Anomalous Dimensions for Boundary Conserved Currents in Holography via the Caffarelli-Silvestri Mechanism for p-forms

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Although it is well known that the Ward identities prohibit anomalous dimensions for conserved currents in local field theories, a claim from certain holographic models involving bulk dilaton couplings is that the gauge field associated with the boundary current can acquire an anomalous dimension. We resolve this conundrum by showing that all the bulk actions that produce anomalous dimensions for the conserved current generate non-local actions at the boundary. In particular, the Maxwell equations are fractional. To prove this, we generalize to p-forms the Caffarelli/Silvestre (CS) extension theorem. In the context of scalar fields, this theorem demonstrates that second-order elliptic differential equations in the upper half-plane in $\mathbb{R}^{n+1}_+$ reduce to one with the fractional Laplacian, $\Delta^\gamma$, when one of the dimensions is eliminated. From the p-form generalization of the CS extension theorem, we show that at the boundary of the relevant holographic models, a fractional gauge theory emerges with equations of motion of the form, $\Delta^\gamma A^t = 0$ with $\gamma \in \mathbb{R}$ and $A^t$ the boundary components of the gauge field. The corresponding field strength $F = d_\gamma A^t = d\Delta^{\gamma-\frac{1}{2}} A^t$ is invariant under $A^t \rightarrow A^t + d_\gamma \Lambda$ with the fractional differential given by $d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$, implying that $[A^t] = \gamma$ which is in general not unity.
I. INTRODUCTION

A text-book problem\[1, 2\] in quantum field theory is to prove that conserved quantities such as the electrical current cannot acquire anomalous dimensions even under renormalization. The basic argument is that the current, $J_\mu$, enters the action through the combination $J_\mu A^\mu$, where $A_\mu$ is the vector potential. As long as the theory remains gauge-invariant, the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, ensures that $[A_\mu] = 1$ and hence the dimension of $J_\mu$ is fixed by the volume factor in the action; that is, $[J_\mu] = d - 1$. Nonetheless, holographic\[3, 4\] bulk models have been constructed in which the dimension of the current at the boundary is arbitrary and hence so is the dimension of associated gauge field, $A_\mu$. Since $A$ is a differential 1-form, precisely what it means for it to acquire an anomalous dimension is unclear. In this note, we show that the operative mechanism for changing the dimension of $A$ in the extant holographic constructions\[3, 4\] is the p-form generalization of the Caffarelli-Silvestre\[5\] mechanism. The conformal version of this theorem can be found in the works of Graham and Zworski\[6\] and Chang and Gonzalez\[7\]. What we prove here is that although the dual theory\[8, 9\] is governed by currents that in principle do not obey the standard local gauge group, they are controlled by a fractional gauge group in which $A \to A + d\gamma \Lambda$, where $d\gamma = (\Delta) \frac{1}{2} d$. We provide an explicit proof of this here.

That there is a fundamental connection between the Caffarelli/Silvestre mechanism and the holographic models that generate anomalous dimensions for the gauge field is ultimately not surprising given that such models are all based on dilaton actions of the form,

$$S = \int d^d x dy \sqrt{-g} Z(\phi) F^2 + \cdots ,$$

(I.1)

where $F$ is the field strength and $y$ is the radial direction. The class of solutions\[3, 4\] that yields the anomalous dimension for the gauge field has the dilaton field scaling as $Z(\phi) \propto y^a$. Consequently, the equations of motion are equivalent to

$$\nabla^\mu (y^a F_{\mu\nu}) = 0.$$  

(I.2)

In the language of differential forms, this equation becomes

$$d(y^a \star dA) = 0,$$

(I.3)

which clearly illustrates that along any slice perpendicular to the radial direction, the standard $U(1)$ gauge transformation applies. To determine what happens at the boundary, we note that these equations are reminiscent of those studied by Caffarelli and Silvestre\[5\] (CS) for the case of a scalar field,

$$\nabla \cdot (y^a \nabla u),$$

(I.4)

which is just a recasting of the (degenerate) elliptic differential equation

$$u(x, y = 0) = f(x)$$

(I.5)

$$\Delta_x u + \frac{a}{y} u_y + u_{yy} = 0.$$  

(I.6)

What they were interested in is what form does this differential equation acquire at the boundary, $y \to 0$. They showed that any equation of this kind satisfies

$$\lim_{y \to 0} (y^a u_y) = C_{d, \gamma} (-\nabla)^\gamma f(x).$$

(I.7)

where $\gamma = (1 - a)/2$. We show here that the same result holds for a differential p-form and hence the boundary action in holographic models that yield anomalous dimensions is of the form,

$$S = \frac{1}{2} \int A_i (-\nabla)^{2\gamma} A^i,$$

(I.8)
thereby giving rise to fractional scaling dimension for $A$. The corresponding field strength is the 2-form,

$$F = d_A A = d\Delta^{\frac{\Delta - 1}{2}} A,$$

with gauge-invariant condition,

$$A \rightarrow A + d_\gamma \Lambda,$$

with

$$d_\gamma \equiv (\Delta)^{\frac{\Delta - 1}{2}} d,$$

which preserves the 1-form nature of the gauge-field with dimension $[A_\mu] = \gamma_\mu$, rather than unity.

