When do nonparametric regressions based on discretely sampled curves work?

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Abstract

In the context of nonparametric regression, we study conditions under which the consistency (and rates of convergence) of estimators built from discretely sampled curves can be derived from the consistency of estimators based on the unobserved whole trajectories. As a consequence, we derive asymptotic results for most of the regularization techniques used in functional data analysis, including smoothing and basis representation.

1. Introduction

Technological progress in collecting and storing data provides datasets recorded at finite grids of points that become denser and denser over time. Although in practice data always comes in the form of finite dimensional vectors, from the theoretical point of view, the classic multivariate techniques are not well suited to deal with data which, essentially, is infinite dimensional and whose observations within the same curve are highly correlated.

From a practical point of view, a commonly used technique to treat this kind of data is to transform the (observed) discrete values into a function via smoothing or a series approximations (see [7], [21], [25, 26, 27], or chapter 9 of [24] and the references therein). For the analysis, we can use the intrinsic infinite dimensional nature of the data and assume the existence of continuous underlying stochastic processes which are observed ideally at every point. In this context, the theoretical analysis is performed on the functional space where they take values (see [14]). In what follows, we will refer to this last setting as the full model.

Nonparametric regression is an important tool in functional data analysis (FDA) which has received considerable attention from different authors in both settings. For the full model, consistency results have been obtained by, among others, [1], [5], [6], [9], [10], [14], [22], and [23]. In particular, [15] (see also the Corrigendum [16]) prove a consistency result close to universality for the kernel (with random bandwidth) estimator. The first contribution of the present paper will be to prove the consistency of the $k$-nearest neighbor with kernel regression.
estimator (Proposition 2) when the full trajectories are observed. This family, considered by [13], combines the smoothness properties of the kernel function with the locality properties of the k-nearest neighbors distances.

Regarding regression when discretized curves are available, [19] study the mean square consistency of the kernel estimator when the sample size as well as the grid size discretization go to infinity. More precisely, from independent realizations of a random process with continuous covariance structure, they estimate the regression function, assuming its smoothness. Under the same assumptions, but using interpolation of the data, [28], in a mainly practical approach, propose a method to estimate the regression function via smoothing splines (see also [20]). More recently, [11] establish minimax rates of convergence of estimators of the mean based on discretized sampled data while [12] establish the minimax rates of convergence for the covariance operator when data are observed on a lattice (see also [18] for the problem of principal components analysis for longitudinal data). In this context it is natural to assess the relation between the ideal nonparametric regression estimator constructed with the entire set of curves and the one computed with the discretized sample. In this direction, we are interested in addressing the following question:

- Under what conditions can the consistency (and rates of convergence) of the estimate computed with the discretized trajectories be derived from the consistency of the estimate based on the full curves?

Clearly, the asymptotic results for estimates computed with the discretized sample will not be a direct consequence of those for the full model. However, we provide reasonable conditions in order to still get the consistency and find rates of convergence of the estimator. In this context we state the results for the well known kernel and k-nearest neighbor with kernel estimators. These results are a consequence of a more general result, which, besides discretization, also includes the cases of regularization via smoothing and basis representation.

This paper is organized as follows: In Section 2 we state the consistency of the k-nearest neighbor with kernel estimator in the infinite dimensional setting (for the full model). In Section 3 we provide conditions for the consistency of the kernel and k-nearest neighbor with kernel estimators when we do not observe the whole trajectories but only a function of them (Theorems 1 and 2). In Section 4 the results for discretization, smoothing and basis representation are obtained as a consequence of Theorems 1 and 2. Proofs are given in Appendices A and B.

2. Two results for the full model

In this section we provide two $L^2$-consistency results for the full model, i.e., when ideally all trajectories are observed at every point of the interval $[0, 1]$. The first one corresponds to kernel estimates, and was obtained in [13], while the second one for k-NN with kernel estimates is derived in the present paper.
Let \((\mathcal{H}, d)\) be a separable metric space and let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be independent identically distributed (i.i.d.) random elements in \(\mathcal{H} \times \mathbb{R}\) with the same law as the pair \((X, Y)\) fulfilling the model:

\[ Y = \eta(X) + e, \quad (1) \]

where the error \(e\) satisfies \(\mathbb{E}_{e|X} (e|X) = 0\) and \(\text{var}_{e|X}(e|X) = \sigma^2 < \infty\). In this context, the regression function \(E(Y|X) = \eta(X)\) can be estimated by

\[ \hat{\eta}_n(X) = \sum_{i=1}^{n} W_{ni}(X)Y_i, \quad (2) \]

where the weights \(W_{ni}(X) = W_{ni}(X, X_1, \ldots, X_n) \geq 0\) and \(\sum_{i=1}^{n} W_{ni}(X) = 1\).

In this paper, we first consider the weights corresponding to the family of kernel estimators given by

\[ W_i(X) = \frac{K\left( \frac{d(X, X_i)}{h_n(X)} \right)}{\sum_{j=1}^{n} K\left( \frac{d(X, X_j)}{h_n(X)} \right)}, \quad (3) \]

where \(K\) is a regular kernel, i.e., there are constants \(0 < c_1 < c_2 < \infty\) such that \(c_1 I_{[0,1]}(u) \leq K(u) \leq c_2 I_{[0,1]}(u)\). Here \(0/0\) is assumed to be 0. In this general setting, [15] proved the following result.

**Proposition 1** (Theorem 5.1 in [15]). Assume that

- \(K1)\) \(K\) is a regular and Lipschitz kernel;
- \(F1)\) \((\mathcal{H}, d)\) is a separable metric space;
- \(F2)\) \(\{(X_i, Y_i)\}_{i \geq 1}\) are i.i.d. random elements with the same law as the pair \((X, Y)\) in \(\mathcal{H} \times \mathbb{R}\) fulfilling model \((1)\);
- \(F3)\) \(\mu\) is a Borel probability measure of \(\mathcal{X}\) and \(\eta \in \mathcal{L}^2(\mathcal{H}, \mu) = \{f : \mathcal{H} \to \mathbb{R} : \int_{\mathcal{H}} f^2(z) d\mu(z) < \infty\}\) is a bounded function which satisfies the Besicovitch condition:

\[ \lim_{\delta \to 0} \frac{1}{\mu(B(X, \delta))} \int_{B(X, \delta)} |\eta(z) - \eta(X)| d\mu(z) = 0, \quad (4) \]

in probability, where \(B(X, \delta)\) is the the closed ball of center \(X\) and radius \(\delta\) with respect to \(d\).

For any sequence \(h_n(x) \to 0\) such that \(\frac{n\mu(B(x, h_n(x)))}{\log n} \to \infty\), for \(x \in \text{supp} \mu\), the estimator given in \((2)\) with weights given in \((3)\) satisfies

\[ \lim_{n \to \infty} \mathbb{E} ((\hat{\eta}_n(X) - \eta(X))^2) = 0. \]
Remark 1. The Besicovitch condition in F3 is a differentiation type condition which, as is well known, in finite dimensional spaces automatically holds for any integrable function \( \eta \). Unfortunately, it is no longer true in infinite dimensional spaces and it can be proved, for instance, that it is necessary in order to get the \( L_1 \)-consistency of uniform kernel estimates (see Proposition 5.1 in [14]). However, it holds in a general setting if, for instance, the function \( \eta \) is continuous.

Remark 2. Note that for \( x \in \text{supp}(\mu) \) the consistency of this estimator holds for every sequence \( \tilde{h}_n(x) \to 0 \) such that \( \tilde{h}_n(x) \geq h_n(x) \), where \( h_n(x) \) is given in Proposition 1 since if \( \tilde{h}_n(x) \geq h_n(x) \), then \( \frac{n\mu(B(x,\tilde{h}_n(x)))}{\log n} \geq \frac{n\mu(B(x,h_n(x)))}{\log n} \to \infty \).

The existence of a sequence verifying \( \frac{n\mu(B(x,h_n(x)))}{\log n} \to \infty \) in Proposition 1 follows from the next lemma.

Lemma 1 (Lemma A.5 in [14]). For any \( x \in \text{supp}(\mu) \), there exists a sequence of positive real numbers \( h_n(x) \to 0 \) such that \( \frac{n\mu(B(x,h_n(x)))}{\log n} \to \infty \).

Let \( H_n(x) \) be the distance from \( x \) to its \( k_n \)-nearest neighbor among \( \{X_1, \ldots, X_n\} \). Recall that the \( k_n \)-nearest neighbor of \( x \) among \( \{X_1, \ldots, X_n\} \) is the sample point \( X_i \) reaching the \( k_n \)-th smallest distance to \( x \) in the sample. Then, when the bandwidth in (3) is given by \( H_n(x) \), we obtain the family of \( k_n \)-nearest neighbor (\( k \)-NN) with kernel estimates. In order to get consistency for this family of estimators, we need to prove that \( H_n(x) \to 0 \), as stated in the following lemma.

