Counting Keith numbers

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Abstract

A Keith number is a positive integer $N$ with the decimal representation $a_1a_2\ldots a_n$ such that $n \geq 2$ and $N$ appears in the sequence $(K_m)_{m \geq 1}$ given by the recurrence $K_1 = a_1, \ldots, K_n = a_n$ and $K_m = K_{m-1} + K_{m-2} + \cdots + K_{m-n}$ for $m > n$. We prove that there are only finitely many Keith numbers using only one decimal digit (i.e., $a_1 = a_2 = \cdots = a_n$), and that the set of Keith numbers is of asymptotic density zero.

1 Introduction

With the number 197, let $(K_m)_{m \geq 1}$ be the sequence whose first three terms $K_1 = 1$, $K_2 = 9$ and $K_3 = 7$ are the digits of 197 and which satisfies the
recurrence $K_m = K_{m-1} + K_{m-2} + K_{m-3}$ for all $m > 3$. Its initial terms are

$$1, 9, 7, 17, 33, 57, 107, 197, 361, 665, \ldots$$

Note that 197 itself is a member of this sequence. This phenomenon was first noticed by Mike Keith and such numbers are now called Keith numbers. More precisely, a number $N$ with decimal representation $a_1a_2\ldots a_n$ is a Keith number if $n \geq 2$ and $N$ appears in the sequence $K^N = (K^N_m)_{m \geq 1}$ whose $n$ initial terms are the digits of $N$ read from left to right and satisfying

$$K^N_m = K^N_{m-1} + K^N_{m-2} + \cdots + K^N_{m-n}$$

for all $m > n$. These numbers appear in Keith’s papers [3] and [4] and they are the subject of entry A007629 in Neil Sloane’s Encyclopedia of Integer Sequences [11] (see also [7], [8] and [9]).

Let $\mathcal{K}$ be the set of all Keith numbers. It is not known if $\mathcal{K}$ is infinite or not. The sequence $\mathcal{K}$ begins

$$14, 19, 28, 47, 61, 75, 197, 742, 1104, 1537, 2208, 2580, 3684, 4788, \ldots$$

In total there are 94 Keith numbers smaller than $10^{29}$ [4]. Recall that a rep-digit is a positive integer $N$ of the form $a(10^n - 1)/9$ for some $a \in \{1, \ldots, 9\}$ and $n \geq 1$; i.e., a number which is a string of the same digit $a$ when written in base 10. Our first result shows that there are only finitely many Keith numbers which are rep-digits.

**Theorem 1.** There are only finitely many Keith numbers which are rep-digits and their set can be effectively determined.

We point out that some authors refer to the Keith numbers as replicating Fibonacci digits in analogy with the Fibonacci sequence $(F_n)_{n \geq 1}$ given by

$$F_1 = 1, F_2 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 1.$$ 

In [5] it is shown that the largest rep-digit Fibonacci number is 55.

The proof of Theorem [1] uses Baker’s type estimates for linear forms in logarithms. It will be clear from the proof that it applies to all base $b$ Keith numbers for any fixed integer $b \geq 3$, where these numbers are defined analogously starting with their base $b$ expansion (see the remark after the proof of Theorem [1]).

For a positive integer $x$ we write $\mathcal{K}(x) = \mathcal{K} \cap [1, x]$. As we mentioned before, $\mathcal{K}(10^{29}) = 94$. A heuristic argument in [4] suggests that $\#\mathcal{K}(x) \gg \log x$, and, in particular, that $\mathcal{K}$ should be infinite. Going in the opposite way, we show that $\mathcal{K}$ is of asymptotic density zero.
Theorem 2. The estimate

\[ \#K(x) \ll \frac{x}{\sqrt{\log x}} \]

holds for all positive integers \( x \geq 2 \).

