Regularity and Green’s relations for the semigroup of partial contractions of a finite chain

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Abstract

Let \([n] = \{1, 2, \ldots, n\}\) be a finite chain and let \(P_n\) be the semigroup of partial transformations on \([n]\). Let \(CP_n = \{\alpha \in P_n : (\text{for all } x, y \in \text{Dom } \alpha) |x\alpha - y\alpha| \leq |x - y|\}\), then \(CP_n\) is a subsemigroup of \(P_n\). In this paper, we give a necessary and sufficient condition for an element in \(P_n\) to be regular and characterize all the Green’s equivalences on the semigroup \(CP_n\).

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1 Introduction and Preliminaries

Let \([n] = \{1, 2, \ldots, n\}\) be a finite chain, a map \(\alpha\) which has domain and range both subsets of \([n]\) is said to be a (partial) transformation. The collection of all partial transformations of \([n]\) is known as the semigroup of partial transformations, usually denoted by \(P_n\). A map \(\alpha \in P_n\) is said to be order preserving (resp., order reversing) if (for all \(x, y \in \text{Dom } \alpha\)) \(x \leq y\) implies \(x\alpha \leq y\alpha\) (resp. \(x\alpha \geq y\alpha\)); is order decreasing if (for all \(x \in \text{Dom } \alpha\)) \(x\alpha \leq x\); is an isometry (i. e., distance preserving) if (for all \(x, y \in \text{Dom } \alpha\)) \(|x\alpha - y\alpha| = |x - y|\); a contraction if (for all \(x, y \in \text{Dom } \alpha\)) \(|x\alpha - y\alpha| \leq |x - y|\). Let

\[CP_n = \{\alpha \in P_n : (\text{for all } x, y \in \text{Dom } \alpha) |x\alpha - y\alpha| \leq |x - y|\}\]

and

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be the subsemigroups of partial contractions and of order preserving partial contractions of \([n]\), respectively.

A general study of these semigroups was first proposed in 2013 by Umar and AlKharousi [16] (a research proposal supported by a grant from The Research Council of Oman - TRC). Umar and AlKharousi [16] proposed among other things, notations for these semigroups and their subsemigroups as such we maintain the same notations in this paper. For standard and basic concepts in semigroup theory, we refer the reader to Howie [7] and Higgins [8].

Regularity and Green’s relations on the semigroup \(\mathcal{P}_n\) and its various subsemigroups have been studied by many authors, see for example, [3] [4] [5] [6] [8] [10] [11] [12] [13] [14] [15] [16]. It is now the case that whenever one encounters a new class of semigroups, the first question usually raised is about its Green’s equivalences. Recently, Zhao and Yang [18] characterized regular elements and all the Green’s equivalences on \(\mathcal{CPO}_n\), where they refer to our “contractions” as “compressions”. However, so far, nothing has been done on regularity and Green’s relations for the new semigroup \(\mathcal{CP}_n\). In this paper, in Section 2, we give necessary and sufficient conditions for an element in \(\mathcal{CP}_n\) to be regular and in Section 3, we describe all the Green’s equivalences. Most of the results concerning regularity and Green’s relations of subsemigroups of \(\mathcal{CP}_n\) can be deduced from the results obtained in this paper. We have demonstrated this assertion in Section 4 by deducing the results of Zhao and Yang [18]. For the remainder of this section we prove some preliminary results that will be needed later.

Let \(\alpha\) be in \(\mathcal{CP}_n\) and let \(\text{Dom } \alpha\), \(\text{Im } \alpha\) and \(h(\alpha)\) denote the domain of \(\alpha\), image of \(\alpha\) and \([\text{Im } \alpha]\), respectively. For \(\alpha, \beta \in \mathcal{CP}_n\), the composition of \(\alpha\) and \(\beta\) is defined as \(x(\alpha \circ \beta) = (x\alpha)\beta\) for any \(x\) in \(\text{Dom } \alpha\). Without ambiguity, we shall be using the notation \(\alpha \circ \beta\) to denote \(\alpha \circ \beta\).

Next, let \(A, B\) be any nonempty subsets of \([n]\). \(A\) is said to precede \(B\) written as \(A < B\) if \(a < b\) for arbitrary \(a \in A, b \in B\) or \(\min A < \min B\). Thus, if \(a < b\) for arbitrary \(a \in A, b \in B\) then \(A < B\) coincides with the natural partial ordering and we can write \(A < B\) instead of \((A < B)\) otherwise we maintain the notation \((A < B)\).

Further, given any transformation \(\alpha\) in \(\mathcal{P}_n\), domain of \(\alpha\) is partitioned into \(p - \text{blocks}\) by the relation \(\ker \alpha = \{(x, y) \in [n] \times [n] : x\alpha = y\alpha\}\), i. e., if

\[
\alpha = \left( \begin{array}{cccc}
A_1 & A_2 & \cdots & A_p \\
x_1 & x_2 & \cdots & x_p \\
\end{array} \right) \in \mathcal{P}_n \quad (1 \leq p \leq n),
\]

then \(A_i (1 \leq i \leq p)\) are equivalence classes under the relation \(\ker \alpha\). Thus, \(a_i\alpha = x_i\) for all \(a_i \in A_i (1 \leq i \leq p)\). The collection of all the equivalence classes of the relation \(\ker \alpha\) is the partition of the domain of \(\alpha\), and is denoted by \(\text{Ker } \alpha\), i. e., \(\text{Ker } \alpha = \{A_1, A_2, \ldots, A_p\}\) and \(\text{Dom } \alpha = A_1 \cup A_2 \cup \ldots \cup A_p\) where \((p \leq n)\). We now have the following lemma.

**Lemma 1.1.** Let \(A\) and \(B\) be any disjoint subsets of \([n]\) then there exist \(a' \in A\) and \(b' \in B\) such that \(|a' - b'| \leq |a - b|\) for all \(a \in A, b \in B\).

**Proof.** Define a set as \((A - B)' = \{|a - b| : a \in A, b \in B\}\), clearly \((A - B)'\) is a subset of \(\mathbb{N}\). The result now follows by the well ordering property of \(\mathbb{N}\).

Now, let \(\alpha\) be as defined in \([8]\) and \(\text{Ker } \alpha = (A_1)_{i \in [p]} = \{A_1 < A_2 < \ldots < A_p\}\) be the partition of \(\text{Dom } \alpha\) ordered with the relation \(\prec\). A subset \(T_\alpha\) of \([n]\) is said to be a transversal of the partition \(\text{Ker } \alpha\) if \(|T_\alpha| = p, |A_i \cap T_\alpha| = 1 (1 \leq i \leq p)\). A transversal \(T_\alpha\) is said to be relatively convex if for all \(x, y \in T_\alpha\) with \(x \leq y\) and if \(x \leq z \leq y\) \((z \in \text{Dom } \alpha)\), then \(z \in T_\alpha\). Notice that every convex transversal is necessarily relatively convex but not vice-versa.
Lemma 1.3. For (iii) if \( A_1 \subseteq A_i \) and \( A_j \subseteq A_i \) for all \( i, j \in \{1, 2, \ldots, p\} \). In other words, a transversal \( T_\alpha \) is admissible if and only if the map \( A_i \mapsto t_i \) \( (t_i \in T_\alpha, i \in \{1, 2, \ldots, p\}) \) is a contraction. Notice that every convex transversal is admissible but not vice-versa.

For the purpose of illustration, consider the following transformations:

\[
\alpha_1 = \left( \begin{array}{cccc}
1, 2, 10, 23 & 4, 12 & 6, 14 & 7, 16, 17 \\
8 & 6 & 4 & 3
\end{array} \right),
\]

\[
\alpha_2 = \left( \begin{array}{cccc}
1, 5, 30 & 2, 12, 10 & 4, 16 & 3, 9, 19 \\
8 & 7 & 9 & 5
\end{array} \right), \quad \alpha_3 = \left( \begin{array}{cccc}
1, 7, 21 & 2, 8, 20 & 3, 9, 19 \\
3 & 4 & 5
\end{array} \right).
\]

The partition \( \operatorname{Ker} \alpha_1 \) has an admissible transversal \( \{2, 4, 6, 7\} \). Next, for the transformation \( \alpha_2 \), none of the transversals of \( \operatorname{Ker} \alpha_2 \) is admissible.

Now consider \( \alpha_3 \). The transversals \( \{1, 2, 3\}, \{7, 8, 9\} \) and \( \{19, 20, 21\} \) are all admissible and convex.

Remark 1.2. We observe the following:

(i) Every convex transversal is an admissible transversal, but the converse is not true;

(ii) Every partition \( \operatorname{Ker} \alpha \), of Dom \( \alpha \) in \( CP_n \) of height 2 has an admissible transversal. This follows from Lemma(1.1);

(iii) Every admissible transversal is relatively convex.

Next, we have the following lemma:

Lemma 1.3. For \( n \geq 4 \), let \( \alpha \in CP_n \) be such that there exists \( k \in \{2, \ldots, p-1\} \) \( (3 \leq p \leq n) \) and \( |A_k| \geq 2 \). If \( A_i < A_j \) \( (i < j) \) for all \( i, j \in \{1, 2, \ldots, p\} \) then the partition \( \operatorname{Ker} \alpha = \{A_1, A_2, \ldots, A_p\} \) of Dom \( \alpha \) has no relatively convex transversal.

