Second-Order Guarantees of Stochastic Gradient Descent in Non-Convex Optimization

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Abstract—Recent years have seen increased interest in performance guarantees of gradient descent algorithms for non-convex optimization. A number of works have uncovered that gradient noise plays a critical role in the ability of gradient descent recursions to efficiently escape saddle-points and reach second-order stationary points. Most available works limit the gradient noise component to be bounded with probability one or sub-Gaussian and leverage concentration inequalities to arrive at high-probability results. We present an alternate approach, relying primarily on mean-square arguments and show that a more relaxed relative bound on the gradient noise variance is sufficient to ensure efficient escape from saddle-points without the need to inject additional noise, employ alternating step-sizes or rely on a global dispersive noise assumption, as long as a gradient noise component is present in a descent direction for every saddle-point.

Index Terms—Stochastic optimization, adaptation, non-convex cost, gradient noise, stationary points.

I. INTRODUCTION

In this work, we consider optimization problems of the form:

\[ w^* \triangleq \arg \min_{w \in \mathbb{R}^d} J(w) \quad (1) \]

where \( J(w) \) is a risk function defined as the expectation of a loss function, i.e.,

\[ J(w) \triangleq \mathbb{E}_x Q(w; x) \quad (2) \]

where the expectation is over the distribution of the data variable \( x \). We wish to study first-order methods for pursuing solutions of (1), i.e., recursions of the form:

\[ w_i = w_{i-1} - \mu \nabla J(w_{i-1}) \quad (3) \]

where \( \nabla J(w_{i-1}) \) denotes some suitable update direction. When the gradient of \( J(\cdot) \) can be evaluated, which in general requires the distribution of \( x \) to be known, then one popular and effective construction is to employ the actual gradient vector:

\[ \nabla J^G(w_{i-1}) \triangleq \nabla J(w_{i-1}) \quad (4) \]

When the distribution of \( x \) is unknown, we can instead can instead rely on the stochastic gradient approximation [1]:

\[ \nabla J^{SG}(w_{i-1}) \triangleq \nabla Q(w_{i-1}, x_i) \quad (5) \]

where \( \nabla Q(w_{i-1}, x_i) \) denotes an instantaneous approximation of \( \nabla J(w_{i-1}) \) based on the realization \( x_i \) observed at time \( i \). For strongly convex cost functions \( J(\cdot) \), both gradient [4] and stochastic gradient [5] implementations of (3) are very well behaved and well studied in the literature – see, e.g., [2], [3] and the references therein. One particular conclusion is that, under suitable conditions on the loss function and data distribution, descent along the true gradient \( \nabla J(w_{i-1}) \) results in linear convergence to the minimizer \( w^* \), while stochastic “descent” along the instantaneous gradient approximation [5] results in a small performance degradation in steady-state for small step-sizes, i.e., \( \limsup_{i \to \infty} \mathbb{E} \| w^* - w_i \|^2 \leq O(\mu) \) [4].

One surprising fact that arises when considering non-convex cost functions is that employing stochastic or perturbed gradient directions is generally beneficial and can in fact improve the ability of an algorithm to escape saddle-points. For example, recursion (3) with true gradients (4) can take exponentially long to escape from saddle-points [5]. However, by simply perturbing the gradient by adding i.i.d. noise will allow the algorithm to escape strict saddle-points in polynomial time [6]. More formally, perturbed gradient descent takes the form [5]:

\[ \nabla J^{PG}(w_{i-1}) \triangleq \nabla J(w_{i-1}) + v_i \quad (6) \]

where \( v_i \) is some i.i.d. perturbation term with positive definite covariance matrix. When the true gradient \( \nabla J(w_{i-1}) \) is unavailable, the perturbation can be added instead to the instantaneous gradient approximation [7]:

\[ \nabla J^{PSG}(w_{i-1}) \triangleq \nabla Q(w_{i-1}, x_i) + v_i \quad (7) \]

In this work, we will study a generic update direction \( \nabla J(w_{i-1}) \) and examine the dynamics of (5) in non-convex environments under conditions that are more relaxed than typically assumed in the recent literature. To this end, we introduce the gradient noise process:

\[ s_i(w_{i-1}) \triangleq \nabla J(w_{i-1}) - \nabla J^{PSG}(w_{i-1}) \quad (8) \]

and write (3) as:

\[ w_i = w_{i-1} - \mu \nabla J(w_{i-1}) - \mu s_i(w_{i-1}) \quad (9) \]

Any particular choice for the gradient estimate \( \nabla J^{PSG}(w_{i-1}) \) will induce a different gradient noise process [8] with varying properties. For example, while employing construction [6] results in i.i.d. gradient noise, a general construction of the form [5] will generally result in a gradient noise process that is no longer i.i.d.

A. Related Works

The results and proof techniques presented in this work are related to our recent works [8], [9], which considered instead...
distributed optimization problems under and absolute variance bound on the gradient noise. The contribution of this current work in relation to these earlier studies is two-fold. First, we focus here solely on the case of single-agent optimization, i.e., on centralized as opposed to decentralized implementations. Second, and more importantly, by limiting our analysis to the single-agent setting, we are able to relax the absolute variance condition employed in [8], [9] to a mixed variance bound consisting of a mixture of relative and absolute components, thus leading to new performance guarantees in the centralized case.

There have of course been several other useful works on non-convex optimization using first-order methods in the literature. The primary focus in these earlier works has been establishing convergence to first-order stationary points, i.e., points where the gradient vanishes so that \( \nabla J(w_{i-1}) = 0 \) as \( i \to \infty \) [10]–[13]. First-order stationarity by itself however, is generally not a sufficient guarantee of a desirable solution since the set of first-order stationary points includes saddle-points and even local maxima. For this reason, in more recent years, there has been growing interest in convergence guarantees that exclude such undesirable first-order stationary points. To do so, one also examines second-order conditions. In particular, recall that second-order stationary points are those where not only the gradient vector is zero, but there are also restrictions on the smallest eigenvalue of the Hessian matrix at their locations [14]. These restrictions, when chosen to exclude local maxima and strict saddle-points can help ensure convergence towards local minima. Actually, under such restrictions, the stationary points can be shown to always correspond to local minima for some functions of interest [6], [15]–[18].

One approach for ensuring convergence to these desirable second-order stationary points is by incorporating second-order information via the Hessian matrix into the update relation [19], [20]. Such a construction helps ensure that a descent direction can be identified even when the gradient vanishes and no longer carries directional information. For many, especially large-scale problems, evaluating the Hessian matrix at every iteration can be prohibitively costly. This fact has spawned a number of works that continue to employ first-order schemes for identifying a descent direction around saddle-points for both deterministic and stochastic optimization [21]–[23].

A second class of methods for the escape from saddle-points exploits the fact that strict saddle-points (defined later) are unstable, in the sense that small perturbations, either induced during initialization [24], [25] or added to the true gradient direction [6], [26], [27], will cause iterates to approach second-order stationary points almost surely. These algorithms require knowledge of the true gradient \( \nabla J(w_{i-1}) \), which generally requires information about the distribution of \( x \). Strategies for stochastic optimization, where instantaneous approximations \( \nabla Q(w_{i-1}, x) \) are employed in place of the true gradient \( \nabla J(w_{i-1}) \) have also been studied recently. The works [28], [29] and [7] consider perturbed stochastic gradients [7] with diminishing and constant step-sizes, respectively, while [30] employs [5] by interlacing small and large step-sizes and the works [8], [9], [31] descend along [5] with constant step-sizes.

This work is most related to these latter references — we shall make a detailed distinction when discussing the modeling conditions below. We also note that a number of recent works consider variance reduced strategies for the setting where \( J(\cdot) \) corresponds to an empirical risk based on a finite number of samples [12], [22], [32]. In contrast, our focus is on the streaming data setting, where the sample size tends to infinity and traditional variance reduction techniques are inapplicable.

II. Modeling Conditions

A. Smoothness Conditions

We employ the following smoothness assumptions.

**Assumption 1 (Lipschitz gradients).** The gradient \( \nabla J(\cdot) \) is Lipschitz, namely, there exists \( \delta > 0 \) such that for any \( x, y \):

\[
\|\nabla J(x) - \nabla J(y)\| \leq \delta \|x - y\| \tag{10}
\]

**Assumption 2 (Lipschitz Hessians).** The cost \( J(\cdot) \) is twice-differentiable and there exists \( \rho \geq 0 \) such that:

\[
\|\nabla^2 J(x) - \nabla^2 J(y)\| \leq \rho \|x - y\| \tag{11}
\]

Assumption [11] is common in the study of gradient algorithms, even for the minimization of convex function [4] and first-order stationarity in non-convex environments [10], [11]. It implies a quadratic upper bound on the cost:

\[
J(y) \leq J(x) + \nabla J(x)^T(y - x) + \frac{\delta}{2}\|y - x\|^2 \tag{12}
\]

uniform lower and upper bounds on the Hessian matrix:

\[
-\delta I \leq \nabla^2 J(x) \leq \delta I \tag{13}
\]

The stronger Assumption [2] is not necessary to establish convergence to first-order stationary points [10]. It is frequently employed to characterize more granularly the dynamics of (stochastic) gradient algorithms around first-order stationary points, both to establish the ability of various gradient algorithms to escape saddle-points [6], [7], [22], [24] or to study the mean-square deviation of stochastic gradient implementations from minimizers in the strongly-convex setting [4]. It implies a tighter upper bound than [12] [19]:

\[
J(y) \leq J(x) + \nabla J(x)^T(y - x) + \frac{1}{2}(y - x)^T\nabla^2 J(x)(y - x) + \frac{\delta}{6}\|y - x\|^3 \tag{14}
\]

B. Gradient Noise Conditions

We shall employ the following conditions on the gradient noise process [3].

**Definition 1 (Filtration).** We denote by \( \mathcal{F}_i \) the filtration generated by the random processes \( w_j \) for all \( j \leq i \):

\[
\mathcal{F}_i \triangleq \{w_0, w_1, \ldots, w_i\} \tag{15}
\]

Informally, \( \mathcal{F}_i \) captures all information that is available about the stochastic processes \( w_j \) up to time \( i \).
Assumption 3 (Gradient noise process). The gradient noise process \cite{8} satisfies:
\[ E \{ s_i(w_{i-1}) | \mathcal{F}_{i-1} \} = 0 \]  
(16)  
\[ E \{ \| s_i(w_{i-1}) \|^4 | \mathcal{F}_{i-1} \} \leq \beta^4 \| \nabla J(w_{i-1}) \|^4 + \sigma^4 \]  
(17)  
for some non-negative constants \( \beta, \sigma \).

