Homogeneity of proper complex equifocal submanifolds

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Abstract

In a symmetric space of non-compact type, the notion of a complex equifocal submanifold (with flat section) is defined. It is conjectured that this notion coincides with the notion of an isoparametric submanifold with flat section. As a subclass of a complex equifocal submanifold, the notion of a proper complex equifocal submanifold is defined. In this paper, we show that all irreducible proper complex equifocal submanifolds of codimension greater than one in a symmetric space of non-compact type are extrinsically homogeneous and hence they occur as principal orbits of complex hyperpolar actions on the symmetric space. The proof is performed by showing the homogeneity of the lifted submanifold of the complexification of the original submanifold to an infinite dimensional anti-Kaehlerian space and inducing the homogeneity of the original submanifold from the homogeneity of the lifted submanifold.

Keywords: proper complex equifocal submanifold, proper anti-Kaehlerian isoparametric submanifold, complex principal curvature, complex curvature distribution

1 Introduction

C.L. Terng and G. Thorbergsson [TT] introduced the notion of an equifocal submanifold in a Riemannian symmetric space, which is defined as a compact submanifold with globally flat and abelian normal bundle such that the focal radii for each parallel normal vector field are constant. This notion is a generalization of isoparametric submanifolds in the Euclidean space and isoparametric hypersurfaces in the sphere or the hyperbolic space. For (not necessarily compact) submanifolds in a Riemannian symmetric space of non-compact type, the equifocality is a rather weak property. So, we [Koi1,2] introduced the
notion of a complex focal radius as a general notion of a focal radius and defined the notion of a complex equifocal submanifold as a submanifold with globally flat and abelian normal bundle such that the complex focal radii for each parallel normal vector field are constant and that they have constant multiplicities. Here we note that the notion of a complex focal radius (hence the notion of the complex equifocality) can be defined for submanifolds in a general symmetric space but, in the case where the ambient symmetric space is of non-negative curvature, all complex focal radii are real, that is, they are focal radii and hence the complex equifocality is equivalent to the equifocality. E. Heintze, X. Liu and C. Olmos [HLO] defined the notion of an isoparametric submanifold with flat section as a submanifold with globally flat and abelian normal bundle such that the sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction. The following fact is known (see Theorem 15 of [Koi2]):

All isoparametric submanifolds with flat section in a symmetric space of non-compact type are complex equifocal and, conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric submanifolds with flat section.

Here the curvature-adaptedness means that, for any normal vector \( v \), \( R(\cdot, v)v \) preserves the tangent space invariantly and it commutes with the shape operator \( A_v \). Furthermore, as its subclass, we [Koi1,2] introduced the notion of a proper complex equifocal submanifold as a complex equifocal submanifold whose complex focal structure has a certain kind of regularity. Here we note that the notion of a proper complex equifocal submanifold in a general symmetric space can be defined but, in the case where the ambient symmetric space is of compact type, it is easy to show that this notion coincides with the notion of a complex equifocal submanifold. In [Koi8], we showed that, for a curvature-adapted complex equifocal submanifold \( M \), it is proper complex equifocal if and only if it admits no non-Euclidean type focal point on the ideal boundary of the ambient symmetric space, where the notion of a non-Euclidean type focal point on the ideal boundary was introduced in [Koi8]. Therefore, for a submanifold \( M \) in a symmetric space of non-compact type, the following two statements are equivalent:

(I) \( M \) is curvature-adapted and proper complex equifocal.

(II) \( M \) is a curvature-adapted isoparametric submanifold with flat section admitting no non-Euclidean type focal point on the ideal boundary.

Let \( G/K \) be a symmetric space of non-compact type and \( H \) be a symmetric subgroup of \( G \) (i.e., there exists an involution \( \sigma \) of \( G \) with \( (\text{Fix } \sigma)_0 \subset H \subset \text{Fix } \sigma \), where \( \text{Fix } \sigma \) is the fixed point group of \( \sigma \) and \( (\text{Fix } \sigma)_0 \) is the identity component of \( \text{Fix } \sigma \). Then the \( H \)-action on \( G/K \) is called a Hermann type action. For this action, the following fact is known ([Koi3]):

Principal orbits of Hermann type actions are curvature-adapted and proper complex equifocal.
For a (general) submanifold in a Riemannian symmetric space of non-compact type, the (non-real) complex focal radii are defined algebraically. We needed to find their geometrical essence. For its purpose, we defined the complexification of the ambient Riemannian symmetric space and the extrinsic complexification of the submanifold as a certain kind of submanifold in the complexified symmetric space, where the original submanifold needs to be assumed to be complete and real analytic. In the sequel, we assume that all submanifolds in the Riemannian symmetric space are complete and real analytic. We [Koi2] showed that the complex focal radii of the original submanifold indicate the positions of the focal points of the complexified submanifold. If the original submanifold is complex equifocal, then the extrinsic complexification is an anti-Kaehlerian equifocal submanifold in the sense of [Koi2]. Also, if the original submanifold is proper complex equifocal, then the complexified one is a proper anti-Kaehlerian equifocal submanifold in the sense of this paper. Thus, the study of an anti-Kaehlerian equifocal (resp. proper anti-Kaehlerian equifocal) submanifold leads to that of a complex equifocal (resp. proper complex equifocal) submanifold. The complexified submanifold is not necessarily complete. In the global research, we need to extend the complexified submanifold to a complete one. In [Koi2] and [Koi6], we constructed the complete extension of the complexified submanifold in mutually different methods. In 1999, E. Heintze and X. Liu [HL2] showed that all irreducible isoparametric submanifolds of codimension greater than one in the (separable) Hilbert space are extrinsically homogeneous, which is the infinite dimensional version of the homogeneity theorem for isoparametric submanifolds in a (finite dimensional) Euclidean space by G. Thorbergsson. Note that the result of Thorbergsson states that all irreducible isoparametric submanifolds of codimension greater than two in a Euclidean space are extrinsically homogeneous. In 2002, by using the result of Heintze-Liu, U. Christ [Ch] showed that all irreducible equifocal submanifolds of codimension greater than one in a simply connected symmetric space of compact type are extrinsically homogeneous. The outline of the proof is as follows. For an irreducible equifocal submanifold $M$ of codimension greater than one in a simply connected symmetric space $G/K$ of compact type, any component $\widetilde{M}$ of the inverse image $(\pi \circ \phi)^{-1}(M)$ of $M$ by the composition of the parallel transport map $\phi : H^0([0,1],g) \to G$ with the natural projection $\pi : G \to G/K$ is an irreducible isoparametric submanifold of codimension greater than one in the Hilbert space $H^0([0,1],g)$, where $H^0([0,1],g)$ is the space of all $L^2$-integrable paths in $g := \text{Lie } G$. By the result of Heintze-Liu, $\widetilde{M}$ is extrinsically homogeneous, that is, $\widetilde{M}$ is an orbit of some subgroup $H$ of the isometry group of $H^0([0,1],g)$. From the $H$-action on $H^0([0,1],g)$, he constructed a subgroup $\overline{H}$ of $G$ such that $\overline{M}$ is an orbit of the $\overline{H}$-action on $G/K$. Hence $M$ is extrinsically homogeneous.

For a complex equifocal submanifold $M$ in a symmetric space $G/K$ of non-compact type, the complete extrinsically complexification $M^c$ of $M$ is defined as an anti-Kaehlerian submanifold in the anti-Kaehlerian symmetric space $G^c/K^c$. Without loss of generality,
we may assume that $K$ (hence $K^c$) is connected and that $G^c$ is simply connected. Let $\pi^c$ be the natural projection of $G^c_\infty$ onto $G^c/K^c$ and $\phi^c : H^0([0,1],g^c) \to G^c$ be the parallel transport map for $G^c$. Then $\tilde{M}^c := (\pi^c \circ \phi^c)^{-1}(M^c)$ is an anti-Kaehlerian isoparametric submanifold in the infinite dimensional anti-Kaehlerian space $H^0([0,1],g^c)$ in the sense of [Koi2]. In particular, if $M$ is a proper complex equifocal, then $\tilde{M}^c$ is a proper anti-Kaehlerian isoparametric submanifold in the sense of [Koi2]. First we prove the following fact for the homogeneity of the lifted submanifold $\tilde{M}^c$:

If $M$ is an irreducible proper complex equifocal submanifold of codimension greater than one in a symmetric space of non-compact type, then the lifted submanifold $\tilde{M}^c$ is extrinsically homogeneous.

By using this fact, we prove the following homogeneity theorem for an irreducible proper complex equifocal submanifold.

**Theorem A.** All irreducible proper complex equifocal submanifolds of codimension greater than one in a symmetric space of non-compact type are extrinsically homogeneous.

Let $G/K$ be a symmetric space of non-compact type and $H$ be a closed subgroup of $G$ which admits an embedded flat submanifold of $G/K$ meeting all $H$-orbits orthogonally. Then the $H$-action is called a complex hyperpolar action. From Theorem A, we can show the following fact.

**Corollary B.** Let $M$ be an irreducible proper complex equifocal submanifold of codimension greater than one in a symmetric space $G/K$ of non-compact type. Then the following statements (i) and (ii) hold:

(i) $M$ occurs as a principal orbit of a complex hyperpolar action on $G/K$.

(ii) If $M$ admits a totally geodesic focal submanifold, then it occurs as a principal orbit of a Hermann type action on $G/K$.

The outline of the discussions in Sections 3 and 4 In Section 3, for a submanifold $M$ as in the statement of Theorem A, we prove that $\tilde{M}^c := (\pi^c \circ \phi^c)^{-1}(M^c)$ is homogeneous by imitating the proof of the homogeneity of an irreducible infinite dimensional isoparametric submanifold of codimension greater than one in a Hilbert space by Heintze-Liu [HL2]. In the proof of the homogeneity of the isoparametric submanifold, Heintze-Liu [HL2] used the curvature distributions on the isoparametric submanifold. On the other hand, $\tilde{M}^c$ is proper anti-Kaehlerian isoparametric submanifold and hence the complex curvature distributions on $\tilde{M}^c$ is defined. See the next section about the definition of the complex curvature distribution. In the proof of the homogeneity of $\tilde{M}^c$, we use the complex curvature distributions. We sketch the outline of the discussion in Section 3. First we recall the generalized Chow’s theorem for the set of reachable points for a certain kind of
family of local vector fields on a Hilbert manifold, which were proved in [HL2]. By using this theorem, we show that the set (which is denoted by $Q(u_0)$) of all the points connected with a fixed point $u_0$ of $\tilde{M}^c$ by piecewise smooth curves each of whose segment is contained in a complex curvature spheres (the integral manifolds of the complex curvature distributions) is dense in $\tilde{M}^c$ (see Proposition 3.2). Next we show the homogeneous slice theorem (Theorem 3.3) for $\tilde{M}^c$ by using the homogeneous slice theorem for $M^c$, which was proved in [Koi6]. Next we construct a holomorphic isometry $F_\gamma$ of $V := H^0([0, 1], \mathfrak{g}^c)$ with $F_\gamma(\gamma(0)) = \gamma(1)$ for a certain kind of piecewise smooth curve $\gamma : [0, 1] \to \tilde{M}^c$ (which is a special one of curves called a $E_i$-horizontal curve) in $\tilde{M}^c$, where we note that $E_i$ is a complex curvature distribution. This holomorphic isometry $F_\gamma$ is constructed by using the fact that, for each linear isometry of the tangent space at a point $u_0$ of $V$ onto the tangent space at another point $u_1$ of $V$, there (uniquely) exists a holomorphic isometry of $V$ whose differential at $u_0$ coincides with the linear isometry (see (3.1)). Here we note that this fact holds because $V$ is an (anti-Kaehlerian) linear space. In the discussion in Section 3, it is a key to show that $F_\gamma$ preserves $\tilde{M}^c$ invariantly (see Proposition 3.5). In this proof, we use the assumptions of the irreducibility of $M$ and codim $M \geq 2$, and the above homogeneous slice theorem for $\tilde{M}^c$ (see the proof of Lemma 3.5.1). Next we show that, for each $u \in Q(u_0)$, there exists a holomorphic isometry $f$ of $V$ with $f(u_0) = u$ and $f(\tilde{M}^c) = \tilde{M}^c$ (see Proposition 3.6). Finally, by using Propositions 3.2 and 3.6, we show that $\tilde{M}^c$ is homogeneous. In Section 4, we induce the homogeneity of $M$ from the homogeneity of $\tilde{M}^c$ by imitating the proof of the homogeneity of an irreducible equifocal submanifold of codimension greater than one in a symmetric space of compact type by Christ [Ch]. Here we sketch the outline of the discussion in Section 4. The group $H^1([0, 1], G^c)$ acts on $V$ as the gauge action. Denote by $\rho( : H^1([0, 1], G^c) \to I_h(V))$ the representation on $V$ associated with this action, where $I_h(V)$ is the holomorphic isometry group of $V$. In the discussion in Section 4, it is a key to show that the group $H' := \{ f \in I_h(V) \mid f(\tilde{M}^c) = \tilde{M}^c \}$ is contained in $\rho(H^1([0, 1], G^c))$. This fact is shown by comparing the spaces of Killing fields corresponding to $H'$ and $\rho(H^1([0, 1], G^c))$ (see the proofs of Lemmas 4.1.1 ~ 4.1.11). Thus the group $H'$ is given as the image $\rho(H)$ of some subgroup $H$ of $H^1([0, 1], G^c)$. Since $\tilde{M}^c$ is homogeneous, we have $\tilde{M}^c = \rho(H) \cdot u_0 \ (u_0 \in \tilde{M}^c)$. We construct a subgroup $\overline{\Pi}$ of $G^c \times G^c$ from $H$ and show $\overline{\Pi} \cdot g_0 = \pi^{c-1}(M^c) \ (g_0 \in \pi^{c-1}(M^c))$. Next we construct a subgroup $\overline{\Pi}_R$ of $G \times G$ from $\overline{\Pi}$ and show $\overline{\Pi}_R \cdot g_0 = \pi^{-1}(M) \ (g_0 \in \pi^{-1}(M))$, where $\pi$ is the natural projection of $G$ onto $G/K$. Furthermore, we construct a subgroup $\overline{\Pi}_R$ of $G$ from $\overline{\Pi}$ and show $\overline{\Pi}_R(g_0K) = M \ (g_0K \in M)$. Thus the homogeneity of $M$ follows.

**Future plan of research** By using (ii) of Corollary B and the equivalences of the above statements (I) and (II), we plan to investigate whether, for an irreducible $C^\omega$-submanifold $M$ in $G/K$ of codimension greater than one, the following statement is true:
$M$ is a principal orbit of a Hermann type action on $G/K$ if and only if $M$ is a curvature-adapted isoparametric submanifold with flat section admitting no non-Euclidean type focal point on $(G/K)(\infty)$. Furthermore, we plan to investigate whether both the conditions of the curvature-adaptedness and the non-existenceness of non-Euclidean type focal point on $(G/K)(\infty)$ are indispensable in this statement. Note that this statement gives a submanifold geometrical characterization of a principal orbit of a Hermann type action.

