Quenched law of large numbers and quenched central limit theorem for multi-player leagues with ergodic strengths

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Abstract

We propose and study a new model for competitions, specifically sports multi-player leagues where the initial strengths of the teams are independent i.i.d. random variables that evolve during different days of the league according to independent ergodic processes. The result of each match is random: the probability that a team wins against another team is determined by a function of the strengths of the two teams in the day the match is played.

Our model generalizes some previous models studied in the physical and mathematical literature and is defined in terms of different parameters that can be statistically calibrated. We prove a quenched – conditioning on the initial strengths of the teams – law of large numbers and a quenched central limit theorem for the number of victories of a team according to its initial strength.

To obtain our results, we prove a theorem of independent interest. For a stationary process \(\xi = (\xi_i)_{i \in \mathbb{N}}\) satisfying a mixing condition and an independent sequence of i.i.d. random variables \((s_i)_{i \in \mathbb{N}}\), we prove a quenched – conditioning on \((s_i)_{i \in \mathbb{N}}\) – central limit theorem for sums of the form \(\sum_{i=1}^{n} g(\xi_i, s_i)\), where \(g\) is a bounded measurable function. We highlight that the random variables \(g(\xi_i, s_i)\) are not stationary conditioning on \((s_i)_{i \in \mathbb{N}}\).

1 Introduction

Multi-player competitions are a recurrent theme in physics, biology, sociology, and economics since they model several phenomena. We refer to the introduction of [BNH07] for various examples of models of multi-player competitions related to evolution, social stratification, business world and other related fields.

Among all these fields, sports are indubitably one of the most famous examples where modelling competitions is important. This is the case for two reasons: first, there is accurate and easily accessible data; second, sports competitions are typically head-to-head, and so they can be viewed as a nice model of multi-player competitions with binary interactions.

Sports multi-player leagues where the outcome of each game is completely deterministic have been studied for instance by Ben-Naim, Kahng, and Kim [BNKK06] (see also references therein and explanations on how this model is related to urn models). Later, Ben-Naim and Hengartner [BNH07] investigated how randomness affects the outcome of multi-player competitions. In their model, the considered randomness is quite simple: they start with \(N\) teams ranked from best to worst and in each game they assume that the weaker team can defeat the stronger team with a fixed probability. They investigate the total number of games needed for the best team to win the championship with high probability.

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A more sophisticated model was studied by Benaîm, Benjamini, Chen, and Lima [BBCL15]: they consider a finite connected graph, and place a team at each vertex of the graph. Two teams are called a pair if they share an edge. At discrete times, a match is played between each pair of teams. In each match, one of the teams defeats the other (and gets a point) with probability proportional to its current number of points raised to some fixed power \( \alpha > 0 \). They characterize the limiting behavior of the proportion of points of the teams.

In this paper we propose a more realistic model for sport multi-player leagues that can be briefly described as follows (for a precise and formal description see Section 1.1.1): we start with \( 2n \) teams having i.i.d. initial strengths. Then we consider a calendar of the league composed by \( 2n - 1 \) different days, and we assume that on each day each team plays exactly one match against another team in such a way that at the end of the calendar every team has played exactly one match against every other team (this coincides with the standard way how calendars are built in real football-leagues for the first half of the season). Moreover, we assume that on each day of the league, the initial strengths of the teams are modified by independent ergodic processes. Finally, we assume that a team wins against another team with probability given by a chosen function of the strengths of the two teams in the day the match is played.

We prove (see Section 1.4 for precise statements) a quenched law of large numbers and a quenched central limit theorem for the number of victories of a team according to its initial strengths. Here quenched means that the results hold a.s. conditioning on the initial strengths of the teams in the league.

### 1.1 The model

We start by fixing some simple and standard notation.

**Notation.** Given \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \), we set \([n] = \{1, 2, \ldots, n\} \) and \([n]_0 = \{0, 1, 2, \ldots, n\} \). We also set \( \mathbb{R}_+ = \mathbb{R} \cap [0, \infty) \).

We refer to random quantities using **bold** characters. Convergence in distribution is denoted by \( \overset{d}{\to} \), almost sure convergence is denote by \( \overset{a.s.}{\to} \), and convergence in probability by \( \overset{P}{\to} \).

Given a collection of sets \( \mathcal{A} \), we denote by \( \sigma(\mathcal{A}) \) and \( \lambda(\mathcal{A}) \) the \( \sigma \)-algebra and the monotone class generated by \( \mathcal{A} \), respectively. Given a random variable \( X \), we denote by \( \sigma(X) \) the \( \sigma \)-algebra generated by \( X \).

For any real-valued function \( f \) we denote with \( f^2 \) the function such that \( f^2(\cdot) = (f(\cdot))^2 \).

We can now proceed with the precise description of our model for multi-player leagues.

**The model.** We consider a league of \( 2n \in 2\mathbb{N} \) teams denoted by \( \{T_i\}_{i \in [2n-1]} \) whose *initial random strengths* are denoted by \( \{s_i\}_{i \in [2n-1]} \in \mathbb{R}^{2n} \).

In the league every team \( T_i \) plays \( 2n - 1 \) matches, one against each of the remaining teams \( \{T_j\}_{j \in [2n-1] \setminus \{i\}} \). Note that there are in total \( \binom{2n}{2} \) matches in the league. These matches are played in \( 2n - 1 \) different days in such a way that each team plays exactly one match every day.

For all \( i \in [2n-1] \setminus \{i\} \), the initial strength \( s_i \) of the team \( T_i \) is modified every day according to a discrete time \( \mathbb{R}_+ \)-valued stochastic process \( \zeta^i = (\zeta^i_j)_{j \in \mathbb{N}} \). More precisely, the strength of the team \( T_i \) on the \( p \)-th day is equal to \( s_i \cdot \zeta^i_p \in \mathbb{R}_+ \).

We now describe the rules for determining the winner of a match in the league. We fix a function \( f : \mathbb{R}_+^2 \to [0, 1] \) that controls the winning probability of a match between two teams given their strengths. When a team with strength \( x \) plays a match against another team with strength \( y \), its probability of winning the match is equal to \( f(x, y) \) and its probability of loosing is equal to \( 1 - f(x, y) \) (we are excluding the possibility of having a draw). Therefore, if the match between the teams \( T_i \) and \( T_j \) is played the \( p \)-th day, then, conditionally on the random variables \( s_i, s_j, \zeta^i_p, \zeta^j_p \), the probability that \( T_i \) wins is \( f(s_i \cdot \zeta^i_p, s_j \cdot \zeta^j_p) \). Moreover, conditionally on the strengths of the teams, the results of different matches are independent.
1.2 Goal of the paper

The goal of this work is to study the model defined above when the number $2n$ of teams in the league is large. We want to look at the limiting behavior of the number of wins of a team with initial strength $s \in \mathbb{R}_+$ at the end of the league. More precisely, given $s \in \mathbb{R}_+$, we assume w.l.o.g. that the team $T_0$ has deterministic initial strength $s$, i.e. $s_0 = s$ a.s., and we set

$$W_n(s) := \text{Number of wins of the team } T_0 \text{ at the end of a league with } 2n \text{ players.}$$

We investigate a quenched law of large numbers and a quenched central limit theorem for $W_n(s)$.

1.3 Our assumptions

In the following two subsections we state some assumptions on the model.

1.3.1 Assumptions for the law of large numbers

We make the following natural assumptions$^1$ on the function $f : \mathbb{R}_+^2 \to [0,1] :$

- $f(x,y)$ is measurable;
- $f(x,y)$ is weakly-increasing in the variable $x$ and weakly-decreasing in the variable $y$.

Recall also that it is not possible to have a draw, i.e. $f(x,y) + f(y,x) = 1$, for all $x, y \in \mathbb{R}_+$.

Before describing our additional assumptions on the model, we introduce some further quantities. Fix a Borel probability measure $\nu$ on $\mathbb{R}_+$; let $\xi = (\xi_\ell)_{\ell \in \mathbb{N}}$ be a discrete time $\mathbb{R}_+$-valued stochastic process such that

$$\xi_\ell \overset{d}{=} \nu, \quad \text{for all } \ell \in \mathbb{N}, \quad (1)$$

and$^2$

$$(\xi_\ell, \xi_k) \overset{d}{=} (\xi_{\ell+\tau}, \xi_{k+\tau}), \quad \text{for all } \ell, k, \tau \in \mathbb{N}. \quad (2)$$

We further assume that the process $\xi$ is weakly-mixing, that is, for every $A \in \sigma(\xi_1)$ and every collection of sets $B_\ell \in \sigma(\xi_\ell)$, it holds that

$$\frac{1}{n} \sum_{\ell=1}^n |\mathbb{P}(A \cap B_\ell) - \mathbb{P}(A)\mathbb{P}(B_\ell)| \overset{n \to \infty}{\longrightarrow} 0. \quad (3)$$

The additional assumptions on our model are the following:

- For all $i \in [2n-1]_0$, the stochastic processes $\xi^i$ are independent copies of $\xi$.
- The initial random strengths $\{s_i\}_{i \in [2n-1]}$ of the teams different than $T_0$ are i.i.d. random variables on $\mathbb{R}_+$ with distribution $\mu$, for some Borel probability measure $\mu$ on $\mathbb{R}_+$.
- The initial random strengths $\{s_i\}_{i \in [2n-1]}$ are independent of the processes $\{\xi^i\}_{i \in [2n-1]}$ and of the process $\xi$.

$^1$The second hypothesis is not needed to prove our results but it is very natural for the model.

$^2$This is a weak-form of the stationarity property for stochastic processes.
1.3.2 Further assumptions for the central limit theorem

In order to prove a central limit theorem, we need to make some stronger assumptions. The first assumption concerns the mixing properties of the process $\xi$. For $k \in \mathbb{N}$, we introduce the two $\sigma$-algebras $\mathcal{A}_k^0 = \sigma(\xi_1, \ldots, \xi_k)$ and $\mathcal{A}_k^\infty = \sigma(\xi_k, \ldots)$ and we define for all $n \in \mathbb{N}$,

$$
\alpha_n = \sup_{k \in \mathbb{N}} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right|.
$$

(4)

We assume that

$$
\sum_{n=1}^{\infty} \alpha_n < \infty.
$$

(5)

Note that this condition, in particular, implies that the process $\xi$ is strongly mixing, that is, $\alpha_n \to 0$ as $n \to \infty$.

Finally, we assume that there exist two sequences $p = p(n)$ and $q = q(n)$ such that:

- $p \xrightarrow{n \to \infty} +\infty$ and $q \xrightarrow{n \to \infty} +\infty$,
- $q = o(p)$ and $p = o(n)$ as $n \to \infty$,
- $np^{-1} \alpha_q = o(1)$,
- $\frac{p}{n} \cdot \sum_{j=1}^{p} j \alpha_j = o(1)$.

Remark 1.1. The latter assumption concerning the existence of the sequences $p$ and $q$ is not very restrictive. For example, simply assuming that $\alpha_n = O\left(\frac{1}{n \log(n)}\right)$ ensures that the four conditions are satisfied for $p = \sqrt{\frac{n}{\log \log n}}$ and $q = \frac{\sqrt{n}}{\log \log n}$. Indeed, in this case, the first two conditions are trivial, the fourth one follows by noting that $\sum_{j=1}^{p} j \alpha_j = O(p)$ thanks to the assumption in Eq. (5), and finally the third condition follows by standard computations.

Remark 1.2. Note also that as soon as $p = O\left(\sqrt{n}\right)$ then the fourth condition is immediately verified. Indeed, $\sum_{j=1}^{p} j \alpha_j \leq \sqrt{p} \sum_{j=1}^{\sqrt{p}} \alpha_j + p \sum_{j=\sqrt{p}}^{p} \alpha_j = o(p)$.

1.4 Main results

1.4.1 Results for our model of multi-player leagues

Let $V, V', U, U'$ be four independent random variables such that $V \overset{d}{=} V' \overset{d}{=} \nu$ and $U \overset{d}{=} U' \overset{d}{=} \mu$. Given a deterministic sequence $s = (s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^\mathbb{N}$, denote by $\mathbb{P}_s$ the law of the random variable $W_n(s) \overset{2n}{\sim} \mathbb{P}$ when the initial strengths of the teams $(T_{i})_{i \in [2n-1]}$ are equal to $s = (s_i)_{i \in [2n-1]}$, i.e. we study $W_n(s) \overset{2n}{\sim}$ on the event

$s_0 = s$ and $(s_i)_{i \in [2n-1]} = (s_i)_{i \in [2n-1]}$.

