Topological Electronic Liquids

Electronic Physics of One Dimension Beyond the One Dimension

Abstract

There is a class of electronic liquids in dimensions greater than one, which show all essential properties of one dimensional electronic physics. These are topological liquids - correlated electronic systems with a spectral flow. Compressible topological electronic liquids are superfluids.

In this paper we present a study of a conventional model of a topological superfluid in two spatial dimensions. This model is thought to be relevant to a doped Mott insulator. We show how the spectral flow leads to the superfluid hydrodynamics and how the Orthogonality Catastrophe affects off-diagonal matrix elements. We also compute the major electronic correlation functions. Among them are the spectral function, the pair wave function and various tunneling amplitudes. To compute correlation functions we develop a method of current algebra - an extension of the bosonization technique of one spatial dimension.

In order to emphasize a similarity between electronic liquids in one dimension and topological liquids in dimensions greater than one, we first review the Frohlich-Peierls mechanism of ideal conductivity in one dimension and then extend the physics and the methods into two spatial dimension.

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1 Introduction

Over last two decades the physics community has been and continues to be fascinated by the phenomena of one-dimensional electronic physics. Being so different from perturbative Fermi Liquid picture of metals, electronic physics in one dimension is robust and in many ways more consistent than the Fermi liquid itself. This tempts one to think that the phenomena we observe in one dimension are more universal than they appear now and must be found beyond the one dimension. This paper is intended to show that there is a class of electronic liquids in dimensions greater than one which exhibits all essential features of one dimensional physics.

First, we must identify a core phenomenon responsible for the peculiar infrared properties of one dimensional electronic systems. We will argue that (i) this is the spectral flow (rather than a restrictive geometry) and moreover, (ii) that an electronic system in any dimension which exhibits the spectral flow, shows essentially the same property as a one dimensional system, and its low energy physics is essentially one dimensional.

We refer to the electronic systems with a spectral flow as topological electronic liquids. Topological electronic liquids are not exotic, in fact they are well known models of quantum field theory driven by an anomalous current algebra (examples are given below). In the last decade the evidence was mounting that topological liquids are relevant to strongly correlated electronic systems and may adequately describe the physics of a doped Mott insulator (another example of a physical system which may fall into a class of topological liquids that are quantum crystals around a melting transition).

In this paper we will discuss compressible topological liquids. They possess an important property: their low energy sector is described by the hydrodynamics of an ideal quantum liquid. In dimensions greater than one, this alone means that compressible topological liquids are superconductors.

The spectral flow phenomenon leads to a number of anomalous features which make the physics of topological liquids different from the physics of the Fermi liquid and make the physics of the superconducting state different from BCS. The most dramatic is perhaps the Orthogonality Catastrophe [10]: an overlap between ground states differ by an odd number of particles tends to zero in a macroscopical system. This means that an electron is not a quasiparticle. While embedding an electron, the system rearranges its quantum state by creating soft modes with a singularly large density of states. As a consequence, most scattering amplitudes are singular at low momentum due to emission of a cloud of soft modes. This dramatically affects the physics of tunneling in the superconducting state.

Here we discuss the physics of the superconducting state of topological liquids and especially concentrating on various tunneling effects.

On the technical side, the theoretical methods developed in one dimensional electronic physics may be extended to higher dimensional topological liquids. Among them is bosonization, a powerful tool used to compute correlation functions and matrix elements. We show how bosonization - a coherent state representation of a current algebra - works for higher dimensional topological fluids.

Running ahead, let us give a simplified picture of the spectral flow phenomena found found in compressible liquids. Let us consider gaped fermions in a static (vector or scalar) potential $V$. Assume that the chemical potential $\mu$ lies in a gap. When the potential $V$ changes adiabatically, the fermionic energy levels shift. Generally the levels can not cross the energy $E = \mu$. However, there are potentials whose adiabatic and arbitrary smooth variation may create some (always even)
number of unoccupied state below \( E = \mu \). or force some occupied levels to cross the level \( E = \mu \). This is the spectral flow or level crossing. To produce a level crossing, a variation of the potential \((\text{a soliton})\) must necessarily be topological. The index theorem then relates the topological charge of the potential and the number of levels crossed. If this simple phenomenon occurs, we conclude that a compressible liquid is a superconductor. Here is a short hierarchical list of familiar models that yield topological liquids:

### One Dimension.  
The Peierls model:

\[
H = \sum_{\sigma=1,2} \psi_\sigma^\dagger (\alpha_x i \partial_x + \beta \pi_1 + i \alpha_x \beta \pi_2) \psi_\sigma
\]  

(1.1)

where the Dirac matrices \( \alpha, \beta \) may be chosen as the Pauli matrices: \( \alpha_x = \sigma_3, \beta = \sigma_1 \) and the modulus of the vector \( (\pi_1, \pi_2) \) is assumed to take a fixed value at infinity.

The soliton here is a kink of the \( \pi(\mathbf{x}) \)-field.

### Two Dimensions.  
Dirac fermions coupled to a vector field field

\[
H = \psi^\dagger (\alpha i \nabla + \beta \tau \pi + \beta m) \psi .
\]  

(1.2)

where \( \tau \) are Pauli matrices. Solitons in this model are hedgehogs (skyrmions) of the vector field \( \pi \).

### Three Dimensions.  
Dirac fermions coupled to vector chiral bosons:

\[
H = \psi^\dagger (i \alpha \nabla + i \gamma_5 \beta \pi \tau + \beta m) \psi .
\]  

(1.3)

Solitons are skyrmions of the vector boson field. The last model is known as a linear \( \sigma \) model of current algebra and describes nuclear forces.

All these models in dimensions greater than one, being treated at fixed chemical potential, are superconductors!

Below we will concentrate on another model of Dirac fermions interacting with a gauge field (not an electromagnetic field):

\[
H = \sum_{\sigma=1,2} \psi_\sigma^\dagger (\alpha (-i \nabla + \mathbf{A}) + \beta m) \psi_\sigma .
\]  

(1.4)

Although the physics of all the models is similar, this particular model is selected because of historical reasons and because one may trace it genesis from a model of doped Mott insulator. Solitons in this model are vortices of the gauge field.

The text is organized as follows. We first review some basic elements of superconductivity with an emphasis on general properties which do not depend on the mechanism (Sec.2). Then we review the Frohlich’s ideal conductivity of the one dimensional model \((\text{[ ]})\) - a one dimensional topological liquid (Sec.3). On this example we illustrate how the spectral flow leads to the hydrodynamics, and how the orthogonality catastrophe affects matrix elements and determines the physics at the Fermi surface. We also review methods of computing a mass shell asymptotic of correlation functions (bosonization) and develop a vertex operator technique to compute matrix elements. This, except the matrix elements, is a standard material. We adopted it for the purpose of generalization to higher spatial dimensions.
In the next section (Sec.4), we extend the methods of Sec.3 to the model (1.4) of topological liquid of two spatial dimension. We try to follow the line of Sec.3 as closely as possible, in order to separate the physics of one dimension which is caused by to a restrictive one dimensional geometry and the general physical aspects of spectral flow.

Finally in Sec.5 we discuss various tunneling mechanisms, where a signature of topological mechanism of superconductivity is the most transparent.

A comment on literature is in order. We shall not give too many references for the one dimensional part of this text. The Frohlich’s conductivity goes back to his original seminal paper of 1955 [14], which impresses even today’s readers. Excellent reviews on the further development of physics of charge density waves are [15]. The part regarding the bosonization and correlation functions has been developed in late 70’s and redeveloped in early 90’s. Many classical papers may be found in an excellent preprint collection [6]. As the more recent paper we refer to [3]. The vast body of Sec.3 is in fact the PhD thesis of the author of 1980. The two-particle matrix element calculation of Sec.3.5 is believed to be new. Regarding the spectral flow, current algebra, anomalies and the geometrical phase there is also excellent preprint collections [5]. Most of the material of the Secs.4 and 5 are based on recent papers of A. Abanov and the author [1, 2]. We do not refer to recent attempts to extend the bosonization to higher dimension as an alternative description of the Fermi liquid, since they do not seem relevant to the subject of Sec.4.

2 Criterion for Superconductivity.

We start from a discussion of the definition, criteria, and implementation of superconductivity. This section is aimed to separate fundamental aspects of the phenomenon of superconductivity from features inherited through a particular (BCS) mechanism.

2.1 Hydrodynamics of an Ideal Liquid Phenomenology of Superconductors

2.1.1 Euler equations for an ideal liquid

The phenomenology of superconductivity starts by identifying the superfluid with an ideal quantum liquid. The hydrodynamics of an ideal liquid are given by two equations [4]— continuity

\[ \partial_t \rho + \nabla j = 0 \] (2.1)

and conservation of momentum (Euler equation)

\[ \partial_t j_i + \frac{\partial}{\partial r_k} \Pi_{ik} = 0, \] (2.2)

where

\[ \Pi_{ik} = p \delta_{ik} + \rho v_i v_k \] (2.3)

is the momentum tensor, and \( \rho, v, j = \rho v, j \) are the density, velocity, current and pressure.

All together they can be assembled in the Hamiltonian form

\[ H = \frac{m \rho v^2}{2} + \epsilon \{ \rho \}, \] (2.4)

\[ [\rho(\mathbf{r}), v(\mathbf{r}')] = -\frac{i}{m} \nabla_r \delta(\mathbf{r} - \mathbf{r}') \] (2.5)
where \(\epsilon\{\rho\}\) is the potential energy of the liquid. In the linear approximation \(\rho - \rho_s \ll \rho_s\) (a superfluid density) and
\[
\epsilon\{\rho\} = \frac{\omega^2}{2} \beta^2, \quad p = v_0^2 (\rho - \rho_s),
\]
where \(\omega\) is a compressibility and \(v_0^2 = \partial p/\partial \rho = \frac{m}{\rho_s \omega}\) is the sound velocity, we obtain the Hamiltonian of linear hydrodynamics
\[
H = \frac{\kappa}{2} [v_0^{-2} j^2 + (\rho - \rho_s)^2] + A^\text{ext} j + A^\text{ext}_0 (\rho - \rho_s)
\]

\[
[\rho(r), j(r')] = -i \rho_s \nabla_r \delta(r - r')
\]
(2.7)

Here and in what follows we use \(\kappa = \omega v_0^{-4} = \partial \epsilon/\partial \rho\) and also refer to it as a compressibility. We add an external electromagnetic potential in order to test a response of the system to an electromagnetic field. These give the basic equation of the superfluid — a quantum metastable state with a current:
\[
\nabla \times j = -\frac{1}{4\pi \lambda_L^2} B^\text{ext}.
\]
(2.8)
\[
v_0^2 \nabla \rho + \partial_t j = \frac{1}{4\pi \lambda_L^2} E^\text{ext}.
\]
(2.9)

The first equation (together with the Maxwell equation \(\nabla B^\text{ext} = 4\pi j\)) implies the Meissner effect, where
\[
\lambda_L = \left(4\pi e^2 \rho_s/mc^2\right)^{-1/2} = \left(\frac{\kappa}{4\pi e^2}\right)^{1/2} \left(\frac{c}{\omega_p}\right) = c/\omega_p
\]
is the London penetration depth (for London’s superconductors) and \(\omega_p = (4\pi \rho_s e^2/m)^2\) is the plasma frequency. The second equation means that the liquid is compressible and has an ideal conductivity \((\sigma(\omega) \sim i\omega^{-1})\).

In the Lorentz gauge \(\nabla A + v_0^{-2} \partial_t A_0 = 0\), these equations acquire familiar form
\[
\nabla \times A = \frac{1}{4\pi \lambda_L^2} B^\text{ext},
\]
(2.10)

The London equations can be also translated into the current-current correlators at small \(\omega\) and \(k\). Varying these equations \(2.8,2.9\) over \(A^\text{ext}\) and taking into account the continuity equation \(2.1\) we get
\[
\langle \rho(q, \omega) \rho(-q, -\omega) \rangle = -\kappa^{-1} \frac{v_0^2 q^2}{\omega^2 - v_0^2 q^2}
\]
(2.11)
\[
\langle j_\perp(-q, -\omega) j_\perp(q, -\omega) \rangle = \kappa^{-1} v_0^2, \quad \nabla j_\perp = 0,
\]
(2.12)
where \(j_\perp\) is the transverse component of the current. This is another way to say that the superfluid is a compressible ideal liquid. The density-density correlator \(2.11\) shows a gapless mode (longitudinal sound) which means that the liquid is an ideal and compressible, i.e. may flow without dissipation.

\(^1\)Parameters \(m\) and \(\rho_s\) being treated separately are fictitious. Only their ratio, i.e. compressibility \(\omega = \frac{m}{v_0^2 \rho_s}\) and the sound velocity \(v_0^2\) are physical and measurable.

\(^2\)The Coulomb interaction between charge carriers makes the liquid incompressible. Below we neglect the Coulomb interaction and consider the electromagnetic field as external.
The second equation (which equivalently reflects the Meissner effect) implies that there is no gapless transverse mode, i.e. that there is no shear modulus. A zero rigidity to shear is the only true definition of a liquid. As a contrary transverse sound is a feature of a solid. For a solid the density-density correlator is the same is in eq. (2.11), whereas for the transversal current correlation one would have

$$\langle j_\perp(q,\omega)j_\perp(-q,-\omega)\rangle = -\kappa^{-1}v_0^2\frac{\omega^2}{\omega^2-v_0^2q^2}. \quad (2.13)$$

In one dimension, there is no true distinction between solid and liquid — there is no room to apply a shear.

### 2.1.2 Displacements and Potentials

Density and current are not cannonical variables. In order to quantize the hydrodynamics canonically, one need to solve continuity equation (2.1). The latter plays a role of a constraint. It can be done through displacement $u$:

$$\rho - \rho_s = -\nabla u, \quad j = \partial_t u, \quad (2.14)$$

which is a cannonical partner of the current

$$[j_i(r),u_k(r')] = -i\frac{\rho_s}{m}\delta_{ik}\delta(r-r'). \quad (2.15)$$

In the linear approximation the displacement $u$ is a free field. In terms of displacements the linear hydrodynamics (2.7) becomes

$$\mathcal{L} = \frac{\kappa}{2}(v_0^{-2}(\partial_t u)^2 - (\nabla u)^2). \quad (2.16)$$

The most familiar origin of superfluidity in quantum systems is a formation of a condensate. In this case the continuity equation can be also solved by setting

$$j = \frac{1}{2m^*}\Psi^*(i\nabla - 2eA^{\text{ext}})\Psi + \text{h.c.},$$

$$\rho = |\Psi|^2, \quad [\Psi^*(r), \Psi(0)] = \delta(r). \quad (2.17)$$

where $\Psi = |\Psi|e^{i\phi}$ is an amplitude of the condensate - a sort of order parameter. Then the hydrodynamics can be written as Gross-Pitaevsky equations $^7$. Substituting (2.17) into (2.6), we get

$$i\partial_t \Psi = -\frac{1}{2m^*}\nabla^2 \Psi + \kappa (|\Psi|^2 - \rho_s)\Psi \quad (2.18)$$

Parameter $m^*$ establishes the second scale for the superconductor - the correlation length $\xi \sim \hbar/m^*v_0$.

If one neglects any inhomogeneity of the modulus $|\Psi|$ , the current becomes a gradient of the phase

$$j = -\frac{\rho_s}{2m^*}\nabla \phi. \quad (2.19)$$

$^3$ The Gross-Pitaevsky equation resembles, but has a little in common with the Ginzburg-Landau equation. The latter is valid near $T_c$. The applicability of the former equation to superconductors, if any, is limited.
If for some reasons one may neglect vorticity (i.e. transversal part of the current), this representation (regardless the equation (2.18)) gives the standard version form of the Josephson effect. It can be seen in the form of point current-current correlator

$$\langle j(\omega)j(-\omega) \rangle \sim \delta(\omega).$$  \tag{2.20}$$

This result does not follow from the hydrodynamics alone, but is rather based on a strong assumption that the phase \( \phi \) has no vorticity and depends on the mechanism. It different in topological superconductors.

In fact vorticity is involved in a general solution of the continuity equation. Let us combine a density and a current to \( d+1 \) vector \( j_\mu = (\rho_0, j_\nu) \). The continuity equation then requires the current to be a 1-form. In dimensions 1, 2, 3 it is respectively

\[
\begin{align*}
    j_\mu &= \partial_\mu \phi + \epsilon_{\mu\nu} \partial_\nu \varphi, \\
    j_\mu &= \partial_\mu \phi + \epsilon_{\mu\lambda\nu} \partial_\nu A_\lambda, \\
    j_\mu &= \partial_\mu \phi + \epsilon_{\mu\lambda\nu\rho} \partial_\nu F_{\lambda\rho},
\end{align*}
\tag{2.21}
\]

where \( F_{\lambda\rho}, A_\lambda, \varphi \) are potentials. In one dimensions the potential \( \varphi \) is also a displacement.

In contrast to the BCS, in topological liquids, potentials are more natural (less singular) fields than BCS phase \( \phi \). This obviously weakens the Josephson effect but does not eliminate it and the superconductivity - the current-current correlator (2.20) has a power law peak around \( \omega = 0 \) with the width of the order of Fermi energy, rather than being a delta function.

2.2 Landau Criterion

Not every electronic liquid is ideal. A Fermi liquid, for example, are dissipative. Their current correlation functions show Landau damping:

\[
\Im m \langle j_\perp(q,\omega)j_\perp(q,\omega) \rangle \sim -\int_{|q+k|>k_f}^{k<k_f} \delta(\omega + \epsilon(k) - \epsilon(k+q)) \, dk \sim -\nu_0 m k_f \frac{\omega}{q},
\tag{2.22}
\]

where \( \nu_0 \) is a density of states. The origins of the dissipation are

(i) gapless particle-hole excitations, and

(ii) leaving the Fermi surface, an electron can easily change the direction of the momentum (the dissipation is due to the integration in (2.22) over the angle between momenta \( k \) and \( k+q \)). For obvious reasons there is no Landau dissipation in the one dimensional case.

To avoid the dissipation a liquid either

(i) **must have no gapless density modes other than longitudinal sound**. In particular the single particle spectrum of the homogeneous liquid must have a gap., and/or

(ii) all scattering channels which change a direction of an electron must be effectively suppressed \( \text{[27]} \).

The first proposition is the subject of the Landau criterion \( \text{[3]} \). In its grotesque form it states that

*the spectrum must contain longitudinal sound and the single particle spectrum must have a gap.*
In the mechanism we discuss below both these sufficient criteria hold true.

All of this is true for a homogeneous liquid. The true check whether a liquid is superconductive may be made only if some non-zero concentration of impurities does not result in resistivity. There is a belief that if the Meissner effect holds in the absence of impurities, a weak disorder does not lead to resistivity, regardless of the mechanism of superconductivity or scattering.

The Landau criterion seems to be sufficient in spatial dimensions greater than one. In one dimension it fails for a very simple reason, namely, in one dimension we cannot distinguish between a liquid and a solid since there is no shear and no Meissner effect. As a result a single impurity pins down the flow in the same way as a single impurity pins down a solid [7].

