THE INDEX MAP IN ALGEBRAIC $K$-THEORY

OLIVER BRAUNLING & MICHAEL GROECHENIG & JESSE WOLFSON

Abstract. In analogy with the index map of Fredholm operators, $\text{Fred}(H) \to K^{\text{top}}$, we develop an algebraic analogue $\text{Index} : \Omega K_{\text{Tate}}(C) \to K_C$, for $C$ an idempotent complete exact category. One of the main motivations for this construction is the connection with canonical central extensions of loop groups, since loop groups and their higher analogues embed into automorphism groups of certain Tate objects. Just like classical loop group theory has features of $K$-theory (e.g. determinant bundles, tame symbol cocycle for Kac-Moody extension), our theory weaves into higher $K$-theory. Notably, the index map can be identified with the boundary map in a $K$-theory localization sequence, and a generalized Waldhausen construction is used to identify a canonically arising $E_1$-structure of the index map. In an appendix we develop a theory of relative Tate objects and relative index maps. As an application of this formalism we give an index-theoretic description of boundary maps in the $G$-theory of Noetherian schemes.

Contents

1. Introduction 2
  1.1. Motivation and Main Result 2
  1.2. Connection to Central Extensions of Loop Groups and Higher Loop Groups 7
  1.3. Connection to the Atiyah-Jänich Theorem in Topological $K$-Theory 8
  1.4. The Index Map, its Properties, and Applications 8
  Acknowledgements 10

2. Preliminaries 10
  2.1. Exact Categories 10
  2.2. Algebraic $K$-Theory 13
  2.3. Boundary Maps in Algebraic $K$-Theory 23

3. The Index Map 26
  3.1. The Categorical Index Map 26
  3.2. The $K$-Theoretic Index Map 27
  3.3. A Combinatorial Model of the Index Map 28
  3.4. The Index Map is an Equivalence 36
  3.5. The Index Map as a Boundary Map 37

4. The $E_1$-Structure of the Index Map 39
  4.1. A Generalized Waldhausen Construction 41

2010 Mathematics Subject Classification. 19D55 (Primary), 19K56 (Secondary).

O.B. was supported by DFG SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties” and Alexander von Humboldt Foundation. J.W. was partially supported by an NSF Graduate Research Fellowship under Grant No. DGE-0824162, by an NSF Research Training Group in the Mathematical Sciences under Grant No. DMS-0636646, and by an NSF Post-doctoral Research Fellowship under Grant No. DMS-1400349. He was a guest of K. Saito at IPMU while this paper was being completed. Our research was supported in part by NSF Grant No. DMS-1303100 and EPSRC Mathematics Platform grant EP/I019111/1.

1
1. Introduction

1.1. Motivation and Main Result. The behaviour of finite-dimensional vector spaces is governed by numerical data such as the dimension of a vector space or the determinant of an automorphism. Although infinite-dimensional spaces lack these fundamental invariants, the world of finite-dimensional vector spaces often casts a shadow onto the world of their infinite-dimensional counterparts.

Many infinite-dimensional vector spaces of interest, for example those arising in connection with arithmetic geometry, can be expressed as an ascending union of certain sub-spaces, so-called lattices, satisfying a commensurability condition. Commensurability requires that the quotient of a nested pair of lattices is of finite dimension, and that two lattices are contained in a common enveloping lattice (respectively have a common sub-lattice). This notion was central for Tate’s work on the residue on curves [Tat68], where it allowed Tate to lift the trace of operators on finite-dimensional spaces to an invariant for endomorphisms which preserve lattices, an invariant which turns out to be the classical residue.

Some years later, Kac–Peterson [KPS1] similarly observed that for every pair of commensurable sub-spaces \((L, L')\), we have a well-defined notion of relative dimension, and relative determinant line. Indeed, choosing a lattice \(N\), contained in \(L\) and \(L'\), we may define

\[
\dim(L : L') = \dim(L'/N) - \dim(L/N),
\]

and

\[
\det(L : L') = \det(L'/N) \otimes \det(L/N)\vee.
\]

In [KPS1], these notions were used to construct a canonical central extension of the group of automorphisms of the ambient vector space which preserve the commensurability class of a chosen lattice. Inspired by Tate’s work, Arbarello–de Concini–Kac [ADCK89] then related this central extension to the tame symbol and used it to deduce Weil reciprocity.

In [Kap], Kapranov embedded the above concepts into the formal framework of dimension and determinantal theories (see [Kap Def. 1.2.2]). A dimension theory on a vector space is a map \(f\), which assigns to every lattice \(L\) an integer, such that we have

\[
f(L') = f(L) + \dim(L : L')
\]

In a sense, this is the deeper reason why the push-forward functoriality of the residue is by the trace.
for every pair of lattices. Similarly, a determinantal theory, assigns to every lattice \( L \) a line \( \Delta(L) \), and to every pair of lattices \( L \) and \( L' \), an isomorphism

\[
\Delta(L') \cong \Delta(L) \otimes \det(L: L')
\]

satisfying a suitable compatibility for triples of lattices.

By definition, dimension and determinantal theories are specified (up to unique equivalence) by the value they assign to a fixed lattice \( L \). This implies that the collection of all dimension theories is a \( \mathbb{Z} \)-torsor, and similarly that the collection of all determinantal theories forms a \( B \mathbb{G}_m \)-torsor (better known as a \( \mathbb{G}_m \)-gerbe). Every symmetry of the infinite-dimensional vector space \( V \) induces an automorphism of these torsors, i.e. gives rise to an integer \( n \in \mathbb{Z} \), respectively a line \( L \in B \mathbb{G}_m \). It is therefore appropriate to consider these constructs as higher categorical analogues of the notion of dimension and determinant in the world of infinite-dimensional vector spaces. As an application of the main results of this paper, we give a precise relation between the appearance of \( \mathbb{Z} \) and \( B \mathbb{G}_m \) in the above examples and their appearance, for fields \( k \), in the identities

\[
K_0(k) = \mathbb{Z}, \quad K_1(k) = \mathbb{G}_m(k).
\]

In fact, dimension and determinantal theories are themselves truncations of a bigger object seeing all \( K \)-theory, the universal \( K \)-theory torsor. The “index map” of this paper’s title classifies this \( K \)-theory torsor. We refer the reader to Section 5.2 for a more detailed discussion of this torsor and its truncations.

A second suggestion of [Kap], crucial to this work, was to consider Lefschetz’s theory of linearly locally compact vector spaces as the natural setting for dimension and determinantal theories. A topological vector space (over a field with the discrete topology) is called discrete, if it is endowed with the discrete topology. The topological dual of a discrete vector space is called linearly compact. Every linearly compact vector space can be represented as a topological product \( \prod_{i \in I} k \). A topological vector space which can be expressed as an extension of a discrete vector space by a linearly compact vector space is called locally linearly compact. The main example is

\[
k((t)) \cong k[[t]] \oplus t^{-1}k[t^{-1}] \cong \bigoplus_{n=0}^{\infty} k \oplus \bigoplus_{n=0}^{\infty} k.
\]

Every linearly compact sub-space with discrete quotient is called a lattice. The definitions ensure that lattices are commensurable, which provides the right setup for the constructions outlined above. Following common usage, we will also refer to locally linearly compact vector spaces as Tate spaces.

Determinantal theories and dimensional torsors have received much attention in recent research. The dimensional torsor appears in Kontsevich’s construction of motivic integration [Kon95], where it is used to renormalize the infinite-dimensional jet scheme. Kapranov–Vasserot use the formalism of determinantal theories in [KV04] in their construction of the chiral de Rham complex in a similar context. Other examples include Beilinson–Bloch–Esnault’s work [BBE02] on de Rham \( \epsilon \)-factors, Drinfeld’s study of infinite-dimensional vector bundles [Dri06], and Chinburg–Pappas–Taylor’s arithmetic higher Riemann-Roch theorem [CPT12]. The last two papers emphasize the importance of developing a general framework, which allows one to replace finite-dimensional vector spaces by objects in more general exact categories, including finitely generated projective modules

\[2\text{ i.e. a one-dimensional vector space over } k.\]
over a not necessarily commutative ring $A$, or (generalizations of) Drinfeld’s infinite-dimensional vector bundles [Dr06]. Beginnings of such a framework have appeared in Previdi [Pre12], and the present article can be understood as a continuation of this project.

The need to work with more general exact categories is further motivated by a desire to extend the above theories to higher Tate spaces. In [AK10], Arkhipov–Kremnizer considered so-called gerbal theories in the context of 2-Tate spaces. The theory of 2-Tate spaces is built from the theory of Tate spaces, in analogy with how the theory of Tate spaces is built from the theory of finite-dimensional vector spaces. Unlike for Tate spaces, the category of 2-Tate vector spaces does not embed into the category of topological vector spaces as a full sub-category. However, it remains the case that every 2-Tate space is endowed with a collection of lattices, which are commensurable in the sense that nested pairs of lattices differ by a 1-Tate space.

In [BGW14], the authors constructed an exact category $\text{Tate}^e(C)$ of elementary Tate objects, where $C$ is an arbitrary exact category. The defining property of elementary Tate objects is that they may be expressed as an extension

$$L \hookrightarrow V \twoheadrightarrow V/L,$$

where $L \in \text{Pro}^a(C)$ is an admissible Pro-object, and $V/L \in \text{Ind}^a(C)$ an admissible Ind-object. We say that any such $L$ is a lattice in $V$.

Let us now list our main results, both generally and in the classical case $C = \text{Vect}_f(k)$ for the sake of exposition. To make the link with [Kap] recall that $\text{Tate}(\text{Vect}_f(k))$ is equivalent to the category of linearly locally compact topological $k$-vector spaces, i.e. the framework used by Kapranov. For an exact category $C$, we denote by $K_C$ the $K$-theory space of $C$.

**Theorem 1.1.** Let $C$ be an idempotent complete exact category. There exists a canonical simplicial exact category $\text{Gr}^e\leq\bullet(C)$ of classifying flags of lattices in $\text{Tate}^e(C)$ and a morphism

$$\xymatrix{ K_{\text{Gr}^e\leq\bullet}(C) \ar[r]^-{\text{Index}} & BK_C, }$$

having the following properties:

(a) (Grassmannian) There is a canonical map $\coprod_{V \in \text{Tate}^e(C)} \text{Gr}(V) \longrightarrow K_{\text{Gr}^e\leq\bullet}(C)$. Classically, $\text{Gr}(k((t)))$ is the semi-infinite (or Sato) Grassmannian [SS83], [Kap, § 1.2].

(b) (Dimension) The 1-truncation $\tau_{\leq 1}\text{Index} : K_{\text{Gr}^e\leq\bullet}(C) \longrightarrow BK_0(C)$ satisfies the natural generalization, to an exact category $C$, of Kapranov’s axioms for a dimension theory of vector spaces. Classically,

$$K_{\text{Gr}^e\leq\bullet}(\text{Vect}_f(k)) \longrightarrow B\mathbb{Z},$$

defines a class in $H^1(K_{\text{Gr}^e\leq\bullet}(\text{Vect}_f(k)), \mathbb{Z})$. The representatives associate a dimension ($\in \mathbb{Z}$) to a lattice.

(c) (Determinant) The 2-truncation $\tau_{\leq 2}\text{Index} : K_{\text{Gr}^e\leq\bullet}(C) \longrightarrow B\text{VC}$, where VC is Deligne’s virtual objects, satisfies the natural generalization of Kapranov’s axioms for a determinant theory. Classically,

$$K_{\text{Gr}^e\leq\bullet}(\text{Vect}_f(k)) \longrightarrow B\text{Pic}_k^\mathbb{Z},$$

defines a class in $H^1(K_{\text{Gr}^e\leq\bullet}(\text{Vect}_f(k)), B\text{Pic}_k^\mathbb{Z})$. The representatives associate a graded $k$-line to a lattice.

(d) (Equivalence) After geometric realization, the above map determines an equivalence

$$|K_{\text{Gr}^e\leq\bullet}(C)| \xrightarrow{\sim} BK_C.$$
(e) (Boundary) There is a canonical equivalence $K_{\text{Tate}^\ast(C)} \xrightarrow{\mathcal{L}} |K_{\text{Gr}^\ast(C)}|$ so that the $(\Omega, B)$
adjunction transforms $\text{Index} \circ \mathcal{L}$ into the negative of the boundary map in the $K$-theory
localization sequence
$$K_C \xrightarrow{\pi} K_{\text{Ind}^\ast(C)} \rightarrow K_{\text{Ind}^\ast(C)/C}.$$ We refer to $\text{Index}$ as the index map. We refer to the data it classifies as the index torsor.

Part (d) shows that the index torsor sees the entire $K$-theory of $C$. This picture appeared as a conjecture in the thesis of L. Previdi [Pre10], and precursors of parts (b) and (c) appeared in [Pre12]. S. Saito [Sai12] has recently constructed an abstract equivalence $K_{\text{Tate}^\ast(C)} \leftrightarrow BK_C$ as a consequence of Schlichting’s Localization Theorem [Sch04]. Part (e) of the theorem shows that $\text{Index} \circ \mathcal{L}$ is equivalent to $-1$ times Saito’s map. Parts (a)-(c) establish the compatibility of the index torsor with well-known existing torsors. But one can truncate less.

**Remark 1.2.** For every idempotent complete exact category $C$, the $n$-truncation
$$\tau_{\leq n} \text{Index} : K_{\text{Gr}^\ast(C)} \rightarrow B\tau_{\leq n}K_C$$
is the classifying map for a higher gerbe with values in truncated $K$-theory.

We shall elaborate on this in Section 5. For now, we observe that this defines a torsor which is classified by an element in $H^1(K_{\text{Gr}^\ast(C)}, B\tau_{\leq n}K_C)$ taking values in an excerpt of the $K$-theory data of $C$. One may visualize a representative as associating to any lattice in the Grassmannian category

\[
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z} \\
\mathbb{Z}
\end{array}
\quad
\begin{array}{c}
\mathbb{G}_m \\
\mathbb{G}_m \\
\mathbb{G}_m
\end{array}
\]

Torsors over Sato Grassmannian obtained from 1- and 2-truncations

the choice of a piece of $K$-theory data of the shape as usually only the quotient of two lattices would carry. The next richer truncation would also respect $K_2$-data.

How does the index torsor operate?

We will also give a precise description of the map $\text{Index}$ in terms of maps of simplicial sets. The 1-truncation, the dimension torsor, as in (1), is easy to describe for an automorphism which restricts to a monomorphism on a lattice. For example, the automorphism

$$t^n \in k((t))^\times$$
on a lattice, say $L := k[[t]]$, restricts to $L \xrightarrow{t^n} L$ for $n \geq 0$. The dimension theory then just maps

$$\left( L \xrightarrow{t^n} L \right) \quad \mapsto \quad \dim \frac{L}{t^n \cdot L}.$$ This also essentially describes the general case: for an automorphism of a Tate object which restricts to a monomorphism on any lattice, the dimension torsor maps it to the cokernel. We show that
in richer truncations seeing more of higher $K$-theory, this picture prolongs to longer chains of monomorphisms. To return to our example, giving $n$ automorphisms

\[ k((t))^\times \sim \rightarrow k((t))^\times \sim \rightarrow \cdots \sim \rightarrow k((t))^\times \]

so that for a lattice $L$ they restrict to a chain of monomorphisms

\[ L \hookrightarrow g_1^{-1}L \hookrightarrow \cdots \hookrightarrow g_n^{-1} \cdots g_2^{-1} g_1^{-1} L, \]

the higher index torsor can be described as a simplicial map sending this to subquotients

\[ \frac{g_1^{-1}L}{L} \hookrightarrow \frac{g_2^{-1}g_1^{-1}L}{L} \hookrightarrow \cdots \hookrightarrow \frac{g_n^{-1} \cdots g_1^{-1}L}{L}. \]

This gives an explicit description of the index torsor in terms of Waldhausen’s $S$-model for $K$-theory. It requires a bit more care to spell this out in full, and we refer the reader to Section 3 for details.

Of course a second key feature is that the above description requires automorphisms to restrict to monics on lattices, which need not be true for a general automorphism. However, $K$-theory and the index torsor provide a systematic prolongation of the above map beyond this condition. In the example of the dimension torsor, this systematic prolongation becomes just the classical formula for the Fredholm index

\[ t^n \in k((t))^\times \mapsto \dim \ker t^n_{|k[[t]]} - \dim \coker t^n_{|k[[t]]}. \]

This also explains the title of the paper.

Here is a decategorification of our results, for readers wanting to avoid Tate categories and preferring a classical language:

**Theorem 1.3.** For a field $k$ there is a canonical factorization

\[ \partial_t : K_{k((t))} \longrightarrow \text{Index}_L^{\text{index}} BK_k \]

of the boundary map coming from the localization of schemes $\star \hookrightarrow \text{Spec } k[[t]] \hookrightarrow \text{Spec } k((t))$. In analogy to Theorem 1.1, this means that

(a) the 1-truncation $K_{k((t))} \longrightarrow B\mathbb{Z}$, i.e. Kapranov’s dimension torsor, induces the valuation

\[ K_1(k((t))) \xrightarrow{\sim} k((t))^\times \longrightarrow \mathbb{Z}, \]

(b) the 2-truncation $K_{k((t))} \rightarrow B\text{Pic}_k^\mathbb{Z}$, i.e. Kapranov’s determinant torsor, on $K_1$, induces the valuation, and on $K_2$, induces the tame symbol

\[ K_2(k((t))) \longrightarrow k((t))^\times \]

\[ \{ f, g \} \longmapsto (-1)^{v(f)v(g)} f^{v(g)} g^{-v(f)}(0) \]

(c) the $n$-truncation, restricted to Milnor $K$-groups, matches the Milnor $K$-boundary map, i.e. has the computationally convenient description

\[ \partial_t \{ t^n, u_2, \ldots, u_n \} = n \{ \overline{u_2}, \ldots, \overline{u_n} \} \quad \partial_t \{ u_1, \ldots, u_n \} = 0 \]

for $u_i \in k[[t]]^\times$.

We defer this result to the sequel [BGWa]. Note that it boils down the theory of dimension/determinantal theories into terms of everyday mathematical practice. In fact, instead of Milnor $K$-symbols of a field $k$, a much broader and stronger variant can be achieved, leading to a generalized Contou-Carrère symbol. However, this requires some care to spell out in detail, so we refer the reader to [BGWa]. Nonetheless, it seems worthwhile to point out here how deeply the index map
is tied into classical considerations. The 3-truncation will add the first generalized tame symbol on 3-symbols, and so forth. Explicit formulae are known, but the complexity increases rapidly with \( n \).

The existing literature mostly focuses on Tate \( R \)-modules or Tate vector spaces. By the generality of allowing exact categories as input, there is no obstruction to generalize this to higher Tate spaces. We abbreviate:

\[
\text{Tate}^n(C) = \text{Tate} \cdots \text{Tate}^{n \text{-times}}(C).
\]

One can now iterate the above theorems and, under vanishing assumptions on negative \( K \)-theory groups, we obtain higher index torsors,

\[
\text{Index}^n : K_{\text{Tate}^n(C)} \to B^n K_C,
\]

natural in \( C \). These higher torsors are also compatible with the well-known existing ones:

**Theorem 1.4.** Let \( k \) be a field. Denote by \( 2\text{-Tate}^{el}(k) \) the category \( \text{Tate}^{el}(\text{Tate}(\text{Vect}_f(k))) \). Then the composition

\[
K_{2\text{-Tate}^{el}(k)} \xrightarrow{\text{Index}^2} B^2 K_k \xrightarrow{\tau \leq 3} B^2 \text{Pic}_k^\mathbb{Z}
\]

classifies Arkhipov and Kremnizer’s determinant 2-gerbe \([AK10]\).

For full generality it is actually better to work with non-connective \( K \)-theory. This removes the assumption of idempotent completeness everywhere, but comes at the price that we have to speak about spectrum-valued torsors. However, once we have made the transition to non-connective \( K \)-theory, it requires little additional work to develop an index map for other localizing invariants, for example topological Hochschild homology. Recent work of Tabuada, and Blumberg–Gepner–Tabuada (e.g. \([BGT13]\)) has constructed a stable \( \infty \)-category of “non-commutative motives” which serves as the universal source of such invariants. In a sequel to this paper, \([BGWB]\), we will adapt the construction of the index map to the setting of non-commutative motives. Using this, we will show that natural transformations between localizing invariants intertwine the respective index maps, thus giving rise to a collection of abstract index theorems.

### 1.2. Connection to Central Extensions of Loop Groups and Higher Loop Groups

The loop group \( LU(n) \) of the finite-dimensional compact Lie group \( U(n) \) admits a natural central extension \( \hat{LU}(n) \) by \( U(1) \) (cf. \([PS86]\)). In fact, this central extension appears throughout the theory of loop groups. The algebraic analogue of \( LU(n) \), the \textit{ind-group} \( \text{GL}_n((t)) \), comes equipped with a comparable structure: a central extension by \( \mathbb{G}_m \).

In both the differential and algebro-geometric worlds, this extension is constructed by embedding the loop group inside the automorphisms of an infinite-dimensional vector space, and restricting a canonical central extension of this automorphism group. For \( LU(n) \), the relevant automorphism group is the \textit{restricted general linear group} \( \text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \) of a Hilbert space with polarization \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \) (cf. \([PS86]\) Chapter 6). Every element \( \phi \in \text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \) induces a Fredholm operator

\[
\phi^+ : \mathcal{H}^+ \longrightarrow \mathcal{H}^+,
\]

which is related to the required central extension \( \hat{\text{GL}}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \) by means of the identity

\[
\hat{\text{GL}}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \big|_{\phi} \cong \left( \bigwedge^{\text{top}} \ker(\phi^+) \otimes \bigwedge^{\text{top}} \text{coker}(\phi^+) \right) \setminus 0.\]

\(^5\text{cf. } [ADCK89]\)
i.e. the fibre of the central extension $\tilde{\text{GL}}_{\text{res}}(H, H^+)$ over an automorphism $\phi$ agrees with the $\mathbb{C}^\times$-torsor associated to the line bundle above.

The algebraic counterpart of $\text{GL}_{\text{res}}(H, H^+)$ is the automorphism group of a Tate vector space $V$. Fixing a lattice in $V$ serves the same purpose as the choice of a polarization for a Hilbert space. As we have seen earlier, determinantal theories for lattices in $V$ give rise to a $\mathbb{G}_m$-gerbe $D_V$. Every automorphism of $V$ induces an automorphism of $D_V$, hence gives rise to a line. The resulting map

$$\text{Aut}(V) \rightarrow B \mathbb{G}_m$$

classifies a central extension $\tilde{\text{Aut}}(V)$ of $\text{Aut}(V)$ by $\mathbb{G}_m$.

The index map of Theorem 1.1 lifts this map to a map

$$\text{Aut}(V) \rightarrow K_C$$

which should be understood as inducing a central extension of $\text{Aut}(V)$ by $\Omega K_C$. One can also consider central extensions of groups by spectra, as we will in [BGW]. The latter extension provides one of the main examples of this theory.

1.3. Connection to the Atiyah-Jänich Theorem in Topological $K$-Theory. In [Ati89] and [Jän65], it was independently shown by Atiyah and Jänich that, for a separable Hilbert space $H$, the space $\text{Fred}(H)$ of Fredholm operators, i.e. operators $F: H \rightarrow H$ with finite-dimensional kernel and cokernel, is homotopy equivalent to the classifying space of topological $K$-theory. This equivalence sends a Fredholm operator $F$ to the element of the topological $K$-theory space given by the difference of finite-dimensional vector spaces $\ker F - \text{coker } F$.

In Section 5.1.1, we explain how the index map of Theorem 1.1 can be viewed as an algebraic counterpart of Atiyah-Jänich’s description of the classifying space of topological $K$-theory $K_C^{\text{top}}$. This viewpoint is justified by the following pair of results.

**Theorem 1.5.**

(a) Let $(H, H^+)$ be a polarized, separable complex Hilbert space. The space of Fredholm operators $\text{Fred}(H^+)$ is equivalent to

$$K_C^{\text{top}} \cong \Omega (B \text{GL}_{\text{res}}(H, H^+))^+,\$$

where the outer superscript $+$ refers to Quillen’s $+$-construction.

(b) Let $R$ be a ring. The $R$-module $R((t))$ admits a canonical structure of an elementary Tate object in the category of finitely generated projective $R$-modules, and we have an equivalence

$$K_R \cong \Omega (B \text{Aut}_{\text{Tate}}(R((t))))^+.$$

We refer the reader to Section 5.1.1 for more details.

1.4. The Index Map, its Properties, and Applications.

1.4.1. The Definition of the Index Map. In [Wal85], Waldhausen exhibited a model of algebraic $K$-theory, defined in terms of simplicial sets, which is central to the present paper. For every exact category $C$, and positive integer $n$, Waldhausen defined an exact category $S_n(C)$, whose objects are chains of admissible monomorphisms

$$X_1 \hookrightarrow \cdots \hookrightarrow X_n$$

and choices of all quotients $X_j/X_i$. This collection of exact categories can be assembled into a simplicial object $S_\bullet(C)$ in the 2-category of exact categories. For a category $D$ we denote by $D^\times$ the maximal groupoid contained in $D$. Moreover, we view every groupoid as a topological space, by means of its classifying space (i.e. the geometric realization of its nerve). Waldhausen realized
that the delooping of the $K$-theory space $K_C$ can be obtained as the geometric realization of the simplicial object in spaces

$$BK_C \cong |S_\bullet(C)^\times|.$$  

Applying the loop space functor $\Omega$, one obtains a description of $K_C \cong \Omega|S_\bullet(C)^\times|$.

For an exact category $C$, there exists a simplicial space $Gr^\leq(C)^\times$, classifying nested chains of lattices in elementary Tate objects

$$L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V.$$  

If $C$ is idempotent complete, then we show in Proposition 3.3 that the geometric realization $|Gr^\leq(C)^\times|$ is equivalent to the classifying space of the groupoid of elementary Tate objects $Tate^el(C)^\times$.

**Definition 1.6.** The index map is defined to be the geometric realization of the map of simplicial groupoids $Gr^\leq(C)^\times \to S_\bullet(C)^\times$, which sends $(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V)$ to $(L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0)$.

Under the equivalence $|Gr^\leq(C)^\times| \simeq Tate^el(C)^\times$, this yields a map $\text{Index}: Tate^el(C)^\times \to BK_C$; it assigns to every elementary Tate object $V \in Tate^el(C)$ a $K_C$-torsor. The construction can be iterated to obtain a map

$$K_\text{Tate}^{el}(C) \to BK_C.$$  

Namely, we consider the bisimplicial groupoid $Gr^\leq(S_\bullet(C))^\times$ and construct a map from $Gr^\leq(S_\bullet(C))$ to the 2-fold Waldhausen $S$-construction $S_\bullet S_\bullet(C)^\times$. We show in Corollary 3.6 that the geometric realization of this map defines a map of infinite loop spaces

$$BK_{Tate}^{el}(C) \to B^2K_C.$$  

1.4.2. The $E_1$-Structure. The index map gives rise to a map between two classifying spaces. Choose $V \in Tate^el(C)$, and consider the natural map $B\text{Aut}(V) \to Tate^el(C)^\times$. We have a map

$$B\text{Aut}(V) \to BK_C.$$  

Applying the loop space functor $\Omega$ we obtain a map of group-like $E_1$-objects (cf. Section 4)

$$\text{Aut}(V) \to K_C.$$  

While this topological definition of this structure is well-defined, it is not directly amenable to computations. We will therefore develop an algebraic approach in Section 3.

If one post-composes the index map with the canonical map $BK_C \to B\pi_0(K_C) = BK_0(C)$, the resulting map of groups

$$\text{Aut}(V) \to K_0(C)$$  

can be described in classical terms. If $L \hookrightarrow V$ is an arbitrary lattice, and $g \in \text{Aut}(V)$ an automorphism, then for any lattice $N \supset L, gL$, we have an element

$$[\text{Index}(g)] = [N/gL] - [N/L] \in K_0(C),$$  

which is well-defined independently of the chosen $N$. If $h \in \text{Aut}(V)$ is a second automorphism, we would like to show that $[\text{Index}(gh)] = [\text{Index}(g)] + [\text{Index}(h)]$. Choose an upper bound $M \supset L, hL, ghL$. We have that

$$[M/ghL] = [M/gL] - [M/hL] + [M/hL] - [M/L] = [\text{Index}(g)] + [\text{Index}(h)].$$  

In order to replace $K_0(C)$ by the space $K_C$ in the argument above, we develop a generalization of Waldhausen’s $S$-construction in Section 4.
1.4.3. A Description of Boundary Morphisms. The Theorems 1.1 and 1.3 already emphasize that the index map is an explicit model for particular boundary maps arising in algebraic K-theory. This point of view can be generalized, by using the framework of relative Tate objects and relative index maps, which we develop in Appendix A. As an application of this relative theory, we obtain an index-theoretic model for an array of boundary maps arising in algebraic K-theory, including coherent sheaves on a Noetherian scheme (see Theorem A.25).

Acknowledgements. We thank Y. Kremnizer for introducing us first to these questions and then to each other. We thank E. Getzler, M. Kapranov, R. Nest and B. Tsygan for helpful conversations. We would like to thank T. Hausel for supporting a visit of the first and the third author to EPF Lausanne, where part of this work was carried out. J.W. also thanks K. Saito and IPMU for the pleasant working conditions while this article was being completed. We would like to thank A. Beilinson and V. Drinfeld for supporting a visit of the first and second author to the University of Chicago, where this paper was completed.

2. Preliminaries

2.1. Exact Categories. The first paragraph of this subsection is devoted to the definition of exact categories, and related notions. In Paragraph 2.1.2 we recall the basics of Ind, Pro, and Tate objects, which form the main players of the present text.

2.1.1. Definitions. For the convenience of the reader, we recall the definition of exact categories below. More details can be found in Bühler’s excellent survey [Büh10].

**Definition 2.1.** We denote by $\mathbf{C}$ an additive category.

(a) A kernel-cokernel pair is given by maps

$$X \hookrightarrow Y \twoheadrightarrow Z$$

with $X \hookrightarrow Y$ being the kernel of $Y \twoheadrightarrow Z$, and $Y \twoheadrightarrow Z$ being the cokernel of $X \hookrightarrow Y$.

(b) The structure of an exact category on $\mathbf{C}$ is given by a class $\mathcal{E}_\mathbf{C}$ of kernel-cokernel pairs. A map $X \hookrightarrow Y$, serving as the kernel in a kernel-cokernel pair in $\mathcal{E}_\mathbf{C}$, is called an admissible monic. Similarly, a map $Y \twoheadrightarrow Z$, which serves as a cokernel in a kernel-cokernel pair of $\mathcal{E}_\mathbf{C}$, is called an admissible epic. The following axioms have to be satisfied:

1. identity morphisms are admissible monics and admissible epics,
2. the composition of two admissible monics is an admissible monic, similarly for admissible epics,
3. pushouts of admissible monics along arbitrary maps exist, and are again admissible monics; similarly, pullbacks of admissible epics along arbitrary maps exist and are again admissible epics.

(c) An additive functor $F: \mathbf{C} \longrightarrow \mathbf{D}$ is called exact, if it maps $\mathcal{E}_\mathbf{C}$ to $\mathcal{E}_\mathbf{D}$. If $F: \mathbf{C} \longrightarrow \mathbf{D}$ is fully faithful and exact, we call it fully exact, if every kernel-cokernel pair $F(X) \hookrightarrow F(Y) \twoheadrightarrow F(Z)$ in $\mathcal{E}_\mathbf{D}$ stems from a kernel-cokernel pair in $\mathbf{C}$.

(d) We note by $\text{Cat}_{\text{ex}}$ the 2-category of exact categories, exact functors, and natural transformations.

We will refer to the kernel-cokernel pairs in $\mathcal{E}_\mathbf{C}$ as short exact sequences or extensions in the exact category $\mathbf{C}$. 
Example 2.2. For a ring $R$ we denote by $P_f(R)$ the additive category of finitely generated projective modules. Considering kernel-cokernel pairs obtained from short exact sequences of $R$-modules, with all constituents being finitely generated projective, we obtain a natural exact structure on this category.

The next definition recalls the notion of being idempotent complete. This notion is reminiscent of the properties of linear projectors, familiar from linear algebra.

Definition 2.3. We say that an additive category $C$ is idempotent complete, if every idempotent splits, i.e. for every morphism $p: X \to X$, satisfying $p^2 = p$, we have an isomorphism $X \cong Y \oplus Z$ taking $p$ to the idempotent $0 \oplus 1_Z$.

Every exact category can be embedded into an idempotent complete exact category, in an essentially unique way.

Proposition 2.4. (cf. [Büh10, Proposition 6.10]) For every exact category $C$ there exists a fully exact embedding $C \hookrightarrow C^e$ into an idempotent complete exact category, which is 2-universal with respect to this property.

Following Schlichting [Sch06, Def. 1.3 \& 1.5], we recall the notion of left s-filtering subcategories of exact categories. The perk of left s-filtering inclusions is that a corresponding quotient can be formed in the 2-category of exact categories. The definition given below is an equivalent, but simpler version, which was communicated to us by Bühler (see [BGW14, App. A] for a reproduction of Bühler’s argument, which compares the definition below with Schlichting’s).

Definition 2.5. Let $C \hookrightarrow D$ be a fully faithful exact functor between exact categories.

(a) The inclusion is left special, if for every object $Z \in C$, and every admissible epic $G \to Z$ in $D$, we have a commutative diagram with exact rows in $D$

\[
\begin{array}{ccc}
X & \to & Y & \to & Z \\
\downarrow & & \downarrow & & \downarrow_{1_Z} \\
F' & \to & G & \to & Z,
\end{array}
\]

with the top row being an extension in $C$. The inclusion is called right special, if $C^{\text{op}} \hookrightarrow D^{\text{op}}$ is left special.

(b) The inclusion is left filtering, if every morphism $Y \to F$ in $D$, with $Y \in C$, factors through an admissible monic $Z \to F$ with $Z \in C$:

\[
\begin{array}{ccc}
Y & \to & F \\
\downarrow & & \downarrow \\
& \uparrow & \\
& Z,
\end{array}
\]

It is called right filtering if $C^{\text{op}} \hookrightarrow D^{\text{op}}$ is left filtering.

(c) It is left s-filtering, if it is both left special and left filtering. It is called right s-filtering if $C^{\text{op}} \hookrightarrow D^{\text{op}}$ is left s-filtering.

With this definition in hand, we are able to recall Schlichting’s exact quotient category $D/C$.

Definition 2.6. Consider a left s-filtering inclusion of exact categories $C \hookrightarrow D$ (see Definition 2.5).
(a) Let $\Sigma_\kappa$ be the class of morphisms in $D$, which are admissible epics with kernel in $C$. The left $s$-filtering condition guarantees that $\Sigma_\kappa$ satisfies has a calculus of left fractions (see [Sch04, Lemma 1.13]3).