2 Basic notions and facts

In this section, we recall basic notions introduced in [Koi1~3]. We first recall the notion of a complex equifocal submanifold introduced in [Koi1]. Let $M$ be an immersed submanifold with abelian normal bundle (i.e., the sectional curvature for each 2-plane in the normal space is equal to zero) of in a symmetric space $N = G/K$ of non-compact type. Denote by $A$ the shape tensor of $M$. Let $v \in T_x M$ and $X \in T_x M$ ($x = gK$). Denote by $\gamma_v$ the geodesic in $N$ with $\gamma_v(0) = v$. The strongly $M$-Jacobi field $Y$ along $\gamma_v$ with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = (P_{\gamma_v|[0,s]} \circ (D^c_{sv} - s D^s_{sv} \circ A_v))(X),$$

where $Y'(0) = \tilde{\nabla}_v Y$, $P_{\gamma_v|[0,s]}$ is the parallel translation along $\gamma_v|[0,s]$ and $D^c_{sv}$ (resp. $D^s_{sv}$) is given by

$$D^c_{sv} = g_* \circ \cos(\sqrt{-1} \text{ad}(sg^{-1}_s v)) \circ g^{-1}_s,$$

$$\left(\text{resp. } D^s_{sv} = g_* \frac{\sin(\sqrt{-1} \text{ad}(sg^{-1}_s v))}{\sqrt{-1} \text{ad}(sg^{-1}_s v)} \circ g^{-1}_s\right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of $G$. All focal radii of $M$ along $\gamma_v$ are obtained as real numbers $s_0$ with $\text{Ker}(D^{c\circ}_{sv} - s_0 D^s_{sv} \circ A_v) \neq \{0\}$. So, we call a complex number $z_0$ with $\text{Ker}(D^{c\circ}_{sv} - z_0 D^s_{sv} \circ A^c_v) \neq \{0\}$ a complex focal radius of $M$ along $\gamma_v$ and call $\dim \text{Ker}(D^{c\circ}_{sv} - z_0 D^s_{sv} \circ A^c_v)$ the multiplicity of the complex focal radius $z_0$, where $D^{c\circ}_{sv}$ (resp. $D^s_{sv}$) is a $\mathbb{C}$-linear transformation of $(T_x N)^c$ defined by

$$D^{c\circ}_{sv} = g^c_* \circ \cos(\sqrt{-1} \text{ad}^c(z_0 g^{-1}_s v)) \circ (g^c_*)^{-1},$$

$$\left(\text{resp. } D^s_{sv} = g^c_* \frac{\sin(\sqrt{-1} \text{ad}^c(z_0 g^{-1}_s v))}{\sqrt{-1} \text{ad}^c(z_0 g^{-1}_s v)} \circ (g^c_*)^{-1}\right),$$

where $g^c_*$ (resp. $\text{ad}^c$) is the complexification of $g_*$ (resp. ad). Here we note that, in the case where $M$ is of class $C^\infty$, complex focal radii along $\gamma_v$ indicate the positions of focal points of the extrinsic complexification $M^c(\rightarrow G^c/K^c)$ of $M$ along the complexified geodesic $\gamma^c_v$, where $G^c/K^c$ is the anti-Kaehlerian symmetric space associated with $G/K$ and $\iota$ is the natural immersion of $G/K$ into $G^c/K^c$. See the final paragraph of this
section about the definitions of $G^c/K^c$, $M^c(\to G^c/K^c)$ and $\gamma_{i,v}$. Also, for a complex focal radius $z_0$ of $M$ along $\gamma_v$, we call $z_0\nu \in (T^1_x M)^c$ a complex focal normal vector of $M$ at $x$. Furthermore, assume that $M$ has globally flat normal bundle (i.e., the normal holonomy group of $M$ is trivial). Let $\tilde{v}$ be a parallel unit normal vector field of $M$. Assume that the number (which may be $\infty$) of distinct complex focal radii along $\gamma_{\tilde{v}}$ is independent of the choice of $x \in M$. Let $\{r_i,x\mid i = 1,2,\cdots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}}$, where $|r_{i,x}| < |r_{i+1,x}|$ or $"|r_{i,x}| = |r_{i+1,x}| \& Re r_{i,x} > Re r_{i+1,x}"$ or $"|r_{i,x}| = |r_{i+1,x}| \& Re r_{i,x} = Re r_{i+1,x} \& Im r_{i,x} = -Im r_{i+1,x} < 0"$. Let $r_i \ (i = 1,2,\cdots)$ be complex valued functions on $M$ defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions $r_i \ (i = 1,2,\cdots)$ complex focal radius functions for $\tilde{v}$. We call $r_i \tilde{v}$ a complex focal normal vector field for $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of $M$, the number of distinct complex focal radii along $\gamma_{\tilde{v}}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on $M$ and it has constant multiplicity, then we call $M$ a complex equifocal submanifold. Let $\phi : H^0([0,1],g) \to G$ be the parallel transport map for $G$, that is, $\phi(u) := g_u(1) \ (u \in H^0([0,1],g))$, where $g_u$ is the element of $H^1([0,1],G)$ with $g_u(0) = e$ and $g_u^{-1}g_u' = u$. See Section 4 of [Koi1] about the definitions of $H^0([0,1],g)$ and $H^1([0,1],G)$. Let $\pi : G \to G/K$ be the natural projection. It follows from Theorem 1 of [Koi2] that, $M$ is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. See Section 2 of [Koi1] about the definition of a complex isoparametric submanifold. In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric (i.e., complex isoparametric and, for each unit normal vector $v$, the complexified shape operator $A^c_v$ is diagonalizable with respect to a pseudo-orthonormal base), then we call $M$ a proper complex equifocal submanifold.

Next we recall the notion of an infinite dimensional proper anti-Kaehlerian isoparametric submanifold. Let $M$ be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space $V$ and $A$ be the shape tensor of $M$. See [Koi2] about the definitions of an infinite dimensional anti-Kaehlerian space and anti-Kaehlerian Fredholm submanifold in the space. Denote by the same symbol $J$ the complex structures of $M$ and $V$. Fix a unit normal vector $v$ of $M$. If there exists $X(\neq 0) \in TM$ with $A_vX = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a $J$-eigenvalue of $A_v$ (or a complex principal curvature of direction $v$) and call $X$ a $J$-eigenvector for $a + bv\sqrt{-1}$. Also, we call the space of all $J$-eigenvectors for $a + b\sqrt{-1}$ a $J$-eigenspace for $a + b\sqrt{-1}$. The $J$-eigenspaces are orthogonal to one another and each $J$-eigenspace is $J$-invariant. We call the set of all $J$-eigenvalues of $A_v$ the $J$-spectrum of $A_v$ and denote it by $\text{Spec}_J A_v$. The set $\text{Spec}_J A_v \setminus \{0\}$ is described as follows:

$$\text{Spec}_J A_v \setminus \{0\} = \{\lambda_i \mid i = 1,2,\cdots\}$$

$$\begin{cases}
|\lambda_i| > |\lambda_{i+1}| & \text{or} \ "|\lambda_i| = |\lambda_{i+1}| \& Re \lambda_i > Re \lambda_{i+1}" \\
|\lambda_i| = |\lambda_{i+1}| \& Re \lambda_i = Re \lambda_{i+1} \& Im \lambda_i = -Im \lambda_{i+1} > 0"
\end{cases}.$$
Also, the \( J \)-eigenspace for each \( J \)-eigenvalue of \( A_v \) other than 0 is of finite dimension. We call the \( J \)-eigenvalue \( \lambda_i \) the \( i \)-th complex principal curvature of direction \( v \). Assume that \( M \) has globally flat normal bundle. Fix a parallel normal vector field \( \tilde{v} \) of \( M \). Assume that the number (which may be \( \infty \)) of distinct complex principal curvatures of direction \( \tilde{v}_x \) is independent of the choice of \( x \in M \). Then we can define functions \( \tilde{\lambda}_i \) \( (i = 1, 2, \ldots) \) on \( M \) by assigning the \( i \)-th complex principal curvature of direction \( \tilde{v}_x \) to each \( x \in M \). We call this function \( \tilde{\lambda}_i \) the \( i \)-th complex principal curvature function of direction \( \tilde{v} \). We consider the following condition:

\text{(AKI)} For each parallel normal vector field \( \tilde{v} \), the number of distinct complex principal curvatures of direction \( \tilde{v}_x \) is independent of the choice of \( x \in M \), each complex principal curvature function of direction \( \tilde{v} \) is constant on \( M \) and it has constant multiplicity.

If \( M \) satisfies this condition (AKI), then we call \( M \) an \textit{anti-Kaehlerian isoparametric submanifold}. Let \( \{e_i\}_{i=1}^{\infty} \) be an orthonormal system of \( T_xM \). If \( \{e_i\}_{i=1}^{\infty} \cup \{Je_i\}_{i=1}^{\infty} \) is an orthonormal base of \( T_xM \), then we call \( \{e_i\}_{i=1}^{\infty} \) a \textit{J-orthonormal base}. If there exists a \( J \)-orthonormal base consisting of \( J \)-eigenvectors of \( A_v \), then \( A_v \) is said to be \textit{diagonalized with respect to the \( J \)-orthonormal base}. If \( M \) is anti-Kaehlerian isoparametric and, for each \( v \in T^1M \), the shape operator \( A_v \) is diagonalized with respect to a \( J \)-orthonormal base, then we call \( M \) a \textit{proper anti-Kaehlerian isoparametric submanifold}. For arbitrary two unit normal vector \( v_1 \) and \( v_2 \) of a proper anti-Kaehlerian isoparametric submanifold, the shape operators \( A_{v_1} \) and \( A_{v_2} \) are simultaneously diagonalized with respect to a \( J \)-orthonormal base. Assume that \( M \) is a proper anti-Kaehlerian isoparametric submanifold. Let \( \{E_i \mid i \in I\} \) be the family of distributions on \( M \) such that, for each \( x \in M \), \( \{(E_i)_x \mid i \in I\} \) is the set of all common \( J \)-eigenspaces of \( A_v \)'s \( (v \in T^1_xM) \). The relation \( TM = \bigoplus_{i \in I} E_i \) holds.

Let \( \lambda_i \) \( (i \in I) \) be the section of \( (T^1M)^* \otimes \mathbb{C} \) such that \( A_v = \text{Re}\lambda_i(v)\text{id} + \text{Im}\lambda_i(v)J \) on \( (E_i)_{\pi(v)} \) for each \( v \in T^1M \), where \( \pi \) is the bundle projection of \( T^1M \). We call \( \lambda_i \) \( (i \in I) \) \textit{complex principal curvatures} of \( M \) and call distributions \( E_i \) \( (i \in I) \) \textit{complex curvature distributions} of \( M \). It is shown that there uniquely exists a normal vector field \( n_i \) of \( M \) with \( \lambda_i(\cdot) = \langle n_i, \cdot \rangle - \sqrt{-1}\langle Jn_i, \cdot \rangle \) (see Lemma 5 of [Koi2]). We call \( n_i \) \( (i \in I) \) the \textit{complex curvature normals} of \( M \). Note that \( n_i \) is parallel with respect to the normal connection \( \nabla^\perp \). Similarly we can define a (finite dimensional) proper anti-Kaehlerian isoparametric submanifold in a finite dimensional anti-Kaehlerian space, its complex principal curvatures, its complex curvature distributions and its complex curvature normals. Set \( l^x_i := (\lambda_i)^{-1}(1) \) \( (i \in I) \). In [Koi2], it has been shown that the focal set of \( (M, x) \) is equal to \( \bigcup_{i \in I} l^x_i \). Denote by \( T^x_i \) the complex reflection of order 2 with respect to the complex hyperplane \( l^x_i \) of \( T^1_xM \) (i.e., the rotation of angle \( \pi \) having \( l^x_i \) as the axis), which is an affine transformation of \( T^1_xM \). Let \( W_x \) be the group generated by \( T^x_i \)'s \( (i \in I) \). According to Proposition 3.7 of [Koi4], \( W_x \) is discrete and it is independent of the choice of \( x \in M \) (up to group isomorphism). Hence we simply denote it by \( W \). We call this group \( W \) the \textit{complex Coxeter group associated
with $M$. According to Lemma 3.8 of [Koi4], $W$ is decomposable (i.e., it is decomposed into a non-trivial product of two discrete complex reflection groups) if and only if there exist two $J$-invariant linear subspaces $P_1$ ($\neq \{0\}$) and $P_2$ ($\neq \{0\}$) of $T^+_xM$ such that $T^+_xM = P_1 \oplus P_2$ (orthogonal direct sum), $P_1 \cup P_2$ contains all complex curvature normals of $M$ at $x$ and that $P_i$ ($i = 1, 2$) contains at least one complex curvature normal of $M$ at $x$.

Next we recall the notion of the extrinsic complexification of a complete $C^\omega$-submanifold in a symmetric space of non-compact type which was introduced in [Koi2]. First we recall the notions of an anti-Kaehlerian symmetric space associated with a symmetric space of non-compact type and an aks-representation. Let $J$ be a parallel complex structure on an even dimensional pseudo-Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ of half index. If $\langle JX, JY \rangle = -\langle X, Y \rangle$ holds for every $X, Y \in TM$, then $(M, \langle \cdot, \cdot \rangle, J)$ is called an anti-Kaehlerian manifold. Let $G/K$ be a symmetric space of non-compact type and $(g, \sigma)$ be its orthogonal symmetric Lie algebra. Let $\mathfrak{g} = \mathfrak{j} + \mathfrak{p}$ be the Cartan decomposition associated with a symmetric pair $(G, K)$. Note that $\mathfrak{j}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is identified with the tangent space $T_eK(G/K)$, where $e$ is the identity element of $G$. Let $(\cdot, \cdot)$ be the $\text{Ad}_G(G)$-invariant non-degenerate inner product of $\mathfrak{g}$ inducing the Riemannian metric of $G/K$, where $\text{Ad}_G$ is the adjoint representation of $G$. Let $G^c$ (resp. $K^c$) be the complexification of $G$ (resp. $K$). Without loss of generality, we may assume that $K^c$ is connected and that $G^c$ is simply connected. The 2-multiple of the real part $\text{Re}(\cdot, \cdot)^c$ of $(\cdot, \cdot)^c$ is the Killing form of $G^c$ regarded as a real Lie algebra. The restriction $2\text{Re}(\cdot, \cdot)^c|_{\mathfrak{p}^c \times \mathfrak{p}^c}$ is an $\text{Ad}(K^c)$-invariant non-degenerate inner product of $\mathfrak{p}^c (= T_eK^c(G^c/K^c))$. Denote by $(\cdot, \cdot)^A$ the $G^c$-invariant pseudo-Riemannian metric on $G^c/K^c$ induced from $2\text{Re}(\cdot, \cdot)^c|_{\mathfrak{p}^c \times \mathfrak{p}^c}$. Define an almost complex structure $J_0$ of $\mathfrak{p}^c$ by $J_0X = \sqrt{-1}X$ ($X \in \mathfrak{p}^c$). It is clear that $J_0$ is $\text{Ad}(K^c)$-invariant. Denote by $\tilde{J}$ the $G^c$-invariant almost complex structure on $G^c/K^c$ induced from $J_0$. It is shown that $(G^c/K^c, (\cdot, \cdot)^A, \tilde{J})$ is an anti-Kaehlerian manifold and a (semi-simple) pseudo-Riemannian symmetric space. We call this anti-Kaehlerian manifold an anti-Kaehlerian symmetric space associated with $G/K$ and simply denote it by $G^c/K^c$. The action $\text{Ad}_{G^c}(K^c)$ on $G^c$ preserves $\mathfrak{p}^c$ invariantly, where $\text{Ad}_{G^c}$ is the adjoint representation of $G^c$. Define a representation $\rho$ of $K^c$ on $\mathfrak{p}^c$ by $\rho(k)(X) := \text{Ad}_{G^c}(k)(X)$ ($k \in K^c, X \in \mathfrak{p}^c$). If $G^c/K^c$ is irreducible, then we call this representation $\rho$ an aks-representation (associated with $G^c/K^c$). Let $X_0$ be a semi-simple element of $\mathfrak{p}^c$, where the semi-simplesness means that the complexification of $\text{ad}_{G^c}(X_0)$ is diagonalizable. If the orbit $\rho(K^c) \cdot X_0$ is principal, then it is a (finite dimensional) proper anti-Kaehlerian isoparametric submanifold (see Lemma 3.5.3). Let $N$ be a complete $C^\omega$-Riemannian manifold. The notion of the adapted complex structure on a neighborhood $U$ of the 0-section of the tangent bundle $TN$ is defined as the complex structure (on $U$) such that, for each geodesic $\gamma : \mathbb{R} \to N$, the restriction of its differential $\gamma_* : T\mathbb{R} = \mathbb{C} \to TN$ to $\gamma_*^{-1}(U)$ is holomorphic. We take $U$ as largely as possible under the condition that $U \cap T_xN$ is a star-shaped
neighborhood of $0_x$ for each $x \in N$, where $0_x$ is the zero vector of $T_x N$. If $N$ is of non-negative curvature, then we have $U = TN$. Also, if all sectional curvatures of $N$ are bigger than or equal to $c$ ($c < 0$), then $U$ contains the ball bundle $T^r N := \{ X \in TN \mid ||X|| < r \}$ of radius $r := \frac{\pi}{2\sqrt{-c}}$. In detail, see [S1,2]. Denote by $J_A$ the adapted complex structure on $U$. The complex manifold $(U, J_A)$ is interpreted as the complexification of $N$. We denote $(U, J_A)$ by $N^c$ and call it the complexification of $N$, where we note that $N^c$ is given no pseudo-Riemannian metric. In particular, in case of $N = R^m$ (the Euclidean space), we have $(U, J_A) = C^m$. Also, in the case where $N$ is a symmetric space $G/K$ of non-compact type, there exists the holomorphic diffeomorphism $\delta$ of $(U, J_A)$ onto an open subset of $G^c/K^c$. Let $M$ be an immersed (complete) $C^\omega$-submanifold in $G/K$. Denote by $f$ its immersion. Let $M^c$ be the complexification of $M$ (defined as above). We shall state the definition of the complexification $f^c : M^c \rightarrow G^c/K^c$ of $f$, where we shrink $M^c$ to a neighborhood of the $0$-section of $TM$ if necessary. For its purpose, we first state the definition of the complexification of a $C^\omega$-curve $\alpha : R \rightarrow G/K$. Let $g = f + p$ be the Cartan decomposition associated with $G/K$ and $W : R \rightarrow p$ be the curve in $p$ with $(\exp W(t))K = \alpha(t)$ ($t \in R$), where we note that $W$ is uniquely determined because $G/K$ is of non-compact type. Since $\alpha$ is of class $C^\omega$, so is also $W$. Let $W^c : D \rightarrow p^c$ ($D$ : a neighborhood of $R$ in $C$) be the holomorphic extension of $W$. We define the complexification $\alpha^c : D \rightarrow G^c/K^c$ of $\alpha$ by $\alpha^c(z) = (\exp W^c(z))K^c$. It is shown that this complexification of a $C^\omega$-curve in $G/K$ is a holomorphic curve in $G^c/K^c$. By using this complexification of a $C^\omega$-curve in $G/K$, we define the complexification $f^c : M^c \rightarrow G^c/K^c$ of $f$ by $f^c(X) := (f \circ \gamma^M_X)^c(\sqrt{-1})$ ($X \in M^c (\subset TM)$), where $\gamma^M_X$ is the geodesic in $M$ with $\gamma^M_X(0) = X$. Here we shrink $M^c$ to a neighborhood of the $0$-section of $TM$ if necessary in order to assure that $\sqrt{-1}$ belongs to the domain of $(f \circ \gamma^M_X)^c$ for each $X \in M^c$. It is shown that the map $f^c : M^c \rightarrow G^c/K^c$ is holomorphic and that the restriction of $f^c$ to a neighborhood $U'$ of the $0$-section of $TM$ is an immersion, where we take $U'$ as largely as possible. Denote by $M^c$ this neighborhood $U'$ newly. Give $M^c$ the pseudo-Riemannian metric induced from that of $G^c/K^c$ by $f^c$. Then $M^c$ is an anti-Kaehlerian submanifold in $G^c/K^c$ immersed by $f^c$. We call this anti-Kaehlerian submanifold $M^c$ immersed by $f^c$ the extrinsic complexification of the submanifold $M$. In [Koi2] and [Koi6], we constructed the complete extension of the extrinsic complexification in different methods in the case where $M$ is proper complex equifocal. We also denote this complete extension by the same symbol $M^c$. Here we note that the extrinsic complexification of a $C^\omega$-pseudo-Riemannian submanifold in a general pseudo-Riemannian manifold has recently defined in [Koi7].