Lemma 2 (Lemma A.4 in [14]). Let \( \mathcal{H} \) be a separable metric space, \( \mu \) a Borel probability measure, and \( \{X_i\}_{i=1}^n \) a random sample of \( \mathcal{X} \). If \( x \in \text{supp}(\mu) \) and \( k_n \) is a sequence of positive real numbers such that \( k_n \to \infty \) and \( k_n/n \to 0 \), then \( H_n(x) \to 0 \).

Although the distance from \( x \) to its \( k_n \)-NN among \( \{X_1, \ldots, X_n\} \) converges to zero, to prove first the consistency of this estimator, we cannot apply directly Proposition 1 because we do not know that \( H_n(x) \) satisfies \( \frac{n\mu(B(x,H_n(x)))}{\log n} \to \infty \). However, as we will see in the next result, we can still prove the mean square consistency of this estimator under the same weak conditions as in Proposition 1 whose proof can be found in the Appendix B.

Proposition 2. Assume K1, F1–F3 hold. Let \( k_n \) be a sequence of positive real numbers such that \( k_n \to \infty \) and \( k_n/n \to 0 \). Then, the estimator given by (3) with weights given in (3) is mean square consistent for any sequence \( h_n(x) \to 0 \) such that \( h_n(x) \geq H_n(x) \), \( x \in \text{supp}(\mu) \).

3. Conditions for consistency when we do not observe the complete trajectory but only a function of them

In this section we will assume that we are not able to observe the whole trajectories \( X_i \) in \( \mathcal{H} \), but only a function of them. Different choices of that function will correspond to discretizations, eigenfunction expansions, or smoothing,
as we will see in Section 4. In this context, the weights of the estimator given in (4) cannot be computed because we do not have a distance \( d \) defined for the discretized sample curves (as a consequence, we do not have the validity of the Besicovitch condition (4) for the discretized data) or a bandwidth \( h_n \). We are interested in defining an estimator and proving its consistency in this setting.

Let us consider the following assumptions:

\( H1 \) \( (\mathcal{H}, d) \) and \( (\mathcal{H}_p, d_p) \) are separable Hilbert spaces and \( F : \mathcal{H} \to \mathcal{H}_p \) is a function such that \( F(\mathcal{X}) = \mathcal{X}_p \);

\( H2 \) \( d_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) is a pseudometric defined by \( d_p(\mathcal{X}, \mathcal{Y}) = d(\mathcal{X}_p, \mathcal{Y}_p) \) such that there exists a sequence \( c_{n, p} \to 0 \) as \( n, p \to \infty \) satisfying

\[
\frac{n^2 \mathbb{E}_X \left( \mathbb{P}^2_{\mathcal{Y}|\mathcal{X}} \left( |d(\mathcal{X}, \mathcal{Y}) - d_p(\mathcal{X}, \mathcal{Y})| \geq c_{n, p} \right) \mathcal{X} \in \text{supp}(\mu) \right) }{\sum_{i=1}^{n} K \left( \frac{d_p(\mathcal{X}_i, \mathcal{X})}{\overline{h}_{n, p}(\mathcal{X})} \right) Y_i} \to 0.
\]

For this estimator, we state the following two asymptotic results.

**Theorem 1.** Assume \( K_1, F_2, F_3, H_1 \) and \( H_2 \) hold for \( (\mathcal{H}, d) \).

(a) For \( x \in \text{supp}(\mu) \), let \( h_n^*(x) \to 0 \) be a sequence of positive real numbers such that \( \frac{c_{n, p}(x, h_n^*(x))}{\log n} \to \infty \). Then, for \( c_{n, p} \) given in \( H_2 \) and \( h_n^*(x) \to 0 \) such that there exists a sequence \( h_n(x) \to 0 \), \( h_n(x) \geq h_n^*(x) \) satisfying:

- \( H3.1 \) \( c_{n, p}/h_n(x) \to 0 \) as \( n, p \to \infty \);
- \( H3.2 \) \( c_{n, p} \leq h_{n, p}(x) - h_n(x) \leq C_2 c_{n, p} \) for \( C_2 \geq 1 \);

we have

\[
\lim_{n, p \to \infty} \mathbb{E} \left( \left( \hat{\eta}_{n, p}(\mathcal{X}) - \eta(\mathcal{X}) \right)^2 \right) = 0.
\]

(b) If in (a) we take \( h_{n, p}(x) \to 0 \) such that there exists a sequence \( h_n(x) \to 0 \), \( h_n(x) \geq H_n(x) \) satisfying assumptions \( H3.1 \) and \( H3.2 \) we also have (7) if \( k_n \to \infty \) and \( k_n/n \to 0 \).

**Remark 3.** Observe that the sequence \( h_n^*(x) \) in Theorem 4 always exists by Lemma 4. In addition, under \( H2 \), it is always possible to choose a sequence \( h_{n, p}(x) \to 0 \) fulfilling the conditions in Theorem 4. Indeed, taking \( h_n(x) = \max\{h_n^*(x), \sqrt{c_{n, p}}\} \) and \( h_{n, p}(x) = h_n(x) + C c_{n, p} \), with \( C \geq 1 \), we have that \( h_n(x) \to 0 \), \( h_{n, p}(x) \to 0 \), \( h_n(x) \geq h_n^*(x) \), and \( H3.1 \) and \( H3.2 \) hold. The same happens if instead of taking \( h_n^*(x) \) we take \( H_n(x) \).
Theorem 2. Under the assumptions of Theorem 1, let \( \gamma_n \to \infty \) as \( n \to \infty \) be such that, as \( n, p \to \infty \),

(a) \( \mathbb{E}_X \left( \gamma_n \left( \frac{c_n}{\pi_n(x)} \right)^2 \right) \to 0 \); 

(b) \( \gamma_n n^2 \mathbb{E}_X \left( \mathbb{E}_{Y|X} \left( |d(X, Y) - d_p(X, Y)| \geq c_{n,p} \left| X \in \text{supp}(\mu) \right| \right) \right) \to 0 \).

Then

\[
\lim_{n \to \infty} \mathbb{E} \left( \gamma_n (\hat{\eta}_n(X) - \eta(X))^2 \right) = 0,
\]

implies

\[
\lim_{n,p \to \infty} \mathbb{E} \left( \gamma_n (\hat{\eta}_{n,p}(X) - \eta(X))^2 \right) = 0.
\]

4. Particular cases

In this section we provide definitions of \( \mathcal{H}_p \) and \( d_p \) for discretization, smoothing, and eigenfunction expansions, which satisfy conditions \( H1 \) and \( H2 \). Then, for any sequence \( h_{n,p}(x) \to 0 \) satisfying \( (H3.1) \) and \( (H3.2) \) in Theorem 1, we get the consistency of \( \hat{\eta}_{n,p} \) as a consequence of the consistency results for \( \hat{\eta}_n \) in the full model.

Consider the case where the elements of the dataset are curves in \( L^2([0, 1]) \) that are only observed at a discrete set of points in the interval \([0, 1]\). More precisely, let us assume that \( \{X_i\}_{i=1}^n \) are observed only at some points: \( (X_i(t_1), \ldots, X_i(t_{p+1})) \) where \( 0 = t_1 < t_2 \leq \ldots < t_{p+1} = 1 \), which for simplicity we will assume are equally spaced, i.e., \( \Delta t = t_{i+1} - t_i = 1/p \). In this case, we will need to require the trajectories to satisfy some regularity condition. More precisely, we will assume that \( \mathcal{X} \) is a random element of \( \mathcal{H} \cong H^1([0, 1]) \), the Sobolev space defined as

\[
H^1([0, 1]) = \{ f : [0, 1] \to \mathbb{R} : f \text{ and } Df \in L^2([0, 1]) \},
\]

where \( Df \) is the weak derivative of \( f \), i.e., \( Df \) is a function in \( L^2([0, 1]) \) which satisfies

\[
\int_0^1 f(t)D\phi(t) \, dt = -\int_0^1 Df(t)\phi(t) \, dt, \quad \forall \phi \in C_0^\infty.
\]

In this space, the norm is defined by

\[
\|f\|_{H^1([0,1])} = \|f\|_{L^2([0,1])} + \|Df\|_{L^2([0,1])}.
\]

In this setting, we will prove consistency for the pseudometrics \( d_p \) given below.
4.1. Discretization

Consider the pseudometric

\[ d_p(X, Y) = d(X^p, Y^p) = \left( \frac{1}{p} \sum_{j=1}^{p} (X(t_j) - Y(t_j))^2 \right)^{1/2}, \]

where \( X^p(t) = F(X)(t) = \sum_{j=1}^{p} \phi_j(t)X(t_j) \) with \( \phi_j(t) = \frac{K(t-t_j)/h}{\sum_{j=1}^{K} K((t-t_j)/h)} \) and \( K \) is a regular kernel supported in \([0, 1]\). In this case, consistency will hold for any sequence \( c_{n,p} \to 0 \) as \( n, p \to \infty \) such that \( n^2P_{X,Y}(\|X\|_H + \|Y\|_H \geq pc_{n,p}) \to 0 \).

