RANDOM FLAG COMPLEXES AND ASYMPTOTIC SYZYGIES

DANIEL ERMAN AND JAY YANG

Abstract. We use the probabilistic method to construct examples of conjectured phenomenon about asymptotic syzygies. In particular, we use the Stanley-Reisner ideals of random flag complexes to construct new examples of Ein and Lazarsfeld’s nonvanishing for asymptotic syzygies and of Ein, Erman, and Lazarsfeld’s conjectural on the asymptotic normal distribution of Betti numbers.

Using the probabilistic method, we produce examples of conjectured phenomenon on asymptotic syzygies. One of these examples, involving normally distributed Betti numbers, is the first known example of a phenomenon conjectured by Ein, Erman, and Lazarsfeld.

Our central construction involves random flag complexes. We use $G \sim G(n, p)$ to denote an Erdős-Rényi random graph on $n$ vertices, where each edge is attached with probability $p$. We turn $G$ into a flag complex by adjoining a $k$-cell to every $(k + 1)$-clique in the graph, and $\Delta \sim \Delta(n, p)$ denotes a flag complex chosen with respect to this distribution. The properties of random flag complexes have been studied extensively in recent years; see [Kah14b] for a survey of recent results. From $\Delta$, Stanley-Reisner theory yields a squarefree monomial ideal $I_\Delta \subseteq k[x_1, x_2, \ldots, x_n]$ [BH93, Chapter 5], and we analyze the Betti numbers of $I_\Delta$.

A recent paper De Loera, Petrović, Silverstein, Stasi, and Wilburne [DLPS+] also produces random monomial ideals via a construction similar to Erdős-Rényi random graphs, and one of their constructions specializes to ours. They study thresholds and the distribution of algebraic invariants in this framework, and they provide an array of results and conjectures.

We are motivated by questions and conjectures about asymptotic syzygies. These questions are generally outside of the range computable in Macaulay2 [M2] or elsewhere, and so there is a lack of known examples. By contrast, results on random flag complexes are generally asymptotic in nature. So by using probabilistic techniques to analyze the syzygies of $I_\Delta$, we produce new examples of behaviors conjectured in [EL12] and [EEL15].

We now summarize Ein and Lazarsfeld’s central result on asymptotic syzygies. For a module $M$, we define $\rho_k(M)$ as the ratio of nonzero entries in the $k$th row of the Betti table:

$$\rho_k(M) := \frac{\#\{i \in [0, \text{pdim}(M)] \text{ where } \beta_{i,i+k}(S_d) \neq 0\}}{\text{pdim}(M) + 1}.$$ 

Under increasingly positive embeddings, [EL12] shows that these densities approach 1.

Theorem 1.1 (Ein-Lazarsfeld, 2012). Let $X$ be a smooth, $d$-dimensional projective variety and let $A$ be a very ample divisor on $X$. For any $n \geq 1$, let $S_n$ be the homogeneous coordinate ring of $X$ embedded by $nA$. For each $1 \leq k \leq d$, $\rho_k(S_n) \to 1$ as $n \to \infty$.

See [EL12, Theorem A] for the sharper result and Figure 1 for an illustration. A similar nonvanishing phenomenon was shown to hold for integral varieties [Zho14, Theorem, p. 2256],

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Figure 1. Each dot represents a known nonzero entry in the Betti table of $\mathbb{P}^3$ embedded by $O(n)$ for $n = 10$. By Ein and Lazarsfeld’s Theorem 1.1, the density of the dots in rows 1, 2, and 3 will approach 1 as $n \to \infty$. Theorem 1.4 shows a similar phenomenon holds for ideals of random flag complexes.

arithmetically Cohen-Macaulay varieties [EEL16, Theorem 3.1], and certain iterated subdivisions of Stanley-Reisner rings [CJKW]. Moreover, experiments in Macaulay2 with different asymptotic families of ideals (graph curves, unions of linear spaces, etc.) suggest that this asymptotic nonvanishing behavior occurs in a broad range of examples. This motivates the following question.

Question 1.2. Let $\{I_n\}$ be a family of ideals where $\text{pdim}(I_n) \to \infty$. Fix some $k$. Under what conditions will $\rho_k(S/I_n) \to 1$ as $n \to \infty$?

While Question 1.2 addresses qualitative expectations about asymptotic syzygies, the corresponding quantitative behavior of asymptotic syzygies was raised in [EEL15]. They introduce a random Betti table model to provide a heuristic for the asymptotic behavior of certain families of Betti tables. Their analysis suggests that, roughly speaking, each row of the Betti table of any very positive embedding displays the pattern of a large Koszul complex [EEL15, Conjecture B and Theorem C]. Or nearly equivalently, they conjecture that the ranks of the syzygy modules of a smooth projective variety become normally distributed (after appropriate rescaling) under increasingly positive Veronese embeddings.

Question 1.3. Let $\{I_n\}$ be a family of ideals where $\text{pdim}(I_n) \to \infty$. Fix some $k$. Under what conditions will the function $i \mapsto \beta_{i,i+k}(S/I_n)$ approach a binomial distribution, as $n \to \infty$?

The only known case of this phenomenon comes from a smooth curve of high degree [EEL15, Proposition A]. Yet in that example, all of the nonzero Betti numbers essentially cluster in a single row, and so that result only depends on the Hilbert function, avoiding the complexity of overlapping Betti numbers shown in Theorem 1.1 and elsewhere.

Our main results provide new families where Questions 1.2 and 1.3 have affirmative answers. We write $f(n) \ll g(n)$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

Theorem 1.4. Fix some $r \geq 1$. Let $\Delta \sim \Delta(n,p)$ with $\frac{1}{n^{1/r}} \ll p \ll 1$. For each $1 \leq k \leq r+1$, we have $\rho_k(S/I_\Delta) \to 1$ in probability.

Saying that $\rho_k(S/I_\Delta) \to 1$ in probability is equivalent to asking that for any $\epsilon > 0$, the probability that $\rho_k(S/I_\Delta) \geq 1 - \epsilon$ goes to 1 as $n \to \infty$. In particular, random flag complexes provide a positive answer to Question 1.2, similar to Theorem 1.1. See Example 5.1.

