NON-ARCHIMEDEAN WHITE NOISE, MASSIVE EUCLIDEAN FIELDS, AND SCHWINGER FUNCTIONS.

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Abstract. We construct $p$-adic Euclidean random fields $\Phi$ over $\mathbb{Q}_p^N$, for arbitrary $N$, these fields are solutions of $p$-adic stochastic pseudodifferential equations of Klein-Gordon type. From a mathematical perspective, the Euclidean fields are generalized stochastic processes parametrized by functions belonging to a nuclear countably Hilbert space, these spaces are introduced in this article, in addition, the Euclidean fields are invariant under the action of certain group of transformations. We also study the Schwinger functions of $\Phi$.

1. Introduction

There are general arguments that suggest that one cannot make measurements in regions of extent smaller than the Planck length $\approx 10^{-33}$ cm, see e.g. [44] and the references therein]. The construction of physical models at the level of Planck scale is a relevant scientific problem and a very important source for mathematical research. In [52]-[53], I. Volovich posed the conjecture of the non-Archimedean nature of the spacetime at the level of the Planck scale. This conjecture has originated a lot research, for instance, in quantum mechanics, see e.g. [48], [50], [57], [58], in string theory, see e.g. [10], [16], [22], [36], [49], [46], [47], and in quantum field theory, see e.g. [20], [35], [38]. For a further discussion on non-Archimedean mathematical physics, the reader may consult [14], [45], [51] and the references therein.

On the other hand, the interaction between quantum field theory and mathematics is very fruitful and deep, see e.g. [17], [18], [25], [55]-[56], among several articles. Let us mention explicitly the connection with arithmetic, see e.g. [25], [33], [38], from this perspective the investigation of quantum fields in the non-Archimedean setting is a quite natural problem.

In this article we introduce a class of non-Archimedean Euclidean fields, in arbitrary dimension, which are constructed as solutions of certain covariant $p$-adic stochastic pseudodifferential equations, by using techniques of white noise calculus. The connection between quantum fields and SPDEs has been studied intensively in the Archimedean setting, see e.g. [3]-[7] and the references therein. A massive non-Archimedean field $\Phi$ is a random field parametrized by a nuclear countably Hilbert space $H_\alpha (\mathbb{Q}_p^N; \alpha, \infty)$, which depend on $(q, l, m, \alpha)$, where $q$ is an elliptic quadratic form, $l$ is an elliptic polynomial, and $m, \alpha$ are positive numbers, here $m$ is the mass of $\Phi$. Heuristically, $\Phi$ is the solution of $(L_\alpha + m^2) \Phi = f$, where $f$ is a generalized Lévy noise, this type of noise is introduced in this article, and

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The operator $L_\alpha + m^2$ is a non-Archimedean analog of a fractional Klein-Gordon operator. At this point, it is useful to compare our construction with the classical one. In the Archimedean case, see e.g. [6]-[7], the elliptic quadratic form $b(\xi) = \xi_1^2 + \cdots + \xi_N^2 \in \mathbb{R}[\xi_1, \cdots, \xi_N]$ is used to define the pseudodifferential fractional Klein-Gordon operator $(-\Delta + m^2)^\alpha$, $m, \alpha > 0$, and also $b(\xi)$ is the quadratic form associated with the bilinear form $\xi_1 \eta_1 + \cdots + \xi_N \eta_N$, which is used in the definition of the Fourier transform on $\mathbb{R}^N$. This approach cannot be carried out in the $p$-adic setting because when $b(\xi)$ is considered as a $p$-adic quadratic form, it is not elliptic for $N \geq 5$, and the ellipticity of $b(\xi)$ is essential to establish the non-negativity of the Green functions in the $p$-adic setting. For this reason, we have to replace $\xi_1^2 + \cdots + \xi_N^2$ by an elliptic polynomial, which is a homogeneous polynomial that vanishes only at the origin, there are infinitely many of them. The symmetries of our Green functions and fields depend on the transformations that preserve $q$ and $l$, thus, in order to have large groups of symmetries we cannot fix $q$, instead, we work with pairs $(q, l)$ that have a large group of symmetries. These are two important difference between the $p$-adic case and the Archimedean one. On the other hand, the spaces $\mathcal{H}_\mathbb{R}(\mathbb{Q}_p^N; \alpha, \infty)$ introduced here are completely necessary to carry out a construction similar to the one presented in [7], [6]. For instance, the Green function attached to $L_\alpha + m^2$ gives rise to a continuous mapping from $\mathcal{H}_\mathbb{R}(\mathbb{Q}_p^N; \alpha, \infty)$ into itself. This fact is not true, if we replace $\mathcal{H}_\mathbb{R}(\mathbb{Q}_p^N; \alpha, \infty)$ by the space of test functions $\mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N)$, which is also nuclear. We want to mention here that [7], [6] have influenced strongly our work, and that our results are non-Archimedean counterparts of some of the results of [6].

The line of research pursued in this article continues, from one side, the investigations on generalized stochastic processes invariant under groups of transformations, see e.g. [32], [34], [28] and, on the other side, it continues the development of the white noise calculus, see e.g. [23], [37], in the non-Archimedean setting. Several types of $p$-adic noise calculi have been studied lately, see e.g. [12], [13], [15], [24], [28], [29], [30], [31], [60]. To the best of our knowledge the non-Archimedean generalized Lévy noises studied here have not been reported before in the literature.

Finally, we want to compare our results with those of [26] and [1]. In [26] the authors constructed non-Gaussian measures on the space $\mathcal{D}(\mathbb{Q}_p^n)$ of distributions on $\mathbb{Q}_p^n$ for $1 \leq n \leq 4$. The measures are absolutely continuous with respect to the free field Gaussian measure and live on the spaces of distributions with a fixed compact support (finite volume case). As we mention before our results cannot be formulated in the space of test functions $\mathcal{D}(\mathbb{Q}_p^n)$, because it is nonmetrizable topological space, and thus, the theory of countably Hilbert nuclear spaces does not apply to it. The author thinks that the results of [26] can be extended by using our spaces $\mathcal{H}_\mathbb{R}(\mathbb{Q}_p^n; \alpha, \infty)$, but more importantly, the author thinks that the problem of infinite volume limit in [26] can be settled by using the results presented here in combination with those of [7]. In [1], three dimensional $p$-adic massless quantum fields were constructed and investigated in connection with the renormalization group. This article motivates the study of spaces of type $\mathcal{H}_\mathbb{R}(\mathbb{Q}_p^N; \alpha, \infty)$ in the contest of massless quantum fields. The author expect to consider these matters and some related in forthcoming articles.
The article is organized as follows. In Section 2 we review the basic aspects of the $p$-adic analysis. In Section 3 we construct the spaces $H_b \left( \mathbb{Q}_p^N; \alpha, \infty \right)$, $H_c \left( \mathbb{Q}_p^N; \alpha, \infty \right)$, which are nuclear countably Hilbert spaces, see Theorem 3.6 and Remark 3.7. We also show that $H_c \left( \mathbb{Q}_p^N; \alpha, \infty \right)$ is invariant under the action of a large class of pseudodifferential operators, see Theorem 3.14. In Section 4 we introduce non-Archimedean analogs of the Klein-Gordon fractional operators and study the properties of the corresponding Green functions, see Proposition 4.1. We also study the solutions of the $p$-adic Klein-Gordon equations in $H_b \left( \mathbb{Q}_p^N; \alpha, \infty \right)$, see Theorem 4.2. In Section 5 we introduce a new class of non-Archimedean Lévy noises, see Theorem 5.2 and Definition 5.3. Section 6 is dedicated to the non-Archimedean quantum fields and its symmetries, see Proposition 6.2, Definitions 6.3-6.4 and Proposition 6.7. Finally, Section 7 is dedicated to the study of the Schwinger functions, see Definition 7.6 and Theorem 7.7. Finally, as an application, we construct a $p$-adic Brownian sheet on $\mathbb{Q}_p^N$, see Theorem 7.8.

2. $p$-adic Analysis: Essential Ideas

In this section we fix the notation and collect some basic results on $p$-adic analysis that we will use throughout the article. For a detailed exposition the reader may consult [2], [42], [51].

2.1. The Field of $p$-adic Numbers. Along this article $p$ will denote a prime number. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm $\left| \cdot \right|_p$, which is defined as

$$\left| x \right|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{-\gamma} \frac{a}{b} \end{cases}$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$-adic order of $x$. We extend the $p$-adic norm to $\mathbb{Q}_p^N$ by taking

$$\left| x \right|_p := \max_{1 \leq i \leq N} \left| x_i \right|_p,$$

for $x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$.

We define $\text{ord}(x) = \min_{1 \leq i \leq N} \{ \text{ord}(x_i) \}$, then $\left| x \right|_p = p^{-\text{ord}(x)}$. The metric space $(\mathbb{Q}_p^N, \left| \cdot \right|_p)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [2], [51].

Any $p$-adic number $x \neq 0$ has a unique expansion $x = p^{\text{ord}(x)} \sum_{j=0}^{+\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the fractional part of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} - \sum_{j=0}^{\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0. \end{cases}$$

For $l \in \mathbb{Z}$, denote by $B_l^N(a) = \{ x \in \mathbb{Q}_p^N : \left| x - a \right|_p \leq p^l \}$ the ball of radius $p^l$ with center at $a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$, and take $B_0^N(0) := B_1^N$. Note that $B_l^N(a) = B_l(a_1) \times \cdots \times B_l(a_n)$, where $B_l(a_i) := \{ x \in \mathbb{Q}_p : \left| x - a_i \right|_p \leq p^l \}$ is the one-dimensional ball of radius $p^l$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^N$ equals the product of $N$ copies of $B_0 := \mathbb{Z}_p$, the ring of $p$-adic integers. We denote by $\Omega(\left| x \right|_p)$ the characteristic function of $B_0^N$. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of $A$. For $l \in \mathbb{Z}$, denote
by $S^N(a) = \{ x \in Q^N_p : ||x-a|| = p^l \}$ the sphere of radius $p^l$ with center at $a = (a_1, \ldots, a_N) \in Q^N_p$, and take $S^N(0) := S^N$.

2.2. The Bruhat-Schwartz space. A complex-valued function $g$ defined on $Q^N_p$ is called locally constant if for any $x \in Q^N_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$g(x + x') = g(x) \text{ for } x' \in B^N_l(x).$$

A function $g : Q^N_p \to \mathbb{C}$ is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}_c(Q^N_p)$, while $\mathcal{D}(Q^N_p)$ denotes the $\mathbb{R}$-vector space of Bruhat-Schwartz functions.

Let $\mathcal{D}_c(Q^N_p)$ (resp. $\mathcal{D}(Q^N_p)$) denote the set of all continuous functionals (distributions) on $\mathcal{D}_c(Q^N_p)$ (resp. $\mathcal{D}(Q^N_p)$).

2.3. Fourier transform. Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in Q_p$. The map $\chi_p(\cdot)$ is an additive character on $Q_p$, i.e. a continuous map from $Q_p$ into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1)$, $y_0, y_1 \in Q_p$.

Let $\mathcal{B}(x, y)$ be a symmetric non-degenerate $Q_p$-bilinear form on $Q^N_p \times Q^N_p$. Thus $q(x) := \mathcal{B}(x, x)$, $x \in Q^N_p$ is a non-degenerate quadratic form on $Q^N_p$. We recall that

$$\mathcal{B}(x, y) = \frac{1}{2} \{ q(x + y) - q(x) - q(y) \}.$$

We identify the $Q_p$-vector space $Q^N_p$ with its algebraic dual $(Q^N_p)^*$ by means of $\mathcal{B}(\cdot, \cdot)$. We now identify the dual group (i.e. the Pontryagin dual) of $(Q^N_p, +)$ with $(Q^N_p)^*$ by taking $x^* (x) = \chi_p(\mathcal{B}(x, x^*))$. The Fourier transform is defined by

$$(\mathcal{F}g)(\xi) = \int_{Q^N_p} g(x) \chi_p(\mathcal{B}(x, \xi)) d\mu(x), \text{ for } g \in L^1,$$

where $d\mu(x)$ is a Haar measure on $Q^N_p$. Let $\mathcal{L}(Q^N_p)$ be the space of continuous functions $g$ in $L^1$ whose Fourier transform $\mathcal{F}g$ is in $L^1$. The measure $d\mu(x)$ can be normalized uniquely in such manner that $(\mathcal{F}(f)g)(x) = g(-x)$ for every $g$ belonging to $\mathcal{L}(Q^N_p)$. We say that $d\mu(x)$ is a self-dual measure relative to $\chi_p(\cdot, \cdot)$. Notice that $d\mu(x) = C(q)d^N x$ where $C(q)$ is a positive constant and $d^N x$ is the Haar measure on $Q^N_p$ normalized by the condition $\text{vol}(B^N_0) = 1$. For further details about the material presented in this section the reader may consult [51].