4.2. Kernel Smoothing

Let us consider now the pseudometric

\[ d_p(X, Y) = d(X^p, Y^p) = \left( \int_0^1 |X^p(t) - Y^p(t)|^2 \, dt \right)^{1/2}, \]

where \( X^p(t) = F(X)(t) = \sum_{j=1}^{p} \phi_j(t)X(t_j) \) with \( \phi_j(t) = \frac{K(t-t_j)/h}{\sum_{j=1}^{K} K((t-t_j)/h)} \) and \( K \) is a regular kernel supported in \([0, 1]\). In this case, consistency will be true for any sequence \( c_{n,p} \to 0 \) as \( n, p \to \infty \) satisfying \( n^2P_{X,Y}(\|X\|_H + \|Y\|_H \geq pc_{n,p}) \to 0 \).

Let us note that if \( E_{X}(\|X\|_{H}^{2}) < \infty \), the consistency for the cases given in Sections 4.1 and 4.2 will hold for any sequence \( c_{n,p} \) such that \( \frac{n}{pc_{n,p}} \to 0 \).

4.3. Eigenfunction expansions

Let \( X, Y \) be i.i.d. random elements on \( H = L^{2}[0, 1] \). Let \( v_1, v_2, \ldots \) be the orthonormal eigenfunctions of the covariance operator \( E_{X}(X(t)X(s)) \) (without loss of generality we have assumed that \( E(X(t)) = 0 \)) associated with the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \) such that

\[ E_{X}(X(t)X(s)) = \sum_{k=1}^{\infty} \lambda_k v_k(t)v_k(s). \]

If \( E(\int X^2(s) \, ds) < \infty \) is finite, using the Karhunen–Loève representation, we can write \( X \) as

\[ X(t) = \sum_{k=1}^{\infty} \left( \int X(s)v_k(s) \, ds \right) v_k(t) = \sum_{k=1}^{\infty} \xi_k v_k(t), \quad (8) \]

with \( E(\xi_k) = 0, E(\xi_k\xi_j) = 0 \) (i.e., \( \xi_1, \xi_2, \ldots \) uncorrelated) and \( \text{var}(\xi_k) = E(\xi_k^2) = \lambda_k = E \left( (\int X(s)v_k(s) \, ds)^2 \right) \). The classical \( L^2 \)-norm in \( H \) can be written as

\[ d(X, Y) = \sqrt{\sum_{k=1}^{\infty} \left( \int (X(t) - Y(t))v_k(t) \, dt \right)^2}. \quad (9) \]
If we consider the truncated expansion of $X$ as given in [14],
\[ X^p(t) = \sum_{k=1}^{p} \left( \int X(s) v_k(s) \, ds \right) v_k(t), \quad (10) \]
we can define the parametrized class of seminorms from the classical $L^2$-norm given by
\[ \|X\|_p = \sqrt{\int (X^p(t))^2 \, dt} = \sqrt{\sum_{k=1}^{p} \left( \int X(t) v_k(t) \, dt \right)^2}, \]
which leads to the pseudometric
\[ d_p(X, Y) = d(X^p, Y^p) = \sqrt{\sum_{k=1}^{p} \left( \int (X(t) - Y(t)) v_k(t) \, dt \right)^2}. \quad (11) \]
In this case, the consistency will hold for any sequence $c_{n,p} \to 0$ such that
\[ \frac{n^2}{c_{n,p}^2} \sum_{k=p+1}^{\infty} \lambda_k \to 0 \text{ as } n, p \to \infty. \]

**Appendix A. Proofs of auxiliary results**

To prove the consistency of the examples given in sections 4.1 and 4.2 we need the following result.

**Proposition 3.** Let $X^p(t) = \sum_{j=1}^{p} \phi_j(t) X(t_j)$ with $\phi_j$ satisfying:

(a) for each $t \in [0, 1]$, $\sum_{j=1}^{p} \phi_j(t) = 1$;

(b) for each $t \in [0, 1]$, $\sum_{j=1}^{p} \phi_j^2(t) \leq C_3$ for some constant $C_3$;

(c) $\text{supp}(\phi_j) \subset [t_{j-m}, t_{j+m}]$ with $m$ independent of $p$.

If $c_{n,p} \to 0$ as $n, p \to \infty$ is such that $n^2 \mathbb{P}_{X,Y}(||X||_H + ||Y||_H \geq pc_{n,p}) \to 0$, $H2$ is fulfilled.

**Proof of Proposition 3.** Using the Fundamental Theorem of Calculus (FTC) (see Theorem 8.2 in [8]) for $H^1([0,1])$, we get
\[
\begin{align*}
    d^2(X^p, X) &= \int_0^1 \left| \sum_{j=1}^{p} X(t_j) \phi_j(t) - X(t) \right|^2 \, dt \\
    &= \int_0^1 \left| \sum_{j=1}^{p} (X(t_j) - X(t)) \phi_j(t) \right|^2 \, dt \quad \text{(by (11))}
\end{align*}
\]
\[
\begin{align*}
= & \int_0^1 \left| \sum_{j=1}^p \left( \int_{t_j}^t D \mathcal{X}(s) \, ds \right) \phi_j(t) \right|^2 \, dt \\
\leq & \int_0^1 \left( \sum_{j=1}^p \left( \int_{t_j}^t D \mathcal{X}(s) \, ds \right)^2 \right) \left( \int_{\text{supp}(\phi_j)} (t) \right) \left( \sum_{j=1}^p \phi_j^2(t) \right) \, dt \\
\lesssim & \int_0^1 \sum_{j=1}^p \left( \int_{t_j}^t (D \mathcal{X}(s))^2 \, ds \right) |t - t_j| \, dt \\
\lesssim & \int_0^1 \sum_{j=1}^p \left( \int_{t_j}^t (D \mathcal{X}(s))^2 \, ds \right) |t - t_j| \, dt \\
= & \sum_{i=1}^p \int_{t_i}^{t_{i+1}} \sum_{j=1}^p \left( \int_{t_j}^t (D \mathcal{X}(s))^2 \, ds \right) |t - t_j| \, dt \\
\lesssim & \sum_{i=1}^p \sum_{j=1}^p \int_{t_i}^{t_{i+1}} (D \mathcal{X}(s))^2 \, ds \\
\lesssim & \frac{m}{p^2} \sum_{i=1}^p \sum_{j=1}^p \int_{t_i}^{t_{i+1}} (D \mathcal{X}(s))^2 \, ds \\
= & \frac{m^2}{p^2} \int_0^1 \sum_{i=1}^p \mathbb{1}_{[t_{i-1}, t_i]}(s) (D \mathcal{X}(s))^2 \, ds \lesssim \frac{1}{p^2} \|\mathcal{X}\|^2_{\mathcal{H}},
\end{align*}
\]

from where we get \(d(\mathcal{X}^p, \mathcal{X}) \lesssim \frac{1}{p} \|\mathcal{X}\|_{\mathcal{H}}\). Analogously we can prove that \(d(\mathcal{Y}^p, \mathcal{Y}) \lesssim \frac{1}{p} \|\mathcal{Y}\|_{\mathcal{H}}\). By triangular inequality,

\[
\begin{align*}
&n^2 \mathbb{E}_\mathcal{X} \left( \mathbb{P}^2_{\mathcal{Y}|\mathcal{X}} \left( |d(\mathcal{X}, \mathcal{Y}) - d_p(\mathcal{X}, \mathcal{Y})| \geq c_{n,p} |\mathcal{X} \in \text{supp}(\mu) \right) \right) \\
\leq & \frac{n^2}{p^2} \mathbb{P}_{\mathcal{X}, \mathcal{Y}} \left( \|\mathcal{X}\|_{\mathcal{H}} + \|\mathcal{Y}\|_{\mathcal{H}} \geq p e_{n,p} \right),
\end{align*}
\]

and therefore, for any \(c_{n,p} \to 0\) such that \(n^2 \mathbb{P}_{\mathcal{X}, \mathcal{Y}} (\|\mathcal{X}\|_{\mathcal{H}} + \|\mathcal{Y}\|_{\mathcal{H}} \geq p e_{n,p}) \to 0\) \(H2\) is fulfilled.

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\(\square\)
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\textbf{Appendix A.0.1. Consistency for the example in Section 4.1}

Since the functions \(\phi_j(t) = \mathbb{I}_{[t_j, t_{j+1}]}(t)\) satisfy trivially conditions \(\clubsuit\)–\(\clubsuit\) of Proposition \(\clubsuit\) \(H2\) is fulfilled and therefore, for any sequence \(h_{n,p}(x) \to 0\) satisfying \((H3.1)\) and \((H3.2)\) in Theorem \(\spadesuit\) we get the consistency of \(\eta_{n,p}\).
Appendix A.0.2. Consistency for the example in Section 4.2

Observe that \( \phi_j(t) = \frac{K(|t-t_j|/h)}{\sum_{i=1}^p K(|t-t_i|/h)} \) satisfies conditions (3)–(5) in Proposition H.3.

