LOCAL RINGS WITH A SELF-DUAL MAXIMAL IDEAL

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Abstract. Let $R$ be a Cohen-Macaulay local ring possessing a canonical module. In this paper we consider when the maximal ideal of $R$ is self-dual, i.e. it is isomorphic to its canonical dual as an $R$-module. Local rings satisfying this condition are called Teter rings, and studied by Teter, Huneke-Vraciu, Ananthnarayan-Avramov-Moore, and so on. On the positive dimensional case, we show such rings are exactly the endomorphism rings of the maximal ideals of some Gorenstein local rings of dimension one. We also provide some connection between the self-duality of the maximal ideal and near Gorensteinness.

1. Introduction

Let $R$ be a Cohen-Macaulay local ring with a canonical module $\omega$. For an $R$-module $M$, we denote by $M^\dagger$ the $R$-module Hom$_R(M, \omega)$. The $R$-module $M$ is called self-dual if there exists an isomorphism $M \cong M^\dagger$ of $R$-modules. Note that the self-duality of $R$-modules is independent of the choice of $\omega$.

Let $R$ and $S$ be artinian local rings such that $S$ maps onto $R$. Denote by $c_S(R)$ the colength $\ell_S(S) - \ell_S(R)$. In the case that $S$ is Gorenstein, the integer $c_S(R)$ is used to estimate homological properties of $R$, for example, see [15, Theorem 7.5]. Ananthnarayan [1] introduced the Gorenstein colength $g(R)$ of an artinian local ring $(R, m, k)$ to be the following integer

$$g(R) := \min\{c_S(R) \mid S \text{ is a Gorenstein artinian local ring mapping onto } R\}.$$

The number $g(R)$ measures how close $R$ is to a Gorenstein ring. Clearly, $g(R)$ is zero if and only if $R$ is Gorenstein. One can see that $g(R) = 1$ if and only if $R$ is non-Gorenstein and $R \cong S/\text{soc}(S)$ for an artinian Gorenstein ring $S$. These rings are called Teter rings. On Teter rings, the following characterization is known, which is an improvement of Teter’s result [24]. This was proved by Huneke-Vraciu [12] under the assumption that $1/2 \in R$ and $\text{soc}(R) \subseteq m^2$, and later Ananthnarayan-Avramov-Moore [2] removed the assumption $\text{soc}(R) \subseteq m^2$. See also the result of Elias-Takatsuji [7].

**Theorem 1.1** (Huneke-Vraciu, Ananthnarayan-Avramov-Moore, Elias-Takatsuji). Let $(R, m, k)$ be an artinian local ring such that either $R$ contains $1/2$ or $R$ is equicharacteristic with $\text{soc}(R) \subseteq m^2$. Then the following are equivalent.

1. $g(R) \leq 1$.
2. Either $R$ is Gorenstein or $m \cong m^\dagger$.
3. Either $R$ is Gorenstein or there exists a surjective homomorphism $\omega \rightarrow m$.

Moreover, Ananthnarayan [1] extended this theorem to the case $g(R) \leq 2$ as follows.

**Theorem 1.2** (Ananthnarayan). Let $(R, m)$ be an artinian local ring. Write $R \cong T/I$ where $(T, m_T)$ is a regular local ring and $I$ is an ideal of $T$. Suppose $I \subseteq m_T^2$ and $1/2 \in R$. Then the following are equivalent.

1. $g(R) \leq 2$. 

2010 Mathematics Subject Classification. 13C14, 13E15, 13H10.

**Key words and phrases.** Gorenstein ring, canonical ideal, birational extension.

The author was supported by JSPS Grant-in-Aid for JSPS Fellows 18J20660.
(2) There exists a self-dual ideal \( a \subseteq R \) such that \( l(R/a) \leq 2 \).

In this paper, we try to extend the notion of Gorenstein colengths and the above results to the case that \( R \) is a one-dimensional Cohen-Macaulay local ring.

For a local ring \((R, m)\), we denote by \( Q(R) \) the total quotient ring of \( R \). An extension \( S \subseteq R \) of local rings is called birational if \( R \subseteq Q(S) \). In this case, \( R \) and \( S \) have same total quotient ring.

Let \((S, n) \subseteq (R, m)\) be an extension of local rings. Suppose \( n = m \cap R \). Then \( S \subseteq R \) is called residually rational if there is an isomorphism \( S/n \cong R/m \) induced by the natural inclusion \( S \to R \). For example, if \( S \subseteq R \) is module-finite and \( S/n \) is algebraically closed, then it automatically follows that \( S \subseteq R \) is residually rational. We introduce an invariant \( bg(R) \) for local rings \( R \) as follows, which is the infimum of Gorenstein colengths in birational maps.

**Definition 1.3.** For a local ring \( R \), we define

\[
bg(R) := \inf \left\{ \ell_S(R/S) \mid S \text{ is Gorenstein and } S \subseteq R \text{ is a module-finite residually rational birational map of local rings} \right\}.
\]

We will state the main results of this paper by using this invariant. The first one is the following theorem, which gives a one-dimensional analogue of Theorem [11].

**Theorem 1.4.** Let \((R, m)\) be a one-dimensional Cohen-Macaulay local ring having a canonical module \( \omega \). Consider the following conditions.

1. \( bg(R) \leq 1 \).
2. Either \( R \) is Gorenstein or there exists a Gorenstein local ring \((S, n)\) of dimension one such that \( R \cong \text{End}_S(n) \).
3. Either \( R \) is Gorenstein or \( m \cong m^1 \).
4. There is a short exact sequence \( 0 \to \omega \to m \to k \to 0 \).
5. There is an ideal \( I \) of \( R \) such that \( I \cong \omega \) (i.e. \( I \) is a canonical ideal of \( R \)) and \( l(R/I) \leq 2 \).

Then the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Leftrightarrow \) (4) \( \Rightarrow \) (5) hold. The direction (5) \( \Rightarrow \) (1) also holds if \( R \) contains an infinite field \( k \) as a subalgebra which is isomorphic to \( R/m \) via the projection \( R \to R/m \), i.e. \( R \) has an infinite coefficient field \( k \subseteq R \).

The existence of a canonical ideal \( I \) of \( R \) with \( \ell_R(R/I) = 2 \) is considered by Dibaei-Rahimi [6]. Using their notion, the condition (5) above is equivalent to the condition that \( \min(S_{CR}) \leq 2 \).

We also remark that Bass’s idea [3] tells us the importance of the endomorphism ring \( \text{End}_S(n) \) of the maximal ideal \( n \) of a Gorenstein local ring \( S \) of dimension one. He showed that any torsion-free \( S \)-module without non-zero free summand can be regarded as a module over \( \text{End}_S(n) \). So we can analyze Cohen-Macaulay representations of \( R \) via the ring \( \text{End}_S(n) \) (see also [16, Chapter 4]).

As a corollary, we can characterize Cohen-Macaulay local rings \( R \) of dimension one having minimal multiplicity and satisfying \( bg(R) \leq 1 \).

**Corollary 1.5.** Let \((R, m)\) be a one-dimensional Cohen-Macaulay local ring. Consider the following conditions.

1. \( bg(R) \leq 1 \) and \( R \) has minimal multiplicity.
2. Either \( e(R) \leq 2 \) or \( R \) is almost Gorenstein with \( bg(R) = 1 \).
3. \( m \cong m^1 \) and \( R \) is almost Gorenstein.
4. \( m \cong m^1 \) and \( R \) has minimal multiplicity.
5. \( R \) is almost Gorenstein and has minimal multiplicity.
6. There exists a Gorenstein local ring \((S, n)\) of dimension one such that \( e(S) \leq \text{edim} S + 1 \) and \( R \cong \text{End}_S(n) \).
7. \( m \cong m^1 \) and \( \rho(R) \leq 2 \).


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Then (1) ⇔ (2) ⇒ (3) ⇔ (4) ⇔ (5) holds. If $R/m$ is infinite, then (5) ⇔ (7) and (6) ⇒ (4) hold. If $R$ has an infinite coefficient field $k \subseteq R$, then all the conditions are equivalent.

