Weak Detection in the Spiked Wigner Model with General Rank

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Abstract

We study the statistical decision process of detecting the presence of signal from a ‘signal+noise’ type matrix model with an additive Wigner noise. We derive the error of the likelihood ratio test, which minimizes the sum of the Type-I and Type-II errors, under the Gaussian noise for the signal matrix with arbitrary finite rank. We propose a hypothesis test based on the linear spectral statistics of the data matrix, which is optimal and does not depend on the distribution of the signal or the noise. We also introduce a test for rank estimation that does not require the prior information on the rank of the signal.

1 Introduction

The spiked Wigner model is one of the most natural low-rank models of ‘signal-plus-noise’ type. In this model, the data matrix is of the form

\[ M = \sqrt{\lambda}XX^T + H, \]

where the spike \( X \) is an \( N \times k \) matrix whose column vectors are \( L^2 \)-normalized, \( H \) is an \( N \times N \) Wigner matrix (see Definition 1.1), and \( \lambda \) corresponds to the signal-to-noise ratio (SNR). In this paper, we focus on the hypothesis test for detecting presence of a signal, which is called the weak detection, from a given spiked Wigner matrix where SNR \( \lambda \) is below a threshold so that a reliable detection is not feasible. We derive the optimal error of the weak detection and propose a test achieving the optimal error, which is universal in the sense that it does not depend on the distributions of the signal or the noise. We also introduce a test to estimate the rank of the spike when the prior information on the rank is not known.

Rank-1 spiked Wigner matrix: In the simplest case of the spiked Wigner model, the signal \( x \) is a vector in \( \mathbb{R}^N \) and the spiked Wigner matrix is of the form

\[ \sqrt{\lambda}xx^T + H. \]

The spectral properties of rank-1 spiked Wigner matrix have been extensively studied in random matrix theory (\cite{26, 16, 12, 9}), and the limits of detection have been investigated in statistical learning theory (\cite{24, 25, 17, 22, 6, 19, 21, 5, 27, 15, 14}). The model is also applied to various problems such as community detection (\cite{11}) and submatrix localization (\cite{10}).

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In the rank-1 spiked Wigner matrix with Gaussian noise, assuming the signal is drawn from a distribution, called the prior, the signal is not reliably detectable if the SNR $\lambda$ is below a certain threshold (22). With the normalization $\|x\|_2 = 1$ and $\|H\| \to 2$ as $N \to \infty$, the threshold for $\lambda$ is 1 for a general class of priors, including spherical, Rademacher, and any i.i.d. prior with a sub-Gaussian bound (27). As asserted by Neyman–Pearson lemma, the likelihood ratio (LR) test is optimal in the sense that it minimizes the sum of the Type-I error and the Type-II error. El Alaoui, Krzakala, and Jordan (15) proved that this sum converges to $\text{erfc} \left( \frac{1}{4} \sqrt{-\log(1-\lambda)} \right)$ when the variance of $H_{ii}$ is 2 and hence $H$ is a Gaussian Orthogonal Ensemble (GOE). Though optimal, the LR test is not efficient, and it is desirable to construct a test that does not depend on information about the distribution of the signal, called prior, which is typically not known in many practical applications. In (14), an optimal and universal test was proposed, which is based on the linear spectral statistics (LSS) of the data matrix, a linear functional defined as

$$L_N(f) = \sum_{i=1}^{N} f(\mu_i)$$

for a given function $f$, where $\mu_1, \cdots, \mu_N$ are the eigenvalues of the data matrix. For other results on the rank-1 spiked Wigner model, we refer to (15, 27, 14) and references therein.

**Main contributions:** In this paper, we analyze the relation between the detectability of the signal and the rank of the spike in the subcritical regime. Let us denote by $H_1$ and $H_2$ the hypotheses

$$H_1 : k = k_1, \quad H_2 : k = k_2$$

for distinct non-negative integers $k_1$ and $k_2$. Our first main result is to prove the optimal error of the hypothesis test between $H_1$ and $H_2$. Adapting the strategies of (19, 15), we study the LR of the spiked Wigner matrices and show that the log-LR converges to a Gaussian whose mean and variance depend on $k_1$ and $k_2$ (see Theorem 2.2). It is then not hard to find the optimal error from the log-LR (see Theorem 2.4). We remark that the optimal error depends on $k_1$ and $k_2$ only through their difference $|k_1 - k_2|$.

After identifying the optimal error, it is natural to consider an efficient algorithm for the test achieving it. We propose a test based on the central limit theorem (CLT) of the LSS analogous to the one introduced in (14). The test is optimal and universal, and the various quantities in it can be estimated from the observed data without any prior knowledge on the signal or the noise. Furthermore, we also show that the proposed test can be improved by adapting the entrywise transformation in (27).

An important issue when analyzing a data matrix modeled by the spiked Wigner matrix is that the rank of the spike (signal) must be known a priori. Viable solutions to resolve this issue in the context of the community detection were suggested in (10, 18), which can work for any spiked Wigner matrices whenever $\lambda \gg 1$. However, their methods, which are spectral in nature, are not applicable in the regime $\lambda < 1$ regardless of the rank of the spike. With the CLT of the LSS, we also introduce a test for rank estimation that does not require the prior information on the rank of the signal.

The main mathematical achievement of the current paper is the CLT for the LSS of spiked Wigner matrices with general ranks. For a rank-1 spiked Wigner matrix, the CLT was first proved for a special spike $\sqrt{N}(1, 1, \ldots, 1)^T$ in (4) and later extended for a general rank-1 spike by a direct interpolation with the special case (14). However, the proof in (4) is not readily extended to the spiked Wigner matrices with higher ranks. In this paper, we overcome the difficulty by introducing an interpolation between the spiked Wigner matrix and the corresponding Wigner matrix without a spike and tracking the change of the LSS.
1.1 Model

The data matrix we consider is a rank-$k$ spiked Wigner matrix, which is defined as follows:

**Definition 1.1** (Wigner matrix). An $N \times N$ symmetric random matrix $H = (H_{ij})$ is a (real) Wigner matrix if $H_{ij}$ ($i, j = 1, 2, \ldots, N$) are independent real random variables such that

- All moments of $H_{ij}$ are finite and $\mathbb{E}[H_{ij}] = 0$ for all $i \leq j$.
- For all $i < j$, $N \mathbb{E}[H_{ij}^2] = 1$, $N^{3/2} \mathbb{E}[H_{ij}^3] = w_3$, and $N^2 \mathbb{E}[H_{ij}^4] = w_4$ for some $w_3, w_4 \in \mathbb{R}$.
- For all $i$, $N \mathbb{E}[H_{ii}^2] = w_2$ for some constant $w_2 \geq 0$.

**Definition 1.2** ((rank-$k$) Spiked Wigner matrix). An $N \times N$ matrix $M = \sqrt{\lambda}XX^T + H$ is a spiked Wigner matrix with a spike $X$ and the signal-to-noise ratio $\lambda$ if $H$ is a Wigner matrix and $X = [x(1), x(2), \ldots, x(k)] \in \mathbb{R}^{N \times k}$ with $x(i) \in \mathbb{R}^N$ and $\|x(i)\|_2 = 1$ for $i = 1, 2, \ldots, k$.

For the analysis of the data matrix with Gaussian noise, we use a spiked Wigner matrix with the following normalization.

**Definition 1.3** ((rank-$k$) Spiked Gaussian Wigner matrix). An $N \times N$ matrix $Y = \sqrt{\lambda}NXX^* + W$ is a spiked Gaussian Wigner matrix with the SNR $\lambda$ with the spike $X^*$ and the signal-to-noise ratio $\lambda$ if $W = \sqrt{NH}$ for a Wigner matrix $H$ and $X^* = [x^*(1), x^*(2), \ldots, x^*(k)] \in \mathbb{R}^{N \times k}$. We assume that the entries of the spike matrix are bounded, centered i.i.d. random variables with unit variance, with a prior distribution $\mathcal{P}$.

1.2 Other related works

The spiked Wigner model can be generalized to $p$-tensor models ($p \geq 3$). With the rank-1 spherical spike, the phase transition was proved in [22, 28] that there exist $\lambda_- \leq \lambda_+$ such that detection is impossible for $\lambda < \lambda_-$ but is possible for $\lambda > \lambda_+$. The tensor models with multiple spikes were considered in [20, 7, 13] where i.i.d. signals are sampled from a joint of centered priors with finite variance, and it was further generalized to the non-symmetric setting in [8].

1.3 Organization of the paper

The rest of the paper is organized as follows: In Section 2 we state the main results on the error of the LR test, algorithms for LSS-based tests, and a test for rank estimation that does not require the prior information on the rank of the signal. In Section 3 we study the LR of the spiked Gaussian Wigner model in detail. In Section 4 we state general results on the CLT for the LSS. We conclude the paper in Section 5 with the summary of our works and future research directions. In Appendix A we consider examples of spiked Wigner matrices with numerical simulations. Technical details of the proofs can be found in Appendix B.

2 Main Results

Recall that we denote by $H_1$ and $H_2$ the hypotheses such that

$$H_1 : k = k_1, \quad H_2 : k = k_2$$

for non-negative integers $k_1 < k_2$. We also denote by $\mathbb{P}_1$ and $\mathbb{P}_2$ the joint probability of the observation, a spiked Wigner matrix (or a spiked Gaussian Wigner matrix), under $H_1$ and $H_2$, respectively.
2.1 LR test for spiked Gaussian Wigner matrices

We first consider the fluctuation of the LR of the model defined in Definition 1.3.

**Definition 2.1** (Likelihood ratio). For a data matrix $Y$ in Definition 1.3, the likelihood ratio (or the Radon–Nikodym derivative) of $P_2$ with respect to $P_1$ is

$$
\mathcal{L}(Y; k_1, k_2) := \frac{dP_2}{dP_1}.
$$

For simplicity, we mainly consider the behavior of $\mathcal{L}(Y; k_1, k_2)$ under $H_2$.