We will also use the notation $\mathcal{F}_x \leftrightarrow g$ and $\hat{g}$ for the Fourier transform of $g$. The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}_c^*(Q^N_p)$ is defined by

$$\langle g, \mathcal{F}[T] \rangle = \langle f, \mathcal{F}[g], T \rangle \text{ for all } g \in \mathcal{D}(Q^N_p).$$

The Fourier transform $f \to \mathcal{F}[T]$ is a linear isomorphism from $\mathcal{D}_c^*(Q^N_p)$ onto itself. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

Remark 2.1. Given $r \in [0, +\infty)$, we denote by $L^r(Q^N_p, d^N x) := L^r$, the $\mathbb{C}$-vector space of all the complex valued functions $g$ satisfying $\int_{Q^N_p} |g(x)|^r d^N x < \infty$. Let denote by $C^{\text{cont}} := C^{\text{cont}}(Q^N_p, \mathbb{C})$, the he $\mathbb{C}$-vector space of all the complex valued functions which are uniformly continuous. We denote by $L^r, C^{\text{cont}}$ the corresponding $\mathbb{R}$-vector spaces.
3. A NEW CLASS OF NON-ARCHIMEDEAN NUCLEAR SPACES

The Bruhat-Schwartz space $\mathcal{D}_c(\mathbb{Q}^N_p)$ is not invariant under the action of the pseudodifferential operators required in this work. In this section we introduce a new type of nuclear countably Hilbert spaces, which are invariant under the action of large class of pseudodifferential operators.

Remark 3.1. We set $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. We denote by $\mathbb{N}$ the set of non-negative integers.

3.1. A class of non-Archimedean nuclear spaces. We define for $f, g$ in $\mathcal{D}_c(\mathbb{Q}^N_p)$ (or in $\mathcal{D}_c(\mathbb{Q}^N_p)$) the following scalar product:

\[
(f, g)_{l, \alpha} := (f, g)_{l} = \int \left( \max \left(1, \|\xi\|_p \right) \right)^{2\alpha l} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\mathbb{N},
\]

for a fixed $\alpha \in \mathbb{R}_+ \setminus \{0\}$ and $l \in \mathbb{Z}$. We also set $\|f\|_{l, \alpha}^2 := \|f\|_{l}^2 = (f, f)_{l}$. Notice that $\|\cdot\|_m \leq \|\cdot\|_n$ for $m \leq n$. Let denote by $\mathcal{H}_R (\mathbb{Q}^N_p; l, \alpha) =: \mathcal{H}_R (l)$ the completion of $\mathcal{D}_c(\mathbb{Q}^N_p)$ with respect to $(\cdot, \cdot)_l$. Then $\mathcal{H}_R (n) \subset \mathcal{H}_R (m)$ for $m \leq n$. We set

\[
\mathcal{H}_R (\mathbb{Q}^N_p; \alpha, +\infty) := \mathcal{H}_R (\mathbb{Q}^N_p; \infty) := \mathcal{H}_R (+\infty) = \bigcap_{l \in \mathbb{N}} \mathcal{H}_R (l).
\]

Notice that $\mathcal{H}_R (0) = L^2_\mathbb{R}$ and that $\mathcal{H}_R (\infty) \subset L^2_\mathbb{R}$. We consider on $\mathcal{H}_R (\infty)$ the family of seminorms $\|\cdot\|_{l, \infty}$, then $(\mathcal{H}_R (\infty), \|\cdot\|_{l, \infty})$ becomes a locally convex space, which is metrizable. Indeed,

\[
d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}
\]

is a metric for the topology of $\mathcal{H}_R (\infty)$ considered as a convex topological space, see e.g. [20]. A sequence $\{f_l\}_{l \in \mathbb{N}}$ in $(\mathcal{H}_R (\infty), d)$ converges to $g \in \mathcal{H}_R (\infty)$, if and only if, $\{f_l\}_{l \in \mathbb{N}}$ converges to $g$ in the norm $\|\cdot\|_l$ for all $l \in \mathbb{N}$. From this observation follows that the topology on $\mathcal{H}_R (\infty)$ coincides with the projective limit topology $\tau_p$. An open neighborhood base at zero of $\tau_p$ is given by the choice of $\epsilon > 0$ and $l \in \mathbb{N}$, and the set

\[
U_{\epsilon, l} := \{f \in \mathcal{H}_R (\infty) : \|f\|_l < \epsilon\}.
\]

Remark 3.2. We denote by $\mathcal{H}_C (l)$, $\mathcal{H}_C (\infty)$ the $\mathbb{C}$-vector spaces constructed from $\mathcal{D}_c(\mathbb{Q}^N_p)$. All the above results are valid for these spaces. We shall use $d$ to denote the metric of $\mathcal{H}_C (\infty)$.

Lemma 3.3. $\mathcal{H}_R (\infty)$ endowed with the topology $\tau_p$ is a countably Hilbert space in the sense of Gel'fand and Vilenkin, see e.g. [20] Chapter I, Section 3.1 or [37] Section 1.2]. Furthermore $(\mathcal{H}_R (\infty), \tau_p)$ is metrizable and complete and hence a Fréchet space.

Proof. By the previous considerations, it is sufficient to show that $(\cdot, \cdot)_{l, \infty}$ is a system of compatible scalar products, i.e. if a sequence $\{f_l\}_{l \in \mathbb{N}}$ of elements of $\mathcal{H}_R (\infty)$ converges to zero in the norm $\|\cdot\|_m$ and is a Cauchy sequence in the norm $\|\cdot\|_n$, then it also converges to zero in the norm $\|\cdot\|_n$. We may assume without loss of generality that $m \leq n$ thus $\|\cdot\|_m \leq \|\cdot\|_n$. By using $f_l \rightharpoonup 0 \in \mathcal{H}_R (m)$ and $f_l \rightharpoonup f \in \mathcal{H}_R (n) \subset \mathcal{H}_R (m)$, we conclude that $f = 0$. \qed
Lemma 3.4. (i) Set \((D_R(Q_p^N), d)\) for the completion of the metric space \((D_R(Q_p^N), d)\).
Then \((D_R(Q_p^N), d) = (H_R(\infty), d)\).
(ii) \((H_R(\infty), d)\) is a nuclear space.

Proof. Set \(f \in (D_R(Q_p^N), d)\), then there exists a sequence \(\{f_n\}_{n \in \mathbb{N}}\) in \((D_R(Q_p^N), d)\) such that \(f_n \to f\) for each \(l \in \mathbb{N}\), i.e. \(f \in \cap_{l \in \mathbb{N}} H_l\). Hence \((D_R(Q_p^N), d) \subset (H_R(\infty), d)\). Conversely, set \(g \in H_R(\infty)\). By using the density of \(D_R(Q_p^N)\) in \(H_R(l)\), and the fact that \(\|\cdot\|_m \leq \|\cdot\|_n\) if \(m \leq n\), we construct a sequence \(\{g_n\}_{n \in \mathbb{N}}\) in \(D_R(Q_p^N)\) satisfying
\[
\|g_n - g\|_l \leq \left\{ \frac{1}{n+1} \right\} \quad \text{for } 0 \leq l \leq n.
\]
Then \(d(g_n, g) \leq \max \left\{ \frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{2^{-n}}{n+1}, 2^{-(n+1)} \right\} \to 0\) as \(n \to \infty\). This fact shows that \(g \in (D_R(Q_p^N), d)\).

(ii) We recall that \(D_C(Q_p^N)\) is a nuclear space, cf. [11] Section 4, and thus \(D_R(Q_p^N)\) is a nuclear space, since any subspace of a nuclear space is also nuclear, see e.g. [43] Proposition 50.1. Now, since the completion of a nuclear space is also nuclear, see e.g. [43] Proposition 50.1, by (i), \(H_R(\infty)\) is a nuclear space.

Remark 3.5. (i) Lemma 3.4 is valid if we replace \(D_R(Q_p^N)\) by \(D_C(Q_p^N)\) and \(H_R(\infty)\) by \(H_C(\infty)\).
(ii) By using the Cauchy-Schwartz inequality, if \(l \geq N\) and \(g \in H_R(l)\), then \(\hat{g} \in L^1\), and thus \(g \in C_\infty^{\text{rad}}\), see also [39] Lemma 16. Therefore \(H_R(\infty) \subset L_2^\infty \cap C_\infty^{\text{rad}}\).
(iii) In the definition of the product \(\langle \cdot, \cdot \rangle_1\), the weight \(\left[ \max \left(1, \|\xi\|_p\right) \right]^{2al}\) can be replaced by \(\left[1 + \|\xi\|_p\right]^{2al}\) or by \(\left[1 + |f(\xi)|_p\right]^{2al}\), where \(f\) is an elliptic polynomial, see Section 4 or [59], and the corresponding norm is equivalent to \(\|\cdot\|_1\).

From Lemmas 3.3-3.4 we obtain the main result of this section.

Theorem 3.6. \(H_R(\infty)\) is nuclear countably Hilbert space.

Remark 3.7. (i) As a nuclear Fréchet space \(H_R(\infty)\) admits a sequence of defining Hilbertian norms \(\|\cdot\|_{m \in \mathbb{N}}\) such that (1) \(\|g\|_m \leq C_m \|g\|_{m+1},\ g \in H_R(\infty)\), with some \(C_m > 0\); (2) the canonical map \(i_{n,n+1} : H_R(n+1) \to H_R(n)\) is of Hilbert-Schmidt type, where \(H_R(n)\) is the Hilbert space associated with \(\|\cdot\|_{n}\), cf. [37] Proposition 1.3.2.
(ii) Let \(H_R^*(l)\) be the dual space of \(H_R(l)\). By identifying \(H_R^*(l)\) with \(H_R(-l)\) and denoting the dual pairing between \(H_R^*(\infty)\) and \(H_R(\infty)\) by \(\langle \cdot, \cdot \rangle\), we have from the results of Gel’fand and Vilenkin that \(H_R^*(\infty) = \bigcup_{l \in \mathbb{N}} H_R^*(l)\). We shall consider \(H_R^*(\infty)\) as equipped with the weak topology.
(iii) Theorem 3.5 is valid for \(H_C(\infty)\).
(iv) There are some well-known results for constructing nuclear countably Hilbert spaces starting with suitable operators, see e.g. [37] Sections 1.2-1.3. In order to use the mentioned results, and taking into account the definition of our norms \(\|\cdot\|_l\), see [37], we have to work with operators of type
\[
A_{\gamma}(\cdot) = \mathcal{F}^{-1} \left( \max \left(1, \|\xi\|_p\right) \right)^{\gamma} \mathcal{F}(\cdot) \quad \text{with } \gamma \in \mathbb{R}.
\]
It is essential to show that \( A_{-\gamma} \) is a Hilbert-Schmidt operator for some \( \gamma > 0 \), see [37] Proposition 1.3.4. By using the orthonormal basis for \( L^2(\mathbb{Q}^N_p) \) constructed by Albeverio and Kozyrev, see [5] Theorem 1, one verifies that the elements of this basis are eigenfunctions of \( A_{-\gamma} \) ‘with infinite multiplicity’, for any \( \gamma > 0 \). This fact was also noted, in dimension one, in [10] Theorem 2. Then \( A_{-\gamma} \) is not a Hilbert-Schmidt operator for any \( \gamma > 0 \). The fact that \( \mathcal{S}^0(\mathbb{R}^N) \) is a nuclear space is established by using the Hamiltonian of the harmonic oscillator and [37] Proposition 1.3.4, see for instance [23] Appendix 5.

The following result will be used later on.

**Lemma 3.8.** For any \( l \in \mathbb{N} \),

\[
\mathcal{H}_C(l) = \left\{ g \in L^2 : \left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} \hat{g} \in L^2 \right\} = \left\{ L^2 \left( \mathbb{Q}_p^N, \left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} d^N \xi \right) \right\}.
\]

A similar result is valid for \( \mathcal{H}_R(l) \).

**Proof.** Take \( f \in \mathcal{H}_C(l) \), then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathcal{D}_C(\mathbb{Q}_p^N) \) such that \( f_n \to f \) in \( \mathcal{H}_C(l) \), i.e. \( \left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} \hat{f}_n \to \hat{f} \) in \( L^2 \). The proof is complete, \( \hat{f} \in L^2 \), i.e. \( f \in L^2 \left( \mathbb{Q}_p^N, \left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} d^N \xi \right) \).

Conversely, take \( f \in L^2 \) such that \( \left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} \hat{f} \in L^2 \). By using the fact that \( \mathcal{D}_C(\mathbb{Q}_p^N) \) is dense in \( L^2 \), there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathcal{D}_C(\mathbb{Q}_p^N) \) such that \( f_n \to f \) in \( L^2 \). Then \( g_n = \frac{f_n}{\left[ \max \left( 1, \| \xi \|_p \right) \right] ^{2l} \hat{f}} \) is the desired sequence.

3.1.1. Some remarks on tensor products. Let \( \mathfrak{X} \) and \( \mathfrak{M} \) be locally convex spaces. We denote by \( \mathfrak{X} \otimes \mathfrak{M} \) their algebraic tensor product. The \( \pi \)-topology on \( \mathfrak{X} \otimes \mathfrak{M} \) is the strongest locally convex topology such that the canonical map \( \mathfrak{X} \times \mathfrak{M} \to \mathfrak{X} \otimes \mathfrak{M} \) is continuous. The completion of \( \mathfrak{X} \otimes \mathfrak{M} \) with respect to the \( \pi \)-topology is called the \( \pi \)-tensor product and is denoted by \( \mathfrak{X} \otimes \pi \mathfrak{M} \). If \( \mathfrak{X} \) and \( \mathfrak{M} \) are nuclear spaces, so \( \mathfrak{X} \otimes \pi \mathfrak{M} \) is nuclear, see [37] Proposition 1.3.7.

