Abstract. For any multi-graph $G$ with edge weights and vertex potential, and its universal covering tree $\mathcal{T}$, we completely characterize the point spectrum of operators $A_T$ on $\mathcal{T}$ arising as pull-backs of local, self-adjoint operators $A_G$ on $G$. This builds on work of Aomoto, and includes an alternative proof of the necessary condition for point spectrum he derived in [Aom91]. Our result gives a finite time algorithm to compute the point spectrum of $A_T$ from the graph $G$, and additionally allows us to show that this point spectrum is contained in the spectrum of $A_G$. Finally, we prove that typical pull-back operators have a spectral delocalization property: the set of edge weight and vertex potential parameters of $A_G$ giving rise to $A_T$ with absolutely continuous spectrum is open and dense.

1. Introduction

Consider a finite graph $G = (V, E)$ and its universal cover $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, together with a covering map $\Xi : \mathcal{T} \to G$. The purpose of this paper is to relate the point spectrum of certain local, periodic, self-adjoint operators on $l^2(V)$ to the combinatorial structure of $G$.

Precise definitions, notation and assumptions about the model in consideration will be discussed below in Section 2, but for now we give a high-level overview of the problem setting. By endowing $G$ with edge weights and a potential on its vertex set, we obtain a natural self-adjoint operator $A_G$ on $l^2(V)$. This framework encompasses Schrödinger operators, weighted adjacency matrices, graph Laplacians and transition matrices for random walks, and the corresponding pull-back of the weights and potential to $\mathcal{T}$ induces an analogous periodic, self-adjoint operator $A_T$ on $l^2(\mathcal{V})$.

The class of operators $A_T$ obtained in this way contains, but is richer than, the periodic Schrödinger operators in one dimension, which are of great relevance to spectral theory and the theory of orthogonal polynomials. The spectra of these $A_T$ are additionally crucial to the study of relative expanders [F03] and, as shown in [BC19], control in a strong sense the spectra of large random lifts of a fixed base graph. Interestingly, despite the many advances in functional analysis, operator algebras and operator theory, many natural questions regarding the spectral properties of $A_T$ are unanswered and seem inaccessible with current techniques. We direct the reader to [ABS20] for a survey of both periodic Jacobi matrices and the difficulty in generalizing to the more generic case considered here.

In this paper we will be concerned with the spectrum of $A_T$, which we denote by $\text{Spec } A_T$, its density of states $\mu$ (a natural and canonical measure on $\text{Spec } A_T$), and most importantly those $\lambda \in \text{Spec } A_T$ for which there exists a corresponding $l^2$ eigenvector—in other words, the point spectrum $\text{Spec}_p A_T$.
Our main result is a set of necessary and sufficient condition on $G$ (including its edge weights and potential) for $\text{Spec}_p A_T$ to be non-empty. In particular, this gives a finite algorithm to compute $\text{Spec}_p A_T$ from $G$. This extends the work of Aomoto, who has already shown the necessary half of our result in [Aom91]. However, our new and elementary argument is essentially different from his, and we build on it to show, surprisingly, that $\text{Spec}_p A_T \subset \text{Spec} A_G$, and to give a lower bound for the multiplicity of each eigenvalue of $A_G$ arising in this way. Finally, we prove that the set of edge weights and potentials for which $A_T$ has point spectrum is a closed set of Lebesgue measure zero. This may be regarded as a spectral delocalization result of the kind long-studied in mathematical physics [Ana18]; see [AISW20] for recent and analogous work in the context of quantum graphs.

**Related Work.** The operators $A_T$ defined here have been studied by several authors with different motivations and levels of generality, and are variously referred to as operators of nearest-neighbor type [Aom91], connected, local, pull-back operators [AFH15] or periodic Jacobi operators [ABS20]; we will use the latter. When $G$ is an unweighted $d$-regular graph (making $A_G$ is its adjacency matrix), classical work of Kesten in the context of Cayley graphs [Kes59], and McKay in the context of random graphs [McK81], proved that $\text{Spec} A_T = [-2\sqrt{d-1}, 2\sqrt{d-1}]$ and that $\mu$ follows what is now called the Kesten-Mackay distribution with parameter $d$. When $G$ is an unweighted $(c,d)$-biregular bipartite graph with $c < d$, Godsil and Mohar showed that $\text{Spec} A_T = \{\lambda \in \mathbb{R} : \sqrt{d-1} - \sqrt{c-1} \leq |\lambda| \leq \sqrt{c-1} + \sqrt{d-1}\} \cup \{0\}$ and that $\mu\{0\} = \frac{d-c}{d+c}$ [GM88]. These results imply that for adjacency matrices, when $G$ is $d$-regular, $A_T$ has no point spectrum, while when $G$ is $(c,d)$-biregular and bipartite, $\text{Spec}_p A_T = \{0\}$.

Subsequent work focused on the properties of $\text{Spec} A_T$ and $\mu$, and their relation to $A_G$, without making any assumptions on $G$; see [A’88, Aom91, Sun92, SS92] as well the more recent [AFH15, BC19, ABS20, GVK19]. Of relevance for the current paper is a result of Avni, Breuer and Simon [ABS20] which states that for any $G$, any edge weights, and any potential, the operator $A_T$ has no singular continuous spectrum. As a corollary one can deduce that the continuous part of $\text{Spec} A_T$ always consists of a finite union of closed non-degenerate intervals, and its singular part is the finite set of eigenvalues of $\text{Spec}_p A_T$. Equivalently, $\mu$ can be decomposed into a measure that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and a finite sum of atomic measures.

The most noteworthy prior result regarding the point spectrum of $A_T$ is the aforementioned work of Aomoto, who in addition to deriving necessary conditions for the presence of point spectrum of $A_T$ deduced a remarkable formula relating $\mu\{\lambda\}$ to the combinatorial structure of $G$ for every $\lambda \in \text{Spec}_p A_T$. He then used these results to show that when $G$ is a $d$-regular graph, regardless of the edge weights and potential, $A_T$ has no point spectrum. This generalizes the case when $G$ is a cycle, which was established by different authors in the mathematical physics literature; see Section 2 of [ABS20] for a discussion and survey. In a different context, Keller, Lenz and Warzel [KLW13] showed that adjacency matrices of certain trees have no point spectrum and that this property is stable under small perturbations of the potential. For our setting, their results imply that if $G$ has a loop at every vertex, and if $A_G$ is the adjacency matrix of $G$, it holds that $A_T$ has no point spectrum.

Our results, stated in Section 3 after the preliminary material in Section 2, recover many of the ones above and provide a pleasant unification and generalization of the literature on point spectra.
2. Preliminaries

2.1. Graphs and Covers. We will work in the general setting of weighted graphs with self-loops and multi-edges. In this setup we will regard a graph as a tuple $G = (V, E, a, b)$, consisting of vertices, edges, edge-weights $a : E \to \mathbb{C}$ and a potential $b : V \to \mathbb{C}$. When it is not clear from the context, we will write $V(G)$ and $E(G)$ to emphasize that we are referring to the set of vertices and edges of $G$. It will be convenient to regard $E$ as a set of directed edges, equipped with a direction-reversing involution $e \mapsto \bar{e}$ with no fixed points, as well as source and terminal maps $\sigma, \tau : E \to V$ so that $\sigma(e) = \tau(\bar{e})$ for every $e \in E$. An edge for which $\sigma(e) = \tau(e)$ and $e = \bar{e}$ is a self-loop, and we refer to the remainder as proper edges.\(^1\) We will also abuse notation and write $\sigma(u)$ and $\tau(u)$ for the sets of directed edges whose source and terminal, respectively, are the vertex $u \in V$.

We say that a graph $H$ covers $G$ if there exists a covering map $\xi : H \to G$, namely a map of vertices and edges which is compatible with the source and terminal maps, preserves potential and edge weights, and is an isomorphism on $\sigma(u)$ and $\tau(u)$ for each vertex $u$. If both are finite, then $|V(G)|$ necessarily divides $|V(H)|$, and we call their ratio $n$ the degree of the cover; equivalently we say that $H$ is an $n$-lift of $G$. Each $n$-lift $H$ may be expressed explicitly by an assignment of permutations to edges $\pi : E \to \mathcal{S}_n$, with the property that $\pi_e^{-1} = \pi_e$ for each edge $e \in E$. Then $V(H) = V(G) \times [n]$—throughout the paper we will use the notation $[n] = \{1, \ldots, n\}$—and for every $e \in E(G)$ and each $i \in [n]$, we include an edge $\bar{e} \in E(H)$ with $\sigma(\bar{e}) = (\sigma(e), i)$ and $\tau(\bar{e}) = (\tau(e), \pi_e(i))$.

