THERE IS NO SEPARABLE UNIVERSAL II$_1$-FACTOR

NARUTAKA OZAWA

Abstract. Gromov constructed uncountably many pairwise non-isomorphic discrete groups with Kazhdan’s property (T). We will show that no separable II$_1$-factor can contain all these groups in its unitary group. In particular, no separable II$_1$-factor can contain all separable II$_1$-factors in it. We also show that the full group $C^*$-algebras of some of these groups fail the lifting property.

1. Results

We recall that a discrete group $\Gamma$ is said to have Kazhdan’s property (T) if the trivial representation is isolated in the dual $\hat{\Gamma}$ of $\Gamma$, equipped with the Fell topology. This is equivalent to that there are a finite subset $E$ of generators in $\Gamma$ and a decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ such that the following is true: if $\pi$ is a unitary representation of $\Gamma$ on a Hilbert space $H$ and $\xi \in H$ is a unit vector with $\varepsilon = \max_{s \in E} \| \pi(s)\xi - \xi \|$, then there is a vector $\eta \in H$ with $\| \xi - \eta \| < f(\varepsilon)$ (in particular $\eta \neq 0$ when $\varepsilon$ is small enough) such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. We refer the reader to [HV] and [V] for the information of Kazhdan’s property (T). We recall that a discrete group $\Gamma$ is said to be quasifinite if all its proper subgroups are finite, and is said to be infinite conjugacy classes (abbreviated to ICC) if all nontrivial conjugacy classes in $\Gamma$ are infinite. We note that a discrete group $\Gamma$ is ICC if and only if its group von Neumann algebra $L\Gamma$ is a factor. We also observe that a group which is quasifinite and ICC has to be simple.

Gromov (Corollary 5.5.E in [G]) claimed that any torsion-free non-cyclic hyperbolic group has a quotient group all of whose proper subgroups are cyclic of prescribed orders (cf. Theorem 3.4 in [V]). This claim was partly confirmed by Olshanskii (Corollary 4 in [O]). Actually, what Olshanskii proved there is that any torsion-free non-cyclic hyperbolic group has a nontrivial quasifinite quotient group. We observe that Olshanskii’s argument gives us the following.

Theorem 1 (Gromov-Olshanskii). Any torsion-free non-cyclic hyperbolic group has uncountably many pairwise non-isomorphic quotient groups all of which are quasifinite and ICC. In particular, there is a discrete group $\Gamma$ with Kazhdan’s property (T) which has uncountably many pairwise non-isomorphic quotient groups $\{\Gamma_\alpha\}_{\alpha \in I}$ all of which are simple and ICC.

Connes conjectured that a discrete group $\Delta$ with Kazhdan’s property (T) and the ICC property is uniquely determined by its group von Neumann algebra $L\Delta$. The following theorem and its corollary, which was suggested by S. Popa, confirm

Date: November 1, 2002.

1991 Mathematics Subject Classification. Primary 46L10; Secondary 20F65.

Key words and phrases. Universal II$_1$-factor, uncountably many II$_1$-factors, lifting property. Partially supported by JSPS Postdoctoral Fellowships for Research Abroad.
Connes’ conjecture for \( \{ \Gamma_\alpha \}_{\alpha \in I} \) “modulo countable sets” and solve Problem 4.4.29 in [S], Conjecture 4.5.5 in [P1] and Problem III.45 in [H]. See also Theorem 1 in [P2] and its remarks.

**Theorem 2.** Let \( \Gamma \) and \( \{ \Gamma_\alpha \}_{\alpha \in I} \) be as in Theorem 1 and let \( M \) be a separable \( \Pi_1 \)-factor. Then, the set

\[ \{ \alpha \in I : \text{the unitary group } U(M) \text{ of } M \text{ contains a subgroup isomorphic to } \Gamma_\alpha \} \]

is at most countable.

Recall that two \( \Pi_1 \)-factors \( M \) and \( N \) are said to be stably equivalent if there are \( n \in \mathbb{N} \) and a projection \( p \in M_n(M) \) such that \( pM_n(M)p \) is isomorphic to \( N \).

**Corollary 3.** Let \( \Gamma \) and \( \{ \Gamma_\alpha \}_{\alpha \in I} \) be as in Theorem 1 and let \( M \) be a separable \( \Pi_1 \)-factor. Then, the set

\[ \{ \alpha \in I : M \text{ contains a subfactor which is stably equivalent to } L\Gamma_\alpha \} \]

is at most countable.

