An algorithm for computing the global basis of an irreducible $U_q(sp_{2n})$-module

Cédric Lecouvey
lecouvey@math.unicaen.fr

Abstract

We describe a simple algorithm for computing the canonical basis of any irreducible finite-dimensional $U_q(sp_{2n})$-module.

1 Introduction

The quantum algebra $U_q(g)$ associated to a semisimple Lie algebra $g$ is the $q$-analogue introduced by Drinfeld and Jimbo of its universal enveloping algebra $U(g)$. Kashiwara [4] and Lusztig [10] have discovered a distinguished basis of $U_q^-(g)$ which projects onto a global crystal basis (Kashiwara) or canonical basis (Lusztig) of each simple finite-dimensional $U_q(g)$-module. When $q$ tends to 1 this basis yields in particular a canonical basis of the corresponding $U(g)$-module.

In this article, we restrict ourselves to the case $g = sp_{2n}$. Denote by $\{\Lambda_1, ..., \Lambda_n\}$ the set of fundamental weights, by $P$ the weight lattice and by $P^+$ the set of dominant weights of $sp_{2n}$. Then for each $\lambda = \sum \lambda_i \Lambda_i \in P^+$ there exists a unique irreducible finite-dimensional $U_q(sp_{2n})$-module $V(\lambda)$. The aim of this article is to describe a simple algorithm for computing the global crystal basis of $V(\lambda)$. This algorithm will be a generalization of the one described by Leclerc and Toffin in [8] for the irreducible $U_q(sl_n)$-modules. In the case of $U_q(sp_{2n})$, an algorithm was only known for the fundamental modules $V(\Lambda_p)$, $p = 1, ..., n$ [11]. We note that Zelevinsky and Retakh [13] have described for every irreducible finite-dimensional $U(sp_4)$-module a so-called good basis. This basis is the specialization at $q = 1$ of the dual of the canonical basis.

Our method is as follows. First we realize $V(\Lambda_p)$ as a subrepresentation of a $U_q(sp_{2n})$-module $W(\Lambda_p)$ whose basis $\{v_C\}$ has a natural indexation in terms of column shaped Young tableaux. This representation $W(\Lambda_p)$ may be regarded as a $q$-analogue of the $p$-th exterior power of the vector representation of $sp_{2n}$. Then we give explicit formulas for the expansion of the global crystal basis of $V(\Lambda_p)$ on the basis $\{v_C\}$. Next we embed $V(\lambda)$ in the tensor product $W(\lambda) = W(\Lambda_1) \otimes \lambda_1 \otimes \cdots \otimes W(\Lambda_n) \otimes \lambda_n$.

The tensor product of the crystal bases of the $W(\Lambda_p)$'s is a natural basis $\{v_\tau\}$ of $V(\lambda)$ indexed by combinatorial objects $\tau$ called tabloids. Then we obtain an intermediate basis of $V(\lambda)$ fixed by the involution $q \mapsto q^{-1}$ and such that the transition matrix from this basis to the global crystal basis of $V(\lambda)$ is unitriangular. Finally we compute the expansion of the canonical basis on the basis $\{v_\tau\}$ via an elementary algorithm. We give as an example the matrix associated to the expansion of the global basis of a weight space of the $U_q(sp_6)$-module $V(\lambda)$ with $\lambda = \Lambda_1 + \Lambda_2 + 2\Lambda_3$.

2 Background

In this section we briefly review the basic facts that we shall need concerning the representation theory of $U_q(sp_{2n})$ and the notions of crystal basis and canonical basis of a $U_q(sp_{2n})$-module. The reader is referred to [6], [5], [1] and [2] for more details.
2.1 The quantum enveloping algebra $U_q(sp_{2n})$

Recall the Dynkin diagram of $sp_{2n}$:

$$
\frac{1}{0} - \frac{2}{0} - \frac{3}{0} \cdots \frac{n-2}{0} - \frac{n-1}{0} \leftrightarrow \frac{n}{0}.
$$

Accordingly, the Cartan matrix $A = (a_{i,j})$ of $sp_{2n}$ is:

$$
\begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& & \ddots & \ddots \\
& & & 2 & -1 \\
& & & -2 & 2
\end{pmatrix}
$$

with rows and columns indexed by $\{1, \ldots, n\}$. Given a fixed indeterminate $q$ set

$$q_i = \begin{cases} q & \text{if } i \neq n \\ q^2 & \text{if } i = n \end{cases}.$$

$$[n]_i = \frac{q^n - q^{-n}}{q_i - q_i^{-1}} \quad \text{and} \quad [n]_i! = [n]_i[n]_i! \cdots [1]_i.$$

The quantized enveloping algebra $U_q(sp_{2n})$ is the associative algebra over $C(q)$ generated by $e_i, f_i, q^{h_i}, q^{-h_i}, i = 1, \ldots, n$, subject to the relations:

$$q^{h_i}q^{-h_i} = q^{-h_i}q^{h_i} = 1,$$

$$q^{h_i}q^{h_j} = q^{h_j}q^{h_i},$$

$$q^{h_i}e_jq^{-h_i} = q^{a_{i,j}}e_j,$$

$$q^{h_i}f_jq^{-h_i} = q^{-a_{i,j}}f_j,$$

$$[e_i, f_i] = \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \delta_{i,j},$$

where

$$t_i = \begin{cases} q^{h_i} & \text{if } i \neq n \\ q^{2h_n} & \text{if } i = n \end{cases}.$$  

\[
\text{if } i \neq j \sum_{k=0}^{1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{i,j}-k)} = \sum_{k=0}^{1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{i,j}-k)} = 0
\]

where $e_i^{(m)} = e_i^m /[m]_i!$ and $f_i^{(m)} = f_i^m /[m]_i!$.

The subalgebra of $U_q(sp_{2n})$ generated by $e_i, f_i, q^{h_n}, q^{-h_i}, i = 1, \ldots, n-1$ is isomorphic to $U_q(sl_n)$, the quantum enveloping algebra of $sl_n$.

The representation theory of $U_q(sp_{2n})$ is closely parallel to that of its classical counterpart $U(sp_{2n})$. The weight lattice $P(U_q(sp_{2n}))$ is the $\mathbb{Z}$-lattice generated by the fundamental weights $\Lambda_1, \ldots, \Lambda_n$. We denote by $P_+$ the set of dominant weights of $U_q(sp_{2n})$ i.e. those of the form $\lambda_1 \Lambda_1 + \cdots + \lambda_n \Lambda_n$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$. Let $M$ be a $U_q(sp_{2n})$-module. For every $\mu \in P$ the subspace

$$M_\mu = \{v \in M, q^{h_i}v = q^{<h_i, \mu>}v, \ i = 1, \ldots, n\}$$

is the weight space of weight $\mu$ of $M$. A vector $v_\lambda$ is said to be of highest weight when $e_i(v_\lambda) = 0$ for $i = 1, \ldots, n$. If $M$ is a finite-dimensional irreducible module, $M$ is a highest weight module that is, contains a highest weight vector $v_\lambda$ of weight $\lambda$ such that $M = U_q(sp_{2n})v_\lambda$. Then $\dim M_\lambda = 1$ and $\lambda$ is a dominant weight. Conversely, for each dominant weight $\lambda \in P_+$, there is a unique finite-dimensional module with highest weight $\lambda$. We denote it by $V(\lambda)$ and we write $v_\lambda$ for a fixed highest weight vector.

Given two $U_q(sp_{2n})$-modules $M$ and $N$, we can define a structure of $U_q(sp_{2n})$-module on $M \otimes N$ by putting:

$$q^{h_i}(u \otimes v) = q^{h_i}u \otimes q^{h_i}v,$$

$$e_i (u \otimes v) = e_i u \otimes t_i^{-1}v + u \otimes e_i v,$$

$$f_i (u \otimes v) = f_i u \otimes v + t_i u \otimes f_i v.$$
2.2 Crystal basis for $U_q(sp_{2n})$-modules

Let $M$ be a finite-dimensional $U_q(sp_{2n})$-module. Fix $i \in \{1, ..., n\}$. Let $u \in M$ and suppose that $u \in M_\mu$. Then $u$ can be written uniquely as a finite sum $\sum_k f_i^{(k)} u_k$ where $u_k \in M_{\mu+\kappa_i}$ and $e_i u_k = 0$. Kashiwara’s operators $\tilde{e}_i$ and $\tilde{f}_i$ are defined by:

$$\tilde{e}_i u = \sum f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum f_i^{(k+1)} u_k.$$  

(10)

Denote by $A$ the subalgebra of $\mathbb{C}(q)$ consisting of the rational functions without pole at $q = 0$. Let $L$ be a free $A$-submodule of $M$ such that $L \otimes \mathbb{C}(q) = M$ and $B$ a basis of the $\mathbb{Q}$-vector space $L/qL$. Write $\pi$ for the canonical projection

$$L \overset{\pi}{\to} L/qL$$

Set $L_\mu = L \cap M_\mu$ and $B_\mu = B \cap L_\mu/qL_\mu$. Then $(L, B)$ is a crystal basis of $M$ at $q = 0$ if the following conditions hold:

$$\begin{align*}
\text{(i)} & \quad L = \bigoplus_{\mu \in P} L_\mu \quad \text{and} \quad B = \bigcup_{\mu \in P} B_\mu, \\
\text{(ii)} & \quad \tilde{e}_i L \subset L \quad \text{and} \quad \tilde{f}_i L \subset L, \\
\text{(iii)} & \quad \tilde{e}_i B \subset B \cup \{0\} \quad \text{and} \quad \tilde{f}_i B \subset B \cup \{0\}, \\
\text{(iv)} & \quad \text{for } b_1, b_2 \in B \quad \text{and} \quad i \in \{1, ..., n\}, \quad \tilde{e}_i b_1 = b_2 \iff b_1 = \tilde{f}_i b_2.
\end{align*}$$  

(11)

Note that the action of $\tilde{e}_i$ and $\tilde{f}_i$ on $L/qL$ is well defined because of (ii). Kashiwara [3] has proved that every finite-dimensional $U_q(sp_{2n})$-module $M$ has a crystal basis. Moreover if $M = V(\lambda)$ is simple, this basis is unique up to an overall scalar factor. We shall denote it by $(L(\lambda), B(\lambda))$.