The fourth-order condition (17) also implies a bound on the second-order moment via Jensen’s inequality:
\[ E \{ \| s_i(w_{i-1}) \|^2 | \mathcal{F}_{i-1} \} \leq \sqrt{\beta^4 \| \nabla J(w_{i-1}) \|^4 + \sigma^4} \]
(18)
where (a) follows from the sub-additivity of the square root. Condition (18) is the same as the one employed in \cite{11} to study first-order stationarity under a diminishing step-size rule and corresponds to a mixture of the absolute and relative noise components appearing in \cite{2}. It is weaker than the condition assumed in works on second-order stationarity. For example, the works \cite{6}, \cite{29} require the gradient noise process to be uniformly bounded for all \( w_i \) with probability one. This condition is relaxed in \cite{7} by requiring the difference \( \nabla J(w_{i-1}) - \nabla Q (w_{i-1}, x_i) \) to be sub-Gaussian and further in \cite{8}, \cite{9} by allowing for a uniform bound on the fourth-order moment. Works that employ bounded or sub-Gaussian gradient perturbations generally rely on concentration relations, which explicitly exploit the bounded or sub-Gaussian nature of the gradient noise process \cite{7}.

In this work, we take a different approach by anchoring our analysis around mean-square arguments. This allows us to track the evolution of the iterates \( w_i \) in the mean-square sense, rather than with high probability and avoid the need for restrictive probability bounds on the gradient noise process. Observe that condition (18) is weaker than a uniform bound on the fourth moment of the gradient noise process, since we allow for a relative component in the form of \( \beta^4 \| \nabla J(w_{i-1}) \|^4 \).

This condition allows for the gradient noise variance to grow away from first-order stationary points and in particular does not enforce a uniform bound on the gradient noise variance as seen from (18). In place of stronger bounds on the gradient noise variance, we employ a smoothness condition on the gradient noise covariance, previously employed for characterizing the mean-square deviation of stochastic gradient algorithms around the minimizer in strongly convex optimization \cite{4}.

Assumption 4 (Lipschitz covariances). The gradient noise process has a Lipschitz covariance matrix, i.e.,
\[ R_i(w_{i-1}) \triangleq E \{ s_i(w_{i-1}) s_i(w_{i-1})^T | \mathcal{F}_{i-1} \} \]
(19)
satisfies
\[ \| R_i(x) - R_i(y) \| \leq \beta_R \| x - y \|^\gamma \]  
(20)
for some \( \beta_R \) and \( 0 < \gamma \leq 4 \).

This condition essentially ensures that the second-order moment of the gradient noise process is approximately invariant so long as the iterates \( w_{i-1} \) remain sufficiently close. From the bound on the aggregate gradient noise variance \cite{13}, we can upper bound the gradient noise covariance as follows:
\[ \| R_i(w_{i-1}) \| \leq E \{ \| s_i(w_{i-1}) s_i(w_{i-1})^T \| | \mathcal{F}_{i-1} \} \]
(18)
\[ \| R_i(w_{i-1}) \| \leq \beta^2 \| \nabla J(w_{i-1}) \|^2 + \sigma^2 \]  
(21)

Before introducing the final assumption, we formally define first and second-order stationary points, similar to prior works on second-order stationary guarantees \cite{6}, \cite{8}, \cite{9}, \cite{19}. We decompose the space \( w \in \mathbb{R}^M \) into four sets.

Definition 2 (Sets). To simplify the notation in the sequel, we introduce following sets:
\[ G \triangleq \{ w : \| \nabla J(w) \|^2 \geq \mu c_2 c_1 (1 + \frac{1}{\pi}) \} \]  
(22)
\[ G^C \triangleq \{ w : \| \nabla J(w) \|^2 < \mu c_2 c_1 (1 + \frac{1}{\pi}) \} \]  
(23)
\[ \mathcal{H} \triangleq \{ w : w \in G^C, \min \{ \nabla^2 J(w) \} \leq -\gamma \} \]  
(24)
\[ \mathcal{M} \triangleq \{ w : w \in G^C, \min \{ \nabla^2 J(w) \} > -\gamma \} \]  
(25)
where \( \gamma \) is a small positive parameter, \( c_1 \) and \( c_2 \) are constants:
\[ c_1 \triangleq 1 - \frac{\delta}{2} (1 + \beta^2) = O(1) \]  
(26)
\[ c_2 \triangleq \frac{\delta}{2} \sigma^2 = O(1) \]  
(27)
and \( 0 < \pi < 1 \) is a parameter to be chosen. Note that \( G^C = \mathcal{H} \cup \mathcal{M} \). We also define the probabilities \( \pi^G \triangleq \Pr \{ w_i \in G \} \), \( \pi^H \triangleq \Pr \{ w_i \in \mathcal{H} \} \) and \( \pi^M \triangleq \Pr \{ w_i \in \mathcal{M} \} \). Then, for all \( i \), we have \( \pi^G + \pi^H + \pi^M = 1 \).

As explained in \cite{8}, \cite{9}, the above definition first decomposes the space \( \mathbb{R}^M \) into the set \( G \), where the squared norm of the gradient is larger than \( O(\mu) \) and its complement \( G^C \). Since the squared norm of the gradient in \( G^C \) is not precisely equal to zero, but nevertheless small for small step-sizes \( \mu \), we refer to these points as approximately first-order stationary. The set of approximate first-order stationary points is further decomposed into those where the Hessian matrix has a strictly negative eigenvalue \( \mathcal{H} \), and those who do not \( \mathcal{M} \). The set of points \( \mathcal{H} \) correspond to approximate strict saddle-points, and are points where a descent direction could be identified from the Hessian matrix. Points in \( \mathcal{M} \) are referred to as approximately second-order stationary, since they are indistinguishable from minima based on first and second-order information.

Assumption 5 (Gradient noise in strict saddle-points). Suppose \( w \) is an approximate strict-saddle point, i.e., \( w \in \mathcal{H} \). Introduce the eigendecomposition of the Hessian matrix as \( \nabla^2 J(w) = V \Lambda V^T \) and let the decomposition:
\[ V = [ V^{\geq 0} V^{< 0} ], \quad \Lambda = \begin{bmatrix} \Lambda^{\geq 0} & 0 \\ 0 & \Lambda^{< 0} \end{bmatrix} \]  
(28)
where \( \Lambda^{\geq 0} \geq 0 \) and \( \Lambda^{< 0} < 0 \). Then, we assume that:
\[ \lambda_{\min} \left( (V^{< 0})^T R_i(w) V^{< 0} \right) \geq \sigma_i^2 \]  
(29)
for some $\sigma_i^2 > 0$ and all $w \in H$.

As explained in [8], [9], assumption [5] is similar to the condition in [30], where alternating step-sizes are employed, and ensures that at every strict saddle-point there is a gradient noise component in a descent direction with non-zero probability. It will be leveraged to establish the ability of recursion [4] to escape strict saddle-points. Note that, in contrast to the global dispersive noise assumption [31], condition (29) is only required to hold locally in the vicinity of strict saddle-points. When there is no prior information, condition (29) can always be guaranteed by choosing the update direction to be the perturbed stochastic gradient direction (7) with $v_i \sim N(0, \sigma_i^2I)$, as is done in [7]. Under this construction, the additional perturbation $v_i$ plays a similar role to ridge regularization, which is frequently added to convex optimization problems to ensure strong convexity and hence improved convergence behavior in the absence of a priori strong convexity guarantees. An alternative construction is to add perturbations selectively, when a saddle-point is detected by calculating the gradient norm, resulting in an algorithm similar to [26].

Remark #1: In order to make the notation more compact, and whenever it is clear from context, we shall omit the argument $w_{i-1}$ from the gradient noise term and write instead $s_i \equiv s_i(w_{i-1})$ with the understanding that the gradient noise at time $i$ is a function of the iterate $w_{i-1}$ at time $i - 1$ in addition to the data $x_i$ at time $i$.

Remark #2: The proof technique used to establish the main theorems in the next section are motivated by the arguments used in the works [8], [9] for distributed optimization in non-convex environments. The main difference is that the arguments need to be adjusted to accommodate the more relaxed relative variance bound (17) in the single-agent case.

III. PERFORMANCE ANALYSIS

A. Preliminary Lemmas

Before proceeding with the analysis, we list some preliminary lemmas, which will be used repeatedly throughout.

Lemma 1 (Conditioning [8]). Suppose $w \in \mathbb{R}^M$ is a random variable measurable by $\mathcal{F}$. In other words, $w$ is deterministic conditioned on $\mathcal{F}$ and $E\{w | \mathcal{F}\} = w$. Then,

$$E\{E\{x | \mathcal{F}\} | w \in \mathcal{S}\} = E\{x | w \in \mathcal{S}\}$$

(30)

for any deterministic set $\mathcal{S} \subseteq \mathbb{R}^M$ and random $x \in \mathbb{R}^M$.

Lemma 2 (A limiting result). For $T, \mu, \delta > 0$ and $k \in \mathbb{Z}_+$ with $\mu < \frac{1}{2b}$, we have:

$$\lim_{\mu \to 0} \left( \frac{(1 + \mu\delta)^k + O(\mu^2)}{(1 - \mu\delta)^{k-1}} \right)^{\frac{\mu}{k}} = e^{-7\delta + 2kT\delta} = O(1)$$

(31)

Proof: This lemma is a minor variation of the result in [8]. The adjusted proof is listed in Appendix A.

B. Large-Gradient Regime

Theorem 1. For sufficiently small step-sizes:

$$\mu \leq \frac{2}{\delta (1 + \beta^2)}$$

(32)

and when the gradient at $w_i$ is sufficiently large, i.e., $w_i \in G$, the stochastic gradient recursion (3) yields descent in expectation in one iteration, namely,

$$E\{J(w_{i+1}) | w_i \in G\} \leq E\{J(w_i) | w_i \in G\} - \mu^2c_2$$

(33)

On the other hand, when $w_i \in M$, we can bound the expected ascent:

$$E\{J(w_{i+1}) | w_i \in M\} \leq E\{J(w_i) | w_i \in M\} + \mu^2c_2$$

(34)

Proof: Appendix [A]

Theorem 1 ensures that, whenever $w_i \in G$, i.e., whenever the gradient is sufficiently large, one can expect descent in one iteration. This descent relation is similar to those used to establish convergence to first-order stationary points [11]. In fact, repeatedly applying (33) would allow us to conclude that $w_i$ must eventually reach $G^C$ with high probability, as long as $J(\cdot)$ is bounded from below. In contrast to strongly convex optimization however, where a small gradient norm always implies vicinity to the global minimizer, first-order stationary points can be arbitrarily far from a local minimum in non-convex surfaces. For this reason, we will proceed to study the behavior around strict-saddle points in the sequel.

