Gravitational decay of the $Z$-boson

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Abstract
We study the decay process of the $Z$ boson to a photon and a graviton. The most general form of the on-shell amplitude, subject to the constraints due to the conservation of the electromagnetic current and the energy-momentum tensor, is determined. The amplitude is expressed in terms of three form factors, two of which are CP-odd while one is CP-even. The latter, which is the only non-zero form factor at the one-loop level, is computed in the standard model and the decay rate is determined.

1 Introduction
As is well known [1], the $Z$-boson cannot decay into two photons. Therefore, the simplest decay of the $Z$-boson to two massless bosons is through the channel

$$Z(p) \rightarrow \gamma(k) + \mathcal{G}(q), \quad (1.1)$$

where $\gamma$ denotes the photon and $\mathcal{G}$ the graviton. In this work we consider this process and calculate the decay rate, assuming the standard electroweak interactions and the canonical gravitational coupling of the standard particles.

We work with the linear theory of gravity, which means that we write the space-time metric in the form

$$g_{\lambda\rho} = \eta_{\lambda\rho} + 2\kappa h_{\lambda\rho}, \quad (1.2)$$

where $h_{\lambda\rho}$ is identified with the graviton field, and the gravitational couplings in the Lagrangian are expanded up to the linear order in $\kappa$. Furthermore, the constant $\kappa$ is defined in terms of Newton’s gravitational constant by

$$\kappa = \sqrt{8\pi G}, \quad (1.3)$$

which is such that the field $h_{\lambda\rho}$ has the properly normalized kinetic energy term in the Lagrangian. This point of view for treating processes involving the gravitational and Standard Model interactions is the same as that employed in some recent works for the calculation of
quantum gravity amplitudes \cite{2, 3}, in which General Relativity is treated as an effective field theory for energies below the Planck scale \cite{4, 5, 6}.

The process is a clean one as far the kinematics is concerned: it involves the photon, with an energy equal to half the Z-boson mass. However, the rate is not sizable, as a simple order-of-magnitude estimate of the decay rate readily shows. The gravitational couplings will involve the factor $\kappa$ in the amplitude, and therefore $G$ in the rate. The $Z$ and the photon couplings provide a factor $\alpha^2$ in the rate. Dimensional arguments then indicate that the decay rate is of the form

$$\Gamma \sim \alpha^2 G M_Z^3,$$

which is of order $10^{-36}$ GeV.

Despite this, we have two motivations for performing this calculation. First, in theories that extend the standard model of electroweak interactions and/or the gravitational interactions, including theories of Lorentz invariance violation, the rate might be different. This calculation can pave the way and serve as a test case for similar calculations in the context of those extended theories. Second, the methods employed here can also be helpful in the calculations of related processes where the $Z$ boson appears as a virtual particle, such as $\nu \bar{\nu} \rightarrow \gamma G$, and it is conceivable that they can have physical relevance in some astrophysical contexts.

The amplitude for the process is determined by a set of one-loop diagrams in perturbation theory, that we divide into two classes. The first class, to which we refer as the fermion loops, consists of the diagrams that contain fermion lines circulating in the loop. The second class consists of the diagrams in which the $W$ bosons circulate in the loops, and in principle the diagrams that contain their corresponding unphysical Higgs bosons and Fadeev-Popov ghosts. Among the various vertices required to compute the diagrams, we need the gravitational couplings of the $W$ boson as well as those of the $Z$ and the photon. We adopt the point of view that, for each gauge boson $V = \gamma, W, Z$, those couplings are determined by the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -\kappa h^{\lambda \rho} T_{\lambda \rho}^{(V)},$$

(1.5)

where $T_{\lambda \rho}^{(V)}$ is the expression for the energy-momentum tensor that is obtained from the linear expansion of the gravitational Lagrangian term

$$\mathcal{L}_g^{(V)} = \sqrt{-g} \mathcal{L}_0^{(V)},$$

(1.6)

with $\mathcal{L}_0$ being the canonical expression for the bilinear part of the Lagrangian. Thus, for example, for the photon,

$$\mathcal{L}_0^{(\gamma)} = \frac{1}{4} F^{\mu \nu} F_{\mu \nu},$$

(1.7)

while for the $W$ and $Z$ it contains the corresponding mass term. This amounts to adopting the unitary gauge for the $W$ and $Z$ gauge bosons and therefore, for consistency, we employ the unitary gauge throughout. The amplitude that is obtained in this way has the following properties, which justify this approach: (1) it satisfies the transversality condition required by the conservation of the electromagnetic current, (2) it satisfies the analogous condition required by the conservation of the energy-momentum tensor, (3) the diagrams yield a finite contribution to the amplitude. In the presentation that follows, we consider the fermion loops first which are simpler, and therefore they allow us to introduce a set of techniques that are useful for treating the more complicated $W$ loops.
The paper is organized as follows. In Sec. 2 we show that Lorentz invariance together with the electromagnetic and gravitational transversality conditions imply that the on-shell amplitude can be expressed in terms of three form factors, two of which are non-zero only if the CP symmetry is broken at some level. The remainder of the paper is devoted to the calculation of the only form factor that is non-zero at the one-loop level. In Sec. 3 we enumerate the fermion loop diagrams and the corresponding Feynman rules, and we verify that they form a gauge invariant set in the sense that their total contribution to the amplitude satisfies the electromagnetic and gravitational transversality conditions. We then proceed to calculate their contribution to the form factors and, as expected, the two CP-odd form factors vanish and only the one that is CP even survives and it is finite. In Sec. 4 we carry out a similar procedure for the $W$ loop diagrams, with analogous results. As already mentioned, the calculations are carried out employing the unitary gauge for the $W, Z$ propagators, together with the gravitational couplings that follow from the canonical expressions for the energy-momentum tensor. Finally, in Sec. 5 we use the results obtained in the previous sections to compute the decay rate. In the course of the calculations we have used several Ward-like identities that relate the various gravitational vertices and other algebraic manipulations, we which have summarized in the appendices.

2 General form of the amplitude

We introduce the off-shell vertex function $F_{\lambda\rho\mu\nu}(q, k)$, which is defined such that the on-shell amplitude for the process is given by

$$\mathcal{M} = \mathcal{E}^{\lambda\rho}(q)\varepsilon^{\nu\sigma}(k)\varepsilon^\mu_Z(p)F_{\lambda\rho\mu\nu}(q, k),$$

(2.1)

where $\mathcal{E}^{\lambda\rho}$ is the polarization tensor of the graviton, while $\varepsilon^\nu$ and $\varepsilon^\mu_Z$ are the polarization vectors for the photon and $Z$, respectively. Since we are calculating the on-shell amplitude, the momentum vectors satisfy the on-shell conditions

$$k^2 = 0,$$

(2.2)

$$q^2 = 0,$$

(2.3)

$$p^2 = M_Z^2,$$

(2.4)

and they are related by momentum conservation

$$p = k + q,$$

(2.5)

which in turn imply the kinematic relation

$$2k \cdot q = M_Z^2.$$

(2.6)

The polarization vectors for the photon and $Z$ satisfy

$$\varepsilon^\mu_Z(p)p_\mu = 0,$$

(2.7)

$$\varepsilon^\nu(k)k_\nu = 0.$$

(2.8)

The polarization tensor for the graviton satisfies the analogous relations,

$$\mathcal{E}^{\lambda\rho}(q)q_\lambda = 0, \quad \mathcal{E}^{\lambda\rho}(q)q_\rho = 0,$$

(2.9)
and in addition it is symmetric and traceless, i.e.,

\[ \mathcal{E}^{\lambda\rho} = \mathcal{E}^{\rho\lambda}, \]  
\[ \mathcal{E}^{\lambda\rho} \eta_{\lambda\rho} = 0. \]  

(2.10) (2.11)

Eq. (2.10) implies that \( F_{\lambda\rho\mu\nu} \) can be defined such that

\[ F_{\lambda\rho\mu\nu} = F_{\rho\lambda\mu\nu}. \]  

(2.12)

In addition, the conservation of the electromagnetic current and the energy-momentum tensor, which are consequences of the electromagnetic and gravitational gauge invariance, imply some additional properties of \( F_{\lambda\rho\mu\nu} \) which will be useful in the explicit calculation of the amplitude. In the rest of this section, we explore the consequences of these conditions.

The fact that the \( Z \) and the graviton are electrically neutral has two implications\(^1\): (i) the conservation of the electromagnetic current yields the condition

\[ k^\nu F_{\lambda\rho\mu\nu} = 0, \]  

(2.13)

(ii) the absence of tree-level diagrams implies that \( F_{\lambda\rho\mu\nu} \) can be expanded around \( k = 0 \). We exploit these properties by writing

\[ F_{\lambda\rho\mu\nu} = T^0_{\lambda\rho\mu\nu} + k^\alpha T^1_{\lambda\rho\mu\nu\alpha}, \]  

(2.14)

where \( T^0_{\lambda\rho\mu\nu} \) is independent of \( k \). Since Eq. (2.13) must be satisfied for all \( k \), it implies that

\[ k^\nu T^0_{\lambda\rho\mu\nu} = 0, \]  
\[ k^\nu k^\alpha T^1_{\lambda\rho\mu\nu\alpha} = 0, \]  

(2.15) (2.16)

which in turn imply

\[ T^0_{\lambda\rho\mu\nu} = 0, \]  
\[ T^1_{\lambda\rho\mu\nu\alpha} = \text{antisymmetric in } \nu \leftrightarrow \alpha. \]  

(2.17)

Therefore, the amplitude can be written as

\[ \mathcal{M} = \mathcal{E}^{\lambda\rho} \varepsilon_\mu \varepsilon_\nu t_{\lambda\rho\mu\nu\alpha}(q, k), \]  

(2.18)

where we have defined

\[ f_{\nu\alpha} \equiv k_\nu \varepsilon_\alpha - k_\alpha \varepsilon_\nu. \]  

(2.19)

and \( t_{\lambda\rho\mu\nu\alpha} \) is some undetermined tensor.

In analogous fashion, the conservation of the energy-momentum tensor yields the condition

\[ q^\lambda j_{\lambda\rho} = 0, \]  
\[ q^\rho j_{\rho\lambda} = 0. \]  

(2.20)

\(^1\)When there are charged particles, Eqs. (2.13) and (2.14) are not valid. A relation analogous to Eq. (2.13) holds when the charged particles are on-shell, but in that case the tree-level contributions (Born diagrams) render the amplitude singular at \( k = 0 \).
where we have defined

\[ j_{\lambda\rho} = \varepsilon^{\nu\sigma} \varepsilon_Z^{\mu} F_{\lambda\rho\mu\nu}(q, k). \]  

(2.21)

Mimicking the argument of the electromagnetic case, we write

\[ j_{\lambda\rho} = j_{\lambda\rho}^0 + j_{\lambda\rho\sigma}^1 q^\sigma + j_{\lambda\rho\sigma\tau}^2 q^\sigma q^\tau, \]  

(2.22)

where \( j_{\lambda\rho}^0 \) and \( j_{\lambda\rho\sigma}^1 \) are independent of \( q \). As before, it follows that the transversality condition requires

\[ j_{\lambda\rho}^0 = 0, \]

\[ j_{\lambda\rho\sigma}^1 = -j_{\sigma\rho\lambda}^1. \]  

(2.23)

In addition, Eq. (2.12) implies that

\[ j_{\rho\lambda\sigma}^1 = j_{\lambda\rho\sigma}^1. \]  

(2.24)

By applying Eqs. (2.23) and (2.24) successively and repeatedly, we arrive at

\[ j_{\lambda\rho\sigma}^1 = -j_{\lambda\rho\sigma}^1, \]  

(2.25)

or in other words \( j_{\lambda\rho\sigma}^1 = 0 \).

Considering now the structure of \( j_{\lambda\rho\sigma\tau}^2 \), we can see that it has the following symmetry properties:

1. It must be symmetric in the indices \( \lambda, \rho \) because of Eq. (2.12).
2. It must be symmetric in the indices \( \sigma, \tau \) because of its definition in Eq. (2.22).
3. Because of Eq. (2.20), it must be antisymmetric when we interchange anyone of the indices \( (\lambda, \rho) \) with anyone of the indices \( (\tau, \sigma) \).

