RESIDUE IN INTERSECTION HOMOLOGY AND $L_p$–COHOMOLOGY

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Abstract. We consider a residue form for a singular hypersurface $K$ with isolated singularities. Suppose there are neighbourhoods of the singular points with coordinates in which hypersurface is described by quasihomogeneous polynomials. We find a condition on the weights under which the norm of the Leray residue form is square integrable. For $\dim K \geq 2$ all simple singularities satisfy this condition. Then the residue form determines an element in intersection homology of $K$. We also obtain a residue class in the cohomology of $K$.

0. Introduction

Let $M$ be a complex manifold of dimension $n + 1$ and let $K$ be a smooth hypersurface. Let $Tub K$ be a tubular neighbourhood of $K$. Consider the diagram:

$$
\begin{array}{c}
H^*(M \setminus K) \xrightarrow{\delta} H^{*+1}(M, M \setminus K) \xrightarrow{H^{*+1}(Tub K, Tub K \setminus K)} \\
\downarrow^\cong \quad [M] \cap \\
H_{2n+1-*}^B(K) \xleftarrow{[K] \cap} \quad H^{-1}(K).
\end{array}
$$

In the diagram $H_{2n+1-*}^B(K)$ denotes Borel–Moore homology, i.e. homology with closed supports. All coefficients are in $\mathbb{C}$. The residue map

$$
res = \tau^{-1} \circ \delta : H^*(M \setminus K) \rightarrow H^{-1}(K)
$$

is defined to be the composition of the differential with the inverse of the Thom isomorphism.

Suppose $K$ is singular. Then there is no Thom isomorphism, but we can define a residue morphism

$$
res : H^*(M \setminus K) \rightarrow H_{2n+1-*}^B(K)
$$

$$
res \omega = [M] \cap \delta \omega
$$

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If $K$ was nonsingular, then this definition would be equivalent to the previous one since $\xi \mapsto [K] \cap \xi$ is Poincaré duality isomorphism and the diagram above commutes. In general there is no hope to lift the residue morphism to cohomology. For $M = \mathbb{C}^{n+1}$ the morphism $res$ is the Alexander duality isomorphism and $[K]\cap$ may be not onto. For the same reason we can’t lift the residue morphism to intersection homology of $K$.

Let $\omega$ be a holomorphic $n+1$–form with a first order pole on $K$. Then a form $Res \omega \in \Omega^n(K \setminus \Sigma_K)$ can be defined; [Le], see §1. We estimate its norm in §4. We assume that $K$ has isolated singularities locally described by quasihomogeneous polynomials. Let the weights for a singular point be $a_1, \ldots, a_{n+1}$. We prove that if

$$\kappa = \sum_{i=0}^{n+1} a_i > 1$$

then the norm of $Res \omega$ is square integrable (and even $L_p$–integrable) for a special choice of a metric. Applying the isomorphism of $L_p$–cohomology and intersection homology (see §3) we conclude that $Res \omega$ determines an element in intersection homology and also in cohomology. The last one may depend on the choice of coordinates. The question of uniqueness in sheaf theoretic set up is discussed in §5. For $\kappa \leq 1$ there is a way to define an obstruction (higher residue) to lift the residue class; see the remark in §6. There are some questions one should state:

1) Is it possible to define residue class in intersection homology for nonisolated general singularities?
2) How to define the number $\kappa$ or other numerical obstruction to lift for an arbitrary, possibly nonisolated singularity?
3) Does the residue class in cohomology depend on the choice of coordinates?

I was involved in investigating multidimensional residues by Professor Bogdan Ziemian (see [Zi]). Private talks and his hand written notes motivated me to deal with this subject. I hope that this paper may be useful in solving partial differential equations. I would also like to thank Professors Zbigniew Marciniak, Piotr Jaworski and Henryk Żołądek for help in preparation of this paper.

**Contents**

§1 Residue form,
§2 Topology of a neighbourhood of a singular point,
§3 $L_p$–cohomology,
§4 Local computation,
§5 Uniqueness of the lift,
§6 Appendix: the $P_8$ singularity.

