IN-MEDIUM EFFECTIVE CHIRAL LAGRANGIANS
AND THE PION MASS IN NUCLEAR MATTER

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ABSTRACT

We argue that the effective pion mass in nuclear matter obtained from chiral effective lagrangians is unique and does not depend on off-mass-shell extensions of the pion fields as e.g. the PCAC choice. The effective pion mass in isospin symmetric nuclear matter is predicted to increase slightly with increasing nuclear density, whereas the effective time-like pion decay constant and the magnitude of the density-dependent quark condensate decrease appreciably. The in-medium Gell-Mann-Oakes-Renner relation as well as other in-medium identities are studied in addition. Finally, several constraints on effective lagrangians for the description of the pion propagation in isospin symmetric, isotropic and homogenous nuclear matter are discussed.

1. Introduction

The suggestion of Kaplan and Nelson\cite{1} that attractive S-wave interactions between kaons and nucleons could lower the effective mass of kaons to the extent that kaons could condense in dense neutron star matter at several times nuclear saturation density has started in recent years a considerable discussion on the behaviour of the Pseudo-Goldstone bosons of strong interaction physics, the kaons and also pions, in dense nuclear matter. Whereas the kaon case is plagued by a lot of additional complications as the role of resonances such as the \( \Lambda(1405) \), which governs low-energy \( K^-p \) scattering, the coupling to the \( \Sigma\pi \) channel and the large size of the kinematical regions over which smoothness assumptions are postulated to hold, the S-wave pion propagation in symmetric nuclear matter is a much cleaner case\cite{2,3,4} and can therefore serve as test ground for applying chiral perturbation theory ideas to finite nuclear densities.

Recently, several authors have claimed that the method used to motivate and describe meson condensation is incorrect\cite{3,5}. They argued that chiral effective lagrangians are inconsistent with current algebra and PCAC\cite{3,5}. Secondly, they claimed that the incorporation of these off-meson-mass-shell amplitudes in the calculation inevitably leads to an effective repulsion which serves to inhibit meson condensation\cite{5}.

In a previous letter\cite{6} we showed that these claims do not hold, as either incomplete chiral lagrangians were considered or the source coupling was done inconsis-
tently. In fact, in Ref.\[6\] it was shown – in line with well-established theorems\[7\] – that the S-wave meson-nucleon scattering amplitudes obtained off-meson-mass-shell are entirely unphysical as they are subject to the choice of the meson field, and are thus not to be viewed as constraints on a theory. Furthermore, it was shown that the effective meson mass in nuclear matter is independent of the choice made for the meson field. This is to be understood as a consequence of a general rule: any physically relevant observable is independent of the choice of meson field variables, as is the case for S-matrix elements\[7\]. This conjecture was supported by two calculations\[6\]: one using a formulation of chiral perturbation theory for which the canonical meson field is to be identified with the divergence of the axial vector current, namely that originating in the work of Gasser and Leutwyler\[8\], and one using the traditional treatment, originally due to Kaplan and Nelson\[1\], in which the meson field is not to be identified with the divergence of the axial vector current.

Below, we discuss how chiral perturbation theory can be applied to the analysis of S-wave pion propagation. We consider tree level lagrangians throughout, working to $\mathcal{O}(Q^2)$. In section 3 we illustrate our results for homogeneous, isotropic, isospin symmetric and spin-unpolarized nuclear matter, and evaluate nucleon operators in the mean field approximation, such that the corresponding results hold modulo nuclear correlation corrections. We work out the in-medium pion mass, the effective pion decay constant, the in-medium quark condensate, the Gell-Mann-Oakes-Renner relation and the PCAC relation in nuclear matter. In section 4 we discuss how the new developments about non-relativistic chiral lagrangians\[9\] and generalizations to four-quark condensates\[10\] can constrain the structure of the in-medium chiral lagrangians.

