Optimization Induced Deep Equilibrium Networks

Xingyu Xie*† Qiuhao Wang*† Zenan Ling†‡ Xia Li‡ Yisen Wang† Guangcan Liu¶
Zhouchen Lin†
zlin@pku.edu.cn

Abstract

Implicit equilibrium models, i.e., deep neural networks (DNNs) defined by implicit equations, have been becoming more and more attractive recently. In this paper, we investigate an emerging question: can an implicit equilibrium model’s equilibrium point be regarded as the solution of an optimization problem? To this end, we first decompose DNNs into a new class of unit layer that is the proximal operator of an implicit convex function while keeping its output unchanged. Then, the equilibrium model of the unit layer can be derived, named Optimization Induced Equilibrium Networks (OptEq), which can be easily extended to deep layers. The equilibrium point of OptEq can be theoretically connected to the solution of its corresponding convex optimization problem with explicit objectives. Based on this, we can flexibly introduce prior properties to the equilibrium points: 1) modifying the underlying convex problems explicitly so as to change the architectures of OptEq; and 2) merging the information into the fixed point iteration, which guarantees to choose the desired equilibrium point when the fixed point set is non-singleton. We show that deep OptEq outperforms previous implicit models even with fewer parameters. This work establishes the first step towards optimization guided design of deep models.

1 Introduction

Recently, the implicit models have gained significant attention, since they have been demonstrated to match or exceed the performance of traditional deep neural networks (DNNs) while consuming much less memory. Instances of such models include Neural ODE [11, 20], whose implicit equation is the ODE of a continuous-time dynamical system, and Deep Equilibrium (DEQ) Model [6, 7], which tries to solve a nonlinear fixed point equation. Instead of specifying the explicit computation procedure, the implicit equilibrium model specifies some conditions that should be held at its solution. Thus the implicit equilibrium model’s output is created by finding a solution to some implicitly defined equation. This is quite suitable to study from the optimization perspective, since most convex problems have their equivalent monotone inclusion form, i.e., finding the zero set of the sum of maximal monotone operators.

Some previous work aims at explaining the forward propagation of DNNs as optimization iterations. Li et al. [18] show that DNN’s activation function is indeed a proximal operator if it is non-decreasing. Other works study unrolled networks for optimization [4, 1], i.e., designing neural networks following the optimization algorithms. However, given a general DNN architecture,
whether there exists an underlying optimization problem remains open. If we can find such an underlying optimization problem explicitly, it may help us further understand the mechanism of DNN and the reasons for its empirical success. We can even introduce the customized property into the DNN by adding some regularization terms to the underlying optimization objective, which may help to inspire more effective DNN architectures. Moreover, when finding the model’s output is equivalent to minimizing a convex objective, we can utilize any optimization algorithm, especially the accelerated ones, to obtain the equilibrium point rather than limited root find methods.

In this paper, we investigate the emerging problem of discovering the underlying optimization problem of DNN. We first reformulate the general multi-layer feedforward DNN and decompose it into the composition of a new class of unit layers while keeping the output unchanged. The unit layer is the proximal operator of an underlying implicit convex function under a mild condition. Thus the underlying convex function determines its behavior completely. We propose the corresponding equilibrium model of the new unit layer, named Optimization Induced Equilibrium Networks (OptEq). By replacing the convex objective with its Moreau envelope, OptEq naturally produces the commonly used skip connection architecture while keeping the equilibrium point unchanged. To strengthen the representation ability, we further propose a deep version of OptEq, which includes the general feedforward DNN as a special case. We also provide the underlying convex optimization problem of deep OptEq, whose objective is the sum of Moreau envelopes of several proper convex functions. Moreover, we can introduce any customized property to the equilibrium of OptEq, e.g., feature disentanglement. We propose two methods for such feature regularization. The first way is to modify the underlying optimization problem directly, which will lead to changes in the architecture of OptEq. Another method is using a modified SAM [26] iteration to select the fixed point with minimum regularization when the fixed point set is non-singleton. In summary, our main contributions include:

- By decomposing the general DNN, we propose a new class of unit layer, which is the proximal operator of a convex function. Then we extract the unit layer to make it an equilibrium model called OptEq, and further propose its deep version. Without further reparameterization, the equilibrium point of OptEq is a solution to an underlying convex problem.
- We propose two methods to introduce customized properties to the model’s equilibrium points. One is inspired by the underlying optimization, and the other is induced by a modified SAM [26] iteration. This is the first time that we can customize deep implicit models in a principled way.
- We conduct experiments on CIFAR-10 for image classification and Cityscapes for semantic segmentation. Deep OptEq significantly outperforms implicit baseline models. Moreover, we also provide several feature regularizations that can significantly improve the generalization.

2 The Proposed Optimization Induced Equilibrium Networks

2.1 Preliminaries and Notations

We provide some definitions that are frequently used throughout the paper. A function \( f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\} \) is proper if the set \( \{x : f(x) < +\infty\} \) is non-empty, where \( \mathcal{H} \) is the Euclidean space. We write l.s.c for short of lower semi-continuous. The subdifferential, proximal operator and Moreau envelope of a proper convex function \( f \) are defined as:

\[
\partial f(x) := \{g \in \mathcal{H} : f(y) \geq f(x) + \langle y - x, g \rangle, \forall y \in \mathcal{H}\},
\]

\[
\text{prox}_f(x) := \{z \in \mathcal{H} : z = \text{argmin} \frac{1}{2\mu} \|u - x\|^2 + f(u)\},
\]

\[
M_f^\mu(x) := \min_u \frac{1}{2\mu} \|u - x\|^2 + f(u),
\]

respectively, where \( \langle \cdot, \cdot \rangle \) is the inner product and \( \|\cdot\| \) is the induced norm. The conjugate \( f^* \) of a proper convex function \( f \) is defined as: \( f^*(y) := \sup \{\langle y, x \rangle - f(x) : x \in \mathcal{H}\} \). For the matrix \( W \), \( \|W\|_2 \) is the operator norm. While for the vector \( \ell_2 \)-norm, we write \( \|\cdot\| \) for simplicity. Given the map \( T : \mathbb{R}^m \to \mathbb{R}^m \), the set of all fixed points of \( T(\cdot) \) is denoted by \( \text{Fix}(T) = \{z \in \mathbb{R}^m : T(z) = z\} \), whose cardinality is \( |\text{Fix}(T)| \).
DEQ is a recently proposed implicit model inspired by the observation on the feedforward DNN (with an input-skip connection): for $k = 1, \ldots, L - 1$,

$$z_{k+1} = \sigma(W_k z_k + U_k x + b_k), \quad y = W_{L+1} z_L,$$

(1)

where $\sigma(\cdot)$ is a non-linear activation function, $W_k \in \mathbb{R}^{n_k \times n_{k-1}}$ and $U_k \in \mathbb{R}^{n_k \times d}$ are learnable weights and $b_k \in \mathbb{R}^{n_k}$ is a bias term. A direct way to obtain the equilibrium point of this system is to consider the fixed point equation: $z^* = \sigma(W z^* + U x + b)$. And we can utilize any root-finding algorithm to solve this equation. Although DEQ may achieve good performance with a smaller number of parameters than DNNs, its superiority heavily relies on the careful initialization and regularization due to the instability issue of the fixed point problems. Some recent work [32] is devoted to solving the instability issue by using a tricky re-parametrization of the weight matrix $W$. However, this may greatly weaken the expressive power of DEQ, see Prop. 8 in [24]. Moreover, in the present equation form, we have difficulty in getting further properties of the equilibrium point.

2.2 One Layer OptEq

Here, we consider an alternative form of the system in Eq. (1).

**Lemma 1 (Universal Hidden Unit).** Given the parameters $\{(W_k, U_k, b_k)\}_{k=1}^L$ of a general DNN in Eq. (1), there exists a set of weights $\{(\overline{W}_k) \in \mathbb{R}^{n_k \times m}\}_{k=0}^L$ with $m \geq 2 \max\{n_k, k = 0, \ldots, L\}$, such that the system in Eq. (1) can be re-written as the following network: for $k = 1, \ldots, L - 1$,

$$z_{k+1} = \overline{W}_k \sigma(\overline{W}_k z_k + U_k x + b_k), \quad y = \overline{W}_{L+1} z_L.$$

(2)

Notably, *without changing the output $y$, any feedforward DNN has the reformulation in Eq. (2). The formal proof can be found in appendix, we present the main idea here:

$$y = W_L \sigma(W_{L-1} \sigma(\cdots W_2 \sigma(W_1 z_0 + U_1 x + b_1) \cdots))
\quad \text{where} \quad W_1 = \overline{W}_1, \quad \ldots, \quad W_L = \overline{W}_L,
\quad \sigma(W z^* + U x + b) = z^* = W^T \sigma(W z^* + U x + b).$$

Hence, the feedforward DNN also inspires an interesting and different equilibrium model Eq. (3). We call it Optimization Induced Equilibrium Networks (OptEq) since it is tightly associated with an underlying optimization problem. As shown in Theorem 1 that follows, the equilibrium point $z^*$ is a solution of a convex problem that has an explicit formulation. From the perspective of optimization, we can easily solve the existence and the uniqueness problems of the fixed point equation, rather than resorting to the cumbersome reparameterization trick in [32]. Most importantly, by studying the underlying optimization problem, we can investigate the properties of the equilibrium point of OptEq. The following theorem formally shows the relation between OptEq and optimization.

**Assumption 1.** The activation function $\sigma : \mathbb{R} \to \mathbb{R}$ is monotone and $\bar{L}_\sigma$-Lipschitz, i.e.,

$$0 \leq \frac{\sigma(a) - \sigma(b)}{a - b} \leq \bar{L}_\sigma, \quad \forall a, b \in \mathbb{R}, \quad a \neq b.$$

**Theorem 1.** If Assumption 7 holds, for one NN layer $f : \mathbb{R}^m \to \mathbb{R}^m$ given by:

$$f(z) := \frac{1}{\mu} W^T \sigma(W z + U x + b),$$

we have $f(z) = \text{prox}_\varphi(z)$ when $\mu \geq \bar{L}_\sigma \|
W\|_2^2$, where

$$\varphi(z) = \psi^*(z) - \frac{1}{2} \|z\|^2, \quad \psi(z) = \frac{1}{\mu} \int_0^1 \sigma(t x + b) dt,$$

in which $\forall a \in \mathbb{R}, \quad \sigma(a) = \int_0^a \sigma(t) dt$, applied element-wisely to vectors, and $1$ is the all one vector. Furthermore, the solution to the fixed point equation $z = f(z)$ is the minimizer of the convex function $\varphi(\cdot)$. 

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Theorem 1 shows that unit layer given in Eq. (3) is a proximal operator of an underlying convex function given by a conjugate function in most cases, and the equilibrium point of OptEq happens to be the minimizer of this function. In the rest part of this paper, for ease of discussion, we focus on the case that \( \mu = 1 \) for OptEq, which may correspond to the assumption \( L_\sigma = 1 \) and \( \| W \|_2 \leq 1 \). In some cases, we can write down the closed form of the optimization objection \( \varphi(\cdot) \). For example, when the weight matrix \( W \) is invertible, and the activation \( \sigma(\cdot) \) is ReLU, i.e., \( \sigma(x) = \max \{ x, 0 \} \), \( \forall x \in \mathbb{R} \), we have:

\[
\varphi(z) = 1^\top \tilde{\sigma}^* (W^{-\top} z) - \langle U x + b, W^{-\top} z \rangle - \frac{1}{2} \| z \|^2,
\]

where \( \tilde{\sigma}^*(x) = \begin{cases} \frac{1}{2} x^2, & x > 0, \\ \infty, & x \leq 0 \end{cases} \) applied element-wisely to vectors.

By Theorem 1 we can quickly obtain the well-posedness of OptEq. In general, any \( W \) that makes the underlying objective \( \varphi \) to be strictly convex will ensure the existence and uniqueness of OptEq’s equilibrium. For example, when \( \| W \|_2 < 1 \), the operator: \( z \mapsto W^\top \sigma(Wz + Ux + b) \) is contractive, therefore the fixed point equation Eq. (3) has a unique solution, i.e., it exists and is unique. What’s more, we show a training strategy that can actually deal with a much more general case — \( |\text{Fix}(T)| \geq 1 \). Please see Sec. 4.2 for more details.

OptEq’s most attractive aspect is that we can introduce customized properties or NN architecture to the model simply by modifying the underlying convex problem. For example, we can naturally introduce the commonly used skip connection structure only by replacing \( \varphi(z) \) with its Moreau envelope \( \alpha M_\varphi^{1/\alpha}(z) \), then OptEq becomes:

\[
z = \alpha W^\top \sigma(Wz + Ux + b) + (1 - \alpha) z.
\]

Note that we do not change the equilibrium point of OptEq but make the operator on the right hand side strongly monotone and invertible, which can stabilize the iteration.

### 2.3 Deep OptEq

Some work claims that one layer implicit equilibrium is enough and improves the model expressiveness by stacking small DEQs to obtain a wide one-layer DEQ, i.e., considering a fixed point problem in a higher dimension. However, in practice, it is difficult to solve a large scale fixed point problem. Hence, multi-layer DEQ, having a one-layer equivalent form in most cases, is a good substitute for the one-layer wide since it improves the model ability without changing the problem scale. Furthermore, multi-layer DEQ has a more powerful expressive ability when the output dimension is fixed. Indeed, as we will show in the next section, the wide one-layer DEQ may be a special case of the multi-layer DEQ in the asymptotic sense after fixing the output dimension. In this subsection, we propose a multi-layer version of OptEq, which is also associated to an underlying optimization problem.

We consider a multi-layer OptEq, where \( x_0 \in \mathbb{R}^{d_x} \) denotes the input, \( z \in \mathbb{R}^m \) denotes the hidden unit, and \( y \in \mathbb{R}^{d_y} \) denotes the output. Namely, deep OptEq follows the implicit equation:

\[
\begin{aligned}
x &= g(x_0, W_0), \\
z &= T(z, x, \theta) := f_L \circ f_{L-1} \circ \cdots \circ f_1(z, x, \theta), \\
y &= W_{L+1} z,
\end{aligned}
\]

where for all \( l \in [1, L] \) and \( \alpha \in (0, 1] \),

\[
f_l(z, x) = \alpha W_l^\top \sigma(W_l z + U_l x + b_l) + (1 - \alpha) z.
\]

(5)

Here, given the set of learnable parameters \( W_0, g(\cdot, W_0) : \mathbb{R}^{d_x} \to \mathbb{R}^d \) is any continuous function which we usually choose as a feature extractor, e.g., shallow NNs. \( \theta = \{ (W_l, U_l, b_l) \}_{l=1}^L \) is the set of all learnable parameters for our equilibrium network, where \( W_l \in \mathbb{R}^{n_l \times m} \), \( U_l \in \mathbb{R}^{n_l \times d} \), \( b_l \in \mathbb{R}^m \) are the learnable weight matrices and bias term, respectively. Note that \( W_{L+1} \in \mathbb{R}^{d_y \times m} \).

\(^{6}\)Besides the forms we show, one may expect a common rule to determine whether a general mapping is a proximal operator or not, which can help to create the implicit equilibrium model from the optimization perspective. We provide the sufficient and necessary conditions in Lemma 5 (see appendix).
is also learnable. \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is the activation function, when the input is multi-dimensional, we apply the function \( \sigma(\cdot) \) element-wise. The hidden unit \( z \) is the equilibrium point of the fixed point equation \( z = T(z, x, \theta) \) when \( x \) and \( \theta \) are given. Without loss of generality, we assume that the feature extractor satisfies a Lipschitz continuity assumption w.r.t. the learnable weight, i.e., 
\[
|g(x_0, W_1) - g(x_0, W_2)| \leq L_g\|W_1 - W_2\|_2.
\]

At the first glance, deep OptEq seems very different from the traditional DNNs. Some negative results on DEQ are shown in previous work [24]: with improper weight re-parameterization, DEQ does not contain any feedforward networks. By contrast, without complicated weight re-parameterization, deep OptEq can include the general feedforward DNN as its special case.

**Lemma 2.** The deep OptEqs contain all feedforward DNNs.

### 3 Recover Optimization Problem from Implicit Equilibrium Models

This section provides our main results on the connection between convex optimization problem and our deep OptEq. In the previous section, we have shown that one layer of DNN is a proximal operator under a mild assumption. However, the composition of multiple proximal operators is not a proximal operator in most cases. Fortunately, we can still recover the underlying optimization problem of deep OptEq, and find that its equilibrium point is a zero point of a convex function’s subdifferential with a weak additional permutation constraint. In addition, we can explicitly provide the optimization objectives corresponding to deep OptEq in some cases. Before providing the main results, we first show the connection between our deep OptEq given in Eq.(4) and a multi-block one-layer OptEq.

**Lemma 3** (Deep OptEq and Multi-block OptEq). If \( z^* := z_0^* \) is an equilibrium point of the equation \( z = f_L \circ f_{L-1} \circ \cdots \circ f_1(z, x, \theta) \), set \( z_1^* := f_1(z_0^*, x, \theta) \), \( z_2^* := f_2 \circ f_1(z_0^*, x, \theta) \), \( \cdots \), \( z_{L-1}^* := f_{L-1} \circ \cdots \circ f_2 \circ f_1(z_0^*, x, \theta) \), then \( \tilde{z}^* := [z_1^*, \cdots, z_{L-1}^*, z_0^*]^T \in \mathbb{R}^{mL} \) is an equilibrium point of the equation:
\[
\tilde{z} = \alpha \tilde{W}^T \tilde{\sigma}(\tilde{W}P\tilde{z} + \tilde{U}x + \tilde{b}) + (1 - \alpha)P\tilde{z}, \tag{6}
\]

where \( \tilde{W} \) is block diagonal and \( P \) is a permutation matrix,
\[
\tilde{W} := \begin{bmatrix}
    W_1 & \cdots & W_L
\end{bmatrix}, \quad P := \begin{bmatrix}
    0 & I & & \\
    I & 0 & & \\
    & & \ddots & \\
    & & & I
\end{bmatrix},
\]
\( \tilde{U} := [U_1, \cdots, U_L]^T \) and \( \tilde{b} := [b_1, \cdots, b_L]^T \) are the concatenated matrix and vector, respectively (see Eq.(13) in appendix for more details).

By Eq.(6), and the necessary and sufficient condition for one DNN layer to be a proximal-like operator (Lemma 3 in appendix), we can further reveal the connection between deep OptEq and optimization under a mild assumption.

**Assumption 2.** Assumption 1 with \( \tilde{L}_\alpha = 1 \) and \( \|W_i\|_2 \leq 1, \forall i \in [1, L] \) hold.

Note that we make this assumption just for the ease of discussion. The assumption is actually unnecessary since we can introduce an additional constant to re-scale the whole operator as did in Theorem 4.

