Well-posedness of the nonlinear Schrödinger equation on the half-plane

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Abstract
The initial-boundary value problem (ibvp) for the nonlinear Schrödinger (NLS) equation on the half-plane with nonzero boundary data is studied by advancing a novel approach recently developed for the well-posedness of the cubic NLS equation on the half-line, which takes advantage of the solution formula produced via the unified transform of Fokas for the associated linear ibvp. For initial data in Sobolev spaces on the half-plane and boundary data in Bourgain spaces arising spontaneously when the linear ibvp is solved with zero initial data, the present work introduces a natural method for proving local well-posedness of nonlinear ibvps in higher dimensions.

Keywords: 2D nonlinear Schrödinger equation, initial-boundary value problem, well-posedness in Sobolev spaces, Bourgain spaces, unified transform method of Fokas, linear space-time estimates, Strichartz estimates
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1. Introduction and results
We consider the following initial-boundary value problem (ibvp) for the two-dimensional nonlinear Schrödinger (NLS) equation on the half-plane

\[ \begin{align*}
\iu u_t + u_{x_1 x_1} + u_{x_2 x_2} \pm |u|^{\alpha - 1} u &= 0, \quad (x_1, x_2, t) \in \mathbb{R}^2 \times (0, T), \\
n(x_1, x_2, 0) &= u_0(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \\
n(x_1, 0, t) &= g_0(x_1, t), \quad (x_1, t) \in \mathbb{R} \times [0, T],
\end{align*} \]

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where $\alpha > 1$, and prove its well-posedness for initial data $u_0(x_1, x_2)$ in the Sobolev space $H^s(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+) \times H^s(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)$ with $s \geq 0$ and boundary data $g_b(x_1, t)$ in a Bourgain-type space that arises naturally when the associated linear ibvp is solved with zero initial datum. This result is proven by advancing into two dimensions a novel approach recently introduced for the well-posedness of ibvps on the half-line [FHM1, FHM2, HM], which exploits the linear solution formulae obtained via Fokas’s unified transform.

Remarkably, the Bourgain space associated with the one-dimensional NLS initial value problem (ivp) on the line emerges spontaneously in our work as the natural space for the boundary data. Furthermore, the regularity in the boundary variables $(x_1, t)$ of the solution of the linear Schrödinger ivp on the plane is described by the aforementioned Bourgain space, thereby extending into higher dimensions the result of [KPV1] on the time regularity of the linear Schrödinger ivp on the line. In this respect, the Bourgain space from the one-dimensional NLS on the line comes forth as the optimal choice for the boundary data space of the two-dimensional NLS on the half-plane.

The NLS equation $iu_t + \Delta u \pm |u|^{\alpha-1} u = 0$ is a universal model, arising in multiple areas of mathematical physics such as nonlinear optics [Aga, Tal], water waves [CSS, HO, Per, Z], plasmas [WW], and Bose–Einstein condensates [CP, CH, KSS, PS]. As such, it has been studied extensively, from many points of view, and in a variety of different contexts. In one space dimension, Zakharov and Shabat showed that the cubic case $\alpha = 3$ is a completely integrable system [ZS] (see also [AKNS]). Thus, under the assumption of sufficient smoothness and decay, they were able to study the associated ivp on the line by analysing the Lax pair of the equation via the inverse scattering transform. For rough data, the NLS ivp has been studied in great detail via harmonic analysis techniques. In particular, Tsutsumi [TsuY2] proved global well-posedness in $L^2(\mathbb{R}^n)$ in the subcritical case $1 < \alpha < 1 + \frac{4}{n}$, Cazenave and Weissler [CazW1] extended this result to the critical case $\alpha = 1 + \frac{4}{n}$ and later to all $\alpha > 1$ for data in $H^s(\mathbb{R}^n)$, $s > 0$, using Besov spaces [CazW2]. Other relevant works include Ginibre and Velo [GV1, GV2], Kato [K1, K2], and Constantin and Saut [CS]. The periodic case turned out to be more challenging and required the 1993 breakthrough of Bourgain [B1], who proved well-posedness for data in $H^s(\mathbb{T})$, $s \geq 0$, by introducing the celebrated $X^{b,s}$ spaces (with $b = \frac{1}{s}$). Further results on well-posedness and ill-posedness in one and higher dimensions for the periodic and the non-periodic ivp are available in [B2, B3, B4, CKSTT, D, KPV2, KV] and in numerous other works in the literature.

Contrary to the ivp, progress towards the rigorous analysis of ibvps for NLS and other nonlinear evolution equations is rather limited. This can be largely attributed to the absence of the Fourier transform and underlying theory in the case of domains with a boundary. Recall, in particular, that the procedure for establishing local well-posedness of a nonlinear problem via a contraction mapping argument is initiated by defining an iteration map through the solution of the associated forced linear problem. In the case of the ivp, this task is straightforward thanks to the Fourier transform and Duhamel’s principle. For ibvps, however, a proper spatial transform is not available and hence a significant challenge is already present at the first step of the analysis. This difficulty is reflected in two independent approaches that were introduced in the early 2000s for showing well-posedness of the Korteweg–de Vries (KdV) equation on the half-line, namely the works of Colliander and Kenig [CK] (in fact, this work is concerned with the generalized KdV equation) and of Bona, Sun and Zhang [BSZ1]. The first approach, which was later improved further for KdV and also adapted for NLS on the half-line by Holmer [H1, H2], is based on expressing the relevant forced linear ibvp as a ‘chain’ of ivps, thus allowing one to take advantage of the powerful Fourier analysis machinery but, at the same time, signifying a departure from the ivp framework. The second approach, which has also been employed for NLS on the half-line [BSZ2], relies on solving the forced linear ibvp via a Laplace transform.
in the temporal variable, in contrast with the spatial (Fourier) transform used in the case of the ivp.

A novel approach was recently introduced for proving well-posedness of ibvps for nonlinear evolution equations. This approach, which has already been implemented for the NLS, KdV and ‘good’ Boussinesq equations on the half-line [FHM1, FHM2, HM], overcomes the lack of a proper spatial transform in the ibvp setting by exploiting the linear solution formulae produced via the unified transform, also known as the Fokas method [F1, F2]. Fokas’s unified transform can be employed for solving linear evolution equations of arbitrary spatial order, supplemented with any type of admissible boundary data (including non-separable ones) and formulated in any number of spatial dimensions. In this light, taking also into account that no classical spatial transform exists for ivps involving linear evolution equations of order higher than two, the unified transform can be regarded as the analogue of the Fourier transform in the case of linear ibvps. As such, it comes forth as the natural way of defining the iteration map to be used for showing well-posedness of nonlinear ibvps via contraction mapping.

The essence of the new approach lies in the analysis of the pure linear ibvp, which corresponds to the case of zero initial datum and zero forcing. Indeed, the correct space for the boundary datum of the nonlinear problem is discovered in the process of estimating the boundedness of the Laplace transform in $L^2$, the extension of these estimates, as given the solution of the linear ivp with respect to the boundary variable(s). In one dimension, this leads to the ivp time estimates previously obtained in [KPV1] (see also theorem 4 in [FHM1] and theorem 3.2); in higher dimensions, it provides the extension of these estimates, as given in theorems 3.1 and 3.3. In a different setting, the regularity of the solution with respect to singularities on the boundary was discussed in [TB].

Before stating our results precisely, we define the function spaces needed. For $s \geq 0$, the half-plane Sobolev space $H^s(\mathbb{R} \times \mathbb{R}^+)$ for the initial data is defined as a restriction of the Sobolev space $H^s(\mathbb{R}^2)$ by

$$H^s(\mathbb{R} \times \mathbb{R}^+) = \left\{ f \in L^2(\mathbb{R} \times \mathbb{R}^+) : \| f \|_{H^s(\mathbb{R} \times \mathbb{R}^+)} = \inf_{F_{\mathbb{R} \times \mathbb{R}^+}} \| F \|_{H^s(\mathbb{R})} < \infty \right\}. \quad (1.2)$$

Throughout this work, we assume that $T < 1$. The space $B^s_T$ for the boundary data is defined as

$$B^s_T = \left\{ g \in L^2(\mathbb{R} \times [0, T]) : \| g \|_{B^s_T} = \| g \|_{X^s_T} + \| g \|_{X^s_T}^{1/4} < \infty \right\}, \quad (1.3a)$$

where for $(\sigma, b) = (0, \frac{2s+1}{2})$ and $(\sigma, b) = (s, \frac{1}{2})$ the two components of the $B^s_T$-norm are given by

$$\| g \|_{X^s_T} = \left( \int_{k_1 \in \mathbb{R}} \left( 1 + k_1^2 \right)^s \| \hat{g}^{x_1}(k_1, t) \|_{H^s(\mathbb{R} \times \mathbb{R})}^2 \right)^{1/2} \quad (1.3b)$$

with $\hat{g}^{x_1}(k_1, t)$ denoting the Fourier transform of $g(x_1, t)$ with respect to $x_1$, i.e.

$$\hat{g}^{x_1}(k_1, t) = \int_{x_1 \in \mathbb{R}} e^{-ik_1 x_1} g(x_1, t) \, dx_1. \quad (1.4)$$

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In fact, as shown in section 2, a straightforward manipulation of the global-in-time counterparts of the norms (1.3) reveals that for $s \geq 0$ the space $B^s_r$ is closely related to the space

$$B^s_r = X^{s,\frac{1}{4}} \cap \chi^{s,\frac{1}{4}}, \quad \|g\|_{B^s_r} \doteq \|g\|_{\chi^{s,\frac{1}{4}}} + \|g\|_{\chi^{s,\frac{1}{4}}},$$

(1.5a)

where $X^{s,b}$ is the usual Bourgain space associated with the NLS ivp on the line, i.e.

$$X^{s,b} = \left\{ g \in L^2(\mathbb{R}_1 \times \mathbb{R}_2) : \|g\|_{X^{s,b}} \doteq \left\| (1 + k^2)^{\frac{s}{2}} (1 + |\tau + k^2|) \frac{1}{4} \hat{g}(k, \tau) \right\|_{L^2(\mathbb{R}_1 \times \mathbb{R}_2)} < \infty \right\}$$

(1.5b)

with $\hat{g}(k, \tau)$ denoting the spatiotemporal Fourier transform of $g(x_1, t)$, that is

$$\hat{g}(k, \tau) = \int_{\mathbb{R}_1} e^{-ikx_1 - it\tau} g(x_1, \tau) \text{d} x_1.$$  

(1.6)

We shall show that the NLS ibvp (1.1) is locally well-posed in the sense of Hadamard, namely that it possesses a unique solution which depends continuously on the prescribed initial and boundary data. We note that for $s > \frac{1}{2}$ the data must satisfy the compatibility condition

$$u_0(x_1, 0) = g_0(x_1, 0) \quad \forall x_1 \in \mathbb{R}.\quad (1.7)$$

For ‘smooth’ data ($s > 1$), the precise statement of our main result is the following.

**Theorem 1.1** (Well-posedness of NLS on the half-plane with smooth data). Suppose $1 < s \leq \frac{1}{2}$ and $\frac{1}{s+1} \in \mathbb{N}$. For initial data $u_0 \in H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)$ and boundary data $g_0 \in B^s_r$ satisfying the compatibility condition (1.7), ibvp (1.1) for NLS on the half-plane has a unique solution $u \in C([0, T^*]; H^s(\mathbb{R}_1 \times \mathbb{R}_2^+))$, which admits the estimate

$$\sup_{t \in [0, T^*]} \|u(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} \leq 2c_s \|u_0\|_D, \quad c_s = c(s) > 0,$$

where $\|u_0\|_D = \|u_0\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} + \|g_0\|_{B^s_r}$ and the lifespan $T^*$ is given by

$$T^* = \min \left\{ T, c_{s,0}\|u_0, g_0\|_D^{-2(\alpha - 1)} \right\}, \quad c_{s,0} = c(s, \alpha) > 0.$$

Moreover, the data-to-solution map $\{u_0, g_0\} \mapsto u$ is locally Lipschitz continuous.

The restriction $s > 1$ in the above theorem allows us to handle the NLS nonlinearity in a straightforward way via the algebra property in $H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)$. For $s \leq 1$, however, the algebra property is not available and, as it is well-known from the NLS ivp on the whole plane, the nonlinearity must be handled in a different way that relies on the celebrated Strichartz estimates. In this case, the solution space for the ivp additionally involves the Bessel potential space $H^{s, b}(\mathbb{R}_2^2)$, which for $s \geq 0$ and $1 < p < \infty$ is defined by

$$H^{s, p}(\mathbb{R}_2^2) = \left\{ f \in L^p(\mathbb{R}_2^2) : \|f\|_{H^{s, p}(\mathbb{R}_2^2)} \doteq \left\| (1 + |k|^2)^{\frac{s}{2}} \tilde{f}(k) \right\|_{L^p(\mathbb{R}_2^2)} < \infty \right\}, \quad (1.8)$$

where $\tilde{f}$ is the Fourier transform of $f$ with respect to $x \in \mathbb{R}^2$, i.e.

$$\tilde{f}(k) = \int_{x \in \mathbb{R}^2} e^{-ikx} f(x) \text{d} x, \quad k \in \mathbb{R}^2.\quad (1.9)$$
where \( kx \equiv k_1 x_1 + k_2 x_2 \). Analogously, for “rough” data \((s \leq 1)\), the data-to-solution map involves the Bessel potential space \( H^{s,p}(\mathbb{R}_1 \times \mathbb{R}_2^+) \), which is defined as a restriction of \( H^{s,q}(\mathbb{R}_1^2) \) similarly to (1.2). Our result provides an ibvp analogue of the classical subcritical theory for the NLS non-periodic ivp established by Tsutsumi \([TsuY2]\) and further developed by Cazenave and Weissler \([CazW2]\). It reads as follows.

**Theorem 1.2** (Well-posedness of NLS on the half-plane with rough data).

(i) For \( 0 \leq s < 1 \) with \( s \neq 1 \), suppose

\[
p = \frac{2\alpha}{1 + (\alpha - 1)s}, \quad q = \frac{2\alpha}{(\alpha - 1)(1 - s)}, \quad \begin{cases} 2 \leq \alpha < \frac{3 - s}{1 - s}, & 0 \leq s < \frac{1}{2}, \\ 2 \leq \alpha < \frac{2 - s}{1 - s}, & \frac{1}{2} \leq s < 1. \end{cases}
\]

Then, the NLS ibvp (1.1) with the compatibility condition (1.7) for \( s > \frac{1}{2} \) has a unique solution

\[
u \in C([0, T^*]; H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)) \cap L^q([0, T^*]; H^{s,p}(\mathbb{R}_1 \times \mathbb{R}_2^+))
\]

which admits the estimate

\[
sup_{t \in [0, T^*]} \|u(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} + sup_{x_2 \in [0, \infty)} \|u(x_2)\|_{H^p_x} + \|u\|_{L^q([0, T^*]; H^{s,p}(\mathbb{R}_1 \times \mathbb{R}_2^+))} \\
\leq 2c_{s,\alpha} \|(u_0, g_0)\|_D,
\]

where \( c_{s,\alpha} = c(s, \alpha) > 0 \), \( \|(u_0, g_0)\|_D = \|u_0\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} + \|g_0\|_{H^s} \) and the lifespan \( T^* \) is given by

\[
T^* = \begin{cases} \min \left\{ T, c_{s,\alpha} \|(u_0, g_0)\|_D \right\}^{\frac{2\alpha}{4\alpha - 1}}, & 0 \leq s < \frac{1}{2}, \\ \min \left\{ T, c_{s,\alpha} \|(u_0, g_0)\|_D \right\}^{\frac{2\alpha}{4\alpha - 1}}, & \frac{1}{2} \leq s < 1. \end{cases}
\]

Moreover, the data-to-solution map \( \{(u_0, g_0)\} \mapsto u \) is locally Lipschitz continuous.

(ii) For \( s = 1 \), suppose

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 < p < \min \left\{ \alpha + 1, \frac{2\alpha}{\alpha - 1} \right\}, \quad \alpha \geq 2.
\]

Then, the NLS ibvp (1.1) has a unique solution

\[
u \in C([0, T^*]; H^1(\mathbb{R}_1 \times \mathbb{R}_2^+)) \cap L^q([0, T^*]; H^{1,p}(\mathbb{R}_1 \times \mathbb{R}_2^+)),
\]

which admits the estimate

\[
sup_{t \in [0, T^*]} \|u(t)\|_{H^1(\mathbb{R}_1 \times \mathbb{R}_2^+)} + sup_{x_2 \in [0, \infty)} \|u(x_2)\|_{H^p_x} + \|u\|_{L^q([0, T^*]; H^{1,p}(\mathbb{R}_1 \times \mathbb{R}_2^+))} \\
\leq 2c_{s,\alpha} \|(u_0, g_0)\|_D
\]
where \( c_{p,0} = c(p, \alpha) > 0 \), \( \| (u_0, g_0) \|_D = \| u_0 \|_{H^1(\mathbb{R}_+ \times \mathbb{R}^+)} + \| g_0 \|_{\mathcal{B}} \) and the lifespan \( T^* \) is given by

\[
T^* = \min \left\{ T, c_{p,0} \| (u_0, g_0) \|_D^{\frac{2(p-1)}{p-\alpha}} \right\}.
\]

Moreover, the data-to-solution map \( \{ u_0, g_0 \} \mapsto u \) is locally Lipschitz continuous.