The proof of Theorem 1.4 uses randomness to find particular subcomplexes of $\Delta$. As we will see in Section 2, the $s$-fold suspension of 2 points has the minimal number of edges possible for a flag complex with $(s+1)$th homology, and it is thus the most likely subcomplex to contribute to the $(s+1)$th row of the Betti table of $S/I_\Delta$. The main step of the proof comes from Theorem 1.7 below, where we show that the bound $\frac{1}{n^{1/s}} \ll p$ is the threshold
Figure 2. We plot the function $i \mapsto \beta_{i,i+1}(S/I_\Delta)$ for a random $\Delta \sim \Delta(10, \frac{1}{20})$ and $\Delta \sim \Delta(15, \frac{1}{30})$, respectively. As predicted by the heuristic of [EEL15], and proven in Theorem 1.5, the Betti numbers are converging to a binomial.

for the existence of this particular subcomplex. Once we have crossed this threshold, we can find this particular subcomplex, and minor variants of it, yielding nonzero Betti numbers throughout nearly the entire $(s+1)$th row.

Next we turn to Question 1.3.

**Theorem 1.5.** Fix a constant $0 < c < 1$ and let $\Delta \sim \Delta(n, \frac{c}{n})$ be a random flag complex. If $\{i_n\}$ is an integer sequence satisfying $i_n = n/2 + o(n)$, and if $C := \frac{1-c}{2}$, then

$$\frac{\beta_{i_n,i_n+1}(S/I_\Delta)}{Cn(n)} \to 1$$

in probability.

By applying a standard binomial-to-normal argument, this provides an example of the phenomenon predicted by [EEL15, Conjecture B].

**Corollary 1.6.** Fix a constant $0 < c < 1$ and let $\Delta \sim \Delta(n, \frac{c}{n})$ be a random flag complex. If $\{i_n\}$ is a sequence of integers converging to $\frac{n}{2} + \frac{a\sqrt{n}}{2}$, then

$$\frac{\sqrt{2\pi}}{(1-c)2^n\sqrt{n}} \cdot \beta_{i_n,i_n+1}(S/I_\Delta) \to e^{-a^2/2}$$

in probability.

The only previously known example of normally distributed Betti numbers comes from smooth curves [EEL15, Theorem A], but in that case, the nonzero Betti numbers are clustered almost entirely in a single row, avoiding the complexity of overlapping Betti numbers. By contrast, for the family of ideals in Theorem 1.5, the Betti numbers are not always clustered in a single row (see Remark 6.1), providing a tighter parallel with, for example, the conjectured behavior of Veroneses of surfaces. In fact, Theorem 1.5 produces the first known families of ideals exhibiting both normally distributed and overlapping Betti numbers.

The following simple computation suggests why the Betti numbers of random flag complexes should follow a binomial distribution. For a subset $\alpha$ of the vertices, we write $\Delta|_\alpha$ for...
the restricted flag complex. Hochster’s formula shows that $\beta_{i,i+1}(S/I_\Delta)$ is the sum over all $\alpha \in \{\binom{n}{i}\}$ of $\dim \tilde{H}_0(\Delta|_{\alpha})$. By linearity of expectations, the expected value of $\beta_{i,i+1}(S/I_\Delta)$ is

$$E[\beta_{i,i+1}(S/I_\Delta)] = \sum_{\alpha \in \{\binom{n}{i}\}} \dim \tilde{H}_0(\Delta|_{\alpha}) = \binom{n}{i} E[\tilde{H}_0(\Delta')]$$

where $\Delta' \sim \Delta(i, c_n)$ is a random flag complex. So if we can control how $E[\tilde{H}_0(\Delta')]$ varies with $i$, then the expected values of the Betti numbers will look like a binomial distribution.

The main issue in proving Theorem 1.5 thus arises in showing convergence in probability, stemming from the fact that $\beta_{i,i+1}(S/I_\Delta)$ is a sum of dependent random variables.

We also prove some results on the algebraic invariants of $S/I_\Delta$. For instance, we prove the following threshold result for individual Betti numbers:

**Theorem 1.7 (Betti Number Thresholds).** Fix $i, v$ with $1 \leq i$ and $i + 1 \leq v \leq 2i$ and let $s := v - i - 1$. Fix some constant $0 < \epsilon \leq \frac{1}{2}$ and let $\Delta \sim \Delta(n, p)$.

1. If $\frac{1}{n^{1/3}} \ll p \leq \epsilon$ then $P[\beta_{i,v}(S/I_\Delta) \neq 0] \to 1$.
2. If $p \ll \frac{1}{n^{1/2}}$ then $P[\beta_{i,v}(S/I_\Delta) = 0] \to 1$.

We bound the regularity of $S/I_\Delta$ in Corollary 5.2. Corollary 7.1 shows that while $S/I_\Delta$ is almost never Cohen-Macaulay, the depth and codimension of $S/I_\Delta$ converge as $n \to \infty$.

This paper is organized as follows. Section 2 provides some essential definitions. Section 4 provides a threshold for the vanishing/nonvanishing of individual Betti numbers, the nonvanishing half of which relies on a variance bound proven Section 3. In Section 5 we use the Betti number threshold to prove Theorem 1.4. In Section 6 we prove Theorem 1.5 and Corollary 1.6. Section 7 contains estimates on the projective dimension of the ideal $I_\Delta$.

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**2. Background and Notation**

We work over an arbitrary field $k$. We write $P[\cdot]$ for the probability of an event and $E[\cdot]$ for the expected value of a random variable.

A flag complex is a simplicial complex obtained from a graph by adjoining a $k$-cell to every $(k + 1)$-clique in the graph. We use $G \sim G(n, p)$ to denote an Erdős-Rényi random graph on $n$ vertices, where each edge is attached with probability $p$, and we use $\Delta \sim \Delta(n, p)$ to denote the corresponding random flag complex. If $H$ is a subset of the $n$ vertices, then we use $\Delta|_H$ for the induced flag complex.

The generators of $I_\Delta$ correspond to the maximal non-faces of $\Delta$ [BH93, Chapter 5], and since $\Delta$ is flag this means that $I_\Delta$ is generated by quadrics. Hochster’s Formula [BH93, Theorem 5.5.1], which relates the Betti table of $S/I_\Delta$ to topological properties of $\Delta$, is our key tool for studying the syzygies of $S/I_\Delta$.

**Remark 2.1.** As discussed in the introduction, our goal is to use the $I_\Delta$ to model asymptotic syzygies. The ideals of high degree Veroneses always admit a quadratic Gröbner
basis [ERT94], and this is one reason why we chose to use random flag complexes. By contrast, models in [DLPS+] often produce ideals with generators in different degrees, and those would thus provide better models for other families of examples.