Here we use the notion of almost Gorenstein rings in the sense of [9]. Also we denote by $e(S)$ the multiplicity of $S$ and by $\text{edim} S$ the embedding dimension of $S$. A Gorenstein ring of dimension one satisfying $e(S) = \text{edim} S + 1$ are called a ring of almost minimal multiplicity or a Gorenstein ring of minimal multiplicity, and studied by J. D. Sally [21]. The invariant $\rho(R)$ is the canonical index of $R$, introduced by Ghezzi-Goto-Hong-Vasconcelos [8].

The second main theorem of this paper is the following, which is a one-dimensional analogue of Theorem 1.2.

**Theorem 1.6.** Let $(R, m)$ be a complete one-dimensional Cohen-Macaulay local ring. Consider the following conditions.

1. $\operatorname{bg}(R) \leq 2$.
2. There exists a self-dual ideal $a \subseteq R$ such that $\ell_R(R/a) \leq 2$.

Then (1) implies (2). The implication (2) ⇒ (1) also holds if $R$ has an infinite coefficient field $k \subseteq R$.

In the view of Theorem 1.4, local rings with self-dual maximal ideal are naturally constructed from Gorenstein local rings, and so their ubiquity is certified. It is interesting to consider how good properties they have, comparing Gorenstein rings. In section 3, we have an observation that a Cohen-Macaulay local ring $(R, m)$ is nearly Gorenstein in the sense of Herzog-Hibi-Stamate [11] if $m$ is self-dual. The converse of this is not true in general, however, we have the following result.

**Theorem 1.7.** Let $(R, m, k)$ be a Cohen-Macaulay local ring of dimension one. Put $B = m : m$. Assume $k$ is infinite.

1. If $B$ is local with Cohen-Macaulay type two and $R$ is nearly Gorenstein, then $R$ is almost Gorenstein and does not satisfy $m \cong m^1$.
2. If $B$ is local with Cohen-Macaulay type three and $R$ is nearly Gorenstein, then either $R$ is almost Gorenstein or $m \cong m^1$.

We will provide a proof of Theorem 1.7 in section 3. One should compare this theorem with the following result of Goto-Matsuoka-Phuong [9, Theorem 5.1].

**Theorem 1.8** (Goto-Matsuoka-Phong). Let $(R, m, k)$ be a Cohen-Macaulay local ring of dimension one. Put $B = m : m$. Then $B$ is Gorenstein if and only if $R$ is almost Gorenstein and has minimal multiplicity.

In section 4, we deal with numerical semigroup rings having self-dual maximal ideal. The definition of UESY-semigroups was given by [19]. These numerical semigroups are exactly the semigroups obtained by adding one element to a symmetric numerical semigroup. We will show that a numerical semigroup ring has self-dual maximal ideal if and only if the corresponding numerical semigroup is UESY. After that, we also prove that the rings of UESY-numerical semigroup have quasi-decomposable maximal ideal. According to [17], an ideal $I$ of $R$ is called quasi-decomposable if there exists a regular sequence $\underline{x} = x_1, \ldots, x_r$ such that $I/(\underline{x})$ is decomposable as an $R$-module. Local rings with quasi-decomposable maximal ideal have some interesting properties; we can classify thick subcategories of the singularity category with some assumption on the punctured spectrum ([17, Theorem 4.5]), and we have results on the vanishings of Ext and Tor ([17, Section 6]).

In section 5, we characterize the endomorphism ring of a local hypersurface of dimension one, using Theorem 1.4.
2. Proof of Theorem 1.4 and 1.6

In this section, we prove Theorem 1.4 and 1.6. Let \((R, \mathfrak{m})\) be a Noetherian local ring with total quotient ring \(Q(R)\). Denote by \(\hat{R}\) the integral closure of \(R\) in \(Q(R)\). A fractional ideal is a finitely generated \(R\)-submodule \(I\) of \(Q(R)\) with \(IQ(R) = Q(R)\) (i.e. \(I\) contains a non zero divisor of \(R\)).

If \(R\) has depth \(\geq 1\), then every \(\mathfrak{m}\)-primary ideal is a fractional ideal of \(R\). For a fractional ideal \(I\) and \(J\), we can naturally identify \(\text{Hom}_R(I, J)\) with the set \(J : I = \{a \in Q(R) \mid aI \subseteq J\}\). In this way, the endomorphism ring \(\text{End}_R(\mathfrak{m})\) of \(\mathfrak{m}\) is identified with the \(R\)-subalgebra \(\mathfrak{m} : \mathfrak{m}\) of \(Q(R)\).

Here we note that the extension \(R \subseteq \mathfrak{m} : \mathfrak{m}\) is module-finite and hence \(\mathfrak{m} : \mathfrak{m}\) is commutative semilocal ring.

We give the following well-known lemma in order to use in the proof of Theorem 1.4.

Lemma 2.1. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension one. Assume \(R\) is not a \(\mathfrak{m}\)-primary ideal. Then

(a) \(R \subseteq \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}\)

(b) \(\ell_R(\mathfrak{m} : \mathfrak{m}/R)\) is equal to the Cohen-Macaulay type \(r(R)\) of \(R\).

(c) There exists a short exact sequence

\[ 0 \to \mathfrak{m}^{\oplus r(R)} \to R^{\oplus r(R)+1} \to \mathfrak{m} : \mathfrak{m} \to 0.\]

In particular, \(\mu_R(\mathfrak{m} : \mathfrak{m}) = r(R) + 1\) and \(\Omega_R(\mathfrak{m} : \mathfrak{m}) = \mathfrak{m}^{\oplus r(R)}\).

Lemma 2.2. Let \((S, \mathfrak{n}) \subseteq (R, \mathfrak{m})\) be a module-finite birational extension of one-dimensional local rings. Assume \(R\) is reflexive as an \(S\)-module. Then we have birational extensions \(S \subseteq R \subseteq n \subseteq R\).

Proof. Note that \(S\) is not a \(\mathfrak{n}\)-primary ideal, and so \(S\) is properly contained in \(n : \mathfrak{n}\) (Lemma 2.1). By the assumption, \(R = S : (S : R)\). Since \(R\) has constant rank one over \(S\) and \(R\) is not isomorphic to \(S\), one has \(R = n : (n : R)\). Therefore, \((n : \mathfrak{n}) \subseteq (n : n)R \subseteq R\).

Now we can explain the proof of the direction 1.4 (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\iff\) (4) \(\iff\) (5) of Theorem 1.4.

Proof of Theorem 1.4 (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\iff\) (4) \(\iff\) (5) (1) \(\Rightarrow\) (2): Assume \(bg(R) \leq 1\). If \(bg(R) = 0\), then \(R\) is Gorenstein, and there is nothing to prove. We may assume \(bg(R) = 1\). Then there is a Gorenstein local ring \((S, \mathfrak{n})\) and module-finite residually rational birational extension \(S \subseteq R\) with \(\ell_S(R/S) = 1\). In particular, \(R\) is Cohen-Macaulay. By the previous lemma, we have \(S \subseteq n : n \subseteq R\). Therefore, it should follows that \(\ell_S(R/n : n) = 0\), in other words, \(R = n : n = n : S\).

(2) \(\Rightarrow\) (3): We may assume \(S\) is not a \(\mathfrak{n}\)-primary ideal. Otherwise \(\mathfrak{R} \cong S\) and hence \(R\) is Gorenstein. Identify \(R\) with \(\mathfrak{n}\). By Lemma 2.1 one has \(\ell_S(R/S) = 1\). Hence we have that the colength \(\ell_S(\mathfrak{m}/\mathfrak{n})\) of the inclusion \(\mathfrak{n} \subseteq \mathfrak{m}\) is less than or equal to 1. It is easy to check that \(\mathfrak{m}/\mathfrak{n}\) is an \(R\)-module and it has dimension one as a vector space over \(R/\mathfrak{m}\). Fix a preimage \(t \in R\) of a basis \(\mathfrak{t}\) of \(\mathfrak{m}/\mathfrak{n}\). Then \(\mathfrak{m} = \mathfrak{n} + Rt\) and \(\mathfrak{m}^2 = \mathfrak{n}^2 + \mathfrak{m}t \subseteq \mathfrak{n}\). This means \(\mathfrak{m} \subseteq S : \mathfrak{m}\). We have another inclusion

\[ S : \mathfrak{m} \subseteq S : n = R.\]

Since \(Rt \not\subseteq S\), it follows that \(S : \mathfrak{m} = \mathfrak{m}\). The fractional ideal \(S : \mathfrak{m}\) is isomorphic to \(\mathfrak{m}^\dagger\) and so we obtain \(\mathfrak{m} \cong \mathfrak{m}^\dagger\).