We note that the case \( s = 1 \) in two as well as in higher dimensions has been considered by Audiard in [Aud], where a global well-posedness result is additionally provided. In fact, the unified-transform-inspired approach of the present work can also deal with dimensions higher than two. However, this aspect is not considered here, as we choose to focus on the two-dimensional results of theorems 1.1 and 1.2 which advance for the first time the unified-transform-inspired approach of [FHM1, FHM2, HM] to higher than one dimensions. Furthermore, we note that \( s = \frac{1}{2} \) is not included in theorem 1.2. The origin of this exclusion can be traced to the adaptations of Sobolev spaces from the half-line to the whole line performed in section 4, which explains why the restriction \( s \neq \frac{1}{2} \) also occurs in the one-dimensional problem for NLS and KdV [BSZ2, ET, H1, H2] but not in the corresponding theory for the ivp.

We note that the vast majority of results in the literature on the Hadamard well-posedness of ibvps for NLS in higher than one spatial dimensions refer to the case of zero (homogeneous) boundary data; see, for example, Brezis and Gallouet [BG], Tsutsumi [TsuY1], Tsutsumi [TsuM1, TsuM2], and Burq, Gérard and Tzvetkov [BGT1, BGT2]. Indeed, to the best of our knowledge, the only well-posedness results available for the non-homogeneous NLS ibvp (1.1) are those in the recent works by Audiard [Aud] and Ran, Sun and Zhang [RSZ]. We emphasize, however, that the results of the present work are established via an entirely different method, namely by advancing into two spatial dimensions the novel approach which was recently introduced for one-dimensional ibvps [FHM1, FHM2, HM] and which relies on Fokas’s unified transform solution formulae. Taking into account the wide range of applicability of the unified transform, the approach developed in our work could be further adapted and generalized for showing well-posedness of ibvps involving other well-known evolution equations in two as well as in higher dimensions. That is, thanks to the universality of the unified transform as a solution method for linear ibvps in any spatial dimension, formulated in a variety of physical domains and with any admissible choice of boundary conditions, the present work introduces a general framework for the local well-posedness of higher-dimensional ibvps. Concrete examples of such ibvps that could be studied via the methodology introduced in this work include the Davey–Stewartson and Kadomtsev–Petviashvili equations formulated on the half-plane with nonzero boundary conditions.

Theorems 1.1 and 1.2 will be established via a contraction mapping argument. Hence, a key role in the analysis will be played by the forced linear analogue of the nonlinear problem (1.1), which reads

\[
\begin{align*}
\tag{1.12a}
i u_t + u_{x_1 x_1} + u_{x_2 x_2} &= f(x_1, x_2, t), & (x_1, x_2, t) &\in \mathbb{R} \times \mathbb{R}^+ \times (0, T), \\
\tag{1.12b}
u(x_1, x_2, 0) &= u_0(x_1, x_2), & (x_1, x_2) &\in \mathbb{R} \times \mathbb{R}^+ \\
\tag{1.12c}
u(x_1, 0, t) &= g_0(x_1, t), & (x_1, t) &\in \mathbb{R} \times [0, T].
\end{align*}
\]

The forced linear ibvp (1.12) will be estimated by taking advantage of the following explicit solution formula obtained via the unified transform of Fokas:
\[ u(x_1, x_2, t) = S[u_0, g_0; f](x_1, x_2, t) \]  

\[ u(x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - \beta k_1^2 + \beta k_2^2} \hat{u}_0(k_1, k_2) dk_1 dk_2 \]  

\[ - \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - \beta k_1^2 + \beta k_2^2} \hat{u}_0(k_1, -k_2) dk_2 dk_1 \]  

\[ - \frac{i}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - \beta k_1^2 + \beta k_2^2} \int_{t' = 0}^t e^{\beta k_1^2 + \beta k_2^2} \]  

\[ \cdot \hat{f}(k_1, k_2, t') dk_1 dk_2 \]  

\[ + \frac{i}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - \beta k_1^2 + \beta k_2^2} \int_{t' = 0}^t e^{\beta k_1^2 + \beta k_2^2} \]  

\[ \cdot \hat{f}(k_1, -k_2, t') dk_1 dk_2 \]  

\[ + \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - \beta k_1^2 + \beta k_2^2} 2k_2 \hat{g}_0(k_1, -k_1^2 - k_2^2) dk_2 dk_1. \]  

In the above formula, \( \hat{u}_0 \) stands for the half-plane Fourier transform of the initial datum \( u_0 \), i.e.

\[ \hat{u}_0(k_1, k_2) = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} e^{-ik_1x_1 - ik_2x_2} u_0(x_1, x_2) dx_2 dx_1, \]  

which is well-defined for \( k_1 \in \mathbb{R} \) and \( k_2 \in \mathbb{C}^+ = \{ k_2 \in \mathbb{C} : \text{Im}(k_2) \leq 0 \} \) due to the fact that \( x_2 \geq 0 \). Moreover, \( \hat{g}_0(k_1, \tau) \) denotes the spatiotemporal transform

\[ \hat{g}_0(k_1, \tau) = \int_{x_1 \in \mathbb{R}} \int_{t = 0}^T e^{-ik_1x_1 - \beta \tau} g_0(x_1, t) dt dx_1, \]

which makes sense for \( k_1 \in \mathbb{R} \) and \( \tau \in \mathbb{C} \) since integration in \( t \) takes place over a bounded interval. Finally, the contour of integration \( \partial D \) is the positively oriented boundary of the first quadrant \( D \) of the complex \( k_2 \)-plane, as shown in figure 1. A concise derivation of formula (1.13), whose homogeneous version was first derived in [F3], is provided in the appendix.

Starting from the unified transform formula (1.13), we shall establish the following result which, apart from being instrumental in the proof of theorem 1.1 for the well-posedness of the nonlinear ibvp (1.1), is interesting in its own right in regard to the forced linear ibvp (1.12).

**Theorem 1.3** (Forced linear Schrödinger on the half-plane). Suppose that \( 1 < s \leq \frac{1}{2} \). Then, the unified transform formula (1.13) defines a solution \( u = S[u_0, g_0; f] \) to the forced linear Schrödinger ibvp (1.12) supplemented with the compatibility condition (1.7), which satisfies the estimate

\[ \sup_{t \in [0, T]} \| u(t) \|_{H^s(\mathbb{R} \times \mathbb{R}^+)} \leq C_s \left( \| u_0 \|_{H^s(\mathbb{R} \times \mathbb{R}^+)} + \| g_0 \|_{H^s(\mathbb{R} \times \mathbb{R}^+)} \right) \]  

\[ + \sqrt{T} \sup_{t \in [0, T]} \| f(t) \|_{H^s(\mathbb{R} \times \mathbb{R}^+)} , \]  

where \( C_s > 0 \) is a constant that depends only on \( s \).
The literature on the ivp for the NLS equation in one and higher dimensions is extensive. Further results on the well-posedness, stability and blow-up behaviour of NLS via hard analysis as well as integrability techniques can be found in [BM, BGT3, CKS, DPC, DZ1, DZ2, FH, GS, KPV3, KTV, LPSS, LP, M, MR, SS, V, W] and the references therein. Moreover, for a thorough introduction to Fokas’s unified transform method we refer the reader to the monograph [F2] and the review articles [DTV, FS, Pel]. In particular, we note that the ibvp on the half-line and on the interval for the cubic NLS equation has been studied via the integrable nonlinear extension of the unified transform by Fokas and collaborators in [FIS] and [F1] respectively. Other works via similar techniques include [F4, FP, LF1, LF2, OY].

Structure of the paper. In section 2, employing the Fokas unified transform formula (1.13) and exploiting the boundedness of the Laplace transform in $L^2$, we estimate the solution of the pure linear Schrödinger ibvp, i.e. of problem (1.12) with zero initial datum and zero forcing. The resulting estimate is key to our analysis, as it reveals the boundary data space $B_{sT}$. In section 3, we show that the $x_1t$-regularity of the solution of both the homogeneous and the forced linear Schrödinger on the whole line is described by the space $B_{sT}$. Then, in section 4, we combine the estimates of the previous sections to obtain theorem 1.3 for the forced linear ibvp (1.12). This result provides the central estimate needed for the proof of theorem 1.1 on the well-posedness of NLS on the half-plane with smooth data, which is carried out via contraction mapping in section 5. Theorem 1.2 for the well-posedness of NLS on the half-plane with rough data as well as the required estimates for the forced linear ibvp analogous to theorem 1.3 are established in sections 6 and 7. Finally, a concise derivation of the unified transform solution formula (1.13) is provided in the appendix.

2. The pure linear ibvp

The essence of the analysis of the forced linear ibvp (1.12) is captured in the simplest genuine such problem, namely an ibvp with zero initial datum and forcing but with a nonzero boundary datum (note that the case of zero boundary datum can be reduced to an ivp). In fact, this model problem can be simplified further by assuming a boundary datum with compact support in the $t$-variable. Thus, we are led to the following problem, which we identify as the pure linear ibvp:

\begin{align}
iv_t + v_{x_1x_1} + v_{x_2x_2} &= 0, & (x_1, x_2, t) &\in \mathbb{R} \times \mathbb{R}^+ \times (0, 2), \\
v(x_1, x_2, 0) &= 0, & (x_1, x_2) &\in \mathbb{R} \times \mathbb{R}^+, \\
v(x_1, 0, t) &= g(x_1, t), & \text{supp}(g) &\subset \mathbb{R}_{x_1} \times (0, 2). 
\end{align} (2.1)
As we shall see below, the estimation of the solution of the pure linear ibvp (2.1) in the usual Sobolev space $H^s(\mathbb{R} \times \mathbb{R}^+)$ reveals the exotic Bourgain-type space $B'^s$ defined by (1.5) and, in turn, the space $B'_{s,c}$ for the boundary datum of the NLS ibvp (1.1). We note that, contrary to the half-line, where placing the initial datum in $H^s(\mathbb{R}^+)$ automatically requires a Sobolev boundary datum (see theorem 5 in [FHM1]), in the case of the half-plane it is not \textit{a priori} clear what is the appropriate space for the boundary datum. In this respect, the pure linear ibvp holds a central role in the analysis of both the linear and the nonlinear problem. Another source of motivation for the boundary data space $B_{s,c}$ is depicted in figure D. Importantly, note that thanks to the support assumption (2.1c) the boundary datum $g$ appears in the above formula through its two-dimensional Fourier transform (1.6) instead of the truncated transform (1.15).

\textbf{Theorem 2.1} (Pure linear ibvp estimate). The solution $v = S[0, g; 0]$ of the pure linear ibvp (2.1) given by formula (2.2) admits the estimate

$$ \sup_{t \in [0, 2]} \| v(t) \|_{H^s(\mathbb{R})} \leq c_s \| g \|_{B'^s}, \quad s \geq 0. \tag{2.3} $$

\textbf{Remark 2.1}. In fact, one can also prove the following estimate:

$$ \sup_{x_2 \in [0, \infty]} \| v(x_2) \|_{B'^s} \leq c_s \| g \|_{B'^s}, \quad s \in \mathbb{R}, \tag{2.4} $$

which allows one to carry out the contraction for the nonlinear ibvp in the space $C([0, T^*]; H^s(\mathbb{R}_{x_1} \times \mathbb{R}^{+}_{x_2})) \cap C(\mathbb{R}_{x_1}; B'_{s,c})$ instead of the Hadamard space of theorem 1.1.

\textbf{Proof of Theorem 2.1}. We decompose formula (2.2) in two parts as $v = v_1 + v_2$ where

$$ v_1(x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - ik_2^2 t + ik_1^2 T} 2k_2 \hat{g}(k_1, -k_1^2 + k_2^2) dk_2 dk_1 \tag{2.5} $$

corresponds to the imaginary axis portion of $\partial D$ and

$$ v_2(x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - ik_2^2 t + ik_1^2 T} 2k_2 \hat{g}(k_1, -k_1^2 + k_2^2) dk_2 dk_1 \tag{2.6} $$

corresponds to the part of $\partial D$ along the real axis. The estimation of $v_2$ is easier, while that of $v_1$ is more challenging and relies crucially on the boundedness of the Laplace transform in $L^2$. 

\textit{Estimation along the real axis}. Let $V_2(x_1, x_2, t)$ be a global-in-space function defined via the two-dimensional Fourier transform

$$ V_2(k_1, k_2, t) = \begin{cases} e^{-ik_1^2 t + ik_2^2 T} 2k_2 \hat{g}(k_1, -k_1^2 - k_2^2), & k_2 > 0, \\ 0, & k_2 \leq 0, \end{cases} $$

where
so that $V_2 |_{x_2 \in \mathbb{R}^+} = v_2$. In turn,

$$\left\| v_2(t) \right\|^2_{L^p(\mathbb{R}^n)} \leq \left\| V_2(t) \right\|^2_{L^p(\mathbb{R}^n)} = \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}^+} (1 + k_1^2 + k_2^2)^s \left| \tilde{V}_2(k_1, k_2, t) \right|^2 dk_2 dk_1.$$ 

Hence, for $s \geq 0$, which implies $(1 + k_1^2 + k_2^2)^s \simeq (1 + k_1^2 + (k_2^2)^s$, we have

$$\left\| v_2(t) \right\|^2_{L^p(\mathbb{R}^n)} \leq \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} (1 + k_1^2 + k_2^2)^s \left| \tilde{g}(k_1, k_2) \right|^2 \, dk_2 \, dk_1$$

and making the substitution $k_2 = \sqrt{-\tau - k_1^2}$ we find

$$\left\| v_2(t) \right\|^2_{L^p(\mathbb{R}^n)} \leq \int_{k_1 \in \mathbb{R}} \int_{\tau = -\infty}^{\infty} (1 + k_1^2 + k_2^2)^s \left| \tilde{g}(k_1, \tau) \right|^2 \, d\tau \, dk_1 \quad (2.7a)$$

$$+ \int_{k_1 \in \mathbb{R}} \int_{\tau = -\infty}^{\infty} (1 + k_1^2 + (k_2^2)^s \left| \tilde{g}(k_1, \tau) \right|^2 \, d\tau \, dk_1. \quad (2.7b)$$

**Estimation along the imaginary axis.** We employ the physical space equivalent Sobolev norm

$$\left\| v_1(t) \right\|^2_{L^p(\mathbb{R}^n)} = \sum_{|\mu| \leq |x|} \left\| \partial^\mu v_1(t) \right\|^2_{L^p(\mathbb{R}^n)} + \sum_{|\mu| = |x|} \left\| \partial^\mu v_1(t) \right\|^2_{L^p(\mathbb{R}^n)}, \quad s \geq 0, \quad (2.8)$$

where $x = (x_1, x_2)$ and $\partial^\mu = \partial^{\mu_1}_1 \partial^{\mu_2}_2$ with $|\mu| = \mu_1 + \mu_2$, and for $\beta = s - |x| \in (0, 1)$ we define

$$\left\| \partial^\mu v_1(t) \right\|^2_{L^p(\mathbb{R}^n)} \equiv \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial^\mu v_1(x, t) - \partial^\mu v_1(y, t)|^2}{|x - y|^{2(1 + \beta)}} \, dy \, dx$$

$$\simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial^\mu v_1(x + z, t) - \partial^\mu v_1(x, t)|^2}{|z|^{2(1 + \beta)}} \, dz \, dx. \quad (2.9)$$

**Estimation of the integer part.** We begin with the estimation of $\left\| \partial^\mu v_1(t) \right\|_{L^p(\mathbb{R}^n)}$ for all $|\mu| \in \mathbb{N}_0$ with $|\mu| \leq |x|$. Differentiating formula (2.5), we have

$$\partial^\mu v_1(x_1, x_2, t) \simeq \int_{x_2 = 0}^{\infty} e^{-k_2 x_2} G(k_2, x_1, t) \, dk_2,$$

where

$$G(k_2, x_1, t) = \int_{k_1 \in \mathbb{R}} e^{ik_1 x_1 - ik_2 x_2 + i k_1^2 + k_2^2} \tilde{g}(k_1, k_2) \, dk_1. \quad (2.10)$$

Thus,

$$\left\| \partial^\mu v_1(t) \right\|^2_{L^p(\mathbb{R}^n)} \simeq \int_{x_1 \in \mathbb{R}} \left\| \mathcal{L} \{ G(k_2, x_1, t) \} \right\|^2_{L^p(\mathbb{R}^n)} \, dx_1,$$
where $\mathcal{L}\{G\}$ denotes the Laplace transform of $G$ with respect to $k_2$, i.e.

$$\mathcal{L}\{G(k_2, x_1, t)\}(x_2) = \int_{k_2=0}^{\infty} e^{-k_2x_2} G(k_2, x_1, t)\,dk_2.$$  

**Lemma 2.1** ($L^2$-boundedness of the Laplace transform). The map

$$\mathcal{L} : \phi \mapsto \int_{k_2=0}^{\infty} e^{-k_2x_2} \phi(k_2)\,dk_2$$

is bounded from $L^2(0, \infty)$ into $L^2(0, \infty)$ with

$$\|\mathcal{L}\{\phi\}\|_{L^2(0, \infty)} \leq \sqrt{\pi}\|\phi\|_{L^2(0, \infty)}.$$  

A proof of lemma 2.1 is available in [FHM1]. Using the relevant estimate for $\phi(k_2) = G(k_2, x_1, t)$, we obtain

$$\|\partial^\mu_x v_1(t)\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+^2)} \leq \sum_{|\mu| \leq |\sigma|} \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} \left| e^{\hat{k}_1x_1 - \hat{k}_2x_2 - \hat{k}_2^\mu k_1^\mu k_2^\nu k_2^\nu} \hat{g}(k_1, -k_1^2 + k_2^2) \right|^2 dk_1 dk_2$$

after also applying Plancherel’s theorem in $x_1$ and $k_1$.

