Linear perturbations in vector inflation and stability issues

Alexey Golovnev
Arnold Sommerfeld Center for Theoretical Physics, Department für Physik, Ludwig Maximilians Universität, Theresienstr. 37, D-80333, Munich, Germany
Alexey.Golovnev@physik.uni-muenchen.de

Abstract

We continue the analysis of perturbations in vector inflation. The dominant theme of this paper is the long wavelength limit of perturbations in small fields inflation and the controversial issue of its linear stability. We explain the nature of longitudinal modes, describe how they evolve, and show that they are not as harmful as it could seem at the first glance. On the other hand, the gravitational waves instability in large fields models is shown explicitly. It strongly limits a potential applicability of the recently proposed \(\delta N\)-type approach to vector inflationary perturbations. Finally, we expose a problem of an extra (gravitational) degree of freedom which appears whenever the vector fields are non-minimally coupled to gravity.

1 Introduction

Recently it became evident that higher spin fields can source inflationary expansion of the Universe \([1, 2, 3]\). At the level of background FRW dynamics, higher spin models go almost identically to the scalar inflaton case. But they also provide a possible account for a large scale anisotropy \([1]\) which is currently of a great phenomenological importance. It motivated many independent researchers to study cosmological perturbations in higher spin inflationary scenarios \([4, 5, 6, 7]\). However, the problem appears to be very complicated due to non-trivial couplings of different types of perturbations (scalar, vector, tensor) to each other even at the linear order \([5]\), with a possible exception of a 3-form inflaton field \([2, 8, 9]\) being dual\(^1\) to some peculiar scalar \([2, 7]\).

In this paper we focus on vector inflation with the action \([1, 4, 5]\)

\[
S = \int \sqrt{-g} \left[ -\frac{R}{2} \left( 1 + \sum_{n=1}^{N} \frac{1}{6} I(n) \right) - \frac{1}{4} \sum_{n=1}^{N} F^{(n)\mu\nu}_F F_{(n)^{\mu\nu}} - \sum_{n=1}^{N} V(I(n)) \right] dx^4
\]

where \(I(n) \equiv -A^{(n)}_{\mu} A^{(n)}_{\mu}\) and \(F^{(n)}_{\mu\nu} \equiv \nabla_{\mu} A^{(n)}_{\nu} - \nabla_{\nu} A^{(n)}_{\mu}\); the vector fields are supposed to be randomly oriented so that the background metric is approximately isotropic with linear perturbations given by

\[
ds^2 = (1 + 2\phi) dt^2 + 2a(t) V_i dt dx^i - a^2(t) (\delta_{ij} - 2\psi_{ij}) dx^i dx^j,
\]

\(V_i^i \equiv 0, h_i^i \equiv 0, h_{ij}^i \equiv 0\). The mass-term inflation corresponds to \(V = -m^2 A_{\mu} A^{\mu} = \frac{m^2}{2} I\). For every single inflaton field the energy-momentum tensor takes the form

\[
T_{\alpha\beta} = \frac{1}{4} F^{\gamma\delta}_{\alpha\beta} \delta_{\alpha\beta} - F^{\alpha\gamma} F_{\beta\gamma} + \left( 2 V_{,I} + \frac{R}{6} \right) A^\alpha A_\beta + V(I) \delta^\alpha_{\beta} + \frac{1}{6} \left( R^\alpha_{,\beta} - \frac{1}{2} \delta^\alpha_{\beta} R \right) A^\gamma A_\gamma + \frac{1}{6} \left( \delta^\alpha_{\beta} \Box - \nabla^\alpha \nabla_\beta \right) A^\gamma A_\gamma
\]

\(^1\)Actually, the duality is flawed by higher time derivatives in the dual action. These higher derivative effects seem to be intrinsically non-linear and do not spoil the linear analysis \([7]\). Probably, they can be consistently eliminated in perturbation theory \([10]\). But it is also quite possible that this problem reveals an actual extra degree of freedom which is discussed for vector fields in Section 5.
which is by itself quite capable of giving the general feeling for why the perturbation theory is so messy. A crucial simplification can be achieved assuming that inflation is driven by small vector fields, \( N B^2 \ll 1 \), \( B \equiv \frac{1}{a(\tau)} \). Anyway, this assumption is almost unavoidable if one wants to ensure stability of gravitational waves \([4]\). In Section 2 we accept it and consider the perturbations for very small values of inflaton fields.

After that we proceed with elucidating some tricky aspects of vector inflation. Namely, in Section 3 we give a thorough analysis of longitudinal modes and associated stability problems \([11, 12]\), and in Section 4 we show the gravitational instability of large fields vector inflation explicitly, for it has recently been doubted in \([9]\). After that, in Section 5 we report a new problem of vector inflation concerning the number of degrees of freedom. And in Section 6 we conclude.

## 2 Linear perturbations in small fields inflation

The full (and horrible) set of linear perturbation equations can be found in \([5]\). Fortunately, we need only a few rudiments of the general perturbation analysis. Every term in \([3]\) can be varied easily, although the whole expression becomes very bulky. It is clear that scalar, vector and tensor perturbations mix with each other because we can contract a background vector \( B_i \) with a perturbation. For example, using the metric \([2]\) we have \( \delta B^2 = 2 B_i \delta B_i + 2B^2 \psi + h_{ij} B_i B_j \). And in another linear relation, \( \delta A^0 = \delta A_0 + \nu_i B_i \), the quantity \( \nu_i B_i \) is also a scalar. From \( A = aB \) and \( H \equiv \frac{\dot{a}}{a} \) it follows that the leading contributions of most of the terms in \([4]\) are proportional to \( H^2 B^2 \), with indices contracted or not. However, at the background level the largest terms cancel each other (with \( \sim \frac{1}{\sqrt{N}} \) accuracy for \( N \) random fields and exactly for the fine-tuned case of a triad), and the background dynamics coincides with that of scalar N-flation \([1]\). As for the fluctuations, one has of course to take \( H^2 B \delta B \) terms into account, and we will see below that they are important.

