Edge-decompositions of $O(m)$-edge-connected graphs into isomorphic copies of a fixed tree of size $m$

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Abstract

In this paper, we show that every $O(m)$-edge-connected simple graph $G$ of size divisible by $m$ with minimum degree at least $2^{O(m)}$ has an edge-decomposition into isomorphic copies of any given tree $T$ of size $m$. Moreover, the minimum degree condition can be dropped for graphs $G$ with girth greater than the diameter of $T$. These results improve two results due to Bensmail, Harutyunyan, Le, Merker, and Thomassé (2017) and Merker (2017) who gave a factorial upper bound on the necessary edge-connectivity.

Keywords: Tree-decomposition; partition-connectivity; edge-connectivity; minimum degree.

1 Introduction

In this paper, graphs may have multiple edges but loops are avoided and simple graphs have neither multiple edges nor loops. Let $G$ be a graph. The vertex set, the edge set, and the minimum degree of $G$ are denoted by $V(G)$, $E(G)$, and $\delta(G)$. The degree $d_G(v)$ of a vertex $v$ is the number of edges of $G$ incident to $v$. The girth of $G$ is the size of a shortest cycle that is denoted by $\text{girth}(G)$. If $G$ has no cycle, the girth is defined to be infinity. Likewise, we denoted by $\text{diam}(T)$ the diameter of $G$ that is the maximum distance between any two vertices of $G$. If $G$ is disconnected, the diameter is defined to be infinity. Note that when $G$ is a tree, the diameter is the size of a longest path. For a partition $P$ of $V(G)$, we denote by $e_G(P)$ the number of edges of $G$ connecting different parts of $P$. A factor of a graph refers to a spanning subgraph, and an even factor is a spanning subgraph with even degrees. For a graph $G$, we denote by $G[X,Y]$ the bipartite induced factor of $G$ with the bipartition $(X,Y)$. In addition, we denote by $G[X]$ the induced factor with the vertex set $X$. The bipartite index $\text{bi}(G)$ of a graph $G$ refers to the minimum number of edges for deleting to make the graph bipartite. For a graph $G$, we say that $G$ is modulo $m$-regular, if its vertices have degrees divisible by $m$. For a bipartite graph $G$, we say that $G$ is modulo semi-$m$-regular, if vertices of one side have degrees divisible by $m$. A graph $G$ is said to be $m$-edge-connected, if for any
some of its vertices. More precisely, there must be a surjective map\ngiving an orientation such that for each vertex \(v\), \(d^+_T(v) \geq \lambda(v)\), where \(\lambda\) is a nonnegative integer-valued function on \(V(G)\) and \(d^+_T(v)\) denotes the out-degree of \(v\) in \(T\). Note that an \((m_1 + m_2, l_1 + l_2)\)-partition-connected graph can be edge-decomposed into an \((m_1, l_1)\)-partition-connected factor and an \((m_2, l_2)\)-partition-connected factor. For two integers \(x\) and \(y\), we say that \(x \equiv y\), if \(x - y\) is divisible by \(k\). Let \(T\) be a tree. A vertex in \(T\) is said to be \textit{internal} if it has degree at least two. We denote by \(\text{int}(T)\) the set of internal vertices of \(T\). A \textit{homomorphic copy} \(T'\) of \(T\) is a graph obtained from \(T\) by identifying some of its vertices. More precisely, there must be a surjective mapping \(f : V(T) \to V(T')\) such that \(uv\) is an edge of \(T\) if and only if \(f(u)f(v)\) is an edge of \(T'\). We say that a graph \(G\) admits a \(T^*\)-\textit{edge-decomposition}, if the edges of \(G\) can be decomposed into homomorphic copies of \(T\). If \(f\) is bijective, then \(T'\) is also called an isomorphic copy of \(T\). We say that a graph \(G\) admits a \(T\)-\textit{edge-decomposition}, if the edges of \(G\) can be decomposed into isomorphic copies of \(T\). If \(\mathcal{A}\) is a \(T^*\)-edge-decomposition (or \(T\)-edge-decomposition), for any pair \(v \in V(G)\) and \(t \in V(T)\), we denote by \(\mathcal{A}(v, t)\) the set of all homomorphic copies in \(\mathcal{A}\) including \(v\) with the preimage \(t\). In addition, we denote by \(\mathcal{A}(v)\) the set of all of homomorphic copies in \(\mathcal{A}\) including \(v\). Let \((A, B)\) be the bipartition of \(T\). For a bipartite graph \(G\) with bipartition \((X, Y)\), we say that \(G\) admits a \((B, Y)\)-\textit{compatible} \(T\)-edge-decomposition (resp. \(T^*\)-edge-decomposition), if every copy of \(T\) its vertices in \(Y\) are image of vertices in \(B\). Consider an edge-coloring of \(T\) with \(m\) colors \(1, \ldots, m\), where \(m = |E(T)|\). Assume that \(G\) admits a \((B, Y)\)-compatible \(T^*\)-edge-decomposition. Let \(F_1, \ldots, F_m\) be a factorization of \(G\) such that every factor \(F_i\) consists of all edges in \(G\) having the color \(i\) in a copy of \(T\). It is not difficult to see that for all \(v \in Y\), \(d_{F_i}(v) = d_{F_j}(v)\), where \(i\) and \(j\) are the colors of two arbitrary edges in \(T\) incident with a common vertex in \(B\). Likewise, for all \(v \in X\), \(d_{F_i}(v) = d_{F_j}(v)\), where \(i\) and \(j\) are the colors of two arbitrary edges in \(T\) incident with a common vertex in \(A\). We call such a factorization \((B, Y)\)-\textit{compatible} \(T\)-\textit{equitable factorization} or (briefly \(T\)-equitable factorization). It was proved in [19, Lemma 3.2] that a bipartite graph admits a compatible \(T\)-equitable factorization if and only if it admits a compatible \(T^*\)-edge-decomposition. Throughout this article, all trees are nontrivial, all variables \(\lambda\) are nonnegative integers, and all variables \(b\) and \(k\) are positive integers.

In 2006 Barát and Thomassen proposed the following interesting conjecture on graph edge-decompositions.

**Conjecture 1.1.** ([2]) For every tree \(T\) of size \(m\), there exists a positive integer \(k_T\) such that every \(k_T\)-edge-connected simple graph of size divisible by \(m\) admits a \(T\)-edge-decomposition.

To prove this conjecture, the first attempts were made by Thomassen (2008) [21, 22] who proved it for the 3-path and the 4-path. Later, Barát and Gerbner (2012) [1] proved it for a small bistar of size 4, and Thomassen (2012, 2013) [23, 24, 25] proved it for every path whose length is a power of two, all stars, and a certain family of bistars. After several years, Botler, Mota, Oshiro, and Wakabayashi (2016, 2017) [5, 6]
confirmed the conjecture for all paths and Merker (2017) [19] confirmed it for trees with diameter at most 4.

Finally, Bensmail, Harutyunyan, Le, Merker, and Thomassé (2017) [4] proved Conjecture 1.1 completely by giving the following factorial upper bound on $k_T$ for trees $T$ of size $m$. The parameter $f(m, \lambda)$ was already defined in [19] which is at least $(\lambda + 1)m^m$ (by reviewing proofs).

**Theorem 1.2.** ([4]) Every $(8m^{2m+3} + 8f(m, m10^5)m) + 12m)$-edge-connected simple graph of size divisible by $m$ admits an edge-decomposition into isomorphic copies of any given fixed tree of size $m$.

For stars, Thomassen (2012) proved a quartic upper bound on $k_T$, and next Lovász, Thomassen, Wu, and Zhang (2013) [18] refined Thomassen’s result by replacing the upper bound by a linear bound.

**Theorem 1.3.** ([18, 13]) Every $(3m − 3)$-edge-connected simple graph of size divisible by $m$ has an $m$-star-edge-decomposition.

For paths, Botler, Mota, Oshiro, and Wakabayashi (2017) [6] presented a quartic upper bound on $k_T$ in bipartite graphs. Next, Bensmail, Harutyunyan, Le, and Thomassé (2018) [3] showed that dependence on edge-connectivity is not inherently necessary by reducing the upper bound to the constant number 24 in graphs with sufficiently large minimum degrees. Finally, Klímošová and Thomassé (2019) [17] improved their result to the following sharp version.

**Theorem 1.4.** ([17]) For every path $P$, there exists a positive integer $l_P$ such that every 3-edge-connected simple graph of size divisible by $|E(P)|$ with minimum degree at least $l_P$ has a $P$-edge-decomposition.

