Mass, gauge conditions and spectral properties of the Sen–Witten and 3-surface twistor operators in closed universes

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Abstract
A non-negative expression, built from the norm of the 3-surface twistor operator and the energy–momentum tensor of the matter fields on a spacelike hypersurface, is found which, in the asymptotically flat/hyperboloidal case, provides a lower bound for the ADM/Bondi–Sachs mass, while on closed hypersurfaces coincides with the first eigenvalue of the Sen–Witten operator. Also in the closed case, its vanishing is equivalent to the existence of non-trivial solutions of Witten’s gauge condition. Moreover, it is vanishing if and only if the closed data set is in a flat spacetime with spatial topology \( S^1 \times S^1 \times S^1 \). Thus, it provides a positive definite measure of the strength of the gravitational field (with physical dimension mass) on closed hypersurfaces, i.e. some sort of the total mass of closed universes.

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1. Introduction

1.1. Three problems

1.1.1. Lower bound for the ADM/Bondi–Sachs masses and the mass of closed universes. The classical energy positivity proofs guarantee that the total energy of asymptotically flat matter+gravity systems, measured both in spatial and null infinities (even in the presence of black holes), is bounded from below by zero [1–4]. Similar but strictly positive lower bound would be provided by the Penrose inequality: the total mass could not be less than the irreducible mass associated with the black holes [5]. (For a review of the status of the Penrose inequality, see e.g. [6].) Recently, Bäckdahl and Valiente-Kroon showed by explicit calculation [7] that in vacuum, asymptotically flat spacetimes the ADM mass can be re-expressed by the
norm of the 3-surface twistor operator [8], whose norm is some form of a geometric invariant of the actual spacelike hypersurface.

This raises the question whether or not the Bondi–Sachs mass can also be expressed in an analogous way, perhaps even in the non-vacuum case. Or, more generally, whether or not other strictly positive lower bounds for the total mass can also be found, even in the absence of black holes. It is known that the Hamiltonian structure of general relativity dictates that mass/energy–momentum (and other ‘conserved’ quantities) should be associated with closed spacelike 2-surfaces (for a review of the various strategies, see e.g. [9]). Hence, strictly speaking, no such quantity can be expected to be associated with a closed spacelike hypersurface (e.g. in a closed universe). Thus, we have the question if a quantity analogous to that behind the lower bounds for the ADM and Bondi–Sachs masses could provide certain notion of the ‘total mass’ in closed universes.

1.1.2. Witten-type gauge conditions in closed universes. In several specific problems (e.g. in the energy positivity proofs, in the Hamiltonian formulation of the theory or in the study of the field equations, in particular, in the evolution problems), it is desirable to reduce the huge gauge freedom of general relativity. Such classical gauge conditions are the Witten [1] or Parker [10] gauges, or Nester’s frame gauge condition [11–13] for spinor or orthonormal frame fields, respectively, on spacelike hypersurfaces. Apparently, while in the asymptotically flat cases these gauge conditions can be imposed and admit non-trivial solutions, explicit calculations indicate that the first two cannot be imposed on special, highly symmetric closed spacelike hypersurfaces.

Thus, the question arises naturally whether the Witten-type gauge conditions can always be imposed, at least on generic closed spacelike hypersurfaces, and if not, then how those can be generalized. However, as far as we know, no such systematic investigation has been devoted to this question. On these hypersurfaces Nester’s gauge condition (in its spinor form) is particularly interesting, because it takes the form of a general eigenvalue problem for a Dirac-type operator, while the former two require the spinor field to be the eigenspinor of a (slightly different) Dirac operator or modified Dirac operator with zero eigenvalue.

1.1.3. Lower bound for the eigenvalues of the Sen–Witten operator in closed universes. The eigenvalue problem for Dirac-type operators appears in another context in geometry and in general relativity. Namely, a promising approach to constructing observables of the gravitational field in general relativity could be based on the spectral analysis of Dirac operators on various submanifolds of the spacetime. For example, the eigenvalues of these operators are such gauge-invariant objects, which are expected to reflect the geometrical properties of the submanifold in question, e.g. in the form of some lower bound for the eigenvalues in terms of other well-known geometrical objects. (For a review of a number of related problems in differential geometry, see e.g. [14], section IV, pp 685–8.)

The first who gave such a lower bound in differential geometry was Lichnerowicz [15]: he showed, in particular, that on any closed Riemannian spin manifold Σ with positive scalar curvature \( \frac{1}{4} \inf \{ R(p) | p \in \Sigma \} \) is a lower bound for the square of the eigenvalues. However, this bound is not sharp: on a metric 2-sphere with radius \( r \) the (positive) eigenvalues are \( \frac{n}{r} \), \( n \in \mathbb{N} \), while on metric spheres the bounds were expected to be saturated. In fact, in the last two decades such sharp lower bounds were found in terms of the scalar curvature [16–19], the more general curvature operator (even in the presence of non-trivial boundary conditions) [20] or the volume [21, 19]. In particular, in dimension \( m \) the sharp lower bound, given by Friedrich [16, 19], is \( \frac{m}{2(m-1)} \inf \{ |R(p)| | p \in \Sigma \} \).
To have significance of these results in general relativity we should be able to link the bounds to well-known concepts of physics, e.g. the objects defined in a natural way on a spacelike hypersurface \( \Sigma \) of a Lorentzian 4-manifold. Such an extension of the pure Riemannian geometrical results to spacelike hypersurfaces in Lorentzian spin manifolds has in fact been given by Hijazi and Zhang [22] using the Sen–Witten operator (acting on Dirac spinors) and the technique of Friedrich: the eigenvalues of the Sen–Witten operator are bounded from below by a certain average of the energy–momentum of the matter fields seen by the observers at rest with respect to the hypersurface.

However, there are non-flat solutions of Einstein’s equations even in the absence of matter fields, in which case the lower bound of [22] is zero, and hence the bound is trivial. Thus, we have the question whether or not an even greater, sharp lower bound for the eigenvalues can be found which is not zero even in the vacuum case. A more ambitious claim is to find an explicit expression for the first eigenvalue itself.

1.2. The aims and results of the paper

The aim of this paper is to answer the questions above. Apparently these problems seem to be independent, and it is only the formalism, e.g. the use of (actually Weyl) spinorial techniques, which make them related to each other. However, this is not the case: we find a non-negative expression \( M \), built from the norm of the 3-surface twistor operator and the energy–momentum tensor, such that (1) in the asymptotically flat and asymptotically hyperboloidal cases it provides a lower bound for the ADM and Bondi–Sachs masses, respectively; (2) on closed spacelike hypersurfaces Witten’s gauge condition can be imposed if and only if it is vanishing, which is also equivalent to the flatness of the spacetime with \( S^1 \times S^1 \times S^1 \) spatial topology and (3) it gives the first eigenvalue of the Sen–Witten operator on closed spacelike hypersurfaces. Thus, \( M \) provides a common generalization of the results of [7] and [22]. Moreover, since this \( M \) is some measure of the strength of the gravitational ‘field’ (and its physical dimension is mass, in contrast e.g. to the so-called Bel–Robinson energy which does not have the correct dimension [9]), this can also be interpreted as some total mass associated with the whole closed spacelike hypersurface.

Technically, one of the two key ingredients in our investigations is the Sen–Witten identity, given for Weyl spinors in [4]. The other is the observation that the decomposition of the derivative of a spinor field (with respect to the Sen connection [23] on the spacelike hypersurface \( \Sigma \)) into its Sen–Witten derivative (which is a Dirac operator built from the Sen connection) and the 3-surface twistor derivative is not only algebraically irreducible, but also is an \( L^2 \)-orthogonal decomposition with respect to the natural global \( L^2 \) scalar product on the space of the spinor fields. This decomposition makes it possible to derive a general, manifestly non-negative expression, which on the asymptotically flat/hyperboloidal hypersurfaces coincides with an appropriate null component of the ADM/Bondi–Sachs energy–momentum in the Witten gauge, and on closed hypersurfaces it coincides with the norm of the Sen–Witten operator. Thus, the eigenvalues of this operator are given by the general expression of the total energy of the matter+gravity systems appearing in Witten’s positive energy proof. The quantity \( M \) above is defined to be the infimum of this general expression on the space of spinor fields satisfying appropriate boundary and normalization conditions. In vacuum the same \( M \) trivially gives a lower bound for the eigenvalues of the 3-surface twistor operator.

As examples, we calculate the first eigenvalue of the Sen–Witten operator on a \( t = \text{const} \) hypersurface of the \( k = 1 \) Friedmann–Robertson–Walker as well as in the spatially closed Bianchi I cosmological spacetimes. The corresponding eigenspinor is nowhere vanishing.
This suggests a possible generalization of Witten’s gauge condition, which is a modification of Nester’s condition: the spinor field should be the eigenspinor of the Sen–Witten operator with the smallest non-negative eigenvalue.

In a mathematically complete analysis of the spectral properties of the Sen–Witten and the 3-surface twistor operators we should clarify some of their functional analytic properties. Since the Sen–Witten operator is elliptic, the general theorems and results in the theory of elliptic p.d.e. could be applied to it. However, the 3-surface twistor operator is only overdetermined elliptic, and hence these theorems cannot be applied to it directly. Also, the Sen–Witten operator acting on Dirac spinors is self-adjoint (see e.g. [20]), but it is not when acts on Weyl spinors. Thus, care is needed in using results in elliptic p.d.e. theory that are also based on the self-adjointness of the elliptic operator. Hence, to have a solid functional analytic ground of our investigations, we must carry out such a systematic analysis. The key observation is that a fundamental estimate for the 3-surface twistor operator can be proven, even though it is not elliptic. This is a consequence of the Sen–Witten identity, Einstein’s equations and the dominant energy condition, i.e. a consequence of our physical assumptions.

In section 2, we review the necessary geometrical background, in particular the Sen connection and the Sen–Witten identity in the form that we use. In section 3, first we derive our fundamental identity for the norm of the derivatives of spinor fields. Then we show that both the ADM and Bondi–Sachs energies can be expressed by the norm of the 3-surface twistor operator and the energy–momentum tensor of the matter fields, and by this expression we introduce a non-negative lower bound $M$ for them. In closed universes the same expression for $M$ (but with different normalization conditions) is suggested as the total mass. It is shown that $M = 0$ is the necessary and sufficient condition of the existence of a non-trivial solution of Witten’s gauge condition, and that this happens precisely when the spacetime is flat with toroidal spatial topology.

Section 4 is devoted to the eigenvalue problem of the Sen–Witten operator. First we discuss the potential difficulties in defining the eigenvalue problem for the Sen–Witten operator acting on Weyl spinors. Then we show that the first eigenvalue is given by $M$, and we discuss how this expression is related to the previously given lower bounds for the eigenvalues. We conclude this section with a remark on the eigenvalue problem for the 3-surface twistor operator and the examples.

The analysis of the mathematical properties of the Sen–Witten and 3-surface twistor operators, acting on Weyl spinors, is given in the appendix. Here we worked in classical Sobolev spaces over closed data sets, but most of the results seem to extend to appropriate weighted Sobolev spaces (over asymptotically flat/hyperboloidal data sets), as well as to the manifold-with-boundary case when the spinor fields are subject to non-trivial (chiral or APS) boundary conditions. Since the notations and the formalism of this analysis are the usual ones in general relativity (rather than in the p.d.e. theory), we hope that this appendix makes the functional analytic techniques available for a wider readership in the general relativity community.

We use the abstract index formalism, and only the boldface indices take numerical values. We adopt the sign conventions of [24]. In particular, the signature of the spacetime metric is $(+,-,-,-)$, and the curvature and Ricci tensors and the curvature scalar are defined by $\mathring{4}R_{\mu\nu\rho\sigma}X^\rho := -\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right)X^\rho$, $\mathring{4}R_{\mu\nu} := \mathring{4}R^\rho_{\mu\nu\rho}$ and $\mathring{4}R := \mathring{4}R_{\mu\nu\rho\sigma}g^{\mu\nu}$. Then Einstein’s equations take the form $\mathring{4}G_{ab} = -\kappa T_{ab}$, where $\kappa := 8\pi G$ with Newton’s gravitational constant $G$. Every manifold and all the geometric structures will be assumed to be smooth.
2. Geometrical preliminaries

2.1. Metrics on bundles over $\Sigma$

Let $\Sigma$ be a smooth orientable spacelike hypersurface, $t^a$ its future-pointing unit normal, and define $P_a^b := \delta_a^b - t^a t_b$. This is the orthogonal projection to $\Sigma$, by means of which the induced (negative definite) 3-metric is defined by $h_{ab} := P_a^c P_b^d g_{cd}$. We assume that the spacetime is space and time orientable, at least on an open neighbourhood of $\Sigma$, in which case $t^a$ can be (and, in what follows, will be) chosen to be globally defined.

Let $\mathcal{V}^a(\Sigma)$ denote the pull back to $\Sigma$ of the spacetime tangent bundle, which decomposes in a unique way into the $g_{ab}$-orthogonal direct sum of the tangent bundle $T \Sigma$ and the normal bundle of $\Sigma$ spanned by $t^a$. $g_{ab}$ is a Lorentzian fibre metric on $\mathcal{V}^a(\Sigma)$, and we call the triple $(\mathcal{V}^a(\Sigma), g_{ab}, P_a^b)$ the Lorentzian vector bundle over $\Sigma$. It is the projection $P_a^b$, as a base point preserving bundle map of $\mathcal{V}^a(\Sigma)$ to itself, which tells us how the tangent bundle $T \Sigma$ is embedded in $\mathcal{V}^a(\Sigma)$. Since both $T \Sigma$ and the normal bundle of $\Sigma$ in $M$ are globally trivializable, $\mathcal{V}^a(\Sigma)$ is also globally trivializable. This implies the existence of a spinor structure also. In general, there might be inequivalent spinor structures on $\mathcal{V}^a(\Sigma)$, labelled by the elements of the first cohomology group of $\Sigma$ with $\mathbb{Z}_2$ coefficients. Let us fix such a spinor structure, and let $\mathcal{S}^A(\Sigma)$ denote the bundle of two-component (i.e. Weyl) spinors over $\Sigma$. We denote the complex conjugate bundle by $\mathcal{S}^A(\Sigma)$. As is usual in general relativity (see e.g. [24]), we identify the Hermitian sub-bundle of $\mathcal{S}^A(\Sigma) \otimes \bar{\mathcal{S}}^A(\Sigma)$ with $\mathcal{V}^a(\Sigma)$. Thus, we can convert tensor indices to pairs of spinor indices and back freely.

On the spinor bundle two metrics are defined: the first is the natural symplectic metric $\varepsilon_{AB}$, while the other is the positive definite Hermitian metric $G_{AB} := \sqrt{2} \varepsilon_{AB}$. (The reason of the factor $\sqrt{2}$ is that for this definition $G^{AB}$, the inverse of $G_{AB}$ defined by $G^{AB} G_{BC} = \delta^A_B$ is just the contravariant form $\varepsilon^{AC} e^{BD} G_{CD}$ of the Hermitian metric, i.e. the Hermitian and the symplectic metrics are compatible.) The Hermitian metric defines the $\mathbb{C}$-linear bundle isomorphisms $\bar{\mathcal{S}}^A(\Sigma) \rightarrow \mathcal{S}^A(\Sigma) : \bar{\lambda}^A \mapsto -G^{AB} \bar{\lambda}^B$ and $\mathcal{S}^A(\Sigma) \rightarrow \mathcal{S}_A(\Sigma) : \lambda^A \mapsto G_A^A \lambda^A$, as well as
\begin{equation}
\langle \lambda_A, \phi_A \rangle := \int_{\Sigma} G^{AB} \lambda_A \phi_A d\Sigma,
\end{equation}
which is a global $L_2$ scalar product on the space $L_2(\Sigma, \mathcal{S}_A)$ of the (square integrable) spinor fields on $\Sigma$. This defines an $L_2$ norm in the standard way: $\|\lambda_A\|_{L_2} := \langle \lambda_A, \lambda_A \rangle$. The scalar product (2.1) and the corresponding norm extend in an obvious way to spinor/tensor fields on $\Sigma$ with an arbitrary index structure. The conventions above ensure that the $L_2$-norm of a spatial tensor field, say $\|T_{ab} \|_{L_2}$, coincides with that of its spinor form $\|T_{AA}^{BB} \|_{L_2}$.

2.2. The Sen connection

The intrinsic Levi-Civita covariant derivative operator, defined on $T \Sigma$, will be denoted by $D_a$. This will be extended to the whole $\mathcal{V}^a(\Sigma)$ by requiring $D_a t_b = 0$. We introduce another connection on $\mathcal{V}^a(\Sigma)$, the so-called Sen connection [23], by $D_a := P_a^c \nabla_b$. Clearly, both $D_a$ and $D_a$ annihilate the fibre metric $g_{ab}$, but the projection is annihilated only by $D_a$. ($D_a$ is a reduction of $D_a$, and the reduction is made by requiring that the projection be annihilated by the covariant derivative operator $D_a$.) The extrinsic curvature of $\Sigma$ in $M$ is $\chi_{ab} := D_a t_b = \chi_{(ab)}$. In terms of $D_a$ and the extrinsic curvature the action of the Sen derivative on an arbitrary cross section $X^a$ of $\mathcal{V}^a(\Sigma)$ is given by
\begin{equation}
D_a X^a = D_a X^a + (\chi_a - t^a \chi_b) X^b.
\end{equation}
The curvature of $D_\Sigma$ is defined by the convention $-F^u_{bcd}v^c w^d := v^c D_b (w^d D_c X^u) - w^d D_c (v^c D_b X^u)$ for any cross section $X^u$ of $\Sigma$ and any $v^c$ and $w^d$ tangent to $\Sigma$. This is just the pull back to $\Sigma$ of the spacetime curvature 2-form, $F^u_{bcd} = 4R^u_{bej}p^e_p^f$, and it can be re-expressed as

$$F_{abcd} = R_{abcd} + \chi_{ac} \chi_{bd} - \chi_{ad} \chi_{bc} + t_a(D_c \chi_{db} - D_d \chi_{cb}) - t_b(D_c \chi_{da} - D_d \chi_{ca}),$$

where $R_{abcd}$ is the curvature tensor of the intrinsic geometry of $(\Sigma, h_{ab})$.

$D_\Sigma$ extends in a natural way to the spinor bundle, and its action on a spinor field is

$$D_\Sigma \lambda = D_\Sigma \lambda - \chi_{aa} \lambda B^B.$$

(4.1)

The commutator of two Sen operators acting on the spinor field $\lambda^A$ is

$$(D_\Sigma D_d - D_d D_\Sigma) \lambda^A = -F^A_{Bcd} \lambda^B - 2 \chi^e [\epsilon_d] D_e \lambda^A,$$

(2.3)

where the curvature $F^A_{Bcd}$ is just the pull back to $\Sigma$ of the anti-self-dual part $4R^A_{Bcd}$ of the spacetime curvature 2-form, which can also be expressed by the (spinor form of the) intrinsic curvature and the extrinsic curvature.