Proof. Let \( \operatorname{Ker} \alpha = \{A_1, A_2, \ldots, A_{k-1}, A_k, A_{k+1}, \ldots, A_p\} \) be partition of Dom \( \alpha \) such that \( |A_k| \geq 2 \) where \( 2 \leq k \leq p-1 \). Suppose \( A_i < A_j \) \( (i < j) \) for all \( i, j \in \{1, 2, \ldots, p\} \).

Suppose by way of contradiction that \( \operatorname{Ker} \alpha \) has a relatively convex transversal \( T_\alpha = \{t_1, t_2, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_p\} \) \( (t_i \in A_i \) for all \( 1 \leq i \leq p) \). Now since \( |A_k| \geq 2 \), it means that there exists \( a_k \in A_k \) with \( a_k \neq t_k \) and \( a_k \notin T_\alpha \). Suppose \( a_k < t_k \). Notice that every element in \( A_{k-1} \) is less than every element in \( A_k \), in particular \( t_{k-1} < a_k < t_k \). This contradicts the fact that \( T_\alpha \) is relatively convex. On the other hand, suppose \( t_k < a_k \). Notice also that \( A_k < A_{k+1} \), thus \( t_k < a_k < t_{k+1} \). This also contradicts the fact that \( T_\alpha \) is relatively convex and hence the result follows.

Corollary 1.4. For \( n \geq 4 \), let \( \alpha \in ORCP_n \) be such that \( (p \geq 3) \) there exists \( k \in \{2, \ldots, p-1\} \) and \( |A_k| \geq 2 \). Then the partition \( \operatorname{Ker} \alpha = \{A_1, A_2, \ldots, A_p\} \) of Dom \( \alpha \) has no relatively convex transversal.

Lemma 1.5. Let \( \alpha \in CP_n \) be such that \( A_i < A_j \) for all \( i < j \) in \( \{1, 2, \ldots, p\} \) \( (p \geq 3) \). If \( |A_i| = 1 \) for all \( 2 \leq i \leq p-1 \) then the partition \( \operatorname{Ker} \alpha \) of Dom \( \alpha \) has an admissible transversal \( T_\alpha \).

Proof. Take \( T_\alpha = \{\alpha_i : 2 \leq i \leq p-1\} \cup \{\max A_1, \min A_p\} \). Clearly, \( \theta = \left( \begin{array}{cccc}
A_1 & a_2 & \ldots & a_{p-1} & A_p \\
\max A_1 & a_2 & \ldots & a_{p-1} & \min A_p
\end{array} \right) \) is a contraction. Hence \( T_\alpha \) is admissible.
A map $\alpha \in \mathcal{P}_n$ is said to be an *isometry* if and only if $|x\alpha - y\alpha| = |x - y|$ for all $x, y \in \text{Dom } \alpha$. If we consider $\alpha$ as defined in (3), then $\alpha$ is an *isometry* if and only if $|x_i - x_j| = |a_i - a_j|$ for all $a_i \in A_i$ and $a_j \in A_j$ ($i, j \in \{1, 2, \ldots , p\}$). Notice that this forces the blocks $A_i$ ($i = 1, \ldots , p$) to be singletons, because $\alpha$ is one--one. In other words $\alpha$ is an *isometry* if and only if Dom $\alpha = \{a_i : 1 \leq i \leq p\} = \{x_i + e : 1 \leq i \leq p\} =$ (Dom $\alpha)\alpha + e$ (called a *translation*) or Dom $\alpha = \{a_i : 1 \leq i \leq p\} = \{x_{p-i+1} + e : 1 \leq i \leq p\} =$ (Dom $\alpha)\alpha + e$ (called a *reflection*) for some integer $e$.

Now let $\alpha \in \mathcal{P}_n$ be as defined in (3) ($1 \leq p \leq n$). Then we have by the definition of contraction the following lemma:

**Lemma 1.6.** An element $\alpha$ in $\mathcal{P}_n$ is a contraction if and only if $|x_i - x_j| \leq |a - b|$ for all $a \in A_i$ and $b \in A_j$ ($i, j \in \{1, 2, \ldots , p\}$).

The next lemma gives a characterization of contractions with an admissible transversal.

**Lemma 1.7.** Let $\alpha \in \mathcal{P}_n$ be such that Ker $\alpha$ has an admissible transversal, $T_\alpha$. Then $\alpha$ is a contraction if and only if $|x_i - x_j| \leq |t_i - t_j|$ for all $t_i, t_j \in T_\alpha$ for all $i, j \in \{1, 2, \ldots , p\}$.

**Proof.** Suppose $\alpha \in \mathcal{P}_n$ is such that Ker $\alpha$ has an admissible transversal, $T_\alpha$. Further, suppose that $\alpha$ is a contraction. Then by Lemma 1.6, $|x_i - x_j| \leq |a_i - a_j|$ for all $a_i \in A_i$ and $a_j \in A_j$ (for all $i, j \in \{1, 2, \ldots , p\}$). Thus in particular, $|x_i - x_j| \leq |t_i - t_j|$.

Conversely, suppose $|x_i - x_j| \leq |t_i - t_j|$ for all $t_i, t_j \in T_\alpha$ for some $T_\alpha$. Notice that $|t_i - t_j| \leq |a_i - a_j|$ for all $a_i \in A_i$ and $a_j \in A_j$ (for all $i, j \in \{1, 2, \ldots , p\}$). Thus $|x_i - x_j| \leq |t_i - t_j| \leq |a_i - a_j|$. The result follows from Lemma 1.6.

**Lemma 1.8.** Let $\alpha \in \mathcal{CP}_n$ and let $A$ be a convex subset of Dom $\alpha$. Then $A\alpha$ is convex.

**Proof.** Let $\alpha \in \mathcal{CP}_n$ such that $A \subseteq \text{Dom } \alpha$ is convex. Suppose by way of contradiction that $A\alpha$ is not convex. That is to say, there exist $x, z \in A\alpha$ with $x < z < y$ for some $z \in [n] \setminus A\alpha$. Let $(z-1)$ and $(z+1)$ be the lower and upper saturations of $z-1$ and $z+1$, respectively. Note that, $x \in (z-1)$ and $y \in (z+1)$. Also note that, $(z-1)\alpha^{-1} \neq \text{Dom } \alpha \neq [z+1]\alpha^{-1}$, but $(z-1)\alpha^{-1} \cup [z+1]\alpha^{-1} = \text{Dom } \alpha$. If $(z-1)\alpha^{-1}$ is convex then, since $(z-1)\alpha^{-1} \neq \text{Dom } \alpha$, there exist either (i) an element $a$ in $(z-1)\alpha^{-1}$ and $a+1$ in $(z+1)\alpha^{-1}$; or (ii) $a$ in $(z-1)\alpha^{-1}$ and $a-1$ in $(z+1)\alpha^{-1}$. Case i: Clearly $aa \leq z-1$ and $(a+1)\alpha \geq z+1$ so that,

$$2 \leq (a+1)\alpha - aa = |(a+1)\alpha - aa| \leq |(a+1) - a| = 1,$$

which is a contradiction. Case ii: it is clear that $aa \leq z-1$ and $(a-1)\alpha \geq y+1$ so that

$$2 \leq (a-1)\alpha - aa = |(a-1)\alpha - aa| \leq |(a-1) - a| = 1,$$

which is another contradiction.

Now if $(z-1)\alpha^{-1}$ is not convex then there exists $a \in (z-1)\alpha^{-1}$ and either $a+1 \in (z+1)\alpha^{-1}$ or $a-1 \in (z+1)\alpha^{-1}$. In the former, we see that $aa \leq z-1$ and $(a+1)\alpha \geq z+1$. Therefore,

$$2 \leq (a+1)\alpha - aa = |(a+1)\alpha - aa| \leq |(a+1) - a| = 1,$$

which is a contradiction. In the latter, we see that $aa \leq z-1$ and $(a-1)\alpha \geq z+1$. Thus,

$$2 \leq (a-1)\alpha - aa = |(a-1)\alpha - aa| \leq |(a-1) - a| = 1,$$

which is another contradiction. Hence the result follows.
Corollary 1.9 ([2], lemma1.2). Let $\alpha \in CT_n$ be such that $|\text{Im } \alpha| = p$. Then $\text{Im } \alpha$ is convex.

Proof. Let $\alpha \in CT_n$. Notice that $\text{Dom } \alpha = [n]$ is convex. Thus, by Lemma(1.8) $|n|\alpha = \text{Im } \alpha$ is convex. □

Next, we have

Lemma 1.10. Let $\alpha \in P_n$ be as defined in (3) ($3 \leq p \leq n$), such that $\text{Ker } \alpha$ has a convex transversal. Then $\alpha$ is a contraction if and only if $T_\alpha = (T_\alpha)\alpha + e$, for some $e \in \mathbb{Z}$.

