Dirichlet Series with Periodic Coefficients
and Their Value-Distribution near the Critical Line

Athanasios Sourmelidis, Jörn Steuding, and Ade Irma Suriajaya

Received July 25, 2020; revised February 26, 2021; accepted June 9, 2021

Abstract—The class of Dirichlet series associated with a periodic arithmetical function $f$ includes the Riemann zeta-function as well as Dirichlet $L$-functions to residue class characters. We study the value-distribution of these Dirichlet series $L(s; f)$ and their analytic continuation in the neighbourhood of the critical line (which is the axis of symmetry of the related Riemann-type functional equation). In particular, for a fixed complex number $a \neq 0$, we find for an even or odd periodic $f$ the number of $a$-points of the $\Delta$-factor of the functional equation, prove the existence of the mean of the values of $L(s; f)$ taken at these points, show that the ordinates of these $a$-points are uniformly distributed modulo one and apply this to show a discrete universality theorem.

DOI: 10.1134/S0081543821040118

1. MOTIVATION AND STATEMENT OF THE MAIN RESULTS

Since the classical works of Euler and Riemann, it is known that the distribution of prime numbers is intimately related to the Riemann zeta-function $\zeta(s)$. In general, many topics in analytic number theory are tied to associated Dirichlet series. With his path-breaking articles, Ivan Matveevich Vinogradov made outstanding contributions in this direction. For example, he established the so far widest zero-free region for $\zeta(s)$ and therefore the up-to-date smallest error term in the prime number theorem [39] (independently obtained by Nikolai Mikhailovich Korobov); moreover, with his treatment of exponential sums, Vinogradov proved that all sufficiently large odd integers satisfy the ternary Goldbach conjecture [38] (which was recently extended by Harald Helfgott to all odd integers $\geq 7$).

In this article we consider generalizations of the Riemann zeta-function and their value-distribution. Given an arithmetical function $f: \mathbb{N} \to \mathbb{C}$, the associated Dirichlet series is a complex-valued function of a complex variable $s := \sigma + it$, given by

$$L(s; f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$ 

Let $q$ be a positive integer. If $f$ is $q$-periodic, which means $f(n+q) = f(n)$ for all positive integers $n$, then the Dirichlet series defining $L(s; f)$ converges absolutely in the right half-plane $\sigma > 1$. 

---

$^a$ Institute of Analysis and Number Theory, TU Graz, Steyrergasse 30, 8010 Graz, Austria.

$^b$ Institute of Mathematics, Würzburg University, Emil-Fischer-Str. 40, 97074 Würzburg, Germany.

$^c$ Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan.

E-mail addresses: sourmelidis@math.tugraz.at (A. Sourmelidis), steuding@mathematik.uni-wuerzburg.de (J. Steuding), adeirmasuriajaya@math.kyushu-u.ac.jp (A. I. Suriajaya).
Moreover, \( L(s; f) \) can be analytically continued to the whole complex plane except for at most a simple pole at \( s = 1 \); and \( L(s; f) \) is regular at \( s = 1 \) if, and only if, the related residue

\[
\frac{1}{q} \sum_{a \mod q} f(a)
\]

vanishes. For technical reasons it is useful to extend \( f \) to be defined on \( \mathbb{Z} \) by periodicity. In addition, \( L(s; f) \) satisfies a functional equation (or identity), namely,

\[
L(1 - s; f) = \left( \frac{q}{2\pi} \right)^s \Gamma(s) \left( e\left( \frac{s}{4} \right) L(s; f^-) + e\left( -\frac{s}{4} \right) L(s; f^+) \right),
\]

(1.1)

where \( e(z) := \exp(2\pi iz) \) and \( f^\pm \) are \( q \)-periodic arithmetical functions defined by

\[
f^\pm(n) = \frac{1}{\sqrt{q}} \sum_{a \mod q} f(a) e\left( \pm \frac{an}{q} \right),
\]

(1.2)

which may be interpreted as discrete Fourier transforms of the functions \( f^\pm \) defined by \( f_+ = f \) and \( f_-(n) = f(-n) \). All these properties follow from similar properties of the Hurwitz zeta-function \( \zeta(s; \alpha) = \sum_{m \geq 0} (m + \alpha)^{-s} \) (with a real parameter \( \alpha \in (0, 1) \)) and a straightforward representation of such \( L(s; f) \) as a sum of Hurwitz zeta-functions with rational parameters \( \alpha = a/q \). This and more were first discovered by Walter Schnee [27] ninety years ago.

We are concerned with even and odd \( q \)-periodic functions \( f \), i.e., \( f = \delta f_- \) with \( \delta = 1 \) for even \( f \) and \( \delta = -1 \) for odd \( f \). In this case (1.1) can be rewritten as

\[
L(s; f) = \Delta(s; f)L(1 - s; f^+),
\]

(1.3)

where

\[
\Delta(s; f) := -i \left( \frac{q}{2\pi} \right)^{1-s} \frac{\Gamma(1 - s)}{\sqrt{q}} \left( e\left( \frac{s}{4} \right) - \delta e\left( -\frac{s}{4} \right) \right).
\]

(1.4)

This setting includes the case of Dirichlet \( L \)-functions associated with (not necessarily primitive) residue class characters \( \chi \mod q \) (see [1] for details). Some of our results generalize previous ones for the Riemann zeta-function [14, 15, 35]; however, the findings concerning uniform distribution and universality are new (although the latter property has been considered in this context earlier by Maxim Korolev and Antanas Laurinčikas [17] for a special case). Our approach is inspired by rather general though deep theorems from the theory of functions due to Émile Picard, Gaston Julia and Rolf Nevanlinna, and our analysis relies mainly on the functional equation (1.3).

In 1879, Picard [22] proved that if an analytic function \( f \) has an essential singularity at a point \( \omega \), then \( f(s) \) takes all possible complex values with at most a single exception (infinitely often) in every neighbourhood of \( \omega \). And in 1919, a little more than one hundred years ago, Julia [12] refined Picard’s great theorem by showing that one can even add a restriction on the angle at \( \omega \) to lie in an arbitrarily small cone (see also [3, Ch. XII, § 4]). For Dirichlet series appearing in number theory, however, it is more natural to consider so-called Julia lines rather than Julia directions. For this and further details we refer to [4, 32, 35].

Given a complex number \( a \), the solutions of the equation

\[
\Delta(s; f) = a
\]

are called the \( a \)-points of \( \Delta(s; f) \), and we denote them as \( \delta_a := \beta_a + i\gamma_a \). We will show that for any fixed \( a \neq 0 \) most of the \( a \)-points are clustered around the critical line \( 1/2 + i\mathbb{R} \), which is, therefore, a Julia line for \( \Delta(s; f) \). Moreover, we prove that the mean of the values \( L(\delta_a; f) \) exists and that every complex number appears as such a limit. This indicates an interesting link between the distribution
of $a$-points of $\Delta(s; f)$ and the values taken by $L(s; f)$. For the case of the Riemann zeta-function and values $a$ from the unit circle, these $a$-points have been studied by Justas Kalpokas and the second author [15] as well as by Kalpokas, Maxim Korolev and the second author [14]. In this situation the $\exp(2i\phi)$-points correspond to intersections of the curve $t \mapsto \zeta(1/2 + it)$ with straight lines $\exp(i\phi) \mathbb{R}$ through the origin.

**Theorem 1.** Let $f$ be an even or odd $q$-periodic function. Then, the function $s \mapsto \Delta(s; f)$ is a meromorphic function with two exceptional values 0 and $\infty$ in the sense of Nevanlinna theory and the critical line $1/2 + i\mathbb{R}$ is a Julia line for $\Delta(s; f)$. Moreover, given a complex number $a \neq 0$, the number $N_a(T; f)$ of $a$-points $\delta_a = \beta_a + i\gamma_a$ of $\Delta(s; f)$ satisfying $0 < \beta_a < 1$ and $0 < \gamma_a < T$ is asymptotically equal to

$$N_a(T; f) = \frac{T}{2\pi} \log \frac{qT}{2\pi e} + O(\log T).$$

The error term here and all error terms in the sequel depend on $a$ and $f$. The condition $0 < \beta_a < 1$ in the above theorem can be relaxed to any open interval centered at $1/2$. This can be easily explained using an asymptotic equation for $\Delta(s; f)$ (appearing in the next section).

**Theorem 2.** Let $f$ be an even or odd $q$-periodic function and let $\delta_a = \beta_a + i\gamma_a$ denote the $a$-points of $\Delta(s; f)$. Then, for every complex number $a \neq 0$ and any $\epsilon > 0$,

$$\sum_{0 < \gamma_a < T, 0 < \beta_a < 1} L(\delta_a; f) = (f(1) + af^+(1)) \frac{T}{2\pi} \log \frac{qT}{2\pi e} + O(T^{1/2+\epsilon}).$$

The horizontal distribution of the $a$-points of $\Delta(s; f)$ is quite regular, as we will see in the next section. We want to provide some additional information regarding their vertical distribution, since such questions arise quite often in zeta-function theory. For example, in the case of $\zeta(s)$, Edmund Landau [19] proved the explicit formula

$$\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O_x(\log T),$$

valid for any $T > 1$ and any positive number $x > 1$, where $\rho := \beta + i\gamma$ denote the non-trivial zeros of $\zeta(s)$ and $\Lambda(x)$ is the von Mangoldt function extended to the whole real line by setting it to be zero if $x$ is not a positive integer. The second author [34] proved a similar result in the case of $a$-points of $\zeta(s)$; namely, for $a \neq 1$,

$$\sum_{0 < \text{Im} \rho_a \leq T} x^{\rho_a} = \frac{T}{2\pi} \left( a(x) - x \Lambda \left( \frac{1}{x} \right) \right) + O_x(T^{1/2+\epsilon}),$$

where $\rho_a$ denotes the so-called non-trivial $a$-points of $\zeta(s)$ and $a(x)$ is a certain computable arithmetical function extended to the whole real line. The authors could not succeed in proving a comparable asymptotic formula in the case of $\delta_a$ but obtained an upper bound for the corresponding sum with respect to the $a$-points of $\Delta$ which is remarkably small compared with (1.6) or (1.7).

In order to formulate this result, we recall some standard notation. The number $[x]$ is the greatest integer which is smaller than or equal to the real number $x$, and $1_S$ denotes the characteristic function of a set $S$.

**Theorem 3.** Let $a \neq 0$ be a fixed complex number and $q$ a fixed positive integer. Then there exists a constant $c > 0$, depending on $a$, such that, for any $0 < x \neq 1$ and any $T$ and $T'$ satisfying $\max\{1, 4\pi/q\} \leq T < T + 1 \leq T' \leq 2T$, we have

$$\sum_{T < \gamma_a < T'} x^{\delta_a} \ll x^{1/2} \left( x^{\epsilon/\log(qT/(2\pi))} + x^{-\epsilon/\log(qT/(2\pi))} \right) \left( \left| \log x \right| + \log T + \frac{\log T}{\left| \log x \right|} \right)$$

$$+ 1_{(1, +\infty)}(x) E_1(x, a, T) - 1_{(0, 1)}(x) E_2(x, a, T),$$
where \( E_1(x, a, T) \) and \( E_2(x, a, T) \) are always zero, unless there is a positive integer

\[
j \leq \left[ \frac{2 \log(qT/(4\pi))}{\log 30} \right]
\]
such that \( x^{1/j} \in [qT/(4\pi), 5qT/(4\pi)] \) or \( x^{-1/j} \in [qT/(4\pi), 5qT/(4\pi)] \), in which case we have

\[
E_1(x, a, T) := \frac{x^{1/2} \log x}{j^{3/2}} \left( \frac{x^{1/(2j)}}{|a|^j} + \frac{x^{-c/(2\log(qT/(2\pi)))}}{30^j} T^{1/2} \right),
\]

\[
E_2(x, a, T) := \frac{x^{1/2} \log x}{j^{3/2}} \left( x^{-1/(2j)} |a|^j + \frac{x^{-c/(2\log(qT/(2\pi)))}}{30^j} T^{1/2} \right).
\]

In particular, if \( x \neq 1 \) is such that \( 4\pi/(qT) \leq x \leq qT/(4\pi) \), then

\[
\sum_{T < \gamma < T'} x^{\delta_\alpha} \ll x^{1/2} \left( 1 + \frac{1}{|\log x|} \right) \log T.
\]