While analysing the possible perturbations in \([3]\) step by step, it is rather tempting to conclude that the curvature perturbations in vector inflation with a large number of fields are ridiculously small. For example, one could argue that any terms in \( \xi_{00} \) of the form of \( \frac{1}{N} \sum \dot{A}_i \delta A^i \) are statistically suppressed as \( N \to \infty \) because the fluctuations \( \delta A \) have arbitrary directions. However, it is not a reliable argument. Indeed, we definitely want to keep the Hubble constant intact in the course of the limiting procedure (or at least the Hubble rate should not diverge). In the mass-term inflation it means that \( A \propto 1/\sqrt{N} \), and despite the \( 1/\sqrt{N} \) statistical suppression the length fluctuation term \( H^2 \sum \dot{A}_i \delta A^i \) has no scaling with \( N \). It shows that one has to look for some other approximations.

Note that this type of naive argument, if it were only correct, would also suppress the perturbations in scalar N-flation in contradiction to the general statement of \([13]\). And the reason for which it does not actually happen is precisely the same as for vectors. Namely, for \( V = \frac{m^2}{2} \sum \phi_i^2 \) we get \( V \propto N \frac{m^2}{2} \phi^2 \) and \( \delta V \propto \sqrt{N} m^2 \phi (\delta \phi) \) where \( (\delta \phi) \) is a typical magnitude (variance) of fluctuations. And then for the relative magnitude of perturbations we have \( \frac{\delta V}{V} \propto \frac{\sqrt{N}}{\sqrt{2}} \left( \frac{\delta \phi}{\phi} \right) \). There is the \( \sqrt{N} \) factor in denominator. However, recall that \( H^2 \propto m^2 N \phi^2 \) and hence \( \phi \propto \frac{H}{\sqrt{N} m} \). Let’s parametrize the wavelengths \( \lambda \) by \( \kappa \equiv (\lambda H)^{-1} \) so that \( \kappa = 1 \) at the horizon scale. Then we have \( \delta \phi_k \propto \kappa H \propto \kappa \sqrt{N} m \phi \) and finally \( \frac{\delta V}{V} \propto \kappa m \). The magnitude of perturbations depends solely on the inflaton mass.

In vector inflation all types of perturbations are mixed, and it is particularly interesting to evaluate the effect of mixing with gravitational waves due to \( H^2 \sum B_i \delta B_j \) contributions to the linear fluctuations of the stress tensor \([3]\). If we assume the fluctuations \( \delta B \) are random (e.g. their directions are not correlated with the background direction of the field) then this term has the usual \( \sqrt{N} \) suppression. So the effect should be of order \( \sqrt{N} H^2 B (\delta B) \). For the mass-term inflation we get \( \delta B_{\kappa} \propto \kappa H \propto \kappa m B \sqrt{N} \), and the relative weight of this perturbation \( \sqrt{N} (\delta B B) \propto \kappa m B^2 N \) is huge when we start inflation at \( B \sim N^{-1/4} \). Some perturbations are completely out of control at the onset of large fields inflation. It is understandable because the whole story of the background dynamics has emerged from statistical cancellation of the leading terms \( H^2 B^2 \) in the energy-momentum tensor, and inflation is initiated when anisotropic corrections are of order one (and actually badly unstable, see Section 4). In new inflation the quantity \( \sqrt{N} \delta B B \propto \kappa H \sqrt{N} B \) can be made relatively small. This term serves as a source for gravity waves provided by inflatons fluctuations. One can control its magnitude by varying the form of potential.

\[ \text{\footnotesize{\textsuperscript{2}This can raise some doubts about the instability of gravitational waves in such inflation as it was deduced neglecting these terms \([12]\). Intuitively it’s hard to believe that some external force can neutralize the effect of the large tachyonic mass. And in fact, this intuition works well, see Section 4.}} \]
and the number of fields in order to produce a desired amount of tensor perturbations which is usually a problem in new inflation (see also [14]).

It is very natural that, under suitable conditions in the small fields limit, the gravity waves disentangle from the other modes. After all, the linear mixing of different perturbations occurs due to the presence of (a random set of) preferred directions. But the vectors which represent the preferred directions become very small and not very important, therefore the mixing of modes is notably weak in this limit [5]. Hence we can approach the dynamics by considering background values of the inflatons as small quantities. (Although they should be larger than the perturbations, of course.) For the equations of motion of the vector fields it means that in the first approximation we are to consider the fixed background geometry, i.e. neglect the gravitational backreaction. Indeed, all other terms in the first-order equations of motion contain variations of metric multiplied by background vector fields, see also [5]. In this limit we substitute the non-perturbed FRW metric (with a fixed but otherwise arbitrary time-dependence of the scale factor) in the action [14] and get for the vector fields [1]:

\[
- \frac{1}{a^2} \Delta A_0 + \left(2V_{,I} + \frac{R}{6}\right) A_0 + \frac{1}{a} \left(\partial_i \dot{B}_i + H \partial_i B_i\right) = 0,
\]

(4)

\[
\dot{B}_i + 3H \dot{B}_i + 2V_{,I} B_i - \frac{1}{a^2} \Delta B_i - \frac{1}{a} \left(\partial_i A_0 + H \partial_i A_0\right) + \frac{1}{a^2} \partial_i \left(\partial_j B_j\right) = 0.
\]

(5)

And there is also a consistency condition

\[
\nabla_\mu \left(2V_{,I} + \frac{R}{6}\right) A^\mu = 0.
\]

Note that for \(H = \text{const}\) one gets a de Sitter background which amounts to setting the slow-roll parameters to zero.

