Research article

A regularity criterion for liquid crystal flows in terms of the component of velocity and the horizontal derivative components of orientation field

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Abstract: In this paper, we establish a regularity criterion for the 3D nematic liquid crystal flows. More precisely, we prove that the local smooth solution \((u, d)\) is regular provided that velocity component \(u_3\), vorticity component \(\omega_3\) and the horizontal derivative components of the orientation field \(\nabla_h d\) satisfy

\[
\int_0^T \left( \|u_3\|_{L^p}^{\frac{2p}{3a}} + \|\omega_3\|_{L^q}^{\frac{2q}{3a}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{3a}} \right) \, dt < \infty,
\]

with \(3 < p \leq \infty, \frac{3}{2} < q \leq \infty, 3 < a \leq \infty\).

Keywords: liquid crystal flow; velocity component; regularity criterion

Mathematics Subject Classification: 35B65, 35Q35, 76A15

1. Introduction

In this paper, we will consider the following three-dimensional (3D) nematic liquid crystal flows:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p &= -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\
\partial_t d + u \cdot \nabla d &= \gamma (\Delta d - f(d)), \\
\nabla \cdot u &= 0, \\
u(x, 0) &= u_0(x), d(x, 0) &= d_0(x),
\end{aligned}
\] (1.1)

where \(u = (u_1, u_2, u_3) \in \mathbb{R}^3\) is the velocity field, \(d = (d_1, d_2, d_3) \in \mathbb{R}^3\) is the macroscopic average of molecular orientation field and \(p\) represents the scalar pressure. The notation \(\nabla d \odot \nabla d\) represents the
3 × 3 matrix of which the \((i, j)\) entry can be denoted by
\[
\sum_{k=1}^{3} \partial_i d_k \partial_j d_k (1 \leq i, j \leq 3),
\]
and
\[
f(d) = \frac{1}{|\eta|^2} (|d|^2 - 1)d.
\]
u_0 is the initial velocity with \( \nabla \cdot u_0 = 0 \), \( d_0 \) is initial orientation vector with \(|d_0| \leq 1\). Here, \( \mu, \lambda, \gamma, \eta \) are all positive constants. And to simplify the presentation, we shall assume that \( \mu = \lambda = \gamma = \eta = 1 \) in this paper.

The hydrodynamic theory of liquid crystals was established by Ericksen and Leslie during 1960s (see [4, 10]). And the system (1.1) is a simplified version of the Ericksen-Leslie model which still retains most of the essential features of the hydrodynamic equations for nematic liquid crystal (see [8]). One of the most significant studies in this area was made by Lin and Liu [9], where they established the existence of global-in-time weak solutions and local-in-time classical solutions. When the orientation field \( d \) equals a constant, the above equations reduce to the incompressible Navier-Stokes equations. For well-known Prodi-Serrin type regularity criterion, people paid much focus on decomposing the integral term about \( u \cdot \nabla u \) and got some improving results based on the components of velocity field \( u \) and the gradient of the velocity field \( \nabla u \), readers can refer to [1–3, 7, 14, 20, 21, 23, 24]. Naturally, these related results were extended to the liquid crystal flows, see [5, 6, 11, 12, 16–19, 22], and references therein. Moreover, these Prodi-Serrin type regularity criteria based on velocity field indicate that the velocity field \( u \) plays a more dominate role than the orientation field \( d \) does on the regularity of solutions to the system (1.1).

