Beyond $k$-induction: Learning from Counterexamples to Bidirectionally Explore the State Space

Mikhail R. Gadelha³, Felipe R. Monteiro², Enrico Steffinlongo¹, Lucas C. Cordeiro³, and Denis Nicole¹

1University of Southampton, United Kingdom
2Federal University of Amazonas, Brazil
3University of Manchester, United Kingdom
esbmc@googlegroups.com

Abstract. We describe and evaluate a novel $k$-induction proof rule called bidirectional $k$-induction ($bkind$), which substantially improves the $k$-induction bug-finding capabilities. Particularly, $bkind$ exploits the counterexamples generated by the over-approximation step to derive new properties and feed them back to the bounded model checking procedure. We also combine an interval invariant generator and $bkind$ to significantly improve the number of correct verification results. Experimental results show that $bkind$ can considerably reduce the verification time compared to the naive $k$-induction proof rule, since it only requires half the number of steps to find a given safety property violation in an unsafe program. The $bkind$ algorithm outperforms 2LS, another state-of-the-art $k$-induction verifier, and produces more than twice correct proofs and about 35% more correct alarms than when analysing a large set of public available benchmarks.

1 Introduction

Software model checking has experienced significant progress in the last two decades, however, one major bottleneck for its practical applications is scalability. In particular, Bounded Model Checking (BMC) is a promising approach to software verification [1], but its application to verify and refute properties in large code bases is limited by the resource requirements of the technique [2]. This happens when BMC techniques unwind all loops and recursive functions up to their given maximum bound or completeness threshold [3], which is typically infeasible for checking realistic programs.

In contrast, a variant called $k$-induction applied to unbounded programs uses BMC as a “component” to prove partial correctness [4]; it has been successfully combined with continuously-refined invariants [5] and to prove that ANSI-C programs do not contain data races [6,7] or that design-time constraints are respected [8]. Additionally, $k$-induction is a well-established technique in hardware verification, where it is applied due to the transition relation present in hardware designs [8,9,10]. Although we can prove partial correctness via induction without fully unwinding a program, state-of-the-art $k$-induction procedures still waste time and resources to falsify properties in programs since they unwind loops and recursion up to the depth that exposes a bug.

Here we describe and evaluate a bidirectional $k$-induction ($bkind$) algorithm, which is an extension of the original $k$-induction [10] that improves its bug-finding capabilities, reducing the number of iterations to find a property violation in half. In practice,
the bkind algorithm performs a bidirectional search for bugs in the program state space to quickly refute properties. Given the current knowledge in software model checking, our extension has not previously been described or evaluated in the literature, but we have already provided preliminary results of this approach on a limited number of small benchmarks [11]. Similar techniques do exist, however, in other domains: Bischoff et al. [12] describe a technique called “target enlargement” which combines binary decisions diagrams (BDDs) and Boolean Satisfiability (SAT) solvers to reduce the time to find property violations in hardware verification, and Bradley et al. introduced “property-directed reachability” (or IC3) procedure for safety verification of systems [13] and have shown that IC3 can scale on certain benchmarks, where k-induction fails to succeed. Jovanović et al. [14] describe a technique called “Property-Directed k-induction” to generate stronger invariants for programs written in the SALLY input language.

In summary, this paper makes the following original contributions. Firstly, we exploit the counterexamples generated by the k-induction proof rule to derive new properties and feed them back to the BMC procedure. Secondly, we combine an interval invariant generator and bkind to significantly improve the number of correct verification results. Lastly, our experimental results show that bkind can considerably reduce the verification time compared to the naïve k-induction proof rule, since it only requires half the number of steps to find a given safety property violation in an unsafe program. Compared to other state-of-the-art k-induction verifier (2LS), bkind produces more than twice correct proofs and about 35% more correct alarms, when analysing a large set of public available benchmarks.

2 Naïve k-Induction Proof Rule

The first version of the k-induction proof rule was proposed by Sheeran et al. [10]. They used BMC algorithms to prove correctness by induction. Consider a program $P$ with a loop and a safety property $\phi(s)$. BMC algorithms can only show that no counterexample exists for a $k$ loop unwindings but not that longer counterexamples do not exist. The $k$-induction proof rule tries to prove by induction that if $\phi$ holds for any given iteration through the loop then $\phi$ holds for the next iteration. In particular, the base case tries to find a counterexample where $\phi$ does not hold and the inductive step tries to prove that there exists no counterexamples. k-induction extends induction by assuming the safety property $k - 1$ times before checking its satisfiability [5], as described in Eq. (1)

$$
\left( \bigwedge_{i=0}^{k-1} \phi(i) \land \forall n : \left( \bigwedge_{i=0}^{k-1} \phi(n + i) \Rightarrow \phi(n + k) \right) \right) \Rightarrow \forall n : \phi(n).
$$

Since k-induction assumes the safety property $\phi$ more than once, a less general case is checked after each iteration, thus it is more likely to succeed [16]. In a previous work [17,18], we extended the k-induction proof rule to check program completeness in a separate step and defined it as an iterative deepening algorithm [19], consisting of three independent checks: base case, forward condition and inductive step.

---

1 Note that Craig interpolants can be used to exploit the SAT/SMT solvers’ ability to produce refutations, i.e., proofs that there is no counterexample of depth less than or equal to $k$ [15].
We describe the algorithms in this paper by assuming that a given program $P$ under verification is a state transition system $M$. In $M$, a state $s \in S$ is a tuple $(m, c)$, where $m$ is the state variable data, $c$ is the state constraint data. A predicate $\text{init}_P(s)$ denotes that $s$ is an initial state, $\text{tr}_P(s_i, s_j) \in T$ is a transition relation from $s_i$ to $s_j$, $\phi(s)$ is the formula encoding for states satisfying a safety property, and $\psi(s)$ is the formula encoding for states satisfying a completeness threshold $[3]$, which is equal to the maximum number of loop iterations occurring in $P$. For convenience, we define an error state $\epsilon$, reachable if there is a property violation in the program $P$. A counterexample $\pi^k$ is a sequence of states of length $k$ from the initial state $s_1$ to $\epsilon$.