A sequence of real numbers \((x_n)_{n \in \mathbb{N}}\) is called uniformly distributed modulo 1 if

\[
\lim_{N \to \infty} \frac{1}{N} \#\{1 \leq n \leq N : \{x_n\} \in [a, b]\} = b - a
\]
for any real numbers \(0 \leq a \leq b \leq 1\), where \( \{x\} := x - \lfloor x \rfloor \) for \( x \geq 0\). Hans Rademacher [23] employed Landau’s formula (1.6) to prove, under the Riemann Hypothesis, that the sequence \((\alpha \gamma)_{\{\gamma > 0\}}\) is uniformly distributed modulo 1, where \( \alpha \neq 0 \) is a real number and \((\gamma)_{\{\gamma > 0\}}\) is the sequence of ordinates of the non-trivial zeros of \(\zeta(s)\) in ascending order (and counted with multiplicities). Peter Elliott [5] and (independently) Edmund Hlawka [11] proved the uniform distribution of \((\alpha \gamma)_{\{\gamma > 0\}}\) unconditionally. The second author [33, 34] proved the uniform distribution of the sequence \((\alpha \Im(\rho_0))_{\{\Im(\rho_0) > 0\}}\), where \((\Im(\rho_0))_{\{\Im(\rho_0) > 0\}}\) is the sequence of ordinates of the non-trivial \(a\)-points of \(\zeta(s)\) in ascending order (counting multiplicities) in the upper half-plane. The common feature of both statements is that they follow from (1.6) and (1.7) and known results on the clustering of the non-trivial zeros (or \(a\)-points) of \(\zeta(s)\) around the critical line. This finally leads to

\[
\sum_{0 < \gamma < T} x^{i\gamma} = o_x(T \log T) = \sum_{0 < \Im(\rho_0) < T} x^{i\Im(\rho_0)}, \tag{1.8}
\]

and uniform distribution follows from a well-known criterion of Hermann Weyl. It is noteworthy that, for fixed \(x\), the best possible bounds known for the sums in (1.8) are of order \(T\). Compared to these results, the bounds given in the following theorem for the case of \(a\)-points of \(\Delta(s; f)\) are very different. This is probably not unexpected since \(\Delta(s; f)\) is not a zeta-function to begin with and the clustering of its \(a\)-points around the critical line has a simple explanation, as we will see in Sections 2 and 3.

**Theorem 4.** Let \(0 < \gamma^{(1)}_a \leq \gamma^{(2)}_a \leq \gamma^{(3)}_a \leq \ldots\) be the sequence of ordinates of all \(a\)-points of \(\Delta(s; f)\) (counted with multiplicities) that have real part in \((0, 1)\), where \(f\) is an odd or even \(q\)-periodic function. Then, for any integer \(N \geq 0\) and any number \(x \neq 1\) satisfying \(4\pi/(qN) \leq x \leq qN/(4\pi)\), we have

\[
\sum_{N < n \leq 2N} x^{i\gamma^{(n)}_a} \ll \left( \frac{1}{|\log x|} + |\log x| \right) \log N.
\]

In particular, the sequence of numbers \(\alpha \gamma^{(n)}_a\), \(n \in \mathbb{N}\), is uniformly distributed modulo 1 for any real number \(\alpha \neq 0\).
Robert Spira [29] showed that the $\Delta$-function appearing in the functional equation for the Riemann zeta-function $\zeta(s) = L(s; 1)$ (with 1 being the function constant 1) satisfies $|\Delta(s; 1)| < 1$ for $1/2 < \sigma < 1$ and $t \geq 10$. This implies along with the identity $\Delta(s; 1)\Delta(1 - s; 1) = 1$ that if $a$ is a complex number from the unit circle, then all but finitely many $a$-points of $\Delta(s; 1)$ have real part equal to $1/2$. Therefore, our theorems generalize some results from [15], as well as recent results by Korolev and Laurinčikas [17] on the uniform distribution of Gram points (the 1-points of $\Delta(s; 1)$). This last article motivates us to prove the following joint universality theorem.

**Theorem 5.** Let $f$ be an even or odd $q$-periodic function, and denote the ordinates of the $a$-points of $\Delta(s; f)$ in ascending order by $\gamma_a^{(n)}$. Let also $\chi_1, \chi_2, \ldots, \chi_J$ be non-equivalent Dirichlet characters and $\psi \neq 0$ be an $r$-periodic arithmetical function with $r \neq 2$. Then, for any compact set with connected complement $K$ inside the strip $1/2 < \sigma < 1$, any $g_1, g_2, \ldots, g_J$ continuous non-vanishing functions on $K$ which are analytic in its interior, any real numbers $z > 0$ and $\xi_p$, indexed by the primes $p \leq z$, and any $\varepsilon > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \#\left\{1 \leq n \leq N : \max_{1 \leq j \leq J} \max_{s \in K} |L(s + i\gamma_a^{(n)}; \chi_j) - g_j(s)| < \varepsilon \right\} > 0$$

and

$$\lim_{N \to \infty} \frac{1}{N} \#\left\{1 \leq n \leq N : \max_{s \in K} |L(s + i\gamma_a^{(n)}; \psi) - h(s)| < \varepsilon \right\} > 0,$$

(1.9)

where $\#A$ denotes the cardinality of a set $A \subseteq \mathbb{R}$ and $\|x\| := \min_{m \in \mathbb{Z}}|m - x|$. In addition, if $\psi$ has period $r \geq 3$ and is not a multiple of a Dirichlet character mod $r$, then $h$ is allowed to have zeros in $K$. In the case when $r = 2$, (1.10) holds for any compact set with connected complement $K$ inside the open set

$$D_0 := \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\} \setminus \left\{\log \left(1 - \frac{\psi(2)}{\psi(1)}\right) + 2k\pi i : k \in \mathbb{Z}\right\},$$

where $\log$ is the principal logarithm.

This is a discrete version of Sergei Voronin’s classical universality theorem for the Riemann zeta-function [40]:

$$\liminf_{T \to \infty} \frac{1}{T} \#\left\{\tau \in [0, T] : \max_{s \in K} |\zeta(s + i\tau) - h(s)| < \varepsilon \right\} > 0$$

(1.11)

(and its simultaneous version [41] for a family of Dirichlet $L$-functions, which is often called joint universality). As a matter of fact, Voronin proved that there exists some real $\tau$ such that $\zeta(s + i\tau)$ is $\varepsilon$-close to $h(s)$ when $s$ ranges in a disc centered at $3/4 + it_0$ of radius $r < 1/4$ (see, for example, [42, Ch. VII]). A few years later Axel Reich [25] and (independently) Bhaskar Bagchi [2] obtained (1.11) (which is implicit in Voronin’s work) and also provided a discrete version with respect to arithmetic progressions

$$\liminf_{T \to \infty} \frac{1}{T} \#\{1 \leq n \leq N : \max_{s \in K} |\zeta(s + i\tau n) - h(s)| < \varepsilon \} > 0,$$

where $d$ is a fixed non-zero real number. Theorem 5 is of similar nature, where the shifts are ordinates of $a$-points of $\Delta(s; f)$. 

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS  Vol. 314  2021
2. PROOF OF THEOREM 1

We first remark that $\Delta(s; f)$ depends only on the period $q$ of $f$ and $\delta = \pm 1$, which determines whether $f$ is an even or odd function. We also observe that if $f$ is an even or odd $q$-periodic arithmetical function, then so are $f^+$ and $\overline{f}$ (the complex conjugate of $f$). In the rest of this paper, we will use the simplifed notation

$$\Delta(s) := \Delta(s; f) = \Delta(s; \overline{f}) = \Delta(s; f^+). \quad (2.1)$$

This function $\Delta(s)$ is the product of an exponential function, the gamma function, and a trigonometric function. It is well known that $\Gamma(z)^{-1}$ is an entire function with only simple zeros at $z = -n$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Hence, by (1.4), $\Delta(s)$ is regular except for simple poles at the positive odd integers (if $\delta = 1$) or at the positive even integers (if $\delta = -1$); moreover, $\Delta(s)$ vanishes exactly at the non-positive even integers (if $\delta = 1$) or at the negative odd integers (if $\delta = -1$). One can show by an application of Rouché’s theorem (as Levinson did in [20] for $\Delta(s; 1)$, the case of the Riemann zeta-function) that there are a few $a$-points of $\Delta(s)$ in a neighbourhood of the real line; their count inside a strip $\{x + iy : |x| \leq r, |y| \leq 1\}$ is $O(r)$ as $r \to \infty$. In comparison with formula (1.5) it follows (along the lines of [32, Sect. 7.3]) that $0$ and $\infty$ are exceptional values of $\Delta(s)$ in the sense of Nevanlinna theory (more precisely, in the sense of the deficiency relation; this means that these values are attained considerably less often than others; see [10, p. x] for details). It remains to prove the Riemann–von Mangoldt-type formula (1.5).

It is an easy consequence of Stirling’s formula,

$$\Gamma(\sigma + it) = \sqrt{2\pi} e^{-\frac{\pi t}{2} - it + \frac{\pi i}{2}} \left( \sigma - \frac{1}{2} \right)^{-\sigma} \left( 1 + O\left( \frac{1}{t} \right) \right), \quad (2.2)$$

and the reflection principle

$$\Gamma(\sigma - it) = \overline{\Gamma(\sigma + it)}, \quad (2.3)$$

both valid uniformly for $t \geq 1$ and $\sigma$ from any strip of bounded width, and (1.4) that

$$\Delta(\sigma + it) = \delta \left( \frac{qt}{2\pi} \right)^{1/2 - (\sigma + it)} \exp\left( i \left( t + \frac{\pi}{4} \right) \right) \left( 1 + O\left( \frac{1}{t} \right) \right), \quad (2.4)$$

as $t \to +\infty$. Hence, $\Delta(\sigma + it)$ tends for $\sigma < 1/2$ to infinity and for $\sigma > 1/2$ to zero as $t \to +\infty$. Thus, the critical line $1/2 + i\mathbb{R}$ divides the upper half-plane into two domains where the limit $\lim_{t \to +\infty} \Delta(\sigma + it)$ exists in the compactified plane $\mathbb{C} \cup \{\infty\}$; on the critical line however the limit does not exist. The behaviour in the lower half-plane is ruled by conjugation,

$$\Delta(\sigma - it) = \delta \overline{\Delta(\sigma + it)} \quad \text{and} \quad \Delta'(\sigma - it) = \delta \overline{\Delta'(\sigma + it)}, \quad (2.5)$$

as follows from (1.4), (2.3) and Cauchy’s integral formula. Near the boundary (critical) line $1/2 + i\mathbb{R}$, however, the distribution of values is rather different. As a matter of fact, $\Delta(s)$ takes every complex value $a \neq 0$ infinitely often there. Writing $a = \Delta(\delta_a) = |a| \exp(i\phi)$ with an $a$-point $\delta_a = \beta_a + i\gamma_a$ of $\Delta$ and comparing with (2.4) implies that

$$|a| = \left( \frac{q\gamma_a}{2\pi} \right)^{1/2 - \beta_a} \left( 1 + O\left( \frac{1}{\gamma_a} \right) \right), \quad \phi \equiv \gamma_a \log \frac{2\pi e}{q\gamma_a} + \frac{\pi}{4} + \frac{1 - \delta \pi}{2} + O\left( \frac{1}{\gamma_a} \right) \mod 2\pi. \quad (2.6)$$

This shows that $\beta_a \to 1/2$ as $\gamma_a \to +\infty$ (and explains a remark from the introduction). In particular, there exists a real number $t_a > 0$, depending only on $a$, $\delta$ and $q$, such that $t_a$ is not an ordinate of any $a$-point of $\Delta$ and $\beta_a \in (0, 1)$ if, and only if, $\gamma_a \geq t_a$; we can actually choose $t_a$ here such that $\beta_a$ is included in any open interval centered at $1/2$, but the way we define it here yields, for instance, $N_a(T; f) = \# \{ \gamma_a : t_a < \gamma_a < T \} + O(1)$. 