The above estimate is very weak. It does not even imply that that sum of the reciprocals of the members of \( K \) is convergent. We leave to the reader the task of finding a better upper bound on \( \#K(x) \). Typographical changes (see the remark after the proof of Theorem 2) show that Theorem 2 also is valid for the set of base \( b \) Keith numbers if \( b \geq 4 \). Perhaps it can be extended also to the case \( b = 3 \). For \( b = 2 \), Kenneth Fan has an unpublished manuscript showing how to construct all Keith numbers (see [4]) and that, in particular, there are infinitely many of them. For example, any power of 2 is a binary Keith number.

Throughout this paper, we use the Vinogradov symbols \( \gg \) and \( \ll \) as well as the Landau symbols \( O \) and \( o \) with their usual meaning. Recall that for functions \( A \) and \( B \) the inequalities \( A \ll B \), \( B \gg A \) and \( A = O(B) \) are all equivalent to the fact that there exists a positive constant \( c \) such that the inequality \( |A| \leq cB \) holds. The constants in the inequalities implied by these symbols may occasionally depend on other parameters. For a real number \( x \) we use \( \log x \) for the natural logarithm of \( x \). For a set \( A \), we use \( \#A \) and \( |A| \) to denote its cardinality.

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2 Preliminary Results

For an integer \( N > 0 \), recall the definition of the sequence \( K^N = (K^N_m)_{m \geq 1} \) given in the Introduction. In \( K^N \) we allow \( N \) to be any string of the digits 0,1,\ldots,9, so \( N \) may have initial zeros. So, for example, \( K^{020} = (0,2,0,2,4,6,12,22,\ldots) \). For \( n \geq 1 \) we define the sequence \( L^n \) as \( L^n = K^M \)
where $M = 11\ldots1$ with $n$ digits 1. In particular, $L^1 = (1,1,1,\ldots)$ and $L^2 = (1,1,2,3,5,8,\ldots)$, the Fibonacci numbers. In the following lemma, which will be used in the proofs of both Theorems 1 and 2, we establish some properties of the sequences $K^N$ and $L^n$.

**Lemma 1.** Let $N$ be a string of the digits $0,1,\ldots,9$ with length $n \geq 1$. If $N$ does not start with 0, we understand it also as the decimal representation of a positive integer.

1. If $N$ has at least $k \geq 1$ nonzero entries, then $K^N_m \geq L^k_{k+m-n}$ holds for every $m \geq n + 1$.

2. If $N$ has at least one nonzero entry, then $K^N_m \geq L^n_{m-n}$ holds for every $m \geq n + 1$. We have $K^N_m \leq 9L^n_m$ for every $m \geq 1$.

3. If $n \geq 3$ and $N = K^N_m$ for some $m \geq 1$ (so $N$ is a Keith number), then $2n < m < 7n$.

4. For fixed $n \geq 2$ and growing $m \geq n + 1$,

$$L^n_m = 2^{n-n-1}(n-1)(1 + O(m/2^n)) + 1$$

where the constant in $O$ is absolute.

**Proof.**

1. By the recurrences defining $K^N$ and $L^k$, the inequality clearly holds for the first $k$ indices $m = n + 1, n + 2, \ldots, n + k$. For $m > n + k$ it holds by induction.

2. We have $K^N_m \geq 1 = L^n_{m-n}$ for $m = n + 1, n + 2, \ldots, 2n$ and the inequality holds. For $m > 2n$ it holds by induction. The second inequality follows easily by induction.

3. The lower bound $m > 2n$ follows from the fact that $K^N$ is nondecreasing and that

$$K^N_{2n} \leq 9L^n_{2n} = 9 \cdot 2^{n-1}(n-1) + 9 < 10^{n-1} \leq N$$

for $n \geq 3$. To obtain the upper bound, note that for $m \geq n$ we have by induction that $L^n_m \geq L^2_{m-n+2} \geq \phi^{m-n}$ where $\phi = 1.61803\ldots$ is the golden ratio. Thus, by part 2,

$$10^n > N = K^N_m \geq L^n_{m-n} \geq \phi^{m-2n}$$
and \( m < (2 + \log 10/ \log \phi)n < 7n \).

