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A HARDY-MOSER-TRUDINGER INEQUALITY

GUOFANG WANG AND DONG YE

Abstract. In this paper we obtain an inequality on the unit disk $B$ in $\mathbb{R}^2$, which improves the classical Moser-Trudinger inequality and the classical Hardy inequality at the same time. Namely, there exists a constant $C_0 > 0$ such that

$$\int_B e^{\frac{4\pi u^2}{|x|^2}} \, dx \leq C_0 < \infty, \quad \forall \ u \in C_0^\infty(B),$$

where

$$H(u) := \int_B |\nabla u|^2 \, dx - \int_B \frac{u^2}{(1 - |x|^2)^2} \, dx.$$ 

This inequality is a two dimensional analog of the Hardy-Sobolev-Maz’ya inequality in higher dimensions, which has been intensively studied recently. We also prove that the supremum is achieved in a suitable function space, which is an analog of the celebrated result of Carleson-Chang for the Moser-Trudinger inequality.

1. Introduction

Let $B$ denote the standard unit disk in $\mathbb{R}^2$. The famous Moser-Trudinger inequality \[28, 32\]

$$\int_B e^{\frac{4\pi u^2}{|x|^2}} \, dx \leq C < \infty, \quad \forall \ u \in H_0^1(B)$$

plays an important role in two dimensional analytic problems. This inequality is viewed as a 2-dimensional analog of the Sobolev inequality. It is optimal in the sense that the constant $4\pi$ in (1) could not be replaced by any larger constant. Its slightly weaker form

$$\frac{1}{2} \int_B |\nabla u|^2 \, dx - 8\pi \log \left( \int_B e^u \, dx \right) \geq -C > -\infty, \quad \forall \ u \in H_0^1(B)$$

has been intensively used in the study of prescribing Gaussian curvature problem, and recently in the study of mean field equation. Here again, the constant $8\pi$ is optimal to have a finite infimum.

There is another important inequality in analysis, the Hardy inequality

$$H(u) := \int_B |\nabla u|^2 \, dx - \int_B \frac{u^2}{(1 - |x|^2)^2} \, dx \geq 0, \quad \forall \ u \in H_0^1(B).$$

In this paper $| \cdot |$ denotes always the Euclidean norm. This result is also optimal in the sense that for any $\lambda > 1$,

$$\inf_{H_0^1(B)} \left[ \int_B |\nabla u|^2 \, dx - \lambda \int_B \frac{u^2}{(1 - |x|^2)^2} \, dx \right] = -\infty.$$ 

Indeed, this inequality holds also for higher dimensions. The Hardy inequality can be improved in the following way. Let $B^n$ denotes the unit ball in $\mathbb{R}^n$ with $n \geq 2$, it is known that there exists $C > 0$ (see \[8\] or Remark 1 below for $n = 2$) such that

$$H(u) \geq C \int_{B^n} u^2 \, dx, \quad \forall \ u \in H_0^1(B^n).$$
Hence \( \|u\|_H := \sqrt{\mathcal{H}(u)} \) defines a norm over \( H_0^1(B^n) \) and the completion of \( C_0^\infty(B^n) \) with respect to the norm \( \| \cdot \|_H \) is a Hilbert space, which is denoted by \( \mathcal{H}(B^n) \). Obviously \( H_0^1(B) \subsetneq \mathcal{H}(B) \). For simplicity, we denote \( \mathcal{H}(B) \) by \( \mathcal{H} \) and \( \| \cdot \|_H \) by \( \| \cdot \| \). In this paper we call \( \mathcal{H} \) the Hardy functional.

In this paper, one of our main objectives is to improve the Moser-Trudinger inequality by combining the Hardy inequality.

**Theorem 1.** There exists a constant \( C_0 > 0 \) such that

\[
\int_B e^{\frac{4\pi^2 u^2}{|x|^2}} \, dx \leq C_0 < \infty, \quad \forall \ u \in \mathcal{H} \setminus \{0\}.
\]

A direct consequence is the following, slightly weaker, but applicable form.

**Corollary 1.** There exists a constant \( C > 0 \) such that

\[
\int_B |\nabla u|^2 \, dx - \frac{1}{2} \int_B \frac{u^2}{(1 - |x|^2)^2} \, dx - 8\pi \log \left( \int_B e^u \, dx \right) \geq -C > -\infty, \quad \forall \ u \in \mathcal{H}.
\]

In the first glimpse these improved inequalities look too strong to be true. But if one compares to the recent work on the Hardy-Sobolev inequality in higher dimensional case, one would speculate that this can be true. The Hardy-Sobolev inequality for higher dimension is also called the Hardy-Sobolev-Maz’ya inequality. Maz’ya proved in [27] (Section 2.1.6, Corollary 3) that there exists a constant \( C > 0 \) such that for any \( u \in H_0^1(B^n) \) with \( n > 2 \),

\[
\int_{B^n} |\nabla u(x)|^2 \, dx - \int_{B^n} \frac{u^2}{(1 - |x|^2)^2} \, dx \geq C \left( \int_{B^n} |u(x)|^{2n} \, dx \right)^{\frac{n-2}{n}}.
\]

This inequality combines naturally the Sobolev inequality and the Hardy inequality. Let \( C_n \) be the best constant such that (6) holds. In some recent works, the constant \( C_n \) has been estimated (see [30] for \( n > 3 \) and [7] for \( n = 3 \)).

- If \( n > 3 \), then \( C_n < S_n \) and the sharp constant \( C_n \) is achieved in \( \mathcal{H}(B^n) \). See [30].
- If \( n = 3 \), then \( C_3 = S_3 \) is not achieved in \( \mathcal{H}(B^n) \). See [7].

Here \( S_n \) denotes the best constant for the Sobolev embedding from \( H^1(\mathbb{R}^n) \) into \( L^{\frac{2n}{n-2}}(\mathbb{R}^n) \). See also [20]. The Moser-Trudinger inequality (1) is considered as the most natural 2-dimensional analog of the Sobolev inequality. Therefore, the work on the Hardy-Sobolev-Maz’ya inequality inspires us to ask if one can also combine the Moser-Trudinger inequality (1) and the Hardy inequality (2) into one single inequality. Theorem 1 gives an affirmative answer to this question and shows that the analog of the Hardy-Sobolev-Maz’ya inequality (6) holds on a two dimensional disk with the same best constant \( 4\pi \) as in the classical Moser-Trudinger inequality (1). We call (4) or (5) a Hardy-Moser-Trudinger inequality. Moreover, we prove that the maximum of the functional in (4) is achieved, which is a Carleson-Chang type result. The original Carleson-Chang [10] result asserts that the maximum of the functional in (1) is achieved, which is one of seminal results in geometric analysis.

**Theorem 2.** There exists \( u_0 \in \mathcal{H} \) such that \( \|u_0\| = 1 \) and

\[
\int_B e^{4\pi u_0^2} \, dx = \max_{u \in \mathcal{H}, \|u\| \leq 1} \int_B e^{4\pi u^2} \, dx = \max_{u \in \mathcal{H} \setminus \{0\}} \int_B e^{4\pi u^2} \, dx.
\]

Note that the supremum is not achieved in \( H_0^1(B) \), see Remark 6 below.

If we denote the best constant in the Moser-Trudinger inequality by \( S_2 \) and the best constant in (4) by \( C_2 \), the results proved in this paper could be stated in a similar form as above

- If \( n = 2 \), then \( C_2 = S_2 = 4\pi \) is achieved in \( \mathcal{H}(B^2) \).
We wonder if this kind of Hardy-Moser-Trudinger inequality holds for more general domains $\Omega \subset \mathbb{R}^2$. For example, let $\Omega \subset \mathbb{R}^2$ be a regular, bounded and convex domain, then (see [8])

$$H_d(u) := \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{4} \int_{B} \frac{u^2}{d(x, \partial \Omega)^2} \, dx > 0, \quad \forall \, u \in H^1_0(\Omega) \setminus \{0\}.$$ 

We propose the following

**Conjecture:** There is a constant $C(\Omega) > 0$ such that

$$\int_{\Omega} e^{4\pi u^2} \, dx \leq C(\Omega) < \infty, \quad \forall \, u \in \mathcal{H}_d(\Omega) \setminus \{0\}.$$ 

Here $\mathcal{H}_d(\Omega)$ denotes the completion of $C^\infty_0(\Omega)$ with the corresponding norm, defined by $\|u\|_{\mathcal{H}_d}^2 = H_d(u)$. The conjecture is true when $\Omega = B$. This follows immediately from Theorem 1, since $H(u) \leq H_d(u)$ for any $u \in C^\infty_0(B)$.

There is another improved Moser-Trudinger inequality on the disk in $\mathbb{R}^2$, which was recently proved and studied in [26, 4].

$$\sup_{u \in H^1_0(B), \|u\| \leq 1} \int_{B} e^{4\pi u^2} - 1 \left(1 - |x|^2\right)^2 \, dx < \infty. \tag{7}$$

For other generalizations of the classical Moser-Trudinger inequality (1), see for instance [2, 15, 19, 22, 23]. See also [14, 31] for a Moser-Trudinger inequality in Kähler geometry. A generalization of (4) to a higher dimensional ball like in [24] and to higher order derivatives like in [1, 6, 17, 25] would be very interesting. For the related higher order equations see also [18].

For the proof of our results, we use an important tool in geometric analysis, the blow-up analysis. Similar approaches were used in [10, 13, 2, 21, 23] to establish 2-dimensional inequalities. The blow-up analysis for elliptic equations related to the classical Moser-Trudinger inequality (1) was initiated in [10, 29, 3].

This proof is different from Moser’s proof for the classical Moser-Trudinger inequality (1). It would be an interesting question if there is a proof similar to Moser’s approach for (4). After symmetrization, as in [28, 10], we need only to deal with a one-dimensional inequality, or equivalently to deal with (4) in the class of nonincreasing, radially symmetric functions (see (8) below). However, it is difficult to us to follow Moser’s proof, even for the weaker inequality (16) below. First, the condition $H(u) \leq 1$ is weaker than the condition $D(u) \leq 1$. Here $D(u)$ is the Dirichlet functional $\int |\nabla u|^2$. Second, the integrand of the Dirichlet energy $D$ is non-negative, while the integrand of the Hardy functional $H$ could be somewhere negative. Hence, unlike in [27], we can only obtain weaker estimates of $u(r)$ in terms of $H(u)$ (see Remark 2 below). Nevertheless, we can show the claim (17), which, together with a weak Moser-Trudinger inequality (see (19) below), implies a weaker Hardy-Moser-Trudinger inequality (see Theorem 3 below). With this weaker inequality we start the blow-up analysis.

