Interaction of Tachyons and Discrete States in $c = 1$ 2-D Quantum Gravity

Yoichiro Matsumura, Norisuke Sakai

Department of Physics, Tokyo Institute of Technology
Oh-okayama, Meguro, Tokyo 152, Japan

and

Yoshiaki Tani

Physics Department, Saitama University
Urawa, Saitama 338, Japan

Abstract

The two-dimensional (2-D) quantum gravity coupled to the conformal matter with $c = 1$ is studied. We obtain all the three point couplings involving tachyons and/or discrete states via operator product expansion. We find that cocycle factors are necessary and construct them explicitly. We obtain an effective action for these three point couplings. This is a brief summary of our study of couplings of tachyons and discrete states, reported at the workshop in Tokyo Metropolitan University, December 4-6, 1991.
Recently the understanding of two-dimensional (2-D) quantum gravity has advanced significantly. There are two main motivations to study the 2-D quantum gravity coupled to matter. Firstly, it is precisely a string theory when the 2-D space-time is regarded as the world sheet for the string. Secondly, it provides a toy model for the quantum gravity in four dimensions. There are two approaches to study the 2-D quantum gravity: the matrix model as a discretized theory and the Liouville theory as a continuum theory [1, 2]. The former can provide a nonperturbative treatment, but is sometimes less transparent in physical terms since it is not in the usual continuum language. In spite of the nonlinear dynamics of the Liouville theory, a method based on conformal field theory has now been sufficiently developed to understand the results of the matrix model and to offer in some cases a more powerful method in computing various quantities. In particular, we can calculate not only partition functions but also correlation functions by using the procedure of the analytic continuation [3, 4].

So far only conformal field theories with central charge $c \leq 1$ have been successfully coupled to quantum gravity. The $c = 1$ model is the richest and the most interesting, and it is in some sense the most easily soluble. From the viewpoint of the string theory, the $c = 1$ model has at least one (continuous) dimension of target space in which strings are embedded. Hence, we can discuss the space-time interpretation in the usual sense in the $c = 1$ model. Since the Liouville (conformal) mode plays a dynamical role if the dimensions of the target space is different from the usual critical dimensions, the theory is called “noncritical” string theory.

It has been observed that the $c = 1$ quantum gravity can be regarded effectively as a critical string theory in two dimensions, since the Liouville field zero mode provides an additional “time-like” dimension besides the obvious single spatial dimension given by the zero mode of the $c = 1$ matter [5]. We have a physical scalar particle corresponding to the center of mass motion of the string. Though it is massless, it is still referred to as a “tachyon” following the usual terminology borrowed from the critical string theory. Since there are no transverse directions, the continuous (field) degrees of freedom are exhausted by the tachyon field. The
partition function for the torus topology was computed in the Liouville theory, and was found to give precisely the same partition function as the tachyon field alone. However, it has been noted that there exist other discrete degrees of freedom in the $c = 1$ matter coupled to the 2-D quantum gravity [6-8]. It has been pointed out that the symmetry group relevant to the dynamics of these discrete states in the $c = 1$ quantum gravity is the area preserving diffeomorphisms whose generators fall into representations of SU(2) [9]. Exploiting the SU(2) symmetry, Klebanov and Polyakov have recently worked out the three point interactions of the discrete states and have proposed an effective action for these discrete states [10].

This paper is a brief report on our study of the interaction of tachyons and discrete states in the $c = 1$ quantum gravity [11]. We have obtained all the possible three point couplings completely including both tachyons and discrete states by using the operator product expansion (OPE) of vertex operators. We have also found that the so-called cocycle factor is needed to obtain the operator product expansion with the proper analytic behaviour.

Let us consider the $c=1$ conformal matter realized by a single bosonic field $X$ coupled to the 2-D quantum gravity. After fixing the conformal gauge $g_{\alpha\beta} = e^{\phi} \hat{g}_{\alpha\beta}$ using the Liouville field $\phi$, the $c = 1$ quantum gravity can be described by the following action on a surface with a boundary [1, 2, 12]

$$S[\hat{g}, X, \phi] = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\hat{g}} \left( \hat{g}^{\alpha\beta} \partial_\alpha X \partial_\beta X + \hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2\sqrt{\alpha'} \hat{R} \phi \right)$$

$$+ 4\alpha' \mu e^{-2\phi/\sqrt{\alpha'}} + \frac{1}{\pi\sqrt{\alpha'}} \int d\hat{s} \left( -\hat{k} \phi + \sqrt{\alpha'} \lambda e^{-\phi/\sqrt{\alpha'}} \right), \quad (1)$$