Let $X_i = (x_i(1), x_i(2), \ldots, x_i(k_2))$ be the $i$-th row vector of $X$. Similarly, we also let $X_i^*$ be the $i$-th row vector of $X^*$. Note that

$$
X_i^* = (x_i^*(1), \ldots, x_i^*(k_2)) \sim P_{\otimes k_2} =: P_0.
$$

Conditioning on $X^*$, from the Gaussianity of $W$ we obtain that

$$
\mathcal{L}(Y; k_1, k_2) = \int e^{-H(x)} dP_0^{\otimes N}(X),
$$

where the Hamiltonian $H(X)$ is given by

$$
-H(X) = \sum_{i<j}^{N} \sum_{n=k_1+1}^{k_2} \left[ \frac{\sqrt{\lambda}}{N} Y_{ij} x_i(n) x_j(n) - \lambda \sum_{m=1}^{k_2} x_i(m) x_j(m) x_i(n) x_j(n) + \frac{\lambda}{2N} x_i(n)^2 x_j(n)^2 \right] + \frac{1}{w_2} \sum_{i=1}^{N} \sum_{n=k_1+1}^{k_2} \left[ \frac{\sqrt{\lambda}}{N} Y_{ii} x_i(n)^2 - \lambda \sum_{m=1}^{k_2} x_i(m)^2 x_i(n)^2 + \frac{\lambda}{2N} x_i(n)^4 \right].
$$

Our first main result is the following theorem.

**Theorem 2.2.** Assume that the prior $P$ is centered with third moment $\kappa$, has unit variance and bounded support for $m = 1, 2, \ldots, k$. Then, there exists $\lambda_c \in (0, 1]$ such that for $\lambda < \lambda_c$

$$
\log \mathcal{L}(Y; k_1, k_2) \Rightarrow N(-\mu, 2\mu) \quad \text{under } H_1,
$$

$$
\log \mathcal{L}(Y; k_1, k_2) \Rightarrow N(\mu, 2\mu) \quad \text{under } H_2,
$$

where

$$
\mu = \mu(k_1, k_2) = \frac{(k_1 - k_2)^2}{4} \left( 1 + \frac{\kappa}{w_2} \right) (- \log(1 - \lambda) - \lambda) + \frac{(k_1 - k_2)^2 \lambda}{2w_2}.
$$

We prove Theorem 2.2 in Section 3.2. With Theorem 2.2 and the Le Cam’s first lemma, we obtain the following corollary by a contiguity argument.

**Corollary 2.3.** Under the assumptions of Theorem 2.2, $P_1$ and $P_2$ are mutually contiguous.

The optimal test minimizing the sum of Type-I and Type-II errors is the LR test as asserted by the Neyman–Pearson lemma. In the LR test, we accept $H_1$ if $\mathcal{L}(Y; k_1, k_2) \leq 1$ and accept $H_2$ if $\mathcal{L}(Y; k_1, k_2) > 1$. The error of such a test is

$$
err^*(k_1, k_2) = P_1(\mathcal{L}(Y; k_1, k_2) \leq 1) + P_2(\mathcal{L}(Y; k_1, k_2) > 1).
$$

In the next theorem, we compute the limiting error of the LR test.
Theorem 2.4. Under the assumptions of Theorem 2.2 if $\lambda < \lambda_c$ then

$$\lim_{N \to \infty} \text{err}^*(k_1, k_2) = \text{erfc} \left( \frac{k_2 - k_1}{4} \sqrt{1 + \frac{\kappa}{w_2}} (-\log(1 - \lambda) - \frac{2\lambda}{w_2}) \right)$$

(2.5)

where \( \text{erfc}(\cdot) \) is the complementary error function defined as \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \).

Proof. Theorem 2.4 is a direct consequence of Theorem 2.2. We refer to Section 3 of [15] and the proof of Theorem 2 of [14] for more detail.

2.2 LSS-based test for spiked Wigner Matrices

We next consider the spiked Wigner model with general noise, without any prior information on the distribution of the spike. We introduce the test statistic \( L_{\lambda} \) defined by

$$L_{\lambda} = -\log \det \left( (1 + \lambda)I - \sqrt{\lambda}M \right) + \frac{\lambda N}{2} + \sqrt{\lambda} \left( \frac{2}{w_2} - 1 \right) \text{Tr} \, M + \lambda^2 \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) (\text{Tr} \, M^2 - N),$$

(2.6)

which was also used in the hypothesis test proposed in [14]. If there is no signal present, it was proved in [2] (see also Section 3.1 of [3]) that

$$L_{\lambda} \Rightarrow \mathcal{N}(m_0, V_0),$$

where

$$m_0 = -\frac{1}{2} \log(1 - \lambda) + \left( \frac{w_2 - 1}{w_4 - 1} - \frac{1}{2} \right) \lambda + \frac{(w_4 - 3)\lambda^2}{4},$$

(2.7)

$$V_0 = -2 \log(1 - \lambda) + \left( \frac{4}{w_2} - 2 \right) \lambda + \left( \frac{2}{w_4 - 1} - 1 \right) \lambda^2.$$

(2.8)

For a rank-\( k \) spiked Wigner matrix, we have the following result.

Theorem 2.5. Let \( M \) be a rank-\( k \) spiked Wigner matrix with a spike \( X \) as in Definition 1.2 with \( 0 < \lambda < 1 \). Denote by \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \) the eigenvalues of \( M \). Then,

$$L_{\lambda} \Rightarrow \mathcal{N}(m_k, V_0),$$

(2.9)

where the variance \( V_0 \) is as in (2.8) and the mean \( m_k \) is given by

$$m_k = m_0 + k \left[ -\log(1 - \lambda) + \left( \frac{2}{w_2} - 1 \right) \lambda + \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) \lambda^2 \right]$$

(2.10)

Proof. Theorem 2.5 directly follows from Theorem 4.2 in Section 4. See Section C in Supplementary Material of [14] for more detail.

We can construct a hypothesis test (between \( H_1 \) and \( H_2 \)) based on Theorem 2.2 which we describe in Algorithm 1. In this test, for a given data matrix \( M \), we compute \( L_{\lambda} \) and compare it with the critical value.
Algorithm 1 Hypothesis test

Data: $M_{ij}$, parameters $w_2, w_4, \lambda$
$L_\lambda \leftarrow$ test statistic in (2.6), $m_\lambda \leftarrow$ critical value in (2.11)

if $L_\lambda \leq m_\lambda$ then
  Accept $H_1$
else
  Accept $H_2$
end if

$m_\lambda$, defined as

$$m_\lambda := \frac{m_{k_1} + m_{k_2}}{2} = -\frac{k_1 + k_2 + 1}{2} \log(1 - \lambda) + \left( \frac{w_2 - 1}{w_4 - 1} + \frac{k_1 + k_2}{w_2} - \frac{k_1 + k_2 + 1}{2} \right) \lambda$$

(2.11)

to accept $H_1$ or $H_2$. We remark that the error of the CLT-based test is minimized with the test statistic $L_\lambda$. See Theorem 4.3.

Theorem 2.6. The error of the test in algorithm 1 converges to

$$\text{erfc} \left( \frac{k_2 - k_1}{4} \sqrt{-\log(1 - \lambda) + \left( \frac{2}{w_2 - 1} \right) \lambda + \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) \lambda^2} \right).$$

Proof. Theorem 2.6 is a direct consequence of Theorem 2.5. See also the proof of Theorem 2.4.

Remark 2.7. In case $w_4 = 3$, we obtain

$$\lim_{N \to \infty} \text{err}(\lambda) = \text{erfc} \left( \frac{k_2 - k_1}{4} \sqrt{-\log(1 - \lambda) + \left( \frac{2}{w_2 - 1} \right) \lambda} \right),$$

which coincides with the limiting error in (2.5) in Theorem 2.4 with $\kappa = 0$. As a function of $\kappa$, the limiting error in (2.5) is maximal when $\kappa = 0$. Thus, considering that we have no information on $\kappa$, our test is optimal when the noise is Gaussian and $\lambda$ is below the construction threshold $\lambda_c$.

2.3 LSS-based test with entrywise transformation

In this section, we apply the entrywise transformation to the data matrix, which was used for rank-1 spiked Wigner matrices to improve the principal component analysis (PCA) for the strong detection [27] or the LSS-based hypothesis test for the weak detection [14]. As in [27, 14], we use the following technical assumption.

Assumption 2.8. For the spike $x$, we assume that $\|x\|_\infty \leq N^{-c}$ for some $c > \frac{3}{8}$.

For the noise, let $P$ and $P_d$ be the distributions of the normalized off-diagonal entries $\sqrt{N}H_{ij}$ and the normalized diagonal entries $\sqrt{N}H_{ii}$, respectively. We assume the following:

1. The density function $g$ of $P$ is smooth, positive everywhere, and symmetric (about 0).

2. The function $h = -g'/g$ and its all derivatives are polynomially bounded in the sense that $|h^{(\ell)}(w)| \leq C_\ell |w|^{C_\ell}$ for some constant $C_\ell$ depending only on $\ell$.