**Theorem 2** (Recovering Optimization Problem from Deep OptEq). If Assumption 2 holds, then any equilibrium point \( \tilde{z}^* \) of Eq.(6) satisfies:
\[
0 \in \partial \Phi(\tilde{z}^*) + (I - P)\tilde{z}^*, \tag{7}
\]
where \( I \) is the identity matrix and \( \Phi(\tilde{z}^*) \) is given by a sequence Moreau envelopes of convex functions \( \{\varphi_i\}_{i=1}^L \) such that \( \text{prox}_{\varphi_i}(z) = W_i^T \sigma(W_i z + U_i x + b_i) \), namely:
\[
\Phi(\tilde{z}) = \sum_{i=1}^L \alpha M_{\varphi_i}^{1-\alpha}(z_i),
\]
where \( z_i \) is the \( i \)-th block of \( \tilde{z}^* \) and \( M_{\varphi_i}^{1-\alpha}(z) \) is the \( \varphi_i \)'s Moreau envelope.
When the block size is 1, i.e., \( L = 1 \), we can immediately obtain that \( 0 \in \partial \Phi(\bar{z}^*) \), namely, the equilibrium point is a solution of a convex optimization problem. So the results provided in Theorem 1 is a special case here. Note that one block does not mean that \( z \) is one-dimensional. Moreover, for two blocks, the deep OptEq is also an optimization solver.

**Corollary 1.** If the block size \( L = 2 \) and Assumption 2 holds, then the equilibrium point \( \bar{z}^* = [z_1^*, z_2^*] ^{\top} \) of Eq.(5) is also a solution to a convex problem:

\[
\min_{z_1, z_2} \left\{ \alpha M_\phi^{-\alpha}(z_1) + \alpha M_\phi^{-\alpha}(z_2) + \frac{1}{2} \| z_1 - z_2 \|^2 \right\}.
\]

For general \( L > 2 \), we can also write down the monotone inclusion equation Eq.(7)’s underlying optimization problem when \( \alpha \to 0 \). Interestingly, this result implies the equivalence between the composited deep models and the wide shallow ones in the asymptotic sense.

**Theorem 3** (Connection between Wide and Deep OptEq). If Assumption 2 holds, and there is at least one \( \| W_i \|_2 < 1 \). Assume \( \bar{z}^*(\alpha) := [z_1^*(\alpha), \cdots, z_L^*(\alpha), \tilde{z}_0^*(\alpha)] ^{\top} \in \mathbb{R}^{mL} \) is the equilibrium point of Eq.(6). When \( \alpha \to 0 \), all \( z_i^*(\alpha) \)s tend to be equal, and the limiting point is the last entry \( y \) of the minimizer \( (x_1, \cdots, x_L, y) \) of the following optimization problem:

\[
\min_{x_1, \cdots, x_L, y} \left\{ \sum_{i=1}^{L} \left( \varphi_i(x_i) + \frac{1}{2} \| x_i - y \|^2 \right) \right\},
\]

where \( \varphi_i(\cdot) \) is the same as that in Theorem 2.

Theorem 3 implies that, when \( \alpha \to 0 \), the equilibrium point of the deep OptEq is the same as the solution of \( Lx = \sum_{i=1}^{L} W_i ^{\top} \sigma(Wix_0 + U_0x + b_i) \), which is a wide one-layer OptEq with multiple blocks. Given the output dimension and the same amount of learnable parameters, the wide multi-block OptEq is actually a special case of deep OptEq in the asymptotic sense. Hence, deep OptEq is more expressive than wide one-layer OptEq.

In general, we can still loosely treat the equilibrium point as a minimizer of an implicit optimization problem, since the only difference between the monotone inclusion equation \( 0 \in \partial \Phi(\bar{z}^*) \) and Eq.(7) is a weak constraint dominated by the operator \( (I - P)(\cdot) \), which aims to reduce the divergence between the multi-blocks.

### 4 Introducing Customized Properties to Equilibrium Points

By employing the underlying optimization problem, we can investigate the potential property of the equilibrium points. A more advanced way to use the connection between (deep) OptEq and optimization is to introduce some customized properties to equilibrium points, i.e., the feature learned by OptEq. Note that none of previous DEQs take into account the regularization of features, which has been proved to be effective both theoretically \([3, 2]\) and empirically \([33]\).

#### 4.1 Underlying Optimization Inspired Feature Regularization

As we mentioned in Sec.2.2, if we replace the underlying optimization objective with its Moreau envelope, OptEq will naturally have a skip-connection structure, which has been adopted in the construction of deep OptEq. Following this idea, when we modify the underlying optimization problem of deep OptEq, it should inspire more network architectures.

An exciting application of Theorem 2 is introducing customized properties to the equilibrium points of deep OptEq, because appending one layer after deep OptEq is equivalent to adding one term to the objective \( \Phi(\cdot) \). Specifically, if we modify \( \Phi(\bar{z}) \) to \( \Phi(\bar{z}) + \mathcal{R}_z(z_L) \), then deep OptEq becomes:

\[
z = T_{\mathcal{R}_z} \circ T(z, x, \theta),
\]

where \( T_{\mathcal{R}_z} = \text{prox}_{\mathcal{R}_z} \) or \( T_{\mathcal{R}_z} = I - \gamma \frac{\partial \mathcal{R}_z}{\partial z} \) when the proximal is hard to calculate. For example, \( \mathcal{R}_z \) will re-scale \( T(z, x, \theta) \)’s output when \( \mathcal{R}_z(\cdot) = \| \cdot \|^2 \), and becomes a shrinkage operator after changing \( \mathcal{R}_z \) to \( \| \cdot \|_1 \). In general, \( \mathcal{R}_z(\cdot) \) can be any convex function that contains the prior information of the feature. In summary, we introduce feature regularization by modifying the underlying optimization problem, which leads to a change of network structure. Once again, studying the implicit equilibrium models from the perspective of optimization shows great superiority.
4.2 SAM Iteration Induced Feature Regularization

In this subsection, we provide another strategy for feature regularization. Note that most previous DEQs are devoted to ensuring a singleton fixed point set, relying on the tricky weight matrix reparameterization. Considering the general case — $|\text{Fix}(T)| \geq 1$, we can choose the equilibrium with desired property by solving the following constrained optimization problem:

$$z^*(x, \theta) := \underset{z \in \text{Fix}(T(x, \theta))}{\text{argmin}} R_z(z), \quad (8)$$

where $R_z(\cdot)$ is the feature regularization that contains the prior information of the feature. Given the training data $(x_0, y_0) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$, the whole training procedure becomes

$$\min_{\theta} \ell(W_{L+1} \cdot z^*(x, \theta), y_0) + R_w(\tilde{\theta}), \quad (9)$$

where $\tilde{\theta} := \{W_0, \theta, W_{L+1}\}$. $x$ is given by Eq. (4), $R_w(\cdot)$ is the regularizer on the parameters, e.g., weight decay, and $\ell : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^+$ is the loss function.

We adopt the SAM fixed point iteration [26] to solve the problem Eq. (8). Starting from any $z_0 \in \mathbb{R}^m$, we consider the following sequence $\{z^k\}_{k \in \mathbb{N}}$:

$$z^k = \beta_k S_{\lambda_k}(z^{k-1}) + (1 - \beta_k) T(z^{k-1}, x, \theta), \quad (10)$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ and $\{\lambda_k\}_{k \in \mathbb{N}}$ are sequences of real numbers in $(0, 1]$, $S_{\lambda}(z) = (1 - \gamma\lambda)z - \gamma \frac{\partial R_z(z)}{\partial z}$. Then we choose the $K$-th iteration $z^K(x, \theta)$ as an approximate of $z^*(x, \theta)$ and put it in the final loss term:

$$\min_{\theta} \ell(W_{L+1} \cdot z^K(x, \theta), y_0) + R_w(\tilde{\theta}). \quad (11)$$

The unrolling term $z^K(x, \theta)$ aggregate information from both $R_z(\cdot)$ and $T$, making the prior information of feature being an inductive bias during training. And our model can be easily trained by any first-order optimization algorithms, e.g., GD, SGD, Adam, etc.

**Remark 1.** SAM iteration [26] needs $R_z(\cdot)$ to be strongly convex, here we use a modified version of SAM which only assumes convexity. Given well-chosen $\{\beta_k\}_{k \in \mathbb{N}}$ and $\{\lambda_k\}_{k \in \mathbb{N}}$, we can prove that the sequence generated by Eq. (10) converges to the point $z^*(x, \theta)$. Furthermore, we prove that the whole training dynamic, using the unrolling SAM strategy with backpropagation (BP), converges with a linear convergence rate. We leave all the convergence results in appendix.

**Remark 2.** If we use the method in Sec [7] to introduce the prior information, there is no need to use SAM iteration again. Therefore, we can let $\beta_k = 0$, $S_{\lambda_k}(\cdot) = 0$, and use the unrolling fixed point iteration strategy during training. Note that we can also use the implicit function theorem (IFT) based training way.

Previous implicit models utilize IFT in training to avoid the storage consumption of forward-propagation. The cost for that is it needs to solve two large-scale linear equations (or perform the matrix inversion directly) during training. Deep OptEqs can be trained both in the unrolling based and IFT based ways. In the sense of BP, the two training ways have different merits and limitations. We provide comparative experiments in Sec [5] and detailed discussion in appendix.

5 Experiments

In this section, we investigate the empirical performance of deep OptEq from three aspects. First, on the image classification problem, we evaluate the performance of deep OptEqs along with our feature regularization strategies. The results trained with different $\alpha$s are also reported. Second, we compare deep OptEqs with previous implicit models and traditional DNNs. Finally, we compare our unrolling-based method with the IFT-based method and investigate the influence of unrolling iteration number $K$. Furthermore, we present the results on Cityscapes for semantic segmentation.

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For the sake of clarity, we utilize one training pair for discussion. In general, all the discussed results hold when we replace the single data point with the whole data set.
Table 1: (a) The testing accuracy (Acc.) of deep OptEq with different settings. We set different \( \lambda \) for \( \mathcal{R}_z^\ast \) and the mean values are taken on the whole feature tensor. (b) Comparisons with previous implicit models.

![Table 1](attachment:image.png)

**5.1 Performance of Different Feature Regularizations**

We construct the deep OptEqs with 5 convolutional layers, using five \( 3 \times 3 \) convolution kernels with channels of 16, 32, 64, 128, 128. In this experiment, we compare the performance of two ways to introduce the feature regularization on CIFAR-10 dataset. We adopt the feature decorrelation as the \( \mathcal{R}_z \) here. The specific formulation can be found in appendix. We also use two norm regularizations to show whether there is a corresponding effect on the learned feature of deep OptEq. Moreover, we show how the hyperparameter \( \alpha \) affects the model performance.

The results are shown in Table 1(a). With the same size of parameters, deep OptEqs beats the general DNN (given in Eq.11) easily. It turns out that there is no linear relationship between the performance and the hyperparameter \( \alpha \). The hyperparameter \( \alpha \) serves as a trade-off between the effect of fixed point equation and the regulation induced operator \( S \). In our setting, with a small initialization for \( \{W_i\} \)s, all weights will stay in a small compact set during training (see proof of Theorem 5). Therefore when \( \alpha \) approaches 1, deep OptEq is an intense contraction (i.e., with a small contractive coefficient), and the SAM iterations will quickly converge to the fixed point, in which case regularization induced operator \( S \) has a limited impact. When \( \alpha \) approaches 0, deep OptEq is to be more like the identity operator, so \( S \) dominates the whole iterations.

The optimization inspired implicit regularization (\( \mathcal{R}_z^\ast \)) is also an efficient feature regularization way since it modifies deep OptEq structure directly. Here we present two \( \mathcal{R}_z^\ast(\cdot) \) candidates: \( \lambda \|\cdot\|_1 \) and \( \lambda \|\cdot\|_2^2 \). The outputs of the feature show decreases in the corresponding norm, and a suitable regularization coefficient can lead to better generalization performances.

**5.2 Comparison with Previous Implicit Models**

In this experiment, we compare deep OptEq with other implicit models, NODEs [11], Augmented NODEs [12], single convolutional Monotone DEQs [32] (short as MON), and classical ResNet-18 [16]. We leave the definition of the HSIC regularization in appendix. Note that deep OptEqs do not require the additional re-parameterization like MONs [32]. For fair comparisons, we construct the deep OptEqs with 5 convolutional layers. In order to construct deep OptEqs with a similar number of parameters as baseline methods, we use five \( 3 \times 3 \) convolution kernels with channels of
16, 32, 64, 64, 128. Moreover, we only use a single convolutional layer as the feature extractor \( g(\cdot) \) for the model with 162k parameters, which is the same as single convolution MONs [32].

The results are shown in Table 1(b). Notably, even without feature regularization trick, our deep OptEqs significantly outperform baseline methods. We highlight the performance of deep OptEqs on CIFAR-10 which outperforms Augmented Neural ODE by 25.1% and MON by 11.0% with fewer parameters. Without adding the number of parameters, feature regularization helps deep OptEq to achieve better performance easily. Notably, HSIC, a feature disentanglement regularization, provides a significant gain for the generalization.

5.3 Efficiency and Approximation Error

We train deep OptEqs by the IFT based way given in Bai et al. [6] and compare the results with the unrolling way. The time for inference and BP is provided, and it is the total time for 80 iteration steps with the batch size being 125 on GPU NVIDIA GTX 1070. The relative residual is averaged over all test batches: \( \|z^K - \mathcal{T}(z^K, x, \theta)\|_2 / \|z^K\|_2 \). For fair comparison, we do not utilize any feature regularization in this experiment. We set \( \alpha = 0.8 \) and let “thd” represent the residual threshold.

| Table 2: Comparison between SAM and IFT based Training (# params 199k) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|
| Method                  | Acc.            | Inference Time  | Back-Prop Time  | Relative Residual |
| Unrolling(K=5)          | 83.52%          | 1.3s            | 1.8s            | 1.22e-02         |
| Unrolling(K=10)         | 87.28%          | 2.2s            | 3.3s            | 4.88e-03         |
| Unrolling(K=20)         | 87.71%          | 3.9s            | 6.4s            | 4.81e-04         |
| Unrolling(K=40)         | 87.83%          | 7.5s            | 12.7s           | 1.20e-05         |
| IFT (thd = 1e-03)       | 87.63%          | 16.3s           | 6.7s            | 7.33e-04         |
| IFT (thd = 1e-02)       | 85.95%          | 15.6s           | 1.3s            | 9.52e-03         |

Although IFT based methods consume much less memory, given the comparable relative residual, the unrolling methods achieve better performance with much less inference and BP time. Note that a loose residual threshold may destroy the IFT based method significantly. We should choose the appropriate training method according to practice. For IFT, the inference time is longer than the back-prop one since the fixed point equation needed to solve during inference is non-linear, which is more challenging than the linear one during BP.

5.4 Cityscapes Semantic Segmentation

In this experiment, we evaluate the empirical performance of our deep OptEq on a large-scale computer vision task: semantic segmentation on the Cityscapes dataset. We construct a deep OptEq with only three weighted layers and channels of 256, 512 and 512. The deep OptEq is used as the “backbone” of the segmentation network. We compare our method with FCN [27] on the Cityscapes test set. We employ the poly learning rate policy to adjust the learning rate, where the initial learning rate is multiplied by \((1 - \text{iter/total\_iter})^{0.9}\) after each iteration. The initial learning rate is set to be 0.01 for both networks. Moreover, momentum and weight decay are set to 0.9 and 0.001, respectively. Note that we only train on finely annotated data. We train the model for 40K iterations, with mini-batch size set as 8. The results on the validation set are shown in Table 3. Notably, our deep OptEq significantly outperforms FCN with a similar number of parameters. Note that in this experiment, we have not introduced any customized property of the feature, so the performance improvement is entirely due to the superiority of the implicit structure of deep OptEq.

| Table 3: Evaluation on the validation set of Cityscapes semantic segmentation. |
|-----------------|-----------------|-----------------|-----------------|
| Method          | mIoU            | mAcc            | aAcc            |
| FCN             | 71.47           | 79.23           | 95.56           |
| deep OptEq      | 74.47           | 81.91           | 95.93           |

6 Conclusions

In this paper, we decompose the feedforward DNN and find a more reasonable basic unit layer, which shows a close relationship with the proximal operator. Based on it, we propose new implicit models, OptEqs, and explore their underlying optimization problems thoroughly. We provide two strategies to introduce customized regularizations to the equilibrium points, and achieve significant performance improvement in experiments. We highlight that by modifying the underlying optimization problems, we can create more effective network architectures. Our work may inspire more interpretable implicit models from the optimization perspective.
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Appendix

This Supplementary material section contains more detailed experimental results with different regularizer settings, the technical proofs of main theoretical results, and some auxiliary lemmas.

A Additional Experimental Results

A.1 HSIC

In this section, we introduce HSIC regularizer, which is a feature disentanglement method.

HSIC is a statistical method to test independence. Compared with the decorrelation method we will present in the following section, HSIC can better capture the nonlinear dependency between random variables. We apply HSIC to the feature space. Many works \cite{30, 10} show that when the features learned by the network are uncorrelated, the model usually obtains a good generalization performance. For any pair of random variables $X = (x_1, \cdots, x_B), Y = (y_1, \cdots, y_B)$, where $B$ is the batch size, we utilize the biased finite-sample estimator of HSIC \cite{28}:

$$\text{HSIC}(X, Y) := (B - 1)^{-2} \text{tr}(K_X H K_Y H),$$

where $K_X$ and $K_Y$ are the kernel matrices w.r.t. Gaussian RBF kernel of $X$ and $Y$, and $H$ is the centering matrix $H = I - B^{-1} 1_B 1_B^T \in \mathbb{R}^{B \times B}$. Following \cite{35}, we aim to eliminate all the correlations between feature maps. To this end, our HSIC regularization is:

$$R_z(Z) = \sum_{1 \leq i < j \leq m} \text{HSIC}(Z_{i,:}, Z_{j,:}).$$

Note that HSIC is a nonparametric regularization term, so it does not increase the parameter size of deep OptEq. The computing cost of HSIC grows as the batch size and feature dimension increase. Some tricks, such as Random Fourier Features approximation \cite{35}, can be applied to speed up the calculation. In addition, Theorem 4 is only guaranteed for convex regularization, while HSIC regularization is non-convex. In this paper, we report the great empirical superiority of HSIC and leave the above issues to future work.

A.2 Different Settings for Regularizers

On the dataset CIFAR-10, this experiment detailedly shows the effect of different settings on regularizer $R_z^1$ (for Sec. 3.2), regularizer $R_z^2$ (for Sec. 4.1), and $\alpha$.

