Transit index of various graph classes

K.M. Reshmi¹* and Raji Pilakkat²

Abstract
Transit of a vertex \(v\) is a graph invariant which was defined as the sum of the length of all shortest paths with \(v\) as an internal vertex. In this paper, transit index for various classes of graph like complete graphs, cycles, wheel graph, friendship graph, crown graph, total graph of a path, comet are computed.

Keywords
Transit of a vertex, Transit Index.

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05C10, 05C12.

1 Department of Mathematics, Government Engineering College, Kozhikode-673005, Kerala, India.
2 Department of Mathematics, University of Calicut, Malappuram-673385, Kerala, India.

*Corresponding author: ¹reshmikm@gmail.com; ²rajipilakkat@gmail.com

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1. Introduction

Graph topological indices are widely studied. They find application in many field of science. Chemical graph Theory and Networking are a few to name. In[8], transit index of a graph was introduced and its correlation with one of the physical property -MON of octane isomers was established. In this paper we compute the transit index for various graph classes and for certain graphs developed from complete graphs.

Throughout \(G\) denotes a simple, connected, undirected graph with vertex set \(V\) and edge set \(E\). For undefined terms we refer [1].

Preliminaries

Definition 1.1. [8] Let \(v \in V\). Then the transit of \(v\) denoted by \(T(v)\) is “the sum of the lengths of all shortest path with \(v\) as an internal vertex” and the transit index of \(G\) denoted by \(TI(G)\) is

\[
TI(G) = \sum_{v \in V} T(v)
\]

Lemma 1.2. [8] \(T(v) = 0\) iff \(\langle N[v]\rangle\) is a clique.

Theorem 1.3. [8] For a path \(P_n\), Transit index is

\[
TI(P_n) = \frac{n(n+1)(n^2-3n+2)}{12}
\]

Definition 1.4. Two vertices \(v_1\) and \(v_2\) of a graph are called transit identical if the shortest paths passing through them are same in number and length.

2. Transit index for various graph classes

2.1 Star

Theorem 2.1. For a star graph \(S_n\), \(TI(S_n) = (n-1)(n-2)\)

Proof. In a star graph on \(n\) vertices, \(n-1\) vertices are pendant vertices. Hence for them \(T(v) = 0\). There are \(C(n-1,2)\) shortest path of length 2 passing through the center vertex. Hence \(TI(S_n) = 2.C(n-1,2) = (n-1)(n-2)\)
2.2 Complete Graphs

Theorem 2.2. For the complete graph $K_n$, transit index is zero.

Proof. For every vertex $v$ in a complete graph $K_n$, $|N[v]| = K_n$, a clique. Hence by lemma[1.2], $TI(K_n) = 0$.

Theorem 2.3. For $n \geq 3$, deleting an edge from $K_n$, increases the transit index by $2(n-2)$.

Proof. The deletion of the edge $e = uv$, makes $u$ and $v$ non-adjacent. Hence every other vertex will be an internal vertex of the shortest path between $u$ and $v$ of length 2. Hence $TI(K_n - e) = 2(n-2)$.

Theorem 2.4. Let $G = K_{p,q}$ where $V = V_1 \cup V_2$ is the bi-partition with $|V_1| = p, |V_2| = q$. Then $TI(G) = pq[p + q - 2]$.

Proof. When $p = 1$ or $q = 1$, the result is obvious.
Let $p, q \geq 2$.
Let $v \in V_1$, Then, $T(v) = 2C(q, 2)$.
If $v \in V_2$, then $T(v) = 2C(p, 2)$.

Hence $TI(G) = 2\sum_{v \in V} T(v)$
$\quad = \sum_{v \in V_1} T(v) + \sum_{v \in V_2} T(v)$
$\quad = 2[pq(q-1) + 2]pq(p-1)]$
$\quad = pq[p + q - 2]$

Theorem[2.4] can be generalised to s-partite graphs as follows.

Theorem 2.5. Let $G$ be the complete s-partite graph [6].

Then $TI(G) = \sum_{i=1}^{s} 2n_i \left[ \sum_{j \neq i} C(n_j, 2) \right]$.

