Quantization of classical integrable systems  
Part II:  
quantization of functions on Poisson manifolds

M. Marino and N. N. Nekhoroshev  
Dipartimento di Matematica, Università degli Studi di Milano,  
via Saldini 50, I-20133 Milano (Italy)  
January 26, 2010

Abstract
In a previous work we have introduced the concept of quasi-integrable  
quantum system. In the present one we determine sufficient conditions un- 
der which, given an integrable classical system, it is possible to construct  
a quasi-integrable quantum system by means of a quantization procedure  
based on the symmetrized product of operators. This procedure will be  
applied to concrete classes of integrable systems in two following papers.

1 Introduction
In a previous paper [1] (see also references therein) we have introduced the con-  
cept of quasi-integrable quantum system. It represents a quantum equivalent  
of the concept of classical integrable system in the noncommutative sense, i.e.  
with a number of functionally independent first integrals generally greater than  
the dimension of the configuration space [2, 3, 4, 5]. As an application of this  
concept, in this paper we discuss about the mathematical basis of the quanti- 
ization by symmetrization. The simplest situation, among those considered in  
the paper, is the following. There exists a one-to-one correspondence between  
given set of functions on an abstract Poisson manifold, and a given set of op- 
erators on some configuration space. This correspondence preserves the linear  
operations, and to the Poisson bracket of two functions it associates the com- 
mutator of the corresponding operators. We then consider the extension of this  
correspondence, which associates with the (commutative) product of functions  
the symmetrized product of the corresponding operators, where the product of  
operators is defined as their composition. We investigate under which condi- 
tions this extension preserves the correspondence between the Poisson brackets  
of functions on one side, and the commutators of the corresponding operators on  
the other. In this way, we find conditions on a classical integrable system which  
allow a corresponding quantum integrable system to be constructed by means  
of symmetrization. The main condition is the existence of a set of functions  
$B = (B_1, \ldots, B_l)$ on a symplectic manifold $M^{2n}$, which is a basis of a finite  
dimensional Lie algebra with respect to Poisson brackets, and moreover has the  
following property. There exists an “integrable” set of functions of the type

1
considered in [1], $F = (F_1, \ldots, F_{2n-k})$, which are polynomials of the functions of the set $B$ with constant coefficients: $F_j = P_j(B)$, $j = 1, \ldots, 2n - k$. Two main cases are considered: the general one and the case in which $\{B_i, B_j\} = c_{ij}$, where $c_{ij}$ are constants. In the general case a sufficient condition for the construction of an integrable quantum system is that the polynomials $P_1, \ldots, P_k$ are of first degree, i.e., $F_1, \ldots, F_k$ are linearly dependent on $B$:

$$F_j = \sum_{i=1}^l d_{ji} B_i + b_j,$$

where $c_{ji}$ and $b_j$ are constants. In the latter case instead, a quadratic dependence on the functions of the set $B$ is sufficient, that is $\deg P_i \leq 2$, $i = 1, \ldots, k$.

A typical example of the second case is when $B$ is the set (or a subset) of the linear canonical coordinates in the symplectic manifold $\mathbb{R}^{2n}$. In this case $l = 2n$. Note that we fix the Planck constant $\hbar = 1$ throughout the article, and correspondingly we never consider the limit $\hbar \to 0$. In particular, we consider only exact quantization, which means that if three functions $A, B, C$ satisfy the equality $\{A, B\} = C$, then their quantizations $\hat{A}, \hat{B}, \hat{C}$ satisfy the exact equality $[\hat{A}, \hat{B}] = \hat{C}$ without any additional term which tends to 0 as $\hbar \to 0$ (as for example in [6, 7]).

In quantum physics, the main source of integrable sets of operators is the quantization of integrable sets of functions on symplectic manifolds. The following statement is generally true: if a classical system is integrable, then the corresponding quantum system is also integrable. This refers to systems which describe real phenomena of nature, because artificial examples of systems which contradict this rule can be apparently constructed. Sometimes this rule has a formal basis. In this paper we consider some significant cases in which, starting from integrable classical systems, one can easily construct integrable quantum systems. A fundamental role in this construction is played by the so-called “symmetrization” of products of operators corresponding to classical functions on a symplectic manifold. Hence we begin with the study of some general properties of symmetrized products in an abstract associative algebra.

## 2 Symmetrized products in an associative algebra

Let us consider an arbitrary associative algebra $\mathfrak{A}$, for example an algebra of linear operators with composition as product. The operation of commutation $[A, B] := AB - BA$ between elements $A$ and $B$ of the algebra $\mathfrak{A}$ transforms this algebra into a Lie algebra. A commutator can be considered as (twice) the “antisymmetrized product” of two elements of the algebra. It is then natural to introduce also a “symmetrized product”.

**Definition 2.1.** The operation which associates with the pair $(A, B)$ the element

$$\text{Sym}_2(A, B) := \frac{1}{2}(AB + BA) \quad (2.1)$$

is called **symmetrized product** of the associative algebra $\mathfrak{A}$. We shall denote this operation between two elements also with the symbol $\diamond$:

$$A \diamond B := \text{Sym}_2(A, B) = B \diamond A.$$
We also call \textit{symmetrized product of the set} $(A_1, A_2, \ldots, A_k)$ of $k$ elements of the algebra $\mathfrak{A}$ the expression
\[ \text{Sym}_k(A_1, A_2, \ldots, A_k) := \frac{1}{k!} \sum_{\pi \in \Pi_k} A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(k)}, \]
where $\Pi_k$ is the set of all $k!$ permutations $\pi$ of $k$ objects.

The symmetrized product is distributive with respect to the sum, but it is not associative. In general we have in fact
\[ A_1 \circ (A_2 \circ A_3) \neq (A_1 \circ A_2) \circ A_3. \]

Moreover, both members of the above inequality are in general different from $\text{Sym}_3(A_1, A_2, A_3)$, see lemma 2.2.

Let us consider the linear space $S'_C = S'^{0, l}_C$ of all commutative polynomials $P = P(B)$ of $l$ variables. By making use of the symmetrized product, it is possible to associate with any element $P \in S'^{0, l}_C$ a noncommutative polynomial $P_{\text{sym}} \in S'_N = S'^{0, l}_N$ (see definition of noncommutative polynomial in \cite{1}) which is symmetrical with respect to any permutation of its $l$ arguments. Consider the linear map $\text{sym}: S'_C \rightarrow S'_N$, which associates with an arbitrary monomial $Q = B_{i_1} B_{i_2} \cdots B_{i_k}$ the symmetrized product
\[ Q_{\text{sym}} := \text{Sym}_k(B_{i_1}, B_{i_2}, \ldots, B_{i_k}). \]

The set of all monomials $Q$ is a basis of the linear space $S'_C$. Therefore the mapping $Q \mapsto Q_{\text{sym}}$ defines a linear map $\text{sym}: P \mapsto P_{\text{sym}}$ on the full space $S'_C$.

\textbf{Definition 2.2.} We call the noncommutative polynomial $P_{\text{sym}} \in S'_N$, associated with the commutative polynomial $P(B) \in S'^{0, l}_C$ by the linear map defined above, the \textit{symmetrization} of the polynomial $P = P(B_1, \ldots, B_l)$.

The abelianization of $P_{\text{sym}}$ (see definition of abelianization in \cite{1}) obviously coincides with $P$. The map $\text{sym}$ defined above is a one-to-one linear map from $S'_C$ to $S'_N$, where $S'_N$ denotes the linear space of all symmetrical noncommutative polynomials of $l$ variables. The restriction to $S'_N$ of the abelianization $\mathcal{T}$ represents the inverse map of $\text{sym}$.

The symmetrization of polynomials will play an essential role in the following of this paper, since it is the main tool which we shall employ for the quantization of classical hamiltonian systems. However, before putting this tool at work, we need preliminarily some important results about the algebraic relations between commutators and symmetrized products in the associative algebra $\mathfrak{A}$. In the following propositions, $A$ and $B$ (with suitable indexes when necessary) will denote generic elements of $\mathfrak{A}$.

\textbf{Proposition 2.1.} \textit{We have}
\[ [A, \text{Sym}_k(B_1, B_2, \ldots, B_k)] = \text{Sym}_k([A, B_1], B_2, \ldots, B_k) + \text{Sym}_k(B_1, [A, B_2], \ldots, B_k) + \cdots + \text{Sym}_k(B_1, B_2, \ldots, [A, B_k]). \] (2.2)

\textit{Proof.} The thesis follows from the algebraic identity
\[ [A, B_1 B_2 \cdots B_k] = [A, B_1] B_2 \cdots B_k + B_1 [A, B_2] B_3 \cdots B_k + \cdots + B_1 B_2 \cdots B_{k-1} [A, B_k], \]
Lemma 2.2. We have

\[ A \diamond (B_1 \diamond B_2) = \text{Sym}_3(A, B_1, B_2) \]

\[ + \frac{1}{12} ([[A, B_1], B_2] + [[A, B_2], B_1]) \]  

(2.3)

and

\[ A \diamond \text{Sym}_3(B_1, B_2, B_3) = \text{Sym}_4(A, B_1, B_2, B_3) \]

\[ + \frac{1}{12} \left\{ B_1 \diamond ([[A, B_2], B_3] + [[A, B_3], B_2]) + B_2 \diamond ([[A, B_1], B_3] + [[A, B_3], B_1]) + B_3 \diamond ([[A, B_1], B_2] + [[A, B_2], B_1]) \right\} . \]

(2.4)

The above identities are particular cases of the general formula

\[ A \diamond \text{Sym}_k(B_1, B_2, \ldots, B_k) = \text{Sym}_{k+1}(A, B_1, B_2, \ldots, B_k) \]

\[ + \sum_{h=1}^{[k/2]} c_h C_{k,h}(A, B_1, B_2, \ldots, B_k), \]  

(2.5)

where \([k/2]\) indicates the integer part of \(k/2\). In the above formula we put

\[ C_{k,h}(A; B_1, B_2, \ldots, B_k) := \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2h}} \text{Sym}_{k-2h+1} \left( \left[ \begin{array}{c} 2h \end{array} \right] \left( A, B_{i_1}, B_{i_2}, \ldots, B_{i_{2h}} \right), B_{j_1}, B_{j_2}, \ldots, B_{j_{k-2h}} \right) , \]  

(2.6)

where the symbol \([2h] \) stands for \(2h\) open square brackets \(([\cdots]\) in a row, and indexes \(j_1, \ldots, j_{k-2h}\) have to be taken so that \((i_1, \ldots, i_{2h}) \cup (j_1, \ldots, j_{k-2h}) = (1, \ldots, k).\) Finally, the coefficients \(c_h\) in (2.5) are determined recursively by the relations

\[ c_1 = \frac{1}{12}, \quad c_h = -\frac{1}{2h+1} \sum_{i=1}^{h-1} c_i c_{h-i} \quad \text{for } h > 1. \]  

(2.7)

It follows in particular that

\[ c_2 = -\frac{1}{720}, \quad c_3 = \frac{1}{30240}, \quad c_4 = -\frac{1}{1209600}, \quad c_5 = \frac{1}{47900160}, \quad \cdots \]

Using (2.7) it can be shown that \(c_h = B_{2h}/(2h)!\), \(h = 1, 2, \ldots\), where \(B_h\) are the so-called Bernoulli numbers.