The Sen–Witten operator, i.e. the Dirac operator built from the Sen connection, is defined to be $D : C^\infty(\Sigma, S^A) \to C^\infty(\Sigma, S^A)$ : $\lambda^A \mapsto D_\Sigma \lambda^A$, where e.g. $C^\infty(\Sigma, S^A)$ denotes the space of the smooth unprimed, contravariant spinor fields on $\Sigma$. Since

$$\langle D_\Sigma \lambda^A, \bar{\phi}_B \rangle = \int_\Sigma D_{AB} \lambda^A G^{AB} \bar{\phi}_B \, d\Sigma + \int_\Sigma \lambda^A G_{AA} (D^{AB} \bar{\phi}_B) \, d\Sigma,$$

the formal adjoint of $D$ is $D^* : C^\infty(\Sigma, \bar{S}_A) \to C^\infty(\Sigma, S^A)$ : $\bar{\phi}_A \mapsto D\lambda^A \bar{\phi}_A$. Thus, writing the latter as $D^* : \bar{\phi}^A \mapsto -D_A \bar{\phi}^A$, we see that $D^*$ is $-1$ times the complex conjugate of $D$. Therefore, for closed $\Sigma$ (or on the space of the spinor fields for which the first integral on the right is vanishing), both $D^* D : \lambda^A \mapsto D\lambda^A D\lambda^B$ and $D D^* : \phi_A \mapsto D_{AB} D^* \phi_B$ are formally self-adjoint and they are essentially complex conjugate of each other. Moreover, since

$$\langle D_{AB} \lambda^A, \bar{\phi}_B \rangle = \int_\Sigma G_{AB} (D^*_B \bar{\phi}_B) (D^*_B \lambda^B) \, d\Sigma + \int_\Sigma G_{AB} \left( (D^*_B \lambda^B) G^{AB} \bar{\phi}_B \right) \, d\Sigma,$$

(2.6)

e.g. for closed $\Sigma$ the operator $D^* D$ is positive: $(D_{AB} \lambda^A, \lambda^C) \geq 0$ for every spinor field $\lambda^A$. Note, however, that while $D^* D : C^\infty(\Sigma, S^A) \to C^\infty(\Sigma, S^A)$ can be extended to be a self-adjoint operator on an appropriate subspace of $H_{1}(\Sigma, S^A)$, where $H_{k}(\Sigma, S^A)$, $k \in \mathbb{N}$, is the $k$th Sobolev space, in subsection 4.1.1 we will see that $D$ (or, more precisely, $iD$) is not self-adjoint in the strict sense. It yields a self-adjoint operator only on the bundle of the Dirac spinors. Thus, $D$ could be considered to be self-adjoint on the Weyl spinors in some generalized sense.

2.3. The Sen–Witten identity

Using commutator (2.5), the square of the Sen–Witten operator can be written as

$$D_{AB} D_{BAB} = D_{AA} \lambda A + \frac{1}{2} \epsilon_{AB} \epsilon_{B} D_{B} D_{B} \lambda A + \frac{1}{2} \epsilon_{AB} \epsilon_{B} D_{B} D_{B} \lambda A + \frac{1}{2} D_{q} D_{q} \lambda A = \frac{1}{2} D_{q} D_{q} \lambda A + \frac{1}{2} \epsilon_{AB} \epsilon_{B} D_{B} D_{B} \lambda A + \frac{1}{2} \epsilon_{AB} \epsilon_{B} D_{B} D_{B} \lambda A.\) (7.6)

(2.7)

The last term can also be written as $\chi^A_{AA} \chi^B \epsilon^B B D_{B} \lambda A$. Using (2.3) and the fact that in three dimensions the curvature tensor can be expressed by the metric $h_{ab}$ and the corresponding Ricci tensor and curvature scalar, a straightforward computation yields that

$$\epsilon_{AB} \epsilon_{B} D_{B} D_{B} \lambda A = -\frac{1}{2} \epsilon_{AC} (R + \chi^2 - \chi_{AC} \chi B^B) + (D_{q} \chi^A_{AA} - D_{AA} \chi) \chi^A C.$$

(2.8)
However, the terms on the right-hand side are precisely the constraint parts of the spacetime Einstein tensor:

\[
\frac{1}{2} (R + \chi^2 - \chi_{ab} \chi^{ab}) = -4 G_{ab} \phi^{ab} = \kappa T_{ab} \phi^{ab} =: \kappa \mu,
\]

which we used Einstein's field equations. The right-hand sides of these formulae define the energy density and the spatial momentum density of the matter fields, respectively, seen by the observer \( t^a \). We will assume that the matter fields satisfy the dominant energy condition, i.e. \( \mu^2 \geq |J_\mu | \). Substituting (2.8)–(2.10) into (2.7), we obtain

\[
2 D^A \mathcal{A}^B = D_A D^B \lambda^A = 2 \mathcal{A}^A \mathcal{A}^B \mathcal{A}^B - \frac{1}{2} t^B \mathcal{A} G^{AB} t_B + \frac{1}{2} t^B \mathcal{A} G^{BA} t_A = 2 \mathcal{A}^A \mathcal{A}^B \mathcal{A}^B.
\]

This equation is analogous to the Lichnerowicz identity [15] (or rather an equation called in the mathematical literature a Weitzenböck-type equation): the square of the Dirac operator is expressed in terms of the Laplacian and the curvature, but here \( \mathcal{A}^A \mathcal{A}^B \mathcal{A}^B \) is not simply the scalar curvature, but a genuine tensorial piece of the curvature. If, on the other hand, the extrinsic curvature is vanishing, then \( \mathcal{A} \) reduces to the Levi-Civita \( \mathcal{A} \), and (2.11) reduces to

\[
2 D^A \mathcal{A}^B = D_A D^B \lambda^A = D_A D^B \lambda^A + 2 R \lambda^A,
\]

which is the genuine Lichnerowicz identity for the three-dimensional intrinsic Dirac operator.

Contracting (2.11) with \( t_{AB} \phi^B \) and using the definitions, equation (2.4) and the fact that \( G^A_B G^B_A = \lambda^A \) acts on vectors tangent to \( \Sigma \) as \( -P^B_B \), we obtain

\[
D_{AA} (2 (\mathcal{A}^B B \phi^B D^A \lambda^B) + 2 t^A (D_A \phi^B) (D_{AB} \phi^B)) = D_{AA} (2 D_{AB} \phi^B) - t_{AB} (D_A \phi^B) (D_{AB} \phi^B) = -t_{AB} (D_A \phi^B) (D_{AB} \phi^B) - \frac{1}{2} t^A \mathcal{A} G_{AB} \lambda^B \phi^B.
\]

Writing the total divergences in a different way, we obtain the Reula–Tod (or the SL(3, C) spinor) form [4] of the Sen–Witten identity:

\[
D_{AA} (2 (\mathcal{A}^B B \phi^B D^A \lambda^B) + 2 t^A (D_A \phi^B) (D_{AB} \phi^B)) = -t_{AB} (D_A \phi^B) (D_{AB} \phi^B) - \frac{1}{2} t^A \mathcal{A} G_{AB} \lambda^B \phi^B.
\]

Clearly, its right-hand side is positive definite for \( \phi^A = \lambda^A \) and matter fields satisfying the dominant energy condition.

3. Energy–momentum and gauge conditions

3.1. Gravitational energy–momentum

3.1.1. The key ingredient: the 3-surface twistor operator Using the unitary spinor form

\[
D_{E E} := G^E D^E = D_{E E} (D_E) \qquad \text{of the Sen derivative operator} \quad D_E \quad \text{(see [25, 26]), the decomposition of the derivative} \quad D_{E A} \quad \text{into its irreducible parts is}
\]

\[
G_{E E} D_{E E} \lambda_A = D_{E E} \lambda_A + \frac{1}{2} \epsilon_{E A} D_{E B} \lambda^B + \frac{1}{2} \epsilon_{E A} D_{E B} \lambda^B = D_{E E} \lambda_A + \frac{1}{2} \epsilon_{E A} G_{E K} \lambda^K + \frac{1}{2} \epsilon_{E A} G_{E K} \lambda^K = D_{E E} \lambda_A + \frac{1}{2} G_{E E} (\epsilon_{E A} \delta^K_B - G_{E K} G_{E A}) \lambda^K = D_{E E} \lambda_A + \frac{1}{2} G_{E E} p^K_L \epsilon_{L A} \lambda^K,
\]

where the first term on the right, \( D_{AB} \lambda_C \), is just the 3-surface twistor derivative of the spinor field [8], while the second is essentially the Sen–Witten operator acting on \( \lambda^A \). Indeed, \( D_{AB} \lambda_C = 0 \) is the purely spatial part in the complete irreducible 3+1 decomposition of the 1-valence spacetime twistor equation \( \nabla_{A B} \lambda_C = 0 \). A straightforward calculation shows that
(3.1) is in fact the pointwise orthogonal decomposition with respect to $G_{AA}$, and hence it is $(\cdot, \cdot)$-orthogonal also. Thus, the $L_2$ scalar product of the derivative of two spinor fields is
\[
\langle D_A \lambda^A, D_B \phi^B \rangle = \langle D_{(AB)} \lambda^C, D_{(AB)} \phi^C \rangle + \frac{1}{3} \langle D_{AA} \lambda^A, D_{AB} \phi^B \rangle. \tag{3.2}
\]

Hence, in particular, the integral of identity (2.13) for $\phi^A = \lambda^A$ gives
\[
\| D_{AA} \lambda^A \|_{L_2}^2 = \frac{3}{4} \| D_{(AB)} \lambda^C \|_{L_2}^2 + \frac{3}{4\sqrt{2}} \int_\Sigma t^a T_{aBB} \lambda^B \lambda^B d\Sigma \tag{3.3}
\]
on a closed $\Sigma$.

### 3.1.2. A lower bound for the ADM and Bondi–Sachs masses.

Suppose for a moment that $\Sigma$ is asymptotically flat or asymptotically hyperboloidal and $\lambda^A$ is asymptotically constant or satisfies the asymptotic twistor equation at the infinity (or infinities) of $\Sigma$, respectively. Let $\lambda^A$ denote the asymptotic value of $\lambda^A$ at infinity. Let us choose $\phi^A = \lambda^A$ in (2.13), and let $P(\phi, \lambda)$ be $2/k$ times of the integral of the total divergence on the left-hand side of (2.13), converted to a 2-surface integral on the boundary $\partial \Sigma$ at infinity. Then $P(\phi, \lambda)$ is just the $2$-surface integral of the Nester–Witten $2$-form at infinity, built from $\lambda^A$, and hence is just the $\lambda^A$ component of the ADM or Bondi–Sachs energy–momentum, respectively. Hence, from (2.13), we obtain that
\[
P(\phi, \lambda) + \frac{4\sqrt{2}}{3k} \| D_{AA} \lambda^A \|_{L_2} = \frac{\sqrt{2}}{\kappa} \| D_{(AB)} \lambda^C \|_{L_2}^2 + \int_\Sigma t^a T_{aBB} \lambda^B \lambda^B d\Sigma. \tag{3.4}
\]

This identity is the basis of (probably the simplest) proof of the positivity of the ADM and Bondi–Sachs energies, as well as of infinitely many different quasi-local energy expressions. (For the key ideas and the references in the quasi-local case, see e.g. [9].) The basic idea is that if $\lambda^A$ is chosen to be a solution to the Witten equation $D_{AA} \lambda^A = 0$ and satisfying appropriate boundary conditions, then the second term on the left-hand side is vanishing, and hence $P(\phi, \lambda)$ is the sum of two manifestly positive definite expressions.

The gauge condition of Parker [10], formulated in terms of Dirac spinors, can also be translated into the language of Weyl spinors: since by the dominant energy condition $t^a T_{aBB}$ is a non-negative Hermitian spinor, it has a uniquely determined non-negative Hermitian square root $S_{AA}$ satisfying $G^{AA} S_{AB} S_{EB} = t^a T_{aBB}$. (In fact, there is a normalized spin frame $(a^A, l^A)$ such that $\sqrt{2} a^A = d^A - r^A$, and the vector $\sqrt{2} l^A := d^A + r^A$ is proportional to the spatial momentum density $J_i := t^a T_{aBB} P_i^B$ of the matter fields. In this frame $t^a T_{aBB} = a a_B l_B + b l_B l_B$, where by the dominant energy condition, $a, b \geq 0$. Then $S_{AA} := \sqrt{a a_B} a_A + \sqrt{b} l_A l_A$ is the square root that we need.) Then Parker’s gauge condition is simply $D_{AA} \psi^A + \gamma S_{AA} \psi^A = 0$ for some real constant $\gamma$. Thus, if this equation admits a solution (with given boundary conditions) and the constant $\gamma$ is chosen to be $3k/4\sqrt{2}$, then
\[
P(\phi, \lambda) = \frac{\sqrt{2}}{\kappa} \| D_{AA} \lambda^A \|_{L_2}.
\]
Now the total energy–momentum of the matter+gravity system is represented by the norm of the 3-surface twistor derivative of the spinor field $\psi^A$ alone.

The explicit calculations of Bäckdahl and Valiente-Kroon showed [7] that in asymptotically flat vacuum spacetimes the ADM mass can be given as the $L_2$-norm of the 3-surface twistor derivative of (appropriately decaying) asymptotically constant spinor fields. The discussion above shows that this result can be recovered as a simple consequence of the Sen–Witten identity (in the Witten gauge), even in the presence of matter. More precisely, the component of both the ADM and Bondi–Sachs energy–momentum with respect to any null vector can be written as the sum of the $L_2$-norm of the 3-surface twistor operator and an energy–momentum term. In addition, recalling that the ADM/Bondi–Sachs energy–momentum is
future pointing and timelike (or zero) under the conditions of the positive energy theorems, for the corresponding ADM/Bondi–Sachs mass \( m \) we have the non-negative lower bound

\[
m := \sqrt{E^2 - |P|^2} = \sqrt{(E - |P|)(E + |P|)} \geq E - |P|, \quad \text{where} \ E \ \text{and} \ P^i, \ i = 1, 2, 3, \text{are}
\]

the ADM/Bondi–Sachs energy and linear (spatial) momentum, respectively, in some global Lorentz frame at infinity, and \(|P|^2 := \sqrt{g_{ij}P^iP^j}\), the length of the latter. However, \( E - |P| \) is just the infimum of \( P(\phi\lambda,\bar{\lambda}) \) on the set of the spinors \( \phi\lambda,\bar{\lambda} \) for which \( \phi^\dagger\phi\bar{\lambda}\bar{\lambda} = 1 \), where \( \phi^\dagger\phi \) is the timelike basis (unit) vector of the Lorentz frame, chosen to be orthogonal to \( \Sigma \) at infinity.

Therefore, the infimum of the right-hand side of (3.4),

\[
\mathcal{M} := \inf \left\{ \frac{\sqrt{\gamma}}{\kappa} \| D_{(A\bar{A})} \mathcal{C} \|_{L^2}^2 + \int_{\Sigma} T_{\alpha\beta}^{\gamma\delta} \bar{\lambda}^\alpha \bar{\lambda}^\beta \bar{\lambda}^\gamma \bar{\lambda}^\delta \, d\Sigma \right\},
\]

provides a non-negative lower bound for the ADM/Bondi–Sachs mass \( m \). Here the infimum is taken on the set of spinor fields satisfying the appropriate boundary and normalization conditions at infinity.

### 3.1.3. \( \mathcal{M} \) as the mass of closed universes

Suppose that \( \Sigma \) is a closed spacelike hypersurface, and introduce \( \mathcal{M} \) by (3.5), but now the infimum is taken on the set of all smooth spinor fields \( \lambda_{\bar{A}} \)

e.g. with norm \( \| \lambda_{\bar{A}} \|_{L^2} = 1 \). This provides a measure of the strength of the gravitational ‘field’, and we interpret this as the total mass of the closed universe. This interpretation is supported by the fact that its physical dimension is mass, and that in the asymptotically flat/hyperboloidal case the same formula gives the ADM/Bondi–Sachs energy in the Witten gauge. Moreover, in subsection 3.2 we will show that \( \mathcal{M} \) is strictly positive, i.e. it is vanishing precisely for the trivial data set with toroidal spatial topology. This property of \( \mathcal{M} \) is analogous to the rigidity part of the positive mass theorems in the asymptotically flat/hyperboloidal case where the vanishing of the total mass implies flatness. (Recall that the vanishing of certain spinorial quasi-local mass expressions is equivalent only to \( pp \)-wave Cauchy development with pure radiative matter fields. The flatness is equivalent to the vanishing of the whole energy–momentum 4-vector, and not only of its Lorentzian length [27–29].)

Although the expression between the curly brackets in (3.5) looks like the \( H_1 \)-Sobolev norm (especially if we write the second term in (3.5) as \( \| S_{A\bar{A}} \lambda_{\bar{A}} \|_{L^2}^2 \)), in general it is only a semi-norm. Indeed, as we will see, this ‘norm’ of the Witten spinor on the trivial data set above is vanishing.

Since \( \mathcal{M} \) was introduced as the infimum of a certain semi-norm of smooth spinor fields, it does not follow \textit{a priori} that a smooth spinor field \( \lambda_{\bar{A}} \) with

\[
\| D_{(A\bar{A})} \mathcal{C} \|_{L^2}^2 = \frac{3}{2\sqrt{2}} \mathcal{M} \| \lambda_{\bar{A}} \|_{L^2}^2,
\]

i.e. with \( \mathcal{M} = \frac{3}{2\sqrt{2}} \| D_{(A\bar{A})} \mathcal{C} \|_{L^2}^2 + \| S_{A\bar{A}} \lambda_{\bar{A}} \|_{L^2}^2 \), should exist. However, we show that such a smooth spinor field does exist.