The universal cover of a connected graph $G$ is the unique (up to isomorphism) infinite tree $\mathcal{T} = (V, E, \alpha, \beta)$ that covers every other cover of $G$. It can be constructed directly in terms of non-backtracking walks on $G$, which are sequences of edges $e_1, e_2, \ldots, e_\ell$ such that, for every $s \in \{1, \ldots, \ell - 1\}$, $\tau(e_s) = \sigma(e_{s+1})$ and $e_s \neq \bar{e}_{s+1}$. If we choose a root vertex $u \in V$, then we may set the vertex set $V$ of $\mathcal{T}$ to be the set of non-backtracking walks on $G$ starting at $u$, with directed edges $E$ whenever one walk is an immediate prefix or suffix of another, and edge weights and potential inherited from the final edge and vertex of the walk, respectively. Up to isomorphism $\mathcal{T}$ is independent of the root choice. This is manifestly a cover of $G$, and we will call the covering map $\Xi$. Finally, we note that $\mathcal{T}$ is finite if and only if $G$ is acyclic—that is, if it does not contain a closed non-backtracking walk. In this case $G = \mathcal{T}$.

Given $G$ and a universal cover $\mathcal{T}$, the latter is endowed with a set of symmetries which act transitively on $V$ by simultaneously permuting the fibres over every vertex.

2.2. Jacobi Operators, Spectra, and the Density of States. Following the convention introduced in [ABS20], the Jacobi operator associated to $G = (V, E, a, b)$ acts on $\eta \in \ell^2(V) \cong \mathbb{C}^{|V|}$ as:

$$
(A_G \eta)(u) = b_u \eta(u) + \sum_{e \in \tau(u)} a_e \eta(\sigma(e)).
$$

(1)

Throughout the paper, we will assume that the edge weights satisfy a conjugate symmetry condition $\bar{a_e} = a_e$ and that the potential $b$ is real—these ensure that $A_G$ is Hermitian, and we will accordingly call such edge weights and potential Hermitian as well.

When $H$ is an $n$-lift of $G$, a standard result characterizes the spectrum of $A_H$, thought of as acting on $\ell^2(V) \otimes \mathbb{C}^n$, which we will always treat as the set of $\mathbb{C}^n$-valued functions on $V$. Let

\(^1\)Some authors additionally include so-called half-loops, which are edges $e$ with $\sigma(e) = \tau(e)$ and $e = \bar{e}$; see [Fri93]. Our results easily extend to this case, but for simplicity we will not consider it here.
We remind the reader that, unlike in the finite dimensional case, \( \operatorname{Spec} \) on the right hand side of (1) denotes the \emph{point spectrum}. In other words, for every bounded measurable function \( f \) on \( V \), there is a function \( \tilde{f} \) on \( \Xi \) related to those of \( \operatorname{Spec} A_T \) on \( V \) by \( \tilde{f}(e) = f(e) \) for every \( e \in E \). The regular representation \( \rho \) of \( G \), and the inherited conjugate symmetry condition \( \Xi(e) = \Xi(e)^{-1} \) guarantees that it is Hermitian. We use \( \operatorname{Spec} A_T \) to denote the \emph{spectrum} of the periodic Jacobi operator, that is

\[
\operatorname{Spec} A_T = \{ \lambda \in \mathbb{C} : (\lambda - A_T)^{-1} \notin B(\ell^2(V)) \}.
\]

We remind the reader that, unlike in the finite dimensional case, \( \lambda \in \operatorname{Spec} A_T \) does not guarantee \( \lambda - A_T \) is invertible. The subset of the spectrum with this additional property—the \emph{point spectrum}—will be our primary concern in this work. We will return to it below.

Once again writing \( T = (V, E, a, b) \) for the universal cover of \( G \), we will call the analogous operator on \( \ell^2(V) \) the \emph{periodic Jacobi operator} of \( T \). Since the edge weights \( a \) and potential \( b \) are related to those of \( G \) by \( a_{\hat{e}} = a_{\Xi(\hat{e})} \) for every \( \hat{e} \in \hat{E} \) and \( b_{\hat{v}} = b_{\Xi(\hat{v})} \) for every \( \hat{v} \in \hat{V} \), finiteness of \( G \), \( a \), and \( b \) ensure that \( A_T \) belongs to the set \( B(\ell^2(V)) \) of bounded operators on \( \ell^2(V) \), and the inherited conjugate symmetry condition \( \Xi(e) = \Xi(e)^{-1} \) guarantees that it is Hermitian. We use \( \operatorname{Spec} A_T \) to denote the \emph{spectrum} of the periodic Jacobi operator, that is

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We remind the reader that, unlike in the finite dimensional case, \( \lambda \in \operatorname{Spec} A_T \) does not guarantee \( \lambda - A_T \) is invertible. The subset of the spectrum with this additional property—the \emph{point spectrum}—will be our primary concern in this work. We will return to it below.

For any \( u \in V(G) \), the quantities \( \langle \delta_{\tilde{u}} , A_T' \delta_{\tilde{u}} \rangle \) for \( \tilde{u} \in \Xi^{-1}(u) \) are real and constant over all \( \tilde{u} \) in the fibre over \( u \), and a routine application of the Riesz representation theorem guarantees an accompanying \emph{spectral measure} \( \mu_u \) on \( \operatorname{Spec} A_T \) associated to each \( u \), satisfying

\[
\langle \delta_{\tilde{u}} , f(A_T) \delta_{\tilde{u}} \rangle = \int_{\operatorname{Spec} A_T} f(x) \, d\mu_u(x) \quad \forall \tilde{u} \in \Xi^{-1}(u)
\]

for every bounded measurable function \( f : \operatorname{Spec} A_T \rightarrow \mathbb{C} \). The \emph{density of states (DOS)} of \( A_T \) is the unique measure obtained by averaging these spectral measures over \( u \in V(G) \):

\[
\mu_u = \frac{1}{|V(G)|} \mu_u.
\]

On several occasions we will use the following well-known facts to relate the empirical spectral measures of finite graphs \( G \) to the densities of states of their universal covers. We provide a proof below, since we were not able to find this result verbatim in the literature.
Lemma 2.1. Let $G$ be a finite graph, $\mathcal{T}$ its universal cover, and $\mu$ the density of states of $A_\mathcal{T}$. There exists a sequence of covers $G_n$ of $G$ whose girth\footnote{The girth of a graph is the length of its shortest cycle.} diverges as $n$ goes to infinity. Moreover, for this sequence, the empirical spectral measures $\mu_n$ of $A_{G_n}$ converges weakly to $\mu$.

Proof. By induction, it suffices to show that for every finite graph $H = (V, E)$ with girth $\text{girth}(H) = p$ and $|E| = 2m$, there exists a finite lift $L$ of $H$ whose girth is strictly larger (the weights and potential are irrelevant here, and we will suppress them). We will construct $L$ as a $2^{m+1}$-lift of $H$, with the following set of permutations $\pi : E \to \mathbb{S}_{2^{m+1}}$. Group the edges in to pairs $(e, \bar{e})$ consisting of an edge and its reversal, and order these $(e_1, \bar{e}_1), \ldots, (e_m, \bar{e}_m)$. Now let $\pi_e$ be the permutation that maps $j \mapsto j + 2^t \mod 2^{m+1}$ for every $j \in [2^{m+1}]$, and let $\pi_{\bar{e}} = \pi^{-1}_e$ as required.

Since girth $L > \text{girth} H$, we need only to show that $L$ contains no cycle of length $p$. Seeking contradiction, assume instead that $e_1, \ldots, e_p$ is a sequence of $p$ directed edges forming a cycle in $L$. Writing $\xi$ for the covering map, and $\xi(e_1), \ldots, \xi(e_p)$ form a cycle in $H$ with length $p$, and since girth $H = p$, they are distinct. The vertices of $L$ are $V \times [2^{m+1}]$, which we regard as a set of $2^{m+1}$ 'layers;' assume $\sigma(e_i)$ is in the $t$th one. Because of how we have arranged the permutations, $\tau(e_{i_p})$ is in layer $t + 2^{i_1} + \cdots + 2^{i_k} \equiv t \mod 2^{m+1}$, because the $i_1, \ldots, i_p$ are distinct and smaller than $m + 1$. Thus $\tau(e_{i_p}) \neq \sigma(e_k)$—a contradiction.