Proof. For any set of elements \(C_0, C_1, \ldots, C_k \in \mathcal{A}\), let us define

\[ A_k(C_0, C_1, C_2, \ldots, C_k) := C_0 \circ \text{Sym}_k(C_1, C_2, \ldots, C_k) \]

\[ - \text{Sym}_{k+1}(C_0, C_1, C_2, \ldots, C_k). \]  

(2.8)

The above expression is obviously symmetric with respect to any permutations of \(C_1, C_2, \ldots, C_k\). By symmetry we have also

\[ C_0 \circ \text{Sym}_k(C_1, C_2, \ldots, C_k) + C_1 \circ \text{Sym}_k(C_0, C_2, \ldots, C_k) \]

\[ + C_2 \circ \text{Sym}_k(C_1, C_0, C_3, \ldots, C_k) + \cdots + C_k \circ \text{Sym}_k(C_1, \ldots, C_{k-1}, C_0) \]

\[ = (k+1)\text{Sym}_{k+1}(C_0, C_1, C_2, \ldots, C_k), \]  

(2.9)
whence

\[ A_k(C_0; C_1, C_2, \ldots, C_k) + A_k(C_1; C_0, C_2, \ldots, C_k) + A_k(C_2; C_1, C_0, C_3, \ldots, C_k) + \cdots + A_k(C_k; C_1, \ldots, C_{k-1}, C_0) = 0. \]  

(2.10)

We shall prove by induction that

\[ A_k(A; B_1, B_2, \ldots, B_k) = \sum_{h=1}^{|k/2|} c_h C_{k,h}(A; B_1, B_2, \ldots, B_k), \]  

(2.11)

which is equivalent to (2.3). For \( k = 2 \) we have by direct calculation

\[
A_2(A; B_1, B_2) = \frac{1}{12} (AB_1B_2 + B_1B_2A + AB_2B_1 + B_2B_1A) \\
- \frac{1}{6} (B_1AB_2 + B_2AB_1) \\
= \frac{1}{12} ([A, B_1]B_2 + B_1[B_2, A] + [A, B_2]B_1 + B_2[B_1, A]) \\
= \frac{1}{12} ([B_1, B_2] + [[A, B_2], B_1]),
\]

which is equivalent to (2.3). This also implies

\[
B_1 \circ (B_2 \circ A) - B_2 \circ (B_1 \circ A) \\
= \frac{1}{12} (2[[B_1, B_2], A] + [[B_1, A], B_2] - [[B_2, A], B_1]) \\
= \frac{1}{4} [[B_1, B_2], A], \tag{2.12}
\]

where the last equality follows from Jacobi identity.

According to (2.8) we can write

\[
A \circ \text{Sym}_k(B_1, B_2, \ldots, B_k) = A \circ (B_1 \circ \text{Sym}_{k-1}(B_2, \ldots, B_k)) \\
- A \circ \text{A}_{k-1}(B_1; B_2, \ldots, B_k). \tag{2.13}
\]

From (2.12), (2.8) and proposition 2.1, it follows that

\[
A \circ (B_1 \circ \text{Sym}_{k-1}(B_2, \ldots, B_k)) \\
= B_1 \circ (A \circ \text{Sym}_{k-1}(B_2, \ldots, B_k)) + \frac{1}{4} [[A, B_1], \text{Sym}_{k-1}(B_2, \ldots, B_k)] \\
= B_1 \circ \text{Sym}_k(A, B_2, \ldots, B_k) + B_1 \circ A_{k-1}(A; B_2, \ldots, B_k) \\
+ \frac{1}{4} \{ \text{Sym}_{k-1}([[A, B_1], B_2], B_3, \ldots, B_k) + (\text{Sym}_{k-1}(B_2, [[A, B_1], B_3], \ldots, B_k) \\
+ \cdots + (\text{Sym}_{k-1}(B_2, \ldots, B_{k-1}, [[A, B_1], B_k])) \}.
\]

Let us substitute this expression into the right-hand side of (2.13), and then symmetrize with respect to \( B_1, \ldots, B_k \). Using (2.8), (2.9) and (2.10) we obtain

\[
(k + 1)A_k(A; B_1, B_2, \ldots, B_k) = B_1 \circ \text{A}_{k-1}(A; B_2, \ldots, B_k) \\
+ B_2 \circ \text{A}_{k-1}(A; B_1, B_3, \ldots, B_k) + \cdots + B_k \circ \text{A}_{k-1}(A; B_1, B_2, \ldots, B_{k-1}) \\
+ \frac{1}{4} \sum_{i_1 \neq i_2} \text{Sym}_{k-1}([[A, B_{i_1}], B_{i_2}], B_{i_1}, \ldots, B_{i_2})), \tag{2.14}
\]

(Continued on page 5)
where \((i_1, i_2) \cup (j_1, \ldots, j_{k-2}) = (1, \ldots, k)\).

Let us now use the hypothesis of induction, and assume that all elements
\(A_k\) have the form \((2.11)\) for \(h = 2, 3, \ldots, k - 1\). We want to begin by showing
that \(A_k(A; B_1, B_2, \ldots, B_k)\) contains the term
\[
c_1 C_{k,1}(A; B_1, B_2, \ldots, B_k)
= \frac{1}{12} \sum_{i_1 \neq i_2} \text{Sym}_{k-1}([A, B_{i_1}], [B_{i_2}], [B_j, B_{j_2}, \ldots, B_{j_{k-2}}])
\tag{2.15}
\]

For simplicity of notation, let us consider in the above sum only the term for
\(i_1 = k, i_2 = k - 1\). Using \((2.8)\) and \((2.10)\) we have
\[
B_{j_1} \circ \text{Sym}_{k-2}([A, B_k], B_{k-1}], B_{j_2}, B_{j_3}, \ldots, B_{j_{k-2}})
+ B_{j_2} \circ \text{Sym}_{k-2}([A, B_k], B_{j_1}], B_{j_3}, \ldots, B_{j_{k-2}})
+ \cdots + B_{j_{k-2}} \circ \text{Sym}_{k-2}([A, B_k], B_{j_1}], B_{j_2}, \ldots, B_{j_{k-3}})
= (k - 2)\text{Sym}_{k-1}([A, B_k], B_{k-1}], B_{j_1}, B_{j_2}, \ldots, B_{j_{k-2}})
- A_{k-2}([A, B_k], B_{k-1}], B_{j_1}, B_{j_2}, \ldots, B_{j_{k-2}}).
\]

Therefore, using the induction hypothesis in \((2.14)\), we easily see that
\(A_k(A; B_1, B_2, \ldots, B_k)\) contains the term
\[
c'_{i_1} \text{Sym}_{k-1}([A, B_k], B_{k-1}], B_{j_1}, B_{j_2}, \ldots, B_{j_{k-2}}),
\]
where
\[
c'_{i_1} = \frac{1}{k + 1} \left[ (k - 2)c_1 + \frac{1}{4} \right].
\]

Using the first of \((2.7)\) it follows that
\[
c'_{i_1} = \frac{1}{12} = c_1.
\]

The full expression \((2.15)\) can then be obtained by repeating the above argument
for all the other terms of the sum.

Similarly, for \(1 < h < \lfloor k/2 \rfloor\), let us consider all the terms on the right-hand
side of \((2.14)\) which can contribute to \(A_k(A; B_1, B_2, \ldots, B_k)\) a piece of the form
\[
\alpha \text{Sym}_{k-2h+1}([A, B_k], B_{k-1}], \ldots, B_{k-2h+1}], B_{j_1}, B_{j_2}, \ldots, B_{k-2h}\),
\tag{2.16}
\]
where \(\alpha\) is a numerical coefficient. For any \(j\) such that \(1 \leq j \leq h\), using \((2.8)\)
and \((2.10)\) we have
\[
B_{j_1} \circ \text{Sym}_{k-2j}([A, B_k], B_{k-1}], \ldots, B_{k-2j+1}], B_{j_2}, B_{j_3}, \ldots, B_{k-2j})
+ B_{j_2} \circ \text{Sym}_{k-2j}([A, B_k], B_{k-1}], \ldots, B_{k-2j+1}], B_{j_3}, B_{j_4}, \ldots, B_{k-2j}) + \cdots
+ B_{k-2j} \circ \text{Sym}_{k-2j}([A, B_k], B_{k-1}], \ldots, B_{k-2j+1}], B_{j_1}, B_{j_3}, \ldots, B_{k-2j-1})
= (k - 2j)\text{Sym}_{k-2j+1}([A, B_k], B_{k-1}], \ldots, B_{k-2j+1}, B_{j_1}, B_{j_3}, \ldots, B_{k-2j})
- A_{k-2j}([A, B_k], B_{k-1}], \ldots, B_{k-2j+1}], B_{j_1}, B_{j_3}, \ldots, B_{k-2j}).
\]

We see that the above expression contains a term of type \((2.16)\) with
\[
\alpha = \begin{cases} -c_{h-j} & \text{if } 1 \leq j < h, \\ k - 2h & \text{if } j = h. \end{cases}
\]
Therefore, we find using (2.13) that $A_k(A; B_1, B_2, \ldots, B_k)$ contains the term
\[
c'_k \text{Sym}_{k-2h+1} \left( [2h, A, B_k], [B_k-1], \ldots, [B_{k-2h+1}], B_1, B_2, \ldots, B_{k-2h} \right),
\]
where
\[
c'_k = \frac{1}{k+1} [-c_1 c_{k-1} - c_2 c_{k-2} - \cdots - c_{k-1} c_1 + (k - 2h)c_h].
\]

Using the second of (2.14) it follows that $c'_h = c_h$, in agreement with (2.11). The lemma is thus completely proved.