C. Escape from Saddle-Points

Beginning at a strict saddle-point $w_i \in H$ and for any $j \geq 0$, we have from (4):

$$w_{i+j+1} = w_{i+j} - \mu \nabla J(w_{i+j}) - \mu s_{i+j+1}(w_{i+j})$$

(35)

Subtracting this relation from $w_i$, we find:

$$w_i - w_{i+j+1} = w_i - w_{i+j} + \mu \nabla J(w_{i+j}) + \mu s_{i+j+1}(w_{i+j})$$

(36)

We shall study the evolution of the deviation $w_i - w_{i+j+1}$ over several iterations $j \geq 0$. For brevity, we define:

$$\bar{w}_{j+1} \equiv w_i - w_{i+j+1}$$

(37)

so that (36) becomes:

$$\bar{w}_{j+1} = \bar{w}_j + \mu \nabla J(w_{i+j}) + \mu s_{i+j+1}(w_{i+j})$$

(38)

From the mean-value theorem we find [4]:

$$\nabla J(w_{i+j}) - \nabla J(w_i) = H_{i+j}(w_{i+j} - w_i) - H_{i+j} \bar{w}_j$$

(39)

where

$$H_{i+j} \equiv \int_0^1 \nabla^2 J((1 - t) w_{i+j} + t w_i) dt$$

(40)

so that (38) can be reformulated to:

$$\bar{w}_{j+1} = (I - \mu H_{i+j}) \bar{w}_j + \mu \nabla J(w_i) + \mu s_{i+j+1}(w_{i+j})$$

(41)
In a manner similar to \([4], [6], [33]\), we replace the random and time-varying matrix \(H_{i+j}\) by the Hessian matrix \(\nabla^2 J(w_i)\) evaluated at the starting point \(i\). This substitution obviously leads to an approximate recursion in place of \((41)\); we shall denote its state vector by \(\tilde{w}_{j+1}\) instead of \(w_{j+1}\), as seen below in \((42)\). The point is that while the Hessian \(\nabla^2 J(w_i)\) is random and depends on the time instance \(i\), it becomes deterministic and constant when conditioning on \(\mathcal{F}_i\) and iterating over \(j \geq 0\). We thus arrive at the following recursion, which we shall refer to as the short-term model: \[
abla^2 J(w_j) + \mu \nabla J(w_j) + \mu s_{i+j+1}(w_{i+j})
abla^2 J(w_{i+j})
abla^2 J(w_{i+j}),
\] where \(
abla^2 J(w_{i+j}) \triangleq w_i - w_{i+j+1}\) \((42)\).

The fact that the driving matrix \(I - \mu \nabla^2 J(w_i)\) is constant for all \(j \geq 0\) ensures that \((42)\) is a more tractable recursion than \((41)\). In order for this model to be useful, however, we need to ensure that the function \(J(w_i)\) evaluated at the iterate of the short-term model carries sufficient information about the actual recursion of interest, i.e., \(J(w_{i+j})\). We begin by establishing a set of deviation bounds over a finite time horizon. These ensure that the iterates \(w'_{i+j}\) and \(w_{i+j}\) remain close for a bounded number of iterations, which will allow us to relate \(J(w_{i+j})\) and \(J(w_{i+j})\) further below.

**Lemma 3 (Deviation bounds).** The following quantities are conditionally bounded:

\[
\mathbb{E}\left\{\left\|\tilde{w}_j\right\|^2 \mid w_i \in \mathcal{H}\right\} \leq O(\mu) \tag{44}
\]

\[
\mathbb{E}\left\{\left\|\tilde{w}_j\right\|^3 \mid w_i \in \mathcal{H}\right\} \leq O(\mu^{3/2}) \tag{45}
\]

\[
\mathbb{E}\left\{\left\|\tilde{w}_j\right\|^4 \mid w_i \in \mathcal{H}\right\} \leq O(\mu^2) \tag{46}
\]

\[
\mathbb{E}\left\{\left\|\tilde{w}_j - \tilde{w}_j'\right\|^2 \mid w_i \in \mathcal{H}\right\} \leq O(\mu^2) \tag{47}
\]

\[
\mathbb{E}\left\{\left\|\tilde{w}_j'\right\|^2 \mid w_i \in \mathcal{H}\right\} \leq O(\mu) \tag{48}
\]

for \(j \leq T\), where \(T\) denotes an arbitrary constant that is independent of the step-size \(\mu\).

**Proof:** Appendix C.

These deviation bounds, along with the smoothness conditions on \(J(\cdot)\) allow us to establish the following corollary.

**Corollary 1 (Short-term model accuracy).** Beginning at \(w_i \in \mathcal{H}\), the short-term model is accurate over a finite horizon \(j \leq T\), i.e.,

\[
\mathbb{E}\{J(w_{i+j}) \mid w_i \in \mathcal{H}\} \leq \mathbb{E}\{J(w_{i+j}) \mid w_i \in \mathcal{H}\} + O(\mu^{3/2}) \tag{49}
\]

for \(j \leq T\), where \(T\) denotes an arbitrary constant that is independent of the step-size \(\mu\).

**Proof:** Appendix D.

We conclude that \(J(\cdot)\) evaluated at the true iterate \(w_{i+j}\) is upper bounded by \(J(\cdot)\) evaluated at the short-term model \(w'_{i+j}\) (up to an approximation error \(O(\mu^{3/2})\) that will turn out to be negligible for small step-sizes), so long as both recursions are initialized at strict-saddle points \(w_i \in \mathcal{H}\).

**Theorem 2 (Descent through strict saddle-points).** Beginning at a strict saddle-point \(w_i \in \mathcal{H}\) and iterating for \(i^*\) iterations after \(i\) with

\[
i^* = \frac{\log \left(2M \frac{\sigma^2}{\sigma_2^2} + 1 + O(\mu)\right)}{\log(1 + 2\mu\tau)} \leq O\left(\frac{1}{\mu\tau}\right) \tag{50}\]

guarantees

\[
\mathbb{E}\{J(w_{i+i^*}) \mid w_i \in \mathcal{H}\} \leq \mathbb{E}\{J(w_{i}) \mid w_i \in \mathcal{H}\} - \frac{\mu}{2} M \sigma^2 + o(\mu) \tag{51}\]

**Proof:** Appendix E.

We conclude that when \(w_i\) reaches an approximately strict-saddle points in \(\mathcal{H}\), where the gradient norm alone is no longer sufficient to guarantee descent in a single iteration, we can nevertheless guarantee descent after \(O(1/\mu)\) iterations. Recall that Theorem 1 guarantees descent for points \(\mathcal{G}\). As such, Theorems 1 and 2 together guarantee (expected) descent whenever \(w_i \not\in \mathcal{M}\) and, as long as \(J(\cdot)\) is bounded from below, ensure that \(w_i\) must eventually reach a point in \(\mathcal{M}\). This argument is formalized in the final theorem.

**Theorem 3.** Suppose \(J(w) \geq J^0\). Then, for sufficiently small step-sizes \(\mu\), we have with probability \(1 - \gamma\), that \(w_{i^*} \in \mathcal{M}\), i.e., \(\|\nabla J(w_{i^*})\|^2 \leq O(\mu)\) and \(\lambda_{\min}(\nabla^2 J(w_{i^*})) \geq -\gamma\) in at most \(i^*\) iterations, where

\[
i^* \leq \frac{(J(w_0) - J^0)}{\mu^2 c^2 \pi} i^* \tag{52}\]

and \(i^*\) denotes the escape time from Theorem 2.

**Proof:** Appendix F.

**IV. Simulation Results**

In this section, we consider a simple example, arising from a single-hidden-layer neural network with a linear hidden layer and a logistic activation function leading into the output layer. The cross-entropy loss for such a structure can be simplified to an equivalent logistic loss \([9]\):

\[
Q(w_1, W_2; \gamma, h) = \log \left(1 + e^{-\gamma w_1 W_2 h}\right) \tag{53}\]

The regularized learning problem can then be formulated as:

\[
J(w_1, W_2) = \mathbb{E}Q(w_1, W_2; \gamma, h) + \frac{\rho}{2} ||w_1||^2 + \frac{\rho}{2} ||W_2||_F^2 \tag{54}\]

The cost surface is depicted in Fig. 11. The cost \(J(\cdot)\) has two local minima in the positive and negative quadrants, respectively, and a single strict saddle-point at \(w_1 = W_2 = 0\). We initialize \(w_{0} = \text{col}\{-0.5, 0.5\}\) and compare the direct stochastic gradient descent implementation \([5]\) with:

\[
\nabla J(w_1, W_2) \triangleq \nabla Q(w_1, W_2; \gamma, h) + s \cdot \text{col}\{1, 1\} \tag{55}\]

where \(s \sim \mathcal{N}(0, 1)\) and the direction \(\text{col}\{1, 1\}\) corresponds to the local descent direction at the strict saddle-point \(w_1 = W_2 = 0\). The particular choice of the direction is informed
The symmetric nature of the loss and initialization result in an equal probability of escaping towards the local minimum in the positive or negative quadrant.

Since the logarithm is continuous over $\mathbb{R}_+$, we have:

\[
\log \left( \lim_{\mu \to 0} \left( \frac{(1 + \mu \delta)^k + O(\mu^2)}{(1 - \mu \delta)^{k-1}} \right)^{\frac{T}{\mu}} \right) \\
= \lim_{\mu \to 0} \log \left( \frac{(1 + \mu \delta)^k + O(\mu^2)}{(1 - \mu \delta)^{k-1}} \right)^{\frac{T}{\mu}} \\
= \lim_{\mu \to 0} \frac{T}{\mu} \left( \log \left( (1 + \mu \delta)^k + O(\mu^2) \right) - (k - 1) \log (1 - \mu \delta) \right) \\
= \lim_{\mu \to 0} \frac{T}{\mu} \left( (k \log ((1 + \mu \delta)) - (k - 1) \log (1 - \mu \delta) \right) \\
= kT \lim_{\mu \to 0} \frac{\log (1 + \mu \delta)}{\mu} - (k - 1)T \lim_{\mu \to 0} \frac{\log (1 - \mu \delta)}{\mu} \\
= (57)
\]

We examine the fraction inside the limit more closely. Since both the numerator and denominator of the fraction approach zero as $\mu \to 0$, we apply L’Hôpital’s rule:

\[
\lim_{\mu \to 0} \frac{\log (1 + \mu \delta)}{\mu} = \lim_{\mu \to 0} \frac{\pm \delta}{1 \pm \mu \delta} = \pm \delta \\
= (58)
\]

Hence, we find:

\[
\lim_{\mu \to 0} \left( \frac{(1 + \mu \delta)^k + O(\mu^2)}{(1 - \mu \delta)^{k-1}} \right)^{\frac{T}{\mu}} = e^{kT \delta + (k-1)T \delta} = e^{-T \delta + 2kT \delta} \\
= (59)
\]