Proof. Suppose $\alpha$ is a contraction whose $\text{Ker } \alpha$ has a convex transversal, $T_\alpha$. Then by Lemma(1.5) we see that $(T_\alpha)\alpha = \text{Im } \alpha$ is convex and so $T_\alpha = (T_\alpha)\alpha + e$, for some $e \in \mathbb{Z}$.

Conversely, suppose $\text{Ker } \alpha$ has a convex transversal $T_\alpha$ such that $T_\alpha = (T_\alpha)\alpha + e$ for some $e \in \mathbb{Z}$. Take $i < j$ with $a_i \in A_i$ and $a_j \in A_j$ ($i, j \in \{1, 2, \ldots, p\}$) then

$$|a_i - a_j| \geq |t_i - t_j| = |(t_i\alpha + e) - (t_j\alpha + e)| = |t_i\alpha - t_j\alpha| = |x_i - x_j|$$

or

$$|a_i - a_j| \geq |t_i - t_j| = |(t_{p-i+1}\alpha + e) - (t_{p-j+1}\alpha + e)| = |t_{p-i+1}\alpha - t_{p-j+1}\alpha| = |x_i - x_j|.$$ 

Thus, the result follows from Lemma(1.6). □

2 Regularity of elements in $CP_n$

An element $\alpha$ in a semigroup $S$ is said to be regular if and only if there exists $a' \in S$ such that $\alpha = aa'a$, if every element of $S$ is regular then the semigroup $S$ is said to be a regular semigroup. Many transformation semigroups were shown to be regular or their regular elements have been characterized [9, 10, 12, 13, 15, 17, 18]. In this section we investigate the regular elements of $CP_n$ where we give a sufficient and necessary condition for an element in the semigroup $CP_n$ and some of its subsemigroups to be regular. Recall that Zhao and Yang [18] characterized regular elements in the semigroup of order preserving partial contractions of a finite chain $OCP_n$, but their characterization depends heavily on order preservedness. Therefore, their characterization of regular elements would not hold in the more general semigroup $CP_n$. To see this consider $\alpha_1 = \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right) \in CP_3$. Denote $A_1 = \{2\}$ and $A_2 = \{1, 3, 5\}$, but $A_1 < A_2$ or $A_2 < A_1$ does not hold, as such we cannot apply the condition of Theorem 2.2 in [18]. Similarly for $\alpha_2 = \left( \begin{array}{ccc} 2 & 1 & 2 \\ 1 & 2 & 3 \end{array} \right) \in CP_{10}$ the conditions of Theorem 2.2 are not satisfied. Thus, we cannot conclude by Theorem 2.2 (2) and (3) in [18] that $\alpha_1$ and $\alpha_2$ are (or are not) regular, respectively. This is due to the fact that $A_i < A_j$ for all $i, j \in \{1, 2, \ldots, p\}$ ($i < j$) does not hold generally for the elements of $CP_n$. But we shall see in this section that these elements are regular.

Now we have the main result of this section:

Theorem 2.1. Let $\alpha$ and $CP_n$ be as defined in (3) and (1), respectively, where $\alpha \in CP_n$. Then $\alpha$ is regular if and only if there exists an admissible transversal $T_\alpha$ of $\text{Ker } \alpha$ such that $|t_j - t_i| = |t_j\alpha - t_i\alpha|$ for all $t_j, t_i \in T_\alpha$ ($i, j \in \{1, 2, \ldots, p\}$). Equivalently, $\alpha$ in $CP_n$ is regular if and only if there exists an admissible transversal $T_\alpha$, such that the map $t_i \mapsto x_i$ ($i \in \{1, 2, \ldots, p\}$) is an isometry.
Proof. Let \( \alpha \in CP_n \) be a regular element. Then there exists \( \gamma \in CP_n \) such that \( \alpha = \alpha \gamma \alpha \). Thus given any \( t \in \{1, 2, \ldots, p\} \), \( x_i = A_i \alpha = (A_i \alpha) \gamma \alpha = (x_i \gamma) \alpha \), i.e., \( x_i \gamma \in A_i \) for all \( 1 \leq t \leq p \). Now suppose by way of contradiction that for all admissible transversals \( T_\alpha \) of \( \text{Ker } \alpha \), there exist \( t_i, t_j \in T_\alpha \) for some \( i, j \in \{1, 2, \ldots, p\} \) such that

\[
|t_i \alpha - t_j \alpha| < |t_i - t_j|.
\]

(4)

Let \( \{C_1, C_2, \ldots, C_m\} \) \((m \leq n)\) be the kernel classes of \( \gamma \) arrange in such a way that \( A_k \alpha \in C_k \) (i.e., \( x_k \in C_k \)) for \( k = 1, 2, \ldots, p \leq m \). Since \( \gamma \in CP_n \), then \( |x_i \gamma - x_j \gamma| \leq |x_i - x_j| \) where \( x_i \in C_i \), \( x_j \in C_j \). Also given any \( x_k \in C_k \), \( x_k \gamma = a_k \) for any \( a_k \in A_k \) \((1 \leq k \leq p)\), in particular \( x_k \gamma = t_k \), where \( t_k \in T_\alpha \). Thus, \( x_i \gamma = t_i \) and \( x_j \gamma = t_j \) where \( t_i, t_j \in T_\alpha \). Observe that using (10), \( |x_i \gamma - x_j \gamma| = |t_i - t_j| > |t_i \alpha - t_j \alpha| = |x_i - x_j| \). This contradicts the fact that \( \gamma \) is an isometry, and hence the result follows.

Conversely, suppose there exists an admissible transversal \( T_\alpha \) such that \( |t_j - t_i| = |t_j \alpha - t_i \alpha| \) for all \( t_j, t_i \in T_\alpha \). Define a map say \( \gamma \), from \( \text{Im } \alpha \) to \( T_\alpha \) by \( x_k \gamma = t_k \) \((1 \leq k \leq p)\), the claim here is that, \( \gamma \) is an isometry. To see this, let \( x_i, x_j \in \text{Dom } \gamma \) \((i, j \in \{1, 2, \ldots, p\})\). Then, \( |x_i \gamma - x_j \gamma| = |t_i - t_j| = |t_i \alpha - t_j \alpha| = |x_i - x_j| \), as such \( \gamma \) is an isometry and for any \( a \in A_k \) \((1 \leq k \leq p)\), \( a \alpha \gamma \alpha = x_k \gamma \gamma = t_k \alpha \alpha = a \alpha \). Thus \( \alpha \) is regular.

As a consequence of the above theorem, we give the following definition. An admissible transversal \( T_\alpha \) is said to be good if there is an isometry from \( T_\alpha \) to \( \text{Im } \alpha \). Thus, an element \( \alpha \) in \( CP_n \) is regular if and only if \( \alpha \) has a good transversal. Moreover, it is not difficult to see that every convex transversal is good. We conclude the section with the following (now) obvious result:

**Corollary 2.2.** The semigroup \( CP_n \) \((n \geq 3)\) is not regular.

## 3 Green’s Relations for the semigroup \( CP_n \)

Let \( S \) be a semigroup and \( a, b \in S \). If \( S^1a = S^1b \) (i.e., \( a \) and \( b \) generate the same principal left ideal) then we say that \( a \) and \( b \) are related by \( \mathcal{L} \) and we write \( (a, b) \in \mathcal{L} \) or \( a \mathcal{L} b \), if \( aS^1 = bS^1 \) (i.e., \( a \) and \( b \) generate the same principal right ideal) then we say \( a \) and \( b \) are related by \( \mathcal{R} \) and we write \( (a, b) \in \mathcal{R} \) or \( a \mathcal{R} b \) and if \( S^1aS^1 = S^1bS^1 \) (i.e., \( a \) and \( b \) generate the same principal two sided ideal) then we say \( a \) and \( b \) are related by \( \mathcal{J} \) and we write \( (a, b) \in \mathcal{J} \) or \( a \mathcal{J} b \). Each of the relations \( \mathcal{L} \), \( \mathcal{R} \) and \( \mathcal{J} \) is an equivalence on \( S \). The relations \( \mathcal{H} = \mathcal{L} \cap \mathcal{R} \) and \( \mathcal{D} = \mathcal{L} \circ \mathcal{R} \) are also equivalences on \( S \). These five equivalences are known as Green’s relations, first introduced by J. A. Green in 1951 [3].

To begin our investigation we introduce the following concept. A subset \( A \) of \([n]\) is said to be translated by an integer \( e \) written as \( A + e \) if \( A + e = \{a + e : a \in A\} \).

For two subsets \( A \) and \( B \) of \([n]\), \( A \) and \( B \) are said to be \( e \) - translates if and only if \( B = A + e \) for some \( e \in \mathbb{Z} \).