Example 2.2. Hochster’s Formula implies that \( \beta_{r+1,2r+2}(S/I_\Delta) \) is the number subcomplexes \( \Delta|_H \subseteq \Delta \) where \( H \) has \( 2r + 2 \) vertices and where \( \tilde{H}_r(\Delta|_H) \neq 0 \). For instance \( \beta_{1,2}(S/I_\Delta) \) is the pairs of disjoint vertices in \( \Delta \), or equivalently it is the number of non-edges of the \( \Delta \). And \( \beta_{2,4}(S/I_\Delta) \) is the number of squares in \( \Delta \). On the other hand, \( \beta_{2,5}(S/I_\Delta) \) counts subcomplexes on five vertices with nonzero \( \tilde{H}_1 \). There are several different types of examples, such as:

![Diagrams](image)

Figure 3. Among flag complexes with nonzero \( r \)th homology, the \( r \)-fold suspension of 2 points, which we denote \( \Diamond_r \), has the fewest edges.



Lemma 2.3. If \( \Delta \) is a flag complex, then \( \beta_{i,j}(S/I_\Delta) = 0 \) for all \( j > 2i \).

Proof. Since \( \Delta \) is flag, \( I_\Delta \) is a monomial ideal generated by quadrics. The Taylor resolution of \( S/I_\Delta \) thus involves monomials of degree 0, 1, or 2 [Pee11, Construction 26.5].

The \( r \)-fold suspension of 2 points plays a key role in our results (for instance, see Remark 3.1), and we denote this flag complex by \( \Diamond_r \). See Figure 3. Since a pair of points is disconnected, we have \( \tilde{H}_0(\Diamond_0) \cong \mathbb{Z} \), and since taking suspensions shifts reduced homology groups up by one degree, we have that \( \tilde{H}_r(\Diamond_r) \cong \mathbb{Z} \). We now prove that any flag complex with nonzero \( r \)th homology will have at least as many vertices and edges as \( \Diamond_r \).

Lemma 2.4. Let \( \Delta \) be a flag complex with \( \tilde{H}_r(\Delta) \neq 0 \).

1. Then \( \Delta \) has at least \( 2r + 2 \) vertices.
2. If \( v \in \Delta \) is a vertex such that \( \tilde{H}_r(\Delta_{\Delta-v}) = 0 \), then \( \deg(v) \geq 2r \).
3. \( \Delta \) has at least \( 2r(r+1) \) edges.

Proof. Write \( n \) for the number of vertices of \( \Delta \). Since \( \tilde{H}_r(\Delta) \neq 0 \), Hochster’s formula implies that \( \beta_{n-r-1,n}(S/I_\Delta) \neq 0 \) [BH93, Theorem 5.5.1]. By Lemma 2.3, we then have \( n \leq 2(n - r - 1) \), which implies (1).

For (2) we consider \( \text{link}_\Delta(v) \). The Mayer-Vietoris sequence gives a long exact sequence

\[
\cdots \to \tilde{H}_r(\text{star}_\Delta(v)) \oplus \tilde{H}_r(\Delta|_{\Delta-v}) \to \tilde{H}_r(\Delta) \to \tilde{H}_{r-1}(\text{link}_\Delta(v)) \to \cdots
\]
and since $v$ is a vertex, $\star_{\Delta}(v)$ is contractable and thus $\widetilde{H}_r(\star_{\Delta}(v)) = 0$. By assumption $\widetilde{H}_r(\Delta|_{\Delta - v}) = 0$, and $\widetilde{H}_r(\Delta) \neq 0$, and so $\widetilde{H}_{r-1}(\text{link}(v)) \neq 0$. But part (1) implies that $\text{link}_{\Delta}(v)$ contains at least $2r$ vertices, and since the vertices of $\text{link}_{\Delta}(v)$ are in bijection with edges of $\Delta$ through $v$, we obtain $\deg(v) \geq 2r$.

For (3), we choose a connected flag complex $\Delta$ with $\widetilde{H}_r(\Delta)$ and where $\Delta$ has the minimal possible number of edges. By minimality and connectedness, each vertex $v$ satisfies $\widetilde{H}_r(\Delta|_{\Delta - v}) = 0$. Combining (1) and (2), we conclude that $\Delta$ has at least $2r(r + 1)$ edges. □

Remark 2.5. The complex $\Box_s$ shows that the bounds in Lemma 2.4 are sharp. □

Remark 2.6. Our proof of Lemma 2.4(1) is purely algebraic, relying on Taylor's resolution. Purely topological proofs are possible as well, but we know of no proof that is as concise. □

3. Variance Bound

In this section we prove a variance bound that is used in our convergence results. The proof is similar to those in [BE76, Theorem 1] and [Kah14a, Lemma 2.2] and elsewhere.

Remark 3.1. We are particularly interested in the appearance of subcomplexes of the form $\Box_s$, as by Lemma 2.4 these are the flag complexes with the fewest edges and nonzero $s$th homology. Since $p$ generally goes to 0 as $n \to \infty$, subcomplexes with fewer edges are more likely to appear, and so we expect these $\Box_s$ to control $(s + 1)$th row of $\beta(S/I_{\Delta})$. □

In $\Box_s$, every vertex has a unique antipodal vertex, and thus as a subgraph of $\Delta$, $\Box_s$ is determined by $s + 1$ pairs of vertices, all distinct.

Definition 3.2. Let $X_s = X_s(n, p)$ denote the random variable for the number of copies of $\Box_s$ appearing as a subgraph of a random graph $G \sim G(n, p)$. If $H$ is a set of $s + 1$ pairs of vertices, all distinct, we define $X_H$ as the indicator random variable for whether the subgraph on $H$ has the form $\Box_s$.

We have $X_s = \sum_H X_H$. We will need to bound the variance $\text{Var}[X_s]$.

Lemma 3.3 (Variance Bound). If $np^{s+\frac{1}{2}} \to \infty$ and $p \leq (1 - p)$, then $\frac{\text{Var}[X_s]}{\mathbb{E}[X_s]^2} \to 0$.

