THE UNIFICATION OF ELECTROMAGNETISM AND GRAVITATION IN THE CONTEXT OF QUANTIZED FRACTAL SPACE TIME

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Abstract

The attempts to unify electromagnetism and gravitation have included the formulations of Herman Weyl and the Kaluza Klein theory with the fifth dimension. More recently there have been fruitful attempts in the domain of Quantum Superstrings and the author’s formulation in terms of Quantum Mechanical Kerr-Newman Black Holes. Though all these appear to be widely divergent approaches, they are shown to have a unified underpinning in the context of quantized fractal space time.

1 Introduction

The problem of the unification of gravitation and electromagnetism has a long and as yet uncompleted history. The very early attempt of Weyl did not find favour as he had incorporated electromagnetism into the field equation, as it were from outside. The ingenious suggestion of Kaluza that a fifth, and somehow suppressed, dimension be introduced also did not find favour,

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though the idea has resurfaced in the theory of Quantum Superstrings[4]. Another approach has been that of Quantum Mechanical Black Holes which recovers the Kerr-Newman metric in the context of quantized fractal space time (QFST)[5, 6, 7]. While there is formal resemblance to the Weyl formulation on the one hand, there is an imaginary shift of coordinates, quite meaningful in the above context, on the other. In the derivation of the Kerr-Newman metric too[8], there is a mysterious imaginary shift which however is not explicable in classical theory. Finally both in Quantum Superstrings and the (QFST) or quantized fractal space time picture, the underlying geometry is non-commutative[9, 10, 4].

So there are in these scenarios several at first sight disparate and even inexplicable strands: the Kaluza-Klein curled up extra dimension, Weyl’s electromagnetism, Quantum Mechanical Black Holes, Quantum Superstrings, the Kerr-Newman metric, non-commutative geometry.... We will now show that in fact all these characteristics have a unified underpinning in the context of quantified fractal space time (QFST).

2 The Kaluza-Klein and Weyl Formulations

Our starting point is the fact that the fractal dimension of a Brownian quantum path is 2, as pointed out by Abbott and Wise, Nottale and others[11, 12]. This was further analysed by the author and it was explained that this is symptomatic of quantized fractal space time and it was shown that in fact the coordinate $x$ becomes $x + i\epsilon t$[13], reminiscent of El Naschie’s complex time[14] and the Hawking-Hartle static time[15]. The complex coordinates or equivalently non-Hermitian position operators are symptomatic of the unphysical zitterbewegung which is eliminated after an averaging over the Compton scale. In this picture the fluctuational creation of particles is taken into account in a consistent cosmological scheme[16].

It is well known that the generalization of complex $x$ coordinate to three dimensions leads to quarternions[17], and the Pauli spin metrics. We next come to a model of an electron as a Quantum Mechanical Black Hole and widely discussed by the author (Cf. example[5, 6]). In this model an electron is a spinning shell (reminiscent of Dirac’s model[18]) of radius equalling the Compton wavelength, and equivalently a Kerr-Newman Black Hole. Within this Compton scale Black Hole we encounter the unphysical
zitterbewegung region of complex (or non-Hermitian) coordinates already alluded to.

Infact electromagnetism was deduced in two ways. The first was by considering an imaginary shift,

\[ x^\mu \rightarrow x^\mu + i a^\mu, \ (a^\mu \sim \text{Compton scale}) \] (1)

in a Quantum Mechanical context. This lead to

\[ i\hbar \frac{\partial}{\partial x^\mu} \rightarrow i\hbar \frac{\partial}{\partial x^\mu} + \frac{\hbar}{a^\mu} \] (2)

and the second term on the right side of (2) was shown to be the electromagnetic vector potential \( A^\mu \),

\[ A^\mu = \frac{\hbar}{a^\mu} \] (3)

The second was by taking into account the fact that at the Compton scale, it is the so called negative energy two spinors \( \chi \) of the Dirac bispinor that dominate where,

\[ \chi \rightarrow -\chi \]

under reflections. This lead to the tensor density property,

\[ \frac{\partial}{\partial x^\mu} \to \Gamma^\mu_{\nu} \] (4)

the second term on the right side of (4) being identified with \( A^\mu \),

\[ A^\mu = h \Gamma^\mu_{\nu} \] (5)

It was pointed out that (5) is formally and mathematically identical to Weyl’s original formulation, except that here it arises due to the purely Quantum Mechanical spinorial behaviour whereas Weyl had put it by hand.

Another early scheme for the unification of gravitation and electromagnetism as referred to earlier was that put forward by Kaluza and Klein in which an extra dimension was introduced and taken to be curled up. This idea has resurfaced in recent years in String Theory.