3. The density function $g_d$ of $P_d$ satisfies the assumptions 1 and 2.
\begin{algorithm}
\textbf{Algorithm 2} Hypothesis test with the entrywise transformation
\begin{itemize}
    \item \textbf{Data:} $M_{ij}$, parameters $w_2, w_4, \lambda$, densities $g, g_d$
    \item $\tilde{M} \leftarrow$ transformed matrix in (2.12), $\tilde{L}_\lambda \leftarrow$ test statistic in (2.13), $\tilde{m}_\lambda \leftarrow$ critical value in (2.17)
    \item if $\tilde{L}_\lambda \leq \tilde{m}_\lambda$ then
        \begin{itemize}
            \item Accept $H_1$
        \end{itemize}
    \item else
        \begin{itemize}
            \item Accept $H_2$
        \end{itemize}
\end{itemize}
\end{algorithm}

Set $h = -g'/g$ and $h_d = -g'_d/g_d$. For a rank-$k$ spiked Wigner matrix $M$ that satisfies Assumption 2.8, we define a matrix $\tilde{M}$ by

\begin{equation}
\tilde{M}_{ij} = \frac{1}{\sqrt{F^HN}} h(\sqrt{NM_{ij}}) \quad (i \neq j), \quad \tilde{M}_{ii} = \sqrt{\frac{w_2}{F^HN}} h_d \left( \sqrt{\frac{N}{w_2}} M_{ii} \right),
\end{equation}

where

\begin{align*}
F^H &= \int_{-\infty}^{\infty} \frac{g'(w)^2}{g(w)} dw, & F^H_d &= \int_{-\infty}^{\infty} \frac{g'_d(w)^2}{g_d(w)} dw.
\end{align*}

The transformation has an effect of changing the SNR from $\lambda$ to $\lambda F^H$, which is an improvement when the noise is non-Gaussian since $F^H \geq 1$ and the equality holds if and only if $P$ is a standard Gaussian. For more detail, we refer to [27, 14].

We denote by $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_N$ the eigenvalues of $\tilde{M}$, and we use the same test statistic $\tilde{L}_\lambda$ as in [14] defined by

\begin{equation}
\tilde{L}_\lambda := -\log \det \left( (1 + \lambda F^H)I - \sqrt{\lambda F^H} \tilde{M} \right) + \frac{\lambda F^H}{2} N \nonumber
\end{equation}

\begin{equation}
+ \sqrt{\lambda} \left( \frac{2\sqrt{F^H_d}}{w_2} - \sqrt{F^H} \right) \text{Tr} \tilde{M} + \lambda \left( \frac{G^H}{w_4 - 1} - \frac{F^H}{2} \right) (\text{Tr} \tilde{M}^2 - N),
\end{equation}

where

\begin{equation}
G^H = \frac{1}{2F^H} \int_{-\infty}^{\infty} \frac{g'(w)^2 g''(w)}{g(w)^2} dw, \quad \tilde{w}_4 = \frac{1}{(F^H)^2} \int_{-\infty}^{\infty} \frac{(g'(w))^4}{(g(w))^2} dw.
\end{equation}

We have the following CLT result for $\tilde{L}_\lambda$ that generalizes Theorem 3 of [14].

\begin{theorem}
For a rank-$k$ spiked Wigner matrix $M$ with $\lambda F^H < 1$ satisfying Assumption 2.8,
\begin{equation}
\tilde{L}_\lambda \Rightarrow \mathcal{N}(\tilde{m}_k, \tilde{V}_0),
\end{equation}

where the mean and the variance are given by
\begin{equation}
\tilde{m}_k = -\frac{1}{2} \log(1 - \lambda F^H) + \left( \frac{(w_2 - 1)G^H}{\tilde{w}_4 - 1} - \frac{F^H}{2} \right) \lambda + \frac{\tilde{w}_4 - 3}{4} (\lambda F^H)^2 
\end{equation}
\begin{equation}
+ k \left[ -\log(1 - \lambda F^H) + \left( \frac{2F^H_d}{w_2} - F^H \right) \lambda + \left( \frac{G^H}{\tilde{w}_4 - 1} - \frac{(F^H)^2}{2} \right) \lambda^2 \right],
\end{equation}
\begin{equation}
\tilde{V}_0 = -2 \log(1 - \lambda F^H) + \left( \frac{4F^H_d}{w_2} - 2F^H \right) \lambda + \left( \frac{2(G^H)^2}{\tilde{w}_4 - 1} - (F^H)^2 \right) \lambda^2.
\end{equation}
\end{theorem}
Algorithm 3 Rank estimation

Data: \( M_{ij}, \) parameters \( w_2, w_4, \lambda \)

\[ L_\lambda \leftarrow \text{test statistic in (2.6)}, \quad m_0 \leftarrow \text{mean in (2.11)}, \quad m_1 \leftarrow \text{mean in (2.10)} \text{ with } k = 1 \]

\[ k' \leftarrow \text{value in (2.18)} \]

if \( L_\lambda \leq (m_0 + m_1)/2 \) then

Set \( k^* = 0 \)

else

Set \( k^* = \lceil k' - 0.5 \rceil \)

end if

Proof. Theorem 2.5 directly follows from Theorem 4.4 in Section 4.

Analogous to Algorithm 1, we propose a test described in Algorithm 2 where we compute \( \tilde{L}_\lambda \) and compare it with \( \tilde{m}_\lambda := (\tilde{m}_{k_1} + \tilde{m}_{k_2})/2. \) \hfill (2.17)

Theorem 2.10. The error of the test in Algorithm 3 converges to

\[ \text{erfc} \left( \frac{k_2 - k_1}{4} \sqrt{- \log(1 - \lambda F^H) + \left( \frac{2 F^H_d}{w_2} - 1 \right) \lambda + \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) \lambda^2} \right). \]

Proof. Theorem 2.6 is a direct consequence of Theorem 2.9. See also the proof of Theorem 2.4.

2.4 Rank estimation

In this section, we explain how we can adapt the test in Algorithm 1 to estimate the rank of the signal when the prior information on the rank is not known. Recall the test statistic \( L_\lambda \) defined in (2.6). As proved in Theorem 2.5, the test statistic converges to a Gaussian random variable with mean \( m_k \) and the variance \( V_0 \), where \( m_k \) is equidistributed with respect to \( k \) and \( V_0 \) does not depend on \( k \). It is then natural to set the best candidate for \( k \), which we call \( k^* \), be the minimizer of the distance \( |L_\lambda - m_k| \). This procedure is equivalent to find the nearest nonnegative integer of the value

\[ k' := \frac{L_\lambda - m_0}{- \log(1 - \lambda) + \left( \frac{2}{w_2} - 1 \right) \lambda + \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) \lambda^2}. \] \hfill (2.18)

The test is described in Algorithm 3 whose probability of error converges to

\[ \mathbb{P}(k = 0) \cdot \mathbb{P} \left( \frac{V_0}{4} \right) + \sum_{i=1}^{\infty} \mathbb{P}(k = i) \cdot \mathbb{P} \left( |Z| > \frac{\sqrt{V_0}}{4} \right) = \left( 1 - \frac{\mathbb{P}(k = 0)}{2} \right) \text{erfc} \left( \frac{1}{4} \sqrt{- \log(1 - \lambda) + \left( \frac{2}{w_2} - 1 \right) \lambda + \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) \lambda^2} \right), \] \hfill (2.19)

where \( Z \) is a standard Gaussian random variable. Note that it depends only on \( \mathbb{P}(k = 0) \).
3 Likelihood Ratio with Gaussian Noise

3.1 Reconstruction threshold

In order to describe the fluctuation of $L(Y; k_1, k_2)$, we need to introduce the reconstruction threshold. For rank-1 spiked model, as discussed in [19, 15], it is possible to detect the signal strongly (reliably) if and only if $\lambda$ is above the reconstruction threshold. We generalize it to our rank-$k$ model as follows. For a $k \times k$ symmetric positive semidefinite matrix $q$, we define a replica symmetric (RS) potential function $F(q)$ by

$$F(q) = -\frac{\lambda}{4} \|q\|_F^2 + \mathbb{E} \left[ \log \int \exp \left( \sqrt{\lambda}(Z, q^{1/2}\xi) + \lambda\xi^T q^* - \frac{\lambda}{2} (q, q^*) \right) d\mathcal{P}^\otimes k(\xi) \right]$$  \hspace{1cm} (3.1)

where $\| \cdot \|_F$ is the Frobenius norm, $\xi^* \sim \mathcal{P}^\otimes k$, and $Z \sim \mathcal{N}(0, I_k)$ is a $k$-dimensional Gaussian random vector independent of $\xi$ and $\xi^*$. We let

$$q^* = \|\arg\max F(q)\|,$$  \hspace{1cm} (3.2)

where the maximizer is over the set of all $k \times k$ symmetric positive semidefinite matrices. As proved in Theorem 12 of [19], for almost all $\lambda > 0$, it turns out that all the maximizers $q$ have the same norm. Note that $q^*$ can be regarded as $q^*(\lambda)$, a function of $\lambda$. We now define the reconstruction threshold for rank-$k$ model as follows:

**Definition 3.1 (Reconstruction threshold for rank-$k$ model).** The reconstruction threshold $\lambda_c^{(k)}$ is defined as

$$\lambda_c^{(k)} := \inf \{ \lambda > 0 : q^*(\lambda) > 0 \}. \hspace{1cm} (3.3)$$

The reconstruction threshold $\lambda_c$ in Theorem 2.2 satisfies that

$$\lambda_c^{(k_2)} \leq \lambda_c \leq 1,$$

and we conjecture that $\lambda_c = \lambda_c^{(k_2-k_1)}$.

3.2 Proof of Theorem 2.2

Following the strategy in [15], we use the characteristic function of the log-LR,

$$\phi_N(\lambda) = \mathbb{E}_{P_2} \left[ e^{is \log L(Y; k_1, k_2)} \right] \hspace{1cm} (3.4)$$

for fixed $s \in \mathbb{R}$.