In [13], Qian established the regularity criterion for system (1.1). That is, if
\[
\int_0^T \|u_3\|_{L^p}^p + \|\omega_3\|_{L^p}^p + \|\partial_3 u_3\|_{L^p}^p \, dt < M, \text{ for some } M > 0
\]
and
\[
\frac{3}{p} + \frac{2}{q} = 1, \quad \frac{3}{a} + \frac{2}{b} = 2, \quad 3 < p \leq \infty, \quad \frac{3}{2} < a \leq \infty
\]
(1.2)
where \( u_3 = (u_1, u_2) \), \( \omega_3 = \partial_3 u_2 - \partial_2 u_1 \), then the solution is regular. Later, Qian [15] proved the following regularity criterion:
\[
\int_0^T \|u_3\|_{L^p}^p + \|\partial_3 u_3\|_{L^p}^p + \|\nabla u d\|_{L^p}^p \, dt < M, \text{ for some } M > 0
\]
and
\[
\frac{3}{p} + \frac{2}{q} = 1, \quad \frac{3}{a} + \frac{2}{b} = 2, \quad 3 < p \leq \infty, \quad \frac{3}{2} < a \leq \infty.
\]
(1.3)

Inspired by the above results, we establish the following regularity criterion:

**Theorem 1.1.** Suppose the initial data \( u_0 \in H^1(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), \( d_0 \in H^2(\mathbb{R}^3) \), and let \( (u, d) \) be a smooth solution to the system (1.1) on \([0, T)\) for some \( 0 < T < \infty \). If \( (u, d) \) satisfies the following condition
\[
\int_0^T \|u_3\|_{L^p}^p + \|\omega_3\|_{L^q}^q + \|\nabla d\|_{L^a}^a \, dt < \infty, \text{ with } 3 < p \leq \infty, \quad \frac{3}{2} < q \leq \infty, \quad 3 < a \leq \infty,
\]
(1.4)
then \((u, d)\) can be extended beyond \(T\).

**Remark 1.1.** In [20], Zhang has decomposed the integral \(\int_{\mathbb{R}^3}(u \cdot \nabla)u \cdot \Delta u \,dx\) into the several integrals containing \(u_3\) and \(\omega_3\) for the Navier-Stokes equation, and the corresponding criterion is
\[
\int_0^T \left( \|u_3\|_{L_p}^{2p} + \|\omega_3\|_{L_q}^{2q} \right) \,dt < \infty, \quad \text{with} \quad 3 < p \leq \infty, \quad \frac{3}{2} < q \leq \infty.
\]
(1.5)

So the condition on \(\partial_3 u_h\) in (1.2) can be removed and the condition on \(\partial_3 u_h\) in (1.3) can be replaced.

And, the regularity condition of orientation field \(d\) is needed to control the term \(\nabla \cdot (\nabla d \odot \nabla d)\) in view of (1.3).

**Remark 1.2.** Compared with the corresponding results (1.2), we replace the conditions on \(\partial_3 u_h\) with \(\nabla d\) because we cannot control the 2-order higher derivatives term \(\nabla \cdot (\nabla d \odot \nabla d)\) by only \(u_3\) and \(\omega_3\).

Compared with (1.3), we reduce 1-order derivative on orientation field \(d\), which improves the result of (1.3).

Throughout this paper, the letter \(C\) means a generic constant which may vary from line to line, and the directional derivatives of a function \(\varphi\) are denoted by \(\partial_i \varphi = \frac{\partial \varphi}{\partial x_i}\) \((i = 1, 2, 3)\).

### 2. Proof of Theorem 1.1

According to the local well-posedness of smooth solution established by Lin and Liu [9], we only need to establish the priori estimates. And we have the following standard \(L^2\) estimate (for example, see [17, p.2-3]):
\[
(||u||_{L^2}^2 + ||\nabla d||_{L^2}^2) + \int_0^T (||\nabla u||_{L^2}^2 + ||\Delta d||_{L^2}^2 + ||d|\nabla d||_{L^2}^2 + \frac{1}{2} |||d|d^2||_{L^2}^2) \,dt \\ \leq C(||u_0||_{L^2}^2 + ||\nabla d_0||_{L^2}^2)\
\]

(2.1)