(3) \(\Rightarrow\) (4): Applying the functor \((-)^\dagger\) to the short exact sequence \(0 \to \mathfrak{m} \to R \to k \to 0\), we see that the resulting exact sequence is \(0 \to \omega \to \mathfrak{m}^\dagger \to \text{Ext}_R^1(k, \omega) \cong k \to 0\). Replacing \(\mathfrak{m}^\dagger\) by \(\mathfrak{m}\), using the assumption \(\mathfrak{m} \cong \mathfrak{m}^\dagger\), we get the desired exact sequence.

(4) \(\Rightarrow\) (3): Applying the functor \((-)^\dagger\) to the short exact sequence \(0 \to \omega \to \mathfrak{m} \to k \to 0\), we get an exact sequence \(0 \to \mathfrak{m}^\dagger \to R \to \text{Ext}_R^1(k, \omega) \cong k \to 0\). Then, the image of \(\mathfrak{m}^\dagger\) in \(R\) must be equal to \(\mathfrak{m}\) and hence one has an isomorphism \(\mathfrak{m}^\dagger \cong \mathfrak{m}\).
(4) $\Rightarrow$ (5): The exact sequence $0 \to \omega \to m \to k \to 0$ yields that there is an ideal $I \cong \omega$ such that the colength $\ell_R(m/I)$ is one. The equality $\ell_R(m/I) = 2$ immediately follows from the above.

(5) $\Rightarrow$ (4): Take an ideal $I \cong \omega$ such that $l(R/I) \leq 2$. If $I = R$, then $R$ is Gorenstein and there is nothing to prove. So we may suppose that $I \subseteq m$. If $I = m$, then $R$ must be since $m \cong \omega$. Now we deal with an assumption that $I \nsubseteq m$. The inequality $l(R/I) \leq 2$ implies that the equality $l(m/I) = 1$. Thus the exact sequence $0 \to I \to m \to k \to 0$ is induced.

All that remains is to show the direction (5) $\Rightarrow$ (1). Let $(R, m)$ be a Noetherian local ring containing a coefficient field $k \cong R/m$. Let $I \subseteq R$ be a fractional ideal such that $\ell_R(R/I) < \infty$. Put $k + I := \{a + b \mid a \in k, b \in I\} \subseteq R$, which is a $k$-subalgebra of $R$. Then, since dim$_k R/(k + I) \leq \ell_R(R/I) < \infty$, $R$ is finitely generated as a $k + I$-module and hence $k + I$ is Noetherian local ring with a maximal ideal $(k + I) \cap m = I$. Thus, the ring extension $k + I \subseteq R$ is module-finite residually rational and birational.

**Lemma 2.3.** Let $(R, m)$ be a one-dimensional Cohen-Macaulay local ring. Assume $R$ has a canonical ideal $I \cong \omega$ such that $l(R/I) = 2$. Put $S = k + I$. Then $S$ is Gorenstein, and the colength $\ell_S(R/S)$ is equal to 1.

**Proof.** $S$ is local with a maximal ideal $n = I$. The extension $S \subseteq R$ is module-finite, residually rational and birational. Since $I$ is a canonical ideal, we have $I : I = R$. Equivalently, $n : n = R$. In particular, the colength $\ell_S(S/R)$ is equal to the Cohen-Macaulay type of $S$ (Lemma 2.1). Since $R$ and $S$ have same residue field $k$, we can see the equalities $\ell_S(S/R) = \ell_S(m/n) = \ell_R(m/I)$. On the other hand, we have

$$\ell_R(m/I) = \ell_R(R/I) - \ell_R(R/m) = 2 - 1 = 1.$$  

It follows that $S$ has Cohen-Macaulay type 1, that is, $S$ is Gorenstein. Moreover, the colength $\ell_S(R/S)$ is equal to $\ell_R(m/I) = 1$.

**Proof of Theorem 1.4** (5) $\Rightarrow$ (1). Assume There is a canonical ideal $I$ such that $\ell_R(R/I) \leq 2$. If $\ell_R(R/I) = 1$, then $I = R$ or $m$. In both of these cases, $R$ should be Gorenstein. Thus we only need to consider the case $\ell_R(R/I) = 2$. By previous lemma, the ring $S := k + I$ is Gorenstein and the colength $\ell_S(R/S)$ is 1. This shows $bg(R) \leq 1$.

We put the following lemma here, which will be used in the proof of Corollary 1.5.

**Lemma 2.4.** Let $(R, m)$ be a Cohen-Macaulay generically Gorenstein local ring of dimension one having a canonical module. Assume $R$ is not a. Then

1. $R$ has minimal multiplicity if and only if $m \cong m : m$.
2. $R$ is almost Gorenstein in the sense of [9] if and only if $m \cong m : m$.

**Proof.** See [18] Proposition 2.5] and [13] Theorem 2.14] respectively.

We give a proof of Corollary 1.5 as follows.

**Proof of Corollary 1.5.** The implications (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow immediately from Lemma 2.4.

(1) $\Leftrightarrow$ (2): In the case $bg(R) = 0$, $R$ is Gorenstein and has minimal multiplicity and thus $e(R) \leq 2$. The converse also holds. Now suppose $bg(R) = 1$. Then by Theorem 1.4, $m$ is isomorphic to $m^\dagger$. Therefore, $R$ has minimal multiplicity if and only if $R$ is almost Gorenstein.

(1) $\Rightarrow$ (3): Clear.

Now assume the residue field $R/m$ is infinite.

(6) $\Rightarrow$ (4): Obviously, $S/n$ is also infinite. If $e(S) \leq edim S$, then $e(S) \leq 2$ and $R$ also has an inequality $e(R) \leq 2$. This says that $R$ is Gorenstein and has minimal multiplicity. So we may assume $e(S) = edim S + 1$. 

(4) $\Rightarrow$ (5): The exact sequence $0 \to \omega \to m \to k \to 0$ yields that there is an ideal $I \cong \omega$ such that the colength $\ell_R(m/I)$ is one. The equality $\ell_R(m/I) = 2$ immediately follows from the above.

(5) $\Rightarrow$ (4): Take an ideal $I \cong \omega$ such that $l(R/I) \leq 2$. If $I = R$, then $R$ is Gorenstein and there is nothing to prove. So we may suppose that $I \subseteq m$. If $I = m$, then $R$ must be since $m \cong \omega$. Now we deal with an assumption that $I \subseteq m$. The inequality $l(R/I) \leq 2$ implies that the equality $l(m/I) = 1$. Thus the exact sequence $0 \to I \to m \to k \to 0$ is induced.

All that remains is to show the direction (5) $\Rightarrow$ (1). Let $(R, m)$ be a Noetherian local ring containing a coefficient field $k \cong R/m$. Let $I \subseteq R$ be a fractional ideal such that $\ell_R(R/I) < \infty$. Put $k + I := \{a + b \mid a \in k, b \in I\} \subseteq R$, which is a $k$-subalgebra of $R$. Then, since dim$_k R/(k + I) \leq \ell_R(R/I) < \infty$, $R$ is finitely generated as a $k + I$-module and hence $k + I$ is Noetherian local ring with a maximal ideal $(k + I) \cap m = I$. Thus, the ring extension $k + I \subseteq R$ is module-finite residually rational and birational.
Take a minimal reduction \((t)\) of \(n\) and a preimage \(\delta \in n^2\) of a generator of the socle of \(S/(t)\). Then \(n^3 = tn^2\) (see [20] Proof of (3.4))), \(n^3 = tn + S\delta\) and \((t) : S n = (t) + S\delta\). Therefore
\[
R \cong \text{End}_S(n) \cong (t) : S n/t = S + S(\delta/t).
\]
Identify \(R\) with \(S + S(\delta/t)\). Since \(R\) is local and \(\delta^2 \in n^4 = t^2n^2\), \((\delta/t)\) cannot be a unit of \(R\). This shows \(m = n + S(\delta/t)\). By this equality, we also have an isomorphism \(R/m \cong S/n\) induced by \(S \subseteq R\). Observe the following equalities
\[
 tm = tn + S\delta = n^2
\]
and
\[
m^2 = (n + S(\delta/t))^2 = n^2 + n(\delta/t) + S(\delta/t)^2.
\]
Then \(\delta^2 \in n^4 = t^2n^2\) implies \(S(\delta/t)^2 \subseteq n^2\), and \(n\delta \subseteq n^3 = tn^2\) implies \(n(\delta/t) \subseteq n^2\). So \(m^2 = n^2 = tm\). This means that \(R\) has minimal multiplicity.