An important consequence of a gaped one particle spectrum and a soft density mode is that the gap opens always at the level of the chemical potential. This means that while adding any even number of particles into the system, the spectrum rearranges itself so that new particles appear below the gap. Then the two particle wave function, i.e., the matrix element

\[ \Delta(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathcal{N} + 2 | c_\uparrow^\dagger(\mathbf{r}_1)c_\downarrow^\dagger(\mathbf{r}_2) | \mathcal{N} \rangle \]  

(2.23)

between the ground states of the system with \( \mathcal{N} \) and \( \mathcal{N} + 2 \) particles is localized, i.e. decays with a distance between particles. This gives rise to the Josephson tunneling.

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The Josephson tunneling amplitude, i.e. the two particle matrix element must not be confused with the order parameter of Sec.2.1.2.

2.3 Implementation of superconductivity.

Superconductivity—a metastable quantum state with a current in a macroscopical system—manifests itself as a particular set of correlations in the ground state:

- (i) **Meissner effect**: no rigidity to shear (an ideal diamagnetism)
  \[ \langle j_\perp(k, \omega) j_\perp(-k, -\omega) \rangle = \frac{1}{(4\pi\lambda_L)^2}, \text{ at } k, \omega \to 0. \]

  (2.24)

- (ii) **Gap in the electronic spectrum**: a singularity \( \omega = \Omega(p) \) closest to the origin of the one-particle Green’s function in the \( \omega \) plane
  \[ G(\omega, p) = \langle c_p^\dagger(\omega, p)c_p(\omega, p) \rangle \]

  (2.25)

- (iii) **Josephson tunneling**: the two particle wave function (the matrix element between the ground states of the system with \( \mathcal{N} \) and \( \mathcal{N} + 2 \) particles) is localized and normalizable
  \[ \Delta(\mathbf{r}_1 - \mathbf{r}_2) = \langle \mathcal{N} + 2 | c_\uparrow^\dagger(\mathbf{r}_1)c_\downarrow^\dagger(\mathbf{r}_2) | \mathcal{N} \rangle \]

  (2.26)

- (iv) **Correlation length**:
  \[ j(k) = \frac{1}{(4\pi\lambda_L)^2}(1 - O(k\xi)^2)A^\text{ext} \]

  (2.27)
These correlations describe very different aspects of the phenomenon: (i, iv) the hydrodynamics of an ideal liquid (density and compressibility), (ii) one particle spectrum, and (iii) two particle matrix element, (iv) correction to the hydrodynamics.

They also set up scales: a penetration depth, correlation length, a tunneling amplitude, a gap and a transition temperature $T_c$.

From a general point of view, there is almost no reasons to relate different quantities and different scales to each other.

Nevertheless, due to the mean field character of the BCS theory many of them turn out to be essentially the same. For instance, the gap (2.23) and two particle matrix element (2.23) appear to be related and the two particle matrix element is identified with the order parameter $\Psi$. Moreover, the inverse correlation length, gap, tunneling amplitude and $T_c$ emerge as the same scale.

This misleading "advantage" of the BCS theory often allows one to draw conclusions about the gap function by looking at the matrix element, about the tunneling based on the hydrodynamics, and about transition temperature based on the gap and vice versa.

However, the gap, Josephson current, and the order parameter are essentially different: the first characterizes the spectrum, the second is a matrix element, determined also by the phase of the wave function, while the third measures correlation between pairs. For instance, in the model we consider below, contrary to the BCS, the pair correlation function

$$\lim_{|(r_1, r_2) - (r_3, r_4)| \to \infty} \langle c_{\uparrow}^{\dagger}(r_1)c_{\downarrow}^{\dagger}(r_2)c_{\downarrow}(r_3)c_{\uparrow}(r_4) \rangle \to 0$$

vanishes due to strong fluctuations of the phase of a pair $c_{\uparrow}^{\dagger}(r_1)c_{\downarrow}^{\dagger}(r_2)$. This however does not mean that there is not a sort of a long range order or superfluid density (2.27).

These quantities start to diverge beyond the mean field level of the BCS theory, although the deviations are perturbative (non singular).

In an electronic liquid where the interaction is strong, one also expects to see a difference between dissimilar implementations of superconductivity. This difference becomes dramatic in topological liquids, where the entire effect of superconductivity is due to peculiar quantum phases of wave functions of the ground state and low energy excitations. The deviations are also dramatic in the cuprate superconductors.

There are several reasons why topological liquids are interesting. First of all they are new electronic liquids, known previously only in one dimension. In higher spatial dimensions they exhibit a fundamentally new mechanism of superconductivity and superfluidity. Secondly, the topological mechanism has been found in models of strongly correlated electronic systems, namely in the doped Mott insulator.

Below we provide a detailed analysis of a model of topological superconductivity.

We start by reviewing the one dimensional case were the topological character of the ground state is the most transparent.

### 3 Frohlich Ideal Conductivity – One Dimension.

In the early days of superconductivity and before BCS, Frohlich [14] noticed that in a one-dimensional metal an incommensurate charge density wave (CDW) slides through the lattice unat-
tenuated. Since it carries an electric charge and since a gap has been developed in the electronic spectrum, Frohlich concluded that the ground state of his system is superconducting. Although a sliding charge density wave indeed contributes to conductivity, the possibility of Frohlich superconductivity in one dimension was considered nothing more than a theoretical curiosity, because of a variety of pinning mechanisms \[7\].

The failure of the charge density wave mechanism in one dimension does not devaluate Frohlich’s ideas, which as we shall see are valid in higher dimensions \[13\].

Moreover, Peierls-Frohlich model reveals the physics of the bosonization - the method which work just as well in higher dimensional topological liquids.

### 3.1 Peierls Instability

Let us now review Frohlich’s ideas (see e.g.,Ref. \[15\]). We start from the Frohlich model of an incommensurate electron-phonon system

\[
H = \sum_{\sigma=1}^{n} (c_{\sigma}^\dagger \left( -\frac{\nabla^2}{2m} - \mu \right) c_{\sigma} + gu(x)c_{\sigma}^\dagger \nabla c_{\sigma}) + \text{an energy of phonons} \tag{3.1}
\]

where \( c_{\sigma} \) are electrons, \( u(x) \) is a displacement (phonons) and \( n \) is the degeneracy of the electronic state (spin, for example).

In one-dimensional electron-phonon systems the Peierls instability causes a lattice displacement

\[
\langle u(x) \rangle = \frac{\Delta_0}{g} \cos(2k_Fx + \varphi) \tag{3.2}
\]

with an amplitude \( \Delta_0 \sim E_f \exp(-\text{const.} / g^2) \ll E_f \) and a period equal to the average distance between particles \( 2\pi/2k_f \). In its turn the phonon modulation causes a modulation of the electronic density \( \rho - \rho_0 \sim \Delta_0 \cos(2k_Fx + \varphi) \). The phase \( \varphi \) determines the position of the charge density wave (CDW) relative to the lattice.

Frohlich noticed that the periodic density fluctuations of electrons are fixed only relative to the lattice (clearly the energy of the incommensurate state does not depend on the constant part of \( \varphi \)) and it can easily travel with some velocity, such that \( \rho \sim \Delta_0 \cos(2k_F(x - vt) + \varphi) \). This could be compensated by changing the phase according to \( \dot{\varphi} = -2k_Fv \). Therefore the current \( j = \rho v = v(nk_F/\pi) \), is

\[
j_x = -n \frac{2\pi}{2\pi} \dot{\varphi}, \tag{3.3}
\]

and from continuity

\[
\rho = \rho_0 + n \frac{2\varphi'}{2\pi}. \tag{3.4}
\]

These are Frohlich’s equations.

If the charge density wave is not pinned it propagates with the dispersion \( \omega = v_0 k \) (where \( v_0 \) needs to be computed) i.e.,

\[
L_c = \frac{n^2}{8\pi v_f} ((\partial_t \varphi)^2 - v_0^2 (\partial_x \varphi)^2), \tag{3.5}
\]

where \( v_f = \frac{\pi m}{\hbar} \neq v_0 \) is the Fermi velocity, and due to the Frohlich equations yields the hydrodynamics \[2\], \[16\].
Since there is a gap in the electronic spectrum, and the only gapless mode is the sliding CDW, Frohlich concluded that his system is superconductive. In fact the CDW is an ideal conductor rather than a superconductor due to pinning mechanisms. The Landau criterion does not work in one dimension - there is no space to flow around an obstacle.

Let us now derive the Frohlich equations formally. The first step is to pick out the fast variables and keep only the slow variables. In this case the slow variables are associated with electrons in the vicinity of two Fermi points $\pm k_F$, 

$$c(x) \sim e^{i k_F x} \psi_L + e^{-i k_F x} \psi_R ,$$

(3.6)

and phonons with momentum close to $2k_f$

$$u(x) \sim \frac{\Delta(x)}{g} \cos(2k_F x + \varphi(x)) ,$$

(3.7)

where $\varphi(x)$ and $\Delta(x)$ are fluctuating fields. In the continuum limit we then obtain the so-called linear $\sigma$-model

$$L = \frac{|\Delta|^2}{g^2 \omega^2} - \frac{|\Delta|^2}{g^2} + \bar{\psi}(i \dot{D} - |\Delta| e^{i \gamma_5 \varphi}) \psi ,$$

(3.8)

where $\dot{D} = \gamma_\mu (i \partial_\mu - A^\text{ext}_\mu)$ and $\gamma_0, \gamma_1, \gamma_5$ are two-dimensional Dirac matrices, and $\omega$ is a characteristic frequency of phonons. The modulus of the phonon field does not fluctuate much and is determined by its mean field value $\Delta_0$. The effective model is

$$L = \sum_{\sigma} \bar{\psi}_\sigma (i \dot{D} - \Delta_0 e^{i \gamma_5 \varphi}) \psi_\sigma ,$$

(3.9)

This is the result of the Peierls instability – a gap $\Delta_0$ has opened at the Fermi level. What is the spectrum of this system? We shall see that the spectrum consists of

- a gapped electronic mode
  $$\varepsilon(k) = \pm \sqrt{\Delta^2 + v_f^2 (k \pm k_f)^2};$$

- a gapless sonic mode - a phase $\varphi$ of distortion. Below we shall see that it is a mode of modulation of density with dispersion $E(k) = v_0 k$;

- a gapped mode of spin density modulation.

The vacuum of the model is infinitely degenerate: the states $|\varphi\rangle$ and $|\varphi + \text{const}\rangle$ have the same energy. Degeneracy leads to a soft sonic mode so that the system is an ideal conductor. Let us stress that position of the gap is always at the Fermi level, so that the spectrum strongly depends on the number of particles (filling factor).

For comparison let us consider a commensurate Peierls model, where the number of electrons (with a given spin) is half the number of lattice sites. A canonical tight-binding model is

$$H = \sum_n \Delta_{n,n+1} (a_n^+ a_{n+1} + h.c) + H_\Delta$$

(3.10)

where $\Delta_{n,n+1}$ is a fluctuating hopping amplitude and $H_\Delta$ is a phonon energy. In the continuum limit the half-filled case is described by the same $\sigma$-model (3.8) but with a real $\Delta$:

$$L = \bar{\psi}(i \dot{D} - i \gamma_5 \Delta(x)) \psi - \frac{\Delta^2}{2g^2} + \frac{|\Delta|^2}{g^2 \omega^2}$$

(3.11)
At exact half filling, the CDW is two-fold commensurate and so the vacuum is two-fold degenerate. In this case there is no soft translational mode since the CDW is commensurate and is pinned by the lattice. The excitations are gapped electrons and kinks of $\Delta(x)$ which connect two-fold degenerate mean field vacua: $\Delta \to \pm \Delta_0$ when $x \to \pm \infty$. The system is an insulator. It becomes an ideal conductor under a doping.

Below we discard the kinetic energy of phonons (the term $\frac{1}{g^2} \frac{\omega^2}{\Delta}$ in eqs. (3.8,3.11)) by sending $g \to \infty$. The energy of phonons then appears as radiative corrections due to exchange by electrons. The advantage of this limit is that the compressibility of the system (i.e. its central charge) and conformal dimensions of all operators will be universal and determined by the number of fermionic flavors $n$. Later we specify that $n$ is equal 2.

3.2 Hydrodynamics

3.2.1 Spectral flow and Zero modes

Let us now add particles to the system. In an incommensurate case the gap must follow the new chemical potential. In other words extra particles will not go into the upper band to occupy the lowest empty state. Instead a doping rearranges the period of the CDW, so as to create one more level in the lower band. This phenomenon is known as it level crossing or a spectral flow. How does it happen?

In the presence of a kink in the phase, $\varphi(\infty) - \varphi(-\infty) = 2\pi$, the electronic spectrum changes in such a way that one extra unoccupied level appears at the top of the occupied band with an energy $E = -\Delta_0$. When we add a particle to the system, it will therefore create a kink in the spatial configuration of $\varphi$ and an extra level, in order to be absorbed by the lower band. Moreover, the density of extra particles is locally and adiabatically bound to the kink as is suggested by eqs.(3.3, 3.4).

\[
\dot{j}_\mu = \frac{n}{2\pi} \epsilon_{\mu\nu} \partial_\nu \varphi.
\] (3.12)

We derive this equation on a more formal basis later on.

The situation in the commensurate case (3.11) is more subtle but is essentially the same. In the presence of the kink the electronic spectrum remains approximately unchanged, except for the appearance of the so-called zero mode, a state with a zero energy, located exactly in the middle of the gap. Indeed, the solution of the Schroedinger equation at a static kink $\Delta(x \to \pm \infty) \to \pm |\Delta_0|$ with minimal energy is the zero mode

\[
\begin{align*}
    i\partial_x \psi_L - i\Delta \psi_R &= E \psi_L \\
    -i\partial_x \psi_R + i\Delta \psi_L &= E \psi_R
\end{align*}
\] (3.13, 3.14)

with minimal energy is the zero mode

\[
\psi_L = \psi_R = e^{-\int_0^x \Delta(x)}; \quad E = 0.
\] (3.15)

The wave function of the zero mode is located in the core of the kink.

While interpreting this result a subtle difference occurs, according to whether we keep the number of particles or the chemical potential fixed. If we fix the number of particles, the zero mode is unoccupied and ready to accept an extra particle. In the case of a fixed chemical potential (equal zero), the zero mode is occupied by $1/2$. To understand this better let us notice that a single
kink (an odd number of kinks) is not compatible with fixed boundary conditions. If the boundary conditions are fixed, say periodic, then the minimum of two kinks (an even number of kinks) is required. However the two kinks may be well separated and can be treated almost independently. Since the topological charge is zero, there is no zero modes, but rather a symmetrical and an antisymmetrical combination of zero modes of independent kinks, split around zero energy. The antisymmetric state will appear below the chemical potential. If the chemical potential is kept at zero, it will be occupied by a particle. Thus we have two kinks per particle, i.e. each kink will be 1/2 occupied.

An adiabatic relation between the density of particles and soliton configuration (the axial current anomaly), similar to eq. (3.4), in this case tells us that the density of extra states is equal to the density of zeros of $\Delta(x)$

$$\rho(x) = \frac{n}{2} \delta(\Delta(x)) \frac{\partial \Delta}{\partial x} .$$

The factor 1/2 reflects the fractional occupation number, discussed above.

Suppose we now dope the system by adding extra particles. The remarkable fact is that the system will lower its energy by creating the number of solitons (zeros in $\Delta(x)$) which is necessary to absorb all dopants. We refer this phenomenon as a topological instability.

If a nonzero density of particles is added, i.e. a nonzero density of kinks is created, an interaction between kinks results in the formation of a narrow band in the middle of the wide gap (a midgap band). The width of this band is of the order of $\Delta_0 \exp(-\text{const} \frac{\rho}{\rho_d - \rho_s})$ where $\rho - \rho_s$ is the doping density and $\rho_s$ the density of the undoped system. This band absorbs all the dopants and is always completely filled.

The passage to an incommensurate case is as follows. As in Frohlich’s case, solitons have a translation mode due to their topological origin: a soliton lattice can slide along the atomic lattice without dissipation. Let $\bar{x}_i$ be the zeros of $\Delta(x)$. Then the density of extra particles (dopants) is $\delta \rho(x) = \frac{1}{2} \sum_i \delta(x - \bar{x}_i)$. Displacement of the positions of zeros around their mean field values, $x_i = \bar{x}_i + \varphi(x_i, t)/2\pi \rho_0$, give rise to fluctuations of the density $\delta \rho(x) = \rho(x) - \rho_0$. According to (3.16) they obey the same Frohlich equations (3.3, 3.4, 3.12).

Each twist of $\varphi$ adds one additional state in the middle of the gap. Therefore, adding $n_e$ extra particles gives rise to the topological charge of $n_e = Q$ where $Q = \int \frac{d\varphi}{2\pi}$. All of this is true in incommensurate cases when the system, after doping, has infinitely degenerate classical vacua. If the doping is a rational number, say, $p/q$, then the number of degenerate vacua is finite, namely $q$. The CDW is generally pinned by an exponentially small potential $\Delta_0^q(\Delta_0/\varepsilon_f)^q - 2 \cos q\phi$.

If the CDW is not pinned, an electric field can easily drag the solitons relative to the crystal. This leads to a current and is given by axial current anomaly equations, which determine the response to an external electromagnetic field. Setting $m = v_0 = 1$, it reads

$$\epsilon_{\mu\nu} \partial_\nu j_\mu = \frac{n}{\pi} E_\text{ext} .$$

These equations are equivalent to the equations of linear hydrodynamics (2.8, 2.9). As we discussed earlier, we cannot distinguish between a solid and a liquid in one dimension. Nevertheless, there is a global version of the Meissner effect (a spectral flow) in one-dimension– a diamagnetic current is generated by a magnetic flux $\Phi_\text{ext}$ being set inside a metallic ring:

$$\int j dx = -\frac{n}{\pi} \Phi_\text{ext} .$$

(3.18)
Combining (3.17) with the Frohlich equations (3.12) and using the relation $j = -\partial L/\partial A^{\text{ext}}$ we obtain a bosonized version of the incommensurate CDW:

$$L_\varphi = \frac{n}{4\pi} \left( \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{2}{\pi} E^{\text{ext}} \varphi \right).$$

(3.19)

The Hamiltonian of the linear hydrodynamics (2.7) in one dimensional literature is known as Sugawara form.

