Complex Dynamic Behaviors of an Impulsively Controlled Predator-prey System with Watt-type Functional Response

HUNKI BAEK
Department of Mathematics Education, Catholic University of Daegu, Gyeongsan, Gyeongbuk, 712-702, South Korea.
e-mail: hkbak@cu.ac.kr

ABSTRACT. In this paper, we consider a discrete predator-prey system with Watt-type functional response and impulsive controls. First, we find sufficient conditions for stability of a prey-free positive periodic solution of the system by using the Floquet theory and then prove the boundedness of the system. In addition, a condition for the permanence of the system is also obtained. Finally, we illustrate some numerical examples to substantiate our theoretical results, and display bifurcation diagrams and trajectories of some solutions of the system via numerical simulations, which show that impulsive controls can give rise to various kinds of dynamic behaviors.

1. Introduction

Impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. Recently, impulsive differential equations are significantly used to study the mathematical properties of an impulsive predator-prey system in population dynamics. Especially, controlling the population of insect pest (prey) has become an increasingly complex issue ([1, 2, 3, 13, 19, 26, 27, 38]).

There are many methods that can be used to help manage insect pests. One of important methods for pest control is chemical control. Pesticides are useful because they quickly kill a significant portion of a pest population. However, there are many deleterious effects associated with the use of chemicals that need to be reduced or eliminated. These include human illness associated with pesticide applications, insect resistance to insecticides, contamination of soil and water, and diminution of

Received April 16, 2015; revised July 19, 2015; accepted September 25, 2015.
2010 Mathematics Subject Classification: 34A37, 34D20, 34H05, 92D25.
Key words and phrases: Predator-prey system, Watt-type functional response, impulsive differential equation, Floquet theory, bifurcation diagrams.
This work was supported by research grants from the Catholic University of Daegu in 2014.
biodiversity. As a result, it is required that we should combine pesticide efficacy tests with other ways of control. For the reason, biological control is presented as one of important alternatives. It is defined as the reduction in pest populations from the actions of other living organisms, often called natural enemies or beneficial species. Virtually all pests have some natural enemies, and the key to successful pest control is to identify the pest and its natural enemies and releasing them at fixed times for pest control. Spraying pesticide can affect natural enemies. But, in some cases, pesticides can be successfully integrated into a biological control strategy with little harming natural enemies([4, 6, 7, 8, 9, 23, 29, 30, 31, 32, 33, 35, 36, 37, 38]).

On the other hand, the relationship between pest and natural enemy can be expressed as a predator(natural enemy)-prey(pest) system mathematically as follows;

\[ \begin{align*}
    x'(t) &= ax(t)(1 - \frac{x(t)}{K}) - yP(x, y), \\
    y'(t) &= -dy(t) + cyP(x, y), \\
    x(0) &= x_0, y(0) = y_0,
\end{align*} \]  

where \( x(t), y(t) \) represent the population density of the prey and the predator at time \( t \), respectively. Usually, \( K \) is called the carrying capacity of the prey. The constant \( a \) is called intrinsic growth rate of the prey. The constants \( c, d \) are the conversion rate and the death rate of the predator, respectively. The function \( P \) is the functional response of the predator which means prey eaten per predator per unit of time.

Many scholars have studied such predator-prey systems with a functional response, such as Holling-type [20, 21, 25], Monod-type [20, 21, 28] and Beddington-type [15, 16, 18], etc. One of well-known function response is of Watt-type, proposed by [34]. The predator-prey system with Watt-type is described as the following differential equation:

\[ \begin{align*}
    x'(t) &= ax(t)(1 - \frac{x(t)}{K}) - b(1 - \exp(-cx(t)y(t)))y(t), \\
    y'(t) &= -dy(t) + e(1 - \exp(-cx(t)y(t)))y(t),
\end{align*} \]

where \( b \) is the maximum number of prey that can be eaten by a predator per unit time. The constant \( c \) is the constant for the decrease in motivation to hunt and \( \gamma \) is a nonnegative constant.

In order to accomplish the aims discussed above, we need to consider the following impulsive differential equation:

\[ \begin{align*}
    x'(t) &= ax(t)(1 - \frac{x(t)}{K}) - b(1 - \exp(-cx(t)y(t)))y(t), \quad t \neq nT, \\
    y'(t) &= -dy(t) + e(1 - \exp(-cx(t)y(t)))y(t), \quad t \neq nT, \\
    x(t^+) &= (1 - p)x(t), \quad t = nT, \\
    y(t^+) &= y(t) + q, \quad t = nT, \\
    (x(0^+), y(0^+)) &= (x_0, y_0),
\end{align*} \]
where $T$ is the period of the impulsive immigration or stock of the predator, $0 < p < 1$ presents the fraction of prey which die due to harvesting or pesticides etc and $q$ is the size of immigration or stock of the predator.