If $(L, B)$ and $(L', B')$ are crystal bases of the finite-dimensional $U_q(sp_{2n})$-modules $M$ and $M'$, then $(L \otimes L', B \otimes B')$ with $B \otimes B' = \{b \otimes b'; b \in B, b' \in B'\}$ is a crystal basis of $M \otimes M'$. The action of $\tilde{e}_i$ and $\tilde{f}_i$ on $B \otimes B'$ is given by:

$$\tilde{f}_i (u \otimes v) = \begin{cases} 
\tilde{f}_i (u) \otimes v \quad \text{if } \varphi_i (u) > \varepsilon_i (v) \\
u \otimes \tilde{f}_i (v) \quad \text{if } \varphi_i (u) \leq \varepsilon_i (v)
\end{cases}$$  

(12)

and

$$\tilde{e}_i (u \otimes v) = \begin{cases} 
 u \otimes \tilde{e}_i (v) \quad \text{if } \varphi_i (u) < \varepsilon_i (v) \\
\tilde{e}_i (u) \otimes v \quad \text{if } \varphi_i (u) \geq \varepsilon_i (v)
\end{cases}$$  

(13)

where $\varepsilon_i (u) = \max\{k; \tilde{e}_i^k (u) \neq 0\}$ and $\varphi_i (u) = \max\{k; \tilde{f}_i^k (u) \neq 0\}$.

The set $B$ may be endowed with a combinatorial structure called the crystal graph of $M$. Crystal graphs are oriented colored graphs with colors $i \in \{1, ..., n\}$. An arrow $a \overset{i}{\to} b$ means that $\tilde{f}_i (a) = b$ and $\tilde{e}_i (b) = a$. The decomposition of $M$ into its irreducible components is reflected into the decomposition of $B$ into its connected components. The crystal graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. A vertex $v^0 \in B$ satisfying $\tilde{e}_i (v^0) = 0$ for $i \in \{1, ..., n\}$ is called a highest weight vertex. The crystal $B(\lambda)$ contains a unique highest weight vertex.

The end of this section is devoted to Kashiwara-Nakashima’s combinatorial description of the crystal graphs of the finite-dimensional irreducible $U_q(sp_{2n})$-modules [4]. It is based on the notion of symplectic tableaux analogous to Young tableaux for type $A$. In the sequel we use De Concini’s version of these tableaux which is equivalent to Kashiwara-Nakashima’s one [5].

Let us consider the totally ordered alphabet

$$\mathcal{C}_n = \{1 < \cdots < n < \pi < \cdots < \top\}.$$
For each letter $x \in \mathcal{C}_n$ we denote by $\text{pred}(x)$ the largest letter $y$ such that $y < x$. A column on $\mathcal{C}_n$ is a Young diagram $C$ of column shape filled from top to bottom by increasing letters of $\mathcal{C}_n$. The height $h(C)$ of a column $C$ is the number of its letters. Set $\mathcal{C}(n, h)$ for the set of columns of height $h$ on $\mathcal{C}_n$ i.e. with letters in $\mathcal{C}_n$. The reading of the column $C \in \mathcal{C}(n, h)$ is the word $w(C)$ of $\mathcal{C}_n$ obtained by reading the letters of $C$ from top to bottom. We will say that a column $C$ contains the pair $(z, \overline{z})$ when $C$ contains the unbarred letter $z \leq n$ and the barred letter $\overline{z} > n$. Let $C_1$ and $C_2$ be two columns. We will write $C_1 \leq C_2$ when $h(C_1) \geq h(C_2)$ and the rows of the tableau $C_1C_2$ weakly increase.

**Definition 2.2.1** Let $C$ be a column and $I_C = \{z_1 > \cdots > z_r\}$ the set of unbarred letters $z$ such that the pair $(z, \overline{z})$ occurs in $C$. The column $C$ is admissible when there exists a set of unbarred letters $J_C = \{t_1 > \cdots > t_r\} \subset \mathcal{C}_n$ such that:

- $t_1$ is the greatest letter of $\mathcal{C}_n$ satisfying: $t_1 < z_1, t_1 \notin C$ and $t_1 \notin C$,
- for $i = 2, \ldots, r$, $t_i$ is the greatest letter of $\mathcal{C}_n$ satisfying: $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$ and $t_i \notin C$.

In this case we write:

- $rC$ for the column obtained from $C$ by changing $\overline{z}_i$ into $\overline{t}_i$ for each letter $z_i \in I_C$,
- $lC$ for the column obtained from $C$ by changing $z_i$ into $t_i$ for each letter $z_i \in I_C$.

A column $C$ on $\mathcal{C}_n$ may be non admissible. For $C = \begin{array}{c}
2 \\
3 \\
3 \\
1 
\end{array}$ it is impossible to find a letter $t < 3$ such that $t \notin C$ and $\overline{t} \notin C$. We write $\mathcal{C}a(n, h)$ for the set of admissible columns of height $h$ on $\mathcal{C}_n$.

Notice that the condition $t_i < \min(t_{i-1}, z_i)$ for $i = 2, \ldots, r$ of the above definition can be replaced by the condition $t_i < z_i$ and $t_i \notin \{t_1, \ldots, t_{i-1}\}$ for $i = 2, \ldots, r$. Indeed, $t_i > t_{i-1}$ contradicts the fact that $t_{i-1}$ is maximal. Then $J_C$ may be regarded as the set of $r$ maximal unbarred letters $\{t_1, \ldots, t_r\}$ such that $t_i < z_i$ and $\{t_i, \overline{t}_i\} \cap C = \emptyset$.

Similarly to the type $A$ case, we can associate to each dominant weight $\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i$ a Young diagram $Y(\lambda)$ having $\lambda_i$ columns of height $i, i = 1, \ldots, n$. By definition, a symplectic tableau $T$ of shape $\lambda$ is a filling of $Y(\lambda)$ by letters of $\mathcal{C}_n$ satisfying the following conditions:

- the columns $C_i$ of $T = C_1 \cdots C_s$ are admissible,
- $rC_i \leq lC_{i+1}$.

The set of symplectic tableaux of shape $\lambda$ will be denoted $\mathbf{ST}(n, \lambda)$. If $T = C_1C_2 \cdots C_r \in \mathbf{ST}(n, \lambda)$, the reading of $T$ is the word $w(T) = w(C_r) \cdots w(C_2)w(C_1)$. From $\mathbb{F}$ and $\mathbb{P}$ we deduce the

**Theorem 2.2.2**

(i): The vertices of $B(\Lambda_p)$ are in one-to-one correspondence with the readings of admissible columns of height $p$.

(ii): The vertices of $B(\lambda)$ are in one-to-one correspondence with the readings of the symplectic tableaux of shape $\lambda$.

More precisely Kashiwara and Nakashima realize $V(\lambda)$ into a tensor power $V(\Lambda_1) \otimes^l$ of the vector representation whose crystal graph is:

$$
1 \xrightarrow{1} 2 \cdots \xrightarrow{n-1} n-1 \xrightarrow{n} \overline{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} 1.
$$

(14)
Let us identify the vertices of the crystal graph $G_n = \bigoplus_l B(\Lambda_l)^ \otimes l$ with the words on $C_n$. The weight of the vertex $b \in G_n$ is defined by
\[
\text{wt}(b) = \sum_{i=1}^n (\varphi_i(b) - \varepsilon_i(b)) \Lambda_i.
\] (15)

$B(\Lambda_p)$ can then be identified with the connected component of $G_n$ whose vertex of highest weight is the reading of the column
\[
C_p^0 = \begin{array}{c}
1 \\
2 \\
\vdots \\
p
\end{array}
\] In this identification, the vertices of $B(\Lambda_p)$ are the readings of the admissible columns of height $p$. If $\lambda = \sum_{p=1}^n \lambda_p \Lambda_p$, $B(\lambda)$ is identified with the connected component whose highest weight vertex is the reading of the symplectic tableau $T_\lambda$ containing $\lambda_p$ columns $C_p^0$ for $p = 1, \ldots, n$. Then the vertices of $B(\lambda)$ are the readings of the symplectic tableaux of shape $\lambda$.

Using Formulas $[12]$ and $[13]$ we obtain a simple rule to compute the action of $\tilde{e}_i$ and $\tilde{f}_i$ on $w \in G_n$ that we will use in Section 4. Consider the subword $w_i$ of $w$ containing only the letters $i+1, i, i, i+1$. Then encode in $w_i$ each letter $i+1$ or $i$ by the symbol $+$ and each letter $i$ or $i+1$ by the symbol $-$. Because $\tilde{e}_i(+) = \tilde{f}_i(-) = 0$ in $B(\Lambda_1) \otimes B(\Lambda_1)$ the factors of type $+-+-$ may be ignored in $w_i$. So we obtain a subword $w_i^{(1)}$ in which we can ignore all the factors $+-$ to construct a new subword $w_i^{(2)}$ etc... Finally we obtain a subword $\rho(w)$ of $w$ of type $\rho(w) = -^r +^s$.

Then we have the

**Rule 2.2.3**

- If $r > 0$, $\tilde{e}_i(w)$ is obtained by changing the rightmost symbol $+$ of $\rho(w)$ into its corresponding symbol $+$ (i.e. $i+1$ into $i$ and $7$ into $i+1$) the others letters of $w$ being unchanged. If $r = 0$, $\tilde{e}_i(w) = 0$

- If $s > 0$, $\tilde{f}_i(w)$ is obtained by changing the leftmost symbol $-$ of $\rho(w)$ into its corresponding symbol $-$ (i.e. $i$ into $i+1$ and $7$ into $i+1$) the others letters of $w$ being unchanged. If $s = 0$, $\tilde{f}_i(w) = 0$.

