Factorization of a class of Toeplitz + Hankel operators and the $A_p$-condition

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Abstract

Let $M(\phi) = T(\phi) + H(\phi)$ be the Toeplitz plus Hankel operator acting on $H^p(T)$ with generating function $\phi \in L^\infty(T)$. In a previous paper we proved that $M(\phi)$ is invertible if and only if $\phi$ admits a factorization $\phi(t) = \phi_-(t)\phi_0(t)$ such that $\phi_-$ and $\phi_0$ and their inverses belong to certain function spaces and such that a further condition formulated in terms of $\phi_-$ and $\phi_0$ is satisfied. In this paper we prove that this additional condition is equivalent to the Hunt-Muckenhoupt-Wheeden condition (or, $A_p$-condition) for a certain function $\sigma$ defined on $[-1,1]$, which is given in terms of $\phi_0$. As an application, a necessary and sufficient criteria for the invertibility of $M(\phi)$ with piecewise continuous functions $\phi$ is proved directly. Fredholm criteria are obtained as well.

1 Introduction

This paper is devoted to continuing the study (started in [2]) of operators of the form

$$M(\phi) = T(\phi) + H(\phi)$$

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acting on the Hardy space $H^p(\mathbb{T})$ where $1 < p < \infty$. Here $\phi \in L^\infty(\mathbb{T})$ is a Lebesgue measurable and essentially bounded function on the unit circle $\mathbb{T}$. The Toeplitz and Hankel operators are defined by

$$
T(\phi) : f \mapsto P(\phi f), \quad H(\phi) : f \mapsto P(\phi(Jf)), \quad f \in H^p(\mathbb{T}),
$$

where is $J$ the following flip operator,

$$
J : f(t) \mapsto t^{-1}f(t^{-1}), \quad t \in \mathbb{T},
$$

acting on the Lebesgue space $L^p(\mathbb{T})$. The operator $P$ stands for the Riesz projection,

$$
P : \sum_{n=-\infty}^{\infty} f_n t^n \mapsto \sum_{n=0}^{\infty} f_n t^n, \quad t \in \mathbb{T},
$$

which is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$, and whose image is $H^p(\mathbb{T})$. The complex conjugate Hardy space $\overline{H^p(\mathbb{T})}$ is the set of all functions $f$ whose complex conjugate belongs to $H^p(\mathbb{T})$. Moreover, we denote by $L^p_{\text{even}}(\mathbb{T})$ the subspace of $L^p(\mathbb{T})$ consisting of all even functions, i.e., functions $f$ for which $f(t) = f(t^{-1})$.

For $\phi \in L^\infty(\mathbb{T})$ we denote by $L(\phi)$ the multiplication operator acting on $L^p(\mathbb{T})$,

$$
L(\phi) : f(t) \mapsto \phi(t)f(t).
$$

Obviously, $T(\phi)$ and $H(\phi)$ can be written as

$$
T(\phi) = PL(\phi)P|_{H^p(\mathbb{T})}, \quad H(\phi) = PL(\phi)JP|_{H^p(\mathbb{T})}.
$$

In a previous paper [2] we proved that for $\phi \in L^\infty(\mathbb{T})$ the operator $M(\phi)$ is a Fredholm operator on the space $H^p(\mathbb{T})$ if and only if the functions $\phi$ admits a certain kind of generalized factorization. Before recalling the underlying definitions, let us state the following simple necessary condition for the Fredholmness of $M(\phi)$ which was also established in [2] Prop. 2.2]. Therein $GL^\infty(\mathbb{T})$ stands for the group of all invertible elements in the Banach algebra $L^\infty(\mathbb{T})$.

**Proposition 1.1** Let $1 < p < \infty$ and $\phi \in L^\infty(\mathbb{T})$. If $M(\phi)$ is Fredholm on $H^p(\mathbb{T})$, then $\phi \in GL^\infty(\mathbb{T})$.

A function $\phi \in L^\infty(\mathbb{T})$ is said to admit a *weak asymmetric factorization* in $L^p(\mathbb{T})$ if it can be written in the form

$$
\phi(t) = \phi_-(t)t^\kappa \phi_0(t), \quad t \in \mathbb{T},
$$

such that $\kappa \in \mathbb{Z}$ and
(i) \((1 + t^{-1})\phi_- \in H^p(\mathbb{T}), (1 - t^{-1})\phi_-^{-1} \in H^q(\mathbb{T})\),

(ii) \(|1 - t|\phi_0 \in L^q_{\text{even}}(\mathbb{T}), |1 + t|\phi_0^{-1} \in L^p_{\text{even}}(\mathbb{T})\).

Here \(1/p + 1/q = 1\). It was proved in [2, Prop. 3.1] that if a factorization exists, then the index \(\kappa\) of the weak asymmetric factorization is uniquely determined and the factors \(\phi_-\) and \(\phi_0\) are uniquely determined up to a multiplicative constant. (In [2] also the notion of a weak antisymmetric factorization in \(L^p(\mathbb{T})\) has been introduced. This notion will play no role in the present paper.)

In order to introduce yet another notion, let \(\mathcal{R}\) stand for the set of all trigonometric polynomials. Under the assumption that \(\phi\) admits a weak asymmetric factorization in \(L^p(\mathbb{T})\) introduce the linear spaces

\[
X_1 = \left\{ (1 - t^{-1})f(t) : f \in \mathcal{R} \right\},
\]

\[
X_2 = \left\{ (1 + t^{-1})\phi_0^{-1}(t)f(t) : f \in \mathcal{R}, f(t) = f(t^{-1}) \right\}.
\]

It is easy to see that \(X_1\) and \(X_2\) are linear subspaces of \(L^p(\mathbb{T})\) and that the space \(X_1\) is dense in \(L^p(\mathbb{T})\). Moreover, it has been proved [2, Lemma 4.1(a)] that

\[
B := L(\phi_0^{-1})(I + J)PL(\phi_-^{-1})
\]

is a well-defined linear (not necessarily bounded) operator acting from \(X_1\) into \(X_2\).