Our paper is organized as follows: In Section 2, by using symmetrization we reduce our problem to a problem with radially nonincreasing functions and study the property of such functions with bounded $H(u)$. We also study Green’s function to the operator

$$\mathcal{L}_H := -\Delta - \frac{1}{\left(1 - |x|^2\right)^2}.$$ 

In Section 3, we prove a subcritical or a weaker Hardy-Moser-Trudinger inequality (16) with constant $(4\pi - \varepsilon)$ for any constant $\varepsilon \in (0, 4\pi)$, and show that (16) is achieved by a function $u_\varepsilon$. In Section 4, we analyze the convergence of family of the extremals $\{u_\varepsilon\}$ as $\varepsilon$ tends to 0, and its blow-up behavior. Finally, Theorems 1 and 2 are proved respectively in Sections 5 and 6, by contradiction arguments.
In the following, $\| \cdot \|_p$ denotes the standard $L^p$ norm for $p \in [1, \infty]$ and $C$ denotes a positive constant, which may change from line to line.

2. Preliminaries

First of all, we use the nonincreasing symmetrization with respect to the standard hyperbolic metric $dv_{Hy} = \frac{ds}{(1-|r|^2)^{p/2}}$ over $B$, which will enable us to reduce our problem to a one-dimensional problem, or equivalently to a problem in the space of nonincreasing, radially symmetric functions.

For any $u \in H^1_0(B)$, let $u_*$ denote the associated radially nonincreasing rearrangement with respect to $dv_{Hy}$. It is well-known that $u_* \in H^1_0(B)$ and $\|\nabla u_\|_2 \leq \|\nabla u\|_2$ (see for example [5]). We know also

$$\int_B \frac{u_*^2}{(1-|x|^2)^2} dx = \int_B u^2 dv_{Hy} = \int_B u^2 dv_{Hy} = \int_B \frac{u^2}{(1-|x|^2)^2} dx.$$  
Thus $H(u) \leq 1$ implies that $H(u_*) \leq 1$. Furthermore, we have

$$\int_B e^{4\pi u_*^2} dx = \int_B e^{4\pi u^2} (1-|x|^2)^2 dv_{Hy} \geq \int_B e^{4\pi u^2} (1-|x|^2)^2 dv_{Hy} = \int_B e^{4\pi u^2} dx.$$  
This follows from the Hardy-Littlewood inequality (see for example [9]) by noticing that the rearrangement of $(1-|x|^2)^2$ is just itself. Therefore we only need to consider nonincreasing, radially symmetric functions. Let

$$\Sigma := \left\{ u \in C^\infty_0(B), u(x) = u(r) \text{ with } r = |x|, \ u' \leq 0 \right\}$$

and $\mathcal{H}_1$ be the closure of $\Sigma$ in $\mathcal{H}$. To prove Theorem 1, we need only to show that

$$(8) \quad \sup_{u \in \mathcal{H}_1, \|u\| \leq 1} \int_B e^{4\pi u^2} dx \leq C_0 < \infty.$$  

Let $u \in \Sigma$, $r = \varphi(t) = \tanh \left( \frac{t}{2} \right)$ and

$$\tilde{u}(t) := u(r) = u \circ \varphi(t).$$

Let $B_r$ be the disk of radius $r$ centered at 0 and $B_r^c = B \setminus B_r$ its complement in $B$. Define

$$H_{\Omega}(u) := \int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{u^2}{(1-|x|^2)^2} dx, \quad \forall \ \Omega \subset B.$$  
It is easy to see that

$$H_{B_r}(u) = 2\pi \int_{t=\varphi^{-1}(t)}^\infty \left( \tilde{u}'^2 - \frac{\tilde{u}^2}{4} \right) \sinh(s)ds, \quad \forall \ r \in (0, 1).$$

Set $v(s) = e^{-\frac{s}{2}} \tilde{u}(s)$. Noting that $v(s) = 0$ for large $s$, integration by parts gives then

$$\frac{H_{B_r}(u)}{2\pi} = \int_t^\infty e^{-s} (v'^2 - vv') \sinh(s)ds = v^2(t) e^{-t} \sinh(t) + \int_t^\infty e^{-2s} v'^2 ds + \int_t^\infty e^{-s} v'^2 \sinh(s)ds.$$  
We obtain, by taking $t \to 0$, i.e. $r \to 0$,

$$(10) \quad \frac{H(u)}{2\pi} = \int_0^\infty e^{-2s} v'^2 ds + \int_0^\infty e^{-s} v'^2 \sinh(s)ds, \quad \forall \ u \in \Sigma.$$  
Consequently, we have

**Lemma 1.** $\mathcal{H}_1$ is embedded continuously in $H^1_{loc}(B) \cap C^{0,\frac{1}{2}}_{loc}(B \setminus \{0\})$. Moreover, for any $p \geq 1$, $\mathcal{H}_1 \subset L^p(B)$ and this embedding is compact.
Proof. Fix any \( r \in (0, 1) \), we have for all \( u \in \Sigma \),

\[
\int_{B_r} u^2 dx + \int_{B_r} |\nabla u|^2 dx = 2\pi \int_0^{\varphi^{-1}(r)} \left[ \bar{u}^2 \frac{1}{4 \cosh^4 \left( \frac{\bar{u}}{2} \right)} + \bar{u}^2 \right] \sinh(s) ds \\
\leq C \int_0^{\varphi^{-1}(r)} \left( \bar{u}^2 + \bar{u}^2 \right) \sinh(s) ds \\
\leq C \int_0^{\varphi^{-1}(r)} \left( v^2 e^{-s} + v'^2 e^{-s} \right) \sinh(s) ds \\
\leq C_r \int_0^{\varphi^{-1}(r)} \left[ v^2 e^{-2s} + v'^2 e^{-s} \sinh(s) \right] ds \\
\leq C_r H(u).
\]

(11)

Here the constant \( C_r \) depends only on \( r \in (0, 1) \). From above we have \( \mathcal{H}_1 \subset \mathcal{H}^1_{loc}(B) \). By the Sobolev embedding theorem, we get

\[
\mathcal{H}_1 \subset \bigcap_{p \geq 1} L^p_{loc}(B) \quad \text{and} \quad \mathcal{H}_1 \subset C^0_{loc}(B \setminus \{0\}).
\]

Furthermore, for any \( r \in (0, 1) \), there exists \( C_r > 0 \) such that \( u(r) \leq C_r H(u) \), \( \forall \ u \in \mathcal{H}_1 \). Since \( u \in \mathcal{H}_1 \) is nonincreasing, \( \mathcal{H}_1 \) is continuously embedded in \( L^p(B) \) for any \( p \geq 1 \). For any bounded sequence \( \{u_k\} \subset \mathcal{H}_1 \), we may assume, by taking a subsequence, that \( u_k \) converges to \( u \) weakly in \( \mathcal{H} \) and a.e. in \( B \) by the following Lemma. 

Lemma 2. Let \( \Omega \subset \mathbb{R}^n \) be of finite measure and \( w_k \) be a sequence of measurable functions converging a.e. in \( \Omega \) to \( w \). Assume that there exists \( q > 1 \) such that \( \{w_k\} \) is bounded in \( L^q(\Omega) \), then \( w_k \) converges to \( w \) in \( L^1(\Omega) \).

Remark 1. Using the symmetrization argument, we can see that \( \mathcal{H} \) is embedded continuously in \( L^p(B) \) for any \( p \in [1, \infty) \).

Moreover, from (9) we get

\[
\frac{H_{B_r}(u)}{2\pi} \geq \frac{v^2(t)}{2} e^{-t} \sinh(t) \geq 0, \quad \forall \ u \in \Sigma, \ r = \tanh \left( \frac{t}{2} \right), \ t > 0.
\]

Since \( \tilde{u}(s) \) is nonincreasing in \( s \) for any \( u \in \Sigma \), from (10) we have

\[
\frac{H(u)}{2\pi} \geq \int_0^t e^{-2s} u^2(s) ds = \int_0^t e^{-s} \tilde{u}^2(s) ds \geq (1 - e^{-t}) \tilde{u}^2(t), \quad \forall \ u \in \Sigma, \ t > 0.
\]

Since \( e^t = \frac{1 + t}{1 - t} \) and \( \sinh(t) = \frac{2e^t}{1 - e^{-2t}} \), the above inequalities imply then

Lemma 3. For any \( u \in \mathcal{H}_1 \),

\[
u^2(r) \leq \frac{1 - r^2}{2\pi r} H_{B_r}(u) \quad \text{and} \quad u^2(r) \leq \frac{r + 1}{4\pi r} H(u), \quad \forall \ r \in (0, 1].
\]

Remark 2. We note that \( H_{B_r}(u) \leq H(u) \) for any \( u \in \mathcal{H}_1 \) and \( r \in (0, 1) \), since \( H_{B_r}(u) \geq 0 \). However, \( H_{B_r}(u) \leq H(u) \) is in general not true. For example \( H_{B_r}(u) < 0 \) if \( u \) is a positive constant on \( B_r \). Therefore we have no a priori control of \( H_{B_r}(u) \) even \( ||u|| \leq 1 \), this is why we estimate \( u(r) \) also by \( H(u) = ||u||^2 \).

Another crucial point in our approach is to handle the Hardy operator

\[
\mathcal{L}_H = -\Delta - \frac{1}{(1 - |x|^2)^2}.
\]

The problem is not trivial because we cannot apply directly the classical theory to \( \mathcal{H} \) due to the potential which is singular on the boundary. Our idea is to separate the study into two parts, and to use the classical theory near the origin and the \( L^2 \) theory in \( \mathcal{H} \) for the exterior part.
Proposition 1. For any \( f \in L^1(B) \cap L^2(B_1^\alpha) \), there exists a unique
\[
v \in \mathcal{H} + W_0^{1,p}(B_1^\alpha) \quad \text{with} \quad p \in (1,2)
\]
such that
\[
\mathcal{L}_H(v) = -\Delta v - \frac{v}{(1-|x|^2)^2} = f \quad \text{in } \mathcal{D}'(B).
\]

Moreover, we can decompose \( v = v_1 + v_2 \) where \( v_1 \in \mathcal{H} \), \( v_2 \in \cap_{p<2} W_0^{1,p}(B_1^\alpha) \) and
\[
\|v_1\| + \|\nabla v_2\|_p \leq C_p\|f\|_1 + C\|f\|_{L^2(B_1^\alpha)} \quad \forall \ p \in (1,2).
\]

Remark 3. Of course, the decomposition \((v_1,v_2)\) is not unique. However the solution \( v \) is uniquely determined. For \( L^2 \) theory with more general singular potentials, see for example [12].