**APPENDIX B**

**PROOF OF LEMMA 1**

Since $J(\cdot)$ has $\delta$-Lipschitz gradients:

\[
J(w_{i+1}) \leq J(w_i) + \nabla J(w_i)^T (w_{i+1} - w_i) + \frac{\delta}{2} \|w_{i+1} - w_i\|^2 \\
= (60)
\]

From (3), we find:

\[
J(w_{i+1}) \\
\leq J(w_i) + \nabla J(w_i)^T \left( -\nabla J(w_i) \right) + \frac{\delta}{2} \|\nabla J(w_i)\|^2 \\
\leq J(w_i) - \mu \nabla J(w_i)^T \nabla J(w_i) - \mu \nabla J(w_i)^T s_{i+1}(w_i) + \frac{\mu^2 \delta}{2} \|\nabla J(w_i) + s_{i+1}(w_i)\|^2 \\
= (61)
\]

Under conditional expectation, we have:

\[
E \{ J(w_{i+1}) \vert \mathcal{F}_i \} \\
\leq J(w_i) - \mu \|\nabla J(w_i)\|^2 - \mu \nabla J(w_i)^T E \{ s_{i+1}(w_i) \vert \mathcal{F}_i \} + \frac{\mu^2 \delta}{2} E \{ \|\nabla J(w_i) + s_{i+1}(w_i)\|^2 \vert \mathcal{F}_i \} \\
= (a) J(w_i) - \mu \left( 1 - \mu \frac{\delta}{2} (1 + \beta^2) \right) \|\nabla J(w_i)\|^2 + \mu^2 \frac{\delta}{2} \sigma^2 \\
= (62)
\]
where $(a)$ follows from (26). Taking expectations conditioned on $w_i \in \mathcal{G}$, we find:

\[
E \{ J(w_{i+1}) | w_i \in \mathcal{G} \} \\
\leq E \{ J(w_i) | w_i \in \mathcal{G} \} - \mu c_1 E \left\{ \| \nabla J(w_i) \|^2 | w_i \in \mathcal{G} \right\} + \mu^2 c_2 \\
\leq E \{ J(w_i) | w_i \in \mathcal{G} \} - \mu c_1 \cdot \frac{c_2}{c_1} \left( 1 + \frac{1}{\pi} \right) + \mu^2 c_2 \\
= E \{ J(w_i) | w_i \in \mathcal{G} \} - \mu^2 c_2 \frac{1}{\pi} \\
\quad \text{(63)}
\]

On the other hand, starting from (62) and taking expectations conditioned on $w_i \in \mathcal{M}$, we have:

\[
E \{ J(w_{i+1}) | w_i \in \mathcal{M} \} \\
\leq E \{ J(w_i) | w_i \in \mathcal{M} \} - \mu c_1 E \left\{ \| \nabla J(w_i) \|^2 | w_i \in \mathcal{M} \right\} + \mu^2 c_2 \\
\quad \text{(a)} \\
\leq E \{ J(w_i) | w_i \in \mathcal{M} \} + \mu^2 c_2 \quad \text{(64)}
\]

where $(a)$ follows since $c_1 = 1 - \mu \frac{\delta}{2} (1 + \beta^2) \geq 0$ whenever $\mu \leq \frac{2}{\delta(1+\beta^2)}$.

**APPENDIX C**

**PROOF OF LEMMA**

We refer to (41). Suppose $j \leq \frac{T}{\mu}$, where $T$ is an arbitrary constant independent of $\mu$. We then have for $j \geq 0$:

\[
E \left\{ \left\| \tilde{w}_{j+1} \right\|^2 | F_{i+j} \right\} \\
\leq E \left\{ \left\| (I - \mu H_{i+j}) \tilde{w}_j + \mu \nabla J(w_i) + \mu s_{i+j+1} \right\|^2 | F_{i+j} \right\} \\
= \frac{1}{1 - \mu \delta} \left\| (I - \mu H_{i+j}) \tilde{w}_j \right\|^2 + \frac{\mu}{\delta} \| \nabla J(w_i) \|^2 \\
+ \mu^2 E \left\{ \| s_{i+j+1} \|^2 | F_{i+j} \right\} \\
\quad \text{(a)} \\
\leq \frac{(1 + \mu \delta)^2}{1 - \mu \delta} \left\| \tilde{w}_j \right\|^2 + \frac{\mu}{\delta} \| \nabla J(w_i) \|^2 \\
+ \mu^2 E \left\{ \| s_{i+j+1} \|^2 | F_{i+j} \right\} \\
\quad \text{(b)}
\]

where $(a)$ follows since $\frac{1}{1 - \mu \delta} = 1 - \mu \frac{\delta}{2} (1 + \beta^2) \geq 0$ when $\mu \leq \frac{2}{\delta(1+\beta^2)}$.

\[
\leq \frac{(1 + \mu \delta)^2}{1 - \mu \delta} \left\| \tilde{w}_j \right\|^2 + \frac{\mu}{\delta} \| \nabla J(w_i) \|^2 \quad \text{(65)}
\]

where $(a)$ follows from the conditional zero-mean property of the gradient noise term in Assumption \[\text{3 (b)}\]

\[
\| a + b \|^2 \leq \frac{1}{\alpha} \| a \|^2 + \frac{1}{\alpha - 1} \| b \|^2 \quad \text{(66)}
\]

with $\alpha = \mu \delta < 1$ and $(c)$ follows from the sub-multiplicative property of norms along with $-\delta I \leq \nabla^2 J(w_i) \leq \delta I$, which follows from the Lipschitz gradient condition in Assumption \[\text{[1]}\].

We can now take expectations over $w_i \in \mathcal{H}$ to obtain:

\[
E \left\{ \left\| \tilde{w}_{j+1} \right\|^2 | w_i \in \mathcal{H} \right\} \\
\leq \frac{(1 + \mu \delta)^2 + O(\mu^2)}{1 - \mu \delta} E \left\{ \left\| \tilde{w}_j \right\|^2 | w_i \in \mathcal{H} \right\} + O(\mu^2) \\
\quad \text{(a)} \\
\leq \frac{(1 + \mu \delta)^2 + O(\mu^2)}{1 - \mu \delta} E \left\{ \left\| \tilde{w}_j \right\|^2 | w_i \in \mathcal{H} \right\} + O(\mu^2) \quad \text{(67)}
\]

where $(a)$ follows from the definition of the set $\mathcal{H}$ \[\text{[24]}\]. Note that, at time $i = 0$, we have:

\[
\tilde{w}_0 = w_i - w_{i+0} = 0 \\
\quad \text{(68)}
\]

and hence the initial deviation is zero, by definition. Iterating, starting at $j = 0$ yields:

\[
E \left\{ \left\| \tilde{w}_j \right\|^2 | w_i \in \mathcal{H} \right\} \\
\leq \sum_{n=0}^{j-1} \frac{(1 + \mu \delta)^2 + O(\mu^2)}{1 - \mu \delta} \| w_i \|^n \cdot O(\mu^2) \\
= \frac{\left( (1 + \mu \delta)^2 + O(\mu^2) \right) \frac{1}{1 - \mu \delta} \| w_i \|^j}{1 - (1 + \mu \delta)^2 + O(\mu^2)} \\
= \frac{\left( (1 + \mu \delta)^2 + O(\mu^2) \right) \frac{1}{1 - \mu \delta} \| w_i \|^j - 1 \left( 1 - (1 + \mu \delta)^2 \right) O(\mu^2)}{1 - (1 + \mu \delta)^2 + O(\mu^2)} \\
= \frac{\left( (1 + \mu \delta)^2 + O(\mu^2) \right) \frac{1}{1 - \mu \delta} \| w_i \|^j - 1 \left( 1 - (1 + \mu \delta)^2 \right) O(\mu)}{3 \delta + \mu \delta^2} \\
= O(\mu) \quad \text{(69)}
\]

where the last line follows from Lemma \[\text{2, after noting that:} \]

\[
\frac{\left( (1 + \mu \delta)^2 + O(\mu^2) \right) \frac{1}{1 - \mu \delta} \| w_i \|^j - 1 \left( 1 - (1 + \mu \delta)^2 \right) O(\mu)}{3 \delta + \mu \delta^2} \\
\leq \frac{\left( (1 + \mu \delta)^2 + O(\mu^2) \right) \frac{1}{1 - \mu \delta} \| w_i \|^j - 1 \left( 1 - (1 + \mu \delta)^2 \right) O(\mu)}{3 \delta + \mu \delta^2}
\]

\[
\quad \text{(69)}
\]
\[
\left( \frac{(1+\mu\delta)^2 + O(\mu^2)}{1-\mu\delta} \right)^2 
\leq \frac{2}{\delta^2}
\]  
(70)

This establishes (44). We proceed to establish a bound on the fourth-order moment. Using the inequality [4]:
\[
\|a+b\|^4 \leq \|a\|^4 + 3\|b\|^4 + 8\|a\|^2\|b\|^2 + 4\|a\|^2\|b\|^2 + 4\|a\|^2\langle a^T b \rangle
\]  
(71)

we have:
\[
E \left\{ \left\| \tilde{w}^i_{j+1} \right\|^4 | F_{i+j} \right\} 
\leq \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^4 
+ 3\mu^4 E \left\{ \| s_{i+j+1} \|^4 | F_{i+j} \right\} 
+ 8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 
\times E \left\{ \| s_{i+j+1} \|^2 | F_{i+j} \right\} 
+ 4\mu \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 
\times \left( (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right)^T 
\times (E \{ s_{i+j+1} | F_{i+j} \}) 
\leq \left( (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right)^4 
+ 3\mu^4 E \left\{ \| s_{i+j+1} \|^4 | F_{i+j} \right\} 
+ 8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 
\times E \left\{ \| s_{i+j+1} \|^2 | F_{i+j} \right\} 
\leq \left( (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right)^4 
+ 3\mu^4 \left\{ \| \nabla J (w_i) \|^4 + \sigma^2 \right\} 
+ 8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 
\times \left\{ \| \nabla J (w_i) \|^2 + \sigma^2 \right\}
\]  
(72)

where in step (a) we dropped cross-terms due to the conditional zero-mean property of the gradient noise in Assumption K. Step (b) follows from the fourth-order conditions on the gradient noise in Assumption [5]. We shall bound each term one by one. From Jensen’s inequality, we find for 0 < \alpha < 1:
\[
\|a+b\|^4 = \frac{1}{\alpha^3} \|a\|^4 + \frac{1}{(1-\alpha)^3} \|b\|^4
\]  
(73)

and hence for the first term on the right-hand side of (72) with \alpha = 1 - \mu\delta and 0 < \mu < \frac{1}{\delta^4}:
\[
\left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^4 
\leq \left( \frac{(1+\mu\delta)^2 + O(\mu^2)}{1-\mu\delta} \right)^2 \| \tilde{w}^j_{i+1} \|^4 + \frac{\mu^4}{(1+\mu\delta)^3} \| \nabla J (w_i) \|^4 
= \left( \frac{(1+\mu\delta)^2 + O(\mu^2)}{1-\mu\delta} \right)^2 \| \tilde{w}^j_{i+1} \|^4 + O(\mu) \| \nabla J (w_i) \|^4
\]  
(74)