Proof. We start by computing

$$\mathbb{E}[X_s^2] = \sum_{\text{pairs } (H, J)} \mathbb{E}[X_H X_J]$$

$$= \sum_{\text{pairs } (H, J)} \mathbb{P}[X_J = 1|X_H = 1] \mathbb{P}[X_H = 1]$$

$$= \sum_H \mathbb{P}[X_H = 1] \sum_J \mathbb{P}[X_J = 1|X_H = 1]$$

Since $\sum_J \mathbb{P}[X_J = 1|X_H = 1]$ is independent of the choice of $H$, we may fix an $H'$ to decouple the factors, yielding

$$= \left( \sum_H \mathbb{P}[X_H = 1] \right) \sum_J \mathbb{P}[X_J = 1|X_{H'} = 1]$$

$$= \mathbb{E}[X_s] \mathbb{E}[X_s|X_{H'} = 1]$$
Since $\text{Var}[X_s] = E[X_s^2] - E[X_s]^2$, the above computation allows us to compute:

\[
\frac{\text{Var}(X_s)}{E[X_s]^2} = \frac{E[X_s|X_H = 1] - E[X_s]}{E[X_s]}
\]

\[
= \frac{\sum_{m=0}^{2s+2} \sum_{|J \cap H| = m} \mathbb{P}[X_J = 1|X_H = 1] - \mathbb{P}[X_J = 1]}{E[X_s]}
\]

If $J$ and $H$ are disjoint or intersect in only a single vertex, then $\mathbb{P}[X_J = 1|X_H = 1] = \mathbb{P}[X_J = 1]$. We can thus ignore the terms with $m = 0$ or $m = 1$ in this sum.

\[
\frac{\sum_{m=2}^{2s+2} \sum_{|J \cap H| = m} \mathbb{P}[X_J = 1|X_H = 1] - \mathbb{P}[X_J = 1]}{E[X_s]}
\]

By Lemma 3.4, we obtain the bound

\[
\leq \frac{\sum_{m=2}^{2s+2} \sum_{|J \cap H| = m} p^{-m(m-1)/2} \mathbb{P}[X_J = 1] - \mathbb{P}[X_J = 1]}{E[X_s]}
\]

Since the probability $\mathbb{P}[X_J = 1]$ does not depend on $J$, we can use the bound from Lemma 3.5 to pull $\mathbb{P}[X_J = 1]/E[X_s]$ outside, and simplify the expression, where $C$ is a constant:

\[
\leq C n^{-2(s+1)} \sum_{m=2}^{2s+2} \sum_{|J \cap H| = m} p^{-m(m-1)/2} - 1
\]

Up to a constant, for a fixed $H$ there are $n^{2(s+1) - m}$ choices of $J$ where $|J \cap H| = m$. Absorbing those constants into our $C$ we get:

\[
\leq C n^{-2(s+1)} \sum_{m=2}^{2s+2} n^{2(s+1) - m} (p^{-m(m-1)/2} - 1)
\]

\[
= C \sum_{m=2}^{2s+2} n^{-m} (p^{-m(m-1)/2} - 1)
\]

\[
\leq C \sum_{m=2}^{2s+2} (np^{(m-1)/2})^{-m}
\]

Since $0 < (m - 1)/2 \leq s + \frac{1}{2}$ we have $np^{(m-1)/2} \to \infty$ by hypothesis. It follows that all of the finitely many terms in the sum go to 0, and thus $\text{Var}(X_s)/E[X_s]^2 \to 0$. \(\square\)

**Lemma 3.4.** Given $J, H \subset \Delta$ such that $|J \cap H| = m$

\[
\mathbb{P}[X_J = 1|X_H = 1] \leq p^{-m(m-1)/2} \mathbb{P}[X_J = 1]
\]

**Proof.** If $X_H = 1$ then the edges in $J \cap H$ are completely determined. If those edges do not match the required edges for $J$, then $\mathbb{P}[X_J = 1|X_H = 1] = 0$. If they do match the required edges, then since the probability of any edge existing or not existing is $p$ or $1-p$, and since $p \leq 1 - p$, we get that $\mathbb{P}[X_J = 1|X_H = 1] \leq p^{-m(m-1)/2} \mathbb{P}[X_J = 1]$ \(\square\)

**Lemma 3.5.** For any fixed $H$, we have $\mathbb{P}[X_H = 1]/E[X_s] \leq C n^{-2(s+1)}$ for some constant $C$. 

Proof. Since \( X_s = \sum_H X_H \) we have \( \mathbb{E}[X_s] = \sum_H \mathbb{P}[X_H = 1] \). But since \( \mathbb{P}[X_H = 1] \) does not depend on \( H \), this amounts to counting the number of possible choices of \( H \). Each \( H \) corresponds to \( s + 1 \) pairs of points in \( \Delta \), of which there \( \frac{1}{(s+1)!} \binom{n}{2,2,\ldots,n-2(s+1)} \) choices. It follows that, for an appropriate constant \( C \), we have \( \mathbb{P}[X_H = 1]/\mathbb{E}[X_s] \leq C n^{-2(s+1)} \).

4. Betti Number Thresholds

In this section, we determine thresholds of nonvanishing for individual Betti numbers. Lemma 2.3 shows that \( \beta_i,v(S/I_\Delta) = 0 \) whenever \( v \leq i \) or \( v \geq 2i \), and Theorem 1.7 computes thresholds in the remaining cases. To prove that theorem, we first bound the expected values of the Betti numbers. For \( \Delta \sim \Delta(n,p) \) we define \( B_{i,v} \) where \( B_{i,v}(\Delta) := \beta_i,v(S/I_\Delta) \).

By convention, when \( s = 0 \) we interpret \( \frac{1}{n/s} \) as a trivial bound.

**Lemma 4.1.** Fix any constant \( 0 < \epsilon < 1 \). Let \( \frac{1}{n/s} \ll p \leq \epsilon \) and \( \Delta \sim \Delta(n,p) \). We have \( \mathbb{E}[B_{s+1,2s+2}] \to \infty \) as \( n \to \infty \).

Proof. By Hochster’s formula [BH93, Theorem 5.5.1], since \( \tilde{H}_s(\Diamond_s) \neq 0 \), we have \( \mathbb{E}[B_{s+1,2s+2}] \geq \sum_H \mathbb{E}[X_H] \) where as in Definition 3.2, \( H \) is a set of \( s + 1 \) pairs of vertices, all distinct. Since each \( \Diamond_s \) involves \( s(2s + 2) \) edges and \( s + 1 \) non-edges, we have

\[
\mathbb{E}[X_H] = \mathbb{P}[X_H = 1] = p^{s(2s+2)}(1-p)^{s+1}.
\]

As in the proof of Lemma 3.5, the number of choices for \( H \) is at least \( C n^{2s+2} \) for some positive constant \( C \), and thus

\[
\mathbb{E}[B_{s+1,2s+2}] = \sum_H \mathbb{E}[X_H] \geq C n^{2s+2} p^{s(2s+2)}(1-p)^{s+1} \geq C'(n p^s)^{2s+2}.
\]

where \( C' = C(1-\epsilon)^{s+1} \). Since \( n p^s \to \infty \) it follows that \( \mathbb{E}[B_{s+1,2s+2}] \to \infty \).

To prove the other threshold, we introduce new random variables.

**Definition 4.2.** Let \( Y^s_n = Y^s_n(n,p) \) be the number of subgraphs with \( m \leq v \) vertices and at least \( ms \) edges. If \( K \) is a subset of \( m \) vertices, we let \( Y^s_K \) be the indicator random variable for whether the subgraph on \( K \) has at least \( ms \) edges.