Therefore,

$$\sum_{|\mu| \leq |\sigma|} \|\partial^\mu_x v_1(t)\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+^2)} \leq \sum_{|\mu| \leq |\sigma|} \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} \left| (k_1^\mu (k_2)^\nu \hat{g}(k_1, -k_1^2 + k_2^2)) \right|^2 dk_1 dk_2$$

$$\approx \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} \left( \sum_{|\mu| = 0} |k_2^\mu k_2^\nu \hat{g}(k_1, -k_1^2 + k_2^2)|^2 dk_1 dk_2$$

$$\approx \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} (1 + k_1^2 + k_2^2)^{|\nu|} k_2^\nu |\hat{g}(k_1, -k_1^2 + k_2^2)|^2 dk_1 dk_2.$$  

(2.11)

Since $(1 + k_1^2 + k_2^2)^{|\nu|} \approx (1 + k_1^s)^s + (k_2^s)^s$ for $s \geq 0$, we infer

$$\sum_{|\mu| \leq |\sigma|} \|\partial^\mu_x v_1(t)\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+^2)} \lesssim \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} (1 + k_1^s)^s |\hat{g}(k_1, -k_1^2 + k_2^2)|^2 dk_1 dk_2$$

$$+ \int_{k_2=0}^{\infty} \int_{k_1 \in \mathbb{R}} (k_2^s)^s |\hat{g}(k_1, -k_1^2 + k_2^2)|^2 dk_1 dk_2.$$  

The change of variable $k_2 = \sqrt{\tau + k_1^2}$ then yields

$$\sum_{|\mu| \leq |\sigma|} \|\partial^\mu_x v_1(t)\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+^2)} \lesssim \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} (1 + k_1^s)^s \left( \tau + k_1^2 \right)^{\frac{1}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1$$

$$+ \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} (\tau + k_1^2)^{s+\frac{1}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1.$$  

(2.12)
Estimation of the fractional part. We proceed to the estimation of \( \| \partial_t^\mu v_1(t) \|_\beta \) where now \( |\mu| = \mu_1 + \mu_2 = |x| \in \mathbb{N}_0 \). By the definition (2.9) of the fractional norm and Plancherel’s theorem in \( x_1 \) and \( k_1 \), we have

\[
\| \partial_t^\mu v_1(t) \|^2_\beta \lesssim \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \int_{z_2 = 0}^{\infty} \frac{1}{|z|^{1+\beta}} \int_{x_1 \in \mathbb{R}} e^{ik_1 x_1} \left[ k_1^\mu e^{-ik_1 t} \cdot \int_{k_2 = 0}^{\infty} \int_{z_2 = 0}^{\infty} e^{-k_2 x_2 + \beta k_2^2 t} \times \left( e^{ik_1 z_2 - k_2 z_2} - 1 \right) k_2^\mu k_2 \overline{g}(k_1, -k^2_1 + k_2^2) dk_2 \right]^2 dx_1 dz_2 \,dk_1.
\]

Thus, using the Laplace transform lemma 2.1 in \( x_2 \) and \( k_2 \), we obtain

\[
\| \partial_t^\mu v_1(t) \|^2_\beta \lesssim \int_{k_1 \in \mathbb{R}} k_1^{2\mu_1} \int_{k_2 = 0}^{\infty} I(k_1, k_2, \beta) \cdot \left| k_2 + k_2^2 \right|^2 \left| \overline{g}(k_1, -k_1^2 + k_2^2) \right|^2 dk_2 \,dk_1,
\]

(2.13)

where

\[
I(k_1, k_2, \beta) = \int_{z_2 = 0}^{\infty} \int_{z_1 = 0}^{\infty} \frac{|e^{ik_1 z_1 - k_2 z_2} - 1|^2}{(z_1^2 + z_2^2)^{1+\beta}} \,dz_2 \,dz_1
\]

\[
= \frac{1}{|k_1| k_2} \int_{z_1 = 0}^{\infty} \int_{z_2 = 0}^{\infty} \frac{|e^{ik_1 z_1} - 1|^2}{(z_1^2 + z_2^2)^{1+\beta}} \,dz_2 \,dz_1.
\]

(2.14)

Lemma 2.2. The integral I defined by (2.14) admits the bound

\[
I(k_1, k_2, \beta) \lesssim (k_1^2 + k_2^2)^{\beta}, \quad \beta \in (0, 1).
\]

Lemma 2.2 is proven after the end of the current proof. Combining estimate (2.15) with inequality (2.13), we deduce

\[
\| \partial_t^\mu v_1(t) \|^2_\beta \lesssim \int_{k_1 \in \mathbb{R}} k_1^{2\mu_1} \int_{k_2 = 0}^{\infty} \left( k_1^2 + k_2^2 \right)^\beta \left( k_2 + k_2^2 \right)^2 \left| \overline{g}(k_1, -k_1^2 + k_2^2) \right|^2 dk_2 \,dk_1.
\]

Therefore, recalling that \( \mu_2 = |\mu| - \mu_1 \), we obtain

\[
\sum_{|\mu| = |x|} \| \partial_t^\mu v_1(t) \|^2_\beta \lesssim \sum_{|\mu| = |x|} \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left( k_1^2 + k_2^2 \right)^\beta \left( k_1^2 + k_2^2 \right)^{2\mu_1} \left( k_2 + k_2^2 \right)^2 \left| \overline{g}(k_1, -k_1^2 + k_2^2) \right|^2 \,dk_2 \,dk_1
\]

\[
\approx \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left( k_1^2 + k_2^2 \right)^\beta \left( k_1^2 + k_2^2 \right)^{|x|} \left( k_2 + k_2^2 \right)^2 \left| \overline{g}(k_1, -k_1^2 + k_2^2) \right|^2 \,dk_2 \,dk_1
\]

\[
= \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left( k_1^2 + k_2^2 \right)^\beta \left( k_1^2 + k_2^2 \right)^{|x|} \left( k_2 + k_2^2 \right)^2 \left| \overline{g}(k_1, -k_1^2 + k_2^2) \right|^2 \,dk_2 \,dk_1.
\]
The above right-hand side can be handled like (2.11) to eventually lead to (2.12). Thus, via definition (2.8) we deduce
\[
\|v_1(t)\|^2_{H^p(\mathbb{R}_+^1 \times \mathbb{R}_+^2)} \leq \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} (1 + k_1^2)^{\frac{p}{2}} (\tau + k_1^2)^{\frac{p}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1 \\
+ \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} (\tau + k_1^2)^{\alpha+\frac{1}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1.
\] (2.16)

Together, estimates (2.7) and (2.16) imply
\[
\|v(t)\|^2_{H^p(\mathbb{R}_+^1 \times \mathbb{R}_+^2)} \leq \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} (1 + k_1^2)^{\frac{p}{2}} |\tau + k_1^2|^{\frac{p}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1 \\
+ \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} |\tau + k_1^2|^{\alpha+\frac{1}{2}} |\hat{g}(k_1, \tau)|^2 d\tau dk_1 \\
\leq \|g\|^2_{X^{p,\alpha}} + \|g\|^2_{X^{p,\alpha+1}} = \|g\|^2_B,
\]
which is the desired estimate (2.3). The proof of theorem 2.1 is complete. □

**Remark 2.2.** The change of variable \(\tau \mapsto \tau - k_1^2\) and the property \(\hat{g}(\tau - a) = e^{i\alpha g(\tau)(\tau)}\) allow us to write the Bourgain \(X^{p,\alpha}\)-norm in the form
\[
\|g\|^2_{X^{p,\alpha}} = \int_{k_1 \in \mathbb{R}} (1 + k_1^2)^{\alpha} \int_{\tau = -k_1^2}^{\infty} |(1 + \tau^2)^{\frac{p}{2}} e^{i\alpha g(\tau)(k_1, \tau)}|^2 d\tau dk_1 \\
= \int_{k_1 \in \mathbb{R}} (1 + k_1^2)^{\alpha} \|e^{i\alpha g(\tau)(k_1, t)}\|^2_{H^p(\mathbb{R}_+^1)} d\tau dk_1,
\] (2.17)
motivating the definition (1.3) for the boundary data space \(B^\gamma\) of the original ibvp (1.1).

**Proof of Lemma 2.2.** Note that \(I\) is even in \(k_1\). Hence, without loss of generality, we estimate \(I\) for \((k_1, k_2) \in \mathbb{R}^+ \times \mathbb{R}^+\). There are two cases to consider: (i) \(k_2 = \lambda k_1, 0 \leq \lambda \leq 1\), and (ii) \(k_1 = \lambda k_2, 0 \leq \lambda \leq 1\). In the first case, starting from (2.14) we have
\[
I = I(k_1, \lambda k_1, \beta) = \left(\frac{k_2}{k_1}\right)^{\beta} \int_{\zeta_1 = 0}^{\infty} \int_{\zeta_2 = 0}^{\infty} \frac{|\hat{e}^{i\lambda g(\zeta_1, \zeta_2)} - 1|^2}{(\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}+\frac{1}{2}+\beta}} d\zeta_1 d\zeta_2 = \left(\frac{k_2}{k_1}\right)^{\beta} J_1(\lambda, \beta).
\]
Switching to polar coordinates \(\zeta_1 = r \cos \theta, \zeta_2 = r \sin \theta\) yields
\[
J_1(\lambda, \beta) = \int_{\theta = 0}^{\pi} \int_{r = 0}^{\infty} \frac{|\hat{e}^{i\beta \cos \theta - \lambda \sin \theta r} - 1|^2}{r^{1+2\beta}} d\theta dr.
\]
For \(r \ll 1\), we have \(\hat{e}^{i\beta \cos \theta - \lambda \sin \theta r} = 1 + (i \cos \theta - \lambda \sin \theta) r + O(r^2)\) thus the integrand of \(J_1\) becomes
\[
\frac{|\hat{e}^{i\beta \cos \theta - \lambda \sin \theta r} - 1|^2}{r^{1+2\beta}} = \frac{1}{r^{2\beta-1}} |(i \cos \theta - \lambda \sin \theta) + O(r)|^2,
\]
which is integrable at \(r = 0\) since \(\beta < 1\). Furthermore, for \(r \gg 1\) the integrand of \(J_1\) becomes
\[
\frac{|\hat{e}^{i\beta \cos \theta - \lambda \sin \theta r} - 1|^2}{r^{1+2\beta}} \leq \frac{(1 + 1)^2}{r^{1+2\beta}}.
\]
which is integrable at \( r = \infty \) since \( \beta > 0 \). Thus, \( J_t(\lambda, \beta) = c_{\lambda, \beta} < \infty \) and hence

\[
I(k_1, \lambda k_1, \beta) \lesssim (k_1^2)\lambda^\beta, \quad 0 \leq \lambda \leq 1. \tag{2.18}
\]

A similar argument shows that

\[
I(\lambda k_2, k_2, \beta) \lesssim (k_2^2)^\lambda, \quad 0 \leq \lambda \leq 1. \tag{2.19}
\]

Combining estimates (2.18) and (2.19) concludes the proof of lemma 2.2. \( \square \)

3. Linear IVP estimates

The pure linear ibvp (2.1) will be combined with appropriate linear ivps to yield the forced linear ibvp (1.12) via the superposition principle. The details of this (de)composition are given in section 4. Hence, theorem 1.3 for the forced linear ibvp will be established by combining the estimate of theorem 2.1 for the pure linear ibvp with suitable estimates for the aforementioned ivps. These estimates are derived below.

**Homogeneous linear ivp estimates.** We begin with the linear Schrödinger ivp

\[
iU_t + U_{x_1 x_1} + U_{x_2 x_2} = 0, \quad (x_1, x_2, t) \in \mathbb{R}^3, \tag{3.1a}
\]

\[
U(x_1, x_2, 0) = U_0(x_1, x_2) \in H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}), \tag{3.1b}
\]

whose solution is given by

\[
U(x_1, x_2, t) = S[U_0; 0](x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - i(t^2 + k_2^2)\beta/2} \mathcal{F}U_0(k_1, k_2) dk_2 dk_1, \tag{3.2}
\]

where \( \mathcal{F}U_0 \) is the Fourier transform of \( U_0 \) on the plane defined by

\[
\mathcal{F}U_0(k_1, k_2) = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}} e^{-ik_1 x_1 - ik_2 x_2} U_0(x_1, x_2) dx_1 dx_2. \tag{3.3}
\]

We shall discover below that the regularity of the solution of ivp (3.1) in the variables \( x_1 \) and \( t \) is described by the boundary data space \( B^s_T \). This result can be regarded as the two-dimensional analogue of the time estimate of [KPV1], which states that the solution of the linear Schrödinger ivp on the line with initial data in \( H^s(\mathbb{R}) \) belongs to \( H^s(\mathbb{R}^+; 0, T) \) as a function of \( t \). Hence, both in one and in two dimensions, the regularity of the linear ivp with respect to the boundary variables is described by the boundary data space of the associated Dirichlet ibvp. The precise statement of our result is the following.

**Theorem 3.1** (Homogeneous linear ivp estimates). The solution \( U = S[U_0; 0] \) of the homogeneous linear Schrödinger ivp (3.1) given by formula (3.2) satisfies the estimates

\[
\sup_{t \in [0, T]} \|U(t)\|_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2})} \leq \|U_0\|_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2})}, \quad s \in \mathbb{R}, \tag{3.4}
\]

\[
\sup_{x_2 \in \mathbb{R}} \|U(x_2)\|_{H^s_t} \leq c_s \|U_0\|_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2})}, \quad s \geq 0. \tag{3.5}
\]

**Remark 3.1.** Recall that \( B^s_T \) emerges naturally in the proof of estimate (2.3) as the boundary data space that allows for the solution of the pure linear ibvp to belong in \( H^s(\mathbb{R} \times \mathbb{R}^+) \). Estimate
\( (3.5) \) indicates an alternative, reverse path for discovering \( B^p_T \), namely by investigating the \( x_1 t \)-regularity of the solution of the linear Schrödinger ivp when the initial datum belongs in \( H^s(\mathbb{R} \times \mathbb{R}) \). Together, estimates \((2.3)\) and \((3.5)\) can be visualized as the upper and lower half of a closed loop from the data space to the solution space and back, and hence confirm that \( B^p_T \) is indeed the correct boundary data space for NLS on the half-plane in the case of Sobolev initial data.

**Proof of Theorem 3.1.** The isometry relation \((3.4)\) follows easily from formula \((3.2)\) and the definition of the Sobolev norm. Concerning estimate \((3.5)\), we note that the function

\[
Q(k_1, x_2, t) = e^{ik_1^2 T} \hat{U}^{s_1}(k_1, x_2, t)
\]

involved in the \( B^s_T \)-norm of \( U \) satisfies for all \( k_1 \in \mathbb{R} \) the one-dimensional ivp

\[
iQ_1 + Q_{x_2} = 0, \quad (x_2, t) \in \mathbb{R} \times (0, T),
\]

\[
Q(k_1, x_2, 0) = \hat{U}^{s_1}_0(k_1, x_2), \quad x_2 \in \mathbb{R}.
\]

Thus, we have the following estimate from theorem 4 of \([FHM1]\):

\[
\|Q(k_1, x_2)\|_{H^{s_1 - \frac{1}{2}}(\mathbb{R})} \lesssim \|\hat{U}^{s_1}_0(k_1)\|_{H^s(\mathbb{R} \times \mathbb{R})}, \quad s \geq -\frac{1}{2}, \quad k_1, x_2 \in \mathbb{R}.
\]

\((3.7)\)

In turn, for all \( x_2 \in \mathbb{R} \) and \( s \geq -\frac{1}{2} \), we infer

\[
\|U(x_2)\|^2_{B^s_T} \lesssim \int_{k_1 \in \mathbb{R}} \|\hat{U}^{s_1}_0(k_1, x_2)\|^2_{H^s(\mathbb{R} \times \mathbb{R})}dk_1 + \int_{k_1 \in \mathbb{R}} (1 + k_1^2)^s \|\hat{U}^{s_1}_0(k_1, x_2)\|^2_{H^s(\mathbb{R} \times \mathbb{R})}dk_1
\]

\[
= \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} \left[ (1 + k_1^2)^s + (1 + k_2^2)^s \right] \|\hat{U}_0(k_1, k_2)\|^2 dk_2 dk_1,
\]

which implies estimate \((3.5)\) upon restricting \( s \geq 0 \). \( \blacksquare \)

**Forced linear ivp estimates.** We continue with the estimation of the forced linear Schrödinger on the whole plane. First, we need a result for the one-dimensional forced ivp

\[
iw_t + w_{xx} = f(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R},
\]

\[
w(x, 0) = 0, \quad x \in \mathbb{R},
\]

\((3.8a)\) \quad \((3.8b)\)

whose solution is given by

\[
w(x, t) = S[0 ; f](x, t) = -\frac{i}{2\pi} \int_{\mathbb{R}} \int_{0}^{t} e^{-ik^2(t-t')} \hat{f}(k, t') dk \, dt'
\]

\[
= -i \int_{t_0}^{t} \left( f(\cdot, t'); 0 \right) (x, t - t') \, dt',
\]

\((3.9a)\) \quad \((3.9b)\)

where \( \hat{f} \) is the whole-line Fourier transform of \( f \) defined by

\[
\hat{f}(k, t) = \int_{\mathbb{R}} e^{-ikx} f(x, t) \, dx
\]

\((3.10)\)
and \( S [ f(x, t'); 0] \) denotes the solution of the one-dimensional homogeneous ivp (3.6) with initial datum \( f(x, t') \).