Now, as in [3] let \( \alpha \) and \( \beta \) in \( CP_n \) be expressed as:

\[
\alpha = \left( \begin{array}{cccc}
A_1 & A_2 & \cdots & A_p \\
x_1 & x_2 & \cdots & x_p \\
\end{array} \right) \quad \text{and} \quad \beta = \left( \begin{array}{cccc}
B_1 & B_2 & \cdots & B_p \\
y_1 & y_2 & \cdots & y_p \\
\end{array} \right) \quad (p \leq n).
\]

(5)

Then \( \text{Dom } \alpha \) and \( \text{Dom } \beta \) are said to be \( e \) - translates if and only if \( A_i = B_i + e \) \((i \in \{1, 2, \ldots, p\})\) for some \( e \in \mathbb{Z} \). Similarly, \( \text{Im } \alpha \) and \( \text{Im } \beta \) are said to be \( e \) - translates if and only if \( \text{Im } \beta = \text{Im } \alpha + e \) (or \( B_i \beta = A_i \alpha + e \)) for all \( 1 \leq i \leq p \), for some \( e \in \mathbb{Z} \).

Next, let \( \alpha \) and \( \beta \in CP_n \) \((1 \leq p \leq n)\) be as defined in [3], then we introduce further the following concept which helps towards characterizing the Green’s \( \mathcal{L} \)-relation in \( CP_n \).
A partition $\text{Ker } \gamma$ (for $\gamma \in \mathcal{P}_n$) is said to be a refinement of the partition $\text{Ker } \alpha$ if $\ker \gamma \subseteq \ker \alpha$. Thus, if $\text{Ker } \gamma = \{A_1, A_2, \ldots, A_s\}$ and $\text{Ker } \alpha = \{A_1, A_2, \ldots, A_p\}$ then $p \leq s$. A refined partition $\text{Ker } \gamma$ of $\text{Ker } \alpha$ is said to be maximum, if $\ker \gamma \subseteq \ker \alpha$ and every refined relation of $\ker \gamma$ say $\ker \theta$ is contained in $\ker \gamma$. Moreover, if there are at least two maximal relations say $\ker \tau_i (i \geq 2)$ contained in $\ker \alpha$, then $\ker \gamma$ is maximum if $\ker \gamma = \bigcap_{i \geq 2} \ker \tau_i$. A maximum refined partition $\text{Ker } \gamma$ of $\text{Ker } \alpha$ is said to be admissible if it has an admissible transversal.

We immediately have the following lemma:

**Lemma 3.1.** For every $\alpha \in \mathcal{CP}_n$, $\text{Ker } \alpha$ has a maximum finer partition say $\text{Ker } \gamma$ (for some $\gamma \in \mathcal{CP}_n$) with an admissible transversal.

**Proof.** Let $\alpha \in \mathcal{CP}_n$ with $\text{Ker } \alpha = \{A_1, A_2, \ldots, A_p\}$ either ordered with the usual ordering or not, so there are two cases to consider:

Case i. Suppose $\text{Ker } \alpha$ is ordered with the usual order. Thus, $\text{Ker } \alpha = \{A_1 < A_2 < \ldots < A_p\}$. If $A_i = \{a_i\}$ for all $i \in \{2, \ldots, p-1\}$ then take $\text{Ker } \gamma = \text{Ker } \alpha = \{A_1 < \{a_2\} < \ldots < \{a_{p-1}\} < A_p\}$ for some $\gamma \in \mathcal{CP}_n$ and observe that if we take $T_\gamma = \{\max A_1 < a_2 < \ldots < a_{p-1} < \min A_p\}$, then the map $\theta = \left( \begin{array}{cccc} A_1 & a_2 & \ldots & a_{p-1} \\ \max A_1 & a_2 & \ldots & \min A_p \end{array} \right)$ is a contraction and hence $\text{Ker } \gamma$ is admissible. Notice that any admissible finer relation of $\ker \alpha$ is contained in the relation $\ker \gamma$. Thus, $\text{Ker } \gamma$ is maximum admissible finer partition.

If $|A_i| > 1$ for some $i \in \{2, \ldots, p-1\}$, then let $C = \bigcup_{i=1}^{p-1} A_i = \{a_2 < \ldots < a_s\}$ (for some $s > p$) and take $\text{Ker } \gamma = \{A_1 < \{a_2\} < \ldots < \{a_s\} < A_p\}$ for some $\gamma \in \mathcal{CP}_n$ and observe that if we take $T_\gamma = \{\max A_1 < a_2 < \ldots < a_s < \min A_p\}$, then the map $\theta = \left( \begin{array}{cccc} A_1 & a_2 & \ldots & a_s \\ \max A_1 & a_2 & \ldots & \min A_p \end{array} \right)$ is a contraction and hence $\text{Ker } \gamma$ is admissible. Notice that any admissible finer relation of $\ker \alpha$ is contained in the relation $\ker \gamma$. Thus, $\text{Ker } \gamma$ is maximum admissible partition.

Case ii. Suppose $\text{Ker } \alpha$ is not ordered, where $\text{Ker } \alpha = \{A_1, A_2, \ldots, A_p\}$.

If $\text{Ker } \alpha$ have a convex transversal, we are done, we take $\text{Ker } \gamma = \text{Ker } \alpha$ and it is maximum and admissible. If $\text{Ker } \alpha$ have a relatively convex transversal which is admissible, we are also done, take $\text{Ker } \gamma = \text{Ker } \alpha$ and it is maximum.

Now suppose $\text{Ker } \alpha$ have no convex or admissible relatively convex transversal. If there exists a convex block $A_k$ of order $j$ ($j < n$) for some $k = 1, \ldots, p$ in the $\text{Ker } \alpha$, we partition $A_k$ into singleton blocks $\{a_{k_1}\}$ and let $\text{Ker } \gamma = \{A_1, \ldots, A_{k-1}, \{a_{k_1}\}, \{a_{k_2}\}, \ldots, \{a_{k_j}\}, A_{k+1}, \ldots, A_p\}$, then we can check whether $\text{Ker } \gamma$ have an admissible transversal, if it does then we are done and if it does not, we then partition $A_{k-1}$ (or $A_{k+1}$) into singleton blocks, and we continue in this fashion until we get one and if all fails then at least we can partition $\text{Ker } \alpha$ into singletons, and the resultant relation is the intersection of all maximal relations contain in $\ker \alpha$ which is maximum and admissible. This complete the proof. □

**Remark 3.2.** If $\alpha$ is a regular element in $\mathcal{CP}_n$, then in view of Theorem 3, $\text{Ker } \alpha$ is a maximum admissible refinement of $\text{Ker } \alpha$, since it has an admissible transversal.

We now give a characterization on the Green’s $L$-relation on $\mathcal{CP}_n$ as follows:

**Theorem 3.3.** Let $\alpha, \beta \in \mathcal{CP}_n$ be as expressed in (3). Then $(\alpha, \beta) \in L$ if and only if $\text{Ker } \alpha$ and $\text{Ker } \beta$ have admissible finer partitions, $\text{Ker } \gamma_1$ and $\text{Ker } \gamma_2$ (for some $\gamma_1$ and $\gamma_2$ in $\mathcal{CP}_n$), respectively, such that there exists either a translation $\tau_i \mapsto \sigma_i$ and $\tau_i \alpha = \sigma_i \beta$ or a reflection $\tau_i \mapsto \sigma_{s-i+1}$ and $\tau_i \alpha = \sigma_{s-i+1} \beta$ for all $i = 1, \ldots, s$ ($s \geq p$), where $A_s = \{\tau_1, \ldots, \tau_s\}$ and $B_\alpha = \{\sigma_1, \ldots, \sigma_s\}$, are the admissible transversals of $\text{Ker } \gamma_1$ and $\text{Ker } \gamma_2$, respectively.
Proof. Let $\alpha, \beta \in \mathcal{CP}_n$ be as expressed in (5) such that $(\alpha, \beta) \in \mathcal{L}$. That is to say there exist $\gamma_1, \gamma_2 \in (\mathcal{CP}_n)^1$ such that

$$\alpha = \gamma_1 \beta \quad \text{and} \quad \beta = \gamma_2 \alpha. \quad (6)$$

Let $\text{Ker} \ \gamma_1 = \{A_1', A_2', \ldots, A_s'\}$ and $\text{Ker} \ \gamma_2 = \{B_1', B_2', \ldots, B_p'\}$ (s $\geq$ p). Whereas $\text{Ker} \ \alpha = \{A_1, A_2, \ldots, A_p\}$ and $\text{Ker} \ \beta = \{B_1, B_2, \ldots, B_p\}$. Clearly $\text{Ker} \ \gamma_1$ and $\text{Ker} \ \gamma_2$ are finer partitions of $\text{Ker} \ \alpha$ and $\text{Ker} \ \beta$, respectively.