We will call the above factorization \([3]\) of \(\phi\) an asymmetric factorization in \(L^p(\mathbb{T})\) if in addition to (i) and (ii) the following condition is satisfied:

(iii) The operator \(B = L(\phi_0^{-1})(I + J)PL(\phi_-^{-1})\) acting from \(X_1\) into \(X_2\) can be extended by continuity to a linear bounded operator acting on \(L^p(\mathbb{T})\).

Clearly, due to the density of \(X_1\) in \(L^p(\mathbb{T})\) an equivalent formulation for condition (iii) is the following statement:

(iii*) There exists a constant \(M\) such that \(\|Bf\|_{L^p(\mathbb{T})} \leq M\|f\|_{L^p(\mathbb{T})}\) for all \(f \in X_1\).

The main result proved in [2, Thm. 6.4] is the following:

**Theorem 1.2** Let \(1 < p < \infty\) and \(\phi \in GL^\infty(\mathbb{T})\). The operator \(M(\phi)\) is a Fredholm operator on \(H^p(\mathbb{T})\) if and only if the function \(\phi\) admits an asymmetric factorization in \(L^p(\mathbb{T})\). In this case, the defect numbers are given by

\[
\dim \ker M(\phi) = \max\{0, -\kappa\}, \quad \dim \ker M^*(\phi) = \max\{0, \kappa\},
\]

where \(\kappa\) is the index of the factorization of \(\phi\).
To formulate the main result of this paper we need the notion of the Hunt-Muckenhoupt-Wheeden condition (or, $A_p$-condition) with respect to the interval $[-1, 1]$.

Let $1 < p < \infty$, $1/p + 1/q = 1$, and let $\sigma : [-1, 1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume in addition that $\sigma \in L^p[-1, 1]$ and $\sigma^{-1} \in L^q[-1, 1]$. We say that $\sigma$ satisfies the $A_p$-condition on $[-1, 1]$ if

$$
\sup_I \frac{1}{|I|} \left( \int_I \sigma^p(x) \, dx \right)^{1/p} \left( \int_I \sigma^{-q}(x) \, dx \right)^{1/q} < \infty,
$$

where the supremum is taken over all subintervals $I$ of $[-1, 1]$. The length of the interval $I$ is denoted by $|I|$. There is an intimate connection between the $A_p$-condition and the boundedness of the singular integral operator, which will be stated later on.

The main result of this paper is the following:

**Theorem 1.3** Let $1 < p < \infty$, $1/p + 1/q = 1$, and $\phi \in L^\infty(\mathbb{T})$. The operator $M(\phi)$ is a Fredholm operator on $H^p(\mathbb{T})$ if and only if the following conditions are satisfied:

(a) $\phi \in GL^\infty(\mathbb{T})$.

(b) The function $\phi$ admits a weak asymmetric factorization in $L^p(\mathbb{T})$,

$$
\phi(t) = \phi_-(t) t^\alpha \phi_0(t), \quad t \in \mathbb{T}.
$$

(c) The function

$$
\sigma(\cos \theta) := |\phi_0^{-1}(e^{i\theta})| \frac{|1 + \cos \theta|^{1/2q}}{|1 - \cos \theta|^{1/2p}}
$$

satisfies the $A_p$-condition.

Moreover, formulas (11) hold in this case.

We note that it is straightforward to prove that condition (b) of the previous theorem implies $\sigma \in L^p[-1, 1]$ and $\sigma^{-1} \in L^q[-1, 1]$.

## 2 Proof of Theorem 1.3

Before we are able to give the proof of Theorem 1.3 we establish some definitions and auxiliary results.

Let $C^\infty[-1, 1]$ stand for the set of all infinitely differentiable functions $f : [-1, 1] \to \mathbb{C}$, and denote by $C_0^\infty[-1, 1]$ the subspace of all functions $f \in C^\infty[-1, 1]$ such that $f(x)$ and all of its derivatives vanish at the endpoints $x = -1$ and $x = 1$,

$$
C_0^\infty[-1, 1] = \left\{ f \in C^\infty[-1, 1] : f^{(n)}(-1) = f^{(n)}(1) = 0 \text{ for all } n \geq 0 \right\}.
$$
The singular integral operator $S_{[-1,1]}$ is defined by the rule

$$(S_{[-1,1]}f)(x) = \frac{1}{\pi i} \int_{-1}^{1} \frac{f(y) - f(x)}{y - x} \, dy, \quad x \in [-1, 1],$$

where the integral has to be understood as the Cauchy principle value. For $f \in C_0^\infty[-1,1]$ the integral exists for each $x \in [-1, 1]$. In fact,

$$(S_{[-1,1]}f)(x) = \frac{1}{\pi i} \int_{-1}^{1} \frac{f(y) - f(x)}{y - x} \, dy + \frac{f(x)}{\pi i} \ln \left( \frac{1 - x}{1 + x} \right), \quad x \in [-1, 1].$$

In particular, $S_{[-1,1]}$ is a well defined linear mapping acting from $C_0^\infty[-1,1]$ into $C^\infty[-1,1]$.

Let $\sigma : [-1, 1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. We denote by $L_\sigma^p[-1,1]$ the space consisting of all Lebesgue measurable functions $f : [-1,1] \to \mathbb{C}$ for which

$$\|f\|_{L_\sigma^p[-1,1]} := \left( \int_{-1}^{1} \sigma^p(x) |f(x)|^p \, dx \right)^{1/p} < \infty.$$  

For certain functions $\sigma$ it is possible to extend the singular integral operator $S_{[-1,1]}$ as defined above on $C_0^\infty[-1,1]$ by continuity to a linear bounded operator acting on the Banach space $L_\sigma^p[-1,1]$. The criteria is related to the $A_p$-condition on $[-1,1]$.

