Riemann-Roch for homotopy invariant $K$-theory and Gysin morphisms

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Abstract

We prove the Riemann-Roch theorem for homotopy invariant $K$-theory and projective local complete intersection morphisms between finite dimensional noetherian schemes, without smoothness assumptions. We also prove a new Riemann-Roch theorem for the relative cohomology of a morphism.

In order to do so, we construct and characterize Gysin morphisms for regular immersions between cohomologies represented by spectra (examples include homotopy invariant $K$-theory, motivic cohomology, their arithmetic counterparts, real absolute Hodge and Deligne-Beilinson cohomology, rigid syntomic cohomology, mixed Weil cohomologies) and use this construction to prove a motivic version of the Riemann-Roch.

Contents

1 Stable homotopy
  1.1 Cohomology and its operations. . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
  1.2 Arithmetic and relative cohomologies . . . . . . . . . . . . . . . . . . . . . . . . 5
  1.3 Orientations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
  1.4 Chern class with support . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
  1.5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16

2 Gysin morphism
  2.1 Regular immersions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  2.2 Functoriality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  2.3 The projective lci case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
  2.4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27

3 Riemann-Roch theorem
  3.1 Applications . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33

A Absolute Hodge cohomology

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Introduction

The original Grothendieck’s Riemann-Roch theorem states that for any proper morphism $f: Y \to X$, between nonsingular quasiprojective irreducible varieties over a field, and any element $a \in K_0(Y)$ of the Grothendieck group of vector bundles the relation

$$\text{ch}(f_!(a)) = f_*(\text{Td}(T_f) \cdot \text{ch}(a))$$

holds (cf. [BS58]). Recall that ch denotes the Chern character, Td the Todd class of the relative tangent bundle and $f_*$ and $f_!$ the direct image in the Chow ring and $K_0$ respectively. Later the result was generalized for locally complete intersection morphisms between singular projective algebraic schemes (cf. [SGA6], [BFM75]).

The extension to higher $K$-theory and schemes over a regular base was proved by Gillet in [Gil81]. The Riemann-Roch theorem proved there is for projective morphisms between smooth quasiprojective schemes. Therefore, note that in the case over a field Gillet’s theorem does not recover the result of [SGA6]. The furthest generalization of the higher Riemann-Roch theorem I know is [Dég14] and [HS15] where Déglise and Holmstrom-Scholbach independently obtained the Riemann-Roch theorem for higher $K$-theory and projective morphisms between regular schemes.

After the work of Cisinski in [Cis13] on Weibel’s homotopy invariant $K$-theory one may apply Voevodsky’s motivic homotopy theory to it. With Cisinski’s result, we present a Riemann-Roch theorem for homotopy invariant $K$-theory $KH$ and projective lci morphisms without smoothness assumptions on the schemes. More concretely, the theorem we prove for motivic cohomology $H_{\text{M}}$ and finite dimensional noetherian schemes is the following:

**Theorem:** Let $f: Y \to X$ be a projective lci morphism and denote $T_f \in K_0(Y)$ the virtual tangent bundle and Td the multiplicative extension of the series given by $\frac{1}{1-e^{-t}}$. Then the diagram

$$
\begin{array}{ccc}
KH(Y)_{\mathbb{Q}} & \xrightarrow{f_*} & KH(X)_{\mathbb{Q}} \\
\text{Td}(T_f) \downarrow & & \downarrow \text{ch} \\
H_{\text{M}}(Y, \mathbb{Q}) & \xrightarrow{f_*} & H_{\text{M}}(X, \mathbb{Q})
\end{array}
$$

commutes. In other words, for $a \in KH(Y)_{\mathbb{Q}}$ we have

$$\text{ch}(f_*(a)) = f_*(\text{Td}(T_f) \cdot \text{ch}(a)).$$

From here, we deduce Riemann-Roch theorems for many cohomologies. In particular, for real absolute Hodge and Deligne Beilinson cohomology, rigid syntomic cohomology and mixed Weil cohomologies such as algebraic de Rham and geometric étale cohomology. Note also that, due to the comparison result with cdh-motivic cohomology in [CD15], motivic cohomology on singular schemes over a base field of exponential characteristic is computed by explicit cycles.

In order to prove this result, the addition we make to the theory is the construction of the Gysin morphism for regular immersions and every cohomology given by spectra. Since its beginnings, the standard construction of the Gysin morphism in motivic homotopy theory relies on the Thom space and the purity isomorphism. However, purity requires smoothness assumptions (cf. [MV99]). Our approach is a different one: We lift the work of Gabber for étale cohomology to the motivic homotopy setting and, thus, obtain Gysin morphisms.
for regular immersions without smoothness assumptions on the schemes. This leads to the construction of new Gysin morphisms for many theories like homotopy invariant $K$-theory, motivic cohomology, real absolute Hodge and Deligne-Beilinson cohomology, rigid syntomic cohomology, motivic cdh-cohomology and any cohomology coming from a mixed Weil theory.

Modules over a cohomology theory are notable geometric and arithmetic invariants. Recall that higher arithmetic $K$-theory and arithmetic motivic cohomology are modules over $K$-theory and motivic cohomology respectively. In addition, for any cohomology the relative cohomology of a morphism is also a module. Note that the cohomology with proper support, the cohomology with support on a closed subscheme, and the reduced cohomology are the relative cohomology of a closed immersion, an open immersion and the projection over a base point respectively.

We deduce from our results new Gysin morphisms and a Riemann-Roch theorem for abstract modules. In particular, we prove a new Riemann-Roch theorem for relative cohomology and finite dimensional noetherian schemes.

**Theorem:** Let $g: T \rightarrow X$ be a morphism of schemes, $f: Y \rightarrow X$ be a projective lci morphism. Denote $g_Y: T \times_X Y \rightarrow Y$, $T_f \in K_0(Y)$ the virtual tangent bundle of $f$ and $KH(g)$ and $H_M(g, \mathbb{Q})$ the relative homotopy invariant $K$-theory and motivic cohomology of $g$ respectively. Assume in addition either $g$ is proper or $f$ is smooth, then the diagram

$$
\begin{array}{ccc}
KH(g_Y)_\mathbb{Q} & \xrightarrow{f_*} & KH(g)_\mathbb{Q} \\
\downarrow \text{Td}(T_f) \text{ch} & & \downarrow \text{ch} \\
H_M(g_Y, \mathbb{Q}) & \xrightarrow{f_*} & H_M(g, \mathbb{Q})
\end{array}
$$

commutes. In other words, for $m \in KH(g_Y)_\mathbb{Q}$ we have

$$
\text{ch}(f_*(m)) = f_*(\text{Td}(T_f) \cdot \text{ch}(m)).
$$

We also deduce the higher arithmetic Riemann-Roch theorem of [HS15] from the Riemann-Roch theorem for modules.

The paper is organized as follows: Section 1 recalls basic facts of the stable homotopy category and cohomologies defined by spectra, in particular, orientations or Chern classes. In section 1.2 we introduce the relative cohomology in the context of motivic stable homotopy theory and construct an absolute spectrum which represents it under some conditions. We also recall Holmstrom-Scholbach’s construction of the higher arithmetic $K$-theory and motivic cohomology. In section 2 we construct Gysin morphisms using Gabber’s ideas for the case of regular immersions and prove the basic properties. We prove the motivic Riemann-Roch theorem in section 3 and an analogous statement for modules (cf. Theorem 3.5 and Theorem 3.7) and obtain the Riemann-Roch theorem for homotopy invariant $K$-theory and for relative cohomologies as a result. The Appendix is devoted to the explicit construction of the real absolute Hodge spectrum using Burgos’s complex (cf. [Bur98]) and to check that this spectrum, as well as the Deligne-Beilinson spectrum of [HS15], represents their cohomologies also in the singular context.

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1 Stable homotopy

All schemes considered are noetherian, finite dimensional and over a fixed base scheme \( S \).

Let \( X \) be a scheme and denote \( \text{SH}(X) \) the stable homotopy category over \( X \) ([Voe98, § 5]). The categories obtained if one varies \( X \) satisfies Grothendieck’s six functor formalism. We will only recall here some of their properties that will be used later. We refer to [Ayo07] and [CD09] for a complete exposition of the construction of \( \text{SH} \) and Grothendieck’s six operations in this context.

Let \( \text{Sm}_X \) be the category of smooth schemes over \( X \). Recall that the category \( \text{SH}(X) \), whose objects are called \( \text{spectra} \), is constructed out of the category of pointed Nisnevich presheaves of simplicial sets on \( \text{Sm}_X \). Any \( X \)-scheme \( T \) defines the simplicial sheaf given by \( \text{Hom}_{\text{Sm}_X}(\_ , T) \), which we still denote \( T \). We denote \( T^+ = T \sqcup \text{pt} \) the pointed simplicial sheaf it defines. Then we have the infinite suspension functor \( \Sigma_\infty : \text{Sm}_X \rightarrow \text{SH}(X) \).

We denote \( \Sigma_\infty T \), or simply \( T \) if no confusion is possible. In the particular case of \( X \) we will use the notation \( /BD_X \in \text{SH}(X) \).

The category \( \text{SH}(X) \) is a monoidal triangulated category. We denote the shift functor as \([1] : \text{SH}(X) \rightarrow \text{SH}(X)\) and the product is denoted by \( \wedge \). Recall that the wedge product is defined in the category of \textit{symmetric spectra} \( \text{Spt}_\Sigma(X) \) whose localization by a suitable model structure is equivalent to \( \text{SH}(X) \).

Recall that we consider the projective line pointed at the infinity. The invertible element \( 1_X(1) := \text{coker}(\Sigma^\infty X \rightarrow \Sigma^\infty \mathbb{P}^1)[-2] \) is called the \textit{Tate object}. For any spectrum \( E \) we denote the \textit{Tate twist} by \( E(1) := E \wedge 1_X(1) \) and denote \( E(q)[p] \) for twisting and shifting \( q, p \in \mathbb{Z} \) times respectively.

For any morphism of schemes \( f : Y \rightarrow X \) there is a pullback functor \( f^* : \text{SH}(X) \rightarrow \text{SH}(Y) \) which is also functorial with respect to \( f \). Moreover, the functor \( f^* \) admits a right adjoint denoted \( f_* \).

The pullback functor \( f^* \) is also monoidal and therefore \( f^*(1_Y) = 1_X \). If \( i : Y \rightarrow X \) is a closed immersion and \( U = X - Y \), then \( 1_Y \simeq i^*(\Sigma^\infty X/U) \) where \( \Sigma^\infty X/U \) stands for the spectrum given by the pointed quotient Nisnevich presheaf \( X/U \). Moreover, the pullback and pushforward functor satisfy the projection formula. In other words, for every \( E \in \text{SH}(Y) \) and \( F \in \text{SH}(X) \) there is a canonical isomorphism \( i_*(E \wedge i^*(F)) \rightarrow i_*(E) \wedge F \).

The stable homotopy category relates to other categories that we will use at some point. In [MV99] Morel and Voevodsky defined the \textit{homotopy category} of schemes \( \mathbf{H}(X) \). This
category is constructed as a localization of an intermediate category $H_*(X)$ called the simplicial homotopy category of schemes. Their pointed counterparts are related to the stable homotopy category via the derived infinite suspension so that there are functors

$$H^*(X) \to H_*(X) \xrightarrow{\Sigma^\infty} SH(X).$$

As an analogy with the topological case, the $\text{Hom}$ in the homotopy category $H_*$ is denoted $[\_\_, \_\_]$.

Finally, recall that the category of Beilinson motives $DM_R(X)$ constructed in [CD09, §14] satisfies Grothendieck’s six functor formalism and it is a full subcategory of $SH(X)_Q$.

### 1.1 Cohomology and its operations.

**Definition 1.1** A ring spectrum is an associative commutative unitary monoid in the stable homotopy category $SH(X)$ of a scheme $X$. Thus, a ring spectrum is a triple $(E, \mu: E \wedge E \to E, \eta: X \to E)$ consisting of a spectrum, the product morphism, and the unit morphism, satisfying the usual conditions of commutative monoids. A morphism of ring spectra is morphism of spectra compatible with the unit and product morphism.

An **absolute ring spectrum** $E$ is a family of ring spectra $E_X$ on $SH(X)$ for every scheme $X$ stable by pullback. In other words, for every morphism $f: Y \to X$ we have fixed an isomorphism $\epsilon_f: f^*E_X \to E_Y$ satisfying the usual compatibility conditions. A **morphism** of absolute ring spectra $\varphi: E \to F$ is a collection of morphism of ring spectra $\varphi_X: E_X \to F_X$ for every scheme $X$ stable by pullback. In other words, for every morphism $f: Y \to X$ we have $\epsilon_f \circ f^*(\varphi_X) = \varphi_Y \circ \epsilon_f^*$. Let $X$ be a scheme. We call a family stable by pullback of spectra $E_Y$ for $Y$ an $X$-scheme an absolute spectra over $X$.

Let $E$ in $SH(X)$ be a ring spectrum, an $E$-**module** is a spectrum $M$ in $SH(X)$ together with a morphism of spectra $v: E \wedge M \to M$ in $SH(X)$ satisfying the usual module condition. Let $\varphi: E \to F$ be a morphism of ring spectra and $(M', v')$ be an $F$-module. A $\varphi$-**morphism** of modules $\Phi: M \to M'$ is a morphism of spectra in $SH(X)$ such that $\varphi' \wedge \varphi \wedge \Phi = \Phi \circ v$.

Let $E$ be an absolute ring spectrum, an $E$**absolute module** is an absolute spectrum $M$ such that $M_X$ is an $E_X$-module for every $X$ and the structural isomorphisms $\epsilon_f$ are isomorphisms of modules. Let $\varphi: E \to F$ be a morphism of absolute ring spectra and $M'$ be an absolute $F$-module. A $\varphi$-**morphism** of absolute modules $\Phi: M \to M'$ is a morphism of absolute spectra such that $\Phi_X$ is a $\varphi_X$-morphism of modules which are stable by pullback. On the following, we may omit the adjective absolute when it is clear by the context and notation.

**Remark 1.2** Every absolute spectrum $E$ is naturally isomorphic to the absolute spectrum obtained by pullback of the spectrum $E_S \in SH(S)$, where $S$ is the base scheme. Instead of considering schemes over a fixed base $S$, one may work over a general category of schemes $S$ without a final object. All definitions and proofs of this paper may be carried into this context using the notion of $S$-absolute spectra (cf. [Dég14]). However, since we will not make use of this generality we have chosen otherwise. In addition, we will abuse notation and call $E_S$ also the absolute spectrum.

**Definition 1.3** Let $X$ be a scheme and $E$ be an absolute spectrum over $X$, we define the $E$-**cohomology** of $X$ to be

$$E^{p,q}(X) = \text{Hom}_{SH(X)}(1_X, E_X(q)[p]) \quad \text{for } (p, q) \in \mathbb{Z}^2$$
and $E(X) = \bigoplus_{p,q} E^{p,q}(X)$. Let $i: Z \to X$ be a closed immersion, we define the $E$-cohomology with support on $Z$ to be

$$E_Z^{p,q}(X) = \text{Hom}_{\text{SH}(X)}(i_*1_Z, E_X(q)[p])$$

for $p, q \in \mathbb{Z}$.

For any $f: T \to X$ and any closed immersion $i: Z \to X$ we define the inverse image of $f$ which maps an element $a: i_*1_Z \to E_X(q)[p]$ in $E_Z^{p,q}(X)$ to the composition

$$f^*(a): f^*i_*1_Z \simeq i'_*1_{Z'} \to f^*(E_X(q)[p]) \simeq E_T(q)[p] \in E_{Z'}^{p,q}(T)$$

where $i': Z' = Z \times_T X \to T$. We denote it $f^*: E_Z(X) \to E_{Z'}(T)$.

**Example 1.4**

- Let $k$ be a perfect field and consider $S = \text{Spec}(k)$. In [CD12, 2.1.4] Cisinski and Déglise defined the notion of mixed Weil theories with coefficients in a field of characteristic zero. In [CD12], every such theory is proved to define a ring spectrum on $\text{SH}(S)_\mathbb{Q}$ stable by pullback and, therefore, an absolute ring spectrum. Recall that there is an algebraic de Rham ring spectrum for $k$ of characteristic zero, an analytic de Rham ring spectrum for $k$ algebraically closed of characteristic zero, and a $\mathbb{Q}_l$ geometric étale ring spectrum for $k$ countable perfect and $l$ a prime different from the characteristic of $k$.

- Let $S = \text{Spec}(\mathbb{Z})$. The $K$-theory absolute ring spectrum $KGL$ is defined in [Voe98] and [Rio10]. By constructions it is periodic, meaning that there are isomorphisms

$$KGL \simeq KGL(i)[2i], \forall i.$$  

It represents Weibel’s homotopy invariant $K$-theory for every scheme (cf. [Cis13, 2.15]), and therefore represents Quillen’s algebraic $K$-theory for regular schemes. We denote the cohomology groups they define as $KH_i$ (\textemdash).

Following [Rio10, 5.3], the $\mathbb{Q}$-localization of the $K$-theory spectrum admits a decomposition induced by the Adams operations, i.e., $KGL_{\mathbb{Q}} = \bigoplus_{i \in \mathbb{Z}} KGL_{\mathbb{Q}}^{(n)}$, where $KGL_{\mathbb{Q}}^{(n)}$ denotes the eigenspaces for the Adams operations. The Beilinson’s absolute motivic cohomology spectrum is defined as $H_{\mathbb{B}} = KGL_{\mathbb{Q}}^{(0)} \in \text{SH}(S)_\mathbb{Q}$ and it is also an absolute ring spectrum. Therefore the Adams operations define an isomorphism

$$\text{ch}: KGL_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{i \in \mathbb{Z}} H_{\mathbb{B}}(i)[2i]$$

(see [Rio10, 5.3.17]). We call this morphism the Chern character since for any regular scheme $X$ it induces the classical higher Chern characters

$$\text{ch}_{r,n}: K_r(X)_\mathbb{Q} \to H^{2n-r}_{\mathcal{M}}(X, \mathbb{Q}(n)).$$

The Chern character is a morphism of absolute ring spectra. In particular $H_{\mathbb{B}}$ is a module over $KGL_{\mathbb{Q}}$.

Finally, Voevodsky’s absolute algebraic cobordism ring spectrum $\text{MGL}$ is constructed out of the Thom spaces of the canonical bundle of Grassmannians ([Voe98]) and its cohomology is called algebraic cobordism.

- In [Spi12] Spitzweck defines the absolute motivic cohomology ring spectrum $H_\Lambda$ with coefficients in $\Lambda$ for schemes over a Dedekind domain $S$. Over a field, this spectrum
coincides with the motivic Eilenberg-MacLane spectrum, so it represents motivic co-

homology. Rationally Spitzweck’s spectrum coincides with the Beilinson’s motivic co-
homology spectrum $H^*_B$. For coherence with notations in $[BMS87]$ we denote the

motivic cohomology groups as $H^*_M(\_\_\_\_, \Lambda(\_\_\_\_))$ for motivic cohomology with coefficients

in $\Lambda$.