There is a precedent for fractional gauge groups in boundary actions. In a spacetime that is asymptotically hyperbolic, Domokos and Gabadze\[10\] considered a bulk action with $F^2$ with a gauge transformation of the standard form along the boundary coordinates but a fractional transformation,

$$A_y \rightarrow A_y + \partial_y^\mu A_y$$

along the radial direction. Here $\partial_y^\mu$ is the fractional derivative. They then integrated out the radial direction and obtained the fractional action, Eq. (I.8), for the boundary components of the gauge field. The relationship between this mechanism and the dilaton approach is that asymptotically the gauge field has an algebraic form at the boundary. Hence, powers of the radial coordinate can be substituted for derivatives. As a result, fractional derivatives along y-coordiante will translate to a bulk coupling of the dilaton form.

II. THE FRACTIONAL LAPLACIAN ON FORMS

Throughout the paper, following standard nomenclature, we will denote by $\Omega^p(M)$ the space of $p$-forms on a manifold $M$. Let us fix the dimension of the manifold to be $n$. We recall a few facts about the Laplacian on manifolds. First the Hodge star operator, $\star : \Omega^p(M) \rightarrow \Omega^{d-p}(M)$ which is defined by requiring that

$$\star (e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-p}},$$

if $\{e_{i_1}, \cdots, e_{i_p}, e_{j_1}, \cdots, e_{j_{d-p}}\}$ is a positive frame of the cotangent bundle $T^*M = \Omega^1(M)$. The inner product on forms is defined by $\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle \star 1$, where $\langle \alpha, \beta \rangle$ is the pointwise scalar product on forms and $\star 1 = dV$, the volume form. The scalar product $\langle \alpha, \beta \rangle$ is readily seen to equal

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta.$$

Also, recall that the adjoint of the differential operator, denoted by $d^*\$, is an operator $d^* : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$ which satisfies the defining property\[3\]

$$(d\alpha, \beta) = (\alpha, d^*\beta).$$

It is a standard fact that the following holds

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1. Here and in the sequel we will be purposely vague about the nature of $M$, that is, whether it is closed or with boundary or compact or not. We will simply assume that the space of forms we take is the subspace of the space of forms that is necessary for integration by parts to hold without boundary terms.
Lemma II.1. One has that on $p$-forms, $d^* = (-1)^{n(p+1)+1} \ast d\ast$.

Let us recall that the Hodge Laplacian on $p$-forms is defined as

$$\Delta = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M).$$

One readily show that if $M = \mathbb{R}^n$ with the standard flat metric $ds^2 = \delta_{ij} dx^i \otimes dx^j$, the calculation of $d^* d\omega$, $dd^* \omega$ and $\Delta \omega$ for a $p$-form $\omega = \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ amounts to

\begin{equation}
d^* \omega = \sum_{\ell=1}^{p} (-1)^{(p+1)(2d-p)+\ell} \frac{\partial \omega_{i_1 \cdots i_p}}{\partial x^{i_\ell}} dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \wedge \cdots \wedge dx^{i_p}, \tag{II.1}
\end{equation}

whence

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\begin{aligned}
d^* d\omega = \sum_{k=1}^{n-p} (-1)^{\ell+1} \frac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x^{i_\ell} \partial x^{j_k}} dx^{j_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \wedge \cdots \wedge dx^{i_p}, \tag{II.3}
\end{aligned}
\end{equation}

Putting equations (II.2) and (II.3) together yields

\begin{equation}
\Delta_e = - \sum_{m=1}^{n} \frac{\partial^2 \omega_{i_1 \cdots i_p}}{(\partial x^m)^2} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{II.4}
\end{equation}

This clearly shows that in general $\Delta$ is an elliptic operator. Furthermore, using the fact that $d^*$ is the adjoint of $d$, one can show that $\Delta$ is a symmetric operator,

$$(\Delta \alpha, \beta) = (\alpha, \Delta \beta).$$

We thus define, following the spectral theorem, the fractional Laplacian on forms as

\begin{equation}
\Delta^a \alpha = \frac{1}{\Gamma(-a)} \int_0^\infty (e^{-t\Delta} \alpha - \alpha) \frac{dt}{t^{1+a}}, \tag{II.5}
\end{equation}

for $a \in (0, 1)$. In fact, this makes sense for any self-adjoint operator and in particular it applies to both $dd^*$ and $d^*d$. Here, the heat semigroup $e^{-t\Delta}$ on forms is defined by requiring that $e^{-t\Delta} \alpha$ be equal to the form $\beta$ which is the solution to the diffusion equation

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} \beta + \Delta \beta = 0, & \text{for } (x,t) \in M \times \mathbb{R}_+ \\
\beta(x,0) = \alpha(x) & \text{for } x \in M.
\end{cases} \tag{II.6}
\end{equation}

Recall, that such solutions always exist if $\omega$ is assumed sufficiently regular ($\omega \in C^{2,\alpha}$ is the correct requirement) is part of the famed theorem of Milgram-Rosenblum\[1 1\] (see Theorem 3.6.1). We next prove a sequence of important facts which will turn out to be useful for later purposes.
Lemma II.2. For any $p$-form $\omega$, one has

$$\Delta^a \omega = \sum_{k=1}^{\infty} \binom{a}{k} (dd^*)^{a-k} (d^* d)^k \omega$$

where $\binom{a}{k}$ are Newton’s binomial coefficients.