**Algorithm 1.1:** The base case.

```plaintext
1 Function base_case($P$):
2   if $\text{init}_P(s_1) \land \bigwedge_{i=1}^{j-1} \text{tr}_P(s_i, s_{i+1}) \land \bigvee_{i=1}^j \neg \phi(s_i)$ then
3       Let $\epsilon = s_j$ such that $\neg \phi(s_j)$;
4       return $[s_1, \ldots, \epsilon]$;
5   else
6       return $\emptyset$;
7  end
```

**Lemma 1 (Base case).** If the function $\text{base\_case}(P)$ returns a sequence of states, then the program is unsafe and the sequence of states is a counterexample.

**Proof.** The execution path returned in line 4 of the Algorithm 1.1 is a counterexample since it is an execution path (ensured by the transition relation $\text{tr}_P$ in line 2) that starts with the first state of the program (ensured by $\text{init}_P(s_1)$ in line 2) and ends with an error state (in line 3); this follows the definition of a counterexample. This is also a non-spurious counterexample because the base case is a precise check: it encodes all reachable states up to $k$ and checks for satisfiability. If the base case returns a real counterexample then the program is unsafe.

**Algorithm 1.2:** The forward condition.

```plaintext
1 Function forward_condition($P$):
2   if $\text{init}_P(s_1) \land \bigwedge_{i=1}^{j-1} \text{tr}_P(s_i, s_{i+1}) \land \neg \psi(s_j)$ then
3       return $[s_1, \ldots, s_j]$;
4   else
5       return $\emptyset$;
6  end
```

**Lemma 2 (Forward condition).** If the function $\text{forward\_condition}(P)$ returns an empty sequence of states then the program is safe.

**Proof.** The forward condition checks if the completeness threshold was reached in current unwound program $P$, i.e., all loops were completely unwound. This is encoded as the completeness threshold property check in line 3 of Algorithm 1.2. In practice, these checks are encoded as unwinding assertions and they check if the termination condition of all loops are satisfiable for the current number of unw windings. This step can prove partial correctness if the base case did not find any bug for the current unwinding since no safety property is checked. We guarantee this precedence in $k$-induction by checking the base case before checking the forward condition. We conclude that if no bug was found by the base case and the completeness threshold holds for the current number of unw windings, all states were explored and the program is safe.
Algorithm 1.3: The inductive step.

Function inductive_{step}(P):
1 if \( \exists n \in \mathbb{N}^+ \land n \neq 1 \land n + j - 1 = n \land \phi(s_i) \land tr_P(s_i, s_{i+1}) \land \neg \phi(s_n + j) \) then
2 Let \( \epsilon = s_n + j \) such that \( \neg \phi(s_n + j) \);
3 return \( [s_n, \ldots, \epsilon] \);
4 else
5 return \( \emptyset \);
6 end

Lemma 3 (Inductive step). If the function inductive_{step}(P) returns an empty sequence of states, then the program is safe.

Proof. Similarly to the forward condition check, the program is safe up to \( k \) loop unwindings because the base case did not find any reachable error state. This is guaranteed in the \( k \)-induction proof rule by running the base case before the inductive step. The inductive step then tries to find any counterexample of length \( k \) in the state space by first assuming that there was no property violation in \( k - 1 \) iterations. The inductive step over-approximates the state space so if no counterexample is found then this is sufficient to prove that there is no reachable bug in the program.

Algorithm 1.4: Naïve \( k \)-induction.

Function kind(P, \( k_{\text{max}}, k \)):
1 if \( k > k_{\text{max}} \) then return unknown;
2 \( P_k := \text{unwinding}(P, k) \);
3 \( \pi := \text{base_case}(P_k) \);
4 if \( \pi \neq \emptyset \) then return \( \pi \);
5 \( \pi := \text{forward_condition}(P_k) \);
6 if \( \pi = \emptyset \) then return \( \emptyset \);
7 \( \pi := \text{inductive_step}(P_k) \);
8 if \( \pi = \emptyset \) then return \( \emptyset \);
9 return kind(P, \( k_{\text{max}}, k + 1 \));

The naïve version of the \( k \)-induction proof rule (shown in Algorithm 1.4) tries to find a property violation or to prove partial correctness for an increasing number of \( k \) loop unwindings. The pre-condition of the algorithm is \( k = 1 \). The unwinding function unwinds the program \( P \), \( k \) times; the function preserves the program behaviour up to \( k \) loop unwindings; if a bug in \( P \) is reachable in \( k \) unwindings, it will be reachable in \( P_k \). If it reaches a maximum number of iterations \( k_{\text{max}} \), the algorithm terminates with an unknown answer.

Theorem 1 (Soundness of the \( k \)-induction proof rule). If the \( k \)-induction proof rule returns: (i) a sequence of states \([s_i, \ldots, s_j]\): the program is unsafe and the sequence is a non-spurious counterexample; (ii) \( \emptyset \): the program is safe; (iii) unknown: the program is safe up to \( k_{\text{max}} \) iterations.

Proof. The first item is ensured by Lemma 1 if there is a property violation reachable after \( k \) unwindings, the program is unsafe and the algorithm terminates returning the counterexample (line 5). The second item is ensured by Lemmas 2 and 3 if no counterexample is found then the program is safe and an empty execution path is returned in lines 7 and 9. Finally, the third item is ensured in line 2 if the algorithm reached the maximum number of defined iterations without terminating, an unknown answer.
is given. We can then conclude that the $k$-induction proof rule always terminates either with a counterexample (if the program is unsafe), an empty execution path (if the program is safe), or with an unknown answer otherwise.

**Theorem 2 (Partial completeness of the $k$-induction algorithm).** If $\exists k : 1 \leq k \leq k_{\text{max}}$ such that the shortest counterexample is $\pi_{\text{min}}^k = [s_1, \ldots, s_k]$ and $s_k = \epsilon$ then the $k$-induction proof rule will find the counterexample in at least $k$ iterations.