**Proposition 2.3.** We have
\[
[A_1 \circ A_2, B_1 \circ B_2] = \text{Sym}_3([A_1, B_1], A_2, B_2) + \text{Sym}_3([A_1, B_2], A_2, B_1)
+ \text{Sym}_3([A_2, B_1], A_1, B_2) + \text{Sym}_3([A_2, B_2], A_1, B_1) - \frac{1}{12} \left( [[A_1, B_1], B_2], A_2 \right)
+ \left( [[A_1, B_2], B_1], A_2 \right) + \left( [[A_2, B_1], B_2], A_1 \right) + \left( [[A_2, B_2], B_1], A_1 \right)
\]
and
\[
\begin{align*}
[A_1 \circ A_2, \text{Sym}_3(B_1, B_2, B_3)] &= \text{Sym}_4(A_1, [A_2, B_1], B_2, B_3)
+ \text{Sym}_4(A_1, [A_2, B_2], B_1, B_3) + \text{Sym}_4(A_1, [A_2, B_3], B_1, B_2)
+ \frac{1}{12} \sum_{\pi \in \Pi_3} \left( [[A_1, B_{\pi(1)}], B_{\pi(2)}] \circ [A_2, B_{\pi(3)}] - [[A_1, B_{\pi(1)}], B_{\pi(2)}], A_2 \circ B_{\pi(3)} \right)
+ A_1 \leftrightarrow A_2,
\end{align*}
\]
where $\Pi_3$ is the set of all 6 permutations $\pi$ of 3 objects, and the symbol $A_1 \leftrightarrow A_2$ means interchanging $A_1$ and $A_2$ in all preceding terms on the right-hand side of the equality. The above identities are particular cases of the general formula
\[
[A_1 \circ A_2, \text{Sym}_k(B_1, B_2, \ldots, B_k)] = \sum_{i=1}^{k} \text{Sym}_{k+1}(A_1, B_1, \ldots, B_{i-1}, [A_2, B_i], B_{i+1}, \ldots, B_k)
- \sum_{h=1}^{\lfloor k/2 \rfloor} c_h D_{k,h}(A_1; A_2; B_1, B_2, \ldots, B_k)
+ \sum_{h=1}^{(k-1)/2} c_h E_{k,h}(A_1; A_2; B_1, B_2, \ldots, B_k) + A_1 \leftrightarrow A_2,
\]
where
\[
D_{k,h}(A_1; A_2; B_1, B_2, \ldots, B_k) := \sum_{i_1 \neq \cdots \neq i_{2h}} \text{Sym}_{k-2h+1} \left( [2h+1, A_1, B_{i_1}], \ldots, [B_{i_{2h}}], A_2, B_{j_1}, \ldots, B_{j_{k-2h}} \right),
\]
and
\[
E_{k,h}(A_1; A_2; B_1, B_2, \ldots, B_k) := \sum_{i_1 \neq \cdots \neq i_{2h+1}} \text{Sym}_{k-2h+1} \left( [2h, A_1, B_{i_1}], \ldots, [B_{i_{2h+1}}], A_2, B_{j_1}, \ldots, B_{j_{k-2h-1}} \right),
\]
and the coefficients $c_h$ are determined recursively by relations (2.7). Indexes $j_1, \ldots, j_{k-2h}$ in (2.20) have to be taken so that $(i_1, \ldots, i_{2h}) = (1, \ldots, k)$. An analogous convention is used in (2.22).

Proof. By applying proposition 2.1 twice and lemma 2.2 we get

$$\begin{align*}
[A_1 \circ A_2, \text{Sym}_k(B_1, B_2, \ldots, B_k)] &= A_1 \circ [A_2, \text{Sym}_k(B_1, B_2, \ldots, B_k)] + A_2 \circ [A_1, \text{Sym}_k(B_1, B_2, \ldots, B_k)] \\
&= \sum_{i=1}^{k} A_1 \circ \text{Sym}_k(B_1, \ldots, B_{i-1}, [A_2, B_i], B_{i+1}, \ldots, B_k) + A_1 \leftrightarrow A_2 \\
&= \sum_{i=1}^{k} \text{Sym}_{k+1}(A_1, B_1, \ldots, B_{i-1}, [A_2, B_i], B_{i+1}, \ldots, B_k) \\
&\quad + \sum_{h=1}^{[k/2]} c_h \sum_{i=1}^{k} C_{k,h}(A_1; B_1, \ldots, B_{i-1}, [A_2, B_i], B_{i+1}, \ldots, B_k) + A_1 \leftrightarrow A_2.
\end{align*}$$

(2.22)

Let us consider all terms which, according to (2.6), are contained in $\sum_{h=1}^{k} C_{k,h}(A_1; \ldots, [A_2, B_i], \ldots)$ for a given value of $h$. Those terms in which $[A_2, B_i]$ does not appear inside the iterated commutators of formula (2.6) give rise to the expressions $E_{k,h}$ in (2.19). The sum of the terms in which $[A_2, B_i]$ appears inside the iterated commutators can instead be considerably simplified by making use of the identity

$$\begin{align*}
&[^{[2h]}A_1, [A_2, B_{i_1}], [B_{i_2}], \ldots, [B_{i_{2h}}]] + [^{[2h]}A_1, [A_2, B_{i_2}], [B_{i_3}], \ldots, [B_{i_{2h}}]] \\
&\quad + \cdots + [^{[2h]}A_1, [B_{i_1}], [B_{i_2}], \ldots, [B_{i_{2h}}]] - [^{[2h+1]}A_1, [B_{i_1}], [B_{i_2}], \ldots, [B_{i_{2h}}], [A_2]].
\end{align*}$$

(2.23)

This identity is derived by splitting each term on the left-hand side according to the formula

$$\begin{align*}
&[^{[m]}A_1, [B_{i_1}], [B_{i_2}], \ldots, [B_{i_{m-1}}], [A_2, B_{i_m}]] \\
&= [^{[m+1]}A_1, [B_{i_1}], [B_{i_2}], \ldots, [B_{i_{m-1}}], [A_2], B_{i_m}] \\
&\quad - [^{[m+1]}A_1, [B_{i_1}], [B_{i_2}], \ldots, [B_{i_{m-1}}], [A_2], B_{i_m}],
\end{align*}$$

which follows from Jacobi identity. The first term on the right-hand side of (2.23) is then cancelled in (2.22) by the corresponding term of $A_1 \leftrightarrow A_2$, whereas the second one contributes to the expressions $D_{k,h}$ in (2.19).

**Corollary 2.4.** Let $(A_1, A_2)$ and $(B_1, B_2, \ldots, B_k)$ be two sets of elements of $\mathfrak{A}$ such that

$$[[A_1, B_j], B_{j'}] = 0 \quad \forall i = 1, 2 \text{ and } \forall j, j' = 1, \ldots, k, \ j \neq j'.$$

Then

$$\begin{align*}
&[A_1 \circ A_2, \text{Sym}_k(B_1, B_2, \ldots, B_k)] \\
&= \sum_{i=1}^{k} \{ \text{Sym}_{k+1}(A_1, B_1, \ldots, B_{i-1}, [A_2, B_i], B_{i+1}, \ldots, B_k) \\
&\quad + \text{Sym}_{k+1}(A_2, B_1, \ldots, B_{i-1}, [A_1, B_i], B_{i+1}, \ldots, B_k) \}.
\end{align*}$$

(2.24)
Proof. In this case, all terms $D_{h,k}$ and $E_{h,k}$ respectively defined by formulas (2.20) and (2.21) are zero.

A typical case, in which the above corollary can be applied, is when $[A_i, B_j]$ is a number (more exactly, a number times the neutral element $I$ of the algebra with respect to the product) $\forall i = 1, 2$ and $\forall j' = 1, \ldots, k$. Under a similar hypothesis, it is also possible to obtain a formula for the product of the symmetrized products of two sets containing arbitrary numbers of elements of the algebra. In order to write this formula in a sufficiently compact form, let us introduce the following notations. For all $m \in \mathbb{N}$ we denote with $N_m := (1, 2, \ldots, m)$ the set of the first $m$ natural numbers. For any $h \leq m$ we denote with $P_{h,m}$ the set of the parts of $N_m$ which contain exactly $h$ elements. For all $I_h \in P_{h,m}$ we denote with $I_h^C$ the complementary set of $I_h$, that is $I_h^C := N_m \setminus I_h$. The set $I_h^C$ obviously contains exactly $m - h$ elements. For any set $I = (i_1, \ldots, i_m) \subset \mathbb{N}$ and any $l \in \mathbb{N}$, we also denote with $l + I$ the set $(l + i_1, \ldots, l + i_m)$.

**Proposition 2.5.** Let $(L_1, \ldots, L_l)$ and $(M_1, \ldots, M_m)$ be two sets of elements of the algebra $\mathfrak{A}$, such that $[L_i, M_j] = d_{ij}$, where $d_{ij}$ is a number for all $i \in N_l$ and all $j \in N_m$. Then

$$
\text{Sym}_l(L_1, \ldots, L_l)\text{Sym}_m(M_1, \ldots, M_m) = \sum_{h=0}^{\min(l,m)} \sum_{I_h \in P_{h,m}} \sum_{J \in \pi(I)} \frac{1}{(h!)^{l-1}(m-1)!} d_{i_1 j_{\pi(1)}} \cdots d_{i_h j_{\pi(h)}} \times \text{Sym}_{l+m-2h}(L_{i_1^C}, \ldots, L_{i_h^C}, M_{j_1^C}, \ldots, M_{j_{m-h}^C}),
$$

(2.25)

where $I = (i_1, \ldots, i_h)$, $J = (j_1, \ldots, j_h)$, $I_h^C = (i_1^C, \ldots, i_{m-h}^C)$ and $J_h^C = (j_1^C, \ldots, j_{m-h}^C)$.