Regularizers. Here is the function we utilize to introduce the customized property to the equilibrium point, see Sec. 4 for more details. For the regularizer $R_z(\cdot)$ (both for $R_z^1$ and $R_z^2$), we set four different settings: (1) $R_z(Z) = \sum_{1 \leq i < j \leq m} \text{HSIC}(Z_{i,:}, Z_{j,:})$; (2) $R_z(z) = \frac{1}{2}||z||^2$; (3) $R_z(z) = 1/\left(||z||^2 + \epsilon\right)$ which is explored in \cite{44}; (4) Decorrelation: for the $B$-batch equilibrium points matrix $Z \in \mathbb{R}^{m \times B}$:

$$R_z(Z) = \frac{1}{2}||DZZ^T D - I||_F^2 := F_D(z),$$

where $D$ is a diagonal matrix whose non-zero entries are $\frac{1}{||z||}$ and $z^i$ is the $i$-th row of the matrix $Z$.

Note that $R_z(Z)$ here aims at reducing redundant information between feature dimensions, which has been discussed in \cite{25, 5}.

Settings. In this experiment, we set $K = 20$ and utilize weight decay to regularize the learnable parameters, i.e., $R_w(\cdot) = \xi ||\theta||^2$, where we choose $\xi = 3\epsilon - 4$. We utilize the commonly used SGD to train the model. We set the learning rate as 0.1 at the beginning and decay by 0.7 after every 20 epochs. And the total training epoch is 200. The batch size is 125 in this experiment. We construct the deep OptEqs with 5 convolutional layers, using five $3 \times 3$ convolution kernels with channels of 16, 32, 64, 128, 128. The total number of learnable parameters is 199k.

Results. The results with different regularizers are presented in Table 4. We can see that either adopting SAM iteration or changing the underlying convex optimization problem both improves classification performance. Note that, given the same type of regularization, modifying the underlying optimization problem, i.e., using $R^*_z$, usually make more improvements. Indeed, to modify the
underlying optimization problem, we need to change the architecture of deep OptEq, which has a more direct impact on the model than turning the training loss by $R_k$. When $\alpha = 0.01$, deep OptEq is almost equivalent to the one-layer wide OptEq (see Theorem 3), which is far outperformed by deep OptEq for $\alpha > 0.1$. Compared with other results, $\alpha = 0.1$ gives a poor result, which implies that the performance is not monotonic to parameter $\alpha$. Fortunately, from the table, setting $\alpha > 0.4$ is a safe choice. We notice that the overall performance of feature disentanglement methods (decoration and HSIC) are better than the other types of regularization terms whether we utilize it as $R^1_k$ or $R^\dagger_k$.

### B Discussion about the Training Methods

The IFT based implicit way utilizes the limited memory to train the model and is insensitive to the equilibrium point finding algorithms. However, it consumes much computation budget to solve the equation during the inference and BP. On the other hand, the way that unrolls the fixed point finding method may induce implicit bias [29, 23] and consumes much memory during training, but it is faster to infer and train. Note that implicit bias is a two-edged sword; The proposed SAM method can aggregate the information from the prior regularization and the fixed point equation. Hence, the implicit bias becomes a controllable inductive bias.

The tradeoff between memory and computing efficiency for the implicit and unrolling training methods is quite common in the other learning community, such as meta-learning [13, 22] and hyper-parameter optimization [21, 19]. Similarly, for DEQ, the two training ways are neither good nor bad. We should choose them in proper circumstances.

### C Convergence Analysis

This section offers the convergence results: (i) the sequence generated by Eq.(10) converges to some point $z^* \in \text{Fix}(T)$ such that $R_z(z^*) \leq R_z(z)$, $\forall z \in \text{Fix}(T)$; (ii) gradient descent can find a global minimum for the approximately implicit model in Eq.(11).

#### C.1 Approximation of Equilibrium Point

Note that we approximate the points $z^*(\theta, x)$ by the iterative steps in Eq.(10). In fact, we take the iterative step by extending an existing algorithm, called Sequential Averaging Method (SAM), which was developed in Hong-Kun [17] for solving a certain class of fixed-point problems, and then was applied to the bi-level optimization problems [26]. However, the existing SAM method can only deal with strongly convex $R_z(z)$. Our method is the first SAM type algorithm that can solve the general convex problem restricted to a nonexpansive operator’s fixed point set. The following theorem provides the formal statement and the required conditions. Since during the forward-propagation, $(\theta, x)$ is fixed, for the sake of convenience, we simplify $T(z, x, \theta)$ as $T(z)$.

**Theorem 4 (Convergence of Modified SAM Iterates).** Suppose that $\nabla R_z(z)$ is $L_z$-Lipschitz, and that for any $\beta \in [0, \frac{1}{2})$, $\lambda \in [0, \frac{L_z}{2}]$, the fixed point set of equation: $z = \beta(z - \gamma(\nabla R_z(z) + \lambda z)) + (1 - \beta)T(z)$ is uniformly bounded by $B^1_\gamma$ (in norm $|| \cdot ||$) w.r.t. $\beta$ and $\lambda$. Suppose that convex function...
Theorem 5 shows that GD converges to a global optimum for any initialization satisfying the boundedness assumption on the set Fix(\(T\)) is mild. Suppose that we have one weight that satisfies \(W_i \leq \zeta < 1\), then the fixed point of \(T\) exists and is unique, and is continuous w.r.t. the parameters \(\theta\). On the other hand, the parameters \(\theta\) will stay in a compact set during training. Therefore, the fixed points are uniformly upper bounded.

**C.2 Global Convergence of Implicit Model**

Most previous works on DEQs lack the convergence guarantees for their training. However, analyzing the learnable parameters’ dynamics is crucial since it may weaken many model constraints and greatly broader the function class that the implicit model can represent. For example, the one-layer DEQ, given in Winston and Kolter [32], maintains the positive definiteness of \(\mathbb{DEQ}\) instead of global space for the DEQ [32].

Theorem 5 shows that GD converges to a global optimum for any initialization satisfying the boundedness assumption within a local region instead of global space for the DEQ [32].

**Theorem 5 (Global Convergence (informal)).** Suppose that the initialized weight \(W_i\)’s singular values are lower bounded away from zero for all \(l \in [1, L + 1]\), and the fixed point set Fix(\(T(\cdot, X, \theta)\)) is non-empty and uniformly bounded for any \(\bar{\theta}\) in a pre-defined compact set. Assume that the activation function is Lipschitz smooth, strongly monotone and 1-Lipschitz. Define constants \(Q_0, Q_1\), and \(Q_3\), which depend on the bounds for initialization parameters, initial loss value, and the data-size. Let the learning rate be \(\eta < \min\{\frac{1}{Q_0}, \frac{1}{Q_1}\}\). If the training data size \(N\) is large enough, then the training loss vanishes at a linear rate as: \(\ell(\bar{\theta}^t) \leq \ell(\bar{\theta}_0)(1 - \eta Q_0)^t\), where \(t\) is the number of iteration. Furthermore, the network parameters also converge to a global minimizer \(\bar{\theta}^*\) at a linear speed: \(\|\bar{\theta}^t - \bar{\theta}^*\| \leq Q_3(1 - \eta Q_0)^t/2\).

Theorem 5 shows that GD converges to a global optimum for any initialization satisfying the boundedness assumption. In general, the lower bounded assumption on singular values is easy to fulfill. With high probability, the weight matrix’s singular values are lower bounded away from zero when it is a rectangle and has independent, sub-Gaussian rows or has independent Gaussian entries, see Thm.4.6.1 and Ex.7.3.4 in [31]. Similar to the remark after Theorem 4 the existence and boundedness assumption on the set Fix(\(T\)) is mild. Suppose that we have one weight that satisfies \(W_i \leq \zeta < 1\), then the fixed point of \(T\) exists and is unique, and is continuous w.r.t. the parameters \(\theta\). On the other hand, the parameters \(\theta\) will stay in a compact set during training. Therefore, the fixed points are uniformly upper bounded.

**D Proofs for the DNN Reformulation**

**D.1 Proof of Lemma 1**

The formal proof for Lemma 1 relies on the following auxiliary lemma.

**Lemma 4.** If \(k \geq 2 \max\{m, n\}\), given any \(W \in \mathbb{R}^{m \times n}\), and a full rank matrix \(A \in \mathbb{R}^{m \times k}\), there exists a full rank matrix \(B \in \mathbb{R}^{n \times k}\), such that \(W = AB^T\).

**Proof.** Considering the full SVD of \(A = U\Sigma V^T\), where \(U \in \mathbb{R}^{m \times m}\), \(\Sigma \in \mathbb{R}^{m \times k}\), and \(V \in \mathbb{R}^{k \times k}\). Let

\[
U^T W := \Omega = \begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\vdots \\
\Omega_m
\end{bmatrix}.
\]

Considering the equation:

\[
\Omega = \Sigma C.
\]
We can easily find that:

\[ C = \begin{bmatrix}
\Omega_1/\sigma_1 \\
\Omega_2/\sigma_2 \\
\vdots \\
\Omega_m/\sigma_m \\
\star
\end{bmatrix}, \]

is a solution, and we let \( \text{rank}(C) = n \) by adjusting \( \star \). We let \( B = C^T V^T \), hence, we can conclude that \( \text{rank}(B) = n \). It is easy to verify that \( W = AB^T \).

Recall that, we have:

\[ y = W_L \sigma \left( W_{L-1} \sigma \left( \cdots W_2 \sigma (W_1 z_0 + U_1 x + b_1) \cdots \right) \right) \]

\[ = \frac{W_L}{W_L^T} \left( \frac{W_{L-1}}{W_{L-1}^T} \sigma \left( \cdots \frac{W_2}{W_2^T} \sigma \left( \frac{W_1}{W_1^T} z_0 + U_1 x + b_1 \right) \cdots \right) \right). \]

Based on the above Lemma [1] the existence of \( \overline{W}_k \in \mathbb{R}^{n_k \times m} \) for all \( k \) can be easily guaranteed by giving a rank \( m \) matrix \( W_L \). Note that \( \forall k, \text{rank}(\overline{W}_k) = m \) and \( m \geq \max\{n_k, n_{k-1}\} \).

### D.2 Proof of Lemma 2

**Proof.** Set \( \alpha = 1 \) and let \( W_1 = 0, U_1 = U_3 = \cdots = U_L = 0 \) and \( b_i = 0 \) for all \( i \), we have:

\[ A_L \sigma \left( A_{L-1} \sigma \cdots \sigma (A_2 \sigma (A_1 x)) \right) = \frac{W_{L+1}}{W_{L+1}^T} \sigma \left( \frac{W_L}{W_L^T} \sigma \left( \cdots \frac{W_3}{W_3^T} \sigma \left( \frac{W_2}{W_2^T} \sigma \left( \frac{W_1}{W_1^T} z_0 + U_1 x + b_1 \right) \cdots \right) \right) \right). \]

The existence of \( \{W_i\}_{i=2}^{L+1} \) can be easily obtained by Lemma [4]. The bias term \( b_i \) can be easily included by changing each layer’s output \( x_i \) to \( [x_i; 1] \) and set \( A_i \) to \( \begin{bmatrix} A_i & b_i \\ 0 & 1 \end{bmatrix} \).

### E Proofs for the Connection between Optimization and OptEq

#### E.1 Conditions to be a Proximal Operator

**Lemma 5** (modified version of Prop. 2 in Gribonval and Nikolova [15]). Consider \( f : \mathcal{H} \rightarrow \mathcal{H} \) defined everywhere. The following properties are equivalent:

(i) there is a proper convex l.s.c function \( \varphi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) such that \( f(z) \in \text{prox}_{\varphi}(z) \) for each \( z \in \mathcal{H} \);

(ii) the following conditions hold jointly:

(a) there exists a convex l.s.c function \( \psi : \mathcal{H} \rightarrow \mathbb{R} \) such that \( \forall y \in \mathcal{H}, f(y) = \nabla \psi(y) \);

(b) \( \| f(y) - f(y') \| \leq \| y - y' \|, \forall y, y' \in \mathcal{H} \).

There exists a choice of \( \varphi(\cdot) \) and \( \psi(\cdot) \), satisfying (i) and (ii), such that \( \varphi(x) = \psi^*(x) - \frac{1}{2}\|x\|^2 \).

**Proof.** (i)⇒(ii): Since \( \varphi(x) + \frac{1}{2}\|x\|^2 \) is a proper l.s.c 1-strongly convex function, then by Thm. 5.26 in Beck [9], i.e, the conjugate function \( f^* \) is \( \frac{1}{\sigma} \)-smooth when \( f \) is proper, closed and \( \sigma \) strongly convex and vice versa. Thus, we have:

\[ \psi(x) := \left[ \varphi(x) + \frac{1}{2}\|x\|^2 \right]^*, \]
is 1-smooth with \( \text{dom}(\psi) = \mathcal{H} \). Then we get:

\[
\begin{align*}
 f(x) & \in \text{argmin}_u \frac{1}{2} \| u - x \|^2 + \varphi(u) = \{ u \mid x \in \partial \varphi(u) + u \} \\
 & = \left\{ u \mid x \in \partial \left( \varphi(u) + \frac{1}{2} \| u \|^2 \right) \right\} \\
 & = \{ u \mid u = \nabla \psi(x) \} = \{ \nabla \psi(x) \}
\end{align*}
\]

where the third equality comes from Thm. 4.20 in Beck [9], which notes that \( y \in \partial f(x) \) is equivalent to \( x \in \partial f^*(y) \). Hence \( f(x) = \nabla \psi(x) \), and 1-smoothness of \( \psi \) implies \( f \) is nonexpansive.

(ii)\(\Rightarrow\)(i): Let \( \varphi(x) = \psi^*(x) - \frac{1}{2} \| x \|^2 \). Since \( \psi \) is 1-smooth, similarly we can conclude: \( \psi^* \) is 1-strongly convex. Hence, \( \varphi \) is convex, and:

\[
\text{prox}_{\varphi}(x) = \text{argmin}_u \left\{ \frac{1}{2} \| u - x \|^2 + \varphi(u) \right\} = \{ u \mid x \in \partial \varphi(u) + u \} = \{ \nabla \psi(x) \} = \{ f(x) \},
\]

which means \( f(x) = \text{prox}_{\varphi}(x) \).

\[ \square \]

E.2 Proof for Theorem 1

Proof. In the proof, w.l.o.g, we let \( \mu = 1 \) for the ease of presentation, and hence let \( \hat{L}_\sigma = 1 \) and \( \| W \| \leq 1 \). Since \( \sum_{i=1}^n \hat{\sigma}(y_i) \), we have \( \nabla \hat{\sigma}(y) = [\sigma(y_1), \ldots, \sigma(y_n)]^T = \sigma(y) \), by the chain rule, \( \nabla \psi(z) = W^T \sigma(Wz + b) = f(z) \).

Since \( \sigma(a) \) is a single-valued function with slope in \([0,1]\), the element-wise defined operator \( \sigma(a) \) is nonexpansive (see the definition in Lemma 5). Combining with \( \| W \|_2 \leq 1 \), operator \( f(z) = W^T \sigma(Wz + b) \) is also nonexpansive.

Due to Lemma 5, we have \( f(z) = \text{prox}_{\varphi}(z) \), and \( \varphi(z) \) can be chosen as \( \varphi(z) = \psi^*(z) - \frac{1}{2} \| z \|^2 \).

\[ \square \]

E.3 Proof of Lemma 3

Proof. We can rewrite deep OptEq \( z = f_L \circ f_{L-1} \circ \cdots \circ f_1(z, \theta) \) in a separated form: let \( z = z_0 \),

\[
\begin{align*}
 z_1 & = \alpha W_1^T \sigma(W_1 z_0 + U_1 x + b_1) + (1 - \alpha) z_0 \\
 z_2 & = \alpha W_2^T \sigma(W_2 z_1 + U_2 x + b_2) + (1 - \alpha) z_1 \\
 & \vdots \\
 z_{L-1} & = \alpha W_{L-1}^T \sigma(W_{L-1} z_{L-2} + U_{L-1} x + b_{L-1}) + (1 - \alpha) z_{L-2} \\
 z_0 & = \alpha W_L^T \sigma(W_L z_{L-1} + U_L x + b_L) + (1 - \alpha) z_{L-1}
\end{align*}
\]

and it also has a compact matrix form:

\[
\begin{bmatrix}
 z_1 \\
 z_2 \\
 \vdots \\
 z_{L-1} \\
 z_0
\end{bmatrix} = \alpha
\begin{bmatrix}
 W_1^T & W_2^T & \cdots & W_L^T
\end{bmatrix}
\begin{bmatrix}
 U_1 \\
 U_2 \\
 \vdots \\
 U_{L-1} \\
 U_L
\end{bmatrix}
\begin{bmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_{L-1} \\
 b_L
\end{bmatrix}
+ (1 - \alpha) \begin{bmatrix}
 z_1 \\
 z_2 \\
 \vdots \\
 z_{L-1} \\
 z_0
\end{bmatrix}
\]

(13)

where \( P = \begin{bmatrix}
 0 & \cdots & I & \cdots & 0 \\
 I & 0 & \cdots & 0 & \cdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots \\
 I & 0 & \cdots & \cdots & 0
\end{bmatrix} \) is a permutation matrix.
Hence a multi-layer deep OptEq is actually a single-layer OptEq with multi-blocks. 

E.4 Proof of Theorem 2

Before proving the main results, we first present an auxiliary lemma.

**Lemma 6** (an extension of Lemma 5). Consider \( f : \mathcal{H} \to \mathcal{H} \) defined everywhere, \( A : \mathcal{H} \to \mathcal{H} \) is any invertible 1-Lipschitz operator. The following properties are equivalent:

(i) there is a proper convex l.s.c function \( \varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\} \) such that \( f(z) \in \arg\min_u \left\{ \frac{1}{2} \|u\|^2 - \langle u, A(z) \rangle + \varphi(u) \right\} \) for each \( z \in \mathcal{H} \);

(ii) the following conditions hold jointly:

(a) there exists a convex l.s.c function \( \psi : \mathcal{H} \to \mathbb{R} \) such that for each \( y \in \mathcal{H} \), \( f(A^{-1}(y)) = \nabla \psi(y) \);

(b) \( f \) is nonexpansive, i.e. \( \|f(y) - f(y')\| \leq \|y - y'\|, \forall y, y' \in \mathcal{H} \).

Moreover, there exists a choice of \( \varphi(\cdot), \psi(\cdot) \), satisfying (i) (ii), such that \( \varphi(x) = \psi^*(x) - \frac{1}{2}\|x\|^2 \).

**Proof.** (i)⇒(ii): Since \( \varphi(x) + \frac{1}{2}\|x\|^2 \) is a proper l.s.c 1-strongly convex function, then \( \psi(x) := \left[ \varphi(x) + \frac{1}{2}\|x\|^2 \right]^* \) is 1-smooth with \( \text{dom}(\psi) = \mathcal{H} \). Note that \( \psi \circ A(\cdot) \) is 1-smooth due to \( A(\cdot) \) is 1-Lipschitz. Moreover, we have:

\[
f(x) \in \arg\min_u \left\{ \frac{1}{2} \|u\|^2 - \langle u, A(x) \rangle + \varphi(u) \right\}
= \{u \mid A(x) \in (\partial \varphi(u) + u)\}
= \left\{ u \mid A(x) \in \partial \left( \varphi(u) + \frac{1}{2}\|u\|^2 \right) \right\}
= \{u \mid u = \nabla \psi(A(x)) \} = \{\nabla \psi(A(x))\},
\]

Hence \( f(x) = \nabla \psi(A(x)) \), and 1-smoothness of \( \psi \circ A(\cdot) \) implies \( f \) is nonexpansive.