where $\alpha'$ is the Regge slope parameter, $\hat{R}$ the scalar curvature, $\hat{k}$ the geodesic curvature along the boundary and $d\hat{s}$ the line element of the boundaries with respect to the reference metric $\hat{g}_{\alpha\beta}$. We have rescaled the Liouville field $\phi$. In this paper we will consider only the bulk (or resonant) correlation functions [7], for which the “energy” and the momentum conjugate to $\phi$ and $X$ respectively are conserved. For such correlation functions we can use the action without the cosmological terms by putting $\mu = \lambda = 0.$

3
There are two types of physical operators. The open string vertex operators are given by line integrals of primary fields with boundary conformal weight one along the boundary, while the closed string vertex operators are given by surface integrals of primary fields with conformal weight \((1, 1)\). It is convenient to set \(\alpha' = 4\) when we discuss the closed (open) string vertex operators. With this convention the integrands of the closed string vertex operators can be constructed by combining the holomorphic operator and the anti-holomorphic operator, both of which have the same form as those of the open string vertex operators. The holomorphic part of the energy-momentum tensor for \(\alpha' = 4\) is given by

\[
T(z) = -\frac{1}{4} (\partial X)^2 - \frac{1}{4} (\partial \phi)^2 - \partial^2 \phi.
\]

From the action, we have correlation functions of \(X\) and \(\phi\) for closed string

\[
\langle X(z, \bar{z})X(w, \bar{w}) \rangle = \langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = -2 \ln |z - w|^2.
\]

in accord with the convention of Klebanov and Polyakov [10].

Let us first consider the holomorphic part of the vertex operator corresponding to the open string vertex operators. They must be a line integral of a primary field of unit conformal weight. The simplest field for such operators is the tachyon vertex operator

\[
\Psi_p^{(\pm)}(z) = e^{ipX(z)} e^{(\pm p-1)\phi(z)}
\]

for an arbitrary real momentum \(p\). For higher levels there are non-trivial primary fields only when the momentum is an integer or a half odd integer. They are primary fields for the “discrete states” [6, 7]. They form SU(2) multiplets and can be constructed as [9, 10]

\[
\Psi_{J,m}(z) = \sqrt{(J + m)! / (2J)! (J - m)!} \oint \frac{du_{J-m}}{2\pi i} H_-(u_{J-m}) \cdots \oint \frac{du_1}{2\pi i} H_-(u_1) \Psi_{J}^{(\pm)}(z),
\]

where \(J = 0, \frac{1}{2}, 1, \cdots; m = -J, -J + 1, \cdots, J\) and \(\Psi_{J}^{(\pm)}(z)\) is the tachyon operator (4) with the momentum \(p = J\). The integrals are along closed contours surrounding
a point \( z \) with \(|u_i| > |u_j|\) for \( i > j \). The field \( H_-(z) \) corresponds to the lowering operator of the SU(2) quantum numbers and is one of the SU(2) currents

\[
H_{\pm}(z) = e^{\pm iX(z)} = \pm \Psi_{1,\pm 1}^{(+)}(z), \quad H_3(z) = \frac{1}{2} i \partial X(z) = -\frac{1}{\sqrt{2}} \Psi_{1,0}^{(+)}(z). \quad (6)
\]

The quantum numbers \( J \) and \( m \) correspond to the “spin” and the magnetic quantum number in SU(2). Actually, the fields \( \Psi_{J,m}^{(\pm)} \) with \( m = \pm J \) are not higher level operators but tachyon operators (4) at integer or half odd integer momenta \( \pm J \).

In ref. [10] the OPEs of the fields for discrete states (5) were obtained using the SU(2) symmetry. Here we make a remark on the analytic property of the OPEs. The OPE of two vertex operators gives a coefficient different in sign depending on the ordering of the two vertex operators. Even if we use the radial ordering of the two vertex operators as usual in conformal field theory, the OPE is not analytic at \(|z| = |w|\). It is desirable to obtain the analytic OPEs since the techniques of conformal field theories make full use of the analyticity. One should multiply the vertex operator (5) by a correction factor as in the vertex operator construction of the affine Kac-Moody algebra [13].