- In $[CD15]$ Cisinski and Dégilde defined an absolute ring spectrum representing the
cdh-motivic cohomology in $\text{SH}(\text{Spec } k)_{\mathbb{Z}/p}$ for $k$ a field of exponential characteristic
$p$. This spectrum is isomorphic to the motivic cohomology spectrum.

- Let $S = \text{Spec}(k)$ for $k$ an arithmetic field (cf. Appendix A.2). In his thesis, Drew
constructed the absolute ring spectrum representing absolute Hodge cohomology with
rational coefficients. His construction also holds for any subfield of the real numbers
$[Dre13, 2.1.8]$. In $[HS15]$ Holmstrom and Scholbach defined the Deligne-Beilinson
ring spectrum $E_{D, X} \in \text{SH}(X)_{\mathbb{Q}}$ for $X$ a smooth $S$-scheme. The absolute ring spectrum
$E_{D, S}$ represents the Deligne-Beilinson cohomology with real coefficients. We include in
Appendix A another direct construction of the real Hodge $E_{AH}$ and the real Deligne-
Beilinson absolute ring spectra.

- Let $K$ be a $p$-adic field, $k$ its residue field and $S = \text{Spec}(k)$. The absolute rigid syntomic
ring spectrum $E_{\text{syn}} \in \text{SH}(S)_{\mathbb{Q}}$, which represents Besser’s rigid syntomic cohomology,
is constructed in $[DM14]$.

- Denote $E$ the absolute ring spectrum defined by a mixed Weil theory as in $[CD12]$ (for
example, algebraic and analytic de Rham or $\mathbb{Q}_l$ geometric étale). Cisinski and Dégilde
constructed a morphism of absolute ring spectra

$$\text{cl}: H^*_B \rightarrow E.$$ 

We call this morphism the cycle class. Therefore any mixed Weil spectrum $E$ is a
module over $H_B$. Note that there is an analogous construction of the cycle class for
Besser’s rigid syntomic, absolute Hodge and Deligne-Beilinson spectra (cf. $[DM13]$ and
$[HS15]$).

**Remark 1.5** In the reference $[HS15]$ the Deligne-Beilinson absolute spectrum is proved to
represent the real Deligne-Beilinson cohomology on smooth schemes asking explicitly for the
nonsmooth case. In the Appendix A we check that it also represent the real Deligne-Beilinson
cohomology for general schemes.

**Definition 1.6** Let $M$ be an absolute $E$-module and denote $\mu: E \land M \rightarrow M$ the structure
morphism. Let $Z \rightarrow Y \rightarrow X$ be closed immersions. We call the refined product to the
morphism

$$M_{\mathbb{P}^q}(Y) \times M_{\mathbb{P}^s}(X) \rightarrow M_{\mathbb{P}^q+r+s}(X)$$

\[(m, a) \mapsto m \cdot a\]

defined as follows: let $m: j_*(\mathbb{I}_Z) \rightarrow M_Y(q)[p]$ and $a: i_*\mathbb{I}_Y \rightarrow E_X(s)[r]$, then

$$m \cdot a: i_*j_*\mathbb{I}_Z \rightarrow i_*M_Y(q)[p] \simeq M_X \land i_*\mathbb{I}_Y(q)[p] \xrightarrow{\mu} M_X \land E_X(q+s)[p+r]$$

Note that the same construction defines a product $E_{\mathbb{P}^q}(Y) \times M_{\mathbb{P}^s}(X) \rightarrow M_{\mathbb{P}^q+r+s}(X)$.
Finally, let us recall a generalization of the morphism of forgetting support.

**Definition 1.7** Let \( \mathcal{M} \) be an absolute \( \mathbb{E} \)-module. Consider \( Z \xrightarrow{j} Y \xrightarrow{i} X \) closed immersions, we define a morphism \( j_\flat: \mathcal{M}_Z(X) \to \mathcal{M}_Y(X) \)
as follows. The adjunction morphism \( \text{ad}: \mathbb{I}_Y \to j_* j^* \mathbb{I}_Y \) defines a morphism \( i_*(\text{ad}): i_* (\mathbb{I}_Y) \to i_* j_* j^* \mathbb{I}_Y = (ij)_* \mathbb{I}_Z \). Let \( a: (ij)_* \mathbb{I}_Z \to \mathcal{M}_X \) be in \( \mathcal{M}_Z(X) \), we define \( j_\flat(a) := i_* (\mathbb{I}_Y) \xrightarrow{i_*(\text{ad})} (ij)_* \mathbb{I}_Z \xrightarrow{a} \mathcal{M}_X \in \mathcal{M}_Y(X) \).

The properties one may expect from the morphism of forgetting support and the former product are summarized in the following result, which comes from [Deg14, 1.2.9] for ring spectra.

**Proposition 1.8** Let \( \mathbb{E} \) be an absolute ring spectrum and \( \mathcal{M} \) be an absolute \( \mathbb{E} \)-module:

1. If \( j: Z \to Y \) and \( i: Y \to X \) are closed immersion then \( i_\flat j_\flat = (ij)_\flat \).

2. Consider the cartesian squares

\[
\begin{array}{ccc}
Z' & \xrightarrow{j'} & Y' & \xrightarrow{f'} & X' \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
Z & \xrightarrow{j} & Y & \xrightarrow{g} & X
\end{array}
\]

where the horizontal arrows are closed immersions. Then, for any \( \rho \in \mathcal{M}_Z(X) \) we have \( f^* j_\flat(\rho) = j'_\flat f^*(\rho) \).

3. With the preceding notations, for any pair \( (a, m) \in \mathbb{E}_Z(Y) \times \mathcal{M}_Y(X) \) we have \( f^*(a \cdot m) = g^*(a) \cdot f^*(m) \) and \( j_\flat(a \cdot m) = j_\flat(a) \cdot m \). Analogous formulas hold classes in \( \mathcal{M}_Z(Y) \times \mathbb{E}_Y(X) \).

4. Consider closed immersions \( T \to Z \to Y \to X \). Then for any triple \( (a, b, m) \in \mathbb{E}_T(Z) \times \mathbb{E}_Z(Y) \times \mathcal{M}_Y(X) \) we have \( a \cdot (b \cdot m) = (a \cdot b) \cdot m \). Analogous formulas hold for classes in \( \mathbb{E}_T(Z) \times \mathcal{M}_Z(Y) \times \mathbb{E}_Y(X) \) and \( \mathcal{M}_T(Z) \times \mathbb{E}_Z(Y) \times \mathbb{E}_Y(X) \).

5. Consider the commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{j'} & Y' \\
\downarrow & & \downarrow \\
Z & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \\
& X
\end{array}
\]

made of closed immersions and such that the square is cartesian. Then for any \( (a, m) \in \mathbb{E}_Z(Y) \times \mathcal{M}_Y(X) \) the relation \( h_\flat (g^*(a) \cdot m) = a \cdot g_\flat(m) \) holds. Analogous formulas hold for classes in \( \mathcal{M}_Z(Y) \times \mathbb{E}_Y(X) \).
1.2 Arithmetic and relative cohomologies

We introduce the notion of relative cohomology of a morphism in the context of stable homotopy theory of schemes and review Holmstrom-Scholbach’s construction the arithmetic counterparts of higher $K$-theory and motivic cohomology. Both constructions define modules over cohomology theories. We use the theory of monoids and modules in model categories. This theory can be found written in the context of motivic homotopy theory in [CD09, §7] and a more accessible summary for $SH$ in [Dég13, §2.2].

**Definition 1.9** We say that a ring spectrum, an absolute ring spectrum, a module, an absolute module, or a morphism is **strict** if it is defined in the category of symmetric spectra $Spt_Σ(X)$ and all diagrams commute in $Spt_Σ(X)$ and not just in $SH(X)$. Thus, a strict ring spectrum $E$ is a commutative monoid in the category $Spt_Σ(X)$ and a morphism of strict ring spectra is morphism of strict ring spectra. Let $E$ be a strict ring spectrum, we denote $E$-mod the category of strict $E$-modules with strict morphisms of modules.

**Example 1.10** The spectra from Example 1.4 representing cohomologies are strict. Find a reference for the algebraic $K$-theory spectrum $KGL$ in [CD09, 13.3.1], for $H_{BU}$ in [CD09, 14.2.6], for mixed Weil theories, Deligne-Beilinson and absolute Hodge spectra one checks that the product and unit morphism from [CD12, 2.1.5] for mixed Weil theories and from [DM14, 1.4.10] for Deligne-Beilinson and absolute Hodge spectra satisfy the monoid axioms. Moreover, the Chern character and the cycle class map are morphism of strict ring spectra [CD09, 14.2.16]. Spitzweck’s motivic cohomology spectrum is also strict ([Spi12, p.4]).

**Remark 1.11** The categories $Mon(X)$ and $E$-mod inherit a model structure from the $A^1$-stable symmetric model structure in $Spt_Σ(X)$. The categories $Ho(Mon(X))$ are well behaved with respect to inverse and direct image as described in [CD09, 7.1.11]. We will use the following fact: let $f: Y → X$ be a morphism of schemes, $E ∈ Mon(Y)$ and $F ∈ Mon(X)$, then $f_*E$ in $SH(X)$ and $f^*E$ in $SH(Y)$ are given by strict ring spectra.

Let $E$ be a strict absolute ring spectrum. The categories $Ho(E_X$-mod) also have good functorial properties. Moreover, they are triangulated categories and the forgetful functor

$$Ho(E_X$-mod) → SH(X)$$

is triangulated.

Let $φ: E → F$ be a morphism of strict absolute ring spectra and $X$ be a scheme. Denote $φ_X: E_X → F_X$ the morphism of strict ring spectra in $Spt_Σ(X)$ and hofib($φ_X$) the homotopy fiber of $φ_X$. The spectrum hofib($φ_S$) in $Spt_Σ(S)$ defines by pullback an absolute spectrum, which we denote hofib($φ$). Recall that the homotopy fiber fits into a distinguished triangle. In other words, for $X$ an $S$-scheme we have that

$$hofib(φ_X) → E_X → F_X → hofib(φ_X)[1].$$

Since $E_X$-mod is a model category triangulated and $φ$ is a morphism of absolute spectra the following result is straightforward.

**Proposition 1.12** Let $φ: E → F$ be a morphism of strict absolute ring spectra. With above notations, hofib($φ$) is a strict absolute $E$-module and for every $S$-scheme $X$ we have hofib($φ)_X = hofib(φ_X)$. 

9
Remark 1.13 Still with above notations, after a replacement we can assume \( \varphi_S \) to be a fibration and \( F_S \) to be fibrant so that hofib(\( \varphi_S \)) fits into a cartesian square

\[
\begin{array}{ccc}
\text{hofib}(\varphi_S) & \longrightarrow & E_S \\
\downarrow & & \downarrow \varphi_S \\
* & \longrightarrow & F_S.
\end{array}
\]

Note that the replacement is functorial so we have a commutative diagram

\[
\begin{array}{ccc}
\text{hofib}(\varphi_S) \wedge \text{hofib}(\varphi_S) & \longrightarrow & E_S \wedge E_S \\
\downarrow & & \downarrow \mu \\
\text{hofib}(\varphi_S) & \longrightarrow & E_S \wedge \phi \longrightarrow F_S.
\end{array}
\]

Therefore, the groups \( \text{Hom}_{\text{SH}(X)}(X, \text{hofib}(\varphi_X)) \) not only are modules over \( E(X) \) but also have an inner product. Note that they do not have a unit. The distinguished triangle (1) gives rise to a long exact sequence

\[
\cdots \rightarrow F^{*-1}(X) \rightarrow \text{Hom}_{\text{SH}(X)}(X, \text{hofib}(\varphi_X)[*]) \rightarrow E^{*-1}(X) \rightarrow F^{*}(X) \rightarrow \cdots
\]

where arrows are compatible with products.

We introduce the relative cohomology in the context of motivic homotopy. Let \( f: Y \rightarrow X \) be a morphism of schemes, then \( f_*E_Y \) represents in \( \text{SH}(X) \) the cohomology of \( Y \). Indeed,

\[
E^{*-1}(Y) = \text{Hom}_{\text{SH}(Y)}(f_1Y, E_Y(*))[*]) = \text{Hom}_{\text{SH}(X)}(f_X, E_X(*)[*]).
\]

Since \( E_Y \simeq f^*E_X \) we have an adjunction morphism \( E_X \rightarrow f_1f^*E_X \). Let us remark two properties of this morphism.

Proposition 1.14 Let \( E \) be an absolute ring spectrum and \( f: Y \rightarrow X \) be a morphism of schemes:

1. The spectrum \( f_*E_Y \) is a ring spectrum. The adjunction \( E_X \rightarrow f_*E_Y \) is a morphism of ring spectra and it induces the inverse image on cohomology \( f^*: E(X) \rightarrow E(Y) \).

2. If in addition \( E \) is strict, then \( f_*E_Y \) is also strict and the adjunction map \( E_X \rightarrow f_*E_X \) is represented by a morphism of strict ring spectra.

\[
\square
\]

Definition 1.15 Let \( E \) be a strict absolute ring spectrum, \( f: Y \rightarrow X \) be a morphism of schemes. Abuse notation and denote hofib\( E(f_X) \) (or simply hofib\( f_X \)) if it is clear by the context) the strict \( E_X \)-module defined as the homotopy fiber of the strict morphism of ring spectra \( E_X \rightarrow f_*E_Y \). We define the relative cohomology of \( f \) to be

\[
E^{p,q}(f) := \text{Hom}_{\text{SH}(X)}(f_X, f_{*E}(f_X)(q)[p]) \quad \text{for } p, q \in \mathbb{Z}.
\]

We also denote hofib\( E(f) \) (or simply hofib\( f \)) the strict absolute \( E \)-module over \( X \) that hofib\( E(f_X) \) defines by pullback.
Remark 1.16 Consider the above notations. Note that, although $E$ is an absolute spectrum, the spectrum $\text{hofib}(f)$ need not to have good functorial properties. More concretely, consider a cartesian square

$$
\begin{array}{ccc}
Y_T & \longrightarrow & Y \\
\downarrow f_T & & \downarrow f \\
T & \underset{g}{\longrightarrow} & X.
\end{array}
$$

Then the spectrum $g^* f_* E_Y$ may not be isomorphic to $f_T_* E_{Y_T}$. In other words, the family of spectra $f_T_* E_{Y_T}$ for $T \to X$ may not define an absolute spectrum. Therefore $\text{hofib}(f_T)$ may not be isomorphic to $\text{hofib}(f)_T = g^* \text{hofib}(f_X)$ so that $\text{hofib}(f)$ may not represent the relative cohomology of $f_T$.

Proposition 1.17 Let $E$ be a strict absolute spectrum, $f: Y \to X$ and $g: T \to X$ be two morphism of schemes. If either $f$ is proper or $g$ is smooth then we have

$$
g^* \text{hofib}(f_X) \simeq \text{hofib}(f_T).
$$

Proof: Denote $g': Y_T \to Y$ and $f_T: Y_T \to T$. It is enough to prove that $g^* f_* E_Y \simeq f_T^* g'^* E_Y$. The result follows from the smooth and proper base change in stable homotopy (cf. [CD09, 1.1.19 and 2.4.50]).

□

Theorem 1.18 Let $f: Y \to X$ be a morphism of regular schemes and denote $K(f)$ the relative algebraic $K$-theory (cf. [Wei13, IV.8.5.3]). Then

$$
K_i(f) = \text{Hom}_{\text{SH}(X)}(\mathbb{1}_X, \text{hofib}_{KGL}(f_X)[-i]) \quad \text{for } i \in \mathbb{Z}.
$$

Proof: We use notation from [CD09]. Recall from [CD09, §3.2] that there is total derived global section functor

$$
\text{RF} : \text{SH}(X) \to \text{Ho}(\text{Spt}_{S^1}).
$$

where $\text{Spt}_{S^1}$ denotes the classic category of $S^1$-spectra of simplicial sets. Recall from [CD09, 13.4] that $K_n(X) = \pi_n(\text{RG}(X, KGL_X)) = \text{Hom}_{\text{SH}(X)}(\mathbb{1}_X, KGL_X)$. Applying the total derived global section functor to the homotopy fiber sequence

$$
\text{hofib}_{KGL}(f_X) \to KGL_X \to f_* KGL_Y
$$

we obtain a (classic) homotopy fiber sequence

$$
\text{RG}(X, \text{hofib}_{KGL}(f_X)) \to \text{RG}(X, KGL_X) \to \text{RG}(X, f_* KGL_Y).
$$

We conclude by recalling that the relative $K$-theory of $f$ is defined in [Wei13, IV.8.5.3] as the classic homotopy fiber.

□

Example 1.19 Let $E$ be a strict absolute ring spectrum. The construction of the relative cohomology generalizes many concrete situations:

1. Let $S = \text{Spec}(k)$ and $p: X \to S$ be the structural morphism. Then $E(p) = \overline{E}(X)$, the reduced cohomology of $X$. 11
2. Let $i: Z \to X$ be a closed immersion with open complement $j: U \to X$. Recall from [CD09 2.3.3] that we have a distinguished triangle
\[ i_*i^!E_X \to E_X \to j_*j^*E_X \to i_*i^!E_X[1]. \]
Therefore $hofib(j) \simeq i_*i^!E_X$ in $\mathbf{SH}(X)$ and $E(j) = E_Z(X)$. The product as $E$-module is the same of Definition 1.6.

3. Consider the above notations. Then we also have a distinguished triangle
\[ j_!j^*E_X \to E_X \to i_*i^*E_X \to j_!j^*E_X[1] \]
so that $hofib(i) \simeq j_!j^*E_X$. Although we have not reviewed it, $j_!j^*E_X$ represents, by definition, the cohomology of $U$ with compact support $\mathbb{E}_c(U)$ in $\mathbf{SH}(X)$.

4. Let $i: Z \to X$ be a closed immersion and consider the blow-up cartesian square
\[
\begin{array}{ccc}
P & \to & B_Z X \\
\pi' \downarrow & & \downarrow \pi \\
Z & \to & X.
\end{array}
\]

It follows from upcoming Corollary 2.6 that if in addition $E$ is oriented (cf. [Il22]), then $E(\pi) = E(P)/E(Z)$. The product as $E(X)$-module is the product through $(i\pi')^*: E(X) \to E(P)$.

5. Let $R$ be a Dedekind domain and $F$ be its field fractions. Denote $K(R) = K(\text{Spec}(R))$, $K(F) = K(\text{Spec}(F))$ and $\gamma: \text{Spec}(F) \to \text{Spec}(R)$ the localization morphism. Then $K_2(\gamma) = \prod_p K_2(R/p)$ where $p$ denote prime ideals of $R$ and $K_1(\gamma) = \prod_p (R/p)^\times$ (cf. [Wei13 III.6.5]).

We review the construction of the arithmetic counterparts of $K$-theory and motivic cohomology of $[HS15]$, which are another example of a homotopy fiber.