Let $\langle , \rangle$ be the Ad($G$)-invariant non-degenerate symmetric bilinear form of $g$ inducing the metric of $G/K$. The Cartan decomposition $g = f \oplus p$ is an orthogonal time-space decomposition of $g$ with respect to $\langle , \rangle$ in the sense of [Koi1]. Set $\langle , \rangle^A := 2\text{Re}\langle , \rangle^c$, where $\langle , \rangle^c$ is the complexification of $\langle , \rangle$ (which is a C-bilinear form of $g^c$). The R-bilinear form $\langle , \rangle^A$ on $g^c$ regarded as a real Lie algebra induces a bi-invariant pseudo-
Riemannian metric of \( G^c \) and a \( G^c \)-invariant anti-Kaehlerian metric on \( G^c/K^c \). It is clear that \( g^c = (f + \sqrt{-1}p) \oplus (-\sqrt{-1}p) \) is an orthogonal time-space decomposition of \( g^c \) with respect to \( \langle , \rangle^A \). For simplicity, set \( g^c_\pm := f + \sqrt{-1}p \). Let \( A \) be the Hilbert Lie group of all absolutely continuous paths \( u : [0, 1] \rightarrow g^c \) such that the weak derivative \( u' \) of \( u \) is \( g^c \)-valued (with respect to \( \langle , \rangle^A \)). Also, let \( H^1([0, 1], G^c) \) be the Hilbert Lie group of all absolutely continuous paths \( g : [0, 1] \rightarrow G^c \) such that the weak derivative \( g' \) of \( g \) is squared integrable (with respect to \( \langle , \rangle^A \)), that is, \( g^{-1}g' \in H^0([0, 1], g^c) \). Let \( \phi^c : H^0([0, 1], g^c) \rightarrow G^c \) be the parallel transport map for \( G^c \), that is, \( \phi^c(u) := g_u(1) \in H^0([0, 1], g^c) \), where \( g_u \) is the element of \( H^1([0, 1], G^c) \) with \( g_u(0) = e \) and \( g_u^{-1}g_u = u \). This map \( \phi^c \) is an anti-Kaehlerian submersion. Set \( P^c(G^c, e \times G^c) := \{ g \in H^1([0, 1], G^c) \mid g(0) = e \} \) and \( \Omega^c : H^1([0, 1], G^c) | g(0) = e \). The group \( H^1([0, 1], G^c) \) acts on \( H^0([0, 1], g^c) \) by gauge transformations, that is,

\[
g \ast u := \text{Ad}(g)u - g'g_u^{-1} \quad (g \in H^1([0, 1], G^c), \ u \in H^0([0, 1], g^c)).
\]

It is shown that the following facts hold:

(i) The above action of \( H^1([0, 1], G^c) \) on \( H^0([0, 1], g^c) \) is isometric,
(ii) The above action of \( P^c(G^c, e \times G^c) \) on \( H^0([0, 1], g^c) \) is transitive and free,
(iii) \( \phi^c(g \ast u) = (L_{g(0)} \circ R_{g(1)}) \phi^c(u) \) for \( g \in H^1([0, 1], G^c) \) and \( u \in H^0([0, 1], g^c) \),
(iv) \( \phi^c : H^0([0, 1], g^c) \rightarrow G^c \) is regarded as a \( \Omega^c(G^c) \)-bundle.
(v) If \( \phi^c(u) = (L_{x_0} \circ R_{x_1}^{-1})(\phi^c(v)) \) \( (u,v \in H^0([0,1], \mathfrak{g}^c), \ x_0, x_1 \in G^c) \), then there exists \( g \in H^1([0,1], G^c) \) such that \( g(0) = x_0, \ g(1) = x_1 \) and \( u = g \ast v \). In particular, it follows that each \( u \in H^0([0,1], \mathfrak{g}^c) \) is described as \( u = g \ast \hat{0} \) in terms of some \( g \in P(G^c, G^c \times e) \).

Let \( \pi^c : G^c \to G^c/K^c \) be the natural projection. It is shown that \( M \) is proper complex equipfocal if and only if \( \tilde{M}^c := (\pi^c \circ \phi^c)^{-1}(M^c) \) is proper anti-Kaehlerian isoparametric. The focal set of \( M^c \) at \( x \in M \) is equal to the image by the normal exponential map (of \( M^c \)) of the focal set (which consists of complex hyperplanes in \( T_u^\perp \tilde{M}^c \)) of \( M^c \) at \( u \in \tilde{M}^c \cap (\pi \circ \phi)^{-1}(x) \) under the identification of \( T_u^\perp \tilde{M}^c \) with \( T_x^\perp M^c \). From this reason, we call the above complex Coxeter group associated with \( \tilde{M}^c \) the complex Coxeter group associated with \( M \).

3 Homogeneity of the lifted submanifold

Let \( M \) be an irreducible proper complex equipfocal submanifold of codimension greater than one in a symmetric space \( G/K \) of non-compact type and \( \tilde{M}^c \) be the lifted submanifold of the complexification of \( M \) to \( H^0([0,1], \mathfrak{g}^c) \). In this section, we shall prove the homogeneity of \( \tilde{M}^c \). First we shall recall the generalized Chow’s theorem, which was proved in [HL2]. Let \( N \) be a (connected) Hilbert manifold and \( \mathcal{D} \) be a set of local (smooth) vector fields which are open sets of \( N \). If two points \( x \) and \( y \) of \( N \) can be connected by a piecwise smooth curve each of whose smooth segments is an integral curve of a local smooth vector field belonging to \( \mathcal{D} \), then we say that \( x \) and \( y \) are \( \mathcal{D} \)-equivalent and we denote this fact by \( x \sim y \). Let \( \Omega^\mathcal{D}(x) := \{ y \in N \mid y \sim x \} \). The set \( \Omega^\mathcal{D}(x) \) is called the set of reachable points of \( \mathcal{D} \) starting from \( x \). Let \( \mathcal{D}^* \) be the set of local smooth vector fields on open sets of \( N \) which is generated by \( \mathcal{D} \) in the following sense: \( \mathcal{D} \subset \mathcal{D}^* \), \( \mathcal{D}^* \) contains the zero vector field and, for any \( X, Y \in \mathcal{D}^* \) and any \( a, b \in \mathbb{R} \), \( aX + bY \) and \( [X, Y] \) (which are defined on the intersection of the domains of \( X \) and \( Y \)) also belong to \( \mathcal{D}^* \). For each \( x \in N \), let \( \mathcal{D}^*(x) := \{ X_x \mid X \in \mathcal{D}^* \text{ s.t. } x \in \text{Dom}(X) \} \). Then the following generalized Chow’s theorem holds.

**Theorem 3.1 ([HL2])** If \( \overline{\mathcal{D}^*(x)} = T_x^\perp N \) for each \( x \in N \), \( \overline{\Omega^\mathcal{D}(x)} = \Omega^\mathcal{D} \) holds for each \( x \in N \), where \( (\cdot) \) implies the closure of \( (\cdot) \).

For simplicity, we set \( V := H^0([0,1], \mathfrak{g}^c) \). Denote by \( A \) the shape tensor of \( \tilde{M}^c \). Let \( \{ E_i \mid i \in I \} \cup \{ E_0 \} \) be the set of all complex curvature distributions of \( \tilde{M}^c \), where \( E_0 \) is one defined by \( (E_0)_u := \bigcap_{v \in T_u^\perp \tilde{M}^c} \text{Ker} \tilde{A}_v \) \( (u \in \tilde{M}^c) \). Also, let \( \lambda_i \) and \( n_i \) be the complex principal curvature and the complex curvature normal corresponding to \( E_i \), respectively. Fix \( u_0 \in \tilde{M}^c \). Denote by \( k_i \) the complex hyperplane \( (\lambda_i)_{u_0}^{-1}(1) \) of \( T_{u_0}^\perp \tilde{M}^c \). Let \( Q(u_0) \) be the set of all points of \( \tilde{M}^c \) connected with \( u_0 \) by a piecewise smooth curve in \( \tilde{M}^c \) each of whose
smooth segments is contained in some complex curvature sphere (which may depend on the smooth segment). By using the above generalized Chow’s theorem, we shall show the following fact.

**Proposition 3.2.** The set \( Q(u_0) \) is dense in \( \tilde{M}^c \).

**Proof.** Let \( \mathcal{D}_E \) be the set of all local (smooth) tangent vector fields on open sets of \( \tilde{M}^c \) which is tangent to some \( E_i \) \((i \neq 0)\) at each point of the domain. Define \( \Omega_{\mathcal{D}_E}(u_0) \), \( \mathcal{D}_E^* \) and \( \mathcal{D}_E(u_0) \) as above. By imitating the proof of Proposition 5.8 of [HL1], it is shown that \( \mathcal{D}_E(u_0) = T_uM^c \) for each \( u \in M^c \). Hence, \( \Omega_{\mathcal{D}_E}(u_0) = \tilde{M}^c \), follows from Theorem 3.1. It is clear that \( \Omega_{\mathcal{D}_E}(u_0) = Q(u_0) \). Therefore we obtain \( Q(u_0) = \tilde{M}^c \). q.e.d.

For each complex affine subspace \( P \) of \( T_{u_0}^\perp \tilde{M}^c \), define \( I_P \) by

\[
I_P := \begin{cases} 
\{ i \in I \mid (n_i)_{u_0} \in P \} & (0 \notin P) \\
\{ i \in I \mid (n_i)_{u_0} \in P \} \cup \{ 0 \} & (0 \in P).
\end{cases}
\]

It is easy to show that \( I_P \) is finite. Define a distribution \( D_P \) on \( \tilde{M}^c \) by \( D_P := \bigoplus_{i \in I_P} E_i \).

It is shown that \((\bigcap_{i \in I_P \setminus \{0\}} l_i) \setminus \bigcup_{i \in I_P \setminus \{0\}} l_i) \neq \emptyset \). Take \( v_0 \in (\bigcap_{i \in I_P \setminus \{0\}} l_i) \setminus \bigcup_{i \in I_P \setminus \{0\}} l_i \). Let \( v \) be a parallel normal vector field on \( \tilde{M}^c \) with \( v_{u_0} = v_0 \). Hence we note that \( v \) is a complex focal normal vector field of \( M^c \). Let \( f_v \) be the focal map for \( v \) (i.e., the end point map for \( v \)), \( F_v \) be the focal submanifold for \( v \) (i.e., \( F_v = f_v(\tilde{M}^c) \)) and \( L_u^{D_P} \) be the leaf of \( D_P \) through \( u \in M^c \). If \( 0 \notin P \), then we have \( L_u^{D_P} = f_v^{-1}(f_v(u)) \). According to the homogeneous slice theorem for the complete complexification of a proper complex equifocal submanifold in [Koi6], we have the following homogeneous slice theorem for \( \tilde{M}^c \).

**Theorem 3.3.** If \( 0 \notin P \), then the leaf \( L_u^{D_P} \) is a principal orbit of the direct sum representation of aks-representations on \( T_{f_v(u)}^\perp F_v \).

**Proof.** Let \( u_1 := f_v(u) \), \( \overline{F}_v := (\pi^c \circ \phi^c)(F_v) \) and \( \overline{v} := (\pi^c \circ \phi^c)_*(v) \), which is well-defined because \( v \) is projectable. It is shown that \( \overline{v} \) is a focal normal vector field of \( M^c \) and that \( \overline{F}_v \) is the focal submanifold of \( M^c \) corresponding to \( \overline{v} \). Denote by \( f_{\overline{v}} \) the focal map for \( \overline{v} \) and set \( \overline{u}_1 := (\pi^c \circ \phi^c)(u_1) \). Set \( L := f_{\overline{v}}^{-1}(u_1) \) and \( \overline{L} := f_{\overline{v}}^{-1}(\overline{u}_1) \), which are leaves of the focal distributions corresponding to \( v \) and \( \overline{v} \), respectively. According to Theorem A of [Koi6], \( \overline{L} \) is the image of a principal orbit of the direct sum representation of aks-representations on \( T_{\overline{u}_1}^\perp \overline{F}_v \) by the normal exponential map \( \exp_{\overline{u}_1} \) of \( \overline{F}_v \) at \( \overline{u}_1 \). On the other hand, under the identification of \( T_{\overline{u}_1}^\perp \overline{F}_v \) with \( T_{u_1}^\perp F_v \), \( \overline{L} \) is the image of \( L \) by \( \exp_{\overline{u}_1} \). Hence it follows that \( L \) is a principal orbit of an aks-representation on \( T_{u_1}^\perp F_v \). Since \( 0 \notin P \) by the assumption, we have \( L_u^{D_P} = L \). Therefore the statement of this theorem follows. q.e.d.
Set \((W_P)_u := u + (D_P)_u \oplus \text{Span}\{(n_i)_u | i \in I_P \setminus \{0\}\}\) \((u \in \tilde{M}^c)\). Let \(\gamma : [0,1] \to \tilde{M}^c\) be a piecewise smooth curve. In the sequel, we assume that the domains of all piecewise smooth curves are equal to \([0,1]\). If \(\gamma(t) \perp (D_P)_{\gamma(t)}\) for each \(t \in [0,1]\), then \(\gamma\) is said to be horizontal with respect to \(D_P\) (or \(D_P\)-horizontal). Let \(\beta_i (i=1,2)\) be curves in \(\tilde{M}^c\). If \(E_{\beta_i(t)} = L_{\beta_i(t)}\) for each \(t \in [0,1]\), then \(\beta_1\) and \(\beta_2\) are said to be parallel. By imitating the proof of Proposition 1.1 in [HL2], we can show the following fact.

**Lemma 3.4.** For each \(D_P\)-horizontal curve \(\gamma\), there exists an one-parameter family \(\{h_{\gamma,t} | 0 \leq t \leq 1\}\) of holomorphic isometries \(h_{\gamma,t} : (W_P)_{\gamma(0)} \to (W_P)_{\gamma(t)}\) such that \(h_{\gamma,t}^{-1}(L_{\gamma(0)}^{D_P}) = L_{\gamma(t)}^{D_P}\) and that, for each \(u \in L_{\gamma(0)}^{D_P}\), \(t \mapsto h_{\gamma,t}^{-1}(u)\) is a \(D_P\)-horizontal curve parallel to \(\gamma\).

**Proof.** First we consider the case of \(0 \notin P\). Take \(v_0 \in \bigcap_{i \in I_P} l_i \setminus \bigcup_{i \in I_P} l_i\). Let \(v\) be the parallel normal vector field of \(\tilde{M}^c\) with \(v_{u_0} = v_0\). Let \(\gamma = f_v \circ \gamma\). Define a map \(h_t : (W_P)_{\gamma(0)} \to V\) by \(h_t(u) := \gamma(t) + \tau_{\gamma(t)}^\perp(\tilde{\gamma}(0)u)\) \((u \in (W_P)_{\gamma(0)}\)\), where \(\tau_{\gamma(t)}^\perp\) is the parallel translation along \(\gamma\) with respect to the normal connection of \(F\). Then it is shown that \(\{h_t | 0 \leq t \leq 1\}\) is the desired one-parameter family (see Fig. 1). Next we consider the case of \(0 \in P\). Take \(v_0 \in \bigcap_{i \in I_P} (\lambda_i)_{u_0}^{-1}(0) \setminus \bigcup_{i \in I_P} (\lambda_i)_{u_0}^{-1}(0)\). Let \(v\) be the parallel normal vector field of \(\tilde{M}^c\) with \(v_{u_0} = v_0\). We define a map \(\nu : \tilde{M}^c \to S^\infty(1)\) by \(\nu(u) := v_u\) \((u \in \tilde{M}^c)\), where \(S^\infty(1)\) is the unit hypersphere of \(V\). Then we have \(\nu_u = -A v_u\) \((u \in \tilde{M}^c)\), where \(A\) is the shape tensor of \(\tilde{M}^c\). If \(i \in I_P\), then we have \(\nu_u((E_i)_u) = -\langle(n_i)_u, v_u(E_i)\rangle_u = 0\) and, if \(i \notin I_P\), then we have \(\nu_u((E_i)_u) = -\langle(n_i)_u, v_u(E_i)\rangle_u = (E_i)_u\). Hence we have \(\text{Ker} \nu_u = (D_P)_u\). Therefore \(D_P\) is integrable and it gives a foliation on \(\tilde{M}^c\). Denote by \(\mathfrak{F}_P\) this foliation and \(D_P^\perp\) the orthogonal complementary distribution of \(\mathfrak{F}_P\). Let \(U\) be a neighborhood of \(\gamma(0)\) in \(L_{\gamma(0)}^{D_P}\) where an element of holonomy along \(\gamma\) with respect to \((\mathfrak{F}_D, D_P^\perp)\) is defined. See [BH] about the definition of an element of holonomy. Let \(\Delta\) be a fundamental domain containing \(u_0\) of the complex Coxeter group of \(\tilde{M}^c\) at \(u_0(\in \tilde{M}^c)\). Denote by \(\Delta_u\) a domain of \(T_u\tilde{M}^c\) given by parallel translating \(\Delta\) with respect to the normal connection of \(\tilde{M}^c\). Set \(\tilde{U} := \bigcup_{u \in U} (\text{Span}\{(n_i)_u | i \in I_P \setminus \{0\}\}) \cap \Delta_u\), which is an open subset of the affine subspace \((W_P)_{\gamma(0)}\). Since an element of holonomy along \(\gamma\) is defined on \(U\), there exists the \(D_P\)-horizontal curve \(\gamma_u\) parallel to \(\gamma\) with \(\gamma_u(0) = u\) for each \(u \in \tilde{U}\). Define a map \(h_t : \tilde{U} \to (W_P)_{\gamma(t)}\) \((0 \leq t \leq 1)\) by \(h_t(u + w) = \gamma_u(t) + \tau_{\gamma_u(0)}^\perp(w)\) \((u \in \tilde{U}, w \in \text{Span}\{(n_i)_u | i \in I_P \setminus \{0\}\}) \cap \Delta_u\) (see Fig. 2). It is shown that \(h_t\) is a holomorphic isometry into \((W_P)_{\gamma(t)}\) (see Lemma 1.2 in [HL2]). Hence \(h_t\) extends to a holomorphic isometry of \((W_P)_{\gamma(0)}\) onto \((W_P)_{\gamma(t)}\). It is shown that this extension is the desired one-parameter family (see Step 3 of the proof of Proposition 1.1 in [HL2]).

q.e.d.