In this paper, we improve Theorem 1.2 by reducing the upper bound on the necessary edge-connectivity to a linear bound and reducing the upper bound on the necessary minimum degree to an exponential bound. In particular, we show that the minimum degree condition can be dropped for graphs $G$ with girth greater than the diameter of $T$.

**Theorem 1.5.** Every $O(m)$-edge-connected simple graph $G$ of size divisible by $m$ with minimum degree at least $2^{O(m)}$ has an edge-decomposition into isomorphic copies of $T$. In particular, the minimum degree condition can be dropped for graphs $G$ with girth greater than the diameter of $T$.

This is also an improvement of a result due to Merker (2017) [19, Theorem 6.2] who provided an implicit factorial upper bound of $8m^{2m+3} + 8f(m, 0) + 12m$ on the necessary edge-connectivity for graphs $G$ with girth greater than the diameter of $T$.

2 Basic tools: partition-connected graphs

In this section, we state some preliminary results and provide a simple proof for each of them based on the following theorem. This result can be proved by a combination of Frank’s Orientation Theorem [9, Theorem
Theorem 2.1. (See [8, 28]) A graph $G$ is $(m, l)$-partition-connected if and only if for every partition $P$ of $V(G)$,

$$e_G(P) \geq m(|P| - 1) + \sum_{v \in P} l(v),$$

where $m$ is a nonnegative integer and $l$ is a nonnegative integer-valued function on $V(G)$.

Let us start with the following theorem which gives a sufficient condition for the existence of partition-connected factors with small degrees on a specified independent set.

Theorem 2.2. ([15]) Let $\varepsilon$ be a real number with $0 < \varepsilon \leq 1$. Let $G$ be a graph with an independent set $X \subseteq V(G)$. If $G$ is $([m/\varepsilon], [l/\varepsilon])$-partition-connected, then it has an $(m, l)$-partition-connected factor $H$ such that for each $v \in X$,

$$d_H(v) \leq \lceil \varepsilon d_G(v) \rceil,$$

where $m$ is a nonnegative integer and $l$ is a nonnegative integer-valued function on $V(G)$.

Proof. For each $v \in X$, define $l'(v) = [(1 - \varepsilon)d_G(v)]$, and for each $v \in V(G) \setminus X$, define $l'(v) = 0$. It is enough to show that $G$ is $(m, l + l')$-partition-connected. This implies that $G$ can be edge-decomposed into an $(m, l)$-partition-connected factor $H$ and a $(0, l')$-partition-connected factor $F$. Thus for each vertex $v \in X$, $d_H(v) = d_G(v) - d_F(v) \leq d_G(v) - l'(v) = \lceil \varepsilon d_G(v) \rceil$. Let $P$ be a partition of $V(G)$, let $S$ be the set of vertices $v$ with $\{v\} \in P$. Since $G$ is $([m/\varepsilon], [l/\varepsilon])$-partition-connected, by Theorem 2.1, we must have $\varepsilon e_G(P) \geq m(|P| - 1) + \sum_{v \in S} l(v)$. Since $S \cap X$ is an independent set, we must also have $(1 - \varepsilon)e_G(P) \geq \sum_{v \in S \cap X} d_G(v)$. Therefore,

$$e_G(P) \geq m(|P| - 1) + \sum_{v \in S} l(v) + \sum_{v \in S \cap X} (1 - \varepsilon)d_G(v) \geq m(|P| - 1) + \sum_{\{v\} \in P} (l(v) + l'(v)).$$

Hence $G$ is $(m, l + l')$-partition-connected by Theorem 2.1, and the proof is completed.

The following theorem gives a sufficient edge-connectivity condition for a graph to be partition-connected.

Theorem 2.3. ([16]) Every $2m$-edge-connected graph $G$ satisfying $d_G(v) \geq 2m + 2l(v)$ for each vertex $v$ is also $(m, l)$-partition-connected (even by deleting at most $m$ arbitrary edges), where $m$ is a nonnegative integer and $l$ is a nonnegative integer-valued function on $V(G)$.

Proof. Let $M$ be a factor of $G$ with size at most $m$ and let $G' = G \setminus E(M)$. Let $P$ be a partition of $V(G)$. Then

$$e_{G'}(P) = \sum_{A \in P} \frac{1}{2} d_{G'}(A) = \sum_{A \in P, |A| \geq 2} \frac{1}{2} d_G(A) - e_M(P) + \sum_{\{v\} \in P} \frac{1}{2} d_G(v) \geq m(|P| - 1) + \sum_{\{v\} \in P} l(v).$$

Thus by Theorem 2.1, the graph $G'$ is $(m, l)$-partition-connected, and the proof is completed. \qed
The following theorem gives a sufficient partition-connectivity condition for the existence of a bipartite partition-connected factor. The edge-connected version of this result was already introduced in [22, Proposition 1].

**Theorem 2.4.** ([10]) Every \((2m, 2l)\)-partition-connected graph \(G\) has a bipartite \((m, l)\)-partition-connected factor \(H\), where \(m\) is a nonnegative integer and \(l\) is a nonnegative integer-valued function on \(V(G)\). In addition, for every vertex set \(A\), \(d_H(A) \geq d_G(A)/2\).

**Proof.** Let \(G\) be a \((2m, 2l)\)-partition-connected graph. Let \(H\) be a bipartite factor of \(G\) with maximum \(|E(H)|\). We claim that \(d_H(A) \geq d_G(A)/2\) for every vertex set \(A\). Suppose, to the contrary, that \(d_H(A) < d_G(A)/2\) for a vertex set \(A\). Define \(X_0 = (X \setminus A) \cup (Y \cap A)\) and \(Y_0 = (Y \setminus A) \cup (X \cap A)\), where \((X, Y)\) is the bipartition of \(H\). It is not difficult to see that the graph \(G[X_0, Y_0]\) is a bipartite factor of \(G\) with more edges than \(H\) which is a contradiction. Let \(P\) be a partition of \(V(H)\). By Theorem 2.1, we must have \(e_G(P) \geq 2m(|P| - 1) + \sum_{v \in P} 2l(v)\). Thus

\[
e_H(P) = \sum_{A \in P} \frac{1}{2} d_H(A) \geq \sum_{A \in P} \frac{1}{4} d_G(A) = \frac{1}{2} e_G(P) \geq m(|P| - 1) + \sum_{v \in P} l(v).
\]

Therefore, by Theorem 2.1, the graph \(H\) is \((m, l)\)-partition-connected, and the proof is completed. \(\square\)

3 Preliminary results: \(T\)-equitable factorizations with large minimum degrees

The following theorem was originally introduced by Merker (2017) for finding edge-decompositions of graphs into homomorphic copies of a fixed tree. Shortly thereafter, it was used for proving Theorem 1.2. In this section, we formulate an explicit linear upper bound on \(f(m, 0)\) provided that the ratio of \(m_b/m_0\) is small enough and all integers \(m_1, \ldots, m_b\) are equal to 2, except possibly \(m_b\). In addition, we strengthen it to guarantee the existence of factors \(G_i\) having large minimum degrees whenever \(G\) itself has large enough minimum degree. This feature plays an important role in the proof of Theorems 1.5.

**Theorem 3.1.** ([19, Theorem 3.3]) For any two positive integers \(m\) and \(\lambda\), there exists a positive integer \(f(m, \lambda)\) such that the following holds: If \(m, m_0, \ldots, m_b\) are positive integers satisfying \(m = \sum_{0 \leq i \leq b} m_i\), then every \(f(m, \lambda)\)-edge-connected bipartite simple graph \(G\) with bipartition \((X, Y)\) in which each vertex in \(X\) has degree divisible by \(m\) can be edge-decomposed into \(\lambda\)-edge-connected factors \(G_0, \ldots, G_b\) satisfying the following properties:

- For each \(v \in X\), \(d_{G_i}(v) = \frac{m_i}{m} d_G(v)\), when \(i \in \{0, \ldots, b\}\).
- For each \(v \in Y\), \(d_{G_i}(v)\) is divisible by \(m_i\), when \(i \in \{1, \ldots, b\}\).
3.1 Almost even factorizations

In this subsection, we improve the edge-connectivity needed in Theorem 3.1 to \(O(\frac{m}{m_0})\) provided that all integers \(m_1, \ldots, m_b\) are equal to the same number 2. Our proof is based on the following recent result about the existence of even factors with restricted degrees.

**Theorem 3.2.** ([11]) Let \(G\) be a graph, let \(b\) be a positive integer, let \(\varepsilon\) be a real number with \(0 < \varepsilon \leq 1\). If \(G\) is odd-\(\left\lceil \frac{1}{\varepsilon} \right\rceil\)-edge-connected, then \(G\) can be edge-decomposed into factors \(G_0, \ldots, G_b\) such that for each vertex \(v\),

- \(|d_{G_i}(v) - \varepsilon d_G(v)| < 2\).
- \(d_{G_i}(v)\) is even and \(|d_{G_i}(v) - \frac{1}{b/m} d_G(v)| < 2\), when \(i \in \{1, \ldots, b\}\).