Let \( \mathcal{L} \) and \( \mathcal{K} \) be Hilbert spaces, we denote by \( \mathcal{L} \otimes \mathcal{K} \) the Hilbert space tensor product of \( \mathcal{L} \) and \( \mathcal{K} \). Now, let \( \mathfrak{X} \) and \( \mathfrak{M} \) be locally convex spaces with defining Hilbertian seminorms \( \{\| \cdot \|_A\} \) and \( \{\| \cdot \|_B\} \) respectively. Let \( \mathfrak{X}_A \) and \( \mathfrak{M}_B \) be the Hilbert spaces associated with \( \| \cdot \|_A \) and \( \| \cdot \|_B \) respectively. Then \( \{\mathfrak{X}_A \otimes \mathfrak{M}_B\}_{A \in \mathfrak{A}, B \in \mathfrak{B}} \) becomes a projective system of Hilbert spaces. If \( \mathfrak{X} \) or \( \mathfrak{M} \) is nuclear, then

\[
(3.2) \quad \mathfrak{X} \otimes \pi \mathfrak{M} \cong \text{proj lim} (\mathfrak{X}_A \otimes \mathfrak{M}_B),
\]

cf. [37] Proposition 1.3.8.

**Lemma 3.9.** \( \mathcal{H}_R(\mathbb{Q}_p^N; \infty) \otimes \pi \mathcal{H}_R(\mathbb{Q}_p^M; \infty) \cong \mathcal{H}_R(\mathbb{Q}_p^N; \infty) \) for \( N, M \in \mathbb{N} \setminus \{0\} \).

By induction \( \mathcal{H}_R(\mathbb{Q}_p^N; \infty) \otimes \pi \cdots \pi \mathcal{H}_R(\mathbb{Q}_p^{N_k}; \infty) \cong \mathcal{H}_R(\mathbb{Q}_p^{N_1}; \infty) \) for \( N_i \in \mathbb{N} \setminus \{0\}, i = 1, \ldots, k.\)
Proof. By using (3.2),

\[(3.3) \quad \mathcal{H}_R \left( Q_p^N; \infty \right) \otimes_\pi \mathcal{H}_R \left( Q_p^M; \infty \right) \cong \operatorname{projlim}_{l,m} \left( \mathcal{H}_R \left( Q_p^N; l \right) \otimes \mathcal{H}_R \left( Q_p^M; m \right) \right). \]

We now recall that
\[
\mathcal{H}_R \left( Q_p^N; l \right) \cong \mathcal{L}_R^2 \left( Q_p^N; \left[ \max \left( 1, \| \xi \|_p \right) \right]^{2al} d^l \xi \right),
\]
then
\[
\mathcal{H}_R \left( Q_p^N; l \right) \otimes \mathcal{H}_R \left( Q_p^M; m \right) \cong \mathcal{L}_R^2 \left( Q_p^{N+M}; \left[ \max \left( 1, \| \xi \|_p \right) \right]^{2al} \left[ \max \left( 1, \| \zeta \|_p \right) \right]^{2am} d^N \xi d^M \zeta \right) := L_0^2 \left( N + M; l, m \right).
\]

Set \((l, m) \geq (l', m') \Leftrightarrow l \geq l' \text{ and } m \geq m', \) and define
\[
j(l,m),(l',m') : L_0^2 \left( N + M;l,m \right) \hookrightarrow L_0^2 \left( N + M;l',m' \right) \text{ for } (l, m) \geq (l', m'),
\]
where \(\hookrightarrow\) denotes a continuous embedding of Hilbert spaces, then
\[
\left\{ L_0^2 \left( N + M;l,m \right) \right\}_{(l,m)}, \left\{ j(l,m),(l',m') \right\}_{(l,m) \geq (l',m')} \text{ forms a projective system and by (3.3),}
\]
\[
\mathcal{H}_R \left( Q_p^N; \infty \right) \otimes_\pi \mathcal{H}_R \left( Q_p^M; \infty \right) \cong \operatorname{projlim}_{l,m} L_0^2 \left( N + M;l,m \right). \]

We now set
\[
L_1^2 \left( N + M;k \right) := \mathcal{L}_R^2 \left( Q_p^{N+M}; \left[ \max \left( 1, \| \xi \|_p, \| \zeta \|_p \right) \right]^{2ak} d^N \xi d^M \zeta \right),
\]
where \(\xi = (\xi_1, \cdots, \xi_N) \in Q_p^N, \zeta = (\zeta_1, \cdots, \zeta_M) \in Q_p^M. \) We now note that
\[(3.4) \quad L_1^2 \left( N + M;l+m \right) \hookrightarrow L_0^2 \left( N + M;l,m \right) \hookrightarrow L_1^2 \left( N + M, \min \{l,m\} \right), \]
for \(l, m \in \mathbb{N}. \)

On the other hand, since for \(l \geq k, \) the inequality \(\| \cdot \|_k \leq \| \cdot \|_l \) implies
\[
i_{k,l} : L_1^2 \left( N + M;l \right) \hookrightarrow L_1^2 \left( N + M,k \right) \text{ for } k \leq l,
\]
thus \(\left\{ L_1^2 \left( N + M;k \right) \right\}_k, \left\{ i_{k,l} \right\}_k \) forms a projective system. The result follows by showing that
\[
\operatorname{projlim}_{(l,m)} L_0^2 \left( N + M;l,m \right) \cong \operatorname{projlim}_{l+m} L_1^2 \left( N + M;l+m \right)
\]
from (3.4). \(\Box\)

**Remark 3.10.** Let \(\mathfrak{X}\) be a nuclear Fréchet space and let \(\mathfrak{M}\) be a Fréchet space. We denote by \(\mathcal{B}(\mathfrak{X}, \mathfrak{M})\) the space of jointly continuous bilinear maps from \(\mathfrak{X} \times \mathfrak{M}\) into \(\mathbb{R},\) and by \(\mathcal{B}_{\text{sep}}(\mathfrak{X}, \mathfrak{M})\) the space of separately continuous bilinear maps from \(\mathfrak{X} \times \mathfrak{M}\) into \(\mathbb{R}.\) The following version of the Kernel Theorem will be used later on:
\[(\mathfrak{X} \otimes_\pi \mathfrak{M})^* \cong \mathcal{B}(\mathfrak{X}, \mathfrak{M}) \cong \mathcal{B}_{\text{sep}}(\mathfrak{X}, \mathfrak{M}),\]
\(\text{cf. [37] Theorem 1.3.10 and Proposition 1.3.12.}\]
3.2. Pseudodifferential operators acting on $\mathcal{H}_C(\infty)$.

**Definition 3.11.** We say that a function $a : \mathbb{Q}_p^N \to \mathbb{R}_+$ is a smooth symbol, if it satisfies the following properties:

(i) $a$ is a continuous function;
(ii) there exists a positive constant $C = C(a)$ such that $a(\xi) \geq C$ for any $\xi \in \mathbb{Q}_p^N$;
(iii) there exist positive constants $C_0$, $C_1$, $\alpha$, $m_0$, with $m_0 \in \mathbb{N}$, such that

\[ C_0 \|\xi\|_p^{\alpha} \leq a(\xi) \leq C_1 \|\xi\|_p^{\alpha} \quad \text{for} \quad \|\xi\|_p \geq p^{m_0}. \]

Given a smooth symbol $a(\xi)$, we attach to it the following pseudodifferential operator:

\[ \mathcal{D}_C(\mathbb{Q}_p^N) \ni g \mapsto A_g, \]

where $(A_g)(x) = \mathcal{F}_\xi^{-1} (a(\xi) \mathcal{F}_x \xi g)$.

**Lemma 3.12.** For any $l \in \mathbb{N}$, the mapping $A : \mathcal{H}_C(l+1) \to \mathcal{H}_C(l)$ is a well-defined continuous mapping between Banach spaces.

**Proof.** Let $g \in \mathcal{D}_C(\mathbb{Q}_p^N)$, then

\[ \|A_g\|^2 \leq \int_{B_{m_0}^N} \left[ \max \left( 1, \|\xi\|_p \right) \right]^{2\alpha l} |a(\xi)|^2 |\hat{g}(\xi)|^2 d^N \xi \]

\[ + C_1^2 \int_{\mathbb{Q}_p^N \setminus B_{m_0}^N} \|\xi\|_p^{2\alpha(l+1)} |\hat{g}(\xi)|^2 d^N \xi \leq \left( \sup_{\xi \in B_{m_0}^N} \left[ a(\xi) \max \left( 1, \|\xi\|_p \right) \right]^{\alpha l} \right)^2 \|g\|^2_0 \]

\[ + C_1^2 \|g\|^2_{l+1} \leq C_2 \|g\|^2_{l+1}. \]

Now, by Lemma 3.8, $A_g \in \mathcal{H}_C(l)$. The result follows from the density of $\mathcal{D}_C(\mathbb{Q}_p^N)$ in $\mathcal{H}_C(l+1)$. \qed

**Lemma 3.13.** For any $l \in \mathbb{N}$, the mapping $A : \mathcal{H}_C(l+1) \to \mathcal{H}_C(l)$ has a continuous inverse defined on $\mathcal{H}_C(l)$. In particular, $A$ is a bi-continuous bijection between Banach spaces.

**Proof.** Take $g \in \mathcal{D}_C(\mathbb{Q}_p^N) \subset \mathcal{H}_C(l)$ and set $u := \mathcal{F}_x^{-1} \left( \frac{\hat{g}(\xi)}{a(\xi)} \right)$. Then $u \in L^2 \cap C^{\text{unif}}$, and it is the unique solution of $Au = g$. Now

\[ \|u\|^2_{l+1} = \int_{B_{m_0}^N} \left[ \max \left( 1, \|\xi\|_p \right) \right]^{2\alpha(l+1)} \frac{|\hat{g}(\xi)|^2}{|a(\xi)|^2} d^N \xi \]

\[ + \int_{\mathbb{Q}_p^N \setminus B_{m_0}^N} \|\xi\|_p^{2\alpha(l+1)} \frac{|\hat{g}(\xi)|^2}{|a(\xi)|^2} d^N \xi \leq C' \|g\|^2_0 + \frac{1}{C_0^2} \int_{\mathbb{Q}_p^N \setminus B_{m_0}^N} \|\xi\|_p^{2\alpha l} |\hat{g}(\xi)|^2 d^N \xi \]

\[ \leq C' \|g\|^2_0 + \frac{1}{C_0^2} \|g\|^2_{l+1} \leq C'' \|g\|^2_{l+1}. \]

By Lemma 3.8, $u \in \mathcal{H}_C(l+1)$ and since $\mathcal{D}_C(\mathbb{Q}_p^N)$ is dense in $\mathcal{H}_C(l)$, the mapping

\[ A^{-1} : \mathcal{H}_C(l) \ni g \mapsto u, \]

is well-defined and continuous. \qed
Theorem 3.14. The mapping $\mathbf{A} : \mathcal{H}_C(\infty) \to \mathcal{H}_C(\infty)$ is a bi-continuous isomorphism of locally convex spaces.

Proof. By Lemma 3.12, $\mathbf{A}$ is a well-defined mapping. In addition, by Lemma 3.13 $\mathbf{A}$ is a bijection in $\mathcal{H}_C(\infty)$ onto itself. To verify the continuity of $\mathbf{A}$, we take a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}_C(\infty)$ such that $g_n \xrightarrow{d} g$, with $g \in \mathcal{H}_C(\infty)$, i.e. $g_n \new{\| \cdot \|_n} g$ for all $m \in \mathbb{N}$. Take $m = l + 1$, $g_n \in \mathcal{H}_C(l+1)$, then by Lemma 3.13 $A_{g_n} \new{\| \cdot \|_{l+1}} A_g \in \mathcal{H}_C(l)$, for any $l \in \mathbb{N}$. The case $A_{g_n} \xrightarrow{\| \cdot \|_0} A_g$ is verified directly. Therefore $A_{g_n} \xrightarrow{d} A_g$. We now show that $A^{-1}$ is continuous. Take a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}_C(\infty)$ as before. By Lemma 3.13 there exists a unique sequence $\{u_n\}_{n \in \mathbb{N}}$ such that $\mathbf{A}u_n = g_n$, with $g_n \in \mathcal{H}_C(l)$ and $u_n \in \mathcal{H}_C(l+1)$, for any $l \in \mathbb{N}$. By verifying that $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy in $\| \cdot \|_m$ for any $m \in \mathbb{N}$, there exists $u \in \mathcal{H}_C(\infty)$ such that $u_n \xrightarrow{d} u$, i.e. $A^{-1}g_n \xrightarrow{d} u$. By the continuity of $\mathbf{A}$, we have $\mathbf{A}u = g$, now by using that $\mathbf{A}$ is a bijection on $\mathcal{H}_C(\infty)$, we conclude that $A^{-1}g_n \xrightarrow{d} A^{-1}g$. □

4. Pseudodifferential Operators and Green Functions

We take $l(\xi) \in \mathbb{Z}_p[\xi_1, \ldots, \xi_N]$ to be an elliptic polynomial of degree $d$, this means that $l$ is homogeneous of degree $d$ and satisfies $l(\xi) = 0 \iff \xi = 0$. There are infinitely many elliptic polynomials. We consider the following elliptic pseudodifferential operator:

$$(L_\alpha h)(x) = \mathcal{F}^{-1}_{\xi \rightarrow x} \left( l(\xi)^\alpha p \mathcal{F}_{x \rightarrow \xi} h \right),$$

where $\alpha > 0$ and $h \in D_C(Q_p^N)$.