In fact, impulsive control methods can be found in almost every field of applied sciences. Theoretical investigations and its application analysis can be found in Bainov and Simeonov[10, 11, 12], Lakshmikantham et al.[22]. Moreover, impulsive differential equations dealing with biological population dynamics are literate in [14, 4, 5, 17, 23, 24, 31, 32, 37]. Especially, the authors in [32] have studied Watt-type predator-prey systems with impulsive perturbations, considering only the impulsive control parameter $q$ in system (1.3) with $p = 0$.

The main purpose of this paper is to investigate the dynamics of system (1.3). In the next section, we introduce some notations which are used in this paper. We study qualitative properties of system (1.3) in Section 3. In fact, we find conditions for the stability of a prey-free periodic solution and for the permanence of system (1.3) by using the Floquet theory. In Section 4, we numerically investigate the effects of impulsive perturbations on inherent oscillation by illustrating bifurcation diagrams and trajectories of solutions of the system.

2. Definitions and Basic Lemmas

In the section, we give some notations, definitions and Lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^2 = \{ x = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \geq 0 \}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand side of system (1.3).

Let $V : \mathbb{R}_+ \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, then $V$ is said to be in a class $V_0$ if

1. $V$ is continuous on $(nT, (n + 1)T] \times \mathbb{R}_+^2$,
   and $\lim_{(t,y) \rightarrow (nT,\infty)} V(t,y) = V(nT^+, x) \text{ exists.}$

2. $V$ is locally Lipschitzian in $x$.

**Definition 2.1.** Let $V \in V_0, (t,x) \in (nT, (n + 1)T] \times \mathbb{R}_+^2$. The upper right derivatives of $V(t,x)$ with respect to the impulsive differential system (1.3) is defined as

$$D^+ V(t,x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t,x)) - V(t,x)].$$

**Definition 2.2.** System (1.3) is permanent if there exist $M \geq m > 0$ such that, for any solution $(x(t), y(t))$ of system (1.3) with $x_0 = (x_0, y_0) > 0$,

$$m \leq \lim_{t \to \infty} \inf x(t) \leq \lim_{t \to \infty} \sup x(t) \leq M \text{ and } m \leq \lim_{t \to \infty} \inf y(t) \leq \lim_{t \to \infty} \sup y(t) \leq M.$$
Remark (1) A solution of system (1.3) is a piecewise continuous function $x: \mathbb{R}_+ \to \mathbb{R}_+^2$, $x(t)$ is continuous on $(nT, (n+1)T]$, $n \in \mathbb{N}$ and $x(nT^+) = \lim_{t \to nT^+} x(t)$ exists.
(2) The smoothness properties of $f$ guarantees the global existence and uniqueness of solution of system (1.3). (See [22] for the details).

The following lemma is obvious.

Lemma 2.3. Let $x(t) = (x(t), y(t))$ be a solution of system (1.3).
(1) If $x(0^+) \geq 0$ then $x(t) \geq 0$ for all $t \geq 0$.
(2) If $x(0^+) > 0$ then $x(t) > 0$ for all $t \geq 0$.

We will use the following important comparison theorem on impulsive differential equations [22].

Lemma 2.4. (Comparison theorem) Suppose $V \in V_0$ and

\begin{equation}
\begin{cases}
D^+V(t, x) \leq g(t, V(t, x)), & t \neq nT, \\
V(t, x(t^+)) \leq \psi_n(V(t, x)), & t = nT, \\
V(0^+, x_0) \leq u_0
\end{cases}
\end{equation}

$g: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $(nT, (n+1)T] \times \mathbb{R}_+$ and for $u(t) \in \mathbb{R}_+, u \in \mathbb{N}$, \(\lim_{(t, y) \to (nT^+, u)} g(t, y) = g(nT^+, u)\) exists, $\psi_n: \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

\begin{equation}
\begin{cases}
u'(t) = g(t, u(t)), & t \neq nT, \\
u(t^+) = \psi_n(u(t)), & t = nT, \\
u(0^+) = u_0,
\end{cases}
\end{equation}

existing on $[0, \infty)$. Then $V(0^+, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t)$, $t \geq 0$, where $x(t)$ is any solution of equation (2.2).

Similar result can be obtained when all conditions of the inequalities in Lemma 2.4 are reversed. Note that if we have some smoothness conditions of $g(t, u(t))$ to guarantee the existence and uniqueness of the solutions for equation (2.2), then $r(t)$ is exactly the unique solution of equation (2.2).

Now, we give the basic properties of the following impulsive differential equation.

\begin{equation}
\begin{cases}y'(t) = -dy(t), & t \neq nT, \\
y(t^+) = y(t) + q, & t = nT, \\
y(0^+) = y_0.
\end{cases}
\end{equation}

Then we can easily obtain the following results.