Let $A$ be an arithmetic ring (cf. Appendix A.2) and $S = \text{Spec}(A)$ and $\eta$ its generic point. Recall from Example [1.4] that we have the Deligne-Beilinson cohomology strict ring spectrum $\mathbb{E}_{D,\eta} \in \mathbf{SH}(\eta)$, which defines a strict absolute ring spectrum $\mathbb{E}_{D,\eta} \in \mathbf{SH}(S)$. Recall from Example [1.4] that we have the cycle class map $\text{cl}: H_{B,\eta} \to \mathbb{E}_{D,\eta}$ which induces a strict map
\[ \varphi: H_{B,S} \to \eta_*H_{B,\eta} ^{n,\text{cl}} \to \eta_*\mathbb{E}_{D,\eta}. \]

Recall that in $\mathbf{SH}(S)$ we have $\text{KGL}_{S,Q} = \bigoplus_{i \in \mathbb{Z}} H_{B,S}(i)[2i]$ (cf. [CD09 §14]). We have a strict map
\[ \oplus(\text{ch}_i \circ \varphi_i): \text{KGL}_{S,Q} \to \eta_*\mathbb{E}_{D,\eta}(i)[2i] \]
where $\varphi_i: H_{B,S}(i)[2i] \to \eta_*\mathbb{E}_{D,\eta}(i)[2i]$.

**Definition 1.20** In above notations, we define the arithmetic motivic cohomology strict absolute spectrum as $\widehat{H}_{B,S} = hofib(\varphi)$. Let $X$ be a smooth $S$-scheme, we denote the cohomology it defines as
\[ \widehat{H}^p_{MB}(X, q) := \text{Hom}_{\mathbf{SH}(X)}(1_X, \widehat{H}_{B,X}(q)[p]) \quad \text{for} \ q, p \in \mathbb{Z}. \]

Analogously, we define the arithmetic $K$-theory strict absolute spectrum as $\widehat{\text{KGL}}_{S,Q} = hofib(\varphi(\text{ch}_i \circ \varphi_i))$. Note that the periodicity of the $K$-theory makes $\widehat{\text{KGL}}$ also periodic. Let $X$ be a smooth $S$-scheme, we denote the cohomology it defines as
\[ \widehat{\text{KH}}^i(X)_Q := \text{Hom}_{\mathbf{SH}(X)}(1_X, \widehat{\text{KGL}}_Q[-i]) \quad \text{for} \ i \in \mathbb{Z}. \]
and we say that $c$ given by $(E, c)$ to the scheme $P$ oriented $c$ we have a natural map $L$-bundle modules respectively, but not rings. Nevertheless, the Chern character $\text{ch}: \text{KGL} \rightarrow \text{H}$ induces an arithmetic Chern character $\hat{\text{ch}}: \hat{\text{KGL}} \rightarrow \hat{H}$ and the square

$$
\begin{array}{ccc}
\hat{\text{KGL}}_Q & \longrightarrow & \text{KGL}_Q \\
\hat{\text{ch}} & & \text{ch} \\
\hat{H}_B & \longrightarrow & \text{H}_B 
\end{array}
$$

commutes.

## 1.3 Orientations

We review the theory of orientations (i.e., Chern classes) for spectra. As in the classical case, they are determined by the first Chern class of the tautological line bundle of projective spaces.

Recall that $I_S(1) = \text{coker}(\Sigma^\infty S \rightarrow \Sigma^\infty \mathbb{P}^1)[-2]$. For any ring spectrum $E$ with unit $\eta: S \rightarrow E_S$ there is a canonical class in $E^{2,1}(\mathbb{P}^1)$ defined as the morphism

$$
P^1 \rightarrow I_S(1)[2] = S \wedge I_S(1)[2] \eta^{\text{Ad}} \rightarrow E_S(1)[2].
$$

By abuse of notation we still denote it $\eta \in E^{2,1}(\mathbb{P}^1)$.

The definition of $E$-cohomology may be extended to general spectra. In particular, recall that the infinite projective space is defined to be $\mathbb{P}^\infty = \varprojlim \mathbb{P}^n$ and we denote

$$
E^{p,q}(\mathbb{P}^\infty) = \text{Hom}_{\text{SH}(X)}(\mathbb{P}^\infty, E_X(q)[p]).
$$

**Definition 1.23** We define an orientation on an absolute ring spectrum $E$ to be a class $c_1 \in E^{2,1}(\mathbb{P}^\infty)$ such that for $i_1: \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ satisfies $i_1^*(c_1) = \eta$. We also say that $E$ is oriented.

Let $X$ be a scheme and $V$ be a locally free $O_X$-module. We call the vector bundle given by $V$ to the scheme $V = \text{Spec}_X(S^*V^*) \rightarrow X$ and the projective bundle given by $V$ to the scheme $\mathbb{P}(V) = \text{Proj}_X(S^*V^*) \rightarrow X$.

Let $BG_m$ be the classifying space for $G_m$-torsors ([MV99, 4.1.16]), due to [MV99, 4.3.7] we have a natural map

$$
\text{Pic}(X) \rightarrow \text{Hom}_{\text{H}_*(X)}(X^+, BG_m) \simeq \text{Hom}_{\text{H}_*(X)}(X^+, \mathbb{P}^\infty) \rightarrow \text{Hom}_{\text{SH}(X)}(\mathbb{I}_X, \mathbb{P}^\infty)
$$

so that any line bundle $L \in \text{Pic}(X)$ defines a morphism $f: X \rightarrow \mathbb{P}_X^\infty$ in $\text{SH}(X)$.

Let $(E, c_1)$ be an oriented absolute ring spectrum and denote $p: \mathbb{P}_X^\infty \rightarrow \mathbb{P}^\infty$. For any line bundle $L$ we have

$$
E^{2,1}(\mathbb{P}^\infty) \xrightarrow{f^*} E^{2,1}(X) \xrightarrow{p^*c_1} c_1(L)
$$

and we say that $c_1(L) := f^*p^*c_1$ is the first Chern class of $L$. 

**Remark 1.21** Note that our definition is written differently from [HS15], where they considered the spectra

$$
\text{hofib}(H_{B,S} \xrightarrow{id \times \eta} H_{B,S} \wedge \eta \in \mathbb{E}_{D,n}) \text{ and } \text{hofib}(KGL_{S} \xrightarrow{id \times \eta} KGL_{S} \wedge \eta \in \mathbb{E}_{D,n}).
$$

Recall that in $\text{SH}(S)_Q$ we have $H_{B,S} \wedge \eta \in \mathbb{E}_{D,n} \simeq \eta \in \mathbb{E}_{D,n}$ (cf. [CD09, 14.2.8]) so that both definitions agree.

**Remark 1.22** By construction, both $\hat{H}_B$ and $\hat{KGL}_Q$ are strict absolute $H_B$ and $KGL_Q$ modules respectively, but not rings. Nevertheless, the Chern character $\text{ch}: \text{KGL}_Q \rightarrow H_B$ induces an arithmetic Chern character $\hat{\text{ch}}: \hat{\text{KGL}}_Q \rightarrow \hat{H}_B$ and the square

$$
\begin{array}{ccc}
\hat{\text{KGL}}_Q & \longrightarrow & \text{KGL}_Q \\
\hat{\text{ch}} & & \text{ch} \\
\hat{H}_B & \longrightarrow & \text{H}_B 
\end{array}
$$

commutes.
Example 1.24 Every example of Example 1.4 representing a cohomology is oriented. We review the references: Mixed Weil theories are oriented in [CD09, 2.2.8], the algebraic K-theory KGL and Beilinson’s motivic cohomology $H_B$ in [CD09, 13.2.2] and [CD09, 14.1.5] respectively, algebraic cobordism $MGL$ in [PPR08, 1.4]. In particular, the orientation of $K$-theory is given by $c_1 KGL(L) = 1 - [L^*]$. Spitzweck’s motivic cohomology spectrum $H_A$ is oriented in [Spi12, 11.1]. In the Appendix A we give an orientation for the absolute Hodge spectrum and the Deligne-Beilinson is done in [HS15, 3.6]. Finally, every cohomology considered in [DM14] is represented by an oriented spectrum (cf. [DM14, 1.4.11,(1) and 2.1.2.(1)]). In particular, Besser’s absolute rigid syntomic spectrum is oriented.

Remark 1.25 Let $\varphi: E \to F$ be a morphism of absolute ring spectra and let $c_1 \in \mathbb{E}^{2,1}(\mathbb{F})$ be an orientation on $E$. Since $\varphi$ is a morphism of rings, it maps the unit $1_S \to \mathbb{E}_S$ onto the unit $1_S \to \mathbb{F}_S$. We conclude that the element $\varphi_{\mathbb{F}}(c_1) \in \mathbb{E}^{2,1}(\mathbb{F})$ is an orientation on $F$.

Remark 1.26 Denote $i_n: \mathbb{P}^n \to \mathbb{P}^\infty$ since $i_n^*(\mathcal{O}(-1)) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$ we have that $i_n^*(c_1) = c_1(\mathcal{O}_{\mathbb{P}^n}(-1))$ and we write $c_1 = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$.

To fix notations, we recall the theory of Chern classes in the context of spectra. Proof of the following result in the context of stable homotopy theory may be found in [Dég14, 2.1.13 and 2.1.22].

Theorem 1.27 (Projective bundle) Let $V \to X$ be a vector bundle of rank $(n+1)$, $E$ an oriented absolute ring spectrum and $x = c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))$. There is a canonical isomorphism

\[ \bigoplus_{i=0}^n \mathbb{E}^{s-2i,s-i}(X) \quad \xrightarrow{\sim} \quad \mathbb{E}^{s,s}(\mathbb{P}(V)) \]

\[ (a_0, \ldots, a_n) \quad \mapsto \quad \sum_i \pi^*(a_i) x^i. \]

\[ \square \]

Definition 1.28 Let $V \to X$ be a vector bundle of rank $n$. We define the $i$-th Chern classes $c_i(V) \in \mathbb{E}^{2i,i}(X)$ for $i = 1, \ldots, n$ as the unique ones satisfying

\[ c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))^n + \sum_{i=1}^n (-1)^i c_i(V) c_1(\mathcal{O}_{\mathbb{P}(V)}(-1))^{n-i} = 0. \]

1.29 Formal (abelian) group laws $F(x, y)$ are certain type of series (cf. [Ada74]). In particular, they satisfy the property that any formal group law $F(x, y)$ is of the form

\[ F(x, y) = x + y + xyf(x, y) \]

for $f(x, y)$ a symmetric series. A formal group law is called additive if $F(x, y) = x + y$. The following is a classic result. See for example [Dég08, 3.7].

Theorem 1.30 Let $E$ be an oriented absolute ring spectrum. There exists a formal group law $F(x, y) \in \mathbb{E}^{s,s}(\mathbb{S}[[x, y]])$ such that

\[ c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) \]

for any line bundles $L_1, L_2$ over $X$. In addition, Chern classes are nilpotent.

\[ \square \]

See [NSO09, 6.2] for a proof of the following result in our context.
Proposition 1.31 Let \((E, c_1)\) be an oriented absolute ring spectrum and denote \(E(\mathbb{P}^\infty) = \prod \mathbb{E}^{p,q}(\mathbb{P}^\infty)\), then

\[ E(\mathbb{P}^\infty) = E(S)[[c_1]]. \]

\[ \square \]

Example 1.32 Every cohomology from Example [3] apart from \(K\)-theory and algebraic cobordism have additive formal group laws. That is to say: motivic cohomology, cohomologies coming from mixed Weil theories, real absolute Hodge and Deligne-Beilinson cohomology and Besser’s rigid syntomic cohomology have additive formal group laws.

Standard arguments yield the following classic formula:

Theorem 1.33 (Whitney sum) Let \(E\) be an oriented absolute ring spectrum and \(0 \to V' \to V \to V'' \to 0\) be a short exact sequence of vector bundles, then we have

\[ c_k(V) = \sum_{i+j=k} c_i(V')c_j(V'') \quad i, j \in \mathbb{N}. \]

\[ \square \]

Proposition 1.34 Let \((E, c_1)\) be an oriented absolute spectrum and \(c_1^{\text{new}}\) be another orientation. Then there exists \(G(t) \in E(S)[[t]]\) with leading coefficient 1 such that for any line bundle \(L\) we have

\[ c_1^{\text{new}}(L) = G(c_1(L))c_1(L). \]

Proof: Since \(\mathbb{P}^\infty \cong B\mathbb{G}_m\), the classifying space for line bundles, it is enough to check the formula for \(x = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))\). Recall that \(E(\mathbb{P}^\infty) = E(S)[[x]]\) and therefore we have \(c_1^{\text{new}}(\mathcal{O}_{\mathbb{P}^\infty}(-1)) = a_0 + a_1x + \ldots = G(x)x\). Finally, both classes satisfy \(i_1^* (c_1(\mathcal{O}_{\mathbb{P}^\infty})) = i_1^* (c_1^{\text{new}}(\mathcal{O}_{\mathbb{P}^\infty})) = \eta \in E^{2,1}(\mathbb{P}^1)\) and we conclude that \(a_0 = 0\) and \(a_1 = 1\).

\[ \square \]

1.35 We recall in our context the multiplicative extensions of a series. Let \(E\) be an absolute oriented ring spectrum and \(F(t) \in E(S)[[t]]^\times\) be an invertible series. Let \(L \to X\) be a line bundle, we set

\[ F(L) := F(c_1(L)) \in E(X) \]

(it is well defined since Chern classes are nilpotent). Let \(V \to X\) be a vector bundle, by the splitting principle we can assume \(V = L_1 + \ldots + L_n\). We put

\[ F_\times(V) := F(L_1) \cdot \ldots \cdot F(L_n) \in E(X) \]

where we consider \(F(L_1) \cdot \ldots \cdot F(L_2)\) as a power series in the symmetric elemental functions of \(c_1(L_1), \ldots, c_1(L_n)\), which are just the Chern classes of \(E\). We call \(F_\times(T)\) the multiplicative extension of \(F\).

Corollary 1.36 Let \((E, c_1)\) be an oriented absolute spectrum and \(c_1^{\text{new}}\) be another orientation on \(E\) and denote \(G(t) \in E(S)[[t]]^\times\) the series such that \(c_1^{\text{new}}(L) = G(c_1(L))c_1(L)\). Then

\[ c_n^{\text{new}}(V) = G_\times(V)c_n(V) \]

for \(V\) a rank \(n\) vector bundle.

\[ \square \]
1.4 Chern class with support

**Definition 1.37** Let $X$ be a scheme and $U = (X - Z) \to X$ be an open subscheme. We call a **pseudo divisor** (trivialized on $U$) a pair $(\mathcal{L}, u)$ consisting of an invertible sheaf $\mathcal{L}$ over $X$ (in the Zariski topology) and a trivialization $u: \mathcal{O}|_U \tilde{\to} \mathcal{L}|_U$. We denote $\text{Pic}_Z(X)$ the group of isomorphism classes of pseudo divisors.

The same arguments of [MV99, §4.1.3] give the following remark:

**Remark 1.38** Let $Z \to X$ be a closed immersion of codimension 1 and $U$ be its open complement. Then there is a map $\text{Pic}_Z(X) \to \text{Hom}_{\text{SH}(X)}(\Sigma^\infty X/U, \mathbb{P}_X^\infty) = \mathbb{E}^2_1(X)$. It follows from above identification that there is a map $\text{Pic}_Z(X) \to \text{Hom}_{\text{SH}(X)}(\Sigma^\infty X/U, \mathbb{P}_X^\infty) \to \text{Hom}_{\text{SH}(X)}(\Sigma^\infty X/U, \mathbb{E}_X(1)[2]) = \mathbb{E}^2_1(X)$. Finally, for an oriented absolute spectrum $\mathbb{E}$ and any line bundle $L$ we have $\text{Pic}_Z(X) \xrightarrow{\varphi^L_{\mathcal{L}, u}} \mathbb{E}^2_1(X)$ and we say that $c_1^Z(L, u) := \varphi^L_{\mathcal{L}, u}(L, u)$ is the first Chern class of $L$ with support on $Z$. We omit the reference to the trivialization when no confusion is possible. The next statement follows from the definition:

**Proposition 1.39** Let $f: X' \to X$ be a morphism of schemes and $(\mathcal{L}, u)$ be a pseudo divisor over the open $X - Z$, then $f^*c_1^Z(L) = c_1^{f^{-1}Z}(f^*L)$.

**Proposition 1.40** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves over $X$ and $u_i: \mathcal{O}|_U \to \mathcal{L}_i|_U$, $i = 1, 2$, trivializations. Denote $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $u = u_1 \otimes u_2$. Let $F(x, y) \in \mathbb{E}(S)[[x, y]]$ be the formal group law of $c_1$ given by Theorem 1.30. Then $c_1^Z(L) = F(c_1^Z(L_1), c_1^Z(L_2))$.

**Proof:** The pseudo divisors $(L_1, u_1)$, $(L_2, u_2)$ and $(L, u)$ correspond to morphism $f_1$, $f_2$, $f: X/U \to \mathbb{P}_X^\infty$ respectively. Denote the Segre embedding $\sigma: \mathbb{P}_X^\infty \times \mathbb{P}_X^\infty \to \mathbb{P}_X^\infty$. By construction, the diagram

$$
\begin{array}{ccc}
X/U & \xrightarrow{f_1 \times f_2} & \mathbb{P}_X^\infty \times \mathbb{P}_X^\infty \\
\downarrow{f} & & \downarrow{\sigma} \\
\mathbb{P}_X^\infty & & 
\end{array}
$$

16
commutes and, taking $E$-cohomology, we have the commutative diagram

\[
\begin{array}{ccc}
E(\mathbb{P}^\infty_X) & \xrightarrow{\sigma^*} & E(\mathbb{P}^\infty_X \times \mathbb{P}^\infty_X) \\
\downarrow f^* & & \downarrow (f_1 \times f_2)^* \\
E_Y(X) & & E_Y(X).
\end{array}
\]

Recall that $E(\mathbb{P}^\infty) = E(S[[t]])$ and $E(\mathbb{P}^\infty \times \mathbb{P}^\infty) = E(S[[u,v]])$ where $t = c_1(O_{\mathbb{P}^\infty}(-1))$, $u = c_1(p_1^*O_{\mathbb{P}^\infty}(-1))$, $v = c_1(p_2^*O_{\mathbb{P}^\infty}(-1))$ and $p_1, p_2$ are the canonical projections. With this notations, the Segre morphism maps $t \mapsto F(u,v)$ where $F$ is the formal group law of the orientation of $\mathbb{E}$. We conclude by the commutativity last diagram.

\[\square\]

2 Gysin morphism

2.1 Regular immersions

We construct the Gysin morphism for a regular immersion following Gabber’s ideas for étale cohomology (see [Fuj02] and [Rio07]). More concretely, we lift the approach of Riou in [Rio07] for étale cohomology to the motivic homotopy setting. The main difference lies in the fact that étale cohomology has additive Chern classes, that is $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ for $L_1$ and $L_2$ line bundles (cf. Paragraph [1.29]), while in the general setting Chern classes are not additive (cf. Example [1.24]).

Gabber’s method reduces the case of general codimension to that of codimension one by means of describing the cohomology of the blow-up (cf. Corollary [1.20]). In order to prove the needed functoriality properties the versatile context of modified blow-up is used.

