PHASE RETRIEVAL OF COMPLEX AND VECTOR-VALUED FUNCTIONS

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Abstract. The phase retrieval problem in the classical setting is to reconstruct real/complex functions from the magnitudes of their Fourier/frame measurements. In this paper, we consider a new phase retrieval paradigm in the complex/quaternion/vector-valued setting, and we provide several characterizations to determine complex/quaternion/vector-valued functions $f$ in a linear space $S$ of (in)finite dimensions, up to a trivial ambiguity, from the magnitudes $\|\phi(f)\|$ of their linear measurements $\phi(f), \phi \in \Phi$. Our characterization in the scalar setting implies the well-known equivalence between the complement property for linear measurements $\Phi$ and the phase retrieval of linear space $S$. In this paper, we also discuss the affine phase retrieval of vector-valued functions in a linear space and the reconstruction of vector fields on a graph, up to an orthogonal matrix, from their absolute magnitudes at vertices and relative magnitudes between neighboring vertices.

1. Introduction

Phase retrieval arises in various engineering fields, such as X-ray crystallography, coherent diffractive imaging and optics [26, 29, 31, 46]. The classical phase retrieval problem is to recover real/complex functions from the magnitudes of their Fourier measurements. Starting from the pioneering work [9] by Balan, Casazza and Edidin, phase retrieval of real vectors $x \in \mathbb{R}^d$ (or complex vectors $x \in \mathbb{C}^d$) from the magnitudes $y = |Ax|$ of their frame measurements has received considerable attention, where $A$ is a measurement matrix, see [5, 8, 11, 14, 27, 32, 36, 50, 55, 56] for historical remarks and additional references. The phase retrieval paradigm has been recently extended to infinite-dimensional setting, where a core problem is to recover real/complex functions $f$ in a linear space of (in)finite dimensions, such as the Paley-Wiener space and shift-invariant spaces, from the magnitudes $|\phi(f)|$ of their linear measurements $\phi(f), \phi \in \Phi$, see [3, 4, 12, 17, 19, 20, 28, 43, 44, 49, 50, 51, 53]. The phase retrieval in the infinite-dimensional setting is fundamentally different from the finite-dimensional setting [4, 12, 17, 28], for instance, phase retrieval in an infinite-dimensional Hilbert space is coherently unstable [12], and the set of phase

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retrieval functions in a real shift-invariant space $S(\psi)$ generated by a compactly supported function $\psi$ is observed to be neither a convex subset of $S(\psi)$ nor its closed subset \cite{17, 19, 20}.

Quaternions form a noncommutative division algebra, and they are used in navigation, robotics and computer vision \cite{6, 37}. In the first part of this paper, we consider the phase retrieval problem whether a complex/quaternion function $f$ on a domain $D$ in a linear space of (in)finite dimensions is determined, up to a unimodular constant and conjugation, from the magnitudes $|f(x)|$, $x \in D$.

An important problem in the dynamic of a fleet of autonomous mobile robots (AMR) is to determine the velocity of each AMR from the absolute speed of AMRs and the relative speed between neighboring AMRs. As the topology of an AMR fleet can be described by a graph $G = (V, E)$, the velocity recovery problem for an AMR fleet becomes whether a vector field $\mathbf{f} = (\mathbf{f}_i)_{i \in V}$ on the graph $G$ can be reconstructed, up to an orthogonal matrix, from its absolute magnitudes $\|\mathbf{f}_i\|$ at all vertices $i \in V$ and relative magnitudes $\|\mathbf{f}_i - \mathbf{f}_j\|$ of neighboring vertices $(i, j) \in E$. In the second part of this paper, we consider the phase retrieval problem whether a vector-valued function $\mathbf{f}$ in a real linear space $S$ of (in)finite dimensions is determined, up to a unitary transformation, from the magnitudes $\|\phi(f)\|$ of their linear measurements $\phi(f)$, $\phi \in \Phi$, and also the reconstruction of a vector field on a undirected graph, up to an orthogonal matrix, from its absolute magnitudes at vertices and relative magnitudes between neighboring vertices. Our characterization in the real scalar setting implies the well-known equivalence between the complement property for linear measurements $\Phi$ and the phase retrieval of linear space $S$ \cite{4, 8, 9, 12}.

Affine phase retrieval arises in holography, data separation and phaseless sampling \cite{16, 17, 22, 23, 40, 41}, and one of its core problems is whether real/complex functions are determined uniquely from their magnitudes of affine linear measurements \cite{27, 30, 34, 38, 44}. In the third part of this paper, we study affine phase retrieval of vector-valued functions.

This paper is organized as follows. In Sections 2 and 3, we consider the phase retrieval of complex/quaternion functions and vector-valued functions respectively. In Section 4, we discuss the affine phase retrieval of vector-valued functions. All proofs are collected in Section 5.

2. PHASE RETRIEVAL OF COMPLEX/QUATERNION FUNCTIONS

Let $\mathcal{C}$ be a complex linear space of functions $f$ on a domain $D$ that is invariant under complex conjugation, i.e. $\overline{f} \in \mathcal{C}$ for all $f \in \mathcal{C}$. Our representative examples of complex conjugate invariant spaces are the complex range space

\begin{equation}
R_\mathbb{C}(\mathbf{A}) = \{\mathbf{Ax}, \ x \in \mathbb{C}^n\}
\end{equation}
of a real matrix $A$ of size $m \times n$, and the complex shift-invariant space
\[(2.2) \quad S_C(\psi) = \left\{ \sum_{k \in \mathbb{Z}} c(k)\psi(\cdot - k), \, c(k) \in \mathbb{C} \text{ for all } k \in \mathbb{Z} \right\} \]
generated by a real-valued function $\psi$ on the real line $\mathbb{R}$ [11, 24, 21].

Given $f \in C$, let
\[(2.3) \quad M_f := \{ g \in C, \ |g(x)| = |f(x)| \text{ for all } x \in D \} \]
contain all functions $g \in C$ that have the same magnitude measurements as
the original function $f$ has on the whole domain $D$. Clearly,
\[(2.4) \quad M_f \supset \{ zf \in C, \, z \in \mathbb{T} \} \cup \{ z\bar{f} \in C, \, z \in \mathbb{T} \}, \]
where $\mathbb{T} = \{ z \in \mathbb{C}, |z| = 1 \}$. We say that a function $f \in C$ is complex conjugate phase retrieval in $C$ if
\[(2.5) \quad M_f = \{ zf \in C, \, z \in \mathbb{T} \} \cup \{ z\bar{f} \in C, \, z \in \mathbb{T} \}, \]
and that the linear space $C$ is complex conjugate phase retrieval if every function $f \in C$ is complex conjugate phase retrieval in $C$, see [25, 35, 44] for complex conjugate phase retrieval of vectors in a finite-dimensional linear space and of entire functions.

In Section 2.1 we characterize complex conjugate phase retrieval of functions $f$ in a complex conjugate invariant space $C$, see Theorems 2.1 and 2.2. Write $A^T = (a_1, \ldots, a_m)$. As an application of Theorem 2.2 we show that the complex range space $R_C(A)$ in (2.1) is complex conjugate phase retrieval if and only if there does not exist a real matrix $X$ of rank at most 2 such that $X^T \neq -X$ and $\text{Tr}(a_i a_i^T X) = 0$ for all $1 \leq i \leq m$, see Corollary 2.4 and [25, Theorem 2.3] for an equivalent formulation.

Let $\mathbb{R}(C)$ be the linear space of all real-valued functions in $C$. By the complex conjugate invariance of the complex linear space $C$, we have
\[(2.6) \quad \mathbb{R}(C) = \{ (f + \bar{f})/2, \, f \in C \}. \]
We say that a function $f \in \mathbb{R}(C)$ is phase retrieval in $\mathbb{R}(C)$ if
\[M_f := \{ g \in \mathbb{R}(C), \ |g(x)| = |f(x)| \text{ for all } x \in D \}\]
contains only two elements $\pm f$, and that the whole real linear subspace $\mathbb{R}(C)$ is phase retrieval if every function $f \in \mathbb{R}(C)$ is phase retrieval in $\mathbb{R}(C)$, i.e., $|f(x)| = |g(x)|, x \in D$ for $f, g \in \mathbb{R}(C)$ only when $f = \pm g$, [3, 17, 19, 20, 39, 43, 52, 54]. In Section 2.2 we show that a complex conjugate phase retrieval function in $C$ has its real part being phase retrieval in $\mathbb{R}(C)$, and hence $\mathbb{R}(C)$ is phase retrieval if $C$ is complex conjugate phase retrieval, see Theorem 2.5 and Corollary 2.6. As an application of Theorem 2.5 we find all complex conjugate phase retrieval functions in a complex shift-invariant space $S_C(h)$ generated by the hat function $h(t) = \max(1 - |t|, 0)$, see Proposition 2.9.

The set $Q_8 = \{ a + bi + cj + dk, \, a, \, b, \, c, \, d \in \mathbb{R} \}$ of quaternions forms a noncommutative division algebra with the multiplication rule for the basis
1, i, j and k given by
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i \text{ and } ki = -ik = j. \]
Quaternions provide a more compact, numerically stable, and efficient representation of orientations and rotations of objects in \( \mathbb{R}^3 \), and they have many engineering applications such as navigation, robotics and computer vision \([6,37]\). To our knowledge, phase retrieval of quaternion-valued functions on a domain \( D \) has not been discussed in literature. In Section 2.3 we generalize the conclusions in Theorems 2.1, 2.2 and 2.5 for complex functions to quaternion-valued functions.

2.1. Complex conjugate phase retrieval. For functions in a complex conjugate invariant linear space, we have the following characterization to the complex conjugate phase retrieval, see Section 5.4 for the proof.

**Theorem 2.1.** Let \( \mathcal{C} \) be a complex linear space of functions on a domain \( D \) that is invariant under complex conjugation, and let \( f \in \mathcal{C} \). Then \( f \) is complex conjugate phase retrieval if and only if there do not exist \( u, v \in \mathcal{C} \) satisfying
\[
(2.7) \quad f = u + v, \\
(2.8) \quad \Re(u(x)\bar{v}(x)) = 0 \quad \text{for all } x \in D, \\
\text{and} \\
(2.9) \quad \Re(u(x)\bar{v}(y) + u(y)\bar{v}(x)) \neq 0 \quad \text{for some } x, y \in D.
\]

We remark that the set of all complex conjugate phase retrieval functions in \( \mathcal{C} \) is a line cone, but it may be neither closed nor convex, cf. \([17]\). Applying Theorem 2.1 we have the following characterization to the complex linear space \( \mathcal{C} \).