Proof. This is a straightforward consequence of the definition and of Newton’s binomial theorem. \qed

Lemma II.3. $\Delta^a d = (dd^*)^a d$.

Proof. Using Lemma II.2 we can write

$$\Delta^a d \omega = \sum_{k=1}^{\infty} \binom{a}{k} (dd^*)^{a-k} (d^* d)^k d \omega$$

but $(d^* d)^k d \omega = 0$ unless $k = 0$, whence

$$\Delta^a d \omega = (dd^*)^a d \omega.$$

\qed

Proposition II.4. For any $a, b \in \mathbb{R}$

$$d(dd^*)^a = 0, \quad d^* (d^* d)^a = 0 \quad \text{and} \quad (d^* d)^b (d^* d)^a = 0$$

and also

$$(d^* d)^a d = 0, \quad (dd^*)^a d^* = 0.$$

Proof. According to the comments following the definition of the fractional Laplacian (Eq. (II.5)), one defines

$$(dd^*)^a \omega = \frac{1}{\Gamma(-a)} \int_0^\infty \left( e^{-tdd^*} \omega - \omega \right) \frac{dt}{t^{1+a}}, \quad (II.7)$$

where $\beta = e^{-tdd^*} \omega$ is the unique solution to

$$\begin{cases} 
\frac{d}{dt} \beta + d d^* \beta = 0, & \text{for } (x, t) \in M \times \mathbb{R}_+ \\
\beta(x, 0) = \omega(x) & \text{for } x \in M, \quad (II.8)
\end{cases}$$

It then follows from this differential equation that $d \beta$ satisfies the equation

$$\begin{cases} 
\frac{d}{dt} (d \beta) = 0, & \text{for } (x, t) \in M \times \mathbb{R}_+ \\
d \beta(x, 0) = d \omega(x) & \text{for } x \in M, \quad (II.9)
\end{cases}$$

having used the fact that $d^2 = 0$ hence $ddd^* \beta = 0$. Clearly equation (II.9) implies that $d(e^{-tdd^*} \omega) = d \omega$ for any $t$ and therefore taking the differential of equation (II.7) implies that

$$d(dd^*)^a \omega = \frac{1}{\Gamma(-a)} \int_0^\infty \left( de^{-tdd^*} \omega - d \omega \right) \frac{dt}{t^{1+a}} = 0.$$
Since the operator $dd^*$ is neither self-adjoint (its adjoint is $d^*d$) nor elliptic (although it is degenerate elliptic with degeneracy at those forms $\beta$ such that $d^*\beta = 0$) the preceding argument needs justification. Specifically, we need to show that equation (II.8) has eternal (i.e., for any $t > 0$) solutions. This is done as follows. Let $\eta$ be the (unique) solution to

$$\begin{cases}
\frac{\partial}{\partial t} \eta + \Delta \eta = 0, & \text{for } (x, t) \in M \times \mathbb{R}^+ \\
\eta(x, 0) = \omega(x) & \text{for } x \in M.
\end{cases} \quad (\text{II.10})$$

Recall that the space of $p$-forms decomposes, according to the Kodaira decomposition as\footnote{To be precise, in the decomposition of Eq. (II.11), one should consider the $L^2$ closure $B_p$ and $B^*_p$ of the spaces $d\Omega^{p-1}(M)$ and $d^*\Omega^{p+1}(M)$ respectively, but elliptic regularity allows us to consider directly smooth forms (instead of $L^2$), whence equation (II.11). Also, as stated our construction works for $M$ compact and closed (i.e., with no boundary). For either non-compact manifolds or for manifolds with boundary, one needs to restrict to the spaces of $L^2$ the $p$-forms with suitable conditions at infinity or at the boundary.}

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus d^*\Omega^{p+1}(M) \oplus H_p, \quad (\text{II.11})$$

where $H_p = \{ \omega \in \Omega^p(M) : \Delta \omega = 0 \} = \{ \omega \in \Omega^p(M) : d\omega = 0 \text{ and } d^*\omega = 0 \}$. Clearly if $\Delta \omega = 0$ or $\omega \in d\Omega^{p-1}(M)$ (so that $d\omega = 0$ in both cases), by taking $d$ on both sides of the diffusion equation and defining $\eta$ (i.e., Eq. (II.10)), one obtains that $d\eta = 0$, whence $\Delta \eta = dd^* \eta$. Therefore, one can take $\beta = \eta$ on $d\Omega^{p-1}(M) \oplus H_p$. On the other hand, if $\omega \in d^*\Omega^{p+1}(M)$, i.e. $\omega = d^*\alpha$ with $\alpha \in d^*\Omega^{p+1}(M)$ (so that $d^*\alpha = 0$), then $dd^*\omega = 0$ and clearly taking $\beta$ constantly equal to $\omega$ solves Eq. (II.8) (thereby proving that $(dd^*)^n \omega = 0$ in this case). In all the cases, we have shown the solution to Eq. (II.8) exists for every $t > 0$.

The proofs that $(d^*d)^a = 0$, $(d^*d)^b(d^*d)^a = 0$ and that $(d^*d)^a d = 0$ and $(dd^*)^a d^* = 0$ are analogous.

\section{Fractional differential}

One of the crucial objects of cohomology theory (essential in geometric quantization) is the notion of the differential of forms, which we generalize here to the fractional differential on a form as follows.