**2.3 Canonical bases for $U_q (sp_{2n})$-modules**

In the sequel we identify $B(\Lambda_p)$ to $\{w(C); C \in Ca(n,p)\}$ and $B(\lambda)$ to $\{w(T); T \in ST(n,\lambda)\}$. Denote by $F \to F$ the involution of $U_q (sp_{2n})$ defined as the ring automorphism satisfying
\[
\varphi = q^{-1}, \quad q^{h_i} = q^{-h_i}, \quad \varphi_i = e_i, \quad \varphi_i = f_i \quad \text{for } i = 1, \ldots, n.
\]

Writing each vector $v$ of $V(\lambda)$ in the form $v = Fv_\lambda$ where $F \in U_q (sp_{2n})$, we obtain an involution of $V(\lambda)$ defined by
\[
\overline{v} = Fv_\lambda.
\]

Let $U^-_\mathbb{Q}$ be the subalgebra of $U_q (sp_{2n})$ generated over $\mathbb{Q}[q,q^{-1}]$ by the $f_i^{(k)}$ and set $V_\mathbb{Q}(\lambda) = U^-_\mathbb{Q}v_\lambda$. We can now state:
Theorem 2.3.1 (Kashiwara)
There exists a unique $\mathbb{Q}[q, q^{-1}]$-basis \{G(T); T \in ST(n, \lambda)\} of $V_Q(\lambda)$ such that:

\[
G(T) \equiv w(T) \mod qL(\lambda),
\]

\[
\frac{G(T)}{G(T)} = G(T).
\]

This basis is called the lower global (or canonical) basis of $V(\lambda)$, and our aim is to calculate it.

3 Fundamental modules

3.1 Marsh’s Algorithm

We review Marsh’s algorithm for computing the global basis of $V(\Lambda_p)$ \[1\]. Let $w(C) \in B(\Lambda_p)$. A letter $x \in C$ is said to be movable if $\text{pred}(x) \notin C$. We define a path in $B(\Lambda_p)$ joining $w(C)$ to $w(C_p^0)$. If $C \neq C_p^0$, let $z$ be the lowest movable letter of $C$. Then we compute a new column $C_1$ as follows:

(i) : if $z = i + 1, \bar{t} \notin C$ or $\bar{t} + 1 \in C$, $C_1 = C - \{i + 1\} + \{i\}$,

(ii) : if $z = i + 1, \bar{t} \in C$ and $\bar{t} + 1 \notin C$, $C_1 = C - \{i + 1\} + \{t + 1, i\}$,

(iii) : if $z = \bar{t} > \pi$, $C_1 = C - \{\bar{t}\} + \{t + 1\}$,

(iv) : if $z = \pi$, $C_1 = C - \{\pi\} + \{n\}$.

Remark 3.1.1 In case (iii) the letters $i$ and $\bar{t}$ can not appear simultaneously in $C$. Otherwise by definition of $z$, the letters $i, i - 1, \ldots, 1$ would appear in $C$ and $C$ would be not admissible.

We will have $w(C_1) = \tilde{e}_1 w(C)$ in cases (i) and (iii), $w(C_1) = \tilde{e}_2 w(C)$ in case (ii) and $w(C_1) = \tilde{e}_n w(C)$ in case (iv). So $C_1$ is an admissible column. Write $w(C_1) = \tilde{e}_1^{p_1} w(C)$. Next we compute similarly $C_2$ from $C_1$ and write $w(C_2) = \tilde{e}_1^{p_2} w(C_1)$. Finally, after a finite number of steps, we will reach $C_p^0$ and we will get $w(C_p^0) = \tilde{e}_1^{p_1} \cdots \tilde{e}_1^{p_r} w(C)$, hence $w(C) = \tilde{f}_1^{p_1} \cdots \tilde{f}_r^{p_r} w(C_p^0)$.

Example 3.1.2 Let $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $n = 3$. We obtain:

In fact $\tilde{f}_1^{p_1}, \ldots, \tilde{f}_r^{p_r}$ are chosen to verify:

\[
f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} v_{\Lambda_p} = \tilde{f}_1^{p_1} \cdots \tilde{f}_r^{p_r} v_{\Lambda_p}.
\]

This implies

Theorem 3.1.3 (Marsh) For any admissible column $C$

\[
G(C) = f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} v_{\Lambda_p},
\]

where the integers $i_1, \ldots, i_r$ and $p_1, \ldots, p_r$ are determined by the algorithm above.
3.2 The representation $W(\Lambda_p)$

It is well known that the fundamental $\mathfrak{sp}_{2n}$-module of weight $\Lambda_p$ may be regarded as an irreducible component of the $p$-th exterior power of the vector representation. This exterior power contains a natural basis indexed by the columns of height $p$. Our aim in this subsection is to obtain a analogous embedding for the $U_q(\mathfrak{sp}_{2n})$-module $V(\Lambda_p)$. We are going to describe a $U_q(\mathfrak{sp}_{2n})$-module $W(\Lambda_p)$ containing an irreducible component isomorphic to $V(\Lambda_p)$ (that we identify with $V(\Lambda_p)$), whose natural basis $\{v_C\}$ is indexed by all columns of $\mathbf{C}(n,p)$. The action of the generators $e_i, f_i$ and $q^{h_i}$ $i = 1, \ldots, n$ of $U_q(\mathfrak{sp}_{2n})$ on each vector $v_C, C \in \mathbf{C}(n,p)$ is easy to describe. So, using Marsh’s algorithm, we will be able to expand the canonical basis of $V(\Lambda_p)$ on this basis.

Consider the two totally ordered alphabets

$$A_n = \{1 < \cdots < n\} \quad \text{and} \quad \overline{A}_n = \{\overline{1} < \cdots < \overline{t}\}.$$ 

Let $E^+_k$ be the vector space of dimension $\binom{n}{k}$ with basis $B^+_k = \{v_C\}$ where $C_+$ runs over the set of columns of height $k$ on $A_n$. We define the action of the operators $e_i, f_i$ and $q^{h_i}$ ($i = 1, \ldots, n-1$) on $E^+_k$ by:

$$q^{h_i}v_{C_+} = \begin{cases} 
q v_{C_+} & \text{if } i \in C_+ \text{ and } i + 1 \notin C_+, \\
q^{-1} v_{C_+} & \text{if } i + 1 \in C_+ \text{ and } i \notin C_+, \\
v_{C_+} & \text{otherwise.}
\end{cases}$$

$$e_i v_{C_+} = \begin{cases} 
0 & \text{if } i + 1 \notin C_+ \text{ or } i \in C_+, \\
v_D & \text{where } D = C_+ - \{i + 1\} + \{i\} \text{ otherwise.}
\end{cases}$$

$$f_i v_{C_+} = \begin{cases} 
0 & \text{if } i \notin C_+ \text{ or } i + 1 \in C_+, \\
v_D & \text{where } D = C_+ - \{i\} + \{i + 1\} \text{ otherwise.}
\end{cases}$$

This endows $E^+_k$ with the structure of a fundamental $U_q(\mathfrak{sl}_n)$-module which may be regarded as a $q$-analogue of the fundamental $\mathfrak{sl}_n$-module $\Lambda^k \mathbf{C}$. Moreover the basis $\{v_C\}$ is then the canonical basis and the crystal basis of $E^+_k$ because the fundamental modules of $U_q(\mathfrak{sl}_n)$ are minuscule. Similarly $E^-_k$ the vector space of dimension $\binom{n}{k}$ with basis $B^-_k = \{v_{C_-}\}$ where $C_-$ runs over the set of columns of height $k$ on $\overline{A}_n$ is a $q$-analogue of the fundamental $\mathfrak{sl}_n$-module $\Lambda^{n-k} \mathbf{C}$ once defined the appropriate action of $e_i, f_i$ and $q^{h_i}$ (obtained by replacing in the right hand sides of the above formulas $i$ by $\overline{i + 1}$ and $i + 1$ by $\overline{i}$). Then $B^-_k = \{v_{C_-}\}$ is again the canonical basis and the crystal basis of $E^-_k$.

Set $\Omega^+ = \bigoplus_{k=0}^n E^+_k$, $\Omega^- = \bigoplus_{k=0}^n E^-_k$ and write $B^+ = \bigcup_{k=0}^n B^+_k$, $B^- = \bigcup_{k=0}^n B^-_k$. Then $B^+ \otimes B^- = \{v_{C^+} \otimes v_{C^-} : v_{C^+} \in B^+, v_{C^-} \in B^-\}$ is a basis of $\Omega^+ \otimes \Omega^-$. For $p \in \{1, \ldots, n\}$, let $W(\Lambda_p)$ be the subspace of $\Omega^+ \otimes \Omega^-$ generated by the vectors $v_{C^+} \otimes v_{C^-}$ such that $h(C^+) + h(C^-) = p$. For each of these vectors we set $v_{C^+} \otimes v_{C^-} = v_C$ where $C$ is the column of height $p$ on $\mathbf{C}_n$ (admissible or not) consisting of the letters of $C_+$ and $C_-$. Then $\{v_C\}$ is a basis of $W(\Lambda_p)$. The action of $U_q(\mathfrak{sl}_n)$ on this basis is deduced from the description of its action on $B^+$ and $B^-$ and from (7), (8) and (9). The formulas below describe this action on the basis vector $v_C$. Set $E = C \cap \{i + 1, i, \overline{i}, i + 1\}$.
then:

\[
\begin{align*}
\text{(i)}: \quad & v_D \text{ with } D = C - \{i\} + \{i + 1\} \text{ if } E = \{i\}, \\
\text{(ii)}: \quad & v_D \text{ with } D = C - \{i\} + \{i + 1\} \text{ if } E = \{\overline{i}, i\}, \\
\text{(iii)}: \quad & v_D \text{ with } D = C - \overline{i + 1} + \{\overline{i}\} \text{ if } E = \{\overline{i + 1}\}, \\
\text{(iv)}: \quad & v_D \text{ with } D = C - \overline{i + 1} + \{\overline{i}\} \text{ if } E = \{\overline{i + 1}, i, i + 1\}, \\
\text{(v)}: \quad & q^{-1}v_D \text{ with } D = C - \{i + 1\} + \{\overline{i}\} \text{ if } E = \{i + 1\}, \\
\text{(vi)}: \quad & v_D \text{ with } D = C - \{i\} + \{i + 1\} \text{ if } E = \{\overline{i}, i\}, \\
\text{(vii)}: \quad & v_{D_1} + qv_{D_2} \text{ with } \begin{cases} D_1 = C - \{i\} + \{i + 1\} \\ D_2 = C - \{\overline{i + 1}\} + \{\overline{i}\} \end{cases} \text{ if } E = \{i + 1\}, \\
\text{(viii)}: \quad & 0 \text{ otherwise.}
\end{align*}
\]

