Fundamental Limits of Lossless Data Compression with Side Information

Lampros Gavalakis∗ Ioannis Kontoyiannis†

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Abstract

The problem of lossless data compression with side information available to both the encoder and the decoder is considered. The finite-blocklength fundamental limits of the best achievable performance are defined, in two different versions of the problem: Reference-based compression, when a single side information string is used repeatedly in compressing different source messages, and pair-based compression, where a different side information string is used for each source message. General achievability and converse theorems are established for arbitrary source-side information pairs, and the optimal asymptotic behaviour of arbitrary compressors is determined for ergodic source-side information pairs. A central limit theorem and a law of the iterated logarithm are proved, describing the inevitable fluctuations of the second-order asymptotically best possible rate, under appropriate mixing conditions. An idealized version of Lempel-Ziv coding with side information is shown to be first- and second-order asymptotically optimal, under the same conditions. Nonasymptotic normal approximation expansions are proved for the optimal rate in both the reference-based and pair-based settings, for memoryless sources. These are stated in terms of explicit, finite-blocklength bounds, that are tight up to third-order terms. Extensions that go significantly beyond the class of memoryless sources are obtained. The relevant source dispersion is identified and its relationship with the conditional varentropy rate is established. Interestingly, the dispersion is different in reference-based and pair-based compression, and it is proved that the reference-based dispersion is in general smaller.

Keywords — Entropy, lossless data compression, side information, conditional entropy, central limit theorem, law of the iterated logarithm, reference-based compression, pair-based compression, nonasymptotic bounds, conditional varentropy, reference-based dispersion, pair-based dispersion

∗Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, U.K. Email: lg560@cam.ac.uk. L.G. was supported in part by EPSRC grant number RG94782.

†Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, U.K. Email: i.kontoyiannis@eng.cam.ac.uk. Web: http://www.eng.cam.ac.uk/profiles/ik355. I.K. was supported in part by a grant from the Hellenic Foundation for Research and Innovation.
1 Introduction

It has long been recognised in information theory [37, 9] that the presence of correlated side information can dramatically improve compression performance. Moreover, in many applications useful side information is naturally present.

Reference-based compression. A particularly important and timely application of compression with side information is to the problem of storing the vast amounts of genomic data currently being generated by modern DNA sequencing technology [32]. In a typical scenario, the genome $X$ of a new individual that needs to be stored is compressed using a reference genome $Y$ as side information. Since most of the time $X$ will only be a minor variation of $Y$, the potential compression gains are large. An important aspect of this scenario is that the same side information – in this case the reference genome $Y$ – is used in the compression of many new sequences $X^{(1)}, X^{(2)}, \ldots$. We call this the reference-based version of the compression problem.

Pair-based compression. Another important application of compression with side information is to the problem of file synchronization [45], where updated computer files need to be stored along with their earlier versions, and the related problem of software updates [41], where remote users need to be provided with newer versions of possibly large software suites. Unlike genomic compression, in these cases a different side information sequence $Y$ (namely, the older version of the specific file or of the particular software) is used every time a new piece of data $X$ is compressed. We refer to this as the pair-based version of the compression problem, since each time a different $(X, Y)$ pair is considered. Other areas where pair-based compression is used include, among many others, the compression of noisy versions of images [35], and the compression of future video frames given earlier ones [2].

In addition to those appearing in work already mentioned above, a number practical algorithms for compression with side information have been developed over the past 25 years. The following are a some representative examples. The most common approach is based on generalisations of the celebrated family of Lempel-Ziv compression methods [40, 46, 44, 15, 8, 13, 16]; see Section 5 for a discussion of some of the underlying fundamentals of these methods. Information-theoretic treatments of problems related to DNA compression with side information have been given, e.g., in [50, 11, 6]. A grammar-based algorithm was presented in [38], turbo codes were used in [1], and a generalization of the context-tree weighting algorithm was developed in [5].

Our main goal in this work is to describe and evaluate the fundamental limits of the best achievable compression performance, when side information is available both at the encoder and the decoder. We derive tight asymptotic as well as nonasymptotic bounds on the optimum rate, under a variety of different assumptions. Moreover, we determine the source dispersion in both the reference-based and the pair-based cases, and we examine the difference between the two.

1.1 Outline of main results

In Section 2 we give precise definitions for the finite-blocklength fundamental limits of reference-based and pair-based compression. We identify the theoretically optimal one-to-one compressor in each case, for arbitrary source-side information pairs $(X, Y) = \{(X_n, Y_n) ; n \geq 1\}$. Moreover, in Theorem 2.5 we show that, for any blocklength $n$, requiring the compressor to be prefix-free imposes a penalty of no more than $1/n$ bits per symbol on the optimal rate.

In Section 3 we state and prove four general, single-shot, achievability and converse results, for the compression of arbitrary sources with arbitrarily distributed side information. Theorems 3.1–3.3 generalize corresponding results established in [25] without side information.
Theorem 3.4, one of the main tools we use later to derive the normal approximation results in Sections 6 and 7, gives new, tight achievability and converse bounds, based on a counting argument.

Before considering the optimal rate in reference-based and pair-based compression with finite blocklengths in Sections 6–8, in Section 4 we examine the best compression performance that can be achieved asymptotically. The results of Section 4 generalize the corresponding asymptotics developed in [22] and [25].

First we show in Theorem 4.3 that, for any jointly stationary and ergodic source-side information pair \( (X, Y) = \{(X_n, Y_n) : n \geq 1\} \), the best asymptotically achievable compression rate is the conditional entropy rate, \( H(X|Y) \), not just in expectation, but also with probability 1. This generalises Kieffer’s corresponding result [21] to the case of compression with side information. Recall that, in this case,

\[
H(X|Y) = \lim_{n \to \infty} \frac{1}{n} H(X^n|Y^n), \quad \text{bits/symbol},
\]

where \( X^n = (X_1, X_2, \ldots, X_n) \), \( Y^n = (Y_1, Y_2, \ldots, Y_n) \), and \( H(X^n|Y^n) \) denotes the conditional entropy of \( X^n \) given \( Y^n \); precise definitions will be given in the following sections.

Furthermore, we show that there is a sequence of random variables \( \{Z_n\} \) such that the description lengths \( \ell(f_n(X^n_1|Y^n_1)) \) of any sequence of compressors \( \{f_n\} \) satisfy a “one-sided” central limit theorem (CLT): Eventually, with probability 1,

\[
\ell(f_n(X^n_1|Y^n_1)) \geq nH(X|Y) + \sqrt{n}Z_n + o(\sqrt{n}), \quad \text{bits},
\]

(1)

where the \( Z_n \) converge to a \( N(0, \sigma^2(X|Y)) \) distribution, and the term \( o(\sqrt{n}) \) is of order negligible compared to \( \sqrt{n} \). The lower bound (1) is established in Theorem 4.7 where it is also shown that it is asymptotically achievable. This means that the best rate that can be obtained by any sequence of compressors has inevitable \( O(\sqrt{n}) \) fluctuations around the conditional entropy rate, and that the size of these fluctuations is quantified by the conditional variance rate,

\[
\sigma^2(X|Y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}( - \log P(X^n_1|Y^n_1) ) ,
\]

where \( \log = \log_2 \). This generalizes the minimal coding variance of [22]. The asymptotic lower bound (1) holds for a broad class of source-side information pairs, including all ergodic Markov chains with positive transition probabilities. Under the same conditions, a corresponding “one-sided” law of the iterated logarithm (LIL) is established in Theorem 4.8, which gives a precise description of the inevitable almost-sure fluctuations above \( H(X|Y) \), for any sequence of compressors.

The proofs of all the results in Section 4 are based on Theorem 4.2 which shows that the description lengths \( \ell(f_n(X^n_1|Y^n_1)) \) of any sequence of compressors \( \{f_n\} \) can never be much smaller than the conditional information density, \( - \log P(X^n_1|Y^n_1) \). This reduces the compression problem to that of establishing the asymptotic behaviour of the conditional information density: In Section 4.3, a corresponding CLT and an LIL are stated for \( - \log P(X^n_1|Y^n_1) \); these, in turn, follow from the almost sure invariance principle for \( - \log P(X^n_1|Y^n_1) \), proved in Appendix A. Theorem A.1 generalizes the corresponding invariance principle established for the (unconditional) information density \( - \log P(X^n_1) \) by Philipp and Stout [34]. Finally, in Theorem 4.11 we give a characterisation of the degenerate case when \( \sigma^2(X|Y) \) is zero.
Section 5 is devoted to the analysis of an idealised version of Lempel-Ziv coding with side information. As in the case of Lempel-Ziv compression without side information [47, 48], the performance of this scheme is essentially determined by the asymptotics of a family of conditional recurrence times, \( R_n = \mathcal{R}_n(X|Y) \). Under appropriate, general conditions on the source-side information pair \((X, Y)\), in Theorem 5.3 we show that the ideal description lengths, \( \log \mathcal{R}_n \), can be well-approximated by the conditional information density \(- \log P(X^n|Y^n)\), and second-order optimality results are derived in Theorem 5.5. The results of this section generalize the corresponding asymptotics without side information established in [31] and [23].

Sections 6 and 7 contain our main results, giving nonasymptotic, normal-approximation expansions to the optimal reference-based rate and the optimal pair-based rate. These expansions give finite-\( n \) upper and lower bounds that are tight up to third-order terms. For the sake of clarity, we first describe the pair-based results of Section 7.

Let \( R^*(n, \epsilon) \) be the best pair-based compression rate that can be achieved at blocklength \( n \), with excess rate probability no greater than \( \epsilon \). For a memoryless source-side information pair \((X, Y)\), in Theorems 7.1 and 7.2 we show that there are finite constants \( C, C' > 0 \) such that, for all \( n \) greater than some \( n_0 \),

\[
- \frac{C'}{n} \leq R^*(n, \epsilon) - \left[ H(X|Y) + \frac{1}{\sqrt{n}} \sigma(X|Y) Q^{-1}(\epsilon) - \frac{\log n}{2n} \right] \leq \frac{C}{n}.
\]

Moreover, explicit expressions are obtained for \( n_0, C, C' \). Here \( Q \) denotes the standard Gaussian tail function, \( Q(z) = 1 - \Phi(z), \ z \in \Re \); for memoryless sources, the conditional entropy rate reduces to \( H(X|Y) = H(X_1|Y_1) \), and the conditional varentropy rate becomes \( \sigma^2(X|Y) = \text{Var}(- \log P(X_1|Y_1)) \).

The bounds in (2) generalize the corresponding no-side-information results in Theorems 17 and 18 of [25]; see also the discussion in Section 1.2 for a description of the natural connection with the Slepian-Wolf problem. Our proofs rely on the general coding theorems of Section 2 combined with appropriate versions of the classical Berry-Esseen bound. An important difference with [25] is that the approximation used in the proof of the upper bound in [25, Theorem 17] does not admit a natural analog in the case of compression with side information. Instead, we use the tight approximation to the description lengths of the optimal compressor given in Theorem 3.4.

Results analogous to (2) are also established in a slightly weaker form for the case of Markov sources in Theorem 8.2, which is the main content of Section 8.

In Section 6 we consider the corresponding problem in the reference-based setting. Given an arbitrary, fixed side information string \( y^n_1 \), let \( R^*(n, \epsilon|y^n_1) \) denote the best pair-based compression rate that can be achieved at blocklength \( n \), conditional on \( Y^n_1 = y^n_1 \), with excess-rate probability no greater than \( \epsilon \). Suppose that the distribution of \( Y \) is arbitrary, and the the source \( X \) is conditionally i.i.d. (independent and identically distributed) given \( Y \). In Theorems 6.4 and 6.5 we prove reference-based analogs of the bounds in (2): There are finite constants \( C(y^n_1), C'(y^n_1) > 0 \) such that, for all \( n \) greater than some \( n_1(y^n_1) \), we have,

\[
- \frac{C'(y^n_1)}{n} \leq R^*(n, \epsilon|y^n_1) - \left[ H_n(X|y^n_1) + \frac{1}{\sqrt{n}} \sigma_n(y^n_1) Q^{-1}(\epsilon) - \frac{\log n}{2n} \right] \leq \frac{C(y^n_1)}{n},
\]

where now the first-order rate is given by,

\[
H_n(X|y^n_1) = \frac{1}{n} \sum_{i=1}^n H(X|Y = y_i),
\]
and the variance $\sigma^2_n(y^n_1)$ is,

$$\sigma^2_n(y^n_1) = \frac{1}{n} \sum_{i=1}^{n} \text{Var}(-\log P(X|y_i)|Y = y_i).$$

Once again, explicit expressions are obtained for $n_1(y^n_1), C(y^n_1)$ and $C'(y^n_1)$. A numerical example illustrating the accuracy of the normal approximation in (3) is shown in Figure 1.

Figure 1: Normal approximation to the reference-based optimal rate $R^*(\epsilon, n | y^n_1)$ for a memoryless side information process $\{Y_n\}$ with Bern(1/3) distribution. The source $\{X_n\}$ has $X|Y = 0 \sim \text{Bern}(0.1)$ and $X|Y = 1 \sim \text{Bern}(0.6)$. The side information sequence is taken to be $y^n_1 = 001001001 \cdots$. The graph shows $R^*(\epsilon, n | y^n_1)$ itself, with $\epsilon = 0.1$, for blocklengths $1 \leq n \leq 500$, together with the normal approximation to $R^*(\epsilon, n | y^n_1)$ given by the three terms in square brackets in (3).

Note that there is an elegant analogy between the bounds in (2) and (3). Indeed, there is further asymptotic solidarity in the normal approximation of the two cases. If $Y$ is ergodic, then for a random side information string $Y_1^n$ we have that, with probability 1,

$$H_n(X|Y^n_1) \to H(X|Y), \quad \text{as } n \to \infty.$$  

But the corresponding variances are different: With probability 1,

$$\sigma^2_n(Y^n_1) \to E\left[\text{Var}(-\log P(X|Y)|Y)\right], \quad \text{as } n \to \infty,$$

which is shown in Proposition 6.1 (i) to be strictly smaller than $\sigma^2(X|Y)$ in general. This admits the intuitively satisfying interpretation that, in reference-based compression, where a single side information string is used to compress multiple source messages, the optimal rate has smaller variability.
Although the approximation bounds in (3) do not directly relate the optimal finite-
rate $R^*(n, \epsilon | y^n)$ to the optimal asymptotic rate $H(X|Y)$, in Theorem 6.6 we give an explicit, finite-
n concentration bound, showing that, for a random side information string $Y^n$, the probability
that $R^*(n, \epsilon | Y^n)$ exceeds $H(X|Y)$ by any $\delta > 0$, is exponentially small in $n$.

Finally, in Section 9 we examine the pair-based dispersion $D(X|Y)$, defined as the limiting
normalised variance of the optimal description lengths of $X$ given $Y$, and the reference-based
dispersion $D(X|y)$ similarly defined for the optimal conditional description lengths, given a fixed
side information sequence $y = y^n$. Theorem 9.2 states that, under fairly general conditions, the pair-based dispersion $D(X|Y)$ is equal to the conditional variance rate $\sigma^2(X|Y)$, and relates $D(X|Y)$ to the behaviour of the pair-based optimal rate $R^*(n, \epsilon)$ as $n \to \infty$ and $\epsilon \to 0$. Analogous results for the reference-based dispersion $D(X|y)$ are established in Theorem 9.4.

1.2 Related work
The finer study of the optimal rate in source coding (without side information) originated in
Strassen’s pioneering work [39], followed more than three decades later by [22] and more recently
by [25]. In addition to the works already described, we also mention that third-order normal
approximation results in universal source coding were obtained in [14].

The most direct relationship of the present development with current and recent work is
in connection with the Slepian-Wolf (SW) problem [37]. Tan and Kosut [42] give a second-
order multidimensional normal approximation to the SW region for memoryless sources, and they show that, up to terms of order $(\log n)/n$, achievable rates are the same as if the side
information were known perfectly at the decoder. Nomura and Han [30] derive the second-order
SW region for general sources. Recently, Chen et al. [7] refined the results of [42] by establishing
inner and outer asymptotic bounds for the SW region, which are tight up to and including
third-order terms. Since, by definition, any SW code is also a pair-based code for our setting,
the achievability result from [7] implies a slightly weaker form of our Theorem 7.1, with an
asymptotic $O(1/n)$ term in place of the explicit $C/n$ in (46). It is interesting to know that this
high level of accuracy in bounding above $R^*(n, \epsilon)$ can be derived both by random coding as
in [7] and by deterministic methods as in Theorem 7.1. The sharpest known SW converse is
obtained in [17] via a linear programming argument.
2 The Optimal Compressor and Fundamental Limits

Let \((X, Y) = \{(X_n, Y_n) : n \geq 1\}\) be a source-side information pair, that is, a pair of arbitrary, jointly distributed sources with finite alphabets \(\mathcal{X}, \mathcal{Y}\), respectively, where \(X\) is to be compressed and \(Y\) is the side information process. Given a source string \(x^n_1 = (x_1, x_2, \ldots, x_n)\) and assuming \(y^n_1 = (y_1, y_2, \ldots, y_n)\) is available to both the encoder and decoder, a fixed-to-variable one-to-one compressor with side information, of blocklength \(n\), is a collection of functions \(f_n\), where each \(f_n(x^n_1|y^n_1)\) takes a value in the set of all finite-length binary strings,

\[
\{0,1\}^* = \bigcup_{k=0}^{\infty} \{0,1\}^k = \{\emptyset, 0, 00, 01, 000, \ldots\},
\]

with the convention that \(\{0,1\}^0 = \{\emptyset\}\) consists of just the empty string \(\emptyset\) of length zero. For each \(y^n_1 \in \mathcal{Y}^n\), \(f_n(\cdot|y^n_1)\) is assumed to be an injective function from \(\mathcal{X}^n\) to \(\{0,1\}^*\), so that the compressed string \(f_n(x^n_1|y^n_1)\) is always uniquely and correctly decodable. The associated description lengths of \(f_n\) are,

\[
\ell(f_n(x^n_1|y^n_1)) = \text{length of } f_n(x^n_1|y^n_1), \quad \text{bits},
\]

where, throughout, \(\ell(s)\) denotes the length, in bits, of a binary string \(s\). For \(1 \leq i \leq j \leq \infty\), we use the shorthand notation \(x^j_i\) for the string \((z_i, z_{i+1}, \ldots, z_j)\), and similarly \(Z^j_i\) for the corresponding collection of random variables \(Z^j_i = (Z_i, Z_{i+1}, \ldots, Z_j)\).