Proof. To simplify notations, define \( \Omega_1 = B_1^\alpha \), \( \Omega_2 = B_1^\beta \) and
\[
a(x) = \frac{1}{(1-|x|^2)^2}.
\]

For the uniqueness of \( v \), we need only to consider the case \( f = 0 \). Let \( v = v_1 + v_2 \) satisfy \( \mathcal{L}_H(v) = 0 \) in \( \mathcal{D}'(B) \) with \( v_1 \in \mathcal{H} \), \( v_2 \in W_0^{1,p}(\Omega_2) \) and \( p > 1 \). We have
\[
\langle v_1, \varphi \rangle_{\mathcal{H}} \leq C\|\varphi\| \leq C\|\nabla \varphi\|_{L^2(\Omega_2)}, \quad \forall \varphi \in C_0^\infty(\Omega_2).
\]

From \( \mathcal{L}_H(v) = 0 \), we have \( -\Delta v_2 - a(x)v_2 \in H^{-1}(\Omega_2) \), the dual space of \( H_0^1(\Omega_2) \). For any \( u \in H_0^1(\Omega_2) \), let \( w(x) = u(\frac{x}{2}) \in H_0^1(B) \). Using the monotonicity of \( a \), we have
\[
0 \leq H(w) \leq \int_{\Omega_2} |\nabla u|^2 dx - \int_B a(x)w^2 dx \leq \int_{\Omega_2} |\nabla u|^2 dx - 4\int_{\Omega_2} a(x)u^2 dx,
\]
and hence \( H_{\Omega_2}(u) \geq \frac{3}{4}\|\nabla u\|^2_2 \) for all \( u \in H_0^1(\Omega_2) \). Therefore the operator \( \mathcal{L}_H \) is coercive in \( H_0^1(\Omega_2) \) and the classical regularity theory implies that \( v_2 \) belongs to \( H_0^1(\Omega_2) \subset H_0^1(B) \subset \mathcal{H} \).

Finally \( v = v_1 + v_2 \in \mathcal{H} \) verifies \( \langle v, \varphi \rangle_{\mathcal{H}} = 0 \) for any \( \varphi \in C_0^\infty(B) \). By a density argument, we have \( v = 0 \), namely there is at most one solution.

For the existence of solutions to (13) with \( f \in L^1(B) \cap L^2(\Omega_1) \), consider first
\[
\mathcal{L}_H(w) = f \quad \text{in } \Omega_2, \quad w = 0 \quad \text{on } \partial\Omega_2.
\]

From the standard elliptic theory, there exists a unique solution \( w \in \cap_{p<2} W_0^{1,p}(\Omega_2) \) and \( \|\nabla w\|_p \leq C_p\|f\|_1 \). Choose a cut-off function \( \Psi \in \Sigma \) such that \( \Psi(r) = 1 \) for \( r \leq \frac{1}{4} \) and \( \Psi(r) = 0 \) for \( r \geq \frac{1}{4} \). It is easy to check that
\[
\mathcal{L}_H((1-\Psi)w) = (1-\Psi)f + 2\nabla w\nabla \Psi + w\Delta \Psi =: f_1 \quad \text{in } \mathcal{D}'(\Omega_2)
\]
with \( f_1 \in L^p(\Omega_2) \), \( \forall \ p \in (1,2) \). Thus we get \((1-\Psi)w \in W^{2,p}(\Omega_2), \forall \ p \in (1,2) \). In particular, from the Sobolev embedding theorem, the extension of \( w \) by 0 lies in \( W^{1,q}(B_1^\alpha) \) for all \( q > 1 \).

Define now \( \Psi_1(x) = \Psi(2x) \) and \( h = (1-\Psi_1)f + 2\nabla w\nabla \Psi_1 + w\Delta \Psi_1 \). It is clear that \( h \in L^2(B) \).

Thus we obtain \( \|h\varphi\|_1 \leq \|h\|_2\|\varphi\|_2 \leq C\|h\|_2\|\varphi\|_2 \) in view of (3) or Remark 1. By the Riesz Theorem, we know that there exists unique \( v_1 = \mathcal{H} \) such that
\[
\langle v_1, \varphi \rangle_{\mathcal{H}} = \int_B h\varphi dx, \quad \forall \varphi \in \mathcal{H}.
\]

It is easy to see that
\[
\|v_1\| \leq C\|h\|_2 \leq C\|f\|_{L^2(\Omega_1)} + C_p\|((1-\Psi)w)\|_{W^{2,p}(\Omega_2)} \leq C_p\|f\|_1 + C\|f\|_{L^2(\Omega_1)}.
\]

Finally, let \( v_2 = w\Psi_1 \), we check readily that \( v = v_1 + v_2 \) is the desired solution. 

\[\blacktriangleleft\]
Using this result, we can define Green’s function associated to the operator $L_H$.

**Proposition 2.** There exists a unique function $G_0 \in H + W^{1,p}_0(B_1)$ with $p \in [1, 2)$ such that

\begin{equation}
L_H(G_0) = \delta_0 \quad \text{in} \ D'(B)
\end{equation}

where $\delta_0$ stands for the Dirac distribution at 0. Moreover, $G_0$ is a radial function and there is a constant $C_G \in \mathbb{R}$ such that for any $\alpha \in (0, 1)$,

\begin{equation}
G_0(r) = -\frac{\ln r}{2\pi} + C_G + O(r^{1+\alpha}) \quad \text{as} \quad r \to 0,
\end{equation}

**Proof.** Let

\begin{equation}
G_2(r) = -\frac{1}{2\pi} \Psi(r) \ln r, \quad F(r) = -\frac{\ln r}{2\pi(1 - r^2)^2} \Psi - \frac{\Psi'}{\pi r} - \frac{\ln r}{2\pi} \Delta \Psi.
\end{equation}

Here $\Psi$ is the same cut-off function as in the previous proof. It is clear that $F \in L^2(B)$. Denote $G_1$ the unique solution in $H$ such that

\begin{equation}
\langle G_1, \varphi \rangle_H = \int_B F \varphi dx, \quad \forall \varphi \in H.
\end{equation}

Clearly, $G_0 = G_2 + G_1$ satisfies equation (14). The uniqueness of $G_0$ is ensured by Proposition 1, which implies then $G_0$ is radial. Since $F$ belongs to $L^p(B)$ for any $p > 1$, the standard elliptic theory yields that $G_1 \in W^{2,p}_{loc}(B) \subset C^{1,\alpha}_{loc}(B)$ for any $\alpha \in (0, 1)$. Hence we have the expansion (15).

**Remark 4.** Since $(1 - \Psi) \frac{\ln r}{2\pi} \in H^1_0(B)$, we have $G_0(r) = -\frac{\ln r}{2\pi} + \overline{G}(r)$ in $B$ with $\overline{G} \in H$.

### 3. A weaker Hardy-Moser-Trudinger inequality

In this Section we prove a weaker form of the Hardy-Moser-Trudinger inequality, or its sub-critical version, which will be used in our proof of Theorem 1.

**Theorem 3.** For any constant $\varepsilon \in (0, 4\pi)$, it holds

\begin{equation}
\sup_{u \in H_1, \|u\| \leq 1} \int_B e^{(4\pi - \varepsilon)u^2} dx < \infty.
\end{equation}

and the supremum is achieved by some $u_\varepsilon \in H_1$.

Define

\begin{equation}
A_u(r) = \frac{1}{\pi r^2} \int_{B_r} \frac{u^2}{(1 - |x|^2)^2} dx.
\end{equation}

**Lemma 4.** Let $u \in H_1$ and $r \in (0, 1)$, we have

\begin{equation}
\pi \left( \frac{1}{2} - r^2 \right) A_u(r) \leq H(u) + \frac{\pi u(r)^2}{1 - r^2}.
\end{equation}

**Proof.** From an elementary inequality

\begin{equation}
(u - b)^2 + b^2 \geq \frac{u^2}{2}, \quad \forall \, u, b \in \mathbb{R},
\end{equation}

we have for any $r \in (0, 1)$,

\begin{equation}
\frac{1}{r^2} \int_{B_r} \frac{(u - b)^2}{(1 - |x|^2)^2} dx \geq \frac{1}{2r^2} \int_{B_r} \frac{u^2}{(1 - |x|^2)^2} dx - \frac{1}{r^2} \int_{B_r} \frac{b^2}{(1 - |x|^2)^2} dx = \pi \frac{A_u(r) - \pi b^2}{1 - r^2}.
\end{equation}
Applying the above formula to \( w(x) = u(rx) - u(r) \in H^1_0(B) \) and \( b = u(r) \), we get
\[
\int_B \frac{w^2}{(1 - |x|^2)^2} \, dx \geq \int_{B_r} \frac{(u - u(r))^2}{r^2(1 - |x|^2)^2} \, dx \geq \frac{\pi}{2} A_u(r) - \frac{\pi u(r)^2}{1 - r^2}.
\]

It follows that, together with the Hardy inequality,
\[
0 \leq H(w) = \int_{B_r} |\nabla u|^2 \, dx - \int_B \frac{w^2}{(1 - |x|^2)^2} \, dx \leq \int_{B_r} |\nabla u|^2 \, dx - \frac{\pi u(r)^2}{1 - r^2}
= H_{B_r}(u) + \pi r^2 A_u(r) - \frac{\pi}{2} A_u(r) + \frac{\pi u(r)^2}{1 - r^2}
\leq H(u) + \frac{\pi u(r)^2}{1 - r^2} - \pi \left( \frac{1}{2} - r^2 \right) A_u(r),
\]
which is just the conclusion.

**Proof of Theorem 3.** By a density argument, we need only to show (16) for the subspace \( \Sigma \). Moreover we only need to show (16) for functions \( u \in \Sigma \).