\[
\begin{align*}
\text{(i)}: \quad & v_D \text{ with } D = C - \{i + 1\} + \{i\} \text{ if } E = \{i + 1\}, \\
\text{(ii)}: \quad & v_D \text{ with } D = C - \{i + 1\} + \{i\} \text{ if } E = \{\overline{i + 1}, i, i + 1\}, \\
\text{(iii)}: \quad & v_D \text{ with } D = C - \overline{i} + \{\overline{i + 1}\} \text{ if } E = \{\overline{i}\}, \\
\text{(iv)}: \quad & v_D \text{ with } D = C - \overline{i} + \{\overline{i + 1}\} \text{ if } E = \{\overline{i}, i, i + 1\}, \\
\text{(v)}: \quad & q^{-1}v_D \text{ with } D = C - \{i\} + \{i\} \text{ if } E = \{i + 1\}, \\
\text{(vi)}: \quad & v_D \text{ with } D = C - \{\overline{i}\} + \{\overline{i + 1}\} \text{ if } E = \{\overline{i}, i\}, \\
\text{(vii)}: \quad & v_{D_1} + qv_{D_2} \text{ with } \begin{cases} D_1 = C - \overline{i} + \{\overline{i + 1}\} \\ D_2 = C - \{i + 1\} + \{i\} \end{cases} \text{ if } E = \{i, i + 1\}, \\
\text{(viii)}: \quad & 0 \text{ otherwise.}
\end{align*}
\]

\[q^{h_i}v_C = q^{<\text{wt}(C), \Lambda_i>} v_C. \tag{21}\]

Now we define an action of the operators \(e_n, f_n\) and \(q^{h_n}\) in order to endow \(W(\Lambda_p)\) with the structure of a \(U_q(sp_{2n})\)-module. Set

\[
\begin{align*}
f_n v_C &= \begin{cases} 0 \text{ if } \ul{\pi} \in C \text{ or } n \notin C \\ v_D \text{ with } D = C - \{n\} + \{\ul{\pi}\} \text{ otherwise} \end{cases}, \tag{22}
\end{align*}
\]

\[
\begin{align*}
e_n v_C &= \begin{cases} 0 \text{ if } \ul{\pi} \notin C \text{ or } n \in C \\ v_D \text{ with } D = C - \{\ul{\pi}\} + \{n\} \text{ otherwise} \end{cases}, \tag{23}
\end{align*}
\]

\[
q^{h_n}v_C = q^{<\text{wt}(C), \Lambda_n>} v_C. \tag{24}
\]

**Lemma 3.2.1** The above actions of \(e_n, f_n\) and \(q^{h_n}\) make \(W(\Lambda_p)\) into a \(U_q(sp_{2n})\)-module.
Proof. It suffices to show that the actions of $f_n, e_n$ and $h_n$ are compatible with (i), (ii), (iii), (iv), (v) and (vi). $W(\Lambda_p)$ is already a $U_q(sl_n)$-module so we can restrict ourselves to the cases where $i = n$ or $j = n$ in these relations. Suppose first $(i,j) = (n,n)$. Then denote by $U_q(sl_2)$ the subalgebra of $U_q(sp_{2n})$ generated by $f_n, e_n$ and $h_n$. Formulas (22), (23) and (24) imply that $W(\Lambda_p)$ is a $U_q(sl_2)$-module and the only non trivial module occurring in its decomposition into irreducible $U_q(sl_2)$-modules is the vector representation of $U_q(sl_2)$. Hence we can suppose $(i,j) \neq (n,n)$. Let $C \in C(n,p)$. For $i \neq n$, we have to establish that:

(i) : $q^n e_i q^{-n} v_C = \begin{cases} q^{-1} e_i v_C & \text{if } i = n - 1 \\ e_i v_C & \text{otherwise} \end{cases}$ and $q^n f_i q^{-n} v_C = \begin{cases} q e_i v_C & \text{if } i = n - 1 \\ f_i v_C & \text{otherwise} \end{cases}$.

(ii) : $q^n e_n q^{-n} v_C = \begin{cases} q^{-1} e_n v_C & \text{if } i = n - 1 \\ e_n v_C & \text{otherwise} \end{cases}$ and $q^n f_n q^{-n} v_C = \begin{cases} q e_n v_C & \text{if } i = n - 1 \\ f_n v_C & \text{otherwise} \end{cases}$.

(iii) : $[e_n, f_i] v_C = [f_n, e_i] v_C = 0$ for $i \neq n - 1$.

(iv) : $[e_n, e_i] v_C = [f_n, f_i] v_C = 0$ for $i \neq n - 1$.

(v) : $\begin{cases} (e_n - 1) e_i^2 - (q^2 + q^{-2}) e_n - 1 + e_i^2 - 1) v_C = 0 \\ (f_n - 1) f_i^2 - (q^2 + q^{-2}) f_n = 1 + f_i^2 - 1) v_C = 0 \end{cases}$.

(vi) : $\begin{cases} e_n e_{n-1} e_i = (q^2 + q^{-2}) e_{n-1} e_i e_n + (q^2 - q^{-2}) e_{n-1} e_i e_n - e_{n-1} e_i v_C = 0 \\ f_n f_{n-1}^2 - (q^2 + q^{-2}) f_n f_{n-1} + f_n^2 f_{n-1} - 1) v_C = 0 \end{cases}$.

When $i \neq n - 1$, the actions the operators $e_n, f_n$ or $h_n$ on $W(\Lambda_p)$ commute with those of the operators $e_i, f_i$ or $h_i$. Moreover we have $e_i^3 v_C = f_i^3 v_C = 0$ and $e_i^2 v_C = f_i^2 v_C = 0$. This implies that the above relations are immediate when $i \neq n - 1$. If $i = n - 1$, we obtain (i) and (ii) by a case by case computation from (20), (19) and (24). Relations (iii) follows from the equalities $e_{n-1} f_n v_C = n e_{n-1} v_C = e_{n-1} f_n v_C = 0$. To establish relations (v) and (vi) it suffices to prove that

$e_n e_{n-1} e_i v_C = f_n f_{n-1} f_n v_C = 0$;

$f_{n-1} f_n f_{n-1} f_n = f_{n-1} f_n f_{n-1} v_C = 0$;

$e_n e_{n-1} e_i v_C = e_n f_{n-1} f_n v_C = 0$.

If $f_n v_C = v_D \neq 0$, then the column $D$ contains $n$ but not $n$ so $D$ is of type (ii), (iii) or (vii) in (10). Then we obtain after a simple computation that $f_n f_{n-1} f_n v_C = 0$. We prove similarly the equality: $e_n e_{n-1} e_i v_C = 0$ which implies (v). From (vii) of formula (10) it follows that $f_n f_{n-1} v_C = 0$. Moreover $f_{n-1} f_n f_{n-1} v_C = 0$ because $f_n f_{n-1} v_C = 0$ or may be written $v_D$ with $D$ a column which is not of the type (vii) of (10). In the same manner we can see that $e_n e_{n-1} v_C = 0$ which implies (vi). $\blacksquare$

$W(\Lambda_p)$ is not an irreducible $U_q(sp_{2n})$-module. Since it is finite-dimensional, it decomposes into a direct sum of irreducible components. It is easy to verify that the vector $v_C$ where $C_p$ is the column defined in (22) is a highest weight vector of weight $\Lambda_p$. Hence setting $v_{\Lambda_p} = v_C$ we can identify $U_q(sp_{2n}) v_{C_p}$ and $V(\Lambda_p)$. Then it is possible to expand explicitly the vectors of the canonical basis of $V(\Lambda_p)$ on the basis $\{v_C; C \in C(n,p)\}$.

Lemma 3.2.2

1. Let $L_p$ be the $A$-submodule of $W(\Lambda_p)$ generated by the vectors $v_C$, $C \in C(n,p)$. Write $w(C)$ for the image of the vector $v_C$, $C \in C(n,p)$ by the projection $\pi_p : L_p \to L_p/qL_p$ and $B_p = \{w(C) ; C \in C(n,p)\}$. Then $(L_p, B_p)$ is a crystal basis of $W(\Lambda_p)$.

2. Set $L(\Lambda_p) = L_p \cap V(\Lambda_p)$ and denote by $B(\Lambda_p)$ the set of images of the vectors $\{v_C; C \in C(n,p)\}$ by $\pi_p$. Then $(L(\Lambda_p), B(\Lambda_p))$ is the crystal basis of $V(\Lambda_p)$.

Proof. 1 : We have to prove that $L_p$ and $B_p$ verify the assertions (i), (ii), (iii) and (iv) of (11). When $i = n$ this follows immediately from formulas (22) and (23). Indeed the actions of $f_n$ and $f_n$ (resp. $e_n$ and $e_n$) coincide on each vector $v_C$, $C \in C(n,p)$. On the other hand, by construction $W(\Lambda_p)$ is a direct sum of tensor products of $U_q(sl_n)$-modules and it follows from the
compatibility of crystals with tensor products that \((L_p, B_p)\) is a crystal basis of \(W(\Lambda_p)\) considered as a \(U_q(sl_n)\)-module. Hence the assertions (i), (ii), (iii) and (iv) are also verified for \(i = 1, \ldots, n - 1\).

Note that the actions of \(\bar{e}_i\) and \(\bar{f}_i\), \(i = 1, \ldots, n\) on \(B_p\) coincide with those given by Rule 2.2.3.