**Definition 2.1** Let $i : Z \to X$ be a regular immersion of codimension 1. The sheaf $\mathcal{I}_Z = \mathcal{L}_Z$ is locally principal and it has a natural trivialization on $X - Y$, and so does its dual. We define the **refined fundamental class** (of $Z$ in $X$) to be

\[\bar{\eta}^X_Z := c_1^X(\mathcal{I}_Z^2) = c_1^X(L_Z) \in E^{2,1}_Z(X)\]

and the **fundamental class** to be $\eta^X_Z := c_1(L_Z) = i_*(c_1^X(L_Z)) \in E^{2,1}(X)$. We define the **refined Gysin morphism** as

\[p_i : \ E^{*,*}(Z) \to E^{*,2,*,+1}_Z(X)\]

\[a \mapsto a \cdot \bar{\eta}^X_Z\]

and the **Gysin morphism** as $i_* : E^{*,*}(Z) \to E^{*,2,*,+1}(X)$, $a \mapsto i_*(a \cdot \bar{\eta}^X_Z)$.

More generally, let $(\mathcal{L}, u : \mathcal{O}|_U \simeq \mathcal{L}|_U)$ be a pseudo divisor where $U = X - Z$. We define the **refined Gysin morphism (given by $(\mathcal{L}, u)$)** as

\[p_{\mathcal{L}} : \ E^{*,*}(Z) \to E^{*,2,*,+1}(X)\]

\[a \mapsto a \cdot c_1^X(\mathcal{L}).\]

**Remark 2.2** Let $Z \hookrightarrow X$ be a regular immersion of codimension 1. Due to Proposition [1.8] it is easy to check that

\[c_1^X(L_Z) \cdot c_1(N_{Z/X}) = c_1^X(L_Z)^2\]

where $c_1^X(L_Z) \in E^{2,1}_Z(X), N_{Z/X} = \text{Spec}(S^{\bullet}(\mathcal{I}_Z/\mathcal{I}_Z^2)^*)$ and $c_1(N_{Z/X}) \in E^{2,1}(Z)$. 17
We now define a refined fundamental class for any closed subscheme \( Z \to X \) and any epimorphism \( F^* \to I_Z/I_Z^2 \) where \( F \) is a locally free \( \mathcal{O}_Z \)-module. This more general context, due to Gabber, is the suitable one to prove the basic properties of the Gysin morphism.

**Definition 2.3** Let \( Z \to X \) be a closed immersion of strictly positive codimension defined by a sheaf of ideals \( I_Z \). Let \( F^* \to I_Z/I_Z^2 \) be an epimorphism of \( \mathcal{O}_Z \)-modules where \( F \) is locally free and consider the \( \mathcal{O}_X \)-graded algebra \( A = \bigoplus A_n \) defined on each degree as the fibre product

\[
\begin{array}{ccc}
A_n & \longrightarrow & I_Z^n \\
\downarrow & & \downarrow \\
S^n F^* & \longrightarrow & I_Z^n/I_Z^{n+1}.
\end{array}
\]

We define the **modified blow-up** as the projective scheme \( B_{Z,F}X := \text{Proj}_X(A) \).

See [Rio07, 2.2.1.3; 2.2.1.4 and 2.1.5] for a proof of the following properties of the modified blow-up:

**Proposition 2.4** Let \( \pi : B_{Z,F}X \to X \) be a modified blowing-up:

1. The epimorphism \( A \to \bigoplus I_Z^n/I_Z^{n+1} \) defines a closed embedding \( B_ZX \to B_{Z,F}X \).
2. If \( Z \to X \) is a regular immersion and \( F^* = I_Z/I_Z^2 \) then \( B_{Z,F}X = B_ZX \) is the classic blow-up \( B_ZX \).
3. Denote \( U = X - Z \), then \( B_{Z,F}X|_{\pi^{-1}(U)} \simeq U \).
4. \( \pi^{-1}(Z) = \mathbb{P}(F) = \text{Proj}_Z(S^*F) \).
5. For any morphism \( p: X' \to X \) there is a canonical morphism

\[
B_{Z',F'}X' \longrightarrow B_{Z,F}X \times_X X'
\]

which is a nil-immersion.

\[ \square \]

**Proposition 2.5** Let \( i : Z \to X \) be a closed immersion and let \( F^* \to I_Z/I_Z^2 \) be an epimorphism of \( \mathcal{O}_Z \)-modules where \( F \) is locally free. Let \( B = B_{Z,F}X \) be the modified blowing up, \( \pi : B \to X \) the canonical morphism and \( P = \mathbb{P}(F) \) the exceptional divisor. Then for any \( q \in \mathbb{Z} \) there is a long exact sequence

\[
\cdots \to E^q_Z(X) \to E^q_P(B) \oplus E^q(Z) \to E^q(P) \to E^{q+1}_Z(X) \to \cdots
\]

The same long exact sequence holds without support.

**Proof:** First recall that we call a cdh-distinguished square any cartesian square

\[
\begin{array}{ccc}
Z' & \to & X' \\
\downarrow & & \downarrow \pi' \\
Z & \to & X
\end{array}
\]
such that \( i \) is a closed immersion, \( \pi \) is proper and defines an isomorphism \( \pi^{-1}(X - Z) \cong X - Z \). Following [CD09] 3.3.8 any cdh-distinguished square gives homotopy bicartesian squares in \( \mathbf{SH}(X) \). In particular, for any absolute spectrum \( E \) the diagram in \( \mathbf{SH}(X) \)

\[
\begin{array}{ccc}
E_X & \longrightarrow & \pi_*E_X' \\
\downarrow & & \downarrow \\
i_*/E_Z & \longrightarrow & i'_*/\pi_*E_Z',
\end{array}
\]

is homotopy bicartesian so that there is a distinguished triangle

\[
E_X \rightarrow \pi_*E_X' \oplus i_*/E_Z \rightarrow i'_*/\pi_*E_Z' \rightarrow E_X[1].
\]

Applying the functor \( \text{Hom}_{\mathbf{SH}(X)}(i_*Z(-q), \quad) \) (or \( \text{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X(-q), \quad) \) for the case without support) to this triangle in the case \( X' = B = \text{Bl}_Z X \) and \( Z' = P = \mathbb{P}(F) \) we conclude.

\[ \square \]

**Corollary 2.6** With the preceding notations let \( p, q \in \mathbb{Z} \), we have a split short exact sequence

\[
0 \longrightarrow E_{Z}^{p,q}(X) \longrightarrow E_{Z}^{p,q}(B) \overset{s}{\rightarrow} E_{Z}^{2n,q}(P)/E_{Z}^{p,q}(Z) \longrightarrow 0.
\]

The same exact sequence holds without support.

**Proof:** The preceding long exact sequence may be rewritten as

\[
\cdots \rightarrow E_{Z}^{p,q}(X) \overset{\pi^*}{\longrightarrow} E_{Z}^{p,q}(B) \overset{i^*}{\longrightarrow} E_{Z}^{p,q}(P)/E_{Z}^{p,q}(Z) \rightarrow \cdots.
\]

Denote \( x = c_1(O_P(-1)) \), by the projective bundle theorem we have

\[
E_{Z}^{p,q}(P)/E_{Z}^{p,q}(Z) = E_{Z}^{p-2,q-1}(Z)x \oplus \cdots \oplus E_{Z}^{p-2(n-1),q-n+1}(Z)x^{n-1}.
\]

Since \( i^*(c_1^p(O_B(-1))) = c_1(O_P(-1)) = x \) we define the section \( s \) by giving its value at the generators \( s(x^i) = (c_1^p(O_B(-1)))^i \) and linearity.

\[ \square \]

2.7 Consider once again the notations of Proposition 2.5. Recall that \( Z \) a is closed subscheme of \( X \), \( B \rightarrow X \) is a modified blowing-up of \( X \) over \( Z \) and \( P = \pi^{-1}(Z) \). We now construct a distinguished class in \( \mathbb{E}_Z(X) \) to define the Gysin morphism.

Although \( P \) is not in general of codimension 1, the invertible sheaf \( L_P = O_B(-1) \) has a canonical trivialization on \( B - P \). Therefore we consider the refined Gysin morphism \( \mathfrak{p}_{O_B(-1)} \) and the diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & E_{Z}^{2n,n}(X) & \overset{\pi^*}{\longrightarrow} \mathbb{E}_{P}^{2n,n}(B) & \overset{i^*}{\longrightarrow} \mathbb{E}_{Z}^{2n,n}(P)/\mathbb{E}_{Z}^{2n,n}(Z) & \longrightarrow & 0 \\
& & \mathfrak{p}_{O_B(-1)} \downarrow & & \downarrow & & \\
& & \mathbb{E}_{Z}^{2n-2,n-1}(P) & & & &
\end{array}
\]

where \( n = \text{rank} \mathcal{F} \). Since \( \Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i} = c_n(F) = 0 \) in \( \mathbb{E}(P)/\mathbb{E}(Z) \), we define

\[
C_{Z,F}^X := \Sigma_0^{n-1}(-1)^{n+1+i}c_i(F)x^{n-i-1} \in \mathbb{E}_{Z}^{2n-2,n-1}(P).
\]
Note that
\[ i^* p_{\mathcal{O}_B(-1)}(\text{Cl}_Z^X) = i^* (c_i^I(\mathcal{O}_B(-1)) (\Sigma_0^{n-1} (-1)^{n+1+i} c_i(F) x^{n-i})) \]
\[ = \Sigma_0^{n-1} (-1)^{n+1+i} c_i(F) x^{n-i} = 0. \]

Therefore, there exists a unique class \( \tilde{\eta}_Z^X \in \mathbb{E}_{2n,n}^Z(X) \) such that \( \pi^* \tilde{\eta}_Z^X = p_{\mathcal{O}_B(-1)}(\text{Cl}_Z^X) \).

**Definition 2.8** Let \( Z \to X \) be a closed subscheme and let \( F^+ \to \mathcal{I}_Z/I_Z^2 \) be an epimorphism of \( \mathcal{O}_Z \)-modules where \( F \) is locally free of rank \( n \). With the preceding notations, we define the **refined fundamental class of \( Z \) in \( X \) modified by \( F \)** to be the unique class \( \tilde{\eta}_Z^X \) in \( \mathbb{E}_{2n,n}^Z(X) \) such that \( p_{\mathcal{O}_B(-1)}(\text{Cl}_Z^X) = \pi^*(\tilde{\eta}_Z^X) \).

In the case \( i: Z \to X \) is a regular immersion of codimension \( n \) and \( F^+ = \mathcal{I}_Z/I_Z^2 \) we call this class the **refined fundamental class of \( Z \) in \( X \)** and denote it \( \tilde{\eta}_Z^X \in \mathbb{E}_{2n,n}^Z(X) \). We define the **refined Gysin morphism** to be
\[ p_i: \mathbb{E}_{n,+}^+(Z) \to \mathbb{E}_{n,+}^{n+2}(X) \]
\[ a \mapsto a \cdot \tilde{\eta}_Z^X \]
and the **Gysin morphism** to be \( i_*: \mathbb{E}_{n,+}^+(Z) \to \mathbb{E}_{n,+}^{n+2}(X), a \mapsto i_*(a \cdot \tilde{\eta}_Z^X) \).

**Proposition 2.9** Let \( E \) be an absolute ring spectrum and \( i: Z \to X \) be a regular immersion of codimension \( n \):

1. Denote \( N_{Z/X} = \text{Spec}(S^* \mathcal{I}_Z/I_Z^2) \) the normal bundle, then
\[ i^* \eta_X^Z = c_n(N_{Z/X}) \in \mathbb{E}_{2n,n}^Z(Z). \]
In particular, let \( V \to X \) be a rank \( n \) vector bundle and \( s_0: X \to V \) be the zero section, then \( s_0^* \eta_X^V = c_n(V) \in \mathbb{E}_{2n,n}^Z(X) \).

2. **Projection formula:** The Gysin morphism \( i_* \) is a morphism of \( \mathbb{E}(X) \)-modules. In other words,
\[ a \cdot i_*(b) = i_*(a^i(b)) \quad \forall \ a \in \mathbb{E}(X), b \in \mathbb{E}(Z). \]

3. Let \( r: X \to Z \) be a retraction of \( i \), then \( i_* \) is a morphism of \( \mathbb{E}(Z) \)-modules with respect to \( r^* \). In other words,
\[ i_*(a \cdot b) = r^*(a) \cdot i_*(b) \quad \forall \ a \in \mathbb{E}(Z). \]
In particular, \( i_*(a) = r^*(a) \cdot \eta_X^Z \).

**Proof:** The first point is a consequence of the definition. The projection formula follows since
\[ i_*(i^*(a) \cdot b) = i_*(i^*(a) \cdot b \cdot \tilde{\eta}_Z^X) = a \cdot i_*(b \cdot \tilde{\eta}_Z^X) = a \cdot i_*(b), \]
where we have used point 6 of Proposition 1.8.

For the last point follows note
\[ r^*(a) \cdot i_*(b) = r^*(a) \cdot i_*(b \cdot \tilde{\eta}_Z^X) = i_*(r^*(a) \cdot b \cdot \tilde{\eta}_Z^X) \]
\[ = i_*(a \cdot b \cdot \tilde{\eta}_Z^X) = i_*(a \cdot b), \]
where we have used point 4 and 6 of Proposition 1.8.\[ \Box \]
Proposition 2.10 The refined fundamental class is stable under base change. In other words, let $p: X' \to X$ be a morphism of schemes, $Z \to X$ a closed subscheme and $\mathcal{F}^* \to \mathcal{I}_Z/\mathcal{I}_Z^2$ an epimorphism of $\mathcal{O}_Z$-modules where $\mathcal{F}$ is locally free, then
\[ p^*(\hat{\eta}^X_{Z,\mathcal{F}}) = \hat{\eta}^X_{(Z),p^*(\mathcal{F})}. \]

Proof: It is enough to check
\[ p^*(Cl^X_{Z,\mathcal{F}}) = Cl^X_{(Z),p^*(\mathcal{F})} \in E^{2r-2,r-1}(p^*(\mathcal{F})) \]
where $r = \text{rank } \mathcal{F}$. This follows from the fact that Chern classes are functorial and the induced morphism $\bar{p}: B_{(Z),p^*(\mathcal{F})} X' \to B_{Z,\mathcal{F}} X$ satisfies
\[ p^* \mathcal{O}_{B_{(Z),p^*(\mathcal{F})} X'}(-1) = \mathcal{O}_{B_{Z,\mathcal{F}} X}(-1). \]

\[ \square \]

Lemma 2.11 With the preceding notations, let $\mathcal{F}_1^* \to \mathcal{F}_2^*$ be an epimorphism of locally free $\mathcal{O}_Z$-modules of constant rank $r'$, $r$ respectively. Denote $K^*$ be the kernel and $F$, $F'$, and $K$ the vector bundles they define. We have the relation
\[ \hat{\eta}^X_{Z,\mathcal{F}} = c_{r'-r}(K)\hat{\eta}^X_{Z,\mathcal{F}'.} \]

Proof: We check that due to the splitting principle we are reduced to prove the case where $r' = r + 1$. Indeed, we can assume the epimorphisms comes is the composition from a sequence of epimorphisms $\mathcal{F}_0^* \to \mathcal{F}_2^* \to \mathcal{F}_3^* \cdots$ each one with kernel of rank 1. If the propositions holds for $r' = r + 1$ then
\[ \hat{\eta}^X_{Z,\mathcal{F}_r} = c_1(K_1)\hat{\eta}^X_{Z,\mathcal{F}_{r+1}} = c_1(K_1)c_1(K_2)\hat{\eta}^X_{Z,\mathcal{F}_{r+2}}. \]
Since $c_1(K_1)c_1(K_2) = c_2(K')$ where $K'$ is the kernel of $F_1 \to F_3$. We conclude repeating this process.

For the case $r = r + 1$ it is enough to check that
\[ j^* Cl^X_{Z,\mathcal{F}} = c_1(K) Cl^X_{Z,\mathcal{F}}. \]

By construction there is short exact sequence $0 \to F \to F' \to K \to 0$ of vector bundles. Recall from Theorem 1.33 that $c_i(F') = c_i(F) + c_1(K)c_{i-1}(F)$ and therefore
\[ j^* Cl^X_{Z,\mathcal{F}} = j^* ((-1)^{r'+1}\sum_{0}^{r'-1}(-1)^i c_i(F') x^{r'-1-i}) \]
\[ = (-1)^{r'}\sum_{0}^{r'}(-1)^i(c_i(F) + c_1(K)c_{i-1}(F)) x^{r'-i} \]
\[ = (-1)^{r'}\sum_{0}^{r'}(-1)^i c_i(F) x^{r'-i} + (-1)^{r'+1} c_2(K)\sum_{0}^{r'-1}(-1)^i c_j(F) x^{r'-1-j} \]
\[ = c_1(K) Cl^X_{Z,\mathcal{F}}. \]

\[ \square \]

In Gabber’s versatile context of modified blow-up the so called key formula (cf. [Ful98, 6.7]) and the more general excess intersection formula are a direct consequence of the definition by Proposition 2.10 and Lemma 2.11.

\[ \text{[1]} \text{Let us remark that this result can be proven without relying in the splitting principle as shown by Riou in } [\text{Hig07}]. \]
Corollary 2.12 (Excess intersection formula) Consider the cartesian square

\[
\begin{array}{ccc}
P & \xrightarrow{j} & X' \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Z & \xrightarrow{i} & X 
\end{array}
\]

where both \(i\) and \(j\) are regular immersions of codimension \(n\) and \(m\) respectively. If \(K = \pi'^*N_{Z/X}/N_{P/X'}\) is the excess vector bundle then

\[
\pi'^*i_*(a) = j_*(c_{n-m}(K)\pi'^*(a)).
\]

Moreover, we have the refined version

\[
\pi'^*p_1(a) = p_j(c_{n-m}(K)\pi'^*(a)).
\]

\[\square\]

Remark 2.13 Due to Corollary 2.6, the key formula and the fact that for a regular closed immersion \(i: Z \to X\) of codimension one we have \(p_1(a) = a \cdot c_1(L_Z)\) characterize the Gysin morphism and the refined Gysin morphism for regular immersions. The general Gysin morphism is characterized by analogous conditions (see upcoming Theorem 2.34 for the complete statement).

2.2 Functoriality

In order for the definition of the Gysin morphism to be of any use it has to be functorial. In other words, if \(Z \xrightarrow{j} Y \xrightarrow{i} X\) are regular immersions then the morphism \((ij)_*\) should be equal to \(i_*j_*\). It is clear that if the classes \(\bar{\eta}_Z^X\) and \(\bar{\eta}_X^Z\) in \(E_Z(X)\) coincide this readily implies the functoriality.

Theorem 2.14 If \(Z \xrightarrow{j} Y \xrightarrow{i} X\) are two regular immersions then

\[
\bar{\eta}_Z^X = \bar{\eta}_X^Y \cdot \bar{\eta}_Z^Y \in E_Z(X). \tag{2}
\]

Proof: Let \(n\) be the codimension of \(j\) and \(m\) that of \(i\), that we may assume constant. We split the proof into two parts:

Lemma 2.15 With the preceding notations, if equation (2) holds for \(m = 1\) then it holds for any \(m\).