For instance, for $l = 2$ and $m = 3$ the above proposition asserts that

$$
\text{Sym}_2(L_1, L_2)\text{Sym}_3(M_1, M_2, M_3) = \text{Sym}_3(L_1, L_2, M_1, M_2, M_3)
+ \frac{1}{2} [d_{11}\text{Sym}_3(L_2, M_2, M_3) + d_{12}\text{Sym}_3(L_2, M_1, M_3)]
+ d_{13}\text{Sym}_3(L_2, M_1, M_2) + d_{21}\text{Sym}_3(L_1, M_2, M_3)
+ d_{22}\text{Sym}_3(L_1, M_1, M_3) + d_{23}\text{Sym}_3(L_1, M_1, M_2)]
+ \frac{1}{4} [(d_{11}d_{22} + d_{12}d_{21})M_3 + (d_{11}d_{23} + d_{13}d_{21})M_2 + (d_{12}d_{23} + d_{13}d_{22})M_1].
$$

Note that no hypothesis has been made on commutators of type $[L_i, L_j]$ and $[M_i, M_j]$.

**Proof.** Let us consider the set of elements $(N_1, \ldots, N_{l+m})$, where $N_i = L_i$ for $i = 1, \ldots, l$ and $N_{l+i} = M_j$ for $j = 1, \ldots, m$. Let $\Pi_{l,m}$ be the set of all permutations $\sigma$ of $l+m$ objects, which do not change the order of both the group of the first $l$ elements and the group of the last $m$ elements. More formally, $\Pi_{l,m}$ is the set of all $(l+m)!/(l!m!)$ permutations $\sigma \in \Pi_{l+m}$ satisfying the following two conditions:

i. $\sigma^{-1}(i) < \sigma^{-1}(j)$ $\forall i, j$ such that $1 \leq i < j \leq l$;

ii. $\sigma^{-1}(l + h) < \sigma^{-1}(l + k)$ $\forall h, k$ such that $1 \leq h < k \leq m$. 


Introducing for $i \in N_l$ and $j \in N_m$ the quantities
\[
\tilde{d}^\sigma_{ij} = \begin{cases} 
0 & \text{if } \sigma^{-1}(i) < \sigma^{-1}(l + j) \\
d_{ij} & \text{if } \sigma^{-1}(i) > \sigma^{-1}(l + j) \n\end{cases},
\] (2.26)
it is easy to prove by induction that, for any $\sigma \in \Pi_{m,l}$,
\[
\mathcal{L}_1 \cdots \mathcal{L}_l \mathcal{M}_1 \cdots \mathcal{M}_m = \sum_{h=0}^{\min(l,m)} \sum_{I \in \Pi_{h,l}} \sum_{J \in \Pi_{h,m}} \sum_{\pi \in \Pi_n} \sum_{\sigma \in \Pi_{m+l}} \tilde{d}^\sigma_{i_{1j_{(1)}}} \cdots \tilde{d}^\sigma_{i_{m},j_{(h)}} 
\times \mathcal{N}_{\sigma(k_1)} \cdots \mathcal{N}_{\sigma(k_{m-2h})},
\] (2.27)
where $I = (i_1, \ldots, i_h)$, $J = (j_1, \ldots, j_h)$, $(\sigma(k_1), \ldots, \sigma(k_{m-2h})) = I^C \cup (l + J^C)$ and $k_1 < k_2 < \ldots < k_{l+m-2h}$. For instance, for $l = 2$, $m = 3$ and $\sigma = (31425)$, the above equality becomes
\[
\mathcal{L}_1 \mathcal{L}_2 \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 = \mathcal{M}_1 \mathcal{L}_1 \mathcal{M}_2 \mathcal{L}_2 \mathcal{M}_3 + d_{11} \mathcal{M}_2 \mathcal{L}_2 \mathcal{M}_3 \\
+ d_{21} \mathcal{L}_1 \mathcal{M}_1 \mathcal{M}_3 + d_{22} \mathcal{M}_1 \mathcal{L}_1 \mathcal{M}_3 + d_{11} d_{22} \mathcal{M}_3.
\]

We are obviously allowed to replace the right-hand side of (2.27) with its average with respect to all $(l + m)!/(l!m!)$ distinct permutations $\sigma \in \Pi_{m,l}$. If we then symmetrize both members of the equation with respect to all possible permutations of the $l$ elements $\mathcal{L}_i$ and the $m$ elements $\mathcal{M}_j$, we obtain on the right-hand side an average over all $(l + m)!$ permutations $\sigma \in \Pi_{l+m}$:
\[
\text{Sym}_l(\mathcal{L}_1, \ldots, \mathcal{L}_l)\text{Sym}_m(\mathcal{M}_1, \ldots, \mathcal{M}_m) \\
= \frac{1}{(l + m)!} \sum_{h=0}^{\min(l,m)} \sum_{I \in \Pi_{h,l}} \sum_{J \in \Pi_{h,m}} \sum_{\pi \in \Pi_n} \sum_{\sigma \in \Pi_{l+m}} \tilde{d}^\sigma_{i_{1j_{(1)}}} \cdots \tilde{d}^\sigma_{i_{l},j_{(h)}} 
\times \mathcal{N}_{\sigma(k_1)} \cdots \mathcal{N}_{\sigma(k_{m-2h})}.\] (2.28)
From this, in order to obtain (2.20), we need only observe that, as a consequence of (2.20),
\[
\frac{1}{(l + m)!} \sum_{\sigma \in \Pi_{l+m}} \tilde{d}^\sigma_{i_{1j_{(1)}}} \cdots \tilde{d}^\sigma_{i_{l},j_{(h)}} \mathcal{N}_{\sigma(i'_1)} \cdots \mathcal{N}_{\sigma(i'_{m-2h})}
= \frac{1}{2^h} d_{i_{1j_{(1)}}} \cdots d_{i_{l},j_{(h)}} \text{Sym}_{l+m-2h}(\mathcal{L}_{i'_1}, \ldots, \mathcal{L}_{i'_{m-2h}}, \mathcal{M}_{j'_1}, \ldots, \mathcal{M}_{j'_{m-2h}}),
\]
where $I^C = (i'_1, \ldots, i'_{m-2h})$ and $J^C = (j'_1, \ldots, j'_{m-2h})$. In fact, $(l + m)!2^{-h}$ is the number of permutations $\sigma \in \Pi_{l+m}$ such that $\sigma^{-1}(i_r) > \sigma^{-1}(l + j_{\pi(r)})$ for all $r \in N_h$.

**Corollary 2.6.** Under the same hypotheses of the preceding proposition, we have
\[
[\text{Sym}_l(\mathcal{L}_1, \ldots, \mathcal{L}_l), \text{Sym}_m(\mathcal{M}_1, \ldots, \mathcal{M}_m)] \\
= \sum_{h \in H} \sum_{I \in \Pi_{h,l}} \sum_{J \in \Pi_{h,m}} \sum_{\pi \in \Pi_n} \frac{1}{2^h} d_{i_{1j_{(1)}}} \cdots d_{i_{l},j_{(h)}} 
\times \text{Sym}_{l+m-2h}(\mathcal{L}_{i'_1}, \ldots, \mathcal{L}_{i'_{m-2h}}, \mathcal{M}_{j'_1}, \ldots, \mathcal{M}_{j'_{m-2h}}),\] (2.29)
where the notation is the same as that of (2.27), and $H$ is the set of all odd integers $p$ such that $1 \leq p \leq \min(l,m)$.
Proof. Since \([M_i, L_j] = -d_{ji}\), it is easy to see that the commutator on the left-hand side of (2.29) is twice the sum of the terms on the right-hand side of (2.25) which correspond to odd values of \(h\).

3  Correspondence by symmetrization between functions on a Poisson manifold and abstract operators

Let us consider the following abstract construction. Let \(M\) be a Poisson manifold. We recall that a Poisson manifold is a manifold such that an operation called “Poisson bracket” is defined in the space of the functions defined on this manifold. This operation has the following properties: it is bilinear, skew-symmetric, it satisfies the identity of Jacobi and the rule of Leibniz. An important example of Poisson manifold is a symplectic manifold \(M = M^{2N}\) of arbitrary dimension \(2N\). Moreover, let \(\mathfrak{A}\) be an associative algebra of operators with composition as product.

Let \(B = (B_1, \ldots, B_l)\) be a set of linearly independent functions on the Poisson manifold \(M\), which generate a finite dimensional Lie algebra with respect to Poisson brackets. Correspondingly, let \(B = (B_1, \ldots, B_l)\) be a set of linearly independent operators which generate a finite Lie subalgebra of \(\mathfrak{A}\). We suppose that the correspondence \(B_i \rightarrow B_i\) defines an isomorphism between Lie algebras.

**Definition 3.1.** Let \(P = P(B)\) be a polynomial of \(l\) variables. We call the operator \(P_{\text{sym}} = P_{\text{sym}}(B) := P_{\text{sym}}(B)|_{B=B}\), where \(P_{\text{sym}}\) is the symmetrization of \(P\) according to definition (2.2) the symmetrization with respect to the operators \(B_1, \ldots, B_l\) of the polynomial \(P = P(B_1, \ldots, B_l)\).

We shall consider initially two different cases, “constant” and “linear”.

1. In the constant case we have
   \[
   \{B_i, B_j\} = c_{ij}, \quad [B_i, B_j] = c_{ij}, \quad i, j = 1, \ldots, l, \quad (3.1)
   \]
   where \(c_{ij} = -c_{ji}\) are constant numbers.

2. In the linear case we have
   \[
   \{B_i, B_j\} = \sum_{k=1}^{l} c_{ij}^k B_k, \quad [B_i, B_j] = \sum_{k=1}^{l} c_{ij}^k B_k, \quad i, j = 1, \ldots, l, \quad (3.2)
   \]
   where \(c_{ij}^k = -c_{ji}^k\) are constant numbers. Note that Jacobi identity implies that
   \[
   \sum_{k=1}^{l} (c_{ij}^k c_{kh}^m + c_{ih}^k c_{kj}^m + c_{jh}^k c_{ki}^m) = 0 \quad \forall i, j, h, m = 1, \ldots, l.
   \]

If \(F = F(B)\) and \(H = H(B)\) are polynomials in \(B\) of arbitrary degree with constant coefficients, then \(\{H, F\} = G\), where \(G\) is a function on \(M\) which too can be represented as a polynomial in \(B\). Such a polynomial representation is unique when \(B = (B_1, \ldots, B_l)\) is a set of polynomially independent functions.
on the Poisson manifold \(M\). Polynomial independence means that if \(P(B) = 0\), where \(P\) is a polynomial with constant coefficients, then \(P = 0\), i.e., all the coefficients of \(P\) are zero. For example, if the functions \(B\) are functionally independent, i.e., their differentials are linearly independent almost everywhere on \(M\), then they are in particular also polynomially independent.