By an argument similar to [17, Eq (2.7)], we have
\[
\int_0^T \left( ||\nabla u||_{L^2}^2 + ||\Delta d||_{L^2}^2 \right) + ||d|\nabla d||_{L^2}^2 + |||d|d^2||_{L^2}^2 \\
= \int_{\mathbb{R}^3} (u \cdot \nabla)u \cdot \Delta u \,dx + \int_{\mathbb{R}^3} \nabla \cdot (\nabla d \odot \nabla d) \cdot \Delta u \,dx \\
- \int_{\mathbb{R}^3} \Delta(u \cdot \nabla d) \cdot \Delta d \,dx - \int_{\mathbb{R}^3} \Delta(|d|^2 d - d) \cdot \Delta d \,dx \\
:= I_1 + I_2 + I_3 + I_4.
\]

(2.2)

In the following part, we estimate the terms above one by one. For \(I_1\) referring to [20, (2.1)–(2.7)], (or see [11]), \(I_1\) can be decomposed as follows:
\[
I_1 = \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \partial_1 u_i \partial_j u_j \partial_k u_l \\
+ \sum_{i,j,k,l=1}^3 \alpha_{12ijkl} \partial_1 u_1 \partial_j u_i \partial_k u_l
\]
where \( \alpha_{mijkl}, 1 \leq m, n \leq 2, 1 \leq i, j, k, l \leq 3 \), are suitable integers. And the purpose is to rewrite \( \partial_m u_n \) by \( u_3 \) and \( \omega_3 \), \( 1 \leq m, n \leq 2 \).

Denoting by \( \Delta_h = \partial_1 \partial_1 + \partial_2 \partial_2 \) the horizontal Laplacian, and \( \mathcal{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}} \) the two-dimension Riesz transformation, it was shown in [20, (2.2)–(2.4)], that

\[
\Delta_h u_1 = -\partial_2 \omega_3 - \partial_1 \partial_3 u_3, \quad \Delta_h u_2 = \partial_1 \omega_3 - \partial_2 \partial_3 u_3.
\]

\[
\partial_m u_1 = \frac{\partial_2}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_1}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \partial_3 u_3 = \mathcal{R}_2 \mathcal{R}_m \omega_3 + \mathcal{R}_1 \mathcal{R}_m \partial_3 u_3,
\]

\[\text{(2.3)}\]

\[
\partial_m u_2 = \mathcal{R}_1 \mathcal{R}_m \omega_3 + \mathcal{R}_2 \mathcal{R}_m \partial_3 u_3, \quad 1 \leq m, n \leq 2.
\]

\[\text{(2.4)}\]

The term \( I_{11} \) can be expressed as

\[
I_{11} = \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \partial_1 u_1 \partial_j u_j \partial_k u_l \, dx
\]

\[
= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} (\mathcal{R}_2 \mathcal{R}_1 \omega_3 + \mathcal{R}_1 \mathcal{R}_1 \partial_3 u_3) \partial_j u_j \partial_k u_l \, dx
\]

\[
= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathcal{R}_2 \mathcal{R}_1 \omega_3 \partial_j u_j \partial_k u_l \, dx
\]

\[
- \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathcal{R}_1 \mathcal{R}_1 u_3 \partial_j \partial_j \partial_k u_l + \partial_i u_i \partial_j \partial_k u_l \, dx,
\]

by (2.3) and integration by parts. Because the Riesz transformation is bounded from \( L^p(\mathbb{R}^2) \) to \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \), we have

\[
I_{11} \leq C \| u_3 \|_{L^p} \| \nabla u \|_{L^\infty}^{\frac{2p}{p-2}} \| \nabla^2 u \|_{L^2} + C \| \omega_3 \|_{L^p} \| \nabla u \|_{L^\infty}^{\frac{2p}{p-2}} \| \nabla^2 u \|_{L^2}^{\frac{p+3}{2}}
\]

\[
\leq C \| u_3 \|_{L^p} \| \nabla u \|_{L^\infty}^{\frac{2p}{p-2}} \| \nabla^2 u \|_{L^2} + C \| \omega_3 \|_{L^p} \| \nabla u \|_{L^\infty}^{\frac{2p}{p-2}} \| \nabla^2 u \|_{L^2}^{\frac{3}{2}}
\]