Moreover, any term in \( j_{\lambda\rho\sigma\tau}^2 \) containing \( q_\lambda \) or \( q_\rho \) vanishes in the amplitude because of the on-shell transversality conditions of Eq. (2.14), and any term containing \( q_\sigma \) or \( q_\tau \) vanishes as well in Eq. (2.22) since \( q^2 = 0 \). Thus none of the Lorentz indices \( \lambda, \rho, \tau, \sigma \) can be attached to the momentum \( q \). Remembering Eq. (2.18), we know that \( j_{\lambda\rho\sigma\tau}^2 \) must have a factor of \( f_{\alpha\beta} \), and of course it also involves a factor of \( \varepsilon_Z^\mu \). Thus, combining these pieces of information, and excluding for the moment possible terms involving the Levi-Civita tensor, it follows that \( j_{\lambda\rho\sigma\tau}^2 \) must be of the form

\[ j_{\lambda\rho\sigma\tau}^2 = F f_{\lambda\tau}^\mu Z_{\rho\sigma} + (\lambda \leftrightarrow \rho), \]  

(2.26)

where \( F \) is some undetermined scalar, and we have introduced the notation

\[ Z_{\rho\sigma} = k_\rho \varepsilon_Z^\sigma - k_\sigma \varepsilon_Z^\rho, \]  

(2.27)

or equivalently

\[ Z_{\rho\sigma} = p_\rho \varepsilon_Z^\sigma - p_\sigma \varepsilon_Z^\rho. \]  

(2.28)
The terms involving the Levi-Civita tensor are most easily enumerated by noticing that all the requirements stated above for \( J_{\alpha \beta \sigma} \) are satisfied if \( f_{\alpha \beta} \) or \( Z_{\alpha \beta} \) in Eq. (2.26) is replaced by its dual.

\[
\tilde{f}_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \mu \nu} f^{\mu \nu}, \\
\tilde{Z}_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \mu \nu} Z^{\mu \nu},
\]

respectively. Thus the most general form for the amplitude is

\[
\mathcal{M} = \mathcal{E}^{\lambda \rho \nu \sigma} q^\sigma q^\tau \left[ F f^*_{\lambda \tau} Z_{\rho \sigma} + F_1 \tilde{f}^*_\lambda Z_{\rho \sigma} + F_2 f^*_\lambda \tilde{Z}_{\rho \sigma} \right] + (\lambda \leftrightarrow \rho),
\]

where \( F_1 \) and \( F_2 \) are two additional Lorentz scalars. Notice also that a term involving both \( \tilde{f} \) and \( \tilde{Z} \) is not included, since the product of two epsilon tensors can be written without it, and all such terms are already exhausted in Eq. (2.30).

It is convenient to write the expression for \( F_{\lambda \rho \mu \nu} \) that follows from the form of the amplitude given in Eq. (2.30). Using the definition of Eq. (2.1) we get

\[
F_{\lambda \rho \mu \nu} = \left( F(k \lambda q_\nu - k \cdot q_\nu \lambda)(k_\rho q_\mu - k \cdot q_\mu \rho) + F_1 [q]_{\lambda \nu}(k_\rho q_\mu - k \cdot q_\mu \rho) + F_2 (k \lambda q_\nu - k \cdot q_\nu \lambda)(q[k]_{\rho \mu}) \right) + (\lambda \leftrightarrow \rho),
\]

where we have used the shorthand notation

\[
[qk]_{\mu \nu} \equiv \epsilon_{\mu \nu \alpha \beta} q^\alpha k^\beta.
\]

3. **Diagrams with fermion loops**

3.1 **Diagrams and Feynman rules**

As already mentioned in the Introduction, we consider first the fermion loop diagrams, which are shown in Fig. 1. The Feynman rules for the various vertices that appear there have been given in the literature [7, 8, 9, 10, 11], and for convenience they are summarized in Fig. 2. Any fermion \( f \) that circulates in the loop must be electrically charged since it is attached to the photon line, and we denote its charge by \( eQ_f \) where \( e \) is the charge of the positron. The gravitational vertex function of the fermion is given by

\[
V_{\lambda \rho}(p, p') = \frac{1}{4} \left[ \gamma_{\lambda}(p + p')_{\rho} + \gamma_{\rho}(p + p')_{\lambda} \right] - \frac{1}{2} \eta_{\lambda \rho} \left[ p + p' - 2m_f \right],
\]

which can also be written as

\[
V_{\lambda \rho}(p, p') = -\frac{1}{2} a_{\lambda \rho \alpha \beta}(p + p')^\alpha \gamma^\beta + m_f \eta_{\lambda \rho},
\]

where

\[
a_{\lambda \rho \mu \nu} = \eta_{\lambda \rho} \eta_{\mu \nu} - \frac{1}{2} \eta_{\lambda \rho} \eta_{\mu \nu}.
\]
In writing this form, we have used the shorthand notation
\[ \eta_{\lambda\rho\mu\nu} \equiv \eta_{\lambda\mu} \eta_{\rho\nu} + \eta_{\lambda\nu} \eta_{\rho\mu} \, . \tag{3.4} \]

The tensor defined in Eq. (3.3) also appears in the coupling of a fermion bilinear to a gauge boson and the graviton. For example, if we denote Feynman rule for the Z-boson coupling to a fermion by \(-ig\tilde{\gamma}_\mu/(2\cos \theta_W)\), where
\[ \tilde{\gamma}_\mu \equiv \gamma_\mu (X_f + Y_f \gamma^5) , \tag{3.5} \]
the fermion-Z-graviton vertex is given by \(-ikga_{\lambda\rho\nu} \tilde{\gamma}^\nu/(2\cos \theta_W)\), as indicated in Fig. 2. In the standard model,
\begin{align*}
X_f &= T_{Lf} - 2Q_f \sin^2 \theta_W , \\
Y_f &= -T_{Lf} , \tag{3.6}
\end{align*}
where \(T_{Lf}\) is the eigenvalue of the diagonal generator of SU(2) acting on the left-chiral component of the fermion.

As emphasized earlier, we employ the unitary gauge for the gauge bosons, which means that their propagators and the gravitational couplings are determined using the canonical form of
Figure 2: Notations for Feynman rules for couplings that appear in Fig. 1. The charge of the fermion is $eQ_f$. Various symbols appearing in this figure have been explained in the text.

the kinetic term in the bilinear part of the Lagrangian. The gravitational vertex function $c_{\lambda\rho\alpha\beta}$ of the photon is then given by

$$c_{\lambda\rho\alpha\beta}(k, k') = \eta_{\lambda\rho}(\eta_{\alpha\beta}k \cdot k' - k_\alpha k_\beta) - \eta_{\alpha\beta}(k_\lambda k'_\rho + k'_\lambda k_\rho)$$

$$+ k_\beta(\eta_{\lambda\alpha}k'_\rho + \eta_{\rho\alpha}k'_\lambda) + k_\alpha(\eta_{\lambda\beta}k_\rho + \eta_{\rho\beta}k_\lambda)$$

$$- k \cdot k'(\eta_{\lambda\alpha}\eta_{\rho\beta} + \eta_{\lambda\beta}\eta_{\rho\alpha}),$$

and it is is useful to note that it satisfies

$$k^\alpha c_{\lambda\rho\alpha\beta}(k, k') = k'^\beta c_{\lambda\rho\alpha\beta}(k, k') = 0.$$  

For our present purposes it is convenient to write it in the more compact form

$$c_{\lambda\rho\alpha\beta}(k, k') = \eta_{\lambda\rho}(\eta_{\alpha\beta}k \cdot k' - k_\alpha k_\beta) - (\eta_{\lambda\rho|\alpha\beta,\mu\nu} - \eta_{\lambda\rho|\alpha\beta,\mu\nu})k^\mu k'^\nu,$$

where we have introduced the notation

$$\eta_{\lambda\rho|\alpha\beta,\mu\nu} \equiv \eta_{\lambda\rho|\alpha\beta}\eta_{\mu\nu} + \eta_{\lambda\rho|\alpha\beta}\eta_{\mu\nu},$$

with $\eta_{\lambda\rho|\mu\nu}$ defined in Eq. (3.4). Similarly, the gravitational vertex function of the Z-boson is given by

$$c_{\lambda\rho\alpha\beta}'(k, k') = c_{\lambda\rho\alpha\beta}(k, k') - M_Z^2 a'_{\lambda\rho\alpha\beta},$$

where

$$a'_{\lambda\rho\alpha\beta} = \eta_{\lambda\rho}\eta_{\mu\nu} - \eta_{\lambda\rho|\mu\nu}.$$
With this choice of vertices, the $Z$-propagator to be used is then given by

$$D^\alpha_\beta(k) = \frac{1}{k^2 - M_Z^2} \left( -\eta^\alpha_\beta + \frac{k^\alpha k^\beta}{M_Z^2} \right). \quad (3.13)$$

For the photon propagator $D^\alpha_\beta(k)$, the equation of motion that follows from the Lagrangian given in Eq. (1.7) determines only the transverse part of the solution, leaving an undetermined longitudinal part. That is, in the presence of a source $J_\mu$, the vector potential is given by

$$A_\alpha = D^\alpha_\beta J^\beta, \quad (3.14)$$

where

$$D^\alpha_\beta(k) = \frac{1}{k^2} \left( -\eta^\alpha_\beta + \frac{k^\alpha k^\beta}{k^2} \right) + D_L k^\alpha k^\beta, \quad (3.15)$$

with $D_L$ being an undetermined scalar function of $k$. The consistency of the equation of motion requires that $J_\alpha$ be conserved, which means that $D_L$ drops out in Eq. (3.14), and in fact that we can set

$$D^\alpha_\beta(k) = -\frac{\eta^\alpha_\beta}{k^2} \quad (3.16)$$

in the solution. The counterpart to this result in the context of our calculation is that, since the photon propagator enters only in the diagram in which the graviton is attached to the external photon line, then by virtue Eq. (3.8) for practical purposes the propagator can be taken as given in Eq. (3.16), which is what we adopt.

### 3.2 The amplitude

We denote by $F^{(f)}_{\lambda\rho\mu\nu}$ the contribution to $F_{\lambda\rho\mu\nu}$ from one fermion $f$ in the loop. Then writing it in the form

$$F^{(f)}_{\lambda\rho\mu\nu} = \frac{\kappa e g}{2 \cos \theta_W} Q_f T^{(f)}_{\lambda\rho\mu\nu}, \quad (3.17)$$

the contributions from the various diagrams to $T^{(f)}_{\lambda\rho\mu\nu}$ are given by

\begin{align*}
T^{(f:a)}_{\lambda\rho\mu\nu} &= i \int \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(l - k) \gamma_\nu S(l) V_{\lambda\rho}(l + q, l) S(l + q) \right], \\
T^{(f:b)}_{\lambda\rho\mu\nu} &= i \int \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(l) V_{\lambda\rho}(l + q, l) S(l + q) \gamma_\nu S(l + p) \right], \\
T^{(f:c)}_{\lambda\rho\mu\nu} &= a_{\lambda\rho\mu\nu} \Pi^\alpha_\beta(p), \\
T^{(f:d)}_{\lambda\rho\mu\nu} &= a_{\lambda\rho\mu\nu} \Pi^\alpha_\nu(k), \\
T^{(f:e)}_{\lambda\rho\mu\nu} &= c_{\lambda\rho\mu\nu}(p, k) D^\alpha_\beta(p) \Pi_{\alpha\beta}(p), \\
T^{(f:f)}_{\lambda\rho\mu\nu} &= c^Z_{\lambda\rho\mu\nu}(p, k) D^\alpha_\beta(k) \Pi_{\beta\nu}(k), \quad (3.18)
\end{align*}

where

$$\Pi_{\mu\nu}(k) = i \int \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu S(l) \gamma_\nu S(l + k) \right]. \quad (3.19)$$
\( \Pi_{\mu\nu}(k) \), which is immediately recognized to be the fermion contribution to the \( \gamma Z \) polarization mixing tensor, satisfies the transversality condition

\[ k^\nu \Pi_{\mu\nu}(k) = 0, \quad (3.20) \]

which is a consequence of electromagnetic gauge invariance and which can be proved explicitly by means of the elementary identity

\[ k^\nu S(l) \gamma_\mu S(l + k) = S(l) - S(l + k). \quad (3.21) \]

This property of \( \Pi_{\mu\nu}(k) \) implies that it is of the form

\[ \Pi_{\mu\nu}(k) = \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k), \quad (3.22) \]

where \( \Pi(k) \) is given by a (logarithmically divergent) integral that can be obtained from Eq. (3.19) but whose precise value, as we will see, is not relevant for our calculation.

