Berenstein-Zelevinsky triangles, elementary couplings and fusion rules

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Abstract

We present a general scheme for describing $\widehat{su}(N)_k$ fusion rules in terms of elementary couplings, using Berenstein-Zelevinsky triangles. A fusion coupling is characterized by its corresponding tensor product coupling (i.e. its Berenstein-Zelevinsky triangle) and the threshold level at which it first appears. We show that a closed expression for this threshold level is encoded in the Berenstein-Zelevinsky triangle and an explicit method to calculate it is presented. In this way a complete solution of $\widehat{su}(4)_k$ fusion rules is obtained.
1. Introduction

Berenstein and Zelevinsky recently made a remarkable contribution to the classic problem of computing $su(N)$ triple tensor product multiplicities [1]. They found these multiplicities to be identical to the number of triangles (hereafter called BZ triangles) defined by a set of non-negative integers satisfying certain conditions. Each BZ triangle is associated to a particular coupling of three integrable highest weight representations of $su(N)$.

On the other hand, every coupling of a given triple tensor product can be decomposed into a product of elementary couplings. This decomposition is unique once the redundancies (syzygies) are eliminated (see for instance [2] and references therein). Here we show that the BZ triangles provide a powerful tool for the description of these elementary couplings. In this framework, elementary couplings are put in correspondence with basic BZ triangles. Then, the decomposition of a coupling into a product of elementary ones is translated into a sum of basic triangles. We provide a method of construction of the set of basic BZ triangles and we observe that syzygies can be characterized in a very simple way.

Next, BZ triangles are used to describe fusion coefficients, or, equivalently, restricted tensor product multiplicities. The latter refer to the truncated tensor product for $U_q(su(N))$ (universal enveloping algebra of the quantum deformations of $su(N)$) when $q$ is a root of unity [3,4,5]. On the other hand, fusion coefficients correspond to the multiplicity of the scalar representation in the triple product of three integrable representations of the Kac-Moody algebra $\widehat{su}(N)_k$ at some fixed level $k$ [6,7,8]. The link between these dual descriptions is $q^{N+k} = 1$ [9]. For definiteness, we will use the language of fusion rules.

The physical framework for fusion rules is conformal field theory. In this context, fusion rules specify the conformal families (with their multiplicities) of the various fields arising in the expansion of the operator product of two fields in given conformal families. Each conformal family is characterized by its lowest state, called a primary field. For the special case of WZNW model with spectrum generating algebra $\hat{g}_k$, primary fields are in one-to-one correspondence with integrable representations of $\hat{g}_k$ [7,10].

Fusion coefficients are uniquely characterized by [11,12]

1. the corresponding tensor product coefficients;
2. the set of minimum levels $\{k_0^{(i)}\}$, at which the various couplings,
labelled by \((i)\), will first appear.

In other words, every fusion coupling can be fully characterized by a BZ triangle and a threshold level \(k_0^{(i)}\). Here we argue that \(k_0^{(i)}\) is encoded in the data of the corresponding BZ triangle.

Our results on fusion rules are presented as a set of observations and conjectures. They are illustrated by two examples: \(\widehat{su}(3)_k\) and \(\widehat{su}(4)_k\).

2. BZ triangles.

An \(su(3)\) BZ triangle, describing a particular coupling associated to the triple product \(\lambda \otimes \mu \otimes \nu\), is a triangular arrangement of nine non-negative integers:

\[
\begin{array}{ccc}
m_{13} & & \\
& n_{12} & l_{23} \\
m_{23} & n_{13} & l_{12} \\
& m_{12} & n_{23} & l_{13}
\end{array}
\] (2.2)

These integers are related to the Dynkin labels of the three integrable highest weights by

\[
\begin{align*}
m_{13} + n_{12} &= \lambda_1 \\
n_{13} + l_{12} &= \mu_1 \\
l_{13} + m_{12} &= \nu_1 \\
m_{23} + n_{13} &= \lambda_2 \\
n_{23} + l_{13} &= \mu_2 \\
l_{23} + m_{13} &= \nu_2
\end{align*}
\] (2.3)

and they further satisfy the so-called hexagon conditions

\[
\begin{align*}
n_{12} + m_{23} &= n_{23} + m_{12} \\
l_{12} + m_{23} &= l_{23} + m_{12} \\
l_{12} + n_{23} &= l_{23} + n_{12}
\end{align*}
\] (2.4)

These last conditions mean that the length of opposite sides in the hexagon are equal, the length of a segment being defined as the sum of its two vertices. An \(su(3)\) BZ triangle is thus composed of one hexagon and three corner points.