Proof. Let $V_1, V_2, \ldots, V_s$ be the partition of the vertex set $V$.
Then no two vertices in $V_i$ are adjacent to each other. But every vertex in $V_i$, $j \neq i$ is adjacent to all vertices of $V_j$.
The shortest paths passing through $v_i$ are those connecting vertices of the same $V_j$ to itself, of length 2. Hence $T(v_i) = 2 \sum_{j \neq i} C(n_j, 2)$

\[ TI(G) = \sum_{v_i \in V_1} T(v_i) + \sum_{v_i \in V_2} T(v_i) + \cdots + \sum_{v_i \in V_s} T(v_i) \]

\[ = \sum_{i=1}^{s} 2n_i \left[ \sum_{j \neq i} C(n_j, 2) \right] \]

Corollary 2.6. If $G$ is the cocktail party graph [5], $TI(G) = 4n(n-1)$.

Proof. In the theorem [2.5], take $n_i = 2, \forall i$ and $s = n$ with $|G| = 2n$.

2.3 Cycle

Theorem 2.7. Let $C_n$ be a cycle with $n$ even. Then

i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$

ii) $TI(C_{n+1}) = \frac{n(n^2-4)(n+1)}{24}$

Proof.

Figure 1. Cycle $C_n$

(i) Consider the vertex $v$ in the figure[1]. The maximum length of the shortest path passing through $v$ is of length $\frac{n}{2}$.
The sum length of the length of the shortest paths originating

\[ \begin{array}{c|c}
\text{from} & \text{is} \\
1 & 2 + 3 + \ldots + \frac{n}{2} \\
2 & 3 + 4 + \ldots + \frac{n}{2} \\
\vdots & \vdots \\
\frac{n}{2} - 1 & \frac{n}{2} \\
\end{array} \]

Hence $T(v) = \left( \frac{n}{2} - 1 \right)(\frac{n}{2}) + \left( \frac{n}{2} - 2 \right)(\frac{n}{2} - 1) + \ldots + 2.1 + 1.0$

Due to symmetry, every vertex in the cycle are transit identical.

Then $TI(C_n) = \frac{n^2(n^2-4)}{24}$, n is even

(ii) Consider $C_{n+1}$, with $n$ even. The maximum length of the shortest path passing through any vertex $v$ remains to be $\frac{n}{2}$.

Hence as in the case of even cycle $T(v) = \left( \frac{n^2-4}{24} \right)$.

\[ TI(C_{n+1}) = (n + 1)T(v) = (n + 1)\left( \frac{n^2-4}{24} \right) \]

2.4 Wheel Graph

The wheel graph [2], $W_{n+1}$ is the graph obtained from $C_n$, $n \geq 3$ by adding a new vertex and making it adjacent to all vertices of $C_n$. 

Theorem 2.8. \( TI(W_{n+1}) = n(n-1), n > 3 \) and for \( n = 3, TI(W_{3+1}) = 0 \)

Proof. Let \( n > 3 \). In \( W_{n+1} \), the diameter is 2. Hence no shortest path is of length more than 2. The vertices on the outer circle \( C_n \) are transit identical. Let \( v \) be one such vertex. The only shortest path passing through it is between its adjacent vertices. \( T(v) = 2 \), for \( v \in C_n \)

Consider the center vertex \( c \). To find its transit we consider the contribution of each edge to it. Every edge on \( C_n \) contributes to \( T(c) \). Hence contribution of each edge to it. Every edge on \( C_n \) contributes only by \( v \) to travel to every vertex other than its adjacent ones. Hence the contribution is \( n(n-3) \), \( T(c) = n(n-3) \)

i.e. \( TI(W_{n+1}) = 2n + n(n-3) = n(n-1) \)

For \( n = 3 \), we get \( W_{3+1} = K_4 \); \( T(c) \) its transit is zero.

2.5 Friendship Graph

The Friendship graph \( [3] \), \( F_n \) is constructed by coalescence of \( n \) copies of cycle \( C_3 \) of length 3, with a common vertex.

Theorem 2.9. \( TI(F_n) = 4n(n-1), |V| = 2n + 1 \)

Proof. In \( F_n \), the diameter is 2. For every vertex \( v \) other than the coalescence vertex, \( \langle N[v] \rangle \) is a clique. Hence \( T(v) = 0 \), by lemma[1.2]. Hence \( TI(F_n) = T(c) \)

The edges of the type \( e' \), as in the figure[3] does not contribute to \( T(c) \). Hence we count the number of times the edges of the type \( e \) is used. The edge \( e \) will be used by the vertex \( v \) to travel to all vertices other than its adjacent ones. Hence contribution of \( e \) is \( 2n + 1 - 3 = 2(n-1) \). There are \( 2n \) such edges.