3.3 The electromagnetic transversality condition

Eq. (3.20) implies

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a)} = k^\nu T_{\lambda\rho\mu\nu}^{(f:d)} = 0, \quad (3.23) \]

and using Eq. (3.22)

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:e)} = 0. \quad (3.24) \]

Then using Eq. (3.24) and remembering the momentum conservation equation Eq. (2.5), we obtain

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a+b)} = i \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{\gamma}_\mu \{ S(l - k) - S(l) \} V_{\lambda\rho}(l, l + q) S(l + q) \right], \]

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a+b)} = i \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{\gamma}_\mu S(l) V_{\lambda\rho}(l, l + q) \{ S(l + q) - S(l + p) \} \right], \quad (3.25) \]

and therefore

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a+b)} = i \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{\gamma}_\mu S(l - k) V_{\lambda\rho}(l + q, l) S(l + q) - \tilde{\gamma}_\mu S(l) V_{\lambda\rho}(l + q, l) S(l + p) \right]. \quad (3.26) \]

Changing the dummy loop momentum in the first term this can be written as

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a+b)} = i \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{\gamma}_\mu S(l) \{ V_{\lambda\rho}(l + p, l + k) - V_{\lambda\rho}(l + q, l) \} S(l + p) \right], \quad (3.27) \]

and using the identity given in Eq. (A.4)

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f:a+b)} = - a_{\lambda\rho\mu\nu} k^\nu \Pi_\mu^\alpha(p) = - k^\nu T_{\lambda\rho\mu\nu}^{(f:c)}. \quad (3.28) \]

Using Eqs. (3.23), (3.24) and (3.28), it follows that

\[ k^\nu T_{\lambda\rho\mu\nu}^{(f)} = 0, \quad (3.29) \]

which establishes the electromagnetic gauge invariance of this set of diagrams, in the sense that their contribution to the amplitude satisfies the requirement due to the conservation of the electromagnetic current.
3.4 The gravitational transversality condition

In similar fashion, we now establish that

$$q^\lambda \varepsilon^{\nu\kappa}(k) \varepsilon_Z^{\mu}(p) F^{(f)}_{\lambda\mu\nu} = 0. \quad (3.30)$$

In contrast with the electromagnetic case, here both the $Z$ and the photon are assumed to be on-shell.

Using the identity

$$S(l)q^\lambda V_{\lambda\rho}(l + q, l)S(l + q) = l_\rho S(l) - (l_\rho + q_\rho)S(l + q) + \frac{1}{8} S(l) \left( \gamma_\rho \gamma_\rho - \not\gamma \gamma_\rho \right) - \frac{1}{8} \left( \gamma_\rho \gamma_\rho - \not\gamma \gamma_\rho \right) S(l + q), \quad (3.31)$$

together with the Dirac-matrix identity

$$\gamma_\alpha \gamma_\beta \gamma_\rho + \gamma_\rho \gamma_\beta \gamma_\alpha = 2 \left( \eta_{\alpha \beta} \gamma_\rho + \eta_{\beta \rho} \gamma_\alpha - \eta_{\alpha \rho} \gamma_\beta \right), \quad (3.32)$$

and making judicious shifts in the loop momenta in various terms we obtain

$$q^\lambda T^{(f:a+b)}_{\lambda\mu\nu} = k_\rho \Pi_{\mu\nu}(k) - p_\rho \Pi_{\mu\nu}(p) + \frac{1}{2} \left( q_\nu \Pi_{\mu\rho}(p) - \eta_{\nu\rho} q^{\alpha} \Pi_{\mu\alpha}(p) \right), \quad (3.33)$$

The definition of $a_{\lambda\rho\mu}$ given in Eq. (3.3) allows us to write

$$q^\lambda T^{(f:c)}_{\lambda\rho\mu} = q_\rho \Pi_{\mu\nu}(p) - \frac{1}{2} q_\nu \Pi_{\mu\rho}(p) - \frac{1}{2} \eta_{\nu\rho} q^{\alpha} \Pi_{\mu\alpha}(p), \quad (3.34)$$

$$q^\lambda T^{(f:d)}_{\lambda\rho\mu} = q_\rho \Pi_{\mu\nu}(k) - \frac{1}{2} q_\nu \Pi_{\mu\rho}(k) - \frac{1}{2} \eta_{\nu\rho} q^{\alpha} \Pi_{\mu\alpha}(k), \quad (3.35)$$

and therefore

$$T^{(f:a+b+c+d)}_{\lambda\mu\nu} = p_\rho \Pi_{\mu\nu}(p) - k_\rho \Pi_{\mu\nu}(p) - \eta_{\nu\rho} q^{\alpha} \Pi_{\mu\alpha}(p) - \eta_{\mu\rho} q^{\alpha} \Pi_{\nu\alpha}(k). \quad (3.36)$$

For diagram Fig. II, using Eq. (2.13) we first find that

$$q^\lambda \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) c_{\lambda\rho\mu\nu}(p, k) = \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) \left[ (k_\alpha \eta_{\mu\nu} - k_\mu \eta_{\nu\alpha}) p^2 + (q_\nu k_\rho - \eta_{\nu\rho} k \cdot q) p_\alpha \right], \quad (3.37)$$

where we have used Eq. (2.8) and the on-shell photon condition $k^2 = 0$. By the transversality property of $\Pi_{\mu\nu}$ we can use

$$p_\alpha D^{\alpha\beta}(p) \Pi_{\mu\beta}(p) \propto p_\alpha \Pi^{\alpha\mu}(p) = 0, \quad (3.38)$$

which implies that

$$q^\lambda \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) T^{(f:e)}_{\lambda\rho\mu\nu} = \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) \left[ (k_\rho \Pi_{\mu\nu}(p) + \eta_{\nu\rho} q^{\alpha} \Pi_{\mu\alpha}(p) \right] . \quad (3.39)$$

Similarly, for the expression involving diagram of Fig. II, we find

$$q^\lambda \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) c_{\lambda\rho\mu\nu}(p, k) = \varepsilon^{\nu\kappa}(k) \varepsilon^{\mu}(p) \left[ (p_\rho \eta_{\mu\alpha} - p_\alpha \eta_{\mu\rho}) (k^2 - M_Z^2) + (q_\mu p_\rho - \eta_{\mu\rho} k \cdot q) k_\alpha \right], \quad (3.40)$$
and using the $Z$-propagator from Eq. (3.13)

$$q^\lambda \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)c_{\lambda\mu\nu\alpha}(p, k)D^\alpha_Z(k) = \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)\left[\eta_{\mu\rho}q^\beta - \eta^\beta_{\mu\rho}\right],$$

which gives

$$q^\lambda \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)T^{(f;\epsilon)}_{\lambda\rho\mu\nu} = \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)\left[\eta_{\mu\rho}q^\alpha\Pi^\alpha_{\rho\nu}(k) - p_\mu\Pi_{\mu\nu}(k)\right].$$

Adding Eqs. (3.36), (3.39) and (3.42), and using Eqs. (2.5) and (3.20), Eq. (3.30) is established.

### 3.5 Calculation of the form factors

The fact that the contribution of the fermion loop diagrams to the amplitude satisfies the electromagnetic and gravitational transversality conditions implies that it must have the structure given in Eq. (2.31). In order to extract the corresponding contribution to the form factors we need not evaluate in full the integral expressions for the various diagrams, and instead we can proceed as follows. The form factors $F_{1,2}$ are easily identified as the coefficients of the terms containing the Levi-Civita tensor. As we will see, no such terms appear so that these form factors are zero. For the form factor $F$, we can fix our attention on the contributions to just one of the terms that appear in Eq. (2.31). To be specific, we choose $Fk_\lambda k_\nu q_\mu q_\nu$. Since, by Eq. (2.7),

$$\varepsilon^\mu_Z(p)k_\mu = -\varepsilon^\mu_Z(p)q_\mu$$

such terms will arise from the terms in the integrals that contain a factor of either $k_\lambda k_\nu q_\mu q_\nu$ or $k_\lambda k_\nu k_\mu q_\nu$. A term of either form will be called a $kkqq$ term for the sake of brevity, and these are the only ones that we need to track in the evaluation of the integrals.

We now consider the contribution from each diagram to that kind of term, using the expressions given in Eq. (3.18). Since $T^{(f;\epsilon)}_{\lambda\rho\mu\nu}$ and $T^{(f;\lambda)}_{\lambda\rho\mu\nu}$ depend only on $p$ and $k$, respectively, neither one contains a $kkqq$ term. Using Eqs. (3.13) and (3.22) it is immediately realized that $T^{(f;\epsilon)}_{\lambda\rho\mu\nu}$ and $T^{(f;\lambda)}_{\lambda\rho\mu\nu}$ are proportional to $c_{\lambda\rho\mu\nu}(p, k)$ and $c^Z_{\lambda\rho\mu\nu}(p, k)$, respectively, neither one of which contains a $kkqq$ term as can be seen simply by looking at their definitions given in Eqs. (3.9) and (3.11).

Thus, we are left with $T^{(f;\lambda)}_{\lambda\rho\mu\nu}$ and $T^{(f;\lambda)}_{\lambda\rho\mu\nu}$ which using the graviton vertex in the form given in Eq. (3.2) and remembering Eqs. (2.10) and (2.11), can be written as

$$T^{(f;\lambda)}_{\lambda\rho\mu\nu} = i\int \frac{d^4l}{(2\pi)^4} \frac{l_\rho \text{Tr} \left[ \gamma_\mu(l - k + m_f)\gamma_\nu(l + m_f)\gamma_\lambda(l - \ell + m_f) \right]}{[(l - k)^2 - m_f^2][(l + q)^2 - m_f^2][(l + m_f)^2 - m_f^2]},$$

$$T^{(f;\lambda)}_{\lambda\rho\mu\nu} = i\int \frac{d^4l}{(2\pi)^4} \frac{l_\rho \text{Tr} \left[ \gamma_\mu(l - \ell + m_f)\gamma_\nu(l + m_f)\gamma_\lambda(l + k + m_f) \right]}{[(l + k)^2 - m_f^2][(l - q)^2 - m_f^2][(l + m_f)^2 - m_f^2]}.$$  

Changing the integration variable from $l$ to $-\ell$ in the integral for $T^{(f;\lambda)}_{\lambda\rho\mu\nu}$, and then using the cyclic property of the trace, together with the relations

$$C^{-1}\gamma^{\mu}_\ell C = -\gamma^\mu_\ell \quad \text{and} \quad C^{-1}S^\top(\ell)C = S(-\ell),$$

we find

$$T^{(f;\lambda)}_{\lambda\rho\mu\nu} = i\int \frac{d^4l}{(2\pi)^4} \frac{l_\rho \text{Tr} \left[ \gamma_\mu(l - k + m_f)\gamma_\nu(l + m_f)\gamma_\lambda(l + \ell + m_f) \right]}{[(l - k)^2 - m_f^2][(l + q)^2 - m_f^2][(l + m_f)^2 - m_f^2]},$$

which gives

$$q^\lambda \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)T^{(f;\lambda)}_{\lambda\rho\mu\nu} = \varepsilon^{\nu^*}(k)\varepsilon^\mu_Z(p)\left[\eta_{\mu\rho}q^\alpha\Pi^\alpha_{\rho\nu}(k) - p_\mu\Pi_{\mu\nu}(k)\right].$$

Adding Eqs. (3.36), (3.39) and (3.42), and using Eqs. (2.5) and (3.20), Eq. (3.30) is established.
and similar ones, it follows that the $\gamma_\mu \gamma_5$ axial current coupling from $T^{(f:a)}_{\lambda\rho\mu\nu}$ and $T^{(f:b)}_{\lambda\rho\mu\nu}$ are opposite, while the vector current term $\gamma_\mu$ is the same. Therefore,

$$T^{(f:a+b)}_{\lambda\rho\mu\nu} = 2iX_f \int \frac{d^4l}{(2\pi)^4} \frac{f_{\lambda\rho\mu\nu}(l)}{[(l - k)^2 - m_f^2][(l + q)^2 - m_f^2](l^2 - m_f^2)}, \quad (3.46)$$

where

$$f_{\lambda\rho\mu\nu}(l) = l_\rho \text{Tr} \left[ \gamma_\mu(\not{l} - \not{k} + m_f)\gamma_\nu(\not{l} + m_f)\gamma_\lambda(\not{l} + \not{q} + m_f) \right]. \quad (3.47)$$