2. Chiral Perturbation Theory and S-wave Pion Propagation

Here, we briefly review the functional integral formulation of chiral perturbation theory developed by Gasser and Leutwyler\[8\], which was extended to include nucleons by Gasser, Sainio and Švarc\[11\]. In this approach, the effective generating functional for QCD Green functions, the vacuum-to-vacuum transition amplitude $\exp(iZ_{\text{QCD}})$, is developed as follows. External color-neutral sources, the isovector vector $v_\mu$, the isovector axialvector $a_\mu$, the isoscalar scalar $s$ and isovector pseudoscalar $p$, are coupled to the corresponding quark currents of the QCD action and assigned chiral SU(2)$_L \times$ SU(2)$_R$ transformations such that the source-extended action is locally chiral invariant. Note the current quark masses of the QCD lagrangian are hidden in the scalar isoscalar coupling $\bar{q}sq$ by $s$ containing a constant piece: $s = s_0 + s'$ where $s_0$ is the quark mass matrix $\mathcal{M}$. The fact that the sources are coupled in this way ensures that chiral QCD-Ward identities are satisfied. The central idea is now that the generating functional for the low-energy effective theory on the hadronic level, $\exp(iZ_{\text{eff}})$, should depend on the same external sources. In this way, one identifies the to-be-determined Ward identities of QCD with those of
the low-energy effective theory. As the effective theory is a that stage – and has to be – completely general (besides its local chiral invariance), further empirical facts are needed in order to constrain the effective generating functional to something useful and workable. Naturally, the effective theory should respect the usual invariances under e.g. Lorentz-transformations, parity- and time-reflection, charge-conjugation and should be also local. Its sources and fields have to be color-neutral. All the hadronic fields entering the effective action are dummy fields as they are integrated over in the generating functional formalism. At low energies they can be limited to the pions (and kaons), as the masses of these Pseudo-Goldstone bosons are much smaller than the masses of all the other hadrons (which are of the order of the chiral symmetry breaking scale $\Lambda \simeq 1$ GeV) and as therefore the low-momentum decoupling theorems for Goldstone bosons hold: at low $Q$ the Goldstone bosons are weakly interacting, the hadronic Green’s functions are dominated by poles due to Pseudo-Goldstone boson exchange, and the vertices in the Goldstone-Goldstone interaction admit a Taylor series expansion in powers of the small momenta $Q$. In Weinberg’s chiral counting scheme[12] the various tree and loop terms are ordered in powers of the external momenta $Q$ and the pion mass $m_\pi$ (or kaon mass $m_K$).

The values of tree-level coefficients follow from empirical input. The final ingredient in standard chiral perturbation theory is the conjecture that the non-vanishing quark condensate $\langle 0 | \bar{q} q | 0 \rangle$ is not only the order parameter of spontaneous breaking of chiral symmetry, but that it is so large that higher quark condensates can be neglected at leading order[10]. This implies (see the discussion in section 4) that the current quark mass matrix $M$ scales as the squared pion mass and that the Weinberg counting is in squared powers of the external momenta, $Q^2$, and in linear powers of the average current quark mass, $\bar{m} = (m_u + m_d)/2$.

At low-energies the hadronic action, $S_{\text{eff}}$, is therefore formulated in terms of a $2\times2$ dummy field $U$ (which transforms linearly under SU(2)$_L \times$ SU(2)$_R$), involving pions. Note that in this formalism, the sources $s$, $p$, $v_\mu$, and $a_\mu$ are coupled from the start to quark bilinears and are therefore directly associated with SU(2)$_L \times$ SU(2)$_R$ currents in quark variables. The PCAC prescription thus emerges naturally as the sources are transcribed to the hadronic level. Similarly, one may introduce nucleons, $N(\bar{N})$, coupled to corresponding external sources, $\bar{\eta}$ ($\eta$), which transform non-linearly under chiral transformations in order to ensure that the coupling terms are (locally) chiral invariant. Note in this case the caveat that the sources $\eta$ and $\bar{\eta}$ are only defined on the hadronic level as there does not exist any simple interpretation on the QCD level of a baryonic source. Furthermore, the generating functional $\exp(i Z_{\text{eff}})$ should not be linked to the vacuum-to-vacuum transition amplitude any longer, but rather to the $n$-nucleon to $n$-nucleon transition amplitude (with $n \geq 1$)[11]. It reads

$$e^{i Z_{\text{eff}}[s,p,v_\mu,a_\mu,\eta,\bar{\eta}]} = \mathcal{N} \int dU dN d\bar{N} e^{\frac{i}{2} \int d^4x \left( \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \eta N + \bar{\eta} \bar{N} \right)} ,$$

(1)
where $N$ is an overall normalization, $U, N, \bar{N}$ are dummy fields, and the lagrangians $\mathcal{L}_{\pi\pi}$ and $\mathcal{L}_{\pi N}$ are dependent on the sources $s, p, v_\mu$ and $a_\mu$. In what follows, we shall use the scalar source $s$ to generate the quark mass matrix, $s = \mathcal{M} = \text{diag}(m_u, m_d)$, and furthermore retain only the source for asymptotic pions, $p$, but set $v_\mu = a_\mu = 0$ (unless otherwise specified). The sources $\bar{\eta}$ and $\eta$ generate one-nucleon in- and out-states. The nucleons are treated in the static fermion formalism\[13\], in which nucleon loops play no role (see also appendix A of Ref.\[14\]), and the nucleon determinant may therefore taken to be unity.

The lagrangian entering in the generating functional \(1\), $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N}$, is to leading order given by the nucleon kinetic energy term, $i\bar{N}(v \cdot \partial)N$ and to subleading order, $O(Q^2)$, by\[11, 13\]

\[
\mathcal{L}^{(2)}_{\pi\pi} = \frac{f^2_\pi}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{f^2_\pi}{4} \text{Tr}(U^\dagger \chi + \chi^\dagger U) ,
\]

\[
\mathcal{L}^{(2)}_{\pi N} = -\frac{\sigma}{4m^2_\pi} \bar{N} N \text{Tr}(U^\dagger \chi + \chi^\dagger U) + c_2 \bar{N}(v \cdot u)^2 N + c_3 \bar{N}(u \cdot u)N ,
\]