In any case, the application of linearity and Leibniz rule to Poisson brackets univocally determines a polynomial representation for \(G = \{H, F\}\). One is therefore free to choose this particular representation, even when the set \(B\) is not polynomially independent. More precisely, let us consider the map \(B : M \to \mathbb{R}^l\) such that \(B(x) = (B_1(x), \ldots, B_l(x)) \forall x \in M\). This map naturally induces a Poisson structure on \(N = \mathbb{R}^l\). For any pair of functions \(f, g \in C^\infty(\mathbb{R}^l)\) and \(\forall y \in \mathbb{R}^l\) we have:

1. In the constant case
   \[
   \{f, g\}_N(y) = \sum_{i,j=1}^l c_{ij} \frac{\partial f}{\partial y_i}(y) \frac{\partial g}{\partial y_j}(y) ;
   \tag{3.3}
   \]
   where \(\{ , \}_N\) denote Poisson brackets on \(N = \mathbb{R}^l\). It is obvious that, in both cases, if \(f\) and \(g\) are polynomial functions, then \(\{f, g\}_N\) is also a polynomial function. It is convenient to denote also with \(B\) (instead of \(y\)) the coordinates on the Poisson manifold \(N\).

2. In the linear case
   \[
   \{f, g\}_N(y) = \sum_{i,j,k=1}^l c_{ij}^{k} y_k \frac{\partial f}{\partial y_i}(y) \frac{\partial g}{\partial y_j}(y) ,
   \tag{3.4}
   \]

where \(\{ , \}_N\) denote Poisson brackets on \(N = \mathbb{R}^l\). It is obvious that, in both cases, if \(f\) and \(g\) are polynomial functions, then \(\{f, g\}_N\) is also a polynomial function. It is convenient to denote also with \(B\) (instead of \(y\)) the coordinates on the Poisson manifold \(N\).

**Definition 3.2.** Let \(F = F(B)\) and \(H = H(B)\) be two polynomials in \(B\). We say that the polynomial \(G = G(B) = \{H, F\}_N\), obtained with the procedure described above, is the Leibniz representation with respect to \(B\) of the Poisson bracket \(\{H, F\}\). Moreover, we call the symmetrization \(G^{\text{sym}}\) of \(G\) with respect to \(B\) the Leibniz symmetrization with respect to \(B\) of \(\{H, F\}\).

Obviously we have \(G(B) \equiv \{H, F\}\) everywhere in the Poisson manifold \(M\). The name “Leibniz representation” is due to the fact that, in the case of polynomials, the expression of \(\{f, g\}_N\), provided by formulas (3.3) or (3.4), can also simply be obtained from (3.1) or (3.2), by repeated application of linearity and Leibniz rule to Poisson brackets.

Our goal in this section is to find classes of polynomial functions \(H(B)\) of the functions \(B\), such that their Poisson brackets with any polynomial function \(F = F(B)\) is converted by the operation of Leibniz symmetrization into the commutator of the corresponding operators:

\[
[H^{\text{sym}}, F^{\text{sym}}] = \{H, F\}^{\text{sym}}_N .
\]

Let us then consider a function \(H\) which has in the two cases constant and linear considered above the following forms:

1. In the constant case \(H\) is a polynomial of degree two in the functions \(B\):
   \[
   H = \sum_{i \leq j} a_{ij} B_i B_j + \sum_j b_j B_j + c .
   \tag{3.5}
   \]
2. In the linear case \( H \) is a polynomial of degree one, that is a linear non-homogeneous function of \( B \):

\[
H = \sum_j b_j B_j + c.
\]  

(3.6)

In the two above equations the coefficients \( a_{ij} \), \( b_j \) and \( c \) are constants.

**Proposition 3.1.** Let one of the two following hypotheses be satisfied:

1. Condition (3.1) on Poisson brackets (constant case) and condition (3.5) on the degree of \( H \);
2. Condition (3.2) on Poisson brackets (linear case) and condition (3.6) on the degree of \( H \).

Suppose also that we have an isomorphism of Lie algebras defined by the correspondence \( B \rightarrow B \).

Let \( F = F(B) \) be an arbitrary polynomial in \( B \). Then the commutator of the operators \( H^\text{sym} \) and \( F^\text{sym} \), obtained by symmetrization of the polynomials \( H \) and \( F \) respectively, is equal to the Leibniz symmetrization of the Poisson bracket of these functions:

\[
[H^\text{sym}, F^\text{sym}] = \{H, F\}^\text{sym}_N.
\]  

(3.7)

In particular, if the Leibniz representation of \( \{H, F\} \) is the vanishing polynomial, i.e., \( \{H, F\}_N = 0 \), then \([H^\text{sym}, F^\text{sym}] = 0\).

**Proof.** It is clearly enough to prove the proposition when the functions \( H \) and \( F \) are monomials. In the general situation the thesis will then follow immediately by linearity. Let us then suppose that \( F = B_{j_1} \cdots B_{j_p} \).

In case 1 let us suppose that \( H = B_h B_k \). Applying corollary 2.4 Leibniz rule for Poisson brackets, and relations (3.4), we get

\[
[H^\text{sym}, F^\text{sym}] = \{B_h \circ B_k, \text{Sym}_m(B_{j_1}, \ldots, B_{j_m})\]
\[
= c_{h_1j_1} \text{Sym}_m(B_k, B_{j_2}, B_{j_3}, \ldots, B_{j_m}) + c_{h_2j_2} \text{Sym}_m(B_k, B_{j_1}, B_{j_3}, \ldots, B_{j_m}) + \cdots + c_{h_{j_m}j_m} \text{Sym}_m(B_k, B_{j_1}, B_{j_2}, \ldots, B_{j_{m-1}})
\]
\[
+ c_{k_1j_1} \text{Sym}_m(B_h, B_{j_2}, B_{j_3}, \ldots, B_{j_m}) + c_{k_2j_2} \text{Sym}_m(B_h, B_{j_1}, B_{j_3}, \ldots, B_{j_m}) + \cdots + c_{k_{j_m}j_m} \text{Sym}_m(B_h, B_{j_1}, B_{j_2}, \ldots, B_{j_{m-1}})
\]
\[
= \{B_h B_k, B_{j_1} \cdots B_{j_m}\}^\text{sym}_N = \{H, F\}^\text{sym}_N.
\]

Similarly, in case 2 let us suppose that \( H = B_i \). Applying proposition 2.1 Leibniz rule to Poisson brackets, and relations (3.2), we get

\[
[H^\text{sym}, F^\text{sym}] = \{B_i, \text{Sym}_p(B_{j_1}, \ldots, B_{j_p})\]
\[
= \sum_k [c_{ij_1}^k \text{Sym}_p(B_k, B_{j_2}, B_{j_3}, \ldots, B_{j_p}) + c_{ij_2}^k \text{Sym}_p(B_i, B_k, B_{j_3}, \ldots, B_{j_p}) + \cdots + c_{ij_p}^k \text{Sym}_p(B_i, B_{j_2}, \ldots, B_{j_{p-1}}, B_k)]
\]
\[
= \{B_i, B_{j_1} \cdots B_{j_p}\}^\text{sym}_N = \{H, F\}^\text{sym}_N.
\]
Remark 3.1. In case 1, if \( H \) has not the form (3.5), then in general equality (3.7) is no longer valid. Let us in fact identify the operators \( L \) and \( M \) of corollary 2.6 with the operators \( B \) of proposition 3.1, and let us calculate \([H^{\text{sym}}, F^{\text{sym}}}\] by means of formula (2.29). One easily sees that \{H, F\} corresponds to the term for \( h = 1 \) on the right-hand side of (2.29). But, if \( H \) contains also terms of degree greater than two in \( B \), then there are in general also terms with \( h \geq 3 \) which contribute to the commutator.

Similarly, equality (3.7) generally fails in case 2 if \( H \) has not the form (3.6). In fact, if \( H \) contains a quadratic term in \( B \), then \([H^{\text{sym}}, F^{\text{sym}}}\] can be calculated by means of proposition 2.3. One then sees that \{H, F\} corresponds to only the first sum on the right-hand side of (2.19). In general, however, also the two remaining sums, containing the terms \( D_{h,k} \) and \( E_{h,k} \), give nonvanishing contributions to the commutator.

Corollary 3.2. Let \( B \) be a set of polynomially independent functions on the Poisson manifold \( M \). Let one of the two following hypotheses be satisfied:

1. Condition (3.1) on Poisson brackets (constant case) and condition (3.5) on the degree of \( H \);
2. Condition (3.2) on Poisson brackets (linear case) and condition (3.6) on the degree of \( H \).

Suppose also that we have an isomorphism of Lie algebras defined by the correspondence \( B \to B \). Let \( F = F(B) \) be a polynomial in \( B \) such that \{H, F\} = 0 everywhere on \( M \). Then

\[
[H^{\text{sym}}, F^{\text{sym}}] = 0.
\]

Proof. Since \( B \) is a polynomially independent set, from the equality \{H, F\} = 0 everywhere on \( M \) it follows that the Leibniz representation of \{H, F\} is the vanishing polynomial. The thesis then follows from proposition 3.1.

Remark 3.2. Let us consider the case in which \( B = (x, p) \) is the set of canonical coordinates in the linear space \( \mathbb{R}^{2n}_x \), and \( B = (x, \hat{p} = \partial/\partial x) \) is its canonical quantization. In this case, proposition 3.1 for case 1 (constant) also directly follows from the useful formula of Moyal brackets \([8]\), which can in turn be derived from corollary 2.6. We write this formula in the following form:

\[
[H^{\text{sym}}, F^{\text{sym}}] = G^{\text{sym}},
\]

where

\[
G = \sum_{k \in \mathbb{N}} \sum_{|\alpha + \beta| = 2k + 1} \frac{(-1)^{|\beta|}}{2^{k|\alpha|} |\beta|!} \partial^{\alpha + \beta} H \partial^{\alpha + \beta} F.
\]

The sum in the above formula is over all vectors \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n^+ \), \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_n^+ \), such that \( |\alpha + \beta| = \sum_{i=1}^n (\alpha_i + \beta_i) = 2k + 1 \). Since \( H \) and \( F \) are polynomials, only a finite number of terms of the sum are different from zero.