By (3.3) and the definition of \( \mathcal{M} \) it follows that \( \frac{3}{2\sqrt{2}} \| D_{(A\bar{A})} \mathcal{C} \|_{L^2}^2 \geq \mathcal{M} \| \lambda_{\bar{A}} \|_{L^2}^2 \) for any smooth spinor field \( \lambda_{\bar{A}} \). Thus, by the definition of infimum there exists a sequence \( \{ \mathcal{C}_i \} \), \( i \in \mathbb{N} \), of smooth spinor fields for which \( \| \lambda_{\bar{A}} \|_{L^2} = 1 \), and \( \mathcal{C}_i := \frac{3}{2\sqrt{2}} \| D_{A\bar{A}} \mathcal{A} \|_{L^2}^2 \to \mathcal{M} \) as a monotonically decreasing sequence. Since the sequence \( \{ \mathcal{C}_i \} \) is bounded, there exists a positive constant \( K \) such that \( \| D_{A\bar{A}} \mathcal{A} \|_{L^2} \leq K \) for any \( i \in \mathbb{N} \). Thus, by the fundamental elliptic estimate for the Sen–Witten operator (see lemma A.1 in the appendix), we have that

\[
\| D_{A\bar{A}} \mathcal{A} \|_{L^2} \leq \sqrt{2} \| D_{A\bar{A}} \mathcal{C}_i \|_{L^2} + \| \mathcal{A} \|_{L^2} \leq \sqrt{2} K + 1,
\]

i.e. the \( L^2 \)-norm of the derivative of the spinor fields \( \mathcal{C}_i \), \( i \in \mathbb{N} \), as a sequence is bounded with the bound \( 1 + \sqrt{2} K \). However, by this boundedness we have the freedom to deform
the spinor fields such that, for any given \( k \in \mathbb{N} \), the \( L_2 \)-norms of their first \( k \) derivatives are also bounded; i.e. there exists a sequence \( \{ \lambda_i^k \} \), \( i \in \mathbb{N} \), of smooth spinor fields such that \( \| \lambda_i^k \|_{L_2} = 1 \), \( L_i := \frac{4\sqrt{2}}{\lambda_i} \| D_{\alpha A} \lambda_i^k \|_{L_2}^2 \rightarrow \mathcal{M} \) as a monotonically decreasing sequence, and \( \| D_{\alpha i} D_{\beta j} \lambda_i^k \|_{L_2} \leq K_2, \ldots, \| D_{\alpha i} \cdots D_{\beta k} \lambda_i^k \|_{L_2} \leq K_k \) for some positive constants \( K_2, \ldots, K_k \) and for all \( i \in \mathbb{N} \). Again, the convergence \( L_i \rightarrow \mathcal{M} \) implies that \( \| D_{\alpha i} \lambda_i^k \|_{L_2} \leq K_1 \) for some constant \( K_1 > 0 \) and for all \( i \in \mathbb{N} \). Therefore,

\[
\| \lambda_i^k \|_{H_k} \leq 1 + K_1 + \cdots + K_k, \quad \forall i \in \mathbb{N},
\]

i.e. the sequence \( \{ \lambda_i^k \} \) is bounded in the Sobolev space \( H_k(\Sigma, S^4) \). Hence, there is a subsequence \( \{ \lambda_{i_j}^k \}, j \in \mathbb{N} \), which converges to some \( \lambda_{i_0}^k \in H_k(\Sigma, S^4) \) in its weak topology. But since \( \{ \lambda_i^k \} \) is bounded and by the Rellich lemma (see appendix A.2) the injection \( H_k(\Sigma, S^4) \rightarrow L_2(\Sigma, S^4) \) is compact, there is a subsequence \( \{ \lambda_{i_{j_n}}^k \}, n \in \mathbb{N} \), which converges to some \( \lambda_{i_0}^k \in L_2(\Sigma, S^4) \) in the strong topology of \( L_2(\Sigma, S^4) \). Since the strong and the weak limits of a sequence must coincide, we conclude that we can find a subsequence of the sequence \( \{ \lambda_i^k \} \) which converges strongly to some \( \lambda^k := \lambda_{i_0}^k \in H_k(\Sigma, S^4) \). Then by the Sobolev lemma (see appendix A.2), \( H_k(\Sigma, S^4) \subset C^{0,1}(\Sigma, S^4) \) holds, and since \( k \) is arbitrary, the spinor field \( \lambda^k \) is smooth.

Finally, since both the \( L_2 \)-norm \( \| \cdot \|_{L_2} : L_2(\Sigma, S^4) \rightarrow [0, \infty) \) and the Sen–Witten operator \( D : H_1(\Sigma, S^4) \rightarrow L_2(\Sigma, S^4) \) are continuous, moreover \( H_k(\Sigma, S^4) \subset H_1(\Sigma, S^4) \), we have that \( \mathcal{M} = \lim_{n \rightarrow \infty} L_{i_n} = \frac{4\sqrt{2}}{\lambda_{i_n}} \lim_{n \rightarrow \infty} \| D_{\alpha A} \lambda_{i_n}^k \|_{L_2}^2 = \frac{4\sqrt{2}}{\lambda_k} \| D_{\alpha A} \lambda^k \|_{L_2}^2 \) holds. Therefore, by (3.3), this yields \( \mathcal{M} = \frac{2}{\kappa^2} \| D_{AB}\lambda_C \|_{L_2}^2 + \| \tilde{S}_{AB} \lambda^k \|_{L_2}^2 \).

Since \( \lambda^k \) in (3.6) is smooth, we can rewrite that as

\[
\langle 2D^{AB}D_{AB}\lambda^B, -\frac{3}{2}\kappa^2 \mathcal{M} \lambda^A, \lambda^A \rangle = 0. \tag{3.7}
\]

Thus, either \( \lambda^k \) is an eigenspinor of \( 2D^*D \) with the eigenvalue \( \frac{3}{2}\kappa^2 \mathcal{M} \), or \( \lambda^k \) is orthogonal to \( \Delta_{AB}^k \lambda^k \), where, for the sake of brevity, we introduced the operator \( \Delta_{AB}^k := 2D^{AB}D_{AB} - \frac{3}{\kappa^2} \mathcal{M} \lambda^k \). We use (3.7) in subsection 4.2.1 to prove that \( \kappa^2 \mathcal{M} \) is, in fact, the first eigenvalue of the (‘square’ of the) Sen–Witten operator.

### 3.2. On Witten-type gauge conditions in closed universes

#### 3.2.1. The local geometry of closed data sets admitting a Witten spinor.

By the existence of a smooth spinor field \( \lambda^k \) for which (3.6) holds, it is clear that Witten’s gauge condition can be imposed on a closed spacelike hypersurface if and only if \( \mathcal{M} = 0 \). (The existence of spinor fields satisfying Parker’s gauge condition can be characterized in a similar way by inf\( \| D_{\alpha A} \lambda_C \|_{L_2} \), \( \| \lambda_C \|_{L_2} = 1 \) = 0. Thus, the Witten and Parker gauge conditions can be imposed only in special geometries.)

Clearly, if \( (\Sigma, h_{ab}, \chi_{ab}) \) is a data set with flat Sen connection \( D_{ab} \), then the Sen–constant spinor fields solve the Witten equation, and hence for such data sets \( \mathcal{M} = 0 \). In the rest of this subsection, we determine the geometry of those closed data sets \( (\Sigma, h_{ab}, \chi_{ab}) \) which admit non-trivial solutions of the Witten equation, i.e. for which \( \mathcal{M} = 0 \).

Thus, suppose that \( \lambda^k \) is a solution of \( D_{\alpha A} \lambda^k \). Then an immediate consequence of (3.2), (3.3) and the dominant energy condition is that \( \lambda^k \) is constant with respect to \( D_{ab} \) on \( \Sigma \), and that \( r^n T_{ab} \lambda^k \lambda^B = 0 \). Since by the dominant energy condition \( T_{ab} \lambda^k \) must be future pointing and non-spacelike or zero, where \( L^a := \lambda^k \lambda^A \), its orthogonality to the timelike \( r^n \) yields that \( T_{ab} L^a = 0 \). Therefore, the algebraic type of the energy–momentum tensor of the matter fields must be of pure radiation with the wave vector \( L^a \). Thus, it must have the form \( T_{ab} = f L_a L_b \) for some non-negative function \( f \). Therefore, the Ricci spinor and the curvature scalar of the spacetime geometry are \( \Phi_{AB} = \frac{1}{\kappa} \int_{\Sigma} L^a \lambda_B \lambda^A \) and \( \Lambda = 0 \), respectively.
Hence, at the points of $\Sigma$, the anti-self-dual part of the spacetime curvature takes the form 

$$-4R_{ABCD} = \Psi_{ABCD} \epsilon_{CD} + \frac{1}{4} \chi_{\lambda \lambda} \lambda \epsilon_{CD}.$$ 

Since $\lambda^4$ is constant, it does not have any zero on $\Sigma$ (see [23]). Thus, $Z^a := P^a_{\beta} \dot{\beta} \dot{\beta}^B$ is a globally defined, nowhere vanishing vector field on $\Sigma$. If $|Z|^2 := -h_{ab}Z^aZ^b$, the positive definite pointwise norm of $Z^a$, then by $D_{\lambda} \lambda^4 = 0$ and the definitions it follows that

$$D_a Z_b = -|Z| \chi_{ab}.$$  \hspace{1cm} (3.8)$$

Thus, in particular, $Z_a$ is a closed 1-form, i.e. locally it has the form $Z_a = D_a u$ for some (locally defined) real function $u$. Hence, through each point $p \in \Sigma$, there is a maximal integral submanifold $S_a$ orthogonal to $Z^a$, which is a 2-surface in $\Sigma$ given locally by $u = \text{const}$. Let us complete the spinor field $\lambda^A$ to be a spin frame $\{\lambda^A, \dot{\lambda}^A\}$, normalized by $\lambda^A \dot{\lambda}^A = 1$, such that $N^a := t^a \dot{t}^a$ is orthogonal to the 2-surfaces $S_a$ and the complex null vectors $M^a := \lambda^A \dot{\lambda}^A$ and $\bar{M}^a := \bar{\lambda}^A \dot{\lambda}^A$ are tangent to $S_a$. Clearly, $M^a$ and $\bar{M}^a$ satisfy the normalization conditions $M^a M_a = M_a M^a = 0$ and $M_a M^a = 1$, and, if we write $Z^a = -|Z| v^a$, then $L^a = |Z|(t^a - v^a)$ and $2|Z| N^a = t^a + v^a$. Note, however, that while $Z^a$ is globally well defined on $\Sigma$, a priori the vectors $M^a$ and $\bar{M}^a$ are only locally defined.

Next we show that the algebraic type of the Weyl spinor at the points of the hypersurface is constant on $\Sigma$ with respect to $D_a$, the commutator (2.5) gives that $\lambda^A R^A_{\lambda \lambda} P^\ell P^\nu = \lambda^A F^A_{\lambda \lambda} P^\ell = 0$, i.e.

$$\lambda^A R^A_{\lambda \lambda} M^a M^a = \lambda^A R^A_{\lambda \lambda} \bar{M}^a \bar{M}^a = \lambda^A R^A_{\lambda \lambda} M^a \bar{M}^a = \lambda^A R^A_{\lambda \lambda} \bar{M}^a M^a = 0.$$ 

Substituting the above form of $4R_{ABCD} D_a$ here and expressing $Z^a$ in terms of the null vectors $\lambda^A \dot{\lambda}^A$ and $t^a \dot{t}^a$, we find that $\Psi_{ABCD} = \Psi_{\lambda \lambda \lambda \lambda} = 0$ for some complex function $\Psi$ on $\Sigma$. Hence, the Weyl spinor is indeed null and $\lambda^A$ is its fourfold principal spinor.

To determine the local geometry of the surfaces $S_a$, first let us calculate their extrinsic curvature in $\Sigma$. Let us recall that $v^a$ is the unit normal to the 2-surfaces along which the function $u$ is increasing. Substituting $Z^a = -|Z| v^a$ into (3.8), for the extrinsic curvature we find that $v_{ab} := \Pi^a \Pi^b D_a v_b = \chi_{a b} \Pi^a \Pi^b = \tau_{ab}$. Here $\Pi^a := P^a + v^a v_b$, the orthogonal projection to the 2-surfaces, by means of which e.g. the induced metric on $S_a$ is $g_{ab} := h_{a b} \Pi^a \Pi^b = -(M_a M_b + \bar{M}_a \bar{M}_b)$. Thus, the two extrinsic curvatures of the 2-surfaces in the spacetime, the $\tau_{ab}$ corresponding to their spacelike normal $u_a$ and the $\tau_{ab}$ corresponding to the timelike normal $t_a$, coincide.

Next we calculate the scalar curvature $\tilde{\mathcal{R}}$ of the intrinsic metric $g_{ab}$ of the 2-surfaces $S_a$. Since $v_{ab} = \tau_{ab}$, the Gauss equation for $\tilde{\mathcal{R}}$ in the spacetime (see e.g. equation (2.7) in [30]) gives that

$$\tilde{\mathcal{R}} = 4 R_{a b c d} q^{a b} q^{c d} - \tau^{2} + \tau_{a b} \tau^{a b} + \frac{1}{2} v_{a b} v^{a b} = 4 R_{a b c d} q^{a b} q^{c d}$$

Thus, the maximal integral submanifolds $S_a$ are intrinsically locally flat 2-surfaces.

3.2.2. The global topology of $\Sigma$. Let us foliate the spacetime in a neighbourhood of $\Sigma$ by spacelike hypersurfaces obtained from $\Sigma$ by Lie dragging it along its own unit timelike normal $t^a$, and consider the Weyl neutrino equation in its 3+1 form with respect to this foliation:

$$0 = \nabla_{\lambda} \lambda^A = \tau^e (\nabla_e \lambda^A) {\lambda^A} + D_{\lambda} \lambda^A.$$ 

It is known that this equation admits a well-posed initial-value formulation, and hence, for any given initial spinor field $\lambda^A$ on $\Sigma$, it has a unique solution (at least in a neighbourhood $\Sigma \times (-\epsilon, \epsilon)$ of $\Sigma$ for some $\epsilon > 0$). In particular, it has a solution for the initial spinor fields
satisfying Witten’s equation. For such spinor fields \( t' \nabla_s \lambda^A = 0 \) holds, and we show that this spinor field is also constant with respect to the spacetime connection. Since \( \mathcal{D}_{N,A} \lambda^A = 0 \) on \( \Sigma \), implies \( D_\lambda \lambda^A = 0 \), we have that on \( \Sigma \),

\[
\begin{align*}
t' \nabla_s (D_{N,A} \lambda^A) &= t' \nabla_s (-t_{N,A} t') \nabla_s \lambda^A + t' p_{A}^{f} \nabla_s \nabla_s \lambda^A \\
&= t' p_{A}^{f} (\nabla_s \nabla_s - \nabla_s \nabla_s) \lambda^A + t' D_{N,A} (\nabla_s \lambda^A) \\
&= t' p_{A}^{f} (\Psi \lambda^A \lambda^B \lambda^C \lambda^F \epsilon_{EF} + \frac{1}{4} \kappa f \lambda^A \lambda^B \lambda^C \lambda^F \epsilon_{EF}) \lambda^A = 0.
\end{align*}
\]

Thus, the spinor field \( \lambda^A \) satisfies the Witten equation on the neighbouring leaves of the foliation, and hence it is also constant with respect to the Sen connection there. However, this, together with \( t' \nabla_s \lambda^A = 0 \), is equivalent to \( \nabla_s \lambda^A = 0 \).

Since \( \lambda^A \) is constant with respect to the spacetime connection, \( \nabla_s L_0 = 0 \) also holds, i.e. \( L_0 \) is a constant null vector field. Thus, the spacetime has a \( pp \)-wave geometry with the wave vector \( L^a \). Hence, \( L^a \) is the tangent of the null geodesic generators of null hypersurfaces \( \mathcal{L} \). The intersection of these null hypersurfaces with \( \Sigma, \mathcal{S} := \mathcal{L} \cap \Sigma \), gives just the maximal integral submanifolds of the previous subsection. Therefore, the 2-surfaces \( \mathcal{S} \) are globally well-defined closed orientable surfaces. Since the induced metric on these 2-surfaces is locally flat, by the Gauss–Bonnet theorem we obtain that the topology of the 2-surfaces \( \mathcal{S} \) is torus: \( \mathcal{S} \approx S^1 \times S^1 \).

The null hypersurfaces \( \mathcal{L} \) can be labelled locally by the value of the function \( u \) for which \( \mathcal{L} \cap \Sigma = S_u \) holds. This yields an extension of \( u \) from open domains in \( \Sigma \) to open domains in \( \Sigma \times (]-\epsilon, \epsilon[) \). Since \( Z_u = D_u u \) is nowhere vanishing on \( \Sigma \), this \( u \) does not have any critical point, i.e. locally \( u \) provides a parametrization of the global foliation of \( \Sigma \) by the intrinsically flat toroidal 2-surfaces \( S_u \).

We show that \( \Sigma \) is homeomorphic to the 3-torus: \( \Sigma \approx S^1 \times S^1 \times S^1 \). As the first step, we show that it is a fibre bundle over \( S^1 \) with typical fibre \( S^1 \times S^1 \). Let \( B := \Sigma/\mathcal{S}, \mathcal{S} \) the set of the 2-surfaces \( S_u \) in \( \Sigma \), and let \( \pi : \Sigma \to B \) be the natural projection. Thus, the points \([p] \in B, \ p \in \Sigma, \) are equivalence classes of points of \( \Sigma \), where \( p \) and \( q \) are considered to be equivalent if \( p, q \in S_u \) for some \( u \). If \((x^1, x^2)\) are local coordinates in a neighbourhood of \( p \in S_u \), then by the non-vanishing of \( Z^\alpha \) these coordinates can be extended along the integral curves of \( Z^\alpha \) to \( Z^\alpha D_\alpha x^1 = Z^\alpha D_\alpha x^2 = 0 \) onto the neighbouring surfaces. Thus, \((u, x^1, x^2)\) forms a local coordinate system on \( \Sigma \), where \( u \) is a local coordinate on \( B \). Then, in these coordinates the projection \( \pi \) is simply \((u, x^1, x^2) \mapsto u, \) which is clearly smooth, and \( B \) is a one-dimensional manifold. In addition, since \( Z^\alpha \) has no zeros, short enough open intervals in \( B \) are obviously local trivialization domains for \( \Sigma \). Hence, \( \pi : \Sigma \to B \) is a smooth fibre bundle with typical fibre \( S^1 \times S^1 \) and \( B \) is a one-dimensional smooth manifold. Finally, since \( \Sigma \) is compact and \( \pi \) is surjective and continuous, \( B \) must also be compact, i.e. \( B \) is topologically \( S^1 \).

To show that \( \Sigma \), as a bundle, is globally trivial, let us cover the base manifold \( B \approx S^1 \) by two overlapping local trivialization domains. These overlap in two disjoint intervals, say \( U_1 \) and \( U_2 \). The transition function on one, \( \psi_1: U_1 \to \text{Diff}(S^1 \times S^1) \), can be chosen to be the constant map \( u \mapsto \text{Id} \), assigning the identity diffeomorphism of \( S^1 \times S^1 \) to every \( u \in U_1 \).