\textbf{Definition II.5.} We define the fractional differential $d_a$ via

$$d_a \omega = \frac{1}{2} \left( d \Delta^{a-1} \omega + \Delta^{a-1} d \omega \right). \quad (\text{II.12})$$

A few Lemmas are useful here.

\textbf{Lemma II.6.} The adjoint of $d_a$, denoted by $d_a^*$, is given by

$$d_a^* = \frac{1}{2} \left( \Delta^a \Delta^{a-1} \omega + \Delta^{a-1} d^* \omega \right). \quad (\text{II.13})$$

\textbf{Proof.} By definition, the (formal) adjoint of $d_a$ has to satisfy

$$\int_M d_a \omega \wedge \alpha = \int_M \omega \wedge d_a^* \alpha. \quad (\text{II.12})$$

The definition of $d_a$ (Eq. (II.12)) implies that

$$\int_M d_a \omega \wedge \alpha = \int_M \frac{1}{2} \left( d \Delta^{a-1} \omega + \Delta^{a-1} d \omega \right) \wedge \alpha = \int_M \frac{1}{2} \left( \Delta^{a-1} \omega \wedge d^* \omega + d \omega \wedge \Delta^{a-1} \alpha \right)$$

$$= \int_M \frac{1}{2} \left( \omega \wedge \Delta^{a-1} d^* \omega + \omega \wedge d^* \Delta^{a-1} \alpha \right).$$
where we have used that $d^*$ is the adjoint of $d$ and the integration by parts formula for the Laplacian:

$$\int_M \Delta^{b} \eta \wedge \beta = \int_M \eta \wedge \Delta^{b} \beta,$$

for forms $\eta$ and $\beta$. Whence, reading from the first to last the equalities, we have

$$\int_M d_a \omega \wedge \alpha dV = \int_M \frac{1}{2} \left( \omega \wedge \Delta^{\frac{n+1}{2}} d^* \omega + \omega \wedge d^* \Delta^{\frac{n+1}{2}} \alpha \right),$$

which proves the Lemma.

In fact, through the use of Proposition II.4, we can simplify the expressions of $d_a$ and $d^*_a$.

Lemma II.7.

$$d_a = \frac{1}{2} \left( d(d^* d) \frac{n+1}{2} \omega + (dd^*) \frac{n+1}{2} d \omega \right)$$

and

$$d^*_a = \frac{1}{2} \left( d^*(dd^*) \frac{n+1}{2} \omega + (d^* d) \frac{n+1}{2} d^* \right).$$

Proof. Straightforward, using Proposition II.4.

Clearly $d_a : \Omega^p \rightarrow \Omega^{p+1}$, i.e., if $\omega$ is a $p$-form, then $d_a \omega$ is a $p+1$-form. Therefore, the question as to whether $d_a$ generates a complex springs to mind. This is answered in the affirmative by the proof of the following Proposition.

Proposition II.8.

$$d_a \circ d_a = 0 \quad \text{and} \quad d^*_a \circ d^*_a = 0.$$  

Proof. Using Lemma II.7 and Proposition II.4 one immediately obtains

$$d_a d_a \omega = \frac{1}{4} \left( d(d^* d) \frac{n+1}{2} d(d^* d) \frac{n+1}{2} \omega + (dd^*) \frac{n+1}{2} d \right) \left( d(d^* d) \frac{n+1}{2} \omega + (dd^*) \frac{n+1}{2} d \right)$$

$$= \frac{1}{4} \left( d(d^* d) \frac{n+1}{2} (d(d^* d) \frac{n+1}{2} + (dd^*) \frac{n+1}{2} d \omega + (dd^*) \frac{n+1}{2} d \omega + (dd^*) \frac{n+1}{2} dd(d^* d) \frac{n+1}{2} \omega \right)$$

$$= 0.$$

The fact that $d_a$ forms a complex leads to the definition of the fractional cohomology groups $H^p_a(M, \mathbb{R}) = \ker d_a / \text{Im} d_a$. The fractional Hodge theorem and the connection of these cohomology groups with the standard DeRham cohomology will be discussed in a future paper. Perhaps more importantly for the purposes of this article is the fact that $d_a$ is the right fractional differential of forms in relation to the Fractional Laplacian of forms, as testified by the fact (which we shall prove in Theorem II.11) that $d_a d^*_a + d^*_a d_a$ is the fractional Laplacian. In order to prove this fact, we first need to establish a few results.

3 The integration by parts formula for the fractional Laplacian on forms is easily proven appealing to the standard integration by parts formula for the fractional Laplacian on functions, aided by a partition of unity argument.
Proposition II.9. For any form $\omega$ and any $b \in \mathbb{R}$, one has
\[ d^*(dd^*)^b = (d^* d)^b d^* \quad d(d^*)^b = (dd^*)^b d, \] (II.14)
whence
\[ d^*(dd^*)^b = (d^* d)^{b+1}, \quad d(d^*)^b = (dd^*)^{b+1}, \quad d^*(dd^*)^b(d^* d) = (d^* d)^{b+c+1}, \] (II.15)
\[ d^*(dd^*)^b d = (d^* d)^{b+1}, d^*(dd^*)^b = (dd^*)^{b+1}. \]