Each pair of indices \(ij, \ i < j\), on the labels of the triangle indicates association with a positive root of \(su(3)\). If \(e_i\) are orthonormal vectors in \(\mathbb{R}^N\), then the positive roots of \(su(N)\) can be represented in the form \(e_i - e_j, \ 1 \leq i < j \leq N\). The triangle encodes three sums of positive roots:

\[
\begin{align*}
\mu + \nu - C\lambda &= \sum_{i<j} l_{ij}(e_i - e_j) \\
\nu + \lambda - C\mu &= \sum_{i<j} m_{ij}(e_i - e_j) \\
\lambda + \mu - C\nu &= \sum_{i<j} n_{ij}(e_i - e_j)
\end{align*}
\] (2.5)
where $C\lambda$ is the weight contragredient (charge conjugate) to the weight $\lambda$. The hexagon relations (2.4) can then be seen as consistency conditions for these three expansions.

For $su(4)$ the BZ triangle is defined in a similar way, in terms of eighteen non negative integers:

\[
\begin{array}{cccccccccccccc}
  m_{14} & n_{12} & l_{34} & m_{24} & n_{13} & l_{23} & n_{23} & l_{24} & m_{34} & n_{14} & l_{12} & n_{24} & l_{13} & n_{34} & l_{14} \\
\end{array}
\]

related to the Dynkin labels by

\[
\begin{align*}
  m_{14} + n_{12} &= \lambda_1 \\
  m_{24} + n_{13} &= \lambda_2 \\
  m_{34} + n_{14} &= \lambda_3 \\
  n_{14} + l_{12} &= \mu_1 \\
  n_{24} + l_{13} &= \mu_2 \\
  n_{34} + l_{14} &= \mu_3 \\
  l_{14} + m_{12} &= \nu_1 \\
  l_{24} + m_{13} &= \nu_2 \\
  l_{34} + m_{14} &= \nu_3
\end{align*}
\]

Furthermore, the $su(4)$ BZ triangle contains three hexagons:

\[
\begin{align*}
  n_{12} + m_{24} &= m_{13} + n_{23} \\
  n_{13} + l_{23} &= l_{12} + n_{24} \\
  n_{24} + n_{34} &= n_{23} + m_{23} + m_{12} + n_{34} \\
  n_{12} + l_{34} &= l_{23} + n_{23} \\
  n_{23} + m_{34} &= n_{24} + m_{23} + m_{12} + n_{34} \\
  m_{24} + l_{23} &= l_{34} + m_{13} \\
  m_{34} + l_{12} &= l_{23} + m_{23} + m_{12} + n_{34}
\end{align*}
\]

The $su(N)$ generalization is obvious; the triangle is built out of $(N - 1)(N - 2)/2$ hexagons and three corner points.

The relation between BZ triangles and tensor product multiplicities is the following: For a fixed triplet $(\lambda, \mu, \nu)$ of highest weights of integrable representations, the number of possible BZ triangles gives the multiplicity of the scalar representation in the triple tensor product $\lambda \otimes \mu \otimes \nu$ [1].

Example 1:

The five BZ triangles associated to the $su(4)$ tensor product $(1, 2, 1)^{\otimes 3}$ are:

\[
\begin{array}{cccccccccccccccc}
  1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
  1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
  1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
  1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
  1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\]
Finding all possible BZ triangles for fixed highest weights $(\lambda, \mu, \nu)$ might appear difficult at first sight. However, once one is found, all other ones are obtained by addition or substraction of few building block triangles incorporating negative entries. This fact was exploited for $su(3)$ in [13]. For $su(4)$, one can show that if $\Delta$ is a BZ triangle for fixed $(\lambda, \mu, \nu)$, then all others are of the form $\Delta + c_1 \delta_1 + c_2 \delta_2 + c_3 \delta_3$, where the $\delta_i$ are, respectively,

$$
\begin{align*}
\bar{1} & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{align*}
$$

(\bar{1} := -1) and the $c_i$ are integers. For example, the five BZ triangles of (2.9) can be expressed as $\{\Delta - (\delta_1 + \delta_2 + \delta_3), \Delta, \Delta - \delta_1, \Delta - \delta_3, \Delta - \delta_2\}$, respectively.

3. Basis for BZ triangles.

From the linearity of the conditions defining the BZ triangles, it is clear that every BZ triangle can be decomposed into a sum of basic triangles whose entries take values in the set \{0, 1\}. We give a construction of a minimal set of basic triangles.