We have succeeded in constructing the necessary correction factor to recover the analyticity. After some lengthy argument using the knowledge of integral cubic lattice, we arrive at the following choice of the cocycle factor [11]

\[
\varepsilon(\alpha_1, \alpha_2) = (-1)^{2J_1(J_2-m_2-1)} \quad \alpha_i = \sqrt{2}(m_i, J_i - 1), \quad i = 1, 2. \quad (7)
\]

The sign of \( J \) in the cocycle factor should be changed according to the sign of \( J \) in the two-vector \( \alpha \) corresponding to the \((−)\) type. It is easy to see that eq. (7) indeed satisfies the cocycle conditions. With this cocycle the correction factor is constructed as [13]

\[
c_{\alpha} = \sum_{\beta \in \Lambda} \varepsilon(\alpha, \beta) \langle \beta | \beta \rangle, \quad (8)
\]

where \( |\beta\rangle \) is an eigenstate of the energy and the momentum with an eigenvalue \( \beta \).
Then the corrected operators
\[
\Psi_{J,m}^{(s)}(z) = \Psi_{J,m}^{(s)}(z)\alpha,
\]
\[\alpha = \begin{cases} \sqrt{2} (m, J - 1) & \text{for } s = + \\ \sqrt{2} (m, -J - 1) & \text{for } s = - \end{cases}\]
(9)
satisfy the OPEs which are analytic in the complex \(z\) plane.

We find that after an appropriate rescaling the corrected operators (9) satisfy the same OPEs as those given in ref. [10]. The non-trivial OPEs are given by
\[
\tilde{\Psi}_{J,m}^{(+)}(z) \tilde{\Psi}_{J,m}^{(+)}(w) \sim \frac{1}{z - w} (J_2 m_1 - J_1 m_2) \tilde{\Psi}_{J+J-1,m_1+m_2}^{(+)}(w),
\]
\[
\tilde{\Psi}_{J,m}^{(+)}(z) \tilde{\Psi}_{J,m}^{(-)}(w) \sim \frac{1}{z - w} (-J_1 m_3 - J_3 m_1) \tilde{\Psi}_{J,m}^{(-)}(w).
\]
(10)

Other OPEs have no singular term. We have used rescaled fields
\[
\tilde{\Psi}_{J,m}^{(+)}(z) = \tilde{N}(J, m) \Psi_{J,m}^{(+)}(z),
\]
\[
\tilde{\Psi}_{J,m}^{(-)}(z) = (-1)^{J(2J-1)+J-m} \tilde{N}(J, m)^{-1} \Psi_{J,m}^{(-)}(z),
\]
(11)
\[
\tilde{N}(J, m) = (2J - 1)! \sqrt{\frac{J}{2}} N(J, m), \quad N(J, m) = \left[ \frac{(J + m)! (J - m)!}{(2J - 1)!} \right]^{\frac{1}{2}}.
\]
(12)

We shall now generalize these results of the OPEs to include tachyon operator (4). We have succeeded to generalize the cocycle operator to the tachyon case, but we merely refer our paper [11] for the full account of the construction and write down only the OPE without the cocycle factors because of lack of space. From the conservation of the energy and the momentum we find that only four non-trivial OPEs are possible:
\[
\Psi_{p_1}^{(+)}(z) \Psi_{p_2}^{(+)}(w) \sim \frac{1}{z - w} F_{p_1 p_2}^{(+)} \tilde{\Psi}_{J_3,1-J_3}^{(-)}(w) \quad (J_3 = -p_1 - p_2 + 1),
\]
\[
\Psi_{p_1}^{(-)}(z) \Psi_{p_2}^{(-)}(w) \sim \frac{1}{z - w} F_{p_1 p_2}^{(-)} \tilde{\Psi}_{J_3,1-J_3}^{(-)}(w) \quad (J_3 = p_1 + p_2 + 1),
\]
\[
\tilde{\Psi}_{J_1,1-J_1}^{(+)}(z) \Psi_{p_2}^{(+)}(w) \sim \frac{1}{z - w} G_{J_1 p_2}^{(+)} \Psi_{p_3}^{(+)}(w) \quad (p_3 = J_1 - 1 + p_2),
\]
\[
\tilde{\Psi}_{J_1,1-J_1}^{(+)}(z) \Psi_{p_2}^{(-)}(w) \sim \frac{1}{z - w} G_{J_1 p_2}^{(-)} \Psi_{p_3}^{(-)}(w) \quad (p_3 = 1 - J_1 + p_2).
\]
(13)
The coefficient in the third and fourth OPE in eq. (13) can be obtained by using the representation (5) for $\Psi^{(+)}_{J_1,J_1-1}$ or the similar expression for $m = 1 - J$ and directly evaluating the OPE

