BOUND S FOR THE LONELY RUNNER PROBLEM VIA LINEAR PROGRAMMING

FELIPE GONÇALVES AND JOÃO P. G. RAMOS

ABSTRACT. In this note we develop a linear programming framework to produce upper and lower bounds for the lonely runner problem.

1. The Lonely Runner Problem

Suppose you are competing in race on a circular track of perimeter $L$ with $n - 1$ other runners. Assume all competitors have distinct constant speeds. The gap of loneliness is the largest length $\ell$ such that at some time $t$ in the future (assuming the race continues forever) the closest runner to you is at distance $\ell$. The lonely runner conjecture states that

$$\ell \geq \frac{L}{n}.$$ 

This problem was introduced independently by Wills [8] (1967) and Cusick [2] (1973) in the context of view obstruction problems. The conjecture is known to be true for $n \leq 7$ runners. Moreover, speeds can be assumed to be distinct integers. For more on the history of partial results see Bohnman et al. [1] (2001), Perarnau & Serra [6] (2016) and Tao [7] (2018).

By Galilean relativity your speed can be assumed to be zero and the conjecture takes the following equivalent formulation: Let $\|x\| = \min_{n \in \mathbb{Z}} |x-n|$ denote the distance to the nearest integer. For a vector $x \in \mathbb{R}^{n-1}$ let

$$\mu(x) = \min\{\|x_1\|, \ldots, \|x_{n-1}\|\}.$$ 

Then for any vector $v = (v_1, \ldots, v_{n-1}) \in \mathbb{Z}^{n-1}$ of distinct integers show that

$$\text{gap}(v) := \max_{t \in \mathbb{T}} \mu(tv) \geq \frac{1}{n}, \quad (conjecture)$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

In general, since $t \mapsto \mu(tv)$ is piece-wise linear with slopes drawn from $\{v_1, \ldots, v_{n-1}\}$, the set of local maxima of $t \mapsto \mu(tv)$ is contained in the intersection of any two line segments. Therefore, if $t$ satisfies $\mu(tv) = \text{gap}(v)$ then $t \in \frac{1}{q}\mathbb{Z}$, where $q$ is a factor of some $(v_j \pm v_i)$ with $i \neq j$, and therefore $\text{gap}(v) = a/b$ where $b$ divides $q$.

Date: October 13, 2020.

1
1.1. The Linear Programming Approach. We want to study the following two problems:

**Problem** (Linear Programming Problems). Fix a sign $\epsilon = \pm 1$. Let $v = (v_1, ..., v_{n-1})$ be a given vector of increasing integer speeds. We want to

\[
\text{Minimize } \epsilon \frac{\hat{f}(0)}{f(0)}
\]

subjected to

(I) $f$ is a non-zero, even and real trigonometric polynomial

\[
f(x) = \sum_{n=-D}^{D} \hat{f}(k)e^{2\pi ikx} = \hat{f}(0) + 2 \sum_{k=1}^{D} \hat{f}(k) \cos(2\pi ikx)
\]

of degree at most $D \geq \max(v_1, ..., v_{n-1})$.

(II) In case $\epsilon = +1$ we ask

\[f(x) \geq 0 \text{ for } 0 \leq x \leq \frac{1}{2}\]

and in case $\epsilon = -1$ we ask

\[f(x) \leq 0 \text{ for } \frac{1}{v_{n-1}+v_{n-2}} \leq x \leq \frac{1}{2}\.\]

(III) $\epsilon \hat{f}(k) \leq 0$ if $k \notin \{0, v_1, ..., v_{n-1}\}$.

We denote by $\Lambda_\epsilon(v)$ the class of trigonometric polynomials satisfying (I), (II) and (III). We write

\[
\lambda_+(v) = \inf_{f \in \Lambda_+(v)} \frac{\hat{f}(0)}{f(0)} \text{ and } \lambda_-(v) = \sup_{f \in \Lambda_-(v)} \frac{\hat{f}(0)}{f(0)}
\]

**Theorem 1.** Let $v = (v_1, ..., v_{n-1})$ be a vector of increasing positive integers. Then

\[\text{gap}(v) \leq \lambda_+(v)\] (1)

and

\[\text{gap}(v) \geq \lambda_-(v)\] (2)

**Theorem 2.** Let $v = (v_1, ..., v_{n-1})$ be a vector of increasing positive integers. Then equality is attained in (1) if one of the following conditions hold:

(i) All $v_i$’s are odd. In this case $f(x) = \cos(\pi v_i x)^2$ is optimal for any $i = 1, ..., n - 1$;

(ii) There exist coprime integers $a, m \geq 1$ such that $a\{1, ..., m - 1\} \subset \{v_1, ..., v_{n-1}\}$ and all integers in $\{v_1, ..., v_{n-1}\} \setminus a\{1, ..., m - 1\}$ are not divisible by $m$. In this case $f(x) = K_m(ax)$ is optimal, where $K_m$ is Fejér’s kernel.\(^{[3]}\)

(iii) There exists integer $a \geq 1$ such that $v = av'$ and $v'$ satisfies condition (i) or (ii). In this case if $f(x)$ is optimal for $v'$ then $f(ax)$ is optimal for $v$.\(^{[i]}\)
In the range \( n \leq 20 \) and \( \max(v_1, \ldots, v_n) \leq 40 \) we have performed a computer search in \( \mathbf{v} \) in conjunction with Gurobi’s linear programming solver \([4]\) to approximate \( \lambda_+(\mathbf{v}) \). The sign conditions of \( f \) was modelled with sampling. This produced reliable numerical approximations to what we believe is the true value of \( \lambda_+(\mathbf{v}) \). In this way we check that the only cases where

\[
\text{gap}(\mathbf{v}) + [\text{very small error}] > \lambda_+(\mathbf{v})
\]

for \( |\mathbf{v}|_\infty \leq 40 \) and \( n \leq 20 \) were the ones contemplated by Theorems 2. This leads to the following conjecture.

**Conjecture.** Equality is attained in \([1]\) if and only if one of the conditions in Theorem 2 hold.

It is unfortunate that the bounds generated for \( \lambda_- (\mathbf{v}) \) do not seem to be nearly as good as the bounds generated by \( \lambda_+ (\mathbf{v}) \), and we believe this is because condition (II) seems to be very strong for \( \epsilon = -1 \). We need high degree polynomials and a large number of sampling points for feasibility of the linear program. In Section 3 we propose an improved version of this lower bound.

We note that proving exact bounds is not hard as if some numerical \( f \) satisfies \( \epsilon f \geq -\delta \) in some region, but \( \epsilon f \) should be nonnegative in that region, then all we have to do is use \( g = f + \epsilon \delta \) as this would be admissible for \( \Lambda_\pm (\mathbf{v}) \) and \( \hat{g}(0)/g(0) = \hat{f}(0)/f(0) + O(\delta) \).

## 2. Proofs for the main results

We start by recalling that Dirichlet’s Approximation Theorem implies the Lonely Runner Conjecture is sharp; a rephrasing of Dirichlet’s theorem is

\[
\max_{t \in \mathbb{T}} \mu(t(1, 2, \ldots, n-1)) = \frac{1}{n}.
\]

The maxima is attained for \( t = a/n \) for a coprime with \( n \). We now observe that inequality \([1]\) is also tight in case \( \mathbf{v} = (1, 2, \ldots, n - 1) \), and Fejér’s kernel

\[
K_n(x) = \frac{1}{n} \left( \frac{\sin(\pi nx)}{\sin(\pi x)} \right)^2 = \sum_{|j|<n-1} (1 - |j|/n) e^{2 \pi i j x}
\]

is the unique optimum. Optimality can easily be checked by hand, while uniqueness (modulo scaling) comes from the proof of Theorem \([1]\). Essentially, because \( f \geq 0 \), we must have \( f(k/n) = f'(k/n) = 0 \) for \( k = 1, \ldots, n \) and Fejér’s kernel is the only even trigonometric polynomial of degree \( n - 1 \) with these properties.
Proof of Theorem 1. Let $\delta = \text{gap}(v) = \mu(tv)$ and $h(x) = (\delta - |x|)_+$ be a hat function. Since $\hat{h}(x) = (\sin(\pi \delta x)/(\pi x))^2$ we have

$$h(x) = \sum_{n \in \mathbb{Z}} \left( \frac{\sin(\pi \delta n)}{\pi n} \right)^2 e^{2\pi inx}.$$ 