(b) Following [Sch04, Def. 1.14], the localization $D[\Sigma_\kappa^{-1}]$ will be denoted by $D/C$. It inherits an exact structure, by considering the images of short exact sequences in $D$ in the additive category $D/C$ ([Sch04, Prop. 1.16]).

2.1.2. Admissible Ind, Pro and Tate Objects. This paragraph provides an informal introduction to Tate objects in exact categories. Full details are given in [BGW14]. Let $C$ be an exact category. In loc. cit., the authors defined, for an infinite cardinal $\kappa$,

1. the exact category $\text{Ind}_\kappa^a(C)$ of admissible Ind-objects in $C$ ([BGW14, Def. 3.3]),
2. the exact category $\text{Pro}_\kappa^a(C)$ of admissible Pro-objects in $C$ ([BGW14, Def. 4.1]),
3. the exact category $\text{Tate}_\kappa^{\text{el}}(C)$ of elementary Tate objects in $C$ ([BGW14, Def. 5.1]), and
4. the exact category $\text{Tate}_\kappa(C)$ of $Tate$ objects in $C$ as the idempotent completion of $\text{Tate}_\kappa^{\text{el}}(C)$ ([BGW14, Def. 5.22]).

The study of these constructions goes back at least to Lefschetz [Lef42, Ch. II.25], Artin–Mazur [AM69a], Beilinson [Be˘ı87], Kato [Kat00], and Kapranov [Kap01]. For $\kappa = \aleph_0$, a recent treatment has also been given by Previdi in [Pre11]. For a ring $R$, Drinfeld [Dri06] has studied a related notion of $Tate$-$R$-module. The category of countably generated $Tate$-$R$-modules in Drinfeld’s sense is equivalent to the category $\text{Tate}_{\aleph_0}(P_f(R))$ ([BGW14, Thm. 5.25]). In general, Drinfeld’s category of $Tate$ modules is a fully exact sub-category of $\text{Tate}(P_f(R))$. For uncountable cardinalities, the authors provide a geometric interpretation in terms of flat Mittag-Leffler modules of the category $\text{Tate}(P_f(R))$ in [BGW14], based on work by Šťovíček and Trlifaj ([BGW14, App. B]).

For the exact category $\text{Vect}_f(k)$ of finitely generated vector spaces over a discrete field $k$, the category $\text{Ind}_\kappa^a(\text{Vect}_f(k))$ is equivalent to the category of discrete vector spaces generated by a basis of cardinality at most $\kappa$. A guiding example is the vector space $k[x] \in \text{Ind}_\kappa^a(\text{Vect}_f(k))$. The category $\text{Pro}_\kappa^a(\text{Vect}_f(k))$ is equivalent to the category of topological duals of discrete vector spaces of cardinality at most $\kappa$. The topological vector space $k[[t]] \cong (k[x])^\vee$ is an important example, where the topology on $k[[t]]$ is the $t$-adic topology, i.e. the finest linear topology such that the sequence $\{t^n\}_{n\in \mathbb{N}}$ converges to $0$. The category $\text{Tate}_\kappa^{\text{el}}(\text{Vect}_f(k))$ is equivalent to the category of topological vector spaces of the form $V \oplus W^\vee$ where $V$ and $W$ are discrete vector spaces of cardinality at most $\kappa$. By definition, this is Lefschetz’s category of locally linearly compact vector spaces [Lef42, Ch. II.25]. The archetypical example is the topological vector space $k((t)) \cong k[[t]] \oplus t^{-1}k[t^{-1}] \in \text{Tate}_\kappa^{\text{el}}(\text{Vect}_f(k))$.

The categories of admissible Ind-objects, admissible Pro-objects, and elementary Tate objects are related by a commuting square of fully exact embeddings

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\text{Ind}_\kappa^a(C)} & \text{Pro}_\kappa^a(C) \\
\downarrow & & \downarrow \\
\text{Tate}_\kappa^{\text{el}}(C) & \xrightarrow{} & \text{Tate}_\kappa(C)
\end{array}
\]

The inclusion functors in this diagram are well-behaved with respect to taking quotients (see Definition 2.5).

3The converse of this statement is true as well, as we learned from private correspondence with Bühler.
Proposition 2.7. ([BGW14, Prop. 3.10, 5.6, 5.8 & 5.27]) Let $C$ be an exact category. The inclusions $C \hookrightarrow \text{Ind}_a^\kappa(C)$ and $\text{Pro}_a^\kappa(C) \hookrightarrow \text{Tate}_a^\kappa(C)$ are left s-filtering. The inclusion $\text{Ind}_a^\kappa(C) \hookrightarrow \text{Tate}_a^\kappa(C)$ is right filtering. The quotient exact categories $\text{Ind}_a^\kappa(C)/C$ and $\text{Tate}_a^\kappa(C)/\text{Pro}_a^\kappa(C)$ are equivalent with respect to the map induced by the inclusion $\text{Ind}_a^\kappa(C), C \hookrightarrow (\text{Tate}_a^\kappa(C), \text{Pro}_a^\kappa(C))$ of pairs of exact categories.

Remark 2.8. We do not know at present if the inclusion $\text{Ind}_a^\kappa(C) \hookrightarrow \text{Tate}_a^\kappa(C)$ is right special.

Following Sato–Sato [SS83], we consider the set of all lattices in an elementary Tate object. The archetypical example of such is the inclusion $k[[t]] \hookrightarrow k((t))$.

We observe that $k[[t]]$ is a Pro-object, and the quotient $\bigoplus_{n \geq 1} k(t^{-n})$ is an Ind-object. This is the defining quality of lattices.

Definition 2.9. Let $V$ be an elementary Tate object in $C$.

1. A lattice $L \hookrightarrow V$ of an elementary Tate object is an admissible sub-object, with $L \in \text{Pro}_a^\kappa(C) \subseteq \text{Tate}_a^\kappa(C)$ and the cokernel $V/L \in \text{Ind}_a^\kappa(C) \subseteq \text{Tate}_a^\kappa(C)$.

2. The Sato Grassmannian $\text{Gr}(V)$ is the partially ordered set of lattices in $V$, where $L_0 \leq L_1$ if there exists a commuting triangle of admissible monics

\[
\begin{array}{ccc}
L_0 & \rightarrow & L_1 \\
\downarrow & & \downarrow \\
V & \leftarrow & \leftarrow
\end{array}
\]

Lattices and the Sato Grassmannian play a key role in our study of Tate objects. Assertion (c) in theorem below, is viewed by the authors as the main result of [BGW14].

Theorem 2.10. ([BGW14 Prop. 6.6, Thm. 6.7]) Let $C$ be an exact category.

(a) Every elementary Tate object in $C$ has a lattice.
(b) The quotient of a lattice by a sub-lattice is an object of $C$.
(c) If $C$ is idempotent complete, and $L_0 \hookrightarrow V$ and $L_1 \hookrightarrow V$ are two lattices in an elementary Tate object $V$, then there exists a lattice $N \hookrightarrow V$ with $L_0, L_1 \leq N$ in $\text{Gr}(V)$. Similarly, $L_0$ and $L_1$ have a common sub-lattice $M \subseteq L_0, L_1$.

The following convention helps us to avoid awkward notation.

Remark 2.11. From now on we consider the infinite cardinal $\kappa$ as fixed, and omit the cardinality bound from the notation, i.e. for an exact category $C$ we denote by $\text{Ind}_a^\kappa(C)$, $\text{Pro}_a^\kappa(C)$, and $\text{Tate}_a^\kappa(C)$ the corresponding exact categories.

The justification for this omission is that algebraic $K$-theory is not sensitive to a change of the cardinality $\kappa$. This follows for instance from Theorem 3.22.

2.2. Algebraic $K$-Theory. The results of this article are formulated in the language of algebraic $K$-theory. Primary references include Quillen [Qui73], Waldhausen [Wal85], and Schlichting [Sch04, Sch06].
2.2.1. The $K$-Theory Space of an Exact Category.

**Definition 2.12.** Let $C$ be an exact category. Denote by $S_\bullet(C)$ the simplicial object in exact categories defined as follows. The $n$-simplices $S_n(C)$ are the exact category with objects given by strings of admissible monics in $C$

$$(X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n)$$

along with choices of quotients $X_j/X_i$ for all $i < j$. The face maps are given by

$$d_i: (X_1 \hookrightarrow \ldots \hookrightarrow X_n) \mapsto (X_1 \hookrightarrow \ldots \hookrightarrow X_{i-1} \hookrightarrow X_{i+1} \hookrightarrow \ldots \hookrightarrow X_n),$$

for $i \geq 1$, and

$$d_0: (X_1 \hookrightarrow \ldots \hookrightarrow X_n) \mapsto (X_2/X_1 \hookrightarrow \ldots \hookrightarrow X_n/X_1).$$

The degeneracy maps $s_i: S_n(C) \to S_{n+1}(C)$ are defined by repeating the $i$-th entry. We refer to this simplicial object as Waldhausen’s $S$-construction.

**Definition 2.13.** For a category $C$ we denote by $C^\times$ the maximal sub-groupoid of $C$, i.e. the groupoid obtained by discarding all non-invertible morphisms from the category $C$.

**Definition 2.14.** For an exact category $C$ we define the $K$-theory space $K_C$ as the loop space

$$K_C = \Omega|S_\bullet(C)^\times|.$$

**Remark 2.15.** The simplicial object $S_\bullet(C)^\times$ has the property that its space of 0-simplices is a singleton. Hence, every 1-simplex induces a loop in the geometric realization. Therefore, we have a map

$$C^\times \cong S_1C^\times \to \Omega|S_\bullet(C)^\times| = K_C,$$

which is natural in $C$.

2.2.2. Additivity. The fundamental property of algebraic $K$-theory is established in the following “Additivity Theorem”. All of the results of this paper can be understood as consequences of the Additivity Theorem combined with Theorem [2.10]

**Theorem 2.16** (Waldhausen’s Additivity Theorem). ([Wal85] Theorem 1.4.2, Proposition 1.3.2(4))

Let $F_1 \hookrightarrow F_2 \to F_3$ be an exact sequence of functors $C_1 \hookrightarrow C_2$. Then the map

$$|S_\bullet F_2|: |S_\bullet(C_1)^\times| \to |S_\bullet(C_2)^\times|$$

is naturally homotopic to

$$|S_\bullet F_1 \oplus S_\bullet F_3|: |S_\bullet(C_1)^\times| \to |S_\bullet(C_2)^\times|.$$

Several equivalent reformulations exist. We will need the following two.

**Definition 2.17** (Waldhausen). Let $D$ be an exact category, and let $C_1$ and $C_2$ be full sub-categories of $D$ which are closed under extensions. Define $E(C_1, D, C_2)$ to be the full sub-category of $E D$ consisting of the exact sequences $X_1 \hookrightarrow Y \to X_2$ with $X_i \in C_i$.

Note that, because $C_1$ and $C_2$ are closed under extensions in $D$, $E(C_1, D, C_2)$ is closed under extensions in $E D$; in particular, it is an exact category.

**Theorem 2.18.** ([Wal85] Theorem 1.4.2, Proposition 1.3.2(1)]) The projection

$$|S_\bullet(E(C_1, D, C_2))^\times| \to |S_\bullet(C_1)^\times| \times |S_\bullet(C_2)^\times|$$

$$(X_1 \hookrightarrow Y \to X_2) \mapsto (X_1, X_2)$$

is a homotopy equivalence.
We also need the following version.

**Theorem 2.19.** Let $A \to B \to C$ be a composable pair of exact functors such that $i$ is fully faithful and induces an equivalence with the full sub-category of $B$ annihilated by $p$. Moreover, assume that $p$ has a left adjoint

$$s: C \to B,$$

such that $ps \cong 1_C$ and such that, for every object $Y \in B$, the co-unit $sp(Y) \to Y$ is an admissible monic with cokernel in $A$. Then, the map

$$i \times s: K_A \times K_C \cong K_B,$$

is an equivalence of spaces.

While this theorem is, without doubt, well-known, we have a chosen a less conventional statement which is convenient for our applications. Therefore, we now give a proof.

**Proof.** Applied to the sequence $C_1 \to \mathcal{E}(C_1, D, C_2) \to C_2$, this theorem gives Theorem 2.18. It remains to deduce the present statement from the formulations above.

We have a well-defined map of spaces $i \times s: K_A \times K_C \to K_B$. By the Whitehead lemma it suffices to show that it establishes an equivalence on all homotopy groups.

The admissible monic of functors

$$sp \hookrightarrow 1_B: B \to B,$$

given by the co-unit of the adjunction $(p, s)$, extends to a short exact sequence

$$sp \hookrightarrow 1_B \to f: B \to B.$$

By construction, $pf = 0$, therefore $f$ can be expressed as $ir$, where $r: B \to A$ is an exact functor. By the Additivity Theorem (Theorem 2.16), we have

$$\pi_i(K(ir) \oplus K(sp)) = \pi_i(K(1_B)).$$

Moreover, the relations $ps = 1_C$ and $ri = 1_A$ imply that we also have

$$\pi_i(K_B) \cong \pi_i(K_A) \times \pi_i(K_C).$$

The Whitehead lemma concludes the proof. □

### 2.2.3. The $K$-Theory Fiber Sequence.

A fundamental consequence of the Additivity Theorem is that an exact functor $C \to D$ determines a natural fibre sequence of $K$-theory spaces. We recall relevant details from [Wal85] here.

**Definition 2.20.** Let $C$ be an exact category. Define the right path-space of $S_\bullet(C)$ to be the simplicial diagram of exact categories $P^n S_\bullet(C)$ with $n$-simplices

$$P^n S_n(C) := S_{n+1}(C),$$

with the face map $d_i$ given by the face map $d_{i+1}$ of $S_\bullet(C)$, and with the degeneracy map $s_i$ given by the degeneracy $s_{i+1}$ of $S_\bullet(C)$.

The face maps $d_0: S_{n+1}(C) \to S_n(C)$ determine a map of simplicial diagrams of exact categories

$$P^n S_\bullet(C) \to S_\bullet(C).$$
Definition 2.21. ([Wal85] Definition 1.5.4) Let \( \text{C} \xrightarrow{f} \text{D} \) be an exact functor. Define the simplicial diagram of exact categories \( S^\bullet_n(f) \) to be the strict pullback
\[
\begin{array}{c}
S^\bullet_n(f) \\
\downarrow \delta \\
S^\bullet_n(S_n(C)) \\
\downarrow f \\
S^\bullet_n(S_n(D))
\end{array}
\]
Explicitly, the \( n \)-simplices \( S^\bullet_n(f) \) consist of the the full sub-category of \( S_n(C) \times S_{n+1}(D) \) on the objects
\[
(Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1})
\]
such that
\[
(f(Y_1) \hookrightarrow \ldots \hookrightarrow f(Y_n)) = (X_2/X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1}/X_1)
\]
for all \( i \geq 1 \). The face and degeneracy maps are the products of the face and degeneracy maps for \( S^\bullet_n(C) \) and \( P^n S^\bullet_n(D) \). The map \( S^\bullet_n(f) \xrightarrow{\delta} S^\bullet_n(S_n(C)) \) is the projection onto the \( S^\bullet_n(C) \)-factor.

The Additivity Theorem implies the following.

Proposition 2.22. (cf. the proof of [Wal85] Proposition 1.5.5) Let \( \text{C} \xrightarrow{f} \text{D} \) be an exact map of exact categories. The map
\[
\begin{array}{c}
S^\bullet_n(f) \\
\downarrow q^r \\
D \times S_n(C),
\end{array}
\]
\[
(Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1}) \mapsto (X_1, Y_1 \hookrightarrow \ldots \hookrightarrow Y_n)
\]
induces an equivalence
\[
|S^\bullet_n(S^\bullet_n(f))| \xrightarrow{\cong} |S^\bullet_n(D)\times| \times |S^\bullet_n(S_n(C))|.
\]

Proof. A right inverse to the above map \( q^r \) is given by the map
\[
S^\bullet_n(C) \times D \xrightarrow{\sigma} S^\bullet_n(f)
\]
sending
\[
(X, Y_1 \hookrightarrow \ldots \hookrightarrow Y_n) \mapsto (Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X \hookrightarrow X \oplus f(Y_1) \hookrightarrow \ldots \hookrightarrow X \oplus f(Y_n)).
\]
It remains to exhibit a homotopy \(|S^\bullet \sigma| \circ |S^\bullet q^r| \cong 1_{|S^\bullet_n(S^\bullet_n(f))|}|. For this, consider the functors
\[
S^\bullet_n(f) \xrightarrow{\alpha} S^\bullet_n(f)
\]
\[
(Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1}) \mapsto (0 \hookrightarrow \ldots \hookrightarrow 0; X_1 \hookrightarrow \ldots \hookrightarrow X_1)
\]
and
\[
S^\bullet_n(f) \xrightarrow{\beta} S^\bullet_n(f)
\]
\[
(Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1}) \mapsto (Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; 0 \hookrightarrow f(Y_1) \hookrightarrow \ldots \hookrightarrow f(Y_n)).
\]
Then we have a natural isomorphism \( \sigma \circ q^r \cong \alpha \oplus \beta \) as well as a short exact sequence of functors
\[
\alpha \hookrightarrow 1_{S^\bullet_n(f)} \to \beta.\]
By the Additivity Theorem (Theorem 2.16), there exists a homotopy
\[ |S\sigma| \circ |Sq| \simeq |S1_{S^\ast}(f)|. \]
We see that \(|Sq|\) is a homotopy equivalence as claimed. \(\square\)

Let \(D^\text{triv}\) denote the constant simplicial diagram on \(D\). The identity map \(D \to D\) extends to an exact map of simplicial diagrams of exact categories
\[ D^\text{triv} \to S^\ast(f). \]
Applying the \(S\)-construction to this map, we obtain an exact map of bisimplicial diagrams of exact categories
\[ S^\ast(D)^{v-\text{triv}} \to S^\ast S^\ast(f), \]
where the superscript \(\text{“}v - \text{triv}\) indicates that the bisimplicial object is constant in the vertical direction.\(^5\) Applying the \(S\)-construction to the map
\[ S^\ast(f) \to S^\ast(C), \]
we obtain an exact map of bisimplicial diagrams of exact categories
\[ S^\ast S^\ast(f) \to S^\ast S^\ast(C). \]
The above maps determine a strictly commuting square
\[ (S^\ast(D)^{x})^{v-\text{triv}} \to S^\ast(S^\ast(f))^{x} \]
\[ \downarrow \quad \downarrow \]
\[ \ast \to S^\ast(S^\ast(C))^{x}. \]

Proposition 2.22 provides the core of the proof of the following.

**Proposition 2.23.** ([Wal85, Proposition 1.5.5]) The geometric realization of the square (3) is cartesian in the \(\infty\)-category of spaces.

**Corollary 2.24.** ([Wal85, Corollary 1.5.6]) Let \(C_1 \overset{f}{\to} C_2 \overset{g}{\to} C_3\) be a sequence of exact functors. Then the square of spaces
\[ |S^\ast(C_2)^{x}| \to |S^\ast(C_3)^{x}| \]
\[ |S^\ast(S^\ast(f))^{x}| \to |S^\ast(S^\ast(gf))^{x}| \]
is cartesian.

If we consider the sequence \(C_1 \overset{1}{\to} C \overset{f}{\to} D\), we obtain the following.

---

\(^5\)We adopt the convention that in a bisimplicial set \(X^\bullet\cdot\), viewed as a first quadrant diagram, the first bullet denotes the horizontal coordinate, while the second denotes the vertical one.
Corollary 2.25. ([Wal85, Corollary 1.5.7]) Let \( \mathcal{C} \rightarrow \mathcal{D} \) be an exact functor. Then the square of spaces

\[
\begin{array}{ccc}
|S_\bullet(C)^x| & \xrightarrow{f} & |S_\bullet(D)^x| \\
\downarrow & & \downarrow \\
|S_\bullet S_r(1C)^x| & \xrightarrow{f} & |S_\bullet S_r(f)^x| \\
\end{array}
\]

is cartesian.

Lemma 2.26. Let \( \mathcal{C} \) be an exact category. Then \( |S_\bullet(1C)^x| = |P^r S_\bullet(C)^x| \simeq * \).

Proof. Denote by \( S_\bullet(C)^x \) the bisimplicial set obtained by taking the nerves of the groupoids \( S_\bullet(C)^x \) in the vertical direction, and denote by \( P^r S_\bullet(C)^x \) the analogue for \( P^r S_\bullet(C)^x \). For each \( m \), the horizontal simplicial set \( P^r S_\bullet(C)^x_m \) is obtained from \( S_\bullet(C)^x_m \) by forgetting the zeroth face and degeneracy maps and shifting all simplicial indices down by one (i.e. \( P^r S_\bullet(C)^x_m = S_\bullet(C)^x_{m+1} \) and the \( i \)th face and degeneracy maps are given by the maps \( d_i, s_i \) on \( S_\bullet(C)^x_{m+1} \)). Recall (e.g. from [Dus01, p. 219]) that the maps \( d_0: P^r S_\bullet(C)^x_m \rightleftharpoons S_\bullet(C)^x_m = *: s_0 \) are the value on \( n \)-simplices of a homotopy equivalence \( P^r S_\bullet(C)^x_m \simeq * \).

Because geometric realization preserves level-wise weak equivalences, we conclude that \( |P^r S_\bullet(C)^x| \simeq * \) is a weak equivalence.

Combining Corollary 2.25 with Lemma 2.26, we obtain the following.

Proposition 2.27 (Waldhausen). Let \( \mathcal{C} \rightarrow \mathcal{D} \) be an exact functor. There exists a natural cartesian square in the \( \infty \)-category of spaces

\[
\begin{array}{ccc}
|S_\bullet(C)^x| & \xrightarrow{f} & |S_\bullet(D)^x| \\
\downarrow & & \downarrow \\
* & \xrightarrow{f} & |S_\bullet S_r(f)^x| \\
\end{array}
\]

If we take \( \mathcal{D} \) to be the zero category, we see that the square (5) induces the inclusion of 1-simplices map

\[
|S_\bullet(C)^x| \longrightarrow \Omega(S_\bullet S_r(C)^x)
\]

of Remark 2.13. The proposition shows that this is in fact an equivalence. By iterating the \( S \)-construction, one sees that the \( K \)-theory space \( K_C \) of an exact category \( C \) is canonically an infinite loop space. We denote the corresponding connective spectrum by \( K_C \).

Remark 2.28. For an exact category \( C \), the connective spectrum \( K_C \) admits a natural non-connective extension \( \mathcal{K}_C \), capturing negative \( K \)-theory groups. In a sequel, we will extend the constructions of the present paper to non-connective \( K \)-theory and similar invariants of exact categories.

We conclude this paragraph by introducing a dual version of the relative \( S \)-construction which plays a role in the applications below.
**Definition 2.29.** Let $C$ be an exact category. Define the left path-space of $S_\bullet(C)$ to be the simplicial diagram of exact categories $P^\ell S_\bullet(C)$ with $n$-simplices

$$P^\ell S_n(C) := S_{n+1}(C),$$

with the face map $d_i$ given by the face map $d_i$ of $S_\bullet(C)$, and with the degeneracy map $s_i$ given by the degeneracy $s_i$ of $S_\bullet(C)$.

The face maps $d_{n+1}: S_{n+1}(C) \longrightarrow S_n(C)$ determine a map of simplicial exact categories

$$P^\ell S_\bullet(C) \longrightarrow S_\bullet(C).$$

**Definition 2.30.** Let $C \xrightarrow{f} D$ be an exact map of exact categories. Define the simplicial diagram of exact categories $S_\bullet(f)$ to be the strict pullback

$$\begin{array}{ccc}
S_\bullet(f) & \longrightarrow & P^\ell S_\bullet(D) \\
\delta \downarrow & & \downarrow \\
S_\bullet(C) & \xrightarrow{f} & S_\bullet(D)
\end{array}$$

Explicitly, the $n$-simplices $S_\bullet(f)$ consist of the the full sub-category of $S_n(C) \times S_{n+1}(D)$ on the objects

$$(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n; X_1 \hookrightarrow \cdots \hookrightarrow X_{n+1})$$

such that

$$(f(Y_1) \hookrightarrow \cdots \hookrightarrow f(Y_n)) = (X_1 \hookrightarrow \cdots \hookrightarrow X_n)$$

for all $i \geq 1$. The face and degeneracy maps are the products of the face and degeneracy maps for $S_\bullet(C)$ and $P^\ell S_\bullet(D)$. The map $S_\bullet(f) \xrightarrow{\delta} S_\bullet(C)$ is the projection onto the $S_\bullet(C)$ factor.

**Lemma 2.31.** Let $C \xrightarrow{f} D$ be an exact functor. Denote by $f^{\text{op}}$ the induced functor on opposite categories, and denote by $t: \Delta \longrightarrow \Delta$ the functor which sends a finite ordinal to its opposite (equivalently, $t^*$ reverses the order of simplices in a simplicial diagram). Then there exists a natural equivalence of simplicial diagrams of exact categories

$$t^* S_\bullet(f^{\text{op}})^{\text{op}} \xrightarrow{\simeq} S_\bullet(f).$$

We emphasize that on the left hand side, we have first replaced $f$ by $f^{\text{op}}$, then applied $S_\bullet(-)$, then taken the opposite categories in the simplicial diagram $S_\bullet(f^{\text{op}})$ and then reversed the orientation of the simplices in this diagram.

**Proof.** We begin by observing that $S_\bullet(f^{\text{op}})^{\text{op}}$ is equivalent to the category consisting of objects

$$(\tilde{Y}; \tilde{X}; \phi) := (Y_n \rightarrow \cdots \rightarrow Y_1; X_{n+1} \rightarrow \cdots \rightarrow X_1; \phi)$$

where $Y_n \rightarrow \cdots \rightarrow Y_1$ is a string of admissible epics in $C$, $X_{n+1} \rightarrow \cdots \rightarrow X_1$ is a string of admissible epics in $D$, and $\phi$ is an isomorphism

$$(f(Y_n) \rightarrow \cdots \rightarrow f(Y_1)) \rightarrow (\ker(X_{n+1} \rightarrow X_1) \rightarrow \cdots \rightarrow \ker(X_2 \rightarrow X_1)).$$

A morphism $(\tilde{Y}^0; \tilde{X}^0; \phi^0) \longrightarrow (\tilde{Y}^1; \tilde{X}^1; \phi^1)$ consists of a collection of morphisms

$$Y_i^0 \longrightarrow Y_i^1$$

$$X_i^0 \longrightarrow X_i^1$$
making all of the appropriate diagrams commute.

Consider the assignment which sends \((Y_\to Y_1; X_\to X_1; \varphi)\) to

\[
(\ker(Y_\to Y_{n-1}) \to \cdots \to \ker(Y_\to Y_1) \to Y_n; \\
\ker(X_{n+1} \to X_n) \to \cdots \to \ker(X_{n+1} \to X_1) \to X_{n+1}; \tilde{\varphi}),
\]

where \(\tilde{\varphi}\) denotes the isomorphism

\[
(\ker(Y_\to Y_{n-1}) \to \cdots \to \ker(Y_\to Y_1)) \cong (\ker(X_{n+1} \to X_n) \to \cdots \to \ker(X_{n+1} \to X_1))
\]

induced, by Noether’s lemma, from \(\varphi\). This extends to an equivalence of categories

\[
S_n^\ell(f^\text{op})^\text{op} \cong S_n^\ell(f),
\]

where the inverse is defined in the analogous manner.

Under this equivalence, the face map \(d_i\) on \(S_n^\ell(C^\text{op} \subset D^\text{op})^\text{op}\) corresponds to the face map \(d_{n-i}\) on \(S_n^\ell(C \subset D)\), while the degeneracy \(s_i\) corresponds to the degeneracy \(s_{n-i}\). Letting \(n\) vary, these equivalences determine an equivalence of simplicial diagrams of exact categories. \(\square\)

**Remark 2.32.** Note that the equivalence of Lemma 2.31 fits into a natural commuting square

\[
\begin{array}{ccc}
(D^\text{op})^\text{op} & \longrightarrow & t^*S_n^\ell(f^\text{op})^\text{op} \\
1 & \downarrow & \cong \\
D & \longrightarrow & S_n^\ell(f).
\end{array}
\]

Taking \(D = 0\) in the lemma above, we obtain the following.

**Corollary 2.33.** There is a natural equivalence \(t^*S_n^\ell(C^\text{op})^\text{op} \cong S_n^\ell(C)\).

Combining this with the results above, we obtain the following.

**Proposition 2.34.** Let \(C \stackrel{f}{\longrightarrow} D\) be an exact functor.

(a) The map

\[
S_n^\ell(f) \longrightarrow S_n(C) \times D,
\]

given on objects by the assignment

\[
(Y_1 \hookrightarrow \cdots \hookrightarrow Y_n; X_1 \hookrightarrow \cdots \hookrightarrow X_{n+1}) \longrightarrow (Y_1 \hookrightarrow \cdots \hookrightarrow Y_n, X_{n+1}/X_n)
\]

induces a homotopy equivalence

\[
|S_n^\ell(f)| \longrightarrow |S_n^\ell(C)| \times |S_n^\ell(D)|.
\]
(b) There exists a natural commuting cube

\begin{align*}
\begin{array}{ccc}
|S_\bullet(C^\text{op})^\times| & \xrightarrow{f^\text{op}} & |S_\bullet(D^\text{op})^\times| \\
\searrow & & \searrow \\
|S_\bullet(C)^\times| & \xrightarrow{f} & |S_\bullet(D)^\times| \\
\downarrow & & \downarrow \\
|S_\bullet(S^\text{r}(1_\text{C})^\times)| & \xrightarrow{\sim} & |S_\bullet(S^\text{r}(f^\text{op})^\times)| \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{ccc}
|S_\bullet(S^\text{r}(1_\text{C})^\times)| & \xrightarrow{\sim} & |S_\bullet(S^\text{r}(f)^\times)| \\
\end{array}
\end{align*}

in which all the diagonal arrows are equivalences. In particular, the front face is cartesian, and its lower left corner is contractible.

Proof. For the first statement, the definition of the map $q^r$ ensures that, under the equivalences of Lemma \[\ref{lemma}\], it is naturally isomorphic to the map

\[S^\text{r}_n(f^\text{op})^\text{op} \xrightarrow{q^\text{r}-\text{op}} S^\text{r}_n(C^\text{op})^\text{op} \times (D^\text{op})^\text{op},\]

where $q^r$ is the map associated to $\xymatrix{C \ar[r]^{f} & D}$ in Proposition \[\ref{prop}\]. Applying the $S$-construction, we obtain a map

\[S_\bullet(S^\text{r}_n(f^\text{op})^\text{op}) \xrightarrow{S_\bullet q^\text{r}-\text{op}} S_\bullet(S^\text{r}_n(C^\text{op})^\text{op}) \times S_\bullet(D).\]

By Corollary \[\ref{cor}\] this map is naturally equivalent to

\[t^\bullet S_\bullet(S^\text{r}_n(f^\text{op})^\text{op}) \xrightarrow{t^\bullet S_\bullet q^\text{r}} t^\bullet S_\bullet(S^\text{r}_n(C^\text{op})^\text{op}) \times S_\bullet(D)^\text{op}.\]

For any groupoid $G$, the assignment (on morphisms) $g \mapsto g^{-1}$ determines a natural equivalence of groupoids $G \xrightarrow{\sim} G^\text{op}$. There is also a canonical natural equivalence $\xymatrix{t^\bullet X \ar[r]^{\sim} & X}$ for any simplicial set $X_\bullet$. Applying both of these equivalences, we see that the previous map fits into a commuting square of spaces

\[|t^\bullet(S_\bullet(S^\text{r}_n(f^\text{op})^\text{op})^\times)| \xrightarrow{\sim} |t^\bullet(S_\bullet(S^\text{r}_n(C^\text{op}) \times D^\text{op})^\text{op})^\times| \xrightarrow{\sim} |S_\bullet(S^\text{r}_n(C^\text{op}) \times D^\text{op})^\text{op}^\times|.\]

Applying the canonical equivalence $|S_\bullet(S^\text{r}_n(C^\text{op}) \times D^\text{op})^\text{op}^\times| \xrightarrow{\sim} |S_\bullet(S^\text{r}_n(C^\text{op}))^\times| \times |S_\bullet(D^\text{op})^\times|$, we see that the bottom map is the equivalence of Proposition \[\ref{prop}\].

The second statement follows by a similar argument. The equivalences $G \simeq G^\text{op}$ and $|X| \simeq |t^\bullet X|$ determine a commuting cube of spaces

\begin{align*}
\begin{array}{ccc}
|S_\bullet(C^\text{op})^\times| & \xrightarrow{f^\text{op}} & |S_\bullet(D^\text{op})^\times| \\
\searrow & & \searrow \\
|t^\bullet(S_\bullet(C^\text{op})^\times)| & \xrightarrow{\sim} & |t^\bullet(S_\bullet(D^\text{op})^\times)| \\
\downarrow & & \downarrow \\
|S_\bullet(S^\text{r}(1_\text{C})^\times)| & \xrightarrow{\sim} & |S_\bullet(S^\text{r}(f^\text{op})^\times)| \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{ccc}
|t^\bullet(S_\bullet(S^\text{r}(1_\text{C})^\times))^\times| & \xrightarrow{\sim} & |t^\bullet(S_\bullet(S^\text{r}(f^\text{op})^\times))^\times| \\
\end{array}
\end{align*}
Applying the equivalence of Corollary 2.33 to the outer $S$, we see that the front face of this square fits into a commuting cube

\[
\begin{array}{c}
|t^*(S(C^{op})^{op})^\times| \\
\Downarrow f^{op} \\
|S(C)^\times| \\
\Downarrow f \\
|S(D)^\times| \\
\end{array}
\]

\[
\begin{array}{c}
|t^*(S(S^r(1C^{op}))^{op})^\times| \\
\Downarrow \simeq \\
|S(S^r(1C^{op})^{op})^\times| \\
\Downarrow \simeq \\
|S(S^r(f^{op})^{op})^\times|. \\
\end{array}
\]

Applying the equivalence of Lemma 2.31 to the inner $S$-construction, we obtain a commuting cube

\[
\begin{array}{c}
|S(C)^\times| \\
\Downarrow 1 \\
|S^r(C)^\times| \\
\Downarrow f \\
|S^r(D)^\times| \\
\Downarrow 1 \\
|S^r(t^*S^r(1C)^\times)| \\
\Downarrow \simeq \\
|S^r(t^*S^r(f)^\times)|. \\
\end{array}
\]

By a final application of the equivalence $|X| \simeq |t^*X|$, we see that the front face of this cube is equivalent to the front face of (6). Composing this equivalence with the cubes above, we obtain the cube (6) as claimed. \hfill \Box

2.2.4. Localization. In [Sch04], Schlichting established a fundamental “Localization Theorem” for the $K$-theory of exact categories.

