Identification of modal parameters in a lightly damped system based on impact vibration testing: Application of exponential window and removal of its effect

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Abstract. The experimental environments for impact tests are simple to construct compared to other vibration tests in experimental modal analysis. However, given that sufficient sampling time is necessary for the response to attenuate, the following two problems arise: The first is that work efficiency is lowered. The second is that the measurement time increases, which makes it more susceptible to noise. To solve these problems, it is common to shorten the sampling time and apply an exponential window to reduce the leakage error of the frequency response function (FRF). Although studies on the influence on the FRF exist, identification accuracy for damping characteristic and the influence on the residue have not discussed. In this study, the influence on the modal properties obtained and the accuracy of the results are clarified using numerical examples based on a circle fit and method using simultaneous equations of the real and imaginary parts of the FRF. Finally, the identification results were compared and made sure that the method using simultaneous equations of the real and imaginary parts of FRF can identify modal properties with high accuracy. This method is effective for improving the working efficiency and reducing the influence of noise.

1. Introduction
Identification of the modal properties to improve the noise and vibration characteristics of mechanical structures is important. Therefore, experimental modal analysis is widely used to identify modal properties such as the natural frequency, damping characteristics, and residue from vibration test results [1][2][3][4]. An experimental modal analysis uses the frequency response function (FRF) obtained from a vibration test. This process is important process because it checks whether the modal properties of the numerical model are correct or not by comparing the vibration phenomenon such as FRF between the experiment and the numerical analysis. During the comparison, the natural frequency and residue are used. However, for a more detailed verification, it is necessary to compare all the modal properties. In the case of a structure with relatively low damping, an impact test using a hammer is advantageous because the influence of the damping force of the contact portion is less. The experimental environments for impact tests are easy to construct [2][5][6][7]. However, given that sufficient sampling time is necessary for the response to attenuate [8][9][10][11][12], the following two problems arise: The first is that work efficiency is lowered. The second is that the measurement time increases, making the response more susceptible to noise. A common approach to addressing these issues is to shorten the sampling time and apply an exponential window to reduce the leakage error of the FRF [2][13]. However,
this method adds numerical attenuation. Although studies on the change of damping characteristics do exist, the identification accuracy for the damping characteristics and the influence on the residue have not been discussed [2][14][15]. With the conventional identification method such as the half power method and the circle fit, identification of the damping characteristic in a lightly damped system is difficult [16][17][18][19]. It is known that the identification accuracy varies depending on the relationship between the frequency resolution and natural frequency [20]. In the half power method, the identification accuracy decreases when the resonance peak of the FRF cannot be read. In the circle fit, the identification accuracy decreases when the accuracy of the approximation to the circle decreases. Furthermore, the identification values of other modal properties changes due to the identification value of the damping characteristic. As a result, the FRF that is reconstructed using the identified modal properties is different from the original FRF.

In response to this problem, Kawamura proposed a more accurate method to identify modal properties [16][17][19]. The aforementioned method uses simultaneous equations of the real and imaginary parts of the FRF. In these simultaneous equations, the natural frequency and damping characteristic are unknown quantities, and they are obtained using the least squares method (LSM). In this method, the identification formula of the modal properties is not the approximation formula. Therefore, the modal characteristics can be accurately identified not only in a lightly damped system, but also in a highly damped system. In addition, this method can identify the modal characteristics without depending on the relationship between the frequency resolution and the natural frequency. Therefore, if the modal properties can be accurately identified from the FRF by applying an exponential window, it is possible to shorten the sampling time in the impact test. Accordingly, it is possible to improve the work efficiency and reduce the influence of the measurement noise response.

In this study, we apply the exponential window to process measurement data obtained from an impact test in a lightly damped system. The influence on the identification result and its accuracy are clarified using numerical examples based on a circle fit and method using simultaneous equations of the real and imaginary parts of the FRF. A single point reference method is used for the numerical example. Subsequently, a hysteretic damping system is modeled. In the remainder of this paper, the modal properties are identified using a circle fit [1][2][3] and the method using simultaneous equations of the real and imaginary parts of the FRF from the FRF [16][17][19][20] of a three degree-of-freedom system with a practical mode density is explained. Then, the identification results using different methods are compared. Finally, we present our conclusions.