Three basic triangles are easily described: they have 0’s everywhere and a single 1 at one of the three corners. The other basic triangles have 0 at each corner and some 1’s distributed among the hexagons such that each hexagon contains at most three 1’s (actually, it can have zero, two or three 1’s), with at least one hexagon being non empty. Every inequivalent irreducible distribution produces an independent basic triangle.
Example 2:

The $su(3)$ basic triangles are

\[ E_1 = (0, 0)(1, 0)(0, 1) \quad E_3 = (1, 0)(0, 1)(0, 0) \quad E_5 = (0, 1)(0, 0)(1, 0) \]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[ E_2 = (1, 0)(0, 0)(0, 1) \quad E_4 = (0, 1)(1, 0)(0, 0) \quad E_6 = (0, 0)(0, 1)(1, 0) \]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

\[ E_7 = (0, 1)(0, 1)(0, 1) \quad E_8 = (1, 0)(1, 0)(1, 0) \]

One can introduce a compact notation to specify the hexagon content as well as the exact position of the 1's inside the hexagon:

\[
2 = \begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \quad 2' = \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array} \quad 2'' = \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[ 3 = \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \quad 3' = \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array} \quad 3'' = \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Example 3:

With this notation and the hexagon ordering $2 \ 3 \ 1$, one can write the full set of basic triangles (that is the full set of elementary couplings) for $su(4)$: (compare with [14,15])

\[ A_1 = 000 \quad C_1 = 220 \quad D_1' = 03'2'' \\
A_2 = 000 \quad C_2 = 2'02' \quad D_2' = 02''3 \\
A_3 = 000 \quad C_3 = 02''02'' \quad D_3' = 2'03' \\
B_3 = 2''00 \quad D_1 = 302' \quad E_1 = 3'30 \\
B_2 = 02'0 \quad D_2 = 3'20 \quad E_2 = 303' \\
B_1 = 002 \quad D_3 = 230 \quad E_3 = 03'3
\]
The underlined zero means that one places a 1 at the corner adjacent to the corresponding hexagon. For instance

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\(000\) corresponds to the coupling \((0,0,1)(1,0,0)(0,0,0)\). Similarly

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\(03'2''\) describes the coupling \((0,1,0)(1,0,0)(1,0,0)\). To illustrate the irreducible character of the elementary couplings, notice that the following is a consistent way of distributing the 1’s in the three hexagons:

\[
\begin{array}{cccccc}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\(3'32\) but it is reducible, e.g.: \(3'32 = 3'30 + 002\).

### 4. Characterization of the syzygies

The decomposition of a general BZ triangle into a sum of basic triangles is unique only after all the redundancies are eliminated. From the point of view of BZ triangles, syzygies can be characterized by a number of adjacent hexagons. Explicitly, the sources of all redundancies for \(su(N)\) (at least for \(su(N < 6)\)) are the following two non-unique decompositions:

\[
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
\end{array}
\]

\(6 = 1 + 1\)

\[
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
\end{array}
\]

\(6 = 3 + 3' = 2 + 2' + 2''\)
and, with $5 = 2 + 3$, $5' = 2 + 3'$, e.g.

$$
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 2 & 0 \\
1 & 0 \\
1 & 1
\end{pmatrix}
$$

(4.3)

$$
55' = 22 + 3'3 = 23 + 3'2
$$

(4.4)

\textbf{Example 4:}

For $su(3)$ there is thus only one syzygy, namely (4.2). In terms of elementary couplings, it reads $E_7E_8 = E_1E_3E_5$. An example of the second type for $su(4)$ is $C_1E_1 = D_3D_2$. The full list contains 15 syzygies, among which only three are of the second type:

$$
\begin{align*}
C_1E_1 & = D_2D_3 \\
B_1C_2C_3 & = D'_2D'_3 \\
D_2E_2 & = B_3C_1D'_3 \\
C_2E_2 & = D_1D'_3 \\
E_1E_2 & = B_3D_3D'_3 \\
D_3E_3 & = B_2C_1D'_2 \\
C_3E_3 & = D'_2D'_1 \\
E_1E_3 & = B_2D_2D'_2 \\
D'_1E_1 & = B_2C_3D_2 \\
B_3C_1C_2 & = D_1D_2 \\
E_2E_3 & = B_1D_1D'_1 \\
D'_2E_2 & = B_1C_3D_1 \\
B_2C_1C_3 & = D_3D'_1 \\
D_1E_1 & = B_3C_2D_3 \\
D'_3E_3 & = B_1C_2D'_1
\end{align*}
$$