**Theorem 2.2.** Let \( \mathcal{C} \) be a complex linear space of functions on a domain \( D \) that is invariant under complex conjugation. Then \( \mathcal{C} \) is complex conjugate phase retrieval if and only if there do not exist \( u, v \in \mathcal{C} \) satisfying (2.8) and (2.9).

Now we consider the application of Theorems 2.1 and 2.2 to the complex range space \( R_{\mathcal{C}}(A) \) of a real matrix \( A \) of size \( m \times n \). Without loss of generality, we assume that
\[
(2.10) \quad \rank(A) = n,
\]
otherwise replacing \( A \) by its submatrix \( \tilde{A} \) of full rank such that \( R_{\mathcal{C}}(\tilde{A}) = R_{\mathcal{C}}(A) \). Write \( A^T = (a_1, \ldots, a_m) \). As the vector \( f \in R_{\mathcal{C}}(A) \subset \mathbb{R}^m \) can be considered as a function on \( D = \{1, \ldots, m\} \), we obtain
\[
\mathcal{M}_f = \{ Ax, \quad |a_i^T x| = |a_i^T x_f|, 1 \leq i \leq m \},
\]
where \( x_f \) is the unique vector in \( \mathbb{C}^n \) by (2.10) such that
\[
(2.11) \quad f = Ax_f.
\]
Therefore a function \( f \in R_C(A) \) is complex conjugate phase retrieval in \( R_C(A) \) if and only if
\[
(2.12) \quad \mathcal{M}_f = \{zAx_f, z \in \mathbb{T}\} \cup \{zA\bar{x}_f, z \in \mathbb{T}\}.
\]
Observe that the linear space of all real symmetric matrices is spanned by \( a_i a_j^T + a_j a_i^T, 1 \leq i, j \leq m \), by (2.10). Hence by Theorem 2.1, we have the following corollary about complex conjugate phase retrieval in \( R_C(A) \).

**Corollary 2.3.** Let \( A \) be a real matrix of size \( m \times n \) satisfying (2.10). Then \( f \in R_C(A) \) is complex conjugate phase retrieval if and only if there do not exist \( x_1, x_2 \in \mathbb{C}^n \) such that \( x_f = x_1 + x_2, X^T \neq -X \), and \( \text{Tr}(a_i a_i^T X) = 0 \) for all \( 1 \leq i \leq m \), where \( x_f \) is the unique vector in \( \mathbb{C}^n \) satisfying (2.11) and
\[
(2.13) \quad X = \frac{1}{2}(\bar{x}_2 x_1^T + x_2 \bar{x}_1^T) = \Re(x_2)\Re(x_1^T) + \Im(x_2)\Im(x_1^T).
\]

Let \( M_{n,2}(\mathbb{R}) \) be the set of all real matrices of size \( n \times n \) with rank at most 2. Observe that a real matrix \( X \) is of the form (2.13) if and only if \( X \in M_{n,2}(\mathbb{R}) \). Therefore we have the following characterization to complex conjugate phase retrieval of the complex range space \( R_C(A) \), see [25, Theorem 2.3] for an equivalent formulation and [56, Theorem 2.1] for phase retrieval of the real linear subspace of \( R_C(A) \).

**Corollary 2.4.** Let \( A \) be a real matrix of size \( m \times n \) satisfying (2.10). Then the complex range space \( R_C(A) \) is complex conjugate phase retrieval if and only if there does not exist \( X \in M_{n,2}(\mathbb{R}) \) such that \( X^T \neq -X \) and \( \text{Tr}(a_i a_i^T X) = 0 \) for all \( 1 \leq i \leq m \).

2.2. Complex conjugate phase retrieval and phase retrieval. Let \( \Re(C) \) be the linear space of all real-valued functions in \( C \). By [17, Theorem 2.1], a real function \( f \in \Re(C) \) is phase retrieval in \( \Re(C) \) if and only if there do not exist nonzero functions \( f_1, f_2 \in \Re(C) \) such that
\[
(2.14) \quad f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0.
\]
The above characterization for phase retrieval in \( \Re(C) \) is closely related to the characterization (2.7), (2.8) and (2.9) with the requirement (2.9) replaced by the nonzero requirement for the functions \( f_1, f_2 \in \Re(C) \). In the following theorem, we introduce a necessary condition for a function to be complex conjugate phase retrieval in \( C \), see Section 5.5 for the proof.

**Theorem 2.5.** Let \( C \) be a complex linear space of functions on a domain \( D \) that is invariant under complex conjugation, and \( \Re(C) \) be its real linear subspace in (2.6). If a function \( f \in C \) is complex conjugate phase retrieval in \( C \), then any function in the real linear space spanned by the real and imaginary parts of \( f \) is phase retrieval in \( \Re(C) \).

As an application of the above theorem, we have the following result.

**Corollary 2.6.** Let \( C \) and \( \Re(C) \) be as in Theorem 2.5. If \( C \) is complex conjugate phase retrieval, then \( \Re(C) \) is phase retrieval.
Let \( A \) be a real matrix of size \( m \times n \) and \( \Omega \) be a linear subspace of \( \mathbb{R}^n \). We say that the subspace \( \Omega \) is phase retrieval if \( |Ax| = |Ay| \) holds for \( x, y \in \Omega \) only when \( y = \pm x \). Applying Theorem 2.5 to the complex range space \( R_C(A) \) in (2.1), we have the following conclusion about the phase retrieval of a real linear subspace.

**Corollary 2.7.** Let \( A \) be an \( m \times n \) real matrix satisfying (2.10). If \( f = Ax_f \in R_C(A) \) is complex conjugate phase retrieval, then the space spanned by the real and imaginary parts of the vector \( x_f \) is phase retrieval.

For a real matrix \( A \) of size \( m \times 2 \), we obtain from Corollary 2.7 that any vector \( x \) in \( \mathbb{R}^2 \) is phase retrieval from its phaseless measurement \( |Ax| \) if \( R_C(A) \) is complex conjugate phase retrieval. In the following proposition, we show that the converse is also true, see Section 5.6 for the proof.

**Proposition 2.8.** Let \( A \) be a real matrix of size \( m \times 2 \) satisfying (2.10). Then any vector \( x \) in \( \mathbb{R}^2 \) is phase retrieval from its phaseless measurements \( |Ax| \) if and only if \( R_C(A) \) is conjugate phase retrieval.

The complex shift-invariant space \( S_C(\psi) \) generated by a compactly supported real function \( \psi \) is not complex conjugate phase retrieval, as functions \( f(t) = z_1 \psi(t) - z_2 \psi(t - N) \in S_C(\psi) \) are not complex conjugate phase retrieval, where \( z_1, z_2 \) are nonzero complex numbers and \( N \) is an integer such that \( \psi(t), \psi(t - N) \) have disjoint supports. By Theorem 2.5, a necessary condition to the complex conjugate phase retrieval of a function \( f(t) = \sum_{k \in \mathbb{Z}} c(k) \psi(t - k) \in S_C(\psi) \) is that for all \( a, b \in \mathbb{R} \), \( \sum_{k \in \mathbb{Z}} (a \Re(c(k) + b \Im(c(k))) \psi(t - k) \) are phase retrieval in the real shift-invariant space

\[
S(\psi) = \left\{ \sum_{k \in \mathbb{Z}} d(k) \psi(\cdot - k), \quad d(k) \in \mathbb{R} \text{ for all } k \in \mathbb{Z} \right\},
\]

see [17]. In the following result, we show that the converse is true when the generator \( \psi \) is the hat function \( h(t) = \max(1 - |t|, 0) \), see Section 5.7 for the proof.

**Proposition 2.9.** A nonzero function \( f = \sum_{k \in \mathbb{Z}} c(k) h(\cdot - k) \in S_C(h) \) is complex conjugate phase retrieval if and only if

\[
(2.15) \quad c(k) \neq 0 \text{ for all } K_- - 1 < k < K_+ + 1
\]

and there exists at most one \( k \in (K_- - 1, K_+) \) such that

\[
(2.16) \quad \Im(c(k)c(k + 1)) \neq 0,
\]

where \( K_- = \inf\{k, c(k) \neq 0\} \) and \( K_+ = \sup\{k, c(k) \neq 0\} \).

**2.3. Quaternion conjugate phase retrieval.** For a quaternion \( q = a + bi + cj + dk \in \mathbb{Q}_8 \), denote its conjugate, real part, and norm by \( q^* = a - bi - cj - dk \), \( \Re(q) = a \), and \( \|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2} \) respectively. A linear space \( W \) of quaternion-valued functions on a domain \( D \) is quaternion conjugate invariant if

\[
(2.17) \quad qf, (qf)^* \in W \text{ for all } q \in \mathbb{Q}_8 \text{ and } f \in W.
\]
For $f \in \mathcal{W}$, define
\begin{equation}
\mathcal{M}_f := \{ g \in \mathcal{W}, \| g(x) \| = \| f(x) \| \text{ for all } x \in D \},
\end{equation}
and write
\begin{equation}
f = \frac{f + f^*}{2} + \frac{-i f - (if)^*}{2}i + \frac{-j f - (jf)^*}{2}j + \frac{-k f - (kf)^*}{2}k
\end{equation}
(2.19) \quad =: f_1 + f_2i + f_3j + f_4k.

Since the linear space $\mathcal{W}$ is quaternion conjugate invariant, we obtain $f_i, 1 \leq i \leq 4$, and their quaternion linear combinations belong to $\mathcal{W}$,
\begin{equation}
q_1 f_1 + q_2 f_2 + q_3 f_3 + q_4 f_4 \in \mathcal{W}
\end{equation}
for all $q_i \in \mathcal{Q}_8, 1 \leq i \leq 4$.

Observe that
\begin{equation}
\left\| \sum_{i=1}^{4} q_i f_i \right\|^2 = \sum_{i=1}^{4} \| q_i \|^2 \| f_i \|^2 + \sum_{1 \leq i < j \leq 4} f_i f_j (q_i q_j^* + q_j q_i^*)
\end{equation}
for all $q_i \in \mathcal{Q}_8, 1 \leq i \leq 4$. Then
\begin{equation}
\left\| \sum_{i=1}^{4} q_i f_i \right\|^2 = \sum_{i=1}^{4} \| f_i \|^2
\end{equation}
if $q_i \in \mathcal{T}_8 = \{ q \in \mathcal{Q}_8, \| q \| = 1 \}, 1 \leq i \leq 4$, are unit quaternions satisfying
\begin{equation}
q_i q_j^* + q_j q_i^* = 0 \text{ for all } 1 \leq i < j \leq 4.
\end{equation}
(2.21)

Therefore
\begin{equation}
\mathcal{M}_f \supset \left\{ \sum_{i=1}^{4} q_i f_i \in \mathcal{W}, q_i \in \mathcal{T}_8, 1 \leq i \leq 4, \text{ satisfies (2.21)} \right\}.
\end{equation}

In this paper, we say that $f \in \mathcal{W}$ is quaternion conjugate phase retrieval in $\mathcal{W}$ if
\begin{equation}
\mathcal{M}_f = \left\{ \sum_{i=1}^{4} q_i f_i \in \mathcal{W}, q_i \in \mathcal{T}_8, 1 \leq i \leq 4, \text{ satisfies (2.21)} \right\},
\end{equation}
(2.22)
and the linear space $\mathcal{W}$ is quaternion conjugate phase retrieval if every function in $\mathcal{W}$ is quaternion conjugate phase retrieval. In this section, we extend the characterizations in Theorems 2.1 and 2.2 for complex-valued functions to quaternion-valued functions, see Section 5.8 for the proof.