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
Before showing (1.5), we use an argument similar to [37, § 9.2] to prove
\[ N_a(T + 1; f) - N_a(T; f) \ll \log T \] (2.7)
for any \( T \geq t_a + 3 \). Indeed, if \( n(r) \) denotes the number of \( a \)-points of \( \Delta(s) \) in the disc with center \( 2 + iT \) and radius \( r \), then
\[ N_a(T + 1; f) - N_a(T; f) \leq n(\sqrt{5}) \ll \int_0^3 \frac{n(r)}{r} \, dr. \]

It follows from Jensen’s formula (see, for example, [36, § 3.61]) and (2.4) that
\[ 3 \int_0^3 n(r) \, r \, dr = \frac{1}{2\pi} \int_0^{2\pi} \log|\Delta(2 + iT + 3e^{i\theta}) - a| \, d\theta - \log|\Delta(2 + iT) - a| \ll \log T, \]
and thus (2.7) holds.

To prove (1.5), we apply the argument principle to the function \( \Delta(s) - a \) and integrate over the counterclockwise oriented rectangle \( \mathcal{C} \) with vertices \( -1 + it_a, 2 + it_a, 2 + iT \) and \( -1 + iT \). This gives
\[ 2\pi iN_a(T; f) = \int_{\mathcal{C}} \frac{\Delta'(s)}{\Delta(s) - a} \, ds; \] (2.8)

since all poles of \( \Delta(s) \) lie on the real line, they do not affect here. In addition, all zeros lie outside \( \mathcal{C} \); hence, we may rewrite the integrand as
\[ \frac{\Delta'(s)}{\Delta(s) - a} = \frac{\Delta'(s)}{-a} \cdot \frac{1}{1 - \Delta(s)/a} \] (2.9)
or
\[ \frac{\Delta'(s)}{\Delta(s) - a} = \frac{\Delta'(s)}{-a} \cdot \frac{1}{1 - \Delta(s)/a}. \] (2.10)

Now, by (2.9) and taking into account (2.4) in combination with another form of Stirling’s formula,
\[ \frac{\Delta'}{\Delta}(\sigma + it) = -\log \frac{qt}{2\pi} + O\left(\frac{1}{t}\right), \] (2.11)
which is also valid for \( t \geq t_a > 0 \) and \( \sigma \) from any strip of bounded width, we obtain, for \( \epsilon > 0 \) and \( \sigma \geq 1/2 + \epsilon \),
\[ \frac{\Delta'(s)}{\Delta(s) - a} \ll \epsilon t^{1/2 - \sigma} \log t. \] (2.12)

Similarly, again by (2.9), we have, for \( \epsilon > 0 \) and \( \sigma \leq 1/2 - \epsilon \),
\[ \frac{\Delta'(s)}{\Delta(s) - a} = \frac{\Delta'}{\Delta}(s) \cdot \left(1 + \sum_{j \geq 1} \left(\frac{a}{\Delta(s)}\right)^j\right) = -\log \frac{qt}{2\pi} + O_\epsilon \left(t^{-1} + t^{\sigma - 1/2} \log t\right); \] (2.13)
the expansion into a geometric series is justified by (2.4).

Next, it follows from (2.12) that the contribution of the integral over the right vertical segment is negligible:
\[ \int_{2 + iT}^{2 + it_a} \frac{\Delta'(s)}{\Delta(s) - a} \, ds \ll 1. \]
Using expression (2.13), however, the contribution of the integral over the left vertical segment leads to
\[ \int_{-1+it_a}^{-1+iT} \frac{\Delta'(s)}{\Delta(s) - a} \, ds = -i \int_{t_a}^{T} \left( -\log \frac{qt}{2\pi} + O\left( \frac{1}{t} \right) \right) \, dt + O(\log T) = iT \log \frac{qT}{2\pi e} + O(\log T). \]

It remains to estimate the integrals on the horizontal segments of $C$. The lower one is trivially bounded:
\[ \int_{-1+it_a}^{2+it_a} \frac{\Delta'(s)}{\Delta(s) - a} \, ds \ll 1, \]
while for the upper one may use the following truncated partial fraction decomposition:
\[ \frac{\Delta'(s)}{\Delta(s) - a} = \sum_{|t - \gamma_a| \leq 1} \frac{1}{s - \delta_a} + O(\log t), \tag{2.14} \]
which is valid for $\sigma \in [-1, 2]$ and $t \geq t_a > 0$. We will show this after finishing the proof of Theorem 1.

In view of (2.14)
\[ \int_{2+iT}^{-1+iT} \frac{\Delta'(s)}{\Delta(s) - a} \, ds = \sum_{|T - \gamma_a| \leq 1} \int_{2+iT}^{-1+iT} \frac{ds}{s - \delta_a} + O(\log T). \]

By the calculus of the residues we obtain
\[ \int_{2+iT}^{-1+iT} \frac{ds}{s - \delta_a} = \left\{ \int_{2+iT}^{2+i(T+2)} + \int_{2+i(T+2)}^{-1+i(T+2)} + \int_{-1+i(T+2)}^{2+it_a} \right\} \frac{ds}{s - \delta_a} - 2\pi i R(\delta_a), \]
where $R(\delta_a)$ is 1 or 0 depending on whether $\delta_a$ lies inside the rectangle described above or not. Recall that $\beta_a \in (0, 1)$. Hence, for every $a$-point with $|T - \gamma_a| \leq 1$,
\[ \int_{2+iT}^{-1+iT} \frac{ds}{s - \delta_a} \ll \int_{T}^{T+2} \frac{dt}{|2 - \beta_a + i(t - \gamma_a)|} + \int_{-1}^{2} \frac{d\sigma}{|\sigma - \beta_a + i(T + 2 - \gamma_a)|} + \int_{T}^{T+2} \frac{dt}{|1 - \beta_a + i(t - \gamma_a)|} + 1 \ll 1. \]

In combination with inequality (2.7), we obtain
\[ \int_{2+iT}^{-1+iT} \frac{\Delta'(s)}{\Delta(s) - a} \, ds \ll \sum_{|T - \gamma_a| \leq 1} 1 + O(\log T) \ll \log T. \]

Finally, we arrive at
\[ \int_{C} \frac{\Delta'(s)}{\Delta(s) - a} \, ds = iT \log \frac{qT}{2\pi e} + O(\log T). \]

Substituting this into (2.8) finishes the proof of (1.5).

It remains to show (2.14). For this purpose, we apply Jacques Hadamard’s theory of functions of finite order (see [30, §5.3]); our reasoning is similar to the case of the Riemann zeta-function (see [37, §9.6]).
As already mentioned, $\Delta(s)$ is analytic except for simple poles at some positive integers. Thus,

$$F(s) := (\Delta(s) - a) \Gamma(1 - s)^{-1}$$

defines an entire function. By Stirling’s formula it follows that $F$ is entire and of order one. Hence, Hadamard’s factorization theorem implies the product representation

$$F(s) = \exp(A + Bs) \prod_{\delta_a} \left(1 - \frac{s}{\delta_a}\right) \exp\left(\frac{s}{\delta_a}\right),$$

where $A$ and $B$ are certain complex constants and the product is taken over all zeros $\delta_a$ of $F(s)$. Taking the logarithmic derivative, we deduce

$$\frac{F'}{F}(s) = B + \sum_{\delta_a} \left(\frac{1}{s - \delta_a} + \frac{1}{\delta_a}\right).$$

Since

$$\frac{F'}{F}(s) = \frac{\Delta'(s)}{\Delta(s) - a} + \frac{\Gamma'}{\Gamma}(1 - s)$$

and $(\Gamma'/\Gamma)(1 - s) \ll \log t$ (also a consequence of Stirling’s formula), we have

$$\frac{\Delta'(s)}{\Delta(s) - a} = \sum_{\delta_a} \left(\frac{1}{s - \delta_a} + \frac{1}{\delta_a}\right) + O(\log t).$$

Setting $s = 2 + it$ and subtracting from the latter expression

$$\frac{\Delta'(2 + it)}{\Delta(2 + it) - a},$$

which is $O(1)$ by (2.12), we obtain

$$\frac{\Delta'(s)}{\Delta(s) - a} = \sum_{\delta_a} \left(\frac{1}{s - \delta_a} - \frac{1}{2 + it - \delta_a}\right) + O(\log t).$$

In view of (2.7), it follows that

$$\sum_{|t - \gamma_a| \leq 1} \frac{1}{2 + it - \delta_a} \ll \sum_{|t - \gamma_a| \leq 1} 1 = N_a(t + 1; f) - N_a(t - 1; f) \ll \log t.$$

A short computation shows that, for any positive integer $n$,

$$\sum_{t + n < \gamma_a \leq t + n + 1} \left(\frac{1}{s - \delta_a} - \frac{1}{2 + it - \delta_a}\right) \ll \sum_{t + n < \gamma_a \leq t + n + 1} \frac{1}{n^2} \ll \frac{\log(t + n)}{n^2}.$$

And since $\sum_{n \geq 1} \log(t + n)/n^2 \ll \log t$, it follows that

$$\sum_{\gamma_a > t + 1} \left(\frac{1}{s - \delta_a} - \frac{1}{2 + it - \delta_a}\right) \ll \log t;$$

obviously, we can bound the sum over the $a$-points $\delta_a$ satisfying $\gamma_a < t - 1$ similarly by the same bound. Consequently, the contribution of the $a$-points distant from $s$ is negligible. This implies (2.14) and concludes the proof of Theorem 1.
3. PROOF OF THEOREM 2

In view of (1.5), it follows that for any given $T_0 > 0$ there exists $T \in [T_0, T_0 + 1)$ such that

$$\min_{\delta_a} |T - \gamma_a| \gg \frac{1}{\log T},$$

where the minimum is taken over all $a$-points $\delta_a$. In the proof of Theorem 1 we have observed that there exists a real number $t_a > 0$ such that $\beta_a \in (0, 1]$ if and only if $\gamma_a > t_a$. Thus integrating over the counterclockwise oriented rectangle $C$ with vertices $2 + it_a, 2 + iT, -\epsilon + iT$ and $-\epsilon + it_a$, where $\epsilon > 0$, we obtain

$$\sum_{0 < \gamma_a < T \atop 0 < \beta_a < 1} L(\delta_a; f) = \frac{1}{2\pi i} \left\{ \int_{2+it_a} + \int_{2+iT} - \int_{-\epsilon+iT} - \int_{-\epsilon+it_a} \right\} \frac{\Delta'(s)}{\Delta(s) - a} L(s; f) \, ds =: \sum_{1 \leq j \leq 4} I_j. \quad (3.2)$$

Since the Dirichlet series coefficients $f(n)$ are bounded, we have $L(s; f) \ll \epsilon$ for $\sigma \geq 1 + \epsilon$. In view of the functional equation (1.3) and the asymptotic formula (2.4), it follows from the Phragmén–Lindelöf principle (which is a kind of maximum principle for unbounded domains) that

$$L(\sigma + it; f) \ll \epsilon + t^{(1-\sigma)/2+\epsilon} \quad \text{for} \quad \sigma \in [0, 2], \quad t \geq x_0 > 0 \quad (3.3)$$

and

$$L(\sigma + it; f) \ll \epsilon t^{1/2-\sigma+\epsilon} \quad \text{for} \quad \sigma \in [-1, 0], \quad t \geq x_0 > 0 \quad (3.4)$$

(see [37, § 5.1] and [36, § 9.41] for the case of $\zeta(s)$; the generalization to $L(s; f)$ is straightforward).

