > \delta \} ds < \delta \right\} = 1,
\]

where, as in Section 2, \( \tilde{T_s f} \) denotes the continuum image of \( T_s f \) at grid size \( h \), \( L^\infty \) is the pseudometric defined in (2.5), and \( Q_{\lambda(m)} \) is the ground state soliton of mass \( m \). In other words, if the initial data is chosen according to the probability measure \( \nu \) and \( \nu \) satisfies SRC with a small error, then the DNLS flow will stay close to the ground state soliton most of the time.

Let \( \mathcal{P} \) be the set of probability measures on \( S \) endowed with the usual weak-* topology. Let \( \mathcal{M} \subseteq \mathcal{P} \) be the set of all ergodic invariant probability measures of the map \( T_1 \) restricted to the set \( S \). By standard results from ergodic theory
and the Choquet representation theorem (see, e.g., remark (2) following theorem 6.10 in [72]), there is a unique probability measure \( \tau \) on \( \mathcal{M} \) such that the uniform distribution \( \mu \) on \( S \) may be represented as

\[
\mu = \int_{\mathcal{M}} \nu \, d \tau(\nu),
\]

in the sense that for all continuous \( \phi : S \to \mathbb{R} \),

\[
\int_S \phi(f) \, d\mu(f) = \int_{\mathcal{M}} \left( \int_S \phi(f) \, d\nu(f) \right) \, d\tau(\nu).
\]

This measure \( \tau \) may be called the “natural probability measure” on \( \mathcal{M} \). Intuitively, it chooses ergodic components proportional to their volume.

**Theorem 3.1.** Suppose that \( 1 < p < 1 + 4/d \). Fixing \( \epsilon, h, \) and \( n \), let \( T_1 \) be the DNLS flow defined above. Fixing \( E \) and \( m \), let \( \nu \) be a random ergodic invariant probability measure for \( T_1 \) chosen according to the “natural probability measure” \( \tau \) on \( \mathcal{M} \). Then for any \( \delta > 0 \),

\[
\lim_{h \to 0} \limsup_{\epsilon \to 0} \limsup_{n \to \infty} P(\nu \text{ satisfies SRC with error } \delta) = 1.
\]

In other words, if \( n \) is large and \( \epsilon \) and \( h \) are appropriately small, then nearly all ergodic components of the DNLS flow on \( S \) satisfy SRC with small error. Theorem 3.1 is proved in Section 24.

### 4 Microcanonical Invariant Measure for the Discrete NLS

The proof of Theorem 2.1 is based on a result for the discrete lattice with fixed grid size \( h \). To state this result, we need some preparation. First, define the mass \( M_h(\nu) \) and the energy \( H_h(\nu) \) of a function \( \nu : \mathbb{Z}^d \to \mathbb{C} \) at grid size \( h \) just as in (2.2) and (2.3), but after replacing \( V_n \) by \( \mathbb{Z}^d \).

Given \( h > 0 \) and \( m \geq 0 \), let \( E_{\text{max}}(m, h) \) and \( E_{\text{min}}(m, h) \) denote the supremum and infimum of possible energies of functions with mass \( m \) at grid size \( h \).

**Theorem 4.1.** Suppose that \( 1 < p < 1 + 4/d \). With the above definitions, for any \( h > 0 \) and \( m > 0 \) we have \( E_{\text{max}}(m, h) = 2dm/h^2 \) and \( -\infty < E_{\text{min}}(m, h) < 0 \). Moreover, the function \( E_{\text{min}} \) satisfies for all positive \( m \) and \( m' \) the strict subadditive inequality

\[
E_{\text{min}}(m + m', h) < E_{\text{min}}(m, h) + E_{\text{min}}(m', h).
\]

Lastly, \( \lim_{h \to 0} E_{\text{min}}(m, h) = E_{\text{min}}(m) \), and the convergence is uniform over compact subsets of \( (0, \infty) \).

The first couple of assertions of Theorem 4.1 are proved in Section 9. The subadditive inequality is proved in Section 16. The convergence argument is more complicated. It follows from Corollary 19.7 in Section 19. The convergence is based on the convergence of discrete solitons to continuum solitons (see Theorem
These scattered results are gathered into a formal proof of Theorem 4.1 in Section 21.

As in Section 2, we define a set of pseudometrics on the space of all complex-valued functions on $\mathbb{Z}^d$. For any $q \in [1, \infty]$, let $\bar{L}^q$ be the pseudometric on $C(\mathbb{Z}^d)$ defined as

$$
\bar{L}^q(u, v) := \inf_{\alpha_0 \in S^1} \inf_{x_0 \in \mathbb{Z}^d} \|u(\cdot) - \alpha_0 v(\cdot + x_0)\|_q,
$$

where $\| \cdot \|_q$ is the usual $L^q$-norm on $C(\mathbb{Z}^d)$.

Let $S(m, h)$ be the set of all functions $f$ with $M_h(f) = m$ and $H_h(f) = E_{\min}(m, h)$. The set $S(m, h)$ will be called the set of discrete ground state solitons with mass $m$ at grid size $h$. Note that, as in the continuum case, a simple Euler-Lagrange argument shows that any discrete ground state soliton must necessarily satisfy the discrete soliton equation

$$
-\omega v = -\Delta v - |v|^{p-1}v
$$

for some $\omega > 0$.

Unlike the continuum case, the discrete ground state soliton for a given mass may not be unique. However, we do know from the following theorem that $S(m, h)$ is nonempty and compact in the $\bar{L}^q$-topologies. Not only that, the set $S(m, h)$ also has an analogue of the so-called orbital stability property (see [50, sec. 1.3]) of continuum solitons: any function that has near-minimal energy must be nearly a soliton. The subadditive inequality from Theorem 4.1, together with the classical concentration-compactness technique ([36]; see also [50, sec. 1.4]), is the key to the proof of this result. (Note that the subadditive inequality is trivial in the continuous case by the formula (1.6).)

**Theorem 4.2.** Suppose that $1 < p < 1 + 4/d$. Let $S(m, h)$ be the set of ground state solitons of mass $m$ at grid size $h$, as defined above. Then for any $m > 0$ and $h > 0$, $S(m, h)$ is nonempty. Moreover, for any sequence of functions $f_k$ such that $M_h(f_k) \to m$ and $H_h(f_k) \to E_{\min}(m, h)$, there is a subsequence $f_{k_j}$ and some $f \in S(m, h)$ such that $f_{k_j}$ converges to $f$ in the $\bar{L}^q$ pseudometric for any $q \in [2, \infty]$.

The main argument for the proof of Theorem 4.2, using a discretization of the concentration-compactness method, is presented in Section 16. The proof is completed in Section 21.

What happens to $S(m, h)$ as $h$ tends to 0? The next theorem answers this question. As $h \to 0$, the set $S(m, h)$ shrinks to a single point, namely, the unique continuum ground state soliton $Q_{\lambda(m)}$ defined in Section 1. For related results on continuum limits of the discrete NLS in one dimension, see [31].

**Theorem 4.3.** Suppose that $1 < p < 1 + 4/d$. Let $m_k$ be a positive sequence converging to some $m > 0$. Let $h_k$ be a positive sequence tending to 0. For each $k$,
let $f_k$ be an element of $S(m_k, h_k)$. Let $\tilde{f}_k$ be the continuum image of $f_k$ at grid size $h$, as defined in Section 2. Let $\tilde{L}^q$ be the pseudometric on $L^q(\mathbb{R}^d)$ defined in Section 2. Then for any $q \in [2, \infty]$, $\lim_{k \to \infty} \tilde{L}^q(\tilde{f}_k, Q_\lambda(m)) = 0$.

The above theorem is proved by showing that for small $h$, discrete ground state solitons can be approximated by smooth functions—and then applying the orbital stability of continuum solitons [50, sec. 1.3]. To show that discrete ground state solitons can be approximated by smooth functions, one has to prove regularity estimates that do not blow up as $h \to 0$. To achieve this, the route taken in this paper is to translate the proof of regularity of continuum solitons (as sketched in [65, prop. B.7]) to the discrete setting and obtain “$h$-free” estimates. This translation necessitates the development of a slew of discrete harmonic analytic results, including fine properties of discrete Green’s functions, discrete Littlewood-Paley decompositions, and a discrete Hardy-Littlewood-Sobolev inequality of fractional integration. The harmonic analytic tools are developed in Section 17, and the regularity of discrete solitons is worked out in Section 18. The convergence argument is presented in Section 19.

Next, recall the random function $f_{e,h,n}$ from Section 2. Observe that by Liouville’s theorem, the law of $f_{e,h,n}$ is an invariant measure for the DNLS flow (3.2) at grid size $h$. In Theorem 2.1, we saw what happens to $f_{e,h,n}$ as $(e, h, nh) \to (0, 0, \infty)$ in a certain manner. On the way to proving Theorem 2.1 we first investigate what happens to $f_{e,h,n}$ as $(e, n) \to (0, \infty)$, fixing $h > 0$. What happens is the following: with high probability, $f_{e,h,n}$ is close to a discrete ground state soliton of mass $m^0$, where $m^0 \in [0, m]$ is determined in a complicated manner by $E$, $m$, and $h$. The following theorem makes this precise:

**Theorem 4.4.** Suppose that $1 < p < 1 + 4/d$. Take any $E \in \mathbb{R}$, $m > 0$, and $h > 0$ such that $E_{\min}(m, h) < E < E_{\max}(m, h)$. Let $f_{e,h,n}$ be a uniform random choice from the set $S(e, h, n)$ defined in (2.4) and extend its domain to $\mathbb{Z}^d$ by defining it to be 0 outside $V_n$. If $E \leq \frac{1}{2} E_{\max}(m, h)$, then there exists a compact set $K \subseteq [0, m]$ such that for any $\delta > 0$ and any $q \in (2, \infty]$,

$$
\lim_{e \to 0} \lim_{n \to \infty} \mathbb{P}(\inf_{m^0 \in K} \inf_{v \in S(m^0, h)} \tilde{L}^q(f_{e,h,n}, v) > \delta) = 0.
$$

Furthermore, the set $K$ can be described as follows: It is the set of all $m^0 \in [0, m]$ that maximize

$$
\log(m - m^0) - \Psi_d\left(\frac{2h^2(E - E_{\min}(m^0, h))}{m - m^0}\right),
$$

where $\Psi_d : \mathbb{R} \to [0, \infty]$ is the function

$$
\Psi_d(\alpha) = \sup_{0 < \gamma < 1} \int_{[0,1]^d} \log\left(1 - \gamma + \frac{4\beta}{\alpha} \sum_{i=1}^d \sin^2(\pi x_i)\right) dx_1 \cdots dx_d
$$

(4.2)
for $\alpha \in (0, 2d)$, $\Psi_d(\alpha) = \Psi_d(4d - \alpha)$ for $\alpha \in (2d, 4d)$, $\Psi_d(2d) = 0$, and $\Psi_d(\alpha) = \infty$ for $\alpha \geq 4d$ and $\alpha \leq 0$. Lastly, if $E \geq \frac{1}{2} E_{\text{max}}(m, h)$, then for any $\delta > 0$,

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbb{P}(\|f_{\epsilon, h, n}\|_\infty > \delta) = 0.$$ 

In a few words, the above theorem says the following: “For the DNLS on a large torus, a typical function with a given mass and energy is close in the $L^\infty$-distance to a soliton with a (possibly) different mass and energy.”

The proof of Theorem 4.4 is divided into several components. The first ingredient, proved in Section 10, is a large-deviation principle for gradients of random functions. The variational problem related to this large-deviation principle is dealt with in Section 11. The proof of the main identity in Theorem 4.4 is separated into two pieces: first an upper bound and then a matching lower bound. The upper bound in the case $E < \frac{1}{2} E_{\text{max}}(m, h)$ is proved in Section 12. The matching lower bound requires us to first prove the exponential decay of discrete solitons. This is done in Section 13, mostly along the lines of the proof of exponential decay of continuum solitons [65, prop. B.7], but with the crucial difference that we now have to deal with discrete Green’s functions. Using the information from Section 13, the lower bound is proved in Section 14. Finally, the case $E \geq \frac{1}{2} E_{\text{max}}(m, h)$ is handled in Section 15. Everything is formally put together to complete the proof of Theorem 4.4 in Section 22.

Last of all, let us indicate how all this leads to the proof of Theorem 2.1. This is quite easy, given Theorems 4.4 and 4.3. We just take the limit $h \to 0$ in the explicit formula given in Theorem 4.4. The main step is to prove that the set $K$ in Theorem 4.4 shrinks to the singleton set $\{0\}$ as $h$ goes to 0. This is done in Section 20. The convergence of discrete solitons to continuum solitons as given by Theorem 4.3 finishes the proof. This argument is formalized in Section 23.

5 Main Ideas in the Proof

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function “uniformly chosen” from the set of functions satisfying $M(f) = m$ and $H(f) = E$, whatever that means. We need to show that for any set $A$ of functions that do not contain the ground state soliton, the chance of $f \in A$ is 0.

Take any $\delta > 0$ and let $V_\delta := \{x : |f(x)| \leq \delta\}$. Then

$$\int_{V_\delta} |f(x)|^{p+1} \, dx \leq \delta^{p-1} \int_{V_\delta} |f(x)|^2 \, dx \leq \delta^{p-1} m.$$ 

Decompose $f$ as $u + v$, where $u = f 1_{V_\delta}$ and $v = f 1_{\mathbb{R}^d \setminus V_\delta}$. The above inequality shows that when $\delta$ is close to 0,

$$H(u) \approx \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx.$$
On the other hand
\[ \text{Vol}(\mathbb{R}^d \setminus V_\delta) \leq \frac{1}{\delta^2} \int_{\mathbb{R}^d \setminus V_\delta} |f(x)|^2 \, dx \leq \frac{m}{\delta^2}. \]

Let us refer to \( v \) and \( u \) as the “visible” and “invisible” parts of \( f \). The last two inequalities show that:
- The visible part is supported on a finite volume set, whose size is controlled by \( \delta \).
- The energy of the invisible part is essentially the same as the \( L^2 \)-norm squared of its gradient times \( \frac{1}{2} \).

The game now is to compute \( P(f \in A) \) by controlling the visible and invisible parts separately. The visible part, being supported on a “small” set, can be analyzed directly. For the invisible part, one has to develop joint large deviations for the mass and the gradient, since the nonlinear term is negligible in the invisible part. Solving the variational problem related to the large-deviation question, one arrives at the conclusion that the visible part must be close to the ground state soliton with high probability.

The main steps in the above program are the following:

1. Develop large-deviation estimates for the invisible part in the finite volume discrete case. This is done in Sections 7–8, 10, 12–15.
2. Analyze the variational problem related to this large-deviation question, and thereby show that with high probability, the visible part has the minimum possible energy for its mass. This is done in Sections 9, 11, and 20.
3. Pass to the infinite volume limit (keeping the grid size fixed) using a discretization of the classical concentration-compactness argument, and show convergence to discrete solitons. This is done in Sections 16 and 21.
4. Develop discrete analogues of harmonic analytic tools (Littlewood-Paley decompositions, Hardy-Littlewood-Sobolev inequality of fractional integration, Gagliardo-Nirenberg inequality, discrete Green’s function estimates, etc.) to prove smoothness estimates for discrete solitons that do not blow up as the grid size \( \to 0 \). This is done in Sections 17 and 18.
5. Use these smoothness estimates, together with the orbital stability of the ground state soliton, to prove convergence of discrete solitons to continuum solitons. This is done in Section 19.

6 Summary of Notation

In this section we summarize the notation that will be used repeatedly in this manuscript. Some of it has already been introduced, and some will be defined in later sections. The summary in this section is for the reader’s convenience.
6.1 Spaces and Norms

For a typical element \( x \in \mathbb{R}^d \), we denote the \( i \)th coordinate of \( x \) by \( x_i \). The usual euclidean norm of a vector \( x \in \mathbb{R}^d \) is denoted by \( |x| \), while the \( \ell^1 \)-norm of \( x \) is denoted by \( |x|_1 \). The same notation is used for norms of vectors in \( \mathbb{Z}^d \).

The \( L^q \)-norms for functions on \( \mathbb{R}^d \) and \( \mathbb{Z}^d \) are defined as usual, and they induce the pseudometrics \( L^q \) defined in (2.5) and (4.1). The \( L^q \)-norm of a function \( v \), whether on \( \mathbb{R}^d \) or on \( \mathbb{Z}^d \), is denoted by \( \| v \|_q \).

Sometimes we use a slightly different \( L^q \)-norm for functions on \( \mathbb{Z}^d \) by combining the usual \( L^q \)-norm with the grid size \( h \) to get an \( L^q;h \)-norm:

\[
\| v \|_{q,h} := h^{d/q} \| v \|_q.
\]

These norms will be used heavily in Sections 17 and 18.

6.2 The Discrete Torus

Assuming that the dimension \( d \) is fixed, the torus \( V_n \) in \( \mathbb{Z}^d \) is the set \( \{0, 1, \ldots, n - 1\}^d \). We say two elements \( x \) and \( y \) in \( V_n \) are neighbors, and write \( x \sim y \), if \( |x - y| = 1 \), where the difference \( x - y \) is computed by subtraction modulo \( n \) in each coordinate.

Sometimes we use \( \partial V_n \) to denote the boundary of the torus when considered as a subset of \( \mathbb{Z}^d \) (without the toric graph structure). In general, \( \partial U \) denotes the boundary of a set \( U \subseteq \mathbb{Z}^d \) or \( U \subseteq V_n \). That is, \( \partial U \) is the set of points in \( U \) that are adjacent to some point outside \( U \). Note that this boundary is different if \( U \) is considered as a subset of the torus \( V_n \) rather than as a subset of \( \mathbb{Z}^d \). Similarly, \( U^c \) denotes the set \( \mathbb{Z}^d \setminus U \) when \( U \) is a subset of \( \mathbb{Z}^d \), whereas \( V_n^c \) denotes \( V_n \setminus U \) when \( U \) is considered as a subset of the torus \( V_n \).

6.3 Mass and Energy

The mass of a function \( v : \mathbb{R}^d \to \mathbb{C} \) is defined as

\[
M(v) = \int_{\mathbb{R}^d} |v(x)|^2 \, dx,
\]

and its energy, in the context of the NLS equation (1.3) with \( \kappa = -1 \), is defined as

\[
H(v) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v(x)|^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^d} |v(x)|^{p+1} \, dx.
\]

When \( v \) is a function on \( \mathbb{Z}^d \), its mass “at grid size \( h \)” is defined as

\[
M_h(v) = h^d \sum_{x \in \mathbb{Z}^d} |v(x)|^2.
\]
Similarly, the energy at grid size $h$ is defined as

\begin{equation}
H_h(v) = \frac{h^d}{2} \sum_{x, y \in \mathbb{Z}^d} \left| \frac{v(x) - v(y)}{h} \right|^2 - \frac{h^d}{p+1} \sum_{x \in \mathbb{Z}^d} |v(x)|^{p+1}.
\end{equation}

For a function $v$ on the torus $V_n$, the mass $M_{h, n}(v)$ and the energy $H_{h, n}(v)$ are defined exactly as in the last two displays except that the sums are now over elements of $V_n$ instead of $\mathbb{Z}^d$.

For a function $v : \mathbb{Z}^d \to \mathbb{C}$, the energy $H_h(v)$ can be decomposed into the gradient component $G_h(v)$ and the nonlinear component $N_h(v)$, defined as the first and second terms on the right-hand side in (6.1), so that $H_h(v) = G_h(v) - N_h(v)$. Similarly, we define $G_{h, n}(v)$ and $N_{h, n}(v)$.

Given a subset $U \subseteq V_n$, we define

$$ G_{h, n}(v, U) = \frac{h^d}{2} \sum_{x, y \in U} \left| \frac{v(x) - v(y)}{h} \right|^2, $$

and $H_{h, n}(v, U)$, $N_{h, n}(v, U)$, and $M_{h, n}(v, U)$ are defined similarly.

### 6.4 Maximum and Minimum Energies

Suppose that $p$ and $d$ are given. For each $m \geq 0$, we define $E_{\text{max}}(m)$ and $E_{\text{min}}(m)$ to be the supremum and infimum of the set of all possible energies for functions with a given mass on $\mathbb{R}^d$, where mass and energy are defined as above. It is not difficult to verify that $E_{\text{max}}(m) = \infty$, and it is known that $E_{\text{min}}(m)$ has the form (1.6) when $p < 1 + 4/d$.

In the discrete case on $\mathbb{Z}^d$, we define $E_{\text{max}}(m, h)$ and $E_{\text{min}}(m, h)$ to be supremum and infimum of the set of all possible energies for functions with mass $m$, where both mass and energy are computed at grid size $h$. We prove a number of things about these quantities in Section 9.

Finally, for functions on the torus $V_n$, we similarly define $E_{\text{max}}(m, h, n)$ and $E_{\text{min}}(m, h, n)$.

Two related functions $E^+$ and $E^-$ are defined Section 11.

The set of functions on $\mathbb{Z}^d$ with mass $m$ at grid size $h$ that minimize energy is denoted by $S(m, h)$.

### 6.5 Notation Related to the Variational Problem

The function $\Psi_d$ defined in (4.2) is of fundamental importance in this manuscript. Two other functions, $\Theta$ and $\hat{\Theta}$, and two related subsets of $\mathbb{R}^2$, $\mathcal{R}(E, m, h)$ and $\mathcal{M}(E, m, h)$, are defined in the beginning of Section 11. All of these are used repeatedly in the manuscript.
6.6 Continuum Image of a Function

Given a function \( v : \mathbb{Z}^d \to \mathbb{C} \), its “continuum image at grid size \( h \)” is defined in Section 2, but let us repeat the definition here. The continuum image at grid size \( h \) is a function \( v : \mathbb{R}^d \to \mathbb{C} \) defined as follows. Given \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \), let \( x = (x_1, \ldots, x_d) \) be the unique point in \( \mathbb{Z}^d \) such that for each \( i \),
\[
x_i \leq y_i / h < x_i + 1,
\]
and let \( \overline{v}(y) := v(x) \). When \( v \) is a function on \( V_n \), we define the continuum image by first defining the function to be 0 on \( \mathbb{Z}^d \setminus V_n \), and then defining the continuum image as above.

7 Comparison with a Gaussian Function

Let \( \mathbf{D} = (\mathbf{D}(x))_{x \in V_n} \) be a collection of i.i.d. complex Gaussian random variables, with \( \mathbb{E}(\mathbf{D}(x)) = \mathbb{E}(\mathbf{D}(x)^2) = 0 \) and
\[
\mathbb{E} \left| \mathbf{D}(x) \right|^2 = \frac{1}{(nh)^d}.
\]
Recall the function \( f_{\epsilon, h, n} \) and the set \( S_{\epsilon, h, n}(E, m) \) defined in Section 2. The following basic lemma connects the properties of \( \phi \) with that of \( f_{\epsilon, h, n} \).

**Lemma 7.1.** Take any \( m > 0 \) and \( E \in \mathbb{R} \). Let \( f = f_{\epsilon, h, n} \) and \( S = S_{\epsilon, h, n}(E, m) \) for simplicity. Then for any \( A \subseteq S \),
\[
\mathbb{P}(f \in A) \leq e^{2nd \epsilon} \frac{\mathbb{P}(\phi \in A)}{\mathbb{P}(\phi \in S)}.
\]

**Proof.** For any measurable \( A \subseteq \mathbb{C}V_n \),
\[
\mathbb{P}(f \in A) = \frac{\text{Vol}(A)}{\text{Vol}(S)}.
\]
Now, \( |M_{h,n}(v) - m| \leq \epsilon \) when \( v \in S \). Therefore,
\[
\mathbb{P}(\phi \in S) = \int_S \frac{(nh)^d e^{-nd M_{h,n}(v)}}{\pi^{nd}} dv \leq (nh^d \pi^{-1})^{nd} e^{-nd (m-\epsilon)} \text{Vol}(S).
\]
Similarly,
\[
\mathbb{P}(\phi \in A) = \int_A \frac{e^{-nd M_{h,n}(v)}}{\pi^{nd} h^{-dn^d/2}} dv \geq (nh^d \pi^{-1})^{nd} e^{-nd (m+\epsilon)} \text{Vol}(A).
\]
This completes the proof.
8 Diagonalizing the Laplacian

Let $\Gamma = (\Gamma(x, y))_{x, y \in V_n}$ be the matrix defined as

$$
\Gamma(x, y) = \begin{cases} 
2d & \text{if } x = y, \\
-1 & \text{if } x \sim y, \\
0 & \text{in all other cases.}
\end{cases}
$$

The matrix $\Gamma$ may be viewed as an operator acting on $\mathbb{C}V_n$ in the natural sense. The action of $\Gamma$ on a function $f : V_n \to \mathbb{C}$ will be denoted by $\Gamma f$. Notice that $\Gamma = -h^2 \Delta$, where $\Delta$ is the discrete Laplacian on the torus $V_n$ with grid size $h$, as defined in (3.1), and that $\Gamma$ is a real symmetric matrix of order $n^d$. In this section, we will write down the spectral decomposition of $\Gamma$.

For two functions $u, v \in \mathbb{C}V_n$, let $(u, v)$ be the standard inner product,

$$(u, v) := \sum_{x \in V_n} u(x)\overline{v(x)},$$

where $\overline{v(x)}$ is the complex conjugate of $v(x)$. Notice that for any $v$,

$$(v, \Gamma v) = (\Gamma v, v) = \sum_{x, y \in V_n \atop x \sim y} |v(x) - v(y)|^2.$$

We use the notation $x_i$ to denote the $i^{th}$ coordinate of a vector $x \in \mathbb{R}^d$.

**Lemma 8.1.** For each $y = (y_1, \ldots, y_d) \in V_n$, let $\rho_y$ be the function

$$
\rho_y(x) := n^{-d/2} e^{i2\pi (y_1x_1 + \cdots + y_dx_d)/n}.
$$

Then the functions $(\rho_y)_{y \in V_n}$ form a complete orthonormal system of eigenfunctions of $\Gamma$, and the eigenvalue corresponding to $\rho_y$ is

$$
\lambda_y := 4 \sum_{i=1}^{d} \sin^2(\pi y_i/n).
$$

**Proof.** To prove orthogonality, first notice that for any $k \in \mathbb{Z}$, $r := e^{i2\pi k/n}$ is an $n^{th}$ root of unity, and hence

$$
\sum_{j=0}^{n-1} e^{i2\pi kj/n} = \sum_{j=0}^{n-1} r^j = \begin{cases} 
0 & \text{if } k \neq 0, \\
n & \text{if } k = 0.
\end{cases}
$$
Thus, for any \( y, y' \in V_n \),

\[
(\rho_y, \rho_{y'}) = n^{-d} \sum_{x_1, \ldots, x_d = 0} \sum_{i=1}^{n-1} e^{i2\pi((y_1-y'_1)x_1 + \cdots + (y_d-y'_d)x_d)/n}
\]

\[
= n^{-d} \prod_{i=1}^{d} \sum_{x_i = 0}^{n-1} e^{i2\pi(y_i-y'_i)x_i/n}
\]

\[
= \begin{cases} 
0 & \text{if } y \neq y', \\
1 & \text{if } y = y'. 
\end{cases}
\]

To show that \( \rho_y \) is an eigenfunction of \( \Gamma \) with eigenvalue \( \lambda_y \), note that for any \( x \in V_n \),

\[
\Gamma \rho_y(x) = \sum_{z \sim x} (\rho_y(x) - \rho_y(z))
\]

\[
= \sum_{i=1}^{d} (2 - e^{i2\pi y_i/n} - e^{-i2\pi y_i/n}) \rho_y(x)
\]

\[
= -\rho_y(x) \sum_{i=1}^{d} (e^{i\pi y_i/n} - e^{-i\pi y_i/n})^2 = 4 \rho_y(x) \sum_{i=1}^{d} \sin^2(\pi y_i/n).
\]

This completes the proof of the lemma.

Let \( R \in \mathbb{C}^{V_n \times V_n} \) be the matrix whose \( y \)-th column is \( \rho_y \) for each \( y \in V \). Note that \( R \) is a unitary matrix. Let \( \Lambda \in \mathbb{C}^{V_n \times V_n} \) be the diagonal matrix whose \( y \)-th diagonal element is \( \lambda_y \). Then

\[
\Gamma = R \Lambda R^*,
\]

where \( R^* \) is the adjoint of \( R \). This is the spectral decomposition of \( \Gamma \).

9 The Set of Possible Energies for a Given Mass

Let \( E_{\min}(m, h, n) \) and \( E_{\max}(m, h, n) \) be the minimum and maximum possible energies at grid size \( h \) of a function \( f : V_n \to \mathbb{C} \) with mass \( m \). Since the map \( f \mapsto H_{h,n}(f) \) is continuous and \( \{ f : M_{h,n}(f) = m \} \) is a compact connected subset of \( \mathbb{C}^{V_n} \), for every \( E \in [E_{\min}(m, h, n), E_{\max}(m, h, n)] \) there exists some \( f \) on \( V_n \) with mass \( m \) and energy \( E \). Recall that \( E_{\min}(m, h) \) and \( E_{\max}(m, h) \) are the infimum and supremum of the set of possible energies of functions with mass \( m \) on the full lattice \( \mathbb{Z}^d \) at grid size \( h \).

**Lemma 9.1.** For any \( m \geq 0 \) and any \( E \in (E_{\min}(m, h), E_{\max}(m, h)) \), there is a function \( f : \mathbb{Z}^d \to \mathbb{C} \) such that \( M_{h}(f) = m \) and \( H_{h}(f) = E \). Moreover, for any \( m \geq 0 \), any function on \( \mathbb{Z}^d \) with mass \( m \) and energy \( E \) (at grid size \( h \)) satisfies

\[
|E| \leq C(p, d, h)(m + m^{(p+1)/2}).
\]
PROOF. If $M_h(f) = m$, then for any $x$, $|f(x)|^2 \leq mh^{-d}$. Therefore,
\[ |f(x)|^{p+1} \leq (mh^{-d})^{(p-1)/2} |f(x)|^2.\]

Thus,
\[
H_h(f) \leq 2dh^{d-2} \sum_{x \in \mathbb{Z}^d} |f(x)|^2 + \frac{(mh^{-d})^{(p-1)/2}h^d}{p+1} \sum_{x \in \mathbb{Z}^d} |f(x)|^2
\]
\[= 2dh^{-2}m + \frac{h^{-d(p-1)/2}m^{(p+1)/2}}{p+1}.\]

This proves the inequality.

Next, take any two functions $f, g \in \mathbb{C}^{\mathbb{Z}^d}$ with $M_h(f) = M_h(g) = m > 0$. If $f = g$ or $f = -g$, then $H_h(f) = H_h(g)$. Suppose that $f \neq \pm g$. For each $\theta \in [0, 1]$, let $f_\theta := \theta f + (1-\theta)g$. Since $M_h(f) = M_h(g)$ and $f \neq -g$, it is easy to see that $f_\theta$ is not the 0 function for any $\theta$. In particular, $M_h(f_\theta) > 0$. Let
\[
w_\theta(x) := \sqrt{\frac{m}{M_h(f_\theta)}} f_\theta(x) \quad \text{for } x \in \mathbb{Z}^d.
\]

Then $M_h(w_\theta) = m$ for all $\theta$. Moreover, it is easy to prove that $H_h(w_\theta)$ varies continuously from $H_h(g)$ to $H_h(f)$ as $\theta$ varies from 0 to 1. This shows that for every $E \in (E_{\min}(m, h), E_{\max}(m, h))$, there is a function $f$ with $M_h(f) = m$ and $H_h(f) = E$. □

**Lemma 9.2.** As $n$ goes to infinity, $E_{\min}(m, h, n)$ tends to $E_{\min}(m, h)$ and $E_{\max}(m, h, n)$ tends to $E_{\max}(m, h)$. Moreover, in both cases, the convergence is uniform on compact subsets of $[0, \infty)$ (for the parameter $m$, keeping $h$ fixed). The functions $E_{\max}(\cdot, h)$ and $E_{\min}(\cdot, h)$ are absolutely continuous on $[0, \infty)$.

**Proof.** Take any $\epsilon > 0$. Let $f \in \mathbb{C}^{\mathbb{Z}^d}$ be a function such that $M_h(f) = m$ and $H_h(f) \leq E_{\min}(m, h) + \epsilon$. For each $n$, define a function $f_n$ on $V_n$ as simply the restriction of $f$ on $V_n$. Then it is easy to see that $M_{h,n}(f_n) \to M_h(f)$ and $H_{h,n}(f_n) \to H_h(f)$ as $n \to \infty$. Since $\epsilon$ is arbitrary, this shows that
\[
\limsup_{n \to \infty} E_{\min}(m, h, n) \leq E_{\min}(m, h).
\]

Fix $n$ and a function $f \in \mathbb{C}^{V_n}$ such that
\[
M_{h,n}(f) = m \quad \text{and} \quad H_{h,n}(f) \leq E_{\min}(m, h, n) + \epsilon.
\]

Define a function $g$ on $\mathbb{Z}^d$ as follows: For each $x \in V_n$, let $f_x$ be the translated function $f_x(y) := f(x + y)$, where the addition is modulo $n$ in each coordinate.
Let $\partial V_n$ denote the boundary of $V_n$ when $V_n$ is considered as a subset of $\mathbb{Z}^d$. Recall the notation $M_{h,n}(f, U)$ from Section 6. Then
\[
\sum_{x \in V_n} M_{h,n}(f_x, \partial V_n) = \sum_{x \in V_n} \sum_{y \in \partial V_n} h^d |f(x + y)|^2 = \sum_{z \in \partial V_n} h^d |f(z)|^2 |\partial V_n| = M_{h,n}(f) |\partial V_n|.
\]
This shows that there exists $x \in V_n$ such that
\[
M_{h,n}(f_x, \partial V_n) \leq M_{h,n}(f) |\partial V_n| n^{-d} \leq C(d, m)n^{-1}.
\]
Take such an $x$ and define $g : \mathbb{Z}^d \to \mathbb{C}$ as
\[
g(y) := \begin{cases} f_x(y) & \text{if } y \in V_n, \\ 0 & \text{otherwise.} \end{cases}
\]
Clearly, $M_h(g) = m$. Since $x$ was chosen so that the “boundary effect” is small, $|H_{h,n}(f_x) - H_{h,n}(f)| \leq C(d, m, n) n^{-1}$. Since $\epsilon$ is arbitrary and $H_{h,n}(f_x) = H_{h,n}(f) \leq \varepsilon_{\min}(m, h, n) + \epsilon$, this shows that
\[
\liminf_{n \to \infty} \varepsilon_{\min}(m, h, n) \geq \varepsilon_{\min}(m, h).
\]
The proof for $\varepsilon_{\max}$ is similar.