Theorem 1.3. (Quenched law of large numbers) Suppose that the assumptions in Section 1.3.1 hold. Fix any $s \in \mathbb{R}_+$. For $\mu^\mathbb{N}$-almost every sequence $s = (s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^\mathbb{N}$, under $\mathbb{P}_s$ the following convergence holds

$$
\frac{W_n(s)}{2n} \xrightarrow{n \to \infty} \ell(s),
$$

where

$$
\ell(s) = \mathbb{E}\left[f(s \cdot V, U \cdot V')\right] = \int_{\mathbb{R}_+^2} f(s \cdot v, u \cdot v') d\nu(v) d\nu(v') d\mu(u).
$$

We now state our second result.
**Theorem 1.4.** (Quenched central limit theorem) Suppose that the assumptions in Section 1.3.1 and Section 1.3.2 hold. Fix any $s \in \mathbb{R}_+$. For $\mu^N$-almost every sequence $s = (s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^N$, under $\mathbb{P}_s$ the following convergence holds

$$\frac{W_n(s) - \mathbb{E}_s[W_n(s)]}{\sqrt{2n}} \xrightarrow{d} \mathcal{N}(0, \sigma(s)^2 + \rho(s)^2),$$

where, for $F_s(x, y) := \mathbb{E}[f(s \cdot x, y \cdot V')]$ and $\tilde{F}_s(x, y) := F_s(x, y) - \mathbb{E}[F_s(V, y)]$,

$$\sigma(s)^2 = \mathbb{E} \left[ F_s(V, U) - (F_s(V, U))^2 \right] = \ell(s) - \mathbb{E} \left[ (F_s(V, U))^2 \right]$$

and

$$\rho(s)^2 = \mathbb{E} \left[ \tilde{F}_s(V, U)^2 \right] + 2 \sum_{k=1}^{\infty} \mathbb{E} \left[ \tilde{F}_s(\xi_1, U) \tilde{F}_s(\xi_{1+k}, U') \right],$$

the last series being convergent.

**Remark 1.5.** The assumption that the initial strengths of the teams $(s_i)_{i \in \mathbb{N}}$ are independent could be relaxed from a theoretical point of view, but we believe that this is a very natural assumption for our model.

### 1.4.2 Originality of our results and analysis of the limiting variance: a quenched CLT for functionals of ergodic processes and i.i.d. random variables

We now comment on the originality of our results and contextualize them in relation to the established literature and prior studies on sums of dependent and not equi-distributed random variables. Additionally, we give some informal explanations on the two components $\sigma(s)^2$ and $\rho(s)^2$ of the variance of the limiting Gaussian random variable in Eq. (6).

We start by noticing that, without loss of generality, we can assume that for every $j \in [2n-1]$, the team $T_0$ plays against the team $T_j$ the $j$-th day of the league. Denoting by $W_j = W_j(s)$ the event that the team $T_0$ wins against the team $T_j$, then $W_n(s)$ rewrites as

$$W_n(s) = \sum_{j=1}^{2n-1} I_{W_j}.$$  

Note that the Bernoulli random variables $(I_{W_j})_{j \in [2n-1]}$ are only independent conditionally on the process $(\xi_j^0)_{j \in [2n-1]}$. In addition, under our assumptions, we have that the conditional parameters of the Bernoulli random variables are given by $\mathbb{P}_s \left( W_j | \xi_j^0, s_j \right) = F_s(s_j, s_j)$. Therefore, under $\mathbb{P}_s$, the random variable $W_n(s)$ is a sum of Bernoulli random variables that are neither independent nor identically distributed. As a consequence, the proofs of the quenched law of large numbers (see Section 2) and of the quenched central limit theorem (see Section 3) do not follow from a simple application of classical results in the literature.

We recall that it is quite simple to relax the identically distributed assumption in order to obtain a central limit theorem: the Lindeberg criterion, see for instance [Bil95, Theorem 27.2], gives a sufficient (and almost necessary) criterion for a sum of independent random variables to converge towards a Gaussian random variable (after suitable renormalization). Relaxing independence is more delicate and there is no universal theory to do it. In the present paper, we combine two well-known theories to obtain our results: the theory for stationary ergodic processes (see for instance [Ros56, Ibr62, IL71, Pel92, Bra07]) and the theory for $m$-dependent random variables (see for instance [HR48, Ber73, RW00]) and dependency graphs (see for instance [PL82, Jan88, BR89, Jan04, FMN20]).
To explain the presence of the two terms in the variance, we start by noticing that using the law of total conditional variance, the conditional variance of \( \mathbb{1}_{W_j} \) is given by
\[
\text{Var}_{x} \left( \mathbb{1}_{W_j} | \xi_j^0 \right) = F_x(\xi_j^0, s_j) - (F_x(\xi_j^0, s_j))^2.
\]
The term \( \sigma(s)^2 \) arises as the limit of
\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \text{Var}_{x} \left( \mathbb{1}_{W_j} | \xi_j^0 \right),
\]
and this is in analogy with the case of sums of independent but not identically distributed Bernoulli random variables. The additional term \( \rho(s)^2 \), on the contrary, arises from the fluctuations of the conditional parameters of the Bernoulli variables, i.e. \( \rho(s)^2 \) comes from the limit of
\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \text{Var} \left( \mathbb{P}_x(W_j | \xi_j^0) \right) = \frac{1}{2n} \sum_{j=1}^{2n-1} \text{Var} \left( F_x(\xi_j^0, s_j) \right).
\]
Note that an additional difficulty arises from the fact that the sums in the last two equations are not independent (but we will show that they are asymptotically independent).

To study the limit in Eq. (9) we prove in Section 3 the following general result that we believe to be of independent interest.

**Theorem 1.6.** Suppose that the assumptions in Eqs. (1) and (2), and in Section 1.3.2 hold. Let \( g : \mathbb{R}^2_+ \to \mathbb{R} \) be a bounded, measurable function, and define \( \tilde{g} : \mathbb{R}^2_+ \to \mathbb{R} \) by \( \tilde{g}(x, y) := g(x, y) - \mathbb{E}[g(V, y)] \). Then, the quantity
\[
\rho^2 := \mathbb{E}[\tilde{g}(V, U)^2] + 2 \cdot \sum_{k=1}^{\infty} \mathbb{E}[\tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U')]
\]
is finite. Moreover, for \( \mu^\mathbb{N} \)-almost every sequence \( (s_i)_{i \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}_+ \), the following convergence holds
\[
\sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \quad \overset{d}{\to} \quad \mathcal{N}(0, \rho^2).
\]

The main difficulty in studying the sum \( \sum_{j=1}^{2n-1} g(\xi_j, s_j) \) is that fixing a deterministic realization \( (s_i)_{i \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}_+ \) imposes that the random variables \( g(\xi_j, s_j) \) are not stationary anymore.

In many of the classical results available in the literature (see references given above) – in addition to moment and mixing conditions – stationarity is assumed, and so we cannot directly apply these results. An exception are [PRW97, Eks14], where stationarity is not assumed but it is assumed a stronger version \(^3\) of our condition in Eq. (5). Therefore, to the best of our knowledge, Theorem 1.6 does not follow from known results in the literature.

### 1.5 Calibration of the parameters of the model and some examples

An interesting feature of our model consists in the fact that the main parameters that characterize the evolution of the league can be statistically calibrated in order for the model to describe real-life tournaments. Such parameters are:

- The function \( f : \mathbb{R}^2_+ \to [0, 1] \) that controls the winning probability of the matches.
- The distribution \( \mu \) of the initial strengths \( (s_i)_i \) of the teams.
- The marginal distribution \( \nu \) of the tilting process \( \xi \).

\(^3\)This is assumption (B3) in [Eks14] that is also assumed in [PRW97]. Since in our case all moments of \( g(\xi_j, s_j) \) are finite (because \( g \) is a bounded function), we can take the parameter \( \delta \) in [Eks14] equal to +\( \infty \) and thus condition (B3) in [Eks14] requires that \( \sum_{\alpha=1}^{\infty} \alpha^2 \alpha_n < +\infty \).
We end this section with two examples. The first one is more theoretical and the second one more related to the statistical calibration.

**Example 1.7.** We assume that:

- \( f(x, y) = \frac{x}{x+y} \);
- for all \( i \in \mathbb{N} \), the initial strengths \( s_i \) are uniformly distributed in \([0, 1]\), i.e., \( \mu \) is the Lebesgue measure on \([0, 1]\);
- the tilting process \( \xi \) is a Markov chain with state space \( \{a, b\} \) for some \( a \in (0, 1), b \in (1, \infty) \), with transition matrix \( \begin{pmatrix} p_a & 1 - p_a \\ 1 - p_b & p_b \end{pmatrix} \) for some \( p_a, p_b \in (0, 1) \). Note that the invariant measure \( \nu = (\nu_a, \nu_b) \) is equal to \( \left( \frac{1-p_b}{2-p_a-p_b}, \frac{1-p_a}{2-p_a-p_b} \right) \).

Under this assumptions, Theorem 1.3 guarantees that for any fixed \( s \in \mathbb{R}^+ \) and for almost every realization \( \vec{s} = (s_i)_{i \in \mathbb{N}} \) of the random sequence \( (s_i)_{i \in \mathbb{N}} \in [0, 1]^\mathbb{N} \), the following convergence holds under \( P_{\vec{s}} \):

\[
\frac{W_n(s)}{2n} \xrightarrow{P} \ell(s),
\]

where

\[
\ell(s) = \int_{\mathbb{R}_+^3} \frac{sv}{sv + uv'} d\nu(v) d\nu(v') d\mu(u) = \sum_{i,j \in \{a,b\}} \frac{s \cdot i}{j} \log \left( 1 + \frac{j}{s \cdot i} \right) \frac{\nu_i \cdot \nu_j}{s \cdot i}.
\]

In particular if \( a = \frac{1}{2}, b = 2, p_a = \frac{1}{2}, p_b = \frac{1}{2} \) then

\[
\ell(s) = \frac{s}{16} \log \left( \frac{(1 + s)^8 (4 + s)(1 + 4s)^{16}}{2^{32} \cdot s^{25}} \right),
\]

whose graph is plotted in Fig. 1.

Figure 1: The graph of the function \( \ell(s) \) in Eq. (12) for \( s \in [0, 1] \).

In addition, Theorem 1.4 implies that for any \( s \in \mathbb{R}_+ \) and for almost every realization \( \vec{s} = (s_i)_{i \in \mathbb{N}} \) of the random sequence \( (s_i)_{i \in \mathbb{N}} \in [0, 1]^\mathbb{N} \), the following convergence also holds under \( P_{\vec{s}} \):

\[
\frac{W_n(s) - \mathbb{E}_{\vec{Z}}[W_n(s)]}{\sqrt{2n}} \xrightarrow{d} N(0, \sigma^2(s)^2 + \rho^2(s)^2),
\]

where \( \sigma^2(s)^2 \) and \( \rho^2(s)^2 \) can be computed as follows. From Eqs. (7) and (8) we know that

\[
\sigma^2(s)^2 = \ell(s) - \mathbb{E} \left[ (F_s(V, U))^2 \right]
\]

and that

\[
\rho^2(s)^2 = \mathbb{E} \left[ \tilde{F}_s(V, U)^2 \right] + 2 \cdot \sum_{k=1}^{\infty} \mathbb{E} \left[ \tilde{F}_s(\xi_1, U) \tilde{F}_s(\xi_{1+k}, U') \right].
\]
Recall that $\ell(s)$ was computed in Eq. (11). Note that

$$F_s(x, y) = E \left[ f \left( s \cdot x, y \cdot V' \right) \right] = \sum_{v' \in \{a, b\}} f \left( s \cdot x, y \cdot v' \right) \nu_{v'}$$

and that

$$F_s(x, y) = F_\epsilon(x, y) - E [F_\epsilon(V, y)] = \sum_{v' \in \{a, b\}} f \left( s \cdot x, y \cdot v' \right) \nu_{v'} - \sum_{(v, v') \in \{a, b\}^2} f \left( s \cdot v, y \cdot v' \right) \nu_{v'} \nu_v.$$ 

Therefore

$$E \left[ (F_s(V, U))^2 \right] = \sum_{v \in \{a, b\}} \left( \int_0^1 \left( \sum_{v' \in \{a, b\}} f \left( s \cdot v, u \cdot v' \right) \nu_{v'} \right)^2 \, du \right) \nu_v$$

and

$$E \left[ (\tilde{F}_s(V, U))^2 \right] = \sum_{w \in \{a, b\}} \left( \int_0^1 \left( \sum_{v' \in \{a, b\}} f \left( s \cdot w, u \cdot v' \right) \nu_{v'} - \sum_{(v, v') \in \{a, b\}^2} f \left( s \cdot v, u \cdot v' \right) \nu_{v'} \nu_v \right)^2 \, du \right) \nu_v.$$ 