The following fundamental limits describe the best achievable performance among one-to-one compressors with side information, in both the reference-based and the pair-based versions of the problem, as described in the Introduction.

**Definition 2.1 (Reference-based optimal compression rate \(R^*(n, \epsilon|y^n_1)\))** For any blocklength \(n\), any fixed side information string \(y^n_1 \in \mathcal{Y}^n\), and any \(\epsilon \in [0,1)\), we let \(R^*(n, \epsilon|y^n_1)\) denote the smallest compression rate that can be achieved with excess-rate probability no larger than \(\epsilon\). Formally, \(R^*(n, \epsilon|y^n_1)\) is the infimum among all \(R > 0\) such that,

\[
\min_{f_n(\cdot|y^n_1)} \mathbb{P}[\ell(f_n(X^n_1|y^n_1)) > nR|Y^n_1 = y^n_1] \leq \epsilon,
\]

where the minimum is over all one-to-one compressors \(f_n(\cdot|y^n_1) : \mathcal{X}^n \to \{0,1\}^*\).

**Definition 2.2 (Pair-based optimal compression rate \(R^*(n, \epsilon)\))** For any blocklength \(n\) and any \(\epsilon \in [0,1)\), we let \(R^*(n, \epsilon)\) denote the smallest compression rate that can be achieved with excess-rate probability, over both \(X^n_1\) and \(Y^n_1\), no larger than \(\epsilon\). Formally, \(R^*(n, \epsilon)\) is the infimum among all \(R > 0\) such that,

\[
\min_{f_n} \mathbb{P}[\ell(f_n(X^n_1|Y^n_1)) > nR] \leq \epsilon,
\]

where the minimum is over all one-to-one compressors \(f_n\) with side information.

**Definition 2.3 (Reference-based excess-rate probability \(\epsilon^*(n, k|y^n_1)\))** For any blocklength \(n\), any fixed side information string \(y^n_1 \in \mathcal{Y}^n\), and any \(k \geq 1\), let \(\epsilon^*(n,k|y^n_1)\) be the best achievable excess-rate probability with rate \(R = k/n\),

\[
\epsilon^*(n,k|y^n_1) = \min_{f_n(\cdot|y^n_1)} \mathbb{P}[\ell(f_n(X^n_1|y^n_1)) \geq k|Y^n_1 = y^n_1],
\]

where the minimum is over all one-to-one compressors \(f_n(\cdot|y^n_1) : \mathcal{X}^n \to \{0,1\}^*\).
Definition 2.4 (Pair-based excess-rate probability $e^*(n, k)$) For any blocklength $n$ and any $k \geq 1$, let $e^*(n, k)$ be the best achievable excess-rate probability with rate $R = k/n$, 

$$e^*(n, k) = \min_{f_n} \Pr \left[ \ell(f_n(X^n_1|Y^n_1)) \geq k \right],$$

where the minimum is over all one-to-one compressors $f_n$ with side information.

Before establishing detailed results on these fundamental limits in the following sections, some remarks are in order.

The optimal compressor $f^*_n$: It is easy to see from Definitions 2.1–2.4 that, in all four cases, the minimum is achieved by the same simple compressor $f^*_n$: For each side information string $y^n_1$, $f^*_n(\cdot|y^n_1)$ is the optimal compressor for the distribution $P(X^n_1 = \cdot|Y^n_1 = y^n_1)$, namely, the compressor that orders the strings $x^n_1$ in order of decreasing probability $P(X^n_1 = x^n_1|Y^n_1 = y^n_1)$, and assigns them codewords from $\{0, 1\}^n$ in lexicographic order; cf. Property 1 in [25].

Equivalence of minimal rate and excess-rate probability. The following relationships are straightforward from the definitions: For any $n, k \geq 1$, any $\epsilon \in [0, 1)$, and all $y^n_1 \in \mathcal{Y}^n$:

$$R^*(n, \epsilon|y^n_1) = \frac{k}{n} \quad \text{iff} \quad e^*(n, k + 1|y^n_1) \leq \epsilon < e^*(n, k|y^n_1),$$

(4)

$$R^*(n, \epsilon) = \frac{k}{n} \quad \text{iff} \quad e^*(n, k + 1) \leq \epsilon < e^*(n, k).$$

(5)

Therefore, we can concentrate on determining the fundamental limits in terms of the rate; corresponding results for the minimal excess-rate probability then follow from (4) and (5).

Prefix-free compressors. Let $R^p_\epsilon(n, \epsilon|y^n_1)$, $R^p_\epsilon(n, \epsilon)$, $e^p_\epsilon(n, k|y^n_1)$ and $e^p_\epsilon(n, k)$ be the corresponding fundamental limits as those in Definitions 2.1–2.4, when the compressors are required to be prefix-free. As it turns out, the prefix-free condition imposes a penalty of at most $1/n$ on the rate.

Theorem 2.5 (i) For all $n, k \geq 1$:

$$e^p_\epsilon(n, k + 1) = \begin{cases} e^*(n, k), & k < n \log |\mathcal{X}| \\ 0, & k \geq n \log |\mathcal{X}|. \end{cases}$$

(6)

(ii) For all $n \geq 1$ and any $0 \leq \epsilon < 1$:

$$R^p_\epsilon(n, \epsilon) \leq R^p_\epsilon(n, \epsilon) \leq R^*(n, \epsilon) + \frac{1}{n}.$$

Throughout the paper, ‘log’ denotes ‘$\log_2$’, the logarithm taken to base 2, and all familiar information theoretic quantities are expressed in bits.

Note that, for any fixed side information string $y^n_1$, analogous results to those in Theorem 2.5 hold for the reference-based versions, $R^p_\epsilon(n, \epsilon|y^n_1)$ and $e^p_\epsilon(n, \epsilon|y^n_1)$, as an immediate consequence of [25, Theorem 1] applied to the source with distribution $P(X^n_1 = \cdot|Y^n_1 = y^n_1)$.

Proof. For part (i) note that, by the above remark, the reference-based analog of (6) in terms of $R^p_\epsilon(n, \epsilon|y^n_1)$ and $e^p_\epsilon(n, \epsilon|y^n_1)$ is an immediate consequence of [25, Theorem 1]. Then, (6) follows directly by averaging over all $y^n_1$. The result of part (ii) follows directly from (i) upon noticing that the analog of (4) also holds for prefix-free codes: $R^p_\epsilon(n, \epsilon) = \frac{k}{n}$ if and only if $e^p_\epsilon(n, k + 1) \leq \epsilon < e^p_\epsilon(n, k)$. □
3 Direct and Converse Theorems for Arbitrary Sources

In this section we briefly describe generalizations and extensions of the nonasymptotic coding theorems in [25] to the case of compression with side information. Consider two arbitrary discrete random variables \((X, Y)\), with joint (PMF) \(P_{X,Y}\), taking values in \(X\) and \(Y\), respectively. For the sake of simplicity we may assume, without loss of generality, that the source alphabet \(X\) is the set of natural numbers \(X = \mathbb{N}\), and that, for each \(y \in Y\), the values of \(X\) are ordered with nonincreasing conditional probabilities given \(y\), so that \(P(X = x | Y = y)\) is nonincreasing in \(x\), for each \(y \in Y\).

Let \(f^* = f^*_1\) be the optimal compressor described in the last section, and write \(P_X\) and \(P_{X|Y}\) for the PMF of \(X\) and the conditional PMF of \(X\) given \(Y\), respectively. The ordering of the values of \(X\) implies that, for all \(x \in X, y \in Y\),

\[
\ell(f^*(x|y)) = \lfloor \log x \rfloor.
\] (7)

The following is a general achievability result that applies to both the reference-based and the pair-based versions of the compression problem:

**Theorem 3.1** For all \(x \in X, y \in Y\),

\[
\ell(f^*(x|y)) \leq -\log P_{X|Y}(x|y),
\]

and for any \(z \geq 0\),

\[
P[\ell(f^*(X|Y)) \geq z] \leq P[-\log P_{X|Y}(X|Y) \geq z].
\]

The first part is an immediate consequence of [25, Theorem 2], applied separately for each \(y \in Y\) to the optimal compressor \(f^*(\cdot | y)\) for the source with distribution \(P_{X|Y}(\cdot | y)\). The second part follows directly from the first.

The next two theorems give general converse results for the pair-based compression problem:

**Theorem 3.2** For any integer \(k \geq 0\) and any \(\tau > 0\), we have:

\[
P[\ell(f^*(X|Y) \geq k)] \geq \sup_{\tau > 0} \{P[-\log P_{X|Y}(X|Y) \geq k + \tau] - 2^{-\tau}\}.
\]

**Proof.** Let \(k \geq 0\) and \(\tau > 0\) be arbitrary, and define,

\[
\mathcal{L} = \{(x, y) \in X \times Y : P_{X|Y}(x|y) \leq 2^{-k-\tau}\}
\]

and

\[
\mathcal{C} = \{(x, y) \in X \times Y : x \in \{1, 2, \ldots, 2^k - 1\}\}.
\]

Then,

\[
P[-\log P_{X|Y}(X|Y) \geq k + \tau] = P_X(Y) \mathcal{L} + P_{X,Y}(\mathcal{C}^c)
\]

\[
\leq P_X(Y) \mathcal{L} + P_{X,Y}(\mathcal{C}^c)
\]

\[
\leq \sum_{y \in Y} P_Y(y)(2^k - 1)2^{-k-\tau} + P[\lceil \log X \rceil \geq k]
\]

\[
\leq 2^{-\tau} + P[\ell(f^*(X|Y)) \geq k],
\]

where the last inequality follows from (7).
Theorem 3.3 For any compressor $f$ and any $\tau > 0$:

$$\mathbb{P}[\ell(f(X|Y)) \leq -\log P_{X|Y}(X|Y) - \tau] \leq 2^{-\tau(|X|) + 1}.$$ 

Once again, Theorem 3.3 is an immediate consequence of [25, Theorem 5], applied separately for each $y \in \mathcal{Y}$ to the optimal compressor $f^*(\cdot|y)$ for the source with distribution $P_{X|Y}(\cdot|y)$, and then averaging over $y$.

Our next result is one of the main tools in the proofs of the achievability results in the normal approximation bounds for $R^*(n,\epsilon|y_n)$ and $R^*(n,\epsilon)$ in Sections 6 and 7. It gives tight upper and lower bounds on the performance of the optimal compressor, that are useful in both the reference-based and the pair-based setting:

Theorem 3.4 For all $x,y$,

$$\ell(f^*(x|y)) \geq \log \left( \mathbb{E} \left[ \frac{1}{P_{X|Y}(X|y)} 1\{P_{X|Y}(X|y) > P_{X|Y}(x|y)\} \mid Y = y \right] \right) - 1,$$

$$\ell(f^*(x|y)) \leq \log \left( \mathbb{E} \left[ \frac{1}{P_{X|Y}(X|y)} 1\{P_{X|Y}(X|y) \geq P_{X|Y}(x|y)\} \mid Y = y \right] \right),$$

where $1_A$ denotes the indicator function of an event $A$, with $1_A = 1$ when $A$ occurs and $1_A = 0$ otherwise.

Proof. Recall from (7) that, for any $k \in \mathbb{N}, y \in \mathcal{Y}$, we have $\ell(f^*(k|y)) = \lfloor \log k \rfloor$. In other words, for any $y$, the optimal description length of the $k$th most likely value of $X$ according to $P_{X|Y}(\cdot|y)$, is $\lfloor \log k \rfloor$ bits. Although there may be more than one optimal ordering of the values of $X$ when there are ties, it is always the case that (given $y$) the position of $x$ is between the number of values that have probability strictly larger than $P_{X|Y}(x|y)$ and the number of outcomes that have probability $\geq P_{X|Y}(x|y)$. Formally,

$$\left\lfloor \log \left( \sum_{x' \in \mathcal{X}} 1\{P_{X|Y}(x'|y) > P_{X|Y}(x|y)\} \right) \right\rfloor \leq \ell(f^*(x|y)) \leq \left\lfloor \log \left( \sum_{x' \in \mathcal{X}} 1\{P_{X|Y}(x'|y) \geq P_{X|Y}(x|y)\} \right) \right\rfloor.$$

The result follows. \qed
4 Pointwise Asymptotics

Suppose \((X, Y)\) is an arbitrary source-side information pair, with values in the finite alphabets \(\mathcal{X}, \mathcal{Y}\). Before examining the fundamental limits \(R^s(n, \epsilon|y_n^1)\) and \(R^s(n, \epsilon)\) for reference-based and pair-based compression in detail in Sections 6–8, in this section we derive refined asymptotic bounds for the description lengths \(\ell(f_n(X^n_1|Y^n_1))\) of arbitrary compressors \(\{f_n\}\), and corresponding achievability results.

**Definition 4.1 (Conditional information density)** For an arbitrary source-side information pair \((X, Y)\), the conditional information density of blocklength \(n\) is the random variable

\[ -\log P(X^n_1|Y^n_1) = -\log P_{X^n_1|Y^n_1}(X^n_1|Y^n_1). \]

The starting point is the following almost sure (a.s.) approximation result between the description lengths \(\ell(f_n(X^n_1|Y^n_1))\) of an arbitrary sequence of compressors and the conditional information density \(-\log P(X^n_1|Y^n_1)\) of an arbitrary source-side information pair \((X, Y)\). When it causes no confusion, we drop the subscripts for PMFs and conditional PMFs, e.g., simply writing \(P(x^n_1|y_1^n)\) for \(P_{X^n_1|Y^n_1}(x^n_1|y_1^n)\) as in the definition above. Recall the definition of the optimal compressors \(\{f_n^*\}\) from Section 2.

**Theorem 4.2** For any source-side information pair \((X, Y)\), and any sequence \(\{B_n\}\) that grows faster than logarithmically, i.e., such that \(B_n/\log n \to \infty\) as \(n \to \infty\), we have:

(a) For any sequence of compressors with side information \(\{f_n\}\):

\[ \liminf_{n \to \infty} \frac{\ell(f_n(X^n_1|Y^n_1)) - [\log P(X^n_1|Y^n_1)]}{B_n} \geq 0, \quad \text{a.s.} \]

(b) The optimal compressors \(\{f_n^*\}\) achieve the above bound with equality.

**Proof.** Fix \(\epsilon > 0\) arbitrary and let \(\tau = \tau_n = \epsilon B_n\). Applying Theorem 3.3 with \(X^n_1, Y^n_1\) in place of \(X, Y\) and \(X^n, Y^n\) in place of \(\mathcal{X}, \mathcal{Y}\), gives,

\[ \mathbb{P}[\ell(f(X^n_1|Y^n_1)) \leq -\log P(X^n_1|Y^n_1) - \epsilon B_n] \leq 2^{\log n - \epsilon B_n(\log |X|) + 1}, \]

which is summable in \(n\). Therefore, by the Borel-Cantelli lemma we have that, eventually, almost surely,

\[ \ell(f(X^n_1|Y^n_1)) + \log P(X^n_1|Y^n_1) > -\epsilon B_n. \]

Since \(\epsilon > 0\) was arbitrary, this implies (a). Part (b) follows from (a) together with the fact that, with probability 1, \(\ell(f_n^*(X^n_1|Y^n_1)) + \log P(X^n_1|Y^n_1) \leq 0\), by Theorem 3.1. \(\square\)

4.1 First-order asymptotics

For any source-side information pair \((X, Y)\), the conditional entropy rate \(H(X|Y)\) is defined as:

\[ H(X|Y) = \lim_{n \to \infty} \frac{1}{n} H(X^n_1|Y^n_1). \]

Throughout \(H(Z)\) and \(H(Z|W)\) denote the discrete entropy of \(Z\) and the conditional entropy of \(Z\) given \(W\), in bits. If \((X, Y)\) are jointly stationary, then the above lim sup is in fact a limit,
and it is equal to \( H(X, Y) - H(Y) \), where \( H(X, Y) \) and \( H(Y) \) are the entropy rates of \((X, Y)\) and of \(Y\), respectively [9]. Moreover, if \((X, Y)\) are also jointly ergodic, then by applying the Shannon-McMillan-Breiman theorem [9] to \(Y\) and to the pair \((X, Y)\), we obtain its conditional version:

\[
- \frac{1}{n} \log P(X^n_1 | Y^n_1) \to H(X | Y), \quad \text{a.s.} \quad (8)
\]

The next result states that the conditional entropy rate is the best asymptotically achievable compression rate, not only in expectation but also with probability 1. It is a consequence of Theorem 4.2 with \(B_n = n\), combined with (8).

**Theorem 4.3** Suppose \((X, Y)\) is a jointly stationary and ergodic source-side information pair with conditional entropy rate \(H(X | Y)\).

(a) For any sequence of compressors with side information \(\{f_n\}\):

\[
\lim \inf_{n \to \infty} \frac{\ell(f_n(X^n_1 | Y^n_1))}{n} \geq H(X | Y), \quad \text{a.s.}
\]

(b) The optimal compressors \(\{f^*_n\}\) achieve the above bound with equality.

### 4.2 Finer asymptotics

The refinements of Theorem 4.3 presented in this section will be derived as consequences of the general approximation results in Theorem 4.2, combined with corresponding refined asymptotics for the conditional information density \(- \log P(X^n_1 | Y^n_1)\). For clarity of exposition these are stated separately, in Section 4.3 below.

The results of this section will be established for a class of jointly stationary and ergodic source-side information pairs \((X, Y)\), that includes all Markov chains with positive transition probabilities. The relevant conditions, in their most general form, will be given in terms of the following mixing coefficients.