It remains to show that \(m \cong m^1\). By Theorem 1.4 it holds that either \(R\) is Gorenstein or \(m \cong m^1\). In the case that \(R\) is Gorenstein, it holds that \(e(R) \leq 2\) and so \(m\) is self-dual by [18, Theorem 2.6].

\((5) \Rightarrow (7)\): Assume \(R\) is almost Gorenstein and has minimal multiplicity. Then we already saw that \(m\) is self-dual. It follows from [9, Theorem 3.16] that \(\rho(R) \leq 2\).

\((7) \Rightarrow (5)\): It is enough to show that \(\rho(R) \leq 2\). Recall that \(\rho(R)\) is the reduction number of a canonical ideal of \(R\) ([8, Definition 4.2]). So if \(\rho(R) \leq 1\), then \(R\) is Gorenstein ([9, Theorem 3.7]). Assume \(\rho(R) = 2\). Combining [6, Theorem 3.7, Proposition 3.8] and Theorem 1.4, we obtain that \(R\) is almost Gorenstein and has minimal multiplicity.

Finally, we deal with the assumption that \(R\) contains an infinite field \(k\) isomorphic to \(R/m\) via \(R \to R/m\).

\((4) \Rightarrow (1)\): This follows directly from Theorem 1.4.

\((1) \Rightarrow (6)\): First we consider the case that \(R\) is Gorenstein (i.e. \(bg(R) = 0\)). In this case, \(e(R) \leq 2\) and \(\text{edim } R \leq 2\) by the assumption. Take a minimal reduction \(Rt\) of \(m\). Then \(m^2 = tm\). In particular, \(\ell_R(m/I) = \ell_R(m/I + m^2) \leq 1\). Put \(I = Rt\) and \(S = k + I\). Then the ring-extension \(S \subseteq R\) is module-finite, residually rational and birational. Since \(I : I = R\) and \(\ell_R(m/I) \leq 1\), we can see that \(S\) is Gorenstein and \(\text{End}_S(I) \cong R\) by the similar argument in the proof of [14] (3) \(\Rightarrow (1)\). Furthermore, one has an equality \(I/I^2 = I^2\), which particularly show that \(S\) has minimal multiplicity, that is, \(e(S) = \text{edim } S\).

Now consider the case that \(bg(R) = 1\). Repeating the proof of Theorem 1.4 (3) \(\Rightarrow (1)\), there is a canonical ideal \(I\) such that if we let \(S = k + I\), then \(S\) is Gorenstein local and \(R \cong \text{End}_S(n)\), where \(n\) is the maximal ideal of \(S\). Since \(R\) is Almost Gorenstein, it was shown in [9, Theorem 3.16] that there is a minimal reduction \(Q = (t) \subseteq I\) of \(I\) in \(R\) such that \(\ell_R(I^2/QI) \leq 1\) and \(QI^2 = I^3\). Then it follows that \(\ell_S(I/QI) \leq 1\). Using [20, Proposition 3.3], the equality \(e(S) = \text{edim } S + 1\) holds.

We give here an example of a ring \(R\) with \(bg(R) = 1\).

**Example 2.5.** Let \(R = k[[t^3, t^4, t^5]]\) and \(S = k[[t^3, t^4]]\) be numerical semigroup rings, where \(k\) is a field. Then the natural inclusion \(S \subseteq R\) is a module-finite birational extension of local rings with the same coefficient field. The colength \(\ell_S(R/S)\) is equal to 1. Since \(R\) is non-Gorenstein and \(S\) is Gorenstein, we have \(bg(R) = 1\).

We now turn to estimate the invariant \(bg(R)\) in general. Suppose there exists a self-dual fractional ideal of \(R\). Then we have an upper bound of \(bg(R)\) as follows.

**Lemma 2.6.** Let \((R, m)\) be a complete one-dimensional Cohen-Macaulay local ring. Assume \(R\) contains an infinite coefficient field \(k \cong R/m\). Let \(I \subseteq R\) be a fractional ideal of \(R\). If \(I\) is self-dual, then we have \(bg(R) \leq \ell(R/I)\). In other words, the following inequality holds
\[
bg(R) \leq \inf\{\ell_R(R/I) \mid I \cong I\}.
\]
Therefore \(\ell\) Example 2.11. 2.10 has positive answer given in Proposition 3.6 if \(bg\) \(R\) since hold true? extension \(S\) R. 

Question 2.9. \(\sigma\) Remark 2.8. Ananthnarayan \([1]\) shows the following inequalities hold for an artinian local ring \(R\).

\[
(2.8.1) \quad \ell_R(R/\omega^*(\omega)) \leq \min\{\ell_R(R/I) \mid I \cong I^\dagger\} \leq g(R).
\]

Here \(\omega^*(\omega)\) is the trace ideal of \(\omega\); see Definition 3.3.

As analogies of these inequalities, the followings are natural questions.

Question 2.9. Let \((R, m)\) be a one-dimensional Cohen-Macaulay local ring. Is an equality

\[
bg(R) \geq \inf\{\ell_R(R/I) \mid I \cong I^\dagger\}
\]

hold true?

Question 2.10. Let \((R, m)\) be a one-dimensional generically Gorenstein local ring. Is an inequality \(\ell_R(R/\omega^*(\omega)) \leq bg(R)\) hold true?

By our main theorems 1.4 and 1.6 Question 2.9 is affirmative for \(R\) with \(bg(R) \leq 2\). Question 2.10 has positive answer given in Proposition 3.6 if \(bg(R) \leq 1\), but there is the following counterexample, even when \(bg(R) = 2\).

Example 2.11. Let \(R\) be the numerical semigroup ring \(k[[H]]\), where \(H = \langle 3, 13, 14 \rangle\). Then \(\ell_R(R/\omega^*(\omega)) = \text{res}(H) = 4\) by \([1]\) Remark 7.14. there is, however, a Gorenstein subring \(S = k[[H']]\) of \(R\), where \(H' = \langle 6, 9, 13, 14, 16, 17 \rangle\). The colength \(\ell_S(R/S)\) is equal to 2, and so this gives a counter-example of Question 2.10.

We now return to prove the Theorem 1.6.

Proof of Theorem 1.6. (2) \(\Rightarrow\) (1): This is a consequence of Lemma 2.6 by letting \(I = a\).

(1) \(\Rightarrow\) (2): In the case \(bg(R) \leq 1\), assertion is followed by Theorem 1.4 So we may assume \(bg(R) = 2\). Take a Gorenstein local ring \((S, n)\) and module-finite residually rational birational extension \(S \subset R\) satisfying \(\ell_S(R/S) = 2\).