Moreover,

$$\sum_{k=1}^{\infty} E \left[ \tilde{F}_s(\xi_1, U) \tilde{F}_s(\xi_{1+k}, U') \right] = \sum_{k=1}^{\infty} \sum_{(i, j) \in \{a, b\}^2} \mathbb{P}(\xi_1 = i, \xi_{1+k} = j) \int_0^1 \tilde{F}_s(i, u) \, du \int_0^1 \tilde{F}_s(j, u') \, du'.$$

It remains to compute $\mathbb{P}(\xi_1 = i, \xi_{1+k} = j)$ for $i, j \in \{a, b\}$. Note that

$$\begin{pmatrix} p_a & 1-p_a \\ 1-p_b & p_b \end{pmatrix} \bigg| \begin{pmatrix} 1 & 1-p_a \\ 1 & 0 \end{pmatrix} \bigg| \begin{pmatrix} p_a & 1-p_a \\ 1 & p_b \end{pmatrix} = : SJS^{-1}.$$ 

Therefore,

$$\begin{align*}
\mathbb{P}(\xi_1 = a, \xi_{1+k} = a) &= \frac{p_b - 1 + (p_a - 1)(p_a + p_b - 1)^k}{p_a + p_b - 2} \cdot \nu_a, \\
\mathbb{P}(\xi_1 = a, \xi_{1+k} = b) &= \frac{(p_a - 1)((p_a + p_b - 1)^k - 1)}{p_a + p_b - 2} \cdot \nu_a, \\
\mathbb{P}(\xi_1 = b, \xi_{1+k} = a) &= \frac{(p_b - 1)((p_a + p_b - 1)^k - 1)}{p_a + p_b - 2} \cdot \nu_b, \\
\mathbb{P}(\xi_1 = b, \xi_{1+k} = b) &= \frac{p_a - 1 + (p_b - 1)(p_a + p_b - 1)^k}{p_a + p_b - 2} \cdot \nu_b.
\end{align*}$$

With some tedious but straightforward computations, we can explicitly compute $\sigma(s)^2$ and $\rho(s)^2$. The graphs of the two functions $\sigma(s)^2$ and $\rho(s)^2$ for $s \in \{0, 1\}$ are plotted in Fig. 2 for three different choices of the parameters $a, b, p_a, p_b$. It is interesting to note that $\sigma(s)^2$ is much larger than $\rho(s)^2$ when $p_a$ and $p_b$ are small, $\sigma(s)^2$ is comparable with $\rho(s)^2$ when $p_a$ and $p_b$ are around 0.9, and $\rho(s)^2$ is much larger than $\sigma(s)^2$ when $p_a$ and $p_b$ are very close to 1.

**Example 1.8.** We collect here some data related to the Italian national basketball league in order to compare some real data with our theoretical results. We believe that it would be interesting to develop a more accurate and precise analysis of real data in some future projects.

In Fig. 3, the rankings of the last 22 national leagues played among exactly 16 teams in the league are shown (some leagues, like the 2011-12 league, are not tracked since in those years the league was not formed by 16 teams). The mean number of points of the 22 collected leagues are (from the team ranked first to the team ranked 16-th):
Figure 2: In green the graph of $\sigma(s)^2$. In red the graph of $\rho(s)^2$. In blue the graph of $\sigma(s)^2 + \rho(s)^2$.

**Left:** The parameters of the model are $a = 1/2, b = 2, p_a = 2/5, p_b = 2/5$. **Middle:** The parameters of the model are $a = 1/2, b = 2, p_a = 92/100, p_b = 92/100$. **Right:** The parameters of the model are $a = 1/2, b = 2, p_a = 99/100, p_b = 99/100$.

| 2018-19 | 2017-18 | 2016-17 | 2015-16 | 2014-15 | 2013-14 | 2012-13 | 2011-12 | 2010-11 | 2009-10 | 2008-09 | 1999-00 | 1993-94 |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 46      | 46      | 46      | 44      | 52      | 50      | 46      | 52      | 58      | 54      | 48      |         |         |
| 40      | 44      | 42      | 42      | 44      | 40      | 44      | 40      | 42      | 42      |         |         |         |
| 40      | 40      | 38      | 40      | 40      | 40      | 40      | 42      | 38      | 42      | 40      |         |         |
| 36      | 40      | 36      | 38      | 38      | 36      | 38      | 34      | 35      | 36      | 40      |         |         |
| 36      | 36      | 34      | 32      | 38      | 36      | 36      | 34      | 34      | 36      | 36      |         |         |
| 34      | 32      | 34      | 32      | 32      | 34      | 36      | 30      | 34      | 34      | 32      |         |         |
| 32      | 32      | 30      | 30      | 30      | 30      | 30      | 30      | 30      | 30      | 32      |         |         |
| 32      | 30      | 30      | 30      | 28      | 30      | 32      | 30      | 28      | 30      | 30      |         |         |
| 32      | 32      | 28      | 28      | 28      | 30      | 28      | 28      | 28      | 30      | 28      |         |         |
| 32      | 30      | 26      | 26      | 26      | 24      | 26      | 28      | 28      | 26      | 24      |         |         |
| 30      | 26      | 26      | 24      | 24      | 24      | 26      | 24      | 24      | 24      | 22      |         |         |
| 28      | 26      | 26      | 24      | 24      | 24      | 22      | 24      | 24      | 22      | 22      |         |         |
| 18      | 20      | 24      | 22      | 22      | 22      | 22      | 22      | 20      | 20      | 22      |         |         |
| 14      | 18      | 22      | 22      | 20      | 22      | 20      | 20      | 20      | 20      | 21      |         |         |
| 12      | 16      | 20      | 22      | 16      | 18      | 20      | 20      | 20      | 20      | 20      |         |         |
| 10      | 14      | 18      | 22      | 15      | 18      | 12      | 16      | 12      | 14      | 12      |         |         |

| 1992-93 | 1991-92 | 1990-91 | 1989-90 | 1988-89 | 1987-88 | 1986-87 | 1985-86 | 1984-85 | 1983-84 | 1982-83 |         |         |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 48      | 44      | 42      | 42      | 42      | 46      | 44      | 52      | 46      | 50      | 44      |         |         |
| 42      | 44      | 40      | 40      | 38      | 42      | 42      | 44      | 46      | 44      | 44      |         |         |
| 38      | 44      | 38      | 38      | 36      | 40      | 40      | 40      | 40      | 36      | 36      | 42      |         |
| 36      | 42      | 36      | 36      | 36      | 36      | 40      | 36      | 36      | 36      | 32      | 36      | 42      |
| 34      | 38      | 34      | 38      | 36      | 36      | 40      | 36      | 36      | 36      | 32      | 36      | 42      |
| 34      | 34      | 34      | 38      | 36      | 36      | 36      | 36      | 36      | 32      | 32      | 40      |         |
| 32      | 30      | 32      | 32      | 34      | 30      | 32      | 34      | 32      | 32      | 32      | 32      | 36      |
| 32      | 30      | 32      | 32      | 34      | 28      | 30      | 32      | 32      | 32      | 30      | 32      | 30      |
| 28      | 28      | 32      | 32      | 34      | 28      | 28      | 30      | 30      | 30      | 28      |         |         |
| 28      | 28      | 30      | 30      | 33      | 28      | 28      | 28      | 28      | 28      | 28      |         |         |
| 28      | 22      | 28      | 30      | 28      | 28      | 28      | 24      | 28      | 26      | 26      |         |         |
| 26      | 22      | 26      | 28      | 22      | 24      | 24      | 22      | 28      | 24      | 24      |         |         |
| 20      | 20      | 24      | 28      | 20      | 22      | 22      | 22      | 26      | 20      | 22      |         |         |
| 18      | 20      | 22      | 16      | 20      | 22      | 22      | 18      | 18      | 18      | 16      |         |         |
| 18      | 20      | 22      | 16      | 18      | 20      | 18      | 16      | 14      | 14      | 12      |         |         |
| 18      | 14      | 8       | 0       | 12      | 12      | 8       | 10      | 10      | 14      | 4       |         |         |

Figure 3: The rankings of the last 22 national italian basket leagues played with exactly 16 teams in the league. In the Italian basket league every team plays two matches against every other team and every victory gives two points.
The diagram of this ranking is given in the left-hand side of Fig. 4.

As mentioned above, an interesting question consists in assessing whether it is possible to describe the behaviour of these leagues by using our model. More precisely, we looked for a function \( f(x, y) \) and two distributions \( \mu \) and \( \nu \) such that the graph of \( \ell(s) \) for \( s \in [0, 1] \) can well approximate the graph in the left-hand side of Fig. 4. We find out that choosing \( \mu \) to be the uniform measure on the interval \([0, 0.999]\), \( \nu = 0.6 \cdot \delta_{0.25} + 0.9 \cdot \delta_{1.3} \), and

\[
f(x, y) := \frac{g(x)}{g(x) + g(y)}, \quad \text{with} \quad g(x) := \log \left(1 - \min \left\{ \frac{x}{1.3}, 0.999 \right\}\right),
\]

then the graph of \( \ell(s) \) for \( s \in [0, 0.999] \), is the one given in the right-hand side of Fig. 4. Note that the two graphs have a similar convexity.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Left: The diagram of the mean number of points of the 22 leagues collected in Fig. 3. On the x-axis the teams are ranked from the weakest to the strongest; on the y-axis the mean number of points are plotted. Right: The graph of \( \ell(s) \) for \( s \in [0.1, 0.999] \) for the specific \( \mu, \nu \) and \( f(x, y) \) given in (and before) Eq. (13).}
\end{figure}

\subsection{Open problems}

We collect some open questions and conjectures that we believe might be interesting to investigate in future projects:

\begin{itemize}
\item Conditioning on the initial strengths of the teams, how many times do we need to run the league in order to guarantee that the strongest team a.s. wins the league? We point out that a similar question was investigated in [BNH07] for the model considered by the authors.
\item In the spirit of large deviations results, we believe it would be interesting to study the probability that the weakest team wins the league, again conditioning on the initial strengths of the teams.
\item Another natural question is to investigate the whole final ranking of the league. We conjecture the following.
\end{itemize}
For a sequence of initial strengths \((s_i)_{i \in [2n-1]}\), we denote by \((\tilde{T}_i)_{i \in [2n-1]}\) the sequence of teams reordered according to their initial strengths \((s_i)_{i \in [2n-1]}\) (from the weakest to the strongest). Set

\[
\tilde{W}_n(i) := \text{Number of wins of the team } \tilde{T}_i \text{ at the end of a league with } 2n \text{ players.}
\]

Let \(\tilde{W}_n(x) : [0,1] \to \mathbb{R}\) denote the piece-wise linear continuous function obtained by interpolating the values \(\tilde{W}_n(i/n)\) for all \(i \in [2n-1]\).

Denote by \(H_\mu(y)\) the cumulative distribution function of \(\mu\) and by \(H_\mu^{-1}(x)\) the generalized inverse distribution function, i.e. \(H_\mu^{-1}(x) = \inf\{y \in \mathbb{R}_+ : H_\mu(y) \geq x\}\).

**Conjecture 1.9.** Suppose that the assumptions in Section 1.3.1 hold with the additional requirement that the function \(f\) is continuous\(^{5}\). For \(\mu^{\mathbb{N}\cup\{0\}}\)-almost every sequence \(\bar{s} = (s_i)_{i \in \mathbb{N}\cup\{0\}} \in \mathbb{R}_+^{\mathbb{N}\cup\{0\}}\) and for every choice of the calendar of the league, under \(\mathbb{P}_x\) the following convergence of càdlàg processes holds

\[
\frac{\tilde{W}_n(x)}{2n} \xrightarrow{P} n \to \infty \ell\left(H_\mu^{-1}(x)\right),
\]

where \(\ell(s)\) is defined as in Theorem 1.3 by

\[
\ell(s) = \mathbb{E}\left[f\left(s \cdot V, U \cdot V'\right)\right] = \int_{\mathbb{R}^3_+} f\left(s \cdot v, u \cdot v'\right) dv(v)du(v')d\mu(u).
\]

Note that the correlations between various teams in the league strongly depend on the choice of the calendar but we believe that this choice does not affect the limiting result in the conjecture above. We refer the reader to Fig. 5 for some simulations that support our conjecture.

We also believe that the analysis of the local limit (as defined in [Bor20]) of the whole ranking should be an interesting but challenging question (and in this case we believe that the choice of the calendar will affect the limiting object). Here, with local limit we mean the limit of the ranking induced by the \(k\) teams – for every fixed \(k \in \mathbb{N}\) – in the neighborhood (w.r.t. the initial strengths) of a distinguished team (that can be selected uniformly at random or in a deterministic way, say for instance the strongest team).