We finally show that, given such a sequence $G_n$ with diverging girth, $\mu_n$ converges weakly to $\mu$. For every fixed positive integer $k$ and each vertex $u$ of $G_n$, the quantity $\langle \delta_u, A_k^{\mathcal{T}} \delta_u \rangle$ is a weighted count of length-$k$ closed walks in $G_n$ starting and ending at $u$. Since the $G_n$ have diverging girth, for $n$ sufficiently large the depth-$k$ neighborhood of $u$ in $G_n$ is identical to that of every $\tilde{u} \in \Xi^{-1}(u)$, and thus this count is eventually constant and equal to $\langle \delta_{\tilde{u}}, A_k^{\mathcal{T}} \delta_{\tilde{u}} \rangle$. Finally, as the $k$th moments of the empirical spectral measures $\mu_n$ are given by normalized traces of $A_k^{\mathcal{T}}$, the method of moments gives us weak convergence to the density of states. \hfill \square

Substantially stronger versions of this result are known but will not be necessary for us; we direct the reader for instance to the recent work of Bordenave and Collins [BC19].

2.3. Point Spectrum and the Aomoto Sets. We will denote the point spectrum of $A_\mathcal{T}$ as

$$\text{Spec}_p A_\mathcal{T} = \{ \lambda \in \mathbb{R} : \text{Ker}(\lambda - A_\mathcal{T}) \neq \{0\} \}.$$  

The following proposition collates several equivalent characterizations of $\text{Spec}_p A_\mathcal{T}$.

Proposition 2.1. Let $G$ be a finite graph. Assume $G$ has at least one cycle, and let $\mathcal{T}$ be its universal cover. Then $\lambda \in \text{Spec}_p A_\mathcal{T}$ if and only if any of the following hold:

(i) $\dim \text{Ker}(\lambda - A_\mathcal{T}) = \infty$

(ii) $\lambda$ is an atom of $\mu$

(iii) For some $u \in V(G)$, $\lambda$ is an atom of $\mu_u$.

(iv) For some $u \in V(G)$, the Cauchy transform

$$S_u(z) = \int_{\text{Spec} A_\mathcal{T}} (z - x)^{-1} d\mu_u(x)$$

has a pole at $\lambda$.

(v) For some $u \in V(G)$, and every $\tilde{u} \in \Xi^{-1}(u)$, there exists $\zeta \in \text{Ker}(\lambda - A_\mathcal{T})$ with $\zeta(\tilde{u}) \neq 0.$
Moreover, the vertices satisfying (iii),(iv), and (v) coincide.

By way of a complicated set of coupled equations satisfied by the Cauchy transforms $S_u$, Aomoto identified a set of vertices of $G$ whose combinatorial structure is instrumental in understanding $\text{Spec}_p(A_T)$ and will be the focus of much of this paper.

**Definition 2.1** (The Aomoto set). Let $G$ be a finite graph and assume that $\lambda \in \text{Spec}_p(A_T)$. We define the Aomoto set of $G$ associated to $\lambda$ to be the set of vertices in $V(G)$ that satisfy the equivalent conditions (iii-v) in Proposition 2.1. This set will be denoted by $X_\lambda(G)^3$.

We will use repeatedly an equivalent form of (v) above: if $u \notin X_\lambda(G)$, then any eigenvector $\eta \in \text{Ker}(\lambda - A_T)$ is identically zero on the fibre over $u$. We will also require a standard identity expressing the mass assigned to $\lambda \in \text{Spec}_p A_T$ by the spectral measure $\mu_u$.

**Lemma 2.2.** Let $G$ be a finite graph, $T$ be its universal cover, and $\lambda \in \text{Spec}_p A_T$. Then if $\mathcal{B}$ is any orthonormal basis for $\text{Ker}(\lambda - A_T)$, for any $u \in V$ and $\tilde{u} \in \Xi^{-1}(u)$,

$$\mu_u(\lambda) = \sum_{\eta \in \mathcal{B}} |\eta(\tilde{u})|^2. \quad (10)$$

Equation (10) follows from a standard application of the Borel functional calculus, where the key observation is that if $f_\xi : \text{Spec}(A_T) \to \mathbb{R}$ is the indicator function of the singleton $\{\lambda\}$ then $f_\xi(A_T)$ is the orthogonal projection onto $\text{ker}(\lambda - A_T)$.

3. **Main Results**

Our first contribution is to strengthen a result of Aomoto [Aom91], by way of a new and more conceptual proof. Its content is to characterize the induced subgraph on $X_\lambda(G)$ for any $\lambda \in \text{Spec}_p A_T$, and relate the mass $\mu(\lambda)$ to the local structure of this subgraph and neighboring vertices. Let us write $\partial X_\lambda(G)$ for the set of vertices outside the Aomoto set but connected to it by an edge, $\text{cc}X_\lambda(G)$ for the number of connected components of the subgraph it induces, and define the index of $\lambda$ as

$$I_\lambda(G) = \text{cc}X_\lambda(G) - |\partial X_\lambda(G)|. \quad (11)$$

Recall that for us a graph $G = (V, E, a, b)$ contains vertices $V$, directed edges $E$, Hermitian edge weights $a : E \to \mathbb{C}$ satisfying $\overline{a_e} = a_e$ and real potential $b : V \to \mathbb{R}$.

**Theorem 3.1.** Let $G$ be a finite graph, $T$ be its universal cover, and $\lambda \in \text{Spec}_p A_T$. Then:

(i) The subgraph induced by $X_\lambda(G)$ is acyclic,

(ii) $\lambda$ is an eigenvalue, with multiplicity one, of the induced Jacobi operator of each connected component of this subgraph, and

(iii) The density of states of $A_T$ satisfies

$$|V(G)| \cdot \mu(\lambda) = I_\lambda(G). \quad (12)$$

\[\text{This set was referred to as } X_\lambda^{(1)}(G) \text{ in [Aom91] and [ABS20]; we have dropped the superscript to lighten notation, and because we will not consider the sets } X_\lambda^{(a)}(G) \text{ for } a \neq 1 \text{ which appear in that work.}\]
Assertion (i), which was claimed without proof in [ABS20], clarifies an ambiguity in Aomoto's result, which did not rule out self-loops or multi-edges in the subgraph induced by $X_d(G)$; (ii) is a new observation, and (iii) is due to Aomoto. Our new proof is combinatorial and linear algebraic, using properties of eigenvectors of Jacobi operators on finite and infinite trees; the question of finding an alternative to Aomoto's original proof explaining the significance of the quantity $I_d(G)$, was posed in [ABS20, Problem 8.1]. The proofs of (i) and (ii) may be found in Section 4, and that of (iii) in Section 5.

We then build on the proof of Theorem 3.1 to prove a number of novel results. First, we show that for any graph $G$, the point spectrum of the periodic Jacobi operator on its universal cover is contained in $\text{Spec } A_G$—with multiplicity bounded in terms of the index $I_d(G)$. In fact, we can further refine this result for the Jacobi operator of any cover $H$ of $G$.

**Theorem 3.2.** Let $G$ be a finite graph, $H$ an $n$-lift of $G$, and $T$ their common universal cover. If $\lambda \in \text{Spec}_p A_T$, then

(i) $\lambda \in \text{Spec } A_G$ with multiplicity at least $|V(G)| \cdot \mu\{\lambda\}$, and

(ii) $\lambda \in \text{Spec } A_{H/G}$ with multiplicity at least $(n - 1)|V(G)| \cdot \mu\{\lambda\}$.

Additionally, we prove a converse to Theorem 3.1, namely that if a graph has a set replicating the structure of the Aomoto set for some $\lambda$, then its universal cover has $\lambda$ in its point spectrum. Let us extend the notation $\partial$ and $cc$ to apply to any subset $X \subset V(G)$.