**Theorem 2.10.** Let $\mathcal{W}$ be a quaternion linear space of functions on a domain $D$ which is invariant under quaternion conjugation. Then the following statements hold.

(i) $f \in \mathcal{W}$ is quaternion conjugate phase retrieval if and only if there do not exist $u, v \in \mathcal{W}$ such that
\begin{equation}
f = u + v,
\end{equation}
(2.23)
\begin{equation}
\Re(u(x)v^*(x)) = 0 \text{ for all } x \in D,
\end{equation}
(2.24)
and
\[
\Re(u(x)v^*(y) + u(y)v^*(x)) \neq 0 \quad \text{for some } x, y \in D.
\]

(ii) The quaternion linear space \( W \) is quaternion conjugate phase retrieval if and only if there do not exist \( u, v \in W \) such that (2.24) and (2.25) hold.

For a quaternion conjugate invariant space \( W \) of functions on a domain \( D \), let
\[
W_\Re = \{ \Re(f), \ f \in W \}
\]
be the linear subspace of \( W \) containing all real functions in \( W \). For \( f = f_1 + 0i + 0j + 0k \in W_\Re \), one may verify that
\[
\left\{ \sum_{i=1}^{4} q_i f_i \in W_\Re, \ q_i \in \mathbb{T}_8, 1 \leq i \leq 4, \text{ satisfy } (2.21) \right\} = \{ \pm f \}.
\]
We say that a function \( f \in W_\Re \) is phase retrieval in \( W_\Re \) if \( M_f \cap W_\Re = \{ \pm f \} \). Therefore we have the following result, cf. Corollary 2.6.

**Corollary 2.11.** Let \( W \) be a quaternion linear space of functions on a domain \( D \) which is invariant under quaternion conjugation. If \( f \in W \) is quaternion conjugate phase retrieval, then \( \Re(f) \) is phase retrieval in \( W_\Re \).

For a quaternion conjugate invariant space \( W \), let
\[
W_\mathbb{C} = \{ \Re(f) + \Re(-if)i, \ f \in W \}
\]
be the linear subspace of \( W \) that contains all complex functions in \( W \). For any complex function \( f = f_1 + f_2i + 0j + 0k \in W_\mathbb{C} \) and \( z \in \mathbb{T} \), one may verify that
\[
\left\{ \sum_{i=1}^{4} q_i f_i \in W_\mathbb{C}, q_i \in \mathbb{T}_8, 1 \leq i \leq 4, \text{ satisfy } (2.21) \right\} = \{ zf_1 \pm f_2 i, \ z \in \mathbb{T} \}.
\]
We say that a complex function \( f \in W_\mathbb{C} \) is complex conjugate phase retrieval in \( W_\mathbb{C} \) if
\[
M_f \cap W_\mathbb{C} = \{ zf, z \in \mathbb{T} \} \cup \{ z\bar{f}, z \in \mathbb{T} \}.
\]
Therefore we have the following result about complex conjugate phase retrieval in \( W_\mathbb{C} \).

**Corollary 2.12.** Let \( W \) be a quaternion linear space of functions on a domain \( D \) which is invariant under quaternion conjugation. If \( f = f_1 + f_2i + f_3j + f_4k \in W \) is quaternion conjugate phase retrieval in \( W \), then \( f_1 + f_2i \) is complex conjugate phase retrieval in \( W_\mathbb{C} \).
3. Phase retrieval of vector-valued functions

Let $\mathcal{H}$ be a real separable Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, and $\mathcal{U}(\mathcal{H})$ be the group of unitary operators $U$ on $\mathcal{H}$ that satisfy

$$U^*U = UU^* = I,$$

where $U^*$ is the adjoint of $U$. We say that a function $f$ on a domain $D$ is $\mathcal{H}$-valued, or vector-valued when unambiguous, if

$$f(x) \in \mathcal{H} \text{ for all } x \in D,$$

and a real linear space $\mathcal{S}$ of vector-valued functions if on a domain $D$ is unitary invariant if

$$(3.1) \quad Uf \in \mathcal{S} \text{ for all } f \in \mathcal{S} \text{ and } U \in \mathcal{U}(\mathcal{H}).$$

Our representative unitary invariant linear spaces are

$$(3.2) \quad \mathcal{C}_{\mathbb{R}^2} = \left\{ \left( \Re(f), \Im(f) \right), f \in \mathcal{C} \right\}$$

associated with a complex conjugate invariant linear space $\mathcal{C}$, the linear space of vector fields on a spatially distributed network [10, 18, 33, 47, 48], and vector-valued reproducing kernel spaces in multi-task learning [7, 15, 42, 45].

We say that a family $\Phi$ of linear measurements on $\mathcal{S}$ is unitary invariant if

$$(3.3) \quad \phi(Uf) = U\phi(f) \in \mathcal{H} \text{ for all } f \in \mathcal{S} \text{ and } U \in \mathcal{U}(\mathcal{H}).$$

For a vector-valued function $f \in \mathcal{S}$ and a unitary invariant family $\Phi$ of linear measurements, let

$$\mathcal{M}_{f,\Phi} = \{ g \in \mathcal{S}, \| \phi(g) \| = \| \phi(f) \| \text{ for all } \phi \in \Phi \}$$

contain all vector-valued functions $g \in \mathcal{S}$ such that $g$ and $f$ have the same magnitude observations. By the unitary invariance of the linear space $\mathcal{S}$ and the family $\Phi$ of linear measurements, we have

$$(3.4) \quad \mathcal{M}_{f,\Phi} \supset \{ Uf, U \in \mathcal{U}(\mathcal{H}) \}, \ f \in \mathcal{S}.$$ 

In this paper, we say that a vector-valued function $f \in \mathcal{S}$ is phase retrieval in $\mathcal{S}$ if the inclusion in (3.4) becomes an equality, i.e.,

$$\mathcal{M}_{f,\Phi} = \{ Uf, U \in \mathcal{U}(\mathcal{H}) \},$$

and the unitary invariant space $\mathcal{S}$ is phase retrieval if every vector-valued function in $\mathcal{S}$ is phase retrieval, see [3, 9, 12, 17, 19, 20, 43, 52, 53, 54] for phase retrieval of scalar-valued functions in various function spaces. In Section 3.1, we characterize the phase retrieval of vector-valued functions in $\mathcal{S}$, see Theorems 3.2 and 3.3. Applying our characterization in the real scalar setting, we have the well known equivalence between the complement property for linear measurements $\Phi$ and the phase retrieval of linear space $\mathcal{S}$ [4, 8, 9, 12], see Corollary 3.6.
Let $\tilde{H}$ be a linear subspace of the Hilbert space $H$, denote the projection from $H$ onto $\tilde{H}$ by $P_{\tilde{H}}$, and set
\begin{equation}
P_{\tilde{H}}S = \{P_{\tilde{H}}f, f \in S\}.
\end{equation}
As an application of Theorems 3.2 and 3.3, we show that the projection $P_{\tilde{H}}f$ of a phase retrieval function $f$ in $S$ is phase retrieval in the projection space $P_{\tilde{H}}S$, see Theorem 3.7 in Section 3.2 and cf. [13, 24] for phase retrieval by projections.

Let $G = (V, E)$ be a simple graph and $f = (f_i)_{i \in V}$ be a $d$-dimensional vector field on $G$, where $f_i \in \mathbb{R}^d, i \in V$. In Section 3.3, we consider the problem whether the vector field $f$ can be reconstructed, up to an orthogonal matrix, from its absolute magnitudes $\|f_i\|$ at all vertices $i \in V$ and relative magnitudes $\|f_i - f_j\|$ of neighboring vertices $(i, j) \in E$. In other words, given any vector field $g = (g_i)_{i \in V}$ satisfying
\begin{equation}
\|g_i\| = \|f_i\| \text{ for all } i \in V
\end{equation}
and
\begin{equation}
\|g_i - g_j\| = \|f_i - f_j\| \text{ for all } (i, j) \in E,
\end{equation}
we can find an orthogonal matrix $U$ of size $d \times d$ such that $g = Uf$. For the case that $G$ is a complete graph, we show in Theorem 3.9 that a vector field $f = (f_i)_{i \in V}$ on $G$ is determined, up to an orthogonal matrix, from its absolute magnitudes $\|f_i\|$ at all vertices $i \in V$ and relative magnitudes $\|f_i - f_j\|$ between vertices $i, j \in V$. For an arbitrary simple graph $G$, given a vector field $f$, we introduce a $d$-simplex graph $G_f = (V_f, E_f)$ associated with $f$, and show in Theorem 3.10 that $f$ is determined, up to an orthogonal matrix, from its absolute magnitudes at vertices and relative magnitudes between vertices when the $d$-simplex graph $G_f$ is connected.

3.1. Phase retrieval for vector-valued setting. For a vector-valued function $f \in S$, by the unitary invariance of linear space $S$ and the family $\Phi$ of linear measurements, we have
\begin{equation}
\langle \phi(g), \tilde{\phi}(g) \rangle = \langle \phi(f), \tilde{\phi}(f) \rangle \text{ for all } \phi, \tilde{\phi} \in \Phi,
\end{equation}
when $g = Uf$ for some $U \in \mathcal{U}(H)$. To consider the phase retrieval of vector-valued functions $f$ from their magnitude measurements $\|\phi(f)\|$, $\phi \in \Phi$, we assume that the converse in (3.8) holds.

Assumption 3.1. Let $f$ be a $H$-valued function in the linear space $S$. Then for any $H$-valued function $g \in S$ satisfying (3.8), there exists $U \in \mathcal{U}(H)$ such that $g = Uf$.

A necessary condition for Assumption 3.1 to hold is that
\begin{equation}
T_\Phi : S \ni f \mapsto \{\phi(f)\}_{\phi \in \Phi} \text{ is injective,}
\end{equation}
or equivalently the null space of the above map is trivial,
\begin{equation}
N_{T_\Phi} = \{0\}.
\end{equation}
In the case that $H$ is finite dimensional, the injectivity in (3.9) is also sufficient for the Assumption 3.1, see Remark 3.4.