Now let $a_i \in A_i'$ and $a_j \in A_j'$ (1 $\leq$ i, j, $\leq$ s) where $A_i' \subseteq A_u$ and $A_j' \subseteq A_v$ for some 1 $\leq$ u, v, $\leq$ s. Then since $\alpha = \gamma_1 \beta$, there exists $b_i' \in B_u$, $b_j' \in B_v$ (for some 1 $\leq$ u, v $\leq$ p) such that $a_i \gamma_1 = b_i'$ and $a_j \gamma_1 = b_j'$, and that $a_i \gamma_1 \beta = b_i' \beta = a_i \alpha$ and $a_j \gamma_1 \beta = b_j' \beta = a_i \alpha$. Since, $\gamma_1$ is a contraction we have;

$$|b_i' - b_j'| = |a_i \gamma_1 - a_j \gamma_1| \leq |a_i - a_j| \quad \text{for all } a_i \in A_i' \quad \text{and} \quad a_j \in A_j' \quad \text{for } i, j \in \{1, \ldots, s\} \quad (7)$$

Now, recall that by Lemma 3 there exists $\tau_i \in A_i'$ and $\tau_j \in A_j'$ such that

$$|\tau_i - \tau_j| \leq |a_i - a_j| \quad \text{for all } a_i \in A_i' \quad \text{and} \quad a_j \in A_j' \quad \text{for } i, j \in \{1, \ldots, s\} \quad (8)$$

Thus, equation (7) and (8) implies that

$$|b_i' - b_j'| \leq |\tau_i - \tau_j|. \quad (9)$$

Similarly, let $b_i \in B_i'$ and $b_j \in B_j'$ (1 $\leq$ i, j, $\leq$ s) where $B_i' \subseteq B_u$ and $B_j' \subseteq B_v$ for some 1 $\leq$ u, v, $\leq$ s. Then since $\beta = \gamma_2 \alpha$, there exist $a_i' \in A_u$, $a_j' \in A_v$ (for some 1 $\leq$ u, v $\leq$ p) such that $b_i \gamma_2 = a_i'$ and $b_j \gamma_2 = a_j'$, and that $b_i \gamma_2 \alpha = a_i' \alpha = b_i \beta$ and $b_j \gamma_2 \alpha = a_j' \alpha = b_i \beta$.

Since, $\gamma_2$ is a contraction we have;

$$|a_i' - a_j'| = |b_i \gamma_2 - b_j \gamma_1| \leq |b_i - b_j| \quad \text{for all } b_i \in B_i' \quad \text{and} \quad b_j \in B_j' \quad \text{for } i, j \in \{1, \ldots, s\} \quad (10)$$

Now, recall that by Lemma 3 there exists $\sigma_i \in B_i'$ and $\sigma_j \in B_j'$ such that

$$|\sigma_i - \sigma_j| \leq |b_i - b_j| \quad \text{for all } b_i \in B_i' \quad \text{and} \quad b_j \in B_j' \quad \text{for } i, j \in \{1, \ldots, s\} \quad (11)$$

Thus, equation (10) and (11) implies that

$$|a_i' - a_j'| \leq |\sigma_i - \sigma_j|. \quad (12)$$

Notice that, since $\sigma_i \in B_i' \subset B_u$ and $\sigma_j \in B_j' \subset B_v$ then from (10) we have

$$|\sigma_i - \sigma_j| \leq |b_i' - b_j'| \leq |\tau_i - \tau_j| \quad (13)$$

and also since $\tau_i \in A_i' \subset A_u$ and $\tau_j \in A_j' \subset A_v$ then from equation (12) we have;

$$|\tau_i - \tau_j| \leq |a_i' - a_j'| \leq |\sigma_i - \sigma_j| \quad (14)$$

Now equation (13) and (14) ensure that

$$|\tau_i - \tau_j| = |\sigma_i - \sigma_j|. \quad (15)$$
This shows that there is an isometry from \( \{ \tau_i \in A'_i : 1 \leq i \leq s \} = A'_s \) and \( \{ \sigma_i \in B'_i : 1 \leq i \leq s \} = B'_s \). Now observe that by equation (3) and (11) the maps \( A_i \mapsto \tau_i \) and \( B_i \mapsto \sigma_i \) for all \( i = 1, \ldots, s \) are contractions. Thus, \( A_s \) and \( B_s \) are admissible and hence \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \) are admissible finer partitions of \( \text{Ker } \alpha \) and \( \text{Ker } \beta \), respectively. And equation (15) shows there is translation \( \tau_i \mapsto \sigma_i \) or a reflection \( \tau_i \mapsto \sigma_{s-i+1} \) (1 \( \leq i \leq s \)). Now we claim that if \( \tau_i \mapsto \sigma_i \) then \( \tau_i \alpha = \sigma_i \beta \), and if \( \tau_i \mapsto \sigma_{s-i+1} \) then \( \tau_i \alpha = \sigma_i \beta \).

Now to show that \( \tau_i \alpha = \sigma_i \beta \) (1 \( \leq i \leq s \)), we suppose by way of contradiction that \( \tau_i \alpha = x_i \) (1 \( \leq i \leq s \)) and that, \( \sigma_{s-u-1} \beta = x_{s-u}, \sigma_{s-u} \beta = x_{s-u-1} \) and \( \sigma_j \beta = x_j \) (1 \( \leq u + 1 \leq j \leq s - u - 2 \) and \( 0 \leq u \leq s - 1 \)) where \( \tau_i \in A_{s-u}, \sigma_i \in A_{s-u-1} \) (1 \( \leq i \leq s \)). Let \( A_\alpha = \{ \tau_1 < \tau_2 < \ldots < \tau_s \} \).

It is clear from (11) that \( A_\alpha \subseteq \text{Dom } \gamma_1 \) and \( B_\beta \subseteq \text{Dom } \gamma_2 \). Thus, \( \tau_{s-u-2}, \tau_{s-u-1}, \tau_{s-u} \in \text{Dom } \gamma_1 \). Notice that \( \alpha = \gamma_1 \beta \), then \( \tau_{s-u-2} \gamma_1 = b_{s-u-2}' \), \( \tau_{s-u-1} \gamma_1 = b_{s-u}' \) and \( \tau_{s-u} \gamma_1 = b_{s-u-1}' \) for any \( b_{s-u-2}' \subseteq B_{s-u-1}', b_{s-u}' \subseteq B_{s-u-1}' \). Thus, in particular, \( \tau_{s-u-2} \gamma_1 = \sigma_{s-u-2}, \tau_{s-u-1} \gamma_1 = \sigma_{s-u-1} \) and \( \tau_{s-u} \gamma_1 = \sigma_{s-u-1} \). Therefore

\[
|\tau_{s-u-2} \gamma_1 - \tau_{s-u-1} \gamma_1| = |\sigma_{s-u-2} - \sigma_{s-u-1}| = |\tau_{s-u-2} - \tau_{s-u-1}| > |\tau_{s-u-2} - \tau_{s-u-1}|.
\] (16)

This contradicts the fact that \( \gamma_1 \) is a contraction, as such \( \tau_i \alpha = \sigma_i \beta \) for all \( 1 \leq i \leq s \). Using the same argument, we can equally show that if \( \tau_i \mapsto \sigma_{s-i+1} \) then \( \tau_i \alpha = \sigma_{s-i+1} \beta \) for all \( 1 \leq i \leq s \).

Conversely, suppose \( \text{Ker } \alpha \) and \( \text{Ker } \beta \) have a maximum admissible finer partitionings, \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \) (for some \( \gamma_1 \) and \( \gamma_2 \) in \( CP_n \)) respectively. Further, let \( A_\alpha = \{ \tau_1, \ldots, \tau_s \} \) and \( B_\beta = \{ \sigma_1, \ldots, \sigma_s \} \) be the admissible transversals of \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \), respectively, such that there exists either a translation \( \tau_i \mapsto \sigma_i \) and \( \tau_i \alpha = \sigma_i \beta \) or a reflection \( \tau_i \mapsto \sigma_{s-i+1} \) and \( \tau_i \alpha = \sigma_{s-i+1} \beta \) for all \( i = 1, \ldots, s \) (\( s \geq p \))

If in the former, \( \tau_i \mapsto \sigma_i \) is a translation and \( \tau_i \alpha = \sigma_i \beta \) (i = 1, \ldots , s), then define \( \gamma_1 = \left( \begin{array}{ccc} A'_1 & A'_2 & \ldots & A'_s \\ \sigma_1 & \sigma_2 & \ldots & \sigma_s \end{array} \right) \) and \( \gamma_2 = \left( \begin{array}{ccc} B'_1 & B'_2 & \ldots & B'_s \\ \tau_1 & \tau_2 & \ldots & \tau_s \end{array} \right) \). Then \( \gamma_1 \) and \( \gamma_2 \) are contractions since for all \( i, j \in \{1, \ldots , s\} \)

\[
|a_i' \gamma_1 - a_j' \gamma_1| = |\sigma_i - \sigma_j| = |\tau_i - \tau_j| \leq |a_i' - a_j'|
\]

and for all \( i, j \in \{1, \ldots , s\} \)

\[
|b_i' \gamma_2 - b_j' \gamma_2| = |\tau_i - \tau_j| = |\sigma_i - \sigma_j| \leq |b_i' - b_j'|
\]

In the later, Suppose that there exists a reflection \( \tau_i \mapsto \sigma_{s-i+1} \) and \( \tau_i \alpha = \sigma_{s-i+1} \beta \) for all \( i = 1, \ldots, s \).