\[
\leq C (\| u_3 \|_{L^p}^{\frac{2p}{p-2}} + \| \omega_3 \|_{L^p}^{\frac{2p}{p-2}}) \| \nabla u \|_{L^2}^2 + \frac{1}{16} \| \Delta u \|_{L^2}^2,
\]

where \( p > 3, q > \frac{3}{2} \).
The similar argument as $I_{11}$ can be used to terms $I_{12}, I_{13}, I_{14}$, therefore it can be deduced that

$$I_1 \leq C(\|u_3\|_{L_p}^{2p} + \|\omega_3\|_{L_p}^{2p})\|\nabla u\|_{L^2}^{2} + \frac{1}{4}\|\Delta u\|_{L^2}^{2}. \quad (2.5)$$

For $I_2$ and $I_3$, by using the fact $\nabla \cdot u = 0$ and integrating by parts several times, we can rewrite it as follows

$$I_2 + I_3 = \int_{\mathbb{R}^3} \sum_{i,k=1}^{3} ((\partial_i \partial_j d_k d_k + \partial_i d_k \partial_j d_k)\Delta u_i
- (\Delta u_i \partial_i d_k \Delta d_k + 2\nabla u_i \partial_i \nabla d_k \Delta d_k + u_i \partial_i \Delta d_k \Delta d_k)) dx

= \int_{\mathbb{R}^3} \sum_{i,k=1}^{3} -2\nabla u_i \partial_i \nabla d_k \Delta d_k dx

= \int_{\mathbb{R}^3} -2 \sum_{j,k=1}^{3} \sum_{i=1}^{2} \partial_j u_i \partial_i \partial_j d_k \Delta d_k dx - \int_{\mathbb{R}^3} 2 \sum_{j,k=1}^{3} \partial_j u_3 \partial_j \partial_k d_k \Delta d_k dx

= I_{21} + I_{22}. $$

$$I_{21} = \int_{\mathbb{R}^3} 2 \sum_{j,k=1}^{3} \sum_{i=1}^{2} \partial_j u_i \partial_i \partial_j d_k \Delta d_k dx + \int_{\mathbb{R}^3} 2 \sum_{j,k=1}^{3} \sum_{i=1}^{2} \partial_j \partial_j u_i \partial_i d_k \Delta d_k dx = I_{211} + I_{212}. $$

Next, employing the Hölder inequality, interpolation inequality and Young’s inequality, we have

$$I_{211} \leq C\|\nabla d\|_{L^{2p/3}} \|\Delta d\|_{L^2} \leq C\|\nabla d\|_{L^{2p/3}} \|\nabla u\|_{L^2} \|\Delta d\|_{L^2} \leq C\|\nabla d\|_{L^{2p/3}} \|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla \Delta d\|_{L^2}^2. \quad (2.6)$$

$$I_{212} \leq C\|\nabla d\|_{L^{2p/3}} \|\Delta d\|_{L^2} \leq C\|\nabla d\|_{L^{2p/3}} \|\nabla \Delta d\|_{L^2} \|\Delta u\|_{L^2} \leq C\|\nabla d\|_{L^{2p/3}} \|\Delta d\|_{L^2} + \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla \Delta d\|_{L^2}^2. \quad (2.7)$$

In the same way, the term $I_{22}$ can be bounded as follows

$$I_{22} = \int_{\mathbb{R}^3} 2 \sum_{j,k=1}^{3} (u_3 \partial_3 \partial_j \partial_j d_k d_k + u_3 \partial_3 \partial_j d_k \partial_j d_k) dx \leq C\|u_3\|_{L^{2p/3}} \|\Delta d\|_{L^2} \|\nabla \Delta d\|_{L^2} \leq C\|u_3\|_{L^{2p/3}} \|\nabla \Delta d\|_{L^2}^2 \|\nabla \Delta d\|_{L^2}^2 \leq C\|u_3\|_{L^{2p/3}} \|\nabla \Delta d\|_{L^2}^2 + \frac{1}{8}\|\nabla \Delta d\|_{L^2}^2. \quad (2.8)$$
Adding the above inequalities (2.6)–(2.8) together, one obtains

\[ I_2 + I_3 \leq C (\|u_3\|^2_{L^\infty} + \|\nabla_d d\|^2_{L^\infty}) (\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}) + \frac{1}{4}\|\Delta u\|^2_{L^2} + \frac{3}{8}\|\nabla \Delta d\|^2_{L^2}. \tag{2.9} \]