(ii)⇒(i): Let \( \varphi(x) = \psi^*(x) - \frac{1}{2}\|x\|^2 \). Since \( \psi \) is 1-smooth, then \( \psi^* \) is 1-strongly convex, and \( \varphi \) is convex. Note that:

\[
\arg\min_u \left\{ \frac{1}{2} \|u\|^2 - \langle u, A(x) \rangle + \varphi(u) \right\} = \{u \mid A(x) \in (\partial \varphi(u) + u)\} = \{\nabla \psi(A(x))\} = \{f(x)\},
\]

which means \( f(x) = \arg\min_u \left\{ \frac{1}{2} \|u\|^2 - \langle u, A(x) \rangle + \varphi(u) \right\} \). 

Now, we are ready to prove the main theorem.
Proof. Denote the right hand side of equation 4.13 as \( F(\tilde{z}) \) (For convenience of notation, we omit \( x, \theta \) here), then

\[
F(\mathbf{P}^T \tilde{z}) = \alpha \left[ \begin{array}{ccc} \mathbf{W}_1^T & \mathbf{W}_2^T & \cdots & \mathbf{W}_L^T \\ \mathbf{U}_1 & \mathbf{U}_2 & \cdots & \mathbf{U}_L \end{array} \right] \sigma \left[ \begin{array}{ccc} \mathbf{W}_1 & \mathbf{W}_2 & \cdots & \mathbf{W}_L \\ \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \mathbf{z}_L \end{array} \right] + \mathbf{z}_0
\]

where \( \mathbf{W}_i^T \) denotes the transpose of \( \mathbf{W}_i \) and \( \mathbf{U}_i \) are matrices.

Given \( \|\mathbf{W}_i\|_2 \leq 1, \forall i \in [1, L] \), \( \nabla \psi_i(z_i) \) is a nonexpansive operator. Then for \( \forall \tilde{z}_1, \tilde{z}_2 \), we have:

\[
\| F(\mathbf{P}^T \tilde{z}_1) - F(\mathbf{P}^T \tilde{z}_2) \|^2 = \sum_{i=1}^{L} \|\alpha(\nabla \psi_i(z_{1,i}) - \nabla \psi_i(z_{2,i})) + (1 - \alpha)(z_{1,i} - z_{2,i})\|^2
\]

\[
\leq \sum_{i=1}^{L} (\alpha \|\nabla \psi_i(z_{1,i}) - \nabla \psi_i(z_{2,i})\| + (1 - \alpha)\|z_{1,i} - z_{2,i}\|)^2
\]

\[
\leq \sum_{i=1}^{L} (\alpha \|z_{1,i} - z_{2,i}\| + (1 - \alpha)\|z_{1,i} - z_{2,i}\|)^2
\]

\[
= \|\tilde{z}_1 - \tilde{z}_2\|^2.
\]

By the results, we have

\[
\| F(\tilde{z}_1) - F(\tilde{z}_2) \| \leq \|\mathbf{P}\tilde{z}_1 - \mathbf{P}\tilde{z}_2\| = \|\tilde{z}_1 - \tilde{z}_2\|
\]

which means \( F \) is nonexpansive. By Lemma 6 let \( \Phi(\tilde{z}) = [\alpha \Psi(\cdot) + (1 - \alpha)\frac{1}{2}\|\cdot\|^2](\tilde{z}) - \frac{1}{2}\|\tilde{z}\|^2 \)

we have:

\[
F(\tilde{z}) \in \arg\min_{\textbf{u}} \left\{ \frac{1}{2}\|\textbf{u}\|^2 - \langle \textbf{u}, \mathbf{P}\tilde{z} \rangle + \Phi(\textbf{u}) \right\}
\]

Hence any fixed point \( \tilde{z} \) of Eq. 4.13 satisfies:

\[
0 \in \partial \Phi(\tilde{z}^*) + (I - \mathbf{P})\tilde{z}^*.
\]

By using Thm. 4.14:

\[
\text{if } h(x) = \alpha f(\frac{x}{\epsilon_x}) \text{ then } h^*(y) = \alpha f^*(y),
\]

Thm. 4.19:

\[
(h_1^* + h_2^*)(x) = \inf_{\textbf{u}} \{h_1(\textbf{u}) + h_2(x - \textbf{u})\},
\]

in Beck [9] and the definition of Moreau envelope:

\[
M_f^\mu(x) = \inf_{\textbf{u}} \left\{ f(\textbf{u}) + \frac{1}{2\mu}\|x - \textbf{u}\|^2 \right\},
\]

18
\( \Phi(z) \) can be formed in terms of \( \varphi_i(z_i) \):

\[
\Phi(z) = \left[ \alpha \Psi(\cdot) + (1 - \alpha) \frac{1}{2} \| \cdot \|^2 \right]^* (z) - \frac{1}{2} \| z \|^2
\]

\[
= \sum_{i=1}^{L} \left\{ \left[ \alpha \psi_i(\cdot) + (1 - \alpha) \frac{1}{2} \| \cdot \|^2 \right]^* (z_i) - \frac{1}{2} \| z_i \|^2 \right\}
\]

\[
= \sum_{i=1}^{L} \left\{ \left[ \alpha \psi_i^{*}(\frac{\cdot}{\alpha})^* + \left( (1 - \alpha) \frac{1}{2} \left\| \frac{\cdot}{1 - \alpha} \right\|^2 \right) \right]^* (z_i) - \frac{1}{2} \| z_i \|^2 \right\}
\]

\[
= \sum_{i=1}^{L} \inf_{y_i} \left\{ \alpha \psi_i^*(y_i) + (1 - \alpha) \frac{1}{2} \left\| y_i - z_i \right\|^2 - \frac{1}{2} \| z_i \|^2 \right\}
\]

\[
= \sum_{i=1}^{L} \inf_{y_i} \left\{ \alpha \left( \varphi_i(y_i) + \frac{1}{2} \left\| y_i \right\|^2 \right) + \frac{1}{2(1 - \alpha)} \left\| \alpha y_i - z_i \right\|^2 - \frac{1}{2} \| z_i \|^2 \right\}
\]

\[
= \sum_{i=1}^{L} \inf_{y_i} \left\{ \alpha \varphi_i(y_i) + \frac{\alpha}{2(1 - \alpha)} \left\| y_i - z_i \right\|^2 \right\}
\]

\[
= \sum_{i=1}^{L} \alpha M_{\varphi_i}^{1-\alpha}(z_i).
\]

\( \square \)

E.5 Proof for Corollary 1

\textit{Proof.} When \( L = 2, \mathbf{I} - \mathbf{P} \) is a symmetric matrix

\[
\begin{bmatrix}
\mathbf{I} & -\mathbf{I} \\
-\mathbf{I} & \mathbf{I}
\end{bmatrix},
\]

and the operator

\[
\begin{bmatrix}
\mathbf{I} & -\mathbf{I} \\
-\mathbf{I} & \mathbf{I}
\end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix},
\]

is the gradient of convex function \( \frac{1}{2}\|z_1 - z_0\|^2 \). Hence the monotone operator splitting equation Eq. (7) is now an first optimality condition of convex optimization problem:

\[
\min_{z_1, z_0} \left\{ \alpha M_{\varphi_1}^{1-\alpha}(z_1) + \alpha M_{\varphi_2}^{1-\alpha}(z_0) + \frac{1}{2} \| z_1 - z_0 \|^2 \right\}.
\]

\( \square \)

E.6 Proof for Theorem 3

For proving the results, we need a lemma from Frigon [14].

\textbf{Lemma 7.} \( \mathcal{H} \) is a Hilbert space, and \( k < 1 \). If \( T_n : \mathcal{H} \rightarrow \mathcal{H} \) is a \( k \)-contraction, for all \( n \in \mathbb{N}^+ \cup \{0\} \), and \( T_n \rightarrow T_0 \) point-wisely. Then the fixed point of \( T_n \) tends to the fixed point of \( T_0 \) when \( n \rightarrow \infty \).

\textit{Proof.} Given \( \text{prox}_{\varphi_i}(z) = W_i^\top \sigma(W_i z + U_i x + b_i) \), \( z_n^\alpha \) is the fixed point of composition equation:

\[
z = \left[ \alpha \text{prox}_{\varphi_1} + (1 - \alpha) I \right] \circ \cdots \circ \left[ \alpha \text{prox}_{\varphi_L} + (1 - \alpha) I \right] (z)
\]
For $\alpha \in (0, 1)$, we get:

$$
\begin{align*}
z &= \frac{\alpha \text{prox}_{\varphi_L} + (1 - \alpha)\mathcal{I} \circ \cdots \circ [\alpha \text{prox}_{\varphi_1} + (1 - \alpha)\mathcal{I}]}{L\alpha} (z) + z \\
&= \left( \frac{(1 - \alpha)L - 1}{L\alpha} + 1 \right) z + \frac{\alpha(1 - \alpha)L^{-1}}{L\alpha} \left( \text{prox}_{\varphi_1} + \cdots + \text{prox}_{\varphi_L} \right) (z) \\
&\quad + \frac{\alpha^2(1 - \alpha)L^{-2}}{L\alpha} \left( \sum_{p>q} \text{prox}_{\varphi_p} \circ \text{prox}_{\varphi_q} \right) (z) + \cdots + \frac{\alpha^L}{L\alpha} \left( \prod_{p=L} \text{prox}_{\varphi_p} \right) (z) \\
&= \left[ \frac{1}{L} \sum_{p=2}^{L} (-1)^{p} \binom{L}{p} \alpha^{p-1} \right] z + \left( \frac{1 - \alpha}{L} \right)^{L-1} \left( \text{prox}_{\varphi_1} + \cdots + \text{prox}_{\varphi_L} \right) (z) \\
&\quad + \frac{\alpha(1 - \alpha)L^{-2}}{L} \left( \sum_{p>q} \text{prox}_{\varphi_p} \circ \text{prox}_{\varphi_q} \right) (z) + \cdots + \frac{\alpha^{L-1}}{L} \left( \prod_{p=L} \text{prox}_{\varphi_p} \right) (z).
\end{align*}
$$

(15)

Note that $z_{\alpha}^L(\alpha)$ is also the fixed point of the above equation.

Denote the right hand side of Eq. (15) as $T_\alpha(z)$, note that $T_0(z)$ is also well-defined now. Estimate the Lipschitz constant of $T_\alpha(z), \alpha \in (0, 1)$:

$$
\text{Lip}(T_\alpha) \leq \frac{1}{L} \sum_{p=2}^{L} (-1)^{p} \binom{L}{p} \alpha^{p-1} + \left( \frac{1 - \alpha}{L} \right)^{L-1} \left( \|W_1\|^2 + \cdots + \|W_L\|^2 \right) \\
&\quad + \frac{\alpha(1 - \alpha)L^{-2}}{L} \left( \sum_{p>q} \|W_p\|^2 \|W_q\|^2 \right) + \cdots + \frac{\alpha^{L-1}}{L} \left( \prod_{p=L} \|W_p\|^2 \right).
$$

Each terms in the right hand side of the above inequality (except the second term) is a polynomial of $\alpha$ with non-zero order for $\alpha$, hence they tend to zero when $\alpha \to 0$. Note that $(1 - \alpha)L \to (1 - L\alpha)$, when $\alpha \to 0$. Thus, the second term tend to $\frac{1}{L} \left( \|W_1\|^2 + \cdots + \|W_L\|^2 \right)$, which is less than 1 by assumption. Hence there is a $\kappa \in (0, 1)$, when $\alpha \in [0, \kappa]$, $\text{Lip}(T_\alpha) < \kappa$.

By using Lemma 7 the fixed point of $T_\alpha$ (i.e. $z_{\alpha}^0(\alpha)$) tends to the fixed point of $T_0$, i.e.

$$
y^* = \frac{\text{prox}_{\varphi_1}(y^*) + \cdots + \text{prox}_{\varphi_L}(y^*)}{L}.
$$

Using first order optimality condition, $(\text{prox}_{\varphi_1}(y^*), \cdots, \text{prox}_{\varphi_L}(y^*), y^*)$ is the minimizer of the following strongly convex problem:

$$
\sum_{l=1}^{L} \left( \vphi_l(x_l) + \frac{1}{2} \|x_l - y\|^2 \right).
$$

Finally, let $z_{\alpha}^1(\alpha)$ be the fixed point of the composition equation:

$$
z = \left[ \alpha \text{prox}_{\varphi_1} + (1 - \alpha)\mathcal{I} \right] \circ \left[ \alpha \text{prox}_{\varphi_L} + (1 - \alpha)\mathcal{I} \right] \circ \cdots \circ \left[ \alpha \text{prox}_{\varphi_2} + (1 - \alpha)\mathcal{I} \right] (z).
$$

A similar argument can be applied to $z_{\alpha}^1(\alpha)$ to show that, when $\alpha \to 0$, $z_{\alpha}^1(\alpha)$ tends to the same $y^*$ defined above, and so do $z_{\alpha}^2(\alpha), \cdots, z_{\alpha}^{L-1}(\alpha)$.

\[\square\]

F  Proofs for the Convergence of SAM

F.1  Auxiliary Lemmas

Next lemma follows Lem. 2.5 in Xu [33].
Lemma 8. If \( a_1 \geq 0, 0 < t_1 < 1, t_2 > 0, 0 < r_1 < 1, r_2 > 0 \), \( \{a_n\} \) is a sequence of non-negative numbers satisfying
\[
a_{k+1} = \left(1 - \frac{r_1}{(k+1)t_1}\right) a_k + \frac{r_1}{(k+1)t_1} r_2.
\]
Then \( \lim_{n \to \infty} a_n = 0 \), and there exists \( B = O(r_1, r_2, t_1, t_2, a_1) \) such that \( \|a_k\| \leq B \).

Proof. Denote \( b_k = \frac{r_1}{(k+1)t_1}, c_k = \frac{r_1}{k+1} \). Since \( 0 < t_1 < 1, 0 < r_1 < 1 \), we have \( 0 < b_k < 1, \sum_{k=1}^{\infty} b_k = \infty \) and:
\[
\prod_{k=1}^{\infty} (1 - b_k) = \exp \left( \sum_{k=1}^{\infty} \ln(1 - b_k) \right) = \lim_{K \to \infty} \exp \left( \sum_{k=1}^{K} \ln(1 - b_k) \right)
\]
\[
\leq \limsup_{K \to \infty} \exp \left( \sum_{k=1}^{K} -b_k \right) = 0.
\]
For any \( \epsilon > 0 \), choose \( N \) big enough such that \( c_k \leq \epsilon, \forall k \geq N \), then for all \( k > N \), by induction, we have,
\[
a_{k+1} = (1 - b_k)a_k + b_k c_k
\]
\[
= (1 - b_k)(1 - b_{k-1})a_{k-1} + (1 - b_k)b_{k-1} c_{k-1} + b_k c_k
\]
\[
= \cdots
\]
\[
= \prod_{j=N}^{k} (1 - b_j)a_N + \sum_{i=N}^{k} \left( \prod_{j=i+1}^{k} (1 - b_j) \right) b_i c_i
\]
\[
\leq \prod_{j=N}^{k} (1 - b_j)a_N + \sum_{i=N}^{k} \left( \prod_{j=i+1}^{k} (1 - b_j) \right) b_i \epsilon
\]
\[
= \prod_{j=N}^{k} (1 - b_j)a_N + \left( 1 - \prod_{j=N}^{k} (1 - b_j) \right) \epsilon
\]
\[
\leq \prod_{j=N}^{k} (1 - b_j)a_N + \epsilon.
\]
Hence \( \limsup_{k \to \infty} a_k \leq \epsilon \), let \( \epsilon \to 0^+ \), we get \( \lim_{n \to \infty} a_n = 0 \). Since every convergence sequence is bounded, there exists \( B = O(r_1, r_2, t_1, t_2, a_1) \) such that: \( \|a_k\| \leq B \). \( \blacksquare \)

Next lemma follows from Prop. 3 in Sabach and Shtern [26].

Lemma 9. \( l(x) \) is a \( L_z \)-smooth convex function (i.e. \( \nabla l(x) \) is \( L_z \)-Lipschitz), and \( \gamma = \frac{1}{1 + z}, 0 < \lambda \leq \frac{2\gamma}{\lambda} \). Then the operator \( S_\lambda(x) = x - \gamma (\nabla l(x) + \lambda x) \) satisfies \( \frac{2}{3} \|x - y\| \leq \|S_\lambda(x) - S_\lambda(y)\| \leq (1 - \frac{2\gamma}{\lambda}) \|x - y\| \).

Proof. Note that \( l(x) + \frac{2}{3} \|x\|^2 \) is \( \lambda \)-strongly convex and \((L_z + \lambda)\)-smooth. By Prop. 3 in Sabach and Shtern [26], it follows that:
\[
\|S_\lambda(x) - S_\lambda(y)\| \leq \sqrt{1 - \frac{2\gamma(2\gamma + \lambda)}{L_z + 2\lambda}} \|x - y\|.
\]

Note that \( \sqrt{1 - \frac{2\gamma(2\gamma + \lambda)}{L_z + 2\lambda}} < \sqrt{1 - \gamma \lambda} < 1 - \frac{2\lambda}{L_z} \), so the operator \( S_\lambda(x) \) is \((1 - \frac{2\lambda}{L_z})\)-contractive.

On the other hand, by Cauchy–Schwarz inequality, \( \langle \nabla l(x) + \lambda x - (\nabla l(y) + \lambda y) \cdot x - y \rangle \leq \frac{2\gamma}{3} \|\nabla l(x) + \lambda x - (\nabla l(y) + \lambda y)\|^2 + \frac{3}{8\gamma} \|x - y\|^2 \), and note that \( S_\lambda(x) - S_\lambda(y) \|^2 = \|x - y\|^2 \).
\[ y^2 - 2\gamma \left( (\nabla l(x) + \lambda x) - (\nabla l(y) + \lambda y), x - y \right) + \gamma^2 \| (\nabla l(x) + \lambda x) - (\nabla l(y) + \lambda y) \|^2, \]

we have:

\[
\|S_\lambda(x) - S_\lambda(y)\|^2 \geq \frac{1}{4}\|x - y\|^2 - \frac{\gamma^2}{3}\|L_x + \lambda^2\|x - y\|^2
\]

\[
\geq \frac{1}{4}\|x - y\|^2 - \frac{\gamma^2}{3}(3L_x^2)^2\|x - y\|^2
\]

\[
\geq \left( \frac{1}{4} - \frac{3}{16}\right)\|x - y\|^2
\]

\[
= \frac{1}{16}\|x - y\|^2,
\]

which is equivalent to \(\|S_\lambda(x) - S_\lambda(y)\| \geq \frac{1}{4}\|x - y\|\).

\[ \square \]

**Lemma 10.** If \(T(z)\) is a nonexpansive map with \(\text{dom}(T) = \mathbb{R}^n\), then the fixed point set of \(T(z)\) is closed and convex. And for any \(x, y, (x - T(x)) - (y - T(y)), x - y \geq 0\).

