SEQUENCES WITH HIGH NONLINEAR COMPLEXITY

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Abstract. We improve lower bounds on the $k$th-order nonlinear complexity of pseudorandom sequences over finite fields and we establish a probabilistic result on the behavior of the $k$th-order nonlinear complexity of random sequences over finite fields.

1. Introduction

Pseudorandom sequences over large finite fields are of interest for simulation methods since such sequences can be transformed easily into sequences of uniform pseudorandom numbers in the unit interval $[0, 1]$ (see [17, Chapter 8]). Another area of applications is cryptography. In order to assess the suitability of a pseudorandom sequence, complexity-theoretic and statistical requirements have to be tested. In practice, both categories of tests—complexity-theoretic and statistical—should be carried out since these two categories are in a sense independent (see e.g. the recent paper [19]). A classical survey article on the testing of pseudorandom sequences in a cryptographic context is [21].

In this paper we focus on the complexity-theoretic analysis of pseudorandom sequences over finite fields. A variety of complexity measures for such sequences is available in the literature. The most common approach is to measure complexity by the shortest length of a feedback shift register that can generate the given sequence. The basic concept of this type is the linear complexity (also called the linear span) where only linear feedback shift registers are considered (see also Remark 3 below). There is a considerable amount of literature on the linear complexity which is surveyed in [18], [23], [25], and the recent handbook article [15].

Far less work has been done on complexity measures referring to feedback shift registers with feedback functions of higher algebraic degree (we may call them “nonlinear complexities”). A complexity measure of this type which has received some attention is the maximum-order complexity due to Jansen [4], [5] (see Remark 2 below). There are also complexity measures for sequences based on pattern counting, such as the Lempel-Ziv complexity (see [7] for the definition and [16] for cryptographic applications). The well-known Kolmogorov complexity is not of practical relevance since it cannot be computed in general for sequences of large length.

This paper contributes to the theory of nonlinear complexities by improving lower bounds on nonlinear complexities of interesting pseudorandom sequences and by establishing a probabilistic result on the behavior of nonlinear complexities of random sequences. In Section 2 we collect the basic definitions. In Sections 3 and 4 we establish complexity bounds for certain explicit inversive sequences and for newly constructed sequences from Hermitian function fields, respectively. Finally, in Section 5 we present the mentioned probabilistic result.

2. Definitions

We write $\mathbb{F}_q$ for the finite field with $q$ elements, where $q$ is an arbitrary prime power. For any positive integer $m$, let $\mathbb{F}_q[x_1, \ldots, x_m]$ be the ring of polynomials over $\mathbb{F}_q$ in the $m$
variables $x_1, \ldots, x_m$. Furthermore, we denote the set of positive integers by $\mathbb{N}$. Now we define nonlinear complexities for sequences of finite length over $\mathbb{F}_q$.

**Definition 1.** Let $s = (s_i)_{i=1}^n$ be a sequence of length $n \geq 1$ over the finite field $\mathbb{F}_q$ and let $k \in \mathbb{N}$. If $s_i = 0$ for $1 \leq i \leq n$, then we define the $k$th-order nonlinear complexity $N^{(k)}(s)$ to be 0. Otherwise, let $N^{(k)}(s)$ be the smallest $m \in \mathbb{N}$ for which there exists a polynomial $f \in \mathbb{F}_q[x_1, \ldots, x_m]$ of degree at most $k$ in each variable such that

$$s_{i+m} = f(s_i, s_{i+1}, \ldots, s_{i+m-1}) \quad \text{for} \quad 1 \leq i \leq n - m. \quad (1)$$

**Remark 1.** For $n = 1$ we have $N^{(k)}(s) = 0$ or 1. For $n \geq 2$ we always have $0 \leq N^{(k)}(s) \leq n - 1$, where the upper bound holds since (1) is satisfied for $m = n - 1$ and $f$ being the constant polynomial $s_n$. Both extreme values 0 and $n - 1$ can occur. This is trivial for 0 by Definition 1. Furthermore, if $s = (s_i)_{i=1}^n$ with $s_i = 0$ for $1 \leq i \leq n - 1$ and $s_n = 1$, then $N^{(k)}(s) = n - 1$, since the assumption $N^{(k)}(s) \leq n - 2$ easily leads to a contradiction.