Lemma 2.5. (1) $y^*(t) = \frac{q \exp(-d(t-nT))}{1-\exp(-dT)}$, $t \in (nT, (n+1)T]$, $n \in \mathbb{N}$ and $y^*(0^+) = \frac{q}{1-\exp(-dT)}$ is a positive periodic solution of system (2.3).
(2) \( y(t) = \left( y(0^+) - \frac{q}{1 - \exp(-dT)} \right) \exp(-dt) + y^*(t) \) is the solution of system (2.3) with \( y_0 \geq 0, t \in (nT, (n+1)T) \) and \( n \in \mathbb{N} \).

(3) All solutions \( y(t) \) of system (1.3) with \( y_0 \geq 0 \) tend to \( y^*(t) \). i.e., \( |y(t) - y^*(t)| \to 0 \) as \( t \to \infty \).

It is from Lemma 2.4 that the general solution \( y(t) \) of equation (2.3) can be synchronized with the positive periodic solution \( y^*(t) \) of equation (2.3) for sufficiently large \( t \) and we can obtain the complete expression for the prey-free periodic solution of system (1.3)

\[ (0, y^*(t)) = \left( 0, \frac{q \exp(-d(t-nT))}{1 - \exp(-dT)} \right) \text{ for } t \in (nT, (n+1)T). \]

To study the stability of the prey-free periodic solution as a solution of system (1.3) we present the Floquet theory for the linear \( T \)-periodic impulsive equation:

\[
\begin{cases}
  x'(t) = A(t)x(t), & t \neq \tau_k, t \in \mathbb{R}, \\
  x(t^+) = x(t) + B_kx(t), & t = \tau_k, k \in \mathbb{Z}.
\end{cases}
\]  

We introduce the following conditions:

- \((H_1)\) \( A(\cdot) \in \text{PC}(\mathbb{R}, C^{n \times n}) \) and \( A(t + T) = A(t)(t \in \mathbb{R}) \), where \( \text{PC}(\mathbb{R}, C^{n \times n}) \) is the set of all piecewise continuous matrix functions which is left continuous at \( t = \tau_k \), and \( C^{n \times n} \) is the set of all \( n \times n \) matrices.

- \((H_2)\) \( B_k \in C^{n \times n}, \det(E + B_k) \neq 0, \tau_k < \tau_{k+1} (k \in \mathbb{Z}) \).

- \((H_3)\) There exists a \( q \in \mathbb{N} \) such that \( B_{k+q} = B_k, \tau_{k+q} = \tau_k + T(k \in \mathbb{Z}) \).

Let \( \Phi(t) \) be a fundamental matrix of equation (2.4), then there exists unique non-singular matrix \( M \in C^{n \times n} \) such that

\[ \Phi(t + T) = \Phi(t)M(t \in \mathbb{R}). \]

By equality (2.5) there corresponds to the fundamental matrix \( \Phi(t) \) and the constant matrix \( M \) which we call the monodromy matrix of equation (2.4) (corresponding to the fundamental matrix of \( \Phi(t) \)).

All monodromy matrices of equation (2.4) are similar and have the same eigenvalues. The eigenvalues \( \mu_1, \cdots, \mu_n \) of the monodromy matrices are called the Floquet multipliers of equation (2.4).

**Lemma 2.6.** ([10]) (Floquet theory) Let conditions \((H_1)-(H_3)\) hold. Then the linear \( T \)-periodic impulsive equation (2.4) is

- (1) stable if and only if all multipliers \( \mu_j (j = 1, \cdots, n) \) of equation (2.4) satisfy the inequality \( |\mu_j| \leq 1 \), and moreover, to those \( \mu_j \) for which \( |\mu_j| = 1 \), there correspond simple elementary divisors;

- (2) asymptotically stable if and only if all multipliers \( \mu_j (j = 1, \cdots, n) \) of equation (2.4) satisfy the inequality \( |\mu_j| < 1 \);

- (3) unstable if \( |\mu_j| > 1 \) for some \( j = 1, \cdots, n \).
3. Main Results

Now, we present a condition which guarantees locally asymptotical stability of the prey-free periodic solution \((0, y^*(t))\).

**Theorem 3.1.** If

\[
aT - bc \left( \frac{q}{1 - \exp(-dT)} \right)^{1-\gamma} (1 - \exp(-d(1 - \gamma)T)) \frac{1}{d(1 - \gamma)} < \ln \frac{1}{1 - p}
\]

then \((0, y^*(t))\) is locally asymptotically stable.