Define \( \gamma_1 = \left( \begin{array}{ccc} A'_1 & A'_2 & \ldots & A'_s \\ \sigma_1 & \sigma_s & \ldots & \sigma_1 \end{array} \right) \) and \( \gamma_2 = \left( \begin{array}{ccc} B'_1 & B'_2 & \ldots & B'_s \\ \tau_1 & \tau_s & \ldots & \tau_1 \end{array} \right) \). Then \( \gamma_1 \) and \( \gamma_2 \) are contractions since for all \( i, j \in \{1, \ldots , s\} \)

\[
|a_i' \gamma_1 - a_j' \gamma_1| = |\sigma_{s-i+1} - \sigma_{s-j+1}| \leq |\tau_i - \tau_j| \leq |a_i' - a_j'|
\]

and for all \( i, j \in \{1, \ldots , s\} \)

\[
|b_i' \gamma_2 - b_j' \gamma_2| = |\tau_{s-i+1} - \tau_{s-j+1}| \leq |\sigma_i - \sigma_j| \leq |b_i' - b_j'|
\]

Now by direct computations, it follows easily that \( \alpha = \gamma_1 \beta \) and \( \beta = \gamma_2 \alpha \). Thus \( (\alpha, \beta) \in \mathcal{L} \). This complete the proof.\]
Lemma 3.4. Let $S = \mathcal{P}_n$ and let $\alpha, \beta, \gamma \in S$ such that $|\text{Im } \alpha| = |\text{Im } \beta|$. If $\alpha = \beta \gamma$ there exists $\gamma' \in S$ such that $|\text{Im } \gamma'| = |\text{Im } \alpha|$ and $\alpha = \beta \gamma'$. 

Proof. Take $\gamma' = \gamma \text{id}_{\text{Im } \alpha}$, then $|\text{Im } \gamma'| = |\text{Im } \alpha|$ and it is easy to see that $\alpha = \beta \gamma'$. □

Next let $\alpha, \beta \in \mathcal{C}P_n$ be as expressed in (5). Then we have the following:

Theorem 3.5. Let $\alpha, \beta \in \mathcal{C}P_n$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\ker \alpha = \ker \beta$ and there exists either a translation $x_i \mapsto y_i$ or a reflection $x_i \mapsto y_{p-i+1}$ $(1 \leq i \leq p)$.

Proof. Suppose $(\alpha, \beta) \in \mathcal{R}$, then there exist $\gamma_1, \gamma_2 \in (\mathcal{C}P_n)^1$ such that

$$\alpha = \beta \gamma_1 \text{ and } \beta = \alpha \gamma_2.$$  

Thus, $\ker \alpha = \ker \beta$ follows easily. Since $\alpha$ and $\beta$ have the same height, then by Lemma 3.4 we can take $\gamma_1$ and $\gamma_2$ of the same height as the height of $\alpha$ and $\beta$, and by (17) $\text{Im } \beta$ must be a transversal of $\text{Ker } \gamma_1$ and $\text{Im } \alpha$ must be a transversal of $\text{Ker } \gamma_2$. Let $\gamma_1 = \left( \begin{array}{cccc} C_1 & C_2 & \cdots & C_p \\ x_1 & x_2 & \cdots & x_p \end{array} \right)$ and $\gamma_2 = \left( \begin{array}{cccc} D_1 & D_2 & \cdots & D_p \\ y_1 & y_2 & \cdots & y_p \end{array} \right)$ $(1 \leq p \leq n)$. Thus $y_i \in C_i$ and $x_i \in D_i$ for all $1 \leq t \leq p$. Since $\gamma_1$ and $\gamma_2$ are contractions, for all $i, j \in \{1, 2, \ldots, p\}$ with $i \neq j$,

$$|x_i - x_j| = |C_i \gamma_1 - C_j \gamma_1| = |y_i \gamma_1 - y_j \gamma_1| \leq |y_i - y_j|$$  

(18)

and

$$|y_i - y_j| = |D_i \gamma_2 - D_j \gamma_2| = |x_i \gamma_2 - x_j \gamma_2| \leq |x_i - x_j|.$$  

(19)

Thus from (18) and (19) we have $|x_i - x_j| = |y_i - y_j|$. This implies that there exists a translation $x_i \mapsto y_i$ or a reflection $x_i \mapsto y_{p-i+1}$ for all $i \in \{1, 2, \ldots, p\}$.

Conversely, suppose that $\ker \alpha = \ker \beta$ and there exists an isometry from $\text{Im } \alpha$ to $\text{Im } \beta$. This implies that there exists either a translation $x_i \mapsto y_i$ or a reflection $x_i \mapsto y_{p-i+1}$ for all $i \in \{1, 2, \ldots, p\}$. If the map is a translation, define $\gamma$ from $\text{Im } \alpha$ to $\text{Im } \beta$ by $x_i \gamma = y_i$; and if it’s a reflection, define $\gamma$ by $x_i \gamma = y_{p-i+1}$ $(1 \leq i \leq p)$. In each case, it is easy to see that $\gamma$ is an isometry.

Now suppose that $x_i \gamma = y_i$. Let $a \in A_i$ $(1 \leq i \leq p)$, then $aa = x_i$ implies $aa \gamma = x_i \gamma = y_i = a \beta$, as such $\alpha = \beta \gamma$.

Similarly, suppose that $x_i \gamma = y_{p-i+1}$. Let $a \in A_i$ $(1 \leq i \leq p)$, then $aa = x_i$ implies $aa \gamma = x_i \gamma = y_{p-i+1} = a \beta$, as such $\alpha = \beta \gamma$. Moreover, since $\gamma$ is an isometry its inverse exists, and therefore $\beta = \alpha \gamma^{-1}$. Hence $\alpha \mathcal{R} \beta$, as required. □

Theorem 3.6. Let $\alpha, \beta \in \mathcal{C}P_n$ be as expressed in (5). Then $(\alpha, \beta) \in \mathcal{D}$ if and only if there exist isometries $\vartheta_1$ from $\text{Ker } \gamma_1$ to $\text{Ker } \gamma_2$ and $\vartheta_2$ from $\text{Im } \alpha$ to $\text{Im } \beta$, where $\text{Ker } \gamma_1$ and $\text{Ker } \gamma_2$ (for some $\gamma_1$ and $\gamma_2$ in $\mathcal{C}P_n$) are maximum admissible finer partitions of $\text{Ker } \alpha$ and $\text{Ker } \beta$, respectively.

Proof. Let $(\alpha, \beta) \in \mathcal{D}$. That is to say there exists $\eta \in (\mathcal{C}P_n)^1$ such that $\alpha \mathcal{L} \eta$ and $\eta \mathcal{R} \beta$. Thus, by Theorem 3.3, $\alpha \mathcal{L} \eta$ implies that there exists an isometry from the refined partition $\text{Ker } \gamma_1$ of $\alpha$ to the refined partition $\text{Ker } \gamma_2$ (for some $\gamma_1, \gamma_2 \in \mathcal{C}P_n$) of $\eta$ and $\tau_1 \alpha = \delta_1 \eta$ or $\tau_1 \alpha = \delta_{x-i+1} \eta$ with $\tau_1 \in A_\alpha$ and $\delta_1 \in C_\eta$ (where $A_\alpha, C_\eta$ denote the admissible transversals of the maximum finer partitions $\text{Ker } \gamma_1$ and $\text{Ker } \gamma_2$, respectively). This implies that $\text{Im } \alpha = \text{Im } \eta$. Furthermore, by Theorem 3.5 $\eta \mathcal{R} \beta$ implies $\ker \eta = \ker \beta$, i.e., $\text{Ker } \eta = \text{Ker } \beta$ and there exists an isometry from $\text{Im } \eta$ to $\text{Im } \beta$. Now since $\text{Ker } \eta = \text{Ker } \beta$ it means that $\text{Ker } \gamma_2$ is the maximum admissible refined partition of $\text{Ker } \beta$. Hence there
exists an isometry from \( \text{Ker} \gamma_1 \) to \( \text{Ker} \gamma_2 \). Note also that, \( \text{Im} \alpha = \text{Im} \eta \) and recall that there exists an isometry from \( \text{Im} \eta \) to \( \text{Im} \beta \), this implies that there exists an isometry from \( \text{Im} \alpha \) to \( \text{Im} \beta \).

Conversely, suppose there exists an isometry \( \vartheta_1 \) from \( \text{Ker} \gamma_1 \) to \( \text{Ker} \gamma_2 \) and also there exists an isometry \( \vartheta_2 \) from \( \text{Im} \alpha \) to \( \text{Im} \beta \). If \( \vartheta_2 \) is a reflection, i. e., \( x_i \vartheta_2 = y_{p-i+1} \) for all \( 1 \leq i \leq p \), then define a map say \( \gamma \) as:

\[
\gamma = \begin{pmatrix} B_1 & B_2 & \ldots & B_p \\ x_p & x_{p-1} & \ldots & x_1 \end{pmatrix}.
\]

Then \( \gamma \) is a contraction and it easily follows from Theorem(3.3) and (3.5) that \( \alpha L \gamma \) and \( \gamma R \beta \). Hence \( (\alpha, \beta) \in D \).

If \( \vartheta_2 \) is a translation, i. e., \( x_i \vartheta_2 = y_i \) for all \( 1 \leq i \leq p \), then define a map say \( \gamma \) as:

\[
\gamma = \begin{pmatrix} B_1 & B_2 & \ldots & B_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix}.
\]

Then it is easy to see that \( \gamma \) is a contraction and it follows from Theorem(3.3) and (3.5) that \( \alpha L \gamma \) and \( \gamma R \beta \). Hence \( (\alpha, \beta) \in D \).

