A SOBOLEV SPACE PROPERTY OF LOGARITHM OF LIPSCHITZ FUNCTIONS

YIFEI PAN

Abstract. For a Lipschitz function $f$ on an open set in $\mathbb{R}^n$, we consider the $L^n$ integrability of the quotient $\frac{|\nabla f|}{|f|}$ over its natural domain of definition.

1. Introduction and Results

In this note, we prove the following seemingly simple result concerning the logarithm of Lipschitz functions.

**Theorem 1.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f$ be a Lipschitz function on $\overline{\Omega}$. Then the function $\log |f(x) - f(a)|$ does not belong to the Sobolev space $W^{1,n}(\Omega)$ for every $a \in \overline{\Omega}$. Furthermore, if the function $\log |f(x)|$ belongs to the Sobolev space $W^{1,n}(\Omega)$, then $f$ never vanishes in the closure of $\Omega$.

A natural way to view this result is to consider a (non-linear) map $T_a(f) = \log |f(x) - f(a)|$ defined on all Lipschitz functions $\text{Lip}(\Omega)$ for each $a \in \overline{\Omega}$. Then

$$T_a(\text{Lip}(\Omega)) \cap W^{1,n}(\Omega) = \emptyset.$$ 

The proof of this result is based on a general blow-up phenomenon, as we shall prove below, for Lipschitz functions. This phenomenon seems to reveal a competition between the sets of critical points and zeros at least for continuously differentiable functions.

**Theorem 1.2.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f$ be a non-constant locally Lipschitz function on $\Omega$. If the zero set $\{x \in \Omega : f(x) = 0\}$ of $f$ is not empty, then

$$\int_{\Omega \setminus f^{-1}(0)} \left| \nabla f(x) \right|^n |f(x)| dx = \infty, \quad (1.1)$$

or equivalently

$$\int_{\Omega \setminus f^{-1}(0)} |\nabla \log |f(x)||^n dx = \infty. \quad (1.2)$$

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Throughout, we denote the gradient $\nabla f$ of a Lipschitz function $f$, which exists a.e. by Rademacher’s theorem, and its norm $|\nabla f|^2 = f^2_1 + \ldots + f^2_n$. Also, it is a well-known fact that a (locally integrable) function is locally Lipschitz if and only if its distributional gradient $\nabla f$ is locally $L^\infty$. On the other hand, any Lipschitz function on a set of a metric measure space can be extended to be a Lipschitz function over the whole space with the same Lipschitz constant; in particularly, for our purpose in this paper, we can always use the extension of a Lipschitz function $f$ in $\Omega$ to one in $\mathbb{R}^n$ with the same Lipschitz constant [4].

An immediate corollary of Theorem 1.2 is a uniqueness theorem of differential inequality of the gradient.

**Corollary 1.3.** Let $f$ be a locally Lipschitz function in an open set $\Omega$ satisfying

$$|\nabla f(x)| \leq V(x)|f(x)|, \ x \in \Omega, \ a.e.,$$

where $V \in L^\infty_{loc}(\Omega)$. If there is a point $x_0 \in \Omega$ such that $f(x_0) = 0$, then $f \equiv 0$ in $\Omega$.

**Corollary 1.4.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f$ be a non-constant locally Lipschitz function on $\Omega$. If the zero set $\{x \in \Omega : f(x) = 0\}$ of $f$ is not empty and

$$\int_{\Omega \setminus f^{-1}(0)} |\nabla \log |f(x)||^p \, dx < \infty.$$

then, $p < n$

**Corollary 1.5.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f \in W^{1,n}(\Omega)$. Then the exponential $e^f$ of $f$ cannot be Lipschitz unless $f$ is Lipschitz and $f \neq 0$ in $\overline{\Omega}$.

Theorem 1.2 is somehow cosmetically related to the following well-known result of J. Bourgain, H. Brezis and P. Mironescu [3].

**Theorem 1.6.** [3] Let $\Omega$ be a connected open set in $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ be a non-constant measurable function. Then

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|^{n+1}} \, dx \, dy = \infty.$$

We make some remarks on the results above, which seem to be easy to state, but not exactly obvious to prove. First the integrability exponent $n$, the dimension of the space is the best possible for the results to hold if we take, for example, the function $f(x) = |x|^2$. Secondly, for positive functions, the integral can be arbitrarily small. In [1] A. Chang, M. Gursky, and T. Wolff proved the following calculus result, which is crucial for them to construct a counterexample of a geometric problem.