After taking expectations conditioned on \(w_i \in H\), we find:
\[
E \left\{ \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^4 | w_i \in H \right\} 
\leq \left( \frac{(1+\mu\delta)^2 + O(\mu^2)}{1-\mu\delta} \right)^2 \| \tilde{w}^j_{i+1} \|^4 + O(\mu^4)
\]  
(75)

For the second term we have, again from (73) with \alpha = \frac{1}{\delta^4}:
\[
3\mu^4 \left\{ \| \nabla J (w_i) \|^4 + \sigma^4 \right\} 
= 3\mu^4 \left\{ \| \nabla J (w_i) + \nabla J (w_{i+j}) - \nabla J (w_i) \|^4 + \sigma^4 \right\} 
\leq 3\mu^4 \left\{ 8 \| \nabla J (w_i) \|^4 + 8 \| \nabla J (w_{i+j}) - \nabla J (w_i) \|^4 + \sigma^4 \right\} 
\leq 3\mu^4 \left\{ 8 \| \nabla J (w_i) \|^4 + 8 \delta^4 \| \tilde{w}^j_{i+1} \|^4 + \sigma^4 \right\}
= O(\mu^4) \| \nabla J (w_i) \|^4 + O(\mu^4) \| \tilde{w}^j_{i+1} \|^4 + O(\mu^4)
\]  
(76)

After taking expectations over \(w_i \in H\) we have:
\[
E \left\{ 3\mu^4 \left\{ \| \nabla J (w_{i+j}) \|^4 + \sigma^4 \right\} | w_i \in H \right\} 
\leq O(\mu^4) E \left\{ \| \nabla J (w_i) \|^4 | w_i \in H \right\} 
+ O(\mu^4) E \left\{ \| \tilde{w}^j_{i+1} \|^4 | w_i \in H \right\} + O(\mu^4)
\]  
(77)

For the last term, we have:
\[
8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 \left\{ \| \nabla J (w_{i+j}) \|^2 + \sigma^2 \right\} 
= 8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 \left\| \nabla J (w_{i+j}) \right\|^2 
+ 8\mu^2 \left\| (I - \mu H_{i+j}) \tilde{w}^j_{i+1} + \mu \nabla J (w_i) \right\|^2 \left\| \nabla J (w_{i+j}) \right\|^2 
\leq 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_{i+j}) \right\|^2 
+ 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_{i+j}) \right\|^2 
= 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_{i+j}) \right\|^2 
+ 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_i) \right\|^2 
\leq 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_{i+j}) \right\|^2 
+ 8\mu^2 \left( \frac{(1+\mu\delta)^2}{1-\mu\delta} \| \tilde{w}^j_{i+1} \|^2 + \frac{\mu}{\delta} \| \nabla J (w_i) \|^2 \right) \left\| \nabla J (w_i) \right\|^2 
\]  
(78)
\[ \leq 8\mu^2 \left( \frac{(1 + \mu\delta)^2}{1 - \mu\delta} \right) \left\| \tilde{w}_j^i \right\|^2 + \frac{4\mu}{\delta} \left\| \nabla J(w_i) \right\|^2 \]
\[ \times \left( 2\left\| \nabla J(w_i) \right\|^2 + 2\delta^2 \left\| \tilde{w}_j^i \right\|^2 \right) \]
\[ + 8\mu^2 \left( \frac{(1 + \mu\delta)^2}{1 - \mu\delta} \right) \left\| \tilde{w}_j^i \right\|^2 + \frac{4\mu}{\delta} \left( \nabla J(w_i) \right)^2 \right) \sigma^2 \]
\[ = O(\mu^2) \left\| \tilde{w}_j^i \right\|^4 + O(\mu^3) \left\| \nabla J(w_i) \right\|^4 \]
\[ + O(\mu^2) \left\| \nabla J(w_i) \right\|^2 \left\| \tilde{w}_j^i \right\|^2 + O(\mu^2) \left\| \tilde{w}_j^i \right\|^2 \]
\[ + O(\mu^3) \left\| \nabla J(w_i) \right\|^2 \right) \tag{78} \]

After taking conditional expectations:
\[ E \left\{ \left( \frac{(1 + \mu\delta)^2}{1 - \mu\delta} \right) \left\| \tilde{w}_j^i \right\|^2 + \frac{4\mu}{\delta} \left\| \nabla J(w_i) \right\|^2 \right\} \]
\[ \times \left( \left\| \nabla J(w_i) \right\|^2 + \sigma^2 \right) \left\| w_i \in \mathcal{H} \right\} \]
\[ \leq O(\mu^2) E \left\{ \left\| \tilde{w}_j^i \right\|^2 \right| w_i \in \mathcal{H} \right\} \]
\[ + O(\mu^3) E \left\{ \left\| \nabla J(w_i) \right\|^4 \right| w_i \in \mathcal{H} \right\} \]
\[ + O(\mu^2) E \left\{ \left\| \nabla J(w_i) \right\|^2 \left\| \tilde{w}_j^i \right\|^2 \right| w_i \in \mathcal{H} \right\} \]
\[ + O(\mu^3) E \left\{ \left\| \nabla J(w_i) \right\|^2 \right| w_i \in \mathcal{H} \right\} \]
\[ \leq O(\mu^2) E \left\{ \left\| \tilde{w}_j^i \right\|^2 \right| w_i \in \mathcal{H} \right\} + O(\mu^3) \cdot O(\mu^2) \]
\[ + O(\mu^2) E \left\{ \left( \frac{(1 + \mu\delta)^2}{1 - \mu\delta} \right) \left\| \tilde{w}_j^i \right\|^2 \right| w_i \in \mathcal{H} \right\} + O(\mu^2) \cdot O(\mu) \]
\[ + O(\mu^3) \cdot O(\mu) \]
\[ \leq O(\mu^2) E \left\{ \left\| \tilde{w}_j^i \right\|^2 \right| w_i \in \mathcal{H} \right\} + O(\mu^3) \tag{79} \]

Returning to (72), after taking expectations over \( w_i \in \mathcal{H} \) on both sides and grouping terms we find:
\[ E \left\{ \left\| \tilde{w}_j^i \right\|^4 \right| w_i \in \mathcal{H} \right\} \]
\[ \leq \left( \frac{(1 + \mu\delta)^4 + O(\mu^2)}{(1 - \mu\delta)^3} \right) E \left\{ \left\| \tilde{w}_j^i \right\|^4 \right| w_i \in \mathcal{H} \right\} + O(\mu^3) \tag{80} \]

Recall again that \( \tilde{w}_j^i = 0 \) and therefore iterating yields:
\[ E \left\{ \left\| \tilde{w}_j^i \right\|^4 \right| w_i \in \mathcal{H} \right\} \]
\[ \leq \left( \sum_{n=0}^{N} \left( \frac{(1 + \mu\delta)^4 + O(\mu^2)}{(1 - \mu\delta)^3} \right)^n \right) O(\mu^3) \]
\[ 1 \quad \frac{(1 + \mu\delta)^4 + O(\mu^2)}{(1 - \mu\delta)^3} O(\mu^3) \]
\[ = \frac{(1 + \mu\delta)^4 + O(\mu^2)}{(1 - \mu\delta)^3} O(\mu^3) \]
\[ = \frac{(1 + \mu\delta)^4 + O(\mu^2) - (1 - \mu\delta)^3 O(\mu^3} \]
where (a) follows from Jensen’s inequality and (b) follows form the Lipschitz Hessian assumption. Returning to (84) and taking norms yields:

\[
\begin{align*}
\| w_{i+j+1} - w'_{i+j+1} \|^2 \\
= \left\| (I - \mu \nabla^2 J(w_i)) (w_{i+j} - w'_{i+j}) \\
+ \mu (H_{i+j} - \nabla^2 J(w_i)) \tilde{w}_j \right\|^2 \\
\leq \frac{1}{1 - \mu \delta} \left[ \| (I - \mu \nabla^2 J(w_i)) (w_{i+j} - w'_{i+j}) \|^2 \\
+ \mu^2 \| (H_{i+j} - \nabla^2 J(w_i)) \tilde{w}_j \|^2 \right] \\
\leq \frac{1}{1 - \mu \delta} \left[ \| (I - \mu \nabla^2 J(w_i)) (w_{i+j} - w'_{i+j}) \|^2 \\
+ \frac{\mu \rho}{\delta} \| (H_{i+j} - \nabla^2 J(w_i)) \tilde{w}_j \|^2 \right] \\
\leq (1 + \mu \delta) \frac{1}{1 - \mu \delta} \left[ \| w_{i+j} - w'_{i+j} \|^2 + \frac{\mu \rho}{\delta} \| \tilde{w}_j \|^4 \right].
\end{align*}
\]

(86)

where (a) again follows from Jensen’s inequality with \( \alpha = 1 - \mu \delta \) and (b) follows from the same inequality with \( \alpha = \frac{1}{2} \). Taking expectations over \( w_i \in \mathcal{H} \) yields:

\[
\begin{align*}
&\mathbb{E}\left\{ \| w_{i+j+1} - w'_{i+j+1} \|^2 | w_i \in \mathcal{H} \right\} \\
\leq \frac{(1 + \mu \delta)^2}{1 - \mu \delta} \mathbb{E}\left\{ \| w_{i+j} - w'_{i+j} \|^2 | w_i \in \mathcal{H} \right\} \\
&+ \frac{\mu \rho}{\delta} \mathbb{E}\left\{ \| \tilde{w}_j \|^4 | w_i \in \mathcal{H} \right\} \\
\leq (1 + \mu \delta) \frac{1}{1 - \mu \delta} \mathbb{E}\left\{ \| w_{i+j} - w'_{i+j} \|^2 + \frac{\mu \rho}{\delta} \| \tilde{w}_j \|^4 \right\}. \\
\end{align*}
\]

(87)

Since both the true and the short-term model are initialized at \( w_i \), we have \( w_{i+0} - w'_{i+0} = 0 \). Iterating and applying the same argument as above leads to:

\[
\mathbb{E}\left\{ \| w_{i+j+1} - w'_{i+j+1} \|^2 \right\} \leq O(\mu^2)
\]

(88)

which is (47).