We shall call a fundamental solution $G(x; m, \alpha)$ of the equation

$$(L_\alpha + m^2) u = h, \text{ with } h \in D_C(Q_p^N), \ m > 0,$$

a Green function of $L_\alpha$. As a distribution on $D_C(Q_p^N)$, the Green function is given by

$$(4.2) \quad G(x; m, \alpha) = \mathcal{F}^{-1}_{\xi \rightarrow x} \left( \frac{1}{|l(\xi)|^\alpha p + m^2} \right).$$

Notice that since

$$(4.3) \quad C_0^\alpha \|\xi\|_p^ad \leq |l(\xi)|^\alpha \leq C_1^\alpha \|\xi\|_p^ad,$$

for some positive constants $C_0, C_1$, cf. [59 Lemma 1],

$$(4.3a) \quad \frac{1}{|l(\xi)|^\alpha p + m^2} \in L^1(Q_p^N, d^N\xi) \text{ for } ad > N,$$

and in this case, $G(x; m, \alpha)$ is an $L^\infty$-function.

4.1. Green functions on $D_C(Q_p^N)$.

Proposition 4.1. The Green function $G(x; m, \alpha)$ verifies the following properties:

(i) the function $G(x; m, \alpha)$ is continuous on $Q_p^N \setminus \{0\}$

(ii) if $ad > N$, then the function $G(x; m, \alpha)$ is continuous;
(iii) for $0 < \alpha d \leq N$, the function $G(x; m, \alpha)$ is locally constant on $\mathbb{Q}_p^N \setminus \{0\}$, and

$$|G(x; m, \alpha)| \leq \begin{cases} C \|x\|_p^{2(\alpha d - N)} & \text{for } 0 < \alpha d < N \\ C_0 - C_1 \ln \|x\|_p & \text{for } N = \alpha d, \end{cases}$$

for $\|x\|_p \leq 1$, where $C, C_0, C_1$ are positive constants,

(iv) $|G(x; m, \alpha)| \leq C_2 \|x\|_p^{-2(\alpha d + N)}$ as $\|x\|_p \to \infty$, where $C_2$ is positive constant;

(v) $G(x; m, \alpha) \geq 0$ on $\mathbb{Q}_p^N \setminus \{0\}$.

Proof. (i) We first notice that

$$G(x; m, \alpha) = \sum_{l=-\infty}^{+\infty} g^{(l)}(x; m, \alpha),$$

where

$$g^{(l)}(x; m, \alpha) = \int_{p^{-1}S^N_0} \frac{\lambda_p(-B(x, \xi))}{|I(\xi)|_p^{\alpha} + m^2} d^N \xi,$$

here $S^N_0 = \{\xi \in \mathbb{Z}_p^N : \|\xi\|_p = 1\}$. For $x \in \mathbb{Q}_p^N \setminus \{0\}$, we take $x = p^{\ord(x)}x_0$, $x_0 = (x_{0,1}, \ldots, x_{0,N})$ with $\|x_0\|_p = 1$. By using the standard basis of $\mathbb{Q}_p^N$ we have

$$B(x, \xi) = p^{\ord(x)}B(x_0, \xi) = p^{\ord(x)} \sum_{i=1}^N \xi_i \left[ \sum_{j=1}^N B_{ij}x_{0,j} \right]$$

$$=: p^{\ord(x)} \sum_{i=1}^N \xi_i A_i(x_0, B)$$

where the $B_{ij} \in \mathbb{Q}_p$, and $\det [B_{ij}] \neq 0$. We set

$$\beta(x_0, B) := \beta = \min_{1 \leq j \leq N} \ord(A_j(x_0, B))$$

and $A_j(x_0, B) := A_j = p^{\beta} \tilde{A}_j$, for $j = 1, \ldots, N$. Hence

$$B(x, \xi) = p^{\ord(x) + \beta} \sum_{i=1}^N \xi_i \tilde{A}_i,$$

with $\left\| (\tilde{A}_1, \ldots, \tilde{A}_N) \right\|_p = 1$.

Furthermore, we notice that

$$\beta \leq \ord(x) + \ord(\det [B_{ij}]).$$

Indeed, by rewriting (4.5) as $p^{\beta}u^T = [B_{ij}]x_0^T$, where $u = (u_1, \ldots, u_N)$ and $\|u\|_p = 1$, we have $x_0^T = \frac{p^{\beta + \alpha}}{\det [B_{ij}]} \left[ B_{ij} \right] u^T$ with $B_{ij} \in \mathbb{Z}_p$ for all $i$ and $j$, and $\alpha = \alpha ([B_{ij}]) \in \mathbb{N}$. By picking $i$-th row: $\ord(x_{0,i}) = \beta + \ord\left( \sum_j B_{ij}u_j \right) - \ord(\det [B_{ij}])$. The announced estimation follows by taking $i$ such that $\ord(x_{0,i}) = \ord(x)$ and by using that $\alpha + \ord\left( \sum_j B_{ij}u_j \right) \geq 0$.

We assert that series (4.4) converges in $D_C^*(\mathbb{Q}_p^N)$. Indeed,

$$\frac{1}{|I(\xi)|_p^{\alpha} + m^2} = \lim_{t \to +\infty} \sum_{j=-t}^{t} \frac{1_{S^N}(\xi)}{|I(\xi)|_p^{\alpha} + m^2} \in D_C^*(\mathbb{Q}_p^N),$$
now, since the Fourier transform is continuous on $D_{C}^{*}(\mathbb{Q}_{p}^{N})$, one gets

$$G(x; m, \alpha) = \lim_{l \to +\infty} \sum_{j=-l}^{l} \mathcal{F}^{-1}_{\xi \to x} \left( \frac{1_{S^N}(\xi)}{|\langle \xi \rangle|^\alpha_p + m^2} \right)$$

$$= \lim_{l \to +\infty} \sum_{j=-l}^{l} g^{(j)}(x; m, \alpha) = \sum_{l=-\infty}^{+\infty} g^{(l)}(x; m, \alpha).$$

Consider now $x \neq 0$, with $\|x\|_p = p^k$, $k \in \mathbb{Z}$. By changing variables as $\xi = p^{-l}z$ in $g^{(l)}(x; m, \alpha)$, one gets

$$g^{(l)}(x; m, \alpha) = p^{lN} \int_{S_{\mathbb{Z}}^N} \chi_p \left( -p^{-l+L}B(x, z) \right) \frac{p^{lda}}{|l(x)|^\alpha_p + m^2} d^Nz.$$

There exists a covering of $S_{\mathbb{Z}}^N$ of the form

$$S_{\mathbb{Z}}^N = \bigcup_{i=1}^{M} \tilde{z}_i + (p^L\mathbb{Z}_p)^n$$

with $\tilde{z}_i \in S_{\mathbb{Z}}^N$ for $i = 1, \ldots, M$, and $L \in \mathbb{N} \setminus \{0\}$, such that

$$|l(x)|^\alpha_p |\tilde{z}_i + (p^L\mathbb{Z}_p)| = |l(\tilde{z}_i)|^\alpha_p \text{ for } i = 1, \ldots, M,$$

cf. [59, Lemma 3]. From this fact and (4.0), one gets

$$g^{(l)}(x; m, \alpha) = p^{lN-LN} \sum_{i=1}^{M} \chi_p \left( -p^{-l+L-k+\beta} \sum_{i=1}^{N} y_i A_i \right) \int_{\mathbb{Z}_p^N} \chi_p \left( -p^{-l+L-k+\beta} \sum_{i=1}^{N} y_i A_i \right) d^Ny.$$

Notice that $g^{(l)}(x; m, \alpha)$ is locally constant for $\|x\|_p = p^k$. We now recall that

$$\int_{\mathbb{Z}_p^N} \chi_p \left( -p^{-l+L-k+\beta} \sum_{i=1}^{N} y_i A_i \right) d^Ny = \begin{cases} 1 & \text{if } l \leq L - k + \beta \\ 0 & \text{if } l \geq L - k + \beta + 1. \end{cases}$$

Hence,

$$g^{(l)}(x; m, \alpha) = 0 \text{ if } l \geq 1 + L - k + \beta$$

and

$$\left| g^{(l)}(x; m, \alpha) \right| \leq \frac{p^{lN-LN}M}{p^{lda} + m^2}, \text{ if } l \leq L - k + \beta,$$

where $\gamma = \min_i |l(\tilde{z}_i)|^\alpha_p > 0$, and since

$$M^{L-k+\beta} \sum_{l=-\infty}^{+\infty} \frac{p^{lN}}{p^{lda} + m^2} < \frac{M^{L-k+\beta}}{m^2} \sum_{l=-\infty}^{+\infty} p^{lN} < \infty,$$

we have that series [4.4] converges uniformly on the sphere $\|x\|_p = p^k$, and equivalently [4.4] converges uniformly on compact subsets of $\mathbb{Q}_p^N \setminus \{0\}$. Therefore, $G(x; m, \alpha)$ is a continuous function on $\mathbb{Q}_p^N \setminus \{0\}$.

(ii) If $N < ad$, estimate (4.12) implies the uniform convergence on $\mathbb{Q}_p^N$ and the continuity of $G(x; m, \alpha)$ on the whole $\mathbb{Q}_p^N$. 
(iii) If \(0 < ad < N\), then by (4.12) for \(\|x\|_p \leq 1\),
\[
|G(x; m, \alpha)| \leq C \sum_{l=-\infty}^{L-k+\beta} p^{\alpha d} L\gamma + m^2 \leq \frac{C}{\gamma} \sum_{l=-\infty}^{L-k+\beta} p^{(N-\alpha d)}
\]
\[
= \frac{C}{\gamma} p^{(L-k+\beta)(N-\alpha d)} \sum_{j=-\infty}^{0} p^j \|x\|_p^{2(\alpha d-N)},
\]
see (4.7). If \(ad = N\), then for \(\|x\|_p \leq 1\), from (4.12),
\[
|G(x; m, \alpha)| \leq \frac{C}{m^2} \sum_{l=-\infty}^{0} p^{L-k+\beta} + \frac{C}{m^2} \sum_{l=1}^{L-k+\beta} 1 = C_0 - C_1 \ln \|x\|_p,
\]
see (4.7).

(iv) Let \(\|x\|_p = p^k\). We notice that if \(\|\xi\|_p \leq p^{-k+\beta}\), then \(\chi_p(-B(x, \xi)) = 1\).
Therefore
\[
G(x; m, \alpha) = G^{(1)}(x; m, \alpha) + G^{(2)}(x; m, \alpha),
\]
where
\[
G^{(1)}(x; m, \alpha) = \sum_{l=-\infty}^{-k+\beta} p^{L-k+\beta} \int_{S_0^N} p^{\alpha d} \|l(z)\|_p^{\alpha} d^N z,
\]
\[
G^{(2)}(x; m, \alpha) = \sum_{l=-k+\beta+1}^{L-k+\beta} p^{L-k+\beta} \int_{S_0^N} \chi_p(-p^{-1}B(x, z)) p^{\alpha d} \|l(z)\|_p^{\alpha} d^N z.
\]
By using the formula
\[
(4.13) \quad \frac{1}{m^2 + t} = m^{-2} - m^{-4} t + O(t^2), \quad t \to 0,
\]
we get
\[
p^{L-k+\beta} \int_{S_0^N} \frac{d^N z}{p^{\alpha d} \|l(z)\|_p^{\alpha} + m^2} = p^{L-k+\beta} \left(1 - p^{-N}\right) m^{-2} - p^{(N+\alpha d)} m^{-4} Z(\alpha) + O(p^{(N+2\alpha d)}),
\]
as \(t \to 0\), where \(Z(\alpha) := \int_{S_0^N} \|l(z)\|_p^{\alpha} d^N z\), hence
\[
G^{(1)}(x; m, \alpha) = \left(1 - p^{-N}\right) m^{-2} \sum_{l=-\infty}^{-k+\beta} p^{L-k+\beta} - Z(\alpha) m^{-4} \sum_{l=-\infty}^{-k+\beta} p^{(N+\alpha d)}
\]
\[
+ O\left(\sum_{l=-\infty}^{-k+\beta} p^{(N+2\alpha d)}\right) = p^{N-2} \|x\|_p^{-N} - \frac{Z(\alpha)m^{-4}p^{(N+\alpha d)}\beta}{1 - p^{-N-\alpha d}} \|x\|_p^{-N-\alpha d}
\]
\[
+ O(\|x\|_p^{-2(N+2\alpha d)}), \text{ as } \|x\|_p \to \infty.
\]
We now consider $G^{(2)}(x; m, \alpha)$, by using (4.13),

$$p^N \int_{S_0^N} \frac{\chi_p \left(-p^{-l} B(x, z)\right)}{p^{l|\alpha|} |I(z)|^\alpha_p + m^2} d^N z = m^{-2} p^N \int_{S_0^N} \chi_p \left(-p^{-l} B(x, z)\right) d^N z$$

$$- m^{-4} p^{l(\alpha+N)} \int_{S_0^N} |I(z)|^\alpha_p \chi_p \left(-p^{-l} B(x, z)\right) d^N z + O(p^{l(N+2\alpha)}).$$