**Theorem 3.3.** Let $G$ be a finite graph, and let $T$ be its universal cover. If some set of vertices $X \subset V(G)$ induces an acyclic subgraph for which every connected component has $\lambda$ in the spectrum of its Jacobi operator and $cc(X) - |\partial X|$ is positive, then

(i) $\lambda \in \text{Spec } A_G$, with multiplicity at least $cc(X) - |\partial X|

(ii) $\lambda \in \text{Spec}_p A_T$, and $|V(G)| \cdot \mu\{\lambda\} \geq cc(X) - |\partial X|.

The proofs of both Theorem 3.2 and Theorem 3.3 follow from a generalization of the latter, Theorem 6.1, which we state and verify in Section 6. The argument proceeds, roughly, by patching together and extending the $\lambda$-eigenvectors on each component of the Aomoto set promised by Theorem 3.1 to global eigenvector of $A_G$ and $A_{H/G}$.

Since the number of forests in a finite graph is finite, and because the conditions in Theorem 3.3 are necessary and sufficient, by searching through through all possible forests in $G$ we trivially obtain a finite time algorithm which can compute the point spectrum of the periodic Jacobi operator of its universal cover.

**Corollary 3.1.** Let $G$ be a finite graph, and let $T$ be its universal cover. There is a finite-time algorithm which, on input $G$, computes $\text{Spec}_p A_T$.

Finally, we use Theorem 3.1 and Theorem 3.3 to argue that point spectrum is rare in a certain sense.

**Theorem 3.4.** Let $G = (V, E)$ be a finite graph with at least one cycle and $T$ be its universal cover. Leaving $V$ and $E$ fixed, the set of Hermitian edge weights and potentials for which $T$ has no point spectrum is open dense in $\mathbb{R}^{V} \oplus \mathbb{C}^{E/2}$ and its complement has Lebesgue measure zero.

The above theorem, which we prove in Section 7 resolves [Aom91, Question 2], which speculated that $\text{Spec}_p A_T$ was only dependent on the combinatorial structure of $G$, and not on the edge
weights and potential. Results in a similar direction were obtained in [KLW12] and [KLW13]. Their results are less general in the sense that require $G$ to have a loop at every vertex and only consider the case $a = 1$, however, they apply to a more general class of trees, namely trees of finite cone type, and to a broader class of potentials on such trees.

4. Acyclic Nature of Aomoto Sets

In this section we will prove the first two assertions of Theorem 3.1, namely that if $\lambda \in \text{Spec}_c \mathcal{T}$, then the Aomoto set $X_0(G)$ is acyclic, and $\lambda$ is an eigenvalue of the induced Jacobi operator on each of its connected components. We begin by generalizing to the infinite case a result of Fielder regarding eigenvectors of finite trees [Fie75, Proposition 1].

**Lemma 4.1.** Let $T$ be a locally finite tree with Hermitian edge weights and potential $a : E(T) \to \mathbb{R}$ and $b : V(T) \to \mathbb{R}$ respectively, and Jacobi operator $A_T$. If $\eta \in \text{Ker}(\lambda - A_T)$ and $\eta(v) \neq 0$ for every vertex $v \in V(T)$, then $\dim \text{Ker}(\lambda - A_T) = 1$.

**Proof.** Choose a root $r$ for $T$, and for each vertex $v$, write $p(v)$ for its unique parent, $T_v$ the infinite sub-tree emanating from $v$ away from its parent and $\eta|_{T_v}$ for the restriction of $\eta$ to the subtree $T_v$. As $T$ is acyclic, it has no multi-edges or self-loops, and there is no ambiguity in writing $a_{uv}$ for the weight of the unique edge with source $u$ and terminal $v$. We then have

$$(\lambda - A_T)\eta|_{T_v} = \sum_{u \in V(T_v)} \lambda \eta(u) \delta_u - \sum_{u \in V(T_v) \setminus \{v\}} (A_T \eta)(u) \delta_u - \left( b_v \eta(v) + \sum_{x : p(x) = v} a_{uv-x} \eta(x) \right) \delta_u - a_{pv} \eta(v) \delta_{p(v)}$$

$$= \lambda \eta(v) \delta_v - (A_T \eta)(v) - a_{pu-p(v)} \eta(p(v)) \delta_v - a_{pv} \eta(v) \delta_{p(v)}$$

$$= a_{uv-p(v)} \eta(p(v)) \delta_v - a_{pv} \eta(v) \delta_{p(v)}$$

for any $v \neq r$. Now, let $\zeta \in \text{Ker}(\lambda - A_T)$. As $A_T$ is self-adjoint and $\lambda$ real,

$$0 = \langle \zeta, (\lambda - A_T)\eta|_{T_v} \rangle = a_{uv-p(v)} \eta(p(v)) \zeta'(v) - a_{pv} \eta(v) \zeta'(p(v)),$$

which implies

$$\frac{\zeta'(v)}{\zeta'(p(v))} = \frac{a_{pv}}{a_{uv-p(v)}} \frac{\eta(v)}{\eta(p(v))}.$$

This identity holds for every $\zeta \in \text{Ker}(\lambda - A_T)$, including $\eta$ itself, so we obtain

$$\frac{\zeta'(v)}{\zeta'(p(v))} = \frac{a_{pv}}{a_{uv-p(v)}} \frac{\eta(v)}{\eta(p(v))} = \frac{|a_{pv}|^2}{|a_{uv-p(v)}|^2} \frac{\eta(v)}{\eta(p(v))},$$

in the final equality we have used conjugate symmetry of the edge weights. Since $\eta|_{\overline{T_v}} = \eta \in \text{Ker}(\lambda - A_T)$, $\zeta$ is unconstrained at the root, and the above equation propagates a condition down the tree that $\zeta = \eta : \zeta(r)/\eta(r)$.

We now prove that the subgraph of $G$ induced by $X_0(G)$ is a forest, and that $\lambda$ is an eigenvalue, with multiplicity one, of the induced Jacobi operator of each of its connected components.

**Proof of Theorem 3.1(i-ii).** Assume $\lambda$ is in the point spectrum of $A_T$, and let $G'$ be a connected component of the subgraph induced by $X_0(G)$. Let $\mathcal{T}'$ be the universal cover of $G'$. If we view $\mathcal{T}'$ as a subgraph of $\mathcal{T}$ then any vector in $\text{Ker}(\lambda - A_T)$ vanishes on the boundary of $\mathcal{T}'$ in $\mathcal{T}$, and thus...
restricts to a $\lambda$-eigenvector of $\mathcal{T}'$. Hence $X_\lambda(G') = V(G')$ by Proposition 2.1(v), and we can now use the following observation, which follows from Zorn’s lemma and appeared as [Nyl98, Lemma 7].

**Observation 4.1.** If $X_\lambda(G') = V(G')$ then there is an $\eta \in \text{Ker}(\lambda - A_{T'})$ satisfying $\eta(u) \neq 0$ for every $u \in V(\mathcal{T}')$.

Combining Observation 4.1 and Lemma 4.1 we conclude finally that $\dim \text{Ker}(\lambda - A_{T'}) = 1$, and thus, by Proposition 2.1(i), that $G'$ is acyclic. This further implies that $\mathcal{T}' = G'$, which proves the second assertion. \qed

In the course of the proof above we showed the following fact, which will be of repeated use throughout the paper.

**Lemma 4.2.** Let $G$ be a finite graph with Hermitian edge weights and potential, with $\lambda \in \text{Spec}_p A_T$ and $T_1, \ldots, T_p$ the Aomoto trees of $G$ associated to $\lambda$. Then, for every $i \in [p]$ there is a unique (up to phase) unit vector $\zeta_i \in \text{Ker}(\lambda - A_{T_i})$ satisfying $\zeta_i(u) \neq 0$ for every $u \in V(T_i)$.

For use in the next section, we record one consequence of the above lemma.

**Observation 4.2.** Let $G$ be a graph with $b = 0$, $\mathcal{T}$ its universal cover, and assume $0 \in \text{Spec}_p A_T$. Then $X_0(G)$ is an independent set in $G$.

**Proof.** By Lemma 4.2, each Aomoto tree of $G$ must have a unique, everywhere nonzero eigenvector in the kernel of its Jacobi operator. On the other hand, a vector in the kernel of a Jacobi operator with potential zero for a tree cannot be nonzero at the parent of a leaf. Thus every Aomoto tree of $G$ is an isolated vertex as desired. \qed

### 5. Aomoto’s Index Formula

In this section we complete the proof of Theorem 3.1 by verifying the formula in equation (12): if $\lambda \in \text{Spec}_p A_T$, then

$$|V(G)| \cdot \mu(\lambda) = I_\lambda(G).$$

Our strategy will be to reduce the problem to the proof of an analogous result on an auxiliary bipartite graph $G'$.