2. Characteristics of the FRF obtained from impact tests and problems of identification accuracy of modal properties in the conventional method

2.1. Influence of exponential window on FRF
The damping vibration in a lightly damped three degree-of-freedom system is simulated. Sufficient sampling time in the impact test is necessary for the response to attenuate; however, the following problems are observed: the work efficiency decreases and the measurement time increases; thus, making the response more susceptible to noise. To address these issues, it is common to shorten the sampling time and apply an exponential window to reduce the leakage error of the FRF [2][13]. In this numerical example, it is assumed that the three independent single degree-of-freedom equations of motion are obtained after performing an eigenvalue analysis. The response of three degree-of-freedom system is obtained as a linear superposition of solutions of each single degree-of-freedom system. The parameters are the modal mass \( m_p \), modal damping coefficient \( c_p \), modal stiffness \( k_p \), modal coordinate \( q_p \), and an external force \( f_p \), all of which are calculated from a modal matrix. The suffix \( p \) is the mode order. Note that \( \eta = 2\zeta \) is established near the resonance point, where \( \zeta \) is the modal damping ratio and \( \eta \) is the structural damping coefficient. First, the right side of equation (1) is taken as the impulse input, and the damping vibration of each mode is obtained by using the Newmark \( \beta \) method (with \( \beta = 1/4 \)). The response of the target system is obtained by superimposing the obtained damping vibrations.
where the modal damping coefficient $c_p$ is expressed by equation (2) using the natural frequency $f_{np}$ and the structural damping coefficient $\eta_p$.

$$c_p = 2\pi f_{np} m_p \eta_p.$$  \hfill (2)

Next, the exponential window, given by equation (3), is applied to the obtained response of the target system. In equation (3), $\alpha$ is a constant that indicates the magnitude of numerical attenuation.

$$w(t) = e^{-\alpha t}.$$  \hfill (3)

Following this, the FRF is obtained using the fast Fourier transformation (FFT), and the modal properties are identified. The displacement data without noise is used. Table 1 shows the modal properties of the target system. $f_{np}$ is the natural frequency, $\eta_p$ is the structural damping coefficient, and $(\phi_{p r}, \phi_{p e})/k_p$ is the residue. The measurement conditions are listed in table 2. The theoretical value of the damped vibration and the response after applying the exponential window for each measurement condition are shown in figure 1. The FRFs are shown in figure 2, where the theoretical value is FRF_{the} and the analysis value is FRF_{fft}. Figure 2 shows that the magnitude of FRF_{fft} is less than FRF_{the}, and a smoother peak is obtained under measurement condition 2.

### Table 1. Modal parameters.

| Mode order | $f_{np}$ [Hz] | $\eta_p$ [-] | $(\phi_{p r}, \phi_{p e})/k_p$ [m/N] |
|------------|---------------|--------------|------------------------------------|
| 1          | 51.9218       | 0.0020       | 9.3959×10^6                        |
| 2          | 145.0391      | 0.0009       | 1.2041×10^6                        |
| 3          | 279.9609      | 0.0008       | 3.2318×10^7                        |

### Table 2. Measurement conditions.

| Case | Exponent $\alpha$ [-] | Sampling time $T$ [s] | Sampling points $N$ [-] | Frequency range $F_s$ [Hz] | Frequency resolution $\Delta f$ [Hz] |
|------|------------------------|-----------------------|-------------------------|-----------------------------|-------------------------------------|
| 1    | 0.4                    | 6.4                   | 16384                   | 1000                        | 0.15625                             |
| 2    | 0.9                    | 6.4                   | 16384                   | 1000                        | 0.15625                             |

![Figure 1. Damped vibration and response after applying the exponential window.](image-url)
2.2. Influence of exponential window on modal properties in circle fit

The modal properties are identified from the FRF in figure 2 using a circle fit [1][2][3], and the elimination of the influence of the exponential window on the structural damping coefficient is investigated. In the case of the input point \( e \) and the reference point \( r \), the compliance \( H_{r,e} \) focusing only on the \( p \)th order mode is expressed using equation (4).

\[
H_{r,e}(f) = \frac{k_p}{1 - \left(\frac{f}{f_{np}}\right)^2 + j\eta_p}.
\]  

(4)

The natural frequency \( f_{np} \) is the midpoint of two points with the maximum distance between the data points in the Nyquist diagram. The structural damping coefficient \( \eta_p \) is determined by calculating the angle \( \Delta\phi \) that the above two points form from the center point, and it is obtained using equation (5). \( \Delta f \) denotes the frequency resolution.

\[
\eta_p = \frac{4\Delta f}{f_{np}\Delta\phi}.
\]  

(5)

Then, to eliminate the influence of the exponential window. The value of \( \eta_p \) obtained in equation (5) is replaced with \( \eta_{ep} \). The original structural damping coefficient of the target system is obtained using equation (6) [2].

\[
\eta_p = \eta_{ep} - \frac{2\alpha}{2\pi f_{np}}.
\]  