### 3.2.2 Bosonization

For illustrative purposes let us rederive the results of the Sec. 3.2.1 by the bosonization procedure. For simplicity let us consider only two fermionic species (spin) $\sigma = \uparrow, \downarrow$. In this approach, fermions $\psi_\sigma$ are treated as soliton of a boson field $\phi_\sigma$:

$$\psi_{L\sigma} \sim a^{-1/2} : e^{i\phi_{L\sigma}} : , \quad \psi_{R\sigma} \sim a^{-1/2} : e^{-i\phi_{R\sigma}} :$$

(3.20)

where $\ldots :$ is the normal ordering and $\phi_{L,R}$ are holomorphic components of the canonical Bose field

$$[\phi_L(x), \phi_L(y)] = -[\phi_R(x), \phi_R(0)] = i\pi \text{sign}(x-y),$$

$$\phi_R(x)\phi_R(0) - \phi_R(x)\phi_R(0) : = \ln \left( \frac{L}{x} \right),$$

(3.21)

where $a$ is a lattice scale and $L$ is the size of the system. The chiral components of currents in terms of chiral bosons are

$$j_{L\sigma} \equiv \psi_{L\sigma}^\dagger \partial_x \psi_{L\sigma} = \frac{1}{2\pi} \partial_x \phi_{L\sigma}, \quad j_{R\sigma} \equiv \psi_{R\sigma}^\dagger \psi_{R\sigma} = \frac{1}{2\pi} \partial_x \phi_{R\sigma}$$

(3.22)

whereas the Hamiltonian of free fermions is given by the Sugawara form:

$$H = i\psi_{L\sigma}^\dagger \partial_x \psi_{L\sigma} - i\psi_{R\sigma}^\dagger \partial_x \psi_{R\sigma} = \frac{1}{4\pi} \left( (\partial_x \phi_{L\sigma})^2 + (\partial_x \phi_{R\sigma})^2 \right).$$

(3.23)

It is instructive to rewrite these formulas in the Hamiltonian formalism, in terms of displacement and its canonical momentum

$$\phi_\sigma = \frac{1}{2\pi} (\phi_{L\sigma} + \phi_{R\sigma}), \quad \Pi_\sigma = \frac{1}{2} \partial_x (\phi_{L\sigma} - \phi_{R\sigma}),$$

$$[\phi_\sigma(x), \Pi_\sigma(0)] = i\delta(x),$$

$$\rho_\sigma(x) = \partial_x \phi_\sigma(x), \quad j_\sigma(x) = \frac{1}{\pi} \Pi_\sigma(x).$$

(3.24)

In these terms the electronic operator (3.6) is

$$c_\sigma(x) \sim \sum_{k_f} e^{i\text{Arg}(k_f)} e^{ik_f x} : e^{i \int x' \Pi_\sigma(x') dx' + i\pi \text{sign}(k_f) \phi_\sigma(x)} :$$

(3.25)

where $k_f = \pm k_f$ and the amplitude of backward scattering is

$$\Delta_0 \bar{\psi} e^{i\gamma \phi} \psi \sim \Delta \sum_\sigma : \cos(2\pi \phi_\sigma + \varphi) :$$

(3.26)


\[ \Delta \sim a^{-1} \Delta_0. \]

Let us notice that the factor \( e^{i k_f x} \) in the equation (3.25) is an inherent part of the second factor. In fact momentum \( \Pi_{\sigma}(x) \) has a constant (\( x \)-independent) part \( \Pi_0 \) which corresponds to a motion of a soliton without changing its configuration. This part is also called zero mode and has to be treated separately. An electron with a momentum close to the Fermi surface corresponds to a state with \( \Pi_0 = k_f \). The factor \( e^{i \text{Arg}(k_f)} = \pm (\text{a relative phase between left and right movers}) \) is more subtle, and not too important in one dimension. In Sec. 4.3.2 we shall see how this factor develops into two spatial dimensions.

### 3.2.3 Hydrodynamics and spin-charge separation

In terms of the bose fields the Lagrangian (3.8) becomes

\[
L = \frac{\pi}{2} (\partial \phi_\uparrow)^2 + \frac{\pi}{2} (\partial \phi_\downarrow)^2 + \Delta (\cos(2\pi \phi_\uparrow + \varphi) + \cos(2\pi \phi_\downarrow + \varphi)) + \frac{1}{\pi} \epsilon_{\mu\nu} A_{\mu}^{\text{ext}} \partial_{\nu}(\phi_\uparrow + \phi_\downarrow) \tag{3.27}
\]

where we introduced an external electromagnetic field to keep track of the response functions. Introducing charge and spin densities

\[
\phi_c = \frac{\phi_\uparrow + \phi_\downarrow}{\sqrt{2}}, \quad \phi_s = \frac{\phi_\uparrow - \phi_\downarrow}{\sqrt{2}} \tag{3.28}
\]

we get

\[
L = \frac{\pi}{2} (\partial \phi_c)^2 + \frac{\pi}{2} (\partial \phi_s)^2 + \Delta \cos(\sqrt{2}\pi \phi_c + \varphi) \cos(\sqrt{2}\pi \phi_s) + \sqrt{2} \epsilon_{\mu\nu} A_{\mu}^{\text{ext}} \partial_{\nu}\phi_c \tag{3.29}
\]

At energy less than \( \Delta \) the field \( \varphi \) follows the field \( -\sqrt{2}\pi \phi_c \) in order to keep the argument of the first cos fixed, i.e., a singlet electronic current adiabatically follows the density of solitons of the phonon field \( j_\mu = \frac{\pi}{2} \epsilon_{\mu\nu} \partial_{\nu} \varphi \). The dynamics of the remaining variables consists of an independent gapped spin density wave and a gapless charge density wave mode:

\[
L = L_c + L_s, \tag{3.30}
\]

\[
L_c = \frac{\pi}{2} (\partial \phi_c)^2 + \sqrt{2} \epsilon_{\mu\nu} A_{\mu}^{\text{ext}} \partial_{\nu}\phi_c, \tag{3.31}
\]

\[
- \sqrt{2}\pi \phi_c = \varphi, \tag{3.32}
\]

\[
L_s = \frac{\pi}{2} (\partial \phi_s)^2 + \Delta \cos(\sqrt{2}\pi \phi_s). \tag{3.33}
\]

The charge density sector is a hydrodynamics with compressibility differs from the compressibility of free fermions by a factor \( 1/2 \).

The lesson we may learn from the Frohlich’s story can be summarized as follows:

If the system has infinitely many topologically distinct degenerate vacua \( |Q\rangle \), then a topological configuration \( \varphi(x, t) \) which transforms one vacuum \( |Q\rangle \) into another \( |Q + 1\rangle \) (\( \varphi(x = \pm \infty, t = 0) = 0 \) and \( \varphi(x = \pm \infty, t = \infty) = 2\pi \)) is a hydrodynamic mode, regardless of whether the particle spectrum is gapped. Then, “the whole system, electrons and solitons, can move through the lattice without being disturbed” [14].
3.3 Correlation functions

3.3.1 One particle electronic Green’s function

The bosonization approach reviewed in the previous section provides an effective way to compute correlation functions. Below we concentrate on the one particle Green’s function

\[ G(t, x) = \langle c^\dagger_\sigma(t, x)c_\sigma(0) \rangle. \]

The Green function consists of two chiral parts

\[
G(x, y) = e^{ik_f(x-y)}\langle \psi^\dagger_R(x)\psi_R(y) \rangle + e^{-ik_f(x-y)}\langle \psi^\dagger_L(x)\psi_L(y) \rangle +
\]

\[
e^{ik_f(x+y)}\langle \psi^\dagger_R(x)\psi_L(y) \rangle + e^{-ik_f(x+y)}\langle \psi^\dagger_L(x)\psi_R(y) \rangle.
\]

As an implication of the Orthogonality Catastrophe, the part which does not conserve momentum vanishes in a macroscopic system \((L \to \infty)\)

\[
\langle \psi^\dagger_R(x)\psi_L(y) \rangle \to 0 \quad (3.34)
\]

Having in mind an extension of this method to higher dimensional topological liquids, let us introduce the following notations: \(\psi_{k_f}(x) = \psi_{L,R}(x), \quad \phi_{k_f}(x) = \phi_{L,R}(x)\) and \(G(k_f, x) = G_{L,R}(x)\) for \(k_f = \pm k_f\). Sometimes we also use \(v_f = k_f/m = \pm v_f\). Then we can write the Green function as

\[
G(x) = \sum_{k_f = \pm k_f} e^{ik_f x} \langle \psi^\dagger_{k_f}(x)\psi_{k_f}(0) \rangle = \sum_{k_f} G(k_f, x). \quad (3.35)
\]

Bosonization decomposes the Green function into spin and charge parts

\[
G_L(x) \sim \langle e^{i\sqrt{2}(\phi^L(x) - \phi^L(0))} \rangle \langle e^{-i\sqrt{2}(\phi^L(x) - \phi^L(0))} \rangle. \quad (3.36)
\]

The charge part is easy to compute, since the charge sector is just an ideal liquid

\[
\frac{1}{2}((\phi_c(k_f, x) - \phi_c(k_f, 0))^2 = \ln((v_f(t - i0) - x)k_f),
\]

\[
\frac{1}{2}((\phi^L_c(x) + \phi^L_c(0))^2 = \ln(\frac{a^2}{L^2}(v_f(t - i0) - x)k_f). \quad (3.37)
\]

The correlation function is determined by the compressibility, which differs from free fermions by a factor 1/2:

\[
\langle e^{i\sqrt{2}(\phi_c(k_f,x)-\phi_c(k_f,0))} \rangle \sim (k_f(v_f(t - i0) - x)^{1/2}. \quad (3.38)
\]

A similar factor

\[
\langle e^{i\sqrt{2}(\phi_c(k_f,x)+\phi_c(k_f,0))} \rangle \quad (3.39)
\]

which occurs in a nondiagonal Green function \((3.34)\) vanishes as \((\frac{1}{L^2}(v_f(t - i0) - x)^{1/2}.\)

The spin part of the propagator can not be calculated in the same manner. However, it can be estimated at the mass shell \(|v_f^{-2}x^2 - t^2|\Delta_0^2 \gg 1\). First of all the Lorentz invariance requires that the Green function \((3.35)\) to be of the form \((v_f(t - i0) - x)^{-1}f((x^2 - (v_f^2t^2))\Delta_0^2)\). This gives the
form of the spin factor in (3.36). It is 
\( f(t - i0) = \frac{1}{2} f((x^2 - (v_f t)^2)\Delta_0^2) \). Then at \( t = 0 \), the spin factor in (3.36) may be asymptotically replaced by the Green function of massive fermions

\[ G^0(k_f, x, t) = \int \frac{d\omega dp}{(2\pi)^2} e^{ix\omega + i\omega t} G^0(\omega, k_f + p), \]  

where

\[ G^0(\omega, k) = \sum_{\nu_f = \pm v_f, k_f = \pm k_f} \frac{\omega - \nu_f(k - k_f)}{\omega^2 - \nu_f^2(k - k_f)^2 - \Delta_0^2 + i\epsilon} \]  

is the free fermionic Green function. As a result the mass shell asymptotic reads

\[ \langle e^{i\sqrt{2}(\phi^L_1(x) - \phi^L_1(0))} \rangle \sim \left( \frac{k_f}{k_f} \right)^{1/2} \left( \frac{x - v_f t + i0}{x + v_f t - i0} \right)^{1/4} G^0_L(x, t) \]  

so that the mass shell behavior of the Green function is

\[ G(k_f, x, t) \sim D(k_f, x, t) G^0(k_f, x, t) \]  

where

\[ D(x, t) \sim \left( \frac{\alpha}{x^2 - v_f^2(t - i0)^2} \right)^{1/4} \]  

is the propagator of the soft modes. In a more general case, velocities of charge and spin waves may be different. Then

\[ D(q, \Omega) \sim \left( \frac{x - v_s(t - i0)}{x + v_s t - i0} \right)^{1/4} e^{i\text{Arg}(k_f)} \]  

where \( \text{Arg}(k_f) = 0, \pi \) is the angle of the Fermi vector. It is instructive to see the Green functions in the momentum representation. It is a convolution of the propagators of free massive fermions and a sonic mode of density modulation

\[ G(\omega, k) = \sum_{k_f = \pm k_f} \int G^0(k - k_f - q, \omega - \Omega) D(q, \Omega) \frac{d\Omega dq}{(2\pi)^2}, \]  

where \( k \) and \( q \) have only two distinct directions along \( k_f = \pm k_f \). Here

\[ D(q, \Omega) \sim 1 \left( \frac{\Omega^2 - v_f^2 q^2 + i0}{\epsilon_f^2} \right)^{1/4} e^{i\text{Arg}(k_f)} \]  

is the propagator of the density wave.

Computing this integral one gets on shell asymptotic of the Green function

\[ G(\omega, k) \sim \sum_{k_f = \pm k_f} \frac{\omega - \nu_f(k - k_f)}{(\omega^2 - v_f^2(k - k_f)^2 - \Delta_0^2 + i0)} \left( \frac{\omega^2 - v_f^2(k - k_f)^2 - \Delta_0^2 + i0}{\epsilon_f^2} \right)^{1/4} e^{i\text{Arg}(k_f)}. \]  

(3.47)
3.3.2 Electronic spectral function

Electronic Green function (3.47) loses its poles at \( \omega = \pm \sqrt{v_f^2(k - k_f)^2 + \Delta_0^2} \). This happens because of the interaction with densities soft modes. Instead of a pole the Green function now has two branch cuts, starting at \( \omega = \pm \sqrt{v_f^2(k - k_f)^2 + \Delta_0^2} \). This means that an electron is no longer an elementary particle. It is a composit object made of the solitons of the charge and spin sectors. It is instructive to write the Lehmann representation for the electronic Green function

\[
G(k, \omega) = \int_0^{\infty} \frac{A(k, E) \omega - E + i0 + B(k, E) \omega + E - i0}{(\omega - E + i0)(\omega + E - i0)} dE. \tag{3.48}
\]

It consists of the convolution of the spectral functions of massive fermions (3.41)

\[
A_0(p, E) = B_0(-p, E) = \sum_{v_f = \pm v_f} (E - v_f p) \delta(E^2 - v_f^2 p^2 - \Delta_0^2) \tag{3.49}
\]

and the spectral function of the soft modes

\[
P(q, \epsilon) = -\frac{1}{\pi} \Im D(q, \epsilon) \sim \epsilon^{-1/2}(\epsilon^2 - v_f^2 q^2)^{-1+1/4}, \text{ at } \epsilon > v_f |q|, \tag{3.50}
\]

so that

\[
A(k, E) = \int_0^{\infty} A_0(k - q, E - \epsilon) P(q, \epsilon) \frac{d\epsilon dq}{(2\pi)^2},
\]

\[
B(k, E) = \int_0^{\infty} B_0(k - q, E - \epsilon) P(q, \epsilon) \frac{d\epsilon dq}{(2\pi)^2}. \tag{3.51}
\]

Computing the integral over energy, we get

\[
A(k, E) \sim \epsilon_f^{-1/2} \int \frac{n_0(k_f + k - q)}{((\sqrt{v_f^2(k_f + k - q)^2 + \Delta_0^2} - E)^2 - v_f^2 q^2)^{1-1/4}} dq, \tag{3.52}
\]

where

\[
n_0(k) = \frac{1}{2} \left(1 - \frac{v_f k}{\sqrt{v_f^2 k^2 + \Delta_0^2}} \right)
\]

is the occupation number of free massive fermions.

The integral in (3.52) goes over the branch cut, which starts from the energy of composit fermions \( E > v_f q + \sqrt{v_f^2 (k_f + k - q)^2 + \Delta_0^2} \) - the energy of a massive particle plus the energy of soft modes.

The eq. (3.52) can be interpreted as follows. A composite particle means that there are many (not just one) states with a given energy and momentum. In addition to the momentum \( k \) and the energy \( E \) the intermediate states with one extra particle are characterized by an additional quantum number \( q \) — the momentum of the soft modes. The amplitude of this state \( |E, k, q\rangle \) is just a free fermionic wave function \( \Psi_0(k - q) \), but the number of these states is

\[
\nu(E, k, q) = \epsilon_f^{-1/2}((\sqrt{v_f^2 (k_f + k - q)^2 + \Delta_0^2} - E)^2 - v_f^2 q^2)^{-1+1/4}, \tag{3.53}
\]

so the spectral function is

\[
A(k, E) = \int |\Psi_0(k_f + k - q)|^2 \nu(E, k, q) dq. \tag{3.54}
\]
The Green function is often used to compute the "occupation number"

\[ n(k - k_f) = \int_0^\infty A(k, E) \frac{dE}{2\pi} \]  

(3.55)

and the "density of states"

\[ \frac{dN(E)}{dE} = \int A(k, E) \frac{dk}{2\pi} . \]  

(3.56)

We have to be cautious in interpretation of these quantities. The spectral function of composite particles describes a tunneling process rather than a density of the spectrum. Indeed,

\[ A(k, E) = \sum_{|N + 1, k \rangle} |\langle N | c(k) | N + 1, k \rangle|^2 \delta(E(N + 1, k) - E_0(N)) \]  

(3.57)

where the sum goes over all excited states of a system with \( N + 1 \) particles with energy \( E(N + 1, k) > E_0(N) \). Here the state \( \langle N \rangle \) is the ground state of the system with \( N \) particles and \( E_0(N) \) is its energy. Were an electron be a particle (an asymptotic state), the matrix element \( \langle N | c(k) | N + 1, k \rangle \) would approaches 1 at \( k \to k_f \). Then at least at the Fermi surface, the spectral function \( A(k, E) \) would describe the density of states \( A(k, E) \sim \sum \delta(E(N + 1, k) - E_0(N)) \). In the case of the topological liquid, however, one particle matrix element vanishes at \( k \to k_f \) (see Sec.(3.5.1)).

As a result, the spectral function reflects the matrix element rather than the spectrum. The spectral function appears in processes with emission and absorption of electrons, such as tunneling, photoemission, etc. In what follows we refer (3.55, 3.56) as tunneling occupation number and tunneling density of states.

The tunneling occupation number is

\[ n(k) = \int n_0(k - q) \nu(k, q) dq \]  

(3.58)

where

\[ \nu(k, q) = \int_{E - \sqrt{v_f^2(k_f + k - q)^2 + \Delta_0^2} > v_f q} \nu(E, k, q) \frac{dE}{2\pi} \sim \epsilon_f^{-1/2} (v_f q)^{-1/2} \text{ at } q \gg \Delta_0 \]  

(3.59)

where the last factor in the integrand is the number on states with a given momentum.

The tunneling occupation number has a broad character of a scale \( k_f \) around the Fermi points. At the region \( k_f \gg |k - k_f| \gg \Delta_0 \), where the gap is not important we get a familiar result for its singular part

\[ n(k) - n(k_f) \sim -\text{const}|k/k_f - 1|^{1/2} \text{sgn}(k - k_f) . \]  

(3.60)

It is a smooth function at \( k = k_f \) and crosses \( k_f \) linearly: \( n(k) - n(k_f) \sim -\text{const}(|k - k_f|) \) at \( |k - k_f| \ll \Delta_0 \). Contrary to the free massive particle case, the tunneling occupation number does not vanish at the Fermi surface. This does not mean that states with \( k > k_f \) are occupied in the ground state. They become occupied in the process of embedding an additional electron into the system.