By using

$$\int_{S_0^N} \chi_p \left(-p^{-l} B(x, z)\right) d^N z =$$

$$\int_{S_0^N} \chi_p \left(-p^{-l-k+\beta} \sum_{j=1}^{N} z_j A_j\right) d^N z = \begin{cases} 1 - p^{-N} & \text{if } l + k - \beta \leq 0 \\ -p^{-N} & \text{if } l + k - \beta = 1 \\ 0 & \text{if } l + k - \beta \geq 2, \end{cases}$$

we get

$$m^{-2} \sum_{l=-k+\alpha+1}^{L-k+\beta} p^N \chi_p \left(-p^{-l} B(x, z)\right) d^N z = -m^{-2} p^{N\beta} \|x\|^{-N}_p,$$

and by using (4.5) - (4.7)

$$- m^{-4} p^{l(\alpha+N)} \int_{S_0^N} |I(z)|^\alpha_p \chi_p \left(-p^{-l} B(x, z)\right) d^N z =$$

$$- m^{-4} p^{l(\alpha+N)-LN} \sum_{i=1}^{M} |I(z_i)|^\alpha_p \chi_p \left(-p^{-l} B(x, z_i)\right) \int_{S_0^N} \chi_p \left(-p^{-l+k+\beta} B(x, z_i)\right) d^N y,$$

where $x = p^{-k} x_0$, $\|x_0\|_p = 1$. By using (4.10)

$$- m^{-4} p^{-LN} \sum_{i=1}^{M} |I(z_i)|^\alpha_p \sum_{l=-k+\beta+1}^{L-k+\beta} \chi_p \left(-p^{-l} B(x_0, z_i)\right) p^{l(\alpha+N)}$$

$$= -m^{-4} p^{\alpha+N-LN} \sum_{i=1}^{M} |I(z_i)|^\alpha_p \sum_{j=0}^{L-1} \chi_p \left(-p^{j-k+\beta+1} B(x, z_i)\right) p^{\alpha(\alpha+N)} \times$$

$$p^{\beta+\alpha-N} \|x\|^{-\alpha-N}_p,$$

therefore

$$G(x; m, \alpha) = G^{(1)}(x; m, \alpha) + G^{(2)}(x; m, \alpha) = -m^{-4} A(x) p^{\beta(\alpha+N)} \|x\|^{-\alpha-N}_p$$

$$+ O(\|x\|^{-2N+2\alpha}_p), \text{ as } \|x\|_p \to \infty,$$
where

\[ A(x) := \frac{Z(\alpha)}{1 - p^{-N - d\alpha}} + p^{d\alpha + N - LN} \left( \sum_{i=1}^{M} (l(\bar{z}_i))_p^L \sum_{j=0}^{L-1} \chi_p \left( -p^{j-k+\beta+1} B(x, \bar{z}_i) \right) \right). \]

(iv) By (i) we have that \( G(x; m, \alpha) \) is a continuous function on \( \mathbb{Q}_p^n \setminus \{0\} \) having expansion (4.4). Thus, it is sufficient to show that

\[ g^{(l)}(x; m, \alpha) \geq 0 \text{ on } \mathbb{Q}_p^n \setminus \{0\}. \]

Take \( t \in \mathbb{Q}_p^\times \), then \( V_t := \{ \xi \in S_j^N : l(\xi) = t \} \) is a \( p \)-adic compact submanifold of the sphere \( S_j^N \). Then, there exists a differential form \( \omega_0 \), the Gel'fand-Leray form, such that \( d\xi_1 \wedge \cdots \wedge d\xi_n = \omega_0 \wedge dt \). Denote the measure corresponding to \( \omega_0 \) as \( \frac{dt}{N} \), then

\[ g^{(l)}(x; m, \alpha) = \int_{\mathbb{Q}_p^\times} \frac{1}{|t|_p + m^2} \left\{ \int_{V_t} \chi_p \left( -B(x, \xi) \right) \frac{d\xi}{dt} \right\} dt. \]

Thus, in order to establish (4.14), it is sufficient to show that

\[ \int_{V_t} \chi_p \left( -B(x, \xi) \right) \frac{d\xi}{dt} = \int_{V_t} \chi_p \left( -p^{ord(x) + \beta} \sum_{i=1}^{N} \xi_i \tilde{A}_i \right) \frac{d\xi}{dt} \geq 0 \text{ for all } x. \]

This last inequality is established as in the proof of Theorem 2 in [59], by using the non-Archimedean Implicit Function Theorem and by performing a suitable change of variables. \( \square \)

4.2. Green functions on \( \mathcal{H}_\mathbb{R}(\infty) \).

**Theorem 4.2.** Let \( \alpha > 0 \), \( m > 0 \), and let \( L_\alpha \) be an elliptic operator. (i) There exists a Green function \( G(x; m, \alpha) \) for the operator \( L_\alpha \), which is continuous and non-negative on \( \mathbb{Q}_p^n \setminus \{0\} \), and tends to zero at infinity. Furthermore, if \( h \in \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \), then \( u(x) = G(x; m, \alpha) * h(x) \) is a solution of (4.1) in \( \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \). (ii) The equation

\[ (L_\alpha + m^2) u = g, \]

with \( g \in \mathcal{H}_\mathbb{R}(\infty) \), has a unique solution \( u \in \mathcal{H}_\mathbb{R}(\infty) \).

**Proof.** (i) We first notice that for any \( h \in \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \), \( u(x) := G(x; m, \alpha) * h(x) \) is a locally constant function because it is the inverse Fourier transform of \( \widehat{\chi_p \left( -B(x, \xi) \right)} \), which is a distribution with compact support. Taking \( u(x) = G(x; m, \alpha) * h(x) \in \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \), we have \( (L_\alpha + m^2) u = h \in \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \). The other results follow from Proposition 4.1.

(ii) Take \( g \in \mathcal{H}_\mathbb{R}(\mathbb{Q}_p^N) \), then by (i), \( u(x) = G(x; m, \alpha) * g(x) \) is a real-valued, locally constant function which is a solution of (4.15) in \( \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^N) \). Now, since
\( \hat{u}(\xi) = \frac{\tilde{g}(\xi)}{|\xi|^\alpha + m^2} \in L^2 \), by using (4.3).

\[
\|u\|_{L^2}^2 + \int_{\mathbb{Q}_p^N \setminus B_\alpha^N} \|\xi\|^{2\alpha(l+d)} |\tilde{g}(\xi)|^2 d^N \xi \\
\leq C \|g\|_0^2 + \frac{1}{C_0^\alpha} \int_{\mathbb{Q}_p^N \setminus B_\alpha^N} \|\xi\|^{2\alpha(l+d)} |\tilde{g}(\xi)|^2 d^N \xi \\
\leq C' \|\xi\|^2, \text{ for } l \in \mathbb{N}.
\]

Then, by Lemma 3.8, \( u \in \mathcal{H}_R(m) \), for \( m \geq d \). In the case, \( 0 \leq m \leq d - 1 \), one gets \( \|u\|_m \leq C'' \|g\|_0^2 \). Therefore \( u \in \mathcal{H}_R(m) \), for \( m \in \mathbb{N} \). The uniqueness of \( u \) follows from Theorem 3.14.

\textbf{Corollary 4.3.} The mapping

\[
\mathcal{H}_R(\infty) \rightarrow \mathcal{H}_R(\infty) \\
g(x) \rightarrow G(x; m, \alpha) * g(x),
\]

is continuous.

\textbf{Proof.} By the proof of Theorem 4.2(ii), and Theorem 3.14,

\[
\mathcal{H}_R(\infty) \rightarrow \mathcal{H}_R(\infty) \\
g \rightarrow \left( L_\alpha + m^2 \right)^{-1} g,
\]

is a well-defined continuous mapping.

\textbf{Remark 4.4.} (i) It is worth to mention that the Archimedean and non-Archimedean Green functions share similar properties, cf. [21, Proposition 7.2.1] and Proposition 4.1 and Theorem 4.2.

(ii) Proposition 4.1 and Theorem 4.2 generalize to arbitrary dimension some results established by Kochubei for Klein-Gordon pseudodifferential operators attached with elliptic quadratic forms cf. [27, Proposition 2.8 and Theorem 2.4].

(iii) Another possible definition for the \( p \)-adic Klein-Gordon operator, with real mass \( m > 0 \), is the following:

\[
\varphi \rightarrow \mathcal{F}^{-1} \left( \left( |\xi|^\alpha + m^2 \right)^\alpha \mathcal{F} \varphi \right).
\]

But, since Proposition 4.1 and Theorem 4.2 remain valid for this type of operators, we prefer to work with (4.7) because this type of operators have been studied extensively in the \( p \)-adic setting.

(iv) Theorem 4.2 shows that \( \mathcal{H}_C(\infty) \) contains distributions.

5. The Generalized White Noise

5.1. Infinitely divisible probability distributions. We recall that an infinitely divisible probability distribution \( P \) is a probability distribution having the property that for each \( n \in \mathbb{N} \) there exists a probability distribution \( P_n \) such that \( P = P_1 * \cdots * P_n \) (\( n \)-times). By the Lévy-Khinchine Theorem, the characteristic function \( C_P \) of \( P \) satisfies

\[
C_P(t) = \int e^{ist} dP(s) = e^{\Phi(t)}, \quad t \in \mathbb{R},
\]

(5.1)
where $\Psi : \mathbb{R} \to \mathbb{C}$ is a continuous function, called the Lévy characteristic of $P$, which is uniquely represented as follows:

\[
\Psi(t) = iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R} \setminus \{0\}} \left( e^{ist} - 1 - \frac{ist}{1 + s^2} \right) dM(s), \quad t \in \mathbb{R},
\]

where $a, \sigma \in \mathbb{R}$, with $\sigma \geq 0$, and the measure $dM(s)$ satisfies

\[
\int_{\mathbb{R} \setminus \{0\}} \min(1, s^2) dM(s) < \infty.
\]

On the other hand, given a triple $(a, \sigma, dM)$ with $a \in \mathbb{R}$, $\sigma \geq 0$, and $dM$ a measure on $\mathbb{R} \setminus \{0\}$ satisfying (5.3), there exists a unique infinitely divisible probability distribution $P$ such that its Lévy characteristic is given by (5.2).

**Remark 5.1.** From now on, we work with infinitely divisible probability distributions which are absolutely continuous with all finite moments. This fact is equivalent to all the moments of the corresponding $M$’s are finite, cf. [7, Theorem 2.3].

Let $N \in \mathbb{N}$ be as before. Let $\mathcal{H}_R(\infty)$ and $\mathcal{H}'_R(\infty)$ be the spaces introduced in Section 3.1. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $\mathcal{H}_R(\infty)$ and $\mathcal{H}'_R(\infty)$. Let $\mathcal{B}$ be the $\sigma$-algebra generated by cylinder sets of $\mathcal{H}_R(\infty)$. Then $(\mathcal{H}_R(\infty), \mathcal{B})$ is a measurable space.

By a characteristic functional on $\mathcal{H}_R(\infty)$, we mean a functional $C : \mathcal{H}_R(\infty) \to \mathbb{C}$ satisfying the following properties:

(i) $C$ is continuous on $\mathcal{H}_R(\infty)$;
(ii) $C$ is positive-definite;
(iii) $C(0) = 1$.

Now, since $\mathcal{H}_R(\infty)$ is a nuclear space, cf. Theorem 3.3, by the Bochner-Minlos Theorem (see e.g. [24]), there exists a one to one correspondence between the characteristic functionals $C$ and probability measures $P$ on $(\mathcal{H}_R(\infty), \mathcal{B})$ given by the following relation

\[
C(f) = \int_{\mathcal{H}_R(\infty)} e^{i\langle f, T \rangle} dP(T), \quad f \in \mathcal{H}_R(\infty).
\]

**Theorem 5.2.** Let $\Psi$ be a Lévy characteristic defined by (5.1). Then there exists a unique probability measure $P_\Psi$ on $(\mathcal{H}_R(\infty), \mathcal{B})$ such that the Fourier transform of $P_\Psi$ satisfies

\[
\int_{\mathcal{H}_R(\infty)} e^{i\langle f, T \rangle} dP_\Psi(T) = \exp \left\{ \int_{\mathbb{Q}_p^N} \Psi(f(x)) d_N x \right\}, \quad f \in \mathcal{H}_R(\infty).
\]

The proof is based on [20] Theorem 6, p. 283] like in the Archimedean case, cf. [7, Theorem 1.1]. However, in the non-Archimedean case the result does not follow directly from [20]. We need some additional results.