**Proof.** It suffices to prove the statement under $H_2$. We first consider the case $w_2 = \infty$ as in the proof of Theorem 7 of [15]. Differentiating $\phi_N$ and applying Stein’s lemma,

$$\phi'_N(\lambda) = \frac{is - s^2}{4} \frac{(k_1 - k_2)^2 \lambda}{1 - \lambda} \phi_N(\lambda) + O(N^{-\frac{1}{2}}). \hspace{1cm} (3.5)$$

(See also Lemma 8 and Proposition 9 of [15].) Since $\varphi(0) = 1$, integrating (3.5) with respect to $\lambda$, we find for any $\lambda < \lambda_c$ and $s \in \mathbb{R}$ that

$$|\phi_N(\lambda) - e^{i(s-s^2)\mu}| = O(N^{-\frac{1}{2}}),$$  \hspace{1cm} (3.6)

where $\mu = \frac{(k_2-k_1)^2}{4} (-\log(1 - \lambda) - \lambda)$. The desired result for $w_2 = \infty$ now directly follows.
In the case \( w_2 < \infty \), we need to add the contribution from the diagonal term. Following the computation in Section 8 of [15], we obtain that

\[
\phi_N' (\lambda) = \frac{i s - s^2}{4} \cdot \frac{(k_1 - k_2)^2 \lambda}{1 - \lambda} \left( 1 + \frac{\kappa}{w_2} \right) \phi_N (\lambda) + \frac{i s - s^2}{2w_2} \cdot \lambda(k_1 - k_2)^2 \phi_N (\lambda) + O(N^{-\frac{1}{2}}),
\]

and we obtain the desired result by integrating (3.7) with respect to \( \lambda \). See Appendix [B] for more detail of the proof.

4 Central Limit Theorems for Spiked Wigner Matrices

In this section, we collect our results on general CLTs for the LSS of spiked Wigner matrices. We begin by defining Chebyshev polynomials of the first kind, which will be used in the CLT statements.

Definition 4.1 (Chebyshev polynomial). The \( n \)-th Chebyshev polynomial (of the first kind) \( T_n \) is a degree \( n \) polynomial defined by

\[
T_0 (x) = 1, \quad T_1 (x) = x, \quad T_{n+1} (x) = 2xT_n (x) - T_{n-1} (x).
\]

We first state a CLT for the LSS that generalizes Theorem 5 of [14].

Theorem 4.2. Assume the conditions in Theorem 2.5. Then, for any function \( f \) analytic on an open interval containing \([-2, 2]\),

\[
\left( \sum_{i=1}^{N} f(\mu_i) - N \int_{-2}^{2} \frac{\sqrt{4 - z^2}}{2\pi} f(z) \, dz \right) \Rightarrow N (m_k(f), V_0(f)).
\]

The mean and the variance of the limiting Gaussian distribution are given by

\[
m_k(f) = \frac{1}{4} (f(2) + f(-2)) \frac{1}{2} \tau_0 (f) + (w_2 - 2) \tau_2 (f) + (w_4 - 3) \tau_4 (f) + k \sum_{\ell=1}^{\infty} \sqrt{\lambda^{\ell}} \tau_{\ell} (f),
\]

\[
V_0 (f) = (w_2 - 2) \tau_1 (f)^2 + 2(w_4 - 3) \tau_2 (f)^2 + 2 \sum_{\ell=1}^{\infty} \ell \tau_{\ell} (f)^2,
\]

where we let

\[
\tau_{\ell} (f) = \frac{1}{\pi} \int_{-2}^{2} T_{\ell} \left( \frac{x}{2} \right) \frac{f(x)}{\sqrt{4 - x^2}} \, dx.
\]

Proof. We adapt the proof of Theorem 5 in [14] with the following change. Instead of interpolating the spiked Wigner matrices \( M \) with the original signal and with the signal with all 1’s considered in [4], we directly interpolate \( M \) and \( H \) and track the change of the mean. Consider the matrix

\[
M(\theta) = \theta \sqrt{XX^T} + H
\]

for \( \theta \in [0, 1] \). The change of the mean in the CLT for \( H \) and the CLT for \( M \) can be computed by tracking the change of the corresponding resolvent, defined as

\[
R(\theta, z) := (M(\theta) - zI)^{-1}
\]
for \( z \in \mathbb{C}^+ \). Following the proof of Theorem 5 in [14], we find that

\[
\frac{\partial}{\partial \theta} \text{Tr} R(\theta, z) = -\sum_{m=1}^{K} \sqrt{\lambda} \frac{\partial}{\partial z} \left( x(m)^T R(\theta, z) x(m) \right) = -k \frac{\partial}{\partial z} \left( \frac{\sqrt{\lambda} s(z)}{1 + \theta \sqrt{\lambda} s(z)} \right) + O(N^{\frac{1}{2}-\epsilon})
\]

\[
= -k \frac{\sqrt{\lambda} s'(z)}{(1 + \theta \sqrt{\lambda} s(z))} + O(N^{\frac{1}{2}-\epsilon})
\]

with high probability for any \( \epsilon > 0 \), where \( s(z) = -\frac{z + \sqrt{z^2 - 4}}{2} \) is the Stieltjes transform of the Wigner semicircle law. Applying Cauchy’s integral formula, and following the computation in the proof of Lemma 4.4 in [4], we then find that the difference between the LSS of \( M \) and the LSS of \( H \) is

\[
k \sum_{\ell=1}^{\infty} \sqrt{\lambda} \tau_\ell(f).
\]

This proves the desired theorem.

Next, we prove that the proposed test in Algorithm 1 achieves the lowest error among all tests based on LSS.

**Theorem 4.3.** Assume the conditions in Theorem 4.2. If \( w_2 > 0 \) and \( w_4 > 1 \), then

\[
\left| \frac{m_{k_2}(f) - m_{k_1}(f)}{\sqrt{V_0(f)}} \right| \leq \left| \frac{m_{k_2} - m_{k_1}}{\sqrt{V_0}} \right|.
\] (4.1)

The equality holds if and only if \( f = C_1 \varphi_\lambda + C_2 \) for some constants \( C_1 \) and \( C_2 \) where

\[
\varphi_\lambda(x) := \log \left( \frac{1}{1 - \sqrt{\lambda} x + \lambda} \right) + \sqrt{\lambda} \left( \frac{2}{w_2} - 1 \right) x + \lambda^2 \left( \frac{1}{w_4 - 1} - \frac{1}{2} \right) x^2.
\]

**Proof.** The theorem easily follows from the proof of Theorem 6 in [14] with applying Theorem 4.2 instead of Theorem 5 in [14].

With the entrywise transformation in Section 2.3, we have the following changes in Theorems 4.2 and 4.3, which we state without detailed proofs.

**Theorem 4.4.** Assume the conditions in Theorem 2.9. For any function \( f \) analytic on an open interval containing \([-2, 2]\),

\[
\left( \sum_{i=1}^{N} f(\tilde{\mu}_i) - N \int_{-2}^{2} \frac{\sqrt{4 - u^2}}{2\pi} f(z) \, dz \right) \Rightarrow N(\tilde{m}_k(f), \tilde{V}_0(f)).
\]

The mean and the variance of the limiting Gaussian distribution are given by

\[
\tilde{m}_k(f) = \frac{1}{4} (f(2) + f(-2)) - \frac{1}{2} \tau_0(f) + k \sqrt{\lambda F_{d}^H} \tau_1(f) + (w_2 - 2 + k\lambda G^H) \tau_2(f) + (w_4 - 3) \tau_4(f) + k \sum_{\ell=4}^{\infty} \sqrt{(\lambda F^H)^\ell} \tau_\ell(f),
\] (4.2)

\[
\tilde{V}_M(f) = V_M(f) = (w_2 - 2) \tau_1(f)^2 + 2(w_4 - 3) \tau_2(f)^2 + 2 \sum_{\ell=4}^{\infty} \ell \tau_\ell(f)^2.
\]
Theorem 4.5. Assume the conditions in Theorem 4.4. Then

\[ \left| (\tilde{m}_{k_2}(f) - \tilde{m}_{k_1}(f))/\sqrt{\tilde{V}_0(f)} \right| \leq (\tilde{m}_{k_2} - \tilde{m}_{k_1})/\sqrt{\tilde{V}_0}. \]  

(4.3)

Here, the equality holds if and only if \( f(x) = C_1 \tilde{\varphi}_\lambda(x) + C_2 \) for some constants \( C_1 \) and \( C_2 \) with

\[ \tilde{\varphi}_\lambda(x) := \log \left( \frac{1}{1 - \sqrt{\lambda} F^H x + \lambda F^H} \right) + \sqrt{\lambda} \left( \frac{2 F^H d}{w_2} - \sqrt{F^H} \right) x + \lambda \left( \frac{G^H}{w_4 - 1} - \frac{F^H}{2} \right) x^2. \]

5 Conclusion and Future Works

In this paper, we considered spiked matrices with general ranks. We computed the error of the likelihood ratio test in case the noise is Gaussian, and proposed a hypothesis test based on the central limit theorem for the linear spectral statistics of the data matrix. We also introduced a test for rank estimation that does not require any prior information on the rank of the signal. When the density of the noise is not known, we improved the test by applying an entrywise transformation.

We believe that the proposed tests can be extended to the spiked rectangular matrices where we may form sample covariance matrices (Gram matrices) and apply the central limit theorem for the linear spectral statistics. This will be discussed in our future works.

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A Examples and Simulations

In Appendix A, we numerically check the errors of the proposed tests in Algorithms 1 and 2 and the test for rank estimation in Algorithm 3 under various settings.

A.1 Spiked Gaussian Wigner model

We consider the simplest case of the spiked Gaussian Wigner model where \( w_2 = 2 \) (i.e., \( H \) is a GOE matrix) and the signal \( x(m) = (x_1(m), x_2(m), \ldots, x_N(m)) \) where \( \sqrt{N} x_i(m) \)'s are i.i.d. Rademacher random variable. Note that the parameters \( w_2 = 2 \) and \( w_4 = 3 \).