In the next theorem, we characterize the phase retrieval of vector-valued functions in a unitary invariant space, see Section 5.1 for the proof.

**Theorem 3.2.** Let $H$ be a real separable Hilbert space, $S$ be a unitary invariant space of $H$-valued functions on the domain $D$, and $\Phi$ be a unitary invariant set of linear measurements satisfying Assumption 3.1. Then $f \in S$ is phase retrieval if and only if there do not exist $u, v \in S$ such that

\begin{align}
&f = u + v, \\
&\langle \phi(u), \phi(v) \rangle = 0 \text{ for all } \phi \in \Phi, \\
&\text{and} \\
&\langle \phi_0(u), \phi_1(v) \rangle + \langle \phi_1(u), \phi_0(v) \rangle \neq 0 \text{ for some } \phi_0, \phi_1 \in \Phi.
\end{align}

Applying Theorem 3.2, we have a characterization to phase retrieval of a unitary invariant space of vector-valued functions.

**Theorem 3.3.** Let $H$ be a real separable Hilbert space, $S$ be a unitary invariant space of $H$-valued functions on the domain $D$, and $\Phi$ be a unitary invariant set of linear measurements satisfying Assumption 3.1. Then the unitary invariant space $S$ is phase retrieval if and only if there do not exist $u, v \in S$ such that (3.12) and (3.13) hold.

**Remark 3.4.** In this remark, we show that in the finite-dimensional setting, i.e., $\dim H < \infty$, a unitary invariant set $\Phi$ of linear measurements satisfies Assumption 3.1 if and only if the map $T_\Phi$ in (3.9) is injective. The necessity is obvious since given any function $g$ in the null space $N_{T_\Phi}$, we have

$$\langle \phi(g), \tilde{\phi}(g) \rangle = 0 \text{ for all } \phi, \tilde{\phi} \in \Phi.$$ 

Now we prove the sufficiency. Let $W_f$ and $W_g$ be the linear subspaces of $H$ spanned by $\phi(f)$ and by $\phi(g), \phi \in \Phi$ respectively. Then the space $W_f$ has a basis $\phi_n(f), 1 \leq n \leq N$. Applying the Gram-Schmidt procedure to the above basis, we can construct an orthonormal basis

$$e_m = \sum_{n=1}^{m} a(m,n) \phi_n(f), 1 \leq m \leq N$$

of the linear space $W_f$, where $a(m,n), 1 \leq n \leq m \leq N$ are functions of $\langle \phi_i(f), \phi_j(f) \rangle, 1 \leq i, j \leq N$. Define

$$\tilde{e}_m = \sum_{n=1}^{m} a(m,n) \phi_n(g), 1 \leq m \leq N.$$ 

Then $\tilde{e}_m, 1 \leq m \leq N$, is an orthonormal basis of the space $W_g$ by (3.8) and (3.14). Define a unitary operator $U$ on $H$ such that

$$Ue_n = \tilde{e}_n, 1 \leq n \leq \dim H,$$
where \{e_n, N + 1 \leq n \leq \dim H\} and \{\tilde{e}_n, N + 1 \leq n \leq \dim H\} are orthonormal bases of orthogonal complements of \(W_f\) and \(W_g\) in \(H\) respectively. For the above unitary operator \(U\), we have

\[\phi(g) = U\phi(f) = \phi(Uf)\]

for all \(\phi \in \Phi\), where the last equality follows from unitary invariance of the linear measurements \(\Phi\). This together with the injectivity hypothesis proves \(g = Uf\) and hence Assumption 3.1 holds.

The functions \(u\) and \(v\) in (3.13) must be nonzero functions. The converse is true for the scalar setting, i.e., \(H = \mathbb{R}\), since

\[\langle \phi_0(u), \phi_1(v) \rangle + \langle \phi_1(u), \phi_0(v) \rangle = \phi_0(u)\phi_1(v) + \phi_1(u)\phi_0(v) = \phi_0(u)\phi_1(v) \neq 0\]

by (3.9) and (3.12), where \(\phi_0, \phi_1 \in \Phi\) are so chosen that \(\phi_0(u) \neq 0\) and \(\phi_1(v) \neq 0\). Therefore by Theorem 3.2, we have the following result, which is established in [17] when \(\Phi\) is the set of point-evaluation functionals.

**Corollary 3.5.** Let \(S\) be a linear space of real functions on the domain \(D\), and \(\Phi\) be a set of linear measurements satisfying (3.9). Then \(f \in S\) is phase retrieval if and only if there do not exist nonzero functions \(u, v \in S\) such that

\[f = u + v\]

and \(\phi(u)\phi(v) = 0\) for all \(\phi \in \Phi\).

For a unitary invariant set \(\Phi\) of linear measurements satisfying Assumption 3.1, the null space of the map T_\Phi in (3.9) contains the zero element only. A strong version about the set \(\Phi\) is its complement property that given any \(\Phi \subset \hat{\Phi}\),

\[\text{either } N_{T_\Phi} = \{0\} \text{ or } N_{T_{\hat{\Phi}\setminus \Phi}} = \{0\}.\]

In the scalar setting, i.e., \(H = \mathbb{R}\), an equivalent formulation of the complement property (3.18) is that there do not exist nonzero functions \(u, v \in S\) such that \(\phi(u)\phi(v) = 0\) for all \(\phi \in \Phi\). Therefore by Corollary 3.5 we have the following result, which is established in [4, 8, 9, 12] for frames in Hilbert/Banach space setting.

**Corollary 3.6.** Let \(S\) be a linear space of real functions on the domain \(D\), and \(\Phi\) be a set of linear measurements such that the map \(T_\Phi\) in (3.9) is injective. Then \(S\) is phase retrieval if and only if \(\Phi\) has the complement property (3.18).

### 3.2. Phase retrieval of projections

Let \(\hat{H}\) be a linear subspace of the Hilbert space \(H\), and \(P_{\hat{H}}\) be the projection from \(H\) onto \(\hat{H}\). In this section, we establish the following result on phase retrieval of the projection of vector-valued functions onto \(\hat{H}\), see Section 5.2 for the proof.
Theorem 3.7. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be a real separable Hilbert space and its linear subspace respectively, $S$ be a unitary invariant space of $\mathcal{H}$-valued functions on the domain $D$, and let $\Phi$ be a unitary invariant set of linear measurements satisfying Assumption 3.1. If $f \in S$ is phase retrieval in $S$, then its projection $P_{\tilde{\mathcal{H}}}f$ onto $\tilde{\mathcal{H}}$ is phase retrieval in $P_{\tilde{\mathcal{H}}}S$, i.e.,

\[(3.19) \quad \mathcal{M}_{P_{\tilde{\mathcal{H}}}f, \Phi} = \{ \tilde{U}P_{\tilde{\mathcal{H}}}f, \tilde{U} \in U(\tilde{\mathcal{H}}) \}.\]

3.3. Phase retrieval of vector fields on graphs. Let $\mathcal{G} = (V, E)$ be a simple graph and $f = (f_i)_{i \in V}$ be a $d$-dimensional vector field on $\mathcal{G}$, where $f_i \in \mathbb{R}^d, i \in V$. First we consider the case that $\mathcal{G}$ is a complete graph. In this case, we obtain from (3.6) and (3.7) that

\[(3.21) \quad \langle g_i, g_j \rangle = \langle f_i, f_j \rangle \quad \text{for all } i,j \in V.\]

Applying (3.21) and following the argument used in Remark 3.4, we can find an orthogonal matrix $U$ of size $d \times d$ such that

\[g_i = Uf_i, \quad i \in V.\]

This leads to the following result.

Theorem 3.9. Let $\mathcal{G} = (V, E)$ be a complete graph. Then a vector field $f = (f_i)_{i \in V}$ can be reconstructed, up to an orthogonal matrix, from its absolute magnitudes $\|f_i\|, i \in V$ and relative magnitudes $\|f_i - f_j\|, i,j \in V$.

From the above theorem, we may conclude that a velocity field on a complete graph is determined, up to an orthogonal matrix, from its absolute speed at vertices and relative speed between vertices. We remark that the conclusion in Theorem 3.9 does not hold for an arbitrary graph. Let $C_n$ be the circulant graph with nodes labeled $0, 1, \ldots, n - 1$ and each node $i$ adjacent to nodes $i \pm 1 \mod n$. One may verify that the vector fields $h = (h_i)_{0 \leq i \leq n - 1}$ on the circulant graph $C_n$ of even order have the same magnitudes $\|h_i\| = 1$ on all vertices $0 \leq i \leq n - 1$ and relative magnitudes $\|h_i - h_{i \pm 1}\| = \sqrt{2}$ at all neighboring vertices, where

\[h_i = \begin{cases} 
\pm(1, 0)^T & \text{if } i \text{ is even} \\
\pm(0, 1)^T & \text{if } i \text{ is odd.} 
\end{cases}\]
For a simple graph \( G = (V,E) \) and a \( d \)-dimensional vector field \( f = (f_i)_{i \in V} \) on \( G \), we define

\[
\Delta(f, G_c) = \left\{ \sum_{i \in V_c} t_i f_i, \sum_{i \in V_c} t_i = 1 \text{ and } 0 \leq t_i \leq 1 \text{ for all } i \in V_c \right\},
\]

where \( G_c = (V_c, E_c) \) is a complete subgraph of order \( d + 1 \). Let \( V_f \) be the set of all complete subgraphs \( G_c \) of order \( d + 1 \) such that \( \Delta(f, G_c) \) is a \( d \)-simplex in \( \mathbb{R}^d \), and \( E_f \) be the set of all pairs of complete subgraphs \( G_c, \tilde{G}_c \in V_f \) of order \( d + 1 \) such that \( \Delta(f, G_c \cap \tilde{G}_c) \) is a \((d-1)\)-simplex and the hyperplane containing \( \Delta(f, G_c \cap \tilde{G}_c) \) does not include the origin. We call the graph \( G_f = (V_f, E_f) \) with the vertex set \( V_f \) and edge set \( E_f \) defined above as the \textit{\( d \)-simplex graph} associated with the vector field \( f \), see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Plotted are the graph \( G \) with vertices marked by blue dots from A to K and edges in black dashed lines, and the 2-simplex graph \( G_f \) with vertices \( \{\Delta ACD, \Delta BCE, \Delta CDE, \Delta CDF, \Delta CEF, \Delta DEF, \Delta DFH, \Delta EFG, \Delta FGH, \Delta GHI, \Delta HIK, \Delta IJK\} \) marked by the red filled triangle located around the centroid and edges between 2-simplices in red solid lines, where we assume that the hyperplane containing two vectors of the vector field \( f \) located at edges of the graph \( G \), except at the edge between vertices \( C \) and \( F \), does not pass through the origin.}
\end{figure}

In the following theorem, see Section 5.3 for the proof, we show that a vector field \( f \) on a simple graph \( G \) is determined, up to an orthogonal matrix, from its absolute magnitudes at vertices and relative magnitudes between neighboring vertices if the \( d \)-simplex graph \( G_f \) is connected.