1. **Residue Form**

We recall the Leray method of defining the residue form [Le]. Let $\omega$ be a smooth closed $k+1$–form on the complement of the set

$$K = \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : s(z_1, \ldots, z_{n+1}) = 0 \} .$$

Suppose that $\omega$ has a first order pole on $K$; i.e. $s \omega$ is a global form on $\mathbb{C}^n$. Choose local coordinates in which $s$ is the first coordinate. This can be done outside the
singularities of $K$. Then $\omega$ can be written locally as

$$\omega = \frac{ds}{s} \wedge r + \eta,$$

where $r$ and $\eta$ do not contain $ds$ and are smooth on $K$. Let $\Sigma_K$ be the singular set of $K$. The form

$$\text{Res} \, \omega = r|_K \in \Omega^k(K \setminus \Sigma_K)$$

is called the residue form of $\omega$. The residue form does not depend on the choice of coordinates and on the function describing $K$. Thus it is defined globally for a hypersurface in a complex manifold. Moreover, $r$ is closed on $K$ and its class in $H^{k-1}(K \setminus \Sigma_K)$ does not depend on the representant of the class $[\omega] \in H^k(M \setminus K)$. For smooth $K$ it represents residue class defined in the introduction multiplied by $2\pi i$; [SS], [Do]. In general the residue can be defined to be a current on $M$ supported by $K$, [BG]. If the singularities are isolated ($n > 1$) then by Poincaré duality for $K \setminus \Sigma$ we have $H_n(K) = H^n(K \setminus \Sigma)$ and $[\text{Res} \, \omega] \in H^n(K \setminus \Sigma)$ also coincides with the homology residue class (multiplied by $2\pi i$) defined in the introduction.

We are particularly interested in holomorphic forms of degree $(n+1, 0)$. Let $\omega$ be such a form. Locally it can be written as

$$\omega = \frac{g}{s} dz_1 \wedge \cdots \wedge dz_{n+1}$$

with $g$ holomorphic. We have

$$ds = \sum_{i=1}^{n+1} \frac{\partial s}{\partial z_i} dz_i.$$  

If $\frac{\partial s}{\partial z_1} \neq 0$ then

$$dz_1 = \left(\frac{\partial s}{\partial z_1}\right)^{-1} \left( ds - \sum_{i=2}^{n+1} \frac{\partial s}{\partial z_i} dz_i \right)$$

and

$$\omega = \frac{g}{s} \left( \frac{\partial s}{\partial z_1}\right)^{-1} \left( ds - \sum_{i=2}^{n+1} \frac{\partial s}{\partial z_i} dz_i \right) \wedge dz_2 \wedge \cdots \wedge dz_{n+1} =$$

$$= \frac{ds}{s} \wedge g \left( \frac{\partial s}{\partial z_1}\right)^{-1} dz_2 \wedge \cdots \wedge dz_{n+1}$$

To see how $\text{Res} \, \omega = g \left( \frac{\partial s}{\partial z_1}\right)^{-1} dz_2 \wedge \cdots \wedge dz_{n+1}$ behaves in the neighbourhood of the singularities let us calculate its norm in the metric coming from the coordinate system:

$$|\text{Res} \, \omega|_K = \left| \frac{ds}{|ds|} \wedge \text{Res} \, \omega \right| = \left| \frac{ds}{|ds|} \wedge g \left( \frac{\partial s}{\partial z_1}\right)^{-1} dz_2 \wedge \cdots \wedge dz_{n+1} \right| =$$

$$= \left| \frac{\partial s}{|ds|} dz_1 \wedge g \left( \frac{\partial s}{\partial z_1}\right)^{-1} dz_2 \wedge \cdots \wedge dz_{n+1} \right| = \frac{g}{|ds|}.$$
We see that \( \text{Res} \omega \) has a pole in singular points.

Suppose \( K \) has isolated singularities. Define \( K^\circ \) to be \( K \) minus the sum of small balls centered at the singular points of \( K \). Let \( j : (K^\circ, \emptyset) \to (K^\circ, \partial K^\circ) \) and \( k : \partial K^\circ \to K^\circ \) be the inclusions. Then for \( \dim K = n > 1 \) we have [Bo]:

\[
IH_n^m(K) = \text{im}(j_* : H_n(K^\circ) \to H_n(K^\circ, \partial K^\circ)) = \\
= \text{im}(j^* : H^n(K^\circ, \partial K^\circ) \to H^n(K^\circ)) = \\
= \ker(k^* : H^n(K^\circ) \to H^n(\partial K^\circ))
\]

The morphism \( IH_n^m(K) \to H_n(K) \cong H^n(K^\circ) \) is just the inclusion. Each class \( \alpha \in H_n(K) \) is determined by a smooth form on the nonsingular part of \( K \). The group \( IH_n^m(K) \) consists of the classes which can be represented by forms with square integrable norms; see §3. Our goal in §4 will be to check whether \( |\text{Res} \omega| \) is square integrable, but first consider the examples.

**Example 1.1.** Let \( s = xy \) and let \( \omega = \frac{1}{s} dx \wedge dy \). Then \( ds = y \, dx + x \, dy \). The residue form is \( \frac{dy}{y} \) for \( x = 0 \) and \( \frac{dx}{x} \) for \( y = 0 \). We see that

\[
K^\circ = (\mathbb{C} \setminus B_\varepsilon) \times \{0\} \cup \{0\} \times (\mathbb{C} \setminus B_\varepsilon)
\]

and \( \text{Res} \omega \) does not belong to \( \ker k^* = IH_n^m(K) \) since the form \( \frac{dy}{y} \) (and \( \frac{dx}{x} \)) is a generator when restricted to the small circle.

