UNIQUENESS OF GRIM HYPERPLANES FOR MEAN CURVATURE FLOWS

DITTER TASAYCO AND DETANG ZHOU

Abstract. In this paper we show that an immersed nontrivial translating soliton for mean curvature flow in $\mathbb{R}^{n+1}$ ($n = 2, 3$) is a grim hyperplane if and only if it is mean convex and has weighted total extrinsic curvature of at most quadratic growth. For an embedded translating soliton $\Sigma$ with nonnegative scalar curvature, we prove that if the mean curvature of $\Sigma$ does not change signs on each end, then $\Sigma$ must have positive scalar curvature unless it is either a hyperplane or a grim hyperplane.

1. Introduction

A mean curvature flow (MCF) in $\mathbb{R}^{n+1}$ is the negative gradient flow of the volume functional, which can be analyzed from the perspective of partial differential equations as shown by Huisken in [4]. MCF is smooth in a short time and singularities must happen over a longer time. According to the rate of blow-up of the second fundamental form $A(t, p)$ of the hypersurface $\Sigma_t$, this finite time singularity $T$ is called type-I, if there exists a constant $C_0$ such that

$$\sup_{p \in \Sigma_t} |A(t, p)|^2 \leq \frac{C_0}{(T - t)}$$

for all $t < T$. Otherwise this finite time singularity is called type-II.

We will deal with translating solitons which are important in study of type-II singularities.

A complete connected isometrically immersed hypersurface $(\Sigma, \Phi)$ in $\mathbb{R}^{n+1}$ is called a translating soliton if its mean curvature vector satisfies

$$\vec{H} = w^\perp,$$

where $w \in \mathbb{R}^{n+1}$ is a unitary vector and $w^\perp$ stands for the orthogonal projection of $w$ onto the normal bundle of $\Phi$. Let $\nu$ denote the unit normal along $\Phi$, then it is equivalent to

$$H = -\langle \nu, w \rangle.$$

In particular, considering $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x) = -\langle x, w \rangle$, then $\nabla f = -w$ and $H = \langle \nabla f, \nu \rangle$, therefore by definition translating solitons are $f$-minimal hypersurfaces. Since MCF is invariant under isometries,

2000 Mathematics Subject Classification. Primary: 53C21; Secondary: 53C42.

The second author was partially supported by CNPq and Faperj of Brazil.
without loss of generality we may suppose that \( w = (0, \ldots, 0, 1) \), then the function \( f \) is defined by \( f(x) = -x_{n+1} \) and the \( L_f \)-stability operator of \( \Sigma \) is given by

\[
L_f = \Delta_f + |A|^2
\]

There are some examples of translating solitons: vertical hyperplanes, grim hyperplanes, translating bowl solitons and translating catenoids. In this article we will give a characterization of grim hyperplanes in dimensions 2 and 3.

Recall that a grim hyperplane in \( \mathbb{R}^{n+1} \) is a hypersurface \( G \) of \( \mathbb{R}^{n+1} \) which can be represented parametrically via the embedding \( \Phi : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n+1} \) defined by

\[
\Phi(t, y_1, \ldots, y_{n-1}) = (t, y_1, \ldots, y_{n-1}, -\ln (\cos t)).
\]

The grim hyperplane \( G \) satisfies the translating soliton equation with \( w = (0, \ldots, 0, 1) \) i.e. it is \( f \)-minimal for \( f(x_1, \ldots, x_{n+1}) = -x_{n+1} \). Also it has positive mean curvature. When \( n = 2 \) or 3, there exists a constant \( C > 0 \) such that

\[
\int_{B_R} |A|^2 e^{-f} \leq CR^2
\]

for all \( R \) sufficiently large. The aim of this article is to prove that indeed the grim hyperplanes are the only ones with these properties when \( n = 2, 3 \).

**Theorem 1.** Let \( \Phi : \Sigma^n \rightarrow \mathbb{R}^{n+1} \) be a translating soliton, with \( n = 2 \) or 3, which is not a hyperplane. Then \( \Sigma \) is a grim hyperplane if and only if \( H \equiv -\langle w, \nu \rangle \geq 0 \) and there exists \( C > 0 \) such that

\[
\int_{B_R} |A|^2 e^{-f} \leq CR^2,
\]

for all \( R \) sufficiently large, where \( B_R \) is the geodesic ball of radius \( R \) and \( f(x) = -\langle x, w \rangle \).