(a) for each \( t \in [0, 1] \), \( \sum_{j=1}^p \phi_j(t) = \sum_{j=1}^p \frac{K(|t-t_j|/h)}{\sum_{i=1}^p K(|t-t_i|/h)} = 1; \)

(b) since \( K \) is nonnegative and \( \frac{K(|t-t_j|/h)}{\sum_{i=1}^p K(|t-t_i|/h)} \leq 1 \), for each \( t \in [0, 1] \), there exists \( C_3 = 1 \) such that,
\[
\sum_{j=1}^p \phi_j^2(t) = \sum_{j=1}^p \left( \frac{K(|t-t_j|/h)}{\sum_{i=1}^p K(|t-t_i|/h)} \right)^2 \leq \sum_{j=1}^p \frac{K(|t-t_j|/h)}{\sum_{i=1}^p K(|t-t_i|/h)} = 1;
\]

(c) \( \text{supp}(\phi_j) = \text{supp}(K(|t-t_j|/h)) = [t_j - h, t_j + h] \), which implies that, for \( h \leq m/p, \text{supp}(\phi_j) \subset [t_{(j-m)}, t_{(j+m)}] \).

This implies that \( H2 \) is fulfilled then, for any sequence \( h_{n,p}(x) \to 0 \) satisfying (H3.1) and (H3.2) in Theorem I, we get the consistency of \( \hat{\eta}_{n,p} \).

Appendix A.0.3. Consistency for the example in Section 4.3

Let us consider the truncated expansion of \( X, \mathcal{X}^p(t) \), given by (10) and the pseudo-metric \( d_p(X, Y) = d(\mathcal{X}^p, \mathcal{Y}^p) \) given by (11). In order to prove H2, let us consider \( c_{n,p} \) such that \( n^{-2} \sum_{k=p+1}^\infty \lambda_k \to 0 \). Using Chebyshev’s Inequality in (5) followed by Cauchy Schwartz, we get
\[
n^{-2} \mathbb{E}_{X,Y} \left( \mathbb{E}_{X|\mathcal{Y}} \left( |d(X,Y) - d_p(X,Y)| \right) \right) \geq c_{n,p} \mathbb{E}_{X|\mathcal{Y}} \left( d(X,Y) \right) \leq \frac{n^2}{c_{n,p}^2} \mathbb{E}_{X,Y} \left( (d(X,Y) - d_p(X,Y))^2 \right). \quad \text{(A.1)}
\]

Now, since \( d(X,Y) \geq d_p(X,Y) \) we have that \( 0 \leq d(X,Y) - d_p(X,Y) = d(X,Y) - d(\mathcal{X}^p, \mathcal{Y}^p) \) and, by triangular inequality \( d(X,Y) \leq d(X,\mathcal{X}^p) + d(\mathcal{X}^p, \mathcal{Y}^p) + d(\mathcal{Y}^p, Y) \) which implies that,
\[
0 \leq d(X,Y) - d_p(X,Y) \leq d(X,\mathcal{X}^p) + d(\mathcal{Y}^p, Y), \quad \text{(A.2)}
\]
and taking squares,
\[
0 \leq (d(X,Y) - d_p(X,Y))^2 \leq (d(X,\mathcal{X}^p) + d(\mathcal{Y}^p, Y))^2 \leq 2 (d^2(X,\mathcal{X}^p) + d^2(\mathcal{Y}^p, Y)).
\]

As a consequence, to proof this proposition it will sufficient to bound \( \mathbb{E}_{X} (d^2(X,\mathcal{X}^p)) \) (equivalently, \( \mathbb{E}_{Y} (d^2(\mathcal{Y}^p, Y)) \)). Since \( v_k \) are orthonormal,
\[
d^2(X,\mathcal{X}^p) = \int \left( X(s) - \sum_{k=1}^p \left( \int \mathcal{X}(t)v_k(t) \, dt \right) v_k(s) \right)^2 \, ds = \sum_{k=p+1}^\infty \left( \int \mathcal{X}(t)v_k(t) \, dt \right)^2.
\]
Then we have,

\[
\mathbb{E}_X\left( d^2(X, X^p) \right) = \mathbb{E}_X\left( \sum_{k=p+1}^{\infty} \left( \int X(t)v_k(t) \, dt \right)^2 \right) = \sum_{k=p+1}^{\infty} \lambda_k. \tag{from (8)}
\]

Analogously we can prove that \( \mathbb{E}_Y\left( d^2(Y, Y^p) \right) = \sum_{k=p+1}^{\infty} \lambda_k. \) Therefore, in (A.1) we get

\[
n^2 \mathbb{E}_X\left( \sum_{i=1}^{n} W_n(X_i) \left| d(X, X_{i}) - d_p(X, Y) \right| \geq c_{n,p} \mid X \in \text{supp}(\mu) \right) \lesssim \frac{n^2}{c_{n,p}^2} \sum_{k=p+1}^{\infty} \lambda_k \to 0.
\]

This implies that \( H2 \) is fulfilled then, for any sequence \( h_{n,p}(x) \to 0 \) satisfying \( (H3.1) \) and \( (H3.2) \) in Theorem 1, we get the consistency of \( \hat{\eta}_{n,p}. \)

Appendix B. Proof of Proposition 2 and Theorems 1 and 2

Here (and hereafter) we will use the notation \( f \lesssim g \) when there exists a constant \( C > 0 \) such that \( f \leq Cg \) and \( f \approx g \) if there exists a constant \( C > 0 \) such that \( f = Cg. \) To prove Proposition 2 we need some preliminary results whose proofs can be found in [15].

**Theorem 3** (Theorem 3.4). If \( \eta \in L^2(\mathcal{H}, \mu) \) and \( \hat{\eta}_n \) is the estimator given in (2) with weights \( W_n(X) = \{W_{n_i}(X)\}_{i=1}^{n} \) satisfying the following conditions:

(i) There is a sequence of nonnegative random variables \( a_n(X) \to 0 \) a.s. such that

\[
\lim_{n \to \infty} \mathbb{E}\left( \sum_{i=1}^{n} W_{n_i}(X) \mathbb{I}_{d(X_i, X) > a_n(X)} \right) = 0;
\]

(ii) \( \lim_{n \to \infty} \mathbb{E}\left( \max_{1 \leq i \leq n} W_{n_i}(X) \right) = 0; \)

(iii) for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \eta^* \) bounded and continuous function fulfilling \( \mathbb{E}_X((\eta(X) - \eta^*(X))^2) < \delta \) we have that

\[
\mathbb{E}\left( \sum_{i=1}^{n} W_{n_i}(X)(\eta^*(X_i) - \eta(X_i))^2 \right) < \epsilon,
\]

then \( \hat{\eta}_n \) is mean square consistent.
Corollary 1 (Corollary 3.3). Let $U_n$ be a sequence of probability weights satisfying conditions (i), (ii) and (iii) of Theorem 3. If $W_n$ is a sequence of weights such that $\sum_{i=1}^{n} W_n(x) = 1$ and, for each $n \geq 1$, $|W_n| \leq MU_n$ for some constant $M \geq 1$, then the estimator given in (3) with weights $W_n(x)$ is mean square consistent.

Lemma 3 (Lemma A.1). Let $H$ be a separable metric space. If $A = \text{supp}(\mu) = \{x \in H : \mu(B(x, \epsilon)) > 0, \forall \epsilon > 0\}$ then $\mu(A) = 1$.

Proof of Proposition 5. Let $x \in \text{supp}(\mu)$ be fixed. Let us observe that, since $K$ is regular, there exist constants $0 < c_1 < c_2 < \infty$ such that, for each $i$,

$$W_{ni}(x) = \frac{K \left( \frac{d(X_i, x)}{h_n(x)} \right)}{\sum_{j=1}^{n} K \left( \frac{d(X_j, x)}{h_n(x)} \right)} \leq \frac{c_2}{c_1} \frac{\mathbb{I}_{\{d(X_i, x) \leq h_n(x)\}}}{\sum_{j=1}^{n} \mathbb{I}_{\{d(X_j, x) \leq h_n(x)\}}} = \frac{c_2}{c_1} U_{ni}(x). \quad (B.1)$$

Let $h_n(x) \to 0$ such that $h_n(x) \geq H_n(x)$ ($H_n(x) \to 0$ by Lemma 2) for $x \in \text{supp}(\mu)$. From (B.1) and Corollary 1 it suffices to prove that the weights $U_{ni}$ satisfy conditions (i), (ii) and (iii) of Theorem 3. To prove (i) let us take $a_n(x) = h_n^{1/2}(x) \to 0$. Then, by Lemma 3

$$\mathbb{E} \left( \sum_{i=1}^{n} U_{ni}(X_i) \mathbb{I}_{\{d(X_i, x) > h_n(x)^{1/2}\}} \right) = \mathbb{E}_X \left( \mathbb{E}_{D_n|X} \left( \mathbb{I}_{\{X \in \text{supp}(\mu)\}} \sum_{i=1}^{n} U_{ni}(X_i) \mathbb{I}_{\{d(X_i, x) > h_n(x)^{1/2}\}} \mid X \in \text{supp}(\mu) \right) \right).$$