The absence of the $\gamma_5$ term implies that there are no terms containing the Levi-Civita tensor, so that there are no contributions to the form factors $F_{1,2}$. Continuing to $D$ dimensions and parameterizing the integrals in the standard fashion we obtain

$$T^{(f:a+b)}_{\lambda\rho\mu\nu} = 4iX_f \int \frac{d^Dl}{(2\pi)^D} \int_0^1 dx \int_0^{1-x} dy \frac{f_a(l + xk - yq)}{[l^2 - m_f^2 + xyM_Z^2]^3}, \quad (3.48)$$

where we have used the on-shell relations given in Eqs. (2.2), (2.3) and (2.6). Evaluating the trace and focusing on the $kkqq$ terms as described above, we get

$$T^{(f:a+b)}_{\lambda\rho\mu\nu} = -64iX_f q_\mu q_\nu k_\lambda k_\rho \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{x^2y(1-x-y)}{[l^2 - m_f^2 + xyM_Z^2]^3} \cdots, \quad (3.49)$$

where we have continued back to four dimensions since the resulting integral is convergent and the ellipses indicate that have omitted all the other terms. Performing the momentum integration and recalling the overall factors in Eq. (3.17), we finally find the contribution from any fermion in the loop to the form factor $F$ to be

$$F^{(f)} = -\frac{\kappa eg}{2\pi^2 \cos \theta_W} Q_f X_f I(m_f/M_Z), \quad (3.50)$$

where

$$I(A) = \int_0^1 dx \int_0^{1-x} dy \frac{x^2y(1-x-y)}{A^2 - xy}. \quad (3.51)$$

### 4 Diagrams with $W$ loops

#### 4.1 Diagrams and Feynman rules

The one-loop diagrams involving the $W$-boson are shown in Fig. 3 and the relevant Feynman rules are summarized in Fig. 4. As stated in the Introduction, we employ the unitary gauge, so that the propagator and gravitational vertex function of the $W$ are given by

$$D_W^{\alpha\beta}(k) = \frac{1}{k^2 - M_W^2} \left( -\eta^{\alpha\beta} + \frac{k^\alpha k^\beta}{M_W^2} \right), \quad (4.1)$$

$$c_W^{\alpha\beta}(k, k') = c_{\lambda\rho\alpha\beta}(k, k') - M_W^2 d_{\alpha\rho\beta}, \quad (4.2)$$

in analogy with the corresponding quantities for the $Z$ boson. The trilinear boson couplings
Figure 3: 1-loop diagrams for the process $Z \rightarrow \gamma + \mathcal{G}$ involving charged gauge bosons in the loop. In the external lines, the thick saw-tooth lines refer to the $Z$-boson, the thin wavy line to the photon, and the braided lines to the graviton.
W_\alpha^+(p) \sim W_\beta^+(p) \\
\frac{ieD_{\alpha\beta}^{-1}}{2}(p)

W_\alpha^+(p_1) \sim W_\beta^+(p_2) \\
-\frac{ieN_{\alpha\beta\gamma}}{2}(p_1, p_2)

W_\alpha^+(p_1) \sim W_\beta^+(p_2) \\
\frac{ieN_{\lambda\rho\alpha\beta}}{2}(p_1, p_2)

W_\alpha^+(p_1) \sim W_\beta^+(p_2) \\
\eta_{\alpha\beta}(q)

W_\alpha^+(p_1) \sim W_\beta^+(p_2) \\
\eta_{\lambda\rho}(q)

Figure 4: Notations for Feynman rules involving the \( W \)-bosons in the unitary gauge and their extensions to include linearized graviton couplings. For the free \( W \)-line, the notation represents the inverse propagator.

Involving the \( W \) have the following form in the momentum space,

\[
\mathcal{L}_{\text{cubic}} = -gw_\alpha W_\beta^+ W_\gamma^0 N_{\alpha\beta\gamma}(p_1, p_2, p_3),
\]

where \( W_\alpha \) is the field which annihilates the \( W^+ \) boson, and \( W_\gamma^0 \) is the field operator for the neutral SU(2) gauge boson. The momenta have been written in the same order as the gauge bosons fields, with \( p_2 \) flowing out of the vertex and the other two flowing in, and

\[
N_{\alpha\beta\gamma}(p_1, p_2, p_3) = \eta_{\beta\gamma}(p_2 + p_3)^\sigma + \eta_{\alpha\sigma}(p_1 - p_3)^\sigma - \eta_{\alpha\beta}(p_1 + p_2)^\sigma.
\]

Since only two of the three momenta are independent in this vertex, we will often use the shorter notation

\[
\tilde{N}_{\alpha\beta\gamma}(p_1, p_2) \equiv N_{\alpha\beta\gamma}(p_1, p_2, p_2 - p_1).
\]

Similarly, the quartic \( WW^+Z \) interaction in the flat space Lagrangian is given by

\[
\mathcal{L}_{\text{quartic}} = -eg \cos \theta_W W_\alpha W_\beta^+ Z_\mu A_\nu R_{\alpha\beta\mu\nu}
\]

where

\[
R_{\alpha\beta\mu\nu} = 2\eta_{\alpha\beta}\eta_{\mu\nu} - \eta_{\alpha\beta}^{\mu\nu}.
\]
Following the linear approximation to the gravitational interactions already outlined, we then obtain the corresponding vertices involving the graviton, characterized by the vertex functions

\[
N_{\lambda\rho\alpha\beta\gamma}(p_1, p_2, p_3) = \eta_{\lambda\rho}N_{\alpha\beta\gamma}(p_1, p_2, p_3) - \eta_{\lambda\rho|\beta\gamma,\alpha\sigma}(p_2 + p_3)\sigma
- \eta_{\lambda\rho|\gamma\alpha,\beta\sigma}(p_1 - p_3)\sigma + \eta_{\lambda\rho|\alpha\beta,\gamma\sigma}(p_1 + p_2)\sigma,
\]

\[
R_{\lambda\rho\alpha\beta\mu\nu} = \eta_{\lambda\rho}R_{\alpha\beta\mu\nu} - 2\eta_{\lambda\rho|\alpha\beta,\mu\nu} + \eta_{\lambda\rho|\alpha\mu,\beta\nu} + \eta_{\lambda\rho|\alpha\nu,\beta\mu},
\]

as indicated in Fig. 4, where the symbol \( \eta \) with six indices was defined in Eq. (3.10).

### 4.2 The amplitude

Using the Feynman rules just discussed, and defining \( T^{(W)}_{\lambda\rho\mu\nu} \) by

\[
F^{(W)}_{\lambda\rho\mu\nu} = \kappa e g \cos \theta_W T^{(W)}_{\lambda\rho\mu\nu},
\]

the contributions to \( T^{(W)}_{\lambda\rho\mu\nu} \) from the diagrams in Fig. 3 are given by

\[
iT^{(W:\text{a})}_{\lambda\rho\mu\nu} = \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l, l + p)D^\alpha_W(l)
\times c^W_{\lambda\rho\sigma\tau}(l + q, l)D^\sigma_W(l + q)\tilde{N}_{\gamma\delta\nu}(l + p, l + q)D^\delta_W(l + p),
\]

\[
iT^{(W:\text{b})}_{\lambda\rho\mu\nu} = \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l - k, l + q)D^\alpha_W(l - k)
\times \tilde{N}_{\gamma\delta\nu}(l, l - k)D^\delta_W(l)\tilde{N}_{\lambda\rho\sigma\tau}(l + p, l + q)D^\sigma_W(l + q),
\]

\[
iT^{(W:\text{c})}_{\lambda\rho\mu\nu} = \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l, l + p)D^\alpha_W(l)\tilde{N}_{\lambda\rho\sigma\tau\nu}(l + p, l, -k)D^\sigma_W(l + p),
\]

\[
iT^{(W:\text{d})}_{\lambda\rho\mu\nu} = \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l + k, l)D^\alpha_W(l)\tilde{N}_{\lambda\rho\sigma\tau\nu}(l, l + k)D^\sigma_W(l + k),
\]

\[
iT^{(W:\text{e})}_{\lambda\rho\mu\nu} = c_{\lambda\rho\sigma\nu}(p, k)D^\gamma_D(p)
\times \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l, l + p)D^\alpha_W(l)\tilde{N}_{\lambda\rho\sigma\tau\gamma}(l + p, l)D^\gamma_D(l + p),
\]

\[
iT^{(W:\text{f})}_{\lambda\rho\mu\nu} = c^{Z}_{\lambda\rho\mu\nu}(p, k)D^\alpha_D(k)
\times \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha\beta\mu}(l, l + k)D^\alpha_W(l)\tilde{N}_{\lambda\rho\sigma\tau\nu}(l, l + k)D^\sigma_W(l + k),
\]

\[
iT^{(W:\text{g})}_{\lambda\rho\mu\nu} = R_{\alpha\beta\mu\nu} \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\lambda\rho\sigma\tau\nu}(l + q, l)\tilde{N}_{\lambda\rho\sigma\tau\gamma}(l + p, l)D^\delta_W(l + q),
\]

\[
iT^{(W:\text{h})}_{\lambda\rho\mu\nu} = R_{\lambda\rho\alpha\beta\mu\nu} \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\lambda\rho\sigma\tau\nu}(l + q, l)\tilde{N}_{\lambda\rho\sigma\tau\gamma}(l + p, l)D^\delta_W(l + q),
\]

\[
iT^{(W:\text{i})}_{\lambda\rho\mu\nu} = c_{\lambda\rho\alpha\nu}(p, k)D^\alpha_D(p)R_{\sigma\tau\mu\beta} \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\lambda\rho\sigma\tau\nu}(l + q, l)\tilde{N}_{\lambda\rho\sigma\tau\gamma}(l + p, l),
\]

\[
iT^{(W:\text{j})}_{\lambda\rho\mu\nu} = c^{Z}_{\lambda\rho\alpha\nu}(p, k)D^\alpha_D(k)R_{\sigma\tau\beta\nu} \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\lambda\rho\sigma\tau\nu}(l + q, l)\tilde{N}_{\lambda\rho\sigma\tau\gamma}(l + p, l).
\]
4.3 The electromagnetic transversality condition

First, using Eq. (3.8) it is immediately seen that
\[ k^\nu T^{(W:e)}_{\lambda \rho \mu \nu} = k^\nu T^{(W:i)}_{\lambda \rho \mu \nu} = 0. \] (4.12)

For the rest, we use the Ward identities that have been given in Appendix A. Thus, using Eq. (A.13)
\[ ik^\nu T^{(W:d)}_{\lambda \rho \mu \nu} = -k^\nu R_{\lambda \rho \beta \mu \nu} \int \frac{d^4l}{(2\pi)^4} D^\alpha_\beta(\lambda), \] (4.13)

and comparing it with the expression for \( T^{(W:h)}_{\lambda \rho \mu \nu} \) in Eq. (4.11) we conclude that
\[ k^\nu T^{(W:d+h)}_{\lambda \rho \mu \nu} = 0. \] (4.14)

Similarly, applying Eq. (A.6), shifting the momentum in one of the resulting terms and then applying Eq. (A.9), we find
\[ k^\nu T^{(W:f+j)}_{\lambda \rho \mu \nu} = 0. \] (4.15)