The following theorem was established in the case of the real line $\mathbb{R}$ rather than the interval $[-1,1]$ first by Hunt, Muckenhoupt and Wheeden \cite{HMW}. The theorem itself follows from the results of Coifman and Fefferman \cite{CF}. For more information about the $A_p$ condition and the boundedness of the singular integral operators on more general curves we refer \cite{HMW}.

**Theorem 2.1** Let $\sigma : [-1, 1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume that $\sigma \in L^p[-1,1]$ and $\sigma^{-1} \in L^q[-1,1]$, where $1 < p < \infty$, $1/p + 1/q = 1$. Then $S_{[-1,1]} : C_0^\infty[-1,1] \to C^\infty[-1,1]$ can be continued by continuity to a linear bounded operator acting on $L_\sigma^p[-1,1]$ if and only if $\sigma$ satisfies the $A_p$-condition on $[-1,1]$.

We remark in connection with the previous theorem that the assumptions that $\sigma$ is nonzero almost everywhere and that $\sigma \in L^p[-1,1]$ imply that $C_0^\infty[-1,1]$ is a dense linear subspace of $L_\sigma^p[-1,1]$.

The proof of this statement is similar to the proof of Lemma \cite{22} below. Let $C^\infty(\mathbb{T})$ stand for the set of all infinitely differentiable functions on $\mathbb{T}$, and let $C_0^\infty(\mathbb{T})$ stand for the space of all $f \in C^\infty(\mathbb{T})$ such that $f(t)$ and all of its derivatives vanish at $t = 1$ and $t = -1$:

$$C_0^\infty(\mathbb{T}) = \left\{ f \in C^\infty(\mathbb{T}) : f^{(n)}(1) = f^{(n)}(-1) = 0 \text{ for all } n \geq 0 \right\}.$$
Let $\varrho : \mathbb{T} \to \mathbb{R}_+$ be a Lebesgue measurable and almost everywhere nonzero function. We denote by $L^p_{\varrho}(\mathbb{T})$ the space of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$
\|f\|_{L^p_{\varrho}(\mathbb{T})} := \left( \frac{1}{2\pi} \int_0^{2\pi} \varrho^p(e^{i\theta}) |f(e^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
$$

Let us remark that the dual space to $L^p_{\varrho}(\mathbb{T})$ can be identified with $L^{q-1}_{\varrho^{-1}}(\mathbb{T})$ by means of the bilinear functional

$$
\langle g, f \rangle := \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) f(e^{i\theta}) \, d\theta
$$

with $f \in L^p_{\varrho}(\mathbb{T})$, $g \in L^{q-1}_{\varrho^{-1}}(\mathbb{T})$, $1 < p < \infty$, $1/p + 1/q = 1$.

**Lemma 2.2** Let $\varrho \in L^p(\mathbb{T})$, $1 < p < \infty$, and assume that $\varrho$ is nonzero almost everywhere. Then $C^\infty_0(\mathbb{T})$ is a dense subspace of $L^p_{\varrho}(\mathbb{T})$.

Proof. We introduce the set

$$
X = \left\{ \varrho f : f \in C^\infty_0(\mathbb{T}) \right\}.
$$

Obviously, $X \subseteq L^p(\mathbb{T})$, which implies that $C^\infty_0(\mathbb{T})$ is a subset of $L^p_{\varrho}(\mathbb{T})$. The assertion that $C^\infty_0(\mathbb{T})$ is a dense subspace in $L^p_{\varrho}(\mathbb{T})$ is equivalent to the statement that $X$ is a dense subspace of $L^p(\mathbb{T})$.

We carry out the proof of this statement in several steps. First we prove that the closure of $X$ contains all functions of the form $f = \varrho g$ with $g \in L^\infty(\mathbb{T})$. Indeed, given $g \in L^\infty(\mathbb{T})$ and $\varepsilon > 0$, there are a subset $M \subset \mathbb{T}$ of Lebesgue measure less than $\varepsilon$ and a sequence $g_n \in C^\infty_0(\mathbb{T})$ such that $g_n \to g$ uniformly on $\mathbb{T} \setminus M$ and $\|g_n\|_{L^\infty(\mathbb{T})} \leq \|g\|_{L^\infty(\mathbb{T})}$. Now we can estimate

$$
\|\varrho g - \varrho g_n\|_{L^p(\mathbb{T})} \leq 2 \|\varrho\|_{L^p(M)} \|g\|_{L^\infty(\mathbb{T})} + \|\varrho\|_{L^p(\mathbb{T})} \|g_n - g\|_{L^\infty(\mathbb{T} \setminus M)}.
$$

Since $\|\varrho\|_{L^p(M)} \to 0$ as $\varepsilon \to 0$ the assertion follows easily.

Next we prove that $L^\infty(\mathbb{T})$ is contained in the closure of $X$. Indeed, given $f \in L^\infty(\mathbb{T})$ we introduce the elements $g_\varepsilon = \varrho_\varepsilon f$ where

$$
\varrho_\varepsilon(t) = \begin{cases} 
\varrho^{-1}(t) & \text{if } 0 < \varrho^{-1}(t) \leq \varepsilon^{-1}, \\
0 & \text{if } \varrho^{-1}(t) > \varepsilon^{-1}.
\end{cases}
$$

Obviously, $\varrho_\varepsilon \in L^\infty(\mathbb{T})$ and hence $g_\varepsilon \in L^\infty(\mathbb{T})$. Now we estimate

$$
\|\varrho g_\varepsilon - f\|_{L^p(\mathbb{T})} = \|(\varrho g_\varepsilon - 1)f\|_{L^p(\mathbb{T})} \leq \|\varrho g_\varepsilon - 1\|_{L^p(\mathbb{T})} \|f\|_{L^\infty(\mathbb{T})}.
$$
The function \(1 - \varrho\varrho_\varepsilon\) is equal to the characteristic function of
\[
K_\varepsilon = \left\{ t \in \mathbb{T} : \varrho^{-1}(t) > \varepsilon^{-1} \right\}.
\]
Since \(\varrho\) is nonzero almost everywhere, the Lebesgue measure of \(K_\varepsilon\) tends to zero as \(\varepsilon \to 0\).
Hence \(\|\varrho\varrho_\varepsilon - 1\|_{L^p(\mathbb{T})} \to 0\) as \(\varepsilon \to 0\). It follows that \(\varrho\varrho_\varepsilon\) approximates \(f\).