From now on we would be interested in only the leading behaviour of the vector fields and therefore assume the geometry to be a pure de Sitter (so that the scalar curvature is constant in time) and neglect the variation of effective mass (it would contain \(V_{,II}\) and extra powers of \(A\)). Then we have \(\nabla_\mu A^\mu = 0\), or more explicitly

\[
\dot{A}_0 + 3H A_0 - \frac{1}{a} \partial_i B_i = 0.
\]

(6)

It allows to simplify the equation (5):

\[
\dot{B}_i + 3H \dot{B}_i + 2V_{,I} B_i - \frac{1}{a^2} \Delta B_i + \frac{2H}{a} \partial_i A_0 = 0,
\]

(7)

and \(2V_{,I}\) can be substituted by a constant mass. We see that in the long wavelength limit all components\(^3\) of \(B_i\) evolve exactly like scalar fields with the mass \(m^2 = 2V_{,I}\). The dominant mode slowly rolls, while the other one fastly decays. And the condition (6) (or, even better, the constraint equation (4)) shows that \(A_0\) decays exponentially in physical time as it was correctly stated in [5]. However, it is not true for the spatial longitudinal mode itself, as it behaves identically to the transverse ones when \(\lambda \to \infty\). It is just its contribution to the constraint equation (6) what goes to zero. And actually, it goes to zero as \(\frac{1}{a}\) (this dependence represents the stretching of waves) so that \(A_0 \propto \frac{1}{a}\) also (more precisely, a little bit different from that due to the slow motion of the field \(B\)). Moreover, the mode which could be erroneously deduced from (6) naively setting the spatial derivatives to zero, namely \(A_0 \propto \frac{1}{a^3}\), is absolutely fake. The consistency condition is necessary but not sufficient for a vector field to satisfy the whole system of equations of motion. From (4) we see that if there is no longitudinal mode, then \(A_0 = 0\) exactly. The superhorizon analysis is always notably subtle in that it is not reliable to neglect the spatial derivatives for evaluation of subleading quantities.

For the Einstein equations we can use the same philosophy and drop the terms which contain two powers of \(B\), like \(B^2 \dot{\psi}\); then, for example, \(\delta (B^2)\) would be given just by \(2B \delta B\). If we could also neglect the gravitational waves and consider the anisotropy as only a small correction, then the number of e-folds in any patch of the Universe would be well-defined and depend only on variations of \(B_i\) (recall that \(A_0\) decays). In this limit and under these assumptions one can use the \(\delta N\)-approach of [6]. However, if the anisotropies grow as fast as for the mass-term inflation then it would be a tricky business even to speak about the number of e-folds. In the limit of small fields the accuracy of \(\delta N\)-approach is determined by the strength of mixing with tensor modes. And as the latter can not be much greater than the tensor-to-scalar

\(^3\) See the next Section for peculiar properties of the longitudinal component which are missed in this limit.
tachyonic vector field is free, one can quantize just three independent fields and the tachyonic mass if the spatial length $\Lambda$ variable given by (8), would have to be very large around these modes which is compatible with negative $|\Lambda|$ length scale of mass at large values of $\Lambda$.

A suitable non-linearity. For example, it is tempting to construct a potential which would give a normal $A$ so for large positive $\Lambda$, it diverges at $k^2 = -m^2 > 0$. At this wavelength the longitudinal mode is not permitted (but $A_0$ is arbitrary) and in the neighbourhood of $k^2 = -m^2$ it involves very large values of $A_0$. As long as the tachyonic vector field is free, one can quantize just three independent fields $A_i$ and forget about unphysical variable $A_0$ but this approach is not suitable for turning on Lorentz invariant interactions.

In principle, the above result looks like a kind of resonant behaviour which one could try to kill by a suitable non-linearity. For example, it is tempting to construct a potential which would give a normal mass at large values of $A_0$. Unfortunately, given the Lorentz invariance, the best we can do is to make it so for large positive $A_0$. Then the differential operator $-\Delta + 2V_j$ would still have zero modes at the length scale of $|m_{eff}|^{-1}$ for negative values of $A^2$ where the field is tachyonic. The temporal component, given by (9), would have to be very large around these modes which is compatible with negative $A^2$ and tachyonic mass if the spatial length $|\Lambda|$ is even large. Thus, the best we can do is to shift the problem.

4Of course, $A_0$ is by itself proportional to the length of longitudinal mode but (in dimensions greater than 1 + 1) we can always play with the lengths of transverse components to make $A^2$ negative.

Finally, we should mention the vector perturbations of the metric. They usually decay as fast as $1/k$ (see for example (15)) obeying the conservation law of angular momentum. The vector inflation is no exception, although the decay could be a bit slower due to vector perturbations of the energy-momentum tensor. In absence of vectorial stress tensor perturbations the spatial part of Einstein equations would give the usual decay. But the right hand side of it contains in vector inflation a source for $V$. The source represents the vortical excitations of vector fields which carry non-zero angular momentum. Therefore, the vector inflatons can produce some vorticity. But apparently, they can not make any kind of vortical instability at least in small fields models because, neglecting the gravitational backreaction, the inflatons evolve in FRW space-time and by themselves obey the angular momentum conservation, so that the stress source for $V$ also decays during expansion in much the same way as a rotating platform slows down if some mass moves on it in the radial direction. The backreaction would allow the vector fields to gain some vorticity by transferring the opposite angular momentum to the gravitational field as a recoil. Therefore we expect that for vector inflation the vector perturbations decay can be a bit slower than the usual $1/k$ law, but not much slower for small fields models. It is hard to make more precise statements on $V$ due to the aforementioned problems of working with subleading quantities in the long wavelength limit.