**Lemma 4.3.** If \( p \ll 1/n^{1/s} \) then \( \mathbb{E}[B_{i,v}] \to 0 \).

Proof. Lemma 2.4 shows that if \( K \) is a minimal subset of vertices of \( \Delta \) such that \( \tilde{H}_s(\Delta|_K) \neq 0 \), then each vertex in \( \Delta|_K \) has degree \( \geq 2s \). In particular, if \( \beta_{i,v}(S/I_\Delta) \neq 0 \), then there must exist some subgraph \( K \) of size at most \( v \) (and with at least \( 2s + 2 \) vertices) where every vertex has degree \( \geq 2s \). It thus suffices to prove that \( \mathbb{E}[Y^s_n] \to 0 \).

We have \( Y^s_v = \sum_{|K| \leq v} Y^s_K \). For a fixed \( K \) with \( |K| = m \), we want to compute the probability that \( \Delta|_K \) has at least \( ms \) edges. We use \( M := \binom{m}{2} \) to denote the maximal number of possible edges. We thus have

\[
\mathbb{P}[Y^s_K = 1] = \sum_{e=ms}^{M} \binom{M}{e} p^e (1-p)^{M-e}.
\]
We then compute:

\[
\mathbf{E}[Y_v^s] = \sum_{m=2s+2}^v \sum_{K:|K|=m} \mathbf{P}[Y_K^s = 1]
\]

\[
= \sum_{m=2s+2}^v \binom{n}{m} \sum_{e=m,s}^M \binom{M}{e} p^e (1-p)^{M-e}
\]

\[
\leq \sum_{m=2s+2}^v \binom{n}{m} \sum_{e=m,s}^M \binom{M}{e} p^e
\]

\[
\leq \sum_{m=2s+2}^v n^m p^m C_{s,m} = \sum_{m=2s}^v (np^s)^m C_{s,m}
\]

However, we can bound \(\sum_{e=m,s}^M \binom{M}{e} p^e \) by a constant \(C_{s,m}\) depending only on \(s\) and \(m\), and we can bound \(\binom{n}{m}\) by \(n^m\). This yields:

\[
\leq \sum_{m=2s}^v n^m p^m C_{s,m} = \sum_{m=2s}^v (np^s)^m C_{s,m}
\]

Finally, since \(np^s \to 0\) by assumption, we conclude that \(\mathbf{E}[Y_v^s] \to 0\). \(\square\)

**Proof of Theorem 1.7.** For statement (1), we first consider the case where \(v = 2i = 2s + 2\). Lemma 4.1 implies that \(\mathbf{E}[B_{s+1,2s+2}] \to \infty\). Thus to prove that \(\mathbf{P}[B_{s+1,2s+2} \neq 0] \to 1\), we may bound the variance of \(B_{s+1,2s+2}\), which is done in Lemma 3.3. Combining this bound with Chebyshev’s Inequality yields the result, as in the proof of [Dur10, Theorem 2.2.4].

We now let \(v < 2i\). The case \(v = 2s + 2\) implies the existence of an \(\diamondsuit_s \subseteq \Delta\) with probability \(1-o(1)\). Fix some vertex \(u \in \diamondsuit_s\). Let \(J\) be the set of vertices \(w \in \Delta\) which don’t lie in \(\diamondsuit_s\) and which are not connected with \(u\). Since the complement of \(\diamondsuit_s\) consists of \(n - (2s + 2)\) vertices, the expected number of vertices in \(J\) is \((n - (2s + 2))(1-p) = n - o(n)\). Moreover, since those conditions are independent, the Weak Law of Large Numbers implies that this happens with high probability. Let \(J' \subseteq J\) be any subset of cardinality \(v-(2s+2)\). Since the only edges in \(\diamondsuit_s \cup J'\) through the vertex \(u\) are the ones from \(\diamondsuit_s\), it follows \(\bar{H}_v(\diamondsuit_s \cup J')\) is still nonzero. Hence \(B_{i,v} \neq 0\) with high probability as desired.

For statement (2), we must show that \(B_{i,v}\) converges to 0 in probability. Hochster’s formula [BH93, Theorem 5.5.1] implies that \(\beta_{i,v}(S/I\Delta)\) is nonzero if and only there is some subset \(K \subseteq \Delta\) with \(|K| = v\) and where \(\bar{H}_{v-i-1}(\Delta|K) \neq 0\). By Lemma 2.4 it suffices to show that \(\mathbf{P}[Y_v^s = 0] \to 1\) for \(s = v - i - 1\). But by Lemma 4.3, we know \(\mathbf{E}[Y_v^s] \to 0\), and since \(Y_v^s \geq 0\) and \(Y_v^s\) takes integer values, this implies that \(\mathbf{P}[Y_v^s = 0] \to 1\). \(\square\)

5. **Ein-Lazarsfeld Asymptotic Nonvanishing of Syzygies**

Whereas Theorem 1.7 provides the nonvanishing thresholds for individual Betti numbers, Question 1.2 asks about the simultaneous nonvanishing of more and more Betti numbers as \(n \to \infty\). However, as we now illustrate, the proof of Theorem 1.7 is sufficiently strong to obtain simultaneous nonvanishing of the various Betti numbers.
Proof of Theorem 1.4. For any \( 0 \leq s \leq r \), the proof of Theorem 1.7 implies the existence of an \( \diamondsuit_s \) with probability \( 1 - o(1) \). Moreover, we can assume with high probability that these all occur simultaneously and that the vertices involved in \( \diamondsuit_0, \diamondsuit_1, \ldots, \diamondsuit_r \) are all disjoint.

