Propagation of Gravitational Waves in Chern-Simons Axion $F(R)$ Gravity

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In this paper we shall study the evolution of cosmological gravitational waves in the context of Chern-Simons axion $F(R)$ gravity. In the case of Chern-Simons axion $F(R)$ gravity there exist spin-0, spin-2 and spin-1 modes. As we demonstrate, from all the gravitational waves modes of the Chern-Simons axion $F(R)$ gravity, only the two tensor modes are affected, while the spin-0 and spin-1 modes are not affected at all. With regard to the two tensor modes, we show that these modes propagate in a non-equivalent way, so the resulting tensor modes are chiral. Notably, with regard to the propagation of the spin-2 graviton modes, the structure of the dispersion relations becomes more complicated in comparison with the Einstein gravity with the Chern-Simons axion, but the resulting qualitative features of the propagating modes are not changed. With regard to the spin-0 and spin-1 modes, the Chern-Simons axion $F(R)$ gravity contains two spin-0 modes and no vector spin-1 mode at all. We also find that for the very high energy mode, both the group velocity and the phase velocity are proportional to the inverse of the square root of the wave number, and therefore the velocities become smaller for larger wave numbers or even vanish in the limit that the wave number goes to infinity.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq,11.25.-w

I. INTRODUCTION

The presence of a dark component of matter in the Universe was assumed early after the first galactic rotation curves appeared. Since then, theoretical physics studies were focused on proposing massive particles weakly interacting with luminous matter, the so-called WIMPs (Weakly Interacting Massive Particles), and there exist examples coming from various theoretical contexts, see for example [1-7]. To be honest, for the moment only indications exist that support the particle nature of dark matter, such as the observational data from the bullet cluster. However, nearly two decades of searches did not result in finding any WIMP. A crucial point to stress is that all the dark matter searches focused in mass ranges from a few MeV up to hundreds of GeV’s. However, only recently the experimentalists focused their interest in searching WIMPs having masses a few eV or even sub-eV.

String theory is up to date the most prominent theory that may describe in a consistent way the quantum theory of gravity. One of the interesting predictions of string theory is the existence of low-mass axion [7,10] particles, other than the QCD axions. The axions are particularly appealing as dark matter candidates, since the mass range that these may have, is not investigated yet, and only the last 5 years experimentalists turned their focus on the axions. There is a plethora of experimental [11–19] and theoretical proposals related to the axions [20–68]. The axions can induce particularly interesting effects in the phenomenology of gravitational theories, since Chern-Simons terms of the form $U(\phi) RR$ are allowed in the theory [69,82]. The Chern-Simons terms produce non-equivalent polarizations in the tensor modes of the underlying gravitational theory [83,84], and in the literature there exist timely studies on chiral gravitational waves [85,92].

In this paper we shall study a general Chern-Simons Axion $F(R)$ gravity [93–99], by mainly focusing on the possibility of generating non-equivalent polarizations in the tensor modes of the gravitational waves. We shall be interested in the primordial gravity waves propagation, assuming the conservation of the gravitational wave in the large scale [54]. Our approach will mainly be quantitative, since we will extract the gravitational wave equations, and we shall study the tensor, vector and scalar modes. As we demonstrate, the situation with the spin-0 scalar mode and spin-1 vector mode, is not changed from the standard $F(R)$ gravity coupled with quintessence-like type scalar field and as we demonstrate, the Chern-Simons axion term does not affect the propagation of these modes at all. Particularly, there exist two propagating scalar modes, and no propagating vector mode. With regard to the two tensor modes, the resulting dispersion relations are more complicated compared with the Einstein gravity with...
the Chern-Simons axion term, but the qualitative features of the propagating modes are not changed, and actually non-equivalent propagation occur in both Einstein Chern-Simons and \( F(R) \) gravity Chern-Simons theories.

This paper is organized as follows: In section II we present the general features of Chern-Simons Axion \( F(R) \) gravity models. In section III we extract the general gravitational wave equations, while in section IV the various gravitational wave modes, and the corresponding polarizations, are studied. Finally, the conclusions follow in the end of the paper.

II. THE CHERN-SIMONS CORRECTED AXION \( F(R) \) GRAVITY

In this paper we shall mainly consider a Axionic Chern-Simons corrected \( F(R) \) gravity model, whose action is given by,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ F(R) - \frac{\omega(\phi)}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + U(\phi) \varepsilon^{\mu\nu\rho\sigma} R^\tau_{\lambda\mu
u} R^\lambda_{\tau\rho\sigma} \right],
\]

where we defined the totally antisymmetric Levi-Civita symbols \( \epsilon_{\mu\nu\rho\sigma} \) and \( \epsilon^{\mu\nu\rho\sigma} \) as follows,

\[
\epsilon_{0123} = -\epsilon^{0123} = 1,
\]

and in addition,

\[
\epsilon_{\mu\nu\rho\sigma} = \eta_{\mu\mu'} \eta_{\nu\nu'} \eta_{\rho\rho'} \eta_{\sigma\sigma'} \epsilon^{\mu'\nu'\rho'\sigma'}.
\]

Then we obtain the following tensors,

\[
\tilde{\epsilon}^{\mu\nu\rho\sigma} \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma}, \quad \equiv g_{\mu\mu'} g_{\nu\nu'} g_{\rho\rho'} g_{\sigma\sigma'} \epsilon^{\mu'\nu'\rho'\sigma'} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}.
\]

We should note that the following tensor identity holds true,

\[
\nabla_\sigma \tilde{\epsilon}^{\eta\eta\rho\xi} = 0.
\]

Since,

\[
\delta \left( \sqrt{-g} U(\phi) \tilde{\epsilon}^{\mu\nu\rho\sigma} R^\tau_{\lambda\mu
u} R^\lambda_{\tau\rho\sigma} \right) = 2\sqrt{-g} U(\phi) \left[ \tilde{\epsilon}^{\eta\eta\nu\rho} R^\tau_{\lambda\rho\xi} + \tilde{\epsilon}^{\eta\eta\nu\rho} R^\tau_{\lambda\xi\rho} \right] \nabla_\rho \nabla_\tau \delta g_{\mu\nu},
\]

upon varying the action \( \Box \) with respect to the metric, we obtain the equations as follows,

\[
0 = \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) + \nabla_\rho \nabla_\nu F'(R) - g_{\mu\nu} \Box F'(R)
\]

\[
+ \frac{1}{2} \left\{ -\frac{\omega(\phi)}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\} g_{\mu\nu} + \frac{\omega(\phi)}{2} \partial_\mu \phi \partial_\nu \phi
\]

\[
+ 2 (g_{\mu\xi} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\xi}) \nabla_\tau \nabla_\rho \left( U(\phi) \tilde{\epsilon}^{\eta\eta\rho\xi} R^\tau_{\lambda\rho\xi} \right).
\]

The equation obtained from the variation of the action with respect to the scalar field \( \phi \) is given by,

\[
0 = \nabla^\mu (\omega(\phi) \partial_\mu \phi) - V'(\phi) + U'(\phi) \epsilon^{\mu\nu\rho\sigma} R^\tau_{\lambda\mu\nu} R^\lambda_{\tau\rho\sigma}.
\]