In connection with Connes’ embedding problem [C], it would be interesting to know whether all (or at least one of) \( \Gamma_\alpha \)’s are embeddable into the unitary group \( U(R^\omega) \) of the ultrapower \( R^\omega \) of hyperfinite \( \Pi_1 \)-factors. Since each \( \Gamma_\alpha \) arises as a limit of hyperbolic groups, we observe that if all hyperbolic groups are embeddable into \( U(R^\omega) \), then so is \( \Gamma_\alpha \). We remark that whether all hyperbolic groups are residually finite (and thus embeddable into \( U(R^\omega) \)) is one of the major open problems in geometric group theory.

Let us consider the category of unital \( C^* \)-algebras and unital completely positive maps. A \( C^* \)-algebra \( A \) is said to be complementary universal for a class \( \mathcal{C} \) of \( C^* \)-algebras if for every member \( B \) of \( \mathcal{C} \), there are unital completely positive maps \( \psi : B \to A \) and \( \varphi : A \to B \) such that \( \varphi \psi = \text{id}_B \). It follows from Kirchberg’s theorem [K2] that any separable \( C^* \)-algebra of not type I is complementary universal for the class of separable nuclear \( C^* \)-algebras. The full group \( C^*_\mathcal{F}_\infty \) of the free group \( \mathcal{F}_\infty \) on countably many generators is complementary universal for the class of separable \( C^* \)-algebras with the lifting property (abbreviated to LP). See [K1] for the information of the LP. It is not known whether there exists a separable complementary universal \( C^* \)-algebra for the class of separable exact \( C^* \)-algebras.

**Theorem 4.** Let \( \Gamma \) and \( \{ \Gamma_\alpha \}_{\alpha \in I} \) be as in Theorem 1 and let \( \mathcal{C} = \{ C^*\Gamma_\alpha : \alpha \in I \} \) or \( \mathcal{C} = \{ C^*_\text{red} \Gamma_\alpha : \alpha \in I \} \). Then, there is no separable complementary universal \( C^* \)-algebra for \( \mathcal{C} \).

From the above discussion, we immediately obtain the following corollary.

**Corollary 5.** The full group \( C^* \)-algebra \( C^*\Gamma_\alpha \) of \( \Gamma_\alpha \) fails the LP for some \( \alpha \in I \).

**Acknowledgment**

The author thanks Professor S. Popa for useful comments. This research was carried out while the author was visiting the University of California Berkeley under the support of the Japanese Society for the Promotion of Science Postdoctoral Fellowships for Research Abroad. He gratefully acknowledges the kind hospitality from UCB.
2. Proofs

Proof of Theorem 4. Since there exists a torsion-free non-cyclic hyperbolic group with Kazhdan’s property (T) (e.g., a co-compact lattice in $Sp(n,1)$ or in $F_4(-20)$), the second part is a straightforward consequence of the first. We just indicate how to modify the proof of Corollary 3 in [8] to obtain the first part of our Theorem 4. Let $G = \{g_1, g_2, \ldots\}$ be a torsion-free non-cyclic hyperbolic group. It follows that $G$ is ICC since the set $E(x)$ of all $x \in G$ whose conjugacy class is finite is a finite subgroup in $G$ (cf. Proposition 1 in [8]).

Recall that the quasifinite quotient group $G'$ was the inductive limit of a sequence of epimorphisms $G = G_0 \to G_1 \to G_2 \to \cdots$. By the construction, every $E(G_i)$ is trivial, or equivalently every $G_i$ is ICC. Hence, manipulating the construction, for every $i$ and $j \leq i$, we can carry at least $i$ mutually distinct elements from the conjugacy class of $g_j$ in $G_i$ injectively into $G'$ unless $g_j = 1$ in $G_i$. This ensures the ICC property of $G'$. In the construction, there has to be infinitely many $i$’s such that $g_i$ is torsion-free in $G_{2i-2}$. For such $i$, we may choose arbitrarily large number for the order of $g_i$ in $G_{2i-1}$ which will be equal to that in $G'$. Combined with a diagonal argument, this implies that there are uncountably many normal subgroups in $G$ all of whose corresponding quotient groups are quasifinite and ICC. Theorem 4 now follows from this result and Lemma III.42 in [9].

Proof of Theorem 5. To prove the theorem by reductio ad absurdum, suppose that $$I_0 = \{\alpha \in I : \mathcal{U}(M) \text{ contains a subgroup isomorphic to } \Gamma_\alpha\}$$ is uncountable. For each $\alpha \in I_0$, let $u_\alpha : \Gamma \to \mathcal{U}(M)$ be a non-trivial homomorphism which factors through $\Gamma_\alpha$. We fix a standard representation of $M$ on $\mathcal{H}$ with a unit cyclic separating trace vector $\xi$ in $\mathcal{H}$. It follows that there are $\delta > 0$ and an uncountable subset $I_1$ of $I_\delta$ such that $\max_{s \in \Gamma} \|u_\alpha(s)\xi - \xi\| > \delta$ for all $\alpha \in I_1$.