**Theorem 3.2** (Forced ivp time estimates in one dimension). For all \( s \in \mathbb{R} \), the solution \( w = S[0; f] \) of the one-dimensional forced linear ivp (3.8) given by formula (3.9) admits the bounds

\[
\|w(x)\|^2_{H^{2/m}(0,T)} \lesssim \int_0^T \|f(t')\|_{H^m(\mathbb{R}_+)}^2 \, dt' + \int_0^T \int_{z=0}^{T-t} \frac{1}{z^{s+2}} \left[ \int_{t'=t}^{t'+z} \|f(t')\|_{H^m(\mathbb{R}_+)} \, dt' \right]^2 \, dz \, dt', \quad \frac{1}{2} \leq s < \frac{3}{2}.
\]

(3.11a)

\[
\|w(x)\|^2_{H^{2/m}(0,T)} \lesssim \int_0^T \|f(t)\|_{H^m(\mathbb{R}_+)}^2 \, dt, \quad -\frac{1}{2} \leq s < \frac{1}{2}, \quad s = \frac{3}{2}.
\]

(3.11b)

**Proof of Theorem 3.2.** Let \( m = \frac{2s+1}{s+2} \) and note that \( 0 \leq m < 1 \) corresponds to \( -\frac{1}{s} \leq s < \frac{3}{2} \). Hence, in this range of \( s \) the physical space equivalent \( H^m(0,T) \)-norm reads

\[
\|w(x)\|^2_{H^m(0,T)} = \|w(x)\|^2_{L^2(0,T)} + \|w(x)\|^2_{H^m(0,T)}, \quad 0 \leq m < 1,
\]

(3.12)

where the fractional part of the norm is defined for \( m \in (0, 1) \) by

\[
\|w(x)\|^2_m = \int_0^T \int_{z=0}^{T-t} \frac{|w(x, t+z) - w(x, t)|^2}{z^{s+2}} \, dz \, dt.
\]

(3.13)

Starting from formula (3.9b) and using Minkowski’s integral inequality and the homogeneous ivp time estimate (3.7), we find

\[
\|w(x)\|_{L^2(0,T)} \lesssim \int_0^T \|S[f(\cdot, t'); 0] (x, t - t')\|_{L^2(0,T)} \, dt' \lesssim \int_0^T \|f(t')\|_{H^{2/m}(\mathbb{R}_+)} \, dt'.
\]

(3.14)

Moreover, employing the Duhamel representation (3.9b), we have

\[
\|w(x)\|_{L^2(0,T)}^2 \lesssim \int_0^T \int_{t'=0}^{t} \left[ S[f(\cdot, t'); 0] (x, t+z-t') - S[f(\cdot, t'); 0] (x, t-t') \right]^2 \, dz \, dt
\]

(3.15a)

\[
+ \int_0^T \int_{t'=0}^{T-t} \left[ S[f(\cdot, t'); 0] (x, t+z-t') \right]^2 \, dz \, dt.
\]

(3.15b)

The term (3.15a) can be estimated via Minkowski’s integral inequality in the \( t' \)- and \( z \)-integrals, followed by the homogeneous ivp time estimate (3.7). Eventually, we find

\[
(3.15a) \lesssim \int_0^T \|f(t')\|_{H^m(\mathbb{R}_+)}^2 \, dt.
\]

(3.16)
For the term (3.15b), we treat the cases \( m > \frac{1}{2} \) and \( m < \frac{1}{2} \) separately. In the former case, using formula (3.9a) and applying Cauchy–Schwarz inequality in \( k \), we have

\[
(3.15b) \lesssim \int_{t=0}^{T} \int_{z=0}^{T-z} \frac{1}{z^{1+2m}} \left( \int_{R} \int_{t'=t}^{t'+z} \left| \tilde{f}(k, t') \right| \, dk \right)^{2} \, dz \, dt
\]

\[
\lesssim \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \int_{R} \left( 1 + k^{2} \right)^{\frac{1}{2}} \left( \int_{t'=t}^{t'+z} \left| \tilde{f}(k, t') \right| \, dk \right)^{2} \, dz \, dt.
\]

where we have made crucial use of the fact that \( m > \frac{1}{2} \) implies \( s > \frac{1}{2} \) and hence \( \int_{R} \frac{dk}{1+k^{2}} < \infty \). Thus, Minkowski’s integral use of the fact that \( m > \frac{1}{2} \) implies \( s > \frac{1}{2} \) and hence \( \int_{R} \frac{dk}{1+k^{2}} < \infty \).

Thus, Minkowski’s integral inequality in \( t' \) and \( k \) yields

\[
(3.15b) \lesssim \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left( \int_{t'=t}^{t'+z} \| f(t') \|_{H(R)} \, dt' \right)^{2} \, dz \, dt. \tag{3.17}
\]

Combining estimates (3.16) and (3.17), for \( \frac{1}{2} < m < 1 \) we deduce the estimate

\[
\| w(x) \|_{m}^{2} \lesssim \int_{t'=0}^{T} \| f(t') \|_{H(R)}^{2} \, dt' + \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left( \int_{t'=t}^{t'+z} \| f(t') \|_{H(R)} \, dt' \right)^{2} \, dz \, dt,
\]

which, together with the \( L^{2} \)-estimate (3.14), implies the bound (3.24a).

For \( m < \frac{1}{2} \), applying Cauchy–Schwarz inequality in \( t' \) and then interchanging the \( t' \) and \( z \)-integrals, we have

\[
(3.15b) \lesssim \int_{t=0}^{T} \frac{1}{z^{1+2m}} \int_{z}^{T} \int_{t'=t}^{t'+z} \left| S \left[ f(\cdot, t'); 0 \right] (x, t + z - t') \right|^{2} \, dt' \, dz \, dt.
\]

Hence, letting \( t \to t - z \) and augmenting the range of integration with respect to \( t \) and \( t' \), we find

\[
(3.15b) \lesssim \int_{t=0}^{T} \frac{1}{z^{1+2m}} \int_{t=z}^{T} \int_{t'=t-z}^{t'\prime} \left| S \left[ f(\cdot, t'); 0 \right] (x, t - t') \right|^{2} \, dt' \, dz \, dt
\]

\[
\lesssim \left( \int_{t=0}^{T} \frac{1}{z^{1+2m}} \, dz \right) \int_{t=0}^{T} \left( S \left[ f(\cdot, t'); 0 \right] (x, t - t') \right)_{L^{2}(0, T)}^{2} \, dt',
\]

so, integrating in \( z \) (recall that \( 2m < 1 \)) and using the homogeneous ivp time estimate (3.7), we obtain

\[
(3.15b) \lesssim \frac{T^{1-2m}}{1 - 2m} \int_{t=0}^{T} \| f(t) \|_{H^{s}(R)}^{2} \, dt. \tag{3.18}
\]

Estimates (3.16) and (3.18) together imply

\[
\| w(x) \|_{m}^{2} \lesssim \int_{t=0}^{T} \| f(t) \|_{H^{s}(R)}^{2} \, dt + \frac{T^{1-2m}}{1 - 2m} \int_{t=0}^{T} \| f(t) \|_{H^{s}(R)}^{2} \, dt, \quad 0 < m < \frac{1}{2},
\]

thus, recalling also the \( L^{2} \)-estimate (3.14) and the fact that \( T < 1 \), we deduce

\[
\| w(x) \|_{H^{m}(0, T)}^{2} \lesssim \frac{1}{1 - 2m} \int_{t=0}^{T} \| f(t) \|_{H^{s}(R)}^{2} \, dt, \quad 0 \leq m < \frac{1}{2},
\]

which is the bound (3.11b) for \( -1 \leq s < \frac{1}{2} \).
Finally, in order to establish \((3.11b)\) for \(s = \frac{3}{2}\) (which corresponds to \(m = 1\)) we need to estimate
\[
\|w(x)\|_{H^1((0,T))}^2 = \|w(x)\|_{L^2(0,T)}^2 + \|\partial_t w(x)\|_{L^2(0,T)}^2.
\]
The first \(L^2\)-norm above was estimated earlier (see \((3.14)\)). Furthermore, differentiating the Duhamel representation \((3.9b)\) with respect to \(t\), we have
\[
\|\partial_t w(x)\|_{L^2(0,T)}^2 \lesssim \|f(x)\|_{L^2(0,T)}^2 + \left\| \int_{t'=0}^t S \left[ \partial_t^2 f(\cdot, t'); 0 \right] (x, t - t') \, dt' \right\|_{L^2(0,T)}^2.
\]
Since \(s > \frac{1}{2}\), the Sobolev embedding theorem in \(x\) implies
\[
\|f(x)\|_{L^2(0,T)}^2 \lesssim \int_{t=0}^T \|f(t)\|_{H^s(R^1)}^2 \, dt \lesssim \int_{t=0}^T \|f(t)\|_{H^s(R^1)}^2 \, dt.
\]
Moreover, similarly to the derivation of \((3.14)\), we have
\[
\left\| \int_{t'=0}^t S \left[ \partial_t^2 f(t'); 0 \right] (x, t - t') \, dt' \right\|_{L^2(0,T)} \lesssim \int_{t'=0}^T \|\partial_t^2 f(t')\|_{H^{s+1}(R^1)} \, dt' \\
\leq \int_{t'=0}^T \|f(t')\|_{H^{s+1}(R^1)} \, dt'.
\]
Therefore, applying Cauchy–Schwarz inequality in \(t'\) we obtain
\[
\|\partial_t w(x)\|_{L^2(0,T)}^2 \lesssim \int_{t=0}^T \|f(t')\|_{H^{s+1}(R^1)} \, dt',
\]
which combined with estimate \((3.14)\) yields the bound \((3.11b)\) for \(s = \frac{3}{2}\).

The proof of theorem 3.2 is complete.

The bounds of theorem 3.2 will now be employed in the estimation of the two-dimensional forced linear Schrödinger ivp with zero initial datum:
\[
iW_t + W_{x_1 x_1} + W_{x_2 x_2} = F(x_1, x_2, t), \quad (x_1, x_2, t) \in \mathbb{R}^2, \quad (3.19a)
\]
\[
W(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad (3.19b)
\]
with solution
\[
W(x_1, x_2, t) = \mathcal{S} [0; F] (x_1, x_2, t)
\]
\[
= -\frac{i}{(2\pi)^2} \int_{t'=0}^t \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - (k_1^2 + k_2^2)(t - t')^2} \hat{F}^s(k_1, k_2, t') \, dk_2 \, dk_1 \, dt'
\]
\[
= -i \int_{t'=0}^t \mathcal{S} \left[ F(\cdot, \cdot, t'); 0 \right] (x_1, x_2, t - t') \, dt',
\]
\[
(3.20a)
\]
\[
(3.20b)
\]
where $\hat{F}^s$ is the Fourier transform of $F$ on the plane defined similarly to (3.3) and $S \{F(x_1, x_2, t); 0\}$ denotes the solution of the homogeneous ivp (3.6) with initial datum $F(x_1, x_2, t')$.

**Theorem 3.3** (Forced ivp estimates). The solution $W = S [0; F]$ of the forced linear Schrödinger ivp (3.8) given by (3.9) satisfies the estimates

\[
\sup_{t \in [0, T]} \|W(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2)} \leq T \sup_{t \in [0, T]} \|F(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2)}, \quad s \in \mathbb{R}, \tag{3.21}
\]

\[
\sup_{x_2 \in \mathbb{R}} \|W(x_2)\|_{B^s_{q, p}} \leq c_s \sqrt{T} \sup_{t \in [0, T]} \|F(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2)}, \quad 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2}. \tag{3.22}
\]

**Proof of Theorem 3.3.** Estimate (3.21) can be derived by employing the Duhamel representation (3.20b) in combination with the space estimate (3.4) for the homogeneous ivp.

Concerning estimate (3.22), we recall that

\[
\|W(x_2)\|_{X^s_{q, p}} = \left( \int_{k_1 \in \mathbb{R}} \|e^{i k_1 x_1} \hat{W}^s(k_1, x_2, t)\|^2_{H^{s+1}(0, T)} \, dk_1 \right)^{\frac{1}{2}}, \tag{3.23a}
\]

\[
\|W(x_2)\|_{X^s_{q, p}} \leq \left( \int_{k_1 \in \mathbb{R}} (1 + k_1^2)^{s} \|\hat{W}^s(k_1, x_2, t)\|^2_{H^{s+1}(0, T)} \, dk_1 \right)^{\frac{1}{2}}, \tag{3.23b}
\]

and note that the function

\[
Q(k_1, x_2, t) = e^{i k_1 x_1} \hat{W}^s(k_1, x_2, t)
\]

does not all $k_1 \in \mathbb{R}$ the one-dimensional ivp

\[
iQ_t + Q_{x_2 x_2} = R(k_1, x_2, t), \quad (x_2, t) \in \mathbb{R}^2, \\
Q(k_1, x_2, 0) = 0, \quad x_2 \in \mathbb{R},
\]

with forcing $R$ given by

\[
R(k_1, x_2, t) = e^{i k_1 x_1} \hat{F}^s(k_1, x_2, t).
\]

Hence, for all $k_1 \in \mathbb{R}$ the function $Q$ admits the bounds of theorem 3.2, i.e.

\[
\|Q(k_1, x_2)\|^2_{H^{s+1}(0, T)} \leq \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{s+3}} \left[ \int_{t'=t}^{T-t} \|R(k_1, t')\|_{H^{s}((0, T)_{x_2})} \, dt' \right]^2 \, dz \, dt + \int_{t=0}^{T} \|R(k_1, t)\|^2_{H^{s}((0, T)_{x_2})} \, dt, \quad \frac{1}{2} < s < \frac{3}{2}, \tag{3.24a}
\]

\[
\|Q(k_1, x_2)\|^2_{H^{s+1}(0, T)} \leq \int_{t=0}^{T} \|R(k_1, t')\|^2_{H^{s}((0, T)_{x_2})} \, dt', \quad \frac{1}{2} < \frac{s}{2} < \frac{3}{2}. \tag{3.24b}
\]
Combining the bound (3.24a) with the norm (3.23a) and then applying Minkowski’s integral inequality in \( k \) and \( t' \), we obtain

\[
\|W(x_2)\|_{X^{2\alpha+1}_T} \lesssim \int_0^T \int_{k_1 \in \mathbb{R}} \left[ \int_{z=0}^{T+\varepsilon} \left( \int_{k_1 \in \mathbb{R}} \|R(k_1, t')\|_{H^1(\mathbb{R})}^2 \right)^\frac{1}{2} dt' \right] dz \frac{1}{z^{\frac{1}{2} + \varepsilon}} \lesssim \int_0^T \left( \int_{k_1 \in \mathbb{R}} \|R(k_1, t')\|_{H^1(\mathbb{R})}^2 \right) dt', \quad \frac{1}{2} < s < \frac{3}{2}.
\]

Hence, noting that

\[
\int_{k_1 \in \mathbb{R}} \|R(k_1, t')\|_{H^1(\mathbb{R})}^2 dk_1 \lesssim \|F(t')\|_{H^1(\mathbb{R})}^2, \quad s \geq 0,
\]

and using Cauchy–Schwarz inequality in \( t' \), we infer the estimate

\[
\|W(x_2)\|_{X^{2\alpha+1}_T} \lesssim \|F\|^2_{L^2([0, T] \times \mathbb{R})} + \int_0^T \|F(t')\|_{H^1(\mathbb{R})}^2 \int_{z=0}^{T} \frac{1}{z^{\frac{1}{2} + \varepsilon}} dz dt'.
\]

\[
\lesssim \|F\|^2_{L^2([0, T] \times \mathbb{R})}, \quad \frac{1}{2} < s < \frac{3}{2}, \quad x_2 \in \mathbb{R}.
\]

In fact, starting from the bound (3.24b) and proceeding as above, we obtain the same estimate also for \(-\frac{1}{2} < s < \frac{1}{2}\) and \( s = \frac{1}{2}\). Therefore,

\[
\|W(x_2)\|_{X^{\alpha+1}_T} \lesssim \|F\|_{L^2([0, T] \times \mathbb{R})}, \quad 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2}, \quad x_2 \in \mathbb{R}. \tag{3.25}
\]

Moreover, combining the bound (3.24b) for \( s = 0 \) with the norm (3.23b), we find

\[
\|W(x_2)\|_{X^{\alpha+1}_T} \lesssim \int_0^T \int_{k_1 \in \mathbb{R}} \left[ (1 + k_1^2)^{\alpha} \|e^{ik_1 t} \hat{F}^s(k_1, x_2, t)\|_{L^2(\mathbb{R})}^2 \right] dk_1 dt,
\]

so by Plancherel’s theorem in \( x_2 \) and \( k_2 \) we infer

\[
\|W(x_2)\|_{X^{\alpha+1}_T} \lesssim \int_0^T \int_{k_1 \in \mathbb{R}} \left[ (1 + k_1^2)^{\alpha} \int_{k_2 \in \mathbb{R}} \|\hat{F}^s(k_1, k_2, t)\|_{L^2(\mathbb{R})}^2 \right] dk_1 dt \lesssim \|F\|_{L^2([0, T] \times \mathbb{R})}^2, \quad s \geq 0, \quad x_2 \in \mathbb{R}. \tag{3.26}
\]

Estimates (3.25) and (3.26) imply estimate (3.22) via the definition of the \( B^1_T \)-norm.

4. The forced linear IBVP: proof of theorem 1.3

We shall now combine the pure linear ibvp theorem 2.1 with the ivp theorems 3.1 and 3.3 in order to estimate the solution of the forced linear ibvp (1.12). To do so, we first decompose this problem into simpler problems that have already been estimated in the preceding sections.