2 : It is clear that \(L(\Lambda_p)\) verifies the conditions of [11]. By theorem 4.2 of [6], we know that

\[
B(\Lambda_p) = \{\bar{f}_{i_1}^{a_1} \cdots \bar{f}_{i_r}^{a_r} w(C^0_p); \ i_1, \ldots, i_r = 1, \ldots, n; \ a_1, \ldots, a_r > 0\} \setminus \{0\}
\]

for \(w(C^0_p)\) is the image of \(v_{\Lambda_p}\) by \(\pi_p\). We have just seen that the actions of \(\bar{e}_i\) and \(\bar{f}_i\), \(i = 1, \ldots, n\) on \(B_p\) coincide with those given by Rule 2.2.3. Hence, by Theorem 2.2.2, \(B(\Lambda_p) = \{w(C); C \in C(a(n, p))\}\.

3.3 Expression of \(\{G(C)\}\) on \(\{v_C\}\)

Let \(C\) be an admissible column of height \(p\) and set \(C = \bar{f}_{i_1}^{p_1} \cdots \bar{f}_{i_r}^{p_r} C^0_p\) with the notation of Section 3.1. Then by Theorem 3.1.3

\[
G(C) = \bar{f}_{i_1}^{p_1} \cdots \bar{f}_{i_r}^{p_r} v_{\Lambda_p}.
\]

We are going to give a combinatorial description of \(G(C)\). Define the \(r\)-tuple \(L_C = (u_1, \ldots, u_r) \subset A_n\) from the \(r\)-tuple \(K_C = (x_1 < \cdots < x_r)\) containing the unbarred letters \(x\) such that \((x, \pi) \subset C\) by:

\[
u_1 < x_1, \{u_1, \pi_1\} \cap C = \emptyset \quad \text{and} \quad u_1 \text{ is maximal,}
\]

for \(i = 2, \ldots, r\), \(u_i < x_i\), \(\{u_i, \pi_i\} \cap (C \cup \{u_1, \ldots, u_{i-1}\}) = \emptyset \) and \(u_i\) is maximal.

Then the letters of \(L_C\) are those of the set \(J_C\) defined in 2.2. Indeed we have seen that \(J_C\) may be regarded as the set of \(r\) maximal unbarred letters \(\{t_1, \ldots, t_r\}\) such that \(t_i < z_i\) and \(\{t_i, \pi_i\} \cap C = \emptyset\). Notice that, in general, the letters of \(L_C\) are not ordered in decreasing order as those of \(J_C\).

Example 3.3.1 \(C = \begin{pmatrix}3 \\ 5 \\ 6 \\ 5 \\ 5 \\ 3 \end{pmatrix}\) is admissible with \(K_C = (3 < 5 < 6)\) and \(L_C = (2, 4, 1)\).

For any subset \(X = \{x_1, \ldots, x_k\} \subset K_C\), let \(C_X\) be the column of \(C(n, p)\) obtained by changing in \(C\) each pair of letters \((x_i, \pi_i)\) into the corresponding pair of letters \((u_{i}, \pi_{i})\). Then, with the above notations we obtain:

Theorem 3.3.2 For any admissible column \(C\) of height \(p\)

\[
G(C) = \sum_{X \subset K_C} q^{\text{card}(X)} v_{C_X}.
\]

Example 3.3.3 With the column \(C\) of the previous example we have:

\[
G(C) = \begin{pmatrix}3 \\ 5 \\ 6 \\ 5 \\ 5 \\ 3 \end{pmatrix} + q \begin{pmatrix}2 \\ 5 \\ 6 \\ 6 \\ 5 \\ 2 \end{pmatrix} + q^2 \begin{pmatrix}2 \\ 4 \\ 5 \\ 4 \\ 3 \\ 2 \end{pmatrix} + \cdots
\]

where we have written for short \(C\) in place of \(v_C\).
Proof. Let $C$ be an admissible column and set $w(C) = \tilde{p}_{i_1} \cdots \tilde{p}_{i_r} w(C'_p)$ with the notation of Section 3. First notice that by (13) and (2) we will have $d(D) = d(C')$ for any column $D$ such that $v_D$ occurs with a non-zero coefficient in the decomposition of $G(C')$ on the basis $\{v_C\}$. We proceed by induction on $r$. The theorem is clear for $r = 0$, i.e. $C = C'_p$. Suppose the result is proved for the admissible column $w(C') = \tilde{p}_{i_2} \cdots \tilde{p}_{i_r} w(C'_p)$, that is:

$$G(C') = \sum_{X' \subseteq K_C} q^{\text{card}(X')} v_{C'_{X'}}.$$ 

When $p_1 = 2$, the letters $i_1$ and $i_1 + 1$ occur in $C'$ but not the letters $\tilde{i}_1$ or $\tilde{i}_1 + 1$. We have the same property for all the columns $C'_{X'}$. Moreover $C = C' - \{i_1, i_1 + 1\} + \{\tilde{i}_1, \tilde{i}_1 + 1\}$, $K_C = K_{C'}$ and $L_C = L_{C'}$. Then $G(C) = f_{i_1} G(C')$ is obtained by replacing in the columns $C'_{X'}$ each pair $(i_1, i_1 + 1)$ by $(\tilde{i}_1, \tilde{i}_1 + 1)$. So the theorem is proved.

When $p_1 = 1$ and $i_1 = n$, $n \in C'_{X'}$, but $n \notin C'_{X'}$. Then $K_C = K_{C'}$, $L_C = L_{C'}$ and $G(C)$ is obtained by replacing all the letters $n$ by letters $\tilde{n}$ into the columns $C'_{X'}$. So the theorem is proved.

When $p_1 = 1$ and $i_1 \neq n$, $C$ can not contain the pair $(i_1 + 1, \tilde{i}_1)$ without containing a letter of $(i_1, i_1 + 1)$ (otherwise $p_1 = 2$). Hence $C'$ may not be of type (v) in (13). Moreover $C'$ is not of type (vi) or (viii) otherwise $f_i w(C') = 0$. Suppose $C'$ of type (i) or (ii). Then $C = C' - \{i_1\} + \{i_1 + 1\}$, $K_C = K_{C'}$ and $L_C = L_{C'}$. Hence

$$G(C) = f_{i_1} (G(C')) = \sum_{X' \subseteq K_C} q^{\text{card}(X')} v_{C'_{X'}} - (i_1) + (i_1 + 1) = \sum_{X' \subseteq K_C} q^{\text{card}(X')} v_{C_{X'}}.$$ 

When $C'$ of type (iii) or (iv) we have $C = C' - \{i_1, i_1 + 1\} + \{\tilde{i}_1\}$, $K_C = K_{C'}$ and $L_C = L_{C'}$. Hence

$$G(C) = f_{i_1} (G(C')) = \sum_{X' \subseteq K_C} q^{\text{card}(X')} v_{C'_{X'}} - (i_1, i_1 + 1) = \sum_{X' \subseteq K_C} q^{\text{card}(X')} v_{C_{X'}}.$$ 

If $C'$ is of type (vii) in (13), the letters $i_1$ and $i_1 + 1$ occur in all the columns $C'_{X'}$ but not the letters $\tilde{i}_1$ or $\tilde{i}_1 + 1$. Then $C = C' - \{i_1\} + \{i_1 + 1\}$, $K_C = K_{C'} + \{i_1 + 1\}$. Note that $i_1 + 1$ is the lowest letter of $K_C$ because $i_1 + 1$ is the lowest movable letter of $C$. Hence $L_C = L_{C'} + \{i_1\}$ and the letter of $L_C$ corresponding to $i_1 + 1 \in K_C$ is $i_1$. We obtain $f_{i_1} v_{C'_{X'}} = v_{C_{X'}} + q v_{C_{X' + (i_1 + 1)}}$ which implies:

$$G(C) = \sum_{X' \subseteq K_C} q^{\text{card}(X')} (v_{C_{X'}} + q v_{C_{X' + (i_1 + 1)}}) = \sum_{X \subseteq K_C} q^{\text{card}(X)} v_{C_{X}}$$

because the parts of $K_C$ are exactly the elements of the set $\{X', X' + \{i_1 + 1\}; X' \subseteq K_{C'}\}$. ■

4 The computation of the canonical basis of $V(\lambda)$

4.1 The representation $W(\lambda)$

Let $\lambda = \sum_{p=1}^n \lambda_p \Lambda_p$ be a dominant weight and write

$$W(\lambda) = W(\Lambda_1) \otimes \lambda_1 \otimes \cdots \otimes W(\Lambda_n) \otimes \lambda_n.$$ 

The natural basis of $W(\lambda)$ consists of the tensor products $v_{C_1} \otimes \cdots \otimes v_{C_1}$ of basis vectors $v_C$ of the previous section. The juxtaposition of the columns $C_1, \ldots, C_r$ is called a tabloid of shape $\lambda$. We can regard it as a filling $\tau$ of the Young diagram of shape $\lambda$, the $i$-th column of which is equal to $C_i$. We shall write $v_\tau = v_{C_1} \otimes \cdots \otimes v_{C_1}$. Note that the columns of a tabloid are not necessarily admissible and there is no condition on the rows. Write $T(n, \lambda)$ for the set of tabloids of shape $\lambda$. The reading of the tabloid $\tau = C_1 \cdots C_r \in T(n, \lambda)$ is the word $w(\tau) = w(C_r) \cdots w(C_1) \in C_n$. The weight of $\tau$ is the weight $v_\tau$. 