In the numerical simulation done in Matlab, we generated 10,000 independent samples of the \( 256 \times 256 \) data matrix \( M \), where we fix \( k_1 = 1 \) (under \( H_1 \)) and vary \( k_2 \) from 2 to 5 (under \( H_2 \)), with the SNR \( \lambda \) varying from 0 to 0.7. To apply Algorithm 1, we compute

\[ L_\lambda = -\log \det ((1 + \lambda) I - \sqrt{\lambda} M) + \frac{\lambda N}{2}. \]  

(A.1)

We accept \( H_1 \) if

\[ L_\lambda \leq \frac{m_{k_1} + m_{k_2}}{2} = -\frac{k_2 + 2}{2} \log(1 - \lambda) \]
Figure 1: The errors from the simulation with Algorithm 1 (solid) versus the limiting errors (A.2) (dashed) for the setting in Section A.1 with $k_2 = 2, 3, 4, 5$.

and accept $H_2$ otherwise. The (theoretical) limiting error of the test is

$$erfc\left(\frac{k_2 - 1}{4\sqrt{-\log(1-\lambda)}}\right).$$ (A.2)

In Figure 1, we compare the error from the numerical simulation and the theoretical error of the proposed algorithm, which show that the numerical errors of the test closely match the theoretical errors.

### A.2 Spiked Wigner model

We next consider a spiked Wigner model with non-Gaussian noise, where the density function of the noise matrix is given by

$$g(x) = g_d(x) = \frac{1}{2\cosh(\pi x/2)} = \frac{1}{e^{\pi x/2} + e^{-\pi x/2}}.$$

(See Example 1 of [14].) We sample $W_{ij} = W_{ji}$ from the density $g$ and let $H_{ij} = W_{ij}/\sqrt{N}$. We again let the signal $x(m) = (x_1(m), x_2(m), \ldots, x_N(m))$ where $\sqrt{N}x_i(m)$’s are i.i.d. Rademacher random variable. Note that the parameters $w_2 = 1$ and $w_4 = 5$. We again perform the numerical simulation 10,000 samples of the $256 \times 256$ data matrix $M$ with the SNR $\lambda$ varying from 0 to 0.6, where we fix $k_1 = 1$ (under $H_1$) and $k_2 = 3$ (under $H_2$).

In Algorithm 1 we compute

$$L_\lambda = -\log \det \left((1 + \lambda)I - \sqrt{\lambda}M\right) + \frac{\lambda N}{2} + \sqrt{\lambda} \mathrm{Tr} M - \frac{\lambda}{4}(\mathrm{Tr} M^2 - N).$$ (A.3)

We accept $H_1$ if

$$L_\lambda \leq \frac{m_{k_1} + m_{k_2}}{2} = -\frac{k_2 + 2}{2}\log(1-\lambda) + \frac{k_2\lambda}{2} - \frac{(k_2-3)\lambda^2}{8}.$$
and accept $H_2$ otherwise. The (theoretical) limiting error of the test is
\[
\text{erfc} \left( \frac{k_2 - 1}{4} \sqrt{-\log(1 - \lambda) + \lambda - \frac{\lambda^2}{4}} \right).
\] (A.4)

We can further improve the test by introducing the entrywise transformation given by
\[
h(x) = -\frac{g'(x)}{g(x)} = \frac{\pi}{2} \tanh\left( \frac{\pi x}{2} \right).
\]
The Fisher information $F^H = \frac{\pi^2}{8}$, which is larger than 1. We thus construct a transformed matrix $\widetilde{M}$ by
\[
\widetilde{M}_{ij} = \frac{2\sqrt{2}}{\pi\sqrt{N}} h(\sqrt{N}M_{ij}) = \sqrt{\frac{2}{N}} \tanh\left( \frac{\pi \sqrt{N}}{2} M_{ij} \right).
\]
If $\lambda > \frac{1}{\pi\sqrt{8}} = \frac{8}{\pi^2}$, we can apply PCA for strong detection of the signal. If $\lambda < \frac{8}{\pi^2}$, applying Algorithm 2, we compute
\[
\bar{L}_\lambda = -\log \det \left( (1 + \frac{\pi^2 \lambda}{8})I - \sqrt{\frac{\pi^2 \lambda}{8}} \widetilde{M} \right) + \frac{\pi^2 \lambda N}{16} + \frac{\pi \sqrt{\lambda}}{2\sqrt{2}} \text{Tr} \, \widetilde{M} + \frac{\pi^2 \lambda}{16} (\text{Tr} \, \widetilde{M}^2 - N).
\]
(Here, $F^H = F^{d}_d = \frac{\pi^2}{8}$, $G^H = \frac{\pi^2}{16}$, and $\bar{w}_4 = \frac{3}{2}$.) We accept $H_1$ if
\[
\bar{L}_\lambda \leq -\frac{k_2 + 2}{2} \log \left( 1 - \frac{\pi^2 \lambda}{8} \right) + \frac{k_2 \pi^2 \lambda}{16} - \frac{(2k_2 + 3)\pi^4 \lambda^2}{512}.
\]
and accept $H_2$ otherwise. The limiting error with entrywise transformation is

$$\text{erfc} \left( \frac{k_2 - 1}{4} \sqrt{-\log\left(1 - \frac{\pi^2 \lambda}{8}\right) + \frac{\pi^2 \lambda}{8}} \right),$$

(A.5)

Since $\text{erfc}(\cdot)$ is a decreasing function and $\frac{\pi^2}{8} > 1$, it is immediate to see that the limiting error in (A.5) is strictly smaller than the limiting error in (A.4).

In Figure 2, we plot the result of the simulation with $k_2 = 3$, which shows that the numerical error from Algorithm 2 is smaller than that of Algorithm 1; both errors closely match theoretical errors in (A.5) and (A.4).

### A.3 Rank Estimation

We again consider the example in Section A.1 and apply Algorithm 3 to estimate the rank of the signal. We again perform the numerical simulation 1,000 samples of the $256 \times 256$ data matrix $M$ with the SNR $\lambda$ varying 0.025 to 0.6 and choose the rank of the signal $k$ uniformly from 0 to 5. We compute the same test statistic

$$L_\lambda = -\log \det \left((1 + \lambda)I - \sqrt{\lambda}M\right) + \frac{\lambda N}{2},$$

(A.6)

and find the nearest nonnegative integer of the value

$$\left\lfloor \frac{L_\lambda}{\log(1 - \lambda)} - \frac{1}{2} \right\rfloor.$$

(A.7)
rounding half down. Since $\mathbb{P}(k = 0) = 0.2$, the limiting error of the estimation is

$$0.9 \cdot \text{erf} \left( \frac{1}{4} \sqrt{-\log(1-\lambda)} \right). \quad (A.8)$$

The result of the simulation can be found in Figure 3 where we compare the error from the estimation (Algorithm 3) and the theoretical error in (A.8).

## B Details of the proof of Theorem 2.2

In Appendix B we provide the details of the proof of our first main result, Theorem 2.2. Throughout Appendix B we consider the following generalization of the spiked Gaussian Wigner matrices.

**Definition B.1** (Generalized spiked Gaussian Wigner matrix). An $N \times N$ matrix

$$Y = \sqrt{\frac{\lambda}{N}} X^* X^{*T} + W \quad (B.1)$$

is a (rank-$k$) spiked Gaussian Wigner matrix with the signal-to-noise ratio (SNR) $\lambda$ with the spike $X^*$ and SNR $\lambda$ if $W = \sqrt{N}H$ for a Wigner matrix $H$ and $X^* = [x^*(1), x^*(2), \ldots, x^*(k)] \in \mathbb{R}^{N \times k}$.

We assume that for $i = 1, 2, \ldots, N$ and $\ell = 1, 2, \ldots, k$ the elements $x_i^*(\ell)$ columns of the $i$-th column of the spike matrix are independent with prior distributions $\mathcal{P}_{x_i}$ on $\mathbb{R}$ having bounded support with mean zero and unit variance. In case $\mathcal{P}_{x_i} = \mathcal{P}$ for $\ell = 1, \ldots, k$, it reduces to the spiked Gaussian Wigner matrix defined in Definition 1.3.

**Notational Remark B.2.** We use the standard big-O and little-o notation: $a_N = O(b_N)$ implies that there exists $N_0$ such that $a_N \leq Cb_N$ for some constant $C > 0$ independent of $N$ for all $N \geq N_0$; $a_N = o(b_N)$ implies that for any positive constant $\varepsilon$ there exists $N_0$ such that $a_N \leq \varepsilon b_N$ for all $N \geq N_0$.

For $X$ and $Y$, which can be deterministic numbers and/or random variables depending on $N$, we use the notation $X = O(Y)$ if for any (small) $\varepsilon > 0$ and (large) $D > 0$ there exists $N_0 \equiv N_0(\varepsilon, D)$ such that $\mathbb{P}(|X| > N\varepsilon | Y) < N^{-D}$ whenever $N > N_0$.

For an event $\Omega$, we say that $\Omega$ holds with high probability if for any (large) $D > 0$ there exists $N_0 \equiv N_0(D)$ such that $\mathbb{P}(\Omega^c) < N^{-D}$ whenever $N > N_0$.

For a sequence of random variables, the notation $\Rightarrow$ denotes the convergence in distribution as $N \to \infty$.