Fix some \( 0 < \epsilon < 1 \). For each \( 0 \leq s \leq r \), fix some vertex \( v \in \diamondsuit_s \). Since the complement of \( \bigcup_{s=0}^{r} \diamondsuit_s \) consists of \( n - O(1) \) vertices, the expected number of vertices \( w \notin \bigcup_{s=0}^{r} \diamondsuit_s \) that are not connected with vertex \( v \) is \( (n - O(1))(1 - p) \geq n - n^{1-\epsilon} \), at least for \( n \) sufficiently large. Since those conditions are independent, the Weak Law of Large Numbers implies that this happens with high probability. Call that set \( J \) and \( J' \subseteq J \) be any subset. Since the only edges in \( \diamondsuit_s \cup J' \) through the vertex \( v \) are the ones from \( \diamondsuit_s \), it follows \( \bar{H}_s(\Delta|_{\diamondsuit_s \cup J'}) \) is still nonzero. Since \( |\diamondsuit_s \cup J'| \) ranges from \( 2s + 2 \) to \( n - n^{1-\epsilon} + 2s + 2 \), it follows that \( \beta_{s+1,s+2}(S/I_\Delta) \neq 0 \) for all \( s \leq i \leq n - n^{1-\epsilon} + s \) with high probability. In particular, with high probability we have

\[
\lim_{n \to \infty} \rho_{s+1}(S/I_\Delta) \geq \lim_{n \to \infty} \frac{n - n^{1-\epsilon} + 1}{n} = 1.
\]

Moreover, since the \( \diamondsuit_s \) involve disjoint vertices, these nonvanishing conditions are independent in \( s \), and we thus obtain the desired convergence of \( \rho_{s+1} \) for all \( s \) simultaneously. \( \square \)

The proof of Theorem 1.4 shows that once we cross the threshold for the appearance of subcomplexes of the form \( \diamondsuit_s \), we get nonvanishing across nearly the entire \( (s+1) \)th row of the Betti table. The appearance of \( \diamondsuit_s \) thus account for why \( \rho_{s+1}(S/I_\Delta) \) goes to 1.

Example 5.1. Here is the Betti table of \( S/I_\Delta \) for a randomly chosen \( \Delta \sim \Delta(18, \frac{1}{18^{0.6}}) \), as computed in Macaulay2 [M2].

\[
\begin{array}{cccccccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\hline
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & 126 & 1203 & 5986 & 19491 & 45278 & 78385 & 103667 & 106356 & 85548 & 54408 & 27541 & 11118 & 3550 & 873 & 156 & 1 \\
& . & . & 1 & 24 & 233 & 1282 & 4568 & 11261 & 19911 & 25743 & 24538 & 17229 & 8815 & 3204 & 786 & 117 & 8 \\
\end{array}
\]

As predicted by Theorem 1.4, the entries in rows 1 and 2 are almost all nonzero. \( \square \)

Though we do not compute a precise threshold for the Castelnuovo-Mumford regularity of \( S/I_\Delta \), we do obtain a linear bound.

Corollary 5.2. Let \( \frac{1}{n^{1/r}} \ll p \ll \frac{1}{n^{2/(2r+1)}} \). Then with high probability \( r + 1 \leq \text{reg}(S/I_\Delta) \leq 2r \).

Proof. Since \( \frac{1}{n^{1/r}} \ll p \) we have that \( \beta_{r+1,2r+2}(S/I_\Delta) \neq 0 \) and thus \( \text{reg}(S/I_\Delta) \geq r \), with high probability. For the other direction, we let \( s = 2r + 1 \) so that \( p \ll \frac{1}{n^{2/r}} \). A simple computation shows that the expected number of \( (s+1) \)-cliques in \( \Delta \) is

\[
\left( \begin{array}{c}
  n \\
  s + 1
\end{array} \right) p^{\frac{s+1}{2}} \leq n^{s+1} \left( \frac{p^{s/2}}{2} \right)^{s+1} \ll n^{s+1} (n^{-1})^{s+1} = 1.
\]

Since the expected number of \( (s+1) \)-cliques goes to zero, it follows that with high probability \( \Delta \) has no subcomplex with \( (s+1) \)th homology and thus \( \text{reg}(S/I_\Delta) < s = 2r + 1 \). \( \square \)

Question 5.3. Does \( \text{reg}(S/I_\Delta) \) converge in probability (with appropriate conditions on \( p \))? More precisely, if \( \frac{1}{n^{1/r}} \ll p \ll \frac{1}{n^{4/(r+1)}} \) does \( \text{reg}(S/I_\Delta) \) converge to \( r + 1 \) in probability?
6. Normal Distribution of Quadratic Strand

In this section, we prove Theorem 1.5 and Corollary 1.6, which show that the entries in the first row of the Betti table of $S/I_\Delta$ look like a normal distribution.

Remark 6.1. For $\Delta$ as in Theorem 1.5, the second row of the Betti table of $S/I_\Delta$ is interesting as well, because $p = c/n$ is on a boundary case for the nonvanishing in Theorem 1.4. In [ER60, Theorem 5b], they prove that the 1-skeleton of $\Delta$ will contain a cycle with probability $1 - \sqrt{1 - e^{(c^2)/2}}$. Among graphs containing at least one cycle, an argument similar to the proof of Theorem 1.4 yields $n - n^{1-\epsilon}$ nonzero entries in the second row of the Betti table of $S/I_\Delta$, and thus in this case, $S/I_\Delta$ will have overlapping Betti numbers throughout two rows, similar to the case of a smooth surface in Theorem 1.1.

Proof of Theorem 1.5. Since the first row of the Betti table only depends on the 1-skeleton of $\Delta$, we abuse notation by equating $\Delta$ with its underlying graph.

We claim that we can safely ignore flag complexes with high unlikely properties, as ruling out those properties will clarify the proofs. To justify this, we start by providing a lower bound for the expected value of $B_{i_n,i_n+1}$. Since $\beta_{i_n,i_n+1}(S/I_\Delta)$ is a sum of multigraded Betti numbers, $B_{i_n,i_n+1}$ is a sum of (dependent) random variables $B_\Delta$ where $\Delta' \sim \Delta(i_n, C_n)$ is a random flag complex on $i_n$ vertices. By linearity of expectations, we have

$$\mathbb{E}[B_{i_n,i_n+1}] = \sum_{\Delta' \in \binom{n}{i_n}} \mathbb{E}[B_{\Delta'}] = \binom{n}{i_n} \mathbb{E}[B_{\Delta'}].$$

We are focused on row 1 of the Betti table, so $\mathbb{E}[B_{\Delta'}]$ is the expected value of $\tilde{H}_0(\Delta')$, which is at least the number of vertices in $\Delta'$ minus the number of edges. Since $\Delta'$ has $i_n = \frac{n}{2} + o(n)$ vertices, and since the expected number of edges in $\Delta'$ is $\frac{c}{n} \cdot \binom{i_n}{2}$, we see that $\mathbb{E}[B_{\Delta'}] \geq \frac{c}{4}n + o(n)$. Thus

$$\left(\frac{c}{4}n + o(n)\right) \binom{n}{i_n} \leq \mathbb{E}[B_{i_n,i_n+1}] \leq \frac{n}{2} + o(n) \binom{n}{i_n},$$

where the inequality on the right follows from the maximal possible value for $\beta_{i_n,i_n+1}(S/I_\Delta)$, which is $i_n \cdot \binom{n}{i_n} = \left(n/2 + o(n)\right) \binom{n}{i_n}$. Flag complexes that arise with probability $o(1)$ can not affect the expected value of $B_{i_n,i_n+1}$ by more than $o(1)$ times this maximal value, and since that produce is $o(\mathbb{E}[B_{i_n,i_n+1}])$, we can safely ignore any flag complexes arising with probability $o(1)$ in our expected value computation.