Applying the above formula to \( w \), we have
\[
0 \leq H(w) = \int_{B_r} |\nabla u|^2 \, dx - \int_B \frac{w^2}{(1 - |x|^2)^2} \, dx \leq \int_{B_r} |\nabla u|^2 \, dx - \frac{\pi u(r)^2}{1 - r^2}
= H_{B_r}(u) + \pi r^2 A_u(r) - \frac{\pi}{2} A_u(r) + \frac{\pi u(r)^2}{1 - r^2}
\leq H(u) + \frac{\pi u(r)^2}{1 - r^2} - \pi \left( \frac{1}{2} - r^2 \right) A_u(r),
\]
which is just the conclusion.

We first claim that there exist two constants \( r_2 \in (0, 1) \) and \( C_1 > 0 \) independent of \( u \in \Sigma \cap \{ u \mid u(0) > 1 \} \) such that
\[
\| \nabla u \|_{L^2(B_{r_2})} \leq 1 \quad \text{and} \quad u(r_2) \leq C_1.
\]

Define
\[
r_1 = \inf \{ r > 0 \mid u(r) \leq 1 \} > 0.
\]
Since \( u(r_1) = 1 \), we have an upper bound of \( r_1, r_1 \leq \frac{1}{2\pi} \) by the second inequality in (12) for \( H(u) \leq 1 \). From Lemma 4 and the fact \( u(r) \geq 1 \) for \( r \leq r_1 \), we know that there exists \( C > 0 \) independent of \( u \) such that \( A_u(r) \leq C u(r)^2 \), \( \forall r \in (0, r_1] \). Using the first estimate in (12), we have, for any \( r \leq r_1 \),
\[
\int_{B_r} |\nabla u|^2 \, dx = H_{B_r}(u) + \pi r^2 A_u(r) \leq 1 - H_{B^c}(u) + C \pi r^2 u(r)^2
\leq 1 - H_{B^c}(u) + \frac{C}{2} r H_{B^c}(u).
\]
Therefore, there exists \( r_2 \in [0, r_1] \) small enough, independent of \( u \), such that
\[
\| \nabla u \|_{L^2(B_{r_2})} \leq 1.
\]
Moreover, Lemma 3 and \( H(u) \leq 1 \) imply that \( u(r_2) \leq C_1 \) for some constant \( C_1 > 0 \) independent of \( u \). This finishes the proof of the claim.

Thanks to the weak Moser-Trudinger inequality (19): for any small \( \varepsilon > 0 \)
\[
\int_B e^{\frac{(4\pi - \varepsilon/2)u^2}{\|\nabla u\|^2}} \, dx \leq C_\varepsilon < \infty, \quad \forall u \in H^1_0(B),
\]
we have
\[
\int_{B_{r_2}} e^{(4\pi - \varepsilon/2)(u(r) - u(r_2))^2} \, dx = \int_B e^{(4\pi - \varepsilon/2)(u(r) - u(r_2))^2} \, dx \leq C_\varepsilon < \infty,
\]
in view of (17). Here \( f_+ = \max \{ f, 0 \} \). It is easy to see for any \( r \leq r_2 \) we have
\[
(4\pi - \varepsilon)u(r)^2 \leq (4\pi - \varepsilon/2)\left[ u(r) - u(r_2) \right]^2 + 2(4\pi - \varepsilon/2)u(r)u(r_2) - \varepsilon/2u(r)^2
\leq (4\pi - \varepsilon/2)\left[ u(r) - u(r_2) \right]^2 + C(\varepsilon),
\]
for some positive constant $C(\varepsilon)$ depending on $C_1$ and $\varepsilon$, but independent of $u$. This yields
\[
\int_B e^{(4\pi-\varepsilon)u^2} \, dx = \int_{B_{r_2}} e^{(4\pi-\varepsilon)u^2} \, dx + \int_{B_{r_2}^c} e^{(4\pi-\varepsilon)u^2} \, dx \\
\leq \int_{B_{r_2}} e^{(4\pi-\varepsilon/2)|u-u(r_2)|^2+C(\varepsilon)} \, dx + \int_{B_{r_2}^c} e^{4\pi-\varepsilon}u(r_2)^2 \, dx \\
\leq e^{C(\varepsilon)} C_\varepsilon + \tau e^{4\pi C} < \infty.
\]
The proof of inequality (16) is completed.

Now we show the achievement of the supremum. Fix $\varepsilon > 0$, consider a maximizing sequence $u_j \in H_1$ for (16) with $\|u_j\| \leq 1$. Recall that $\| \cdot \|$ is the norm in $H$. By taking a subsequence, we may assume that $u_j$ converges to $u_\varepsilon \in H$ weakly in $H$. Thus $\|u_\varepsilon\| \leq 1$. By Lemma 1, we may assume also that $u_j$ converges to $u_\varepsilon$ a.e. in $B$.

Using (16) with $\varepsilon/2$, we see that $e^{(4\pi-\varepsilon)u_j^2}$ is bounded in $L^q(B)$ for some $q > 1$. Lemma 2 implies that $e^{(4\pi-\varepsilon)u_j^2}$ converges in $L^1(B)$, that is,
\[
\int_B e^{(4\pi-\varepsilon)u_j^2} \, dx = \lim_{j \to \infty} \int_B e^{(4\pi-\varepsilon)u^2} \, dx,
\]
and hence the supremum of (16) is attained by $u_\varepsilon$. Clearly we must have $\|u_\varepsilon\| = 1$.

\begin{remark}
Instead of the weak Moser-Trudinger inequality (19) one can certainly use the Moser-Trudinger inequality (1) in the proof of Theorem 3. Since the proof of the weak Moser-Trudinger inequality is very elementary (see [27]), we use it here to make the proof of Theorem 3 also elementary. In the proof, the claim (17) is crucial. We do not know if there is a very elementary proof of Theorem 3, similar to the original proof of Moser in [27], without first showing the claim. Remark that the proof of the Hardy-Moser-Trudinger inequality, Theorem 1, does not follow from this argument, since the constant $C(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.
\end{remark}

\section{4. Blow-up analysis}

For any $\varepsilon \in (0, 4\pi)$, let $u_\varepsilon$ be the maximizer obtained by Theorem 3. In this Section we consider the convergence of the sequence $\{u_\varepsilon\}$ when $\varepsilon$ goes to zero.

Suppose that $\|u_\varepsilon\|_\infty = u_\varepsilon(0)$ does not go to infinity as $\varepsilon$ tends to 0. Namely then there exists $\varepsilon_j \to 0$ such that $\|u_{\varepsilon_j}\|_\infty \leq C$. It is easy to see that in this case, up to a subsequence $u_{\varepsilon_j}$ converges weakly to $u_0 \in H_1$ in $H$ and a.e. in $B$, with $\|u_0\| \leq 1$ and $u_0 \in L^\infty(B)$. Let $w \in H$, $\|w\| \leq 1$,
\[
\int_B e^{(4\pi-\varepsilon_j)w^2} \, dx \leq \int_B e^{(4\pi-\varepsilon)u_{\varepsilon_j}^2} \, dx, \quad \text{for any } j \in \mathbb{N}.
\]
Applying respectively monotone and dominated convergence Theorem, we have
\[
\int_B e^{4\pi w^2} \, dx = \lim_{j \to \infty} \int_B e^{(4\pi-\varepsilon_j)w^2} \, dx \leq \lim_{j \to \infty} \int_B e^{(4\pi-\varepsilon_j)u_{\varepsilon_j}^2} \, dx = \int_B e^{4\pi u_0^2} \, dx < \infty.
\]
In other words, $u_0$ realizes the finite maximum of the Hardy-Moser-Trudinger functional. Therefore both Theorems 2 and 1 are proved in this case.

In the following, we will suppose the contrary, i.e. $\lim_{\varepsilon \to 0} \|u_\varepsilon\|_\infty = \infty$ and perform a blow-up analysis as in [2, 23]. Since $u_\varepsilon \in H_1$ is a maximizer, there exists $\lambda_\varepsilon > 0$ such that
\[
\mathcal{L}_H(u_\varepsilon) = -\Delta u_\varepsilon - \frac{u_\varepsilon}{(1-|x|^2)^2} = \lambda_\varepsilon u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} \quad \text{in } \mathcal{D}'(B).
\]
(20)

From Lemma 1, (16) and $\Delta u_\varepsilon \in L^q_{loc}(B)$ for some $q \in (1, 2)$, we get $u_\varepsilon \in W^{2,q}_{loc}(B)$ from the standard regularity theory. By the Sobolev embedding theorem in dimension two, we have that $u_\varepsilon$ is continuous in $B$, and hence $u_\varepsilon \in C(\overline{B})$. Here the continuity up to $\partial B$ follows from $u_\varepsilon \in H_1$. 

Indeed, $u_\varepsilon$ is the so called $H$–solution of (20) over $B$, in the spirit of Dávila and Dupaigne [12]. Using $u_\varepsilon$ as a test function, we have

$$
\lambda\int_B u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \|u_\varepsilon\|^2 = 1.
$$

Remark 6. Notice that $u_\varepsilon$ does not belong to $H^1_0(B)$. This is due to Theorem III in [8], because $u_\varepsilon \in C(\overline{B})$ and

$$
a(x) - \frac{1}{4d(x, \partial B)^2} = \frac{1}{(1-r^2)^2} - \frac{1}{4(1-r)^2} = \frac{1}{1-r} \times \frac{(3+r)}{4(1+r)^2} = O \left( d(x, \partial B)^{-1} \right).
$$

Suppose that Theorem 1 does not hold true, then

$$
\lim_{\varepsilon \to 0} \int_B e^{(4\pi-\varepsilon)u_\varepsilon^2}dx = \infty = \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{\infty}.
$$

Since $\|u_\varepsilon\| = 1$, there exist weakly convergent subsequences in $H$. Note that from now on, for simplicity, we do not distinguish between convergence and subconvergence. Assume $u_\varepsilon \to u_0 \in \mathcal{H}_1$ weakly in $H$.

Lemma 5. We have $u_0 \equiv 0$.

Proof. Suppose the contrary, then there is $r_0 \in (0, \frac{1}{2})$ such that $u_0(r_0) > 0$.