**Remark 2.** In Definition 1 it suffices to consider $1 \leq k \leq q - 1$. This follows from the well-known fact that, as a map, any polynomial $f : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$ can be represented by a polynomial over $\mathbb{F}_q$ in $m$ variables of degree at most $q - 1$ in each variable (see [8, pp. 368–369]). For $k \geq q - 1$ all nonlinear complexities $N^{(k)}(s)$ of a fixed $s$ are the same and equal to the maximum-order complexity $M(s) = N^{(q-1)}(s)$ introduced by Jansen [4], [5]. Connections between the Lempel-Ziv complexity and the maximum-order complexity were studied in [4], [9], [10].

**Remark 3.** One may also consider the nonlinear complexity $L^{(k)}(s)$ where in Definition 1 we replace “of degree at most $k$ in each variable” by “of total degree at most $k$”. It is then trivial that $L^{(k)}(s) \geq N^{(k)}(s)$ for any $k$ and $s$. Note that $L^{(1)}(s)$ is not quite the same as the linear complexity $L(s)$ of $s$, since in the definition of $L(s)$ we accept only homogeneous linear polynomials $f \in \mathbb{F}_q[x_1, \ldots, x_m]$ as feedback functions in (1), whereas in the definition of $L^{(1)}(s)$ we accept also linear polynomials with constant term. We have $L(s) \geq L^{(1)}(s) \geq L(s) - 1$ for any $s$, where the first inequality is trivial and the second inequality follows from a remark in [8, p. 401]. In particular, any lower bound on $L^{(1)}(s)$, like in Corollaries 1 and 2 and in Theorem 1 below, is also a lower bound on the linear complexity $L(s)$.

In order to define nonlinear complexities for infinite sequences, we proceed in analogy to the step from the linear complexity to the linear complexity profile (see [15]), namely by considering nonlinear complexities of finite-length initial segments of a given infinite sequence.

**Definition 2.** Let $S = (s_i)_{i=1}^\infty$ be an infinite sequence over $\mathbb{F}_q$. Then for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we define $N^{(k)}_n(S) = N^{(k)}(s_n)$ and $L^{(k)}_n(S) = L^{(k)}(s_n)$, where $s_n = (s_i)_{i=1}^n$.

### 3. Complexity bounds for explicit inversive sequences

We first consider sequences of finite length that belong to the family of explicit inversive pseudorandom sequences introduced in [13]. Let $e$ be a primitive element of $\mathbb{F}_q$, i.e., a generator of the cyclic multiplicative group $\mathbb{F}_q^\ast$ of nonzero elements of $\mathbb{F}_q$, and choose an element $a \in \mathbb{F}_q^\ast$. Let $s = (s_i)_{i=1}^{q-2}$ be the sequence over $\mathbb{F}_q$ defined by

$$s_i = (ae^i - a)^{-1} \quad \text{for} \quad 1 \leq i \leq q - 2. \quad (2)$$

**Theorem 1.** Let $s = (s_i)_{i=1}^{q-2}$ be the sequence over $\mathbb{F}_q$ defined by (2). Then for any integer $k$ with $1 \leq k \leq q - 1$ we have

$$N^{(k)}(s_n) \geq (n - 1)/(k + 1) \quad \text{for} \quad 1 \leq n \leq q - 2,$$

where $s_n = (s_i)_{i=1}^n$. 

Proof. Since the kth-order nonlinear complexity is invariant under the termwise multiplication of a sequence by an element from \( \mathbb{F}_q^* \), we can assume that \( a = 1 \). The result is trivial for \( n = 1 \), and so we can also assume that \( 2 \leq n \leq q - 2 \). Suppose that \( f \in \mathbb{F}_q[x_1, \ldots, x_m] \) with \( 1 \leq m \leq n - 1 \) is a polynomial of degree at most \( k \) in each variable such that
\[
s_{i+m} = f(s_i, s_{i+1}, \ldots, s_{i+m-1}) \quad \text{for } 1 \leq i \leq n - m.
\]
Thus, we have
\[
(3) \quad - \frac{1}{e^{i+m} - 1} + f \left( \frac{1}{e^i - 1}, \frac{1}{e^{i+1} - 1}, \ldots, \frac{1}{e^{i+m-1} - 1} \right) = 0 \quad \text{for } 1 \leq i \leq n - m.
\]