4. We write \( L^n_m \) in the form
\[
L^n_m = (2^{m-n-1} - d(m))(n-1) + 1
\]
and prove by induction on \( m \) that for \( m \geq n + 1 \),
\[
0 \leq d(m) < m2^{m-2n}.
\]
This will prove the claim.

It is easy to see by the recurrence that \( L^n_{n+1}, L^n_{n+2}, \ldots, L^n_{2n+1} \) are equal, respectively, to \( 2^0(n-1) + 1, 2^1(n-1) + 1, \ldots, 2^n(n-1) + 1 \). So \( d(m) = 0 \) for \( n + 1 \leq m \leq 2n + 1 \) and the claim holds. For \( m \geq 2n + 1 \),
\[
L^n_m = L^n_{m-1} + L^n_{m-2} + \cdots + L^n_{m-n}
\]
\[
= \sum_{k=1}^{n} \left((2^{m-n-1-k} - d(m-k))(n-1) + 1\right)
\]
\[
= \left(2^{m-n-1} - 2^{m-2n-1} + 1 - \sum_{k=1}^{n} d(m-k)\right)(n-1) + 1
\]
and the induction hypothesis give
\[
0 \leq d(m) = 2^{m-2n-1} - 1 + \sum_{k=1}^{n} d(m-k)
\]
\[
< 2^{m-2n-1} + (m - 1)\sum_{k=1}^{n} 2^{m-2n-k}
\]
\[
< m2^{m-2n}.
\]

\( \square \)

In part 4, if \( m \) is roughly of size \( 2^n \) and larger then the error term swallows the main term and the asymptotics is useless. Indeed, the correct asymptotics of \( L^n_m \) when \( m \to \infty \) is \( cm^\alpha \) where \( c > 0 \) is a constant and \( \alpha < 2 \) is the only positive root of the polynomial \( x^n - x^{n-1} - \cdots - x - 1 \). But for \( m \) small relative to \( 2^n \), say \( m = O(n) \) (ensured for Keith numbers by part 3), this “incorrect” asymptotics of \( L^n_m \) is very precise and useful, as we shall demonstrate in the proofs of Theorems 1 and 2.

In the proof of Theorem 1 we will apply also a lower bound for a linear form in logarithms. The following result can be deduced from Corollary 2.3 of [6].
Lemma 2. Let \( A_1, \ldots, A_k \), \( A_i > 1 \), and \( n_1, \ldots, n_k \) be integers, and let \( N = \max\{|n_1|, \ldots, |n_k|, 2\} \). There exist positive absolute constants \( c_1 \) and \( c_2 \) (which are effective), such that if
\[
\Lambda = n_1 \log A_1 + n_2 \log A_2 + \cdots + n_k \log A_k \neq 0,
\]
then
\[
\log |\Lambda| > -c_1 c_2 \log A_1 \cdots (\log A_k) \log N.
\]

For the proof of Theorem 2 we will need an upper bound on sizes of antichains (sets of mutually incomparable elements) in the poset (partially ordered set)
\[
P(k, n) = (\{1, 2, \ldots, k\}^n, \leq_p)
\]
where \( \leq_p \) is the product ordering
\[
a = (a_1, a_2, \ldots, a_n) \leq_p b = (b_1, b_2, \ldots, b_n) \iff a_i \leq b_i \text{ for } i = 1, 2, \ldots, n.
\]
We have \( |P(k, n)| = k^n \) and for \( k = 2 \) the poset \( P(2, n) \) is the Boolean poset of subsets of an \( n \)-element set ordered by inclusion. The classical theorem of Sperner (see [1] or [2]) asserts that the maximum size of an antichain in \( P(2, n) \) equals to the middle binomial coefficient \( \left( \binom{n}{\lfloor n/2 \rfloor} \right) \). In the next lemma we obtain an upper bound for any \( k \geq 2 \).