**Proposition 1.7.** [1, page 144] Suppose $n \geq 3$. Then for any $R > 0, A, B > 0, \epsilon > 0$, there are $\delta > 0$ and a smooth radial function $\psi : \mathbb{R}^n \to (0, +\infty)$ such that $\psi(x) = A$ when $|x| \geq R, \psi(x) = B$ when $|x| \leq \delta$, $\min(A, B) \leq \psi \leq \max(A, B)$, and

$$\int_{|x| \leq R} \left| \psi^{-1} \frac{d\psi}{dx^i} \right|^n < \epsilon \quad 1 \leq i \leq n.$$
This example is, equivalently, saying that there are positive functions for which the integral
\[ \int_{|x|<R} |\nabla \log |\psi(x)||^n \, dx \]
can be made arbitrarily small. On the other hand, Theorem 4.1 has the following interesting interpretation as an extension result: for every Lipschitz \( f \) and every \( a \in \Omega \), the function \( \frac{\nabla f(x)}{|f(x)-f(a)|} \) cannot be extended to a function in \( L^n(\Omega) \). We will actually prove that the integral in (4.1) diverges around any point of the boundary of \( f^{-1}(f(a)) \). Hence \( \frac{\nabla f(x)}{|f(x)-f(a)|} \) cannot be extended to an \( L^n \) function beyond its natural domain of definition \( \Omega \setminus f^{-1}(f(a)) \).

As a curious consequence, given any closed set \( A \subset \mathbb{R}^n \), it is possible to construct a continuous function on \( \mathbb{R}^n \setminus A \) that cannot be extended locally to a \( L^n \) function past any point of the boundary of \( A \). Indeed, one just needs to consider the quotient \( \frac{\nabla f}{|f|} \), where \( f \) is a smooth function in \( \mathbb{R}^n \) whose zeroset is \( A \). Note that such an \( f \) exists by the Whitney extension theorem [2].

2. Proof of Theorem 1.2

We first begin with a uniqueness theorem for Lipschitz functions in one real variable over intervals which is tailored for applications in polar coordinates in order to prove the results in this paper. Recall that a Lipschitz function in \([0,1]\) is differentiable almost everywhere and satisfies \( f(b) - f(a) = \int_a^b f'(x) \, dx \) for any \( a,b \in [0,1] \).

**Lemma 2.1.** Let \( \varphi \) be a Lipschitz function over \([0,1]\) with \( \varphi(0) = 0 \). Assume that there exists \( p \geq 1 \) and a non-negative function \( \lambda \in L^p(0,1) \) such that
\[ |\varphi'(x)| \leq \lambda(x) |\varphi(x)| \frac{x^{1-p}}{x^p} \quad \forall x \in (0,1), \text{a.e.} \quad (2.1) \]
Then \( \varphi \equiv 0 \) in \([0,1]\).