\( T(c) = 4n(n-1) \)

i.e. \( TI(F_n) = 4n(n-1), |V| = 2n + 1 \).

2.6 Crown Graph

A crown graph \([4]\) is the unique \( n - 1 \) regular graph with \( 2n \) vertices, obtained from the complete bipartite graph \( K_{n,n} \) by deleting a perfect matching. Or it is the graph with vertices as two sets \( \{u_i\} \) and \( \{v_i\} \), with an edge from \( u_i \) to \( v_j \) whenever \( i \neq j \)

\[ \text{Figure 2. Wheel graph } W_{n+1} \]

\[ \text{Figure 3. Friendship Graph } F_n \]

\[ \text{Figure 4. Crown graph} \]

Theorem 2.10. For the Crown graph \( G \), \( TI(G) = 2n(n^2 - 1) \).

Proof. Let the bipartition be \( V = U \cup V \), with \( V = \{v_1, v_2, \ldots, v_n\} \) and \( U = \{u_1, u_2, \ldots, u_n\} \). Consider a vertex of \( V \), say \( v_k \). Note that \( d(u_i, v_i) = 3 \) and \( d(u_i, v_j) = 2, i \neq j \). The shortest path through \( v_k \) are those connecting \( v_i \) to \( v_j, i \neq j \) of length 2 and those connecting \( v_i \) to \( u_i \) of length 3. Hence \( T(v_k) = 2C(n-1, 2) + 3(n-1) = n^2 - 1 \). In this graph every vertex is transit identical. \( \therefore TI(G) = 2n(n^2 - 1) \).

2.7 Snake Graph

The triangular snake graph can be viewed as the graph formed by replacing every edge of \( P_n \) by a triangle, thus adding \( n - 1 \) vertices and \( 2(n-1) \) edges.

Theorem 2.11. If \( G \) is the triangular snake graph of a path on \( 2n - 1 \) vertices, \( TI(G) = TI(P_n) + 2|n-2|(n-1)n(n+1) \).

Proof. Let \( v_1, v_2, \ldots, v_n \) denote the vertices of the path \( P_n \). The newly added vertices are named as \( u_1, u_2, \ldots, u_{n-1} \). For every \( u_i \), \( \langle N[u_i] \rangle \) is a clique. Hence \( T(u_i) = 0, \forall i \), by lemma[1.2]. Also \( \langle N[v_1] \rangle, \langle N[v_n] \rangle \) are cliques. \( \therefore T(v_1) = T(v_n) = 0 \).

Hence we need to compute only the transit of \( v_i \) for \( 1 < i < n \). The transit of these vertices are due to path connecting \( v_i \) among themselves, path connecting \( u_i \) among themselves and paths connecting \( v_i \) to \( u_i \), i.e. \( TI(G) = TI(P_n) + I \),
Let \( v_k \) be any vertex of \( P_n \). Then the increase in \( T(v_k) \) is due to the paths connecting the vertices on the left of it to the newly added vertices. This can be computed as
\[
T(P_{n+1}) = T(P_n) + \frac{mn(n^2 - 1)}{3}
\]
Hence the theorem.

**Remark 2.13.** Applying the recursive formula for a path, \( T(P_{n+1}) = T(P_n) + \frac{mn(n^2 - 1)}{3} \), the transit of a comet \( G \) of Theorem 2.12 can be expressed as, \( T(G) = mT(P_{n-1}) - (m-1)I(T(P_n)) \).

### 3. Transit index for some graphs derived from complete graph

**Theorem 3.1.** Let \( G \) be the graph obtained by attaching a pendant edge to one of the vertices of a complete graph, i.e. \( |V(G)| = |V(K_n)| + 1 \) and \( |E(G)| = |E(K_n)| + 1 \). Then \( T(G) = 2(n-1) \)

**Proof.** Let the new vertex be \( v \) and the vertex to which it is attached be \( u \). Then for every vertex in \( G \) other than \( u \), \( N[v_i] \)

is a clique. Hence transit is zero. There are \( n-1 \) paths of length 2 connecting \( v \) to vertices of \( K_n - \{u\} \), passing through \( v \).
\[
\therefore T(G) = 2(n-1)
\]