To prove uniform convergence on compact subsets of $[0, \infty)$, we will first prove it for compact subsets of $(0, \infty)$. We will show that the collection of functions $(\varepsilon_{\min}(\cdot, h, n))_{n \geq 1}$ is equi-Lipschitz-continuous on any compact subinterval of the positive reals and apply the Arzelà-Ascoli theorem. This will also prove absolute continuity of the function $\varepsilon_{\min}$ on $(0, \infty)$. Continuity at 0 follows from Lemma 9.1. The result for $\varepsilon_{\max}$ follows similarly.

Take any $n$ and $0 < a < b$. Take $a \leq m' < m \leq b$. Let $f$ be an energy minimizing function (at grid size $h$) of mass $m$ on the torus $V_n$. Let $f' := (m'/m)^{1/2} f$. Then $M_{h,n}(f') = m'$. Note that
\[
H_{h,n}(f') = (m'/m) G_{h,n}(f) + (m'/m)^{(p+1)/2} N_{h,n}(f).
\]
Thus,
\[
|H_{h,n}(f') - H_{h,n}(f)| \leq
\]
\[
|(m'/m) - 1| G_{h,n}(f) + |(m'/m)^{(p+1)/2} - 1| N_{h,n}(f).
\]
Since $N_{h,n}(f)$ and $G_{h,n}(f)$ can both be bounded above by
\[
C_1(p, d, h)m C_2(p, d, h),
\]
this shows that
\[
\varepsilon_{\min}(m', h, n) \leq \varepsilon_{\min}(m, h, n) + C(a, b, p, d, h)m' - m.
\]
Similarly, taking $g'$ such that $M_{h,n}(g') = m'$ and $H_{h,n}(g') = \varepsilon_{\min}(m', h, n)$, and letting $g := (m/m')g'$, it follows that $\varepsilon_{\min}(m, h, n)$ is bounded above by
This proves uniform convergence on compact subsets of \((0, \infty)\). To prove uniform convergence on compact subsets of \([0, \infty)\), one simply notices that the uniform bound on the energy given in Lemma 9.1 holds for functions on \(V_n\) as well (the bound will be the same, independent of \(n\)). \(\square\)

Recall that we say \(a_j \sim b_j\) as \(j \to \infty\) if \(a_j\) and \(b_j\) are sequences that satisfy 
\[
\lim_{j \to \infty} a_j = b_j.
\]

**Lemma 9.3.** Let \(c_j\) be a sequence such that for some \(C > 0\) and \(\alpha \geq 0\), \(c_j \sim C j^\alpha\) as \(j \to \infty\). Then for any \( \beta \geq 0\),
\[
\sum_{j=0}^{k-1} c_j (k-j)^\beta \sim C k^{\alpha+\beta+1} \int_0^1 x^\alpha (1-x)^\beta \, dx \quad \text{as } k \to \infty.
\]

**Proof.** Writing
\[
\sum_{j=0}^{k-1} c_j (k-j)^\beta = k^{\alpha+\beta+1} \frac{1}{k} \sum_{j=0}^{k-1} (c_j/k^\alpha)(1-j/k)^\beta,
\]

note that
\[
\left| \frac{1}{k} \sum_{j=0}^{k-1} (c_j/k^\alpha)(1-j/k)^\beta - C \frac{1}{k} \sum_{j=0}^{k-1} (j/k)^\alpha (1-j/k)^\beta \right| \leq \frac{1}{k} \sum_{j=0}^{k-1} \frac{|c_j - C j^\alpha|}{k from \(c_j/k^\alpha\)).
\]

Since \(c_j/C j^\alpha \to 1\) as \(j \to \infty\), the above bound tends to 0 as \(k \to \infty\). Riemann sum approximation gives
\[
\lim_{k \to \infty} \frac{C}{k} \sum_{j=0}^{k-1} (j/k)^\alpha (1-j/k)^\beta = C \int_0^1 x^\alpha (1-x)^\beta \, dx.
\]
This completes the proof. \(\square\)

**Lemma 9.4.** Suppose that \(1 < p < 1 + 4/d\). Then for any \(m > 0\) and \(A > 0\),
\[
\sup_{0<h\leq A} E_{\text{min}}(m, h) < 0.
\]

**Proof.** Fix \(m > 0\). For each positive integer \(k\), define a function \(f_k\) as follows: If \(|x|_1 \geq k\), let \(f_k(x) = 0\). If \(|x|_1 < k\), let
\[
f_k(x) = A_k (k-|x|_1),
\]
where \(A_k\) is a positive constant such that \(M_h(f_k) = m\). Since the number of vertices \(x\) with \(|x|_1 = j\) asymptotes to \(C(d)j^{d-1}\) as \(j \to \infty\), it follows from
Lemma 9.3 that
\[ A_k^2 \sim C(d, m)h^{-d}k^{-(d+2)} \quad \text{as } k \to \infty. \]

Note that if \( x \) and \( y \) are neighboring points, then \( f_k(x) \neq f_k(y) \) if and only if one of them has \( \ell^1 \)-norm \( j \) and the other has \( \ell^1 \)-norm \( j+1 \) for some \( 0 \leq j < k \). And in that case,
\[ |f_k(x) - f_k(y)| = A_k. \]

There are \( \sim C(d)j^{d-1} \) such pairs as \( j \to \infty \). Thus, again by Lemma 9.3,
\[ G(f_k) \sim C(d)h^{d-2}A_k^2 \sum_{j=0}^{k-1} j^{d-1} \sim C(d, m)h^{d-2}A_k^2k^d \sim C(d, m)(hk)^{-2}. \]

Again by Lemma 9.3,
\[
\begin{align*}
\sum_{x \in \mathbb{Z}^d} h^d \sum_{j=0}^{k-1} |f_k(x)|^{p+1} &= h^d \sum_{j=0}^{k-1} \sum_{x:|x|=j} |f_k(x)|^{p+1} \\
&\sim C(d)h^dA_k^{p+1} \sum_{j=0}^{k-1} j^{d-1}(k-j)^{p+1} \\
&\sim C(p, d, m)h^{d-d(p+1)/2}k^{-(d+2)(p+1)/2}k^{p+d+1} \\
&= C(p, d, m)(hk)^{-d(p-1)/2}.
\end{align*}
\]

Thus, for all \( k \),
\[ H(f_k) \leq C_1(p, d, m)(hk)^{-2} - C_2(p, d, m)(hk)^{-d(p-1)/2}. \]

If \( p < 1 + 4/d \), then \( d(p-1)/2 < 2 \). It is now easy to see from the above bound that if \( h \leq A \) for some constant \( A \), then
\[ E_{\min}(m, h) \leq \inf_k H(f_k) \leq -C(p, d, m, A) < 0, \]

which concludes the proof.

\[ \square \]

**Lemma 9.5.** For any \( m > 0 \), \( E_{\max}(m, h) = 2dm/h^2 \).

**Proof.** Fix \( n \). Let \( \Gamma, R, \) and \( \Lambda \) be the matrices from Section 8. Let
\[ \lambda_{\max} := \max_{y \in V_n} \lambda_y = \max_{y \in V_n} \sum_{i=1}^d 4 \sin^2(\pi y_i/n). \]
Clearly, $\lambda_{\text{max}} \leq 4d$ always, and $\lambda_{\text{max}} \to 4d$ as $n \to \infty$. Note that for any $f \in \mathbb{C}V_n$ with $M_{h,n}(f) = m$,

$$H_{h,n}(f) \leq \frac{h^{d-2}}{2} \sum_{x,y \in V_n, x \sim y} |f(x) - f(y)|^2$$

$$= \frac{h^{d-2}}{2} (f, \Gamma f) \leq \frac{h^{d-2}}{2} \lambda_{\text{max}} \sum_{x \in V_n} |f(x)|^2 = \frac{\lambda_{\text{max}} m}{2h^2}.$$ 

Thus, $E_{\text{max}}(m, h, n) \leq 2dm/h^2$.

Next, let $f := \sqrt{mh^{-d}} \rho_{[n/2]}$, where $(\rho_y)_{y \in V_n}$ are the eigenfunctions defined in Section 8 and $[n/2]$ is the element of $V_n$ whose components are all equal to the integer part of $n/2$. Note that $|f(x)| = \sqrt{mh^{-d} n^{-d}/2}$ for each $x$. Therefore, $M_{h,n}(f) = m$ and

$$H_{h,n}(f) = \frac{h^{d-2}}{2} \sum_{x,y \in V_n, x \sim y} |f(x) - f(y)|^2 - \frac{h^d}{p+1} \sum_{x \in V_n} |f(x)|^{p+1}$$

$$= \frac{h^{d-2}}{2} (f, \Gamma f) - \frac{h^d}{p+1} (mh^{-d} (p+1/2)n^{-d(p-1)/2}$$

$$= \frac{h^{d-2}\lambda_{[n/2]}}{2} \sum_{x \in V_n} |f(x)|^2 - \frac{h^d}{p+1} (mh^{-d} (p+1/2)n^{-d(p-1)/2}.$$

Since $\lambda_{[n/2]} \to 4d$ as $n \to \infty$ and the second term goes to 0, this shows that

$$\liminf_{n \to \infty} E_{\text{max}}(m, h, n) \geq \frac{2dm}{h^2}.$$

By Lemma 9.2, this completes the proof. 

10 Large Deviations for the Gradient

Recall the function $\Psi_d$ defined in the statement of Theorem 4.4. The following proposition summarizes some important properties of this function:

**Proposition 10.1.** The function $\Psi_d$ has the following properties: it is continuous in $(0, 4d)$, it is strictly decreasing in $(0, 2d]$ and strictly increasing in $[2d, 4d)$, $\Psi_d(2d) = 0$, and

$$\lim_{\alpha \downarrow 0} \Psi_d(\alpha) = \infty = \lim_{\alpha \uparrow 4d} \Psi_d(\alpha).$$

In particular, $\Psi_d$ is a continuous function from $\mathbb{R}$ into $[0, \infty]$.

**Proof.** It is obvious from the definition that $\Psi$ is strictly decreasing in $(0, 2d)$ and by symmetry, strictly increasing in $(2d, 4d)$. To extend the monotonicity up
to the point $2d$, we have to show that $\Psi(\alpha) > 0$ for all $\alpha < 2d$. For each $\alpha \leq 2d$ and $0 < \gamma < 1$ define

$$K_\alpha(\gamma) := \int_{[0,1]^d} \log \left(1 - \gamma + \frac{4\gamma}{\alpha} \sum_{i=1}^{d} \sin^2(\pi x_i)\right) dx_1 \cdots dx_d,$$

so that

$$\Psi_d(\alpha) = \sup_{0<\gamma<1} K_\alpha(\gamma).$$

By dominated convergence, $K_\alpha$ is continuous on $[0, 1)$ and differentiable in $(0, 1)$, and has a right derivative at 0. Moreover, $K_\alpha(0) = 0$. A simple computation shows that $K_\alpha'(0) = 2d/\alpha - 1 > 0$ if $\alpha < 2d$. Thus if $\alpha < 2d$, then $K_\alpha$ is strictly increasing at 0 and therefore by (10.1), $\Psi_d(\alpha)$ is positive. By symmetry, $\Psi_d(\alpha)$ is positive for $\alpha \in (2d, 4d)$ also.

Next, let $\alpha_n$ be a sequence converging to $\alpha \in (0, 2d)$. By dominated convergence, $K_{\alpha_n} \to K_\alpha$ pointwise in $[0, 1)$. It is easily seen that $K_{\alpha_n}$ is a strictly concave function for every $n$. By concavity and pointwise convergence, it follows that $\sup_{0<\gamma<1} K_{\alpha_n}(\gamma)$ converges to $\sup_{0<\gamma<1} K_\alpha(\gamma)$, proving that $\Psi_d$ is continuous at $\alpha$. To show continuity at $2d$, follow the same argument and simply note that $K_{2d}$ attains its maximum at 0, since $K_{2d}$ is concave in $[0, 1)$ and $K_{2d}'(0) = 0$.

Lastly, note that for any fixed $\gamma \in (0, 1)$, $K_\alpha(\gamma) \to \infty$ as $\alpha \to 0$. This proves that $\lim_{\alpha \downarrow 0} \Psi_d(\alpha) = \infty$. By symmetry, $\lim_{\alpha \uparrow 4d} \Psi_d(\alpha) = \infty$. \hfill \Box

Next, let $\xi = (\xi(x))_{x \in V_n}$ be a random function chosen uniformly from the unit sphere

$$\left\{ u \in \mathbb{C}^{V_n} : \sum_{x \in V_n} |u(x)|^2 = 1 \right\}.$$

The following theorem is a large-deviation result for the gradient of $\xi$ that is of fundamental importance in the rest of the manuscript:

**Theorem 10.2.** For each $\alpha \in (0, 2d)$,

$$\lim_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \leq \alpha \right) = -\Psi_d(\alpha).$$

Moreover, for any $\delta > 0$, the same limit holds for

$$\frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \leq \alpha, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right).$$

The same conclusions hold if $\alpha \in (2d, \infty)$ and the “$\leq \alpha$” is replaced by “$\geq \alpha$” in both expressions.
PROOF OF THEOREM 10.2 IN THE CASE $0 < \alpha < 2d$. Fix $\alpha \in (0, 2d)$ and a positive integer $n$. Let $\phi$ be the Gaussian random function defined in Section 7. Let $\Gamma$, $R$, and $\Lambda$ be the matrices defined in Section 8. Let

$$\tau := R^* \phi,$$

where $R^*$ is the adjoint of $R$. Since $R$ is a unitary matrix, $\tau$ has the same distribution as $\phi$. Moreover,

$$\sum_{y \in V_n} |\tau(y)|^2 = \sum_{x \in V_n} |\phi(x)|^2$$

and

$$\sum_{y \in V_n} \lambda_y |\tau(y)|^2 = (\phi, \Gamma \phi) = \sum_{x, y \in V_n} |\phi(x) - \phi(y)|^2.$$

Let

$$\xi(x) := \frac{\phi(x)}{(\sum_{y \in V_n} |\phi(y)|^2)^{1/2}}.$$

Then $\xi$ is uniformly distributed on the unit sphere of $\mathbb{C} V_n$. Moreover, by (10.3) and (10.4),

$$\sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 = \frac{\sum_{x, y \in V_n} |\phi(x) - \phi(y)|^2}{\sum_{x \in V_n} |\phi(x)|^2} = \frac{\sum_{y \in V_n} \lambda_y |\tau(y)|^2}{\sum_{y \in V_n} |\tau(y)|^2}.$$

Let $\eta(y) := n^d h^d |\tau(y)|^2$. Then $\eta(y)$ is an exponential random variable with mean 1 and the $\eta(y)$’s are independent. For any $\alpha \in (0, 2d)$,

$$\mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \leq \alpha \right) = \mathbb{P}\left( \sum_{y \in V_n} (\lambda_y - \alpha)|\tau(y)|^2 \leq 0 \right) = \mathbb{P}\left( \sum_{y \in V_n} (\lambda_y - \alpha)\eta(y) \leq 0 \right).$$

Thus, for any $\theta \in [0, 1/\alpha)$,

$$\mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \leq \alpha \right) \leq \mathbb{E}(e^{-\theta \sum_{y \in V_n} (\lambda_y - \alpha)\eta(y)}) = \prod_{y \in V_n} \frac{1}{1 + \theta(\lambda_y - \alpha)}.$$
Note that we need the restriction that $\theta < 1/\alpha$ since $\lambda_0 = 0$. Now,

\begin{equation}
(10.5) \quad -\frac{1}{n^d} \log \prod_{y \in V_n} \frac{1}{1 + \theta(\lambda_y - \alpha)} = \frac{1}{n^d} \sum_{y_1, \ldots, y_d = 0}^{n-1} \log \left(1 - \theta \alpha + 4\theta \sum_{i=1}^d \sin^2(\pi y_i / n)\right) =: I_n(\theta).
\end{equation}

The sequence of functions $I_n$ converges pointwise to the function $I$ on $[0, 1/\alpha)$, where

$$I(\theta) = \int_{[0,1]^d} \log \left(1 - \theta \alpha + 4\theta \sum_{i=1}^d \sin^2(\pi x_i)\right) dx_1 \cdots dx_d.$$ 

This shows that for each $\theta \in [0, 1/\alpha)$,

\begin{equation}
(10.6) \quad \limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n, x \sim y} |\xi(x) - \xi(y)|^2 \leq \alpha \right) \leq -I(\theta).
\end{equation}

Now, $I(\theta) = K_\alpha(\theta \alpha)$, where $K_\alpha$ is defined in the proof of Proposition 10.1. By (10.5) and (10.6), this shows that

\begin{equation}
(10.7) \quad \limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n, x \sim y} |\xi(x) - \xi(y)|^2 \leq \alpha \right) \leq -\sup_{\theta \in (0,1/\alpha)} I(\theta) = -\sup_{\theta \in (0,1/\alpha)} K_\alpha(\theta \alpha) = -\Psi_d(\alpha).
\end{equation}

This proves the upper bound in the case $0 < \alpha < 2d$. Next, we establish the matching lower bound.

As noted in Proposition 10.1, $K_\alpha$ is a concave function and is strictly increasing at 0 if $\alpha < 2d$. Thus, the maximum of $I$ in $[0, 1/\alpha)$ must be achieved either inside $(0, 1/\alpha)$ or as $\theta \to 1/\alpha$. The first case holds if $I'(1/\alpha) < 0$, and the second happens if $I'(1/\alpha) \geq 0$. The proof of the lower bound is different in the two cases.

**Case 1.** $I'(1/\alpha) < 0$. In this case, there is a unique $\theta^* \in (0, 1/\alpha)$ where $I$ is maximum. Fix any $\epsilon > 0$, and let $\theta'$ be a point so close to $\theta^*$ from the right that $I'(\theta') \in (-\epsilon, 0)$. Fixing $n$, let $\eta' = (\eta'(y))_{y \in V_n}$ be a collection of independent random variables where $\eta'(y)$ is an exponential random variable with mean $1/(1 + \theta'(\lambda_y - \alpha))$. Assume that $\eta'$ is defined on the same probability space as all other variables and is independent of everything else.

Recall the definition (10.2) of $\tau$. Let $\sigma(y) := \tau(y)/|\tau(y)|$. Since $\tau(y)$ is a complex Gaussian random variable with mean 0, $\sigma(y)$ and $|\tau(y)|$ are independent random variables. Note that

$$\tau(y) = \sigma(y)|\tau(y)| = \sigma(y) \sqrt{\eta'(y)}.$$
Define a random vector $\xi' = (\xi'(x))_{x \in V_n}$ as follows: Take the variables $\sigma(y)$ and $\eta'(y)$ defined above, and let

$$\tau'(y) := \sigma(y) \sqrt{\frac{\eta'(y)}{n^d h^d}}.$$

Let $\phi' := R\tau'$, and let

$$\xi'(x) := \frac{\phi'(x)}{(\sum_{y \in V_n} |\phi'(y)|^2)^{1/2}} = \frac{\phi'(x)}{(\sum_{y \in V_n} |\tau'(y)|^2)^{1/2}}.$$

Note that $\tau'$, just like $\tau$, is a collection of independent complex Gaussian random variables with $\mathbb{E}(\tau'(y)) = \mathbb{E}(\tau'(y)^2) = 0$ for each $y$, but unlike $\tau$,

$$\mathbb{E}|\tau'(y)|^2 = \frac{1}{n^d h^d (1 + \theta'(\lambda_y - \alpha))}. \tag{10.8}$$

Consequently, $\phi'$ is a complex Gaussian random vector satisfying

$$\mathbb{E}|\phi'(x)|^2 \leq \frac{1}{n^d h^d (1 - \theta' \alpha)} \quad \text{for all } x, \tag{10.9}$$

but its components are not independent.

Note that the map that takes $(\sigma, \eta)$ to $\xi$ is the same map as the one that takes $(\sigma, \eta')$ to $\xi'$. Therefore a simple change-of-measure computation shows that for any measurable set $A \subseteq \mathbb{C} V_n$,

$$\mathbb{P}(\xi \in A) = \mathbb{E}(\rho(\eta') \mathbb{1}_{\{\xi' \in A\}}),$$

where

$$\rho(\eta') = e^{\sum_{y \in V_n} \theta'(\lambda_y - \alpha) \eta'(y)} \prod_{y \in V_n} \frac{1}{1 + \theta'(\lambda_y - \alpha)}.$$

Thus, if we fix $\delta > 0$ and let $E$ be the event

$$E := \left\{ -4n^d \epsilon \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 0, \text{ and } \max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n} |\tau'(x)|^2 \right\},$$

...
then
\[
\mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \leq \alpha, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right)
\]
\[
= \mathbb{E}\left( \rho(\eta') \left\{ \sum_{x, y \in V_n} |\xi'(x) - \xi'(y)|^2 \leq \alpha, \max_{x \in V_n} |\xi'(x)|^2 \leq n^{-d(1-\delta)} \right\} \right)
\]
\[
= \mathbb{E}\left( \rho(\eta') \left\{ \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 0, \max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n} |\tau'(x)|^2 \right\} \right)
\]
\[
\geq \mathbb{E}(\rho(\eta') |E|) \geq e^{-4n^d \theta' \varepsilon} \mathbb{P}(E) \prod_{y \in V_n} \frac{1}{1 + \theta'(\lambda_y - \alpha)}.
\]

Now, if \( I_n \) is the function defined in (10.5), a simple computation gives
\[
I_n' (\theta') = \frac{1}{n^d} \sum_{y \in V_n} \frac{\lambda_y - \alpha}{1 + \theta'(\lambda_y - \alpha)}.
\]
Thus,
\[
\mathbb{E}\left( \frac{1}{n^d} \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \right) = I_n' (\theta').
\]
But by independence,
\[
\text{Var}\left( \frac{1}{n^d} \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \right) \leq \frac{C(\alpha, \theta')}{{n^d}}.
\]
In particular, the variance tends to 0 as \( n \to \infty \). Since \( I_n' (\theta') \to I'(\theta) \in (-\varepsilon, 0) \), this shows that as \( n \to \infty \),
\[
P\left( -4n^d \varepsilon \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 0 \right) \to 1.
\]
By the Gaussian nature of \( \phi' \) and (10.9), for any \( \delta' > 0 \),
\[
\lim_{n \to \infty} \mathbb{P}\left( n^d h^d \max_{x \in V_n} |\phi'(x)|^2 \leq n^{\delta'd} \right) = 1,
\]
and similarly by (10.8) and the independence of the coordinates of \( \tau' \),
\[
\frac{1}{n^d} \sum_{y \in V_n} n^d h^d |\tau'(y)|^2 \to \int_{[0,1]^d} \frac{1}{1 - \theta' \alpha + 4\theta' \sum_{i=1}^d \sin^2(\pi x_i)} dx_1 \cdots dx_d > 0
\]
in probability as \( n \to \infty \).
Combining the last two displays and choosing $\delta' \in (0, \delta)$, we get
\begin{equation}
\lim_{n \to \infty} P\left( \max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n} |\tau'(x)|^2 \right) = 1.
\end{equation}
Therefore, by (10.10), (10.11), and (10.12), $P(E) \to 1$ as $n \to \infty$, and hence
\begin{equation}
\liminf_{n \to \infty} \frac{1}{n^d} \log P\left( \sum_{x,y \in V_n, x \sim y} \frac{|\xi(x) - \xi(y)|^2}{\alpha}, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right) \geq -4\theta' \epsilon + \lim_{n \to \infty} \frac{1}{n^d} \sum_{y \in V_n} \log(1 + \theta'(\lambda_y - \alpha)) = -4\theta' \epsilon - I(\theta').
\end{equation}

Since $\epsilon$ is arbitrary and $\theta' \to \theta^*$ as $\epsilon \to 0$, this shows that when $\alpha \in (0, 2d)$ and $I'(\alpha) < 0$,
\begin{equation}
\liminf_{n \to \infty} \frac{1}{n^d} \log P\left( \sum_{x,y \in V_n, x \sim y} \frac{|\xi(x) - \xi(y)|^2}{\alpha}, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right) \geq -I(\theta^*) = -\Psi_d(\alpha).
\end{equation}

\textit{Case 2.} $I'(1/\alpha) \geq 0$. Take any $\epsilon > 0$ and let $\theta'$ solve $1 - \theta' \alpha = \epsilon$. Fix $n$ and define $\eta'$, $\tau'$, $\xi'$, and $\delta$ as in the previous case. Then as before, we arrive at the inequality (10.10), with $E$ being the same event. It suffices, as before, to show that
\begin{equation}
\liminf_{n \to \infty} \frac{1}{n^d} \log P(E) \geq 0.
\end{equation}

Let
\begin{equation*}
X_n := \frac{1}{n^d} \sum_{y \in V_n \setminus \{0\}} (\lambda_y - \alpha) \eta'(y)
\end{equation*}
and
\begin{equation*}
Y_n := \frac{(\lambda_0 - \alpha) \eta'(0)}{n^d} = -\frac{\alpha \eta'(0)}{n^d}.
\end{equation*}

As before, it is easy to argue that $\text{Var}(X_n) \to 0$ as $n \to \infty$. Again,
\begin{equation*}
\mathbb{E}(X_n) = \frac{1}{n^d} \sum_{y \in V_n \setminus \{0\}} \frac{\lambda_y - \alpha}{1 + \theta'(\lambda_y - \alpha)} \to I'(\theta') \quad \text{as } n \to \infty.
\end{equation*}

Consequently, $X_n \to I'(\theta')$ in probability as $n \to \infty$.

Next let $\eta''(0)$ be an independent copy of $\eta'(0)$. Let $\eta''$ be the vector in $C V_n$ whose $0^\text{th}$ component is $\eta''(0)$ and $\eta''(y) = \eta'(y)$ for every $y \neq 0$. Let $\tau''$ and $\phi''$ be obtained from $(\sigma, \eta'')$ in the same way that $\tau'$ and $\phi'$ were obtained from $(\sigma, \eta')$. 
Then note that \( \phi'' \) is independent of \( \eta'(0) \). Since the elements of the matrix \( R \) are bounded in absolute value by \( n^{-d/2} \), and from definition we have

(a) \( \phi'' = R \phi', \phi' = R \tau' \),
(b) \( \tau''(0) = \sigma(0) \sqrt{n \eta'(0)/(nh)^d} \), \( \tau'(0) = \sigma(0) \sqrt{n \eta'(0)/(nh)^d} \), and
(c) \( \tau''(y) = \tau'(y) \) for all \( y \neq 0 \),

therefore for all \( x \in V_n \),

\[
\text{(10.15)} \quad |\phi''(x) - \phi'(x)| \leq n^{-d/2}(nh)^{-d/2}|(\eta''(0))^{1/2} - (\eta'(0))^{1/2}|.
\]

Fix \( \delta' \in (0, \delta) \) and some \( \epsilon' \) so small that \(-d(1 - \delta') + \epsilon' < -d(1 - \delta)\), and define four events:

\[
E_1 := \{|X_n - I'(\theta')| \leq \epsilon\},
\]

\[
E_2 := \{-I'(\theta') - 3\epsilon \leq Y_n \leq -I'(\theta') - \epsilon\}
= \left\{ \frac{n^d (I'(\theta') + \epsilon)}{\alpha} \leq \eta'(0) \leq \frac{n^d (I'(\theta') + 3\epsilon)}{\alpha} \right\},
\]

\[
E_3 := \{(nh)^d \max_{x \in V_n} |\phi''(x)|^2 \leq n^{\delta'd}, \frac{1}{n^d} \sum_{x \in V_n\setminus\{0\}} (nh)^d |\tau''(x)|^2 \geq n^{-\epsilon'}\}
\]

\[
E_4 := \{ |\eta''(0)| \leq n^d \}.
\]

Then by (10.15), \( E_2 \cap E_3 \cap E_4 \) implies that for each \( x \),

\[
|\phi'(x)|^2 \leq 2|\phi''(x)|^2 + 2|\phi'(x) - \phi''(x)|^2 \leq (nh)^{-d} (2n^{\delta'd} + C(\alpha, \epsilon)).
\]

Since \( \tau''(x) = \tau'(x) \) for all \( x \neq 0 \), this shows that for all \( n \geq C(\alpha, \epsilon, \delta', \epsilon') \),
\( E_2 \cap E_3 \cap E_4 \) implies

\[
\max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta')} + \epsilon' \sum_{x \in V_n\setminus\{0\}} |\tau'(x)|^2 \leq n^{-d(1-\delta')} \sum_{x \in V_n} |\tau'(x)|^2.
\]

Again, \( E_1 \cap E_2 \) implies \(-4\epsilon \leq Y_n \leq 0\), which is the same as

\[
-4n^d \epsilon \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 0.
\]

Thus, for all \( n \geq C(\alpha, \epsilon, \delta', \epsilon', \delta) \), \( E_1 \cap E_2 \cap E_3 \cap E_4 \) implies \( E \). So, to show (10.14), it suffices to show that

\[
\text{(10.16)} \quad \liminf_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \geq 0.
\]

Since \( \eta'(0) \) is an exponential random variable with mean \( 1/\epsilon \), it is easy to see that

\[
\mathbb{P}(E_2) \geq \frac{2n^d \epsilon^2}{\alpha} \exp\left(-\frac{\epsilon n^d (I'(\theta') + 3\epsilon)}{\alpha}\right).
\]
As argued above, \( X_n \to 1'(\theta') \) in probability and therefore \( \mathbb{P}(E_1) \to 1 \) as \( n \to \infty \). The probability of \( E_4 \) tends to 1 trivially, and \( \mathbb{P}(E_3) \to 1 \) by the same logic that led to (10.12). Thus, \( \mathbb{P}(E_1 \cap E_3 \cap E_4) \to 1 \). Lastly, observe that the events \( E_1, E_3, \) and \( E_4 \) are jointly independent of \( E_2 \). Combining all of these observations, we get

\[
\liminf_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) = \liminf_{n \to \infty} \frac{1}{n^d} \left( \log \mathbb{P}(E_1 \cap E_3 \cap E_4) + \log \mathbb{P}(E_2) \right) \\
\geq -\frac{\epsilon (I'(\theta') + 3\epsilon)}{\alpha}.
\]

Since \( \epsilon \) is arbitrary and \( 0 \leq I'(\theta') \leq I'(0) = 2d - \alpha \) for all \( \theta \) (by concavity and the assumption that \( I'(1/\alpha) \geq 0 \)), this proves (10.16) and hence (10.14), leading to the proof of (10.13) when \( I'(1/\alpha) \geq 0 \). Combining (10.7) and (10.13), the proof of Theorem 10.2 for \( \alpha \in (0, 2d) \) is complete. \( \square \)

**Proof of Theorem 10.2 in the case** \( 2d < \alpha < \infty \). The proof is very similar to the previous case, with a few important modifications. Let all notation be as before. Note that for any \( \alpha \in (2d, 4d) \),

\[
\mathbb{P} \left( \sum_{x,y \in V_n} |\xi(x) - \xi(y)|^2 \geq \alpha \right) = \mathbb{P} \left( \sum_{y \in V_n} (\lambda_y - \alpha) \tau(y)^2 \geq 0 \right) \\
= \mathbb{P} \left( \sum_{y \in V_n} (\lambda_y - \alpha) \eta(y) \geq 0 \right).
\]

Thus, for any \( \theta \in [0, 1/(4d - \alpha)) \),

\[
\mathbb{P} \left( \sum_{x,y \in V_n} |\xi(x) - \xi(y)|^2 \geq \alpha \right) \leq \mathbb{E}(e^{\theta \sum_{y \in V_n} (\lambda_y - \alpha) \eta(y)}) \\
= \prod_{y \in V_n} \frac{1}{1 - \theta(\lambda_y - \alpha)}.
\]

The condition \( \theta < 1/(4d - \alpha) \) ensures that the right-hand side makes sense, since \( \lambda_y \) is uniformly bounded above by \( 4d \). Now,

\[
-\frac{1}{n^d} \log \prod_{y \in V_n} \frac{1}{1 - \theta(\lambda_y - \alpha)} = \\
\frac{1}{n^d} \sum_{y_1, \ldots, y_d=0}^{n-1} \log \left( 1 + \theta \alpha - 4\theta \sum_{i=1}^{d} \sin^2(\pi y_i/n) \right) =: J_n(\theta),
\]
and the sequence of functions $J_n$ converges pointwise to the function $J$ on the interval $[0, 1/(4d - \alpha))$, where

$$J(\theta) = \int_{[0,1]^d} \log \left( 1 + \theta \alpha - 4\theta \sum_{i=1}^d \sin^2(\pi x_i) \right) dx_1 \cdots dx_d.$$  

This shows that for each $\theta \in [0, 1/(4d - \alpha))$,

$$\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P} \left( \sum_{x, y \in V_n \atop x \sim y} |\xi(x) - \xi(y)|^2 \geq \alpha \right) \leq -J(\theta),$$  

and therefore

$$\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P} \left( \sum_{x, y \in V_n \atop x \sim y} |\xi(x) - \xi(y)|^2 \geq \alpha \right) \leq - \sup_{\theta \in (0,1/(4d - \alpha))} J(\theta).$$  

Now, putting $\gamma = (4d - \alpha)\theta$ gives

$$\sup_{\theta \in (0,1/(4d - \alpha))} J(\theta)$$

\[= \sup_{\gamma \in (0,1)} J(\gamma/(4d - \alpha)) \]

\[= \sup_{\gamma \in (0,1)} \int_{[0,1]^d} \log \left( 1 + \frac{\gamma \alpha}{4d - \alpha} - \frac{4\gamma}{4d - \alpha} \sum_{i=1}^d \sin^2(\pi x_i) \right) dx_1 \cdots dx_d \]

\[= \sup_{\gamma \in (0,1)} \int_{[0,1]^d} \log \left( 1 - \gamma + \frac{4\gamma}{4d - \alpha} \sum_{i=1}^d \cos^2(\pi x_i) \right) dx_1 \cdots dx_d.\]

The above expression is the same as $\Psi_d(4d - \alpha)$ except that we have cosine instead of sine. However, this does not matter, since we can break up the hypercube $[0, 1]^d$ into a union of smaller hypercubes like $[a_1, a_1 + 1/2] \times \cdots \times [a_d, a_d + 1/2]$, where each $a_i \in \{0, 1/2\}$, and then, within each hypercube, replace cosine by sine by a change of variable $y_i = 1/2 - x_i$ when $a_i = 0$ and $y_i = 3/2 - x_i$ when $a_i = 1/2$. Thus,

\[\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P} \left( \sum_{x, y \in V_n \atop x \sim y} |\xi(x) - \xi(y)|^2 \geq \alpha \right) \leq \sup_{\theta \in (0,1/(4d - \alpha))} J(\theta) = -\Psi_d(4d - \alpha) = -\Psi_d(\alpha).\]

Next, we turn our attention to the lower tail. The function $J$ is continuous in the interval $[0, 1/(4d - \alpha))$ and differentiable in $(0, 1/(4d - \alpha))$. Moreover, $J$ is differentiable from the right at 0 and differentiable from the left at $1/(4d - \alpha)$, and
an easy computation gives \( J'(0) = \alpha - 2d \). Since \( \alpha > 2d \) and \( J \) is continuous at 0, this shows that \( J \) must be strictly increasing in a neighborhood of 0. It is easy to check that \( J \) is a concave function. Since \( J \) is increasing at 0, its maximum in \([0, 1/(4d - \alpha)]\) must be achieved either inside \((0, 1/(4d - \alpha))\) or as \( \theta \to 1/(4d - \alpha) \). The first case holds if \( J'(1/(4d - \alpha)) < 0 \), and the second happens if \( J'(1/(4d - \alpha)) \geq 0 \). Just as before, the proof of the lower bound is different in the two cases.