By Lemma 1, $u_\varepsilon$ tends to $u_0$ in $C_{loc}(B \setminus \{0\})$ (since the embedding of $C^{0,\frac{1}{2}}$ into $C^0$ is compact), and hence $u_\varepsilon(r_0) \geq \delta > 0$ for $\varepsilon$ small enough. Using Lemma 4, we have $A_{u_\varepsilon}(r) \leq C u_\varepsilon(r)^2$ for any $r \leq r_0$ when $\varepsilon$ is small enough, for $u_\varepsilon(r) \geq u_\varepsilon(r_0) \geq \delta$ and $H(u_\varepsilon) = 1$. Hence we have

$$
\int_{B_r} |\nabla u_\varepsilon|^2 dx = 1 - H_{B_r}(u_\varepsilon) + \pi r^2 A_{u_\varepsilon}(r) \leq 1 - \frac{2\pi r}{1-\varepsilon^2} u_\varepsilon(r)^2 + Cr^2 u_\varepsilon(r)^2, \quad \forall r \leq r_0.
$$

There exists then $r_1 \in (0, r_0)$ and $\eta > 0$ such that for $\varepsilon$ small, $\|\nabla u_\varepsilon\|_{L^2(B_{r_1})} \leq 1 - \eta < 1$. By the Moser-Trudinger inequality (1), we have

$$
\int_{B_{r_1}} e^{\frac{4\pi}{\varepsilon}(u_\varepsilon - b_\varepsilon)^2} dx \leq C_{MT}, \quad \text{where} \quad b_\varepsilon = u_\varepsilon(r_1).
$$

Similarly as in the proof for (16), using $\lim_{\varepsilon \to 0} b_\varepsilon = u_0(r_1) < \infty$, we can conclude that $\|e^{4\pi u_\varepsilon^2}\|_1 \leq C < \infty$ for small enough $\varepsilon$. This contradicts (22). Hence $u_0 \equiv 0$.

Applying Lemma 5 and Lemma 1, we know that $u_\varepsilon \in \mathcal{H}_1$ converges uniformly to $0$ in $\overline{B_\varepsilon}$ for $r > 0$. Thus we will concentrate now our attention on the behavior of $u_\varepsilon$ near the origin. Define

$$
M_\varepsilon = u_\varepsilon(0) = \max u_\varepsilon \quad \text{and} \quad \nu_\varepsilon^2 = \frac{e^{(\varepsilon-4\pi)M_\varepsilon^2}}{\lambda_\varepsilon M_\varepsilon^2}.
$$

Using (16), there holds

$$
\lambda_\varepsilon^{-1} \int_B u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2}dx \leq \int_{B_{\frac{1}{2}}} u_\varepsilon^2 e^{(4\pi-\varepsilon)u_\varepsilon^2}dx + C \leq M_\varepsilon^2 e^{(2\pi-\varepsilon)M_\varepsilon^2} \int_B e^{2\pi u_\varepsilon^2}dx + C
$$

$$
\leq M_\varepsilon^2 e^{(2\pi-\varepsilon)M_\varepsilon^2} \sup_{\|u\| \leq 1} \int_B e^{2\pi u^2}dx + C
$$

$$
\leq CM_\varepsilon^2 e^{(2\pi-\varepsilon)M_\varepsilon^2} + C.
$$

It follows that (recall that $\lim_{\varepsilon \to 0} M_\varepsilon = \infty$)

$$
\nu_\varepsilon^2 M_\varepsilon^2 = \frac{e^{(\varepsilon-4\pi)M_\varepsilon^2}}{\lambda_\varepsilon} \leq CM_\varepsilon e^{-2\pi M_\varepsilon^2},
$$

indicating the behavior of $u_\varepsilon$ near the origin.
and hence $\lim_{\varepsilon \to 0} r_\varepsilon M_\varepsilon = 0$ and $\lim_{\varepsilon \to 0} r_\varepsilon = 0$. Define $v_\varepsilon(x) = u_\varepsilon(r_\varepsilon x)$ and $\xi_\varepsilon(x) = M_\varepsilon [v_\varepsilon(x) - M_\varepsilon]$. A direct calculation leads to

$$
-\Delta \xi_\varepsilon = \frac{v_\varepsilon}{M_\varepsilon} e^{(4\pi - \varepsilon) (v_\varepsilon^2 - M_\varepsilon^2)} + \frac{r_\varepsilon^2 M_\varepsilon^2}{(1 - r_\varepsilon^2 |x|^2)^2 M_\varepsilon} v_\varepsilon \quad \text{in } \mathcal{D}'(B_{r_\varepsilon^{-1}}).
$$

For any $R > 0$, $-\Delta \xi_\varepsilon = O(1)$ in $B_R$ for small $\varepsilon$, since $0 \leq v_\varepsilon \leq M_\varepsilon$. By $\xi_\varepsilon(0) = 0$, the standard elliptic estimate implies that $\xi_\varepsilon$ converges in $C^1_{loc}(\mathbb{R}^2)$ to $\xi$. Therefore we have

$$
v_\varepsilon - M_\varepsilon = \frac{\xi_\varepsilon}{M_\varepsilon} \to 0, \quad \frac{v_\varepsilon}{M_\varepsilon} \to 1 \quad \text{and} \quad v_\varepsilon^2 - M_\varepsilon^2 = 2\xi_\varepsilon + \frac{\xi_\varepsilon^2}{M_\varepsilon^2} \to 2\xi \quad \text{in } C^1_{loc}(\mathbb{R}^2).
$$

By taking $\varepsilon \to 0$ in (24), we know that $\xi \in C^1(\mathbb{R}^2)$ satisfies

$$
-\Delta \xi = e^{8\pi \xi} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).
$$

Combining the facts $\xi(0) = 0$, $\xi$ is radially symmetric and nonincreasing with respect to $r$, we can deduce that

$$
\xi(x) = -\frac{1}{4\pi} \ln(1 + \pi r^2), \quad \int_{\mathbb{R}^2} e^{8\pi \xi} dx = 1.
$$

Note that all solutions of (26) with $e^{8\pi \xi} \in L^1(\mathbb{R}^2)$ were classified in [11].

From the above analysis we understand the behavior of the sequence $\{u_\varepsilon\}$ near the blow-up point 0, more precisely in $B_{r_\varepsilon} R$ for any large, but fixed $R > 0$. Let $L > 1$ and $R > 0$ large. We divide the disk $B$ into three parts: the interior part $B_{r_\varepsilon} R$, the outer part

$$
\{Lu_\varepsilon \leq M_\varepsilon\} := \left\{ x \in B \mid u_\varepsilon(x) \leq \frac{M_\varepsilon}{L} \right\}
$$

and the neck region

$$
\{Lu_\varepsilon \geq M_\varepsilon\} \setminus B_{r_\varepsilon} R := \left\{ x \in B \setminus B_{r_\varepsilon} R \mid u_\varepsilon(x) \geq \frac{M_\varepsilon}{L} \right\}.
$$

To analyze $\{u_\varepsilon\}$ in the outer part and the neck region, let us denote $u_{\varepsilon, L} = \min(u_\varepsilon, \frac{M_\varepsilon}{L})$. We have then

**Lemma 6.** For any $L > 1$, $\limsup_{\varepsilon \to 0} H(u_{\varepsilon, L}) \leq L^{-1}$.

**Proof.** Consider $\zeta_{\varepsilon, L} = u_\varepsilon - u_{\varepsilon, L} = (u_\varepsilon - \frac{M_\varepsilon}{L})_+$. Fix $R > 0$. Using $\zeta_{\varepsilon, L}$ as a test function to equation (20), we have

$$
\int_B |\nabla \zeta_{\varepsilon, L}|^2 dx - \int_B \frac{u_\varepsilon \zeta_{\varepsilon, L}}{(1 - |x|^2)^2} dx = \lambda_\varepsilon \int_B \zeta_{\varepsilon, L} u_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} dx
$$

$$
\geq \lambda_\varepsilon \int_{B_{r_\varepsilon} R} \zeta_{\varepsilon, L} u_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} dx
$$

$$
= \int_{B_R} \left( \frac{v_\varepsilon}{M_\varepsilon} - \frac{1}{L} \right) + \frac{v_\varepsilon}{M_\varepsilon} e^{(4\pi - \varepsilon)(v_\varepsilon^2 - M_\varepsilon^2)} dx
$$

$$
\to \left(1 - \frac{1}{L}\right) \int_{B_R} e^{8\pi \xi} dx,
$$

when $\varepsilon \to 0$. The convergence in (28) is ensured by (25). Recall that $r_\varepsilon$ is defined by (23). Moreover, one can check easily that

$$
H(u_{\varepsilon, L}) = H(u_\varepsilon) - \int_B |\nabla \zeta_{\varepsilon, L}|^2 dx + \int_B \frac{u_\varepsilon \zeta_{\varepsilon, L}}{(1 - |x|^2)^2} dx,
$$

which, together with (28) and (27), completes the proof of the Lemma, if we let $R \to \infty$. \(\blacksquare\)
Using our subcritical inequality (16) to functions \( \frac{L_{u,L}}{\varepsilon} \), we get
\[
\int_B e^{\pi L^2 u^2_{\varepsilon,L} \varepsilon} dx \leq C < \infty, \quad \text{for } \varepsilon \text{ small enough.}
\]

Furthermore,

**Lemma 7.** We have
\[
\lim_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon}^2 = 0
\]

and
\[
\lim_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon} \int_B u_{\varepsilon} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx = 1.
\]

**Proof.** Let us first estimate \( \| e^{(4\pi-\varepsilon)u_\varepsilon^2} \|_1 \). Fix \( L > 2 \). We deduce
\[
I_{\varepsilon} := \int_{\{L_{u_{\varepsilon}} \leq M_{\varepsilon}\}} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leq \int_B e^{(4\pi-\varepsilon)u_{\varepsilon,L}^2} dx \to \pi, \quad \text{as } \varepsilon \to 0.
\]

The convergence follows from the facts that \( u_{\varepsilon,L} \to 0 \) a.e. in \( B \), estimate (29) and Lemma 2. Using once again the uniform convergence of \( u_{\varepsilon} \) to zero in \( B_{c_r} \) for any \( r \in (0,1) \), we have
\[
\int_{\{L_{u_{\varepsilon}} \leq M_{\varepsilon}\}} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \geq \int_{B_{c_r}} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \to \pi(1-r^2), \quad \text{as } \varepsilon \to 0.
\]

Taking \( r \to 0 \), we have that \( \lim_{\varepsilon \to 0} I_{\varepsilon} = \pi \). On the other hand, there holds
\[
J_{\varepsilon} := \int_{\{L_{u_{\varepsilon}} \geq M_{\varepsilon}\}} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leq \frac{L^2}{\lambda_{\varepsilon} M_{\varepsilon}^2} \int_{\{L_{u_{\varepsilon}} \geq M_{\varepsilon}\}} \lambda_{\varepsilon} u_{\varepsilon}^2 e^{(4\pi-\varepsilon)u_\varepsilon^2} dx
\]
\[
\leq \frac{L^2}{\lambda_{\varepsilon} M_{\varepsilon}^2} \int_B \lambda_{\varepsilon} u_{\varepsilon}^2 e^{(4\pi-\varepsilon)u_\varepsilon^2} dx
\]
\[
= \frac{L^2}{\lambda_{\varepsilon} M_{\varepsilon}^2}.
\]

Finally, we have
\[
\infty = \lim_{\varepsilon \to 0} \int_B e^{(4\pi-\varepsilon)u_\varepsilon^2} dx = \lim_{\varepsilon \to 0} (I_{\varepsilon} + J_{\varepsilon}) \leq \pi + \limsup_{\varepsilon \to 0} \frac{L^2}{\lambda_{\varepsilon} M_{\varepsilon}^2},
\]

which implies \( \liminf_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon}^2 = 0 \). This argument is in fast valid for any subsequence. Hence we obtain (30).