**Proof.** If \(\text{Fix}(T)\) is empty, then it is closed and convex. If \(\text{Fix}(T)\) is non-empty, for any \(x, y \in \text{Fix}(T)\), let \(z = \theta x + (1 - \theta)y\), where \(\theta \in (0, 1)\). Since \(T\) is nonexpansive, we have:

\[
\begin{align*}
\|T(z) - x\| &= \|T(z) - T(x)\| \leq \|z - x\| = (1 - \theta)\|x - y\|
\end{align*}
\]

So the triangle inequality,

\[
\|x - y\| \leq \|T(z) - x\| + \|T(z) - y\| \leq (1 - \theta)\|x - y\| + \theta\|x - y\| = \|x - y\|
\]

holds with equality. So \(T(z)\) lies on the line segment between \(x, y\), and \(\|T(z) - y\| = \theta\|x - y\|, \|T(z) - x\| = (1 - \theta)\|x - y\|\) hold, which means \(T(z) = \theta x + (1 - \theta)y = z\), i.e., \(z \in \text{Fix}(T)\).

The inequality \(\langle (x - T(x)) - (y - T(y)), x - y \rangle \geq 0\) follows directly form Cauchy–Schwarz inequality.

\[ \square \]

**F.2 Proof for Theorem 4**

**Proof.** Since \(\lambda_k \leq \frac{L_z}{2}\), by Lemma 9, operator \(z \mapsto z - \gamma(\nabla R_z(z) + \lambda_k z)\) is contractive, so is the operator \(z \mapsto \beta_k(z - \gamma(\nabla R_z(z) + \lambda_k z)) + (1 - \beta_k)T(z)\), hence the latter operator has a unique fixed point \(\tilde{z}^{+1}\).

By assumption, \(\|\tilde{z}^{+1}\| \leq B^*_1\). Note that in our setting, \(\nabla R_z(\cdot)\) and \(T(\cdot)\) are continuous, so we have \(\|T(\tilde{z}^{+1})\| + \|\tilde{z}^{+1} - \gamma(\nabla R_z(\tilde{z}^{+1}))\| \leq B^*_2\). First we estimate the difference of two successive fixed point:

\[
\|\tilde{z}^k - \tilde{z}^{k+1}\|
\]

\[
= \|\beta_k(\tilde{z}^k - \gamma(\nabla R_z(\tilde{z}^k) + \lambda_k \tilde{z}^k)) + (1 - \beta_k)T(\tilde{z}^k)
\]

\[
- \beta_{k+1}(\tilde{z}^{k+1} - \gamma(\nabla R_z(\tilde{z}^{k+1}) + \lambda_{k+1} \tilde{z}^{k+1})) - (1 - \beta_{k+1})T(\tilde{z}^{k+1})\|
\]

\[
\leq \|\beta_k(\tilde{z}^k - \gamma(\nabla R_z(\tilde{z}^k) + \lambda_k \tilde{z}^k)) - \beta_k(\tilde{z}^{k+1} - \gamma(\nabla R_z(\tilde{z}^{k+1}) + \lambda_k \tilde{z}^{k+1}))\|
\]

\[
+ \|\beta_{k+1}(\tilde{z}^{k+1} - \gamma(\nabla R_z(\tilde{z}^{k+1}) + \lambda_{k+1} \tilde{z}^{k+1})) - \beta_{k+1}(\tilde{z}^{k+1} - \gamma(\nabla R_z(\tilde{z}^{k+1}) + \lambda_{k+1} \tilde{z}^{k+1}))\|
\]

\[
+ \|((1 - \beta_k)T(\tilde{z}^k) - T(\tilde{z}^{k+1}))\|
\]

\[
\leq \beta_k(1 - \frac{\gamma \lambda_k}{2})\|\tilde{z}^k - \tilde{z}^{k+1}\| + (1 - \beta_k)\|\tilde{z}^k - \tilde{z}^{k+1}\| + B^*_2\|\beta_k - \beta_{k+1}\| + B^*_1\|\beta_k \lambda_k - \beta_{k+1} \lambda_{k+1}\|
\]

\[
=(1 - \frac{\gamma \beta_k \lambda_k}{2})\|\tilde{z}^k - \tilde{z}^{k+1}\| + B^*_2\|\beta_k - \beta_{k+1}\| + B^*_1\|\beta_k \lambda_k - \beta_{k+1} \lambda_{k+1}\|.
\]

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We get:

\[
\frac{\|z^k - z^{k+1}\|}{2\beta_{k+1} \lambda_{k+1}} \leq \frac{4B_2^*}{\gamma^2} |\beta_k - \beta_{k+1}| + \frac{4B_1^*}{\gamma^2} |\beta_k \lambda_k - \beta_{k+1} \lambda_{k+1}| + \frac{4B_1^*}{\gamma^2} |\beta_k \lambda_k - \beta_{k+1} \lambda_{k+1}|
\]

\[
= \frac{4B_2^*}{\gamma^2} |k^\rho - (k + 1)^{-\rho}| + \frac{4B_1^*}{\gamma^2} |k^\rho - c - (k + 1)^{-\rho} - c| + \frac{4B_1^*}{\gamma^2 \eta^2} k^{-\rho}(k + 1)^{-\rho + c - 1} k^\rho (k + 1)^{\rho + c}
\]

\[
\leq \frac{4B_2^* \rho}{\gamma^2 \eta^2} k^{-\rho - 1} k^{\rho + c} (k + 1)^{\rho + c} + \frac{4B_1^*(\rho + c) 2^\rho c}{\gamma^2 \eta^2} k^{\rho + c - 1}
\]

\[
\leq \frac{4B_2^* \rho + \rho + 1}{\gamma^2 \eta^2} k^{\rho + 2c - 1} + \frac{4B_1^*(\rho + c) 2^\rho c}{\gamma^2 \eta^2} k^{\rho + 2c - 1}
\]

\[
:= B_3^* k^{\rho + 2c - 1}.
\]

By the iteration equation:

\[z^{k+1} = \beta_{k+1} (z^k - \gamma (\nabla R_z (z^k) + \lambda_{k+1} z^k)) + (1 - \beta_{k+1}) T(z^k),\]

we have:

\[
\|z^{k+1} - z^k\| = \|\beta_{k+1} (z^k - \gamma (\nabla R_z (z^k) + \lambda_{k+1} z^k)) + (1 - \beta_{k+1}) T(z^k)\| - \|\beta_{k+1} (z^{k+1} - \gamma (\nabla R_z (z^{k+1}) + \lambda_{k+1} z^{k+1})) + (1 - \beta_{k+1}) T(z^{k+1})\| \leq \beta_{k+1} \|z^k - \gamma (\nabla R_z (z^k) + \lambda_{k+1} z^k)\| - (1 - \beta_{k+1}) \|z^{k+1} - z^k\|.
\]

Since \(\eta \leq \sqrt{2L}\), so \(\eta^2 \leq 1\). Let \(r_1 = \frac{\eta^2}{2}, r_2 = B_3^*, t_1 = \rho + c \in (0, 1), t_2 = 1 - \rho - 2c > 0\), and \(a_k = \|z^k - \bar{z}^k\|\), by Lemma 6 we have: \(\|z^k - \bar{z}^k\| \rightarrow 0\) as \(k \rightarrow \infty\) and \(\|z^k - \bar{z}^k\| \leq B_3^* = O(z^1, \rho, c, B_3^*).

Next, we denote the only minimizer of convex function \(R_z(x)\) on compact convex set \(\text{Fix}(T)\) as \(\bar{z}\), which is bounded by \(B_3^*\). Note that the convexity follows from Lemma 10. We now prove the final results by contradiction. Since the sequence \(\{\bar{z}^k\}\) is bounded by \(B_3^*\), if it does not converge to \(\bar{z}\), there exists \(\delta > 0\) and a sub-convergent \(\{\bar{z}^{k_j}\}\) such that:

\[
\|\bar{z}^{k_j} - \bar{z}\| \geq \delta, \quad \forall j \in \mathbb{N}^+,
\]

and

\[
\bar{z}^{k_j} \rightarrow \bar{z}', \quad \|\bar{z}' - \bar{z}\| \geq \delta.
\]

By the fixed point equation and Lemma 10 we have

\[
\bar{z}^{k_j} = \beta_{k_j} (\bar{z}^{k_j} - \gamma (\nabla R_z (\bar{z}^{k_j}) + \lambda_{k_j} \bar{z}^{k_j})) + (1 - \beta_{k_j}) T(\bar{z}^{k_j})
\]

\[
\Rightarrow \gamma (\nabla R_z (\bar{z}^{k_j}) + \lambda_{k_j} \bar{z}^{k_j}) = \frac{1 - \beta_{k_j}}{\beta_{k_j}} (T(\bar{z}^{k_j}) - \bar{z}^{k_j})
\]

\[
\Rightarrow \langle \gamma (\nabla R_z (\bar{z}^{k_j}) + \lambda_{k_j} \bar{z}^{k_j}), z - \bar{z}^{k_j} \rangle = \frac{1 - \beta_{k_j}}{\beta_{k_j}} \langle T(\bar{z}^{k_j}) - \bar{z}^{k_j} - (T(z) - z), z - \bar{z}^{k_j} \rangle \geq 0, \quad \forall z \in \text{Fix}(T)
\]

\[
\Rightarrow \langle \nabla R_z (\bar{z}^{k_j}) + \lambda_{k_j} \bar{z}^{k_j}, z - \bar{z}^{k_j} \rangle \geq 0, \quad \forall z \in \text{Fix}(T).
\]
Let \( j \to \infty \), we get:
\[
\langle \nabla R_z(\bar{z}'), z - \bar{z}' \rangle \geq 0, \quad \forall z \in \text{Fix}(T),
\]
which is equivalent to \( R_z(\bar{z}') \leq R_z(z), \forall z \in \text{Fix}(T) \). Namely \( \bar{z}' = \bar{z} \), which is a contradiction.
Thus, the sequence \( \{\bar{z}^k\} \) converge to \( \bar{z} \). Combining the result \( \|z^k - \bar{z}^k\| \to 0 \), we have:
\[
\bar{z}^k \to \bar{z}.
\]

We finish the proof. \( \square \)

### G Proofs for Linear Convergence Training

In the proof, we will consider the whole data set, i.e., \( Z^k \in \mathbb{R}^{m \times N} \), where \( N \) is the training data size. We denote by \( z^k \defeq \text{vec}(Z^k) \), where \( \text{vec}(Z) \in \mathbb{R}^{mN} \) is the vectorization of the matrix \( Z \in \mathbb{R}^{m \times N} \). \( Z^k \) is the \( k \)-th iterates of the sequence generated by Eq. (10) applied on the data matrix. Then, we denote:
\[
Z^{(k,l)} \defeq f_1 \circ \cdots \circ f_1(Z^k; Z, \theta), \quad \text{and} \quad z^{(k,l)} \defeq \text{vec}(Z^{(k,l)}).
\]

\( z^{(k,l)} \) and \( z^k \) all depend on the parameter \( \theta \). However, for the sake of brevity, we omit the mark \( \theta \) when the meaning of the symbol is clear. We let \( I_n \) be the \( n \times n \) identity matrix. For the output of the network, we let \( y \defeq \text{vec}(W_{L+1}Z^K) \).

Note that we have
\[
S_{\lambda_k}(Z, W_{L+1}) = Z - \gamma(\nabla R_z(Z) + \lambda_k Z), \quad \text{and} \quad \gamma < \frac{1}{2L_2 + 1}.
\]

And we let \( s^k \defeq \text{vec}(S(Z^k, W_{L+1})) \), then:
\[
z^{k+1} = \beta_{k+1} \text{vec}(S_{\lambda_k}(Z^k, W_{L+1})) + (1 - \beta_{k+1})T(z^k, x, \theta) = \beta_{k+1}s^k + (1 - \beta_{k+1})z^{(k,L)}.
\]

We let:
\[
\mathcal{V}_\gamma(\cdot) \defeq (1 - \gamma \lambda_k)\mathcal{I}(\cdot) - \gamma \nabla R_z(\cdot), \quad \text{and denote by} \quad G^{(k,l)} \defeq \sigma(W_iZ^{(k,l-1)} + U_iX + b_i).
\]

Note that by our setting on \( \gamma \) and \( \lambda_k \), we can easily conclude that \( \|\mathcal{V}_\gamma\|_2 < 1 \). Moreover, due to Lemma\( \square \), we also get the lower bound: \( \|\mathcal{V}_\gamma\|_2 \geq \frac{1}{\gamma} \).

We also denote:

\[
D^{(k,l)} \defeq \begin{bmatrix}
\tilde{D}_1 \\
\vdots \\
\tilde{D}_N
\end{bmatrix}, \quad \tilde{D}_j \defeq \begin{bmatrix}
d_{1j} \\
\cdots \\
d_{mj}
\end{bmatrix},
\]

where
\[
d_{ij} = \sigma'(W_iZ^k_j + U_iX + b_i)
\]
is the \((i,j)\)-th entry of the derivative matrix
\[
\sigma'(W_iZ^{(k,l)} + U_iX + b_i) \in \mathbb{R}^{m \times N}.
\]

We let:
\[
A(k, l_2, l_1) \defeq \prod_{l=l_1}^{l_2} \left( \alpha(I_N \otimes W_i^T)D^{(k,l)}(I_N \otimes W_i) + (1 - \alpha)I_{mN} \right),
\]
where \( \otimes \) is the Kronecker product.

For Theorem\( \square \) we consider the square loss, i.e.,
\[
\ell(\tilde{\theta}) = \ell(y, y^0) = \frac{1}{2}\|y - y_0\|^2.
\]

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G.1 Auxiliary Lemmas

We first offer several auxiliary lemmas.

**Lemma 11.** The following results hold:

\[
\begin{align*}
\frac{\partial z^{k+1}}{\partial \text{vec}(W_l)} &= \sum_{k=1}^{k} \left( \prod_{q=k+1}^{k+1} (\beta_{q+1} (I_N \otimes V_q) + (1 - \beta_{q+1}) A(q, 1)) \right) A(k, l, 1) \frac{\partial \text{vec} \left( f_l(Z(k,l-1), X) \right)}{\partial \text{vec}(W_l)}, \\
\frac{\partial z^{k+1}}{\partial \text{vec}(U_l)} &= \sum_{k=1}^{k} \left( \prod_{q=k+1}^{k+1} (\beta_{q+1} (I_N \otimes V_q) + (1 - \beta_{q+1}) A(q, 1)) \right) A(k, l, 1) \frac{\partial \text{vec} \left( f_l(Z(k,l-1), X) \right)}{\partial \text{vec}(U_l)}, \\
\frac{\partial z^{k+1}}{\partial \text{vec}(b_l)} &= \sum_{k=1}^{k} \left( \prod_{q=k+1}^{k+1} (\beta_{q+1} (I_N \otimes V_q) + (1 - \beta_{q+1}) A(q, 1)) \right) A(k, l, 1) \frac{\partial \text{vec} \left( f_l(Z(k,l-1), X) \right)}{\partial \text{vec}(b_l)},
\end{align*}
\]

(16)

where

\[
\begin{align*}
\frac{\partial \text{vec} \left( f_l(Z(k,l), X) \right)}{\partial \text{vec}(W_l)} &= \alpha \left( G(k,l) \otimes I_m \right) K^{(n_1, m)} + \alpha (I_N \otimes W_l^T) D^{(k,l)} \left( (Z^{(k,l)})^T \otimes I_{n_1} \right), \\
\frac{\partial \text{vec} \left( f_l(Z(k,l), X) \right)}{\partial \text{vec}(U_l)} &= \alpha (I_N \otimes W_l^T) D^{(k,l)} (X^T \otimes I_{n_1}), \quad \frac{\partial \text{vec} \left( f_l(Z(k,l), X) \right)}{\partial \text{vec}(b_l)} = \alpha (I_N \otimes W_l^T) D^{(k,l)} 1_N.
\end{align*}
\]

(17)

here $1_N$ is a $N$-dimensional all-one vector.