where $u_\mu = iu^\dagger \partial_\mu U u^\dagger$, $U = u^2 = \exp(i\pi^a / f_\pi)$, $\chi = 2B(s + ip)$, and $v_\mu$ is the four-velocity of the nucleon which reduces to $v_\mu = (1, 0, 0, 0)$ in the rest frame of the nucleon. The empirical values\[16\] of $f^2_\pi$ (and the later to be used $m^2_\pi$) include $O(Q^2)$ corrections to the corresponding quantities at tree level. They are used here for notational convenience as we anyhow neglect corrections to the lagrangian of $O(Q^2)$ and higher. The constants $\sigma, c_2$ and $c_3$ are linear in the quark masses and therefore of order $O(Q^2)$. The constant $\sigma$ is to be identified with the sigma term, $\sigma(t=0)$, which also serves to increase the nucleon mass over that in the SU(2) chiral limit, $m_N = m_0 + \sigma$, where $m_0 \approx 890$ MeV, using $\sigma = 45$ MeV\[16\]. Thus the sigma term is fixed to be positive. We do not write down the Weinberg (vector) term explicitly in the lagrangian above, as it does not enter in the S-wave pion propagation in isospin-symmetric matter to be discussed in the next section.

Expanding $U$ to second order in $\pi^a$, we find

\[
\mathcal{L}_{\text{eff}} = \bar{N}(iv \cdot \partial - \sigma)N + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2} m^2_\pi \pi^2
+ \frac{1}{f^2_\pi} \left( \frac{1}{2} \sigma \pi^2 + c_2(v \cdot \partial \pi)^2 + c_3(\partial_\mu \pi)^2 \right) \bar{N} N + j^a \pi^a \left( 1 - \frac{\sigma \bar{N}N}{f^2_\pi m^2_\pi} \right) .
\]

The pseudoscalar source is $j^a = 2Bf_\pi p^a$, in terms of the original source $p^a (p=p^a \tau^a)$, and of the “quark condensate” $-2f^2_\pi B = -2f^2_\pi m^2_\pi / (m_u + m_d)$, as follows from the quadratic expansion. Since Green functions are obtained by taking functional derivatives of the generating functional with respect to the source $j^a$, the nontrivial coupling of the source to the pion field in Eq.\[3\] plays an important role in the consistent\#1

\#1The pion decay constant $f_\pi = 93$ MeV and the pion mass $m_\pi = 139$ MeV.
description of the pion-off-shell S-wave \( \pi N \) amplitude, see \[1\]. From the lagrangian \( \[1\] \) we find the isospin even scattering length, \( a^+_{\pi N} \):

\[
a^+_{\pi N} = \left\{ 4\pi f^2_\pi (1 + m_\pi/m_N) \right\}^{-1} \left( 2(c_2 + c_3)m^2_\pi + \sigma \right) + \mathcal{O}(m_\pi^3). \tag{5}
\]

Empirically, \( a^+_{\pi N} = -0.0083m_\pi^{-1} \) \[17\], corresponding to a repulsive interaction. Using \( \sigma \approx 45 \) MeV, we find \( (c_2 + c_3)m^2_\pi \approx -26 \) MeV. Improved values for the constants can be found by including loop corrections \[13\]. The first corrections of \( \mathcal{O}(m^2_\pi) \) result from finite loop terms. This is the reason why in nuclear matter all quantities of order \( \mathcal{O}(Q^2) \) get their first correction already at \( \mathcal{O}(Q^3) \) and not at \( \mathcal{O}(Q^4) \) as their free-space analogs.

### 3. The Effective Meson Mass in Nuclear Matter

Given the problems encountered in extending chiral perturbation theory from the meson to the baryon sector, it is not surprising that a rigorous formulation of the expansion in nuclear matter has not yet been found. The presence of an additional scale (the Fermi momentum of nucleons), the breaking of Lorentz-invariance, and nuclear correlations, add new levels of complexity to the formulation of a chiral expansion. As a first step, one simply uses a free space chiral expansion, such as those outlined above and evaluates nucleon operators at the mean field level, and consequently works with the action to linear order in density: namely, \( S = \int d^4x \mathcal{L}^{(2)}(\rho) \) with

\[
\mathcal{L}^{(2)}(\rho) = \frac{f^2_\pi}{4} \left( g^{\mu\nu} + \frac{D^{\mu\nu}}{f^2_\pi} \right) \text{Tr}(\partial_\mu U \partial_\nu U^\dagger) + \frac{f^2_\pi}{4} \left( 1 - \frac{\sigma\rho}{f^2_\pi m^2_\pi} \right) \text{Tr}(U^\dagger \chi + \chi^\dagger U), \tag{6}
\]
and \( D^{\mu\nu} \equiv 2c_2\pi^{\mu\nu} + 2c_3g^{\mu\nu} \), as follows from Eqs.\[2\] and \[3\]. One may raise the question of the role of the pion interpolating field in this context. We will argue that the basic idea, established rigorously in the case of free space scattering \[2\], that physically relevant observables are independent of the choice of field variables, also holds in nuclear matter. In this case, the relevant observable is the position of the pole of the pion propagator in symmetric nuclear matter \[3\]. The pole position is often referred to as the effective mass, which we shall also do here.