We begin with the vertical integrals in (3.2). Since the range of integration of $I_1$ lies in the half-plane $\sigma > 1/2$, it follows from (2.12) in combination with (3.3) that

$$I_1 \ll \int_{t_a}^{T} t^{-3/2} \log t \, dt \ll 1. \quad (3.5)$$

The integral $I_3$ is over a line segment in $\sigma < 1/2$. It follows from the functional equation (1.3) and relation (2.1) that

$$L(1-s; f) = \Delta(1-s)L(s; f^+) = \Delta(1-s)\Delta(s)L(1-s; (f^+)^+). \quad (3.6)$$

Here we observe an instance of the Fourier inversion formula, namely, $(f^+)^+ = \delta f$. In order to see that, we compute via (1.2) that

$$(f^\pm)^+(n) = \frac{1}{\sqrt{q}} \sum_{a \mod q} f^\pm(a) e\left(\frac{an}{q}\right) = \frac{1}{q} \sum_{a,b \mod q} f(b) e\left(\frac{\pm b a}{q}\right) e\left(\frac{a n}{q}\right)$$

$$= \frac{1}{q} \sum_{b \mod q} f(b) \sum_{a \mod q} e\left(\frac{a(n \pm b)}{q}\right) = f(\mp n),$$

by the orthogonality relation for additive characters (or simply using geometric series). This proves

$$(f^\pm)^+ = f_{\mp} = \delta f. \quad (3.7)$$

Inserting this into (3.6) leads to

$$\Delta(1-s)\Delta(s) = \delta. \quad (3.8)$$
Thus using (2.13), we get
\[
\mathcal{I}_3 = -\frac{1}{2\pi i} \int_{-\epsilon + it_a}^{-\epsilon + iT} \left( \frac{a'}{\Delta}(s) \left( 1 + \frac{a}{\Delta(s)} + \sum_{j \geq 2} \left( \frac{a}{\Delta(s)} \right)^j \right) L(s; f) \right) ds
\]
\[
= \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} \left( \frac{a'}{\Delta}(1-s) \left( 1 + \delta a \Delta(s) + \sum_{j \geq 2} (\delta a \Delta(s))^j \right) \right) L(1-s; f) ds =: \sum_{1 \leq \ell \leq 3} \mathcal{J}_\ell. \tag{3.9}
\]

We estimate at first \( \mathcal{J}_1 \) by computing its conjugate \( \overline{\mathcal{J}_1} \). The functional equation (1.3), relations (2.1) and (2.5), estimate (2.11) and the Dirichlet series representation of \( L(s; f^+) \) in the half-plane \( \sigma > 1 \) yield
\[
\overline{\mathcal{J}_1} = -\frac{1}{2\pi i} \int_{t_a}^{T} \left( \frac{-a'}{\Delta}(-\epsilon + it) \Delta(-\epsilon + it) L(1 + \epsilon - it; f^+)(-i) \right) dt
\]
\[
= \frac{1}{2\pi i} \int_{t_a}^{T} \left( \frac{-a'}{\Delta}(-\epsilon + it) \right) \delta \Delta(-\epsilon - it) L(1 + \epsilon + it; f^+) i dt
\]
\[
= \delta \int_{t_a}^{T} \left( \log \frac{qT}{2\pi} + O\left( \frac{1}{T} \right) \right) d \left( \frac{1}{2\pi i} \int_{1+\epsilon+i}^{1+\epsilon+i} \Delta(1-s) L(s; f^+) ds \right). \tag{3.10}
\]

We evaluate the inner integral by applying Gonek’s lemma: Suppose that \( \sum_{n=1}^{\infty} a(n)n^{-s} \) converges for \( \sigma > 1 \) where \( a(n) \ll n^\epsilon \) for any \( \epsilon > 0 \). Let \( \omega = \pm 1 \) and \( \epsilon > 0 \). Then we have
\[
\frac{1}{2\pi i} \int_{1+\epsilon+i}^{1+\epsilon+i} \left( \frac{q}{2\pi} \right)^s \Gamma(s) \exp\left( \frac{\pi is}{2} \right) \sum_{n=1}^{\infty} \frac{a(n)}{n^s} ds
\]
\[
= \begin{cases} 
\sum_{n \leq \tau q/2\pi} a(n) \exp\left( -2\pi i \frac{n}{q} \right) + O_{\epsilon}(\tau^{1/2+\epsilon}) & \text{if } \omega = -1, \\
O_{\epsilon}(1) & \text{if } \omega = 1.
\end{cases}
\]

A proof of this lemma follows along the lines of [31, Lemma 3]. A slightly weaker statement was originally proved by Gonek [7, Lemma 5] (and there exists a preversion of it in [37, §7.4]). Since
\[
\Delta(1-s) = \left( \frac{q}{2\pi} \right)^s \frac{\Gamma(s)}{\sqrt{q}} \left( \delta e\left( \frac{s}{4} \right) + e\left( -\frac{s}{4} \right) \right)
\]
by the definition (1.4) of \( \Delta \), this gives here
\[
\frac{1}{2\pi i} \int_{1+\epsilon+i}^{1+\epsilon+i} \Delta(1-s) L(s; f^+) ds = \frac{1}{\sqrt{q}} \sum_{n \leq \tau q/2\pi} f^+(n) e\left( -\frac{n}{q} \right) + O(\tau^{1/2+\epsilon});
\]
we can simplify the right-hand side: taking into account the \( q \)-periodicity, we find
\[
\sum_{n \leq \tau q/(2\pi)} f^+(n) e\left( -\frac{n}{q} \right) = \sum_{a \mod q} f^+(a) e\left( -\frac{a}{q} \right) \sum_{n \leq \tau q/(2\pi)} e\left( \frac{a}{q} \frac{n}{\tau q/2\pi} \right) = \sum_{a \mod q} f^+(a) e\left( \frac{a}{q} \right) \left( \frac{\tau}{2\pi} + O(1) \right)
\]
\[
= \sqrt{q} f^+(1) \frac{\tau}{2\pi} + O(1) = \sqrt{q} \delta f(1) \frac{\tau}{2\pi} + O(1).
\]
This leads via (3.10) to
\[ J_1 = f(1) \frac{T}{2\pi} \log \frac{qT}{2\pi} + O(T^{1/2+\epsilon}). \] (3.11)

Next we consider
\[ J_2 = \frac{a}{2\pi\delta i} \int_{1+\epsilon-it_a}^{1+\epsilon-it} \frac{\Delta'}{\Delta} (1-s) \Delta(s) \Delta(1-s) L(s; f^+) \, ds. \]

In view of (3.8) this equals \( a/(2\pi) \) times the conjugate of
\[
\left. \int_{t_a}^T \frac{\Delta'}{\Delta}(-\epsilon+it)L(1+\epsilon-it; f^+) \, dt = \int_{t_a}^T \left( \log \frac{qt}{2\pi} + O\left(\frac{1}{t}\right) \right) \sum_{n \geq 1} \frac{f^+(n)}{n^{1+\epsilon+it}} \, dt \right.
\]
\[ = f^+(1) \int_{t_a}^T \log \frac{qt}{2\pi} \, dt + \sum_{n \geq 2} \frac{f^+(n)}{n^{1+\epsilon}} J_n(T) + O(1), \]

where we have used the absolute convergence in \( \sigma > 1 \) and \( J_n(T) \) for \( n \geq 2 \) is given by
\[ J_n(T) := \int_{t_a}^T \log \frac{qt}{2\pi} \exp(-it \log n) \, dt. \]

To bound this integral, we will use the first derivative test: \textit{Given real functions } \( F \) \textit{and } \( G \) \textit{on } \([a,b]\) \textit{such that } \( G(t)/F'(t) \) \textit{is monotonic and either } \( F'(t)/G(t) \geq M > 0 \) \textit{or } \( F'(t)/G(t) \leq -M < 0 \), \textit{we have}
\[ \left. \int_a^b G(t) \exp(iF(t)) \, dt \ll \frac{1}{M}. \right. \]

This is essentially a classical lemma from [37, §4.3]. It leads to the estimate \( J_n(T) \ll \log T/\log n \) for \( n \geq 2 \). Hence, the contribution of the tail of the Dirichlet series is negligible and we get
\[ J_2 = a f^+(1) \frac{T}{2\pi} \log \frac{qT}{2\pi} + O(\log T). \] (3.12)

Finally, we have to consider the third integral of the tail of the geometric series. For this aim we apply (2.4), (2.11) and (3.4) and get
\[ J_3 \ll \int_{t_a}^T \log t \sum_{j \geq 2} t^{-j(1/2+\epsilon)} t^{1/2+\epsilon} \, dt \ll T^{1/2+\epsilon}. \]

This together with (3.11) and (3.12) substituted into (3.9), in combination with (3.5), shows that the vertical integrals contribute
\[ I_1 + I_3 = (f(1) + a f^+(1)) \frac{T}{2\pi} \log \frac{qT}{2\pi} + O(T^{1/2+\epsilon}), \] (3.13)

which is already the main term. It remains to consider the horizontal integrals in (3.2). Although they can be treated in the same way as in [35], we sketch the details.

The integral \( I_2 \) in (3.2) can be rewritten with the aid of the truncated partial fraction decomposition (2.14) as
\[ I_2 = \frac{1}{2\pi i} \int_{2+iT}^{2-iT} \left( \sum_{|T-\gamma_i| \leq 1} \frac{1}{s - \delta_a} + O(\log T) \right) L(s; f) \, ds. \]
Taking into account (3.1), we notice that $1/|s - \delta_a| \ll \log T$. Hence, utilizing (2.7), (3.3) and (3.4), we can show

$$I_2 \ll (\log T)^2 \left\{ \int_{-\epsilon}^{0} + \int_{0}^{1} + \int_{1}^{2} \right\} |L(\sigma + it; f)| d\sigma \ll_{\epsilon} (\log T)^2 \left\{ T^{1/2+\epsilon} + T^{1/2+\epsilon} + 1 \right\} \ll_{\epsilon} T^{1/2+\epsilon},$$

where $\epsilon$ at different places may take different values. This is the bound for the horizontal integrals.

Combining it with (3.13), we arrive via (3.2) at the asymptotic formula of the theorem. Taking into account (3.3) and (3.4), we may replace the chosen $T$ (with respect to (3.1)) with a general $T \geq t_0$ at the expense of an error of order $T^{1/2+\epsilon}$ (as follows from (3.3)). This concludes the proof of Theorem 2.

4. PROOF OF THEOREM 3

We first note that (2.4) implies

$$\Delta(\sigma + it) \asymp \left( \frac{qt}{2\pi} \right)^{1/2-\sigma}, \quad t \geq t_0 > 0,$$

while (2.6) yields

$$\beta_a = \frac{1}{2} - \frac{\log|a|}{\log q\gamma_a/(2\pi)} + O \left( \frac{1}{\gamma_a \log(q\gamma_a/(2\pi))} \right)$$

for any $\gamma_a \geq T_0 := \max\{1,4\pi/q\}$.

Let $T \geq T_0$, $x > 1$ and

$$\alpha = \alpha(T) := \frac{1}{2} + \frac{c}{\log(qT/(2\pi))},$$

where $c > 0$ is a sufficiently large constant depending on $a$ and satisfying

$$\left| \beta_a - \frac{1}{2} \right| \leq \frac{c}{2\log(qT/(2\pi))}$$

for any $a$-point with $\gamma_a \geq T \geq T_0$, as follows from (4.2), and

$$|\Delta(\alpha + it)| \leq K \left( \frac{qt}{2\pi} \right)^{-c/\log(qT/(2\pi))} \leq Ke^{-c} < \frac{|a|}{30},$$

$$|\Delta(\alpha + it)|^{-1} = |\Delta(1 - \alpha + it)| \geq L \left( \frac{qt}{2\pi} \right)^{c/\log(qT/(2\pi))} \geq Le^{c} > 30|a|$$

for any $t \geq T \geq T_0$, where $K$ and $L$ are absolute constants coming from (4.1) and we can assume without loss of generality that $K > 1$ is sufficiently large and $L = 1/K$. After choosing a suitable $K$, we then take $c$ sufficiently large. These show that $\Delta(\alpha + it) \neq a$ and $\Delta(1 - \alpha + it) \neq a$ when $t \geq T$.