(6)

The residue \((\phi_{r,\phi,\phi_e})/k_p\) is identified from the radius of the modal circle. In this numerical example, the number of data points adopted for identification is 10 points on the low and high frequency sides, with the resonance frequency as the center. The modal properties are identified based on the FRF in figure 2, and the relative errors with respect to the true values are listed in table 3. Table 3 shows that the errors in the structural damping coefficient and the residue are very large for both measurement conditions. However, the error in the structural damping coefficient under measurement condition 2 is smaller than that in condition 1. This is because \( \Delta\phi \) decreases as additional numerical attenuation is introduced. Furthermore, the error in the residue under measurement condition 2 is also smaller than
that in condition 1. This is because the accuracy of identifying the radius of the modal circle is improved by adding a larger numerical attenuation. When a large numerical attenuation is added, the number of points constituting the modal circle increases. Figure 3 shows the FRF_{ana} that was reconstructed using these identified values. As shown, FRF_{ana} is smaller than FRF_{the} by 2 dB in the first and second modes for both measurement conditions. Particularly, in measurement condition 1, there is a large difference between FRF_{ana} and FRF_{the}. The reproducibility of FRF_{ana} depends on the value of the structural damping coefficient. Based on the above discussion, the identification accuracy of the natural frequency for the circle fit is acceptable; however, there are large errors in the structural damping coefficient and residue, which may be unacceptable in some cases.

| Table 3. Relative errors [%] in circle fit. |
|--------------------------------------|
| Mode order | (a) Measurement condition 1 | (b) Measurement condition 2 |
|------------|-----------------|-----------------|
|            | $f_{np}$ [Hz]   | $f_{np}$ [Hz]   |
| 1          | -0.06           | -0.06           |
| 2          | -0.03           | -0.03           |
| 3          | 0.01            | 0.01            |
|            | $\eta_p$ [-]    | $\eta_p$ [-]    |
| 1          | -42.78          | -28.85          |
| 2          | -40.83          | -22.44          |
| 3          | -11.76          | -8.34           |
|            | $k_p$ [m/N]     | $k_p$ [m/N]     |
| 1          | -21.70          | -28.85          |
| 2          | -22.44          | -26.76          |
| 3          | -8.34           | -6.42           |

3. Identification of modal properties using simultaneous equations of real and imaginary parts of FRF

In Section 2, the modal properties were identified using the circle fit. In this section, mode characteristics are identified using simultaneous equations of the real and imaginary parts of the FRF [16][17][19]. The outline of this method is shown below. In equation (4), which expresses compliance $H_{r,e}$, the separation of the real and imaginary parts of the equation can be expressed as follows:

$$
\text{Re}\{H_{r,e}(f)\} = \frac{\varphi_{r,p} \varphi_{r,p}}{k_p} \left(1 - \frac{f^2}{f_{np}^2}\right)^2, \quad \text{Im}\{H_{r,e}(f)\} = \frac{\varphi_{r,p} \varphi_{r,p}}{k_p} \left(-\eta_p\right) \left(1 - \frac{f^2}{f_{np}^2}\right)^2 + \eta_p^2.
$$

(a) Measurement condition 1

(b) Measurement condition 2

Figure 3. Theoretical value FRF_{the} and analysis value FRF_{ana} in circle fit.
Next, equation (8) is obtained from the ratio of the real part to the imaginary part of equation (7). Using certain compliance data around the natural frequency \( f_{np} \), the simultaneous equation could be obtained as follows:

\[
\frac{\text{Im}\{H_{r,e}(f)\}}{\text{Re}\{H_{r,e}(f)\}} = \frac{1}{f_{np}^2} \left[ \frac{\text{Im}\{H_{r,e}(f)\}}{\text{Re}\{H_{r,e}(f)\}} \right] f^2 - \eta_p. \tag{8}
\]

\( f_{np} \) and \( \eta_p \) are unknown parameters. When equation (8) is constructed with more than two measured frequencies, the unknown parameters can be identified using the least-squares method (LSM). Although we explained compliance, equation (8) is applicable for mobility and accessibility as well. The identified structural damping coefficient is corrected using equation (6). The residue can be obtained using equation (9). The coefficient matrix is calculated using the identified values of \( f_{np} \) and \( \eta_p \) [20], where \( f_a \) and \( f_b \) represent the frequencies at the beginning and the end of the analysis, respectively.