The expression (3) is not satisfied for \( n \geq 4 \) (see Proposition 1), thus Theorem 1 is sharp in this sense.

It has been known that if \( H \geq 0 \) on a translating soliton \( \Sigma \), then either \( H \equiv 0 \) on \( \Sigma \) and \( \Sigma \) is a hyperplane, or \( H > 0 \) everywhere on \( \Sigma \). Note that both hyperplane and grim hyperplane has vanishing scalar curvature. In [6], Martín-Sávás-Halilaj-Šmoczyk proved that flat hyperplane and grim hyperplane are the only translating soliton with vanishing scalar curvature. It would be interesting to ask if the following is true.

**Problem:** Let \( \Sigma \) be a translating soliton with nonnegative scalar curvature \( S \). Is it true that either \( S \equiv 0 \) on \( \Sigma \) and \( \Sigma \) is a hyperplane or grim hyperplane, or \( S > 0 \) everywhere on \( \Sigma \)?

This problem is related to a result proved by Huang-Wu in [3]. Let \( M \) be a closed embedded \( n \)-dimensional hypersurface in \( \mathbb{R}^{n+1} \) with nonnegative scalar curvature. Let \( M_t \) be a solution to the mean curvature flow with
initial hypersurface $M$. Then the scalar curvature of $M_t$ is strictly positive for all $t > 0$.

For complete embedded translating solitons, we have

**Theorem 2.** Let $(Σ^n, g)$ be a embedded translating soliton with nonnegative scalar curvature $S$. Assume $H$ does not change signs on each end. Then either $Σ$ is a hyperplane or a grim hypersurface; or $Σ$ has positive scalar curvature.

2. **Total weighted extrinsic curvature**

In this section we will give the asymptotic properties of the total weighted extrinsic curvatures of grim hyperplanes. We have

$$\partial_t = \sec(t) (\cos t, 0, \cdots, 0, \sin t).$$

We choose the unit normal $ν$ to $G$ to be $ν = (\sin t, 0, \cdots, 0, −\cos t)$. A little computation shows that $\nabla \partial_t ν = (\cos t) \partial_t ν$ and $\nabla \partial_{y_i} ν = 0$ $(1 ≤ i ≤ n − 1)$.

Then the principal curvatures are $λ_1 = \cos t, λ_2 = \cdots = λ_n = 0$, thus on the coordinates $t, y_1, \ldots, y_{n−1}$ the mean curvature only depends on $t$ and is given by $H(t) = \cos t$. Since $t ∈ (−\frac{\pi}{2}, \frac{\pi}{2})$, we have the norm of the second fundamental form is given by

$$|A| (t) = |H(t)|.$$

Now, consider the function $f : \mathbb{R}^{n+1} → \mathbb{R}$ defined by $f(x) = −x_{n+1}$, then

$$\langle \nabla f, ν \rangle = \cos t = H(t).$$

**Proposition 1.** The Grim Hyperplane $G$ in $\mathbb{R}^{n+1}$ satisfies

$$\lim_{R→+∞} \frac{1}{R^{n−1}} \int_{B_R} |A|^2 e^{-f} = |B^{n−1}(1)| π,$$

where $B_R$ is the geodesic ball with center at $0$ and radius $R$ and $B^{n−1}(1)$ is the open ball in $\mathbb{R}^{n−1}$ of radius $1$ and center at the origin.

**Proof of Proposition 1**. Observe that $f$ and the metric on $G$ in the coordinates $t, y_1, \ldots, y_{n−1}$ are given by

$$f(t) = \ln(\cos t)$$

and

$$g = \sec^2(t) dt^2 + dy_1^2 + \cdots + dy_{n−1}^2.$$ 

Thus

$$r = \int_0^t \sec(ξ) dξ = −\ln \left( \tan \left( \frac{1}{2} (\frac{π}{2} − t) \right) \right),$$

we have $t = \frac{π}{2} − η(r)$, where $η(r) = 2 \arctan(e^r)$. Then

$$g = dr^2 + dy_1^2 + \cdots + dy_{n−1}^2.$$ 

Besides that $|A|$ and $f$ in the coordinates $r, y_1, \cdots, y_{n−1}$ are given by

$$|A|(r) = \sin(η(r)),$$
and
\[ f(r) = \ln(\sin(\eta(r))). \]