Proof. By definition (cf. Eq. (II.7)) one knows that
\[ (dd^*)^b \omega = -\Gamma(-b) \int_0^\infty \left( e^{-tdd^*} - \omega \right) \frac{dt}{t^{1+b}}, \] (II.16)
where $\beta = e^{-tdd^*}$ is the unique solution to
\[ \begin{cases} \frac{\partial}{\partial t} \beta + dd^* \beta = 0, & \text{for } (x,t) \in M \times \mathbb{R}_+ \\ \beta(x,0) = \omega(x), & \text{for } x \in M \end{cases} \] (II.17)
Taking $d^*$ of this equation produces
\[ \begin{cases} \frac{\partial}{\partial t} (d^* \beta) + d^* d(d^* \beta) = 0, & \text{for } (x,t) \in M \times \mathbb{R}_+ \\ d^* \beta(x,0) = d^* \omega(x), & \text{for } x \in M \end{cases} \] (II.18)
which shows that $d^* (e^{-tdd^*} \omega) = e^{-tdd^*} d^* \omega$ and therefore calculating $d^*$ of both sides of Eq. (II.16), we obtain
\[ d^*(dd^*)^b \omega = \frac{1}{\Gamma(-b)} \int_0^\infty \left( e^{-tdd^*} - d^* \omega \right) \frac{dt}{t^{1+b}} = (d^* d)^b (d^* \omega). \] (II.19)
The second formula in Eq. (II.14) is proven analogously and the formulae in Eq. (II.15) are consequences of Eq. (II.14).

We can now make the following observation which readily follows from the Proposition above.

Remark II.10. Equations (II.14) in Proposition II.9 allow us to write the fractional differential as
\[ d^a = d \Delta^{\frac{a}{2}} \] or equivalently \[ d^a = \Delta^{\frac{a}{2}} d \] and the analogous statements for $d^*_a$.

We are now ready to prove

Theorem II.11.
\[ d^*_a d^a + d^a d^*_a = (\Delta)^a. \]

Proof. This is a simple calculation aided by Proposition II.9. \qed

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\footnote{Again, here one needs to argue that the solution to the diffusion equation for $d^* d$ has (a unique) solution. This is done in the same way as in Lemma II.4 and specifically the arguments surrounding Eq. (II.10).}
B. Fractional Curvature

We now use the definition of the fractional differential $d_a$ to define the fractional curvature of a connection on a principal bundle $P \to M$. Let $G$ be the Lie group of the bundle and $\mathfrak{g}$ its Lie algebra. We fix an open covering $\{U_i\}$ of $M$. More precisely, given a connection $D$, we write it locally on each $U_i$ as $D = d + A_i$ (as customary), where $A_i$ is a $Lie(G)$-valued 1-form, and then define the fractional connection to be the one modeled after $d_a$ such that,

$$D_a \phi = (d + A_i) \Delta^{\frac{a-1}{2}} \phi = d_a \phi + A_i \Delta^{\frac{a-1}{2}} \phi.$$

Given a connection $A$ we will also denote its corresponding covariant fractional differentiation by

$$D_{a,A} = (d + A) \Delta^{\frac{a-1}{2}}.$$

We then define the curvature to be,

$$F_{D_a} = D_{a}^2,$$

which one readily reduces to

$$F_{D_a} = d_a A + [A, A].$$

Here, writing $A = A_j dx^j$ in local coordinates, with $A_j$ elements of the Lie algebra of $G$, as customary, one denotes

$$[A, A] = [A_i, A_j] \ dx^i \wedge dx^j,$$

where $[A_i, A_j]$ is the commutator. The Gauge group is given by sections of $Aut(P)$ which act on fractional sections via

$$s^* D_a = s^{-1} \circ D_a \circ s.$$

In the case in which $G = U(1)$, then

$$F_{D_a} = d_a A$$

and the Gauge group (identified as the sections of the form $s = e^\Lambda$) acts as

$$A \to A + d_a A.$$

III. BRIEF REMARKS ON FORMS ON MANIFOLDS WITH BOUNDARY

Let $M$ be a manifold with smooth boundary $\partial M$. Let $\nu$ the unit normal to $\partial M$.

The natural generalization of the Dirichlet condition to forms is given by

$$[Relative \ B.C.] \begin{cases} \omega \mid_{\partial M} = 0 \\ d^* \omega \mid_{\partial M} = 0 \end{cases} \quad \text{(III.1)}$$

and the generalization of the Neumann condition is

$$[Absolute \ B.C.] \begin{cases} i_\nu \omega \mid_{\partial M} = 0 \\ i_\nu d\omega \mid_{\partial M} = 0 \end{cases} \quad \text{(III.2)}$$

Here, given any vector field $V$ and a $p$-form $\omega$, $i_V \omega$ indicates the $(p-1)$-form determined by

$$i_V \omega(X_1, \cdots, X_{p-1}) = \omega(X_1, \cdots, X_{p-1}, V),$$

for arbitrary vector fields $X_1, \cdots, X_{p-1}$. It is a standard fact that for either of these boundary conditions, the integration by parts (or Green’s formula) holds

$$\int (\Delta \omega, \alpha) = \langle d\omega, d\alpha \rangle + \langle d^* \omega, d^* \alpha \rangle. \quad \text{(III.3)}$$

It is also a well known fact that the Hodge star operator interchanges these two conditions.
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\[(\Delta \omega, \alpha) = (d\omega, d\alpha) + (d^* \omega, d^* \alpha). \tag{IV.3}\]

It is also a well known fact that the Hodge star operator interchanges these two conditions.