Lemma 3. If \( k \geq 2, n \geq 1 \) and \( X \subset P(k, n) \) is an antichain to \( \leq_p \), then
\[
|X| < \frac{(k/2) \cdot k^n}{n^{1/2}}.
\]

Proof. We proceed by induction on \( k \). For \( k = 2 \) this bound holds by Sperner’s theorem because
\[
\binom{n}{\lfloor n/2 \rfloor} < \frac{2^n}{n^{1/2}}
\]
for every \( n \geq 1 \). Let \( k \geq 3 \) and \( X \subset P(k, n) \) be an antichain. For \( A \) running through the subsets of \( [n] = \{1, 2, \ldots, n\} \), we partition \( X \) in the sets \( X_A \) where \( X_A \) consists of the \( u \in X \) satisfying \( u_i = k \iff i \in A \). If we delete from all \( u \in X_A \) all appearances of \( k \), we obtain (after appropriate relabelling of coordinates) a set of \( |X_A| \) distinct \((n - |A|)\)-tuples from \( P(k - 1, n - |A|) \) that must be an antichain to \( \leq_p \). Thus, by induction, for \( |A| < n \) we have
\[
|X_A| < \frac{((k-1)/2) \cdot (k-1)^{n-|A|}}{(n-|A|)^{1/2}}
\]
and $|X_{[n]}| \leq 1$. Summing over all $A$s and using the inequality $\sqrt{n/m} \leq (n+1)/(m+1)$ (which holds for $1 \leq m \leq n$) and standard properties of binomial coefficients, we get

$$|X| = \sum_{A \subseteq [n]} |X_A|$$

$$< 1 + \sum_{i=0}^{n-1} \binom{n}{i} \frac{(k-1/2) \cdot (k-1)^{n-i}}{(n-i)^{1/2}}$$

$$= \frac{1}{\sqrt{n}} \left( \sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \binom{n}{i} \sqrt{n/(n-i)} \cdot (k-1)^{n-i+1} \right)$$

$$\leq \frac{1}{\sqrt{n}} \left( \sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \frac{n+1}{n-i+1} (k-1)^{n-i+1} \right)$$

$$< \frac{k^{n+1}}{2\sqrt{n}}.$$ 

\[ \square \]

We conclude this section with three remarks as to the last lemma.

1. Various generalizations and strengthenings of Sperner’s theorem were intensively studied, see, e.g., the book of Engel and Gronau [2]. Therefore, we do not expect much originality in our bound.

2. It is clear that for $k = 2$ the exponent $1/2$ of $n$ in the bound of Lemma 3 cannot be increased. The same is true for any $k \geq 3$. We briefly sketch a construction of a large antichain when $k = 3$; for $k > 3$ similar constructions can be given. For $k = 3$ and $n = 3m \geq 3$ consider the set $X \subseteq P(3, n)$ consisting of all $u$ which have $i$ 1s, $n-2i$ 2s and $i$ 3s, where $i = 1, 2, \ldots, m = n/3$. It follows that $X$ is an antichain and that

$$|X| = \sum_{i=1}^{m} \binom{n}{i, i, n-2i} = \sum_{i=1}^{m} \frac{n!}{(i!)^2(n-2i)!}.$$ 

By the usual estimates of factorials, if $m - \sqrt{n} < i \leq m$ then

$$\binom{n}{i, i, n-2i} \gg \binom{n}{m, m, m} \gg \frac{3^n}{n}.$$ 

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Hence \( X \) is an antichain in \( P(3,n) \) with size

\[
|X| \gg \sqrt{n} \cdot \frac{3^n}{n} = \frac{3^n}{\sqrt{n}}.
\]

3. For composite \( k \) we can decrease the factor \( k/2 \) in the bound of Lemma \[1\] Suppose that \( k = lm \) where \( l \geq m \geq 2 \) are integers and let \( X \subset P(k,n) \) be an antichain. We associate with every \( u \in X \) the pair of \( n \)-tuples \((v^u, w^u)\in P(m,n) \times P(l,n)\) defined by \( v_i^u = u_i - m[u_i/m] + m \) and \( w_i^u = [u_i/m], 1 \leq i \leq n \). Note that the pair \((v^u, w^u)\) uniquely determines \( u \) and that if \( w^u = w^{u'} \) then \( v^u \) and \( v^{u'} \) are incomparable by \( \leq_p \). Thus, by Lemma \[3\] for fixed \( w \in P(l,n) \) there are less than \((m/2)m^n/\sqrt{n}\) elements \( u \in X \) with \( w^u = w \). The number of \( w \)'s is at most \( |P(l,n)| = l^n \). Hence

\[
|X| < \frac{(m/2) \cdot m^n}{n^{1/2}} \cdot l^n = \frac{(m/2) \cdot k^n}{n^{1/2}}.
\]