Denoting by \( \| \cdot \| \) the standard norm of \( \mathbb{R}^{n-1} \), we have
\[ B_R = \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : r^2 + \|y\|^2 \leq R^2 \right\} \]
\[ = \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : -\sqrt{R^2 - \|y\|^2} \leq r \leq \sqrt{R^2 - \|y\|^2}, \|y\| \leq R \right\}. \]

Since \(-\eta'(r) = \sin(\eta(r))\) is an even function, then
\[
\int_{B_R} |A|^2 e^{-f} = \int_{\{\|y\| \leq R\}} \left[ \int_{-\sqrt{R^2 - \|y\|^2}}^{\sqrt{R^2 - \|y\|^2}} \sin(\eta(r)) \, dr \right] \, dy
\]
\[ = \int_{\{\|y\| \leq R\}} \left[ \pi - 2\eta\left(\sqrt{R^2 - \|y\|^2}\right) \right] \, dy
\]
\[ = \pi |B^{n-1}(1)| R^{n-1} - 2 \int_0^R \left( \int_{S^{n-2}_\rho} \eta\left(\sqrt{R^2 - \rho^2}\right) \, dA \right) \, d\rho
\]
\[ = \pi |B^{n-1}(1)| R^{n-1} - 2 \text{area}(S^{n-2}) \int_0^R \eta\left(\sqrt{R^2 - \rho^2}\right) \rho^{n-2} \, d\rho.
\]

where we have used the co-area formula. Now, letting \( \rho = R \cos \theta \) and using the fact \( \text{area}(S^{n-2}) = (n-1) |B^{n-1}(1)| \), we have
\[
\frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| \left[ \pi - 2(n-1) F_{n-1}(R) \right],
\]
where
\[
F_{n-1}(R) = \int_0^{\pi/2} \eta(R \cos \theta) \sin^{n-2} \theta \cos \theta d\theta.
\]

Observe that
\[
\lim_{R \to +\infty} \eta(R \cos \theta) \sin^{n-2} \theta \cos \theta = 0 \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right].
\]

Fixing \( R > 0 \), we have \( |\eta(R \cos \theta) \sin^{n-2} \theta \cos \theta| \leq \frac{\pi}{2} \sin^{n-2} \theta \cos \theta \) for all \( \theta \in [0, \pi/2] \). Besides that
\[
\int_0^{\pi/2} \sin^{n-2} \theta \cos \theta d\theta = 1/(n-1).
\]

Then \( \lim_{R \to +\infty} F_{n-1}(R) = 0 \), and hence by (\ref{eq:5}), we get
\[
\lim_{R \to +\infty} \frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| \pi.
\]
3. Proofs of Theorem 1 and Theorem 2

We begin this section with the following lemma which is in a form more general than we need. The lemma may have its independent interest.

**Lemma 1.** Assume that on a complete weighted manifold \( (M, \langle \cdot, \cdot \rangle, e^{-f} dv_{M}) \), the functions \( u, v \in C^2(M) \), with \( u > 0 \) and \( v \geq 0 \) on \( M \), satisfy

\[
\Delta f u + q(x) u \leq 0 \quad \text{and} \quad \Delta f v + q(x) v \geq 0,
\]

where \( q(x) \in C^0(M) \). Suppose that there exists a positive function \( \kappa > 0 \) on \( \mathbb{R}^+ \) satisfying the nonincreasing condition

\[
\int^{+\infty} \frac{t}{\kappa(t)} dt = +\infty,
\]

such that

\[
\int_{B_R} v^2 e^{-f} \leq \kappa(R)
\]

for all \( R \). Then there exists a constant \( C \) such that \( v = Cu \).

**Remark 1.** Without loss of generality, we can assume \( \kappa(t) \geq Ct^2 \). Some examples of \( \kappa(t) \) are \( Ct^2 \), \( Ct^2 \log(1 + t^2) \), \( Ct^2 \log(1 + t) \log \log(3 + t) \), \( \cdots \).