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Fig. 1.

in fact non-compact

in fact non-compact

Fig. 2.
Fix $i_0 \in I \cup \{0\}$. Denote by $\Phi_{i_0}(u_0)$ the group of holomorphic isometries of $(W_P)_{u_0}$ generated by
\[
\{h_{\gamma,1}^{E_{i_0}} \mid \gamma : E_{i_0}^l \to -\text{horizontal curve s.t.} \gamma(0), \gamma(1) \in L_u^{E_{i_0}}\},
\]
where $L_u^{E_{i_0}}$ is the integral manifold of $E_{i_0}$ through $u$. Also, denote by $\Phi_{i_0}^0(u_0)$ the identity component of $\Phi_{i_0}(u_0)$ and $\Phi_{i_0}^0(u_0)_{u_0}$ the isotropy subgroup of $\Phi_{i_0}(u_0)$ at $u_0$. Give $\Phi_{i_0}^0(u_0)$ the metric associated with its representation on $(W_P)_{u_0}$. Take $X \in \text{Lie} \Phi_{i_0}^0(u_0) \subset \text{Lie} \Phi_{i_0}^0(u_0)_{u_0}$, where Lie$(\cdot)$ is the Lie algebra of $(\cdot)$. Set $g(t) := \exp tX$ and $\gamma(t) := g(t)u_0$, where exp is the exponential map of $\Phi_{i_0}^0(u_0)$. It is clear that $\gamma$ is a curve in $L_u^{E_{i_0}}$. Hence $\gamma$ is an $E_i$-horizontal curve for $i \in I$ with $i \neq i_0$. We define a holomorphic isometry $F_\gamma$ of $V$ by $F_\gamma(\gamma(0)) = \gamma(1)$ and
\[
(F_\gamma)_{\gamma(0)} = \begin{cases} 
g(1)_{\gamma(0)} & \text{on } (E_{i_0})_{\gamma(0)} \\
h_{\gamma,1}^{E_{i_0}}(0) & \text{on } (E_i)_{\gamma(0)} (i \neq i_0) \\
\tau_{\gamma}^l & \text{on } T_{\gamma(0)}^l \tilde{M}^c
\end{cases}
\]
In similar to Theorem 4.1 of [HL2], we have the following fact.

**Proposition 3.5.** The holomorphic isometry $F_\gamma$ preserves $\tilde{M}^c$ invariably (i.e., $F_\gamma(\tilde{M}^c) = \tilde{M}^c$). Furthermore, it preserves $E_i$ ($i \in I$) invariably (i.e., $F_{\gamma*}(E_i) = E_i$).

To show this proposition, we prepare lemmas. By imitating the proof (P163~166) of Proposition 3.1 in [HL2], we can show the following fact.

**Lemma 3.5.1.** Let $N$ and $\tilde{N}$ be irreducible proper anti-Kaehlerian isoparametric submanifolds of complex codimension greater than one in an infinite dimensional anti-Kaehlerian space. If $N \cap \tilde{N} \neq \emptyset$ and, for some $p_0 \in N \cap \tilde{N}$, $T_{p_0}N = T_{p_0}\tilde{N}$ and there exists a complex affine line $l_0$ of $T_{p_0}N = T_{p_0}\tilde{N}$ such that $L_{p_0}D^l_{l_0} = \tilde{L}_{p_0}D^l_{l_0}$ for any complex affine line $l$ of $T_{p_0}N$ with $l \neq l_0$, then $N = \tilde{N}$ holds, where $L_{p_0}D^l_{l_0}$ (resp. $\tilde{L}_{p_0}D^l_{l_0}$) is the leaf through $p_0$ of the integrable distribution $D_l$ (resp. $\tilde{D}_l$) on $N$ (resp. $\tilde{N}$) defined as above for $l$.

*Proof.* Let $\{E_i \mid i \in I\} \cup \{E_0\}$ (resp. $\{\tilde{E}_i \mid i \in \tilde{I}\} \cup \{\tilde{E}_0\}$) be the set of all complex curvature distributions of $N$ (resp. $\tilde{N}$). Let $Q_0(p_0)$ (resp. $\tilde{Q}_0(p_0)$) be the set of all points of $N$ (resp. $\tilde{N}$) connected with $p_0$ by a piecewise smooth curve in $N$ (resp. $\tilde{N}$) each of whose smooth segments is contained in some complex curvature sphere or some integral manifold of $E_0$ (resp. $\tilde{E}_0$), where $E_0$ (resp. $\tilde{E}_0$) is the distribution on $N$ (resp. $\tilde{N}$) corresponding to the above distribution $E_0$ on $M^c$. Take any $p \in Q_0(p_0)$. There exists a sequence $\{p_0, p_1, \cdots, p_l (= p)\}$ such that, for each $j \in \{1, \cdots, l\}$, $p_j \in (\cup_{i \in I} L_{p_{j-1}}^{E_i}) \cup L_{p_{j-1}}^{E_0}$. Assume
that there exists \( j_0 \in \{1, \ldots, l\} \) such that \( p_{j_0} \in L_{p_{j_0}}^{E_{i_0}} \) for some \( i_0 \in I \) with \((n_{i_0})_{p_0} \in l_0\).

Since \( N \) is irreducible (hence the complex Coxeter group associated with \( N \) is irreducible) and \( \text{codim}_c N \geq 2 \), there exists a complex curvature normal \( n_{i_1} \) such that \((n_{i_1})_{p_0} \) and \((n_{i_0})_{p_0} \) are linearly independent, \( \langle (n_{i_1})_{p_0}, (n_{i_0})_{p_0} \rangle \neq 0 \) and that \((n_{i_1})_{p_0} \) does not belong to \( l_0 \). Denote by \( l_{a_{i_1}} \) the affine line in \( T_{p_0}^\perp N \) through \((n_{i_0})_{p_0} \) and \((n_{i_1})_{p_0} \), and set \( D_{a_{i_1}} := D_{l_{a_{i_1}}} \) for simplicity. According to Theorem 3.3, \( L_{p_{j_0}}^{D_{a_{i_1}}} \) is an irreducible proper anti-Kaehlerian isoparametric submanifold in \((W_{a_{i_1}})_{p_{j_0}} \) of complex codimension two. Hence, by the anti-Kaehlerian version of Theorem D of [HOT], \( p_{j_0} \) can be joined to \( p_{j_0} \) by a piecewise smooth curve each of whose smooth segments is tangent to one of \( E_i \)'s \((i \in I \) s.t. \((n_{i})_{p_0} \in l_{a_{i}} \) and \((n_{i})_{p_0} \neq (n_{i_0})_{p_0} \)). Therefore, we can take a sequence \( \{p_0, p_1', \ldots, p_{l'} (=p)\} \) such that, for each \( j \in \{1, \ldots, l'\} \), \( p_j' \in \left( \bigcup_{i \in I \text{ s.t. } (n_{i})_{p_0} \notin l_0} L_{p_{j-1}}^{E_i} \right) \cup L_{p_{j-1}}^{E_0} \). Hence it follows from Lemma 3.5.2 (see below) that \( p_1' \in \tilde{Q}_0(p_0), p_2' \in \tilde{Q}_0(p_1'), \ldots, p_{l'}-2 \in \tilde{Q}_0(p_{l'-2}) \) and \( p \in \tilde{Q}_0(p_{l'-1}) \) inductively. Therefore we have \( p \in \tilde{Q}_0(p_0) \). From the arbitrariness of \( p \), it follows that \( Q_0(p_0) \subseteq \tilde{Q}_0(p_0) \). Similarly we can show \( \tilde{Q}_0(p_0) \subseteq Q_0(p_0) \). Thus we obtain \( Q_0(p_0) = \tilde{Q}_0(p_0) \) and hence \( \overline{Q_0(p_0)} = \overline{\tilde{Q}_0(p_0)} \). Let \( D^0_E \) (resp. \( \tilde{D}^0_E \)) be the set of all local (smooth) vector fields of \( N \) (resp. \( \tilde{N} \)) which is tangent to some \( E_i \) (resp. \( \tilde{E}_j \)) (where \( i \) may be equal to 0) at each point of the domain. For \( D^0_E \) and \( p \in N \) (resp. \( \tilde{D}_E \) and \( \tilde{p} \in \tilde{N} \)), define \( \Omega_{D^0_E}(p), (D^0_E)^* \) and \( (D^0_E)^*(p) \) (resp. \( \Omega_{\tilde{D}_E}(\tilde{p}), (\tilde{D}_E)^* \) and \( (\tilde{D}_E)^*(\tilde{p}) \)) as the quantities corresponding to the above \( \Omega_D(x), D^* \) and \( \tilde{D}^*(x) \). Since \( (D^0_E)^*(p) = (E_0)p \oplus (\oplus_{i \in I} (E_i)p) = T_pN \) for each \( p \in \tilde{N} \), it follows from Theorem 3.1 that \( \overline{\Omega_{D^0_E}(p_0)} = N \). Similarly, we have \( \overline{\Omega_{\tilde{D}_E}(p_0)} = \tilde{N} \). Also, it is clear that \( \Omega_{D^0_E}(p_0) = \tilde{Q}_0(p_0) \) and \( \Omega_{\tilde{D}_E}(p_0) = \tilde{Q}_0(p_0) \). Therefore we obtain \( N = \tilde{N} \).

q.e.d.

**Lemma 3.5.2.** Let \( \tilde{N}, \tilde{N}, p_0 \) and \( l_0 \) be as in Lemma 3.5.1. Then, for any \( p \in L_{p_0}^{E_0} \cup \left( \bigcup_{i \in I \text{ s.t. } (n_{i})_{p_0} \notin l_0} L_{p_0}^{E_i} \right) \), we have \( T_pN = T_p\tilde{N} \) and \( L_p^l = L_p^{\tilde{l}} \) for any complex affine line \( l \) of \( T_{p_0}^\perp N \) with \( l \neq l_0 \).

Proof. First we consider the case where \( p \in L_{p_0}^{E_i} \) for some \( i \) with \((n_{i})_{p_0} \notin l_0 \). Then, from the assumption, we have \( p \in L_{p_0}^{E_i} = L_{p_0}^{E_i} \) and hence \( L_{p}^{E_i} = L_{p}^{E_i} \). Let \( l \) be a complex affine line of \( T_{p_0}^\perp N \) with \( l \neq l_0 \). Assume that \((n_{i})_{p_0} \in l_0 \). Then we have \( p \in L_{p_0}^{E_i} \subseteq L_{p_0}^{D_i} \). Since \( l \neq l_0 \), it follows from the assumption that \( L_{p_0}^{D_i} = L_{p_0}^{D_i} \). Hence we have \( L_{p}^{D_i} = L_{p}^{D_i} \). Assume that \((n_{i})_{p_0} \notin l_0). Take a curve \( \gamma : [0, 1] \rightarrow L_{p_0} \) with \( \gamma(0) = p_0 \) and \( \gamma(1) = p \). Since
Let $N$ be a proper anti-Kaehlerian isoparametric submanifold in a finite dimensional anti-Kaehlerian space and $\{E_1, \cdots, E_k\}$ be the set of all complex curvature distributions of $N$. We can define the holomorphic isometry of the anti-Kaehlerian space corresponding to the above $F_\gamma$. Denote by the same symbol $F_\gamma$ this holomorphic isometry. In similar to Lemma 4.2 in [HL2], we have the following fact.

**Lemma 3.5.3.** If $N$ is a principal orbit through a semi-simple element of an aks-representation, then $N$ is proper anti-Kaehlerian isoparametric and $F_\gamma(N) = N$ holds.

**Proof.** First we note that any irreducible (semi-simple) anti-Kaehlerian symmetric space is regarded as the complexification of an irreducible Riemannian symmetric space of non-compact type. Let $L/H$ be an irreducible Riemannian symmetric space of non-compact type. Denote by $\mathfrak{l}$ (resp. $\mathfrak{h}$) the Lie algebra of $L$ (resp. $H$). Let $\theta$ be the Cartan involution of $L$ with $(\text{Fix}\theta)_0 \subset H \subset \text{Fix}\theta$ and denote by the same symbol $\theta$ the involution of $\mathfrak{l}$ associated with $\theta$. Set $q := \text{Ker}(\theta + \text{id})$, which is identified with the tangent space $T_{\theta H}(L/H)$. The complexification $q^c$ is identified with the tangent space $T_{\theta H^c}(L^c/H^c)$ of the associated anti-Kaehlerian symmetric space $L^c/H^c$. Let $\rho$ be the aks-representation associated with $L^c/H^c$ and $N$ be a principal orbit of the representation $\rho$ through a semi-simple element $w(\in q^c)$, that is, $N = \rho(H^c) \cdot w$. Denote by $A$ the shape tensor of $N$. Let $\mathfrak{a}$ be a Cartan subspace of $q^c$ containing $w$. The space $\mathfrak{a}$ contains the maximal split abelian
subspace \( a_v := a \cap q \) of vector-type and \( a = a_v^c \) holds. For each \((R\text{-})\)linear function \( \alpha \) on \( a_v \) (i.e., \( \alpha \in a_v^c \)), we set
\[
a_v^c := \{ X \in q^c \mid \text{ad}(a)^2(X) = \alpha(a)^2X \ (\forall a \in a_v) \}.
\]
Set \( \Delta := \{ \alpha \in a_v^c \mid a_v^c \neq \{0\} \} \), which is called the root system with respect to \( a_v \). Then we have the root space decomposition
\[
q^c = a + \sum_{\alpha \in \Delta_+} a_v^c,
\]
where \( \Delta_+ \) is the positive root system under some lexicographical ordering of \( \Delta \) and we note that \( a \) is equal to the centralizer of \( a_v \) in \( q^c \). For each \( \alpha \in \Delta_+ \), the complexification \( \alpha^c \) is regarded as a \( C \)-linear function on \( a \) and we have
\[
a_v^c = \{ X \in q^c \mid \text{ad}(a)^2(X) = \alpha^c(a)^2X \ (\forall a \in a) \}.
\]
Since \( N \) is a principal orbit and hence \( w \) is a regular element, we have \( \alpha^c(w) \neq 0 \) for any \( \alpha \in \Delta_+ \). Under the identification of \( a \) with \( T_{w}^\perp N \), \( \alpha^c \) is regarded as a \( C \)-linear function on \( T_{w}^\perp N \), which is denoted by \( \overline{\alpha^c} \). Easily we can show
\[
A_v^c|_{a_v^c} = -\frac{\overline{\alpha^c}(v)}{\alpha^c(w)} \text{id} \ (\alpha \in \Delta_+)
\]
for any \( v \in T_{w}^\perp N \). Let \( \lambda_{\alpha^c,w} \) be the parallel section of the \( C \)-dual bundle \((T_{w}^\perp N)^*\) of \( T_{w}^\perp N \) with \( \lambda_{\alpha^c,w}^* = -\frac{\overline{\alpha^c}(w)}{\alpha^c(w)} \). It is clear that \( N \) is a proper anti-Kaehlerian isoparametric submanifold having \( \{ \lambda_{\alpha^c,w} \mid \alpha \in \Delta_+ \} \) as the set of all complex principal curvatures. Denote by \( E_{\alpha^c} \) the complex curvature distribution for \( \lambda_{\alpha^c,w} \). Take \( v_0 \in (\lambda_{\alpha_0^c,w})^{-1}(1) \setminus \bigcup_{\alpha \in \Delta_+ \text{ s.t. } \alpha \neq \alpha_0} (\lambda_{\alpha^c,w})^{-1}(1) \) and set \( F := \rho(H^c) \cdot v_0 \), which is a focal submanifold of \( N \) whose corresponding focal distribution is equal to \( E_{\alpha_0^c} \). We have the relations \( h^c = 3h^c(a_v) + \sum_{\alpha \in \Delta_+} h^c_{\alpha} \) and \( T_{v_0}^\perp F = a + a_v^c \), where \( 3h^c(a_v) \) is the centralizer of \( a_v \) in \( h^c \) and \( h^c_{\alpha} := \{ X \in h^c \mid \text{ad}(a)^2X = \alpha(a)^2X \ (\forall a \in a_v) \} \). Denote by \( H_{v_0}^c \) (resp. \( H_{v_0}^e \)) the isotropy group of the \( H^c \text{-} \)action at \( w \) (resp. \( v_0 \)) and by \( h_{v_0}^c \) (resp. \( h_{v_0}^e \)) the Lie algebra of \( H_{v_0}^c \) (resp. \( H_{v_0}^e \)). Then we have \( h_{v_0}^c = 3h_{v_0}^c(a_v) \) and \( h_{v_0}^e = 3h_{v_0}^e(a_v) + h_{v_0}^c \). For the restriction of the \( \rho(H_{v_0}^e) \text{-} \)action on \( q^c \) to \( T_{v_0}^\perp F \) is called the slice representation of the action at \( v_0 \). It is shown that this slice representation coincides with the normal holonomy group action of \( F \) at \( v_0 \) and \( \rho(H_{v_0}^c) \cdot w = L_{v_0}^{E_{v_0}^c} \). Set \( \Phi(v_0) := \rho(H_{v_0}^c) \) and \( \Phi(w) := \rho(H_{v_0}^c) \). The leaf \( L_{v_0}^{E_{v_0}^c} \) is regarded as the quotient manifold \( \Phi(v_0)/\Phi(w) \). The holomorphic isometry \( F_\gamma \) in the statement is given as follows. Take \( X = \text{ad}_e(\overline{X}) \in \text{Lie} \Phi(v_0) \oplus \text{Lie} \Phi(w) \), where \( \overline{X} \in h_{v_0}^c \), and set \( g(t) := \exp_{\Phi(v_0)}(tX) \) and \( \gamma(t) := g(t) \cdot w \), where \( t \in [0,1] \). Then
$F_\gamma$ is given by $F_\gamma(w) = \gamma(1)$, $F_{\gamma + w}|(E_{\alpha_0})_w = g(1)_{\gamma + w}|(E_{\alpha_0})_w$, $F_{\gamma + w}|(E_{\alpha_0})_w = h_{E_\gamma}^{\alpha_0}|(E_{\alpha_0})_w$ ($\alpha \in \Delta_+ \text{ s.t. } \alpha \neq \alpha_0$) and $F_{\gamma + w}|_{T^*_\gamma N} = \tau_\gamma^+$, where $h_{E_\gamma}^{\alpha_0}$ is defined as in Lemma 3.4 and $\tau_\gamma^+$ is the parallel translation along $\gamma$ with respect to the normal connection of $N$. Easily we can show $(F_\gamma)_\gamma = g(1)_\gamma$. Hence, since both $F_\gamma$ and $g(1)$ are affine transformations of $\mathfrak{q}_\gamma^c$, they coincide with each other. Therefore, we obtain $F_\gamma(N) = g(1)(\rho(H^c) \cdot w) = \rho(\exp_{L^c}(\overline{\alpha})))(\rho(H^c) \cdot w) = N$.