Case 1. \( J'(1/(4d - \alpha)) < 0 \). There is a unique \( \theta^* \in (0, 1/(4d - \alpha)) \) where \( J \) is maximum in this case. Fix any \( \epsilon > 0 \), and let \( \theta' \) be a point so close to \( \theta^* \) from the right that \( J'((\theta')) \in (-\epsilon, 0) \). Given \( n \), let \((\eta'_{(y)})_{y \in V_n}\) be a collection of independent random variables where \( \eta'_{(y)} \) has the exponential distribution with mean \( 1/(1 - \theta'((\lambda_y - \alpha))) \). (Note the minus sign in front of \( \theta' \), which was plus in the case \( \alpha < 2d \)). As before, assume that \( \eta' \) is defined on the same probability space as all other variables and is independent of everything else.

Given \( \eta' \), define \( \tau', \phi', \) and \( \xi' \) as before. Then all the properties of these vectors are the same as before except that now
\[
\mathbb{E} |\tau'(y)|^2 = \frac{1}{(nh)^d(1 - \theta'((\lambda_y - \alpha))}
\]
and
\[
\mathbb{E} |\phi'(x)|^2 \leq \frac{1}{(nh)^d(1 - \theta'((4d - \alpha))}
\]
for all \( x \in V_n \).

Defining
\[
\rho(\eta') = e^{-\sum_{y \in V_n} \theta'((\lambda_y - \alpha)) \eta'(y)} \prod_{y \in V_n} \frac{1}{1 - \theta'((\lambda_y - \alpha))},
\]
fix \( \delta > 0 \) and let \( E \) be the event
\[
E := \left\{ 0 \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 4n\delta \epsilon, \quad \max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n} |\tau'(x)|^2 \right\}.
\]
Then as before we arrive at the inequality
\[
(10.18) \quad \mathbb{P} \left( \sum_{x, y \in V_n, x \neq y} |\xi(x) - \xi(y)|^2 \geq \alpha, \quad \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right) \geq \frac{1}{4\theta' n^d \epsilon} \mathbb{P}(E) \prod_{y \in V_n} \frac{1}{1 - \theta'((\lambda_y - \alpha))}.
\]

Exactly as before, we can now argue that
\[
\frac{1}{n^d} \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \to -J'((\theta')) \quad \text{in probability as } n \to \infty.
\]
Since $J'(\theta) \in (-\epsilon, 0)$, this shows that as $n \to \infty$,
\begin{equation}
\mathbb{P}\left( 0 \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 4n^d \epsilon \right) \to 1.
\end{equation}

The inequality (10.12) continues to be valid, and therefore, by (10.18), (10.19), and (10.12),
\[
\liminf_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \geq \alpha, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right)
\geq -4\epsilon + \lim_{n \to \infty} \frac{1}{n^d} \sum_{y \in V_n} \log(1 - \theta'(\lambda_y - \alpha))
= -4\epsilon - J(\theta').
\]

Since $\epsilon$ is arbitrary and $\theta' \to \theta^*$ as $\epsilon \to 0$, this shows (as in (10.17)) that when $\alpha \in (2d, 4d)$ and $J'(1/(4d - \alpha)) < 0$,
\begin{equation}
\liminf_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}\left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \geq \alpha, \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \right) \geq -\Psi_d(\alpha).
\end{equation}

**Case 2.** $J'(1/(4d - \alpha)) \geq 0$. Take any $\epsilon > 0$ and let $\theta'$ solve $1 - \theta'(4d - \alpha) = \epsilon$. Fix $n$ and define $\eta'$, $\tau'$, and $\xi'$ as in the previous case. Let $\lfloor n/2 \rfloor$ be the vector in $\mathbb{C}^{V_n}$ whose components are all equal to the integer part of $n/2$. Then as before, we arrive at the inequality (10.18), with $E$ being the same event. Let
\[
X_n := \frac{1}{n^d} \sum_{y \in V_n \setminus \lfloor n/2 \rfloor} (\lambda_y - \alpha) \eta'(y),
\]
\[
Y_n := \frac{(\lambda_{\lfloor n/2 \rfloor} - \alpha) \eta'(\lfloor n/2 \rfloor)}{n^d}.
\]

Note that $\lambda_{\lfloor n/2 \rfloor} \to 4d$ as $n \to \infty$.

As before, it is easy to argue that $\text{Var}(X_n) \to 0$ as $n \to \infty$. Again,
\[
\mathbb{E}(X_n) = \frac{1}{n^d} \sum_{y \in V_n \setminus \lfloor n/2 \rfloor} \frac{\lambda_y - \alpha}{1 - \theta'(\lambda_y - \alpha)} \to -J'(\theta') \text{ as } n \to \infty.
\]

Consequently, $X_n \to -J'(\theta')$ in probability as $n \to \infty$.

Next let $\eta''(\lfloor n/2 \rfloor)$ be an independent copy of $\eta'(\lfloor n/2 \rfloor)$. Let $\eta''$ be the vector in $\mathbb{C}^{V_n}$ whose $\lfloor n/2 \rfloor$th component is $\eta''(\lfloor n/2 \rfloor)$ and $\eta''(y) = \eta'(y)$ for every $y \neq \lfloor n/2 \rfloor$. Let $\tau''$ and $\phi''$ be obtained from $(\sigma, \eta'')$ the same way as $\tau'$ and $\phi'$ were
obtained from $(\sigma, \eta')$. Then note that $\phi''$ is independent of $\eta'(0)$. Exactly as we proved (10.15), it follows that for all $x \in V_n$,

$$|\phi''(x) - \phi'(x)| \leq n^{-d/2} (nh)^{-d/2} ([\eta''([n/2])]^{1/2} - ([\eta([n/2])]^{1/2})/2).$$

Fix $\delta' \in (0, \delta)$ and some $\epsilon'$ so small that $-d(1-\delta') + \epsilon' < -d(1-\delta)$, and define four events:

$$E_1 := \{|X_n + J'(\theta')| \leq \epsilon\},$$

$$E_2 := \{J'(\theta') + \epsilon \leq Y_n \leq J'(\theta') + 3\epsilon\},$$

$$E_3 := \left\{\frac{n^d (J'(\theta') + \epsilon)}{\lambda_{[n/2]} - \alpha} \leq \eta'([n/2]) \leq \frac{n^d (J'(\theta') + 3\epsilon)}{\lambda_{[n/2]} - \alpha}\right\},$$

$$E_4 := \{\eta''([n/2]) \leq n^d\}.$$

Then $E_2 \cap E_3 \cap E_4$ implies that for each $x$,

$$|\phi'(x)|^2 \leq 2|\phi''(x)|^2 + 2|\phi'(x) - \phi''(x)|^2 \leq (nh)^d (2n^{\delta'} + C(\alpha, \epsilon)).$$

Since $\tau''(x) = \tau'(x)$ for all $x \neq [n/2]$, this shows that for $n \geq C(\alpha, \epsilon, \delta', \delta')$, $E_2 \cap E_3 \cap E_4$ implies

$$\max_{x \in V_n} |\phi'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n \setminus \{0\}} |\tau'(x)|^2 \leq n^{-d(1-\delta)} \sum_{x \in V_n} |\tau'(x)|^2.$$

Again, $E_1 \cap E_2$ implies $0 \leq X_n + Y_n \leq 4\epsilon$, which is the same as

$$0 \leq \sum_{y \in V_n} (\lambda_y - \alpha) \eta'(y) \leq 4n^d \epsilon.$$

Thus, for $n \geq C(\alpha, \epsilon, \delta', \delta')$, $E_1 \cap E_2 \cap E_3 \cap E_4$ implies $E$. So it suffices to find a lower bound for the probability of $E_1 \cap E_2 \cap E_3 \cap E_4$. Since $\eta'([n/2])$ is an exponential random variable with mean

$$1 \over 1 - \theta'(\lambda_{[n/2]} - \alpha)$$

and $\lambda_{[n/2]} \to 4d$ as $n \to \infty$, and by definition of $\theta'$, $1 - \theta'(4d - \alpha) = \epsilon$, it is easy to see that

$$\lim_{n \to \infty} \frac{1}{n^d} \log P(E_2) \geq - \frac{\epsilon (J'(\theta') + 3\epsilon)}{4d - \alpha}.$$

The proof is now completed exactly as for the lower tail in the case $\alpha \in (0, 2d)$. \qed
The following theorem is the main result of this section.

**Theorem 10.3.** For each \( \epsilon > 0 \) and \( \alpha > 0 \),
\[
\lim_{n \to \infty} \frac{1}{n^d} \log \mathbb{P} \left( \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 - \alpha \leq \epsilon \right) = -\Psi_{d,\epsilon}(\alpha),
\]
where
\[
\Psi_{d,\epsilon}(\alpha) = \begin{cases} 
\Psi_d(\alpha + \epsilon) & \text{if } \alpha \leq 2d - \epsilon, \\
\Psi_d(\alpha - \epsilon) & \text{if } \alpha \geq 2d + \epsilon, \\
0 & \text{if } 2d - \epsilon < \alpha < 2d + \epsilon.
\end{cases}
\]
In particular, \( \Psi_{d,\epsilon} \) converges uniformly to \( \Psi_d \) on compact subsets of \((0, 4d)\) as \( \epsilon \to 0 \). As in Theorem 10.2, the same limit holds if we include the additional requirement that \( \max_{x \in V_n} |\xi(x)|^2 \leq n^{-d(1-\delta)} \).

**Proof.** The proof is obvious from Theorem 10.2 and Proposition 10.1. \( \square \)

### 11 The Variational Problem

For each \( E \geq 0, m \geq 0 \) and \( h > 0 \) define
\[
\Theta(E, m, h) := \log m - \Psi_d \left( \frac{2h^2 E}{m} \right).
\]
When \( m = 0 \), the right-hand side is interpreted as \(-\infty\). With this definition, it is easy to verify that for fixed \( h \), \( \Theta \) is a continuous function from \([0, \infty) \times [0, \infty) \) into \([0, \infty) \).

Given \( m_0 > 0 \) and \( E_{\min}(m, h) < E_0 < E_{\max}(m, h) \), let \( \mathcal{M}(E_0, m_0, h) \) denote the set of all \((E, m)\) that maximize \( \Theta(E, m, h) \) in the set
\[
\mathcal{R}(E_0, m_0, h) = \{(E, m) : 0 \leq m \leq m_0, \max\{E^-(m, h), 0\} \leq E \leq E^+(m, h)\},
\]
where
\[
E^-(m, h) := E_0 - E_{\max}(m_0 - m, h),
E^+(m, h) := E_0 - E_{\min}(m_0 - m, h).
\]
Define
\[
\widehat{\Theta}(E_0, m_0, h) := \max_{(E, m) \in \mathcal{R}(E_0, m_0, h)} \Theta(E, m, h).
\]
The following lemma lists some important properties of the sets \( \mathcal{R} \) and \( \mathcal{M} \).

**Lemma 11.1.** Suppose \( E_{\min}(m_0, h) \leq E_0 < d m_0 / h^2 \). Then the set \( \mathcal{R}(E_0, m_0, h) \) is a nonempty compact subset of \( \mathbb{R}^2 \), and so is \( \mathcal{M}(E_0, m_0, h) \). Moreover, any \((E, m) \in \mathcal{M}(E_0, m_0, h)\) satisfies \( m \in (0, m_0] \) and \( E = E^+(m, h) \), where \( E^+ \) is defined in (11.2) above.
Proof. From Lemma 9.1 and Lemma 9.2, it follows that $\mathcal{R}(E_0, m_0, h)$ is simply a region enclosed between the graphs of two continuous functions on a closed interval and therefore is a compact subset of $\mathbb{R}^2$. It is clearly nonempty. By continuity of $\Theta$, this shows that $\mathcal{M}(E_0, m_0, h)$ is also compact and nonempty.

It is obvious that $m > 0$, since $\Theta(E, m, h) = -\infty$ if $m = 0$. Let $E^-$ and $E^+$ be as in (11.2). Let $E^*(m, h) := d m / h^2$. Lemma 9.5 implies that for any $m \in [0, m_0]$,

$$E^-(m, h) = E_0 - \frac{2d(m_0 - m)}{h^2} < \frac{d m_0}{h^2} - \frac{2d(m_0 - m)}{h^2} = \frac{d m}{h^2} - \frac{d(m_0 - m)}{h^2} \leq E^*(m, h).$$

Moreover, $E^*(m, h)$ is nonnegative and hence

$$E^*(m, h) \geq \max\{E^-(m, h), 0\}.$$ 

Now fix any $m \in [0, m_0]$. If $E^+(m, h) < 0$, then there exists no $E$ such that $(E, m) \in \mathcal{R}(E_0, m_0, h)$, and hence no $E$ such that $(E, m) \in \mathcal{R}(E_0, m_0, h)$. So assume that $E^+(m, h) \geq 0$. By Proposition 10.1, $\Theta(E, m, h)$ increases strictly as $E$ increases from 0 to $E^*(m, h)$, and then starts decreasing as $E$ increases further. Therefore, if we impose the restriction that $(E, m) \in \mathcal{R}(E_0, m_0, h)$, then for fixed $m$, $\Theta(E, m, h)$ is maximized at $\min\{E^+(m, h), E^*(m, h)\}$.

Suppose that $(E, m) \in \mathcal{M}(E_0, m_0, h)$ is such that $E = E^*(m, h) < E^+(m, h)$. This is clearly not true if $m = m_0$, since $E^*(m_0, h) = d m_0 / h^2 > E_0 = E^+(m_0, h)$. We claim that this is impossible even if $m < m_0$. Indeed, if this is true for some $m < m_0$, then since $E^\text{min}$ is a continuous function by Lemma 9.2, we can choose a slightly larger $m' > m$ such that $E^*(m', h) < E^+(m', h)$. But then $E^*(m', h) \in [E^-(m', h), E^+(m', h)]$, and

$$\Theta(E^*(m', h), m', h) = \log m' - \Psi_d \left( \frac{2h^2 E^*(m', h)}{m'} \right) = \log m' > \log m = \Theta(E, m, h),$$

showing that $(E, m)$ cannot belong to $\mathcal{M}(E_0, m_0, h)$.

12 Upper Bound

Fix $h > 0$, and some $E_0 \in \mathbb{R}$ and $m_0 > 0$. The numbers $p, d, h, E_0$, and $m_0$ will be fixed throughout this section and will be called the “fixed parameters.” Any constant that depends only on the fixed parameters will be denoted simply by $C$ instead of $C(p, d, h, E_0, m_0)$. If the constant depends on additional parameters $a, b, \ldots$, then it will be denoted by $C(a, b, \ldots)$.

Recall the random function $\phi$ defined in Section 7 and the objects $\mathcal{M}$, $\mathcal{R}$, $\Theta$, and $\bar{\Theta}$ defined in Section 11.
Take any positive integer $n$. For any $\delta > 0$, we will call a function $f \in \mathbb{C}^V_n$ a $\delta$-soliton if there exists a $g \in \mathbb{C}^V_n$ such that

\begin{enumerate}[(a)]
    \item $\|f - g\|_{\infty} \leq \delta$, and
    \item there exists $(E^*, m^*) \in \mathcal{M}(E_0, m_0, h)$ such that
        \begin{align*}
            |(E_0 - E^*) - H_{h,n}(g)| &\leq \delta \\
            |(m_0 - m^*) - M_{h,n}(g)| &\leq \delta.
        \end{align*}
\end{enumerate}

**Theorem 12.1.** For arbitrary $\epsilon, \delta \in (0, 1)$, let $B = B(\epsilon, \delta, n)$ be the event

$\{ |H_{h,n}(\phi) - E_0| \leq \epsilon, |M_{h,n}(\phi) - m_0| \leq \epsilon, \phi \text{ is not a } \delta\text{-soliton} \}$.

Then for any fixed $\delta \in (0, 1)$,

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \mathbb{P}(B(\epsilon, \delta, n))}{n^d} < 1 - m_0 + \tilde{\Theta}(E_0, m_0, h).$$

The strict inequality is the main point of the above theorem. Let us now embark on the proof of Theorem 12.1. We begin with two simple technical lemmas.

**Lemma 12.2.** Let $z_1, \ldots, z_k$ be standard complex Gaussian random variables where $k \geq 2$. Let $S := \sum_{i=1}^k |z_i|^2$. Then for any $x \geq 2$ and $0 < y \leq x/2$,

(12.1) \hspace{1cm} \mathbb{P}(|S - x| \leq y) = \exp(k + k \log(x/k) - x + R(x, y, k)).

where

$$|R(x, y, k)| \leq C \log k + C \log x + \frac{Cky}{x} + y,$$

where $C$ is a universal constant. Moreover,

(12.2) \hspace{1cm} \mathbb{P}(S \leq x) \leq \exp(k + k \log(x/k)).$

**Proof.** The random variable $S$ has a Gamma density with parameters $k$ and 1. Explicitly, the density function of $S$ is

$$\rho(t) = \frac{t^{k-1} e^{-t}}{(k-1)!}.$$

Let $T(x, k) := k + k \log(x/k) - x$. Note that for any $t \in [x - y, x + y]$,

$$|\log \rho(t) - T(x, k)|$$

$$\leq |\log(k - 1)! - (k \log k - k)| + |\log t| + k |\log t - \log x| + |x - t|$$

$$\leq C \log k + C \log x + \frac{Cky}{x} + y.$$
(The assumption that \( y \leq x/2 \) was used to bound second and third terms.) Further, note that

\[
\mathbb{P}(|S - x| \leq y) = \int_{x-y}^{x+y} \rho(t)dt = e^{T(x,k)} \int_{x-y}^{x+y} e^{\log \rho(t) - T(x,k)} dt.
\]

Using the bound from the previous display finishes the proof of the first part of the lemma.

To prove the second part, note that

\[
\mathbb{P}(S \leq x) = \int_0^x \frac{t^{k-1}e^{-t}}{(k-1)!} dt \leq \int_0^x \frac{t^{k-1}}{(k-1)!} dt = \frac{x^k}{k!} - \frac{x^k}{k^k e^{-k}}.
\]

This completes the proof. \( \square \)

**Lemma 12.3.** Let \( \Psi_{d,\epsilon} \) be the function defined in Theorem 10.5. Then for any \( \alpha \in [0, \infty) \) and any \( L > 0 \),

\[
\Psi_{d,\epsilon}(\alpha) \geq \min\{\Psi_d(\alpha), L\} - a(\epsilon, L, d),
\]

where \( a(\epsilon, L, d) \) is a quantity that depends only on \( \epsilon, L, \) and \( d \) (and not on \( \alpha \)) such that for any fixed \( L > 0 \), \( \lim_{\epsilon \to 0} a(\epsilon, L, d) = 0. \)

**Proof.** In this proof, \( a(\epsilon, L, d) \) will denote any constant with the properties described above.

By the properties of \( \Psi_d \) listed in Proposition 10.1 there exist \( 0 < c_1 < c_2 < 4d \) such that \( \Psi_d(\alpha) > L \) whenever \( \alpha \notin [c_1, c_2] \). Fix \( c_1' \) and \( c_2' \) such that \( 0 < c_1' < c_1 < c_2 < c_2' < 4d \). Note that \( c_1, c_2, c_1', \) and \( c_2' \) can be chosen depending only on \( L \) and \( d \). By the uniform convergence of \( \Psi_{d,\epsilon} \) on compact sets, if \( \alpha \in [c_1', c_2'] \),

\[
\Psi_{d,\epsilon}(\alpha) \geq \Psi_d(\alpha) - a(\epsilon, L, d).
\]

On the other hand, by the definition of \( \Psi_{d,\epsilon} \), it is easy to see that if \( \alpha \notin [c_1', c_2'] \), for all sufficiently small \( \epsilon \) (depending only on \( L \) and \( d \)), \( \Psi_{d,\epsilon}(\alpha) > L \). In particular, if \( \alpha \notin [c_1', c_2'] \), then

\[
\Psi_{d,\epsilon}(\alpha) \geq L - a(\epsilon, L, d).
\]

The proof is completed by combining the two cases. \( \square \)

For a subset \( U \) of \( V_n \), recall that \( U^c \) denotes the set \( V_n \setminus U \), and \( \partial U \) denotes the set of all vertices in \( U \) that are adjacent to some vertex in \( U^c \). Recall also the definitions of \( M_{h,n}(f, U) \), \( H_{h,n}(f, U) \), \( N_{h,n}(f, U) \), and \( G_{h,n}(f, U) \) from Section 9. The following lemma shows that the nonlinear component of any \( f \) must come from a small region.
LEMMA 12.4. Take any \( f \in \mathbb{C}^V \). Let \( m := M_{h,n}(f) \). Then, given any \( \epsilon \in (0, 1) \), there exists a nonempty subset \( U \) of \( V \) such that

1. \( |U| \leq \frac{4d^d m}{h^d \epsilon^{d+1}} \),
2. \( |f(x)| \leq \epsilon \) for all \( x \in U^c \),
3. \( N_{h,n}(f, U^c) \leq \epsilon^{p-1} m \), and
4. \( M_{h,n}(f, \partial U \cup \partial U^c) \leq 2\epsilon m \).

PROOF. Let \( U_1 \) be the subset of \( V \) on which \( f \) is bigger than \( \epsilon \). If this set is empty, let \( U_1 \) be any singleton subset of \( V \). Then note that \( f \leq \epsilon \) outside \( U_1 \), and

\[
 h^d \sum_{x \in U_1^c} |f|^{p+1} \leq \epsilon^{p-1} h^d \sum_{x \in U_1^c} |f|^2 \leq \epsilon^{p-1} m.
\]

Also note that

\[
|U_1| \leq \frac{h^d \sum_{x \in U_1} |f(x)|^2}{h^d \epsilon^2} \leq \frac{m}{h^d \epsilon^2}.
\]

Define \( U_2, U_3, \ldots \) as follows: For each \( i \geq 2 \), let \( U_i \) be the set of vertices that are either in \( U_{i-1} \) or adjacent to a vertex in \( U_{i-1} \). Note that \( U_{i-1} \subseteq U_i \) for each \( i \). Let \( W_i := \partial U_i \). Then \( W_1, W_2, \ldots \) are disjoint sets, and \( W_{i+1} = \partial U_i^c \) for each \( i \geq 1 \).

Thus, for any \( k \),

\[
\min_{1 \leq i \leq k} h^d \sum_{x \in \partial U_i \cup \partial U_i^c} |f(x)|^2 = \min_{1 \leq i \leq k} h^d \sum_{x \in W_i \cup W_i^c} |f(x)|^2 \leq \frac{2h^d k+1}{k} \sum_{i=1}^{k+1} \sum_{x \in W_i} |f(x)|^2 \leq \frac{2m}{k}.
\]

Since each element of \( U_k \) is within \( \ell^1 \)-distance \( k-1 \) from some element of \( U_1 \), and the \( \ell^1 \)-ball of radius \( k-1 \) around any vertex has \( \leq (2k-1)^d \) points,

\[
\max_{1 \leq i \leq k} |U_i| = |U_k| \leq (2k-1)^d |U_1|.
\]

Choose an integer \( k \) such that \( 1/\epsilon \leq k \leq 2/\epsilon \). The proof is completed by choosing an \( i \) between 1 and \( k \) that minimizes \( h^d \sum_{x \in \partial U_i \cup \partial U_i^c} |f(x)|^2 \) and defining \( U \) to be the set \( U_i \). \( \square \)

Given any \( f \in \mathbb{C}^V \), recall that \( \mathcal{U}(f, \epsilon) \) denotes the set of all nonempty subsets of \( V \) that satisfy conditions (i) through (iv) of Lemma 12.4. Then Lemma 12.4 says that \( \mathcal{U}(f, \epsilon) \) is nonempty for any \( \epsilon > 0 \).

LEMMA 12.5. For any \( \epsilon \in (0, 1) \), \( E \geq 0 \), and \( m \geq 0 \), let \( K = K(\epsilon, n, E, m) \) be the event

\[
\{ |M_{h,n}(\phi) - m_0| \leq \epsilon, |H_{h,n}(\phi) - E_0| \leq \epsilon, \text{ and for some } U \in \mathcal{U}(\phi, \epsilon), |M_{h,n}(\phi, U^c) - m| \leq \epsilon, |G_{h,n}(\phi, U^c) - E| \leq \epsilon \}.
\]
Then for any $L > 0$,
\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(K(\epsilon, n, E, m))}{n^d} \leq \max\{1 - m_0 + \Theta(E, m, h), C - L\} + a(\epsilon, L),
\]
where $C$ depends only on the fixed parameters, and $a(\epsilon, L)$ is a quantity that depends only on $\epsilon, L$, and the fixed parameters such that for any fixed $L > 0$, $\lim_{\epsilon \to 0} a(\epsilon, L) = 0$. In particular, $a(\epsilon, L)$ does not depend on $E$ and $m$.

**Proof.** Fix $L > 0$, $E \geq 0$, $m \geq 0$, and $\epsilon \in (0, 1)$. Throughout this proof, $a(\epsilon, L)$ will denote any constant with the properties outlined in the statement of the lemma, and $o(1)$ will denote any constant that depends only on $\epsilon, E, m, L, n$, and the fixed parameters that goes to 0 as $n \to \infty$ while keeping the other parameters fixed.

Choose a positive integer $n$ and a set $U \subseteq V_n$. For notational simplicity, define
\[
\begin{align*}
M &= M_{h,n}(\phi, U), \\
H &= H_{h,n}(\phi, U), \\
M' &= M_{h,n}(\phi, U^c), \\
G' &= G_{h,n}(\phi, U^c).
\end{align*}
\]
Let $\phi'$ be an independent copy of $\phi$, and define
\[
\tau(x) := \begin{cases} 
\phi'(x) & \text{if } x \in U, \\
\phi(x) & \text{if } x \in U^c.
\end{cases}
\]
Note that $\tau$ has the same distribution as $\phi$ and is independent of $(\phi(x))_{x \in U}$.

Let $K_0$ be the event
\[
\{ |M_{h,n}(\phi) - m| \leq \epsilon, |H_{h,n}(\phi) - E_0| \leq \epsilon, |M' - m| \leq \epsilon, \\
|G' - E| \leq \epsilon, U \in \mathcal{U}(\phi, \epsilon) \}.
\]
Note that it is possible for $K_0$ to happen only if $m \leq M' + \epsilon \leq M_{h,n}(\phi) + \epsilon \leq m_0 + 2\epsilon \leq C$ and $E \leq G' + \epsilon \leq CM' + \epsilon \leq C$. Therefore we will assume these upper bounds on $E$ and $m$ in what follows. We will also assume that
\[
|U| \leq \frac{4^d(m_0 + \epsilon)}{h^d \epsilon^{d+2}},
\]
since without this condition, the event $K_0$ is impossible. Let $K_1$ be the event
\[
K_1 := \left\{ M_{h,n}(\tau, U) \leq \frac{2|U|}{h^d} \right\}.
\]
Since $M + M' = M_{h,n}(\phi)$, therefore if $K_0$ happens, then
\[
|M - (m_0 - m)| \leq |M + M' - m_0| + |M' - m| \leq 2\epsilon.
\]
Again, if $K_0 \cap K_1$ happens, then
\[
|M_{h,n}(\tau) - m| \leq |M' - m| + M_{h,n}(\tau, U)
\]
\[
\leq \epsilon + \frac{2|U|}{h^d} \leq \epsilon + C \epsilon^{-d+2} n^{-d} =: \epsilon_1,
\]
(12.6)
and

\[ |G_{h,n}(\tau) - E| \leq |G' - E| + G_{h,n}(\tau, U) + \frac{h^{d-2}}{2} \sum_{x \in U, y \in U^c} |\tau_x - \tau_y|^2 \]

(12.7)

\[ \leq \epsilon + C \sum_{x \in U} |\tau_x|^2 + C \sum_{x \in U^c} |\phi_x|^2 \]

\[ \leq C\epsilon + C\epsilon^{-(d+2)}n^{-d} =: \epsilon_2. \]

Let

\[ p_1 := \mathbb{P}(\{|M - (m_0 - m)| \leq 2\epsilon\}). \]

\[ p_2 := \mathbb{P}(\{|M_{h,n}(\tau) - m| \leq \epsilon_1, |G_{h,n}(\tau) - E| \leq \epsilon_2\}). \]

Note that \( K_0 \) and \( K_1 \) are independent events, and there is a positive universal constant \( C_0 \) such that \( \mathbb{P}(K_1) \geq 1/C_0. \) Thus, by (12.5), (12.6), (12.7), and the independence of \( M \) and \( \tau, \)

(12.8)

\[ \mathbb{P}(K_0) \leq C_0\mathbb{P}(K_0)\mathbb{P}(K_1) = C_0\mathbb{P}(K_0 \cap K_1) \leq C_0p_1p_2. \]

Define

\[ \xi(x) := \frac{\tau(x)}{(\sum_{y \in V_n} |\tau(y)|^2)^{1/2}}. \]

Then \( \xi \) is uniformly distributed on the unit sphere of \( \mathbb{C}^V_n. \) Note that

\[ 2h^{2-d}G_{h,n}(\xi) = \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 = \frac{h^d}{M_{h,n}(\tau)} \sum_{x, y \in V_n} |\tau(x) - \tau(y)|^2 \]

\[ = \frac{2h^2G_{h,n}(\tau)}{M_{h,n}(\tau)}. \]

Thus, if \( |M_{h,n}(\tau) - m| \leq \epsilon_1 \) and \( |G_{h,n}(\tau) - E| \leq \epsilon_2, \) then (since \( m \leq C \) and \( E \leq C, \) as observed before),

\[ \left| 2h^{2-d}G_{h,n}(\xi) - \frac{2h^2E}{m} \right| = 2h^2\left| \frac{G_{h,n}(\tau)}{M_{h,n}(\tau)} - \frac{E}{m} \right| \]

\[ \leq C\epsilon_1 + C\epsilon_2 \leq (m - \epsilon_1)m =: \epsilon_3, \]

provided that \( m > \epsilon_1. \) If \( m \leq \epsilon_1, \) define \( \epsilon_3 = \infty. \) Now, it is a simple probabilistic fact that \( \xi \) and \( M_{h,n}(\tau) \) are independent. Hence \( p_2 \leq p_3p_4, \) where

\[ p_3 := \mathbb{P}(\{|M_{h,n}(\tau) - m| \leq \epsilon_1\}). \]

\[ p_4 := \mathbb{P}\left( \left| 2h^{2-d}G_{h,n}(\xi) - \frac{2h^2E}{m} \right| \leq \epsilon_3 \right). \]
Thus, from (12.8) we have

\[(12.9) \quad P(K_0) \leq C_0 p_1 p_3 p_4.\]

Our next task is to get upper bounds for \(p_1, p_3,\) and \(p_4\). To bound \(p_1\), we consider two cases. First, if \(m_0 - m > 2\epsilon\), then we apply (12.1) from Lemma 12.2 (with \(k \leq C(\epsilon), x = n^d(m_0 - m),\) and \(y = n^\epsilon\)) to get

\[\log \frac{p_1}{n^d} = -(m_0 - m) + \epsilon + o(1),\]

where recall that the notation \(o(1)\) stands for a quantity depending only on \(\epsilon, E, m, L, n,\) and the fixed parameters that goes to 0 as \(n \to \infty\) with all else fixed. In particular, \(o(1)\) does not depend on our choice of \(U\).

Next, if \(m_0 - m \leq 2\epsilon\), apply (12.2) from Lemma 12.2 (with \(k \leq C(\epsilon)\) and \(x = 3n^\epsilon\)) to get

\[\log \frac{p_1}{n^d} = o(1).\]

Combining the two cases, we get

\[(12.10) \quad \log \frac{p_1}{n^d} \leq -(m_0 - m) + 3\epsilon + o(1).\]

We deal with \(p_3\) similarly. If \(m \geq 2\epsilon^{1/4}\), we apply (12.1) with \(k = n^d, x = n^d m,\) and \(y = n^d \epsilon_1\) to get

\[(12.11) \quad \log \frac{p_3}{n^d} = 1 + \log m - m + C \epsilon^{3/4} + o(1).\]

When \(m < 2\epsilon^{1/4}\), we apply (12.2) with \(k = n^d\) and \(x = 6n^d \epsilon\) to get

\[(12.12) \quad \log \frac{p_3}{n^d} \leq 1 + C \log \epsilon + o(1).\]

Again, if \(m \geq 2\epsilon^{1/4}\), then \(\epsilon_3 \leq C \epsilon^{1/2} + o(1)\). Therefore by Theorem 10.3 we have that if \(m \geq 2\epsilon^{1/4}\), then

\[\log \frac{p_4}{n^d} \leq -\Psi_d C \epsilon^{1/2} \left( \frac{2h^2 E}{m} \right) + o(1).\]

By Lemma 12.3, this gives

\[(12.13) \quad \log \frac{p_4}{n^d} \leq \max \left\{ -\Psi_d \left( \frac{2h^2 E}{m} \right), -L \right\} + a(\epsilon, L) + o(1),\]

where recall that \(a(\epsilon, L)\) stands for a quantity that depends only on \(\epsilon, L,\) and the fixed parameters that goes to 0 as \(\epsilon \to 0\) for any fixed \(L\). In particular, \(a(\epsilon, L)\) does not depend on \(E, m, n,\) or \(U\).
Combining (12.9), (12.10), (12.11), and (12.13) and the observation that \( m \leq C \), we see that when \( m \geq 2\epsilon^{1/4} \), we have
\[
\frac{\log \mathbb{P}(K_0)}{n^{d}} \leq \frac{\log p_1 + \log p_3 + \log p_4}{n^{d}} \leq 1 - m_0 + \log m + \max\left\{-\Psi_d\left(\frac{2h^2E}{m}\right), -L\right\} + a(\epsilon, L) + o(1)
\]
\[
\leq \max\{1 - m_0 + \Theta(E, m, h), C - L\} + a(\epsilon, L) + o(1).
\]
On the other hand, since \( C \log \epsilon \leq -L + a(\epsilon, L) \), and \( \log p_4 \leq 0 \) and \( m \leq C \), it follows from (12.9), (12.10), and (12.12) that when \( m < 2\epsilon^{1/4} \),
\[
\frac{\log \mathbb{P}(K_0)}{n^{d}} \leq \frac{\log p_1 + \log p_3}{n^{d}} \leq C - L + a(\epsilon, L) + o(1).
\]
Combining the last two displays, we see that for all \( \epsilon \in (0, 1) \), \( n \geq 1 \), \( E \geq 0 \), \( m \geq 0 \), and \( U \) satisfying (12.4), we have
\[
\frac{\log \mathbb{P}(K_0(\epsilon, n, U, E, m))}{n^{d}} \leq \max\{1 - m_0 + \Theta(E, m, h), C - L\} + a(\epsilon, L) + o(1).
\]
Now note that \( K \) can be written simply as
\[
K = \bigcup_{\substack{U \subseteq V_n \quad |U| \leq 4^d(m_0 + 2\epsilon)\frac{\epsilon}{n^{d \epsilon^{1/2}}}}} K_0(\epsilon, n, U, E, m).
\]
Since there are at most \( e^{C(\epsilon)\log n} \) terms in the above union, this completes the proof of the lemma. \( \square \)

**Lemma 12.6.** Fix \( n \) and let \( K = K(\epsilon, n, E, m) \) be the event defined in Lemma 12.5.

If \( K \) happens, then there exists a function \( \eta \) on \( V_n \) such that

(a) \( \|\phi - \eta\|_{\infty} \leq \epsilon \),
(b) \( |(E_0 - E) - H_{h,n}(\eta)| \leq 2\epsilon \), and
(c) \( |(m_0 - m) - M_{h,n}(\eta)| \leq C\epsilon + C\epsilon^{p-1} \).