We now assume that the geometric background is a Friedmann-Robertson-Walker (FRW) spacetime with flat spatial part,

\[
ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,
\]

and we also assume that the scalar field \( \phi \) depends solely on the cosmological time \( t \). In the FRW background, we obtain,

\[
\Gamma^i_{ij} = a^2 H \delta_{ij}, \quad \Gamma^i_{jt} = \Gamma^i_{ij} = H \delta^i_j, \quad \Gamma^i_{jk} = \Gamma^i_{jk},
\]
Then we can find the equations corresponding to the perturbed Einstein equation, which are presented in the Appendix metric is

\[ a^2 \phi^2 (\delta_{ij} - \delta_{ij}) \]

\[ R_{ij} = -\left( \dot{H} + H^2 \right) a^2 \delta_{ij}, \quad R_{ijkl} = a^4 H^2 (\delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj}) , \]

\[ R_{tt} = -3 \left( \dot{H} + H^2 \right), \quad R_{ij} = a^2 \left( \dot{H} + 3H^2 \right) \delta_{ij}, \quad R = 6\dot{H} + 12H^2; \]

other components = 0,

\[ (10) \]

and from these we obtain the following equations,

\[ 0 = -\frac{1}{2} F(R_0) + 3 \left( \dot{H}^2 + \dot{\dot{H}} \right) F'(R_0) - 18 \left( 4H^2 \dot{H} + H \dot{\dot{H}} \right) F''(R_0) + \frac{\omega(\phi)}{4} \dot{\phi}^2 + \frac{V(\phi)}{2}, \]

\[ 0 = \frac{1}{2} F(R_0) - \left( \dot{H} + 3H^2 \right) F'(R_0) + 24 \left( 4H^2 \dot{H} + \dot{\dot{H}}^2 + 2H \dot{\dot{H}} \right) F''(R_0) + 36 \left( 4H \dot{H} + H \dot{\theta} \right)^2 F'''(R_0) \]

\[ + \frac{\omega(\phi)}{4} \dot{\phi}^2 - \frac{V(\phi)}{2} . \]

\[ (11) \]

Here \( R_0 = 12H^2 + 6\dot{H} \) and we have neglected the contributions from matter perfect fluids. We should note that the term containing the scalar coupling term to the Chern-Simons function, namely, \( U(\phi) \), does not contribute to the above equations in \( (11) \). We may choose \( \phi \) to be the cosmological time \( t \), that is, \( \phi = t \). Then the equations in \( (11) \) can be rewritten as,

\[ \omega(\phi) = -4H F'(R_0) - 4 \left( 12H^2 \dot{H} + 12\dot{H} + 15H \dot{\dot{H}} \right) F''(R_0) - 72 \left( 4H \dot{H} + \dot{\dot{H}} \right)^2 F'''(R_0), \]

\[ V(\phi) = F(R_0) - 2 \left( 2\dot{H} + 3H^2 \right) F'(R_0) + \left( 168H^2 \dot{H} + 24\dot{H}^2 + 66H \dot{\dot{H}} \right) F''(R_0) + 36 \left( 4H \dot{H} + \dot{\dot{H}} \right)^2 F'''(R_0) . \]

\[ (12) \]

Then if we choose,

\[ \omega(\phi) = -4f(\phi) F'(R_f) - 4 \left( 12f(\phi)^2 f'(\phi) + 12f'(\phi)^2 + 15f(\phi) f''(\phi) \right) F''(R_f) \]

\[ - 72 \left( 4f(\phi)f'(\phi) + f''(\phi) \right)^2 F'''(R_f), \]

\[ V(\phi) = F(R_f) - 2 \left( 2f'(\phi) + 3f(\phi)^2 \right) F'(R_f) + \left( 168f(\phi)^2 f'(\phi) + 24f'(\phi)^2 + 66f(\phi) f''(\phi) \right) F''(R_f) \]

\[ + 36 \left( 4f(\phi)f''(\phi) + f''(\phi) \right)^2 F'''(R_f), \]

\[ R_f := 12f(\phi)^2 + 6f'(\phi). \]

\[ (13) \]

a solution of the equations in \( (11) \) is given by \( H = f(t) \) and \( \phi = t \).

### III. GRAVITATIONAL WAVE EQUATIONS

We now investigate the propagation of gravitational waves in the Chern-Simons Axion \( F(R) \) gravity. In order to study the propagation of the gravitational waves, we consider the perturbation of Eq. \( (7) \), from the background whose metric is \( g_{\mu\nu}^{(0)}, \)

\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} , \quad \phi = \phi^{(0)} + \varphi, \]

\[ (14) \]

Then we can find the equations corresponding to the perturbed Einstein equation, which are presented in the Appendix in Eq. \( (14) \) due to the extended analytic form of these have. On the other hand, Eq. \( (8) \) gives,

\[ 0 = -\frac{1}{2} g^{(0)\mu\nu} h_{\mu\nu} \nabla^{(0)\rho} \left( \omega \left( \phi^{(0)} \right) \partial_\rho \phi^{(0)} \right) - \nabla^{(0)\rho} \left( h^{(0)\mu\nu} \omega \left( \phi^{(0)} \right) \partial_\mu \phi^{(0)} \right) + \frac{1}{2} \nabla^{(0)\rho} \left( g^{(0)\mu\nu} h_{\mu\nu} \omega \left( \phi^{(0)} \right) \partial_\rho \phi^{(0)} \right) \]

\[ + 2U' \left( \phi^{(0)} \right) \left[ \epsilon^{(0)\mu\nu\rho\sigma} R^{(0)\tau\sigma}_{\xi\eta} + \epsilon^{(0)\mu\nu\rho\tau} R^{(0)\sigma\mu}_{\xi\eta} \right] \nabla^{(0)\rho} \nabla^{(0)\sigma} h_{\mu\nu} - \frac{1}{2} g^{(0)\mu\nu} h^{(0)\rho\sigma} U' \left( \phi^{(0)} \right) \epsilon^{(0)\mu\nu\rho\sigma} R_{\lambda\mu\nu} R^{(0)\lambda}_{\tau\rho\sigma} \nabla^{(0)\tau} \nabla^{(0)\lambda} \phi^{(0)} \]

\[ + \nabla^{(0)\mu} \left( \omega' \left( \phi^{(0)} \right) \varphi \partial_\mu \phi^{(0)} \right) + \nabla^{(0)\nu} \left( \omega' \left( \phi^{(0)} \right) \partial_\nu \phi^{(0)} \right) - V'' \left( \phi^{(0)} \right) \varphi + U'' \left( \phi^{(0)} \right) \varphi \theta^{(0)\mu\nu\rho\sigma} R^{(0)\tau}_{\lambda\mu\nu} R^{(0)\lambda}_{\tau\rho\sigma} \varphi^{(0)\mu\nu\rho\sigma} R_{\lambda\mu\nu} R^{(0)\lambda}_{\tau\rho\sigma} . \]

\[ (15) \]