V. THE EXTENSION THEOREM

Here we show that there is a form of the Caffarelli-Silvestri theorem that works also for fractional Laplacians on forms.

Specifically we show

**Theorem V.1.** (Caffarelli-Silvestri for forms) Let $\omega$ a $p$-form in $\mathbb{R}^n$ which is in the domain the Hodge Laplacian $\Delta_x = d_x d_x^* + d_x^* d_x : \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^n)$. Let $\alpha \in \Omega^p(\mathbb{R}^n \times \mathbb{R}_+)$ be a bounded solution to the extension problem

\[
\begin{cases} 
\d(y^a d^a \alpha) + d^*(y^a d\alpha) = 0 \in M \times \mathbb{R}_+ \\
\alpha |_{\partial M} = \omega \text{ and } d^* \alpha |_{\partial M} = d_x^* \omega.
\end{cases} \tag{V.1}
\]

Then

\[
\lim_{y \to 0} y^{1-2a} i_{\nu} d\alpha = C_{n,a}(\Delta)^a \omega, \tag{V.2}
\]

with $\nu = \frac{\partial}{\partial y}$, for some positive constant $C_{n,a}$

**Proof.** The moral of the strategy would be to make use the fact that we have shown that in $\mathbb{R}^n$ the fractional Laplacian on forms is given by

\[
(\Delta_x)^a \alpha = \left( -\sum_{m=1}^n \frac{\partial^2}{\partial x^m \partial x^m} \right)^a \alpha \bigg|_{x^1 \cdots x^n} \ dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{V.3}
\]

The way in which we implement the proof is to show, more directly, that the equation on forms reduces to Caffarelli and Silvestri equations on components.
In order to proceed, we choose coordinates \(x^1, \ldots, x^n\) on \(\mathbb{R}^n \times \mathbb{R}_+\). Also, if we write \(\alpha = \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \alpha_{0 \ell_1 \ldots \ell_{p-1}} dy \wedge dx^{\ell_1} \wedge \cdots \wedge dx^{\ell_{p-1}},\) a straightforward calculation yields

\[
d\alpha = \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial y} dy \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^p \frac{\partial \alpha_{0 \ell_1 \ldots \ell_{p-1}}}{\partial x^k} dy \wedge dx^{\ell_1} \wedge \cdots \wedge dx^{\ell_{p-1}} \wedge dx^k.
\]

hence

\[
i_* d\alpha = \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial y} dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \sum_{k=1}^p \frac{\partial \alpha_{0 \ell_1 \ldots \ell_{p-1}}}{\partial x^k} dx^{\ell_1} \wedge \cdots \wedge dx^{\ell_{p-1}} \wedge dx^k.
\]

This shows that in order to prove the theorem, we merely need to show that

\[
\lim_{y \to 0} y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial y} = C_{n,a} (-\Delta)^a \omega_{i_1 \ldots i_p}, \quad \text{and} \quad \lim_{y \to 0} y^a \frac{\partial \alpha_{0 \ell_1 \ldots \ell_{p-1}}}{\partial x^k} = 0.
\]

For the purposes of the next few calculations we set \(y = x^{n+1}\) so as to make the notation less cumbersome. From equation (II.1) it follows immediately that

\[
d(y^n d^* \alpha) = \sum_{\ell=1}^{p} (-1)^\ell \frac{\partial}{\partial x^{i_\ell}} \left( y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^{i_\ell}} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_p}
\]

\[+ \sum_{\ell=1}^{p} \sum_{k=1}^{n+1-p} (-1)^\ell \frac{\partial}{\partial x^k} \left( y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^{i_\ell}} \right) dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{V.4}\]

and

\[
d^*(y^n \varpi) = \sum_{k=1}^{n+1-p} (-1)^k \frac{\partial}{\partial x^k} \left( y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^k} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_p}
\]

\[+ \sum_{k=1}^{n+1-p} \sum_{\ell=1}^{p} (-1)^{\ell+1} \frac{\partial}{\partial x^{i_\ell}} \left( y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^k} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \wedge \cdots \wedge dx^{i_p}. \tag{V.5}\]

Putting equations (V.4) and (V.5) together yields

\[
d(y^n d^* \alpha) + d^*(y^n \varpi) = -\sum_{\ell=1}^{n+1} (-1)^\ell \frac{\partial}{\partial x^{i_\ell}} \left( y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x^{i_\ell}} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{V.6}\]

Observing that the righthand side of equation (V.6) is none other than \(\text{div}(y^n \nabla \alpha_{i_1 \ldots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p}\) we can then write equation (V.4) as

\[
\begin{cases}
\text{div}(y^n \nabla \alpha_{i_1 \ldots i_p}) = 0 & \text{in } M \times \mathbb{R}_+ \\

(\alpha_{i_1 \ldots i_p})_{\vert \partial M} = \omega_{i_1 \ldots i_p} \quad \text{and} \quad d^* \alpha_{\vert \partial M} = d^* \omega.
\end{cases} \tag{V.7}\]