(4.5)

\section{5. Fusion coefficients and threshold levels.}

Recall that a fusion triple product refers to a product (denoted by $\times$) of three integrable highest weight representations of a Kac-Moody algebra $\hat{g}$ at some positive integer level $k$. Such representations are characterized by a highest weight whose Dynkin labels are integers and satisfy the inequality $(\lambda, \theta) \leq k$, where $\theta$ is the longest root. To $\lambda$ we then associate the affine weight $\hat{\lambda}$ obtained by the addition of a zeroth Dynkin label $\lambda_0 = (\lambda, \theta) - k$. Hence all the Dynkin labels of the highest weight of an integrable representation, including the zeroth one, must be non-negative integers. The fusion coefficients $N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}}$ gives the multiplicity of the scalar representation in the triple product $\hat{\lambda} \times \hat{\mu} \times \hat{\nu}$.

The method of generating functions for fusion rules [11] as well as the depth rule [7,12], suggest that an efficient description of fusion coefficients consists in specifying the minimum level at which every coupling is first allowed (see also [13]). We label the couplings by an index $(i)$ running from 1 to $N_{\lambda\mu\nu}$, and denote the threshold level associated to the coupling
(i) as $k_0^{(i)}$. We will assume that the couplings are ordered such that $k_0^{(i)} \leq k_0^{(i+1)}$. Then the relation between fusion coefficients and the data $N_{\lambda\mu\nu}$ and $\{k_0^{(i)}\}$ is

$$ N_{\lambda\mu\nu}^{(k)} = \begin{cases} \max(i) & \text{such that } k \geq k_0^{(i)} \text{ and } N_{\lambda\mu\nu} \neq 0 \\ 0 & \text{if } k < k_0^{(1)} \text{ or } N_{\lambda\mu\nu} = 0. \end{cases} \quad (5.2) $$

For later reference, we recall how the group of outer automorphisms $O(\hat{g})$ acts on fusion coefficients [16]:

$$ N_{\lambda\mu\nu}^{(k)} = N_{AA',A''\hat{\lambda}}^{(k)} \quad \text{if } AA'A'' = 1 \quad (5.3) $$

$A, A'$ and $A''$ are three elements of $O(\hat{g})$. For $\hat{su}(N)$, any element of the outer automorphism group can be written as a power of $a$, defined as

$$ a\hat{\lambda} = a[\lambda_0, ..., \lambda_{n-1}] = [\lambda_{n-1}, \lambda_0, \lambda_1, ..., \lambda_{n-2}] \quad (5.4) $$

(We use square brackets when the zeroth Dynkin label is included and parentheses otherwise.)

6. Elementary couplings for fusion rules.

An arbitrary fusion coupling $(\hat{\lambda} \times \hat{\mu} \times \hat{\nu})^{(i)}$ may be expressed as a product of the elementary fusion couplings $\mathcal{F}_j$:

$$ (\hat{\lambda} \times \hat{\mu} \times \hat{\nu})^{(i)} = \prod_j \mathcal{F}_j^{f_j}. \quad (6.2) $$

Before the syzygies are taken into account, this decomposition is not unique. Each elementary coupling $\mathcal{F}_j$ has a threshold level $k_0(\mathcal{F}_j)$. In reference [11] it was conjectured that there exists a choice of elementary couplings, and a way to eliminate their syzygies, such that the decomposition (6.2) is unique, and

$$ k_0^{(i)} = \sum_j f_j k_0(\mathcal{F}_j) \quad (6.3) $$

Here we assume the validity of this conjecture.
Let us now describe the set \( \{F_j\} \). First, it contains all elementary couplings for tensor products, which we denote by \( \{E_j\} \). Their \( k_0 \) value is easily computed using tensor product multiplicities and the affine Weyl group [8]. The results always turn out to be given by:

\[
k_0(E_j) = \left\lfloor \frac{|E_j|}{2} \right\rfloor \tag{6.4}
\]

where \(|E_j|\) is the sum of the Dynkin labels of the three weights in the coupling \( E_j \), and \( \lfloor \cdot \rfloor \) stands for the integer part.