Since one may think, that the example is degenerated (\( K \) is not normal and \( \dim K = 1 \)) let us consider another example.

**Example 1.2.** Let \( s \) be a singularity of the type \( P_8 \):

\[
s(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3.
\]

The residue class \( \text{Res}(\frac{1}{s} dz_1 \wedge dz_2 \wedge dz_3) \) has no lift to intersection homology; see the Appendix.

To show that \( [\text{Res} \omega] \in IH_n^m(K) = \ker(k^* : H^n(K^\circ) \to H^n(\partial K^\circ)) \) one has to integrate the residue form over each \( n \)-cycle \( X \subset \partial K^\circ = K \cap S_\varepsilon \). Since the mapping \( f \) restricted to \( S_\varepsilon \) is a fibration in a neighbourhood of \( f^{-1}(0) \cap S_\varepsilon \) one can find a continuous family of cycles \( X_t \subset f^{-1}(t) \cap S_\varepsilon \) with \( X_0 = X \). The function \( t \mapsto \int_{X_t} s\omega/\,ds \) is holomorphic (single-valued) function. It can be expanded in a series \( \sum \alpha a_\alpha t^{\alpha} \). By [ArII p.261] we can assume \( -\alpha + 1 \leq d \) where \( d \) is the distance of the Newton diagram of \( s \) (see [ArII p. 140]). In the case of quasihomogeneous \( s \) with weights \( a_1, \ldots, a_{n+1} \) (see §4) we have \( d = -(a_1 + \cdots + a_{n+1}) \). Thus to deduce that \( \int_{X_t} \text{Res} \omega = \int_{X_0} s\omega/\,ds = 0 \) one should take \( s \) with \( a_1 + \cdots + a_{n+1} > 1 \). This is exactly the condition obtained in §4 by applying \( L_p \)-methods. If \( \alpha = 0 \) occurs in the Taylor expansion of \( \int_{X_t} s\omega/\,ds \) then 0 is a spectral number of the singular point. Properties (and definition) of spectra were discussed in papers of Varchenko e.g. [ArII], [Va] and Steenbrink e.g. [St1], [St2]. This way we have:
Conclusion 1.3. If \( 0 \) does not belong to the spectra of the singular points of \( K \) then each residue class lifts to the intersection homology of \( K \).

From the topological point of view this statement can be partially explained by the fact that spectral numbers multiplied by \( 2\pi i \) are logarithms of the eigenvalues of the monodromy. Thus if \( 0 \) is not in the spectrum, then \( 1 \) is not an eigenvalue of the monodromy and \( K \) is a rational manifold; see the next paragraph. To check that \( 0 \) is not a spectral number for the most of singularities see the table [ArII, page 275]. This shows that in general a residue class lies in intersection homology. The \( L_p \)-method of lift presented in the rest of the paper is not so general but in addition we obtain a concrete lift of residue class to the cohomology of \( K \).

2. TOPOLOGY OF A NEIGHBOURHOOD OF A SINGULAR POINT

Let us assume that \( 0 \in \mathbb{C}^{n+1} \) is an isolated singular point of a hypersurface \( K \). Intersect \( K \) with a ball of a small radius. Then the set \( L = S_t \cap K \) is called the link of the singular point. Milnor [Mi] gave the precise description of the topology of \( L \). It is \( 2n - 1 \) dimensional manifold with nonzero homology only in dimensions \( 0, n - 1, n \) and \( 2n - 1 \). Let \( h_+ \) be the monodromy acting on the homology of the Milnor fiber and let \( \Delta(t) \) be its characteristic polynomial.

Theorem 2.1. [Mi], [Hi]. Let \( n > 2 \). The link of an isolated singular point of \( s: \mathbb{C}^{n+1} \to \mathbb{C} \) is homeomorphic to a sphere if and only with \( \Delta(1) = \pm 1 \). The link is a rational homology sphere if and only if \( \Delta(1) \neq 0 \), i.e. \( 1 \) is not a eigenvalue of the monodromy.

Milnor describes in his book a recipe for computing \( \Delta(t) \) of quasihomogeneous polynomials. We restrict our attention to the case of simple and unimodal parabolic (simply elliptic) singularities. All these types may be represented by quasihomogeneous polynomials. Our choice is motivated by the following:

Theorem 2.2. [ArI]. Every singularity is simple (i.e. it is of the type: \( A_k, D_k, E_6, E_7, E_8 \)) or it is adjacent to one of the unimodal parabolic type (i.e. to \( P_8, X_9 \) or \( J_{10} \)).