**Proof.** This is ensured by always starting the $k$-induction algorithm as defined in Algorithm 1.4 with one loop unwinding; it always increments the number of loop unwindings by one (line 10). Furthermore, if the program is unsafe, neither the forward condition nor the inductive step will terminate the verification before the counterexample is found. Also note that, if a property violation requires zero loop unwindings (e.g., a property violation before a loop), the $k$-induction proof rule will still unwind the program once but the base case will find the property violation since it checks all states reachable with one loop unwinding (line 2).

### 2.1 Why is the $k$-induction proof rule naïve?

The inductive step assumption of all possible sequences of $k$ iterations is what makes the $k$-induction proof rule naïve; these sequences often include large unreachable regions of the state space. Safety properties might not hold in these regions of the state space but they are irrelevant for the safety of the program. For example, when verifying the safe program in Fig. 1a using $k$-induction, the inductive step will try to prove that the program is safe for all possible values that both input and $s$ variables can assume; this will result in a series of spurious counterexamples since these variables only assume a small range of values. In fact, the version of the $k$-induction proof rule as presented in Algorithm 1.4 assumes that all counterexamples produced by the inductive step are spurious, even if they are not.

Let us use an illustrative example to show the verification process using $k$-induction. First, consider the safe program in Fig. 1a. The property violation is reachable if the transition condition $[\text{input} = 5 \land s > 5]$ holds and, since the state space is over-approximated, there are several states that will satisfy this condition. In this case, the $k$-induction as defined in Algorithm 1.4 will eventually reach the maximum number of loop unwindings and terminate with an unknown answer because the base case will not find a property violation (the program is safe), the completeness threshold will never be reached since the program contains an infinite loop and the inductive step will keep finding spurious counterexamples. Now, let us consider the unsafe program in Fig. 1b. The $k$-induction as defined in Algorithm 1.4 will need at least five iterations until a counterexample is found by the base case. During these iterations, both the forward condition and the inductive step are executed and any reasoning performed in these steps are discarded but what if the inductive step finds an actual partial counterexample? This useful information (a partial counterexample) is ignored as all counterexamples found by the inductive step are assumed to be spurious.

When using the naïve $k$-induction to verify the programs in Fig. 1 it will either produce an unknown result or will discard useful information.
unsigned int s = 1;

while (1) {
    unsigned int input = _VERIFIER_nondet_int();
    if (input > 5) {
        return 0;
    } else if (input == 1 && s == 1) {
        s = 2;
    } else if (input == 2 && s == 2) {
        s = 3;
    } else if (input == 3 && s == 3) {
        s = 4;
    } else if (input == 4 && s == 4) {
        s = 5;
    } else if (input == 5 && s > 5) {
        // property violation
        _VERIFIER_error();
    }
}
}

(a) Simplified safe program.

unsigned int s = 1;

while (1) {
    unsigned int input = _VERIFIER_nondet_int();
    if (input > 5) {
        return 0;
    } else if (input == 1 && s == 1) {
        s = 2;
    } else if (input == 2 && s == 2) {
        s = 3;
    } else if (input == 3 && s == 3) {
        s = 4;
    } else if (input == 4 && s == 4) {
        s = 5;
    } else if (input == 5 && s >= 5) {
        // property violation
        _VERIFIER_error();
    }
}

(b) Simplified unsafe program.

Fig. 1: Simplified illustrative examples extracted from SV-COMP’18 encoding an event-condition-action (ECA) system [20]. The program in Fig. 1a is safe since the property violation is unreachable while the program in Fig. 1b is unsafe since the property violation is reachable after at least 5 iterations.

3 Learning from Counterexamples to Bidirectionally Explore the State Space

The \( k \)-induction proof rule can be applied to solve various verification problems [6,7,8], but it can be further improved by taking advantage of two important observations: (1) partial counterexamples are ignored: useful counterexamples may be generated by the inductive step and they are ignored by the algorithm; (2) unconstrained state space: the inductive step may find spurious counterexamples if the over-approximation is unconstrained. Several authors address the latter by generating program invariants to rule out unreachable regions of the state space, either as a pre-processing step where invariants are introduced in the program before [21,22] or during the verification [5,23,24]. Our algorithm is the first to address the former in the context of software verification.

3.1 Bidirectional bug-finding using \( k \)-induction

The \( b \text{kind} \) algorithm extends the bug-finding capabilities of the \( k \)-induction proof rule by performing two alternating bug searches, one forward (i.e., from the initial state \( s_1 \)) and one backward (i.e., from any error state \( \epsilon \)) and stopping if the forward search finds a state in a counterexample produced by the backward search. Our proposed algorithm is similar to the bidirectional search algorithm from the graph theory field [25]. This new algorithm relies on two checks from the \( k \)-induction proof rule to implement the searches. The base case is the forward search, since it tries to find a counterexample \( \pi^k = [s_1, \ldots, \epsilon] \), while the inductive step is the backward search and tries to find any
partial counterexample $\pi^k = [s_i, \ldots, \epsilon]$. We shall refer to the base case and inductive step as forward and backward searches, respectively.

To perform the forward search we need to extend the base case as shown in Algorithm 1.5. First we define a new function $\text{starts\_counterexample}$ that given a state $s$ and a counterexample $\pi$, returns $true$ if $s \in \pi$ otherwise returns $false$; this function will be used to perform the bidirectional search. The first condition in the new algorithm (line 10) is the same condition in the base case from the original $k$-induction proof rule and returns a counterexample if a bug was found in $k$ iterations. The second (and new) condition (line 13) uses the function $\text{starts\_counterexample}$ to check if any of the states reachable by the base case start the counterexample $\pi_{\text{back}}$ found by the backward search. If this holds, the new base case function returns the execution path found by the forward search concatenated with the counterexample found by the backward search (line 14). The “·” operator concatenates two sequences. If no bug is found, the algorithm returns an empty sequence.