Now, we are in a position to significantly refine Theorem 3.1 for a particular case. It should be pointed out that a weaker version of the special cases \(m_0 \in \{1, 2\}\) and \(l = 0\) of this result were already proved in [6, Lemma 4.3 and Corollary 4.5].

**Corollary 3.3.** Let \(m, m_0, \ldots, m_b\) be positive integers satisfying \(m = \sum_{0 \leq i \leq b} m_i\) and \(m_1 = \cdots = m_b = 2\). Let \(G\) be a bipartite graph with bipartition \((X, Y)\) in which each vertex in \(X\) has degree divisible by \(m\), and let \(l\) be a nonnegative integer-valued function on \(V(G)\). If \(G\) is \(\left\lfloor \frac{m}{m_0} \right\rfloor\)-partition-connected, then it can be edge-decomposed into factors \(G_0, \ldots, G_b\) such that

- For each vertex \(v\), \(d_{G_i}(v) \geq m_i l(v)\), when \(i \in \{0, \ldots, b\}\).
- For each \(v \in X\), \(d_{G_i}(v) = \frac{m_i}{m} d_G(v)\), when \(i \in \{0, \ldots, b\}\).
- For each \(v \in Y\), \(d_{G_i}(v)\) is even, when \(i \in \{0, \ldots, b\}\).

Moreover by replacing the weaker condition \(d_{G_0}(v) \geq m_0 l(v) - 1\), it is suffices that \(G\) is odd-\(\left\lceil \frac{1}{m_0} \right\rceil\)-edge-connected and for each vertex \(v\), \(d_G(v) \geq m l(v)\).

**Proof.** By applying Theorem 3.2 with \(\varepsilon = m_0/m\), the graph \(G\) can be edge-decomposed into factors \(G_0, \ldots, G_b\) such that for each vertex \(v\), \(d_{G_0}(v) \geq \left\lfloor \frac{m}{m_0} d_G(v) \right\rfloor - 1\) and for all \(i \in \{1, \ldots, b\}\), \(d_{G_i}(v)\) is even and \(\left\lceil \frac{2m}{m} d_G(v) \right\rceil - 1 \leq d_{G_i}(v) \leq \left\lceil \frac{2m}{m} d_G(v) \right\rceil + 1\). Note that \(\frac{1}{b/m} = \frac{2b/m}{b} = \frac{2}{m}\). Since \(d_G(v) \geq \left\lfloor \frac{m}{m_0} \right\rfloor + m l(v)\), we must have \(d_{G_0}(v) \geq \left\lfloor \frac{m}{m_0} \left( \left\lfloor \frac{m}{m_0} \right\rfloor + m l(v) \right) \right\rfloor - 1 \geq m_0 l(v)\). If \(x \in X\), then one can conclude that \(d_{G_i}(x) = \frac{2m}{m} d_G(x) \geq 2l(x) = m_i l(v)\), because both integers \(d_{G_i}(x)\) and \(\frac{2m}{m} d_G(x)\) have the same parity. This implies that \(d_{G_0}(x) = \frac{m}{m_0} d_G(x)\). In addition, if \(y \in Y\), then \(d_{G_i}(y) \geq \left\lceil \frac{2m}{m} d_G(y) \right\rceil - 1 \geq 2l(y) - 1\), because \(d_G(y) \geq m l(y)\). Since both integers \(d_{G_i}(y)\) and \(2l(y)\) have the same parity, we must have \(d_{G_i}(y) \geq 2l(y)\). Hence the assertion holds. \(\square\)
3.2 Equitable factorizations with large degrees

In this subsection, we improve the edge-connectivity needed in Theorem 3.1 to $O(\frac{m}{m_0} \sum_{1 \leq i \leq k, m_i \neq 2} m_i)$. Before doing so, let us recall the following two lemmas from [14, 12]. The first one is an improved version of Lemma 4.1 in [19] and the second one is inspired by Lemma 4.3 in [19]. (Note that the proof of Lemma 4.3 in [19] needs to be repaired in some parts because degrees of vertices of $G$ in $A$ are not necessarily divisible by $m^2$).

Lemma 3.4. ([14, 19]) Let $G$ be a bipartite graph with bipartition $(X, Y)$ in which each vertex in $X$ has even degree. Let $f$ be an integer-valued function on $Y$ satisfying $\sum_{v \in Y} f(v) \equiv \frac{k}{2}|E(G)|$. If $G$ is $(3k - 3)$-edge-connected, then there exists a factor $H$ of $G$ such that

- For each $v \in X$, $d_H(v) = \frac{1}{2}d_G(v)$.
- For each $v \in Y$, $d_H(v) \equiv f(v)$ and $|d_H(v) - \frac{1}{2}d_G(v)| < k$.

Lemma 3.5. ([12]) Let $G$ be a bipartite graph with bipartition $(X, Y)$ and let $f$ be an integer-valued function on $V(G)$ satisfying $\sum_{v \in Y} f(v) \equiv k \sum_{v \in X} f(v)$. Let $F, F_0,$ and $T$ be three edge-disjoint factors of $G$ such that for each $v \in X$,

$$d_F(v) + \frac{1}{2}d_T(v) \leq f(v) \leq d_G(v) - (d_{F_0}(v) + \frac{1}{2}d_T(v)).$$

If $T$ is $(3k - 3)$-edge-connected, then there exists a factor $H$ of $G$ including $F$ excluding $F_0$ such that

- For each $v \in X$, $d_H(v) = f(v)$.
- For each $v \in Y$, $d_H(v) \equiv f(v)$.

Proof. We repeat the proof in [12] to be self-contained. For convenience, let us define $G_0 = G \setminus E(F_0)$. By the assumption, if for a vertex $v \in X$, $d_T(v)$ is odd, then $d_F(v) + \frac{1}{2}d_T(v) < d_G(v) - \frac{1}{2}d_T(v)$ and so $d_T(v) + d_F(v) \neq d_G(v)$. Thus there is a factor $T'$ of $G_0 \setminus E(F)$ including $T$ such that for each $v \in X$, $d_{T'}(v) = d_T(v) + 1$ when $d_T(v)$ is odd, and $d_{T'}(v) = d_T(v)$ when $d_T(v)$ is even. By the assumption, for each $v \in X$, we have $d_F(v) \leq f(v) - d_T'(v)/2 \leq d_G(v) - d_T'(v)$. Therefore, there exists a factor $F'$ of $G_0 \setminus E(T')$ including $F$ such that for each $v \in X$, $d_{F'}(v) = f(v) - d_T'(v)/2$. For each $v \in Y$, define $f'(v) = f(v) - d_{F'}(v)$. According to the assumption, we must have $\sum_{v \in Y} f'(v) = \sum_{v \in Y} (f(v) - d_{F'}(v)) \equiv \sum_{v \in X} (f(v) - d_{F'}(v)) = \frac{k}{2}d_T'(v) = \frac{1}{2}|E(T')|$. Since $T'$ is $(3k - 3)$-edge-connected, by Lemma 3.4, there exists a factor $H'$ of $T'$ such that for each $v \in X$, $d_{H'}(v) = \frac{1}{2}d_T'(v)$, and for each $v \in Y$, $d_{H'}(v) \equiv f'(v)$ and $|d_{H'}(v) - \frac{1}{2}d_T'(v)| < k$. It is not hard to see that $H' \cup F'$ is the desired factor that we are looking for. \square

The following theorem develops Corollary 6.10 in [12] to a partition-connected version.
Theorem 3.6. Let $m$, $m_0$, and $m_1$ be three positive integers satisfying $m = m_0 + m_1$. Let $G$ be a bipartite graph with bipartition $(X, Y)$ in which each vertex in $X$ has degree divisible by $m$. For $i \in \{0, 1\}$, let $\lambda_i$ be a nonnegative integer and let $l_i$ be a nonnegative integer-valued function on $V(G)$. If $G$ is $(\lceil \frac{3m m_1}{2 \min\{m_0, m_1\}} \rceil + \lceil \frac{m \lambda_1}{m_0} \rceil + \frac{m l_0}{m_0} + \lceil \frac{m l_1}{m_1} \rceil)$-partition-connected, then $G$ can be edge-decomposed into two factors $G_0$ and $G_1$ such that

- $G_i$ is $(\lambda_i, l_i)$-partition-connected, when $i \in \{0, 1\}$.
- For each $v \in X$, $d_{G_i}(v) = \frac{m_i}{m} d_G(v)$, when $i \in \{0, 1\}$.
- For each $v \in Y$, $d_{G_i}(v)$ is divisible by $m_1$.