**Proof.** Suppose that \( K \) has happened. Choose any \( U \in \mathcal{U}(\phi, \epsilon) \) satisfying the conditions of \( K \) and let \( M, H, M', \) and \( G' \) be as in (12.5). Let \( \eta \) be the (random) function
\[
\eta(x) := \begin{cases} 
\phi(x) & \text{if } x \in U, \\
0 & \text{if } x \in U^c.
\end{cases}
\]

(12.14)
Then
\[ \| \phi - \eta \|_\infty = \max_{x \in U^c} |\phi(x)| \leq \epsilon. \]

Next, note that
\[
|M_{h,n}(\eta) - (m_0 - m)| = |M - (m_0 - m)| = |M_{h,n}(\phi) - M' - (m_0 - m)| \leq 2\epsilon
\]
and
\[
|H_{h,n}(\eta) - (E_0 - E)| \\
\leq |H_{h,n}(\phi) - E_0| + |G' - E| + |H_{h,n}(\phi) - G' - H_{h,n}(\eta)| \\
\leq 2\epsilon + C \sum_{x \in \partial U \cup \partial U^c} |\phi(x)|^2 + C \sum_{x \in U^c} |\phi(x)|^{p+1} \\
\leq C \epsilon + C \epsilon^{p-1}.
\]

This completes the proof. \(\square\)

**Lemma 12.7.** Let \(K(\epsilon, n, E, m)\) be as in Lemma 12.5. Given any \(m_1 > 0\), and \(E_1 > 0\), there exists \(C_0 = C_0(E_1, m_1)\) and \(C_1\) such that if \(m \geq m_1\), \(E \geq E_1\), \(\epsilon \leq C_0\), and \((E, m)\) is at \(\ell^\infty\)-distance greater than \(C_1 \epsilon + C_1 \epsilon^{p-1}\) from the set \(\mathcal{R}(E_0, m_0, h)\), then for all \(n \geq C_2(E_1, m_1, \epsilon)\), the event \(K(\epsilon, n, E, m)\) is impossible.

**Proof.** If \(K(\epsilon, n, E, m)\) happens, then by Lemma 12.6, there exists a function \(\eta \in \mathbb{C}^{V_n}\) such that
\[
|\eta| \leq \epsilon, \\
|E_0 - E| = |H_{h,n}(\eta)| \leq C \epsilon + C \epsilon^{p-1}.
\]

Let \(E' := E_0 - H_{h,n}(\eta)\) and \(m' := m_0 - M_{h,n}(\eta)\). Then the above inequalities may be rewritten as
\[
|m - m'| \leq 2\epsilon, \\
|E - E'| \leq C \epsilon + C \epsilon^{p-1}.
\]

Now, if \(m \geq m_1\) and if \(\epsilon\) is small enough depending only on \(m_1\) and the fixed parameters, then by (12.17),
\[
m' = m_0 - M_{h,n}(\eta) \geq m - 2\epsilon \geq m_1 - 2\epsilon \geq 0.
\]
But by definition, \(m' \leq m_0\). Therefore, \(m' \in [0, m_0]\). Next, note that
\[
E_{\min}(M_{h,n}(\eta), h, n) \leq H_{h,n}(\eta) \leq E_{\max}(M_{h,n}(\eta), h, n),
\]
which is the same as
\[
E_0 - E_{\max}(m_0 - m', h, n) \leq E' \leq E_0 - E_{\min}(m_0 - m', h, n).
\]
Again, if $E \geq E_1$, then for sufficiently small $\epsilon$ (depending only on $m_1$ and the fixed parameters), (12.18) implies that

$$E' = E_0 - H_{h,n}(\eta) \geq E_1 - C \epsilon - C \epsilon^{p-1} \geq 0.$$  

Combining the last two displays, we see that if $E \geq E_1$, $m \geq m_1$, and $\epsilon$ is sufficiently small, then

$$\max\{E_0 - \max_{m_0 - m', h, n}, 0\} \leq E' \leq E_0 - \min_{m_0 - m', h, n},$$

By Lemma 9.2 and the characterization (11.14) of the set $\mathcal{R}(E_0, m_0, h)$ from Section 11, the equations (12.20), (12.21), and (12.22) show that if $\epsilon \leq C_0(E_1, m_1)$ and $n \geq C_2(E_1, m_1, \epsilon)$, then the event $K(\epsilon, n, E, m)$ implies that the point $(E', m')$ is within distance $\epsilon$ from the set $\mathcal{R}(E_0, m_0, h)$. By (12.19), the proof is done. □

**Lemma 12.8.** For any $\epsilon \in (0, 1)$ and any closed set $A \subseteq [0, \infty]^2$, let $F = F(\epsilon, n, A)$ be the event

$$\left\{|M_{h,n}(\phi) - m_0| \leq \epsilon, |H_{h,n}(\phi) - E_0| \leq \epsilon, \text{ and for some } U \in \mathcal{U}(\phi, \epsilon), (G_{h,n}(\phi, U^c), M_{h,n}(\phi, U^c)) \in A\right\}.$$

Then for any such set $A$,

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log P(F(\epsilon, n, A))}{n^d} \leq 1 - m_0 + \Theta(E_0, m_0, h, A),$$

where

$$\Theta(E_0, m_0, h, A) := \sup_{(E, m) \in A \cap \mathcal{R}(E_0, m_0, h)} \Theta(E, m, h).$$

**Proof.** If $F(\epsilon, n, A)$ happens, then $M_{h,n}(\phi, U^c)$ and $G_{h,n}(\phi, U^c)$ are bounded by constants depending only on the fixed parameters. Also, $\mathcal{R}(E_0, m_0, h)$ is a bounded set. Hence we can assume without loss of generality that $A$ is contained in some bounded region determined by the fixed parameters.

Then $A$ is compact, and therefore there is a minimal collection $C(\epsilon)$ of points in $A$ such that the union of $\ell^\infty$-balls of radius $\epsilon$ around these points covers $A$, and the size of this collection is bounded by a constant depending only on $\epsilon$ and the fixed parameters. Then we have

$$F(\epsilon, n, A) \subseteq \bigcup_{(E, m) \in C(\epsilon)} K(\epsilon, n, E, m),$$

and therefore

$$\frac{\log P(F(\epsilon, n, A))}{n^d} \leq \frac{\log |C(\epsilon)|}{n^d} + \max_{(E, m) \in C(\epsilon)} \frac{\log P(K(\epsilon, n, E, m))}{n^d}.$$  

Fix $L > 0$. Fix $E_1 > 0$ and $m_1 > 0$ so small that whenever $E < E_1$ or $m < m_1$,

$$1 - m_0 + \Theta(E, m, h) \leq -L.$$  

(It is easy to see that this is possible, by first choosing $m_1$ so small that $1 - m_0 + \log m_1 \leq -L$, and then choosing $E_1$ depending on $m_1$.) Let $C_0 = C_0(E_1, m_1)$
be the constant from Lemma \ref{lem:constant1} and assume that \( \epsilon \leq C_0 \). Let \( C_1 \) be the second constant from Lemma \ref{lem:constant1}. Let \( C' (\epsilon) \) be the set of points \((E, m) \in \mathcal{C}(\epsilon)\) that are at \( \ell^\infty \)-distance \( \leq C_1 \epsilon + C_1 \epsilon^{p-1} \) from the set \( \mathcal{R}(E_0, m_0, h) \) and satisfy \( E \geq E_1 \) and \( m \geq m_1 \). Let \( \mathcal{C}'(\epsilon) \) be the set of points in \((E, m) \in \mathcal{C}(\epsilon)\) such that \( E < E_1 \) or \( m < m_1 \). Let \( \mathcal{C}''(\epsilon) \) be the set of all remaining points in \( \mathcal{C}(\epsilon) \). Then by Lemma \ref{lem:constant1} for each \((E, m) \in \mathcal{C}''(\epsilon)\),

\[
\lim_{n \to \infty} \frac{\log \mathbb{P}(K(\epsilon, n, E, m))}{n^d} = -\infty.
\]

For any \((E, m) \in \mathcal{C}''(\epsilon)\), Lemma \ref{lem:upper_bound} and the inequality \eqref{ineq:upper_bound} imply that

\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(K(\epsilon, n, E, m))}{n^d} \leq \max \left\{ 1 - m_0 + \max_{(E, m) \in \mathcal{C}'(\epsilon)} \Theta(E, m, h), C - L \right\} + a(\epsilon, L)
\]

\[
\leq C - L + a(\epsilon, L),
\]

where \( a(\epsilon, L) \) is a quantity depending only on \( \epsilon, L \), and the fixed parameters such that for each fixed \( L > 0 \), \( \lim_{\epsilon \to 0} a(\epsilon, L) = 0 \).

Finally, again by Lemma \ref{lem:upper_bound} for any \((E, m) \in \mathcal{C}'(\epsilon)\),

\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(K(\epsilon, n, E, m))}{n^d} \leq \max \left\{ 1 - m_0 + \max_{(E, m) \in \mathcal{C}'(\epsilon)} \Theta(E, m, h), C - L \right\} + a(\epsilon, L).
\]

Combining the last three displays with \eqref{ineq:lower_bound}, we get

\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(F(\epsilon, n, A))}{n^d} \leq \max \left\{ 1 - m_0 + \max_{(E, m) \in \mathcal{C}'(\epsilon)} \Theta(E, m, h), C - L \right\} + a(\epsilon, L).
\]

Now, by the definition of \( \mathcal{C}'(\epsilon) \) and the continuity of \( \Theta \), it follows easily that

\[
\limsup_{\epsilon \to 0} \max_{(E, m) \in \mathcal{C}'(\epsilon)} \Theta(E, m, h) \leq \tilde{\Theta}(E, m, h, A).
\]

Therefore,

\[
\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \mathbb{P}(F(\epsilon, n, A))}{n^d} \leq \max \{ 1 - m_0 + \tilde{\Theta}(E, m, h, A), C - L \}.
\]

Since \( L \) was arbitrary, this completes the proof. \( \square \)

**Lemma 12.9.** Take any \( \epsilon, \delta \in (0, 1) \). Let \( A_\delta \) be the set

\[
\{(E, m) \in [0, \infty)^2 : \max \{|E - E^*|, |m - m^*|\} \geq \delta/2 \text{ for all } (E^*, m^*) \in \mathcal{M}(E_0, m_0, h)\}.
\]

If \( \epsilon \) is sufficiently small depending only on \( \delta \) and the fixed parameters, then for any \( n \), \( B(\epsilon, \delta, n) \implies F(\epsilon, n, A_\delta) \).
Suppose that $B(\epsilon, \delta, n)$ has happened. Recall that $\mathcal{U}(\phi, \epsilon)$ is always nonempty by Lemma \[12.4\] choose any $U \in \mathcal{U}(\phi, \epsilon)$. Let $E := G_{h,n}(\phi, U^0)$ and $m := M_{h,n}(\phi, U)$.

Suppose that $|E - E^*| < \delta/2$ and $|m - m^*| < \delta/2$ for some $(E^*, m^*) \in \mathcal{M}(E_0, m_0, h)$. We prove that this is impossible by arriving at a contradiction.

In the situation described above, the event $K(\epsilon, n, E, m)$ has happened. Thus, there is a function $\phi$ satisfying the conditions (a), (b), and (c) of Lemma \[12.6\]. Consequently, we have

\[
\begin{align*}
\text{(a1)} \quad & \|\phi - \eta\|_{\infty} \leq \epsilon, \\
\text{(b1)} \quad & |(E_0 - E^*) - H_{h,n}(\eta)| \leq 2\epsilon + \delta/2, \text{ and} \\
\text{(c1)} \quad & |(m_0 - m^*) - M_{h,n}(\eta)| \leq C\epsilon + C\epsilon^{p-1} + \delta/2.
\end{align*}
\]

If $\epsilon$ is sufficiently small depending only on $\delta$ and the fixed parameters, then this shows that $\phi$ is a $\delta$-soliton, giving the desired contradiction. □

**Proof of Theorem** \[12.1\] Fix $\delta \in (0, 1)$. Choose $\epsilon \in (0, 1)$ small enough to satisfy the criterion of Lemma \[12.9\]. Then for all $n$,

$$\mathbb{P}(B(\epsilon, \delta, n)) \leq \mathbb{P}(F(\epsilon, n, A_\delta)).$$

By the continuity of $\Theta$ and the fact that $A_\delta$ is a closed set not intersecting the region where $\Theta$ attains its maximum in $\mathcal{R}(E_0, m_0, h)$, it follows that

$$\widehat{\Theta}(E_0, m_0, h, A_\delta) < \widehat{\Theta}(E_0, m_0, h).$$

Lemma \[12.8\] now completes the proof. □

**13 Exponential Decay of Solitons**

Fix $n$. Suppose that $q \in \mathbb{C}V_n$ is a ground state soliton; that is, it minimizes energy among all functions with a given mass. Then by standard Euler-Lagrange theory, $q$ satisfies

$$-\omega q = -\Delta q - |q|^{p-1}q$$

for some real number $\omega$. This can be rewritten as

$$2dh^{-2} + \omega)q(x) - h^{-2} \sum_{y: y \sim x} q(y) = |q(x)|^{p-1}q(x) \quad \text{for all } x \in V_n.$$

Let $m := M_{h,n}(q)$. The following theorem shows that $q$ must be exponentially decaying outside a small set:

**Theorem 13.1.** There exists a subset $U$ of $V_n$ whose size can be bounded by a number depending only on $m, h, p, and d$ such that for all $x \in V_n$,

$$|q(x)| \leq Ae^{-bd_U(x)},$$

where $D_U(x)$ is the $\ell^1$-distance of $x$ from $U$, that is, the minimum of $|y - x|_1$ over all $y \in U$, and $A, b$ are positive constants depending only on $m, h, p, and d$. Here $y - x$ means the difference of $y$ and $x$ modulo $n$ in each coordinate.
The proof of Theorem 13.1 follows closely the outline of the proof of exponential decay in the continuum case, as given in [65, prop. B.7]. The main difference is that in the discrete case, we have to deal with discrete Green’s functions. The proof is divided into several lemmas.

**Lemma 13.2.** If \( p < 1 + 4/d \), then for any \( m > 0, h > 0, \) and \( n \geq C(p, d, h, m) \),
\[
\omega > -\frac{E_{\min}(m, h)}{m} > 0.
\]

**Proof.** Multiplying both sides of (13.1) by \( q(x) \) and summing over \( x \in V_n \), we get
\[
\omega \sum_{x \in V_n} |q(x)|^2 = -2dh^{-2} \sum_{x \in V_n} |q(x)|^2 + h^{-2} \sum_{\substack{x, y \in V_n \\tilde{x} = \tilde{y}}} (q(x)q(y) + q(x)\overline{q(y)})
\]
\[+ \sum_{x \in V_n} |q(x)|^{p+1}
\]
\[= -h^{-2} \sum_{\substack{x, y \in V_n \\tilde{x} = \tilde{y}}} |q(x) - q(y)|^2 + \sum_{x \in V_n} |q(x)|^{p+1}
\]
\[\geq -h^{-2} \sum_{\substack{x, y \in V_n \\tilde{x} = \tilde{y}}} |q(x) - q(y)|^2 + \frac{2}{p + 1} \sum_{x \in V_n} |q(x)|^{p+1}
\]
\[= -2h^{-d} E_{\min}(m, h, n).\]

Note that by Lemma 9.2, \( E_{\min}(m, h, n) \to E_{\min}(m, h) \) as \( n \to \infty \) and by Lemma 9.4, \( E_{\min}(m, h) < 0 \). This completes the proof. \( \square \)

**Lemma 13.3.** Let \( r := 2d/(2d + \omega h^2) \). Let \( p(x, y, k) \) be the probability that a simple symmetric random walk on the torus \( V_n \) starting at \( x \) at step \( 0 \) is at \( y \) at step \( k \). Then the soliton \( q \) satisfies for all \( x \in V_n \) the identity
\[
q(x) = \frac{h^2}{2d} \sum_{y \in V_n} \sum_{k=0}^{\infty} r^{k+1} p(x, y, k)|q(y)|^{p-1}q(y).
\]

**Proof.** Given \( q \) satisfying (13.1), let \( f \) be the function on the right-hand side in the above display. (Note that the series converges because \( r < 1 \) by Lemma 13.2.) Our goal is to show that \( q = f \). First, note that
\[
\sum_{z:z \sim x} f(z) = \frac{h^2}{2d} \sum_{y \in V_n} \sum_{k=0}^{\infty} r^{k+1} |q(y)|^{p-1}q(y) \left( \sum_{z:z \sim x} p(z, y, k) \right).
\]
By the translation invariance of the torus, it is easy to see that
\[
\sum_{z:z \sim x} p(z, y, k) = \sum_{w:w \sim y} p(x, w, k).
\]
Again, note that the random walk can be at \( y \) at time \( k + 1 \) if and only if it was at some neighbor of \( y \) at time \( k \) and moved to \( y \) at the \( (k + 1) \)th step. Therefore,

\[
p(x, y, k + 1) = \frac{1}{2d} \sum_{w: w \sim y} p(x, w, k).
\]

Combining the last three displays, we get

\[
\sum_{z: z \sim x} f(z) = h^2 \sum_{y \in V_n} \sum_{k=0}^{\infty} r^{k+1} p(x, y, k + 1)|q(y)|^{p-1} q(y)
\]

\[
= \frac{h^2}{r} \sum_{y \in V_n} \sum_{k=0}^{\infty} r^{k+1} p(x, y, k)|q(y)|^{p-1} q(y)
\]

\[
- h^2 \sum_{y \in V_n} p(x, y, 0)|q(y)|^{p-1} q(y)
\]

\[
= (2d + \omega h^2) f(x) - h^2 |q(x)|^{p-1} q(x).
\]

Comparing this with (13.1) shows that for all \( x \in V_n \),

\[
(2d + \omega h^2) (f(x) - q(x)) - \sum_{y: y \sim x} (f(y) - q(y)) = 0.
\]

In other words, \((\omega I - \Delta)(f - q) = 0\), where \( I \) is the identity matrix in \( \mathbb{C}^{V_n \times V_n} \).

Since \( \Delta \) is a negative semidefinite operator (Lemma \[8.1\]) and \( \omega > 0 \) by Lemma \[13.2\], \( \omega I - \Delta \) is nonsingular. This shows that \( q = f \) and completes the proof. \( \square \)

For any \( \delta > 0 \), let

\[
U_\delta := \{ x \in V_n : |q(x)| > \delta \}.
\]

For \( x \in V_n \), let \( D_\delta(x) \) denote the \( \ell^1 \)-distance of \( x \) from \( U_\delta \), that is, the minimum of \( |x - y|_1 \) over all \( y \in U_\delta \). Here, as usual the difference \( x - y \) is computed modulo \( n \) in each coordinate.

**Lemma 13.4.** For each \( x \in V_n \) and \( \delta > 0 \),

\[
|q(x)| \leq C_0 r D_\delta(x) + \omega^{-1} \delta^{p-1} \sum_{y \in V_n} r^{y-x|_1} |q(y)|,
\]

where \( C_0 = \omega^{-1} m^{p+1} h^{-d(p-1)} \delta^{-2} \) and \( r = 2d/(2d + \omega h^2) \).

**Proof.** A random walk that starts at \( x \) at time 0 cannot reach \( y \) before time \( |y - x|_1 \). Thus, \( p(x, y, k) = 0 \) for all \( k < |y - x|_1 \). By Lemma [13.3] this gives

\[
|q(x)| \leq \frac{h^2}{2d} \sum_{y \in V_n} |q(y)|^p \left( \sum_{k=|y-x|_1}^{\infty} r^{k+1} \right)
\]

\[
= \omega^{-1} \sum_{y \in V_n} r^{y-x|_1} |q(y)|^p.
\]
Now, if \( y \notin U_\delta \), then \(|q(y)|^p \leq \delta^{p-1} |q(y)|\). On the other hand, if \( y \in U_\delta \), then \(|y - x|_1 \geq D_\delta(x)\). But again, \(|q(x)|^2 \leq h^{-d} m\) for all \( x \) and

\[
|U_\delta| \leq \frac{h^d}{h \delta^2} \sum_{x \in U_\delta} |q(x)|^2 \leq \frac{m}{h \delta^2}.
\]

Thus,

\[
\sum_{y \in V_n} r^{\frac{1}{2}|y-x|_1} |q(y)|^p \leq \frac{m}{h \delta^2} r^{D_\delta(x)} (h^{-d} m)^p + \delta^{p-1} \sum_{y \notin U_\delta} r^{\frac{1}{2}|y-x|_1} |q(y)|.
\]

This completes the proof. \( \square \)

**Proof of Theorem 13.1.** Define

\[
B(x) := C_0 \max_{y \in V_n} r^{\frac{1}{2} x-y|_1 + D_\delta(y)},
\]

where \( C_0 \) is the constant from Lemma 13.4. Note that \( B(x) \) is never 0, \( B(x) \geq C_0 r^{D_\delta(x)} \) for all \( x \), and for all \( x, y \),

\[
B(y) = C_0 \max_{z \in V_n} r^{\frac{1}{2} x-z|_1 + D_\delta(z)}
\]

\[
\leq C_0 \max_{z \in V_n} r^{\frac{1}{2} x-z|_1 - \frac{1}{2} y-x|_1 + D_\delta(z)}
\]

\[
= r^{-1} r^{\frac{1}{2} x-y|_1} B(x).
\]

Let \( K \) be the smallest number such that \(|q(x)| \leq KB(x)\) for all \( x \in V_n\). Since \( B \) is never 0 on \( V_n \) and \( V_n \) is a finite set, \( K \) must be finite. By Lemma 13.4 Lemma 13.2 and the above observations,

\[
|q(x)| \leq C_0 r^{D_\delta(x)} + K \omega^{-1}\delta^{p-1} \sum_{y \in V_n} r^{\frac{1}{2} y-x|_1} B(y)
\]

\[
\leq B(x) + KB(x) \omega^{-1}\delta^{p-1} \sum_{y \in V_n} r^{\frac{1}{2} y-x|_1}
\]

\[
\leq B(x) + KB(x) \delta^{p-1} C(p, d, h, m).
\]

If \( \delta \) is chosen so small that \( \delta^{p-1} C(p, d, h, m) \leq 1/2 \), then the above inequality implies that

\[
K \leq 1 + \frac{K}{2}.
\]

In other words, \( K \leq 2 \). Thus, with such a choice of \( \delta \),

\[
|q(x)| \leq 2C_0 \max_{y \in V_n} r^{\frac{1}{2} x-y|_1 + D_\delta(y)}
\]
for all \( x \in V_n \). To complete the proof, note that for any \( y \),

\[
\frac{1}{2} |y - x|_1 + D_\delta(y) \geq \frac{1}{2} |y - x|_1 + \frac{D_\delta(y)}{2} \geq \frac{1}{2} |y - x|_1 + \frac{D_\delta(x) - |y - x|_1}{2} = \frac{D_\delta(x)}{2}.
\]

Thus, \(|q(x)| \leq 2C_0 r^{-1} D_\delta(x)/2\) for all \( x \in V_n \). \( \square \)

### 14 Lower Bound

Fix \( h > 0 \) and some \( E_0 \in \mathbb{R} \) and \( m_0 > 0 \) such that

(14.1) \( E_{\text{min}}(m_0, h) < E_0 < \frac{dm_0}{h^2} \).

Let \((E^*, m^*)\) be a point in \( \mathcal{M}(E_0, m_0, h) \). By Lemma \[11.1\] \( m^* \) is strictly positive. The numbers \( p, d, h, E_0, m_0, E^*, \) and \( m^* \) will be fixed throughout this section and will be called the “fixed parameters.” Any constant that depends only on the fixed parameters will be denoted simply by \( C \) instead of \( C(p, d, h, E_0, m_0, E^*, m^*) \). If the constant depends on additional parameters \( a, b, \ldots \), then it will be denoted by \( C(a, b, \ldots) \).

Recall the random function \( \phi \) defined in Section \[7\] and the objects \( \mathcal{M}, \mathcal{R}, \Theta, \) and \( \widehat{\Theta} \) defined in Section \[11\].

**Theorem 14.1.** Assume the condition \( (14.1) \). For arbitrary \( \epsilon \in (0, 1) \), let \( B_0 = B_0(\epsilon, n) \) be the event

\[
\{|H_{h,n}(\phi) - E_0| \leq \epsilon, \ |M_{h,n}(\phi) - m_0| \leq \epsilon\}.
\]

Then

\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \log \frac{\mathbb{P}(B_0(\epsilon, n))}{n^d} \geq 1 - m_0 + \widehat{\Theta}(E_0, m_0, h).
\]

Fix \( n \) and let \( f^* \) be an element of \( \mathbb{C}^{V_n} \) such that \( M_{h,n}(f^*) = m_0 - m^* \) and \( H_{h,n}(f^*) = E_{\text{min}}(m_0 - m^*, h, n) \). Then \( f^* \) is a ground state soliton for the DNLS on \( V_n \). Fix \( \epsilon > 0 \). Let \( U \) be a subset of \( V_n \) such that

(14.2) \( M_{h,n}(f^*, U^c \cup \partial U) \leq \epsilon^2 \),

where, as before, \( U^c = V_n \setminus U \). By Theorem \[13.1\] there exists a \( U \) satisfying the above property such that \(|U| \leq C(\epsilon)\).

Let \( z \) be a point chosen uniformly at random from \( V_n \). Let \( \phi' \) be an independent copy of \( \phi \) and define

\[
\gamma(x) := \begin{cases} 
\phi'(x) & \text{if } x \in z + U, \\
\phi(x) & \text{otherwise}.
\end{cases}
\]

Here \( z + U \) is the translate of \( U \) by \( z \) on the torus \( V_n \); that is, the addition is modulo \( n \) in each coordinate.
Let $U'$ be the set of all points that are either in $U$ or adjacent to some point in $U$. In other words, $U' = U \cup \partial U^c$. Define the following events:

\begin{align*}
A_1 &:= \{ |M_{h,n}(\phi) - m^*| \leq \varepsilon^2 \}, \\
A_2 &:= \left\{ \left| \frac{2h^2G_{h,n}(\phi)}{M_{h,n}(\phi)} - \frac{2h^2E^*}{m^*} \right| \leq \varepsilon^2, \right. \\
&\left. \quad \max_{x \in V_n} |\phi(x)|^2 \leq n^{-d/2} \sum_{x \in V_n} |\phi(x)|^2 \right\}, \\
A_3 &:= \{ M_{h,n}(\phi, z + U') \leq 4m^*(2d + 1)n^{-d} |U'| \}, \\
A_4 &:= \{ |\phi'(x) - f^*(x - z)| \leq n^{-2d} \text{ for all } x \in z + U \}.
\end{align*}

Finally, let $A = A(\epsilon, n) := A_1 \cap A_2 \cap A_3 \cap A_4$.

**Lemma 14.2.** Let $A$ and $\gamma$ be defined as above. We claim that if $A$ happens, then

\begin{equation}
|M_{h,n}(\gamma) - m_0| \leq 2\varepsilon^2 + C(\epsilon)n^{-d}
\tag{14.3}
\end{equation}

and

\begin{equation}
|H_{h,n}(\gamma) - E_0| \leq C\varepsilon^2 + b(\epsilon, n),
\tag{14.4}
\end{equation}

where $b(\epsilon, n)$ is a number depending only on the fixed parameters $\epsilon$ and $n$ such that $\lim_{n \to \infty} b(\epsilon, n) = 0$.

**Proof.** Suppose that $A$ has happened. To prove (14.3), note that

\begin{equation}
M_{h,n}(\gamma) = M_{h,n}(\phi, z + U^c) + M_{h,n}(\phi', z + U).
\tag{14.5}
\end{equation}

By $A_1$ and $A_3$,

\begin{equation}
|M_{h,n}(\phi, z + U^c) - m^*|
\leq |M_{h,n}(\phi) - m^*| + M_{h,n}(\phi, z + U)
\leq |M_{h,n}(\phi) - m^*| + M_{h,n}(\phi, z + U')
\leq \varepsilon^2 + C(\epsilon)n^{-d}.
\tag{14.6}
\end{equation}

Now if $A_4$ holds, then by the inequality

\begin{equation}
|a^r - b^r| \leq r|a - b| \max\{a^{r-1}, b^{r-1}\} \leq r|a - b|(a + |a - b|)^{r-1}
\end{equation}

that holds for any $a, b > 0$ and $r > 1$, we see that for any $x \in U$ and any $r > 1$,

\begin{equation}
|f^*(x)|^r - |\phi'(z + x)|^r \leq C(r)n^{-2d}.
\tag{14.7}
\end{equation}

Thus by $A_4$, (14.2), and the fact that $M_{h,n}(f^*) = m_0 - m^*$, we have

\begin{equation}
|M_{h,n}(\phi', z + U) - (m_0 - m^*)|
\leq |M_{h,n}(f^*, U) - (m_0 - m^*)| + C|U|n^{-2d}
\leq M_{h,n}(f^*, U^c) + C(\epsilon)n^{-2d}
\leq \varepsilon^2 + C(\epsilon)n^{-2d}.
\tag{14.8}
\end{equation}
Combining (14.5), (14.6), and (14.8) gives

$$|M_{h,n}(y) - m_0| \leq 2\epsilon^2 + C(\epsilon)n^{-d}.$$  

This proves (14.3). Next, note that

$$H_{h,n}(y) = H_{h,n}(\phi, z + U^\circ) + H_{h,n}(\phi', z + U)$$
$$\quad + \frac{h^{d-2}}{2} \sum_{x \in z + U, y \in z + U^\circ} |\phi'(x) - \phi(y)|^2.$$  

(14.9)

By (14.2), (14.7), and $A_4$,

$$\sum_{x \in z + \partial U} |\phi'(x)|^2 \leq \sum_{x \in \partial U} |f^*(x)|^2 + C(\epsilon)n^{-2d}$$
$$\leq C\epsilon^2 + C(\epsilon)n^{-2d}.$$  

By $A_3$,

$$\sum_{x \in z + \partial U^c} |\phi(x)|^2 \leq C(\epsilon)n^{-d}.$$  

The last two displays imply that

$$\sum_{x \in z + \partial U^c} |\phi(x)|^2 \leq C(\epsilon)n^{-d}.$$  

(14.10)

By $A_4$ and (14.7),

$$|H_{h,n}(\phi', z + U) - H_{h,n}(f^*, U)| \leq C(\epsilon)n^{-2d}.$$  

Again from (14.2), it follows easily that

$$|H_{h,n}(f^*, U) - H_{h,n}(f^*)| \leq C\epsilon^2.$$  

From the last two displays, we have

$$|H_{h,n}(\phi', z + U^\circ) - H_{h,n}(f^*)| \leq C\epsilon^2 + C(\epsilon)n^{-2d}.$$  

(14.11)

Next, note that

$$H_{h,n}(\phi, z + U^\circ) = G_{h,n}(\phi, z + U^\circ) - N_{h,n}(\phi, z + U^\circ).$$  

By $A_3$,

$$|G_{h,n}(\phi, z + U^\circ) - G_{h,n}(\phi)| \leq CM_{h,n}(\phi, z + U') \leq C(\epsilon)n^{-d}.$$  

Again by $A_1$ and $A_2$ and the fact that $m^* > 0$, it follows that if $\epsilon$ is sufficiently small (depending only on the fixed parameters), then

$$|G_{h,n}(\phi) - E^*| \leq C\epsilon^2.$$
Lastly, note that by $A_2$, 
\[ \sum_{x \in z + U^c} |\phi(x)|^{p+1} \leq (\max_{x \in V_n} |\phi(x)|^{p-1}) \sum_{x \in V_n} |\phi(x)|^2 \]
\[ \leq Cn^{-d(p-1)/2}. \]

The last four displays combine to give
\[ |H_{h,n}(\phi, z + U^c) - E^*| \leq C\varepsilon^2 + C(\epsilon)n^{-d} + Cn^{-d(p-1)/2}. \]

Combining (14.9), (14.10), (14.11), and (14.12) we get
\[ |H_{h,n}(\gamma) - (H_{h,n}(f^*) + E^*)| \leq C\varepsilon^2 + C(\epsilon)n^{-d} + Cn^{-d(p-1)/2}. \]

By Lemma 9.2, $H_{h,n}(f^*) \to E_{\min}(m_0 - m^*, h)$ as $n \to \infty$. On the other hand by Lemma 11.1, $E_{\min}(m_0 - m^*, h) = E_0 - E^*$. This completes the proof. \hfill \Box

**Lemma 14.3.** Let $A = A(\epsilon, n)$ be the event defined immediately before the statement of Lemma 14.2. Then
\[ \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log P(A(\epsilon, n))}{n^d} = 1 - m_0 + \hat{\Theta}(E_0, m_0, h). \]

(Note that the definition of $A(\epsilon, n)$ involves our choice of $f^*$; the above result holds for any sequence of choices of $f^*$ as $n \to \infty$.)