q.e.d.

By using Lemmas 3.5.1 and 3.5.3, we shall prove Proposition 3.5.

**Proof of Proposition 3.5.** Since $M$ is an irreducible proper complex equifocal submanifold, $\widetilde{M}^\rho$ is a full irreducible proper anti-Kaehlerian isoparametric submanifold. Since $F_\gamma$ is an isometry, so is also $\widetilde{M}^\rho := F_\gamma(\widetilde{M}^\rho)$. Let $\{E_{\alpha}'| i \in I \} \cup \{E_{\alpha}'\}$ be the set of all complex curvature distributions on $\widetilde{M}^\rho$ and $n_{\alpha}'$ be the complex curvature normal corresponding to $E_{\alpha}'$. From the definition of $F_\gamma$, it follows that $\gamma(1) \in \widetilde{M}^\rho \cap \widetilde{M}^\rho$, $T_{\gamma(1)}\widetilde{M}^\rho = T_{\gamma(1)}\widetilde{M}^\rho$, complex curvature normals of $\widetilde{M}^\rho$ coincide with those of $\widetilde{M}^\rho$ at $\gamma(1)$ and that complex curvature spheres of $\widetilde{M}^\rho$ through $\gamma(1)$ coincides with those of $\widetilde{M}^\rho$ through $\gamma(1)$. Let $l_0$ be the complex affine line through $0$ and $(n_{i_0})_{\gamma(1)}$, that is, $l_0 := \text{Span}_{\mathbb{C}}\{(n_{i_0})_{\gamma(1)}\}$. Let $l$ be any complex affine line of $T_{\gamma(1)}\widetilde{M}^\rho$ with $l \neq l_0$. Now we shall show that $L_{\gamma(1)}^D = L_{\gamma(1)}^D$, where $D_l$ (resp. $D_l^\rho$) is the distribution on $\widetilde{M}^\rho$ (resp. $\widetilde{M}^\rho$) defined as above for $l$. First we consider the case of $(n_{i_0})_{\gamma(1)} \in l$. Then we have $0 \not\in l$. If there does not exist $i_1 \neq i_0 \in I$ with $(n_{i_1})_{\gamma(1)} \in l$, then we have $L_{\gamma(1)}^D = L_{\gamma(1)}^E = L_{\gamma(1)}^D = L_{\gamma(1)}^D$. Assume that there exists $i_1 \neq i_0 \in I$ with $(n_{i_1})_{\gamma(1)} \in l$. Let $v$ be a complex focal normal vector field of $\widetilde{M}^\rho$ such that the corresponding focal distribution is equal to $D_l$. Since $0 \not\in l$, it follows from Theorem 3.3 that $L_{\gamma(1)}^D$ is a principal orbit of an aks-representation on $T_{F_\overline{\alpha}(\gamma(1))}^{\perp}$. Since $(n_{i_0})_{\gamma(1)}, (n_{i_1})_{\gamma(1)} \in l$ and $0 \not\in l$, $(n_{i_0})_{\gamma(1)}$ and $(n_{i_1})_{\gamma(1)}$ are $\mathbf{C}$-linear independent. Hence the aks-representation is the one associated with some anti-Kaehlerian symmetric space of complex rank two. If $L_{\gamma(1)}^D$ is reducible, then the complex Coxeter group associated with $L_{\gamma(1)}^D$ is reducible. Hence it follows from Lemma 3.8 of [Koi4] that $(n_{i_0})_{\gamma(1)}$ and $(n_{i_1})_{\gamma(1)}$ are orthogonal and that, for any complex curvature normal $n$ of $L_{\gamma(1)}^D$, $n_{\gamma(1)}$ is contained in $l_0 \cup \text{Span}_{\mathbb{C}}\{(n_{i_0})_{\gamma(1)}\}$. Also, since $n_{\gamma(1)} \in l$, $n_{\gamma(1)}$ is equal to $(n_{i_0})_{\gamma(1)}$ or $(n_{i_1})_{\gamma(1)}$. Hence the set of all complex curvature normals of $L_{\gamma(1)}^D$ is equal to $\{n_{i_0}|_{L_{\gamma(1)}^E}, n_{i_1}|_{L_{\gamma(1)}^E}\}$. This implies that $L_{\gamma(1)}^D$ is congruent to the (extrinsic) product of complex spheres $L_{\gamma(1)}^E$ and $L_{\gamma(1)}^E$. Similarly $L_{\gamma(1)}^D$ is congruent to the (extrinsic) product of
holomorphic isometry. Hence we have
\[ F_\gamma(L_{u_0}^{E_1}) = F_\gamma(L_{u_0}^{E_0}) \times F_\gamma(L_{u_0}^{E_1}) = L_{\gamma(1)}^{E_0} \times L_{\gamma(1)}^{E_1} = L_{\gamma(1)}^{D_i}. \]

On the other hand, we have \( F_\gamma(L_{u_0}^{D_i}) = L_{\gamma(1)}^{D_i} \). Therefore we obtain \( L_{\gamma(1)}^{D_i} = L_{\gamma(1)}^{D_i} \). If \( L_{\gamma(1)}^{D_i} \) is irreducible, then it follows from Lemma 3.5.3 that \( F_\gamma(L_{\gamma(1)}^{D_i}) = (F_\gamma(W_{\gamma(1)})(L_{\gamma(1)}^{D_i}) = L_{\gamma(1)}^{D_i} \).

Hence we obtain \( L_{\gamma(1)}^{D_i} = L_{\gamma(1)}^{D_i} \). Next we consider the case of \((n_{u_0})_{\gamma(1)} \notin l\). Then, since \( \gamma \) is \( D_i \)-horizontal, we have \( F_\gamma(L_{u_0}^{D_i}) = L_{\gamma(1)}^{D_i} \) and hence \( L_{\gamma(1)}^{D_i} = L_{\gamma(1)}^{D_i} \). Therefore, from Lemma 3.5.1, we obtain \( F_\gamma(\widetilde{M^c}) = \widetilde{M^c} \). Furthermore, from this relation, we can show \((F_\gamma)_i(E_i) = E_i (i \in I) \) easily.

q.e.d.

By using Proposition 3.5, we prove the following fact.

**Proposition 3.6.** For any \( u \in Q(u_0) \), there exists a holomorphic isometry \( f \) of \( V \) such that \( f(u_0) = u \), \( f(\widetilde{M^c}) = \widetilde{M^c} \) (hence \( f_*(E_i) = E_i (i \in I) \), \( f(Q(u_0)) = Q(u_0) \) and that \( f_{*u_0}|_{T_{u_0}^{\widetilde{M^c}}} \) coincides with the parallel translation along a curve in \( M^c \) starting from \( u_0 \) and terminating to \( u \) with respect to the normal connection of \( \widetilde{M^c} \).

**Proof.** Take a sequence \( \{u_0, u_1, \ldots, u_k (= u)\} \) of \( Q(u_0) \) such that, for each \( i \in \{0, 1, \ldots, k - 1\} \), \( u_i \) and \( u_{i+1} \) are some complex curvature sphere \( S_i^c \) of \( M^c \). Furthermore, for each \( i \in \{0, 1, \ldots, k - 1\} \), we take the geodesic \( \gamma_i : [0, 1] \to S_i^c \) with \( \gamma_i(0) = u_i \) and \( \gamma_i(1) = u_{i+1} \), and define a holomorphic isometry \( F_{\gamma_i} \) of \( V \) as in (3.1). Set \( f := F_{\gamma_{k-1}} \circ \cdots \circ F_{\gamma_1} \circ F_\gamma \). According to the definition of \( F_{\gamma_i} (i = 0, 1, \ldots, k - 1) \) and Proposition 3.5, \( f \) preserves \( \widetilde{M^c} \) invariantly and the restriction of \( f_{*u_0} \) to \( T_{u_0}^{\widetilde{M^c}} \) coincides with the parallel translation with respect to the normal connection of \( \widetilde{M^c} \). Also, since \( f \) preserves complex curvature spheres invariantly, it is shown that \( f \) preserves \( Q(u_0) \) invariantly. Thus \( f \) is the desired holomorphic isometry.

q.e.d.

By using Propositions 3.2 and 3.6, we shall prove the homogeneity of \( \widetilde{M^c} \).

**Proposition 3.7.** \( \widetilde{M^c} \) is extrinsically homogeneous.

**Proof.** Take any \( \widehat{u} \in \widetilde{M^c} \). Since \( Q(u_0) = \widetilde{M^c} \) by Proposition 3.2, there exists a sequence \( \{u_k\}_{k=1}^{\infty} \) in \( Q(u_0) \) with \( \lim_{k \to \infty} u_k = \widehat{u} \). According to Proposition 3.6, for each \( k \in \mathbb{N} \), there
exists a holomorphic isometry $f_k$ of $V$ with $f_k(u_0) = u_k$, $f_k(\widetilde{M}^c) = \widetilde{M}^c$, $f_k(Q(u_0)) = Q(u_0)$ and $f_k(L_{u_0}^{E_i}) = L_{u_k}^{E_i}$ $(i \in I)$.

(Step I) First we shall show that, for each $i \in I$, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ such that $\{f_{kj}|_{L_{u_0}^{E_i}}\}_{j=1}^{\infty}$ pointwisely converges to some holomorphic isometry of $L_{u_0}^{E_i}$ onto $L_{u_k}^{E_i}$. Let $(L_{u_0}^{E_i})_R$ be the compact real of the complex sphere $L_{u_0}^{E_i}$ satisfying $(T_{u_0}(L_{u_0}^{E_i}))_R, JT_{u_0}(L_{u_0}^{E_i}))_R = 0$. Also, let $(L_{u_k}^{E_i})_R$ and $(L_{u_k}^{E_i})_R$ be the same kind of compact reals of $L_{u_k}^{E_i}$ and $L_{u_k}^{E_i}$, respectively. Clearly we have $f_k((L_{u_0}^{E_i}))_R) = (L_{u_k}^{E_i})_R$. Let $\widetilde{M}^c/\mathcal{F}_i$ be the leaf space of the foliation $\mathcal{F}_i$ consisting of the integral manifolds of $E_i$ and $\psi_i : M^c \to \widetilde{M}^c/\mathcal{F}_i$ be the quotient map. Take a sufficiently small tubular neighborhood $U$ of $L_{u_i}^{E_i}$ in $V$ such that $L_{u_i}^{E_i} \subset U$ for each $u \in U$. Let $m_i := \dim E_i$. Take a base $\{e_1, \cdots, e_{m_i}\}$ of $T_{u_0}(L_{u_0}^{E_i})_R$ such that the norms $\|e_1\|, \cdots, \|e_{m_i}\|$ are sufficiently small and $\tilde{u}_a := \exp(e_a)$ $(a = 1, \cdots, m_i)$, where $\exp$ is the exponential map of $(L_{u_0}^{E_i})_R$. Since $\lim_{k \to \infty} f_k(u_0) = \tilde{u}$ and $(L_{u_0}^{E_i})_R$'s $(u \in U)$ are compact, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ such that $\{f_{kj}(\tilde{u}_a)\}_{j=1}^{\infty}$ converges pointwisely to some holomorphic isometry $\tilde{f}$ of $(L_{u_0}^{E_i})_R$ onto $(L_{u_k}^{E_i})_R$ satisfying $\tilde{f}(u_0) = \tilde{u}$ and $\tilde{f}(\tilde{u}_a) = \tilde{u}_a$ $(a = 1, \cdots, m_i)$. It is clear that $\tilde{f}$ is uniquely extended to a holomorphic isometry of $L_{u_0}^{E_i}$ onto $L_{u_k}^{E_i}$. Denote by $f$ this holomorphic extension. It is easy to show that $\{f_{kj}|_{L_{u_0}^{E_i}}\}_{j=1}^{\infty}$ pointwisely converges to $f$.

(Step II) Next we shall show that, for each fixed $w \in Q(u_0)$, there exists a subsequence $\{f_{kj}\}_{j=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ such that $\lim_{j \to \infty} f_{kj}(w)$ exists. There exists a sequence $\{u_1, \cdots, u_m(=w)\}$ in $Q(u_0)$ such that, for each $j \in \{1, \cdots, m\}$, $u_j$ is contained in some complex curvature sphere $L_{u_{j-1}}^{E_j}$. For simplicity, assume that $m = 3$. From the fact in Step I, there exists a subsequence $\{f_{kj}\}_{j=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ (in Step I) such that $\{f_{kj}|_{L_{u_0}^{E_j}}\}_{j=1}^{\infty}$ pointwisely converges to some holomorphic isometry $f^1$ of $L_{u_0}^{E_j}$ onto $L_{u_k}^{E_j}$. Again, from the fact in Step I, there exists a subsequence $\{f_{kj}^2\}_{j=1}^{\infty}$ of $\{f_{kj}^1\}_{j=1}^{\infty}$ such that $\{f_{kj}^2|_{L_{u_1}^{E_j}}\}_{j=1}^{\infty}$ pointwisely converges to some holomorphic isometry $f^2$ of $L_{u_1}^{E_j}$ onto $L_{u_k}^{E_j}$. Again, from the fact in step I, there exists a subsequence $\{f_{kj}^3\}_{j=1}^{\infty}$ of $\{f_{kj}^2\}_{j=1}^{\infty}$ such that $\{f_{kj}^3|_{L_{u_2}^{E_j}}\}_{j=1}^{\infty}$ pointwisely converges to some holomorphic isometry $f^3$ of $L_{u_2}^{E_j}$ onto $L_{u_k}^{E_j}$. In particular, we have
Also we have
\[ \lim_{j \to \infty} f_{k_j}^3(w) = f^3(w). \]

(Step III) Let \( W \) be the affine span of \( \tilde{M}^e \). Next we shall show that there exists a subsequence \( \{ f_{k_j} \}_{j=1}^\infty \) of \( \{ f_k \}_{k=1}^\infty \) such that \( \{ f_{k_j}|w \}_{j=1}^\infty \) pointwisely converges to some holomorphic isometry of \( W \). Take a countable subset \( B := \{ w_j \mid j \in \mathbb{N} \} \) of \( Q(u_0) \) with \( \overline{B} = Q(u_0) = (M^e) \). According to the fact in Step II, there exists a subsequence \( \{ f_{k_j} \}_{j=1}^\infty \) of \( \{ f_k \}_{k=1}^\infty \) such that \( \lim_{j \to \infty} f_{k_j}(w_1) \) exists. Again, according to the fact in Step II, there exists a subsequence \( \{ f_{k_j} \}_{j=1}^\infty \) of \( \{ f_k \}_{j=1}^\infty \) such that \( \lim_{j \to \infty} f_{k_j}(w_2) \) exists. In the sequel, we take subsequences \( \{ f_{k_j} \}_{j=1}^\infty \) (\( l = 3, 4, 5, \cdots \)) inductively. It is clear that \( \lim_{j \to \infty} f_{k_j}(w) \) exists for each \( l \in \mathbb{N} \), that is, \( \{ f_{k_j}|B \}_{j=1}^\infty \) pointwisely converges to some map \( f \) of \( B \) into \( \tilde{M}^e \). Since each \( f_{k_j} \) is a holomorphic isometry, \( f \) is extended to a holomorphic isometry of \( \tilde{M}^e \). Denote by \( \tilde{f} \) this extension. It is clear that \( \{ f_{k_j}|\tilde{M}^e \}_{j=1}^\infty \) pointwisely converges to \( \tilde{f} \). Furthermore, since each \( f_{k_j} \) is an affine transformation and hence the restriction \( f_{k_j}|W \) of \( f_{k_j} \) to \( W \) is a holomorphic isometry of \( W \), \( \tilde{f} \) is extended to a holomorphic isometry of \( W \).