**Proof.** The local stability of the periodic solution \((0, y^*(t))\) of system (1.3) may be determined by considering the behavior of small amplitude perturbations of the solution. Let \((x(t), y(t))\) be any solution of system (1.3). Define \(x(t) = u(t), y(t) = y^*(t) + v(t)\). Then they may be written as

\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \quad 0 \leq t \leq T,
\]

where \(\Phi(t)\) satisfies

\[
\frac{d\Phi}{dt} = \begin{pmatrix} a - bc \gamma^*(t) & 0 \\ ec \gamma^*(t) & -d \end{pmatrix} \Phi(t)
\]

and \(\Phi(0) = I\), the identity matrix. The linearization of the third and fourth equation of system (1.3) becomes

\[
\begin{pmatrix} u(nT^+)) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1 - p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}.
\]

Note that all eigenvalues of \(S = \begin{pmatrix} 1 - p & 0 \\ 0 & 1 \end{pmatrix} \Phi(T)\) are \(\mu_1 = \exp(-dT) < 1\) and \(\mu_2 = (1 - p) \exp(\int_0^T a - bc \gamma^*(t)^{1-\gamma} dt)\). Since

\[
\int_0^T \gamma^*(t)^{1-\gamma} dt = \left( \frac{q}{1 - \exp(-dT)} \right)^{1-\gamma} \frac{1}{d(1 - \gamma)} (\exp(d(1 - \gamma)T) - 1) \frac{1}{d(1 - \gamma)}
\]

we have

\[
\mu_2 = (1 - p) \exp \left( aT - bc \left( \frac{q}{1 - \exp(-dT)} \right)^{1-\gamma} (1 - \exp(-d(1 - \gamma)T)) \frac{1}{d(1 - \gamma)} \right)
\]

The condition \(|\mu_2| < 1\) is equivalent to the equation

\[
aT - bc \left( \frac{q}{1 - \exp(-dT)} \right)^{1-\gamma} (1 - \exp(-d(1 - \gamma)T)) \frac{1}{d(1 - \gamma)} < \ln \frac{1}{1 - p}.
\]
According to Lemma, \((0, y^*(t))\) is locally asymptotically stable.

\[\square\]

We show that all solutions of system (1.3) are uniformly ultimately bounded.

**Proposition 3.2.** There is an \(M > 0\) such that \(x(t), y(t) \leq M\) for all \(t\) large enough, where \((x(t), y(t))\) is a solution of system (1.3).

**Proof.** Let \(x(t) = (x(t), y(t))\) be a solution of system (1.3) and let \(V(t, x) = ex(t) + by(t)\). Then \(V \in V_0\), if \(t \neq n\tau\)

\[D^+ V + \beta V = -\frac{e a}{K} x(t)^2 + e (a + \beta) x(t) + b (\beta - d) y(t),\]

and \(V(n\tau^+) = V(n\tau) + q\). Clearly, the right hand of (3.4), is bounded when \(0 < \beta < d\). Thus we can choose \(0 < \beta_0 < d\) and \(M_0 > 0\) such that

\[\begin{align*}
D^+ V &\leq -\beta_0 V + M_0, t \neq n\tau, \\
V(n\tau^+) &= V(n\tau) + q.
\end{align*}\]

From Lemma 2.4, we can obtain that

\[V(t) \leq (V(0^+) - \frac{M_0}{\beta_0}) \exp(-\beta_0 t) + \frac{p(1 - \exp(-(n + 1)\beta_0\tau))}{1 - \exp(-\beta_0\tau)} \exp(-\beta_0(t - n\tau) + \frac{M_0}{\beta_0}) \quad \text{for} \quad t \in (n\tau, (n + 1)\tau).\]

Therefore, \(V(t)\) is bounded by a constant for sufficiently large \(t\). Hence there is an \(M > 0\) such that \(x(t), y(t) \leq M\) for a solution \((x(t), y(t))\) with all \(t\) large enough.

\[\square\]

**Theorem 3.3.** System (1.3) is permanent if

\[\begin{align*}
aT - bc \left(\frac{q}{1 - \exp(-dT)}\right)^{1-\gamma} (1 - \exp(-d(1-\gamma)T)) \frac{1}{d(1-\gamma)} &> \ln \frac{1}{1-p}.
\end{align*}\]

**Proof.** Let \((x(t), y(t))\) be any solution of system (1.3) with \((x_0, y_0) > 0\). From Proposition 3.2, we may assume that \(x(t) \leq M, y(t) \leq M\), \(t \geq 0\) and \(M > \left(\frac{a}{bc}\right)\frac{1}{\tau\gamma}\).

Let \(m_2 = \frac{p \exp(-dT)}{1 - \exp(-dT)} - \epsilon_2, \epsilon_2 > 0\). From Lemma 2.4, clearly we have \(y(t) \geq m_2\) for all \(t\) large enough. Now we shall find an \(m_1 > 0\) such that \(x(t) \geq m_1\) for all \(t\) large enough. We will do this in the following two steps.