**Proof.** Let $A_1, A_2, A_3,$ and $A_4$ be as in Lemma 14.2. Write
\[ P(A) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)P(A_4 \mid A_1 \cap A_2 \cap A_3). \]

For the first term, simply note that by Lemma 12.2,
\[ P(A_1) = \exp(n^d + n^d \log m^* - n^d m^* + n^d \hat{o}(\epsilon, n)), \]
where $o(\epsilon, n)$ is a term depending only on $\varepsilon, n$, and the fixed parameters such that
\[ \lim_{\epsilon \to 0} \lim_{n \to \infty} |o(\epsilon, n)| = 0. \]

Next, define
\[ \xi_x := \frac{\phi_x}{(\sum_{y \in V} |\phi_y|^2)^{1/2}}. \]
Then, as in the proof of Lemma 12.5, $\xi$ is uniformly distributed on the unit sphere of $\mathbb{C}^{V_n}$, is independent of $M_{h,n}(\phi)$, and satisfies
\[ 2h^2 G_{h,n}(\phi) = \sum_{x, y \in V_n} |\xi(x) - \xi(y)|^2 \]
\[ \text{and} \]
\[ \max_{x \in V_n} |\phi(x)|^2 \leq \max_{x \in V_n} |\xi(x)|^2. \]
Consequently by Theorem [10.3] and Proposition [10.1]
\[ P(A_2 \mid A_1) = P(A_2) = \exp(-n^d \Psi_d \left( \frac{2h^2 E^*}{m^*} \right) + n^d o(e, n)). \]

Let \( z \) be as in the proof of Lemma [14.2]. Note that since \( z \) is uniformly distributed on \( V_n \) and is independent of \( \phi \), by Markov's inequality
\[ P(A_3 \mid \phi) \leq \frac{\mathbb{E}(M_{h,n}(\phi, z + U') \mid \phi)}{4m^*(2d + 1)n^{-d}|U|} = \frac{|U'|M_{h,n}(\phi)}{n^d} \cdot \frac{n^d}{4m^*(2d + 1)|U|} \leq \frac{M_{h,n}(\phi)}{4m^*}. \]
Thus, if \( A_1 \) happens and \( \epsilon \) is sufficiently small (depending only on the fixed parameters), then \( P(A_3 \mid \phi) \geq 1/2. \) Consequently,
\[ 1 \geq P(A_3 \mid A_1 \cap A_2) \geq 1/2. \]

Lastly, note that the coordinates of \( \phi' \) are i.i.d. complex Gaussian with probability density function \((nh)^d \pi^{-1} \exp(-(nh)^d |x|^2)\) and are independent of \( z \) and \( \phi \), and \(|U| \leq C(\epsilon); \) therefore by [14.2],
\[ P(A_4 \mid \phi, z) = P(A_4) = \exp\left(-n^d \sum_{x \in U} |f^*(x)|^2 + n^d o(e, n)\right) \]
\[ = \exp(-n^d (m_0 - m^*) + n^d o(e, n)). \]
Consequently, the same is true for \( P(A_4 \mid A_1 \cap A_2 \cap A_3) \). Combining the above estimates finishes the proof. \( \square \)

**Proof of Theorem [14.1]** By Lemma [14.2], if \( \epsilon \) is sufficiently small (depending only on the fixed parameters) and \( n \) sufficiently large (depending only \( e \) and the fixed parameters), then the event \( A(\epsilon, n) \) implies the event \( B_0(\epsilon, n) \), but with \( \phi \) replaced by \( \gamma \). However, since \( \gamma \) has the same distribution as \( \phi \), by Lemma [14.3] this completes the proof of Theorem [14.1]. \( \square \)

### 15 The Radiating Case

Fix \( h > 0 \) and some \( E_0 \in \mathbb{R} \) and \( m_0 > 0 \) such that
\[ \frac{d m_0}{h^2} \leq E_0 < \frac{2d m_0}{h^2}. \]
As in Section [12], the numbers \( p, d, h, E_0, \) and \( m_0 \) will be fixed throughout this section and will be called the “fixed parameters.” Any constant that depends only on the fixed parameters will be denoted simply by \( C \) instead of \( C(p, d, h, E_0, m_0) \). If the constant depends on additional parameters \( a, b, \ldots, \), then it will be denoted by \( C(a, b, \ldots) \).

Recall the random function \( \phi \) defined in Section [7].
Theorem 15.1. Fix $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$. Let $A_0$ be the event
\[
\{|M_{h,n}(\phi) - m_0| \leq \epsilon, \ |H_{h,n}(\phi) - E_0| \leq \epsilon\}.
\]
Then for any fixed $\delta \in (0, 1),$
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log \mathbb{P}(\max_{x \in V_n} |\phi(x)| > \delta \mid A_0)}{n^d} < 0.
\]

The rest of this section is devoted to the proof of this theorem. Define
\[
\xi_x := \frac{\phi(x)}{\left(\sum_{y \in V_n} |\phi(y)|^2\right)^{1/2}}.
\]

Then $\xi$ is uniformly distributed on the unit sphere of $\mathbb{C}^V_n$. Let
\[
\alpha_0 := \frac{2h^2 E_0}{m_0},
\]
so that $\alpha_0 \in [2d, 4d)$. Define three events:
\[
A_1 := \{|M_{h,n}(\phi) - m_0| \leq \epsilon^2\}.
\]
\[
A_2 := \left\{\left|\sum_{x \sim y \in V_n} |\xi(x) - \xi(y)|^2 - \alpha_0\right| \leq \epsilon^2\right\}.
\]
\[
A_3 := \{|\max_{x \in V_n} |\xi(x)|^2 \leq n^{-d/2}\}.
\]

Lemma 15.2. If $\epsilon < C$ and $n > C(\epsilon)$, then $A_1 \cap A_2 \cap A_3 \implies A_0$.

Proof. Note that
\[
\left|\sum_{x \sim y \in V_n} |\xi(x) - \xi(y)|^2 - \alpha_0\right| = 2h^2 \left|\frac{G_{h,n}(\phi)}{M_{h,n}(\phi)} - \frac{E_0}{m_0}\right|.
\]

This shows that if $\epsilon < C$, then $A_1 \cap A_2$ implies that
\[(15.1) \quad |G_{h,n}(\phi) - E_0| \leq \epsilon^{3/2}.
\]

Now, $A_1 \cap A_3$ implies
\[
\max_{x \in V_n} |\phi(x)|^2 \leq n^{-d/2} \sum_{x \in V_n} |\phi(x)|^2 \leq C n^{-d/2}
\]
and hence
\[
N_{h,n}(\phi) \leq C \max_{x \in V_n} |\phi(x)|^2 \left(\frac{p-1}{2}\right) \sum_{x \in V_n} |\phi(x)|^2
\]
\[
\leq C n^{-d(p-1)/4}.
\]

Therefore if $n > C(\epsilon)$, then $N_{h,n}(\phi) \leq \epsilon^{3/2}$. Combining this with (15.1) completes the proof. \(\square\)
Lemma 15.3.
\[
\liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log \mathbb{P}(A_0)}{n^d} \geq 1 + \log m_0 - m_0 - \Psi_d(\omega_0).
\]

Proof. The random variable \( M_{h,n}(\phi) \) and the random vector \( \xi \) are independent. Therefore by Lemma 15.2 for \( \epsilon < C \) and \( n > C(\epsilon) \),
\[
(15.2) \quad \mathbb{P}(A_0) \geq \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2 \cap A_3).
\]
By Theorem 10.3
\[
(15.3) \quad \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log \mathbb{P}(A_2 \cap A_3)}{n^d} = -\Psi_d(\omega_0).
\]
By Lemma 12.2
\[
(15.4) \quad \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log \mathbb{P}(A_1)}{n^d} = 1 + \log m_0 - m_0.
\]
Combining (15.2), (15.3), and (15.4) proves the lemma.

Proof of Theorem 15.1. Define an event \( A_4 \) as
\[
A_4 := \{ \max_{x \in V_n} |\phi(x)| \geq \delta \}.
\]
We have to show that
\[
\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \mathbb{P}(A_0 \cap A_4) - \log \mathbb{P}(A_0)}{n^d} < 0.
\]
If \( |H_{h,n}(\phi) - E_0| \leq \epsilon \) and \( \max_{x \in V_n} |\phi(x)| \geq \delta \), then
\[
G_{h,n}(\phi) = H_{h,n}(\phi) + \frac{h^d}{p+1} \sum_{x \in V_n} |\phi(x)|^p + 1 
\geq E_0 - \epsilon + C \delta^{p+1}.
\]
Therefore, \( A_0 \cap A_4 \) implies
\[
\sum_{x,y \in V_n, x \sim y} |\xi(x) - \xi(y)|^2 = \frac{2h^2 G_{h,n}(\phi)}{M_{h,n}(\phi)} 
\geq \frac{2h^2 (E_0 - \epsilon + C \delta^{p+1})}{m_0 + \epsilon}.
\]
Thus, there is a constant \( C_0 \) depending only on the fixed parameters such that if \( \epsilon < C(\delta) \), then \( A_0 \cap A_4 \) implies the event \( A_1 \cap A_5 \), where
\[
A_5 := \left\{ \sum_{x,y \in V_n, x \sim y} |\xi(x) - \xi(y)|^2 \geq a_0 + C_0 \delta^{p+1} \right\}.
\]
Moreover, by the independence of $M_{h,n}(\phi)$ and $\xi$, the events $A_1$ and $A_5$ are independent. Therefore, if $\epsilon < C(\delta)$, then

$$\mathbb{P}(A_0 \cap A_4) \leq \mathbb{P}(A_1 \cap A_5) = \mathbb{P}(A_1)\mathbb{P}(A_5).$$

Theorem 10.2 shows that for any $\epsilon < C(\delta)$,

$$\lim_{n \to \infty} \frac{\log \mathbb{P}(A_5)}{n^d} = -\Psi_d(\alpha_0 + C_0 \delta^{p+1}),$$

and we already know the limit of $n^{-d} \log \mathbb{P}(A_1)$ from (15.4). Since $\Psi_d$ is a strictly increasing function in the interval $[2d, 4d]$ by Proposition 10.1, a combination of the above inequality and Lemma 15.3 proves the theorem. \qed

16 Discrete Concentration-Compactness

Fix $h > 0$ and $m > 0$. Let $f_n$ be a sequence of functions on $\mathbb{Z}^d$ such that $M_h(f_n) = m$ for all $n$ and $H_h(f_n) \to E_{\min}(m, h)$ as $n \to \infty$. The following theorem is the main result of this section:

**Theorem 16.1.** There is a subsequence $n_k$ of natural numbers and a sequence of points $y_k \in \mathbb{Z}^d$ such that the sequence of functions $g_k$ defined as

$$g_k(x) := f_{n_k}(y_k + x)$$

converges to a limit function $g$ in the $L^q$-norm for every $q \in [2, \infty]$. The function $g$ has mass $m$ and energy $E_{\min}(m, h)$.

The proof of Theorem 16.1 uses the well-known concentration-compactness argument (see [50, sec. 1.4] and references therein). The main new challenge in the discrete case is that the properties of $E_{\min}(m, h)$ are not as well understood as those of $E_{\min}(m)$.

For each $x \in \mathbb{Z}^d$ and each positive integer $R$, let $B(x, R)$ denote the $\ell^1$-ball of radius $R$ around $x$ in $\mathbb{Z}^d$. For any two positive integers $n$ and $R$, define the “concentration function”

$$\rho_n(R) := \sup_{x \in \mathbb{Z}^d} \sum_{y \in B(x, R)} h^d |f_n(x)|^2.$$

Note that $\rho_n(R)$ is a nondecreasing function of $R$ for each $n$. Let

$$\mu := \lim_{R \to \infty} \liminf_{n \to \infty} \rho_n(R).$$

Clearly, there are sequences $R_k$ and $n_k$ increasing to infinity such that

$$\lim_{k \to \infty} \rho_{n_k}(R_k) = \mu.$$

It is easy to fix $R_k$ such that $R_k$ is even for each $k$. Passing to a subsequence if necessary (using a diagonal argument), we may assume that

$$\rho(R) := \lim_{k \to \infty} \rho_{n_k}(R)$$

exists for each positive integer $R$. 
Lemma 16.2. \( \mu = \lim_{k \to \infty} \rho_{n_k}(R_k) = \lim_{k \to \infty} \rho_{n_k}(R_k/2) = \lim_{R \to \infty} \rho(R). \)

Proof. First observe that from the monotonicity of \( \rho_{n_k}, \)
\[
\limsup_{k \to \infty} \rho_{n_k}(R_k/2) \leq \limsup_{k \to \infty} \rho_{n_k}(R_k) = \mu.
\]
On the other hand, for each \( R, \)
\[
\rho(R) = \liminf_{k \to \infty} \rho_{n_k}(R) \geq \liminf_{n \to \infty} \rho_n(R),
\]
and thus
\[
\lim_{R \to \infty} \rho(R) \geq \mu.
\]
Lastly, for any \( R, \) we have \( R_k/2 \geq R \) for all \( k \) large enough, and thus
\[
\rho_{n_k}(R_k/2) \geq \rho_{n_k}(R).
\]
Let \( k \to \infty \) gives that for all \( R, \)
\[
\liminf_{k \to \infty} \rho_{n_k}(R_k/2) \geq \liminf_{k \to \infty} \rho_{n_k}(R) = \rho(R).
\]
Letting \( R \to \infty \) completes the proof. \( \square \)

Since the function \( \rho_n \) is bounded between 0 and \( m \) for every \( n, \) therefore \( 0 \leq \mu \leq m. \)

Lemma 16.3. \( \mu > 0. \)

Proof. Suppose that \( \mu = 0. \) Let all notation be as in Lemma 16.2. Then from Lemma 16.2 \( \lim_{R \to \infty} \rho(R) = 0. \) But \( \rho \) is a nondecreasing, nonnegative function. Therefore \( \rho(R) = 0 \) for every \( R. \) In particular, \( \rho(1) = 0 \) and hence \( \lim_{k \to \infty} \rho_{n_k}(1) = 0. \) This implies that
\[
\lim_{k \to \infty} \sup_{x \in \mathbb{Z}^d} |f_{n_k}(x)| = 0,
\]
and therefore
\[
\lim_{k \to \infty} (H_h(f_{n_k}) - G_h(f_{n_k})) = \lim_{k \to \infty} \frac{h^d}{p+1} \sum_{x \in \mathbb{Z}^d} |f_{n_k}(x)|^{p+1} \leq \lim_{k \to \infty} (\sup_{x \in \mathbb{Z}^d} |f_{n_k}(x)|)^{p-1} \frac{M_h(f_{n_k})}{p+1} = 0.
\]
In particular,
\[
\liminf_{k \to \infty} H_h(f_{n_k}) \geq 0,
\]
contradicting \( \lim_{k \to \infty} H_h(f_{n_k}) = E_{\min}(m, h) < 0 \) (Lemma 9.4). \( \square \)

Lemma 16.4. There is a sequence of points \( y_n \) in \( \mathbb{Z}^d \) such that
\[
\liminf_{n \to \infty} |f_n(y_n)| > 0.
\]
PROOF. Let all notation be as in Lemma 16.2. By Lemma 16.3, \[
0 < \mu = \lim_{R \to \infty} \liminf_{n \to \infty} \rho_n(R).
\]
Thus, there is some \( R \) such that
\[
\liminf_{n \to \infty} \rho_n(R) > 0.
\]
Fix such an \( R \). Let \( x_n(R) \) be a point such that
\[
\rho_n(R) = \sum_{z \in B(x_n(R), R)} h^d |f_n(z)|^2.
\]
(The existence of such a point follows easily from the assumption that \( M_h(f^n) = m < \infty \).) Since \( R \) is fixed, this shows that
\[
\liminf_{n \to \infty} \frac{\sum_{z \in B(x_n(R), R)} |f_n(z)|^2}{|B(x_n(R), R)|} > 0.
\]
But there exists a point \( y_n \in B(x_n(R), R) \) such that
\[
|f_n(y_n)|^2 \geq \frac{\sum_{z \in B(x_n(R), R)} |f_n(z)|^2}{|B(x_n(R), R)|}.
\]
This completes the proof. \( \square \)

**Lemma 16.5.** If \( a \) and \( b \) are positive real numbers and \( \alpha > 1 \), then
\[
(a + b)^{\alpha} \geq a^{\alpha} + b^{\alpha}.
\]
Moreover, for each \( 0 < c_1 < c_2 \), there is a positive constant \( C = C(c_1, c_2, \alpha) \) such that whenever \( a, b \in [c_1, c_2] \),
\[
(a + b)^{\alpha} \geq a^{\alpha} + b^{\alpha} - C.
\]
PROOF. The first inequality is a simple consequence of
\[
\sup_{0 < x < 1} (x^\alpha + (1-x)^\alpha) = 1
\]
by putting \( x = a/(a+b) \). The second assertion follows similarly. \( \square \)

**Lemma 16.6.** For any positive \( m, m' \),
\[
E_{\min}(m, h) + E_{\min}(m', h) > E_{\min}(m + m', h).
\]
PROOF. Let \( f_n \) and \( g_n \) be sequences of functions such that \( M_h(f^n) = m \) and \( M_h(g^n) = m' \) for all \( n \), and
\[
\lim_{n \to \infty} H_h(f_n) = E_{\min}(m, h), \quad \lim_{n \to \infty} H_h(g_n) = E_{\min}(m', h).
\]
Since for any function \( f \), \( M_h(|f|) = M(f) \) and \( H_h(|f|) \leq H_h(f) \) by the triangle inequality, we may assume that the functions \( f_n \) and \( g_n \) are nonnegative real
valued. By Lemma 16.4, there exist sequences of points \( y_n \) and \( z_n \) and a positive real number \( \epsilon \) such that

\[
\liminf_{n \to \infty} f_n(y_n) > \epsilon, \quad \liminf_{n \to \infty} g_n(z_n) > \epsilon.
\]

Define functions \( v_n \) as

\[
v_n(x) := f_n(x + y_n) + ig_n(x + z_n),
\]

where \( i = \sqrt{-1} \). Since \( f_n \) and \( g_n \) are real valued, it is clear that \( M_h(v_n) = m + m' \) and \( G_h(v_n) = G_h(f_n) + G_h(g_n) \) for each \( n \). Now note that since \( p > 1 \), Lemma 16.5 implies that for all \( x \),

\[
|v_n(x)|^{p+1} = |f_n(x + y_n) + ig_n(x + z_n)|^{p+1} \\
= ((f_n(x + y_n))^2 + (g_n(x + z_n))^2)^{(p+1)/2} \\
\geq |f_n(x + y_n)|^{p+1} + |g_n(x + z_n)|^{p+1}.
\]

If \( n \) is large enough, then \( f_n(y_n) > \epsilon \) and \( g_n(z_n) > \epsilon \). Since \( M_h(f_n) = m \) and \( M_h(g_n) = m' \) for each \( n \), therefore the sequences \( f_n(y_n) \) and \( g_n(z_n) \) are also uniformly bounded above. Therefore by Lemma 16.5, there is a positive constant \( C \) such that for all \( n \),

\[
|v_n(0)|^{p+1} = ((f_n(y_n))^2 + (g_n(z_n))^2)^{(p+1)/2} \\
\geq |f_n(y_n)|^{p+1} + |g_n(z_n)|^{p+1} - C.
\]

Combining all of the above observations, it follows that

\[
\limsup_{n \to \infty} H_h(v_n) < \lim_{n \to \infty} (H_h(f_n) + H_h(g_n)).
\]

Since \( M_h(v_n) = m + m' \) for all \( n \), this completes the proof of the lemma. \( \square \)

**Lemma 16.7.** \( \mu = m \).

**Proof.** By Lemma 16.3 we know that \( \mu > 0 \). We also know from definition that \( 0 \leq \mu \leq m \). So we only have to eliminate the case \( 0 < \mu < m \). Suppose that this is true. Let \( x_n(R) \) be as in the proof of Lemma 16.4. Then we can write

\[
f_{n_k} = u_k + v_k + w_k
\]

with

\[
u_k(x) = f_{n_k}(x)1_{\{|x-x_{n_k}(R_k/2)\leq R_k/2\}}, \quad v_k(x) = f_{n_k}(x)1_{\{|x-x_{n_k}(R_k/2)\geq R_k\}},
\]

\[
w_k(x) = f_{n_k}(x)1_{\{|R_k/2<|x-x_{n_k}(R_k/2)|\leq R_k\}}.
\]
Then note that by Lemma 16.2,

\[ M_h(w_k) = \sum_{x \in B(x_nk(R_k/2), R_k)} h^d |u_{nk}(x)|^2 - \sum_{x \in B(x_nk(R_k/2), R_k/2)} h^d |u_{nk}(x)|^2 \leq \rho_{nk}(R_k) - \sum_{x \in B(x_nk(R_k/2), R_k/2)} h^d |u_{nk}(x)|^2 = \rho_{nk}(R_k) - \rho_{nk}(R_k/2) \to 0 \text{ as } k \to \infty. \]

By Lemma 16.2 it also follows that

\[ M_h(u_k) = \sum_{x \in B(x_nk(R_k/2), R_k/2)} h^d |f_{nk}(x)|^2 = \rho_{nk}(R_k/2) \to \mu \text{ as } k \to \infty. \]

Thus, \( \lim_{k \to \infty} M_h(v_k) = m - \mu. \) From similar arguments using Lemma 16.2 it follows that

\[ (16.1) \quad \lim_{k \to \infty} \left( H_h(f_{nk}) - (H_h(u_k) + H_h(v_k)) \right) = 0. \]

By the continuity of \( E_{\min}(\text{Lemma 9.2}) \),

\[ \liminf_{k \to \infty} H_h(u_k) \geq E_{\min}(\mu, h), \quad \liminf_{k \to \infty} H_h(v_k) \geq E_{\min}(m - \mu, h). \]

Therefore by (16.1),

\[ \liminf_{k \to \infty} H_h(f_{nk}) \geq E_{\min}(\mu, h) + E_{\min}(m - \mu, h). \]

By the initial assumption that \( H_h(f_n) \to E_{\min}(m, h), \) this shows that

\[ E_{\min}(m, h) \geq E_{\min}(\mu, h) + E_{\min}(m - \mu, h), \]

contradicting Lemma 16.6.

\[ \square \]

**Proof of Theorem 16.1.** By Lemma 16.7 we know that

\[ \lim_{R \to \infty} \liminf_{n \to \infty} \rho_n(R) = m. \]

Choose \( R_0 \) so large that

\[ \liminf_{n \to \infty} \rho_n(R_0) > m/2. \]

Let \( x_n(R) \) be as in the proof of Lemma 16.4. Then for all sufficiently large \( n \),

\[ \sum_{x \in B(x_n(R_0), R_0)} h^d |f_n(x)|^2 > m/2. \]

Fix \( \epsilon \in (0, m/2) \). Let \( R_\epsilon \) be so large that

\[ \liminf_{n \to \infty} \rho_n(R_\epsilon) > m - \epsilon. \]

Then for all \( n \) sufficiently large,

\[ \sum_{x \in B(x_n(R_\epsilon), R_\epsilon)} h^d |f_n(x)|^2 > m - \epsilon. \]
Since \( m - \epsilon + m/2 > m \), and \( M(h, f_n) = m \) for all \( n \), for sufficiently large \( n \), the balls \( B(x_n(R_0), R_0) \) and \( B(x_n(R_\epsilon), R_\epsilon) \) cannot be disjoint. In particular,
\[
|x_n(R_0) - x_n(R_\epsilon)|_1 \leq R_0 + R_\epsilon,
\]
and therefore
\[
B(x_n(R_\epsilon), R_\epsilon) \subseteq B(x_n(R_0), R_0 + 2R_\epsilon).
\]
Thus, if we define
\[
v_n(x) := f_n(x + x_n(R_0)),
\]
then for every \( \epsilon \in (0, m/2) \), there is a sufficiently large integer \( S_\epsilon \) such that for all sufficiently large \( n \),
\[
\sum_{x : |x| > S_\epsilon} h^d |v_n(x)|^2 \leq \epsilon.
\]
Thus, the sequence \( v_n \) is compact in \( L^2(\mathbb{Z}^d) \) and therefore has a convergence subsequence \( v_n_k \), which we call \( g_k \). Let \( g \) denote the limit of \( g_k \) in \( L^2 \). Since the convergence is in \( L^2 \), it follows automatically that \( M(h, g) = m \). Again, since the \( L^\infty \)-norm of a function on \( \mathbb{Z}^d \) is bounded above by its \( L^2 \)-norm, it follows that \( \|g_k - g\|_\infty \) also goes to 0. Therefore, since for any \( q \in (2, \infty) \),
\[
\sum_{x \in \mathbb{Z}^d} |g_k(x) - g(x)|^q \leq (\sup_{x \in \mathbb{Z}^d} |g_k(x) - g(x)|)^{q-2} \sum_{x \in \mathbb{Z}^d} |g_k(x) - g(x)|^2,
\]
it follows that \( g_k \) converges to \( g \) in \( L^q \) for any \( q \in (2, \infty) \). This implies, in particular, that \( H_h(g_k) \rightarrow H_h(g) \).

17 Harmonic Analysis on the Lattice

In this section \( p \) will not denote the nonlinearity parameter in the NLS. Instead, it will typically play the role of the \( p \) in the \( L^p \)-norm.

We define the \( L^p \)-norm for functions on \( \mathbb{Z}^d \) at grid size \( h \) as follows:
\[
\|f\|_{p,h} := \left(h^d \sum_{x \in \mathbb{Z}^d} |f_x|^p\right)^{1/p} = h^{d/p} \|f\|_p.
\]
It may seem strange to define a new norm by multiplying the standard \( L^p \)-norm by a constant; the purpose of the definition is to ensure that the constants in the discrete analogues of classical inequalities (that we develop below) do not depend on the grid size \( h \). Note also that \( \|f\|_{p,h} = \|\tilde{f}\|_p \), where \( \tilde{f} \) is the continuum image of \( f \) at grid size \( h \).

Similar inequalities were developed by Ladyzhenskaya [33] in the context of the “finite-difference method.” However, since I could not find in [33] exactly what I needed (in particular, some delicate analyses of discrete Green’s functions and a discrete version of the Hardy-Littlewood-Sobolev inequality of fractional integration), I decided to go ahead with my own derivations.
The grid size $h$ will be fixed throughout this section. We will assume that $h \in (0, 1)$ to avoid complications arising out of large values of $h$, in which we are not interested since we eventually want to send $h$ to 0.

### 17.1 Convolutions

We define the convolution of two functions $f$ and $g$ on $\mathbb{Z}^d$ at grid size $h$ as

$$(f * g)(x) := h^d \sum_{y \in \mathbb{Z}^d} f(y) g(x - y).$$

Although the notation does not explicitly include the grid size, it will be understood from the context.

### 17.2 Young's Inequality

Below we state and prove the discrete analogue of Young’s inequality for convolutions at grid size $h$. The proof is exactly the same as in the continuous case; the important thing is that the constant does not depend on the grid size.

**Proposition 17.1.** Let $f, g$ be complex-valued functions on $\mathbb{Z}^d$. Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Then for any $h > 0$,

$$\|f * g\|_{r, h} \leq \|f\|_{p, h} \|g\|_{q, h}.$$  

**Proof.** Let $\alpha = (r - p)/r$ and $\beta = (r - q)/r$ so that $\alpha$ and $\beta$ both belong to the interval $[0, 1]$. Let $p_1 = p/\alpha$ and $p_2 = q/\beta$ so that $p_1, p_2 \in [1, \infty]$. Note that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1.$$

Let $u = f * g$. By Hölder’s inequality,

$$|u(x)| = h^d \left| \sum_{y \in \mathbb{Z}^d} f(x - y) g(y) \right|$$

$$\leq h^d \sum_{y \in \mathbb{Z}^d} |f(x - y)|^{1-\alpha} |g(y)|^{1-\beta} |f(x - y)|^\alpha |g(y)|^\beta$$

$$\leq h^d \left( \sum_{y \in \mathbb{Z}^d} |f(x - y)|^{(1-\alpha)\alpha} |g(y)|^{(1-\beta)\beta} \right)^{1/r}$$

$$\times \left( \sum_{y \in \mathbb{Z}^d} |f(x - y)|^{\alpha p_1} \left( \sum_{y \in \mathbb{Z}^d} |g(y)|^{\beta p_2} \right)^{1/p_2} \right)^{1/r}$$

$$= \left( h^d \sum_{y \in \mathbb{Z}^d} |f(x - y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} \right)^{1/r} \|f\|_{\alpha p_1, h} \|g\|_{\beta p_2, h}.$$
Taking the $r^{th}$ power and summing over $x$ gives
\[
\|u\|_r^r \leq \| f \|_{(1-\alpha)r, r, h} (1-\alpha)r, \| g \|_{(1-\beta)r, r, h} (1-\beta)r, \| f \|_{\alpha p_1, h} \| g \|_{\beta p_2, h}.
\]
Since $(1-\alpha)r = \alpha p_1 = p$ and $(1-\beta)r = \beta p_2 = q$, this completes the proof. □

17.3 Fourier Transform
Take any $h > 0$ and let $K = 1/h$. For a function $f : \mathbb{Z}^d \to \mathbb{C}$ with finite $L^1$-norm, we define the Fourier transform of $f$ at grid size $h$ as
\[
\hat{f}(\xi) := h^d \sum_{x \in \mathbb{Z}^d} f(x) e^{i \pi h x \cdot \xi}, \quad \xi \in [-K, K]^d,
\]
where $i = \sqrt{-1}$. Again, the notation does not explicitly include the grid size; it is to be understood from the context. It is easy to verify the inversion formula
\[
f(x) = 2^{-d} \int_{[-K,K]^d} \hat{f}(\xi) e^{-i \pi h x \cdot \xi} d\xi.
\]
With the above definition, if $u = f * g$ (at grid size $h$), then
\[
\hat{u}(\xi) = \hat{f}(\xi) \hat{g}(\xi),
\]
provided that $f$ and $g$ are in $L^1 \cap L^2$. Another easy fact is the Plancherel identity
\[
\| f \|_{2,h}^2 = 2^{-d} \int_{[-K,K]^d} |\hat{f}(\xi)|^2 d\xi.
\]

17.4 Littlewood-Paley Decomposition
Fix $h > 0$ and let $K = 1/h$, as in the preceding subsection. Let $\gamma_0 : [-K, K] \to [0, 1]$ be a smooth (i.e., $C^\infty$) function, that is, 1 in $[-1, 1]$ and 0 outside $(-2, 2)$. Define $\gamma : [-K, K]^d \to [0, 1]$ as
\[
\gamma(\xi) := \prod_{i=1}^d \gamma_0(\xi_i).
\]
Note that $\gamma$ is simply $\hat{g}$, where $g : \mathbb{Z}^d \to \mathbb{C}$ is the function
\[
g(x) = 2^{-d} \int_{[-K,K]^d} \gamma(\xi) e^{-i \pi h x \cdot \xi} d\xi
\]
\[
= 2^{-d} \prod_{i=1}^d \int_{-2}^2 \gamma_0(\xi_i) e^{-i \pi h x_i \xi_i} d\xi_i = \prod_{i=1}^d \varphi(\pi h x_i),
\]
where $\varphi : \mathbb{R} \to \mathbb{C}$ is the function
\[
\varphi(x) = \frac{1}{2} \int_{-2}^2 \gamma_0(t) e^{-ixt} dt.
\]
For any \( a \in (0, K] \), let \( \gamma_a : [-K, K]^d \to [0, 1] \) be the function \( \gamma(\xi/a) \). A computation similar to the above shows that \( \gamma_a \) is the Fourier transform of \( g_a \), where

\[
g_a(x) = a^d \prod_{i=1}^{d} \varphi(a\pi h x_i).
\]

**Lemma 17.2.** For any \( a \in (0, K] \) and any \( p \in [1, \infty] \),

\[
C_1(p, d)a^{d(p-1)/p} \leq \|g_a\|_{p,h} \leq C_2(p, d)a^{d(p-1)/p},
\]

where \( C_1(p, d) \) and \( C_2(p, d) \) are constants depending only on \( p \) and \( d \).

**Proof.** Note that for any \( p \in [1, \infty] \),

\[
\|g_a\|_{p,h}^p = h^d a^{dp} \left( \sum_{x \in \mathbb{Z}} |\varphi(a\pi h x)|^p \right)^d.
\]

From formula (17.2), the properties of \( \gamma_0 \) and standard results about oscillatory integrals (see, e.g., [61, chap. VIII, prop. 1]) it follows that \( \varphi(x) \) decays faster than \( |x|^{-\alpha} \) for any \( \alpha \) as \( |x| \to \infty \). Moreover, \( \varphi \) is a continuous function. Therefore, as \( ah \to 0 \),

\[
a\pi h \sum_{x \in \mathbb{Z}} |\varphi(a\pi h x)|^p \to \int_{\mathbb{R}} |\varphi(u)|^p \, du \in (0, \infty).
\]

This shows that

\[
\frac{C_1(p)}{ah} \leq \sum_{x \in \mathbb{Z}} |\varphi(a\pi h x)|^p \leq \frac{C_2(p)}{ah},
\]

which completes the proof. \( \square \)

For each \( a \), let \( P_a f \) be the function whose Fourier transform is

\[
(\gamma_a(\xi) - \gamma_{a/2}(\xi)) \hat{f}(\xi).
\]

In other words, \( P_a f = (g_a - g_{a/2}) * f \). Now, for any nonzero \( \xi \in [-K, K]^d \),

\[
\sum_{j=0}^{\infty} (\gamma_{2^{-j}K}(\xi) - \gamma_{2^{-(j+1)}K}(\xi)) = 1,
\]

which shows that for functions with suitable decay at infinity (e.g., functions with bounded support)

\[
f = \sum_{j=0}^{\infty} P_{2^{-j}K} f.
\]

This is the discrete analogue (at grid size \( h \)) of the classical Littlewood-Paley decompositions (see, e.g., [65, app. A]).
LEMMA 17.3. For any \( h > 0 \), any \( 1 \leq p \leq q \), any \( a \in (0, K] \), and any \( f : \mathbb{Z}^d \to \mathbb{C} \),
\[
\| P_a f \|_{q,h} \leq C(p, q, d) a^{\frac{d}{q} - \frac{d}{p}} \| f \|_{p,h}.
\]

PROOF. By Young’s inequality (Lemma 17.1), for any \( 1 \leq p \leq q \) and any \( a \in (0, K] \),
\[
\| P_a f \|_{q,h} \leq (\| g_a \|_{s,h} + \| g_a/2 \|_{s,h}) \| f \|_{p,h},
\]
where
\[
\frac{1}{s} = \frac{1}{q} + 1 - \frac{1}{p}.
\]
The proof is now easily completed using Lemma 17.2.

17.5 Gagliardo-Nirenberg Inequality

The goal of this subsection is to prove a version of the Gagliardo-Nirenberg inequality for the lattice at grid size \( h \). (For the well-known continuum version, see, e.g., [65, app. A].) The result is stated as Proposition 17.6 below. To prepare for this, we first need some definitions and lemmas.