**Definition 4.4** Suppose \(Z = \{Z_n; n \in \mathbb{Z}\}\) is a stationary process on a finite alphabet \(\mathcal{Z}\). For \(-\infty \leq i \leq j \leq \infty\), let \(\mathcal{F}^i_j\) denote the \(\sigma\)-algebra generated by \(Z^i_j\). For \(d \geq 1\), define:

\[
\alpha(Z)(d) = \sup \{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| ; \ A \in \mathcal{F}^i_j, B \in \mathcal{F}^\infty_d \},
\]

\[
\gamma(Z)(d) = \max_{z \in \mathcal{Z}} \mathbb{E}( | \log \mathbb{P}(Z_0 = z|Z_{-\infty}^{-1}) - \log \mathbb{P}(Z_0 = z|Z_{-d}^{-1}) | ).
\]

Note that if \(Z\) is an ergodic Markov chain of order \(k\), then \(\alpha(Z)(d)\) decays exponentially fast [4], and \(\gamma(Z)(d) = 0\) for all \(d \geq k\). Moreover, if \((X, Y)\) is a Markov chain with all positive transition probabilities, then \(\gamma(Y)(d)\) also decays exponentially fast; cf. [12, Lemma 2.1].

Throughout this section we will assume that the following conditions hold:

**Assumption (M).** The source-side information pair \((X, Y)\) is stationary and satisfies one of the following three conditions:

(a) \((X, Y)\) is a Markov chain with all positive transition probabilities; or

(b) \((X, Y)\) as well as \(Y\) are \(k\)th order, ergodic Markov chains; or
(c) \((X, Y)\) is jointly ergodic and satisfies the following mixing conditions:\(^1\)
\[
\begin{align*}
\alpha(X,Y)(d) &= O(d^{-336}), \\
\gamma(X,Y)(d) &= O(d^{-48}), \\
\gamma(Y)(d) &= O(d^{-48}).
\end{align*}
\]

In view of the discussion following Definition 4.4, (a) \(\Rightarrow (c)\) and (b) \(\Rightarrow (c)\). Therefore, all results stated under assumption (M) will be proved under the weakest set of conditions, namely, that (9)–(11) hold.

**Definition 4.5** For a source-side information pair \((X,Y)\), the conditional varentropy rate is:
\[
\sigma^2(X|Y) = \limsup_{n \to \infty} \frac{1}{n} \text{Var}(\alpha(X^n_1|Y^n_1)) = \lim_{n \to \infty} \frac{1}{n} \text{Var}\left(\alpha(X^n_1|Y^n_1)\right).
\]

Under the above assumptions, the lim sup in (12) is in fact a limit. Lemma 4.6 is proved in the Appendix.

**Lemma 4.6** Under assumption (M), the conditional varentropy rate \(\sigma^2(X|Y)\) is:
\[
\sigma^2(X|Y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}(\alpha(X^n_1|Y^n_1)) = \lim_{n \to \infty} \frac{1}{n} \text{Var}\left(\alpha(X^n_1|Y^n_1)\right).
\]

Our first main result of this section is a “one-sided” central limit theorem (CLT), which states that the description lengths \(\ell(f_n(X^n_1|Y^n_1))\) of an arbitrary sequence of compressors with side information, \(\{f_n\}\), are asymptotically at best Gaussian, with variance \(\sigma^2(X|Y)\). Recall the optimal compressors \(\{f^*_n\}\) described in Section 2.

**Theorem 4.7** (CLT for codelengths) Suppose \((X,Y)\) satisfy assumption (M), and let \(\sigma^2 = \sigma^2(X|Y) > 0\) denote the conditional varentropy rate (12). Then there exists a sequence of random variables \(Z_n : n \geq 1\) such that:

(a) For any sequence of compressors with side information, \(\{f_n\}\), we have,
\[
\liminf_{n \to \infty} \left[ \frac{\ell(f_n(X^n_1|Y^n_1)) - H(X^n_1|Y^n_1)}{\sqrt{n}} - Z_n \right] \geq 0, \quad \text{a.s.,}
\]
where,
\[
Z_n \to N(0, \sigma^2), \quad \text{in distribution, as } n \to \infty.
\]

(b) The optimal compressors \(\{f^*_n\}\) achieve the lower bound in (13) with equality.

---

\(^1\)Our source-side information pairs \((X, Y)\) are only defined for \((X_n, Y_n)\) with \(n \geq 1\), whereas the coefficients \(\alpha(Z)(d)\) and \(\gamma(Z)(d)\) are defined for two-sided sequences \(\{Z_n : n \in \mathbb{Z}\}\). But this does not impose an additional restriction, since any one-sided stationary process can be extended to a two-sided one by the Kolmogorov extension theorem [3].
Proof. Letting \( Z_n = \left[ - \log P(X^n | Y^n) \right] / \sqrt{n} \), \( n \geq 1 \), and taking \( B_n = \sqrt{n} \), both results follow by combining the approximation results of Theorem 4.2 with the corresponding CLT for the conditional information density in Theorem 4.9.

Our next result is in the form of a “one-sided” law of the iterated logarithm (LIL) which states that, with probability 1, the description lengths of any compressor with side information will have inevitable fluctuations of order \( \sqrt{2\sigma^2 n \log_2 \log_2 n} \) bits around the conditional entropy rate \( H(X|Y) \); throughout, \( \log_e \) denotes the natural logarithm to base \( e \).

**Theorem 4.8 (LIL for codelengths)** Suppose \((X, Y)\) satisfy assumption (M), and let \( \sigma^2 = \sigma^2(X|Y) > 0 \) denote the conditional varentropy rate (12). Then:

(a) For any sequence of compressors with side information, \( \{f_n\} \), we have:

\[
\limsup_{n \to \infty} \frac{\ell(f_n(X^n | Y^n)) - H(X^n | Y^n)}{\sqrt{2n \log_2 \log_2 n}} \geq \sigma, \quad \text{a.s.,}
\]

and

\[
\liminf_{n \to \infty} \frac{\ell(f_n(X^n | Y^n)) - H(X^n | Y^n)}{\sqrt{2n \log_2 \log_2 n}} \geq -\sigma, \quad \text{a.s.}
\]

(b) The optimal compressors \( \{f_n^*\} \) achieve the lower bounds in (14) and (15) with equality.

Proof. Taking \( B_n = \sqrt{2n \log_2 \log_2 n} \), the results of the theorem again follow by combining the approximation results of Theorem 4.2 with the corresponding LIL for the conditional information density in Theorem 4.10.

Remarks.

1. Although the results in Theorems 4.7 and 4.8 are stated for one-to-one compressors \( \{f_n\} \), they remain valid for the class of prefix-free compressors. Since prefix-free codes are certainly one-to-one, the converse bounds in Theorem 4.7 (a) and 4.8 (a) are valid as stated, while for the achievability results it suffices to consider compressors \( f^n \) with description lengths \( \ell(f^n(x^n | y^n)) = \left[ - \log P(x^n | y^n) \right] \), and then apply Theorem 4.9.

2. Theorem 4.7 says that the compression rate of any sequence of compressors \( \{f_n\} \) will have at best Gaussian fluctuations around \( H(X|Y) \),

\[
\frac{1}{n} \ell(f_n^*(X^n | Y^n)) \approx N \left( H(X|Y), \frac{\sigma^2(X|Y)}{n} \right), \quad \text{bits/symbol},
\]

and similarly Theorem 4.8 says that, with probability 1, the description lengths will have inevitable fluctuations of approximately \( \pm \sqrt{2n\sigma^2 \log_2 \log_2 n} \) bits around \( nH(X|Y) \).

As both of these vanish when \( \sigma^2(X|Y) = 0 \), we note that, if the source-side information pair \((X, Y)\) is memoryless, so that \( \{(X_n, Y_n)\} \) are independent and identically distributed, then the conditional varentropy rate reduces to,

\[
\sigma^2(X|Y) = \text{Var}(\log P(X_1|Y_1)),
\]

which is equal to zero if and only if, for each \( y \in \mathcal{Y} \), the conditional distribution of \( X_1 \) given \( Y_1 = y \) is uniform on a subset \( \mathcal{X}_y \subset \mathcal{X} \), where all the \( \mathcal{X}_y \) have the same cardinality; see Section 6.1 for a more detailed discussion.

In the more general case when both the pair process \((X, Y)\) and the side information \( Y \) are Markov chains, necessary and sufficient conditions for \( \sigma^2(X|Y) \) to be zero are established in Section 4.4.
4.3 The conditional information density

Here we show that the conditional information density itself, \(- \log P(X^n_t | Y^n_t)\), satisfies a CLT and a LIL. The next two theorems are consequences of the almost sure invariance principle established in Theorem A.1, in the Appendix.

**Theorem 4.9 (CLT for the conditional information density)** Suppose \((X, Y)\) satisfy assumption (M), and let \(\sigma^2 = \sigma^2(X | Y) > 0\) denote the conditional varentropy rate (12). Then, as \(n \to \infty\):

\[
\frac{- \log P(X^n_t | Y^n_t) - H(X^n_t | Y^n_t)}{\sqrt{n}} \to N(0, \sigma^2), \quad \text{in distribution.} \tag{16}
\]

**Proof.** The conditions (10) and (11), imply that, 

\[
\lim_{n \to \infty} \frac{- \log P(X^n_t | Y^n_t) - nH(X | Y)}{\sqrt{n}} = 0 \quad \text{in distribution.} \tag{17}
\]

Let \(D = D([0, 1], \mathbb{R})\) denote the space of cadlag (right-continuous with left-hand limits) functions from \([0, 1]\) to \(\mathbb{R}\), and define, for each \(t \geq 0\),

\[S(t) = \log P(X^n_{1\lfloor t \rfloor} | Y^n_{1\lfloor t \rfloor}) + tH(X | Y),\]

where \(\lfloor \cdot \rfloor\) denotes the greatest integer function. Then Theorem A.1 implies that, as \(n \to \infty\),

\[\frac{1}{\sigma \sqrt{n}} S_n(t) ; t \in [0, 1] \to \{B(t) ; t \in [0, 1]\}, \quad \text{weakly in } D,
\]

where \(\{B(t)\}\) is a standard Brownian motion; see, e.g., [34, Theorem E, p. 4]. In particular, this implies that,

\[\frac{1}{\sigma \sqrt{n}} S_n(1) \to B(1) \sim N(0, 1), \quad \text{in distribution,}
\]

which is exactly (17). \(\square\)

**Theorem 4.10 (LIL for the conditional information density)** Suppose \((X, Y)\) satisfy assumption (M), and let \(\sigma^2 = \sigma^2(X | Y) > 0\) denote the conditional varentropy rate (12). Then:

\[
\lim_{n \to \infty} \sup_{n \to \infty} \frac{- \log P(X^n_t | Y^n_t) - H(X^n_t | Y^n_t)}{\sqrt{2n \log e \log_e n}} = \sigma, \quad \text{a.s.,} \tag{18}
\]

and

\[
\lim_{n \to \infty} \inf_{n \to \infty} \frac{- \log P(X^n_t | Y^n_t) - H(X^n_t | Y^n_t)}{\sqrt{2n \log e \log_e n}} = -\sigma, \quad \text{a.s.} \tag{19}
\]

**Proof.** As in the proof of (16), it suffices to prove (18) with \(nH(X | Y)\) in place of \(H(X^n_t | Y^n_t)\). But this is immediate from Theorem A.1, since, for a standard Brownian motion \(\{B(t)\}\),

\[\lim_{t \to \infty} \frac{B(t)}{\sqrt{2t \log e \log_e t}} = 1, \quad \text{a.s.,}
\]

see, e.g., [20, Theorem 11.18]. And similarly for (19). \(\square\)
4.4 Markov chains

Here we provide a characterization of the case when the conditional varentropy rate is zero, under additional assumptions on the source-side information process \((X, Y)\). Theorem 4.11 is proved in Appendix B.

**Theorem 4.11** Suppose that the pair \((X, Y)\) as well as \(Y\) are stationary, irreducible and aperiodic Markov chains, with conditional entropy rate \(H = H(X|Y)\).

The conditional varentropy rate \(\sigma^2 = \sigma^2(X|Y)\) is zero if and only if there is a function \(g : X \times Y \rightarrow (0, \infty)\) such that,

\[
P[(X_{i+1}, Y_{i+1}) = (x_2, y_2) | (X_i, Y_i) = (x_1, y_1)] = \frac{2^{-H} g(x_1, y_1)}{g(x_2, y_2)},
\]

for all \((x_1, x_2, y_1, y_2) \in X^2 \times Y^2\) such that the the transition probability in the numerator in (20) is nonzero.

Equivalently, \(\sigma^2 = 0\) if and only if there is a \(q > 0\) such that, for every pair of strings \((x_1^{n+1}, y_1^{n+1}) \in X^{n+1} \times Y^{n+1}\) that starts and ends at the same state \((x_1, y_1) = (x_{n+1}, y_{n+1}) = (s, t) \in X \times Y\), we have,

\[
\frac{P[X_1^{n+1} = x_1^{n+1}, Y_1^{n+1} = y_1^{n+1} | X_1 = s, Y_1 = t]}{P[Y_1^{n+1} = y_1^{n+1} | Y_1 = t]} = \text{either } q^n \text{ or } 0,
\]

with the convention that \(0/0 = 0\).
5 Idealized LZ Compression with Side Information

Consider the following idealized version of Lempel-Ziv-like compression with side information. For a given source-side information pair \((X, Y) = \{(X_n, Y_n) : n \in \mathbb{Z}\}\), the encoder and decoder both have access to the infinite past \((X_{-\infty}^0, Y_{-\infty}^0)\) and to the current side information \(Y_1^n\). The encoder describes \(X^n\) to the decoder as follows. First she searches for the first appearance of \((X_1^n, Y_1^n)\) in the past \((X_{-\infty}^0, Y_{-\infty}^0)\), that is, for the first \(r \geq 1\) such that, \((X_{-r+1}, Y_{-r+1}) = (X_1^n, Y_1^n)\). Then she counts how many times \(Y_1^n\) appears in \(Y_{-\infty}^0\) between locations \(-r+1\) and 0, namely, how many indices \(1 \leq j < r\) there are, such that \(Y_{-j+1}^{j+n} = Y_1^n\). Say there are \((R_n - 1)\) such js. She describes \(X_1^n\) to the decoder by telling him to look at the \(R_n\)th position where \(Y_1^n\) appears in the past \(Y_{-\infty}^0\), and read off the corresponding \(X\) string.

This description takes \(\approx \log R_n\) bits, and, as it turns out, the resulting compression rate is asymptotically optimal: As \(n \to \infty\), with probability 1,

\[
\frac{1}{n} \log R_n \to H(X|Y), \quad \text{bits/symbol.} \tag{22}
\]

Moreover, it is second-order optimal, in that it achieves equality in the CLT and LIL bounds given in Theorems 4.7 and 4.8 of Section 4.

Our purpose in this section is to make these statements precise. We will prove (22) as well as its CLT and LIL refinements, generalizing the corresponding results for recurrence times without side information in [23].

The use of recurrence times in understanding the Lempel-Ziv (LZ) family of algorithms was introduced by Willems [47] and Wyner and Ziv [48, 49]. In terms of practical methods for compression with side information, Subrahmanya and Berger [40] proposed a side information analog of the sliding window LZ algorithm [51], and Uyematsu and Kuzuoka [46] proposed a side information version of the incremental parsing LZ algorithm [52]. The Subrahmanya-Berger algorithm was shown to be asymptotically optimal in [15] and [16]. Different types of LZ-like algorithms for compression with side information were also considered in [44] and [13].

Throughout this section, we assume \((X, Y)\) is a jointly stationary and ergodic source-side information pair, with values in the finite alphabets \(\mathcal{X}, \mathcal{Y}\), respectively. We use bold lowercase letters \(x, y\) without subscripts to denote infinite realizations \(x_{-\infty}^\infty, y_{-\infty}^\infty\) of \(X, Y\), and the corresponding bold capital letters \(X, Y\) without subscripts to denote the entire process, \(X = X_{-\infty}^\infty, Y = Y_{-\infty}^\infty\).

The main quantities of interest are the recurrence times defined next.

**Definition 5.1 (Recurrence times)** For a realization \(x\) of the process \(X\), and \(n \geq 1\), define the repeated recurrence times \(R_n^{(j)}(x)\) of \(x_1^n\), recursively, as:

\[
R_n^{(1)}(x) = \inf \{i \geq 1 : x_{-i+1}^{-i+n} = x_1^n\},
\]

\[
R_n^{(j)}(x) = \inf \{i > R_n^{(j-1)}(x) : x_{-i+1}^{-i+n} = x_1^n\}, \quad j > 1.
\]

For a realization \((x, y)\) of the pair \((X, Y)\) and \(n \geq 1\), the joint recurrence time \(R_n(x, y)\) of \((x_1^n, y_1^n)\) is defined as,

\[
R_n(x, y) = \inf \{i \geq 1 : (x, y)_{-i+1}^{-i+n} = (x, y)_1^n\},
\]

and the conditional recurrence time \(R_n(x|y)\) of \(x_1^n\) among the appearances \(y_1^n\) is:

\[
R_n(x|y) = \inf \{i \geq 1 : x_{-i+1}^{-i+n} = x_1^n, y_{-i+1}^{-i+n} = y_1^n\}.
\]
An important tool in the asymptotic analysis of recurrence times is Kac’s Theorem [18]. Its conditional version in Theorem 5.2 was first established in [15] using Kakutani’s induced transformation [19, 36].

**Theorem 5.2 (Conditional Kac’s theorem)** [15] Suppose \((X, Y)\) is a jointly stationary and ergodic source-side information pair. For any pair of strings \(x^n_1 \in \mathcal{X}^n, y^n_1 \in \mathcal{Y}^n\):

\[
\mathbb{E}[\mathcal{R}_n(X|Y)|X^n_1, Y^n_1 = y^n_1] = \frac{1}{P(x^n_1|y^n_1)}.
\]

The following result states that we can asymptotically approximate \(\log \mathcal{R}_n(X|Y)\) by the conditional information density not just in expectation as in Kac’s theorem, but also with probability 1. Its proof is in Appendix C.