As alluded to above, in the nucleon mean field approximation we set \( \langle \bar{N}N \rangle = \rho \) (we also approximate the vector density by the scalar density). Note that from \( \[1\] \) it is immediately apparent that since \( U \), and therefore \( \pi \), is a dummy variable, any observable must be independent of this field as long as it is correctly normalized and has a nonvanishing matrix element between the pion and the vacuum. At tree level we have \( Z[j; \rho] = S[\pi] + \int d^4x j^a(x)\pi^a(x)(1 - \sigma\rho/f^2_\pi m^2_\pi) \). Thus the tree-level effective action, \( \Gamma[\phi_{\pi}; \rho] = Z[\phi_{\pi}; \rho] - \int d^4x j\phi_{\pi} = S[(1 - \sigma\rho/f^2_\pi m^2_\pi)^{-1} \phi_{\pi}] \), reads

\[
\Gamma[\phi_{\pi}; \rho] = \frac{1}{2} \int d^4x \left( 1 - \frac{\sigma\rho}{f^2_\pi m^2_\pi} \right)^{-2} \partial_\mu\phi_{\pi}\partial_\nu\phi_{\pi} \left( g^{\mu\nu} + \frac{D^{\mu\nu}}{f^2_\pi} \right) - m^2_\pi \left( 1 - \frac{\sigma\rho}{f^2_\pi m^2_\pi} \right) \phi^2_{\pi}. \tag{6}
\]

\( \#^2Z[j; \rho] := Z^{\text{eff.}}_{M,j} \) \[M, j, 0, 0, 0, 0\] and the so-called classical pion field: \( \phi^c_{\pi} = (1 - \sigma\rho/f^2_\pi m^2_\pi)\pi^a. \)
from which we obtain the in-medium charged pion propagator

\[
D(q, \rho) = \frac{i \left(1 - \frac{\sigma \rho}{f_\pi^2 m_\pi^2} \right)^2}{q^2 - m_\pi^2 + \frac{f_\pi^2}{f_\pi^2} \sigma + 2c_2 (v \cdot q)^2 + 2c_3 q^2} + O(m_\pi^3). \tag{7}
\]

Evaluating the poles of the propagator we find the effective pion mass \(m_\pi^*(\rho) := \omega^2(k=0; \rho)\) in symmetric nuclear matter to be

\[
m_\pi^*(\rho) = m_\pi^2 \left\{1 - \rho/(f_\pi^2 m_\pi^2)\right\} / \left(1 + 2(c_2 + c_3) \rho / f_\pi^2 \right) + O(m_\pi^3) \tag{8}
\]

where in Eq. (8) we explicitly show the prediction to linear order in density. Eq. (8) gives \(m_\pi^*(\rho) = m_\pi - 2\pi \kappa_\pi^+ \rho / m_R\), to linear order in density, where \(m_R\) is the reduced mass of the pion-nucleon system. The behaviour of the effective mass to linear order in density is as expected from the lowest order optical potential, without reference to chiral perturbation theory. The effective mass receives additional contributions over those given by Eq. (8) at higher than linear order in density, from factors such as those mentioned at the beginning of this section.

The fact that the lagrangian \(\mathcal{L}_{\text{eff}}\) embodies the PCAC choice of the interpolating pion field appears in the residue of the pion propagator, not in the effective pion mass. A different off-shell choice for the pion field implies a change in the source coupling in (4) (e.g. \(j^a_{\pi} \pi^a \) instead of \(j^a_{\pi} \pi^a (1 - \sigma \rho / f_\pi^2 m_\pi^2) \)) and results to a different residuum (e.g. 1 instead of \((1 - \sigma \rho / f_\pi^2 m_\pi^2)^2\) of the propagator (7), but leaves the pole position invariant. We therefore conclude that there is no discrepancy in the effective mass obtained from theories using different interpolating pion fields.

The relation of the effective pion mass in symmetric nuclear matter, to the Gell-Mann-Oakes-Renner (GMOR) relation

\[
f_\pi^2 m_\pi^2 = -\frac{m_u + m_d}{2} \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle + O(m_\pi^4). \tag{10}
\]

Solving for \(B\) and using the relation \(B(m_u+m_d) = m_\pi^2\) one thus obtains the GMOR relation. To study the GMOR relation with respect to the in-medium pion mass, we use the lagrangian \(\mathcal{L}_{\text{eff}}\) in the nucleon mean field approximation, Eq. (6).
Since matter breaks Lorentz-invariance (but still keeps rotational invariance, if it is isotropic), it is convenient to separate space and time components \[20\] via
\[
f_t^2(\rho) = f_\pi^2 \left( 1 + \frac{D^{\mu\nu} \rho}{f_\pi^2} \right) + \mathcal{O}(m_\pi).
\] (11)