- As mentioned above, we believe that it would be interesting to develop a more accurate and precise analysis of real data in order to correctly calibrate the parameters of our model collected at the beginning of Section 1.5.

## 2 Proof of the law of large numbers

The proof of Theorem 1.3 follows from the following result using standard second moment arguments.

**Proposition 2.1.** We have that

\[
\mathbb{E}\left[\frac{W_n(s)}{2n}\bigg| (s_i)\right] \xrightarrow{a.s.} n \to \infty \ell(s), \quad \text{and} \quad \mathbb{E}\left[\left(\frac{W_n(s)}{2n}\right)^2\bigg| (s_i)\right] \xrightarrow{a.s.} n \to \infty \ell(s)^2.
\]

\(^{5}\)The assumption that \(f\) is continuous might be relaxed.
For all \( (\ell, j) \) we recall (see Section 1.4.2) that we assumed that, for every \( n \in [2n-1] \), the team \( T_0 \) plays against the team \( T_j \) the \( j \)-th day and we denoted by \( W_{j} = W_j(s) \) the event that the team \( T_0 \) wins the match. In particular, \( W_{n}(s) = \sum_{j=1}^{2n-1} \mathbb{I}_{W_j} \) and

\[
\mathbb{P}\left( W_j \mid s_j, \xi^0_j, \xi^j_j \right) = f(s \cdot \xi_0^j, s_j \cdot \xi^j_j). \tag{14}
\]

**Proof of Proposition 2.1.** We start with the computations for the first moment. From Eq. (14) we have that

\[
\mathbb{E}\left[ W_{n}(s) \mid s_i \right] = \sum_{j=1}^{2n-1} \mathbb{P}(W_j \mid s_i) = \sum_{j=1}^{2n-1} \mathbb{E}\left[ f(s \cdot \xi_0^j, s_j \cdot \xi^j_j) \mid s_j \right].
\]

Since for all \( j \in [2n-1] \), \( \xi_j^0 \) and \( s_j \) are independent, \( \xi_0^j \overset{d}{=} V \), and \( \xi_j^j \overset{d}{=} V' \), we have

\[
\mathbb{E}\left[ f(s \cdot \xi_0^j, s_j \cdot \xi^j_j) \mid s_j \right] = G_s(s_j),
\]

where \( G_s(x) = \mathbb{E}[f(s \cdot V, x \cdot V')] \). By the Law of large numbers, we can conclude that

\[
\mathbb{E}\left[ W_{n}(s) \right] = \frac{1}{2n} \sum_{j=1}^{2n-1} G_s(s_j) \xrightarrow{n \to \infty} \mathbb{E}[G_s(U)] = \ell(s).
\]

We now turn to the second moment. Since for all \( i, j \in [2n-1] \) with \( i \neq j \), conditioning on \( (s_i, s_j, \xi_i^0, \xi_i^j, \xi_j^0, \xi_j^j) \) the events \( W_i \) and \( W_j \) are independent, we have that, using Eq. (14),

\[
\mathbb{E}\left[ W_{n}(s)^2 \right] = \mathbb{E}\left[ W_{n}(s) \mid s_i \right] + \sum_{\ell, j=1}^{2n-1} \mathbb{P}(W_\ell \cap W_j \mid s_\ell, s_j)
\]

\[
= \mathbb{E}\left[ W_{n}(s) \mid s_i \right] + \sum_{\ell, j=1}^{2n-1} \mathbb{E}\left[ f(s \cdot \xi_\ell^\ell, s_\ell \cdot \xi_\ell^\ell) f(s \cdot \xi_j^0, s_j \cdot \xi_j^j) \mid s_\ell, s_j \right].
\]

For all \( \ell, j \in [2n-1] \) with \( \ell \neq j \), \( s_\ell, s_j, \xi_\ell^\ell \) and \( \xi_j^j \) are mutually independent and independent of \( (\xi_\ell^\ell, \xi_j^j) \). In addition, \( (\xi_\ell^\ell, \xi_j^j) \overset{d}{=} (\xi_\ell, \xi_j) \) and \( \xi_\ell^\ell \overset{d}{=} \xi_\ell^j \overset{d}{=} V' \). Thus, we have that

\[
\mathbb{E}\left[ f(s \cdot \xi_\ell^\ell, s_\ell \cdot \xi_\ell^\ell) f(s \cdot \xi_j^0, s_j \cdot \xi_j^j) \mid s_\ell, s_j \right] = \mathbb{E}[F_s(\xi_\ell, s_\ell) F_s(\xi_j, s_j) | s_\ell, s_j],
\]

The rest of this section is devoted to the proof of Proposition 2.1.
where we recall that \( F_s(x,y) = \mathbb{E}[f(s \cdot x, y \cdot V')] \). Simple consequences of the computations done for the first moment are that

\[
\mathbb{E} \left[ \frac{W_n(s)}{n^2} \right]_{n \to \infty} \xrightarrow{a.s.} 0 \quad \text{and} \quad \mathbb{E} \left[ \frac{\sum_{j=1}^{2n-1} F_s^2(\xi_j, s_j)}{n^2} \right]_{n \to \infty} \xrightarrow{a.s.} 0.
\]

Therefore, we can write

\[
\mathbb{E} \left[ \left( \frac{W_n(s)}{2n} \right)^2 (s_i) \right] = \mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} F_s(\xi_\ell, s_\ell) F_s(\xi_j, s_j) (s_i) \right] + o(1)
\]

\[
= \mathbb{E} \left[ \left( \frac{1}{2n} \sum_{j=1}^{2n-1} F_s(\xi_j, s_j) \right)^2 (s_i) \right] + o(1), \quad (15)
\]

where \( o(1) \) denotes a sequence of random variables that a.s. converges to zero. We now need the following result, whose proof is postponed at the end of this section.

**Proposition 2.2.** For all bounded, measurable functions \( g : \mathbb{R}^2_+ \to \mathbb{R}_+ \), we have that

\[
\mathbb{E} \left[ \left( \frac{1}{2n} \sum_{j=1}^{2n-1} g(\xi_j, s_j) \right)^2 (s_i) \right] \xrightarrow{a.s.} \mathbb{E} [g(V, U)^2].
\]

From Eq. (15) and the proposition above, we conclude that

\[
\mathbb{E} \left[ \left( \frac{W_n(s)}{2n} \right)^2 (s_i) \right] \xrightarrow{a.s.} \mathbb{E} [F_s(V, U)^2] = \ell(s)^2.
\]

This concludes the proof of Proposition 2.1. \(\square\)

It remains to prove Proposition 2.2. We start with the following preliminary result.

**Lemma 2.3.** For every quadruple \((A, A', B, B')\) of Borel subsets of \(\mathbb{R}_+\), we have that

\[
\mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{1}_{A \times B}(\xi_\ell, s_\ell) \mathbb{1}_{A' \times B'}(\xi_j, s_j) (s_i) \right] \xrightarrow{a.s.} \mathbb{E} [\mathbb{1}_{A \times B}(V, U)] \mathbb{E} [\mathbb{1}_{A' \times B'}(V, U)].
\]

**Proof.** Note that since the process \( \xi \) is independent of \((s_i)_i\),

\[
\mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{1}_{A \times B}(\xi_\ell, s_\ell) \mathbb{1}_{A' \times B'}(\xi_j, s_j) (s_i) \right] = \left( \frac{1}{2n} \right)^2 \sum_{\ell=1}^{2n-1} \mathbb{1}_{B}(s_\ell) \cdot \left( \frac{1}{2n} \right)^2 \sum_{j=1}^{2n-1} \mathbb{P}(\xi_\ell \in A, \xi_j \in A') \mathbb{1}_{B'}(s_j). \quad (16)
\]

For all \( \ell \in [2n-1] \), we can write

\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{P}(\xi_\ell \in A, \xi_j \in A') \mathbb{1}_{B'}(s_j)
\]

\[
= \mathbb{P}(V \in A) \mathbb{P}(V \in A') \frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{1}_{B'}(s_j)
\]

\[
+ \frac{1}{2n} \sum_{j=1}^{2n-1} (\mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A')) \mathbb{1}_{B'}(s_j).
\]

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First, from the Law of large numbers we have that
\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{1}_{B'}(s_j) \overset{a.s., n \to \infty}{\to} \mu(B'). \tag{18}
\]

Secondly, we show that the second sum in the right-hand side of Eq. (17) is negligible. We estimate
\[
\left| \frac{1}{2n} \sum_{j=1}^{2n-1} \left( \mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right) \mathbb{1}_{B'}(s_j) \right|
\]

\[
\leq \frac{1}{2n} \sum_{j=1}^{2n-1} \left| \mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|
\]

\[
= \frac{1}{2n} \sum_{j=1}^{\ell-1} \left| \mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|
\]

\[
+ \frac{1}{2n} \sum_{j=\ell}^{2n-1} \left| \mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|
\]

Using the stationarity assumption in Eq. (2), the right-hand side of the equation above can be rewritten as follows
\[
\frac{1}{2n} \sum_{j=2}^{\ell} \left| \mathbb{P}(\xi_j \in A, \xi_1 \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|
\]

\[
+ \frac{1}{2n} \sum_{j=1}^{2n-\ell} \left| \mathbb{P}(\xi_1 \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|.
\]

Therefore, we obtain that
\[
\left| \frac{1}{2n} \sum_{j=1}^{2n-1} \left( \mathbb{P}(\xi_\ell \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right) \mathbb{1}_{B'}(s_j) \right|
\]

\[
\leq \frac{1}{2n} \sum_{j=1}^{2n-1} \left| \mathbb{P}(\xi_1 \in A', \xi_j \in A) - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|
\]

\[
+ \frac{1}{2n} \sum_{j=1}^{2n} \left| \mathbb{P}(\xi_1 \in A, \xi_j \in A') - \mathbb{P}(V \in A) \mathbb{P}(V \in A') \right|.
\]

The upper bound above is independent of \( \ell \) and tends to zero because the process \( \xi \) is weakly-mixing (see Eq. (3)); we can thus deduce from Eqs. (17) and (18) that, uniformly for all \( \ell \in [2n - 1] \),
\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{P}(\xi_\ell \in A, \xi_j \in A') \mathbb{1}_{B'}(s_j) \overset{a.s., n \to \infty}{\to} \mathbb{P}(V \in A) \mathbb{P}(V \in A') \mu(B'). \tag{19}
\]

Hence, from Eqs. (16) and (19) and the Law of large numbers, we can conclude that
\[
E \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{1}_{A \times B}(\xi_\ell, s_\ell) \mathbb{1}_{A' \times B'}(\xi_j, s_j) \bigg| (s_i)_i \right] \overset{a.s., n \to \infty}{\to} \mathbb{P}(V \in A) \mu(B) \mathbb{P}(V \in A') \mu(B') = E \left[ \mathbb{1}_{A \times B}(V, U) \right] E \left[ \mathbb{1}_{A' \times B'}(V, U) \right].
\]
We now show that we can extend the result of Lemma 2.3 to all Borel subsets of \( \mathbb{R}^2_+ \), denoted by \( \mathcal{B}(\mathbb{R}^2_+) \). We also denote by \( \mathcal{R}(\mathbb{R}^2_+) \) the sets of rectangles \( A \times B \) of \( \mathbb{R}^2_+ \) with \( A, B \in \mathcal{B}(\mathbb{R}_+) \).

We recall that we denote by \( \sigma(A) \) and \( \lambda(A) \) the sigma algebra and the monotone class generated by a collection of sets \( A \), respectively.