Proof. For each $v \in X$, define $f(v) = \frac{m_1}{m} d_G(v)$, and for each $v \in Y$, define $f(v) = 0$. Obviously, $\sum_{v \in Y} f(v) \equiv 0 \not\equiv \sum_{v \in X} f(v)$. By the assumption, we can decompose $G$ into three factors $H_0$, $H_1$, and $H$ such that $H_i$ is $(\lceil \frac{m \lambda_i}{m_0} \rceil, \lceil \frac{m l_i}{m_1} \rceil)$-partition-connected and $H$ is $\lceil \frac{m (3m_1 - 1)}{2 \min\{m_0, m_1\}} \rceil$-tree-connected. According to Theorem 2.2, the graph $H_i$ has a $(\lambda_i, l_i)$-partition-connected factor $F_i$ such that for each $v \in X$, $d_{F_i}(v) \leq \lceil \frac{m_0}{m} d_{H_i}(v) \rceil$. In addition, the graph $H$ has a $(3m_1 - 1)$-tree-connected factor $T'$ such that for each $v \in X$, $d_{T'}(v) \leq \lceil \frac{2 \min\{m_0, m_1\}}{m} d_H(v) \rceil$. Let $T$ be a $(3m_1 - 3)$-tree-connected factor of $T'$ such that for each vertex $v$, $d_T(v) \leq d_{T'}(v) - 2$. Therefore, for each $v \in X$, $d_{F_i}(v) + \frac{1}{2} d_T(v) \leq \lceil \frac{m_0}{m} d_{H_i}(v) \rceil + \lceil \frac{m_0}{m} d_H(v) \rceil \leq \lceil \frac{m_0}{m} d_G(v) \rceil = f(v)$. Similarly, for each $v \in X$, $d_{F_0}(v) + \frac{1}{2} d_T(v) \leq \lceil \frac{m_0}{m} d_{H_0}(v) \rceil + \lceil \frac{m_0}{m} d_H(v) \rceil \leq \lceil \frac{m_0}{m} d_G(v) \rceil = d_G(v) - f(v)$.

By Lemma 3.5, the graph $G$ has a factor $G_1$ including $F_1$ excluding $F_0$ such that for each $v \in X$, $d_{G_i}(v) = f(v) = \frac{m_i}{m} d_G(v)$, and for each $v \in Y$, $d_{G_i}(v) \equiv f(v) = 0$. It is enough to set $G_0 = G \setminus E(G_1)$ to complete the proof.

Now, we are in a position to significantly refine Theorem 3.1 for another particular case. In fact, this version allows us to select $m_b$ in Corollary 3.3 arbitrarily but by increasing the required partition-connectivity.

Corollary 3.7. Let $m, m_0, m_1, \ldots, m_b$ be positive integers satisfying $m = \sum_{0 \leq i \leq b} m_i$ and $m_1 = \cdots = m_{b-1} = 2$. Let $G$ be a bipartite graph with bipartition $(X, Y)$ in which each vertex in $X$ has degree divisible by $m$, and let $l$ be a nonnegative integer-valued function on $V(G)$. If $G$ is $(\lceil \frac{3m m_0}{2 \min\{m-m_b, m_b\}} + \frac{2m}{m_0} \rceil, 2ml)$-partition-connected, then $G$ can be edge-decomposed into factors $G_0, \ldots, G_b$ such that

- For each vertex $v$, $d_{G_i}(v) \geq m_i l(v)$, when $i \in \{0, \ldots, b\}$.
- For each $v \in X$, $d_{G_i}(v) = \frac{m_i}{m} d_G(v)$, when $i \in \{0, \ldots, b\}$.
- For each $v \in Y$, $d_{G_i}(v)$ is divisible by $m_i$, when $i \in \{1, \ldots, b\}$.

In addition, by ignoring the condition $d_{G_i}(v) \geq m_i l(v)$, it suffices that $G$ is $(\lceil \frac{3m m_0}{2 \min\{m-m_b, m_b\}} + \frac{2m}{m_0} \rceil, ml)$-partition-connected.
We may assume that each such that for each vertex \( v \) of \( G \), the graph \( G \) can be edge-decomposed into factors \( G_0, \ldots, G_{b-1} \) such that for each vertex \( v \), \( d_{G_i}(v) \geq m_i l(v) \), for each \( v \in X \), \( d_{G_i}(v) = \frac{m_i}{m_0} d_G(v) \) and \( d_{G_i}(v) = \frac{m_i}{m_0} d_G(v) \), when \( i \in \{0, \ldots, b-1\} \), and for each \( v \in Y \), \( d_{G_i}(v) \) is even when \( i \in \{1, \ldots, b-1\} \). For proving the last statement, it is enough to set \( l_b = 0 \). Hence the assertion holds. \( \square \)

**Lemma 3.8.** ([27]) Every bipartite graph \( G \) can be edge-decomposed into \( m \) factors \( F_1, \ldots, F_m \) such that for each \( v \in V(F_i) \) with \( 1 \leq i \leq m \),

\[
|d_{F_i}(v) - d_G(v)/m| < 1.
\]

Consequently, if \( G \) is a modulo \( m \)-regular bipartite graph, then it can be decomposed into factors \( G_1, \ldots, G_b \) such that for each \( v \in V(G_i) \), \( d_{G_i}(v) = \frac{m_i}{m} d_G(v) \), where \( m_1, \ldots, m_b \) are positive integers satisfying \( m = \sum_{1 \leq i \leq b} m_i \).

**Proof.** To prove the consequence, it is enough to set \( G_i \) to be the union of \( m_i \) edge-disjoint factors \( F_j \) so that all of \( G_i \) are edge-disjoint. Note that for each vertex \( v \), \( d_{F_j}(v) = \frac{1}{m} d_G(v) \) and hence \( d_{G_i}(v) = \frac{m_i}{m} d_G(v) \).

\( \square \)

The following corollary enables us to select \( m_i \) in Corollary 3.3 arbitrarily but by increasing the required partition-connectivity.

**Corollary 3.9.** Let \( m, m_0, m_1, \ldots, m_b \) be positive integers satisfying \( m = \sum_{0 \leq i \leq b} m_i \) and let \( m'_b = \sum_{1 \leq i \leq b, i \neq 2} m_i \geq 2 \). Let \( G \) be a bipartite graph with bipartition \((X, Y)\) in which each vertex in \( X \) has degree divisible by \( m \), and let \( l \) be a nonnegative integer-valued function on \( V(G) \). If \( G \) is \( \left( \left\lfloor \frac{3mn}{2m_0} \right\rfloor \right) \)-partition-connected, then \( G \) can be edge-decomposed into factors \( G_0, \ldots, G_b \) such that

- For each vertex \( v \), \( d_{G_i}(v) \geq m_i l(v) \), when \( i \in \{0, \ldots, b\} \).
- For each \( v \in X \), \( d_{G_i}(v) = \frac{m_i}{m_0} d_G(v) \), when \( i \in \{0, \ldots, b\} \).
- For each \( v \in Y \), \( d_{G_i}(v) \) is divisible by \( m_i \), when \( i \in \{1, \ldots, b\} \).

**Proof.** We may assume that \( m_i = 2 \) for all \( 1 \leq i \leq p \), and \( m'_b = \sum_{p<i \leq b} m_i \). By Corollary 3.7, the graph \( G \) can be edge-decomposed into \( p + 2 \) factors \( G_0, \ldots, G_p \) and \( H \) such that for each vertex \( v \), \( d_{G_i}(v) \geq m_i l(v) \) when \( i \in \{0, \ldots, p\} \), for each \( v \in X \), \( d_{G_i}(v) = \frac{m_i}{m} d_G(v) \) when \( i \in \{0, \ldots, p\} \), and for each \( v \in Y \), \( d_{G_i}(v) \) is even when \( i \in \{1, \ldots, p\} \). In addition, for each vertex \( v \), \( d_H(v) \geq m'_b l(v) \), for each \( v \in X \), \( d_H(v) = \frac{m_i}{m} d_G(v) \), and for each \( v \in Y \), \( d_H(v) \) is divisible by \( m'_b \). By Lemma 3.8, the graph \( H \) can be edge-decomposed into factors \( G_{p+1}, \ldots, G_b \) such that for each vertex \( v \), \( d_{G_i}(v) = \frac{m_i}{m'_b} d_H(v) \), where
To prove Theorem 4.1, let $T$ be a tree of size $m$ and let $m_0, \ldots, m_b$ be positive integers satisfying $m = \sum_{0 \leq i \leq b} m_i$. Assume that there is a partition $B_0, \ldots, B_b$ of one partite set $B$ of $T$ such that $B_0$ consists of all leaf vertices in $B$ and $m_i = \sum_{v \in B_i} d_T(v)$. Let $G$ be a bipartite graph with bipartition $(X, Y)$ in which each vertex in $X$ has degree divisible by $m$. Then $G$ admits a $(B, Y)$-compatible $T^*$-edge-decomposition $A^*$, if it can be edge-decomposed into factors $G_0, \ldots, G_b$ satisfying the following properties:

- For each $v \in X$, $d_{G_i}(v) = \frac{m_i}{m} d_G(v)$, when $i \in \{0, \ldots, b\}$.
- For each $v \in Y$, $d_{G_i}(v)$ is divisible by $m_i$, when $i \in \{1, \ldots, b\}$.