3 The problem of longitudinal components

At a finite wavelength, the equation (4) and the scalar part of equation (7) describe the evolution of temporal and longitudinal components of vector fields. Note that (4) is a constraint equation, and therefore we consider only one degree of freedom per every vector field (see, however, Section 5) which is actually suspected to be badly unstable for tachyonic masses [11, 12]. It was claimed to be a ghost which always play with the lengths of transverse components to make $\delta N$ formalism is applicable to at least a few percent accuracy for viable models, and we can be sure in the nearly flat primordial spectrum of curvature perturbations if the gravitational waves are tame. Moreover, $\theta_{00}$ (see [5]) depends then only on the length fluctuations $B_i \partial B_j$ (and their time derivatives) which explains why the results of [6] are identical to that for scalar N-flation. All the relevant variations in the slow-roll regime come from changing the scalar argument $A^2$ of the potential. In general, precision of these results can be roughly estimated as above, and beyond this accuracy the perturbations differ from that of scalar inflation, although the detailed predictions are model dependent and presumably require numerical methods.
to the region of very large fields. It is not possible to completely exercise the instability with modified potential, at least without abandoning Lorentz invariance or introducing new degrees of freedom. The Hamiltonian analysis also confirms this result.

### 3.2 Hamiltonian analysis

As usual, the conjugate momenta are defined as $\pi_\mu = \frac{\partial L}{\partial \dot{A}_\mu}$ and given by $\pi_i = F_{0i}$ and $\pi_o = 0$ (primary constraint). For the mass-term potential the Hamiltonian density reads

$$\mathcal{H} = \frac{1}{2} \pi_i^2 + (\partial_i A_0) \pi_i + \frac{1}{4} F_{ij} F_{ij} + \frac{m^2}{2} (A_i^2 - A_0^2).$$

Commutation of the primary constraint $\pi_o$ with the Hamiltonian gives the secondary constraint $\partial_i \pi_i + m^2 A_0 = 0$ which allows for elimination of unphysical variable $A_0$:

$$\mathcal{H} = \frac{1}{2} \pi_i^2 + \frac{1}{2m^2} (\partial_i \pi_i)^2 + \frac{1}{4} F_{ij} F_{ij} + \frac{m^2}{2} A_i^2.$$  

If $m^2 > 0$ then all the terms are positive, while for tachyonic masses the second and the fourth terms are negative. The latter is just the ordinary tachyonic potential but the former contains momenta and therefore looks like a ghost. However, the secondary constraint shows that for the physical states it equals just to $\frac{m^2}{2} A_i^2$:

$$\mathcal{H} = \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F_{ij} + \frac{m^2}{2} (A_i^2 + A_0^2).$$

There is no problem in the infrared because $A_0 \to 0$ if $k \to 0$, and no problem in the ultraviolet because $A_0 \sim \frac{1}{k^2} \propto \mathcal{O}(1)$ when $k \to \infty$ so that another term $\frac{1}{4} F_{ij} F_{ij} \propto k^2 A_i^2$ wins the game. It is not a genuine ghost as it presents a problem only in a limited range of $k^2$ near $-m^2$ in a sharp contrast with real ghosts which go worse and worse in the ultraviolet.

For a general potential the secondary constraint is $\partial_i \pi_i + 2 V, I A_0 = 0$, and in terms of physical variables the Hamiltonian becomes quite complicated since we have to use $A_0 = -\frac{\partial V, I}{2V, I}$ in the argument of $V$. However, we can go the opposite way again and exclude $\partial_i \pi_i$:

$$\mathcal{H} = \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F_{ij} + \frac{A_0^2}{2V, I} + V(I).$$

Now one can easily convince himself that it would be rather difficult to make this Hamiltonian positive definite because at some point we need to pass from $V, I < 0$ to $V, I > 0$.

So, we do not understand the theory completely. But it is not a fatal problem since we do not necessarily have an infinite phase space volume in the ultraviolet for the large $A_0$ instability. And the crucial property of inflationary space-times is that the wavelengths are changing with time and every mode dwells at a singular point $k^2 = -m^2$ only for a single instant of time. Now we proceed to show that it solves the instability problem at least at the level of classical equations of motion using the test field approximation in the simplest case of de Sitter geometry.

### 3.3 Longitudinal mode in vector inflation

For a longitudinal mode ($k_i B_i = k B$) we use the constraint $\pi_i$ to find the closed form of the equation of motion $\dot{B}$ in de Sitter space ($\frac{H}{\alpha} = -2H^2$):

$$\dot{B} + \left(3H + \frac{2H^2 k^2}{\alpha^2} + m^2 - 2H^2\right)\dot{B} + \left(\frac{k^2}{\alpha^2} + m^2 + \frac{2H^2 k^2}{\alpha^2} + m^2 - 2H^2\right) B = 0. \quad (9)$$

At short wavelengths there is no big difference between transverse and longitudinal modes. The problem appears when the wavelength is near the value for which $\frac{k^2}{\alpha^2} = 2H^2 - m^2$. It’s time to pay for introducing the tachyonic effective mass $-2H^2$. We need to understand the properties of solutions in this region. Let’s neglect the mass $m$ of the inflaton for simplicity as it much smaller than $H$ in the slow roll regime.
and pick up a wave which was well under the horizon with \( \frac{k^2}{a^2} = 4H^2 \) at some instant of time \( t_0 \). Then \( \frac{k^2}{a^2} = 4H^2e^{-2H(t-t_0)} \) and \( [6] \) takes the form:

\[
\left( 2 - e^{2H(t-t_0)} \right) \ddot{B} + \left( 10H - 3H e^{2H(t-t_0)} \right) \dot{B} + 8H^2 e^{-2H(t-t_0)} B = 0.
\]

With a new time variable \( \tau = 2H(t-t_0) - \ln 2 \) we get

\[
2(1 - e^\tau) \ddot{B} + (5 - 3e^\tau) \dot{B} + e^{-\tau} B = 0. \quad (10)
\]

The critical point of crossing the singularity is at \( \tau = 0 \). The coefficient in front of \( \ddot{B} \) vanishes at this point and all trajectories are tangent there to a one-parameter family of curves \( \ddot{B} = -\frac{\dot{B}}{2} \). This behaviour is stable because if \( \ddot{B} \neq -\frac{\dot{B}}{2} \) at small \( \tau < 0 \) then the second derivative \( \ddot{B} \sim \frac{\dot{B}}{\tau} \) has an appropriate sign to correct the trajectory.