For \( i = 1, \ldots, d \), let \( e_i \) denote the \( i \)th coordinate vector. Let \( \nabla_i \) denote the discrete derivative operator in the \( i \)th coordinate direction, defined as
\[
(17.3) \quad \nabla_i f(x) := \frac{f(x + e_i) - f(x)}{h}.
\]
Note that \( \nabla_i f \) is simply the convolution of \( f \) with \( \delta_i \), where \( \delta_i \) is the function
\[
\delta_i(x) = \begin{cases} 
-h^{-d-1} & \text{if } x = 0, \\
h^{-d-1} & \text{if } x = -e_i, \\
0 & \text{for all other } x.
\end{cases}
\]

LEMMA 17.4. For any \( \xi \in [-K, K]^d \),
\[
\sum_{i=1}^d |\hat{\nabla}_i f(\xi)|^2 \geq C|\xi|^2 |\hat{f}(\xi)|^2,
\]
where \( |\xi| \) is the euclidean norm of \( \xi \) and \( C \) is a positive universal constant.

PROOF. A straightforward verification shows that
\[
\hat{\nabla}_i f(\xi) = \frac{e^{-i\pi h \xi} - 1}{h} \hat{f}(\xi).
\]
There is a positive constant \( C_0 \) such that for all \( \theta \in [-\pi, \pi] \),
\[
|1 - e^{-i\theta}| \geq C_0 |\theta|.
\]

Therefore, if \( \xi \in [-K, K] \), then
\[
|\hat{\nabla}_i f(\xi)| \geq C |\xi| |\hat{f}(\xi)|.
\]
This completes the proof of the lemma.
Lemma 17.5. For any \( a \in (0, K) \) and any \( f \in L^2(\mathbb{Z}^d) \),
\[
\| P_a f \|_{2,h} \leq C a^{-1} G_h(f),
\]
where \( C \) is a universal constant.

Proof. If \( |\xi| \leq a/2 \), then \( |\xi_i| \leq a/2 \) for each \( i \), and hence
\[
\gamma_a(\xi) = \gamma(\xi/a) = 1 = \gamma(2\xi/a) = \gamma_{a/2}(\xi).
\]
On the other hand, if \( |\xi| > 2\sqrt{d}a \), then \( |\xi_i| > 2a \) for some \( i \), and therefore
\[
\gamma_{a}(\xi) = \gamma(\xi/a) = 0 = \gamma(2\xi/a) = \gamma_{a/2}(\xi).
\]
Combining these facts with Lemma 17.4, Lemma 17.3, and the Plancherel identity (17.1), we get
\[
\| P_a f \|_{2,h}^2 = \int_{[-K,K]^d} |\widehat{P_a f}(\xi)|^2 d\xi
= \int_{[-K,K]^d} |(\gamma_a(\xi) - \gamma_{a/2}(\xi)) \hat{f}(\xi)|^2 d\xi
\leq C a^{-2} \int_{a/2 \leq |\xi| \leq 2\sqrt{d}a} \sum_{i=1}^d |(\gamma_a(\xi) - \gamma_{a/2}(\xi)) \nabla_i \hat{f}(\xi)|^2 d\xi
\leq C a^{-2} \sum_{i=1}^d \| P_a \nabla_i f \|_{2,h}^2 \leq C a^{-2} \sum_{i=1}^d \| \nabla_i f \|_{2,h}^2 = C a^{-2} G_h(f).
\]
This completes the proof. \( \square \)

The following proposition may be called the Gagliardo-Nirenberg inequality for the lattice with grid size \( h \). The important thing, as usual, is that the constant does not depend on \( h \).

Proposition 17.6. Take any \( 2 < q \leq \infty \) and let \( \theta \in (0, 1) \) solve
\[
(17.4) \quad \frac{1}{q} = \frac{1}{2} - \frac{\theta}{d}.
\]
Then for any \( f \in L^q(\mathbb{Z}^d) \cap L^2(\mathbb{Z}^d) \), we have
\[
\| f \|_{q,h} \leq C(q, d) \| f \|_{2,h}^{1-\theta} G_h(f)^{\theta/2}.
\]
PROOF. Then for any \( f \), by Lemma 17.3

\[
\| f \|_{q,h} \leq \sum_{j=0}^{\infty} \| P_{2^{-j} K} f \|_{q,h}
\]

\[
\leq C(q, d) \sum_{j=0}^{\infty} (2^{-j} K)^{\frac{d}{2} - \frac{d}{q}} \| P_{2^{-j} K} f \|_{2,h}
\]

\[
= C(q, d) \sum_{j=0}^{\infty} (2^{-j} K)^{\theta} \| P_{2^{-j} K} f \|_{2,h}.
\]

Again by Lemma 17.3

\[
\| P_{2^{-j} K} f \|_{2,h} \leq \| f \|_{2,h},
\]

and by Lemma 17.5 there is a universal constant \( C_0 \) such that for all \( j \geq 0 \)

\[
\| P_{2^{-j} K} f \|_{2,h} \leq C_0 2^j h G_h(f)^{1/2}.
\]

Let \( j_0 \in \mathbb{Z} \) be the unique integer such that

\[
C_0 2^{j_0 - 1} h G_h(f)^{1/2} \leq \| f \|_{2,h} \leq C_0 2^{j_0} h G_h(f)^{1/2}.
\]

Then note that

\[
\sum_{j > j_0} 2^{-\theta j} K^\theta \leq C(q, d) 2^{-\theta j_0} K^\theta \leq C(q, d) \left( \frac{G_h(f)^{1/2}}{\| f \|_{2,h}} \right)^\theta,
\]

and similarly

\[
\sum_{j \leq j_0} 2^{(1-\theta) j} K^\theta h \leq C(q, d) 2^{(1-\theta) j_0} h^{1-\theta}
\]

\[
\leq C(q, d) \left( \frac{\| f \|_{2,h}}{G_h(f)^{1/2}} \right)^{1-\theta}.
\]

Combining (17.5), (17.6), (17.7), (17.8), and (17.9) we get

\[
\| f \|_{q,h} \leq C(q, d) \sum_{j=0}^{\infty} (2^{-j} K)^{\theta} \| P_{2^{-j} K} f \|_{2,h}
\]

\[
\leq C(q, d) \sum_{j=j_0+1}^{\infty} (2^{-j} K)^{\theta} \| f \|_{2,h}
\]

\[
+ C(q, d) \sum_{j=-\infty}^{j_0} (2^{-j} K)^{\theta} 2^j h G_h(f)^{1/2}
\]

\[
\leq C(q, d) \| f \|_{2,h}^{1-\theta} G_h(f)^{\theta/2}.
\]

This completes the proof of the discrete Gagliardo-Nirenberg inequality. \( \square \)
17.6 Green’s Function

Let $\Delta$ be the Laplacian operator on $\mathbb{C}^{\mathbb{Z}^d}$ with grid size $h$, i.e.,

$$\Delta f(x) = \frac{1}{h^2} \sum_{y : y \sim x} (f(y) - f(x)).$$

**Lemma 17.7.** Let $I$ be the identity operator on $\mathbb{C}^{\mathbb{Z}^d}$ and $\omega$ be any positive real number. For any $u \in L^2(\mathbb{Z}^d)$, the unique solution to the equation

$$(\omega I - \Delta) f = u$$

is given by $f = g * u$, where $g$ is the discrete Green’s function

$$(17.10) g(x) = \frac{h^{2-d}}{2^d} \sum_{k=0}^{\infty} r^{k+1} p(x, k).$$

where $r = 2d/(2d + \omega h^2)$ and $p(x, k)$ is the probability that a $d$-dimensional simple symmetric random walk started at the origin is at $x$ at time $k$.

**Proof.** Note that $\Delta$ is negative semidefinite, since

$$\langle f, \Delta f \rangle = -\frac{1}{2h^2} \sum_{x,y \in \mathbb{Z}^d, x \neq y} |f_x - f_y|^2.$$

Thus for any positive $\omega$, $\omega I - \Delta$ is a positive definite operator. In particular, given a function $u \in L^2(\mathbb{Z}^d)$, there can be at most one solution of

$$(\omega I - \Delta) f = u.$$

To show that $g * u$ is a solution, one proceeds exactly as in the proof of Lemma 13.3. \qed

The Green’s function is an indispensable tool in classical harmonic analysis. While the continuum Green’s functions are relatively simple objects, the discrete ones are more complicated. The purpose of this subsection is to derive some careful estimates for the discrete Green’s functions.

**Lemma 17.8.** Let $p(x, k)$ be as in Lemma 17.7. Then for all $x \in \mathbb{Z}^d$ and $k \geq 0$,

$$p(x, k) \leq C(d) e^{-|x|^2/2k} (1 + k)^{-d/2},$$

where we interpret $|x|^2/2k$ as $\infty$ if $x \neq 0$ and $k = 0$, and as $0$ if $x = 0$ and $k = 0$.

**Proof.** Suppose we are given $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, $k \geq 0$, and $k_1, \ldots, k_d$ summing to $k$. If we know that for each $i = 1, \ldots, d$, the walk has taken a total of $k_i$ steps along the $i^{th}$ coordinate axis, then the number of ways that the walk can be at $x$ at step $k$ is exactly

$$\prod_{i=1}^{d} \binom{k_i}{(k_i + x_i)/2},$$
where the combinatorial term is understood to be 0 if \( k_i + x_i \) is odd or if \( x_i \not\in [-k_i, k_i] \). Therefore,

\[
p(x, k) = (2d)^{-k} \sum_{0 \leq k_1, \ldots, k_d \leq k \atop k_1 + \cdots + k_d = k} \frac{k!}{k_1! \cdots k_d!} \prod_{i=1}^{d} \left( \frac{k_i}{(k_i + x_i)/2} \right)
\]

(17.11)

\[
= \sum_{0 \leq k_1, \ldots, k_d \leq k \atop k_1 + \cdots + k_d = k} \frac{k! d^{-k}}{k_1! \cdots k_d!} \prod_{i=1}^{d} \left( \frac{k_i}{(k_i + x_i)/2} \right) 2^{-k_i}.
\]

Suppose \( k \) balls are dropped independently and uniformly at random into \( d \) boxes. Let \( K_i \) be the number of balls falling into the \( i \)th box. Then for any \( 0 \leq k_1, \ldots, k_d \leq k \) such that \( k_1 + \cdots + k_d = k \),

\[
\Pr(K_1 = k_1, \ldots, K_d = k_d) = \frac{k! d^{-k}}{k_1! \cdots k_d!}.
\]

Therefore,

(17.12) \[
p(x, k) = \mathbb{E} \left( \prod_{i=1}^{d} \left( \frac{K_i}{(K_i + x_i)/2} \right) 2^{-K_i} \right).
\]

A simple computation using Stirling’s formula (e.g., the matching upper and lower bounds in [53]) shows that there is a universal constant \( C \) such that for any integers \( a \geq 1 \) and \( b \in [-a, a] \) such that \( a + b \) is even,

\[
\log \left( \left( \frac{a}{(a+b)/2} \right) 2^{-a} \right) \leq C - \frac{1}{2} \log a - \frac{a + b + 1}{2} \log \left( 1 + \frac{b}{a} \right) - \frac{a - b + 1}{2} \log \left( 1 - \frac{b}{a} \right).
\]

It is easy to verify that for any \( x \in (-1, \infty) \), \( \log(1 + x) \geq x - x^2/2 \). Applying this to control the logarithms on the right-hand side of the above expression, one gets

\[
\left( \frac{a}{(a+b)/2} \right) 2^{-a} \leq C e^{-b^2/2a} \sqrt{a}.
\]

This holds for \( a \geq 1 \). To include \( a = 0 \), one modifies the inequality slightly to get

(17.13) \[
\left( \frac{a}{(a+b)/2} \right) 2^{-a} \leq C e^{-b^2/2a} \sqrt{1+a}.
\]
Using this bound in (17.12) and applying Hölder’s inequality, we get

\[ p(x, k) \leq C(d) \mathbb{E}\left( e^{-\sum_{i=1}^{d} x_i^2/2K_i} \prod_{i=1}^{d} (1 + K_i)^{-1/2} \right) \]

\[ \leq C(d) e^{-|x|^2/2k} \mathbb{E}\left( \prod_{i=1}^{d} (1 + K_i)^{-1/2} \right) \]

\[ \leq C(d) e^{-|x|^2/2k} \left( \mathbb{E}(1 + K_i)^{-d/2} \right)^{1/d}. \]

(17.14)

(The interpretation for \( k = 0 \) is as in the statement of the lemma.)

Now, each \( K_i \) has a binomial distribution with parameters \( k \) and \( 1/d \). Therefore by Hoeffding’s inequality [26],

\[ \mathbb{P}(K_i \leq k/2d) \leq e^{-C(d)k}, \]

which clearly shows that for any \( r > 0 \),

\[ \mathbb{E}(1 + K_i)^{-r} \leq \mathbb{P}(K_i \leq k/2d) + (1 + k/2d)^{-r} \]

\[ \leq C(d, r)(1 + k)^{-r}. \]

(17.15)

Plugging this into (17.14) proves the lemma. □

**Lemma 17.9.** Suppose that \( f : [0, \infty) \rightarrow [0, \infty) \) is a nonincreasing function and \( g : [0, \infty) \rightarrow [0, \infty) \) is a nondecreasing function. Then

\[ \sum_{k=0}^{\infty} f(k + 1) g(k) \leq \int_{0}^{\infty} f(t) g(t) dt. \]

More generally,

\[ \sum_{x \in \mathbb{Z}^d} f(|x| + \sqrt{d}) g(|x|) \leq \int_{\mathbb{R}^d} f(|y|) g(|y|) dy. \]

**Proof.** For the first part, simply observe that when \( t \in [k, k + 1) \), \( f(t) \geq f(k + 1) \) and \( g(t) \geq g(k) \), so that \( f(k + 1) g(k) \leq f(t) g(t) dt \).

For the second part, let \( \mathbb{Z}_+ = \{0, 1, \ldots\} \) and \( \mathbb{R}_+ = [0, \infty) \). Then

\[ \sum_{x \in \mathbb{Z}^d} f(|x| + \sqrt{d}) g(|x|) \leq 2^d \sum_{x \in \mathbb{Z}_+^d} f(|x| + \sqrt{d}) g(|x|) \]

and

\[ \int_{\mathbb{R}^d} f(|y|) g(|y|) dy = 2^d \int_{\mathbb{R}_+^d} f(|y|) g(|y|) dy. \]
so that it suffices to prove
\[ \sum_{x \in \mathbb{Z}_+^d} f(|x| + \sqrt{d})g(|x|) \leq \int_{\mathbb{R}_+^d} f(|y|)g(|y|)dy. \]

Take any \( x \in \mathbb{Z}_+^d \). Let \( B(x) \) be the set \( \{ y \in \mathbb{R}^d : x_i \leq y_i < x_i + 1, i = 1, \ldots, d \} \).

The sets \( B(x) \) are pairwise disjoint and all have volume 1. Moreover, on the set \( B(x) \),
\[ f(|x| + \sqrt{d})g(|x|) \leq \int_{B(x)} f(|y|)g(|y|)dy. \]

This completes the proof of the lemma.

**Lemma 17.10.** Given \( \omega > 0 \), let \( r = 2d/(2d + \omega h^2) \), as in Lemma [17.7]. Take any \( \beta > 0, \alpha \in \mathbb{R} \), and \( x \in \mathbb{Z}^d \). Let
\[ S := \sum_{k=0}^{\infty} r^{k+1}e^{-\beta|x|^2/k} (1+k)^{-\alpha/2}. \]

The following bounds hold:

(a) If \( \alpha > 2 \), then
\[ S \leq C(d, \alpha, \beta)(\sqrt{d} + |x|)^{2-\alpha} e^{-C(d, \beta, \omega)h|x|}. \]

(b) If \( \alpha = 2 \) then
\[ S \leq C(d, \alpha, \beta)(C(\beta) + C(\beta)|\log(h(\sqrt{d} + |x|)))|e^{-C(d, \beta, \omega)h|x|}. \]

(c) If \( \alpha < 2 \), then
\[ S \leq C(d, \alpha, \beta)h^{\alpha-2} e^{-C(d, \beta, \omega)h|x|}. \]

In all three cases, \( C(d, \beta, \omega) \) is an increasing function of \( \omega \).

**Proof.** First, assume that \( x \neq 0 \). For any \( y \geq 0 \),
\[ \log(1 + y) = \int_0^y \frac{1}{1+z} \, dz \geq \frac{y}{1+y}. \]

Consequently, for any \( K > 0 \) and \( x \in [0, 1] \),
\[ \frac{1}{1+Kx} = e^{-\log(1+Kx)} \leq e^{-Kx/(1+Kx)} \leq e^{-Kx/(1+K)}. \]

Since \( h \in (0, 1) \), the above inequality shows that
\[ (17.16) \quad r = \frac{2d}{2d + \omega h^2} = \frac{1}{1 + \frac{\omega h^2}{2d h^2}} \leq e^{-C(d, \omega)h^2}. \]

where \( C(d, \omega) \) is an increasing function of \( \omega \).
By Lemma \[17.8\], Lemma \[17.9\] and the inequality \((17.16)\),

\[
S \leq \int_0^\infty r^t e^{-r|x|^2/t} t^{-\alpha/2} \, dt \\
\leq \int_0^\infty e^{-C(d, \omega) h^2 t - \beta |x|^2/t} t^{-\alpha/2} \, dt.
\]

 Applying the change of variable \(u = |x|^2/t\) in the above integration, we get

\[
S \leq |x|^{2-\alpha} \int_0^\infty \exp \left( -\frac{A}{u} - \frac{\beta u}{2} \right) u^{(\alpha-4)/2} \, du,
\]

where \(A = C(d, \omega) h^2 |x|^2\). The inequality \(a^2 + b^2 \geq 2ab\) shows that for all \(u > 0\),

\[
\frac{A}{2u} + \frac{\beta u}{2} \geq \sqrt{A \beta} =: B.
\]

Consequently,

\[
S \leq |x|^{2-\alpha} e^{-B} \int_0^\infty \exp \left( -\frac{A}{2u} - \frac{\beta u}{2} \right) u^{(\alpha-4)/2} \, du.
\]

When \(\alpha > 2\), the integrand may be bounded by a constant that depends only on \(\beta\), \(\alpha\), and \(d\) (and not \(x\)), by simply dropping the \(A/2u\) term.

Next, consider the case \(\alpha = 2\). Then

\[
\int_0^\infty \exp \left( -\frac{A}{2u} - \frac{\beta u}{2} \right) u^{-1} \, du \leq \int_0^\infty e^{-\beta u/2} u^{-1} \, du \\
\leq C(\beta) + C(\beta) |\log A|,
\]

and by the change of variable \(z = A/u\),

\[
\int_0^\infty \exp \left( -\frac{A}{2u} - \frac{\beta u}{2} \right) u^{-1} \, du \leq \int_0^\infty e^{-z/2} \frac{1}{z} \, dz \\
\leq C + C |\log A|.
\]

Finally, consider the case \(\alpha < 2\). Again by the change of variable \(z = A/u\),

\[
\int_0^\infty \exp \left( -\frac{A}{2u} - \frac{\beta u}{2} \right) u^{(\alpha-4)/2} \, du \leq A^{(\alpha-2)/2} \int_0^\infty e^{-z/2} z^{-\alpha/2} \, dz \\
\leq C(\alpha) A^{(\alpha-2)/2}.
\]

This completes the proofs of all three cases when \(x \neq 0\). When \(x = 0\), the steps are essentially the same, but simpler. \(\square\)

**Proposition 17.11.** Suppose that \(d \geq 3\). Given \(\omega > 0\), define \(g\) as in \((17.10)\). Then for all \(x \in \mathbb{Z}^d\),

\[
g(x) \leq C(d) h^{2-d} (\sqrt{d} + |x|)^{2-d} e^{-C(d, \omega) h|x|},
\]

where \(C(d, \omega)\) is an increasing function of \(\omega\) (when \(d\) is fixed).
PROOF. First, suppose that \( x \neq 0 \). Then by Lemma 17.7 and Lemma 17.8,

\[
g(x) \leq C(d)h^{2-d} \sum_{k=0}^{\infty} r^{k+1} e^{-|x|^2/2k} (1 + k)^{-d/2}.
\]

Lemma 17.10 now completes the proof. \( \square \)

COROLLARY 17.12. Suppose that \( d \geq 3 \) and \( g \) is the Green’s function defined in (17.10). Then for any \( t \in [1, d/(d-2)) \), \( \|g\|_{t,h} \leq C(t, d, \omega) \), where \( C(t, d, \omega) \) is a decreasing function of \( \omega \).

PROOF. Let \( A = 2\sqrt{d} \). By Proposition 17.11,

\[
\|g\|_{t,h}^t \leq h^d \sum_{x \in \mathbb{Z}^d} \frac{C(t, d)(h^{2-d}(\sqrt{d} + |x|)^2)^t}{|x| \leq A/h} + \sum_{x \in \mathbb{Z}^d} \frac{C(t, d)e^{-C_0(d, \omega)h|x|} t}{|x| > A/h},
\]

where \( C_0(d, \omega) \) is an increasing function of \( \omega \). By Lemma 17.9 and the assumption that \( h < 1 \),

\[
\begin{align*}
h^d \sum_{x \in \mathbb{Z}^d} e^{-C_0(d, \omega)h|x|} t &= e^{C_0(d, \omega)t\sqrt{d}} h^d \sum_{x \in \mathbb{Z}^d} e^{-C_0(d, \omega)h(|x| + \sqrt{d})t} 1_{\{|x| > A/h\}} \\
&\leq e^{C_0(d, \omega)t\sqrt{d}} h^d \int_{\mathbb{R}^d} e^{-C_0(d, \omega)h|y|} t 1_{\{|y| > A/h\}} dy \\
&= e^{C_0(d, \omega)t\sqrt{d}} \int_{\mathbb{R}^d} e^{-C_0(d, \omega)|z|} t 1_{\{|z| > A\}} dz \\
&= e^{C_0(d, \omega)t\sqrt{d}} \int_A^{\infty} u^{d-1} e^{-C_0(d, \omega)ut} du.
\end{align*}
\]

Since \( A = 2\sqrt{d} \), it is easy to see that the last expression above can be bounded by a constant \( C(t, d, \omega) \) that is a decreasing function of \( \omega \).

Again by Lemma 17.9,

\[
\begin{align*}
h^d \sum_{x \in \mathbb{Z}^d} \frac{C(t, d)(h^{2-d}(\sqrt{d} + |x|)^2)^t}{|x| \leq A/h} &\leq C(t, d)h^d \int_{\mathbb{R}^d} (h|y|)^{(2-d)t} 1_{\{|y| \leq A/h + \sqrt{d}\}} dy = \\
\end{align*}
\]
\begin{equation*}
= C(t, d) \int_{\mathbb{R}^d} |z|^{(2-d)t} 1_{\{|z| \leq A+h \sqrt{d}\}} \frac{d}{dz} z
= C(t, d) \int_0^{A+h \sqrt{d}} u^{d-1+(2-d)t} \frac{d}{du} u.
\end{equation*}

Since $h < 1$ and $d - 1 + (2 - d)t \geq -1 + \epsilon$ for some positive $\epsilon = \epsilon(t, d)$, this shows that the integrand is bounded by $C(t, d)$ and completes the proof of the lemma. \hfill \Box

### 17.7 Hardy-Littlewood-Sobolev Inequality

Take any $\omega > 0$ and let $g$ be the Green’s function defined in (17.10). Given a function $u \in L^2(\mathbb{Z}^d)$, let $f$ be the unique solution to

\[(\omega I - \Delta) f = u,
\]

so that by Lemma [17.7] $f = g * u$. The following proposition is a discrete analogue of the Hardy-Littlewood-Sobolev theorem of fractional integration (see [60 chap. V, sec. 1.2]).

**Proposition 17.13.** Let $f$ and $u$ be as above and suppose that $d \geq 3$. Let $1 < p < q < \infty$ satisfy

\[\frac{1}{q} > \frac{1}{p} - \frac{2}{d}.
\]

Then $\|f\|_{q, h} \leq C(p, q, d, \omega) \|u\|_{p, h}$, where $C(p, q, d, \omega)$ is a decreasing function of $\omega$.

This theorem is actually quite a bit simpler than the classical Hardy-Littlewood-Sobolev theorem, which includes the endpoint case $1/q = 1/p - 2/d$. Including the endpoint requires a somewhat delicate argument using the Marcinkiewicz interpolation theorem, which we can afford to avoid.

**Proof.** Let $s = (0, 2)$ satisfy

\[\frac{1}{q} = \frac{1}{p} - \frac{s}{d}.
\]

Let $t = d/(d - s)$. Then by Young’s inequality,

\[\|f\|_{q, h} \leq \|g\|_{t, h} \|u\|_{p, h}.
\]

Since $t < d/(d - 2)$, by Corollary [17.12] $\|g\|_{t, h} \leq C(p, q, d, \omega)$, where $C(p, q, d, \omega)$ is a decreasing function of $\omega$. This completes the proof. \hfill \Box
17.8 Derivatives of the Green’s Function

Recall the discrete derivative operator $\nabla_i$ defined in (17.3). The following proposition gives an estimate on the size of $\nabla_i g$, where $g$ is the discrete Green’s function defined in (17.10).

**Proposition 17.14.** Given $\omega > 0$, define $g$ as in (17.10). If $d \geq 2$, then for all $x \in \mathbb{Z}^d$ and all $1 \leq i \leq d$,

$$|\nabla_i g(x)| \leq C(d) h^{1-d} (\sqrt{d} + |x|)^{1-d} e^{-C(d,\omega) h|x|},$$

where $C(d,\omega)$ is an increasing function of $\omega$. When $d = 1$,

$$|\nabla_1 g(x)| \leq C (1 + \log(h(1 + |x|))) e^{-C(\omega) h|x|},$$

where $C(\omega)$ is an increasing function of $\omega$.

To prove this proposition, we first need to introduce some notation. For each nonnegative integer $k$, let

$$P_k := \{(k_1, \ldots, k_d) \in \mathbb{Z}^d : 0 \leq k_1, \ldots, k_d \leq k, \ k_1 + \cdots + k_d = k\}.$$

For any $k \geq 0$ and any $(k_1, \ldots, k_d) \in P_k$ and $(x_1, \ldots, x_d) \in \mathbb{Z}^d$, let

$$\psi(k; k_1, \ldots, k_d; x_1, \ldots, x_d) := \frac{k! d^{-k}}{k_1! \cdots k_d!} \prod_{j=1}^d \left(\frac{k_j}{(k_j + x_j)/2}\right)^{2-k_j}.$$

Here, as usual, we interpret $\binom{q}{a}$ as 0 if either $a$ or $b$ is not a nonnegative integer, or if $b > a$. In other words, for the above expression to be nonzero, it is necessary and sufficient that for all $j$, $x_j$ has the same parity as $k_j$ and satisfies $|x_j| \leq k_j$.

**Lemma 17.15.** For any $k \geq 1$, any $1 \leq i \leq d$, any $(k_1, \ldots, k_d) \in P_{k-1}$, and any $(x_1, \ldots, x_d) \in \mathbb{Z}^d$, we have

$$\psi(k - 1; k_1, \ldots, k_d; x_1, \ldots, x_d) = \psi(k; k_1, \ldots, k_i + 1, \ldots, k_d; x_1, \ldots, x_i + 1, \ldots, x_d) \frac{(k_i + x_i + 2)d}{k}.$$

**Proof.** If $k_j + x_j$ is odd for some $j$, then both sides are 0. So assume that $k_j$ has the same parity as $x_j$ for each $j$. Similarly, if $|x_j| > k_j$ for some $j \neq i$, then both sides are 0. So assume that $|x_j| \leq k_j$ for all $j \neq i$.

If $|x_i| > k_i$, then the left side is 0, and there are three possibilities for the right side. First, $x_i$ may be equal to $-k_i - 2$. In that case, $k_i + x_i + 2 = 0$ and so the right side is 0. The other possibilities are that $x_i < -k_i - 2$ or $x_i \geq k_i + 1$. In both of these cases, $|x_i + 1| > k_i + 1$, and hence the right side is 0. Note that the case $x_i = -k_i - 1$ is excluded because $x_i$ has the same parity as $k_i$. Therefore in all three cases we have equality of the two sides. So we may now safely assume that $|x_i| \leq k_i$.

At this point, we have that both sides are nonzero. Verifying the identity is now a simple algebraic exercise.
Proof of Proposition 17.14. Take any $x \in \mathbb{Z}^d$ such that $\sum_{i=1}^{d} x_i$ is even. Then $p(x, k) = 0$ for all odd $k$ and $p(x + e_i, k)$ is 0 for all even $k$, where $e_i$ denotes the $i^{th}$ coordinate vector. Thus, from the expression (17.10),

$$g(x + e_i) - g(x) = h \left( \frac{h^{1-d}}{2d} \left( \sum_{k \text{even}} r^{k+1} p(x, k) - \sum_{k \text{odd}} r^{k+1} p(x + e_i, k) \right) \right)$$

(17.17)

$$= \frac{h^{1-d}}{2d} \sum_{k \text{even}} r^{k+1} (1 - r) p(x, k) + \frac{h^{1-d}}{2d} \sum_{k \text{odd}} r^{k+1} (p(x, k - 1) - p(x + e_i, k)).$$

Now fix some odd $k$ and some $1 \leq i \leq d$. Then by formula (17.11),

(17.18) $p(x, k - 1) = \sum_{(k_1, \ldots, k_d) \in P_{k-1}} \psi(k - 1; k_1, \ldots, k_d; x_1, \ldots, x_d)$

and

(17.19) $p(x + e_i, k) = \sum_{(k_1, \ldots, k_d) \in P_k} \psi(k; k_1, \ldots, k_d; x_1, \ldots, x_i + 1, \ldots, x_d)$.

Let $P'_k$ be the set of all $(k_1, \ldots, k_d) \in P_k$ with $k_i \neq 0$. It is easy to see that the map

$$(k_1, \ldots, k_d) \mapsto (k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_d)$$

is a bijection between $P_{k-1}$ and $P'_k$. Thus, by Lemma 17.15

$$p(x, k - 1) = \sum_{(k_1, \ldots, k_d) \in P_{k-1}} \psi(k - 1; k_1, \ldots, k_d; x_1, \ldots, x_d)$$

$$= \sum_{(k_1, \ldots, k_d) \in P_k} \psi(k; k_1, \ldots, k_i + 1, \ldots, k_d; x_1, \ldots, x_i + 1, \ldots, x_d) \frac{(k_i + x_i + 2)d}{k}$$

$$= \sum_{(k_1, \ldots, k_d) \in P'_k} \psi(k; k_1, \ldots, k_d; x_1, \ldots, x_i + 1, \ldots, x_d) \frac{(k_i + x_i + 1)d}{k}.$$

By (17.18) and (17.19), this shows that

$$|p(x, k - 1) - p(x + e_i, k)| \leq \sum_{(k_1, \ldots, k_d) \in P_k} \psi(k; k_1, \ldots, k_d; x_1, \ldots, x_i + 1, \ldots, x_d) \left| \frac{(k_i + x_i + 1)d}{k} \right| 1_{\{k_i \neq 0\}} - 1.$$
Let $K_1, \ldots, K_d$ be the random variables defined in the proof of Lemma 17.8. Then by the above inequality and (17.13), we get

$$|p(x, k - 1) - p(x + e_i, k)|$$

$$\leq \mathbb{E}
\left(2^{-k} \left| \frac{(K_i + x_i + 1)}{k} 1_{\{K_i \neq 0\}} - 1 \right| \prod_{j=1}^{d} \left(1 + \frac{K_j}{1 + (K_j)^{-1/2}}\right)\right)$$

$$\leq C(d)e^{-|x|^2/2k}\mathbb{E}
\left(\left| \frac{(K_i + x_i + 1)}{k} 1_{\{K_i \neq 0\}} - 1 \right| \prod_{j=1}^{d} (1 + K_j)^{-1/2}\right).$$

Now observe the following:

- By (17.15), for all $r > 0$, $\mathbb{E}(1 + K_i)^{-r} \leq C(d, r)(1 + k)^{-r}$. By Hölder’s inequality, this gives

$$\mathbb{E}
\left(\left| \frac{(K_i + x_i + 1)}{k} 1_{\{K_i \neq 0\}} - 1 \right| \prod_{j=1}^{d} (1 + K_j)^{-1/2}\right) \leq (1 + k)^{-d/2}\mathbb{E}
\left(\left| \frac{(K_i + x_i + 1)}{k} 1_{\{K_i \neq 0\}} - 1 \right|^{d+1}\right)^{1/(d+1)}.$$

- Since $|x_i|/k \leq (|x|)/\sqrt{k}k^{-1/2}$,

$$\left(\mathbb{E}
\left| \frac{(K_i + x_i + 1)}{k} 1_{\{K_i \neq 0\}} - 1 \right|^{d+1}\right)^{1/(d+1)} \leq \frac{|x|}{\sqrt{k}}k^{-1/2} + \frac{d}{k} + \left(\mathbb{E}
\left| \frac{K_i d}{k} 1_{\{K_i \neq 0\}} - 1 \right|^{d+1}\right)^{1/(d+1)}.$$

- Since $K_i$ is a binomial random variable with parameters $k$ and $1/d$, it follows by Hoeffding’s tail bound [26] that $\mathbb{P}(K_i = 0) \leq e^{-C(d)k}$. Moreover, $K_i \leq k$. Thus,

$$\left(\mathbb{E}
\left| \frac{K_i d}{k} 1_{\{K_i \neq 0\}} - 1 \right|^{d+1}\right)^{1/(d+1)} \leq e^{-C(d)k} + \left(\mathbb{E}
\left| \frac{K_i d}{k} - 1 \right|^{d+1}\right)^{1/(d+1)}.$$

- Again by Hoeffding’s bound, for any $r > 0$,

$$\mathbb{E}
\left| \frac{K_i d}{k} - 1 \right|^r \leq C(d, r)k^{-r/2}.$$

Combining all of the above, we see that for any odd $k$,

(17.20) $|p(x, k - 1) - p(x + e_i, k)| \leq C_1(d)k^{-d+1/2}e^{-C_2(d)|x|^2/k}.$
Using this estimate and Lemma [17.8] to bound the right-hand side in (17.17) and applying Lemma [17.10] we get that for \( d \geq 2 \),
\[
|\nabla_i g(x)| \leq C(d)h^{3-d} (\sqrt{d} + |x|)^{2-d} e^{-C(d, \omega) h|x|} \\
+ C(d)h^{1-d} (\sqrt{d} + |x|)^{1-d} e^{-C(d, \omega) h|x|},
\]
where \( C(d, \omega) \) is an increasing function of \( \omega \). This proves Proposition [17.14] when \( d^* \geq 2 \).