We know from (1.5) that, for any $T_0 \leq T < T + 1 \leq T' \leq 2T$, we can find $T_1 \in [T, T + 1/2)$ and $T_2 \in (T' - 1/2, T']$ such that

$$\min_{\ell = 1,2} \min_{\delta_a} |T_{\ell} - \gamma_a| \gg \frac{1}{\log T}.$$  

If $C$ is the positively oriented rectangular contour with vertices $\alpha + iT_1$, $\alpha + iT_2$, $1 - \alpha + iT_2$ and $1 - \alpha + iT_1$, then (1.5), (4.3) and the calculus of residues yield

$$\sum_{T < \gamma_a < T'} x^{\delta_a} = \sum_{T_1 < \gamma_a < T_2} x^{\delta_a} + \sum_{T < \gamma_a < T_1} x^{\delta_a} + \sum_{T_2 < \gamma_a < T'} x^{\delta_a}$$

$$= \frac{1}{2\pi i} \int_{C} \frac{\Delta'(s)}{\Delta(s) - a} x^s ds + O(x^\alpha \log T).$$  

4. PROOF OF THEOREM 3
We break the integral
\[
\frac{1}{2\pi i} \int_C \frac{\Delta'(s)}{\Delta(s) - a} x^s \, ds
\]
down into
\[
\frac{1}{2\pi i} \left( \int_{\alpha+iT_2}^{1-\alpha+iT_2} + \int_{\alpha+iT_1}^{1-\alpha+iT_1} + \int_{1-\alpha+iT_2}^{\alpha+iT_1} \right) \frac{\Delta'(s)}{\Delta(s) - a} x^s \, ds =: \sum_{1 \leq j \leq 4} I_j.
\]

In view of (2.14) and (4.5), for \( \ell = 1, 2 \) we have
\[
\int_{\alpha+iT_\ell}^{\alpha+iT_\ell} \frac{\Delta'(s)}{\Delta(s) - a} x^s \, ds = \int_{1-\alpha+iT_\ell}^{\alpha+iT_\ell} \left( \sum_{|t-\gamma_a| \leq 1} \frac{1}{s - \delta_a} + O(\log t) \right) x^s \, ds
\]
\[
= \int_{1-\alpha+iT_\ell}^{\alpha+iT_\ell} \sum_{|t-\gamma_a| \leq 1} \frac{1}{s - \delta_a} x^s \, ds + O(x^\alpha),
\]

since
\[
\int_{\alpha+iT_\ell}^{\alpha+iT_\ell} O(\log t) x^s \, ds \ll \log T \int_{1-\alpha}^{\alpha} x^\sigma \, d\sigma \ll x^\alpha (2\alpha - 1) \log T \ll x^\alpha.
\]

Meanwhile,
\[
\int_{1-\alpha+iT_\ell}^{\alpha+iT_\ell} \sum_{|t-\gamma_a| \leq 1} \frac{1}{s - \delta_a} x^s \, ds \ll \sum_{|T_\ell - \gamma_a| \leq 1} \int_{1-\alpha}^{\alpha} \frac{x^\sigma}{|\alpha - \beta_a + i(T_\ell - \gamma_a)|} \, d\sigma
\]
\[
\ll x^\alpha (2\alpha - 1) \log T \sum_{|T_\ell - \gamma_a| \leq 1} 1 \ll x^\alpha \log T.
\]

Therefore,
\[
I_2, I_4 \ll x^\alpha \log T. \tag{4.7}
\]

We now estimate the vertical integrals \( I_1 \) and \( I_3 \). For \( \sigma = \alpha \), we use (2.10) and thus
\[
\frac{\Delta'(s)}{\Delta(s) - a} = \frac{\Delta'(s)}{-a} \frac{1}{1 - \Delta(s)/a} = \frac{\Delta'(s)}{-a} \left( 1 + \sum_{j \geq 1} \left( \frac{\Delta(s)}{a} \right)^j \right);
\]

here the first inequality in (4.4) allows us to expand the second factor into a geometric series. Applying this, we find
\[
I_1 = -\frac{x^\alpha}{2\pi a} \int_{T_1}^{T_2} \Delta'(\alpha + it) \left( \sum_{0 \leq j < m} \left( \frac{\Delta(\alpha + it)}{a} \right)^j + \sum_{j \geq m} \left( \frac{\Delta(\alpha + it)}{a} \right)^j \right) x^it \, dt. \tag{4.9}
\]

Here we pick \( m \) depending on \( T \), large enough to bound the last term in the integral trivially. By (4.1) and again the first inequality in (4.4),
\[
\sum_{j \geq m} \left( \frac{\Delta(\alpha + it)}{a} \right)^j = \frac{\left( \frac{\Delta(\alpha + it)}{a} \right)^m}{1 - \Delta(\alpha + it)/a} \ll \frac{30}{29} \left( \frac{1}{30} \right)^m. \tag{4.10}
\]
We see that if

The first term of the integrand can be estimated using (2.4), for which we obtain

Here we substituted the value of $\alpha$ or

Since $I$ we can rewrite

It follows then by (4.10) that

We estimate

Here we substituted the value of $\alpha$ and used the first inequality in (4.4). It then follows that we can discard the last term in $I_1$ as

Since

we can rewrite $I_1$ as

We estimate $I_{1j}$ for $1 \leq j \leq m$. Integrating by parts, we obtain with the aid of (2.4) and the first inequality in (4.4)

We see that if $|O(t^{-1})| < D/t$ for some $D > 1$, then

or

$$(1 + O(t^{-1}))^j = 1 + O((2D)^j t^{-1}).$$
In view of (4.4), we then have, for a sufficiently large constant $K$,

$$I_{1j} \ll \log x \left| \int_{T_1}^{T_2} \left( \frac{qt}{2\pi} \right)^{\left( -\epsilon/\log(qT/(2\pi)) - it \right) j} \exp(it) \left( x^{1/j} \right)^{it} \frac{j}{t} \right|$$

$$+ \log x \int_{T_1}^{T_2} \left( 2D \left( \frac{qt}{2\pi} \right)^{\left( -\epsilon/\log(qT/(2\pi)) \right) j} \right)^{-1} \frac{j}{t} \left( \frac{|a|}{30} \right)^{j}$$

$$\ll q^{-jc/\log(qT/(2\pi))} \log x \left| \int_{T_1}^{T_2} \left( \frac{t}{2\pi} \right)^{\left( -jc/\log(qT/(2\pi)) \right) j} \exp \left[ -ijt \log \left( \frac{qt}{2\pi x^{1/j}e} \right) \right] \left( \frac{|a|}{30} \right)^{j}$$

$$+ \left( 2D \frac{|a|}{30K} \right)^{j} \log x \int_{T_1}^{T_2} t^{-1} \frac{j}{t} \left( \frac{|a|}{30} \right)^{j}$$

$$\ll \frac{\log x}{j} \left( \frac{q}{j} \right)^{\left( -jc/\log(qT/(2\pi)) \right)} |J| + (1 + \log x) \left( \frac{|a|}{30} \right)^{j},$$

(4.13)

where

$$J := \int_{jT_1}^{jT_2} \exp \left[ -it \log \left( \frac{qt}{2\pi x^{1/j}e} \right) \right] \left( \frac{t}{2\pi} \right)^{1/2 - ic/\log(qT/(2\pi)) - 1/2} \frac{j}{t}.$$

To estimate $J$, we employ another form of Gonek’s lemma (see [7, Lemma 2]): For large $A$ and $A < r \leq B \leq 2A$,

$$\int_{A}^{B} \exp \left[ it \log \left( \frac{t}{re} \right) \right] \left( \frac{t}{2\pi} \right)^{a-1/2} \frac{j}{t} = (2\pi)^{1-a} r^a \exp \left( -i \left( r - \frac{\pi}{4} \right) \right) \frac{j}{t}.$$ 

where $a$ is a fixed real number and

$$E(r, A, B) \ll A^{a-1/2} + \frac{A^{a+1/2}}{|A - r| + A^{1/2}} + \frac{B^{a+1/2}}{|B - r| + B^{1/2}}.$$

It is easily seen that this holds uniformly in $a$ from a bounded interval (the implicit constants depend on the limits of this interval). We then apply this, by taking complex conjugate, to $J$ with $a = 1/2 - j/\log(qT/(2\pi))$, $r = 2\pi x^{1/j}/q$, $A = jT_1$ and $B = jT_2$:

$$J \ll (2\pi)^{1/2 + j/\log(qT/(2\pi))} \left( \frac{2\pi x^{1/j}}{q} \right)^{1/2 - j/\log(qT/(2\pi))} \frac{j}{t} \left( jT_1 \right)^{1/jc/\log(qT/(2\pi))} \frac{j}{t} \left( jT_2 \right)^{1/jc/\log(qT/(2\pi))} \frac{j}{t} \left( jT_1 + jT_2 \right) \left( \frac{2\pi x^{1/j}}{q} \right)^{1/2 - j/\log(qT/(2\pi))}$$

$$+ \left( jT_1 \right)^{1/jc/\log(qT/(2\pi))} + \left( jT_2 \right)^{1/jc/\log(qT/(2\pi))} + \frac{\left( jT_1 \right)^{1/jc/\log(qT/(2\pi))}}{|jT_1 - 2\pi x^{1/j}/q| + \sqrt{jT_1}} + \frac{\left( jT_2 \right)^{1/jc/\log(qT/(2\pi))}}{|jT_2 - 2\pi x^{1/j}/q| + \sqrt{jT_2}}$$

$$\ll \left( \frac{2}{q} \right)^{1/2 - j/\log(qT/(2\pi))} x^{1/(2j) - c/\log(qT/(2\pi))} \left( \frac{2\pi x^{1/j}}{q} \right)$$

$$+ \left( jT \right)^{jc/\log(qT/(2\pi))} E_0(x, j, T),$$

where $E_0(x, j, T)$ is a bounded function.
where
\[ E_0(x, j, T) = \begin{cases} O(\sqrt{J}) & \text{if } \frac{2\pi x^{1/j}}{q} \in \left( \frac{T}{2}, \frac{5T}{2} \right), \\ O(1) & \text{otherwise}. \end{cases} \] (4.14)

Recall that \( T \asymp T_1 \asymp T_2 \) and \( j \ll \log T \). Therefore, \( E_0 \) does not restrictively depend on \( T_1 \) and \( T_2 \).

Applying this to (4.13), we have
\[
I_{1j} \ll \frac{\log x}{\sqrt{jq}} x^{1/(2j)-c/\log(qT/(2\pi))} \frac{1}{(T/2,5T/2)} \left( \frac{2\pi x^{1/j}}{q} \right)
+ \frac{\log x}{j} \left( (qT)^{-c/\log(qT/(2\pi))} \right)^j E_0(x, j, T) + (1 + \log x) \left( \frac{|a|}{30} \right)^j.
\]

Hence, in view of (4.12) and (4.4), we obtain
\[
I_1 \ll x^{1/2} \log x \sum_{1 \leq j \leq m} \frac{x^{1/(2j)}}{j^{3/2} |a|^j} \frac{1}{(T/2,5T/2)} \left( \frac{2\pi x^{1/j}}{q} \right)
+ x^\alpha \log x \sum_{1 \leq j \leq m} \frac{1}{j^2 |a|^j} \left( (2\pi)^{-c/\log(qT/(2\pi))} \right)^j E_0(x, j, T)
+ x^\alpha (1 + \log x) \sum_{1 \leq j \leq m} \frac{1}{j \cdot 30^j} + x^\alpha.
\] (4.15)

Observe that the intervals
\[ \left( \left( \frac{qT}{4\pi} \right)^{1/j}, \left( \frac{5qT}{4\pi} \right)^{1/j} \right), \quad 1 \leq j \leq m, \]

are pairwise disjoint whenever
\[ j < \frac{1}{\log 5} \log \frac{qT}{4\pi}. \]

Comparing this with (4.11), we see that
\[ m \leq \frac{2}{\log 30} \log \frac{qT}{4\pi} < \frac{1}{\log 5} \log \frac{qT}{4\pi} \]
for all \( T \geq T_0 \). By this construction, the first sum in (4.15) can have at most only one term, namely,
\[
\frac{x^{1/(2j_x)}}{j_x^{3/2} |a|^{j_x}}
\]
for the unique, if any, \( j_x \leq m \) such that \( 2\pi x^{1/j_x} / q \in (T/2,5T/2) \). Similar reasoning also applies to \( E_0 \) of the second sum in (4.15), and it follows in view of (4.14) that
\[
\sum_{1 \leq j \leq m} \frac{1}{j^2 \cdot 30^j} E_0(x, j, T) \ll \sum_{1 \leq j \leq m, j \neq j_x} \frac{1}{j^2 \cdot 30^j} + \frac{T^{1/2}}{j_x^{3/2} \cdot 30^{j_x}} \ll 1 + \frac{T^{1/2}}{j_x^{3/2} \cdot 30^{j_x}}.
\]

The last sum in (4.15) is trivially \( O(1) \). Collecting these estimates, we conclude that
\[
I_1 \ll x^\alpha (1 + \log x) + E_1(x, a, T), \] (4.16)
where \( E_1(x, a, T) \) is defined to be equal to
\[
\log\frac{x}{j_x^{3/2}} \left( \frac{x^{1/2 + 1/(2j_x)}}{|a|^{j_x}} + \frac{x^\alpha T^{1/2}}{30x} \right) \quad (4.17)
\]
if such \( j_x \) exists, and zero otherwise. Observe that if \( T \geq T_0 \geq 4\pi/q \), then for any \( x > 1 \) we have \( E_1(x, a, T) \geq 0 \), while for \( 0 < x < 1 \) we always have \( E_1(x, a, T) = 0 \).