**Theorem 5.3** Suppose \((X, Y)\) is a jointly stationary and ergodic source-side information pair. For any sequence \(\{c_n\}\) of non-negative real numbers such that \(\sum_n n^{2-c_n} < \infty\), we have:

\[
\begin{align*}
(i) \quad & \log \mathcal{R}_n(X|Y) - \log \left(\frac{1}{P(X^n_1|Y^n_1)}\right) \leq c_n, \quad \text{eventually a.s.} \\
(ii) \quad & \log \mathcal{R}_n(X|Y) - \log \left(\frac{1}{P(X^n_1|Y^n_1, Y^0_{-\infty}, X^n_{-\infty})}\right) \geq -c_n, \quad \text{eventually a.s.} \\
(iii) \quad & \log \mathcal{R}_n(X|Y) - \log \left(\frac{P(Y^n_1|Y^0_{-\infty})}{P(X^n_1, Y^n_1|Y^0_{-\infty}, X^n_{-\infty})}\right) \geq -2c_n, \quad \text{eventually a.s.}
\end{align*}
\]

Next we state the main consequences of Theorem 5.3 that we will need. Recall the definition of the coefficients \(\gamma^{(Z)}(d)\) from Section 4.2. Corollary 5.4 is proved in Appendix C.

**Corollary 5.4** Suppose \((X, Y)\) are jointly stationary and ergodic.

(a) If, in addition, \(\sum_d \gamma^{(X,Y)}(d) < \infty\) and \(\sum_d \gamma^{(Y)}(d) < \infty\), then for any \(\beta > 0\):

\[
\log[\mathcal{R}_n(X|Y)P(X^n_1|Y^n_1)] = o(n^{\beta}), \quad \text{a.s.}
\]

(b) In the general jointly ergodic case, we have:

\[
\log[\mathcal{R}_n(X|Y)P(X^n_1|Y^n_1)] = o(n), \quad \text{a.s.}
\]

From part (b) combined with the Shannon-McMillan-Breiman theorem as in (8), we obtain the result (22) promised in the beginning of this section:

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathcal{R}_n(X|Y) \to H(X|Y), \quad \text{a.s.}
\]

This was first established in [15]. But at this point we have already done the work required to obtain much finer asymptotic results for the conditional recurrence time.

For any pair of infinite realizations \((x, y)\) of \((X, Y)\), let \(\{\mathcal{R}^{(x|y)}(t) : t \geq 0\}\) be the continuous-time path, defined as:

\[
\begin{align*}
\mathcal{R}^{(x|y)}(t) &= 0, \quad \text{for } t < 1, \\
\mathcal{R}^{(x|y)}(t) &= \log \mathcal{R}_{\lfloor t \rfloor}(x|y) - \lfloor t \rfloor H(X|Y), \quad \text{for } t \geq 1.
\end{align*}
\]

The following theorem is a direct consequence of Corollary 5.4 (a) combined with Theorem A.1 in the Appendix. Recall assumption (M) from Section 4.2.
Theorem 5.5 Suppose \((X, Y)\) satisfy assumption \((M)\), and let \(\sigma^2 = \sigma^2(X \mid Y) > 0\) denote the conditional varentropy rate. Then \(\{R(X \mid Y)(t)\}\) can be redefined on a richer probability space that contains a standard Brownian motion \(\{B(t) : t \geq 0\}\) such that, for any \(\lambda < 1/294:\)

\[
R(X \mid Y)(t) - \sigma B(t) = O(t^{1/2-\lambda}), \quad \text{a.s.}
\]

Two immediate consequences of Theorem 5.5 are the following:

Theorem 5.6 (CLT and LIL for the conditional recurrence times) Suppose \((X, Y)\) satisfy assumption \((M)\) and let \(\sigma^2 = \sigma^2(X \mid Y) > 0\) denote the conditional varentropy rate. Then:

\(\log R_n(X \mid Y) - H(X^n_1 \mid Y^n_1)\)

\[
\frac{\sqrt{n}}{\sqrt{2 \ln \log \log n}} \rightarrow N(0, \sigma^2), \quad \text{in distribution, as } n \to \infty.
\]

\(\limsup_{n \to \infty} \frac{\log R_n(X \mid Y) - H(X^n_1 \mid Y^n_1)}{\sqrt{2n \ln \log \log n}} = \sigma, \quad \text{a.s.}
\)
6 Normal Approximation for Reference-Based Compression

In this section we give explicit, finite- \( n \) bounds on the reference-based optimal rate \( R^\ast(n, \epsilon|y^n_1) \).

Suppose the source and side information, \((X, Y)\), consist of independent and identically distributed (i.i.d.) pairs \(\{(X_n, Y_n)\}\), or, more generally, that \((X, Y)\) is a conditionally-i.i.d. source-side information pair, i.e., that the distribution of \(Y\) is arbitrary, and for each \(n\), given \(Y^n_1 = y^n_1\), the random variables \(X^n_i\) are conditionally i.i.d.,

\[
P(X^n_1 = x^n_1|Y^n_1 = y^n_1) = \prod_{i=1}^n P_{X|Y}(x_i|y_i), \quad x^n_1 \in X, y^n_1 \in Y,
\]

for a given family of conditional PMFs \(P_{X|Y}(\cdot|\cdot)\).

We will use the following notation. For any \(y \in Y\), we write,

\[
H(X|y) = - \sum_{x \in X} P_{X|Y}(x|y) \log P_{X|Y}(x|y),
\]

for the entropy of the conditional distribution of \(X\) given \(Y = y\), and,

\[
V(y) = \text{Var}[\log P_{X|Y}(X|Y = y)].
\]

For a side information string \(y^n_1 \in Y^n\), we denote,

\[
H_n(X|y^n_1) = \frac{1}{n} \sum_{j=1}^n H(X|y_j),
\]

\[
\sigma^2_n(y^n_1) = \frac{1}{n} \sum_{j=1}^n V(y_j).
\]

The upper and lower bounds developed in Theorems 6.5 and 6.4 below say that, for any conditionally-i.i.d. source-side information pair \((X, Y)\) and any side information string \(y^n_1\), the reference-based optimal compression rate,

\[
R^\ast(n, \epsilon|y^n_1) = H_n(X|y^n_1) + \frac{\sigma^2_n(y^n_1)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} + O\left(\frac{1}{n}\right),
\]

with explicit bounds on the \(O(1/n)\) term, where \(Q\) denotes the standard Gaussian tail function \(Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz\).

As described in the Introduction, \(R^\ast(n, \epsilon|y^n_1)\) is the best achievable rate with excess-rate probability no more than \(\epsilon\), with respect to a fixed side information string \(y^n_1\). Although the description of \(R^\ast(n, \epsilon|y^n_1)\) in (26) is quite detailed and accurate up to and including the third-order term, both the rate \(H_n(X|y^n_1)\) and the “dispersion” \(\sigma^2_n(y^n_1)\) depend on \(y^n_1\), so it is natural to ask if anything can be said about the behaviour of \(R^\ast(n, \epsilon|y^n_1)\) for “typical” \(y^n_1\)’s. Indeed, in Theorem 6.6 we show that if the source-side information pair \((X, Y)\) is i.i.d., then \(R^\ast(n, \epsilon|Y^n_1) \approx H(X|Y)\) with high probability; roughly speaking, there are positive constants \(c, c', \delta^*\) such that, for all \(n\),

\[
\mathbb{P}[R^\ast(n, \epsilon|Y^n_1) > H(X|Y) + \delta] \leq \begin{cases} e^{-nc^2 \delta^2}, & \text{for } 0 < \delta \leq \delta^*, \\ e^{-nc'}, & \text{for } \delta > \delta^*. \end{cases}
\]

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6.1 Preliminaries

Suppose for now that \((X_n, Y_n) = \{(X_n, Y_n)\}\) is an i.i.d. source-side information pair, with all \((X_n, Y_n)\) distributed as \((X, Y)\), with joint PMF \(P_{X,Y}\) on \(\mathcal{X} \times \mathcal{Y}\). In this case, the conditional entropy rate \(H(X|Y)\) is simply \(H(X|Y)\) and the conditional varentropy rate \((12)\) reduces to the conditional varentropy of \(X\) given \(Y\),

\[
\sigma^2(X|Y) = \text{Var}\left[-\log P_{X|Y}(X|Y)\right],
\]

where \(P_{X|Y}\) denotes the conditional PMF of \(X_n\) given \(Y_n\). As in earlier sections, we will drop the subscripts of PMFs when they can be understood unambiguously from the context.

First we state some simple properties for the conditional varentropy. We write \(\hat{H}_X(Y)\) for the random variable,

\[
\hat{H}_X(Y) = -\sum_{x \in \mathcal{X}} P_{X|Y}(x|Y) \log P_{X|Y}(x|Y).
\] (27)

**Proposition 6.1** Suppose \((X, Y)\) is an i.i.d. source-side information pair, with each \((X_n, Y_n) \sim (X, Y)\). Then:

(i) The conditional varentropy can also be expressed:

\[
\sigma^2(X|Y) = \mathbb{E}[V(Y)] + \text{Var}[\hat{H}_X(Y)].
\]

(ii) \(\mathbb{E}[V(Y)] = 0\) if and only if, for each \(y \in \mathcal{Y}\), \(P_{X|Y}(x|y)\) is uniform on a (possibly singleton) subset of \(\mathcal{X}\).

(iii) \(\sigma^2(X|Y) = 0\) if and only if there exists \(k \in \{1, 2, \ldots, |\mathcal{X}|\}\), such that, for each \(y \in \mathcal{Y}\), \(P_{X|Y}(x|y)\) is uniform on a subset \(\mathcal{X}_y \subset \mathcal{X}\) of size \(|\mathcal{X}_y| = k\).

**Proof.** For (i) we have,

\[
\sigma^2(X|Y) = \text{Var}\left[-\log P(X|Y)\right] = \mathbb{E}[(\log P(X|Y))^2] - H(X|Y)^2
\]

\[
= \mathbb{E}[(\log P(X|Y))^2] - \mathbb{E}[\hat{H}_X(Y)^2] + \mathbb{E}[\hat{H}_X(Y)^2] - H(X|Y)^2
\]

\[
= \mathbb{E}\{\mathbb{E}[(\log P(X|Y))^2]|Y] - \hat{H}_X(Y)^2\} + \text{Var}[\hat{H}_X(Y)]
\]

\[
= \mathbb{E}[V(Y)] + \text{Var}[\hat{H}_X(Y)].
\]

Parts (ii) and (iii) are straightforward from the definitions. \(\square\)

An important technical tool in what follows will be the classical Berry-Esséen bound. The two relevant versions of this result are stated in Theorems 6.2 and 6.3 below.

**Theorem 6.2 (I.i.d. Berry-Esséen bound)** [33, 26] Let \(\{Z_k\}\) be i.i.d. random variables with zero mean, unit variance, and finite third moment, and let \(F_n\) denote the distribution function of \(\frac{\sum_{k=1}^n Z_k}{\sqrt{n}}\). Then,

\[
\sup_x |F_n(x) - \Phi(x)| \leq \frac{\mathbb{E}[|Z_1|^3]}{2\sqrt{n}}, \quad \text{for all } n \geq 1,
\]

where \(\Phi\) denotes the standard normal distribution function.
Theorem 6.3 (Non-i.i.d. Berry-Esséen bound) [10] Let \( \{Z_k\} \) be independent, zero mean random variables, with:
\[
\sigma_k^2 = \mathbb{E}[Z_k^2], \quad \rho_k = \mathbb{E}[|Z_k|^3] < \infty, \quad k \geq 1.
\]

Put,
\[
s_n = \sum_{k=1}^{n} \sigma_k^2, \quad r_n = \sum_{k=1}^{n} \rho_k,
\]
and let \( F_n \) denote the distribution function of \( \frac{\sum_{k=1}^{n} Z_k}{\sqrt{s_n}} \). Then:
\[
\sup_x |F_n(x) - \Phi(x)| \leq \frac{6r_n}{s_n^{3/2}}, \quad \text{for all } n \geq 1.
\]

6.2 Direct and converse bounds

Before stating our main results we note that, if \( \sigma_n^2(y_n^1) \) were equal to zero for some side information sequence \( y_n^1 \), then each source symbol would be known (both to the encoder and decoder) to be uniformly distributed on some subset of \( X \), so the compression problem would be rather trivial. To avoid these degenerate cases, we assume that \( \sigma_n^2(y_n^1) > 0 \) in Theorems 6.4 and 6.5.

Theorem 6.4 (Converse for \( R^*(n, \epsilon|y_n^1) \)) Suppose \((X, Y)\) is a conditionally-i.i.d. source-side information pair. For any \( 0 < \epsilon < \frac{1}{2} \), the reference-based optimal compression rate satisfies,
\[
R^*(n, \epsilon|y_n^1) \geq H_n(X|y_n^1) + \frac{\sigma_n(y_n^1)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} - \frac{1}{n} \eta(y_n^1),
\]
for all,
\[
n > \frac{(1 + 6m_3 \sigma_n^3(y_n^1))^2}{4(Q^{-1}(\epsilon) \phi(Q^{-1}(\epsilon)))^2},
\]
and any side information string \( y_n^1 \in \mathcal{Y}^n \) such that \( \sigma_n^2(y_n^1) > 0 \), where \( \phi \) is the standard normal density, \( H_n(X|y_n^1) \) and \( \sigma_n^2(y_n^1) \) are given in (24) and (25),
\[
m_3 = \max_{y \in \mathcal{Y}} \mathbb{E}[| - \log P(X|y) - H(X|y)|^3],
\]
and,
\[
\eta(y_n^1) = \frac{\sigma_n^3(y_n^1) + 6m_3}{\phi(Q^{-1}(\epsilon)) \sigma_n^2(y_n^1)}.
\]

Note that, by the definitions in Section 2, Theorem 6.4 obviously also holds for prefix-free codes, with \( R^*_p(n, \epsilon|y_n^1) \) in place of \( R^*(n, \epsilon|y_n^1) \).
Proof. Since, conditional on \( y^n_1 \), the random variables \( X^n_1 \) are independent, we have,

\[
P \left[ - \log P(X^n_1 | y^n_1) \geq \sum_{i=1}^{n} H(X | y_i) + \sqrt{n} \sigma_n(y^n_1) Q^{-1}(\epsilon) - \eta(y^n_1) \mid Y^n_1 = y^n_1 \right]
\]

\[
= P \left[ \sum_{i=1}^{n} \left( - \log P(X_i | y_i) - H(X | y_i) \right) \geq \frac{\eta(y^n_1)}{\sigma_n(y^n_1) \sqrt{n}} \right] \geq Q^{-1}(\epsilon) - \frac{\eta(y^n_1)}{\sigma_n(y^n_1) \sqrt{n}} \mid Y^n_1 = y^n_1
\]

\[
\geq Q \left( Q^{-1}(\epsilon) - \frac{\eta(y^n_1)}{\sigma_n(y^n_1) \sqrt{n}} \right) - 6 \frac{m_3}{\sigma_n^3(y^n_1) \sqrt{n}}
\]

\[
\geq \epsilon + \frac{\eta(y^n_1)}{\sigma_n(y^n_1) \sqrt{n}} \phi(Q^{-1}(\epsilon)) - 6 \frac{m_3}{\sigma_n^3(y^n_1) \sqrt{n}}
\]

\[
= \epsilon + \frac{1}{\sqrt{n}},
\]

where (30) follows from the Berry-Esséen bound in Theorem 6.3, (31) follows from the fact that,

\[
Q(\alpha) - Q(\alpha - \Delta) \leq \Delta \phi(\alpha), \quad \text{for } 0 < \alpha < \frac{\Delta}{2},
\]

and (32) follows from the definition of \( \eta(y^n_1) \). Putting \( \alpha = Q^{-1}(\epsilon) \) and \( \Delta = \Delta(y^n_1) = \frac{\eta(y^n_1)}{\sqrt{n}} \), (28) is sufficient for (33) to hold.

Since we condition on the fixed side information sequence \( y^n_1 \), [25, Theorem 4] applies, with \( \tau = \frac{1}{2} \log n \), where we replace \( X \) by \( X^n_1 \) with PMF \( P_{X^n_1 | Y^n_1} (\cdot | y^n_1) \). Thus, putting, \( K_n = \sum_{i=1}^{n} H(X | y_i) + \sigma_n(y^n_1) \sqrt{n} Q^{-1}(\epsilon) - \eta(y^n_1) \), yields,

\[
P \left[ \ell^*(f^n_1 | X^n_1 | y^n_1) \right] \geq K_n - \frac{\log n}{2} \right] \geq P \left[ - \log P(X^n_1 | y^n_1) \geq K_n | Y^n_1 = y^n_1 \right] - \frac{1}{\sqrt{n}} \geq \epsilon,
\]

and the claimed bound follows.

Next we derive an upper bound to \( R^*(n, \epsilon | y^n_1) \) that matches the lower bound in Theorem 6.4 up to and including the third-order term. Note that, in view of Theorem 2.5, the result of Theorem 6.5 also holds for prefix-free codes, with \( R_p^*(n, \epsilon | y^n_1) \) and \( \zeta_n(y^n_1) + 1 \) in place of \( R^*(n, \epsilon | y^n_1) \) and \( \zeta_n(y^n_1) \), respectively.