(Step 1) Since

\[aT - bc \left(\frac{q}{1 - \exp(-dT)}\right)^{1-\gamma} (1 - \exp(-d(1-\gamma)T)) \frac{1}{d(1-\gamma)} > \ln \frac{1}{1-p},\]

we can choose \(m_3 > 0, \epsilon_1 > 0\) small enough such that \(\delta \equiv \frac{qm_3}{m_3 + bm_2^2} < d\)

and \(R = (1 - p) \exp\left(\int_0^r a - \frac{a}{K} m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma} dt\right) > 1\), where \(u^*(t) =\)
\[
q \exp((-d + \delta)(t - nT)) - 1 - \exp((-d + \delta)T) = t \in (nT, (n + 1)T] \text{ and } n \in \mathbb{N}.
\]
Now we can prove that \( x(t) < m_3 \) cannot hold for all \( t \). Otherwise, we can get \( y'(t) \leq y(t)(-d + \delta) \). By Lemma 2.4, we have \( y(t) \leq u(t) \) and \( u(t) \to u^*(t) \), \( t \to \infty \), where \( u(t) \) is the solution of
\[
\begin{aligned}
&u'(t) = (-d + \delta)u(t), \quad t \neq nT, \\
&u(t^+) = u(t) + q, \quad t = nT \\
&u(0^+) = y_0.
\end{aligned}
\]
(3.6)
Then there exists \( T_1 > 0 \) such that \( y(t) \leq u(t) \leq u^*(t) + \epsilon_1 \). Since \( 1 - \exp\left( -\frac{cx(t)}{y(t)^\gamma} \right) \leq \frac{cx(t)}{y(t)^\gamma} \), we obtain that
\[
x'(t) = x(t)\left( a - \frac{a}{K}x(t) \right) - b \left( 1 - \exp\left( -\frac{cx(t)}{y(t)^\gamma} \right) \right) y(t) \geq x(t) \left( a - \frac{a}{K}m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma} \right) \geq x(t) \left( a - \frac{a}{K}m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma} \right) \text{ for } t \geq T_1 \text{ and } t \neq nT.
\]
Let \( N_1 \in \mathbb{N} \) and \( N_1 T \geq T_1 \). We have, for \( n \geq N_1 \),
\[
\begin{aligned}
x'(t) &\geq x(t) \left( a - \frac{a}{K}m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma} \right), \quad t \neq nT, \\
x(t^+) &\geq (1 - p)x(t), \quad t = nT.
\end{aligned}
\]
Integrating (3.7) on \((nT, (n + 1)T] \text{ for } n \geq N_1 \), we obtain
\[
x((n + 1)T) \geq x(nT^+) \exp \left( \int_{nT}^{(n+1)T} a - \frac{a}{K}m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma} dt \right) = x(nT)R.
\]
Then \( x((N_1 + k)T) \geq x(N_1 T) R^k \to \infty \) as \( k \to \infty \) which is a contradiction. Hence there exists a \( t_1 > 0 \) such that \( x(t_1) \geq m_3 \).

(Step 2) If \( x(t) \geq m_3 \) for all \( t \geq t_1 \), then we are done. If not, we may let \( t^* = \inf_{t > t_1} \{ x(t) < m_3 \} \). Then \( x(t) \geq m_3 \) for \( t \in [t_1, t^*) \). If \( t^* \neq nT \) for all \( n \in \mathbb{N} \) and \( x(t) \) is continuous then \( x(t^*) = m_3 \). If \( t^* = n_0 T \) for some \( n_0 \in \mathbb{N} \), let \( t^{**} = t^* - \epsilon_0 \), where \( \epsilon_0 \) is small enough, then \( x(t^{**}) \geq m_3 \). Without loss of generality, we may assume that \( t^* \neq nT \) for all \( n \in \mathbb{N} \). Suppose that \( t^* \in [n_1 T, (n_1 + 1)T) \) for some \( n_1 \in \mathbb{N} \). Select \( n_2, n_3 \in \mathbb{N} \) such that \( n_2 T > \frac{\log \left( \frac{c_1}{M + p} \right)}{-d + \delta} \) and \( (1 - p)n_2 + 1 \) \( \exp((n_2 + 1)\sigma T) \) \( R^{n_3} > 1 \), where \( \sigma = a - \frac{a}{K}m_3 - bcM^{1-\gamma} < 0 \). Let \( T' = n_2 T + n_3 T \). There are two possible cases for \( t \in (t^*, (n_1 + 1)T) \).

Case 1) \( x(t) < m_3 \) for all \( t \in (t^*, (n_1 + 1)T) \).