For \( \sigma = 1 - \alpha \), we use (2.9) and the corresponding (2.13) to write
\[
\frac{\Delta'(s)}{\Delta(s) - a} = \frac{\Delta'(s)}{\Delta(s)} \cdot \frac{1}{1 - \alpha/\Delta(s)} = \frac{\Delta'(s)}{\Delta(s)} \left[ 1 + \sum_{j \geq 1} \left( \frac{a}{\Delta(s)} \right)^j \right]; \quad (4.18)
\]
here the second inequality in (4.4) allows us to expand the second factor into a geometric series. Thus the left vertical integral \( I_3 \) can be decomposed as follows:
\[
I_3 = \frac{1}{2\pi i} \int_{1 - \alpha +iT_1}^{1 - \alpha +iT_2} \frac{\Delta'(s)}{\Delta(s) - a} x^s \, ds
= \frac{1}{2\pi i} \int_{1 - \alpha +iT_1}^{1 - \alpha +iT_2} \frac{\Delta'(s)}{\Delta(s)} x^s \, ds + \frac{1}{2\pi i} \int_{1 - \alpha +iT_2}^{1 - \alpha +iT_1} \frac{\Delta'(s)}{\Delta(s)} x^s \sum_{j \geq 1} \left( \frac{a}{\Delta(s)} \right)^j \, ds =: I_{31} + I_{32}.
\]

Integrating by parts, we obtain in view of (2.11)
\[
I_{31} = \frac{x^{1-\alpha}}{2\pi} \int_{T_1}^{T_2} \left( -\log \frac{qt}{2\pi} + O \left( \frac{1}{t} \right) \right) x^it \, dt
= \frac{x^{1-\alpha}}{2\pi i \log x} \left( -\log \frac{qt}{2\pi} + O \left( \frac{1}{t} \right) \right) \bigg|_{T_1}^{T_2} - \frac{x^{1-\alpha}}{2\pi i \log x} \int_{T_1}^{T_2} O \left( \frac{1}{t} \right) \, dt \ll \frac{x^{1-\alpha} \log T}{\log x}. \quad (4.19)
\]

In the case of \( I_{32} \), it follows from (3.8) that
\[
I_{32} = -\frac{1}{2\pi i} \int_{-iT_2}^{-iT_1} \frac{\Delta'(s)}{\Delta(s)} x^{1-s} \sum_{j \geq 1} \left( \frac{a}{\Delta(1-s)} \right)^j \, ds
= \frac{1}{2\pi i} \int_{-iT_2}^{-iT_1} \frac{\Delta'(s)}{\Delta(s)} x^{1-s} \sum_{j \geq 1} \left( \frac{a \Delta(s)}{\delta} \right)^j \, ds
= -\frac{x^{1-\alpha}}{2\pi} \int_{T_1}^{T_2} \frac{\Delta'(s)}{\Delta(s)} x^it \sum_{j \geq 1} \left( \frac{a \Delta(s)}{\delta} \right)^j \, dt,
\]
or from (2.5)
\[
I_{32} = -\frac{x^{1-\alpha}}{2\pi} \int_{T_1}^{T_2} \frac{\Delta'(s)}{\Delta(s)} x^it \sum_{j \geq 1} \left( \frac{\pi \Delta(s)}{\delta} \right)^j \, dt
= -\frac{\pi x^{1-\alpha}}{2\pi} \int_{T_1}^{T_2} \frac{\Delta'(s)}{\Delta(s)} x^it \sum_{j \geq 0} \left( \frac{\pi \Delta(s)}{\delta} \right)^j \left( \frac{1}{x} \right)^{it} \, dt. \quad (4.20)
\]

To estimate now \( I_{32} \), we proceed exactly as in the estimation of \( I_1 \) in (4.9), where we have \( 1/\pi \) instead of \( a \), \( x^{1-\alpha} \) instead of \( x^\alpha \) and \( y := 1/x \) instead of \( x \). We can then derive, as in (4.9)–(4.16),
\[
I_{32} \ll x^{1-\alpha}(1 + \log x) + E_2(x, a, T), \quad (4.21)
\]
where \( E_2(x, a, T) \) is defined to be equal to

\[
\log x \left( x^{1/2 - 1/(2q_v)} |a|^{j_v} + \frac{x^{1-\alpha} T^{1/2}}{30^\nu} \right)
\]  

(4.22)

if such \( j_v \) exists, and zero otherwise. Observe that if \( T \geq T_0 \geq 4\pi/q \), then for any \( x > 1 \) we always have \( E_2(x, a, T) = 0 \), while for \( 0 < x < 1 \) we have \( E_2(x, a, T) \leq 0 \).

Collecting the estimates in (4.6), (4.7), (4.16), (4.19) and (4.21), we obtain

\[
\sum_{T < \gamma_a < T'} x^{\delta_a} \ll x^\alpha \left( \log x + \log T + \frac{\log T}{\log x} \right) + E_1(x, a, T)
\]

for any \( x > 1 \) and any \( T_0 \leq T < T + 1 \leq T' \leq 2T \).

The case of \( 0 < x < 1 \) follows from (2.5), (3.8) and the above estimate. Indeed, relations (2.5) and (3.8) imply that a complex number \( z \) is an \( \alpha \)-point of \( \Delta(s) \) (where \( a \neq 0 \)) if and only if the complex number \( 1 - \varpi \) is a \( b \)-point of \( \Delta \), where \( b := 1/\varpi \). Thus if \( 0 < x < 1 \), then

\[
\frac{1}{x} \sum_{T < \gamma_a < T'} x^{\delta_a} = \sum_{T < \gamma_a < T'} \left( \frac{1}{x} \right)^{1-\delta_a} = \sum_{T < \gamma_a < T'} \left( \frac{1}{x} \right)^{1-\delta_a},
\]

or

\[
\sum_{T < \gamma_a < T'} x^{\delta_a} \ll x^{1-\alpha} \left( -\log x + \log T - \frac{\log T}{\log x} \right) + x \left( E_1 \left( \frac{1}{x}, b, T \right) \right).
\]

By symmetry, it follows easily from (4.17) and (4.22) that

\[
x E_1 \left( \frac{1}{x}, b, T \right) = -E_2(x, a, T) \geq 0.
\]

Hence,

\[
\sum_{T < \gamma_a < T'} x^{\delta_a} \ll (x^\alpha + x^{1-\alpha}) \left( |\log x| + \log T + \frac{\log T}{|\log x|} \right) + 1_{(1,+\infty)}(x) E_1(x, a, T) - 1_{(0,1)}(x) E_2(x, a, T)
\]

for any \( 0 < x \neq 1 \) and any \( T_0 \leq T < T + 1 \leq T' \leq 2T \).

The last statement of the theorem follows by the above construction. If \( x \neq 1 \) is such that \( 4\pi/(qT) \leq x \leq qT/(4\pi) \), then \( E_1(x, a, T) = E_2(x, a, T) = 0 \) and

\[
\sum_{T < \gamma_a < T'} x^{\delta_a} \ll x^{1/2} \left( x^{c/|\log(qT)/(2\pi)|} + x^{-c/|\log(qT)/(2\pi)|} \right) \left( |\log x| + \log T + \frac{\log T}{|\log x|} \right)
\]

\[
\ll x^{1/2} \left( 1 + \frac{1}{|\log x|} \right) \log T.
\]

5. PROOF OF THEOREM 4

The Riemann–von Mangoldt type formula (1.5) implies that

\[
N_a(T; f) \sim \frac{T \log T}{2\pi} \quad \text{and} \quad \gamma_a^{(n)} \sim \frac{2\pi n}{\log n}.
\]

(5.1)

Therefore, there is a \( K \in \mathbb{N} \) such that \( 2^{K-1} \leq \gamma_a^{(2n)}/\gamma_a^{(n)} \leq 2^K \) for all \( n \in \mathbb{N} \). If we now set \( \delta_a^{(n)} := \beta_a^{(n)} + \iota \gamma_a^{(n)} \) to be the \( \alpha \)-point of \( \Delta \) with ordinate \( \gamma_a^{(n)} \), then for any integers \( 1 \leq N < M \leq 2N \) and any positive number \( x \neq 1 \), we have

\[
\sum_{N < n \leq M} x^{\delta_a^{(n)}} = \sum_{k \leq K} \sum_{2^{k-1} \gamma_a^{(N)} < \gamma_a \leq \min\{\gamma_a^{(M)}, 2^k \gamma_a^{(N)}\}} x^{\delta_a},
\]

(5.2)
where the inner sum on the right-hand side is zero when the set of indices of its summands is empty. Since $4\pi/(qN) \leq x \leq qN/(4\pi)$, (5.1), (5.2) and Theorem 3 yield

$$\sum_{N<n\leq M} x^{\delta_a(n)} \ll x^{1/2} \left( 1 + \frac{1}{|\log x|} \right) \log N.$$ 

We also have

$$x^{-\beta_a(M)} \ll x^{-1/2 \pm c/(\log(qN/(2\pi)))} \ll x^{-1/2}$$

for any $M > N$. Using this and Abel’s summation formula (summation by parts), we obtain

$$\sum_{N<n<2N} x^{\gamma_a(n)} = \sum_{N<n\leq 2N} x^{-\beta_a(n)} x^{\delta_a(n)}$$

$$\ll x^{-\beta_a(2N)} \left| \sum_{N<n<2N} x^{\delta_a(n)} \right| + \max_{N<M<2N} \left| \sum_{N<n\leq M} x^{\delta_a(n)} \right| \sum_{N<M<2N} \left| x^{-\beta_a(M+1)} - x^{-\beta_a(M)} \right|$$

$$\ll \left( 1 + \frac{1}{|\log x|} \right) \log N \left( 1 + \sum_{N<M<2N} \left| x^{\beta_a(M+1) - \beta_a(M)} - 1 \right| \right).$$

(5.3)

By (2.6) and (4.2),

$$\frac{\beta_a(M+1) - \beta_a(M)}{\log|a|} = \left[ \left( \log \frac{q_\alpha(M+1)}{2\pi} \right)^{-1} - \left( \log \frac{q_\alpha(M)}{2\pi} \right)^{-1} + O\left( \frac{1}{\gamma_a(M) \log \gamma_a(M)} \right) \right]$$

and $4\pi/(qN) \leq x \leq qN/(4\pi)$. Hence, with the aid of (5.1) we can show

$$\sum_{N<M<2N} \left| x^{\beta_a(M+1) - \beta_a(M)} - 1 \right|$$

$$\ll |\log x| \sum_{N<M<2N} \left[ \left( \log \frac{q_\alpha(M+1)}{2\pi} \right)^{-1} - \left( \log \frac{q_\alpha(M)}{2\pi} \right)^{-1} + \frac{1}{\gamma_a(M) \log \left( q_\alpha(M)/(2\pi) \right)} \right]$$

$$\ll |\log x| \left[ \left( \log \frac{\gamma_a(N+1)}{2\pi} \right)^{-1} - \left( \log \frac{\gamma_a(2N-1)}{2\pi} \right)^{-1} + \frac{N}{\gamma_a(N) \log \gamma_a(N)} \right]$$

$$\ll |\log x| \left( \log \gamma_a(N) \right)^{-2} + 1 \ll |\log x|.$$ 

This and inequality (5.3) lead for fixed $x$ to

$$\sum_{N<n\leq 2N} x^{\gamma_a(n)} \ll \left( \frac{1}{|\log x|} + |\log x| \right) \log N = o_x((\log N)^2),$$

(5.4)

which is the first statement of the theorem.

Let now $\alpha \neq 0$ be a real number and $k \neq 0$ be an integer. If $x = \exp(2\pi \alpha k)$, then for every $N \geq N_x := 4\pi x/q$ and $K_x := [\log(N/N_x)/\log 2]$ we have

$$\sum_{n\leq N} e(k\alpha \gamma_a(n)) = \sum_{n\leq N/2K_x} e(k\alpha \gamma_a(n)) + \sum_{k \leq K_x} \sum_{N/2^k < n \leq N/2^{k-1}} x^{\gamma_a(n)}$$

$$= O(N_x) + \sum_{k \leq K_x} o_x((\log N)^2) = o_x((\log N)^3),$$

which is the second statement of the theorem.
as follows from (5.4). Since $k \in \mathbb{Z} \setminus \{0\}$ can be chosen arbitrarily, the sequence $\alpha \gamma_a^{(n)}$, $n \in \mathbb{N}$, satisfies Weyl's criterion (see [43]): A sequence $(x_n)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if

$$\sum_{n \leq N} e(kx_n) = o(N)$$

for any integer $k \neq 0$. This concludes the proof of the theorem.