**Theorem 6.5 (Achievability for \( R^*(n, \epsilon | y^n_1) \))** Let \((X, Y)\) be a conditionally-i.i.d. source-side information pair. For any \( 0 < \epsilon \leq \frac{1}{2} \), the reference-based optimal compression rate satisfies,

\[
R^*(n, \epsilon | y^n_1) \leq H_n(X | y^n_1) + \frac{\sigma_n^2(y^n_1)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} + \frac{1}{n} \zeta_n(y^n_1),
\]

for all,

\[
n > \frac{36m_3^2}{e^2 \sigma_n^4(y^n_1)},
\]

and any side information string \( y^n_1 \in Y^n \) such that \( \sigma_n^2(y^n_1) > 0 \), where \( H_n(X | y^n_1) \) and \( \sigma_n^2(y^n_1) \) are given in (24) and (25), \( m_3 \) is given in (29), and,

\[
\zeta_n(y^n_1) = \frac{6m_3}{\sigma_n^3(y^n_1) \phi \left( \Phi^{-1} \left( \frac{Q^{-1}(\epsilon)}{\sqrt{n} \sigma_n(y^n_1)} \right) + \frac{6m_3}{\sigma_n(y^n_1)} \right) + \log \left( \frac{\log \epsilon}{2 \sqrt{2 \pi} \sigma_n^2(y^n_1)} + \frac{12m_3}{\sigma_n^3(y^n_1)} \right)}.
\]
Proof. Given \( \epsilon \) and \( y^n_1 \), let \( \beta_n = \beta_n(y^n_1) \) be the unique constant such that,

\[
\begin{align*}
P\left[-\log P(X^n_1|y^n_1) \leq \log \beta_n|Y^n_1 = y^n_1\right] &\leq 1 - \epsilon, \\
P\left[-\log P(X^n_1|y^n_1) < \log \beta_n|Y^n_1 = y^n_1\right] &< 1 - \epsilon,
\end{align*}
\]
and write \( \lambda_n \) for its normalized version,

\[
\lambda_n = \frac{\log \beta_n - \sum_{i=1}^{n} H(X|y_i)}{\sqrt{n}}.
\]

Using the Berry-Esséen bound in Theorem 6.3, yields,

\[
1 - \epsilon \leq P\left[ \frac{-\log P(X^n_1|y^n_1) - \sum_{i=1}^{n} H(X|y_i)}{\sigma_n(y^n_1)\sqrt{n}} \leq \lambda_n \bigg| Y^n_1 = y^n_1 \right] \leq \Phi(\lambda_n) + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}},
\]
and,

\[
1 - \epsilon > P\left[ \frac{-\log P(X^n_1|y^n_1) - \sum_{i=1}^{n} H(X|y_i)}{\sigma_n(y^n_1)\sqrt{n}} < \lambda_n \bigg| Y^n_1 = y^n_1 \right] \geq \Phi(\lambda_n) - \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}}.
\]

Define,

\[
\lambda = \Phi^{-1}(1 - \epsilon) = Q^{-1}(\epsilon).
\]

For \( n \) satisfying (34), we have,

\[
\Phi(\lambda) + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}} < 1,
\]
so, using (37) and a first-order Taylor expansion, we obtain,

\[
\begin{align*}
\lambda_n &\leq \Phi^{-1}\left(\Phi(\lambda) + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}}\right) \\
&= \lambda + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}}(\Phi^{-1})'(\xi_n) \\
&= \lambda + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}}(\Phi^{-1}(\xi_n)),
\end{align*}
\]
for some \( \xi_n = \xi_n(y^n_1) \) between \( \Phi(\lambda) \) and \( \Phi(\lambda) + 6m_3/\sigma_n^3(y^n_1)\sqrt{n} \). Since \( \epsilon \leq \frac{1}{2} \), we have \( \lambda \geq 0 \) and \( \Phi(\lambda) \geq \frac{1}{2} \), so that \( \xi_n \geq \frac{1}{2} \). Also, since \( \Phi^{-1}(t) \) is strictly increasing for all \( t \) and \( \phi \) is strictly decreasing for \( t \geq 0 \), from (38) we get,

\[
\lambda_n \leq \lambda + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}} \times \frac{1}{\phi(\Phi^{-1}(\Phi(\lambda) + \frac{6m_3}{\sigma_n^3(y^n_1)\sqrt{n}}))}.
\]

On the other hand, from the discussion in the proof of Theorem 3.4, together with (36), we conclude that,

\[
P \left[ \ell(f^n_{\pi}(X^n_1|y^n_1)) > \log \left( \sum_{x^n_1 \in \mathcal{X}^n} I\{P(x^n_1|y^n_1) \geq \frac{1}{\sqrt{n}}\} \right) \bigg| Y^n_1 = y^n_1 \right] \leq \epsilon,
\]

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Note that where we used Theorem 6.3 twice. Hence,

\[
R^*(n, \epsilon | y^n_1) \leq \frac{1}{n} \log \left( \sum_{x^n_1 \in X^n} \mathbb{1}_{\{ P(x^n_1 | y^n_1) \geq \frac{1}{2m^2} \}} \right)
\]

\[
= \frac{1}{n} \log \left( \mathbb{E} \left[ 2^{-\log P(X^n_1 | y^n_1) \mathbb{1}_{\{ \log P(X^n_1 | y^n_1) \leq \log \beta_n \}} \left| Y^n_1 = y^n_1 \right. \right] \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} H(X_i | y_i) + \lambda_n \frac{\sigma_n(y^n_1)}{\sqrt{n}} + \frac{1}{n} \log \alpha_n, \quad (40)
\]

where,

\[
\alpha_n = \mathbb{E} \left[ 2^{\log \beta_n + \log P(X^n_1 | y^n_1) \mathbb{1}_{\{ \log \beta_n + \log P(X^n_1 | y^n_1) \geq 0 \}} \left| Y^n_1 = y^n_1 \right. \right]
\]

\[
= \mathbb{E} \left[ 2^{\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - Z_n) \mathbb{1}_{\{ \sigma_n(y^n_1) \sqrt{\pi} (\lambda_n - Z_n) \geq 0 \}} \left| Y^n_1 = y^n_1 \right. \right],
\]

and,

\[
Z_n = \frac{1}{\sigma_n(y^n_1) \sqrt{n}} \left[ - \log P(X^n_1 | y^n_1) - \sum_{i=1}^{n} H(X_i | y_i) \right]
\]

\[
= \frac{1}{\sigma_n(y^n_1) \sqrt{n}} \sum_{i=1}^{n} \left[ - \log P(X_i | y_i) - H(X_i | y_i) \right].
\]

Note that \( Z_n \) has zero mean and unit variance. Let,

\[
\tilde{\alpha}_n = \mathbb{E} \left[ 2^{\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - Z) \mathbb{1}_{\{ \sigma_n(y^n_1) \sqrt{\pi} (\lambda_n - Z) \geq 0 \}} \right],
\]

where \( Z \) is a standard normal random variable. Then,

\[
\tilde{\alpha}_n = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi n \sigma^2_n(y^n_1)}} 2^{-x} \exp \left\{ - \frac{(x - \lambda_n \sigma_n(y^n_1) \sqrt{\pi})^2}{2n\sigma^2_n(y^n_1)} \right\} dx \leq \frac{\log e}{\sqrt{2\pi n \sigma^2_n(y^n_1)}}.
\]

Denoting by \( F_n(t) \) the distribution function of \( Z_n \), and integrating by parts,

\[
\alpha_n = \int_{-\infty}^{\lambda_n} 2^{-\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - t) F_n(t) dt
\]

\[
= F_n(\lambda_n) - (\log e) \int_{-\infty}^{\lambda_n} F_n(t) \sigma_n(y^n_1) \sqrt{n} 2^{-\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - t) dt
\]

\[
= \tilde{\alpha}_n + F_n(\lambda_n) - (\log e) \sigma_n(y^n_1) \sqrt{n} \int_{-\infty}^{\lambda_n} (F_n(t) - \Phi(t)) 2^{-\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - t) dt
\]

\[
\leq \tilde{\alpha}_n + \frac{6m_3}{\sqrt{n} \sigma^3_n(y^n_1)} + \frac{6m_3}{\sqrt{n} \sigma^2_n(y^n_1)} (\log e) \int_{-\infty}^{\lambda_n} 2^{-\sigma_n(y^n_1)} \sqrt{\pi} (\lambda_n - t) dt
\]

\[
\leq \tilde{\alpha}_n + \frac{12m_3}{\sqrt{n} \sigma^2_n(y^n_1)} + \frac{12m_3}{\sigma^3_n(y^n_1)}, \quad (41)
\]

where we used Theorem 6.3 twice.

The claimed bound follows from (39), (40), and (41).
6.3 Concentration for the reference-based optimal rate

Here we derive a finite-$n$ deviation bound, showing that the probability that the rate $R^*(n, \epsilon|Y^n)$ with respect to a random side information string, will exceed $H(X|Y)$, is exponentially small. By Theorem 2.5, the same result holds in the prefix-free case for $R^*_p(n, \epsilon|Y^n)$, with $\delta + 1/n$ in place of $\delta$.

**Theorem 6.6 (Concentration for $R^*(n, \epsilon|Y^n)$)** Suppose $(X, Y)$ is an i.i.d. source-side information pair. Then, for any $0 < \epsilon \leq 1/2$, and any $\delta > 0$, we have,

$$P[R^*(n, \epsilon|Y^n) \geq H(X|Y) + \delta] \leq \exp\left\{-\frac{n}{32}\left[\min\left\{\frac{\bar{v}}{v^*}, \frac{\delta}{\log |X|}\right\}^2 - \frac{32\log e}{n}\right]\right\},$$

where $\bar{v} = \mathbb{E}[V(Y)]$ and $v^* = \max_y V(y)$, with $V$ defined in (23), for all,

$$n > \max\left\{4\left[\frac{6m_3}{\delta\left(\bar{v}/2\right)^3/\log \phi} + \log \left(\frac{\log e}{\sqrt{\pi \bar{v}}} + 12\right)\right], \frac{16v^*(Q^{-1}(\epsilon))^2}{\delta^2}, \frac{36m_3^2}{(\bar{v}/2)^3}, \frac{288m_3^2}{e^2\bar{v}^3}\right\},$$

(42)

where $m_3$ is given in (29).

**Proof.** For all $n \geq 1$ let,

$$V_n = \sum_{i=1}^{n} [\bar{V}(Y_i) + \bar{v}] = n\bar{v} - n\sigma_n^2(Y^n),$$

and,

$$W_n = \sum_{i=1}^{n} [\hat{H}_X(Y_i) - H(X|Y)].$$

Both $V_n$ and $W_n$ are sums of i.i.d., zero mean random variables, that are bounded with probability 1 as,

$$|\bar{V}(Y_i) + \bar{v}| \leq 2v^*,$$

and,

$$|\hat{H}_X(Y_i) - H(X|Y)| \leq 2\log |X|.$$ 

Therefore, by the Azuma-Hoeffding inequality [28],

$$P[V_n \geq n\bar{v}/2] \leq e^{-n\bar{v}^2/(32v^*^2)},$$

(43)

and,

$$P[W_n \geq n\delta/2] \leq e^{-n\delta^2/(32\log^2 |X|)}.$$ 

(44)
Using Theorem 6.5, we can bound,
\[
P[R^*(n, \epsilon|Y^n) \geq H(X|Y) + \delta] \leq P[R^*(n, \epsilon|Y^n) \geq H(X|Y) + \delta; \sigma_n^2(Y^n) \geq \bar{v}/2] + P[\sigma_n^2(Y^n) < \bar{v}/2] \\
\leq P \left[ \frac{1}{n} \sum_{i=1}^{n} \hat{H}_X(Y_i) + \frac{\sigma_n(Y^n)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} + \frac{\zeta_n(Y^n)}{n} \geq H(X|Y) + \delta; \sigma_n^2(Y^n) \geq \bar{v}/2 \right] \\
+ P[V_n > n\bar{v}/2].
\]

From (35) we see that the random variable \( \zeta_n(Y^n) \) is bounded on the event \( \{\sigma_n^2(Y^n) \geq \bar{v}/2\} \), and after some tedious computations we find that, for \( n \) satisfying (42),
\[
P[R^*(n, \epsilon|Y^n) \geq H(X|Y) + \delta] \leq P \left[ \frac{1}{n} \sum_{i=1}^{n} \hat{H}_X(Y_i) \geq H(X|Y) + \frac{\delta}{2} \right] + P \left[ V_n \geq n\bar{v}/2 \right] \\
= P \left[ W_n \geq n\delta/2 \right] + P \left[ V_n \geq n\bar{v}/2 \right] \\
\leq \exp \left\{ -n \left[ \min \left\{ \frac{\bar{v}^2}{32\delta^2}, \frac{\delta^2}{32 \log^2 |X|} \right\} - \frac{\log_e 2}{n} \right] \right\},
\]
where the last inequality follows by bounding the sum of the two probabilities by twice the maximum of the bounds in (43) and (44). \qed
7 Normal Approximation for Pair-Based Compression

Here we give upper and lower bounds to the pair-based optimal compression rate $R^*(n, \epsilon)$, analogous to those presented in Theorems 6.4 and 6.5 for the reference-based optimal rate: For an i.i.d. source-side information pair $(X, Y)$, the result in Theorems 7.1 and 7.2 state that,

$$R^*(n, \epsilon) = H(X|Y) + \frac{\sigma(X|Y)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} + O\left(\frac{1}{n}\right),$$

(45)

with explicit upper and lower bounds for the $O(1/n)$ term.

Recall the discussion in the Introduction comparing (45) with the corresponding expansion (26) in the reference-based case. In particular, we note that, for large $n$, we typically have $H_n(X|y^n_1) \approx H(X|Y)$, but $\sigma^2_n(y^n_1) < \sigma^2(X|Y)$.

**Theorem 7.1 (Achievability for $R^*(n, \epsilon)$)** Let $(X, Y)$ be an i.i.d. source-side information pair, with conditional varentropy rate $\sigma^2 = \sigma^2(X|Y) > 0$. For any $0 < \epsilon \leq \frac{1}{2}$, the pair-based optimal compression rate satisfies,

$$R^*(n, \epsilon) \leq H(X|Y) + \frac{\sigma(X|Y)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} + \frac{C}{n},$$

(46)

for all

$$n > \frac{4\sigma^2}{B^2 \phi(Q^{-1}(\epsilon))^2} \times \left[ \frac{B^2}{2\sqrt{2\pi e \sigma^2}} + \frac{\psi^2}{(1 - \frac{1}{2\pi})^2 \bar{v}^2} \right]^2,$$

(47)

where $\bar{v} = \mathbb{E}[V(Y)]$ and $\psi^2 = \text{Var}(V(Y))$, with $V$ defined in (23),

$$C = \log \left( \frac{2}{\bar{v}^{3/2}} + \frac{24m_3(2\pi)^{3/2}}{\bar{v}^{3/2}} \right) + B,$$

$m_3$ is given in (29), and,

$$B = \mathbb{E}\left[ \frac{[- \log P(X|Y) - H(X|Y)]^3}{\sigma^2(\phi(Q^{-1}(\epsilon))} \right].$$

As in Section 6, we note that, in view of Theorem 2.5, the result of Theorem 7.1 remains true for prefix-free codes, with $R^*_p(n, \epsilon)$ and $C + 1$ in place $R^*(n, \epsilon)$ and $C$, respectively.

**Proof.** We will use the achievability part of Theorem 3.4. For each $i$, take $X_i, \bar{X}_i$ to be conditionally independent versions of $X$ given $Y = Y_i$, and define,

$$S_n = \frac{1}{\sigma_n(Y^n_1)\sqrt{n}} \sum_{i=1}^{n} [- \log P(X_i|Y_i) - \hat{H}_X(Y_i)],$$

$$\bar{S}_n = \frac{1}{\sigma_n(Y^n_1)\sqrt{n}} \sum_{i=1}^{n} [- \log P(\bar{X}_i|Y_i) - \hat{H}_X(Y_i)],$$

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\hat{H}_X(Y_i) - H(X|Y)],$$

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where \( \sigma_n^2(Y^n) \), \( \hat{H}_X(Y) \) are defined in (25) and (27), respectively. For any \( K > 0 \) the upper bound in Theorem 3.4 gives,

\[
\mathbb{P} \left[ \ell(f_n^*(X_1^n | Y_1^n)) > K \right] 
\leq \mathbb{P} \left[ \mathbb{E} \left( \frac{1}{P(X_1^n | Y_1^n)} I_{\{P(X_1^n | Y_1^n) \geq P(X_1^n | Y_1^n)\}} X_1^n, Y_1^n \right) > 2K \right]
\leq \mathbb{P} \left[ \mathbb{E} \left( 2^{\sqrt{n}\sigma_n(Y_1^n \hat{S}_n \{\hat{S}_n \leq S_n\})} | X_1^n, Y_1^n \right) > 2K - \sum_{i=1}^{\infty} H(X_i) \right],
\]

and taking \( K = K_n = nH(X | Y) + \sigma(X | Y) \sqrt{n}Q^{-1}(\epsilon) - \log \sqrt{n} + C \),

\[
\mathbb{P} \left[ \ell(f_n^*(X_1^n | Y_1^n)) > K_n \right] 
\leq \mathbb{P} \left[ \mathbb{E} \left( 2^{\sqrt{n}\sigma_n(Y_1^n \hat{S}_n - Q^{-1}(\epsilon)) \{\hat{S}_n \leq S_n\}} | X_1^n, Y_1^n \right) > \frac{1}{\sqrt{n}} 2^{-\sqrt{n}} \left[ T_n - (\sigma - \sigma_n(Y_1^n))Q^{-1}(\epsilon) \right] + C \right].
\] (48)

For the conditional expectation, writing \( \sigma_n \) for \( \sigma_n(Y_1^n) \) for clarity, we have,

\[
\mathbb{E} \left( 2^{\sqrt{n}\sigma_n(\hat{S}_n - Q^{-1}(\epsilon)) \{\hat{S}_n \leq S_n\}} | X_1^n, Y_1^n \right)
= \sum_{k=0}^{\infty} \mathbb{E} \left( 2^{\sqrt{n}\sigma_n(\sqrt{n\sigma_n}S_n - k < \sqrt{n\sigma_n} \hat{S}_n \leq \sqrt{n\sigma_n}S_n - k)} | X_1^n, Y_1^n \right)
\leq 2^{\sqrt{n}\sigma_n(S_n - Q^{-1}(\epsilon))} \sum_{k=0}^{\infty} 2^{-k} \mathbb{P} \left[ \sqrt{n\sigma_n}S_n - k < \sqrt{n\sigma_n}S_n \leq \sqrt{n\sigma_n}S_n - k | X_1^n, Y_1^n \right].
\] (49)