**Proof.** Note that $f_l(Z(k,l), X) = \alpha W_l^T \sigma (W_l Z(k,l) + U_l X + b_l) + (1 - \alpha) Z^{(k,l)}$. Hence, we have:

\[
\begin{align*}
\frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec}(Z^{(k,l)})} &= \alpha (I_N \otimes W_l^T) D^{(k,l)} (I_N \otimes W_l) + (1 - \alpha) I_{mN},
\end{align*}
\]

where $\otimes$ is the Kronecker product. Then, we can get:

\[
\begin{align*}
\frac{\partial T(z^k, x, \theta)}{\partial \text{vec}(z^k)} &= \prod_{l=1}^{L} (\alpha (I_N \otimes W_l^T) D^{(k,l)} (I_N \otimes W_l) + (1 - \alpha) I_{mN}),
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial z^{k+1}}{\partial z^k} &= \beta_{k+1} \frac{\partial S(z^k, W_{L+1})}{\partial z^k} + (1 - \beta_{k+1}) \frac{\partial T(z^k, x, \theta)}{\partial z^k} \\
&= \beta_{k+1} (I_N \otimes (I_{d_N} - \gamma W_{L+1}^T W_{L+1})) + (1 - \beta_{k+1}) \frac{\partial T(z^k, x, \theta)}{\partial z^k} \\
&= \beta_{k+1} (I_N \otimes V_{\gamma}) + (1 - \beta_{k+1}) \prod_{l=1}^{L} (\alpha (I_N \otimes W_l^T) D^{k,l} (I_N \otimes W_l) + (1 - \alpha) I_{mN}) \\
&= \beta_{k+1} (I_N \otimes V_{\gamma}) + (1 - \beta_{k+1}) A(k, L, 1).
\end{align*}
\]

On the other hand, we have:

\[
\begin{align*}
\frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec}(W_l)} &= \alpha \left( \sigma (W_l Z^{(k,l)} + U_l X + b_l) \right)^T \otimes I_m \right) K^{(n_1, m)} + \alpha (I_N \otimes W_l^T) D^{(k,l)} \left( (Z^{(k,l)})^T \otimes I_{m_1} \right), \\
&= \alpha \left( (G(k,l))^T \otimes I_m \right) K^{(n_1, m)} + \alpha (I_N \otimes W_l^T) D^{(k,l)} \left( (Z^{(k,l)})^T \otimes I_{m_1} \right),
\end{align*}
\]

where $K^{(n_1, m)} \in \mathbb{R}^{n_1m \times n_1m}$ is the commutation matrix such that $K^{(n_1, m)} \text{vec}(W) = \text{vec}(W^T)$. And

\[
\begin{align*}
\frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec}(U_l)} &= \alpha (I_N \otimes W_l^T) D^{(k,l)} (X^T \otimes I_{m_1}), \\
\frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec}(b_l)} &= \alpha (I_N \otimes W_l^T) D^{(k,l)} 1_N.
\end{align*}
\]
Thus, we get:
\[ \frac{\partial z^{k+1}}{\partial \text{vec}(W_l)} = \frac{\partial z^{k+1}}{\partial \text{vec}(W_l)} + \frac{\partial z^{k+1}}{\partial \text{vec}(W_l)} = (\beta_{k+1}(I_N \otimes \mathcal{V}_\gamma) + (1 - \beta_{k+1})A(k, 1)) \frac{\partial z^k}{\partial \text{vec}(W_l)} + A(k, l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(W_l)} \]

\[ = \sum_{k=1}^k \left( \prod_{q=k+1}^k (\beta_{q+1}(I_N \otimes \mathcal{V}_\gamma) + (1 - \beta_{q+1})A(q, 1)) \right) A(k, l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(W_l)} \]

Similarly, we can also obtain:
\[ \frac{\partial z^{k+1}}{\partial \text{vec}(U_l)} = \sum_{k=1}^k \left( R_k A(k, l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(U_l)} \right), \]

and
\[ \frac{\partial z^{k+1}}{\partial \text{vec}(B_l)} = \sum_{k=1}^k \left( R_k A(k, l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(B_l)} \right), \]

where
\[ R_k = \left( \prod_{q=k+1}^k (\beta_{q+1}(I_N \otimes \mathcal{V}_\gamma) + (1 - \beta_{q+1})A(q, 1)) \right). \]

Similar to the proof in the previous lemma, we can easily obtain
\[ \begin{align*}
    \frac{\partial z^{(k,l')}}{\partial \text{vec}(W_l)} &= 1_{\{l' > l\}} A(k, l', l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(W_l)} + A(k, l', 1) \frac{\partial z^{k-1}}{\partial \text{vec}(W_l)}, \\
    \frac{\partial z^{(k,l')}}{\partial \text{vec}(U_l)} &= 1_{\{l' > l\}} A(k, l', l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(U_l)} + A(k, l', 1) \frac{\partial z^{k-1}}{\partial \text{vec}(U_l)}, \\
    \frac{\partial z^{(k,l')}}{\partial \text{vec}(B_l)} &= 1_{\{l' > l\}} A(k, l', l + 1) \frac{\partial \text{vec}(f_l(z^{(k,l-1)}, X))}{\partial \text{vec}(B_l)} + A(k, l', 1) \frac{\partial z^{k-1}}{\partial \text{vec}(B_l)}, \end{align*} \tag{19} \]

where \( \frac{\partial z^{(k-1)}}{\partial \text{vec}(W_l)} \) and \( \frac{\partial \text{vec}(f_l(z^{(k-1)}, X))}{\partial \text{vec}(W_l)} \) are given in Eq.(16) and Eq.(17) in the previous lemma, and \( 1_{\{l' > l\}} \) is the indicator function. Before providing the lemmas, we present two assumptions commonly used in the following lemmas.

**Assumption 3** (Compact Set of Parameters). Given the learnable parameters
\[ \theta = \{(W_l, U_l, b_l)\}_{l=1}^L, \quad \forall l \in [1, L], \]
we assume \( \|W_l\|_2 \leq 1 \) and \( \max \{\|U_l\|_2, \|b_l\|_2\} \leq B_{U,b} \). Moreover, we let \( \|X\|_F \leq B_x \) and \( \|W_{L+1}\|_2 \leq B_L \).

**Assumption 4** (Existence and boundedness). Let \( T_{\beta, \lambda}(z, X, \theta) = \beta(z - \gamma(\nabla R_z(z) + \lambda z)) + (1 - \beta)T(z, X, \theta) \). For any learnable parameters \( \theta \) and \( X \) satisfy Assumption 3 and \( \beta \in [0, \frac{1}{2}], \lambda \in [0, \frac{L}{2}], \gamma \in [0, \frac{1}{2}], \gamma = \frac{1}{2}, \lambda \in [0, \frac{L}{2}], \\forall z \in \text{Fix}(T_{\beta, \lambda}(\cdot, X, \theta)), \|z\|_2 \leq B^*_1 \). Without loss of generality, we also let \( \|z^1\| \leq B^*_1 \).

We now show the uniform boundedness of \( z^k \) and \( z^{(k,l)} \).

**Lemma 12.** If Assumption 3 and Assumption 4 hold, and \( \alpha \leq \frac{\ln 2}{2L} \), then:
\[ \|z^k\| \leq B^* \quad \text{and} \quad \|z^{(k,l)}\| \leq 3B^*, \]

here \( B^* = \mathcal{O}(B_c, B_{U,b}, B^*_1) \), and \( B^*_1 \) is the uniform bound for any \( z \in \text{Fix}(T_{\beta, \lambda}(\cdot, X, \theta)) \), as shown in Assumption 4.
Proof. We first give the boundedness for $z^k$. Following the proof of Theorem 3, we have $\|z^k - z^k^*\| \leq B^*_1 = O(\beta^\gamma)$. Note that $B^*_1 = O(\beta_1^\gamma, \beta_1^\gamma, \rho, \epsilon, \gamma, \eta), B^*_1 = O(B_1, B_{U_{1, l}}, B_1^*), \rho, \epsilon, \gamma, \eta$ are constants, and $\|z^k\| \leq B^*_{1, l}$ by Assumption 4 so $B^*_1 = O(B_1, B_{U_{1, l}}, B_1^*)$.

And by the definition of $z^k$, it is the fixed point of equation $z = \beta_k (z - \gamma (\nabla R_x(z) + \lambda_k z)) + (1 - \beta_k)T(z, X, \theta)$, therefore $\|z^k\| \leq B^*_{1, l}$.

Thus, we have

$$\|z^k\| \leq \|z^k - z\| + \|z\| \leq B^*_{1, l} + B^*_1 = O(B_1, B_{U_{1, l}}, B_1^*).$$

On the other hand, we can suppose that $\|f_1 \circ \cdots \circ f_1(0)\| \leq B^*_1 = O(B_1, B_{U_{1, l}}), \forall l \in [1, L]$, since they are continuously depended on $W, U, b$ and $\alpha$. Let $B^* = \max\{B^*_1 + B^*_1, B^*_1\}$.

Note that $z^{(k, l)} = f_1 \circ \cdots \circ f_1(z^k)$, and each $f_k(z) = z + \alpha (W^*_k \sigma(W_k x + U_k x + b) - z)$ is $(1 + 2\alpha)$-Lipschitz. Therefore

$$\|z^{(k, l)}\| \leq \|f_1 \circ \cdots \circ f_1(z^k) - f_1 \circ \cdots \circ f_1(0)\| + \|f_1 \circ \cdots \circ f_1(0)\| \leq (1 + 2\alpha^*)\|z^k - 0\| + B^* \leq \exp\{2\alpha L\} B^* + B^* \leq 2B^* + B^* = 3B^*.$$

For $\zeta \in (0, 1)$, we say an operator $f$ is $\zeta$-averaged if $f = (1 - \zeta)I + \zeta S$ for some nonexpansive operator $S$.

Lemma 13. If Assumption 3 holds, given any learnable parameters satisfy Assumption 3 the function $f := f_1 \circ \cdots \circ f_1(\cdot, x)$ is averaged for all $l \in [1, L]$.

Proof. Note that:

$$\frac{\partial \text{vec}(f_1(Z, X))}{\partial z} = \alpha (I_N \otimes W_l^T) D(I_N \otimes W_l) + (1 - \alpha)I_{mN}.$$

Hence, we have:

$$\left\| \frac{\partial \text{vec}(f_1(Z, X))}{\partial z} \right\| \leq \alpha \|W_l\|_2^2 \|D\|_2 + (1 - \alpha) \leq 1,$$

where the last inequality comes from Assumption 1 and Assumption 3. Hence the function $f_1(\cdot, x)$ is averaged for any parameters satisfy Assumption 3. Moreover, by Proposition 4.46 in [8], the composition of $l$ operators which are averaged with respect to the same norm is also averaged, i.e., $f_1 \circ \cdots \circ f_1(\cdot, x)$ is averaged when each $f_i(\cdot)$ is averaged.}

We now provides the bounds for $A_{k, l}$ and $\frac{\partial \text{vec}(f_i(Z^{(k, l)}, X))}{\partial \text{vec}(\cdot)}$.

Lemma 14. If Assumption 3 and Assumption 1 hold, then:

$$\begin{align*}
\left\| A_{k, l} \right\|_2 & \leq 1, \\
\left\| \prod_{q=k+1}^k (\beta_{q-1}(I_N \otimes \nu_q) + (1 - \beta_{q-1})A(q, 1)) \right\| & \leq 1,
\end{align*}$$

\begin{align}
\left\| \frac{\partial \text{vec}(f_i(Z^{(k, l)}, X))}{\partial \text{vec}(W_l)} \right\| & \leq \alpha(\sqrt{\text{min}}(\sigma(0) + 6B^* + 2B_{U_{1, l}}B_x)), \\
\left\| \frac{\partial \text{vec}(f_i(Z^{(k, l)}, X))}{\partial \text{vec}(U_l)} \right\| & \leq \alpha B_x, \\
\left\| \frac{\partial \text{vec}(f_i(Z^{(k, l)}, X))}{\partial \text{vec}(b_l)} \right\| & \leq \alpha \sqrt{N}.
\end{align}

Proof. First of all, we get:

$$\left\| A_{k, l} \right\|_2 \leq \prod_{l=1}^L \left\| \alpha (I_N \otimes W_l^T) \right\|_2 \left\| D(k, l)(I_N \otimes W_l) + (1 - \alpha)I_{mN} \right\| \leq 1,$$

$$\left\| \prod_{l=1}^L \left\| \alpha \left\| D(k, l) \right\|_2 \right\|_2 \left\| W_l \right\|_2^2 + (1 - \alpha) \right\| \leq 1,$$
where the last inequality comes form Assumption 3 and Assumption 1. Note that \( \gamma < \frac{1}{2^{l+1}} \), hence we have \( \| \mathcal{V}_\gamma \|_2 \leq 1 \). Thus, we have:

\[
\left\| \prod_{q=k+1}^k (\beta_{q+1}(I_N \otimes \mathcal{V}_\gamma) + (1 - \beta_{q+1})A(q,1)) \right\|_2 \leq \prod_{q=k+1}^k (\beta_{q+1}\|\mathcal{V}_\gamma\|_2 + (1 - \beta_{q+1})) \leq 1,
\]

where the first inequality we utilize \( \left\| A(k,\bar{d}) \right\|_2 \leq 1 \). By Assumption 1, we can easily obtain that \( \sigma(a) \leq a + \sigma(0), \forall a \in \mathbb{R} \). Hence, for \( G^{(k,l)} := \sigma(W_i Z^{(k,l-1)} + U_i X + b_l) \), we have:

\[
\left\| G^{(k,l)} \right\|_F \leq \sqrt{mn_l} \sigma(0) + \left\| W_i Z^{(k,l-1)} + U_i X + b_l \right\|_F \\
\leq \sqrt{mn_l} \sigma(0) + \left\| Z^{(k,l)} \right\|_F + B_{U_b} \| X \|_F + \sqrt{N} B_{U_b} \\
\leq \sqrt{mn_l} \sigma(0) + 3B^* + 2B_{U_b} B_x,
\]

where, w.l.o.g, we use \( \| X \|_F = \Theta(\sqrt{N}) \) in the last inequality. Then, by Eq. (17), we get:

\[
\left\| \frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec} (W_i)} \right\|_2 \leq \alpha \left( \left\| G^{(k,l)} \right\|_2 + \left\| W_i \right\|_2 \left\| D^{(k,l)} \right\|_2 \left\| Z^{(k,l)} \right\|_2 \right) \leq \alpha (\sqrt{mn_l} \sigma(0) + 6B^* + 2B_{U_b} B_x),
\]

where we utilize \( \| K^{(n,m)} \|_2 = 1 \). Similarly, we have:

\[
\left\| \frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec} (U_i)} \right\|_2 \leq \alpha B_x, \quad \text{and} \quad \left\| \frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec} (b_l)} \right\|_2 \leq \alpha \sqrt{N}.
\]

\[
\square
\]

**Lemma 15.** If Assumption 3 and Assumption 1 hold, then:

\[
\left\| \frac{\partial z^k}{\partial \text{vec}(W_i)} \right\|_2 \leq (k - 1)\alpha(\sqrt{mn_l} \sigma(0) + 6B^* + 2B_{U_b} B_x), \\
\left\| \frac{\partial z^k}{\partial \text{vec}(U_i)} \right\|_2 \leq (k - 1)\alpha B_x, \\
\left\| \frac{\partial z^k}{\partial \text{vec}(b_l)} \right\|_2 \leq (k - 1)\alpha \sqrt{N}, \\
\left\| \frac{\partial z^{k'}}{\partial \text{vec}(W_i)} \right\|_2 \leq (1_{l' > l} + k - 2)\alpha(\sqrt{mn_l} \sigma(0) + 6B^* + 2B_{U_b} B_x), \\
\left\| \frac{\partial z^{k'}}{\partial \text{vec}(U_i)} \right\|_2 \leq (1_{l' > l} + k - 2)\alpha B_x, \\
\left\| \frac{\partial z^{k'}}{\partial \text{vec}(b_l)} \right\|_2 \leq (1_{l' > l} + k - 2)\alpha \sqrt{N}.\tag{21}
\]

Moreover, we get:

\[
\left\| \text{vec} \left( \nabla_{W_i} \ell(y, y_0) \right) \right\|_2 \leq (K - 1)\alpha(\sqrt{mn_l} \sigma(0) + 6B^* + 2B_{U_b} B_x)B_L \| y - y_0 \|, \\
\left\| \text{vec} \left( \nabla_{U_i} \ell(y, y_0) \right) \right\|_2 \leq (K - 1)\alpha B_x B_L \| y - y_0 \|, \\
\left\| \text{vec} \left( \nabla_{b_l} \ell(y, y_0) \right) \right\|_2 \leq (K - 1)\alpha \sqrt{N} B_l \| y - y_0 \|.\tag{22}
\]

**Proof.** By Eq. (16), we have:

\[
\left\| \frac{\partial z^k}{\partial \text{vec}(W_i)} \right\|_2 \leq \sum_{k=1}^{k-1} \left\| \frac{\partial \text{vec} \left( f_l(Z^{(k,l)}, X) \right)}{\partial \text{vec} (W_i)} \right\|_2 \leq (k - 1)\alpha(\sqrt{mn_l} \sigma(0) + 6B^* + 2B_{U_b} B_x).
\]

where we use the results in Eq. (20). Similarly, by Eq. (16), we can also have:

\[
\left\| \frac{\partial z^k}{\partial \text{vec}(U_i)} \right\|_2 \leq (k - 1)\alpha B_x, \quad \text{and} \quad \left\| \frac{\partial z^k}{\partial \text{vec}(b_l)} \right\|_2 \leq \sqrt{N}(k - 1).\alpha.
\]

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By Eq. (19), an immediate consequence of these bounds is:
\[
\left\| \frac{\partial z^{(k,l')}}{\partial \text{vec}(W_l)} \right\|_2 \leq (1_{l'>l} + k - 2) \alpha (\sqrt{m_l} \sigma(0) + 6B^* + 2B_{U_3}B_x).
\]
Similarly, we also get:
\[
\left\| \frac{\partial z^{(k,l')}}{\partial \text{vec}(U_l)} \right\|_2 \leq (1_{l'>l} + k - 2) \alpha B_x, \quad \text{and} \quad \left\| \frac{\partial z^{(k,l')}}{\partial \text{vec}(b_l)} \right\|_2 \leq (1_{l'>l} + k - 2) \sqrt{N} \alpha.
\]
Note that, we already have:
\[
\left\| \frac{\partial \ell(y, y_0)}{\partial \text{vec}(W_l)} \right\|_2 \leq (K - 1) \alpha (\sqrt{m_l} \sigma(0) + 6B^* + 2B_{U_3}B_x) \left\| y - y_0 \right\|.
\]
Hence, by Eq. (20), we can easily get:
\[
\left\| \text{vec} \left( \nabla W, \ell(y, y_0) \right) \right\| \leq (K - 1) \alpha (\sqrt{m_l} \sigma(0) + 6B^* + 2B_{U_3}B_x) B_L \left\| y - y_0 \right\|.
\]
Similarly, we have:
\[
\left\{ \begin{array}{l}
\left\| \text{vec} \left( \nabla U_l, \ell(y, y_0) \right) \right\| \leq (K - 1) \alpha B_L \left\| y - y_0 \right\|, \\
\left\| \text{vec} \left( \nabla b_l, \ell(y, y_0) \right) \right\| \leq (K - 1) \alpha \sqrt{N} B_L \left\| y - y_0 \right\|.
\end{array} \right.
\]

**Lemma 16.** We let for all \( l \in [1, L] \), \( \sigma_{\min}(W_l) \geq \sigma_m \). If \( \sigma' (\cdot) \geq \kappa > 0 \) and Assumption 3 hold, then:
\[
\left\| \text{vec} \left( \nabla \ell(y, y_0) \right) \right\|^2 \geq \sum_{l=1}^{L} \left\| \text{vec} \left( \nabla b_l, \ell(y, y_0) \right) \right\|^2 \geq cK^2 L^2 \kappa^2 \sigma_m^2 \sigma_{\min}^2(W_{L+1}) \left\| y - y_0 \right\|^2,
\]
for some \( 0 < c < 1 \).