Now we list the families of simple singularities and the corresponding characteristic polynomials. The table contains answers to the following questions:

a) Is the link homeomorphic to a sphere? (For \( n = 2 \) — is it a homology sphere?)

b) Is it a rational sphere?

| Singularity type | \( k \) | \( n \) | characteristic polynomial | a) | b) |
|------------------|--------|--------|----------------------------|----|----|
| \( A_k \): \( z_1^{k+1} + \sum_{i=2}^{n+1} z_i^2 \) | odd, even | odd, even | \( \pm(t^k - t^{k-1} + \cdots \pm 1) \) | no | no |
| \( D_k \): \( z_1^2 z_2 + z_2^{k-1} + \sum_{i=3}^{n+1} z_i^2 \) | \( \geq 4 \) | odd, even | \( \pm(t - 1)(t^{k-1} - (-1)t^k) \) | no | no |
| \( E_6 \): \( z_1^3 + z_1^2 + \sum_{i=3}^{n+1} z_i^2 \) | odd, even | odd, even | \( \pm(t + 1)(t^{k-1} + 1) \) | yes | yes |
| \( E_7 \): \( z_1^3 + z_1^2 + \sum_{i=3}^{n+1} z_i^2 \) | odd, even | odd, even | \( -(t - 1)(t^6 + t^3 + 1) \) | no | no |
| \( E_8 \): \( z_1^3 + z_2^5 + \sum_{i=3}^{n+1} z_i^2 \) | odd, even | odd, even | \( -(t + 1)(t^6 + t^3 + 1) \) | yes | yes |
The unimodal parabolic singularities as follows:\(^1\):

| Singularity type | \(n\) | characteristic polynomial | a) | b) |
|------------------|------|--------------------------|----|----|
| \(P_8: z_1^3 + z_2^3 + z_3^3 + a z_1 z_2 z_3 + \sum_{i=4}^{n+1} z_i^2\) | odd | \((t^3 + 1)(t^2 - t + 1)\) | no | yes |
| even | \((t^3 - 1)(t^2 + t + 1)\) | no | no |
| \(X_9: z_1^4 + z_2^4 + a z_1^2 z_2^2 + \sum_{i=3}^{n+1} z_i^2\) | odd | \(-(t^4 - 1)^2(t - 1)\) | no | no |
| even | \(-(t^4 - 1)^2(t + 1)\) | no | no |
| \(J_{10}: z_1^2 + z_2^2 + a z_1 z_2 + \sum_{i=3}^{n+1} z_i^2\) | odd | \((t^6 - 1)(t^3 + 1)(t - 1)\) | no | no |
| even | \((t^6 - 1)(t^3 - 1)(t + 1)\) | no | no |

We see that the link of a singular point often happens to be a rational homology sphere. If it is the case then \(K = \{s = 0\}\) is a rational homology manifold and the Poincaré duality map

\[
PD : H^k(K; \mathbb{C}) \rightarrow H_{2n-k}(K; \mathbb{C})
\]

is an isomorphism. Thus each residue class lifts to cohomology. For other cases there is no chance to construct the uniform lift of the residue morphism. We will study only the residues of meromorphic forms with a first order pole on \(K\).

3. \(L_p\)-COHOMOLOGY

To show that the residue form on the nonsingular part of \(K\) determines an element in intersection homology we apply the isomorphism which was suggested in [BGM]:

**Theorem 3.1.** [Ch], [We1]. Let \(X\) be a pseudomanifold equipped with a Riemannian metric on the nonsingular part. Assume that this metric is concordant with a conelike structure of the pseudomanifold. Then \(H^*_p(X_0)\), the \(L_p\)-cohomology of the nonsingular part, is isomorphic to the intersection homology with respect to the perversity which is the largest perversity strictly smaller then the function \(F(i) = \frac{i}{p}\).

Concordance with the conelike structure means that each singular point has a neighbourhood which is quasisometric to the metric cone over the link, i.e. to \(cL_x = L_x \times [0,1]/L_x \times \{0\}\) with the metric \(t^2 dx^2 + dt^2\). The intersection homology of a pseudomanifold \(K\) with isolated singularities is either \(H^{2n-*}_{BM}(K)\) or the image of the Poincaré duality map \(im(PD : H^{2n-*}(K) \rightarrow H^{BM}_*(K))\). The case depends on the value of the perversity for \(2n\). If the dimension is one then we should take the normalization of \(K\) instead of \(K\). The perversity associated with \(p \in \left(2, 2 + \frac{2}{n-1}\right)\) is the middle perversity \(m\) and \(m(2n) = n - 1\). Thus we obtain:

**Corollary 3.2.** If a hypersurface \(K\) with isolated singularities is equipped with a conelike metric then

\[
H^p_{(p)}(K \setminus \Sigma K) \simeq \begin{cases} 
H_{BM}^p(K) & \text{for } 1 + \frac{1}{2n-1} \leq p < 2 \\
im PD & \text{for } 2 \leq p < 2 + \frac{2}{n-1} \\
H^p(K) & \text{for } 2 + \frac{2}{n-1} \leq p.
\end{cases}
\]

\(^1\)The number \(a\) is such that: \(a^3 + 27 \neq 0\) for \(P_8\), \(a^2 \neq 4\) for \(X_9\) and \(4a^3 + 27 \neq 0\) for \(J_{10}\).
In §4 we construct a suitable conelike metric and estimate the norm of a residue form for every $p > 1$. We show that it is $L_p$–integrable for a wide class of isolated singularities including all simple singularities. In this way we obtain a lift of the residue class to intersection homology.

4. LOCAL COMPUTATION

Recall that we say that a polynomial is quasihomogeneous with weights $a_1, \ldots, a_{n+1}$, if it is a sum of monomials $\prod z_i^{k_i}$ such that $\sum_{i=1}^{n+1} k_i a_i = 1$. The homogeneous polynomial of degree $d$ is quasihomogeneous with weights $a_i = \frac{1}{d}$.

We show the following:

**Theorem 4.1.** Let $s$ be a quasihomogeneous polynomial in $n + 1$ variables with weights $a_1, \ldots, a_{n+1}$. Assume that 0 is an isolated critical point of $s$. If

$$\kappa = \sum_{i=2}^{n+1} a_i > 1$$

then there exists a conelike metric on $\mathbb{C}^{n+1}$ such that the norm of the residue form

$$\text{Res} \left( \frac{g \, dz_1 \wedge \cdots \wedge dz_{n+1}}{s} \right) \in \Omega^{n,0}(\{s = 0\} \setminus \{0\})$$

is $L_p$–integrable.

**Proof.** We choose $m \in \mathbb{R}$ and parametrize $\mathbb{C}^{n+1}$ by the homeomorphism:

$$(u_1, \ldots, u_{n+1}) \mapsto (u_1|u_1|^{m a_1-1}, \ldots, u_{n+1}|u_{n+1}|^{m a_{n+1}-1}).$$

The set $\Phi^{-1}(K)$ is conical. We estimate the norm of the residue form

$$r = \text{Res} \left( \frac{g \, dz_1 \wedge \cdots \wedge dz_{n+1}}{s} \right) = \left( \frac{g \, dz_2 \wedge \cdots \wedge dz_{n+1}}{\frac{\partial s}{\partial z_1}} \right)$$

in the metric induced by this parametrization. The norm $|\Phi^* (dz_i)|_u$ is (real) homogeneous of degree $m a_i - 1$, the denominator $\frac{\partial s}{\partial z_1}(\Phi(u))$ is homogeneous of degree $m - m a_1$. Thus the norm $|\Phi^* r|_u$ is bounded by a homogeneous function of degree

$$\sum_{i=2}^{n+1} (m a_i - 1) - (m - m a_i) = \sum_{i=1}^{n+1} m a_i - n + m = m \left[ \sum_{i=2}^{n+1} a_i - 1 \right] - n.$$

Then $\int_{\{|u|=r\} \cap K} |\Phi^* r|^p d\nu$ is bounded by a homogeneous function of degree

$$\alpha = p \left\{ m \left[ \sum_{i=2}^{n+1} a_i - 1 \right] - n \right\} + 2n - 1 = p m (\kappa - 1) + (2 - p) n - 1$$

If $p = 2$ then we see that this function is integrable. For $p > 2$ one must take $m$ large enough so that $\alpha > -1$. \qed
Corollary 4.2. If $K$ has quasihomogeneous singularities with $\kappa > 1$ then the residue form defines an element in $L_p$-cohomology of $K$ for a suitable metric.

Proof. In each singular point we choose $m$ such that $m(\kappa - 1) > (p - 2)n$. Then the residue form is $L_p$-integrable with respect to the conelike metric constructed in the proof of the Theorem 4.1. Hence it defines an element in $L_p$-cohomology. □

Remarks. In the proof of the Theorem 4.1. we can use the function $e^{-\frac{a_i}{|z_i|^p}}$ as well as $|z_i|^{m a_i}$ ($m$ large). As a result we get the same condition for weights. Note that this condition is fulfilled if the Hessian of $s$ is of rank at least 2 and $n \geq 2$. Then $s$ has either a term $z_iz_j$ or $z_i^2 + z_j^2$ so $a_i + a_j = 1$ and the other summands in $\kappa$ are nonzero. Practically the theorem shows that we can integrate the residue form over chains which are regular enough i.e. which enter singular points along the cone lines.