Let \( \alpha \) and \( \beta \) be regular elements in \( CP_n \) and be as expressed in (3). Then as a consequence of Theorem(3.3), (3.5), (3.6) and Remark(3.2) we have:

**Corollary 3.7.** Let \( \alpha, \beta \in CP_n \) be regular elements.

(i) \( (\alpha, \beta) \in L \) if and only \( \text{Im} \alpha = \text{Im} \beta \).

(ii) \( (\alpha, \beta) \in R \) if and only \( \text{Ker} \alpha = \text{Ker} \beta \).

(iii) \( (\alpha, \beta) \in D \) if and only \( x_i = y_i \) or \( x_i = y_{p-i+1} \) for some \( e \in \mathbb{Z} \).

## 4 Semigroup of order reversing partial contractions

We recall that a map \( \alpha \in P_n \) is said to be order preserving if (for all \( x, y \in \text{Dom} \alpha \)) \( x \leq y \) implies \( x\alpha \leq y\alpha \). The collection of all order preserving contractions of a finite chain \( [n] \) is denoted by \( OCP_n = \{ \alpha \in CP_n : \forall x, y \in \text{Dom} \alpha \} \) and is a subsemigroup of \( CP_n \). In 2013, Zhao and Yang [18] studied this semigroup, where they referred to our “contractions” as “compressions” and they characterized the Green’s equivalences and gave a necessary and sufficient condition for an element to be regular. In this section, we deduce the regularity and Green’s relations characterizations of this semigroup from the results already obtained for the larger semigroup \( CP_n \). However, before we do that, we establish the following crucial lemma.

**Lemma 4.1.** Let \( \alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix} \in ORCP_n \) (1 \( \leq p \leq n \)). Then the following statements are equivalent:

(i) \( \max A_1 - x_1 = \min A_p - x_p = d \) and \( A_i = \{x_i + d\} \) (i = 2,...,p−1), or \( \max A_1 - x_p = \min A_p - x_1 = d \) and \( A_i = \{x_{p-i+1} - d\} \) (i = 2,...,p−1);

(ii) \( \text{Ker} \alpha \) has a good transversal.

**Proof.** Suppose (i) holds. In the former, it means that \( \text{Ker} \alpha = \{A_1 < x_2 + d < \ldots < x_{p-1} + d < A_p\} \). Take \( T_\alpha = \{\max A_1 < x_2 + d < \ldots < x_{p-1} + d < \min A_p\} \). Then clearly \( T_\alpha \) is a relatively convex transversal.
of Ker $\alpha$ and the map $\theta = \left( \begin{array}{cccc} A_1 & x_2 + d & \ldots & x_{p-1} + d \\ \max A_1 & x_2 + d & \ldots & x_{p-1} + d \\ \min A_p & x_p + d & \ldots & x_p \end{array} \right)$ is clearly a contraction. Thus, $T_\alpha$ is admissible. Next, define a map say $\gamma$ as:

$$\gamma = \left( \begin{array}{cccc} \max A_1 & x_2 + d & \ldots & x_{p-1} + d \\ x_1 & x_2 & \ldots & x_{p-1} \end{array} \right).$$

Clearly, $\gamma$ is an isometry since $\gamma = \alpha|_{T_n}$ and $d = \max A_1 - x_1 = (x_i + d) - x_i = \min A_p - x_p$ ($i = 2, \ldots, p-1$).

Hence $T_\alpha$ is good.

In the latter, Ker $\alpha = \{A_p < \{x_{p-1} - d\} < \{x_{p-2} - d\} < \ldots < \{x_2 - d\} < A_1\}$. Take $T_\alpha = \{\max A_1 < x_{p-1} - d < x_{p-2} - d < \ldots < x_2 - d < \min A_p\}$. Then clearly $T_\alpha$ is a relatively convex transversal of Ker $\alpha$ and the map $\theta' = \left( \begin{array}{cccc} A_p & x_{p-1} - d & \ldots & x_2 - d \\ \max A_1 & x_{p-1} - d & \ldots & x_2 - d \end{array} \right)$ is a contraction. Thus, $T_\alpha$ is admissible.

Next, define a map say $\gamma'$ as

$$\gamma' = \left( \begin{array}{cccc} \max A_1 & x_{p-1} - d & \ldots & x_2 - d \\ x_1 & x_2 & \ldots & x_{p-1} \end{array} \right).$$

Clearly, $\gamma'$ is a reflection of $\gamma$ which is also an isometry. Thus, $T_\alpha$ is good, as required.

Conversely, suppose $T_\alpha$ is good. This means that $T_\alpha = \{t_1, t_2, \ldots, t_p\}$ is an admissible relatively convex transversal of Ker $\alpha$ with $1 \leq \max A_1 = t_1 < t_2 < \ldots < t_p = \min A_p \leq n$ and the map $t_i \mapsto x_i$ ($1 \leq i \leq p$) is an isometry. If $1 \leq x_1 < x_2 < \ldots < x_p \leq n$ then $\min A_p - \max A_1 = |\max A_p - \max A_1| = |t_p - t_1| = |x_p - x_1| = x_p - x_1$ i.e., $\min A_p - \max A_1 = x_p - x_1$ or $\min A_p - x_p = \max A_1 - x_1 = d$.

Notice that: $|t_i - t_j| = |x_i - x_j|$ ($i, j \in \{2, \ldots, p-1\}$). This means that if (without loss of generality) $i < j$ then $t_j - t_i = x_j - x_i$ which implies $t_j - x_j = t_i - x_i = d$ if and only if $t_i = x_i + d$. Thus, $A_i = \{x_i + d\}$ ($2 \leq i \leq p - 1$). The same result is obtained if $n \geq x_1 > x_2 > \ldots > x_p \geq 1$.

Next, in view of the above lemma, we deduce the corresponding results for regularity of elements in the semigroups of order preserving partial contractions and order reversing partial contractions, ORCP$_n$ and ORCP$_n$, respectively, from Theorem (2.1).

**Corollary 4.2.** Let $S = \text{ORCP}_n$ and $\alpha \in S$. If $|\text{Im} \alpha| \geq 3$, then $\alpha$ is regular if and only if either $\min A_p - x_p = \max A_1 - x_1 = d$ and $A_i = \{x_i + d\}$ or $\min A_p - x_1 = \max A_1 - x_p = d$ and $A_i = \{x_{p-i+1} + d\}$, for $i = 2, \ldots, p - 1$.

As a consequence of the above corollary we have:

**Corollary 4.3 (18, Theorem 2.2 (3)).** Let $S = \text{OCP}_n$ and $\alpha \in S$. If $|\text{Im} \alpha| \geq 3$, then $\alpha$ is regular if and only if either $\min A_p - x_p = \max A_1 - x_1 = d$ and $A_i = \{x_i + d\}$, for $i = 2, \ldots, p - 1$.

We conclude the characterizations of the regular elements in $S = \text{ORCP}_n$ with the following (now) obvious result:

**Corollary 4.4.** The semigroup ORCP$_n$ ($n \geq 3$) is not regular.

Next, we deduce the characterizations of Green’s equivalences obtained in [18] from our results in the previous section. First, let us prove the following lemma.

**Lemma 4.5.** Let $\alpha, \beta$ in ORCP$_n$ be as expressed in (5). Then the following statements are equivalent:
(i) Let \( A_1 - \max B_1 = d \). If \( \min A_p - \min B_p = d \) and \( A_i = B_i + d \) for all \( i = 2, \ldots, p-1 \) then \( \alpha \) and \( \beta \) are of the same kernel type \( \text{denoted as } \alpha^{\text{Ker } \beta} \) in \([13]\).

(ii) There exists a contraction from \( \text{Ker } \gamma_1 = \{ A_1 < a_2 < \ldots < a_s < A_p \} \) to \( \text{Ker } \gamma_2 = \{ B_1 < b_2 < \ldots < b_s < B_p \} \), where \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \) are the maximum admissible refined partitions of \( \text{Ker } \alpha \) and \( \text{Ker } \beta \), respectively. Moreover, \( \bigcup_{i=2}^{p-1} A_i = \{a_2, \ldots, a_s\} \) and \( \bigcup_{i=2}^{p-1} B_i = \{b_2, \ldots, b_s\} \) for some \( s \geq p-2 \) where \( A_s = \{\max A_1 < a_2 < \ldots < a_s < \min A_p\} \) and \( B_s = \{\max B_1 < b_2 < \ldots < b_s < \min B_p\} \) are admissible transversals of \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \), respectively.

Proof. Suppose (i) holds. Define a map \( \gamma \) from \( \text{Ker } \gamma_1 \) to \( \text{Ker } \gamma_2 \) by

\[
a \gamma = \begin{cases} 
  \max B_1, & \text{if } a \in A_1; \\
  a - d, & \text{if } a = a_i, \ i \in \{2, \ldots, s-2\}; \\
  \min B_p, & \text{if } a \in A_p.
\end{cases}
\]

Clearly \( \gamma \) is well defined and we claim that \( \gamma \) is a contraction. To see this, let \( x, y \in \text{Dom } \gamma = A_1 \cup \{a_2\} \cup \ldots \cup \{a_{s-2}\} \cup A_p \). Then there are five cases to consider:

Case i: If \( x = y \) then \( |x - y| = 0 = |x - x| \leq |x - y| \).