Proof: The case \(m = 0\) is trivial so that assume \(m > 0\). Consider \(B = B_Y X\) the blow-up of \(Y\) in \(X\) and denote \(P = \mathbb{P}(N_{Y/X})\), \(P' = \mathbb{P}(N_{Y/X}|Z)\). We have the diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{j'} & P \\
\downarrow{p'} & & \downarrow{p} \\
Z & \xrightarrow{\pi'} & Y \\
\uparrow{\pi} & & \downarrow{\pi} \\
 & & X 
\end{array}
\]

where both squares are cartesian, \(P \to B\) is a regular closed immersion of codimension 1 and \(j'\) is of codimension \(n\). Since the morphism

\[
E_{2(n+m)}(X) \xrightarrow{\pi'} E_{2(n+m)}(B)
\]
is injective (cf. Corollary 2.6) it is enough to check in $\mathbb{P}^{2(n+m),n+m}(B)$ the relation. Denote $K^*$ the kernel of the epimorphism $p^*\mathcal{I}_Z/\mathcal{I}_Z^2 \to \mathcal{I}_P/\mathcal{I}_P^2$ and $K$ its associated vector bundle. Using that the refined fundamental class are stable under base change (Proposition 2.10), the formula from Lemma 2.11 and the equation (2) for $m = 1$ we get
\[
\pi^* \bar{\eta}^X = \bar{\eta}^B_{P',\pi^*N_{Z/X}} = c_{n-1}(K)\bar{\eta}^B_{P'} = c_{n-1}(K)\bar{\eta}^B_{P'}\bar{\eta}^B_{P'}.
\]

Now, consider the commutative diagram of vector bundles on $P'$

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & K & \to & j'^*K' & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & p'^*N_{Y/X} & \to & j'^*p'^*N_{Y/X} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & N_{P'/P} & \to & j'^*N_{P'/P} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}
\]

where $K'^*$ is the vector bundle associated to the kernel of $p'^*\mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{I}_P/\mathcal{I}_P^2$. Taking into account that $j'^*K' \sim K$ and using Proposition 2.10 and Lemma 2.11 once again we conclude
\[
\pi^* \bar{\eta}^X = c_{n-1}(K')\bar{\eta}^P_{P'} = (c_{n-1}(K)\bar{\eta}^B_{P'})\bar{\eta}^P_{P'} = \bar{\eta}^Y_{Y'},\pi^* \bar{\eta}^Y_{Y'} = \pi^* (\bar{\eta}^X_{Y'}) = \pi^* \bar{\eta}^X_{Y'}\pi^* \bar{\eta}^Y_{Y'}.
\]

\[\square\]

**Lemma 2.16** With the preceding notations, the equation (3) is true for $m = 1$.

*Proof:* Denote $P = \mathbb{P}(N_{Z/X})$, $P' = \mathbb{P}(N_{Z/Y})$, $n = \text{codim}_Z Z$ and $Y, Z$ for the blow-up of $Z$ in $Y$ and $X$ respectively. Consider the commutative diagram

\[
\begin{array}{ccccccc}
P' & \xrightarrow{b} & Y_Z & \xrightarrow{w} & \pi^{-1}(Y) & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \xrightarrow{g} & \pi^{-1}(Y) & \xrightarrow{j} & Y & \xrightarrow{\pi} & X
\end{array}
\]

where every square is cartesian. Since $\pi^*: E^{2n,2,n+1}_p(X) \to E^{2n+2,n+1}_p(X_Z)$ is injective (cf. Corollary 2.6) it is enough to prove
\[
\pi^* \bar{\eta}^X = \pi^* \bar{\eta}^Y \pi^* \bar{\eta}^Y
\]

where $\pi^* \bar{\eta}^X \in E^{2n,2,n+1}_p(X_Z)$, $\pi^* \bar{\eta}^Y \in E^{2n,n}(\pi^{-1}(Y))$ and $\pi^* \bar{\eta}^Y \in E^{2,1}_{p-1}(Y)(X_Z)$.

We make the explicit computations of these two terms. This is the point where we separate from [Rio07], since we do not assume Chern classes to be additive as in the case of étale cohomology.
Let $\mathcal{I}_X^Y$ be the sheaf of ideals of $Y$ in $X$ and $L_Y$ the line bundle associated to its dual. To begin with,

$$\pi^*\eta^Y = c_1^{\pi^{-1}Y}(\pi^*L_Y) = c_1^{\pi^{-1}Y}(LP \otimes L_Y) = c_1^{\pi^{-1}Y}(O_{X_Z}(-1) \otimes L_Y)$$

$$= c_1^{\pi^{-1}Y}(O_{X_Z}(-1)) + c_1^{\pi^{-1}Y}(O_{X_Z}(-1))c_1^{\pi^{-1}Y}(L_Y)f$$

where $f \in \mathbb{E}**(S)[x, y]$ is the series given by the formal group law (cf. \[1.28\] and \[1.40\]). Therefore, the right hand side of equation \[3\] is the sum of the preceding three terms multiplied by $\pi^*\eta^Y$.

We compute each one of those three terms. From now on, we use the notation $u = c_1^y(O_{X_Z}(-1)) \in \mathcal{E}^{2,1}_Z$. For the first term

$$\pi^*\eta^Y \cdot c_1^{\pi^{-1}Y}(O_{X_Z}(-1)) = g^*\pi^*\eta^Y \cdot u = c_n(\pi^*N_{Z/Y})u = I_1$$

where the first equality is due to 6 of Proposition \[1.8\] applied to the map $g_y: \mathbb{E}_P(X_Z) \rightarrow \mathbb{E}_\pi^{-1}Y(X_Z)$ and the second one to $g^*\pi^* = \pi^*j^*$ together with Proposition \[2.9\]. For the second term

$$\pi^*\eta^Y \cdot c_1^{\pi^{-1}Y}(L_Y) = \pi^*\eta^Y \cdot w_3c_2^z(L_Y) = v_y(w^*\pi^*\eta^Y \cdot c_1^z(L_Y))$$

$$= v_y((-1)^n+1 \sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})c_i(\mathcal{O}_P(-1))n^{-1-i})c_1^{\pi^{-1}Y}(O_{X_Z}(-1))c_1^z(L_Y))$$

$$= v_y((-1)^n+1 \sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})c_i(\mathcal{O}_Z(-1))n^{-1-i})c_1^z(L_Y))$$

$$= (-1)^n+1 \sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})u^{n-i}c_1(L_Y)) = I_2$$

is due to Proposition \[1.8\] for $w_3: \mathbb{E}_Z(X_Z) \rightarrow \mathbb{E}_\pi^{-1}Y(X_Z)$, Proposition \[1.39\] and Corollary \[2.7\]. For the third and last term

$$\pi^*\eta^Y c_1^{\pi^{-1}Y}(O_{X_Z}(-1))f$$

$$= (-1)^n+1 \sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})u^{n-i}c_1(L_Y) c_1^{\pi^{-1}Y}(O_{X_Z}(-1))f$$

$$= (-1)^n+1 \sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})u^{n-i+1}c_1(L_Y) f = I_3$$

where we use the preceding computation.

Consider the short exact sequence

$$0 \rightarrow K^* \rightarrow \mathcal{I}_Z^X \rightarrow \mathcal{I}_Z^X \rightarrow 0$$

With it, we compute the other side of the equation \[3\]:

$$\pi^*\eta_Z^Y = (-1)^n \sum_{j=0}^n (-1)^j c_j(N_{Z/X})c_j(\mathcal{O}_P(-1))^{n-j})u$$

$$= (-1)^n \sum_{j=0}^n (-1)^j (c_j(N_{Z/Y})c_j(K))u^{n+1-j}$$

24
Note that \( I^P = K \otimes \mathcal{O}_P(-1) \) and therefore \( K = \mathcal{I}^{X \otimes} \otimes \mathcal{O}_P(1) \) so that \( c_1(K) = c_1(L_Y \otimes \mathcal{O}_P(-1)) \).

\[
\sum_{j=0}^{n} (-1)^{n+j}[c_j(N_{Z/Y}) + c_{j-1}(N_{Z/Y})c_1(L_{Y}) + c_1(\mathcal{O}_P(-1)) + c_1(L_{Y})c_1(\mathcal{O}_P(-1)f)]u^{n+1-j}
\]

Therefore, \( \pi^*\eta^X \) is the sum of three terms:

\[
\sum_{j=0}^{n} (-1)^{n+j}c_j(N_{Z/Y}) + c_{j-1}(N_{Z/Y})c_1(L_{Y})u^{n+1-j} = \\
= \sum_{j=0}^{n} (-1)^{n+j}c_j(N_{Z/Y})u^{n+1-j} + \sum_{i=0}^{n-1} (-1)^{n+i+1}c_i(N_{Z/Y})u^{n+1-i} = \\
= c_n(N_{Z/Y})u = I_1
\]

which is given by Proposition 2.9 and the definition of the Chern class,

\[
\sum_{j=0}^{n} (-1)^{n+j}c_{j-1}(N_{Z/Y})c_1(L_{Y})u^{n+1-j} = \\
= (-1)^{n+1}\sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})u^{n-i}c_1(L_{Y}) = I_2
\]

and finally

\[
\sum_{j=0}^{n} (-1)^{n+j}[c_{j-1}(N_{Z/Y})c_1(L_{Y})c_1(\mathcal{O}_{X_2}(-1)f)]u^{n+1-j} = \\
= (-1)^{n+1}\sum_{i=0}^{n-1} (-1)^i c_i(N_{Z/Y})u^{n+1-i}c_1(L_{Y})f] = I_3.
\]

\( \square \)

**Example 2.17** Let \( V \to X \) be a vector bundle of rank \( n \) and \((E, c_1)\) be an oriented absolute ring spectrum. The **Thom class** of \( V \) is defined to be

\[
t(V) := \sum_{i=0}^{n} (-1)^i c_i(V)x^i \in \mathbb{E}^{2n,n}(\bar{V})
\]

where \( x = c_1(\mathcal{O}_X(-1)) \) and \( \bar{V} = \mathbb{P}(V \oplus 1) \). It has being standard in motivic homotopy theory since its beginning to define fundamental classes out of Thom classes. More concretely, denote \( s_0 : X \to V \) the zero section. Its fundamental class was, by definition, \( t(V) \). Therefore, the unicity of Gysin morphisms in the context of regular schemes (cf. [Dég14]) proves that for a regular scheme \( X \) then

\[
\eta^V_X = t(V).
\]

**2.18** Let us check that \( t(V) \) coincides with \( \eta^V_X \) for arbitrary schemes. In order to do so we recall some facts of the theory of Thom classes. For convenience of the reader we recall the definitions. We define the **Thom space** of \( V \) as

\[
\text{Th}(V) = V/V - 0 \simeq \bar{V}/\mathbb{P}(V).
\]
Its cohomology fits into a long exact sequence

$$\ldots \rightarrow \mathbb{E}^{**}(\text{Th}(V)) \xrightarrow{\pi^*} \mathbb{E}^{**}(\bar{V}) \rightarrow \mathbb{E}^{**}(V) \rightarrow \ldots$$

where, from Theorem 1.24, the third arrow is always a split epimorphism. Since \( t(V) \) is zero in \( \mathbb{E}(\mathbb{P}(V)) \), we call the refined Thom class to the unique element

$$\bar{t}(V) \in \mathbb{E}(\text{Th}(V)) \simeq \mathbb{E}_X(\bar{V}) = \mathbb{E}_X(V)$$

such that \( \pi^*(\bar{t}(V)) = t(V) \). Clearly, proving that \( \bar{t}(V) \) coincides with \( \bar{\eta}^V_X \) is equivalent to proving that \( \bar{t}(V) \) coincides with \( \bar{\eta}^V_X \).

One last technical recall (cf. [Deg14] for example): if \( 0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0 \) is exact, the refined Thom classes satisfy

$$\bar{t}(V) = \bar{t}(V')\bar{t}(V'') \in \mathbb{E}_X(V).$$

Here \( \bar{t}(V_V) \) denotes \( V \) considered as a bundle over \( V' \) and the product is that of Definition 1.6.

**Proposition 2.19** Let \( V \rightarrow X \) be a vector bundle and denote \( \bar{V} \) its projective completion. Then

$$\bar{\eta}^V_X = t(V).$$

**Proof:** It is clear that this formula is equivalent to its refined counterpart, \( \bar{\eta}^V_X = \bar{t}(V) \). Due to Theorem 2.14 and the previous remark it is enough to prove it for the case of a line bundle \( V = L \).

In this case \( t(L) = c_1(L) - c_1(\mathcal{O}_L(-1)) \) and \( \eta^X_L = c_1(\mathcal{I}^*) \), where \( \mathcal{I} \) stands for the sheaf of ideals of the zero section in \( L \). This sheaf may be computed explicitly: the composition \( \mathcal{O}_L(-1) \rightarrow L \oplus \mathcal{O} \rightarrow L \) of the canonical morphism and the projection is an isomorphism out of the zero section, which induces \( L^* \oplus \mathcal{O}_L(-1) \simeq \mathcal{I} \rightarrow \mathcal{O} \). Consider the canonical short exact sequence

$$0 \rightarrow \mathcal{O}_L(-1) \rightarrow L \oplus \mathcal{O} \rightarrow Q \rightarrow 0$$

where \( Q \) is the canonical quotient bundle. Hence \( L = \bigwedge^2(L \oplus \mathcal{O}) = Q \otimes \mathcal{O}_L(-1) \) so that 

$$Q = L \otimes \mathcal{O}_L(1) = \mathcal{I}^*.$$  

To conclude note that \( c_1(L \oplus \mathcal{O}) = c_1(L) \). Also by the additivity of Chern classes

$$c_1(L \oplus \mathcal{O}) = c_1(Q) + c_1(\mathcal{O}(-1)) \tag{2}$$

Let \( \mathbb{E} \) be an absolute ring spectrum and \( \mathbb{M} \) be an absolute \( \mathbb{E} \)-module. Note that \( \mathbb{M} \) does not have a unit. As a consequence there are no fundamental nor Chern classes in the \( \mathbb{M} \)-cohomology. However, \( \mathbb{M}(X) \) is an \( \mathbb{E}(X) \)-module and therefore we can still multiply classes in the \( \mathbb{M} \)-cohomology by fundamental classes and Chern classes of the \( \mathbb{E} \)-cohomology. This suffices to define the Gysin morphism.

**Definition 2.20** Let \( i: Z \rightarrow X \) be a regular immersion. We define the Gysin morphism \( i_* \) and the refined Gysin morphism \( p_i \) in the \( \mathbb{M} \)-cohomology to be

\[
\begin{align*}
i_* & : \quad \mathbb{M}^{p,q}(Z) \rightarrow \mathbb{M}^{p+2n,q+n}(X), \\
p_i & : \quad \mathbb{M}^{p,q}(Z) \rightarrow \mathbb{M}^{p+2n,q+n}(X), \quad m \cdot \eta^X_Z \mapsto m \cdot \bar{\eta}^X_Z.
\end{align*}
\]

\(^2\)The scheme which parametrises the sections of \( V \rightarrow V'' \) is a torsor (of group \( \text{Hom}(V'', V') \)) and the pullback of the short exact sequence is naturally split there.

\(^3\)Note that we already computed a more general formula in the proof Lemma 2.16 from which this equality follows.

\(^4\)I am thankful to J. Riou who pointed me out a mistake in this computation on an earlier version of this text.
We readily deduce from the case of ring spectra the following properties for modules:

**Theorem 2.21** Let $E$ be an oriented absolute ring spectrum, $M$ an absolute $E$-module and $i: Z \to X$ a regular immersion.

1. **Functoriality:** Let $j: Y \to Z$ be a regular immersion, then $(ij)_* = i_*j_*$. 
2. **Projection formula:** The Gysin morphism is $E(X)$-linear. In other words,

$$a \cdot i_*(m) = i_*(i^*(a) \cdot m) \quad \forall \ a \in E(X), \ m \in M(Z).$$

Note that an analogous formula also holds for $n \in M(X)$ and $b \in E(Z)$.

3. Denote $n = \text{codim}_X Z$, we have

$$i^*i_*(m) = c_n(N_{Z/X}) \cdot m \quad \forall \ m \in M^{2n,n}(Z).$$

4. Let $r: X \to Z$ be a retraction of $i$, then the Gysin morphism $i_*$ is $M(Z)$-linear (with $r^*$). In other words,

$$i_*(m) = r^*(m) \cdot \eta_X^Z \quad \forall \ m \in M(Z).$$

5. **Excess intersection formula:** Consider a cartesian square

\[
\begin{array}{ccc}
P & \xrightarrow{j} & X' \\
\pi' \downarrow & & \downarrow \pi \\
Z & \xrightarrow{i} & X
\end{array}
\]

where both $i$ and $j$ are regular immersions of codimension $n$ and $m$ respectively. Denote $K = \pi^*N_{Z/X}/N_{P/X'}$, the excess vector bundle, then

$$\pi^*i_*(m) = j_*(c_{n-m}(K) \cdot \pi'^*(m)) \quad \text{and}$$

$$\pi^*p_*(m) = p_*(c_{m-n}(K) \cdot \pi'^*(m)) \quad \forall \ m \in M(Z).$$

\[
\square
\]

### 2.3 The projective lci case

The main reference we have used for this section is [Dég08 p.5] where Dégilse thoroughly studied the Gysin morphism. However, the reference works on the smooth case and on a general category of premotives satisfying certain axioms, which do not hold for $SH$. Nevertheless, the arguments still hold *mutatis mutandis* in our context.

We first observe that the fundamental class of the diagonal $\eta_{\Delta,n} = \eta_{\Delta,n}^{P^\times \times P^\times} \in E(P_X^n) \otimes E(P_X^n)$ defines a nondegenerate bilinear pairing. The duality it defines (similar to Euclidean spaces) allows us to define the Gysin morphism for the projection $p_! : P_X^\times \to X$ of a projective space onto its base as the dual of the inverse image $p^*_X : E(X) \to E(P_X^n)$ (cf. Definition [22]). We then prove that the Gysin morphism for projective lci morphism, without smoothness assumptions, have all usual properties. We deduce from the case of ring spectra a Gysin morphism for modules.
2.22 Denote $E(P^n_X)^\vee = \text{Hom}_{E(X)-\text{mod}}(E(P^n_X), E(X))$ the dual of $E(P^n_X)$ as $E(X)$-module, $p_X: P^n_X \rightarrow X$ the canonical projection and $(p^n_X)^\vee: E(P^n_X)^\vee \rightarrow E(X)^\vee \simeq E(X)$ the transpose of the inverse image. Recall that from the projective bundle theorem we have $E(P^n_X \times_X P^n_X) \simeq E(P^n_X) \otimes_{E(X)} E(P^n_X)$. Denote $\Delta_n: P^n_X \rightarrow P^n_X \times_P^n_X$ the diagonal embedding. The fundamental class of the diagonal $\eta_{\Delta_n} \in E(P^n_X) \otimes E(P^n_X)$ defines a symmetric $E(X)$-bilinear pairing on $E(P^n_X)^\vee$, and therefore a polarity $\Phi: E(P^n_X)^\vee \rightarrow E(P^n_X), \omega \mapsto (\omega \otimes 1)(\eta_{\Delta_n}) = \eta_{\Delta_n}(\omega, \ )$.