**Lemma 2.4.** For all \( C, C' \in \mathcal{B}(\mathbb{R}^2_+) \), we have that

\[
E \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{I}_C(\xi_{\ell}, s_j) \mathbb{I}_{C'}(\xi_j, s_j) (s_i)_{i=1} \right] \overset{a.s.}{\underset{n \to \infty}{\to}} E[\mathbb{I}_C(V, U)] E[\mathbb{I}_{C'}(V, U)].
\]

**Proof.** We first fix a rectangle \( A \times B \in \mathcal{R}(\mathbb{R}^2_+) \) and we consider the set

\[
\mathcal{A}_{A \times B} := \{ C \in \mathcal{B}(\mathbb{R}^2_+) \mid \text{Eq. (20) holds with } C' = A \times B \}.
\]

By Lemma 2.3, we have \( \mathcal{R}(\mathbb{R}^2_+) \subseteq \mathcal{A}_{A \times B} \subseteq \mathcal{B}(\mathbb{R}^2_+) \). If we show that \( \mathcal{A}_{A \times B} \) is a monotone class, then we can conclude that \( \mathcal{A}_{A \times B} = \mathcal{B}(\mathbb{R}^2_+) \). Indeed, by the monotone class theorem (note that \( \mathcal{R}(\mathbb{R}^2_+) \) is closed under finite intersections), we have that

\[
\mathcal{B}(\mathbb{R}^2_+) = \sigma(\mathcal{R}(\mathbb{R}^2_+)) = \lambda(\mathcal{R}(\mathbb{R}^2_+)) \subseteq \mathcal{A}_{A \times B}.
\]

The equality \( \mathcal{A}_{A \times B} = \mathcal{B}(\mathbb{R}^2_+) \) implies that Eq. (20) holds for every set in \( \mathcal{B}(\mathbb{R}^2_+) \times \mathcal{R}(\mathbb{R}^2_+) \). Finally, if we also show that for any fixed Borel set \( C^* \in \mathcal{B}(\mathbb{R}^2_+) \), the set

\[
\mathcal{A}_{C^*} := \{ C' \in \mathcal{B}(\mathbb{R}^2_+) \mid \text{Eq. (20) holds with } C = C^* \}
\]

is a monotone class, then using again the same arguments that we used in Eq. (21) (note that \( \mathcal{R}(\mathbb{R}^2_+) \subseteq \mathcal{A}_{C^*} \), thanks to the previous step) we can conclude that Eq. (20) holds for every pair of sets in \( \mathcal{B}(\mathbb{R}^2_+) \times \mathcal{B}(\mathbb{R}^2_+) \), proving the lemma. Therefore, in order to conclude the proof, it is sufficient to show that \( \mathcal{A}_{A \times B} \) and \( \mathcal{A}_{C^*} \) are monotone classes.

We start by proving that \( \mathcal{A}_{A \times B} \) is a monotone class:

- Obviously \( \mathbb{R}^2_+ \in \mathcal{A}_{A \times B} \).
- If \( C, D \in \mathcal{A}_{A \times B} \) and \( C \subseteq D \), then \( D \setminus C \in \mathcal{A}_{A \times B} \) because \( \mathbb{I}_{D \setminus C} = \mathbb{I}_D - \mathbb{I}_C \).
- Let now \( (C_m)_{m \in \mathbb{N}} \) be a sequence of sets in \( \mathcal{A}_{A \times B} \) such that \( C_m \subseteq C_{m+1} \) for all \( m \in \mathbb{N} \). We want to show that \( C := \bigcup_m C_m \in \mathcal{A}_{A \times B} \), i.e. that

\[
E \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{I}_{C_m}(\xi_{\ell}, s_j) \mathbb{I}_{A \times B}(\xi_j, s_j) (s_i)_{i=1} \right] \overset{a.s.}{\underset{m \to \infty}{\to}} E[\mathbb{I}_C(V, U)] E[\mathbb{I}_{A \times B}(V, U)].
\]

Since \( \mathbb{I}_C = \lim_{m \to \infty} \mathbb{I}_{C_m} \), then by monotone convergence we have for all \( n \in \mathbb{N} \),

\[
E \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{I}_{C_m}(\xi_{\ell}, s_j) \mathbb{I}_{A \times B}(\xi_j, s_j) (s_i)_{i=1} \right] \overset{a.s.}{\underset{m \to \infty}{\to}} \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} E[\mathbb{I}_C(\xi_{\ell}, s_j) \mathbb{I}_{A \times B}(\xi_j, s_j) s_{\ell}, s_j].
\]

We also claim that:

(a) For all \( m \in \mathbb{N} \),

\[
\frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} E[\mathbb{I}_{C_m}(\xi_{\ell}, s_j) \mathbb{I}_{A \times B}(\xi_j, s_j) s_{\ell}, s_j] \overset{a.s.}{\underset{n \to \infty}{\to}} E[\mathbb{I}_{C_m}(V, U)] E[\mathbb{I}_{A \times B}(V, U)].
\]
(b) The convergence in Eq. (23) holds uniformly for all \( n \in \mathbb{N} \).

Item (a) holds since \( C_m \in \mathcal{A}_{A \times B} \). Item (b) will be proved at the end. Items (a) and (b) allow us to exchange the following a.s.-limits as follows

\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} I_{C_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) \right] \left( s_i \right)
\]

\[
= \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{E} \left[ I_{C_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) s_\ell, s_j \right]
\]

\[
= \lim_{m \to \infty} \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} \mathbb{E} \left[ I_{C_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) s_\ell, s_j \right]
\]

\[
= \lim_{m \to \infty} \mathbb{E} \left[ I_{C_m} (V, U) \right] \mathbb{E} \left[ I_{A \times B} (V, U) \right] = \mathbb{E} \left[ I_C (V, U) \right] \mathbb{E} \left[ I_{A \times B} (V, U) \right],
\]

where the last equality follows by monotone convergence. This proves Eq. (22) and concludes the proof (up to proving item (b)) that \( \mathcal{A}_{A \times B} \) is a monotone class.

**Proof of item (b).** Since \( I_{C \setminus C_m} = I_C - I_{C_m} \), it is enough to show that

\[
\sup_n \mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} I_{C \setminus C_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) \right] \left( s_i \right) \xrightarrow{a.s. m \to \infty} 0.
\]

We set \( D_m := C \setminus C_m \), and define

\[
(D_m)^s := \{ x \in \mathbb{R}_+ | (x, s) \in D_m \}, \quad \text{for all } s \in \mathbb{R},
\]

\[
\pi_Y(D_m) := \{ y \in \mathbb{R}_+ | \exists x \in \mathbb{R}_+ \text{ s.t. } (x, y) \in D_m \}.
\]

Since \( \xi_\ell \overset{d}{=} V \) for all \( \ell \in [2n - 1] \), then for all \( \ell, j \in [2n - 1] \), a.s.

\[
\mathbb{E} \left[ I_{D_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) s_\ell, s_j \right] = \mathbb{P} (\xi_\ell \in (D_m)^s, \xi_j \in A | s_\ell) \mathbb{I}_{\pi_Y(D_m)} (s_\ell) \mathbb{I}_B (s_j) \leq \mathbb{P} (\xi_\ell \in (D_m)^s | s_\ell) = \mathbb{P} (V \in (D_m)^s | s_\ell) . \quad (24)
\]

Therefore a.s.

\[
\sup_n \mathbb{E} \left[ \frac{1}{4n^2} \sum_{\ell,j=1}^{2n-1} I_{C \setminus C_m} (\xi_\ell, s_\ell) I_{A \times B} (\xi_j, s_j) \right] \leq \mathbb{P} (V \in (C \setminus C_m)^s | s_\ell).
\]

Now the sequence \( \mathbb{P} (V \in (C \setminus C_m)^s | s_\ell) \) is a.s. non-increasing (because \( C_m \subseteq C_{m+1} \)) and hence has an a.s. limit. The limit is non-negative and its expectation is the limit of expectations which is 0 because \( C := \bigcup_m C_m \). This completes the proof of item (b).

It remains to prove that \( \mathcal{A}_{C^*} \) is also a monotone class. The proof is similar to the proof above, replacing the bound in Eq. (24) by

\[
\mathbb{E} \left[ I_{C^*} (\xi_\ell, s_\ell) I_{D_m} (\xi_j, s_j) s_\ell, s_j \right] = \mathbb{P} (\xi_\ell \in (C^*)^s, \xi_j \in (D_m)^s | s_\ell, s_j) \mathbb{I}_{\pi_Y(C^*)} (s_\ell) \mathbb{I}_{\pi_Y(D_m)} (s_j) \leq \mathbb{P} (\xi_j \in (D_m)^s | s_j).
\]

This completes the proof of the lemma. \( \square \)
We now generalize the result in Lemma 2.4 to all bounded and measurable functions, hereby proving Proposition 2.2.

**Proof of Proposition 2.2.** We further assume that \( g : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) is non-negative, the general case following by standard arguments. Fubini’s theorem, together with the fact that \( g(x, y) = \int_0^\infty \mathbb{1}_{\{z \leq g(x,y)\}} \, dz \), yields

\[
\mathbb{E} \left[ \left( \frac{1}{2n} \sum_{j=1}^{2n-1} g(\xi_j, s_j) \right)^2 \right] = \mathbb{E} \left[ \frac{1}{4n^2} \sum_{j=1}^{2n-1} \int_0^\infty \mathbb{1}_{\{z \leq g(\xi_j, s_j)\}} \, dz \int_0^\infty \mathbb{1}_{\{t \leq g(\xi_j, s_j)\}} \, dt \right] \mathbb{E} \left[ \left( s_i \right)_i \right]
\]

where \( A(s) = \{(x, y) \in \mathbb{R}_+^2 | g(x, y) \geq s\} \). By Lemma 2.4, for almost every \((z, t) \in \mathbb{R}_+^2\)

\[
\frac{1}{4n^2} \sum_{j=1}^{2n-1} \mathbb{E} \left[ \mathbb{1}_{A(z)}(\xi_j, s_j) \mathbb{1}_{A(t)}(\xi_j, s_j) \right] \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{1}_{A(z)}(\mathbf{V}) \mathbb{E} \left[ \mathbb{1}_{A(t)}(\mathbf{V}) \right] \right] \right] .
\]

Since the left-hand side is bounded by 1, we can conclude by dominated convergence that

\[
\mathbb{E} \left[ \left( \frac{1}{2n} \sum_{j=1}^{2n-1} g(\xi_j, s_j) \right)^2 \right] \xrightarrow[\mathbb{P}]{} \mathbb{E} \left[ g(\mathbf{V}, \mathbf{U}) \right]^2,
\]

completing the proof of the proposition. \(\square\)

### 3 Proof of the central limit theorems

In this section, we start by proving Theorem 1.4 using Theorem 1.6 and then we prove the latter result.

**Proof of Theorem 1.4.** We set \( X_n := \frac{W_n(s) - 2n}{2\sqrt{n}} \). In order to prove the convergence in Eq. (6), it is enough to show that, for every \( t \in \mathbb{R} \),

\[
\mathbb{E}_q \left[ e^{itX_n} \right] = e^{-\frac{t^2}{2} \left( \sigma(s)^2 + \rho(s)^2 \right)},
\]

where we recall that \( \sigma(s)^2 = \mathbb{E} [F_s(\mathbf{V}, \mathbf{U}) - F_s^2(\mathbf{V}, \mathbf{U})] \) and \( \rho(s)^2 = \rho^2 \). Note that \( \sigma(s)^2 + \rho(s)^2 \) is finite thanks to Theorem 1.6 and the fact that \( F_s - F_s^2 \) is a bounded and measurable function.

Recalling that \( W_n(s) = \sum_{j=1}^{2n-1} \mathbf{1}_{W_j} \), \( W_j \) being the event that the team \( T_0 \) wins against the team \( T_j \), and that, conditioning on \( (\xi_j^{(0)})_{r \in [2n-1]} \), the results of different matches are independent, we have that

\[
\mathbb{E}_q \left[ e^{itX_n} \right] = e^{-\sqrt{2n} \mathbb{E}_q [W_n(s)] \cdot it} \cdot \mathbb{E}_q \left[ e^{\frac{it}{\sqrt{2n}} \sum_{j=1}^{2n-1} \mathbf{1}_{W_j}} \right] = e^{-\sqrt{2n} \mathbb{E}_q [W_n(s)] \cdot it} \cdot \mathbb{E}_q \left[ e^{\frac{it}{\sqrt{2n}} \sum_{j=1}^{2n-1} \mathbf{1}_{W_j} (\xi_j^{(0)})_{r \in [2n-1]}} \right] = e^{-\sqrt{2n} \mathbb{E}_q [W_n(s)] \cdot it} \cdot \mathbb{E}_q \left[ \prod_{j=1}^{2n-1} \mathbb{E}_q \left[ e^{\frac{it}{\sqrt{2n}} \mathbf{1}_{W_j} (\xi_j^{(0)})_{r \in [2n-1]}} \right] \right].
\]
Since, by assumption, we have that $\mathbb{P}_{\sigma} \left( W_j \xi_j^0, \xi_j^0 \right) = f(s \cdot \xi_j^0, s_j \cdot \xi_j^0)$ and, for all $j \in [2n - 1]$, $\xi_j^0$ is independent of $\xi_j^0$, $\xi_j^0 \overset{d}{=} \xi_j$ and $\xi_j^0 \overset{d}{=} \mathbf{V}'$, we have that