Finally, it we note that \(L^\infty(\mathbb{T})\) is dense in \(L^p(\mathbb{T})\). \(\square\)

Let \(Q = I - P\) stand for the complementary projection to the Riesz projection \(P\). Moreover, define the operators
\[
G = \frac{1}{2}(I + J)(P - Q)(I - J), \tag{22}
\]
\[
G^* = \frac{1}{2}(I - J)(P - Q)(I + J). \tag{23}
\]
We think of \(G\) and \(G^*\) as linear mappings acting from \(C^\infty_0(\mathbb{T})\) into \(C^\infty(\mathbb{T})\). Notice in this connection that \(P\) and \(Q\) map \(C^\infty_0(\mathbb{T})\) into \(C^\infty(\mathbb{T})\).

**Proposition 2.3** Let \(1 < p < \infty\) and \(1/p + 1/q = 1\). Assume that \(\phi \in GL^\infty(\mathbb{T})\) admits a weak asymmetric factorization \(\phi(t) = \phi_-(t)t^\kappa\phi_0(t)\) in \(L^p(\mathbb{T})\). Then the following is equivalent:

(i) The operator \(B := L(\phi_0^{-1})(I + J)PL(\phi_-^{-1}) : X_1 \to X_2\) can be continued by continuity to a linear bounded operator acting on \(L^p(\mathbb{T})\).

(iii) The operator \(G^* : C^\infty_0(\mathbb{T}) \to C^\infty(\mathbb{T})\) can be continued by continuity to a linear bounded operator acting on \(L^q_{\omega^{-1}}(\mathbb{T})\).

(iii) The operator \(G : C^\infty_0(\mathbb{T}) \to C^\infty(\mathbb{T})\) can be continued by continuity to a linear bounded operator acting on \(L^p_{\omega}(\mathbb{T})\).

Therein \(\omega(t) = |\phi_0^{-1}(t)|\), and \(X_1\) and \(X_2\) are defined by (7) and (8), respectively.

Proof. (i)\(\Leftrightarrow\)(ii): Assertion (i) is equivalent to the fact that there exists a constant \(M > 0\) such that
\[
\|L(\phi_0^{-1})(I + J)PL(\phi_-^{-1})f\|_{L^p(\mathbb{T})} \leq M \|f\|_{L^p(\mathbb{T})}
\]
for all \(f \in X_1\). From the definition of \(L^p_{\omega}(\mathbb{T})\) it follows that the last inequality can be rewritten as
\[
\|(I + J)PL(\phi_-^{-1})f\|_{L^p_{\omega}(\mathbb{T})} \leq M \|f\|_{L^p(\mathbb{T})}.
\]
Since $C_0^\infty(\mathbb{T})$ is dense in $L^q_{\omega-1}(\mathbb{T})$ by Lemma 2.2, we obtain that this is in turn equivalent to the statement that
\[ |\langle g, (I + J)PL(\phi^{-1})f \rangle| \leq M \|g\|_{L^q_{\omega-1}(\mathbb{T})} \|f\|_{L^p(\mathbb{T})} \]
for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Next notice that
\[ \langle g, (I + J)PL(\phi^{-1})f \rangle = \langle P(I + J)g, L(\phi^{-1})f \rangle = \langle L((\phi^{-1})^*)P(I + J)g, f \rangle \]
by noting that $L(\phi^{-1})f \in L^q(\mathbb{T})$. Hence the above is equivalent to the statement that
\[ |\langle L((\phi^{-1})^*)P(I + J)g, f \rangle| \leq M \|g\|_{L^q_{\omega-1}(\mathbb{T})} \|f\|_{L^p(\mathbb{T})} \]
for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Since $X_1$ is dense in $L^p(\mathbb{T})$ we can reformulate this by saying that
\[ \|L((\phi^{-1})^*)P(I + J)g\|_{L^q(\mathbb{T})} \leq M \|g\|_{L^q_{\omega-1}(\mathbb{T})} \]
for all $g \in C_0^\infty(\mathbb{T})$. Because $\phi^{-1}(t) = \phi_0(t)\phi^{-1}(t)$ the latter can be rewritten as
\[ \|P(I + J)g\|_{L^q_{\omega-1}(\mathbb{T})} \leq M \|g\|_{L^q_{\omega-1}(\mathbb{T})}. \]
Since $\omega(t) = \omega(t^{-1})$ the operator $(I + J)$ is bounded on $L^q_{\omega-1}(\mathbb{T})$. Moreover, since $P - Q = 2P - I$ we can conclude that the latter is equivalent to
\[ \|(P - Q)(I + J)g\|_{L^q_{\omega-1}(\mathbb{T})} \leq M \|g\|_{L^q_{\omega-1}(\mathbb{T})} \tag{24} \]
for all $g \in C_0^\infty(\mathbb{T})$. Noting that
\[ G^* = \frac{1}{2}(I - J)(P - Q)(I + J) = (P - Q)(I + J) \]
completes the proof of (i)⇔(ii).