Below we list singularities with computed weights and with the $\kappa$ numbers. They coincide with ‘the oscillation indicators’ from [ArII].

| Type | Weights | $\kappa$ |
|------|---------|----------|
| $A_k$ | $\frac{1}{k+1}, \frac{1}{2}, \frac{1}{2}, \ldots$ | $n + \frac{1}{k+1}$ |
| $D_k$ | $\frac{k-2}{2k-2}, \frac{1}{k-1}, \frac{1}{2}, \frac{1}{2}, \ldots$ | $n + \frac{1}{2(k-1)}$ |
| $E_6$ | $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$ | $n + \frac{1}{12}$ |
| $E_7$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \ldots$ | $n + \frac{1}{18}$ |
| $E_8$ | $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$ | $n + \frac{1}{30}$ |
| $P_3$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ | $\frac{2}{3}$ |
| $X_9$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ | $\frac{2}{3}$ |
| $J_{10}$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ | $\frac{2}{3}$ |

We see that for all simple singularities we have $\kappa > 1$ for $n \geq 2$. For unimodal parabolic singularities one should take $n \geq 3$. The Example 1.2. (see Appendix) shows, that for $P_3$, $n = 2$, the residue class has no lift to the intersection homology. The lift of $Res\omega$ to cohomology we may call a regularization, that is giving a meaning to the symbol $\int_X Res\omega$ where $X$ is a cycle intersecting singularities of $K$. Certain regularization of residue form was described in [Zi].

5. Uniqueness of the lift

Denote by $i$ the inclusion $M \setminus K \hookrightarrow M$. Let $\Omega^\bullet_{M \setminus K}$ be the sheaf of complex-valued forms on $M \setminus K$. It is a soft resolution of the constant sheaf $\mathbb{C}_{M \setminus K}$. Thus $Ri_*\mathbb{C}_{M \setminus K} = i_\ast\Omega^\bullet_{M \setminus K}$. The inclusion $i$ induces a distinguished triangle.

$$
\begin{array}{c}
\mathbb{C}_{M} \\
\downarrow^{+1} \\
\downarrow \leftarrow \\
R\Gamma_{K} \mathbb{C}_{M} \\
\end{array}
\xrightarrow{\quad \sim}

Ri_*\mathbb{C}_{M \setminus K} = i_\ast\Omega^\bullet_{M \setminus K}.
$$

By taking the cohomology we get the long exact sequence of the pair $(M, M \setminus K)$. The stalk of $R\Gamma_{K} \mathbb{C}_{M}$ is:

$$
\mathcal{H}^j_x(R\Gamma_{K} \mathbb{C}_{M}) \simeq H^j_x(B_x, B_x \setminus K) \xleftarrow{\left(B_x \cap [B_x] \right)} H^{2n+2-j}_{B_M}(K \cap B_x),
$$

8
where $B_x$ is a small ball around $x$. Moreover, the whole sheaf $R\Gamma_K \mathcal{C}_M$ is isomorphic (with a shift of degrees) to the dualizing sheaf:

$$R\Gamma_K \mathcal{C}_M [2n + 2] \simeq \mathbb{D}_K .$$

Thus we get the residue morphism (Grothendieck residue)

$$res : i_* \Omega^\bullet_{M \setminus K} [2n + 1] = R^1 i_* \mathcal{C}_M \setminus K \simeq R\Gamma_K \mathcal{C}_M [2n + 2] \simeq \mathbb{D}_K$$

which is an isomorphism on the cohomology sheaves for $j \neq -(2n + 1)$

$$\mathcal{H}^j_x (i_* \Omega^\bullet_{M \setminus K} [2n + 1]) \xrightarrow{res} \mathcal{H}^j_x (\mathbb{D}_K) .$$

Let $\mathcal{O}^{(n+1)}_1$ be the sheaf of meromorphic forms of the type $(n+1,0)$ with poles of order 1 on $K$. Its sections are described locally by the formula $\frac{s}{g} dz_1 \wedge \cdots \wedge dz_{n+1}$, where $s$ defines $K$ and $g \in \mathcal{O}_M$. In the previous chapter we have constructed a conelike metric, for which (under the assumption $\kappa > 1$) Leray residue forms have $p$–integrable norms:

$$\text{Res} \left( \mathcal{O}^{(n+1)}_1 \right) \subset L^p(2) \simeq IC^\bullet_0$$

and

$$\text{Res} \left( \mathcal{O}^{(n+1)}_1 \right) \subset L^p(p) \simeq IC^\bullet_0$$

for $p > 2 + \frac{2}{n-1}$. A question arises: are these morphisms independent on the metric? To be precise, let us consider the sequence of the canonical morphisms and the obstruction sheaves [GM, §5.5]:

$$\begin{array}{l}
\mathbb{C}_K[n] +1 & \xrightarrow{} & IC^\bullet_0 \xleftarrow{+1} & IC^\bullet_m \xleftarrow{+1} & IC^\bullet_t \xleftarrow{+1} & \mathbb{D}_K \\
S_1 & S_2 & S_3 & S_4
\end{array}$$