\[
G^{(+)}_{J_1p_2} = \frac{\Gamma(1 - 2p_2)}{2\Gamma(-2p_3)} = (-1)^{2J_1-1} \frac{\tilde{N}(p_3,p_3)}{N(p_2,p_2)} p_2,
\]
\[
G^{(-)}_{J_2p_2} = (-1)^{J_1(2J_1-1)} \frac{\Gamma(1 + 2p_2)}{2\Gamma(2p_3)} = (-1)^{J_1(2J_1-1)} \frac{\tilde{N}(p_2,p_2)}{N(p_3,p_3)} p_3,
\]

where $\tilde{N}(p, p) = \frac{1}{2} \Gamma(1 + 2p)$. To obtain the coefficient of the first OPE in eq. (13), we apply the operator $\oint \frac{du}{2\pi i} H_-(u)$ to both hand sides of the equation, where the integration contour surrounds both of $z$ and $w$. The coefficient of the second OPE in eq. (13) can be obtained similarly by applying $\oint \frac{du}{2\pi i} H_+(u)$. We find

\[
(-1)^{J_3(2J_3-1)} F^{(+)}_{p_1p_2} = \frac{\Gamma(1 - 2p_1)}{2\Gamma(2p_2)} = \left[ \tilde{N}(p_1,p_1)\tilde{N}(p_2,p_2) \right]^{-1} \frac{\pi p_1p_2}{2\sin(2\pi p_1)},
\]
\[
(-1)^{J_3(2J_3-1)} F^{(-)}_{p_1p_2} = \frac{\Gamma(1 + 2p_2)}{2\Gamma(-2p_1)} = \frac{2}{\pi} \tilde{N}(p_1,p_1)\tilde{N}(p_2,p_2) \sin(2\pi p_1).
\]

The coefficients of the OPE determine the three-point correlation functions of the physical operators, which can be summarized by the effective action. Introducing a variable $g_{J,m}^{(s)} (s = \pm)$ for each discrete state, the cubic terms of the effective action for discrete states determined by the OPEs (10) are [10]

\[
S_3 = \frac{1}{2} \sum_{J_1,m_1,J_2,m_2} (J_2m_1 - J_1m_2) f^{ABC} g^{(-)A}_{J_1,J_2-1,-m_1-m_2} g^{(+)B}_{J_1,m_1} g^{(+)C}_{J_2,m_2} \int d\phi,
\]

where we have introduced the Chan-Paton index $A$ in the adjoint representation of some Lie algebra and have factored out the Liouville volume $\int d\phi$.

In ref. [10] it was shown that the terms in the cubic interaction (16) which depend only on the integer modes $g_{J,m}^{(s)A} (J, m \in \mathbb{Z})$ can be written in a compact
form by introducing a scalar field on $\mathbb{R} \times S^2$

$$\Phi_0(\phi, \theta, \varphi) = \sum_{s, A, J, m} T^A g_{j,m}^{(s)A} M^s(J, m) D^J_{m0}(\varphi, \theta, 0) e^{(sJ-1)\phi}. \quad (17)$$

Here, $T^A$ are the representation matrices of the Lie algebra, $D^J_{m0}$ are components of the SU(2) rotation matrix [14] and $M^s(J, m)$ are the normalization factor

$$D^J_{mm'}(\varphi, \theta, \psi) = \langle J m | e^{-i\varphi J_z} e^{-i\theta J_y} e^{-i\psi J_z} | J m' \rangle, \quad (18)$$

$$M^+(J, m) = \frac{N(J, m)N(J, 0)}{J}, \quad M^+(J, m) = \frac{(-1)^m J(2J+1)}{4\pi N(J, m)N(J, 0)}, \quad (19)$$

The effective action can be written in terms of the field $\Phi_0$ using $x^i = (\theta, \varphi)$

$$S^{(1)}_3 = \int d\phi e^{2\phi} \int_{S^2} d\theta d\varphi \epsilon^{ij} \text{Tr} \left( \Phi_0 \frac{\partial \Phi_0}{\partial x^i} \frac{\partial \Phi_0}{\partial x^j} \right). \quad (20)$$