#### 5.1. Constructing the Auxiliary Graph

Let $T_1, \ldots, T_p$ be the Aomoto trees of $G = (V, E, a, b)$ associated to $\lambda$, write $\mathcal{F}_i$ for the set of disjoint copies of $T_i$ in $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mathcal{a}, \mathcal{b})$ obtained by lifting $T_i$, and let $\mathcal{F} = \bigcup_{i=1}^p \mathcal{F}_i$. Note that $\mathcal{F}$ is a subforest of $\mathcal{T}$ and all of its subtrees are isomorphic to some Aomoto tree of $G$.

By Lemma 4.2 there is for each $T_i$ a unique (up to phase) vector $\zeta_i \in \text{Ker}(\lambda - A_{T_i})$ with unit norm and nonzero entries. Take any $\eta \in \text{Ker}(\lambda - A_T)$. For every $S \in \mathcal{F}$, by definition of the Aomoto set it holds that $\eta$ is zero on all vertices in $\partial V(S)$. Hence, the restriction of $\eta$ to any $S \in \mathcal{F}_i$ induces an eigenvector of $A_{T_i}$. This implies that $\eta$ can be decomposed as

$$\eta = \sum_{S \in \mathcal{F}} a_S \zeta_S,$$

(13)
We now construct $G'$ with Aomoto trees.\footnote{Aomoto tree: vertex outside $e$ to consider identically zero on the fibre over it is injective; it remains only to show that $\text{Ker}$.}

Observation 5.1.\footnote{Observation 5.1. in (13), any $\xi \in \ell^2(V(T))$ are inclusions of the $\lambda$-eigenvectors of each Aomoto tree: $\zeta_\xi(v) = \begin{cases} \xi(S(v)) & \text{if } v \in V(S) \text{ and } S \in F_i. \\ 0 & \text{otherwise} \end{cases}$}

We now construct $G' = (V', E', a', b')$; the process is summarized in Figure 1. First, $V'$ is obtained from $V$ by deleting every vertex outside $X_i(G) \cup \partial X_i(G)$, and contracting each Aomoto tree $T_i$ to a single vertex $t_i$. Write $\{t_1, ..., t_p\} = U \subset V'$, and identify $\partial X_i(G)$ with $\partial U$. Now, for each $v \in \partial U = \partial X_i(G)$, each edge $e \in \tau(v) \subset E$ whose source is in a tree $T_i$, include an edge $e' \in E'$ with $\tau(e') = v$ and $\sigma(e') = t_i$, and set its weight as $a_{e'} = a_e \zeta_\xi(\sigma(e))$.\footnote{The vector $\eta'$ is a unit vector and since the map is an isometry it is injective; it remains only to show that $\text{Ker}(\lambda - A_T)$ is mapped to $\text{Ker} A_T$. The vector $\eta'$ is identically zero on the fibre over $\partial U$ and thus $(A_T, \eta')(u) = 0$ for any $u \in (\Xi')^{-1}(U)$. It remains only to consider $v \in (\Xi')^{-1}(\partial U)$, which as above we may identify with $\Xi^{-1}(\partial X_i(G)) \subset V$. For each edge $e \in \tau(v) \subset E$ write $S_e$ for the tree in $F$ to which $\sigma(e)$ belongs, so that the reweighting in (14) gives}

This process is mirrored to construct an edge $f' \in E'$ from any $f \in \sigma(v) \subset E$ whose terminal is in $T_i$; no other edges are included. Finally, the potential $b'$ is identically zero.

We have arranged things so that $G'$ is bipartite, with connected components $G'_1, ..., G'_m$, whose respective covers we will denote $T'_1, ..., T'_m$. We may also construct a new infinite graph $T' = (V', E', a', b')$ from $T$ analogously to the construction of $G'$ from $G$: by deleting the fibres over any vertex outside $X_i(G) \cup \partial X_i(G)$, contracting each tree $S \in F' \subset T'$ into a single vertex $u_S$, including for each $e \in E$ with ends in $\Xi^{-1}(X_i(G))$ and $\Xi^{-1}(\partial X_i(G))$ a corresponding edge $e' \in E'$ with ends in the contraction if $\Xi^{-1}(X_i(G))$ and its boundary, and reweighting any such edge according to (14). This $T'$ consists of countably many copies of each $T_j$, and is a cover of $G'$ via a map $\Xi'$; note that $(\Xi')^{-1}(U) = \{u_S : S \in F\}$, the contraction of $\Xi^{-1}(X_i(G))$. With this setup and the decomposition in (13), any $\eta \in \text{Ker}(\lambda - A_T)$ gives rise to a vector $\eta' \in \ell^2(V(T'))$ in a natural way:

$$\eta = \sum_{S \in F} a_S \zeta_S \mapsto \eta' = \sum_{S \in F} a_S \delta_{u_S}. \quad (15)$$

**Observation 5.1.** The map $\eta \mapsto \eta'$ is an isometric inclusion of $\text{Ker}(\lambda - A_T)$ in $\text{Ker} A_T$.\footnote{Proof: Preservation of norm is immediate since $\zeta_S$ is a unit vector and since the map is an isometry it is injective; it remains only to show that $\text{Ker}(\lambda - A_T)$ is mapped to $\text{Ker} A_T$. The vector $\eta'$ is identically zero on the fibre over $\partial U$ and thus $(A_T, \eta')(u) = 0$ for any $u \in (\Xi')^{-1}(U)$. It remains only to consider $v \in (\Xi')^{-1}(\partial U)$, which as above we may identify with $\Xi^{-1}(\partial X_i(G)) \subset V$. For each edge $e \in \tau(v) \subset E$ write $S_e$ for the tree in $F$ to which $\sigma(e)$ belongs, so that the reweighting in (14) gives}
\( a' = a \zeta_\mathcal{S}(\sigma(e)) \). As the potential \( b' \) is identically zero and \( \eta \) and \( \eta' \) vanish outside the fibres over \( X_i(G) \) and \( U \) respectively, we have

\[
(\mathcal{A}_T \cdot \eta')(v') = \delta(v) \eta'(v') + \sum_{e \in \tau(v') \cap E} a'_e \eta'(\sigma(e)) = \sum_{e \in \tau(v) \cap E; \sigma(e) \in \Xi^{-1}(X_i(G))} a_e \zeta_\mathcal{S}(\sigma(e)) a_\mathcal{S} = \delta(v) \eta(v) + \sum_{e \in \tau(v) \cap E} a_e \eta(\sigma(e)) = (\mathcal{A}_T \eta)(v) = \lambda \eta(v) = 0.
\]

In the third line, note that some edges in \( \tau(v) \subset E \) have a source outside of the fibre over \( X_i(G) \), but that \( \eta \) is identically zero there. \( \square \)

Immediately from this observation, we can conclude that \( 0 \in \text{Spec}_\rho \mathcal{A}_T \). Moreover, as \( \mathcal{T}' \) is comprised of disjoint copies of the \( \mathcal{T}_j' \)'s, \( \mathcal{A}_T \) restricts to \( \mathcal{A}_{T_j} \) on each one, and thus \( 0 \in \text{Spec}_\rho \mathcal{A}_{T_j} \) for at least one \( \mathcal{T}_j \). Our next observation characterizes the associated Aomoto set on \( G'_j \). Recall that \( G' \) is bipartite with vertex classes \( U \) and \( \partial U \), and let us write \( U_j \) and \( \partial U_j \) for the corresponding classes of vertices in each connected component \( G'_j \).

**Observation 5.2.** \( X_0(G'_j) = U_j \).

**Proof.** By Proposition 2.1, the definition of the map \( \eta \mapsto \eta' \), and Observation 5.1, we immediately have the inclusion \( U_j \subset X_0(G'_j) \), since any \( \eta \in \text{Ker}(\lambda - \mathcal{A}_T) \) maps to \( \eta' \) supported only on the fibre over \( U_j \). On the other hand, from Lemma 4.2, we know that \( X_0(G'_j) \) is an independent set, and \( U_j \) is a maximal independent set in \( G'_j \) by definition of \( \partial U_j \). \( \square \)

Finally, we can strengthen Observation 5.1

**Observation 5.3.** The map \( \eta \mapsto \eta' \) gives an isomorphism between \( \text{Ker}(\lambda - \mathcal{A}_T) \) and \( \text{Ker} \mathcal{A}_{T_j} \).