**Proposition 2.35.** (Schlichting [Sch04] Lemma 2.3) Let $C \subset D$ be the inclusion of an idempotent complete, right $s$-filtering sub-category. Consider the map of simplicial diagrams of categories

\[
S^r(C \subset D) \xrightarrow{q} D/C
\]

given by the assignment

\[
(Y_1 \leftarrow \ldots \leftarrow Y_n; X_1 \leftarrow \ldots \leftarrow X_{n+1}) \mapsto X_{n+1}.
\]

Then all of the diagonal arrows in the commuting cube of spaces are equivalences

\[
\begin{array}{c}
|S(C)^\times| \\
\Downarrow 1 \\
|S^r(C)^\times| \\
\Downarrow f \\
|S^r(D)^\times| \\
\Downarrow 1 \\
|S^r(S^r(1C)^\times)| \\
\Downarrow \simeq \\
|S^r(S^r(f)^\times)| \\
\end{array}
\]

Combined with Proposition 2.33, this implies the following.
**Proposition 2.36.** Let \( C \subset D \) be the inclusion of an idempotent complete, left s-filtering sub-category. Consider the map of simplicial diagrams of categories

\[
S^\ell(C \subset D) \xrightarrow{q} D / C
\]

given by the assignment

\[
(Y_1 \hookrightarrow \ldots \hookrightarrow Y_n; X_1 \hookrightarrow \ldots \hookrightarrow X_{n+1}) \mapsto X_{n+1}.
\]

Then all of the diagonal arrows in the commuting cube of spaces are equivalences

\[
\begin{array}{ccc}
|S^\bullet(\mathbb{C})^\times| & \xrightarrow{f} & |S^\bullet(\mathbb{D})^\times| \\
|S^\bullet(\mathbb{C})^\times| & \downarrow & |S^\bullet(\mathbb{D})^\times| \\
|S^\bullet S^\ell(1_C)^\times| & \xrightarrow{g} & |S^\bullet S^\ell(f)^\times| \\
& \downarrow & \downarrow \\
& |S^\bullet(\mathbb{D} / C)^\times| & \xrightarrow{\partial} |S^\bullet(\mathbb{D} / C)^\times|
\end{array}
\]

**Proof.** Because \( C \subset D \) is left s-filtering, \( C^{\text{op}} \subset D^{\text{op}} \) is right s-filtering. We can now compose the cube of Proposition 2.35 with the cube (6) of Proposition 2.34 to obtain a cube of the form above. By tracing through the construction, we see that the equivalence in this cube

\[
S^\ell(C \subset D) \xrightarrow{q} D / C
\]

is given by the map \( q \) above. \( \square \)

Combined with Propositions 2.27 and 2.34, these propositions give the following.

**Theorem 2.37** (Schlichting’s Localization Theorem). Let \( C \subset D \) be the inclusion of an idempotent complete, left or right s-filtering sub-category. Then the square of spaces

\[
\begin{array}{ccc}
K_C & \xrightarrow{} & K_D \\
\downarrow & & \downarrow \\
* & \xrightarrow{} & K_D / C
\end{array}
\]

is cartesian.

### 2.3. Boundary Maps in Algebraic \( K \)-Theory.

By the universal property of cartesian squares, the Localization Theorem associates, to a left or right s-filtering sub-category \( C \subset D \), a boundary map

\[
\Omega K_{D / C} \xrightarrow{\partial} K_C.
\]

For our applications, we will need to be able to describe this map in some detail. We therefore use this section to explain how the results of Waldhausen which we recalled above lead to an explicit description of this boundary map.

From the proof of the Localization Theorem, we see that the \( K \)-theory localization sequence is equivalent to the looping of the Waldhausen fibration sequence. To avoid a proliferation of \( \Omega \)s on the page, we will describe the boundary map

\[
\Omega |S^\bullet S^\ell(f)^\times| \xrightarrow{\partial} |S^\bullet(\mathbb{C})^\times|. \tag{7}
\]
Note that the boundary map in $K$-theory is obtained by applying $\Omega$ to this map. Note also that, by Proposition 2.34, our description will immediately imply an analogous description of the map

$$\Omega|S_\bullet S^\ell(f)^\times| \xrightarrow{\partial} |S_\bullet(C)^\times|.$$ 

**Proposition 2.38.** Let $\xymatrix{C \ar[r]^f & D}$ be an exact functor. The boundary map (2.3) fits into a natural, commuting diagram

(8) 

$$\xymatrix{ |S^\ell(f)^\times| \ar[r]^\delta \ar[d]_{\simeq} & |S_\bullet(C)^\times| \ar[d]_{\simeq} \\
\Omega|S_\bullet S^\ell(f)^\times| \ar[r]_{\Omega|S_\bullet \delta|} \ar[d]_{\partial} & \Omega|S_\bullet S_\bullet(C)^\times| \ar[d]_{\simeq} \\
|S_\bullet(C)^\times| \ar[d]_{\simeq} \\
|S_\bullet(C)^\times| \ar[d]_{\simeq} \\
|S_\bullet(C)^\times| \ar[d]_{\simeq} \\
|S_\bullet(C)^\times|}$$

The vertical equivalences in the rectangle are those induced by the inclusions of horizontal 1-simplices $S_\bullet(C) \xrightarrow{1} S_1 S_\bullet(C)$ and $S^\ell(f) \xrightarrow{1} S_1 S^\ell(f)$. The equivalence in the triangle is inverse to the equivalence induced by the inclusion of vertical 1-simplices $S_\bullet(C) \xrightarrow{1} S_\bullet S_1(C)$. Because the two inclusion of 1-simplices maps are canonically equivalent, the composition of the right vertical maps is canonically equivalent to the identity $1|S_\bullet(C)^\times|$. In particular, the boundary map (2.3) is canonically naturally equivalent to the map $|S^\ell(f)^\times| \xrightarrow{\delta} |S_\bullet(C)^\times|$.

**Proof.** By Corollary 2.25 the commuting square of exact functors

$$\xymatrix{ C \ar[r]^1 \ar[d]_f & C \ar[d] \\
D \ar[r] & 0}$$

determines a commuting cube of spaces

in which the left and right faces are cartesian. By Lemma 2.26 the lower rear corners of this diagram are contractible. Therefore, by the universal property of cartesian squares, this diagram
determines a commuting square of spaces

\[
\begin{array}{ccc}
\Omega|S_\bullet S_r(f)^\times| & \xrightarrow{\Omega|S_\bullet \delta|} & \Omega|S_\bullet S_\bullet(C)^\times| \\
\partial & \downarrow \approx & \\
|S_\bullet(C)^\times| & \xrightarrow{1} & |S_\bullet(C)^\times|
\end{array}
\]

or equivalently, a commuting triangle as in (8). As Waldhausen observed in [Wal85, Lemma 1.5.2], the equivalence induced by the right face of the above cube is inverse to the equivalence

\[
|S_\bullet(C)^\times| \xrightarrow{\approx} \Omega|S_\bullet S_\bullet(C)^\times|
\]

induced by the inclusion of vertical 1-simplices \(S_\bullet(C) \rightarrow S_\bullet S_1(C)\).

Similarly, by Corollary 2.25, the commuting square of exact functors

\[
\begin{array}{ccc}
S_\bullet(f) & \delta & S_\bullet(C) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}
\]

determines a commuting cube of spaces

\[
\begin{array}{ccc}
|S_\bullet(f)^\times| & \delta & |S_\bullet(C)^\times| \\
\downarrow & \downarrow & \downarrow & \\
|S_\bullet(1S_\bullet(f))^\times| & \xrightarrow{\ast} & |S_\bullet(1S_\bullet(C))^\times| \\
\downarrow & \downarrow & \downarrow & \\
|S_\bullet S_\bullet(f)^\times| & \xrightarrow{\delta} & |S_\bullet S_\bullet(C)^\times|
\end{array}
\]

in which the left and right faces are cartesian. By Lemma 2.26 the lower rear corners are naturally contractible. Therefore, by the universal property, this cube determines a commuting square

\[
\begin{array}{ccc}
|S_\bullet(f)^\times| & \delta & |S_\bullet(C)^\times| \\
\downarrow & \approx & \downarrow \\
\Omega|S_\bullet S_\bullet(f)^\times| & \xrightarrow{\Omega|S_\bullet \delta|} & \Omega|S_\bullet S_\bullet(C)^\times|
\end{array}
\]

This gives the upper square in (8). By the same reasoning as in [Wal85, Lemma 1.5.2], the vertical equivalences in the square above are the maps induced by the inclusion of horizontal 1-simplices \(S_\bullet(C) \rightarrow S_1 S_\bullet(C)\) and \(S_\bullet(f) \rightarrow S_1 S_\bullet(f)\).

To compare the two maps \(|S_\bullet(C)^\times| \xrightarrow{\approx} \Omega|S_\bullet S_\bullet(C)^\times|\) produced above, it suffices to observe that, after applying the diagonal functor from bisimplicial to simplicial objects, we have an equality of commuting squares

\[
\begin{array}{ccc}
S_\bullet(C) & \rightarrow & 0 \\
\downarrow & = & \downarrow \\
d(S_\bullet S_\bullet(1_C)) & \xrightarrow{d} & d(S_\bullet S_\bullet(C))
\end{array}
\]

\[
\begin{array}{ccc}
S_\bullet(C) & \rightarrow & 0 \\
\downarrow & = & \downarrow \\
d(S_\bullet(1_{S_\bullet(C)})) & \xrightarrow{d} & d(S_\bullet S_\bullet(C)).
\end{array}
\]

Because \(|X| \approx |d(X)|\) for any bisimplicial set \(X\), we conclude that the two maps are equivalent. □
Corollary 2.39. If $C \subset D$ is the inclusion of an idempotent complete, right s-filtering sub-category, then the map

$$|S^\bullet_*(C \subset D)^\times\rangle \xrightarrow{\delta} |S^\bullet_*(C)^\times\rangle$$

is canonically equivalent to the boundary map

(9) $$\Omega |S^\bullet_*(D/C)^\times\rangle \xrightarrow{\partial} |S^\bullet_*(C)^\times\rangle$$

associated to the localization sequence

(10) $$|S^\bullet_*(C)^\times\rangle \xrightarrow{\delta} |S^\bullet_*(D/C)^\times\rangle$$

Similarly, by tracing through the proof of the above, we see that if $C \subset D$ is the inclusion of an idempotent complete, left s-filtering sub-category, then the map

$$|S^\bullet_\ell_*(C \subset D)^\times\rangle \xrightarrow{\delta} |S^\bullet_\ell_*(C)^\times\rangle$$

is canonically equivalent to $-1$ times the boundary map (9).

3. The Index Map

In this section, we use Theorem 2.10 to define the index map $\Omega K_{\text{Tate}^\ell(C)} \xrightarrow{\text{Index}} K_C$. In Theorem 3.19 and Corollary 3.20, we produce an explicit combinatorial model for this map. Using the Additivity Theorem, we show in Theorem 3.22 that the index map is an equivalence. We conclude this section with Theorem 3.25, where we relate the index map to the boundary map associated by the Localization Theorem to the left s-filtering inclusion $C \subset \text{Ind}^a(C)$.

3.1. The Categorical Index Map. For $n \geq 0$, denote by $[n]$ the partially ordered set $\{0 < \ldots < n\}$, and denote by $\text{Fun}([n], C)$ the category of functors from the partially ordered set $[n]$, viewed as a category, to a category $C$.

**Definition 3.1.** Let $C$ be an exact category. Define the Sato complex $Gr^\prec_\bullet(C)$ to be the simplicial diagram of exact categories with

1. $n$-simplices $Gr^\prec_\bullet(C)$ given by the full sub-category of $\text{Fun}([n+1], \text{Tate}^a(C))$ consisting of sequences of admissible monics

$$L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V$$

where, for all $i$, $L_i \hookrightarrow V$ is the inclusion of a lattice\(^6\)

2. face maps are given by the functors

$$d_i(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) := (L_0 \hookrightarrow \cdots \hookrightarrow L_{i-1} \hookrightarrow L_{i+1} \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V),$$

3. and degeneracy maps are given by the functors

$$s_i(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) := (L_0 \hookrightarrow \cdots \hookrightarrow L_i \hookrightarrow L_i \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V).$$

The simplicial object $Gr^\prec_\bullet(C)$ allows us to introduce the index map.

\(^6\)To see that this is an exact category, observe that because $\text{Pro}^a(C)$ and $\text{Ind}^a(C)$ are closed under extensions in $\text{Tate}^a(C)$, $Gr^\prec_\bullet(C)$ is closed under extensions in $\text{Fun}([n+1], \text{Tate}^a(C))$. 
Definition 3.2. Let $C$ be an exact category. The categorical index map is the span of simplicial maps

\[ \text{Tate}_{el}(C) \leftarrow \text{Gr}^{< \infty}(C) \xrightarrow{\text{Index}} S_{\bullet}(C), \]

where the left-facing arrow is given on $n$-simplices by the assignment

\[ (L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto V, \]

and $\text{Index}$ is given on $n$-simplices by the assignment

\[ (L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0). \]

3.2. The $K$-Theoretic Index Map. In this section, we explain how the categorical index map determines an index map in $K$-theory.

Proposition 3.3. Let $C$ be an idempotent complete exact category. Then the map from $\text{Gr}^{\leq \infty}(C)$ to $\text{Tate}_{el}(C)$ of (11) induces an equivalence

\[ |\text{Gr}^{\leq \infty}(C)| \xrightarrow{\simeq} \text{Tate}_{el}(C). \]

Remark 3.4. The fibres of the augmentation are by definition equivalent to the nerve of a filtered partially ordered set. Hence, the map induced after taking the geometric realization, has contractible fibres (the classifying space of a filtered partially ordered set is contractible). This idea lies at the heart of the following proof.

Proof of Proposition 3.3. Geometric realizations are colimits in the $\infty$-category of spaces. Similarly, the groupoid $\text{Gr}^{\leq \infty}(C)$ is the Grothendieck construction of the set-valued functor $\text{Gr}^{\leq \infty}(C) \to \text{Set}$, i.e. it is a colimit in the category of groupoids.

Commuting the two colimits we obtain the following equivalence

\[ |\text{Gr}^{\leq \infty}(C)| \simeq \lim_{V \in \text{Tate}_{el}(C)} |\text{Gr}^{\leq \infty}(V)|. \]

For a fixed Tate object, the geometric realization $|\text{Gr}^{\leq \infty}(V)|$ computes the classifying space of the category corresponding to the partially ordered set $\text{Gr}(V)$. Because the partially ordered set $\text{Gr}(V)$ is directed (Theorem 2.10(c)), its classifying space is contractible. This implies that

\[ |\text{Gr}^{\leq \infty}(C)| \simeq \lim_{\text{Tate}_{el}(C)} \{\ast\} \simeq \text{Tate}_{el}(C) \]

and hence concludes the proof. $\square$

Corollary 3.5. Let $C$ be idempotent complete. The categorical index map determines a map

\[ \text{Index}: \text{Tate}_{el}(C) \xrightarrow{\simeq} |S_{\bullet}(C)| \simeq BK_{C}. \]

Moreover, using the canonical equivalence $S_k(\text{Tate}_{el}(C)) \simeq \text{Tate}_{el}(S_k(C))$ [BGW14, Prop. 5.10], we also obtain the following.

Corollary 3.6. Let $C$ be idempotent complete. The categorical index map determines a map of infinite loop spaces

\[ B\text{Index}: |S_{\bullet}(\text{Tate}_{el}(C))| \longrightarrow |S_{\bullet}S_{\bullet}(C)| \]
which fits into a commuting triangle

\[
\begin{array}{ccc}
\text{Tate}^\text{el}(C)^x & \xrightarrow{\text{Index}} & \Omega|S^\text{el}(C)^x| \\
\downarrow & & \downarrow \Omega B \text{Index} \\
\Omega|S^\text{el}(C)^x| & & \\
\end{array}
\]

Proof. Recall from [Wal85, p. 329] that the infinite loop space structures on $|S^\text{el}(C)^x|$ and $|S^\text{el}(C)^x|$ are given by the chain of equivalences

\[
|S^{x^n}(\text{Tate}^\text{el}(C))^x| \xrightarrow{\simeq} \Omega|S^{x^{n+1}}(\text{Tate}^\text{el}(C))^x|,
\]

and

\[
|S^{x^{n+1}}(C)^x| \xrightarrow{\simeq} \Omega|S^{x^{n+2}}(C)^x|,
\]

which are determined by the inclusions of 1-simplices

\[
\Delta^1 \times S^{x^n}(\text{Tate}^\text{el}(C))^x \longrightarrow S^{x^{n+1}}(\text{Tate}^\text{el}(C))^x,
\]

and

\[
\Delta^1 \times S^{x^{n+1}}(C)^x \longrightarrow S^{x^{n+2}}(C)^x.
\]

For each $n$, we have a map

\[
|S^{x^n}(\text{Tate}^\text{el}(C))^x| \xrightarrow{\simeq} |\text{Tate}^\text{el}(S^{x^n}(C))^x| \xrightarrow{\simeq} |\text{Gr}^x_<(S^{x^n}(C))^x| \longrightarrow |S^xS^{x^n}(C)^x|.
\]

By inspection, these maps fit into a commuting square

\[
\begin{array}{ccc}
S^{x^n}(\text{Tate}^\text{el}(C))^x & \xrightarrow{\text{Index}} & |S^xS^{x^n}(C)^x| \\
\downarrow & & \downarrow \Omega B \text{Index} \\
\Omega|S^{x^{n+1}}(\text{Tate}^\text{el}(C))^x| & & \Omega|S^xS^{x^{n+1}}(C)^x|
\end{array}
\]

for each $n$. For $n = 0$, this square gives the triangle (13). Taken together for all $n$, these squares show that (12) is a map of infinite loop spaces. \[\square\]

3.3. A Combinatorial Model of the Index Map. In this section, we introduce convenient simplicial models of $|\text{Gr}^x_<(C)^x|$ and $|S^x(C)^x|$ which allow us to construct the map

\[
|\text{Tate}^\text{el}(C)^x| \longrightarrow |S^x(C)^x|
\]

as an explicit simplicial map from the nerve of $\text{Tate}^\text{el}(C)^x$.

Remark 3.7. We pause for a moment to explain the data exhibited by such a map.

1. From the perspective which we adopt in this section, the combinatorial model of the index map can be understood as a universal computation of indices, symbols, and higher torsion invariants of automorphisms of elementary Tate objects. In order to define the computation, we require a sequence of auxiliary choices. Theorem 2.10 ensures that the data required for
these choices exist, while the framework of simplicial homotopy theory ensures that the end-result is independent of the choices.\footnote{We conclude this section with a sample computation which explains the name of the index map.}

(2) From another perspective, this simplicial map encodes an $E_1$-map

$$\text{Aut}(V) \longrightarrow K_C$$

for all elementary Tate objects $V$. More precisely, recall (e.g. [AJ09, Chapter V]) that there exists a model structure, essentially due to Kan, on the category $\text{sSet}_0$ of reduced simplicial sets, i.e. simplicial sets having a unique vertex. There is also a model structure, essentially due to Moore, on the category $\text{sGrp}$ of simplicial groups, and a Quillen equivalence $G: \text{sSet}_0 \leftrightarrow \text{sGrp}: W$, with $G \dashv W$. At the level of the underlying $\infty$-categories, this equivalence corresponds to the adjoint equivalence $\Omega \dashv B_{\bullet}$ between the $\infty$-category of group-like $E_1$-spaces and the $\infty$-category of pointed connected spaces. Both the nerve $N_{\bullet} \text{Aut}(V)$ of the group $\text{Aut}(V)$ and the model we use of $BK_C$ are reduced simplicial sets, so the construction falls within this classical framework.

However, there exists a different, and, for many purposes, more natural approach to $E_1$-objects in an $\infty$-category, essentially due to Segal. In Section 4, we develop a formalism which allows us to produce an efficient Segal-style model for this $E_1$-map in the $\infty$-category of spaces.

(3) From a third perspective, which we develop in Section 5.2, this simplicial map can be understood as specifying the data of a $K_C$-torsor $T \longrightarrow \text{Tate}_{\ast}(C)^{\times}$, or as specifying, for each elementary Tate object $V$, a $K_C$-torsor $T|_V$ with a coherent action of $\text{Aut}(V)$.

**Notation 3.8.** Throughout this section, $I \subset [n]$ will denote a non-empty sub-set.

Given a groupoid $G$ and a functor $F_{\bullet}: G \longrightarrow \text{sSet}$, denote by $\int_G F_{\bullet}$ the Grothendieck construction of $F_{\bullet}$. Explicitly, $\int_G F_{\bullet}$ is the simplicial diagram of categories whose category of $n$-simplices is the usual Grothendieck construction of $F_n$.

Given an elementary Tate object $V$, denote by $\text{Gr}^<=_{\bullet}(V)$ the nerve of the partially ordered set $\text{Gr}(V)$. As we observed in the proof of Proposition 3.3, the assignment $V \mapsto \text{Gr}^<=_{\bullet}(V)$ defines a functor

$$\text{Gr}^<=_{\bullet}: \text{Tate}^{et}(C)^{\times} \longrightarrow \text{sSet}$$

with $\int_{\text{Tate}^{et}(C)^{\times}} \text{Gr}^<=_{\bullet} = \text{Gr}^<=(C)^{\times}$. Starting with this observation, we now introduce several constructions which allow us to define a simplicial model for the inverse of the equivalence $|\text{Gr}^<=_{\bullet}(C)^{\times}| \overset{\simeq}{\longrightarrow} \text{Tate}^{et}(C)^{\times}$.

### 3.3.1. Subdivision and Kan’s Ex$^1$.

**Definition 3.9.** The subdivision of the linearly ordered set $[n]$, denoted $\text{sd}([n])$, is the partially ordered set consisting of all non-empty sub-sets $I \subset [n]$, ordered by inclusion.

By taking the nerve of $\text{sd}([n])$, we obtain a functor

$$\Delta \longrightarrow \text{sSet}$$

\footnote{Though we do not pursue this here, we also expect that, given two sequences of such auxiliary choices, one can directly construct a homotopy between the resulting simplicial maps by a sequence of similar choices.}
The left Kan extension of this functor along the Yoneda embedding gives a functor
\[ \text{sd} : \text{sSet} \rightarrow \text{sSet}. \]

**Example 3.10.** The simplicial set \( \text{sd} \Delta^1 \) consists of two 1-simplices \( x_0 \) and \( x_1 \) glued at their ends
\[
\bullet \rightarrow \bullet \longleftrightarrow \bullet
\]

**Definition 3.11.** Let \( X_\bullet \) be a simplicial set. Define \( \text{Ex}^1(X)_\bullet \) to be the simplicial set whose \( n \)-simplices are given by
\[
\text{Ex}^1(X)_n := \text{hom}_{\text{sSet}}(\text{sd} \Delta^n, X_\bullet).
\]

**Example 3.12.** The example above shows that a 1-simplex of \( \text{Ex}^1(X) \) consists of two 1-simplices \( x_0 \) and \( x_1 \) of \( X \) glued at their ends.

The assignment \( I \mapsto \max(I) \) defines a natural map of partially ordered sets
\[ \text{sd}([n]) \rightarrow [n]. \]
This extends to a natural transformation \( \text{sd}(-) \rightarrow 1(-) \), which, in turn, defines a natural transformation
\[ 1(-) \rightarrow \text{Ex}^1(-). \]

The following is one of the foundational results of simplicial homotopy theory.

**Lemma 3.13 (Kan).** ([Kan57], cf. [GJ09, Theorem III.4.6]) Let \( X_\bullet \) be a simplicial set. The map
\[ X_\bullet \rightarrow \text{Ex}^1(X)_\bullet \]
is a weak equivalence.

When \( X_\bullet \) is the nerve of a poset \( P \), \( \text{Ex}^1(X)_\bullet \) admits a particularly simple description.

**Example 3.14.** Let \( P \) be a partially ordered set. Denote by \( P_\bullet \) its nerve. Then
\[ \text{Ex}^1(P)_n \cong \{(x_I)_{I \subseteq [n]} \mid x_I \in P \text{ for all } I, \text{ and } x_I < x_J \text{ for } I \subseteq J\}. \]
The face \( i \)th face of a simplex \( (x_I)_{I \subseteq [n]} \) is given by the functor
\[ J \mapsto x_{d_i(J)} \]
Conversely, the \( i \)th degeneracy of a simplex \( (x_I)_{I \subseteq [n]} \) is given by the functor
\[ J \mapsto x_{s_i(J)}. \]

Under the inclusion \( \text{Set} \hookrightarrow \text{Cat} \), we obtain a functor
\[ \text{sd}^h : \text{sSet} \rightarrow \text{sSet} \hookrightarrow \text{sCat}. \]

**Definition 3.15.** Let \( X_\bullet : \Delta^{\text{op}} \rightarrow \text{Cat} \) be a simplicial category. Define \( \text{Ex}^{1,h}(X)_\bullet \) to be the simplicial category with
\[ \text{Ex}^{1,h}(X)_n := \text{Fun}_{\text{sCat}}(\text{sd}^h \Delta^n, X_\bullet). \]
From the definition, we have \( \text{ob} \text{Ex}^{1,h}(X)_\bullet \cong \text{Ex}^1(\text{ob}(X)_\bullet) \). Morphisms in \( \text{Ex}^{1,h}(X)_\bullet \) are morphisms of diagrams. This forms the basis of the following lemma.

**Lemma 3.16.** Let \( G \) be a groupoid, and let \( F_\bullet : G \rightarrow \text{sSet} \) be a functor taking values in simplicial sets. Let \( \int_G F_\bullet \) denote the simplicial diagram in groupoids obtained from the Grothendieck construction. Then
\[ \text{Ex}^{1,h}(\int_G F_\bullet) = \int_G \text{Ex}^1 F_\bullet. \]
Proof. From the definition, objects of the category $\text{Ex}^{1, h} \int_{G} F_n$ are pairs $(V, L)$, where $V \in G$ and $L \in \text{Ex}^{1, h} (F(V))_n$. A morphism $(V_0, L_0) \xrightarrow{g} (V_1, L_1)$ in $\text{Ex}^{1, h} \int_{G} F_n$ consists of a morphism $V_0 \xrightarrow{g} V_1$ in $G$ such that $F(g)(L_0) = L_1$. We see that $\text{Ex}^{1, h} \int_{G} F_n$ is, by definition, the category $\int_{G} \text{Ex}^{1, h} (F)_n$. A similar exercise shows the equality of face and degeneracy maps. □

Lemma 3.17. Let $X_\bullet$ be a simplicial diagram of categories. Then the map

$$X_\bullet \rightarrow \text{Ex}^{1, h} X_\bullet$$

induces a weak equivalence

$$|X| \xrightarrow{\sim} |\text{Ex}^{1, h} X|.$$ 

Proof. Given a simplicial diagram of categories $X_\bullet$, let $X_{\bullet, \bullet}$ be the bisimplicial set obtained by taking the nerve (in the vertical direction) of the categories $X_n$. Unpacking the definition, we see that the $\text{Ex}^{1, h} X_{\bullet, \bullet}$ is the bisimplicial set with

$$(\text{Ex}^{1, h} X)_{\bullet, n} = \text{Ex}^{1} X_{\bullet, n}.$$ 

Similarly, the map

$$X_{\bullet, \bullet} \rightarrow \text{Ex}^{1, h} X_{\bullet, \bullet}$$

is the map of bisimplicial sets given on horizontal $n$-simplices by

$$X_{\bullet, n} \rightarrow \text{Ex}^{1} X_{\bullet, n}.$$ 

By Lemma 3.13, this is a weak equivalence for all $n$. □

3.3.2. The Diagonal of the Grothendieck Construction. Let $G$ be a groupoid, and let $F_\bullet : G \rightarrow \text{sSet}$ be a functor taking values in simplicial sets. Let $\int_{G} F_\bullet$ denote the simplicial diagram in groupoids obtained from the Grothendieck construction. Taking the nerves of these groupoids (in the vertical direction), we obtain a bisimplicial set $\int_{G} F_{\bullet, \bullet}$.

Concretely, $(n, m)$-simplices of $\int_{G} F_{\bullet, \bullet}$ consist of a string of isomorphisms in $G$

$$x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} x_m$$

along with elements $y_i \in F_n(x_i)$ such that $F(g_i)(y_i) = y_{i+1}$. Because all the $g_i$ are isomorphisms, we see that $(n, m)$-simplices of $\int_{G} F_{\bullet, \bullet}$ are equivalent to tuples

$$(x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} x_m, y) \in N_{n}G \times F_n(x_m).$$

Applying the diagonal functor, we see that $n$-simplices of $d(\int_{G} F_\bullet)$ consist of tuples

$$(x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} x_n, y) \in N_{n}G \times F_n(x_n).$$

The degeneracy $s_i$ is the product of the $i^{th}$ degeneracy maps in $F_\bullet(x_n)$ and $N_\bullet G$. For $i < n$, the face map $d_i$ is given by the product of the $i^{th}$ face maps in $F_\bullet(x_n)$ and $N_\bullet G$, while the face map $d_n$ is given by

$$d_n(x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} x_n, y) := (x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} x_{n-1}, F(g_n)^{-1}(y)).$$

In the case $\text{Ex}^{1, h} G_{\bullet, \bullet} (\mathbb{C}^\times)^{\times}$, the previous description combines with Lemma 3.16 and Example 3.14 to give the following.
Lemma 3.18. Let \( C \) be an exact category. Then
\[
|Gr^\leq_n(C)^\times| \simeq |d(Ex^{1,h}Gr^\leq_n(C)^\times)|.
\]
Further, the \( n \)-simplices of the simplicial set \( d(Ex^{1,h}Gr^\leq_n(C)^\times) \) consist of tuples
\[
(V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} V_n, \{L_I\}_{I \subset [n]}) \in N_n \text{Tate}^{el}(C)^\times \times \text{Ex}^1 Gr^\leq_n(V_n).
\]
In this description, the degeneracy \( s_i \) is the product of the \( i \)-th degeneracy maps in \( N_n \text{Tate}^{el}(C)^\times \) and \( \text{Ex}^1 Gr^\leq_n(V_n) \). For \( i < n \), the face map \( d_i \) is given by the product of the \( i \)-th face maps in \( N_n \text{Tate}^{el}(C)^\times \) and \( \text{Ex}^1 Gr^\leq_n(V_n) \), while the face map \( d_n \) is given by
\[
d_n(V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} V_n, \{L_I\}_{I \subset [n]}) := (V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} V_{n-1}, \{g_n^{-1}L_{d^n(I)}\}_{I \subset [n-1]}).
\]

3.3.3. A Combinatorial Model.

Theorem 3.19. Let \( C \) be an idempotent complete exact category. A section of the map
\[
d(Ex^{1,h}Gr^\leq_n(C)^\times) \to N_n \text{Tate}^{el}(C)^\times.
\]
is constructed according to the following induction.

1. For each \( V \in \text{Tate}^{el}(C)^\times \), choose a lattice \( L_{0,[0]} \hookrightarrow V \).

2. For the inductive step, suppose that for each \( k < n \), for each non-degenerate simplex
\[
\mathcal{G} = (V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_k} V_k) \in N_k \text{Tate}^{el}(C)^\times,
\]
and for each non-empty subset \( I \subset [k] \), we have specified a collection of lattices \( L_{k,I}(\mathcal{G}) \hookrightarrow V_k \) satisfying:

(a) if \( I \subset d^i([k-1]) \subset [k] \) for \( i < k \), then
\[
L_{k,I}(\mathcal{G}) = L_{k-1,s^i(I)}(d_i \mathcal{G}),
\]
(b) if \( I \subset d^k([k-1]) \subset [k] \), then
\[
L_{k,I}(\mathcal{G}) = L_{k-1,s^k(I)}(d_k \mathcal{G}),
\]

(c) and, for all \( I \subset J \subset [k] \), \( L_{k,I}(\mathcal{G}) \) is a sub-lattice of \( L_{k,J}(\mathcal{G}) \).

Then, let
\[
\mathcal{G} = (V_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} V_n) \in N_n \text{Tate}^{el}(C)^\times
\]
be a non-degenerate simplex, let \( I \subset [n] \) be a proper, non-empty subset, and let \( i \) be any number such that \( I \subset d^i([n-1]) \subset [n] \). Define
\[
L_{n,I}(\mathcal{G}) := \begin{cases} 
L_{n-1,s^i(I)}(d_i \mathcal{G}) & \text{if } i < n \\
g_nL_{n-1,s^{n-1}(I)}(d_n \mathcal{G}) & \text{if } i = n.
\end{cases}
\]

Then \( L_{n,I}(\mathcal{G}) \) is a well-defined lattice of \( V_n \), independent of the choice of \( i \), and we complete the inductive step by choosing a lattice \( L_{n,[n]}(\mathcal{G}) \hookrightarrow V_n \) which contains \( L_{n,I}(\mathcal{G}) \) as a sub-lattice for all proper, non-empty subsets \( I \subset [n] \).

The induction constructs a collection of lattices \( \{L_{n,I}(\mathcal{G}) \hookrightarrow V_n\}_{I \subset [n]} \) for every \( n \) and every non-degenerate simplex \( \mathcal{G} \in N_n \text{Tate}^{el}(C)^\times \). These families are such that the assignment
\[
\mathcal{G} \mapsto (\mathcal{G}, \{L_{n,I}(\mathcal{G}) \hookrightarrow V_n\}_{I \subset [n]})
\]
defines a right inverse
\[ N \cdot \text{Tate}^{el}(C)^{\times} \xrightarrow{\mathcal{L}} d(\text{Ex}^{1,h} G_{\bullet}^{\leq}(C)^{\times} \cdot) \]
to the map \([14]\).

**Proof.** For ease of notation, we will leave the elementary Tate object \(V_n\) implicit in the course of the proof, e.g. we write \([18]\) as \[ \bar{g} \mapsto (\bar{g}, \{ L_{n,I}(\bar{g}) \}_{I \subset [n]}). \]

For the proof, we need to establish the following claims:

1. That for a proper, non-empty sub-set \(I \subset [n]\), \(L_{n,I}(\bar{g})\) is well-defined.
2. That we can make the choices of \(L_{n,[n]}(\bar{g})\) required for the induction.
3. That, for each non-degenerate \(n\)-simplex \(\bar{g} \in N_n \text{Tate}^{el}(C)^{\times}\), the collection \(\{ L_{n,I}(\bar{g}) \}_{I \subset [n]}\) of the inductive step satisfies the inductive hypothesis for the next stage.
4. That the assignment \([18]\) defines a simplicial map.