**Example 5:**

Directly from eq. (6.4) one has:

\[
\begin{align*}
\widehat{su}(3) & : \quad k_0(E_i) = 1 \\
\widehat{su}(4) & : \quad k_0(A_i) = k_0(B_i) = k_0(C_i) = k_0(D_i) = k_0(D'_i) = 1; \quad k_0(E_i) = 2
\end{align*}
\tag{6.5}
\]

Now consider the affine extension of the couplings \( \{E_j\} \) at level \( k_0(E_j) \), which we denote as \( \widehat{E}_j \). At least for \( su(N \leq 5) \), all possible actions of the outer automorphism group on these \( \widehat{E}_j \)'s produce the remaining elements in the set \( \{F_j\} \). Write the finite form of these extra elements as \( A\widehat{E}_j \). Clearly one has \( k_0(A\widehat{E}_j) = k_0(E_j) \). The augmented set is not minimal, however, since many couplings of the form \( A\widehat{E}_j \) can be decomposed into products of those in the set \( \{E_j\} \).

For \( su(N \leq 5) \), the minimal set of fusion elementary couplings includes \( \{E_j\} \), the set of tensor product elementary couplings (with \( k_0(E_j) \) given by (6.4)). All other fusion elementary couplings may be obtained from \((1,0,...,0,1)(1,0,...,0,1)(1,0,...,0,1)\) (with \( k_0 = 2 \)) by the action of the outer automorphism group, with one weight fixed. For \( su(2) \) and \( su(3) \), the set \( \{E_j\} \) is sufficient.

**Example 6:**

For \( su(3) \), the coupling \((1,1)(1,1)(1,1)\) at level 2 can be obtained from \( E_7E_8 \). For \( su(4) \), among the set \( \{E_j\} \), there is \((0,1,0)(0,1,0)(1,0,1)\) which by eq.(6.4) has \( k_0 = 2 \). At level \( k = 2 \) one then has the elementary fusion coupling \([1,0,1,0] \times [1,0,1,0] \times [0,1,0,1]\). Another allowed fusion coupling is \( a^3[1,0,1,0] \times a[1,0,1,0] \times [0,1,0,1] \) whose finite part yields the coupling \((1,0,1)(1,0,1)(1,0,1)\). This last coupling has then \( k_0 = 2 \). The other possible actions of the outer automorphism group do not produce new
independent elementary fusion couplings. Now \((1, 0, 1)^{\otimes 3}\) has two decompositions, \(A_1A_2A_3\) and \(C_1C_2C_3\), which both give \(k_0 = 3\) (since \(k_0(A_i) = k_0(C_i) = 1\)). The idea is thus to forbid \(C_1C_2C_3\), and replace it by a new elementary coupling with \(k_0 = 2\), denoted say by \(F\).

7. Threshold level of arbitrary couplings.

We now present our main conjecture, which leads to an explicit expression for \(k_0(i)\) in terms of the entries in the BZ triangle associated to the coupling \((i)\). (Given the existence of fusion elementary couplings, the following conjecture can be viewed as a sharpened version of the conjecture in [11] mentioned in the previous section.)

Conjecture: Before eliminating syzygies,

\[
k_0(\lambda \times \mu \times \nu)^{(i)} = \min(\sum j f_j k_0(F_j)) ,
\]

where the minimum is taken over all possible decompositions \(\prod_j F_j^{f_j}\).

This conjecture can also be rephrased in terms of the specification of a set of forbidden couplings (eliminating then the syzygies), such that the \(k_0\) value of a coupling can be obtained by the sum of its component elementary couplings. With redundancies of the form \(\prod_i F_i^{e_i} = \prod_i F_i'^{e'_i}\) one eliminates the product of couplings with highest values of \(k_0\). In other words, if \(\sum e'_i k_0(F'_i) \geq \sum e_i k_0(F_i)\), one eliminates the product \(\prod_i F_i'^{e'_i}\). When they have the same value of \(k_0\), which product is forbidden is immaterial.

Example 7:

For \(su(3)\), the only redundancy is \(E_7E_8 = E_1E_3E_5\) with left hand side having \(k_0 = 2\), and right hand side \(k_0 = 3\). Hence one should eliminate \(E_1E_3E_5\). For \(su(4)\), one eliminates all the products on the l.h.s of the first nine syzygies given in (4.5); for the remaining six (and as far as the calculation of \(k_0\) is concerned), the choice is arbitrary.
8. Threshold level in terms of BZ triangle data: \( \hat{s}u(3)_k \) and \( \hat{s}u(4)_k \).