**Proof.** We noted above that \( \mathcal{A}_{T_j} \) decomposes as a direct sum of the Jacobi operators on the copies of \( \mathcal{T}_j' \) comprising \( \mathcal{T}' \). By applying Observation 5.2 separately to each copy, any \( \theta \in \text{Ker} \mathcal{A}_{T_j} \) is supported only on \( (\Xi')^{-1}(U) \). Thus the adjoint of the map \( \eta \mapsto \eta' \) takes any vector \( \theta \in \text{Ker} \mathcal{A}_{T_j} \) to one in \( l^2(V(\mathcal{T})) \):

\[
\theta = \sum_{S \in \mathcal{F}} a_\mathcal{S} \delta_{us} \leftrightarrow \sum_{S \in \mathcal{F}} a_\mathcal{S} \zeta_\mathcal{S}.
\]

Once again this is clearly injective and norm-preserving, and a parallel argument to Observation 5.1 shows that it takes \( \text{Ker} \mathcal{A}_{T_j} \) into \( \text{Ker}(\lambda - \mathcal{A}_T) \). \( \square \)

We can finally relate the density of states of \( \mathcal{A}_T \) to those of the \( \mathcal{A}_{T_j} \).

**Observation 5.4.** Let \( T \) be an Aomoto tree in \( G \), and \( t \in V(G'_j) \) its contraction in a component \( G'_j \) of \( G' \). Writing \( \mu_t \) for the spectral measure of \( t \) in \( \mathcal{A}_{T_j} \), and for \( \mu_v \) for the spectral measure of \( \mathcal{A}_T \) for each \( v \in V(T) \subset V(G) \), we have

\[
\sum_{v \in V(T)} \mu_v \{ \lambda \} = \mu_t \{ 0 \}.
\]
Proof. Choose a copy $\tilde{T}$ of $T$ in its fibre in $\mathcal{T}$, and let $\tilde{t}$ be the contraction of $\tilde{T}$ in $\mathcal{T}_j \subset \mathcal{T}$. By construction, for each $\eta \in \ker(\lambda - A_T)$,

$$\eta'(\tilde{t})^2 = \sum_{v \in V(\tilde{T})} \eta(v)^2.$$ 

Now, let $\mathfrak{B}_j$ be an orthonormal basis of $\ker A_{T_j}$. By Observation 5.3 this is the image of some orthonormal set $\mathfrak{B}$ in $\ker(\lambda - A_T)$. In particular, recalling our construction of $\mathcal{T}'$ from $\mathcal{T}$ by deleting vertices and contracting Aomoto trees, our chosen copy of $\mathcal{T}_j$ contains $\tilde{T}$. Moreover, $\mathfrak{B}_j$ is an orthonormal basis for the orthogonal projection of $\ker(\lambda - A_T)$ to the subspace of $\ell^2(\mathcal{V})$ supported on the vertices of $\mathcal{T}_j$, and we can therefore augment $\mathfrak{B}_j$ to an orthonormal basis $\mathfrak{B}$ of $\ker(\lambda - A_T)$, whose additional vectors vanish on $\mathcal{T}_j$.

We now use Lemma 2.2 to compute

$$\sum_{v \in V(\mathcal{T})} \mu_v(\lambda) = \sum_{v \in V(\tilde{T})} \sum_{\eta \in \mathfrak{B}} \eta(v)^2 = \sum_{v \in V(\tilde{T})} \sum_{\eta \in \mathfrak{B}_j} \eta(v)^2 = \sum_{\eta \in \mathfrak{B}_j} \eta'(\tilde{t})^2 = \sum_{\eta \in \mathfrak{B}_j} \mu'(\{0\}).$$

\[\square\]

5.2. Analyzing the Auxiliary Graph. This section is devoted to the final observation of our proof:

Observation 5.5. Fix $j \in [m]$ and assume that $0 \in \text{Spec}_p A_{T_j}$. Then

$$\sum_{v \in U_j} \mu'_j(\{0\}) = I_0(G'_j).$$

This will finish the proof, as combining Observations 5.4 and 5.5 and recalling the construction of $G'$ gives

$$|V(G)| \cdot \mu(\lambda) = \sum_{u \in X_\mu(G)} \mu_u(\lambda) = \sum_{v \in U} \mu_v(\{0\})$$

$$= \sum_{j \in [m]} I_0(G'_j) = \sum_{j \in [m]} |U_j| - |\partial U_j|$$

$$= |U| - |\partial U| = \text{cc}X_\mu(G) - |\partial X_\mu(G)|$$

$$= I_0(G).$$

Proof of Observation 5.5. Let $\mu'$ be the DOS of $A_{T_j}$. Let $L_1, L_2, \ldots$ be a sequence of finite lifts of $G'_j$ with covering maps $\xi_n : L_n \to G'_j$. By Lemma 2.1 we may choose the $L_n$ with girth going to infinity. Since $G'_j$ is bipartite with zero potential, the Jacobi matrices $A_{L_n}$ and $A_{T_j}$ have the following block structure

$$A_{L_n} = \begin{pmatrix} 0 & Z_n^T \\ Z_n & 0 \end{pmatrix} \quad \text{and} \quad A_{T_j} = \begin{pmatrix} 0 & Z_{\infty}^T \\ Z_{\infty} & 0 \end{pmatrix},$$

where for $n \in \mathbb{N} \cup \{\infty\}$ the domain and range of $Z_n$ correspond to the fibers of $\partial U_j$ and $U_j$ respectively. Note that $A_{L_n} = Z_n^T Z_n \oplus Z_n Z_n^T$ and $A_{T_j} = Z_{\infty} Z_{\infty} \oplus Z_{\infty} Z_{\infty}^T$.

Let $\mu_{Z_n Z_n^T}$ and $\mu_{Z_{\infty} Z_{\infty}^T}$ be the empirical spectral distributions of $Z_n Z_n^T$ and $Z_{\infty} Z_{\infty}^T$ respectively. Fix a positive integer $k$ and note that, since $L_n$ is bipartite, the terms in $\text{tr}(Z_n^T Z_n)^k$ are in one-to-one correspondence with the closed walks of length $2k$ in $L_n$ that start and end at the same vertex in $\xi_n^{-1}(\partial U_j)$. Moreover, by the girth assumption, for large enough $n$ it holds that the value of the diagonal entries of $A_{L_n}^{2k}$ are constant on each fiber $\xi_n^{-1}(v)$ for every $v \in V(G'_j)$ and coincide with
the respective diagonal entries of $A^2_T$. Hence, if we write $v_\nu$ for the spectral measure of $u$ for the operator $A^2_T$, then by the method of moments $\mu_{Z_n^T z_n^T}$ and $\mu_{Z_n z_n^T}$ converge weakly to

$$v_{\partial U_j} = \frac{1}{|\partial U_j|} \sum_{v \in \partial U_j} v_\nu \quad \text{and} \quad v_{U_j} = \frac{1}{|U_j|} \sum_{v \in U_j} v_\nu.$$  

(16)

Since $X_0(G'_j) = U_j$, equation (16) implies $v_{\partial U_j} \{ 0 \} = 0$. If it were the case that $|\partial U_j| > |U_j|$, we would have by standard properties of matrices that the spectrum of $Z_n^T z_n$ is equal to that of $Z_n z_n^T$, plus the eigenvalue zero with multiplicity at least $|\partial U_j| - |U_j|$, and thus

$$\frac{|\partial U_j| - |U_j|}{|\partial U_j|} \leq \limsup_{n \to \infty} \mu_{Z_n z_n^T} \{ 0 \} \leq v_{\partial U_j} \{ 0 \} = 0,$$

a contradiction. Thus $I_0(G'_j) \geq 0$. Applying the same matrix property a second time, we have

$$\mu_{Z_n z_n^T} = \left( 1 - \frac{I_0(G'_j)}{|U_j|} \right) \mu_{Z_n^T z_n} + \frac{I_0(G'_j)}{|U_j|} \delta_0.$$

By weak convergence, and as compact measures are determined by their moments,

$$v_{U_j} = \left( 1 - \frac{I_0(G'_j)}{|U_j|} \right) v_{\partial U_j} + \frac{I_0(G'_j)}{|U_j|} \delta_0,$$

and thus

$$\sum_{v \in U_j} \mu'_v \{ 0 \} = |U_j| v_{U_j} \{ 0 \} = \left( |U_j| - I_0(G'_j) \right) v_{\partial U_j} \{ 0 \} + I_0(G'_j) \delta_0 \{ 0 \} = I_0(G'_j).$$  

$\square$

### 6. A Generalized Converse to Aomoto’s Theorem

In this section we will prove the following generalization of Theorem 3.3, and use it to prove one our other main contribution, Theorem 3.2.