**Lemma 5.3.** The function $\int_{\mathbb{Q}_p^N} \Psi(f(x)) d_N x$ is continuous on $\mathcal{H}_R(\infty)$.

**Proof.** By (5.2) and Lemma 3.4(i), it is sufficient to show that

\[
f \to \int_{\mathbb{Q}_p^N} \int_{\mathbb{R} \setminus \{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1 + s^2} \right) dM(s) d_N x
\]
is continuous in \((D_{\mathbb{R}}(\mathbb{Q}^N_{p}), d)\). We take a sequence \(\{f_n\}_{n \in \mathbb{N}}\) in \(D_{\mathbb{R}}(\mathbb{Q}^N_{p})\) such that \(f_n \xrightarrow{d} f\), then \(f_n \xrightarrow{L^2} f\), and by passing to a subsequence if necessary, \(f_n \rightarrow f\) almost everywhere. The result follows from the Dominated Convergence Theorem, since the integrand in [4] is dominated by \(2 |f|_{L^\infty} |s| \cdot 1_{\text{supp}(f)}(x) 1_{\mathbb{R}\setminus\{t\}}(s)\), by using \(|e^{st}(x)| - 1| \leq |sf(x)|\), \(s \in \mathbb{R}, x \in \mathbb{Q}^N_{p}\), which is integrable with respect to \(dM(s)d^N x\) because \(M\) is a bounded measure with finite moments.

Set \(L(f) := \exp \left\{ \int_{\mathbb{R}^p} \Psi(f(x)) d^N x \right\}\) for \(f \in \mathcal{H}_\mathbb{R}(\infty)\).

**Proposition 5.4.** The function \(L(f)\) is positive-definite if and only if \(e^{\Psi(t)}\) is positive-definite for every \(s > 0\).

**Proof.** Suppose that \(L(f)\) is positive-definite, i.e.

\[
(5.5) \quad \sum_{j,k=1}^m L(f_j - f_k) z_j \overline{z}_k \geq 0 \text{ for } j, k \in \mathcal{H}_\mathbb{R}(\infty), \ z_j, \ z_k \in \mathbb{C}, \ j, k = 1, \ldots, m.
\]

Take \(f_j(x) = t_j \Omega \left( p^{-l} \|x - x_0\|_p \right)\), \(t_j \in \mathbb{R}\), for \(j = 1, \ldots, m, \ l \in \mathbb{Z}\), and \(x_0 \in \mathbb{Q}^N_{p}\), then

\[
\sum_{j,k=1}^m L(f_j - f_k) z_j \overline{z}_k = \sum_{j,k=1}^m \exp \left( \int_{\|x-x_0\|_p \leq p^l} \Psi(t_j - t_k) d^N x \right) z_j \overline{z}_k
\]

\[
= \sum_{j,k=1}^m \left\{ \exp \left[ p^{N\Psi}(t_j - t_k) \right] \right\} z_j \overline{z}_k \geq 0.
\]

This proves that \(e^{p^{N\Psi}(t)}\) is a positive-definite function. Now, \(t \rightarrow \frac{1 - e^{p^{N\Psi}(t)}}{p^{N}}\) is negative-definite, cf. [9, Corollary 7.7], for every \(l \in \mathbb{N}\). Furthermore, by [9, Proposition 7.4 (i)], \(\lim_{t \to -\infty} \frac{1 - e^{p^{N\Psi}(t)}}{p^{N}} = -\Psi(t)\) is a negative-definite function, and since the negative-definite functions form a cone, \(-\Psi(t)\) is negative-definite for every \(s > 0\). Finally, by the Schoenberg Theorem, cf. [9, Theorem 7.8], \(e^{\Psi(t)}\) is positive-definite for every \(s > 0\).

We now assume that \(e^{\Psi(t)}\) is positive-definite for every \(s > 0\). In order to prove [5.7], by Lemma 5.3, it is sufficient to take \(f_j, \ f_k \in D_{\mathbb{R}}(\mathbb{Q}^N_{p})\), for \(j, k = 1, \ldots, m\). Consider the matrix \(A = [a_{ij}]\) with \(a_{ij} := \exp \left( \int_{\mathbb{Q}^N_{p}} \Psi(f_i(x) - f_j(x)) d^N x \right)\). We have to show that \(A\) is positive-definite. Take \(B^N_n\) such that \(\text{supp}(f_i) \subseteq B^N_n\) for \(i = 1, \ldots, m\). Then \(a_{ij} = \exp \left( \int_{B^N_n} \Psi(f_i(x) - f_j(x)) d^N x \right)\). By using that each \(f_i(x)\) is a locally constant function, and that \(B^N_n\) is an open compact set, there exists a finite covering \(B^N_n = \bigcup_{l=1}^L B^N_{n_l}(\overline{x_l})\) such that \(f_i(x) \big|_{B^N_{n_l}(\overline{x_l})} = f_i(\overline{x_l})\). Hence \(a_{ij} = \prod_{l=1}^L \exp \left( p^{Nn_l'} \alpha_l \right)\) with \(\alpha_l := \Psi(f_i(\overline{x_l}) - f_j(\overline{x_l}))\). By a Schur Theorem, cf. [20, Theorem in p. 277], \(A\) is positive-definite if the matrix \(\left[ \exp \left( p^{Nn_l'} \alpha_l \right) \right]\) is positive-definite, which follows from the fact that \(e^{\Psi(t)}\) is positive-definite for every \(s > 0\). □

5.1.1. **Proof of Theorem 5.7**. By [20, Theorem 1, p. 273], \(L(f) \neq 0\) is the characteristic functional of a generalized random process (a random field in our terminology) with independent values at every point, if and only if: (A) \(L\) is positive-definite and (B) for any functions \(f_1(t), f_2(t) \in \mathcal{H}_\mathbb{R}(\infty)\) whose product vanishes, it verifies
that \(L(f_1 + f_2) = L(f_1)L(f_2)\). The verification of condition B is straightforward. Condition A is equivalent to \(e^{s\Psi(t)}\) is positive-definite for every \(s > 0\), cf. Proposition 5.3. By [20, Theorem 4 in p. 279 and Theorem 3 in p. 189], this last condition turns out to be equivalent to the fact that \(\Psi\) has the form (5.2).

5.2. Non-Archimedean generalized white noise measures.

**Definition 5.5.** We call \(P_\Psi\) in Theorem 5.2 a generalized white noise measure with \(Lévy\) characteristic \(\Psi\) and \((\mathcal{H}_R^\ast(\infty), \mathcal{B}, P_\Psi)\) the generalized white noise space associated with \(\Psi\). The associated coordinate process

\[ F : \mathcal{H}_R(\infty) \times \left( \mathcal{H}_R^\ast(\infty), \mathcal{B}, P_\Psi \right) \to \mathbb{R} \]

defined by \(f(T) = (f, T), f \in \mathcal{H}_R(\infty), T \in \mathcal{H}_R^\ast(\infty),\) is called generalized white noise.

The generalized white noise \(F\) is composed by three independent noises: constant, Gaussian and Poisson (with jumps given by \(M\)) noises, see Remark 1.3 in [0].

6. Euclidean random fields as convoluted generalized white noise

6.1. Construction.

**Definition 6.1.** Let \((\Omega, \mathcal{F}, P)\) be a given probability space. By a generalized random field \(\Phi\) on \((\Omega, \mathcal{F}, P)\) with parameter space \(\mathcal{H}_R(\infty)\), we mean a system

\[ \{ \Phi(g, \omega) : \omega \in \Omega \}_{g \in \mathcal{H}_R(\infty)} \]

of random variables on \((\Omega, \mathcal{F}, P)\) having the following properties:

(i) \(P\{ \omega \in \Omega : \Phi(c_1g_1 + c_2g_2, \omega) = c_1\Phi(g_1, \omega) + c_2\Phi(g_2, \omega) \} = 1\), for \(c_1, c_2 \in \mathbb{R}, g_1, g_2 \in \mathcal{H}_R(\infty)\);

(ii) if \(g_n \to g\) in \(\mathcal{H}_R(\infty)\), then \(\Phi(g_n, \omega) \to \Phi(g, \omega)\) in law.

The coordinate process in Definition 5.5 is a random field on the generalized white noise space \((\mathcal{H}_R^\ast(\infty), \mathcal{B}, P_\Psi)\), because property (i) is fulfilled pointwise and property (ii) follows from the following fact:

\[ \lim_{n \to \infty} P_\Psi \left\{ T \in \mathcal{H}_R^\ast(\infty) : |(g_n - g, T)| < \epsilon \right\} = 1. \]

Indeed, since \(\mathcal{H}_R^\ast(\infty)\) is the union of the increasing spaces \(\mathcal{H}_R^\ast(l)\), there exists \(l_0 \in \mathbb{N}\) such that \(T \in \mathcal{H}_R^\ast(l_0)\), and thus \(|(g_n - g, T)| \leq \|T\|_{-l_0} \|g_n - g\|_{l_0} \leq \|T\|_{-l_0}\), for \(n\) big enough. Now, (6.1) follows by the Dominated Convergence Theorem. Therefore \(\lim_{n \to \infty} P_\Psi \left\{ T \in \mathcal{H}_R^\ast(\infty) : |(g_n - g, T)| \geq \epsilon \right\} = 0\).

We now recall that \((Gf)(x) := G(x; m, \alpha) * f(x)\) gives rise to a continuous mapping from \(\mathcal{H}_R(\infty)\) into itself, cf. Corollary 4.3. Thus, the conjugate operator \(\tilde{G} : \mathcal{H}_R^\ast(\infty) \to \mathcal{H}_R^\ast(\infty)\) is a measurable mapping from \((\mathcal{H}_R^\ast(\infty), \mathcal{B})\) into itself.

The generalized white noise measure \(P_\Psi\) on \((\mathcal{H}_R^\ast(\infty), \mathcal{B})\) associated with a \(Lévy\) characteristic \(\Psi\) was introduced in Definition 5.5. We set \(P_\Psi\) to be the image probability measure of \(P_\Psi\) under \(\tilde{G}\), i.e. \(P_\Phi\) is the measure on \((\mathcal{H}_R^\ast(\infty), \mathcal{B})\) defined by

\[ P_\Phi(A) = P_\Psi \left( \tilde{G}^{-1}(A) \right), \text{ for } A \in \mathcal{B}. \]
Proposition 6.2. The Fourier transform of \( P \) is given by

\[
\int_{\mathcal{H}^*_R(\infty)} e^{i\langle f, T \rangle} dP_T(T) = \exp \left\{ \int_{\mathcal{Q}_p^N} \Psi \left\{ \int_{\mathcal{Q}_p^N} G(x - y; m, \alpha) f(y) d^N y \right\} d^N x \right\},
\]

for \( f \in \mathcal{H}_R(\infty) \).

Proof. For \( f \in \mathcal{H}_R(\infty) \), by (6.2) and Theorem 5.2 we get that

\[
\int_{\mathcal{H}^*_R(\infty)} e^{i\langle f, T \rangle} dP_T(T) = \int_{\mathcal{H}^*_R(\infty)} e^{i\langle \tilde{G}f, T \rangle} dP_T(T) = \int_{\mathcal{H}^*_R(\infty)} e^{i\langle Gf, T \rangle} dP_T(T)
\]

\[
= \exp \left\{ \int_{\mathcal{Q}_p^N} \Psi \left\{ \int_{\mathcal{Q}_p^N} G(x - y; m, \alpha) f(y) d^N y \right\} d^N x \right\}.
\]

\[\square\]

By Proposition 6.2, the associated coordinate process

\[
\Phi : \mathcal{H}_R(\infty) \times \left( \mathcal{H}^*_R(\infty), \mathcal{B} \right) \to \mathbb{R}
\]

given by \( \Phi(f, T) = \langle Gf, T \rangle \), \( f \in \mathcal{H}_R(\infty) \), \( T \in \left( \mathcal{H}^*_R(\infty), \mathcal{B} \right) \), is a random field on \( \left( \mathcal{H}^*_R(\infty), \mathcal{B}, P_\Phi \right) \). In fact, \( \Phi \) is nothing but \( \tilde{G}f \) which is defined by

\[
\tilde{G}f (f, T) = I \langle Gf, T \rangle, \quad f \in \mathcal{H}_R(\infty), \quad T \in \mathcal{H}^*_R(\infty).
\]

It is useful to see \( \Phi \) as the unique solution, in law, of the stochastic equation

\[
(L_\alpha + m^2) \Phi = I,
\]

where \( (L_\alpha + m^2) \Phi (f, T) := \Phi \left( (L_\alpha + m^2) f, T \right) \), for \( f \in \mathcal{H}_R(\infty) \), \( T \in \mathcal{H}^*_R(\infty) \). We note that the correctness of this last definition is a consequence of Corollary 4.11 and Theorem 4.12 (ii).