To prove (31), we estimate the integral over three parts separately. First, we have
\[
\lambda_{\varepsilon} M_{\varepsilon} \int_{\{L_{u_{\varepsilon}} \leq M_{\varepsilon}\}} u_{\varepsilon} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leq \lambda_{\varepsilon} M_{\varepsilon}^2 I_{\varepsilon} \to 0, \quad \text{as } \varepsilon \to 0.
\]

Moreover, for any \( R > 0 \), we get
\[
\lambda_{\varepsilon} M_{\varepsilon} \int_{B_{cR}} u_{\varepsilon} e^{(4\pi-\varepsilon)u_\varepsilon^2} dx = \int_{B_R} \frac{\varepsilon}{M_{\varepsilon}} e^{(4\pi-\varepsilon)(v_\varepsilon^2-M_{\varepsilon}^2)} dx \to \int_{B_R} e^{8\pi \xi} dx.
\]
The proof of (31) is completed by tending $R$ to $\infty$.}

Let $g_\varepsilon = M_\varepsilon u_\varepsilon$. It is clear that $g_\varepsilon$ satisfies the following equation

\begin{equation}
L_H(g_\varepsilon) = \lambda_\varepsilon g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} \quad \text{in } \mathcal{D}'(B).
\end{equation}

(31) and its proof shows that $\lambda_\varepsilon g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2}$ converges to the Dirac operator $\delta_0$ in $\mathcal{D}'(B)$. This suggests that $g_\varepsilon$ should tend to the corresponding Green’s function $G_0$, which is confirmed as follows.

**Proposition 3.** When $\varepsilon \to 0$, the family $\{g_\varepsilon\}$ converges to $G_0$ in $W^{1,p}_0(B)$ weakly for $p \in (1,2)$, strongly in $L^q(B)$ for all $q \geq 1$ and also in $C(\overline{B}^c_\varepsilon)$, $\forall \, r \in (0,1)$. Here $G_0$ is defined by Proposition 2.

**Proof.** Since $g_\varepsilon \in \mathcal{H}$, using Proposition 1 on (33) we know that there exist $k_\varepsilon$ and $h_\varepsilon$ such that $g_\varepsilon = h_\varepsilon + k_\varepsilon$ with $h_\varepsilon \in \mathcal{H}$ and $k_\varepsilon \in \cap_{p < 2} W^{1,p}_0(B^\frac{1}{2})$ satisfying that for any $p \in (1,2)$,

$$
\|h_\varepsilon\| + \|\nabla k_\varepsilon\|_p \leq C_p \|\lambda_\varepsilon g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2}\|_1 + C \|\lambda_\varepsilon g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2}\|_{L^2(\Omega_1)} \quad \text{where } \Omega_1 = B^c_\varepsilon.
$$

Since $u_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2}$ tends to zero uniformly in $\Omega_1$, $\|h_\varepsilon\| + \|\nabla k_\varepsilon\|_p$ are uniformly bounded for $\varepsilon$ small. Thus we see (by taking a subsequence) that $h_\varepsilon$ converges weakly to $h_0$ in $\mathcal{H}$ and $k_\varepsilon$ converges weakly to $k_0$ in $W^{1,p}_0(B^\frac{1}{2})$ for $p \in (1,2)$.

Let $g_0 = h_0 + k_0 \in \mathcal{H} + W^{1,p}_0(B^\frac{1}{2})$. Since $\lambda_\varepsilon g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} \to \delta_0$, we have $L_H(g_0) = \delta_0$ in $\mathcal{D}'(B)$. Proposition 2 implies $g_0 = G_0$ and we obtain all statements about the convergence of $\{g_\varepsilon\}$ to $G_0$.

\section{5. Proof of Theorem 1}

**Proof of Theorem 1.** Suppose that Theorem 1 does not hold. Let $\rho \in (0,1)$ be a small constant, which will be determined later. Thanks to Proposition 3, we have

\begin{equation}
\lim_{\varepsilon \to 0} M_\varepsilon u_\varepsilon(\rho) = G_0(\rho), \quad \lim_{\varepsilon \to 0} \int_{B^\rho} a(x)M^2_\varepsilon u^2_\varepsilon dx = \int_{B^\rho} a(x)G^2_0 dx =: J_1(\rho).
\end{equation}

Recall that $a(x) = (1 - |x|^2)^{-2}$. By equation (33), we get

$$
H_{B^\rho}(g_\varepsilon) = H(g_\varepsilon) - H_{B^\rho}(g_\varepsilon) = \lambda_\varepsilon \int_{B^\rho} g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} dx - \int_{\partial B^\rho} \frac{\partial g_\varepsilon}{\partial \nu} g_\varepsilon d\sigma.
$$

It is clear that the first term goes to 0 when $\varepsilon \to 0$. From (31) and (34), we have

$$
- \int_{\partial B^\rho} \frac{\partial g_\varepsilon}{\partial \nu} g_\varepsilon d\sigma = -g_\varepsilon(\rho) \int_{B^\rho} \Delta g_\varepsilon dx = g_\varepsilon(\rho) \left( \int_{B^\rho} a(x)g_\varepsilon dx + \lambda_\varepsilon \int_{B^\rho} g_\varepsilon e^{(4\pi - \varepsilon)u_\varepsilon^2} dx \right)
$$

$$
\to G_0(\rho) \left( \int_{B^\rho} a(x)G_0 dx + 1 \right) =: J_2(\rho).
$$
Finally we have
\begin{equation}
\int_{B_\rho} |\nabla u_\varepsilon|^2 dx = 1 - H_{B_\rho}(u_\varepsilon) + \int_{B_\rho} a(x)u_\varepsilon^2 dx = 1 - \frac{1}{M_\varepsilon^2} \left[ J_2(\rho) - J_1(\rho) + a\varepsilon(1) \right],
\end{equation}
where \( \lim_{\varepsilon \to 0} a\varepsilon(1) = 0 \), for any fixed \( \rho > 0 \).

Furthermore, using the expansion of \( G_0 \), we can show that
\[ J_2(\rho) \sim -\frac{\ln \rho}{2\pi} \quad \text{and} \quad J_1(\rho) \sim \frac{\rho^2 (\ln \rho)^2}{4\pi} \quad \text{as} \ \rho \to 0. \]

Hence there is \( \rho > 0 \) small enough such that \( J_2(\rho) - J_1(\rho) > 0 \). Fixing such a \( \rho \), it is easy to see that (35) implies
\[ \int_{B_\rho} |\nabla u_\varepsilon|^2 dx < 1 \quad \text{for} \ \varepsilon \ \text{small enough}. \]

Applying the classical Moser-Trudinger inequality (1) to \([u_\varepsilon - u_\varepsilon(\rho)]_+ \in H^1_0(B)\), we get
\[ \int_B e^{4\pi |u_\varepsilon - u_\varepsilon(\rho)|^2} dx = \int_B e^{4\pi |u_\varepsilon - u_\varepsilon(\rho)|^2} dx \leq C_{MT}. \]

On the other hand, there holds
\[ u_\varepsilon^2(r) = [u_\varepsilon(r) - u_\varepsilon(\rho)]^2 + 2u_\varepsilon(r)u_\varepsilon(\rho) - u_\varepsilon^2(\rho) \leq [u_\varepsilon(r) - u_\varepsilon(\rho)]^2 + 2M_\varepsilon u_\varepsilon(\rho). \]

Therefore, letting \( \varepsilon \) tend to 0, we have
\[ \int_B e^{4\pi u_\varepsilon^2} dx = \int_{B_\rho} e^{4\pi u_\varepsilon^2} dx + \int_{B_\rho} e^{4\pi u_\varepsilon^2} dx \leq \int_{B_\rho} e^{4\pi |u_\varepsilon - u_\varepsilon(\rho)|^2} dx + 8\pi M_\varepsilon u_\varepsilon(\rho) dx + \pi e^{4\pi u_\varepsilon(\rho)^2} \]
\[ \leq e^{8\pi u_\varepsilon(\rho) C_{MT}} + \pi e^{4\pi u_\varepsilon(\rho)^2} \]
\[ = e^{8\pi G_0(\rho) C_{MT}} + \pi. \]

This contradicts obviously the hypothesis (22), hence the Hardy-Moser-Trudinger inequality must hold true.

\[ \square \]

6. PROOF OF THEOREM 2

Proof of Theorem 2. Let \( u_\varepsilon \) be the maximizer given by Theorem 3. We will prove the Theorem also by a contradiction argument and follow ideas in [10, 2, 23]. We know that we only need to exclude the case \( \lim_{\varepsilon \to 0} \|u_\varepsilon\|_\infty = \infty \). By contradiction we assume \( \lim_{\varepsilon \to 0} \|u_\varepsilon\|_\infty = \infty \).

Define
\[ T_\varepsilon = \int_B e^{(4\pi - \varepsilon)u_\varepsilon^2} dx = \max_{u \in \mathcal{H}, ||u|| \leq 1} \int_B e^{(4\pi - \varepsilon)u^2} dx, \quad \forall \ \varepsilon \in (0, 4\pi). \]

It is easy to see that \( T_\varepsilon \) is increasing and
\[ \lim_{\varepsilon \to 0} T_\varepsilon = \sup_{u \in \mathcal{H}, ||u|| \leq 1} \int_B e^{4\pi u^2} dx := T_0. \]

By Theorem 1, \( T_0 < \infty \). It is trivial to see that \( T_0 > \pi \).