Conditional on \( (X_1^n, Y_1^n) \), the only randomness in the probabilities in (49) and (50) is in \( S_n \) via \( \hat{S}_n \), and we can apply the Berry-Esséen bound of Theorem 6.3 twice to bound their difference, resulting in,

\[
2^{\sqrt{n}\sigma_n(S_n - Q^{-1}(\epsilon))} \sum_{k=0}^{\infty} 2^{-k} \left\{ \mathbb{P} \left[ \hat{S}_n \leq S_n - \frac{k}{\sqrt{n\sigma_n}} | X_1^n, Y_1^n \right] - \mathbb{P} \left[ \hat{S}_n \leq S_n - \frac{(k + 1)}{\sqrt{n\sigma_n}} | X_1^n, Y_1^n \right] \right\}.
\] (50)

Hence, combining all the estimates in (48)–(51),

\[
\mathbb{P} \left[ \ell(f_n^*(X_1^n | Y_1^n)) > K_n \right] \leq \mathbb{P} \left[ 2^{\sqrt{n}\sigma(U_n - Q^{-1}(\epsilon))} \left( \frac{2}{\sqrt{2\pi\sigma_n}} + \frac{24m_3}{\sigma_n^3} \right) > 2^C \right],
\]

where we have defined,

\[
U_n = \frac{1}{\sigma} \mathbb{E} \left[ \sigma_n S_n + T_n \right] = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} \left[ - \log P(X_i | Y_i) - H(X | Y) \right].
\]

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And we can further bound,
\[
P[\ell(f^*_n(X^n|Y^n)) > K_n] \leq P[2^{\sigma^2/(2\sqrt{n}(Q-1)(\epsilon))} \left(\frac{2}{\sqrt{\pi}} + \frac{24(2\pi)^{3/2}/m_3}{\sigma^2} \right) > 2^C] + P[\sigma^2(Y^n_i) < \frac{\bar{v}}{2\pi}] . \tag{52}
\]

For the first probability in (52) we have,
\[
P\left[2^{\sigma^2/(2\sqrt{n}(Q-1)(\epsilon))} 2^C > 2^C \right] = P\left[U_n > Q^{-1}(\epsilon) + \frac{B}{\sigma\sqrt{n}}\right] \leq Q\left(Q^{-1}(\epsilon) + \frac{B}{\sigma\sqrt{n}}\right) + \mathbb{E}[-\log P(X|Y) - H(X|Y)]^3, \tag{53}
\]
where we used the Berry-Esséen bound in Theorem 6.2 for the normalized partial sum \(U_n\) of the i.i.d. random variables \{-\log P(X_i|Y_i)\} with mean \(H(X|Y)\) and variance \(\sigma^2\). And a second-order Taylor expansion of \(Q\), using the fact that, \(0 \leq Q''(x) = x\phi(x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{e}\), for all \(x \geq 0\), gives,
\[
P\left[2^{\sigma^2/(2\sqrt{n}(Q-1)(\epsilon))} 2^C > 2^C \right] \leq \epsilon - B\phi(Q^{-1}(\epsilon)) + \frac{B^2}{2\sqrt{2\pi}} + \frac{\mathbb{E}[-\log P(X|Y) - H(X|Y)]^3}{2\sigma^2\sqrt{n}}. \tag{53}
\]
For the second probability in (52), a simple application of Chebyshev’s inequality gives,
\[
P\left[\frac{1}{n} \sum_{i=1}^{n} V(Y_i) < \frac{\bar{v}}{2\pi}\right] \leq \frac{\psi^2}{n(1 - \frac{1}{2\pi})^2\bar{v}^2} . \tag{54}
\]
After substituting the bounds (53) and (54) in (52), simple algebra shows that, for all \(n\) satisfying (47), the probability is \(\leq \epsilon\), completing the proof. \(\square\)

Next we prove a corresponding converse bound. Once again we observe that, by the definitions in Section 2, Theorem 7.2 also holds for \(R^*_p(n, \epsilon)\) in the case of prefix-free codes.

**Theorem 7.2 (Converse for \(R^*(n, \epsilon)\))** Let \((X, Y)\) be an i.i.d. source-side information pair, with conditional varentropy rate \(\sigma^2 = \sigma^2(X|Y) > 0\). For any \(0 < \epsilon < \frac{1}{2}\), the pair-based optimal compression rate satisfies,
\[
R^*(n, \epsilon) \geq H(X|Y) + \frac{\sigma(X|Y)}{\sqrt{n}}Q^{-1}(\epsilon) - \frac{\log n}{2n} - \frac{C}{n},
\]
for all,
\[
n > \frac{C^2}{4(Q^{-1}(\epsilon))^2\sigma^2}, \tag{55}
\]
where,
\[
C = \mathbb{E}[| - \log P(X|Y) - H(X|Y)]^3 + 2\sigma^3 \frac{2\sigma^2\phi(Q^{-1}(\epsilon))}{2\sigma^2\phi(Q^{-1}(\epsilon))}.
\]
Proof. Using the Berry-Essén bound in Theorem 6.3, we have,
\[
P \left[ - \log P(X^n_1 | Y^n_1) \geq nH(X | Y) + \sqrt{n}\sigma Q^{-1}(\epsilon) - C \right]
\]
\[
= P \left[ \frac{1}{\sigma \sqrt{n}} \sum_{i=1}^{n} \left[ - \log P(X_i | Y_i) - H(X | Y) \right] \geq Q^{-1}(\epsilon) - \frac{C}{\sigma \sqrt{n}} \right]
\]
\[
\geq Q \left( Q^{-1}(\epsilon) - \frac{C}{\sigma \sqrt{n}} \right) - \frac{\mathbb{E} \left[ - \log P(X | Y) - H(X | Y) \right]^3}{2\sigma^3 \sqrt{n}},
\]
and using the simple earlier bound (33), noting that (55) implies that the condition in (33) is satisfied,
\[
P \left[ - \log P(X^n_1 | Y^n_1) \geq nH(X | Y) + \sqrt{n}\sigma Q^{-1}(\epsilon) - C \right]
\]
\[
\geq \epsilon + \frac{C}{\sigma \sqrt{n}} \phi(Q^{-1}(\epsilon)) - \frac{\mathbb{E} \left[ - \log P(X | Y) - H(X | Y) \right]^3}{2\sigma^3 \sqrt{n}}
\]
\[
= \epsilon + \frac{1}{\sqrt{n}}
\]
Now applying the general converse result in Theorem 3.2 with \( \tau = \tau_n = \frac{1}{2} \log n \) and \( X^n_1 \) in place of \( X \),
\[
P \left[ \ell(f^n_*(X^n_1 | Y^n_1)) \geq nH(X | Y) + \sqrt{n}\sigma Q^{-1}(\epsilon) - C - \frac{\log n}{2} \right]
\]
\[
\geq P \left[ - \log P(X^n_1 | Y^n_1) \geq nH(X | Y) + \sqrt{n}\sigma Q^{-1}(\epsilon) - C - \frac{1}{\sqrt{n}} \right] \geq \epsilon,
\]
and the claimed bound follows. \( \square \)
8 Normal Approximation for Markov Sources

In this section we consider extensions of the normal approximation bounds for the optimal rate in Sections 6 and 7, to the case of Markov sources. Note that the results of Section 6 for the reference-based optimal rate \( R^*(n, \epsilon) \) apply not only to the case of i.i.d. source-side information pairs \((X, Y)\), but much more generally to arbitrary side-information sources \(Y\) as long as \(X\) is conditionally i.i.d. given \(Y\). This is a broad class including, among others, all hidden Markov models \(X\). For this reason, we restrict our attention here to the pair-based optimal rate \( R^*(n, \epsilon) \).

As discussed in the context of compression without side information [25], the Berry-Esséen bound for Markov chains is not known to hold at the same level of generality as in the i.i.d. case. In fact, even for restricted class of reversible chains where an explicit Berry-Esséen bound is known [27], it involves constants that are larger than those in the i.i.d. case by more than four orders of magnitude, making any resulting bounds significantly less relevant in practice.

Therefore, in the Markov case we employ a general result of Nagaev [29] that does not lead to explicit values for the relevant constants, but which applies to all ergodic Markov chains. For similar reasons, rather than attempting to generalize the rather involved proof of the achievability result in Theorem 7.1, we choose to illustrate a much simpler argument that leads to a weaker bound, not containing the third-order \((\log n)/2n\) term as in (46).

Throughout this section we consider a source-side information pair \((X, Y)\) which is an ergodic, \(k\)th order Markov chain, with conditional entropy rate \(H = H(X|Y)\) and conditional varentropy rate \(\sigma^2 = \sigma^2(X|Y) > 0\) as in Lemma 4.6, and we also assume that the side information process \(Y\) itself is an ergodic, \(k\)th order Markov chain. The main probabilistic tool we will need in the proof of Theorem 8.2 is the following normal approximation bound for the conditional information density \(-\log P(X^n_1|Y^n_1)\); it is proved in Appendix D.

**Theorem 8.1 (Berry-Esséen bound for the conditional information density)** There exists a finite constant \(A > 0\) such that, for all \(n \geq 1\):

\[
\sup_{z \in \mathbb{R}} \left| P\left[ \frac{-\log P(X^n_1|Y^n_1) - nH}{\sigma \sqrt{n}} > z \right] - Q(z) \right| \leq \frac{A}{\sqrt{n}}.
\]

As with the corresponding results for memoryless sources, Theorems 7.1 and 7.2 in Section 7, we observe that both (56) and (57) in our next result, Theorem 8.2 below, remain valid for \(R^p(n, \epsilon)\) in the case of prefix-free codes.

**Theorem 8.2 (Normal approximation for Markov sources)** For any \(\epsilon \in (0, 1/2)\), there are finite constants \(C, C'\) and integers \(N, N'\) such that,

\[
R^*(n, \epsilon) \leq H(X|Y) + \frac{\sigma(X|Y)}{\sqrt{n}} Q^{-1}(\epsilon) + \frac{C}{n}, \quad \text{for all } n \geq N,
\]

and,

\[
R^*(n, \epsilon) \geq H(X|Y) + \frac{\sigma(X|Y)}{\sqrt{n}} Q^{-1}(\epsilon) - \frac{\log n}{2n} - \frac{C'}{n}, \quad \text{for all } n \geq N'.
\]
Note that the only reason we do not give explicit values for the constants $C, C', N, N'$ in (56) and (57) is because of the unspecified constant in the Berry-Esséen bound. In fact, for any class of Markov chains for which the constant $A$ in Theorem 8.1 is known explicitly, we can take,

$$C = \frac{2A\sigma}{\phi(Q^{-1}(\epsilon))}, \quad C' = \frac{\sigma(A+1)}{\phi(Q^{-1}(\epsilon))},$$

and,

$$N = \frac{2A^2}{\pi e(\phi(Q^{-1}(\epsilon)))^4}, \quad N' = \left(\frac{A+1}{Q^{-1}(\epsilon)\phi(Q^{-1}(\epsilon))}\right)^2.$$

**Proof.** Let $A$ be the constant of Theorem 8.1.

Taking $C = 2A\sigma/\phi(Q^{-1}(\epsilon))$ and $K_n = nH + \sigma\sqrt{n}Q^{-1}(\epsilon) + C$, the general achievability bound in Theorem 3.1 gives,

$$\Pr[\ell(f_n^*(X^n|Y^n)) \geq K_n] \leq \Pr[-\log P(X^n|Y^n) \geq K_n] \leq Q\left(Q^{-1}(\epsilon) + \frac{C}{\sigma\sqrt{n}}\right) + \frac{A}{\sqrt{n}},$$

where the second inequality follows from Theorem 8.1. Since $Q''(x) \leq \frac{1}{\sqrt{2\pi e}}, x \geq 0$, a second-order Taylor expansion for $Q$ yields,

$$\Pr[\ell(f_n^*(X^n|Y^n)) \geq K_n] \leq \epsilon - \frac{C}{\sigma\sqrt{n}} \left(\phi(Q^{-1}(\epsilon)) - \frac{C}{2\sigma\sqrt{2\pi e}} - \frac{A\sigma}{C}\right) \leq \epsilon,$$

where the last inequality holds for all,

$$n \geq \frac{2A^2}{\pi e(\phi(Q^{-1}(\epsilon)))^4}.$$

This proves (56).

For the converse, taking $C' = \sigma(A+1)/\phi(Q^{-1}(\epsilon))$ and $K_n = nH + \sigma\sqrt{n}Q^{-1}(\epsilon) - (\log n)/2 - C'$, and $\tau = (\log n)/2$, the general converse bound in Theorem 3.2 gives,

$$\Pr[\ell(f_n^*(X^n|Y^n)) \geq K_n] \geq \Pr\left[\frac{-\log P(X^n|Y^n) - nH}{\sigma\sqrt{n}} \geq Q^{-1}(\epsilon) - \frac{C'}{\sqrt{n}}\right] - \frac{1}{\sqrt{n}} \geq Q\left(Q^{-1}(\epsilon) - \frac{A+1}{\phi(Q^{-1}(\epsilon))\sqrt{n}}\right) - \frac{A+1}{\sqrt{n}},$$

where the second bound follows from Theorem 8.1. Finally, using a simple first-order Taylor expansion of $Q$, and noting that $\phi(x)$ is nonincreasing for $x \geq 0$ and that,

$$Q^{-1}(\epsilon) - \frac{A+1}{\phi(Q^{-1}(\epsilon))\sqrt{n}} \geq 0,$$

for $\epsilon \in (0, 1/2)$ and,

$$n \geq \left(\frac{A+1}{Q^{-1}(\epsilon)\phi(Q^{-1}(\epsilon))}\right)^2,$$

yields that $\Pr[\ell(f_n^*(X^n|Y^n)) \geq K_n] > \epsilon$. This gives (57) and completes the proof. \hfill \square
9 Dispersion

9.1 Pair-based dispersion

In analogy with the source dispersion for the problem of lossless compression without side information \([25, 43]\), for an arbitrary source-side information pair \((X, Y)\) we define the pair-based dispersion \(D(X|Y)\) as the limiting variance of the optimal description lengths of \(X\) given side information \(Y\).

**Definition 9.1** The pair-based dispersion \(D(X|Y)\) of a source-side information pair \((X, Y)\) is:

\[
D(X|Y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}[\ell(f_n^*(X^n_1|Y^n))].
\]

The following result shows that the pair-based dispersion coincides with the varentropy rate in the Markov case; Theorem 9.2 it is the natural side information analog of \([25, \text{Theorem 23}]\).

**Theorem 9.2** Suppose that both the pair \((X, Y)\) and \(Y\) itself are irreducible and aperiodic Markov chains, with conditional entropy rate \(H(X|Y)\) and conditional varentropy rate \(\sigma^2(X|Y)\). Then:

\[
D(X|Y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}[\ell(f_n^*(X^n_1|Y^n))] = \sigma^2(X|Y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}(- \log P(X^n_1|Y^n)) < \infty. \tag{58}
\]

If, moreover, \(\sigma^2(X|Y)\) is nonzero, then:

\[
D(X|Y) = \sigma^2(X|Y) = \lim_{n \to \infty} \lim_{\ell \to 0} n \left( \frac{R^*(n, \epsilon) - H(X|Y)}{Q^{-1}(\epsilon)} \right)^2. \tag{59}
\]

**Proof.** Denote, for brevity, \(\ell_n = \ell(f_n^*(X^n_1|Y^n)), \quad \tau_n = - \log P(X^n_1|Y^n)\) and \(H_n = H(X^n_1|Y^n)\). For (58), in view of Lemma 4.6, it suffices to show that:

\[
\lim_{n \to \infty} \frac{1}{n} \left| \text{Var}(\ell_n) - \text{Var}(\tau_n) \right| = 0. \tag{60}
\]

Proceeding as in the proof of \([25, \text{Theorem 22}]\), noting that \(H_n = E(\tau_n)\), we have,

\[
\text{Var}(\ell_n) = E \left[ (\ell_n - \tau_n + (\tau_n - H_n) - E(\ell_n - \tau_n))^2 \right] = E((\ell_n - \tau_n)^2) + E((\tau_n - H_n)^2) - (E(\ell_n - \tau_n))^2 + 2E((\ell_n - \tau_n)(\tau_n - H_n)).
\]

Thus, by the Cauchy-Schwarz inequality,

\[
\left| \text{Var}(\ell_n) - \text{Var}(\tau_n) \right| = \left| E((\ell_n - \tau_n)^2) - E((\ell_n - \tau_n)^2) + 2E((\ell_n - \tau_n)(\tau_n - H_n)) \right| \\
\leq 2E((\ell_n - \tau_n)^2) + 2\sqrt{E((\ell_n - \tau_n)^2)} \sqrt{\text{Var}(\tau_n)}. \tag{61}
\]

Note that, since \((X, Y)\) is an ergodic Markov chain, there exists a finite constant \(C > 0\) such that, with probability 1, \(\tau_n \leq Cn\). Thus, for an arbitrary \(\tau_n > 0\) to be specified later, and recalling that \(\ell_n \leq \tau_n\) by Theorem 3.1,

\[
E((\ell_n - \tau_n)^2) = E((\ell_n - \tau_n)^2)1_{\{\ell_n \geq \tau_n - \tau_n\}} + E((\ell_n - \tau_n)^2)1_{\{\ell_n < \tau_n - \tau_n\}} \\
\leq \tau_n^2 + C^2n^2P[\ell_n < \tau_n - \tau_n].
\]

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and using Theorem 3.3,
\[ E(\ell_n - \iota_n)^2 \leq \tau_n^2 + C^2 n^2 2^{-\tau_n} ([\log |X|^n] + 1) \leq \tau_n^2 + C' n^3 2^{-\tau_n}. \]

Substituting this into (61), and choosing \( \tau_n = 3 \log n \), yields,
\[ \frac{1}{n} |\text{Var}(\ell_n) - \text{Var}(\iota_n)| \leq \frac{2[9(\log n)^2 + C']}{n} + 2 \left[ \frac{9(\log n)^2 + C'}{n} \right]^{1/2} \left[ \frac{\text{Var}(\log P(X_n^n | Y_n^n))}{n} \right]^{1/2}, \]

which, using Lemma 4.6, tends to zero as \( n \to \infty \), proving (60) as required.