The explicit form of the \((t,t), (i,j), \) and \((t,i)\) components of the modified Einstein equations \( \delta G_{\mu\nu} = 0 \) from Eq. \( (14) \) in the FRW background \( (9) \) are given as follows,

\[ \delta G_{t \ t} = g^{(0)\mu\nu} \delta G_{\mu\nu} \]
presented in the Appendix in Eq. (45) due to their extended form. The corresponding Levi-Civita symbol, which is totally antisymmetric tensor, in three dimensions as follows,

$$H'' = 2 \ddot{g} - \frac{1}{2} \left[ \omega (\phi^{(0)}) \phi^{(0)} - V (\phi^{(0)}) \right] + \frac{1}{2} \left[ \omega (\phi^{(0)}) \phi^{(0)} - V (\phi^{(0)}) \right] \phi^{(0)} \varphi + V' (\phi^{(0)}) \varphi$$

We define the Levi-Civita symbol, which is totally antisymmetric tensor, in three dimensions as follows,

$$\epsilon_{xyz} \equiv a^3, \quad \epsilon^{ijk} \equiv g^{(0)\mu \nu} g^{(0)\rho \sigma} \epsilon_{\mu \nu \rho \sigma} = a^{-6} \epsilon^{ijk}.$$ (17)

Then we can obtain the non-zero components of the perturbed Einstein tensor. Specifically, the components $\delta G_{ij}$ are presented in the Appendix in Eq. (45) due to their extended form. The corresponding $\delta G_{ij}$ components are,

$$\delta G_{ij} \equiv g^{(0)\mu \nu} \delta G_{\mu \nu}$$

We should note that all the terms coming from the last term in Eq. (14) identically vanish. A more explicit form of Eq. (15) is given by,

$$0 = \frac{1}{2} g^{(0)\mu \nu} h_{\mu \nu} \left( \partial_k + 3H \right) \left( \omega (\phi^{(0)}) \phi^{(0)} - \partial_k \left( h^{(0)} \omega (\phi^{(0)}) \phi^{(0)} \right) - 3H h^{(0)} \partial_k \left( \omega (\phi^{(0)}) \phi^{(0)} \right) \right)$$
\[-\frac{1}{2}(\partial_t + 3H) \left( g^{(0)\mu\nu} h_{\mu\nu} \partial_t \phi^{(0)} \right) \]
\[-(\partial_t + 3H) \left( \omega \left( \phi^{(0)} \right) \partial_t \phi^{(0)} \right) - (\partial_t + 3H) \left( \omega \left( \phi^{(0)} \right) \partial_t \phi^{(0)} \right) + a^3 \omega \left( \phi^{(0)} \right) \partial^k \partial_k \phi - V'' \left( \phi^{(0)} \right) \phi. \quad (19)\]

IV. POLARIZATIONS OF GRAVITATIONAL WAVES

The most important study in the Chern-Simons Axion $F(R)$ gravity is related to the polarization modes of the gravitational waves. We now consider the following modes,

- Spin 2 tensor mode
  \[ \dot{h}_{ij}, \quad h_{tt} = h_{tt} = 0, \quad h^i_i = 0, \quad \partial^j h_{ij} = 0, \quad (i = x, y, z), \quad \varphi = 0. \quad (20)\]

- Spin 1 vector mode
  \[ A_i \equiv \partial^j h_{ji}, \quad \partial^i \partial^j h_{ij} = \partial^i A_i = 0, \quad h_{tt} = h_{tt} = 0, \quad h^i_i = 0, \quad (i, j = x, y, z), \quad \varphi = 0. \quad (21)\]

- Spin 0 scalar mode
  \[ h = h^i_i, \quad h_{ij} = \partial_i \partial_j B - \frac{1}{3} \delta^{ij}_{kl} \partial^k \partial_l B, \quad \varphi. \quad (22)\]

Then the tensor $h_{ij}$ can be decomposed as follows,

\[ h_{ij} = \dot{h}_{ij} + \frac{1}{2} (\partial^k \partial_k)^{-1} (\partial_i A_j + \partial_j A_i) + \frac{1}{2} g^{(0)ij} h + \partial_i \partial_j B - \frac{1}{3} \delta^{ij}_{kl} \partial^k \partial_l B, \quad (23)\]

which gives,

\[ B = \frac{3}{2} (\partial^k \partial_k)^{-2} \partial^i \partial^j h_{ij} - \frac{1}{2} (\partial^k \partial_k)^{-1} h \quad \text{or} \quad \partial^i \partial^j h_{ij} = \frac{2}{3} (\partial^k \partial_k)^2 B + \frac{1}{3} \partial^k \partial_k h. \quad (24)\]

For all the above modes we considered, we have implicitly chosen the gauge condition $h_{t\mu} = 0$.

A. Spin 2 tensor mode

Let us study in some detail the spin 2 tensor mode. In the case of spin 2 mode, we find that $\delta G^t_i$ in (16) and $\delta G^t_i$ in Eq. (18) trivially vanish. In addition $\delta G^t_i = \delta G^t_i = 0$ and Eq. (19) is also trivially satisfied. On the other hand, $\delta G^j_i$ in (15) has the following form,

\[ \delta G^j_i = -\epsilon^{klm} \left( \delta^i_{k} \delta^j_{lm} + \delta^i_{n} \delta^j_{km} \right) \left\{ 4H \dot{U} + 2 \dot{U} \right\} \partial_t \partial_i \dot{h}^m_m + 2 \dot{U} \partial_t \partial_i \dot{h}^2_m \right\} \\
+ F'' \left\{ -3 \dot{H} \partial_i \dot{h}^j_j - \frac{1}{2} \partial_i \partial_j \dot{h}^j_j + \frac{1}{2} \partial_i \partial_j \dot{h}^i_j \right\} \\
+ F'' \left\{ -12 \dot{H} \partial_i \dot{h}^j_j + 3 \partial_i \dot{h}^j_j \right\} - 12 \dot{H} \dot{h}^2_j \right\} \\
+ F'' \left\{ -36 \dot{H} \dot{h}^2_j \right\} - 288 \dot{H} \dot{h}^j_j - 576 \dot{H}^2 \dot{h}^2_j \right\} + \frac{1}{2} \left( \omega \left( \phi^{(0)} \right) \right)^2 - V \left( \phi^{(0)} \right) \right\} \hat{h}^j_j. \quad (25)\]

We should note that the obtained equation (25) is a second order differential equation with respect to the cosmic time $t$, although the original equation (17) or (15) is a fourth order difference equation. This is not curious, because the $F(R)$ gravity action can be rewritten in the form of a scalar-tensor theory, that is, the rewritten action is the sum of the Einstein-Hilbert action and the action of the scalar field with potential. So in effect, the spin-two mode originates from the Einstein-Hilbert part, which gives the standard scalar equation, that is, the second order differential equation. The existence of $U$-terms in (25) gives the mixing of $+$-mode and $x$-mode and the dispersion relation of the left-polarized mode is different from that of the right-polarized mode (see also [10], for example). We should
note that there appear terms including the first derivative with respect to the cosmic time, \( \partial_t \hat{h}_j \), which generate an enhancement or dissipation of the gravitational wave. By using (11), we may rewrite (25) as follows,

\[
\delta G^i_j = - \epsilon^{klm} \left( \delta^i_k g^j_n + \delta^i_n g^j_k \right) \left\{ \left( 4H U + 2 \dot{U} \right) \partial_t \hat{h}^n_m + 2 \dot{U} \partial_t \partial^2 \hat{h}^n_m \right\}
- \frac{1}{2} F \dot{h}^i_j + F' \left\{ - \frac{3}{2} H \partial_t \hat{h}^i_j - \frac{1}{2} \partial_t^2 \hat{h}^i_j + \frac{1}{2} \partial^2 \partial_t \hat{h}^i_j + \left( \dot{H} + 3 H^2 \right) \hat{h}^i_j \right\}
+ F'' \left\{ 3 \dot{H} \partial_t \hat{h}^i_j + 12 H \partial_t \partial^2 \hat{h}^i_j - \left( 42 H \dot{H} + 168 H^2 \dot{H} + 48 H^2 + 48 H \ddot{H} \right) \hat{h}^i_j \right\}
+ F''' \left\{ - 72 \ddot{H}^2 - 576 \dddot{H} \dot{H} H - 1152 \dddot{H}^2 H^2 \right\} \hat{h}^i_j ,
\]