(ii)⇔(iii): Since $C_0^\infty(\mathbb{T})$ is dense in both $L^p_\omega(\mathbb{T})$ and $L^q_{\omega-1}(\mathbb{T})$ by Lemma 2.2, it is easily seen that (ii) is equivalent to the statement that
\[ |\langle G^*g, f \rangle| \leq M \|g\|_{L^p_\omega(\mathbb{T})} \|f\|_{L^q_{\omega-1}(\mathbb{T})} \]
for all $f, g \in C_0^\infty(\mathbb{T})$. Moreover, (iii) is equivalent to the statement that
\[ |\langle g, Gf \rangle| \leq M \|g\|_{L^p_\omega(\mathbb{T})} \|f\|_{L^q_{\omega-1}(\mathbb{T})} \]
for all \( f, g \in C_0^\infty(\mathbb{T}) \). Since \( \langle G^* g, f \rangle = \langle g, Gf \rangle \) the result follows. \( \square \)

Let \( C_{\text{even}}^\infty(\mathbb{T}) \) stand for the space of all functions \( f \in C^\infty(\mathbb{T}) \) which are even, i.e., for which \( f(t) = f(t^{-1}), \; t \in \mathbb{T} \). Moreover, introduce the operators

\[
U : \hat{f}(x) \in C_0^\infty[-1,1] \mapsto f(e^{i\theta}) := \hat{f}(\cos \theta) \in C_0^\infty(\mathbb{T}),
\]

\[
V : f(t) \in C_{\text{even}}^\infty(\mathbb{T}) \mapsto \hat{f}(\cos \theta) := f(e^{i\theta}) \in C^\infty[-1,1],
\]

and let

\[
\chi(e^{i\theta}) = \begin{cases} 
1 & \text{if } 0 < \theta < \pi, \\
-1 & \text{if } \pi < \theta < 2\pi.
\end{cases}
\]

Clearly, the image of \( U \) is the set of even functions defined on \( \mathbb{T} \).

**Proposition 2.4** For \( \hat{f} \in C_0^\infty[-1,1] \) we have

\[
S_{[-1,1]} \hat{f} = \frac{1}{2} VL((1 + t^{-1})^{-1}) GL(\chi(1 + t^{-1})) U \hat{f}.
\]

**Proof.** Given \( \hat{f} \in C_0^\infty[-1,1] \), we introduce the functions

\[
f = L(\chi(1 + t^{-1})) U \hat{f}, \quad g = \frac{1}{2} Gf, \quad \hat{g} = VL((1 + t)^{-1}) g.
\]

Notice that

\[
G = \frac{1}{2} (I + J)(P - Q)(I - J) = (P - Q)(I - J).
\]

Hence \( g(t) = t^{-1}g(1/t) \) and it follows that \( L((1 + t^{-1})^{-1})g \) is an even function. Moreover, it is easily seen that \( Jf = -f \) whence it follows that \( g = \frac{1}{2} Gf = (P - Q)f \). It is well known that the singular integral operator \( S = P - Q \) on \( \mathbb{T} \) can be written as

\[
(Sf)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(s) - f(t)}{s - t} ds + f(t)
\]

for functions \( f \in C^\infty(\mathbb{T}) \). From this we deduce the relations

\[
f(e^{i\theta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta - \varphi)}} d\varphi + f(e^{i\theta}),
\]

\[
g(e^{i\theta}) = \chi(e^{i\theta}) + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta - \varphi)}} d\varphi + f(e^{i\theta}),
\]

\[
\hat{g}(\cos \theta) = (1 + e^{-i\theta})^{-1} g(e^{i\theta}).
\]
We split the integral appearing in the second equation into two parts integrating on $[0, \pi]$ and $[-\pi, 0]$, respectively, and make a change of variables $\varphi \mapsto -\varphi$ in the second integral. This gives

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \left( \frac{f(e^{i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta - \varphi)}} + \frac{f(e^{-i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta + \varphi)}} \right) d\varphi + f(e^{i\theta}).$$

Since $f(e^{-i\varphi}) = -e^{i\varphi} f(e^{i\varphi})$ and since

$$\frac{1}{1 - e^{i(\theta - \varphi)}} - \frac{e^{i\varphi}}{1 - e^{i(\theta + \varphi)}} = \frac{(1 + e^{-i\theta})(e^{i\varphi} - 1)}{2(\cos \varphi - \cos \theta)},$$

$$\frac{1}{1 - e^{i(\theta - \varphi)}} + \frac{1}{1 - e^{i(\theta + \varphi)}} = \frac{e^{i\varphi} + e^{-i\varphi} - 2e^{-i\theta}}{2(\cos \varphi - \cos \theta)},$$

it follows that

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \left( \frac{(1 + e^{-i\theta})(e^{i\varphi} - 1)f(e^{i\varphi})}{2(\cos \varphi - \cos \theta)} - \frac{(e^{i\varphi} - e^{-i\varphi})f(e^{i\theta})}{2(\cos \varphi - \cos \theta)} \right) d\varphi$$

$$- \frac{f(e^{i\theta})}{\pi} \int_0^\pi \frac{e^{-i\varphi} - e^{-i\theta}}{\cos \varphi - \cos \theta} d\varphi + f(e^{i\theta}).$$

If we assume $0 < \theta < \pi$, we obtain

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \frac{(1 + e^{-i\theta})(e^{i\varphi} - e^{-i\varphi})(\hat{f}(\cos \varphi) - \hat{f}(\cos \theta))}{2(\cos \varphi - \cos \theta)} d\varphi$$

$$- \frac{f(e^{i\theta})}{\pi} \int_0^\pi \frac{\sin \varphi - \sin \theta}{\cos \varphi - \cos \theta} d\varphi.$$

The first integral is equal to $(1 + e^{-i\theta})$ times

$$\frac{i}{\pi} \int_0^\pi \frac{(\hat{f}(\cos \varphi) - \hat{f}(\cos \theta)) \sin \varphi}{\cos \varphi - \cos \theta} d\varphi = \frac{1}{\pi i} \int_{-1}^1 \frac{\hat{f}(y) - \hat{f}(\cos \theta)}{y - \cos \theta} dy.$$

The second integral equals $f(e^{i\theta})$ times

$$\frac{1}{\pi i} \int_0^\pi \cot \left( \frac{\varphi + \theta}{2} \right) d\varphi = \left[ \frac{2}{\pi i} \ln \sin \left( \frac{\varphi + \theta}{2} \right) \right]_{\varphi = 0}^\pi = \frac{1}{\pi i} \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right).$$