**APPENDIX D**

**PROOF OF LEMMA 1**

Recall that \( J(\cdot) \) has \( \delta \)-Lipschitz gradients, which implies:

\[
J(w_{i+j}) \leq J(w'_{i+j}) + \nabla J(w'_{i+j}) ^T (w_{i+j} - w'_{i+j}) \\
+ \frac{\delta}{2} \| w_{i+j} - w'_{i+j} \|^2.
\]

(89)

In the vicinity of saddle-points, we can refine the upper bound (89) by taking expectations conditioned on \( w_i \in \mathcal{H} \):

\[
\begin{align*}
&\mathbb{E}\left\{ J(w_{i+j}) | w_i \in \mathcal{H} \right\} \\
\leq &\mathbb{E}\left\{ J(w'_{i+j}) | w_i \in \mathcal{H} \right\} \\
&+ \mathbb{E}\left\{ \nabla J(w'_{i+j}) ^T (w_{i+j} - w'_{i+j}) | w_i \in \mathcal{H} \right\} \\
&+ \frac{\delta}{2} \mathbb{E}\left\{ \| w_{i+j} - w'_{i+j} \|^2 | w_i \in \mathcal{H} \right\} \\
\leq &\mathbb{E}\left\{ J(w'_{i+j}) | w_i \in \mathcal{H} \right\} \\
&+ \mathbb{E}\left\{ \nabla J(w'_{i+j}) ^T (w_{i+j} - w'_{i+j}) | w_i \in \mathcal{H} \right\} \\
&+ \frac{\delta}{2} \mathbb{E}\left\{ \| w_{i+j} - w'_{i+j} \|^2 | w_i \in \mathcal{H} \right\} \\
\leq &\mathbb{E}\left\{ J(w'_{i+j}) | w_i \in \mathcal{H} \right\} \\
&+ \frac{\delta}{2} \mathbb{E}\left\{ \| w_{i+j} - w'_{i+j} \|^2 | w_i \in \mathcal{H} \right\} \\
&+ \frac{\rho}{\delta} \mathbb{E}\left\{ \| \tilde{w}_j \|^4 | w_i \in \mathcal{H} \right\}. \\
\end{align*}
\]

(90)

**APPENDIX E**

**PROOF OF THEOREM 1**

The argument generally mirrors the proof to [9] Theorem 1] after accounting for the relative variance bound [17] by noting that, around first-order stationary points, the relative component \( \beta |\nabla J(w_i)|^4 \) will necessarily be small.

From Corollary 1 we have:

\[
\mathbb{E}\left\{ J(w_{i+j}) | w_i \in \mathcal{H} \right\} \leq \mathbb{E}\left\{ J(w'_{i+j}) | w_i \in \mathcal{H} \right\} + O(\mu^{3/2}),
\]

(92)

so long as \( j \leq \frac{\epsilon}{\mu} \). We can hence proceed by study- ing \( \mathbb{E}\left\{ J(w'_{i+j}) | \mathcal{H} \right\} \) and will add the approximation error \( O(\mu^{3/2}) \) to the end result. From (13) we find:

\[
J(w'_{i+j}) \leq J(w_{i}) - \nabla J(w_{i}) ^T \tilde{w}_j + \frac{\delta}{2} \| w_{i+j} - w'_{i+j} \|^2 \\
+ \frac{\rho}{\delta} \| \tilde{w}_j \|^4.
\]

(93)

We will bound each term appearing on the right-hand side. From (42) we find after conditioning on \( \mathcal{F}_{i+j} \):

\[
\begin{align*}
&\mathbb{E}\left\{ \tilde{w}_{i+j+1} | \mathcal{F}_{i+j} \right\} \\
&= (I - \mu \nabla^2 J(w_i)) \tilde{w}_j + \mu \nabla J(w_i) + \mu \mathbb{E}\left\{ s_{i+j+1} | \mathcal{F}_{i+j} \right\} \\
&= (I - \mu \nabla^2 J(w_i)) \tilde{w}_j + \mu \nabla J(w_i)
\end{align*}
\]

(94)
Note that $\mathcal{F}_{i+j}$ denotes the information captured in $w_{k,j}$ up to time $i + j$, while $\mathcal{F}_i$ denotes the information available up to time $i$. Hence:

$$\mathcal{F}_{i+j} = \mathcal{F}_i \cup \text{filtration } \{w_{k,i+1}, \ldots, w_{k,i+j}\}$$  \hfill (95)

Hence, taking expectation of (94) conditioned on $\mathcal{F}_i$ removes the elements in filtration $\{w_{k,i+1}, \ldots, w_{k,i+j}\}$ contained in $\mathcal{F}_i$ and yields:

$$\mathbb{E}\left\{ \tilde{w}_{i+1}^j | \mathcal{F}_i \right\} = \left( I - \mu \nabla^2 J(w_i) \right) \mathbb{E}\left\{ \tilde{w}_{i}^j | \mathcal{F}_i \right\} + \mu \nabla J(w_i)$$  \hfill (96)

Since $\tilde{w}_0^j = 0$, iterating starting at $j = 0$ yields:

$$\mathbb{E}\left\{ \tilde{w}_{i}^j | \mathcal{F}_i \right\} = \mu \left( \sum_{k=1}^{j} \left( I - \mu \nabla^2 J(w_i) \right)^{k-1} \right) \nabla J(w_i)$$  \hfill (97)

This allows us to bound the linear term appearing in (93) as:

$$- \mathbb{E}\left\{ \nabla J(w_i)^T \tilde{w}_{i}^j | \mathcal{F}_i \right\} = - \nabla J(w_i)^T \mathbb{E}\left\{ \tilde{w}_{i}^j | \mathcal{F}_i \right\}$$

$$= - \mu \nabla J(w_i)^T \left( \sum_{k=1}^{j} \left( I - \mu \nabla^2 J(w_i) \right)^{k-1} \right) \nabla J(w_i)$$

$$= - \mu \nabla J(w_i)^T \left( \sum_{k=1}^{j} \left( I - \mu \nabla^2 J(w_i) \right)^{k-1} \right) \nabla J(w_i)$$  \hfill (98)

To study the quadratic term in (93), we introduce the eigenvalue decomposition of the Hessian around the iterate at time $i$:

$$\nabla^2 J(w_i) \triangleq V_i \Lambda_i V_i^T$$  \hfill (99)

which motivates the transformation:

$$\left\| \tilde{w}_{i+1}^j \right\|_{\nabla^2 J(w_i)}^2 = \left\| \tilde{w}_{i+1}^j \right\|_{V_i \Lambda_i V_i^T}^2$$

$$= \left\| V_i^T \tilde{w}_i - V_i^T \tilde{w}_{i+1}^j \right\|_{\Lambda_i}^2$$

$$= \left\| \tilde{w}_{i+1}^j \right\|_{\Lambda_i}^2$$  \hfill (100)

where we introduced:

$$\tilde{w}_{i+1}^j \triangleq V_i^T \tilde{w}_{i+1}^j$$  \hfill (101)

Under this transformation, recursion (42) is also diagonalized, yielding:

$$\tilde{w}_{i+1}^j \triangleq V_i^T \tilde{w}_i^j$$

$$= V_i^T \left( I - \mu \nabla^2 J(w_i) \right) V_i V_i^T \tilde{w}_i^j$$

$$+ \mu V_i^T \nabla J(w_i) + \mu V_i^T \tilde{w}_{i+1}^j$$

$$= (I - \mu \Lambda_i) \tilde{w}_i^j + \mu \nabla J(w_i) + \mu \tilde{w}_{i+1}^j$$  \hfill (102)

with $\nabla J(w_i) \triangleq V_i^T \nabla J(w_i)$ and $\tilde{w}_{i+1}^j \triangleq V_i^T \tilde{w}_{i+1}^j$. Applying the same transformation to the conditional mean recursion (96), and subtracting the transformed conditional mean on both sides of (102), we find:

$$\tilde{w}_{i+1}^j - \mathbb{E}\left\{ \tilde{w}_{i+1}^j | \mathcal{F}_i \right\}$$

$$= (I - \mu \Lambda_i) \left( \tilde{w}_i^j - \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right) + \mu \tilde{w}_{i+1}^j$$  \hfill (103)

which allows us to cancel the driving term involving the gradient. For brevity, define the (conditionally) centered random variable:

$$\tilde{w}_i^j = \tilde{w}_i^j - \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\}$$  \hfill (104)

so that:

$$\tilde{w}_{i+1}^j = \left( I - \mu \Lambda_i \right) \tilde{w}_i^j + \mu \tilde{w}_{i+1}^j$$  \hfill (105)

Before proceeding, note that we can express:

$$\mathbb{E}\left\{ \left\| \tilde{w}_i^j \right\|_{\nabla^2 J(w_i)}^2 | \mathcal{F}_i \right\}$$

$$= \mathbb{E}\left\{ \left\| \tilde{w}_i^j - \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right\|_{\nabla^2 J(w_i)}^2 | \mathcal{F}_i \right\}$$

$$= \mathbb{E}\left\{ \left\| \tilde{w}_i^j - \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right\|_{\Lambda_i}^2 \right\}$$  \hfill (106)

Hence, we have:

$$\mathbb{E}\left\{ \left\| \tilde{w}_i^j \right\|_{\nabla^2 J(w_i)}^2 | \mathcal{F}_i \right\}$$

$$= \mathbb{E}\left\{ \left\| \tilde{w}_i^j \right\|_{\Lambda_i}^2 \right\}$$  \hfill (107)

In order to make claims about $\mathbb{E}\left\{ \left\| \tilde{w}_i^j \right\|_{\nabla^2 J(w_i)}^2 | \mathcal{F}_i \right\}$, we need to establish a bound on $\left\| \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right\|_{\Lambda_i}^2$. We have:

$$\left\| \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right\|_{\Lambda_i}^2$$

$$\leq \left\| \mathbb{E}\left\{ \tilde{w}_i^j | \mathcal{F}_i \right\} \right\|_{\Lambda_i}^2$$

$$\leq \mu^2 \left( \sum_{k=1}^{j} \left( I - \mu \Lambda_i \right)^{k-1} \right) \nabla J(w_i)$$

$$\leq \mu^2 \nabla J(w_i)^T \left( \sum_{k=1}^{j} \left( I - \mu \Lambda_i \right)^{k-1} \right) \nabla J(w_i)$$

$$\times \left( \sum_{k=1}^{j} \left( I - \mu \Lambda_i \right)^{k-1} \right) \nabla J(w_i)$$  \hfill (108)