**Theorem 6.1.** Let $G$ be a finte graph, $T$ its universal cover, $U : E(G) \to U(n)$ a set of unitary-valued edge weights satisfying $U^*_e = U_e$ for every $e \in E(T)$, and $A_{G,U}$ the unitary-weighted Jacobi operator acting on $\eta \in l^2(V(G)) \otimes \mathbb{C}^n$ as

$$(A_{G,U} \eta)(v) = b_v \eta(v) + \sum_{e \in T(v)} a_e U_e \eta(\sigma(e)) \in \mathbb{C}^n.$$

If some set of vertices $X \subset V(G)$ induces an acyclic subgraph, every component of which has $\lambda$ in the spectrum of its induced unitary-weighted Jacobi operator and $cc(X) - |\partial X| > 0$, then $\lambda \in \text{Spec} A_{G,U}$ with multiplicity at least $n(cc(X) - |\partial X|)$.

We begin with a lemma regarding unitary-weighted Jacobi operators of finite trees.

**Lemma 6.1.** Let $T$ be a finite tree, $U : E(T) \to U(n)$ a set of unitary-valued edge weights satisfying $U^*_e = U_e$ for every $e \in E(T)$, and $A_{T,U}$ the associated unitary-weighted Jacobi operator. If $\lambda \in \text{Spec} A_T$, then $\lambda \in \text{Spec} A_{T,U}$ with multiplicity at least $n$.

**Proof.** As in the proof of Lemma 4.1, we will choose a root $r$ of $T$, for each vertex $v$ write $p(v)$ for its unique parent and $c(v)$ for its set of children, and, since $T$ is acyclic, write $v \leftarrow u$ for the unique edge with source $u$ and terminal $v$. By absorbing $\lambda$ into the potential, it suffices to study
We claim that \( \eta \in \ker A_T \); we will produce a subspace of dimension \( n \) contained in \( \ker A_{T,U} \).

Fix a vector \( \zeta_0 \in \mathbb{C}^n \) and set \( \zeta(r) = \zeta_0 \). For each vertex \( v \in V(T) \), let \( y_v \) denote the directed edges in the unique shortest path from \( v \) to \( r \), set

\[
\zeta(v) = \prod_{e \in y_v} U_e \cdot \eta(v) \cdot \zeta_0
\]

We claim that \( \zeta \in \ker A_{T,U} \); since \( \zeta_0 \) was arbitrary, this will complete the proof.

At the root, we have

\[
(A_{T,U} \zeta)(r) = b_r \eta(r) \zeta_0 + \sum_{u \in \Gamma(r)} a_{r \rightarrow u} U_{r \rightarrow u} \eta(u) \zeta_0 = \left( b_r \eta(r) + \sum_{u \in \Gamma(r)} a_{r \rightarrow u} \eta(u) \right) \zeta_0 = 0,
\]

since \( U_{r \rightarrow u} U_{u \rightarrow r} = 1 \) and \( \eta \in \ker A_T \). Similarly, for any other vertex \( v \in V(T) \), conjugate symmetry of the unitary weights gives us

\[
(A_{T,U} \zeta)(v) = b_v \prod_{e \in y_v} U_e \eta(v) \zeta_0 + a_{v \rightarrow p(v)} U_{v \rightarrow p(v)} \prod_{e \in y_p(v)} U_e \eta(p(v)) \zeta_0 + \sum_{u \in \Gamma(v)} a_{v \rightarrow u} U_{v \rightarrow u} \prod_{e \in y_v} U_e \eta(u) \zeta_0
\]

\[
= \left( b_v + a_{v \rightarrow p(v)} \eta(p(v)) + \sum_{u \in \Gamma(v)} a_{v \rightarrow u} \eta(u) \right) \prod_{e \in y_v} U_e \zeta_0
\]

\[= 0. \]

We can now proceed with the proof.

**Proof of Theorem 6.1.** For any Aomoto tree \( T \) of \( G \), the induced Jacobi operator \( A_T \) has \( \lambda \) in its spectrum. By Lemma 6.1, the induced unitary-weighted Jacobi operator \( A_{T,U} \) thus satisfies \( \dim \ker (\lambda - A_{T,U}) \geq n \), and therefore the space

\[
\bigoplus_{T \in X_\lambda(G)} \ker (\lambda - A_{T,U}) \subset \ell^2(X_\lambda(G)) \otimes \mathbb{C}^n \subset \ell^2(V) \otimes \mathbb{C}^n
\]

has dimension \( n \) \( ccX_\lambda(G) \). We will show that it contains a subspace of dimension \( nI_\lambda(G) \) which is itself contained in \( \ker (\lambda - A_{G,U}) \).

For each \( v \in X_\lambda(G) \), let \( \Pi_v : \ell^2(V) \otimes \mathbb{C}^n \to \ell^2(v) = \mathbb{C}^n \) be the orthogonal projection to the \( \mathbb{C}^n \)-valued functions in \( \ell^2(V) \otimes \mathbb{C}^n \) supported on \( v \). For each \( u \in \partial X_\lambda(G) \), there is an operator

\[
\phi_u = \sum_{e \in \Gamma(u)} a_e U_e \Pi_{\sigma(e)} : \bigoplus_{T \in X_\lambda(G)} \ker (\lambda - A_{T,U}) \to \ell^2(v) = \mathbb{C}^n
\]

and we define

\[
\phi = \bigoplus_{u \in \partial X_\lambda(G)} \phi_u : \bigoplus_{T \in X_\lambda(G)} \ker (\lambda - A_{T,U}) \to \ell^2(\partial X_\lambda(G)) \cong \mathbb{C}^{n|\partial X_\lambda(G)|}.
\]

Counting dimensions, \( \dim \ker \phi \geq nI_\lambda(G) \), and we will show that \( \ker \phi \subset \ker (\lambda - A_{G,U}) \).

Let \( \zeta \in \ker \phi \); since the latter is a subspace of \( \ell^2(X_\lambda(G)) \otimes \mathbb{C}^n \subset \ell^2(V) \otimes \mathbb{C}^n \), we have \( \zeta(u) = 0 \) for every \( u \in X_\lambda(G) \). This immediately gives \( (\lambda - A_{G,U}) \zeta)(u) = 0 \) for any \( u \) outside the Aomoto set and its boundary, as \( \zeta \) is identically zero on \( u \) and its neighbors. On the other hand, if \( u \) belongs
to some tree $T$ in the Aomoto set, then because $\text{Ker } \phi \subset \bigoplus_{T \subset X_0(G)} \text{Ker}(\lambda - A_{T,U})$ and $\zeta$ vanishes on $\partial X_0(G)$, we have
\[
\left((\lambda - A_{G,U})\zeta\right)(u) = \lambda\zeta(u) + b_u\zeta(u) + \sum_{e \in \tau(u)} a_e U_e \zeta(\sigma(e)) = \lambda\zeta(u) + b_u\zeta(u) + \sum_{e \in \tau(u)} a_e U_e \zeta(\sigma(e)) = (\lambda - A_{T,U})\zeta(u) = 0
\]

It remains to check that $(\lambda - A_{G,U})\zeta(u) = 0$ when $u \in \partial X_0(G)$, which will follow from $\zeta \in \text{Ker } \phi$. In particular, using a final time that $\zeta$ is supported only on the Aomoto set, if $u \in \partial X_0(G)$ we have
\[
\left((\lambda - A_{G,U})\zeta\right)(u) = \lambda\zeta(u) + b_u\zeta(u) + \sum_{e \in \tau(u)} a_e U_e \zeta(\sigma(e)) = \sum_{e \in \tau(u)} a_e U_e \zeta(\sigma(e)) = (\phi\zeta)(u) = 0.
\]

Theorems 3.2 and 3.3 now follow easily.