The tunneling density of states behaves similarly

\[ \frac{dN(E)}{dE} = \int A(k, E) \frac{dk}{2\pi} \sim \epsilon_f^{-1/2} \int_{\Delta_0}^{E} \frac{dN_0(\epsilon)}{(E - \epsilon)^{1/2}} \]  

(3.61)
where \(dN_0/de = \epsilon/\sqrt{\epsilon^2 - \Delta_0^2}\) is the density of states of free fermions. The density of states shows an asymmetric broad (\(\sim \epsilon_f\)) peak. It decays from the peak toward the threshold \(E = \Delta_0\) as
\[
\frac{dN(E)}{dE} \sim \text{const}|E/\epsilon_f|^{1/2}, \quad \text{at} \quad \epsilon_f \gg E - \Delta_0 \gg \Delta_0
\]
In contrast to free massive particles, the tunneling density of states is not singular at \(E = \Delta_0\). It remains smooth at the threshold and approaches it linearly (\(\sim E - \Delta_0\)). Among the interesting features of the spectral function, one is worth mentioning: the spectral function is determined by two scales, \(\Delta_0\) and \(\epsilon_f\), rather than just \(\Delta_0\). The second scale is the signature of the orthogonality catastrophe and of the composite nature of the electron.

### 3.3.3 Axial current anomaly

One can reproduce the result of the last section by performing an anomalous transformation
\[
\psi_{L\sigma} = e^{i\frac{1}{2}\varphi_L}\chi_{L\sigma}, \quad \psi_{R\sigma} = e^{-i\frac{1}{2}\varphi_R}\chi_{L\sigma},
\]
where \(\varphi_L\) and \(\varphi_R\) are holomorphic and antiholomorphic parts of the field \(\varphi\), so that \((\partial_t \pm v_f \partial_x)\varphi_{L,R} = 0\) and \(\varphi_L + \varphi_R = 2\varphi\). Then formally the field \(\varphi\) may be absorbed by the chiral gauge transformation \(A_{L,R}^{\text{ext}} \to A_{L,R}^{\text{ext}} + \partial_x \varphi_{L,R}\) so that it disappears from the Lagrangian \((3.9)\). This is, however, not true due to the axial current anomaly. To correct it one must add to the Lagrangian the hydrodynamics of zero modes, namely the term \(\frac{1}{2}2(\partial_x^2)^2\), and connect it with charge density \((3.12, 3.32)\). One can see this immediately by comparing a bosonized version of the Lagrangian before and after the anomalous transformation. The bosonization treats an anomalous axial symmetry correctly.

The operator \(\chi\) carries no charge \([\chi, \phi]\) = 0 and is a soliton operator of the spin sector. Due to the constraint \((3.32)\), its bosonized version is \(\chi^\uparrow \sim a^{-1/2} : e^{\frac{i}{\sqrt{2}}:\phi_1}\) and \(\chi^\downarrow \sim a^{-1/2} : e^{-\frac{i}{\sqrt{2}}:\phi_1}\). In its turn the vertex operator
\[
V_c(k_f, x) \equiv e^{\pm \frac{i}{\sqrt{2}}\text{Arg}(k_f)\phi(k_f, x)}:
\]
annihilates a soliton in the charge sector.

In terms of \(\chi\) particles, the Green function has the form
\[
G(x) = \sum_{k_f} \langle \chi_{k_f, \sigma}^\dagger(x) V^{-1}_c(k_f, x) V_c(k_f, 0) \chi_{k_f, \sigma}(0) \rangle,
\]
so the mass shell asymptotic of the Green function \((3.46)\) becomes transparent. At equal times, where there is no complications with the Lorentz factor \((3.42)\) one may treat charge and spin sectors independently
\[
G(x) = \sum_{k_f} \langle \chi_{k_f, \sigma}^\dagger(x) \chi_{k_f, \sigma}(0) \rangle D(x)
\]
where
\[
D(x) = \langle V^{-1}_c(k_f, x) V_c(k_f, 0) \rangle
\]
is given by \((3.43)\). This reproduces the result of eq. \((3.43)\).

Under the constraint \((3.32)\) the vertex operator becomes
\[
V_c(k_f, x) = e^{\pm \frac{i}{\sqrt{2}}\text{Arg}(k_f)\phi(k_f, x)}:
\]
It carries no spin but annihilates a unit charge \([V_c(x), \rho(y)] = V_c(x)\delta(x - y)\).
3.3.4 Pair correlation function and a long range order

**Pair correlation function.** Among many possible two-particle correlation functions, let us consider the pair correlation function

\[ K(r_1, r_2, r_3, r_4) = \langle c_1^\dagger(r_1)c_1^\dagger(r_2)c_3(r_3)c_4(r_4) \rangle \]  

(3.68)

It illustrates a difference between the tunneling amplitude \( \langle 2.23 \rangle \), the order parameter and the long range order.

The function consists of harmonics \( e^{ik_f(r_1 \pm r_2 - r_3 \pm r_4)} \). Let us pick up one harmonic, say, \( e^{ik_f(r_1 - r_2 - r_3 + r_4)} \), which corresponds to a correlation between pairs located at \( r_1, r_2 \) and \( r_3, r_4 \):

\[ K(r_1, r_2, r_3, r_4) = e^{ik_f(r_1 - r_2 - r_3 + r_4)} \langle \psi_{R_1}^\dagger(r_1)\psi_{L_1}^\dagger(r_2)\psi_{R_4}(r_3)\psi_{L_4}(r_4) \rangle. \]  

(3.69)

Bosonization gives

\[ K(r_1, r_2, r_3, r_4) = a^{-2}e^{ik_f(r_1 - r_2 - r_3 + r_4)} \langle e^{\frac{i}{\sqrt{2}}\phi_{R,c}(r_1) - \phi_{L,c}(r_2)} e^{-\frac{i}{\sqrt{2}}\phi_{R,c}(r_3) - \phi_{L,c}(r_4)} \rangle \]

\[ \langle e^{\frac{i}{\sqrt{2}}\phi_{R,s}(r_1) + \phi_{L,s}(r_2)} e^{-\frac{i}{\sqrt{2}}\phi_{R,s}(r_3) + \phi_{L,s}(r_4)} \rangle \]  

(3.70)

Let us assume that the size of a pair is less than the distance between pairs \( |r_1 - r_2| \sim |r_3 - r_4| \ll R = |r_1 + r_2 - r_3 - r_4| \) and let us consider the dependence of the correlation function on \( R \). We have

\[ K(R) = \langle \psi_{R_1}^\dagger(R)\psi_{L_1}^\dagger(L)\psi_{R_4}(0)\psi_{L_4}(0) \rangle \sim a^{-2} \langle e^{i\sqrt{2}\int_0^R \pi_c(r)dr} e^{-i\pi\sqrt{2}(\phi_s(R) - \phi_s(0))} \rangle \]  

(3.71)

where \( 2\pi\phi = \phi_R + \phi_L, 2\pi = \partial_x(\phi_R - \phi_L) \). The spin sector factor approaches to a constant at \( R \to \infty \)

\[ \langle e^{-i\pi\sqrt{2}(\phi_s(R) - \phi_s(0))} \rangle \sim 1 \]  

(3.72)

whereas the charge sector factor can be computed by means of eqs. (3.31, 3.32)

\[ K(R) \sim \frac{k_f}{R} \]  

(3.73)

The correlation between pairs decays, but not as fast as a similar correlation for free particles \( \sim R^{-2} \). A decay of the pair correlation function must not be misleadingly considered to be a signature of the absence of superfluidity in one dimension. As we discussed above, superfluidity indeed does not exist in one dimension, but for a different reason. The reason is a pinning by impurities, rather than a decay of the correlation function \( K(R) \). In the next sections we show that in spite of the decay of \( K(R) \) not only the superfluid density, but also the matrix element \( \langle c_1^\dagger(r_1)c_1^\dagger(r_2) \rangle \) does not vanish.

**A long range order.** It is easy to construct an object which carries the electric charge \( 2e \) and exhibits a long range order. For instance, the operator \( \Psi(r_1, r_2) = c_1^\dagger(r_1)V_c(r_1)V_c(r_2)c_4^\dagger(r_2) \) shows the long range order:

\[ \tilde{K}(R) = \langle \Psi^\dagger(r_1, r_2)\Psi(r_3, r_4) \rangle \sim \langle e^{i\sqrt{2}(\phi_{R,s}(r_1) + \phi_{L,s}(r_2))} e^{-i\sqrt{2}(\phi_{R,s}(r_3) + \phi_{L,s}(r_4))} \rangle \sim 1 \]  

(3.74)

at \( R \to \infty \). This long range order indicates the Peierls instability at \( 2k_f \).
3.4 Vertex operators and local singlet state

3.4.1 Vertex operators and operator algebra

In this section we rederive the results for the Green function (obtained by the bosonization in the previous section) by means of the Hamiltonian approach. This approach, although less automatic than bosonization, has certain advantages: it clarifies the physical picture and can be lifted to higher dimensions.

Let us introduce a vertex operator of the spin sector

$$V_s(k_f, x) = e^{\sqrt{2} \phi_s(k_f, x)}.$$ (3.75)

This operator is charge neutral \([\phi_c, V_s] = 0\) but creates spin 1/2 state:

$$[S^3(x), V_s(y)] = -\frac{1}{2} V_s(x) \delta(x - y)$$ (3.76)

$$[S^3(x), (V_s(y))^{-1}] = \frac{1}{2} (V_s(x))^{-1} \delta(x - y)$$ (3.77)

where \(S^3(x) = \frac{1}{\sqrt{2\pi}} \partial_x \phi_s\) is a spin density. The charge vertex operator, then, as a composition of the electron and an antikink of the spin density. The latter removes the spin from the electron operator.

The vertex operator obeys an operator algebra.

$$[\psi_{\sigma, k_f}(x), \sqrt{2} \phi_s(k_f, 0)] = \sigma \ln \left( \frac{k_f x}{k_f L} \right)$$ (3.78)

and therefore

$$\psi_{k_f, \sigma}(x) V_s(y) = V_s(y) \psi_{k_f, \sigma}(x) \left( \frac{x - y}{L} \right)^{\sigma/2} e^{i \frac{\sigma}{2} \text{Arg}(k_f)}$$ (3.79)

where \(\text{Arg}(k_f) = 0, \pi\) is the angle of the Fermi vector and \(\sigma = \pm\). Being written in this form the equations (3.79) have a straightforward generalization to higher dimensions.

3.4.2 Local singlet state

Let us consider a system with periodic boundary conditions and even number \(N\) of particles, such that the ground state is a singlet \(|N\rangle\). Now let us attempt to embed an extra spin up electron. According to (3.12) a state with one extra particle requires a change of boundary condition (one extra particle requires a half of a kink \(\phi(x = \infty) - \phi(x = -\infty) = \pi\)) from periodic to antiperiodic. Also, the spin of a state with odd number of particles and antiperiodic boundary conditions will be 1/2. Technically it will be convenient to consider the one particle tunneling process by keeping the boundary conditions periodic. One has to add two particles (compatible with the boundary conditions) but separate them in order to consider the particles independently. The new state remains a singlet. Moreover, the spin localized in the vicinity of each electron bound to a soliton is also zero. Following Laughlin we refer it as a local singlet.

How does one create the local singlet state \(|N + 1\rangle\)? One can try to act with the operator \(\psi_\uparrow(k_f, x)\). The state \(\psi_\uparrow(k_f, x)|N\rangle\), however, has spin 1/2. To create a local singlet state we must remove the spin from the electron by creating a proper 1/2 vortex of spin density acting by the
vertex operator $V_s$. To do this, let us first write the electronic operator $\chi_\sigma$ in terms of modes of free massive Dirac particles

$$\chi_\sigma(x) = \int (\Phi_0^{-}(k, x) a_{\sigma}(k) + \Phi_0^{(+)}(k, x) a_{\sigma}^{\dagger}(k)) dk$$

(3.80)

where

$$\Phi_0^{(+)}(k, r) = u_{k-k_f} e^{ikr} + v_{k-k_f} e^{i(k-2k_f)r},$$

(3.81)

$$\Phi_0^{-}(k, r) = v_{k-k_f} e^{ikr} - u_{k-k_f} e^{i(k-2k_f)r}.$$  

(3.82)

are Dirac wave functions of positive (negative) energy $E_p = \sqrt{p^2 + \Delta_0^2}$ and $u_p$ and $v_p$ are

$$u_p = \sqrt{\frac{1}{2} \left( 1 + \frac{|p|}{E_p} \right)},$$

(3.83)

$$v_p = \sqrt{\frac{1}{2} \left( 1 - \frac{|p|}{E_p} \right)}.$$  

(3.84)

In these terms we can write the operator which annihilates (creates) a singlet state with one extra particle with momentum $p$ relative to the Fermi surface is a composition of an electron and the vertex operator

$$\alpha(k_f, x) = a_{\uparrow}(k_f, x) : V_s(k_f, x) := a_{\downarrow}(k_f, x) : (V_s(k_f, x))^{-1} :$$

$$\sim \left( \frac{q}{\pi} \right)^{1/2} : V_s(k_f, x) a_{\uparrow}(k_f, x) :$$

(3.85)

The vertex operator in (3.85) removes the spin from the electron by creating a soliton and binding the soliton and electron. One can reinterpret the eq. (3.85): an electronic operator is a composition of two operators which create half of a soliton in the spin and charge sectors. The vertex operator annihilates the spin-half-soliton. Therefore the operator $\alpha$ creates one half of the charge soliton and inserts a particle. This gives the true ground state with one extra particle:

$$|N + 1, k_f + q\rangle \sim \int a_{\uparrow}^{\dagger}(k_f, r) e^{-i(k_f + q) r} dr |N\rangle$$

(3.86)

It carries a unit charge but no spin.

### 3.5 Matrix elements

The operator algebra and the constructed local singlet state allow one to compute matrix elements. We start with the one-particle matrix element.

#### 3.5.1 One particle amplitude and Orthogonality Catastrophe

Using eqs. (3.85, 3.86) we can write the one particle matrix element as a correlation function in the ground state with $N$ particles.

$$\langle N| c_{\uparrow}(x)| N + 1, k_f + q\rangle \sim \int \frac{a}{L}^{1/2} \langle N| \psi_{\uparrow}(k_f, y) V_{s}^{\dagger}(k_f, y)| N\rangle e^{i(k_f - y) - iqy} dy \sim$$

$$e^{i\frac{\pi}{2} \text{Arg}(k_f)} \left( \frac{a}{x} \right)^{1/2} \Phi_0^{(+)}(k_f, x) \langle N| V_s^{\dagger}(0)| N\rangle$$

(3.87)
Here we used the operator algebra of the Sec 3.4.1. The matrix element of the spin vertex operator on the ground state is $\langle \mathcal{N} \mid V_s(0) \mid \mathcal{N} \rangle = 1$, and we have

$$\langle \mathcal{N} \mid c_\uparrow(k) \mid \mathcal{N} + 1, k \rangle \sim e^{i\frac{\pi}{2} \text{Arg}(k_f)} \int \Phi_0^{(+)}(k_f, k - q) \nu(q) dq \quad (3.88)$$

where $\nu(k, q)$ is the number of states with a given momentum (3.59).

Similarly, the wave function of the one particle eigenstate, i.e., the one particle matrix element with a given energy and momentum is

$$\langle \mathcal{N} \mid c_\uparrow(k) \mid \mathcal{N} + 1, k, E \rangle \sim e^{i\frac{\pi}{2} \text{Arg}(k_f)} \int \Phi_0^{(+)}(k_f, k - q) \nu(q, k, E) dq \quad (3.89)$$

where $\nu(q, k, E)$ is given by the eq.(3.53).

The overlap between the wave functions of the ground states of system with $\mathcal{N}$ and $\mathcal{N} + 1$ particles suppressed by the factor $|k/k_f - 1|^{1/2}$ at $|k - k_f| \gg \Delta_0$. At $|k - k_f| \ll \Delta_0$ the matrix element, is nonzero, but small $\sim (\Delta_0/\varepsilon_f)^{1/2}$. This phenomenon is known as an orthogonality catastrophe. A physical reason for this is that a spectral flow is always even degenerated. A state with one extra particle fractionally (1/2) occupies the zero mode created by a soliton only fractionally 1/2. Any fractionally occupied state is orthogonal to fully occupied states.

Similar arguments may be applied to any odd particle amplitudes. They vanish as a power of $k - k_f$ the size of the system $1/L$ or a gap $\Delta$, whatever is smaller.

Matrix elements between states with an even number of particles are different. Low energy states with even number of particles are nondegenerate (completely filled) — an overlap between nondegenerate states is nonzero.

It is instructive to rewrite the matrix element (3.88) in different forms. It can be written as an expectation value over the ground state with $\mathcal{N} + 1$ particles of the vertex operator of the spin sector:

$$\langle \mathcal{N} \mid c_\uparrow(k) \mid \mathcal{N} + 1, k \rangle \sim (a/L)^{1/2} \int dx e^{ikx} \Phi_0^{(+)}(k_f, x) \langle \mathcal{N} + 1, k \mid V_s^\dagger(k_f, x) \mid \mathcal{N} + 1, k \rangle \quad (3.90)$$

Using the operator algebra, we have

$$\langle \mathcal{N} \mid V_s^\dagger(k_f, x) \mid \mathcal{N} + 1, k \rangle \sim e^{ikx} \left( \frac{L}{x} \right)^{1/2} \langle \mathcal{N} \mid V_s^\dagger(k_f, x) \mid \mathcal{N} \rangle \quad (3.91)$$

The latter matrix element is 1. This shows that the vertex operators have different matrix elements on states with odd and even number of particles.

Another instructive form is found by virtue of the vertex operator of the charge sector. According to (3.63, 3.64)

$$c_\sigma(x) = V_c^{-1}(k_f, x) \chi_\sigma(k_f, x) \quad (3.92)$$

so, we have

$$\langle \mathcal{N} \mid c_\sigma(k) \mid \mathcal{N} + 1, k \rangle \sim \int dx e^{ikx} \langle \mathcal{N} \mid V_c(k_f, x) V_c^\dagger(k_f, 0) \mid \mathcal{N} \rangle \langle \mathcal{N} \mid \chi_{k_f, \sigma}(x) \mid \mathcal{N} + 1 \rangle \quad (3.93)$$

where the matrix element of operator $\chi_\sigma$ is given by eq. (3.81), and the correlator of charge vertex operators is (3.44).
3.5.2 Two particle matrix element

Let us use the same technique to compute a two particle matrix element:

\[ \Delta(x - y) = \varepsilon_{\sigma\sigma'} \langle N | c_\sigma(x) c_{\sigma'}(y) | N + 2 \rangle \]  

(3.94)

As we shall see the result will be different from the one particle case. There are three related differences. First of all one can embed two particles with a zero total momentum by putting them on the opposite sides of the Fermi surface. Secondly, a two particle state can be a singlet even without the help of solitons. Finally, by adding two particles we create and fill completely the zero mode. As a result the orthogonality catastrophe would not show up for the two particle case.