We shall order the eigenvalues of $\nabla^2 J(w_i)$, such that its eigendecomposition has a block structure:

$$V_i = \left[ \begin{array}{cc} V_i^{>0} & V_i^{=0} \end{array} \right], \quad \Lambda_i = \left[ \begin{array}{cc} \Lambda_i^{>0} & 0 \\ 0 & \Lambda_i^{<0} \end{array} \right]$$  \hfill (109)

with $\delta I \geq \Lambda_i^{>0} \geq 0$ and $\Lambda_i^{<0} < 0$. Note that since $\nabla^2 J(w_i)$ is random, the decomposition itself is random as well. Nevertheless, it exists with probability one. We also decompose the transformed gradient vector with appropriate dimensions:

$$\nabla J(w_i) = \text{col} \left\{ \nabla J(w_i)^{>0}, \nabla J(w_i)^{<0} \right\}$$  \hfill (110)
We can then decompose \( (108) \):

\[
\| \mathbb{E} \{ \mathbf{w}' \mid \mathcal{F}_i \} \|_{\Lambda_i}^2 = \mu^2 \nabla J(w_i)^T \left( \sum_{k=1}^{j} (I - \mu \Lambda_k)^{k-1} \right) \Lambda_i \\
= \mu^2 \nabla J(w_i)^T \left( \sum_{k=1}^{j} (I - \mu \Lambda_k > 0)^{k-1} \right) \Lambda_i^{> 0} \\
\times \left( \sum_{k=1}^{j} (I - \mu \Lambda_k < 0)^{k-1} \right) \nabla J(w_i)^{< 0} \\
+ \mu^2 \nabla J(w_i)^{< 0} \left( \sum_{k=1}^{j} (I - \mu \Lambda_k < 0)^{k-1} \right) \Lambda_i^{< 0} \\
\times \left( \sum_{k=1}^{j} (I - \mu \Lambda_k < 0)^{k-1} \right) \nabla J(w_i)^{< 0}.
\]

(113)

Comparing (111) to (98), we find that we can bound:

\[
- \mathbb{E} \{ \nabla J(w_i)^T \mathbf{w}' \mid \mathcal{F}_i \} + \| \mathbb{E} \{ \mathbf{w}' \mid \mathcal{F}_i \} \|_{\Lambda_i}^2 \leq 0
\]

(114)

To recap, we can simplify (93) as:

\[
\mathbb{E} \{ J(w_{i+j}) \mid \mathcal{F}_i \} \\
\leq J(w_i) + \frac{1}{2} \mathbb{E} \left\{ \| \mathbf{w}' \|^2 \mid \mathcal{F}_i \right\} + \frac{1}{6} \mathbb{E} \left\{ \| \mathbf{w}' \|^3 \mid \mathcal{F}_i \right\}
\]

(115)

We proceed with the now simplified quadratic term. We square both sides of (105) under an arbitrary diagonal weighting matrix \( \Sigma_i \), deterministic conditioned on \( w_i \) and \( w_{i+j} \), to obtain:

\[
\| \mathbf{w}' \|_{\Sigma_i}^2 = \|(I - \mu \Lambda_i) \mathbf{w}' + \mathbf{s}_{i+j+1} \|_{\Sigma_i}^2 \\
= \|(I - \mu \Lambda_i) \mathbf{w}' + \mathbf{s}_{i+j+1} \|_{\Sigma_i}^2 \\
+ 2 \mu \mathbf{w}'^T (I - \mu \Lambda_i) \mathbf{s}_{i+j+1} \\
+ \mu^2 \text{Tr} \left( V_i \Sigma_i V_i^T R_s (w_i) \right) \\
+ \mu^2 \text{Tr} \left( V_i \Sigma_i V_i^T \left( R_s (w_{i+j}) - R_s (w_i) \right) \right)
\]

(116)

Note that upon conditioning on \( \mathcal{F}_{i+j} \), all elements of the cross-term, aside from \( \mathbf{s}_{i+j+1} \), become deterministic, and as such the term disappears when taking expectations. We obtain:

\[
\mathbb{E} \left\{ \| \mathbf{w}'_{i+j+1} \|^2 \mid \mathcal{F}_{i+j} \right\} \\
\leq \|(I - \mu \Lambda_i) \mathbf{w}'_{i+j} \|_{\Sigma_i}^2 + \mu^2 \mathbb{E} \left\{ \| \mathbf{s}_{i+j+1} \|^2 \mid \mathcal{F}_{i+j} \right\} \\
+ \mu^2 \text{Tr} \left( V_i \Sigma_i V_i^T R_s (w_{i+j}) \right) \\
+ \mu^2 \text{Tr} \left( V_i \Sigma_i V_i^T \left( R_s (w_{i+j}) - R_s (w_i) \right) \right)
\]

(117)

We proceed to bound the last two terms. First, we have:

\[
\text{Tr} \left( V_i \Sigma_i V_i^T \left( R_s (w_{i+j}) - R_s (w_i) \right) \right) \\
\leq \| V_i \Sigma_i V_i^T \| \| R_s (w_{i+j}) - R_s (w_i) \| \\
\leq \rho (\Sigma_i) \beta \| \mathbf{w}'_{i+j} \|^2
\]

(118)

where (a) follows from Cauchy-Schwarz, since \( \text{Tr}(A^T B) \) is an inner product over the space of symmetric matrices, and hence, \( \| \text{Tr}(A^T B) \| \leq \| A \| \| B \| \), and (b) follows from Assumption 4.

For the second term, we have:

\[
\| \mathbf{w}'_{i+j} \|^2_{\Lambda_i, \Sigma_i, \Lambda_i} \leq \rho (\Sigma_i, \Lambda_i) \| \mathbf{w}'_{i+j} \|^2 \\
\leq \delta^2 \rho (\Sigma_i) \| \mathbf{w}'_{i+j} \|^2
\]

(119)
We conclude that

\[
E \left\{ \left\| \bar{w}_{j+1}^i \right\|^2_{\Sigma_j} | \mathcal{F}_i \right\} = E \left\{ \left\| \bar{w}_{j}^i \right\|^2_{\Sigma_j - 2 \mu \Lambda_i \Sigma_j} | \mathcal{F}_i \right\} + \mu^2 \text{Tr} \left( V_i \Sigma_i V_i^T R_i \left( w_i \right) \right) + \mu^2 \rho \left( \Sigma_i \right) E \left\{ q_{i+n} | \mathcal{F}_i \right\} \]

where

\[
q_{i+j} \triangleq \beta_{ij} \left\| \bar{w}_j^i \right\|^2 + \delta^2 \left\| \bar{w}_j^i \right\|^2
\]

For brevity, we define

\[
D \triangleq I - 2 \mu \Lambda_i
\]

\[
Y \triangleq V_i^T R_i \left( w_i \right) V_i
\]

With these substitutions we obtain:

\[
E \left\{ \left\| \bar{w}_{j+1}^i \right\|^2_{\Sigma} | \mathcal{F}_i \right\} = E \left\{ \left\| \bar{w}_j^i \right\|^2_{\Sigma} | \mathcal{F}_i \right\} + \mu^2 \text{Tr} \left( \Sigma_Y \right) + \mu^2 \rho \left( \Sigma_i \right) E \left\{ q_{i+n} | \mathcal{F}_i \right\} \]

At \( j = 0 \), we have \( \bar{w}_0^i = 0 \). Letting \( \Sigma_j = \Lambda_i D^j \), we can iterate to obtain:

\[
E \left\{ \left\| \bar{w}_{j+1}^i \right\|^2_{\Lambda_i} | \mathcal{F}_i \right\} = \mu^2 \sum_{n=0}^j \text{Tr} \left( \Lambda_i D^n Y \right) + \mu^2 \rho \left( \Lambda_i D^n \right) \cdot E \left\{ q_{i+n} | \mathcal{F}_i \right\}
\]

\[
= \mu^2 \text{Tr} \left( \Lambda_i \left( \sum_{n=0}^j D^n \right) Y \right) + \mu^2 \sum_{n=0}^j \rho \left( \Lambda_i D^n \right) \cdot E \left\{ q_{i+n} | \mathcal{F}_i \right\}
\]

since \( \bar{w}_{i+j}^i = \bar{w}_i \) at \( j = 0 \). Our objective is to show that the first term on the right-hand side yields sufficient descent (i.e., will be sufficiently negative), while the second term is small enough to be negligible. To this end, we again make use of the structured eigendecomposition [109]. We have:

\[
\mu^2 \text{Tr} \left( \Lambda_i \left( \sum_{n=0}^j D^n \right) V_i^T R_i \left( w_i \right) V_i \right)
\]

\[
\overset{(a)}{=} \mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right) \times \left( V_i^{0^2} \right)^T R_i \left( w_i \right) V_i^{0^2}
\]

\[
+ \mu^2 \text{Tr} \left( \Lambda_i < 0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i < 0 \right)^n \right) \times \left( V_i^{0^2} \right)^T R_i \left( w_i \right) V_i^{0^2}
\]

\[
\overset{(b)}{=} \mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right)
\]

\[
\leq \mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right) \leq \mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right)
\]

where in (a) we decomposed the trace since \( \Lambda_i \left( \sum_{n=0}^j D^n \right) \) is a diagonal matrix, (b) applies \(- \Lambda_i < 0 = \Lambda_i < 0 \), where in (a) we decomposed the trace since \( \Lambda_i \left( \sum_{n=0}^j D^n \right) \) is a diagonal matrix and applied \(- \Lambda_i < 0 = \Lambda_i < 0 \). Step (c) follows from \( \text{Tr}(A) \lambda_{\min}(B) \leq \text{Tr}(AB) \leq \text{Tr}(A) \lambda_{\max}(B) \) which holds for \( A = A^T, B = B^T \geq 0 \), and (c) follows from the bounded covariance property [21] and Assumption 5. For the positive term, we have:

\[
\mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right) \left( \beta^2 \| \nabla J(\omega_i) \|^2 + \sigma^2 \right)
\]

\[
\overset{(c)}{=} \mu^2 \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right) \left( \beta^2 \| \nabla J(\omega_i) \|^2 + \sigma^2 \right)
\]

\[
\overset{(d)}{=} \mu^2 \text{Tr} \left( \Lambda_i^0 \left( 2 \mu \Lambda_i^0 \right)^{-1} \right) \left( \beta^2 \| \nabla J(\omega_i) \|^2 + \sigma^2 \right)
\]

where (a) follows since \( I - 2 \mu \Lambda_i^0 \) is elementwise non-negative, (b) follows from \( \sum_{n=0}^j A^n = \left( I - A \right)^{-1} \) and (c) follows since \( \nabla^2 J(\omega_i) \) is of dimension \( M \). Hence, under expectation:

\[
\mu^2 E \left\{ \text{Tr} \left( \Lambda_i^0 \sum_{n=0}^j \left( I - 2 \mu \Lambda_i^0 \right)^n \right) \right\} \times \left( \beta^2 \| \nabla J(\omega_i) \|^2 + \sigma^2 \right) \left| \left. w_i \in \mathcal{H} \right\{ \right\}
\]

\[
\leq \mu^2 M \left( \beta^2 \mathbb{E} \left\{ \| \nabla J(\omega_i) \|^2 \left| \omega_i \in \mathcal{H} \right\{ \right) + \sigma^2 \right)\]

\[
\leq \mu^2 M \left( \beta^2 \cdot O(\mu) + \sigma^2 \right) = \mu^2 M \sigma^2 + O(\mu^2)
\]
For the negative term, we have under expectation conditioned on $w_i \in \mathcal{H}$:

$$
\text{E} \left\{ \text{Tr} \left( (-\Lambda_i) \left( \sum_{n=0}^{j} (I - 2\mu \Lambda_i^{-1})^n \right) \right) \sigma_i^2 \mid w_i \in \mathcal{H} \right\}
$$

$$(a) \geq E \left\{ \tau \left( \sum_{n=0}^{j} (1 + 2\mu \tau)^n \right) \sigma_i^2 \mid w_i \in \mathcal{H} \right\}
$$

$$(b) = \tau \left( \sum_{n=0}^{j} (1 + 2\mu \tau)^n \right) \sigma_i^2 \leq \frac{1}{\mu^2} \left( (1 + 2\mu \tau)^{j+1} - 1 \right) \sigma_i^2 \quad (129)
$$