Consider the rational function
\[
(4) \quad R(z) = - \frac{1}{e^m z - 1} + f \left( \frac{1}{z - 1}, \frac{1}{ez - 1}, \ldots, \frac{1}{e^{m-1} z - 1} \right) \in \mathbb{F}_q(z).
\]
Since \( 1 \leq m < q - 1 \), we have \( e^m \neq e^i \) for \( 0 \leq i \leq m - 1 \). Therefore \( e^{-m} \) is not a pole of \( f(1/(z - 1), 1/(ez - 1), \ldots, 1/(e^{m-1} z - 1)) \), and so \( R(z) \neq 0 \in \mathbb{F}_q(z) \). Write \( R(z) \) in reduced form as \( R(z) = v(z)/w(z) \) with \( v(z), w(z) \in \mathbb{F}_q[z], v(z) \neq 0, w(z) \neq 0 \), and \( \gcd(v(z), w(z)) = 1 \). From (3) we get \( R(e^i) = 0 \) for \( 1 \leq i \leq n - m \). Therefore \( v(z) \) has at least \( n - m \) zeros, and so \( \deg(v(z)) \geq n - m \). On the other hand, the definition of \( R(z) \) in (4) implies that \( \deg(v(z)) \leq \deg(w(z)) \leq km + 1 \), and so \( km + 1 \geq n - m \). This yields \( m \geq (n - (km + 1)) \), which is the desired bound.

Corollary 1. Let \( s = (s_i)_{i=1}^{q-2} \) be the sequence over \( \mathbb{F}_q \) defined by (2). Then for any integer \( k \) with \( 1 \leq k \leq q - 1 \) we have
\[
L^{(k)}(s_n) \geq (n - 1)/(k + 1) \quad \text{for } 1 \leq n \leq q - 2,
\]
where \( s_n = (s_i)_{i=1}^{n} \).

Proof. This follows from Theorem 2 and Remark 3.

For \( k = 1 \), it follows from Corollary 1 and an inequality in Remark 3 that for the linear complexity \( L(s_n) \) of the initial segment \( s_n = (s_i)_{i=1}^{n} \) in Corollary 1 we have \( L(s_n) \geq (n - 1)/2 \) for \( 1 \leq n \leq q - 2 \). This improves on the lower bound \( L(s_n) \geq (n - 1)/3 \) shown in [13, Theorem 1].

Now we consider infinite periodic sequences belonging to the family of explicit inversive pseudorandom sequences introduced in [13]. Let \( d \) be a positive divisor of \( q - 1 \) with \( d < q - 1 \) and let \( b \) be an element of order \( d \) of the multiplicative group \( \mathbb{F}_q^* \). Such an element can be obtained as \( u = e^{(q-1)/d} \), where \( e \) is a primitive element of \( \mathbb{F}_q \). Furthermore, choose \( b, c \in \mathbb{F}_q^* \) such that \( cb^{-1} \) does not belong to the cyclic subgroup of \( \mathbb{F}_q^* \) generated by \( u \). Then we define the sequence \( S = (s_i)_{i=1}^{\infty} \) by
\[
(5) \quad s_i = (bu^i - c)^{-1} \quad \text{for all } i \geq 1.
\]
Note that the sequence \( S \) is periodic with least period \( d \).

Theorem 2. Let \( S = (s_i)_{i=1}^{\infty} \) be the sequence over \( \mathbb{F}_q \) defined by (5). Then for any integer \( k \) with \( 1 \leq k \leq q - 1 \) we have
\[
N^{(k)}_n(S) \geq \min \{(n - 1)/(k + 1), (d - 1)/k \} \quad \text{for all } n \geq 1.
\]

Proof. Since the kth-order nonlinear complexity is invariant under the termwise multiplication of a sequence by an element from \( \mathbb{F}_q^* \), we can assume that \( b = 1 \) and that \( c \) does not belong to the cyclic subgroup of \( \mathbb{F}_q^* \) generated by \( u \). We can also assume that \( n \geq 2 \) and \( N^{(k)}_n(S) < (d - 1)/k \), for otherwise the result is trivial. Suppose that \( f \in \mathbb{F}_q[x_1, \ldots, x_m] \) with \( 1 \leq m \leq n - 1 \) and \( m < (d - 1)/k \) is a polynomial of degree at most \( k \) in each variable such that
\[
s_{i+m} = f(s_i, s_{i+1}, \ldots, s_{i+m-1}) \quad \text{for } 1 \leq i \leq n - m.
\]
Thus, we have
\[
(6) \quad - \frac{1}{u^{i+m} - c} + f \left( \frac{1}{u^i - c}, \frac{1}{u^{i+1} - c}, \ldots, \frac{1}{u^{i+m-1} - c} \right) = 0 \quad \text{for } 1 \leq i \leq n - m.
\]