Our task is to find a two-parametric family of solutions for equation \( (10) \). From the previous analysis it is clear that one possible solution contains no longitudinal mode at all when \( \frac{k^2}{a^2} = 2H^2 \). We take \( B(\tau = 0) = \dot{B}(\tau = 0) = 0 \) and construct the solution in the form of power series \( B = \tau^2 + \sum_{n \geq 3} C_n \tau^n \).

The first term solves the equation up to \( O(\tau^3) \)-terms, and \( O(\tau^3) \)-corrections give \( C_3 = -\frac{7}{3} \) and so on. This solution is rather smooth around the problematic point due to absence of longitudinal mode, it corresponds to \( A_0 = 0 \) at \( \tau = 0 \).

The second solution is more interesting. We write it down as \( B = 1 - \frac{\tau}{2} + \sum_{n \geq 2} D_n \tau^n \) which gives finite values of \( A_0 \) due to cancellation of two first-order zeros in \( (8) \). The first two terms solve \( (10) \) at the level of \( O(\tau) \), then \( D_2 \) is undetermined because at the level of \( O(\tau^2) \) it solves the equation by itself, thus one can take \( D_2 = 0 \) and proceed with \( D_3 \). The full two-parameter family of solutions is given by \( B = \alpha - \frac{\tau}{2} + \beta \tau^2 + \sum_{n \geq 3} C_n \tau^n \) where \( \alpha \) and \( \beta \) are arbitrary constants while \( C_n \)'s should be determined one by one in terms of \( \alpha \) and \( \beta \).

In order to understand the properties of the second solution around \( \tau = 0 \) we note that, unlike for the first one, the main players for it are \( B \) and \( \dot{B} \). If we neglect the \( \dot{B} \)-term in \( (10) \), then the resulting equation can be solved explicitly in elementary functions. We present the solution in terms of the physical time:

\[
B = C e^{\frac{3}{2} e^{-2H(t-t_0)} - \frac{3}{10} \frac{1}{t_0}}.
\]

One can easily check that at the moment of time \( t - t_0 = \frac{\ln 2}{2} \) we have \( \dot{B} = -BH \) as required. It is also straightforward to estimate the value of \( A_0 \) at the same time:

\[
A_0 \sim \frac{(\dot{B} + HB)e^{H(t-t_0)}}{(2 - e^{2H(t-t_0)})H} \sim \frac{C}{20}
\]

which is clearly not too large (although it is the value of a \( \frac{1}{10} \) fraction), and no catastrophe happens. The \( t \to \infty \) asymptotic is completely stable too, the amplitude goes to a constant value (because the inflaton mass was neglected) of the same order of magnitude as it was when we picked it up at \( t = t_0 \). (Of course, one should properly rearrange the signs for \( t > t_0 + \frac{1}{2H}(\ln 10 - \ln \frac{\ln 2}{2}) \). The solutions can be evolved through the dangerous point smoothly.

One could make the above analysis without neglecting the mass in \( (11) \). It would not change the qualitative results, but the coefficients would not be so nice. The passage through the singular point (with zero coefficient in front of \( \dot{B} \)) gets shifted in time, but the signs of all the terms around this point are the same as before. Instead of the \( B = -HB \) behaviour at the crossing point, we would find \( \dot{B} = -H \left( 1 + \frac{n^2}{H^2} \right) B \), and in the infinite time limit the longitudinal mode would exhibit the slow roll evolution \( B \sim e^{-\frac{n^2}{H^2} t} \). Of course, when taking into account the inflaton mass, one should better use the actual FRW-metric instead of the pure de Sitter one, see \( [10] \).

There is no visible signature for the linear theory breakdown. Note however that the amplitude changes from its initial value to zero, and then to values of opposite sign at a time scale of order of one e-fold. Of course, this is just the amplitude of some Fourier modes around the horizon, so that it does not imply anything catastrophic, but in general the time variations of \( A^2 \) in the argument of potential term can be somehow more important than usually. It means that non-linearity of potential (changing of
effective mass) could play some role, and therefore considerable non-Gaussianities in primordial spectrum could be produced.

When the present article was in preparation, another work has appeared on arXiv, namely [16], in which the same problem is analysed. The Authors of [16] also came to the conclusion that the linear longitudinal modes safely pass through the point of \( k^2 + M^2_{\text{eff}} = 0 \), at least in the case of a test field in FRW-Universe which is dubbed the zero vector vev in their work. They also argue that in other situations it is not the case: they claim that even a single vector field with non-vanishing vev in Bianchi I space-time develops a singularity of linear modes at this point. It is quite contrary to our general intuition which assumes that inclusion of the metric perturbations in (10) should not make the transition harder, recall that it is basically governed by the sign of the coefficient in front of \( \dot{B} \). We would even expect that the transition should be softer in presence of metric perturbations since not only different modes but also different parts of the space would pass through this point at different instants of time. However, the linear instability was inferred in [16] from numerical simulations. The Authors do not expain their numerics explicitly, but one can safely guess that they evolve the longitudinal mode from deep inside the horizon directly solving the equations of motion by standard methods which involve explicit determination of the second time derivatives of the physical variables via the other terms in equations. It basically amounts to solving our equation (10) for \( \ddot{\varphi} \) and using this value at each step to construct the numerical solution. We suspect that, probably, this numerics fails to give a meaningful answer for a very complicated system of equations near the point at which the expression for the highest derivative terms is singular. On the other hand, one should also take care about a possible anisotropic instability in the system (see the Section 4), which indeed could change a lot.