In this case we will show that there exists \( t_2 \in [(n_1 + 1)T, (n_1 + 1)T + T') \) such that \( x_2(t_2) \geq m_3 \). Suppose not. i.e., \( x(t) < m_3 \), for all \( t \in [(n_1 + 1)T, (n_1 + 1 + n_2 + n_3)T] \).
Then \( x(t) < m_3 \) for all \( t \in (t^*, (n_1 + 1 + n_2 + n_3)T] \). By (3.6) with \( u((n_1 + 1)T^+) = y((n_1 + 1)T^+) \), we have

\[
u(t) = \left( u((n_1 + 1)T^+) - \frac{q}{1 - \exp(-d + \delta)} \right) \exp((-d + \delta)(t - (n_1 + 1)T)) + u^*(t)
\]

for \( t \in (nT, (n + 1)T], n_1 + 1 \leq n \leq n_1 + n_2 + n_3 \). So we get \( |u(t) - u^*(t)| \leq (M + q) \exp((-d + \delta)nT) \) for \( n \in I_{n_1 + 1} \) and hence \( y(t) \leq u(t) \leq u^*(t) + \epsilon_1 \) for \( t \in [(n_1 + 1 + n_2)T, (n_1 + 1 + n_2 + n_3)T] \), which implies (3.7) holds on \( [(n_1 + 1 + n_2)T, (n_1 + 1 + n_2 + n_3)T] \).

As in step 1, we have

\[
x((n_1 + 1 + n_2 + n_3)T) \geq x_2((n_1 + 1 + n_2)T)R^{n_3}.
\]

Since \( y(t) \leq M \), for \( t \in (t^*, (n_1 + 1 + n_2)T] \), we obtain

\[
\begin{aligned}
x'(t) &\geq x(t) \left( a - \frac{a}{R} m_3 - bcM^{1-\gamma} \right), t \neq nT, \\
x(t^+) &\equiv (1 - p)x(t), t = nT.
\end{aligned}
\]

Integrating it on \( [t^*, (n_1 + 1 + n_2)T] \) we get

\[
x((n_1 + 1 + n_2)T) \geq m_3(1 - p)R^{n_2} \exp(\sigma(n_2 + 1)T).
\]

Thus \( x((n_1 + 1 + n_2 + n_3)T) \geq m_3(1 - p)^{n_2+1} \exp(\sigma(n_2 + 1)T)R^{n_3} > m_3 \) which is a contradiction. Now, let \( t = \inf_{t > t^*}\{ x(t) \geq m_3 \} \). Then \( x(t) \leq m_3 \) for \( t^* \leq t < t^* \) and \( x(t^*) = m_3 \). Thus (3.8) holds for \( t \in [t^*, t^*] \). By the integration of it on \( [t^*, t^*] \), we can get that \( x(t) \geq x(t^*) \exp(\sigma(t - t^*)) \geq m_3(1 - p)^{1 + n_2 + n_3} \exp(\sigma(1 + n_2 + n_3)T) \equiv m_1 \).

Case 2) There is a \( t' \in (t^*, (n_1 + 1)T] \) such that \( x_2(t') \geq m_3 \). Let \( t = \inf_{t > t'}\{ x(t) \geq m_3 \} \). Then \( x(t) \leq m_3 \) for \( t \in [t^*, t^*] \) and \( x(t) = m_3 \). Also, (3.8) holds for \( t \in [t^*, t^*] \). Integrating the equation on \( [t^*, t] \), we can get that \( x(t) \geq x(t^*) \exp(\sigma(t - t^*)) \geq m_3 \exp(\sigma T) \geq m_1 \).

Thus in both case the similar argument can be continued since \( x(t) \geq m_3 \) for some \( t > t_1 \). This completes the proof. \( \square \)

**Remark** Let \( q_{\text{max}} = (1 - \exp(-dT)) \left( \frac{d(1-\gamma)(\alpha T \ln(1-p))}{(1-\exp(-d(1-\gamma)T))} \right)^{1-\gamma} \). From Theorem 3.1 and Theorem 3.3, we know that the prey-free periodic solution is locally asymptotically stable if \( q > q_{\text{max}} \) and otherwise, the prey and predator can coexist. Thus \( q_{\text{max}} \) plays a role of a critical value that discriminates between stability and permanence.

4. Numerical Simulation

In this section, we will study dynamic behaviors of system (1.3) by means of numerical simulation because the continuous system (1.3) cannot be solved explicitly. Especially, we investigate the influence of impulsive perturbations numerically.
For this, we fix the parameters as follows:

\[(4.1) \quad a = 4.0, K = 2.0, b = 1.0, c = 5.5, d = 0.2, e = 9.0, \gamma = 0.4, p = 0.2.\]

Figure 1: Dynamical behavior of system (1.3). (a) Phase portrait of a T-period solution for \( q = 0 \). (b) Phase portrait of a T-period solution for \( q = 0.2 \).

Figure 2: Dynamical behavior of system (1.3) with \( q = 1.0 \). (a) The trajectory of \( x \) is plotted. (b) The trajectory of \( y \) is plotted.