Denote by \( \tilde{f} \) this holomorphic extension. It is clear that \( \{ f_{k_j}|W \}_{j=1}^\infty \) pointwisely converges to \( \tilde{f} \).

(Step IV) Let \( f \), \( \tilde{f} \) and \( \tilde{f} \) be as in Step III. It is clear that \( \tilde{f} \) is extended to a holomorphic isometry of \( V \). Denote by \( \hat{f} \) this extension. We have
\[ \hat{f}(\tilde{M}^e) = \tilde{f}(\tilde{M}^e) = \tilde{M}^e. \]
Also we have
\[ \hat{f}(u_0) = f(u_0) = \lim_{j \to \infty} f_{k_j}(u_0) = \lim_{j \to \infty} u_{k_j} = \hat{u}. \]
Let \( H \) be the closed subgroup of \( I_h(V) \) generated by all holomorphic isometries of \( V \) preserving \( \tilde{M}^e \) invariantly, where \( I_h(V) \) is the holomorphic isometry group of \( V \). Then it follows from the above fact (together with the arbitrariness of \( \hat{u} \)) that \( H \cdot u_0 = \tilde{M}^e \).
\[ \text{q.e.d.} \]

4 Proofs of Theorem A and Corollary B

In this section, we shall prove Theorem A and Corollary B. Let \( M \) be an irreducible proper complex equifocal submanifold in a symmetric space \( G/K \) of non-compact type. Assume that the codimension of \( M \) is greater than one. Let \( M^e \) be the (complete extrinsic) complexification of \( M \), \( \tilde{M}^e := \pi^e(M^e) \) and \( \tilde{M}^e := (\pi^e \circ \phi^e)^{-1}(M^e) \), where \( \pi^e \) is the natural
projection of $G^c$ onto $G^c/K^c$ and $\phi^c$ is the parallel transport map for $G^c$. Without loss of
generality, we may assume that $K^c$ is connected and that $G^c$ is simply connected. Hence
both $\tilde{M}^c$ and $\tilde{M}^c$ are connected. For simplicity, set $V := H^0([0,1], G^c)$. According to
Proposition 3.7, $\tilde{M}^c$ is extrinsically homogeneous, that is, there exists a closed connected
subgroup $H$ of the anti-Kaehlerian transformation group (i.e., the holomorphic isometry
group) $I_h(V)$ of $V$ having $\tilde{M}^c$ as an orbit. Let $\rho : H^1([0,1], G^c) \to I_h(V)$ be the repre-
sentation of $H^1([0,1], G^c)$ defined by $\rho(g) := g \cdot (g \in H^1([0,1], G^c))$. In the proof of
Theorem A, it is key to show the following fact.

**Proposition 4.1.** The above group $H$ is a subgroup of $\rho(H^1([0,1], G^c))$.

To prove this proposition, we prepare some lemmas. Let $\mathcal{K}^h$ be the Lie algebra of all
holomorphic Killing fields on $V$ (i.e., the Lie algebra of $I_h(V)$) and $\mathcal{K}^h_{\tilde{M}^c}$ be the Lie algebra
of all holomorphic Killing fields on $V$ which are tangent to $\tilde{M}^c$ along $\tilde{M}^c$. For $K \in \mathcal{K}^h$,
we define a map $F_K : \Omega_e(G^c) \to g^c$ by $F_K(g) := \phi^c_{\psi_0}((g \ast \cdot)_{\ast} K)_{\psi_0}$. For this map $F_K$, we
have the following fact.

**Lemma 4.1.1.** (i) For $g \in \Omega_e(G^c)$, $F_K(g) = \int_0^1 \text{Ad}(g)(K_{g^{-1} \ast \psi_0}) dt$.

(ii) If $K \in \mathcal{K}^h_{\tilde{M}^c}$, then the image of $F_K$ is contained in $T_e \tilde{M}^c$.

**Proof.** Let $\{\psi_s\}_{s \in \mathbb{R}}$ be the one-parameter transformation group associated with $K$. For
each $g \in \Omega_e(G^c)$, we have

$$(g \ast \cdot)_{\ast} K)_{\psi_0} = \frac{d}{ds}|_{s=0} g \ast (\psi_s(g^{-1} \ast \psi_0))$$

$$= \frac{d}{ds}|_{s=0} (\text{Ad}(g)(\psi_s(g^{-1} \ast \psi_0)) - g'g^{-1}) = \text{Ad}(g)(K_{g^{-1} \ast \psi_0}).$$

On the other hand, we have $\phi^c_{\psi_0}(u) = \int_0^1 u(t) dt \ (u \in T_e \tilde{M}^c)$ (see Lemma 6 of [Koi2]).
Hence we obtain the relation in (i). Since $g \in \Omega_e(G^c)$, it maps each fibre of $\phi^c$ to oneself.
Hence, if $K \in \mathcal{K}^h_{\tilde{M}^c}$, then $(g \ast \cdot)_{\ast} K \in \mathcal{K}^h_{\tilde{M}^c}$. In particular, we have
$((g \ast \cdot)_{\ast} K)_{\psi_0} \in T_e \tilde{M}^c$. Therefore we obtain $F_K(g) \in \phi^c_{\psi_0}(T_e \tilde{M}^c) = T_e \tilde{M}^c$. q.e.d.

For $v \in H^1([0,1], G^c)$, we define a vector field $K^v$ on $V$ by $({K^v})_u := [v, u] - v' \ (u \in V)$. Let $\{\exp sv \mid s \in \mathbb{R}\}$ be the one-parameter subgroup of $H^1([0,1], G^c)$ associated with $v$. Then the holomorphic Killing field associated with the one-parameter transformation
group $\{\rho(\exp sv) \mid s \in \mathbb{R}\} \text{ of } V$ is equal to $K^v$. Thus we have $K^v \in \mathcal{K}^h$. For $K^v$, we have
the following fact.

**Lemma 4.1.2.** The map $F_{K^v}$ is a constant map.
Proof. Since \( \rho(\exp sv) \) maps the fibres of \( \phi^c \) to them, for \( u_1, u_2 \in V \) with \( \phi^c(u_1) = \phi^c(u_2) \), we have \( \phi^c(\rho(\exp sv)(u_1)) = \phi^c(\rho(\exp sv)(u_2)) \) and hence \( \phi^c_*(\phi^c)^t(u_1) = \phi^c_*(\phi^c)^t(u_2) \). Take \( g_1, g_2 \in \Omega_\epsilon(G^c) \). Since \( g_i \cdot \cdot \cdot \) maps each fibre of \( \phi^c \) to oneself, we have \( F_{K^v}(g_i) = \phi^c_{\ast 0}((\rho(g_i)_\ast (K^v))_0) = \phi^c_{\ast 0}((K^v)_{g_i^{-1}1}) \). Also, we have \( \phi^c(g_1^{-1}10) = \phi^c(g_2^{-1}10)(e) \). Hence we have \( \phi^c_{\ast 0}((K^v)_{g_1^{-1}1}) = \phi^c_{\ast 0}((K^v)_{g_2^{-1}1}) \). Therefore we obtain \( F_{K^v}(g_1) = F_{K^v}(g_2) \). Thus \( F_{K^v} \) is a constant map. q.e.d. 

Also we have the following fact for \( F_K \).

**Lemma 4.1.3.** (i) The map \( K \mapsto F_K \) is linear.

(ii) \( F_K[g_1g_2] = F_{\rho(g_2)} \cdot K[g_1] \) \((g_1, g_2 \in \Omega_\epsilon(G^c) \)).

(iii) \( (dF_K)_g \circ (dR_g)_\varepsilon = (dF_{\rho(g)} \cdot K)_\varepsilon \) \((g \in \Omega_\epsilon(G^c) \)).

(iv) If \( K_u = Au + b \) \((u \in V) \) for some linear transformation \( A \) of \( V \) and some \( b \in V \), then \( (dF_K)_\varepsilon(u) = \int_0^1 (A + \text{ad}(b))u' \, dt \), where \( \tilde{b}(t) := \int_0^1 b(t) \, dt \).

(v) If \( \overline{K} = K + K^v \), then \( F_{\overline{K}} = F_K \) is a constant map.

Proof. The statements (i) \~ (iii) are trivial. The statement (iv) is shown by imitating the proof of Theorem 2.2 of [Ch]. For \( u \in \Omega_\epsilon(G^c) \subset H^1([0, 1], \mathfrak{g}^c) \), it follows from (iv) that

\[
(d(F_{\overline{K}} - F_K))_\varepsilon(u) = (dF_{K^v})_\varepsilon(u) = \int_0^1 (\text{ad}(v) + \text{ad}(-v))u' \, dt = 0.
\]

This implies that \( F_{\overline{K}} - F_K \) is a constant map. Thus the statement (v) follows. q.e.d. 

By imitating the proof of Theorem 2.2 of [Ch], we can show the following fact in terms of Lemmas 4.1.1\~4.1.3.

**Lemma 4.1.4.** Let \( K \) be a holomorphic Killing field on \( V \) given by \( K_u := [v, u] - b \) \((u \in V) \) for some \( v, b \in V \). If \( K \in K_{\mathcal{E}}^b \), then we have \( v \in H^1([0, 1], \mathfrak{g}^c) \) and \( b = v' \).

Proof. First we consider the case where \( G^c \) is simple. Set \( \overline{K} := K - K\tilde{b} \) and \( w := v - \tilde{b} \), where \( \tilde{b}(t) = \int_0^t b(t) \, dt \). From \( K = \text{ad}(w) \), we have

\[
(\rho(g)_\ast \overline{K})_u = (\text{Ad}(g)\overline{K})_u = \text{Ad}(g)([w, g^{-1}u]) = [\text{Ad}(g)w, u - g\cdot 0] \quad (u \in V).
\]
From this relation and (i) of Lemma 4.1.1, we have

\[
(dF_{\rho(g),\overline{K}})\varepsilon(u) = \frac{d}{ds}|_{s=0} F_{\rho(g),\overline{K}}(\exp su)
= \frac{d}{ds}|_{s=0} \int_{0}^{1} \text{Ad}(\exp su)((\rho(g),\overline{K})\exp(-su)=0)dt
= \int_{0}^{1} \text{ad}(u)(\text{Ad}(g\overline{K}))_{u}dt
= [[u, (\text{Ad}(g\overline{K}) \circ u)]_{\overline{0}} - \int_{0}^{1} [u', (\text{Ad}(g\overline{K}) \circ u)]dt
= \int_{0}^{1} [(\text{Ad}(g\overline{K}) \circ u, u')dt
= \int_{0}^{1} [[\text{Ad}(g)w, u] - [\text{Ad}(g)w, g \ast \overline{0}], u']dt
(4.1)
\]

for \( u \in V \). For simplicity, set \( \eta := [\text{Ad}(g)w, u] - [\text{Ad}(g)w, g \ast \overline{0}] \). According to (ii) of Lemma 4.1.1, we have \( \text{Im} F_{\overline{K}} \subset T_{\phi}M^{c} \) and hence \( \dim(\text{Span Im} F_{\overline{K}}) \leq \dim T_{\phi}M^{c} < \dim g^{c} \). Since \( F_{\overline{K}} - F_{K} \) is a constant map by (v) of Lemma 4.1.3, we have \( \dim(\text{Span Im} F_{\overline{K}}) = \dim(\text{Span Im} F_{\overline{K}}) < \dim g^{c} \), that is, \( g^{c} \cap \text{Span Im} F_{\overline{K}} \neq \{0\} \). Take \( X(\neq 0) \in g^{c} \cap \text{Span Im} F_{\overline{K}} \). Take any \( g \in \Omega_{\epsilon}(G^{c}) \) and any \( u \in T_{\phi}(\Omega_{\epsilon}(G^{c})) \). By using (iii) of Lemma 4.1.3 and (4.1), we have

\[
\langle (dF_{\overline{K}})_{g}((dR_{g})\varepsilon(u)), X \rangle^{A} = \langle (dF_{\rho(g),\overline{K}})\varepsilon(u), X \rangle^{A}
= \int_{0}^{1} \langle \eta, u' \rangle, X \rangle^{A}dt = - \int_{0}^{1} \langle u', [\eta, X] \rangle^{A}dt = - \langle u', [\eta, X] \rangle^{0} = 0,
\]

where \( \langle \, \rangle^{A} \) and \( \langle \, \rangle^{0} \) are as in Section 2. From the arbitrariness of \( u \), it follows that \( [\eta, X] \) is horizontal with respect to \( \phi^{c} \). Since \( G^{c} \) has no center, there exists \( Y \in g^{c} \) with \( [X, Y] \neq 0 \). Set \( Z := [X, Y] \). Since \( [\eta, X] \) is horizontal, it is a constant path. Hence it follows from \( \langle [\eta, Z]^{A} = \langle [\eta, X], Y \rangle^{A} \) that \( [\eta, Z]^{A} \) is constant. By differentiating \( \langle [\eta, Z]^{A} \) with respect to \( t \), we have

\[
\langle [\text{Ad}(g)w, u], Z \rangle^{A} = \langle [\text{Ad}(g)w, g \ast \overline{0}], Z \rangle^{A} \quad (u \in V).
\]

Since \( g^{c} \) has no center, there exists \( W \in g^{c} \) with \( [Z, W] \neq 0 \). Since \( G^{c} \) is simple, \( \text{Ad}(G^{c})[Z, W] \) is full in \( g^{c} \). Hence there exist \( h_{1}, \ldots, h_{2m} \in G^{c} \) such that \( \{\text{Ad}(h_{1})[Z, W], \ldots, \text{Ad}(h_{2m})[Z, W]\} \) is a base of \( g^{c} \) (regarded as a real Lie algebra), where \( m := \dim_{C}G^{c} \). For a sufficiently small \( \epsilon > 0 \), we take \( g_{i} \in H^{1}([0, 1], G^{c}) \) with \( g_{i}|_{(\epsilon, 1-\epsilon)} = h_{i} \) \( (i = 1, \ldots, 2m) \). From (4.2), we have

\[
\langle w, \text{Ad}(h_{i})[Z, W] \rangle^{A}|_{(\epsilon, 1-\epsilon)} = \langle w, \text{Ad}(g_{i})[Z, W] \rangle^{A}|_{(\epsilon, 1-\epsilon)}
= -\langle [\text{Ad}(g_{i}^{-1})w, W], Z \rangle^{A}|_{(\epsilon, 1-\epsilon)} = -\langle [\text{Ad}(g_{i}^{-1})w, g_{i}^{-1} \ast \overline{0}], Z \rangle^{A}|_{(\epsilon, 1-\epsilon)}
= \langle [\text{Ad}(h_{i}^{-1})(w|_{(\epsilon, 1-\epsilon)}), (g_{i}^{-1})'(g_{i}^{-1})^{-1}|_{(\epsilon, 1-\epsilon)}], Z \rangle^{A} = 0.
\]

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In particular, we have \( \langle w, \text{Ad}(h_i)[Z, W] \rangle^A \mid_{\varepsilon = 1} = 0 \) \((i = 1, \ldots, 2m)\). From the arbitrariness of \(\varepsilon\), it follows that \(\langle w, \text{Ad}(h_i)[Z, W] \rangle^A = 0 \) \((i = 1, \ldots, 2m)\). Hence we obtain \(w = 0\), that is, \(v = \tilde{b} \in H^1([0, 1], g^c)\) (hence \(b = v')\).