(26)

Just for simplicity, we consider the case that \( \dot{U} \) and \( H \), and therefore, \( F, F', F'' \), and \( F''' \) can be regarded as constants. Then Eq. (26) can be reduced as follows,

\[
\delta G^i_j = - \epsilon^{klm} \left( \delta^i_k g^j_n + \delta^i_n g^j_k \right) \dot{U} \left\{ 4H \partial_t \hat{h}^n_m + 2 \partial_t \partial^2 \hat{h}^n_m \right\}
- \frac{1}{2} F \dot{h}^i_j + F' \left\{ - \frac{3}{2} H \partial_t \hat{h}^i_j - \frac{1}{2} \partial_t^2 \hat{h}^i_j + \frac{1}{2} \partial^2 \partial_t \hat{h}^i_j + 3 H^2 \hat{h}^i_j \right\} ,
\]

(27)

We consider the plane wave propagating in the \( z \)-direction with the wave number \( k \) and frequency \( \omega \), \( \hat{h}^i_j = h^{(0)i}_j e^{-i \omega t + ik z} \) with constants \( h^{(0)j}_i \). Then Eq. (20) tells \( h^{(0)z}_j = h^{(0)i}_j = 0 \) and in effect we find,

\[
\delta G^x = 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} h^{(0)x}_y + \left\{ - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) \right\} h^{(0)x}_x ,
\]

\[
\delta G^y = - 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} h^{(0)y}_y + \left\{ - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) \right\} h^{(0)y}_y ,
\]

\[
\delta G^y_x = 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} h^{(0)x}_y + \left\{ - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) \right\} h^{(0)y}_x ,
\]

\[
\delta G^x_x = 0 .
\]

(28)

Usually we consider the following two modes, namely the + mode where \( h^+ = h^{(0)x}_{y} = - h^{(0)y}_{x} \) and the × mode where \( h_{\times} = h^{(0)x}_{y} = h^{(0)y}_{x} \). In terms of this mode, the non-trivial equations in (28) take the following forms,

\[
0 = 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} h^{(0)}_{\times} + \left\{ - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) \right\} h^{(0)}_{\times} ,
\]

\[
0 = - 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} h^+ + \left\{ - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) \right\} h^+ .
\]

(29)

The above equations indicate that there should always be a mixing between the + mode and × mode and they should appear in the forms of the left-handed or right handed mode, where \( h^+ = \pm h_{\times} \). The equations in (29) give also the following dispersion relation,

\[
0 = \pm 2 \omega k \dot{U} \left\{ 4H - 2i \omega \right\} - \frac{1}{2} F + F' \left( \frac{3}{2} i \omega H + \frac{1}{2} \omega^2 - \frac{1}{2} k^2 + 3 H^2 \right) .
\]

(30)

The terms including \( i \omega \) in (30) come from the terms including \( \partial_t \hat{h}^i_j \) in (25), which generate the enhancement or dissipation of the gravitational wave. In the case of the Chern-Simons axion Einstein gravity (22), we have \( F' = 1 \) and \( F = R \sim 24 H^2 \). Therefore the qualitative structure of the dispersion relation and the left- and right-handed modes are not so changed from those corresponding to the Chern-Simons axion Einstein gravity. In case of the Chern-Simons axion \( F(R) \) gravity, \( F(R) \) depends on the time coordinate \( t \) and therefore the full solution of the gravitational wave in (27) becomes rather complicated. We should note that the polarization of the gravitational wave in the early Universe also affects the polarization of CMB, and specifically the E-mode and B-modes, see for example (101).
We now investigate the dispersion relation (29) in more detail. Eq. (29) can be solved as,

$$\omega = -\left(\frac{3}{2} F^t H + \delta_{LR} 8 k \dot{U} H\right) \pm \sqrt{F^t^2 k^2 - \frac{32}{3} F^t^2 H^2 + 64 k^2 \dot{U}^2 H^2 + F^t F + i \delta_{LR} \left(72 k F^t \dot{U} H^2 - 8 k \dot{U} F - 8 k \dot{U} F^t k^3\right)} \frac{2 \left(\frac{\dot{U}'}{F^t} - \delta_{LR} 4 i k \dot{U}\right)}{2}.$$  

(31)

Here $\delta_{LR} = \pm 1$ for left or right-handed mode. For the high energy mode, for which $k \gg H$ and by neglecting the contribution from the Chern-Simons term, that is, $k \ll \frac{F^t}{U}$, we obtain $\omega \sim \pm k$. Therefore the propagating speed of the gravitational wave is not changed. When $\omega$ is real number, $\omega$ should be positive and we should choose the plus sign $+ \pm$ in (31). We should note, however, that for the very high energy mode when $\dot{U}$ does not vanish, $\dot{U} \neq 0$, that is, the mode $k \gg H$ and $k \gg \frac{F^t}{U}$. Eq. (30) has the form $0 = -\delta_{LR} 4 i \omega^2 k \dot{U} - \frac{\dot{U}^2}{F^t} k^2$ and therefore we find $\omega^2 \sim -\delta_{LR} i \frac{\dot{U}'}{F^t} k$, which is rather strange dispersion relation. By assuming that the real part of $\omega$ is positive we obtain $\omega = e^{\pm \frac{\pi}{4} \sqrt{|\frac{\dot{U}'}{F^t}| k}}$. Therefore there always appear an amplified and a decaying mode. Furthermore the group velocity $v_g$ and the phase velocity $v_p$ are given by $v_g \equiv \frac{\dot{U}}{\dot{U}'} \times \frac{1}{V_k}$ and $v_p \equiv \frac{\dot{U}'}{F^t} \times \frac{1}{V_k}$, which become smaller for larger $k$ and much smaller than the velocity of light.