We are now going to consider a more general case, in which we get rid of the hypothesis that the functions of set \( B \) form a closed Lie subalgebra with respect to Poisson brackets. This new case, to which we shall refer as case 3 (general), can be precisely formulated in the following way.
i. Suppose that there exists a set \( T \) of \( m \) linearly independent functions, \( T = (T_1, \ldots, T_m) \), \( T_i : M \to \mathbb{R} \) \( \forall i = 1, \ldots, m \), and a subset \( B \subseteq T \), \( B = (B_1, \ldots, B_l) \), with \( B_i = T_i \) for \( i = 1, \ldots, l \), \( l \leq m \), such that the Poisson brackets between the functions of set \( B \) are:

\[
\{ B_i, B_j \} = \sum_{s=1}^{m} c_{ij}^s T_s, \quad 1 \leq i < j \leq l ,
\]

(3.8)

where \( c_{ij}^s = -c_{ji}^s \) are constant coefficients. Note that we do not suppose that the functions of set \( T \) are dependent on \( B \). Moreover, neither the functions of set \( B \) nor the functions of set \( T \) are assumed to be closed with respect to Poisson brackets. However, in practical situations one encounters sometimes sets of functions \( B \) satisfying nonlinear Poisson bracket relations. In such cases one is able to write formulas of the form (3.8), in which the elements of set \( T \) are nonlinear functions of \( B \).

ii. Suppose also that there exist a set \( T \) of \( m \) elements of an associative Lie algebra \( A \) of operators, and a subset \( B \subseteq T \), \( B = (B_1, \ldots, B_l) \), with \( B_i = T_i \) for \( i = 1, \ldots, l \), such that the commutation relations between the operators of set \( B \) are

\[
[B_i, B_j] = \sum_{s=1}^{m} c_{ij}^s T_s, \quad 1 \leq i < j \leq l ,
\]

(3.9)

where \( c_{ij}^s \) are the same constants coefficients as in formula (3.8).

In this case the map \( T : M \to \mathbb{R}^m \) defined by the set \( T \) does not induce a Poisson structure on the set \( N = \mathbb{R}^m \). Nevertheless, by means of a simple modification of formula (3.3), one is able to define an operation \( \{ , \} \) \( N \), which associates with any pair of functions \( f, g \in C^\infty(\mathbb{R}^l) \) a function \( \{ f, g \}_N \in C^\infty(\mathbb{R}^m) \). More precisely, \( \forall y \in N = \mathbb{R}^m \) we define

\[
\{ f, g \}_N(y) = \sum_{i,j=1}^{l} \sum_{k=1}^{m} c_{ij}^k y_k \frac{\partial f}{\partial y_i}(y) \frac{\partial g}{\partial y_j}(y) .
\]

(3.10)

Obviously, if \( f \) and \( g \) are polynomial functions, then \( \{ f, g \}_N \) is also a polynomial function. It may be convenient in this case to denote with \( T \) (instead of \( y \)) the coordinates on the Poisson manifold \( N = \mathbb{R}^m \), and with \( B \) the first \( l \) components of \( T \).

**Definition 3.3.** Let \( F = F(B) \) and \( H = H(B) \) be two polynomials in \( B \). We say that the polynomial \( G = G(T) = \{ H, F \}_N \), obtained with the procedure described above, is the *Leibniz representation with respect to \( T \) of the Poisson bracket \( \{ H, F \} \).* Moreover, we call the symmetrization \( G^{sym} \) of \( G \) with respect to \( T \) the *Leibniz symmetrization with respect to \( T \) of \( \{ H, F \} \).

Obviously we have \( G(T) \equiv \{ H, F \} \) everywhere in the Poisson manifold \( M \). Also in this case, the name “Leibniz representation” is due to the fact that, in the case of polynomials, the expression of \( \{ f, g \}_N \), provided by formula (3.10), can also simply be obtained from (3.8) by repeated application of linearity and Leibniz rule to Poisson brackets.
Proposition 3.3. In the general case 3 specified by hypotheses i and ii given above, see formulas (3.8)–(3.9), consider the function

\[ H(B) = \sum_{j=1}^{l} b_j B_j + c \]  

(3.11)

and the corresponding operator

\[ \mathcal{H} = \sum_{j=1}^{l} b_j B_j + c. \]  

(3.12)

The coefficients \( b_j \) and \( c \) are the same in the two formulas above. Let \( F = F(B) \) be an arbitrary polynomial, and let \( F_{\text{sym}} \) denote its symmetrization with respect to \( B \). Then the commutator of the operators \( \mathcal{H} \) and \( F_{\text{sym}} \) is equal to the Leibniz symmetrization with respect to \( T \) of the Poisson bracket \( \{ H, F \} \):

\[ \left[ \mathcal{H}, F_{\text{sym}} \right] = \left[ H, F \right]_{N} \text{sym} = \left\{ H, F \right\}_{N} \text{sym}. \]  

(3.13)

In particular, if the Leibniz representation of \( \{ H, F \} \) with respect to \( T \) is the vanishing polynomial, i.e., \( \{ H, F \}_N = 0 \), then \( \left[ \mathcal{H}, F_{\text{sym}} \right] = 0 \).

Proof. The proof is similar to that of proposition 3.1 for case 2. For \( H = B_i \) and \( F = B_{j_1} \cdots B_{j_p} \) we have

\[
\begin{align*}
\left[ \mathcal{H}, F_{\text{sym}} \right] &= \left[ B_i, \text{Sym}_p(B_{j_1}, \ldots, B_{j_p}) \right] \\
&= \sum_k [e_{ij_1}^k \text{Sym}_p(T_k, T_{j_2}, \ldots, T_{j_p}) + e_{ij_2}^k \text{Sym}_p(T_{j_1}, T_k, T_{j_3}, \ldots, T_{j_p})] \\
&\quad + \cdots + e_{ij_p}^k \text{Sym}_p(T_{j_1}, T_{j_2}, \ldots, T_{j_{p-1}}, T_k)] \\
&= \left\{ B_i, B_{j_1} \cdots B_{j_p} \right\}_{N} \text{sym} = \left\{ H, F \right\}_{N} \text{sym}.
\end{align*}
\]

The general result follows by linearity. \( \square \)

The following corollary is the analogue of corollary 3.2.

Corollary 3.4. Let hypotheses i–ii above be satisfied, see formulas (3.8)–(3.9). Consider the function \( H \) and the operator \( \mathcal{H} \) given by formulas (3.11) and (3.12) respectively. In addition, suppose that \( T \) is a set of polynomially independent functions on the Poisson manifold \( M \). Let \( F = F(B) \) be a polynomial in \( B \) such that \( \{ H, F \} = 0 \), and let \( F_{\text{sym}} \) denote its symmetrization with respect to \( B \). Then

\[ \left[ \mathcal{H}, F_{\text{sym}} \right] = 0. \]  

(3.14)

Remark 3.3. Very often the set \( T \) is locally closed with respect to Poisson brackets, that is locally Poisson-closed. (If this set is not closed, usually it makes sense to consider its Poisson closure, i.e., its extension by means of finite iterations of Poisson brackets.) Let us consider for simplicity a set \( T \) which is globally Poisson-closed, i.e., there exist globally uniquely defined functions \( c_{ij} \) of \( m \) variables such that \( \{ T_i, T_j \} = c_{ij}(T) \) at all points of \( M \), \( 1 \leq i < j \leq m \). In this case, using the map \( T : M \to \mathbb{R}^m \), defined by the set \( T = (T_1, \ldots, T_m) \), one can transfer the Poisson structure of \( M \) on its image \( T(M) \subseteq \mathbb{R}^m \). With respect to this Poisson structure induced on \( N := T(M) \), the map \( T \) becomes a
Poisson map. We recall that a map $T : M \to N$, where $M$ and $N$ are Poisson manifolds, is called a Poisson map if

$$\{f, g\}_N \circ T = \{f \circ T, g \circ T\}_M$$

for any pair of functions $f, g \in C^\infty(N)$. Here $\{,\}_M$ and $\{,\}_N$ denote Poisson brackets in the two Poisson manifolds $M$ and $N$ respectively, and the symbol $\circ$ is used to indicate the composition of maps. When $T$ is a globally Poisson-closed set of functions on $M$, the above formula univocally determines a Poisson structure on $N$. The Poisson bracket of two functions $f$ and $g$ on $N$ takes the form

$$\{f, g\}_N(y) := \sum_{i,j=1}^m c_{ij}(y) \frac{\partial f}{\partial y^i}(y) \frac{\partial g}{\partial y^j}(y)$$

for all $y = (y_1, \ldots, y_m) \in N \subseteq \mathbb{R}^m$.

An example of Poisson-closed set of functions on a Poisson manifold $M$ is the set of functions $B = (B_1, \ldots, B_l)$ in the linear case 2. The set $B$ is a basis of the $l$-dimensional Lie algebra $\mathfrak{g}$ defined by the linear Poisson bracket relations

$$[\mathfrak{g}, \mathfrak{g}] = \{\mathfrak{g}, \mathfrak{g}\}.$$

In this case, the set $B$ defines the Poisson map $B : M \to N$, where the Poisson manifold $N$ is isomorphic to the $l$-dimensional linear space $\mathbb{R}^l$. The manifold $N$ can be identified with the Lie co-algebra $\mathfrak{g}^*$, that is with the space which is the linear conjugate to the Lie algebra $\mathfrak{g}$: $N = \mathfrak{g}^*$. The Poisson structure on $N$ transferred from $M$ can be written in intrinsic terms: for any two functions $g, f \in C^\infty(\mathfrak{g}^*)$ and any point $\xi \in \mathfrak{g}^*$, one defines $\{g, f\}_\xi := ([dg]_\xi, df|_\xi, \xi)$, where the differentials $dg|_\xi$ and $df|_\xi$ are elements of the Lie algebra $\mathfrak{g}$, and $[\cdot, \cdot]$ is the commutator in this Lie algebra. This means that any Lie co-algebra has a natural structure of Poisson manifold. It has to be noted that the elements of set $B$, considered as functions on the Poisson manifold $\mathfrak{g}^*$, are always functionally independent, even when they are not functionally independent as functions on the Poisson manifold $M$.