Given $\epsilon > 0$, let $x \in \text{supp}(\mu)$ be fixed. Since $h_n(x) \to 0$, there exists $N_1 = N_1(x)$ such that if $n \geq N_1$, $\mathbb{I}_{\{h_n(x)^{1/2} < d(x, x) \leq h_n(x)\}} = 0$ for all $i$ and consequently,

$$\mathbb{E}_{D_n} \left( \sum_{j=1}^{n} \mathbb{I}_{\{d(x, x) \leq h_n(x)\}} \sum_{i=1}^{n} \mathbb{I}_{\{h_n(x)^{1/2} < d(x, x) \leq h_n(x)\}} \right) < \epsilon.$$ 

In addition, $\sum_{j=1}^{n} \mathbb{I}_{\{h_n(x)^{1/2} < d(x, x) \leq h_n(x)\}} \leq 1$ from what follows that,

$$\mathbb{E}_{D_n} \left( \frac{1}{\sum_{j=1}^{n} \mathbb{I}_{\{d(x, x) \leq h_n(x)\}}} \sum_{i=1}^{n} \mathbb{I}_{\{h_n(x)^{1/2} < d(x, x) \leq h_n(x)\}} \right) \leq 1.$$ 

Therefore, by the dominated convergence theorem we have that condition (i) is satisfied. Now, since $h_n(x) \geq H_n(x)$,

$$\sum_{j=1}^{n} \mathbb{I}_{\{d(X_j, x) \leq h_n(x)\}} \geq \sum_{j=1}^{n} \mathbb{I}_{\{d(X_j, x) \leq H_n(x)\}} = k_n \to \infty.$$ 

Therefore,

$$\max_{1 \leq i \leq n} U_{ni}(x) \leq \max_{1 \leq i \leq n} \frac{1}{\sum_{j=1}^{n} \mathbb{I}_{\{d(X_j, x) \leq h_n(x)\}}} \leq \frac{1}{k_n} \to 0,$$
from what we derive (ii) using the dominated convergence theorem. It remains to verify that condition (iii) holds. Since \( \eta \in L^2(\mathcal{H}, \mu) \) which is separable and complete, there exists \( \eta^* \) continuous and bounded such that, for all \( \delta > 0 \),
\[
E_X((\eta(X) - \eta^*(X))^2) < \delta.
\]
Then,
\[
\mathbb{E}_n \left( \sum_{i=1}^n U_{ni}(X)(\eta^*(X_i) - \eta(X_i))^2 \right)
\]

\[
= \mathbb{E}_X \left( \mathbb{E}_{\mathcal{D}_n \mid X} \left( \sum_{i=1}^n U_{ni}(X)(\eta^*(X_i) - \eta(X_i))^2 \mid X \in \text{supp} \, (\mu) \right) \right).
\]

Let \( x \in \text{supp} \,(\mu) \) be fixed. From [13], Lemma A.7, for any nonnegative bounded measurable function \( f \), we have
\[
\mathbb{E}_{\mathcal{D}_n} \left( \sum_{i=1}^n U_{ni}(x)f(X_i) \right) \leq 12 \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} f(y) \, d\mu(y).
\]
Then, applying the inequality to \( f(X_i) = (\eta^*(X_i) - \eta(X_i))^2 \) we get
\[
\mathbb{E}_{\mathcal{D}_n} \left( \sum_{i=1}^n U_{ni}(x)(\eta^*(X_i) - \eta(X_i))^2 \right)
\]

\[
\leq \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} (\eta^*(y) - \eta(y))^2 \, d\mu(y)
\]

\[
\leq \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} (\eta^*(y) - \eta^*(x))^2 \, d\mu(y)
\]

\[
+ \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} (\eta^*(x) - \eta(x))^2 \, d\mu(y)
\]

\[
+ \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} (\eta(x) - \eta(y))^2 \, d\mu(y)
\]

\[
= f_{1,n}(x) + f_{2,n}(x) + f_{3,n}(x).
\]

This part will be complete if we show that the expectation with respect to \( \mathcal{X} \) of these three functions converges to zero. For this, let \( \epsilon > 0 \) and \( \delta \leq \epsilon \). Since \( \eta^* \) is continuous, there exists \( r = r(x, \epsilon) > 0 \) such that if \( d(x, y) < r \) then \( |\eta^*(x) - \eta^*(y)| < \epsilon \). On the other hand, since \( h_n(x) \to 0 \), for that \( r(x, \epsilon) > 0 \), there exists \( N_2 = N_2(x, r(x, \epsilon)) \) such that if \( n \geq N_2 \), \( h_n(x) < r \). Then, \( f_{1,n}(x) = \frac{1}{\mu(B(x, h_n(x)))} \int_{B(x, h_n(x))} (\eta^*(y) - \eta^*(x))^2 \, d\mu(y) < \epsilon \) for \( n \geq N_2 \) and in addition it is bounded so, by the dominated convergence theorem we have that
\[
E_X(f_{1,n}(\mathcal{X})) \to 0.
\]

For the second term, since \( \delta \leq \epsilon \), we have that
\[
E_X(f_{2,n}(\mathcal{X})) = E_X((\eta(\mathcal{X}) - \eta^*(\mathcal{X}))^2) < \epsilon.
\]
Finally, since $\eta$ is bounded,
\[
\mathbb{E}_X(f_{3,n}(X)) \leq \mathbb{E}_X\left( \frac{1}{\mu(B(X,h_n(X)))} \int_{B(X,h_n(X))} |\eta(X) - \eta(y)| \, d\mu(y) \right),
\]
which converge to zero if the bounded random variables
\[
\frac{1}{\mu(B(X,h_n(X)))} \int_{B(X,h_n(X))} |\eta(X) - \eta(y)| \, d\mu(y)
\]
converge to zero in probability. To see this, let $\lambda > 0$ be fixed. For every $\delta_0 > 0$,
\[
\mathbb{P}_X\left( \frac{1}{\mu(B(X,h_n(X)))} \int_{B(X,h_n(X))} |\eta(X) - \eta(y)| \, d\mu(y) > \lambda \right)
\]
\[
\leq \mathbb{P}_X\left( h_n(X) > \delta_0 \right) + \sup_{\delta \leq \delta_0} \mathbb{P}_X\left( \frac{1}{\mu(B(X,\delta))} \int_{B(X,\delta)} |\eta(X) - \eta(y)| \, d\mu(y) > \lambda \right).
\]
Since $h_n(X) \to 0$ a.s. the first term converges to zero while the second term does thanks to the truth of the Besicovitch condition (1).

Proof of Theorem 1.

Proof of (a): Let us define $D_n = \{X_1, \ldots, X_n\}$ and $C_n = \{Y_1, \ldots, Y_n\}$. In order to prove the mean square consistency, we consider
\[
\mathbb{E}_X(\hat{\eta}_{n,p}(X) - \eta(X))^2) = \mathbb{E}_X(\mathbb{E}_{D_n,C_n|X}(\hat{\eta}_{n,p}(X) - \eta(X))^2|X))
\]
Let $x \in \text{supp } (\mu)$ be fixed. To simplify the notation, we set $\mathbb{E}(\cdot) = \mathbb{E}_{D_n,C_n|X}(\cdot)$. Then, for a particular $h_n(x) \geq h_n^*(x)$ to be defined later, let us define the theoretical quantities
\[
K\left( \frac{d(x,X_i)}{h_n(x)} \right) \doteq K_i(x) \doteq K_i \quad \text{and} \quad K\left( \frac{d_p(x,X_i)}{h_n^*(x)} \right) \doteq K_{i,p}(x) \doteq K_{i,p},
\]
and as in (9),
\[
\frac{K_i}{\sum_{j=1}^n K_j} \doteq W_i \quad \text{and} \quad \frac{K_{i,p}}{\sum_{j=1}^n K_{j,p}} \doteq W_{i,p}.
\]
Let us consider the following auxiliary unobservable quantities
\[
\hat{\eta}_n(x) = \sum_{i=1}^n W_i Y_i, \quad \eta_n(x) = \sum_{i=1}^n W_i \eta(X_i), \quad \text{and} \quad \eta_{n,p}(x) = \sum_{i=1}^n W_{i,p} \eta(X_i).
\]
Then we have,
\[
\hat{\eta}_{n,p}(x) - \eta(x) = [\hat{\eta}_{n,p}(x) - \eta_{n,p}(x)] + [\eta_{n,p}(x) - \eta_n(x)] + [\eta_n(x) - \hat{\eta}_n(x)] + [\hat{\eta}_n(x) - \eta(x)]
\]
\[ \sum_{i=1}^{n} W_{i,p} (Y_i - \eta(\mathcal{X}_i)) + \sum_{i=1}^{n} (W_{i,p} - W_i) \eta(\mathcal{X}_i) + \sum_{i=1}^{n} W_i (\eta(\mathcal{X}_i) - Y_i) + [\tilde{\eta}_n(x) - \eta(x)] \]