Next we consider the case where \(G^c\) is not simple. Let \(G^c = G^c_1 \times \cdots \times G^c_k\) be the irreducible decomposition of \(G^c\) and \(g_i^c\) be the Lie algebra of \(G^c_i\) \((i = 1, \ldots, k)\). Let \(g^c_K\) be the maximal ideal of \(g^c\) such that the orthogonal projection of \(w = v - \tilde{b}\) onto the ideal is a constant path, where we note that any ideal of \(g^c\) is equal to the direct sum of some \(g_i^c\)'s and hence it is a non-degenerate subspace with respect to \(\langle , \rangle^A\). Let \(g^c_K\), \(\tilde{g}^c_{\tilde{K}}\) be the ideal corresponding to \(g^c_K\), \(\tilde{g}^c_{\tilde{K}}\) defined for \(\tilde{K}\). Since \(\tilde{K}_u = [v - \tilde{b}, u]\), we have \(g^c_{\tilde{K}} = g^c_K\). Now we shall show

\[(g^c_K)^\perp \subset T_{e\tilde{M}^c},\]

where \((g^c_K)^\perp\) is the orthogonal complement of \(g^c_K\) in \(g^c\) with respect to \(\langle , \rangle^A\). Let \(V_i := \text{Span } g^c_{\tilde{K}}\) \((i = 1, \ldots, k)\). It is clear that \(V = V_1 \oplus \cdots \oplus V_k\) (orthogonal direct sum).

The Killing field \(K\) is decomposed into \(K = K_1 + \cdots + K_k\), where \(K_i\) is a Killing field on \(V_i\) \((i = 1, \ldots, k)\). For \(g \in \Omega_c(G^c_i)\), we have

\[F_{K_i}(g) = \int_0^1 \text{Ad}(g)(\langle (K_i)_{g^{-1}t}, \tilde{b} \rangle)dt = \int_0^1 \text{Ad}(g)(\langle (K_i)_{g^{-1}t}, \tilde{b} \rangle)dt = F_{K_i}(g)\]

by (i) of Lemma 4.1.1 and \(\text{Ad}(g)K_j = 0\) \((j \neq i)\). Thus we have \(\text{Span } F_{K_i} = \text{Span } K_i \subset \Omega_c(G^c_i) \subset g^c_i\).

From this relation and (i) of Lemma 4.1.3, we have

\[
\text{Span } \text{Im } F_{K_i} = \text{Span } (\text{Im } F_{K_1} + \cdots + \text{Im } F_{K_k}) = \bigoplus_{i=1}^k \text{Span } \text{Im } F_{K_i}.
\]

Let \(v = \sum_{i=1}^k v_i\) and \(\tilde{b} = \sum_{i=1}^k \tilde{b}_i\), where \(v_i, \tilde{b}_i \in V_i\) \((i = 1, \ldots, k)\). Since \(F_{K_i}(g) = \int_0^1 \text{Ad}(g)[v_i - \tilde{b}_i, g^{-1}t, 0]dt\) \((g \in \Omega_c(G^c_i))\), \(\text{Im } F_{K_i} = 0\) when \(v_i - \tilde{b}_i = 0\) and \(\text{Im } F_{K_i} = g^c_i\) when \(v_i - \tilde{b}_i \neq 0\). Therefore we have \(\text{Span } \text{Im } F_{K_i} = \bigoplus_{i \in \{1, \ldots, k\} \text{ s.t. } v_i - \tilde{b}_i = 0} g^c_i\). On the other hand, since \(g_i^c\) is irreducible, \((g_i^c)^\perp = g_i^c\) when \(v_i - \tilde{b}_i = 0\) and \((g_i^c)^\perp = \{0\}\) when \(v_i - \tilde{b}_i \neq 0\). Hence we have \(g^c_{\tilde{K}} = \bigoplus_{i \in \{1, \ldots, k\} \text{ s.t. } v_i - \tilde{b}_i = 0} g^c_i\). Therefore we obtain \(\text{Span } \text{Im } F_{\tilde{K}} = (g^c_{\tilde{K}})^\perp\). Since \(F_{\tilde{K}} - F_K\) is a constant map by (v) of Lemma 4.1.3 and \(0 \in \text{Im } F_{\tilde{K}}\), we have \(\text{Span } F_{\tilde{K}} \subset \text{Span } F_K\).

Hence \((g^c_{\tilde{K}})^\perp = (g^c_{\tilde{K}})^\perp \subset T_{e\tilde{M}^c}\) follows from (ii) of Lemma 4.1.1. Thus (4.3) is shown.

Next we shall show that \((R_g)^*(g^c_{\tilde{K}})^\perp \subset T_{g\tilde{M}^c}\) for any \(g \in \tilde{M}^c\). Fix \(g \in \tilde{M}^c\). Define \(\tilde{g} \in H^1([0, 1], G^c)\) with \(\tilde{g}(0) = e\) and \(\tilde{g}(1) = g\) by \(\tilde{g}(t) := \text{exp}_{c \tilde{M}} tX\) for some \(X \in g^c\). Since \(\phi^c \circ \rho(\tilde{g}) = R_g^{-1} \circ \phi^c\), we have \((\phi^c)^{-1}(R_g^{-1} (\tilde{M}^c)) = \rho(\tilde{g})(M^c)\). Since \(\rho(\tilde{g})_* K\) is tangent to
\[ \rho(\hat{g})(\hat{M}^c), \text{ in similar to } (4.3), \text{ we have} \]
\[ (g^c_{\rho(\hat{g}), K})^{\perp} \subset T_e R_{\hat{g}}^{-1}(\hat{M}^c) = (R_{\hat{g}})^{-1}(T_{\hat{g}}\hat{M}^c). \]

We have
\[ (\rho(\hat{g}), K)_u = (\text{Ad}(\hat{g})K)_u = [\text{Ad}(\hat{g})v, u] - [\text{Ad}(\hat{g})v, \hat{g} \ast \hat{0}] - \text{Ad}(\hat{g})b. \]

Denote by \( \text{pr}_{g^c_K} \) the orthogonal projection of \( g^c \) onto \( g^c_K \). By noticing that \( \text{Ad}(\hat{g}) \) preserves each \( g^c_v \) (hence \( g^c_K \)) invariantly and that \( \hat{g} \ast \hat{0} = -X = -\text{Ad}(\hat{g})X \), we have
\[ \frac{d}{dt} \text{pr}_{g^c_K} \left( \text{Ad}(\hat{g})v - [\text{Ad}(\hat{g})v, \hat{g} \ast \hat{0}] - \text{Ad}(\hat{g})b \right) = \frac{d}{dt} \text{pr}_{g^c_K} \left( \text{Ad}(\hat{g}) (v - \hat{b}) + \text{Ad}(\hat{g})\hat{b} - [\text{Ad}(\hat{g})v, \hat{g} \ast \hat{0}] - \text{Ad}(\hat{g})b \right) = \text{Ad}(\hat{g})[X, \text{pr}_{g^c_K}(v - \hat{b})] + \text{Ad}(\hat{g})[X, \text{pr}_{g^c_K}(\hat{b})] + \text{Ad}(\hat{g})\text{pr}_{g^c_K}(b) + \text{pr}_{g^c_K}[\text{Ad}(\hat{g})v, X] - \text{Ad}(\hat{g})\text{pr}_{g^c_K}(b) = (\text{pr}_{g^c_K} \circ \text{Ad}(\hat{g}))(\{X, v - \hat{b}\} + [X, \hat{b}] + [v, X]) = 0. \]

Thus \( \text{pr}_{g^c_K} (\text{Ad}(\hat{g})v - [\text{Ad}(\hat{g})v, \hat{g} \ast \hat{0}] - \text{Ad}(\hat{g})) \) is a constant path. This fact together with (4.5) implies \( g^c_K \subset g^c_{\rho(\hat{g}), K} \). By exchanging the roles of \( K \) and \( (\hat{g} \ast \cdot), K \), we have \( g^c_{\rho(\hat{g}), K} \subset g^c_K \). Thus we obtain \( g^c_K = g^c_{\rho(\hat{g}), K} \). Therefore \( (R_{\hat{g}})_* (g^c_K)^{\perp} \subset T_{\hat{g}}\hat{M}^c \) follows from (4.4).

Since \( (R_{\hat{g}})_* (g^c_K)^{\perp} \subset T_{\hat{g}}\hat{M}^c \) for any \( g \in \hat{M}^c \) and \( g^c_K \) is an ideal of \( g^c \), we have \( \hat{M}^c = \hat{M}^c \times G^c_K \subset G^c_K \times G^c_K (= G^c) \), where \( G^c_K := \text{exp}_{G^c}(g^c_K) \) and \( G^c_K := \text{exp}_{G^c}((g^c_K)^{\perp}) \).

Since \( \hat{M}^c \) is irreducible and \( \dim \hat{M}^c < \dim G^c \), we have \( (g^c_K)^{\perp} = \{0\} \), that is, \( g^c_K = g^c \).

This implies that \( v - \hat{b} \) is a constant path. Therefore we obtain \( b = v' \).

Let \( \mathfrak{o}_{AK}(V) \) be the Lie algebra of all continuous skew-symmetric complex linear transformations of \( (V, \langle \cdot, \cdot \rangle^A, \hat{J}) \). Let \( \mathfrak{g} = \mathfrak{f} + \mathfrak{p} \) be the Cartan decomposition, \( \mathfrak{g}^c := \mathfrak{f} + \sqrt{-1}\mathfrak{p}, \mathfrak{g}^c_+ := \sqrt{-1}\mathfrak{f} + \mathfrak{p} \) and \( H^0_{\pm, c} := H^0([0, 1], \mathfrak{g}^c_+) \). Give \( \mathfrak{o}_{AK}(V) \) the operator norm topology with respect to \( \langle \cdot, \cdot \rangle_{A, H^0_{\pm, c}} \). Any holomorphic Killing field \( K \) on \( V \) is described as \( K_u = A v + b \) for some \( A \in \mathfrak{o}_{AK}(V) \) and some \( b \in V \). Thus \( \mathcal{K}^h \) is identified with \( \mathfrak{o}_{AK}(V) \times V \). Give \( \mathcal{K}^h = \mathfrak{o}_{AK}(V) \times V \) the product topology of the above topology on \( \mathfrak{o}_{AK}(V) \) and the original topology of \( V \).

**Lemma 4.1.5.** The set \( \mathcal{K}^h_{\hat{M}^c} \) is closed in \( \mathcal{K}^h \).

**Proof.** Denote by \( \overline{\mathcal{K}^h_{\hat{M}^c}} \) the closure of \( \mathcal{K}^h_{\hat{M}^c} \) in \( \mathcal{K}^h \). Take \( K \in \overline{\mathcal{K}^h_{\hat{M}^c}} \). Then there exists a sequence \( \{K_n\}_{n=1}^{\infty} \) in \( \mathcal{K}^h_{\hat{M}^c} \) with \( \lim_{n \to \infty} K_n = K \). Let \( (K_n)_u = A_n v + b_n \) and \( K_u = A v + b \),
Therefore we have \( \lim_{n \to \infty} K_n = K \), we have \( \lim_{n \to \infty} A_n = A \) and hence \( \lim_{n \to \infty} A_n u = Au \) \((u \in V)\). Also, \( \lim_{n \to \infty} b_n = b \) holds. Hence we have \( \lim_{n \to \infty} (K_n)_u = K_u \) \((u \in V)\). For each \( u \in \tilde{M}^c \), denote by \( \text{pr}_u^\perp \) the orthogonal projection of \( V \) onto \( T_u^\perp \tilde{M}^c \). Since \( \dim T_u^\perp \tilde{M}^c < \infty \), \( \text{pr}_u^\perp \) is a compact operator. Hence, since \( \text{pr}_u^\perp ((K_n)_u) = 0 \) for all \( n \), we obtain \( \text{pr}_u^\perp (K_u) = 0 \). From the arbitrariness of \( u \), \( K \in \mathcal{K}_M^h \) follows. Therefore we obtain \( \mathcal{K}_M^h = \mathcal{K}_M^h \). q.e.d.

Take \( v \in V \) and set \( \tilde{v}(t) := \int_0^t v(t)dt \). Define \( g_n \in H^1([0, 1], G^c) \) \((n \in \mathbb{N})\) by \( g_n(t) := \exp_{G^c}(n\tilde{v}(t)) \) and \( K_n^v \in \mathcal{K}_h \) \((n \in \mathbb{N})\) by \( K_n^v := \frac{1}{n} \rho(g_n)_* K \). Let \( K_u = Au + b \) and \( (K_n^v)_u = A_n^v u + b_n^v \). Then we have

\[
(K_n^v)_u = \frac{1}{n} \text{Ad}(g_n) (K_{g_n^{-1} u}) = \frac{1}{n} \text{Ad}(g_n) (A(g_n^{-1} u) + b) \\
= \frac{1}{n} \text{Ad}(g_n) \circ A \circ (g_n^{-1} u) + \frac{1}{n} \text{Ad}(g_n) (A(g_n^{-1} \hat{0}) + b)
\]

and hence

\[
A_n^v = \frac{1}{n} \text{Ad}(g_n) \circ A \circ (g_n^{-1} u) \text{ and } b_n^v = \frac{1}{n} \text{Ad}(g_n) (A(g_n^{-1} \hat{0}) + b).
\]

For \( \{K_n^v\}_{n=1}^\infty \), we have the following fact.

**Lemma 4.1.6.** If \( K \in \mathcal{K}_M^h \) and \( v \in H_0^0 \) with \( \int_0^1 v(t)dt = 0 \), then there exists a subsequence of \( \{K_n^v\}_{n=1}^\infty \) converging to the zero vector field.

**Proof.** Take \( u \in V \). Let \( u = u_- + u_+ \) \((u_- \in g_-, u_+ \in g_+)\). Then we have

\[
\langle \text{Ad}(g_n) u_\pm \rangle(t) = \text{Ad}(\exp_{G^c}(n\tilde{v}(t))) u_\pm (t) = \exp(\text{ad}(n\tilde{v}(t))) u_\pm (t) \in g_\pm
\]

because \( \tilde{v}(t) \in g^c \) by the assumption. Hence we have

\[
\langle \text{Ad}(g_n) u_\pm \rangle(t) = \langle \text{Ad}(g_n) u_\pm \rangle_0^A = -(\langle \text{Ad}(g_n) u_- \rangle_0^A + \langle \text{Ad}(g_n) u_+ \rangle_0^A) = -(u_- u_0^A + u_+ u_0^A) = \langle u_0 \rangle_0^A.
\]

Therefore we have \( \|A_n^v\|_{\text{op}} = \frac{1}{n} \|A\|_{\text{op}} \to 0 \) \((n \to \infty)\) and

\[
\|b_n^v\|_{0, H_0^0} \leq \frac{1}{n} \left( \|\text{Ad}(g_n^{-1} \hat{0})\|_{0, H_0^0} + \|b\|_{0, H_0^0} \right) = \|A\tilde{v}\|_{0, H_0^0} \to \|A\tilde{v}\|_{0, H_0^0} \quad (n \to \infty),
\]

where \( \| \cdot \|_{\text{op}} \) is the operator norm of \( \rho(V) \) with respect to \( \langle , \rangle^A_{0, H_0^0} \) and \( \| \cdot \|_{0, H_0^0} \) is the norm of \( V \) associated with \( \langle , \rangle^A_{0, H_0^0} \). Since \( \{K_n^v\}_{n=1}^\infty \) is bounded, there exists a
convergent subsequence \( \{K_{n_j}^v\}_{j=1}^\infty \) of \( \{K_n^v\}_{n=1}^\infty \). Set \( K_0 := \lim_{n \to \infty} K_n^v \). From \( \lim_{n \to \infty} A_n^v = 0 \), \( K_0 \) is a parallel Killing field on \( V \). From \( \int_0^1 v(t)dt = 0 \), we have \( g_n \in \Omega_c(G^c) \) and hence \( \rho(g_n)(M^c) = \bar{M}^c \). This fact together with \( K \in \mathcal{K}_{M^c}^h \), deduces \( K_n^v \in \mathcal{K}_{M^c}^h \). Hence have \( K_0 \in \mathcal{K}_{M^c}^h \). According to Lemma 4.1.5, we have \( \mathcal{K}_{M^c}^h = \mathcal{K}_{M^c}^h \). Therefore we have \( K_0 \in \mathcal{K}_{M^c}^h \). Thus, since \( K_0 \) is parallel and \( K_0 \in \mathcal{K}_{M^c}^h \), it follows from Lemma 4.1.4 that \( K_0 = 0 \). This completes the proof.

q.e.d.

On the other hand, we have the following fact.

**Lemma 4.1.7.** Let \( K \in \mathcal{K}_{M^c}^h \) and \( f \in H^0([0, 1], C)(= H^0([0, 1], R^2)) \) with \( f(0) = f(1) \) and \( X \in \mathfrak{g}^c \). Then we have \( A(fX) = [X, v] \) for some \( v \in V \), where \( A \) is the linear part of \( K \).

