Description of Quasi-Equilibrium States in N-body Self-gravitating System

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Abstract. We present our recent progress on the characterization of self-gravitating N-body system away from the thermal equilibrium. We show that the transient state of out-of-equilibrium system can be approximately described by the stellar polytropic distribution with time-varying polytrope index. The time-scale of quasi-equilibrium states can be analytically estimated by means of the generalized variational method, which reproduces the N-body result quite well.

1. Introduction

Due to the long-range attractivity of the Newton gravity, the long-term evolution of stellar self-gravitating system driven by the two-body encounter shows various peculiar behaviors. In particular, one of the most important consequences is the thermodynamic instability so-called gravothermal catastrophe, which has been discovered by Antonov [1] and later investigated by Lynden-Bell and Wood [2]. The presence of thermodynamic instability implies that no stable thermal equilibrium exists for highly dense clusters. Therefore, the stellar self-gravitating system is essentially a non-equilibrium system and thereby the description of out-of-equilibrium state is a central issue in the subject of stellar dynamics. Here, we present a recent progress on the characterization of non-equilibrium states of stellar self-gravitating system.

2. Antonov problem

To discuss the out-of-equilibrium states, we consider the same situation as investigated in classic papers [1, 2], the so-called Antonov problem (see Fig.1). That is, we treat a self-gravitating N-body system confined by the adiabatic wall of a sphere, where the adiabatic wall represents a perfectly reflecting boundary which preserves the total energy of the system. In this setup, Antonov and Lynden-Bell and Wood originally considered the stability of the thermal equilibrium (isothermal state) characterized by the Boltzmann-Gibbs entropy. Since we are especially concerned with the characterization of the evolutionary sequences away from the thermal equilibrium, standard thermostatistical treatment becomes inadequate. Hence, we take the following three approaches: (i) thermostatistical approach by means of the non-extensive entropy [3]; (ii) dynamical approach based on the N-body simulation [4, 5]; (iii) kinetic-theory approach using the Fokker-Planck model for stellar dynamics [6]. In what follows, we mainly focus on the treatment of (ii) and (iii) and present recent results.
3. N-body simulations
Let us first describe the dynamical approach based on N-body simulations. Without loss of
generality, we set the units to $G = M = r_e = 1$. All the particles are assumed to have the
same mass $m = 1/N$, where $N$ is the total number of the particles. The initial conditions for
particle data were whole created by a random realization of the stellar models. We deal with
the two kinds of the initial stellar models: (a) stellar polytropes, whose distribution functions
are characterized by the power-law function, i.e., $f(\varepsilon) = A(\varepsilon_0 - \varepsilon)^{n-3/2}$ with
$\varepsilon = v^2/2 + \phi(r)$ and $n$ being polytrope index; (b) the stellar models with cusped density profile, which cannot be
described by the power-law distribution. We are specifically concerned with collisional aspects
of N-body dynamics, the timescale of which is much longer than the two-body relaxation time.
For this purpose, we utilized a special-purpose hardware, GRAPE-6, which is especially designed
to accelerate the gravitational force calculations for collisional N-body system. The details of
the numerical simulations are described in Ref.[5] (see also Ref.[4]).

The overview of the simulation results is as follows. For the initial conditions (a), the stellar
polytropic distribution gradually changes with time on time-scales of two-body relaxation.
Focusing on the evolutionary sequence, we found that the transient state starting from the
initial stellar polytrope can be remarkably characterized by a sequence of stellar polytropes
with time-varying polytrope index $n$. Interestingly, this is even true for a certain class of the
initial conditions (b). That is, even starting from the non-power-law initial distribution, the
system soon settles into a stellar polytropic state and follows the equilibrium sequence of stellar
polytropes for a long time.