**Decomposition into simpler problems.** Problem (1.12) can be expressed as the superposition of the homogeneous linear ibvp

\[
iu_t + u_{x_1} + u_{x_2} = 0, \quad (x_1, x_2, t) \in \mathbb{R} \times \mathbb{R}^+ \times (0, T). \tag{4.1a}
\]
Then, problem \((U)\) where the function decompositions as follows:

\[
u(x_1, x_2, 0) = u_0(x_1, x_2) \in H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+), \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+,
\]

\[
u(x_1, 0, t) = g_0(x_1, t) \in B_t^1, \quad (x_1, t) \in \mathbb{R} \times [0, T],
\]

and the forced linear ibvp with zero initial and boundary data

\[
iu_t + u_{x_1 x_1} + u_{x_2 x_2} = f \quad u(x_1, x_2, 0) = 0, \quad u(x_1, 0, t) = 0,
\]

where \(f \in C([0, T]; H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+))\).

The above decomposition has decoupled the forcing \(f\) from the data \(u_0, g_0\). Next, we will perform further decompositions in order to separate the data from each other.

In particular, let the initial datum \(U_0\) of the homogeneous linear ivp \((3.1)\) be defined as a whole-plane extension of the half-plane initial datum \(u_0\) of ibvp \((4.1)\) such that

\[
\|U_0\|_{H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)} \leq c \|u_0\|_{H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)}.
\]

Then, problem \((4.1)\) can be expressed as the superposition of ivp \((3.1)\) and the following homogeneous linear ibvp with zero initial datum:

\[
iu_t + u_{x_1 x_1} + u_{x_2 x_2} = 0, \quad u(x_1, x_2, 0) = 0, \quad u(x_1, 0, t) = 0,
\]

where the function \(U(x_1, 0, t)\) involved in the boundary condition is the solution \(U = S[U_0; 0]\) of ivp \((3.1)\) evaluated at \(x_2 = 0\).

In addition, let the forcing \(F\) of the forced linear ivp \((3.19)\) be defined as a whole-plane extension of the half-plane forcing \(f\) of ibvp \((1.12)\) such that

\[
\|F(t)\|_{H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)} \leq c \|f(t)\|_{H'(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)} \quad t \in [0, T].
\]

Then, problem \((4.2)\) can be written as the superposition of ivp \((3.19)\) and the following homogeneous linear ibvp with zero initial datum:

\[
iu_t + u_{x_1 x_1} + u_{x_2 x_2} = 0, \quad u(x_1, x_2, 0) = 0, \quad u(x_1, 0, t) = 0,
\]

where the boundary datum \(W(x_1, 0, t)\) is obtained by evaluating the solution \(W = S[0; F]\) of ivp \((3.19)\) at \(x_2 = 0\).

In summary, the solution of the forced linear ibvp \((1.12)\) can be analysed into the respective solutions of the four component problems \((3.1), (3.19), (4.4)\) and \((4.6)\) involved in the above decompositions as follows:

\[
S[u_0, g_0; f] = S[U_0; 0]_{x_2 \in \mathbb{R}^+} + S[0; F]_{x_2 \in \mathbb{R}^+} + S[0, G_0; 0] - S[0, W|_{x_2=0}; 0].
\]
The first two terms on the right-hand side of (4.7) have been estimated in theorems 3.1 and 3.3 respectively. Also, the remaining two terms can be estimated via theorem 2.1 as their associated problems, namely ibvps (4.4) and (4.6), essentially correspond to different versions of the pure linear ibvp (2.1). Indeed, both of these problems are homogeneous, with zero initial datum, and with boundary datum in \( B_f \), since for \( 0 \leq s \leq \frac{1}{2}, s \neq \frac{1}{2} \), estimates (3.5), (3.22), (4.3) and (4.5) imply

\[
\| G_0 \|_{B_f^s} \lesssim \| g_0 \|_{B_f^s} + \| u_0 \|_{H^p(\mathbb{R}_+^4 \times \mathbb{R}_+^8)}^*, \tag{4.8}
\]

\[
\| W \|_{s=0} \|_{B_f^s} \lesssim \sqrt{T} \sup_{t \in [0, T]} \| f(t) \|_{H^p(\mathbb{R}_+^4 \times \mathbb{R}_+^8)}^*, \tag{4.9}
\]

Moreover, thanks to the compatibility condition (1.7) and the initial conditions (3.1b) and (3.19b), for \( s > \frac{1}{2} \) the boundary data \( G_0 \) and \( W \big|_{s=0} \) vanish at \( t = 0 \) for all \( x_1 \in \mathbb{R} \). Hence, ibvps (4.4) and (4.6) can be treated simultaneously by analysing the following problem:

\[
\begin{align*}
&\iota u_t + u_{x_1 x_1} + u_{x_2 x_2} = 0, \\
&\quad (x_1, x_2, t) \in \mathbb{R} \times \mathbb{R}^+ \times (0, T), \tag{4.10a} \\
&u(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+, \tag{4.10b} \\
&u(x_1, 0, t) = Q_0(x_1, t) \in B_f^s, \quad (x_1, t) \in \mathbb{R}_+ \times [0, T], \tag{4.10c}
\end{align*}
\]

where the boundary datum satisfies the compatibility condition

\[
Q_0(x_1, 0) = 0, \quad s > \frac{1}{2}, x_1 \in \mathbb{R}. \tag{4.11}
\]

Next, we will reduce the estimation of ibvp (4.10) to that of the pure linear ibvp (2.1). Let

\[
q_0(k_1, t) = e^{i k_1 t} \hat{Q}_0^s(k_1, t) \tag{4.12}
\]

and note that \( q_0 \in H^{\frac{2s+1}{2}}(0, T) \) since \( Q_0 \in B_f^s \). Let \( E \) be an extension of \( q_0 \) from \([0, T]\) to \( \mathbb{R} \), such that

\[
\| E(k_1) \|_{H^{\frac{2s+1}{2}}(\mathbb{R}_+)} \leq c \| q_0(k_1) \|_{H^{\frac{2s+1}{2}}(0, T)}, \quad s \in \mathbb{R}, \ k_1 \in \mathbb{R}. \tag{4.13}
\]

Consider the function \( E_\theta(k_1, t) = \theta(t) E(k_1, t) \in H^{\frac{2s+1}{2}}(\mathbb{R}_+) \) with \( \theta \in C^\infty_0(\mathbb{R}), \theta \equiv 1 \) on \([-1, 1], \theta \equiv 0 \) on \((-2, 2)^c \) and \( \| \theta \|_{L^\infty(\mathbb{R})} = 1 \). Since \( T < 1 \), we have \( E_\theta = q_0 \) on \( \mathbb{R}_+ \times [0, T] \). Furthermore, combining estimate (4.13) with lemma 2 of [HH], we find

\[
\| E_\theta(k_1) \|_{H^{\frac{2s+1}{2}}(\mathbb{R}_+)} \leq c_\theta \| q_0(k_1) \|_{H^{\frac{2s+1}{2}}(0, T)}, \quad s > -\frac{3}{2}, k_1 \in \mathbb{R}. \tag{4.14}
\]
Also, thanks to condition (4.11), for \( s > \frac{1}{2} \) we have \( E(k_1, 0) = q_0(k_1, 0) = 0 \). In addition, \( \text{supp}(E_0) \subset \mathbb{R}_+^1 \times (-2, 2) \) so that, in particular, \( E_0(k_1, 2) = 0 \). Thus, \( E_0 \in H^{\frac{3s+1}{2}}_0(0, 2) \) for all \( s > \frac{1}{2} \) and hence theorem 11.4 of [LM] implies that the extension

\[
q(k_1, t) = \begin{cases} 
E_0(k_1, t), & t \in (0, 2), \\
0, & t \in (0, 2)^c
\end{cases}
\]  

(4.15)

belongs to \( H^{\frac{2s+1}{2}}(\mathbb{R}_+) \) with \( \|q(k_1)\|_{H^{\frac{2s+1}{2}}(\mathbb{R}_+)} \leq c_s \|E_0(k_1)\|_{H^s_0(0, 2)} \) for all \( s \geq -\frac{1}{2}, \frac{2s+1}{4} \notin \mathbb{Z} + \frac{1}{2} \). Therefore, in view of estimate (4.14) we infer

\[
\|q(k_1)\|_{H^{\frac{2s+1}{2}}(\mathbb{R}_+)} \leq c_s \|q_0(k_1)\|_{H^s_0(0, 2)}, \quad s \geq -\frac{1}{2}, \frac{2s+1}{4} \notin \mathbb{Z} + \frac{1}{2}, \quad k_1 \in \mathbb{R}.
\]  

(4.16)

Overall, we have constructed an extension \( q \) of \( q_0 \) that satisfies the bound (4.16) and has support \( \text{supp}(q) \subset \mathbb{R}_+^1 \times (0, 2) \). Thus, defining the boundary datum \( g \) of the pure linear ibvp (2.1) via its \( x_1 \)-Fourier transform by

\[
\hat{g}_s^{\pm_1}(k_1, t) = e^{-ik_1^2t}q(k_1, t), \quad (k_1, t) \in \mathbb{R}^2,
\]  

(4.17)

which implies in particular that

\[
\hat{g}_s^{\pm_1}(k_1, t) = e^{-ik_1^2t}q_0(k_1, t), \quad (k_1, t) \in \mathbb{R} \times [0, T],
\]  

(4.18)

and recalling the definition (4.12) of \( q_0 \), we conclude that ibvp (4.10) is embedded inside the pure linear ibvp (2.1), i.e. \( S[0, Q_0; 0] = S[0, g; 0]_{k \in [0, T]} \). Also, importantly, the bound (4.16) together with (4.12), (4.18) and the writing (2.17) guarantee that

\[
\|g\|_{W^s} \leq \|Q_0\|_{B^s_1}, \quad s \geq -\frac{1}{2}, \frac{2s+1}{4} \notin \mathbb{Z} + \frac{1}{2}.
\]  

(4.19)

Therefore, thanks to theorem 2.1 we deduce the estimate

\[
\sup_{t \in [0, T]} \|S[0, Q_0; 0](t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} \leq c_s \|Q_0\|_{B^s_1}, \quad s \geq 0, \frac{2s+1}{4} \notin \mathbb{Z} + \frac{1}{2}.
\]  

(4.20)

**Proof of Theorem 1.3.** At this point, all four components of the superposition (4.7) have been estimated. In particular, estimate (3.4) of theorem 3.1 and inequality (4.3) imply

\[
\sup_{t \in [0, T]} \|S[U_0; 0](t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} \leq c \|u_0\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)^1}, \quad s \in \mathbb{R},
\]

estimate (3.21) of theorem 3.3 together with inequality (4.5) and the fact that \( T < 1 \) yield

\[
\sup_{t \in [0, T]} \|S[0, F](t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)} \leq c \sqrt{T} \sup_{t \in [0, T]} \|f(t)\|_{H^s(\mathbb{R}_1 \times \mathbb{R}_2^+)^1}, \quad s \in \mathbb{R},
\]
estimate (4.20) and inequality (4.8) with \( Q_0 = G_0 \) imply

\[
\sup_{r \geq 0} \left\| S [0, G_0; 0] (t) \right\|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} \leq c_s \left( \| u_0 \|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} + \| g_0 \|_{\mathfrak{F}^s} \right), \quad s > 0, \quad 2s + 1 \notin \mathbb{Z} + \frac{1}{2}.
\]

and estimate (4.20) together with inequality (4.9) with \( Q_0 = W_{|\alpha|=0} \) yield

\[
\sup_{r \geq 0} \left\| S [0, W_{|\alpha|=0}; 0] (t) \right\|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} \leq c_r \sqrt{T} \sup_{r \geq 0} \left\| f(t) \right\|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)}, \quad s > 0, \quad 2s + 1 \notin \mathbb{Z} + \frac{1}{2}.
\]

Combining the above four estimates, we deduce estimate (1.16) for the forced linear Schrödinger ibvp (1.12).

5. Well-posedness of NLS on the half-plane: proof of theorem 1.1

Using the forced linear ibvp estimate (1.16) and the contraction mapping theorem, we shall now establish well-posedness of the NLS ibvp (1.1) for smooth data \( s > 1 \) in the sense of Hadamard, i.e. we shall show that there exists a unique solution to this problem that depends continuously on the initial and boundary data.

Existence and uniqueness. Setting \( f = \pm |u|^{\alpha-1} u \) in the unified transform formula (1.13) for the solution \( S [u_0, g_0; f] \) of the forced linear ibvp (1.12) gives rise to the following iteration map for the NLS ibvp (1.1):

\[
u \mapsto \Phi u = \Phi_{u_0, g_0} u = S [u_0, g_0; \pm |u|^{\alpha-1} u]. \tag{5.1}
\]

We shall show that this map is a contraction in the space

\[
X = C([0, T^{*}); H^s (\mathbb{R}^+ \times \mathbb{R}^+)) \tag{5.2}
\]

for some appropriate lifespan \( T^{*} \in (0, T] \) to be determined.

Showing that the map \( u \mapsto \Phi u \) maps \( X \) into \( X \). For \( 1 < s \leq \frac{3}{2} \), estimate (1.16) implies

\[
\left\| \Phi u \right\|_X \leq c_s \left( \| u_0 \|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} + \| g_0 \|_{\mathfrak{F}^s} + \sqrt{T} \sup_{r \geq 0} \left\| \Phi (u^{\alpha-1} u) \right\|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} \right),
\]

where we have written \( |u|^{\alpha-1} = (u \bar{u})^{\frac{\alpha-1}{2}} \) since \( \frac{\alpha-1}{2} \in \mathbb{N} \). Thus, by repeated use of the algebra property in \( H^s (\mathbb{R}^+ \times \mathbb{R}^+ ; \mathbb{C}) \) we obtain

\[
\left\| \Phi u \right\|_X \leq c_s \left( \| u_0 \|_{H^s (\mathbb{R}^+ ; \mathbb{R}^+)} + \| g_0 \|_{\mathfrak{F}^s} + \sqrt{T} \| u \|_X \right). \tag{5.3}
\]

Let \( B(0, r) = \{ u \in X : \| u \|_X \leq r \} \) be a ball centred at 0 with radius \( r = 2c_r \| (u_0, g_0) \|_{\mathbb{C}^2} \). For \( u \in B(0, r) \), estimate (5.3) implies

\[
\left\| \Phi u \right\|_X \leq \frac{r}{2} + c_r \sqrt{T} r^\alpha.
\]
Hence, choosing
\[ T^* \leq \frac{1}{4c^2 r^{2(\alpha - 1)}} \]  
(5.4)
e nsures that \( \Phi u \in B(0, \alpha) \) whenever \( u \in B(0, r) \).

Showing that the map \( u \mapsto \Phi u \) is a contraction in \( X \). We shall show that
\[ \| \Phi u_1 - \Phi u_2 \| \leq \frac{1}{2} \| u_1 - u_2 \| \]
(5.5)for any \( u_1, u_2 \in B(0, r) \) and \( 1 < s \leq \frac{4}{3} \). Noting that
\[ \Phi u_1 - \Phi u_2 = S \left[ 0, 0; \pm \left( |u_1|^{\alpha - 1} u_1 - |u_2|^{\alpha - 1} u_2 \right) \right] \]and using estimate (1.16), we obtain
\[ \| \Phi u_1 - \Phi u_2 \| \leq c_i \sqrt{T^*} \sup_{\tau \in [0, T]} \| \left( |u_1|^{\alpha - 1} u_1 - |u_2|^{\alpha - 1} u_2 \right) (\tau) \|_{H^2(\mathbb{R}_1 \times \mathbb{R}_2^+)} . \]  
(5.6)
Since \( \frac{\alpha - 1}{\alpha} \in \mathbb{N} \), we can write \( |u_1|^{\alpha - 1} u_1 - |u_2|^{\alpha - 1} u_2 = (u_1 \bar{u}_1)^{\alpha - 1} u_1 - (u_2 \bar{u}_2)^{\alpha - 1} u_2 \), which is a polynomial that vanishes when \( u_1 = u_2 \). Hence, we must have
\[ |u_1|^{\alpha - 1} u_1 - |u_2|^{\alpha - 1} u_2 = P(u_1, \bar{u}_1, u_2, \bar{u}_2) (u_1 - u_2) + Q(u_1, \bar{u}_1, u_2, \bar{u}_2) (u_1 - u_2) , \]  
(5.7)for some polynomials \( P \) and \( Q \) of degree \((\alpha - 1)\). Combining inequality (5.6) with the writing (5.7) and the algebra property in \( H^s(\mathbb{R}_1 \times \mathbb{R}_2^+) \), we find
\[ \| \Phi u_1 - \Phi u_2 \| \leq c_i \sqrt{T^*} c_{\alpha}^{\alpha - 1} \| u_1 - u_2 \|_X \]  
(5.8)
Hence, choosing
\[ T^* \leq \frac{1}{4c^2 r^{2(\alpha - 1)}} = c_{\alpha, \alpha} \| (u_0, \bar{u}_0) \|_{D^s}^{2(\alpha - 1)}, \quad c_{\alpha, \alpha} = \left( \frac{c^2 2^{2\alpha} 2^{\alpha}}{s} \right)^{-1}, \]  
(5.9)
ensures that the contraction inequality (5.5) is satisfied for all \( u_1, u_2 \in B(0, r) \).

Since \( c_{\alpha, \alpha} > 1 \), the contraction condition (5.9) is stronger than condition (5.4). Overall, for \( T^* \in (0, T) \) satisfying (5.9), the map \( u \mapsto \Phi u \) defined by (5.1) is a contraction on the ball \( B(0, r) \subset X \). Hence, by the contraction mapping theorem \( u \mapsto \Phi u \) has a unique fixed point in \( B(0, r) \). Equivalently, the integral equation \( u = \Phi u \) for the solution of the NLS ibvp (1.1) has a unique solution \( u \in B(0, r) \subset X \).