All arguments and properties obtained for \( u_\varepsilon \) in the previous Section are true, except two points. One is the proof of the fact that the weak limit \( u_0 \) is 0 and another is the property (30).

For the former point one can argue as follows. Fixing \( \rho \in (0, 1) \), we see that for any \( L > 1 \), \( u_{\varepsilon,L} = u_\varepsilon \) in \( B_\rho^c \) for \( \varepsilon \) small enough, because \( u_\varepsilon \) is uniformly bounded in \( B_\rho^c \) by Lemma 1 and \( \lim_{\varepsilon \to 0} M_\varepsilon = \infty \). It follows, together with Lemma 6,
\[ \limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(B_\rho^c)} = \limsup_{\varepsilon \to 0} \|u_{\varepsilon,L}\|_{L^\infty(B_\rho^c)} \leq C_\rho \limsup_{\varepsilon \to 0} H(u_{\varepsilon,L}) \leq \frac{C_\rho}{L}. \]

Letting \( L \) tend to \( \infty \), we have \( u_0 = 0 \) in \( B_\rho^c \). Since \( \rho > 0 \) is arbitrary, thus \( u_0 = 0 \).
The latter is no longer true. In fact, now we have

**Lemma 8.** \( \lim_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon}^2 = (T_0 - \pi)^{-1} \).

**Proof.** By the same argument as in the proof of (32), we get

\[
T_0 = \lim_{\varepsilon \to 0} \varepsilon T_\varepsilon \leq \pi + \limsup_{\varepsilon \to 0} \frac{L^2}{\lambda_{\varepsilon} M_{\varepsilon}^2}, \quad \forall L > 2,
\]

which implies that \( \limsup_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon}^2 < \infty \) since \( T_0 > \pi \). Hence \( \lim_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon} = 0 \). Let \( p_1 = \frac{L^2}{4} > 1 \) (as \( L > 2 \)). The estimate (29) and \( \|u_\varepsilon\|_{q} \to 0 \) (for any \( q \geq 1 \)) imply that as \( \varepsilon \to 0 \),

\[
\int_{\{L_{u_\varepsilon} \leq M_{\varepsilon}\}} u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \leq \|e^{(4\pi-\varepsilon)u_\varepsilon^2 \cdot L}\|_{p_1} \|u_\varepsilon\|_{q_1} \to 0 \quad \text{where} \quad \frac{1}{p_1} + \frac{1}{q_1} = 1.
\]

The same argument shows that

\[
\lambda_{\varepsilon} M_{\varepsilon} \int_{\{L_{u_\varepsilon} \leq M_{\varepsilon}\}} u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} dx \to 0, \quad \text{as} \quad \varepsilon \to 0
\]

and (31) holds true. Thus Proposition 3 remains true. Let \( \Omega_{\rho, r} = B_\rho \setminus \overline{B_r} \) with \( 0 < r < \rho < 1 \). Since \( \mathcal{L}_{H}(G_0) = 0 \) in \( \Omega_{\rho, r} \subset \mathbb{R}^2 \), the Pohozaev identity yields (recall that \( a(x) = (1 - |x|^2)^{-2} \))

\[
\int_{\Omega_{\rho, r}} \text{div}[a(x)\frac{x}{2}G_0^2(x)dx - \pi\int G_0^2(s) + a(s)sG_0^2(s)]_r^\rho = 0.
\]

Using the expansion (15) and tending \( r \to 0 \), we have

\[
\int_{B_\rho} \text{div}[a(x)\frac{x}{2}G_0^2(x)dx - \pi\rho^2 G_0^2(\rho) - \pi a(\rho)\rho^2 G_0^2(\rho) = -\frac{1}{4\pi}, \quad \forall \rho \in (0, 1).
\]

Similarly, applying the Pohozaev identity to \( \mathcal{L}_{H}(u_\varepsilon) = \lambda_{\varepsilon} u_\varepsilon e^{(4\pi-\varepsilon)u_\varepsilon^2} \) in \( B_\rho \) and multiplying by \( M_{\varepsilon}^2 \), we obtain that for any \( \rho \in (0, 1) \),

\[
\int_{B_\rho} \text{div}[a(x)\frac{x}{2}g_\varepsilon^2(x)dx - \pi\rho^2 g_\varepsilon^2(\rho) - \pi a(\rho)\rho^2 g_\varepsilon^2(\rho) = \lambda_{\varepsilon} M_{\varepsilon}^2 \left[ \pi\rho^2 \frac{e^{(4\pi-\varepsilon)u_\varepsilon(\rho)^2}}{4\pi - \varepsilon} - \int_{B_\rho} e^{(4\pi-\varepsilon)u_\varepsilon^2 dx} \right].
\]

Finally, since \( g_\varepsilon \) converges to \( G_0 \) in \( C^1_{\text{loc}}(B \setminus \{0\}) \) and \( L^2(B) \), by the standard elliptic theory and Proposition 3, we obtain, for any \( \rho \in (0, 1) \) (as \( \limsup_{\varepsilon \to 0} \lambda_{\varepsilon} M_{\varepsilon}^2 < \infty \)),

\[
\lambda_{\varepsilon} M_{\varepsilon}^2 \left[ \int_{B_\rho} e^{(4\pi-\varepsilon)u_\varepsilon^2 dx} - \pi \rho^2 \right] \to 1, \quad \text{as} \quad \varepsilon \to 0.
\]

Taking \( \rho \to 1 \) and using the uniform convergence of \( u_\varepsilon \) to 0 in \( \overline{B_r} \) for \( r > 0 \), we have the conclusion of the Lemma.

To get a contradiction, we proceed as in [10, 23]. We first claim a Carleson-Chang type result.

**Lemma 9.** If \( \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{\infty} = \infty \), then \( T_0 \leq \pi(1 + e^{(4\pi-\varepsilon)C_G}) \) where \( C_G \) is given by (15).

**Proof.** Fix \( L > 2 \) and let \( R > 0 \). Using (29) and Lemma 2, we have the estimate for the exterior region as follows:

\[
\int_{\{L_{u_\varepsilon} \leq M_{\varepsilon}\}} e^{(4\pi-\varepsilon)u_\varepsilon^2 dx} \leq \int_{B} e^{(4\pi-\varepsilon)u_\varepsilon^2 L} dx \to \pi, \quad \text{as} \quad \varepsilon \to 0.
\]
On the neck region, there holds
\[
\int_{\{ Lu \geq M\} \setminus B_{r_0} R} e^{(4\pi - \varepsilon)u^2} dx \leq \frac{L^2}{\lambda_\varepsilon M^2} \int_{\{ Lu \geq M\} \setminus B_{r_0} R} \lambda_\varepsilon u_\varepsilon^2 e^{(4\pi - \varepsilon)u_\varepsilon^2} dx
\]
\[
\leq \frac{L^2}{\lambda_\varepsilon M^2} \int_{B \setminus B_{r_0} R} \lambda_\varepsilon u_\varepsilon^2 e^{(4\pi - \varepsilon)u_\varepsilon^2} dx
\]
\[
= \frac{L^2}{\lambda_\varepsilon M^2} \left( 1 - \int_{B_{r_0} R} \lambda_\varepsilon u_\varepsilon^2 e^{(4\pi - \varepsilon)u_\varepsilon^2} dx \right)
\rightarrow L^2(T_0 - \pi) \left( 1 - \int_{B_R} e^{8\pi\xi} dx \right).
\]

For the integral over the interior region $B_{r_0} R$, fix a small constant $\rho \in (0, 1)$. By (35), we know that
\[
\int_{B_{r_0} \setminus B_{\rho}} |\nabla u_\varepsilon|^2 dx = 1 - \frac{E_\rho + o_\varepsilon(1)}{M^2},
\]
where $\lim_{\varepsilon \to 0} o_\varepsilon(1) = 0$ and
\[
E_\rho := J_2(\rho) - J_1(\rho) = G_0(\rho) + G_0(\rho) \int_{B_{r_0}} a(x)G_0 dx - \int_{B_{r_0}} a(x)G_0 dx.
\]
Let $\ell_\varepsilon(x) = ||\nabla u_\varepsilon||^2_{L^2(B_{r_0})} [u_\varepsilon(x) - u_\varepsilon(\rho)]$. Clearly, $\ell_\varepsilon \in H^1_0(B_{r_0})$, $||\nabla \ell_\varepsilon||_2 = 1$ and $\ell_\varepsilon$ converges weakly to 0 in $D'(B_{r_0})$. Using a result of Carleson-Chang [10], there holds
\[
\limsup_{\varepsilon \to 0} \int_{B_{r_0}} \left( e^{4\pi\ell_\varepsilon^2} - 1 \right) dx \leq \pi e \rho^2.
\]
Moreover, we know $M_\varepsilon^{-1} \ell_\varepsilon \to 1$ uniformly in $B_{r_0} R$ since $M_\varepsilon^{-1} u_\varepsilon \to 1$ uniformly in $B_R$ by (25). Therefore, we have
\[
u^2(x) = \left[ \ell_\varepsilon(x) + u_\varepsilon(\rho) \right]^2 \left[ \nabla u_\varepsilon \right]_{L^2(B_{r_0})}^2
\]
\[
\left[ \ell_\varepsilon(x) + u_\varepsilon(\rho) \right]^2 \left[ \nabla u_\varepsilon \right]_{L^2(B_{r_0})}^2 \times \left[ 1 - M_\varepsilon^{-2} E_\rho + o_\varepsilon(M_\varepsilon^{-2}) \right]
\]
\[
= \ell_\varepsilon^2(x) + 2\ell_\varepsilon(x) M_\varepsilon^{-1} G_0(\rho) - \ell_\varepsilon^2(x) M_\varepsilon^{-2} E_\rho + o_\varepsilon(1)
\]
\[
= \ell_\varepsilon^2(x) + 2G_0(\rho) - E_\rho + o_\varepsilon(1),
\]
where $o_\varepsilon(1)$ tends to 0 uniformly in $B_{r_0} R$ as $\varepsilon$ goes to 0. It implies that, together with (36),
\[
\limsup_{\varepsilon \to 0} \int_{B_{r_0} R} e^{(4\pi - \varepsilon)u_\varepsilon^2} dx \leq \limsup_{\varepsilon \to 0} \int_{B_{r_0} R} \left( e^{4\pi\ell_\varepsilon^2} - 1 \right) dx
\]
\[
\leq e^{8\pi G_0(\rho) - 4\pi E_\rho} \limsup_{\varepsilon \to 0} \int_{B_{r_0} R} \left( e^{4\pi\ell_\varepsilon^2} - 1 \right) dx
\]
\[
\leq e^{8\pi G_0(\rho) - 4\pi E_\rho} \limsup_{\varepsilon \to 0} \int_{B_\rho} \left( e^{4\pi\ell_\varepsilon^2} - 1 \right) dx
\]
\[
\leq \pi \rho^2 e^{1 + 8\pi G_0(\rho) - 4\pi E_\rho}.
\]
Combining the three parts of estimation and letting $R$ tend to $\infty$, we conclude
\[
T_0 = \lim_{\varepsilon \to 0} \int_{B} e^{(4\pi - \varepsilon)u_\varepsilon^2} dx \leq \pi + \pi \rho^2 e^{1 + 8\pi G_0(\rho) - 4\pi E_\rho}, \quad \text{for any small } \rho > 0.
\]
Using the expansion (15), we have
\[
\frac{1}{2\pi} \ln \rho + 2G_0(\rho) - E_\rho \to C_G, \quad \text{as } \rho \to 0.
\]
Hence it follows $T_0 \leq \pi (1 + e^{1+4\pi C_G})$. \[\square\]
We complete the proof of Theorem 2 with the following lower bound estimate, which contradicts Lemma 9.