Case ii: If \( x \in A_1 \) and \( y = a_i (2 \leq i \leq s-2) \) then \( |x - y| = |\max B_1 - (y - d)| = |y - (\max B_1 + d)| = |y - \max A_1| \leq |x - y| = |x - y| \).

Case iii: If \( x \in A_1 \) and \( y \in A_p \) then \( |x - y| = |\max B_1 - \min B_p| = |(\max A_1 - d) - (\min A_p - d)| = |\max A_1 - \min A_p| \leq |x - y| \).

Case iv: If \( x = a_i \) \((2 \leq i \leq s-2)\) and \( y \in A_p \) then \( |x - y| = |(x - d) - \min B_p| = |\min B_p + d - x| = |\min A_p - x| \leq |y - x| \).

Case v: If \( x = a_i, y = a_j \) \((i, j \in \{2, \ldots, s-2\})\) then \( |x - y| = |(x - d) - (y - d)| \leq |x - y| \). Thus, \( \gamma \) is a contraction.

Conversely, suppose there exists a contraction from \( \text{Ker } \gamma_1 = \{ A_1 < a_2 < \ldots < a_s < A_p \} \) to \( \text{Ker } \gamma_2 = \{ B_1 < b_2 < \ldots < b_s < B_p \} \), where \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \) are the maximum admissible refined partitions of \( \text{Ker } \alpha \) and \( \text{Ker } \beta \), respectively. Moreover, \( \bigcup_{i=2}^{p-1} A_i = \{a_2, \ldots, a_s\} \) and \( \bigcup_{i=2}^{p-1} B_i = \{b_2, \ldots, b_s\} \) for some \( s \geq p-2 \) where \( A_s = \{\max A_1 < a_2 < \ldots < a_s < \min A_p\} \) and \( B_s = \{\max B_1 < b_2 < \ldots < b_s < \min B_p\} \) are admissible transversals of \( \text{Ker } \gamma_1 \) and \( \text{Ker } \gamma_2 \), respectively. Further, let \( \max A_1 - \max B_1 = d = \min A_p - \min B_p \). Thus, the map defined as

\[
\theta = \left( \begin{array}{ccccc}
\max A_1 & a_2 & \ldots & a_{s-2} & \min A_p \\
\max B_1 & b_2 & \ldots & b_{s-2} & \min B_p
\end{array} \right)
\]

is a translation. Hence, \( a_i = b_i + d \) for all \( 2 \leq i \leq s-2 \). Note that \( \max A_1 - \max B_1 = d = \min A_p - \min B_p \). This shows that, \( \gamma_1 \) and \( \gamma_2 \) are of the same kernel type which implies that \( \alpha \) and \( \beta \) are of same kernel type.

In view of the above result, we deduce the following corollaries to Theorems \([3.3], [3.4], [3.6] \) and \([3.7]\), respectively.

**Corollary 4.6.** Let \( \alpha, \beta \in \text{ORCP}_n \). Then \( (\alpha, \beta) \in \mathcal{L} \) if and only if \( \Im \alpha = \Im \beta \text{ and } \alpha^{\text{Ker } \beta} \).

Proof. The result for elements in \( \text{ORCP}_n \) of height 1 is obvious. Thus, without loss of generality we may suppose that \( |\Im \alpha| \geq 2 \) and let \( (\alpha, \beta) \in \mathcal{L} \in \text{ORCP}_n \). Notice that \( \text{Ker } \alpha = \{ A_1 < \ldots < A_p \} \) and
\textbf{Ker} $\beta = \{B_1 < \ldots < B_p\}$. It follows from Theorem \ref{thm:3.3} (i) that there exists a translation (if $\alpha$ and $\beta$ are order preserving) $A_n \mapsto B_\beta$ and $t_i \alpha = t_i' \beta$ ($1 \leq i \leq s$) or a reflection (if $\alpha$ or $\beta$ is order reversing) $A_n \mapsto B_\beta$ and $t_i \alpha = t_i' \beta$ ($1 \leq i \leq s$) where $t_i \in \text{Adm}$, $t_i' \in \text{Adm}$, and $A_n$, $B_\beta$ are admissible transversals of the refined partitions of $\text{Ker} \alpha$ and $\text{Ker} \beta$, respectively. Thus, by Lemma \ref{lem:4.9} we have $\alpha \beta$ and $t_i \alpha = t_i' \beta$ or $t_i \alpha = t_i' \beta$ ($1 \leq i \leq s$) implies $\text{Im} \alpha = \text{Im} \beta$.

Conversely, suppose that $\text{Im} \alpha = \text{Im} \beta$. Note that if $\text{Im} \alpha = \{x_1 < x_2 < \ldots < x_p\} = \{y_1 < y_2 < \ldots < y_p\} = \text{Im} \beta$ then $x_i = y_i$ for all $i = 1, \ldots, p$. From Lemma \ref{lem:4.5}, we see that $\alpha \beta$ implies there exists a contraction from the maximum admissible refined partitions $\text{Ker} \gamma_1$ to $\text{Ker} \gamma_2$ (for some $\gamma_1$, $\gamma_2 \in CP_n$) of $\text{Ker} \alpha$ to $\text{Ker} \beta$, respectively and that $A_n \alpha = B_i \beta$ for all $i = 1, \ldots, p$.

Furthermore, if $\text{Im} \alpha = \{x_1 < x_2 < \ldots < x_p\} = \{y_1 > y_2 > \ldots > y_p\} = \text{Im} \beta$ then $x_i = y_{p-i+1}$ for all $i = 1, \ldots, p$. From Lemma \ref{lem:4.5}, we see that $\alpha \beta$ implies there exists a contraction from the maximum admissible refined partitions $\text{Ker} \gamma_1$ to $\text{Ker} \gamma_2$ (for some $\gamma_1$, $\gamma_2 \in CP_n$) of $\text{Ker} \alpha$ to $\text{Ker} \beta$, respectively and that $A_n \alpha = B_p \beta$ for all $i = 1, \ldots, p$.

Thus in any case by Theorem \ref{thm:3.3} (i) $(\alpha, \beta) \in \mathcal{L}$.

Next, let $\alpha, \beta \in \mathcal{ORCP}_n$ be as expressed in \ref{eq:3.3}. Then we have the following:

**Corollary 4.7.** Let $\alpha, \beta \in \mathcal{ORCP}_n$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\text{ker} \alpha = \text{ker} \beta$ and there exists a translation $x_i \mapsto y_i$, or a reflection $x_1 \mapsto y_{p-i+1}$ for all $1 \leq i \leq p$.

**Proof.** The result follows directly from Theorem \ref{thm:3.3} (ii).

**Corollary 4.8.** Let $\alpha$ and $\beta \in \mathcal{ORCP}_n$. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if there exist translations (or reflections) $\vartheta_1$ and $\vartheta_2$ from $\text{Ker} \gamma_1$ to $\text{Ker} \gamma_2$ and from $\text{Im} \alpha$ to $\text{Im} \beta$, respectively.

**Proof.** It follows directly from Theorem \ref{thm:3.3}.

In view of the above result, we deduce the following corollaries to Corollaries \ref{cor:4.6}, \ref{cor:4.7} and \ref{cor:4.8}, respectively.

**Corollary 4.9** \textbf{(18), Theorem 3.1.} Let $\alpha, \beta \in \mathcal{OC}_n$. Then $(\alpha, \beta) \in \mathcal{L}$ if and only if $\text{Im} \alpha = \text{Im} \beta$ and $\alpha \beta$.

Next, let $\alpha, \beta \in \mathcal{OC}_n$ be as expressed in \ref{eq:3.3}. Then we have the following:

**Corollary 4.10** \textbf{(18), Theorem 3.2.} Let $\alpha, \beta \in \mathcal{OC}_n$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\text{ker} \alpha = \text{ker} \beta$ and there exists a translation $x_i \mapsto y_i$, for all $1 \leq i \leq p$.

**Corollary 4.11** \textbf{(18), Theorem 3.3.} Let $\alpha$ and $\beta \in \mathcal{OC}_n$. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if there exist translations $\vartheta_1$ from $\text{Ker} \gamma_1$ to $\text{Ker} \gamma_2$ and $\vartheta_2$ from $\text{Im} \alpha$ to $\text{Im} \beta$.

**Proof.** It follows directly from Corollary \ref{cor:4.8} together with the fact that we only need the existence of translations from $\text{Ker} \gamma_1$ to $\text{Ker} \gamma_2$ and from $\text{Im} \alpha$ to $\text{Im} \beta$.

**Corollary 4.12** \textbf{(18), Theorem 3.5.} Let $\alpha$ and $\beta \in \mathcal{OC}_n$ be regular elements. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if $x_i = y_i + e$ $(i = 1, 2, \ldots, p)$ for some $e \in \mathbb{Z}$.

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