\[
\mathbb{E}_\sigma \left[ e^{it \mathbf{X}_n} \right] = e^{-\sqrt{2 \pi} \mathbb{E}_\sigma [W_n(s)^2] \cdot E} \left[ \prod_{j=1}^{2n-1} \mathbb{E} \left[ 1 + \left( e^{it \delta} - 1 \right) f(s \cdot \xi_j^0, s_j \cdot \xi_j^0) \right] \xi_j^0 \right] \]

\[
= e^{-\sqrt{2 \pi} \mathbb{E}_\sigma [W_n(s)^2] \cdot E} \left[ \prod_{j=1}^{2n-1} \left( 1 + \left( e^{it \delta} - 1 \right) \cdot F_s(\mathbf{\xi}(s), s_j) \right) \right],
\]

where we recall that $F_s(x, y) = \mathbb{E} \left[ f(s \cdot x, y \cdot \mathbf{V}') \right]$. Rewriting the last term as

\[
e^{-\sqrt{2 \pi} \mathbb{E}_\sigma [W_n(s)^2] \cdot E} \left[ \sum_{j=1}^{2n-1} \log \left( 1 + \left( e^{it \delta} - 1 \right) \cdot F_s(\mathbf{\xi}(s), s_j) \right) \right],
\]

and observing that

\[
\sum_{j=1}^{2n-1} \log \left( 1 + \left( e^{it \delta} - 1 \right) \cdot F_s(\mathbf{\xi}(s), s_j) \right)
\]

\[
= \sum_{j=1}^{2n-1} \left( \frac{it}{\sqrt{2 \pi}} \cdot F_s(\mathbf{\xi}(s), s_j) - \frac{t^2}{4n} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) + O \left( \frac{1}{n} \right) \right)
\]

\[
= \frac{it}{\sqrt{2 \pi}} \sum_{j=1}^{2n-1} F_s(\mathbf{\xi}(s), s_j) - \frac{t^2}{2} \cdot \frac{1}{2n} \sum_{j=1}^{2n-1} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) + O \left( \frac{1}{n} \right),
\]

we obtain that the characteristic function $\mathbb{E}_\sigma \left[ e^{it \mathbf{X}_n} \right]$ is equal to

\[
e^{O \left( \frac{1}{\sqrt{n}} \right)} \cdot e^{-\frac{t^2}{2} \sigma(s)^2} \cdot \mathbb{E} \left[ e^{\frac{it}{\sqrt{2 \pi}} \sum_{j=1}^{2n-1} F_s(\mathbf{\xi}(s), s_j) - 2n \mathbb{E}_\sigma [W_n(s)]} \right], e^{-\frac{t^2}{2} \sigma(s)^2} \left( 1 - \frac{1}{2n} \sum_{j=1}^{2n-1} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) \right).
\]

Now we set

\[
A_n := e^{\frac{it}{\sqrt{2 \pi}} \sum_{j=1}^{2n-1} F_s(\mathbf{\xi}(s), s_j) - 2n \mathbb{E}_\sigma [W_n(s)]},
\]

\[
B_n := e^{-\frac{t^2}{2} \sigma(s)^2} \left( 1 - \frac{1}{2n} \sum_{j=1}^{2n-1} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) \right),
\]

obtaining that $\mathbb{E}_\sigma \left[ e^{it \mathbf{X}_n} \right] = e^{O \left( \frac{1}{\sqrt{n}} \right)} \cdot e^{-\frac{t^2}{2} \sigma(s)^2} \left( \mathbb{E} [A_n] - \mathbb{E} [A_n(1 - B_n)] \right)$. Hence, Eq. (25) holds if we show that

1. $\mathbb{E} [A_n] \rightarrow e^{-\frac{t^2}{2} \sigma(s)^2}$,

2. $\mathbb{E} [A_n(1 - B_n)] \rightarrow 0$.

Item 1 follows from Theorem 1.6. For Item 2, since $|A_n| = 1$, we have that

\[
|\mathbb{E} [A_n(1 - B_n)]| \leq \mathbb{E} [1 - B_n] = 0.
\]

Recalling that $\sigma(s)^2 = \mathbb{E} \left[ F_s(\mathbf{V}, U) - F_s^2(\mathbf{V}, U) \right]$, and that $\xi_j \overset{d}{=} \mathbf{V}$ for all $j \in [2n - 1]$, we have that

\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) - \sigma(s)^2 =
\]

\[
\frac{1}{2n} \sum_{j=1}^{2n-1} \left( F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right) - \frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{E} \left[ F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right]
\]

\[
+ \frac{1}{2n} \sum_{j=1}^{2n-1} \mathbb{E} \left[ F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right] - \mathbb{E} \left[ F_s(\mathbf{\xi}(s), s_j) - F_s^2(\mathbf{\xi}(s), s_j) \right] \overset{P}{\rightarrow} 0,
\]
where for the limit we used once again Theorem 1.6 and similar arguments to the ones already used in the proof of Proposition 2.1. Since the function \( e^{-t^2/2} \) is continuous and the random variable \( \frac{1}{2n} \left( \sum_{j=1}^{2n-1} F_j(\xi_j, s_j) - F_j^2(\xi_j, s_j) \right) - \sigma(s)^2 \) is bounded, we can conclude that \( \mathbb{E} \left[ |1 - B_n| \right] \to 0 \). This ends the proof of Theorem 1.4.

The rest of this section is devoted to the proof of Theorem 1.6. We start by stating a lemma that shows how the coefficients \( \alpha_n \) defined in Eq. (4) control the correlations of the process \( \xi \).

**Lemma 3.1** (Theorem 17.2.1 in [IL71]). Fix \( \tau \in \mathbb{N} \) and let \( X \) be a random variable measurable w.r.t. \( \mathcal{A}_1^{k} \) and \( Y \) a random variable measurable w.r.t. \( \mathcal{A}_k^{\infty} \). Assume, in addition, that \( |X| < C_1 \) almost surely and \( |Y| < C_2 \) almost surely. Then

\[
|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]| \leq 4 \cdot C_1 \cdot C_2 \cdot \alpha_{\tau}.
\]

We now focus on the behaviour of the random variables \( \sum_{j=1}^{2n-1} g(\xi_j, s_j) \) appearing in the statement of Theorem 1.6. It follows directly from Proposition 2.2 and Chebyshev’s inequality that, for \( \mu^n \)-almost every sequence \((s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^\mathbb{N} \),

\[
\frac{1}{2n} \sum_{j=1}^{2n-1} g(\xi_j, s_j) \overset{P}{\rightarrow} \mathbb{E}[g(V, U)].
\]

We aim at establishing a central limit theorem. Recalling the definition of the function \( \tilde{g} \) in the statement of Theorem 1.6, we note that, for all \( j \in \mathbb{N} \),

\[
\tilde{g}(\xi_j, s_j) = g(\xi_j, s_j) - \mathbb{E}[g(V, s_j)],
\]

and so \( \mathbb{E}[\tilde{g}(\xi_j, s_j)] = 0 \). Define

\[
\rho_{g,n}^2 = \text{Var} \left( \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \right) = \text{Var} \left( \sum_{j=1}^{2n-1} g(\xi_j, s_j) \right).
\]

The following lemma shows that the variance \( \rho_{g,n}^2 \) is asymptotically linear in \( n \) and proves the first part of Theorem 1.6.

**Lemma 3.2.** The quantity \( \rho_{g}^2 \) defined in Eq. (10) is finite. Moreover, for \( \mu^n \)-almost every sequence \((s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^\mathbb{N} \), we have that

\[
\rho_{g,n}^2 = 2n \cdot \rho_{g}^2 \cdot (1 + o(1)).
\]

**Proof.** We have that

\[
\rho_{g,n}^2 = \text{Var} \left( \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \right) = \mathbb{E} \left[ \left( \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \right)^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{2n-1} \tilde{g}^2(\xi_j, s_j) \right] + 2 \cdot \mathbb{E} \left[ \sum_{i=1}^{2n-2} \sum_{j=i+1}^{2n-1} \tilde{g}(\xi_i, s_i) \tilde{g}(\xi_j, s_j) \right] = \sum_{j=1}^{2n-1} \mathbb{E} \left[ \tilde{g}^2(\xi_j, s_j) \right] + 2 \cdot \sum_{i=1}^{2n-2} \sum_{k=1}^{2n-1-i} \mathbb{E} \left[ \tilde{g}(\xi_i, s_i) \tilde{g}(\xi_{i+k}, s_{i+k}) \right].
\]

Using similar arguments to the ones already used in the proof of Proposition 2.1, we have that

\[
\sum_{j=1}^{2n-1} \mathbb{E} \left[ \tilde{g}^2(\xi_j, s_j) \right] = 2n \cdot \mathbb{E} \left[ \tilde{g}^2(V, U) \right] + o(n).
\]
We now show that
\[
\lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{2n-2} \sum_{k=1}^{2n-1-i} \mathbb{E} \left[ \tilde{g} (\xi_i, s_i) \tilde{g} (\xi_{i+k}, s_{i+k}) \right] = \sum_{k=1}^{\infty} \mathbb{E} \left[ \tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U') \right]. \tag{26}
\]

First we show that the right-hand side is convergent. We start by noting that from Fubini’s theorem and Lemma 3.1,
\[
\mathbb{E} \left[ \mathbb{E} \left[ \tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U') \right] \mid \xi_1, \xi_{1+k} \right] = \int_{\mathbb{R}^2} \mathbb{E} \left[ \tilde{g}(\xi_1, x)\tilde{g}(\xi_{1+k}, y) \right] d\mu(x)d\mu(y) \leq 4 \cdot \alpha_k.
\]

Therefore, thanks to the assumption in Eq. (5), we have that
\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ \tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U') \right] = \sum_{k=1}^{\infty} \mathbb{E} \left[ \mathbb{E} \left[ \tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U') \right] \mid \xi_1, \xi_{1+k} \right] \leq \sum_{k=1}^{\infty} 4 \cdot \alpha_k < \infty. \tag{27}
\]

Now we turn to the proof of the limit in Eq. (26). Using the stationarity assumption for the process \( \xi \) in Eq. (2), we can write
\[
\frac{1}{2n} \sum_{i=1}^{2n-2} \sum_{k=1}^{2n-1-i} \mathbb{E} \left[ \delta (\xi_i, s_i) \delta (\xi_{i+k}, s_{i+k}) \right] = \frac{1}{2n} \sum_{k=1}^{2n-2} \sum_{i=1}^{2n-1-k} \mathbb{E} \left[ \delta (\xi_i, s_i) \delta (\xi_{i+k}, s_{i+k}) \right].
\]

Using a monotone class argument similar to the one used for the law of large numbers, we will show that the right-hand side of the equation above converges to
\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ \tilde{g}(\xi_1, U)\tilde{g}(\xi_{1+k}, U') \right].
\]

We start, as usual, from indicator functions. We have to prove that for all quadruplets \((A, A', B, B')\) of Borel subsets of \(\mathbb{R}_+\), it holds that
\[
\sum_{k=1}^{2n-2} \frac{1}{2n} \sum_{i=1}^{2n-1-k} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_i, s_i) \mathbb{I}_{A' \times B'} (\xi_{i+k}, s_{i+k}) \right]
\rightarrow \sum_{k=1}^{\infty} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, U) \mathbb{I}_{A' \times B'} (\xi_{1+k}, U') \right] < \infty, \tag{28}
\]

where for every rectangle \(R\) of \(\mathbb{R}_+^2\),
\[
\mathbb{I}_R (x, y) := \mathbb{I}_R (x, y) - \mathbb{E} \left[ \mathbb{I}_R (\xi_1, y) \right].
\]