Putting the pieces together we obtain

$$g(e^{i\theta}) = \frac{(1 + e^{-i\theta})}{\pi i} \int_{-1}^1 \frac{\hat{f}(y) - \hat{f}(\cos \theta)}{y - \cos \theta} dy + \frac{f(e^{i\theta})}{\pi i} \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right).$$
Note that the above assumption \(0 < \theta < \pi\) is not an essential restriction since \(\hat{g}\) is determined by the formula
\[ \hat{g}(\cos \theta) = (1 + e^{-i\theta})^{-1} g(e^{i\theta}), \quad 0 < \theta < \pi. \]

It follows that
\[ \hat{g}(x) = \frac{1}{\pi i} \int_{-1}^{1} \frac{\check{f}(y) - \check{f}(x)}{y - x} dy + \frac{\check{f}(x)}{\pi i} \ln \left( \frac{1 - x}{1 + x} \right). \]

This is equal to the singular integral operator \(S_{[-1,1]}\) applied to the function \(\check{f}\). Hence \(\hat{g} = S_{[-1,1]} \check{f}\) which is the assertion.

Now we are able to present the proof of Theorem 1.3.

Proof of Theorem 1.3  It is obvious from Proposition 1.1 and Theorem 1.2 that the Fredholmness of \(M(\phi)\) implies assertions (a) and (b).

Hence it is sufficient to prove the following. If conditions (a) and (b) are fulfilled, then \(M(\phi)\) is Fredholm if and only if \(\sigma\) satisfies the \(A_p\)-condition.

Under these assumptions we deduce from Theorem 1.2 that the Fredholmness of \(M(\phi)\) on \(H^p(\mathbb{T})\) is equivalent to the existence of a bounded continuation of the operator \(B = L(\phi_0^{-1})(I + J)PL(\phi_0^{-1}) : X_1 \to X_2\) on \(L^p(\mathbb{T})\). We apply Proposition 2.3 and see that this existence is equivalent to the condition that
\[ \|Gg\|_{L^p(\mathbb{T})} \leq M \|g\|_{L^p(\mathbb{T})} \]
for all \(g \in C_0^\infty(\mathbb{T})\) where \(\omega(t) := |\phi_0^{-1}(t)|\).

Next we claim that this, in turn, is equivalent to the condition that
\[ \|Gg\|_{L^p(\mathbb{T})} \leq M \|g\|_{L^p(\mathbb{T})} \]
for all \(g \in C_0^\infty(\mathbb{T})\) for which \(Jg = -g\). In order to prove the non-trivial part of this equivalence, we decompose an arbitrary given \(g \in C_0^\infty(\mathbb{T})\) into \(g = g_1 + g_2\) where \(g_1 = \frac{1}{2}(I + J)g\) and \(g_2 = \frac{1}{2}(I - J)g\). The function \(g_1\) lies in the kernel of \(G\) while \(g_2(t) = (g(t) - t^{-1}g(t^{-1}))/2\) belongs to \(C_0^\infty(\mathbb{T})\) and satisfies the relation \(Jg_2 = -g_2\). Moreover, since \(\omega\) is an even function, the operator \((I - J)/2\) is bounded on \(L^p_0(\mathbb{T})\). We obtain the estimate
\[ \|Gg\|_{L^p(\mathbb{T})} = \|Gg_2\|_{L^p(\mathbb{T})} \leq M \|g_2\|_{L^p(\mathbb{T})} \leq M \|g\|_{L^p(\mathbb{T})}, \]
proves the claim.

Next we remark that the operator \(L(\chi(1 + t^{-1}))U\) maps the space \(C_0^\infty[-1,1]\) onto the subspace of functions \(g \in C_0^\infty(\mathbb{T})\) satisfying \(Jg = -g\). This allows us to make the substitution \(g = L(\chi(1 + t^{-1}))Uf\) with \(f \in C_0^\infty[-1,1]\). We obtain the equivalent estimate
\[ \|GL(\chi(1 + t^{-1}))Uf\|_{L^p_0(\mathbb{T})} \leq M \|L(\chi(1 + t^{-1}))Uf\|_{L^p(\mathbb{T})}. \]
for all \( f \in C_0^\infty[-1,1] \). Clearly, the last estimate can be written in the form
\[
\|VL((1 + t^{-1})^{-1})GL(\chi(1 + t^{-1})Uf)\|_{L^p_{[-1,1]}} \leq M \|f\|_{L^p_{[-1,1]}}
\]
for all \( f \in C_0^\infty[-1,1] \), where
\[
\sigma(\cos \theta) = \frac{\omega(e^{i\theta})|1 + e^{-i\theta}|}{\sqrt{2}|\sin \theta|^{1/p}} = \left|\phi_0^{-1}(e^{i\theta})\right| \frac{|1 + \cos \theta|^{1/q}}{|1 - \cos \theta|^{1/p}}.
\]

Along with Proposition 2.4 and Theorem 2.1 this completes the proof. \( \square \)

### 3 Application to piecewise continuous functions

We now apply the previous results in order to obtain necessary and sufficient conditions for the operator \( M(\phi) \) to be invertible or Fredholm on \( H^p(\mathbb{T}) \) for piecewise continuous functions \( \phi \) with finitely many jumps. These results have already been established in [2] (and in [1] for the case \( p = 2 \)). The proofs given in [1, 2] relies on the results establish in [7] by help of Banach algebra methods. The proof which we will give here is more direct and relies entirely on the factorization methods developed here and in [2] in connection with the \( A_p \)-condition.