In particular, if \( k \) is a power of 2 then \( |X| < k^n/\sqrt{n} \) for every antichain \( X \subset P(k,n) \).

3 The proof of Theorem \[1\]

Let \( N = a(10^n - 1)/9 = aa \ldots a, 1 \leq a \leq 9 \), be a rep-digit. Since \( K^n = aL^n \), \( N \) is a Keith number if and only if the repunit \( M = (10^n - 1)/9 = 11 \ldots 1 \) is a Keith number. Suppose that \( M \) is a Keith number: for some \( m \) we have

\[
M = \frac{10^n - 1}{9} = L_m = 2^{m-n-1}(n-1) \left(1 + O\left(\frac{m}{2^n}\right)\right),
\]

where the asymptotics was proved in Lemma \[4\]. We rewrite this relation as

\[
\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = \frac{1}{9(n-1)2^{m-n-1}} + O\left(\frac{m}{2^n}\right).
\]

Since \( 2n < m < 7n \) by Lemma \[3\], we get

\[
\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = O\left(\frac{n}{2^n}\right).
\]
Because $5^n > 9(n - 1)$ for every $n \geq 1$, the left side is always non-zero. Writing it in the form $e^\Lambda - 1$ and using that $e^\Lambda - 1 = O(\Lambda)$ (as $\Lambda \to 0$), we get

$$0 \neq \Lambda = (2n + 1 - m) \log 2 + n \log 5 - \log(9(n - 1)) \ll \frac{n}{2^n}.$$  

Taking logarithms and applying Lemma 2, we finally obtain

$$-d(\log n)^2 < \log |\Lambda| < c(\log n - n \log 2)$$

where $c, d > 0$ are effectively computable constants. This implies that $n$ is effectively bounded and completes the proof of Theorem 1.

Remark. The same argument shows that for every integer $b \geq 3$ there are only effectively finitely many base $b$ rep-digits, i.e., positive integers of the form $a(b^n - 1)/(b - 1)$ with $a \in \{1, \ldots, b-1\}$, which are base $b$ Keith numbers. Indeed, we argue as for $b = 10$ and derive the equation

$$\frac{b^n}{(b-1)(n-1)2^{m-n-1}} - 1 = O(n/2^n).$$

In order to apply Lemma 2 we need to justify that the left side is not zero. If $b$ is not a power of 2, it has an odd prime divisor $p$, and $p^n$ cannot be cancelled, for big enough $n$, by $(b - 1)(n - 1)$. If $b \geq 3$ is a power of 2, then $b - 1$ is odd and has an odd prime divisor, which cannot be cancelled by the rest of the expression.

4 The proof of Theorem 2

For an integer $N > 0$, we denote by $n$ the number of its digits: $10^{n-1} \leq N < 10^n$. We shall prove that there are $\ll 10^n/\sqrt{n}$ Keith numbers with $n$ digits; it is easy to see that this implies Theorem 2. There are only few numbers with $n$ digits and $\geq n/2$ zero digits: their number is bounded by

$$\sum_{i \geq n/2} \binom{n}{i} 9^{n-i} \leq n 2^n 9^{n/2} = n6^n \ll (10^n)^{0.8}.$$  

Hence it suffices to count only the Keith numbers with $n$ digits, of which at least half are nonzero.