For the first claim, suppose that \(i < j < n\) are in \([n] \setminus I\) (i.e. \(I \subset d^i([n - 1]) \cap d^j([n - 1]) \subset [n]\)). We claim that
\[ L_{n-1,s^i(I)}(d_j \bar{g}) = L_{n-1,s^j(I)}(d_i \bar{g}). \]
By assumption, \(s^j(I) \subset d^i([n - 2])\), while \(s^i(I) \subset d^{j-1}([n - 2])\). Therefore, by the inductive hypothesis and by the simplicial identities, we have
\[ L_{n-1,s^i(I)}(d_j \bar{g}) = L_{n-2,s^{i-1}s^i(I)}(d_{j-1}d_i \bar{g}) = L_{n-1,s^j(I)}(d_i \bar{g}) \]
as asserted. It remains to show the case where \(j = n\). For this, we first suppose that \(i < n - 1\). We claim that
\[ L_{n-1,s^i(I)}(d_n \bar{g}) = g_n L_{n-1,s^{n-1}(I)}(d_n \bar{g}). \]
By assumption, \(s^{n-1}(I) \subset d^i([n - 2])\), while \(s^i(I) \subset d^{n-1}([n - 2])\). Therefore, by the inductive hypothesis and by the simplicial identities, we have
\[ g_n L_{n-1,s^{n-1}(I)}(d_n \bar{g}) = g_n L_{n-2,s^{n-2}(I)}(d_n d_i \bar{g}) = g_n L_{n-2,s^{n-2}s^{n-1}(I)}(d_n d_{i-1} \bar{g}) \]
as asserted.

Finally, suppose that \(i = n - 1\). Note that, because the maps \(s^{n-2}\) and \(s^{n-1}\) are equal on \([n] \setminus \{n - 1, n\}\), our assumption on \(I\) implies that \(s^{n-2}(I) = s^{n-1}(I) \subset [n - 1]\). Then we claim that
\[ L_{n-1,s^{n-1}(I)}(d_{n-1} \bar{g}) = g_n L_{n-1,s^{n-1}(I)}(d_{n-1} \bar{g}). \]
By assumption, \(s^{n-1}(I)\) is contained in \(d^{n-1}([n - 2]) \subset [n - 1]\). Therefore, by the inductive hypothesis, by the simplicial identities, and by the equality \(s^{n-2}(I) = s^{n-1}(I)\), we have
\[ g_n L_{n-1,s^{n-1}(I)}(d_{n-1} \bar{g}) = g_n L_{n-2,s^{n-2}(I)}(d_{n-1} d_{n-1} \bar{g}) = g_n L_{n-2,s^{n-2}(I)}(d_{n-1} d_{n-1} \bar{g}) = L_{n-1,s^{n-1}(I)}(d_{n-1} \bar{g}). \]
We have established the first claim.
The second claim follows, by Theorem 2.10, from our assumption that \( C \) is idempotent complete. Indeed, for every elementary Tate object \( V \), a lattice \( L_{0, [g]} \to V \) exists. For the induction step, the lattices \( L_{n, i}(\mathcal{J}) \to V_n \) are given, and it remains to choose a lattice \( L_{n, \{g\}}(\mathcal{J}) \to V_n \) containing all of these as proper sub-lattices. Such a lattice exists by Theorem 2.10 (there are finitely many non-empty sub-sets of \( [n] \)).

We now turn to the third claim and verify that for each non-degenerate simplex \( \mathcal{J} \in N_n \text{ Tate}^x(C)^x \), the collection \( \{L_{n, i}(\mathcal{J})\}_{I \subseteq [n]} \) satisfies the inductive hypotheses. Because we have already shown that the \( L_{n, i}(\mathcal{J}) \) are well-defined, the first two inductive hypotheses are satisfied by definition. It remains to show that if \( I \subseteq J \), then \( L_{n, i}(\mathcal{J}) \) is a sub-lattice of \( L_{n, j}(\mathcal{J}) \). For \( J = [n] \) this follows by definition. Now suppose we are given a proper, non-empty subset \( J \subseteq [n] \). Suppose there exists \( i < n \) such that \( J \subseteq d^e([n - 1]) \subseteq [n] \). Then for any \( I \subseteq J \), we have

\[
L_{n, i}(\mathcal{J}) = L_{n-1, s^i(J)}(d_i \mathcal{J}),
\]
and

\[
L_{n, i}(\mathcal{J}) = L_{n-1, s^i(J)}(d_i \mathcal{J}).
\]

By the inductive hypothesis, because \( s^i(I) \subseteq s^i(J) \), we see that \( L_{n, i}(\mathcal{J}) \) is a sub-lattice of \( L_{n, j}(\mathcal{J}) \). It remains to consider \( J = d^e([n - 1]) \). For any \( I \subseteq J \), we have

\[
L_{n, i}(\mathcal{J}) = g_n L_{n-1, [n-1]}(d_n \mathcal{J}),
\]
and

\[
L_{n, i}(\mathcal{J}) = g_n L_{n-1, s^{i-1}(J)}(d_n \mathcal{J}).
\]

By the inductive hypothesis \( L_{n-1, s^{i-1}(J)}(d_n \mathcal{J}) \) is a sub-lattice of \( L_{n, [n-1]}(d_n \mathcal{J}) \). We conclude that \( L_{n, i}(\mathcal{J}) \) is a sub-lattice of \( L_{n, j}(\mathcal{J}) \), as required.

We conclude the proof by showing that we have indeed defined a simplicial map. A map of simplicial sets is uniquely determined by its value on non-degenerate simplices. Therefore, it is enough to check that the above assignment respects the face maps. For this, we begin by showing that, for all \( i < n \), we have

\[
d_i(\mathcal{J}, \{L_{n, i}(\mathcal{J})\}_{I \subseteq [n]}) = (d_i \mathcal{J}, \{L_{n-1, i}(d_i \mathcal{J})\}_{J \subseteq [n-1]}).
\]

From Example 3.14 and Lemma 3.15 we see that the left hand side is equal to

\[
(d_i \mathcal{J}, \{L_{n, i}(\mathcal{J})\}_{I \subseteq d^e([n-1]) \subseteq [n]}),
\]

where the collection \( \{L_{n, i}(\mathcal{J})\}_{I \subseteq d^e([n-1]) \subseteq [n]} \) denotes the \((n-1)\)-simplex of \( \text{Ex}^1(G_{\mathcal{J}}^{\leq x}(V)) \) which sends \( J \subseteq [n-1] \) to \( L_{n, d^e(J)}(\mathcal{J}) \). On the other hand, by the inductive definition and the simplicial identities, we have that

\[
L_{n, d^e(J)}(\mathcal{J}) := L_{n-1, d^e(J)}(d_i \mathcal{J}) = L_{n-1, J}(d_i \mathcal{J}).
\]

This establishes the equality (10).

We must also show that

\[
d_n(\mathcal{J}, \{L_{n, i}(\mathcal{J})\}_{I \subseteq [n]}) = (d_n \mathcal{J}, \{L_{n-1, i}(d_n \mathcal{J})\}_{J \subseteq [n-1]}).
\]

---

\(^8\) e.g. this follows from the definition of the \( n \)-skeleton functors, and the fact that every simplicial set is the union of its \( n \)-skeleta.
From Example 3.14 and the description of Lemma 3.18 we see that the left hand side is equal to
\[(d_n \mathcal{G}, \{g_n^{-1} L_{n,I}(\mathcal{G})\}_{I \subseteq d^n([n-1]) \subseteq [n]}),\]
where the collection \(\{g_n^{-1} L_{n,I}(\mathcal{G})\}_{I \subseteq d^n([n-1]) \subseteq [n]}\) denotes the \((n-1)\)-simplex of \(Ex^1 \Gamma(V)\) which sends \(J \subseteq [n-1]\) to \(g_n^{-1} L_{n,d^n(J)}(\mathcal{G})\). On the other hand, by the inductive definition and the simplicial identities, we have that
\[g_n^{-1} L_{n,d^n(J)}(\mathcal{G}) := g_n^{-1} g_n L_{n-1,s^n-1 d^n J} (d_n \mathcal{G}) = L_{n-1,J}(d_n \mathcal{G}).\]
This establishes the equality (17) and we therefore conclude that the map is simplicial. \(\square\)

**Corollary 3.20.** Let \(C\) be idempotent complete. Let \(\mathcal{L}\) be a map as in the previous theorem. Then the geometric realization of the composite

\[(18) \quad N_\bullet \text{Tate}^d(C) \xrightarrow{\rho} d(Ex^{1,h} \Gamma(V)) \xrightarrow{\text{Index}} d(Ex^{1,h} S_\bullet(C)),\]

is equivalent to the index map.

**Proof.** By Lemma 3.18 we see that \(|\mathcal{L}|\) gives a left inverse to the equivalence

\[|\Gamma(V)| \xrightarrow{\sim} |\text{Tate}^d(C)|.\]

Similarly, by Lemma 3.17 and the equivalence \(|X_{\bullet, \bullet}| \simeq |d(X)|\), we see that

\[|d(Ex^{1,h}(\text{Index}))| \simeq |\text{Index}| : |\Gamma(V)| \xrightarrow{\sim} |S_\bullet(C)|.\]

The corollary now follows from the definition of the index map. \(\square\)

The corollary represents a universal calculation of symbols and higher torsion invariants for automorphisms of Tate objects. As an example of the information this contains, we now compute the value of the map

\[B \text{Aut}(V) = \pi_1(|\text{Tate}^d(C)|, V) \xrightarrow{\text{Index}} \pi_1(|S_\bullet(C)|) = K_0(C).\]

The construction above shows that, once we have chosen a lattice \(L \hookrightarrow V\) (denoted \(L_{0,[0]}\) above), we have

\[\mathcal{L}(e) = (e, \{L\}_{I \subseteq [1]}),\]

where the collection \(\{L\}_{I \subseteq [1]}\) denotes the 1-simplex of \(Ex^1 \Gamma(V)\) which sends \(I \subseteq [1]\) to \(L\). Therefore, \(\text{Index}(e)\) consists of two copies of the degenerate 1-simplex joined at their ends. We conclude

\[|\text{Index}(e)| = 0 \in K_0(C)\]
as expected.

Now let \(g \in \text{Aut}(V)\) be a non-trivial automorphism. The construction above shows that, having chosen \(L\), to define \(\mathcal{L}(g)\), it suffices to choose a lattice \(N\) which contains both \(L\) and \(gL\) as sub-lattices (we denote \(N\) by \(L_{1,[1]}(g)\) above). The map \(\mathcal{L}(g)\) sends \(g\) to the loop in \(d(Ex^{1,h} \Gamma(V))\), given by

\[(g, gL \xrightarrow{\bullet} \mathcal{N} \xrightarrow{\bullet} L) \in N_1 \text{Tate}^d(C) \times Ex^1 \Gamma(V).\]

Applying the categorical index map, we obtain the loop in \(d(Ex^{1,h} S_\bullet(C))\) given by

\[\bullet \xrightarrow{\frac{N}{gL}} \bullet \xrightarrow{\frac{N}{L}} \bullet.\]
Passing to \( \pi_1 \), this gives
\[
[\text{Index}(g)] = [N/gL] - [N/L] \in K_0(\mathbb{C}).
\]
Corollary \[3.20\] ensures that this value is independent of our choices (as one can also check directly).

**Example 3.21.** Let \( k \) be a field. The ring of Laurent series \( k((t)) \), has a canonical structure of an elementary Tate vector space over \( k \). An invertible element \( f = \sum_{i=0}^{\infty} a_i t^i \in k((t))^\times \) gives an automorphism of the Tate module \( k((t)) \) which takes the lattice \( k[[t]] \subset k((t)) \) to the lattice \( t^n k[[t]] \). Taking \( L_{0,[0]} = k[[t]] \) and \( L_{1,[1]}(g) = t^{\min \{ n \}} k[[t]] \), we conclude that
\[
[\text{Index}(f)] = \begin{cases} 
- k(t^{-1}, \ldots, t^{-n}) & \text{if } n < 0 \\
0 & \text{if } n = 0 \\
k(1, t, \ldots, t^{n-1}) & \text{if } n > 0
\end{cases}
\]
where \( [k(\ldots)] \) denotes the class of the \( k \)-vector space with generators \((\ldots)\). In particular, if we identify \( \pm n \) with \( \pm [k^n] \in K_0(k) \) for \( n \in \mathbb{N} \), we have \( [\text{Index}(f)] = n \in K_0(k) \). So, in this example, \( \pi_0 \) of the index map recovers the winding number of a non-vanishing formal Laurent series \( f \).

3.4. The Index Map is an Equivalence. Our goal in this section is to prove the following theorem.

**Theorem 3.22.** Let \( C \) be an idempotent complete exact category. Then the \( K \)-theoretic index map is an equivalence.

**Remark 3.23.** The existence of this equivalence has been suspected for some time (e.g. \[Pre10\]). S. Saito \[Sat12\] has previously established an abstract equivalence \( \Omega K_{\text{Tate}}(C) \simeq K_C \) as a consequence of Schlichting’s Localization Theorem \[Sch11\]. We deduce the present theorem from Waldhausen’s Additivity Theorem \[Wal85\]. In Section \[3.3\], we show that the \( K \)-theoretic index map is equivalent to \(-1\) times Saito’s equivalence. In Section \[4.1\], we explain how the present theorem can be understood as the analogue for algebraic \( K \)-theory of the equivalence, due to Atiyah and Jänich, between the space of Fredholm operators on a separable complex Hilbert space and the classifying space of topological complex \( K \)-theory.

**Proof of Theorem 3.22.** It suffices to prove that, for each \( n \), the map
\[
\text{Gr}_n^<(C) \xrightarrow{\text{Index}} S_n(C)
\]
induces an equivalence
\[
|S_* (\text{Gr}_n^<(C))^\times | \xrightarrow{\sim} |S_* (S_n(C))^\times |.
\]
For \( n = 0 \), observe that \( \text{Gr}_0^<(C) = \mathcal{E}(\text{Pro}^a(C), \text{Tate}^d(C), \text{Ind}^a(C)) \). Therefore, by the Additivity Theorem (Theorem \[2.13\]), we have an equivalence
\[
|S_* (\text{Gr}_0^<(C))^\times | \xrightarrow{\sim} |S_* (\text{Pro}^a(C))^\times | \times |S_* (\text{Ind}^a(C))^\times |.
\]
The Eilenberg swindle shows that the right hand side is contractible. We conclude that the map
\[
|S_* (\text{Gr}_0^<(C))^\times | \xrightarrow{\sim} |S_* (S_0(C))^\times |
\]
is an equivalence.

Next, consider the functor
\[
\text{Gr}_n^<(C) \rightarrow \mathcal{E}(\text{Pro}^a(C), \text{Gr}_n^<(C), S_n^a(C \subset \text{Ind}^a(C)))
\]
which sends \((L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V)\) to

\[
\begin{array}{c}
L_0 \
L_1 \
\vdots \\
L_n \\
0 \\
L_1/L_0 \
\vdots \\
L_n/L_0 \\
V/L_0
\end{array}
\]

A quick check shows that this is an equivalence of exact categories. By the Additivity Theorem (Theorem 2.16) and by Proposition 2.34, we conclude that the map

\[
Gr_{\leq n}(C) \to \text{Pro}^a(C) \times S_n(C) \times \text{Ind}^a(C)
\]

\((L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_0, L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0, V/L_n)\)

induces an equivalence

\[
|S_*(Gr^\leq_n(C))^\times| \xrightarrow{\sim} |S_*\text{Pro}^a(C)^\times| \times |S_*(S_n(C))^\times| \times |S_*(\text{Ind}^a(C))^\times|.
\]

By the Eilenberg swindle, the projection

\[
|S_*\text{Pro}^a(C)^\times| \times |S_*S_n(C)^\times| \times |S_*(\text{Ind}^a(C))^\times| \xrightarrow{\sim} |S_*(S_n(C))^\times|
\]

is an equivalence. The map

\[
|S_*(Gr^\leq_n(C))^\times| \xrightarrow{\text{Index}} |S_*(S_n(C))^\times|
\]

is the composite of the above two maps, and is therefore an equivalence. \(\square\)

**Corollary 3.24.** Let \(C\) be an idempotent complete exact category, and let \(\kappa \leq \kappa'\) be a pair of infinite cardinals. Then the exact functor \(\text{Tate}^e_\kappa(C) \to \text{Tate}^e_{\kappa'}(C)\) induces an equivalence in \(K\)-theory

\[
\text{Tate}^e_\kappa(C) \xrightarrow{\sim} \text{Tate}^e_{\kappa'}(C).
\]

**Proof.** We have a commuting diagram

\[
\begin{array}{ccc}
\text{Tate}^e_\kappa(C) & \xrightarrow{\text{Index}} & S_*(C) \\
\downarrow & & \downarrow 1 \\
\text{Tate}^e_{\kappa'}(C) & \xrightarrow{\text{Index}} & S_*(C)
\end{array}
\]

After applying \(|S_*(-)|\), all of the horizontal arrows become equivalences (by Proposition 3.3 and Theorem 3.22). The corollary now follows from the 2 of 3 property for equivalences. \(\square\)

**3.5. The Index Map as a Boundary Map.**

**Theorem 3.25.** Let \(C\) be an idempotent complete exact category. Let

\[
\partial : \Omega K_{\text{Ind}^a(C)/C} \to K_C
\]
be the boundary map in the \( K \)-theory localization sequence associated to the left \( s \)-filtering embedding \( C \subset \text{Ind}^a(C) \). Then the \( K \)-theoretic index map fits into a natural commuting triangle

\[
\begin{array}{c}
\Omega K_{\text{Tate}^e(C)} \\
\downarrow \text{Index} \\
\Omega K_{\text{Ind}^a(C)/C}
\end{array} \quad \begin{array}{c}
\downarrow \text{Index} \\
\delta
\end{array}
\rightarrow K_C
\]

where the left vertical map is given by the canonical functor \( \text{Tate}^e(C) \rightarrow \text{Ind}^a(C)/C \) which sends an elementary Tate object \( V \) to \( V/L \) for any choice of lattice \( L \hookrightarrow V \).

The proof of Saito’s Delooping Theorem \([Sai12]\) implies the following.

**Corollary 3.26.** Let \( C \) be idempotent complete. The index map is canonically equivalent to \(-1\) times Saito’s delooping

\[
\Omega K_{\text{Tate}^e(C)} \xrightarrow{\sim} K_C
\]

**Proof of Theorem 3.25.** The assignment

\[
(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0; L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0 \hookrightarrow V/L_0)
\]

determines a map of simplicial diagrams of categories

\[
\text{Gr}^\leq_\bullet(C) \longrightarrow S^\bullet_\bullet(C \subset \text{Ind}^a(C))
\]

which fits into a 2-commuting triangle

\[
\begin{array}{c}
\text{Gr}^\leq_\bullet(C) \\
\downarrow \text{Index} \\
S^\bullet_\bullet(C \subset \text{Ind}^a(C))
\end{array} \quad \begin{array}{c}
\downarrow \delta
\end{array}
\rightarrow S^\bullet_\bullet(C)
\]

We also have a 2-commuting square

\[
\begin{array}{c}
\text{Tate}^e(C) \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Gr}^\leq_\bullet(C) \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Ind}^a(C)/C \\
\downarrow
\end{array} \quad \begin{array}{c}
S^\bullet_\bullet(C \subset \text{Ind}^a(C))
\end{array}
\]

where the bottom horizontal map is the restriction to 1-simplices of the equivalence appearing in the proof of Corollary 2.39, and where the left vertical map is the functor described above.

Applying the \( S \)-construction, we obtain a commuting diagram of spaces

\[
\begin{array}{c}
|S^\bullet_\bullet(\text{Tate}^e(C))| \\
\downarrow
\end{array} \quad \begin{array}{c}
|S^\bullet_\bullet(\text{Gr}^\leq_\bullet(C))| \\
\downarrow \text{Index}
\end{array} \quad \begin{array}{c}
|S^\bullet_\bullet(\text{Ind}^a(C)/C)| \\
\downarrow
\end{array}
\rightarrow \begin{array}{c}
|S^\bullet_\bullet(\text{Ind}^a(C)/C)\times| \\
\downarrow
\end{array} \quad \begin{array}{c}
|S^\bullet_\bullet(\text{Ind}^a(C))\times| \\
\downarrow \text{Index}
\end{array}
\]

By inverting the left-facing equivalences and taking the double loop spaces, we obtain a contractible space of commuting triangles of the form (19). □
4. The $E_1$-Structure of the Index Map

In [Seg74], Segal introduced a definition which, in the hands of Thomason [Tho79], Rezk [Rez01], Lurie [Lur] and many others, has become fundamental to the study of $E_1$-objects (a.k.a. $A_\infty$-objects or homotopy coherent associative monoids) in a homotopical setting.

**Definition 4.1.** Let $\mathcal{C}$ be an $\infty$-category with finite products. For each $n$, let

$$\iota_n : \coprod_{i=1}^n [1] \longrightarrow [n]$$

 denote the disjoint union of the maps $[1] = \{0 < 1\} \overset{\sim}{\longrightarrow} \{i - 1 < i\} \subset [n]$. A Segal object in $\mathcal{C}$ is a simplicial object $X_\bullet \in \text{Fun}(\Delta^{op}, \mathcal{C})$ such that, for $n \geq 2$, the map

$$\iota_n^* X_n \longrightarrow X_1 \times X_0 \cdots \times X_0 X_1,$$

is an equivalence. We define the $\infty$-category of Segal objects in $\mathcal{C}$ to be the full sub-category of $\text{Fun}(\Delta^{op}, \mathcal{C})$ consisting of the Segal objects.

A reduced Segal object $X_\bullet$ is a Segal object with $X_0 \cong *$. Similarly, the $\infty$-category of reduced Segal objects is the full sub-category of $\text{Fun}(\Delta^{op}, \mathcal{C})$ consisting of the reduced Segal objects.

As pioneered by Thomason [Tho79], reduced Segal objects provide the fundamental formalism for studying $E_1$-objects in a homotopical setting (i.e. in an $\infty$-category): all other formalisms for $E_1$-objects are shown to be equivalent to the formalism of reduced Segal objects. As a recent example of this phenomenon, we have the following.

**Proposition 4.2.** ([Lur, Prop. 4.1.2.6]) Let $\mathcal{C}$ be an $\infty$-category admitting finite products. There exists a canonical equivalence between $E_1$-objects in $\mathcal{C}$, and the $\infty$-category of reduced Segal objects in $\mathcal{C}$.

For our purposes, the two most relevant examples of reduced Segal objects in $\text{Spaces}$ are

1. The nerve $B_\bullet \text{Aut}(V)$ of the automorphism group of an elementary Tate object $V$, and
2. The $K$-theory $K_{S_\bullet(C)}$ of the simplicial exact category $S_\bullet(C)$.

In Section 3, we constructed the index map

$$B \text{Aut}(V) \simeq |B_\bullet \text{Aut}(V)| \xrightarrow{\text{Index}} |S_\bullet(C)^\times| \simeq BK_\mathcal{C}.$$  

This encodes an $E_1$-map of the loop spaces

$$\text{Aut}(V) \longrightarrow K_\mathcal{C},$$

which in turn amounts to a coherent collection of homotopies

$$\text{Index}(g_1) + \text{Index}(g_2) \simeq \text{Index}(g_1 g_2).$$

For many applications, e.g. [BGWa], we need to be able to access and manipulate these homotopies in detail. In Section 3.3, we remarked that the combinatorial index map can be viewed as describing this $E_1$-map in a Kan-style approach to $E_1$-objects. Abstractly, the Kan-style approach is equivalent to the Segal-style approach. In this section, we make this abstract equivalence concrete by exhibiting a map of reduced Segal objects

$$B_\bullet \text{Aut}(V) \longrightarrow K_{S_\bullet(C)}$$

whose geometric realization is the index map.

---

9 This is a direct consequence of Waldhausen’s Additivity Theorem 2.19.
4.1. A Generalized Waldhausen Construction. As we explained in Section 2, Waldhausen’s treatment of algebraic $K$-theory \cite{Wal85} hinges on two simplicial exact categories, denoted by $S_\bullet(C)$, and $S'_\bullet(f)$ respectively, where $C$ is an exact category and $f: C \to D$ is an exact functor. The simplicial object $S_\bullet(C)$ associates to every finite non-empty totally ordered set $[k]$ the exact category $S_k(C)$, which consists of functors $[k] \to C$, sending every arrow in $[k]$ to an admissible monic.

The results of this section require an extension of these functors $S_\bullet(C), S'_\bullet(f): \Delta^{op} \to \text{Cat}_{ex}$ to the category of filtered finite partially ordered sets. If $I$ is a finite, filtered, partially ordered set, it makes sense to consider the exact category of $I$-diagrams in an exact category $C$. By definition, an admissible $I$-diagram in $C$ is a functor $F: I \to C$, sending every morphism $x \leq y$ in $I$ to an admissible monic $F(x) \hookrightarrow F(y)$.

In this section we will relativize this construction for a pair of exact categories $C \subset D$, such that $C$ is extension-closed in $D$.

Definition 4.3. We denote by $\text{Cat}_{pair}^{ex}$ the 2-category of pairs of exact categories $C \subset D$, such that $C$ is an extension-closed sub-category of $D$. Objects in this category will also be referred to using the notation $(D, C)$.

In Subsection 4.2 we then use the generalized Waldhausen construction to give a treatment of the $E_1$-structure of the index map.

4.1.1. Partially Ordered Sets and Related Structures. The current paragraph contains several definitions of combinatorial nature. We first describe how to depict a partially ordered set by means of an oriented graph.

Definition 4.4. Let $I$ be a partially ordered set. We denote by $\Gamma(I)$ the directed graph given by the set underlying $I$ as set of vertices, and intervals $a < b$ as edges. We denote the set of directed edges of $\Gamma(I)$ by $E(I)$.

Example 4.5. For the ordinal $[2]$ we obtain

for the oriented graph $\Gamma([2])$. While this graph is more traditionally drawn as the boundary of a 2-simplex, the present depiction is chosen to highlight the maximal tree.

We will work with finite, filtered, partially ordered sets with base points (which are chosen to be minimal elements). The definition is given below.

Definition 4.6. A based, finite filtered, partially ordered set is a pair $(I; x_0, \ldots, x_k)$, where $I$ is a finite partially ordered set with a final element, and $(x_0, \ldots, x_k)$ is a tuple of minimal elements in
A morphism of based partially ordered sets is a map of pairs
\[(f, \sigma): (I; x_0, \ldots, x_k) \rightarrow (I'; y_0, \ldots, y_m),\]
where \(f: I \rightarrow I'\) is a map of partially ordered sets, \(\sigma: [m] \rightarrow [k]\) is a map of finite ordinals, and where \(f(x_i) = y_{\sigma(i)}\). The category of based, finite, filtered, partially ordered sets will be denoted by \(\text{poSet}_{\text{filt}}^f\).

The assumption of finiteness is crucial for the inductive proofs that are given later, but could eventually be relaxed.

Some arguments require us to choose a maximal tree in \(\Gamma(I)\) with good properties. We therefore have the following definition.

**Definition 4.7.** Let \(\Gamma\) be an oriented graph. A maximal tree \(T \subset \Gamma\) is said to be admissible, if for every pair of vertices \((x, y)\), there exists a vertex \(z\), and a unique oriented path from \(x\) to \(z\), respectively \(y\) to \(z\) within \(T\).

The following examples help to clarify this definition.

**Example 4.8.**

The tree on the left is admissible, while the one on the right is not.

**Example 4.9.** Let \(I\) be a finite, filtered, partially ordered set. An admissible tree \(T \subset \Gamma(I)\) always exists. Indeed, let \(m \in I\) denote the final element. Then the tree \(T\) given by the union of all edges \((x, m)\) for \(x \in I\), is admissible.

The definition below introduces the concept of a framing of a based partially ordered set.

**Definition 4.10.** A framed partially ordered set is a triple \((I, E(T), x_0, \ldots, x_k)\), where \(E(T) \subset E(I)\) is the set of edges of an admissible maximal tree, and \((I; x_0, \ldots, x_k)\) is a based, finite, filtered, partially ordered set. The category of framed, partially ordered sets \(\text{poSet}_{\text{fr}, \text{filt}}^f\) is the category with framed, partially ordered sets as objects, and morphisms
\[
\phi: (I, E(T), x_0) \rightarrow (I', E(T'), x'_0),
\]
where \(\phi: I \rightarrow I'\) is a map of partially ordered sets, mapping the base points bijectively onto each other, and satisfying \(\phi(T) \subset \phi(T')\). We denote by
\[
\phi_T: E(T) \rightarrow E(T')_+ = E(T) \cup \{\star\}
\]
the map which sends \(e \in E(T)\) either to its image \(\phi(e) \in E(T')\), or, if \(\phi(e)\) consists of a single point, to the base point \(\star\).

---

10It is important to note that the base points are not assumed to be pairwise distinct.
4.1.2. Pairs of Exact Categories and Diagrams. For every partially ordered set $I$ we have an associated category. For notational convenience, we will not distinguish between these.

**Definition 4.11.** Let $(D,C) \in \mathbf{Cat}^\text{pair}$ be a pair of exact categories. Let $I$ be a partially ordered set. An admissible $I$-diagram in $(D,C)$ is a functor $I \longrightarrow D$, sending each arrow in $I$ to an admissible monic in $D$ with cokernel an object of $C$. We denote the exact category of such functors by $\text{Func}_C(I,D)$.

The following example serves as a motivation for this definition.

**Example 4.12.** We observe that $\text{Func}_C([n],D) = S_n^C(C \subset D)$ (see Definition 2.30).

In Definition 4.10 we introduced the concept of framed, partially ordered sets. Recall the map $\phi_T : E(T) \longrightarrow E(T')$. By abuse of notation we will also use the symbol $\phi_T$ to denote the unique map of pointed sets

$E(T)_+ \longrightarrow E(T')_+$.

Note that, for every object $X$ in a pointed $\infty$-category $\mathcal{C}$ with finite coproducts, we have a natural functor

$\prod ? X : (\text{Set}_+^{\text{fin}})^{\text{op}} \longrightarrow \mathcal{C}$.

An inductive argument allows us to establish the following lemma. The choice of a maximal tree $T \subset \Gamma(I)$ should be understood as an analogous to choosing a basis for a vector space.

**Lemma 4.13.** Let $(I;E(T), x_0, \ldots, x_k)$ be a framed, partially ordered set. We denote by $T \subset \Gamma(I)$ an admissible maximal tree of $\Gamma(I)$. Then there exists an equivalence

$\phi(T) : K_{\text{Func}}(I,D) \cong K_D \times K_C^{\times E(T)}$.

Moreover, this equivalence can be seen as a natural equivalence of functors

$K_{\text{Func}}(-,-) \cong K_- \times K_-^{\times E(-)} : \mathbf{Cat}^\text{pair}_{ex} \times (\text{poSet}^\text{fr,fil}_t)^{\text{op}} \longrightarrow \text{Spaces}$.

Although the lemma is stated for a framed partially ordered set with base points $x_0, \ldots, x_k$, we actually only need the zeroth base point $x_0$. An inspection of the proof below shows that all the other base points could be discarded.

**Proof of Lemma 4.13.** For every $e = (y_i \leq y_{i+1}) \in E(T)$ we denote by $X_e = F(y_{i+1})/F(y_i)$. We have an exact functor

$\text{Func}_C(I,D) \longrightarrow D \times C^{E(T)}$,

which sends $F : I \longrightarrow D$ to $(F(x_0), (X_e)_{e \in E(I)})$. This map defines a natural transformation between the functors

$\text{Fun}_-(-,-) \times (-)^{E(-)} : \mathbf{Cat}^\text{pair}_{ex} \times (\text{poSet}^\text{fr,fil}_t)^{\text{op}} \longrightarrow \text{Cat}_{ex}$.

Applying the functor $K_- : \mathbf{Cat}_{ex} \longrightarrow \text{Spaces}$, we obtain the natural transformation $\phi(T)$. It remains to show that $\phi(T)$ is an equivalence for each triple $(I,D,C)$. We will use induction on the cardinality of $I$ to show this. As a warmup, we begin with the case that $I$ is a totally ordered set. Without loss of generality we may identify it with $\{0 < \cdots < n\}$. Moreover, in the totally ordered case, there is only one possible choice for the framing $(T,x_0)$. The induction is anchored to the case $n = 0$, i.e. the case of the singleton set, which is evidently true.
Assume that $\phi(T)$ has been shown to be an equivalence for totally ordered sets of cardinality $< n$. We denote by $I'$ the framed partially ordered set defined by the subset $\{0 < \cdots < n - 1\}$.

The restriction functor $\text{Fun}_C(I, D) \rightarrow \text{Fun}_C(I', D)$ sits in a short exact sequence of exact categories

$$
\mathbb{C} \hookrightarrow \text{Fun}_C(I, D) \twoheadrightarrow \text{Fun}_C(I', D),
$$

where we send $X \in \mathbb{C}$ to $(0 \hookrightarrow \cdots \hookrightarrow 0 \hookrightarrow X) \in \text{Fun}_C(I, D)$. We also have a splitting, given by

$$
\text{Fun}_C(I', D) \twoheadrightarrow \text{Fun}_C(I, D),
$$

which sends $(Y_0 \hookrightarrow \cdots \hookrightarrow Y_{n-1})$ to $(Y_0 \hookrightarrow \cdots \hookrightarrow Y_{n-1} \hookrightarrow Y_{n-1})$. By means of the Additivity Theorem 2.19, we conclude

$$
K_{\text{Fun}_C(I, D)} \cong K_{\text{Fun}_C(I', D)} \times K_{\mathbb{C}}.
$$

Applying the inductive hypothesis to $\text{Fun}_C(I', D)$, we conclude the assertion for totally ordered sets.

The proof for general $I$ also works by induction on the number of elements. If $I$ is not totally ordered, but of cardinality $n + 1$, we may decompose our framed partially ordered set $(I, T) = (I' \cup (I'', T''))$, where $I''$ is totally ordered, $I' \cap I'' = \{\max I''\}$, and $x_0 \in I'$. Consider for example the graph:

![Example graph]

where edges belonging to $I''$ have been drawn as squiggly lines.

There exists a positive integer $1 \leq k \leq n$, such that $I'' \cong \{0 < \cdots < k\}$. The restriction functor from $I$-diagrams to $I'$-diagrams belongs to a short exact sequence of exact categories

$$
\text{Fun}_C(I'' \setminus \{\max I''\}, \mathbb{C}) \hookrightarrow \text{Fun}_C(I, D) \twoheadrightarrow \text{Fun}_C(I', D),
$$

where the left hand side is seen as the exact category of morphisms $(Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots \hookrightarrow Y_{k-1})$, which extends to an $I$-diagram by sending the object $Y_{k-1}$ to every vertex in $I'$. This short exact sequence is split by the functor

$$
\text{Fun}_C(I', D) \twoheadrightarrow \text{Fun}_C(I, D),
$$

which extends an $I'$-diagram to an $I$-diagram, by sending each vertex $y$ of $I''$ to the object $\max I'' \in I' \cap I''$ (with the identity morphisms as admissible epimorphisms between them). The Additivity Theorem 2.19 yields

$$
K_{\text{Fun}_C(I, D)} \cong K_{\text{Fun}_C(I', D)} \times K_{\mathbb{S}_k(\mathbb{C})}.
$$

Using the induction hypothesis, we see that the first component is equivalent to $K_{\mathbb{D}} \times K_{\mathbb{C}} \times E(T')$, and the second component to $K_{\mathbb{C}} \times K_{E(T'')}$. This proves the assertion.
4.1.3. The Index Space. Let \((I; x_0, \ldots, x_k)\) be a based, finite, filtered, partially ordered set (Definition 4.6). Together with a pair of exact categories \(C \subset D\), such that \(C\) is extension-closed in \(D\), we define the \textit{index space}, which is the recipient of a map from \(K_{\text{Fun}(I,D)}\). It can be thought of as measuring the difference between the base points.