Note also that one more problem with linear analysis was pointed out in [16]. The linear solution diverges when \( M^2_{\text{eff}} \equiv m^2 + \frac{\dot{B}^2}{2} = 0 \), approximately at the end of inflation. We want to explain here the reason for that, which is actually quite simple. The consistency equation \( \nabla^\mu \left( M^2_{\text{eff}} A^\mu \right) = 0 \) reduces at this point to \( \frac{dM^2_{\text{eff}}}{dt} A^0 = 0 \). And it clearly can not be satisfied with general initial conditions. This is a real problem which occurs only once at the exit from inflation. And again, it signals the lack of fundamental understanding of the nature of the vector fields in the model. At this (and only this) instant of time the gauge freedom exists which makes the number of degrees of freedom ill-defined. The longitudinal mode becomes infinite exactly at the point at which it fails to be physical.

### 4 Instability of large fields inflation

In Ref. [4] it was shown that large fields vector inflation is badly unstable with respect to gravitational waves, i.e. small anisotropies grow in it much too fast. It was argued in [3] that this conclusion is just an artifact of the linear approximation technique in the Jordan frame, while in the Einstein frame everything should be stable because the interaction of gravity with matter has the normal form there. However, we would like to remind that, roughly speaking, only two thirds of the instability reported in [4] came from the \( \frac{\dot{B}}{2} A^2 \) term in the action, while a remaining one third came from the kinetic term of the vector fields which is conformally invariant. Of course, for this instability to develop we had to stabilize the fields \( B \) (to ensure the slow roll), but once it is done, the instability is there in both frames. In this Section we analyse the behaviour of anisotropies in a manner which is somewhat closer to \( \delta N \)-formalism.

We assume that in a separate patch of the Universe the Hubble rate experienced a sudden jump in only one direction, \( \dot{H}_z = \dot{H} + h \). The metric would be the axially symmetric Bianchi I type

\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - h^2(t)dz^2
\]

with the new components of the Einstein tensor: \( G^0_0 = H^2_a + 2H_a H_b = 3H^2 + 2H_h \), \( G^0_a = G^0_b = \dot{H}_a + \dot{H}_b + H^2_a + H^2_b + H_a H_b = 2\dot{H} + 3H^2 + \dot{h} + 3HH + h^2 \) and \( G^z_z = 2\dot{H} + 3H^2 \). If we also assume that the energy-momentum tensor remains isotropic, then subtracting \( G^z_z \) from \( G^z_z \) gives the equation \( h + 3Hh + h^2 = 0 \) which shows that anisotropies are being washed out (as they should be according to the general theorem of Wald, [17]). If \( H \approx \text{const} \) then \( h \propto \frac{1}{a} \) for small \( h \). One could worry about large negative jumps which could become stable at \( h \approx -3H \), but it would already correspond to a phantom matter with energy density \( -3H^2 \). And in any case, large anisotropies are not generally expected to decay since, for example, Kasner solutions are known.

We claim that in vector inflation non-zero \( h \) would render the energy-momentum tensor anisotropic too. In the homogeneous limit and with the slow-roll assumption, the main contribution to anisotropic

\[^7\text{See also the last Hamiltonian of the Section 3.2.}\]
part of $T^{ij}_T$ comes from $\frac{1}{6}A_iA^i$, $-F_{0i}F^{0j}$ and $\frac{1}{6}G^{ij}_T A^2$ terms. The latter one renormalizes the gravitational constant, and we forget about it for a moment. But the first two terms give important contributions to $T^{xx}_x - T^{zz}_z$. With a natural definition $B_z \equiv \frac{1}{6}$ we get for them

$$-\sum \frac{R}{6} (B_x^2 - B_z^2) \approx 2NH^2 \left( \langle B_x^2 \rangle - \langle B_z^2 \rangle \right),$$

$$\sum (H_x^2 B_x^2 - H_z^2 B_z^2) \approx NH^2 \left( \langle B_x^2 \rangle - \langle B_z^2 \rangle \right) - 2NHh^2 \frac{B^2}{3}$$

if both $h$ and $\langle B_x^2 \rangle - \langle B_z^2 \rangle$ are small. And we also have to take the subleading contribution to $-F_{0i}F^{0j}$ into account, namely $2 \sum (H_x B_x B_z - H_z B_z B_z)$. It is easy to derive the equations of motion of vector fields:

$$\ddot{B}_x + (2H_a + H_b) \dot{B}_x + \left( m^2 + \frac{R}{6} + \dot{H}_a + H_a^2 + H_a H_b \right) B_x = 0,$$

$$\ddot{B}_z + (2H_a + H_b) \dot{B}_z + \left( m^2 + \frac{R}{6} + \dot{H}_b + 2H_a H_b \right) B_z = 0,$$

which show that in the slow roll regime we have

$$\ddot{B}_x - \dot{B}_z \approx \frac{\dot{H}_b - \dot{H}_a + H_a H_b - H^2}{2H_a + H_b} B \approx \frac{\dot{h} + H h}{3H} B$$

for a pair of identical fields in directions of $x$ and $z$ axes. And therefore

$$2 \sum (H_x B_x B_z - H_z B_z B_z) \approx 2HB \left( \frac{N}{3} \cdot \frac{\dot{h} + H h}{3H} \right) B \approx \frac{2NB^2}{9} (\dot{h} + H h).$$

Note that at the moment of turning on the fluctuation of the Hubble rate we do not change the value of the scale factor itself, therefore we may keep both $A$ and $B$ independent of the spatial direction of the field. Then the anisotropy of $T^{ij}_T$ initially equals to $-2NHh \frac{B^2}{3} + \frac{2NB^2}{9} (\dot{h} + H h)$. Now we have to recall that the $\frac{1}{6}G^{ij}_T A^2$ term effectively divides the gravitational constant by $1 + \frac{NB^2}{6}$, and the resulting effective anisotropy would be $-4Hh + \frac{4}{3}(\dot{h} + H h)$ in the large fields limit. It has a drastic effect of making the linear equation for $h$ unstable: $-\frac{1}{3} \dot{h} + \frac{17}{3} H h = 0$.