In addition, for any pair $v \in X$ and $t \in A$, $|A^*(v, t)| = d_G(v)/m$, and for any pair $v \in Y$ and $t \in B_i$, $|A^*(v, t)| - d_{G_i}(v)/m_i < 1$. In particular, $G$ admits such a $(B, Y)$-compatible $T$-edge-decomposition, if $G$ is simple and one of the following conditions holds:

- $\text{girth}(G) > \text{diam}(T)$.
- $\delta(G_i) \geq m_i^{10^{50m}}$, when $i \in \{0, \ldots, b\}$.

**Proof.** By Lemma 3.8, the graph $G_i$ has a factorization $F_{i,1}, \ldots, F_{i,m_i}$ such that for any $v \in X$ and $j \in \{1, \ldots, m_i\}$, $d_{F_{i,j}}(v) = \frac{1}{m_i} d_{G_i}(v) = \frac{1}{m} d_G(v)$, and for any $v \in Y$ and $j \in \{1, \ldots, m_i\}$, $\lfloor \frac{1}{m_i} d_{G_i}(v) \rfloor \leq d_{F_{i,j}}(v) \leq \lfloor \frac{1}{m_i} d_{G_i}(v) \rfloor$. Consequently, $d_{F_{i,j}}(v) = \frac{1}{m_i} d_{G_i}(v)$ when $i \neq 0$. Therefore, all factors $F_{i,j}$ form a $(B, Y)$-equitable factorization $F_1, \ldots, F_m$ for $G$ (corresponding to an edge-coloring of $T$ with colors $\{1, \ldots, m\}$). Thus by Lemma 3.2 in [19], the graph $G$ admits a $(B, Y)$-compatible $T^*$-edge-decomposition satisfying the theorem. Note that if $\text{girth}(G) > \text{diam}(T)$, then this $T^*$-edge-decomposition must automatically be a $(B, Y)$-compatible $T$-edge-decomposition. In addition, if the minimum degree of every $G_i$ is at least $m_i^{10^{50m}}$, then the minimum degree of every $F_i$ must be at least $10^{50m}$. Thus by Theorem 7 in [4], the graph $G$ admits a $T$-edge-decomposition satisfying the theorem. \qed
4.2 Bipartite graphs with degree divisible by \(m\) in one side

The following theorem shows an application of Corollary 3.7 and Theorem 4.1.

**Theorem 4.2.** Let \(T\) be a tree of size \(m\) with bipartition \((A, B)\). Let \(G\) be a bipartite simple graph with bipartition \((X, Y)\) in which vertices in \(X\) have degrees divisible by \(m\). Then \(G\) admits a \((B, Y)\)-compatible \(T\)-edge-decomposition, if at least one of the following properties holds:

- \(\text{girth}(G) > \text{diam}(T)\) and \(G\) is \([\frac{3}{2}m]\)-tree-connected.
- \(\text{diam}(T) = 3\) and \(G\) is \([\frac{3}{2}m]\)-tree-connected.
- \(G\) is \(([\frac{3}{2}m], 2m10^{50m})\)-partition-connected.

Moreover, if \(([\frac{3}{2}m], 4ml)\)-partition-connected, then \(G\) admits a \((B, Y)\)-compatible \(T^*\)-edge-decomposition such that for any compatible pair \(v \in V(G)\) and \(t \in V(T)\), there are \(l(v)\) homomorphic copies of \(T\) including \(v\) with the preimage \(t\), where \(l\) is a nonnegative integer-valued function on \(V(G)\).

**Proof.** Assume that \(\text{diam}(T) = 3\) and let \(m_1\) the degree of an internal vertex \(v\) of \(T\) and let \(m_0 = m - m_1\). We select \(v\) such that \(m_1 \leq (m + 1)/2\). This yields that \(m_1 = \min\{m_0, m_1\}\) or \(m_0 = m_1 - 1 = (m - 1)/2\). Therefore, the graph \(G\) must be \(([\frac{3}{2}m_{m_1} - 1]\)-tree-connected. By Theorem 2.2, the graph \(G\) has a \((3m_1 - 3)\)-tree-connected factor \(H\) such that for each \(v \in X\), \(d_H(v) \leq \frac{2\min\{m_0, m_1\}}{m}d_G(v)\). Note that \(0 < \frac{2\min\{m_0, m_1\}}{m} < 1\). By Lemma 3.5, the graph \(G\) has an edge-decomposition into two factors \(G_0\) and \(G_1\) such that for each \(v \in X\), \(d_{G_0}(v) = \frac{m_0}{m}d_G(v)\) and for each \(v \in Y\), \(d_{G_1}(v)\) is divisible by \(m_1\). Since \(\text{girth}(G) \geq 4 > \text{diam}(T)\), by Theorem 4.1, the graph \(G\) admits a \((B, Y)\)-compatible \(T\)-edge-decomposition.

Now, assume that \(G\) is \(([\frac{3}{2}m], 2ml)\)-partition-connected. We may assume \(A \cap I(T)\) and \(B \cap I(T)\) are not empty. Otherwise, \(T\) must be a star and so the assertion follows from Theorem 1.3. Let \(B_0, \ldots, B_b\) be the partition of \(B\) such that \(B_0\) consists all leaf vertices of \(T\) in \(B\), every \(B_i\) with \(0 \leq i < b\) consists of one vertex of \(T\) in \(B\) with degree \(2\), and \(B_b\) either consists of all vertices of \(T\) in \(B\) with degree at least \(3\) or consists of one vertex of \(T\) in \(B\) with degree \(2\). Let \(m_i = \sum_{v \in B_i} d_T(v)\). Notice that \(m_b\) denotes be the number of leaf vertices of \(T\) in \(B\), and \(m_1 = \cdots = m_{b-1} = 2\), and \(b - 1\) or \(b\) denotes the number of vertices of \(T\) with degree \(2\) in \(B\). Note also that \(m = m_b \geq m_0\). Since \(T\) contains \(\max_{v \in A \cap I(T)}(d_T(v) - 2) + \max_{v \in B \cap I(T)}(d_T(v) - 2) + 2\) leaf vertices, there are at least \(\max_{v \in A \cap I(T)}(d_T(v) - 2) + 1\) leaf vertices in \(A\) or at least \(\max_{v \in B \cap I(T)}(d_T(v) - 2) + 1\) leaf vertices in \(B\). We may therefore assume that \(m_0 \geq \sum_{v \in B \cap I(T)}(d_T(v) - 2) + 1\). If \(m_b \neq 2\), then

\[
3(m_0 - 1) \geq \sum_{v \in B \cap I(T)}(d_T(v) - 2) = \sum_{v \in B_b} 3(d_T(v) - 2) \geq \sum_{v \in B_b} d_T(v) = m_b,
\]

which implies that (regardless of \(m_b = 2\) or not)

\[
\left\lceil \frac{9}{2}m \right\rceil \geq \max\{\frac{3m}{2}, \frac{3m(m_0 - 3)}{2m_0}\} + \frac{2m}{m_0} \geq \frac{3mm_b}{2\min\{m_b, m - m_b\}} + \frac{2m}{m_0}
\]
Thus by Corollary 3.7, we can decompose $G$ into factors $G_0, \ldots, G_b$ such that for each vertex $v$, $d_{G_i}(v) \geq m_i l(v)$ when $i \in \{0, \ldots, b\}$, for each $v \in X$, $d_{G_i}(v) = \frac{m_i}{m} d_G(v)$ when $i \in \{0, \ldots, b\}$, and for each $v \in Y$, $d_{G_i}(v)$ is divisible by $m_i$ when $i \in \{1, \ldots, b\}$. Therefore, by Theorem 4.1, the graph $G$ admits a $(B, Y)$-compatible $T^*$-edge-decomposition such that for any pair $v \in X$ and $t \in A$ and any pair $v \in Y$ and $t \in B_i$, there are $l(v)$ homomorphic copies of $T$ including $v$ with the preimage $t$. Hence $G$ admits a $(B, Y)$-compatible $T^*$-edge-decomposition with the desired properties. If $girth(G) > diam(T)$ (regardless of $l = 0$ nor not), then every homomorphic copy of $T$ must be an isomorphic copy and so $G$ admits a $(B, Y)$-compatible $T$-edge-decomposition automatically. If $l = 10^{50m}$ and $G$ is a simple graph, then according to Theorem 4.1, $G$ admits a $(B, Y)$-compatible $T$-edge-decomposition. Hence the theorem holds. \hfill \Box

The following theorem reduces the edge-connectivity needed in Theorem 4.2 for a special type of trees.