Let \(B\) be the ring \(n : n\). By Lemma 2.2 and Lemma 2.1 we have \(\ell_S(B/S) = 1\) and \(S \subseteq B \subseteq R\). Therefore \(\ell_B(R/B) = 1\) and \(B\) is local. Let \(m_B\) be the maximal ideal of \(B\) and fix a preimage
t ∈ R of a basis ℓ of the one-dimensional vector space R/B over B/m_B. By the relation m_Bt ⊆ B yields that t ∈ B : m_B = m_B : m_B. Therefore R = B + Bt ⊆ m_B : m_B. In particular, Rm_B ⊆ m_B. This says that m_B is an ideal of R. Since bg(R) = 1, m_B is a self-dual ideal of B by Theorem [1,4]. Thus, it is also self-dual as R-module. One can also have equalities

\[ \ell_R(R/m_B) = \ell_B(R/B) + \ell_B(B/m_B) = 2. \]

\[ \Box \]

Remark 2.12. Let (R, m) be a one-dimensional local ring. Assume R is complete, equicharacteristic and bg(R) = n < ∞. If there exists a Cohen-Macaulay local ring (B, m_B) with bg(B) = 1 and module-finite residually rational birational extensions B ⊆ R ⊆ m_B such that \( \ell_B(R/B) + 1 \leq n \). Then, by the same argument of proof of Theorem 1.6, it follows that m_B is a self-dual ideal of R satisfying \( \ell_R(R/m_B) \leq n \). In this case, Question 2.9 is affirmative for R.

3. THE SELF-DUALITY OF THE MAXIMAL IDEAL

In this section, we collect some properties of local rings (R, m) with m ≅ m^\dagger.

Lemma 3.1. Let (R, m) be a Cohen-Macaulay local ring with a canonical module. Assume m ≅ m^\dagger. Then

1. \( \dim R \leq 1 \).
2. Let \( x \in m \setminus m^2 \) be a non-zero divisor of R. Then \( R/(x) \) also has self-dual maximal ideal.
3. \( \edim(R) = \tau(R) + 1 \).

Proof. Suppose \( \dim R \geq 2 \) and \( \omega \) is a canonical module of R. Applying \((-)^\dagger\) to the exact sequence \( 0 \to m \to R \to k \to 0 \), we get an exact sequence

\[ 0 \to \text{Hom}_R(k, \omega) \to m^\dagger \to \omega \to \text{Ext}^1_R(k, \omega). \]

By the assumption \( \dim R \geq 2 \) yields that \( \text{Hom}_R(k, \omega) = 0 = \text{Ext}^1_R(k, \omega) \) and hence \( m^\dagger \cong R \). From the isomorphism \( m \cong m^\dagger \), it follows that m is principal ideal. This shows that \( \dim R \leq 1 \), which is a contradiction. Thus, it must be \( \dim R \leq 1 \).

When \( \dim R \geq 2 \), the maximal ideal m cannot be self-dual. However, we suggest the following generalization of the self-duality of the maximal ideal in higher dimensional case.

Proposition 3.2. (R, m) be a non-Gorenstein Cohen-Macaulay local ring of dimension \( d > 0 \) having an infinite residue field. Assume R has a canonical ideal I satisfying \( e(R/I) = 2 \). Then there is a regular sequence \( \underline{x} = x_1, \ldots, x_d-1 \) such that \( R/\langle \underline{x} \rangle \) has self-dual maximal ideal.

Proof. Since \( R/I \) is Cohen-Macaulay of dimension \( d - 1 \), we can take a minimal reduction \( \underline{y} = y_1, \ldots, y_d-1 \) of the maximal ideal m/I in R/I. Then the length \( l(R/I)/(\underline{y}) \) is equal to \( e(R/I)(\underline{y}) \). Let \( \underline{x} = x_1, \ldots, x_d-1 \) be a preimage of \( \underline{y} \) in R. As I is unmixed, we can take \( \underline{x} \) as a regular sequence in R. The tensor product \( I' = I \otimes R/\langle \underline{x} \rangle \) is naturally isomorphic a canonical ideal of \( R' = R/\langle \underline{x} \rangle \). The quotient \( R'/I' \) has length \( l(R/I + \underline{x}) = l(R/I)/(\underline{x}) \leq 2 \). Therefore \( R' \) has self-dual maximal ideal by Theorem [1,4].

Example 3.3. Let \( R = k[[x^3, x^2y, xy^2, y^3]] \) be the third Veronese subring of \( k[[x, y]] \). Then \( I = (x^3, x^2y)R \) is a canonical ideal of R. The quotient ring \( R/I \) is isomorphic to \( k[[s, t]]/(s^2) \), and hence \( e(R/I) = 2 \).

Go back to the subject on self-duality of the maximal ideal. Recall the notion of trace ideal of an R-module and nearly Gorensteinness of local rings (see [11]).
**Definition 3.4.** Let $R$ be a commutative ring. For an $R$-module $M$, the *trace ideal* $M^*(M)$ of $M$ in $R$ is defined to be the ideal $\sum_{f \in \text{Hom}_R(M,R)} \text{Im} f \subseteq R$.

A Cohen-Macaulay local ring $(R, \mathfrak{m})$ with a canonical module $\omega$ is called nearly Gorenstein if $\omega^*(\omega) \supseteq \mathfrak{m}$.

**Lemma 3.5.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. The following are equivalent.

1. $R$ is nearly Gorenstein.
2. There is a surjective homomorphism $\omega^{\oplus n} \to \mathfrak{m}$ for some $n$.

Moreover, if $\dim R \leq 1$, then we can add the following conditions.

3. There is a short exact sequence $0 \to \mathfrak{m}^{\dagger} \to R^{\oplus n} \to M \to 0$ for some $n$ and maximal Cohen-Macaulay module $M$.
4. There is a short exact sequence $0 \to \mathfrak{m}^{\dagger} \to \mathfrak{m}^{\oplus n} \to M \to 0$ for some $n$ and maximal Cohen-Macaulay module $M$.

**Proof.** (1) $\iff$ (2): By the definition of trace ideals, there is a surjection $\omega^{\oplus n} \to \omega^*(\omega)$ for some $n$. So the equivalence immediately follows.

Now assume $\dim R \leq 1$. Then the maximal ideal $\mathfrak{m}$ is maximal Cohen-Macaulay as an $R$-module. So the condition (2) is equivalent to that there is a short exact sequence $0 \to M \to \omega^{\oplus n} \to \mathfrak{m} \to 0$ for some $n$ and maximal Cohen-Macaulay module $M$. Taking the canonical duals, the equivalence of (2) and (3) follows.

We turn the equivalence of (3) and (4). We may assume $R$ is not a discrete valuation ring, and hence $\mathfrak{m}^{\dagger}$ is not a free $R$-module. So it follows from.

**Proposition 3.6.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. Assume $\mathfrak{m} \cong \mathfrak{m}^{\dagger}$. Then

1. $R$ is nearly Gorenstein.
2. If $R$ is non-Gorenstein and $2$ is invertible in $R$, then $R$ is G-regular.

**Proof.** We already saw that $\dim R \leq 1$ from Lemma 3.5.

(1) In the case of $\dim R = 0$, we have a short exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ and hence we can apply Lemma 3.5 (3) $\Rightarrow$ (1).

In the case of $\dim R = 1$, we may assume $R$ is not a . Applying Lemma 3.5 to the short exact sequence in Theorem 1.4 (4), it follows that $R$ is nearly Gorenstein.

(2) In the case that $\dim R = 0$, the statement is proved in [22, Corollary 3.4]. So we may assume $\dim R = 1$. Take $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ a non-zero divisor. Applying $(-)^{\dagger}$ to the exact sequence $0 \to \mathfrak{m} \to \mathfrak{m}/x\mathfrak{m} \to 0$, we get an exact sequence

$$0 \to \mathfrak{m}^{\dagger} \to \mathfrak{m}^{\dagger} \to \text{Ext}^1_R(\mathfrak{m}/x\mathfrak{m}, \omega) \to 0.$$ 

This shows that $\mathfrak{m}^{\dagger}/x\mathfrak{m}^{\dagger} \cong \text{Ext}^1_R(\mathfrak{m}/x\mathfrak{m}, \omega) \cong \text{Hom}_{R/(x)}(\mathfrak{m}/x\mathfrak{m}, \omega/x\omega)$. Therefore, by $\mathfrak{m} \cong \mathfrak{m}^{\dagger}$, $\mathfrak{m}/x\mathfrak{m}$ is self-dual as an $R/(x)$-module. On the other hand, there is a direct sum decomposition $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}/x \oplus k$ of $R/(x)$-modules, where $\mathfrak{m}/x$ is the maximal ideal of $R/(x)$. Since $k$ is self-dual over $R/(x)$, the $R/(x)$-module $\mathfrak{m}/x$ must be self-dual by the Krull-Schmidt theorem. $R/(x)$ is G-regular by [22, Corollary 3.4]. It follows from [23, Proposition 4.2] that $R$ is also G-regular.