Algorithm 1.5: The base case used in the $bkind$ algorithm.

```
Function $\text{starts\_counterexample}(s, \pi)$:
1. if $\pi \neq \emptyset \land s \in \pi$ then
2. return $true$;
3. else
4. return $false$;
5. end

Function $\text{bkind\_base\_case}(P, \pi_{\text{back}})$:
6. $\text{BC} := \text{init\_P}(s_1) \land \bigwedge_{i=1}^{k} \text{tr\_P}(s_i, s_{i+1})$;
7. if $\text{BC} \land \bigvee_{i=1}^{k} \neg \phi(s_i)$ then
8. Let $\epsilon = s_i$ such that $\neg \phi(s_i)$;
9. return $[s_1, \ldots, \epsilon]$;
10. else if $\text{BC} \land \exists i: 1 \leq i \leq k \land \text{starts\_counterexample}(s_i, \pi_{\text{back}})$ then
11. return $[s_1, \ldots, s_{i-1}] \cdot \pi_{\text{back}}$;
12. else
13. return $\emptyset$;
14. end
```

Lemma 4 (Base case). If Algorithm 1.5 returns a sequence of states $[s_i, \ldots, s_j]$, this is a non-spurious counterexample.

**Proof.** The first condition in the new base case (line 10) is identical to the condition in the original base case and Lemma 1 ensures that this is a real counterexample. We only need to prove that the execution path returned in line 14 is a counterexample. The returned sequence is a counterexample since it is a concatenation of an execution path starting from the initial state in the state space with a counterexample. We know that the sequence of states in the concatenation is an execution path because of the transition relation and $\text{init\_P}(s_1)$ ensures that it starts from the initial state in the state space (both in line 9). Lemma 5 guarantees that $\pi_{\text{back}}$ is a partial counterexample. Finally, this is a non-spurious counterexample because of the partial order property of the state space: this is sufficient to allow the concatenation of the execution path and the counterexample.
The \textit{bkind} algorithm is shown in Algorithm 1.6. Similarly to the naïve $k$-induction algorithm, the \textit{bkind} algorithm tries to either find a property violation or to prove partial correctness for an increasing number of $k$ unwindings. The pre-conditions of the algorithm are $\pi = \emptyset$ and $k = 1$. If it reaches a maximum number of iterations $k_{\text{max}}$, the algorithm terminates with an \textit{unknown} answer. The novel contribution in the new \textit{bkind} algorithm is the bidirectional bug-finding technique. We use the counterexample produced by the backward search in the previous iteration, and check if it is reachable by the forward search in the next iteration (line 4).

\textbf{Lemma 5 (Partial counterexample from the inductive step).} If Algorithm 1.3 returns a sequence of states $[s_1, \ldots, s_j]$, this is a partial counterexample of length $k$.

\textbf{Proof.} This is a partial counterexample because it is an execution path (the transition relation in line 2 ensures that) and the last state in the path is an error state (ensured in line 3). Finally, the counterexample has length $k$ because the inductive step always tries to find a counterexample of length $k$; this is performed by checking if the property violation is reachable in $k$ iterations, assuming that it holds for $k - 1$ iterations.

\textbf{Theorem 3 (Partial completeness of the \textit{bkind} algorithm).} If $\exists k : 1 \leq k \leq k_{\text{max}}$ such that the shortest counterexample is $\pi_{\text{min}} = [s_1, \ldots, s_k]$ and $s_k = \epsilon$ then the $k$-induction proof rule will find the counterexample in at least $\lceil \frac{k}{2} \rceil + 1$ iterations.

\textbf{Proof.} In order to prove this theorem, we assume that the inductive step always returns the same non-spurious partial counterexample for every $k$; we will show how to give partial guarantees to this assumption in Sec. 3.2.

First, we show that no more than $\lceil \frac{k}{2} \rceil + 1$ iterations are required to find a property violation. By contradiction, assume that the number of iterations required to find the property violation is greater than $\lceil \frac{k}{2} \rceil + 1$.

Let us assume a $\pi_{\text{min}}^k$ where $k$ is even. In the iteration $\frac{k}{2}$, the new base case will have explored all states up to $\frac{k}{2}$ and the inductive step will have provided a partial counterexample $\pi_{\text{back}}^\frac{k}{2}$. In the iteration $\frac{k}{2} + 1$, the new base case will not reach any state in $\pi_{\text{back}}^\frac{k}{2}$ if either the counterexample $\pi_{\text{back}}$ is spurious (i.e., contradicting our initial assumption) or the counterexample $\pi_{\text{back}}$ is not spurious and there is at least one state $s_u \in \pi_{\text{min}}^k$ such that $\pi_{\text{min}}^k = [s_1, \ldots, s_{\frac{k}{2} + 1}] \cdot [s_u, \ldots] \cdot \pi_{\text{back}}^{\frac{k}{2} + 1}$, which has a length greater than $k$ contradicting our assumption about the length of $\pi_{\text{min}}^k$.

Now, assume a $\pi_{\text{min}}^k$ where $k$ is odd. The proof is similar to the one where $k$ is even, except that we consider the sequences at iteration $\frac{k-1}{2}$. If the counterexample is
not found in the iteration \( \frac{k-1}{2} + 1 \), then either the counterexample is spurious or a greater number of iterations is required to find the property violation, contradicting our initial assumptions. We then generalize and conclude that no more than \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \) iterations are required to find a property violation.

Now we show that at least \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \) iterations are required to find a property violation. By contradiction, assume that the number of iterations required to find the property violation is less than \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \). This means that either there exists a smaller counterexample that was not found the base case, which violates Lemma 4, or there is a state \( s_v \in \pi^k_{\text{min}} \) such that \( \pi^k_{\text{min}} = [s_1, \ldots, s_v] \cdot [s_{v+1}, \ldots, s_k] \) which has a length smaller than \( k \) contradicting our assumption about the length of \( \pi^k_{\text{min}} \). Given that at least \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \) iterations are needed to find the property violation and no more than \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \) iterations are needed to find the property violation, we can conclude that the \textit{bkind} algorithm will find a counterexample \( \pi^k_{\text{min}} \) in exactly \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \) iterations.

**Theorem 4 (Soundness of the \textit{bkind} algorithm).** If the \textit{bkind} algorithm returns: (i) a sequence of states \([s_i, \ldots, s_j]\): the program is unsafe and the sequence is a non-spurious counterexample; (ii) \( \emptyset \): the program is safe; (iii) unknown: the program is safe up to \( k_{\text{max}} \) iterations.

**Proof.** The proof follows the same proof as the Theorem 2, except that the first item is ensured by Lemma 4 instead of Lemma 1.