Therefore, using the CS theorem, we have that

\[
\lim_{y \to 0} y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial y} = C_{n,a} (-\Delta)^a \omega_{i_1 \ldots i_p}, \tag{V.8}\]

which proves that

\[
\lim_{y \to 0} y^a i_* d\alpha = (\Delta)^a \omega,
\]
since by (elliptic) regularity of solutions to equation (V.9)

$$\lim_{y \to 0} y^a \frac{\partial \alpha \ell_1 \cdots \ell_{p-1}}{\partial y} = 0.$$  

Another (more invariant) way to proceed is as follows. We decompose the equations for $\alpha$ by separating out (aided by Kodaira’s decomposition) the form $\alpha$ into two parts $\alpha = \alpha_1$ and $\alpha_2$ such that $d\alpha_1 = 0$ and $d^*\alpha_2 = 0$. We now focus on the equations that $\alpha_2$ satisfies. $\alpha_1 = 0$ and by abuse of notation, we write $\alpha$ for $\alpha_2$. So, we can first make the assumption that $d^* x \omega = 0$, which because of the Dirichlet boundary condition implies that $d^* \alpha |_{\partial M} = 0$. We make now the observation that from Eqs. (II.2) and (II.3), it follows that

$$\left\{ \begin{array}{l} d^*(y^a d\alpha) = 0 \in M \times \mathbb{R}_+ \\ \alpha |_{\partial M} = \omega \text{ and } d^* \alpha |_{\partial M} = 0, \end{array} \right. \quad \text{(V.9)}$$

with $d^* \omega = 0$. In this case, using Eq. (V.3), we can rewrite Eq. (V.9) as

$$\left\{ \begin{array}{l} \text{div}(y^a \nabla \alpha_{i_1 \cdots i_p}) = 0 \in M \times \mathbb{R}_+ \\ (\alpha_{i_1 \cdots i_p}) |_{\partial M} = \omega_{i_1 \cdots i_p} \text{ and } d^* \alpha |_{\partial M} = 0. \end{array} \right. \quad \text{(V.10)}$$

The rest is as above.

VI. HOLOGRAPHIC SCALING

We recall that in [3] and [4] one calculates the effective holographic theories in order to study the IR regime of boundary strongly-coupled theories. The significant part of the effective action is then

$$S = \int d^{d+2}x \sqrt{-g} \left[ R - \frac{\partial \phi^2}{2} - \frac{Z(\phi)}{4} F^2 + V(\phi) \right], \quad \text{(VI.1)}$$

where the quantities $Z$ and $V$ are taken to have the asymptotics

$$\left\{ \begin{array}{l} Z(\phi) \rightarrow \phi \to \infty Z_0 e^{\gamma \phi} \\ V(\phi) \rightarrow \phi \to \infty V_0 e^{-\delta \phi}. \end{array} \right. \quad \text{(VI.2)}$$

In the special but all-telling case

$$\left\{ \begin{array}{l} Z(\phi) = Z_0 e^{\gamma \phi} \\ V(\phi) = V_0 e^{-\delta \phi}, \end{array} \right. \quad \text{(VI.3)}$$

yields the following field equations

$$R_{\mu \nu} + \frac{2}{\phi} F_{\mu \rho} F^\rho_{\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \quad \text{(VI.4)}$$

$$+ \frac{Z}{\phi} \left[ \frac{1}{2} (\partial \phi)^2 - V - \frac{2}{\phi} F^2 \right] = 0, \quad \text{(VI.5)}$$

$$\square \phi = \frac{1}{2} Z(\phi) F^2 + 2 - V'(\phi), \quad \text{(VI.6)}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} Z(\phi) F^{\mu \nu}) = 0. \quad \text{(VI.7)}$$

We concentrate on Maxwell’s equations,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} Z(\phi) F^{\mu \nu}) = 0. \quad \text{(VI.8)}$$

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One class of solutions found in \[4\] is
\[
ds^2 = r^{2\theta} \left( \frac{L^2 dr^2 + dR^2(r)}{r^2} - \frac{dt^2}{r^{2\theta}} \right), \quad A = Q_0 r^{\zeta - \xi - z} dt, \quad \phi = r^{\pm \kappa},
\]
\[
L_2 V_0 = (d - 1 + z - \theta)(d + z - \theta) + (z - 1)\xi, \quad Q_0^2 = \frac{2(z - 1)}{Z_0(z - \zeta + \xi)}, \tag{VI.9}
\]
\[
\kappa = \sqrt{2(1 - z)(\zeta - \xi) + \frac{2}{d} \theta (\theta - d)}, \quad \delta \kappa = \pm \frac{2\theta}{d}
\]
\[
\gamma \kappa = \pm 2 \left( \frac{1}{d} \theta - \zeta + \xi \right), \quad \epsilon = \gamma - \delta.
\]

Significantly in all the solutions found in \[3, 4, 12\], the dilaton is such that \(e^\phi = r^{\pm \kappa}\), which as remarked in the introduction is identical to the Domokos/Gabadze\[10\] mechanism. Consequently, the main application of this note is incarnated in the fact that a consequence of Theorem V.1 is that this action induces the fractional Maxwell equations at the boundary
\[
\Delta^a A^t = 0,
\]
where \(A^t\) is the tangential (boundary) component and \(a\) is determined by \(\kappa\). This is of course a straightforward application of the afore-mentioned theorem, given that Eq. (VI.8) is of the form of Theorem V.1. Consequently, the dimension of the gauge field at the boundary is indeed anomalous: \([A^t] = a\). Analogously the associated current also has an anomalous dimension: \([J] = d - 1 - a\).