Taking squares and expectation in \( D_n, C_n \) we have

\[ \mathbb{E} \left( (\tilde{\eta}_{n,p}(x) - \eta(x))^2 \right) \lesssim \mathbb{E} \left( \left( \sum_{i=1}^{n} (W_{i,p} - W_i)(Y_i - \eta(\mathcal{X}_i)) \right)^2 \right) + \mathbb{E} \left( \left( \sum_{i=1}^{n} (W_{i,p} - W_i) \eta(\mathcal{X}_i) \right)^2 \right) + \mathbb{E} \left( ([\tilde{\eta}_n(x) - \eta(x)])^2 \right) \]

\[ \doteq I + II + III. \]

By Proposition 11 and Remark 2 (since \( h_n(x) \to 0 \) and \( h_n(x) \geq h^n_n(x) \)), taking expectation on \( \mathcal{X} \) we have that term III converges to zero. For the first term we have,

\[ I \approx \mathbb{E} \left( \left( \sum_{i=1}^{n} (W_{i,p} - W_i)(Y_i - \eta(\mathcal{X}_i)) \right)^2 \right) \]

\[ = \mathbb{E} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (W_{i,p} - W_i)(W_{j,p} - W_j)e_i e_j \right) \quad (Y_i - \eta(\mathcal{X}_i) = e_i) \]

\[ = \mathbb{E} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (W_{i,p} - W_i)(W_{j,p} - W_j) \mathbb{E}_{C_n | D_n} (e_i e_j | D_n) \right) \]

\[ = \mathbb{E} \left( \sum_{i=1}^{n} |W_{i,p} - W_i|^2 \mathbb{E}_{C_n | D_n} (e_i^2 | D_n) \right) \quad \text{(cond. ind.)} \]

\[ = \sigma^2 \mathbb{E} \left( \sum_{i=1}^{n} |W_{i,p} - W_i|^2 \right). \]

On the another hand, since \( \eta \) is bounded, in II we have

\[ II = \mathbb{E} \left( \left( \sum_{i=1}^{n} (W_{i,p} - W_i) \eta(\mathcal{X}_i) \right)^2 \right) \lesssim \mathbb{E} \left( \left( \sum_{i=1}^{n} |W_{i,p} - W_i| \right)^2 \right). \]
We will see that terms I and II converge to zero splitting the sum in different pieces.

(1) Let \( A_1 \doteq \{ i : d_p(x, \mathcal{X}_i) > h_{n,p}(x) \} \). Observe that in this case \( W_{i,p} = 0 \) since \( K \) is supported in \([0, 1]\). In addition, if we also consider the set \( \{ i : d(x, \mathcal{X}_i) > h_n(x) \} \), for the same reason we would have \( W_i = 0 \). Therefore, \( A_1 \equiv \{ i : d(x, \mathcal{X}_i) \leq h_n(x), d_p(x, \mathcal{X}_i) > h_{n,p}(x) \} \) and since \( |W_i| \leq 1 \) we get

\[
I_{A_1} = E \left( \sum_{i=1}^{n} |W_i|^2 \mathbb{I}_{\{i \in A_1\}} \right) \leq E \left( \sum_{i=1}^{n} \mathbb{I}_{\{i \in A_1\}} \right),
\]

and,

\[
II_{A_1} = E \left( \left( \sum_{i=1}^{n} |W_i| \mathbb{I}_{\{i \in A_1\}} \right)^2 \right) \leq E \left( \sum_{i=1}^{n} \mathbb{I}_{\{i \in A_1\}} \right)^2 = C_{A_1}. \quad (B.2)
\]

Observe that the i.i.d. random variables \( \mathbb{I}_{\{i \in A_1\}} \) have a Bernoulli distribution with parameter

\[
p_1 = P_{X_1}(d_p(x, \mathcal{X}_1) > h_{n,p}(x), d(x, \mathcal{X}_1) \leq h_n(x))
\]

\[
\leq P_{X_1}(d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1) \geq h_{n,p}(x) - h_n(x))
\]

\[
\leq P_{X_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}). \quad \text{(by H3.2)}
\]

As a consequence, the random variable \( Z = \sum_{i=1}^{n} \mathbb{I}_{\{i \in A_1\}} \) has Binomial distribution with parameters \( n_1 \leq n \) and \( p_1 \). Therefore, since \( E(Z) = n_1 p_1 \leq n p_1 \) we have

\[
I_{A_1} \leq E(Z) \leq nP_{X_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}), \quad (B.3)
\]

and, since \( E(Z^2) = n_1 p_1 (1 - p_1) + n_1^2 p_1^2 \leq n p_1 + (n_1 p_1)^2 \),

\[
II_{A_1} \leq C_{A_1} \leq E(Z^2) \leq nP_{X_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p})
\]

\[
+ (nP_{X_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}))^2. \quad (B.4)
\]

(2) Let \( A_2 \doteq \{ i : d_p(x, \mathcal{X}_i) \leq h_{n,p}(x) \} \).

(2.1) Let \( A_{21} \doteq A_2 \cap \{ i : d(x, \mathcal{X}_i) > 3h_n(x) \} \). Observe that in this case \( W_i = 0 \) since \( K \) is supported in \([0, 1]\). Then, since \( \forall i, |W_{i,p}| \leq 1 \) we get

\[
I_{A_{21}} = E \left( \sum_{i=1}^{n} |W_{i,p}|^2 \mathbb{I}_{\{i \in A_{21}\}} \right) \leq E \left( \sum_{i=1}^{n} \mathbb{I}_{\{i \in A_{21}\}} \right),
\]

and

\[
II_{A_{21}} = E \left( \left( \sum_{i=1}^{n} |W_{i,p}| \mathbb{I}_{\{i \in A_{21}\}} \right)^2 \right) \leq E \left( \sum_{i=1}^{n} \mathbb{I}_{\{i \in A_{21}\}} \right)^2. \quad (B.5)
\]
Now, the i.i.d. random variables $\mathbb{I}_{\{i \in A_{21}\}}$ have Bernoulli distribution with parameter

$$p_2 = P_{\mathcal{X}_1}(d_p(x, \mathcal{X}_1) \leq h_{n,p}(x), d(x, \mathcal{X}_1) > 3h_n(x))$$

$$\leq P_{\mathcal{X}_1}(d(x, \mathcal{X}_1) - d_p(x, \mathcal{X}_1) \geq 3h_n(x) - h_{n,p}(x)).$$

As a consequence, the random variable $Z = \sum_{i=1}^n \mathbb{I}_{\{i \in A_{21}\}}$ has Binomial distribution with parameters $n_2 \leq n$ and $p_2$. But from (H3.1), for $n$ large enough (that can depend on $x$), $h_n(x) \geq \left(\frac{1 + C_{22}}{2}\right)c_{n,p}$ which, together with H3.2 implies that

$$3h_n(x) - h_{n,p}(x) \geq 2h_n(x) - C_2c_{n,p} \geq c_{n,p},$$

and then, for $n$ large enough (that can depend on $x$),

$$p_2 \leq P_{\mathcal{X}_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}).$$

Therefore, since $E(Z) = n_2p_2 \leq np_2$ we have

$$I_{A_{21}} \lesssim E(Z) \leq np_{\mathcal{X}_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}), \quad (B.6)$$

and since $E(Z^2) = n_2p_2(1 - p_2) + n_2^2p_2^2 \leq np_2 + (np_2)^2$,

$$II_{A_{21}} \lesssim E(Z^2) \leq np_{\mathcal{X}_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p})) + (np_{\mathcal{X}_1}(|d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| \geq c_{n,p}))^2. \quad (B.7)$$

(2.2) Let $A_{22} = A_2 \cap \{i : d(x, \mathcal{X}_i) \leq 3h_n(x)\}$. In this case we write,

$$W_{i,p} - W_i = \frac{K_{i,p}}{\sum_{j=1}^n K_{j,p}} - \frac{K_i}{\sum_{j=1}^n K_j}$$

$$= \frac{K_{i,p}}{\sum_{j=1}^n K_{j,p}} - \frac{K_i}{\sum_{j=1}^n K_j} + \frac{K_i}{\sum_{j=1}^n K_j} - \frac{K_i}{\sum_{j=1}^n K_j}$$

$$= (K_{i,p} - K_i) \frac{1}{\sum_{j=1}^n K_{j,p}} + K_i \frac{\sum_{j=1}^n (K_{j,p} - K_{j,p})}{\sum_{j=1}^n K_{j,p}}$$

$$= (K_{i,p} - K_i) \frac{1}{\sum_{j=1}^n K_{j,p}} + W_i \frac{\sum_{j=1}^n (K_{j,p} - K_{j,p})}{\sum_{j=1}^n K_{j,p}}.$$