### 3.2 Constraint Generation Using Interval Analysis

Here we use invariants to constraint the state and rule out unreachable states evaluated by the inductive step [5,22,23,24]. Fig. 2 shows an example of the usage of invariants (dashed line) to constraint the state space. The invariants reduce the number of states explored by the backward search by constraining the over-approximation. Similarly to Rocha et al. [22], we perform a static program analysis prior to loop unwinding and estimate the intervals that a variable can assume. In contrast to Rocha et al., we do not rely on external tools and implement the invariant generation as a pre-processing step of the verification. In particular, we use the abstract interpretation component from CProver [26]. The invariant generation algorithm uses an abstract domain based on expressions over intervals, such that every constraint \( c \) in a state \( s \) is a map \( \text{var} \rightarrow 2^{128} \times 2^{128} \) that maps interval constraints to every variable \( \text{var} \).

In order to use the invariants, we need to extend the inductive step as shown in Algorithm 1.7. The algorithm is similar to the inductive step as defined in Algorithm 1.3 but it now takes an extra argument: a set of invariants \( \phi \). These invariants will constraint the state space and filter unreachable states from the inductive step check, thus reducing the number of spurious path that might be explored.

**Algorithm 1.7:** The new inductive step with invariants.

1. \textbf{Function} \textit{inductive\_step\_invariants} \((P, \phi)\);
2. \hspace{1em} if \( \exists n \in \mathbb{N}^+ . \phi(s_n) \land \text{tr}_{P}(s_n, s_{n+1}) \)
3. \hspace{2em} then Let \( c = s_{n+1} \) such that \( \phi(s_{n+1}) \land \neg \phi(s_{n+1}) \)
4. \hspace{2em} return \( [s_n, \ldots, c] \);
5. \hspace{1em} else
6. \hspace{2em} return \( \emptyset \).
7. \textbf{end}
Fig. 2: Visual representation of the state space of a program, where the states inside the dashed line are the reachable states and the ones outside are unreachable states. An unconstrained over-approximation of the program assumes that all the states are reachable which might contain spurious counterexamples (counterexamples that lead to unreachable error states). An invariant is a filter of states: a strong enough invariant will remove unreachable error states from reasoning, allowing the inductive step to prove the program correctness or to find non-spurious partial counterexamples.

Lemma 6 (Inductive step with invariants). If \( \text{inductive\_step\_invariants}(P) \) returns an empty sequence of states the program is safe.

Proof. This follows the same reasoning of the Lemma 5. The program is safe up to \( k \) iterations because the base case did not find any property violation and that the inductive step over-approximates the state space when it tries to find a property violation. The new inductive step with invariants will constraint the over-approximation to be closer to the reachable state space of the program. Lemma 5 ensures that the \( k \)-induction proof rule can use the new inductive step with invariants. Theorem 1 must change if invariants are used to prove partial correctness; Lemmas 5 and 6 ensure the theorem is sound.

Lemma 7 (Partial counterexample from the inductive step with invariants). If Algorithm 1.7 returns a sequence of states \( [s_i, \ldots, s_j] \), this is a partial counterexample of length \( k \).

Proof. This follows the same reasoning of the Lemma 5. We know this is a counterexample because of the sequence of transitions defined by \( \text{tr}_P \) and that the last state is an error state. Again, the invariants here will only constraint the state space so the over-approximation is closer to the set of reachable states of the program. The Lemma 7 is defined so the new inductive step with invariants can be used with the \( \text{bkind} \) algorithm. Theorem 1 needs to be changed if invariants are used to prove correctness to use Lemma 7 instead of Lemma 6.
3.3 Why is the bkind algorithm more efficient than k-induction?

First, consider that we wish to verify the safe program in Fig. 1a using the bkind algorithm. The state transition system is analyzed and the following intervals are estimated based on the assignments:

\[ \varphi = (input \geq 0, input \leq UINT \text{MAX}, s \geq 1, s \leq 5) \]

The invariants are introduced in the program and are sufficient to prove that the program is safe with two loop unw windings: there will be no counterexample of size two that leads to a property violation. Now, consider that we wish to verify the unsafe program in Fig. 1b. Here, the same set of constraints are introduced in the program but now the inductive step will find a counterexample that satisfies

\[ \text{input} \geq 0 \land \text{input} \leq UINT \text{MAX} \land s \geq 1 \land s \leq 5 \land \text{input} = 5 \land s = 5 \]

This is the program state prior to the error state; the reachability of this state is introduced in the program as a new property and checked in the base case. This is then extended further back for every loop iteration, effectively performing the backward search. In conclusion, bkind can correctly verify both programs in Fig. 1; the program in Fig. 1a can be proven to be safe and the program in Fig. 1b requires fewer number of steps to find the property violation.

4 Experimental Evaluation

The experimental evaluation of the bkind algorithm and the invariant generation in our software model checker (ESBMC [27,28]) consists of three parts. In Sec. 4.1 we describe the experimental objectives and present the benchmarks used to evaluate the bkind algorithm. In Sec. 4.2 we compare our bkind algorithm and the invariant generation with the naïve k-induction, while in Sec. 4.3 we compare the bkind algorithm with invariants against another state-of-the-art BMC tool that uses k-induction and invariant generation to verify ANSI-C programs, 2LS [23]. The tools are compared in terms of number of refuted bugs and verification time. We provide a virtual machine with all the binaries and scripts to reproduce our results in www.esbmc.org.

4.1 Experimental Objectives and Setup

Our experimental evaluation aims to answer three research questions:

RQ1 (soundness) Does our approach provide correct results?
RQ2 (performance I) Does our approach improve results compared to the naïve k-induction?
RQ3 (performance II) How does our approach compare against other k-induction verifiers?

We use 5591 benchmarks from SV-COMP’18 to evaluate the algorithms described in this paper. The benchmarks were extracted from the subcategories Arrays, BitVectors, ControlFlow, ECA, Floats, Heap, Loops, ProductLines, Sequentialized and Systems_DeviceDriversLinux64. The remaining categories were excluded because they use features that our k-induction does not support (e.g., termination, recursion, and concurrency). When verifying those programs, ESBMC disables the inductive step and uses only the base case and the forward condition, thus they are not included here. Out of the 5591 benchmarks, 4134 are safe while 1457 are unsafe programs.