**Definition 4.14.**

(a) For a based, finite, filtered, partially ordered set \((I; x_0, \ldots, x_k)\) we denote by \(I^\Delta\) the partially ordered set obtained by identifying the base points. Cofunctoriality of \(\text{Fun}(\cdot, C)\) yields a forgetful functor \(\text{Fun}_C(I^\Delta, D) \to \text{Fun}_C(I, D)\).

(b) For an exact category \(D\), let \(K_D\) be the connective \(K\)-theory spectrum. We denote by \(I^\Delta_{x_0, I,D}\) the space underlying \((i.e. \Omega^\infty)\) of the cofibre of the morphism\(^{11}\)

\[
K_{\text{Fun}(I^\Delta, D)} \to K_{\text{Fun}(I, D)}
\]

By the functoriality of cofibres, this gives rise to a functor

\[
\text{Idx} : \text{Cat}^{\text{pair}} \times (\text{poSet}^{\text{filt}})^{\text{op}} \to \text{Spaces}.
\]

We refer to \(I^\Delta_{x_0, I,D}\) as the \textit{Index-space} of the pair \((D, C)\) relative to \((I; x_0, \ldots, x_k)\).

The index space is to a large extent independent of \(I\), as guaranteed by its functorial nature in Definition 4.14(b). We record this observation in the next two results. In Proposition 4.20 we will further refine this statement.

**Lemma 4.15.** Let \(C \hookrightarrow D\) be an extension-closed exact sub-category of an exact category \(D\). We consider an injective morphism of finite, based, filtered, partially ordered sets, in the sense of Definition 4.6,

\[
(I; x_0, \ldots, x_k) \to (I'; y_0, \ldots, y_k),
\]

which induces a bijection of base points \((i.e. \text{on base points, it corresponds to the identity map } [k] \to [k])\). Then, the induced morphism of index spaces

\[
\text{Idx}_{C,I,D} : \text{Idx}_{C,I,D} \to \text{Idx}_{C,I,D}
\]

is an equivalence.

**Proof.** By virtue of Lemma 4.13 the choice of an admissible maximal tree \(T\) in \(I\) induces an equivalence of \(K\)-theory spaces

\[
K_{\text{Fun}(I,D)} \cong K_D \times K_C^{\otimes E(T)}.
\]

Recall from Definition 4.14 that \(I^\Delta\) denotes the finite, based, filtered, partially ordered set obtained by identifying all base points. We can choose \(T\) in a way, such that its image \(T^\Delta\) in \(I^\Delta\) is also an admissible tree. For instance, we could take the tree given by the edges \((x, m), \) where \(m = \max I, \) and \(x\) runs through the elements of \(I \setminus \{m\}\). We denote by \(e_i\) the (unique) edge of \(T\) which contains \(x_i\). By construction, the edges \(e_i\) map to the same edge in \(T^\Delta\), and we denote this edge by \(e\). We can apply the functoriality of Lemma 4.13 to obtain the commutative square of connective \(K\)-theory spectra

\[
\begin{array}{ccc}
K_{\text{Fun}(I^\Delta, D)} & \xrightarrow{\cong} & K_{\text{Fun}(I, D)} \\
\cong & & \cong \\
K_D \oplus K_C^{\otimes E(T^\Delta)} & \xrightarrow{\alpha} & K_D \oplus K_C^{\otimes E(T)},
\end{array}
\]

\(^{11}\)The long exact sequence of homotopy groups implies that this cofibre is again a connective spectrum.
where the morphism $\alpha$ is given by the identity $1_{K_{\mathcal{C}}}$ for edges in $E(T) \setminus \{e_0, \ldots, e_k\}$, and given by the diagonal map

$$\Delta_{K_{\mathcal{C}}}: K_{\mathcal{C}} \to K_{\mathcal{C}}^{\oplus (k+1)},$$

for the component $e$. In particular, we see that $\text{cofib}(\alpha) \cong \text{cofib}(\Delta_{K_{\mathcal{C}}}).$

The same analysis applies to $I'$. Because we can choose an admissible maximal tree $T$ in $I$ which extends to an admissible maximal tree $T'$ in $I'$, we see that $\text{cofib}(K_{\text{Func}(J, D)} \to K_{\text{Func}(I, D)})$ is equivalent to

$$\text{cofib}(\Delta_{K_{\mathcal{C}}}: K_{\mathcal{C}} \to K_{\mathcal{C}}^{k+1}) \cong \text{cofib}(K_{\text{Func}(J, D)} \to K_{\text{Func}(I', D)}).$$

The restriction functor $\text{Idx}_{\mathcal{C}, J} D \to \text{Idx}_{\mathcal{C}, I'} D$ is defined independently of any choices. The admissible maximal trees $T$ and $T'$ only play a role in verifying that this map is an equivalence. We therefore see that we have a canonical equivalence between $\text{Idx}_{\mathcal{C}, I} D$ and $\text{Idx}_{\mathcal{C}, I'} D$.

**Definition 4.16.** For every positive integer $k$ we have an object $B[k] = (B[k]; b_0, \ldots, b_k) \in \text{poSet}^{\text{filt}}$. Given by the set of non-empty intervals in the ordinal $[k]$. An interval is understood to be a subset $J \subset [k]$ with the property that $x \leq y \leq z$ and $x, z \in J$ implies that $y \in J$. The base points $(b_i)_{i=0, \ldots, k}$ are given by the singletons $\{i\}$.

We have drawn the filtered partially ordered set $B[2]$ below.

![Diagram of B[2]](image)

**Definition 4.17.** For an arbitrary $I = (I, x_0, \ldots, x_k)$ in $\text{poSet}^{\text{filt}}$, we denote by $I^B = (I^B; x_0, \ldots, x_k)$ the based, finite, filtered, partially ordered set given by $I \cup B[k]$, where we identify the base points $b_i = x_i$ and where we extend the inductive ordering of $I$ to $I^B$ by demanding $x \leq y$, for all $x \in B[k]$ and $y \in I \setminus \{x_0, \ldots, x_k\}$. To summarize the previous construction, we obtain $I^B$ from $I$ by gluing on a copy of $B[k]$ to $I$, with all new elements being $\leq$ than elements in $I$. This process is functorial in $I$, we denote the resulting functor by

$$(-)^B: \text{poSet}^{\text{filt}} \to \text{poSet}^{\text{filt}}.$$

The inclusion $I \subset I^B$ gives rise to a natural transformation of functors

$$1_{\text{poSet}^{\text{filt}}} \Rightarrow (-)^B.$$

The category $\text{poSet}^{\text{filt}}$ satisfies the following property: for two objects $(I; x_0, \ldots, x_k)$ and $(I'; y_0, \ldots, y_k)$ we can find an $(I''; z_1, \ldots, z_k)$, containing sub-objects isomorphic to $I$ and $I'$ (respecting base points). Combining this observation with the lemma proven above, we obtain a complete description of index spaces.

**Corollary 4.18.** Let $(I, x_0, \ldots, x_k)$ be a based, finite, filtered, partially ordered set with pairwise distinct base points. Then, the index space of the pair $(D, C)$ is equivalent to

$$K_{S_k(C)} \cong K^{\times k}_C.$$

This equivalence is functorial in the pair $C \subset D$, where $C$ is extension-closed in $D$, and it is contravariantly functorial in the based filtered partially ordered set $I$. Moreover, if $M_\bullet$ is a simplicial
object in \( \text{poSet}^\text{filt} \) such that, for every non-negative integer \( k \), \( M_k \) has \( k + 1 \) pairwise distinct base points, then we have an equivalence of simplicial spaces

\[
\text{Idx}_{C,M} D \cong K_{S_k(C)}.
\]

Proof. Lemma 4.14 implies that we have a canonical equivalence

\[
\text{Idx}_{C,I} D \cong \text{Idx}_{C,I^n} D \cong \text{Idx}_{C,B[k]} D.
\]

To conclude the argument, we have to show that \( \text{Idx}_{C,B[k]} D \cong K_{S_k(C)} \). This equivalence will be shown to be induced by the exact functor

\[
S_k(C) \longrightarrow \text{Fun}_C(B[k], D),
\]

sending \((0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k)\) to the functor \( F \) in \( \text{Fun}_C(B[k], D) \), which maps the interval \([i, j]\) to the object \( X_j \). We draw the resulting diagram for \( k = 2 \) to illustrate the idea behind the definition:

\[
\begin{array}{ccc}
0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_2
\end{array}
\]

Alluding to Lemma 4.13, one can prove with the help of the right choice of admissible maximal tree in \( B[k] \) that the induced map of index spaces is indeed an equivalence. We choose to work with the naive admissible maximal tree \( T \) in \( B[k] \), uniquely defined by the property that for every non-maximal element there is a unique edge in \( T \) connecting it with the maximum. The image of \( T \) in \( B[k]^\Delta \), i.e. the partially ordered set obtained by identifying the base points \( b_0, \ldots, b_k \) (see Definition 4.14), is also an admissible maximal tree. We can therefore apply Lemma 4.13 to analyze the map of spaces

\[
K_{\text{Fun}_C(B[k]^\Delta, D)} \longrightarrow K_{\text{Fun}_C(B[k], D)}.
\]

Doing so, we obtain a commutative diagram of connective \( K \)-theory spectra (as in the proof of Lemma 4.15)

\[
\begin{array}{ccc}
K_{\text{Fun}_C(B[k]^\Delta, D)} & \longrightarrow & K_{\text{Fun}_C(B[k], D)} \\
\downarrow & & \downarrow \\
K_D \oplus K_C^{\oplus E(T^\Delta)} & \longrightarrow & K_D \oplus K_C^{\oplus E(T)},
\end{array}
\]

where the morphism \( \alpha \) agrees with the identity \( 1_{K_C} \) for edges in \( E(T) \setminus \{e_0, \ldots, e_k\} \), and with the diagonal map

\[
\Delta_{k_C} : K_C \longrightarrow K_C^{\oplus (k+1)},
\]

for the component \( e \). This is the same map arising in the proof of Lemma 4.15 and we have

\[
\text{Idx}_{C,B[k]} D \cong \Omega^\infty \text{cofib}(K_C^\Delta \rightarrow K_C^{\{b_0, \ldots, b_k\}}) \cong K_C^{\times k},
\]

where the last equivalence is defined as the inverse to the composition

\[
K_C^{\times k} \overset{i}{\longrightarrow} K_C^{\{b_0, \ldots, b_k\}} \longrightarrow \Omega^\infty \text{cofib}(K_C^\Delta \rightarrow K_C^{\{b_0, \ldots, b_k\}}),
\]
where the map $i$ is the inclusion of $K_C^{\times k}$ into $K_C^{\{b_0, \ldots , b_k\}}$ which misses the $K_C^{\{b_0\}}$-factor. In particular, we see that $i$ corresponds to the map of $K$-theory spaces induced by the functor $C^{\times k} \rightarrow C^{\{b_0, \ldots , b_k\}}$ given by the inclusion of the last $k$ factors.

Recall that we have $K_{S_k(C)} \cong K_C^{\times k}$, with respect to the map induced by the exact functor

$$C^{\times k} \longrightarrow S_k(C),$$

sending

$$(X_1, \ldots , X_k) \mapsto (0 \hookrightarrow X_1 \hookrightarrow X_1 \oplus X_2 \hookrightarrow \cdots \hookrightarrow X_1 \oplus \cdots \oplus X_k).$$

Composing the functors

$$K_C^{\times k} \longrightarrow K_{S_k(C)} \longrightarrow K_{\text{Func}(B[k], D)} \longrightarrow \text{Idx}_{C,B[k]} D \longrightarrow K_C^{\times k}$$

we obtain the identity, as can be checked on the level of exact categories: we have a commutative diagram of exact functors

$$\xymatrix{ D \times C^{\{b_0, \ldots , b_k\}} \ar[d] & C^{\times k} \ar[d] \ar[r] & S_k(C) \ar[d] & \text{Func}(B[k], D) \ar[d] \ar[r] & \text{Idx}_{C,B[k]} D \ar[d] \ar[r] & K_C^{\times k} \ar[d] \ar[l] }$$

where the right vertical functor sends $F$ to $(F(b_0), F([k])/F(b_1), \ldots , F([k])/F(b_k))$. The composition of exact functors represented by the diagonal arrow is equivalent to the inclusion of the last $k$ factors in $C^{\times k+1}$. Applying $K$-theory, and juxtaposing with (22), we obtain a commutative diagram of spaces

$$\xymatrix{ K_D \times K_C^{\{b_0, \ldots , b_k\}} \ar[r] \ar[d] & \Omega^\infty \text{cofib}(K_C \xrightarrow{\Delta_{K_C}} K_C^{\{b_0, \ldots , b_k\}}) \ar[d]^\cong \ar[l] \ar[d]^\cong \ar[l] & K_{S_k(C)} \ar[d] \ar[r] & K_{\text{Func}(B[k], D)} \ar[d] \ar[r] & \text{Idx}_{C,B[k]} D \ar[l] \ar[l] }$$

As we observed in (20), the composition of the arrows on the top agrees with the equivalence $\text{Idx}_{C,B[k]} D \cong K_C^{\times k}$.

To conclude the argument it suffices to establish the last claim. The functoriality of the index space construction guarantees that $\text{Idx}_{C,M^B} D$ is a well-defined simplicial space. Since the construction $I \mapsto I^B$ is functorial, we obtain a well-defined simplicial object $M^B_\bullet$, which acts as a bridge between $\text{Idx}_{C,M^B} D$ and $\text{Idx}_{C,B[\bullet]} D$, i.e. according to Lemma 4.15, we have equivalences

$$\text{Idx}_{C,M^B} D \cong \text{Idx}_{C,M^B[\bullet]} D \cong \text{Idx}_{C,B[\bullet]} D.$$

It therefore suffices to show that $\text{Idx}_{C,B[\bullet]} D \cong K_{S_k(C)}$ as simplicial spaces. Since the map $\text{Id}_{C,B[\bullet]} D$ is clearly a map of simplicial objects in exact categories, and a map of simplicial objects is an equivalence if it is a levelwise equivalence, we may conclude the proof.

4.1.4. Rigidity of the Pre-Index Map. In this paragraph we record a consequence of Lemma 4.15 which we will refer to as the rigidity of the pre-index map. In order to formulate the result, we have to introduce a localization of the category poSet$_{\text{filt}}$. 

\[\text{THE INDEX MAP IN ALGEBRAIC K-THEORY \hspace{47ex} 47} \]
Lemma 4.19. Consider the class of morphisms $W$ in the category $\text{poSet}^{\text{filt}}$ which consists of maps $(I \to I', [k] \xrightarrow{\phi} [k'])$ such that $\phi: [k] \to [k']$ is an isomorphism. We denote by $\text{poSet}^{\text{filt}}[W^{-1}]$ the $\infty$-category obtained by localization at $W$. This localization is canonically equivalent to the category $\Delta$ of finite non-empty ordinals, by means of the functor

$$\text{base}: \text{poSet}^{\text{filt}} \to \Delta,$$

which sends the pair $(I, (x_0, \ldots, x_k))$ to $[k]$. The functor $B[\bullet]: \Delta \to \text{poSet}^{\text{filt}}$ (Definition 4.16) is an inverse equivalence

$$\Delta \to \text{poSet}^{\text{filt}}[W^{-1}].$$

Proof. Note that we have $\text{base} \circ B[\bullet] \simeq \text{id}_\Delta$.

The universal property of localization of $\infty$-categories implies that the functor $\text{base}$ induces a functor

$$\widetilde{\text{base}}: \text{poSet}^{\text{filt}}[W^{-1}] \to \Delta.$$

In particular, we obtain a natural equivalence

$$\widetilde{\text{base}} \circ B[\bullet] \simeq \text{id}_\Delta.$$

Similarly, we recall from the proof of Corollary 4.18 that we have a natural transformation

$$\text{id}_{\text{poSet}^{\text{filt}}} \to (-)^B: \text{poSet}^{\text{filt}} \to \text{poSet}^{\text{filt}},$$

as well as $B[\bullet] \circ \text{base} \to (-)^B$. Putting these two natural transformations together, we obtain a zigzag

$$\text{id}_{\text{poSet}^{\text{filt}}} \to (-)^B \leftarrow B[\bullet] \circ \text{base},$$

which induces a natural equivalence of functors

$$\text{id}_{\text{poSet}^{\text{filt}}[W^{-1}]} \simeq B[\bullet] \circ \widetilde{\text{base}}.$$

We conclude that $B[\bullet]$ and $\widetilde{\text{base}}$ are mutually inverse equivalences of $\infty$-categories (in fact this shows that the $\infty$-category $\text{poSet}^{\text{filt}}[W^{-1}]$ is equivalent to a category).\qed

We use this localization to draw the following porism from the proof of Corollary 4.18.

Proposition 4.20. The functor $\text{idx}: \text{Cat}_{\text{pair}}^{\text{ex}} \times \text{poSet}^{\text{filt} \text{op}} \to \text{Spaces}$, of Definition 4.14, descends along the localization $\text{poSet}^{\text{filt}} \to \text{poSet}^{\text{filt}}[W^{-1}]$ of Lemma 4.19. In particular, by virtue of the equivalence

$$\text{poSet}^{\text{filt}}[W^{-1}] \cong \Delta,$$

we see that $\text{idx}$ induces a functor

$$\text{Cat}_{\text{pair}}^{\text{ex}} \times \Delta^{\text{op}} \to \text{Spaces}.$$
Proof of Proposition 4.20. We have seen, in Lemma 4.15, that every inclusion $I \subset I'$ which restricts to a bijection on base points induces an equivalence of index spaces

$$\llbracket \text{dx}, I \rrbracket \cong \llbracket \text{dx}, I' \rrbracket.$$

As in the proof of Corollary 4.18 we observe that the zigzag of inclusions

$$I \subset I \supset B[\text{base}(I)]$$

yields a zigzag of equivalences of index spaces. In particular, we see that the functor $\llbracket \text{dx} \rrbracket$ is equivalent to $\llbracket \text{dx} \rrbracket \circ B[\bullet] \circ \text{base}$. In particular, it factors through the map base: $\text{poSet}^{\text{filt}} \rightarrow \Delta$. □

In Subsection 4.3 we sketch a construction of index spaces for infinite filtered sets, using the rigidity property as main ingredient.

4.1.5. Three Examples for the Structure of the Pre-Index Map. In order to shed some light on the abstract constructions introduced above, we take a look at a few concrete examples. This serves a purely expository purpose, and we will only refer to the results of this paragraph to illustrate the theory. The first example is a simple lemma illustrating that the ostensible complexity of the definitions above can be avoided if $C = D$.

Example 4.22. Let $C$ be an exact category. Then, for every based, filtered, partially ordered set $(I; x_0, \ldots, x_k)$, the index map

$$\text{Fun}_C(I, C) \rightarrow \llbracket \text{dx}, I \rrbracket C \cong K_C^{\times k}$$

is equivalent to the map

$$F \mapsto (F(x_1) - F(x_0), \ldots, F(x_k) - F(x_{k-1})),$$

where we view $F(x_i)$ as a point in the $K$-theory space $K_C$ and we use the subtraction operation stemming from the infinite loop space structure of $K$-theory spaces (which is well-defined, up to a contractible space of choices).

Proof. Without loss of generality we may assume that $I = B[k]$. The map of the assertion is homotopic to the zero map on $\text{Fun}_C^B, I \rrbracket C$: for every diagram in this category, we have a choice of identifications $F(x_0) \cong \cdots \cong F(x_k)$. In particular, we obtain a well-defined map $\llbracket \text{dx}, I \rrbracket C \rightarrow K_C^{\times k}$, and it suffices to check that we have a commutative triangle

$$\begin{array}{ccc}
K_C^{\times k} & \rightarrow & \llbracket \text{dx}, I \rrbracket C \\
\downarrow & & \downarrow \\
K_C^{\times k} & \rightarrow & K_C^{\times k}
\end{array}$$

where the horizontal arrow is the one of (24). One sees that the diagonal arrow is equivalent to the map sending $(X_1, \ldots, X_k) \in C^{\times k}$ to

$$(X_1 - 0, (X_1 \oplus X_2) - X_1, (X_1 \oplus X_2 \oplus X_3) - (X_1 \oplus X_2), \ldots, (X_1 \oplus \cdots \oplus X_k) - (X_1 \oplus \cdots \oplus X_{k-1})),$$

This map is homotopic to the identity $1_{K_C^{\times k}}$. □
Example 4.23. Let \( I \) be a based, finite, filtered, partially ordered set such that the \( k \) base points are pairwise distinct. We denote the unique maximal element of \( I \) by \( m \). Then, the pre-index map

\[
\text{Fun}_C(I, D) \longrightarrow K^\times_k C
\]

can be expressed as

\[
(F(m)/F(x_0) - F(m)/F(x_1), \ldots, F(m)/F(x_{k-1}) - F(m)/F(x_k)).
\]

Proof. Without loss of generality we may assume that \( I = B[k] \). As in the proof of Example 4.22 we have to establish the existence of a commutative triangle

\[
\begin{array}{c}
K^\times_k C \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\text{Id}_{C, I} D \quad \text{Id}_{C, I} D
\end{array}
\begin{array}{c}
K^\times_k C
\end{array}
\]

where the horizontal arrow actually factors through \( \text{Id}_{C, I} C \). In particular, we see that the assertion follows from Example 4.22 once we have shown that for every diagram \( F \in \text{Fun}_C(I, D) \) the maps \( F(m)/F(x_i) - F(m)/F(x_{i+1}) \) and \( F(x_{i+1}) - F(x_i) \) are naturally homotopic. This follows directly from the basic properties of algebraic \( K \)-theory. \( \square \)

Example 4.24. Let \( I = B[2] \) with its three base points \( b_0, b_1, \) and \( b_2 \). It contains three copies of \( B[1] \), indexed by the set of unordered pairs of distinct elements in \( \{b_0, b_1, b_2\} \). We denote these inclusions by \( \phi_{ij}: B[1] \longrightarrow B[2] \). For every \( F \in \text{Fun}_C(I, D) \), we have a contractible space of homotopies

\[
\phi_{01}^* F + \phi_{12}^* F \simeq \phi_{02}^* F
\]

in \( K_C \cong K_{S_1(C)} \cong \text{Id}_{C, B[1]} D \).

Proof. We will construct these homotopies as homotopies of loops in \( K_C \cong \Omega[K_{S_1(C)}] \). By Corollary 4.18 for every simplicial object \( M_* \) in \( \text{poSet}_{\text{filt}}^{\text{from}} \) with \( k + 1 \) base points in level \( k \), we have a map of simplicial spaces

\[
(\text{Fun}_C(M_*, D))^\times \longrightarrow K_{S_*}(C).
\]

We apply this observation to the degenerate simplicial object \( M_* \), which agrees with \( B[k] \) for \( k \leq 2 \), and satisfies \( M_k = B[2] \) for \( k \geq 2 \), with the last base point \( x_2 \) repeated \( k - 2 \) times in \( M_k \). In particular, a diagram \( F \) gives rise to a 2-simplex of the left hand side

\[
\begin{array}{c}
\phi_{01}^* F \\
\phi_{12}^* F
\end{array}
\begin{array}{c}
\phi_{02}^* F
\end{array}
\]

with boundary faces \( \phi_{01}^* F, \phi_{12}^* F, \) and \( \phi_{02}^* F \). Since \( K_{S_0(C)} \cong 0 \), every 1-simplex induces an element of \( \Omega[K_{S_*}(C)] \). The geometric realization of the triangle above yields a contractible space of homotopies between the loops \( \phi_{01}^* F \cdot \phi_{12}^* F \) and \( \phi_{02}^* F \). \( \square \)
The existence of such a homotopy is not surprising. Indeed, passing to $K_0$, this statement amounts to the simple observation that we have the identity
\[ F(x_{01})/F(x_0) - F(x_{12})/F(x_1) + F(x_{12})/F(x_1) = F(x_{02})/F(x_0) - F(x_{02})/F(x_2). \]

The pre-index provides a natural contractible space of choices for this homotopy. We return to this at the end of this section.

4.2. The Index Map for Tate Objects Revisited. We now apply the generalized Waldhausen construction to produce a simplicial map
\[ N \cdot \text{Tate}^{el}(C) \times \rightarrow K_{S\bullet}(C) \]
whose geometric realization is equivalent to the index map. For any elementary Tate object $V$, by pre-composing (25) with the map
\[ B \cdot \text{Aut}(V) \rightarrow N \cdot \text{Tate}^{el}(C) \times \]
we obtain a map of reduced Segal objects in $\text{Spaces}$
\[ B \cdot \text{Aut}(V) \rightarrow K_{S\bullet}(C) \]
which encodes the $E_1$-structure of the map (20) in the Segal approach to algebra in $\infty$-categories.

Recall that $d(-)$ denotes the functor which associates to a bisimplicial object in an $\infty$-category the simplicial object given by restriction along the diagonal $\Delta^{op} \times \Delta^{op} \rightarrow \Delta^{op}$. As above, denote by $\text{Ex}^{1,h}G_{\mathcal{S}\bullet}(C)^{\times}$ the bisimplicial set obtained by taking the nerve (in the vertical direction) of the simplicial diagram of groupoids $\text{Ex}^{1,h}G_{\mathcal{S}\bullet}(C)^{\times}$. In Theorem 3.19 we constructed a simplicial map
\[ L : N \cdot \text{Tate}^{el}(C) \times \rightarrow d(\text{Ex}^{1,h}G_{\mathcal{S}\bullet}(C)^{\times}). \]
In this subsection we use the index space formalism to produce a map
\[ \Theta : \text{Ex}^{1,h}G_{\mathcal{S}\bullet}(C)^{\times} \rightarrow K_{S\bullet}(C). \]
We treat this as a map of bisimplicial spaces and apply the functor $d(-)$, which yields a map
\[ d(\text{Ex}^{1,h}G_{\mathcal{S}\bullet}(C)^{\times}) \rightarrow K_{S\bullet}(C). \]
Here, we use that the diagonal simplicial object of the bisimplicial object $K_{S\bullet}(C)$, which is vertically constant, agrees with the simplicial object $K_{S\bullet}(C)$. By post-composition of this map with $L$, we obtain a simplicial map
\[ N \cdot \text{Tate}^{el}(C) \times \rightarrow K_{S\bullet}(C), \]
and hence also a morphism of reduced Segal objects $B \cdot \text{Aut}(V) \rightarrow K_{S\bullet}(C)$ for every elementary Tate object $V$.

4.2.1. The Definition of the Map $\Theta$. We begin with two technical observations.

Remark 4.25. \(a\) The Grothendieck construction turns a simplicial set $M\bullet$ into a category $\tilde{M}\bullet \rightarrow \Delta^{op}$ over the opposite category of finite non-empty ordinals. We have a canonical equivalence
\[ M\bullet \cong \lim_{\tilde{M}\bullet/\Delta^{op}} \{\bullet\}, \]
where we take a fibrewise colimit (respectively a left Kan extension) on the left hand side over the constant, singleton-valued diagram indexed by $\tilde{M}$.
(b) Let \( I \) be a partially ordered set, and let \( I^* \) denote the nerve of \( I \). For \((D, C) \in \text{Cat}_{\text{pair}}^{ex} \), and an admissible diagram \( F \in \text{Func}(I, D) \) we have a canonical map \( \text{Ex}^1 I^* \rightarrow \text{Func}(\text{sd}(\bullet), D)^x \). This map is constructed as the fibrewise colimit of the map between constant diagrams

\[
\{\bullet\}_{\text{Ex}^1 I^*/\Delta^{ex}} \rightarrow \{\text{Func}(\text{sd}(\bullet), D)^x\}_{\text{Ex}^1 I^*},
\]

which sends the point \( \bullet \) to the restriction of the diagram \( F : I \rightarrow D \) along the map \( \text{sd}([n]) \rightarrow I \).

We can now define the following morphism.

**Definition 4.26.** Let \( V \) vary over the groupoid \( \text{Tate}^e_l(C)^x \). We consider the canonical map

\[
\text{Ex}^1 \text{Gr}_{\bullet}^e(V) \rightarrow \text{Func}(\text{sd}(\bullet), D)^x
\]

from Remark 4.25, we obtain a natural transformation of diagrams indexed by \( \text{Tate}^e_l(C)^x \):

\[
\{\text{Ex}^1 \text{Gr}_{\bullet}^e(V)\}_{\text{Tate}^e_l(C)^x} \rightarrow \{K_{S_1}(C)\}.
\]

By virtue of the universal property of colimits (since the right hand side is a constant diagram), we obtain a morphism

\[
\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x \rightarrow K_{S_1}(C).
\]

We observe that for every \( n \) we have that \( (\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x)_n \) is the colimit in the \( \infty \)-category of spaces of the simplicial object \( (\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x)_n \). In particular, we have a map

\[
(\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x)_n \rightarrow (\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x)^x_n.
\]

**Definition 4.27.** Let \( \Theta : \text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x \rightarrow K_{S_1}(C) \) be defined to be the composition

\[
\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x \rightarrow \text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x \rightarrow K_{S_1}(C),
\]

where the map on the right hand side has been constructed in Definition 4.26, and the morphism on the left hand side is the canonical augmentation of \( \text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x \), obtained by taking the geometric realization in the vertical direction.

As we described at the beginning of this sub-section, we have now constructed all pieces we need to construct the \( E_1 \)-structure of the index map.

4.2.2. Comparison with the Index Map.

**Corollary 4.28.** Let \( C \) be idempotent complete. Then the index map is equivalent to the geometric realization of the map

\[
N_* \text{Tate}^e_l(C)^x \xrightarrow{\Theta} d(\text{Ex}^1 \text{Gr}_{\bullet}^e(C)^x) \rightarrow K_{S_1}(C).
\]

In particular, the composition

\[
N_* \text{Tate}^e_l(C)^x \rightarrow K_{S_1}(C)
\]

is a map of Segal objects in spaces, and, for any elementary Tate object \( V \), the restriction of this map along the map \( B_* \text{Aut}(V) \rightarrow N_* \text{Tate}^e_l(C)^x \) encodes the \( E_1 \)-structure of the index map.
Proof. This follows from the commutative diagram

\[
\begin{array}{ccc}
\text{Index} & \Downarrow & \\
N^\bullet \text{Tate}^\bullet(C)_\bullet^\times & \overset{\mathcal{L}}{\longrightarrow} & \text{d(Ex}^{1,h} \text{Gr}^\leq(C)_\bullet^\times) \Theta \longrightarrow \text{K}_S^\bullet(C).
\end{array}
\]

The left hand side commutative triangle has been constructed in Theorem 3.19. We will show the existence of the commutative triangle on the right at the end of this proof.

Applying the functor \(| - |\) of geometric realization, the vertical arrow

\[|\text{d(Gr}^\leq(C)_\bullet^\times)| \longrightarrow |\text{d(Ex}^{1,h} \text{Gr}^\leq(C)_\bullet^\times)|\]

becomes an equivalence. Therefore, the geometric realization of \(\Theta \circ \mathcal{L}\) is equivalent to the index map \(\text{Tate}^\bullet(C)_\bullet^\times \longrightarrow |\text{K}_S^\bullet(C)|\).

It remains to construct the commutative triangle, relating \(\Theta\) and \(\text{Index}\). We claim that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}^\geq(C)_\bullet^\times & \Downarrow & \\
\text{Ex}^{1,h} \text{Gr}^\leq(C)_\bullet^\times & \Theta \longrightarrow & \text{K}_S^\bullet(C).
\end{array}
\]

By Definition 4.27 the composition \(\text{Gr}^\geq(C)_\bullet^\times \longrightarrow \text{Ex}^{1,h} \text{Gr}^\leq(C)_\bullet^\times \longrightarrow \text{K}_S^\bullet(C)\) is equivalent to the levelwise colimit of the map of constant diagrams

\[
\{
\ast \} \longrightarrow \{\text{Func}(\text{sd}(\bullet)), \text{Pro}^a(C)^\times\} \longrightarrow \{\text{Idx}_{\text{C, sd}(\bullet)}, \text{Pro}^a(C)\} \longrightarrow \{\text{Gr}^\geq(V)/\Delta^a\} \longrightarrow \{\text{Gr}^\leq(V)/\Delta^a\},
\]

where \(\ast\) is sent to an admissible diagram \(\text{sd([n])} \longrightarrow \text{[n]} \longrightarrow \text{Pro}^a(C)\), obtained by restriction of an admissible \([n]\)-diagram in \((\text{Pro}^a(C), C)\). Recall that the latter corresponds to a chain of admissible monics

\[Y_0 \hookrightarrow \ldots \hookrightarrow Y_n,
\]

and in particular, we have an admissible epic, in \(\text{Func}(\text{sd([n])), \text{Pro}^a(C)\), from the restriction of this chain to the admissible \(\text{sd([n])}\)-diagram obtained by restricting the admissible \([n]\)-diagram

\[
\text{0} \hookrightarrow \text{Y}_i/\text{Y}_0 \hookrightarrow \ldots \hookrightarrow \text{Y}_n/\text{Y}_0.
\]

The kernel of the admissible epic relating the two diagrams lies in \(\text{Func}(\text{sd([n])}^\Delta, \text{Pro}^a(C)\), hence the resulting images in \(\text{Idx}_{\text{C, sd([n])}}, \text{Pro}^a(C)\) are naturally equivalent.

The admissible diagram obtained by restricting \(\text{27}\) along \(\text{sd([n])} \longrightarrow \text{[n]}\) extends to \(\text{sd([n])}^B\), by sending the interval \([i, j]\) to \(Y_i/Y_0\). And the resulting \(B[n]\)-subdiagram lies in the image of the functor \(S^\bullet(C) \longrightarrow \text{Func}(B[n], \text{Pro}^a(C)\), which induces the equivalence \(\text{Idx}_{\text{C, } S^\bullet(C)} \equiv \text{K}_S^\bullet(C)\). Hence we conclude that we have a commutative square

\[
\begin{array}{ccc}
\text{Gr}^\geq(C)_\bullet^\times & \Downarrow & \\
\text{Ex}^{1,h} \text{Gr}^\leq(C)_\bullet^\times & \Theta \longrightarrow & \text{K}_S^\bullet(C).
\end{array}
\]

This implies commutativity of \(\text{24}\).
4.3. A Sketch of an Alternative Approach to $E_1$-structures. Let $\text{poSet}^{\text{filt}}$ denote the category of possibly infinite based sets $I$, together with a choice of base points $(x_0, \ldots, x_k) \in I^k$. Note that we do not impose the condition that the base points are minimal in $I$.