**Proof.** Recall that, we already have:
\[
\frac{\partial z^{k+1}}{\partial \text{vec}(b_l)} = \sum_{k=1}^{k} \left( R_{\tilde{k}} \Lambda(k, L, l + 1) \frac{\partial \text{vec} \left( f_l(Z(k, l, l-1), X_l) \right)}{\partial \text{vec}(b_l)} \right),
\]
where
\[
R_{\tilde{k}} = \left( \prod_{q=k+1}^{k} (\beta_{q+1}(I_N \otimes V_q) + (1 - \beta_{q+1})A(q, 1)) \right).
\]
First of all, we note that
\[
\left\| V_q \right\|_2 \geq \frac{1}{4}.
\]
On the other hand, for any \( k \in [K] \), we have:
\[
A(k, L, l + 1) \geq (1 - \alpha)^L - l.
\]
Hence, when \( \alpha K L < 1 \), we have
\[
\left\| \prod_{q=k+1}^{k} (\beta_{q+1}(I_N \otimes V_q) + (1 - \beta_{q+1})A(q, L, l)) \right\|_2 \geq \prod_{q=k+1}^{k} \left( \frac{\beta_{q+1}}{4} + (1 - \beta_{q+1})(1 - \alpha L) \right) \geq (1 - \alpha L)^{(k-\tilde{k})}.
\]
We observe that
\[
\sum_{k=1}^{k} (1 - \alpha L)^{(k-\tilde{k})} \geq \frac{1 - (1 - \alpha L)^k}{\alpha L} \geq c_1 k,
\]
\[
\sum_{k=1}^{k} (1 - \alpha L)^{(k-\tilde{k})} \geq \frac{1 - (1 - \alpha L)^k}{\alpha L} \geq c_1 k,
\]
for some $0 < c_1 < 1$. And for any $k \in [K]$, 
\[
\alpha(I_N \otimes W_i^T)D^{(k,l)}1_N \geq \alpha \sigma_m \kappa \sqrt{N}.
\]
Then by Eq. (16) and Eq. (17), we can obtain:
\[
\left\| \frac{\partial z^K}{\partial \text{vec}(b_l)} \right\|_2^2 \geq K^2(1 - \alpha)^{2(L-l)} \alpha^2 \kappa^2 \sigma_m^2 N.
\]
Observing that:
\[
\sum_{l=1}^{L} (1 - \alpha)^{2L-l} = \frac{1 - (1 - \alpha)^{2L}}{1 - (1 - \alpha)^2} \geq c_2 L,
\]
for some $0 < c_2 < 1$. Hence, we have:
\[
\sum_{l=1}^{L} \left\| \frac{\partial z^K}{\partial \text{vec}(b_l)} \right\|_2^2 \geq cK^2 L \alpha^2 \kappa^2 \sigma_m^2 N,
\]
for some $0 < c < 1$. Finally, we can conclude that:
\[
\sum_{l=1}^{L} \| \nabla \ell(y, y_0) \|_2^2 \geq \| \nabla b_l \ell(y, y_0) \|_2^2 \geq cK^2 L \alpha^2 \kappa^2 \sigma_m^2 N \sigma_{\min}(W_{L+1}) \| y - y_0 \|^2.
\]

Before proceeding, let us consider how to calculate $\frac{\partial z^k}{\partial x}$.

**Lemma 17.** If Assumption 2 and Assumption 3 hold, then:
\[
\left\| \frac{\partial z^{k+1}}{\partial x} \right\|_2 \leq \alpha KLBu_b.
\]
Moreover, if $g(X_0, W)$ is smooth and $L_g$-Lipschitz continuous w.r.t $W$, we can obtain:
\[
\| \text{vec}(\nabla \ell(y, y_0)) \| \leq \alpha KLBu_b B_L L_g \| y - y_0 \|.
\]

**Proof.**
\[
\frac{\partial z^{k+1}}{\partial x} = \frac{\partial z^{k+1}}{\partial x} + \sum_{l=1}^{L} \left( A(k, L, l) \frac{\partial \text{vec}(f_i(Z^{(k,l)}, X))}{\partial x} \right)
\]
\[
= \beta_{k+1}(I_N \otimes V_q) + (1 - \beta_{k+1}) A(k, L, 1) \frac{\partial z^{k-1}}{\partial x} + \sum_{l=1}^{L} \left( A(k, L, l) \frac{\partial \text{vec}(f_i(Z^{(k,l)}, X))}{\partial x} \right)
\]
\[
= \sum_{k=1}^{K} \left( \prod_{q=k+1}^{K} (\beta_{q+1}(I_N \otimes V_q) + (1 - \beta_{q+1}) A(q, L, 1)) \right) \sum_{l=1}^{L} \left( A(k, L, l) \frac{\partial \text{vec}(f_i(Z^{(k,l)}, X))}{\partial x} \right),
\]
where the second inequality comes from Eq. (13) and
\[
\frac{\partial \text{vec}(f_i(Z^{(k,l)}, X))}{\partial x} = \alpha(I_N \otimes W_i^T)D^{(k,l)}(I_N \otimes U_l).
\]
By the bounds given in Eq. (20), we can get:
\[
\left\| \frac{\partial \text{vec}(f_i(Z^{(k,l)}, X))}{\partial x} \right\|_2 \leq \alpha Bu_b.
\]
Hence, we can immediately get:
\[
\left\| \frac{\partial z^{k+1}}{\partial x} \right\|_2 \leq \alpha KLBu_b.
\]
Note that:
\[
\vec{\nabla}_W \ell(y, y_0) = \left( \frac{\partial z^K}{\partial x} \right)^\top \left( I_N \otimes W_{L+1}^\top \right) (y - y_0).
\]
Thus, we can conclude that:
\[
\|\vec{\nabla}_W \ell(y, y_0)\| \leq \alpha KLB_{U_0}B_L\|y - y_0\|.
\]

**Lemma 18.** If Assumption[7] and Assumption[5] hold, then:
\[
\left\| \frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} \right\|_2 \leq 2K\gamma B^*B_L.
\]
Moreover, we can obtain:
\[
\left\| \vec{\nabla}_W \ell(y, y_0) \right\| \leq 3K\gamma B^*\|y - y_0\|.
\]

**Proof.** We already have:
\[
\frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} = \frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} + H^K
\]
where we let
\[
H^K := -\gamma K \left( (W_{L+1}Z^K)^\top \otimes I_m \right) K^{(d_q, m)}(I_N \otimes W_{L+1}^\top) \left( Z^K \otimes I_{d_q} \right).
\]
Hence, we can obtain:
\[
\left\| \frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} \right\| \leq 2K\gamma B^*B_L,
\]
where we utilize the bounds in Assumption[3]. Note that:
\[
\vec{\nabla}_W \ell(y, y_0) = \left( \frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} \right)^\top \left( I_N \otimes W_{L+1}^\top \right) (y - y_0) + (z^K \otimes I_{d_q}) (y - y_0).
\]
Thus, we can conclude that:
\[
\left\| \vec{\nabla}_W \ell(y, y_0) \right\| \leq 3K\gamma B^*B_L^2\|y - y_0\| \leq 3K\gamma B^*\|y - y_0\|.
\]

In the following lemma, we consider the Lipschitz continuity, we let \( \tilde{\theta} := \{W_{L+1}, \theta, W_0\} \). Moreover, we denote:
\[
J_{z^K} := \left[ \frac{\partial z^K}{\partial \vec{\text{vec}}(W_{L+1})} \frac{\partial z^K}{\partial \vec{\text{vec}}(W_L)} \frac{\partial z^K}{\partial \vec{\text{vec}}(U_L)} \frac{\partial z^K}{\partial \vec{\text{vec}}(b_L)} \ldots \right] \left[ \frac{\partial z^K}{\partial \vec{\text{vec}}(W_0)} \right].
\]

**Lemma 19.** Let Assumption[1] hold. We assume that the activation function is \( L_a \)-smooth, and \( g(X_0; W) \) is smooth and \( L_\gamma \)-Lipschitz continuous w.r.t. \( W \), given two parameters \( \tilde{\theta}^a \) and \( \tilde{\theta}^b \) satisfying Assumption[2] we have:
\[
\begin{align*}
\|z^k(\theta^a) - z^k(\theta^b)\|_2 & \leq \left( 2\sqrt{LK}C_L \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|,
\|z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b)\|_2 & \leq \left( 2\sqrt{LK}C_L \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|,
\|D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b)\|_2 & \leq L_a \left( C_D \|\Delta \theta\| + B_{U_0}L_\sigma \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|,
\|A(k, l_1, l_2, \theta^a) - A(k, l_1, l_2, \theta^b)\|_2 & \leq \left( 4\sqrt{KL}C_L \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|,
\|J_{z^K}(\theta^a) - J_{z^K}(\theta^b)\|_2 & \leq \left( 9\sqrt{LB^*}C_L + 2\sqrt{3L_\gamma K^2B^*L_\sigma C_D} \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|,
\end{align*}
\]
where
\[
\begin{align*}
\|\Delta \theta\| := & \left( \|W_0^b - W_0^a\|_2 + \|U_0^a - U_0^b\|_2 + \|b_0^a - b_0^b\| \right),
C_D := & \max \{ 3B^*, B_z \}, \quad \text{and} \quad C_z := \left( \min_1 \sigma(0) + 6B^* + 3B_{U_0}B_z \right).
\]
Proof. Due to the smoothness of the activation function, $z^k$ is a continuous function of $\theta$ on the compact set given in Assumption 3. By the mean value theorem in high dimension, we can have:

$$\|z^k(\theta^a) - z^k(\theta^b)\|_2 \leq \left\| \left[ \frac{\partial z^k(\theta)}{\partial \text{vec}(U_L)} \frac{\partial z^k(\theta)}{\partial \text{vec}(U_L)} \ldots \frac{\partial z^k(\theta)}{\partial \text{vec}(W_{\alpha})} \right] \|_{2,\infty} \|\theta^a - \theta^b\|, $$

where $\hat{\theta} = \theta^a + \xi (\theta^b - \theta^a)$ for some $\xi \in (0, 1)$. Due to the convexity of the bounded ball in Euclidean space, we can conclude that $\hat{\theta}$ also satisfy Assumption 3. Hence, by Eq. (21), we can get:

$$\|z^k(\theta^a) - z^k(\theta^b)\|_2 \leq \left( L(k - 1)^2 \alpha^2 (C_z^2 + B_z^2 + N) + \left\| \frac{\partial z^k(\hat{\theta})}{\partial \text{vec}(U_L)} \right\|_{2,\infty} \right)^{\frac{1}{2}} \|\theta^a - \theta^b\|.$$ 

where we utilize the bound in Eq. (24) in the penultimate inequality, and note that $C_z = \Theta(B_{UB}\sqrt{N}) \gg L$. Hence, we have:

$$\|z^k(\theta^a) - z^k(\theta^b)\|_2 \leq \left( 2\alpha \sqrt{L} K C_z L_g \right) \|\theta^a - \theta^b\|.$$ 

Similarly, by Eq. (19) and Eq. (21), we can have:

$$\|z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b)\| \leq \left( 2\alpha \sqrt{L} K C_z L_g \right) \|\theta^a - \theta^b\|.$$ 

Due to the activation function is $L_\sigma$-smooth, we have:

$$\left\| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \right\|_2 \leq \left\| \sigma' \left( W^a_i Z^{(k,l)}(\theta^a) + U^a_i X^a + b_i^a \right) - \sigma' \left( W^b_i Z^{(k,l)}(\theta^b) + U^b_i X^b + b_i^b \right) \right\| \leq L_\sigma \| W \|_2 \left\| Z^{(k,l)}(\theta^a) - Z^{(k,l)}(\theta^b) \right\|_2 + \| U \|_2 \left\| Z^{(k,l)}(\theta^a) - Z^{(k,l)}(\theta^b) \right\|_2 + \| U \|_2 \| X^a - X^b \|_2 + \sqrt{N} \| b_i^a - b_i^b \| \leq L_\sigma \left( 3B \| W \|_2 \| Z^{(k,l)}(\theta^a) - Z^{(k,l)}(\theta^b) \|_2 + \| U \|_2 \| X^a - X^b \|_2 + \sqrt{N} \| b_i^a - b_i^b \| \right)$$

where

$$C_D := \max \{ 3B^*, B_x \}, \quad \| \Delta \theta \| := \left( \| W^a_i - W^b_i \|_2 + \| U^a_i - U^b_i \|_2 + \| b_i^a - b_i^b \| \right).$$
Based on this upper bound, by the telescoping sum, we can easily have:

\[
\| A(k, l_1, l_2, \theta^a) - A(k, l_1, l_2, \theta^b) \|_2 \\
\leq \sum_{l=1}^{l_2} \left( \prod_{i=l+1}^{l_2} T^{(k,i)}(\theta^a) \right) \left( T^{(k,l)}(\theta^a) - T^{(k,l)}(\theta^b) \right) \prod_{j=l-1}^{l_1} T^{(k,i)}(\theta^b) \right) \|_2 \\
\leq \sum_{l=1}^{l_2} \left( \| T^{(k,l)}(\theta^a) - T^{(k,l)}(\theta^b) \|_2 \\
\leq \sum_{l=1}^{l_2} \alpha \left( 2 \| W_i^a - W_i^b \|_2 \| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \|_2 + \| W_i^a \|_2 \| W_i^b \|_2 \| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \|_2 \right) \\
\leq \sum_{l=1}^{l_2} \alpha \left( 2 \| W_i^a - W_i^b \|_2 + \| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \|_2 \right). 
\]

where we note that for \( \prod_{j=l-1}^{l_1} \), \( j \) runs in the reverse order, we also let

\[
T^{(k,i)} := \left( \alpha (I_N \otimes W_i^T) D^{(k,i)}(I_N \otimes W_i) + (1 - \alpha) I_m \right),
\]

and we use that \( \| T^{(k,l)} \|_2 \leq 1 \) from Eq. (20) for the second inequality. We note that:

\[
\left( 2 \| W_i^a - W_i^b \|_2 + \| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \|_2 \right) \\
\leq C_D' L_\sigma \| \Delta \theta_i \| + B_{UB} L_\sigma L_g \| W_0^a - W_0^b \|_2 + L_\sigma \| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \| 
\]

where

\[
C_D' := \max \{ 3B^* + 2/L_\sigma, B_x \}. 
\]

Hence, we can obtain:

\[
\| A(k, l_1, l_2, \theta^a) - A(k, l_1, l_2, \theta^b) \|_2 \\
\leq \alpha L_\sigma \sum_{l=1}^{l_2} \left( C_D' \| \Delta \theta_i \| + B_{UB} L_g \| W_0^a - W_0^b \|_2 + \| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \| \right) \\
\leq \sqrt{3(l_1 - l_2)} \alpha L_\sigma C_D' \| \theta^a - \theta^b \| + \\
\alpha L_\sigma \sum_{l=1}^{l_2} \left( B_{UB} L_g \| W_0^a - W_0^b \|_2 + \| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \| \right) \\
\leq \left( \alpha L_\sigma L \left( B_{UB} L_g + 2\alpha \sqrt{LKC_z L_g} \right) + \alpha \sqrt{3L_\sigma C_D'} \right) \| \theta^a - \theta^b \| \\
\leq \left( 3\alpha^2 L^{3/2} K L_\sigma C_z L_g + \alpha \sqrt{3L_\sigma C_D'} \right) \| \theta^a - \theta^b \| \\
\leq \left( 3\alpha \sqrt{L_\sigma C_z L_g} + \alpha \sqrt{3L_\sigma C_D'} \right) \| \theta^a - \theta^b \| \\
\leq \left( 4\alpha \sqrt{L_\sigma C_z L_g} \right) \| \theta^a - \theta^b \|.
\]

where we use \( \alpha KL \ll 1 \), w.l.o.g. and recall that:

\[
C_z := (\sqrt{m_i}(\sigma(0) + 6B^* + 3B_{UB}B_x)). 
\]

Now, we consider the Lipschitz continuity for \( G^{(k,l)} \), similar to \( D^{(k,l)} \):

\[
\| G^{(k,l)}(\theta^a) - G^{(k,l)}(\theta^b) \|_2 \\
\leq \left\| \left( W_i^a z^{(k,l)}(\theta^a) + U_i^a x^a + b_i^a \right) - \left( W_i^b z^{(k,l)}(\theta^b) + U_i^b x^b + b_i^b \right) \right\|_F \\
\leq L_\sigma \left( C_D \| \Delta \theta_i \| + B_{UB} L_g \| W_0^a - W_0^b \|_2 + \| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \| \right). 
\]

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where we use the fact that the activation is 1-Lipschitz continuous in the first inequality. Therefore, by Eq. [17], we get:

\[
\frac{1}{\alpha} \left\| \frac{\partial \text{vec} \left( f_i(Z^{(k,l)}(\theta^a), X^a) \right)}{\partial \text{vec} (W_i)} - \frac{\partial \text{vec} \left( f_i(Z^{(k,l)}(\theta^b), X^b) \right)}{\partial \text{vec} (W_i)} \right\| \\
\leq \left\| G^{(k,l)}(\theta^a) - G^{(k,l)}(\theta^b) \right\|_2 \\
+ \left( B^* \left\| W_a^a - W_i^b \right\|_2 + B^* \left\| D^{(k,l)}(\theta^a) - D^{(k,l)}(\theta^b) \right\|_2 + \left\| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \right\| \right)
\]

\[
\leq L_\alpha (B^* + 1) \left\| \Delta \theta \| + B_{Ub} L_g \left\| W_a^a - W_i^b \right\|_2 + \left\| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \right\| + B^* \left\| W_a^a - W_i^b \right\|_2 + \left\| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \right\|
\]

\[
\leq 2L_\alpha B^* \left( C_D \| \Delta \theta \| + B_{Ub} L_g \left\| W_a^a - W_i^b \right\|_2 + \left\| z^{(k,l)}(\theta^a) - z^{(k,l)}(\theta^b) \right\| \right).
\]