Recall that we only consider the case $k_1 = 0$ and $k_2 = k$. Let $\mathcal{P}_0 := \mathcal{P}_{x_1} \otimes \cdots \otimes \mathcal{P}_{x_k}$. By Bayes’s rule,

$$d\mathbb{P}_2(X|Y) = \frac{e^{-H^*(X)} d\mathbb{P}_0^{\otimes N}(X)}{\int e^{-H^*(X)} d\mathbb{P}_0^{\otimes N}(X)}.$$  

For a positive integer $n$ and a function $f : (\mathbb{R}^{N \times k})^{n+1} \to \mathbb{R}$, the Gibbs average of $f$ with respect to $H$ is defined as

$$\left\langle f(X^{(1)}, \ldots, X^{(n)}, X^*) \right\rangle := \frac{\int f(X^{(1)}, \ldots, X^{(n)}, X^*) d\mathbb{P}_0^{\otimes N}(X^{(1)}) \cdots d\mathbb{P}_0^{\otimes N}(X^{(n)})}{(\int e^{-H^*(X)} d\mathbb{P}_0^{\otimes N}(X))^n}. \quad (B.2)$$

The variables $X^{(\ell)}, \ell = 1, \ldots, n$ are oftentimes called *replicas*, which are random samples independently...
drawn from the posterior. Following [15], we let

\[ R_{\ell,\ell'}(m,s) := x^{(\ell)}(m) \cdot x^{(\ell')}(s) = \frac{1}{N} \sum_{i=1}^{N} x_i^{(\ell)}(m) x_i^{(\ell')}(s) \quad (B.3) \]

for \( \ell, \ell' = 1, \ldots, n \) and \( m, s = 1, 2, \ldots, k \). The overlap for the rank-\( k \) model is a \( k \times k \) matrix

\[ R_{k,1,\ast} = \frac{1}{N} X^{(1)T} X^* = \frac{1}{N} \sum_{i=1}^{N} X_i^{(1)} X_i^{*T} = [R_{1,\ast}(m,s)]_{1 \leq m,s \leq k}. \quad (B.4) \]

We remark that the Nishimori property [23] holds for this model; the \((n+1)\)-tuples \((X^{(1)}, \ldots, X^{(n)}, X^{(n+1)})\) and \((X^{(1)}, \ldots, X^{(n)}, X^*)\) have the same distribution under \(E_{P_2} \langle \cdot \rangle\), which generalizes the property in Section 4.2 of [15]. In particular, under \(P_2\), the distribution of the overlap \(R_{k,1,\ast}\) between a replica and the spike is equal to that of the overlap \(R_{k,2,\ast}\) between two replicas.

Recall that our proof of Theorem 2.2 is based on the fact that the characteristic function of the log LR converges to a Gaussian. By differentiating the characteristic function \(\phi_N\) defined in (3.4), we find that the following generalization of Proposition 9 in [15] directly implies Theorem 2.2:

**Proposition B.3.** For all \(\lambda < \lambda_c\) and \(s \in \mathbb{R}\), there exists \(K = K(\lambda,s) < \infty\) such that

\[ \mathbb{E} \left[ (N \langle (R_{1,\ast}(\ell,m))^2 \rangle - \langle x_N(\ell)^2x_N^*(m)^2 \rangle) e^{is \log \mathcal{L}} \right] = \left( \frac{\lambda}{1-\lambda} + \frac{\lambda}{1-\lambda} \frac{\kappa_m}{w_2} \right) \mathbb{E} \left[ e^{is \log \mathcal{L}} \right] + O(N^{-\frac{1}{2}}), \]

where \(\kappa_m = \mathbb{E}_{P_{\ast,m}} [X^3]\).

In the rest of Appendix B we prove Proposition B.3

### B.1 Preliminary bounds

As in [15], we apply the interpolation trick for the proof of Proposition B.3. We collect a few results that will be repeatedly used in the proof. In what follows, we use the notation

\[ R_{\ell,\ell'}(m,s) = \frac{1}{N} \sum_{i=1}^{N-1} x_i^{(\ell)}(m) x_i^{(\ell')}(s), \]

and let

\[ R_{k,\ell} = [R_{\ell,\ell'}(m,s)]_{1 \leq m,s \leq k}. \]
Let \( \{H_t^k : t \in [0,1]\} \) be the family of interpolating Hamiltonians defined by

\[
-H_t^k(X) := \sum_{1 \leq i < j \leq N-1} \left( \sqrt{\frac{\lambda}{N}} W_{ij} X_i^T X_j + \frac{\lambda}{N} X_i^T X_j X_i^* X_j^* - \frac{\lambda}{2N} (X_i^T X_j)^2 \right) + \frac{1}{w_2} \sum_{i=1}^{N-1} \left( \sqrt{\frac{\lambda}{N}} W_{ii} X_i^T X_i + \frac{\lambda}{N} X_i^T X_i X_i^* X_i^* - \frac{\lambda}{2N} (X_i^T X_i)^2 \right) + \frac{1}{w_2} \sqrt{\frac{\lambda}{N}} W_{NN} X_N^T X_N + \frac{1}{w_2} \frac{\lambda}{N} X_N^T X_N X_N^* X_N^* - \frac{1}{w_2} \frac{\lambda}{2N} (X_N^T X_N)^2.
\]

We denote by \( \langle \cdot \rangle_t \) the corresponding Gibbs average and use the following notation of Talagrand:

\[
\nu_t(f) := \mathbb{E}(f)_t,
\]

for a function \( f \) of \( n \) replicas \( X^{(\ell)} \), \( \ell = 1, \cdots, n \). We set \( \nu_1 \equiv \nu \). By definition, \( H_t^k = H \). Further, we notice that the variable \( x_N \) decouples from the other variables at \( t = 0 \).

**Lemma B.4.** Let \( f \) be a function of \( n \) replicas \( X^{(1)}, \cdots, X^{(n)} \) and \( X^* \). Then

\[
\nu_t^k(f) = \sum_{m,s=1}^{k} \nu_t(f, m, s)
\]

where

\[
\nu_t(f, m, s) = \frac{\lambda}{2} \sum_{1 \leq \ell \neq \ell' \leq n} \nu_t(R_{\ell, \ell'}^- (m, s)y^{(\ell)}(m)y^{(\ell')}(s)f) - \lambda n \sum_{\ell=1}^{n} \nu_t(R_{\ell, n+1}^- (m, s)y^{(\ell)}(m)y^{(n+1)}(s)f) + \lambda \sum_{\ell=1}^{n} \nu_t(R_{\ell, n+1}^- (m, s)y^{(\ell)}(m)y^*(s)f) - \lambda n \nu_t(R_{n+1, n+2}^- (m, s)y^{(n+1)}(m)y^*(s)f) + \lambda n \frac{n(n+1)}{2} \nu_t(R_{n+1, n+2}^- (m, s)y^{(n+1)}(m)y^{(n+2)}(s)f) + \frac{1}{w_2} \frac{\lambda}{2N} \sum_{1 \leq \ell \neq \ell' \leq n} \nu_t(y^{(\ell)}(m)^2 y^{(\ell')}(s)^2 f) - \frac{1}{w_2} \frac{\lambda n}{2N} \sum_{\ell=1}^{n} \nu_t(y^{(\ell)}(m)^2 y^{(n+1)}(s)^2 f) + \frac{1}{w_2} \frac{\lambda}{2N} \sum_{\ell=1}^{n} \nu_t(y^{(\ell)}(m)^2 y^*(s)^2 f) - \frac{1}{w_2} \frac{\lambda n}{2N} \nu_t(y^{(n+1)}(m)^2 y^*(s)^2 f) + \frac{1}{w_2} \frac{\lambda n(n+1)}{2N} \nu_t(y^{(n+1)}(m)^2 y^{(n+2)}(s)^2 f),
\]

and we have written \( y = x_N \).

**Proof.** The proof follows from the Gaussian integration by parts. See, e.g., [29].
Lemma B.5. If $f$ is a bounded nonnegative function, then for all $t \in [0, 1]$,

$$\nu_t(f) \leq K(\lambda, n)\nu(f).$$

Proof. By Grönwall’s inequality, it suffices to show for all $t \in [0, 1]$ that

$$|\nu_t'(f)| \leq K(\lambda, n)\nu_t(f),$$

which follows from Lemma B.4 and that all the variables and the overlaps are bounded. □

### B.2 Generalization of Proposition 9 in [15]

Following the proof of Proposition 9 in [15], we consider self-consistency relations among various quantities. More precisely, we prove that for any $\lambda < 1$,

$$(1 - \lambda)N\mathbb{E} \left[ \langle (R_{1,\ast}(\ell, m))^2 \rangle e^{i\lambda \mathcal{L}} \right] = \mathbb{E} \left[ \langle x_N(\ell)^2 x_N(m)^2 \rangle e^{i\lambda \mathcal{L}} \right] + \frac{\lambda N\kappa m}{w_2} \mathbb{E} \left[ e^{i\lambda \mathcal{L}} \right] + \delta,$$  \hspace{1cm} (B.5)

and

$$\mathbb{E} \left[ \langle x_N(\ell)^2 x_N(m)^2 \rangle e^{i\lambda \mathcal{L}} \right] = \mathbb{E} \left[ e^{i\lambda \mathcal{L}} \right] + \delta,$$  \hspace{1cm} (B.6)

where $|\delta| \leq K(\lambda)N\mathbb{E} \langle |R_{1,\ast}(\ell, m)|^3 \rangle$. We note that the main challenge of the proof is to obtain the optimal convergence rate of the third moment of each diagonal elements of overlap matrix $R_{1,\ast}$, under $\mathbb{E}[\cdot]$, which is $O(N^{-3/2})$. Once we have the optimal rate, we find $\delta = O(N^{-1/2})$ and the desired result can be obtained as in Section 6 of [15]. For example, (B.5) can be proved by the cavity computation with the family of interpolating Hamiltonians $H^k_t$ for $t \in [0, 1]$ and associated functions

$$X(t) := \exp \left( is \log \int e^{-H^k_t(X)} d\mathcal{P}_0 \otimes N(X) \right),$$  \hspace{1cm} (B.7)

and

$$\varphi(t) := N\mathbb{E} \left[ \langle x_N(\ell)x_N(m)R_{1,\ast}(\ell, m) \rangle_{\cdot} X(t) \right],$$  \hspace{1cm} (B.8)

where $\langle \cdot \rangle_\cdot$ is the Gibbs average under the Hamiltonian $H^k_t$. Then we obtain $\varphi(0) = 0$ and

$$\varphi'(0) = \lambda N\mathbb{E} \left[ \langle x_N(\ell)^2 x_N(m)^2 (R_{1,\ast}^{-1}(\ell, m))^2 \rangle_0 X(0) \right] \nonumber \quad + \frac{\lambda}{w_2} \mathbb{E} \left[ \langle x_N(\ell)^3 x_N(m)^3 R_{1,\ast}^{-1}(\ell, m) \rangle_0 X(0) \right].$$

We now return to the proof of the optimal rate of convergence of the third moment of the overlap matrix. The proof is based on the following result on the convergence of the fourth moment.