For \(I_4\), we have

\[
I_4 \leq \int_{\mathbb{R}^3} |\Delta d|^2 + \Delta(|d|^2 d) \cdot \Delta dd x
\leq \|\Delta d\|^2_{L^2} + C (\|\Delta |d|^2\|_{L^2} \|d\|_{L^5} \|\Delta d\|_{L^3} + \|\Delta d\|_{L^1} \|d\|_{L^6} \|\Delta d\|_{L^3})
\leq \|\Delta d\|^2_{L^2} + C \|\Delta d\|_{L^1} \|\Delta d\|_{L^3}
\leq \|\Delta d\|^2_{L^2} + C \|\Delta d\|_{L^2} \|\nabla \Delta d\|_{L^2}
\leq C \|\Delta d\|^2_{L^2} + \frac{1}{8}\|\nabla \Delta d\|^2_{L^2}. \tag{2.10} \]

Hence, inserting (2.5), (2.9) and (2.10) into (2.2) yields

\[
\frac{d}{dt} (\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}) + \|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2}
\leq C (1 + \|u_3\|^2_{L^p} + \|\omega_3\|^2_{L^q} + \|\nabla_d d\|^2_{L^p}) (\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}),
\]

and it could be derived by Gronwall inequality that

\[
\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2} + \int_0^T (\|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2}) dt
\leq (\|\nabla u_0\|^2_{L^2} + \|\Delta d_0\|^2_{L^2}) \exp \left\{ \int_0^T C (1 + \|u_3\|^2_{L^p} + \|\omega_3\|^2_{L^q} + \|\nabla_d d\|^2_{L^p}) dt \right\}.
\]

Then the proof of Theorem 1.1 is completed.

3. Conclusions

In this paper, we prove a regular criterion of solution for the 3D nematic liquid crystal flows via velocity component \(u_3\), vorticity component \(\omega_3\) and the horizontal derivative components of the orientation field \(\nabla_d d\), and we hope that the condition on \(\nabla_d d\) will be removed in future study.

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Conflict of interest

All authors declare no conflict of interest in this paper.
References

1. C. S. Cao, Sufficient conditions for the regularity to the 3D Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, **26** (2010), 1141–1151. http://dx.doi.org/10.3934/dcbs.2010.26.1141

2. C. S. Cao, E. S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations, *Indiana Univ. Math. J.*, **57** (2008), 2643–2661.

3. B. Q. Dong, Z. F. Zhang, The BKM criterion for the 3D Navier-Stokes equations via two velocity components, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 2415–2421. https://doi.org/10.1016/j.nonrwa.2009.07.013

4. J. L. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Rational Mech. Anal.*, **9** (1962), 371–378. https://doi.org/10.1007/BF00253358

5. J. S. Fan, B. L. Guo, Regularity criterion to some liquid crystal models and the Landau-Lifshitz equations in $\mathbb{R}^3$, *Sci. China Ser. A-Math.*, **51** (2008), 1787–1797. https://doi.org/10.1007/s11425-008-0013-3

6. W. J. Gu, B. Samet, Y. Zhou, A regularity criterion for a simplified non-isothermal model for nematic liquid crystals, *Funkcial. Ekvac.*, **63** (2020), 247–258.