Consider the rational function
\[
(7) \quad R(z) = -\frac{1}{u^m z - c} + f \left( \frac{1}{z - c}, \frac{1}{uz - c}, \ldots, \frac{1}{u^{m-1}z - c} \right) \in \mathbb{F}_q(z).
\]

Since \(1 \leq m < (d - 1)/k \leq d - 1\), we have \(u^m \neq u^i\) for \(0 \leq i \leq m - 1\). Therefore \(cu^{-m}\) is not a pole of \(f(1/(z - c), 1/(uz - c), \ldots, 1/(u^{m-1}z - c))\), and so \(R(z) \neq 0 \in \mathbb{F}_q(z)\). Write \(R(z)\) in reduced form as \(R(z) = v(z)/w(z)\) with \(v(z), w(z) \in \mathbb{F}_q[z]\), \(v(z) \neq 0\), \(w(z) \neq 0\), and \(\gcd(v(z), w(z)) = 1\). From (6) we get \(R(u^i) = 0\) for \(1 \leq i \leq n - m\). Therefore \(v(z)\) has at least \(\min\{n - m, d\}\) zeros, and so \(\deg(v(z)) \geq \min\{n - m, d\}\). On the other hand, the definition of \(R(z)\) in (7) implies that \(\deg(v(z)) \leq \deg(w(z)) \leq km + 1\), and so
\[
km + 1 \geq \min\{n - m, d\}.
\]

Now \(m < (d - 1)/k\) yields \(km + 1 < d\), and so we must have \(\min\{n - m, d\} = n - m\). Therefore \(km + 1 \geq n - m\), hence \(m \geq (n - 1)/(k + 1)\), and the proof is complete. \(\square\)

**Corollary 2.** Let \(S = (s_i)_{i=1}^\infty\) be the sequence over \(\mathbb{F}_q\) defined by (5). Then for any integer \(k\) with \(1 \leq k \leq q - 1\) we have
\[
L_n^{(k)}(S) \geq \min\{(n - 1)/(k + 1), (d - 1)/k\} \quad \text{for all } n \geq 1.
\]

**Proof.** This follows from Theorem 2 and Remark 4. \(\square\)

The lower bounds on nonlinear complexities in Theorem 2 and Corollary 2 are better than those for the periodic sequences over \(\mathbb{F}_q\) (inversive generators, quadratic exponential generators, general nonlinear generators) shown in [3] and [12]. The exact value of the linear complexity of any finite-length initial segment of the sequence defined by (6) is known from [14, Corollary 7]. Distribution properties and structural properties of this sequence were investigated in [23].

4. **Sequences obtained from Hermitian function fields**

The length of the sequence (2) over \(\mathbb{F}_q\) has order of magnitude \(q\) and the period length of the sequence (5) over \(\mathbb{F}_q\) has an order of magnitude at most \(q\). In this section, we construct finite-length sequences over \(\mathbb{F}_q\) with high nonlinear complexity for which the length has an order of magnitude larger than \(q\). This new construction of sequences uses the theory of global function fields. We follow the monographs [20] and [22] with regard to the notation and terminology for global function fields.

Let \(F/\mathbb{F}_q\) be a global function field with full constant field \(\mathbb{F}_q\). We write \(\mathbb{P}_F\) for the set of places of \(F\). Let \(\deg(P)\) denote the degree of the place \(P \in \mathbb{P}_F\). If \(\deg(P) = 1\), then we speak of a rational place of \(F\). Let \(\nu_P\) be the normalized discrete valuation corresponding to \(P \in \mathbb{P}_F\). For a divisor \(D\) of \(F\), let \(\mathcal{L}(D)\) be the Riemann-Roch space associated with \(D\). We note that \(\mathcal{L}(D)\) is a finite-dimensional vector space over \(\mathbb{F}_q\). Let \(\deg(D)\) denote the degree of the divisor \(D\). By the Riemann-Roch theorem [22, Theorem 1.5.17] we have
\[
\dim(\mathcal{L}(D)) = \deg(D) + 1 - g \quad \text{whenever } \deg(D) \geq 2g - 1,
\]
where \(g\) is the genus of \(F\). For \(P \in \mathbb{P}_F\) and \(h \in F\) with \(\nu_P(h) \geq 0\), we write \(h(P)\) for the residue class of \(h\) modulo \(P\) (see [22, p. 6]). If \(P\) is a rational place, then \(h(P) \in \mathbb{F}_q\).