Applying Eq. (A.6) to \( T^{(W:a)}_{\lambda \rho \mu \nu} \) and \( T^{(W:b)}_{\lambda \rho \mu \nu} \) we obtain
\[
\begin{align*}
ik^\nu T^{(W:a+b)}_{\lambda \rho \mu \nu} &= \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha \beta \mu}(l-k, l+q)(D^\delta_W(l-k) - D^\delta_W(l))c^W_{\lambda \rho \gamma \delta}(l + q, l)D^\beta_W(l + q) \\
&+ \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha \beta \mu}(l, l+p)D^\gamma_W(l)c^W_{\lambda \rho \sigma \tau}(l + q, l)(D^\beta_W(l + q) - D^\beta_W(l + p)) \\
&= \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\alpha \beta \mu}(l, l+p)D^\delta_W(l)(c^W_{\lambda \rho \gamma \delta}(l + p, l + k) - c^W_{\lambda \rho \gamma \delta}(l + q, l))D^\beta_W(l + p) \\
&+ \int \frac{d^4l}{(2\pi)^4} (\tilde{N}_{\alpha \beta \mu}(l, l+p) - \tilde{N}_{\alpha \beta \mu}(l-k, l+q))D^\gamma_W(l)c^W_{\lambda \rho \sigma \tau}(l + q, l)D^\beta_W(l + q).
\end{align*}
\] (4.16)

From this, and using Eqs. (A.7) and (A.9), it follows that
\[ k^\nu T^{(W:a+b+c+g)}_{\lambda \rho \mu \nu} = 0, \] (4.17)

which complemented by Eqs. (4.14) and (4.15) establishes the property
\[ k^\nu T^{(W)}_{\lambda \rho \mu \nu} = 0. \] (4.18)

4.4 The gravitational transversality condition

In similar fashion, here we establish that \( T^{(W)}_{\lambda \rho \mu \nu} \) satisfies
\[ q^\lambda e^{\nu \tau}(k)\varepsilon_{Z}^\mu(p)T^{(W)}_{\lambda \rho \mu \nu} = 0. \] (4.19)
In the formulas that we obtain below for the contraction of $q$ with the various amplitudes $T_{\lambda\rho\mu\nu}^{(W:x)}$, we omit writing the polarization vectors of the photon and the $Z$ to simplify the notation, but it should be understood that the relations are valid in general only when the contractions with the polarization vectors as indicated in Eq. (4.19) are taken, and the on-shell conditions for the photon and the $Z$ are imposed.

We start with the identity involving the $W$ gravitational vertex and the propagator in the unitary gauge,

$$D_W^{\alpha\beta'}(l + q) q^\lambda c_{\lambda\alpha\beta}(l + q, l) D_W^{\beta\gamma'}(l) = l_\rho D_W^{\alpha\beta'}(l) - (l_\rho + q_\rho) D_W^{\alpha\beta'}(l + q) + \eta_{\rho\sigma} \left( q^{\alpha'} D_W^{\beta\sigma'}(l) + q^{\beta'} D_W^{\alpha\sigma'}(l + q) \right), \quad (4.20)$$

which resembles Eq. (3.31) for fermions. Using Eq. (4.20), we obtain

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:g)} = R_{\alpha\beta\mu\nu} \int \frac{d^4l}{(2\pi)^4} \left( l_\rho D_W^{\alpha\beta}(l) - (l_\rho + q_\rho) D_W^{\alpha\beta}(l + q) + \eta_{\rho\sigma} q^{\alpha} D_W^{\beta\sigma}(l + q) + \eta_{\rho\sigma} q^{\beta} D_W^{\alpha\sigma}(l) \right). \quad (4.21)$$

The first two terms cancel, as can be shown by making a change of variables in the second one. Using Eq. (4.7), the remaining two terms can be written as

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:g)} = \int \frac{d^4l}{(2\pi)^4} \left( (4\eta_{\mu\nu} q^\alpha D_{\alpha\rho}^W(l) - 2q_\mu D_{\nu\rho}^W(l) - 2q_\nu D_{\mu\rho}^W(l) \right). \quad (4.22)$$

Next, using the definition in Eq. (4.19), it is straightforward to write

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:h)} = \int \frac{d^4l}{(2\pi)^4} \left( q_\rho R_{\alpha\beta\mu\nu} D_{\alpha\beta}^W(l) - 2 \left( \eta_{\rho\lambda} q_\lambda + \eta_{\mu\rho} q_\mu \right) \eta_{\alpha\beta} D_{\alpha\beta}^W(l) - 4\eta_{\mu\nu} q^\alpha D_{\alpha\rho}^W(l) + 2q_\alpha \left( \eta_{\rho\lambda} D_{\alpha\lambda}^W(l) + \eta_{\mu\rho} D_{\alpha\mu}^W(l) \right) + 2q_\mu D_{\nu\rho}^W(l) + 2q_\nu D_{\mu\rho}^W(l) \right). \quad (4.23)$$

Further, using Eq. (4.31), we obtain

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:i)} = \left[ k_\rho R_{\sigma\tau\mu\nu} - \eta_{\rho\lambda} k_\lambda R_{\sigma\tau\mu\beta} - (q_\alpha k_\beta - \eta_{\nu\rho} k \cdot q) \frac{p^\beta}{p^2} R_{\sigma\tau\mu\beta} \right] \int \frac{d^4l}{(2\pi)^4} D_{\sigma\tau}^W(l). \quad (4.24)$$

Similarly, using Eq. (4.31), we can write

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:j)} = \left[ \eta_{\rho\mu} q^\alpha R_{\sigma\tau\alpha\nu} - p_\alpha R_{\sigma\tau\mu\nu} \right] \int \frac{d^4l}{(2\pi)^4} D_{\sigma\tau}^W(l), \quad (4.25)$$

and summing up Eqs. (4.22), (4.23), (4.24) and (4.25), we obtain

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:g+h+i+j)} = 2\eta_{\rho\mu} p^\alpha \int \frac{d^4l}{(2\pi)^4} D_{\alpha\mu}^W(l) - (q_\alpha k_\rho - \eta_{\nu\rho} k \cdot q) \frac{p^\beta}{p^2} R_{\sigma\tau\mu\beta} \int \frac{d^4l}{(2\pi)^4} D_{\sigma\tau}^W(l). \quad (4.26)$$

Regarding $T_{\lambda\rho\mu\nu}^{(W:c)}$, by means of Eq. (4.17) we can write

$$i\eta^\lambda T_{\lambda\rho\mu\nu}^{(W:c)} = \int \frac{d^4l}{(2\pi)^4} N_{\alpha\beta\mu}(l, l + p) D_{\alpha\beta}^W(l) D_{\alpha\beta}^W(l + p) \left[ -k_\rho N_{\sigma\tau\nu}(l + p, l) + l_\rho q^\delta R_{\sigma\tau\nu\delta} + p_\rho N_{\sigma\tau\nu}(l + k, l) - \eta_{\rho\sigma} q^\delta N_{\delta\tau\nu}(l + k, l) - \eta_{\rho\sigma} q^\delta N_{\delta\tau\nu}(l + p, l) \right], \quad (4.27)$$

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and Eq. 3.31 allows us to write the result for \(q^\mu T^{(W:e)}_{\lambda\rho\mu\nu}\) as the sum of the two terms,

\[
\begin{align*}
  i q^\lambda T^{(W:e1)}_{\lambda\rho\mu\nu} &= (k_\rho \eta_\nu^\delta - k^\delta \eta_{\rho\nu}) \int \frac{d^4l}{(2\pi)^4} N_{\sigma\tau\delta}(l + p, l) N_{\alpha\beta\mu}(l, l + p) D_W^{\alpha\tau}(l) D_W^\beta_\sigma(l + p), \\
  i q^\lambda T^{(W:e2)}_{\lambda\rho\mu\nu} &= -(q_\mu k_\rho - \eta_{\mu\rho} k \cdot q) \frac{p^\gamma}{p^2} \int \frac{d^4l}{(2\pi)^4} N_{\alpha\beta\mu}(l, l + p) N_{\sigma\tau\gamma}(l + p, l) D_W^{\alpha\tau}(l) D_W^\beta_\sigma(l + p) \\
  &= (q_\mu k_\rho - \eta_{\mu\rho} k \cdot q) \frac{p^\delta}{p^2} R_{\sigma\tau\mu\delta} \int \frac{d^4l}{(2\pi)^4} D_W^\sigma_\tau(l), \quad (4.28)
\end{align*}
\]

where we have applied Eq. (A.12) in the last step. If we consider the sum \(q^\mu T^{(W:c+e1)}_{\lambda\rho\mu\nu}\), the term from \(q^\mu T^{(W:e1)}_{\lambda\rho\mu\nu}\) containing \(k_\rho\) cancels an identical term from Eq. (4.27). The remaining term from \(q^\mu T^{(W:e1)}_{\lambda\rho\mu\nu}\) combines with the term proportional to \(\eta_{\rho\nu}\) from Eq. (4.27), and applying Eq. (A.12) they yield

\[
\eta_{\rho\nu} p^\delta R_{\sigma\tau\mu\delta} \int \frac{d^4l}{(2\pi)^4} D_W^\sigma_\tau(l). \quad (4.29)
\]

Substituting the expression for \(R_{\alpha\beta\mu\delta}\) given in Eq. (4.11), the term proportional \(p^\mu\) vanishes due to Eq. (4.29) and the remainder cancels one of the terms of Eq. (4.26). The other term of Eq. (4.26) is canceled by the \(q^\mu T^{(W:e2)}_{\lambda\rho\mu\nu}\) contribution, and in this way we obtain

\[
\begin{align*}
  i q^\lambda T^{(W:c+e+g+h+i+j)}_{\lambda\rho\mu\nu} &= \int \frac{d^4l}{(2\pi)^4} N_{\alpha\beta\mu}(l, l + p) D_W^{\alpha\tau}(l) D_W^\beta_\sigma(l + p) \left[ l_\rho q^\delta R_{\sigma\tau\nu\delta} \\
  &\quad + p_\rho N_{\sigma\tau\nu}(l + k, l) - \eta_{\rho\sigma} q^\delta N_{\delta\tau\nu}(l + k, l) \\
  &\quad - \eta_{\rho\tau} q^\delta N_{\sigma\delta\nu}(l + p, l + q) \right]. \quad (4.30)
\end{align*}
\]

For the contraction of \(T^{(W:f)}_{\lambda\rho\mu\nu}\) we use Eq. (3.41) to write it in the form

\[
\begin{align*}
  i q^\lambda T^{(W:f)}_{\lambda\rho\mu\nu} &= (\eta_{\mu\nu} q^\delta - p_\rho q^\delta) \int \frac{d^4l}{(2\pi)^4} N_{\alpha\beta\delta}(l, l + k) N_{\sigma\tau\nu}(l + k, l) D_W^{\alpha\tau}(l) D_W^{\beta_\sigma}(l + k), \quad (4.31)
\end{align*}
\]

and applying the identity of Eq. (3.5)

\[
\begin{align*}
  i q^\lambda T^{(W:d)}_{\lambda\rho\mu\nu} &= \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\sigma\tau\nu}(l + k, l) D_W^{\alpha\tau}(l) D_W^{\beta_\sigma}(l + k) \left[ (p_\rho - l_\rho) \tilde{N}_{\alpha\beta\mu}(l, l + k) \\
  &\quad + (p_\rho - l_\rho) \tilde{N}_{\alpha\beta\mu}(l - q, l - q + k) - k_\rho \tilde{N}_{\alpha\beta\mu}(l, l + p) - \eta_{\rho\alpha} q^\delta \tilde{N}_{\delta\beta\mu}(l - q, l + k) \\
  &\quad - \eta_{\rho\beta} q^\delta \tilde{N}_{\alpha\delta\mu}(l, l + p) - \eta_{\rho\mu} q^\delta \tilde{N}_{\alpha\beta\delta}(l, l + k) \right]. \quad (4.32)
\end{align*}
\]