When \( d = 1 \), we use (17.20), (17.17), (17.10), and Lemma 17.8 to get
\[
|\nabla_i g(x)| \leq C h^2 \sum_{k=0}^{\infty} r^{k+1} e^{-|x|^2/2k} (1 + k)^{-1/2} \\
+ C \sum_{k=0}^{\infty} r^{k+1} e^{-|x|^2/2k} (1 + k)^{-1}.
\]
Lemma [17.10] now completes the proof. \( \square \)

**Corollary 17.16.** For any \( d \) and any \( 1 \leq i \leq d \),
\[
\|\nabla_i g\|_{1,h} \leq C(d, \omega),
\]
where \( C(d, \omega) \) is a decreasing function of \( \omega \).

**Proof.** This lemma follows easily from Proposition [17.16] and Lemma [17.9] \( \square \)

## 18 Regularity of Discrete Solitons

Suppose that \( 1 < p < 1 + 4/d \). As in Section [17] assume that \( h \in (0, 1) \). Take any \( m > 0 \) and let \( f \in L^2(\mathbb{Z}^d) \) be a ground state soliton of mass \( m \) for the discrete NLS on \( \mathbb{Z}^d \) under grid size \( h \). That is, \( f \) minimizes energy among all functions of mass \( m \). By Theorem 16.1 it is easy to see that at least one such function exists. Also, by elementary Euler-Lagrange techniques, it follows that there is an \( \omega > 0 \) such that \( f \) satisfies the soliton equation \((\omega I - \Delta) f = |f|^{p-1} f\), where \( I \) is the identity operator and \( \Delta \) is the discrete Laplacian on \( \mathbb{Z}^d \) at grid size \( h \) as defined in (3.1).

The purpose of this section is to prove regularity properties of discrete solitons. The main goal will be to prove that the smoothness bounds remain uniformly bounded as the grid size goes to 0. This is necessary for proving convergence to continuum solitons. The proof follows more or less the sketch of the proof of regularity for continuum solitons (see [65, prop. B.7]), but using the discrete estimates from Section 17.

In this section, \( C \) will denote any positive constant that depends only on \( p, d, \) and \( m \). In particular, \( C \) will not depend on \( h \). Moreover, we impose the additional condition that \( C \) is uniformly bounded as \( m \) ranges over any given compact subinterval of \( (0, \infty) \). We will call this the uniform boundedness condition. If \( C \) depends on additional parameters \( a, b, \ldots \), then it is denoted as \( C(a, b, \ldots) \).
Lemma 18.1. $\omega \geq 1/C$.

Proof. Following exactly the same steps as in the proof of Lemma 13.2, we arrive at the inequality

$$\omega \geq -\frac{E_{\min}(m,h)}{m}.$$  

The uniform boundedness condition holds due to Lemma 9.4. □

Lemma 18.2. $G_h(f) \leq C$.

Proof. Taking $q = p + 1$ and

$$\theta = \frac{d(p-1)}{2(p+1)},$$

the discrete Gagliardo-Nirenberg inequality (Proposition 17.6) implies that

$$\|f\|_{p+1,h}^{p+1} \leq C \|f\|_{2,h}^{(1-\theta)(p+1)} G_h(f)^{\theta/(p+1)/2} \leq C G_h(f)^{d(p-1)/4}.$$  

(Note that, since $h > 0$, $f \in L^2$ implies that $f \in L^q$. Also, it is easy to verify using the condition $1 < p < 1 + 4/d$ that $\theta \in (0,1)$. Lastly, note that the uniform boundedness condition on $C$ is clearly satisfied.) Now, $H_h(f) = E_{\min}(m,h) < 0$ by Lemma 9.4. Consequently, by the previous display,

$$G_h(f) = H_h(f) + N_h(f) \leq C \|f\|_{p+1,h}^{p+1} \leq C G_h(f)^{d(p-1)/4}.$$  

Since $1 < p < 1 + 4/d$, or, equivalently, $d(p-1)/4 \in (0,1)$, this shows that $G_h(f) \leq C$. □

Lemma 18.3. For all $q \in [2,\infty]$, $\|f\|_{q,h} \leq C(q)$.

Proof. The case when $q = 2$ is already known from the assumption that $M_h(f) = m$. First, assume that $q \in (2,\infty)$. If $d \leq 2$, then for any such $q$, we can find $\theta \in (0,1)$ satisfying (17.4), and therefore by the discrete Gagliardo-Nirenberg inequality and Lemma 18.2, it follows that $\|f\|_{q,h} \leq C(q)$ for all $2 < q < \infty$.

Next, suppose that $d \geq 3$. By the discrete Gagliardo-Nirenberg inequality (Proposition 17.6) and Lemma 18.2, we have that $\|f\|_{q,h} \leq C(q)$ for all $2 \leq q < 2d/(d-2)$.

Define a sequence $q_k$ as follows: Let $q_0 := 2d/(d-2)$. Note that the condition $p < 1 + 4/d$ implies that $q_0 > p + 1$. For each $k$, let $q_{k+1}$ satisfy

$$\frac{1}{q_{k+1}} = \frac{p}{q_k} - \frac{2}{d}$$

unless the right-hand side is nonpositive, in which case let $q_{k+1} = \infty$. If $q_k = \infty$ for some $k$, then this definition implies that $q_j = \infty$ for all $j \geq k$. 
Note that if \( q_k \) is finite for all \( k \), then for each \( k \) we must have
\[
\frac{1}{q_k} = \frac{p^k}{q_0} - \frac{2}{d}(1 + p + \cdots + p^{k-1}) = \frac{p^k}{2d} \left(d - 2 - \frac{4}{p} \sum_{i=0}^{k-1} p^{-i}\right).
\]
But as \( k \to \infty \),
\[
\frac{4}{p} \sum_{i=0}^{k-1} p^{-i} \to \frac{4}{p-1} > d - 2.
\]
This shows that \( q_k \) must become infinity at some finite \( k \).

Next, note that \( q_k \) is an increasing sequence. This is easily proved by induction as follows. Suppose that \( q_k \geq q_0 = 2d/(d - 2) \). If \( q_{k+1} = \infty \), there is nothing to prove. Otherwise, by the condition \( p < 1 + 4/d \),
\[
\frac{1}{q_k} - \frac{1}{q_{k+1}} = \frac{2}{d} - \frac{p-1}{q_k} \geq \frac{2}{d} - \frac{(p-1)(d-2)}{2d} > 0.
\]
Take any \( k \) such that \( q_k < \infty \). Suppose we have proved that for all \( q \in [2, q_k) \),
\[
(18.1) \quad \|f\|_{q,h} \leq C(q).
\]
Choose arbitrary \( q \in [q_k, q_{k+1}) \). Then \( 1 < q_k/p < q < \infty \), and
\[
\frac{1}{q} > \frac{1}{q_{k+1}} \geq \frac{p}{q_k} - \frac{2}{d}.
\]
Take \( q' \in [2, q_k) \) so close to \( q_k \) that
\[
\frac{1}{q} > \frac{p}{q'} - \frac{2}{d}.
\]
Let \( f_0 := |f|^{p^{-1}} \). Since \( f = (\omega I - \Delta)^{-1} f_0 \) and \( \omega \geq 1/C \) by Lemma [18.1], it follows by the discrete Hardy-Littlewood-Sobolev inequality (Proposition [17.13]) that
\[
\|f\|_q \leq C(q,q')\|f_0\|_{q'/p}
\]
\[
= C(q,q')\|f\|^{p}_{q'} \leq C(q,q').
\]
However, \( q' \) can be chosen depending only on \( q \), \( p \), and \( d \). Since the sequence \( q_k \) increases to infinity, this proves by induction that \( \|f\|_{q,h} \leq C(q) \) for all \( q \in (2, \infty) \).

(An important thing to note is that we crucially used the fact that the constant in Proposition [17.13] is a decreasing function of \( \omega \), in conjunction with Lemma [18.1] to conclude that \( C(q) \) may be chosen to depend on \( \omega \) only through \( m \) and not on the actual value of \( \omega \), and that moreover, \( C(q) \) satisfies the uniform boundedness condition.)
Next, consider the case $q = \infty$. Take any $r > d/(d-1)$, so that $r' < d/(d-2)$. Then by Corollary 17.12 Young’s inequality, Lemma 18.1 and what we have already proved above,

$$\|f\|_{\infty,h} = \|g \ast f_0\|_{\infty,h} \leq \|g\|_{r',h} \|f_0\|_{r,h} \leq \|g\|_{r',h} \|f\|_{pr,h} \leq C.$$ 

This completes the proof. \qed

**Lemma 18.4.** For any $q \in [2, \infty]$ and any $1 \leq i, j \leq d$,

$$\|\nabla_i f\|_{q,h} \leq C(q) \quad \text{and} \quad \|\nabla_i \nabla_j f\|_{q,h} \leq C(q).$$

**Proof.** Since the ground state soliton minimizes energy, and the function $|f|$ satisfies $M_h(f) = M_h(|f|)$ and $H_h(|f|) \leq H_h(f)$ by the triangle inequality, we must have $H_h(|f|) = H_h(f)$. Consequently, $\|f(x) - f(y)\| = |f(x) - f(y)|$ for each neighboring pair of points $(x, y)$, which shows that $f$ must be of the form $f(x) = \alpha f_1(x)$ for some constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and some function $f_1 : \mathbb{Z}^d \to [0, \infty)$. Therefore, without loss of generality we will assume in this proof that $f$ is a nonnegative function.

By Young’s inequality, Lemma 18.3, and the observation that $\nabla_i$ is a convolution operator, we have that for any $q \in [2, \infty]$,

$$\|\nabla_i f\|_{q,h} = \|(\nabla_i g) \ast f^p\|_{q,h} \leq \|\nabla_i g\|_{1,h} \|f^p\|_{q,h} \leq C(q) \|\nabla_i g\|_{1,h}. \quad (18.2)$$

Next, note that by the inequality $|ap - bp| \leq \max\{|pa^{p-1}, pb^{p-1}|\}|a - b|$ for nonnegative $a$ and $b$ (which follows by the mean value theorem) and fact that $\|f\|_{\infty,h} \leq C$ from Lemma 18.3 we have that for all $x$,

$$|\nabla_i f^p(x)| \leq C |\nabla_i f(x)|.$$ 

In particular, by Lemma 18.3 this implies that for all $q \in [2, \infty]$,

$$\|\nabla_i f^p\|_{q,h} \leq C \|\nabla_i f\|_{q,h} \leq C(q). \quad (18.3)$$

By the commutativity of convolution operators,

$$\nabla_i \nabla_j f = (\nabla_i g) \ast (\nabla_j f^p).$$

Thus, by Young’s inequality and (18.3), we see that for any $q \in [2, \infty]$,

$$\|\nabla_i \nabla_j f\|_{q,h} \leq \|\nabla_i g\|_{1,h} \|\nabla_j f^p\|_{q,h} \leq C(q) \|\nabla_i g\|_{1,h}. \quad (18.4)$$

Applying Corollary 17.16 (in conjunction with Lemma 18.1) to (18.2) and (18.4) completes the proof. The uniform boundedness condition follows from the monotonicity of the constant in Corollary 17.16. \qed
19 Continuum Limit of Discrete Solitons

In this section we establish that discrete solitons converge to continuum solitons as the grid size goes to 0 provided that the nonlinearity is mass-subcritical.

Assume that $1 < p < 1 + 4/d$. Fix $h > 0, m > 0$, and let $f$ be a ground state soliton of mass $m$ for the DNLS system (3.2) on $\mathbb{Z}^d$ at grid size $h$. Let $Q_{\lambda(m)}$ be the unique ground state soliton of mass $m$ for the continuum system (1.3). Let $\tilde{f}$ be the continuum image of $f$ at grid size $h$, as defined in Section 2.

**Theorem 19.1.** For any $q \in [2, \infty]$, $L^q(\tilde{f}, Q_{\lambda(m)}) \leq C(p, d, m, q, h)$, where $C(p, d, m, q, h)$ satisfies, for any $0 < m_0 \leq m_1 < \infty$ and any fixed $p, d, m,$ and $q$,

$$\lim_{h \to 0} \sup_{m_0 \leq m \leq m_1} C(p, d, m, q, h) = 0.$$  

The same bound also holds for $|H_{h}(f) - E_{\text{min}}(m)|$ (with the modification that there is no $q$).

In the following, $C$ will denote any positive constant that depends only on $p, d$, and $m$ satisfying the uniform boundedness condition defined in Section 18; that is, $C$ remains uniformly bounded as $m$ varies over a compact subinterval of $(0, \infty)$, with $p$ and $d$ fixed. If $C$ depends on additional parameters $a, b, \ldots$, then it will be written as $C(a, b, \ldots)$. We will also adopt the convention that $o(1)$ denotes any constant that depends only on $p, d, m$, and $h$ satisfies (19.1).

Let $w : \mathbb{R} \to \mathbb{R}$ be the function

$$w(\tau) := \begin{cases} 1 - |\tau| & \text{if } |\tau| \leq 1, \\ 0 & \text{if } |\tau| > 1. \end{cases}$$

Extend $w$ to $\mathbb{R}^d$ as

$$w(t_1, \ldots, t_d) := w(t_1)w(t_2) \cdots w(t_d).$$

Define a function $f^c : \mathbb{R}^d \to \mathbb{C}$ as

$$f^c(y) := \sum_{x \in \mathbb{Z}^d} f(x)w\left(\frac{y - hx}{h}\right).$$

**Lemma 19.2.** The function $f^c$ is absolutely continuous. If $x \in \mathbb{Z}^d$ and $y = hx + ht$ for some $t \in (0, 1)^d$, then

$$f^c(y) = \sum_{s \in \{0, 1\}^d} f(x + s) \prod_{i=1}^d t_i^{s_i} (1 - t_i)^{1-s_i}.$$
If $\partial_i f^c$ denotes the partial derivative of $f$ in the $i^{th}$ coordinate, then for $x$ and $y$ as above,

$$\partial_i f^c(y) = \sum_{s \in \{0,1\}^d \atop s_i = 0} \nabla_i f(x + s) \prod_{1 \leq j \leq d \atop j \neq i} t_j^{s_j} (1 - t_j)^{1-s_j}.$$ 

**PROOF.** Since $w$ is absolutely continuous with bounded support, it follows easily from the definition of $f^c$ that $f^c$ is an absolutely continuous function on $\mathbb{R}^d$. Take $x$ and $y$ as in the statement of the lemma. Take any $\epsilon_2 \mathbb{Z}^d$, and note that $w((y - h \epsilon)/h)$ is nonzero if and only if $|y_i/h - z_i| < 1$ for $i = 1, \ldots, d$. Since $y_i/h = x_i + t_i$ for some $t_i \in (0, 1)$ for each $i$, $w((y - h \epsilon)/h) \neq 0$ if and only if each $z_i$ is either $x_i$ or $x_i + 1$. In other words,

$$f^c(y) = \sum_{s \in \{0,1\}^d} f(x + s) w\left(\frac{y - hx - hs}{h}\right) = \sum_{s \in \{0,1\}^d} f(x + s) \prod_{i=1}^d w(t_i - s_i).$$ 

An easy verification shows that when $s_i \in \{0,1\}$, $w(t_i - s_i) = t_i^{s_i} (1 - t_i)^{1-s_i}$. This completes the proof of the first identity. For the second, note that by the previous display,

$$\partial_i f^c(y) = \frac{1}{h} \frac{\partial}{\partial t_i} f^c(hx + ht) = \frac{1}{h} \sum_{s \in \{0,1\}^d} f(x + s) \frac{\partial}{\partial t_i} \left(\prod_{j=1}^d w(t_j - s_j)\right).$$

Now,

$$\frac{\partial}{\partial t_i} w(t_i - s_i) = \begin{cases} 1 & \text{if } s_i = 1, \\ -1 & \text{if } s_i = 0. \end{cases}$$

This proves the second identity. \qed

**LEMMA 19.3.** The function $f^c$ satisfies

$$|M_h(f) - M(f^c)| \leq C h \quad \text{and} \quad |H_h(f) - H(f^c)| \leq C h^{2/(p+1)}.$$ 

**PROOF.** For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, let $B(x)$ be the cube

$$B(x) := \{y \in \mathbb{R}^d : x_i \leq y_i/h < x_i + 1, \ i = 1, \ldots, d\}.$$ 

Another way to represent $B(x)$ is $\{hx + ht : t \in [0, 1)^d\}$. If $y$ is a point in the interior of $B(x)$, then Lemma[19.2] shows that $f^c(y)$ is a convex combination of $\{f(x + s) : s \in \{0,1\}^d\}$. Therefore, for any $r \in [2, \infty),

$$(19.2) \quad |f^c(y)|^r \leq \max_{s \in \{0,1\}^d} |f(x + s)|^r \leq \sum_{s \in \{0,1\}^d} |f(x + s)|^r.$$
Consequently,
\[
\int_{B(x)} |f^c(y)|^r \, dy \leq h^d \sum_{s \in \{0,1\}^d} |f(x + s)|^r.
\]

Summing over \(x\) gives
\[
\int_{\mathbb{R}^d} |f^c(y)|^r \, dy = \sum_{x \in \mathbb{Z}^d} \int_{B(x)} |f^c(y)|^r \, dy
\leq 2^d h^d \sum_{x \in \mathbb{Z}^d} |f(x)|^r = 2^d \|f\|_{r,h}.
\]

By Lemma 18.3 the bounds (19.2) and (19.3) show that for all \(r \in \mathbb{R}^d\)
\[
\|f^c\|_r \leq C(r).
\]

Next, note that
\[
|f^c(y) - f(x)| \leq \max_{s \in \{0,1\}^d} |f(x + s) - f(x)|
\leq \sum_{s \in \{0,1\}^d} \sum_{j=1}^d h|\nabla_j f(x + s)|.
\]

Recall that if \(y \in B(x)\), then \(\tilde{f}(y) = f(x)\). Therefore by inequality (19.5) and Lemma 18.2,
\[
\int_{\mathbb{R}^d} (f^c(y) - \tilde{f}(y))^2 \, dy = \sum_{x \in \mathbb{Z}^d} \int_{B(x)} (f^c(y) - f(x))^2 \, dy
\leq Ch^2 G_h(f) \leq Ch^2.
\]

It is easy to see that for all \(r\), \(\|\tilde{f}\|_r = \|f\|_{r,h}\). Therefore, for any \(r \in \mathbb{R}^d\), by the \(L^\infty\)-bounds from (19.4) and Lemma 18.3 and the above inequality,
\[
\|f^c\|_r - \|\tilde{f}\|_r \leq \|f^c - \tilde{f}\|_r
\leq \|f^c - \tilde{f}\|_2 \leq 2^d \leq C(r)h^{2/r}.
\]

Using this bound, and again applying the \(L^r\)-bounds from inequality (19.4) and Lemma 18.3, we get
\[
\|f^c\|_r - \|\tilde{f}\|_r \leq C(r)h^{2/r}.
\]

Next, let \(\partial_i f^c\) be the partial derivative of \(f^c\) in the \(i^{th}\) coordinate. By Lemma 19.2,
\[
|\partial_i f^c(y) - \nabla_i f(x)| \leq \max_{s \in \{0,1\}^d} |\nabla_i f(x + s) - \nabla_i f(x)|
\leq \sum_{s \in \{0,1\}^d} \sum_{j=1}^d h|\nabla_j \nabla_i f(x + s)|.
\]
Let $\partial_i f(x) = \frac{\partial_i f^c(y) - \partial_i f(y)}{h}$ be the function that is identically equal to $\nabla_i f(x)$ in the interior of the box $B(x)$ and arbitrarily defined on the boundaries. Then the above inequality, together with Lemma [18.4] implies that

$$\int_{\mathbb{R}^d} (\partial_i f^c(y) - \partial_i f(y))^2 \, dy = \sum_{x \in \mathbb{Z}^d} \int_{B(x)} (\partial_i f^c(y) - \nabla_i f(x))^2 \, dy \leq C h^2 \max_{1 \leq i \leq d} \|\nabla_i \nabla_i f\|_{2,h}^2 \leq C h^2.$$

Clearly for all $r$, $\|\partial_i f\|_r = \|\nabla_i f\|_{r,h}$. Therefore by the above inequality and Lemma [18.4] $\|\partial_i f^c\|_2$ and $\|\partial_i f\|_2$ are both bounded by $C$. Consequently, again applying the previous display,

$$\|\|\partial_i f^c\|_2^2 - \|\partial_i \bar{f}\|_2^2\| \leq C \|\partial_i f^c\|_2 - \|\partial_i \bar{f}\|_2 \leq C \|\partial_i f^c - \partial_i \bar{f}\|_2 \leq C h.$$

Using the above bound and (19.6), we get

$$|M_h(f) - M(f^c)| = \|\bar{f}\|_2^2 - \|f^c\|_2^2 \leq C h$$

and

$$|H_h(f) - H(f^c)| \leq \|\partial_i \bar{f}\|_2^2 - \|\partial_i f^c\|_2^2 + \|\bar{f}\|_{p+1}^2 - \|f^c\|_{p+1}^2 \leq C h^{2/(p+1)}.$$

This completes the proof of the lemma. □

Let $Q$ be the unique, positive radially symmetric solution of (1.7). Let $Q_0 := Q_{\lambda(m)}$ be the continuum ground state soliton of mass $m := M(f^c)$. Let $\hat{Q} : \mathbb{Z}^d \to \mathbb{R}$ be the function $\hat{Q}(x) := Q(hx)$.

**Lemma 19.4.** Recall the $o(1)$ convention introduced immediately after the statement of Theorem [19.1] Then

$$|M(Q') - M_h(\hat{Q})| \leq o(1) \quad \text{and} \quad |H(Q') - H_h(\hat{Q})| \leq o(1).$$

**Proof.** The function $Q$ is a Schwartz function (see the remark following the proof of proposition B.7 in [65, app. B]). In other words, $Q$ is a $C^\infty$-function, and all its derivatives decay faster than any polynomial. This provides ample regularity of $Q$, and together with the scaling relation (1.8), this completes the proof of this lemma. □

**Lemma 19.5.** $|H(f^c) - H(Q')| \leq o(1)$. 

PROOF. Let $\alpha$ be a number such that $\alpha \hat{Q}$ has the same mass as $f$. In other words,

$$\alpha = \frac{M_h(f)}{\sqrt{M_h(Q)}}.$$  

By Lemma 19.3 $|M_h(f) - M(f_c)| \leq o(1)$, and by Lemma 19.4 $|M(Q') - M(\hat{Q})| \leq o(1)$. But by definition of $Q'$, $M(Q') = M(f_c)$. Thus,

(19.8) $|\alpha - 1| \leq o(1)$.

Now recall the following:

- $Q$ is a Schwarz function (see the remark in the proof of Lemma 19.4).
- $|M(f_c) - m| \leq o(1)$ by Lemma 19.3.
- $Q'$ is related to $Q$ by the scaling relation (1.8).

Combining the above, it follows easily that for all $r \in [2, \infty)$ and $1 \leq i \leq d$, $\|\hat{Q}\|_{r,h} \leq C(r)$ and $\|\nabla_i \hat{Q}\|_{r,h} \leq C(r)$. Together with (19.8), this implies that

$$|H(\alpha \hat{Q}) - H(\hat{Q})| \leq o(1).$$

Since $f$ is a discrete ground state soliton, $H_h(\alpha \hat{Q}) \geq H_h(f)$. Combining this with the previous display and Lemmas 19.3 and 19.4 we get

$$H(Q') \leq H(f_c) \leq H_h(f) + o(1) \leq H_h(\alpha \hat{Q}) + o(1) \leq H_h(\hat{Q}) + o(1) \leq H(Q') + o(1).$$

This completes the proof of the lemma.

LEMMA 19.6. For any $q \in [2, \infty]$ and any absolutely continuous $v : \mathbb{R}^d \to \mathbb{C}$ such that $v \in L^2(\mathbb{R}^d)$ and $\nabla v$ is uniformly bounded, we have

$$\|v\|_q \leq C(d, q) \|v\|_2^{\frac{q}{2} (1 - \frac{1}{2}) + \frac{2}{d}} \|\nabla v\|_{\infty}^{\frac{d}{2} (1 - \frac{1}{2})},$$

where $\|\nabla v\|_{\infty} := \max_{1 \leq i \leq d} \|\partial_i v\|_{\infty}$.

PROOF. If $\|\nabla v\|_{\infty} = 0$, then since $v \in L^2$, $v$ must be 0 almost everywhere. So let us assume that $\|\nabla v\|_{\infty} \in (0, \infty)$.

Take any $x_0 \in \mathbb{R}^d$. Let $B$ be the ball of radius $r$ around $x_0$, where

$$r = \left(\frac{\|v\|_2}{\|\nabla v\|_{\infty}}\right)^{2/(d+2)}.$$

Note that

$$\int_B |v(x)|^2 \frac{d\chi}{\text{Vol}(B)} \leq \frac{\|v\|_2^2}{\text{Vol}(B)},$$

which shows that there exists $y \in B$ such that

$$|v(y)|^2 \leq \frac{\|v\|_2^2}{\text{Vol}(B)} = C(d) r^{-d} \|v\|_2^2.$$
Since \( \|x_0 - y\| \leq r \), this shows that
\[
|v(x_0)| \leq |v(y)| + \|x_0 - y\| \|\nabla v\|_{\infty}
\leq C(d)r^{-d/2}\|v\|_2 + Cr\|\nabla v\|_{\infty}
\leq C(d)\|v\|_2^{2/(d+2)}\|\nabla v\|_{d/(d+2)}.
\]
Since this is true for every \( x_0 \), the right-hand side is a bound for \( \|v\|_{\infty} \). To complete the proof, note that for any \( q \in (2, \infty) \), \( \|v\|^q \leq \|v\|_{\infty}^{q-2} \|v\|_2^2 \).

**Proof of Theorem 19.1.** By Lemma 19.3,
\[
(19.9) \quad |M(f^c) - m| = |m' - m| \leq o(1).
\]
Again by Lemma 19.5, \( |H(f^c) - H(Q')| \leq o(1) \). But \( H(Q') = E_{\min}(m') \), \( |m' - m| \leq o(1) \), and \( E_{\min} \) is a continuous function by (1.6); consequently,
\[
(19.10) \quad |H(f^c) - E_{\min}(m)| \leq o(1).
\]
Together with Lemma 19.3, this proves that \( |H_h(f) - E_{\min}(m)| \leq o(1) \).

The bounds (19.9) and (19.10), together with the orbital stability of ground state solitons (see, e.g., [50, prop. 3]) imply that
\[
\tilde{L}^2(f^c, Q_{\lambda(m)}) \leq o(1).
\]
By (19.7) and Lemma 18.4, \( \|\partial_i f^c\|_{\infty} \leq C \) for each \( i \). Therefore by Lemma 19.6,
\[
\tilde{L}^q(f^c, Q_{\lambda(m)}) \leq o(1) \quad \text{for each} \quad q \in [2, \infty).
\]

**Corollary 19.7.** For every fixed \( 0 < m_0 \leq m_1 < \infty \),
\[
\lim_{h \to 0} \sup_{m_0 \leq m \leq m_1} |E_{\min}(m, h) - E_{\min}(m)| = 0.
\]

**Proof.** For every \( m > 0 \) and \( h > 0 \), let \( f_{m,h} \) be a discrete ground state soliton of mass \( m \) for the DNLS at grid size \( h \). Then \( E_{\min}(m, h) = H_h(f_{m,h}) \). Theorem 19.1 completes the proof.

### 20 Continuum Limit of the Variational Problem

Recall the objects \( \Theta, \mathcal{M}, \) and \( \mathcal{R} \) defined in Section 11 and the function \( \Psi_d \) defined in (4.2). The following theorem shows that as the grid size goes to 0, the set \( \mathcal{M}(E_0, m_0, h) \) converges to the single point \((0, E_0 - E_{\min}(m_0))\):

**Theorem 20.1.** Fix \( m_0 > 0 \) and \( E_{\min}(m_0) < E_0 < \infty \). For each \( h > 0 \), let \( (E_h, m_h) \) be an element of \( \mathcal{M}(E_0, m_0, h) \). Then
\[
\lim_{h \to 0} m_h = 0 \quad \text{and} \quad \lim_{h \to 0} E_h = E_0 - E_{\min}(m_0).
\]

**Proof.** Recall the functions \( E^+(m, h) \) and \( E^-(m, h) \). If \( h \) is sufficiently small, then \( E_0 < dm_0/h^2 \) and by Lemma 19.7, \( E_0 \geq E_{\min}(m_0, h) \). Therefore by Lemma 11.1 when \( h \) is sufficiently small,
\[
E_h = E^+(m_h, h) = E_0 - E_{\min}(m_0 - m_h, h).
\]
We will prove that \( m_h \to 0 \) as \( h \to 0 \) by a subsequence argument. Then by the above identity and Lemma 19.7 the limit for \( E_h \) will be automatically established. Let \( m_h \) tend to a point \( m' \in [0, m_0] \) along a sequence \( h_i \to 0 \). Then by Corollary 19.7

\[
\lim_{i \to \infty} E_{h_i} = E' := E_0 - E_{\min}(m_0 - m') \geq 0.
\]

We will show that \( m' = 0 \) by the method of contradiction. Suppose that \( m' > 0 \).

Fix \( m'' \in (0, m') \) and let

\[
E'' := E_0 - E_{\min}(m_0 - m'').
\]

Note that \( E_{\min} \) is strictly decreasing (by Lemma 16.6 and Lemma 9.4); therefore

\[
(20.1) \quad E'' > E' \geq 0.
\]

Also, defining

\[
E_i'' := E_0 - E_{\min}(m_0 - m'', h_i) = E^+(m'', h_i),
\]

we see that by Corollary 19.7

\[
(20.2) \quad \lim_{i \to \infty} E_i'' = E''.
\]

Fix \( \gamma \in (0, 1) \) and note that for all \( i \),

\[
\Theta(E_{h_i}, m_{h_i}, h_i) - 2 \log h_i \leq \log E_{h_i} - \int_{[0,1]^d} \log \left( \frac{(1 - \gamma)h_i^2 E_{h_i}}{m_{h_i}} + 2\gamma \sum_{j=1}^d \sin^2(\pi x_j) \right) dx_1 \cdots dx_d,
\]

and therefore (since \( m' > 0 \)),

\[
\limsup_{i \to \infty} \Theta(E_{h_i}, m_{h_i}, h_i) - 2 \log h_i \leq \log E' - C(\gamma),
\]

where

\[
C(\gamma) := \int_{[0,1]^d} \log \left( 2\gamma \sum_{j=1}^d \sin^2(\pi x_j) \right) dx_1 \cdots dx_d.
\]

Since this is true for all \( \gamma \in (0, 1) \), we can take \( \gamma \to 1 \) in the above bound and get

\[
(20.3) \quad \limsup_{i \to \infty} \Theta(E_{h_i}, m_{h_i}, h_i) - 2 \log h_i \leq \log E' - C(1).
\]

On the other hand, for any \( i \), (20.2) gives

\[
\Theta(E_i'', m'', h_i) - 2 \log h_i = \log E_i'' - \sup_{0<\gamma<1} \int_{[0,1]^d} \log \left( \frac{(1 - \gamma)h_i^2 E_i''}{m''} + 2\gamma \sum_{j=1}^d \sin^2(\pi x_j) \right) dx_1 \cdots dx_d \geq
\]
\begin{align*}
&\geq \log E''_i - \int_{[0,1]^d} \log \left( \frac{\hbar^2 E''_i}{m''} + 2 \sum_{j=1}^d \sin^2(\pi x_j) \right) dx_1 \cdots dx_d \\
\rightarrow \log E'' \sim C(1) &\text{ as } i \to \infty.
\end{align*}

Therefore, by (20.3) and (20.1), this shows that for all sufficiently large $i$,

$$\Theta(E''_i, m'', h_i) > \Theta(E_{h_i}, m_{h_i}, h_i).$$

But by the definition of $E''_i$, we know that $E_{h_i} > E_{h_i}$, contradicting the definition of $E_{h_i}$ as a maximizer of $\Theta(E, m, h)$ in $\mathcal{R}(E_0, m_0, h)$. □

21 Proofs of Theorems 4.1, 4.2, and 4.3

**Proof of Theorem 4.1** Lemma 9.5 gives the formula for $E_{\text{max}}(m, h)$, while Lemma 9.4 and Lemma 9.1 show that $-\infty < E_{\text{min}}(m, h) < 0$. The subadditive inequality follows from Lemma 16.6. Lastly, Corollary 19.7 shows that $E_{\text{min}}(m, h) \to E_{\text{min}}(m)$ as $h \to 0$ and that the convergence is uniform over compact subsets of $(0, \infty)$. □

**Proof of Theorem 4.2** Take any sequence of function $f_k$ on $\mathbb{Z}^d$ such that $M_h(f_k) \to m$ and $H_h(f_k) \to E_{\text{min}}(m, h)$ as $k \to \infty$. Let $\alpha_{k}$ be a constant such that $M(\alpha_{k} f_k) = m$. Then $\alpha_{k} \to 1$ and therefore $H_h(\alpha_{k} f_k)$ also tends to $E_{\text{min}}(m, h)$. Theorem 16.1 now guarantees the existence of a subsequence $\alpha_{k_j} f_{k_j}$ converging to a limit $f \in S(m, h)$ in the $L^q$ pseudometric for every $q \in [2, \infty]$. Now, $f_k$ is uniformly $L^2$-bounded, and hence uniformly $L^q$-bounded for every $q \in [2, \infty]$. Therefore, since $\alpha_{k_j} \to 1$, $f_{k_j}$ also tends to $f$ in $L^q$. This proves compactness of $S(m, h)$. To prove that the set is nonempty, simply note that by the definition of $E_{\text{min}}(m, h)$, there exists a sequence $f_k$ satisfying $M_h(f_k) = m$ for all $k$ and $H_h(f_k) \to E_{\text{min}}(m, h)$. □

**Proof of Theorem 4.3** This is a direct consequence of Theorem 19.1. □

22 Proof of Theorem 4.4

To be notationally compatible with the theorems of Sections 12, 14, and 15 we will write $E_0$ instead of $E$ and $m_0$ instead of $m$. As in those sections, the numbers $p, d, h, E_0,$ and $m_0$ will be fixed throughout this section and will be called the “fixed parameters.” Any positive constant that depends only on the fixed parameters will be denoted simply by $C$, instead of $C(p, d, h, E_0, m_0)$. If the constant depends on additional parameters $a, b, \ldots$, then it will be denoted by $C(a, b, \ldots)$.

Fix $n$ and recall the definition of $\delta$-soliton from Section 12. Recall also the random function $\phi$ defined in Section 7 and the objects $\Theta$, $\mathcal{M}$, and $\mathcal{R}$ defined in
Lastly, recall that the set $S(m, h)$ denotes the set of discrete ground state solitons of mass $m$ at grid size $h$.