**Lemma 10.** There holds $T_0 > \pi(1 + e^{1+4\pi C\gamma})$.

**Proof.** The proof is a direct verification by choosing suitable test functions as in [10]. Thanks to the blow-up analysis, we will consider a family $f_\varepsilon$ such that $f_\varepsilon$ looks like $M^{-1}_\varepsilon G_0$ outside a very small region of 0 and $M^{-1}_\varepsilon \xi(r^{-1}) + M_\varepsilon$ near the origin where $\xi$ is given by (27). For $\varepsilon > 0$ small, define

$$f_\varepsilon(r) = \begin{cases} \beta_\varepsilon + \frac{\xi(\varepsilon^{-1}r) + \gamma_\varepsilon}{\beta_\varepsilon} & \text{if } r \leq \varepsilon R_\varepsilon \\
G_0(r) & \text{if } \varepsilon R_\varepsilon \leq r \leq 1 \end{cases} \quad \text{with } R_\varepsilon = -\ln \varepsilon.$$

Here $\beta_\varepsilon$ and $\gamma_\varepsilon$ are constants to be chosen later. First, choose $\gamma_\varepsilon$ such that

$$G_0(\varepsilon R_\varepsilon) = \frac{\beta_\varepsilon + \xi(\varepsilon^{-1}r) + \gamma_\varepsilon}{\beta_\varepsilon},$$

which makes functions $f_\varepsilon$ continuous. Using the expansion of $G_0$, as $\varepsilon \to 0$, we have

$$4\pi (\beta_\varepsilon^2 + \gamma_\varepsilon) = -2\ln (\varepsilon R_\varepsilon) + 4\pi C_G + \ln(1 + \pi R^2_\varepsilon) + o(\varepsilon R_\varepsilon)$$

(37)

$$= -2\ln \varepsilon + 4\pi C_G + \ln \pi + O(\varepsilon R^{-2}_\varepsilon).$$

Clearly, $f_\varepsilon \in \mathcal{H}$. Now we estimate $\|f_\varepsilon\|$. Let $0 < r < \rho < 1$, by the equation of $G_0$,

$$H_{B_\rho \setminus B_r}(G_0) = \int_{\partial(B_\rho \setminus B_r)} G_0 \frac{\partial G_0}{\partial \nu} d\sigma = -2\pi r G_0(r)G'_0(r) + 2\pi \rho G_0(\rho)G'_0(\rho) \leq -2\pi r G_0(r)G'_0(r),$$

since $G_0$ is decreasing by the comparison principle. Taking $\rho \to 1$, we get

$$H_{B_\varepsilon R_\varepsilon}(f_\varepsilon) = -\frac{2\pi \varepsilon R_\varepsilon}{\beta_\varepsilon^2} G_0(\varepsilon R_\varepsilon)G'_0(\varepsilon R_\varepsilon)$$

$$\leq -\frac{2\pi \varepsilon R_\varepsilon}{\beta_\varepsilon^2} \int_{B_{\varepsilon R_\varepsilon}} \frac{\partial G_0}{\partial \nu} d\sigma$$

$$= \frac{G_0(\varepsilon R_\varepsilon)}{\beta_\varepsilon^2} \left(1 + \int_{B_{\varepsilon R_\varepsilon}} a(x) G_0 dx\right)$$

$$= \frac{1}{4\pi \beta_\varepsilon^2} \left[-2\ln(\varepsilon R_\varepsilon) + 4\pi C_G + o(\varepsilon R_\varepsilon)\right].$$

On the other hand, we have

$$\int_{B_{\varepsilon R_\varepsilon}} |\nabla f_\varepsilon|^2 dx = \frac{1}{\beta_\varepsilon^2} \int_{B_{\varepsilon R_\varepsilon}} |\nabla \xi(\varepsilon^{-1}x)|^2 dx = \frac{1}{\beta_\varepsilon^2} \int_{B_r} |\nabla \xi|^2 dx$$

$$= \frac{1}{4\pi \beta_\varepsilon^2} \left[\ln(1 + \pi R^2_\varepsilon) - 1 + \frac{1}{1 + \pi R^2_\varepsilon}\right],$$

and hence

$$H(f_\varepsilon) \leq H_{B_{\varepsilon R_\varepsilon}}(f_\varepsilon) + \int_{B_{\varepsilon R_\varepsilon}} |\nabla f_\varepsilon|^2 dx \leq \frac{1}{4\pi \beta_\varepsilon^2} \left[-2\ln \varepsilon + 4\pi C_G - 1 + \ln \pi + O\left(R^{-2}_\varepsilon\right)\right].$$

(38) \hspace{1cm} H(f_\varepsilon) \leq \frac{1}{4\pi \beta_\varepsilon^2} \left[-2\ln \varepsilon + 4\pi C_G - 1 + \ln \pi + O\left(R^{-2}_\varepsilon\right)\right].

We choose $\beta_\varepsilon > 0$ such that $H(f_\varepsilon) = 1$. The estimate (38) leads to (recall that $R_\varepsilon = -\ln \varepsilon$)

$$4\pi \beta_\varepsilon^2 \leq -2\ln \varepsilon + 4\pi C_G - 1 + \ln \pi + O\left(R^{-2}_\varepsilon\right) = O\left(|\ln \varepsilon|\right), \ \text{as } \varepsilon \to 0.$$

From (37), it follows

$$4\pi \gamma_\varepsilon \geq 1 + O\left(R^{-2}_\varepsilon\right), \quad \beta_\varepsilon = O\left(|\ln \varepsilon|^{1/2}\right) \ \text{as } \varepsilon \to 0.$$
Now we estimate $\|e^{4\pi f_2^2}\|_1$. Using $e^{t} \geq 1 + t$ in $\mathbb{R}$ and (39), we have

$$
\int_{B_{\varepsilon R_{\varepsilon}}} e^{4\pi f_2^2} dx \geq \pi - \pi(\varepsilon R_{\varepsilon})^2 + \frac{4\pi}{\beta_\varepsilon^2} \int_{B_{\varepsilon R_{\varepsilon}}} G_0^2 dx = \pi + \frac{4\pi}{\beta_\varepsilon^2} \left[ \int_B G_0^2 dx + o(\varepsilon R_{\varepsilon}) - \frac{\varepsilon^2 R_{\varepsilon}^2 \beta_\varepsilon^2}{4} \right]
$$

Moreover, in $B_{\varepsilon R_{\varepsilon}}$, by (37) and (39), there holds

$$
4\pi f_2^2(r) = 4\pi \left( \beta_\varepsilon + \xi \frac{e^{-1}r + \gamma_\varepsilon}{\beta_\varepsilon} \right)^2 \geq 4\pi \beta_\varepsilon^2 + 8\pi \gamma_\varepsilon + 8\pi \xi (e^{-1}r)
$$

$$
= 4\pi (\beta_\varepsilon^2 + \gamma_\varepsilon) + 4\pi \gamma_\varepsilon + 8\pi \xi (e^{-1}r)
$$

$$
\geq -2\ln \varepsilon + 4\pi C_G + \ln \pi + 1 + 8\pi \xi (e^{-1}r) + O \left( R_{\varepsilon}^{-2} \right).
$$

The above estimate is uniform in $B_{\varepsilon R_{\varepsilon}}$. Consequently we have

$$
\int_{B_{\varepsilon R_{\varepsilon}}} e^{4\pi f_2^2} dx \geq e^{-2\ln \varepsilon + 4\pi C_G + \ln \pi + 1 + O \left( R_{\varepsilon}^{-2} \right)} \int_{B_{\varepsilon R_{\varepsilon}}} e^{8\pi \xi (e^{-1}r)} dx
$$

$$
= \pi e^{4\pi C_G + 1 + O \left( R_{\varepsilon}^{-2} \right)} \int_{B_{\varepsilon R_{\varepsilon}}} e^{8\pi \xi dx}
$$

$$
= \pi e^{4\pi C_G + 1} \left[ 1 + O \left( R_{\varepsilon}^{-2} \right) \right],
$$

where (27) has been used in the last equality. Finally since $R_{\varepsilon}^{-2} \beta_\varepsilon^2 = o_\varepsilon(1)$, it holds

$$
\int_B e^{4\pi f_2^2} dx \geq \pi + \frac{4\pi}{\beta_\varepsilon^2} \left[ \int_B G_0^2 dx + o_\varepsilon(1) \right] + \pi e^{4\pi C_G + 1} \left[ 1 + O \left( R_{\varepsilon}^{-2} \right) \right]
$$

$$
= \pi + \pi e^{4\pi C_G + 1} + \frac{4\pi}{\beta_\varepsilon^2} \left[ \int_B G_0^2 dx + o_\varepsilon(1) \right].
$$

By choosing a small $\varepsilon > 0$, we conclude readily $T_0 \geq \|e^{4\pi f_2^2}\|_1 > \pi + \pi e^{4\pi C_G + 1}$. Hence we finish the proof of Lemma 10, and hence the proof of Theorem 2. $\blacksquare$

**Remark 7.** Like $u_\varepsilon$, the maximizer $u_0 \in H_1$ given by Theorem 2 cannot belong to $H_0^1(B)$.

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