**Definition 4.29.** For $(I; x_0, \ldots, x_k) \in \text{poSet}^{\text{filt}}$, and $(D, C) \in \text{Cat}_{\text{ex}}^{\text{pair}}$ we define $\text{Idx}_{C, I} D$ as the limit

$$\lim_{I'} \text{Idx}_{C, I'} D,$$

where $I'$ ranges over the filtered category of finite based sets $(I'; x_0, \ldots, x_k)$ together with a map of based sets $(I'; x_0, \ldots, x_k) \longrightarrow (I; x_0, \ldots, x_k)$, which corresponds to $\text{id}_{I^k}$. This yields a functor

$$\text{Idx}: \text{poSet}^{\text{filt}} \longrightarrow \text{Cat}_{\text{ex}}^{\text{pair}} \longrightarrow \text{Spaces}.$$

Since the category we are taking the colimit over in Definition 4.29 is co-filtered, and for a morphism $I' \longrightarrow I''$ the induced map of index spaces

$$\text{Idx}_{C, I''} D \longrightarrow \text{Idx}_{C, I'} D$$

is an equivalence, we are taking an inverse limit over a co-filtered system of equivalences. Hence, we have a canonical equivalence of index spaces $\text{Idx}_{C, I} D \cong \text{Idx}_{C, I'} D$. This implies at once that the rigidity property (Proposition 4.20) holds as well for objects in $\text{poSet}^{\text{filt}}$.

**Definition 4.30.** Let $\text{Gr}_\bullet(C)^\times$ denote the Grothendieck construction of the functor $\text{Tate}\text{ex}(C)^\times \longrightarrow \text{sSet}$, which sends $V \in \text{Tate}\text{ex}(C)^\times$ to the simplicial set of (unordered) tuples of lattices in $\text{Gr}(V)$, i.e. an $n$-simplex in $\text{Gr}_\bullet(C)^\times$ is given by the data $(V; L_0, \ldots, L_n)$, where $V \in \text{Tate}\text{ex}(C)^\times$, and each $L_i$ denotes a lattice in $V$.

Imitating Definition 4.29 we may construct a map

$$\text{Gr}_\bullet(C)^\times \longrightarrow K_{S_4(C)},$$

where we send $(V; L_0, \ldots, L_k)$ to $(\text{Gr}(V); L_0, \ldots, L_k) \in \text{poSet}^{\text{filt}}$, and compute the index of the tautological diagram $\text{Gr}(V) \longrightarrow \text{Pro}_\bullet(C)$, which sends $L \in \text{Gr}(V)$ to the corresponding Pro-object. The following result can be established by almost the same proof as for Corollary 4.28. We therefore leave its proof to the reader.

**Corollary 4.31.** Choose a lattice $L \in \text{Gr}(V)$ for each elementary Tate object. This gives rise to a section $N_\bullet \text{Tate}\text{ex}(C)^\times \longrightarrow d(N_\bullet \text{Gr}_\bullet(C)^\times)$ of the canonical map $N_\bullet \text{Gr}_\bullet(C)^\times \longrightarrow N_\bullet \text{Tate}\text{ex}(C)^\times$. The geometric realization of the composition

$$N_\bullet \text{Tate}\text{ex}(C)^\times \longrightarrow d(N_\bullet \text{Gr}_\bullet(C)^\times) \longrightarrow K_{S_4(C)}$$

agrees with the map $\text{Ind} : \text{Tate}\text{ex}(C)^\times \longrightarrow B K C$, thus provides an explicit model for a map of $E_1$-objects $\text{Aut}(V) \longrightarrow K C$ for each elementary Tate object $V$.

5. Perspectives and Applications

5.1. Comparison with Index Theory in Topological $K$-Theory. In this section, we relate the index map to similar constructions defined in the context of index theory for Fredholm operators on Hilbert space.
5.1.1. The $K$-Theory of Tate Objects as an Analogue of Fredholm Operators. Let $\mathcal{H}$ be a complex separable Hilbert space, e.g. $L^2(S^1; \mathbb{C})$. Recall that a bounded operator $A : \mathcal{H} \to \mathcal{H}$ is Fredholm if $\dim \ker(A) < \infty$ and $\dim \coker(A) < \infty$. Denote by $\text{Fred}(\mathcal{H})$ the space of Fredholm operators (topologized as a subspace of the space of bounded operators on $\mathcal{H}$). The space $\text{Fred}(\mathcal{H})$ is endowed with a tautological complex

$$\text{Index}^* \longrightarrow \text{Fred}(\mathcal{H})$$

whose fibre at $A \in \text{Fred}(\mathcal{H})$ is the complex $\mathcal{H}A \to \mathcal{H}$.

**Theorem 5.1** (Atiyah [Ati89], Jänich [Jän65]). The complex $\text{Index}^* \longrightarrow \text{Fred}(\mathcal{H})$ is perfect, i.e. its restriction to any compact subspace $X \subset \text{Fred}(\mathcal{H})$ is quasi-isomorphic to a bounded complex of finite-dimensional topological vector bundles.

A perfect complex $E^* \longrightarrow X$ on a space $X$ defines a map

$$X \longrightarrow K^*_\text{top}$$

from $X$ to the classifying space of topological complex $K$-theory, sending $x \in X$ to $\chi(E|_x)^{\mathbb{C}}$. In particular, the tautological perfect complex defines a map

$$\text{Fred}(\mathcal{H}) \overset{\text{Index}}{\longrightarrow} K^*_\text{top},$$

sending $A \in \text{Fred}(\mathcal{H})$ to $\chi(\mathcal{H}A \to \mathcal{H})$.

**Theorem 5.2** (Atiyah, Jänich). The map $\text{Fred}(\mathcal{H}) \overset{\text{Index}}{\longrightarrow} K^*_\text{top}$ is an equivalence.

Let $R$ be a ring. The Sato Grassmannian $\text{Gr}(R((t)))$ can be understood as an analogue of the space $\text{Fred}(\mathcal{H})$. We have a map

$$\pi_{R[[t]]} : R((t)) \longrightarrow R[[t]],$$

given by forgetting the principal part of a formal Laurent series. Utilizing this map, we observe that a lattice $L \subset R((t))$ corresponds to the operator

$$L \overset{\pi_{R[[t]]}[L]}{\longrightarrow} R[[t]].$$

This operator has finite-dimensional kernel and cokernel, so (30) allows us to think of lattices as algebraic Fredholm operators. In the Hilbert space setting, this identification of a lattice with an operator defines a weak equivalence between $\text{Fred}(\mathcal{H})$ and the Segal–Wilson analogue of the Sato Grassmannian (c.f. [PS86, Chapter 7]).

We now consider the tautological complex of $R$-modules

$$\gamma^* \longrightarrow \text{Gr}(R((t))),$$

We can see this directly as follows. Denote by $\text{Perf}^\text{top}$ the stack which assigns to a space $X$ its $(\infty)$-category of topological perfect complexes $\text{Perf}^\text{top}(X)$. Under the Yoneda embedding, a perfect complex $E^* \longrightarrow X$ is equivalent to a map $X \longrightarrow \text{Perf}^\text{top}(-)^x$. We obtain the map $X \longrightarrow K^*_\text{top}$ by composing $X \longrightarrow \text{Perf}^\text{top}(\mathcal{H})^x$ with the canonical map $\text{Perf}^\text{top}(-)^x \longrightarrow K^*_\text{Perf}^\text{top}(\mathcal{H})$, followed by the equivalence $K_{\text{Perf}^\text{top}}(-) \simeq K^*_\text{top}$. The formula in terms of the Euler characteristic follows from the definitions.
whose fibre at a lattice is the complex (30). Just as above, this perfect complex corresponds to a classifying map

\[ \text{Gr}(R((t))) \longrightarrow \text{Perf}(R)^{\times}, \]

where now Perf(R) is the classifying stack of perfect complexes of R-modules. Composing with the map Perf(R) \times \longrightarrow K_R, we obtain an analogue of (29)

\[ \text{Gr}(R((t))) \longrightarrow K_R. \]

However, from the perspective of the Atiyah–Jänich theorem, it is a crude analogue of (29): because the source is a presheaf of sets while the target is a presheaf of spaces, it cannot possibly be an equivalence.

A richer analogue of (29) exists. The Sato Grassmannian \( \text{Gr}(R((t))) \) is a torsor for the group Ind-scheme \( \text{Aut}(R((t))) \). In particular, we can view an automorphism \( g \in \text{Aut}(R((t))) \) as a Fredholm operator by the assignment

\[ g \mapsto gR[[t]] \xrightarrow{\pi_R[[t]]} R[[t]] \]

where \( gR[[t]] \) denotes the translate of the lattice \( R[[t]] \) under \( g \). The analogue of \( \text{Aut}(R((t))) \) in the Hilbert space setting is the “restricted general linear group” \( \text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \) of a polarized Hilbert space. The Segal–Wilson Grassmannian is a homogeneous space for this group, and the analogue of (31) induces a weak equivalence between the restricted general linear group and the space of Fredholm operators on \( \mathcal{H}^+ \). This justifies us in viewing \( \text{Aut}(R((t))) \) as a richer algebraic analogue of the space of Fredholm operators. Tracing through the discussion above, we obtain a map

\[ \text{Aut}(R((t))) \longrightarrow K_R, \]

sending \( g \) to \( \chi(gR[[t]] \xrightarrow{\pi_R[[t]]} R[[t]]) \). The same reasoning as above shows that this map cannot be an equivalence.

However, as we now explain, the infinite loop space \( \Omega K_{\text{Tate}^{\text{et}}(\mathbb{C})} \) should be understood as an analogue of \( \text{Fred}(\mathcal{H}^+) \). A detailed argument follows from the +-construction.

Recall that the perfect radical of a group \( G \) is the largest proper subgroup \( P \subset G \) such that \([P, P] = P\). Every group has a perfect radical (c.f. [Wei13, Remark 4.1.5]), and the perfect radical is a normal subgroup.

**Definition 5.3.** Let \( X \) be a connected space. A +-construction on \( X \) is a map \( X \longrightarrow X^+ \) such that the induced map on integral homology is an isomorphism and such that the kernel of the induced map \( \pi_1(X) \longrightarrow \pi_1(X^+) \) is the perfect radical of \( \pi_1(X) \).

A theorem of Quillen (c.f. [Ger73, Theorem 2.1]) shows that a +-construction exists and is unique up to homotopy equivalence.

**Proposition 5.4.** Let \( R \) be a ring. The canonical map

\[ \Omega(B \text{Aut}(R((t))))^+ \longrightarrow \Omega K_{\text{Tate}^{\text{et}}(R)} \]

is an equivalence.

---

13 The restricted general linear group consists of bounded invertible operators whose commutator with the projection onto \( \mathcal{H}^+ \) is Hilbert–Schmidt. For a longer discussion of this group and the Grassmannian, see [PS86, Chapter 6].
Proof. We show that \( \Omega(B\text{Aut}(R((t))))^+ \simeq \Omega K_{\text{Tate}^e}(R) \). By Corollary 3.24 this will imply the result for uncountable Tate objects.

For any ring, the category \( \text{Tate}^e(R) \) of countable elementary Tate modules is split exact [BGW14 Prop. 5.20]. So, its \( K \)-theory as an exact category is equivalent to the \( K \)-theory of the symmetric monoidal groupoid \( \text{Tate}^e(R)^\times \).

Following Weibel [Wei13 Theorem 4.4.10], to characterize the \( K \)-theory of a symmetric monoidal category \( (\mathcal{S}, \otimes) \) in terms of the \( \pm \)-construction, it suffices to show that

1. for any \( a, b \in \mathcal{S} \), the canonical map \( \text{Aut}(a) \to \text{Aut}(a \otimes b) \) is an injection, and
2. there exists a sequence of objects \( \{s_i\}_{i=0}^\infty \subset \mathcal{S} \), such that for every \( b \in \mathcal{S} \), there exists \( b' \in \mathcal{S} \) such that \( b \otimes b' \cong \otimes_{i=0}^n s_i \) for some \( n \).

Given such a sequence \( \{s_i\} \), define \( \text{Aut}(\mathcal{S}) := \lim_{\to} \text{Aut}(\otimes_{i=0}^n s_i) \). Then

\[
\Omega(B\text{Aut}(\mathcal{S}))^+ \cong \Omega K_{\mathcal{S}}.
\]

For \( (\mathcal{S}, \otimes) = (\text{Tate}^e(R), \otimes) \), the first condition is immediately satisfied. For the second, it suffices to observe that every countable elementary Tate module is a direct summand of \( R((t)) \) (see [BGW14, Prop. 5.21]). Taking \( s_0 := R((t)) \) and \( s_i = 0 \) for \( i > 0 \), we obtain a sequence of the desired form and conclude the result. \( \square \)

Lemma 5.5. The canonical map \( B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \to (B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+))^+ \) is a weak equivalence.

Proof. This is an immediate consequence of the definition of the \( \pm \)-construction and the isomorphism \( \pi_1(B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+)) \cong \mathbb{Z} \). Recall that this isomorphism arises from the sequence of isomorphisms

\[
\pi_1(B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+)) \cong \pi_0(\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+)) \cong \pi_0(\text{Fred}(\mathcal{H}^+)),
\]

and from the fact that the classical index map \( A \mapsto \dim(\ker(A)) - \dim(\coker(A)) \) induces a bijection between \( \pi_0(\text{Fred}(\mathcal{H}^+)) \) and \( \mathbb{Z} \) (c.f. [Dou98 Theorem 5.35]). \( \square \)

Along with the canonical equivalence \( B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \to \Omega B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \), this gives the following.

Corollary 5.6. The canonical map \( \text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \to (\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+))^+ \) is a weak equivalence.

Combined with the equivalence \( \text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+) \) and \( \text{Fred}(\mathcal{H}^+) \), we see that \( \Omega(B\text{GL}_{\text{res}}(\mathcal{H}, \mathcal{H}^+))^+ \) is another model for \( \text{Fred}(\mathcal{H}^+) \), and that \( \Omega K_{\text{Tate}^e}(\mathcal{C}) \) and, by extension, \( \Omega K_{\text{Tate}^e}(\mathcal{R}) \) is its algebraic analogue.

5.1.2. The Index Map and Perfect Complexes. We now give an alternative account of the index map for a split exact, idempotent complete category \( \mathcal{C} \). Throughout this section we work with \( \text{countably generated} \) Tate objects. The category \( \text{Tate}^e(\mathcal{C}) \) is split exact if \( \mathcal{C} \) is [BGW14 Proposition 5.20].

Let \( V \in \text{Tate}^e(\mathcal{C}) \), and \( L \in \text{Gr}(V) \) a lattice in \( V \). We choose a splitting \( \pi: V \to L \); as a first step we define a complex \( (L' \xrightarrow{\pi|_L'} L) \) for every lattice \( L' \in \text{Gr}(V) \), analogous to the one of \([30]\).
Definition 5.7. Let $C$ be a idempotent complete, split exact category. We denote by $\text{Perf}(C)$ the $\infty$-category of perfect complexes in $C$, i.e. the $\infty$-categorical localization of $\text{Ch}^b(C)$ at the quasi-isomorphisms. The inverse of the equivalence of $K$-theory spaces\(^{14}\) induced by the exact functor $C \rightarrow \text{Perf}(C)$ will be denoted by $\chi : K_{\text{Perf}(C)} \xrightarrow{\simeq} K_C$.

The $\infty$-category $\text{Perf}(C)$ is stable. In [Kel99, Sect. 2], Keller, using different notation, describes its homotopy category $\text{Ho}(\text{Perf}(C)) \cong D^b(C)$ as the Verdier localization of the pre-triangulated dg-category $\text{Ch}^b(C)$ at the localizing sub-category of acyclic complexes. Since both dg-categories give rise to a stable $\infty$-category by means of the dg-nerve functor $N_{dg}$ (see [Lur, Sect. 1.3.1]), their quotient in stable $\infty$-categories can be identified with $\text{Perf}(C)$. We can now define the object $(L' \xrightarrow{\pi|L'} L)$.

Definition 5.8. We denote by $\text{Gr}_{(L,L')}((V))$ the codirected set of lattices $N$ in $V$, which are contained in $L$ and $L'$. It parametrizes a diagram in the category $\text{Ch}^b(C)$, which sends $N$ to $(L'/N \xrightarrow{\pi|L'} L/N)$ (concentrated in degrees 0 and 1 respectively). By virtue of the functor $\text{Ch}^b(C) \rightarrow \text{Perf}(C)$, we obtain a $\text{Gr}_{(L,L')}((V))$-diagram in $\text{Perf}(C)$. One sees that all inclusions $N \hookrightarrow N'$ induce equivalent objects in $\text{Perf}(C)$, since the fibre of the resulting map of cones, is equivalent to $\text{fib}(N'/N \xrightarrow{id} N'/N) \cong 0$.

Since $\text{Gr}_{(L,L')}((V))$ is cofiltering, the limit of this system is canonically equivalent to the complex $(L'/N \rightarrow L/N)$ for all $N \in \text{Gr}_{(L,L')}((V))$. We define $(L' \xrightarrow{\pi|L'} L)$ to be

$$\lim_{\text{Gr}_{(L,L')}((V))} (L'/N \rightarrow L/N) \in \text{Perf}(C).$$

After having clarified the notation, we are able to give the following interpretation of the index map. We emphasize however that the theorem below ignores the $E_1$-structure, and therefore is not a panacea to avoid the lengthy discussion of Section 4. We hope to return to a description of the $E_1$-structure in terms of perfect complexes in future work.

Proposition 5.9. Let $C$ be an idempotent complete, split exact category, $V \in \text{Tate}_{\aleph_0}(C)$ and let $L \rightarrow V$ be a lattice. Fix a splitting $\pi_L : V \rightarrow L$ of the inclusion $L \hookrightarrow V$. The composition

$$\text{Aut}(V) \xrightarrow{\Omega} K_{\text{Tate}_{\aleph_0}(C)} \xrightarrow{\text{Ind}_{\text{enh}}} K_C$$

is equivalent to the map

$$\text{Aut}(V) \xrightarrow{K} K_C,$$

which sends $g$ to $\chi(gL \xrightarrow{\pi_L} L)$ (cf. Definition 5.7).

Proof. We claim that we have a bicartesian square

$$
\begin{array}{ccc}
(gL \xrightarrow{\pi_L} L) & \rightarrow & gL/N \\
\downarrow & & \downarrow \\
0 & \rightarrow & L/N,
\end{array}
$$

\(^{14}\)cf. Gillet–Waldhausen’s theorem [TT90 Thm. 1.11.7].
in $\text{Perf}(\mathbb{C})$, where $N \in Gr_{(L,gL)}(\mathbb{C}; V)$ is a lattice contained in both $L$ and $gL$. To see this, we observe that $(gL \to L)[1]$ is equivalent to $\text{cone}(gL/N \to L/N)$.

The cone of a morphism of cochain complexes is a model for the cofibre. In particular, $(gL \to L) \cong \text{cone}(gL/N \to L/N)[-1]$ is a model for the fibre. We obtain the asserted bicartesian square.

By the definition of $K$-theory, this implies $\chi(gL \to L) \cong \text{cone}(gL/N \to L/N)[-1]$ is a model for the fibre. We obtain the asserted bicartesian square.

We can summarize the above discussion as follows. For an elementary Tate object $V$ in an idempotent complete, split exact category $\mathbb{C}$, a lattice $L \to V$ gives rise to a commuting square

$$\begin{array}{ccc}
\text{Aut}(V) & \to & Gr(V) \\
\downarrow & & \downarrow \\
\Omega K_{\text{Tate}^e(\mathbb{C})} & \to & K_{\mathbb{C}},
\end{array}$$

where the top horizontal map sends $g$ to the lattice $gL$; and the right hand side vertical arrow maps the lattice $L'$ to $\chi(L' \to L)$. The lower left, upper right and upper left corners are all algebraic analogues of $\text{Fred}(\mathcal{H})$, while each map to $K_{\mathbb{C}}$ is an algebraic analogue of the map $\chi$.

By Theorem 3.22, the bottom horizontal map, i.e. the index map, is an equivalence $\text{Index} : \Omega K_{\text{Tate}^e(\mathbb{C})} \cong K_{\mathbb{C}}$.

We view this as an algebraic analogue of the Atiyah-J"{a}nich equivalence between the space of Fredholm operators and the classifying space of topological complex $K$-theory (Theorem 5.2).

5.2. $K$-Theory Torsors. Let $R$ be a ring. Denote by $\text{Tate}_{el}^e(R) := \text{Tate}_{el}^e(P_f(R))$ the category of elementary Tate $R$-modules. In this section, we interpret the index map $K_{\text{Tate}^e_{el}}(R) \xrightarrow{\text{Index}} BK_R$ as the classifying map of a universal $K_R$-torsor over $K_{\text{Tate}^e_{el}}(R)$, and we explain how classical dimension and determinantal torsors arise as truncations of this $K_R$-torsor. We hope that this will provide the beginning of a satisfactory answer to [Dri06, Problem 5.5.3] as well as shed light on the relation between the various torsors arising in [Kap], [Pre12], [BBE12], and [AK10]. In [San12], Saito gives a construction of a torsor, using the abstract delooping equivalence $K_{\text{Tate}^e(\mathbb{C})} \cong BK_{\mathbb{C}}$ from [San12]. Our treatment via the index map yields the dual of the torsor classified by Saito’s map, and in a form which can be directly compared to the classical constructions of dimensional and determinantal torsors.

Recall that the category of countable Tate $R$-modules, $\text{Tate}_{el}(R)$, is the idempotent completion of the category of elementary Tate $R$-modules $\text{Tate}_{el}^e(R)$. The assignments $R \mapsto \text{Tate}_{el}(R)$ and $R \mapsto \text{Tate}_{el}^e(R)$ are functorial with respect to flat base change, and we view them as defining presheaves of categories on the Nisnevich site of rings.

From Drinfeld, we see that the inclusion $\text{Tate}_{el}^e(R) \hookrightarrow \text{Tate}_{el}(R)$ is Nisnevich-locally an equivalence [Dri06, Theorem 3.4], and further that $\text{Tate}_{el}^e(\mathbb{C})$ is a Nisnevich sheaf [Dri06, Theorem 3.3]. In other words, $\text{Tate}_{el}^e(\mathbb{C})$ is the Nisnevich sheafification of $\text{Tate}_{el}^e(\mathbb{C})$. 

We view this as an algebraic analogue of the classification of Fredholm operators with countable determinantal torsors as truncations of the $K_R$-torsor classified by Drinfeld’s map.

We summarize the above discussion as follows.

$$\begin{array}{c}
\text{Aut}(V) \to Gr(V) \\
\downarrow \\
\Omega K_{\text{Tate}^e(\mathbb{C})} \to K_{\mathbb{C}},
\end{array}$$

where the top horizontal map sends $g$ to the lattice $gL$; and the right hand side vertical arrow maps the lattice $L' \to L$. The lower left, upper right and upper left corners are all algebraic analogues of $\text{Fred}(\mathcal{H})$, while each map to $K_{\mathbb{C}}$ is an algebraic analogue of the map $\chi$.

By Theorem 3.22, the bottom horizontal map, i.e. the index map, is an equivalence $\text{Index} : \Omega K_{\text{Tate}^e(\mathbb{C})} \cong K_{\mathbb{C}}$.

We view this as an algebraic analogue of the Atiyah-J"{a}nich equivalence between the space of Fredholm operators and the classifying space of topological complex $K$-theory (Theorem 5.2).

5.2. $K$-Theory Torsors. Let $R$ be a ring. Denote by $\text{Tate}_{el}^e(R) := \text{Tate}_{el}^e(P_f(R))$ the category of elementary Tate $R$-modules. In this section, we interpret the index map $K_{\text{Tate}^e_{el}}(R) \xrightarrow{\text{Index}} BK_R$ as the classifying map of a universal $K_R$-torsor over $K_{\text{Tate}^e_{el}}(R)$, and we explain how classical dimension and determinantal torsors arise as truncations of this $K_R$-torsor. We hope that this will provide the beginning of a satisfactory answer to [Dri06, Problem 5.5.3] as well as shed light on the relation between the various torsors arising in [Kap], [Pre12], [BBE12], and [AK10]. In [San12], Saito gives a construction of a torsor, using the abstract delooping equivalence $K_{\text{Tate}^e(\mathbb{C})} \cong BK_{\mathbb{C}}$ from [San12]. Our treatment via the index map yields the dual of the torsor classified by Saito’s map, and in a form which can be directly compared to the classical constructions of dimensional and determinantal torsors.

Recall that the category of countable Tate $R$-modules, $\text{Tate}_{el}(R)$, is the idempotent completion of the category of elementary Tate $R$-modules $\text{Tate}_{el}^e(R)$. The assignments $R \mapsto \text{Tate}_{el}(R)$ and $R \mapsto \text{Tate}_{el}^e(R)$ are functorial with respect to flat base change, and we view them as defining presheaves of categories on the Nisnevich site of rings.

From Drinfeld, we see that the inclusion $\text{Tate}_{el}^e(R) \hookrightarrow \text{Tate}_{el}(R)$ is Nisnevich-locally an equivalence [Dri06, Theorem 3.4], and further that $\text{Tate}_{el}^e(\mathbb{C})$ is a Nisnevich sheaf [Dri06, Theorem 3.3]. In other words, $\text{Tate}_{el}^e(\mathbb{C})$ is the Nisnevich sheafification of $\text{Tate}_{el}^e(\mathbb{C})$. 

We summarize the above discussion as follows. For an elementary Tate object $V$ in an idempotent complete, split exact category $\mathbb{C}$, a lattice $L \to V$ gives rise to a commuting square

$$\begin{array}{ccc}
\text{Aut}(V) & \to & Gr(V) \\
\downarrow & & \downarrow \\
\Omega K_{\text{Tate}^e(\mathbb{C})} & \to & K_{\mathbb{C}},
\end{array}$$

where the top horizontal map sends $g$ to the lattice $gL$; and the right hand side vertical arrow maps the lattice $L' \to L$. The lower left, upper right and upper left corners are all algebraic analogues of $\text{Fred}(\mathcal{H})$, while each map to $K_{\mathbb{C}}$ is an algebraic analogue of the map $\chi$.

By Theorem 3.22, the bottom horizontal map, i.e. the index map, is an equivalence $\text{Index} : \Omega K_{\text{Tate}^e(\mathbb{C})} \cong K_{\mathbb{C}}$.

We view this as an algebraic analogue of the Atiyah-J"{a}nich equivalence between the space of Fredholm operators and the classifying space of topological complex $K$-theory (Theorem 5.2).
The same holds after we pass to $K$-theory. From Thomason–Trobaugh \cite[Theorem 10.8]{TT90} and the local vanishing of $K_\infty$ \cite[Theorem 3.7]{Drinfeld06}, we know that the Nisnevich sheafification of $\mathcal{B}K_R$ is given by $\Omega^\infty \Sigma K_R$, where $K_R$ is the non-connective $K$-theory spectrum of $R$. Denote the Nisnevich sheafification of a presheaf $F$ of infinite loop spaces by $L(F)$ \cite{Drinfeld06}. Drinfeld’s observations imply that the natural map $K_{Tate_{k_0}}(R) \longrightarrow L(K_{Tate_{k_0}}(R))$ extends to a map

$$K_{Tate_{k_0}}(R) \longrightarrow L(K_{Tate_{k_0}}(R)).$$

Sheafifying the index map, we obtain a natural map

$$K_{Tate_{k_0}}(R) \longrightarrow L(K_{Tate_{k_0}}(R)) \longrightarrow \Omega^\infty \Sigma K_R.$$

By Theorem \cite[3.22]{Drinfeld06}, this map gives natural isomorphisms on $\pi_i$ for $i > 0$. By \cite[Theorem 3.6]{Drinfeld06}, we also have that it gives an isomorphism on $\pi_0$. We conclude the following.

**Proposition 5.10.** The index map extends to an equivalence of Nisnevich sheaves of infinite loop spaces

$$K_{Tate_{k_0}}(-) \xrightarrow{\text{Index}} \Omega^\infty \Sigma K_{\infty}.$$

Below, we explain how the 1 and 2-truncations of this map give rise to the dimension and determinantal torsors of $[\text{Kap}]$.

We can also consider the category $\text{2-Tate}^e_{k_0}(R)$ of elementary 2-Tate $R$-modules, defined by $\text{2-Tate}^e_{k_0}(R) := \text{Tate}^e_{k_0}(\text{Tate}_{k_0}(R))$ (cf. \cite[Section 7]{BGW14}). In this setting, the index map takes the form

$$K_{2,\text{Tate}^e_{k_0}}(R) \xrightarrow{\text{Index}} \mathcal{B}K_{\text{Tate}^e_{k_0}}(R).$$

Post-composing with \eqref{34}, we obtain a natural map

$$K_{2,\text{Tate}^e_{k_0}}(R) \xrightarrow{\text{Index}^2} \mathcal{B} \Omega^\infty \Sigma K_R.$$

We will explain how the 3-truncation of this map gives rise to the 2-gerbe of \cite{AKI06}. We also discuss the relationship to a gerbe, which was conjectured to exist in \cite{FZ12}, on the Sato Grassmannian $\text{Gr}(R((t_1))((t_2)))$ of the 2-Tate $R$-module $R((t_1))((t_2))$.

**5.2.1. The Index $K$-Theory Torsor for Elementary Tate Objects.** We begin by considering the general case of an idempotent complete exact category $\mathcal{C}$. In Corollary \ref{33} we introduced the map

$$\text{Index} : \text{Tate}^e(\mathcal{C})^\times \longrightarrow \mathcal{B}K_{\mathcal{C}} \cong [S^\times_{\mathcal{C}}],$$

by replacing $\text{Tate}^e(\mathcal{C})^\times$ by the geometric realization $|Gr^\infty_{\mathcal{C}}(\mathcal{C})^\times|$, and using the natural map of simplicial groupoids

$$Gr^\infty_{\mathcal{C}}(\mathcal{C})^\times \longrightarrow S^\times_{\mathcal{C}},$$

which sends $(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V)$ to $(L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0)$. Let

$$\mathcal{T} \longrightarrow \text{Tate}^e(\mathcal{C})^\times$$

be the $K_{\mathcal{C}}$-torsor classified by this map, and let $\mathcal{T}|_V$ be the fibre of this torsor at an elementary Tate object $V$. We now explain how the index map allows us to give an elementary description of a part of the space of sections of $\mathcal{T}|_V$.

\footnote{N.b. the $\infty$-category of sheaves of infinite loop spaces is the $\infty$-categorical localization of the $\infty$-category of presheaves of such at the local equivalences. The $L$ stands for “localization” functor.}
We begin by observing that $V$ determines a commuting square

$$
\begin{array}{ccc}
Gr_{\bullet}^<(V) & \longrightarrow & Gr_{\bullet}^<(C)^x \\
\downarrow & & \downarrow \\
* & \longrightarrow & Tate_q^<(C)^x.
\end{array}
$$

By Theorem 2.10 (and its corollary, Proposition 3.3), the vertical maps induce equivalences after realization.

We also note that the contractibility of $P^r S_\bullet (C)^x$ (Lemma 2.26) implies that the map

$$
P^r S_\bullet (C)^x \longrightarrow S_\bullet (C)^x
$$

becomes equivalent, after realization, to the universal $K_C$-torsor

$$
* \longrightarrow B K_C.
$$

Therefore, from our construction of the index map, we see that every commuting triangle of the form

$$
\begin{array}{ccc}
Gr_{\bullet}^<(V) & \longrightarrow & P^r S_\bullet (C)^x \\
f & & \downarrow \\
Gr_{\bullet}^<(C)^x & \longrightarrow & S_\bullet (C)^x
\end{array}
$$

determines a section of the $K_C$-torsor $T|_V$. Unpacking the definitions, we see that sections of this form admit a very classical description.

**Lemma 5.11.** The data of a map $f : Gr_{\bullet}^<(V) \longrightarrow P^r S_\bullet (C)^x$, fitting into a triangle of the form (36), consists of:

1. a map $f : Gr(V) \longrightarrow \text{ob } C$, and
2. for each nested sequence of lattices $L_0 \hookrightarrow \cdots \hookrightarrow L_n$, a sequence $f(L_0) \hookrightarrow \cdots \hookrightarrow f(L_n)$ of admissible monics in $C$ such that the assignment of monics to monics is functorial and such that

$$
f(L_1)/f(L_0) \hookrightarrow \cdots \hookrightarrow f(L_n)/f(L_0) = \text{Index}(L_0 \hookrightarrow \cdots \hookrightarrow L_n) = L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0.
$$

From the perspective of $K$-theory, we see that $f$ encodes a map

$$
Gr(V) \longrightarrow K_C
$$

along with a coherent collection of homotopies

$$
f(L_0) + L_1/L_0 \simeq f(L_1)
$$

for every nested pair of lattices $L_0 \hookrightarrow L_1$. In particular, the coherence conditions are neatly encoded in the higher simplices of the $S$-construction.

We can also observe, as a consequence of Theorem 2.11, that such a map $f$ is determined, up to homotopy, by its value on a single lattice. Indeed, for any two lattices $L$ and $L'$, there exists a common enveloping lattice $N$. Any such $N$ determines a homotopy

$$
f(L') \simeq f(L) + N/L - N/L',
$$
and the machinery of Sections 5.3 and 4 provides a systematic way of relating the homotopies associated to different choices of $N$.

It is worth noting, that without further work, we cannot say for certain that all sections of the $K_C$-torsor $T|_V$ are of the above form. However, for several truncations of this torsor of classical interest, we can describe the entire space of sections in these terms. In the following sections, we examine these truncated torsors for $C = P_f(R)$ in more detail. For a related discussion of these truncations in the abstract setting, see [Pre12] 16.

5.2.2. The Dimension Torsor. In this section we relate the $1$-truncation of $[34]$ to the dimension torsor of $[Kap]$ (see also $[Dri06]$). By descent, it suffices to treat the restriction of (34) to $5.2.2$. The Dimension Torsor.

We consider an elementary Tate $R$-module $V \in \operatorname{Tate}_{R_0}(R)$. The rank of finitely-generated projective $R$-modules defines a natural map

$$K_0(R) \longrightarrow \mathbb{Z}^{\pi_0(\text{Spec } R)}.$$ 

For ease of notation, we assume from now on that Spec $R$ is connected. The general case can be recovered by Zariski sheafification.

The Tate $R$-module $V$ gives rise to the $\mathbb{Z}$-torsor, given by the set

$$T(V) = \{ f : \operatorname{Gr}(V) \longrightarrow \mathbb{Z} | f(L_1) = f(L_0) + \operatorname{rk}(L_1/L_0), \forall (L_0 \leq L_1) \in \operatorname{Gr}_1^T(V) \}.$$ 

Such $f$ are called dimension theories in $[Kap]$. We see that $T(V)$ is a $\mathbb{Z}$-torsor as follows. First, the action of $k \in \mathbb{Z}$ on $T(V)$ is defined pointwise, by $f \mapsto f + k$. Second, one uses that any two lattices $L_0$ and $L_1$ admit a common upper bound to show that a function $f \in T(V)$ is determined by its value at a single lattice $f(L)$.