Adding these results and using Eq. (3.3),

\[
\begin{align*}
  i q^\lambda T^{(W:d+1)}_{\lambda\rho\mu\nu} &= \int \frac{d^4l}{(2\pi)^4} N_{\sigma\tau\nu}(l + k, l) D_W^{\alpha\tau}(l) D_W^{\beta_\sigma}(l + k) \left[ l_\rho q^\delta R_{\alpha\beta\mu\delta} \\
  &\quad - k_\rho N_{\alpha\beta\mu}(l, l + p) - \eta_{\rho\alpha} q^\delta \tilde{N}_{\delta\beta\mu}(l - q, l + k) - \eta_{\rho\beta} q^\delta \tilde{N}_{\alpha\delta\mu}(l, l + p) \right]. \quad (4.33)
\end{align*}
\]
Finally, for $T_{\lambda \rho \mu \nu}^{(W:a)}$ and $T_{\lambda \rho \mu \nu}^{(W:b)}$ we use Eq. (4.20) once again. By redefining the integration variable in certain terms, the results can be written as

$$
 iq^{\lambda} T_{\lambda \rho \mu \nu}^{(W:a)} = \int \frac{d^4 l}{(2\pi)^4} \left[ N_{\alpha \beta \mu}(l, l + p) N_{\tau \nu \tau}(l + p, l + q) D_{W}^{\beta \sigma}(l + p) \left( l_{\rho} D_{W}^{\alpha \tau}(l) + \eta^{\alpha \delta} q^{\tau} D_{\delta \rho}^{W}(l) \right) 
 + N_{\alpha \beta \mu}(l - q, l + k) N_{\tau \nu \tau}(l + k, l) D_{W}^{\beta \sigma}(l + k) \left( - l_{\rho} D_{W}^{\alpha \tau}(l) + \eta^{\tau \delta} q^{\alpha} D_{\delta \rho}^{W}(l) \right) \right],
$$

(4.34)

$$
 iq^{\lambda} T_{\lambda \rho \mu \nu}^{(W:b)} = \int \frac{d^4 l}{(2\pi)^4} N_{\alpha \beta \mu}(l, l + p) N_{\tau \nu \tau}(l + k, l) D_{W}^{\beta \sigma}(l + k) \left[ (l_{\rho} + k_{\rho}) D_{W}^{\alpha \tau}(l + k + q) 
 - (l_{\rho} + p_{\rho}) D_{W}^{\beta \sigma}(l + p + q) + \eta^{\beta \delta} q^{\sigma} D_{\delta \rho}^{W}(l + p) + \eta^{\sigma \delta} q^{\beta} D_{\delta \rho}^{W}(l + k) \right].
$$

(4.35)

Adding Eqs. (4.30), (4.33), (4.34) and (4.35), and applying the identities of Eqs. (B.3) and (B.4), we verify that all the terms cancel, which proves Eq. (4.19).

### 4.5 Calculation of the form factors

As we have already argued, the virtue of having proved that $T_{\lambda \rho \mu \nu}^{(W)}$ satisfies the electromagnetic and gravitational transversality conditions is that we are now assured that it has the structure given in Eq. (2.31). Furthermore, there cannot be any contribution to the factors $F_1$ and $F_2$ since the Feynman rules do not involve the Levi-Civita tensor. Therefore, to calculate the contribution to the form factor $F$, we can just look at the $kkqq$ terms, as we did in the fermion case. Fortunately, as we now show, only $T_{\lambda \rho \mu \nu}^{(W:a)}$ and $T_{\lambda \rho \mu \nu}^{(W:b)}$ produce such terms, and they are in fact equal so we need to evaluate only one of them.

Since $R_{\alpha \beta \mu \delta}$ is independent of momentum, the amplitude of $T_{\lambda \rho \mu \nu}^{(W:g)}$ does not depend on $k$, and consequently does not produce a $kkqq$ term. The same argument applies to the amplitude $T_{\lambda \rho \mu \nu}^{(W:i)}$, which is independent of $k$ as well as $q$. Next, consider the amplitudes $T_{\lambda \rho \mu \nu}^{(W:f)}$ and $T_{\lambda \rho \mu \nu}^{(W:j)}$. Their sum can be written in the form

$$
 T_{\lambda \rho \mu \nu}^{(W:f)} + T_{\lambda \rho \mu \nu}^{(W:j)} = c^{Z}_{\lambda \rho \mu \nu}(p, k) D_{Z}^{\alpha \beta}(k) \Pi_{\beta \nu}^{(W)}(k),
$$

(4.36)

where $\Pi_{\beta \nu}^{(W)}$ denotes the contribution of $W$-loops to the $\gamma Z$ polarization mixing tensor, represented by the diagrams shown in Fig. 5. This tensor, as can be readily verified explicitly, is transverse and therefore it has the same form given in Eq. (3.22). We can then apply the same argument given in Section 3.5 to dismiss the $T_{\lambda \rho \mu \nu}^{(f:e)}$ and $T_{\lambda \rho \mu \nu}^{(f:i)}$ amplitudes in the fermion case,
to conclude that the combination $T^{(W:f)}_{\lambda\rho\mu
u} + T^{(W:i)}_{\lambda\rho\mu
u}$ does not give any $kkqq$ term, and a similar argument holds for $T^{(W:e)}_{\lambda\rho\mu
u} + T^{(W:i)}_{\lambda\rho\mu
u}$ as well.

Considering $T^{(W:d)}_{\lambda\rho\mu
u}$, the $q$-dependence can come only from the factor $N_{\lambda\rho\alpha\beta\mu}(l, l + k, p)$ in the integrand after we substitute $p = k + q$. Since

$$N_{\lambda\rho\alpha\beta\mu}(l, l + k, k + q) = (\eta_{\lambda\rho}|_{\mu\alpha,\beta\sigma} - \eta_{\lambda\rho}|_{\beta\mu,\alpha\sigma}) q^\sigma + \text{terms independent of } q,$$

we obtain

$$iT^{(W:d)}_{\lambda\rho\mu\nu} = \left(\eta_{\lambda\rho}|_{\mu\alpha,\beta\sigma} - \eta_{\lambda\rho}|_{\beta\mu,\alpha\sigma}\right) q^\sigma \int \frac{d^4l}{(2\pi)^4} D_W^{\alpha\tau}(l) \tilde{N}_{\sigma\tau\nu}(l + k, l) D_W^{\beta\sigma}(l + k) + \cdots,$$

where the dots denote $q$-independent terms and therefore do not contain any $kkqq$ terms. The remaining integral in this equation yields a three-index tensor that depends only on $k$. Since $k^2 = 0$, the only contribution from the integral that yields a term with four powers of momenta in $T^{(W:d)}_{\lambda\rho\mu\nu}$ contains the factor $k^\alpha k^\beta k^\nu$. Eq. (4.37) then implies that it does not contribute to the amplitude, and thus we conclude that $T^{(W:d)}_{\lambda\rho\mu\nu}$ does not yield any $kkqq$ term. The same conclusion is reached for $T^{(W:c)}_{\lambda\rho\mu\nu}$ as well, for which it is only necessary to note that $T^{(W:d)}_{\lambda\rho\mu\nu}(p, k) = T^{(W:c)}_{\lambda\rho\mu\nu}(-k, -p)$, Eq. (4.39).

Thus, only $T^{(W:a)}_{\lambda\rho\nu\mu}$ and $T^{(W:b)}_{\lambda\rho\mu\nu}$ can give the required type of terms. Furthermore, by changing the integration variable from $l$ to $-l + k$ in the expression for $T^{(W:b)}_{\lambda\rho\mu\nu}$ given in Eq. (4.11) and using the symmetry properties of the couplings involved, it follows that

$$T^{(W:a)}_{\lambda\rho\mu\nu} = T^{(W:b)}_{\lambda\rho\mu\nu}.$$

Therefore we need to evaluate only one of these two, and we choose $T^{(W:b)}_{\lambda\rho\mu\nu}$. We first write the amplitude it in the form

$$iT^{(W:b)}_{\lambda\rho\mu\nu} = \int \frac{d^4l}{(2\pi)^4} \Delta_W(l - k) \Delta_W(l) \Delta_W(l + q) S_{\lambda\rho\mu\nu},$$

where we have defined

$$\Delta_W(l) = \frac{1}{l^2 - M_W^2},$$

and

$$S_{\lambda\rho\mu\nu} = \tilde{N}_{\alpha\beta\mu}(l - k, l + q) \tilde{N}_{\sigma\tau\nu}(l, l - k) c_W^{\rho\gamma}(l + q, l)$$

$$\times \left[ -\eta^{\alpha\tau} + \frac{(l - k)^\alpha (l - k)^\tau}{M^2_W} \right] \left[ -\eta^{\sigma\gamma} + \frac{l^\sigma l^\gamma}{M^2_W} \right] \left[ -\eta^{\beta\delta} + \frac{(l + q)^\beta (l + q)^\gamma}{M^2_W} \right]$$

$$\equiv S^{(0)}_{\lambda\rho\mu\nu} + S^{(2)}_{\lambda\rho\mu\nu} + S^{(4)}_{\lambda\rho\mu\nu} + S^{(6)}_{\lambda\rho\mu\nu}. \quad (4.43)$$

In the last step, we divide the terms in $S_{\lambda\rho\mu\nu}$ into four terms, each one labeled by an index that denotes the number of inverse powers of $M_W$ that it contains.
We consider first the term which has an overall factor of $1/M_W^6$,

$$S_{\lambda\rho\mu}^{(6)} = \frac{1}{M_W^6} (l-k)^\alpha (l-k)^\gamma l^\sigma l^\beta (l+q)^\beta (l+q)^\gamma$$
$$\times \tilde{N}_{\alpha\beta\mu}(l-k,l+q)\tilde{N}_{\sigma\tau\nu}(l,l-k)c_{\lambda\rho\gamma\delta}^W(l+q,l). \quad (4.44)$$

To reduce this term and the other ones, we use the following identities

\begin{align*}
a^\alpha \tilde{N}_{\alpha\beta\gamma}(a,b) &= P_{\beta\gamma}(b-a) - P_{\beta\gamma}(b), \\
b^\beta \tilde{N}_{\alpha\beta\gamma}(a,b) &= P_{\gamma\alpha}(b-a) - P_{\gamma\alpha}(a), \\
a^\alpha b^\beta \tilde{N}_{\alpha\beta\gamma}(a,b) &= a \cdot (b-a)(b-a) - (b-a)^2 a_\gamma, \quad (4.45)
\end{align*}

where

$$P_{\alpha\beta}(l) = -l^2 \eta_{\alpha\beta} + l_\alpha l_\beta.$$ \quad (4.46)

The identities in Eq. (4.45), which are similar to the relations given in Eq. (A.5), follow directly from Eqs. (4.4) and (4.5). Applying them to the particular combination that appears in Eq. (4.44), we obtain

$$l^\sigma (l-k)^\tau \tilde{N}_{\sigma\tau\nu}(l,l-k) = l \cdot kk_{\nu} - k^2 l_\nu,$$ \quad (4.47)

which shows that $S_{\lambda\rho\mu}^{(6)}$ does not contribute to the amplitude after using the on-shell conditions for the photon given in Eqs. (2.8) and (2.2).

$S_{\lambda\rho\mu}^{(4)}$ contains three terms, with an overall factor of $1/M_Z^4$. One of them vanishes due to Eq. (4.47). For the others, we use the definition of Eq. (4.2) and the identity of Eq. (3.8) to obtain another useful identity

$$l^\sigma c_{\lambda\rho\gamma\delta}^W(l,l') = -M_Z^2 l^\sigma a'_{\lambda\rho\gamma\delta}, \quad (4.48)$$

which allows us to write

$$S_{\lambda\rho\mu}^{(4)} = \frac{1}{M_Z^2} a'_{\lambda\rho\gamma\delta} \tilde{N}_{\alpha\beta\mu}(l-k,l+q)\tilde{N}_{\sigma\tau\nu}(l,l-k)$$
$$\times \left[ \eta^\alpha (l-k)^\alpha (l-k)^\gamma + \eta^{\alpha\gamma} l^\delta (l+q)^\beta (l+q)^\gamma \right], \quad (4.49)$$

where in the last step we have used the definition of $c_{\lambda\rho\gamma\delta}^W$ from Eq. (4.2), along with the identity of Eq. (3.8). Applying the identities in Eq. (4.45), and omitting any term that does not contribute to the amplitude due to the on-shell conditions for the photon and the Z, the expression for $S_{\lambda\rho\mu}^{(4)}$ reduces to

$$S_{\lambda\rho\mu}^{(4)} = \frac{1}{M_Z^2} a'_{\lambda\rho\gamma\delta}(l+q)^\gamma \left[ M_Z^2 (l-k)_\mu P_{\nu}^\delta (l) + l^\delta \left( M_Z^2 \eta_{\alpha\mu} + P_{\alpha\mu}(l-k) \right) P_{\nu}^\alpha (l-k) \right]. \quad (4.50)$$