From Figure 1(a), we can figure out that there exists a limit cycle of system (1.3) when \( q = 0 \). It follows from Theorem 3.1 that the prey-free periodic solution \((0, y^*(t))\) is locally asymptotically stable provided that \( q > q_{\text{max}} = 0.8517 \). A typical prey-free periodic solution of system (1.3) is exhibited in the Figure 2(a) and (b), where we observe how the variable \( y(t) \) oscillates in a stable cycle. In contrast, the prey \( x(t) \) rapidly decreases to zero. On the other hand, if the amount \( q \) of releasing species is smaller than \( q_{\text{max}} \), then prey and predator can coexist on a stable positive periodic solution (Figure 1(b)) and system (1.3) can be permanent which follows from Theorem 3.1. In Figure 3, we display a bifurcation diagram for prey and predator populations as \( q \) increasing from 0 to 1 with initial value \( x_0 = (1.0, 1.0) \). The resulting bifurcation diagram clearly shows that system (1.3) has rich dynamics.
An impulsively controlled Watt-type predator-prey system

Figure 3: Bifurcation diagrams of system (1.3) for $q$ when $0 < q < 1.0$. (a) $x$ is plotted. (b) $y$ is plotted.

Figure 4: Coexistence of prey and predator when $q = 0.58$. (a) A solution with initial value $(1, 1)$. (b) A solution with initial value $(0.04, 0.6)$.

including cycles, periodic doubling bifurcation, chaotic bands, periodic window, period-halving bifurcation, etc. In Figure 3, solutions with period $T$ are still stable for $q < 0.3885$. When $q > 0.3885$, they become unstable and solutions with period 3 begin to appear. Figure 3 illustrates an evidence for cascade of period doubling bifurcations leading to chaos when $0.4915 < q < 0.53$. We can capture a typical chaotic attractor when $q = 0.7$. (Figure 5(a)). We can also find that there exist sudden changes in Figure 3 when $q \approx 0.407, 0.4529, 0.58$ and $0.6071$. Furthermore, they can lead to non-unique attractors. Specially, there exist two attractors when $q = 0.58$, shown in Figure 4. These results show that just one parameter can give rise to multiple attractors. Narrow periodic windows and wide periodic windows are intermittently scattered. At the end of the chaotic region, there is a cascade of period-halving bifurcation from chaos to one cycle. (see Figure 5). Periodic halving is the flip bifurcation in the opposite direction.

The results we obtain in this paper show that the impulsive perturbations have significant effects on the stable limit cycle of system (1.2) and make the dynamics of system (1.3) more complicated.
Figure 5: Period-halving bifurcation from chaos to cycle. (a) Chaotic attractor for $q = 0.7$. (b) Phase portrait of a $4T$-period solution for $q = 0.74$. (c) Phase portrait of a $2T$-period solution for $q = 0.78$. (d) Phase portrait of a $T$-period solution for $q = 0.84$.

References

[1] J. F. M. Al-Omari, Stability and optimal harvesting in lotka-volterra competition model for two-species with stage structure, Kyungpook Math. J., 47(1)(2007), 31-56.
[2] J. F. Andrews, A mathematical model for the continuous culture of macroorganisms utilizing inhibitory substrates, Biotechnol. Bioeng., 10(1968), 707-723.
[3] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics: Ratio-dependence, J. Theor. Biol., 139(1989), 311-326.
[4] H. Baek, Dynamics of an impulsive food chain system with a Lotka-Volterra functional response, J. of the Korean Society for Industrial and Applied Mathematics, 12(3)(2008), 139-151.
[5] H. Baek, A food chain system with Holling-type IV functional response and impulsive perturbations, Computers and Mathematics with Applications, 60(2010), 1152-1163.
[6] H. Baek, On the dynamical behavior of a two-prey one-predator system with two-type functional responses, Kyungpook Math. J., 53(4)(2013), 647-660.
[7] H. Baek and Y. Do, Stability for a holling type iv food chain system with impulsive perturbations, Kyungpook Math. J., 48(3)(2008), 515-527.
[8] H. Baek and C. Jung, Extinction and permanence of a holling I type impulsive predator-prey model, Kyungpook Math. J., 49(4)(2009), 763-770.
[9] H. Baek and H. H. Lee, Permanence of a three-species food chain system with impulsive perturbations, Kyungpook Math. J., 48(3)(2008), No. 3, 503-514.
[10] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, vol. 66, of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Science & Technical, Harlo, UK, 1993.
[11] D. D. Bainov and M. B. Dimitrova, Oscillatory and asymptotic properties of solutions of nonlinear impulsive differential equations of third order with retarded argument, Kyungpook Math. J., 39(1)(1999), 111-118.
An impulsively controlled Watt-type predator-prey system

[12] D. D. Bainov and M. B. Dimitrova, A. B. Dishliev, Nonoscillatory solutions of a class of impulsive differential equations of n-th order with retarded argument, Kyungpook Math. J., 39(1)(1999), 33-46.