Starting from Eq.(6) and using the same method as to derive Eq.(10), we obtain the density-dependent quark condensate
\[
\langle \bar{u}u + \bar{d}d \rangle_\rho = \langle 0|\bar{u}u + \bar{d}d|0 \rangle \frac{1}{2} \sigma_\rho f_t^2(\rho) + \mathcal{O}(m_\pi),
\] (12)
a result which is, in fact, model-independent \[21\]. Eq.(12), when combined with the effective pion mass as given in Eq.(8), yields the in-medium GM OR relation in the nucleon mean-field approximation:
\[
f_t^2(\rho) m_\pi^2(\rho) = -\langle \bar{u}u + \bar{d}d \rangle_\rho + \mathcal{O}(m_\pi^3),
\] (13)

(Other discussions of the GMOR relation at finite density are given in Ref.[22].) It is therefore only the time-component of the coupling constant, \(f_t^*\), that enters in the GMOR relation at finite density. As a function of density, \(f_t^*\) and \(-\langle \bar{u}u + \bar{d}d \rangle_\rho\) decrease appreciably (to about two-thirds of their vacuum value at nuclear matter density), whereas \(m_\pi^*\) increases, though very slowly.

It is worthwhile to check that \(f_t^*(\rho)\), as given by Eq.(11), agrees with the definition in terms of the axial current coupling to the pion in matter,
\[
\langle 0|\bar{q} \gamma_\mu (1/2) \tau^a q|\pi^b \rangle_\rho = ip_0 \delta_{ab} f_t^*(\rho) + \mathcal{O}(m_\pi^3).
\] (14)

As the pion state in the medium, the ket \(|\pi^b \rangle_\rho\), is not known, the expectation value is evaluated from the axial vector two-point function where the information about the pion state doesn’t enter because of closure. The axial vector correlator reads at zero nuclear density (see Ref.[8]):
\[
\frac{\delta^2 Z_{\text{eff}}}{\delta a_\mu^a (\pm q) \delta a_\nu^b (q)} |_{a=v=p=0; s=\mathcal{M}} = i \int dx e^{iq(x-y)} \langle 0|TA_\mu^a(x)A_\nu^b(y)|0 \rangle = \delta^{ab} \left\{ g_{\mu\nu} f_\pi^2 + \frac{g_{\mu\nu} q_\mu q_\nu}{m_\pi^2 - q^2} \right\} + \mathcal{O}(q^2).
\] (15)

Eq.(15), and other correlators, may be evaluated at finite density, in the mean field approximation, by re-instating the general sources \(s\), \(v^\mu\) and \(a^\mu\) in \(Z_{\text{eff}}\), expanding the action to second order in the pionic field, and integrating out the pionic degrees of freedom. The second order variation with respect to the external sources then gives the two-point function. For example, one finds that – up to \(\mathcal{O}(m_\pi)\) corrections – the variation with respect to \(a_0 \) gives an in-medium correlator of the same form as the
time-time-component of Eq. (15), but with \( f_\pi^2 \) replaced by \( f_\pi^2(\rho) \), Eq. (11), and \( m_\pi \) replaced by the effective mass \( m_\pi^\ast \). Thus the desired equivalence is established. This result is independent of the off-shell extension of the pion field, as the pseudoscalar sources, \( p^a \), do not enter in this calculation.

In an analogous manner one may arrive at other relations valid at finite density. Evaluation of the in-medium pseudoscalar correlator results in

\[
g_\pi^\ast^2(\rho) = (2B_\pi f_\pi )^2 \left( 1 - \frac{\sigma\rho}{f_\pi^2 m_\pi^2} \right)^2 \left( 1 + D^{00} \rho f_\pi^2 \right) + \mathcal{O}(m_\pi) ,
\]

( where \( g_\pi \) in the vacuum is defined as \( g_\pi \delta^{ab} \equiv \langle 0 | \bar{q} i \gamma^5 \tau^a q | \pi^b \rangle \) ). This result is dependent on the off-shell extension of the pion field, as the calculation explicitly involves functional derivatives with respect to the pseudoscalar source, \( p^a \). The axialvector-pseudoscalar correlator is \( p^a \)-dependent, as well. Using Eqs. (12) and (8) one can then check that the finite-density version of the PCAC relation [8] holds under the source-coupling of (4):

\[
f_\pi^\ast(\rho) m_\pi^\ast^2(\rho) = \frac{m_u + m_d}{2} g_\pi^\ast(\rho) + \mathcal{O}(m_\pi^3) .
\]