**Theorem 4.3.** Let $T$ be a tree of size $m$ with bipartition $(A, B)$. Assume that $B$ consists of $m_0$ leaves and some vertices with degree 2, where $0 < m_0 < m$. Let $G$ be a 2-edge-connected bipartite simple graph in which vertices in one side have degrees divisible by $m$. Then $G$ admits a $(B, Y)$-compatible $T$-edge-decomposition, if at least one of the following properties holds:

- $G$ is odd-$\lceil \frac{m}{m_0} \rceil$-edge-connected and $girth(G) > diam(T)$.
- $G$ is odd-$\lceil \frac{m}{m_0} \rceil$-edge-connected and $\delta(G) \geq m 10^{51m}$.

**Proof.** Let $(X, Y)$ be a bipartition of $G$ in which vertices in $X$ have degrees divisible by $m$. For convenience, let us define $m_1 = \cdots = m_b = 2$ where $b = (m - m_0)/2$. Since $G$ is odd-$\lceil \frac{m}{m_0} \rceil$-edge-connected and $\delta(G) \geq m 10^{51m}$, by applying Corollary 3.3, the graph $G$ can be edge-decomposed into factors $G_0, \ldots, G_b$ such that for each vertex $v$, $d_{G_i}(v) \geq m_i 10^{51m} - 1 \geq m_i 10^{50m}$ when $i \in \{0, \ldots, b\}$, for each $v \in X$, $d_{G_i}(v) = \frac{m_i}{m} d_G(v)$ when $i \in \{0, \ldots, b\}$, and for each $v \in Y$, $d_{G_i}(v)$ is even when $i \in \{1, \ldots, b\}$. Thus the assertion follows from Theorem 4.1. The proof of the first assertion is also similar. \hfill \Box

### 4.3 Bipartite graphs with large girth or minimum degree

The following lemma is an important tool for decomposing graphs into two highly partition-connected modulo semi-regular factors $G_1$ and $G_2$ which was originally introduced in [25, Proposition 2] with a weaker version. Note that two similar versions are also established in [5, Lemma 2.5], and [19, Proposition 6].

**Lemma 4.4.** Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$. Let $\lambda$ be a nonnegative integer, let $l$ be a nonnegative integer-valued function on $V(G)$. If $G$ is $(2m - 2 + 2\lambda, 2l)$-partition-connected and $|E(G)|$ is divisible by $m$, then $G$ admits an edge-decomposition into two $(\lambda, l)$-partition-connected factors $G_1$ and $G_2$ such that for each $v \in V_i$, $d_{G_i}(v)$ is divisible by $m$.

**Proof.** Decompose $G$ into three factors $H_1$, $H_2$, and $H$ such that $H_i$ is $(\lambda_i, l_i)$-partition-connected and $H$ is $(2m - 2)$-tree-connected. For each $v \in V_i$, define $p(v) = -d_{H_i}(v)$ (modulo $m$). Since $|E(G)|$ is divisible by
m, we must have \( |E(H)| = |E(G)| - |E(H_1)| - |E(H_2)| \overset{\text{def}}{=} \sum_{v \in V_1} p(v) + \sum_{v \in V_2} p(v) = \sum_{v \in V(H)} p(v) \). Thus by Corollary 4.5 in [13], the graph \( H \) has an orientation such that for each vertex \( v \), \( d_H^+(v) \overset{\text{def}}{=} p(v) \). Let \( F_i \) be the factor of \( H \) consisting of all edges directed away from \( V_i \). It is easy to check that \( G_1 = H_1 \cup F_1 \) and \( G_2 = H_2 \cup F_2 \) are the desired factors that we are looking for.

The following theorem develops Theorem 4.2 to arbitrary bipartite graphs but requires to increase the necessary partition-connectivity by a constant factor.

**Theorem 4.5.** Let \( T \) be a tree of size \( m \). A bipartite simple graph \( G \) admits a \( T \)-edge-decomposition, if at least one of the following conditions holds:

- girth\((G) > \text{diam}(T) \) and \( G \) is \( 11m \)-tree-connected.
- \( \text{diam}(T) = 3 \) and \( G \) is \( 5m \)-tree-connected.
- \( G \) is \((11m, 4m10^{50m})\)-partition-connected.

Moreover, every \((11m, 4ml)\)-partition-connected bipartite graph \( G \) admits a \( T^* \)-edge-decomposition such that for any pair \( v \in V(G) \) and \( t \in V(T) \), there are \( l(v) \) homomorphic copies of \( T \) including \( v \) with the preimage \( t \), where \( l \) is a nonnegative integer-valued function on \( V(G) \).

**Proof.** Let \((V_1, V_2)\) be the bipartition of \( V(G) \). Assume that \( \text{diam}(T) = 3 \). By Lemma 4.4, we can decompose \( G \) into two \([4m] \)-tree-connected factors \( G_1 \) and \( G_2 \) such that for each \( v \in V_i \), \( d_{G_i}(v) \) is divisible by \( m \). Therefore, by Theorem 4.2, the graph \( G_i \) admits a \( T \)-edge-decomposition and so does \( G \). Now, assume that \( G \) is \((11m, 4ml)\)-partition-connected. By Lemma 4.4, we can decompose the graph \( G \) into two factors \( G_1 \) and \( G_2 \) such that \( G_i \) is \((\lfloor \frac{9}{2}m \rfloor, 2ml)\)-partition-connected for each \( v \in V_i \), \( d_{G_i}(v) \) is divisible by \( m \). Therefore, by Theorem 4.2, the graph \( G_i \) admits a \((B, V_i)\)-compatible \( T^* \)-edge-decomposition such that for any pair \( v \in V_i \) and \( t \in A \) and any pair \( v \in V(G) \setminus V_i \) and \( t \in B \), there are \( l(v) \) homomorphic copies of \( T \) including \( v \) with the preimage \( t \). Hence \( G \) admits a \( T^* \)-edge-decomposition with the desired properties. If girth\((G) > \text{diam}(T) \) (regardless of \( l = 0 \) nor not), then every homomorphic copy of \( T \) must be an isomorphic copy and so \( G \) admits a \( T \)-edge-decomposition automatically. If \( l = 10^{50m} \) and \( G \) is a simple graph, then according to Theorem 4.1, the graph \( G_i \) admits a \( T \)-edge-decomposition and so does \( G \). Hence the theorem holds.

5 Graphs with sufficiently large girth

In this section, we are going to develop the first item of Theorem 4.5 to arbitrary graphs. Before stating this result, let us establish the following lemma for working with graphs with small bipartite index.
Lemma 5.1. Let $G$ be a simple graph with a factor $M$, and let $T$ be a tree of size $m$. If $\text{girth}(G) > \text{diam}(T)$ and $\delta(G) \geq |E(M)| + 2m$, then $M$ can be extended to a subgraph of $G$ having a $T$-edge-decomposition. In addition, the assertion holds for all trees when $\delta(G) \geq 2|E(M)| + 2m^2$.

Proof. We may assume there is no copy of $T$ in $M$. Otherwise, we delete it from $M$ and use induction. Let $t_0$ be a fixed internal vertex of $T$. We inductively select a copy $M_i$ of a subtree of $T$ including $t_0$ in the remaining factor of $M$ with the maximum size and delete it from $M$. Following [1, Page 5], we say that a vertex in $M_i$ is unsaturated if its degree in $M_i$ is less than the degree of its preimage in $T$. Let $M_j$ be the first subtree including $v$ as an unsaturated vertex. If $M_j$ has another subtree including $v$ as an unsaturated vertex and $vv' \in E(M_j)$, then according to the maximality of $M_j$, we must have $v' \in V(M_j)$. Thus there are at most $|V(M_j)| - 1$ subtrees including $v$ as an unsaturated vertex. In addition, there is at most one subtree including $v$ as an unsaturated vertex when $\text{girth}(G) > \text{diam}(T)$. Let $r = 1$ when $\text{girth}(G) > \text{diam}(T)$ and let $r = m - 1$ otherwise. In other words, in both cases, are at most $r$ subtrees including $v$ as an unsaturated vertex.