We will first show that the characterization of \( A^\mu \) in (3) is identical to a Kaluza Klein formulation. Then we will show that equations (4) and (5) really denote the fact that the geometry around an electron is non-integrable.
Finally we will show that in fact both (2) or (3) and (4) or (5) are the same formulations. We first observe that the transformation (5) can be written as,

\[ x^i \rightarrow x^i + \alpha_{i5} x^5 \]  

(6)

where \( \alpha_{i5} \) in (6) will represent a small shift from the Minkowski metric \( g_{ij} \), and \( i, j = 1, 2, 3, 4, 5 \), \( x^5 \) being a fifth coordinate introduced for purely mathematical conversion. Owing to (6), we will have,

\[ g_{ij} dx^i dx^j \rightarrow g_{ij} dx^i dx^j + (g_{ij} \alpha_j^5) dx^i dx^5 \]  

(7)

In Kaluza's formulation,

\[ A_\mu \propto g_\mu^5 \]  

(8)

Comparison of (6) and (7) with (1) and (3) shows that indeed this is the case. That is, the formulation given in (1) and (2) could be thought of as introducing a fifth curled up dimension, as in the Kaluza-Klein theory. To see why the Quantum Mechanical formulation (4) and (5) corresponds to Weyl's theory, we start with the effect of an infinitesimal parallel displacement of a vector [Bergmann].

\[ \delta a^\sigma = -\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu \]  

(9)

As is well known, (9) represents the extra effect in displacements, due to the curvature of space - in a flat space, the right side would vanish. Considering partial derivatives with respect to the \( \mu^{th} \) coordinate, this would mean that, due to (9)

\[ \frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu \]  

(10)

The second term on the right side of (10) can be written as:

\[ -\Gamma^\lambda_{\mu\nu} g^\nu_\lambda a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma \]

where we have utilized the property that in the above formulation (Cf.refs.[5,6,7]),

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]
\( \eta_{\mu\nu} \) being the Minkowski metric and \( h_{\mu\nu} \), a small correction whose square is neglected.

That in, (10) becomes,

\[
\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu} \tag{11}
\]

The relation (11) is the same as the relation (4). We will next show the correspondence between (11) or (5) or (4) and (3) or (2). To see this simply we note that the geodesic equation is,

\[
\dot{u}^\mu \equiv \frac{du^\mu}{ds} = \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma \tag{12}
\]

We also use the fact that in the Quantum Mechanical Black Hole model referred to, we have [5]

\[
u^\mu = c \quad \text{for} \quad \mu = 1, 2 \quad \text{and} 3,
\]

while,

\[
|\dot{u}^\mu| = |u^\mu| \frac{mc^2}{h}
\]

So, from (12) we get,

\[
\Gamma^\mu_{\nu\mu} = \frac{1}{a^\nu}, |a^\nu| = \frac{\hbar}{mc}
\]

This establishes the required identity.

3 Quantized Fractal Space time and Quantum Superstrings

It was shown [9] that the quantized fractal space time referred to really leads to a non-commutative geometry, not surprisingly:

\[
[x, y] = 0(l^2), \ [x, p_x] = i\hbar[1 + l^2], \ [t, E] = i\hbar[1 + \tau^2], \cdots \tag{13}
\]

It was shown earlier [9] that these relations directly lead to the Dirac equation: Quantized fractal space time or the above relation (13) are the underpinning for Quantum Mechanical spin or the Quantum Mechanical Black Hole, that is ultimately equations like (2) or (3) or (4) or (5).
It is also true that both the Kaluza Klein formulation and the non commutative geometry \((\text{13})\) hold in the theory of Quantum Superstrings. Infact we get from here a clue to the mysterious six extra curled up dimensions of Quantum Superstring Theory. For this we observe that \((\text{13})\) gives an additional contribution to the Heisenberg Uncertainity Principle and we can easily deduce

\[
\Delta p \Delta x \sim \hbar l^2
\]

Remembering that at this Compton scale

\[
\Delta p \sim mc
\]

It follows that

\[
\Delta x \sim l^3 \; \text{(14)}
\]

as \(l \sim 10^{-11} \text{cms}\) for the electron we recover from \((\text{14})\) the Planck Scale, as well as a rationale for the peculiar fact that the Planck Scale is the cube of the electron Compton scale.

More importantly, what \((\text{14})\) shows is, that at this level, the single dimension along the \(x\) axis shows up as being three dimensional. That is there are two extra dimensions, in the unphysical region below the Compton scale. As this is true for the \(y\) and \(z\) coordinates also, there are a total of six curled up or unphysical or inaccessible dimensions in the context of the preceding section.