**Proposition B.6.** For all $\lambda < \lambda_c$ and $m, s = 1, \ldots, k$, there exists a constant $K = K(\lambda) < \infty$ such that

$$\mathbb{E} \langle (R_{1,\ast}(m, s))^4 \rangle \leq \frac{K}{N^2}.$$  

Note that we immediately obtain from Proposition B.6 and Hölder’s inequality that

$$\mathbb{E} \langle (R_{1,\ast}(m, s))^3 \rangle \leq K(\lambda) \mathbb{E} \langle (R_{1,\ast}(m, s))^4 \rangle^{3/4} \leq \frac{K(\lambda)}{N^{3/2}}.$$
B.3 Overlap convergence

It remains to prove Proposition B.6, which asserts the convergence of overlaps to zero under $\mathbb{P}_2$. We begin by proving the following estimate on the diagonal elements of the overlap matrix $R_{1,s}^k$: for all $\varepsilon > 0$,

$$\max_{1 \leq m \leq k} \mathbb{E}(\mathbb{1}\{|R_{1,s}(m, m)| \geq \varepsilon\}) \leq Ke^{-cN}. \quad (B.9)$$

With this convergence in probability, we prove the estimate on the second moment,

$$\mathbb{E}(|R_{1,s}(m, m)|^2) \leq \frac{K}{N},$$

which will result in the conclusion of Proposition B.6.

We also use the following estimate for the off-diagonal elements of the overlap matrix.

Lemma B.7. For any $p \geq 1$,

$$\mathbb{E}(|R_{1,s}(m, s)|^{2p}) \leq (\mathbb{E}(|R_{1,s}(m, m)|^{2p}))^{1/2} (\mathbb{E}(|R_{1,s}(s, s)|^{2p}))^{1/2}. \quad (B.10)$$

Proof. We first expand the $2p$-th moment as

$$\mathbb{E}(|R_{1,s}(m, s)|^{2p}) = \frac{1}{N^{2p}} \mathbb{E} \left( \sum_{i_1, i_2, \ldots, i_{2p} = 1}^N x_{i_1}(m)x_{i_2}(m) \cdots x_{i_{2p}}(m) \cdot x_{i_1}^*(s)x_{i_2}^*(s) \cdots x_{i_{2p}}^*(s) \right)$$

$$= \frac{1}{N^{2p}} \sum_{i_1, i_2, \ldots, i_{2p} = 1}^N \mathbb{E} \left( x_{i_1}(m)x_{i_2}(m) \cdots x_{i_{2p}}(m) \cdot x_{i_1}^*(s)x_{i_2}^*(s) \cdots x_{i_{2p}}^*(s) \right).$$

Thus, by Schwartz inequality,

$$\mathbb{E}(|R_{1,s}(m, s)|^{2p}) \leq (\mathbb{E}(|R_{1,2}(m, m)|^{2p}))^{1/2} (\mathbb{E}(|R_{1,s}(s, s)|^{2p}))^{1/2}$$

$$= (\mathbb{E}(|R_{1,s}(m, m)|^{2p}))^{1/2} (\mathbb{E}(|R_{1,s}(s, s)|^{2p}))^{1/2}.$$

Here, the last equality follows from Nishimori property. \qed

We next introduce the precise statement for (B.9).

Proposition B.8. For all $0 < \lambda_0 < \lambda < \lambda_c$ and $\varepsilon > 0$, there exist constants $K = K(\lambda, \varepsilon) \geq 0$, $c = c(\lambda, \varepsilon, \mathbb{P}_0) \geq 0$ such that

$$\max_{1 \leq m \leq k} \mathbb{E}(\mathbb{1}\{|R_{1,s}(m, m)| \geq \varepsilon\}) \leq Ke^{-cN}.$$

Proof. It directly follows from Proposition 7 in [13] (with the choice $p = 2$ case). \qed

B.4 Proof of Proposition B.6

Adapting the strategy in [13], we use the following lemmas for the proof:

Lemma B.9. For any $p \geq 1$ and $m, s = 1, 2, \ldots, k$,

$$| (R_{1,s}(m, s))^{p+1} - (R_{1,s}^{-}(m, s))^{p+1} | \leq \frac{C(p)}{N} (|R_{1,s}(m, s)|^p + |R_{1,s}^{-}(m, s)|^p)$$
Proof. It readily follows from an elementary inequality that
\[
|x^{p+1} - y^{p+1}| \leq p |x-y| (|x|^p + |y|^p)
\]
for any \(x, y \in \mathbb{R}\) and \(p \geq 1\).

Lemma B.10. For a positive integer \(p\), suppose that there exists a constant \(C \geq 1\) such that
\[
\nu((R_{1,s}(m,s))^{2j}) \leq \frac{C}{N^j}
\]
for any \(0 \leq j \leq p\) and \(m, s = 1, 2, \ldots, k\). Then,
\[
\nu((R_{1,s}(m,s))^{2p}) \leq \frac{C'(p)}{N^p}.
\]

Proof. Since \(R_{1,s}(m,s) = R_{1,s}(m,s) - \frac{1}{N} x_N(m) x_N^*(s)\), from the binomial expansion
\[
\nu((R_{1,s}(m,s))^{2p}) \leq \sum_{j=0}^{2p} \binom{2p}{j} \left( \frac{C(p)}{N} \right)^{2p-j} \nu(|R_{1,s}(m,s)|^j |x_N(m) x_N^*(s)|^{2p-j})
\]
(B.11)
\[
\leq \sum_{j=0}^{2p} \binom{2p}{j} \frac{C(p)}{N^{2p-j}} \nu(|R_{1,s}(m,s)|^j).
\]
Then, from the assumption of the lemma,
\[
\nu((R_{1,s}(m,s))^{2p}) \leq C \sum_{j=0}^{2p} \binom{2p}{j} \left( \frac{C(p)}{N} \right)^{2p-j} \left( \frac{1}{\sqrt{N}} \right)
\]
(B.12)
\[
\leq C \left( \frac{C(p)}{N} + \frac{1}{\sqrt{N}} \right)^{2p} \leq \frac{C'(p)}{N^p},
\]
where we used the Schwarz inequality \(\nu(|R_{1,s}(m,s)|^j) \leq \nu(|R_{1,s}(m,s)|^{2j})^{1/2}\).

From Lemma [B.7] we notice that the off-diagonal entries of the overlap matrix can be bounded by the diagonal entries in the sense of \(2p\)-th moments, i.e.,
\[
\mathbb{E}((R_{1,s}(m,s))^{2p}) \leq \max_{1 \leq \ell \leq k} \mathbb{E}((R_{1,s}(\ell,\ell))^{2p})
\]
(B.13)
for any \(1 \leq m, s \leq k\). Thus, we only focus on the diagonal entries of the overlap matrix as in the following lemma:

Lemma B.11. There exist nonnegative continuous functions \(C_1, C_2,\) and \(C_3\) of \(\lambda\), independent of \(N\), satisfying
\[
\max_{1 \leq m \leq k} \nu((R_{1,s}(m,m))^2) \leq C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,s}(m,m)))^{1/2} \max_{1 \leq m \leq k} \nu((R_{1,s}(m,m))^2)^{1/2}
\]
\[
+ \lambda \max_{1 \leq m \leq k} \nu((R_{1,s}(m,m))^2)
\]
\[
+ \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,s}(m,m))^2)^{1/2} + \frac{C_3(\lambda)}{N}.
\]
(B.14)