With probability $1 - o(1)$ a random $\Delta \sim \Delta(n, C_n)$ with $c < 1$ will be the disjoint union of trees and components with a single cycle [FK16, p. 31]. Moreover, since the expected number of cycles is constant when $c < 1$, we conclude that with high probability, $\Delta$ has at most $n^{1-\epsilon}$ cycles for any fixed $0 < \epsilon < 1$. We thus restrict attention to the case where $\Delta$ is the disjoint of trees and at most $n^{1-\epsilon}$ components each with a single cycle. We denote this restricted distribution of flag complexes by $\tilde{\Delta}(n, C_n)$ and henceforth choose $\Delta \sim \tilde{\Delta}(n, C_n)$.

To prove the main result, we introduce several auxiliary random variables. For a simplicial complex $\Delta$, we now set $E(\Delta)$ to be the number of edges in $\Delta$ and we define $C(\Delta)$ to be the number of cycles in $\Delta$. Finally, for a pair of vertices $e \in \binom{n}{2}$, we define $Z_e$ to be the indicator random variable of whether that pair of vertices is an edge in $\Delta$. 
With this notation, and using our assumption that $\Delta$ is a disjoint union of trees and components containing a single cycle, we have

$$B_{i_n,i_{n+1}} = \sum_{\alpha \in \binom{[n]}{i_n}} i_n - E(\Delta|\alpha) + C(\Delta|\alpha)$$

Ignoring the cycles, we get

$$\geq \sum_{\alpha \in \binom{[n]}{i_n}} i_n - E(\Delta|\alpha) = \binom{n}{i_n} i_n - \sum_{\alpha \in \binom{[n]}{i_n}} E(\Delta|\alpha).$$

We may rewrite the righthand sum in terms of individual edges to obtain

$$= \binom{n}{i_n} i_n - \sum_{e \in \binom{[n]}{2}} \left( \binom{n}{i_n-2} Z_e \right).$$

But $E(\Delta)$ is the sum of the $Z_e$, and thus we have:

$$= \binom{n}{i_n} i_n - \binom{n}{i_n} \left( \binom{n}{i_n-2} E(\Delta) \right).$$

By a similar argument, but where we do not ignore $C(\Delta|\alpha)$, we can use the fact that $\Delta$ has at most $n^{1-\epsilon}$ cycles to obtain an upper bound $B_{i_n,i_{n+1}} \leq \left( \binom{n}{i_n} i_n - \binom{n}{i_n-2} (E(\Delta) - n^{1-\epsilon}) \right).$

(6.2)  $$\binom{n}{i_n} i_n - \binom{n}{i_n-2} E(\Delta) \leq B_{i_n,i_{n+1}} \leq \binom{n}{i_n} i_n - \binom{n}{i_n-2} \left( E(\Delta) - n^{1-\epsilon} \right).$$

We have $\binom{n}{i_n-2} = \binom{n}{i_n} \frac{i_n(i_n-1)}{(n-i_n+2)(n-i_n+1)}$ and since $i_n = n/2 + o(n)$ this yields that $\binom{n}{i_n-2} = \binom{n}{i_n}(1 + o(1))$. Applying this to Equation (6.2) yields:

$$\binom{n}{i_n} \left( i_n - (1 + o(1)) E(\Delta) \right) \leq B_{i_n,i_{n+1}} \leq \binom{n}{i_n} \left( i_n - (1 + o(1)) \left( E(\Delta) - n^{1-\epsilon} \right) \right).$$

Recall that $C := \frac{1-c}{2}$. We then divide through by $\frac{1}{Cn(i_n)}$. Rewriting $i_n = n/2 + o(n)$, and absorbing the $n^{1-\epsilon}$ term into the $o(n)$ the lefthand and righthand bounds have the same form, and we obtain

$$\frac{B_{i_n,i_{n+1}}}{Cn(i_n)} = \frac{(n/2) - E(\Delta) + o(n) + o(1)E(\Delta)}{Cn}$$

Since $E(\Delta)$ is a sum of independent random variables, one for each potential edge, the variance is $\binom{n}{2} p(1-p)$. We may thus apply a Weak Law of Large Number, for instance [Dur10, Theorem 2.2.4] with $b_n = Cn$, to see that $E(\Delta)$ converges to its expected value $\frac{c(n-1)}{2}$ in probability. This implies that $\frac{B_{i_n,i_{n+1}}}{Cn(i_n)} \rightarrow 1$ in probability.
Proof of Corollary 1.6. Let $C = \frac{1-c}{2}$. Using Theorem 1.5(2) and the normal approximation of the binomial distribution, e.g. [Boas (8.3), p. 762], we obtain that

$$\beta_{i_n, i_{n+1}}(S/I_\Delta) \sim C_n \binom{n}{i_n} \sim C_n \frac{2^{n+1}}{\sqrt{2\pi n}} e^{-a^2/2}.$$ 

Therefore we have

$$\frac{\sqrt{2\pi n}}{C_n 2^{n+1}} \beta_{i_n, i_{n+1}}(S/I_\Delta) = \frac{\sqrt{2\pi}}{(1-c)2^n \sqrt{n}} \beta_{i_n, i_{n+1}}(S/I_\Delta) \sim e^{-a^2/2}.$$ 

Since the righthand side is a constant, we have convergence in probability. □

Conjecture 6.3. In cases where Theorem 1.4 yields nonvanishing Betti numbers in row $k$, we conjecture that the $k$th row of the Betti table will be normally distributed, in a manner similar to Corollary 1.6.

7. Projective Dimension Estimates

We conclude with a corollary about Cohen-Macaulayness. For many values of $p$, we show that $S/I_\Delta$ will essentially never be Cohen-Macaulay. However, while the projective dimension almost never equals the codimension of $S/I_\Delta$, with high probability the ratio of these quantities converges to 1 as $n \to \infty$.

Corollary 7.1. For any $k \geq 1$, and any $p$ satisfying $\frac{1}{n^{2/3}} \ll p \ll \left(\frac{\log(n)}{n}\right)^{2/(k+3)}$ we have that

$$\frac{\text{codim}(S/I_\Delta)}{\text{pdim}(S/I_\Delta)} \to 1 \text{ in probability, yet the probability that } S/I_\Delta \text{ is Cohen-Macaulay goes to } 0.$$ 

First we prove a quick lemma bounding the dimension of $\Delta$.