On the other hand, also by the telescoping sum, we have:

\[
\left\| \prod_{q=k+1}^{k} (\beta_{q+1} (I_N \otimes V_{q}^a) + (1 - \beta_{q+1}) A (\theta^a)) - \prod_{q=k+1}^{k} (\beta_{q+1} (I_N \otimes V_{q}^b) + (1 - \beta_{q+1}) A (\theta^b)) \right\|_2
\]

\[
\leq \sum_{q=k+1}^{k} \left( 2\beta_{q+1} \gamma B_L \left\| W_{L+1}^a - W_{L+1}^b \right\|_2 + (1 - \beta_{q+1}) \left\| A(q, L, 1, \theta^a) - A(q, L, 1, \theta^b) \right\|_2 \right)
\]

\[
\leq k \left\| W_{L+1}^a - W_{L+1}^b \right\|_2 + \sum_{q=k+1}^{k} \left\| A(q, L, 1, \theta^a) - A(q, L, 1, \theta^b) \right\|_2,
\]

where we utilize \( \gamma B_L^2 < 1 \) in the last inequality. Since for any \( k, l_1 \) and \( l_2 \), \( \| A(\theta^a) - A(\theta^b) \|_2 \)
share the same upper bound, in the following proof, we omit the index \( (k, l_1, l_2) \). Combining all things together, by Eq. [16], we have:

\[
\left\| \frac{\partial z^K(\theta^a)}{\partial \text{vec}(W_i)} - \frac{\partial z^K(\theta^b)}{\partial \text{vec}(W_i)} \right\|_2
\]

\[
\leq K \left( (2K^2 \alpha C_Z) \left\| W_{L+1}^a - W_{L+1}^b \right\|_2 + \left\| A(\theta^a) - A(\theta^b) \right\|_2 \right) + \\
\left\| \frac{\partial \text{vec} \left( f_i(Z^{(k,l)}(\theta^a)) \right)}{\partial \text{vec} (W_i)} - \frac{\partial \text{vec} \left( f_i(Z^{(k,l)}(\theta^b)) \right)}{\partial \text{vec} (W_i)} \right\|_2
\]

\[
\leq 2\alpha K^3 C_Z \left\| W_{L+1}^a - W_{L+1}^b \right\|_2 + 2\alpha K B^* L_g B_{Ub} L_g \left\| W_0^a - W_0^b \right\|_2 + 2\alpha K B^* L_g C_D \| \Delta \theta \| + \\
(2\alpha L_g B^* 2\sqrt{L} K C_Z L_g + 4\alpha K \sqrt{L} L_g C_Z L_g) \| \tilde{\theta}^a - \tilde{\theta}^b \|
\]

\[
\leq 2\alpha K^3 C_Z \left\| W_{L+1}^a - W_{L+1}^b \right\|_2 + 2\alpha K B^* L_g B_{Ub} L_g \left\| W_0^a - W_0^b \right\|_2 + 2\alpha K B^* L_g C_D \| \Delta \theta \| + \\
(8\alpha^2 K^2 \sqrt{L} B^* L_g C_Z L_g) \| \tilde{\theta}^a - \tilde{\theta}^b \|
\]

\[
\leq 2\alpha K B^* L_g C_D \| \Delta \theta \| + (9\alpha^2 K \sqrt{L} B^* L_g C_Z L_g) \| \tilde{\theta}^a - \tilde{\theta}^b \|
\]

where, w.l.o.g, we utilize the fact \( K^3 \sqrt{L} \leq B^* \). An Immediate consequence we can get is:

\[
\left\| J_{\alpha K}(\tilde{\theta}^a) - J_{\alpha K}(\tilde{\theta}^b) \right\|_2
\]

\[
\leq \sum_{l=1}^{L} \left( \left\| \frac{\partial z^K(\theta^a)}{\partial \text{vec}(W_l)} - \frac{\partial z^K(\theta^b)}{\partial \text{vec}(W_l)} \right\|_2 + \left\| \frac{\partial z^K(\theta^a)}{\partial \text{vec}(U_l)} - \frac{\partial z^K(\theta^b)}{\partial \text{vec}(U_l)} \right\|_2 + \left\| \frac{\partial z^K(\theta^a)}{\partial \text{vec}(b_l)} - \frac{\partial z^K(\theta^b)}{\partial \text{vec}(b_l)} \right\|_2 \right)
\]

\[
\leq L \left( 9\alpha^2 K \sqrt{L} B^* L_g C_Z L_g \right) + 2\sqrt{L} \alpha K B^* L_g C_D \| \tilde{\theta}^a - \tilde{\theta}^b \|
\]

\[
\leq 9\alpha \sqrt{L} B^* L_g C_Z L_g + 2\sqrt{L} \alpha K B^* L_g C_D \| \tilde{\theta}^a - \tilde{\theta}^b \|
\]

We now finish the proof. \( \square \)
### G.2 Proof for Theorem 5

Before presentation the main results, we assume a mild condition for initialization.

**Assumption 5 (Initial conditions).** We assume the initialized parameters of OptDeq in Eq. (4) satisfy:

\[
\begin{align*}
\| W_0^0 \|_2 & \leq \frac{3}{4}, \quad \sigma_{\min}(W_0^0) \geq \left(\frac{1}{4} + \sigma_m\right), \\
\max \{ \| U^0_0 \|_2, \| b^0_0 \| \} & \leq \frac{B_{Ub}}{2}, \quad \| W_0^0 \|_2 \leq \frac{B_x}{2L_g}, \\
\| W_{L+1}^0 \|_2 & \leq \frac{B_L}{2}, \quad \sigma_{\min}(W_{L+1}^0) > 0,
\end{align*}
\]

where \( L_g \) is the Lipschitz constant of the function \( g(\cdot) \) and \( \sigma_{\min}(\cdot) \) is the smallest singular value of a matrix.

Due to the extractor \( g(X, \cdot) \) is \( L_g \)-Lipschitz continuous, the initial conditions is equivalent to:

\[
\begin{align*}
\| W_0^0 \|_2 & \leq \frac{3}{4}, \quad \sigma_{\min}(W_0^0) \geq \left(\frac{1}{4} + \sigma_m\right), \\
\max \{ \| U^0_0 \|_2, \| b^0_0 \| \} & \leq \frac{B_{Ub}}{2}, \\
\| W_{L+1}^0 \|_2 & \leq \frac{B_L}{2}, \quad \sigma_{\min}(W_{L+1}^0) > 0,
\end{align*}
\]

Now, we are ready to prove the main theorem. We denote

\[
\begin{align*}
C_D & := \max \{ 3B^*, B_x \}, \\
C_z & := (\sqrt{\min_i \sigma(0)} + 6B^* + 3B_{Ub}B_x) \\
B_J & := 2\alpha \sqrt{LKC_z}, \\
C_J & := \left(9\alpha \sqrt{L}B^* L_{\sigma} C_z L_g + 2\sqrt{3L} \alpha KB^* L_{\sigma} C_D\right), \\
Q_0 & := \frac{1}{4} K^2 L^2 \alpha^2 \sigma_m^2 N \sigma_{\min}(W_{L+1}^0), \\
Q_1 & := \left(C_J \sqrt{\ell(\tilde{\theta}^0)} + 2\alpha \sqrt{LKC_z L_g B_J B_L} + B_J \sqrt{\ell(\tilde{\theta}^0)}\right), \\
Q_2 & := \frac{10C_z B_J L_g}{Q_0},
\end{align*}
\]

where \( \ell(\tilde{\theta}^0) \) is the training loss at initialization.

**Theorem (Global Convergence).** Suppose Assumption 4 and Assumption 5 hold. Assume that the activation function is \( L_{\sigma} \)-Lipschitz smooth, strongly monotone and \( 1 \)-Lipschitz continuous. Let the learning rate be \( \eta < \min \left\{ \frac{1}{Q_0}, \frac{1}{Q_1} \right\} \). If the training data size \( N \) is in the order:

\[
\sqrt{N} = \Omega \left( \frac{B_L \sqrt{\ell(\tilde{\theta}^0)}}{\kappa^2 \sigma_m^2 \sigma_{\min}(W_{L+1}^0)} \right),
\]

then the training loss vanishes at a linear rate as:

\[
\ell(\tilde{\theta}^t) \leq \ell(\tilde{\theta}^0)(1 - \eta Q_0)^t,
\]

where \( t \) is the number of iteration. Furthermore, the network parameters also converge to a global minimizer \( \tilde{\theta}^* \) at a linear speed:

\[
\| \tilde{\theta}^t - \tilde{\theta}^* \| \leq Q_2 (1 - \eta Q_0)^{t/2}.
\]
Proof. We already have $B^*, B_x < C_z = \mathcal{O}(B_{\theta} \sqrt{N})$. Hence, when the data size $N$ is large enough, i.e., when

$$\sqrt{N} = \Omega \left( \frac{B_L \sqrt{\ell(\theta^0)}}{\kappa^2 \sigma_m^2 \sigma_{\min}^2(W_{L+1}^0)} \right),$$

we have:

$$\frac{2 \alpha K C_z B_L}{Q_0} \| y^0 - y_0 \| = \mathcal{O} \left( \frac{B_L \sqrt{\ell(\theta^0)}}{\kappa^2 \sigma_m^2 \sigma_{\min}^2(W_{L+1}^0)} \right) = \mathcal{O}(1).$$

Hence, w.l.o.g we let:

$$\frac{2 \alpha K C_z B_L}{Q_0} \| y^0 - y_0 \| \leq \frac{1}{4}.$$

We denote the index $t$ to represent the iteration number during training, i.e., $\{W_t^0, \theta_t, W_t^b\}$ is the learnable parameters at the $t$-th iteration. We show by induction that, for every $t > 0$, $t \in [1, \tilde{t}]$ and $t \in [1, L]$, the following holds:

$$\begin{cases}
\| W_t^0 \|_2 \leq 1, & \max \{ \| U_t^0 \|_2, \| b_t^0 \| \} \leq B_{\theta}, \\
\| W_{L+1}^t \|_2 \leq B_L, & \| X^t \|_F \leq B_x
\end{cases}$$

By the initial condition, Eq.(32) holds for $t = 0$ clearly. We now suppose that Eq.(32) holds for all iterations from $0$ to $\tilde{t}$, and show the claim for iteration $\tilde{t} + 1$. Note that for every $t \in [1, L]$ and $t \in [1, \tilde{t}]$, we have:

$$\begin{align*}
\| W_{t+1} - W_0 \|_2 & \leq \sum_{i=1}^{t} \| W_{t+1}^i - W_i \|_2 \leq \eta \sum_{i=1}^{t} \| \text{vec} \left( \nabla W_i \ell(\theta^i) \right) \| \\
& \leq \eta K \alpha C_z B_L \sum_{i=1}^{t} \| x^i \|_2 \leq \frac{1}{Q_0} \alpha K C_z B_L (1 - s^2) \frac{1}{\sqrt{1 - s}} \| y^0 - y_0 \| \\
& \leq \frac{2 Q_0 \alpha K C_z B_L}{Q_0} \| y^0 - y_0 \| \leq \frac{1}{4},
\end{align*}$$

where we use the bound in Eq.(32) and let $s := (1 - \eta Q_0)^{1/2}$, the last inequality comes from the initial condition on $Q_0$, see Eq.(31). Similarly, when the data size is large enough, we can also have:

$$\begin{align*}
\| U_{t+1} - U_0 \|_2 & \leq \frac{2 Q_0 \alpha K B_x B_L}{Q_0} \| y^0 - y_0 \| < 1/3 \leq \frac{B_{\theta}}{2}, \\
\| b_{t+1}^0 - b_0^0 \|_2 & \leq \frac{2 Q_0 \alpha K \sqrt{N} B_L}{Q_0} \| y^0 - y_0 \| < 1/3 \leq \frac{B_{\theta}}{2}.
\end{align*}$$

And by Eq.(24) and Eq.(26)

$$\begin{align*}
\| x_{t+1}^0 - x_0 \| & \leq L_g \| W_{t+1}^0 - W_0^0 \|_F \leq \frac{2 Q_0 \alpha K B_{\theta} B_L L_g^2}{Q_0} \| y^0 - y_0 \| < B_x, \\
\| W_{L+1}^t - W_0^0 \|_2 & \leq \frac{2 Q_0 \alpha K B_x B_L L_g^2}{Q_0} \| y^0 - y_0 \| < \frac{1}{2} \sigma_{\min}(W_{L+1}^0).
\end{align*}$$

Thus by Weyl’s inequality, we obtain:

$$\begin{cases}
\| W_{t+1} \|_2 \leq 1, & \max \{ \| U_t \|_2, \| b_t \|_2 \} \leq B_{\theta}, \\
\| W_{L+1}^t \|_2 \leq B_L, & \| X^t \|_F \leq B_x
\end{cases}$$

$$\begin{cases}
\sigma_{\min}(W_t) \geq \sigma_m, & \sigma_{\min}(W_{L+1}^t) \geq \frac{1}{2} \sigma_{\min}(W_{L+1}^0).
\end{cases}$$

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We now provide a Lipschitz constant for the gradient of the loss function. Given two parameters $\tilde{\theta}^a$ and $\tilde{\theta}^b$ such that satisfies Assumption 3 and has the bounds $\ell(\tilde{\theta}^a) \leq \ell(\tilde{\theta}^b)$ and $\ell(\tilde{\theta}^a) \leq \ell(\tilde{\theta}^b)$, we have

\[
\left\| \nabla \ell(\tilde{\theta}^a) - \nabla \ell(\tilde{\theta}^b) \right\| \\
= \left\| J_{\sigma}^T(\tilde{\theta}^b)(I_N \otimes (W_{L+1}^b)^\top)(y^a - y_0) - J_{\sigma}^T(\tilde{\theta}^a)(I_N \otimes (W_{L+1}^a)) (y^b - y_0) \right\| \\
\leq \left\| J_{\sigma}^T(\tilde{\theta}^a) \right\|_2 \left\| \ell(\tilde{\theta}^a) + B_J \left\| W_{L+1}^a - W_{L+1}^b \right\| \sqrt{\ell(\tilde{\theta}^a) + B_J B_F^2} \left\| x^a - x^b \right\| \right\| \\\n\leq \left( C_J \sqrt{\ell(\tilde{\theta}^a) + B_J \sqrt{\ell(\tilde{\theta}^a)}} + 2\alpha \sqrt{\tilde{L}K C_z L_B} B_J B_L + B_J \sqrt{\ell(\tilde{\theta}^a)} \right) \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\| = Q_1 \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\| ,
\]

where we use the bound in Eq. (29) in the second and third inequality, and the bound from Eq. (21) and Eq. (26) for

\[
\left\| J_{\sigma}^T \right\| \leq \left( 3LK^2\alpha^2 C_z^2 + \alpha^2 K^2 L^2 B^2 U_B + 4K^2\eta^2(B^*)^2 B^2_F \right)^{\frac{1}{2}} \lesssim 2\alpha \sqrt{L}K C_z := B_J.
\]

When $\eta \leq 1/Q_1$, the Lipschitz bound $\left\| \nabla \ell(\tilde{\theta}^a) - \nabla \ell(\tilde{\theta}^b) \right\| \leq Q_1 \left\| \tilde{\theta}^a - \tilde{\theta}^b \right\|$ implies that:

\[
\ell(\tilde{\theta}^{t+1}) \leq \ell(\tilde{\theta}^t) + \left\langle \nabla \ell(\tilde{\theta}^t), \tilde{\theta}^{t+1} - \tilde{\theta}^t \right\rangle + \frac{Q_1}{2} \left\| \tilde{\theta}^{t+1} - \tilde{\theta}^t \right\|^2 \\
\leq \ell(\tilde{\theta}^t) - \eta \left\| \nabla \ell(\tilde{\theta}^t) \right\|^2 + \frac{Q_1}{2} \eta \left\| \nabla \ell(\tilde{\theta}^t) \right\|^2 \\
\leq \ell(\tilde{\theta}^t) - \frac{\eta}{2} \left\| \nabla \ell(\tilde{\theta}^t) \right\|^2 + \frac{Q_1}{2} \eta \left\| \nabla \ell(\tilde{\theta}^t) \right\|^2 \\
\leq \ell(\tilde{\theta}^t) - \frac{\eta}{2} \left( K^2 L \alpha^2 \sigma^2 \sigma^2 m N \sigma^2 \min (W_{L+1}^t) \| y^t - y_0 \|^2 \right) \\
\leq \left( 1 - \frac{\eta}{4} K^2 L \alpha^2 \sigma^2 \sigma^2 m N \sigma^2 \min (W_{L+1}^t) \right) \ell(\tilde{\theta}^t) = \left( 1 - \eta Q_0 \right) \ell(\tilde{\theta}^t),
\]

where the third inequality comes from the fact Eq. (23) and recall that

\[
Q_0 := \frac{1}{4} K^2 L \alpha^2 \sigma^2 \sigma^2 m N \sigma^2 \min (W_{L+1}^t).
\]

So far, we have proven the hypothesis in Eq. (32).

We start to show that the sequence $\{\tilde{\theta}^t\}_{t=1}^{\infty}$ is a Cauchy sequence. Given any $\epsilon > 0$ and the index $r > 0$, we chose two indices $j > i \geq r$. Then, we have:

\[
\left\| \tilde{\theta}^j - \tilde{\theta}^i \right\| \leq \left\| W_{L+1}^j - W_{L+1}^i \right\|_F + \left\| \tilde{\theta}^j - \tilde{\theta}^i \right\| + \left\| W_0^j - W_0^i \right\|_F \\
\leq \eta \sum_{s=i+1}^{j-1} \left( \left\| W_{L+1}^{s+1} - W_{L+1}^s \right\|_F + \left\| \tilde{\theta}^{s+1} - \tilde{\theta}^s \right\| + \left\| W_0^{s+1} - W_0^s \right\|_F \right) \\
\leq \eta \sum_{s=i+1}^{j-1} \left( \left\| W_{L+1}^{s+1} - W_{L+1}^s \right\|_F + \sum_{t=1}^{L} \left( \left\| W_{t+1}^{s+1} - W_{t+1}^t \right\| + \left\| U_{t+1}^{s+1} - U_{t+1}^t \right\| + \left\| b_{t+1}^{s+1} - b_{t+1}^t \right\| \right) + \left\| W_0^{s+1} - W_0^s \right\|_F \right) \\
\leq \sum_{s=i+1}^{j-1} \eta \| y^s - y_0 \| (3C_z B_L + B_U L_B B_L + 3K B^*) \\
\leq (1 - \eta Q_0)^{j-i/2} \left( \sum_{s=0}^{j-i} (1 - \eta Q_0)^{s/2} \| y^0 - y_0 \| \right) \eta (3C_z B_L + B_U L_B B_L + 3K B^*) \\
\leq (1 - \eta Q_0)^{j-i/2} (3C_z B_L + B_U L_B B_L + 3K B^*) \left( \frac{1}{Q_0} (1 - s^2) \frac{1 - s^{j-i}}{1 - s} \right) \| y^0 - y_0 \| \\
\leq (1 - \eta Q_0)^{j-i/2} \frac{2}{Q_0} (3C_z B_L + B_U L_B B_L + 3K B^*) \| y^0 - y_0 \|. 
\]
where (a) comes from Eq. (22), Eq. (25) and Eq. (27) and the assumption \( \alpha KL < 1 \), and in (b) we set \( s = \sqrt{1 - \eta Q_0} \). Note that \( (1 - \eta Q_0)^{i/2} \leq (1 - \eta Q_0)^{r/2} \) and thus we can select a sufficiently large \( r \) such that \( \| \tilde{\theta}^i - \tilde{\theta}^j \| \leq \epsilon \). Hence, we can conclude that \( \{ \tilde{\theta}^i \}_{i=1}^{\infty} \) is a Cauchy sequence, and thus has a convergent point \( \tilde{\theta}^* \). Due to the continuity, we have:

\[
\ell(\tilde{\theta}^*) = \ell(\lim_{t \to \infty} \tilde{\theta}^t) = \lim_{t \to \infty} \ell(\tilde{\theta}^t) = 0,
\]

where the last equality comes from Eq. (32). Hence, \( \tilde{\theta}^* \) is a global minimizer, and the rate of convergence is:

\[
\| \tilde{\theta}^i - \tilde{\theta}^* \| = \lim_{j \to \infty} \| \tilde{\theta}^i - \tilde{\theta}^j \| \leq (1 - \eta Q_0)^{i/2} Q_2,
\]

note that

\[
\frac{2}{Q_0} (3C_z B_L + B_{U_0} L_g B_L + 3K B^*) \| y^0 - y_0 \| \leq \frac{10C_z B_L L_g}{Q_0} := Q_2.
\]

We now finish the whole proof. \( \square \)