Now let \(H/\mathbb{F}_q\) be the Hermitian function field over \(\mathbb{F}_q\) which exists whenever \(q\) is a square, say \(q = \ell^2\) with a prime power \(\ell\). The Hermitian function field \(H/\mathbb{F}_q\) can be defined explicitly by \(H = \mathbb{F}_q(x, y)\) with \(y^\ell + y = x^{\ell+1}\). The function field \(H/\mathbb{F}_q\) has exactly \(\ell^3 + 1\) rational places and genus \(g = \ell(\ell - 1)/2\). A summary of the properties of \(H/\mathbb{F}_q\) can be found in [22].
Lemma 6.4.4. We single out the rational place $P_{\infty} \in \mathbb{P}_H$ which is defined as the unique pole of $x$.

Let $\mathcal{G} = \text{Aut}(H/\mathbb{F}_q)$ be the group of field automorphisms of the Hermitian function field $H/\mathbb{F}_q$ that fix the elements of $\mathbb{F}_q$. We refer to [26, Section II] for a summary of the properties of the group $\mathcal{G}$. If $\sigma \in \mathcal{G}$ and $P \in \mathbb{P}_H$, then the set $\sigma(P) := \{ \sigma(h) : h \in P \}$ is again a place of $H$. We have the following simple facts (see [22, Section 8.2] and [26, Lemma 2.1]).

**Lemma 1.** For any $\sigma \in \mathcal{G} = \text{Aut}(H/\mathbb{F}_q)$, $P \in \mathbb{P}_H$, and $h \in H$ we have:

(i) $\deg(\sigma(P)) = \deg(P)$;

(ii) $\nu_{\sigma(P)}(\sigma(h)) = \nu_{\sigma(P)}(h)$;

(iii) $\sigma(h)(\sigma(P)) = h(P)$ if $\nu_{\sigma(P)}(h) \geq 0$.

Now, using the same notation as in [26, Lemma 2.2], let $\phi$ be the element of $\mathcal{G}$ determined by

$$\phi(x) = ex, \quad \phi(y) = e^{\ell+1}y,$$

where $e$ is a primitive element of $\mathbb{F}_q$. Then according to [26, Lemma 2.2], the rational place $P_{\infty}$ of $H$ satisfies $\phi(P_{\infty}) = P_{\infty}$, and under the action of $\phi$ on $\mathbb{P}_H$ there are $\ell$ orbits each containing exactly $q-1$ distinct rational places of $H$. We denote these $(q-1)\ell$ distinct rational places of $H$ occurring altogether in these $\ell$ orbits by

$$Q, \phi(Q), \ldots, \phi^{q-2}(Q), P_1, \phi(P_1), \ldots, \phi^{q-2}(P_1), \ldots, P_{\ell-1}, \phi(P_{\ell-1}), \ldots, \phi^{q-2}(P_{\ell-1}).$$

By (9) we have $\dim(\mathcal{L}((2g-1)P_{\infty} + Q)) = g+1$ and $\dim(\mathcal{L}((2g-1)P_{\infty})) = g$, and so we can choose an element $h \in \mathcal{L}((2g-1)P_{\infty} + Q) \setminus \mathcal{L}((2g-1)P_{\infty})$. Then we consider the sequence $s = (s_i)_{i=1}^M$ over $\mathbb{F}_q$ of length $M := (q-1)(\ell - 1)$ given by

$$s = (h(P_1), h(\phi(P_1)), \ldots, h(\phi^{q-2}(P_1)), \ldots, h(P_{\ell-1}), h(\phi(P_{\ell-1})), \ldots, h(\phi^{q-2}(P_{\ell-1}))).$$

The choice of $h$ guarantees that all terms of the sequence $s$ are well defined. Note that the length $M$ of $s$ has order of magnitude $q^{3/2}$.

**Theorem 3.** Let $H/\mathbb{F}_q$ be the Hermitian function field over $\mathbb{F}_q$ with $q = \ell^2$ for some prime power $\ell$. Let $s = (s_i)_{i=1}^M$ with $M = (q-1)(\ell - 1)$ be the sequence over $\mathbb{F}_q$ defined by (9). Then for any integer $k$ with $1 \leq k \leq q-1$ we have

$$N^{(k)}(s_n) \geq \frac{(q-1)[n/(q-1)] - 1}{\ell(\ell - 1)k + [n/(q-1)]} \quad \text{for } 1 \leq n \leq M,$$

where $s_n = (s_i)_{i=1}^n$.