\[
\begin{bmatrix}
1 - \left( \frac{f_a}{f_{n1}} \right)^2 - \eta_1^2 \\
1 - \left( \frac{f_a}{f_{n2}} \right)^2 + \eta_2^2 \\
1 - \left( \frac{f_{a+M}}{f_{n1}} \right)^2 - \eta_1^2 \\
1 - \left( \frac{f_{a+M}}{f_{n2}} \right)^2 + \eta_2^2 \\
\vdots \\
1 - \left( \frac{f_a}{f_{n1}} \right)^2 - \eta_1^2 \\
1 - \left( \frac{f_a}{f_{n2}} \right)^2 + \eta_2^2 \\
\vdots \\
1 - \left( \frac{f_b}{f_{n1}} \right)^2 - \eta_1^2 \\
1 - \left( \frac{f_b}{f_{n2}} \right)^2 + \eta_2^2
\end{bmatrix}
\begin{bmatrix}
\varphi_{r,1}
\varphi_{r,2}
\varphi_{r,p}
\end{bmatrix}
\begin{bmatrix}
\varphi_{r,1}
\varphi_{r,2}
\varphi_{r,p}
\end{bmatrix}
\begin{bmatrix}
1 - \left( \frac{f_a}{f_{np}} \right)^2 - \eta_p^2 \\
1 - \left( \frac{f_a}{f_{np}} \right)^2 + \eta_p^2 \\
\vdots \\
1 - \left( \frac{f_b}{f_{np}} \right)^2 - \eta_p^2 \\
1 - \left( \frac{f_b}{f_{np}} \right)^2 + \eta_p^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{Re}\{H_{r,e}(f_a)\} + \text{Im}\{H_{r,e}(f_a)\} \\
\text{Re}\{H_{r,e}(f_{a+M})\} + \text{Im}\{H_{r,e}(f_{a+M})\} \\
\vdots \\
\text{Re}\{H_{r,e}(f_b)\} + \text{Im}\{H_{r,e}(f_b)\}
\end{bmatrix}
\]

In this numerical example, the number of data points adopted for identification is 10 points on the low and high frequency sides, with the resonance frequency as the center. The modal properties are identified based on the FRF in figure 2, and the relative errors with respect to the true values are listed in table 4. The identification accuracy in this method is sufficiently high compared with that of the circle fit, as shown in table 4. In particular, the structural damping coefficient has a maximum error of \(-0.6\%\) for measurement condition 2. The residue has a maximum error of \(-2.1\%\) for measurement condition 1. The FRF reconstructed using the identified values is shown in figure 4. Figure 4 shows that FRF can...
Table 4. Relative errors [%] in method using simultaneous equations of the real and imaginary parts of an FRF.

| Mode order | \( f_{np} \) [Hz] | \( \eta_p \) [-] | \( k_p \) [m/N] | \( f_{np} \) [Hz] | \( \eta_p \) [-] | \( k_p \) [m/N] |
|------------|-----------------|----------------|-----------------|-----------------|----------------|-----------------|
| 1          | 0.01            | 0.22           | -2.07           | 0.00            | 0.09           | -1.91           |
| 2          | 0.01            | 0.33           | -1.49           | 0.00            | -0.16          | -1.70           |
| 3          | 0.00            | -0.40          | -0.81           | 0.00            | -0.60          | -0.86           |

Figure 4. Theoretical value \( FRF_{the} \) and analysis value \( FRF_{ana} \) in method using simultaneous equations of the real and imaginary parts of an FRF.

4. Identification accuracy of modal properties for shorter sampling time

In Sections 2 and 3, the modal properties were identified by applying two exponential windows for attenuation with a sampling time of 6.4 s. However, the measurement time can be further shortened for measurement condition 2. Therefore, the number of sampling points is set to be less than that shown in Table 2. Then, as shown in Table 5, the sampling time is shortened and the frequency resolution is expanded. Using the conventional identification method, the identification accuracy of the damping characteristic changes significantly depending on the relationship between the frequency resolution and the natural frequency. Conversely, if identification accuracy of the damping characteristics does not change, the identification method is useful for shortening the time required for the impact test. The modal properties are identified using the two methods explained above. Then, the identification accuracies are compared.