At the same time the anisotropy of matter distribution grows according to vector fields equations of motion:

$$\frac{d}{dt} \cdot 3NH^2 \left( \langle B_x^2 \rangle - \langle B_z^2 \rangle \right) \approx 3NH^2 \frac{N}{3} \cdot 2B \cdot \frac{\dot{h} + H h}{3H} B \approx \frac{2NHBN^2}{3} \left( \dot{h} + H h \right),$$

and in this case tends to stabilize the fluctuation. However, if we differentiate the anisotropic part of Einstein equations with respect to time:

$$\frac{d}{dt} \left( -\frac{1}{3} \dot{h} + \frac{17}{3} H h \right) \approx \left( 1 + \frac{NB^2}{6} \right)^{-1} \cdot \frac{2NB^2}{3} \left( \dot{h} + H h \right),$$

we get a simple approximate equation $\dot{h} - 5H \dot{h} + 12H^2 h = 0$ with initial condition $\dot{h}(0) = 17Hh(0)$. Its solution exhibits a fast exponential growth with oscillations.

Admittedly, the geometric effects which overturned the sign of $\dot{h}$ can be changed by, for example, taking anisotropic distributions of the vector fields. But even if one can force $\dot{h}$ to decay initially, then after a small period of time (before the perturbation could finally decay) one generically gets a considerable amount of anisotropy in the term which does not depend on the current value of $h$, namely $3NH^2 \left( \langle B_x^2 \rangle - \langle B_z^2 \rangle \right)$. This conclusion is robust. It can either prevent the Hubble rate jump from dropping at zero value and make it growing with the opposite sign, or change the sign of $\dot{h}$ before the fluctuation could reach the zero, depending on the coefficients in other terms. In any case, a small anisotropic fluctuation in the Hubble law is badly unstable for the large fields vector inflation.

5 An extra degree of freedom

Now we want to report on a new problem with vector inflation. A naive counting of independent propagating degrees of freedom would give the number of $3N + 2$ for $N$ vector fields and one graviton. Equations
also contain the second time derivatives of ψ. But it is usually not a dynamical quantity since the temporal component of the Einstein equations contains no second time derivatives and can be (and should be) regarded as a constraint which determines ψ completely. (It actually corresponds to Newtonian potential defined by distribution of masses in non-relativistic limit.) However, for vector inflation it is not the case beyond the linear approximation since \( T_{00} \) has a term with \( R A_0^2 \) and the scalar curvature depends on \( \dot{\psi} \).

On the other hand, the field equation (4) is also no longer a constraint due to precisely the same reason. And it is not clear how to find those linear combinations of equations which would give constraints in the Jordan frame.

To find out the actual number of degrees of freedom we need to perform a conformal transformation to the Einstein frame: \( \tilde{g}_{\mu\nu} = e^{2\rho} g_{\mu\nu} \), \( \tilde{R} = e^{-2\rho} \left( R - 6 \Box \rho - 6 \left( \nabla \rho \right)^2 \right) \), \( \tilde{A}_\mu^{(n)} = A_\mu^{(n)} \) with the conformal weight

\[
\rho = \frac{1}{2} \ln \left( 1 + \frac{1}{6} \sum_{n=1}^{N} I_{(n)} \right).
\]

It transforms the original action to

\[
S = \int dx^4 \sqrt{-g} \left( -\frac{R}{2} + \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V - \sum_n \frac{1}{4} F_{\mu\nu}^{(n)} F_{\alpha\beta}^{(n)} g^{\mu\alpha} g^{\nu\beta} \right)
\]

where

\[
\varphi = \sqrt{\frac{3}{2}} \cdot \ln \left( 1 + \frac{1}{6} \sum_{n=1}^{N} I_{(n)} \right)
\]

and the new potential is given by

\[
V = \left( 1 + \frac{1}{6} \sum_m I_{(m)} \right)^2 \cdot \sum_n V \left( 1 + \frac{1}{6} \sum_p I_{(p)} \right)^{-1} \cdot I_{(n)}
\]

so that the vector fields are intricately interacting now. This potential looks scary, and one can think of the special case \( V = \frac{m^2}{2} I \) for which it is just

\[
V = \frac{m^2}{2} \left( 1 + \frac{1}{6} \sum_m I_{(m)} \right) \cdot \sum_n I_{(n)}.
\]

Anyway, our discussion does not depend on a potential. It refers to any non-minimally coupled vector fields (and should be suitable for higher forms too).

We first consider the new action (11) with \( N = 1 \) in a flat (Minkowski) space-time. Canonical momenta are given by \( \pi_0 = 4 A_0 \varphi_1 \left( A_0 \dot{A}_0 - A_j \dot{A}_j \right) \) and \( \pi_i = -4 A_i \varphi_1 \left( A_0 \dot{A}_0 - A_j \dot{A}_j \right) + \dot{A}_i - \partial_i A_0 \).

This system of linear differential equations for velocities has a determinant

\[
-(4\varphi_1^2)^3 A_0^3 \left( A_1^2 A_2^2 + A_2^2 A_3^2 + A_3^2 A_1^2 \right)
\]

which is not zero if \( A_0 \neq 0 \) and therefore is solvable under this assumption, and no constraints are there. Thus, the vector field acquires a fourth degree of freedom whenever the longitudinal mode is on. The reason is easy to understand analysing the equation of motion

\[
2 A^\mu \frac{\partial \varphi}{\partial I^\mu} \left( \Box \varphi \right) + \nabla_\mu F^{\mu\nu} + 2 A^\mu \frac{\partial V}{\partial I^\mu} = 0.
\]

Its temporal component is not a constraint due to the time derivatives in \( \Box \varphi \).

For an arbitrary number of degrees, evaluation of the determinant is a tedious procedure. \( 3N + 1 \) degrees of freedom are guaranteed, but a priori there could be even more of them, up to \( 4N \). However,

---

8One notable exception is a treatment by Weinberg [18] who casts the equations into a form \( R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha \) which contains time derivatives in the temporal component too. (Weinberg has also an overall minus sign in the right hand side due to the opposite sign convention for the Ricci tensor.) Of course, a proper linear combination of these equations restores the constraint.