Setting \(S_n := \sum_{k=1}^{n} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, U) \mathbb{I}_{A' \times B'} (\xi_{1+k}, U') \right]\), we estimate
\[
\left| S_{\infty} - \sum_{k=1}^{2n-2} \frac{1}{2n} \sum_{i=1}^{2n-1-k} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_i, s_i) \mathbb{I}_{A' \times B'} (\xi_{i+k}, s_{i+k}) \right] \right|
\leq |S_{\infty} - S_{2n-2}| + \left| S_{2n-2} - \sum_{k=1}^{2n-2} \frac{1}{2n} \sum_{i=1}^{2n-2} \sum_{k=1}^{2n-2} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_i, s_i) \mathbb{I}_{A' \times B'} (\xi_{i+k}, s_{i+k}) \right] \right|
+ \sum_{k=1}^{2n-2} \frac{1}{2n} \sum_{i=2n-1-k}^{2n-2} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_i, s_i) \mathbb{I}_{A' \times B'} (\xi_{i+k}, s_{i+k}) \right]. \tag{29}
\]
Clearly, the first term in the right-hand side of the inequality above tends to zero, being the tail of a convergent series (the fact that $S\infty < \infty$ follows via arguments already used for Eq. (27)). For the last term, we notice that, using Lemma 3.1,

$$\left| \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, s_i) \mathbb{I}_{A' \times B'} (\xi_{1+k}, s_{i+k}) \right] \right| \leq 4 \cdot \alpha_k,$$

and thus

$$\left| \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, s_i) \mathbb{I}_{A' \times B'} (\xi_{1+k}, s_{i+k}) \right] \right| \leq \frac{1}{2n} \sum_{k=1}^{n} \sum_{i=2n-1-k}^{2n-2} \left| \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, s_i) \mathbb{I}_{A' \times B'} (\xi_{1+k}, s_{i+k}) \right] \right| \leq \frac{1}{2n} \sum_{k=1}^{n} 4k \cdot \alpha_k,$$

which converges to 0 as $n$ goes to infinity by the assumption in Eq. (5) and the same arguments used in Remark 1.2. It remains to bound the second term. Expanding the products and recalling that $V, V', U, U'$ are independent random variables such that $\xi_1 \overset{d}{=} \xi_{1+k} \overset{d}{=} V \overset{d}{=} V'$ and $U \overset{d}{=} U' \overset{d}{=} \mu$, we have that

$$S_{2n-2} = \sum_{k=1}^{n} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, U) \mathbb{I}_{A' \times B'} (\xi_{1+k}, U') \right]$$

$$= \sum_{k=1}^{n} \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, U) \mathbb{I}_{A' \times B'} (\xi_{1+k}, U') \right] - \mathbb{E} \left[ \mathbb{I}_{A \times B} (V, U) \mathbb{I}_{A' \times B'} (V', U') \right]$$

$$= \sum_{k=1}^{n} \mu (B) \cdot \mu (B') \cdot \left( \mathbb{E} \left[ \mathbb{I}_{A \times A'} (\xi_1, \xi_{1+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times A'} (V, V') \right] \right).$$

Similarly, we obtain

$$\mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, s_i) \mathbb{I}_{A' \times B'} (\xi_{1+k}, s_{i+k}) \right]$$

$$= \mathbb{E} \left[ \mathbb{I}_{A \times B} (\xi_1, s_i) \mathbb{I}_{A' \times B'} (\xi_{1+k}, s_{i+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times B} (V, s_i) \mathbb{E} \left[ \mathbb{I}_{A' \times B'} (V', s_{i+k}) \right] \right]$$

$$= \mathbb{E} \left[ \mathbb{I}_{A \times B} (s_i, s_{i+k}) \right] \cdot \left( \mathbb{E} \left[ \mathbb{I}_{A \times A'} (\xi_1, \xi_{1+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times A'} (V, V') \right] \right).$$

Therefore the second term in the right-hand side of Eq. (29) is bounded by

$$\sum_{k=1}^{n} \left( \mathbb{E} \left[ \mathbb{I}_{A \times A'} (\xi_1, \xi_{1+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times A'} (V, V') \right] \right) \cdot \left| \mathbb{E} [B] \mu (B) - \frac{1}{2n} \sum_{i=1}^{n} \mathbb{I}_{B \times B'} (s_i, s_{i+k}) \right|. \ (30)$$

Using Proposition A.1 we have that

$$\sup_{k \in [2n-2]} \left| \mathbb{E} \left[ \mathbb{I}_{A \times A'} (\xi_1, \xi_{1+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times A'} (V, V') \right] \right| \overset{a.s.}{\rightarrow} 0.$$

In addition, using once again Lemma 3.1 and the assumption in Eq. (5) we have that

$$\sum_{k=1}^{n} \left| \mathbb{E} \left[ \mathbb{I}_{A \times A'} (\xi_1, \xi_{1+k}) \right] - \mathbb{E} \left[ \mathbb{I}_{A \times A'} (V, V') \right] \right| < \infty.$$

The last two equations imply that the bound in Eq. (30) tends to zero as $n$ tends to infinity for $\mu^N$-almost every sequence $(s_i)_{i \in \mathbb{N}} \in \mathbb{R}^N$, completing the proof of Eq. (28).

In order to conclude the proof of the lemma, it remains to generalize the result in Eq. (28) to all bounded and measurable functions. This can be done using the same techniques adopted to prove the law of large numbers, therefore we skip the details. □
We now complete the proof of Theorem 1.6.

**Proof of Theorem 1.6.** Recalling that \( \tilde{g}(x, y) := g(x, y) - \mathbb{E}[g(V, y)] \) and thanks to Lemma 3.2, it is enough to show that
\[
\frac{1}{\rho_{g,n}} 2n-1 \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \overset{d}{\to} \mathcal{N}(0, 1). \tag{31}
\]

The difficulty in establishing this convergence lies in the fact that we are dealing with a sum of random variables that are neither independent nor identically distributed. We proceed in two steps. First, we apply the Bernstein’s method, thus we reduce the problem to the study of a sum of “almost” independent random variables. More precisely, we use the decay of the correlations for the process \( \xi \) to decompose \( \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \) into two distinct sums, in such a way that one of them is a sum of “almost” independent random variables and the other one is negligible (in a sense that will be specified in due time). After having dealt with the lack of independence, we settle the issue that the random variables are not identically distributed using the Lyapounov’s condition.

We start with the first step. Recall that we assume the existence of two sequences \( p = p(n) \) and \( q = q(n) \) such that:

- \( p \xrightarrow{n \to \infty} +\infty \) and \( q \xrightarrow{n \to \infty} +\infty \),
- \( q = o(p) \) and \( p = o(n) \) as \( n \to \infty \),
- \( np^{-1}\alpha_q = o(1) \),
- \( \frac{p}{n} \cdot \sum_{j=1}^{p} j\alpha_j = o(1) \).

As said above, we represent the sum \( \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) \) as a sum of nearly independent random variables (the *big blocks* of size \( p \), denoted \( \beta_i \) below) alternating with other terms (the *small blocks* of size \( q \), denoted \( \gamma_i \) below) whose sum is negligible. We define \( k = \lfloor (2n - 1)/(p + q) \rfloor \).

We can thus write
\[
\sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) = \sum_{i=0}^{k-1} \beta_i + \sum_{i=0}^{k} \gamma_i,
\]
where, for \( 0 \leq i \leq k - 1 \),
\[
\beta_i = \beta_i(\tilde{g}, n) := \sum_{j=ip+iq+1}^{(i+1)p+iq} \tilde{g}(\xi_j, s_j), \quad \gamma_i = \gamma_i(\tilde{g}, n) := \sum_{j=(i+1)p+iq+1}^{(i+1)p+(i+1)q} \tilde{g}(\xi_j, s_j),
\]
and
\[
\gamma_k = \gamma_k(\tilde{g}, n) := \sum_{j=kp+kq+1}^{2n-1} \tilde{g}(\xi_j, s_j).
\]

Henceforth, we will omit the dependence on \( \tilde{g} \) and on \( n \) simply writing \( \beta_i \) and \( \gamma_i \), in order to simplify the notation (whenever it is clear). Setting \( H'_n := \frac{1}{\rho_{g,n}} \sum_{i=0}^{k-1} \beta_i \) and \( H''_n := \frac{1}{\rho_{g,n}} \sum_{i=0}^{k} \gamma_i \),

we can write
\[
\frac{1}{\rho_{g,n}} 2n-1 \sum_{j=1}^{2n-1} \tilde{g}(\xi_j, s_j) = H'_n + H''_n.
\]

The proof of Eq. (31) now consists of two steps. First, we show that \( H''_n \xrightarrow{P} 0 \), and secondly we show that the characteristic function of \( H''_n \) converges to the characteristic function of a standard Gaussian random variable. Then, we can conclude using standard arguments.
We start by proving that $H''_n \overset{P}{\to} 0$. By Chebyshev’s inequality and the fact that $\mathbb{E}[H''_n] = 0$, it is enough to show that $\mathbb{E}[(H''_n)^2] \to 0$ as $n \to \infty$. We can rewrite $\mathbb{E}[(H''_n)^2]$ as

$$\frac{1}{\rho_{g,n}^2} \cdot \mathbb{E} \left[ \left( \sum_{i=0}^{k} \gamma_i \right)^2 \right] = \frac{1}{\rho_{g,n}^2} \left( \mathbb{E} \left[ \sum_{i=0}^{k-1} \gamma_i^2 \right] + \mathbb{E} \left[ \sum_{i,j=0}^{k-1} \gamma_i \gamma_j \right] + 2 \mathbb{E} \left[ \sum_{i=1}^{k-1} \gamma_i \gamma_k \right] \right).$$

Note that, by definition of $\gamma_i$ and using Lemma 3.1 once again, for $i \neq j$, we have the bounds

$$\mathbb{E} [\gamma_i \gamma_j] \leq q^2 \cdot \alpha_{p(i-j)}; \quad \text{(32)}$$

and

$$\mathbb{E} [\gamma_i \gamma_k] \leq q \cdot (p + q) \cdot \alpha_{p(k-i)}.$$

Moreover, by the same argument used in Lemma 3.2, we have that

$$\mathbb{E} [\gamma_i^2] = \rho_{g,n}^2 \cdot q \cdot (1 + o(1)) = O(q)$$

and

$$\mathbb{E} [\gamma_k^2] = O(p + q) = O(p).$$

Hence, using Lemma 3.2,

$$\frac{1}{\rho_{g,n}^2} \mathbb{E}\left[ \sum_{i=0}^{k-1} \gamma_i^2 \right] = O \left( \frac{kq}{n} \right) = O \left( \frac{q}{p} \right) = o(1)$$

and

$$\frac{1}{\rho_{g,n}^2} \mathbb{E} [\gamma_k^2] = o(1).$$

Using Eq. (32), we also have that

$$\frac{1}{\rho_{g,n}^2} \mathbb{E}\left[ \sum_{i,j=0,i\neq j}^{k-1} \gamma_i \gamma_j \right] \leq \frac{2}{\rho_{g,n}^2} \sum_{j=0}^{k-1} \sum_{i=j+1}^{k-1} q^2 \cdot \alpha_{p(i-j)} = \frac{2}{\rho_{g,n}^2} \sum_{j=0}^{k-1} \sum_{m=1}^{k-j} q^2 \cdot \alpha_{pm} \leq \frac{2kq^2}{\rho_{g,n}^2} \cdot \sum_{m=1}^{\infty} \alpha_m = o(1),$$

where in the last inequality we used that, since $\alpha_n$ is decreasing,

$$\sum_{m=1}^{\infty} \alpha_m \leq \frac{1}{p} \sum_{s=1}^{\infty} \alpha_s \leq \frac{1}{p} \cdot \sum_{s=1}^{\infty} \alpha_s,$$

Analogously, we can prove that

$$\frac{2}{\rho_{g,n}^2} \mathbb{E}\left[ \sum_{i=0}^{k-1} \gamma_i \gamma_k \right] = o(1),$$

concluding the proof that $H''_n \overset{P}{\to} 0$.

Now we turn to the study of the limiting distribution of $H'_n$. We have that, for $t \in \mathbb{R}$,

$$\exp \left\{ it H'_n \right\} = \exp \left\{ \frac{it}{\rho_{g,n}} \sum_{i=0}^{k-1} \beta_i \right\}.$$
We now look at \( \exp \left\{ \frac{it}{\rho_{g,n}} \sum_{i=0}^{k-2} \beta_i \right\} \) and \( \exp \left\{ \frac{it}{\rho_{g,n}} \beta_{k-1} \right\} \). We have that the first random variable is measurable with respect to \( A_{k-1}^{(k-1)p+(k-2)q} \) and the second one is measurable with respect to \( A_{k-1}^{\infty} \). So, by Lemma 3.1,

\[
\left| \mathbb{E} \left[ \exp \left\{ \frac{it}{\rho_{g,n}} \sum_{i=0}^{k-2} \beta_i \right\} \right] - \mathbb{E} \left[ \exp \left\{ \frac{it}{\rho_{g,n}} \sum_{i=0}^{k-2} \beta_i \right\} \right] \right| \leq 4 \cdot \alpha_q.
\]

Iterating, we get

\[
\left| \sum_{i=0}^{k-1} \mathbb{E} \left[ \exp \left\{ \frac{it}{\rho_{g,n}} \beta_i \right\} \right] - \prod_{i=0}^{k-1} \mathbb{E} \left[ \exp \left\{ \frac{it}{\rho_{g,n}} \beta_i \right\} \right] \right| \leq 4 \cdot (k - 1) \cdot \alpha_q,
\]

the latter quantity tending to 0 as \( k \to \infty \) thanks to the assumptions on the sequences \( p \) and \( q \). The last step of the proof consists in showing that, as \( n \to \infty \),

\[
\prod_{i=0}^{k-1} \mathbb{E} \left[ \exp \left\{ \frac{it}{\rho_{g,n}} \beta_i \right\} \right] \to e^{i \frac{t^2}{2}}.
\]

(33)

Consider a collection of independent random variables \( X_{n,i}, n \in \mathbb{N}, i \in [k-1]_0 \), such that \( X_{n,i} \) has the same distribution as \( \frac{1}{\rho_{g,n}} \beta_i(n) \). By [Bil95, Theorem 27.3], a sufficient condition to ensure that \( \sum_{i=0}^{k-1} X_{n,i} \to \mathcal{N}(0,1) \) and so verifying Eq. (33), is the well-known Lyapounov's condition:

\[
\lim_{n \to \infty} \frac{1}{\rho_{g,n}^2} \sum_{i=0}^{k-1} \mathbb{E} \left[ Y_{n,i}^{2+\delta} \right] = 0, \quad \text{for some } \delta > 0,
\]

(34)

where \( Y_{n,i} := X_{n,i} \cdot \rho_{g,n} \overset{d}{=} \sum_{j=ip+iq}^{(i+1)p+iq} \tilde{g} (\xi_j, s_j) \). The condition is satisfied with \( \delta = 2 \) thanks to the following lemma.