**Example 3.7.** Let $R = k[[t^4, t^5, t^7]]$. Then $R$ is almost Gorenstein local ring of dimension one. Therefore, $R$ is G-regular and nearly Gorenstein. On the other hand, $R$ does not have minimal multiplicity, and hence $\mathfrak{m}$ is not self-dual. This shows that the converse of Proposition 3.6 doesn’t hold true in general.

**Example 3.8.** The associated graded ring $\text{gr}_\mathfrak{m}(R)$ of a local ring $(R, \mathfrak{m})$ with self-dual maximal ideal need not be Cohen-Macaulay, for example, $R = k[[t^4, t^5, t^{11}]]$. 
We use the notion of minimal faithful modules. The definition of them is given in below.

**Definition 3.9.** Let $R$ be a commutative ring. An $R$-module $M$ is called *minimal faithful* if it is faithful and no proper submodule or quotient module is faithful.

**Example 3.10.** For an artinian local ring $R$, the $R$-module $R$ and a canonical module $\omega$ of $R$ (i.e. injective hull of the residue field) are minimal faithful.

The following is proved by Bergman [4 Corollary 2].

**Lemma 3.11 (Bergman).** Let $A, B$ and $C$ be finite-dimensional vector spaces over a field $k$. and $f: A \times B \to C$ be a bilinear map. Assume the following conditions.

1. any nonzero element $a$ of $A$ induces a nonzero map $f(-, a): B \to C$
2. any proper submodule $i: B' \to B$, there is a nonzero element $a \in A$ such that $f(i(\cdot), a): B' \to C$ is a zero map.
3. any proper quotient module $p: C \to C'$ there is a nonzero element $a \in A$ such that the map $p \circ f(-, a): B \to C'$ is a zero map.

Then $\dim_k A \geq \dim_k B + \dim_k C - 1$.

**Lemma 3.12.** Let $R$ be a commutative ring, $n$ be a positive integer, $M, N$ be $R$-modules and $f = [f_1, \ldots, f_n]: N \to M^\oplus n$ be an $R$-homomorphism. Then $f$ is injective if and only if for any nonzero element $a \in \soc(N)$, there exists $i$ such that $f_i(a) \neq 0$.

**Lemma 3.13.** Let $(R, m, k)$ be an artinian local ring and $M, N$ be finitely generated faithful $R$-modules. Assume $M$ is minimal faithful. If $n$ is a positive integer such that exists an injective map $f = [f_1, \ldots, f_n]: N \to M^\oplus n$, then the $k$-subspace $B$ of $\Hom_k(N, M) \otimes_R k$ generated by the image of $f_1, \ldots, f_n$ has a dimension exactly equal to $n$ over $k$.

**Proof.** We only need to show $\dim_k B \geq n$. Assume there is an equation $f_1 = a_1 f_2 + \cdots + a_n f_n + g$ for some $a_1, \ldots, a_n \in R$ and $g \in m \Hom(R(N, M)$. Then for any element $a \in \soc(N)$, $g(a) = 0$. So $n \geq 2$ and $f(a) \neq 0$ implies there exists $i \geq 2$ such that $f_i(a) \neq 0$. This particular says that the homomorphism $[f_2, \ldots, f_n]: N \to M^\oplus n$ also an injection by Lemma 3.12 which is a contradiction to our assumption on $n$.

The following lemma is a generalization of the result of Gulliksen [10 Lemma 2].

**Lemma 3.14.** Let $(R, m, k)$ be an artinian local ring and $M, N$ be finitely generated faithful $R$-modules. Assume $M$ is minimal faithful. If there exists an injective homomorphism $N \to M^\oplus n$ for some $n$, then $\dim_k \soc(M) \leq \dim_k \soc(N)$ and equality holds if and only if $N \cong M$.

**Proof.** Let $n$ be the minimal integer such that there is an injective map $N \to M^\oplus n$. Take a injective map $N \to M^\oplus n$ and set $B$ the $k$-subspace of $\Hom_R(N, M) \otimes_R k$ generated by the image of $f_1, \ldots, f_n$. Then $\dim_k B = n$ by Lemma 3.13. By letting $A = \soc(N)$ and $C = \soc(M)$, we have a bilinear map $A \times B \to C$ over $k$ satisfying the assumption (1) and (2) of Lemma 3 in view of Lemma 3.12 and 3.13. We also verify the condition (3) of Lemma as follows. Assume (3) is not satisfied. Then there is a subspace $C'$ of $C$ that any nonzero element $a$ of $A$ induces a nonzero map $p \circ f(-, a): B \to C'$, where $p: C \to C/C'$ is the natural surjection.

Since $C/C' \subseteq M/C'$ as an $R$-module, we obtain an injective map $g: N \to (M/C')^\oplus m$, where $q: M \to M/C'$ is also the natural surjection. Since $N$ is faithful, there is an injective map $h$ from $R$ to some copies $N^\oplus m$ of $N$. Taking a composition of $h$ and $g^\oplus m$, one has an injective map from $R$ to $(M/C')^\oplus mn$. In particular, $M/C'$ is a faithful $R$-module, which contradicts the assumption that $M$ is minimal faithful.

Therefore, we can apply Lemma 3 and get an equality $\dim \soc(N) = \dim \soc(M) \geq n-1 \geq 0$. If the equalities hold, then $n = 1$ and $N$ is isomorphic to a submodule of $M$. By the minimality of $M$, one has $N \cong M$. ■
We also give some basic properties of minimal faithful modules.

**Lemma 3.15.** Let \((R, m, k)\) be an artinian local ring. Then

1. Any minimal faithful \(R\)-module is indecomposable.
2. Assume \(R\) has Cohen-Macaulay type at most three. Then \(\ell_R(R) \leq \ell_R(M)\) for all faithful \(R\)-module \(M\). In particular, a faithful \(R\)-module \(M\) is minimal faithful if \(\ell_R(M) = \ell_R(R)\).

**Proof.** (1): Let \(M\) be a minimal faithful \(R\)-module, and assume that \(M\) decomposes as direct sum \(M = M_1 \oplus M_2\) of \(R\)-modules. The faithfulness of \(M\) yields that \(\text{Ann}(M_1) \cap \text{Ann}(M_2) = 0\). Take minimal generators \(x_1, \ldots, x_n\) of \(M_1\) and \(y_1, \ldots, y_m\) of \(M_2\). Without loss of generality, we may assume \(n \leq m\). Then the submodule \(N\) of \(M = M_1 \oplus M_2\) generated by the elements \(x_1 + y_1, \ldots, x_n + y_m, 0 + y_{n+1}, \ldots, 0 + y_m\) is proper and faithful. This contradicts that \(M\) is minimal faithful. (2): This follows by [5] Theorem 1. ■

**Definition 3.16.** Let \((R, m, k)\) be a commutative ring. A fractional ideal \(I\) of \(R\) is called closed [5] if the natural homomorphism \(R \to \text{Hom}_R(I, I)\) is an isomorphism.

**Example 3.17.** Let \((R, m, k)\) be a one-dimensional Cohen-Macaulay local ring. Set \(B = m : m\). Then \(m\) is closed as a fractional ideal of \(B\).

**Lemma 3.18.** Let \((R, m, k)\) be a one-dimensional Cohen-Macaulay local ring having a canonical module and \(I\) be a fractional ideal of \(R\). Then the following are equivalent.

1. \(I\) is closed.
2. \(I^\dagger\) is closed.
3. There is a surjective homomorphism \(I^\oplus n \to \omega\) for some \(n\).
4. There is a short exact sequence \(0 \to R \to I^\oplus n \to M \to 0\) for some \(n\) and maximal Cohen-Macaulay \(R\)-module \(M\).

**Proof.** See [5] Proposition 2.1. Note that (4) follows by the canonical dual of (3), since \(I\) is Cohen-Macaulay as an \(R\)-module. ■

**Theorem 3.19.** Let \((R, m, k)\) be a Cohen-Macaulay local ring of dimension one. Put \(B = m : m\). Assume \(k\) is infinite.