**Proof.** The result is trivial for $n < q-1$, and so we can assume that $n \geq q-1$. Since $N^{(k)}(s_n)$ is a nondecreasing function of $n$, we can also assume that $n$ is a multiple of $q-1$, say $n = (q-1)r$ with $r \in \mathbb{N}$ and $r \leq \ell - 1$. Now we fix such an $n$. We claim that $s_n$ is not the zero sequence. For otherwise there exist $n$ rational places $Q_1, \ldots, Q_n$ of $H$ different from $P_{\infty}$ and $Q$ that are zeros of $h$. This implies that $h \in \mathcal{L}(D)$ with

$$D := (2g-1)P_{\infty} + Q - Q_1 - \cdots - Q_n.$$ But

$$\deg(D) = 2g - n = \ell(\ell - 1) - n \leq \ell(\ell - 1) - (q-1) = -\ell + 1 < 0,$$

and so $h = 0$ by [20, Corollary 3.4.4]. This is a contradiction to the fact that $h \notin \mathcal{L}((2g-1)P_{\infty})$ by the choice of $h$.

Thus we have $N^{(k)}(s_n) \geq 1$. If $N^{(k)}(s_n) \geq q-1$, then the lower bound in the theorem holds trivially. Hence we can assume that $N^{(k)}(s_n) \leq q-2$. Suppose that $f \in \mathbb{F}_q[x_1, \ldots, x_m]$ with $1 \leq m \leq q-2 \leq n-1$ is a polynomial of degree at most $k$ in each variable such that

$$s_{i+m} = f(s_i, s_{i+1}, \ldots, s_{i+m-1}) \quad \text{for } 1 \leq i \leq n-m.$$
By applying (11) only for \( i = (q-1)(j-1)+t+1 \) with \( j = 1, \ldots, r \) and \( t = 0,1,\ldots,q-m-2 \), we obtain
\[-h(\phi^{t+m}(P_j)) + f(h(\phi^t(P_j)), h(\phi^{t+1}(P_j)), \ldots, h(\phi^{t+m-1}(P_j))) = 0\]
for \( 1 \leq j \leq r \) and \( 0 \leq t \leq q-m-2 \). Lemma \( \text{(iii)} \) yields
\[h(\phi^{t+b}(P_j)) = h(b(\phi^t(P_j))) = \phi^{-b}(h(\phi^t(P_j)))\]
for \( 1 \leq j \leq r \) and all integers \( t \geq 0 \) and \( b \geq 0 \), and so
\[- \phi^{-m}(h)(\phi^t(P_j)) + f\left( h(\phi^t(P_j)), \phi^{-1}(h)(\phi^t(P_j)), \ldots, \phi^{-(m-1)}(h)(\phi^t(P_j)) \right) = 0\]
for \( 1 \leq j \leq r \) and \( 0 \leq t \leq q-m-2 \).

Consider the element
\[w = -\phi^{-m}(h) + f\left( h, \phi^{-1}(h), \ldots, \phi^{-(m-1)}(h) \right) \in H.\]
We have \( \nu_Q(h) = -1 \) by the choice of \( h \), hence \( \nu_{\phi^{-m}(Q)}(\phi^{-m}(h)) = -1 \) by Lemma \( \text{(ii)} \), and so the place \( \phi^{-m}(Q) \) is a pole of \( \phi^{-m}(h) \). On the other hand, for \( b = 0,1,\ldots,m-1 \), the place \( \phi^{-m}(Q) \) is not a pole of \( \phi^{-b}(h) \) (use again Lemma \( \text{(ii)} \) and the choice of \( h \)), and so \( \phi^{-m}(Q) \) is not a pole of \( f\left( h, \phi^{-1}(h), \ldots, \phi^{-(m-1)}(h) \right) \). Hence we must have \( w \neq 0 \).