B. Spin 1 Vector Mode

In the case of the spin 1 mode, we find $\delta G^t_i$ in (16) vanishes and Eq. (19) is also trivially satisfied, again, $\delta G^t_i = 0$. On the other hand, $\delta G^i_j$ in (18) has the following form,

$$\delta G^i_j = \frac{2 \dot{U}}{a^2} \epsilon_{ikl} \partial^k \partial^i \dot{A}^l - \frac{1}{2} F^t \partial^i \dot{A}^l = 0,$$  

(32)

and $\delta G^j_i$ in Eq. (15) has the following form,

$$\delta G^j_i = \epsilon^{klm} \left(\delta_{ij} g^{(0)}_{lm} + \delta_{ij} g^{(0)}_{jm} \right) \left[- \left\{4 \dot{H} \dot{U} + 2 \dot{U} \right\} \partial_i \partial_j h^{(A)} \partial^m - 2 \dot{H} \partial_i \partial^m \right]$$

$$+ F^t \left\{- \frac{3}{2} \dot{H} \partial_i h^{(A)} \partial^m + \frac{1}{2} \partial^m h^{(A)} \partial_i \partial_j - \frac{1}{2} \left(\partial^i A_j + g^{(0)} \partial^i \partial_j A_m \right) \right\}$$

$$+ F^t \left\{ \dot{H} \left[-42 \dddot{H} h^{(A)} \partial_i + 3 \partial_i h^{(A)} \partial_j - 2 \dot{H} \dddot{h}^{(A)} h^{(A)} \partial^i + \dddot{h}^{(A)} \partial^i \right] \right\}$$

$$+ F^t \left\{ -36 \dddot{H} h^{(A)} \partial_i - 288 \dddot{H} H h^{(A)} \partial^i - 576 \dddot{H} H h^{(A)} \partial^i \right\} + \frac{1}{2} \left\{ \omega \left(\phi^{(0)}\right) \left(\phi^{(0)}\right)^2 - V \left(\phi^{(0)}\right) \right\} h^{(A)} \partial_i \partial_j \right\} = 0,$$  

(33)

where $h^{(A)}_{ij}$ is,

$$h^{(A)}_{ij} = \frac{1}{2} \left(\partial^k h^{(A)} \partial^i \partial^k \partial^j \right) \left(\partial_i A_j + \partial_j A_i \right).$$  

(34)

We now discuss the qualitative implications of Eq. (32). When $U = 0$ as in the standard $F(R)$ gravity, we obtain $\dot{A} = 0$. In effect, there is no time evolution of $A_i$, or no propagating mode and therefore $A_i$ is determined by the initial conditions consistent with (33). When $U \neq 0$, since there is a rotational symmetry, we may consider the plane wave propagating in the $z$-direction with the wave number $k$, $A_i = \alpha_i(t)e^{ikz}$. Eq. (21) also indicates that $A_z = \alpha_z(t) = 0$. Then the $i = z$ component in Eq. (32) is trivially satisfied and we obtain the following non-trivial equations,

$$0 = -\frac{2 i k \dot{U}}{a} \partial_t \left(\alpha^{-2} \alpha_y(t)\right) + \frac{1}{2} F^t \partial_t \alpha_x(t),$$

$$0 = \frac{2 i k \dot{U}}{a} \partial_t \left(\alpha^{-2} \alpha_x(t)\right) + \frac{1}{2} F^t \partial_t \alpha_y(t),$$

which can be rewritten in a matrix form as follows,

$$\left(\frac{2 i k \dot{U}}{a} \partial_t - \frac{1}{2} \frac{F^t}{a^2} \right) \left(\begin{array}{c} \partial_x \\ \partial_y \end{array}\right) = \left(\frac{2 i k \dot{U}}{a} \partial_t \right) \left(\begin{array}{c} \alpha_x \\ \alpha_y \end{array}\right) = \frac{2 i k \dot{U} H \alpha_x}{a^3} \left(\begin{array}{c} -\alpha_x \\ \alpha_y \end{array}\right).$$  

(36)
As an example, we consider the case that $H$, $\frac{\dot{H}}{a}$, and $F'$ are constant. Then by assuming $\alpha_x = \alpha_x^{(0)} e^{-i\omega t}$ and $\alpha_y = \alpha_y^{(0)} e^{-i\omega t}$ with constants $\omega$, $\alpha_x^{(0)}$ and $\alpha_y^{(0)}$, we obtain,

\[
\begin{pmatrix}
\frac{2kU}{a} (-i\omega + H) \\
\frac{1}{2} F' \end{pmatrix} \begin{pmatrix}
\alpha_x^{(0)} \\
\alpha_y^{(0)}
\end{pmatrix} = 0.
\]  

(37)

In order that Eq. (37) has non-trivial solutions for $\alpha_x^{(0)}$ and $\alpha_y^{(0)}$, the determinant of the matrix should vanish, and this constraint gives the following dispersion relation,

\[
0 = \frac{4k^2 \dot{U}^2}{a^6} (\omega^2 + H^2) + \frac{1}{4} F'^2,
\]  

(38)

which indicates that $\omega$ must be purely imaginary and therefore there is no propagating mode. The above result is true for the high frequency mode, even if $H$, $\frac{\dot{H}}{a}$, and $F'$ are not constant. Therefore, there is no propagating mode of Spin 1, which is consistent with the standard requirement coming from the general covariance.

C. Spin 0 scalar mode

Let us now consider the spin-0 mode, which is also present in the pure $F(R)$ gravity. For the spin-0 mode, we find,

\[
\delta G^{it} = \frac{1}{3} F' \left\{ (\partial^k \partial_t)^2 B - \partial^k \partial_t h \right\}
+ F' \left\{ \dot{H} \left[ -12H h - 3 \partial_t h \right] + H \left[ -48 H^2 h - 12H \partial_t h - 3 \partial_t^2 h - 2 (\partial^k \partial_t h)^2 B + 2 \partial^k \partial_t h \right]
- 12 H^3 \partial_t h + H^2 \left( 9 \partial_t^2 h - 6 (\partial^k \partial_t h)^2 B + 6 \partial^k \partial_t h \right) + H \left( 3 \partial_t^3 h + (\partial_t + 2H) \left( 2 (\partial^k \partial_t h)^2 B - 6 \partial^k \partial_t h \right) \right)
- \partial^i \partial_t \partial_t^2 h - \frac{2}{3} \partial^i \partial_t \left( (\partial^k \partial_t h)^2 B - \partial^k \partial_t h \right) \right\}
+ F'' \left\{ \dot{H} \left[ 72H^2 \partial_t h + 18H \left( \partial_t^2 h + 12 \left( (\partial^k \partial_t h)^2 B - \partial^k \partial_t h \right) \right) \right]
+ H \left[ 288 H^3 \partial_t h + 72 H^2 \left( \partial_t^2 h + \left( \frac{2}{3} (\partial^k \partial_t h)^2 B + \frac{1}{3} \partial^k \partial_t h \right) - \partial^i \partial_t h \right) \right]
- \frac{1}{2} \left\{ \omega' \left( \phi^{(0)} \right)^2 \varphi + \omega \left( \phi^{(0)} \right) \dot{\phi}^{(0)} \right\} \right\}
= 0,
\]  