The triangles in the diagram are distinguished in the derived category. We regard these sheaves as sheaves on $M$ supported by $K$. The cohomology of the links is nonzero only in dimensions $0$, $n-1$, $n$ and $2n-1$, so for arbitrary perversity the sheaf $IC^\bullet_0$ is isomorphic to:

1) $IC^\bullet_0$ if $p(2n) < n-1$,
2) $IC^\bullet_m$ if $p(2n) = n-1$,
3) $IC^\bullet_t$ if $p(2n) > n-1$.

The obstruction sheaves $S_2$ and $S_3$ are supported by the singular points and

$$\mathcal{H}^{-n-1}_x (S_2) = IH^m_{n+1} (cL_x) = H_n (L_x) \quad \text{and} \quad \mathcal{H}^i_x (S_2) = 0 \quad \text{for} \quad i \neq -(n+1)$$

$$\mathcal{H}^{-n}_x (S_3) = IH^m_n (cL_x) = H_{n-1} (L_x) \quad \text{and} \quad \mathcal{H}^i_x (S_3) = 0 \quad \text{for} \quad i \neq -n .$$
The obstruction sheaves $S_1$ and $S_4$ are also supported by $\Sigma_K$ and concentrated in one dimension:

\[
\mathcal{H}^{2n}_x(S_0) = \tilde{H}^0(L_x) \quad \text{and} \quad \mathcal{H}^{i}_x(S_0) = 0 \quad \text{for} \quad i \neq -2n
\]
\[
\mathcal{H}^0_x(S_4) = \tilde{H}_0(L_x) \quad \text{and} \quad \mathcal{H}^{i}_x(S_4) = 0 \quad \text{for} \quad i \neq 0
\]

Applying the functor $R\text{Hom}(\mathcal{O}_1^{(n)}[n], -)$ to the diagram above we get distinguished triangles and long exact sequences. Replacing $R^0\text{Hom}$ by $\text{Hom}_D$ — homomorphisms in the derived category we obtain:

\[
\text{Hom}_D(\mathcal{O}_1^{(n)}[n], \mathbb{C}_K[2n]) \xrightarrow{\simeq} \text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_0^\bullet)
\]
\[
\bigoplus_{x \in \Sigma} \text{Hom}(\mathcal{O}_1^{(n)}[x], H_n(L_x)) \rightarrow \text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_0^\bullet) \xrightarrow{\text{epi}} \text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_m^\bullet)
\]
\[
\text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_m^\bullet) \xrightarrow{\text{mono}} \text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_t^\bullet) \rightarrow \bigoplus_{x \in \Sigma} \text{Hom}(\mathcal{O}_1^{(n)}[x], H_{n-1}(L_x))
\]
\[
\text{Hom}_D(\mathcal{O}_1^{(n)}[n], IC_t^\bullet) \xrightarrow{\simeq} \text{Hom}_D(\mathcal{O}_1^{(n)}[n], D_K^\bullet).
\]

In this way we see that:

**Proposition 5.1.** The lift of the residue morphism to $\mathcal{O}_1^{(n)}[n] \rightarrow IC_m^\bullet$ is unique. If such a lift exists then there exists a lift to $\mathbb{C}_K[2n]$, which is not unique in general.

The Proposition 5.1. is not a surprise since on the cohomology level we have

\[
IH_m^n(K) = \text{im} \left( PD : H^n(K) \rightarrow H_n(K) \right)
\]

for $n > 1$. We do not know if the lift to $IC_0^\bullet$ essentially depends on the choice of a metric. We remind that a metric depends on the choice of coordinates in which the singularity is quasihomogeneous. The metric was determined by the weights.

**Example 5.2.** Consider the polynomial $s(x, y) = xy + y^{100} + z^2 + t^2$ it is quasihomogeneous with weights $99/100$, $1/100$, $1/2$ and $1/2$. This is a Morse singularity (i.e. of type $A_1$), and one can change coordinates so that $s(x', y') = x'^2 + y'^2 + z'^2 + t^2$. Then all weights are $1/2$.