Fig.2 shows the representative result taken from the N-body simulation (run n3A, see Table
1 of Ref.[5]). We plot the snapshots of one-particle distribution function $f(\varepsilon)$ starting from the
initial conditions (a). To be precise, the initial distribution was chosen as stellar polytropic
distribution with index $n = 3$ and the density contrast defined by $D \equiv \rho_{\text{core}}/\rho_{\text{edge}}$ was set to
$D = 10^4$. In this plot, just for illustrative purpose, each output result is artificially shifted to the
two-digits below. Only the final output with $t = 30t_{\text{rh,i}}$ represents the correct scales. Here, the
quantity $t_{\text{rh,i}}$ denotes initial half-mass relaxation time [see Eq.(27) of [5]]. Solid lines represents
the initial stellar polytrope with $n = 3$ and the other lines indicate the fitting results to the
stellar polytrope by varying the polytrope index $n$.

Clearly, the stellar polytropes quantitatively characterize the evolutionary sequence of the
simulation results. The fitting results are remarkably good until $t \approx 30t_{\text{rh,i}}$. Afterwards, the
system enters the gravothermally unstable regime and finally undergoes the core-collapse. A
closer look at the low-energy part of the distribution function $f(\varepsilon)$ reveals that the simulation
results partly resemble the exponential form rather than the power-law function. Nevertheless, it
is remarkable that the stellar polytropes as simple power-law distribution globally approximate
the simulation results in a quite good accuracy. In Fig.3, the polytrope indices are estimated at
each time step and the resultant values are plotted together with the fitting results of $N = 4K$
and $N = 8K$. Apart from the fluctuations, the fitted values of the polytrope index monotonically
Figure 2. Snapshots of one-particle distribution function as function of specific energy of particle, \( \varepsilon = v^2/2 + \phi(r) \), taken from the N-body simulation with number of particle \( N = 2,048 \) (run n3A).

Figure 3. Time evolution of the polytrope index fitted to N-body simulations in the case of run n3A: \( N = 2,048 \) (solid); \( N = 4,096 \) (long-dashed); \( N = 8,192 \) (short-dashed).

increase and the growth rates of \( n \) normalized by half-mass relaxation time almost coincide with each other, consistent with the fact that the quasi-equilibrium sequence is evolved via two-body relaxation.

4. Characterizing the time-scale of quasi-equilibrium state

The N-body simulations imply that quasi-equilibrium behavior approximately described by the stellar polytropes with time-varying index \( n \) may be a unique character of the out-of-equilibrium states in self-gravitating N-body system. We then wish to understand the time-scale of quasi-equilibrium states in an analytical manner. For this purpose, we next focus on the kinetic-theory approach and treat the Fokker-Planck (F-P) model for stellar dynamics.

F-P model describes the energy diffusion process driven by the two-body relaxation in phase space. The main assumption in the F-P model is that the orbit of each particle in spherical stellar system is dominantly affected by the weak encounter. Assuming the isotropy of velocity distribution, the (orbit-averaged) F-P equation can be written as [7]

\[
\left( \frac{\partial f}{\partial t} \right)_q = \left( \frac{\partial \Pi}{\partial q} \right)_t : \Pi(q,t) = \Gamma \int f f' \min(q, q') \left( \frac{\partial \ln f}{\partial \varepsilon} - \frac{\partial \ln f'}{\partial \varepsilon'} \right) d\varepsilon' \tag{1}
\]

with \( \Gamma = (4\pi Gm)^2 \ln \Lambda \). Here, \( q \) is the volume of phase space where the energy is less than \( \varepsilon \), i.e., \( q(\varepsilon) = (16\pi^2/3) \int [2(\varepsilon - \phi(r))]^{3/2} r^2 \, dr \). The function \( \Pi \) represents the flux of stars crossing into the volume \( q \) in phase space. Eq.(1) must be solved with Poisson equation self-consistently.

It is important to notice that despite the fact that F-P equation is not self-adjoint, there exists a generalized variational principle to obtain an approximate solution. It is an extension of the classical variational principle to the non-self-adjoint problems and is based on the concept of a local potential [8]. Inagaki and Lynden-Bell [7] showed that local potential of F-P equation (1) is given by

\[
\Phi(f, f_0) = \int dq \left( \frac{\partial f_0}{\partial t} \right)_q \ln f + \frac{\Gamma}{4} \int \int f_0 f_0' \min(q, q') \left( \frac{\partial \ln f}{\partial \varepsilon} - \frac{\partial \ln f'}{\partial \varepsilon'} \right) d\varepsilon d\varepsilon' \tag{2}
\]
Note that the local potential (2) has the following properties: (i) the condition that $\Phi$ is minimum with respect to $f$ for fixed $f_0$, and the subsidiary condition $f = f_0$ lead to the original differential equation; (ii) $\Phi$ is absolutely minimum at a solution $f_0$, i.e., $\Delta \Phi = \Phi(f, f_0) - \Phi(f_0, f_0) > 0$.