[13] D. D. Bainov and I. M. Stamova, Global stability of the solutions of impulsive functional differential equations, Kyungpook Math. J., 39(2)(1999), 239-249.

[14] M. A. Basudan, On population growth model with density dependence, Kyungpook Math. J., 41(1)(2001), 127-136.

[15] J. R. Beddington, Mutual interference between parasites or predator and its effect on searching efficiency, J. Animal Ecol., 44(1975), 331-340.

[16] J. Cost, Comparing predator-prey models qualitatively and quantitatively with ecological time-series data, PhD-Thesis, Institute National Agronomique, Paris-Grignon, 1998.

[17] M. B. Dimitrova, Criteria for oscillation of impulsive differential equations of first order with deviating argument, Kyungpook Math. J., 40(1)(2000), 29-37.

[18] M. Fan and Y. Kuang, Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional response, J. of Math. Anal. and Appl., 295(2004), 15-39.

[19] S. Gakkhar and R. K. Naji, Chaos in seasonally perturbed ratio-dependent prey-predator system, Chaos, Solitons and Fractals, 15(2003), 107-118.

[20] C. S. Holling, The functional response of predator to prey density and its role in mimicry and population regulations, Mem. Ent. Sec. Can., 45(1965), 1-60.

[21] S. -B. Hsu and T. -W. Huang, Global stability for a class of predator-prey systems, SIAM J. Appl. Math., 55(3)(1995), 763-783.

[22] V Lakshmikantham, D. Bainov, P. Simeonov, Theory of Impulsive Differential Equations, World Scientific Publisher, Singapore, 1989.

[23] B. Liu, Y. Zhang and L. Chen, Dynamic complexities in a Lotka-Volterra predator-prey model concerning impulsive control strategy, Int. J. of Bifur. and Chaos, 15(2)(2005), 517-531.

[24] Y. Liu and Z. Li, Second order impulsive neutral functional differential inclusions, Kyungpook Math. J., 48(1)(2008), 1-14.

[25] S. Ruan and D. Xiao, Global analysis in a predator-prey system with non-monotonic functional response, SIAM J. Appl. Math., 61(4)(2001), 1445-1472.

[26] E. Saez and E. Gonzalez-Olivares, Dynamics of a predator-prey model, SIAM J. Appl. Math., 59(5)(1999), 1867-1878.

[27] G. T. Skalski and J. F. Gilliam, Functional responses with predator interference: viable alternatives to the Holling type II mode, Ecology, 82(2001), 3083-3092.

[28] W. Sokol and J. A. Howell, Kinetics of phenol oxidation by ashed cell, Biotechnol. Bioeng., 23(1980), 2039-2049.

[29] G. Tr. Stamov, Almost periodic processes in ecological systems with impulsive perturbations, Kyungpook Math. J., 49(2)(2009), 299-312.

[30] G. i Tr. Stamov, Second method of lyapunov for existence of almost periodic solutions for impulsive integro-di erential equations, Kyungpook Math. J., 43(2)(2003), 221-231.
[31] W. Wang, H. Wang and Z. Li, *The dynamic complexity of a three-species Beddington-type food chain with impulsive control strategy*, Chaos, Solitons and Fractals, **32**(2007), 1772-1785.

[32] X. Wang, W. Wang and X. Lin, *Chaotic behavior of a Watt-type predator-prey system with impulsive control strategy*, Chaos, Solitons and Fractals, **37**(3)(2008), 706-718.

[33] X. Wang and Z. Li, *Global attractivity and oscillations in a nonlinear parabolic equation with delay*, Kyungpook Math. J., **48**(4)(2008), 593-611.

[34] K. E. F Watt, *A mathematical model for the effect of densities of attacked and attacking species on the number attacked*, Can. Entomol., **91**(1959), 129C144.

[35] R. Xu, M. A. J. Chaplain and F. A. Davidson, *Periodic solutions of a delayed population model with one predator and two preys*, Kyungpook Math. J., **44**(4)(2004), 519-535.

[36] F. Zhang, *Anti-periodic boundary value problem for impulsive differential equations with delay*, Kyungpook Math. J., **48**(4)(2008), 553-558.

[37] S. Zhang, L. Dong and L. Chen, *The study of predator-prey system with defensive ability of prey and impulsive perturbations on the predator*, Chaos, Solitons and Fractals, **23**(2005), 631-643.

[38] S. Zhang, D. Tan and L. Chen, *Chaos in periodically forced Holling type IV predator-prey system with impulsive perturbations*, Chaos, Solitons and Fractals, **27**(2006), 980-990.