**Proposition 2.23** The polarity $\Phi$ defined above is an isomorphism.

**Proof:** The matrix that defines $\Phi$ is the same as $\eta_{\Delta_n}$. One can check that, as in the smooth case, we have

$$\eta_{\Delta_n} = \sum_{r,s=0}^n a_{r,s} x_n^r \otimes x_n^s$$

where

$$a_{r,s} = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}$$

(4)

where $x_n = c_1(O_{P^n_X}(-1))$ (cf. [Nav16b, 2.2]).

\[\Box\]

**Definition 2.24** Let $E$ be an oriented absolute ring spectrum and denote $p_X: P^n_X \rightarrow X$ the natural projection, we define the **direct image** of $p_X$ to be$p_X^*: E(P^n_X)^\vee \rightarrow E(P^n_X)^\vee \rightarrow E(X)^\vee \simeq E(X)$.

**Remark 2.25** Denote $p: P^n \rightarrow S$, since the square

```
\begin{array}{ccc}
P^n & \rightarrow & P^n \times P^n \\
\downarrow & & \downarrow \pi \\
P^n & \rightarrow & P^n \times P^n
\end{array}
```

is transversal, the excess intersection formula gives $\eta_{\Delta_nX} = \pi^* \eta_{\Delta_n}$ and we have

$$p_X^* = p^* \otimes 1_X.$$
Remark 2.27 The previous lemma applied to our case \( \eta = \eta_\Delta \) and \( f = p_X^*: E(\mathbb{P}_X^n) \to E(X) \) concludes
\[ \Phi(p_X^*) = p_X^*(1) = 1, \]
which totally determines \( p_X^* \).

Lemma 2.28 Let \( \Delta_n: \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n = \mathbb{P}^n \times \mathbb{P}^n \) be the diagonal embedding. Then \( p_{2*}\Delta_\ast = 1_{P^n} \).

Proof: Recall that \( \{1, x = c_1(O_{\mathbb{P}^n}(-1)), \ldots, x^n\} \) is a basis of \( E(\mathbb{P}^n) \). From last lemma applied to \( f = p^*, \omega = p_* \) and \( \eta = \Delta_\ast(1) \) we have
\[ 1 = p^*(1) = (p_\ast \otimes 1)(\eta_\Delta) = p_{2*}(\Delta_\ast(1)). \]
For the rest of elements of the basis we conclude by observing that
\[ p_{2*}(\Delta_\ast(x_j)) = p_{2*}(\Delta_\ast(\Delta_\ast(1) \otimes x_j)) = p_{2*}(1 \otimes x_j) \cdot p_{2*}(\Delta_\ast(1)) = x_j \]
since \( p_{2*} = p_\ast \otimes 1_{p^n*} \).

\[ \square \]

Corollary 2.29 Let \( s: X \to \mathbb{P}_X^n \) be a section of \( p_X: \mathbb{P}_X^n \to X \). Then \( p_{X*} s_\ast = 1_X \).

Proof: Note that preceding lemma also holds for a projective space over a general base \( \mathbb{P}_X^n \). Consider the cartesian squares
\[
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{s} & \mathbb{P}^n \times X \xrightarrow{p_X} X \\
\downarrow & & \downarrow \quad \downarrow \\
\mathbb{P}^n_X & \xrightarrow{p_{X*}} & \mathbb{P}^n_X \times \mathbb{P}^n_X \\
\Delta & \xrightarrow{\Delta} & \mathbb{P}^n_X \times \mathbb{P}^n_X \xrightarrow{p_{X*}} \mathbb{P}^n_X.
\end{array}
\]
We may apply Corollary 2.12 to the left square so that \( s^* s_\ast = \Delta_\ast(1_{\mathbb{P}^n_X} \times s)^\ast \). From Definition 2.24 we also have that \( p_{X*}(1_{\mathbb{P}^n_X} \times s)^\ast = p_{\mathbb{P}^n_X*} \). Together with the previous lemma we deduce
\[ s^* = p_\ast s_\ast s^\ast. \]
Since \( s^\ast \) is surjective we conclude that \( p_{X*} s_\ast = 1_X \).

\[ \square \]

Lemma 2.30 Let \( i: Z \to X \) be a regular immersion and consider the cartesian diagram
\[
\begin{array}{ccc}
\mathbb{P}^n_Z & \xrightarrow{k} & \mathbb{P}^n_X \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i} & X
\end{array}
\]
Then \( k_\ast = (1_{\mathbb{P}^n} \times i)_\ast = 1_{\mathbb{P}^n} \otimes i_\ast \) and \( p_{X*} k_\ast = i_\ast p_{Z*} \).

Proof: For the first claim, applying the excess intersection formula and the projective bundle theorem (Corollary 2.12, Theorem 1.27) we get that the diagram
\[
\begin{array}{ccc}
E(\mathbb{P}^n) \otimes_{E(\mathbb{S})} E(Z) & \xrightarrow{k_\ast} & E(\mathbb{P}^n) \otimes_{E(\mathbb{S})} E(X) \\
p_{Z*} & & p_X^* \\
E(Z) & \xrightarrow{i_*} & E(X)
\end{array}
\]

29
commutes so that \( k_*(1 \otimes a) = 1 \otimes i_*(a) \) for \( a \in \mathcal{E}(Z) \). Applying the projection formula from Proposition 2.9 we get that \( k_*(b \otimes 1) = (b \otimes 1) \cdot k_*(1 \otimes 1) = b \otimes i_*^Z \) for \( b \in \mathcal{E}(\mathbb{P}^n) \) so we conclude \( k_* = (1_{\mathbb{P}^n} \times i)_* = 1_{\mathbb{P}^n} \otimes i_* \). From here and the previous definition the formula \( p_* k_* = i_* p_{Z*} \) follows.

\[\square\]

**Theorem 2.31** Consider a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{k} & \mathbb{P}^n_X \\
\downarrow{i} & & \downarrow{p} \\
\mathbb{P}^m_X & \xrightarrow{q} & X
\end{array}
\]

where \( i \) and \( k \) are regular immersions of codimension \( r \) and \( s \) respectively and \( p \) and \( q \) are the natural projections. Then, \( p_* k_* = q_* i_* \).

**Proof:** Consider the following commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{k} & \mathbb{P}^n_X \\
\downarrow{i} & & \downarrow{p} \\
\mathbb{P}^m_X & \xrightarrow{q} & X
\end{array}
\]

Since it is clear that \( p_* q'_* = q_* p'_* \) it is enough to prove that \( p'_* v_* = i_* \). For that case, denote \( T = \mathbb{P}^m_X \), \( v = \rho \times i \) where \( \rho: Y \to \mathbb{P}^n \) and consider

\[
\begin{array}{ccc}
Y & \xrightarrow{v} & \mathbb{P}^m_X \\
\downarrow{i} & & \downarrow{\pi} \\
\mathbb{P}^m_Y & \xrightarrow{i} & \mathbb{P}^m_T
\end{array}
\]

where \( i, s = \rho \times 1_Y, l \) and \( v \) are regular immersions. By the functoriality of Theorem 2.14 we have \( v_* = l_* s_* \) and by the previous Lemma 2.30 we also have \( \pi_* s_* = 1_Y \) and \( p'_* l_* = i_* \pi_* \). Considering all together we conclude

\[ p'_* v_* = p'_* l_* s_* = i_* \pi_* s_* = i_* \]

\[\square\]

**Definition 2.32** We define an \( X \)-scheme \( Y \to X \) to be a **local complete intersection** (lci) if it locally admits a factorization by a regular immersion into \( \mathbb{A}^n_X \) (SGA6 VIII 1.1]). Given our conventions, a projective lci morphism \( f: Y \to X \) admits a factorization of the form \( Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X \) where \( i \) is a regular closed immersion (cf. SGA6 VIII 1.2) and \( p \) is the canonical projection.

Let \( E \) be an absolute ring spectrum, \( f: Y \to X \) a projective lci morphism and \( Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} Y \) be a factorization. We define the **direct image** of \( f \) as \( f_* := p_* i_* \) (by Theorem 2.31 it does not depend on the choice of factorization).
Let $M$ be an absolute $\mathbb{E}$-module and $p_X : \mathbb{P}^n_X \to X$ be the natural projection. We define the direct image $p_X \ast$ in the $M$-cohomology as the morphism

\[ \mathbb{M}(\mathbb{P}^n_X) \xrightarrow{p_X \ast} M(X) \xrightarrow{\rho} \mathbb{E}(S) \otimes_{\mathbb{E}(S)} \mathbb{M}(X). \]

Let $f : Y \to X$ be a projective lci morphism and $f = p i$ be a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ where $i$ is a regular immersion and $p$ is the natural projection. We define the direct image of $f$ as $f_* := p_* i_*$.

Finally, let us remark the main properties the direct image in this context.

**Theorem 2.33** Let $f : Y \to X$ be a projective lci morphism, $\mathbb{E}$ be an absolute oriented ring spectrum and $M$ be an absolute $\mathbb{E}$-module:

1. **Functoriality:** If $g : Z \to Y$ is another projective lci morphism then
   \[ (fg)_* = f_* g_* : M(Z) \to M(X). \]

2. **Excess intersection formula:** Consider a cartesian square

   \[
   \begin{array}{c}
   Y' \\
   \downarrow^g \\
   X' \\
   \downarrow^p \\
   Y \\
   \downarrow^f \\
   X
   \end{array}
   \]

   where $f$ and $g$ are projective lci morphisms of codimension $r$ and $s$ respectively. Choose a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of $f$ and denote $K = q^* N_{Y/\mathbb{P}^n_X}/N_{Y'/\mathbb{P}^n_X}$.

   Then
   \[ p^* f_*(m) = g_*(c_{r-s}(K) \cdot q^*(m)) \quad \forall \ m \in M(Y). \]

3. **Projection formula:** $f_*$ is a morphism of $\mathbb{E}(X)$-modules. That is to say,
   \[ f_*(f^*(a) \cdot m) = a \cdot f_*(m) \quad \forall \ a \in \mathbb{E}(X), \ m \in M(Y). \]

**Proof:** We prove the case of ring spectra and the case of modules follows straightforwardly.

For the first point consider factorizations $Z \xrightarrow{i} \mathbb{P}^m_X \xrightarrow{q} X$ of $f \circ g$ and $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of $f$. We can compute explicitly the base change

\[
\begin{array}{c}
Y' \\
\xrightarrow{i'} \mathbb{P}^m_X \times \mathbb{P}^n_X \\
\downarrow^\pi \\
Y \\
\xrightarrow{i} \mathbb{P}^n_X
\end{array}
\]

Recall that the definition of $K$ in the previous proposition does not depend on the choice of factorization (cf. Fulisse 6.6).
as \( Y' = (\mathbb{P}^n_X \times_X \mathbb{P}^n_X) \times_{\mathbb{P}^n_X} Y = \mathbb{P}^m_Y \). If we denote \( j = v \times (fg) : Z \rightarrow \mathbb{P}^n \times X \) and consider \( k = v \times g : Z \rightarrow \mathbb{P}^n \times Y \) it fits into a commutative diagram

The preceding lemmas allow to conclude the functoriality since

\[
(f_* g_*) = p_* i_* \pi'_* k_* - q_* = q_* = (fg)_*.
\]

For the excess intersection formula recall the factorization \( Y \xrightarrow{j} \mathbb{P}^n_X \xrightarrow{p'} X \). Changing base on the regular immersion we get a cartesian diagram

which has the same excess bundle \( K \) and where \( j \) and \( i \) are regular immersions. Hence by Corollary \ref{cor:excess_intersection} for any \( a \in E(Y) \) the relation

\[
\pi^* i_*(a) = j_*(c_{n-m}(K) \cdot q^*(a))
\]

holds. Consider the diagram

then \( p^* \pi_* = \pi'_* p'^* \) and we conclude.

For the projection formula consider the commutative diagram

where \( \gamma_f \) denotes the graphic of \( f \) and \( \Delta \) denotes the diagonal. Since \( \Delta \) is transversal to \( f \times 1_X \) we may apply the excess intersection formula: for any \( a \in E(X) \) and \( b \in E(Y) \) we have that

\[
\Delta^*(f \times 1_X)_*(b \times a) = \Delta^*((f, b) \times a) = f_*(b) \cdot a
\]

equals

\[
f_* \gamma_f^*(b \times a) = f_*(b \cdot f^*(a)).
\]

As a result of the construction we can characterize direct images. The result develops \cite[2.5]{Pan09} for smooth \( k \)-schemes and \cite[3.3.1]{Deg14} for regular schemes.
Theorem 2.34 Let \((E, c_1)\) be an oriented absolute ring spectrum, there exists a unique way of assign for any projective lci morphisms \(f: Y \to X\) a group morphism \(f_*: E(Y) \to E(X)\) satisfying the following properties:

1. Functoriality: \((fg)_* = f_*g_*\).
2. Normalization: For regular immersions \(i: Y \to X\) of codimension one they satisfy \(i_*(a) = i_!(a \cdot c_1^Y(L_Y))\).
3. Key formula: If \(i: Y \to X\) is a regular immersion of codimension \(n\), \(\pi_*: B_Y X \to X\) is the blowing-up of \(Y\) in \(X\) with exceptional divisor \(j: P(N_Y/X) \to B_Y X\), we have \(\pi^*i_*(a) = j_*(c_{n-1}(K) \cdot \pi^*(a))\).
4. Projection formula: They are \(E(X)\)-linear, i.e., \(f_*(f^*(a) \cdot b) = a \cdot f_*(b)\) for \(a \in E(X)\) and \(b \in E(Y)\).

When considering regular immersions, properties 2 and 3 characterize them.

Proof: The functoriality property reduces the proof to the case of regular immersions and the projection of a projective space onto its base. The case of closed immersions follows directly from properties 2, 3 and the long exact sequence of the blow-up (cf. Corollary 2.6).

For the projection of a projective space we apply the excess formula to the commutative square

\[
P^n \xrightarrow{p_X} X \xrightarrow{p} S.
\]

Together with the projective bundle theorem \([1,27]\) and the projection formula we obtain that \(p_{X*} = p_* \otimes 1_X\). The functoriality property implies the commutativity of the triangle

\[
\begin{array}{ccc}
E(P^n) & \xrightarrow{\Delta_*} & E(P^n) \otimes E(P^n) \\
\downarrow^{1_{E(P^n)}} & & \downarrow^{p_*} \\
& E(P^n). &
\end{array}
\]

The argument from Remark 2.27 shows that \(p_* \in E(P^n)\)'s uniquely determined by its image through the polarity, which is \(\Phi(p_*) = (p_* \otimes 1)(\eta_\Delta) = 1\).

Remark 2.35 By the previous theorem the direct image constructed here coincides with classic constructions for cohomologies represented by spectra of \([1,4]\) (for instance cf. \([\text{Deg14}, 3.3.4]\)). Let us explicitly mention the case of higher \(K\)-theory: all properties apart from normalization are proved for Quillen’s \(K\)-theory in \([\text{Qui73}]\) and normalization is proved in \([\text{Tho93}]\).

3 Riemann-Roch theorem

We devote this section to prove the motivic Riemann-Roch theorem in the context of the algebraic stable homotopy category. We deduce a Riemann-Roch theorem for modules. The classic Chern character is replaced in the general setting by a morphism of oriented spectra \(\varphi: (E, c_1) \to (\mathbb{F}, \bar{c}_1)\). For clarity in the exposition, overlined morphisms and elements will refer to the \(\mathbb{F}\)-cohomology.
Theorem 3.1 Let \( \varphi: (E, c_1) \rightarrow (F, \bar{c}_1) \) be a morphism of oriented absolute ring spectra such that \( \varphi_{P\infty}(c_1) = \bar{c}_1 \) and let \( f: Y \rightarrow X \) be a projective lci morphism, then the diagram

\[
\begin{array}{ccc}
E(Y) & \xrightarrow{f_*} & E(X) \\
\downarrow \varphi_Y & & \downarrow \varphi_X \\
F(Y) & \xrightarrow{\bar{f}_*} & F(X)
\end{array}
\]

commutes. In other words, for \( a \in E(Y) \)

\[ \varphi_X(f_*(a)) = \bar{f}_*(\varphi_Y(a)). \]

Proof: Following the standard approach, it is enough to check the theorem for a regular immersion \( i: X \rightarrow P^n_Y \) and the projection \( p: P^n_X \rightarrow Y \).

Lemma 3.2 (Regular immersions) Theorem 3.1 holds for a regular immersion.

Proof: Since \( \varphi \) preserves the orientation it also preserves Chern classes of vector bundles. It follows that \( \varphi(\eta^n_Y) = \bar{\eta}^n_Y \) since fundamental classes were defined in terms of Chern classes. We conclude that \( \varphi_Xi_* = i_*\varphi_Z. \) \( \square \)

Lemma 3.3 (Projection) Theorem 3.1 holds for the canonical projection \( p: P^n_X \rightarrow X \).