**Continuity of the data-to-solution map.** Finally, we shall show that the data-to-solution map
\[ H^s(\mathbb{R}_1 \times \mathbb{R}_2^+) \times B_T^s \ni (u_0, \bar{u}_0) \mapsto u \in X \]
(5.10)is locally Lipschitz continuous.
Let \((u_0, g_0)\) and \((w_0, h_0)\) be two pairs of data lying inside a ball \(B_r \subset D\) of radius \(r > 0\) centred at a distance \(a\) from 0. Denote by \(u = \Phi_{u_0, g_0} u\) and \(w = \Phi_{w_0, h_0} w\) the corresponding solutions to the NLS ibvp \((1.1)\) and by \(T_u\) and \(T_w\) their respective lifespans, which are given by

\[
T_u = \min \left\{ T, c_{s, \alpha} \| (u_0, g_0) \|_D^{\frac{2(\alpha - 1)}{\alpha}} \right\}, \quad T_w = \min \left\{ T, c_{s, \alpha} \| (w_0, h_0) \|_D^{\frac{2(\alpha - 1)}{\alpha}} \right\}.
\]

Since \(\max \{ \| (u_0, g_0) \|_D, \| (w_0, h_0) \|_D \} \leq a + r\) and \(\alpha > 1\), it follows that

\[
\min \{ T_u, T_w \} \geq \max \{ T, c_{s, \alpha} (a + r)^{\frac{2(\alpha - 1)}{\alpha}} \} = T_c.
\]

Hence, both solutions are guaranteed to exist for \(0 \leq t \leq T_c\).

The common lifespan \(T_c\) gives rise to the space \(X_c = C([0, T_c]; H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+))\). We shall now determine the radius \(r_c\) of a ball \(B(0, r_c) \subset X_c\) such that \(u, w \in B(0, r_c)\) and

\[
\| u - w \|_{X_c} \leq 2c_r \| (u_0 - w_0, g_0 - h_0) \|_D.
\]

Recall that \(u\) and \(w\) are fixed points of the map \(\Phi\) in the spaces \(X_u\) and \(X_w\) defined by \((5.2)\) with \(T^*\) replaced by \(T_u\) and \(T_w\) respectively. Hence, since \(X_u, X_w \subseteq X_c\) by the definition of \(T_c\), \(u = w\) is a fixed point of \(\Phi\) in \(X_c\). Thus, using estimate \((1.16)\) together with the writing \((5.7)\) and the algebra property in \(H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+)\), we obtain

\[
\| u - w \|_{X_c} = \| S [u_0 - w_0, g_0 - h_0, \pm (|u|^{\alpha - 1}u - |w|^{\alpha - 1}w)] \|_{X_c} \leq c_r \| (u_0 - w_0, g_0 - h_0) \|_D + c_s c_r \sqrt{T} \| u - w \|_{X_c}.
\]

For \(r_c = 2c_r (a + r)\), estimate \((5.13)\) implies inequality \((5.12)\) which in turn establishes local Lipschitz continuity of the data-to-solution map. The proof of theorem \(1.1\) is complete.

**Remark 5.1.** Estimates \((2.4), (3.5)\) and \((3.22)\) give rise to the following additional estimate for the forced ibvp \((1.12)\):

\[
\sup_{s \in [0, \infty)} \| u(s) \|_{B^s_y} \leq c_r \left( \| u_0 \|_{H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+)} + \| g_0 \|_{B^s_y} + \sqrt{T} \sup_{t \in [0, T]} \| f(t) \|_{H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+)} \right),
\]

\[
0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2},
\]

which, as noted in remark \(2.1\), can be combined with estimate \((1.16)\) in order to show well-posedness of the NLS ibvp \((1.1)\) in the finer space \(C([0, T^*]; H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+)) \cap C(\mathbb{R}^d_1; B^s_y)\) instead of the Hadamard space of theorem \(1.1\).

**6. Well-posedness of NLS on the half-plane with rough data**

For rough data \((s \leq 1)\), we note that the space \(H^s(\mathbb{R}^d_1 \times \mathbb{R}^d_2^+)\) is not an algebra. Thus, as it is well-known from the NLS ivp theory, showing well-posedness of NLS in this case requires additional linear estimates of Strichartz type beyond the Hadamard estimate \((1.16)\) of theorem \(1.3\). For the forced linear ibvp \((1.12)\), these estimates are as follows.

**Theorem 6.1** (Strichartz estimates for the forced linear ibvp). Suppose

\[
\frac{1}{q} + \frac{1}{p} = \frac{1}{2}, \quad 2 \leq p < \infty.
\]
Then, the unified transform formula (1.13) defines a solution $u = S[u_0, g_0; f]$ to the forced linear Schrödinger ivp (1.12) (with the compatibility condition (1.7) for $s > 1$), which admits the estimates
\[
\sup_{t \in [0, T]} \|u(t)\|_{H^p(R_1 \times R_2^+)} + \sup_{x_2 \in [0, \infty)} \|u(x_2)\|_{H^s} + \|u\|_{L^p([0, T]; H^p(R_1 \times R_2^+))} \\
\leq c_{s,p} \left( \|u_0\|_{H^p(R_1 \times R_2^+)} + \|g_0\|_{H^s} + c_{s,p} \begin{cases} \|f\|_{L^p([0, T]; H^p(R_1 \times R_2^+))}, & 0 \leq s < \frac{1}{2}, \\ \|f\|_{L^2([0, T]; H^p(R_1 \times R_2^+))}, & \frac{1}{2} < s \leq \frac{3}{2}, \end{cases} \right)
\]
(6.2)

where $c_{s,p} = c(s, p) > 0$ is a constant depending only on $s$ and $p$.

Theorem 6.1 will be proven by employing the linear superposition (4.7) via the procedure followed in section 4. Thus, we first recall the Strichartz estimates for the linear Schrödinger ivp.

**Theorem 6.2** (Strichartz estimates for the linear Schrödinger ivp). Suppose $s \in \mathbb{R}$ and let $(p, q)$ and $(\rho, \gamma)$ satisfy (6.1). Then, the solution $U = S[U_0; 0]$ of the homogeneous linear Schrödinger ivp (3.1) given by (3.2) satisfies
\[
\|U\|_{L^p([0, T]; H^p(R_1 \times R_2^+))} \leq c_{s,p}\|U_0\|_{H^p(R_1 \times R_2^+)},
\]
(6.3)

Moreover, the solution $W = S[0; F]$ of the forced linear Schrödinger ivp (3.19) given by (3.20) satisfies
\[
\|W\|_{L^p([0, T]; H^p(R_1 \times R_2^+))} \leq c_{s,p}\|F\|_{L^p([0, T]; H^s(R_1 \times R_2^+))},
\]
(6.4)

The estimates of theorem 6.2 were first derived in [S1]. Their proofs can also be found in [Caz] and [Tao]. We continue with an analogous estimate for the pure linear ibvp (2.1).

**Theorem 6.3** (Strichartz estimates for the pure linear ibvp). Suppose $s \geq 0$. Then, for any $(p, q)$ satisfying (6.1) the solution $v = S[0; g; 0]$ of the pure linear ibvp (2.1) given by formula (2.2) admits the estimate
\[
\|v\|_{L^p([0, T]; H^p(R_1 \times R_2^+))} \leq c_{s,p}\|g\|_{B^p}.
\]
(6.5)

The proof of theorem 6.3 is provided in section 7. In the remaining of the current section, we combine estimates (6.3), (6.4) and (6.5) to deduce estimate (6.2), which is then used to establish the well-posedness of the NLS ibvp (1.1) for $s \leq 1$ via contraction mapping (theorem 1.2).

**Proof of Theorem 6.1.** Estimates (2.35) and (2.36) imply
\[
\sup_{x_2 \in \mathbb{R}} \|W(x_2)\|_{H^s} \leq c_{s,p} \|F\|_{L^p([0, T]; H^p(R_1 \times R_2^+))}, \quad 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2}.
\]
(6.6)

Thus, inequality (4.9) can be replaced by
\[
\|W\|_{L^p([0, T]; H^p(R_1 \times R_2^+))} \leq c_{s,p} \|F\|_{L^p([0, T]; H^p(R_1 \times R_2^+))}, \quad 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2}.
\]
(6.7)

Moreover, the Strichartz estimate (6.4) for $(p, q) = (p, \gamma) = (2, \infty)$ implies
\[
\sup_{t \in [0, T]} \|W(t)\|_{H^p(\mathbb{R}^2)} \leq \|F\|_{L^1([0, T]; H^p(\mathbb{R}^2))}, \quad s \in \mathbb{R}.
\]
(6.8)
Therefore, similarly to the proof of theorem 1.3 at the end of section 4, and using in addition estimate (5.14), we obtain

\[
\sup_{t \in [0,T]} \| S[u_0, g_0; f](t) \|_{H^s(\mathbb{R} \times \mathbb{R}_+^2)} + \sup_{x_2 \in [0,\infty)} \| S[u_0, g_0; f](x_2) \|_{B^s_p}
\leq c_s \left( \| u_0 \|_{H^s(\mathbb{R} \times \mathbb{R}_+^2)} + \| g_0 \|_{B^s_p} + \| f \|_{L^2([0,T],H^s(\mathbb{R} \times \mathbb{R}_+^2))} \right), \quad 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2}.
\]

(6.9)

Furthermore, the Strichartz estimates (6.3), (6.4), (6.5) together with inequalities (4.3), (4.5), (4.8), (6.7) and the fact that \( T < 1 \) imply

\[
\| S[u_0, g_0; f] \|_{L^p([0,T],H^p(\mathbb{R} \times \mathbb{R}_+^2))} \leq c_{s,p} \left( \| u_0 \|_{H^s(\mathbb{R} \times \mathbb{R}_+^2)} + \| g_0 \|_{B^s_p} + \| f \|_{L^2([0,T],H^s(\mathbb{R} \times \mathbb{R}_+^2))} \right)
\]

(6.10)

for all \((p, q)\) satisfying condition (6.1) and all \( 0 \leq s \leq \frac{3}{2}, \quad s \neq \frac{1}{2} \).

In fact, for \( 0 \leq s < \frac{1}{2} \) the term \( S[0, W|_{x_2=0}; 0] \) involved the superposition (4.7) can be estimated in a different way that leads to a slight improvement of estimate (6.10). More specifically, instead of directly handling this term via estimate (6.5) and inequality (6.7), we rearrange the relevant unified transform formula (cf. (1.13)) into

\[
S[0, W|_{x_2=0}; 0](x, t) = -i \int_0^T S[0, S[F(t'); 0]|_{x_2=0}; 0](x, t - t') dt'.
\]

(6.11)

Then, noting that the extension of \( S[F(t'); 0]|_{x_2=0}(x_1, t) \) by zero outside \([0, T]\) is continuous if and only if \( 0 \leq s < \frac{1}{2} \) (see theorem 11.4 in [LM]) and employing estimate (6.5) with boundary datum \( g \) equal to this extension, we eventually find

\[
\| S[0, S[F(t'); 0]|_{x_2=0}; 0](x, t - t') \|_{L^p([0,T],H^p(\mathbb{R} \times \mathbb{R}_+^2))} \lesssim \| S[F(t'); 0](x, t)|_{x_2=0} \|_{B^s_p},
\]

\[
0 \leq s < \frac{1}{2}.
\]

This inequality is then combined with estimate (3.5) to yield via (6.11) the estimate

\[
\| S[0, W|_{x_2=0}; 0] \|_{L^p([0,T],H^p(\mathbb{R} \times \mathbb{R}_+^2))} \lesssim \| F \|_{L^1([0,T],H^p(\mathbb{R}^2))}, \quad 0 \leq s < \frac{1}{2},
\]

which in turn implies the following alternative form of (6.10) for \( 0 \leq s < \frac{1}{2} \):

\[
\| S[u_0, g_0; f] \|_{L^p([0,T],H^p(\mathbb{R} \times \mathbb{R}_+^2))} \leq c_{s,p} \left( \| u_0 \|_{H^s(\mathbb{R} \times \mathbb{R}_+^2)} + \| g_0 \|_{B^s_p} + \| f \|_{L^1([0,T],H^s(\mathbb{R} \times \mathbb{R}_+^2))} \right).
\]

(6.12)

Estimates (6.10) and (6.12) complete the proof of theorem 6.1.  

■
Having shown the linear estimate (6.2), we proceed to establish theorem 1.2 for the well-posedness of the NLS ibvp (1.1) in the case of rough data. As in the case of smooth data (theorem 1.1), the proof is done via a contraction mapping argument.

**Proof of Theorem 1.2.** For \( p, q, \alpha \) satisfying (1.10), we define the space

\[
Y = C([0, T^*]; H^s(\mathbb{R}^3 \times \mathbb{R}_t^+)) \cap L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+)),
\]

\[
\|u\|_Y = \sup_{t \in (0, T^*)} \|u(t)\|_{H^s(\mathbb{R}^3 \times \mathbb{R}_t^+)} + \|u\|_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))},
\]

(6.13) (6.14)

We shall show that for an appropriate choice of lifespan \( T^* \in (0, T] \) the map \( u \mapsto \Phi u \) defined by (5.1) is a contraction in \( Y \). Key to this task will be the linear estimate (6.2) after setting \( f = \pm |u|^{\alpha-1} u \). In this connection, since the algebra property is not available in \( H^s(\mathbb{R}^3 \times \mathbb{R}_t^+) \) for \( s \leq 1 \), the term \( \|u|^{\alpha-1} u\|_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} \) will be handled via the following result.

**Proposition 6.1** (Handling the nonlinearity). For \( p, q, \alpha \) satisfying (1.10) when \( 0 \leq s < 1 \), \( s \neq \frac{1}{2} \) and (1.11) when \( s = 1 \), we have

\[
\|u|^{\alpha-1} u - u_1 \|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} \lesssim T^{-\frac{\alpha \omega}{4}} \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))},
\]

(6.15a)

\[
\|u|^{\alpha-1} u - u_2 \|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} \lesssim T^{-\frac{\alpha \omega}{4}} \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))},
\]

(6.15b)

and

\[
\|u\|^{\alpha-1} u - u_1 \|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} \lesssim T^{-\frac{\alpha \omega}{4}} \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} + \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))},
\]

(6.16a)

\[
\|u\|^{\alpha-1} u - u_2 \|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} \lesssim T^{-\frac{\alpha \omega}{4}} \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))} + \|u\|^\alpha_{L^q([0, T^*]; H^p(\mathbb{R}^3 \times \mathbb{R}_t^+))},
\]

(6.16b)

Results similar to those of proposition 6.1 are standard in the NLS ivp theory and, in fact, also appear in the half-line ibvp [H2]. For fractional values \( 0 < s < 1 \), the proof of estimates (6.15) reduces to showing the following result:

\[
\|D^s(|\varphi|^{\alpha-1} \varphi)\|_{L^2(\mathbb{R}^3)} \lesssim \|\varphi\|^{\alpha-1}_{H^s(\mathbb{R}^3)}, \quad 0 < s < 1, \quad \alpha \geq 1,
\]

(6.17)

where \( D^s\varphi(k) = |k|^s \hat{\varphi}(k) \). Inequality (6.17) is established by using fractional versions of the chain rule (proposition 3.1 in [ChWn]) and the Sobolev–Gagliardo–Nirenberg inequality (theorem 1.3.4 in [CazH] and corollary 1.5 in [HMOw]). The proof of estimates (6.16) additionally makes use of the fact that for \( Z_\lambda = \lambda \varphi_1 + (1-\lambda) \varphi_2 \) with \( \lambda \in [0, 1] \) we can write

\[
|\varphi_1|^{\alpha-1} \varphi_1 - |\varphi_2|^{\alpha-1} \varphi_2 = \alpha \frac{1}{2} \int_{\lambda=0}^1 |Z_\lambda|^{\alpha-1} d\lambda (\varphi_1 - \varphi_2) + \alpha \frac{1}{2} \int_{\lambda=0}^1 |Z_\lambda|^{\alpha-3} Z_\lambda^{2} d\lambda (\varphi_1 - \varphi_2).
\]

(6.18)
Relation (6.18) follows from the mean value theorem and provides a stronger version of the well-known inequality
\[ \left| \left| \varphi_1^{(n-1)} \varphi_1 - \varphi_2^{(n-1)} \varphi_2 \right| \right| \lesssim \left( \left| \varphi_1^{(n-1)} + |\varphi_2^{(n-1)}| \right| \right) \left| \varphi_1 - \varphi_2 \right|, \quad \alpha \geq 1. \] (6.19)

Relations (6.18) and (6.19) can be combined with the aforementioned fractional versions of the chain rule and the Sobolev–Gagliardo–Nirenberg inequality as well as with a fractional version of the product rule (proposition 3.3 in [ChWn]) to yield estimates (6.16). For the integer values \( s = 0 \) and \( s = 1 \), proposition 6.1 follows via independent, simpler arguments. Estimate (6.2) for \( f = \pm |u|^{\alpha-1} u \) combined with estimates (6.15) yield
\[ \left| \left| \Phi u \right| \right|_{\mathcal{H}} \lesssim \left| \left| u_0 \right| \right|_{\mathcal{H}(\mathbb{R}^d \times \mathbb{R}^d)} + \left| \left| u_0 \right| \right|_{\mathcal{H}} + T^{\frac{2\alpha}{\alpha}} \left| \left| u \right| \right|_{\mathcal{H}}, \quad 0 \leq s < \frac{1}{2}, \] (6.20a)
\[ \left| \left| \Phi u \right| \right|_{\mathcal{H}} \lesssim \left| \left| u_0 \right| \right|_{\mathcal{H}(\mathbb{R}^d \times \mathbb{R}^d)} + \left| \left| u_0 \right| \right|_{\mathcal{H}} + T^{\frac{2\alpha}{\alpha}} \left| \left| u \right| \right|_{\mathcal{H}}, \quad \frac{1}{2} \leq s \leq 1. \] (6.20b)

Moreover, estimate (6.2) for \( u_0 = g_0 = 0 \) and \( f = \pm |u_1|^{\alpha-1} u_1 - |u_2|^{\alpha-1} u_2 \) together with estimates (6.16) imply
\[ \left| \left| \Phi u_1 - \Phi u_2 \right| \right|_{\mathcal{H}} \lesssim T^{\frac{2\alpha}{\alpha}} \left( \left| \left| u_1 \right| \right|_{\mathcal{H}}^{\alpha-1} + \left| \left| u_2 \right| \right|_{\mathcal{H}}^{\alpha-1} \right) \left| \left| u_1 - u_2 \right| \right|_{\mathcal{H}}, \quad 0 \leq s < \frac{1}{2}, \] (6.21a)
\[ \left| \left| \Phi u_1 - \Phi u_2 \right| \right|_{\mathcal{H}} \lesssim T^{\frac{2\alpha}{\alpha}} \left( \left| \left| u_1 \right| \right|_{\mathcal{H}}^{\alpha-1} + \left| \left| u_2 \right| \right|_{\mathcal{H}}^{\alpha-1} \right) \left| \left| u_1 - u_2 \right| \right|_{\mathcal{H}}, \quad \frac{1}{2} \leq s \leq 1. \] (6.21b)

Estimates (6.20) and (6.21) are similar to (5.3) and (5.8) respectively. Thus, existence and uniqueness of solution for the NLS ibvp (1.1) in the range \( 0 \leq s \leq 1, s \neq \frac{1}{2} \), as well as Lipschitz continuity of the data-to-solution map follow via contraction mapping along the lines of section 5. The proof of theorem 1.2 is complete. \[ \blacksquare \]

7. Strichartz estimates for the pure linear IBVP: proof of theorem 6.3

Recall that the solution \( v = S[0, t; 0] \) of the pure linear ibvp (2.1) has been written in the form \( v = v_1 + v_2 \) with \( v_1 \) and \( v_2 \) given by expressions (2.5) and (2.6) and corresponding, respectively, to the imaginary and real axis components of the complex contour \( \partial D \) involved in the unified transform formula (2.2).