**Proof.** Since \( (\mathfrak{g}^c)_c = \mathfrak{g}^c \), we suffice to show in the case where \( X \in \mathfrak{g}^c \). First we consider the case of \( f \in H^1([0, 1], C) \). Denote by \( f' \) the weakly derivative of \( f \). Let \( A(fX)(t) = u_1(t) + u_2(t) \) \( (u_1(t) \in \text{Ker ad}(X) \) and \( u_2(t) \in \text{Im ad}(X)) \), and \( u_i(t) = u_i^-(t) + u_i^+(t) \) \( (u_i^-(t) \in \mathfrak{g}^c, u_i^+(t) \in \mathfrak{g}^c_i) \) \( (i = 1, 2) \) and \( b(t) = b^-(t) + b^+(t) \) \( (b^-(t) \in \mathfrak{g}^c, b^+(t) \in \mathfrak{g}^c_i) \). Let \( g_n(t) := \exp_{G^c}(nfX) \). From (4.6) and \( \text{Ad}(g_n)|_{\text{Ker ad}(X)} = \text{id} \), we have

\[
\begin{align*}
\langle b_n^fX, u_1 \rangle^A_{0, H^c_{H^c}} &= \langle u_1, u_1 \rangle^A_{0, H^c_{H^c}} + \langle \text{Ad}(g_n)(u_2 + \frac{b}{n}), u_1 \rangle^A_{0, H^c_{H^c}} \\
&= \langle u_1, u_1 \rangle^A_{0, H^c_{H^c}} + \frac{1}{n} \langle b, u_1 \rangle^A_{0, H^c_{H^c}} + \langle u_1, u_1 \rangle^A_{0, H^c_{H^c}} \quad (n \to \infty).
\end{align*}
\]

According to Lemma 4.1.6, there exists a subsequence \( \{K_{n_j}^fX\}_{j=1}^\infty \) of \( \{K_n^fX\}_{n=1}^\infty \) converging to the zero vector field because of \( K \in \mathcal{K}_{M^c}^h \), \( f'X \in H^0_{H^c} \) and \( \int_0^1 (fX)(t)dt = (f(1) - f(0))X = 0 \). Since \( \lim_{j \to \infty} b_{n_j}^fX = 0 \), we have \( \langle u_1, u_1 \rangle^A_{0, H^c_{H^c}} = 0 \), that is, \( u_1 = 0 \). Thus we see that \( A(fX)(t) \in \text{Im ad}(X) \) holds for all \( t \in [0, 1] \). That is, we have \( A(fX) = [X, v] \) for some \( v \in V \). Next we consider the case where \( f \) is a general element of \( V \). Since \( H^1([0, 1], \mathfrak{g}^c) = V \), there exists a sequence \( \{f_n\}_{n=1}^\infty \) in \( H^1([0, 1], \mathfrak{g}^c) \) with \( \lim_{n \to \infty} f_n = f \). Then, since \( A \) is continuous, we have \( A(fX)(t) = \lim_{n \to \infty} A(f_nX)(t) \in \text{Im ad}(X) \) for all \( t \in [0, 1] \). That is, we have \( A(fX) = [X, v] \) for some \( v \in V \). This completes the proof.

q.e.d.
According to Lemma 2.10 of [Ch], we have the following fact.

**Lemma 4.1.8.** If $B : g^c \rightarrow g^c$ is a linear map of the form $BX = [\mu(X), X] \ (X \in g^c)$ for some map $\mu : g^c \rightarrow g^c$, then $\mu$ is a constant map.

By using lemmas 4.1.7 and 4.1.8, we can show the following fact.

**Lemma 4.1.9.** Let $A, f, X$ and $v$ be as in Lemma 4.1.7. Then $v$ is independent of the choice of $X$.

**Proof.** Denote by $v_{f,X}$ the above $v$. Define a linear map $B^1_{f} : g^c \rightarrow g^c$ by $B^1_{f}(X) := A(fX)(t)_{g^c} \ (X \in g^c)$ and a linear map $B^2_{X} : g^c \rightarrow g^c$ by $B^2_{X}(X) := \sqrt{-1}(A(fX)(t)_{g^c}) \ (X \in g^c)$, where $(\cdot)_{g^c}$ is the $g^c$-component of $(\cdot)$. Since $B^1_{f}(X) = [X, v_{f,X}(t)_{g^c}]$ and $B^2_{X}(X) = [X, \sqrt{-1}(v_{f,X}(t)_{g^c})]$, it follows from Lemma 4.1.8 that, for each $t \in [0,1]$, $v_{f,X}(t)_{g^c}$ and $v_{f,X}(t)_{g^c}$ are independent of the choice of $X \in g^c$. Hence $v_{f,X}$ is independent of the choice of $X \in g^c$. Since $g^c$ has no center, $v_{f,\sqrt{-1}X} = v_{f,X}$ for any $X \in g^c$. Therefore $v_{f,X}$ is independent of the choice of $X \in g^c$. q.e.d.

According to this lemma, for a fixed $K \in K^{h}_{M^c}$, the above $v$ depends on only $f$. So we denote this by $v_f$.

**Lemma 4.1.10.** For any $f \in H^0([0,1], C)$ with $f(0) = f(1)$, we have $v_f = f v_1$.

**Proof.** Let $(\cdot, \cdot)^c$ be the complexification of the $\text{Ad}(G)$-invariant non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of $g$ inducing the metric of $G/K$. Fix $\alpha \in \Delta$ and $n \in \mathbb{N} \cup \{0\}$. Define $H_\alpha \in h$ by $(H_\alpha, \cdot)^c = \alpha(\cdot)$ and $c_\alpha := \frac{2n\pi \alpha}{\alpha(H_\alpha)}$. Define $g \in H^1([0,1], G^c)$ by $g(t) := \exp_{G^c}(tc_\alpha H_\alpha) \ (0 \leq t \leq 1)$. It is clear that $g \in \Omega_\epsilon(G^c)$. Let $\mathbf{K} := \rho(g)^{-1}K$. Since $\rho(g)(M^c) = M^c$, we have $\mathbf{K} \in K^{h}_{M^c}$. Let $\mathbf{R} = \mathbf{K} + \mathbf{H}$. From Lemmas 4.1.7 and 4.1.9, there exists $\bar{v}_f \in V$ such that $\mathbf{A}(fX) = [X, \bar{v}_f]$ for each $f \in H^0([0,1], C)$ with $f(0) = f(1)$ and each $X \in g^c$, and $\bar{v}_f$ depends on only $f$. It is easy to show that $\mathbf{A} = \text{Ad}(g)^{-1} \circ A \circ \text{Ad}(g)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $g^c$ and $g^c = \mathfrak{h} + \sum_{\alpha \in \Delta} g^c_\alpha$ be the root space decomposition with respect to $\mathfrak{h}$. Let $X \in \mathfrak{h}$ and $X_\alpha \in g^c_\alpha$. Then we have

$$[\text{Ad}(g)v_1, X] = [\text{Ad}(g)v_1, \text{Ad}(g)X] = -\text{Ad}(g)(\mathbf{A}X) = -A(\text{Ad}(g)X) = -AX = [v_1, X].$$
It follows from the arbitrariness of $X$ that $\text{Im}(\text{Ad}(g)v_1 - v_1) \subset \mathfrak{h}$. Also we have

\[
\begin{align*}
[\text{Ad}(g)v_1, X_\alpha] &= \exp(2n\pi\sqrt{-1}t)^{-1}[\text{Ad}(g)v_1, \text{Ad}(g)X_\alpha] \\
&= -\exp(2n\pi\sqrt{-1}t)^{-1}\text{Ad}(g)(AX_\alpha) = -\exp(2n\pi\sqrt{-1}t)^{-1}A(\text{Ad}(g)X_\alpha) \\
&= \exp(2n\pi\sqrt{-1}t)^{-1}\text{Ad}(\exp(2n\pi\sqrt{-1}t)X_\alpha) \\
&= \exp(2n\pi\sqrt{-1}t)^{-1}[v_{\exp(2n\pi\sqrt{-1}t)}, X_\alpha]
\end{align*}
\]

and hence

\[
\begin{align*}
[\text{Ad}(g)v_1 - \exp(2n\pi\sqrt{-1}t)^{-1}v_{\exp(2n\pi\sqrt{-1}t)}, X_\alpha] &= 0,
\end{align*}
\]

that is,

$$
\text{Im}\left(\text{Ad}(g)v_1 - \exp(2n\pi\sqrt{-1}t)^{-1}v_{\exp(2n\pi\sqrt{-1}t)}\right) \subset \mathfrak{z}_g(\mathfrak{g}_c).
$$

Therefore we have

$$
\text{Im}\left(\exp(2n\pi\sqrt{-1}t)v_1 - v_{\exp(2n\pi\sqrt{-1}t)}\right) \subset \mathfrak{h} + \mathfrak{z}_g(\mathfrak{g}_c).
$$

From the arbitrariness of $\alpha$, we obtain

$$
\text{Im}\left(\exp(2n\pi\sqrt{-1}t)v_1 - v_{\exp(2n\pi\sqrt{-1}t)}\right) \subset \mathfrak{h} + \bigcap_{\alpha \in \Delta} \mathfrak{z}_g(\mathfrak{g}_c) = \mathfrak{h}.
$$

Take another Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}_c$ with $\mathfrak{h}' \cap \mathfrak{h} = \{0\}$. Similarly we can show

$$
\text{Im}\left(\exp(2n\pi\sqrt{-1}t)v_1 - v_{\exp(2n\pi\sqrt{-1}t)}\right) \subset \mathfrak{h}'
$$

and hence

$$
v_{\exp(2n\pi\sqrt{-1}t)} = \exp(2n\pi\sqrt{-1}t)v_1.
$$

Take any $f \in H^0([0, 1], \mathbb{C})$. Let $f = \sum_{n=-\infty}^{\infty} c_n \exp(2n\pi\sqrt{-1}t)$ be the Fourier’s expansion of $f$. Then, since $A$ is continuous and linear, we have $A(fX) = \sum_{n=-\infty}^{\infty} c_n A(\exp(2n\pi\sqrt{-1}t)X)$. Also, we have $\text{ad}(X)(fv_1) = \sum_{n=-\infty}^{\infty} c_n \text{ad}(X)(\exp(2n\pi\sqrt{-1}t)v_1)$. Hence we obtain

\[
\begin{align*}
[X, v_f] &= A(fX) = \sum_{n=-\infty}^{\infty} c_n A(\exp(2n\pi\sqrt{-1}t)X) \\
&= \sum_{n=-\infty}^{\infty} c_n [X, v_{\exp(2n\pi\sqrt{-1}t)}] = [X, fv_1].
\end{align*}
\]

Since $\mathfrak{g}_c$ has no center, it follows from the arbitrariness of $X$ that $v_f = fv_1$. q.e.d.
Lemma 4.1.11. Let \( K \) be a holomorphic Killing field on \( V \) given by \( K_u = Au + b \) \( (u \in V) \) for some \( A \in \mathfrak{o}_K(V) \) and \( b \in V \). If \( K \in K^h_M^c \), then we have \( A = \text{ad}(v) \) for some \( v \in V \).

Proof. According to Lemmas 4.1.7 and 4.1.10, there exists \( v \in V \) such that \( A(fX) = [v, fX] \) for any \( f \in H^0([0,1], C) \) with \( f(0) = f(1) \) and any \( X \in \mathfrak{g}^c \). Take any \( u \in V \). Let \( u = \sum_{n=-\infty}^{\infty} e_n \exp(2n\pi \sqrt{-1} t) \) be the Fourier’s expansion of \( u \). Then, since \( A \) is continuous and linear, we have

\[
Au = \sum_{n=-\infty}^{\infty} A(e_n \exp(2n\pi \sqrt{-1} t))
\]

\[
= \sum_{n=-\infty}^{\infty} [v, e_n \exp(2n\pi \sqrt{-1} t)] = [v, u].
\]

Thus we obtain \( A = \text{ad}(v) \). q.e.d.

By using Lemmas 4.1.4 and 4.1.11, we shall prove Proposition 4.1.

Proof of Proposition 4.1. Let \( H \) be as in the statement of Proposition 4.1. We shall compare the Lie algebra (which is denoted by \( \text{Lie} H \)) of \( H \) with the Lie algebra \( \rho_*(H^0([0,1], \mathfrak{g}^c)) \) of \( \rho(H^1([0,1], \mathfrak{g}^c)) \). Take any \( K \in \mathcal{H}(\mathcal{H}^0([0,1], \mathfrak{g}^c)) \). Let \( \{h_s\} \) be a one-parameter transformation group of \( H \) generating \( K \). Since \( h_s \) preserves \( \mathcal{M}^c \) invariantly, \( K \) is tangent to \( \mathcal{M}^c \) along \( \mathcal{M}^c \), that is, \( K \in K_{\mathcal{M}^c}^h \). Hence, it follows from Lemmas 4.1.4 and 4.1.11 that \( K = K^v \) for some \( v \in V \). Thus we have \( K \in \rho_*(H^0([0,1], \mathfrak{g}^c)) \). Therefore \( \text{Lie} H \subset \rho_*(H^0([0,1], \mathfrak{g}^c)) \) follows. That is, \( H \subset \rho(H^1([0,1], \mathfrak{g}^c)) \) follows. q.e.d.

By using Proposition 4.1, we shall prove Theorem A.

Proof of Theorem A. Without loss of generality, we may assume \( \hat{0} \in \mathcal{M}^c \). Let \( M \) be as in the statement of Theorem A. According to Propositions 3.7 and 4.1, we have \( \mathcal{M}^c = \rho(H) \cdot \hat{0} \) for some (connected) subgroup \( H \) of \( H^1([0,1], \mathfrak{g}^c) \). Let \( \mathcal{H} \) be a closed connected subgroup of the anti-Kaehlerian transformation group \( I_h(G^c) \) generated by \( \{L_{h(0)} \circ R_{h(1)}^{-1} \mid h \in H\} \).

Since \( \phi \circ \rho(h) = (L_{h(0)} \circ R_{h(1)}^{-1}) \circ \phi \) for each \( h \in H \), we have \( \mathcal{M}^c = \mathcal{H} \cdot e \). Set \( \mathcal{M} := \pi^{-1}(M) \), where \( \pi \) is the natural projection of \( G \) onto \( G/K \) and \( e \) is the identity element of \( G^c \).

Since \( \mathcal{M} \) is a component of \( \mathcal{M}^c \cap ((G \times G) \cdot ((e,e)\Delta G^c)) \) containing \( e \) and \( (\mathcal{H} \cap (G \times G)) \cdot ((e,e)\Delta G^c) \) is a complete open submanifold of \( \mathcal{M}^c \cap ((G \times G) \cdot ((e,e)\Delta G^c)) \), \( \mathcal{M} \) is a component of \( (\mathcal{H} \cap (G \times G)) \cdot ((e,e)\Delta G^c) \), where we note that \( G^c = (G^c \times G^c)/\Delta G^c \) (hence \( e = (e,e)\Delta G^c \)) and that \( (G \times G) \cdot ((e,e)\Delta G^c) = G(\subset G^c) \). Therefore we have \( \mathcal{M} = (\mathcal{H} \cap (G \times G))_0 \cdot ((e,e)\Delta G^c) \), where \( (\mathcal{H} \cap (G \times G))_0 \) is the identity component of
\( \overline{\pi} \cap (G \times G) \). Set \( \overline{\pi}_R := (\overline{\pi} \cap (G \times G))_0 \). Since \( \hat{M} \) consists of fibres of \( \pi \), we have 
\( (\overline{\pi}_R \cup (e \times K)) \cdot (\langle e, e \rangle \triangle G^c) = \hat{M} \), where \( (\overline{\pi}_R \cup (e \times K)) \) is the group generated by 
\( \overline{\pi}_R \cup (e \times K) \). Denote by the same symbol \( \overline{\pi}_R \) the group \( \langle \overline{\pi}_R \cup (e \times K) \rangle \) newly. 

Set \( (\overline{\pi}_R)_i := \{ g_i \in G \mid (g_1, g_2) \in \overline{\pi}_R \} \) \( (i = 1, 2) \), \( \overline{\pi}_R \cdot (e, e) \triangle G^c \) and 
\( (\overline{\pi}_R)_2 := \{ g \in G \mid (e, g) \in \overline{\pi}_R \} \). It is clear that \( (\overline{\pi}_R)_i \) is a normal subgroup of \( \overline{\pi}_R \).

From \( e \times K \subset \overline{\pi}_R \), we have \( K \subset (\overline{\pi}_R)_2 \). Since \( K \subset (\overline{\pi}_R)_2 \subset (\overline{\pi}_R)_2 \subset G \) and \( K \) is a maximal connected subgroup of \( G \), we have \( (\overline{\pi}_R)_2 = K \) or \( G \) and \( (\overline{\pi}_R)_2 = K \) or \( G \). Suppose that \( (\overline{\pi}_R)_2 = G \). Then we have \( \hat{M} = G \) and hence \( M = G/K \). Thus a contradiction arises. Hence we have \( (\overline{\pi}_R)_2 = K \). Since \( K \) is not a normal subgroup of \( G \) and it is a normal subgroup of \( (\overline{\pi}_R)_2 \), we have \( (\overline{\pi}_R)_2 \neq G \). Therefore we have \( (\overline{\pi}_R)_2 = K \) and hence \( \overline{\pi}_R \subset G \times K \). Set \( \overline{\pi}_R := \{ g \in G \mid (\{g\} \times K) \cap \overline{\pi}_R \neq \emptyset \} \).

Then, since \( \hat{M} = \overline{\pi}_R \cdot (\langle e, e \rangle \triangle G^c) \) and \( M = \pi(\hat{M}) \), we have \( M = \overline{\pi}_R(eK) \). Thus \( M \) is extrinsically homogeneous.

q.e.d.

Next we prove Corollary B in terms of Theorem A.

**Proof of Corollary B.** Let \( M \) be as in the statement of Corollary B. According to Theorem A, \( M \) is extrinsically homogeneous. Hence it follows from Theorem A of [Koi5] that it occurs as a principal orbit of a complex hyperpolar action. Furthermore, if it admits a totally geodesic focal submanifold, then it follows from Corollary D (anf Remark 1.1) of [Koi5] that it occurs as a principal orbit of a Hermann type action. q.e.d.

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