We have succeeded to generalize this construction to the terms containing half odd integer modes as well as integer modes. We introduce two spinor fields $\Phi_{\frac{1}{2}}$ and $\Phi_{-\frac{1}{2}}$ on $\mathbb{R} \times S^2$ for half odd integer modes $g_{j,m}^{(s)A}$ ($J, m \in \mathbb{Z} + \frac{1}{2}$)

$$\Phi_{\mu}(\phi, \theta, \varphi) = \sum_{s, A, J, m} T^A g_{j,m}^{(s)A} M^{s\mu}(J, m) D^J_{m\mu}(\varphi, \theta, 0) e^{(sJ-1)\phi} \left( \mu = \pm \frac{1}{2} \right), \quad (21)$$

where

$$M^{+\mu}(J, m) = \frac{N(J, m)N(J, \frac{1}{2})}{J + \frac{1}{2}}, \quad M^{-\mu}(J, m) = \frac{(-1)^{m+\mu} 2J(J+1)}{4\pi N(J, m)N(J, \frac{1}{2})}. \quad (22)$$

Note that $\Phi_{\frac{1}{2}}$ and $\Phi_{-\frac{1}{2}}$ have the same coefficients $g_{j,m}^{(s)A}$ and are not independent. In order to write down the effective action in terms of these fields we need covariant
derivatives on $S^2$ acting on spinor fields $\Phi_{\mu}$. They are given by

$$\nabla_{\pm} = \mp \partial_{\theta} - \frac{1}{\sin \theta} (i \partial_{\varphi} - \mu \cos \theta)$$

when acting on $\Phi_{\mu}$. The effective action can be written as

$$S_{3}^{(2)} = \int d\phi e^{2\phi} \int_{S^2} d\theta d\varphi \sin \theta \text{Tr} \left( \Phi_0 \left[ \nabla_{+} \Phi_{-\frac{1}{2}}, \nabla_{-} \Phi_{\frac{1}{2}} \right] \right).$$

The sum of eqs. (20) and (24) gives the complete cubic terms for the discrete states (16).

Apart from the special case of the compact boson $X$ with the self-dual radius, we have tachyons with momenta other than integer or half odd integer which should be included in the effective action. The OPE results (13) can be summarized as two types of terms in the effective action involving tachyons: two tachyons with the same chirality (+) or (−) couple to the single discrete state of the (+) type. We have succeeded to write down the local effective action involving tachyons, for which we refer our paper [11].

References

1. J. Distler and H. Kawai, *Nucl. Phys.* B321 (1989) 509; J. Distler, Z. Hlousek and H. Kawai, *Int. J. of Mod. Phys.* A5 (1990) 391; 1093; F. David, *Mod. Phys. Lett.* A3 (1989) 1651.

2. N. Seiberg, *Prog. Theor. Phys. Suppl.* 102 (1990) 319.

3. M. Goulian and M. Li, *Phys. Rev. Lett.* 66 (1991) 2051.

4. P. Di Francesco and D. Kutasov, *Phys. Lett.* B261B (1991) 385; Y. Kitazawa, *Phys. Lett.* B265B (1991) 262; N. Sakai and Y. Tanii, *Prog. Theor. Phys.* 86 (1991) 547; V.S. Dotsenko, Paris preprint PAR–LPTHE 91–18 (1991).

5. J. Polchinski, *Nucl. Phys.* B324 (1989) 123; *Nucl. Phys.* B346 (1990) 253.
6. D.J. Gross, I.R. Klebanov and M.J. Newman, *Nucl. Phys.* **B350** (1991) 621.

7. A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 635.

8. N. Sakai and Y. Tanii, Tokyo Inst. of Tech. and Saitama preprint TIT/HEP–173, STUPP–91–120 (1991); TIT/HEP–179, STUPP–91–122 (1991).

9. E. Witten, Princeton preprint IASSNS–HEP–91/51 (1991).

10. I.R. Klebanov and A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3273.

11. Y. Matsumura, N. Sakai and Y. Tanii, Tokyo Inst. of Tech. and Saitama preprint TIT/HEP–186, STUPP–92–124 (1992).

12. M. Bershadsky and D. Kutasov, Princeton and Harvard preprint PUPT–1283, HUTP–91/A047 (1991); Y. Tanii and S. Yamaguchi, Saitama preprint STUPP–91–121 (1991), to appear in *Mod. Phys. Lett.* **A**.

13. P. Goddard and D. Olive, in *Vertex Operators in Mathematics and Physics*, eds. J. Lepowsky et al., (Springer, Heidelberg, 1985) p. 51; *Int. J. of Mod. Phys.* **A1** (1986) 303.

14. M.E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, New York, 1957).