**Estimation along the real axis.** Consider the globally defined function
\[ V_2(x, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ikx - |k|^2 t} \hat{\psi}(k) dk, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \]
where the function \( \psi(x) \) is defined via its Fourier transform by \( \hat{\psi}(k) = \mathcal{H}(k_2)2k_2\delta(k_1, -k_1^2 - k_2^2) \) with \( \mathcal{H} \) denoting the Heaviside function. Observe that \( V_2|_{x^2 \leq R^2} = v_2 \) and, furthermore, that \( V_2 \) satisfies the linear Schrödinger ivp on the whole plane with initial datum equal to \( \psi \). Hence, the homogeneous Strichartz estimate (6.3) implies
\[ \left| \left| V_2 \right| \right|_{\mathcal{L}(\mathbb{R}^2; \mathcal{H}^{\mu}(\mathbb{R}^2))} \lesssim \left| \left| \psi \right| \right|_{\mathcal{H}^{\mu}(\mathbb{R}^2)}, \quad s \in \mathbb{R}, \] (7.1)
for any \((p, q)\) satisfying condition (6.1). Furthermore, by Plancherel’s theorem, for \( s \geq 0 \) we have
\[ \|\psi\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k_1 \in \mathbb{R}} \int_{k_2=-\infty}^{\infty} (1 + \left|k_1^2\right|^2) \mathcal{H}(k_2) 2k_2 \hat{\psi}(k_1, -k_1^2 - k_2^2) \, dk_2 \, dk_1 \]

where \( \mathcal{H} \) is the Heaviside function. Then, expression (7.3) for \( v_1 \) can be written in the form

\[ v_1(x_1, x_2, t) \approx \int_{k_1 \in \mathbb{R}} \int_{k_2=0}^{\infty} e^{ik_1 x_1 - k_2 x_2 - i(k_1^2 + k_2^2)t} \hat{\psi}(k_1, k_2) \, dk_2 \, dk_1 \]

so the change of variable \( k_2 = \sqrt{-\tau - k_1^2} \) yields

\[ \|\psi\|_{L^2(\mathbb{R}^2)}^2 \approx \int_{k_1 \in \mathbb{R}} \int_{\tau=-\infty}^{-k_1^2} (1 + k_1^2) |\tau + k_1^2|^2 \left|\hat{\psi}(k_1, \tau)\right|^2 \, d\tau \, dk_1 \]

Combining this bound with estimate (7.1), we obtain

\[ \|v_2\|_{L^p(\mathbb{R}^n; L^q(\mathbb{R}^n \times \mathbb{R}^n_\tau))} \lesssim \|g\|_{L^q}, \quad s \geq 0, \tag{7.2} \]

for all pairs \((p, q)\) satisfying (6.1).

**Estimation along the imaginary axis.** First we shall consider \( s = 0 \), then we shall deal with \( s \in \mathbb{N} \), and finally we shall treat \( s \geq 0 \) by interpolating between integer values of \( s \).

**The case \( s = 0 \).** Let the function \( \psi \) be defined via its Fourier transform with respect to \( x \in \mathbb{R}^2 \) by

\[ \hat{\psi}(k_1, k_2) = \mathcal{H}(k_2) 2k_2 \hat{\psi}(k_1, -k_1^2 + k_2^2) \tag{7.3} \]

where \( \mathcal{H} \) is the Heaviside function. Then, expression (2.5) for \( v_1 \) can be written in the form

\[ v_1(x_1, x_2, t) \approx \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} \psi(y_1, y_2) K(x_1, x_2, y_1, y_2, t) \, dy_2 \, dy_1 \tag{7.4} \]

with

\[ K(x_1, x_2, y_1, y_2, t) = \int_{k_1 \in \mathbb{R}} \int_{k_2=0}^{\infty} e^{ik_1 (x_1-y_1) - k_2 (x_2-y_2) - i(k_1^2 + k_2^2)t} \hat{\psi}(k_1, k_2) \, dk_2 \, dk_1. \tag{7.5} \]

According to the duality formulation of the \( L^p(\mathbb{R}^n; L^p(\mathbb{R}^n \times \mathbb{R}^n_\tau)) \)-norm,

\[ \|v_1\|_{L^p(\mathbb{R}^n; L^p(\mathbb{R}^n \times \mathbb{R}^n_\tau))} = \sup_{\|\varphi\|_{L^p(\mathbb{R}^n; L^p(\mathbb{R}^n_\tau))} = 1} \left\{ \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}} \int_{t \in \mathbb{R}^+} v_1(x_1, x_2, t) \varphi(x_1, x_2, t) \, dx_2 \, dx_1 \, dt \right\}. \tag{7.6} \]
By the writing (7.4), we have
\[
\left| \int_{\mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} v_1(x_1, x_2, t) \varphi(x_1, x_2, t) \, dx_2 \, dx_1 \, dt \right|
\]
\[
\overset{\approx}{=} \left| \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} \psi(y_1, y_2) \cdot \left( \int_{\mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} K(x_1, x_2, y_1, y_2, t) \varphi(x_1, x_2, t) \, dx_2 \, dx_1 \, dt \right) \, dy_2 \, dy_1 \right|
\]
so by Cauchy–Schwarz inequality in \( y_1 \) and \( y_2 \) we obtain
\[
\left| \int_{\mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} v_1(x_1, x_2, t) \varphi(x_1, x_2, t) \, dx_2 \, dx_1 \, dt \right| \lesssim M \| \psi \|_{L^2(\mathbb{R}^2)},
\]
(7.7)
where
\[
M = \left( \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} K(x_1, x_2, y_1, y_2, t) \varphi(x_1, x_2, t) \, dx_2 \, dx_1 \, dt \right|^2 \, dy_2 \, dy_1 \right)^{\frac{1}{2}}.
\]
Next, we estimate \( M^2 \), which can be rearranged to
\[
M^2 = \int_{\mathbb{R}} \int_{t' \in \mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^+} \varphi(x_1, x_2, t) \overline{\varphi(z_1, z_2, t')} N \, dz_2 \, dz_1 \, dx_2 \, dx_1 \, dt \, dt'
\]
(7.8)
with
\[
N = N(x_1, x_2, z_1, z_2, t, t') = \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} K(x_1, x_2, y_1, y_2, t) \overline{K(z_1, z_2, y_1, y_2, t')} \, dy_2 \, dy_1.
\]
Substituting for \( K \) via (7.5), we have
\[
N = \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 = 0}^{\infty} e^{-i\lambda_1 z_1 - \lambda_2 z_2 + i\lambda_1^2 \theta - \lambda_2^2 \theta'} \left[ \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} e^{i\lambda_1 y_1 + i\lambda_2 y_2} \right. \\
\left. \cdot R(x_1, x_2, t, y_1, y_2) \, dy_2 \, dy_1 \right] \, d\lambda_2 \, d\lambda_1,
\]
(7.9)
where
\[
R(x_1, x_2, t, y_1, y_2) = \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} e^{i(k_1 x_1 - k_2 x_2 - i\lambda_1^2 \theta - \lambda_2^2 \theta')} \, dk_2 \, dk_1.
\]
However, \( R \) is actually the Fourier transform with respect to \( k_1 \) and \( k_2 \) of the function
\[
F(x_1, x_2, t, k_1, k_2) = \mathcal{H}(k_2) e^{i(k_1 x_1 - k_2 x_2 - i\lambda_1^2 \theta - \lambda_2^2 \theta')}, \quad (k_1, k_2) \in \mathbb{R}^2.
\]
Therefore, (7.9) can be written as
\[ N = \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 = 0}^{\infty} e^{-\lambda_1 z_1 - \lambda_2 z_2 + i(\lambda_1^2 - \lambda_2^2) t} \left[ \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} e^{i\lambda_1 y_1 + i\lambda_2 y_2} \right] F(x_1, x_2, t, y_1, y_2) \, dy_2 \, dy_1 \, d\lambda_2 \, d\lambda_1, \]

which by the Fourier inversion formula reduces to

\[ N \simeq \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 = 0}^{\infty} e^{-\lambda_1 z_1 - \lambda_2 z_2 + i(\lambda_1^2 - \lambda_2^2) t} \, F(x_1, x_2, t, \lambda_1, \lambda_2) \, d\lambda_2 \, d\lambda_1 \]

\[ \simeq \int_{\lambda_1 \in \mathbb{R}} \int_{\lambda_2 = 0}^{\infty} e^{-\lambda_1 z_1 - \lambda_2 z_2 + i(\lambda_1^2 - \lambda_2^2) t} \cdot e^{i\lambda_1 x_1 - i\lambda_2 x_2 - i(\lambda_1^2 - \lambda_2^2) t} \, d\lambda_2 \, d\lambda_1 \]

\[ = I(x_1 - z_1, t - \tau)J(x_2 + z_2, t - \tau), \]

where

\[ I(x_1, t) = \int_{\lambda_1 \in \mathbb{R}} e^{i\lambda_1 x_1} \, d\lambda_1, \quad J(x_2, t) = \int_{\lambda_2 = 0}^{\infty} e^{-\lambda_2 x_2 + i\lambda_2^2 t} \, d\lambda_2. \quad (7.10) \]

In turn, expression (7.8) for \( M^2 \) becomes

\[ M^2 \simeq \int_{r \in \mathbb{R}} \int_{t \in \mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^+} \varphi(x_1, x_2, t) \varphi(z_1, z_2, \tau) \]

\[ \cdot I(x_1 - z_1, t - \tau)J(x_2 + z_2, t - \tau) \, dz_2 \, dx_2 \, dx_1 \, d\tau \, dr \]

\[ \leq \int_{r \in \mathbb{R}} \int_{t \in \mathbb{R}} \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}^+} \left| \varphi(x_1, x_2, t) \right| \left( \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^+} I(x_1 - z_1, t - \tau)J(x_2 + z_2, t - \tau) \varphi(z_1, z_2, \tau) \, dz_2 \, dz_1 \right) \, dx_2 \, dx_1 \, d\tau \, dr. \]

Hence, by Hölder’s inequality in \( x_1 \) and \( x_2 \) we have

\[ M^2 \lesssim \int_{r \in \mathbb{R}} \| \varphi(t) \|_{L^p(R_{x_1} \times R_{z_1}^{+})} \]

\[ \cdot \left( \int_{t \in \mathbb{R}} \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^+} I(x_1 - z_1, t - \tau)J(x_2 + z_2, t - \tau) \, dz_2 \, dz_1 \right) \, d\tau \, dr, \quad (7.11) \]
while an additional Hölder's inequality in $t$ together with the fact that $\|\varphi\|_{L^p(R;L^q(\mathbb{R}_1 \times \mathbb{R}^n_+))} = 1$ imply

$$M^2 \lesssim \int_{t' \in \mathbb{R}} \left\| \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^n_+} I(x_1 - z_1, t - t') J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \, dz_1 \right\|_{L^p(\mathbb{R})} \, dt'. $$

(7.12)

Therefore, the estimation of $M^2$ reduces to that of

$$Q_p(t, t') = \left\| \int_{z_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^n_+} I(x_1 - z_1, t - t') J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \, dz_1 \right\|_{L^p(\mathbb{R}_1 \times \mathbb{R}^n_+)}^2,$$

which, similarly to the Strichartz estimates for the linear Schrödinger ivp on $\mathbb{R}^n$, will be done by interpolating between $p = 2$ and $p = \infty$.

For $p = 2$, recalling the definition (7.10) of $I$ we have

$$Q^2 = \int_{x_2 \in \mathbb{R}^n_+} \int_{x_1 \in \mathbb{R}} \int_{\lambda_1 \in \mathbb{R}} e^{i\lambda_1 x_1} \cdot e^{-i\lambda_1 (t-t')} \int_{x_2 \in \mathbb{R}^n_+} \int_{z_2 \in \mathbb{R}^n_+} e^{-i\lambda_1 z_2} J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \, dz_1 \, d\lambda_1 \, dx_2.$$

Successive applications of Plancherel's theorem, first between $x_1$ and $\lambda_1$ and then between $\lambda_1$ and $z_1$, yield

$$Q^2 = \frac{1}{2\pi} \int_{x_2 \in \mathbb{R}^n_+} \int_{x_1 \in \mathbb{R}} \int_{\lambda_1 \in \mathbb{R}} e^{-i\lambda_1 z_2} J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \, dz_1 \, d\lambda_1 \, dx_2$$

$$= \int_{x_2 \in \mathbb{R}^n_+} \int_{x_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^n_+} J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \, dz_1 \, dx_2.$$

Thus, substituting for $J$ via (7.10) we have

$$Q^2 = \int_{x_2 \in \mathbb{R}^n_+} \int_{x_1 \in \mathbb{R}} \int_{\lambda_2 = 0}^{\infty} e^{-i\lambda_2 z_2} \cdot e^{i\lambda_2 (t-t')} \int_{x_2 \in \mathbb{R}^n_+} e^{-i\lambda_2 z_2} \varphi(z_1, z_2, t') dz_2 \, d\lambda_2 \, dz_2 \, dx_2,$$

so that using the Laplace transform boundedness lemma 2.1 twice, first between $\lambda_2$ and $x_2$ and then between $z_2$ and $\lambda_2$, we obtain

$$Q^2 \lesssim \int_{x_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^n_+} \left| \varphi(z_1, z_2, t') \right|^2 \, dz_2 \, dz_1$$

$$\lesssim \int_{x_1 \in \mathbb{R}} \int_{z_2 \in \mathbb{R}^n_+} \left| \varphi(z_1, z_2, t') \right|^2 \, dz_2 \, dz_1 = \left\| \varphi(t') \right\|_{L^2(\mathbb{R} \times \mathbb{R}^n_+)}^2. $$

(7.13)

For $p = \infty$, we have

$$Q_\infty = \sup_{x_2 \in \mathbb{R}^n_+} \left| I(v, t - t') \ast \left( \int_{z_2 \in \mathbb{R}^n_+} J(x_2 + z_2, t - t') \overline{\varphi(z_1, z_2, t')} dz_2 \right) (x_1) \right|_{L^\infty(\mathbb{R}_1)}.$$

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Hence, Young’s convolution inequality implies

\[ Q_\infty \lesssim \|I(x_1, t - t')\|_{L^\infty(R^{1+1})} \sup_{x_2 \in \mathbb{R}^+} \left\| \int_{z_2 \in \mathbb{R}^+} J(x_2 + z_2, t - t') \varphi(x_1, z_2, t') \, dz_2 \right\|_{L^1(R^{1+1})}. \]

Also, completing the square in (7.10) we find \( I(x_1, t) = e^{-i \text{sgn}(t) \frac{t^2}{2}} \sqrt{\pi} e^{\frac{1}{4} |t|^{-1}}. \) Therefore,

\[ Q_\infty \lesssim |t - t'|^{-\frac{1}{2}} \sup_{x_2 \in \mathbb{R}^+} |J(x_2 + z_2, t - t')| \|\varphi(t')\|_{L^1(R^{1+1})}. \]

**Lemma 7.1.** For all \( x_2 \geq 0 \) and all \( t \in \mathbb{R} \setminus \{0\} \), the integral \( J_2 \) defined by \( (7.10) \) admits the bound \( |J(x_2, t)| \leq c|t|^{-\frac{1}{2}} \) with \( c > 0 \) a constant independent of \( x_2 \) and \( t \).