**Theorem 3.10.** Let \( G = (V,E) \) be a simple finite graph, \( f = (f_i)_{i \in V} \) be a vector field on the graph \( G \) with its \( d \)-simplex graph denoted by \( G_f \). If the \( d \)-simplex graph \( G_f \) is connected and for any vertex \( i \in V \) there exists a complete subgraph \( G_c = (V_c, E_c) \) of order \( d + 1 \) such that \( i \in V_c \) and \( \Delta(f, G_c) \)
is a $d$-simplex, then the vector field $\mathbf{f}$ is determined, up to an orthogonal matrix, from its absolute magnitudes $\|f_i\|, i \in V$ at vertices and relative magnitudes $\|f_i - f_j\|, (i, j) \in E$ of neighboring vertices.

The $d$-simplex graph $\mathcal{G}_\mathbf{f}$ is defined when the vector field $\mathbf{f}$ is given. In the following remark, we show that the $d$-simplex graph $\mathcal{G}_\mathbf{f}$ can also be constructed from the available absolute magnitudes $\|f_i\|, i \in V$ and relative magnitudes $\|f_i - f_j\|, (i, j) \in E$.

**Remark 3.11.** For a complete subgraph $\mathcal{G}_c = (V_c, E_c)$ of order $d + 1$, one may verify that the convex set $\Delta(f, \mathcal{G}_c)$ is a $d$-simplex if and only if $c_i, i \in V_c$, are affinely independent in the sense that

$$
\sum_{i \in V_c} c_i f_i = 0 \quad \text{and} \quad \sum_{i \in V_c} c_i = 0 \quad \text{implies} \quad c_i = 0 \quad \text{for all} \quad i \in V_c,
$$

or equivalently vectors $f_i - f_{i_0} \in \mathbb{R}^d, i \in V_c \setminus \{i_0\}$, are linearly independent for some $i_0 \in V_c$, or equivalently the Gramian matrix $(\langle f_i, f_j \rangle)_{i,j \in V_c}$ is strictly positive definite on the hyperplane

$$
\left\{ (c_i)_{i \in V_c}, \sum_{i \in V_c} c_i = 0 \right\} \subset \mathbb{R}^{d+1}.
$$

Observe that entries in the Gramian matrix are given by

$$
\langle f_i, f_j \rangle = \frac{1}{2} (\|f_i\|^2 + \|f_j\|^2 - \|f_i - f_j\|^2), i, j \in V_c.
$$

Then the vertex set $V_\mathcal{G}$ of the $d$-simplex graph $\mathcal{G}_\mathbf{f}$ can be constructed, from the absolute magnitudes $\|f_i\|, i \in V$ and relative magnitudes $\|f_i - f_j\|, (i, j) \in E$.

Given a pair of complete subgraphs $\mathcal{G}_c = (V_c, E_c)$ and $\tilde{\mathcal{G}}_c = (\tilde{V}_c, \tilde{E}_c)$ in the vertex set $V_\mathcal{G}$, one may verify that $\Delta(f, \mathcal{G}_c \cap \tilde{\mathcal{G}}_c)$ is a $(d-1)$-simplex if and only if $V_c \cap \tilde{V}_c$ has cardinality $d$, and also that the hyperplane containing $\Delta(f, \mathcal{G}_c \cap \tilde{\mathcal{G}}_c)$ does not include the origin if and only if $f_i, i \in V_c \cap \tilde{V}_c$, are linearly independent or equivalently the Gramian matrix $(\langle f_i, f_j \rangle)_{i,j \in V_c \cap \tilde{V}_c}$ is strictly positive definite. This together with (3.24) implies that edges in the $d$-simplex graph $\mathcal{G}_\mathbf{f}$ are determined from the available absolute magnitudes $\|f_i\|, i \in V$ and relative magnitudes $\|f_i - f_j\|, (i, j) \in E$.

## 4. Affine phase retrieval of vector-valued functions

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{S}$ be a linear space of $\mathcal{H}$-valued functions on a domain $D$, $\Phi$ be a family of linear measurements, and let $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ be a family of reference vectors in $\mathcal{H}$ for each linear measurement. For any $f \in \mathcal{S}$, we define

$$
\mathcal{A}_f = \{ g \in \mathcal{S}, \|\phi(f) + b_{\phi,i}\| = \|\phi(g) + b_{\phi,i}\|, \phi \in \Phi, 1 \leq i \leq N \}.
$$

We say that $f \in \mathcal{S}$ is affine phase retrieval in $\mathcal{S}$ if $f$ is uniquely determined from its phaseless affine measurements $\|\phi(f) + b_{\phi,i}\|, \phi \in \Phi, 1 \leq i \leq N$, i.e.,

$$
(4.1) \quad \mathcal{A}_f = \{ f \},
$$
and the whole space $S$ is affine phase retrieval if every vector-valued function in $S$ is affine phase retrieval, see [27, 30, 33, 38] for affine phase retrieval of complex/real functions (vectors). In this section, we consider the affine phase retrieval of vector-valued functions in a linear space, see Theorems 4.1 and 4.3.

For all $f, g \in S$ having the same phaseless affine measurements, i.e.,

$$
\|\phi(f) + b_{\phi,i}\| = \|\phi(g) + b_{\phi,i}\| \text{ for all } \phi \in \Phi \text{ and } 1 \leq i \leq N,
$$

we have

$$
\langle \phi(f - g), \phi(f + g) + 2b_{\phi,i} \rangle = 0, \phi \in \Phi, 1 \leq i \leq N.
$$

This leads to the following characterization to affine phase retrieval of a linear space.

**Theorem 4.1.** Let $S$ be a linear space of $H$-valued functions on a domain $D$, $\Phi$ be a family of linear measurements on $S$, and let $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ be a family of reference vectors in $H$. Then the linear space $S$ is affine phase retrieval if and only if for any $u \in S$, the linear map $T_u$ defined by

$$
(4.2) \quad T_u : S \ni v \mapsto (\langle \phi(v), \phi(u) + b_{\phi,i} \rangle)_{\phi \in \Phi, 1 \leq i \leq N}
$$

is injective.

Applying Theorem 4.1 to the complex range space $R_C(A)$ in (2.1), we obtain a conclusion given in [27, Theorem 3.1].

**Corollary 4.2.** Let the real matrix $A$ have full rank $n$. Then any function $y = (y_1, \ldots, y_m)^T \in R_C(A)$ is determined from its affine measurements $|y_k + b_{k,i}|, 1 \leq k \leq m, 1 \leq i \leq N$, if and only if for any $u = (u_1, \ldots, u_m)^T \in R_C(A)$ there does not exist a nonzero vector $v = (v_1, \ldots, v_m)^T \in R_C(A)$ such that

$$
\Re(v_k^*(u_k + b_{k,i})) = 0 \text{ for all } 1 \leq k \leq m \text{ and } 1 \leq i \leq N.
$$

We remark that the above characterization also holds for the real linear subspace of $R_C(A)$ in (2.1), cf. [27, Theorem 2.1].

By Theorem 4.1, the verification of affine phase retrieval of a linear space $S$ reduces to checking the injectivity of $T_u$ in (4.2) for all $u \in S$. In the following theorem, see Section 5.9 for the proof, we provide a simpler characterization to affine phase retrieval when there are multiple reference vectors for each linear measurement, i.e., $N \geq 2$.

**Theorem 4.3.** Let $N \geq 2$, $S$ be a linear space of $H$-valued functions on a domain $D$, $\Phi$ be a family of linear measurements on $S$, and let $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ be a family of reference vectors in $H$. Then a sufficient condition for affine phase retrieval of the linear space $S$ is injectivity of the linear map

$$
(4.3) \quad T : S \ni v \mapsto (\langle \phi(v), b_{\phi,i} - b_{\phi,j} \rangle)_{\phi \in \Phi, 1 \leq i,j \leq N}.
$$
The injectivity of the map $T$ is necessary for affine phase retrieval of the linear space $\mathcal{S}$ if one group of the reference vectors is the linear measurement of a function $f_0 \in \mathcal{S}$, i.e., there exists $1 \leq i_0 \leq N$ such that

$$(4.4) \quad b_{\phi,i_0} = \phi(f_0), \quad \phi \in \Phi.$$ 

A necessary condition for affine phase retrieval of the linear space $\mathcal{S}$ is that the map $T_{\Phi}$ in (3.9) is injective. The above necessary condition is also sufficient when the reference vectors $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ satisfies

$$(4.5) \quad \mathcal{H} = \text{span}\{b_{\phi,i} - b_{\phi,j}, 1 \leq i, j \leq N\} \quad \text{for all } \phi \in \Phi,$$

as the map $T$ in (4.3) is injective under the above assumption. Therefore applying Theorem 4.3, we have the following equivalence between affine phase retrieval of the linear space $\mathcal{S}$ and injectivity of the map $T_{\Phi}$, cf. [27] for the scalar setting.

**Corollary 4.4.** Let $N \geq 2$, $\mathcal{S}$ be a linear space of vector-valued functions on a domain $D$, $\Phi$ be a family of linear measurements on $\mathcal{S}$, and let $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ be a family of reference vectors in $\mathcal{H}$ satisfying (4.5). Then $\mathcal{S}$ is affine phase retrieval if and only if the map $T_{\Phi}$ in (3.9) is injective.

In the scalar setting, i.e., $\mathcal{H} = \mathbb{R}$, the requirement (4.5) is satisfied only if $F$ is an empty set, where $F = \{\phi \in \Phi, \sum_{i,j=1}^{n} |b_{\phi,i} - b_{\phi,j}|^2 = 0\}$. Applying Theorem 4.3, we have the following slight generalization of Corollary 4.4 in the scalar setting.

**Corollary 4.5.** Let $N \geq 2$, $\mathcal{S}$ be a linear space of scalar-valued functions on a domain $D$, $\Phi$ be a family of linear measurements on $\mathcal{S}$, and let $B_{\Phi,N} = \{b_{\phi,i}, \phi \in \Phi, 1 \leq i \leq N\}$ be a family of reference real numbers. Then $\mathcal{S}$ is affine phase retrieval if there does not exist a nonzero function $f$ such that $\phi(f) = 0$ for all $\phi \notin F$.

5. Proofs

In this section, we collect the proofs of Theorems 3.2, 3.7, 3.10, 2.1, 2.5, 2.10 and 4.3 and Propositions 2.8 and 2.9.