Proof: Applying Theorem 3.1 to the diagonal embedding \( \Delta_n: P^n_X \rightarrow P^n_X \times P^n_X \) we obtain that \( \varphi_{P^n_X \times P^n_X} \) preserves the fundamental class of the diagonal. Recall from Definition 2.22 that \( p_* = \Phi^{-1}(p^*)^\vee \) where \( \Phi: E(P^n_X) \rightarrow E(P^n_X)^\vee \) is the polarity defined by the fundamental class of the diagonal (cf. Paragraph 2.22). Since \( \varphi \) commute with inverse images the diagram

\[
\begin{array}{ccc}
E(P^n_X) & \xrightarrow{\Phi^{-1}} & E(P^n_X)^\vee \\
\downarrow \varphi_{P^n_X} & & \downarrow \varphi_X \\
F(P^n_X) & \xrightarrow{\bar{\Phi}^{-1}} & F(P^n_X)^\vee
\end{array}
\]

is made of commutative squares. \( \square \)

Lemma 3.4 (Change of direct image) Let \( (E, c_1) \) be an oriented absolute ring spectrum. Let \( c_1^{\text{new}} = G(c_1) \cdot c_1 \) be a new orientation (cf. Proposition 1.34) and denote \( G^{-1}_X \) the multiplicative extension of \( G^{-1} \in E(S)[[t]] \) (cf. Paragraph 1.35). Let \( f: Y \rightarrow X \) be a projective lci morphism and denote \( T_f = i^*T_{P^n_Y} - N_i \in K_0(Y) \) the virtual tangent bundle of \( f = p \circ i \), then

\[ f_*^{\text{new}}(a) = f_*(G^{-1}_X(T_f) \cdot a) \quad \forall \ a \in E(Y). \]

Proof: We have to check that the right hand side of the equality satisfies the conditions of Theorem 2.34 In the case of the canonical projection \( p: P^n_X \rightarrow X \) we only have to check the projection formula, which is immediate. For the case of regular immersions \( i: Y \rightarrow \)

---

I am grateful to J. Riou for an observation that simplified this proof as it is now.
1.34 In this case is the formal inverse \( -c \) consider Corollary 2.6 notations and recall that we denote the canonical quotient bundle by \( K = \pi^* N_{Y/X}/\mathcal{O}_P(-1) \). We then have

\[
\pi^* i_*^{new}(a) = \pi^* i_* (G_{G}^{-1}(-N_{Y/X}) \cdot a) = j_* (c_{d-1}(K) \cdot G_{G}^{-1}(-K - \mathcal{O}_P(-1)) \cdot \pi^*(a)) = j_*^{new} (c_{d-1}^{new}(K) \cdot \pi^*(a))
\]

where the last equality comes from Corollary 1.34.

\[ \square \]

**Theorem 3.5 (Motivic Riemann-Roch)** Let \( \varphi : (\mathbb{E}, c_1) \to (\mathbb{F}, c_1) \) be a morphism of oriented absolute spectra. Denote \( G \in \mathbb{F}[S][[t]] \) the series such that \( \varphi(c_1) = G(c_1) \cdot c_1 \in \mathbb{F}(\mathbb{P}_G) \) and \( G_{G}^{-1} \) be the multiplicative extension of \( G^{-1} \). Let \( f : Y \to X \) be a projective lc where \( f = p \circ i \) and denote \( T_f = i^*T_p \cdot N_i \in K_0(Y) \) the virtual tangent bundle. Then the diagram

\[
\begin{array}{ccc}
E(Y) & \xrightarrow{f_*} & E(X) \\
\downarrow G_{G}^{-1}(T_f)\varphi_Y & & \downarrow \varphi_X \\
F(Y) & \xrightarrow{f_*} & F(X)
\end{array}
\]

commutes. In other words, for \( a \in E(Y) \) we have

\[
\varphi_X(f_*(a)) = \bar{f}_* (G_{G}^{-1}(T_f) \cdot \varphi_Y(a)).
\]

**Proof:** We define \( \bar{c}_1^{new} = \varphi_{G}^{-1}(c_1) = G(c_1) \cdot c_1 \) that gives a direct image \( \bar{f}_*^{new} \) satisfying \( \varphi_X(f_*(c_1)) = \bar{f}_*^{new} (\varphi_Y(c_1)) \) due to Theorem 3.1 We conclude recalling Lemma 3.4.

\[ \square \]

**Example 3.6** Consider the identity \( \text{Id} : (\mathbb{E}, c_1) \to (\mathbb{E}, c_1) \) between a ring spectrum with two different orientations \( c_1 \) and \( \bar{c}_1 \). The explicit computations of \( G^{-1} \) is a classic subject on formal group laws (cf. [Dég14] §5.2 for a review in the context of the Riemann-Roch theorem). The simplest example being

\[
\bar{c}_1(L) = -c_1(L^*).
\]

Recall from the proof of Proposition 2.19 that the canonical short exact sequence \( 0 \to \mathcal{O}_{P^1}(-1) \to \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1} \to \mathcal{Q} \to 0 \) satisfies that \( Q = \mathcal{O}_{P^1}(1) \) so we have \( c_1(\mathcal{O}_{P^1}(1)) = -c_1(\mathcal{O}_{P^1}(-1)) \) for any orientation. Therefore the class \( \bar{c}_1(\mathcal{O}_{P^1}(-1)) \) as defined above is always an orientation. If \( F \) is the formal group law of \( c_1 \) then the series \( G \) of Proposition 1.34 in this case is the formal inverse \( \mu \) of \( F \), i.e., the series satisfying \( F(x, \mu(x)) = 0 \). However, it is much easier to compute Chern classes explicitly by the splitting principle and the projective bundle theorem obtaining

\[
\bar{c}_1(E) = (-1)^i c_i(E^*) \quad \text{and} \quad \sum_{i=0}^{n} (-1)^i c_i(E^*)y^{n-i} = 0 \in \mathbb{E}(\mathbb{P}(E))
\]

where \( y = c_1(\mathcal{O}_{P(E)}(1)) \).

The Riemann-Roch theorem for modules is a direct consequence of the case of rings.
Theorem 3.7 Let $\varphi: (E, c_1) \to (F, c_1)$ be a morphism of oriented absolute spectra. Denote $G \in F[S||t]]$ the series such that $\varphi(c_1) = G(c_1) \cdot c_1 \in F(F_S^\infty)$ and $G_\infty^{-1}$ be the multiplicative extension of $G^{-1}$. Let $\Phi: M \to M'$ be a $\varphi$-morphism of absolute modules and $f: Y \to X$ be a projective lci where $f = p \circ i$ and denote $T_f = i^*T_p - N_i \in K_0(Y)$ the virtual tangent bundle. Then the diagram

\[
\begin{align*}
& M(Y) \xrightarrow{f_*} M(X) \\
& G_\infty^{-1}(T_f) \Phi_Y \downarrow \quad \Phi_X \\
& M'(Y) \xrightarrow{f_*} M'(X)
\end{align*}
\]

commutes. In other words, for $m \in M(Y)$ we have

\[\Phi_X(f_*(m)) = \Phi_Y(G_\infty^{-1}(T_f) \cdot m).\]

Recall that in Section 1.2 we constructed different examples of modules. Let us concretely remark the case of the relative cohomology.

Theorem 3.8 Let $\varphi: E \to F$ be a morphism of strict oriented absolute ring spectra. Let $g: T \to X$ be a morphism of schemes, $f: Y \to X$ be a projective lci morphism and denote $g_Y: T \times_X Y \to Y$ and $T_f \in K_0(Y)$ the virtual tangent bundle of $f$. Assume in addition either $g$ is proper or $f$ is smooth, then the diagram

\[
\begin{align*}
& E(g_Y) \xrightarrow{f_*} E(g) \\
& G_\infty^{-1}(T_f)^\varphi \downarrow \quad ^\varphi \\
& F(g_Y) \xrightarrow{f_*} F(g)
\end{align*}
\]

commutes. In other words, for $m \in E(g_Y)$ we have

\[\varphi(f_*(m)) = f_*(G_\infty^{-1}(T_f) \cdot \varphi(m)).\]

Proof: We have a commutative diagram

\[
\begin{align*}
& \text{hofib}_E(g) \xrightarrow{i} E_X \xrightarrow{Rg_*g^*E_X} \\
& \Phi_X | \downarrow \quad \varphi_X \\
& \text{hofib}_F(g) \xrightarrow{\Phi} F_X \xrightarrow{Rg_*g^*F_X}
\end{align*}
\]

so that there exists a $\varphi_X$-morphism of modules $\Phi_X: \text{hofib}_E(g) \to \text{hofib}_F(g)$. We conclude by applying Theorem 3.7 and Proposition 1.17.

Example 3.9 We review some concrete examples of this formula. Let $\varphi: E \to F$ a morphism of strict absolute ring spectra:

- Let $i: Z \to X$ be a closed immersion and consider the cartesian square

\[
\begin{array}{ccc}
P & \xrightarrow{c} & B_Z X \\
\downarrow \pi' & & \downarrow \pi \\
Z & \xrightarrow{i} & X
\end{array}
\]

36
where $B_Z X$ denotes the blow-up of $Z$ in $X$. Recall from Example 1.19 that $E(\pi) \simeq E(\pi') = E(P)/E(Z)$. We deduce from the Riemann-Roch theorem for modules that the square

$$
\begin{array}{ccc}
E(P)/E(Z) & \xrightarrow{i_*} & E(P)/E(Z) \\
G_\chi^{-1}(-N_{Z/X}) & \downarrow & \varphi \\
\mathbb{F}(P)/\mathbb{F}(Z) & \xrightarrow{i_*} & \mathbb{F}(P)/\mathbb{F}(Z)
\end{array}
$$

commutes. Note that $i_*(m) = m \cdot (i \circ \pi')^*(\eta_X^2) = m \cdot c_n(N_{Z/X})$. Recall from Corollary 1.36 that $c_n(N_{Z/X}) = G_\chi^{-1}(-N_{Z/X}) \cdot \tilde{c}_n(N_{Z/X})$ so the formula also follows from the Riemann-Roch theorem for rings.

- Let $S = \text{Spec}(k)$ be a point. Denote $g: X \to S$ be a $k$-scheme and $f: Y \to S$ be a smooth projective $k$-scheme. We have that $E(g) = \bar{E}(X) = E(X)/E(S)$ and that $E(g_Y) = E(X \times Y)/E(Y)$. Then the square

$$
\begin{array}{ccc}
E(X \times Y)/E(Y) & \xrightarrow{f_*} & \bar{E}(X) \\
G_\chi^{-1}(T_X) & \downarrow & \varphi \\
\mathbb{F}(X \times Y)/\mathbb{F}(Y) & \xrightarrow{\bar{f}_*} & \bar{F}(X)
\end{array}
$$

commutes.

Now assume both $E$ and $F$ satisfy the Künneth formula. Then $E(X \times Y)/E(Y) = \bar{E}(X) \otimes E(Y)$ and

$$f_* = 1 \otimes f_* : \bar{E}(X) \otimes E(Y) \to \bar{E}(X) \otimes E(S) = \bar{E}(X)$$

and $\bar{f}_* = 1 \otimes \bar{f}_*$. In this case this formula also follows from the Riemann-Roch theorem for ring spectra.

- **Residual Riemann-Roch:** Let $i: Z \to X$ be a closed immersion of open complement $j: U \to X$. Recall that $E(j) = E_Z(X)$. Note that the morphism of modules $\varphi: \text{hofib}_E(j) \to \text{hofib}_F(j)$ fits into a morphism of distinguished triangles. We deduce that the square

$$
\begin{array}{ccc}
E(U) & \xrightarrow{\delta} & E_Z(X) \\
\varphi & \downarrow & \varphi \\
\mathbb{F}(U) & \xrightarrow{\delta} & F_Z(X)
\end{array}
$$

where $\delta$ denotes the connecting, commutes. If both $Z$ and $X$ are smooth then we have the purity isomorphism $E(Z) \simeq E_Z(X)$ and we deduce Déglise’s residual Riemann-Roch formula (cf. [Dég14, 4.2.3]): The square

$$
\begin{array}{ccc}
E(U) & \xrightarrow{\delta_{P_1}} & E(Z) \\
\varphi & \downarrow & \varphi \\
\mathbb{F}(U) & \xrightarrow{\delta_{P_1}} & \mathbb{F}(Z)
\end{array}
$$

commutes.
Let $i: Z \to X$ be a closed immersion of open complement $j: U \to X$. We have that $E(i) = E_c(U)$. Let $f: Y \to X$ be a projective lci morphism, the square

\[
\begin{array}{ccc}
E_c(U \times_X Y) & \xrightarrow{f_*} & E_c(U) \\
G_{\chi}^{-1}(T_f) \varphi & \downarrow \varphi & \downarrow \varphi \\
F_c(U \times_X Y) & \xrightarrow{\bar{f}_*} & F_c(U),
\end{array}
\]

commutes.

### 3.1 Applications

The main application we are interested in is the Grothendieck-Riemann-Roch theorem for higher $K$-theory and the Riemann-Roch theorem for the relative cohomology of a morphism. We afterwards review some other Riemann-Roch type formulas and the arithmetic Riemann-Roch theorem. Recall from Example 1.4 that the Chern character is a morphism of strict absolute ring spectra $\text{ch}: KGL_{\mathbb{Q}} \to \bigoplus_i H_{BU}(i)[2i]$. Denote $T_d$ the multiplicative extension of the Todd series $\frac{t}{1-e^{-t}}$ (cf. Paragraph 1.35).

**Theorem 3.10 (Riemann-Roch)** Let $f: Y \to X$ be a projective lci where $f = p \circ i$ and denote $T_f := i^* T_p - N_i \in K_0(Y)$ the virtual tangent bundle. Then the diagram

\[
\begin{array}{ccc}
KH(Y, \mathbb{Q}) & \xrightarrow{f_*} & KH(X, \mathbb{Q}) \\
\text{ch} & \Downarrow \text{ch} & \Downarrow \text{ch} \\
H_M(Y, \mathbb{Q}) & \xrightarrow{f_*} & H_M(X, \mathbb{Q})
\end{array}
\]

commutes. In other words,

\[
\text{ch}(f_*(a)) = f_*(T_d(T_f) \cdot \text{ch}(a)).
\]

**Proof:** The result follows from Theorem 3.5 applied to $\text{ch}$. Recall that $\text{ch}(L) = e^{c_1^{H_{BU}}(L)}$, that $c_1^{KGL}(L) = 1 - L^*$ and that $c_1^{H_\mathbb{B}}$ is additive so, in particular, $c_1^{H_\mathbb{B}}(O_{\mathbb{P}^n}(1)) = -c_1^{H_\mathbb{B}}(O_{\mathbb{P}^n}(-1))$. Denote $x = c_1^{H_\mathbb{B}} = c_1^{H_{BU}}(O_{\mathbb{P}^n}(-1))$ and $y = c_1^{KGL} = c_1^{KGL}(O_{\mathbb{P}^n}(-1))$, we have

\[
\text{ch}(y) = 1 - e^{-x} = x \cdot \frac{1 - e^{-x}}{x}.
\]

Therefore $G = \frac{1 - e^{-x}}{x}$ and $G^{-1} = T_d$ the multiplicative extension of $\frac{t}{1-e^{-t}}$.

**Remark 3.11** Note that in the case over a base field of exponential characteristic due to comparison result of [CD15] this theorem applies to $cdh$-motivic cohomology.

A general Riemann-Roch statement as in [Gil81] follows from the fact that Beilinson motivic cohomology spectrum is universal for spectra with additive orientations (cf. [CD09] 14.2.16 and [Dég14 5.3.9]).
Proposition 3.12 Let $(E, c_1)$ be an oriented absolute ring spectrum in $\text{SH}(S)_\mathbb{Q}$ with $c_1$ additive. Then there exists a unique morphism of absolute spectra

$$\varphi: H_B \to E.$$ 

Moreover, the morphism satisfies that $\varphi_{\mathbb{P}^\infty}(c_1^H) = c_1 \in E^{2,1}(\mathbb{P}^\infty)$.

Recall that some examples of oriented absolute ring spectra with additive orientations are those coming from real absolute Hodge and Deligne-Beilinson cohomology, rigid syntomic cohomology, and mixed Weil theories. Let now $S = \text{Spec}(k)$ for $k$ a perfect field for mixed Weil theories, a field of characteristic zero for real absolute Hodge and Deligne-Beilinson cohomology, or a residue field of a $p$-adic field for rigid syntomic cohomology. The next result follows from the Riemann-Roch theorem and Proposition 3.12:

Corollary 3.13 Let $H$ denote either real absolute Hodge cohomology, real Deligne-Beilinson cohomology, rigid syntomic cohomology or any cohomology coming from a mixed Weil theory. Let $S$ be as above so that $H$ is defined and let $f: Y \to X$ be a projective lci morphism of $S$-schemes. Then, with previous notations, the diagram

$$
\begin{array}{ccc}
K H(Y)_\mathbb{Q} & \xrightarrow{f_*} & K H(X)_\mathbb{Q} \\
\text{Td}(T_f) \downarrow & & \downarrow \text{ch} \\
H(Y) & \xrightarrow{f_*} & H(X)
\end{array}
$$

commutes. In other words, for $a \in K H(Y)_\mathbb{Q}$ we have

$$\text{ch}(f_*(a)) = f_*(\text{Td}(T_f) \cdot \text{ch}(a)).$$

Another general type of morphism of oriented absolute ring spectra to which the motivic Riemann-Roch theorem applies are those coming from algebraic cobordism $\text{MGL}$. Recall that $\text{MGL}$ is the universal oriented absolute ring spectrum (see [Vez01]).

Proposition 3.14 Let $(E, c_1)$ be an oriented absolute ring spectrum. Then there exists a unique morphism of absolute ring spectra

$$\varphi: \text{MGL} \to E$$

such that $\varphi(c_1^{\text{MGL}}) = (c_1) \in E^{2,1}(\mathbb{P}^\infty)$.

Since this morphism preserves the orientation then Theorem 3.1 applies to them.

Corollary 3.15 Let $(E, c_1)$ be an oriented absolute ring spectrum and $f: Y \to X$ be a projective lci morphism. Then, with previous notations, for $a \in \text{MGL}(Y)$ we have

$$\varphi_X(f_*(a)) = f_*(\varphi_Y(a)).$$

We apply the Riemann-Roch theorem for modules 3.7 to the examples we described in Section 1.2.
Theorem 3.16 Let $g: T \to X$ be a morphism of schemes, $f: Y \to X$ be a projective lci morphism and denote $g_Y : T \times_X Y \to Y$ and $T_f \in K_0(Y)$ the virtual tangent bundle of $f$. Assume in addition either $g$ is proper or $f$ is smooth, then the diagram

$$
\begin{array}{ccc}
KH(g_Y)_\mathbb{Q} & \xrightarrow{f_*} & KH(g)_\mathbb{Q} \\
\text{Td}(T_f) \cdot \text{ch} & & \text{ch} \\
H_M(g_Y, \mathbb{Q}) & \xrightarrow{f_*} & H_M(g, \mathbb{Q})
\end{array}
$$

commutes. In other words, for $m \in KH(g_Y)_\mathbb{Q}$ we have

$$\text{ch}(f_*(m)) = f_*(\text{Td}(T_f) \cdot \text{ch}(m)).$$

□

We also obtain the arithmetic Riemann-Roch theorem of [HS15] as a consequence of the Riemann-Roch theorem for modules.

Theorem 3.17 (Arithmetic Riemann-Roch) Let $f: Y \to X$ be a projective morphism between smooth schemes over an arithmetic ring and $T_f \in K_0(Y)$ the virtual tangent bundle. Then the diagram

$$
\begin{array}{ccc}
\hat{KH}(Y)_\mathbb{Q} & \xrightarrow{f_*} & \hat{KH}(X)_\mathbb{Q} \\
\text{Td}(T_f) \cdot \text{ch} & & \text{ch} \\
\hat{H}_M(Y, \mathbb{Q}) & \xrightarrow{f_*} & \hat{H}_M(f, \mathbb{Q})
\end{array}
$$

commutes. In other words, for $m \in \hat{KH}(Y)_\mathbb{Q}$ we have

$$\hat{\text{ch}}(f_*(m)) = f_*(\text{Td}(T_f) \cdot \hat{\text{ch}}(m)).$$

□

A Absolute Hodge cohomology

In this Appendix we apply a theorem of Déglise and Mazzari to give direct construction of the (real) absolute Hodge spectrum representing absolute Hodge cohomology with real coefficients with no smoothness assumption. Since in [HS15] the authors asked explicitly if the Deligne-Beilinson spectrum represented the Deligne-Beilinson cohomology on singular schemes we also prove it for the Deligne-Beilinson spectrum.

We refer to Drew’s thesis [Dre13] for the original construction of the (rational) absolute Hodge spectrum and a more complete treatment on the subject. We refer to [Bei83], [Jan88], [Bur98] and [Bei85], [EV88], [Bur94] for more details of the following constructions for absolute Hodge and Deligne-Beilinson cohomology respectively.