Given a function $f : V \rightarrow \mathbb{C}$, let $f^e : \mathbb{Z}^d \rightarrow \mathbb{C}$ be the extension of $f$ to $\mathbb{Z}^d$, defined as

$$f^e(x) = \begin{cases} f(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

Having defined $f^e$, say that $f$ is an “improved $\delta$-soliton” if there exists $g : \mathbb{Z}^d \rightarrow \mathbb{C}$ such that

(a) $\|f^e - g\|_\infty \leq \delta$, and
(b) there exists $(E^*, m^*) \in \mathcal{M}(E_0, m_0, h)$ such that $|(E_0 - E^*) - H_h(g)| \leq \delta$ and $|(m_0 - m^*) - M_h(g)| \leq \delta$.

**Lemma 22.1.** If $E_{\min}(m_0, h) < E_0 < d m_0 / h^2$, then the set $K$ in the statement of Theorem 4.4 can be alternatively described as

$$K = K(h) := \{m' \in [0, m] : m' = m_0 - m^* \text{ for some } (E^*, m^*) \in \mathcal{M}(E_0, m_0, h)\}.$$ 

**Proof.** Suppose that $m' \in [0, m_0]$ maximizes

$$q(m) := \log(m_0 - m) - \Psi_d \left( \frac{2h^2(E_0 - E_{\min}(m, h))}{m_0 - m} \right).$$

Let $m^* = m_0 - m'$ and $E^* = E_0 - E_{\min}(m', h) = E^+(m^*, h)$. Note that $E^+(m^*, h)$ is necessarily positive, for otherwise $q(m')$ would be $-\infty$, which is impossible since it maximizes $q(m)$ in the interval $[0, m_0]$ and $q(m) > -\infty$ for $m$ sufficiently close to $m_0$. This shows that $(E^*, m^*) \in \mathcal{R}(E_0, m_0, h)$. Now take any $(E, m) \in \mathcal{M}(E_0, m_0, h)$. Then by Lemma 11.1, $E = E^+(m, h)$, and therefore

$$\Theta(E, m, h) = \log m - \Psi_d \left( \frac{2h^2(E_0 - E_{\min}(m_0 - m, h))}{m} \right) = q(m_0 - m) \leq q(m') = \Theta(E^*, m^*, h).$$

Thus, $(E^*, m^*) \in \mathcal{M}(E_0, m_0, h)$.

Next, suppose that we are given $m'$ such that $m' = m_0 - m^*$ for some $(E^*, m^*) \in \mathcal{M}(E_0, m_0, h)$. By Lemma 11.1

$$\Theta(E^*, m^*, h) = q(m').$$

Take any $m \in [0, m_0]$. We have to show that $q(m') \geq q(m)$. If $q(m) = -\infty$, there is nothing to prove. If not, then

$$0 < E_0 - E_{\min}(m, h) = E^+(m_0 - m, h).$$

Let $m_1 := m_0 - m$ and $E_1 := E^+(m_0 - m, h)$. The above display proves that $\max\{E^-(m_1, h), 0\} \leq E_1 = E^+(m_1, h)$.
and hence \((E_1, m_1) \in \mathcal{R}(E_0, m_0, h)\). Thus,
\[
q(m') = \Theta(E^*, m^*, h) \geq \Theta(E_1, m_1, h) = q(m).
\]
This completes the proof of the lemma.

**Lemma 22.2.** For any \(\eta > 0\), there exists \(\delta = \delta(\eta) > 0\) depending only on \(\eta\) and the fixed parameters (and not on \(n\)) such that if \(f : V_n \to \mathbb{C}\) is an improved \(\delta\)-soliton, then there exists \(v \in S(m', h)\) for some \(m' \in K\) such that \(\tilde{L}^\infty(f^e, v) \leq \eta\).

**Proof.** We will argue by contradiction. Suppose that the statement of the theorem is false. Then there exists an \(\eta > 0\) such that for every positive integer \(k\), there exist \(n_k\) and a function \(f_k : V_{n_k} \to \mathbb{C}\) such that \(f_k\) is an improved \(k^{-1}\)-soliton, but \(\tilde{L}^\infty(f^e_k, v) > \eta\) for all \(v \in \bigcup_{m' \in K} S(m', h)\).

For each \(k\), let \(g_k\) be a function on \(\mathbb{Z}^d\) satisfying the requirements (a) and (b) in the definition of improved \(\delta\)-soliton (with \(\delta = k^{-1}\) and \(f = f_k\)). Let \((E_k^*, m_k^*)\) be the corresponding element of \(\mathcal{M}(E_0, m_0, h)\). We will show that \(f_k^e\) approaches an element of \(\bigcup_{m' \in K} S(m', h)\) in \(\tilde{L}^\infty\) pseudometric through a subsequence that will give us the necessary contradiction.

Since \(\mathcal{M}(E_0, m_0, h)\) is a compact set (Lemma 11.1), we may assume without loss of generality that \((E_k^*, m_k^*)\) approaches a limit \((E^*, m^*) \in \mathcal{M}(E_0, m_0, h)\) as \(k \to \infty\). Let \(E' := E_0 - E^*\) and \(m' := m_0 - m^*\). Then \(M_{h}(g_k) \to m'\) and \(H_{h}(g_k) \to E'\). But by Lemma 11.1, \(E' = E_{\min}(m', h)\). Therefore, by Theorem 4.2, \(g_k\) approaches some \(g \in S(m', h)\) in the \(\tilde{L}^\infty\) pseudometric through a subsequence. Since \(\|f_k^e - g_k\|_{\tilde{L}^\infty} \to 0\) as \(k \to \infty\), this shows that \(f_k^e\) also approaches \(g\) in the \(\tilde{L}^\infty\) pseudometric through the same subsequence. But by Lemma 22.1, \(m' \in K\). This completes the argument.

Given a function \(f : V_n \to \mathbb{C}\), let \(f_\tau\) denote a random translate of \(f\), that is,
\[
f_\tau(x) := f(x + \tau),
\]
where \(\tau\) is uniformly distributed on \(V_n\) and the addition on the right-hand side is addition modulo \(n\) in each coordinate.

**Lemma 22.3.** If \(f\) is a \(\delta\)-soliton, then, provided that \(n > C(\delta)\),
\[
\mathbb{P}(f_\tau\text{ is an improved } 2\delta\text{-soliton}) \geq 1 - C n^{-1}.
\]

**Proof.** Let \(g : V_n \to \mathbb{C}\) be a function satisfying the requirements (a) and (b) in the definition of \(\delta\)-soliton. Then clearly \(f_\tau\) is also a \(\delta\)-soliton, with \(g_\tau\) serving the role of \(g\). Let \(\partial V_n\) denote the boundary of \(V_n\) in \(\mathbb{Z}^d\). Since \(\tau\) is uniformly distributed on \(V_n\), it is easy to see that
\[
\mathbb{E}(M_{h,n}(g_\tau, \partial V_n)) = \frac{|\partial V_n|}{|V_n|} M_{h,n}(g) \leq C n^{-1}.
\]
Therefore by Markov’s inequality,
\[
\mathbb{P}(M_{h,n}(g_\tau, \partial V_n) > n^{-1/2}) \leq C n^{-1/2}.
\]
Now, $M_h(g^\xi_T) = M_{h,n}(g_T)$ and $N_h(g^\xi_T) = N_{h,n}(g_T)$. Also, it is easy to verify that
\[
|G_h(g^\xi_T) - G_{h,n}(g_T)| \leq CM_h,n(g_T, \partial V_n).
\]
Thus, if $M_{h,n}(g_T, \partial V_n) \leq n^{-1/2}$, then $f_T$ is an improved $\delta'$-soliton, where $\delta' = \delta + C n^{-1/2}$. By (22.2), this completes the proof. □

**Proof of Theorem 4.4.** First, assume that
\[
E_{\min}(m_0, h) < E_0 < \frac{1}{2} E_{\max}(m_0, h).
\]
By Lemma 9.5, $E_{\max}(m_0, h) = 2d m_0 / h^2$. Therefore, we are in the setting of Theorem 14.1. Let
\[
S := S_{\varepsilon,h,n}(E_0, m_0) = \{v \in \mathbb{C}^V : |M_{h,n}(v) - m_0| \leq \varepsilon, |H_{h,n}(v) - E_0| \leq \varepsilon\}.
\]
as defined in Section 2. Let $f = f_{\varepsilon,h,n}$ be a random function chosen uniformly from $S$. Let
\[
A := \{v \in S : v \text{ is not a } \delta\text{-soliton}\}.
\]
Then by Theorem 12.1 and Theorem 14.1,
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log \mathbb{P}(\phi \in A) - \log \mathbb{P}(\phi \in S)}{n^d} < 0.
\]
By Lemma 7.1 this shows that
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log \mathbb{P}(f \in A)}{n^d} < 0.
\]
In particular,
\[
(22.4) \quad \lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(f \text{ is a } \delta\text{-soliton}) = 1.
\]
But by Lemma 22.3,
\[
\mathbb{P}(f_T \text{ is an improved } 2\delta\text{-soliton}) \geq (1 - C n^{-1}) \mathbb{P}(f \text{ is a } \delta\text{-soliton}).
\]
However, $f_T$ has the same distribution as $f$. Combined with (22.4), this shows that
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(f \text{ is an improved } 2\delta\text{-soliton}) = 1.
\]
Since this is true for any $\delta > 0$, Lemma 22.2 completes the proof of Theorem 4.4 for the $\hat{L}_1$ pseudometric under the condition (22.3).

To prove the result for general $q \in (2, \infty)$, simply observe that for any such $q$ and any $v : \mathbb{Z}^d \to \mathbb{C}$,
\[
(22.5) \quad \|v\|_q^q \leq \|v\|_\infty^{q-2} \|v\|_2^2.
\]
and that $\|f\|_2 \leq C$ by definition.

When $E_0 \geq \frac{1}{2} E_{\max}(m_0, h)$, Theorem 4.4 is a direct consequence of Theorem 15.1 and Lemma 7.1. □
23 Proof of Theorem 2.1

As in Section 22, we will write $E_0$ instead of $E$ and $m_0$ instead of $m$. The convention about the notation $C$ will also be the same.

Let $K(h)$ be defined as in (22.1). Fix $q \in (2, \infty]$ and $\delta > 0$.

**Lemma 23.1.** Whenever $h < C(q, \delta)$, for all $v \in \bigcup_{m' \in K(h)} S(m', h)$,

$$\bar{L}^q(\bar{v}, Q_{\lambda(m_0)}) \leq \delta/2,$$

where $\bar{v}$ is the continuum image of $v$ at grid size $h$.

**Proof.** Take a sequence $h_k$ decreasing to 0. For each $k$, let $v_k$ be an element of $\bigcup_{m' \in K(h_k)} S(m', h_k)$. Let $m'_k := M_{h_k}(v_k)$ and $m_k := m_0 - m'_k$. By definition of $v_k$, $m'_k \in K(h_k)$, and by definition of $K(h_k)$, there exists $E_k$ such that $(E_k, m_k) \in \mathcal{M}(E_0, m_0, h)$. Therefore by Theorem 20.1 $\lim_{k \to \infty} m_k = 0$. Consequently, $M_{h_k}(v_k)$ tends to $m_0$. Since we know that $v_k$ is a discrete ground state soliton for every $k$, by Theorem 4.3

$$\lim_{k \to \infty} \bar{L}^q(\bar{v}_k, Q_{\lambda(m_0)}) = 0.$$

A simple argument by contradiction now completes the proof. \hfill \Box

**Proof of Theorem 2.1.** Fix $q \in (2, \infty]$ and $\delta > 0$. Take any $h$ and any $v \in \bigcup_{m' \in K(h)} S(m', h)$.

Note that

$$\bar{L}^q(\bar{f}_{e,h,n}, Q_{\lambda(m_0)}) \leq \bar{L}^q(\bar{f}_{e,h,n}, \bar{v}) + \bar{L}^q(\bar{v}, Q_{\lambda(m_0)}) = \bar{L}^q(f_{e,h,n}, v) + \bar{L}^q(\bar{v}, Q_{\lambda(m_0)}).$$

Thus, if $\bar{L}^q(\bar{v}, Q_{\lambda(m_0)}) \leq \delta/2$ for all $v \in \bigcup_{m' \in K(h)} S(m', h)$, then

$$\mathbb{P}(\bar{L}^q(\bar{f}_{e,h,n}, Q_{\lambda(m_0)}) > \delta) \leq \mathbb{P}(\inf_{m' \in K(h)} \inf_{v \in S(m', h)} \bar{L}^q(f_{e,h,n}, v) > \delta/2).$$

Now, $E_{\min}(m_0) < E_0 < \infty$ by assumption. Therefore, Corollary 19.7 and Lemma 9.5 show that for all sufficiently small $h$,

$$E_{\min}(m_0, h) < E_0 < \frac{dm_0}{h^2} = \frac{1}{2} E_{\max}(m_0, h).$$

Therefore by Theorem 4.4 and Lemma 23.1 for all $h < C(q, \delta)$,

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbb{P}(\bar{L}^q(\bar{f}_{e,h,n}, Q_{\lambda(m_0)}) > \delta) = 0.$$

This completes the proof of the first part of Theorem 2.1. For the second part, first fix $q \in (2, \infty]$ and $\delta > 0$. Fix some $k > 0$. Choose $h_k$ so small that $h_k < 1/k$ and (23.1) holds with $h = h_k$. Given $h_k$, choose $\epsilon_k$ so small that $\epsilon_k < 1/k$ and

$$\lim_{n \to \infty} \mathbb{P}(\bar{L}^q(\bar{f}_{\epsilon_k,h_k,n}, Q_{\lambda(m_0)}) > \delta) < 1/k.$$
Finally, given $h_k$ and $\epsilon_k$, choose $n_k$ so large that $n_kh_k > k$ and

$$\mathbb{P}(\tilde{L}^q(f_{\epsilon_k,h_k,n_k}, Q\lambda(m_0)) > \delta) < 2/k.$$  

This shows that

$$\lim_{k \to \infty} \mathbb{P}(\tilde{L}^q(f_{\epsilon_k,h_k,n_k}, Q\lambda(m_0)) > \delta) = 0.$$  

Since such a sequence $(\epsilon_k, h_k, n_k)$ exists for any $\delta > 0$ (with $q$ fixed), one can extract a sequence that works simultaneously for all $\delta > 0$ by a diagonal argument. In particular, there exists a sequence $(\epsilon_k, h_k, n_k)$ such that for all $\delta > 0$,

$$\lim_{k \to \infty} \mathbb{P}(\tilde{L}^\infty(f_{\epsilon_k,h_k,n_k}, Q\lambda(m_0)) > \delta) = 0.$$  

But again, for any $v : \mathbb{R}^d \to \mathbb{C}$ and any $q \in (2, \infty)$, we have the inequality (22.5). Since the $L^2$-norm of $f_{\epsilon,h,n}$ is uniformly bounded (because $\|f_{\epsilon,h,n}\|_2^2 = M_{h,n}(f_{\epsilon,h,n}) \in [m_0 - \epsilon, m_0 + \epsilon]$), this completes the proof of the second assertion of Theorem 2.1. \hfill \Box

### 24 Proof of Theorem 3.1

Take any $v \in \mathcal{M}$ and any bounded measurable function $\phi$ on $S$. By Birkhoff’s ergodic theorem (see, e.g., [72, theorem 1.14]) and the ergodicity of the map $T_1$ with respect to the measure $\nu$,

$$v\left\{f \in S : \lim_{l \to \infty} \frac{1}{l} \sum_{r=1}^{l} \phi(T_r f) = \int_{S} \phi(v)d\nu(v) \right\} = 1.$$  

Now fix $\phi$ to be the function

$$\phi(f) = \int_{0}^{1} \frac{1}{L^\infty(T_t+rf, Q\lambda(m)) > \delta} dt.$$  

Then for any $r$,

$$\phi(T_r f) = \int_{0}^{1} \frac{1}{L^\infty(T_{t+r} f, Q\lambda(m)) > \delta} dt$$

$$= \int_{r}^{r+1} \frac{1}{L^\infty(T_t f, Q\lambda(m)) > \delta} dt.$$  

Consequently,

$$\frac{1}{l} \sum_{r=1}^{l} \phi(T_r f) = \frac{1}{l} \int_{0}^{l} \frac{1}{L^\infty(T_t f, Q\lambda(m)) > \delta} dt.$$  

The above steps show that for any $v \in \mathcal{M}$,

(24.1) if $\int_{S} \phi(v)d\nu(v) < \delta$, then $v$ satisfies SRC with error $\delta$.  

For each $j$, let $g_j : \mathbb{R} \to \mathbb{R}$ be the function

$$g_j(x) = \begin{cases} 
1 & \text{if } x > \delta + j^{-1}, \\
y & \text{if } x = \delta + yj^{-1} \text{ for some } y \in [0, 1], \\
0 & \text{if } x < \delta.
\end{cases}$$

Note that (a) $g_j$ is a continuous function, (b) $g_j(x) \leq 1_{\{x > \delta\}}$ for all $x$ and $j$, and (c) for each $x$, $g_j(x)$ increases to $1_{\{x > \delta\}}$ as $j \to \infty$.

Define the map $\xi : S \to \mathbb{R}$ as

$$\xi(v) := \tilde{L}^\infty(\nu, Q_{\lambda(m)}).$$

It is easy to see that $\xi$ is a continuous function on $S$. For each $j$, let

$$\phi_j(v) := \int_0^1 g_j(\xi(T_t v)) dt.$$

By our previous observations about $g_j$ and the continuity of $\xi$ and $T_t$ (as remarked in Section 3), we see that (a) for each $j$, $\phi_j$ is a continuous function taking value in $[0, 1]$, (b) $\phi_j(v) \leq \phi(v)$ for each $v$ and $j$, and (c) for each $v$, $\phi_j(v)$ increases to $\phi(v)$ as $j \to \infty$.

By (3.3) and the continuity of $\phi_j$, we have that for each $j$,

$$\int_M \left( \int_S \phi_j(v) d\nu(v) \right) d\tau(v) = \int_S \phi_j(v) d\mu(v).$$

(24.2)

Since $\phi_j \leq \phi$, therefore

$$\int_S \phi_j(v) d\mu(v) \leq \int_S \phi(v) d\mu(v)$$

for all $j$. Since $\phi_j \to \phi$ pointwise, by Fatou’s lemma from measure theory and (24.2) and (24.3),

$$\int_M \left( \int_S \phi(v) d\nu(v) \right) d\tau(v) \leq \liminf_{j \to \infty} \int_S \phi_j(v) d\mu(v) \leq \int_S \phi(v) d\mu(v)$$

(24.4)

Again, since $T_t$ preserves $\mu$,

$$\int_S \phi(v) d\mu(v) = \int_0^1 \int_{L^\infty(T_t v, Q_{\lambda(m)}) > \delta} dt d\mu(v)$$

$$= \int_0^1 \int_S 1_{L^\infty(T_t v, Q_{\lambda(m)}) > \delta} d\mu(v) dt =$$
\[ \int_0^1 \int_{S} 1_{\{ \tilde{L} \sim (\bar{v}, Q_{\lambda(m)}) > \delta \}} d\mu(v) dt \]

(24.5)

\[ \int_{S} 1_{\{ \tilde{L} \sim (\bar{v}, Q_{\lambda(m)}) > \delta \}} d\mu(v). \]

By (24.4), this gives

(24.6) \[ \int_{\mathcal{M}} \left( \int_{S} \phi(v) d\nu(v) \right) d\tau(v) \leq \int_{S} 1_{\{ \tilde{L} \sim (\bar{v}, Q_{\lambda(m)}) > \delta \}} d\mu(v). \]

But by Theorem 2.1

(24.7) \[ \lim_{h \to 0} \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{S} 1_{\{ \tilde{L} \sim (\bar{v}, Q_{\lambda(m)}) > \delta \}} d\mu(v) = 0. \]

The displays (24.1), (24.6), and Markov’s inequality show that

\[ \tau\{ v \in \mathcal{M} : v \text{ satisfies SRC with error } \delta \} \]

\[ \geq \tau\{ v \in \mathcal{M} : \int_{S} \phi(v) d\nu(v) < \delta \} \]

\[ \geq 1 - \frac{1}{\delta} \int_{\mathcal{M}} \left( \int_{S} \phi(v) d\nu(v) \right) d\tau(v) \]

\[ \geq 1 - \frac{1}{\delta} \int_{S} 1_{\{ \tilde{L} \sim (\bar{v}, Q_{\lambda(m)}) > \delta \}} d\mu(v). \]

Together with (24.7), this completes the proof of Theorem 3.1.

Acknowledgment. The author thanks Pierre Germain, Partha Dey, Terence Tao, Jalal Shatah, Phil Sosoe, Kay Kirkpatrick, Julien Barré, Lai-Sang Young, Carlos Kenig, Persi Diaconis, Fraydoun Rezakhanlou, Stefano Olla, Raghu Varadhan, and the anonymous referee for useful comments. The manuscript of Raphaël [50] (which was brought to my attention by Pierre Germain) and Tao’s book [65] have been of immeasurable help.

Bibliography

[1] Arnol’d, V. I. Matematicheskie metody klassicheskoj mekhaniki. Third edition. “Nauka”, Moscow, 1989.

[2] Berestycki, H.; Gallouët, T.; Kavian, O. Équations de champs scalaires euclidiens non linéaires dans le plan. C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 5, 307–310.

[3] Berestycki, H.; Lions, P.-L. Existence d’ondes solitaires dans des problèmes nonlinéaires du type Klein-Gordon. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 7, A395–A398.
[4] Bludov, Y. V.; Konotop, V. V.; Akhmediev, N. Matter rogue waves. *Phys. Rev. A* **80** (2009), no. 3, 033610, 5 pp. [doi:10.1103/PhysRevA.80.033610]

[5] Bourgain, J. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166** (1994), no. 1, 1–26.

[6] Bourgain, J. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.* **176** (1996), no. 2, 421–445.

[7] Bourgain, J. On nonlinear Schrödinger equations. *Les relations entre les mathématiques et la physique théorique*, 11–21. Institut des hautes études scientifiques, Bures-sur-Yvette, France, 1998.

[8] Bourgain, J. Invariant measures for NLS in infinite volume. *Comm. Math. Phys.* **210** (2000), no. 3, 605–620. [doi:10.1007/s002200050792]

[9] Brydges, D. C.; Slade, G. Statistical mechanics of the 2-dimensional focusing nonlinear Schrödinger equation. *Comm. Math. Phys.* **182** (1996), no. 2, 485–504.

[10] Burq, N.; Tzvetkov, N. Invariant measure for a three dimensional nonlinear wave equation. *Int. Math. Res. Not. IMRN* **2007** (2007), no. 22, Art. ID rnm108, 26 pp. [doi:10.1093/imrn/rnm108]

[11] Burq, N.; Tzvetkov, N. Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.* **173** (2008), no. 3, 449–475. [doi:10.1007/s00222-008-0124-z]

[12] Burq, N.; Tzvetkov, N. Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.* **173** (2008), no. 3, 477–496. [doi:10.1007/s00222-008-0123-0]

[13] Cazenave, T. *An introduction to nonlinear Schrödinger equations*. Textos de Metodes Matemáticos, 22. Instituto de Matemática, Rio de Janeiro, 1989.

[14] Chatterjee, S. A note about the uniform distribution on the intersection of a simplex and a sphere. Preprint, 2010. arXiv:1011.4043

[15] Chatterjee, S.; Kirkpatrick, K. Probabilistic methods for discrete nonlinear Schrödinger equations. *Comm. Pure Appl. Math.* **65** (2012), no. 5, 727–757. [doi:10.1002/cpa.21388]

[16] Coffman, C. V. Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. *Arch. Rational Mech. Anal.* **46** (1972), 81–95. [doi:10.1007/BF00250684]

[17] Collander J.; Oh, T. Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. *Duke Math. J.* **161** (2012), no. 3, 367–414. [doi:10.1215/00127094-1507400]

[18] Duyckaerts, T.; Kenig, C.; Merle, F. Classification of radial solutions of the focusing, energy-critical wave equation. Preprint, 2012. arXiv:1204.0031

[19] Eckhaus, W. The long-time behaviour for perturbed wave-equations and related problems. *Trends in applications of pure mathematics to mechanics* (Bad Honnef, 1985), 168–194. Lecture Notes in Physics, 249. Springer, Berlin, 1986. [doi:10.1007/BFb0016391]

[20] Eckhaus, W.; Schuur, P. The emergence of solitons of the Korteweg-de Vries equation from arbitrary initial conditions. *Math. Methods Appl. Sci.* **5** (1983), no. 1, 97–116. [doi:10.1002/mmna.1670050108]

[21] Erdős, L.; Schlein, B.; Yau, H.-T. Derivation of the cubic nonlinear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.* **167** (2007), no. 3, 515–614. [doi:10.1007/s00222-006-0022-1]

[22] Erdős, L.; Schlein, B.; Yau, H.-T. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. of Math. (2)* **172** (2010), no. 1, 291–370. [doi:10.4007/annals.2010.172.291]

[23] Flach, S.; Kladko, K.; MacKay, R. S. Energy thresholds for discrete breathers in one-, two-, and three-dimensional lattices. *Phys. Rev. Lett.* **78** (1997), 1207–1210. [doi:10.1103/PhysRevLett.78.1207]

[24] Ginibre, J.; Velo, G. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Funct. Anal.* **32** (1979), no. 1, 1–32. [doi:10.1016/0022-1236(79)90076-4]
[25] Glimm, J.; Jaffe, A. *Quantum physics. A functional integral point of view*. Second edition. Springer, New York, 1987. doi:10.1007/978-1-4612-4728-9

[26] Hoeffding, W. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* 58 (1963), 13–30.

[27] Kato, T. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 46 (1987), no. 1, 113–129.

[28] Kenig, C. E.; Merle, F. Global well-posedness, scattering and blow-up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case. *Invent. Math.* 166 (2006), no. 3, 645–675. doi:10.1007/s00222-006-0011-4

[29] Kevrekidis, P. G. *The discrete nonlinear Schrödinger equation. Mathematical analysis, numerical computations and physical perspectives*. Springer Tracts in Modern Physics, 232. Springer, Berlin, 2009. doi:10.1007/978-3-540-89199-4

[30] Killip, R.; Visan, M. The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. *Amer. J. Math.* 132 (2010), no. 2, 361–424. doi:10.1353/ajm.0.0107

[31] Kirkpatrick, K.; Lenzmann, E.; Staffilani, G. On the continuum limit for discrete NLS with long-range lattice interactions. *Comm. Math. Phys.* 317 (2013), no. 3, 563–591. doi:10.1007/s00220-012-1621-x

[32] Kirkpatrick, K.; Schlein, B.; Staffilani, G. Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics. *Amer. J. Math.* 133 (2011), no. 1, 91–130. doi:10.1353/ajm.2011.0004

[33] Ladyzhenskaya, O. A. *The boundary value problems of mathematical physics*. Applied Mathematical Sciences, 49. Springer, New York, 1985.

[34] Lebowitz, J. L.; Mouniaix, P.; Wang, W.-M. Approach to equilibrium for the stochastic NLS. (English summary) *Comm. Math. Phys.* 321 (2013), no. 1, 69–84. doi:10.1007/s00220-012-1632-7

[35] Lebowitz, J. L.; Rose, H. A.; Speer, E. R. Statistical mechanics of the nonlinear Schrödinger equation. *I. Statist. Phys.* 50 (1988), no. 3–4, 657–687. doi:10.1007/BF01026495

[36] Lions, P.-L. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1 (1984), no. 2, 109–145.

[37] McKea, H. P. Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger. *Comm. Math. Phys.* 168 (1995), no. 3, 479–491.

[38] McKea, H. P.; Vaninsky, K. L. Brownian motion with restoring drift: the petit and micro-canonical ensembles. *Comm. Math. Phys.* 160 (1994), no. 3, 615–630.

[39] McKea, H. P.; Vaninsky, K. L. Action-angle variables for the cubic Schrödinger equation. *Comm. Pure Appl. Math.* 50 (1997), no. 6, 489–562. doi:10.1002/(SICI)1097-0312(199706)50:6<489::AID-CPA1>3.0.CO;2-4

[40] McKea, H. P.; Vaninsky, K. L. Cubic Schrödinger: the petit canonical ensemble in action-angle variables. *Comm. Pure Appl. Math.* 50 (1997), no. 7, 593–622. doi:10.1002/(SICI)1097-0312(199707)50:7<593::AID-CPA1>3.0.CO;2-2

[41] Miura, R. M. The Korteweg-de Vries equation: a survey of results. *SIAM Rev.* 18 (1976), no. 3, 412–459. doi:10.1137/1018076

[42] Nahmod, A. R.; Rey-Bellet, L.; Sheffield, S.; Staffilani, G. Absolute continuity of Brownian bridges under certain gauge transformations. *Math. Res. Lett.* 18 (2011), no. 5, 875–887.

[43] Nakanishi, K.; Schlag, W. *Invariant manifolds and dispersive Hamiltonian evolution equations*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2011. doi:10.4171/095

[44] Novokšenov, V. Ju. Asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for a nonlinear Schrödinger equation. *Dokl. Akad. Nauk SSSR* 251 (1980), no. 4, 799–802.

[45] Oh, T. Invariant Gibbs measures and a.s. global well posedness for coupled KdV systems. *Differential Integral Equations* 22 (2009), no. 7–8, 637–668.
Oh, T. Invariance of the white noise for KdV. *Comm. Math. Phys.* **292** (2009), no. 1, 217–236. doi:10.1007/s00220-009-0856-7

Oh, T. Invariance of the Gibbs measure for the Schrödinger-Benjamin-Ono system. *SIAM J. Math. Anal.* **41** (2009/10), no. 6, 2207–2225. doi:10.1137/080738180

Oh, T.; Quastel, J. On invariant Gibbs measures conditioned on mass and momentum. *J. Math. Soc. Japan* **65** (2013), no. 1, 13–35. doi:10.2969/jmsj/06510013

Oh, T.; Quastel, J.; Valkó, B. Interpolation of Gibbs measures with white noise for Hamiltonian PDE. *J. Math. Pures Appl. (9)* **97** (2012), no. 4, 391–410. doi:10.1016/j.matpur.2011.11.003

Raphaël, P. On the singularity formation for the non linear Schrödinger equation. *Evolution equations. Proceedings of the Clay Mathematics Institute Summer School held in Zürich, June 23–July 18, 2008, 269–323. Clay Mathematics Proceedings, 17.* American Mathematical Society, Providence, R.I.; Clay Mathematics Institute, Cambridge, Mass., 2014.

Rider, B. Fluctuations in the thermodynamic limit of focussing cubic Schrödinger. *J. Statist. Phys.* **113** (2003), no. 3–4, 575–594. doi:10.1023/A:1026072819239

Rider, B. C. On the \( \infty \)-volume limit of the focusing cubic Schrödinger equation. *Comm. Pure Appl. Math.* **55** (2002), no. 10, 1231–1248. doi:10.1002/cpa.10043

Robbins, H. A remark on Stirling’s formula. *Amer. Math. Monthly* **62** (1955), 26–29. doi:10.2307/2308012

Rumpf, B. Simple statistical explanation for the localization of energy in nonlinear lattices with two conserved quantities. *Phys. Rev. E* **69** (2004), no. 1, 016618, 5 pp. doi:10.1103/PhysRevE.69.016618

Rumpf, B.; Newell, A. C. Coherent structures and entropy in constrained, modulationally unstable, nonintegrable systems. *Phys. Rev. Lett.* **87** (2001), no. 5, 054102, 4 pp. doi:10.1103/PhysRevLett.87.054102

Schuur, P. C. *Asymptotic analysis of soliton problems.* An inverse scattering approach. Lecture Notes in Mathematics, 1232. Springer, Berlin, 1986.

Segur, H. The Korteweg-de Vries equation and water waves. Solutions of the equation. I. *J. Fluid Mech.* **59** (1973), 721–736. doi:10.1017/S0022112073001813

Segur, H. Asymptotic solutions and conservation laws for the nonlinear Schrödinger equation. II. *J. Mathematical Phys.* **17** (1976), no. 5, 714–716.

Soffer, A. Soliton dynamics and scattering. *International Congress of Mathematicians. Vol. III*, 459–471. European Mathematical Society, Zürich, 2006.

Stein, E. M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J., 1970.

Stein, E. M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, N.J., 1993.

Stein, E. M.; Weiss, G. *Introduction to Fourier analysis on Euclidean spaces.* Princeton Mathematical Series, 32. Princeton University Press, Princeton, N.J., 1971.

Strauss, W. A. *Nonlinear wave equations.* CBMS Regional Conference Series in Mathematics, 73. Published for the Conference Board of the Mathematical Sciences, Washington, D.C.; by the American Mathematical Society, Providence, R.I., 1989.

Tao, T. On the asymptotic behavior of large radial data for a focusing nonlinear Schrödinger equation. *Dyn. Partial Differ. Equ.* **1** (2004), no. 1, 1–48.

Tao, T. *Nonlinear dispersive equations.* CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, R.I., 2006.

Tao, T. A (concentration-)compact attractor for high-dimensional nonlinear Schrödinger equations. *Dyn. Partial Differ. Equ.* **4** (2007), no. 1, 1–53.

Tao, T. Why are solitons stable? *Bull. Amer. Math. Soc. (N.S.)* **46** (2009), no. 1, 1–33. doi:10.1090/S0273-0979-08-01228-7
[68] Thomann, L.; Tzvetkov, N. Gibbs measure for the periodic derivative nonlinear Schrödinger equation. *Nonlinearity* **23** (2010), no. 11, 2771–2791. doi:10.1088/0951-7715/23/11/003

[69] Tzvetkov, N. Invariant measures for the nonlinear Schrödinger equation on the disc. *Dyn. Partial Differ. Equ.* **3** (2006), no. 2, 111–160.

[70] Tzvetkov, N. Invariant measures for the defocusing nonlinear Schrödinger equation. *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 7, 2543–2604.

[71] Tzvetkov, N. Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation. *Probab. Theory Related Fields* **146** (2010), no. 3-4, 481–514. doi:10.1007/s00440-008-0197-z

[72] Walters, P. *An introduction to ergodic theory*. Graduate Texts in Mathematics, 79. Springer, New York–Berlin, 1982.

[73] Weinstein, M. I. Excitation thresholds for nonlinear localized modes on lattices. *Nonlinearity* **12** (1999), no. 3, 673–691. doi:10.1088/0951-7715/12/3/314

[74] Zakharov, V. E. Stability of periodic waves of finite amplitude on a surface of deep fluid. *J. Appl. Mech. Tech. Phys.* **9** (1968), 190–194.

[75] Zakharov, V. E.; Shabat, A. B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP* **34** (1972), no. 1, 62–69.; translated from Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134.

[76] Zhidkov, P. E. An invariant measure for the nonlinear Schrödinger equation. *Dokl. Akad. Nauk SSSR* **317** (1991), no. 3, 543–546; translation in *Soviet Math. Dokl.* **43** (1991), no. 2, 431–434.

SOURAV CHATTERJEE
Department of Statistics
Stanford University
390 Serra Mall, Sequoia Hall
Stanford, CA 94305
USA
E-mail: souravc@stanford.edu

Received June 2012.