Then the different bundles are in a one-to-one correspondence with the homotopy classes of the other transition function \( \psi_2: U_2 \to \text{Diff}(S^1 \times S^1) \). Since \( U_2 \) is a connected interval, there are two such homotopy classes: one is the homotopy class of the maps into the orientation preserving component of \( \text{Diff}(S^1 \times S^1) \), while the other is the homotopy class of the maps into the orientation changing component. Since, however, the second yields a bundle whose total space is not orientable while \( \Sigma \) is orientable, we conclude that \( \psi_2: U_2 \to \text{Diff}(S^1 \times S^1) \) is homotopic with the constant, identity map \( u \mapsto \text{Id} \). The corresponding bundle is therefore the globally trivial one: \( \Sigma \approx S^1 \times S^1 \times S^1 \). (For the classification of bundles over \( n \)-spheres, and also for the related ideas, see e.g. [31], section 18, p 96.)
3.2.3. The line element. To determine the form of the spacetime metric (on a neighbourhood of \( \Sigma \)), we can adapt the strategy of \cite{29} (developed originally for the quasi-local case) to the present closed case. The main points of this analysis are as follows:

First, let us combine the surface coordinates \((x^1, x^2)\) into the complex coordinate \(\zeta = \frac{1}{\sqrt{2}}(x^1 - ix^2)\) on the 2-surfaces, and then complete the local spatial coordinate system \((u, \xi, \zeta)\) on \(\Sigma\) to a local coordinate system \((u, \xi, \zeta, v)\) in a neighbourhood of \(\Sigma\). (Since topologically \(\Sigma\) is a 3-torus, the domain of the spatial coordinates is finite, say \(0 \leq x^1, x^2 < 2\pi\) and \(0 \leq u < u_+\).) Here the coordinate \(v\) is the affine parameter along the future-pointing null geodesic generators of the null hypersurfaces \(\mathcal{L}\), measured from \(\Sigma\). In these coordinates \(L^a = g^{ab} \nabla_b \mu = (\frac{\partial}{\partial \zeta})^a\) and \(M^a = (\frac{\partial}{\partial v})^a\), and \(M^a\) and \(M^a\) are already globally defined.

Since \(L^a = (\frac{\partial}{\partial \zeta})^a\) is a Killing vector, neither \(\Psi\) nor \(f\), the only \textit{a priori} not zero components of the curvature, depends on \(v\). In the coordinates \((u, \xi, \zeta, v)\) the line element of the spacetime metric takes the form \(d\mathcal{L}^2 = 2H du^2 + 2 du dv + 2(G d\xi + \bar{G} d\bar{\xi}) du - 2 d\xi d\bar{\xi}\), where \(H = H(u, \xi, \bar{\zeta})\) is a real and \(G = G(u, \xi, \bar{\zeta})\) is a complex function. Both are globally defined on \(\Sigma\). Since \(0 = F^a_{\lambda\beta} u^\lambda = \lambda^A D_\lambda M_a = e^\alpha D_\alpha M_a\), and its right-hand side is just a Christoffel symbol that can be calculated from the line element, it yields that \((\partial G/\partial \xi) = (\partial \bar{G}/\partial \bar{\xi})\).

Recalling that in the complex coordinates the flat Laplacian on \(\Sigma\) is \(\partial^2/\partial \xi \partial \bar{\xi}\), this implies that, apart from a purely \(u\)-dependent additive term, there is a uniquely determined globally defined \textit{real} function \(V = V(u, \xi, \bar{\zeta})\) such that \((\partial^2 V/\partial \xi \partial \bar{\xi}) = -(\partial G/\partial \bar{\xi})\). Therefore, \(G_0 := G + (\partial V/\partial \bar{\xi})\) is a globally defined \textit{anti-holomorphic} function, and hence by Liouville’s theorem it does not depend on \(\xi\) and \(\bar{\xi}\), i.e. \(G_0 = G_0(u)\). (In the language of differential forms \(G = G_0 - (\partial V/\partial \bar{\xi})\) is just the Hodge decomposition of the closed 1-form on \(\Sigma\) with the components \((\sqrt{2} G_0 + \bar{G}), -\sqrt{2}(G - G)\) and an exact form.)

Finally, by the transformation \((u, \xi, \bar{\zeta}) \mapsto (u, \xi, \bar{\zeta}, v + V(u, \xi, \bar{\zeta}))\) of the coordinates, we can change the form of the line element such that in the new form \(G_0\) appears in place of \(G\), and the new \(H\) is also denoted by \(H\). In these new coordinates for the only non-zero components of the Weyl and Ricci spinors, respectively, we have the expressions

\[
\Psi = \frac{\partial^2 H}{\partial \xi^2}, \quad \frac{1}{2} \kappa f = \frac{\partial^2 H}{\partial \xi \partial \bar{\xi}} \tag{3.9}
\]

However, the second is a Poisson equation for \(H\) on the \textit{closed} \(\Sigma\), which can have a solution only if the 2-surface integral of its source term, \(\frac{1}{2} \kappa f\), is vanishing. Thus, by \(f \geq 0\) (i.e. the dominant energy condition) this implies \(f = 0\), and hence \(H\) must be a real harmonic function on \(\Sigma\). Hence, \(H = H(u)\) and, by the first equation of (3.9), \(\Psi = 0\) follows. Therefore, all the \textit{not a priori} vanishing components of the spacetime curvature tensor have also been shown to be vanishing. Thus, the spacetime is locally flat with toroidal spatial topology.

3.3. On Nester’s gauge condition in closed universes

By the results of subsection 3.2, Witten’s gauge condition cannot be imposed in non-flat closed universes. One such alternative condition might be that of Nester [11–13]. (For a recent discussion of the full gauge condition, namely the additional requirement of the non-vanishing of the solutions of this equation, see e.g. [33].) In the present Weyl spinor formalism the equation underlying Nester’s gauge condition is

\[
i G_A^B D_B B \lambda^A = \frac{1}{2\kappa} \beta \lambda_A \tag{3.10}
\]
for some real constant $\beta$. In terms of the Sen connection this takes the form $D_{AA} \lambda^A = \frac{1}{2\sqrt{2}}(\chi - 2i\beta)G_{AA} \lambda^A$. Substituting this into (3.3) and using the definition of $\|$, we obtain that

$$\beta^2 \geq \frac{3}{2\sqrt{2}} \| \chi \| - \frac{1}{4} \sup \{ \chi^2(p) : p \in \Sigma \}.$$

(3.11)

Thus, Nester’s gauge condition can be imposed only for those $\beta$ which satisfy inequality (3.11). However, this gauge condition in the form (3.10) is an eigenvalue problem for the Dirac operator built from the intrinsic Levi-Civita covariant derivative operator $D_e$. Thus, the results on the eigenvalue problem (3.10) can be obtained from the corresponding results for the Sen–Witten operator, derived in the following section, by the formal substitution $\chi_{ab} = 0$.

4. The eigenvalue problem

4.1. The eigenvalue problem for the Sen–Witten operator

4.1.1. An attempt with Weyl spinors. According to the general theory of spinors (see e.g. the appendix of [32]) in three dimensions the spinors have two components; moreover, the Sen–Witten operator maps cross sections of $\mathbb{S}^1(\Sigma)$ to cross sections of the complex conjugate bundle $\bar{\mathcal{S}}_A(\Sigma)$, it seems natural to define the eigenvalue problem by the unitary spinor form [25, 26] of $D_{AA}$ according to

$$iD^A_A \psi_B = -\frac{1}{\sqrt{2}} \beta \psi_A.$$

(4.1)

(The choice for the apparently ad hoc coefficient $-1/\sqrt{2}$ in front of the eigenvalue $\beta$ yields the compatibility with the known standard results in special cases. See also (3.10).) However, it is desirable that the Hermitian metric be compatible with the connection in the sense that $D_e G_{AA} = 0$. Unfortunately, since $D_e G_{AA}$ is $\sqrt{2}$ times the extrinsic curvature of $\Sigma$, in general this requirement cannot be satisfied. As a consequence, in general the eigenvalue $\beta$ is not real. In fact, a straightforward calculation (by elementary integration by parts) gives that

$$\beta \| \psi_A \|^2_{L_2} = \bar{\beta} \| \psi_A \|^2_{L_2} + i \int_{\Sigma} \chi G_{AA} \psi^A \bar{\psi}^A \ d\Sigma + i\sqrt{2} \int_{\Sigma} D_{AA} (\psi^A \bar{\psi}^A) \ d\Sigma.$$

(4.2)

This implies that, even if $\Sigma$ is closed (which will be assumed in the rest of this paper), the imaginary part of $\beta$ is proportional to the integral of the mean curvature $\chi$ weighted by the pointwise norm $G_{AA} \psi^A \bar{\psi}^A$, which is not zero in general. This indicates that the operator $iD^A_A : C^\infty(\Sigma, \mathcal{S}_A) \to C^\infty(\Sigma, \bar{\mathcal{S}}_A)$ is not even formally self-adjoint, and hence it cannot be made self-adjoint in the strict sense. In fact, for any two smooth spinor fields $\lambda_A$ and $\mu_A$ we have

$$i(D^A_A \lambda_B, \mu_A) = (\lambda_A, iD^A_A \mu_B) = \frac{1}{\sqrt{2}} \int_{\Sigma} \chi G_{AA} \lambda^A \bar{\mu}^A \ d\Sigma.$$

4.1.2. The definition with Dirac spinors. This difficulty raises the question whether we can find a slightly different definition of the eigenvalue problem for the Sen–Witten operator yielding real eigenvalues. To motivate this, observe that although the base manifold $\Sigma$ is only three dimensional, the connection $D_e$ is four dimensional in its spirit, as originally it is defined on the Lorentzian vector bundle $\mathbb{V}^e(\Sigma)$. Since its fibres are four dimensional, the corresponding spinors are the four-component Dirac spinors. Hence, we should define the eigenvalue problem for the Sen–Witten operator in terms of the Dirac spinors.

Recall that a Dirac spinor $\Psi^a$ is a pair of Weyl spinors $\lambda^A$ and $\bar{\mu}^\alpha$, written as a column vector

$$\Psi^a = \begin{pmatrix} \lambda^A \\ -\bar{\mu}^\alpha \end{pmatrix}.$$
and adopting the convention $\alpha = A \oplus A', \beta = B \oplus B'$, etc. Its derivative $\mathcal{D}_c \Psi^\alpha$ is the column vector consisting of $\mathcal{D}_c \lambda^A$ and $\mathcal{D}_c \bar{\mu}^A$. If Dirac’s $\gamma$-‘matrices’ are denoted by $\gamma_{c\beta}$, then one can consider the eigenvalue problem

$$i\gamma_{c\beta}^\alpha \mathcal{D}^\alpha \Psi^\beta = \alpha \Psi^\alpha. \quad (4.4)$$

Explicitly, with the representation

$$\gamma_{c\beta}^\alpha = \sqrt{2} \begin{pmatrix} 0 & \varepsilon_E \delta^A_E \\ \varepsilon_E B^A_E & 0 \end{pmatrix} \quad (4.5)$$

(same as e.g. [24], p 221), this is just the pair of equations

$$i\mathcal{D}_A^\lambda \lambda_A = -\frac{\alpha}{\sqrt{2}} \bar{\mu}_A, \quad i\mathcal{D}_A^\lambda \bar{\mu}_A = -\frac{\alpha}{\sqrt{2}} \lambda_A. \quad (4.6)$$

These imply that both the unprimed and the primed Weyl spinor parts of $\Psi^\alpha$ are eigenspinors of $2^D D$ and its complex conjugate, respectively, with the same eigenvalue:

$$2\mathcal{D}^A \mathcal{D}_A B^\lambda = \alpha^2 \lambda^A, \quad 2\mathcal{D}^A \mathcal{D}_A B^\lambda = \alpha^2 \bar{\mu}^A. \quad (4.7)$$

Then by (2.6) $0 \leq 2 \|\mathcal{D}^A \mathcal{D}_A B^\lambda \| \leq \alpha^2 \|\lambda^A\| I_A$, i.e. the eigenvalues $\alpha$ are real. Conversely, the pair ($\alpha^2$, $\lambda^A$) is a solution of the eigenvalue problem for $2^D D$ with non-zero real $\alpha$, then $\pm(\alpha, \Psi^\lambda)$ is a Dirac eigenspinor with the eigenvalue $\alpha$, where $\bar{\mu}^A := \mp i\sqrt{2/\alpha})i\mathcal{D}^A \lambda_A$ are solutions of the eigenvalue problem (4.4).

By (4.6) $\Psi^\lambda = (\lambda^A, \bar{\mu}^A)$ is a Dirac eigenspinor with the eigenvalue $\alpha$ precisely when $(\lambda^A, -\bar{\mu}^A)$ is a Dirac eigenspinor with the eigenvalue $-\alpha$. In the language of Dirac spinors this is formulated in terms of the chirality, represented by the so-called $\gamma_5$-matrix, denoted here by

$$\eta^\alpha_\beta := \frac{1}{4!} \varepsilon_{abcd} \gamma_{abc} \gamma_\beta \gamma_\alpha = i \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^\alpha_\beta \end{pmatrix} \quad (4.8)$$

See appendix II of [32]. Since this is anti-commuting with $\gamma_{c\beta}^\alpha$, from (4.4) we obtain that $i\gamma_{c\beta}^\alpha \mathcal{D}^\alpha (\eta^\alpha_\beta \Psi^\beta) = -\alpha (\eta^\alpha_\beta \Psi^\beta)$. Thus, if $\Psi^\alpha$ is a Dirac eigenspinor with eigenvalue $\alpha$, then, in fact, $\eta^\alpha_\beta \Psi^\beta$ is a Dirac eigenspinor with the eigenvalue $-\alpha$. On the other hand, if there are Dirac eigenspinors with definite chirality, then they belong to the kernel of the Sen–Witten operator. Indeed, Dirac spinors with definite chirality have the structure either $(\lambda^A, 0)$ or $(0, \bar{\mu}^A)$, which, by (4.6), yields that $\mathcal{D}^A \lambda^A = 0$ or $\mathcal{D}^A \bar{\mu}^A = 0$, respectively. Therefore, this notion of chirality cannot be used to decompose the space of the eigenspinors with the given eigenvalue. Its role is simply to take a Dirac eigenspinor with the eigenvalue $\alpha$ to a Dirac eigenspinor with the eigenvalue $-\alpha$.

By the reality of the eigenvalues both the unprimed spinor part $\lambda^A$ and the complex conjugate of the primed spinor part $\bar{\mu}^A$ of $\Psi^\alpha$ are eigenspinors of $2^D D$ with the same eigenvalue $\alpha$. This raises the question if the eigenvalue problem can be restricted by $\lambda^A = \mu^A$, i.e. by requiring the Dirac eigenspinors $\Psi^\alpha$ to be Majorana spinors also. However, (4.6) implies that in this case $\alpha$ would have to be purely imaginary or zero, i.e. the Sen–Witten operator does not have genuine, non-trivial Majorana eigenspinors.

Finally, we consider the special case in which the extrinsic curvature is vanishing. In this case $\mathcal{D}_c = \mathcal{D}_e$, and let us consider the eigenvalue problem defined by (4.1). Then $i\mathcal{G}^A_B \mathcal{D}_A^B (i\mathcal{G}^B_C \mathcal{D}_B^C \psi_C) = \frac{1}{2} \beta^2 \psi_A$. However, by $\mathcal{D}_e \mathcal{G}_{AA} = 0$ we can write

$$\beta^2 \psi_A = -2 \mathcal{G}^A_B \mathcal{G}^B_C \mathcal{D}_A^B (D_B^C \psi_C) = -2 \mathcal{G}^A_B \mathcal{G}^B_C \mathcal{D}_B^C \psi_C = -2 \mathcal{G}_{AA} \mathcal{G}^A_B \mathcal{D}_B^C \psi_C = -2 \mathcal{D}_A^A \mathcal{D}_A^B \psi_B. \quad (4.9)$$

Thus, the pair $(\beta, \psi_A)$ is a solution of the eigenvalue problem for $2^D D$, and hence we may write $\beta = \alpha$ and $\psi_A = \lambda_A$. Then $\alpha \mu_A = -i \sqrt{2} \mathcal{D}_A^A \lambda_A = i \sqrt{2} \mathcal{G}^A_B \mathcal{G}_B^C \mathcal{D}_A^B \phi_B = i \sqrt{2} \mathcal{G}^A_B (\frac{1}{2} i \alpha \lambda_A) = \alpha \mathcal{G}_{AA} \lambda_A$, i.e. the primed spinor part $\bar{\mu}_A$ of the Dirac eigenspinor
just \( G_{M} \lambda^{A} \). Hence, in the special case of the vanishing extrinsic curvature, the eigenvalue problems (4.1) and (4.4) coincide.

Therefore, in the general case, there is no way to reduce the eigenvalue problem (4.4) to a simpler one with a first-order operator and only with a single Weyl spinor. We must study either the eigenvalue problem with the first-order Sen–Witten operator but with Dirac spinors, or with a single Weyl spinor but with the second-order operator \( D^{*} \). In this paper we choose the second strategy, and, for the sake of completeness, in the appendix we prove and summarize the key theorems on the functional analytic properties of \( D^{*} \). In particular, \( D^{*} \) is a non-negative, self-adjoint operator with a pure discrete spectrum.

4.2. Lower bounds for the eigenvalues of the Sen–Witten and 3-surface twistor operators

4.2.1. \( \lambda \) as the first eigenvalue of \( D^{*} \). Suppose that \( \lambda^{A} \) is an eigenspinor of \( 2D^{*} \) with the eigenvalue \( \alpha^{2} \). Then since we assumed that \( \Sigma \) is closed, (2.6), (2.13) and (4.7) yield that

\[
\alpha^{2} \| \lambda^{A} \|_{L^{2}}^{2} = \| D_{\nu} \lambda^{A} \|_{L^{2}}^{2} + \frac{\kappa}{\sqrt{2}} \int_{\Sigma} t^{a} T_{abbb} \lambda^{B} \bar{\lambda}^{B} \ d\Sigma,
\]

implying a lower bound for the eigenvalue \( \alpha^{2} \):

\[
\alpha^{2} \geq \frac{\kappa}{\sqrt{2} \| \lambda^{A} \|_{L^{2}}^{2}} \int_{\Sigma} t^{a} T_{abbb} \lambda^{B} \bar{\lambda}^{B} \ d\Sigma \geq \frac{1}{2} \kappa \inf_{\Sigma} \int_{\Sigma} t^{a} T_{abbb} \lambda^{B} \bar{\lambda}^{B} \ d\Sigma.
\]

Here the infimum is taken on the set of the smooth, future-pointing null vector fields \( l^{\nu} \) on \( \Sigma \). However, this bound is certainly not sharp: in the special case of the vanishing extrinsic curvature the nominator is the integral of \( \frac{1}{4} R_{\mu \nu} l^{\mu} \) (see equations (2.9) and (2.10)), yielding Lichnerowicz's bound \( \frac{1}{4} \inf \{ R(n) \} \) instead of Friedrich's sharp bound \( \frac{3}{8} \inf \{ R(n) \} \).