The two particle matrix element is

\[ \Delta(x - y) = \sum_{k_f} \langle N | c_\uparrow(x) c_\downarrow(y) \alpha \dagger(k_f, x) \alpha \dagger(-k_f, y) | N \rangle e^{i k_f(x - y)} \]  

(3.95)

or, using (3.85)

\[ \Delta(x - y) \sim \sum_{k_f} e^{i k_f(x - y)} \langle N | \left( e^{i k_f(x - y)} \psi_{L, \uparrow}(x) \psi_{R, \downarrow}(y) + e^{-i k_f(x - y)} \psi_{R, \uparrow}(x) \psi_{L, \downarrow}(y) \right) \]  

\[ V_s(k_f, x) V_s^{-1}(-k_f, y) a \dagger(k_f, x) a \dagger(-k_f, y) | N \rangle \]  

(3.96)

Proceeding the same way as in the previous section by using the operator algebra (3.79), we find

\[ \Delta(x) \sim \frac{\sin(k_f x)}{(k_f x)^{1/2}} \Delta_0(x) \]  

(3.97)

where

\[ \Delta_0(x) = v_f \int \text{sign}(k - k_f) \Phi_0^{(+)}(k, x) \Phi_0^{(+)*}(k, x) dk = v_f \int e^{ipx} \frac{\Delta_0}{\sqrt{v_f^2 p^2 + \Delta_0^2}} dp \]  

(3.98)

is the matrix element for free massive particles. Notice that the latter has the form of the BCS wave function. In the one dimension it is the modified Bessel function

\[ \Delta_0(x) = \Delta_0 K_0(\Delta_0 x). \]  

(3.99)

This formula gives the universal part of the matrix element as a function of \( v_f(k - k_f)/\Delta_0 \) at \( |k - k_f| \ll k_f \):

\[ \frac{\Delta(p + k_f) - \Delta(k_f)}{\Delta(k_f)} \sim \text{sign}(p) \begin{cases} \text{const} + \left( \frac{\Delta_0}{v_f |p|} \right)^{1/2} \ln \left( \frac{v_f |p|}{\Delta_0} \right) & v_f |p| \gg \Delta_0 \\ v_f |p|/\Delta_0 & v_f |p| \ll \Delta_0 \end{cases} \]

The rest of the function, including its value at the Fermi surface \( \Delta_0(k_f) \sim (\Delta_0 \varepsilon_f)^{1/2} \) is not universal and is determined by processes with large momentum transfer. A qualitative graph of \( \Delta(k) \) has a broad non-universal asymmetric peak deep inside the Fermi-surface and approaches the Fermi surface according to (3.100). At \( |k - k_f| \sim \Delta_0 \), it changes behaviour and crosses the Fermi surface linearly. As we can see \( \Delta(k) \) is drastically different from \( \Delta_0(k) \).

Let us comment that there is no contradiction in the fact that the correlation function \( K(R) = \langle \psi_{R, \uparrow}(0) \psi_{L, \downarrow}(R) \psi_{R, \uparrow}(0) \psi_{L, \downarrow}(0) \rangle \) vanishes (3.71, 3.73), while the matrix element \( \Delta(R) = \langle \psi_{R, \uparrow}(0) \psi_{L, \downarrow}(R) \rangle \)
does not. The reason is that the "clusterization theorem" which connects the long range order and the matrix element is valid only for a true condensed system where all states with a nonzero momentum are separated from the ground state by a gap or for a system where the matrix element between an electron and the soft modes is small, like in BCS. Neither is the case in our system. In the Lehmann expansion of

\[ K(R) = \sum_P \sum_\phi e^{iPR} |\langle \mathcal{N} | \psi_{R\uparrow}^\dagger (0) \psi_{L\downarrow}^\dagger (0) | \mathcal{N} + 2; \phi; P \rangle|^2 \]

...the are many intermediate states with the same given arbitrary momentum \( P \). They are characterized by the profile of solitons \( \phi(x) \), i.e. by configuration of soft modes, and have a singular density, similar to (3.53). A sum of oscillating factors decays with \( R \).

3.6 Concluding remarks

As we have seen the Frohlich-Peierls models provide all necessary spectral features of a superconductor:

- the electronic liquid is compressible - the superfluid density is a smooth function of chemical potential;
- one particle spectrum is gapped;
- a pair wave function is localized and normalizable;
- there is a long range order, although the order parameter differs from \( \langle c_{\uparrow}^\dagger c_{\downarrow}^\dagger \rangle \).

However, Frohlich’s electronic state is not a superconductor, but rather an ideal conductor, because of the strength of pinning in one dimension - an arbitrary small concentration of disorder generates a resistivity.

Topological liquids in higher dimensions discussed in the next section are free from this flaw. Pinning plays no role as long the system is a liquid.

4 Topological superconductivity- Two Dimensions

In this section we discuss the simplest model of the topological fluid in two spatial dimensions. Its physics is very close to the one dimensional example of the previous section. To stress the similarity, we try to follow the line of Sec.3 as much as possible. The two dimensional model is not as well developed as the one dimensional model. In particular the identification of fast and slow modes and the passage to a continuum field theory from a microscopic lattice model are not as straightforward as in the case of the Peierls model. We therefore start directly from the continuum model. For the account of attempts to derive this model from the doped Mott insulator, see [1, 2].

4.1 Topological liquids and topological instability

Let us consider an electronic liquid where the interaction between electrons is mediated by an electrically neutral bosonic field, that can form a point-like spatial topological configuration (a
soliton). Let us suppose that in a sector with zero topological charge the electronic spectrum has a gap $\Delta_0$. Assume now that the system exhibit a spectral flow. This means that in the presence of a static soliton the electronic spectrum differs from an unperturbed one by an additional state just at the top of the valence band or within the gap—a so-called zero mode or a midgap state \[19\]. If the zero mode is separated from the spectrum, its wave function is localized around the core of the topological defect. In case when the level is attached to a band, the wave function decays as a power law away from the center of the soliton. A general argument \[20\] suggests that the midgap state always has an even degeneracy. This degeneracy eventually leads to a proper flux quantization and below it is assumed to be twofold.

Now let us add an even number of extra electrons with the concentration $\delta$. They may occupy a new state at the Fermi level of the conduction band. It costs the energy of the gap plus the Fermi energy $\Delta_0 + \mu$ per particle, where $\mu$ is a chemical potential. Alternatively, the system may create a topological configuration and a number of zero modes in order to accommodate all extra particles. The energy of this state is the soliton mass plus exponentially small corrections due to the interactions between zero modes. If the latter energy is less than $\mu + \Delta_0$ then every two extra electrons added to the system create a soliton and then completely fill a zero mode, rather than occupy the Fermi level of the state with zero topological charge. As a result the total number of solitons in the ground state is equal to half of the total number of electrons in the system.

Formally it means that, contrary to the Landau Fermi-liquid picture, the expansion of the energy in a small smooth variation of chemical potential $\delta \mu(r)$ with a fixed topological density $F(r)/2\pi$ has a linear term in $\delta \mu(r)$:

$$\delta E(\mu) = -\int \delta \mu(r) \rho(r) \, dr + \int \delta \mu(r) K(\mu, \mathbf{r} - \mathbf{r}') \frac{F(r')}{2\pi} \, dr' + O(\delta \mu^2). \quad (4.1)$$

with a non vanishing zero harmonic of the kernel $\int K(\mu, \mathbf{r}) \, dr \neq 0$. The linear term in chemical potential is known as Chern-Simons term.

The minimum amount of energy is achieved if the variation of density follows the variation of the topological charge

$$\rho(r) - \rho_s = \int K(\mu, \mathbf{r} - \mathbf{r}') \frac{F(r')}{2\pi} \, dr'. \quad (4.2)$$

On doping, electrons create and occupy zero mode states to minimize their energy, thus giving a non zero value to the topological charge (compare with \(3.4\)):

$$\delta = \left( \int K(\mu, \mathbf{r}) \, dr \right) \int \frac{F(r)}{2\pi} \, dr. \quad (4.3)$$

This is a topological instability.

This is already sufficient to conclude about superconductivity — the chemical potential always lies in a gap. The arguments are the same as in one dimensional case and can be borrowed directly from Fröhlich’s paper \[14\]. The position of the topological excitation is not fixed relative to the crystal lattice. Therefore, a pair of electrons bound to a topological excitation can easily slide through the system (and carry an electric current). It slides without attenuation, since the state is completely filled and is separated by the gap from unoccupied electronic states. As a result, the low energy physics of density fluctuations is described by the hydrodynamics of a liquid of zero modes \(2.7\). Indeed, due to a gap, the system has a finite rigidity to a local change of the density of topological charge:

$$\delta E = \frac{\kappa}{8\pi^2} \int F^2(r) \, dr$$

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This together with (4.2) yields the compressibility (2.7) and the hydrodynamics. In dimensions higher than one the hydrodynamics already implies the Meissner effect (2.8, 2.9) and superconductivity.

A standard example of the theory which exhibits both spectral flow and a topological instability and therefore superconductivity is the Dirac Hamiltonian with a fixed chemical potential

\[ H = \sum_{\sigma=1,2} \psi_\sigma^\dagger (v f \alpha (-i \nabla + A) \psi_\sigma + \Delta_0 \beta + \mu(r)) \psi_\sigma, \]  

(4.4)

Here \( \alpha = (\alpha_x, \alpha_y) \) and \( \beta \) are \( 2 \times 2 \) Dirac matrices: \( \{ \alpha_x, \alpha_y \} = 0, \beta = -i \alpha_x \alpha_y \) and the gauge field \( A \) mediates the interaction between electrons (it is not an electromagnetic field). For a relation between this model and doped Mott insulator see [2].

This Hamiltonian has \( \frac{2}{2\pi} \int F(r) dr \) zero modes (the flux is directed up). Wave functions of zero modes are:

\[ \Phi(r) = e^{-i \int A(r') \cdot dr' - \beta \int A(r') \times dr' \Phi_0(\bar{z})}, \]

\[ \beta \Phi_0(\bar{z}) = -\Phi_0(\bar{z}), \]

\[ \alpha \nabla \Phi_0 = 0, \]

where \( \Phi_0 \) is any polynomial of degree \( \frac{1}{2\pi} \int F(r) dr - 1 \) [24]. If the soliton has a unit topological charge \( \Phi_0 \) is a constant.

This single fact implies that the energy at \( \mu = 0 \) has a linear term (4.1) in chemical potential

\[ \int K(\mu = 0, r) dr = 2 \]

. There are other models of topological superconductivity in any dimensions (Sec.1), which we do not discuss here.

Let us comment on the relation between the topological mechanism of superconductivity and superconductivity in a system of anyons [16, 17]. The models become very similar after projection onto the low energy sector. Then the relation (4.2) can be imposed as a constraint rather than as a result of minimization of the energy. The projection can be done by introducing a Lagrangian multiplier \( A_0 \) for the relation (4.2) and commutation relations

\[ [A_x(r), A_y(r')] = 2\pi i \varepsilon_{ij} K^{-1}(r - r'), \]  

(4.5)

or by adding the Chern-Simons term with the kernel \( K \) to the Lagrangian. Also at large gap and a small concentration \( \delta \), one may replace the Dirac Hamiltonian with the Pauli Hamiltonian, so that

\[ H = \frac{1}{2\Delta_0} (-i \nabla + A)^2 + A_0 \]  

(4.6)

In anyon model the kernel \( K(r) \) is replaced by \( 2\delta(r) \) (even at \( \mu \neq 0 \)), so that the relation between topological charge and the density (4.4) becomes local \( \rho(r) = \frac{2F(r)}{2\pi} \). This simplification

\footnote{This mechanism is very different from a condensation of pairs bound to a polaron. In contrast, in the topological mechanism the electric charge of a pair is only partially localized at the core of the topological soliton. Although the number of zero modes is equal to the topological charge of the soliton, a part of the charge is smoothly distributed throughout the rest of the system.}

\footnote{The same model, written with a fixed number of particles looks different. It requires an additional commutation \([A_i(r), A_j(0)] = 2\pi i \varepsilon_{ij} K^{-1}(r) \) or a proper Chern-Simons term in the Lagrangian formulation.}
results in a generation of transversal electric currents or an "internal" magnetic field by light or by inhomogeneous electric charge. For the same reason, the Meissner effect in the anyon model per se exists only at zero frequency, zero momentum, zero temperature, infinitesimal magnetic field etc. An origin of these unphysical consequences is the topological constraint (4.2) is taken locally and instantaneously. In topological superconductors the kernel $K(\mu, r)$ is determined self-consistently.

4.2 Hydrodynamics

4.2.1 Compressible charge liquid

The hydrodynamics of a superfluid (2.16) can easily be obtained from the model (4.4). Let us see how the energy (4.1) of a spin singlet state changes under smooth variations of “electric” $E = \partial_t A - \nabla \delta \mu$ and “magnetic” $F = \nabla \times A$ fields. To keep track of spin variations we add an external field $A_3$ to the Hamiltonian (4.4)

\[ i \nabla - A_3 \rightarrow i \nabla - A - A_3 \sigma_3. \]

To obtain a linear hydrodynamics it is sufficient to compute radiative corrections in the Gaussian approximation. Let us now fix the chemical potential and set the total spin to be zero. The result consists of two separate parts—the spin sector and the charge sector:

\[ \delta E = \delta E_c + \delta E_s, \]

In the Coulomb gauge $\nabla A = 0$ the density of energy is

\[ \delta E_c = \frac{\Pi}{2} \left( v_f E^2 + F^2 \right) + \mu (K \frac{F}{2\pi} - \delta \rho), \]

\[ \delta E_s = \frac{\Pi}{2} \left( v_f \left( \partial_t A^3 \right)^2 + (\nabla \times A^3)^2 \right) + \frac{K}{2\pi} A_0 \nabla \times A^3, \]

where polarization operators

\[ k^2 \Pi(\omega, k) = \langle j_\perp(\omega, k) j_\perp(-\omega, -k) - j_\parallel(\omega, k) j_\parallel(-\omega, -k) \rangle = \]

\[ i \left( \frac{k_0 - \omega}{k} - \delta_{ij} \right) \int \alpha_i G_0(p) \alpha_j G_0(k - p) \frac{dp}{(2\pi)^3}, \]

\[ \omega K(\omega, k) = i \langle j_\parallel(\omega, k) j_\perp(-\omega, -k) \rangle = i \epsilon_{ij} \int \alpha_i G_0(p) \alpha_j G_0(k - p) \frac{dp}{(2\pi)^3}, \]

are current-current correlators ($j = j_\perp + j_\parallel$) of free Dirac massive fermions and $G_0(p) = (p_0 - \alpha p - \beta \Delta_0)^{-1}$ is the Green function of massive Dirac particles.

Minimization over $\delta \mu$ gives the relations (4.2) for the charge sector. Substituting this relation into the rest of (4.8) we get

\[ \delta E_c = \frac{2\pi}{K} \Pi \left( (\delta \rho)^2 + v_f^2 j^2 \right) \]

This gives the hydrodynamics of the ideal liquid (2.7) provided that the propagators $(2\pi)^2 K^{-2}(\omega, k) \Pi(\omega, k)$ are nonzero and analytical at $k, \omega \rightarrow 0$.

The latter is indeed true. The off-diagonal propagator $K(\omega, k)$ fixes the number of zero modes of Dirac operator per unit flux, so $K(0) = 2$, whereas $\Pi(\omega, k)$ describes vacuum polarization of gapped particles. Some caution and proper regularization is necessary, since the integrals diverge at large $p$, but otherwise the calculations are standard

\[ \Pi \sim \frac{v^2}{\Delta_0} (1 + O(k^2/\Delta_0^2)) \]

\[ K = 2 + O(k^2/\Delta_0^2). \]
4.2.2 Incompressible chiral spin liquid

In the spin sector (4.9) the story is different. The total spin is kept to be zero. Therefore, the Chern-Simons term (the last term in (4.9)) remains in the spin sector and results in the hydrodynamics of an incompressible spin liquid. Writing $\partial_t u_s = -\delta L/\delta A^3$, we obtain the hydrodynamics of a topological spin liquid:

$$L_s = 2\pi u_s \times \partial_t u_s.$$  \hspace{1cm} (4.15)

By combining (4.8, 4.9) we obtain the hydrodynamics of the topological superconductor:

The hydrodynamics consists of two independent fluids $L = L_c + L_s$: a compressible charged liquid

$$\langle u^\| (\omega, k), u^\| (-\omega, -k) \rangle = \kappa^{-1} \frac{v_f^2}{\omega^2 - v_f^2 k^2},$$

$$\langle u^\perp (\omega, k), u^\perp (-\omega, -k) \rangle = \kappa^{-1} \frac{v_f^2}{\omega^2},$$ \hspace{1cm} (4.16)

and an incompressible topological (chiral) spin liquid:

$$\langle u^\parallel_s (\omega, k), u^\perp_s (-\omega, -k) \rangle = \frac{i}{4\pi\omega},$$

where $u^\parallel, u^\perp$ are the longitudinal and transversal parts of the displacement $u_i(k) = \frac{k_i}{\kappa} u^\| + \epsilon^{ij} k_j u^\perp$. Eqs. (4.16) imply compressibility of the charge fluid and the Meissner effect. The hydrodynamics (4.13) of the spin sector is equivalent to the hydrodynamics of the FQHE fluid (see e.g., [21, 22]).

One of the direct consequences of the chiral nature of the spin liquid is that the spin liquid generates spin edge current. Similar to the FQHE the spin excitations are suppressed in the bulk but develop a spontaneous spin edge current with the level $k = 2$ current algebra. Let us stress that the spin edge current is the only hydrodynamical manifestation of spontaneous parity breaking. Contrary to a number of claims scattered through the literature, the spontaneous parity breaking is invisible in the charge sector even for the systems with an odd number of layers, but is found in the spin sector in a system with a boundary. A similar phenomenon takes place in the one dimensional Peierls model with open ends. In this case there are free spin excitations on the ends, although the bulk spin excitations are gapped.

4.2.3 London penetration depth

Equation (4.14) gives the scale of compressibility and the London penetration length. From (4.12, 4.14) we have $\kappa = \hbar^2 v_f^2 / \Delta_0$. This result is very different from the conventional BCS theory. In the BCS the compressibility and the penetration length are determined by the plasma frequency (a classical object, independent of $\hbar$ and the gap in the spectrum). It is given by the atomic parameters and the density of conducting electrons $\lambda_0^p = c/\omega_p$. In contrast the compressibility of the topological liquid described by the model (4.4) is determined by the gap and has a quantum origin. The reason for this difference is the same as for the orthogonality catastrophe — conducting electrons are not quasiparticles. Only a small fraction of the conducting electrons are
involved in the supercurrent $\rho_s/\rho_0 \sim (\Delta_0/\epsilon_f)^{d-1}$, where $\rho_0 \sim (k_f/\hbar)^d$ is the Fermi density (the volume of the Fermi sphere) and $d$ is a spatial dimension.