### 4. Non-relativistic Chiral Lagrangians

Note the statements of the last section about the S-wave pion propagation in homogeneous, isotropic nuclear matter result from an \( \mathcal{O}(Q^2) \) chiral lagrangian to linear order in the density in the mean-field approximation. Already at this order the presence of the nuclear matter background distinguishes a preferred rest frame for the pion propagation and therefore Lorentz-invariance is broken. This manifests itself e.g. in distinct values of the time-like and space-like effective pion decay constant, \( f_\pi^\ast(\rho) \) and \( f_\pi^\ast(\rho) \), respectively. Thus in trying to build up a chiral perturbation theory for nuclear matter – valid for even higher densities – one cannot any longer insist on Lorentz-invariance as one of the preconditions. Rather it is has to be replaced by the left-over three-dimensional Euclidean rotational invariance, in case the background matter is still isotropic and homogenous. The question is whether there exists a modified chiral perturbation theory under such non-relativistic preconditions. Fortunately, in a different context, namely solid state physics, such non-relativistic chiral perturbation theory is already known, see Leutwyler’s work [9] and references therein. In Ref. [9] an effective field theory is constructed which is relevant for the low energy analysis of spontaneously broken symmetries in the non-relativistic domain and which applies to any system for which Goldstone modes are the only excitations without energy gap. Let us assume that the corresponding hamiltonian is symmetric with respect to a Lie group \( G \), whereas the ground state is only invariant under the subgroup \( H \subset G \). Therefore the effective theory involves \( \text{dim}G - \text{dim}H \) real fields which we will still refer to as ‘pion’ fields, \( \pi^a \), where small Latin indices \( a = 1, \ldots, \text{dim}G - \text{dim}H \) denote the components of the effective field.
The leading order effective lagrangian (without explicit symmetry breaking) is there-

rally invariant. The term \( L \) implicitly density- or background-dependent of

where the lagrangian \( L^{(nt,ns)} \) is of \( \mathcal{O}(\omega^{nt}, |q|^{ns}) \) (with \( nt \) and \( ns \) positive integer), is
chiral symmetric and rotational invariant. The last point implies that \( ns \) has to be
even. The decoupling of the Goldstone bosons at low energies and momenta excludes
the effective lagrangian \( L_{\text{eff}}^{(0,0)} \) (which contains only pion fields, but no derivatives).
The leading order effective lagrangian (without explicit symmetry breaking) is there-
fore of \( \mathcal{O}(\omega, |q|^0) \) and collects only pion vertices with one time derivative as well as
those with one insertion of the time-like external source \( f_0^A \) which is counted as
\( \mathcal{O}(\omega^1) \). It reads \( L_{\text{eff}}^{(1,0)} = c_0(\pi)\pi^a + e_A(\pi)f_0^A \). The quantities \( c_0(\pi) \) and \( e_A(\pi) \) are
implicitly density- or background-dependent of \( \mathcal{O}(1) \) and make the lagrangian chiral-
rally invariant. The term \( L_{\text{eff}}^{(1,0)} \) cannot occur in Lorentz-invariant effective theories.
The next corrections are of \( \mathcal{O}(\omega^2, |q|^0) \) and \( \mathcal{O}(\omega^0, |q|^2) \), respectively \[9\]:

\[
L_{\text{eff}}^{(2,0)} = \frac{1}{2} g_{ab}(\pi) \pi^a \pi^b + \bar{h}_{ab}(\pi) f_0^A \pi_a + \frac{1}{2} k_{AB}(\pi) j_0^A j_0^B \\
L_{\text{eff}}^{(0,2)} = -\frac{1}{2} g_{ab}(\pi) \partial_i \pi^a \partial_i \pi^b - h_{ab}(\pi) j_0^A \partial_i \pi^a - \frac{1}{2} k_{AB}(\pi) j_0^A j_0^B ,
\]

where \( j_i^A \) is a space-like source of \( \mathcal{O}(|q|^1) \) and \( \bar{g}_{ab}(\pi), g_{ab}(\pi), \bar{k}_{AB}(\pi), k_{AB}(\pi) \) etc.
are ‘time-like’ and ‘space-like’ metric tensors in the coset space and group space,
respectively. Lorentz-invariance would imply the coincidence of the barred with the
unbarred terms \((c \equiv 1)\). Note that \( e_A(\pi) \) at \( \pi = 0 \) gives a term in the effective lagrangian
which is linear in the external source and hence determines the expectation value
of the charge density in the ground state, \( e_A(0) = \langle gs | J_0^A(x) | gs \rangle \). For non-
abelian symmetries the charge densities transform non-trivially under the group \( G \)
such that their expectation values can serve as order parameters. If the charge den-
sity acquires a non-zero expectation value, as e.g. the spin-density in the ferromagnet,
the first order effective lagrangian is non-vanishing and the dispersion follows the
usual non-relativistic law, \( \omega \propto q^2 \). If, however, \( e_A(0) = 0 \) and therefore the charge
density does not acquire a non-zero expectation value, as e.g. for an antiferromagnet,
the effective lagrangian \( L_{\text{eff}}^{(1,0)} \) vanishes identically (because of the equation of motion
and chiral Ward identity \[8\]) and the dispersion law reads \( \omega \propto |q| \). The existence
of a non-trivial source for the charge density is needed for building up an \((nt, ns)\)
effective lagrangian with odd integer values for \( nt \). Transcribing these principles
to the low-energy pion propagation in an isotropic and homogenous nuclear matter
background, we see that we can expect the ‘ferromagnetic’ dispersion in case we have a non-vanishing external isovector source, i.e. the propagation in isospin non-symmetric nuclear matter where the presences of the isovector-density implies that the Weinberg-term (an $O(\omega)$ term) is governing the pion evolution. However, for isospin symmetric nuclear matter, all external isovector sources are zero such that Weinberg-type terms are ineffective and the pion propagation – in the chiral limit – has to follow the ‘antiferromagnetic dispersion’ law of lagrangians of proto-type (13) and (24). Rewriting (13) and (24) in the standard quaternion-formulation, one gets the following structure [9]