Then,

$$I_{A_{22}} \leq E\left(\sum_{i=1}^n |W_{i,p} - W_i|^2 \mathbb{I}_{\{i \in A_{22}\}}\right)$$

$$\leq E\left(\sum_{i=1}^n |K_{i,p} - K_i|^2 \mathbb{I}_{\{i \in A_{22}\}}\left(\frac{1}{\sum_{j=1}^n K_{j,p}}\right)^2\right).$$

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and,

\[ II_{A_{22}} = \mathbb{E} \left( \sum_{i=1}^{n} |W_{i,p} - W_{i}\mathbb{1}_{\{i \in A_{22}\}}|^2 \right) \]

\[ \lesssim \mathbb{E} \left( \sum_{i=1}^{n} |K_{i,p} - K_i| \mathbb{1}_{\{i \in A_{22}\}}^2 \sum_{j=1}^{n} \sum_{j=1}^{n} (K_j - K_{j,p})^2 \right) \]

\[ \lesssim \mathbb{E} \left( \sum_{i=1}^{n} |K_{i,p} - K_i| \mathbb{1}_{\{i \in A_{22}\}}^2 \sum_{j=1}^{n} \sum_{j=1}^{n} (K_j - K_{j,p})^2 \right) \]

\[ \lesssim \mathbb{E} \left( \sum_{i=1}^{n} |K_{i,p} - K_i| \mathbb{1}_{\{i \in A_{22}\}}^2 \sum_{j=1}^{n} \sum_{j=1}^{n} (K_j - K_{j,p})^2 \right) \]

\[ \lesssim \mathbb{E} \left( \sum_{i=1}^{n} |K_{i,p} - K_i| \mathbb{1}_{\{i \in A_{22}\}}^2 \sum_{j=1}^{n} \sum_{j=1}^{n} (K_j - K_{j,p})^2 \right) \]

Observe that if \( \sum_{i=1}^{n} \mathbb{1}_{\{i \in A_{22}\}} = 0 \) then \( \forall i, \mathbb{1}_{\{i \in A_{22}\}} = 0 \) so in this case, \( I_{A_{22}} \) and \( II_{A_{22}} \) are zero. Then, in what follows we will assume that \( \sum_{i=1}^{n} \mathbb{1}_{\{i \in A_{22}\}} \neq 0 \). Since \( K \) is Lipschitz and we are only considering the indexes \( i \) such that \( d_p(x, \mathcal{X}_i) \leq h_{n,p}(x) \) we get,

\[ |K_{i,p} - K_i| = \left| K \left( \frac{d_p(x, \mathcal{X}_i)}{h_{n,p}(x)} \right) - K \left( \frac{d(x, \mathcal{X}_i)}{h_n(x)} \right) \right| \]

\[ \leq \left| \frac{d_p(x, \mathcal{X}_i)}{h_{n,p}(x)} - \frac{d(x, \mathcal{X}_i)}{h_n(x)} \right| \]

\[ = \left| \frac{d_p(x, \mathcal{X}_i)h_n(x) - d(x, \mathcal{X}_i)h_{n,p}(x)}{h_{n,p}(x)h_n(x)} \right| \]

\[ \leq \frac{|d_p(x, \mathcal{X}_i) - d(x, \mathcal{X}_i)|}{h_n(x)} + \frac{d_p(x, \mathcal{X}_i)|h_n(x) - h_{n,p}(x)|}{h_n(x)h_{n,p}(x)} \]
\[
\frac{|d_p(x, X_i) - d(x, X_i)|}{h_n(x)} + \frac{c_{n,p}}{h_n(x)} \leq (\text{by H3.2})
\]

Therefore,
\[
I_{A_{22}}^1 \leq \frac{1}{h_n^2(x)} E \left( \sum_{i=1}^{n} \frac{|d_p(x, X_i) - d(x, X_i)|^2}{(\sum_{j=1}^{n} I_{\{j \in A_{22}\}})^2} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \tag{B.10}
\]

and,
\[
II_{A_{22}}^1 \leq \frac{1}{h_n^2(x)} E \left( \sum_{i=1}^{n} \frac{|d_p(x, X_i) - d(x, X_i)| \cdot I_{\{i \in A_{22}\}}}{(\sum_{j=1}^{n} I_{\{j \in A_{22}\}})^2} \right) \tag{B.11}
\]

\[
\leq \frac{1}{h_n^2(x)} E \left( \sum_{i=1}^{n} \frac{|d_p(x, X_i) - d(x, X_i)| \cdot I_{\{i \in A_{22}\}}}{(\sum_{j=1}^{n} I_{\{j \in A_{22}\}})^2} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2.
\]

\[\text{2.2.1 Let } A_{221} \doteq A_{22} \cap \{i : |d_p(x, X_i) - d(x, X_i)| \leq c_{n,p}\}. \text{ In this case, by (H3.1) we get}
\]
\[
I_{A_{221}}^1 \doteq \frac{c_{n,p}^2}{h_n^2(x)} \left( \sum_{i=1}^{n} I_{\{i \in A_{22}\}} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \tag{B.12}
\]

\[
\leq \left( \frac{c_{n,p}}{h_n(x)} \right)^2,
\]

and
\[
II_{A_{221}}^1 \doteq \frac{c_{n,p}^2}{h_n^2(x)} E \left( \sum_{i=1}^{n} I_{\{i \in A_{22}\}} \right) \tag{B.13}
\]

\[
\leq \left( \frac{c_{n,p}}{h_n(x)} \right)^2.
\]
(2.2.2) Let $A_{222} = A_{22} \cap \{ i : |d_p(x, X_i) - d(x, X_i)| > c_{n,p} \}$. Let us define the i.i.d. random variables $Z_i = d_p(x, X_i) - d(x, X_i)$, $i = 1, \ldots, n$. Since $d_p(x, X_i) \leq h_{n,p}(x)$ and $d(x, X_i) \leq 3h_n(x)$ we have that $|Z_i| \leq h_{n,p}(x) + 3h_n(x)$. Observe that, from (H3.2) and (H3.1), respectively, for $n$ large enough (that can depend on $x$) we have

$$h_{n,p} \leq h_n(x) + C_2 c_{n,p} \leq C h_n(x).$$

Which implies that, for $n$ large enough, $|Z_i| \leq C h_n(x)$. Therefore,

$$I_{A_{222}} = \frac{1}{h_n^2(x)} \mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_i| \leq C h_n(x)\}} \right)$$

$$\leq \frac{1}{h_n^2(x)} \mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_i| \leq C h_n(x)\}} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \quad (#A_{222} \leq n)$$

$$\lesssim \frac{n}{h_n(x)} \mathbb{E} \left( |Z_1|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq C h_n(x)\}} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \quad (|Z_1| \lesssim h_n(x))$$

On the other hand,

$$II_{A_{222}} = \frac{1}{h_n^2(x)} \mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_i| \leq C h_n(x)\}} \right)$$

$$\leq \frac{1}{h_n^2(x)} \mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_i| \leq C h_n(x)\}} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \quad (#A_{222} \leq n)$$

$$\lesssim \frac{n}{h_n(x)} \mathbb{E} \left( |Z_1|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq C h_n(x)\}} \right) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \quad (|Z_1| \lesssim h_n(x))$$

Observe that, for $i \neq j$, $Z_i$ is independent of $Z_j$ then,

$$\mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_i| \leq C h_n(x)\}} \right)^2$$

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and, with the same inequality in (B.16),

\[
\begin{align*}
E & \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |Z_i||Z_j| \mathbb{1}_{\{i:c_{n,p} \leq |Z_i| \leq Ch_n(x)\}} \mathbb{1}_{\{j:c_{n,p} \leq |Z_j| \leq Ch_n(x)\}} \right) \\
= & \mathbb{E} \left( \sum_{i=1}^{n} |Z_i|^2 \mathbb{1}_{\{i:c_{n,p} \leq |Z_i| \leq Ch_n(x)\}} \right) \\
+ & \mathbb{E} \left( \sum_{i=1}^{n} \sum_{j=1 \neq j}^{n} |Z_i||Z_j| \mathbb{1}_{\{i:c_{n,p} \leq |Z_i| \leq Ch_n(x)\}} \mathbb{1}_{\{j:c_{n,p} \leq |Z_j| \leq Ch_n(x)\}} \right) \\
\leq & n \mathbb{E} \left( |Z_1|^2 \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \\
+ & n^2 \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \\
\leq & nh_n(x) \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \\
+ & n^2 \left( \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \right)^2.
\end{align*}
\]

Using this bound in (B.15) we get,

\[
II_{A_{222}} \leq \frac{n}{h_n(x)} \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \\
+ \frac{n^2}{h_n^2(x)} \left( \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \right)^2 + \left( \frac{c_{n,p}}{h_n(x)} \right)^2.
\]  