Bearing in mind that the gap also determines the correlation length $\xi \sim \hbar v_f/\Delta_0$, we may express the penetration depth through the correlation length, atomic parameters and an electronic wave length $\ell = 2\pi \hbar/k_f$. We write them for spatial dimensions $d = 2, 3$.

$$\frac{\lambda_L}{\lambda^0_L} = \left(\frac{\xi}{\ell}\right)^{\frac{d-1}{2}}, \quad d = 2, 3. \quad (4.17)$$

and the Ginzburg-Landau parameter is

$$\frac{\lambda_L}{\xi} = \frac{\lambda^0_L}{\xi}\left(\frac{\xi}{\ell}\right)^{\frac{d-1}{2}}, \quad (4.18)$$

where $\lambda^0_L = c/\omega_p$. The Ginzburg-Landau parameter becomes universal (independent of a gap) at $d = 3$:

$$\frac{\lambda_L}{\xi} = \frac{\lambda^0_L}{\xi} = (c/v_f)^{1/2}(4\pi e^2/\hbar c)^{-1/2} \approx 0.1 (c/v_f)^{1/2} \sim 10^2. \quad (4.19)$$

Topological superconductors are of the London type.

### 4.3 Electron as a composite object

#### 4.3.1 Electronic operator

The most difficult part of the theory is to determine a relation between the true electron operator $c_\sigma$ and the “spinned” fermion $\psi_\sigma$ of (4.4).

There are several reasons why $\psi$ differs from a physical electronic operator. First of all electronic states are gauge invariant, whereas $\psi$ is not. Moreover, without a gauge field (i.e. without interaction) $\psi_\sigma$ being a Dirac spinor has $1/2$ - orbital momentum and is double valued. Electronic states are single valued and must have an integer orbital momentum. Another problem is that an electronic excitation carries a typical momentum of the order of $k_f$, while typical momenta of Dirac particles are close to zero. These difficulties do not occur while studying the hydrodynamics, but arise in matrix elements.

Below we conjecture the form of the electronic operator based on plausible physical arguments. In fact this relation can be derived (under certain assumptions) from a microscopical model of the doped Mott insulator (see e.g., [2]). We do not discuss it here.

The requirements for the electronic states are

(i) the electron is gauge invariant (in respect to mediating gauge field), i.e. remains unchanged under a non-singular gradient transformation $A \rightarrow A + \nabla \Lambda$;

(ii) In a sector with completely filled zero modes, i.e., when the flux and the number of particles obey the topological constraint (4.2), a charged singlet excitation is a spatial scalar, i.e. its wave function has a zero orbital moment $l = 0$;

(iii) Since an electronic liquid is compressible, the most essential electronic modes have momenta $k \sim (2\pi/\delta)^{1/2} \equiv k_f$.  

---

6This result is specific to the model [4,5]. Other models of topological superconductors, describing commensurate doped insulator, like the model of [6], give a superfluid density varying in the range from $\delta = \rho_0$ (a doping) to $\rho_0 \sim (\Delta_0/\epsilon_f)^{d-1}$.  

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We find the electronic operator in three steps. The first step is the use of rotational invariance. Let us consider a wave function of the free Dirac field in two spatial dimensions. In the basis, where α-matrices are \( \alpha_x = \sigma_3, \alpha_y = -\sigma_2, \beta = \sigma_1 \), the solution to the Dirac equation with momentum \( \mathbf{p} \) is

\[
e^{i \mathbf{p} \mathbf{r}} e^{-\frac{i}{2} \beta \text{Arg}(\mathbf{p})} \begin{pmatrix} u_p \\ v_p \end{pmatrix}, \quad E = +E_p = \sqrt{p^2 + \Delta_0^2}
\]

(4.20)

for the positive energy, and

\[
e^{i \mathbf{p} \mathbf{r}} e^{-\frac{i}{2} \beta \text{Arg}(\mathbf{p})} \begin{pmatrix} v_p \\ -u_p \end{pmatrix}, \quad E = -E_p = -\sqrt{p^2 + \Delta_0^2}
\]

(4.21)

for the negative energy, where again \( \text{Arg}(\mathbf{p}) \) is an angle of the momentum \( \mathbf{p} \), relative to the \( x \)-axis and

\[
u_p = \sqrt{\frac{1}{2}(1 + \frac{p^2}{E_p})}
\]

(4.22)

\[
u_p = \sqrt{\frac{1}{2}(1 - \frac{p^2}{E_p})}
\]

(4.23)

We choose the basis such that \( u_p \) and \( v_p \) to be the BCS wave functions and agree with one dimensional case (3.83).

The spinor carries an angular momentum \( l = 1/2 \). We unwind the Dirac field by a chiral rotation

\[
\psi_\sigma(p) \rightarrow e^{\frac{i}{2} \beta \text{Arg}(\mathbf{p})} \psi_\sigma(p).
\]

(4.24)

This singular transformation has a clear physical sense—it projects the spinor wave function onto a direction of the momentum \( \mathbf{p} \). Indeed, the chiral transformation \( e^{\frac{i}{2} \beta \text{Arg}(\mathbf{p})} \) in two spatial dimensions, where \( \beta = -i/2 [\alpha_x, \alpha_y] \) is equivalent to a spatial rotation of the momentum \( \mathbf{p} \) by the angle \( \text{Arg}(\mathbf{p}) \) which aligns the momentum \( \mathbf{p} \) along the \( x \)-axis of the coordinate system. Now, without topological gauge fluctuations, the transformed operator is a spatial scalar.

The next step is to boost the fermion to the Fermi surface. In the chosen basis the upper and the lower components of the Dirac field \( \psi_\sigma = (\psi_\sigma^{(1)}, \psi_\sigma^{(2)}) \) correspond to the states propagating forward and backward along the vector \( \mathbf{p} \). To construct an electronic operator we shift the momentum of the upper component by the Fermi vector \( \mathbf{k}_f = k_f \mathbf{e}_p \) directed along the momentum \( \mathbf{p} \): \( \mathbf{p} \rightarrow \mathbf{k} = \mathbf{k}_f + \mathbf{p} \) and the momentum of the lower component by \( -\mathbf{k}_f \): \( \mathbf{p} \rightarrow \mathbf{k} - 2\mathbf{k}_f = -\mathbf{k}_f + \mathbf{p} \)

\[
\begin{pmatrix}
\tilde{c}_\sigma(\mathbf{k}_f + \mathbf{p}) \\
\tilde{c}_\sigma(-\mathbf{k}_f + \mathbf{p})
\end{pmatrix}
\sim e^{\frac{i}{2} \beta \text{Arg}(\mathbf{k}_f)} \psi_\sigma(p) = \begin{pmatrix}
\cos \frac{\text{Arg}(\mathbf{k})}{2} \psi_\sigma^{(1)} + i \sin \frac{\text{Arg}(\mathbf{k})}{2} \psi_\sigma^{(2)} \\
i \sin \frac{\text{Arg}(\mathbf{k})}{2} \psi_\sigma^{(1)} + \cos \frac{\text{Arg}(\mathbf{k})}{2} \psi_\sigma^{(2)}
\end{pmatrix}.
\]

(4.25)

Here we used \( \tilde{c}_\sigma \) to indicate that the gauge field has not been taken into account yet.

Formally, the chiral rotation \( e^{\frac{i}{2} \beta \text{Arg}(\mathbf{k}_f)} \) may be also understood in the following way. The wave function of the Dirac field depends explicitly on the choice of α-matrices, i.e. on the choice of holomorphic coordinates in a plane. The chiral transformation aligns holomorphic coordinates relative to each point of the “Fermi surface”, i.e. sets up the momentum dependent α-matrices

\[
\alpha_{\mathbf{k}_f} = e^{\frac{i}{2} \beta \text{Arg}(\mathbf{k}_f)} \alpha e^{-\frac{i}{2} \beta \text{Arg}(\mathbf{k}_f)}.
\]

In terms of \( \tilde{c} \) the free Dirac Hamiltonian describes isotropic backward scattering

\[
H = \int \left\{ \xi_k \tilde{c}_\sigma^\dagger(k) \tilde{c}_\sigma(k) + \Delta_0 \left( \tilde{c}_\sigma^\dagger(k) \tilde{c}_\sigma(k - 2\mathbf{k}_f) + \text{h.c.} \right) \right\} \, dk,
\]

(4.26)
where $\xi_k = v_f(k-k_f)$. This clarifies the origin of the model. It arises as the result of $2k_f$ instability of some correlated electronic system.

Let us now consider solution of the Dirac equation in the presence of the 1/2-vortex located at $r_0$:

$$A_i = \frac{1}{2} \epsilon_{ij} \frac{r_j - r_{0j}}{(r-r_0)^2}$$

(4.27)

The wave function in $\psi$ representation is

$$\left( \begin{array}{c} 1 \\ -1 \end{array} \right) \frac{1}{(z-z_0)^{1/2}},$$

where $z = x + iy$. This soliton describes a single particle state at the chemical potential. It is double valued and can not be treated as an electronic wave function. To construct the electronic wave function we must first write it in the momentum representation:

$$\left( \begin{array}{c} 1 \\ -1 \end{array} \right) \int e^{ip \cdot r} e^{-\frac{i}{2} \text{Arg}(r-r_0)} \frac{1}{|r - r_0|^{1/2}} d r_0 = \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{ip \cdot r} e^{-\frac{i}{2} \text{Arg}(p)} \int \frac{e^{ipz} e^{-\frac{i}{2} \text{Arg}(p)}}{z_p^{1/2}} d z,$$

(4.28)

where

$$z_p = p \cdot r + ip \times r$$

(4.29)

is the holomorphic coordinate relative to the vector $p$, and then make a chiral rotation \((4.24, 4.25)\).

This gives

$$e^{ikr} e^{-i \text{Arg}(k)} \int \frac{e^{i(k-k_f)z}}{z^{1/2}} d z.$$

(4.30)

Now the wave function is single valued, translational invariant but acquires a spatial orbital moment $l = 1$. We conjecture that it is the wave function of electronic zero mode.

### 4.3.2 Bosonization in two dimensions

The results of the previous section can be written in terms of bosonization. Let us introduce a chiral bosonic field $\Phi(k, r)$ similar to \((3.21)\):

$$\Phi_\sigma(k, r), \Phi_\sigma(k, 0) - : \Phi_\sigma(k, r), \Phi_\sigma(k, 0) := \ln \left( \frac{L_z k}{z_k} \right),$$

(4.31)

where $z_k$ is given by \((4.29)\).

This field is consistent with Chern-Simons commutation relations and the topological constraint \((4.2)\)

\[\partial_z \Phi_\sigma(k, r) = -2\pi i u_\sigma\]

(4.32)

\[[u_\sigma(r) \times u_\sigma(0)] = i \delta(r),\]

(4.33)

where

\[u_\sigma = k \cdot u_\sigma - ik \times u_\sigma\]

(4.34)

\(^7\)Let us notice that commutation relations between vector potentials and displacements depend on whether we consider the theory at fixed number of particles or fixed chemical potential. In the latter case displacements commute. If the number of particles is fixed they do not, and the commutator is given by eq. \((4.33)\).
As in one dimensional case the operator
\[ V_\sigma(k, r) = :e^{\Phi_\sigma(k, r)} : \] (4.35)
correctly reproduces a Dirac fermion in the sector of zero modes \(4.28\).

Assembling all pieces we obtain the bosonized version of physical electron in the sector of zero modes
\[ c_\sigma(r) \sim \int e^{-ikr} e^{-i\text{Arg}(k)} V_\sigma(k, r) \Psi(k) dk \] (4.36)
where \(\Psi(k)\) is the operator which creates a zero mode with momentum \(k\).

Eqs. \((4.32 - 4.36)\) are the bosonization formulas in two spatial dimension. They express the electronic operator through its displacement. These formulas are to be compared with the bosonization formula \((3.25)\) in one dimension.

The vertex operator \(V_\sigma(k, r)\) has a simple meaning. As is in one dimensions \[23\] it creates a flux quantum and a zero mode in the state with the spin \(1/2\). It is gauge invariant and on the subspace of zero modes it obeys two relations
\[ V_\sigma^{-1} \alpha(-i\nabla + A) V_\sigma = -i\alpha \nabla, \] (4.37)
\[ [F(r'), V_\sigma(r)] = 2\pi V_\sigma(r) \delta(r - r'). \] (4.38)

### 4.3.3 Vertex operators

As in one dimensional case we will need an operator algebra for the vertex operator \(V_\sigma\) and two additional vertex operators of the charge and spin sectors. Let
\[ \Phi_c = \frac{1}{2}(\Phi^+ + \Phi^-), \] \[ \Phi_s = \frac{1}{2}(\Phi^+ - \Phi^-) \] (4.39)
and
\[ V_c = :e^{\Phi_c} :, \] \[ V_s = :e^{\Phi_s} :. \] (4.41)

The vertex operator of the spin sector \(V_s(z)\) creates a soliton of the spin displacement \(u_s\) and removes spin down from the site \(z\):
\[ [V_s(z), \nabla u_s(z')] = V_s(z) \delta(z - z'). \] (4.43)

In contrast, the vertex operator of the charge channel \(V_c(z)\) creates a flux of the gauge field (the spin chirality), but commutes with displacements
\[ [F(z'), V_c(z)] = 2\pi V_c(z) \delta(z' - z), \] \[ [u(z'), V_c(z)] = 0. \] (4.44)

Due to the commutation relations \(4.33\) vertex operators obey the operator algebra, similar to the operator algebra of the one-dimensional case \((4.41)\). The difference is that the operator
algebra is localized in each point of the Fermi surface and the holomorphic coordinate is chosen to be relative to the Fermi momentum:

$$V_s(k_f, z)c_\sigma(z') \sim \left(\frac{z_{k_f} - z'_{k_f}}{L}\right)^{\sigma/2}c_\sigma(z')V_s(k_f, z), \quad (4.45)$$

$$V_c(k_f, z)c_\sigma(z') \sim \left(\frac{z_{k_f} - z'_{k_f}}{L}\right)^{1/2}c_\sigma(z')V_c(k_f, z). \quad (4.46)$$

Below we also use a two-particle vertex operator $\mu(u, w)$ which creates a soliton and antisoliton of the spin displacement

$$\mu(u, w) = V_s(u)V_s^{-1}(w). \quad (4.47)$$

The operator algebra gives

$$\mu(u', w')c_\uparrow^\dagger(u)c_\downarrow(u) = \frac{(u' - u)^{1/2}(w' - w)^{1/2}}{(u - w)} \times c_\uparrow^\dagger(u)c_\downarrow(w)\mu(u', w') \quad (4.48)$$

and if the points $u, u'$ and $w, w'$ coincide

$$c_\uparrow(u)c_\downarrow(w)\mu(u, w) \sim \frac{a}{(u - w)}\mu(u, w)c_\uparrow(u)c_\downarrow(w). \quad (4.49)$$

Two composite objects $V_s(z)c_\uparrow(z)$ and $V_s(z)c_\downarrow(z)$ are singlets but carry electric charge.

4.4 Electronic spectral function

The bosonization and operator algebra provides a recipe for computing the most interesting Green functions. The calculations are similar to the one dimensional case (Sec. 3.3.2).

We start from the spectral function. According to (4.36) the electronic Green function at equal time is

$$n(r) = \langle c_\sigma^\dagger(r)c_\sigma(0) \rangle \sim \int e^{ikr}\langle k|V_c^\dagger(k, r)V_c(k, 0)|k\rangle\langle k|V_s^\dagger(k, r)V_s(k, 0)|k\rangle dk, \quad (4.50)$$

where the state $|k\rangle = \Psi(k)|0\rangle$ is a zero mode state with momentum $k$. The first factor describes the amplitude of soft modes of density modulation in the state $|k\rangle$ whereas the second $n_0(k)$ is the number of gapped excitations with spin 1/2. The spin excitations are gapped and their matrix elements are regular. At large $r \gg v_f/\Delta_0$ one can approximate the spin factor in (4.50) by the occupation number of massive particles

$$n_0(k) = \frac{1}{2}(1 - \frac{v_f(k - k_f)}{\sqrt{v_f^2(k - k_f)^2 + \Delta_0^2}}), \quad (4.51)$$

so that the tunneling occupation number becomes

$$n(r) = \langle c_\sigma^\dagger(r)c_\sigma(0) \rangle \sim \int e^{ikr}\langle k|V_c^\dagger(k, r)V_c(k, 0)|k\rangle n_0(k), \quad (4.52)$$

The operator algebra gives the conformal block for the matrix element of the soft modes:

$$D_k(r) = \langle k|V_c^\dagger(k, r)V_c(k, 0)|k\rangle \sim \frac{1}{k_f^2}e^{i\theta(k, r)} = \frac{e^{i\theta(k, r)}}{k_f^2}, \quad (4.53)$$
where $\theta(k, r)$ is the angle between vectors $k$ and $r$. As the result we have a "tomographic" representation for the occupation number

$$n(r) \sim \frac{1}{r} \int e^{ikr \cos \theta + i\theta} n_0(k) d\theta dk.$$  \hfill (4.54)

In momentum representation the tunneling occupation number is

$$n(k) \sim \int D_k(q) n_0(k - q) dq,$$  \hfill (4.55)

where the propagator

$$D_k(q) = \frac{k}{q \cdot k + iq \times k} = \frac{1}{q} e^{i(\text{Arg}k - \text{Arg}q)}$$  \hfill (4.56)

is a holomorphic function relative to $k$.

As is in to the one dimensional case, the Green function may be obtained by replacing $n_0(k)$ and $D_k(q)$ in the eq. (4.55) by

$$G_0(\omega, k) = \frac{\omega - \xi_k}{\omega^2 - \xi_k^2 - \Delta_0^2},$$  \hfill (4.57)

$$D_k(\omega, q) = \frac{1}{\omega^2 - v_f^2 q^2} e^{i(\text{Arg}k - \text{Arg}q)},$$  \hfill (4.58)

so that

$$G(\Omega, k) \sim \int G_0(\Omega - \omega, k - q) D_k(\omega, q) dq d\omega.$$  \hfill (4.59)

and the spectral function

$$A(k, E) \sim \int e^{i(\text{Arg}k - \text{Arg}q)} \frac{n_0(k - q)}{(\sqrt{v_f^2(k_f + k - q)^2 + \Delta_0^2 - E)^2 - v_f^2 q^2}) dq}$$  \hfill (4.60)

where the integral goes over the domain $E > v_f q + \sqrt{v_f^2(k_f + k - q)^2 + \Delta_0^2}$ and the vector $k_f$ of the length $k_f$ is directed along $k$.

The behaviour of the tunneling occupation number and the density of states close to the Fermi surface is linear (modulo logarithmic corrections)

$$n(k) - n(k_f) \sim -\text{const} |k/k_f - 1| \text{sgn}(k - k_f) \mathcal{O}(\ln \frac{k - k_f}{k_f}); \quad \text{at } v_f(k - k_f) \gg \Delta_0.$$  \hfill (4.61)

As in the one dimensional case (but contrary to the free massive particle case), the tunneling occupation number does not vanish at the Fermi surface.