All experiments were conducted on IRIDIS4, the supercomputer from the University of Southampton [29]. The computer nodes used are equipped with Intel Sandybridge processors running at 2.6GHz and 24GB of RAM. We used Boolector as the SMT backend for all the verification tasks. For each benchmark, we set time and memory limits of 900 seconds and 15GB,
respectively, as per the competition definitions. Finally, given the large amount of data involved in the experiments, we used four groups to present the results: *Correct proofs* is the number of correct positive results (i.e., the tool reports SAFE correctly), *Correct alarms* is the number of correct negative results (i.e., the tool reports UNSAFE correctly), *Incorrect proofs* is the number of incorrect positive results (i.e., the tool reports SAFE incorrectly), *Incorrect alarms* is the number of incorrect false results (i.e., the tool reports UNSAFE incorrectly).

### 4.2 Comparison of $k$-induction-based approaches

Here, we evaluate five different $k$-induction proof rules: “original naïve $k$-induction” (the first version implemented in ESBMC [18]), “ naïve $k$-induction” (the $k$-induction proof rule described in Sec. 2), “ naïve $k$-induction + invariants” (the $k$-induction algorithm described in Sec. 2 and the invariants described in Sec. 3.2), “bkind” (the *bkind* algorithm described in Sec. 3.1) and “bkind + invariants” (the *bkind* algorithm described in Sec. 3.1 and the invariants described in Sec. 3.2).

Fig. 3: Results of the $k$-induction-based algorithms in ESBMC for all SV-COMP’18 benchmarks with different configurations.

Fig. 3 shows the results of using the $k$-induction-based approaches to verify the benchmarks from SV-COMP. First, let us compare the results of the original naïve $k$-induction [18] and the current naïve $k$-induction: the number of correct proofs and correct alarms increased by 25% and 20%, respectively, while the number of incorrect proofs and incorrect alarms decreased by 91% and 92%, respectively. Most of the wrong results in the original $k$-induction came from the fact the original algorithm (1) could not reason about early loop exits (e.g., a *break* inside a loop) and (2) would assume wrong safety conditions in the inductive step due to implementation bugs. Furthermore, the original $k$-induction did not support floating-point encoding which resulted in 45 incorrect alarms in the *Floats* category and did not have the clang frontend, thus it could not verify about 500 benchmarks due to parsing errors.

Now let us compare the current $k$-induction proof rule against the new *bkind* algorithm (with and without invariants). First, we notice that the invariants increase the number of correct proofs...
for both the $k$-induction and $bkind$ in about 7%. This, however, comes at a cost: due to bugs in our implementation, the number of incorrect proofs are almost 4 times higher when invariants are used in combination with the algorithms (from 10 to 38). In particular, our algorithm does not track intervals of variables changed through pointers and neither if the intervals are defined in terms of other variables. The number of incorrect results, however, is still low: we only report incorrect proofs in about 2.5% of the 1457 incorrect benchmarks.

Note that all the approaches report similar correct alarms with or without invariants; this is expected in the $k$-induction proof rule since the invariants are supposed to only improve the correctness proof. The $bkind$ verification results could have found a larger number of bugs than the $k$-induction proof rule. Indeed, $bkind$ finds bugs in benchmarks that could not be found by the $k$-induction algorithm but in the end it reported a slightly fewer number of correct alarms and a larger number of incorrect alarms. An in-depth analysis of the wrong results showed that (1) when the invariants are incorrect, the $bkind$ algorithm ends up finding an incorrect counterexample and (2) when the program contains arrays, the algorithm ends up generating incomplete partial counterexamples, which also lead to incorrect alarms. Despite the number of incorrect results, however, the wrong alarms only amount to 0.1% of the 4134 correct benchmarks analysed by the $k$-induction approaches. These numbers allow us to partially affirm our research question RQ1: the new $bkind$ algorithm provides correct results for a large set of benchmarks. There are some programs where $bkind$ will provide incorrect results but it is due to bugs in our implementation.

The total verification time is 1, 253, 015s for “original naïve $k$-induction”, 1, 654, 147s for “naïve $k$-induction”, 1, 522, 851s for “naïve $k$-induction + invariants”, 1, 664, 357s for “$bkind$”, and 1, 504, 208s for “$bkind$ + invariants”. First, let us evaluate the original and the current naïve $k$-induction: the original one is 25% faster the current $k$-induction; this can be easily explained due to the limitations in the program: a number of benchmarks are not parsed by the tool and the greater number of incorrect results allow the original $k$-induction to finish the analysis faster. Regarding the $k$-induction and $bkind$ without invariants, the latter is slightly slower (0.6%); this is expected since the inductive step is most likely to find spurious counterexamples. The slowdown in $bkind$ is the impact of introducing spurious verification conditions and it is negligible.

The results when invariants are used, however, are much better. The verification time decreases considerably in these benchmarks, making both algorithms 10% faster. The naïve $k$-induction algorithm takes 1, 654, 147s to verify all the programs which is equivalent to 19.1 days of continuous processing, while the $bkind$ algorithm with invariants takes 1, 504, 208s or 17.4 days. The $bkind$ algorithm with invariants speeds up the verification by almost two full days in our experiments. These results allow us to affirm our RQ2: the $bkind$ algorithm with invariants improves the performance over the naïve $k$-induction by giving more correct results in less time.

### 4.3 Comparison to a state-of-the-art $k$-induction verifier 2LS

We now compare the $bkind$ algorithm with invariants against 2LS v0.6.0, another state-of-the-art bounded model checker with support for $k$-induction. In particular, 2LS uses the $k$Ind algorithm [23] and combines the $k$-induction proof rule with continuous invariant generation. We used the same configuration from SV-COMP’18 in which 2LS is configured to generated interval constraints similar to the ones generated by ESBMC.