We see clearly then that it is the Caffarelli/Silvestre mechanism that accounts for the generation of anomalous dimensions in the boundary “gauge” theory in holographic constructions. When dilatons are absent from the bulk, that is \(a = 1\), the standard result obtains\[8, 9, 13\] in which anomalous dimensions are absent.

### VII. GAUGE GROUP AND CURRENTS

In the standard \(U(1)\) gauge theory, the local action of the gauge group is given by transforming the complex sections in the Dirac Lagrangian (or any other gauge theory) via \(\phi \rightarrow e^{i\xi} \phi\) for some (real) function \(\xi\). Such transformations are the correct ones in terms of the covariant derivative \(D\), locally given by \(d + iqA\), in that \(D\phi\) transforms as \(e^{i\xi} D\phi\), provided that \(A\) transforms as \(A_\mu \rightarrow A_\mu + \partial_\mu \xi\). A global symmetry would be given by a similar transformation \(\phi \rightarrow e^{i\rho} \phi\), where \(\rho \in \mathbb{R}\) is a real number. This symmetry would clearly leave the Lagrangian unchanged, and just change the phase of \(\phi\) by a constant factor. It would also leave the connection unchanged (and as a consequence not change any gauge). A local transformation of the form \(e^{iq\xi} \phi\) leaves unchanged the Dirac Lagrangian
\[
\mathcal{L} = -\overline{\psi} \gamma_\mu A^\mu \phi - m \overline{\phi} \phi.
\]

If \(A\) were a classical \(U(1)\) Gauge field, then \(\phi \rightarrow \phi' = e^{iq\xi} \phi\) is exactly what one needs and in order for the Lagrangian to be unchanged under \(A \rightarrow A' = A + d\xi\) as the Gauge transformation on the connection 1-forms \(A\). If \(A\) is a fractional field though, we need to take a local transformation on matter that respects the fractional Gauge transformation,
\[
A \rightarrow A' = A + d_a \xi.
\]

In order to achieve this, since \(d_a = d(\Delta)^{\frac{d_a}{d}}\), we simply take as a local transformation
\[
\phi \rightarrow \phi' = e^{iq(\Delta)^{\frac{d_a}{d}} \xi} \phi. \tag{VII.1}
\]
Since $D_A = d - iqA$, one readily calculates that with this local action, $D_A$ transforms under such local transformations as

$$D_A \rightarrow e^{iq(-\Delta)^{-\frac{1}{2}}\xi}D_{A'},$$  \hspace{1cm} (VII.2)

where $A' = A + da\xi$, thus leaving the Dirac Lagrangian unchanged. This is readily seen by the calculation

$$(d + A')\phi' = (d + A')\left(e^{iq(-\Delta)^{-\frac{1}{2}}\xi}\phi\right) = e^{iq(-\Delta)^{-\frac{1}{2}}\xi}(d\phi + iqda\xi\phi + A'\phi).$$  \hspace{1cm} (VII.3)

We can turn this into the discussion which essentially appeared in [14] by considering the 1-form $\alpha$ (which is $a_\mu$ in [14]) defined by

$$\Delta^a\alpha = A$$  \hspace{1cm} (VII.4)

and then consider the Lagrangian

$$L = -\bar{\psi}D_\alpha\psi - m\bar{\psi}\psi,$$

where now $D_\alpha = d - iq\alpha$, and $q$ is unitless. Then, if $\Lambda$ is defined through

$$\Delta^\frac{q}{2}\xi = \Lambda,$$  \hspace{1cm} (VII.5)

we have that $\alpha$ transforms after the local action $\psi \rightarrow e^{iq\Lambda}\psi$, as

$$\alpha \rightarrow \alpha' = \alpha - iq\Lambda.$$

On a related note, the dual fields in the boundary theory[8, 9] are currents. As such, these are differential forms with no local gauge group action. What we have shown in this paper is that the currents in the boundary theory for a bulk theory containing a dilaton coupling are currents generated by a fractional “gauge” theory.

**VIII. FINAL REMARKS**

We have developed the notion of fractional differentiation of p-forms via the fractional Laplacian. Such an operator naturally appears anytime elliptic differential equations are recast in a spacetime with one lower dimension. What we have shown here is that the same is true for p-forms. Since all holographic constructions[3, 4] to date result in equations of motion that are identical to those that underlie the p-form generalization of the CS extension theorem, fractional Maxwell equations naturally result at the boundary. This result then lends credence to work[14] which was based on the intuition that anomalous dimensions for gauge fields results in a non-local gauge-invariant condition. This work lays plain the precise form of the non-locality involves the fractional Laplacian ala

$$A_t \rightarrow A_t + d,\Lambda$$

with $d_\gamma \equiv (\Delta^{-\frac{q}{2}}d$ rather than the standard fractional derivative. That the boundary theory yielding an anomalous dimension must involve the fractional Laplacian is not unexpected since the Ward identities explicitly preclude anomalous dimensions from purely local gauge theories. What this work demonstrates is that anomalous dimensions for gauge fields are a signature that the quantum field theory is necessarily a boundary theory on some manifold.

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