(39)

\[
\delta G^{ij} = \epsilon^{klm} \left( \delta^{i} (g_{jm}^{(0)} + \delta^i g_{jm}) \right) \left[ - \left\{ 4 H \dot{U} + 2 \ddot{U} \right\} \partial_t \partial_t h^{(S)} m + 2 \dot{U} \partial_t \partial_t h^{(S)} m \right]
+ \frac{1}{2} \dot{U} \left( \partial^k \partial_t \partial_t h^{(S)} m - \partial_t \partial_t \partial_t h^{(S)} m \right)
F' \left\{ \dot{H} \left[ \frac{3}{2} \partial_t h \delta^i j - \frac{3}{2} \partial_t h^{(S)} i \right] \right\}
- \frac{1}{2} \partial_t^2 h^{(S)} j + \left( \frac{1}{2} \partial_t^2 h - \partial^k \partial_t h + \frac{2}{3} (\partial^k \partial_t h)^2 B + \frac{1}{3} \partial^k \partial_t h \right) \delta^i j + \frac{1}{2} \partial^k \partial_t h^{(S)} i j + \frac{1}{2} \partial^i \partial_t h
- \frac{1}{2} \left( \partial^i \partial_t h^{(S)} k j + g^{(0) \delta^i k} g^{(0) \partial^i \partial_t h^{(S)} k} m \right)
\right. \right. 
+ F'' \left\{ \ddot{H} \left[ -42 H h^j + (-12H + \partial_t h) \delta^i j + 3 \partial_t h^{(S)} i j \right] - 24 \dot{H}^2 24h^{(S)} i j \right. \right. 
+ H \left[ -72 H^2 h^{(S)} i j + 12 \partial_t h^{(S)} i j + (-48 H^2 h - 8H \partial_t h) \delta^i j + \left( 7 \partial_t^2 h + 2 \partial^k \partial_t h - 2 (\partial^k \partial_t h)^2 B \right) \delta^i j \right]
- 12 H^3 \delta^i j + H^2 \left[ 5 \partial_t^2 h + 2 \partial^k \partial_t h - 2 (\partial^k \partial_t h)^2 B \right] \delta^i j 
+ H \left[ 4 \partial^i \partial_t \partial_t h + \left( 6h - 2 \partial^k \partial_t \partial_t h - 2 (\partial_t + 4H) \left( \frac{2}{3} (\partial^k \partial_t h)^2 B + \frac{1}{3} \partial^k \partial_t h \right) \right) \delta^i j \right]
+ \partial^i \partial_t \partial_t \partial_t h - \partial^k \partial_t \partial_t h
\]
\[+ (\partial_t h - 2 \partial^k \partial_k \partial_t^2 h)\]
\[+ (\partial_t + 4H)^2 \left( \frac{2}{3} (\partial^k \partial_k)^2 B + \frac{1}{3} \partial^k \partial_k h - \partial_t \partial_t \left( \frac{2}{3} (\partial^k \partial_k)^2 B + \frac{1}{3} \partial^k \partial_k h \right) \right) \delta^j_i \}
\[+ F' \left\{ \frac{\dot{H}}{H} \left[ 2AH \partial_t h + 6d_l^2 h + 4 \left( (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right) \right] \delta^i_j - 36 \dot{H}^2 \delta^i_j \}
\[+ \dot{H} \left[ \delta^i_j + 144H^2 \partial_t h \delta^i_j + H \left( 84 \partial_t^2 h + 8 (\partial^k \partial_k)^2 B - 8 \partial^k \partial_k h \right) \delta^i_j \right] + \left( 12 \partial_t^2 h - 12 (\partial_t + 2H) \left( (\partial^k \partial_k)^2 B + \frac{1}{3} \partial^k \partial_k h \right) \right) \delta^i_j \}
\[+ H^2 \left[ -576 H^2 \delta^i_j + 288 \partial_t h + 24 \partial_t^2 h + 16 (\partial^k \partial_k)^2 B - 16 \partial^k \partial_k h \right] \delta^i_j \}
\[+ \dot{H} \left[ 192 \partial_t^3 h + 240 \partial_t^2 h + H^2 \left( -32 (\partial^k \partial_k)^2 B + 32 \partial^k \partial_k h \right) + 48H \partial_t^2 h + 32 \partial_t h \left( (\partial^k \partial_k)^2 B + \partial^k \partial_k h \right) \right] \delta^i_j \}
\[+ F'' \left\{ \dot{H} \left[ 14AH h + 36 \partial_t^2 h + 24 (\partial^k \partial_k)^2 B - 24 \partial^k \partial_k h \right] \delta^i_j \right\}
\[+ \dot{H} \left[ 1152 H^2 h + 288 \partial_t^2 h + 192 \left[ (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right] \right] \delta^i_j \}
\[+ H^2 \left[ 2304H^3 h + 576 H^2 \partial_t^2 h + 384 H \left( (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right) \right] \delta^i_j \right\}
\[+ \frac{1}{2} \left\{ \frac{\omega (\phi^{(0)})}{2} \left( \phi^{(0)} \right)^2 - V \left( \phi^{(0)} \right) \right\} \delta^i_j \]
\[+ \frac{1}{2} \left\{ \frac{\omega' (\phi^{(0)})}{2} \left( \phi^{(0)} \right)^2 \phi - \omega \left( \phi^{(0)} \right) \phi' \right\} \delta^i_j \right\} \]
\[= 0 , \quad (40) \]
\[\delta G_{ij} = \frac{2 \dot{U}}{a^2} \epsilon_{ikl} \partial^k \partial^m \partial^m \partial^l \delta^{(S)} h_{ij} + F' \left\{ \frac{1}{2} \partial_t \partial^i \partial^j h - \frac{1}{2} \partial^i \partial^j \delta^{(S)} h_{ik} \right\}
\[+ F'' \left\{ -4 \dot{H} \partial_t \partial_t h + 4H^2 \partial_t \partial_t h + H \left[ -3 \partial_t \partial_t^2 h + 2 \left( (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right) \right] \right\}
\[+ \partial_t \partial_t^2 h + \frac{2}{3} \partial_t (\partial_t + 2H) \left( (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right) \right\}
\[+ F'' \left\{ \dot{H} \left[ -24 \partial_t h \partial_t \partial_t h - 6 \partial_t \partial_t^2 h - 4 \partial_t \partial_t \left( (\partial^k \partial_k)^2 B + (\partial^k \partial_k h) \right) \right] \right\}
\[+ \dot{H} \left[ -96 H^2 \partial_t h + 24 \partial_t \partial_t^2 h + 16 \left( (\partial^k \partial_k)^2 B - \partial^k \partial_k h \right) \right] \right\}
\[+ \frac{\omega (\phi^{(0)})}{2} \phi^{(0)} \partial_t \phi \]
\[= 0 , \quad (41) \]
where \( h_{ij}^{(S)} \) is equal to,
\[h_{ij}^{(S)} = \frac{1}{3} g_{ij}^{(0)} h + \partial_t \partial_t h - \frac{1}{3} g_{ij}^{(0)} \partial^k \partial_k h . \quad (42) \]
Eq. (19) takes the following form,
\[0 = \frac{1}{2} \left( \partial_t \partial_t + 3H \right) \left( \omega \left( \phi^{(0)} \right) \phi^{(0)} \right) - \frac{1}{2} \left( \partial_t + 3H \right) \left( \left( h \omega \left( \phi^{(0)} \right) \phi^{(0)} \right) \right)
\[- \left( \partial_t + 3H \right) \left( \omega \left( \phi^{(0)} \right) \phi^{(0)} \right) \right) - \left( \partial_t + 3H \right) \left( \omega \left( \phi^{(0)} \right) \phi^{(0)} \right) + a^2 \omega \left( \phi^{(0)} \right) \partial^k \partial_k \phi - V'' \left( \phi^{(0)} \right) \phi . \quad (43) \]
Since the Chern-Simons term does not contribute to the above equations, as in the \( F(R) \) gravity with a scalar field, there appear two propagating scalar modes. It is notable, and expected though, that the Chern-Simons term affects solely the tensor gravitational wave modes, and not the spin-0 mode.
V. SUMMARY