To find the sharp bound, let us use (3.2) with \( \phi^{A} = \lambda^{A} \) in (4.9). We obtain

\[
\alpha^{2} \| \lambda^{A} \|_{L^{2}}^{2} = \frac{3}{2} \| D_{(AB)C} \lambda^{C} \|_{L^{2}}^{2} + \frac{3}{2} \kappa \int_{\Sigma} t^{a} T_{abbb} \lambda^{B} \bar{\lambda}^{B} \ d\Sigma,
\]

from which the lower bound

\[
\alpha^{2} \geq \frac{3}{2} \kappa \inf_{\Sigma} \int_{\Sigma} t^{a} T_{abbb} \lambda^{B} \bar{\lambda}^{B} \ d\Sigma
\]

follows. In the special case of the vanishing extrinsic curvature the expression on the right is not less than Friedrich’s sharp lower bound, and hence \( \mathfrak{M} \) also provides a sharp lower bound for the eigenvalues.

However, now we show that \( \frac{3}{2} \kappa \mathfrak{M} \) is not only a lower bound, but it is just the smallest eigenvalue of \( 2D^{*} \). In fact, since \( 2D^{*} \) has a purely discrete spectrum with eigenvalues \( \alpha^{2} \), the corresponding eigenspinors \( \lambda^{A} \) form a basis in \( L_{2}(\Sigma, S^{A}) \). (The different eigenspinors corresponding to the same eigenvalue with higher multiplicity are chosen to be orthogonal to each other. See appendix A.5.) Thus, if \( \lambda^{A} \) is a smooth spinor field that satisfies (3.6) (or, equivalently, (3.7)), then we can write \( \lambda^{A} = \sum_{c \in [0, \infty)} c_{a}^{A} \lambda^{A} \) for some complex coefficients \( c_{a}^{A} \). Substituting this form of \( \lambda^{A} \) into (3.7), we obtain

\[
0 = \langle \Delta^{A} \rho^{B} \bar{\lambda}^{B}, \lambda^{A} \rangle = \sum_{c \in [0, \infty)} |c_{a}^{A}|^{2} \left( \alpha^{2} - \frac{3}{2} \kappa \mathfrak{M} \right) \| \lambda^{A} \|_{L^{2}}^{2}.
\]

Since by (4.11) \( \alpha^{2} \geq \frac{3}{2} \kappa \mathfrak{M} \) holds for all eigenvalues \( \alpha^{2} \), this implies that \( \frac{3}{2} \kappa \mathfrak{M} \) is just the smallest eigenvalue and \( \lambda^{A} \) is a corresponding eigenvalue; otherwise \( \lambda^{A} \) would have to be vanishing.
The bound on the right-hand side of (4.11) has been given in [22], though the line of derivation was different. Here we show how the bound of [22] can be obtained in the present formalism. The key object is the one-parameter family of differential operators

$$\tilde{T}_s \lambda^A := D_s \lambda^A + s P^A_{\alpha} D_{\alpha} \lambda^B,$$

(4.12)

labelled by the real parameter $s$. Then a direct calculation gives

$$\| \tilde{T}_s \lambda^A \|_{L^2}^2 - 2s \left( 1 + \frac{3}{4} \right) \| D_{\alpha} \lambda^A \|_{L^2}^2 = \| D_{\alpha} \lambda^A \|_{L^2}^2,$$

by means of which from (4.9) we obtain

$$\left( 1 + s + \frac{3}{4} \right) \alpha^2 \| \lambda^A \|_{L^2}^2 = \| \tilde{T}_s \lambda^A \|_{L^2}^2 + \frac{\kappa}{2 \sqrt{2}} \int_\Sigma t^z T_{\alpha \beta \gamma} \lambda^\alpha B^\beta B^\gamma \mathrm{d} \Sigma.$$

(4.13)

Thus, the norm $\| \tilde{T}_s \lambda^A \|_{L^2}$ as a function of the parameter $s$ on the right has a minimum precisely when $\left( 1 + s + \frac{3}{4} \right)$ has, i.e. at $s = -\frac{3}{4}$. With this substitution (4.13) gives (4.10), and hence the bound (4.11). Indeed, the $L_2$-norm of $\tilde{T}_s \lambda^A$ with the critical value of the parameter, $s = -\frac{3}{4}$, is just the $L_2$-norm of the 3-surface twistor operator: $\| \tilde{T}_s \lambda^A \|_{L^2} = \| D_{(AB)C} \lambda_C \|_{L^2}$. Therefore, the role of the parameter $s$ in (4.12) is to change the Sen–Witten part in the $\langle \cdot, \cdot \rangle$-orthogonal decomposition (3.1) of the Sen derivative $\tilde{T}_s \lambda^A$. For the critical value, $s = -\frac{3}{4}$, the ‘Sen–Witten content’ of $\tilde{T}_s$ is zero, i.e. the Sen–Witten operator built from $\tilde{T}_s$ is vanishing: $\tilde{T}_s \lambda^A = 0$ for any spinor field $\lambda^A$.

4.2.2. On the eigenvalue problem for the 3-surface twistor operator. The formal adjoint $T^* : C^\infty(\Sigma, S_{(ABC)}) \rightarrow C^\infty(\Sigma, S_{\lambda})$ of the 3-surface twistor operator is $T^* : \phi_{ABC} \mapsto + D^{BC} \phi_{ABC}$, where

$$\pm D_{AB} \lambda_C := D_{AB} \lambda_C \pm \frac{1}{\sqrt{2}} \chi_{ABC} D \lambda_D,$$

the self-dual/anti-self-dual Sen connection, and $D_{AB} = - D_{BA}$. Here $D_{AB}$ and $\chi_{ABC}$ are the unitary spinor form of the intrinsic Levi-Civita derivative operator and the extrinsic curvature, respectively, and $C^\infty(\Sigma, S_{(ABC)})$ denotes the space of the totally symmetric three-index spinor fields on $\Sigma$.

Repeating the calculations of subsection 2.2 for the 3-surface twistor operator, it is easy to see that $T^* T$ is formally self-adjoint and positive. (It might be worth noting that the elliptic operator on which the analysis of [7] is based is just $T^* T$.) Moreover, by the definition of the formal adjoint, (3.3) can be rewritten in the form

$$\frac{4\sqrt{2}}{3\kappa} \| D_{AB} \lambda_C \|_{L^2}^2 = \frac{\sqrt{2}}{\kappa} \langle + D^{BC} D_{(AB)C}, \lambda_A \rangle + \int_\Sigma t^z T_{\alpha \beta \gamma} \lambda^\alpha B^\beta B^\gamma \mathrm{d} \Sigma - \frac{\sqrt{2}}{\kappa} + D^{BC} D_{(AB)C} + t^z T_{\alpha \beta \gamma} G_{\alpha \beta} B^\gamma \lambda_B \lambda_A \| \lambda^A \|_{L^2}^2,$$

(4.14)

The functional analytic properties of $T^* T$ are proven in the appendix, and are summarized in appendix A.5: it is a positive, self-adjoint operator with a pure point spectrum. We define the eigenvalue problem for the 3-surface twistor operator by the convention $2^\tau D^{BC} D_{(AB)C} = \tau^2 \lambda_A$, and hence in vacuum $\tau^2 \geq \sqrt{2}\kappa \mathcal{M}$.

However, equation (4.14) suggests the introduction of the operator $\mathcal{M} : \text{Dom}(T^* T) \rightarrow L^2(\Sigma, S_{\lambda})$ defined by the expression in the first argument of the scalar product in the second line of (4.14). Clearly, this is the operator that is ‘behind’ the lower bound $\mathcal{M}$, which is $T^* T$ ‘perturbed’ by a bounded, positive zeroth-order operator. Hence, the key functional analytic
properties of $\mathcal{M}$ are the same as those of $T^*T$. If we define the eigenvalue problem for $\mathcal{M}$ by 
\[ M_A^B \lambda_B = \mu^2 \lambda_A, \]
then by (4.14) and the definition of $\mathcal{M}$
\[ \frac{4\sqrt{2}}{3\kappa} \| D_{\alpha \beta} \lambda^\alpha \|^2_{L^2} = \langle M_A^B \lambda_B, \lambda_A \rangle = \mu^2 \| \lambda_A \|^2_{L^2} \geq \mathcal{M} \| \lambda_A \|^2_{L^2}, \]
and hence $\mu^2 \geq \mathcal{M}$. Choosing $\lambda_A$ to be a spinor field that saturates the inequality on the right-hand side (i.e. that satisfies (3.7)), expanding it in terms of the eigenspinors of $\mathcal{M}$ and repeating the argumentation of the previous subsection, we find that $\mathcal{M}$ is just the smallest eigenvalue of $\mathcal{M}$ and $\lambda_A$ is a corresponding eigenspinor.

4.3. Examples

First, let $\Sigma$ be a $t = \text{const}$ spacelike hypersurface in a $k = 1$ Friedmann–Robertson–Walker cosmological spacetime. It is homeomorphic to $S^3$, the intrinsic metric $h_{ab}$ is the standard 3-sphere metric with scalar curvature $R = \text{const}$ and the extrinsic curvature is $\chi_{ab} = \frac{1}{\chi} h_{ab}$ with $\chi = \text{const}$. For this data set $r^a G_{ab} R^b = 0$ and $-r^a \phi^b G_{ab} = \frac{1}{\chi} R + \frac{1}{\chi} \chi^2 = \text{const}$, and hence the lower bound on the right-hand side of (4.11) is $\frac{1}{\chi} R + \frac{1}{\chi} \chi^2$. On the other hand, we know that this example with $\chi = 0$ saturates the inequality of Friedrich, i.e. the smallest eigenvalue of the (Riemannian) eigenvalue problem $2D^{AA} D_{AB} \lambda^B = \beta^2 \lambda^A$ is just $\beta^2 = \frac{1}{\chi} R$.

We show that the corresponding eigenspinor is also an eigenspinor of $2D^{AA} D_{AB}$, and the corresponding eigenvalue saturates both inequalities of (4.11). In fact, since $\chi = \text{const}$, $2D^{AA} D_{AB} \lambda^B = 2D^{AA} D_{AB} \lambda^B + \frac{1}{\chi} \chi^2 \lambda^A$ holds, and hence for the smallest eigenvalue of $2D^* D$ we obtain $\alpha^2 = \beta^2 + \frac{1}{\chi} \chi^2$, just the lower bound on the right of (4.11). This shows, in particular, that the Witten equation does not have any non-trivial solution. The extrinsic curvature shifted both Friedrich’s lower bound and the smallest Riemannian eigenvalue by the same positive term $\frac{1}{\chi} \chi^2$. It is easy to see that the 3-surface twistor operator annihilates this eigenspinor: since it is annihilated by the Riemannian 3-surface twistor operator and $D_{AB} \lambda^C = D_{AB} \lambda^C + \frac{1}{\chi} \chi \left( 2 \varepsilon_{BC} \lambda_A - \varepsilon_{AB} \lambda^C \right)$ holds, $D_{(AB} \lambda^C) = 0$ follows. Thus, the first eigenvalue of $2D^* D$ coincides with $\frac{1}{\sqrt{2}} \chi \mathcal{M}$. The corresponding eigenspinor has constant components in the spin frame adapted to the globally defined left-invariant orthonormal triad on $\Sigma$, and hence, in particular, it has no zeros.

In the Bianchi I cosmological model with toroidal spatial topology the solutions of $D_{AA} \lambda^A = 0$ are the spatially constant spinor fields. Clearly, these spinor fields also solve $2D^{AA} D_{AB} \lambda^B = \frac{1}{\chi} \chi^2 \lambda^A$, i.e. the smallest eigenvalue of $2D^* D$ is $\frac{1}{\chi} \chi^2$. Thus, in particular, the Witten equation has a non-trivial solution precisely when $\chi = 0$. Since by the Hamiltonian constraint (2.9) we have that $\chi^2 = 2\kappa \mu + \chi_{ab} \chi^{ab}$, this, together with the dominant energy condition, implies $\chi_{ab} = 0$ and $\mu = 0$, and hence the flatness of the spacetime. (For a review of the various Bianchi cosmological models, see e.g. [34].)

The general results of section 4 and the specific properties of the eigenspinors discussed in these examples raise the possibility of generalizing Witten’s gauge condition, which, at the same time, is a modified version of Nester’s gauge condition: let the spinor field be the eigenspinor of the Sen–Witten operator with the smallest non-negative eigenvalue.

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The author is grateful to Florian Beyer, Jörg Frauendiener and Péter Vecsernyés for the discussions on various parts of this paper, and special thanks to Juan A Valiente-Kroon for the discussions on his related results and ideas. This work was partially supported by the Hungarian Scientific Research Fund (OTKA) grant K67790.
Appendix. An analysis of $\mathcal{D}$ and $\mathcal{T}$

In this appendix we recall some elementary properties of the Sen–Witten and 3-surface twistor operators (appendix A.1), quote the key Sobolev embedding theorems (appendix A.2) and derive the estimates (appendix A.3) that we use to prove various properties of the operators $\mathcal{D}, \mathcal{T}, \mathcal{D}^*\mathcal{D}$ and $\mathcal{T}^*\mathcal{T}$ (appendices A.4 and A.5). In particular, since $\mathcal{T}$ is only overdetermined elliptic, the standard results of the theory of elliptic operators cannot be applied to it directly. However, as a consequence of the Sen–Witten identity, Einstein’s equation and the dominant energy condition, we can derive a fundamental estimate for $\mathcal{T}$. As a consequence of this, we can prove that $\mathcal{T}$ shares most of the properties of the elliptic $\mathcal{D}$. In some of these proofs we followed the logic of the proofs of some of the analogous statements for the Dirac operator acting on Dirac spinors given in [20].

A.1. Elementary analytic properties of $\mathcal{D}$ and $\mathcal{T}$

The principal symbol of $\mathcal{D}$ is isomorphism, while the symbol of the 3-surface twistor operator $\mathcal{T}$ is only injective, but not surjective. Thus, while the former is elliptic, the latter is only overdetermined elliptic (see e.g. [35], p 462). By (3.2) both $\mathcal{D}: C^\infty(\Sigma, S^4) \to C^\infty(\Sigma, \bar{S}_A)$ and $\mathcal{T}: C^\infty(\Sigma, S_A) \to C^\infty(\Sigma, \bar{S}_{(ABC)})$ are bounded in the $H_1$-Sobolev norm, defined\(^1\) by $\|A\|_{H_1} := \|\lambda^A\|_{L^2} + \|D\lambda^A\|_{L^2}$. Hence, these operators can be extended in a unique way to be the bounded linear operators $\mathcal{D}: H_1(\Sigma, S^4) \to L_2(\Sigma, \bar{S}_A)$ and $\mathcal{T}: H_1(\Sigma, S_A) \to L_2(\Sigma, \bar{S}_{(ABC)})$, respectively. Since $\mathcal{D}$ annihilates $\varepsilon_{AB}$, it is natural to identify $H_1(\Sigma, S_A^4)$ with $H_1(\Sigma, S_A)$ via $\lambda^A \mapsto \lambda^A_\varepsilon$. On the other hand, since $D_{AA}G_{BB} = \sqrt{2}\varepsilon_{AB}$, it does not seem useful to identify the complex conjugate Sobolev space $H_1(\Sigma, \bar{S}_A)$ with $H_1(\Sigma, \bar{S}_A^4)$ via $G_{AA}$, and hence we keep them different.

In subsection 2.2 we calculated the formal adjoint $\mathcal{D}^*$ of $\mathcal{D}$, and we found that it is essentially ($-1$ times) of the complex conjugate of $\mathcal{D}$ itself. Thus, although $i\mathcal{D}$ appears to be formally self-adjoint at first glance, as we saw in subsection 4.1.1, as a consequence of $D_{AA}G_{BB} = \sqrt{2}\varepsilon_{AB} \neq 0$, strictly speaking it is not even symmetric. The principal symbol of the formal adjoint $\mathcal{T}^*$ of the 3-surface twistor operator (determined in subsection 4.2.2) is only surjective but not injective, and hence it is not elliptic either. It is only underdetermined elliptic.

A.2. The Sobolev embedding theorems

The embedding theorems that we need state how the various function spaces over the same compact domain are related to each other. Since we need only the Sobolev spaces based on the $L_2$ norm, here we concentrate only on these special spaces. An extended discussion of these theorems and the related concepts can be found e.g. in [36, 37].

**Theorem A.1.** Let $E(M)$ be a vector bundle over a compact, $m$-dimensional manifold $M$. Suppose that the space of its cross sections is endowed with a global Hermitian scalar product; we denote the $k$th Sobolev space of its cross sections by $H_k(M, E)$ and adopt the convention $H_0(M, E) := L^2(M, E)$.

\(^1\) Strictly speaking, the usual definition of the Sobolev norms (that we also adopt here) is dimensionally not consistent with the physical view that we can add quantities only with the same physical dimension. In fact, the dimensionally correct definition of the $H_k$-Sobolev norm would be $\|\lambda\|_{L^2} + L \|D\lambda\|_{L^2} + \cdots + L^k \|D_{\varepsilon_1 \cdots \varepsilon_k} \lambda\|_{L^2}$, where $L$ is a positive constant with length physical dimension. Since in classical physics there is no such universal constant, the $H_k$-Sobolev norms for $k \geq 1$ are not canonically defined. Therefore, it is only the $L_2$, but not the $H_k$-Sobolev norms for $k \geq 1$, that can have a physical meaning. Consequently, the operator norm of the (already) bounded operators $\mathcal{D}$ and $\mathcal{T}$ coming from the $H_1$-norm does not seem to have a physical meaning.
Lemma A.2. For that also.

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(1) Let $k \geq 1$, $l > 0$ be integers. Then the injection $i : H_{k+l}(M, E) \to H_{k}(M, E)$ is a dense, compact and continuous embedding (i.e. for any $\phi \in H_{k}(M, E)$ and $\epsilon > 0$ there exists a cross section $\psi \in H_{k+l}(M, E)$ such that $\|\phi - \psi\|_{H_{k}} < \epsilon$, and if $\{\phi_{i}\}$, $i \in \mathbb{N}$, is any sequence in $H_{k+l}(M, E)$ for which $\|\phi_{i}\|_{H_{k+l}} \leq 1$, then there is a subsequence of $\{\phi_{i}\}$ which is convergent in $H_{k}(M, E)$).

(2) The injection $i : H_{k+[\frac{l}{2}]+1}(M, E) \to C^{k}(M, E)$ is a dense, compact and continuous embedding, where $[\frac{l}{2}]$ denotes the integer part of $\frac{1}{2} \dim M$, and the norm on $C^{k}(M, E)$ is the $C^{k}$-supremum norm.