5.1. Proof of Theorem 3.2. First the necessity. Suppose, on the contrary, that there exist $u, v \in \mathcal{S}$ such that (3.11), (3.12) and (3.13) hold. Set $g = u - v$. From (3.11) and (3.12), we obtain

$$\|\phi(g)\|^2 = \|\phi(u)\|^2 + \|\phi(v)\|^2 - 2\langle \phi(u), \phi(v) \rangle = \|\phi(u)\|^2 + \|\phi(v)\|^2 + 2\langle \phi(u), \phi(v) \rangle = \|\phi(f)\|^2$$

for all $\phi \in \Phi$. This proves that

$$(5.1) \quad g \in \mathcal{M}_{f,\Phi}.$$
By [3.13], we have
\[ \langle \phi_0(f), \phi_1(f) \rangle - \langle \phi_0(g), \phi_1(g) \rangle = 2(\langle \phi_0(u), \phi_1(v) \rangle + \langle \phi_1(u), \phi_0(v) \rangle) \neq 0 \]
for some \( \phi_0, \phi_1 \in \Phi \). Hence \( g \neq Uf \) for all \( U \in \mathcal{U}(\mathcal{H}) \). This together with (5.1) contradicts to the phase retrieval of function \( f \).

Now the sufficiency. Suppose, on the contrary, that there exists \( g \in \mathcal{M}_{f, \Phi} \) such that \( g \neq Uf \) for all \( U \in \mathcal{U}(\mathcal{H}) \). Then there exist \( \phi_0, \phi_1 \in \Phi \) by Assumption 3.1 such that
\[ (5.2) \quad \langle \phi_0(g), \phi_1(g) \rangle \neq \langle \phi_0(f), \phi_1(f) \rangle. \]
Set \( u = (f + g)/2 \) and \( v = (f - g)/2 \in \mathcal{S} \). One may verify that \( f = u + v \),
\[ \langle \phi(u), \phi(v) \rangle = \frac{1}{4}(\langle \phi(f), \phi(f) \rangle - \langle \phi(g), \phi(g) \rangle) = 0 \]
for all \( \phi \in \Phi \) by the assumption that \( g \in \mathcal{M}_{f, \Phi} \), and
\[ (5.4) \quad \langle \phi_0(u), \phi_1(v) \rangle + \langle \phi_0(v), \phi_1(u) \rangle = \frac{1}{4}(\langle \phi_0(f), \phi_1(f) \rangle - \langle \phi_0(g), \phi_1(g) \rangle) \neq 0 \]
by (5.2). This contradicts to the hypothesis in the sufficiency.

5.2. Proof of Theorem 3.7. By \( (I - P_{\tilde{H}}) \pm P_{\tilde{H}} \in \mathcal{U}(\mathcal{H}) \), the unitary invariance of the linear space \( \mathcal{S} \) and the set \( \Phi \) of linear measurements, we have
\[ (5.3) \quad P_{\tilde{H}}f \in \mathcal{S} \]
and
\[ (5.4) \quad \phi((I - 2P_{\tilde{H}})f) = (I - 2P_{\tilde{H}})\phi(f) \]
for all \( f \in \mathcal{S} \) and \( \phi \in \Phi \). Therefore
\[ (5.5) \quad P_{\tilde{H}}\mathcal{S} = \{ h \in \mathcal{S}, \ h(x) \in \tilde{H} \ \text{for all} \ x \in D \} \subset \mathcal{S} \]
and
\[ (5.6) \quad \phi(P_{\tilde{H}}f) = P_{\tilde{H}}\phi(f) \in \tilde{H} \ \text{for all} \ f \in \mathcal{S}. \]

For any real unitary operator \( U \in \mathcal{U}(\tilde{H}) \), \( \phi \in \Phi \) and \( g \in P_{\tilde{H}}\mathcal{S} \), we have
\[ (5.7) \quad Ug(x) \in \tilde{H} \ \text{for all} \ x \in D, \]
\[ (5.8) \quad Ug = (U + (I - P_{\tilde{H}}))g \in \mathcal{S}, \]
and
\[ (5.9) \quad \phi(Ug) = \phi((UP_{\tilde{H}} + I - P_{\tilde{H}})g) = (UP_{\tilde{H}} + I - P_{\tilde{H}})\phi(g) = U\phi(g) \]
by (5.4), (5.6) and unitary invariance of the linear space \( \mathcal{S} \) and the set \( \Phi \) of linear measurements. Therefore \( P_{\tilde{H}}\mathcal{S} \) is unitary invariant,
\[ (5.10) \quad UP_{\tilde{H}}\mathcal{S} \subset P_{\tilde{H}}\mathcal{S}, \ U \in \mathcal{U}(\tilde{H}), \]
and the set \( \Phi \) of linear measurements is also unitary invariant for the linear space \( P_{\tilde{H}}\mathcal{S} \) by (5.5), (5.7), (5.8) and (5.9).
Take an \( f \in \mathcal{S} \). Suppose, on the contrary, that \( P_{\tilde{H}}f \) is not phase retrieval. Applying Theorem 3.2 to the linear space \( P_{\tilde{H}}\mathcal{S} \), we know that there exist \( u, v \in P_{\tilde{H}}\mathcal{S} \) satisfying (3.11), (3.12) and (3.13). Define
\[
\tilde{u} = (I - P_{\tilde{H}})f + u \quad \text{and} \quad \tilde{v} = v \in \mathcal{S}.
\]
From (3.11), (3.12), (3.13), (5.4), (5.6), (5.11) and the projection property of \( P_{\tilde{H}} \), we have \( \tilde{u} + \tilde{v} \),
\[
\langle \phi(\tilde{u}), \phi(\tilde{v}) \rangle = \langle (I - P_{\tilde{H}})\phi(f), \phi(v) \rangle + \langle \phi(u), \phi(v) \rangle = 0
\]
for all \( \phi \in \Phi \), and
\[
\langle \phi_0(\tilde{u}), \phi_1(\tilde{v}) \rangle + \langle \phi_1(\tilde{u}), \phi_0(\tilde{v}) \rangle = \langle (I - P_{\tilde{H}})\phi_0(f) + \phi_0(u), \phi_1(v) \rangle + \langle (I - P_{\tilde{H}})\phi_1(f) + \phi_1(u), \phi_0(v) \rangle
\]
\[
= \langle \phi_0(u), \phi_1(v) \rangle + \langle \phi_1(u), \phi_0(v) \rangle \neq 0
\]
for the linear measurements \( \phi_0, \phi_1 \) in (3.12). Therefore \( f \) is not phase retrieval in \( \mathcal{S} \) by Theorem 3.2. This contradicts to our hypothesis on the function \( f \) and hence completes the proof.

5.3. Proof of Theorem 3.10. To prove Theorem 3.10, we need a technical lemma about isomorphism of two \( d \)-simplices.

**Lemma 5.1.** Let \( x_i \in \mathbb{R}^d, 0 \leq i \leq d \), be affinely independent. If \( y_i, 0 \leq i \leq d \), satisfy
\[
\|y_i\| = \|x_i\|, \ 0 \leq i \leq d
\]
and
\[
\|y_i - y_j\| = \|x_i - x_j\|, \ 0 \leq i, j \leq d,
\]
then there exists an orthogonal matrix \( U \in \mathcal{U}(\mathbb{R}^d) \) such that
\[
y_i = Ux_i, \ 0 \leq i \leq d.
\]

**Proof.** For the completeness of this paper, we include a proof. By (5.12) and (5.13), we have
\[
\langle y_i, y_j \rangle = \langle x_i, x_j \rangle, \ 0 \leq i, j \leq d.
\]
Let \( U \) be the linear transform on \( \mathbb{R}^d \) determined by
\[
U(x_i - x_0) = y_i - y_0, \ 1 \leq i \leq d,
\]
which is well-defined by the linear independence of \( x_i - x_0, 1 \leq i \leq d \). For any \( x = \sum_{i=1}^d c_i(x_i - x_0) \in \mathbb{R}^d \), we have
\[
\|Ux\|^2 = \left\| \sum_{i=1}^d c_i(y_i - y_0) \right\|^2 = \sum_{i,j=1}^d c_i c_j \langle y_i - y_0, y_j - y_0 \rangle
\]
\[
= \sum_{i,j=1}^d c_i c_j \langle x_i - x_0, x_j - x_0 \rangle = \|x\|^2
\]
by (5.15) and (5.16). This proves that $U$ is an orthogonal matrix in $U(\mathbb{R}^d)$. Therefore by (5.16) and (5.17) it suffices to prove that $y_0 = Ux_0$, or equivalently

\[(5.18) \quad x_0 = U^T y_0.\]

By (5.15), (5.16) and (5.17), we have

\[
\langle x_i - x_0, x_0 \rangle = \langle y_i - y_0, y_0 \rangle = \langle U(x_i - x_0), y_0 \rangle = \langle x_i - x_0, U^T y_0 \rangle
\]

for all $1 \leq i \leq d$. This together with the linear independence of $x_i - x_0, 1 \leq i \leq d$, proves (5.18) and completes the proof. □

Proof of Theorem 3.11. Let $G_1 = (V_1, E_1)$ be a complete subgraph of order $d + 1$ such that $\Delta(f, G_1)$ is a $d$-simplex. Let $g = (g_i)_{i \in V}$ be a vector field satisfying (3.6), (3.7). By Lemma 5.1, there exists an orthogonal matrix $U$ such that

\[(5.19) \quad Uf_i = g_i, i \in V_1.\]

Define $\tilde{g} = U^T g$. One may verify that the new vector field $\tilde{g} = (\tilde{g}_i)_{i \in V}$ still satisfies (3.6) and (3.7) and moreover, it coincides with the original vector field on vertices in the subgraph $G_1$,

\[(5.20) \quad \tilde{g}_i = f_i, i \in V_1.\]

Now we prove that the vector fields $\tilde{g}$ and $f$ are the same, i.e.,

\[(5.21) \quad \tilde{g}_i = f_i, i \in V.\]

By the covering property for $d$-simplices, and the connectivity of $d$-simplex graph $G_f$, it suffices to prove that

\[(5.22) \quad \tilde{g}_i = f_i, i \in V_1 \cup V_2,\]

where $G_2 = (V_2, E_2)$ be a complete subgraph of order $d + 1$ such that $\Delta(f, G_2)$ is a neighboring $d$-simplex of $\Delta(f, G_1)$. By the assumption on the $d$-simplex $\Delta(f, G_2)$, we have that $V_1 \cap V_2$ has $d$ vertices and $f_i, i \in V_1 \cap V_2$, form a basis for $\mathbb{R}^d$, see Remark 3.11. For all $i \in V_1 \cap V_2$ and $k \in V_2 \setminus V_1$, we obtain from (3.6), (3.7) and (5.20) that

\[
\langle \tilde{g}_k, f_i \rangle = \langle \tilde{g}_k, g_i \rangle = \frac{1}{2}(\|\tilde{g}_k\|^2 + \|\tilde{g}_i\|^2 - \|\tilde{g}_k - \tilde{g}_i\|^2)
\]

\[(5.23) \quad = \frac{1}{2}(\|f_k\|^2 + \|\tilde{f}_i\|^2 - \|\tilde{f}_k - \tilde{f}_i\|^2) = \langle f_k, f_i \rangle.\]