In summary, we have investigated the gravitational wave in the context of Chern-Simons axion $F(R)$ gravity. For the spin-0 scalar mode and the spin-1 vector mode, we demonstrated that for these modes, the situation is not changed from the standard $F(R)$ gravity coupled with quintessence type scalar field, and in addition, the Chern-Simons axion term does not affect the propagation of these modes. This result was also known for the case of Chern-Simons axion Einstein gravity, as it was shown in Ref. [83]. Actually, the Chern-Simons term does not affect the scalar perturbations at all, and it affects solely the tensor perturbations. As a result, we have two propagating scalar modes and no propagating vector mode. With regard to the propagation of the spin-2 graviton mode, the structure of the dispersion relations become more complicated in comparison with the Einstein gravity with the Chern-Simons term does not affect the propagation of these modes. This result was also known for the case of Chern-Simons gravity from the standard $F(k)$ and proportional to the square root of the wave number $k$. Then both the group velocity and the propagating velocity become smaller for larger $k$, and even vanish in the limit of $k \to \infty$.

Finally let us note that in the present work we have found two propagating scalar modes, with the one being the scalar mode which appears commonly in the context of higher derivative gravity [102], and with the other being the pseudo-scalar mode corresponding to the axion scalar. Usually the scalar mode does not mix with the pseudo-scalar mode, but if the parity symmetry is broken by the non-trivial value of the Chern-Simons term, a mixing can occur in general.

Acknowledgments

This work is supported by MINECO (Spain), FIS2016-76363-P, and by project 2017 SGR247 (AGAUR, Catalonia) (S.D.O). This work is also supported by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas “Cosmic Acceleration” No. 15H05890 (S.N.) and the JSPS Grant-in-Aid for Scientific Research (C) No. 18K03615 (S.N.). The work of A.P. is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. The work of A.P. was also supported by the Russian Foundation for Basic Research Grant No 19-02-00496.

Appendix: Detailed Form of Perturbed Einstein Tensor Components

In this Appendix we present the detailed form of the tensor expressions needed in the text, but have quite extended form. The full expression of the perturbed Einstein tensor is,

$$\delta G_{\mu\nu} = \frac{1}{2} F(R(0)) h_{\mu\nu} - \frac{1}{2} \left( \nabla_{[\mu}^{(0)} \nabla_{\nu]}^{(0)} \right) h_{\rho\mu} + \nabla_{\mu}^{(0)} \nabla_{\nu}^{(0)} \rho h_{\mu\rho} - \Box^{(0)} h_{\mu\nu} - \nabla_{\rho}^{(0)} \nabla_{\mu}^{(0)} \left( g^{(0)} \rho^{\lambda} h_{\rho\lambda} \right)$$

$$- 2 R^{(0)} \nabla_{\lambda}^{(0)} h_{\rho\lambda} + R^{(0)} \rho h_{\mu\rho} + R^{(0)} \rho h_{\mu\rho} F'''(R(0))$$

$$+ \frac{1}{2} g_{\mu
u}^{(0)} F''(R(0)) \left( - h_{\rho\sigma} R^{(0)} \rho^{\sigma} + \nabla_{\rho}^{(0)} \nabla_{\mu}^{(0)} h_{\rho\sigma} - \Box^{(0)} \left( g^{(0)} \rho^{\sigma} h_{\rho\sigma} \right) \right)$$

$$+ \left( - R^{(0)} h_{\mu\nu} + \nabla_{\mu}^{(0)} \nabla_{\nu}^{(0)} g_{\mu
u}^{(0)} \Box^{(0)} \right) \left( F''(R(0)) - h_{\rho\sigma} R^{(0)} \rho^{\sigma} + \nabla_{\rho}^{(0)} \nabla_{\mu}^{(0)} h_{\rho\sigma} - \Box^{(0)} \left( g^{(0)} \rho^{\sigma} h_{\rho\sigma} \right) \right)$$

$$+ \frac{1}{2} g^{(0)} \nabla_{\mu\nu}^{(0)} h_{\nu\mu} + \nabla_{\mu}^{(0)} h_{\nu\mu} - \nabla_{\mu}^{(0)} h_{\nu\mu} \nabla_{\lambda}^{(0)} h_{\mu\lambda} \partial_{\rho} F''(R(0))$$

$$+ g_{\mu
u}^{(0)} g_{\rho\sigma}^{(0)} g^{(0)} \rho^{\sigma} h_{\rho\sigma} \nabla_{\tau}^{(0)} \nabla_{\eta}^{(0)} F''(R(0)) - \frac{1}{2} g_{\mu
u}^{(0)} g_{\rho\sigma}^{(0)} g^{(0)} \rho^{\sigma} g^{(0)} \nabla_{\tau}^{(0)} h_{\sigma\lambda} + \nabla_{\eta}^{(0)} h_{\rho\lambda} - \nabla_{\mu}^{(0)} h_{\rho\lambda} \nabla_{\lambda}^{(0)} h_{\rho\lambda} \partial_{\rho} F''(R(0))$$