6.2. Symmetries. Given a polynomial \( a(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n] \) and \( g \in \text{GL}_N(\mathbb{Q}_p) \), we say that \( g \) preserves \( a \) if \( a(\xi) = a(g(\xi)) \), for all \( \xi \in \mathbb{Q}_p^N \). By simplicity, we use \( gx \) to mean \( [g_{ij}] x^T \), \( x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N \), where we identify \( g \) with the matrix \([g_{ij}]\).

Definition 6.3. Let \( q(\xi) \) be the elliptic quadratic form used in the definition of the Fourier transform, see Section 2.2, and let \( l(\xi) \) be the elliptic polynomial that appears in the symbol of the operator \( L_\alpha \), see Section 4. We define the homogeneous Euclidean group of \( \mathbb{Q}_p^N \) relative to \( q(\xi) \) and \( l(\xi) \), denoted as \( E_0(\mathbb{Q}_p^N; q, l) := E_0(\mathbb{Q}_p^N) \), as the subgroup of \( \text{GL}_N(\mathbb{Q}_p) \) whose elements preserve \( q(\xi) \) and \( l(\xi) \) simultaneously. We define the inhomogeneous Euclidean group, denoted as \( E(\mathbb{Q}_p^N; q, l) := E(\mathbb{Q}_p^N) \), to be the group of transformations of the form \( (a, g) : x = a + gx \), for \( a, x \in \mathbb{Q}_p^N \), \( g \in E_0(\mathbb{Q}_p^N) \).

Notice that \( (a, g)^{-1} x = g^{-1} (x - a) \).

Let \( (a, g) \) be a transformation in \( E(\mathbb{Q}_p^N) \), the action of \( (a, g) \) on a function \( f \in \mathcal{H}_R(\infty) \) is defined by

\[
((a, g) f)(x) = f \left((a, g)^{-1} x\right), \quad \text{for } x \in \mathbb{Q}_p^N,
\]

\[\Box\]
and on a functional $T \in \mathcal{H}_R^\ast (\infty)$, by

$$\langle f, (a,g)T \rangle := \langle (a,g)^{-1} f, T \rangle,$$

for $f \in \mathcal{H}_R (\infty)$.

The action on a random field $\Phi$ is defined by

$$((a,g)\Phi)(f, T) = \Phi ((a,g)^{-1} f, T), \text{ for } f \in \mathcal{H}_R (\infty), \ T \in \mathcal{H}_R^\ast (\infty).$$

**Definition 6.4.** By Euclidean invariance of the random field $\Phi$ we mean that the laws of $\Phi$ and $(a, g)\Phi$ are the same for each $(a, g) \in E (\mathbb{Q}_p^N)$, i.e. the probability distributions of $\{\Phi (f, \cdot) : f \in \mathcal{H}_R (\infty)\}$ and $\{(a, g)\Phi (f, \cdot) : f \in \mathcal{H}_R (\infty)\}$ coincide for each $(a, g) \in E (\mathbb{Q}_p^N)$.

We say that $\mathcal{G}$ is $(a, g)$-invariant for some $(a, g) \in E (\mathbb{Q}_p^N)$, if $(a, g)\mathcal{G} = \mathcal{G} (a, g)$. If $\mathcal{G}$ is invariant under all $(a, g) \in E (\mathbb{Q}_p^N)$, we say that $\mathcal{G}$ is Euclidean invariant.

**Remark 6.5.** Let $f \in \mathcal{D}_C (\mathbb{Q}_p^N)$ and let $(a, g) \in E (\mathbb{Q}_p^N)$. Then

$$(6.3) \quad \mathcal{F}_{x \rightarrow \xi} \left[ f \left( (a, g)^{-1} x \right) \right] = \chi_p \left( \mathfrak{B} (a, \xi) \right) \mathcal{F}_{x \rightarrow \xi} \left[ f \right] (g^{-1} \xi).$$

Indeed, by taking $g^{-1} (x - a) = y$,

$$\int_{\mathbb{Q}_p^N} f \left( g^{-1} (x - a) \right) \chi_p \left( \mathfrak{B} (x, \xi) \right) d \mu (x) = \chi_p \left( \mathfrak{B} (a, \xi) \right) \int_{\mathbb{Q}_p^N} f (y) \chi_p \left( \mathfrak{B} (gy, \xi) \right) d \mu (y).$$

The formula follows from

$$\mathfrak{B} (gy, \xi) = \frac{1}{2} \left\{ q (gy + \xi) - q (gy) - q (\xi) \right\} = \frac{1}{2} \left\{ q (y + g^{-1} \xi) - q (y) - q (g^{-1} \xi) \right\} = \mathfrak{B} (y, g^{-1} \xi).$$

By the density of $\mathcal{D}_C (\mathbb{Q}_p^N)$ in $\mathcal{D}_C^\ast (\mathbb{Q}_p^N)$, formula (6.3) holds in $\mathcal{D}_C^\ast (\mathbb{Q}_p^N)$.

**Lemma 6.6.** $\mathcal{G}$ is Euclidean invariant.

**Proof.** We first notice that the mapping $f (x) \rightarrow f \left( (a, g)^{-1} x \right)$ is continuous from $\mathcal{H}_R (\infty)$ into $\mathcal{H}_R (\infty)$, for any $(a, g) \in E (\mathbb{Q}_p^N)$, then $\mathcal{G} (a, g), (a, g)\mathcal{G} : \mathcal{H}_R (\infty) \rightarrow \mathcal{H}_R (\infty)$ are continuous, cf. Corollary 6.3 and in order to show that

$$(a, g)\mathcal{G} (f) = (\mathcal{G} (a, g)) (f),$$

it is sufficient to take $f \in \mathcal{D}_R (\mathbb{Q}_p^N)$. Now, since

$$((a, g)\mathcal{G} (f)) (x) = (a, g) (G (x; m, \alpha) * f (x)) = (G * f) \left( (a, g)^{-1} x \right),$$

and

$$(\mathcal{G} (a, g)) (f) (x) = \mathcal{G} \left( f \left( (a, g)^{-1} x \right) \right) = G (x; m, \alpha) * f \left( (a, g)^{-1} x \right),$$

we should show that $(G * f) \left( (a, g)^{-1} x \right) = G (x; m, \alpha) * f \left( (a, g)^{-1} x \right)$. We establish this formula in $\mathcal{D}_C^\ast (\mathbb{Q}_p^N)$ by using the Fourier transform. Indeed,

$$\mathcal{F} \left[ (G * f) \left( (a, g)^{-1} x \right) \right] = \chi_p \left( \mathfrak{B} (a, \xi) \right) \mathcal{F} \left[ (G * f) \right] (g^{-1} \xi) =$$

$$\chi_p \left( \mathfrak{B} (a, \xi) \right) \frac{\mathcal{F} \left[ f \right] (g^{-1} \xi)}{|I(g^{-1} \xi)|_p^a + m^2} = \chi_p \left( \mathfrak{B} (a, \xi) \right) \frac{\mathcal{F} \left[ f \right] (g^{-1} \xi)}{|I(\xi)|_p^a + m^2},$$

for any $(a, g) \in E (\mathbb{Q}_p^N)$.

Thus, $\mathcal{G}$ is Euclidean invariant.
Proposition 6.7. The random field $\Phi = \tilde{g}f$ is Euclidean invariant.

Proof. By Bochner-Minlos Theorem, it is sufficient to show that
\[ C_\Phi (f) = C_{(a, g)\Phi} (f), \quad \text{for } f \in \mathcal{H}_\mathbb{R} \text{ (}\infty\text{) and for every } (a, g) \in E (\mathbb{Q}_p^N). \]
Indeed,
\[
C_{(a, g)\Phi} (f) = \int_{\mathcal{H}_\mathbb{R} \text{ (}\infty\text{)}} e^{i((a, g)\Phi)(f, T)} dP_\Phi (T) = \int_{\mathcal{H}_\mathbb{R} \text{ (}\infty\text{)}} e^{i((a, g)^{-1}f, \tilde{g}T)} dP_\Phi (T)
\]
\[
= \int_{\mathcal{H}_\mathbb{R} \text{ (}\infty\text{)}} e^{i((\tilde{g}(a, g)^{-1}f), T)} dP_\Phi (T) = \exp \left\{ \int_{\mathbb{Q}_p^N} \Psi \left( \tilde{g} \left( (a, g)^{-1}f \right) (x) \right) d^N x \right\},
\]
cf. Theorem 5.2 and Corollary 4.3. Now, since $\tilde{g} \left( (a, g)^{-1}f \right) (x) = (a, g)^{-1} \left( \tilde{g} (f) \right) (x)$ (cf. Lemma 6.5), and $(a, g)$ preserves $d^N x$, we have
\[
C_{(a, g)\Phi} (f) = \exp \left\{ \int_{\mathbb{Q}_p^N} \Psi \left( \tilde{g} (f) \left((a, g) x\right)\right) d^N x \right\} = \exp \left\{ \int_{\mathbb{Q}_p^N} \Psi \left( \tilde{g} (f) (x)\right) d^N x \right\}
\]
\[ = C_\Phi (f), \quad \text{cf. Proposition 6.2.} \]

6.3. Some additional remarks and examples. In the Archimedean case the symmetric bilinear form used in the definition of the Fourier transform has associated a quadratic form which is exactly the symbol of the Laplacian when it is considered as a pseudodifferential operator. This approach cannot be carried out in the $p$-adic setting. Indeed, the quadratic form $\xi_1^2 + \cdots + \xi_N^2$ associated to the bilinear form $\sum_i \xi_i x_i$ does not give rise to an elliptic operator if $N \geq 5$. This is the reason why in the $p$-adic setting we need two different polynomials $q (\xi)$ and $l (\xi)$. In order to have a ‘non-trivial’ group of symmetries, i.e. $E_0 (\mathbb{Q}_p^N, q, l) \neq 1$, the polynomials $q (\xi)$ and $l (\xi)$ should be related ‘nicely’. To illustrate this idea we give two examples.

Example 6.8. Take $N = 4$ and $l_4 (\xi) = \xi_1^2 - s \xi_2^2 - p \xi_3^2 + s \xi_4^2$, where $s \in \mathbb{Z} \backslash \{0\}$ is a quadratic non-residue, i.e. $\left( \frac{s}{p} \right) = -1$. We take $q_4 (\xi) = l_4 (\xi)$, i.e. $\mathcal{B}_4 (x, \xi) = \xi_1 x_1 - s \xi_2 x_2 - p \xi_3 x_3 + s \xi_4 x_4$. In this case, $E_0 (\mathbb{Q}_p^4, l_4, l_4)$ equals
\[
O (l_4) = \left\{ g \in GL_4 (\mathbb{Q}_p) : g^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & s \end{bmatrix} g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \right\},
\]
the orthogonal group of \( l_4 \).

**Example 6.9.** Take \( N = 5 \), \( B_5 (x, \xi) = \xi_1 x_1 - s \xi_2 x_2 - p \xi_3 x_3 + s \xi_4 x_4 + \xi_5 x_5 \), and \( l_5 (\xi) = (\xi_1^2 - s \xi_2^2 - p \xi_3^2 + s \xi_4^2 - \tau \xi_5^2) \), where \( s \) is as in Example 6.9 and \( \tau \notin [Q_p^\times]^2 \), where \( [Q_p^\times]^2 \) denotes the group of squares of \( Q_p^\times \). We notice that

\[
\left\{ \begin{bmatrix} g & 0 \\ 0 & [I_{1 \times 1}] \end{bmatrix} : g \in O (l_4) \right\} \subseteq E_0 (Q_p^5, q_5, l_5).
\]

**Example 6.10.** Take \( B (x, \xi) = \sum_{i=1}^N \xi_i x_i \). Then the theory developed so far can be applied to pseudodifferential operators of type

\[
g \to F^{-1} \left( \| \xi \|^\alpha_p F g \right), \text{ or } g \to F^{-1} \left( \sum_{i=1}^N |\xi_i|_p \right)^\alpha F g, \text{ with } \alpha > 0.
\]

We notice that the group of permutations of the variables \( \xi_1, \cdots, \xi_N \) preserve \( \| \xi \|^\alpha_p \) and \( \left( \sum_{i=1}^N |\xi_i|_p \right)^\alpha \).

7. **Schwinger Functions**

**Remark 7.1.** **Gâteaux derivatives.** Let \( y \in \mathcal{H}_R (\infty) \) be fixed. Assume that \( F : \mathcal{H}_R (\infty) \to \mathbb{R} \). For \( x \in \mathcal{H}_R (\infty) \) consider \( \lambda \to F (x + \lambda y) \) on \( \mathbb{R} \). If this function is differentiable at \( \lambda = 0 \), we say that \( F \) is Gâteaux differentiable at \( x \) in the direction \( y \) and denote

\[
\frac{d}{d\lambda} F (x + \lambda y) \big|_{\lambda = 0}.
\]

We define the partial derivative of the functional \( F \) with respect to \( g \in \mathcal{H}_R (\infty) \) as

\[
\frac{\partial}{\partial g} F = D_g F.
\]

**Definition 7.2.** Let \( g_1, \cdots, g_m \in \mathcal{H}_R (\infty) \). We define \( M_m^f \), the \( m \) th moment of the random field \( F \), by

\[
M_m^f (g_1 \otimes \cdots \otimes g_m) = \int_{\mathcal{H}_R^m (\infty)} \langle g_1, T \rangle \cdots \langle g_m, T \rangle dP_\Psi (T), \quad m \in \mathbb{N} \setminus \{0\},
\]

and \( M_0^f := 1 \).