As shown in Fig. [4] ESBMC with the $bkind$ algorithm and invariants produces more than two times the number of correct proofs and about 35% more correct alarms compared to 2LS when analysing the same set of benchmarks. An in-depth analysis of the results show that 2LS aborts the verification of various benchmarks in the Systems_DevicesDriversLinux64 category with the message “Irreducible control flow not supported”. Alternatively, ESBMC is able to prove the correctness of 1249 benchmarks in this category, greatly improving our results. 2LS provides much fewer incorrect results when compared to ESBMC in this set of benchmarks. 2LS always had a strong focus in invariant generation since its first version; their last version in SV-COMP18
extended it even further by introducing invariant generation for termination proofs and pointer safety. The invariant generation in ESBMC is still in its first version and needs improvements.

The total verification time is 1, 266, 890s for 2LS and 1, 504, 208s for ESBMC with “bkind + invariants”. Here, 2LS is about 15% faster than ESBMC even if we do not consider the Systems_DeviceDriversLinux64 category where 2LS would abort early in the verification. In particular, 2LS was more than 10 times faster in two categories, Arrays and Heap, most likely due to the stronger invariants generated by the tool which allowed it to prove correctness faster; ESBMC would simply run out of time in a large number of benchmarks in these categories. ESBMC with bkind and invariants is not the fastest verification tool using k-induction but it is the one with the highest number of correct results. These results allow us to answer the RQ3: our novel contribution is a improvement over the state-of-the-art verification using k-induction, which can report more correct results for a large number of different benchmarks than other existing approaches.

5 Related Work

The k-induction method is gaining popularity in the software verification community. Donaldson et al. described a verification tool called Scratch to detect data races during Direct Memory Access (DMA) in the CELL BE processor from IBM [7], using k-induction. Properties are automatically inserted in the program to model the behaviour of the memory control-flow and the algorithm tries to find violation of those properties or prove that they hold indefinitely. The method also requires the code to be manually annotated with loop invariants, whereas our approach automatically generates and adds them to the program. Finally, the tool is able to prove the absence of data races, but it is restricted to verify that specific class of problems for a particular type of hardware, while our approach is evaluated over a more general class of programs.

Donaldson further described two tools for proving correctness of programs: K-Boogie and K-Inductor [8]. The former is an extension of the Boogie language, aimed to prove correctness (using k-induction) of programs written in a number of languages (e.g., Boogie and Spec), while the latter is a BMC tool for C programs. Both K-Boogie and K-Inductor use a k-induction proof
rule; the completeness threshold is not separately checked and relies only on the inductive step to prove correctness. Their \( k \)-induction has a pre-processing step, but while we introduce invariants during the pre-processing, their approach removes all nested loops leaving only non-nested loops. They compare the results of K-inductor with Scratch and show that the new approach maintains the same number of correctly verified programs while being faster. Similar to the prior work [7], the programs need to be manually changed to add loop invariants while we do it automatically.

Malík et al. [23] describe 2LS, a C/C++ SAT-based BMC. 2LS is a tool developed using the CProver framework [30] and combines a \( k \)-induction proof rule with abstract interpretation (AI). As CBMC [30], 2LS uses SAT solvers but instead of a fixed unwind approach, 2LS uses an incremental BMC approach, where it first checks for property violations for a given bound, then tries to generate (and refine) invariants using AI and then builds a proof using \( k \)-induction. Their \( k \)-induction, called \( kIkI \) [24], is similar to the one implemented in ESBMC, but adds an extra step to generate and refine invariants. In contrast to our invariant generation that only supports interval domains, 2LS supports several abstract domains for numerical values and a shape domain for pointers. 2LS offers approaches to prove non-termination, while ESBMC has no algorithm to prove non-termination and can only prove termination by checking the unwinding assertions.

Bischoff et al. [12] propose a methodology to use BDDs and SAT solvers for the verification of programs in a bidirectional form similar to our \( bkind \) algorithm. In their work, they refer to the technique as target enlargement: the property violation is “enlarged” by checking if the states around the property violation are reachable. The BDDs are responsible for the target enlargement, collecting the under-approximate reachable state sets, followed by the SAT-based verification with the newly computed sets. They implemented the technique in the Intel BOolean VErifier and showed that the verification time of a set of public benchmarks was up to five times smaller. Compared to this work, we only use \( k \)-induction and SMT solvers; the inductive step in the \( k \)-induction is responsible for enlarging the target and the SMT solver checks for satisfiability.

Jovanović et al. [14] present a reformulation of IC3, separating the reachability checking from the inductive reasoning. They further replace the regular induction proof rule by the \( k \)-induction and show that it provides more concise invariants. The authors implemented the algorithm in the SALLY model checker using Yices2 to do the forward search and MathSAT5 to do the backward search. They showed that the new algorithm can solve a number of real-world benchmarks at least as fast as other approaches. Compared to this work, our \( bkind \) uses consecutive BMC calls to find a solution. We implement our approach independent of solvers and it can be used with any SMT solver supported by ESBMC; both searches are done with the same solver.

6 Conclusions

We have described the \( k \)-induction proof rule and a novel contribution that extended its bug-finding capabilities. The new algorithm, called bidirectional \( k \)-induction or \( bkind \), was implemented in ESBMC and evaluated in a large set of benchmarks. \( k \)-induction is a powerful verification technique implemented in several different tools and was successfully used to verify a large number of different programs and properties. Here, we proposed and evaluated a novel way to exploit the \( k \)-induction proof rule, where useful information can be extracted from the various checks in the algorithm and can be used to improve the results of the algorithm.