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Let $L_{\lambda}$ be the $A$-submodule of $W(\lambda)$ generated by the vectors $v_\tau$, $\tau \in T(n, \lambda)$. We identify the image of the vector $v_\tau$ by the projection $\pi_{\lambda} : L_{\lambda} \to L_{\lambda}/qL_{\lambda}$ with the word $w(\tau)$. The pair $(L_{\lambda}, B_{\lambda} = \{w(\tau) \mid \tau \in T(n, \lambda)\})$ is then a crystal basis of $W(\lambda)$. Indeed by Lemma 3.2.2 it is the tensor product of the crystal bases of the representations $W(\Lambda_p)$ occurring in $W(\lambda)$. Denote by $v_\lambda$ the tensor product in $W(\lambda)$ of the highest weight vectors of each $W(\Lambda_p)$. We identify $V(\lambda)$ with the submodule of $W(\lambda)$ of highest weight vector $v_\lambda$. Then, with the above notations, $v_\lambda = v_{T_{\lambda}}$, where $T_{\lambda}$ is the symplectic tableau of shape $\lambda$ whose $k$-th row is filled by letters $k$. By Theorem 4.2 of [8], we know that

$$B(\lambda) = \{ \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} w(T_{\lambda}); \; i_1, \ldots, i_r = 1, \ldots, n; \; a_1, \ldots, a_r > 0 \} - \{ 0 \}. $$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$, $i = 1, \ldots, n$ on $B_{\lambda}$ coincide with those given by Rule (2.2.3) because it is true on each $B_p$, $p = 1, \ldots, n$. Hence, by Theorem (2.2.2), $B(\lambda) = \{ w(T); T \in ST(n, \lambda) \}$. For each vector $G(T)$ of the canonical basis of $V(\lambda)$ we will have:

$$G(T) \equiv v_T \mod qL_{\lambda}. $$

The aim of this section is to describe an algorithm computing the decomposition of the canonical basis $\{G(T); T \in ST(n, \lambda)\}$ onto the basis $\{v_\tau; \tau \in T(n, \lambda)\}$ of $W(\lambda)$.

Let $w_1 = x_1 \cdots x_l$ and $w_2 = y_1 \cdots y_k$ be two distinct words on $C_n$ with the same length and $k$ the lowest integer such that $x_k \neq y_k$. Write $w_1 \leq w_2$ if $x_k \leq y_k$ in $C_n$ and $w_1 > w_2$ else that is, $\leq$ is the lexicographic order. Then we endow the set $T(n, \lambda)$ with a total ordering by setting:

$$\tau_1 \leq \tau_2 \iff w(\tau_1) \leq w(\tau_2). $$

Notice that for any tabloid $\tau \in T(n, \lambda)$, we have $T_{\lambda} \leq \tau$.

We are going to compute the canonical basis $\{G(T); T \in ST(n, \lambda)\}$ in two steps. First we obtain an intermediate basis $\{A(T); T \in ST(n, \lambda)\}$ which is fixed by the involution $(\tau)$(condition (17)). Next we correct it in order to have condition (16). This second step is easy because we can prove that the transition matrix from $\{A(T); T \in ST(n, \lambda)\}$ to $\{G(T); T \in ST(n, \lambda)\}$ is uniprincipal once the symplectic tableaux and the tabloids are ordered by $\leq$.

We start with a general Lemma analogous to Lemma 4.1 of [8]. Let $\nu$ be a tabloid and $i, m$ two integers such that $f_i^m v_\nu \neq 0$. Suppose that the vector $v_\nu$ appears in the decomposition of $f_i^m v_\nu$ on the basis $\{v_\tau; \tau \in T(n, \lambda)\}$ with a non zero coefficient $\kappa_\tau$. Then the tabloid $\tau$ is obtained from $\nu$ by changing $m$ occurrences of letters of $\{i+1, i\}$ into the corresponding letters of $\{i, i+1\}$. Set $w(\nu) = x_1 \cdots x_l$ and denote by $(x_{i_1}, \ldots, x_{i_m})$, $i_1 < \cdots < i_m$ the $m$-tuple of letters of $w(\nu)$ modified to obtain $w(\tau)$. By formula (17), the vector $v_\tau$ will appear $m!$ times in $f_i^m v_\nu$ according to the $m!$ permutation of $(i_1, \ldots, i_m)$. For each permutation $(i_{\sigma(1)}, \ldots, i_{\sigma(m)})$ of $(i_1, \ldots, i_m)$ write $q_i^{N(\sigma)}$ for the power of $q_i$ appearing when we compute $w(\tau)$ by changing the letters $x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(m-1)}}, x_{i_{\sigma(m)}}$ of $w(\nu)$ in this order. Then we obtain by induction on the length $l(\sigma)$ of the permutation $\sigma$:

$$N(\sigma) = N(\text{id}) + 2l(\sigma). $$

So the coefficient of $v_\nu$ in $f_i^m v_\nu$ is equal to

$$\kappa_\tau = \sum_{\sigma} q_i^{N(\text{id}) + 2l(\sigma)} = q_i^{N(\text{id}) + \frac{m(m+1)}{2}} [m]_i. \tag{25} $$

Hence the coordinates of $f_i^m v_\nu = f_i^m v_\nu/[m]_i$ belong to $\mathbb{N}[q,q^{-1}]$. The following lemma follows by induction on $s$:

**Lemma 4.1.1** Let $v \in V(\lambda)$ be a vector of the type

$$v = f_{i_1}^{(r_1)} \cdots f_{i_s}^{(r_s)} v_\lambda \tag{26} $$

where $(i_1, \ldots, i_s)$ and $(r_1, \ldots, r_s)$ are two sequences of integers. Then the coordinates of $v$ on the basis $\{v_\tau; \tau \in T(n, \lambda)\}$ belong to $\mathbb{N}[q,q^{-1}]$. 

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4.2 The basis \{A(T)\}

The basis \{A(T)\} will be a monomial basis, that is, a basis of the form

\[ A(T) = f_{i_1}^{(r_1)} \cdots f_{i_s}^{(r_s)} v_\lambda. \] (27)

By Lemma 4.1.1, the coordinates of \(A(T)\) on the basis \(\{v_\tau\}\) of \(W(\lambda)\) belong to \(\mathbb{N}[q, q^{-1}]\). Let \(T = C_1 \cdots C_l \neq T_\lambda \in \text{ST}(n, \lambda)\). To find the two sequences of integers \((i_1, \ldots, i_s)\) and \((r_1, \ldots, r_s)\) associated to \(T\), we proceed as follows. Let \(C_k\) be the rightmost column of \(T\) such that \(C_k\) is not of highest weight, \(x\) the lowest movable letter of \(C_k\) and \(i \in \{1, \ldots, n\}\) such that \(\bar{e}_i(x) \neq 0\). Denote by \(y\) the rightmost letter of \(w(T)\) such that \(y \in \{\bar{i}, i+1\}\) and the factor \(x \cdots y\) of \(w(T)\) contains no letter of \(\{\bar{i}, i+1\}\). Set \(r_1\) for the number of letters of \(\{\bar{i}, i+1\}\) in \(x \cdots y\). Then \(i_1\) is defined to be \(i\) and \(T_1\) is defined to be the tabloid obtained by changing in \(x \cdots y\) each letter \(\bar{i}\) into \(i + 1\), and each letter \(i + 1\) into \(i\) if \(i \neq n\), and each letter \(\bar{\pi}\) into \(n\) if \(i = n\).

**Lemma 4.2.1** \(T_1 \in \text{ST}(n, \lambda)\).

Then we do the same with \(T_1\) getting a new symplectic tableau \(T_2\) and a new integer \(i_2\). And so on until the tableau \(T_s\) obtained is equal to \(T_\lambda\). Notice that we can not write \(w(T_1) = \bar{e}_i w(T)\) in general, that is, our algorithm does not provide a path in the crystal graph \(B(\lambda)\) joining the vertex \(w(T)\) to the vertex of highest weight \(w(T_\lambda)\).

**Proof.** (of Lemma 4.2.1).

Set \(T_1 = D_1 \cdots D_l\). If \(T\) does not contain a letter of \(\{\bar{i}, i+1\}\) we may write \(w(T_1) = \bar{e}_i w(T)\) by Rule 2.2.3. So the lemma is true in this case. Otherwise, let \(C_m\) be the rightmost column of \(T\) containing a letter of \(\{\bar{i}, i+1\}\).

The letter \(x \in \{\bar{i}, i+1\}\) is movable in \(C_k\) so there is no letter \(\bar{i+1}\) or \(i\) to the left of \(x\) in \(w(C_k)\). Indeed if \(x = \bar{i}\), then \(i \notin C_k\) by Remark 3.1.1. There is no letter \(i+1\) or \(i\) to the left of \(x\) in \(w(T)\). Indeed the columns to the right of \(C_k\) are of highest weight so can not contain the barred letter \(\bar{i+1}\). Moreover if \(i\) appears in \(w(T)\) to the left of \(x\) in a column word \(w(C')\), the letters \(1, 2, \ldots, i \in C_k\) because \(T\) is a symplectic tableau. So the letter \(i+1\) can not be movable in \(C_k\) and if \(i \in C_k\) the column \(C_k\) is not admissible. Set \(T' = C_{m+1} \cdots C_l\) and \(T'_1 = D_{m+1} \cdots D_l\). Rule 2.2.3 implies that \(w(T'_1) = \bar{e}_i w(T')\) with \(p_1 \leq r_1\). Hence \(T'_1\) is a symplectic tableau.

Suppose first that \(C_m = D_m\) (hence \(y \notin C_m\) and \(p_1 = r_1\)). Then the columns \(C_1, \ldots, C_m\) are not modified when \(T_1\) is computed from \(T\). So it suffices to prove that \(rC_m \leq lD_{m+1}\). If \(\bar{e}_i w(C_m) = 0\), \(\bar{e}_i^{r_1} w(C_m T') = \bar{e}_i^{r_1} w(T') w(C_m) = w(C_m T'_1)\) is a symplectic tableau hence \(rC_m \leq lD_{m+1}\). So we can suppose \(\bar{e}_i C_m \neq 0\). Then \(i \neq n\) and the column \(C_m\) is necessarily of type (iv) in (20). Write \(C_m^{(i)}\) for the admissible column such that \(w(C_m^{(i)}) = \bar{e}_i w(C_m)\). Then we have \(rC_m = r(C_m^{(i)})\) and \(rC_m \leq lD_{m+1}\). Indeed by Formulas 12 and 13 \(\bar{e}_i^{r_1+1} w(C_m T') = w(T') \bar{e}_i w(C_m) = w(C_m T'_1)\) and \(\bar{e}_i^{r_1+1} w(C_m T')\) is a symplectic tableau.