Let $A_0$ be a collection of some edge-disjoint copies $T_1, \ldots, T_n$ of subtrees of $T$ such that every $T_i$ contains $M_i$, and for every vertex $v$ and every $t \in V(T)$, there are at most $r$ subtrees in $A_0$ including $v$ with the preimage $t$. According to the construction, the collection of all $M_i$ is a candidate for $A_0$. We consider $A_0$ with the maximum size of $T_1 \cup \cdots \cup T_n$. We claim that $A_0$ is the desired $T$-edge-decomposition. Otherwise, there is a subtree $T_i \in A_0$ having an unsaturated vertex $x$ so that its degree in $T_i$ is less than the degree of its preimage $t_x$ in $T$. Let $t'_x$ be the vertex in $V(T)$ incident with $t_x$ such that $t_x$ lies in the path in $T$ connecting $t_0$ to $t'_x$. Let $u_1, \ldots, u_p$ be the neighbours of $x$ in $G$ such that every edge $xu_i$ is not contained in subtrees of $A_0$. By the assumption on $A_0$, for every vertex $v$ and every $t \in V(T)$, there are at most $r$ subtrees in $A_0$ including $v$ with the preimage $t$. Thus there are at least $d_G(x) - d_M(x) - 2mr + 1$ edges incident with $x$ in $G$ which are different from the edges of subtrees in $A_0$ (whether in $M$ or not). This means that $p \geq d_G(x) - d_M(x) - 2mr + 1$. If there is at most $r - 1$ subtrees in $A_0$ including $u_j$ with the preimage $t'_x$, then we add the edge $xu_j$ to $T_i$ to obtain a new collection $A_0$ with larger size satisfying the desired properties, which is a contradiction. Thus for every vertex $u_j$, there are at least $r$ subtrees $T_{i,j}, \ldots, T_{r,j}$ in $A_0$ including $u_j$ with the preimage $t'_x$. Obviously, all subtrees $T_{i,j}$ must be distinct. Thus we have at least $pr$ edges of $M$. Therefore, $|E(M)| \geq pr \geq r(\delta(G) - d_M(x) - 2mr + 1)$. If $r = m - 1$, then $2|E(M)| + 2m^2 > \delta(G)$, which is a contradiction. If $\text{girth}(G) > \text{diam}(T)$, then every subtree $T_{i,j}$ does not contain edges of $M$ incident with $x$. In this case, we have at least $pr + d_M(x)$ edges of $M$ and so $|E(M)| \geq pr + d_M(x) \geq \delta(G) - 2m + 1$, which is again a contradiction. \hfill \Box

We also need the following lemma for working with graphs with large bipartite index.

Lemma 5.2. ([11]) If $G$ is a $(2m - 2 + 3\lambda)$-tree-connected graph of size divisible by $m$ and $b_i(G) \geq \lambda$, then every vertex $v$ can be split into two vertices $v_1$ and $v_2$ such that resulting graph $H$ is a bipartite $\lambda$-tree-connected graph with bipartition $(V_1, V_2)$ for which $V_i = \{v_i : v \in V(G)\}$ and for each $v \in V(G)$, $d_H(v_1)$ is divisible by $m$.  

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Now, we are in a position to prove the second part of Theorem 1.5 by giving an explicit upper bound on the necessary edge-connectivity.

**Theorem 5.3.** Let $T$ be a tree of size $m$. Let $G$ be a 31$m$-edge-connected simple graph with $\delta(G) \geq 35m$. If $|E(G)|$ is divisible by $m$ and $\text{girth}(G) > \text{diam}(T)$, then $G$ admits a $T$-edge-decomposition.

**Proof.** For notational simplicity, let us write $\lambda = \lfloor \frac{9}{2}m \rfloor$. By Theorem 2.3, the graph $G$ is $(2m - 2 + 3\lambda, 2m)$-partition-connected. Let $(V_1, V_2)$ be a bipartition of $V(G)$ with $e_G(V_1) + e_G(V_2) = b_i(G)$. Let $M = G[V_1] \cup G[V_2]$. First assume that $b_i(G) \leq \lambda$ and so $|E(M)| \leq \lambda$. Since $|E(M)| \leq \lambda$, it is not difficult to check that we can decompose $G$ into two factors $G_1$ and $G_2$ such that $G_1$ is $(2m - 2 + 2\lambda)$-tree-connected and $G_2$ is $(\lambda, 2m)$-partition-connected and it contains $M$ as well. More precisely, one can first decompose $G$ into $2m - 2 + 3\lambda$ spanning trees and a $(0, 2m)$-partition-connected factor and next select $2m - 2 + 2\lambda$ spanning trees excluding the edges $M$ in order to construct the graph $G_1$. By applying Lemma 5.1 to the graph $G_2$, there are some edge-disjoint copies of $T$ in $G_2$ covering the edges of $M$. Note that $\delta(G_2) \geq \lambda + 2m \geq |E(M)| + 2m$. By deleting those copies from $G$ the remaining graph is an 11-tree-connected bipartite graph of size divisible by $m$ with the bipartition $(V_1, V_2)$. Thus by Theorem 4.5, it admits a $T$-edge-decomposition and so does $G$.

Now, assume that $b_i(G) \geq \lambda$. Since $G$ is $(2m - 2 + 3\lambda)$-tree-connected, by Lemma 5.2, every vertex $v$ can be split into two vertices $v_1$ and $v_2$ such that resulting graph $H$ is a bipartite $\lambda$-tree-connected graph with bipartition $(V_1, V_2)$ for which $V_t = \{v_i : v \in V(G)\}$ and for each $v \in V(G)$, $d_H(v_1)$ is divisible by $m$. Thus by Theorem 4.2, the bipartite graph $H$ admits a $T$-edge-decomposition and so $G$ admits a $T^*$-edge-decomposition. Since $\text{girth}(G) > \text{diam}(T)$, this $T^*$-edge-decomposition must therefore be a $T$-edge-decomposition. Hence the proof is completed. \qed

**Remark 5.4.** Note that the minimum degree condition in Theorem 5.3 can be dropped and the necessary edge-connectivity in Theorem 6.4 can also be reduced to 31$m$. Moreover, we can develop Theorem 5.3 to the case that $\text{girth}(G) = \text{diam}(T)$ or $\text{diam}(T) \leq 4$ in $O(m)$-edge-connected graphs with minimum degree at least $O(m^2)$. The proofs need more complicated arguments. We present details in a forthcoming paper.

6 Graphs with sufficiently large minimum degrees

In this section, we are going to develop the third item of Theorem 4.5 to arbitrary graphs.

6.1 Method 1: extending subtrees using subgraphs

Thomassen (2013) [25] and independently Barát and Gerbner (2014) [1] showed that it is enough to prove Conjecture 1.1 for bipartite graphs. Barát and Gerbner used the following lemma in their proof which implicitly appeared in the proof of Theorem 2 in [1].
Lemma 6.1. ([1]) Let $T$ be a tree of size $m$. Let $G$ be a simple graph having nested bipartite factors $F_m \subseteq \cdots \subseteq F_1$ with the same bipartition $(X, Y)$. If $\delta(F_m) \geq m^3$ and for each $v \in V(F_1)$ satisfying $1 \leq i \leq m - 1$, $d_{F_{i+1}}(v) \leq \frac{1}{m}d_{F_i}(v)$, then there exist edge-disjoint copies of $T$ in $G[X] \cup F_1$ containing all edges of $G[X]$.

We here present a simpler proof for Theorem 1.5 based on Lemma 6.1 but by replacing a factorial upper bound on the necessary minimum degree.

Theorem 6.2. Let $T$ be a tree of size $m$. If $G$ is a $44m$-edge-connected simple graph of size divisible by $m$ and $\delta(G) \geq m^{200m}$, then $G$ admits a $T$-edge-decomposition.