Using the identity

$$P_{\alpha\mu}(l-k) P_{\nu}^\alpha (l-k) = -(l-k)^2 P_{\mu\nu}(l-k), \quad (4.51)$$
and noticing that we can make the replacements
\[ a'_{\lambda\rho\gamma\delta}(l + q)^{\gamma} \rightarrow -\eta_{\lambda\delta}l_{\rho} - \eta_{\rho\delta}l_{\lambda}, \]
\[ P_{\nu}^{\delta}(l) \rightarrow i^{\delta}l_{\nu}, \tag{4.52} \]
because the neglected terms give a vanishing contribution to the amplitude when the on-shell graviton condition is imposed, we obtain
\[ S_{\lambda\rho\mu\nu}^{(4)} = -\frac{2}{M_{W}^{2}}l_{\lambda}l_{\rho}(l - k)_{\mu}l_{\nu}\left[2M_{Z}^{2} - (l - k)^{2}\right] + \cdots, \tag{4.53} \]
where the dots denote terms which cannot produce any \( kkqq \) term. This can be reduced further by writing
\[ (l - k)^{2} = \Delta_{W}^{2}(l - k) + M_{W}^{2}, \tag{4.54} \]
and remembering that \( S_{\lambda\rho\mu\nu}^{(4)} \) is to be substituted in Eq. (4.41). The \( \Delta_{W}^{2}(l - k) \) term then cancels with the \( \Delta_{W}(l - k) \) in Eq. (4.41), and the resulting integral, which depends only on \( q \), does not produce a \( kkqq \) type term. Thus the relevant part of \( S_{\lambda\rho\mu\nu}^{(4)} \) is just
\[ S_{\lambda\rho\mu\nu}^{(4)} = \left(2 - \frac{4M_{Z}^{2}}{M_{W}^{2}}\right)l_{\lambda}l_{\rho}(l - k)_{\mu}l_{\nu} + \cdots. \tag{4.55} \]
\( S_{\lambda\rho\mu\nu}^{(2)} \), which from Eq. (4.43) is given by,
\[ S_{\lambda\rho\mu\nu}^{(2)} = \frac{1}{M_{W}^{2}}\tilde{N}_{\alpha\beta\mu}(l - k, l + q)\tilde{N}_{\sigma\tau\nu}(l, l - k)\epsilon_{\lambda\rho\gamma\delta}^{W}(l + q, l)\]
\[ \times \left[\eta^{\alpha\tau}\eta^{\sigma\delta}(l + q)^{\delta}(l + q)^{\gamma} + \eta^{\alpha\tau}\eta^{\beta\gamma}l^{\sigma}l^{\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}(l - k)^{\alpha}(l - k)^{\gamma}\right], \tag{4.56} \]
can be treated in similar fashion. By using Eqs. (3.8), (4.45) and (4.52), it can be reduced to
\[ S_{\lambda\rho\mu\nu}^{(2)} = \left(P_{\mu}^{\gamma}(p) - P_{\mu}^{\gamma}(l - k)\right)\tilde{N}_{\sigma\tau\nu}(l, l - k)\left(\eta_{\lambda}^{\gamma}l_{\rho} + \eta_{\rho}^{\gamma}l_{\lambda}\right)\]
\[ + \tilde{N}_{\alpha\beta\mu}(l - k, l + q)\left(P_{\nu}^{\delta}(k) - P_{\nu}^{\delta}(l - k)\right)\left(\eta_{\lambda}^{\delta}l_{\rho} + \eta_{\rho}^{\delta}l_{\lambda}\right)\]
\[ + \frac{1}{M_{W}^{2}}\left(P_{\mu}^{\gamma}(p) - P_{\mu}^{\gamma}(l + q)\right)\left(P_{\nu}^{\delta}(k) - P_{\nu}^{\delta}(l)\right)\epsilon_{\lambda\rho\gamma\delta}^{W}(l + q, l) + \cdots, \tag{4.57} \]
which can be simplified further by neglecting the terms that do not contribute to the on-shell amplitude and focusing on the \( kkqq \) type terms. Omitting the rest of those details, we finally obtain, for the relevant part of \( S_{\lambda\rho\mu\nu}^{(2)} \),
\[ S_{\lambda\rho\mu\nu}^{(2)} = 6l_{\lambda}l_{\rho}(l - k)_{\mu}l_{\nu}. \tag{4.58} \]
Adding Eqs. (4.55) and (4.58) we then obtain
\[ S_{\lambda\rho\mu\nu}^{(2+4)} = 4\left(2 - \frac{M_{Z}^{2}}{M_{W}^{2}}\right)l_{\lambda}l_{\rho}(l - k)_{\mu}l_{\nu} + \cdots, \tag{4.59} \]
and it should be remembered that $S^{(6)}_{\lambda\rho\mu\nu}$ does not contribute to the $kkqq$ terms, so that it need not be considered. Substituting Eq. (4.59) into Eq. (4.41) and parameterizing the integral in the standard way, we obtain the result from this part,

$$T_{\lambda\rho\mu\nu}^{(W:b)} \bigg|_{(2+4)} = -8i \left( 2 - \frac{M_Z^2}{M_W^2} \right) \int \frac{d^Dl}{(2\pi)^D} \int_0^1 dx \int_0^{1-x} dy \frac{f_{\lambda\rho\mu\nu}(l + xk - yq)}{[l^2 - M_W^2 + xyM_Z^2]^3} + \cdots, \quad (4.60)$$

where now

$$f_{\lambda\rho\mu\nu}(l) = l_{\lambda\rho}(l - k)_{\mu\nu}. \quad (4.61)$$

Choosing the $kkqq$ terms as we have indicated earlier, we obtain the contribution from this part to the $kkqq$ term,

$$T_{\lambda\rho\mu\nu}^{(W:b)} \bigg|_{(2+4)} = 8i \left( 2 - \frac{M_Z^2}{M_W^2} \right) k_{\lambda\mu}k_{\rho\nu}$$

$$\times \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{x^2y(1 - x - y)}{[l^2 - M_W^2 + xyM_Z^2]^3} + \cdots. \quad (4.62)$$

where we have continued the integral to four dimensions since it is finite.

We now work the $S^{(0)}_{\lambda\rho\mu\nu}$ term in Eq. (4.43), which is given by

$$S^{(0)}_{\lambda\rho\mu\nu} = -\eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\gamma\delta}\tilde{N}_{\alpha\beta\mu}(l - k, l + q)\tilde{N}_{\sigma\tau\nu}(l, l - k)c_{\lambda\mu\nu}^W(l + q, l). \quad (4.63)$$

From the definition of the cubic couplings in Eq. (4.4), and using Eq. (3.38), the part that can possibly contain $kkqq$ terms is

$$S^{(0)}_{\lambda\rho\mu\nu} = \left[ \eta^{\alpha\nu}(l + k + 2q)_{\alpha} - 2\eta^{\alpha\nu}(k + q)_{\gamma} - 2\eta^{\alpha\gamma}(l + q)_{\mu} \right]$$

$$\times \left[ 2\eta_{\mu\nu}k_{\delta} - \eta^{\nu}_\delta(l + k)_{\alpha} + 2\eta^{\delta\alpha}l_{\nu} \right] c_{\lambda\rho\gamma\delta}(l + q, l) + \cdots,$$

where we have also made use of Eqs. (2.8) and (3.33). In addition, we have noticed that the term proportional to $M_W^2$ in $c_{\lambda\rho\gamma\delta}^W$ does not produce a term quartic in the momenta, since the cubic couplings are linear in momenta. From the definition in Eq. (3.38), and excluding all terms which either vanish on contraction with the polarization factors or do not contribute to the $kkqq$ type terms because they do not have enough factors of uncontracted momenta, we find that we can substitute

$$c_{\lambda\rho\gamma\delta}(l + q, l) = \left[ -\eta_{\gamma\delta}l_{\lambda\rho} + \eta_{\lambda\gamma}l_{\rho}(l + q)_{\delta} + \eta_{\lambda\delta}l_{\gamma}(l + q)_{\rho} \right] + (\lambda \leftrightarrow \rho). \quad (4.65)$$

The rest of the calculation is just straightforward algebra, and the result obtained is

$$S^{(0)}_{\lambda\rho\mu\nu} = 16l_{\lambda\rho}(l + q)_{\mu}l_{\nu} + 8(k_{\lambda\rho} + l_{\lambda\rho})[l_{\mu}q_{\nu} - q_{\mu}l_{\nu}] + \cdots. \quad (4.66)$$

Substituting this expression into Eq. (4.41) we then obtain

$$T_{\lambda\rho\mu\nu}^{(W:b)} \bigg|_{(0)} = 32i k_{\lambda\mu}k_{\rho\nu} q_{\nu} \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{x^2y(1 - x - y)}{[l^2 - M_W^2 + xyM_Z^2]^3} + \cdots, \quad (4.67)$$
which has to be added to the contribution obtained in Eq. (4.62) to obtain the complete expression for the $kkqq$ terms in $T^{(W:b)}_{\lambda\rho\mu\nu}$. Remembering that $T^{(W:a)}_{\lambda\rho\mu\nu}$ and $T^{(W:b)}_{\lambda\rho\mu\nu}$ give identical contributions, and recalling the overall factors defined in Eq. (4.10), we obtain the $W$ contribution to the form factor $F$,

$$F^{(W)} = \frac{\kappa e g \cos \theta_W}{4\pi^2 M_Z^2} \left( 6 - \frac{M_Z^2}{M_W^2} \right) I(M_W/M_Z),$$

(4.68)

where $I(A)$ has been defined in Eq. (3.51).

## 5 Decay rate and discussion

The on-shell amplitude is parameterized by the three form factors defined in Eq. (2.31). Our calculations in Sec. 3 and Sec. 4 show, first of all, that

$$F_1 = F_2 = 0,$$

(5.1)

which is a consequence of CP invariance. The terms containing $F_1$ and $F_2$ in Eq. (2.31) contain the Levi-Civita tensor, and are therefore odd under time reversal and under CP. In the standard electroweak model, CP violation enters any amplitude only through the charged current interactions of fermions, which do not appear in the one-loop amplitudes for the present process. Hence, at the one-loop level, the amplitude is CP conserving, and Eq. (5.1) reflects that fact.

The only non-vanishing form factor in one-loop is $F$, for which the results given in Eqs. (3.50) and (4.68) are combined to give

$$F = \frac{\kappa e g}{4\pi^2 M_Z^2 \cos \theta_W} \left[ \cos^2 \theta_W \left( 6 - \frac{1}{\cos^2 \theta_W} \right) I(M_W/M_Z) - 2 \sum_f Q_f X_f I(m_f/M_Z) \right].$$

(5.2)

where $I$ is the integral defined in Eq. (3.51). The integral cannot be performed analytically, but we can make some approximations that are sufficient for our purposes. In the denominator of the integrand the combination $x y$ has the maximum value $\frac{1}{4}$ within the range of integration. We thus calculate the integral in two extreme cases,

$$I(A) = \begin{cases} 
-\frac{1}{24} & \text{for } A \ll \frac{1}{4}, \\
\frac{1}{360 A^2} & \text{for } A \gg \frac{1}{4}.
\end{cases}$$

(5.3)

For all the fermions except the top quark, we use the first form, whereas for the top quark and the $W$ in the loop, we use the second. Thus,

$$F = \frac{\kappa e g}{4\pi^2 M_Z^2 \cos \theta_W} \times \left[ \frac{1}{360} \left( 6 - \frac{1}{\cos^2 \theta_W} \right) + \frac{5}{12} \left( 1 - \frac{2M_Z^2}{15m_t^2} \right) - \frac{10}{9} \sin^2 \theta_W \left( 1 - \frac{M_Z^2}{75m_t^2} \right) \right]$$