6. **APPENDIX: THE $P_8$ SINGULARITY**

Consider a singularity of type $P_8$:

\[
s(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3
\]

and let

\[
\omega = \frac{1}{s} dz_1 \wedge dz_2 \wedge dz_3.
\]

Then

\[
r = \frac{1}{3z_1^2} dz_2 \wedge dz_3
\]
for $z_1 \neq 0$. We want to check if $r|_L = 0 \in H^2(L)$, where $L = S^5 \cap K$. The radius of the sphere does not matter as $s$ is homogeneous. To calculate the cohomology of $L$ we apply the Gysin exact sequence of the fibration

$$S^1 \hookrightarrow L \xrightarrow{p} L/S^1 \subset \mathbb{P}^2.$$  

The projectivization $L/S^1$ of $L$ is a cubic curve in the projective plane $\mathbb{P}^2$, so it is a topological 2-dimensional torus. We obtain the sequence:

$$\rightarrow H^0(L/S^1) \xrightarrow{\cup e} H^2(L/S^1) \xrightarrow{p^*} H^2(L) \xrightarrow{\int_p} H^1(L/S^1) \xrightarrow{\cup e} H^3(L/S^1) = 0,$$

where the morphism $\int_p$ is the integration along the fibers of the projection $p$. The bundle $L \xrightarrow{p} L/S^1$ is the restriction of the tautological bundle $S^5 \to \mathbb{P}^2$. Thus the Euler class of $p$ is the restriction of the generator of $H^2(\mathbb{P}^2)$. Hence the evaluation of the Euler class $\langle e, [L/S^1] \rangle = \deg s = 3$. Thus rationally $\int_p$ in the Gysin sequence is an isomorphism and the necessary and sufficient condition to lift is vanishing of $\left[ \int_p R e \omega_L \right] \in H^1(L/S^1)$. We will show that this element does not vanish. Let

$$U_1 = \{ [z_1 : z_2 : z_3] \in \mathbb{P}^2 : z_1 \neq 0 \} = \{ [1 : y_2 : y_3] \in \mathbb{P}^2 : y_2, y_3 \in \mathbb{C} \} \simeq \mathbb{C}^2.$$  

The tautological bundle $\tilde{p} : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ restricted to $U_1$ is trivial:

$$\tilde{p}^{-1}(U_1) \simeq \mathbb{C}^* \times \mathbb{C}^2$$  

$$(z_1, z_2, z_3) \mapsto \left( z_1, \left( \frac{z_2}{z_1}, \frac{z_3}{z_1} \right) \right)$$  

$$(y_1, y_1y_2, y_1y_3) \mapsto (y_1, y_2, y_3)$$  

We write $r$ in $y$-coordinates:

$$r = \frac{1}{3y_1^3}(y_1dy_2 + y_2dy_1) \wedge (y_1dy_3 + y_3dy_1) = \frac{dy_1}{3y_1}(y_2dy_3 - y_3dy_2) + \frac{1}{3}(dy_2 \wedge dy_3).$$  

We integrate it over each fiber

$$p^{-1}([1, y_2, y_3]) = \{ (y_1, y_1y_2, y_1y_3) : |y_1|^2(1 + |y_2|^2 + |y_3|^3) = 1 \};$$

$$\zeta = \int_p r = \int_p \left[ \frac{dy_1}{3y_1}(y_2dy_3 - y_3dy_2) + \frac{1}{3}(dy_2 \wedge dy_3) \right] = \frac{2}{3}\pi i(y_2dy_3 - y_3dy_2).$$  

The form $\zeta$ does not vanish on $L/S^1$ since

$$\zeta \wedge d(1 + y_2^3 + y_3^3) = \frac{2}{3}\pi i(y_2dy_3 - y_3dy_2) \wedge 3(y_2^2dy_2 + y_3^2dy_3) = -\frac{2}{3}\pi i(y_2^3 + y_3^3)dy_2 \wedge dy_3,$$

and it is equal $\frac{2}{3}\pi i dy_2 \wedge dy_3$ on $L/S^1$. It is a harmonic form, so its class in cohomology is nontrivial. The conclusion is that $Res \omega \notin ker (k^* : H^n(K^\circ) \to H^n(\partial K^\circ))$, so it has no lift to intersection homology.

**Remark.** The method of the example can be used to show that

$$\left[ Res \left( \frac{1}{s}dz_1 \wedge \cdots \wedge dz_{n+1} \right) \right] = 0 \in H^n(K \setminus \{0\})$$

for quasihomogeneous polynomial with $\kappa \neq 1$. For an arbitrary $\omega = \frac{q}{s}dz_1 \wedge \cdots \wedge dz_{n+1}$ one can define an obstruction in $H^{n-1}(L/S^1)$ vanishing if and only if the $1 - \kappa$ weighted homogeneous part of $g$ vanishes. This obstruction vanishes if and only if the residue class lifts to intersection homology [We2]. The method of obstruction does not give a lift to cohomology. It only shows that there exists one.
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