The tunneling density of states has a similar behaviour. It shows an asymmetric broad ($\sim \epsilon_f$) peak and decays from the peak toward the threshold $E = \Delta_0$ as

$$\frac{dN(E)}{dE} = \int A(k, E) \frac{dk}{2\pi} \sim \left(\frac{E}{\epsilon_f}\right) \mathcal{O}(\ln \frac{E}{\epsilon_f}), \quad \text{at } \epsilon_f \gg E - \Delta_0 \gg \Delta_0.$$  \hfill (4.62)

In contrast to free massive particles but similar to the one dimensional case, the tunneling density of states has lost its singularity $(E^2 - \Delta_0^2)^{1/2}$. It remains smooth at the threshold and
approaches it linearly \((\sim E - \Delta_0)\). As in one dimension the spectral function is determined by two scales, \(\Delta_0\) and \(\epsilon_f\), rather than just \(\Delta_0\). The second scale is the signature of the orthogonality catastrophe and of the composite nature of the electron.

These formulas are to be compared with the one dimensional spectral function of the Sec. 3.3.2. They are similar except that in one dimension the integral over the Fermi surface must be replaced by the sum over two Fermi points. While integrating over the Fermi surface special attention should be paid to the phase factor. This factor is not important in one dimensional case because the Fermi surface is not connected. In higher dimensions this factor establishes phase coherence between different points of the Fermi surface.

4.5 Matrix elements

The technique to compute the matrix elements also has been illustrated on the one dimensional case example (Sec. 3.5.2). We adopt similar arguments for the two dimensional case to compute the two particle wave function:

\[
\Delta(x - y) = \varepsilon_{\sigma\sigma'} \langle \mathcal{N} | c_{\sigma}(x) c_{\sigma'}(y) | \mathcal{N} + 2 \rangle. \tag{4.63}
\]

4.5.1 Pair wave function

First we must understand the nature of the ground state with two extra particles \(|\mathcal{N} + 2\rangle\). This state is created by a composite operator which creates a flux and places particles into zero modes created by the flux. Creating a flux also generates spin waves. They are gapped. Let us first assume that the gap \(\Delta_0\) is very large and so that we can neglect spin excitations. Then the vertex operator \(V_c\) of the charge sector creates a flux and a zero mode which gets occupied by a particle.

Let us start from the two-particle state with zero momentum:

\[
|\mathcal{N} + 2\rangle \sim \int dr \, dr' \, dk \, e^{2i \text{Arg}(k) (r - r')} V_c^{-1}(k, r) V_{c'}^{-1}(-k, r') \Psi_{\uparrow}(k) \Psi_{\downarrow}(-k) | \mathcal{N} \rangle. \tag{4.64}
\]

A physical meaning of the composite operator which creates the ground state with an extra particle is more transparent in terms of gauge invariant electronic operator \((1.37)\). Using the tomographic representation we construct the two-particle state \((4.64)\) which is composed of the two electrons and the operator \((1.47)\), creating a vortex and an antivortex of spin density at the points of electron insertions:

\[
|\mathcal{N} + 2\rangle \sim \varepsilon_{\sigma\sigma'} \int dr \, dr' \, dk \,
\times V_s(k, r) V_s^{-1}(-k, r') c_{\sigma}(k, r) c_{\sigma'}(-k, r) | \mathcal{N} \rangle. \tag{4.65}
\]

Electrons with opposite spins see each other with vortices of the opposite angular momenta \(l = \pm 1\) (as it has been first noticed in Ref. [25]). This gives an angular momentum \(l = 2\) for the pair.

The singlet two-particle matrix element has the form

\[
\Delta(r) \sim \int dk \, e^{-2i \text{Arg}(k)} e^{i kr} \times \langle \mathcal{N} + 2 | V_s(r) V_s^{-1}(0) | \mathcal{N} + 2 \rangle.
\]

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Thus the two-particle matrix element is given by the correlation function of the vertex operators of the spin channel computed in the ground state with two extra particles. The operator algebra

\[ V_s(z)V_s^{-1}(z') \sim \frac{\theta}{z-z'} : V_s(z)V_s^{-1}(z') : \text{ implies } \langle \mathcal{N} + 2| V_s(k, r) V_s^{-1}(k, 0) \mathcal{N} + 2 \rangle \sim \frac{1}{k_f r} e^{i\theta|kr|}, \]

where \( \theta(k, r) \) is the angle between vectors \( k \) and \( r \). As a result

\[ \Delta(r) \sim \frac{1}{k_f r} e^{-2i\text{Arg}(r)} \int e^{-i\theta|kr|} e^{ikr} dk \] (4.66)

The matrix element (4.66) is obtained in the limit of a very large gap \( \Delta_0 \to \infty \). This approximation is sufficient in order to analyze the transformation properties and the angular dependence of the tunneling amplitude. If the gap is not very large, an embedding of two extra particles creates a gapped spinon-antispinon excitations. These excitations do not interact with the zero mode of the charged sector and their wave functions (at \( kr \gg 1 \)) are just the wave functions of an unperturbed theory (4.26)

\[ \Phi_0^+(k, r) = u_{k-k_f}e^{ikr} + v_{k-k_f}e^{i(k-2kr)r}, \] (4.67)

\[ \Phi_0^-(k, r) = v_{k-k_f}e^{ikr} - u_{k-k_f}e^{i(k-2kr)r}. \] (4.68)

Here the first (second) function corresponds to a positive (negative) energy. Notice, that they are the same as in one dimensional case (3.81, 3.82).

The wave function of a spinon - antispinon pair with opposite momenta is

\[ \Delta_{BCS}(k, r_1 - r_2) \text{ = sign}(k - k_f) \Phi_0^+(k, r_1)\Phi_0^{-*}(k, r_2) \] (4.69)

\[ = e^{ik(r_1-r_2)} u_{k-k_f}v_{k-k_f} \sim \frac{\Delta_0}{\sqrt{\xi^2 + \Delta_0^2}} \] (4.70)

where the only translational invariant term retained.

Let us notice that in contrast to Cooper’s pairing mechanism, the gap in the topological mechanism of superconductivity is generated via backward scattering and is of insulator nature. Nevertheless, the wave function of a singlet spin excitation with zero relative momentum is the same as the BCS wave function, and is mainly determined by the Lorentz invariance.

To obtain matrix elements for a finite gap, we have to replace the factor \( e^{ikr} \) in eqs.(4.66) with the spinon-antispinon wave function (4.69)

\[ \Delta(r) \sim e^{-2i\text{Arg}(r)} \frac{1}{k_f r} \int e^{-i\theta|kr|} \Delta_{BCS}(k, r) dk. \] (4.70)

It is rather straightforward to take into account the frequency dependence of the two-particle matrix element. In order to do it one must replace \( D_k(q) \) in the eq. (4.70) by (4.58) and \( \Delta_{BCS} \) by

\[ \Delta_{BCS}(\omega, k) = \frac{\Delta_0}{\omega^2 - (\epsilon(k) - \mu)^2 - \Delta_0^2}, \] (4.71)

so that

\[ \Delta(\Omega, k) \sim e^{-2i\text{Arg}(k)} \int d\Omega d\omega \]

\[ \times \Delta_{BCS}(\Omega - \omega, k - q) D_k(\omega, q). \] (4.72)

These results lead to a number of important consequences.
4.5.2 Orthogonality catastrophe and angle dependence of the tunneling amplitude.

Complex \(d\)-wave state

(i) The angular dependence of the pair wave function and of the tunneling amplitude. The pair wave function consists of two vortices of the same charge—one comes from the wave function of the zero mode of the flux created for a spin up particle by a spin down particle (the factor \(e^{-i \text{Arg}(r)}\) in (4.70)). Another vortex \(e^{-i \text{Arg}(k)}\) is located at the center of the Fermi surface. The pair wave function forms the \(d\)-wave \((l = 2)\) irreducible complex representation of the rotational group. Similarly the one particle matrix element carries \(l = 1\) orbital moment. The angular dependence of tunneling in anyon superconductors has been also studied in Refs. [8].

(ii) Orthogonality catastrophe. The overlap between two ground state wave functions with \(N\) and \(N + 2\) electrons does not vanish in a large system.

This is not the case for a one- (or any odd number) particle matrix element. An attempt to embed a single electron in the system leads to a half-occupied zero mode state. As a result this state turns out to be almost orthogonal to all other states with the same spin and number of particles. The matrix element \(|N| c_\sigma(k + q)|N + 1, k, q\rangle\) vanishes at \(q \to 0\).

(iii) Tomographic representation. Eq. (4.70) consists of the integral over the Fermi surface and may be viewed as a “tomographic” representation of the matrix element

\[
\Delta(r) \sim e^{-i2\text{Arg}(r)} \int dk D_k(r)\Delta_{\text{BCS}}(k, r)
\]

where the propagator \(D_k(r)\) is given by (4.53). The tomographic representation in electronic liquids has been anticipated in [26, 27]. The new feature is that the relative phase of electron pairs at different points of the Fermi surface are correlated by the factor \(e^{i\text{Arg}k}\) in the propagator of soft modes (4.53).

(iv) Bremsstrahlung. It is instructive to rewrite eq. (4.73) in the momentum representation

\[
\Delta(k) \sim e^{-i2\text{Arg}(k)} \int \Delta_{\text{BCS}}(k - q)D_k(q)dq
\]

where the propagator \(D_k(q)\) is given by (4.56) and

\[
\Delta_{\text{BCS}}(k) = \frac{\Delta_0}{\sqrt{(\epsilon(k) - \mu)^2 + \Delta_0^2}}
\]

Similar formula stays for the tunneling occupation number (4.55).

These representations clarify the physics of the topological superconductor. An insertion of two particles in the spin singlet state with relative momentum \(k\) close to \(k_f\) emits soft modes of transversal spin current with the propagator \(D_{k-k_f}(q)\). As a result of this: (i) ground states differing by an odd number of particles are orthogonal (for a related arguments for a one dimensional spin chain, see [28]); (ii) the BCS wave function is dressed by soft transversal modes. This is in line with 1D physics and the bremsstrahlung effect of QED. The new feature is the phase of the matrix element of the soft mode (4.56) which is the angle relative to the Fermi momentum of the electron. By contrast, in the BCS the density modulations are not individual excitations but are composed from Cooper pairs. The interaction between density modulations and pairs—the major effect of the topological mechanism—vanishes in the BCS.
(iv) **Momentum dependence of the two-particle wave function and tunneling amplitude.** The momentum dependence of the amplitude of the pair wave function $|\Delta(k)|$ is drastically different from the BCS \(|1.69\). In the vicinity of Fermi surface $|k - k_f| \ll k_f$ the integrals \(|4.74\) can be computed:

$$
|\Delta(k)| - |\Delta(k_f)| \approx \begin{cases} 
  v_f(k - k_f)/\Delta_0 & v_f|k - k_f| \ll \Delta_0 \\
  \text{sgn}(k - k_f)\log(v_f|k - k_f|/\Delta_0) & v_f|k - k_f| \gg \Delta_0 
\end{cases}
$$

The result gives the universal dependence of the pair wave function on $v_f(k - k_f)/\Delta_0$. The constant $|\Delta(k_f)|$ is not universal and depends on states far away from the Fermi surface.

In contrast to the BCS gap function, the pair wave function is asymmetric around the Fermi surface. It is peaked at scale

$$
v_f(k - k_f) \approx \epsilon_f \left(\log \frac{\epsilon_f}{\Delta_0}\right)^{-1},
$$

which is much greater than $\Delta_0$. Also, the pair wave function has a logarithmic branch cut at $v_f(k - k_f) = \pm i\Delta_0$ in contrast to square root singularity of the BCS function. This indicates that transversal spin current soft modes are emitted in the process of tunneling.

### 4.6 Pair Correlation Function and Long Range Order

A non vanishing two particle wave function does not mean that the pair correlation function

$$
K(r_1, r_2, r_3, r_4) = \langle c_\uparrow^\dagger(r_1)c_\downarrow^\dagger(r_2)c_\uparrow(r_3)c_\downarrow(r_4)\rangle
$$

shows long range order. We already observed this result in one dimensional case (Sec.\(|3.3.4\)).

Indeed, if we assume that size of a pair is less than the distance between pairs $|r_1 - r_2| \sim |r_3 - r_4| \ll |R| = |r_1 + r_2 - r_3 - r_4|$ and consider the dependence of the correlation function on $R$, we have

$$
K = \int \langle c_\uparrow^\dagger(k, r_1)c_\downarrow^\dagger(-k, r_2)c_\uparrow(p, r_3)c_\downarrow((-p, r_4))\rangle dk dp \sim \\
\int dke^{ik(r_1 - r_2)}e^{-ik(r_3 - r_4)}\langle e^{-\Phi_s(k, r_1) - \Phi_s(-k, r_2) + \Phi_s(k, r_3) + \Phi_s(-k, r_4)}\times \\
\langle e^{-\Phi_s(k, r_1) + \Phi_s(-k, r_2) + \Phi_s(k, r_3) - \Phi_s(-k, r_4)}\rangle
$$

$$
(4.77)
$$

As in the one dimensional case, the spin sector factor approaches a constant at $R \to \infty$, whereas the charge sector factor decays. As a result the pair correlation function decays as well:

$$
K(r_1, r_2, r_3, r_4) \sim \frac{1}{R^2} \int e^{ik(r_1 - r_2)}e^{-ik(r_3 - r_4)}e^{2i\theta(k, R)} dk
$$

$$
(4.78)
$$

In contrast to the BCS, the decay of the pair correlation function does not mean that a superconducting state is destroyed by fluctuations. Nor does it mean that the tunneling amplitude \(|4.63\) vanishes (see the comment in the end of Sec.\(|3.3.4\)). The decay rather indicates a strong interference effect between pairs. Similar to the Berezinsky-Kosterlitz-Thouless transition, the rate of the decay reflects the value of the superfluid density.

It is easy to construct an object which carries the electric charge $2e$ and shows a long range order. As in one dimensional case (see \(|3.74\)) it can be written as $\Psi(r_1, r_2) =$

42
This long range order indicates a gap opening at $2k_f$ rather than a superfluid order.

A striking difference to the BCS is a relation between the phase of the tunneling amplitude and the phase which parameterizes the current \( e^{-i k (r_1 - r_2)} \). In the BCS these phases are the same. In topological superconductor there is no direct relation between the phase of two particle matrix element \( \langle \Phi_s(k, r_1) + \Phi_s(-k, r_2) \rangle \), the pair correlation functions \( \langle \Phi_c(k, r_3) - \Phi_s(-k, r_4) \rangle \), and \( \Phi_c(k, r_3) - \Phi_s(-k, r_4) \) at \( R \to \infty \). (4.79)

5 Tunneling and orthogonality catastrophe

Tunneling effects are most peculiar in the presence of an orthogonality catastrophe. Tunneling experiments seem to be the best instruments to search for a signature of the topological mechanism. Below we briefly discuss various tunneling effects.

5.1 Josephson Tunneling

The Josephson current through the junction (at zero bias voltage) between two superconductors, of which one is conventional, is given by the well known formula [13]

\[
I = -3m i \sum_{kp} T_{kp} T_{-k-p} \frac{\Delta_1^*}{E_k} \int_0^\infty d\omega \frac{F(p, \omega)}{2\pi} \left[ \frac{F(p, -\omega)}{\omega - E_k + i\eta} - \frac{F(p, -\omega)}{\omega - E_k - i\eta} \right],
\]

(5.1)

where \( T_{kp} \) is a transmission amplitude of the junction, \( E_k = \sqrt{\Delta_1^2 + \epsilon_k^2} \) is the pair spectrum of the conventional superconductor and \( F(p, \omega) \) is the spectral function of the superconductor of interest

\[
F(p, \omega) = 2\pi i e^{i\sigma'} \sum |N| |c_\sigma(p)| |N + 1| \langle N + 1 | c_{\sigma'}(-p) | N + 2 \rangle \delta(\omega - \epsilon_p).
\]

(5.2)

Here the sum goes over all quantum states with one extra particle (\( \epsilon_p \) is the energy of an intermediate state). If the spectrum is symmetric with respect to adding or removing a particle, i.e., \( F(p, \omega) \) is an odd function of \( \omega \), we obtain

\[
I = -2 \Im m i \sum_{kp} T_{kp} T_{-k-p} \frac{\Delta_1^*}{E_k} \int_0^\infty d\omega \frac{F(p, \omega)}{2\pi} \frac{d\omega}{\omega - E_k}.
\]

(5.3)

In the following we assume for simplicity that the transmission amplitude \( T_{kp} \) is strongly peaked at \( k, p \) close to the direction normal to the junction. This simplification should not change the phase dependence of the Josephson current although it can change the value of the critical current. Assuming that the gap in the superconductor of interest \( \Delta_0 \) is bigger than the one in the conventional superconductor \( \Delta_0 \gg \Delta_1 \) we obtain:

\[
I \sim |T|^2 \nu_0 |\Delta_1| \sum_{|p_n|} \int_0^\infty \frac{d\omega}{\omega} \sin(\phi_0 - \phi(p_n, \omega)) |F(p_n, \omega)|,
\]

(5.4)
where $p_n$ is the component of momentum normal to the surface of the junction and averaging over $p_n$ is determined by the actual form of the transition amplitude. In this formula $\phi_0$ and $\nu_0$ are the phase and density of states of the conventional superconductor and $\phi(p, \omega)$ is the phase of the $F$-function (5.2) of the superconductor of interest.

In the BCS theory the $F$-function has a peak at the gap $\omega \sim \Delta_0$, such that the width of the peak is also of the order of the gap:

$$\int dp F_{\text{BCS}}(p, \omega) \sim \frac{\Delta_0}{\sqrt{\omega^2 - \Delta_0^2}}. \quad (5.5)$$

The peak gives the major contribution to the integral (5.4). It selects a characteristic energy of the intermediate state $\epsilon_q \sim \Delta_0$ and gives rise to the traditional BCS picture of tunneling: a pair decays into two electrons while tunneling, so electrons tunnel independently. Short time processes (at $\omega \sim \epsilon_f$) do not contribute to the integral (5.4).

The situation is drastically different in the orthogonality catastrophe environment [11]. In a topological superconductor an individual matrix element $\langle N | c_\sigma(p) | N + 1 \rangle$ acquires an additional factor $1/L k_f$, where $L$ is the size of the system and therefore vanishes in a macroscopic sample. In other words the ground states with $N$ and $N + 1$ particles are almost orthogonal. Nevertheless, the tunneling, i.e., a matrix element between states with $N$ and $N + 2$ particles is nonzero due to a large number of low-energy intermediate states contributing to the sum (5.4). A result of this is that the pair spectral function (5.2) acquires an additional factor $\omega/\epsilon_f$. In contrast to BCS at $\omega \gg \Delta_0$ we have

$$\int dp F(p, \omega) \sim \frac{\Delta_0}{\omega} \left( \frac{\omega}{\epsilon_f} \right) O(\ln \frac{\omega}{\Delta_0}) \sim \frac{\Delta_0}{\epsilon_f} O(\ln \frac{\omega}{\Delta_0}). \quad (5.6)$$

Therefore, the characteristic scale of the spectral function is shifted to the ultraviolet and becomes of the order of the Fermi energy—much larger than the scale of the gap: the integral (5.4) is saturated by $\omega \sim \epsilon_f \gg \Delta_0$. This means that a pair remains intact during the tunneling and the tunneling amplitude is determined by the equal time value of $F(p, t = 0)$, i.e., by the matrix element $\Delta(r)$ of an instantaneous creation of a pair (2.26). Let us notice that the correction to the spectral function in the eq. (5.6) $- (\omega/\epsilon_f)^\alpha$ at $\alpha = 1$ is just marginal. Were $\alpha$ is less than 1 the time of the tunneling would be of the order of $\Delta_0^{-1}$.