7. X. J. Jia, Y. Zhou, Remarks on regularity criteria for the Navier-Stokes equations via one velocity component, *Nonlinear Anal.: Real World Appl.*, **15** (2014), 239–245. https://doi.org/10.1016/j.nonrwa.2013.08.002

8. F. H. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, *Commun. Pure Appl. Math.*, **42** (1989), 789–814. https://doi.org/10.1002/cpa.3160420605

9. F. H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Commun. Pure Appl. Math.*, **48** (1995), 501–537. https://doi.org/10.1002/cpa.3160480503

10. F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.*, **28** (1968), 265–283. https://doi.org/10.1007/BF00251810

11. Q. Li, B. Q. Yuan, Blow-up criterion for the 3D nematic liquid crystal flows via one velocity and vorticity components and molecular orientations, *AIMS Math.*, **5** (2020), 619–628. https://doi.org/10.3934/math.2020041

12. Q. Liu, J. H. Zhao, S. B. Cui, A regularity criterion for the three-dimensional nematic liquid crystal flow in terms of one directional derivative of the velocity, *J. Math. Phys.*, **52** (2011), 033102. https://doi.org/10.1063/1.3567170

13. C. Y. Qian, Remarks on the regularity criterion for the nematic liquid crystal flows in $\mathbb{R}^3$, *Appl. Math. Lett.*, **274** (2016), 679–689. https://doi.org/10.1016/j.aml.2015.11.007

14. C. Y. Qian, A remark on the global regularity for the 3D Navier-Stokes equations, *Appl. Math. Comput.*, **57** (2016), 126–131. https://doi.org/10.1016/j.amc.2016.01.016

15. C. Y. Qian, A further note on the regularity criterion for the 3D nematic liquid crystal flows, *Appl. Math. Comput.*, **290** (2016), 258–266. https://doi.org/10.1016/j.amc.2016.06.011

16. R. Y. Wei, Y. Li, Z. A. Yao, Two new regularity criteria for nematic liquid crystal flows, *J. Math. Anal. Appl.*, **424** (2015), 636–650. https://doi.org/10.1016/j.jmaa.2014.10.089
17. B. Q. Yuan, Q. Li, Note on global regular solution to the 3D liquid crystal equations, *Appl. Math. Lett.*, **109** (2020), 106491. https://doi.org/10.1016/j.aml.2020.106491

18. B. Q. Yuan, C. Z. Wei, BKM’s criterion for the 3D nematic liquid crystal flows in Besov spaces of negative regular index, *J. Nonlinear Sci. Appl.*, **10** (2017), 3030–3037. http://dx.doi.org/10.22436/jnsa.010.06.17

19. B. Q. Yuan, C. Z. Wei, Global regularity of the generalized liquid crystal model with fractional diffusion, *J. Math. Anal. Appl.*, **467** (2018), 948–958. https://doi.org/10.1016/j.jmaa.2018.07.047

20. Z. J. Zhang, Serrin-type regularity criterion for the Navier-Stokes equations involving one velocity and one vorticity component, *Czech. Math. J.*, **68** (2018), 219–225. https://doi.org/10.21136/CMJ.2017.0419-16

21. Z. J. Zhang, F. Alzahrani, T. Hayat, Y. Zhou, Two new regularity criteria for the Navier-Stokes equations via two entries of the velocity Hessian tensor, *Appl. Math. Lett.*, **37** (2014), 124–130. https://doi.org/10.1016/j.aml.2014.06.011

22. L. L. Zhao, F. Q. Li, On the regularity criteria for 3-D liquid crystal flows in terms of the horizontal derivative components of the pressure, *J. Math. Rese. Appl.*, **40** (2020), 165–168.

23. Y. Zhou, M. Pokorný, On the regularity to the solutions of the Navier-Stokes equations via one velocity component, *Nonlinearity*, **23** (2010), 1097–1107.

24. Z. J. Zhang, W. J. Yuan, Y. Zhou, Some remarks on the Navier-Stokes equations with regularity in one direction, *Appl. Math.*, **64** (2019), 301–308. https://doi.org/10.21136/AM.2019.0264-18