If $f \in \Lambda_+(v)$ we obtain

$$\delta \hat{f}(0) = \delta \hat{f}(0) + 2 \sum_{j=1}^{n-1} \hat{f}(v_j)h(tv_j) \geq \sum_{k=-D}^{D} \hat{f}(k)h(tk) = \sum_{k \in \mathbb{Z}} \left( \frac{\sin(\pi \delta k)}{\pi k} \right)^2 f(kt) \geq \delta^2 f(0)$$

which proves the upper bound. For the lower bound ($\epsilon = -1$), first recall that $t = p/q$, $\delta = a/b$ and $b$ divides $q$, while $q$ is a factor of some $v_j \pm v_i$ with $j > i$ (both fractions in lowest terms). In particular $q \leq v_{n-1} + v_{n-2}$. If $f \in \Lambda_-(v)$ we obtain

$$\delta \hat{f}(0) = \delta \hat{f}(0) + 2 \sum_{j=1}^{n-1} \hat{f}(v_j)h(tv_j) \leq \sum_{k=-D}^{D} \hat{f}(k)h(tk)$$

$$= \delta^2 f(0) + \sum_{k \in \mathbb{Z} \setminus b\mathbb{Z}} \left( \frac{\sin(\pi \delta k)}{\pi k} \right)^2 f\left(\frac{k}{q}\right)$$

However, since $\{kp/q \mod 1 : k \in \mathbb{Z}_+ \setminus b\mathbb{Z}\} = \frac{1}{q}\{1, ..., q-1\} \setminus b\mathbb{Z}$, $q \leq v_{n-1} + v_{n-2}$ and $f(x) \leq 0$ for $1/(v_{n-1} + v_{n-2}) \leq x \leq 1/2$, then $f(kp/q) \leq 0$ for $k \in \mathbb{Z} \setminus b\mathbb{Z}$. This concludes the proof. □

This proof is inspired by the analytic proof of Dirichlet’s approximation theorem due to Montgomery [5] (1994). We observe that equality is attained in (1) or (2) if and only if there is $f \in \Lambda_s(v)$ such that:

(a) $\hat{f}(k) = 0$ if $k \notin \{v_1, ..., v_{n-1}\}$;
(b) For some $t = p/q$ that is a global maxima of $t \mapsto \mu(tv)$, where $\text{gap}(v) = a/b$ and $b$ divides $q$ (both fractions in lowest terms) we have $f(kp/q) = 0$ if $k \in \{1, ..., q-1\} \setminus b\mathbb{Z}$.

Proof of Theorem 2. Condition (i) is easy to check because when all $v_i$’s are odd we have $\text{gap}(v) = 1/2$ while $\hat{f}(0) = 1/2$ and $f(0) = 1$ for $f(x) = \cos(\pi v_i x)^2$. Next, if condition (ii) holds, then letting $u = (1, ..., m-1)$ and $w = \{v\} \setminus \{au\}$ (abusing notation) we obtain

$$\mu(tv) = \min(\mu(tau), \mu(tw)) \leq \mu(tau) \leq \frac{1}{m}.$$ 

On the other hand, since $a$ is coprime with $m$ we have $\mu(\frac{1}{m}au) = \frac{1}{m}$, and since $\frac{1}{m}w$ has no integer coordinate we have $\mu(\frac{1}{m}w) \geq \frac{1}{m}$. Therefore $\mu(\frac{1}{m}v) = \frac{1}{m}$ and we obtain that $\text{gap}(v) = \frac{1}{m}$. Now it is easy to check that $f(x) = K_m(ax)$ belongs to $\Lambda_+(v)$ and is optimal. Condition (iii) is trivial. □
3. Improved lower bounds

Let $V_q$ be the class of vectors $v$ of increasing positive integers such that the global maxima of $t \mapsto \mu(tv)$ is attained at some point $t = p/q \in (0, 1/2)$ (in lowest terms). By the proof of Theorem 1 we see that if we let $\Lambda_-(v, q)$ be the class of functions satisfying the above conditions (I), (III) and

\[(\Pi') \ f(x) \leq 0 \text{ for } \frac{1}{q} \leq x \leq \frac{1}{2},\]

then

\[
\lambda_-(v, q) := \sup_{f \in \Lambda_-(v, q)} \frac{\hat{f}(0)}{f(0)} \leq \text{gap}(v)
\]

for any $v \in V_q$. For instance $(1, ..., n-1) \in V_n$, but some other examples of vectors in $V_n$ can be extracted from Goddyn and Wong [3], (2006). They present conditions for $v$ to be tight, that is, $\text{gap}(v) = \frac{1}{n}$. Some of these tight vectors characterized in [3, Theorem 2.3] belong to $V_n$, for instance:

\[
(1, 2, ..., n-3, n-1, 2n-4) \in V_n \text{ if } n = 2 \pmod{2 \cdot 3}
\]

\[
(1, 2, ..., n-4, n-2, n-1, 2n-6) \in V_n \text{ if } n = 3 \pmod{2 \cdot 3 \cdot 5}.
\]