**Theorem 3.2.** Let \( G \) be the graph obtained by attaching a pendant edge to every vertex of \( K_n \), then \( T(G) = 5n(n-1) \)

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( K_n \) and \( u_1, u_2, \ldots, u_n \), be the vertices attached to \( v_1, v_2, \ldots, v_n \) respectively. Since \( u_i \) is pendant vertices \( T(u_i) = 0, \forall i \). The shortest path passing through \( v_i \) is either \( u_i v_i \) paths or \( u_i u_j \) paths of length 2 and 3 respectively. Hence \( T(v_i) = 2(n-1) + 3(n-1) \)
\[
\therefore T(G) = 5n(n-1)
\]

**Theorem 3.3.** Let \( G \) be the graph obtained by merging a vertex of \( K_n \) and \( K_m \), i.e. \( |V(G)| = n+m-1 \) and \( |E(G)| = |E(K_n)| + |E(K_m)| \). Then \( T(G) = 2(n-1)(m-1) \)

**Proof.** Let \( v \) be coalescence vertex. For every vertex \( u \) of \( G \) other than \( v \), \( T(u) = 0 \), as \( N[u] \) is a clique. The shortest paths
passing through $v$ are those connecting the $n - 1$ vertices of $K_n$ with $m - 1$ vertices of $K_m$, each of length 2. Hence $TI(G) = T(v) = 2(n - 1)(m - 1)$.

**Theorem 3.4.** Let $G$ be the graph formed by merging a vertex of $K_n$ with a vertex of $C_m$.

Then $TI(G) = TI(C_m) + \sum_{k=1}^{m} (\frac{m-1}{2} + \frac{m}{2}) + (\frac{m}{2} - k^2) + \frac{(m-1)(m+4)(m+2)m}{12}$, if $m$ is even and $TI(G) = TI(C_m) + \sum_{k=1}^{m} (\frac{m-1}{2} + \frac{m}{2}) + (\frac{m}{2} - k^2) \frac{(m-1)(m+1)(m+3)}{12}$, if $m$ is odd.

**Proof.** Let us denote the coalescence vertex by $v$.

**Case 1** [m even]

Clearly, $TI(G) = TI(C_m) + TI(K_n) + I$, where $I$ denote the increments in transit due to merging of graphs. The transit for vertices in $K_n$ remains zero, except for $v$. The vertex at the distance $\frac{m}{2}$ from $v$ on $C_m$ has no increment. Let $v_k$ denote the $k$th vertex on $P_1$, $v_1$ being $v$. For $1 < k < \frac{m}{2}$, the increment for $v_k$ due to the shortest paths from vertices on its right to vertices of $K_n$ including $v$. This can be computed as

$$I = \left[(\frac{m}{2} + 1)(\frac{m}{2} + 2) - k - k^2\right] \frac{(n-1)}{2}$$

Now due to similar positions, $TI(v_k)$, $TI(v_{m-k+2})$ are transit identical.

Hence we have $I = \sum_{k=1}^{\frac{m}{2}} I_k = \frac{1}{2} \sum_{k=1}^{\frac{m}{2}} \left[(\frac{m}{2} + 1)(\frac{m}{2} + 2) - k - k^2\right] \frac{(n-1)}{2}$

For $1 < k < \frac{m-1}{2},$

$$I = \left[(\frac{m}{2} + 1)(\frac{m}{2} + 2) - k - k^2\right] \frac{(n-1)}{2}$$

**Case 2** [m odd]

Let $v_k$ denote the $k$th vertex on $P_1$, $v_1 = v$. For $1 < k < \frac{m-1}{2},$

$$I = \left[(\frac{m}{2} + 1)(\frac{m}{2} + 2) - k - k^2\right] \frac{(n-1)}{2}$$

Thus, $TI(G) = TI(C_m) + \frac{1}{2} \sum_{k=1}^{\frac{m}{2}} \left[(\frac{m}{2} + 1)(\frac{m}{2} + 2) - k - k^2\right] \frac{(n-1)}{2}$

**4. Conclusion**

In this paper, transit index for various graph classes and for graphs obtained from complete graphs are computed. In future, authors are planning to extend the study to sub-division graphs, graph products and various graphs of importance in chemical graph theory and communication networks.

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