In particular, the \( bkind \) algorithm uses information extracted from the inductive step to shorten the number of steps required to find a property violation; with strong enough invariants the \( bkind \) algorithm requires roughly half of the number of loop unwindings a BMC algorithm requires to find a property violation. We have implemented an interval invariant generator that runs as a pre-processing step: invariants are automatically introduced in the program and, although the implementation has some bugs, it strengthens the \( bkind \) algorithm results. Our results show that our \( bkind \) with invariants can considerably reduce the verification time of a large number of benchmarks: in our experiments this is equivalent to almost two days reduction in the verification time.
References

[1] Biere, A., Cimatti, A., Clarke, E., Zhu, Y.: Symbolic Model Checking Without BDDs. In: Tools And Algorithms For The Construction And Analysis Of Systems. Volume 1633 of LNCS. (1999) 193–207
[2] Beyer, D.: Software Verification With Validation Of Results (Report On SV-COMP 2017). In: Tools And Algorithms For The Construction And Analysis Of Systems. Volume 10206 of LNCS. (2017) 331–349
[3] Kroening, D., Ouaknine, J., Strichman, O., Wahl, T., Worrell, J.: Linear Completeness Thresholds For Bounded Model Checking. In: Computer-Aided Verification. Volume 6806 of LNCS. (2011) 557–572
[4] Bradley, A.R., Manna, Z.: The Calculus Of Computation - Decision Procedures With Applications To Verification. Springer (2007)
[5] Beyer, D., Dangl, M., Wendler, P.: Boosting $k$-Induction With Continuously-Refined Invariants. In: Computer-Aided Verification. Volume 9206 of LNCS. (2015) 622–640
[6] Donaldson, A., Haller, L., Kroening, D., Rümmer, P.: Software Verification Using $k$-Induction. In: Static Analysis Symposium. (2011) 351–368
[7] Donaldson, A., Kroening, D., Rümmer, P.: SCRATCH: A Tool For Automatic Analysis Of DMA Races. In: Symposium On Principles And Practice Of Parallel Programming. (2011) 311–312
[8] Eén, N., Sörensson, N.: Temporal Induction By Incremental SAT Solving. Electronic Notes in Theoretical Computer Science 89(4) (2003) 543–560
[9] Große, D., Le, H., Drechsler, R.: Induction-Based Formal Verification Of SystemC TLM Designs. In: Workshop On Microprocessor Test And Verification. (2009) 101–106
[10] Sheeran, M., Singh, S., Stälmarck, G.: Checking Safety Properties Using Induction And A SAT-Solver. In: Formal Methods In Computer-Aided Design. (2000) 108–125
[11] Gadelha, M.Y.R., Monteiro, F.R., Cordeiro, L.C., Nicole, D.A.: Towards Counterexample-guided $k$-Induction For Fast Bug Detection. In: ACM Joint European Software Engineering Conference And Symposium On The Foundations Of Software Engineering. (2018)
[12] Bischoff, G.P., Brace, K.S., Cabodi, G., Nocco, S.and Queir, S.: Exploiting Target Enlargement And Dynamic Abstraction Within Mixed BDD And SAT Invariant Checking. Electronic Notes in Theoretical Computer Science 119(2) (2005) 33–49
[13] Hassan, Z., Bradley, A.R., Someni, F.: Better Generalization In IC3. In: Formal Methods In Computer-Aided Design, IEEE (2013) 157–164
[14] Jovanović, D., Dutertre, B.: Property-directed $k$-induction. In: Formal Methods In Computer-Aided Design. (2016) 85–92
[15] McMillan, K.L.: Interpolation And Model Checking. In: Handbook Of Model Checking. Springer (2018) 421–446
[16] Wahl, T.: The $k$-induction Principle. http://www.ccs.neu.edu/home/wahl/Publications/k-induction.pdf (2013) [Online; accessed September-2018].
[17] Morse, J., Cordeiro, L.C., Nicole, D.A., Fischer, B.: Handling unbounded loops with ES-BMC 1.20 - (competition contribution). In: TACAS
[18] Gadelha, M.Y.R., Ismail, H.I., Cordeiro, L.C.: Handling Loops In Bounded Model Checking Of C Programs Via $k$-induction. International Journal on Software Tools for Technology Transfer 19(1) (2017) 97–114
[19] Russell, S.J., Norvig, P.: Artificial Intelligence: A Modern Approach. 2nd edn. Pearson Education (2003)
[20] Cano, J., Delaval, G., Rutten, E.: Coordination Of ECA Rules By Verification And Control. In: Coordination Models And Languages, Berlin, Heidelberg, Springer Berlin Heidelberg (2014) 33–48
[21] Rocha, H., Ismail, H., Cordeiro, L.C., Barreto, R.S.: Model Checking Embedded C Software Using $k$-Induction And Invariants. In: SBESC. (2015) 90–95

[22] Rocha, W., Rocha, H., Ismail, H., Cordeiro, L.C., Fischer, B.: DepthK: A $k$-Induction Verifier Based On Invariant Inference For C Programs - (Competition Contribution). In: Tools And Algorithms For The Construction And Analysis Of Systems. (2017) 360–364

[23] Malík, V., Martiček, Š., Schrammel, P., Srivas, M., Vojnar, T., Wahlung, J.: 2LS: Memory Safety And Non-termination. In: Tools And Algorithms For The Construction And Analysis Of Systems, Cham, Springer International Publishing (2018) 417–421

[24] Brain, M., Joshi, S., Kroening, D., Schrammel, P.: Safety Verification And Refutation By $k$-Invariants And $k$-Induction. In: Static Analysis. (2015) 145–161

[25] Sturtevant, N.R., Felner, A.: A Brief History And Recent Achievements In Bidirectional Search. In: Conference On Artificial Intelligence, AAAI Press (2018)

[26] Kroening, D.: CProver Manual. [http://www.cprover.org/cprover-manual/](http://www.cprover.org/cprover-manual/) (2018) [Online; accessed September-2018].

[27] Cordeiro, L.C., Fischer, B., Marques-Silva, J.: SMT-Based Bounded Model Checking For Embedded ANSI-C Software. IEEE Transactions on Software Engineering 38(4) (2012) 957–974

[28] Gadelha, M.R., Monteiro, F.R., Morse, J., Cordeiro, L.C., Fischer, B., Nicole, D.A.: ES-BMC 5.0: An Industrial-Strength C Model Checker. In: Automated Software Engineering, ACM (2018) 888–891

[29] of Southampton, U.: The Iridis Compute Cluster. [https://www.southampton.ac.uk/isolutions/staff/iridis.page](https://www.southampton.ac.uk/isolutions/staff/iridis.page) (2018) [Online; accessed September-2018].

[30] Clarke, E., Kroening, D., Lerda, F.: A Tool For Checking ANSI-C Programs. In: Tools And Algorithms For The Construction And Analysis Of Systems, Volume 2988 of LNCS. (2004) 168–176