$$+ \frac{1}{2} \left( - \frac{\omega^{(0)}}{2} g^{(0)} \rho^{\sigma} \partial_{\rho} \phi^{(0)} \partial_{\sigma} \phi^{(0)} - \left( \phi^{(0)} \right)^{2} \right) h_{\mu\nu} + \frac{\omega^{(0)}}{4} g_{\mu
u}^{(0)} h_{\rho\sigma} \partial^{(0)} \phi^{(0)} \partial^{(0)} \phi^{(0)}$$
\[+ 2 \left( h_{\mu \epsilon} g_{\sigma \epsilon}^{(0)} + h_{\mu \sigma} g_{\epsilon \epsilon}^{(0)} + g_{\mu \epsilon}^{(0)} h_{\sigma \epsilon} + g_{\mu \sigma}^{(0)} h_{\epsilon \epsilon} \right) \varepsilon^{(0)} \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \]
\[+ 2 \left( g_{\mu \epsilon}^{(0)} g_{\sigma \epsilon}^{(0)} + g_{\mu \sigma}^{(0)} g_{\epsilon \epsilon}^{(0)} \right) \left\{ \frac{1}{2} g^{(0) \alpha \beta} h_{\alpha \beta} \varepsilon^{(0)} \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \right\} \]
\[+ \varepsilon^{(0) \zeta_{\eta \epsilon} h_{\alpha \beta} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \]
\[- \frac{1}{2} g^{(0) \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \]
\[\left( \frac{\varepsilon^{(0) \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \right)}{2} + \frac{1}{2} \left( \frac{\varepsilon^{(0) \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \right)}{5} \right) \frac{1}{2} g_{\mu \epsilon}^{(0)} g_{\sigma \epsilon}^{(0)} g_{\epsilon \epsilon}^{(0)} \right) \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \]
\[\delta G_{ij} = g_{(0) \mu \nu} \delta G_{\mu \nu} \]
\[\delta G_{ij} = \varepsilon^{klm} \left( \delta^{k} \delta^{l} \delta^{m} \right) - \left\{ \left( 4 H \dot{U} + 2 \ddot{U} \right) \partial_{\epsilon} h^{n}_{m} + 2 \dot{U} \partial_{\epsilon} \dot{h}^{n}_{m} + 2 \ddot{U} \partial_{\epsilon} h^{n}_{m} \right\} \]
\[\delta G_{ij} = - \frac{1}{2} \frac{\varepsilon^{(0) \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \right)}{2} + \frac{1}{2} \left( \frac{\varepsilon^{(0) \zeta_{\eta \epsilon} \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \right)}{5} \right) \frac{1}{2} g_{\mu \epsilon}^{(0)} g_{\sigma \epsilon}^{(0)} g_{\epsilon \epsilon}^{(0)} \right) \delta_{\tau \sigma} \left( U \left( \phi^{(0)} \right) R^{(0)} \right) \zeta_{\eta \epsilon} \]
\[\delta G_{ij} = 0. \]
\[
+ (\partial_0^2 (3h_t' + 6h_k^0) + 5\partial_0^2 \partial_k h_t' h_t' - 2\partial_0^2 \partial_k \partial_l h_t' h_l' - 2\partial_0^2 \partial_0 \partial_l h_{kl} - 4\partial_0 \partial_k \partial_l h_{kl}) \delta_{l,j}) \\
+ \partial_0^2 \partial_j (\partial_0^2 h_t^0 - \partial_0 \partial_k h_t' + \partial_0 \partial_k h_{kl} + 2\partial_0 \partial_l h_{kl}) \\
+ (\partial_0^2 h_t^0 + 2\partial_0 \partial_k h_t' h_t' + \partial_0^2 \partial_0 \partial_l h_l' - 2\partial_0^2 \partial_0 \partial_l h_{kl} \\
+ \partial_0^2 \partial_0^2 h_{kl} + \partial_0^2 \partial_0 \partial_l h_{kl} + \partial_0^2 \partial_0 \partial_l h_{kl} - \partial_0^2 \partial_0 \partial_l h_{kl}) \delta_{l,j}) \\
+F'' \left\{ \ddot{H} \left[ -36\dot{H}h_t^0 - 72H^2h_t' + H \left( \partial_l (18h_t^0 + 24h_k^0) + 24\partial_0 h_k^0 \right) \\
+ 6\partial_0^2 h_k^0 - 60\partial_0 \partial_k \left( h_t' + h_l' \right) + 6\partial_0^2 h_{kl} + 120\partial_0 \partial_l h_{kl} \right] \delta_{l,j} \\
+ \ddot{H}^2 \left[ -36h_t^0 - 36\dot{H}h_t' + H \left( \partial_l (252h_t^0 + 144h_k^0) + 48\partial_0 h_k^0 \right) \delta_{l,j} \\
+ H \left( \partial_0^2 (-36h_t^0 + 84h_k^0) - 12\partial_0 h_k^0 \partial_0 (h_t' + h_k^0) + 120\partial_0^2 h_{kl} + 72\partial_0 \partial_0 h_k^0 \right) \delta_{l,j} \\
+ \left( (\partial_0^2 h_k^0 + 24\partial_0^2 h_k^0) - 12\partial_0 h_k^0 \partial_0 (h_t' + h_k^0) + 120\partial_0^2 h_k^0 \delta_{l,j} \\
+ \ddot{H}^2 \left[ -576H^2h_t' + \{ -2016H^2h_t' + H \left( \partial_l (504h_t^0 + 288h_k^0) + 288\partial_0^2 h_k^0 + 24\partial_0^2 h_k^0 \partial_0 (h_t' + h_k^0) + 24\partial_0^2 h_k^0 \partial_0 (h_t' + h_k^0) \right) \delta_{l,j} \\
+ \ddot{H} \left[ -576H^2h_t' + H^3 \left( \partial_l (504h_t^0 + 192h_k^0) - 192\partial_0 h_k^0 \right) \\
+ H^2 \partial_0^2 (204h_t^0 + 240h_k^0) + H^3 \left( (48\partial_0 h_k^0 (h_t' + h_k^0) - 48\partial_0^2 h_{kl} + 96\partial_0 h_k^0 \right) \\
+ 48\partial_0^2 h_k^0 + H \left( 96\partial_0^2 h_k^0 \left( -48\partial_0 h_k^0 \partial_0 (h_t' + h_k^0) + 48\partial_0^2 h_{kl} \right) \delta_{l,j} \right) \\
+ F'' \left\{ \ddot{H}^2 \left[ -216H^2h_t' - 64H^2h_t' + H \left( \partial_0 (108h_t^0 + 144h_k^0) \right) \\
+ 144\partial_0 h_k^0 + 36\partial_0^2 h_k^0 - 36\partial_0^2 h_k^0 \partial_0 (h_t' + h_k^0) + 36\partial_0^2 h_{kl} + 72\partial_0 \partial_0 h_k^0 \right) \delta_{l,j} \\
- 1728\ddot{H} \ddot{H} h_t' \delta_{l,j} + \dddot{H} \ddot{H} \left[ -3456H^3 h_t' + H^2 \left( \partial_l (864h_t^0 + 1152h_k^0) + 1152\partial_0 h_k^0 \right) \\
+ 288\partial_0^2 h_k^0 + 288H \left( -\partial_0 \partial_0 (h_t' + h_k^0) + \partial_0^2 \partial_0 h_{kl} + 20\partial_0 \partial_0 h_k^0 \right) \delta_{l,j} - 3456H^3 H^2 h_t' \delta_{l,j} \\
+ \ddot{H} \ddot{H} \left[ -6912H^4 h_t' + H^3 \left( \partial_l (1728h_t^0 + 2304h_k^0) + 2304\partial_0 h_k^0 \right) \\
+ 576H^2 \partial_0^2 h_k^0 + 576H^2 \left( -\partial_0 \partial_0 (h_t' + h_k^0) + \partial_0^2 \partial_0 h_{kl} + 20\partial_0 \partial_0 h_k^0 \right) \delta_{l,j} \right) \\
+ \left( \frac{\omega}{4} \left( \frac{\phi(0)}{2} \right)^2 h_{tj} + \frac{1}{2} \left( \frac{\omega}{2} \left( \frac{\phi(0)}{2} \right) - V \left( \frac{\phi(0)}{2} \right) \right) \right) h_{tj} \\
+ \frac{1}{2} \left( \frac{\omega'}{2} \left( \frac{\phi(0)}{2} \right)^2 - \phi(0) \phi(0) \right)^2 \varphi + \phi(0) \phi(0) \phi(0) \phi(0) - V \left( \frac{\phi(0)}{2} \right) \phi(0) \right) \delta_{l,j} \right) \\
= 0.
\]

(45)

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