Nonlocal integro-differential problems of multi-dimensional wave processes

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Abstract. Nonlocal boundary-value problems for one-dimensional parabolic second-order equations with integral conditions on the lateral boundary are considered in the articles by A.I. Kozhanov. Higher-dimensional integro–differential equations of higher order with integral conditions on the lateral boundary were not studied earlier. The existence and uniqueness theorems of regular solutions are proven. The method of regularization and the method of continuation in a parameter are employed to establish solvability. In the present article we study integro–differential equations with integral conditions on the lateral boundary and prove existence theorems for regular solutions.

1. Introduction
Nonlocal boundary value problems for parabolic and hyperbolic equations are actively studied last time but mainly classical second order equations are examined (see [1–4]). We refer to the articles [5,6], where pseudoparabolic and pseudohyperbolic equations with an integral condition on the lateral boundary are studied. In the present article we study integro–differential equations with integral conditions on the lateral boundary and prove existence theorems for regular solutions.

2. Formulation of the problem
Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth (for simplicity, infinitely differentiable) boundary $\Gamma$, $Q$ is the cylinder $\Omega \times (0, T)$ ($0 < T < +\infty$), $S = \Gamma \times (0, T)$ is its lateral boundary, $f(x, t)$ is a function defined in $Q$, $u_0(x)$, $u_1(x)$ are functions defined in $\Omega$, $N(t)$ is a function defined on $[0, T]$, the functions $K_1(x, y, t)$ and $K_2(x, y, t)$ are defined for $x \in \overline{\Omega}$, $y \in \overline{\Omega}$, and $t \in [0, T]$.

Boundary value problem I: find a solution $u(x, t)$ to the equation

$$Lu \equiv (Au)_t - \Delta u = f(x, t), \quad Au = \int_0^t N(t - \tau)u(x, \tau) \, d\tau,$$

in $Q$ satisfying the conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$
\[ u(x, t)|_{(x, t) \in S} = \int_{\Omega} K_1(x, y, t)u(y, t)dy|_{(x, t) \in S}, \]

Boundary value problem II: find a solution \( u(x, t) \) to the equation (1) in \( Q \) satisfying (2) and the condition

\[ \frac{\partial u(x, t)}{\partial \nu(x)}|_{(x, t) \in S} = \int_{\Omega} K_2(x, y, t)u(y, t)dy|_{(x, t) \in S}, \]

where \( \nu(x) = (\nu_1, \ldots, \nu_n) \) is the interior normal to \( \Gamma \) at a given point.

3. Solvability of the boundary value problem I

The equation (1) yields

\[ N(0)u_t(x, t) + N'(0)u(x, t) + \int_0^t N''(t - \tau)u(x, \tau) d\tau - \Delta u = f(x, t). \]

Taking (2) into account, for \( t = 0 \), we obtain the following nonlocal boundary value problem for determining \( u_0(x) \):

\[ -\Delta u_0(x) + N'(0)u_0(x) + N(0)u_1(x) = f(x, 0), \]

\[ u_0(x)|_{x \in \Gamma} = \int_{\Omega} K_1(x, y, 0)u_0(y)dy|_{x \in \Gamma}. \]

Define an operator \( M_1 \) by the formula

\[ (M_1u)(x, t) = u(x, t) - \int_{\Omega} K_1(x, y, t)u(y, t)dy. \]

The conditions on the operator \( M_1 \) are as follows: \( M_1 \) is continuously invertible as an operator from \( L_2(\Omega) \) in \( L_2(\Omega) \) for \( t \in [0, T] \) and there exist positive constants \( m_1, m_2 \) such that

\[ m_1 \int_{\Omega} u^2(x, t) dx \leq \int_{\Omega} [M_1u(x, t)]^2 dx \leq m_2 \int_{\Omega} u^2(x, t) dx \]  (3)

for every \( t \in [0, T] \) and \( u(x, t) \in L_2(Q) \).

Let \( V = W_{2,0}^2(Q) \). Put \( LM_1u(x, t) - M_1Lu(x, t) = \Phi(x, t, u), \quad w = M_1u. \) We have

\[ \Phi(x, t, u) = \int_{\Omega} [\Delta_x K_1(x, y, t) - N(0)K_{1t}(x, y, t)]u(y, t)dy - \int_{\Omega} K_1(x, y, t)\Delta_y u(y, t)dy. \]

We introduce the notation

\[ P_1 = \max_{t \in [0, T]} \int_{\Omega} K_1^2(x, y, \tau) dx dy. \]
Theorem 1. Assume that the condition (3) holds and
\[ K_1(x, y, t) \in C^2(\Omega \times \Omega \times [0, T]), \quad N(t) \in C^2[0, T], \quad N(0) > 0, \]
there exists \( \delta_0 \in (0; 1/2) \) such that
\[ 1 > 4\delta_0^2 + \frac{p_1}{\delta_0 m_1} \quad \text{and} \quad f(x, t) \in L_2(Q). \]
Then the boundary value problem \( I \) has a unique solution \( u(x, t) \) from the space \( V \).