Now suppose that \(C_m \neq D_m\) (hence \(y \in C_m\) and \(i \neq n\)). We must have \(y = i+1 \in C_m\), \(i \notin C_m\) and \(i+1 \in C_m\) because \(C_m\) contains a letter of \(\{\bar{i+1}, i\}\). We have \(r_1 = p_1 + 1\) and \(\bar{e}_i^{r_1} w(C_m T') = w(T') \bar{e}_i w(C_m) = w(D_{m} T'_1)\) is a symplectic tableau. So \(rD_m \leq lD_{m+1}\) and it suffices to shows that \(rC_m \leq lD_m\). The column \(C_m\) is necessarily of type (ii) or (v) in (20) which implies that \(lC_m = lC_m^{(i)} = lD_m\) where \(C_m^{(i)}\) is the column with reading \(\bar{e}_i w(C_m)\). Then \(rC_m \leq lC_m = lD_m\).

**Example 4.2.2** For \(T = \begin{array}{ccc} 2 & 2 & 3 \\ 3 & 3 & \end{array}\) and \(n = 3\), we obtain successively

\[
\begin{array}{ccc}
2 & 2 & 2 \\
3 & 3 & \end{array}, \begin{array}{ccc}
1 & 1 & 1 \\
3 & 3 & \end{array}, \begin{array}{ccc}
1 & 1 & 1 \\
3 & 3 & \end{array}, \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \end{array}, \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \end{array}.
\]

\[A(T) = f_2 f_1^{(3)} f_3 f_2^{(2)} f_3 T_\lambda.\]
Proposition 4.2.3 The expansion of $A(T)$ on the basis $\{v_\tau; \tau \in T(n, \lambda)\}$ of $W(\lambda)$ is of the form

$$A(T) = \sum_\tau \alpha_{\tau, T}(q)v_\tau$$

where the coefficients $\alpha_{\tau, T}(q)$ satisfy:

(i): $\alpha_{\tau, T}(q) \neq 0$ only if $\tau$ and $T$ have the same weight,
(ii): $\alpha_{\tau, T}(q) \in \mathbb{N}[q, q^{-1}]$ and $\alpha_{\tau, T}(q) = 1$,
(iii): $\alpha_{\tau, T}(q) \neq 0$ only if $\tau \leq T$.

Proof. (i) is a straightforward consequence of the definition of $A(T)$. By Lemma 4.1.1, we know that $\alpha_{\tau, T}(q) \in \mathbb{N}[q, q^{-1}]$. By induction the proposition will be proved if we show that (ii) and (iii) hold for $T$ as soon as they hold for $T_1$ with $A(T) = \sum_{i=1}^n A(T_1)$ (the notations are those of Lemma 4.2.4). If the vector $v_\tau$ occurs in $A(T)$, the tabloid $\tau$ is obtained from a tabloid $\tau_1$ labelling a vector $v_{\tau_1}$ occurring in $A(T_1)$ by changing $r_1$ letters + of $\{i + 1, i\}$ into the corresponding letters $\tau$ of $\{\tau, i + 1\}$.

It follows from the definition of $T_1$ that the tableau $T$ is obtained by changing the $r_1$ rightmost letters + of $T_1$ (i.e. the leftmost letters + of $w(T_1)$) into the corresponding letters −. Hence $v_\tau$ appears in $f_{i_1}^{(r_1)}v_{T_1}$ with a non zero coefficient. Now suppose that there exists $\tau_1 \neq T_1$ such that $v_{\tau_1}$ appears in $A(T_1)$ and $v_T$ appears in $f_{i_1}^{(r_1)}v_{T_1}$. Let $w$ be the factor of the words $w(\tau_1)$ and $w(T_1)$ of maximal length such that there exist two words $u, u'$ and two letters $x \neq y$ satisfying:

$$w(\tau_1) = wxu \text{ and } w(T_1) = wyu'$$

(28)

We must have $x < y$ because $\tau_1 < T_1$. The letter $x$ is necessarily modified when $T$ is obtained from $\tau_1$. Otherwise we have $x \in w(T)$. But $w(T)$ can also be computed from $w(T_1)$. So the letter of $w(T)$ occurring at the same place as the letter $y$ in $w(T_1)$ is $\geq y$: it can not be $x$. This implies that $x$ is letter + and $y$ is its corresponding letter −. Hence $w$ contains the $r_1$ letters + changed to $a$ when $T$ is obtained from $T_1$. We derive a contradiction because in this case there are $r_1 + 1$ letters + changed into $a$ when $T$ is obtained from $\tau_1$. Hence $v_T$ can only appear in $f_{i_1}^{(r_1)}v_{T_1}$.

With the notation of (27) we will have $N(id) = -\frac{r_1(r_1+1)}{2}$ because there is no letters − to the left of the letters + modified in $w(T_1)$ to obtain $w(T)$. Then by (28) the coefficient of $v_T$ in $f_{i_1}^{(r_1)}v_{T_1}$ is equal to 1 which proves (ii) for the coefficient of $v_{T_1}$ in $A(T_1)$ is 1.

Consider $v_\tau$ appearing in $A(T)$ and suppose that the tabloid $\tau$ is obtained from the tabloid $\tau_1$ such that $v_{\tau_1}$ appears in $A(T_1)$ by changing $r_1$ letters + into their corresponding letters −. Let $\tau'$ be the tabloid obtained by changing in $w(\tau_1)$ the $r_1$ leftmost letters + (not immediately followed by their corresponding letters −) by letters −. We are going to prove that $\tau' \leq T$ which implies the proposition because $\tau \leq \tau'$. If $\tau_1 = T_1$ then $\tau' = T$. So we can suppose $\tau_1 \neq T_1$ and decompose the words $w(\tau_1), w(T_1)$ as in (28) with $x < y$. If $\tau' \geq T$, there is a letter + in $w$ (that we write $z_+$) which is changed into its corresponding letter − when $\tau'$ is obtained from $\tau_1$ but is not modified when $T$ is obtained from $T_1$. If we write $w = w_1z_+w_2$ where $w_1$ and $w_2$ are words of $C_\alpha$, we have:

$$w(\tau_1) = w_1z_+w_2xu \text{ and } w(T_1) = w_1z_+w_2yu'.$$

Then by definition of $T_1, w_1$ contains the $r_1$ letters + changing to − to obtain $T$. This contradicts the definition of $\tau'$. So (iii) is true. ■

It follows from (iii) that the vectors $A(T)$ are linearly independent in $V(\lambda)$. This implies that $\{A(T); T \in ST(n, \lambda)\}$ is a $\mathbb{Q}(q)$-basis of $V(\lambda)$. Indeed by Theorem 2.2.2, we know that $\dim V(\lambda) = \text{card}(ST(n, \lambda))$. As a consequence of (27), we obtain that $A(T) = A(T)$. Note that, by definition of Marsh’s algorithm, the bases $\{A(T)\}$ and $\{G(T)\}$ coincide when $\lambda$ is a fundamental weight.

4.3 From $\{A(T)\}$ to $\{G(T)\}$

Let us write

$$G(T) = \sum_\tau d_{\tau, T}(q)v_\tau$$

where the $d_{\tau, T}(q)$ satisfy:

(i): $d_{\tau, T}(q) \neq 0$ only if $\tau$ and $T$ have the same weight,
(ii): $d_{\tau, T}(q) \in \mathbb{N}[q, q^{-1}]$ and $d_{\tau, T}(q) = 1$,
(iii): $d_{\tau, T}(q) \neq 0$ only if $\tau \leq T$.
We are going to describe a simple algorithm for computing the rectangular matrix of coefficients

\[ D = [d_{\tau,T}(q)], \quad \tau \in \mathbf{T}(n, \lambda), \quad T \in \mathbf{ST}(n, \lambda). \]

**Lemma 4.3.1** The coefficients \( d_{\tau,T}(q) \) belong to \( \mathbb{Q}[q] \). Moreover \( d_{\tau,T}(0) = 0 \) if \( \tau \neq T \) and \( d_{T,T}(0) = 1 \).

**Proof.** Recall that \( \{G(T)\} \) is a basis of \( \mathbb{V}_Q(\lambda) = U_Q^- v_{\lambda} \). This implies that the vectors of this basis are \( \mathbb{Q}[q,q^{-1}]-\text{linear combinations of vectors of the type considered in Lemma 4.1.1.} \) In particular \( d_{\tau,T} \in \mathbb{Q}[q,q^{-1}] \). By condition (10), \( d_{\tau,T}(q) \) must be regular at \( q = 0 \) and

\[
d_{\tau,T}(q) \equiv \begin{cases} 0 \mod q & \text{if } \tau \neq T \\ 1 \mod q & \text{otherwise} \end{cases}
\]

So \( d_{\tau,T}(q) \) belong in fact to \( \mathbb{Q}[q] \) and the Lemma is true. \( \blacksquare \)

Let us write

\[
G(T) = \sum_{S \in \mathbf{ST}(n,\lambda)} \beta_{S,T}(q) A(S)
\]

the expansion of the basis \( \{G(T)\} \) on the basis \( \{A(T)\} \). We have the following lemma analogous to Lemma 4.3 of \( \text{[3]} \):

**Lemma 4.3.2** The coefficients \( \beta_{S,T}(q) \) of (29) satisfy:

(i): \( \beta_{S,T}(q) = \beta_{S,T}(q^{-1}) \),

(ii): \( \beta_{S,T}(q) = 0 \) unless \( S \leq T \),

(iii): \( \beta_{T,T}(q) = 1 \).