Let a finite subset $E$ of generators in $\Gamma$ and a function $f$ be as in the above definition of Kazhdan’s property (T). Take $\varepsilon > 0$ small enough so that $2f(\varepsilon) < \delta$. Since $\mathcal{H}$ is separable and $I_1$ is uncountable, there are distinct $\alpha$ and $\beta$ in $I_1$ such that $\max_{s \in E} \|u_\alpha(s)\xi - u_\beta(s)\xi\| < \varepsilon$. We consider the unitary representation $\pi : \Gamma \ni s \mapsto u_\alpha(s)J_\beta(s)J = \mathbb{B}(\mathcal{H})$, where $J$ is the canonical conjugation on $\mathcal{H}$ associated with $M$ and $\xi$. Then, we have $\max_{s \in E} \|\pi(s)\xi - \xi\| < \varepsilon$. It follows from Kazhdan’s property (T) of $\Gamma$ that there is a vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| < f(\varepsilon)$ such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. Let $\Delta = \{s \in \Gamma : u_\alpha(s)\eta = \eta\}$. It is easy to see that $\Delta$ is a subgroup of $\Gamma$ and that $\Delta$ contains the normal subgroups ker $u_\alpha$ and ker $u_\beta$. Since $\Gamma_\alpha$ and $\Gamma_\beta$ are simple and ker $u_\alpha$ and ker $u_\beta$ are distinct, we actually have $\Delta = \Gamma$. It follows that $\max_{s \in \Delta} \|u_\alpha(s)\xi - \xi\| < 2f(\varepsilon) < \delta$, which is absurd. \hfill \Box

Proof of Corollary 6. It is not difficult to see that if $L\Gamma_\alpha$ is isomorphic to a (not necessarily unital) subfactor of $M$, then $\Gamma_\alpha$ is isomorphic to a subgroup of $\mathcal{U}(M)$. Therefore, it follows from Theorem 5 that $$\{\alpha \in I : \mathbb{M}_n(M) \text{ contains a (not necessarily unital) subfactor isomorphic to } L\Gamma_\alpha\}$$ is at most countable for every $n \in \mathbb{N}$, and the conclusion follows. \hfill \Box

Proof of Theorem 7. We only deal with the case where $\mathcal{C} = \{C^*\Gamma_\alpha : \alpha \in I\}$. To prove the theorem by reductio ad absurdum, suppose that there is a separable $C^*$-algebra $A$ which is complementary universal for $\mathcal{C}$. We fix unital completely positive maps $\psi_\alpha : C^*\Gamma_\alpha \to A$ and $\varphi_\alpha : A \to C^*\Gamma_\alpha$ such that $\varphi_\alpha\psi_\alpha = \text{id}_{C^*\Gamma_\alpha}$. 
Let $E \subset \Gamma$ be a finite set of generators of $\Gamma$ containing 1 and let $u_\alpha(s)$ be the unitary element in $C^*\Gamma_\alpha$ corresponding to $s \in \Gamma$. Let $\varepsilon > 0$ be arbitrary. Since $A$ is separable and $I$ is uncountable, there are distinct $\alpha$ and $\beta$ in $I$ such that $\max_{s \in E} \|\psi_\alpha(u_\alpha(s)) - \psi_\beta(u_\beta(s))\| < \varepsilon$. It follows that denoting the left regular representation of $\Gamma_\alpha$ by $\lambda_\alpha$, we have

$$\| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(u_\alpha(s)) \otimes u_\beta(s) \|_{C^*_\text{red} \Gamma_\alpha \otimes_{\max} C^* \Gamma_\beta} \geq \| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(u_\alpha(s)) \otimes \varphi_\beta(u_\beta(s)) \|_{C^*_\text{red} \Gamma_\alpha \otimes_{\max} C^* \Gamma_\alpha}$$

$$\geq \| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(u_\alpha(s)) \otimes u_\alpha(s) \|_{C^*_\text{red} \Gamma_\alpha \otimes_{\max} C^* \Gamma_\alpha} - \varepsilon$$

$$= 1 - \varepsilon.$$ 

Since $\Gamma$ has Kazhdan's property (T), if we choose $\varepsilon > 0$ sufficiently small, this implies that the trivial representation of $\Gamma$ is weakly contained in $C^*_\text{red} \Gamma_\alpha \otimes_{\max} C^* \Gamma_\beta$ (cf. Proposition 4.9 in [V]). Reasoning in the same way as the proof of Theorem 2, one can show that the trivial representation is weakly contained in $C^*_\text{red} \Gamma_\alpha$. This is absurd.

\[ \square \]

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