$$L_{\text{eff}}^{m.f.} = \frac{F_1^2}{4} \text{Tr}(\partial_0 U \partial_0 U^\dagger) - \frac{F_2^2}{4} \text{Tr}(\partial_i U \partial_i U^\dagger) + O(\omega^4, \omega^2 |q|^2, |q|^4)$$

with two effective coupling constants $F_1$ and $F_2$ such that the dispersion law corresponds to a massless particle (as there is no explicit symmetry breaking included) moving with the velocity $v = F_2/F_1$: $\omega(q) = v|q| + \cdots$. (Note $c \equiv 1$.) Now, our mean-field lagrangian (6) can be recasted into the following form

$$L_{\text{eff}}^{m.f.} = \frac{f_t^2(\rho)}{4} \text{Tr}(\partial_0 U \partial_0 U^\dagger) - \frac{f_s^2(\rho)}{4} \text{Tr}(\partial_i U \partial_i U^\dagger) + \frac{f_\pi^2}{4} \frac{B^*(\rho)}{B} \text{Tr}(U \chi^\dagger + U^\dagger \chi) + O(m_\pi^3; \rho^\nu |\nu>1) .$$

(22)

Note (22) is consistent to $O(Q^2)$ (and in the chiral limit) with the general expression of the leading-order non-relativistic antiferromagnetic effective lagrangian. Thus the in-medium lagrangian for pion propagation in symmetric nuclear matter is to this order the transcription of the standard Lorentz-invariant non-linear $\sigma$-model to a spatially rotational-invariant generalization with two – in general – different density-dependent coefficients. In the same line of thought one would naively expect that the next corrections correspond to the generalizations of the ten SU(2) vacuum terms of $O(Q^4)$ (see Ref.[8]) to eighteen terms – again with density-dependent coefficients – which would scale as $O(\omega^4)$, $O(\omega^2 |q|^2)$, $O(|q|^4)$, $O(\omega^2 m_\pi^2)$, $O(|q|^2 m_\pi^2)$ or $O(m_\pi^4)$, respectively. However, the mean-field lagrangian predicts that the first corrections already appear at order $O(m_\pi^3)$ which naively seems to be in contradiction with the general scheme based on isospin-symmetric background matter: First, the non-existence of isospin-symmetry breaking sources together with the chiral symmetry excludes kinetic lagrangians of odd order in $n_t$ (odd terms in $n_s$ are anyhow excluded because of the spatial rotational symmetry of the isotropic background). Second, 1-loop corrections to the 2nd order lagrangian have to be of $O(\omega^4)$, $O(|q|^4)$ or $O(\omega^2 |q|^2)$ and therefore of $O(m_\pi^4)$ (if $\omega$ and $|q|$ are counted as $O(m_\pi)$). How can $O(m_\pi^5)$ correction terms show up in the general scheme? We know they have to as they already are present on the simple mean-field level to linear order in the density. The answer is we need a $L^{(3)}$ lagrangian which cannot be purely kinetic and which is not
present in the standard formulation of chiral perturbation theory \[8\]. Fortunately, chiral perturbation theory can be generalized to incorporate even such terms \[10\]. One of the preconditions on standard chiral perturbation is that the parameter \(B\) (defined in the scalar-pseudoscalar source \(\chi = 2B(s + ip)\)) is of order of the chiral symmetry breaking scale \(\Lambda \simeq 1\ \text{GeV}\). In other words, as \(B\) determines the two-quark condensate, the precondition is that the magnitude of \(\langle \bar{q}q \rangle\) is large compared with the four quark condensate such that the GMOR relation holds and that therefore \(m_\pi^2 \propto \hat{m} = (m_u + m_d)/2\). Then the latter relation implies that \(\hat{m}\) has to be counted as \(\mathcal{O}(Q^2)\). However, as we have seen from the model-independent relation \[13\], the in-medium quark condensate decreases rather rapidly: Already at nuclear matter densities it drops to about two-thirds of its vacuum value. Thus, it is not obvious any longer that the in-medium four-quark condensate can be safely neglected at \(\mathcal{O}(Q^2)\). In Ref.\[10\] an extension to the GMOR relation is discussed; i.e. terms of order \(\langle 0|\bar{q}q\bar{q}q|0\rangle\) are taken into account so that the pion mass square is parameterized as \(m_\pi^2 = \hat{m}B_0 + \hat{m}^2A_0\), where \(A_0\) is related to the four-quark condensate. In the vacuum (and for SU(2)) \(B_0/2A_0 \gg \hat{m}\), so the second term is of no importance. But if one now goes to finite density, the quark condensate \(-\langle \bar{q}q \rangle\rho\) decreases in value. Thus the effect of the four-quark condensate doesn’t need to be small any longer compared with the two-quark condensate. This has consequences for an in-medium effective chiral lagrangian: In the vacuum, and for SU(2), it is not necessary to include in the leading-order effective lagrangian the four-quark condensate directly (as in Ref.\[10\]) as the two-quark condensate is large and doesn’t change. But in the medium the two-quark condensate changes and decreases in value (it can even become vanishingly small), such that one has to offer to the in-medium effective lagrangian the possibility that the four-quark condensate determines the effective pion mass, too. This demands a) the generalization of the \(L^{(2)}\) terms (to include three additional contributions) and b) the inclusion of eleven distinct \(L^{(3)}\) terms as first correction, see Ref.\[10\]; both changes will modify the r.h.s. of the GMOR-like relation. In general, the expansion of the vacuum effective lagrangian