Since $\operatorname{Gr}(-) : \operatorname{Tate}_{R_0}^d(R)^{\times} \longrightarrow \text{Set}$ is a functor, we see that the construction $T(V)$ is functorial as well:

$$T(-) : \operatorname{Tate}_{R_0}^d(R)^{\times} \longrightarrow B\mathbb{Z}.$$ 

Here $B\mathbb{Z}$ denotes the groupoid of $\mathbb{Z}$-torsors.

For every elementary Tate object $V$ we have a map $B\operatorname{Aut}(V) \longrightarrow \operatorname{Tate}^d(C)^{\times}$. We conclude therefore the existence of a $\mathbb{Z}$-torsor on $B\operatorname{Aut}(V)$, classified by the map

$$B\operatorname{Aut}(V) \longrightarrow B\mathbb{Z}.$$ 

**Proposition 5.12.** The map $\operatorname{Tate}_{R_0}^d(R)^{\times} \longrightarrow B\mathbb{Z}$, obtained as the composition of the Index map $\operatorname{Tate}_{R_0}^d(R)^{\times} \xrightarrow{\text{Index}} BK_R$ with the map $BK_R \xrightarrow{B(\operatorname{rk})} B\mathbb{Z}$ (induced by the rank of finitely generated projective modules), is equivalent to the map $T(-)$ classifying the dimensional torsor.

**Proof.** The map $B(\operatorname{rk}) : BK_R \longrightarrow B\mathbb{Z}$ is equivalent to the geometric realization of the map

$$S^*_4(C)^{\times} \longrightarrow B^*_\mathbb{Z},$$ 

which sends $(0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_k)$ to $(\operatorname{rk}(X_1), \operatorname{rk}(X_2/X_1), \ldots, \operatorname{rk}(X_k/X_{k-1}))$. Consequently, $B(\operatorname{rk}) \circ \text{Index}$ is equivalent to the geometric realization of

$$\operatorname{Gr}^S_\bullet(P_f(R))^{\times} \longrightarrow B^*_\mathbb{Z},$$ 

which sends $(V; L_0 \leftrightarrow \cdots \leftrightarrow L_k)$ to $(\operatorname{rk}(L_1/L_0), \ldots, \operatorname{rk}(L_k/L_{k-1}))$. This map induces an augmentation of $\operatorname{Gr}^S_\bullet(P_f(R))^{\times}$ by $B\mathbb{Z}$, hence a $\mathbb{Z}$-torsor on the simplicial space $\operatorname{Gr}^S_\bullet(P_f(R))^{\times}$.

\[16\text{Note that we replace Previdi’s condition that } C \text{ satisfies } "\text{AIC} + \text{AIC}^{opp}\text{" with the condition that } C \text{ is idempotent complete. Mutatis mutandi, Previdi’s discussion of dimension and determinant torsors now applies to the 1 and 2-truncations of the } K_C\text{-torsor considered here.} \]
Since $B_0 \mathbb{Z} = \{\ast\}$ is a singleton, the $\mathbb{Z}$-torsor above trivializes when pulled back to $Gr^<_{\mathbb{Z}}(P_f(R))^\times$. Hence, every choice of a lattice $L \subset V$ induces a trivialization. Given a nested pair of lattices $L \leq L'$, the two corresponding trivializations differ precisely by $\text{rk}(L' / L)$. We therefore obtain for the space of sections of the torsor $B(\text{rk}) \circ \text{Index}$ over a connected component $B \text{Aut}(V) \subset \text{Tate}^{cl}_{\mathbb{R}_0}(R)^\times$ the set of functions $f: Gr(P_f(R), V) \to \mathbb{Z}$, satisfying $f(L') = f(L) + \text{rk}(L' / L)$ for all nested pairs $L \leq L'$ of lattices in $V$. This is precisely the definition of the torsor $T(V)$.

5.2.3. The Determinant Torsor. We now investigate the 2-truncation of (34) and relate it to the graded determinant torsor of [Kap] (see also [BBE02] and [Dri06]).

Denote by $\text{Pic}_R^Z$ the symmetric monoidal groupoid of graded lines $\text{Pic}^Z_R$. A graded line is a pair $(L, n)$, where $L$ is an invertible $R$-module, and $n: \text{Spec } R \to \mathbb{Z}$ a Zariski-locally constant function. The usual symmetry constraint of tensoring $R$-modules

$$\phi_{L,M} : L \otimes M \simeq M \otimes L$$

will be modified by a sign:

$$(L, n) \otimes (M, m) \simeq (L \otimes M, n + m) \xrightarrow{(-1)^mn \phi_{L,M}} (M \otimes L, m + n) \simeq (M, m) \otimes (L, n).$$

For $M \in P_f(R)$, the assignment $M \mapsto (\Lambda_{\text{top}}^M, \text{rk}(M))$ extends to a symmetric monoidal map

(37) \quad $\det^Z: K_R \to \text{Pic}^Z_R$.

We have a natural morphism

$$\text{Tate}^{cl}_{\mathbb{R}_0}(R)^\times \longrightarrow K_{\text{Tate}^{cl}_{\mathbb{R}_0}(R)} \xrightarrow{\text{Index}} BK_R.$$ 

By composition with the graded determinant of equation (37), we obtain a morphism

$$B(\det^Z) \circ \text{Index}: \text{Tate}^{cl}_{\mathbb{R}_0}(R)^\times \longrightarrow B \text{Pic}^Z_R.$$ 

The target is a sheaf of groupoids, so this extends to a morphism

(38) \quad $B(\det^Z) \circ \text{Index}: \text{Tate}_{\mathbb{R}_0}(R)^\times \longrightarrow B \text{Pic}^Z_R$.

In particular, we see that $\text{Tate}^{cl}_{\mathbb{R}_0}(R)^\times$ is endowed with a canonical $\text{Pic}^Z_R$-torsor.

**Definition 5.13.** Define the determinant torsor $\mathcal{D}_R \longrightarrow \text{Tate}_{\mathbb{R}_0}(R)^\times$ to be the $\text{Pic}^Z_R$-torsor classified by the map of (38).

**Proposition 5.14.** Let $R$ be a ring, and let $V$ be an elementary Tate $R$-module. The space of sections $\Gamma(\text{Spec}(R), \mathcal{D}_R|_{\{V\}})$ of the restriction of the determinant torsor to $\{V\} \in \text{Tate}_{\mathbb{R}_0}(R)^\times$ is equivalent to the space of sections of $\Delta: Gr(V) \longrightarrow \text{Pic}^Z_R$ equipped with a coherent collection of equivalences

(39) \quad $\Delta(L') \simeq \Delta(L) \otimes \det^Z(L' / L)$

for every nested pair of lattices $L \hookrightarrow L'$. The coherence condition amounts to the commutativity of the diagram

(40) \quad $\Delta(L'') \longrightarrow \Delta(L') \otimes \det^Z(L'' / L')$

$$\Delta(L) \otimes \det^Z(L'' / L) \longrightarrow \Delta(L) \otimes \det^Z(L'/L) \otimes \det^Z(L'' / L'),$$

\footnote{The sign in front of $\phi_{L,M}$ ensures that this map is symmetric monoidal.}
for all \((L \leq L' \leq L'') \in \mathcal{G}_R(V)\). In particular, \(\mathcal{D}_R|_{\{V\}}\) is equivalent to the determinant torsor of \(V\) as described in [Dr106, Section 5.2].

**Proof.** The proof is analogous to Proposition 5.12, we will therefore only sketch the general argument. The composition \(B(\det^2) \circ \text{Index}\) is equivalent to the geometric realization of the map

\[ Gr^\leq_\bullet (P_f(R))^\times \longrightarrow B_\bullet(\text{Pic}^\leq_R), \]

sending \(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V\) to \((\det^2(L_1/L_0), \ldots, \det^2(L_n/L_{n-1}))\). Using the canonical augmentation of \(B_\bullet(\text{Pic}^\leq_R)\), we obtain an augmentation of \(Gr^\leq_\bullet (P_f(R))^\times \longrightarrow B\text{Pic}^\leq_R\), hence a \(\text{Pic}^\leq_R\)-torsor on the simplicial space \(Gr^\leq_\bullet (P_f(R))^\times\). Since \(Gr^\leq_0(P_f(R))^\times\) factors through \(B_0(\text{Pic}^\leq_R) = \{\star\}\), the corresponding \(\text{Pic}^\leq_R\)-torsor is trivialized on the cover \(Gr^\leq_0(P_f(R))^\times \longrightarrow \text{Tate}^{\leq_0}_R(R)^\times\). To determine the space of sections, it suffices to describe the descent conditions for the space of sections of the trivial torsor on \(Gr^\leq_0(P_f(R))^\times\). By inspection, we see that the trivializations corresponding to a nested pair of lattices \(L \leftrightarrow L' \leftrightarrow V\) differ by \(\det^2(L'/L)\), as claimed in (39). The coherence condition (40) follows as well, but we need to show that no further condition has to be imposed. To see this we observe the following: since the space of sections is a groupoid, it is determined by restricting the simplicial object to the 2-skeleton. Hence, no further coherence conditions appear. \(\blacksquare\)

**Remark 5.15.** In the previous section, we observed that the rank of finitely generated projective modules defines a map

\[ B\text{Aut}(V) \longrightarrow B\mathbb{Z} \]

for any countable Tate module \(V\). Applying \(\Omega(-)\), we obtain a group homomorphism

\[ \nu: \text{Aut}(V) \longrightarrow \mathbb{Z}. \]

Similarly, for each countable Tate \(R\)-module \(V\), the map

\[ B\text{Aut}(V) \longrightarrow B\text{Pic}^\leq_R, \]

induces a monoidal map

\[ \text{Aut}(V) \longrightarrow \text{Pic}^\leq_R. \]

These maps are in correspondence with graded central extensions of \(\text{Aut}(V)\). In particular, we see that the map \(\nu\) extends to a graded central extension of \(\text{Aut}(V)\).

5.2.4. The Determinant 2-Gerbe. Let \(V\) be an elementary 2-Tate module. We now explain how one can recover Arkhipov–Kremnizer’s 2-gerbe of gerbal theories (cf. [AK10]) as a truncation of (33).

Let \(R\) be a ring, and let \(V\) be an elementary 2-Tate module. The map

\[ B\text{Aut}(V) \longrightarrow \text{2-Tate}^{\leq_2}(R)^\times \overset{\text{Index}}{\longrightarrow} B\Omega^\infty \Sigma \mathbb{K}_R \overset{B^2\det^2}{\longrightarrow} B^2\text{Pic}^\leq_R \]

classifies a 2-gerbe on \(B\text{Aut}(V)\). By construction, this map arises as the geometric realization of the simplicial map

\[ Gr^\leq_\bullet(V)^\times \longrightarrow S_\bullet \text{Tate}(R)^\times \longrightarrow B_\bullet B\text{Pic}^\leq_R, \]

where the second map sends \((L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V)\) to \((\mathcal{D}_R|_{L_1/L_0}, \ldots, \mathcal{D}_R|_{L_n/L_{n-1}})\). The space of sections of this 2-gerbe is equivalent to the space of maps \(\Delta: Gr^\leq_\bullet(V) \longrightarrow B\text{Pic}^\leq_R\) equipped with a coherent collection of equivalences

\[ \Delta(L') \simeq \Delta(L) \otimes \mathcal{D}_R|_{L'/L} \]

for a nested pair of lattices \(L \leftrightarrow L'\). In particular, when \(R\) is a field, \(\mathcal{G}_R|_{\{V\}}\) is equivalent to the graded determinant 2-gerbe of the 2-Tate vector space \(V\) introduced by Arkhipov and Kremnizer.
Arkhipov and Kremnizer used the determinant 2-gerbe of a 2-Tate vector space to construct a higher central extension of $\text{Aut}(V)$. As remarked in loc. cit., because their formalism only allowed them to consider 2-Tate vector spaces, they were forced to view $\text{Aut}(V)$ as a discrete group. The above construction allows us to define the central extension of the group scheme $\text{Aut}(V)$. Namely, the extension is the algebraic 2-group given by the central term in the fibre sequence (in the category of stacks over $\text{Spec}(R)$).

\[
\text{Pic}_R^Z \longrightarrow \Omega G_R \big|_{B \text{Aut}(V)} \longrightarrow \text{Aut}(V).
\]

We denote this algebraic 2-group below by $\hat{\text{Aut}}(V)$. This generalizes the constructions of [OZ13, Sect. 5.2] to automorphism groups of arbitrary elementary 2-Tate objects.

5.2.5. The Determinant Gerbe on the Grassmannian of a 2-Tate Module. In [FZ12], Frenkel and Zhu describe a conjectural determinant gerbe on the Sato Grassmannian of a 2-Tate $R$-module in connection with a geometric representation theory of double loop groups. We give a rigorous construction of this gerbe in this section, and use it to construct a basic categorical representation of the aforementioned algebraic 2-group $\hat{\text{Aut}}(V)$.

Let $V$ be an elementary 2-Tate module. In Section 5.1.1, we saw how a choice of lattice $L \in \text{Gr}(V)$ gives rise to the square

\[
\begin{array}{ccc}
\text{Aut}(V) & \longrightarrow & \text{Gr}(V) \\
\downarrow & & \downarrow \\
\Omega K_{2\text{-Tate}}(R) & \longrightarrow & K_{\text{Tate}}(R)
\end{array}
\]

Combining this with the index map and the determinant, we obtain a map

\[
\text{Gr}(V) \longrightarrow K_{\text{Tate}}(R) \longrightarrow B \text{Pic}_R^Z.
\]

The construction guarantees that, up to equivalence, this map is independent of the choice of lattice $L$.

**Definition 5.16.** Let $V$ be an elementary 2-Tate $R$-module. The determinant gerbe on the Sato Grassmannian $\text{Gr}(V)$ is the $\text{Pic}_R^Z$-torsor $\mathcal{G}_V \longrightarrow \text{Gr}(V)$ classified by the map.

**Remark 5.17.** In the present approach, the construction of the determinant gerbe follows from the results of [BGW14] on 2-Tate modules and from the construction of the index map.

We can give a more explicit description of the determinant gerbe. Fix a lattice $L \in \text{Gr}(V)$. From the construction, we see that the fibre of the determinant gerbe at a lattice $L'$ is canonically equivalent to

\[
\mathcal{G}_V \big|_{(L')} \simeq D_R^\vee|_{(L'/N)} \otimes_{\text{Pic}_R^Z} D_R|_{(L'/N)}
\]

where $N$ is any common sub-lattice of $L$ and $L'$, $D_R$ is the determinant torsor on $\text{Tate}(R)^\vee$, $D_R^\vee$ is its dual, and the tensor product of $\text{Pic}_R^Z$-torsors is as described in [Beil09, Sect. 2.2]. Equivalently, $\mathcal{G}_V \big|_{(L')} \simeq \text{Hom}_{\text{Pic}_R^Z}(D_R|_{(L'/N)}, D_R|_{(L'/N)})$.

Given a Tate $R$-module $M$, Drinfeld [Drin06, Section 5.4], following Beilinson–Bloch–Esnault [BBE02], describes the determinant torsor $D_R|_{(M)}$ in terms of invertible modules for a Clifford algebra $\text{Cl}_M$. In more detail, the duality pairing gives a symmetric bilinear form on the Tate module $M \oplus M^\vee$. 
Denote by $\text{Cl}_M$ the $\mathbb{Z}$-graded Clifford algebra associated to this form, with the grading given by placing $M$ in degree 1, and $M^\vee$ in degree $-1$. We similarly consider graded modules for $\text{Cl}_M$.

**Definition 5.18.** Let $M$ be a Tate $R$-module, and let $\text{Cl}_M$ be the graded Clifford algebra described above. A graded Clifford module $F$ is a $\mathbb{Z}/2$-graded $R$-module, with a graded action of $\text{Cl}_M$, which is continuous with the respect to the discrete topology on $F$. We say that $F$ is a graded Fermion module if the functor $(-) \otimes_R F$, from $\mathbb{Z}/2$-graded $R$-modules to graded $\text{Cl}_M$-modules, is an equivalence of categories.

Denote by $\text{Ferm}^Z(M)$ the groupoid of graded Fermion modules for $\text{Cl}_M$. The groupoid $\text{Ferm}^Z(M)$ carries a natural action of $\text{Pic}^Z_R$ given by tensoring a graded Fermion module with a graded line.

**Proposition 5.19.** (Drinfeld [Dri06] Section 5.4, following Beilinson–Bloch–Esnault [BBE02]) Let $M$ be a Tate $R$-module. The groupoid $\text{Ferm}^Z(M)$ is a $\text{Pic}^Z_R$-torsor. For a graded Fermion module $F$, and a lattice $L \subset M$, denote the annihilator of $L \oplus L^\vee \subset M \oplus M^\vee$ in $F$ by $\Delta_F(L)$. Then $\Delta_F(L) \subset F$ is a graded line, with the grading inherited from that of $F$. The assignment $L \mapsto \Delta_F(L) \in \text{Pic}^Z_R$ defines a graded determinantal theory, and the assignment $F \mapsto \Delta_F$ induces an equivalence $\text{Ferm}^Z(M) \xrightarrow{\sim} \text{Cl}_R$ of $\text{Pic}^Z_R$-torsors.

Now let $V$ be a $2$-Tate $R$-module, and $L \subset V$ a lattice, as above. The proposition allows us to reformulate the fibre of the determinant gerbe $G_V$ at a lattice $L'$ as

$$G_V|_{(L')} \simeq \text{Hom}_{\text{Pic}^Z_R}(\text{Ferm}^Z_{L/N}, \text{Ferm}^Z_{L'/N}),$$

where $N$ is any common sub-lattice of $L$ and $L'$. From the perspective of categories of modules, $\text{Ferm}^Z_M$ is the groupoid of indecomposables in the category $\text{Mod}^*_{\text{Cl}_M}$ of semi-simple $\text{Cl}_M$-modules. A homomorphism of $\text{Pic}^Z_R$-torsors is necessarily an equivalence, so a homomorphism $\text{Ferm}^Z_{L/N} \longrightarrow \text{Ferm}^Z_{L'/N}$ corresponds to an equivalence of $R$-linear categories

$$\text{Mod}^*_{\text{Cl}_L/N} \xrightarrow{\sim} \text{Mod}^*_{\text{Cl}_{L'/N}}.$$  

According to Morita theory, the groupoid of such equivalences is equivalent to the category of invertible $\text{Cl}_{L/N} - \text{Cl}_{L'/N}$-bimodules.

Let $V$ be a $2$-Tate $R$-module, and let $G_V \longrightarrow \text{Gr}(V)$ be the determinant gerbe on $\text{Gr}(V)$. Fix a lattice $L \subset V$, the fibre $G_V|_{(L')}$ at any lattice $L' \subset V$ is equivalent to the $\text{Pic}^Z_R$-torsor of invertible $\text{Cl}_{L/N} - \text{Cl}_{L'/N}$-bimodules, where $N$ is any common sub-lattice of $L$ and $L'$.

As described in Section 5.2.3 the homomorphism $\text{Aut}(V) \longrightarrow B \text{Pic}^Z_R$ gives rise to a central extension $\widehat{\text{Aut}(V)}$ of the group scheme $\text{Aut}(V)$ by the stack $\text{Pic}^Z_R$. The constructions guarantee that $\widehat{\text{Aut}(V)}$ acts on the determinant gerbe $G_V$ and that this action lifts the action of $\text{Aut}(V)$ on $\text{Gr}(V)$.

Denote by $P^Z_f(R)$ the category of finitely generated $\mathbb{Z}$-graded projective $R$-modules. The category $P^Z_f(R)$ carries a natural action of $\text{Pic}^Z_R$ and the homotopy quotient

$$P^Z_f(R) // \text{Pic}^Z_R \longrightarrow B \text{Pic}^Z_R$$

defines a bundle of exact categories. The pullback of this bundle along the map $\text{Gr}(V) \longrightarrow B \text{Pic}^Z_R$ defines a bundle of exact categories $\mathcal{P}_V \longrightarrow \text{Gr}(V)$. As with $G_V$, the construction ensures that $\widehat{\text{Aut}(V)}$ acts on $\mathcal{P}_V$ and that this action lifts the action of $\text{Aut}(V)$ on $\text{Gr}(V)$. In particular, we obtain an $R$-linear representation of $\text{Aut}(V)$ on the category of global sections of $\mathcal{P}_V \longrightarrow \text{Gr}(V)$.
Definition 5.20. Let \( V, \hat{\text{Aut}}(V) \) and \( \mathcal{P}_V \) be as above. The basic representation of \( \hat{\text{Aut}}(V) \) is given by the action \( \hat{\text{Aut}}(V) \circ \Gamma(\text{Gr}(V), \mathcal{P}_V) \).

This picture suggests a more concrete description of the fibres of \( \mathcal{P}_V \xrightarrow{\sim} \text{Gr}(V) \). Let \( V \) be a 2-Tate \( R \)-module, and let \( \mathcal{P}_V \xrightarrow{\sim} \text{Gr}(V) \) be the bundle of exact categories associated to \( \mathcal{G}_V \) as above. Given a lattice \( L \subset V \), it seems plausible that the fibre of \( \mathcal{P}_V \big|_{\{L\}} \) at any lattice \( L' \subset V \) is equivalent to the category of semi-simple \( \text{Cl}_{L/N} - \text{Cl}_{L'/N} \)-bimodules.

Frenkel and Zhu [FZ12] have previously constructed a basic representation of the \( \mathbb{C} \)-points of \( \text{Aut}(V) \), when \( V \) is the 2-Tate space \( \mathbb{C}((s))(t) \) over \( \mathbb{C} \) (they denote the \( \mathbb{C} \)-points of \( \text{Aut}(V) \) by \( \mathbb{GL}(V) \)). In their formulation, the basic representation consists of an action of \( \text{Aut}(V) \) on a category of semi-simple modules for a Clifford algebra associated to a lattice of \( V \). We expect the following: let \( V \) be an elementary 2-Tate space over \( \mathbb{C} \). The category of \( \mathbb{C} \)-points of the representation \( \Gamma(\text{Gr}(V), \mathcal{P}_V) \) of the algebraic 2-group \( \hat{\text{Aut}}(V) \) is equivalent to the basic representation of the \( \mathbb{C} \)-points of \( \hat{\text{Aut}}(V) \) constructed in [FZ12].

5.3. Links to More Classical Formulations. In this brief section we relate the works of Arbarello–de Concini–Kac [ADCK89] and Brylinski [Bry97 §2] to the present formulation.

We will follow the presentation and notation of Brylinski. Let \( k \) be a field and \( E \) a \( k \)-vector space (of infinite dimension to be interesting). Two sub-spaces \( F_1, F_2 \subset E \) are called commensurable if \( F_1 \cap F_2 \) is of finite codimension in both \( F_1 \) and \( F_2 \). This is easily seen to be an equivalence relation. Brylinski now fixes a commensurability equivalence class \( S \). He next defines a category \( C \) with objects \( \text{ob}C := S \), i.e. all representatives of the commensurability class. The morphisms are defined as \( \text{hom}_C(F_1, F_2) := (F_1 \mid F_2) \setminus \{0\} \), where \( (F_1 \mid F_2) \) is a vector space, constructed manually and pinned down uniquely by the following properties:

Proposition 5.21 (Arbarello–de Concini–Kac, [ADCK89 §4]). There exists a unique way to assign a one-dimensional \( k \)-vector space \( (F_1 \mid F_2) \) to a pair of objects \( F_1, F_2 \in C \), along with isomorphisms

\[
\omega(F_1 \mid F_2 \mid F_3) := (F_1 \mid F_2) \otimes_k (F_2 \mid F_3) \xrightarrow{\sim} (F_1 \mid F_3),
\]

so that the following normalizations are satisfied:

1. For \( F_1 \subset F_2 \) we have \( (F_1 \mid F_2) = \bigwedge^{\text{top}}(F_2/F_1) \).
2. For \( F_1 \subset F_2 \subset F_3 \) in \( S \) the isomorphism \( \omega(F_1 \mid F_2 \mid F_3) \) is the canonical one, i.e. the one coming from

\[
\bigwedge^{\text{top}}(F_2/F_1) \otimes \bigwedge^{\text{top}}(F_3/F_2) \rightarrow \bigwedge^{\text{top}}(F_3/F_1)
\]

\[v_1 \wedge \cdots \wedge v_n \otimes w_1 \wedge \cdots \wedge w_m \mapsto \tilde{v}_1 \wedge \cdots \wedge \tilde{w}_m,
\]

where \( \tilde{\cdot} \) refers to arbitrarily chosen lifts.
3. Given \( F_1 \subset F_2 \subset F_3 \subset F_4 \) in \( S \) the diagram of isomorphisms

\[
(F_1 \mid F_2) \otimes (F_2 \mid F_3) \otimes (F_3 \mid F_4) \rightarrow (F_1 \mid F_3) \otimes (F_3 \mid F_4)
\]

\[
(F_1 \mid F_2) \otimes (F_2 \mid F_4) \rightarrow (F_1 \mid F_4)
\]

commutes.

Brylinski defines \( \text{GL}(E, S) \) as the subgroup of \( \text{GL}(E) \) which preserves the fixed commensurability class \( S \). Next, he employs Verdier’s notion of a group action on a category [Bry97 Definition 1.1] and shows that \( \text{GL}(E, S) \) acts on \( C \) in this sense. He shows that such an action gives rise to a
certain group extension [Bry97] Prop. 1.2. Brylinski now applies these constructions to \( E := k((t)) \) and picks for \( S \) the commensurability class of \( k[[t]] \). As a result, he obtains a group extension

\[
1 \to k^\times \to \widetilde{GL(E, S)} \to GL(E, S) \to 1.
\]

He shows that for \( f, g \in k((t))^\times \subset GL(E, S) \) the commutator of lifts \( \tilde{f}, \tilde{g} \in \widetilde{GL(E, S)} \) evaluates to be \((f^{\nu(0)}/g^{\nu(f)})(0) \in k^\times \), where \( \nu(\cdot) \) denotes the \( t \)-valuation of \( k((t)) \), see [Bry97] Prop. 2.4. Note that this agrees with the formula of the tame symbol, up to sign. Let us explain the relation to the theory of the present paper:

Of course \( E := k((t)) \) is naturally a 1-Tate vector space, and equivalently a locally linearly compact \( k \)-vector space, and \( k[[t]] \) is a lattice in it. Note that by [BGW14] Prop. 6.6 for any two lattices \( F_1 \subset F_2 \) the quotient \( F_2/F_1 \) is finite-dimensional, so all lattices are commensurable in the classical sense. Since every automorphism \( \varphi \in \text{Aut}_{\text{Tate}}(E) \) sends lattices to lattices, it follows that it sends \( k[[t]] \) to a commensurable sub-space. Therefore, we get a canonical inclusion

\[
\text{Aut}_{\text{Tate}}(E) \subseteq GL(E, S).
\]

Next, as always in this paper, we identify Brylinski’s category \( C \) with its nerve. By construction there are morphisms between any two objects, therefore \( \pi_0 C = 0 \). Moreover, for any object its automorphism group is canonically isomorphic to

\[
\text{Hom}_C(F, F) = (F \mid F) \setminus \{0\} = (F/F)^{\wedge 0} \setminus \{0\} = k^\times,
\]

since the top exterior power of the zero space canonically identifies with \( k \). Hence, \( \pi_1 C = k^\times \), regardless of the base point. In fact, we may identify \( C \), viewed as a groupoid, with the fundamental groupoid of its nerve. It is indeed a homotopy 1-type. Therefore, \( C \) is actually nothing but a concrete model for the Eilenberg-MacLane space \( Bk^\times \). Brylinski’s action of \( GL(E, S) \) on the category \( C \) is nothing but a group action on the nerve, so taking the (homotopy) quotient modulo the group action, we get the fibre sequence

\[
GL(E, S) \to C \to C/\!\!/GL(E, S).
\]

However, as we take the homotopy quotient (as opposed to the naive quotient), this construction is invariant under switching to equivalent homotopy types. Hence, we may just as well speak of

\[
GL(E, S) \to Bk^\times \to Bk^\times/\!\!/GL(E, S) \xrightarrow{(*)} BGL(E, S),
\]

where \((*)\) denotes the classifying map of this \( GL(E, S) \)-principal bundle over \( C \), or equivalently is just the standard continuation of a fibre sequence to the right. The last three terms form a fibre sequence itself, showing that \( Bk^\times/\!\!/GL(E, S) \) is itself a connective homotopy 1-type. It is determined by its fundamental group, \( GL(E, S) := \pi_1(Bk^\times/\!\!/GL(E, S)) \),

\[
Bk^\times \to B\widetilde{GL(E, S)} \to BGL(E, S).
\]

We remind the reader that such a fibre sequence is equivalent to the statement that \( \{13\} \) is a group extension. The classifying map of this fibre bundle is \( BGL(E, S) \to B(Bk^\times) \), or equivalently a cohomology class \( H^2(GL(E, S), k^\times) \), which one may pull back along \( \{14\} \). Pulling it back even further along \( k((t))^\times \subset GL(E, S) \) one can unwind that the cocycle description of this extension reduces to the commutator description of [Bry97] Prop. 2.4 or [ADCK89] 5. We remark that this also explains why Brylinski’s construction gives the tame symbol, but not with the correct sign.
Namely, the homotopy type $Bk^\times$ can be represented explicitly as the nerve of the ordinary Picard category

$$\text{Pic}_k := \{\text{one-dimensional } k\text{-vector spaces } | \text{isos}\}.$$ 

Again this category is a connected groupoid. However, as was observed by P. Deligne in 1987 [Del87, §4], the straightforward construction of the ungraded determinant line of Proposition 5.21 displays certain inconvenient artifacts. Notably, (42) can be rephrased relating the top exterior power of the middle term of a short exact sequence to its outer terms, but for

$$0 \to A \to A \oplus B \to B \to 0 \quad 0 \to B \to A \oplus B \to A \to 0$$

one immediately sees that both the left- and right-hand side object of the isomorphism $\bigwedge^{\text{top}} A \otimes \bigwedge^{\text{top}} B \xrightarrow{\sim} \bigwedge^{\text{top}} (A \oplus B)$ have canonical isomorphisms to their counterpart with $A, B$ interchanged. However, these are not compatible with the isomorphism (for example $a \otimes b \mapsto a \wedge b$ if $A, B$ are one-dimensional vector spaces, so if we exchange $A$ and $B$, we get an additional sign $-1$. But if $A$ is two-dimensional, $a_1 \wedge a_2 \otimes b \mapsto a_1 \wedge a_2 \wedge b$, swapping them preserves the sign). He found that this can elegantly be resolved by “remembering” the dimension of $A, B$ and replacing the tensor product $\otimes$ by a graded-commutative variant. In the language of categories this is Deligne’s Picard groupoid of graded lines, and in terms of the nerve it just means that we get $\mathbb{Z}$ connected components, one for each grading (dimension). Deligne also understood that this is nothing but the 1-truncation of algebraic $K$-theory $\text{Pic}_k^\mathbb{Z} \simeq \tau_{\leq 1} K_k$. In fact, Deligne went further and already gave a description in concrete terms of $\tau_{\leq 1} K_C$ for an arbitrary exact category; this is Deligne’s category of virtual objects [Del87, §4]. There is a functor of groupoids, forgetting the grading, and inducing a morphism of nerves

$$\text{Pic}_k^\mathbb{Z} \longrightarrow \text{Pic}_k \quad (= Bk^\times).$$

However, this functor does not preserve the tensor product, so that it is not a functor of Picard categories. It does not preserve the $E_1$-structure, so although we may apply $B$ to either side, this functor will not induce a morphism between the two spaces; the relation between the two is more subtle than that. F. Muro and A. Tonks have since extended Deligne’s category of virtual objects to Waldhausen categories [MT07]; there is also a construction due to M. Breuning directly for triangulated categories [Bre11]. All these constructions are inspired from earlier work of F. Knudson and D. Mumford [KM76], [Knu02] on the determinant in cohomology, in turn addressing an issue raised by a famous letter of A. Grothendieck (written in 1973).

### Appendix A. Relative Tate Objects and the Relative Index Map

As an appendix, we introduce relative versions of Ind, Pro, and Tate objects. This will allow us to give an index-theoretic description of boundary maps in algebraic $K$-theory.

#### A.1. Relative Tate Objects.

We begin with two lemmas on ordinary admissible Ind-objects.

**Lemma A.1.** Let $C$ and $D$ be exact categories, and let $C \hookrightarrow D$ be an exact, fully faithful embedding. Then $C \simeq \text{Ind}^a(C) \cap D \subset \text{Ind}^a(D)$.

**Proof.** Let $X \in \text{Ind}^a(C)$ be the colimit of an admissible Ind-diagram $X : I \longrightarrow C$. Let $Y \in D$ such that there exists $Y \xrightarrow{\sim} X$ in \text{Ind}^a(D). By the definition of morphisms in \text{Ind}^a(D), there exists $i \in I$,
such that we have a factorization

\[
\begin{array}{ccc}
Y & \sim & X \\
\downarrow & & \downarrow \\
X_i & \rightarrow & X
\end{array}
\]

The diagonal arrow is an admissible monic in \(\text{Ind}^a(C)\) by construction ([BGW14 Lemma 3.11]); and the commutativity of the above diagram implies that it is also an (not necessarily admissible) epic. It is therefore an isomorphism. \(\square\)

The next lemma is a slight generalization of [BGW14 Proposition 5.8(1)]; mutatis mutandi, the proof is the same.

**Lemma A.2.** Let \(D\) be an exact category, and let \(C \subset D\) be a right s-filtering sub-category. Then for any short exact sequence in \(\text{Ind}^a(D)\)

\[
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0
\]

we have that \(Y \in \text{Ind}^a(C)\) if and only if \(X\) and \(Z\) are.

We now define relative \(\text{Ind}, \text{Pro}\) and \(\text{Tate}\) objects.

**Definition A.3.** Let \(D\) be an exact category, and let \(C \subset D\) be an extension-closed sub-category.

(a) Define the category of relative admissible \(\text{Ind}\)-objects \(\text{Ind}_\kappa^a(D, C)\) to be the full sub-category of \(\text{Ind}^a(D)\) consisting of objects that admit a presentation by an admissible \(\text{Ind}\) diagram \(X: I \rightarrow D\) (cf. [BGW14 Def. 3.2]) such that for all \(i < j\) in \(I\), we have \(X_j/X_i \in C\).