These statements are the Sobolev embedding theorems. The first is analogous to the inclusion of the space of the $C^{k}$ differentiable functions in the space of $C^{l}$ differentiable ones. By the second statement cross sections with $H_{k+[\frac{l}{2}]+1}$ control are $C^{k}$ cross sections in the classical sense. The compactness properties of the embeddings are known as the Rellich lemma.

A.3. Elliptic estimates

We assume that the Einstein equations hold and that the matter fields satisfy the dominant energy condition. The next estimates are proven under these assumptions.

The first of these, the so-called fundamental elliptic estimate, both for the Sen–Witten and the 3-surface twistor operators, are simple consequences of the definitions and equations (3.2) and (3.3).

**Lemma A.1.** There is a positive constant $C$ such that for any $\lambda^{A} \in H_{1}(\Sigma, S^{A})$ the inequalities

$$\|\lambda^{A}\|_{H_{1}} \leq \sqrt{2}\|D_{A}\lambda^{A}\|_{L_{2}} + \|\lambda^{A}\|_{L_{2}}, \quad \|\lambda^{A}\|_{H_{1}} \leq \sqrt{\frac{\kappa}{2}}\|D_{(AB\lambda_{C})}\|_{L_{2}} + C\|\lambda^{A}\|_{L_{2}},$$

(A.1)

hold.

Here the constant $C$ can be given explicitly: it is $C = 1 + \sqrt{\frac{\kappa}{2\dim \Sigma}}$, where $T$ is a positive constant such that $\sigma^{B}T_{BA}$, as a pointwise non-negative Hermitian scalar product on the spinor spaces, is not greater than $T G_{AA}$ everywhere on the compact $\Sigma$. Thus, although the 3-surface twistor operator is not elliptic, it is only overdetermined elliptic, by the Sen–Witten identity, Einstein’s equation and the dominant energy condition we do have an estimate for it. For the sake of simplicity, we also call it a fundamental elliptic estimate.

The second estimate, the so-called elliptic regularity estimate, is, in some sense, a generalization of the previous one. Again, though $T$ is not elliptic, we have an estimate for that also.

**Lemma A.2.** There are positive constants $C_{1}$, $C_{2}$ and $C_{3}$ such that for any $\lambda^{A} \in H_{k}(\Sigma, S^{A})$, $k \geq 1$, for which $D_{A}\lambda^{A} \in H_{k}(\Sigma, S_{A})$ or $D_{(AB\lambda_{C})} \in H_{k}(\Sigma, S_{(ABC)})$ is also true, the inequalities

$$\|\lambda^{A}\|_{H_{k+1}} \leq C_{1}\|D_{A}\lambda^{A}\|_{H_{k}} + C_{1}\|\lambda^{A}\|_{H_{k}}, \quad \|\lambda^{A}\|_{H_{k+1}} \leq C_{2}\|D_{(AB\lambda_{C})}\|_{H_{k}} + C_{3}\|\lambda^{A}\|_{H_{k}},$$

(A.2)

respectively, hold.

**Proof.** Since $\Sigma$ is orientable and three dimensional, it is parallelizable, and hence we can find a globally defined $h_{ab}$-orthonormal dual frame field $(e^{i}, \theta_{i}^{A})$, $i = 1, 2, 3$, on $\Sigma$. Then by the triangle inequality

$$\|D_{f}D_{e_{1}} \cdots D_{e_{t}}\lambda^{A}\|_{L_{2}} = \|D_{f}(\theta_{e_{1}}^{b} \cdots \theta_{e_{t}}^{b} e_{1}^{i} \cdots e_{t}^{i} D_{f_{1}} \cdots D_{f_{l}}\lambda^{A})\|_{L_{2}}$$

$$\leq \|(D_{f}\theta_{e_{1}}^{b})e_{f_{1}}^{i}D_{f_{1}} e_{1}^{i} \cdots e_{t}^{i} D_{f_{l}}\lambda^{A}\|_{L_{2}} + \cdots + \|(D_{f}\theta_{e_{t}}^{b})e_{f_{1}}^{i} \cdots e_{t}^{i} D_{f_{l}}\lambda^{A}\|_{L_{2}} + \|D_{f_{1}} \cdots D_{f_{l}}\lambda^{A}\|_{L_{2}},$$

(A.3)
Since $\Sigma$ is compact, there exists a constant $\tilde{C}_1 > 0$ such that $H^\tilde{\theta} := h^{ab} h^{cd}(D_a \theta^1_j)(D_b \theta^1_i)$, as a quadratic form, is nowhere greater than $\tilde{C}_1^2 \delta^{ij}$ on $\Sigma$. Thus, the first $k$ terms on the right of (A.3) can be estimated in this way to obtain
\[\|D_j D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} \leq k \tilde{C}_1 \|D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} + \left( \sum_{i_1, \ldots, i_k=1}^3 \|D_j (e_1^{i_1} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A)\|_{L_2}^2 \right)^{\frac{1}{2}}, \tag{A.4}\]
where we rewrote the last term also using the definition of the $L_2$ norm and the orthonormality of the dual frame field $\{e_i^a, \theta_a^1\}$. The next step is the use of the fundamental elliptic estimate in the last term on the right of (A.4). Thus, at this point, the detailed proof splits according to the two basic estimates, but the spirit of the proof in the two cases is the same. Here we present the detailed proof only in the case of the 3-surface twistor operator.

Thus, in the last term of (A.4), let us use the estimate
\[\|D_j (e_1^{i_1} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A)\|_{L_2} \leq \frac{k}{\sqrt{2}} \|D_j (e_1^{i_1} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A)\|_{L_2} + \frac{1}{\sqrt{2}} \|\lambda\|_{L_2}, \tag{A.5}\]
coming from (3.2) and (3.3), and where $T$ is given in the text following lemma A.1. Substituting this estimate into the second term on the right of (A.4), using the triangle inequality again and the orthonormality of the frame field $\{e_i^a\}$, we obtain
\[\|D_j D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} \leq \left( k \tilde{C}_1 + \sqrt{\frac{2^k}{2}} \frac{T^{\frac{1}{2}}}{\sqrt{2}} \right) \|D_j \cdots D_{e_k} \lambda^A\|_{L_2} + \frac{1}{\sqrt{6}} \left( \sum_{i_1, \ldots, i_k=1}^3 \|D_j (e_1^{i_1} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A)\|_{L_2}^2 \right)^{\frac{1}{2}}. \tag{A.6}\]

The spinor field in the second term on the right is the totally symmetric part (in the spinor indices) of
\[D_{AB} (e_1^{i_1} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A) = e_1^{i_1} \cdots e_k^{i_k} (D_{AB} D_{e_1} \cdots D_{e_k} \lambda^A) + (D_{AB} e_1^{i_1}) e_1^{i_2} \cdots e_k^{i_k} \times (D_{e_1} \cdots D_{e_k} \lambda^A) + \cdots + (D_{AB} e_k^{i_k}) e_1^{i_1} \cdots e_{k-1}^{i_{k-1}} (D_{e_1} \cdots D_{e_k} \lambda^A). \tag{A.6}\]
The square of the $L_2$-norm of the second term on the right is
\[\|(D_{AB} e_1^{i_1}) e_1^{i_2} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} = \int_{\Sigma} G^{AM} G^{MN} (D_{AB} e_1^{i_1}) (D_{AB} e_1^{i_2}) e_1^{i_3} \cdots e_k^{i_k} \times (D_{e_1} \cdots D_{e_k} \lambda^A) (D_{e_1} \cdots D_{e_k} \lambda^A) G^{CC} d\Sigma. \]
However, there exists a positive constant $\tilde{C}_2$ such that $G^{AM} G^{MN} (D_{AB} e_1^{i_1}) (D_{AB} e_1^{i_2}) e_1^{i_3} \cdots e_k^{i_k} \lambda^A$, as a ‘multi-quadratic form’, is not less than $\tilde{C}_2^2 (-h^c_{ij} \cdots (-h^c_{ij})$ on $\Sigma$. With this bound we have that
\[\|(D_{AB} e_1^{i_1}) e_1^{i_2} \cdots e_k^{i_k} D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} \leq \tilde{C}_2 \|D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2}. \]
Clearly, we have similar estimates for the last $k - 1$ terms on the right-hand side of (A.6) also. Substituting all these into (A.5) and using the triangle inequality, we obtain
\[\|D_j D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} \leq \left( k \tilde{C}_1 + \sqrt{\frac{2^k}{2}} \frac{T^{\frac{1}{2}}}{\sqrt{2}} + \sqrt{\frac{3}{2}} \right) \|D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2} + \sqrt{\frac{3}{2}} \left( \sum_{i_1, \ldots, i_k=1}^3 \|e_1^{i_1} \cdots e_k^{i_k} D_{AB} D_{e_1} \cdots D_{e_k} \lambda^A\|_{L_2}^2 \right)^{\frac{1}{2}}. \]
Lemma A.3. \( k \) such that
\[
\|D_f D_{e_1} \cdots D_{e_k} \lambda^A\|_{L^2} \leq C \|D_f D_{e_1} \cdots D_{e_k} \lambda^A\|_{L^2} + C_2 \|D_{(AB)} D_{e_1} \cdots D_{e_k} \chi\|_{L^2}, 
\]
(A.7)
where we have relabelled the constants.

Finally, let us write the spinor field in the last term of estimate (A.7) as
\[
D_{AB} D_{e_1} \cdots D_{e_k} \lambda^C = (D_{AB} D_{e_1} - D_{e_1} D_{AB}) (D_{e_2} \cdots D_{e_k} \lambda^C) + D_{e_1} (D_{AB} D_{e_2} \cdots D_{e_k} \chi),
\]
as, express the commutator in terms of the curvature \( F_{A B C} \). Then by repeating this substitution, in finite steps we obtain that
\[
D_{(AB)} D_{e_1} \cdots D_{e_k} \chi\|_{L^2}
\]
is the sum of \( D_{e_1} \cdots D_{e_k} (D_{AB} \lambda^C) \) and terms which contain at most the \( k \)th-order derivative of \( \lambda^A \), and the coefficients of the latter are built from the curvature, the extrinsic curvature and their (at most \( (k - 1) \)st order) derivatives. Thus, by the compactness of \( \Sigma \) and the appropriate smoothness of the geometry there exist positive constants \( \tilde{C}_0, \tilde{C}_1, \ldots, \tilde{C}_k \) such that
\[
\|D_{(AB)} D_{e_1} \cdots D_{e_k} \chi\|_{L^2} \leq \tilde{C}_0 \|\lambda^A\|_{L^2} + \tilde{C}_1 \|D \lambda^A\|_{L^2} + \cdots + \tilde{C}_k \|D_{e_1} \cdots D_{e_k} \chi\|_{L^2},
\]
which, together with (A.7) and the definition of the \( H_2 \)-Sobolev norm, yield the desired estimate.

The \( H_1 \)-norm of the spinor fields not belonging to the kernel of \( D \) or \( T \) can also be estimated by the \( L_2 \)-norm of these operators acting on the spinor fields. This lemma is the adaptation of a result of [20], given for the Riemannian Dirac operator acting on Dirac spinors, to the Sen–Witten and 3-surface twistor operators acting on Weyl spinors.

**Lemma A.3.** There are positive constants \( C_0 \) and \( C_0' \) such that the inequalities
\[
\|\lambda^A\|_{H_1} \leq C_0 \|D_{(AB)} \lambda^A\|_{L^2}, \quad \forall \lambda^A \in \ker(D)^\perp \cap H_1(\Sigma, S^A),
\]
(A.8)
\[
\|\lambda^A\|_{H_1} \leq C_0' \|D_{(AB)} \chi\|_{L^2}, \quad \forall \lambda_A \in \ker(T)^\perp \cap H_1(\Sigma, S_A)
\]
(A.9)

**Proof.** The proofs for \( (\ker(D))^\perp \) and \( (\ker(T))^\perp \) are essentially the same; thus, in the present proof \( (\ker(T))^\perp \) may be any of them.

Let us define \( S := \{ \hat{\lambda}^A \in (\ker(D))^\perp \cap H_1(\Sigma, S^A) \mid \|\hat{\lambda}^A\|_{L^2} = 1 \} \), and the function \( F : H_1(\Sigma, S^A) \to [0, \infty) \) to be \( \|D_{(AB)} \hat{\lambda}^A\|_{L^2} \) or \( \|D_{(AB)} \chi\|_{L^2} \) in the two cases, respectively. Since \( D \), \( T \) and the norm \( \|\cdot\|_{L^2} \) are continuous, the function \( F \) is also continuous. First we show that the restriction of this function to \( S \) has a strictly positive minimum, and there is a spinor field \( \hat{\phi}^A \) in \( S \) for which \( F \) takes this minimum value.

Thus, let \( F_0 := \liminf\{ F(\hat{\lambda}^A) \mid \hat{\lambda}^A \in S \} \geq 0 \), and let \( \{\hat{\phi}^A_i\} \), \( i \in \mathbb{N} \), be a sequence in \( S \) such that \( \{F(\hat{\phi}^A_i)\} \) is monotonically decreasing and \( \lim_{i \to \infty} F(\hat{\phi}^A_i) = F_0 \). Since \( \{F(\hat{\phi}^A_i)\} \) is monotonically decreasing, it is bounded: \( \exists K > 0 \) such that \( F(\hat{\phi}^A_i) \leq K \) for all \( i \in \mathbb{N} \). Thus, by the fundamental elliptic estimates for \( D \) and \( T \) (Lemma A.1),
\[
\|\hat{\phi}^A_i\|_{H_1} \leq \sqrt{2} \|D_{(AB)} \hat{\phi}^A_i\|_{L^2} + \|\hat{\phi}^A_i\|_{L^2} \leq \sqrt{2} K + 1,
\]
\[
\|\hat{\phi}^A_i\|_{H_1} \leq \sqrt{2} \|D_{(AB)} \hat{\phi}^A_i\|_{L^2} + C \|\hat{\phi}^A_i\|_{L^2} \leq \sqrt{2} K + C,
\]
where \( C \) is the positive constant of lemma A.1, i.e. \( \{\hat{\phi}^A_i\} \) is a bounded sequence in \( H_1(\Sigma, S^A) \). Thus, there is a subsequence \( \{\hat{\phi}^A_k\} \), \( k \in \mathbb{N} \), of \( \{\hat{\phi}^A_i\} \) which converges in the weak topology of \( H_1(\Sigma, S^A) \) to some \( \hat{\phi}^A \in H_1(\Sigma, S^A) \).
On the other hand, by the Rellich lemma the injection $H_1(\Sigma, S^4) \to L_2(\Sigma, S^4)$ is compact, and hence by the boundedness of the sequence $\{\phi^A_{i_l}\}$ in the $H_1$-norm there is a subsequence $\{\hat{\phi}^A_{i_l}\}$, $l \in \mathbb{N}$, which converges in the $L_2$-norm to some $\hat{\phi}^A \in L_2(\Sigma, S^4)$. However, since the strong and the weak limits must be the same, we obtain that $\{\hat{\phi}^A_{i_l}\}$ converges to some $\hat{\phi}^A := \hat{\phi}^A = \hat{\phi}^A \in H_1(\Sigma, S^4)$ in the $L_2$-norm. Moreover, since the norm is continuous, $\|\hat{\phi}^A\|_{L_2} = \lim_{l \to \infty} \|\hat{\phi}^A_{i_l}\|_{L_2} = 1$. Also, since $\hat{\phi}^A \in (\ker)^{\perp} \cap H_1(\Sigma, S^4)$ and $\hat{\phi}^A \to \hat{\phi}^A$, the spinor field $\hat{\phi}^A$ is orthogonal to $\ker$. Thus, $\hat{\phi}^A \in S$. Finally, by the continuity of $F$, $F(\hat{\phi}^A) = \lim_{l \to \infty} F(\hat{\phi}^A_{i_l}) = F_0$. However, by $\|\hat{\phi}^A\|_{L_2} = 1$ and $\hat{\phi}^A \in (\ker)^{\perp}$, this cannot be zero, i.e. $F_0 > 0$.

Therefore, there exist positive constants $F_0$ and $F_0^*$ such that $0 < F_0 < \|D_A \hat{\lambda}^A\|_{L_2}$ and $0 < F_0^* < \|D_{(ABC)} \hat{\lambda}^A\|_{L_2}$ for any $\hat{\lambda}^A \in S$, i.e. for any spinor field $\lambda^A \in (\ker)^{\perp} \cap H_1(\Sigma, S^4)$ or $\lambda^A \in (\ker(T))^{\perp} \cap H_1(\Sigma, S_A)$, respectively, one has

$$\|\hat{\lambda}^A\|_{L_2} \leq \frac{1}{F_0} \|D_A \hat{\lambda}^A\|_{L_2}, \quad \|\hat{\lambda}^A\|_{L_2} \leq \frac{1}{F_0^*} \|D_{(ABC)} \hat{\lambda}^A\|_{L_2}.$$ Combining these with the corresponding fundamental elliptic estimates we obtain the estimates of the lemma, where $C_0 = \sqrt{2} + 1/F_0$ and $C_0^* = \sqrt{2} + (C/F_0^*)$.

\medskip

\textbf{A.4. Kernels and ranges, domains and Fredholm properties}

First we prove statements on the structure of the kernel and range of the Sen–Witten and Cauchy. Therefore, $\dim \ker \lambda^A \in (\ker)^{\perp} \cap H_1(\Sigma, S^4)$ or $\lambda^A \in (\ker(T))^{\perp} \cap H_1(\Sigma, S_A)$, respectively, one has

\medskip

\textbf{Proposition A.1.} $\ker(D)$ and $\ker(T)$ are finite dimensional and their elements are smooth spinor fields.

\begin{proof}
We prove only $\dim \ker(T) < \infty$; the proof of $\dim \ker(D) < \infty$ is similar. Suppose, on the contrary, that $\ker(T)$ is infinite dimensional, and let $\{\lambda^A_{i_j}\}, i \in \mathbb{N}$, be a sequence in $\ker(T)$ such that $\langle \lambda^A_{i_j}, \lambda^A_{j}\rangle = \delta^i_j$, i.e. for example an $L_2$-orthonormal basis in $\ker(T)$. Then by the fundamental elliptic estimate (lemma A.1) $\|\lambda^A_{i_j}\|_{H_1} \leq C\|\lambda^A_{i_j}\|_{L_2} = C$, i.e. in particular this sequence is bounded in $H_1(\Sigma, S_A)$. Since by the Rellich lemma (theorem A.1(1)) the injection $H_1(\Sigma, S_A) \to L_2(\Sigma, S_A)$ is compact, there is a subsequence $\{\lambda^A_{i_k}\}, k \in \mathbb{N}$, of $\{\lambda^A_{i_j}\}$ which is convergent in $L_2(\Sigma, S_A)$. Hence, it would have to be Cauchy. Since, however, $\|\lambda^A_{i_k} - \lambda^A_{j_k}\|_{L_2}^2 = (\lambda^A_{i_k} - \lambda^A_{j_k})^2 = 2$ holds for $i \neq j$, no subsequence of $\{\lambda^A_{i_j}\}$ could be Cauchy. Therefore, $\dim \ker(T) < \infty$.