We restrict to piecewise continuous functions with a finite number of discontinuities because these functions can be written in a convenient manner which is useful in many instances. It is well known that any piecewise continuous and nonvanishing function with a finite number of discontinuities at the points \( \theta_1, \ldots, \theta_R \) can be written as a product
\[
\phi(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)})
\]
where \( b \) is a nonvanishing continuous function and
\[
t_{\beta}(e^{i\theta}) = \exp(i\beta(\theta - \pi)), \quad 0 < \theta < 2\pi.
\]
Notice that the parameters \( \beta_1, \ldots, \beta_R \) in this formula are uniquely determined up to an additive integer. In fact,
\[
\frac{\phi(e^{i\theta_r} - 0)}{\phi(e^{i\theta_r} + 0)} = \frac{t_{\beta_r}(1 - 0)}{t_{\beta_r}(1 + 0)} = \exp(2\pi i \beta_r).
\]
Moreover, the formula
\[
t_{\beta+n}(t) = (-t)^n t_{\beta}(t), \quad t \in \mathbb{T},
\]
holds for \( n \in \mathbb{Z} \).
The parameters in the representation (29) are useful to decide Fredholmness and invertibility. For example, the Toeplitz operators $T(\phi)$ with a piecewise continuous function $\phi$ with finitely many jump discontinuities is invertible on $H^p(\mathbb{T})$ if and only if the function $\phi$ can be represented in the form (29) with $-1/q < \Re \beta_r < 1/p$ and the winding number of $b$ equal to zero. If the operator is Fredholm with Fredholm index $\kappa$, then the zero is replaced by $-\kappa$. ([4], Chapter 6)

Before stating the analogue of this result for Toeplitz plus Hankel operators $M(\phi)$ we have to establish the following auxiliary result.

**Lemma 3.1** Let $1 < p < \infty$, $1/p + 1/q = 1$, $-1 = x_0 < x_1 < \ldots < x_R < x_{R+1} = 1$ and

$$\sigma(x) = \prod_{r=0}^{R+1} |x - x_r|^\alpha_r$$

(33)

If $-1/p < \alpha_r < 1/q$ for each $0 \leq r \leq R + 1$, then $\sigma \in L^p[-1,1]$, $\sigma^{-1} \in L^q[-1,1]$ and $\sigma$ satisfies the $A_p$-condition.

Proof. This can be verified straightforwardly. $\blacksquare$

The promised result is the following:

**Theorem 3.2** Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose that $\phi$ has finitely many jump discontinuities. Then $M(\phi)$ on is Fredholm on $H^p(\mathbb{T})$ if and only if $\phi$ can be written in the form

$$\phi(e^{i\theta}) = b(e^{i\theta})t_{\beta^+}(e^{i\theta})t_{\beta^-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta^+_r}(e^{i(\theta-\theta_r)})t_{\beta^-_r}(e^{i(\theta+\theta_r)})$$

(34)

where $b$ is a continuous nonvanishing function on $\mathbb{T}$, the numbers $\theta_1, \ldots, \theta_R \in (0, \pi)$ are distinct, and

(i) $-1/q < \Re (\beta^+_r + \beta^-_r) < 1/p$ for each $1 \leq r \leq R$,

(ii) $-1/2 - 1/2q < \Re \beta^+ < 1/2p$ and $-1/2q < \Re \beta^- < 1/2 + 1/2p$.

Moreover, in this case,

$$\dim \ker M(\phi) = \max\{0, \text{wind} (b)\}, \quad \dim \ker M^*(\phi) = \max\{0, \text{wind} (b)\}.$$  \hspace{1cm} (35)

Proof. In the first step we prove that $M(\psi)$ is a Fredholm operator on $H^p(\mathbb{T})$ with Fredholm index zero if $\psi$ is of the form

$$\psi(e^{i\theta}) = t_{\beta^+}(e^{i\theta})t_{\beta^-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta^+_r}(e^{i(\theta-\theta_r)})t_{\beta^-_r}(e^{i(\theta+\theta_r)})$$

(36)
and the parameters satisfy the conditions (i) and (ii). In regard to Theorem 1.3 it suffices to construct a weak asymmetric factorization of $\psi$ and to prove that the corresponding weight $\sigma$ satisfies the $A_p$-condition. For this purpose we introduce the functions

$$\eta_\beta(t) = (1 - t)^\beta, \quad \xi_\beta(t) = (1 - t^{-1})^\beta.$$ 

Notice that $t_\beta(t) = \eta_\beta(t)\xi_{-\beta}(t)$. Then we can factor $\psi(t) = \psi_-(t)\psi_0(t)$ with

$$\psi_-(e^{i\theta}) = \xi_{-2\beta} + (e^{i\theta})\xi_{-2\beta} - (e^{i(\theta - \pi)})$$

$$\times \prod_{r=1}^{R} \xi_{-\beta_r^+ - \beta_r^-} (e^{i(\theta - \theta_r)})\xi_{-\beta_r^+ - \beta_r^-} (e^{i(\theta + \theta_r)}),$$

$$\psi_0(e^{i\theta}) = \eta_{\beta} + (e^{i\theta})\eta_{\beta} - (e^{i(\theta - \pi)})\eta_{\beta} - (e^{i(\theta - \pi)})$$

$$\times \prod_{r=1}^{R} \eta_{\beta_r^+} (e^{i(\theta - \theta_r)})\eta_{\beta_r^-} (e^{i(\theta + \theta_r)})\eta_{\beta_r^-} (e^{i(\theta - \theta_r)}).$$

Because of conditions (i) and (ii), it can be checked straightforwardly that the function

$$(1 + e^{-i\theta})\psi_-(e^{i\theta}) = \xi_{-2\beta} + (e^{i\theta})\xi_{-2\beta} + (e^{i(\theta - \pi)})$$

$$\times \prod_{r=1}^{R} \xi_{-\beta_r^+ - \beta_r^-} (e^{i(\theta - \theta_r)})\xi_{-\beta_r^+ - \beta_r^-} (e^{i(\theta + \theta_r)}),$$

belongs to $\overline{H^p(\mathbb{T})}$ and the function

$$(1 - e^{-i\theta})\psi_-^{-1}(e^{i\theta}) = \xi_{2\beta} + (e^{i\theta})\xi_{2\beta} - (e^{i(\theta - \pi)})$$

$$\times \prod_{r=1}^{R} \xi_{\beta_r^+ + \beta_r^-} (e^{i(\theta - \theta_r)})\xi_{\beta_r^+ + \beta_r^-} (e^{i(\theta + \theta_r)}),$$

belongs to $\overline{H^q(\mathbb{T})}$. From the fact that $\psi_0$ is even and that $\psi_0(t) = \psi_-(t)^{-1}\psi(t)$, it follows that the function $\psi_0$ fulfills all the necessary conditions in regard to a weak asymmetric factorization. Hence $\psi(t) = \psi_-(t)\psi_0(t)$ is indeed a weak asymmetric factorization with index zero.