The value of \( k_0 \) associated to the \( su(3) \) BZ triangle (2.2) as calculated from (7.2), is

\[
k_0 = \max \{ m_{13} + \mu_1 + \mu_2, n_{13} + \nu_1 + \nu_2, l_{13} + \lambda_1 + \lambda_2 \}
\]

(8.2)

Similarly, the threshold level of the \( su(4) \) BZ triangle (2.6) is

\[
k_0 = \max \{m_{14} + n_{14} + k_0(\Delta_3),
\]

\[
l_{14} + m_{14} + k_0(\Delta_2),
\]

\[
n_{14} + l_{14} + k_0(\Delta_1),
\]

\[
l_{14} + m_{14} + n_{14} + \left[ \lambda_2 + \mu_2 + \nu_2 + l_{23} + m_{23} + n_{23} + 1 \right] / 2 \}
\]

(8.3)

where \( k_0(\Delta_i) \) refers to the value of \( k_0 \) for the \( su(3) \) BZ triangle circumscribing the hexagon of type \( i = \frac{1}{2} \frac{1}{3} \) (see Example 3). These values of \( k_0 \) are computed using (8.2).

The formula for \( su(3) \) was proved in [12]. The derivation of the \( su(4) \) formula is a straightforward, albeit complicated, generalization of that for \( su(3) \). An alternative approach for \( su(3) \) is presented in [13].

Example 8:

The values of \( k_0 \) for the five BZ triangles of example 1 are respectively \{6, 5, 5, 5, 5\}. Notice that the value of \( k_0 \) of two triangles is not additive. For instance the second BZ triangle of the example 1 has \( k_0 = 5 \) while its double

\[
\begin{array}{cccc}
0 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
0 & 2 & 2 & 2 & 2 & 2 & 0
\end{array}
\]

(8.4)

has \( k_0 = 9 \). This of course is due to the possible contractions induced by the syzygies. For example, adding the triangle with decomposition \( C_1C_2 \) \( (k_0 = 2) \) to that corresponding to \( C_3 \) \( (k_0 = 1) \) yields a triangle associated to \( F \), with also \( k_0 = 2 \).

Acknowledgment
We thank R.T Sharp for useful discussions and D. Sénéchal for making available to us his computer program for fusion rules.

REFERENCES

1. A.D. Berenstein and A.Z. Zelevinsky, J. Algebraic Combinatorics 1 (1992) 7.
2. J. Patera and R.T. Sharp, in Lecture Notes in Physics, vol. 84, Springer Verlag, New York 1979.
3. V. Pasquier and H. Saleur, Nucl. Phys. B330 (1990) 523.
4. P. Furlan, A. Ganchev and V.B. Petkova, Nucl. Phys. B343 (1990) 205
5. F. Goodman and H. Wenzl, Adv. Math. 82 (1990) 244.
6. E. Verlinde, Nucl. Phys. B300 (1988) 389.
7. D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493.
8. M.A. Walton, Nucl. Phys. B340 (1990) 777; Phys. Lett. B241 (1990) 365; V. Kač, Infinite dimensional Lie algebras, 3rd ed. (Cambridge University Press, 1990).
9. L. Alvarez-Gaumé, C. Gomez and G. Sierra, Phys. Lett. B (1989) 142.
10. V.G.Knizhnik and A. Zamolodchikov, Nucl. Phys. B247 (1984) 83.
11. C.J. Cummins, P. Mathieu and M.A. Walton, Phys. Lett. B254 (1991) 390; L. Bégin, P. Mathieu and M.A. Walton, J. Phys. A: Math. Gen. 25 (1992) 135.
12. A.N. Kirillov, P. Mathieu, D. Sénéchal and M.A. Walton, preprint LAVAL-PHY-20/92 (LETH-PHY-2/92), 8/92, to appear in Nucl. Phys. B; preprint LETH-PHY-9/92 (LAVAL-PHY-23/92), 9/92, contributed to the proceedings of the XIXth International Colloquium on Group Theoretical Methods in Physics, Salamanca, Spain, 29/6-4/7,1992.
13. L. Bégin, P. Mathieu and M.A. Walton, Mod. Phys. Lett. A, Vol. 7 , No. 35 (1992) 3255.
14. C.J. Cummins, M. Couture and R.T. Sharp, J. Phys. A: Math. Gen. 23 (1990) 1929.
15. R.T. Sharp and D. Lee, Revista Mexicana de Fisica 20 (1971) 203-215.
16. J. Fuchs and D. Gepner, Nucl. Phys. B294 (1987) 30; J. Fuchs, Nucl. Phys. B (Proc. Suppl.) 6 (1989) 157.