A.1 Let $X$ be a smooth complex variety. We can find a proper complex variety $\bar{X}$ and an open embedding $j: X \to \bar{X}$ such that $D = \bar{X} - X$ is a normal crossing divisor. We denote by $A^*_X(\log D)$ the complex of smooth differential forms with logarithmic singularities along $D$ (cf. [Bur97]). Taking limit over all suitable compactifications we define

$$A^*_\log(X) = \lim A^*_X(\log D)$$
the complex of smooth differential forms with logarithmic singularities along infinity. The complex $A^*_\log(X)$ has a natural filtration $W$ which assigns weight zero to the sections of $A^i(X)$ and weight one to the sections $dz_i/z_i$ and $d\bar{z}_i/z_i$. The complex $A^*_\log(X)$ also has a natural Hodge filtration $F$, as well as a subcomplex $A^*_\log,\log(X)$ of differential forms invariant under complex conjugation. Therefore it defines an $\mathbb{R}$-Hodge complex. The absolute Hodge cohomology of $X$ with real coefficients is defined as

$$H^p_{\text{AH}}(X, \mathbb{R}(q)) = H^p(\tilde{\Gamma}(A^*_\log(X), q))$$

where

$$\tilde{\Gamma}(A^*_\log(X), q) = \text{cone}((2\pi i)^q \tilde{W}2q A^*_\log,\mathbb{R}(X) \oplus \tilde{W}2q \cap F^q A^*_\log(X) \to \tilde{W}2q A^*_\log(X))[-1]$$

and $\tilde{W}$ denotes the decalé filtration of $W$ (cf. [Del71 1.1.2]).

The real Deligne-Beilinson cohomology is obtained by ignoring the weight filtration. That is to say, we define it as

$$H^p_{\text{D}}(X, \mathbb{R}(q)) = H^p(\Gamma(A^*_\log(X), q))$$

where

$$\Gamma(A^*_\log(X), q) = \text{cone}((2\pi i)^q A^*_\log,\mathbb{R}(X) \oplus F^q A^*_\log(X) \to A^*_\log(X))[-1].$$

Both the real absolute Hodge and the Deligne-Beilinson cohomology can also be computed by means of the Thom-Whitney simple introduced in [Nav87]. Following [Bur98], the Thom-Whitney simple has a concrete description in these cases. Denote $L^*_i$ the differential graded commutative $\mathbb{R}$-algebra of algebraic forms over $\mathbb{A}_k^\mathbb{R}$, then the Thom-Whitney simple $\tilde{\Gamma}_{\text{TW}}(A^*_\log(X), q)$ for the real absolute Hodge cohomology is the subcomplex of

$$(2\pi i)^q \tilde{W}2q A^*_\log,\mathbb{R}(X) \oplus \tilde{W}2q \cap F^q A^*_\log(X) \oplus (L^*_i \otimes \tilde{W}2q A^*_\log(X))$$

made by elements $(a, b, \omega)$ such that $\omega(0) = a$ and $\omega(1) = b$. The differential is the natural one on each summand. These complex satisfy that

$$H^p_{\text{AH}}(X, \mathbb{R}(q)) = H^p(\tilde{\Gamma}_{\text{TW}}(A^*_\log(X), q)).$$

**Definition A.2** An arithmetic field is a triple $(k, \Sigma, \text{Fr})$ where $k$ is a field, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a set of embeddings $k \to \mathbb{C}$ and $\text{Fr} : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ is a $\mathbb{C}$-antilinear involution such that the image of $k$ in $\mathbb{C}^\Sigma$ by $\sigma_1 \times \cdots \times \sigma_n$ is invariant under $\text{Fr}$. We call Frobenius to the map $\text{Fr}$. If $X$ is a $k$-scheme we write $X_\Sigma = \prod_i (X \times_{\sigma_i} \text{Spec } k)$, which is naturally a complex variety.

Let $X$ be a smooth scheme over an arithmetic field. The Frobenius $\text{Fr}$ induces a $\mathbb{C}$-linear action on $A^*_\log(X^\Sigma)$ by taking the action it induces on a compactification $X^\Sigma$ together with the complex conjugation. That is to say, $\text{Fr} f(x) = \bar{f}(\text{Fr}(x))$. This action is compatible with the weight and Hodge filtration. We consider

$$\tilde{\Gamma}(A^*_\log(X^\mathbb{R}), q) = \tilde{\Gamma}(A^*_\log(X_\Sigma), q)^{\text{Fr}}, \; \Gamma(A^*_\log(X^\mathbb{R}), q) = \Gamma(A^*_\log(X_\Sigma), q)^{\text{Fr}}.$$

We denote the cohomology they define as

$$H^p_{\text{D}}(X^\mathbb{R}, \mathbb{R}(q)) = H^p(\tilde{\Gamma}(A^*_\log(X^\mathbb{R}), q)), \; H^p_{\text{AH}}(X^\mathbb{R}, \mathbb{R}(q)) = H^p(\Gamma(A^*_\log(X^\mathbb{R}), q)).$$

As before, their respective Thom-Whitney simple also compute the cohomology. In the case of the complex for absolute Hodge cohomology we denote it $\tilde{\Gamma}_{\text{TW}}(A^*_\log(X^\mathbb{R}), q)$.

In [HS13] Holmstrom and Scholbach proved that there exist an absolute spectrum $E_D \in \mathbf{SH}(S)_\mathbb{Q}$ that represents the Deligne-Beilinson cohomology for smooth schemes. In other
words, there exist an absolute spectrum $E_D$ such that for every $X$ smooth and every integers $p, q$ we have

$$E_D^{p,q}(X) = H^p_D(X, \mathbb{R}(q)).$$

This argument has been vastly generalized by Déglise and Mazzari in [DM14, 1.4.10] by giving sufficient conditions for a family of presheaves $X \mapsto F_i(X)$ for $i \in \mathbb{N}$ so that the cohomology they define is represented by an absolute ring spectrum. That is to say, they give sufficient conditions on $(F_i)_{i \in \mathbb{N}}$ so that there exist an absolute ring spectrum $E_F$ satisfying

$$H^p(F_q(X)) = E_F^{p,q}(X).$$

A.3 In order to check that the family $X \mapsto \tilde{\Gamma}_{TW}(A^*_\log(X_\mathbb{R}), i)$ satisfy the hypothesis of loc. cit. let us introduce some notation. Consider $\mathbb{C}$ as an arithmetic field with $\Sigma = \{\text{Id}, \sigma\}$ where $\sigma$ denotes the complex conjugation. Denote $c: \mathbb{R} \to \tilde{\Gamma}_{TW}(A^*_\log(\mathbb{G}_m\mathbb{R}), 1)[1]$ the section given by

$$(d\frac{z}{z} + d\frac{\bar{z}}{\bar{z}})(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}})i, (1-x)(\frac{dz}{z} + d\frac{\bar{z}}{\bar{z}}) + x(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}})i + (\ln z \bar{z} + i \ln \frac{z}{\bar{z}})dx$$

where the first term is in $(2\pi i)^2 \hat{W}_2 A_{\log, 0, \infty}((\mathbb{P}^1)_\mathbb{R})$, the second belongs to $\hat{W}_2 \cap F_1 A_{\log, 0, \infty}((\mathbb{P}^1)_\mathbb{R})$ and the third belongs to $L^*_1 \otimes W_2 A_{\log, 0, \infty}((\mathbb{P}^1)_\mathbb{R})$. For a general arithmetic field we still denote $c$ the section defined by taking $c$ on each component of $\mathbb{G}_m\Sigma$.

Also recall that a distinguished square is a commutative cartesian diagram

$$\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}$$

in $\text{Sm}/S$ such that $Y \to X$ is an open immersion and $X' \to X$ is étale and induces an isomorphism $(X' \setminus Y')_{\text{red}} \to (X \setminus Y)_{\text{red}}$. We say that a complex of presheaves of $R$-modules $F$ on $\text{Sm}/S$ has the Brown-Gersten property with respect to the Nisnevich topology if for every distinguished square the diagram

$$\begin{array}{ccc}
F(X) & \longrightarrow & F(X') \\
\downarrow & & \downarrow \\
F(Y) & \longrightarrow & F(Y')
\end{array}$$

is a homotopy pullback square in the category of complexes of $R$-modules.

**Proposition A.4** Let $k$ be an arithmetic field and denote $S = \text{Spec}(k)$. Consider the family of presheaves $X \mapsto \tilde{\Gamma}_{TW}(A^*_\log(X_\mathbb{R}), i)$ on $\text{Sm}/S$ together with the section $c: \mathbb{R} \to \tilde{\Gamma}_{TW}(A^*_\log(\mathbb{G}_m\mathbb{R}), 1)[1]$ defined above:

1. They form an $\mathbb{N}$-commutative graded monoid (cf. [DM14, 1.4.9]) in the category of complexes of $\mathbb{R}$-linear presheaves on the category of affine and smooth $S$-schemes.
2. They have the Brown-Gersten property with respect to the Nisnevich topology.
3. They are homotopy invariant, i.e., $H^p_{\text{AH}}(A^*_X, \mathbb{R}(q)) = H^p_{\text{AH}}(X, \mathbb{R}(q))$. 

42
4. If \( \bar{c} \in H^1_{AH}(G_m, \mathbb{R}(1)) \) denotes the class of \( c \), then for any smooth scheme \( X \) and any integers \( p, q \) the map

\[
H^p_{AH}(X_\mathbb{R}, \mathbb{R}(q)) \rightarrow H^{p+1}_{AH}((X \times G_m)_\mathbb{R}, \mathbb{R}(q+1))/H^p_{AH}(X_\mathbb{R}, \mathbb{R}(q))
\]

where \([ \_ ]\) denotes the class defined in the quotient, is an isomorphism.

5. If \( u: G_m \rightarrow G_m \) is the inverse map of the group scheme \( G_m \) and \( c' \) is the image of \( c \) in \( H^1_{AH}(G_m, \mathbb{R}(1)) \) then \( u^*(c') = -c' \).

Proof: Although these properties are well known for experts let us review them for the sake of completeness. The Thom-Whitney simple has a well defined associative and commutative product (cf. \[Bur97\] §6), from which point 1 follows.

For point 2, first notice that the Brown-Gersten property is stable by direct sums and cones of maps. Therefore it is enough to prove it for \( \tilde{W}_q A_{log, \mathbb{R}} \), \( W_{2q} \cap F^{q} A_{log} \) and \( \tilde{W}_q A_{log} \). The étale descent for \( A_{log, \mathbb{R}} \), \( F^{q} A_{log} \) and \( A_{log} \) may be found in \[HS15\] 2.9, from which the Brown-Gersten property follows. For the weight filtration, consider a distinguished square as in \[A.3\]. We have a short exact sequence

\[
0 \rightarrow A_{log}(X) \rightarrow A_{log}(X') \bigoplus A_{log}(Y) \rightarrow A_{log}(Y') \rightarrow 0
\]

as well as for \( A_{log, \mathbb{R}} \) and \( F^{q} A_{log} \). These morphisms are strict both for the Hodge and the decalé Weight filtration, so the Brown-Gersten property readily follows for \( \tilde{W}_q A_{log, \mathbb{R}} \), \( W_{2q} \cap F^{q} A_{log} \) and \( \tilde{W}_q A_{log} \). We conclude by taking invariants on the action induced by the Frobenius.

For point 4 note that the absolute Hodge cohomology of \( G_m \) is null apart from the groups \( H^0(G_m, \mathbb{R}(0)) = \mathbb{R} \) and \( H^1(G_m, \mathbb{R}(1)) = \mathbb{R} \). It is easy to see from the definition of absolute Hodge cohomology that group \( H^{p+1}_{AH}(X_\mathbb{R}, \mathbb{R}(q+1)) \)

\[
(H^p_{AH}(X_\mathbb{R}, \mathbb{R}(q)) \otimes H^1_{AH}(G_m, \mathbb{R}(1))) \oplus (H^{p+1}_{AH}(X_\mathbb{R}, \mathbb{R}(q+1)) \otimes H^0_{AH}(G_m, \mathbb{R}(0)))
\]

from which point 4 follows. Finally, a direct computation concludes point 5.

Corollary A.5 Denote \( E_{AH} \) the oriented absolute ring spectrum constructed in \[DM14\] 1.4.10 out of the presheaves \( \big( T_{TW}(A_{log}(-, \mathbb{R}), i) \big)_{i \in \mathbb{N}} \), which we call the absolute Hodge spectrum. Then \( E_{AH} \) represents real absolute Hodge cohomology on smooth schemes. In other words, for any smooth \( S \)-scheme \( X \) and any \( p, q \geq 0 \)

\[
E^{p,q}_{AH} = H^p_{AH}(X_\mathbb{R}, \mathbb{R}(q)).
\]

Let us now prove that the absolute Hodge spectrum represents absolute Hodge cohomology for general schemes. The same method will apply also for the Deligne-Beilinson cohomology.

A.6 Let \( Z \) be complex variety, following \[Del74\] 8.3 we can find a diagram \( \bar{X}' \hookrightarrow X' \rightarrow Z \) so that \( X' \) and \( \bar{X}' \) are simplicial complex varieties, \( p \) satisfies cohomological descent (in particular, it is a hypercovering for the \( h \)-topology), \( \bar{X}' \) is proper smooth and \( \bar{X}' - X' \) is a normal crossing divisor. If \( \bar{X}' \hookrightarrow X' \rightarrow Z \) is a second diagram we have the isomorphisms
$H_{AH}(X, \mathbb{R}) \simeq H_{AH}(X', \mathbb{R})$ and $H_{D}(X, \mathbb{R}) \simeq H_{D}(X', \mathbb{R})$ for both the absolute Hodge and Deligne-Beilinson cohomology of those simplicial varieties. Therefore, we call the absolute Hodge and the Deligne-Beilinson cohomology of $Z$ to

$$H_{AH}(Z, \mathbb{R}) := H_{AH}(X, \mathbb{R}) \quad H_{D}(Z, \mathbb{R}) := H_{D}(X, \mathbb{R})$$

This construction is compatible with the Frobenius action so we define the groups $H_{AH}(Z, \mathbb{R}) := H_{AH}(X, \mathbb{R})$ and $H_{D}(Z, \mathbb{R}) := H_{D}(X, \mathbb{R})$.

Recall that any rational oriented absolute ring spectrum is also a Beilinson motive (cf. [CD09, 14.2.16]). The category of Beilinson motives $\text{DM}_B(S)$ satisfies the $h$-descent (cf. [CD09, 3.1]). This implies that if $X$ is a scheme and $X \to X$ is a $h$-hypercover then for any oriented absolute ring spectrum we have

$$E(X) = E(X),$$

where $E(X)$ denotes the cohomology of the simplicial scheme (cf. [CD09 §3.1] or [DM14, §2.2.1] for $\text{DM}_B$).

**Corollary A.7** Let $E_{AH}$ and $E_{D}$ be the real absolute Hodge and Deligne-Beilinson spectra, then for any scheme $Z$ over an arithmetic field and every $p, q \geq 0$ we have

$$E^{p,q}_{AH}(Z) = H^{p}_{AH}(Z, \mathbb{R}(q)) \quad \text{and} \quad E^{p,q}_{D}(Z) = H^{p}_{D}(Z, \mathbb{R}(q)).$$

**References**

[Ada74] J.F. Adams: *Stable homotopy and generalised homology*. University of Chicago Press (1974)

[Ayo07] J. Ayoub: *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique I-II*. Astérisque, no. 314-315 (2007)

[Bei83] A. Beilinson: *Notes on absolute Hodge cohomology*. Applications of algebraic $K$-theory to Algebraic Geometry and Number Theory (S. Bloch, ed.), Contemporary Mathematics, vol. 55, AMS, 35-68 (1983)

[Bei85] A. Beilinson: *Higher regulators and values of $L$-functions*. J. Soviet Math. 30, 2036-2070 (1985)

[BFM75] P. Baum, W. Fulton and R. MacPherson: *Riemann-Roch for singular varieties*. Publ. Math. IHES, no. 45, 101-145 (1975)

[BMS87] A. Beilinson, R. MacPherson and V. Schechtman: *Notes on motivic cohomology*. Duke Math Journal, vol. 54, 2, 679-710 (1987)

[BS58] A. Borel and J.P. Serre: *Le théorème de Riemann-Roch*. Bull. Soc. Math. France, 86, 97-136 (1958)

[Bur94] J.I. Burgos: *A $C^\infty$-logarithmic Dolbeaut complex*. Compositio Math. 92, 61-86 (1994)

[Bur97] J.I. Burgos: *Arithmetic Chow rings and the Deligne-Beilinson cohomology*. J. Algebraic Geom; 6 (2), 335-377 (1997)
[Nav87] V. Navarro Aznar: *Sur la théorie de Hodge-Deligne*. Invent. Math. 90, 11-76 (1987)

[Nav16] A. Navarro: *The Riemann-Roch theorem and Gysin morphism in arithmetic geometry*. PhD. Dissertation (2016)

[Nav16b] A. Navarro: *On Grothendieck’s Riemann-Roch theorem*. arXiv:1603.06740 (2016)

[Pan04] I. Panin: *Riemann-Roch theorems for oriented cohomologies*. Axiomatic, enriched and motive homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 261-333 (2004)

[Pan09] I. Panin: *Oriented cohomology theories of algebraic varieties II*. Homology, Homotopy Appl. 11, no. 1, 349-405 (2009)

[PPR08] I. Panin, K. Pimenov and O. Röndigs: *A universality theorem for Voevodsky’s algebraic cobordism spectrum*. Homology, Homotopy Appl. 10, no. 2, 211-226 (2008)

[Qui73] D. Quillen: *Higher algebraic K-theory, I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, pp. 85-147. Lecture Notes in Math., Vol. 341. (1973)

[Rio07] J. Riou: *Exposé XVI Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents*. arXiv:1207.3648 (2007)

[Rio10] J. Riou: *Algebraic K-theory, \(\mathbb{A}^1\)-homotopy and Riemann-Roch theorems*. J. Topol. 3, no. 2, 229-264 (2010)

[SGA6] P. Berthelot, A. Grothendieck, L. Illusie: *Séminaire de Géométrie Algébrique du Bois Marie - 1966-67 - Théorie des intersections et théorème de Riemann-Roch - (SGA 6)*. Lecture notes in mathematics, vol. 225 (1971)

[Spi12] M. Spitzweck: *A commutative \(\mathbb{P}^1\)-spectrum representing motivic cohomology over Dedekind domains*. arXiv:1207.4078 (2012)

[Tho93] R.W. Thomason: *Les K-groupes d’un schéma éclaté et une formule d’intersection excédentaire*, Invent. Math. 112, no. 1, 195-215 (1993)

[Vez01] G. Vezzosi: *Brown-Peterson spectra in the stable \(\mathbb{A}^1\)-homotopy theory*. Rend. Sem. Mat. Univ. Padova 106, 47-64 (2001)

[Voe98] V. Voevodsky: *\(\mathbb{A}^1\)-homotopy theory*. Doc. Math. ICM 1998, vol. I, 579-604 (1998)

[Wei13] C. Weibel: *The K-book*. Grad. Studies in Math; Vol. 145 (2013)

[Wei89] C. Weibel: *Homotopy invariant K-theory*. Contemporary Mathematics, vol. 83, 461-488 (1989)