Proof. We may assume that $m \geq 2$. By Theorem 2.3, the graph $G$ is $(22m, 2l)$-partition-connected, where $l = 2m10^{50m} + 2(m + 1)^{m+2}$. According to Theorem 2.4, there is a bipartition $(V_1, V_2)$ of $V(G)$ such that $G[V_1, V_2]$ is $(11m, l)$-partition-connected. Decompose $G[V_1, V_2]$ into a $(11m, 2m10^{50m})$-partition-connected graph $G_0$ and two $(0, (m + 1)^{m+2})$-partition-connected factors $H_1$ and $H_2$. By applying lemma 3.8 with $k = 1/(m + 1)$ repeatedly, we can find nested factors $F_{m, i} \subseteq \cdots \subseteq F_{1, i}$ of $H_i$ such that for each $v \in V(F_s)$ with $1 \leq s \leq m - 1$,

$$(m + 1)^{m-s+2} \leq \left\lfloor \frac{1}{m+1}d_{F_s}(v) \right\rfloor \leq d_{F_{s+1}}(v) \leq \left\lceil \frac{1}{m+1}d_{F_s}(v) \right\rceil.$$ 

Note that $\delta(F_m) \geq (m + 1)^3$. In addition, $d_{F_{i+1}}(v) \leq \frac{1}{m+1}d_{F_i}(v) + 1 \leq \frac{1}{m}d_{F_i}(v)$, since $\delta(F_s) \geq (m + 1)^2$.

Therefore, by Lemma 6.1, there exist some edge-disjoint copies of $T$ in $G[V_i] \cup H_i$ containing all the edges of $G[V_i]$. Let $H$ be the factor of $G$ consisting of the edges of $H_1$ and $H_2$ which did not appear in those copies of $T$. By Theorem 4.5, the bipartite graph $G_0 \cup H$ admits a $T$-edge-decomposition and so does $G$. □

6.2 Method 2: extending subtrees using isomorphic trees

In order to improve the lower bound on the minimum degree in Theorem 6.2 to an exponential bound, we need to recall Lemma 10 in [4] for the special case $\varepsilon = 1/2$ as follows.

Lemma 6.3. ([4]) Let $T$ be a tree of size $m$ and let $\delta$ be a real number with $0 < \delta < 1$. Let $G$ be a bipartite simple graph of size divisible by $m$. If $G$ admits a $T^*$-edge-decomposition $A^*$ such that for any pair $v \in V(G)$ and $t \in V(T)$, $|A(v,t)| \geq (10m)^{18}2^6\delta^{-6}$, then $G$ contains a factor $H$ having a $T$-edge-decomposition $A$ satisfying the following properties:

- For any pair $v \in V(G)$ and $t \in V(T)$, $|A(v,t)| \geq \frac{1}{2}|A^*(v,t)|$.
- For any triple $v, w \in V(G)$ and $t \in V(T)$ with $v \neq w$, $|A(v,t) \cap A(w)| \leq \delta|A(v,t)|$.

Now, we are in a position to prove the first part of Theorem 1.5 by giving explicit upper bounds on the necessary edge-connectivity and minimum degree.
Let $T$ be a tree of size $m$. If $G$ is a $88m$-edge-connected simple graph of size divisible by $m$ and $\delta(G) \geq 2^{200m}$, then $G$ admits a $T$-edge-decomposition.

**Proof.** We may assume that $m \geq 2$. Note that $\delta(G) \geq 2^{200m} \geq 32ml_0 + 16ml + 88m + 4$, where $l_0 = (10m)^{18l^3}(2m^3)^l$ and $l = 10^{20m}$. By Theorem 2.3, the graph $G$ is $(44m, 16ml0 + 8ml + 2)$-partition-connected. According to Theorem 2.4, there is a bipartition $(V_1, V_2)$ of $V(G)$ such that $G[V_1, V_2]$ is $(22m, 8ml0 + 4ml + 1)$-partition-connected. We decompose $G[V_1, V_2]$ into an $(11m, 8ml0 + 1)$-partition-connected factor $G_1$ an $(11m, 4ml)$-partition-connected factor $G_2$. We may assume that $G_1$ is $(11m, 8ml0)$-partition-connected and its size is divisible by $m$ by transferring at most $m$ edges from $G_1$ to $G_2$ (if necessary). Thus by Theorem 4.5, the graph $G_1$ admits a $T^*$-edge-decomposition $A^*$ such that for any pair $v \in V(G)$ and $t \in V(T)$, $|A^*(v, t)| \geq 2l_0$. By applying Lemma 6.3 with setting $\delta = 1/2m^3$, the graph $G_1$ contains a factor $G'_1$ having a $T$-edge-decomposition $A$ such that for any pair $v \in V(G)$ and $t \in V(T)$, $|A(v, t)| \geq \frac{1}{2}|A^*(v, t)| \geq l_0$, and for any triple $v, w \in V(G)$ and $t \in V(T)$, $|A(w) \cap A(v, t)| \leq \frac{1}{2m^3}|A(v, t)|$. For a copy $T'$ of a subtree of $T$ and $v \in V(T')$, we denote below by $t(T', v)$ the preimage of $v$. Note that $t(T', v) \in V(T)$ while $v \in V(T')$.

First, we arbitrarily delete copies of $T$ from $G[V_1] \cup G[V_2]$ as long as possible. Let $t_0$ be a fixed internal vertex of $T$. Next, in the $i$-th step, we select a copy $T_i$ of a subtree of $T$ including $t_0$ in the remaining factor of $G[V_1] \cup G[V_2]$ with the maximum size and delete it from $G$. We say that a vertex in $T_i$ is unsaturated if its degree in $T_i$ is less than the degree of its preimage in $T$. Let $T_j$ be the first subtree including $v$ as an unsaturated vertex. If $T_j$ is another subtree including $v$ as an unsaturated vertex and $vw \in E(T_j)$, then according to the maximality of $T_j$, we must have $v' \in V(T_j)$. Thus there are at most $m$ (more precisely, at most $V(T_j)$) subtrees including $v$ as an unsaturated vertex.

Let $S$ be the set of all pairs $(T, v)$ such that $v$ is an unsaturated vertex in a removed subtree $T$. Define $P$ to be the bipartite simple graph with $V(P) = S \cup A$ such that every $(T, v) \in S$ is adjacent to all elements in $A(v, t(T, v)) \subseteq A$. Since every vertex is unsaturated in at most $m$ removed subtrees, every $T' \in A$ is adjacent with at most $m|I(T)|$ elements in $S$ in the bipartite graph $P$. Thus by Lemma 3.8, there is a factor $M$ in $P$ such that for all $s \in S$, $d_M(s) \geq \lfloor \frac{1}{m|I(T)|} \rfloor d_P(s)$ and for all $a \in A$, $d_M(a) \leq \lfloor \frac{1}{m|I(T)|} \rfloor d_P(a) \leq 1$. This means that for every $(T, v) \in S$, there is a subset $B(T, v)$ of $A(v, t(T, v))$ such that $|B(T, v)| \geq \frac{1}{m|I(T)|} |A(v, t(T, v))| - 1$ and all of $B(T, v)$ are disjoint. We are going to extend every removed subtree $T$ to a copy of $T$ using the edges of trees in sets $B(T, v)$, where $v$ is an unsaturated vertex in $T$. Let $u_1, \ldots, u_n$ be unsaturated vertices of $T$. We construct inductively nested trees $T_i$ which are isomorphic to copies of subtrees of $T$. In addition, $v_1, \ldots, v_n$ are unsaturated in them. First, we set $T_0 = T$. Let $i \in \{0, \ldots, n\}$. For notational simplicity, let us write $t = t(T, v_i)$. Thus

$$|B(T, v_i) \setminus \cup_{v \in V(T_i) \setminus \{v_i\}} (A(v_i, t) \cap A(v_i))| \geq \frac{1}{m|I(T)|} |A(v_i, t)| - 1 - \frac{m}{2m^3} |A(v_i, t)| \geq \frac{1}{2m^3} |A(v_i, t)| - 1 > 0.$$  

Thus there is a tree $T_i \in B(T, v_i)$ such that $V(T_i) \cap V(T_i) = \{v_i\}$. We here denote by $T(t_0, t)$ the component of $T - e$ including $t$, where $e$ is the unique edge incident with $t$ lying in the path of $T$ connecting $t_0$ and $t$. Let $T_i'$ be the subtree of $T_i$ consisting of the images of the edges of $T(t_0, t)$. Now, we define $T_i = T_{i-1} \cup T_i'$. By repeating this process, we obtain a tree $T_n$ which is isomorphic to $T$ and it includes $T$. Since all of $B(T, v)$ are disjoint, all of such trees $T_n$ are edge-disjoint.
In other words, there are some edge-disjoint copies of $T$ in $G[V_1] \cup G[V_2] \cup G_1$ covering all of the edges of $G[V_1] \cup G[V_2]$. If we delete these copies, the resulting graph $G'$ contains $G_2$ and its size must be divisible by $m$. So, $G'$ is $(11m, 4m10^{50m})$-partition-connected. Therefore, by Theorem 4.5, the bipartite graph $G'$ admits a $T$-edge-decomposition and so does $G$. Hence the proof is completed.

□

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