To prove the claim, we consider the following auxiliary boundary value problem: find a solution \( w(x, t) \) to the equation
\[ Lw = g(x, t) + \lambda \Phi_1(x, t, w), \quad (4) \]
in \( Q \) satisfying the conditions
\[ w(x, t)|_{S} = 0, \quad w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in \Omega, \quad (5) \]
where
\[ w_0(x) = u_0(x) - \int_{\Omega} K_1(x, y, 0)u_0(y) \, dy, \]
\[ w_1(x) = u_1(x) - \int_{\Omega} \left[ K_{1t}(x, y, 0)u_0(y) + K_1(x, y, 0)u_1(y) \right] \, dy, \]
\[ \Phi_1(x, t, w) = \Phi(x, t, M_1^{-1}w). \]

Solvability of the boundary value problem (4), (5) in the class \( W = \{v(x, t) : v(x, t) \in V, \quad w(x, t) = M_1 v(x, t) \in V\} \) for every function \( g(x, t) \) from the space \( L_2(Q) \) is proven by the method of continuation in a parameter [7].

To justify an a priori estimate, multiply (4) written in the variables \( x \) and \( \tau \) by \( w - \Delta w \) and integrate the result over \( \Omega \) and with respect to \( \tau \) from 0 to \( t \). The result is the equality
\[ \int_0^t \int_{\Omega} Lw (w - \Delta w) \, dx \, d\tau = \int_0^t \int_{\Omega} (g + \lambda \Phi_1) (w - \Delta w) \, dx \, d\tau. \]
Integrating by parts and taking the boundary conditions (5) into account, applying the Young inequality, the Gronwall lemma, we obtain the a priori estimate
\[ \int_0^t \int_{\Omega} \left[ w^2 + \sum_{i=1}^n w_{x_i}^2 + (\Delta w)^2 \right] \, dx \, d\tau \leq K_0 \int_0^T \int_{\Omega} g^2(x, t) \, dx \, dt \]
with a positive constant \( K_0 \) depending on \( N(t) \), the numbers \( T \) and \( \Omega \).

It is obvious that a similar estimate holds for the function \( u(x, t) \), i.e., we have
\[ \|u\|_V \leq K_1 \|w\|_V \leq K_2 \|g\|_{L_2(Q)} \]
where \( K_1 \) and \( K_2 \) are positive constants depending on the same parameters as \( K_0 \) does.

Hence, the boundary value problem (4), (5) will be solvability in the class \( W \).
4. Solvability of the boundary value problem II

Let \( K_3(x, y, t) \) be a function defined on the set \( \Omega \times \Omega \) such that, for \((x, y, t) \in \Sigma \equiv (\Gamma \times \Gamma \times (0, T))\), the equalities

\[
\frac{\partial K_3(x, y, t)}{\partial \nu(x)} = K_2(x, y, t),
\]

hold, where the variables \( y \) and \( t \) are parameters. Using the function \( K_3(x, y, t) \) we define an operator \( M_2 \) and a function \( \tilde{\Phi}(x, t, u) \) by the formulas

\[
(M_2 u)(x, t) = u(x, t) - \int_{\Omega} K_3(x, y, t) u(y, t) \, dy,
\]

\[
\tilde{\Phi}(x, t, u) = L M_2 u(x, t) - M_2 Lu(x, t).
\]

The operator \( M_2 \) takes a function \( u(x, t) \) into \( \tilde{w} = M_2 u \). Define the initial function \( \tilde{w}(x, 0) = w_0(x), \tilde{w}_t(x, 0) = w_1(x) \) as follows:

\[
w_0(x) = u_0(x) - \int_{\Omega} K_3(x, y, 0) u_0(y) \, dy,
\]

\[
w_1(x) = u_1(x) - \int_{\Omega} [K_3(x, y, 0) u_0(y) + K_3(x, y, 0) u_1(y)] \, dy.
\]

The operator \( M_2 \) is continuously invertible as an operator from \( L^2(\Omega) \) in \( L^2(\Omega) \) for all \( t \in [0, T] \) and there exist positive constants \( m_3 \) and \( m_4 \) such that

\[
m_3 \int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} [M_2 u(x, t)]^2 \, dx \leq m_4 \int_{\Omega} u^2(x, t) \, dx
\]

for all \( t \in [0, T] \) and \( u(x, t) \in L^2(Q) \).

As above, we introduce the notation

\[
Q_1 = \max_{t \in [0, T]} \int_{\Omega} \int_{\Omega} K_3^2(x, y, \tau) \, dxdy.
\]

**Theorem 2.** Let the conditions (6) hold and

\[
K_3(x, y, t) \in C^2(\overline{\Omega} \times \overline{\Omega} \times [0, T]), \quad N(t) \in C^2[0, T], \quad N(0) > 0,
\]

there exists \( \delta_0 \in (0; 1/2) \) such that

\[
1 > 4 \delta_0^2 + \frac{Q_1}{\delta_0^2 m_1} \quad \text{and} \quad f(x, t) \in L^2(Q).
\]

Then the boundary value problem II has a unique solution \( u(x, t) \) from the space \( V \).

5. CONCLUSION

We consider second-order integro-differential equations with integral conditions on the lateral boundary. Existence and uniqueness theorems for regular solutions are proved by the method of continuation with respect to a parameter and a priori estimates. The methods used in Theorems 1 and 2 allow conditions for the smallness of functions. \( K_1(x, y, t), K_3(x, y, t) \) replace with symmetry conditions \( K_i(x, y, t) = K_i(y, x, t) = 0 \) and conditions \( K_i(x, y, t) = K_{iy}(x, y, t) = 0 \) \( (i = 1, 3) \) for \( y \in \Gamma \).
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