**Definition 3.4.** A Casimir function on the Lie co-algebra $\mathfrak{g}^*$ is an invariant of the co-adjoint action of the local Lie group $G$ associated with the Lie algebra $\mathfrak{g}$. It means that $f : \mathfrak{g}^* \to \mathbb{R}$ is a Casimir function if $(df|_\xi, \tau_\alpha|_\xi) = 0$ $\forall \xi \in \mathfrak{g}^*$ and $\forall b \in \mathfrak{g}$, where $\tau_\alpha \in \mathfrak{g}^*$ is defined by the formula $(a, \tau_\alpha|_\xi) = ([b, a]|_\xi) \forall a \in \mathfrak{g}$. Hence, for any other function $g : \mathfrak{g}^* \to \mathbb{R}$ we can write $\{g, f\}_\xi = ([dg|_\xi, df|_\xi], \xi) = (df|_\xi, \tau_\alpha|_\xi) = 0$, with $b = dg|_\xi$. It follows that Casimir functions are in involution with any other function defined on the Lie co-algebra.

**Remark 3.4.** It is known that any function $H : M \to \mathbb{R}$ on a Poisson manifold $M$ defines univocally a vector field $X_H$ on $M$, which is tangent to the symplectic leaves of this manifold $M$. (In local coordinates $y$ on $M$, this vector field $X_H$ is defined by the system of differential equations $\dot{y} = \{H, y\}$.) This vector field $X_H$ is called hamiltonian with hamiltonian function $H$. As in the case of symplectic manifolds, one has $[X_H, X_F] = X_{\{H, F\}}$ for any two functions $H$ and $F$ on any Poisson manifold $M$, where the left-hand side represents the Lie bracket of vector fields. So one has a natural homomorphism of the Lie algebra of functions on the Poisson manifold $M$ to the Lie algebra of vector fields on $M$. In particular the structure of a finite dimensional Lie subalgebra of functions on $M$ transfers to the vectors fields corresponding to these functions.
Let us consider again a finite-dimensional Lie co-algebra $\mathfrak{g}^*$, with the intrinsic structure of Poisson manifold described above. Any linear function $H : \mathfrak{g}^* \to \mathbb{R}$ can be considered as an element of the Lie algebra $\mathfrak{g}$ and can be written in the form $H = H_a$. In this case, the phase flow of the Hamiltonian vector field $X_H$, $H = H_a$, coincides with a local one-parametric Lie subgroup of the Lie group $G$ with co-adjoint action on the Lie co-algebra $\mathfrak{g}^*$: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$. Here $G$ is the local Lie group corresponding to the Lie algebra $\mathfrak{g}$, and this subgroup of $G$ corresponds to the element $a \in \mathfrak{g}$. If $a \neq 0$, this means that the straight line tangent to this subgroup at the neutral element $e \in G$ contains the vector $a \in \mathfrak{g}$.

Propositions 3.1 and 3.3 are particular cases of the following more general proposition. Let us suppose that we have, in addition to the sets $T$ and $\mathcal{T}$, another set of functions $D = (D_1, \ldots, D_s)$ on the Poisson manifold $M$, and another set of operators $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_s)$ of the associative algebra $\mathfrak{A}$, such that $\{D_i, D_j\} = \{D_i, \mathcal{T}_k\} = 0$, $[\mathcal{D}_i, \mathcal{D}_j] = [\mathcal{D}_i, \mathcal{T}_k] = 0$ for $i, j = 1, \ldots, s$ and $k = 1, \ldots, m$. In cases 1 and 2 one can take $T = B, \mathcal{T} = \mathcal{B}$ and $m = l$. In case 1 let us also suppose that the coefficients $a_{ij}, b_j$ and $c$ of formula (3.1), are polynomial functions of $D$. Similarly, in cases 2 and 3, let us suppose that the coefficients $b_j$ and $c$ of formulas (3.6) and (3.11), and the coefficients $c^{ij}$ of formulas (3.2) and (3.5), are polynomial functions of $D$.

For this situation we now generalize the definition of symmetrization. Consider the linear space $P[D, T]$ of all polynomials $P = P(D, T)$ of $s$ variables $D$ and $m$ variables $T$. Consider the linear map $\text{sym}: P[D, T] \to \mathfrak{A}$, which associates with an arbitrary polynomial $Q = D_1^{\alpha_1} \cdots D_s^{\alpha_s} T_{i_1} \cdots T_{i_k}$ the operator

$$Q^\text{sym} := D_1^{\alpha_1} \cdots D_s^{\alpha_s} \text{Sym}_{k}(T_{i_1}, \ldots, T_{i_k}).$$

Here the $\alpha_i$ are integers, $\alpha_i \geq 0$. The monomials $Q$ form a basis in the linear space $P[D, T]$. Hence the mapping $Q \mapsto Q^\text{sym}$ defines a linear map $P \mapsto P^\text{sym}$ on the full space $P[D, T]$.

**Definition 3.5.** We call the operator $P^\text{sym} = P^\text{sym}(D, T) \in \mathfrak{A}$, associated with the polynomial $P(D, T)$ by the linear map defined above, the symmetrization with respect to the operators $T_1, \ldots, T_l$ of the polynomial $P = P(D, T)$.

We observe that the functions of $D$ can practically be treated as constant in all the algebraic manipulations that were necessary to prove propositions 3.1 and 3.3. Therefore:

**Proposition 3.5.** In this more general situation, propositions 3.1 and 3.3 remain valid. The symmetrization of all terms of formulas (3.7) and (3.13) is now performed with respect to the operators of the set $\mathcal{T}$.

## 4 Application to quantization

The term “quantization” is here used for convenience. We will discuss the construction, from functions defined on the Poisson manifold $M$, of differential operators acting on functions defined on the manifold $K$. These operators are not necessarily connected with quantum mechanics, for example we can consider the linear operator with nonconstant coefficients which defines the heat
equation. We suppose that in this construction the Poisson brackets of functions are converted into Lie brackets of operators which are the “quantization” of these functions. As a rule, the manifold $M$ is the cotangent bundle of $K$, $M = T^* K$, but this is not necessary. This transformation of functions into operators is based on the assumption that there exists a correspondence between some finite-dimensional algebras of functions on $M$ and of operators on $K$. This correspondence under some assumptions is extended to products of functions which correspond to the composition of operators.

So, let us consider a particularly interesting case for us, in which $B_1, \ldots, B_l$ are linear differential operators in the variables $x = (x_1, \ldots, x_n)$. More precisely, using the notation of [1], we suppose that $B_i \in \mathcal{O} = \mathcal{O}_K$, $i = 1, \ldots, l$, where $K$ is a domain of $\mathbb{R}^n$. Let these operators form a Lie algebra with respect to commutators. As above, we also suppose the existence of a set $B = (B_1, \ldots, B_l)$ of functions on the Poisson manifold $M$, which forms a basis of a Lie algebra with respect to Poisson brackets. We suppose that there exists an isomorphism $\mathcal{I}$ of Lie algebras, which is defined by the correspondence $Q : B_i \rightarrow B_i$, $i = 1, \ldots, l$, between their bases. Let us consider an arbitrary polynomial $P$ of $l$ variables, and let $P_{\text{sym}}(B)$ be the symmetrization of $P$ with respect to the operators $B$, see definition 3.1.

**Definition 4.1.** In this case, the correspondence $Q$ and the isomorphism $\mathcal{I}$ defined by $Q$ is called basic quantization. The operator $P_{\text{sym}}(B)$ is called quantization by symmetrization of the polynomial $P(B)$ of variables $B = (B_1, \ldots, B_l)$, or symmetric quantization of $P(B)$ with respect to the set $B$.

Let us consider the important example $M^{2n} = \mathbb{R}_{x,p}^{2n}$, $B = (x, p)$ and $B = (x, \hat{p})$, where $\hat{p} = \partial/\partial x$. We have

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, x_j\} = \delta_{ij} \quad (4.1)$$

and

$$[x_i, x_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, x_j] = \delta_{ij} \quad (4.2)$$

for $i, j = 1, \ldots, n$, where $\delta_{ij}$ is the Kronecker symbol. Hence the sets $(x, p)$ and $(x, \hat{p})$ are bases in two $2n$-dimensional Lie algebras, and the correspondence between these bases defines an isomorphism of Lie algebras.

**Definition 4.2.** In this case the basic quantization, given by the correspondence $Q_c : (x, p) \rightarrow (x, \hat{p})$ and the isomorphism $\mathcal{I}_c = \mathcal{I}(Q_c)$, is called canonical quantization.

Note that canonical quantization corresponds to the standard quantization of the set $(x, p)$ according to the definition given in [1]. However, the standard quantization of a polynomial $P(x, p)$ in general does not coincide with its symmetric quantization $P_{\text{sym}}(x, \partial/\partial x)$ constructed according to definition 4.1.

Let us suppose that $B = (B_1, \ldots, B_l)$ is a set of arbitrary functions on some Poisson manifold $M$. Consider the set of operators $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r; \mathcal{F}_{r+1}, \ldots, \mathcal{F}_s)$ on $\mathbb{R}^n_x$, which are obtained from a given basic quantization $Q : B \rightarrow \mathcal{B}$, $B = (B_1, \ldots, B_l)$, by symmetrization (with respect to $B$) of the set of polynomials $P = (P_1, \ldots, P_r; P_{r+1}, \ldots, P_s)$, $P_i = P_i(B)$. We can thus write $\mathcal{F}_i = P_i^{\text{sym}}(B)$. We suppose that

$$\{P_i(B), P_j(B)\}_N = 0, \quad i = 1, \ldots, r, \quad j = 1, \ldots, s, \quad (4.3)$$
where \( \{ , \}_N \) denote the Leibniz representation with respect to \( B \) of Poisson brackets, see definition 3.2. Consider the case 1 (constant):

\[ \{ B_i, B_j \} = c_{ij}. \]

**Proposition 4.1.** Let us suppose additionally that the polynomials of the set \( P \) satisfy at least one of the two following conditions:

a. \( \deg P_i(B) \leq 2, \ i = 1, \ldots, r. \)

b. \( \deg P_j(B) \leq 2, \ j = 2, \ldots, s. \)

Here and in next proposition \( \deg \) means degree with respect to \( B \).