To prove smoothness, suppose that $\lambda^A \in \ker(T) \subset H_1(\Sigma, S_A)$. Then the elliptic regularity estimate (lemma A.2) yields that $\lambda^A \in H_2(\Sigma, S_A)$. This yields that $\lambda^A \in H_2(\Sigma, S_A)$, ... etc, i.e. that $\lambda^A \in H_2(\Sigma, S_A)$ for any $k \in \mathbb{N}$. Then by the Sobolev lemma (theorem A.1(2)) this yields that $\lambda^A \in C^\infty(\Sigma, S_A)$. \hfill \qed

\end{proof}

\medskip

\textbf{Proposition A.2.} $\text{Im}(D) \subset L_2(\Sigma, \hat{S}_A)$ and $\text{Im}(T) \subset L_2(\Sigma, S_{(ABC)})$ are closed subspaces.

\begin{proof}
Let $\{\phi^A_{i_l}\}$ and $\{\phi^A_{ABC}\}, i \in \mathbb{N}$, be Cauchy sequences in $\text{Im}(D)$ and $\text{Im}(T)$, respectively. Thus, we may assume that

$$\phi^A_{i_l} = D_A \lambda^A_{i_l}, \quad \phi^A_{(ABC)} = D_{(ABC)} \mu^A_{(ABC)}.$$
where \( \lambda^A_i \in (\ker(D))^\perp \cap H_1(\Sigma, S^4) \) and \( \mu^A_i \in (\ker(T))^\perp \cap H_1(\Sigma, S_A) \). Then by lemma A.3

\[
\begin{align*}
\| \lambda^A_i - \lambda^A_j \|_{H^2} &\leq C_0 \| D_{A\Lambda} \lambda^A_i - D_{A\Lambda} \lambda^A_j \|_{L^2} = C_0 \| \phi^A_i - \phi^A_j \|_{L^2}, \\
\| \mu^A_i - \mu^A_j \|_{H^2} &\leq C_0 \| D_{(AB)\mu^A_i} - D_{(AB)\mu^A_j} \|_{L^2} = C_0 \| \phi^A_{ABC} - \phi^A_{ABC} \|_{L^2},
\end{align*}
\]

and hence \( \{\lambda^A_i\} \) and \( \{\mu^A_i\} \) are Cauchy sequences in \( H_1(\Sigma, S^4) \). Thus, they converge strongly to some \( \lambda^A \in H_1(\Sigma, S^4) \) and \( \mu^A \in H_1(\Sigma, S_A) \), respectively. Therefore, since \( D \) and \( T \) are continuous, \( \phi^A_i = D_{A\Lambda} \lambda^A_i \rightarrow D_{A\Lambda} \lambda^A \in \text{Im}(D) \) and \( \phi^A_{ABC} = D_{(AB)\mu^A_i} \rightarrow D_{(AB)\mu^A} \in \text{Im}(T) \) when \( i \to \infty \), i.e. \( \text{Im}(D) \) and \( \text{Im}(T) \) are closed.

By proposition A.2 and lemma A.3, it is easy to show that \( D \) and \( T \), as densely defined operators from \( L_2(\Sigma, S^4) \) to \( L_2(\Sigma, S_A) \) and to \( L_2(\Sigma, S_{(ABC)}) \), respectively, are \textit{closed operators}. (Recall that a linear operator \( T : \text{Dom}(T) \subset X \to Y \) from the Banach space \( X \) to the Banach space \( Y \) is called \textit{closed} if for every Cauchy sequence \( x_i \in \text{Dom}(T) \) for which \( T x_i \) is also convergent (with the limit points \( x \in X \) and \( y \in Y \), respectively), \( x \in \text{Dom}(T) \) and \( T x = y \) follow. This is also equivalent to the statement that the graph, \( G(T) := \{ (x, T x) : x \in \text{Dom}(T) \} \), of \( T \) is a closed subspace of \( X \times Y \). See [39], p 164, and in particular problem 5.15 on p 165.)

Since the Sen–Witten operator with the extended domain, \( D : H_1(\Sigma, S^4) \to L_2(\Sigma, \bar{S}_{A}) \), is bounded with respect to the \( H_1(\Sigma, S^4) \) and \( L_2(\Sigma, \bar{S}_{A}) \) norms, at first sight it seems natural to consider its adjoint to be the uniquely determined bounded dual operator from \( L_2(\Sigma, \bar{S}_{A}) \) to \( H_1(\Sigma, S^4) \). However, this notion of adjoint would depend on the \( H_1 \)-Sobolev norm, which, as we noted, does not have a well-defined physical meaning. Moreover, this would not be an extension of the formal adjoint \( D^* \) given explicitly on the smooth spinor fields by \( \bar{\mu}_A \mapsto D^{M\Lambda} \bar{\mu}_A \). Thus, we do not follow this strategy.

Recall that the formal adjoint was introduced by using only the (physically meaningful) \( L_2 \)-norms, with respect to which \( D \) is \textit{not} bounded. Thus, \( D \) can be extended only to a proper dense subspace of \( L_2(\Sigma, S^4) \), but not to the whole of \( L_2(\Sigma, S^4) \). Therefore, though we still consider \( D \) to be extended to be a map \( H_1(\Sigma, S^4) \to L_2(\Sigma, \bar{S}_{A}) \), but from the point of view of its adjoint we consider \( H_1(\Sigma, S^4) \) only to be a dense subspace of \( L_2(\Sigma, S^4) \) and we do not use its \( H_1 \)-Sobolev norm. Then, according to functional analysis (see e.g. [39], p 167), the domain of the adjoint \( D^* \) of \( D : H_1(\Sigma, S^4) \to L_2(\Sigma, \bar{S}_{A}) \) is defined by

\[
\text{Dom}(D^*) := \{ \bar{\mu}_A \in L_2(\Sigma, \bar{S}_{A}) \mid \exists \nu^A \in L_2(\Sigma, S^4) : (D_{A\Lambda} \lambda^A, \bar{\mu}_A) = (\lambda^A, \nu^A) \forall \lambda^A \in H_1(\Sigma, S^4) \}.
\] (A.10)

Here the spinor field \( \nu^A \) is uniquely determined and is necessarily orthogonal to \( \ker(D) \), and the adjoint operator \( D^* \) is defined to be the map \( \bar{\mu}_A \mapsto \nu^A \). The following statement justifies this choice for the domain of \( D \) and the notion of the adjoint: although \( D \) is \textit{not} a formally self-adjoint operator on \( C^\infty(\Sigma, S^4) \) (since e.g. the domain and range spaces consist of the cross sections of different vector bundles), the complex conjugate of its extended domain, i.e. \( H_1(\Sigma, \bar{S}_{A}) \), is just the domain of the adjoint \( D^* \).

**Proposition A.3.** \textit{The domain of the adjoint} \( D^* : \text{Dom}(D^*) \to L_2(\Sigma, S^4) \) \textit{of} \( D \) \textit{is just the complex conjugate of the domain of} \( D \), i.e. \( \text{Dom}(D^*) = H_1(\Sigma, \bar{S}_{A}) \).

\textbf{Proof.} First recall that, for any \( \lambda^A, \mu^A \in H_1(\Sigma, S^4), (D_{A\Lambda} \lambda^A, \bar{\mu}_A) = (\lambda^A, D^{M\Lambda} \bar{\mu}_A) \) holds. Thus, if \( \bar{\mu}_A \in H_1(\Sigma, \bar{S}_{A}) \), then \( D^{M\Lambda} \bar{\mu}_A \in L_2(\Sigma, S^4) \), and hence with the notation \( \nu^A := D^{M\Lambda} \bar{\mu}_A \) one has that \( (D_{A\Lambda} \lambda^A, \bar{\mu}_A) = (\lambda^A, \nu^A) \) for any \( \lambda^A \in H_1(\Sigma, S^4) \), i.e. by definition (A.10) of \( \text{Dom}(D^*) \), \( \bar{\mu}_A \in \text{Dom}(D^*) \). Therefore, \( H_1(\Sigma, \bar{S}_{A}) \subset \text{Dom}(D^*) \).
Conversely, let $\tilde{\mu}_i \in \text{Dom}(D^*)$. Since $H_1(\Sigma, \tilde{\Sigma}_A) \subset L_2(\Sigma, \tilde{\Sigma}_A)$ is dense, there exists a sequence $\{\tilde{\mu}_i\}$ in $\text{Dom}(D^*) \cap H_1(\Sigma, \tilde{\Sigma}_A)$, $i \in \mathbb{N}$, such that $\tilde{\mu}_i \to \tilde{\mu}_A$ in the $L_2$-norm as $i \to \infty$. Moreover,

$$\langle \lambda^A, D^{AA} \tilde{\mu}_i \rangle = \langle D^{AA} \lambda^A, \tilde{\mu}_i \rangle \to \langle D^{AA} \lambda^A, \tilde{\mu}_A \rangle \quad \text{if} \quad i \to \infty.$$  

By $\tilde{\mu}_i \in \text{Dom}(D^*)$ and the definition of $\text{Dom}(D^*)$ there exists $\nu^A \in L_2(\Sigma, S^A)$ such that the limit on the right-hand side has the form $\langle \lambda^A, \nu^A \rangle$, i.e., $\langle \lambda^A, D^{AA} \tilde{\mu}_i - \nu^A \rangle \to 0$ for any $\lambda^A \in H_1(\Sigma, S^A)$ if $i \to \infty$. But since $H_1(\Sigma, S^A) \subset L_2(\Sigma, S^A)$ is dense, this also implies that $\langle \nu^A, D^{AA} \tilde{\mu}_i - \nu^A \rangle \to 0$ for any $\nu^A \in L_2(\Sigma, S^A)$ if $i \to \infty$, i.e., $D^{AA} \tilde{\mu}_i \to \nu^A$ in the weak topology of $L_2(\Sigma, S^A)$.

Since every weakly convergent sequence is bounded, there exist positive constants $K_1$ and $K_2$ such that $\|\tilde{\mu}_i\|_{L_2} \leq K_1$ and $\|D^{AA} \tilde{\mu}_i\|_{L_2} \leq K_2$ for any $i \in \mathbb{N}$. By the fundamental elliptic estimate for the Sen–Witten operator applied to the complex conjugate spinors these imply that the sequence $\{\tilde{\mu}_i\}$ is bounded in the $H_1$-Sobolev norm as well. Thus, the sequence $\{\tilde{\mu}_i\}$ contains a subsequence $\{\tilde{\mu}_k\}$, $k \in \mathbb{N}$, which converges weakly in $H_1(\Sigma, \tilde{\Sigma}_A)$ to some $\tilde{\mu}_{\infty} \in H_1(\Sigma, \tilde{\Sigma}_A)$. Since, however, the sequence $\{\tilde{\mu}_i\}$ was assumed to converge strongly to $\tilde{\mu}_A$, the strong and the weak limits must coincide. Thus, we obtained that $\tilde{\mu}_A = \tilde{\mu}_{\infty} \in H_1(\Sigma, \tilde{\Sigma}_A)$, i.e., that $\text{Dom}(D^*) \subset H_1(\Sigma, \tilde{\Sigma}_A)$. 

In many applications, the Fredholm property of operators plays a key role. (For a very readable review of the theory of Fredholm operators, see e.g. [38], chapters VI and VII.) Although the Fredholm property of $D$ follows from the general elliptic theory, here we give a simple, direct proof.

Proposition A.4. $D : H_1(\Sigma, S^A) \to L_2(\Sigma, \tilde{\Sigma}_A)$ is a Fredholm operator with zero analytic index.

**Proof.** By proposition A.1 $\ker(D)$ is finite dimensional and by proposition A.2 $\text{Im}(D)$ is closed. Thus, we need to show only that $\text{coker}(D) := L_2(\Sigma, \tilde{\Sigma}_A)/\text{Im}(D)$ is finite dimensional.

A spinor field $\lambda^A$ belongs to $\ker(D)$ precisely when $\langle D_{\Sigma A} \lambda^A, \tilde{\mu}_A \rangle = 0$ for all $\tilde{\mu}_A \in L_2(\Sigma, \tilde{\Sigma}_A)$. Since, however, $H_1(\Sigma, \tilde{\Sigma}_A) \subset L_2(\Sigma, \tilde{\Sigma}_A)$ is dense, $\lambda^A \in \ker(D)$ is equivalent even to $0 = \langle D_{\Sigma A} \lambda^A, \phi_A \rangle = \langle \lambda^A, D^{AA} \phi_A \rangle$ for all $\phi_A \in H_1(\Sigma, \tilde{\Sigma}_A)$. Since by proposition A.3 $H_1(\Sigma, \tilde{\Sigma}_A)$ is just the domain of the adjoint operator $D^*$, we have that $\lambda^A \in \ker(D)$ precisely when $\lambda^A \in (\text{Im}(D^*))^\perp$, i.e. $\ker(D) = (\text{Im}(D^*))^\perp$. Thus, by complex conjugation, we obtain from this that $\ker(D^*) = (\text{Im}(D))^\perp$, and hence that $\text{Im}(D)^\perp$ is finite dimensional. Recalling that $\text{Im}(D)$ is closed, it is clear that $\text{Im}(D)^\perp$ is isomorphic to $L_2(\Sigma, \tilde{\Sigma}_A)/\text{Im}(D) := \text{coker}(D)$. Since $\dim \ker(D) = \dim \text{coker}(D)$, the index is vanishing.

Proposition A.3 implies another important result, namely the following decomposition of the space $C^\infty(\Sigma, S^A)$ and its $L_2$ closure (yielding also the corresponding decomposition of their complex conjugate spaces).

Proposition A.5.

$$L_2(\Sigma, S^A) = \ker(D) \oplus \text{Im}(D^*), \quad C^\infty(\Sigma, S^A) = \ker(D) \oplus \text{Im}(D^*)_{|C^\infty}. \quad (A.11)$$

**Proof.** Since $\dim(D)$ is finite dimensional, it is closed even in $L_2(\Sigma, S^A)$, and hence the orthogonal decomposition $L_2(\Sigma, S^A) = \ker(D) \oplus (\ker(D))^\perp$ is well defined. Clearly, $\lambda^A \in \ker(D)$ is equivalent to $(D_{\Sigma A} \lambda^A, \omega_A) = 0$ for any $\omega_A \in L_2(\Sigma, \tilde{\Sigma}_A)$. Since $H_1(\Sigma, \tilde{\Sigma}_A) \subset L_2(\Sigma, \tilde{\Sigma}_A)$ is dense, $\lambda^A \in \ker(D)$ is still equivalent to $0 = \langle D_{\Sigma A} \lambda^A, \tilde{\mu}_A \rangle = \langle \lambda^A, D^{AA} \tilde{\mu}_A \rangle$ for any $\tilde{\mu}_A \in H_1(\Sigma, \tilde{\Sigma}_A)$, i.e. $\ker(D) = (\text{Im}(D^*))^\perp$. Since $\text{Im}(D) \subset L_2(\Sigma, \tilde{\Sigma}_A)$ is closed
and \( \mathcal{D}^* \) is minus the complex conjugate of \( \mathcal{D} \), this implies that \((\ker(\mathcal{D}))^\perp = ( \text{Im}(\mathcal{D}^*))^\perp = \text{Im}(\mathcal{D}^*) = \text{Im}(\mathcal{D}) \). (Here the overline denotes closure in the \( L_2 \)-norm topology.)

To prove the second decomposition, recall that the elements of \( \ker(\mathcal{D}) \) are smooth, and let \( \lambda^A \in C_0^\infty(\Sigma, S^A) \). Let \( \lambda^A = \lambda^A_0 + \lambda^A_1 \) be the orthogonal decomposition corresponding to \( L_2(\Sigma, S^A) = \ker(\mathcal{D}) \oplus \text{Im}(\mathcal{D}^*) \). Then \( \lambda^A_1 = \lambda^A - \lambda^A_0 \) is smooth, and there exists \( \tilde{\mu}_{A'} \in H_1(\Sigma, \tilde{S}_{A'}) \) such that \( \lambda^A_1 = D^{A'} \tilde{\mu}_{A'} \). Since \( D^{A'} \tilde{\mu}_{A'} \) is smooth, it belongs to \( H_1(\Sigma, \tilde{S}_{A'}) \), and hence, by the Sobolev lemma (theorem A.1(2)), that \( \tilde{\mu}_{A'} \in C^\infty(\Sigma, \tilde{S}_{A'}) \).

Following the general rule (see [39], p 167), the domain \( \text{Dom}(T^*) \) of the adjoint of the 3-surface twistor operator is defined by

\[
\text{Dom}(T^*) := \{ \phi_{ABC} \in L_2(\Sigma, S_{(ABC)}) : \exists v_A \in L_2(\Sigma, S_A) : \langle D_{(AB)C}, \phi_{ABC} \rangle = \langle \lambda_A, v_A \rangle \ \forall \lambda_A \in H_1(\Sigma, S_A) \}. \tag{A.12}
\]

(Note that \( v_A \) here is necessarily orthogonal to \( \ker(T) \).) Repeating the first part of the proof of proposition A.3, we can see at once that \( H_1(\Sigma, S_{(ABC)}) \subset \text{Dom}(T^*) \), and hence \( \text{Dom}(T^*) \) is dense in \( L_2(\Sigma, S_A) \). (Moreover, it is clear that this \( T^* \) is the extension of the formal adjoint of \( T \) introduced in subsection 4.3.) However, the proof of the inclusion in the opposite direction fails, because we do not have a fundamental elliptic-type estimate for \( T^* \) (that we did have for \( T \)).

Since the cokernel of \( T \) is infinite dimensional, it is not Fredholm. Similarly, \( T^* \) is not Fredholm either, because it has an infinite-dimensional kernel. However, we have that

**Proposition A.6.** \( T^* : \text{Dom}(T^*) \subset L_2(\Sigma, S_{(ABC)}) \to L_2(\Sigma, S_A) \) is a closed operator and \( T^{**} = T \).