Lemma 7.1 follows from a straightforward application of van der Corput’s lemma (see corollary 1.1 in [LP]) and yields

\[ Q_\infty \lesssim |t - t'|^{-1} \|\varphi(t')\|_{L^1(R^{1+1})}.\]  

(7.14)

Interpolating between (7.13) and (7.14) via the Riesz–Thorin interpolation theorem, we deduce

\[ Q_p(t, t') \lesssim |t - t'|^{\frac{1}{p} - 1} \|\varphi(t')\|_{L^p(R^{1+1})}, \quad 2 \leq p \leq \infty. \]

(7.15)

Having estimated \( Q_p \), we return to the estimation of \( M^2 \). For \( p = 2 \), we combine (7.15) with (7.11) and recall that \( \|\varphi\|_{L^p(R^{1+1})} = 1 \) to infer

\[ M^2 \lesssim \int_{t \in \mathbb{R}} \|\varphi(t)\|_{L^2(R^{1+1})} \left( \int_{t' \in \mathbb{R}} \|\varphi(t')\|_{L^2(R^{1+1})} \, dt' \right) \, dt = 1. \]

(7.16)

For \( 2 < p \leq \infty \), we use (7.15) in (7.12) to obtain

\[ M^2 \lesssim \left\| \int_{t \in \mathbb{R}} |t - t'|^{\frac{1}{p} - 1} \|\varphi(t')\|_{L^p(R^{1+1})} \, dt' \right\|_{L^q(R^{1+1})}. \]

Since \( p > 2 \), the \( L^q \)-norm above can be estimated via the Hardy–Littlewood–Sobolev fractional integration theorem [LP]. This step restricts \( p < \infty \) (cf condition (6.1)) and yields

\[ M^2 \lesssim \|\varphi\|_{L^p(R^{1+1})} \|\varphi\|_{L^q(R^{1+1})}, \quad \frac{1}{r} = \frac{1}{q} + \frac{2}{p}. \]

Requiring that \( r = q' \), which is equivalent to \( \frac{1}{q} + \frac{1}{p} = \frac{1}{2} \) and hence holds by condition (6.1), we get

\[ M \lesssim \|\varphi\|_{L^p(R^{1+1})} = 1, \quad 2 < p < \infty. \]

(7.17)

Estimates (7.16) and (7.17) imply \( M \lesssim 1 \) for all \( (p, q) \) satisfying condition (6.1). Combining this bound with (7.6) and (7.7), we deduce

\[ \|u_1\|_{L^q(R^{1+1})} \lesssim \|\psi\|_{L^2(R^{1+1})}. \]

(7.18)
Furthermore, recalling the definition (7.3) of $\psi$ and using Plancherel’s theorem and the change of variable $k_2 = \sqrt{\tau + k_1^2}$, we find

$$
\|\psi\|_{L^2(\mathbb{R}^2_+)} \simeq \left( \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left|2k_2 \hat{g}(k_1, -k_1^2 + k_2^2)\right|^2 \, dk_2 \, dk_1 \right)^{\frac{1}{2}}
$$

$$
\simeq \left( \int_{k_1 \in \mathbb{R}} \int_{\tau = -k_1^2}^{\infty} \left|\tau + k_1^2\right|^{\frac{1}{2}} |\hat{g}(k_1, \tau)|^2 \, d\tau \, dk_1 \right)^{\frac{1}{2}} \lesssim \|g\|_{X_{\frac{1}{2}, \frac{1}{2}}}.
$$

Therefore, for all pairs $(p, q)$ satisfying condition (6.1) we conclude that

$$
\|v_1\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \|g\|_{X_{\frac{1}{2}, \frac{1}{2}}}.
$$

(7.19)

The case $s \in \mathbb{N}$. According to a classical result by Calderón, the Bessel potential space $H^{s,p}(\mathbb{R}^2)$ coincides with the Sobolev space $W^{s,p}(\mathbb{R}^2)$ for all $s \in \mathbb{N}_0$ and $1 < p < \infty$. Thus,

$$
\|v_1(t)\|_{H^{s,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+)} \simeq \|v_1(t)\|_{W^{s,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+)} = \sum_{|\mu| \leq s} \|\partial^\mu_t v_1(t)\|_{L^p(\mathbb{R}^1_+ \times \mathbb{R}^2_+)}, \quad s \in \mathbb{N}_0.
$$

(7.20)

For any $|\mu| \in \mathbb{N}_0$, differentiating expression (2.5) for $v_1$ we obtain

$$
\partial^\mu_t v_1(x_1, x_2, t) \simeq \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} e^{ik_1x_1 - k_2x_2 - \imath k_1^2 t} \hat{\psi}_\mu(k_1, k_2) \, dk_2 \, dk_1
$$

(7.21)

with the function $\hat{\psi}_\mu$ defined similarly to (7.3) by $\hat{\psi}_\mu(k_1, k_2) = \mathcal{H}(k_2) k_1^{\mu_1} k_2^{\mu_2} \hat{g}(k_1, -k_1^2 + k_2^2)$. Observe that expression (7.21) is entirely analogous to (2.5) but with $\hat{\psi}_\mu$ in place of $\hat{\psi}$. Hence, for any $(p, q)$ satisfying (6.1), we employ estimate (7.18) to deduce

$$
\|\partial^\mu_t v_1\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \|\psi_\mu\|_{L^2(\mathbb{R}^2)},
$$

which via Plancherel’s theorem becomes

$$
\|\partial^\mu_t v_1\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left(k_1^2\right)^{\mu_1} \left(k_2^2\right)^{\mu_2} |\hat{g}(k_1, -k_1^2 + k_2^2)|^2 \, dk_2 \, dk_1.
$$

Therefore, in view of definition (7.20) we have

$$
\|v_1\|_{L^2(\mathbb{R}; H^{s,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \sum_{|\mu| \leq s} \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left(k_1^2\right)^{\mu_1} \left(k_2^2\right)^{\mu_2} |\hat{g}(k_1, -k_1^2 + k_2^2)|^2 \, dk_2 \, dk_1,
$$

which similarly to (2.11) and (2.12) yields

$$
\|v_1\|_{L^p(\mathbb{R}; H^{s,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \sum_{|\mu| \leq s} \|\partial^\mu_t v_1\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \|g\|_{X_{\frac{s}{2}, \frac{s}{2}}},
$$

(7.22)

for all $(p, q)$ satisfying (6.1).

The case $s > 0$ and $s \notin \mathbb{N}$. The validity of estimate (7.22) can be extended to all $s \geq 0$ by interpolation. In particular, any $s \geq 0$ can be expressed as $s = (1 - \beta) \lfloor s \rfloor + \beta \lfloor s \rfloor - 1$, $\beta \in [0, 1]$. Moreover, estimate (7.22) implies

$$
\|v_1\|_{L^p(\mathbb{R}; H^{s,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \|g\|_{B^{s,p}}, \quad \|v_1\|_{L^p(\mathbb{R}; H^{s+1,p}(\mathbb{R}^1_+ \times \mathbb{R}^2_+))} \lesssim \|g\|_{B^{s+1,p}},
$$

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i.e. \( v_1 \) is a continuous linear operator from \( B^{[s]} \) to \( L^q(\mathbb{R}; H^{[\xi, p]}(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)) \) as well as from \( B^{[s]+1} \) to \( L^q(\mathbb{R}; H^{[\xi, p]}(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)) \). Thus, according to theorem 5.1 of [LM], \( v_1 \) is continuous from \( [B^{[s]}, B^{[s]+1}]_\beta = B^\beta \) to \( [L^q(\mathbb{R}; H^{[\xi, p]}(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)), L^q(\mathbb{R}; H^{[\xi, p]}(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+))]_\beta = L^q(\mathbb{R}; H^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+)) \) (the fact that \( [B^{[s]}, B^{[s]+1}]_\beta = B^\beta \) follows by a straightforward application of Hölder’s inequality) allowing us to conclude that

\[
\|v_1\|_{L^q(\mathbb{R}; H^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}^+))} \lesssim \|g\|_{H^p}, \quad s \geq 0,
\]

for all \((p, q)\) satisfying (6.1). Estimates (7.2) and (7.23) complete the proof of theorem 6.3.

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Appendix A. Solution of the forced linear IBVP via Fokas’s unified transform

We provide a concise derivation of the unified transform formula (1.13) for the forced linear Schrödinger ibvp (1.12) under the assumption of smooth initial and boundary values. For a more detailed derivation, we refer the reader to [F3], where formula (1.13) was first obtained.

Let \( \tilde{u} \) satisfy the adjoint of the linear Schrödinger equation, i.e. \( \tilde{u}_t - \tilde{u}_{x_1 x_1} - \tilde{u}_{x_2 x_2} = 0 \). Multiplying this equation by the solution \( u \) of the forced linear Schrödinger equation (1.12a), and adding to it equation (1.12a) multiplied by \( \tilde{u} \), we arrive at the divergence form \( i(\tilde{u}u)_t + (\tilde{u}_{x_1} - \tilde{u}_x u)_{x_1} + (\tilde{u}_{x_2} - \tilde{u}_x u)_{x_2} = \tilde{u} f \). Setting \( \tilde{u}(x_1, x_2, t) = e^{-ik_1 x_1 - ik_2 x_2 + it^2 + k_2^2} u(k_1, k_2, 0) \), \( k_1, k_2 \in \mathbb{C} \), and integrating with respect to \( x_1 \) and \( x_2 \) while assuming that \( u \to 0 \) as \( |x_1|, x_2 \to \infty \), we obtain

\[
i \left( e^{ik_1^2 + k_2^2 t} \tilde{u}(k_1, k_2, t) \right) = -e^{ik_1^2 + k_2^2 t} \left[ ik_1 \tilde{u}(k_1, 0, t) + ik_2 \tilde{u}(0, k_1, t) \right] + e^{ik_1^2 + k_2^2 t} \tilde{f}(k_1, k_2, t),
\]

(A.1)

where \( \tilde{u} \) and \( \tilde{f} \) are the half-plane Fourier transforms of \( u \) and \( f \) defined by (1.14) while \( \tilde{u}^{x_1} \) denotes the Fourier transform of \( u \) with respect to \( x_1 \) defined as (1.4). Note that \( \tilde{u} \) and \( \tilde{f} \) are analytic as functions of \( k_2 \) in \( \mathbb{C}^- = \{ k_2 \in \mathbb{C} : \text{Im}(k_2) < 0 \} \) as a consequence of a Paley–Wiener theorem (see [S2], theorem 7.2.4). Integrating (A.1) with respect to \( t \) gives rise to the global relation

\[
e^{ik_1^2 + k_2^2 t} \tilde{u}(k_1, k_2, t) = \tilde{u}_0(k_1, k_2) + ik_1 \tilde{g}_1(k_1, k_1^2 + k_2^2, t) - k_2 \tilde{g}_0(k_1, k_1^2 + k_2^2, t) - \int_{t=0}^{t} e^{ik_1^2 + k_2^2 \tau} \tilde{f}(k_1, k_2, \tau) d\tau, \quad (k_1, k_2) \in \mathbb{R} \times \mathbb{C}^-,
\]

(A.2)
where for \( g_j(x_1, t) = \partial_t^j u(x_1, 0, t), j = 0, 1, \) we define

\[
\tilde{g}_j(k_1, k_2^2) = \int_{t=0}^{t'} e^{ik_1 x_1 + ik_2 x_2 - ik_2 t' + ik_2 t} \tilde{g}_j(k_1, t') dt', \quad j = 0, 1.
\]

Inverting (A.2) by means of the usual inverse Fourier transform, we obtain

\[
u(x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - ik_2 t + ik_2 t} \tilde{u}(k_1, k_2) dk_2 dk_1
\]

\[
- \frac{i}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - ik_2 t + ik_2 t} \int_{t'=0}^{t'} e^{ik_2 t'} f(k_1, k_2, t') dt' dk_2 dk_1
\]

\[
+ \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1 x_1 + ik_2 x_2 - ik_2 t + ik_2 t} [i \tilde{g}_1(k_1, k_2^2 + k_2, t)
\]

\[
- k_2 \tilde{g}_0(k_1, k_1^2 + k_2, t)] dk_2 dk_1.
\]

(A.3)

The representation (A.3) is not an explicit solution formula for the forced linear ibvp (1.12) because it involves the unknown Neumann boundary value \( u_{x_2}(x_1, 0, t) \) through the transform \( \tilde{g}_1 \). However, it turns out that \( \tilde{g}_1 \) can be eliminated from (A.3) in favour of known quantities.

In particular, note that since \( x_2 \geq 0 \) and \( t \geq t' \) the exponential \( e^{ik_1 x_1 - ik_2 (t - t')} \) is bounded for all \( k_2 \in \mathbb{C}^+ \) where \( D \) here denotes the first quadrant of the complex \( k_2 \)-plane. Thus, exploiting the analyticity of \( \tilde{g}_1 \) for all \( k_2 \in \mathbb{C} \) (which follows via a Paley–Wiener theorem similar to theorem 7.2.4 of [S2]) we apply Cauchy’s theorem in the second quadrant of the complex \( k_2 \)-plane to write the \( k_2 \)-integral in the last term of (A.3) as

\[
\int_{k_2 \in \partial D} e^{ik_1 x_1 - ik_2} [i \tilde{g}_1(k_1, k_1^2 + k_2, t) - k_2 \tilde{g}_0(k_1, k_1^2 + k_2, t)] dk_2 - \lim_{\rho \to \infty} I(\rho, k_1, x_2, t),
\]

where \( \partial D \) is the positively oriented boundary of \( D \) depicted in figure 1 and for the quarter circle \( \gamma_\rho^+ = \{ \rho e^{i\theta}, \frac{\pi}{2} \leq \theta \leq \pi \} \) shown in figure A.1 we define

\[
I(\rho, k_1, x_2, t) = \int_{k_2 \in \gamma_\rho^+} e^{ik_1 x_1 - ik_2} [i \tilde{g}_1(k_1, k_1^2 + k_2, t) - k_2 \tilde{g}_0(k_1, k_1^2 + k_2, t)] dk_2.
\]

(A.4)

Integrating by parts with respect to \( t' \) in the definitions of \( \tilde{g}_0 \) and \( \tilde{g}_1 \), we obtain

\[
(A.4) \leq \frac{C_{\tilde{g}_0, \tilde{g}_1}(k_1)}{x^2} \rho \left( 1 - e^{-\rho x_2} \right) \frac{\rho (1 - e^{-\rho x_2})}{|\rho^2 - k_1^2|},
\]

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where, recalling that we work under the assumption of smooth boundary values,
\[ C_{g_0,g_1}(k_1) = \|g_1\|_{L^\infty(0,T)} + \|\mathcal{G}(k_1)\|_{L^\infty(0,T)} + \|\partial_k \mathcal{G}(k_1)\|_{L^\infty(0,T)} + \|\partial_{kk} \mathcal{G}(k_1)\|_{L^\infty(0,T)} < \infty. \]

Hence, the integral (A.4) vanishes as \( \rho \to \infty \) and, in turn, the integral representation (A.3) reads

\[
\begin{align*}
  u(x_1, x_2, t) &= \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \tilde{u}_0(k_1, k_2) dk_2 \, dk_1 \\
  &- i \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \int_{\rho=0}^\infty e^{ik_1^2 + ik_2^2 \rho^2} f(k_1, k_2, \rho) \, d\rho \, dk_2 \, dk_1 \\
  &+ i \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \left[ i \hat{g}_1(k_1, k_1^2 + k_2^2, t) \right] \, dk_2 \, dk_1. 
\end{align*}
\] (A.5)

Next, note that under the transformation \( k_2 \mapsto -k_2 \) the global relation (A.2) yields the identity

\[
\begin{align*}
  e^{ik_1^2 + ik_2^2} \tilde{u}(k_1, -k_2, t) &= \tilde{u}_0(k_1, -k_2) + i \hat{g}_1(k_1, k_1^2 + k_2^2, t) + k_2 \hat{g}_0(k_1, k_1^2 + k_2^2, t) \\
  &- i \int_{\rho=0}^\infty e^{ik_1^2 + ik_2^2 \rho^2} f(k_1, -k_2, \rho) \, d\rho, \quad (k_1, k_2) \in \mathbb{R} \times \mathbb{C}^+. 
\end{align*}
\] (A.6)

Thanks to the fact that \( \int_{k_2 \in \partial D} e^{ik_2x_2} \tilde{u}(k_1, -k_2, t) \, dk_2 = 0 \) due to the analyticity and exponential decay of the integrand inside \( D \), we are able to solve identity (A.6) for \( \hat{g}_1 \) and substitute the resulting expression in (A.5) to obtain the explicit solution formula

\[
\begin{align*}
  u(x_1, x_2, t) &= \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \tilde{u}_0(k_1, k_2) dk_2 \, dk_1 \\
  &- \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \tilde{u}_0(k_1, -k_2) dk_2 \, dk_1 \\
  &- i \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \mathbb{R}} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \int_{\rho=0}^\infty e^{ik_1^2 + ik_2^2 \rho^2} f(k_1, k_2, \rho) \, d\rho \, dk_2 \, dk_1 \\
  &+ i \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} \int_{\rho=0}^\infty e^{ik_1^2 + ik_2^2 \rho^2} f(k_1, -k_2, \rho) \, d\rho \, dk_2 \, dk_1 \\
  &+ \frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 \in \partial D} e^{ik_1x_1 + ik_2x_2 - ik_1^2/2 - ik_2^2/2} 2k_2 \hat{g}_0(k_1, k_1^2 + k_2^2, t) \, dk_2 \, dk_1. 
\end{align*}
\] (A.7)

Finally, exploiting once again analyticity and exponential decay in \( D \), we infer that

\[
\int_{k_2 \in \partial D} e^{ik_2x_2} \hat{g}(k_1, t) \, dk_2 = 0
\]

and hence the solution formula (A.7) can be written in the equivalent form (1.13), which is the one convenient for proving linear estimates.
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