In order to calculate the corresponding weight function (13) consider

$$\psi_0(e^{i\theta}) = |1 - e^{i\theta}|^{2\beta^+} |1 + e^{i\theta}|^{2\beta^-}$$

$$\times \prod_{r=1}^{R} |1 - e^{i(\theta - \theta_r)}|^{\beta_r^+ + \beta_r^-} |1 - e^{i(\theta + \theta_r)}|^{\beta_r^+ + \beta_r^-} t_{\beta_r^+ - \beta_r^-} (e^{i(\theta - \theta_r)}) t_{\beta_r^+ - \beta_r^-} (e^{i(\theta + \theta_r)}),$$

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and observe that $|1 - e^{i\theta}| = (2 - 2 \cos \theta)^{1/2} = 2 \sin(\frac{\theta}{2})$ and $2 \sin \frac{\theta - \theta_r}{2} \sin \frac{\theta + \theta_r}{2} = \cos \theta_r - \cos \theta$. Hence

$$\psi_0^{-1}(e^{i\theta}) = \sigma_0(\cos \theta)(1 - \cos \theta)^{-\beta^+}(1 + \cos \theta)^{-\beta^-} \prod_{r=1}^R |\cos \theta - \cos \theta_r|^{-\beta^+_r - \beta^-_r},$$

where $\sigma_0(x) \in GL^\infty(\mathbb{T})$ is a function which comes from collecting the terms $t_{\frac{\beta^+_r - \beta^-_r}{2}}(e^{i(\theta - \theta_r)})$, $t_{\frac{\beta^+ - \beta^-}{2}}(e^{i(\theta + \theta_r)})$ and certain constants. It follows that $\sigma$ evaluates to

$$\sigma(x) = |\sigma_0(x)|(1 - x)^{-\Re \beta^+ - 1/2\rho}(1 + x)^{-\Re \beta^- + 1/2\rho} \prod_{r=1}^R |x - \cos \theta_r|^{-\Re \beta^+_r - \Re \beta^-_r}.$$  \hspace{1cm} (37)

It suffices to apply Lemma 3.1 in order to see that $\sigma$ satisfies the $A_p$-condition.

In the second step we prove that $M(\phi)$ is a Fredholm operator if $\phi$ is given by (34) with conditions (i) and (ii) being fulfilled and if the function $b$ is continuous and nonvanishing. We can write $\phi(t) = b(t)\psi(t)$ where $\psi$ is as above. From well-known identities for Toeplitz and Hankel operators,

$$T(\phi) = T(b)T(\psi) + H(b)H(\tilde{\psi}),$$

$$H(\phi) = T(b)H(\psi) + H(b)T(\tilde{\psi}),$$

where $\tilde{\psi}(t) = \psi(t^{-1})$, it follows that

$$M(\phi) = T(b)M(\psi) + H(b)M(\tilde{\psi}).$$

Under the assumption on $b$ the operator $H(b)$ is compact and the operator $T(b)$ is Fredholm with Fredholm index equal to $-\text{wind}(b)$. Since we have just proved that $M(\psi)$ is Fredholm with Fredholm index zero, it follows that $M(\phi)$ is Fredholm with Fredholm index equal to $-\text{wind}(b)$.

Hence we have proved the “if” part of the theorem and also computed the Fredholm index of $M(\phi)$. Now we apply Theorem 1.2 with formula (10). This formula implies that the Fredholm index is equal to $-\kappa$, where $\kappa$ is the index of the asymmetric factorization of $\phi$. Hence $\kappa = \text{wind}(b)$ and formula (35) follows.

In the last step we are going to prove the “only if” part of the theorem. It is settled by a well-known perturbation argument. Suppose that $M(\phi)$ is a Fredholm operator with index $\kappa$, say. We conclude from Proposition 1.1 that $\phi \in GL^\infty(\mathbb{T})$. Since $\phi$ has only a finite number of jump discontinuities, this implies that $\phi$ can be written in the form (29) or in the form (32) if we put some of the $\beta$-parameters equal to zero if necessary. Therein the function $b$ is continuous and nonvanishing on $\mathbb{T}$. Moreover, due to formula (31) and (32) we can choose the $\beta$-parameters to satisfy the conditions
Assume contrary to what we want to prove, namely, that in at least one instance we have equality in the above conditions (i*) and (ii*). We are going to perturbate the $\beta$-parameters (and thus the function $\phi$) in two different ways in order to arrive at a contradiction. Remark that since $M(\phi)$ is assumed to be Fredholm, the Fredholm index is constant with respect to any small perturbation. We first perturbate by replacing all $\beta$-parameters by $\beta - \varepsilon$ where $\varepsilon > 0$ is sufficiently small. This turns the conditions (i*) and (ii*) into (i) and (ii).

Applying the “if” part of the theorem with the formula for the Fredholm index, it follows that $\kappa = -\text{wind} (b)$. In the second perturbation we do the same substitution except for one of the instances of equality in (i*) and (ii*) where we replace the corresponding $\beta^\pm$ by $\beta^\pm - 1 + \varepsilon$, or, the corresponding $\beta^+_r + \beta^-_r$ by $\beta^+_r + \beta^-_r - 1 + \varepsilon$, respectively. Moreover, we have to replace $b(t)$ by $t b(t)$ times a certain constant due to formula (32). This is again a small perturbation of $\phi$, which leaves the Fredholm index unchanged. The corresponding parameters fulfill (i) and (ii), but we obtain $\kappa = -\text{wind} (t b(t)) = -1 - \text{wind} (b)$ contradicting the above formula. This completes the proof of the “only if” part.

\[\Box\]

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