6. PROOF OF THEOREM 5

If $\chi$ is a Dirichlet character mod $r$ and $Q > 0$, we define the truncated and twisted Euler product

$$s \mapsto L_Q(s; \chi) := \prod_{p \leq Q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for every $s \in \mathbb{C}$ with $\sigma > 0$, where $p$ will denote from here on a prime number.

The proof of (1.9) is similar to Reich’s proof [25] for the discrete universality of $\zeta(s)$, and we will not repeat it here. Instead we employ the following theorem which highlights the necessary conditions a sequence $(x_n)_{n \in \mathbb{N}}$ has to meet in order to derive universality: Let $\chi_1, \ldots, \chi_J$ be pairwise non-equivalent Dirichlet characters. Let also $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that the sequence of vectors

$$\left(\frac{x_n \log p}{2\pi}\right)_{p \in \mathcal{M}}, \quad n \in \mathbb{N},$$

(6.2)

is uniformly distributed modulo 1 for any finite set of primes $\mathcal{M}$, and

$$\lim_{Q \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{N \leq n \leq 2N} |L(s + ix_n; \chi_j) - L_Q(s + ix_n; \chi_j)|^2 = 0, \quad j = 1, \ldots, J,$$

(6.3)

uniformly in compact subsets of the strip $1/2 < \sigma < 1$. Then for any compact subset with connected complement $K$ of this strip, any $g_1, \ldots, g_J$ continuous non-vanishing functions on $K$ and analytic in its interior, any $z > 0$ and $\xi_p$, $p \leq z$, real numbers, and any $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \left\{1 \leq n \leq N: \max_{1 \leq j \leq J} \max_{s \in K} |L(s + ix_n; \chi_j) - g_j(s)| < \varepsilon \right\} > 0.$$

For the proof see [28, Theorem 3.1].

The definition of uniform distribution of a multidimensional sequence (sequence of vectors) is analogous to the one-dimensional case (sequence of numbers), and so we omit the details here. However, we will use an equivalent statement of it (see [18, Ch. I, Theorem 6.3]): A sequence $(\xi_n)_{n \in \mathbb{N}}$ of vectors from $\mathbb{R}^\ell$ (for some $\ell \in \mathbb{N}$) is uniformly distributed modulo 1 if, and only if, for every $h \in \mathbb{Z}^\ell \setminus \{0\}$, the sequence $\langle h, \xi_n \rangle$, $n \in \mathbb{N}$, is uniformly distributed modulo 1, where $\langle \cdot, \cdot \rangle$ is the standard inner product.

Therefore, if $\mathcal{M}$ is a finite set of primes, then the sequence

$$\left(\frac{\gamma_a^{(n)} \log p}{2\pi}\right)_{p \in \mathcal{M}}, \quad n \in \mathbb{N},$$

is uniformly distributed modulo 1 if and only if the sequence

$$\gamma_a^{(n)} \sum_{p \in \mathcal{M}} h_p \frac{\log p}{2\pi}, \quad n \in \mathbb{N},$$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
is uniformly distributed modulo 1 for any \((h_p)_{p \in \mathcal{M}} \in \mathbb{Z}^{\mathcal{M}} \setminus \{0\}\). But this follows immediately from Theorem 4 and the unique factorization of integers into primes. Thus, \(\gamma_a^{(n)}\), \(n \in \mathbb{N}\), satisfies condition (6.2).

To prove that \(\gamma_a^{(n)}\), \(n \in \mathbb{N}\), also satisfies condition (6.3), we will use the following approximate functional equation:

\[
L(s; \chi_j) = \sum_{n \leq X} \chi_j(n)n^{-s} + \Delta(s; \chi_j) \sum_{n \leq y} \chi_j^-(n)n^{s-1} + O(X^{-\sigma} \log(y + 2) + \tau^{1/2-\sigma} y^{\sigma-1}) \quad (6.4)
\]

valid for \(s = \sigma + i\tau\) with \(0 < \sigma < 1\), \(\tau \geq \tau_0 > 0\), and \(X, y > 0\) such that \(2\pi Xy = q_j\tau\), where \(q_j\) is the modulus of \(\chi_j\). This formula follows from a result due to Vivek Rane [24]. Observe also that \(L_Q(s; \chi_j)\) can be written as an absolutely convergent Dirichlet series

\[
L_Q(s; \chi_j) = \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} \quad (6.5)
\]

for any \(\sigma > 0\) and \(Q > 0\), as follows from (6.1) by expanding each factor of the truncated Euler product into a geometric series.

Now let \(K\) be a compact subset of the strip \(\sigma_1 \leq \sigma < 1\) for some \(\sigma_1 \in (1/2, 1)\). Then equations (6.4) and (6.5) imply, for every \(s = \sigma + it \in K\), any \(Q > 0\), any sufficiently large \(N \in \mathbb{N}\) and for \(X = q_j N/(4\pi)\), that

\[
\sum_{N \leq n \leq 2N} \left| L(s + i\gamma_a^{(n)}; \chi_j) - L_Q(s + i\gamma_a^{(n)}; \chi_j) \right|^2 \\
\ll \sum_{N \leq n \leq 2N} \left| \sum_{m \leq X} \chi_j(m) \right|^2 + \sum_{N \leq n \leq 2N} \left| \sum_{m > X} \chi_j(m) \right|^2 \\
+ \sum_{N \leq n \leq 2N} \left| \Delta(s + i\gamma_a^{(n)}; \chi_j) \right|^2 \left| \sum_{m \leq y} \chi_j^-(m) \right|^2 \\
+ \sum_{N \leq n \leq 2N} O\left( X^{-2\sigma} (\log(y + 2))^2 + (t + \gamma_a^{(n)})^{1-2\sigma} y^{2(\sigma-1)} \right), \quad (6.6)
\]

where \(\sum'\) denotes the sum over integers \(m\) for which there is no prime \(p \leq Q\) with \(p \mid m\), and \(\sum''\) denotes the sum over integers \(m\) which are divisible only by primes \(p \leq Q\). We denote the terms on the right-hand side of (6.6) by \(S_1, S_2, S_3\) and \(S_4\), respectively, and we prove that each of them is at most \(O(NQ^{1-2\sigma})\) as \(N\) tends to infinity. In what follows, asymptotics and limits are not taken with respect to the parameter \(t\) since \(t\) represents the imaginary part of complex numbers \(s\) from a compact set \(K\). The implicit constants also depend of course on \(K\) and the finitely many moduli \(q_1, \ldots, q_J\), but they are negligible in our proof. Recall that \(\gamma_a^{(n)} \sim n/\log n\). Then, we have

\[
S_4 \ll \sum_{N \leq n \leq 2N} \left[ N^{-2\sigma} \left( \log\left( \frac{\gamma_a^{(n)}}{N} + 2 \right) \right)^2 + (\gamma_a^{(n)})^{1-2\sigma} \left( \frac{\gamma_a^{(n)}}{N} \right)^{2(\sigma-1)} \right] \\
\ll N \left[ N^{-2\sigma_1} + N^{1-2\sigma_1} \log N \right] = o(N),
\]
while (2.4) implies that

$$S_3 \ll \sum_{N \leq n \leq 2N} (\gamma_a(n))^{1-2\sigma} \left( \sum_{m \leq y} \frac{1}{m^{1-\sigma}} \right)^2 \ll N (\gamma_a(N))^{1-2\sigma_1} \left( \frac{\gamma_a(N)}{N} \right)^2 \ll N \left( \frac{N}{\log N} \right)^{1-2\sigma_1} (\log N)^{-2} = o(N).$$

To estimate $S_2$, we first observe that its inner sum is a tail of an absolutely convergent series. Therefore,

$$S_2 \ll \sum_{N \leq n \leq 2N} \left( \sum_{m>X} \frac{1}{m^\sigma} \right)^2 \ll N \left( \sum_{m>N/\log N} \frac{1}{m^\sigma} \right)^2 = o(N).$$

Lastly,

$$S_1 = \sum_{m_1, m_2 \leq X} \frac{1}{(m_1 m_2)^\sigma} \left( \frac{m_2}{m_1} \right)^{it} \sum_{N \leq n \leq 2N} \left( \frac{m_2}{m_1} \right)^{i\gamma_a(n)} \ll N \sum_{m \leq X} \frac{1}{m^{2\sigma_1}} + \sum_{1 \leq m_1 < m_2 \leq X} \left( \frac{m_2}{m_1} \right)^{\sigma_1} \sum_{N \leq n \leq 2N} \left( \frac{m_2}{m_1} \right)^{i\gamma_a(n)}. \quad (6.7)$$

The first term on the right-hand side of (6.7) is $O(NQ^{1-2\sigma_1})$. For the second term observe that, for any $1 \leq m_1 < m_2 \leq X$, we have $1 < m_2/m_1 \leq q_2 N/(4\pi)$. Therefore, Theorem 4 implies that this term is bounded from above by

$$\sum_{1 \leq m_1 < m_2 \leq X} \frac{\log N}{(m_1 m_2)^\sigma_1} \left( \left( \log \frac{m_2}{m_1} \right)^{-1} + \log \frac{m_2}{m_1} \right) \ll X^{2-2\sigma_1} \log X \log N \ll N^{2-2\sigma_1} (\log N)^2 = o(N).$$

Collecting the above estimates we finally arrive at

$$\limsup_{N \to \infty} \frac{1}{N+1} \sum_{N \leq n \leq 2N} \left| L(s + i\gamma_a(n); \chi_j) - L_Q(s + i\gamma_a(n); \chi_j) \right|^2 \ll Q^{1-2\sigma_1} \quad \text{uniformly in } K \text{ and arbitrary } Q > 0.$$ 

Taking $Q$ to infinity shows that the sequence $\gamma_a(n)$, $n \in \mathbb{N}$, also satisfies (6.3) and thus (1.9) holds.

We prove (1.10) from (1.9). We employ techniques introduced by Bagchi [2], Gonek [6] as well as Jürgen Sander and the second author [26]. Let $\psi \neq 0$ be an $r$-periodic arithmetical function. If $r = 1$, then

$$L(s; \psi) = \psi(1) \zeta(s)$$

and (1.10) holds, since $\zeta(s)$ is $L(s; \chi_0)$, where $\chi_0$ is the Dirichlet character mod 1 and we apply (1.9) only to this character.

The case $r = 2$ is rather special; here some cases need a restriction on the range of approximation. This observation is due to Jerzy Kaczorowski [13] and has recently been discussed by Philipp Muth and the second author [21]. Since $\psi(1) \neq \psi(2)$, for any $\sigma > 1$, we have

$$L(s; \psi) = \sum_{n=1}^{\infty} \frac{\psi(1)}{(2n-1)^s} + \sum_{n=1}^{\infty} \frac{\psi(2)}{(2n)^s} = \left( \psi(1) + \frac{\psi(2) - \psi(1)}{2^s} \right) \zeta(s) = P(s) \zeta(s).$$
The latter holds for all \( s \in \mathbb{C} \) by analytic continuation. Observe here that \( P(s) \) is analytic and bounded in any half-plane \( \sigma \geq \sigma_0 \) and

\[
P(s + i\tau) - P(s) \ll \left\| \frac{\log 2}{2\pi} \right\|	ag{6.8}
\]

uniformly in \( \sigma \geq \sigma_0 \) and \( \tau \in \mathbb{R} \). Additionally, \( P(s) \) is zero-free in the open set

\[
D_0 := \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\} \setminus \left\{ \log \left( 1 - \frac{\psi(2)}{\psi(1)} \right) + 2k\pi i : k \in \mathbb{Z} \right\}.
\]

Therefore, if we assume that \( K \subseteq D_0 \) and set

\[
g(s) := \frac{h(s)}{P(s)},
\]

then, from (1.9), we have

\[
\#\left\{ 1 \leq n \leq N : \max_{s \in K} \left| \zeta(s + i\gamma^{(n)}_a) - g(s) \right| < \eta \text{ and } \left\| \gamma^{(n)}_a \frac{\log 2}{2\pi} \right\| < \eta \right\} > c(\eta)N
\]

for any \( \eta > 0 \) and any sufficiently large \( N \gg \eta \), where \( c(\eta) > 0 \) is constant. For those \( n \) from the set described on the left-hand side above, in combination with (6.8), it also follows that

\[
\max_{s \in K} |L(s + i\gamma^{(n)}_a; \psi) - h(s)| < \max_{s \in K} |P(s + i\gamma^{(n)}_a)| \max_{s \in K} \left| \zeta(s + i\gamma^{(n)}_a) - g(s) \right|
\]

\[
+ \max_{s \in K} |g(s)| \max_{s \in K} |P(s + i\gamma^{(n)}_a) - P(s)| \ll \eta.
\]

Taking \( 0 < \eta \ll \varepsilon \) sufficiently small, we obtain (1.10) with the restriction we imposed on \( K \).