**Proof of Lemma 1.** Set \( w = \frac{v}{u} \), then \( v = uw \), thus by (7) we get

\[
\Delta f v = w \Delta f u + 2 \langle \nabla w, \nabla u \rangle + u \Delta f w \leq -w(qu) + 2 \langle \nabla w, \nabla u \rangle + u \Delta f w = -qv + 2 \langle \nabla w, \nabla u \rangle + u \Delta f w.
\]

Then

\[
\Delta f w \geq -2 \langle \nabla w, \nabla (\ln u) \rangle.
\]

On the other hand, let \( \varphi \in C^2_0(M) \), then by (7), we have

\[
\int_M \varphi^2 |\nabla w|^2 e^{-f} = \int_M \langle \varphi^2 \nabla w, \nabla w \rangle e^{-f}
\]

\[
= \int_M \langle \nabla (\varphi^2 w) , \nabla w \rangle e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f}
\]

\[
= - \int_M \varphi^2 w (\Delta f w) e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f}
\]

\[
\leq 2 \int_M \varphi^2 w \langle \nabla w, \nabla (\ln u) \rangle e^{-f} - 2 \int_M \varphi w \langle \nabla \varphi, \nabla w \rangle e^{-f}
\]

\[
= 2 \int_M \langle \varphi \nabla w, w (\varphi \nabla (\ln u) - \nabla \varphi) \rangle
\]

\[
\leq \frac{1}{2} \int_M \varphi^2 |\nabla w|^2 e^{-f} + 2 \int_M w^2 |\varphi \nabla (\ln u) - \nabla \varphi|^2 e^{-f}.
\]
Then
\[(10) \quad \int_M \varphi^2 |\nabla w|^2 e^{-f} \leq 4 \int_M w^2 |\varphi \nabla \ln u - \nabla \varphi|^2 e^{-f} \quad \forall \varphi \in C^2_o(M).\]

If \(\psi \in C^\infty_\circ(M)\), then \(\varphi = \psi u \in C^2_o(M)\). Besides that, a little computation shows
\[\varphi \nabla \ln u - \nabla \varphi = - (\nabla \psi) u,\]

Thus, from (10), we have
\[\int_M \psi^2 u^2 |\nabla w|^2 e^{-f} \leq 4 \int_M w^2 |\nabla \psi|^2 u^2 e^{-f} \quad \forall \psi \in C^\infty_\circ(M).\]

Define functions \(\beta, \xi\) on \([0, +\infty)\) as
\[\beta(t) := \int_0^t \frac{\tau}{\kappa(\tau)} d\tau,\]
and \(\xi\) is the inverse function of \(\beta\). From the hypothesis we know \(\beta'\) is nonincreasing and \(\xi'\) is nondecreasing functions on \([0, +\infty)\). Now, we now choose a cutoff function
\[\psi_R(x) = \begin{cases} 1, & \text{on } B_{\xi(R)}; \\ 2 - \frac{\beta(r(x))}{R}, & \text{on } B_{\xi(2R)} \setminus B_{\xi(R)}; \\ 0, & \text{on } M \setminus B_{\xi(2R)}. \end{cases}\]

where \(r(x) = d(x, p)\), \(p \in M\) is a fixed point and \(B_R\) is the geodesic ball with radius \(R\) and center \(p\). We see that \(|\nabla \psi_R| = \frac{\beta'(r)}{R} = \frac{r}{R\kappa(r)}\). Then, by [S], we get
\[\int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} = \int_{B_{\xi(R)}} \psi_{R}^2 u^2 |\nabla w|^2 e^{-f} \leq \int_M \psi_{R}^2 u^2 |\nabla w|^2 e^{-f} \leq 4 \int_M v^2 |\nabla \psi_R|^2 e^{-f} = 4 \int_{B_{\xi(2R)} \setminus B_{\xi(R)}} v^2 |\nabla \psi_R|^2 e^{-f} = \frac{4}{R^2} \int_{\xi(R)} \left(\beta'(s)\right)^2 \int_{\partial B_s} v^2 e^{-f} dAd\tau.\]

Here we have used co-area formula. For convenience, we write \(V(s) = \int_{B_s} v^2 e^{-f} dV\). Therefore
\[V(s) = \int_0^s \int_{\partial B_r} v^2 e^{-f} dAd\tau \leq \kappa(s),\]
and
\[
\int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} \leq \frac{4}{R^2} \int_{\xi(R)}^{(2R)} (\beta'(s))^2 V'(s) ds
\]

\[
= \frac{4}{R^2} \left[ V(s)(\beta'(s))^2 \left( \frac{\xi(2R)}{\xi(R)} - \int_{\xi(R)}^{\xi(2R)} 2V(s)(\beta'(s)) d\beta'(s) \right) \right]
\]

\[
\leq \frac{4}{R^2} \left[ V(s)(\beta'(s))^2 \left( \frac{\xi(2R)}{\xi(R)} - 2 \int_{\xi(R)}^{\xi(2R)} s d\beta'(s) \right) \right]
\]

\[
\leq \frac{4}{R^2} \left[ V(s)(\beta'(s))^2 \left( \frac{\xi(2R)}{\xi(R)} - 2s\beta'(s) \right) \right]
\]

\[
\leq \frac{4}{R^2} \left[ V(s)(\beta'(s))^2 \left( \frac{\xi(2R)}{\xi(R)} - 2s\beta'(s) \right) + R \right]
\]