9In the slow roll regime the conformal factor changes slowly with time, so that the scalar quantities \( A^2 \) and \( F^2 \) remain almost frozen in both frames. The new kinetic term \( (\nabla \varphi)^2 \) plays the role of non-minimal coupling ensuring the slow roll conditions.
we can check that there is only one extra degree of freedom at any value of \( N \) by uncovering \( N - 1 \) constraints. It is easy to do. Suppose, we have two fields with equations of motion \( \text{[12]} \). Let’s multiply the temporal component of the first equation by \( 2 A^0_{(2)} \partial_0 \), the temporal component of the second equation by \( 2 A^0_{(1)} \partial_0 \), and subtract them from each other. Time derivatives are cancelled, and we get a constraint. Then we can play the same game with other fields. Clearly, there are \( N - 1 \) independent relations of this sort.

We see that whenever at least one of the fields has \( A_0 \neq 0 \), an extra degree of freedom turns on and completely decouples again in absence of longitudinal modes. It can’t be seen at all at the level of background dynamics. Although this mode should not play any role in the linear perturbation analysis around the state with \( A_0 = 0 \), it means that at the fundamental level the theory is not very well defined. The problematic degree of freedom is essentially gravitational because it is one for all the fields and stems from non-minimal coupling term due to time derivatives in \( R \). Perhaps, it could be regulated by explicitly turning it on even in the \( A_0 \to 0 \) limit, for example with a small \( R^2 \)-correction to Einstein gravity.

6 Conclusions

An interesting possibility of driving inflation with higher spin fields is nowadays being emergent. It appears that this idea can be perfectly realised at the background level with almost no complications \( \text{[1 2 3]} \) and with a benefit of a natural large scale anisotropy suggested by some recent observations \( \text{[19]} \). But already at the level of linear perturbations the full set of equations of motion \( \text{[5]} \) becomes almost analytically untractable (with a possible exception of 3-form inflation \( \text{[2 7 8 9]} \)). And also some problems have been reported both before \( \text{[11 12 16 20]} \) and now, in this paper, see Section 5. However, we have shown that although there is obviously a serious lack of understanding the nature of vector inflatons at the fundamental level (let alone the problem of UV-completion), nevertheless one can consistently use small fields inflation as an effective theory of the early cosmological evolution.

Very recently, a first calculation of non-Gaussianities for inflation with vector fields has appeared, see \( \text{[21]} \). It is claimed that vector fields can produce a high level of non-Gaussianity. The analysis in \( \text{[21]} \) is performed in the usual \( \delta N \)-formalism of \( \text{[6]} \) with all its potential shortcomings, but the possibility is nevertheless very interesting. Note also that among the things which could not been taken into account in \( \text{[21]} \) is the tricky evolution of longitudinal modes. We have seen above that this evolution is very peculiar, and therefore a general intuition would imply that it also can make a contribution to non-Gaussian features. Of course, this issue needs a further investigation.

We conclude that there are many unresolved fundamental issues about vector (and higher spin) inflation, but at the level of effective description it is a viable candidate for the theory of inflationary epoch. Moreover, it could provide interesting insights into quite a few problems of current phenomenological interest. A great deal of further progress remains to be done and, from our perspective, worth to be done in the field of understanding vector inflation.

The Author is grateful to Vitaly Vanchurin, Viatcheslav Mukhanov, David Lyth, and Cristiano Germani for very useful discussions. This work was supported in part by the Cluster of Excellence EXC 153 “Origin and Structure of the Universe”.

References

[1] A. Golovnev, V. Mukhanov, V. Vanchurin, JCAP06(2008)009; arXiv:0802.2068
[2] C. Germani, A. Kehagias, arXiv:0902.3667
[3] T. Koivisto, D. Mota, C. Pitrou, JHEP09(2009)092; arXiv:0903.4158
[4] A. Golovnev, V. Mukhanov, V. Vanchurin, JCAP11(2008)018; arXiv:0810.4304
[5] A. Golovnev, V. Vanchurin, Phys.Rev.D79 (2009), 103524; arXiv:0903.2977
[6] K. Dimopoulos, M. Karciauskas, D. Lyth, Y. Rodriguez, JCAP05(2009)013; arXiv:0809.1055
[7] C. Germani, A. Kehagias, arXiv:0908.0001
[8] T. Koivisto, N. Nunes, arXiv:0907.3883
[9] T. Koivisto, N. Nunes, arXiv:0908.0920.

[10] D.A. Eliezer, R.P. Woodard, Nucl.Phys.B325 (1989), 389.

[11] B. Himmetoglu, C. Contaldi, M. Peloso, Phys.Rev.Lett.102 (2009), 111301.

[12] B. Himmetoglu, C. Contaldi, M. Peloso, Phys.Rev.D79 (2009), 063517.

[13] A. Linde, V. Mukhanov, M. Sasaki, JCAP10(2005)002; arXiv:astro-ph/0509015.

[14] V. Mukhanov, A. Vikman, JCAP02(2006)004: arXiv:astro-ph/0512066.

[15] V. Mukhanov. Physical Foundations of Cosmology. CUP, 2005.

[16] B. Himmetoglu, C. Contaldi, M. Peloso, arXiv:0909.3524.

[17] R.W. Wald, Phys.Rev.D40 (1983), 2118.

[18] S. Weinberg. Cosmology. OUP, 2008.

[19] H.K. Eriksen, F.K. Hansen, A.J. Banday, K.M. Gorski, P.B. Lilje, Astrophys.J.605 (2004), 14.

[20] T. Chiba, JCAP08(2008)004: arXiv:0805.4660.

[21] C.A. Valenzuela-Toledo, Y. Rodriguez, D.H. Lyth, arXiv:0909.4064.