If \( r \geq 3 \), then \( \phi(r) \geq 2 \), where \( \phi \) is the Euler totient function. Assuming that \( \psi \) is a multiple of a Dirichlet character mod \( r \), we can work as in the case of \( r = 1 \). If \( \psi \) is not such a multiple, then for every \( s \in \mathbb{C} \)

\[
L(s; \psi) = \frac{1}{r^s} \sum_{n=1}^{r} \psi(n) \zeta\left( s; \frac{n}{r} \right) = \frac{1}{r^s} \sum_{n=1}^{r} \psi(n) \frac{\phi(r)}{\phi(r)} \sum_{i=1}^{\phi(r)} \chi_i(n) L(s; \chi_i)
\]

\[
= \sum_{i=1}^{\phi(r)} \left( \frac{1}{\phi(r)} \sum_{n=1}^{r} \psi(n) \chi_i(n) \right) L(s; \chi_i),
\]

(6.9)

where \( \chi_i, i = 1, 2, \ldots, \phi(r) \), are the Dirichlet characters mod \( r \). The expression of \( L(s; \psi) \) as a linear combination of Hurwitz zeta-functions with rational parameters follows first for \( \sigma > 1 \), where there are absolutely convergent Dirichlet series representations of them, and then by analytic continuation to the whole complex plane. The expression of a Hurwitz zeta-function with rational parameter as a linear combination of Dirichlet \( L \)-functions follows from the orthogonality relation of the characters. Now if we set

\[
c_i := \frac{1}{\phi(r)} \sum_{n=1}^{r} \psi(n) \chi_i(n), \quad i = 1, 2, \ldots, \phi(r),
\]

then at least two of them, say \( c_1 \) and \( c_2 \), are non-zero by assumption. If we define the quantity \( M_h := 1 + \max_{s \in K} |h(s)| \) and the functions

\[
g_1(s) := \frac{h(s) + M_h}{c_1}, \quad g_2(s) := -\frac{M_h}{c_2}, \quad g_i(s) := \eta, \quad i = 3, \ldots, \phi(r),
\]

(6.10)
for a given $\eta > 0$, then $g_i, i = 1, 2, \ldots, \phi(r)$, are non-zero continuous functions on $K$ which are analytic in its interior and

$$h(s) = \sum_{i=1}^{\phi(r)} c_i g_i(s) - \sum_{i=3}^{\phi(r)} c_i g_i(s) = \sum_{i=1}^{\phi(r)} c_i g_i(s) - \eta(\phi(r) - 3) \sum_{i=3}^{\phi(r)} c_i. \quad (6.11)$$

Since $\chi_i, i = 1, 2, \ldots, \phi(r)$, are non-equivalent Dirichlet characters, by (1.9) we obtain

$$\sharp \left\{ 1 \leq n \leq N : \max_{1 \leq i \leq \phi(r)} \max_{s \in K} \left| L(s + i\gamma_{\alpha}^{(n)}; \chi_i) - g_i(s) \right| < \eta \right\} > c(\eta)N$$

for any sufficiently large $N \gg \eta$. In this case, for those $n$ from the set described on the left-hand side above, in combination with (6.9) and (6.10), it also follows that

$$\max_{s \in K} \left| L(s + i\gamma_{\alpha}^{(n)}; \psi) - h(s) \right| \leq \sum_{i=1}^{\phi(r)} |c_i| \max_{s \in K} \left| L(s + i\gamma_{\alpha}^{(n)}; \chi_i) - g_i(s) \right| + \eta(\phi(r) - 3) \sum_{i=3}^{\phi(r)} |c_i| \ll \eta.$$

Taking $0 < \eta \ll \varepsilon$ sufficiently small, we obtain (1.10). Observe that in this case $h(s)$ is allowed to have zeros.

ACKNOWLEDGMENTS

The authors are grateful for the comments of the anonymous referee.

FUNDING

The first author is supported by the Austrian Science Fund, project Y-901, and the third author is supported by JSPS KAKENHI grant no. 18K13400.

REFERENCES

1. T. M. Apostol, “Dirichlet L-functions and primitive characters,” Proc. Am. Math. Soc. 31, 384–386 (1972).
2. B. Bagchi, “The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series,” PhD Thesis (Indian Stat. Inst., Calcutta, 1981).
3. R. B. Burckel, An Introduction to Classical Complex Analysis (Birkhäuser, Basel, 1979), Vol. 1.
4. T. Christ, “Value-distribution of the Riemann zeta-function and related functions near the critical line,” PhD Thesis (Würzburg Univ., Würzburg, 2014); arxiv:1405.1553 [math.NT].
5. P. D. T. A. Elliott, “The Riemann zeta function and coin tossing,” J. Reine Angew. Math. 254, 289–304 (1972).
6. S. M. Gonek, “Analytic properties of zeta and L-functions,” PhD Thesis (Univ. Michigan, Ann Arbor, 1979).
7. S. M. Gonek, “Mean values of the Riemann zeta-function and its derivatives,” Invent. Math. 75, 123–141 (1984).
8. S. M. Gonek, “An explicit formula of Landau and its applications to the theory of the zeta-function,” in A tribute to Emil Grosswald: Number Theory and Related Analysis (Am. Math. Soc., Providence, RI, 1993), Contemp. Math. 143, pp. 395–413.
9. J.-P. Gram, “Note sur les zéros de la fonction $\zeta(s)$ de Riemann,” Acta Math. 27, 289–304 (1903).
10. W. K. Hayman, Meromorphic Functions (Clarendon Press, Oxford, 1964).
11. E. Hlawka, “Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen,” Österr. Akad. Wiss., Math.-naturw. Kl., S.-Ber., Abt. II, 184, 459–471 (1975).
12. G. Julia, “Sur quelques propriétés nouvelles des fonctions entières ou méromorphes (premier mémoire),” Ann. Sci. Éc. Norm. Supér., Sér. 3, 36, 93–125 (1919).
13. J. Kaczorowski, “Some remarks on the universality of periodic L-functions,” in New Directions in Value-Distribution Theory of Zeta and L-Functions: Proc. Conf., Würzburg, 2008, Ed. by R. Steuding and J. Steuding (Shaker Verlag, Aachen, 2009), pp. 113–120.
14. J. Kalpokas, M. A. Korolev, and J. Steuding, “Negative values of the Riemann zeta function on the critical line,” Mathematika 59 (2), 443–462 (2013).
15. J. Kalpokas and J. Steuding, “On the value-distribution of the Riemann zeta-function on the critical line,” Moscow J. Comb. Number Theory 1 (1), 26–42 (2011).

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
16. M. A. Korolev, “Gram’s law in the theory of the Riemann zeta-function. Part 1,” Proc. Steklov Inst. Math. 292 (Suppl. 2), 1–146 (2016) [transl. from Sovrem. Probl. Mat. 20, 3–161 (2015)].
17. M. Korolev and A. Laurinčikas, “A new application of the Gram points,” Aequationes Math. 93 (5), 859–873 (2019).
18. L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences (Dover Publ., Mineola, NY, 2006).
19. E. Landau, “Über die Nullstellen der Zetafunktion,” Math. Ann. 71, 548–564 (1912).
20. N. Levinson, “Almost all roots of $\zeta(s) = a$ are arbitrarily close to $\sigma = 1/2$,” Proc. Natl. Acad. Sci. USA 72 (4), 1322–1324 (1975).
21. P. Muth and J. Steuding, “Joint value-distribution of Dirichlet series associated with periodic arithmetical functions,” Funct. Approx., Comment. Math. 62 (1), 63–79 (2020).
22. É. Picard, “Sur les fonctions analytiques uniformes dans le voisinage d’un point singulier essentiel,” C. R. Acad. Sci. Paris 89, 745–747 (1879).
23. H. A. Rademacher, “Fourier analysis in number theory,” in Collected Papers of Hans Rademacher (MIT Press, Cambridge, 1974), Vol. II, pp. 434–458.
24. V. V. Rane, “On an approximate functional equation for Dirichlet $L$-series,” Math. Ann. 264, 137–145 (1983).
25. A. Reich, “Werteverteilung von Zetafunktionen,” Arch. Math. 34, 440–451 (1980).
26. J. Sander and J. Steuding, “Joint universality for sums and products of Dirichlet $L$-functions,” Analysis (München) 26 (3), 295–312 (2006).
27. W. Schnee, “Die Funktionalgleichung der Zetafunktion und der Dirichletsehen Reihen mit periodischen Koeffizienten,” Math. Z. 31, 378–390 (1930).
28. A. Sourmelidis, “Universality and hypertranscendence of zeta-functions,” PhD Thesis (Würzburg Univ., Würzburg, 2020).
29. R. Spira, “An inequality for the Riemann zeta function,” Duke Math. J. 32, 247–250 (1965).
30. E. M. Stein and R. Shakarchi, Complex Analysis (Princeton Univ. Press, Princeton, NJ, 2003).
31. J. Steuding, “Dirichlet series associated to periodic arithmetic functions and the zeros of Dirichlet $L$-functions,” in Analytic and Probabilistic Methods in Number Theory: Proc. 3rd Int. Conf. in Honour of J. Kubilius, Palanga, 2001, Ed. by A. Dubickas et al. (TEV, Vilnius, 2002), pp. 282–296.
32. J. Steuding, Value-Distribution of L-Functions (Springer, Berlin, 2007), Lect. Notes Math. 1877.
33. J. Steuding, “The roots of the equation $\zeta(s) = a$ are uniformly distributed modulo one,” in Analytic and Probabilistic Methods in Number Theory: Proc. 5th Int. Conf. in Honour of J. Kubilius, Palanga, 2011, Ed. by A. Laurinčikas et al. (TEV, Vilnius, 2012), pp. 243–249.
34. J. Steuding, “One hundred years uniform distribution modulo one and recent applications to Riemann’s zeta-function,” in Topics in Mathematical Analysis and Applications, Ed. by Th. M. Rassias and L. Tóth (Springer, Cham, 2014), Springer Optim. Appl. 94, pp. 659–698.
35. J. Steuding and A. I. Suriajaya, “Value-distribution of the Riemann zeta-function along its Julia lines,” Comput. Methods Funct. Theory 20 (3–4), 389–401 (2020).
36. E. C. Titchmarsh, The Theory of Functions, 2nd ed. (Oxford Univ. Press, Oxford, 1939).
37. E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., rev. by D. R. Heath-Brown (Clarendon Press, Oxford, 1986).
38. I. M. Vinogradov, “A new estimation of a certain sum containing primes,” Mat. Sb. 2 (5), 783–792 (1937).
39. I. M. Vinogradov, “A new estimate for the function $\zeta(1 + it)$,” Izv. Akad. Nauk SSSR, Ser. Mat. 22, 161–164 (1958).
40. S. M. Voronin, “Theorem on the ‘universality’ of the Riemann zeta-function,” Math. USSR, Izv. 9 (3), 443–453 (1975) [transl. from Izv. Akad. Nauk SSSR, Ser. Mat. 39 (3), 457–486 (1975)].
41. S. Voronin, “On the functional independence of Dirichlet $L$-functions,” Acta Arith. 27, 493–503 (1975).
42. S. M. Voronin and A. A. Karatsuba, The Riemann Zeta-Function (Fizmatlit, Moscow, 1994); Engl. transl.: A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function (W. de Gruyter, Berlin, 1992), De Gruyter Expo. Math. 5.
43. H. Weyl, “Über die Gleichverteilung von Zahlen mod. Eins,” Math. Ann. 77, 313–352 (1916).