Since
\[
V(s)(\beta'(s))^2 = V(s)\beta'(s)\beta'(s) \leq s\beta'(s),
\]
and \( \beta'(s) = \frac{\beta}{\kappa(s)} \), thus Remark \( \Box \) implies these terms are bounded, hence when \( R \to +\infty \), all the terms on the right hand side go to zero. So we get
\[
\int_M u^2 |\nabla w|^2 e^{-f} = 0.
\]

Then \( \nabla w \equiv 0 \), thus there is a constant \( C \) such that \( w \equiv C \) and hence \( v = Cu \). \( \square \)

**Definition 1.** A two-sided translating soliton \( \Sigma \) is said to be stable if
\[
\int_{\Sigma} \left[ |\nabla \varphi|^2 - |A|^2 \varphi^2 \right] e^{-f} d\sigma \geq 0 \quad \text{for all} \quad \varphi \in C_0^\infty (\Sigma).
\]

As a consequence of Lemma \( \Box \) we have the following:

**Corollary 1.** Let \( \Phi : \Sigma^m \to \mathbb{R}^{n+1} \) be a stable translating soliton and let \( \omega \in C^2 (\Sigma) \) be a positive solution of the stability equation
\[
(12) \quad \Delta_f \omega + |A|^2 \omega = 0.
\]

Moreover, if \( H \geq 0 \) and there exists a constant \( C > 0 \) such that
\[
(13) \quad \int_{B_R} H^2 e^{-f} \leq CR^2 \quad \text{for all} \quad R \text{ large enough}.
\]

Then there exists a constant \( \tilde{C} \) such that \( H = \tilde{C} \omega \). In particular, if \( H \neq 0 \), then \( \tilde{C} \in \mathbb{R} \setminus \{0\} \) and \( H > 0 \).

Now, we include here a result due to Li and Wang (\[ \overline{5} \]) which will be needed in the proof our main theorem.
Lemma 2. Suppose $\Sigma$ is complete and there exists a nonnegative function $\varphi : \Sigma \to \mathbb{R}$, not identically zero, such that $(\Delta_f + q)(\varphi) \leq 0$. Then $\Delta_f + q$ is stable.

Proof. Let $\Omega$ be a compact subdomain in $\Sigma$ and let $u$ be the first eigenfunction satisfying
\begin{equation}
\begin{cases}
(\Delta_f + q)u = -\lambda_1(\Omega)u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}
We may assume that $u \geq 0$ on $\Omega$. From regularity of $u$ and Hopf Lemma, we have
\begin{itemize}
\item $u > 0$ in the interior of $\Omega$.
\item $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where $\nu$ is the outward unit normal of $\partial\Omega$.
\end{itemize}
Thus, integration by parts on $u$ and $\varphi$ and also the hypothesis, we have
\begin{equation}
\int_{\Omega} u(\Delta_f \varphi) e^{-f} - \int_{\Omega} \varphi (\Delta_f u) e^{-f} = \int_{\partial\Omega} u \frac{\partial \varphi}{\partial \nu} e^{-f} - \int_{\partial\Omega} \varphi \frac{\partial u}{\partial \nu} e^{-f}
= -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi e^{-f} \geq 0.
\end{equation}
From hypothesis and (14), we have
\begin{equation}
\begin{cases}
\Delta_f \varphi + Q\varphi \leq 0, \\
\Delta_f u + Qu = -\lambda_1(\Omega) u.
\end{cases}
\end{equation}
Since $u > 0$, multiplying the first inequality of (16) by $u$ and the second equation by $-\varphi$, and finally both by $e^{-f}$, we have
\begin{equation}
u(\Delta_f \varphi) e^{-f} - \varphi (\Delta_f u) e^{-f} \leq \lambda_1(\Omega)(\varphi u) e^{-f}
\end{equation}
Since both $u > 0$ and $\varphi \geq 0$ are not identically zero, then combining (17) with (15), we have $\lambda_1(\Omega) \geq 0$ for all compact subdomains of $\Sigma$, then $\lambda_1(f, Q) \geq 0$, therefore $\Delta_f + q$ is stable.