(b) Define the category of relative admissible \(\text{Pro}\)-objects \(\text{Pro}_\kappa^a(D, C)\) by

\[
\text{Pro}_\kappa^a(D, C) := (\text{Ind}_\kappa^a(D^{\text{op}}, C^{\text{op}}))^{\text{op}}.
\]

(c) Define the category of relative elementary \(\text{Tate}\) objects \(\text{Tate}_\kappa^{el}(D, C)\) to be the category \(\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(D), C)\).

(d) For \(C\) and \(D\) idempotent complete, define the category of relative Tate objects \(\text{Tate}_\kappa(D, C)\) to be the category \(\text{Tate}_\kappa^{el}(D, C)^{\text{ic}}\).

**Remark A.4.** In the language of Definition A.3, the category of elementary Tate objects in \(C\) can be written as

\[
\text{Tate}_\kappa^{el}(C) = \text{Ind}_\kappa^a(\text{Pro}_\kappa^a(C), C).
\]

**Lemma A.5.** For any cardinal \(\kappa\), \(\text{Ind}_\kappa^a(D, C)\) is closed under extensions in \(\text{Ind}_\kappa^a(D)\). Similarly, \(\text{Pro}_\kappa^a(D, C)\) is closed under extensions in \(\text{Pro}_\kappa^a(D)\) and \(\text{Tate}_\kappa^{el}(D, C)\) is closed under extensions in \(\text{Ind}_\kappa^a(\text{Pro}_\kappa^a(D))\).

**Corollary A.6.** The categories \(\text{Ind}_\kappa^a(D, C), \text{Pro}_\kappa^a(D, C)\) and \(\text{Tate}_\kappa^{el}(D, C)\) are exact categories.

**Proof of Lemma A.5.** The statements about relative Pro and relative elementary Tate objects are special cases of the statement about relative Ind-objects. In all cases, the lemma follows from the straightening construction for exact sequences [BGW14 Prop. 3.12] and the fact that \(C\) is closed under extensions in \(D\).

In more detail, consider an exact sequence in \(\text{Ind}_\kappa^a(D)\)

\[
0 \rightarrow \hat{X} \rightarrow \hat{Y} \rightarrow \hat{Z} \rightarrow 0
\]
with \( \hat{X} \) and \( \hat{Z} \) in \( \text{Ind}^a(D, C) \). Let

\[
X : J \to D, \quad \text{and} \\
Z : I \to D
\]

be admissible relative Ind-diagrams. The straightening construction for exact sequences (Proposition 3.12 of [BGW14]) shows that there exists a directed partially ordered set \( K \) with final maps \( K \to J \) and \( K \to I \) such that the exact sequence above is isomorphic to the colimit of an admissible Ind-diagram of exact sequences

\[
\xymatrix{ K & K & K \\
& K \\
& I \ar[u] \ar[u] }
\]

For any map \( i \leq j \) in \( K \), the 3 \times 3-Lemma [Buh10, Cor. 3.6] shows that we have a diagram with exact rows and columns

\[
\xymatrix{ X_j & Y_j & Z_j \\
X_i & Y_i & Z_i \ar[u] \\
X_i/X_i & Y_i/Y_i & Z_i/Z_i \ar[u] }
\]

Because \( X : J \to D \) is an admissible relative Ind-diagram, so is \( K \to J \xrightarrow{X} D \), and similarly for \( K \to I \xrightarrow{Z} D \). In particular, \( X_j/X_i \) and \( Z_j/Z_i \) are both objects in \( C \). Because \( C \) is closed under extensions in \( D \), we conclude that \( Y_j/Y_i \) is also in \( C \), and thus that \( Y : K \to D \) is an admissible relative Ind-diagram.

Henceforth, we abide by Remark 2.11 and therefore omit the cardinality bound \( \kappa \) from our notation.

A.1.1. **Examples.** Let \( X \) be a Noetherian scheme, and let \( Z \subset X \) be a closed sub-scheme. Denote by \( j : X \setminus Z \to X \) the inclusion of the complement of \( Z \). Denote by \( \text{Coh}_Z(X) \) the full sub-category of \( \text{Coh}(X) \) consisting of coherent sheaves with set-theoretic support in \( Z \). Denote by \( \text{QCoh}(X, \text{Coh}_Z(X)) \) the full sub-category of \( \text{QCoh}(X) \) consisting of quasi-coherent sheaves whose restriction to \( X \setminus Z \) is coherent.

**Proposition A.7.** There exists an exact equivalence

\[
\text{QCoh}(X, \text{Coh}(X \setminus Z)) \xrightarrow{\sim} \text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X)),
\]

and this equivalence fits into a commuting square

\[
\xymatrix{ \text{QCoh}(X, \text{Coh}(X \setminus Z)) & \text{QCoh}(X) \\
\text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X)) \ar[u] \ar[u] & \text{Ind}^a(\text{Coh}(X)) \ar[u] \ar[u] }
\]
Proof. Recall that, because $X$ is Noetherian, there is an exact equivalence $\text{QCoh}(X) \simeq \text{Ind}^a(\text{Coh}(X))$ which sends a quasi-coherent sheaf $F$ to the Ind-object represented by the admissible Ind-diagram of coherent subsheaves of $F$.

Because $\text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X))$ is a fully exact sub-category of $\text{Ind}^a(\text{Coh}(X))$, it suffices to show that a quasi-coherent sheaf is in $\text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X))$ if and only if its pullback to $X \setminus Z$ is coherent. The “only if” is clear.

Let $F$ be represented by an Ind-diagram $F: I \to \text{Coh}(X)$. Suppose the pullback $j^*F$ is coherent. Then there exists a final sub-diagram $J \subset I$ such that the diagram

$$
\begin{array}{ccc}
J & \hookrightarrow & I \\
\downarrow & & \downarrow \\
\text{Coh}(X) & \xrightarrow{j^*} & \text{Coh}(X \setminus Z)
\end{array}
$$

is isomorphic to a constant diagram. In particular, for all $j < k$ in $J$, the cokernel $F_k/F_j$ has set-theoretic support in $Z$. We conclude that $\text{QCoh}(X, \text{Coh}(X \setminus Z)) \subset \text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X))$. □

Corollary A.8. There exists a 2-commuting diagram

$$
\begin{array}{ccc}
\text{Coh}(X \setminus Z) & \xrightarrow{j_*} & \text{QCoh}(X) \\
\downarrow & & \downarrow \\
\text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X)) & \xrightarrow{\simeq} & \text{Ind}^a(\text{Coh}(X))
\end{array}
$$

Proof. Because $X \setminus Z$ is open in $X$, the push-forward $j_*$ gives an exact functor

$$
j_*: \text{Coh}(X \setminus Z) \to \text{QCoh}(X).
$$

Because the co-unit of the adjunction $j_* \dashv j^*$ is an isomorphism, we see that $j_*$ factors through $\text{QCoh}(X, \text{Coh}(X \setminus Z))$. □

Proposition A.9. There exists an exact functor

$$
\text{C}_Z: \text{Coh}(X) \to \text{Pro}^a(\text{Coh}_Z(X)).
$$

Proof. For all $r \geq 1$, let $\iota_r: Z^r \to X$ denote the inclusion of the $r$th-order formal neighborhood of $Z$ in $X$. For $F \in \text{Coh}(X)$, define

$$
\text{C}_Z(F) := \lim_r \iota_r, *, t^*_r F
$$

By inspection, the transition maps $\iota_r, *, t^*_r F \to \iota_{r-1}, *, t^*_{r-1} F$ are admissible epics. Therefore, the assignment $F \mapsto \text{C}_Z(F)$ defines a functor

$$
\text{C}_Z: \text{Coh}(X) \to \text{Pro}^a(\text{Coh}_Z(X)).
$$

By the Artin–Rees Lemma (e.g. [AM69b, Proposition 10.12]), this functor is exact. □

Corollary A.10. There exists a 2-commuting diagram of exact maps

$$
\begin{array}{ccc}
\text{Coh}_Z(X) & \xrightarrow{1} & \text{Coh}(X) \xrightarrow{\text{C}_Z} \text{QCoh}(X, \text{Coh}(X \setminus Z)) \\
\downarrow & & \downarrow \\
\text{Coh}_Z(X) & \xrightarrow{\text{T}_Z} \text{Pro}^a(\text{Coh}_Z(X)) & \xrightarrow{\text{Tate}^e(\text{Coh}_Z(X)).}
\end{array}
$$

Remark A.11. The “T” in $\text{T}_Z$ in the diagram above stands for “Toeplitz”, cf. Example [3.21].
Proof. The definition of $\mathbf{C}_Z$ ensures that if $F \in \text{Coh}_Z(X)$, then the Pro-object $\mathbf{C}_Z(F)$ is represented by the constant Pro-diagram on $F$. This accounts for the left square. For the right square, we observe that $\mathbf{C}_Z$ gives an exact functor of pairs

$$\text{Coh}(X, \text{Coh}_Z(X)) \to \text{Pro}^a(\text{Coh}_Z(X), \text{Coh}_Z(X)).$$

The exact functor $T_Z$ is the corresponding map

$$\text{QCoh}(X, \text{Coh}(X \setminus Z)) \to \text{Ind}^a(\text{Coh}(X), \text{Coh}_Z(X)) \to \text{Ind}^a(\text{Pro}^a(\text{Coh}_Z(X), \text{Coh}_Z(X))) =: \text{Tate}^{el}(\text{Coh}_Z(X)).$$

This concludes the construction and the proof. □

Now let $W \subset Z$ be a closed sub-scheme.

Corollary A.12. There exists a 2-commuting diagram of exact maps

$$\text{Coh}_W(X) \to \text{Coh}(X) \to \text{Qcoh}(X, \text{Coh}(X \setminus W)).$$

Proof. The proof is the same as for the previous corollary, once we observe that, for $W \subset Z$ and $F \in \text{Coh}_W(X)$, we have that $\mathbf{C}_Z(F)$ is represented by the constant Pro-diagram on $F$. □

A.1.2. Properties. The next lemma is a slight generalization of [BGW14, Proposition 3.14]; mutatis mutandii the proof is the same.

Lemma A.13. For $k \geq 0$, there exist canonical equivalences

$$\text{Ind}^a(S_k \text{D}, S_k \text{C}) \to S_k(\text{Ind}^a(\text{D}, \text{C})),
\text{Pro}^a(S_k \text{D}, S_k \text{C}) \to S_k(\text{Pro}^a(\text{D}, \text{C})),
\text{Tate}^{el}(S_k \text{D}, S_k \text{C}) \to S_k \text{Tate}^{el}(\text{D}, \text{C}).$$

Definition A.14.

1. For $V \in \text{Tate}^{el}(\text{D}, \text{C})$ we say that an admissible monic $L \to V$ is a relative lattice, if $L \in \text{Pro}^a(\text{D})$, and $V/L \in \text{Ind}^a(\text{C})$.
2. For two relative lattices $L, L'$, we say that $L \leq L'$, if the inclusion $L \to V$ factors through $L' \to V$ via an admissible monic $L \to L'$.
3. Define the relative Sato Grassmannian $\text{Gr}_\text{C}(V)$ to be the partially ordered set of relative lattices of $V$.

Proposition A.15. If $\text{D}$ is idempotent complete, then the relative Sato Grassmannian $\text{Gr}_\text{C}(V)$ is a directed partially ordered set.

---

18Because $\text{Pro}^a(\text{D})$ is left s-filtering in $\text{Tate}^{el}(\text{D})$, the inclusion $\text{Pro}^a(\text{D}) \subset \text{Tate}^{el}(\text{D}, \text{C})$ reflects admissible monics and epics (cf. [BGW14, Appendix A]).
Proof. Abuse notation and let $V: I \to \text{Pro}^a(D)$ be an admissible relative Tate diagram representing $V$. Note that, by definition, for all $i \in I$, $V_i \in \text{Gr}_C(V)$. Now let $L_1, L_2 \in \text{Gr}_C(V)$. Because $\text{Pro}^a(D)$ is left filtering in $\text{Tate}^{el}(D, C)$, there exists $i \in I$ such that we have a commuting triangle

$$
\begin{array}{ccc}
L_1 \oplus L_2 & \to & V_i \\
\downarrow & & \downarrow \\
V & \to & V
\end{array}
$$

in $\text{Tate}^{el}(D, C)$. If $D$ is idempotent complete, then Lemma 6.9 of [BGW14] shows that for $a = 1, 2$, the map $L_a \to V_i$ is an admissible monic in $\text{Pro}^a(D)$.

**Lemma A.16.** Let $C \subset D$ be a right $s$-filtering sub-category. Let $V \in \text{Tate}^{el}(D, C)$ and let $L_1 \leq L_2 \in \text{Gr}_C(V)$. Then $L_2/L_1 \in C$.

**Proof.** By [BGW14], Proposition 6.6, we know that $L_2/L_1 \in D$. Lemma A.13 and Noether’s Lemma show that $L_2/L_1 \in \text{Ind}^a(C)$. By Lemma A.1 we have $L_2/L_1 \in C$. □

We record the following observation about the role of the right filtering assumption.

**Lemma A.17.** Let $D$ be idempotent complete, and let $C \subset D$ be a left or right filtering sub-category. Then $C$ is idempotent complete.

**Proof.** By duality, it suffices to prove the statement for right filtering. Let $X \to X$ be an idempotent in $C$. Because $D$ is idempotent complete, there exists $Y \in D$ such that $Y = \ker(p)$. Because $C$ is right filtering, the monic $Y \to X$ factors through an admissible epic $Y \to W$ with $W \in C$. The map $Y \to W$ is therefore a monic, admissible epic, and thus an isomorphism. We conclude that $Y \in C$ and that $C$ is idempotent complete. □

**Proposition A.18.** Let $C \subset D$ be a sub-category which is closed under extensions. The inclusions $D \hookrightarrow \text{Ind}^a(D, C)$, and $\text{Pro}^a(D, C) \hookrightarrow \text{Tate}^{el}(D, C)$ are left $s$-filtering. The inclusions $\text{Ind}^a(D, C) \hookrightarrow \text{Ind}^a(D, C)$ and $\text{Ind}^a(D, C) \hookrightarrow \text{Tate}^{el}(D, C)$ induce exact equivalences

$$
\text{Ind}^a(D, C)/C \xrightarrow{\sim} \text{Ind}^a(D, C)/D \xrightarrow{\sim} \text{Tate}^{el}(D, C)/\text{Pro}^a(D).
$$

**Proof.** We first show the inclusions are left $s$-filtering. The second inclusion is a special case of the first. For the first, we observe that $D$ is left special in $\text{Ind}^a(D, C)$ because $D$ is left special in $\text{Ind}^a(D)$ [BGW14, Lemma 2.18]. Further, $D$ is left filtering in $\text{Ind}^a(D, C)$ for the same reason it is in $\text{Ind}^a(D)$, namely given any admissible relative $\text{Ind}$-diagram $Y: I \to D$, and given any $X \to \tilde{Y}$

there exists $i \in I$ such that $X$ factors through the admissible monic $Y_i \to \tilde{Y}$.

We now establish the equivalences. The inverse equivalence $\text{Ind}^a(D, C)/D \to \text{Ind}^a(D, C)/C$ is defined as follows. Let $\text{Dir}^a(D, C) \subset \text{Dir}^a(D)$ be the full sub-category of relative admissible $\text{Ind}$-diagrams indexed by directed partially ordered sets with an initial object. A slight modification of the proof of [BGW14, Proposition 5.11] shows that $\text{Ind}^a(D, C)$ is equivalent to the localization $\text{Dir}^a(D, C)[W^{-1}]$, where $W$ is the sub-category of final maps as in [BGW14, Proposition 3.15]. By inspection, the assignment

$$
(I \xrightarrow{X} D) \mapsto (I \xrightarrow{X/X_a} C)
$$
induces a functor \( \text{Dir}^a(D, C) \longrightarrow \text{Ind}^a(C) \), such that the induced functor \( \text{Dir}^a(D, C) \longrightarrow \text{Ind}^a(C)/C \) factors through the localization \( \text{Dir}^a(D, C) \longrightarrow \text{Ind}^a(D, C) \). By inspection, this functor is the desired inverse for the first functor of (45). To see that it is exact, apply the straightening construction for exact sequences (cf. [BGW14, Proposition 3.12]).

Mutatis mutandis, the proof of Proposition 5.28 of [BGW14] defines an exact functor

\[
\text{Tate}^a(D, C) \longrightarrow \text{Ind}^a(C)/C
\]

which factors through the localization \( \text{Tate}^a(D, C)/\text{Pro}^a(D) \). By inspection, this functor is inverse to the canonical map \( \text{Ind}^a(C)/C \longrightarrow \text{Tate}^a(D, C)/\text{Pro}^a(D) \).

The previous lemma now combines with the 2 of 3 property for equivalences to imply that the second functor in (45) is an equivalence as well. □

**Proposition A.19.** Let \( D \) be idempotent complete and let \( C \subset D \) be a right s-filtering sub-category. Then \( C \subset \text{Pro}^a(D) \) is right s-filtering, and the inclusion \( \text{Pro}^a(D) \subset \text{Tate}^a(D, C) \) induces an exact equivalence

\[
\text{Pro}^a(D)/C \xrightarrow{\cong} \text{Tate}^a(D, C)/\text{Ind}^a(C).
\]

*Proof.* The definition of right s-filtering implies that a composition of right s-filtering embeddings is again right s-filtering. Therefore, our assumption on \( C \) together with the fact that \( D \hookrightarrow \text{Pro}^a(D) \) is right s-filtering (cf. [BGW14, Theorem 4.2(2)]) implies that \( C \subset \text{Pro}^a(D) \) is right s-filtering.

The quotient \( \text{Pro}^a(D)/C \) is defined to be the localization of \( \text{Pro}^a(D) \) at the class of admissible monics with cokernel in \( C \), and we define a sequence in \( \text{Pro}^a(D)/C \) to be exact if and only if it is isomorphic to the image of an exact sequence in \( \text{Pro}^a(D) \). Similarly, define \( \text{Tate}^a(D, C)/\text{Ind}^a(C) \) to be the localization of \( \text{Tate}^a(D, C) \) at the class of admissible monics with cokernel in \( \text{Ind}^a(C) \), and define a sequence to be exact if and only if it is isomorphic to the image of an exact sequence in \( \text{Tate}^a(D, C) \). Using Lemma A.16 the same argument as for [BGW14 Proposition 5.29] shows that the assignment

\[
V \mapsto L
\]

(sending a relative Tate object to a relative lattice) extends to an exact functor

\[
\text{Tate}^a(D, C) \longrightarrow \text{Pro}^a(D)/C.
\]

To see that this factors through \( \text{Tate}^a(D, C)/\text{Ind}^a(C) \), let

\[
V_0 \hookrightarrow V_1 \twoheadrightarrow Z
\]

be a short exact sequence of relative Tate objects with \( Z \in \text{Ind}^a(C) \). By the universal property of localizations, it suffices to show that (46) sends the map \( V_0 \hookrightarrow V_1 \) to an isomorphism in \( \text{Pro}^a(D)/C \).

To check this, we let \( L_0 \hookrightarrow V_0 \) be a relative lattice. By the definition of morphisms in \( \text{Tate}^a(D, C) \), the inclusion

\[
L_0 \hookrightarrow V_1
\]

factors through a relative lattice \( L_1 \hookrightarrow V_1 \). Therefore, the functor (46) sends the map \( V_0 \hookrightarrow V_1 \) to \( L_0 \longrightarrow L_1 \). We claim that this map is an isomorphism in \( \text{Pro}^a(D)/C \), i.e. that it is an admissible monic in \( \text{Pro}^a(D)/C \) with cokernel in \( C \).

By Lemma A.16 it suffices to show that the admissible monic \( L_0 \hookrightarrow V_1 \) is a relative lattice. This follows from Noether’s lemma and Lemma A.2. Indeed, we have a short exact sequence in \( \text{Tate}^a(D, C) \)

\[
V_0/L_0 \hookrightarrow V_1/L_0 \twoheadrightarrow V_1/V_0.
\]

By assumption \( V_0/L_0 \) and \( V_1/V_0 \) are both in \( \text{Ind}^a(C) \). Therefore \( V_1/V_0 \) is as well.
We have shown that (46) induces an exact functor
\[ \text{Tate}^{el}(D, C) / \text{Ind}^a(C) \longrightarrow \text{Pro}^a(D)/C. \]
From the definitions, we see that this is an inverse to the map \( \text{Pro}^a(D)/C \longrightarrow \text{Tate}^{el}(D, C)/\text{Ind}^a(C) \).

\[\text{A.2. The Relative Index Map.}\]

We now give a variant of the index map for relative Tate objects and relate it to boundary maps in algebraic K-theory.

**Definition A.20.** Let \( C \subset D \) be an extension-closed sub-category.

1. For \( n \geq 0 \), define \( \text{Gr}_n \subset (D, C) \) to be the full sub-category of \( \text{Fun}([n+1], \text{Tate}^{el}(D, C)) \) consisting of sequences of admissible monics
   \[ L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V \]
   where, for all \( i \), \( L_i \hookrightarrow V \) is the inclusion of a relative lattice (cf. Definition [A.14] \[15\]).

2. Define the relative Sato complex \( \text{Gr}_n \subset (D, C) \) to be the simplicial diagram of exact categories with \( n \)-simplices \( \text{Gr}_n \subset (D, C) \), with face maps \( d_i \) given by omitting the \( i^{th} \) relative lattice, and with degeneracy maps \( s_i \) given by repeating it.

Lemma [A.16] now allows for the following definition.

**Definition A.21.** Let \( D \) be idempotent complete and let \( C \subset D \) be a right s-filtering sub-category. The categorical relative index map is the span of simplicial maps

\[ \text{Tate}^{el}(D, C) \leftarrow \text{Gr}_\bullet \subset (D, C) \xrightarrow{\text{Index}} S_\bullet(C), \]

where the left-facing arrow is given on \( n \)-simplices by the assignment
\[ (L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \hookrightarrow V, \]
and Index is given on \( n \)-simplices by the assignment
\[ (L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0). \]

We have an analogue of Proposition [3.3] in this setting.

**Proposition A.22.** Let \( D \) be idempotent complete, and let \( C \subset D \) be an extension-closed sub-category. Then the augmentation \( \text{Gr}_\bullet \subset (D, C) \longrightarrow \text{Tate}^{el}(D, C) \) of (47) induces an equivalence
\[ |\text{Gr}_\bullet(D, C)| \cong \text{Tate}^{el}(D, C). \]

**Proof.** Proposition [A.15] implies that the relative Grassmannian is a directed partially ordered set. Just like in the proof of Proposition [3.3] we conclude that \( |\text{Gr}_\bullet(D, C)| \cong \text{Tate}^{el}(D, C). \)

Along with Lemma [A.13] this implies an analogue of Corollary [3.6]

**Corollary A.23.** Let \( D \) be idempotent complete, and let \( C \subset D \) be a right s-filtering sub-category. The categorical relative index map determines a map of infinite loop spaces

\[ B\text{Index}: |S_\bullet \text{Tate}^{el}(D, C)| \longrightarrow |S_\bullet S_\bullet(C)| \]

\[15\]To see that this is an exact category, observe that because \( \text{Pro}^a(D) \) and \( \text{Ind}^a(C) \) are closed under extensions in \( \text{Tate}^{el}(D, C) \), \( \text{Gr}_n \subset (D, C) \) is closed under extensions in \( \text{Fun}([n+1], \text{Tate}^{el}(D, C)) \).
which fits into a commuting square

\[
\begin{array}{ccc}
\text{Tate}^\text{el}(D, C)^\times & \xrightarrow{\text{Index}} & |S_s(C)^\times| \\
\downarrow & & \downarrow \simeq \\
\Omega|S_s\text{Tate}^\text{el}(D, C)^\times| & \xrightarrow{\Omega\text{Index}} & \Omega|S_sS_s(C)^\times|
\end{array}
\]

We refer to the looping of the bottom horizontal map as the $K$-theoretic relative index map.

**Theorem A.24.** Let $D$ be idempotent complete, and let $C \subset D$ be a right $s$-filtering sub-category. Then the $K$-theoretic relative index map fits into a commuting diagram

\[
\begin{array}{ccc}
\Omega K_{\text{Tate}^\text{el}(D, C)} & \xrightarrow{\Omega^2 \text{Index}} & K_{C} \\
\downarrow & & \downarrow \\
\Omega K_{D/C} & \xrightarrow{\partial} & \\
\end{array}
\]

**Proof.** The categorical relative index map fits into a 2-commuting diagram

\[
\begin{array}{ccc}
\text{Tate}^\text{el}(D, C) & \xleftarrow{\text{Gr}^< (D, C)} & \text{Gr}^< (S_s D, S_s (C)) \\
\downarrow & & \downarrow \\
\text{Pro}^s(D)/C & \xleftarrow{S_s^r (C \subset \text{Pro}^s(D))} & S_s(C) \\
\downarrow & & \downarrow \\
D/C & \xleftarrow{S_s^r (C \subset D),} & \\
\end{array}
\]

and the map $\text{Gr}^< (D, C) \rightarrow S_s^r (C \subset \text{Pro}^s(D))$ is given on $n$-simplices by the assignment

\[(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_1/L_0 \hookrightarrow \cdots L_n/L_0; L_0 \hookrightarrow \cdots \hookrightarrow L_n) .\]

Similarly, using Lemma [A.13] we see that there exists a 2-commuting diagram

\[
\begin{array}{ccc}
S_s(\text{Tate}^\text{el}(D, C))^\times & \xleftarrow{\text{Gr}^< (S_s D, S_s (C))^\times} & \\
\downarrow & & \downarrow \\
S_s(\text{Pro}^s(D)/C)^\times & \xleftarrow{S_sS_s^r (C \subset \text{Pro}^s(D))^\times} & S_sS_s(C)^\times \\
\downarrow & & \downarrow \\
S_s(D/C)^\times & \xleftarrow{S_sS_s^r (C \subset D)^\times} & \\
\end{array}
\]
Geometrically realizing and taking the double loop spaces, we obtain a commuting diagram

\[
\begin{align*}
\Omega K_{\text{Tate}^a(D, C)} &\xrightarrow{\cong} \Omega^2 |Gr^{\leq} (S_*D, S_*C)\times| \\
\Omega K_{\text{Pro}^a(D)/C} &\xrightarrow{\cong} \Omega^2 |S_*S^*C \subset \text{Pro}^a(D)^\times| \\
\Omega K_{D/C} &\xrightarrow{\cong} \Omega^2 |S_*S^*C \subset D^\times|.
\end{align*}
\]

Note that the lower two left-facing maps are equivalences by Schlichting’s Proposition 2.35. After inverting the left-facing equivalences, we obtain a commuting diagram

\[
\begin{align*}
\Omega K_{\text{Tate}^a(D, C)} &\xrightarrow{} K_C \\
\Omega K_{\text{Pro}^a(D)/C} &\xrightarrow{} K_C \\
\Omega K_{D/C} &\cong
\end{align*}
\]

By Corollary 2.39, it suffices to prove that the map

\[
\Omega K_{\text{Tate}^a(D, C)} \to \Omega K_{\text{Pro}^a(D)/C}
\]

is an equivalence. We derive this from Proposition A.19 as follows. To wit, consider the commuting diagram

\[
\begin{align*}
C &\to \text{Ind}^a(C) &\to \text{Ind}^a(C)/C \\
\text{Pro}^a(D) &\to \text{Tate}^c(D, C) &\to \text{Tate}^c(D, C)/\text{Pro}^a(D) \\
\text{Pro}^a(D)/C &\cong \text{Tate}^c(D, C)/\text{Ind}^a(C) &\to 0,
\end{align*}
\]

where the equivalences are those of Propositions A.19 and A.18. Applying $K$-theory, we obtain a 3 × 3-diagram
in the stable $\infty$-category of spectra. Note that, of the entries in the diagram, only $K_C$ has non-vanishing $\pi_0$. Along with Theorem 2.37, this shows that all rows in this diagram are fibre-cofibre sequences and that the outer two columns are fibre-cofibre sequences as well. Because the homotopy category suffices to detect fibre-cofibre sequences, we conclude, by the $3 \times 3$-lemma for triangulated categories, that the middle column is a fibre-cofibre sequence.

The Eilenberg swindle implies that $K_{\text{Ind}^a}(C) \cong 0$, and we conclude that $K_{\text{Tate}^a}(D,C) \cong K_{\text{Tate}^a}(D,C)/K_{\text{Pro}^a}(D)/C$ as claimed. \hfill $\Box$

**Corollary A.25.** Let $X$ be a Noetherian scheme, and let $W \subset Z \subset X$ be a flag of closed subschemes. Then the diagram of abelian categories

```
Coh(W) → Coh(Z) → Coh(Z \ W)
```

```
Coh(W)(X) → Coh(Z)(X) → Coh(Z)(X)/Coh(W)(X)
```

determines a commuting triangle

```
ΩK_{Coh(Z \ W)} \quad \partial \\
ΩK_{Coh}(Coh(Z), Coh(W)(X)) \quad \Omega^2 BIndex \\
ΩK_{Coh(Z)}(X)/Coh(W)(X)
```

**Proof.** We have a nested chain of Serre sub-categories $Coh(W)(X) \subset Coh(Z)(X) \subset Coh(Z)(X)$. By [Sch04, Example 1.7], Serre sub-categories are right s-filtering. Therefore, we can take $D = Coh(Z)(X)$ and $C = Coh(W)(X)$ and apply Theorem A.24 to obtain a commuting triangle

```
ΩK_{Coh(Z)}(X)/Coh(W)(X) \quad \partial \\
ΩK_{Coh(Z)}(X)/Coh(W)(X) \quad \Omega^2 BIndex \\
K_{Coh}(X)
```

By devissage [Qui73, Section 5], we have equivalences $K_{Coh}(W) \simeq K_{Coh}(Z) \simeq K_{Coh}(X)$ and $K_{Coh}(Z) \simeq K_{Coh}(X)$. We also have an equivalence $Coh(Z)/Coh(W)(Z) \simeq Coh(Z \ W)$ [Gab62, Chapter 5]. Together with Quillen’s localization sequence, these equivalences determine an equivalence $\Omega K_{Coh(Z)}(X)/Coh(W)(X) \simeq \Omega K_{Coh}(Z \ W)$, which is compatible with the boundary maps of the localization sequences. Applying these equivalences to the commuting triangle above, we obtain the commuting triangle of the desired type. \hfill $\Box$

### References

- [ADCK89] E. Arbarello, C. De Concini, and V. G. Kac, *The infinite wedge representation and the reciprocity law for algebraic curves*, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 171–190. MR 1013132 (90i:22034) 1.1, 3, 5.3, 5.21, 5.3
- [AK10] Sergey Arkhipov and Kobi Kremnitzer, *2-gerbes and 2-Tate spaces*, Arithmetic and geometry around quantization, Progr. Math., vol. 279, Birkhäuser Boston Inc., Boston, MA, 2010, pp. 23–35. MR 2656941 (2011g:22036) 1.1, 1.4, 5.2, 2.2, 5.2.4
- [AM69a] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics, No. 100, Springer-Verlag, Berlin-New York, 1969. MR 0245577 (39 #6833) 2.1, 2
80 OLIVER BRAUNLING & MICHAEL GROECHENIG & JESSE WOLFSON

[AM69b] M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1969.

[Ati89] M. F. Atiyah, K-theory, second ed., Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989, Notes by D. W. Anderson. MR 1043170 (90m:18011)

[BBE02] Alexander Beilinson, Spencer Bloch, and Hélène Esnault, $\epsilon$-factors for Gauss-Manin determinants, Mosc. Math. J. 2 (2002), no. 3, 477–532. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR 1989870 (2004m:14051)

[Bei87] A. A. Beilinson, How to glue perverse sheaves, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 42–51. MR 923134 (89b:14028)

[Be˘ı87] A. Beilinson, How to glue perverse sheaves, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 42–51. MR 923134 (89b:14028)

[Bei09] A. Beilinson, $\epsilon$-Factors for the Period Determinants of Curves, Motives and Algebraic Cycles, American Mathematical Society, 2009.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, A universal characterization of higher algebraic K-theory, Geom. Topol. 17 (2013), no. 2, 733–838. MR 3070515

[BGWa] Oliver Braunling, Michael Groechenig, and Jesse Wolfson, Generalized Contou-Carrère symbols and reciprocity laws for higher dimensional varieties.

[BGWb] , The index map and non-commutative motives.

[BGW14] , Tate objects in exact categories, arXiv:1402.4969, 02 2014.

[Bre11] Manuel Breuning, Determinant functors on triangulated categories, J. K-Theory 8 (2011), no. 2, 251–291. MR 2842932

[Bry97] Jean-Luc Brylinski, Central extensions and reciprocity laws, Cahiers Topologie Géom. Différentielle Catég., 38 (1997), no. 3, 193–215. MR 1474565 (2000f:11150)

[B"uh10] Theo Bühler, Exact categories, Expo. Math. 28 (2010), no. 1, 1–69. MR 2606234 (2011e:18020)

[CPT12] T. Chinburg, G. Pappas, and M. J. Taylor, Higher adeles and non-abelian Riemann-Roch.

[Del87] P. Deligne, Le déterminant de la cohomologie, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 93–177. MR 902592 (89b:32038)

[Dou98] Ronald G. Douglas, Banach algebra techniques in operator theory, second ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998. MR 1634900 (99c:47001)

[Dri06] Vladimir Drinfeld, Infinite-dimensional vector bundles in algebraic geometry: an introduction, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 263–304. MR 2181808 (2007d:14038)

[Dus01] J.W. Duskin, Simplicial matrices and the nerves of weak $n$-categories. I. Nerves of bicategories, Theory Appl. Categ., 9 (2001), 198–308.

[FZ12] E. Frenkel and X. Zhu, Gerbal representations of double loop groups, Int. Math. Res. Not. (2012), no. 17, 3929–4013.

[Gab62] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323–448.

[Ger73] S. M. Gersten, Higher K-theory of rings, Algebraic K-theory, I: Higher K-theories (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972), Springer, Berlin, 1973, pp. 3–42. Lecture Notes in Math., Vol. 341. MR 0382398 (52 #3282)

[GJ99] Kuo-Jung Lee and J.-P. Toulkeridis, Simplicial homotopy theory, Cambridge University Press, Cambridge, 2006. MR 2271027 (2007k:55001)

[Jan65] Klaus Jänich, Vektorraumbündel und der Raum der Fredholm-Operatoren, Math. Ann. 161 (1965), 129–142. MR 0190946 (32 #5206)

[Kan57] D.M. Kan, On c.s.s. complexes, Amer. J. Math. 79 (1957), 449–476.

[Kap] M. Kapranov, Semisimple symmetric powers, arXiv:0107089.

[Kap01] M. Kapranov, Double affine Hecke algebras and 2-dimensional local fields, J. Amer. Math. Soc. 14 (2001), no. 1, 239–262 (electronic). MR 1800352 (2001k:20007)

[Kat00] Kazuya Kato, Existence theorem for higher local fields, Invitation to higher local fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 165–195. MR 1804933 (2002e:11170)