The second part of the theorem, equation (59), is immediate from the upper and lower bounds in Theorem 8.2.

\[ \square \]

9.2 Reference-based dispersion

Finally, in the reference-based case, we define:

Definition 9.3 The reference-based dispersion \( D(X | y) \) of a source \( X \) with respect to a side information string \( y = y_1^\infty \) is:
\[ D(X | y) = \limsup_{n \to \infty} \frac{1}{n} \text{Var}[\ell(f_n^*(X_n^1 | y_n^1))]. \]

Our last result relates the reference-based dispersion to the “individual sequence varentropy” \( \sigma_n^2(y_n^1) \) defined in (25). Recall also the definition of the empirical conditional rate \( H_n(X | y_1^n) \) in (24) and of the conditional variance function \( V(y) \) in (23).

Theorem 9.4 Suppose the side information process \( Y \) is stationary and ergodic, and that the pair \( (X,Y) \) is conditionally i.i.d.

(i) With probability 1, as \( n \to \infty \):
\[ H_n(X | Y_1^n) \to H(X_1 | Y_1), \quad \text{and} \quad \sigma_n^2(y_1^n) \to \mathbb{E}[V(Y_1)] < \infty. \]  \hfill (62)

(ii) Let \( y = y_1^\infty \) be one of the (almost all) realizations of \( Y \) such that (62) holds. Then:
\[ D(X | y) = \lim_{n \to \infty} \frac{1}{n} \text{Var}[\ell(f_n^*(X_1^n | y_1^n))] = \lim_{n \to \infty} \sigma_n^2(y_1^n) = \lim_{n \to \infty} \frac{1}{n} \text{Var}(\log P(X_1^n | y_1^n)) < \infty. \]

(iii) Let \( y = y_1^\infty \) be any realization of \( Y \) as in (ii). If \( \mathbb{E}[V(Y_1)] \) is nonzero, then:
\[ D(X | y) = \lim_{n \to \infty} \sigma_n^2(y_1^n) = \lim_{\epsilon \to 0} \lim_{n \to \infty} n \left( \frac{R^*(n,\epsilon | y_1^n) - H_n(X | y_1^n)}{Q^{-1}(\epsilon)} \right)^2. \]

Proof. Part (i) is an immediate consequence of the ergodic theorem. For part (ii), the proof of [25, Theorem 22] applies under our assumptions with no changes. Part (iii) follows from the upper and lower bounds in Theorems 6.4 and 6.5. \[ \square \]
Appendices

A  Invariance Principle for the Conditional Information Density

This Appendix is devoted to the proof of Theorem A.1, which generalizes the corresponding almost sure invariance principle of Philipp and Stout [34, Theorem 9.1] for the (unconditional) information density $-\log P(X_1^n)$.

Theorem A.1 Suppose $(X, Y)$ is a jointly stationary and ergodic process, satisfying the mixing conditions (9)–(11). For $t \geq 0$, let,

$$S(t) = \log P(X_{\lfloor t \rfloor}^t | Y_{\lfloor t \rfloor}^t) + tH(X|Y).$$

Then the following series converges:

$$\sigma^2 = E[\log P(X_0, Y_0 | X_{-\infty}^{-1}, Y_{-\infty}^{-1}) + H(X|Y)]^2$$

$$+ 2 \sum_{k=1}^{\infty} E\left\{ \log P(X_0, Y_0 | X_{-\infty}^{-1}, Y_{-\infty}^{-1}) + H(X|Y) \log P(X_k, Y_k | X_{-\infty}^{k-1}, Y_{-\infty}^{k-1}) + H(X|Y) \right\}. $$

If $\sigma^2 > 0$, then, without changing its distribution, we can redefine the process $\{S(t) ; t \geq 0\}$ on a richer probability space that contains a standard Brownian motion $\{B(t) ; t \geq 0\}$, such that,

$$S(t) - \sigma B(t) = O(t^{1-\lambda}), \quad a.s.,$$

as $t \to \infty$, for each $\lambda < 1/294$.

To simplify the notation, we write $h = H(X|Y)$ and define,

$$f_j = \log \left( \frac{P(X_j, Y_j | X_{-\infty}^{j-1}, Y_{-\infty}^{j-1})}{P(Y_j | Y_{-\infty}^{j-1})} \right), \quad j \geq 0,$$

so that, for example, the variance $\sigma^2$ in the theorem becomes,

$$\sigma^2 = E[(f_0 + h)^2] + 2 \sum_{k=1}^{\infty} E[(f_0 + h)(f_k + h)].$$

Lemma A.2 If $\sum_d \gamma^{(X,Y)}(d) < \infty$ and $\sum_d \gamma^{(Y)}(d) < \infty$ then, as $n \to \infty$:

$$\sum_{k=1}^{n} f_k - \log P(X_1^n | Y_1^n) = O(1), \quad a.s.$$

Proof.  Let,

$$g_j = \log \left( \frac{P(X_j, Y_j | X_{-\infty}^{j-1}, Y_{-\infty}^{j-1})}{P(Y_j | Y_{-\infty}^{j-1})} \right), \quad j \geq 2,$$

and,

$$g_1 = \log \left( \frac{P(X_1, Y_1)}{P(Y_1)} \right) = \log P(X_1|Y_1).$$
We have, for $k \geq 2$,
\[
\mathbb{E}[f_k - g_k] \leq \mathbb{E} \left| \log P(X_k, Y_k | X_{-\infty}^{k-1}, Y_{-\infty}^{k-1}) - \log P(X_k, Y_k | X_1^{k-1}, Y_1^{k-1}) \right|
\]
\[
+ \mathbb{E} \left| \log P(Y_k | X_{-\infty}^{k-1}) - \log P(Y_k | Y_1^{k-1}) \right|
\]
\[
\leq \sum_{x,y} \mathbb{E} \left| \log P(X_k = x, Y_k = y | X_{-\infty}^{k-1}, Y_{-\infty}^{k-1}) - \log P(X_k = x, Y_k = y | X_1^{k-1}, Y_1^{k-1}) \right|
\]
\[
+ \sum_y \mathbb{E} \left| \log P(Y_k = y | Y_{-\infty}^{k-1}) - \log P(Y_k = y | Y_1^{k-1}) \right|
\]
\[
\leq |X||Y| \gamma(X,Y)(k-1) + |Y| \gamma(Y)(k-1).
\]
Therefore, $\sum_{k=1}^{\infty} \mathbb{E}|f_k - g_k| < \infty$, and by the monotone convergence theorem we have,
\[
\sum_{k=1}^{\infty} |f_k - g_k| < \infty, \quad \text{a.s.}
\]
Hence, as $n \to \infty$,
\[
\left| \sum_{k=1}^{n} f_k - \log P(X_1^n | Y_1^n) \right| \leq \sum_{k=1}^{n} |f_k - g_k| = O(1), \quad \text{a.s.}
\]
as claimed. \hfill \square

The following bounds are established in the proof of [34, Theorem 9.1]:

**Lemma A.3** Suppose $Z = \{Z_n : n \in \mathbb{Z}\}$ is a stationary and ergodic process on a finite alphabet, with entropy rate $H(Z)$, and such that $\alpha(Z)(d) = O(d^{-336})$ and $\gamma(Z)(d) = O(d^{-48})$, as $d \to \infty$.

Let $f_k^{(Z)} = \log P(Z_k | Z_{-\infty}^{k-1})$, $k \geq 0$, and put $\eta_n^{(Z)} = f_n^{(Z)} + H(Z)$, $n \geq 0$. Then:

1. For each $r > 0$, $\mathbb{E} \left[ |f_0^{(Z)}| r \right] < \infty$.

2. For each $r \geq 2$ and $\epsilon > 0$,
\[
\mathbb{E} \left[ |f_0^{(Z)} - \log P(Z_0 | Z_{-\infty}^{k-1})| r \right] \leq C(r, \epsilon)(\gamma(Z)(k))^{1/2 - \epsilon},
\]
where $C(r, \epsilon)$ is a constant depending only on $r$ and $\epsilon$.

3. For a constant $C > 0$ independent of $n$, $\|\eta_n^{(Z)}\|_4 \leq C$.

4. Let $\eta_{n,t}^{(Z)} = \mathbb{E} [\eta_n^{(Z)} | F_{n-t}^n]$. Then, as $\ell \to \infty$:
\[
\|\eta_n^{(Z)} - \eta_{n,t}^{(Z)}\|_4 = O(\ell^{-11/2}).
\]

Note that, under the assumptions of Theorem A.1, the conclusions of Lemma A.3 apply to $Y$ as well as to the pair process $(X, Y)$.

**Lemma A.4** For each $r > 0$, we have, $\mathbb{E} [ |f_0| r ] < \infty$. 

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Proof. Simple algebra shows that,

\[ f_0 = f_0(X, Y) - f_0(Y). \]

Therefore, by two applications of Lemma A.3, part 1,

\[ \|f_0\|_r \leq \|f_0(X, Y)\|_r + \|f_0(Y)\|_r < \infty. \]

The next bound follows from Lemma A.3, part 2, upon applying the Minkowski inequality.

Lemma A.5 For each \( r \geq 2 \) and each \( \epsilon > 0 \),

\[ \|f_0 - \log \left( \frac{P(X, Y|X_{k-1}, Y_{k-1})}{P(Y|X_{k-1})} \right) \|_r \leq C_1(r, \epsilon) \left[ \gamma(X, Y)(k) \right]^{1 + \frac{2\epsilon}{2r}} + C_2(r, \epsilon) \left[ \gamma(Y)(k) \right]^{1 + \frac{2\epsilon}{2r}}. \]

Lemma A.6 As \( N \to \infty \):

\[ \mathbb{E} \left\{ \left( \sum_{k \leq N} (f_k + h) \right)^2 \right\} = \sigma^2 N + O(1). \]

Proof. First we examine the definition of the variance \( \sigma^2 \). The first term in (66),

\[ \|f_0 + h\|_2^2 \leq (\|f_0\|_2 + h)^2 < \infty, \]

is finite by Lemma A.4. For the series in (66), let, for \( k \geq 0 \),

\[ \phi_k = \log \left( \frac{P(X_k, Y_k|X_{k-1}, Y_{k-1})}{P(Y_k|X_{k-1})} \right), \]

and write,

\[ \mathbb{E}(f_0 + h)(f_k + h) = \mathbb{E}(f_0 + h)(f_k - \phi_k) + \mathbb{E}(f_0 + h)(\phi_k + h). \quad (67) \]

For the first term in the right-hand side above, we can bound, for any \( \epsilon > 0 \),

\[ |\mathbb{E}(f_0 + h)(f_k - \phi_k)| \leq \|f_0 + h\|_2 \|f_k - \phi_k\|_2 \leq \|[f_0 + h]\|_2 \|f_k - \phi_k\|_2 \leq AC_1(2, \epsilon) \left[ \gamma(X, Y)(|k/2|) \right]^{\frac{1}{4} - \frac{1}{2}\epsilon} + AC_2(2, \epsilon) \left[ \gamma(Y)(|k/2|) \right]^{\frac{1}{4} - \frac{1}{2}\epsilon}, \]

where (a) follows by the Cauchy-Schwarz inequality, and (b) follows by Lemmas A.4, and A.5, with \( A = \|f_0\|_2 + h < \infty \). Therefore, taking \( \epsilon > 0 \) small enough and using the assumptions of Theorem A.1,

\[ |\mathbb{E}(f_0 + h)(f_k - \phi_k)| = O(k^{-12 + 24\epsilon}) = O(k^{-3}), \quad \text{as } k \to \infty. \quad (68) \]
For the second term in (67), we have that, for any \( r > 0, \| \phi_k \|_r < \infty \), uniformly over \( k \geq 1 \) by stationarity. Also, since \( f_0, \phi_k \) are measurable with respect to the \( \sigma \)-algebras generated by \((X_{-\infty}^0, Y_{-\infty}^0)\) and \((X_{k/2}^\infty, Y_{k/2}^\infty)\), respectively, we can apply [34, Lemma 7.2.1] with \( p = r = s = 3 \), to obtain that,

\[
|\mathbb{E}(f_0 + h)(\phi_k + h)| \leq 10\| f_0 + h \|_3\| \phi_k + h \|_3\alpha^3(k/2)^{1/3},
\]

where \( \alpha(k) = \alpha(X,Y)(k) = O(k^{-48}) \), as \( k \to \infty \), by assumption. Therefore, a fortiori,

\[
\mathbb{E}(f_0 + h)(f_k + h) = O(k^{-3}),
\]

and combining this with (68) and substituting in (67), implies that \( \sigma^2 \) in (66) is well defined and finite.

Finally, we have that, as \( N \to \infty \),

\[
\mathbb{E} \left\{ \left[ \sum_{k \leq N} (f_k + h) \right]^2 \right\} = N\mathbb{E}(f_0 + h)^2 + 2\sum_{k=0}^{N-1} (N-k)\mathbb{E}(f_0 + h)(f_k + h)
\]

\[
= N\sigma^2 - 2\sum_{k=1}^{N-1} k\mathbb{E}(f_0 + h)(f_k + h) - 2N\sum_{k=N}^{\infty} \mathbb{E}(f_0 + h)(f_k + h)
\]

\[
= \sigma^2 N + O(1),
\]

as required.

\[\square\]

**Proof of Lemma 4.6.** Lemma A.6 states that the limit,

\[
\lim_{n \to \infty} \frac{1}{n} \mathrm{Var} \left( -\log \left( \frac{P(X_1^n, Y_1^n|X_{-\infty}^0, Y_{-\infty}^0)}{P(Y_1^n|Y_{-\infty}^0)} \right) \right). \tag{69}
\]

exists and is finite. Moreover, by Lemma A.5, after an application of the Cauchy-Schwarz inequality, we have that, as \( n \to \infty \),

\[
\mathbb{E} \left\{ \left[ \sum_{k \leq n} \log \left( \frac{P(X_k, Y_k|X_1^{k-1}, Y_1^{k-1})}{P(Y_k|Y_1^{k-1})} \right) - \log \left( \frac{P(X_k, Y_k|X_{-\infty}^k, Y_{-\infty}^k)}{P(Y_k|Y_{-\infty}^k)} \right) \right]^2 \right\} = O(1),
\]

therefore,

\[
\frac{1}{n} \left\{ \mathrm{Var} \left( -\log P(X_1^n|Y_1^n) \right) - \mathrm{Var} \left( -\log \left( \frac{P(X_1^n, Y_1^n|X_{-\infty}^0, Y_{-\infty}^0)}{P(Y_1^n|Y_{-\infty}^0)} \right) \right) \right\} = o(1).
\]

Combining this with (69) and the definition of \( \sigma^2 \), completes the proof.

\[\square\]

**Proof of Theorem A.1.** Note that we have already established the fact that the expression for the variance converges to some \( \sigma^2 < \infty \). Also, in view of Lemma A.2, it is sufficient to prove the theorem for \( \{S(t)\} \) instead of \( \{S(t)\} \), where:

\[
\tilde{S}(t) = \sum_{k \leq t} (f_k + h), \quad t \geq 0.
\]
This will be established by an application of [34, Theorem 7.1], once we verify that conditions (7.1.4), (7.1.5), (7.1.6), (7.1.7) and (7.1.9) there are all satisfied.

For each $n \geq 0$, let $\eta_n = f_n + h$, where $f_n$ is defined in (65) and $h$ is the conditional entropy rate. First we observe that, by stationarity,

\[
E[\eta_n] = E \left[ \log \left( \frac{P(X_n, Y_n | X_{-\infty}^{n-1}, Y_{-\infty}^{n-1})}{P(Y_n | Y_{-\infty}^{n-1})} \right) + H(X | Y) \right] \\
= E \left[ \log P(X_0, Y_0 | X_{-\infty}^{1}, Y_{-\infty}^{1}) \right] + H(X, Y) - E \left[ \log P(Y_0 | Y_{-\infty}^{1}) \right] - H(Y) \\
= 0,
\]

(70)

where $H(X, Y)$ and $H(Y)$ denote the entropy rates of $(X, Y)$ and $Y$, respectively [9]. Observe that, in the notation of Lemma A.3, $\eta_n = \eta_n(X, Y) - \eta_n(Y)$, and $\eta_{nl} = \eta_n(X, Y) - \eta_n(Y)$. By Lemma A.3, parts 3 and 4, there exist a constant $C$, independent of $n$ such that,

\[
\|\eta_n\|_4 \leq C < \infty,
\]

(71)

and,

\[
\|\eta_n - \eta_{nl}\|_4 = O(\ell^{-11/2}).
\]

(72)

And from Lemma A.6 we have,

\[
E \left\{ \left( \sum_{n \leq N} \frac{1}{\sigma} \eta_n \right)^2 \right\} = N + O(1).
\]

(73)

From (70)–(73) and the assumption that $\alpha^{(X,Y)}(d) = O(d^{-336})$, we have that all of the conditions (7.1.4), (7.1.5), (7.1.6), (7.1.7) and (7.1.9) of [34, Theorem 7.1] are satisfied for the random variables $\{\eta_n/\sigma\}$, with $\delta = 2$. Therefore, $\{\tilde{S}(t) : t \geq 0\}$ can be redefined on a possibly richer probability space, where there exists a standard Brownian motion $\{B(t) : t \geq 0\}$, such that, as $t \to \infty$:

\[
\frac{1}{\sigma} \tilde{S}(t) - B(t) = O(t^{1/2-\lambda}), \quad \text{a.s.}
\]

By Lemma A.2, this completes the proof. \hfill \Box

## B Proof of Theorem 4.11

We define the Markov chain $\{Z_n = (X_n, Y_n, X_{n+1}, Y_{n+1}) : n \geq 1\}$, with state space $Z = \{(w, v) \in (X \times Y)^2 : P_{(X_2, Y_2)|(X_1, Y_1)}(v|w) > 0\}$, which is also stationary, irreducible and aperiodic. Consider the function, $F : Z \to \mathbb{R}$ defined by

\[
F((x', y'), (x, y)) = -\log \left( \frac{P_{(X_2, Y_2)|(X_1, Y_1)}(x', y'|x, y)}{P_{Y_2|Y_1}(y'|y)} \right).
\]

Observe that the mean of $F(Z_t)$ is,

\[
E[F(Z_t)] = E[-\log P_{(X_2, Y_2)|(X_1, Y_1)}] - E[-\log P_{(Y_2|Y_1)}] = H(X, Y) - H(Y) = H(X | Y).
\]

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and that the partial sums of $F(Z_i)$ are,

$$\sum_{i=1}^{n} F(Z_i) = -\log \left( \frac{P(X^n, Y^n|X_1, Y_1)}{P(Y^n|Y_1)} \right).$$

In view of Lemma 4.6, the conditional varentropy rate can also be expressed as,

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var} \left( \sum_{i=1}^{n} F(Z_i) \right).$$

We will show that: (a) $\sigma^2 = 0$ implies (20); (b) (20) implies (21); and (c) (21) implies $\sigma^2 = 0$.