1. If \(B\) is local with Cohen-Macaulay type two and \(R\) is nearly Gorenstein, then \(R\) is almost Gorenstein and does not satisfy \(m \cong m^\dagger\).
2. If \(B\) is local with Cohen-Macaulay type three and \(R\) is nearly Gorenstein, then either \(R\) is almost Gorenstein or \(m \cong m^\dagger\).

**Proof.** Take a minimal reduction \((t)\) of \(m_B\). Since \(m\) and \(m^\dagger\) has constant rank one as a \(B\)-module, \(\ell_B(B/tB) = e(B) = \ell_B(m/tm) = \ell_B(m^\dagger/tm^\dagger)\). As \(B/tB\) has Cohen-Macaulay type less than or equal to three in both case (1) and (2), Lemma 3.15 ensures that \(m/tm\) and \(m^\dagger/tm^\dagger\) are minimal faithful over \(B/tB\). Consider the exact sequence

\[
0 \to m^\dagger \to m^\oplus n \to M \to 0
\]

as in Lemma 3.15. Then \(\phi \in \text{Hom}_R(m^\dagger, m^\oplus n) = \text{Hom}_B(m^\dagger, m^\oplus n)\). Therefore \(M = \text{Coker} \phi\) is also a \(B\)-module and it is torsion-free over \(B\) as well as over \(R\). Moreover, the above sequence is an exact sequence of \(B\)-modules and \(B\)-homomorphisms. Tensoring \(B/tB\) to this sequence, we have a short exact sequence

\[
(3.19.1) \quad 0 \to m^\dagger/tm^\dagger \xrightarrow{\phi \otimes B/tB} (m/tm)^\oplus n \to M/tM \to 0
\]

of \(B/tB\)-modules.

1. Applying Lemma 3.14 to the sequence (3.19.1), we obtain the inequalities

\[
1 \leq r_B(m) \leq r_B(m^\dagger) < r_B(B) = 2.
\]
So one has either $soc(m/tm) = 1$ or $soc(m^\dagger/tm^\dagger) = 2$. In the former case, $m^\dagger$ must be a cyclic $B$-module and hence $m^\dagger \cong B$. $R$ is almost Gorenstein. In the case $soc(m^\dagger/tm^\dagger) = 2$, $m$ must be a cyclic $B$-module by a similar argument. Thus we have $m \cong B$. Then $R$ has minimal multiplicity. $B$ has type one. Contradiction.

(2): Applying Lemma 3.14 to the sequence (3.19.4), we obtain the inequalities

$$1 \leq r_B(m) \leq r_B(m^\dagger) < r_B(B) = 3.$$ 

In the case $1 = r_B(m)$, it follows by same argument as in (1) that $m^\dagger \cong B$ and $R$ is almost Gorenstein. So we only need to consider the case $r_B(m) = r_B(m^\dagger) = 2$. In this case, $m/tm$ should be isomorphic to $m^\dagger/tm^\dagger$ by lemma 3.14. Put $\phi = [\phi_1, \ldots, \phi_n]^\dagger: m^\dagger \to m^{\dagger \otimes n}$ and so $\phi \otimes B/tB = [\phi_1 \otimes B/tB, \ldots, \phi_n \otimes B/tB]^\dagger$. Consider the canonical dual $(\phi \otimes B/tB)^\dagger: (m/tm)^{\dagger \otimes n} \to (m/tm)$, which is surjective. Since $m/tm$ is indecomposable (Lemma 3.15), the Nakayama’s lemma indicates that $jac(End(m/tm)) \cdot (m/tm) \neq m/tm$. Therefore, one of the endomorphism $(\phi_1 \otimes B/tB)^\dagger, \ldots, (\phi_n \otimes B/tB)^\dagger$ of $m/tm$ must be not contained in $jac(End(m/tm))$, otherwise $(\phi \otimes B/tB)$ cannot be surjective. This means that one of the $\phi_1 \otimes B/tB, \ldots, \phi_n \otimes B/tB$ is an isomorphism. Say $\phi_i \otimes B/tB$ is an isomorphism. Then the $B$-homomorphism $\phi_i: m^\dagger \to m$ is also surjective. Both $m$ and $m^\dagger$ have constant rank, $\phi_i$ must be an isomorphism. This shows that $m \cong m^\dagger$.

**Corollary 3.20.** Let $(R, m, k)$ be a complete Cohen-Macaulay local ring of dimension one with a canonical module. Assume $B :\subset m : m$ is local and $k$ is infinite. If $R$ is nearly Gorenstein with multiplicity $e(R) \leq 4$, then either $R$ is almost Gorenstein or $m \cong m^\dagger$.

**Proof.** Take a minimal reduction ($t$) of $R$. Then the multiplicity $e(B, B) = \ell_R(B/tB)$ of $B$ as an $R$-module is equal to 4, provided $B$ has a constant rank as an $R$-module. Deduce

$$4 = \ell_R(B/tB) \geq \ell_B(B/tB) \geq r(B),$$

where $r(B)$ is the Cohen-Macaulay type of $B$. Now we can apply Theorem 1.7 and attain the desired statement.

**Example 3.21.** Let $R = k[[t^5, t^6, t^7]]$. Then $R$ is nearly Gorenstein and has multiplicity 5, however, neither $R$ is almost Gorenstein nor $m \cong m^\dagger$.

## 4. Numerical semigroup rings

In this section, we deal with the numerical semigroup rings $(R, m)$ having an isomorphism $m \cong m^\dagger$. We begin the section with recalling preliminaries on numerical semigroup. Let $H \subset \mathbb{N}$ be a numerical semigroup. The set of pseudo-Frobenius numbers $PF(H)$ of $H$ is consisting of integers $a \in \mathbb{N} \setminus H$ such that $a + b \in H$ for any $b \in H \setminus \{0\}$. Then the maximal element $F(H)$ of $PF(H)$ is the Frobenius number of $H$. Set $H' = H \cup \{F(H)\}$. Then $H'$ is also a numerical semigroup. A numerical semigroup of the form $H' = H \cup \{F(H)\}$ for some symmetric numerical semigroup $H$ is called a UESY-semigroup (unitary extension of a symmetric semigroup), which is introduced in [19]. Note that $F(H)$ is a minimal generator of $H' = H \cup \{F(H)\}$. For a numerical semigroup $H$ and a field $k$, the numerical semigroup ring of $H$ over $k$ is the subring $k[[t^a \mid a \in H]]$ of $k[[t]]$, where $t$ is an indeterminate.

**Lemma 4.1.** Let $H$ be a numerical semigroup, $k$ is an infinite field and $(R, m)$ is the numerical semigroup ring $k[[H]]$. Then the following are equivalent.

1. $m$ is self-dual.
2. $H$ is a UESY-semigroup.

**Proof.** (1) $\Rightarrow$ (2): In the case that $H$ is symmetric, $e(R) \leq 2$. Then it can easily shown that $H$ is UESY.
We may assume that $H$ is not symmetric. By Theorem 4.2, there is a Gorenstein local subring $(S, n)$ of $R$ such that $R = n : n$. Take a value semigroup $v(S)$ of $S$, where $v$ is the normalized valuation of $k[[t]]$, that is, $v$ takes $t$ to $1 \in \mathbb{Z}$. Then $H = v(R)$, and $v(S)$ is symmetric by the result of Kunz [14]. Since $Rn \subseteq n$, $v(S) \subseteq H \subseteq v(S) \cup \{F(v(S))\}$. Therefore, $H$ should be equal to $v(S) \cup \{F(v(S))\}$. In particular, $H$ is UESY.