**Proof.** Let \( \delta = \sup\{d \in [0,1] | \varphi \equiv 0 \text{ in } [0,d]\} \). By the fundamental theorem for Lipschitz functions and Hölder’s inequality, we have
\[ |\varphi(x)| \leq \int_\delta^x |\varphi'(t)| \, dt \leq \left( \int_\delta^x |\varphi'(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_\delta^x 1^q \, dt \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (2.2) \]
Hence
\[ |\varphi(x)|^p \leq x^\frac{p}{q} \int_\delta^x |\varphi'(t)|^p \, dt = x^{p-1} \int_\delta^x |\varphi'(t)|^p \, dt. \quad (2.3) \]
We multiply both sides of (2.3) by the function \( \lambda^p(x) \). Note that we can assume without loss of generality that \( \lambda \) is non-vanishing. Indeed, if that is not the case, we can just replace \( \lambda \) with \( 1 + \lambda \). We obtain
\[ \lambda^p(x) |\varphi(x)|^p \leq x^{p-1} \lambda^p(x) \int_\delta^x |\varphi'(t)|^p \, dt. \quad (2.4) \]
Let $s \in (\delta, 1)$. Integrating in the variable $x$ on both sides of (2.4) gives
\[
\int_\delta^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx \leq \int_\delta^s \left( \lambda^p(x) \int_\delta^x |\varphi'(t)|^p \, dt \right) \, dx.
\]  
(2.5)
Since $x \leq s$, then (2.5) implies
\[
\int_\delta^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx \leq \left( \int_\delta^s \lambda^p(x) \, dx \right) \left( \int_\delta^s |\varphi'(x)|^p \, dx \right).
\]  
(2.6)
By (2.1), we have that
\[
\int_\delta^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx \leq \left( \int_\delta^s \lambda^p(x) \, dx \right) \left( \int_\delta^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx \right).
\]  
(2.7)
Since $\varphi$ is Lipschitz and $\varphi(0) = 0$, we have that $|\varphi(x)| \leq C|x|$ for some $C$ for $x \in [0, 1]$, and therefore the function $|\varphi(x)|^p x^{1-p}$ is bounded. Since $\lambda \in L^p(0, 1)$, we conclude that
\[
\int_\delta^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx < \infty \quad \forall s \in (\delta, 1).
\]  
(2.8)
and we can find a sequence of points $s_j \in (\delta, 1), s_j \to \delta$, such that
\[
\int_{s_j}^s \lambda^p(x) |\varphi(x)|^p x^{1-p} \, dx \neq 0 \quad \forall j.
\]  
(2.9)
Equation (2.7) then yields
\[
1 \leq \int_{s_j}^s \lambda^p(x) \, dx \quad \forall j.
\]  
(2.10)
Since $\lambda \in L^p(0, 1)$, then letting $s_j \to \delta$ in (2.10) leads to a contradiction. \qed

Rademacher’s theorem says $\varphi$ is Lipschitz function is differentiable almost everywhere. The following simple, but useful lemma tells where the square of a Lipschitz function could be differentiable.

**Lemma 2.2.** Let $f$ be a locally Lipschitz function in a domain. Then the square function $g = f^2$ is differentiable wherever $f$ vanishes and in fact $\nabla g(x) = 0$ there.

**Proof.** Let $x_0$ be such that $f(x_0) = 0$. We claim that $g = f^2$ is differentiable at $x_0$ and $\nabla g(x_0) = 0$. Indeed, by the definition of differentiability,
\[
\lim_{x \to x_0} \frac{g(x) - g(x_0) - 0 \cdot (x - x_0)}{|x - x_0|} = \lim_{x \to x_0} \frac{f^2(x)}{|x - x_0|} = 0,
\]
where we have used $f(x) = f(x) - f(x_0) = O(|x - x_0|)$ because of $f$ being locally Lipschitz at $x_0$. \qed