By (5.23) and the basis property for $f_i, i \in V_1 \cap V_2$, we obtain $\tilde{g}_k = f_k$. This proves (5.22) and completes the proof. □
5.4. **Proof of Theorem 2.1.** Let $C_{\mathbb{R}^2}$ be as in (3.2), and define the linear map $A : C \rightarrow C_{\mathbb{R}^2}$ by

$$Af = \left(\Re f \ \Im f\right), \ f \in C.$$ 

Then $A$ is one-to-one and onto, and

\begin{equation}
\langle Af(x), Ag(y) \rangle = \Re(f(x)\bar{g}(y))
\end{equation}

for all $f, g \in C$ and $x, y \in D$. Also one may verify that

$$Ag = U(Af)$$

for some orthogonal matrix $U$ of size $2 \times 2$ if and only if either $g = zf$ or $g = z\bar{f}$ for some $z \in \mathbb{T}$. This together with the complex conjugate invariance of the complex linear space $C$ implies that the real linear space $C_{\mathbb{R}^2}$ is invariant under all real orthogonal transformations on $\mathbb{R}^2$. Therefore by (5.24) and Theorem 3.2, the proof of Theorem 2.1 reduces to establishing Assumption 3.1 for the family $\Phi = \{\delta_x, x \in D\}$ of the point-evaluation measurements on $C_{\mathbb{R}^2}$, which can be reformulated as the equivalence between the following two statements for $f, g \in C$:

(i) $g = zf$ or $g = z\bar{f}$ for some $z \in \mathbb{T}$.

(ii) $\Re(f(x)f(y) - g(x)\bar{g}(y)) = 0$ for all $x, y \in D$.

The implication (i)$\implies$(ii) is obvious. Now we prove that (ii)$\implies$(i). Applying (ii) with $y$ replaced by $x \in D$, we have

\begin{equation}
|g(x)|^2 = |f(x)|^2, \ x \in D.
\end{equation}

Therefore the conclusion (i) follows from (5.25) if $f$ is a zero function. For a nonzero function $f$, without loss of generality, there exists $x_0 \in D$ such that

\begin{equation}
0 \neq f(x_0) = g(x_0) \in \mathbb{R}
\end{equation}

by (5.25), otherwise replacing $f$ by $z_1f$ and $g$ by $z_2g$ respectively, where $z_1, z_2 \in \mathbb{T}$. Applying (ii) with $y$ replaced by $x_0$, we obtain that

\begin{equation}
\Re f(x) = \Re g(x), \ x \in D.
\end{equation}

Set

\begin{equation}
D_1 = \{x \in D, \ g(x) = f(x)\} \text{ and } D_2 = \{x \in D, \ g(x) = \bar{f}(x)\}.
\end{equation}

Then

\begin{equation}
D_1 \cup D_2 = D.
\end{equation}

by (5.25) and (5.27). The statement (i) is proved if either $D_1 = D$ or $D_2 = D$. Now we consider the case that

\begin{equation}
D_1 \neq D \text{ and } D_2 \neq D.
\end{equation}

By (5.29) and (5.30), $D_1 \setminus D_2 \neq \emptyset$ and $D_2 \setminus D_1 \neq \emptyset$. Taking $x_1 \in D_1 \setminus D_2$ and $y_1 \in D_2 \setminus D_1$, we obtain from (5.28) that

$$\Re(f(x_1)\bar{f}(y_1) - g(x_1)\bar{g}(y_1)) = \Re(f(x_1)(\bar{f}(y_1) - f(y_1))) = 2\Re(f(x_1)\Im f(y_1)) \neq 0,$$
which contradicts to the statement (ii).

5.5. **Proof of Theorem 2.5.** Take a nonzero function \( g \) in the real linear space spanned by the real and imaginary parts of \( f \). Then there exists a nonzero complex number \( z \) such that \( g = zf + \bar{z}f \). Without loss of generality, we assume that \( z = 1 \), as \( zf \) is conjugate phase retrieval in \( C \). Suppose, on the contrary, that \( g = f + \bar{f} \) is not phase retrieval in \( \mathbb{R}(C) \). Then there exist nonzero functions \( f_1, f_2 \in \mathbb{R}(\mathbb{C}) \) by (2.14) such that

\[
(5.31) \quad g = f_1 + f_2 \quad \text{and} \quad f_1f_2 = 0.
\]

Write

\[
(5.32) \quad f = u + v,
\]

where

\[
(5.33) \quad u = \frac{f - \bar{f} + f_1}{2} \quad \text{and} \quad v = \frac{f_2}{2}.
\]

By (5.31) and (5.33), we have

\[
(5.34) \quad \mathbb{R}(u(x)\bar{v}(x)) = \frac{1}{4} \mathbb{R}((f(x) - \bar{f}(x) + f_1(x))f_2(x)) = \frac{1}{4} f_1(x)f_2(x) = 0
\]

for all \( x \in D \), where the second equality holds as \( i(f(x) - \bar{f}(x)) \), \( f_1(x) \), \( f_2(x) \in \mathbb{R} \) for all \( x \in D \). Take \( x_0, y_0 \in D \) such that

\[
(5.35) \quad f_1(x_0) \neq 0 \quad \text{and} \quad f_2(y_0) \neq 0.
\]

Then

\[
\mathbb{R}(u(x_0)\bar{v}(y_0) + u(y_0)\bar{v}(x_0)) = \frac{f_1(x_0)f_2(y_0) + f_1(y_0)f_2(x_0)}{4} = \frac{f_1(x_0)f_2(y_0)}{4} \neq 0
\]

by (5.31), (5.33) and (5.35). By (5.32), (5.34), (5.36) and Theorem 2.1, we obtain that \( f \) is not complex conjugate phase retrieval, which is a contradiction.

5.6. **Proof of Proposition 2.8.** The sufficiency follows from Corollary 2.7. Now we prove the necessity. Write \( A^T = (a_1, \ldots, a_m) \). By the assumption, the matrix \( A \) has the complement property [9]. Hence the matrix \( A \) has rank 2 and the linear space of all real symmetric matrices is spanned by \( a_i a_j^T + a_j a_i^T, 1 \leq i, j \leq m \). By Corollary 2.4, it suffices to prove that \( a_i a_j^T + a_j a_i^T, 1 \leq i, j \leq m \), are linear combinations of \( a_k a_l^T, 1 \leq k \leq m \). By the complement property, without loss of generality, we assume that \( \text{span} \{a_1, a_2\} = \mathbb{R}^2 \) and \( a_3 = \lambda a_1 + \mu a_2 \) for some nonzero real numbers \( \lambda \) and \( \mu \). Then

\[
(5.37) \quad a_1 a_2^T + a_2 a_1^T = (\lambda \mu)^{-1} a_3 a_3^T - \lambda \mu^{-1} a_1 a_1^T - \mu^{-1} a_2 a_2^T.
\]
By the assumption on \(a_1\) and \(a_2\), there exist \(\lambda_i, \mu_i, 1 \leq i \leq m\) such that \(a_i = \lambda_i a_1 + \mu_i a_2\), which together with (5.37) implies that
\[
a_i a_j^T + a_j a_i^T = 2\lambda_j a_1 a_j^T + (\lambda_i \mu_j + \lambda_j \mu_i)(a_1 a_2^T + a_2 a_1^T) + 2\mu_i \mu_j a_1 a_1^T
\in \text{span} \{a_l a_l^T, l = 1, 2, 3\}, \quad 1 \leq i, j \leq m.
\]

This completes the proof.

5.7. **Proof of Proposition 2.9.** To prove Proposition 2.9, we recall a characterization for a phase retrieval function in \(S(h)\).

**Lemma 5.2.** [17] Theorem 3.2 Let \(g(t) = \sum_{k \in \mathbb{Z}} d(k)h(t-k) \in S(h), d(k) \in \mathbb{R}\). Then \(g\) is phase retrieval in \(S(h)\) if and only if
\[
d(k) \neq 0 \quad \text{for all} \quad K_- - 1 < k < K_+ + 1
\]
where \(K_- = \inf\{k, \ d(k) \neq 0\}\) and \(K_+ = \sup\{k, \ d(k) \neq 0\}\).

**Proof of Proposition 2.9. \(\implies\):** Suppose, on the contrary, that (2.15) does not hold. Then there exist \(k_0, k_1, k_2 \in (K_- - 1, K_+ + 1)\) such that
\[
k_0 < k_1 < k_2, \quad c(k_0)c(k_2) \neq 0 \quad \text{and} \quad c(k_1) = 0.
\]

Take \(z_1 \in \mathbb{T}\) with
\[
\Re(z_1 c(k_0)) \neq 0 \quad \text{and} \quad \Re(z_1 c(k_2)) \neq 0.
\]
Then
\[
\Re(z_1 f(t)) = \sum_{k < k_1} \Re(z_1 c(k))h(t-k) + \sum_{k > k_1} \Re(z_1 c(k))h(t-k)
\]
is not phase retrieval in \(S(h)\) by Lemma 5.2, which is a contradiction by Theorem 2.5. This completes the proof of (2.15).

For any \(l_0 \in (K_- - 1, K_+ + 1)\), it follows from Theorem 2.5 that
\[
\Re(i \bar{c}(l_0) f(t)) = \left(\sum_{k < l_0} + \sum_{k > l_0}\right) \Re(i \bar{c}(l_0) c(k)) h(t-k)
\]
is phase retrieval in \(S(h)\). Hence by Lemma 5.2 either \(\Re(i \bar{c}(l_0) c(k)) = 0\) for all \(k < l_0\) or \(\Re(i \bar{c}(l_0) c(k)) = 0\) for all \(k > l_0\). This proves (2.16).

\(\iff\): We prove the sufficiency by two cases.