**Lemma 3.3.** Under the assumptions of Theorem 1.6, we have that for \( \mu_n \)-almost every sequence \( (s_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^\mathbb{N} \), uniformly for all \( i \in [k-1]_0 \),

\[
\mathbb{E} \left[ (Y_{n,i})^4 \right] = O \left( p^2 \cdot \sum_{j=1}^{p} j \alpha_j \right).
\]

Before proving the lemma above above, we explain how it implies the condition in Eq. (34) with \( \delta = 2 \). By Lemma 3.2, we have that \( \rho_{g,n}^2 = 2n \cdot \rho_g^2 \cdot (1 + o(1)) \), thus

\[
\frac{1}{\rho_{g,n}^2} \sum_{i=0}^{k-1} \mathbb{E} \left[ Y_{n,i}^4 \right] \leq C \cdot \frac{k \cdot p^2 \cdot \sum_{j=1}^{p} j \alpha_j}{4n^2 \cdot \rho_g^4} \to 0,
\]

where for the limit we used the fact that \( \frac{k \cdot p}{n} \to 1 \) and that, by assumption, \( \frac{p \cdot \sum_{j=1}^{p} j \alpha_j}{n} \to 0 \).

We conclude the proof of Theorem 1.6 by proving Lemma 3.3. Let \( A_0 := [p] \) and \( A_i := [(i+1)p+iq] \setminus [ip+iq] \) for \( i \geq 1 \). Note that \( |A_i| = p \) for all \( i \geq 0 \). We have that

\[
\mathbb{E} \left[ (Y_{n,i})^4 \right] = \mathbb{E} \left[ \left( \sum_{j=ip+iq+1}^{(i+1)p+iq} \tilde{g} (\xi_j, s_j) \right)^4 \right]
\]

\[
= O \left( \sum_{j \in A_i} \mathbb{E} \left[ \tilde{g}^4 (\xi_j, s_j) \right] + \sum_{j,k \in A_i, j \neq k} \mathbb{E} \left[ \tilde{g}^2 (\xi_j, s_j) \tilde{g}^2 (\xi_k, s_k) \right] + \sum_{j,k \in A_i, j \neq k} \mathbb{E} \left[ \tilde{g}^3 (\xi_j, s_j) \tilde{g} (\xi_k, s_k) \right] \right)
\]

\[
+ \sum_{j,k,l \in A_i, j \neq k \neq l} \mathbb{E} \left[ \tilde{g} (\xi_j, s_j) \tilde{g} (\xi_k, s_k) \tilde{g} (\xi_l, s_l) \right] + \sum_{j,k,l,m \in A_i, j \neq k \neq l \neq m} \mathbb{E} \left[ \tilde{g} (\xi_j, s_j) \tilde{g} (\xi_k, s_k) \tilde{g} (\xi_l, s_l) \tilde{g} (\xi_m, s_m) \right].
\]
The fact that $\tilde{g}$ is bounded and the decay of the correlations will give us some bounds for each of these terms. First of all, since $\tilde{g}$ is bounded, we have that $\sum_{j \in A_i} \mathbb{E}\left[\tilde{g}^4(\xi_j, s_j)\right] = O(p)$, $\sum_{j,k \in A_i, j \neq k} \mathbb{E}\left[\tilde{g}^2(\xi_j, s_j) \tilde{g}^2(\xi_k, s_k)\right] = O\left(p^2\right)$, and $\sum_{j,k \in A_i, j \neq k} \mathbb{E}\left[\tilde{g}^3(\xi_j, s_j) \tilde{g}(\xi_k, s_k)\right] = O\left(p^2\right)$.

We now look at the fourth addend. We have that

$$\sum_{j,k,l \in A_i, j < k < l} \mathbb{E}\left[\tilde{g}^2(\xi_j, s_j) \tilde{g}(\xi_k, s_k) \tilde{g}(\xi_l, s_l)\right] = O\left(\sum_{l \in A_i} \sum_{k = p+iq+1}^{l-1} \sum_{j = p+iq+1}^{k} \alpha_{l-k}\right) = O\left(p^2\right),$$

since $\sum_{i=1}^{\infty} \alpha_i < +\infty$ by assumption and $|A_i| = p$. Finally, we estimate the last addend. We have that

$$\sum_{j,k,l,m \in A_i, j < k < l < m} \mathbb{E}\left[\tilde{g}(\xi_j, s_j) \tilde{g}(\xi_k, s_k) \tilde{g}(\xi_l, s_l) \tilde{g}(\xi_m, s_m)\right] = O\left(\sum_{j,k,l,m \in A_i, j < k < l < m} \min\{\alpha_{k-j}, \alpha_{m-l}\}\right).$$

We analyse the last expression. Since the sequence $(\alpha_n)_{n \in \mathbb{N}}$ is decreasing, we see that

$$\sum_{j,k,l,m \in A_i, j < k < l < m} \min\{\alpha_{k-j}, \alpha_{m-l}\} = \sum_{j \in A_i} \sum_{x=1}^{p-j} \sum_{l \in A_i, l > j+x} \sum_{y=1}^{p-l} \min\{\alpha_x, \alpha_y\}$$

$$= \sum_{j \in A_i} \sum_{x=1}^{p-j} \sum_{l \in A_i, l > j+x} \sum_{y=1}^{x} \alpha_x + \sum_{j \in A_i} \sum_{x=1}^{p-j} \sum_{l \in A_i, l > j+x} \sum_{y=x+1}^{p-l} \alpha_y.$$

Since $|A_i| = p$ we have that

$$\sum_{j \in A_i} \sum_{x=1}^{p-j} \sum_{l \in A_i, l > j+x} x \alpha_x \leq p^2 \cdot \sum_{x=1}^{p} x \alpha_x,$$

and that

$$\sum_{j \in A_i} \sum_{x=1}^{p-j} \sum_{l \in A_i, l > j+x} \sum_{y=x+1}^{p-l} \alpha_y \leq p^2 \cdot \sum_{y=1}^{p} y \alpha_y,$$

from which we conclude that

$$\sum_{j,k,l,m \in A_i, j \neq k \neq l \neq m} \mathbb{E}\left[\tilde{g}(\xi_j, s_j) \tilde{g}(\xi_k, s_k) \tilde{g}(\xi_l, s_l) \tilde{g}(\xi_m, s_m)\right] = O\left(p^2 \cdot \sum_{j=1}^{p} j \alpha_j\right).$$

This concludes the proof of Lemma 3.3, and hence of Theorem 1.6 as well.
A  A uniform estimate for central limit theorems of weakly-correlated random variables

Proposition A.1. Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of i.i.d. real-valued random variables. Let also \(f : \mathbb{R}^2 \to \mathbb{R}\) be a bounded measurable function. Then

\[
\sup_{k \in [n]} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i, X_{i+k}) - f(X_1, X_2) \right| \xrightarrow{a.s.} n \to \infty 0.
\]

Proof. Let \(A > 0\) be such that \(f(x, y) \leq A\) for all \(x, y \in \mathbb{R}^2\) and set \(S_{n,k} := \sum_{i=1}^{n} f(X_i, X_{i+k})\).

From \(^6\) [FMN20, Proposition 5], we have that for any \(x > 0\) and \(k \in [n],\)

\[
P \left( |S_{n,k} - \mathbb{E}[S_{n,k}]| \geq x \right) \leq 2 \exp \left\{ -\frac{x^2}{18A^2 \cdot n} \right\}.
\]

Using this bound together with a union bound, we get that for any \(\varepsilon > 0,\)

\[
P \left( \sup_{k \in [n]} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i, X_{i+k}) - f(X_1, X_2) \right| \geq \varepsilon \right) \leq 2n \exp \left\{ -\frac{n \cdot \varepsilon^2}{18A^2} \right\}.
\]

Then the statement of the proposition follows using a standard Borel-Cantelli argument. \(\square\)

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References

[BBCL15] M. Benaïm, I. Benjamini, J. Chen, and Y. Lima. A generalized Pólya’s urn with graph based interactions. Random Structures Algorithms, 46(4):614–634, 2015.

[Ber73] K. N. Berk. A central limit theorem for \(m\)-dependent random variables with unbounded \(m\). Ann. Probability, 1:352–354, 1973.

[Bil95] P. Billingsley. Probability and measure. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, third edition, 1995. A Wiley-Interscience Publication.

[BNH07] E. Ben-Naim and N. Hengartner. Efficiency of competitions. Physical Review E, 76(2):026106, 2007.

[BNKK06] E. Ben-Naim, B. Kahng, and J. S. Kim. Dynamics of multi-player games. Journal of Statistical Mechanics: Theory and Experiment, 2006(07):P07001, 2006.

[Bor20] J. Borga. Local convergence for permutations and local limits for uniform \(\rho\)-avoiding permutations with \(|\rho| = 3\). Probab. Theory Related Fields, 176(1-2):449–531, 2020.

[BR89] P. Baldi and Y. Rinott. On normal approximations of distributions in terms of dependency graphs. Ann. Probab., 17(4):1646–1650, 1989.

\(^{6}\)Note that the assumptions of [FMN20, Proposition 5] are satisfied: indeed by [FMN20, Theorem 5] we have that for all \(r \geq 1\) the \(r\)-th cumulant of \(S_{n,k}\), denoted by \(K^{(r)}(S_{n,k})\) is bounded by \(K^{(r)}(S_{n,k}) \leq n \cdot 4^{r-1} \cdot r^{-2} \cdot A^r\).
[Bra07] R. C. Bradley. *Introduction to strong mixing conditions. Vol. 1.* Kendrick Press, Heber City, UT, 2007.

[Eks14] M. Ekström. A general central limit theorem for strong mixing sequences. *Statist. Probab. Lett.*, 94:236–238, 2014.

[FMN20] V. Féray, P.-L. Méliot, and A. Nikeghbali. Graphons, permutons and the Thoma simplex: three mod-Gaussian moduli spaces. *Proc. Lond. Math. Soc. (3)*, 121(4):876–926, 2020.

[HR48] W. Hoeffding and H. Robbins. The central limit theorem for dependent random variables. *Duke Math. J.*, 15:773–780, 1948.

[Ibr62] I. A. Ibragimov. Some limit theorems for stationary processes. *Teor. Verojatnost. i Primenen.*, 7:361–392, 1962.

[IL71] I. A. Ibragimov and Y. V. Linnik. *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman.

[Jan88] S. Janson. Normal convergence by higher semi-invariants with applications to sums of dependent random variables and random graphs. *Ann. Probab.*, 16(1):305–312, 1988.

[Jan04] S. Janson. Large deviations for sums of partly dependent random variables. *Random Structures Algorithms*, 24(3):234–248, 2004.

[Pel92] M. Peligrad. On the central limit theorem for weakly dependent sequences with a decomposed strong mixing coefficient. *Stochastic Process. Appl.*, 42(2):181–193, 1992.

[PL82] M. B. Petrovskaya and A. M. Leontovich. The central limit theorem for a sequence of random variables with a slowly growing number of dependences. *Teor. Veroyatnost. i Primenen.*, 27(4):757–766, 1982.

[PRW97] D. N. Politis, J. P. Romano, and M. Wolf. Subsampling for heteroskedastic time series. *J. Econometrics*, 81(2):281–317, 1997.

[Ros56] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.*, 42:43–47, 1956.

[RW00] J. P. Romano and M. Wolf. A more general central limit theorem for $m$-dependent random variables with unbounded $m$. *Statist. Probab. Lett.*, 47(2):115–124, 2000.