Now we study the zeros and poles of \( w \). First of all, it follows from (11) that all the \( (q-m-1)r \) distinct places \( \phi^j(P_j) \), \( 1 \leq j \leq r \), \( 0 \leq t \leq q-m-2 \), are zeros of \( w \). Therefore the degree of the zero divisor \( (w)_0 \) of \( w \) satisfies
\[\deg((w)_0) \geq (q-m-1)r.\]

By the choice of \( h \) and Lemma \( \text{(ii)} \), the only possible poles of \( w \) are the rational places \( P_\infty, Q, \phi(Q), \ldots, \phi^{g-2}(Q) \). Note that \( \nu_{P_\infty}(h) \geq -(2g-1) \). Since \( P_\infty \) is invariant under \( \phi \), Lemma \( \text{(ii)} \) shows that \( \nu_{P_\infty}(\phi^{-b}(h)) \geq -(2g-1) \) for any integer \( b \geq 0 \). It follows that \( \nu_{P_\infty}(w) \geq -(2g-1)km \). Now we determine the possible poles of \( w \) in the set \( Q = \{Q, \phi(Q), \ldots, \phi^{g-2}(Q)\} \) of rational places. The only pole of \( h \) in \( Q \) is \( Q \) and its pole order is 1. Furthermore, for any integer \( b \) with \( 1 \leq b \leq m \leq q-2 \), Lemma \( \text{(ii)} \) shows that the only pole of \( \phi^{-b}(h) \) in \( Q \) is \( \phi^{-b}(Q) = \phi^{g-1-b}(Q) \) and its pole order is 1. Altogether, the degree of the pole divisor \( (w)_\infty \) of \( w \) satisfies
\[\deg((w)_\infty) \leq (2g-1)km + km + 1 = 2gkm + 1.\]
Now \( \deg((w)_\infty) = \deg((w)_0) \) by a fundamental identity for algebraic function fields (see \[22\] Theorem 1.4.11), and so
\[2gkm + 1 \geq \deg((w)_\infty) = \deg((w)_0) \geq (q-m-1)r.\]

It follows that
\[m \geq \frac{(q-1)r - 1}{2gk + r},\]
which completes the proof of the theorem (recall that we assumed without loss of generality that \( n = (q-1)r \)).

Note that the lower bound on \( \Lambda^{(k)}(s_n) \) in Theorem \[3\] is of order of magnitude \( n/(qk) \). If \( n \) is of the maximal order of magnitude \( q^{3/2} \), then the lower bound is of order of magnitude \( q^{1/2}/k \). In contrast to Section \[5\] we can obtain a better lower bound for the nonlinear complexity \( L^{(k)} \) (see Remark \[3\]) of the sequence \( (\phi) \) than that implied by Theorem \[3\].

**Theorem 4.** Let \( H/\mathbb{F}_q^\ell \) be the Hermitian function field over \( \mathbb{F}_q \) with \( q = \ell^2 \) for some prime power \( \ell \). Let \( s = (s_i}_{i=1}^M \) with \( M = (q-1)(\ell-1) \) be the sequence over \( \mathbb{F}_q \) defined by \[9\]. Then for any integer \( k \geq 1 \) we have
\[L^{(k)}(s_n) \geq \frac{(q-1)n/(q-1) - (\ell^2 - \ell - 1)k - 1}{k + [n/(q-1)]} \quad \text{for} \ 1 \leq n \leq M,\]
where \( s_n = (s_i}_{i=1}^n \).
Proof. We proceed exactly as in the proof of Theorem 3. The only difference is that now
\[ \nu_p(w) \geq -(2g-1)k \] since the polynomial \( f \) has total degree at most \( k \). Therefore
\[ \deg((w)_\infty) \leq (2g-1)k + km + 1, \]
and this yields the desired result. \( \square \)

For small \( k \) and for \( n \) of a larger order of magnitude than \( q \), the lower bound on \( L^{(k)}(s_n) \) in Theorem 4 is of order of magnitude \( q \).

5. A PROBABILISTIC RESULT

Let \( \mu_q \) be the uniform probability measure on \( \mathbb{F}_q \) which assigns the measure \( 1/q \) to each element of \( \mathbb{F}_q \). Let \( \mathbb{F}_q^\infty \) be the sequence space over \( \mathbb{F}_q \) and let \( \mu_q^\infty \) be the complete product probability measure on \( \mathbb{F}_q^\infty \) induced by \( \mu_q \). We say that a property of sequences \( S \in \mathbb{F}_q^\infty \) holds \( \mu_q^\infty \)-almost everywhere if it holds for a set of sequences \( S \) of \( \mu_q^\infty \)-measure 1. We may view such a property as a typical property of a random sequence over \( \mathbb{F}_q \).