Step (a) makes use of the fact that $(-\Lambda_i^{-1}) \left( \sum_{n=0}^{j} (I - 2\mu \Lambda_i^{-1})^n \right)$ is a diagonal matrix, where all elements are non-negative. Hence, its trace can be bounded by any of its diagonal elements:

$$
\text{Tr} \left( (-\Lambda_i^{-1}) \left( \sum_{n=0}^{j} (I - 2\mu \Lambda_i^{-1})^n \right) \right) \geq \tau \left( \sum_{n=0}^{j} (1 + 2\mu \tau)^n \right)
$$

$$\geq \tau \left( \sum_{n=0}^{j} (1 + 2\mu \tau)^n \right) \sigma_i^2 \quad (130)
$$

In (b) we dropped the expectation since the expression is no longer random, and (c) is the result of a geometric series. We return to the full expression (126) and find:

$$
\mu^2 E \left\{ \text{Tr} \left( \Lambda_i \left( \sum_{n=0}^{j} D^n \right) V_i^T R_s (w_i) V_i \right) \mid w_i \in \mathcal{H} \right\}
$$

$$\leq \frac{\mu}{2} M \sigma^2 + O(\mu^2) - \frac{\mu}{2} \left( (1 + 2\mu \tau)^{j+1} - 1 \right) \sigma_i^2 \leq -\frac{\mu}{2} M \sigma^2 \quad (a)
$$

$$(b) \leq -\frac{\mu}{2} M \sigma^2 \quad (131)
$$

where (a) holds if, and only if,

$$
\frac{\mu}{2} M \sigma^2 + O(\mu) - \frac{\mu}{2} \left( (1 + 2\mu \tau)^{j+1} - 1 \right) \sigma_i^2 \leq -\frac{\mu}{2} M \sigma^2
$$

$$\iff 2M \sigma^2 \sigma_i^2 + O(\mu) + 1 \leq (1 + 2\mu \tau)^{j+1}
$$

$$\iff \log \left( 2M \sigma^2 \sigma_i^2 + O(\mu) \right) \leq (j + 1) \log (1 + 2\mu \tau)
$$

$$\iff \frac{\log \left( 2M \sigma^2 \sigma_i^2 + O(\mu) \right)}{\log (1 + 2\mu \tau)} \leq j + 1
$$

$$\iff \frac{\log \left( 2M \sigma^2 \sigma_i^2 + O(\mu) \right)}{O(\mu \tau)} \leq j + 1 \quad (132)
$$

Applying this relation to (125) and taking expectations over $w_i \in \mathcal{H}$, we obtain:

$$
E \left\{ \left\| \bar{w}_{i+1} \right\|^2 \mid w_i \in \mathcal{H} \right\}
$$

$$\leq \mu^2 \sum_{n=0}^{i^*} E \left\{ \text{Tr} \left( \Lambda_i D^n \right) \cdot E \left\{ q_{i+n} \mid \mathcal{F}_i \right\} \mid w_i \in \mathcal{H} \right\}
$$

$$- \frac{\mu}{2} M \sigma^2 \quad (134)
$$

We now bound the perturbation term:

$$
\mu^2 \sum_{n=0}^{i^*} E \left\{ (\rho (\Lambda_i D^n) \cdot E \left\{ q_{i+n} \mid \mathcal{F}_i \right\} \mid w_i \in \mathcal{H} \right\}
$$

$$\leq \mu^2 \sum_{n=0}^{i^*} E \left\{ (\rho (\delta I + 2\mu \delta I)^n) \cdot E \left\{ q_{i+n} \mid \mathcal{F}_i \right\} \mid w_i \in \mathcal{H} \right\}
$$

$$= \mu^2 \sum_{n=0}^{i^*} \delta (1 + 2\mu \delta)^n \cdot E \left\{ q_{i+n} \mid w_i \in \mathcal{H} \right\}
$$

$$\mu^2 \sum_{n=0}^{i^*} \delta (1 + 2\mu \delta)^n \cdot \left( \beta R \text{E} \left\{ \left\| \bar{w} \right\|^2 \mid w_i \in \mathcal{H} \right\} + \delta^2 \text{E} \left\{ \left\| \bar{w} \right\|^2 \mid w_i \in \mathcal{H} \right\} \right) \quad (135)
$$

where (a) follows from Lemma 2. We conclude:

$$E \left\{ \left\| \bar{w}_{i+1} \right\|^2 \mid w_i \in \mathcal{H} \right\} \leq -\frac{\mu}{2} M \sigma^2 + o(\mu) \quad (136)
$$

Returning to (115), we find:

$$E \left\{ J(w_{i+j}) \mid w_i \in \mathcal{H} \right\}
$$

$$\leq E \left\{ J(w_i) \mid w_i \in \mathcal{H} \right\} + \frac{1}{2} E \left\{ \left\| \bar{w} \right\|^2 \mid w_i \in \mathcal{H} \right\}
$$

$$+ \frac{\rho}{6} E \left\{ \left\| \bar{w} \right\|^3 \mid w_i \in \mathcal{H} \right\}
$$

$$E \left\{ J(w_i) \mid w_i \in \mathcal{H} \right\} - \frac{\mu}{2} M \sigma^2 + o(\mu) \quad (137)
$$

and with (22) we prove the result.

**APPENDIX F**

**PROOF OF THEOREM 3**

We define the stochastic process:

$$t(k) = \begin{cases} t(k) + 1, & \text{if } w_{t(k)} \in T, \\ t(k) + 1, & \text{if } w_{t(k)} \in M, \\ t(k) + i_s, & \text{if } w_{t(k)} \in H. \end{cases} \quad (138)
$$

where $t(0) = 0$. From Theorem [1] we have:

$$E \left\{ J(w_{t(k)}) - J(w_{t(k-1)}) \mid w_{t(k)} \in T \right\}
$$

$$= E \left\{ J(w_{t(k)}) - J(w_{t(k+1)}) \mid w_{t(k)} \in T \right\}
$$

$$\geq \mu^2 \frac{\delta^2}{\tau} \quad (139)$$
and

\[ E \{ J(w_t(k)) - J(w_t(k+1)) \mid w_t(k) \in M \} \]
\[ = E \{ J(w_t(k)) - J(w_t(k+1)) \mid w_t(k) \in M \} \geq -\mu^2c_2 \quad (140) \]

while Theorem 2 ensures:

\[ E \{ J(w_t(k)) - J(w_t(k+1)) \mid w_t(k) \in H \} \]
\[ = E \{ J(w_t(k)) - J(w_t(k+1)) \mid w_t(k) \in H \} \geq \frac{\mu}{2}M \sigma^2 - o(\mu) \quad (141) \]

Together, they yield:

\[ E \{ J(w_t(k)) - E J(w_t(k+1)) \}
\[ = E \{ J(w_t(k)) - E J(w_t(k+1)) \mid w_t(k) \in G \} \cdot \pi^G_{t(k)} + E \{ J(w_t(k)) - E J(w_t(k+1)) \mid w_t(k) \in H \} \cdot \pi^H_{t(k)} + E \{ J(w_t(k)) - E J(w_t(k+1)) \mid w_t(k) \in M \} \cdot \pi^M_{t(k)} \geq \frac{\mu}{2}M \sigma^2 - o(\mu) \cdot \pi^H_{t(k)} - \mu^2c_2 \cdot \pi^M_{t(k)} \]

Suppose \( \pi^M_{t(k)} \leq 1 - \pi \) for all \( i \). Then \( \pi^G_{t(k)} + \pi^H_{t(k)} \geq \pi \) and

\[ E \{ J(w_t(k)) - E J(w_t(k+1)) \}
\[ \geq \frac{\mu}{2}M \sigma^2 \cdot (\pi - \pi^H_{t(k)}) + \left( \frac{\mu}{2}M \sigma^2 - o(\mu) \right) \cdot \pi^H_{t(k)} - \mu^2c_2 \cdot (1 - \pi) \]
\[ = \mu^2c_2 \pi + \left( \frac{\mu}{2}M \sigma^2 - \mu^2c_2 \cdot o(\mu) \right) \pi^H_{t(k)} \]
\[ \geq \mu^2c_2 \pi \quad (143) \]

where (a) holds whenever \( \frac{\mu}{2}M \sigma^2 - \mu^2c_2 \cdot o(\mu) \geq 0 \), which holds whenever \( \mu \) is sufficiently small. We hence have by telescoping:

\[ J(w_0) - J^* \geq E J(w_{t(0)}) - E J(w_t(k)) \]
\[ = E J(w_{t(1)}) - E J(w_t(k)) + E J(w_{t(1)}) - E J(w_t(2)) + \ldots + E J(w_{t(k-1)}) - E J(w_t(k)) \]
\[ \geq \mu^2c_2 \pi k \quad (144) \]

Rearranging yields:

\[ k \leq \frac{J(w_0) - J^*}{\mu^2c_2 \pi} \quad (145) \]

We conclude by definition of the stochastic process \( t(k) \):

\[ i = t(k) \leq k \cdot i^* \leq \frac{J(w_0) - J^*)}{\mu^2c_2 \pi} i^* \quad (146) \]
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