(5.4)

where we have used the mass relation $M_W = M_Z \cos \theta_W$. Using $e = g \sin \theta_W$ and $\sin^2 \theta_W = 0.23$, this gives

$$F = 0.4 \frac{\kappa e^2}{4\pi^2 M_Z^2}.$$

(5.5)
The decay rate is determined straightforwardly from Eq. (2.31). Using the familiar polarization sum formulas for the photon and the $Z$, as well as the corresponding one for the graviton\[12],

\[
\sum_{\text{pol}} \varepsilon^{\nu*}(k)\varepsilon^\nu(k) = -\eta^{\nu\nu'}, \tag{5.6}
\]

\[
\sum_{\text{pol}} \varepsilon_Z^\mu(p)\varepsilon_Z^{\mu'}(p) = -\eta^{\mu\mu'} + \frac{p^\mu p^{\mu'}}{M_Z^2}, \tag{5.7}
\]

\[
\sum_{\text{pol}} \varepsilon^{\lambda\rho*}(q)\varepsilon^{\lambda\rho'}(q) = \frac{1}{2} \left( \eta^{\lambda\lambda'}\eta^{\rho\rho'} + \eta^{\lambda\rho'}\eta^{\lambda\rho} - \eta^{\lambda\rho}\eta^{\lambda\rho'} \right), \tag{5.8}
\]

we find

\[
\Gamma = \frac{M_Z^7 F^2}{96\pi}, \tag{5.9}
\]

and from Eq. (5.9)

\[
\Gamma = 0.1 \frac{\alpha^2 GM_Z^3}{\pi^2}. \tag{5.10}
\]

This result confirms our expectation in Eq. (1.4) about the smallness of the rate. However, as already mentioned in the Introduction, one of the primary motivations for performing the calculation was to understand some of the intricacies involved and resolve some of the technical complications in a way that can be used in similar, perhaps more complicated, calculations. In this sense, the proof of the general form of the amplitude and its parameterization given in Eq. (2.31), together with all the identities, manipulations and tricks that we have used both in the explicit calculation of the form factors as well as in the proofs of the consistency conditions (the electromagnetic and gravitational transversality conditions) are useful in their own right, independently of the fact that we have applied them in the particular context of the $Z$ decay. It is particularly enlightening the fact that the use of the unitary gauge did not lead to any of the inconsistencies that are sometimes attributed to using that gauge. In fact, as we showed, the diagrams calculated with this gauge give an amplitude that is consistent with the transversality conditions implied by the electromagnetic and gravitational gauge invariance, which in turn allowed us to determine the amplitude and calculate the relevant form factor in a systematic and consistent fashion. By having considered this simpler system, it has allowed us to understand and develop some techniques that we believe can be useful for considering more complicated processes.

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Appendices

A Ward-Takahashi identities

The simplest Ward-Takahashi (WT) identity familiar to us through QED, which relates the fermion two-point function to the fermion-photon vertex. Diagrammatically, it can be written as

$$ -eQ \left[ \begin{array}{c} l + r \\ l \\ \end{array} \right] - \left[ \begin{array}{c} l + s \\ l \\ \end{array} \right] = r^\nu \left[ \begin{array}{c} r^\nu \\ l \\ \end{array} \right] , $$

(A.1)

where the fermion has charge $eQ$. The tree-level version of this equality was presented in Eq. (3.21). After the introduction of gravity, vertices involving gravitons appear in the theory. In a similar fashion, one can now prove the WT identity

$$ -eQ \left[ \begin{array}{c} l + t \\ l + r \\ \end{array} \right] - \left[ \begin{array}{c} l + s \\ l \\ \end{array} \right] = r^\nu \left[ \begin{array}{c} r^\nu \\ l \\ \end{array} \right] . $$

(A.2)

In this and other relations in this appendix, we use the shorthand

$$ t = r + s , $$

(A.3)

where $r$ and $s$ are the momenta of the photon and the graviton lines respectively, both considered outgoing. Recalling the definition of the vertices given in Fig. 2, we see that at tree level, the diagrammatic identity of Eq. (A.2) implies the relation

$$ V^\lambda_\rho(l + r + s, l + r) - V^\lambda_\rho(l + s, l) = -a^\lambda_\rho_\alpha_\beta^{\gamma} r^{\alpha_\beta} , $$

(A.4)

which can be easily checked from the expressions for these vertices appearing in Eqs. (3.1) and (3.12).

The general nature of these WT identities are now clear. For the charged $W^+$ bosons, we can write similar identities. Of course, these results depend on the gauge condition, and involves the unphysical scalar fields as well as ghost fields. In the unitary gauge, however, the unphysical scalar fields and the ghost fields are not present, and the relation look particularly simple. Here, we summarize some relations of this sort. In the diagrams which appear within the equations below, the horizontal lines denote $W^+$, where the left line has incoming momentum and the right line has outgoing. The $Z$-line appears in a saw-tooth pattern, with an inward momentum $p$. The momentum convention for the photon and the graviton lines have already been stated.

First, we can have the identity from a diagram similar to that in Eq. (A.1), with the fermion lines replaced by the $W^+$-lines. At the tree level, this will imply

$$ D^{-1}_{\alpha_\beta}(l + r) - D^{-1}_{\alpha_\beta}(l) = r^\gamma \tilde{N}_{\alpha_\beta\gamma}(l + r, l) . $$

(A.5)

This can also be written as

$$ r^\gamma D^{\alpha_\beta}(l + r)\tilde{N}_{\alpha_\beta\gamma}(l + r, l)D^{\beta_\gamma}(l) = D^{\gamma\tau}(l) - D^{\gamma\tau}(l + r) . $$

(A.6)
Similarly, replacing the fermion lines by $W^+$-lines in Eq. (A.2), we obtain another WT identity, which in the tree-level reads

$$c^W_{\lambda\rho\alpha\beta}(l + r + s, l + r) - c^W_{\lambda\rho\alpha\beta}(l + s, l) = - r^\gamma N_{\lambda\rho\alpha\beta\gamma}(l + r + s, l, -r).$$  

(A.7)

We can have an extra $Z$-boson present in all diagrams involved in the identity of Eq. (A.5), which will give us the relation

$$- e \begin{pmatrix} \left( \begin{array}{c} l \\ l + t \end{array} \right) - \left( \begin{array}{c} l - r \\ l + s \end{array} \right) \right) = r^\delta \begin{pmatrix} \left( \begin{array}{c} l \\ l + r \end{array} \right) \end{pmatrix}. \quad (A.8)$$

At the tree level, this implies

$$\tilde{N}_{\alpha\beta\gamma}(l, l + r + s) - \tilde{N}_{\alpha\beta\gamma}(l - r, l + s) = - r^\delta R_{\alpha\beta\gamma\delta}. \quad (A.9)$$

We can also write down a similar WT identity involving the graviton, viz.,

$$- e \begin{pmatrix} \left( \begin{array}{c} l \\ l + r \end{array} \right) - \left( \begin{array}{c} l - r \\ l \end{array} \right) \right) = k^\nu \begin{pmatrix} \left( \begin{array}{c} l \\ \nu \end{array} \right) \end{pmatrix}. \quad (A.10)$$

The tree-level version of this identity reads

$$N_{\lambda\rho\alpha\beta\mu}(l, l + r, r + s) - N_{\lambda\rho\alpha\beta\mu}(l - r, l, r + s) = - r^\nu R_{\lambda\rho\alpha\beta\mu\nu}. \quad (A.11)$$

It is also instructive to see how some of these contractions behave under the integration over the loop momentum. For example, using Eq. (A.6), we can write

$$r^\gamma \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\sigma\tau\delta}(l, l + r) D_{W}^{\sigma\tau}(l + r) \tilde{N}_{\alpha\beta\gamma}(l + r, l) D_{W}^{\beta\tau}(l)$$

$$= \int \frac{d^4l}{(2\pi)^4} \tilde{N}_{\sigma\tau\delta}(l, l + r) \left[ D_{W}^{\sigma\tau}(l) - D_{W}^{\sigma\tau}(l + r) \right]$$

$$= \int \frac{d^4l}{(2\pi)^4} \left[ \tilde{N}_{\sigma\tau\delta}(l, l + r) - \tilde{N}_{\sigma\tau\delta}(l - r, l) \right] D_{W}^{\sigma\tau}(l)$$

$$= - r^\alpha R_{\sigma\tau\delta\alpha} \int \frac{d^4l}{(2\pi)^4} D_{W}^{\sigma\tau}(l), \quad (A.12)$$

where we have used Eq. (A.9) in arriving at the last step. Through an exactly similar kind of argument, we can prove the relation

$$r^\gamma \int \frac{d^4l}{(2\pi)^4} N_{\lambda\rho\sigma\tau\delta}(l, l + r, r + s) D_{W}^{\sigma\tau}(l + r) N_{\alpha\beta\gamma}(l + r, l, -r) D_{W}^{\beta\tau}(l)$$

$$= - r^\alpha R_{\lambda\rho\sigma\tau\delta\alpha} \int \frac{d^4l}{(2\pi)^4} D_{W}^{\sigma\tau}(l). \quad (A.13)$$
Some other relations between various couplings

In Appendix A, we considered the contraction of various vertices with the photon momentum. In this Appendix, we are considering contractions of a more general kind, in particular relations involving the graviton momentum.

From the definitions of these vertices in Eqs. (4.4) and (4.7), it is easy to see that one can write a relation between the cubic and the quartic gauge couplings:

\[ \tilde{N}_{\alpha\beta\gamma}(a, b) = a^\delta R_{\gamma\alpha\beta\delta} + b^\delta R_{\beta\gamma\alpha\delta}. \]  

(B.1)

It is easy to see that Eq. (A.9) can be derived from this relation by making use of the identity

\[ R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0, \]  

(B.2)

which follows trivially from the expression for the quartic gauge coupling in Eq. (4.7). The following identities, used at different stages of the calculation, can also be derived from Eq. (B.1):

\[ \tilde{N}_{\alpha\beta\gamma}(a, b) - \tilde{N}_{\alpha\beta\gamma}(a, b-r) = r^\delta R_{\beta\gamma\alpha\delta}, \]  

(B.3)

\[ \tilde{N}_{\alpha\beta\gamma}(a, b) - \tilde{N}_{\alpha\beta\gamma}(a-r, b) = r^\delta R_{\gamma\alpha\beta\delta}. \]  

(B.4)

Another important relation involves the contraction of the coupling \( N_{\lambda\rho\alpha\beta\gamma} \) with the graviton momentum. From the definition in Eq. (4.8), it easily follows that

\[ q^\lambda N_{\lambda\rho\alpha\beta\gamma}(a, a + b + q) = (q_\rho - a_\rho + b_\rho) \tilde{N}_{\alpha\beta\gamma}(a, a + b) \]
\[ + a_\rho \tilde{N}_{\alpha\beta\gamma}(a - q, a + b - q) - b_\rho \tilde{N}_{\alpha\beta\gamma}(a, a + b + q) \]
\[ - q^\delta (\eta_{\rho\alpha} \tilde{N}_{\delta\beta\gamma}(a - q, a + b) + \eta_{\rho\beta} \tilde{N}_{\alpha\delta\gamma}(a, a + b + q)) \]
\[ + \eta_{\rho\gamma} \tilde{N}_{\alpha\beta\delta}(a, a + b). \]  

(B.5)

Sometimes alternative forms of this identity is more useful, such as

\[ q^\lambda N_{\lambda\rho\alpha\beta\gamma}(a, a + b + q) = q_\rho \tilde{N}_{\alpha\beta\gamma}(a, a + b) + q^\delta (a_\rho R_{\alpha\beta\gamma\delta} - b_\rho R_{\beta\gamma\alpha\delta}) \]
\[ - q^\delta (\eta_{\rho\alpha} \tilde{N}_{\delta\beta\gamma}(a - q, a + b) + \eta_{\rho\beta} \tilde{N}_{\alpha\delta\gamma}(a, a + b + q)) \]
\[ + \eta_{\rho\gamma} \tilde{N}_{\alpha\beta\delta}(a, a + b), \]  

(B.6)

or

\[ q^\lambda N_{\lambda\rho\alpha\beta\gamma}(a, a + b + q) = (b_\rho + q_\rho) \tilde{N}_{\alpha\beta\gamma}(a, a + b) - b_\rho \tilde{N}_{\alpha\beta\gamma}(a, a + b + q) + q^\delta a_\rho R_{\alpha\beta\gamma\delta} \]
\[ - q^\delta (\eta_{\rho\alpha} \tilde{N}_{\delta\beta\gamma}(a - q, a + b) + \eta_{\rho\beta} \tilde{N}_{\alpha\delta\gamma}(a, a + b + q)) \]
\[ + \eta_{\rho\gamma} \tilde{N}_{\alpha\beta\delta}(a, a + b), \]  

(B.7)

which can be derived from Eq. (B.5) by using the identities of Eqs. (B.3) and (B.4).
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