(a). Suppose $\sigma^2 = 0$. Then the general variance characterization in [24, Proposition 2.4] tells us that there is a function $G : Z \to \mathbb{R}$ such that,

$$P[Z_{i+1} \in S_z|Z_i = z] = 1, \quad \text{for all } z \in Z,$$

where the sets $S_z$ are,

$$S_z = \{ z' \in Z : G(z') = G(z) - [F(z) - \mathbb{E}[F(Z_1)]].$$

This means that for any pair of states $z, z'$ of the form $z = (x_1, y_1, x_2, y_2), z' = (x_2, y_2, x_3, y_3),$$

$$-\log \frac{P[(X_{i+1}, Y_{i+1}) = (x_2, y_2)|(X_i, Y_i) = (x_1, y_1)]}{P[Y_{i+1} = y_2|Y_i = y_1]} = G(x_1, y_1, x_2, y_2) - G(x_2, y_2, x_3, y_3).$$

(74)

Now fix $z = (x_1, y_1, x_2, y_2) \in Z$. Since (74) holds for all $(x_3, y_3)$, it must be the case that $G$ depends only on its first two arguments. Letting $G(x, y) = G(x, y, x', y')$, for arbitrary $x', y'$, (74) can be written,

$$-\log \frac{P[(X_{i+1}, Y_{i+1}) = (x_2, y_2)|(X_i, Y_i) = (x_1, y_1)]}{P[Y_{i+1} = y_2|Y_i = y_1]} = G(x_1, y_1) - G(x_2, y_2).$$

(75)

Now (20) follows with $g(x_j, y_j) = 2^{-G(x_j, y_j)}, j = 1, 2$.

(b). A simple computation shows that (20) leads to a telescoping product which leads to (21) with $q = 2^{-H}$.

(c). Finally, suppose (21) holds. Then,

$$P(X^n_i|Y^n_i) = \frac{P(X^n, Y^n)}{P(Y^n)} = \frac{P(X_1, Y_1)}{P(Y_1)} \times \frac{P(X_{n-1}, Y_{n-1}|X_1, Y_1)}{P(Y_{n-1}|Y_1)}$$

$$= \frac{P(X_1, Y_1)}{P(Y_1)} \times \frac{P(X_{n-1}, Y_{n-1}|X_1, Y_1)}{P(Y_{n-1}|Y_1)} \times \frac{P(X_n, Y_n|X_{n-1}, Y_{n-1})}{P(Y_n|Y_{n-1})}$$

$$\times \frac{E[P_{y_{n-1}}(Y_{n-1}|Y_1)]}{E(P_{X_{n-1}, Y_{n-1}}(X_{n-1}, Y_{n-1})|X_1, Y_1)^*},$$

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and so, by the Borel-Cantelli lemma, 

$$- \log P(X^n_1|Y^n_1) = - \log \frac{P(X_1,Y_1)(X^n_{-1} * X_1, Y^n_{-1} * Y_1)}{P(Y_1(Y^n_{-1} * Y_1))} - \log \left\{ \frac{P(X_1,Y_1)P(X_n,Y_n|X_{n-1},Y_{n-1})}{P(Y_n|Y_{n-1})} \times \mathbb{E} \left( P_{Y_{n-1}}(Y_1|Y_{n-1})|Y_1 \right) \right\},$$

where the first term is $-n \log q$ and the second term is bounded. Hence, 

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var}[- \log P(X^n_1|Y^n_1)] = 0,$$

as required. \hfill \Box

### C Recurrence Times Proofs

In this appendix we provide the proofs of some of the more technical results in Section 5. First we establish the following generalization of [9, Lemma 16.8.3].

**Lemma C.1** Suppose $(X, Y)$ is an arbitrary source-side information pair. Then, for any sequence $\{t_n\}$ of non-negative real numbers such that $\sum_n 2^{-t_n} < \infty$, we have:

$$\log \frac{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} \geq -t_n, \quad \text{eventually a.s.}$$

**Proof.** Let $B(X^n_{-\infty}, Y^n_{-\infty}) \subset Y^n$ denote the support of $P(\cdot|X^n_{-\infty}, Y^n_{-\infty})$. We can compute,

$$\mathbb{E} \left( \frac{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} \right) = \mathbb{E} \left( \frac{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} \left| Y^n_{-\infty}, X^n_{-\infty} \right. \right)$$

$$= \mathbb{E} \left( \sum_{y^n_1 \in B(X^n_{-\infty}, Y^n_{-\infty})} \frac{P(y^n_1|Y^n_{-\infty}, X^n_{-\infty})}{P(y^n_1|Y^n_{-\infty}, X^n_{-\infty})} P(y^n_1|Y^n_{-\infty}, X^n_{-\infty}) \right) \leq 1.$$

By Markov’s inequality,

$$\mathbb{P} \left[ \log \left( \frac{P(Y^n_1|Y^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} \right) > t_n \right] = \mathbb{P} \left[ \frac{P(Y^n_1|Y^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} > 2^{t_n} \right] \leq 2^{-t_n},$$

and so, by the Borel-Cantelli lemma,

$$\log \frac{P(Y^n_1|Y^n_{-\infty})}{P(Y^n_1|Y^n_{-\infty}, X^n_{-\infty})} \leq t_n, \quad \text{eventually a.s.,}$$

as claimed. \hfill \Box
Proof of Theorem 5.3. Let $K > 0$ arbitrary. By Markov’s inequality and Kac’s theorem,
\[
\mathbb{P}(\mathcal{R}_n(X|Y) > K \mid X_1^n = x_1^n, Y_1^n = y_1^n) \leq \frac{\mathbb{E}\left(\mathcal{R}_n(X|Y) \mid X_1^n = x_1^n, Y_1^n = y_1^n\right)}{K} = \frac{1}{KP(x_1^n|y_1^n)}.
\]
Taking $K = 2^{c_n}/P(X_1^n|Y_1^n)$, we obtain,
\[
\mathbb{P}(\log[\mathcal{R}_n(X|Y)P(X_1^n | Y_1^n)] > c_n \mid X_1^n = x_1^n, Y_1^n = y_1^n) = \mathbb{P}\left(\mathcal{R}_n(X|Y) > \frac{2^{c_n}}{P(X_1^n|Y_1^n)} \right) \leq 2^{-c_n}.
\]
Averaging over all $x_1^n \in \mathcal{X}^n$, we have,
\[
\mathbb{P}(\log \mathcal{R}_n(X|Y)P(X_1^n|Y_1^n) > c_n) \leq 2^{-c_n},
\]
and the Borel-Cantelli lemma gives (i).

For (iii) we first note that the probability,
\[
\mathbb{P}(\log[\mathcal{R}_n(X|Y)P(X_1^n | Y_1^n, X_0^n, Y_0^n)] < -c_n \mid Y_1^n = y_1^n, X_0^n = x_0^n, Y_0^n = y_0^n) \quad (76)
\]
is the probability, under $P(X_1^n = \cdot | Y_1^n = y_1^n, X_0^n = x_0^n, Y_0^n = y_0^n)$, of those $z_1^n$ such that,
\[
P(X_1^n = z_1^n | X_0^n, Y_0^n) < \frac{2^{-c_n}}{\mathcal{R}_n(x_0^n * z_1^n|y_0^n)},
\]
where ‘$*$’ denotes the concatenation of strings. Let $G_n = G_n(x_0^n, y_0^n) \subset \mathcal{X}^n$ denote the set of all such $z_1^n$. Then the probability in (76) is,
\[
\sum_{z_n \in G_n} P(z_1^n | x_0^n, y_0^n) \leq \sum_{z_n \in G_n} \frac{2^{-c_n}}{\mathcal{R}_n(x_0^n * z_1^n|y_0^n)} \leq 2^{-c_n} \sum_{z_n \in \mathcal{X}^n} \frac{1}{\mathcal{R}_n(x_0^n * z_1^n|y_0^n)}.
\]
Since both $x_0^n$ and $y_0^n$ are fixed, for each $j \geq 1$, there is exactly one $z_1^n \in \mathcal{X}^n$, such that $\mathcal{R}_n(x_0^n * z_1^n|y_0^n) = j$. Thus, the last sum is bound above by,
\[
\sum_{j=1}^{\left|\mathcal{X}^n\right|} \frac{1}{j} \leq Dn,
\]
for some positive constant $D$. Therefore, the probability in (76) is bounded above by $Dn2^{-c_n}$, which is independent of $x_0^n, y_0^n$ and, by assumption, summable over $n$. Hence, after averaging over all infinite sequences $x_0^n, y_0^n$, the Borel-Cantelli lemma gives (ii).

For part (iii) we have, eventually, almost surely,
\[
\log \left[\mathcal{R}_n(X|Y) \frac{P(X_1^n, Y_1^n | X_0^n, Y_0^n)}{P(Y_1^n | Y_0^n)}\right] = \log \left[\mathcal{R}_n(X|Y) \frac{P(X_1^n, Y_1^n | X_0^n, Y_0^n)P(Y_1^n | X_0^n, Y_0^n)}{P(Y_1^n | Y_0^n)}\right] = \log[\mathcal{R}_n(X|Y)P(X_1^n, Y_1^n | X_0^n, Y_0^n)] + \log \left[\frac{P(Y_1^n | X_0^n, Y_0^n)}{P(Y_1^n | Y_0^n)}\right] \geq -2c_n,
\]
where the last inequality follows from (ii) and Lemma C.1, and we have shown (iii). \qed
Proof of Corollary 5.4. If we take $c_n = \epsilon n^\beta$ in theorem 5.3, with $\epsilon > 0$ arbitrary, we get from (i) and (iii),
\[
\limsup_{n \to \infty} \frac{1}{n^\beta} \log[\mathcal{R}_n(X|Y)P(X^n_1|Y^n_1)] \leq 0, \quad \text{a.s.}
\]
and
\[
\liminf_{n \to \infty} \frac{1}{n^\beta} \log \left[ \mathcal{R}_n(X|Y) \frac{P(X^n_1, Y^n_1|X^0_{-\infty}, Y^0_{-\infty})}{P(Y^n_1|Y^0_{-\infty})} \right] \geq 0, \quad \text{a.s.}
\]
Hence, to prove (a) it is sufficient to show that, as $n \to \infty$,
\[
\log P(X^n_1|Y^n_1) - \log \left[ \frac{P(X^n_1, Y^n_1|X^0_{-\infty}, Y^0_{-\infty})}{P(Y^n_1|Y^0_{-\infty})} \right] = O(1), \quad \text{a.s.,}
\]
which is exactly Lemma A.2 in Appendix A.

To prove (b), taking $\beta = 1$ in (77) and (78), it suffices to show that,
\[
\lim_{n \to \infty} \left\{ \frac{1}{n} \log P(X^n_1|Y^n_1) - \frac{1}{n} \log \left( \frac{P(X^n_1, Y^n_1|X^0_{-\infty}, Y^0_{-\infty})}{P(Y^n_1|Y^0_{-\infty})} \right) \right\} = 0, \quad \text{a.s.}
\]
But the first term converges almost surely to $-H(X|Y)$ by the Shannon-McMillan-Breiman theorem, as in (8), and the second term is,
\[
-\frac{1}{n} \sum_{i=1}^{n} \log P(X_i, Y_i|X^i_{-\infty}, Y^i_{-\infty}) + \frac{1}{n} \sum_{i=1}^{n} \log P(Y_i|Y^i_{-\infty}),
\]
which, by the ergodic theorem, converges almost surely to,
\[
-\mathbb{E}[\log P(X_0, Y_0|X^0_{-\infty}, Y^0_{-\infty})] + \mathbb{E}[\log P(Y_0|Y^0_{-\infty})] = H(X, Y) - H(Y) = H(X|Y).
\]
This completes the proof. \qed

D Proof of Theorem 8.1

As before, we omit the subscripts from PMFs and conditional PMFs when there is no confusion. For any pair of strings $(x_1^{n+k}, y_1^{n+k}) \in (\mathcal{X} \times \mathcal{Y})^{n+k}$ such that $P(x_1^{n+k}, y_1^{n+k}) > 0$, we have,
\[
-\log P(x_1^n|y_1^n) = \log \left( \frac{P(y_1^k) \prod_{j=k+1}^{n} P(y_j|x_{j-1}^k)}{P(x_1^k, y_1^k) \prod_{j=k+1}^{n} P(x_j, y_j|x_{j-1}^k)} \right)
\]
\[
= \sum_{j=k+1}^{k+n} \log \left( \frac{P(y_j|x_{j-1}^k)}{P(x_j, y_j|x_{j-1}^k)} \right)
\]
\[
- \log \left( \frac{P(x_1^k, y_1^k) \prod_{j=n+1}^{n+k} P(x_j, y_j|x_{j-1}^k)}{P(y_1^k) \prod_{j=n+1}^{n+k} P(y_j|x_{j-1}^k)} \right)
\]
\[
= \sum_{j=1}^{n} f(x_j^{j+k}, y_j^{j+k}) + \Delta_n,
\]
(79)
where \( f : (X \times Y)^{k+1} \to \mathbb{R} \) is defined on,
\[
S = \left\{ (x_1^{k+1}, y_1^{k+1}) \in (X \times Y)^{k+1} : P(x_{k+1}, y_{k+1} | x_1^k, y_1^k) > 0 \right\},
\]
by,
\[
f(x_1^{k+1}, y_1^{k+1}) = \log \left( \frac{P(y_1^{k+1}) \prod_{j=1}^{n+k} P(y_j | y_{j-1}^{j-k})}{P(x_1^{k+1}) \prod_{j=1}^{n+k} P(x_j | x_{j-1}^{j-k}, y_{j-1}^{j-k})} \right),
\]
and,
\[
\Delta_n = \log \left( \frac{P(y_1^n) \prod_{j=1}^{n+k} P(y_j | y_{j-1}^{j-k})}{P(x_1^n) \prod_{j=1}^{n+k} P(x_j | x_{j-1}^{j-k}, y_{j-1}^{j-k})} \right).
\]
Then, clearly,
\[
\delta = \max |\Delta_n| < \infty,
\]
where the maximum is over all nonzero-probability strings.

Let \( Z = \{Z_n\} \) denote the first-order Markov chain defined by taking overlapping \((k+1)\)-blocks in the joint process,
\[
Z_n = ((X,Y)_n, (X,Y)_{n+1}, ..., (X,Y)_{n+k}).
\]
Since \((X,Y)\) is irreducible and aperiodic, so is \(Z\), so it has a unique stationary distribution \(\pi\).
Let \(\tilde{Z}_n\) denote a stationary version of \(\{Z_n\}\), with the same transition probabilities as \(\{Z_n\}\) and with \(\tilde{Z}_1 \sim \pi\). Since \(f\) is bounded we can apply [29, Theorem 1] to obtain that there exists a finite constant \(A_1\) such that, for all \(n \geq 1\),
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{\sum_{j=1}^{n} f(X_j, Y_j) - nH}{\sigma \sqrt{n}} > z \right] - Q(z) \right| \leq A_1 \frac{\sqrt{n}}{\sigma},
\]
where \(H\) can also be expressed as \(H = \mathbb{E}[f(\tilde{Z}_1)]\), and since the function \(f\) is bounded and the distribution of the chain \(\{Z_n\}\) converges to the stationary distribution exponentially fast, the conditional varentropy is also given by,
\[
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{j=1}^{n} [f(\tilde{Z}_j) - H] \right)^2,
\]
and it coincides with the expression in Lemma 4.6.

For \(z \in \mathbb{R}\), define,
\[
F_n(z) = \mathbb{P} \left[ \frac{- \log P(X_1^n | Y_1^n) - nH}{\sigma \sqrt{n}} > z \right],
\]
\[
G_n(z) = \mathbb{P} \left[ \frac{\sum_{j=1}^{n} f(X_j, Y_j) - nH}{\sigma \sqrt{n}} > z \right].
\]
Since \(F_n\) and \(G_n\) are non-increasing, (79), (80), and (81) yield,
\[
F_n(z) \geq G_n \left( z + \frac{\delta}{\sigma \sqrt{n}} \right) \geq Q \left( z + \frac{\delta}{\sigma \sqrt{n}} \right) - A_1 \frac{\sqrt{n}}{\sigma} \geq Q(z) - A \frac{\sqrt{n}}{\sigma}.
\]
uniformly in $z$, where in the last inequality we used a simple first-order Taylor expansion for $Q$, with $A = A_1 + \frac{\delta}{\sqrt{2\pi}}$. Similarly, we have,

$$F_n(z) \leq G_n\left(z - \frac{\delta}{\sigma\sqrt{n}}\right) \leq Q\left(z - \frac{\delta}{\sigma\sqrt{n}}\right) + \frac{A_1}{\sqrt{n}} \leq Q(z) + \frac{A}{\sqrt{n}},$$

uniformly in $z$. \qed
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