**Theorem 3.** Let $v = (v_1, ..., v_{n-1}) \in V_q$. Then equality is attained in (4) if one of the following conditions hold:

(i) There exist integer $a \geq 1$ coprime with $q$ such that $a \{1, ..., q-1\} \subset \{v_1, ..., v_{n-1}\}$ and all integers in $\{v_1, ..., v_{n-1}\} \setminus a \{1, ..., q-1\}$ are not divisible by $q$. In this case

\[
f(x) = K_q(ax) \frac{1 - \cos(\pi/q)^2}{(\cos(\pi ax)^2 - \cos(\pi/q)^2)}
\]

is optimal, where $K_q$ is Fejer’s kernel [3].

(ii) There exists an integer $a \geq 1$ such that $v = av'$ and $v'$ satisfies condition (i). In this case if $f(x)$ is optimal for $v'$ then $f(ax)$ is optimal for $v$.

**Proof.** Assume condition (i). By the same discussion in the proof of Theorem 2 we have $\text{gap}(v) = 1/q$. It is easy to see that $f$ satisfies conditions (I),(II’),(III). Moreover its mass equals the mass of $f(x/a)$, which in turn, by exact Gaussian quadrature, equals the mass of $K_q(x)$, which is 1. Also $f(0) = q$. This shows optimality. Condition (ii) is trivial. \(\square\)

3.1. **Numerics.** We used Gurobi [4] to compute $\lambda_+(v)$ and $\lambda_-(q, v)$ for 2429 different velocity vectors $v = (v_1, ..., v_5)$ selected randomly from $0 < v_1 < ... < v_5 \leq 50$. In Figure 1 we plot points $(x, y)$ which are numerical approximations to $(\lambda_-(v, q), \text{gap}(v))$ in blue dots, $(\lambda_+(v), \text{gap}(v))$ in yellow triangles and $(\lambda_+(v), \text{gap}(v))$ in green squares. We took $q = \text{denominator}(t_{\text{max}})$ where $\mu(t_{\text{max}}v) = \text{gap}(v)$, and $t_{\text{max}}$ is the smallest with this property. The diagonal blue line is $x = y$ and the gray vertical line is $x = 1/6$. We note that
we get much better lower bounds if we know a priori that $v \in V_q$, as the yellow triangles are clearly much closer to the line $x = y$ than the blue dots. The plot appears to have some interesting emergent structures: rays of triangles and parabola-like green structures.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{}
\end{figure}

\section*{Acknowledgements}

The first author is thankful to Jeffrey Vaaler for helpful comments. F.G. acknowledges support from the Deutsche Forschungsgemeinschaft through the Collaborative Research Center 1060.

\section*{References}

[1] T. Bohman, R. Holzman, D. Kleitman, Six lonely runners, In honor of Aviezri Fraenkel on the occasion of his 70th birthday. Electron. J. Combin. 8 (2001), no. 2, Research Paper 3, 49 pp.

[2] T. W. Cusick, View obstruction problems, Aequationes Math. 9 (1973), 165–170.

[3] L. Goddyn, E. B. Wong, Tight instances of the lonely runner, Integers 6 (2006), A38.

[4] Gurobi Optimization, LLC, \textit{Gurobi Optimizer Reference Manual} (2020).

[5] H. L. Montgomery, Ten lectures on the interface between analytic number theory and harmonic analysis, CBMS Regional Conference Series in Mathematics, 84 (1994).

[6] G. Perarnau, O. Serra, Correlation among runners and some results on the lonely runner conjecture, Electron. J. Combin. 23 (2016), no. 1, Paper 1.50, 22 pp.

[7] T. Tao, Some remarks on the lonely runner conjecture, Contrib. Discrete Math. 13 (2018), no. 2, 1–31.
[8] J. M. Wills, Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen, Monatsch. Math. 61 (1967), 263-269.

Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Email address: goncalve@math.uni-bonn.de

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

Email address: joao.ramos@math.ethz.ch