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1 Since $\Phi : \Sigma^n \to \mathbb{R}^{n+1}$ is a translating soliton, then the mean curvature $H$ satisfies $\Delta_f H + |A|^2 H = 0$ (see Proposition 3 in [1]). Since $H \geq 0$ and $\Sigma$ is a non-planar translating soliton, then $H$ is not identically zero, thus by Lemma 2 $\Sigma$ is stable and hence the weighted version of a result by Fischer-Colbrie and Schoen [2] guarantees there exists a non-constant positive $C^2$-function $\omega$ on $\Sigma$ such that
\begin{equation}
\Delta_f \omega + |A|^2 \omega = 0.
\end{equation}
As $\frac{H^2}{n} \leq |A|^2$ and $|A|$ satisfies (3), then
\begin{equation}
\int_{B_R} H^2 e^{-f} \leq nC R^2.
\end{equation}
Then, by Corollary 1 and the condition that $H \geq 0$ and not identically zero, there is a constant $C_1 > 0$ such that

$$(20) \quad H = C_1 \omega.$$ 

In particular $H > 0$ everywhere on $\Sigma$. On the other hand, the Simons equation (see [1] or [6]) implies that

$$(21) \quad |A| \left( \Delta f |A| + |A|^2 |A| \right) = |\nabla A|^2 - |\nabla |A||^2 \geq 0.$$ 

Since $|A|$ satisfies (3), then by Lemma 1 \exists $C_2 \geq 0$ such that

$$(22) \quad |A| = C_2 \omega.$$ 

Besides that $\Sigma^n$ is not a hyperplane, then $|A|$ is not identically zero, thus $C_2 > 0$. Then by (20) and (22) we have $|A|^2 H^{-2} = \text{constant} > 0$. In particular this function attains its local maximum on $\Sigma$. Theorem B in [6] says that $\Sigma$ is a grim hyperplane if and only if the function $|A|^2 H^{-2}$ attains a local maximum. Therefore $\Sigma$ is a grim hyperplane. $\square$ 

We now prove Theorem 2.

**Proof of Theorem 2.** To prove Theorem 2 we will need a result of Huang-Wu [3]. Denote by $M_+$ a connected component of $\{ p \in M, H \geq 0 \text{ at } p \}$ that contains a point of positive mean curvature. We say that the mean curvature $H$ changes signs through $\Gamma$ if $\Gamma$ is a connected component of $\partial M_+$ and $\Gamma$ intersects the boundary of a connected component of $M \setminus \partial M_+$. Theorem 2 of Huang-Wu [3] \ \ \ $S \geq 0$, says that if $H$ changes sign along $\Gamma$ then $\Gamma$ is unbounded set. Since we have assumed that $H$ does not changes signs at infinity, $H$ has a sign. Hence either

1. $H \equiv 0$, or
2. $H \geq 0$ but does not vanish at least one point.

In case (1), $\Sigma$ must be a hyperplane.

In case (2), if there is point $p \in \Sigma$, such that $S(p) = 0$ then $|A|^2 = H^2 - S \leq H^2$ and equality holds at $p$. Therefore the function $|A|^2 H^2$ is well defined and attains its maximum at $p$. By Theorem B in [6] it must be a grim hyperplane. $\square$

**References**

[1] Xu Cheng, Tito Mejia, and Detang Zhou, *Simons-Type Equation for $f$-Minimal Hypersurfaces and Applications*, J. Geom. Anal 25 (2015), no. 4, 2667–2686.

[2] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. 33 (1980), no. 2, 199–211.

[3] Lan-Hsuan Huang and Damin Wu, *Hypersurfaces with nonnegative scalar curvature*, J. Differential Geom. 95 (2013), no. 2, 249–278, MR3128984

[4] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. 20 (1984), no. 1, 237–266, MR772132

[5] Peter Li and Jiaping Wang, *Weighted Poincaré inequality and rigidity of complete manifolds*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 6, 921–982, DOI 10.1016/j.ansens.2006.11.001 (English, with English and French summaries). MR2316978
[6] Francisco Martín, Andreas Savas-Halilaj, and Knut Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Calc. Var. Partial Differential Equations 54 (2015), no. 3, 2853–2882, DOI 10.1007/s00526-015-0886-2. MR3412395

**Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, RJ 24020, Brazil**

*E-mail address: ditter.y.t@gmail.com*

*E-mail address: zhou@impa.br*