**Case 1:** \(3\{c(k) c(k+1)\} = 0\) for all \(k \in (K_- - 1, K_+)\).

Without loss of generality, we assume that \(c(k) \in \mathbb{R}\) for all \(k \in \mathbb{Z}\), otherwise replacing \(f\) by \(zf\) for some \(z \in \mathbb{T}\). Let \(g \in S_C(h)\) such that
\[
|g(t)| = |f(t)|, \quad t \in \mathbb{R}.
\]
Take \(k_0 \in \mathbb{Z}\) with \(f(k_0) \neq 0\). Without loss of generality, we assume that
\[
g(k_0) = f(k_0),
\]
otherwise replacing \(g\) by \(\tilde{g} = \frac{f(k_0)}{g(k_0)} g\). By (5.41) with \(t\) replaced by \(k_0 + u \in k_0 + [0, 1]\),
\[
|f(k_0)(1-u) + f(k_0+1)u|^2 = |g(k_0)(1-u) + g(k_0+1)u|^2, \quad u \in [0, 1].
\]
This together with \( 0 \neq g(k_0) = f(k_0) \in \mathbb{R} \) and \( f(k_0 + 1) \in \mathbb{R} \) implies that \( g(k_0 + 1) = f(k_0 + 1) \). Applying the above argument inductively, we have
\[
(5.43) \quad g(k) = f(k) \text{ for all } k \in (k_0, K_+ + 1).
\]

Applying (5.41) with \( t \) replaced by \( k_0 - u \in k_0 + [-1, 0] \) and using \( 0 \neq g(k_0) = f(k_0) \in \mathbb{R} \) and \( f(k_0 - 1) \in \mathbb{R} \), we can show that \( g(k_0 - 1) = f(k_0 - 1) \). Thus we have the following conclusion by induction,
\[
(5.44) \quad g(k) = f(k), k \in (K_- - 1, k_0).
\]

By (5.41), we have
\[
(5.45) \quad g(k) = f(k) = 0, k \notin (K_- - 1, K_+ + 1).
\]

Combining (5.42)–(5.45) proves \( g = f \), which completes the proof.

**Case 2:** There exists \( k_0 \in (K_- - 1, K_+) \) such that \( \Im(c(k_0)c(k_0 + 1)) \neq 0 \) and \( \Im(c(k)c(k + 1)) = 0 \) for all \( k \in (K_- - 1, K_+) \setminus \{k_0\} \).

Without loss of generality, we assume that \( c(k) \in \mathbb{R} \) for all \( k \leq k_0 \), otherwise replacing \( f \) by \( zf \) for some \( z \in \mathbb{T} \). Let \( g \in S_C(h) \) such that (5.41) holds. Without loss of generality, we assume that \( 0 \neq g(k_0) = f(k_0) \in \mathbb{R} \), otherwise replacing \( g \) by \( \tilde{g} = \frac{f(k_0)}{g(k_0)} g \). Following the argument used in Case 1, we can show that
\[
(5.46) \quad g(k) = f(k) \in \mathbb{R}, k \leq k_0.
\]

Similarly, with \( t \) replaced by \( k_0 + u \in k_0 + [0, 1] \) in (5.41) and \( 0 \neq g(k_0) = f(k_0) \in \mathbb{R} \), we obtain
\[
\Re g(k_0 + 1) = \Re f(k_0 + 1) \quad \text{and} \quad |g(k_0 + 1)| = |f(k_0 + 1)|.
\]

Hence either
\[
(5.47) \quad g(k_0 + 1) = f(k_0 + 1)
\]
or
\[
(5.48) \quad g(k_0 + 1) = \bar{f}(k_0 + 1).
\]

For the case that (5.47) holds, we can follow the argument used in Case 1 to prove
\[
(5.49) \quad g(k) = f(k), k \geq k_0 + 1.
\]

Similarly for the case that (5.48) holds, we have
\[
(5.50) \quad g(k) = \bar{f}(k), k \geq k_0 + 1.
\]

Combining (5.46), (5.49) and (5.50), we conclude that either \( g(k) = f(k) \) for all \( k \in \mathbb{Z} \) or \( g(k) = \bar{f}(k) \) for all \( k \in \mathbb{Z} \). Therefore either \( g = f \) or \( g = \bar{f} \). This completes the proof of conjugate phase retrieval for Case 2 and completes the proof of sufficiency. \( \square \)
5.8. **Proof of Theorem 2.10.** The Conclusion (ii) follows from the first conclusion. Then it suffices to prove the Conclusion (i). Let the linear subspace \( W_\mathbb{R} \subset \mathcal{W} \) contain all real functions in \( \mathcal{W} \), and \( W_\mathbb{R}^4 \) be the product space of \( W_\mathbb{R} \). Obviously, \( W_\mathbb{R}^4 \) is unitary invariant. By (2.19) and the isomorphism between \( \mathbb{Q} \) and \( \mathbb{R}^4 \), the map \( T_W \) defined by

\[
T_W : W_\mathbb{R}^4 \ni (f_1, f_2, f_3, f_4)^T \mapsto f_1 + f_2i + f_3j + f_4k \in \mathcal{W}
\]

is an isomorphism between \( W_\mathbb{R}^4 \) and \( \mathcal{W} \), and for any \( f, g \in W_\mathbb{R}^4 \),

\[
\langle f(x), g(y) \rangle = \Re(T_W(f)(x)T_W(g)^*(y)), x, y \in D.
\]

Then it suffices to prove that \( f = (f_1, f_2, f_3, f_4)^T \in W_\mathbb{R}^4 \) is phase retrieval in \( W_\mathbb{R}^4 \) if and only if \( T_W(f) \) is quaternion conjugate phase retrieval in \( \mathcal{W} \), which in turn reduces to prove

(5.51)

\[
\{ T_W(Uf), U \in \mathcal{U}(\mathbb{R}^4) \} = \left\{ \sum_{i=1}^{4} q_i f_i \in \mathcal{W}, q_i \in \mathbb{T}, 1 \leq i \leq 4, \text{satisfies } (2.21) \right\}.
\]

Write \( U = (u_{ij})_{1 \leq i,j \leq 4} \), then

\[
T_W(Uf) = \sum_{j=1}^{4} u_{1j} f_j + \sum_{j=1}^{4} u_{2j} f_j i + \sum_{j=1}^{4} u_{3j} f_j j + \sum_{j=1}^{4} u_{4j} f_j k
\]

\[
= \sum_{j=1}^{4} (u_{1j} + u_{2j} i + u_{3j} j + u_{4j} k) f_j =: \sum_{j=1}^{4} p_j f_j.
\]

As \( U \) is an orthogonal matrix, we have

\[
||p_j||^2 = |u_{1j}|^2 + |u_{2j}|^2 + |u_{3j}|^2 + |u_{4j}|^2 = 1
\]

for all \( 1 \leq j \leq 4 \), and

\[
p_ip_j^* + p_j p_i^* = 2\Re(p_ip_j^*) = 2(u_{1i}u_{1j} + u_{2i}u_{2j} + u_{3i}u_{3j} + u_{4i}u_{4j}) = 0
\]

for \( 1 \leq i < j \leq 4 \). This proves that

(5.52)

\[
\{ T_W(Uf), U \in \mathcal{U}(\mathbb{R}^4) \} \subset \left\{ \sum_{i=1}^{4} q_i f_i \in \mathcal{W}, q_i \in \mathbb{T}, 1 \leq i \leq 4, \text{satisfies } (2.21) \right\}.
\]

Let \( q_i \in \mathbb{T}, 1 \leq i \leq 4 \) satisfy \( (2.21) \). Write \( q_i = u_{1i} + u_{2i}i + u_{3i}j + u_{4i}k, 1 \leq i \leq 4 \). Then it follows from the assumption on \( q_i, 1 \leq i \leq 4 \) that

\[
u_{1i}u_{1j} + u_{2i}u_{2j} + u_{3i}u_{3j} + u_{4i}u_{4j} = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}
\]
Hence $U = (u_{ij})_{1 \leq i,j \leq 4}$ is an orthogonal matrix on $\mathbb{R}^4$. Moreover, we have that

$$\sum_{i=1}^{4} q_i f_i = \left( \sum_{j=1}^{4} u_{1j} f_j \right) + \left( \sum_{j=1}^{4} u_{2j} f_j \right) i + \left( \sum_{j=1}^{4} u_{3j} f_j \right) j + \left( \sum_{j=1}^{4} u_{4j} f_j \right) k = T_{W}(Uf).$$

This proves that

$$\left\{ \sum_{i=1}^{4} q_i f_i \in \mathcal{W}, q_i \in \mathbb{T}_8, 1 \leq i \leq 4, \text{satisfies } (2.21) \right\} \subset \{ T_{W}(Uf), U \in \mathcal{U}(\mathbb{R}^4) \}.$$

Combining (5.52) and (5.53) proves (5.51) and completes the proof.

5.9. Proof of Theorem 4.3. For we prove the sufficiency. Take $f \in \mathcal{S}$ and $g \in \mathcal{A}_f$. Then for all $1 \leq i,j \leq N$ and $\phi \in \Phi$, we have

$$\langle \phi(f), b_{\phi,i} - b_{\phi,j} \rangle = \frac{\| \phi(f) + b_{\phi,i} \|^2 - \| \phi(g) + b_{\phi,j} \|^2 - \|b_{\phi,i} \|^2 - \|b_{\phi,j} \|^2}{2}$$

$$= \frac{\| \phi(f) + b_{\phi,i} \|^2 - \| \phi(f) + b_{\phi,j} \|^2 - \|b_{\phi,i} \|^2 - \|b_{\phi,j} \|^2}{2}$$

$$= \langle \phi(f), b_{\phi,i} - b_{\phi,j} \rangle.$$

Therefore by the injectivity of the map $T$ in (4.3), we obtain that $g = f$, which proves the affine phase retrieval of the linear space $\mathcal{S}$.

Now we prove the necessity. Let $f_0 \in \mathcal{S}$ and $1 \leq i_0 \leq N$ be as in (4.4). Suppose, on the contrary, that the linear map $T$ in (4.3) is not injective. Then there exists a nonzero function $h_0 \in \mathcal{S}$ such that

$$\langle \phi(h_0), b_{\phi,i} - b_{\phi,j} \rangle = 0 \text{ for all } \phi \in \Phi \text{ and } 1 \leq i,j \leq N.$$

Set $f = -f_0 + h_0/2$ and $g = -f_0 - h_0/2 \in \mathcal{S}$. Then $g \neq f$ and $g \in \mathcal{N}_f$, since for all $\phi \in \Phi$ and $1 \leq i \leq N$,

$$\| \phi(f) + b_{\phi,i} \|^2 - \| \phi(g) + b_{\phi,i} \|^2$$

$$= \| \phi(h_0)/2 + b_{\phi,i} - b_{\phi,i} \|^2 - \| \phi(h_0)/2 + b_{\phi,i} \|^2$$

$$= 2\langle \phi(h_0), b_{\phi,i} - b_{\phi,i} \rangle = 0 \text{ by } (4.4) \text{ and } (5.54).$$

This contradicts to the affine phase retrieval of the linear space $\mathcal{S}$.

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