Theorem 5. Let \( k \) be an integer with \( 1 \leq k \leq q-1 \). Then \( \mu_q^\infty \)-almost everywhere we have
\[ \liminf_{n \to \infty} \left( N_n^{(k)}(S) - \frac{\log n}{\log(k+1)} \right) \geq 0. \]

Proof. For \( m, n \in \mathbb{N} \) with \( m \leq n-1 \), let \( T_n^{(k)}(m) \) be the number of sequences \( s \) of length \( n \) over \( \mathbb{F}_q \) with \( N_n^{(k)}(s) \leq m \). Each sequence \( s = (s_i)_{i=1}^n \) counted by \( T_n^{(k)}(m) \) is (not necessarily uniquely) determined by a polynomial \( f \in \mathbb{F}_q[x_1, \ldots, x_m] \) of degree at most \( k \) in each variable and by initial values \( s_1, \ldots, s_m \) of the recursion (11). Since the number of possibilities for \( f \) is \( q^{(k+1)m} \), we have
\[ T_n^{(k)}(m) \leq q^{(k+1)m} + m \quad \text{for } 1 \leq m \leq n-1. \]

Now fix \( \varepsilon > 0 \) and put
\[ b_n = \frac{\log n}{\log(k+1)} \] and
\[ A_n = \{ S \in \mathbb{F}_q^\infty : N_n^{(k)}(S) \leq b_n \} \]
for \( n = 1, 2, \ldots \).

Then \( 1 \leq [b_n] \leq n-1 \) for sufficiently large \( n \), and so (12) yields
\[ \mu_q^\infty(A_n) = q^{-n}T_n^{(k)}([b_n]) \leq q^{(k+1)b_n - n} \]
for sufficiently large \( n \). Now for some \( 0 < \delta < 1 \) we have
\[ (k+1)^b_n + b_n - n < n \left( \frac{1}{(k+1)^\varepsilon} + \frac{\log n}{n \log(k+1)} - 1 \right) < -\delta n \]
for sufficiently large \( n \), and so \( \sum_{n=1}^\infty \mu_q^\infty(A_n) < \infty \). Then the Borel-Cantelli lemma (see [11, Lemma 3.14] and [11, p. 228]) shows that the set of all \( S \in \mathbb{F}_q^\infty \) for which \( S \in A_n \) for infinitely many \( n \) has \( \mu_q^\infty \)-measure 0. In other words, \( \mu_q^\infty \)-almost everywhere we have \( S \in A_n \) for at most finitely many \( n \). It follows then from the definition of \( A_n \) that \( \mu_q^\infty \)-almost everywhere we have
\[ N_n^{(k)}(S) > b_n = \frac{\log n}{\log(k+1)} - \varepsilon \]
for sufficiently large \( n \). This means that \( \mu_q^\infty \)-almost everywhere we have
\[ \liminf_{n \to \infty} \left( N_n^{(k)}(S) - \frac{\log n}{\log(k+1)} \right) \geq -\varepsilon. \]
By applying this for all \( \varepsilon = 1/r \) with \( r \in \mathbb{N} \) and noting that the intersection of countably many sets of \( \mu_q^\infty \)-measure 1 has again \( \mu_q^\infty \)-measure 1, we obtain the result of the theorem. \( \square \)
Remark 4. For $k = q - 1$, Theorem 5 says that $\mu^\infty_q$-almost everywhere the maximum-order complexity $N_n^{(q-1)}(S)$ (see Remark 2) grows at least like $(\log n) / (\log q)$ as $n \to \infty$. This is in good accordance with the result of Jansen [4] (see also [2] and [6]) that the expected value of $N_n^{(q-1)}(S)$ behaves asymptotically like $(\log n) / (\log q)$, up to an absolute constant. On the basis of these results, it may be conjectured that $\mu^\infty_q$-almost everywhere we have

$$\lim_{n \to \infty} \frac{N_n^{(q-1)}(S)}{\log n} = C_q$$

for some constant $C_q > 0$ depending only on $q$. A similar behavior may be conjectured for $N_n^{(k)}(S)$ with $1 \leq k < q - 1$, where $C_q$ is replaced by a constant $C_{q,k} > 0$ depending only on $q$ and $k$. In view of this heuristic that the expected order of magnitude of $N_n^{(k)}(S)$ for random sequences $S$ is $\log n$, it is clear that the sequences considered in Sections 3 and 4 can be said to have high nonlinear complexity.

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