We need to compute the expectation \( \mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) \) which is,

\[
\begin{align*}
\mathbb{E} \left( |Z_1| \mathbb{1}_{\{c_{n,p} \leq |Z_1| \leq Ch_n(x)\}} \right) &= \int_{c_{n,p}}^{h_n(x)} \mathbb{P}(|Z_1| > t) \, dt \\
\leq & \mathbb{P}(|Z_1| > c_{n,p}) \int_{c_{n,p}}^{h_n(x)} dt \\
\leq & \mathbb{P}(|Z_1| > c_{n,p}) h_n(x).
\end{align*}
\]

Therefore, with this inequality in (B.13) we have

\[
I_{A_{222}} \leq n \mathbb{P}(|Z_1| > c_{n,p}) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2
\]  

\[
= n \mathbb{P}(|d_p(x, X_1) - d(x, X_1)| > c_{n,p}) + \left( \frac{c_{n,p}}{h_n(x)} \right)^2,
\]

and, with the same inequality in (B.16),

\[
II_{A_{222}} \leq n \mathbb{P}(|Z_1| > c_{n,p}) + \left( n \mathbb{P}(|Z_1| > c_{n,p}) \right)^2 + \left( \frac{c_{n,p}}{h_n(x)} \right)^2
\]

\[
= n \mathbb{P}(|d_p(x, X_1) - d(x, X_1)| > c_{n,p}) + \left( n \mathbb{P}(|d_p(x, X_1) - d(x, X_1)| > c_{n,p}) \right)^2 + \left( \frac{c_{n,p}}{h_n(x)} \right)^2.
\]
Then, with (B.12) and (B.17) in (B.10) we get
\[ I_{A_{22}}^{1} \lesssim \left( \frac{c_{n,p}}{h_{n}(x)} \right)^{2} + n\mathbb{P}(|d_{p}(x, X_{1}) - d(x, X_{1})| > c_{n,p}). \] (B.19)
and, with (B.13) and (B.18) in (B.11),
\[ II_{A_{22}}^{1} \lesssim \left( \frac{c_{n,p}}{h_{n}(x)} \right)^{2} + n\mathbb{P}(|d_{p}(x, X_{1}) - d(x, X_{1})| > c_{n,p}) \]
(B.20)
\[ + \left( n\mathbb{P}(|d_{p}(x, X_{1}) - d(x, X_{1})| > c_{n,p}) \right)^{2}. \]

On the other hand, observe that \( I_{A_{22}}^{2} = \mathbb{E}\left( \left( \frac{\sum_{j=1}^{n} |K_{j} - K_{j,p}|}{\sum_{j=1}^{n} K_{j,p}} \right)^{2}\right) \). Since \( A_{22}^{1} = \{ j : d(x, X_{j}) > 3h_{n}(x) \} \cup \{ j : d_{p}(x, X_{j}) > h_{n,p}(x) \} \) we can write,
\[ \sum_{j=1}^{n} |K_{j} - K_{j,p}| \leq \sum_{j=1}^{n} K_{j} - K_{j,p} \sum_{j=1}^{n} \frac{\mathbb{1}(j \in A_{22})}{K_{j,p}} \]
\[ + \sum_{j=1}^{n} |K_{j} - K_{j,p}| \mathbb{1}(j : d(x, X_{j}) > 3h_{n}(x)) \]
\[ + \sum_{j=1}^{n} |K_{j} - K_{j,p}| \mathbb{1}(j : d_{p}(x, X_{j}) > h_{n,p}(x)) \sum_{j=1}^{n} K_{j,p}. \]

Using that \( K \) is regular and that \( \sum_{j=1}^{n} K_{j,p} \geq 1 \) (this is since \( A_{2} \neq \emptyset \) we get,
\[ I_{A_{22}}^{3} = \mathbb{E}\left( \left( \frac{\sum_{j=1}^{n} |K_{j} - K_{j,p}|}{\sum_{j=1}^{n} K_{j,p}} \right)^{2}\right) \]
\[ \lesssim II_{A_{22}}^{1} + \mathbb{E}\left( \left( \sum_{j=1}^{n} W_{j,p} \mathbb{1}(j : d_{p}(x, X_{j}) \leq h_{n,p}(x), d(x, X_{j}) > 3h_{n}(x)) \right)^{2}\right) \]
\[ + \frac{\sum_{j=1}^{n} K_{j} \mathbb{1}(j : d_{p}(x, X_{j}) > h_{n,p}(x))}{\sum_{j=1}^{n} K_{j,p}} \]
\[ \lesssim II_{A_{22}}^{1} + II_{A_{21}} + \mathbb{E}\left( \left( \sum_{j=1}^{n} \mathbb{1}(j : d_{p}(x, X_{j}) > h_{n,p}(x), d(x, X_{j}) \leq h_{n}(x)) \right)^{2}\right) \]
\[ \leq II_{A_{22}}^{1} + II_{A_{21}} + C_{A_{1}}, \]
where \( II_{A_{22}}^{1} \) was defined in (B.9), \( II_{A_{21}} \) in (B.8), and \( C_{A_{1}} \) in (B.2). Then, from (B.20), (B.7), and (B.4) we have
\[ I_{A_{22}}^{3} \lesssim \left( \frac{c_{n,p}}{h_{n}(x)} \right)^{2} + n\mathbb{P}(|d_{p}(x, X_{1}) - d(x, X_{1})| > c_{n,p}) \] (B.21)

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Therefore, with (B.19) and (B.21) in (B.8) we have

\[ I_{A_22} \lesssim \left( \frac{c_{n,p}}{h_n(x)} \right)^2 + n \mathbb{P} \left( |d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| > c_{n,p} \right) \]

and with (B.20) and (B.21) in (B.9),

\[ II_{A_22} \lesssim \left( \frac{c_{n,p}}{h_n(x)} \right)^2 + n \mathbb{P} \left( |d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| > c_{n,p} \right) \]

Finally, to complete the proof of this result (i.e. that \( I \) and \( II \) converge to zero) we need to show that the expectation on \( \mathcal{X} \) of

\[ \left( \frac{c_{n,p}}{h_n(x)} \right)^2 + n \mathbb{P} \left( |d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1)| > c_{n,p} \right) \]

converges to zero. In order to show it, recall that from \( H2 \) we have

\[ n^2 \mathbb{E}_X \left( \mathbb{P}_{\mathcal{X}_1|x} \left( |d_p(\mathcal{X}, \mathcal{X}_1) - d(\mathcal{X}, \mathcal{X}_1)| \geq c_{n,p} \right) \big| \mathcal{X} \in \text{supp}(\mu) \right) \to 0, \]

and consequently, by Cauchy Schwartz inequality

\[ n \mathbb{E}_X \left( \mathbb{P}_{\mathcal{X}_1|x} \left( |d_p(\mathcal{X}, \mathcal{X}_1) - d(\mathcal{X}, \mathcal{X}_1)| \geq c_{n,p} \right) \big| \mathcal{X} \in \text{supp}(\mu) \right) \to 0. \]

In addition, from (H3.1) and the Dominated Convergence Theorem we have

\[ \mathbb{E}_X \left( \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \right) \to 0. \]

Therefore, taking expectation with respect to \( \mathcal{X} \) in (B.3), (B.4), (B.6), (B.7), (B.22), and (B.23), we prove Part (1) of the Theorem.

Proof of (b): The only difference with item (a) is the convergence of term \( III \) to zero which is ensured by Proposition 2.

Proof of Theorem 2: Let \( \gamma_n \to \infty \) as \( n \to \infty \) a sequence such that, as \( n, p \to \infty \),

\[ \mathbb{E}_X \left( \gamma_n \left( \frac{c_{n,p}}{h_n(x)} \right)^2 \right) \to 0 \]

and,

\[ \gamma_n n^2 \mathbb{E}_X \left( \mathbb{P}_{\mathcal{X}_1|x} \left( |d(\mathcal{X}, \mathcal{Y}) - d_p(\mathcal{X}, \mathcal{Y})| \geq c_{n,p} \big| \mathcal{X} \in \text{supp}(\mu) \right) \right) \to 0. \]

From proof of Theorem 1 we get,

\[ \mathbb{E} \left( \gamma_n (\tilde{\eta}_{n,p}(\mathcal{X}) - \eta(\mathcal{X}))^2 \right) \lesssim \gamma_n n \mathbb{E}_X \left( \mathbb{P}_{\mathcal{X}_1} \left( d_p(x, \mathcal{X}_1) - d(x, \mathcal{X}_1) \geq c_{n,p} \right) \right) \]
\[ + \mathbb{E}_\mathcal{X} \left( \gamma_n \left( \frac{c_{n,p}}{h_n(X)} \right)^2 \right) \]
\[ + \mathbb{E} \left( \gamma_n (\hat{\eta}_{n,p}(X) - \eta(X))^2 \right), \]

from what follows that,
\[
\lim_{n,p \to \infty} \mathbb{E} \left( \gamma_n (\hat{\eta}_{n,p}(X) - \eta(X))^2 \right) = 0.
\]

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