Lemma 7.2. If $p \leq \epsilon$ for some $0 < \epsilon < 1$ then $P[\dim \Delta \geq \epsilon \cdot n] \to 0$ as $n \to \infty$.

Proof. The dimension of $\Delta$ is the size of the largest $k$-clique in $\Delta$. Let $N := \binom{n}{k}$. The expected number of $k$-cliques in $\Delta$ is $Np^N \leq Ne^N$, which goes to zero as $n \to \infty$. □

Note that [BE76, Theorem 1] provides a much sharper estimate of the dimension of $\Delta$, though we will not need that.

Proof of Corollary 7.1. Lemma 7.2 shows that $\dim \Delta = o(n)$ with high probability. By Auslander-Buchsbaum, this implies that

$$n - o(n) \leq \text{codim}(S/I_\Delta) \leq \text{pdim}(S/I_\Delta) \leq n.$$ 

Thus the ratio between $\text{pdim}(S/I_\Delta)$ and $\text{codim}(S/I_\Delta)$ goes to 1 in probability.

For the statement on Cohen-Macaulayness, using Reisner’s Criterion [BH93, Corollary 5.3.9] it suffices to show that there exists a vertex $v \in \Delta$ and an integer $i < \dim (\text{link}_\Delta(v))$ where $\hat{H}_i(\text{link}_\Delta(v)) \neq 0$. Note that for $\Delta \sim \Delta(n, p)$ and a vertex $v$, the link of $v$ is itself a random flag complex, namely $\text{link}_\Delta(v) \sim \Delta(np, p)$.

For convenience we write $m := np$. In terms of $m$ we can rewrite the left hand side of the original constraints on $p$ as $\frac{1}{m^2} \ll p$. For the right hand side of the constraint, since
\[ \frac{1}{n} \ll p, \text{ we have } \log(m) \sim \log(n) \text{ so we get } p \ll \left( \frac{\log(m)}{m} \right)^{2/(k+1)} \sim \left( \frac{\log(m)}{m} \right)^{2/(k+1)}. \] Thus the constraints in terms of \( m \) are

\[ \frac{1}{m^2} \ll p \ll \left( \frac{\log(m)}{m} \right)^{2/(k+1)}. \]

For \( 1 \leq t \leq k \), we consider the interval \( \frac{1}{m^{2/t}} \ll p \ll \left( \frac{\log(m)}{m} \right)^{2/(t+1)}. \) Since \( \frac{1}{m^{2/(t+1)}} \ll \left( \frac{\log(m)}{m} \right)^{2/(t+1)} \), the successive intervals overlap, and it suffices to show that for each of these intervals \( \Delta \) is not Cohen-Macaulay with probability approaching 1.

First let us consider the case where \( t \geq 2 \). Setting \( i := \lfloor t/2 \rfloor \) and applying [Kah14a, Theorem 1.1] we have \( \pi_i(\text{link}_\Delta(v)) \neq 0 \) with probability \( 1 - o(1) \). Since \( \frac{1}{m^{2/t}} \ll p \), there exist \((t+1)\)-cliques and thus \( \dim(\text{link}_\Delta(v)) \geq t \) with probability \( 1 - o(1) \). Together these implies that \( \Delta \) is not Cohen-Macaulay with probability \( 1 - o(1) \).

We now consider the case \( t = k = 1 \), where we have \( \frac{1}{m^2} \ll p \ll \frac{\log(m)}{m} \). Thus we apply [ER59, Theorem 1] to get \( \tilde{\pi}_0(\text{link}_\Delta(v)) \neq 0 \) with probability \( 1 - o(1) \). On the other hand, since \( \frac{1}{m^2} \ll p \), we have 2-cliques and thus \( \dim(\text{link}_\Delta(v)) \geq t \) with probability \( 1 - o(1) \). \( \square \)

**Remark 7.3.** If \( p \ll \frac{1}{n} \), then with high probability \( \Delta \) is a forest [FK16, Theorem 2.1]. Thus by Reisner’s Criterion [BH93, Corollary 5.3.9] \( S/I_\Delta \) is Cohen-Macaulay in this case. \( \square \)

**References**

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[CJKW] Aldo Conca, Martina Juhnke-Kubitzke, and Volkmar Welker, *Asymptotic syzygies of Stanley- Reisner rings of iterated subdivisions*. arXiv:1411.3695.

[DLPS] Jesús A. De Loera, Sonja Petrović, Lily Silverstein, Despina Stasi, and Dane Wilburne, *Random Monomial Ideals*. arXiv:1701.07130.

[Dur10] Rick Durrett, *Probability: theory and examples*, 4th ed., Cambridge Series in Statistical and Probabilistic Mathematics, vol. 31, Cambridge University Press, Cambridge, 2010.

[EEL16] Lawrence Ein, Daniel Erman, and Robert Lazarsfeld, *A quick proof of nonvanishing for asymptotic syzygies*, Algebr. Geom. 3 (2016), no. 2, 211–222.

[EEL15] Lawrence Ein and Robert Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Invent. Math. 190 (2012), no. 3, 603–646.

[ERT94] David Eisenbud, Alyson Reeves, and Burt Totaro, *Initial ideals, Veronese subrings, and rates of algebras*, Adv. Math. 109 (1994), no. 2, 168–187.

[ER60] P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 5 (1960), 17–61 (English, with Russian summary).

[ER59] *On random graphs. I*, Publ. Math. Debrecen 6 (1959), 290–297.

[FK16] Alan Frieze and Michal Karoński, *Introduction to Random Graphs*, Cambridge University Press, 2016.

[Kah14a] Matthew Kahle, *Sharp vanishing thresholds for cohomology of random flag complexes*, Ann. of Math. (2) 179 (2014), no. 3, 1085–1107.

[Kah14b] Matthew Kahle, *Topology of random simplicial complexes: a survey*, Algebraic topology: applications and new directions, Contemp. Math., vol. 620, Amer. Math. Soc., Providence, RI, 2014, pp. 201–221.
[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at \texttt{http://www.math.uiuc.edu/Macaulay2/}. ↑1, 4, 10

[Pee11] Irena Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011. ↑5

[Zho14] Xin Zhou, *Effective non-vanishing of asymptotic adjoint syzygies*, Proc. Amer. Math. Soc. 142 (2014), no. 7, 2255–2264. ↑1

\textbf{Daniel Erman}: \textsc{Department of Mathematics, University of Wisconsin, Madison, Wisconsin, 53706, United States of America}; derman@math.wisc.edu

\textbf{Jay Yang}: \textsc{Department of Mathematics, University of Wisconsin, Madison, Wisconsin, 53706, United States of America}; yangjay@math.wisc.edu