Further, \(C_1\) is nondecreasing and \(C_1(0) = 0\).
Proof. By the symmetry between variables, we have
\[
\nu((R_{1,*}(m,m))^2) = \nu(x_N(m)x_N^*(m)(R_{1,*}(m,m))) = \nu(x_N(m)x_N^*(m)(R_{1,-*}(m,m))) + \Delta
\]
where \(\Delta = \nu(x_N(m)x_N^*(m)((R_{1,*}(m,m)) - (R_{1,-*}(m,m))))\). Then, by Lemmas [B.9] and [B.10]
\[
|\Delta| \leq C\nu \left( (R_{1,*}(m,m)) - (R_{1,-*}(m,m)) \right) \leq \frac{C'}{N}.
\]
Thus,
\[
\nu((R_{1,*}(m,m))^2) \leq \nu(x_N(m)x_N^*(m)(R_{1,-*}(m,m))) + \frac{C'(p)}{N}.
\]
To estimate the first term of right-hand side, we let \(f = x_N(m)x_N^*(m)(R_{1,-*}(m,m))\) and apply Lemma [B.4]. Note that \(\nu_0(f) = 0\) since \(\mathcal{P}_0\) is centered. Then by Taylor’s theorem,
\[
\nu((R_{1,*}(m,m))^2) \leq \nu(f) + \frac{C'}{N} \leq \nu'_0(f) + \frac{1}{2} \sup_{0 \leq t \leq 1} |\nu''(f)| + \frac{C'}{N}.
\]
From Lemma [B.4] with \(n = 1\), we also have that
\[
\nu'_0(f) = -\sum_{m',s'=1}^k \lambda \nu_0(A(1,2,m',s',m)) + \sum_{m',s'=1}^k \lambda \nu_0(A(1,*,m',s',m))
- \sum_{m',s'=1}^k \lambda \nu_0(A(2,*,m',s',m)) + \sum_{m',s'=1}^k \lambda \nu_0(A(2,3,m',s',m))
- \sum_{m',s'=1}^k \frac{1}{(w_2)^2} \frac{\lambda}{2N} \nu_0(B(1,2,m',s',m)) + \sum_{m',s'=1}^k \frac{1}{w_2} \frac{\lambda}{N} \nu_0(B(1,*,m',s',m))
- \sum_{m',s'=1}^k \frac{1}{w_2} \frac{\lambda}{N} \nu_0(B(2,*,m',s',m)) + \sum_{m',s'=1}^k \frac{1}{(w_2)^2} \frac{\lambda}{N} \nu_0(B(2,3,m',s',m))
= \lambda \nu_0((R_{1,*}(m,m))^2) + \frac{1}{w_2} \frac{\lambda \kappa^2}{N} \nu_0((R_{1,-*}(m,m))),
\]
where
\[
A(a,b,m',s',m) = y^{(a)}(m')y^{(b)}(s')y^{(1)}(m)y^*(m)(R_{a,b}(m',s'))(R_{1,-*}(m,m))
\]
and
\[
B(a,b,m',s',m) = y^{(a)}(m')y^{(b)}(s')y^{(1)}(m)y^*(m)(R_{1,-*}(m,m)).
\]
Obviously, the second term is bounded by \(\frac{1}{\lambda} \nu((R_{1,*}(m,m)))\). Moreover, by using Lemma [B.4] we can observe that \(\nu''(f)\) is represented by linear combination of functions of the following forms:
\begin{itemize}
  \item \(\lambda^2 \nu(R_{\ell_1,\ell_2}(m_1,s_1)R_{\ell_3,\ell_4}(m_2,s_2)y^{(\ell_1)}(m_1)y^{(\ell_2)}(s_1)y^{(\ell_3)}(m_2)y^{(\ell_4)}(s_2)f)\)
  \item \(\lambda^2 \nu(R_{\ell_1,\ell_2}(m_1,s_1)y^{(\ell_1)}(m_1)y^{(\ell_2)}(s_1)y^{(\ell_3)}(m_2)y^{(\ell_4)}(s_2)f)\)
  \item \(\lambda^2 \nu(y^{(\ell_1)}(m_1)y^{(\ell_2)}(s_1)y^{(\ell_3)}(m_2)y^{(\ell_4)}(s_2)f)\)
\end{itemize}
where \(\ell_1 \neq \ell_2\) and \(\ell_3 \neq \ell_4\). For the terms of the first form, we have for any \(1 \leq \ell_1 \neq \ell_2 \leq n, 1 \leq \ell_3 \neq \ell_4 \leq n\).
and $1 \leq m_1, s_1, m_2, s_2 \leq k$ that

$$
\nu_t([R^+_1(m, m)] [R^+_{1, t_2}(m_1, s_1)] [R^+_{1, t_2}(m_2, s_2)])
\leq \nu_t([R^+_1(m, m)]^2)^{1/4} \nu_t([R^+_{1, t_2}(m_1, s_1)]^4)^{1/4} \nu_t([R^+_{1, t_2}(m_2, s_2)]^4)^{1/4}
\leq C \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^3)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2},
$$

where we used the generalized Hölder’s inequality, the Nishimori property, Lemma B.11 and (B.13). The other terms are obviously $O(N^{-1})$. Further, by Lemma B.9, for any $h \in \mathbb{N}$

$$
\nu([R^+_1(m, m)]^{|h+1}| \leq \nu([R^+_1(m, m)]^h) + \frac{C}{N} \nu([R^+_1(m, m)]^h) + \nu([R^+_1(m, m)]^h)). \tag{B.15}
$$

Thus,

$$
|\nu_t'(f)| \leq \frac{C_1' \nu([R^+_1(m, m)]^3) + \frac{C}{N} \nu([R^+_1(m, m)]^2) + \frac{C}{N}}{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2}
\leq C_1' \nu([R^+_1(m, m)]^3) + \frac{C_1'' \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2}
\leq \frac{C_1'' \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3'' \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3 \nu([R^+_1(m, m)]^2)^{1/2}}{N}.

With the same argument with $f = (R^+_1(m, m))^2$, we also get

$$
\nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2}
\leq \frac{C_1'' \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2}
\leq \frac{C_1'' \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3'' \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3 \nu([R^+_1(m, m)]^2)^{1/2}}{N}.

In particular,

$$
|\nu_t''([R^+_1(m, m)]^2)| \leq \frac{C_1'' \nu([R^+_1(m, m)]^3)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2}
\leq \frac{C_2 \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3'' \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3 \nu([R^+_1(m, m)]^2)^{1/2}}{N}.

since $\nu_0([R^+_1(m, m)]^2) \leq \frac{C}{N} \nu([R^+_1(m, m)]^2) \leq \frac{C}{N}$. Thus, we get the desired bound.

From Lemma B.11, we obtain the following lemma.

**Lemma B.12.** For $\lambda < \lambda_c$,

$$
\limsup_{N \to \infty} N \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2) < \infty \tag{B.17}
$$

**Proof.** Assume the contrary. Using Lemma B.11, we get

$$
\max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2) \leq (C_1' + \lambda) \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)
\leq \frac{C_2 \nu([R^+_1(m, m)]^2)^{1/2} \max_{1 \leq m \leq k} \nu([R^+_1(m, m)]^2)^{1/2} + \frac{C_3 \nu([R^+_1(m, m)]^2)^{1/2}}{N}.

Set $\lambda_0 = \sup \{r \in (0, 1) \mid C_1'(r) + r < \lambda < \lambda_c\}$. Note that $\lambda_0 > 0$ since $C_1'(r) + r$ is nondecreasing and $C_1'(0) = 0$. We consider the following cases:
Case 1. If $\lambda \in [0, \lambda_0)$,

$$(1 - C_1' \lambda - \lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \leq \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N}. \quad (B.18)$$

Thus

$$\begin{align*}
(1 - C_1' R_{1,*}(m, m))^2 & \leq C_2(\lambda) + \sqrt{N} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \quad (B.19) \end{align*}$$

if $\max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) > 0$. Therefore, for some subsequence $N_k$, the right-hand side is bounded but the left-hand side diverges to infinity as $N_k \to \infty$, contradiction.

Case 2. If $\lambda \in [\lambda_0, \lambda_\ast)$, then we can apply Proposition [B.8]. Recall that

$$\max_{1 \leq m \leq k} \mathbb{E} \{I \{|R_{1,*}(m, m)| \geq \epsilon\} \leq K e^{-cN}.$$ 

Thus, by Lemma [B.11]

$$\begin{align*}
\max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) & \leq C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \mathbb{1}\{|R_{1,*}(m, m)| \geq \epsilon\} \leq \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N} \\
& + \lambda \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) + \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N} \\
& \leq C_4(\lambda) \nu(I \{|R_{1,*}(m, m)| \geq \epsilon\}) + \epsilon^{1/2} C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \\
& \quad + \lambda \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) + \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N} \\
& \leq C_4(\lambda) \nu(I \{|R_{1,*}(m, m)| \geq \epsilon\}) + \epsilon^{1/2} C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \\
& \quad + \lambda \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) + \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N}.
\end{align*}$$

Hence, if we take $\epsilon > 0$ so small that $\epsilon^{1/2} C_1(\lambda) + \lambda < 1$, which is possible since $\lambda < \lambda_\ast \leq 1$, we get

$$\begin{align*}
(1 - \lambda - \epsilon^{1/2} C_1(\lambda)) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) & \leq C_4(\lambda) \nu(I \{|R_{1,*}(m, m)| \geq \epsilon\}) + \epsilon^{1/2} C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \\
& \quad + \lambda \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) + \frac{C_2(\lambda)}{\sqrt{N}} \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2)^{1/2} + \frac{C_3(\lambda)}{N} \\
& \leq C_4(\lambda) Ke^{-cN} + \epsilon^{1/2} C_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \quad (B.20)
\end{align*}$$

We can draw the same conclusion as in Case 1.

Lemma [B.12] implies that for any $1 \leq m, s \leq k$

$$\nu((R_{1,*}(m, m))^2) \leq \max_{1 \leq m \leq k} \nu((R_{1,*}(m, m))^2) \leq \frac{C}{N}$$

for some constant $C > 0$, which is the estimate on the second moment that we want. The corresponding result for the fourth moment can be proved in a similar manner, and we only state the series of lemmas that lead us to the conclusion.
Lemma B.13. For \( N \geq 1 \),
\[
\max_{1 \leq m \leq k} \nu((R_{1,*}(m,m))^4) \\
\leq K_1(\lambda) \max_{1 \leq m \leq k} \nu((R_{1,*}(m,m))^4)^{1/2} \max_{1 \leq m \leq k} \nu(|R_{1,*}(m,m)|^5)^{1/2} \\
+ \lambda \max_{1 \leq m \leq k} \nu((R_{1,*}(m,m))^4) + \frac{K_2(\lambda)}{N} \max_{1 \leq m \leq k} \nu((R_{1,*}(m,m))^4)^{1/2} + \frac{K_3(\lambda)}{N^2}
\] (B.21)

where \( K_1, K_2 \) and \( K_3 \) are nonnegative continuous functions of the SNR \( \lambda \) and are independent of \( N \). Further, \( K_1 \) is nondecreasing and \( K_1(0) = 0 \).

Lemma B.14. For \( \lambda < \lambda_c \),
\[
\limsup_{N \to \infty} N^2 \max_{1 \leq m \leq k} \nu((R_{1,*}(m,m))^4) < \infty.
\] (B.22)

Remark B.15. The argument we used for the estimate on the fourth moment can be applied to prove similar estimates on higher moments with the different Hölder conjugate.

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