**Proof.** See proof of Lemma 4.3 in \( \text{[3]} \). \( \blacksquare \)

Let \( T_\lambda = T^{(1)} \bowtie T^{(2)} \bowtie \cdots \bowtie T^{(t)} \) be the sequence of tableaux of \( \mathbf{ST}(n, \lambda) \) ordered in increasing order. We have \( G(T_\lambda) = A(T_\lambda) \), i.e. \( G(T^{(1)}) = A(T^{(1)}) \). By the previous lemma, the transition matrix \( M \) from \( \{A(T)\} \) to \( \{G(T)\} \) is upper unitriangular once the two bases are ordered with \( \leq \). Since \( \{G(T)\} \) is a \( \mathbb{Q}[q,q^{-1}] \) basis of \( \mathbb{V}_Q(\lambda) \) and \( A(T) \in \mathbb{V}_Q(\lambda) \), the entries of \( M \) are in \( \mathbb{Q}[q,q^{-1}] \). Suppose by induction that we have computed the expansion on the basis \( \{v_\tau; \tau \in \mathbf{T}(n,\lambda)\} \) of the vectors

\[
G(T^{(1)}), \ldots, G(T^{(i)})
\]

and that this expansion satisfies \( d_{v_{\tau},T^{(p)}}(q) = 0 \) if \( \tau \triangleright T^{(p)} \) for \( p = 1, \ldots, i \). The inverse matrix \( M^{-1} \) is also upper unitriangular with entries in \( \mathbb{Q}[q,q^{-1}] \). So we can write:

\[
G(T^{(i+1)}) = A(T^{(i+1)}) - \gamma_i(q) G(T^{(i)}) - \cdots - \gamma_1(q) G(T^{(1)}).
\] (30)

It follows from condition (17) and Proposition 4.2.3 that \( \gamma_m(q) = \gamma_m(q^{-1}) \) for \( m = 1, \ldots, i \).

By Lemma 4.3.3, the coordinate \( d_{T^{(i)},T^{(i+1)}}(q) \) of \( G(T^{(i+1)}) \) on the vector \( v_{T^{(i)}} \) belongs to \( \mathbb{Q}[q] \), \( d_{T^{(i)},T^{(i+1)}}(0) = 0 \) and the coordinate \( d_{T^{(i)},T^{(i)}}(q) \) of \( G(T^{(i)}) \) on the vector \( v_{T^{(i)}} \) is equal to 1. Moreover \( v_{T^{(i)}} \) can only occur in \( A(T^{(i+1)}) - \gamma_i(q) G(T^{(i)}) \). If

\[
\alpha_{T^{(i)},T^{(i+1)}}(q) = \sum_{j=-r}^{s} a_j q^j \in \mathbb{N}[q,q^{-1}]
\]

then we will have

\[
\gamma_i(q) = \sum_{j=-r}^{0} a_j q^j + \sum_{j=r+1}^{0} a_{-j} q^j \in \mathbb{N}[q,q^{-1}].
\]
Next if the coefficient of $v_{T(i-1)}$ in $A(T^{(i+1)}) - \gamma_i(q)G(T^{(i)})$ is equal to

$$\sum_{j=-l}^{k} b_j q^j$$

using similar arguments we obtain

$$\gamma_{i-1}(q) = \sum_{j=-l}^{0} b_j q^j + \sum_{j=1}^{l} b_{-j} q^j,$$

and so on. So we have computed the expansion of $G(T^{(i+1)})$ on the basis $\{v_\tau\}$ and this expansion satisfies $d_{\tau,T^{(i+1)}}(q) = 0$ if $\tau \ntriangleright T^{(i+1)}$. Finally notice that $\gamma_s(q) \in \mathbb{Z}[q, q^{-1}]$ for all $s$ by Proposition 4.2.3.

By construction of $A(T)$, it is possible to write the basis $\{A(T)\}$ in terms of the basis $\{v_\tau\}$. Using the above, it is then possible to determine the $\gamma_i(q)$ and thus write the $G(T)$ in terms of the basis $\{v_\tau\}$. We have proved that:

**Theorem 4.3.3**  Let $T \in \text{ST}(n, \lambda)$. Then $G(T) = \sum d_{\tau,T}(q)v_\tau$ where the coefficients $d_{\tau,T}(q)$ satisfy:

(i): $d_{\tau,T}(q) \in \mathbb{Z}[q],$

(ii): $d_{\tau,T}(q) = 1$ and $d_{\tau,T}(0) = 0$ for $\tau \neq T,$

(iii): $d_{\tau,T}(q) \neq 0$ only if $\tau$ and $T$ have the same weight, and $\tau \preceq T$.

**5 Examples**

All the vectors occurring in our calculations are weight vectors. So we can use our algorithm to compute the canonical basis of a single weight space. We give below the matrix obtained for the 12-dimension weight space of the $U_q(sp_6)$-module $V(4, 3, 2)$ (i.e. $\lambda = \Lambda_1 + \Lambda_2 + 2\Lambda_3$) corresponding to the weight $\mu = (0, 3, 0)$. Its columns and rows are respectively labelled by the symplectic tableaux and by the tabloids of weight $\mu$ ordered from left to right and top to bottom in decreasing order for $\preceq$. Those tabloids which are symplectic tableaux have been written in bold style.
|    | 1121 | 1331 | 1331 | 1322 | 1222 | 1332 | 1332 | 1322 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 |
|----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|    | 22   | 22   | 22   | 32   | 22   | 22   | 31   | 21   | 22   | 21   | 21   | 21   | 21   | 21   |
| 1121 | 1    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 331  | q^2  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 22   | q    | 1    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 1331 | q^4  | q^2  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 332  | q    | q    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 22   | q^3  | q    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 1332 | q^6  | q^5  | q^3  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 332  | .    | q^2  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 22   | .    | .    | q^4  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 1332 | .    | .    | q^5  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 332  | .    | .    | .    | q^6  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 22   | .    | .    | .    | .    | q    | .    | .    | .    | .    | .    | .    | .    | .    | .    |
| 1332 | .    | .    | .    | .    | .    | q    | .    | .    | .    | .    | .    | .    | .    | .    |
| 332  | .    | .    | .    | .    | .    | .    | q    | .    | .    | .    | .    | .    | .    | .    |
| 22   | .    | .    | .    | .    | .    | .    | .    | q    | .    | .    | .    | .    | .    | .    |
| 12   | .    | .    | .    | .    | .    | .    | .    | .    | q    | .    | .    | .    | .    | .    |
| 1332 | .    | .    | .    | .    | .    | .    | .    | .    | .    | q    | .    | .    | .    | .    |
| 332  | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | q    | .    | .    | .    |
| 22   | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | q    | .    | .    |
| 12   | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | q    | .    |
| 12   | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | q    |
| 12   | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    | .    |

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|     | 1121 | 1351 | 1351 | 1331 | 1322 | 1222 | 1322 | 1322 | 1322 | 1332 | 1332 | 1333 | 1333 |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 331 | 22   | 32   | 32   | 31   | 31   | 31   | 31   | 31   | 31   | 31   | 31   | 31   | 31   |
| 22  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  | q^3  |
| 331 | 22   | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  | q^4  |
| 12  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  | q^5  |
| 21  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  | q^6  |
| 1122| q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  |
| 321 | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  |
| 312 | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  |
| 22  | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 |
| 312 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 |
| 22  | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 |
| 1131 | 1331 | 1331 | 1331 | 1332 | 1332 | 1332 | 1332 | 1332 | 1333 | 1333 |
|------|------|------|------|------|------|------|------|------|------|------|
| 22   | 22   | 22   | 32   | 22   | 22   | 31   | 21   | 22   | 21   | 21   |
| 332  | 332  | 332  | 332  | 332  | 332  | 332  | 332  | 332  | 332  | 332  |
| q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  | q^7  |
| q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  | q^8  |
| q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  | q^9  |
| q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 | q^10 |
| q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 | q^11 |
| q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 | q^12 |
|      | 1132 | 1331 | 1331 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 22   | 22   | 22   | 32   | 22   | 22   | 32   | 22   | 22   | 31   | 22   | 32   | 22   | 21   |
| 322  | .    | .    | q^{7} | .    | .    | q^{5} | .    | .    | q^{6} | .    | .    | q^{5} | .    |
| 332  | .    | .    | q^{8} | .    | .    | q^{6} | .    | .    | q^{7} | .    | .    | q^{5} | .    |
| 333  | .    | .    | q^{6} | .    | .    | q^{5} | .    | .    | q^{4} | .    | .    |      |      |
| 322  | .    | .    | q^{8} | .    | .    | q^{7} | .    | .    | q^{6} | .    | .    | q^{5} | .    |
| 332  | .    | .    | q^{7} | .    | .    | q^{5} | .    | .    | q^{4} | .    | .    |      |      |
| 333  | .    | .    | q^{6} | .    | .    |      |      |      |      |      |      |      |      |
| 322  | .    | .    | q^{7} | .    | .    | .    |      |      |      |      |      |      |      |
| 332  | .    | .    | q^{8} | .    | .    | .    |      |      |      |      |      |      |      |
| 333  | .    | .    | q^{6} | .    | .    | .    |      |      |      |      |      |      |      |
| 322  | .    | .    | q^{7} | .    | .    | .    |      |      |      |      |      |      |      |
| 332  | .    | .    | q^{8} | .    | .    | .    |      |      |      |      |      |      |      |
| 333  | .    | .    | q^{6} | .    | .    | .    |      |      |      |      |      |      |      |
| 322  | .    | .    | q^{7} | .    | .    | .    |      |      |      |      |      |      |      |
| 332  | .    | .    | q^{8} | .    | .    | .    |      |      |      |      |      |      |      |
| 333  | .    | .    | q^{6} | .    | .    | .    |      |      |      |      |      |      |      |
|       | 1123 | 1333 | 1333 | 1333 | 1222 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 1123  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 231   |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 12    |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 1333  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 21    |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 3133  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 3133  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 3133  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 3133  |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 322   |      |      |      |      |      |      |      |      |      |      |      |      |      |
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22
|   | 1121 | 1331 | 1331 | 1331 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 | 1332 |
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Note that all the coefficients $d_{\tau,T}(q)$ of the above matrix are in $\mathbb{N}[q]$. This not true in general. For example, consider the canonical basis of the weight space of the $U_q(sp_8)$-module $V(1,1,1,1)$ corresponding to the weight $(0,0,0,0)$. Then for $T = \begin{pmatrix} 1 & 4 \\ 3 & 4 \\ 4 & 3 \\ 4 & 1 \end{pmatrix}$ there are two coefficients $d_{\tau,T}(q) \notin \mathbb{N}[q]$ in $G(T)$. More precisely, we have $d_{\tau_1,T}(q) = -q^4$ and $d_{\tau_2,T}(q) = -q^4$ for $\tau_1 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 3 & 4 \\ 2 & 1 \end{pmatrix}$ and $\tau_2 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 3 \\ 1 & 2 \end{pmatrix}$.

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