Let $N$ be a Keith number with $n \geq 3$ digits, at least half of them nonzero. So, $N = R^N_m$ for some index $m \geq 1$. By Lemma 3, $2n < m < 7n$ and we
may use the asymptotics in Lemma 1.4. Setting $k = \lfloor n/2 \rfloor$ and using the
inequality in Lemma 1.1, we get

$$10^n > N = K_m^N \geq L_{k+m-n}^k.$$ 

Lemma 1.4 gives that for big $n$,

$$L_{k+m-n}^k > \frac{2^{m-n-1}(k-1)}{2} > \frac{2^{m-n}n}{12}.$$ 

On the other hand, the second inequality in Lemma 1.2 and Lemma 1.4 give, for big $n$,

$$10^{n-1} \leq N = K_m^N \leq 9L_m^n < 9 \cdot 2^{m-n}n.$$ 

Combining the previous inequalities, we get

$$\frac{10^n}{90} < 2^{m-n}n < 12 \cdot 10^n.$$ 

This implies that, for $n > n_0$, the index $m$ attains at most 12 distinct values

and

$$m = (1 + \log 10 / \log 2 + o(1))n = (\kappa + o(1))n.$$ 

Now we partition the set $S$ of considered Keith numbers (with $n$ digits, at least half of them nonzero) in blocks of numbers $N$ having the same value of the index $m$ and the same string of the first (most significant) $k = \lfloor n/2 \rfloor$ digits. So, we have at most $12 \cdot 10^k$ blocks. We show in a moment that the numbers in one block $B$, when regarded as $(n-k)$-tuples from $P(10, n-k)$, form an antichain to $\leq_p$. Assuming this, Lemma 3 implies that $|B| < 10^{n-k+1}/2\sqrt{n-k}$. Summing over all blocks, we get

$$|S| < 12 \cdot 10^k \cdot \frac{10^{n-k+1}}{2\sqrt{n-k}} \ll \frac{10^n}{\sqrt{n}},$$

which proves Theorem 2.

To show that $B$ is an antichain, we suppose for the contradiction that $N_1$ and $N_2$ are two Keith numbers from $B$ with $N_1 <_p N_2$. Let $M = N_2 - N_1$ and $M^* = 00 \ldots 0M \in P(10, n)$ (we complete $M$ to a string of length $n$ by adding initial zeros). It follows that $M$ has at most $n-k$ digits and $M < 10^{n-k}$. On the other hand, by the linearity of recurrence and by $N_1 <_p N_2$, we have

$$M = N_2 - N_1 = K_m^{N_2} - K_m^{N_1} = K_m^{M^*}.$$
Since $M^*$ has some nonzero entry, the first inequality in Lemma 1.2 and Lemma 1.4 give, for big $n$, 

$$K_{m}^{M^*} \geq L_{m-n}^{n} > 2^{m-2n-2n}.$$ 

Thus 

$$10^{n-k} = 10^{n-[n/2]} > M > 2^{m-2n-2n}.$$ 

Using the above asymptotics of $m$ in terms of $n$, we arrive at the inequality 

$$\exp((\frac{1}{2} \log 10 + o(1))n) > \exp((\kappa \log 2 - 2 \log 2 + o(1))n) = \exp((\log 5 + o(1))n)$$ 

that is contradictory for big $n$ because $10^{1/2} < 5 = 10/2$. This finishes the proof of Theorem 2. \hfill \Box

Remark. The above proof generalizes, with small modifications, to all bases $b \geq 4$. We replace base 10 by $b$, modify the proof accordingly, and have to satisfy two conditions. First, in the beginning of the proof we delete from the numbers with $n$ base $b$ digits those with $> \alpha n$ zero digits, for some constant $0 < \alpha < 1$. In order that we delete negligibly many, compared to $b^n$, numbers, we must have $2 \cdot (b-1)^{1-\alpha} < b$. Second, for the final contradiction we need that $b^\alpha < b/2$. For $b \geq 5$, both conditions are satisfied with $\alpha = 1/2$, as in case $b = 10$. For $b = 4$ they are satisfied with $\alpha = 0.49$, say. However, for $b = 3$ they cannot be satisfied by any $\alpha$. Thus, the case $b = 3$ seems to require more substantial changes.

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