Using the local potential, Takahashi and Inagaki [9] have tried to obtain the self-similar solution of F-P equation. Later, Takahashi [10] utilized it to obtain an accurate numerical solution of F-P equation. Here, the generalized variational method is applied to derive the time evolution of polytrope index $n(t)$. To do this, we adopt the polytropic form of the distribution function $f(\varepsilon, t) = A(t)\{\varepsilon_0(t) - \varepsilon\}^{n(t)-3/2}$ as the trial function for $f$. Substituting this trial function into Eq.(2), we consider the variation of $\Phi$ with respect to $n$ with $f_0$ being held fixed. Note that this variation should be done under keeping the energy and the mass fixed. After that, we set $f$ equal to $f_0$. The resultant equation to be solved is

$$\left. \frac{\partial \Phi}{\partial n} \right|_{f=f_0} = 0. \quad \implies \dot{n}(t) = \mathcal{F}[n(t); E, M, \varepsilon_0].$$

The explicit form of the function $\mathcal{F}$ is slightly complicated and includes some integrals (see [6]). Nevertheless, Eq.(3) is nothing but the ordinary differential equation and it can be easily solved by using mathematical software.

As an illustration, the evolution equation for polytrope index $n$ is numerically solved with the same initial condition as in the $N$-body run n3A (Fig.2) and the result is plotted in Fig. 3 (dotted line indicated by arrow). The time evolution obtained from the variational method successfully reproduces the simulation results quite well until the system enters the gravothermally unstable regime. The one remarkable point is that the evolution equation (3) consistently explains the monotonic growth of $n$. Further, the functional form of Eq.(3) indicates the presence of the slow-down phase before entering the unstable regime, which is clearly seen in Fig. 3. We note that this slow-down phase is intimately related to the marginal stability boundary of the quasi-equilibrium state, characterized by the non-extensive thermostatistics [3]. This may give an interesting suggestion for the physical reality of non-extensive thermostatistics.

5. Summary
We have presented a recent progress on the characterization of self-gravitating $N$-body system. Particularly focusing on the long-term evolution, we have numerically studied the quasi-equilibrium properties of the $N$-body systems in the setup of Antonov problem. We have tried to characterize the time-scale of quasi-equilibrium sequences by utilizing the generalized variational principle of F-P equation. Although the present paper did not address the thermostatistical approach by means of the non-extensive entropy, the analysis based on F-P model suggests a possible relationship between the kinetic-theory approach and the thermostatistical approach for the description of the stability boundary of the quasi-equilibrium states. This point will be discussed in details in future publication [6].

References
[1] Antonov V A 1962 Vest. Leningrad Gros. Univ. 7 135
[2] Lynden-Bell D, Wood R 1968 Mon.Not.R.Astron.Soc. 138 495
[3] Taruya A, Sakagami M 2002 Physica A 307 185; 2003 Physica A 318 387; 2003 Physica A 322 285
[4] Taruya A, Sakagami M 2005 Phys.Rev.Lett. 90 181101
[5] Taruya A, Sakagami M 2005 Mon.Not.R.Astron.Soc. 364 990
[6] Taruya A, Sakagami M, Okamura T 2005 in preparation
[7] Inagaki S, Lynden-Bell D 1990 Mon.Not.R.Astron.Soc. 244 254
[8] Glansdorff P, Prigogine I 1971 Thermodynamic Theory of Structure, Stability and Fluctuations (London, Wiley-Interscience) Chap.X
[9] Takahashi K, Inagaki S 1992 Publ.Astron.Soc.Japan 44 623
[10] Takahashi K, 1993 Publ.Astron.Soc.Japan 45 233