\[
\mathcal{L}^{\text{vac}}_{\text{eff}} = \sum_{\nu=2,4,6,...} \sum_{k,l \geq 0} \frac{1}{2(k+l+\nu)} \mathcal{L}^{(\nu)}_{2k;l} \quad \text{with} \quad \mathcal{L}^{(\nu)}_{2k;l} \sim Q^{2k}(B_0 \hat{m})^l
\]

should be replaced in isospin symmetric, isotropic and homogenous matter by the expansion

\[
\bar{\mathcal{L}}_{\text{eff}}(\rho) = \sum_{\nu=2,3,5,...} \sum_{j,k,l,n \geq 0} \bar{\mathcal{L}}^{(\nu)}_{2j;2k;l,n}(\rho) \quad \text{with} \quad \bar{\mathcal{L}}^{(\nu)}_{2j;2k;l,n}(\rho) \sim \omega^{2j} |q|^{2k}(B^*(\rho)\hat{m})^l\hat{m}^n.
\]

Thus a) the coefficients of the various lagrangians are density-dependent, b) Lorentz-invariance is broken down to just spatial rotational invariance such that \(Q^2\)-dependent contributions split into \(\omega^2\)- and \(q^2\)-dependent ones, and c) also totally new terms
appear which have no counter parts in the vacuum sector of standard chiral perturbation theory. One might speculate that the occurrence of an $\mathcal{O}(Q^3)$ correction in the mean-field approximation is already a signal or precursor of the change in the relation between $m_\pi^2(\rho)$ and the current quark mass matrix $\mathcal{M}$ from that given by the vacuum GMOR relation. This phenomenon has eventually to take place as with increasing density the effective pion comes closer and closer to chiral restoration. In the framework of a linear $\sigma$-model its mass has to approach the effective mass of the $\sigma$ (the chiral partner of the pion). The latter mass scales as $\mathcal{O}(\hat{m})$ and not as $\mathcal{O}(\sqrt{\hat{m}})$. Stated differently, with increasing density the pion should behave more and more like a ‘normal’ meson (like the $\rho$-meson or $\omega$) as chiral symmetry becomes more and more restored. If the pion behaves more ‘normal’ in this respect, its effective mass has also to scale more ‘normally’.

In summary, additional $\mathcal{L}^{(3)}(\rho)$ terms are needed in nuclear matter to incorporate the changed scaling behavior of the pion and the fact that the in-medium quark condensate decreases in value. This effect is already present at linear order in density: In Ref. [15] finite $\mathcal{O}(m_\pi^3)$ loop terms were found which ‘renormalize’ the isospin even scattering length and which lead to $\mathcal{O}(Q^3)$ corrections in the mean-field approximation. The new $\mathcal{L}^{(3)}(\rho)$ terms and the additional $\mathcal{L}^{(2)}(\rho)$ terms of the then necessary generalization of chiral perturbation theory (see Ref. [10]) will make it rather unlikely that the GMOR relation prevails at high nuclear densities. In fact, the vanishing of the in-medium quark condensate which is linked to the vanishing of the PCAC-source coupling in (4) and the vanishing of the residuum of the in-medium propagator (7) should be rather interpreted as a signal that the standard chiral perturbation theory is pushed beyond its limits of applicability. The theory has to be generalized. Unfortunately, all the coefficients which enter the generalized in-medium chiral perturbation theory are density-dependent with a priori unknown free functional forms, furthermore the generalized theory incorporates terms which do not have any vacuum analogs. So, the only chance of getting constraints on these free density-dependent coefficients is from the rather scarce astrophysical input and perhaps also from heavy-ion-scattering data which, however, are “polluted” by temperature dependences. In practice, the virtues of the generalized in-medium effective lagrangians are rather the constraints which they pose on hadronic models applied to the pion (or kaon) propagation in the nuclear medium, since chiral perturbation theory gives intercorrelations between data and therefore model-independent results which specific to-be-tested models might either follow or not.

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#4The theory must be generalized even further, in case an in-matter resonance pole becomes comparable with the pion effective mass. Note, however, because of the S-wave nature of the pion propagation additional cuts are not important, at least at low densities.
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