## 4 Discussion and Conclusion

If we start with equations \((\text{1})\) to \((\text{3})\) which were related to QFST (Quantized Fractal space time) and the non-commutative relation \((\text{13})\) we obtain a unification of electromagnetism and gravitation. On the other hand if we consider the spinorial behaviour of the Dirac wave function, we get \((\text{4})\) or \((\text{5})\). The former has been seen to be the same as the Kaluza formulation while the latter is formally similar to the Weyl formulation - but in this case \((\text{5})\) is not put in by hand. Rather it is a Quantum Mechanical consequence. We have thus shown that these two approaches are the same. The extra dimensions are thus seen to be confined to the unphysical Compton scale - classically speaking they are curled up or inaccessible.

In a sense this is not surprising. The bridge between the two approaches was
the Kerr-Newman metric which uses, though without a clear physical meaning in classical theory, the transformation (1). The reason why an imaginary shift is associated with spin is to be found in the Quantum Mechanical zitterbewegung and the consequent QFST.

Wheeler remarked [19], “the most evident shortcoming of the geometrodynamical model as it stands is this, that it fails to supply any completely natural place for spin 1/2 in general and for the neutrino, in particular”, while ”it is impossible to accept any description of elementary particles that does not have a place for spin half.” Infact the bridge between the two is the transformation (1). It introduces spin half into general relativity and curvature to the electron theory, via the equation (3) or (4).

In this context it is interesting to note that El Naschie has given the fractal formulation of gravitation [20].

Thus apparently disparate concepts like the Kaluza Klein and Weyl formulations, Quantum Mechanical Black Holes, quantized fractal space time and QSS are seen to have a harmonious overlap, in the context of QFST with its roots in the fluctuational creation of particles [21].

References

[1] Bergmann, P.G., ”Introduction to the Theory of Relativity”, Prentice-Hall (New Delhi), 1969, p162ff, p.245ff.

[2] Einstein, A., ”The Meaning of Relativity”, Oxford & IBH, New Delhi, 1965, pp.93-94.

[3] Kaluza, Th., ”On the unification problem in Physics” in ”An Introduction to Kaluza-Klein Theories”, Ed., H.C. Lee, World Scientific, Singapore, 1984, p.1ff.

[4] Witten, W., Physics Today, April 1996, pp.24-30.

[5] Sidharth, B.G., Ind. J. Pure and Applied Phys., 35 (7), 1997, 456.

[6] Sidharth, B.G., Int.J.Mod.Phys.A., 13(15), 1998, pp2599-2612.

[7] Sidharth, B.G., Gravitation & Cosmology, Vol.4, No.2 (14), 1998, pp.1-5.
[8] Newman, E.T., J. Math. Phys. 14, 1, pp.102ff, 1973.

[9] Sidharth, B.G., Chaos, Solitons & Fractals, 2000, 11(8), 1269-1278.

[10] Ne'eman, Y., in "Proceedings of Frontiers of Fundamental Physics", Eds.B.G. Sidharth and A. Burinskii, Universities Press, Hyderabad, 1998, p.83-96.

[11] Abbott L.F., and Wise, M.B., AMJ Phys., (1981) 49, 37-39.

[12] Nottale, L., "Fractal Space-Time and Microphysics, World Scientific, Singapore, 1993, p.110-190.

[13] Sidharth, B.G., "Space Time as a Random Heap", to appear in Chaos, Solitons & Fractals.

[14] El Naschie, M.S., On conjugate time and information in relativistic quantum theory, Chaos, Solitons & Fractals, 1995, 5, 1551-1555.

[15] Prigogine, I., "End of Uncertainty", Free Press, New York, 1997, p.170.

[16] Sidharth, B.G., "Issues in Quantized Fractal Space Time", to appear in Chaos, Solitons and Fractals.

[17] Sachs, M., "General Relativity and Matter" (1982), D. Reidel Publishing Company, Holland, p.45ff.

[18] Dirac, P.A.M., Proc.Roy.Soc (London) A268, 1962, p.57.

[19] Misner, C.W., Thorne, K.S., and Wheeler, J.A., "Gravitation", W.H. Freeman, San Francisco, 1973, pp.448, 1200.

[20] El Naschie, M.S., Chaos, Solitons and Fractals, 8(11),1997, p.1865-1872.

[21] Sidharth, B.G., "Fluctuation and Interaction", to appear in Chaos, Solitons & Fractals.