In the coordinate representation the Josephson current is

$$I \sim \Im e^{-i\phi_0} \Delta(k_f n) \sim \sin(\phi_0 - \phi(k_f n, \omega = 0)). \quad (5.7)$$

In the corner-SQUID-junction geometry [9] one can directly measure the difference of phases of $\Delta$ between two faces of the superconducting crystal. If the faces of two contacts have a relative angle $\theta$, then, according to (4.73, 4.74) the phase shift in tunneling amplitude will be twice the angle $\phi(k_f n_1) - \phi(k_f n_2) = 2\theta$:

$$\frac{\Delta(k_f n_1)}{\Delta(k_f n_2)} = e^{2i\theta}. \quad (5.8)$$

The instantaneous character of tunneling and the power laws (5.6) are known from one-dimensional electronic systems where the orthogonality catastrophe comes to its own. In addition, in 2D it also leads to the angular dependence of the two-particle amplitude.

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8 A similar phenomenon has been discussed by Chakravarty and Anderson in the context of interlayer tunneling in cuprate superconductors [1]
5.2 One particle tunneling

The one particle tunneling current is also strongly affected by the orthogonality catastrophe. One particle tunneling current from a metal or a superconductor with a small gap is determined by the tunneling spectral function (4.62)

\[ I_{\text{dir}} = 2|T|^2 \nu_0 \int_{\Delta_0} e^V dE dN(E) \frac{dE}{dE}, \tag{5.9} \]

where \( \nu_0 \) is the density of states of the normal metal (or low gap superconductor). In the BCS, \( \frac{dN(E)}{dE} \sim \frac{E}{\sqrt{E^2 - \Delta_0^2}} \) and the direct current is given by a familiar formula \( I \sim \sqrt{(eV)^2 - \Delta_0^2} \).

In the topological superconductor the one-particle Green function acquires a branch cut, rather than a pole on the threshold and the tunneling density of state loses its singularity at the threshold (3.62). This leads to a suppression of direct current close to the threshold bias voltage. With a logarithmic accuracy we will obtain almost quadratic \( I - V \) behavior well above the threshold

\[ \frac{dI}{dV} \sim V, \quad \text{at } eV \gg \Delta_0. \tag{5.10} \]

It peaks at \( \Delta_0 \ln \epsilon_f/\Delta_0 \).

5.3 N-S Tunneling

The conductance between normal metal and a superconductor does not vanish for the voltage below the gap due to the pair tunneling: an electron incident on the NS boundary captures another electron and generates a hole in the metal and pair in the superconductor (Andreev scattering). If one assumes that tunneling Hamiltonian for the pair tunneling is

\[ H_{NS} = \Gamma C_\uparrow(r_1)C_\downarrow(r_2)c_\uparrow^\dagger(r_3)c_\downarrow^\dagger(r_4), \]

then the tunneling current of this process is given by

\[ I = e|\Gamma|^2 \nu_0^2 \int_0^{eV} d\omega (eV - \omega) \int \Im K(\omega,q) dq \tag{5.11} \]

where

\[ K(\omega,q) = \int e^{i\omega t + i q R} \langle c_\uparrow^\dagger(r_1,t)c_\downarrow^\dagger(r + r_1,t)\rangle \langle c_\uparrow R + r_1,0)\rangle \langle c_\downarrow R' + r_1,0)\rangle dR dt \tag{5.12} \]

is the pair correlation function. Its dependence on positions of electrons \( r, r' \) within pairs is taken care by the tunneling matrix element. At small voltage only small pairs contribute \( r, r' \ll R \) to the tunneling current.

In the BCS, the pair correlation exhibits a long range order, i.e., \( \Im K(\omega,q) \sim \rho_s \delta(q) \delta(\omega) \), the integral in (5.11) is saturated at \( q, \omega = 0 \) and gives a nonzero conductance \( I \sim \sigma_0 V \), where \( \sigma_0 \sim (|\Gamma|^2 e^2 \nu_0^2 \rho_s) \).

In contrast, in the topological superconductor, the pair propagator is determined by soft collective density fluctuations. Extending the result of Sec.4.6, we get \( K(\omega,q) \sim (\omega^2 - v_f^2 q^2)^{1/2} \). This gives a nonlinear \( I - V \) characteristic

\[ I \sim \sigma_0 V \frac{eV}{\epsilon_f}, \quad \text{at } eV \ll \Delta_0. \]
This result implies zero NS conductance and a suppression of the Andreev scattering by the orthogonality catastrophe.

6 Concluding Remarks

6.1 Physics of one dimension beyond the one dimension.

The goal of this paper is to show that one dimensional electronic physics is not restricted to one spatial dimension. The origin of almost all phenomena we have observed in one dimension is the spectral flow, rather than a restricted geometry. A very similar physics takes place in topological electronic liquids, i.e. in compressible liquids with a charge spectral flow.

One of the features of one dimensional physics - the Sugawara form of the stress energy tensor and the associated current algebra, being lifted to a higher spatial dimension, translates into superconductivity. With a great deal of generality we argued that a spectral flow in a compressible liquid with a fixed chemical potential inevitably leads to the hydrodynamics of an ideal liquid (a generalization of the Sugawara form of one dimension) and to a superconducting ground state (topological mechanism of superconductivity).

Another lesson may be learned from one dimension - the orthogonality catastrophe is also a consequence of the spectral flow. It affects drastically all processes of the topological superconductor involving a change of number of particles. Among them one particle spectral properties (e.g. photoemission) and all sorts of tunneling (Josephson tunneling, NS two particle tunneling and Andreev scattering, one particle tunneling).

Neither electrons nor Cooper pairs are elementary excitations in topological liquids. The insertion of one electron (any non-singlet state) drastically changes the ground state of the system, so that the matrix element between two ground states with \( \mathcal{N} \) and \( \mathcal{N} + 1 \) electrons vanishes in a macroscopic system. The same is true for any ground states which differ by an odd number of particles. On the contrary, the matrix elements of two particles in a singlet state do not vanish but are significantly modified by the interaction with soft modes of density modulations. In particular, the poles of Green functions on the mass shell are replaced by branch cuts.

The one-dimensional physics provides tools to compute matrix elements and Green functions in topological fluids in dimension greater than one. This is bosonization. Actual particles, having asymptotic states, are solitons of nonlinear charge density and spin density modulations. They can be represented as coherent states of the charge and spin densities. Their matrix elements are composition of holomorphic functions, no matter what the spatial dimension is, and can be treated in a similar way to what used in one dimension. One is the method is bosonization, presented in Sec. 4.3. Another (more general) method is to consider the matrix elements as conformal blocks of a certain conformal field theory and a current algebra. Let us notice that electronic liquids without a spectral flow, say the Fermi liquid, can not be bosonized, i.e. their bosonized form is not adequate to physics of Fermi liquid ground state. The reason for this is a considerable phase space of scattering at the Fermi surface (leading to a dissipation). Contrary in topological liquids the scattering and emission of a soft mode are asymptotically forward. This reduces the phase space

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9This result will not be affected by the Coulomb interaction, provided that plasmon frequency is smaller than the gap. Effects of interaction between Cooper pairs and soft modes of density modulation on various tunneling mechanisms in conventional BCS and layered superconductors has been studied in \[29\].
of scattering and eliminates dissipation. On a general basis one can say that only superconductors can be bosonized.

6.2 Topological superconductor versus BCS superconductor

The orthogonality catastrophe is a source of major differences between a topological and BCS superconductors:

(i) The tunneling amplitude or the ground state pair wave function \( \Delta(r_1, r_2) = \langle N + 2 | c_\uparrow^\dagger(r_1) c_\downarrow^\dagger(r_2) | N \rangle \) differs from the gap function. The former is complex and its phase \( 2 \text{Arg}(r_1 - r_2) \) depends on the direction of the pair, whereas the gap may be isotropic. The amplitude of the pair wave function and the gap function are also different. The scale of the former is determined by the gap and also by the Fermi scale.

(ii) Contrary to BCS the gradient of the phase of the pair wave function in a current state \( \Delta(r_1, r_2) = |\Delta| e^{i\phi} \) does not determine the current \( j \neq -\frac{|\Delta|^2}{2m} \nabla \phi \). The relation between the phase and the current is nonlocal and retarded. The pair wave function does not play the role of an order parameter.

(iii) The pair correlation function \( \langle c_\uparrow^\dagger(r_1) c_\downarrow^\dagger(r + r_1) c_\downarrow(R + r'_1) c_\uparrow(R + r' + r_1) \rangle \) does not have a long range order but decays with the separation between pairs at \( R \to \infty \). Nevertheless, the superfluid density and the tunneling amplitude do not vanish.

(iv) The pair spectral function \( \Delta(k) \) has a broad structure of the amplitude of the pair wave function in momentum space around what one may call Fermi surface. The width of this structure is of the order of \( k_f/\log(\epsilon_f/\Delta_0) \) which is much bigger than \( \Delta_0/v_f \), the width of the peak of the BCS wave function.

(v) The origin of the gap is the \( 2k_f \) instability rather than a condensation of pairs as in BCS.

(vi) Tunneling: The time of Josephson tunneling is determined not only by the gap (as in BCS) but also by the Fermi scale and is much shorter than \( \Delta^{-1}_0 \). A pair remains intact while tunneling. One particle tunneling is suppressed above the threshold. The amplitude of Andreev scattering vanishes at the Fermi surface, so that a conductance of the NS tunneling. \( I-V \) characteristic of the NS tunneling becomes nonlinear.

(vii) The noise spectrum \( \langle j(\omega)j(-\omega) \rangle \) of a point Josephson junction is expected to have a power law peak, rather than a narrow peak of the BCS superconductors (we do not study the noise spectrum here).

(viii) One particle spectral function has a branch cut rather than a pole i.e., \( Z \)-factor (a residue) as well as the one particle matrix element vanishes at the Fermi surface. The one particle spectral function (tunneling density of state) has a broad peak shifted inside the Fermi surface and vanishes with a (weak) singularity at the Fermi surface.

Ironically, the quantities listed above did not enjoy a detailed measurements even in standard superconductors.

6.3 Parity and time reversal symmetry breaking.

There have been misconceptions in the literature regarding parity symmetry breaking of the ground state of two-dimensional topological liquid. The spatial parity and time reversal symmetry are
simultaneously broken in the ground state of the model considered. This reflects the chiral nature of zero modes. However the broken symmetry does not exhibit itself in every physical quantity. Also there is no easy ways to detect this symmetry breaking experimentally, if any at all. The reason is that, although the time reversal symmetry is broken, there are no spontaneous local electric currents either in the bulk or on the edge in any steady state. One can see this from the hydrodynamics of the charge sector (2.16), but this fact remains valid beyond the low frequency range. Even more general, all diagonal singlet matrix elements are parity and time reversal even. Therefore, one should not expect to observe the time reversal symmetry breaking by measuring Faraday rotation [30] and muon spin relaxation [31] — early experimental searches for a signature of parity symmetry breaking.

Off-diagonal or non-singlet matrix elements are a different matter. The broken time reversal symmetry is explicit in the complex d-wave tunneling amplitude. Another manifestations of the broken symmetry can be seen in the spin sector. Among them are an expectation value of the spin chirality $S \cdot \nabla S \times \nabla S$ and a novel feature—edge spin current. This is a two-dimensional version of the known phenomenon in 1D. A spin chain with gapped bulk spin excitations develops gapless spin excitations at the edges. In a two-dimensional spin liquid edge excitations are chiral spin currents. Edge magnetic excitations have been observed in spin chains [32]. One may expect to find these soft edge spin excitations in model systems with an enhanced boundary (say, for example, an array of superconducting islands).

Parity symmetry breaking is not an inherent property of the topological fluids, but rather a property of the "irreducible" model. By doubling the number of fermionic components one can easily construct a model with no time reversal or parity symmetry breaking. The most natural way to do this is to include an even number of layers. The parity alternates between odd and even layers [33]. Genuine parity breaking takes place only in systems with odd number of layers and minimal number of fields.

6.4 Topological mechanism and superconductivity of cuprates

Being a theory of doped Mott insulators, topological superconductivity has a number of features observed in cuprate superconductors. A detailed discussion of the experimental data from the point of view of topological superconductor is far beyond the scope of this text. We only mention the corner-SQUID-junction experiment [4] in which the relative phase of tunneling amplitudes (2.23) on different faces of a single crystal of YBCO has been measured. It is found to be $\pi$ in accord with d-wave superconductivity and apparently in agreement with the topological mechanism (4.19).

A critically inclined reader faces difficulties to determine whether known London type superconductors are BCS superconductors or topological liquids.

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7 Appendix. Fermionic number and Solitons

For completeness, let us review a well known calculation of the fermionic number \( \eta \) (as known as Atiah-Patodi-Singer invariant \( \eta \) ) is a number of unoccupied states appeared as the result of adiabatically created topological configuration of the potential.

Since without a potential the spectrum to the Dirac operator is symmetric, this number also measures the difference between the number of states with positive and negative energy \( \eta = N_+ - N_- \) (spectral asymmetry). If \( N = N_+ + N_- \) is the total number of levels, then the number of negative levels would be \( N_- = (N - \eta)/2 \). The “regularized” number of negative levels, i.e., the change of the negative levels due to the potential, \( \Delta N = N_- - \frac{\eta}{2} = -\frac{1}{2}\eta \), is the fermionic number:

\[
\Delta N = -\frac{1}{2}\text{Tr}[\text{sign } H] .
\]  

This divergent quantity has to be regularized. One of the standard ways of regularization is to replace it by

\[
\Delta N = -\frac{1}{2\pi} \int \text{Tr}[H H^2 + z^2]\,dz
\]  

The integrand now converges.

**One Dimension.**

We compute the fermionic number for a more general Peierls model

\[
H = \alpha_x i\partial_x + \beta \pi_1 + i\alpha_x \beta \pi_2
\]  

where the Dirac matrices \( \alpha_x, \beta \) may chosen as the Pauli matrices: \( \alpha_x = \sigma_3, \beta = \sigma_1 \). We also assume that the modulus of the vector \( (\pi_1, \pi_2) \) takes a fixed value at infinity.

The soliton here is a field \( \vec{\pi}(x) \) which forms a homotopy class \( \pi_1(S^1) \). Its topological charge is

\[
Q = -\int \frac{\vec{\pi} \times \partial \vec{\pi}}{\vec{\pi} \cdot \vec{\pi}} \frac{dx}{2\pi} .
\]  

With a given value of the \( |\vec{\pi}| \) at infinity \( Q \) is an integer number.

The square of the Hamiltonian is

\[
H^2 = (i\partial_x)^2 + \vec{\pi}^2 + \varepsilon_{ab}\sigma_a \partial_x \pi_b, \quad a, b = 1, 2 .
\]  

Let us now expand the integrand of (7.2) in \( \partial \vec{\pi} \). In the first non-vanishing order of \( |\vec{\pi}(\infty)| \to \infty \) we have

\[
\Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr} \left( \frac{\sigma^3 i\partial + \vec{\sigma} \vec{\pi}}{(i\partial)^2 + \vec{\pi}^2 + \pi^2} \frac{1}{(i\partial)^2 + \vec{\pi}^2 + \pi^2} \right)
\]  

This is the only order which contributes to the fermion number. The trace in spinor space gives

\[
\Delta N = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dz \text{Tr} \left( \frac{\pi^1 \partial \pi^2 - \pi^2 \partial \pi^1}{[(i\partial)^2 + \vec{\pi}^2 + \pi^2]^2} \right) .
\]  

The trace in momentum space is conveniently calculated in the plane wave basis,

\[
\Delta N = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dz \int \frac{dp}{2\pi} \varepsilon_{\mu \nu} \pi^\mu \partial \pi^\nu \frac{1}{[p^2 + \vec{\pi}^2 + \pi^2]^2} .
\]  

\[49\]
Finally the integrals over $p$ and $z$ give the topological charge (7.4) of kinks

$$\Delta N = -\frac{1}{2\pi} \int dx \epsilon_{\mu\nu} \frac{\pi^\mu \partial_{\pi^\nu}}{\pi^2} = Q,$$

(7.9)

the topological charge (7.4) of kinks. The number of extra states induced by a soliton therefore equals the topological charge of the soliton. If $\pi_1 + i\pi_2 = \Delta_0 e^{i\varphi}$, we obtain a global version of the Frohlich relation (3.12). The model with $\pi_1 = 0$ deserves special interest. This is commensurate Peierls model ($\pi_2 = \Delta$). It is defined by

$$H = \alpha_x i \partial_x + i\gamma_5 \Delta .$$

(7.10)

The spectrum of the Hamiltonian is symmetric. This means that if there is a state $\psi_E$ with an energy $E$ then there is always a state $\beta \psi$ with the energy $-E$, except for $E \neq 0$. It follows from the fact that the Hamiltonian anticommute with the matrix $\beta$.

Now the soliton is a kink with $\Delta(-\infty) = -\Delta(+\infty) = \Delta_0$. Setting $\pi_1 = 0$ in (7.9), we obtain

$$Q = -\frac{1}{2} \int \delta(\Delta) \partial \Delta dx .$$

(7.11)

In this case an extra state appears in the middle of the gap (zero mode). The kink does not respect periodic boundary conditions. As a result of this a zero mode state may accommodate only $1/2$ of the particle. In a system with a periodic boundary conditions kinks may appear only together with antikinks, so the total number of states remains an integer.

**Two Dimensions.**

Calculations are similar in two dimensional case

$$H = \sum_{\sigma=1,2} \psi^\dagger_\sigma [\alpha(i\nabla + A) + \beta m] \psi_\sigma .$$

(7.12)

The square of its Hamiltonian is

$$H^2 = (i\nabla + A)^2 + m^2 - \beta(\nabla \times A).$$

(7.13)

Let us expand the integrand in (7.2) in terms of $\beta(\nabla \times A)$. The first and only non-vanishing order gives:

$$\Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \text{Tr} \left( \frac{\alpha(i\nabla + A) + \beta \mathbf{m}}{(i\nabla + A)^2 + m^2 + z^2} \beta(\nabla \times A) \times \frac{1}{(i\nabla + A)^2 + m^2 + z^2} \right).$$

(7.14)

The trace leaves the integral over $p$ and $z$:

$$\Delta N = -\frac{2}{\pi} \int_{-\infty}^{+\infty} dz \int d^2 x \int \frac{d^2 p}{(2\pi)^2} \frac{m(\nabla \times A)}{[(p + A)^2 + m^2 + z^2]^2} .$$

(7.15)

As a result we obtain the relation between the fermion number and the topological charge of the vortices (1.2, 1.3)

$$\Delta N = -\frac{1}{2\pi} \text{sign} m \int d^2 x (\nabla \times A) .$$

(7.16)
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