**Proof.** Since the defining equation of \( \text{Dom}(T^*) \) in (A.12) is just the condition that the inverse graph of \( -T^* \), defined by \( G(-T^*) := \{ (-T^* \phi, \phi) | \phi \in \text{Dom}(T^*) \} \subset L_2(\Sigma, S_A) \times L_2(\Sigma, S_{(ABC)}) \), is the annihilator of the graph of \( T \), it is always closed. Hence, \( T^* \) is a closed operator (like every adjoint operator, see [39], p 168). \( T^{**} = T \) follows from the reflexivity of the \( L_2 \) spaces and the fact that \( T \) is closed. \( \square \)

Although \( \ker(T^*) \) is infinite dimensional, and we do not have an estimate for \( T^* \) analogous to (A.9), the kernel and the range of \( T^* \) are closed:

**Proposition A.7.** \( \ker(T^*) \subset L_2(\Sigma, S_{(ABC)}) \) and \( \text{Im}(T^*) \subset L_2(\Sigma, S_A) \) are closed subspaces.

**Proof.** Let \( \{ \phi_{ABC}^i \}, i \in \mathbb{N} \), be a Cauchy sequence in \( \ker(T^*) \) with respect to the \( L_2 \)-norm, and let \( \phi_{ABC} \) be its limit. Then \( T^* \phi_{ABC}^i = 0 \) for all \( i \in \mathbb{N} \), i.e. both \( \{ \phi_{ABC}^i \} \) and \( \{ T^* \phi_{ABC}^i \} \) are Cauchy and \( T^* \phi_{ABC} \to 0 \). But since the operator \( T^* \) is closed, \( \phi_{ABC} \in \text{Dom}(T^*) \) and \( T^* \phi_{ABC} = 0 \) follow, implying that \( \phi_{ABC} \in \ker(T^*) \). Thus, \( \ker(T^*) \) is closed. The statement that \( \text{Im}(T^*) \) is closed is a direct consequence of the general closed range theorem of Banach (see e.g. [40], p 205), adapted to Hilbert spaces, and the fact that \( T \) is a densely defined closed operator. \( \square \)

As a consequence of this proposition, we have decomposition theorems analogous to the first statement of proposition A.5.

**Proposition A.8.**
\[
L_2(\Sigma, S_A) = \ker(T) \oplus \text{Im}(T^*), \quad L_2(\Sigma, S_{(ABC)}) = \text{Im}(T) \oplus \ker(T^*). \tag{A.13}
\]
Proof. To prove the first, recall that $\ker(T)$ is finite dimensional (proposition A.1), and hence it is closed in $L_2(\Sigma, S_\lambda)$. Hence, we have the well-defined $L_2$-orthogonal decomposition $L_2(\Sigma, S_\lambda) = \ker(T) \oplus (\ker(T))^\perp$, and we will show that $(\ker(T))^\perp = \text{Im}(T^*)$, and hence by the previous proposition the statement follows. Thus, suppose that $\lambda A \in \ker(T)$. Then $0 = \langle D(AB\lambda c_1), \phi_{ABC} \rangle = \langle \lambda A, +D^R \phi_{ABC} \rangle$ for any $\phi_{ABC} \in \text{Dom}(T^*)$, i.e. $\ker(T) \subset (\text{Im}(T^*))^\perp$. Conversely, let $\lambda A \in (\text{Im}(T^*))^\perp \cap H_1(\Sigma, S_\lambda)$. Then $0 = \langle \lambda A, +D^R \phi_{ABC} \rangle = \langle D(AB\lambda c_1), \phi_{ABC} \rangle$ for any $\phi_{ABC} \in \text{Dom}(T^*)$. Since, however, $\text{Dom}(T^*) \subset L_2(\Sigma, S_{\lambda(ABC)})$ is dense, this implies that $D(AB\lambda c_1) = 0$, i.e. $\lambda A \in \ker(T)$. Hence, we obtained that $(\text{Im}(T^*))^\perp \cap H_1(\Sigma, S_\lambda) \subset \ker(T) \subset (\text{Im}(T^*))^\perp$. Taking its closure and recalling that $\ker(T) \subset L_2(\Sigma, S_\lambda)$ is closed and that $H_1(\Sigma, S_\lambda) \subset L_2(\Sigma, S_\lambda)$ is dense, we obtain that $\ker(T) = (\text{Im}(T^*))^\perp$, i.e. that $\text{Im}(T^*) = (\text{Im}(T^*))^\perp = (\ker(T))^\perp$.

The proof of the second decomposition is similar: since $\text{Im}(T) \subset L_2(\Sigma, S_{\lambda(ABC)})$ is closed (proposition A.2), there is the decomposition $L_2(\Sigma, S_{\lambda(ABC)}) = \text{Im}(T) \oplus (\text{Im}(T))^\perp$, and we prove that $(\text{Im}(T))^\perp = \ker(T^*)$. If $\phi_{ABC} \in \ker(T^*)$, then $0 = \langle \lambda A, +D^R \phi_{ABC} \rangle = \langle D(AB\lambda c_1), \phi_{ABC} \rangle$ for any $\lambda A \in H_1(\Sigma, S_\lambda)$, and hence $\ker(T^*) \subset (\text{Im}(T))^\perp$. Conversely, let $\phi_{ABC} \in (\text{Im}(T))^\perp \cap \text{Dom}(T^*)$. Then $0 = \langle D(AB\lambda c_1), \phi_{ABC} \rangle = \langle \lambda A, +D^R \phi_{ABC} \rangle$ for any $\lambda A \in H_1(\Sigma, S_\lambda)$. Since $H_1(\Sigma, S_\lambda) \subset L_2(\Sigma, S_\lambda)$ is dense, this implies that $+D^R \phi_{ABC} = 0$, i.e. that $(\text{Im}(T))^\perp \cap \text{Dom}(T^*) \subset \ker(T^*) \subset (\text{Im}(T))^\perp$. Recalling that $\text{Dom}(T^*) \subset L_2(\Sigma, S_{\lambda(ABC)})$ is dense and that $\ker(T^*)$ is closed by the previous proposition, the closure of this line of inclusions yields that $(\text{Im}(T))^\perp = \ker(T^*) = \ker(T^*)$. □

Since $\ker(T)$ is finite dimensional, repeating the proof of the second statement in proposition A.5, one can show that $\text{C}^\infty(\Sigma, S_\lambda) = \ker(T) \oplus \text{Im}(T^*)|_{C^\infty}$ also holds.

A.5. The second order operators $D^*D$ and $T^*T$

The domain of $D^*D$ and $T^*T$ will be defined, respectively, by

\[
\text{Dom}(D^*D) := \{\lambda A \in H_1(\Sigma, S^d) | D(AB\lambda) \phi_{ABC} \in \text{Dom}(D^*)\},
\]

\[
\text{Dom}(T^*T) := \{\lambda A \in H_1(\Sigma, S_\lambda) | D(AB\lambda c_1) \phi_{ABC} \in \text{Dom}(T^*)\},
\]

where $\text{Dom}(D^*) = H_1(\Sigma, S_{\lambda d})$ (see proposition A.3) and $H_1(\Sigma, S_{\lambda(ABC)}) \subset \text{Dom}(T^*)$.

Lemma A.4.

\[
\ker(D^*D) = \ker(D), \quad \ker(T^*T) = \ker(T); \quad \text{(A.14)}
\]

\[
\text{Im}(D^*) = \text{Im}(D^*D), \quad \text{Im}(T^*) = \text{Im}(T^*T). \quad \text{(A.15)}
\]

Proof. The inclusion $\ker(D) \subset \ker(D^*D)$ is obviously true. To prove the inclusion in the opposite direction, suppose that $\lambda A \in \ker(D^*D)$. Then $0 = \langle D^A D^R D^R, \lambda A \rangle = \langle D(AB\lambda) \phi_{ABC} \rangle$, i.e. $D(AB\lambda) = 0$. Hence, $\ker(D^*D) \subset \ker(D)$. The proof for the 3-surface twistor operator is similar.

To prove (A.15) for the Sen–Witten operator, we should use the first of the decompositions (A.11), and for the 3-surface twistor operator we should use the first of (A.13). Since the two proofs are similar, we give the proof only for the first:

\[
\text{Im}(D^*) = \{D^A \mu_A | \mu_A \in H_1(\Sigma, S_{\lambda d})\}
\]

\[
= \{D^A \mu_A | \mu_A \in H_1(\Sigma, S_{\lambda d}) \cap (\ker(D^*) \oplus \text{Im}(D))\}
\]

\[
= \{D^A \mu_A | \mu_A \in H_1(\Sigma, S_{\lambda d}) \cap \ker(D)\}
\]

\[
= \{D^A \mu_A | \mu_A \in H_1(\Sigma, S_{\lambda d}) \cap \text{Im}(D)\}
\]

\[
= \{D^A D_{AB} R | \phi_{A} \in H_1(\Sigma, S^d), D_{AB} R \phi_{A} \in H_1(\Sigma, S_{\lambda d})\} = \text{Im}(D^*D). \quad \square
\]
An immediate consequence of this lemma is the decomposition \( L_2(\Sigma, S_A) = \ker(D^*D) \oplus \text{Im}(D^*D) = \ker(T^*T) \oplus \text{Im}(T^*T) \), where all the subspaces are closed.

One of the key properties of the second-order operators is their Fredholm property.

**Proposition A.9.** \( D^*D \) and \( T^*T \) are positive, self-adjoint Fredholm operators.

**Proof.** Positivity. Since both \( D^*D \) and \( T^*T \) are positive operators on the space of the smooth spinor fields, which is dense in their domain, the operators are also positive on \( \text{Dom}(D^*D) \) and \( \text{Dom}(T^*T) \), respectively.

Self-adjointness. We should show that \( \text{Dom}(D^*D) = \text{Dom}(D^*D)^* \) and \( \text{Dom}(T^*T) = \text{Dom}(T^*T)^* \). Since the two proofs are similar, we prove it only for \( T^*T \).

Suppose that \( \chi_A \in \text{Dom}(T^*T) \), i.e. \( \chi_A \in H_1(\Sigma, S_A) \) such that \( D(A\beta\lambda C), D(AB\beta C) \in \text{Dom}(T^*T) \). Thus, there exists a spinor field \( v_A \in L_2(\Sigma, S_A) \) such that \( \langle D(A\beta\lambda C), D(AB\beta C) \rangle = \langle \lambda_A, v_A \rangle \) for any \( \lambda_A \in \text{Dom}(T^*T) = H_1(\Sigma, S_A) \). Since, however, Dom\((T^*T) \subset \text{Dom}(T) \), we have that \( \langle (D^*D C)D(AB\beta C), \chi_A \rangle = \langle D(AB\beta C), D(A\beta\lambda C) \rangle = \langle \lambda_A, v_A \rangle \) for any \( \lambda_A \in \text{Dom}(T^*T) \). Thus, \( \chi_A \in \text{Dom}(T^*T)^* \), i.e. \( \text{Dom}(T^*T) \subset \text{Dom}(T^*T)^* \).

Conversely, suppose that \( \chi_A \in \text{Dom}(T^*T)^* \). Then there exists a spinor field \( v_A \in (\ker(T^*T))^\perp \) such that \( \langle (D^*D C)D(AB\beta C), \chi_A \rangle = \langle \lambda_A, v_A \rangle \) for any \( \lambda_A \in \text{Dom}(T^*T) \). However, by \((A.14)\) and the first decomposition in \((A.13)\) \( \ker(T^*T)^\perp = (\ker(T))^\perp = \text{Im}(T) \), and hence \( v_A = D(AB\beta C) \phi_{ABC} \) for some \( \phi_{ABC} \in \text{Dom}(T)^* \). On the other hand, by the second decomposition in \((A.13)\), we have that \( \text{Dom}(T^*T) = \text{Dom}(T)^* \cap (\ker(T^*T)^* \oplus \text{Im}(T)) \), and therefore \( \phi_{ABC} = \phi_{ABC}^0 + D(AB\beta C) \omega_A \) for some \( \omega_A \in \text{Dom}(T) = H_1(\Sigma, S_A) \) and \( \phi_{ABC}^0 \in \ker(T^*T) \). Hence, \( v_A = D(AB\beta C) \phi_{ABC}^0 \), by means of which \( \langle (D^*D C)D(AB\beta C), \chi_A \rangle = \langle \lambda_A, v_A \rangle = \langle (D^*D C)D(AB\beta C), \omega_A \rangle \), i.e. \( \langle (D^*D C)D(AB\beta C), \chi_A - \omega_A \rangle = 0 \). Thus, \( \omega_A^0 := \chi_A - \omega_A \in (\text{Im}(T^*T))^\perp = (\text{Im}(T))^\perp = \ker(T) \subset C^\infty(\Sigma, S_A) \), where we used \((A.15)\), \((A.13)\) and proposition A.1. Consequently, \( \chi_A = \omega_A + \omega_A^0 \in H_1(\Sigma, S_A) \) and \( \chi_A \in \text{Dom}(T^*T)^* \), i.e. \( \chi_A \in \text{Dom}(T^*T)^* \).

The Fredholm property. We prove this only for \( T^*T \); the proof for \( D^*D \) is similar. By proposition A.1 and \((A.14)\) \( \ker(T^*T) = \ker(T) \) is finite dimensional. By \((A.15)\) we have that \( \text{coker}(T^*T) := \text{coker}(\text{Dom}(T^*T)) = \ker(T) \oplus \text{Im}(T^*T)/\text{Im}(T) \approx \ker(T) \), which is also finite dimensional. By proposition A.7 \( \text{Im}(T^*T) \) is closed, and hence by \((A.15)\) \( \text{Im}(T^*T)^* \) is also closed. Therefore, \( T^*T \) is Fredholm.

Finally, we clarify the spectral properties of \( 2D^*D \) and \( 2T^*T \). Thus, let \( E_{e_1} \) and \( E_{e_2} \) denote the space of their eigenspinors with the eigenvalues \( e_1 \) and \( e_2 \), respectively. Adapting an analogous theorem of [20] to the present operators, we have our last statement.

**Proposition A.10.** The resolvent operators of \( 2D^*D \) and \( 2T^*T \) are compact.

**Proof.** We prove the statement only for \( 2T^*T \). The proof for \( 2D^*D \) is similar. First we show that \( 2T^*T \) does not have any eigenvalue in the interval \((0, 2/C_0^2)\), where \( C_0 \) is the positive constant in lemma A.3.

Let \( \tau^2 > 0 \) and suppose that \( \lambda_A \in E_{e_1} \). Then by \( \lambda_A \in \text{Dom}(T^*T) \) and \( \ker(T^*T) = \ker(T) \), we have that \( \lambda_A \in (\ker(T))^\perp \cap H_1(\Sigma, S_A) \), and hence by lemma A.3 \( \frac{\tau^2}{\|B\|^2}\|\lambda_A\|_{L_2}^2 = \left\{ \frac{\tau^2}{\|B\|^2}\|\lambda_A\|_{L_2}^2 \right\} \geq (1/C_0)^2\|\lambda_A\|_{L_2}^2 \geq (1/C_0)^2\|\lambda_A\|_{H_1}^2 \geq 2\|\lambda_A\|_{L_2}^2 \geq 2\|\lambda_A\|_{L_2}^2 \). If, however, \( \tau^2 < C_0^2 \), then this implies that \( \lambda_A = 0 \), i.e. that \( E_{e_2} = \emptyset \).

Next, let us define the operator \( \Delta := 2T^*T - \tau^2 I : \text{Dom}(T^*T) \rightarrow L_2(\Sigma, S_A) \), which is continuous and, by \( E_{e_1} = \emptyset, \ker(\Delta) = \emptyset \). (Here \( I \) denotes the identity operator.) Let us define
$W := (\Im(\Delta))^+ \subset L_2(\Sigma, S_\lambda)$, and suppose that this is not empty. Then, if $\omega_A \in W$, it follows that

$$0 = \{\Delta_A^\mu \lambda_B, \omega_A\} = \{2^+ D_{AB}^\mu \lambda_C, \omega_A\} - \tau^2 \lambda_A$$

for any $\lambda_A \in \text{Dom}(\Delta) = \text{Dom}(T^* T)$, i.e. $\{2^+ D_{AB}^\mu \lambda_C, \omega_A\} = (\lambda_A, 1/2 \tau^2 \omega_A)$ holds. However, this means that $W \subset \text{Dom}((T^* T)^+) = \text{Dom}(T^* T)$ and $2^+ D_{AB}^\mu \lambda_C = 1/2 \tau^2 \omega_A$, which contradicts $E_{\tau^2} = \emptyset$. Hence, $W = \emptyset$, i.e. $\Im(\Delta) = L_2(\Sigma, S_\lambda)$.

Thus, $\Delta : \text{Dom}(T^* T) \to L_2(\Sigma, S_\lambda)$ is a continuous bijection. But then by the open mapping theorem (see e.g. [38], p 107) $\Delta$ is a topological vector space isomorphism, admitting a continuous inverse $\Delta^{-1} : L_2(\Sigma, S_\lambda) \rightarrow \text{Dom}(T^* T) \subset H_1(\Sigma, S_\lambda)$. Therefore, there exists a positive constant $K_2$ such that $\|\Delta^{-1} \omega_B\|_H_1 \leq K_2 ||\omega_A||_{L_2}$ for any $\omega_A \in L_2(\Sigma, S_\lambda)$. Hence, if $\{\omega_B\}_i \subset \mathbb{N}$, is a bounded sequence in $L_2(\Sigma, S_\lambda)$, then $\{\Delta^{-1} \omega_B\}_i$ is bounded in $H_1(\Sigma, S_\lambda)$. But by the Rellich lemma the inclusion $H_1(\Sigma, S_\lambda) \subset L_2(\Sigma, S_\lambda)$ is compact, implying that there is a sequence $\{\Delta^{-1} \omega_B\}_k$, $k \in \mathbb{N}$, which is convergent in $L_2(\Sigma, S_\lambda)$. Therefore, the resolvent $\Delta^{-1}$ of $2T^* T - \tau^2 I$, as a bounded linear operator $L_2(\Sigma, S_\lambda) \to L_2(\Sigma, S_\lambda)$, is compact. $\Box$

Applying the results on the spectral properties of compact operators (see e.g. [38], theorem 16, p 114) to the resolvent, and recalling how the spectra of the operator and its resolvent are related to each other (see e.g. [39], p 187), we obtain (see e.g. [41], p 196) that (1) the spectrum of $2T^* D$ is purely discrete with the only accumulation point at infinity, (2) there is a positive constant $c$ such that for the 4th eigenvalue, $\omega^2 \geq c k^{1/4}$, (3) the space $E_{\omega^2}$ of the eigenspinors with eigenvalue $\omega^2$ is finite dimensional and the eigenspinors are smooth, (4) the spaces $E_{\omega^2}, E_{\beta^2}$ with $\omega^2 \neq \beta^2$ are orthogonal to each other and (5) $L_2(\Sigma, S_\lambda) = \oplus_{\omega^2 \in (0, \infty)} E_{\omega^2}$.

(For the proof, see e.g. [38, 39, 41]. However, we note that the bound for the growth rate of the eigenvalues given in [41], p 196, can be increased, and the bound given in (2) above is this greater one.) Clearly $2T^* T$ has similar spectral properties.

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