**Lemma 2.3.** Let $A$ be a measurable set of measure zero in the unit ball in $\mathbb{R}^n$. Then for almost all $\omega \in S^{n-1}$, the unit sphere, the set of intersection of $A$ with the ray $\{r\omega : 0 \leq r \leq 1\}$ is of measure zero in the line measure.
Proof. Let $\chi_A$ be the characteristic function of the set $A$. We have $0 = |A| = \int_{|x|<1} \chi_A(x)dx = \int_{S^{n-1}} \int_0^1 \chi_A(r\omega) r^{n-1}drd\omega$. By Fubini’s theorem, we conclude that for almost all $\omega \in S^{n-1}$, \[
abla f(x) = 0\] whenever the function is 0. For the rest of the proof we assume $\nabla f(x) = 0$ whenever $f = 0$. Let \[V(x) = \begin{cases} \frac{\nabla f(x)}{f(x)} & x \in \Omega \setminus Z \\ 0 & x \in Z. \end{cases} \] (2.12)
Note that $V$ is a measurable function in $\Omega$. Assume now by contradiction that (2.11) is false. Then \[
abla f(x) \text{ is the origin and the radius } r_0 \text{ is equal to } 1. \] Hence \[
abla f(x) \text{ is the origin and the radius } r_0 \text{ is equal to } 1. \] Hence \[
abla f(x) = 0 \text{ whenever } f = 0. \]
Let \[
abla f(x) = 0 \text{ whenever } f = 0. \]
and therefore \[V \in L^n(\Omega). \] Choose a point $x_0 \in \partial Z$ and a ball $B(x_0, r_0)$ of radius $r_0 > 0$ centered at $x_0$ such that $B(x_0, r_0) \subset \Omega$. We assume, after a translation and scaling, that $x_0$ is the origin and the radius $r_0$ is equal to 1. Hence \[
abla f(x) = 0 \text{ whenever } f = 0. \]
(2.14)
Since the integral (2.14) is finite, then Fubini’s theorem implies that for almost all $\omega \in S^{n-1}$ we have
\[
abla f(x) = 0 \text{ whenever } f = 0. \]
(2.15)
By Rademacher’s theorem, we set $A$ to be the set where $\nabla f(x)$ does not exist at $x$ so that $A$ is of measure zero. Choose $\omega_0 \in S^{n-1}$ is such that (2.15) holds, that is, $V(r\omega)r^{n-1} \in L^d(0, 1)$. and at the same time, by Lemma 2.3 we can choose the same $\omega_0 \in S^{n-1}$ such that $\nabla f(x)$ exists a.e. on the line (ray) $\{r\omega_0\}$. Let \[\varphi(t) := f(t\omega_0), \quad t \in [0, 1]. \]
It is evident that $\varphi(t)$ is Lipschitz in $t$ because of $f$ locally Lipschitz. Then for differential points of $f$, and applying the chain rule there we have
\[ \varphi'(t) = \nabla f(t_0) \cdot \omega_0 \]
which implies, for a.e. $t$,
\[ |\varphi'(t)| \leq |\nabla f(t_0)|. \]
(2.16)
By the definition of $V$, we have
\[ |\varphi'(t)| \leq V(t_0)|f(t_0)| = V(t_0)|\varphi(t)| \quad \text{for } f(t_0) \neq 0. \]
(2.17)
However, when $f(t_0) = 0$, we have by the observation at the beginning of the proof, $\nabla f(r_0) = 0$ and therefore $\varphi'(t) = 0$. Hence we have shown that
\[ |\varphi'(t)| \leq V(t_0)|\varphi(t)| = V(t_0)t^{-\frac{n-1}{n}} \]
holds for a.e. $t$. By Lemma 2.1, with $\lambda(t) = V(t_0)t^{\frac{n-1}{n}}$ and $p = n$, $\varphi(t) \equiv 0$, and it implies $f \equiv 0$ since $\omega_0$ is arbitrary off a measure zero and $x_0$ in the boundary of $Z$, a contradiction.

3. Proof of Theorem 1.1

Here before we prove Theorem 1.1, which is a simple consequence of Theorem 1.2, we recall that a function belongs to $W^{1,p}(\Omega)$ if $f \in L^p(\Omega)$ and its weak derivative $\nabla f$ belongs to $L^p(\Omega)$. Now we prove Theorem 1.1. If $\log|f(x) - f(a)|$ does not belong to $L^n(\Omega)$, we are done. If $\log|f(x) - f(a)|$ belongs to $L^n(\Omega)$, then apply Theorem 1.2 to conclude $\nabla \log|f(x) - f(a)|$ does not belong to $L^n(\Omega)$.

For the second part, first we extend $f$ to be a Lipschitz function defined in $\mathbb{R}^n$. If $f(x)$ vanishes at an point in the closure of $\Omega$, we apply Theorem 1.2 to the extended function to get a contradiction.

4. Generalization to Lipschitz mappings

All results proved so far can be generalized to Lipschitz mappings. First we say $f : \Omega \rightarrow \mathbb{R}^m$ is a Lipschitz mapping if each of the components is a Lipschitz function. Now we can state the mapping version of Theorem 1.2.

**Theorem 4.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f : \Omega \rightarrow \mathbb{R}^m$ be a non-constant locally Lipschitz mapping on $\Omega$. If the common zero set $\{x \in \Omega : f(x) = 0\}$ of the mapping $f$ is not empty, then
\[ \int_{\Omega \setminus f^{-1}(0)} \left| \frac{\nabla f(x)}{f(x)} \right|^n dx = \infty, \]
(4.1)
where we have $|\nabla f|^2 = |\nabla f_1|^2 + \ldots + |\nabla f_m|^2$ if $f = (f_1, \ldots, f_m)$. 
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*Department of Mathematical Sciences, Purdue University Fort Wayne, 2101 East Coliseum Boulevard, Fort Wayne, IN 46805, USA

Email address: pan@pfw.edu*