**Proof of Theorem 3.2.** By Theorem 3.1, the Aomoto set satisfies the hypotheses of Theorem 6.1, and if $H$ is an $n$-lift of $G$, both $A_G$ and $A_{H/G}$ are unitary-weighted Jacobi operators for $G$—the former with weights taking values in $U(1)$ and the latter in $U(n-1)$ by the discussion in Section 2.2. Thus $\lambda \in \text{Spec } A_G$ with multiplicity at least
\[
\text{cc}(X_0(G)) - |\partial X_0(G)| = I_0(G) = |V(G)| \cdot \mu(\{\lambda\})
\]
and similarly $\lambda \in \text{Spec } A_{H/G}$ with multiplicity at least $(n - 1)|V(G)| \cdot \mu(\{\lambda\})$, as desired.

**Proof of Theorem 3.3.** Assertion (i) is a special case of Theorem 6.1. For (ii), let $G_n$ be the sequence of lifts of $G$ promised in Lemma 2.1, whose empirical spectral measures $\mu_{G_n}$ converge weakly to the density of states $\mu$. Applying Theorem 6.1 to each $A_{G_n}$, viewed again as a unitary-weighted Jacobi operator on $G$, the empirical spectral measures $\mu_{G_n}$ satisfy $\mu_{G_n}(\{\lambda\}) \geq \frac{\text{cc}(X) - |\partial X|}{V(G)}$. As these converge weakly to $\mu$, we have
\[
\mu(\{\lambda\}) \geq \frac{\text{cc}(X) - |\partial X|}{V(G)}.
\]

We conclude this section by illustrating how one may combine Theorem 3.1 and Theorem 3.3 to deduce the Aomoto set of a given graph $G$.

**Example 6.1.** Let $G$ be the complete bipartite graph $K_{c,d}$. Assume $d > c$ and denote by $V_c$ and $V_d$ the vertex components of $G$ having $c$ and $d$ vertices respectively. We will first analyze the case when $b = 0$ and $a$ is any Hermitian edge weighting. It is easy to see that $V_d$ is a set satisfying the conditions of Theorem 3.3 for $\lambda = 0$. Hence $\mu\{0\} \geq \frac{d-c}{d+c}$. Then by Theorem 3.1, $X_0(G) \neq \emptyset$ and moreover $I_0(G) \geq d - c$. Now, by Observation 4.2, $X_0(G)$ is an independent set in $G$, which together with the previous observations implies that in fact $X_0(G) = V_d$. So by Theorem 3.1, when $b = 0$, $\mu\{0\} = \frac{d-c}{d+c}$ and the vectors in $\ker(A_T)$ are supported on the fibre of $V_d$, which extends a result of Godsil and Mohar [GM88]. In the more general case when $b$ is not necessarily 0, the existence and location of eigenvalues of $A_T$ depend on the particular choice of $b$, and moreover by Theorem 3.4 we know that one may choose $b$ such that $A_T$ has no point spectrum. This discussion resolves Problems 8.6 and 8.7 posed in [ABS20].
7. Spectral Delocalization for $A_T$

We are now ready to provide a proof of Theorem 3.4. The fact that the set mentioned in the theorem is open dense will follow from Theorem 3.3, while the fact that its complement is of measure zero will follow from Theorem 3.1. This result can be interpreted as an almost surely spectral delocalization result, since it implies that under a random absolutely continuous perturbation (with respect to the Lebesgue measure) of the edge weights and potential of $G$, the spectrum of $A_T$ becomes purely absolutely continuous.

Proof of Theorem 3.4. Let $P \subset \mathbb{R}^{|V|} \oplus \mathbb{C}^{|E|/2}$ be the set of parameters for which $A_T$ has point spectrum, and let $\mathcal{A}(G)$ be the family of vertex sets $X \subset V$ that induce an acyclic subgraph of $G$ with the property that $\text{cc}(X) - |\partial X| > 0$. For every $X \in \mathcal{A}(G)$ denote the forest induced by $X$ by $F_X$ and let $P_X \subset \mathbb{R}^{|V|} \oplus \mathbb{C}^{|E|/2}$ be the set of parameters for which all the Jacobi matrices of the trees in $F_X$ have a common eigenvalue. Note that Theorem 3.1 implies that

$$P \subset \bigcup_{X \in \mathcal{A}(G)} P_X. \quad (17)$$

Now fix $X \in \mathcal{A}(G)$ and let $T_1, ..., T_p$ be the trees in $F_X$. Since $X$ is not a tree and $\text{cc}(X) - |\partial X| > 0$ we know that $p \geq 2$. For any Hermitian edge weights $a : E \to \mathbb{C}$, let $x = \Re(a)$ and $y = \Im(a)$, that is, for every $e \in E$ we have $a_e = x_e + iy_e$ with $x_e, y_e \in \mathbb{R}$. With this setup, the real and imaginary parts of the characteristic polynomial of each $A_{T_i}$, which we will denote by $P(x, y, b, z)$ and $Q_i(x, y, b, z)$, can be viewed as polynomials with real coefficients in $x, y, b, u$ and a variable $z$.

If $(a, b) \in P_X$, then $P_1(x, y, b, z)$ and $P_2(x, y, b, z)$, viewed as polynomials in $z$, have a common root, so their resultant $R(P_1(x, y, b, z), P_2(x, y, b, z))$ with respect to $z$ vanishes. In other words

$$P_X \subset \{(a, b) \in \mathbb{R}^{|V|} \oplus \mathbb{C}^{|E|/2} : R(P_1(x, y, b, z), P_2(x, y, b, z)) = 0\}. \quad (18)$$

On the other hand, $R(P_1(x, y, b, z), P_2(x, y, b, z))$ is a polynomial with real coefficients in the $x_e, y_e$ and $b_u$. Using the identification $\mathbb{C}^{|E|/2} \cong \mathbb{R}^{|E|/2}$, the set on the right hand of (18) is the zero locus in $\mathbb{R}^{|V| + |E|}$ of a polynomial with real coefficients, and hence it is a finite union of real algebraic varieties, so it has Lebesgue measure zero. Since $\mathcal{A}(G)$ is finite, by (17) we get that $P$ is contained in a finite union of sets of measure zero, so $P$ itself has Lebesgue measure zero.

To see that $P^c$ is open, we begin by noting that Theorem 3.3 (ii) implies that the reverse containment of (17) holds, and hence $P = \bigcup_{X \in \mathcal{A}(G)} P_X$. So, if $(a, b) \in P^c$, for every $X \in \mathcal{A}(G)$, the Jacobi matrices of the trees in $F_X$, with labels in $(a, b)$, do not have a common eigenvalue. Fix $(a, b) \in P^c$ and define

$$S = \bigcup_{X \in \mathcal{A}(G)} \bigcup_{T \in F_X} \text{Spec } A_T.$$

As $F_X$ is finite for each of the finitely many $X \in \mathcal{A}(G)$, we may safely define $\Delta > 0$ to be the smallest distance between two distinct points in $S$. We will show that if $(a', b') \in \mathbb{R}^{|V|} \oplus \mathbb{C}^{|E|/2}$ satisfies $\|(a, b) - (a', b')\|_2 < \Delta/2$ then $(a', b') \in P^c$.

Assume otherwise. Then there exists an $X \in \mathcal{A}(G)$ such that the Jacobi matrices with parameters in $(a', b')$ of the trees in $F_X$ have a common eigenvalue $\lambda$. Let $T_1, ..., T_p$ be the trees in $F_X$ with parameters in $(a, b)$ and let $T'_1, ..., T'_p$ denote the same trees but with parameters in $(a', b')$. For every $i$ let $\lambda_i$ be the closest point in $\text{Spec } A_{T_i}$ to $\lambda$. Since $(a, b) \in P^c$ we have $\lambda_i \neq \lambda_j$ for some $i, j$. On the other hand since $\|A_{T_i} - A_{T'_i}\| \leq \|A_{T_i} - A_{T'_i}\|_F \leq \|(a, b)\|_2 < \Delta/2$ and similarly $\|A_{T_j} - A_{T'_j}\| < \Delta/2$,
the triangle inequality and Weyl’s inequality together imply
\[ |\lambda_i - \lambda_j| \leq |\lambda_i - \lambda| + |\lambda_j - \lambda| < \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta, \]
contradicting the definition of $\Delta$. 

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