**Lemma 7.3.** (i) Let

\[
C_f^T : \mathcal{H}_R (\infty) \to \mathbb{R}, \quad g \mapsto \int_{Q_p^N} \Psi (g (x)) d^N x.
\]

Then partial derivatives of all orders of \( C_f^T \) exist everywhere on \( \mathcal{H}_R (\infty) \). For \( g_1, \cdots, g_m \in \mathcal{H}_R (\infty) \), we have

\[
\frac{1}{v_m} \frac{\partial^m}{\partial g_m \cdots \partial g_1} C_f^T \big|_{0} = c_m \int_{Q_p^N} g_1 \cdots g_m d^N x,
\]
where
\[
c_1 := a + \int_{\mathbb{R} \smallsetminus \{0\}} \frac{s^2}{1+s} dM(s), \quad c_2 := \sigma^2 + \int_{\mathbb{R} \smallsetminus \{0\}} s^2 dM(s),
\]
(7.2)
\[
c_m := \int_{\mathbb{R} \smallsetminus \{0\}} s^m dM(s), \quad \text{for } m \geq 3.
\]

(ii) Take \( C_F = \exp C_T \). Then \( C_F \) has partial derivatives of any order, and the moments of \( \hat{\sigma} \) satisfy
\[
M_{\hat{\sigma}}^m (g_1 \otimes \cdots \otimes g_m) = \frac{1}{i^m} \frac{\partial^m}{\partial g_{m} \cdots \partial g_1} C_F |_{0} .
\]
(7.3)

**Proof.** (i) See the proof of Lemma 3.1 in [6]. (ii) The formula follows from Theorem 5.2 and (i), by
\[
\frac{\partial^m}{\partial g_{m} \cdots \partial g_1} \int_{\mathcal{H}_R(\infty)} e^{i(g, T)} d\mathcal{P}_{\Phi}(T) = i^m M_{\hat{\sigma}}^m (g_1 \otimes \cdots \otimes g_m).
\]
This last formula follows from the Dominated Convergence Theorem by using that \(|e^{i(g, T)}| = 1\) and that \( \mathcal{P}_{\Phi} \) is a probability measure. \( \square \)

**Lemma 7.4** ([6 Corollary 3.5]). Let \( V \) be a vector space. Let \( g : V \to \mathbb{C} \) be infinitely often differentiable on \( V \), in the sense of (7.1), such that \( g(0) = 0 \). Let \( f \) the exponential function. Then \( f^{(k)} \circ g(0) = 1 \) for all \( k \in \mathbb{N} \). Let \( P^{(m)} \) be the collection of all partitions \( I \) of \( \{1, \cdots, m\} \) into disjoint sets. It follows that for \( v_1, \cdots, v_m \in V \), we get
\[
\frac{\partial^m}{\partial v_m \cdots \partial v_1} \exp g |_{0} = \sum_{I \in P^{(m)}} \prod_{\{j_1, \cdots, j_l\} \in I} \frac{\partial^l}{\partial v_{j_1} \cdots \partial v_{j_l}} g |_{0} .
\]

**Proposition 7.5.** Set \( g_1, \cdots, g_m \in \mathcal{H}_R(\infty) \). Then
\[
M_{\hat{\sigma}}^m (g_1 \otimes \cdots \otimes g_m) = \sum_{I \in P^{(m)}} \prod_{\{j_1, \cdots, j_l\} \in I} c_l \int_{\mathbb{Q}_p^m} g_{j_1} \cdots g_{j_l} d^N x.
\]

**Proof.** The formula follows from (7.3), \( C_F = \exp C_T \), Lemma 7.3 (i) and Lemma 7.4. \( \square \)

**Definition 7.6.** Set \( g_1, \cdots, g_m \in \mathcal{H}_R(\infty) \). We define the \( m \)–th Schwinger function \( S_m \) of \( \Phi \) as the \( m \)–th moment of \( \Phi \), i.e.
\[
S_m (g_1 \otimes \cdots \otimes g_m) = \int_{\mathcal{H}_R(\infty)} (g_1, T) \cdots (g_m, T) d\mathcal{P}_{\Phi}(T), \ m \in \mathbb{N} \smallsetminus \{0\},
\]
with \( S_0 := 1 \).

**Theorem 7.7.** The Schwinger functions \( S_m \) defined above are symmetric and Euclidean invariant functionals in \( \mathcal{H}_R(\mathbb{Q}_p^m; \infty) \) for \( m \geq 1 \). Furthermore for
\(g_1, \cdots, g_m \in \mathcal{H}(\mathbb{Q}_p^N, \infty)\) we have

\[
S_m (g_1 \otimes \cdots \otimes g_m) = \sum_{l \in \mathcal{P}(m)} \prod_{(j_1, \cdots, j_l) \in l} c_l \int_{\mathbb{Q}_p^N} \left( \prod_{k=1}^l G(x; m, \alpha) * g_{j_k} (x) \right) d^N x,
\]

where the constants \(c_l\) were defined in (7.2).

**Proof.** The symmetry follows from Definition 7.6. To establish the Euclidean invariance, we proceed as follows. By (6.4)-(7.3),

\[
\frac{1}{\nu} \frac{\partial^m}{\partial f_m \cdots \partial f_1} C_\Phi (f) \big|_0 = \frac{1}{\nu} \frac{\partial^m}{\partial f_m \cdots \partial f_1} C_{(a,g)} \Phi (f) \big|_0,
\]

for any \(f, f_1, \cdots, f_m \in \mathcal{H}(\mathbb{Q}_p^N, \infty)\) and \((a, g) \in \mathcal{E} (\mathbb{Q}_p^N)\). Therefore

\[
\int_{\mathcal{H}_b^* (\infty)} \langle g_1, T \rangle \cdots \langle g_m, T \rangle d P_\Phi (T) = \int_{\mathcal{H}_b^* (\infty)} \langle g_1, T \rangle \cdots \langle g_m, T \rangle d P_{(a,g) \Phi} (T),
\]

\(m \in \mathbb{N} \setminus \{0\}\) and any \((a, g) \in \mathcal{E} (\mathbb{Q}_p^N)\).

We now establish (7.5). By using \(P_\Phi = P_\psi \circ (\tilde{g})^{-1}\), the right hand side of (7.4) is equal to

\[
\int_{\mathcal{H}_b^* (\infty)} \langle g_1, \tilde{g} T \rangle \cdots \langle g_m, \tilde{g} T \rangle d P_\Psi (T) = \int_{\mathcal{H}_b^* (\infty)} \langle G * g_1, T \rangle \cdots \langle G * g_m, T \rangle d P_\Psi (T),
\]

with \(G = G(x; m, \alpha)\). Now, the formula follows from Proposition 7.6.

We set \(S_m^T\) for the mapping

\[
g_1 \otimes \cdots \otimes g_l \rightarrow c_l \int_{\mathbb{Q}_p^N} \left( \prod_{i=1}^l G * g_i \right) d^N x,
\]

where \(l \in \mathbb{N} \setminus \{0\}\). We show that \(S_m^T\) is a \(l\)-linear form which is continuous in each of its arguments, i.e. separately continuous. We first show the case \(l = 2\). By Corollary 4.3 \(G * g_1, G * g_2 \in L^2_{\mathbb{R}}\), then

\[
B(g_1, g_2) := \int_{\mathbb{Q}_p^N} (G * g_1) (G * g_2) d^N x = \int_{\mathbb{Q}_p^N} \left( \tilde{G} (\xi) \tilde{g}_1 (\xi) \right) \left( \tilde{G} (\xi) \tilde{g}_2 (\xi) \right) d^N \xi.
\]

We now pick \(k \in \mathbb{N} \setminus \{0\}\), use \(\left| \tilde{G} (\xi) \right| \leq \frac{1}{m^k}\) and the Cauchy-Schwartz inequality, to get

\[
|B(g_1, g_2)| \leq \frac{1}{m^k} \int_{\mathbb{Q}_p^N} \left[ \frac{\left| \tilde{g}_1 (\xi) \right|}{\left[ \max_{1, \|\xi\|_p} \right]^{\alpha_k}} \left[ \left( \max_{1, \|\xi\|_p} \right)^{\alpha_k} \left| \tilde{g}_2 (\xi) \right| \right] d^N \xi \leq \frac{1}{m^{\alpha_k}} \|g_1\|_0 \|g_2\|_k,
\]
i.e. $B(g_1, \cdot)$ is continuous. We now established the case $l \geq 3$ by using induction on $l$. By using that $\hat{g}_i \in L^1_R \cap L^2_R$, for $i = 1, \cdots, l - 1$, cf. Corollary 3.3 and Remark 3.3 (ii), and that $\left| \hat{G}\hat{g}_i \right| \leq \frac{1}{i} \| \hat{g}_i \|$, for $i = 1, \cdots, l - 1$, we have $\hat{G}\hat{g}_i \in L^1_R \cap L^2_R$, for $i = 1, \cdots, l - 1$, now by applying the Young inequality repeatedly,

$$\hat{H}(\xi) := \hat{G}\hat{g}_1(\xi) \ast \cdots \ast \hat{G}\hat{g}_{l-1}(\xi) \in L^2_R.$$

Then $H(\xi) = \prod_{i=1}^{l-1} G \ast g_i \in L^2_R$. We now apply the argument in the case $l = 2$ to $\prod_{i=1}^{l-1} G \ast g_i \in L^2_R$ and $G \ast g_l$ to show that $B(g_1, \cdots, g_{l-1}, \cdot)$ is continuous for fixed $g_1, \cdots, g_{l-1} \in \mathcal{H}_R(Q_p \mathcal{N} ; \infty)$. Hence $S^T_l$ is a $l$-linear form which is continuously continuous on $\mathcal{H}_R(\infty) \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} \mathcal{H}_R(Q_p \mathcal{N} ; \infty)$, $(l - 1)$-times. Now, since $\mathcal{H}_R(Q_p \mathcal{N} ; \infty)$ is a Fréchet space, $S^T_l$ is $l$-linear continuous, see e.g. Proposition 1.3.11 in [37]. We now apply the Kernel Theorem inductively on $l$, see Remark 3.11 and Lemma 3.10 to get

$$S^T_l \in \mathcal{B}(\mathcal{H}_R(Q_p(l-1)\mathcal{N} ; \infty), \mathcal{H}_R(Q_p \mathcal{N} ; \infty)) \cong \mathcal{H}_R(Q_p(l-1)\mathcal{N} ; \infty) \otimes_{\text{alg}} \mathcal{H}_R(Q_p \mathcal{N} ; \infty).$$

We now consider $\prod_{(j_1, \cdots, j_l) \in S^T_l}. \text{ By induction on the cardinality of } I, \text{ and by using the Kernel Theorem and Lemma 3.10, we show that } \prod_{(j_1, \cdots, j_l) \in S^T_l} \in \mathcal{H}_R(Q_p(l-1)\mathcal{N} ; \infty). \text{ Therefore } S_m \in \mathcal{H}_R(Q_p m \mathcal{N} ; \infty). \Box$

### 7.1. The $p$-adic Brownian sheet on $Q_p \mathcal{N}$.

As an application of the results developed in this section, we present a construction of the Wiener process with time variable in $Q_p \mathcal{N}$. In this section we take $\Psi$ with $a = 0$ and $M = 0$ in (5.2). Thus, the generalized white noise $F$ in Definition 5.3 is Gaussian with mean zero. Given $t = (t_1, \cdots, t_N) \in Q_p \mathcal{N}$, we set for $x \in Q_p$,

$$1_{[0,t]}(x) := \begin{cases} 1 & \text{if } \| x \|_p \leq \| t \|_p \\ 0 & \text{otherwise.} \end{cases}$$

We also set $W(t) := \{ W(t, \cdot) \}_{t \in Q_p \mathcal{N}} = \{ F(1_{[0,t]}(\cdot)) \}_{t \in Q_p \mathcal{N}}$. We call the process $W(t)$ with values in $\mathbb{R}$, the $p$-adic Brownian sheet on $Q_p \mathcal{N}$. The following result follows from Theorem 5.2 and Lemma 7.3.

**Theorem 7.8.** The process $W(t)$ has the following properties:

(i) $W(0) = 0$ almost surely;

(ii) the process $W(t)$ is Gaussian with mean zero;

(iii) $E[W(t)W(s)] = \begin{cases} \min(\| t \|_p, \| s \|_p) & \text{if } t \neq 0 \text{ and } s \neq 0 \\ 0 & \text{if } t = 0 \text{ or } s = 0; \end{cases}$

(iv) let $t_1, t_2, t_3, t_4$ in $Q_p \mathcal{N}$ such that $\| t_1 \|_p \leq \| t_2 \|_p < \| t_3 \|_p \leq \| t_4 \|_p$, then $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent.

Bikulov and Volovich [13], and Kamizono [24], constructed Brownian motion with $p$-adic time. In the case $N = 1$, our covariance function does not agree with the one given in [13, 24], thus, in this case our result gives a different stochastic process.
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