(2) $\Rightarrow$ (1): Describe $H$ as $H = H' \cup \{F(H')\}$ with a symmetric numerical semigroup $H'$. Set $S = k[[H']]$. Then $\text{End}_S(m_S)$ is isomorphic to $R$. Thus by our theorem (Theorem 1.4), the maximal ideal $m$ of $R$ is self-dual.

\begin{proposition}
Let $H = \langle a_1, \ldots, a_n \rangle$ be a symmetric numerical semigroup minimally generated by $\{a_i\}$ with $2 < a_1 < a_2 < \cdots < a_n$ and $H' := H \cup \{F(H)\}$. Put $S = k[[H]]$ over an infinite field $k$ and $R = k[[H']]$. Then the maximal ideal of $R$ is quasi-decomposable.
\end{proposition}

\begin{proof}
Denote by $m_R$ the maximal ideal of $R$. We will prove that the maximal ideal $m_R^{(t^{a_1})}$ of $R/t^{a_1}$ has a direct summand $I$ generated by the image of $t^{F(H)}$, and $I \cong k$ as an $R$-module. Since $F(H)$ is a minimal generator of $m_R$, it is enough to show that $m_R^{F(H)} \subseteq t^{a_1}R$. So what we need to show is that $F(H) + a_i - a_1 \in H$ for all $i \neq 1$ and $2F(H) - a_1 \in H$. These follow by the fact that $F(H)$ is the largest number in $\mathbb{N} \setminus H$ and the inequalities $a_i - a_1 > 0$ and $F(H) - a_1 > 0$.
\end{proof}

5. Further characterizations

The goal of this section is to give characterizations of local rings $R$ such that there exists a one-dimensional local hypersurface $(S, n)$ such that $R \cong \text{End}_S(n)$.

\begin{proposition}
Let $(R, m)$ be a Cohen-Macaulay local ring of dimension one. Assume that $R$ has a canonical module and infinite coefficient field $k$. Then the followings are equivalent.

(1) There is a local hypersurface $(S, n)$ such that $R \cong \text{End}_S(n)$.

(2) $e(R) \leq 2$, or $R$ has type $2$ and a canonical ideal $I$ such that $I^2 = mI$ and $\ell_R(I/I^2) = 2$.

(3) $e(R) \leq 2$, or $R$ has embedding dimension $3$, and a canonical ideal $I$ such that $I^2 = m^2$.
\end{proposition}

\begin{proof}
(1) $\Rightarrow$ (2): Assume $e(R) > 2$ and $R$ satisfies (1). Then $R$ is not Gorenstein, and $I := n$ is a canonical ideal of $R$. Since $S$ is a hypersurface and not a, $\ell_R(I/I^2) = \ell_S(n/n^2) = 2$. It forces the equality $I^2 = mI$, since $I$ is not a principal ideal.

(2) $\Rightarrow$ (1): Consider the case that $e(R) \leq 2$. Then by the proof of Corollary 1.5 (1) $\Rightarrow$ (6), there is a Gorenstein local ring $(S, n)$ such that $R \cong \text{End}_S(n)$ and $e(S) = \text{edim} \, S$. In particular, $e(S) \leq 2$ and $S$ is a hypersurface. Now consider the case that $R$ has type $2$ and a canonical ideal $I$ such that $I^2 = mI$ and $\ell_R(R/I) = 2$. One has equalities $\ell_R(I/I^2) = \ell_R(I/mI) = 2$. Put $S := k + I$. Then $S$ is Gorenstein local with a maximal ideal $n := I$, and $R \cong \text{End}_S(n)$ (Lemma 2.3). We can compute the embedding dimension $\text{edim} \, S$ as follows:

$$\text{edim} \, S = \ell_S(n/n^2) = \ell_R(I/I^2) = 2.$$ 

Therefore, $S$ should be a hypersurface. (2) $\Rightarrow$ (3): We may assume $R$ has type $2$. By the implication (2) $\Rightarrow$ (1), we can calculate the embedding dimension of $R$ as $\text{edim} \, R \leq \text{edim} \, S + 1 = 3$, where $(S, n)$ is a hypersurface with $R \cong \text{End}_S(n)$. Since $R$ is not Gorenstein, $S$ should be equal to $3$. This means $\ell_R(m/m^2) = 3$. On the other hand, one has

$$\ell_R(m/m^2) = \ell_R(m/I) + \ell_R(I/I^2) = 1 + \ell_R(I/mI) = 1 + 2 = 3.$$ 

So the inclusion $I^2 \subseteq m^2$ yields that $I^2 = m^2$. The direction (3) $\Rightarrow$ (2) also follows by similar calculations.
\end{proof}

\begin{question}
For a Cohen-Macaulay local ring $(R, m)$ of dimension one, when is there a local complete intersection $(S, n)$ with an isomorphism $R \cong \text{End}_S(n)$?
\end{question}
Acknowledgments. The author is grateful to his supervisor Ryo Takahashi for giving him kind advice throughout the paper, and to Osamu Iyama for his helpful comments on Theorem 1.4. The author is also grateful to Luchezar Avramov for useful comments.

References

[1] H. Ananthnarayan, The Gorenstein colength of an Artinian local ring. J. Algebra 320 (2008), no. 9, 3438-3446.
[2] H. Ananthnarayan; L.L. Avramov; W.F. Moore, Connected sums of Gorenstein local rings. J. Reine Angew. Math. 667 (2012), 149-176.
[3] H. Bass, On the ubiquity of Gorenstein rings, Math. Zeitschrift 82 (1963), 8-28.
[4] G. M. Bergman, Minimal faithful modules over Artinian rings. Publ. Mat. 59 (2015), no. 2, 271-300.
[5] J. Brennan; W. Vasconcelos, On the structure of closed ideals, Math. Scand. 88 (2001), no. 1, 3-16.
[6] M. T. Dibaei; M. Rahimi, Rings with canonical reduction. arXiv:1712.00755.
[7] J. Elias; M. Silva Takatui, On Teter rings. Proc. Roy. Soc. Edinburgh Sect. A 147 (2017), no. 1, 125-139.
[8] L. Ghezzi; S. Goto; J. Hong; W. V. Vasconcelos, Invariants of Cohen-Macaulay rings associated to their canonical ideals. Journal of Algebra. 489, 506-528.
[9] S. Goto; N. Matsuoka; T. Phuong, Almost Gorenstein rings. J. Algebra 379 (2013), 355-381.
[10] Tor H. Gulliksen, On the length of faithful modules over Artinian local rings, Math. Scand. 31 (1972) 78-82.
[11] J. Herzog; T. Hibi, D. I. Stamate, The trace of the canonical module. arXiv:1612.02723.
[12] C. Huneke; A. Vraciu, Rings which are almost Gorenstein. Pacific J. Math., 225 (2006) no. 1, 85–102.
[13] T. Kobayashi, Syzygies of Cohen-Macaulay modules over one dimensional Cohen-Macaulay local rings, Preprint (2017), arXiv:1710.02673.
[14] E. Kunz, The value-semigroup of a one-dimensional Gorenstein ring, Proc. Amer. Math. Soc. 25 (1970), 748-751.
[15] A. Kustin; A. Vraciu, Totally reflexive modules over rings that are close to Gorenstein. arXiv:1705.05714.
[16] G. J. Leuschke; R. Wiegand, Cohen-Macaulay Representations, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012.
[17] S. Nasseh; R. Takahashi, Local rings with quasi-decomposable maximal ideal, Math. Proc. Cambridge Philos. Soc. (to appear).
[18] A. Ooishi, On the self-dual maximal Cohen-Macaulay modules, J. Pure Appl. Algebra 106 (1996), no. 1, 93–102.
[19] J. C. Rosales, Numerical semigroups that differ from a symmetric numerical semigroup in one element. Algebra Colloq. 15 (2008), no. 1, 23-32.
[20] J. D. Sally, Tangent cones at Gorenstein singularities, Comp. Math. 40 (1980), 167–175.
[21] J. D. Sally, Cohen-Macaulay local rings of embedding dimension e+d, J. Algebra 83 (1983), 393–408.
[22] J. Striuli; A. Vraciu, Some homological properties of almost Gorenstein rings, Commutative algebra and its connections to geometry, 201-215, Contemp. Math., 555, Amer. Math. Soc., Providence, RI, 2011.
[23] R. Takahashi, On G-regular local rings, Comm. Algebra 36 (2008), 4472-4491.
[24] W. Teter, Rings which are a factor of a Gorenstein ring by its socle. Inventiones Math, 23 (1974), 153–162.
[25] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990.

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