Then the following two statements are true:

1. \( [F_i, F_j] = 0, \ i = 1, \ldots, r, \ j = 1, \ldots, s. \)

2. Let us additionally suppose that we can extract two sets of polynomials \((P'_1, \ldots, P'_r) \subseteq (P_1, \ldots, P_r)\) and \((P''_{r+1}, \ldots, P'_s) \subseteq (P_{r+1}, \ldots, P_s)\), where \( r' + s' = 2n \), such that the operators of the set \( F' = (F'_1, \ldots, F'_{r'}; F''_{r'+1}, \ldots, F'_s) \) are quasi-independent, see definition of quasi-independence in [1].

Then the set of operators \( F' \) is quasi-integrable with \( r' \) central integrals \((F'_1, \ldots, F'_{r'}). \)

Let us now consider the case 2 (linear):

\[ \{ B_i, B_j \} = \sum_{k} c^i_{kj} B_k. \]

**Proposition 4.2.** Under the same hypotheses stated before proposition 4.1, let us suppose that the polynomials of the set \( P \) satisfy at least one of the two following conditions:

a. \( \deg P_i(B) \leq 1, \ i = 1, \ldots, r. \)

b. \( \deg P_j(B) \leq 1, \ j = 2, \ldots, s. \)

Then the same statements 1 and 2 of proposition 4.1 are true.

Let us finally consider case 3 (general), see hypotheses 3.3 before definition 3.3. In this case we have two sets of functions \( B \) and \( T \), such that \( B \subseteq T \), and two sets of operators \( B \) and \( T \), such that \( B \subseteq T \). These functions and operators satisfy the relations (3.9)–(3.11). Furthermore, we have a set of polynomials \( P = (P_1, \ldots, P_r; P_{r+1}, \ldots, P_s), \ P_i = P_i(B), \) which satisfy relations (4.3), where \( \{ , \}_N \) now denote the Leibniz representation with respect to \( T \) of Poisson brackets, see definition 3.3. Finally, we have a set of operators \( F \) which are obtained from \( P \) by symmetrization: \( F_i = P_i^{sym}(B) \), on the basis of the correspondence \( Q: B \rightarrow B \). The following proposition is a generalization of proposition 4.2 in two different respects. One does not require that the functions of either the set \( B \) or \( T \) satisfy linear Poisson brackets relations, and one does not require that such sets be closed with respect to Poisson brackets.

**Proposition 4.3.** Under the above hypotheses, let us additionally suppose that the polynomials of the set \( P \) satisfy at least one of the two conditions a or b of proposition 4.2. Then the same statements 1 and 2 of proposition 4.1 are true.

Propositions 4.1–4.2 easily follow from proposition 3.1, while the last proposition 4.3 follows from proposition 3.3.
Remark 4.1. Let us suppose that the functions $B_1, \ldots, B_l$ are polynomially independent almost everywhere in $M$. In this case, the equality $\{P_i(B), P_j(B)\} = 0$ in $M$ implies that the Leibniz representation of $\{P_i(B), P_j(B)\}$ is the vanishing polynomial, i.e., $\{P_i(B), P_j(B)\}_N = 0$. Hence, in propositions 4.1, 4.2 one can replace the hypothesis $\{P_i(B), P_j(B)\}_N = 0$ with $\{P_i(B), P_j(B)\} = 0$. The same statement is true for proposition 4.3 if the functions $T_1, \ldots, T_m$ are polynomially independent almost everywhere in $M$.

Let us consider the case of linear relations $\{B_i, B_j\} = \sum_{s=1}^l c_{ij}^s B_s$, see formula (29). Let $g$ be the Lie algebra formed by linear combinations of the functions of set $B$. We know already (see section 4) that the co-algebra $g^*$, i.e., the linear space conjugated to $g$, has a natural structure of Poisson manifold. As above, let us suppose that there is a set of linear operators $B = (B_1, \ldots, B_l)$ such that $[B_i, B_j] = \sum_{s=1}^l c_{ij}^s B_s$, with the same constants $c_{ij}^s$. Let the polynomial $C = C(B)$, i.e., $C : g^* \to \mathbb{R}$, be some invariant of the co-adjoint representation on $g^*$ of the local group $G$ which corresponds to the Lie algebra $g$.

Corollary 4.4. In this case the operator $C_{\text{sym}}$ commutes with all operators of set $B$: $[C_{\text{sym}}, B] = 0$.

Proof. Taking into account that any invariant of the co-adjoint representation of a group is a Casimir function on $g^*$, we have $\{C, B_i\}_N = 0$ for all functions $B_i$ of set $B$, or shortly $\{C, B\}_N = 0$, where $\{, \}_N$ denotes Poisson bracket on the co-algebra $g^*$. The thesis then follows from proposition 4.2 case b. 

Obviously, corollary 4.4 also implies that $[C_{\text{sym}}, P(B)] = 0$ for any noncommutative polynomial $P \in S_N^{\text{sym}}$.

Remark 4.2. According to the results of [1], if a set $(F_1, \ldots, F_k; F_{k+1}, \ldots, F_{2n-k})$ of operators is quasi-integrable, then the set $(MF_1, \ldots, MF_k; MF_{k+1}, \ldots, MF_{2n-k})$ of the main parts of their symbols $F_i := F_i^{\text{sym}}$ is also integrable in the usual classical sense. Therefore, in the situation of statement 2 of propositions 4.1, 4.3 $(r' + s' = 2n)$, the set $MF'$ of the main parts of the symbols of the operators $F_i^{\text{sym}}$, $i = 1, \ldots, s'$, must be integrable in the classical sense.

Note that classical integrability of a set $(F_1, \ldots, F_k; F_{k+1}, \ldots, F_{2n-k})$ implies that this set is (locally) closed with respect to Poisson brackets. This means that locally, in some neighborhood of almost every point, one has $\{F_i, F_j\} = f_{ij}(F)$, where $f_{ij}$ are some functions of $2n - k$ variables and $i, j = 1, \ldots, 2n - k$ (see remark 4.3). In particular, in the situation considered above, the set of functions $MF'$ is integrable and hence Poisson-closed.

Corollary 4.5. Let us suppose that the hypotheses of one of propositions 4.2, 4.3 are true, with $r + s = 2n$, and that the operators $F$ are quasi-independent. Let $H$ be an operator of class $O$ and let the set $\{H, F_1, \ldots, F_k\}$ be globally dependent. For example, $H = S(F_1, \ldots, F_k)$, where $S$ is an arbitrary polynomial of $k$ variables, that is $S \in S_k^0$. We suppose also that $[H, F_i] = 0$ for each $i = 1, \ldots, 2n - k$. In this case, $H$ is a quasi-integrable operator with $k$ central operators.

Remark 4.3. The consideration of general case 3 allows one to go outside the frame of finite Lie algebras of functions on symplectic manifolds, and correspondingly of linear differential operators. Operators forming a basis of a finite Lie algebra usually are quadratic with respect to $(x, \partial/\partial x)$, or linear with respect...
to $\partial/\partial x$. The consideration of nonlinear commutation relations gives in principle the possibility of dealing with operators of more general type. An example of nonlinear commutation relations, arising from the quantization of a classical system of nonlinear resonant oscillators, will be considered in a following paper.

Remark 4.4. The class of systems to which the last corollary can be applied is too small for many physical applications. For example, one has often to deal with functions which are not polynomials. Let us consider the situation of canonical quantization, where $B$ is the set of canonical coordinates $(p, x)$ (constant case). In this case it is often necessary to consider functions of $x$ which are not polynomials. Sometimes one also considers functions of $p$ such as $\exp(ip)$, and also functions of the form $f(x_1, p_2)$. Of course, if one considers operators of class $O$, that is polynomials with respect to $p$, then the only non polynomial functions one can meet are of the form $f(x)$. In this case it is necessary to use an additional property of the operators under consideration. Namely, suppose there exist some subsets $\tilde{B}^i \subseteq B$ such that $[\tilde{B}^i, \tilde{B}^j] = 0$ for each $i = 1, \ldots, l$. This means that all operators of the set $\tilde{B}^i$ are pairwise commuting. Suppose also that for functions $f$ of some class (which contains also non polynomials functions) the operator $f(\tilde{B}^i)$ is well defined. Then, $[f(\tilde{B}^i), g(\tilde{B}^j)] = 0$ for any functions $f, g$ of this class, and if $[B_j, \tilde{B}^i] = 0$ for some $B_j \in B$, we have $[B_j, f(\tilde{B}^i)] = 0$.

Let us suppose that there exist also a set of functions $D = (D_1, \ldots, D_s)$ and a set of operators $\hat{D} = (\hat{D}_1, \ldots, \hat{D}_s)$ with the properties considered in proposition 3.5.

Corollary 4.6. In this case, statement 1 of propositions 4.1–4.3 remains true when the constants $c_{ij}$ in formula (4.1) or $c^k_{ij}$ in formulas (3.2) or (3.8) are replaced by functions $c_{ij}(D)$ or $c^k_{ij}(D)$ respectively, and the coefficients of polynomials $P_i(B)$ are functions of $\hat{D}$. The symmetrization of these polynomials is performed with respect to operators $\mathcal{T}$ according to definition 3.5.

The proof is an easy application of proposition 3.5.

References

[1] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part I: quasi-integrable quantum systems, arXiv:1001.4685 (2010).

[2] N. N. Nekhoroshev, Action-angle variables and their generalizations, Trans. Moscow Math. Soc. 26, 180–198 (1972).

[3] V. I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag (New York), 1978.

[4] A. T. Fomenko, Differential Geometry and Topology, Consultants Bureau (New York), 1987.

[5] F. Fassò, Superintegrable Hamiltonian Systems: Geometry and Perturbations, Acta Appl. Math. 87, 93–121 (2005).

[6] V. Chari and A. Pressley, A guide to Quantum Groups, Cambridge University Press, 1994.
[7] L. Faddeev, A mathematician’s view of the development of physics, in *Miscellanea mathematica*, P. H. Hilton, F. Hirzebruch and R. Remmert eds., Berlin Heidelberg Springer-Verlag, 1991, pp. 119–127.

[8] J. E. Moyal, Quantum mechanics as a statistical theory, *Proc. Camb. Phil. Soc.* 45, 99–124 (1949).