Table 5. Measurement conditions.

| Case | Exponent \( \alpha \) [-] | Sampling time \( T \) [s] | Sampling points \( N \) [-] | Frequency range \( F_s \) [Hz] | Frequency resolution \( \Delta f \) [Hz] |
|------|--------------------------|--------------------------|--------------------------|---------------------------|--------------------------|
| 3    | 0.9                      | 3.2                      | 8192                     | 1000                      | 0.31250                  |
The modal properties are identified from the FRF in figure 2 (b) for different frequency resolutions. The number of data points adopted for identification is five points on the low and high frequency sides, with the resonance frequency as the center. The reason for reducing the number of data points is to make the analysis frequency range that is centered at the resonance frequency equal to that in measurement condition 2. The relative errors with respect to the true values are shown in table 6. The identification accuracy of the natural frequency is sufficiently high for both methods, as shown in table 6. However, in the first method, the error in the structural damping coefficient exceeds 100% and the error in the residue increased. The identification accuracy of the modal properties decreased significantly due to the change in the frequency resolution. The FRF reconstructed using these identified values is shown in figure 5. In figure 5(a), FRF ana is shown to be less than FRF the by at most 6 dB. The reproducibility was less than that in figure 3 (b). However, with the second method, the maximum error in the structural damping coefficient is ~0.7%. The error slightly increased compared to that before the change in the frequency resolution. However, the accuracy is sufficient. Further, the maximum error in the residue is ~0.7%. There is no significant difference in the FRF ana in figure 4 (b) and figure 5 (b). Hence, the method using simultaneous equations of the real and imaginary parts of the FRF does not significantly reduce the identification accuracy, even if the relationship between the natural frequency and the frequency resolution changes. This characteristic is useful for acquiring data with shorter measurement times.

| Mode order | (a) Circle fit | (b) Method using simultaneous equations of the real and imaginary parts of an FRF |
|------------|----------------|---------------------------------------------------------------------------------|
|            | fnp [Hz]       | ηp [-]                           | k_p [m/N]              | fnp [Hz]       | ηp [-]                           | k_p [m/N]              |
| 1          | -0.21          | 145.54                           | -39.33                 | 0.02           | 0.65                             | -0.34                  |
| 2          | -0.08          | 135.76                           | -42.84                 | 0.01           | 0.66                             | -0.24                  |
| 3          | 0.04           | -42.30                           | -19.37                 | 0.01           | -0.07                            | -0.65                  |

Figure 5. Theoretical value FRF the and analysis value FRF ana.

5. Conclusion
We applied an exponential window to process the measurement data obtained from impact tests in a lightly damped system. The influence of the exponential window on the identification result and its
accurately were clarified using numerical examples based on different identification methods. The model used in the numerical example is a three degree-of-freedom system with a practical mode density. The modal properties were identified using a circle fit and a method using simultaneous equations of the real and imaginary parts of the FRF. Finally, the identification results were compared and the findings are summarized below.

- It is difficult to accurately identify the modal properties in a lightly damped system using the circle fit, which is affected by the relationship between the frequency resolution and the natural frequency. However, the method using simultaneous equations of the real and imaginary parts of FRF can identify modal properties with high accuracy, and does not rely on the relationship between the natural frequency and frequency resolution.
- When the mode circle is used, the relative errors of the structural damping coefficient and residue decrease when the numerical attenuation is added for the same frequency resolution condition. This is because the accuracy of identifying the modal circle improves as the number of points that constitute the modal circle are increased.
- All the modal properties can be identified with a high accuracy using the simultaneous equations of the real and imaginary parts of the FRF. The FRF reconstructed using these identified values can reproduce the FRF obtained from the experiment accurately.
- The method using simultaneous equations of the real and imaginary parts of FRF is useful for obtaining the modal properties in shorter measurement times. This method is effective for improving the working efficiency and reducing the influence of noise.

References
[1] Nagamatsu A 1985 Modal analysis (Tokyo: Baifukan) (in Japanese)
[2] Nagamatsu A 1993 Introduction to Modal analysis (Tokyo: Baifukan) (in Japanese)
[3] Ewins D J 2000 Modal testing: theory, practice, and application (Hertfordshire: Research Studies Press Ltd)
[4] Jimin H and Zhi-Fang F 2001 Modal Analysis (Milton Keynes: Butterworth Heinemann)
[5] William G H and David L B 1997 Impulse technique for structural frequency response testing. Reprinted from Sound and Vibration November
[6] Brian J S and Mark H R 1999 Experimental modal analysis (Tokyo: Baifukan) (in Japanese)
[7] Reynolds P and Pavic A 2000 Impulse hammer versus shaker excitation for the modal testing of building floors Experimental techniques 24(3) 39
[8] Randall R B 1987 Frequency analysis (Denmark: B & K Ltd.)
[9] McConnell K G 1995 Vibration Testing: Theory and Practice (New York: Wiley)
[10] Roy P K and Ganesan N 1995 Transient response of a cantilever beam subjected to an impulse load. J. of Sound and Vibration 183 (5) 873–90
[11] Ahn S J and Jeong W B 2002 The improvement of muti-dof impulse response spectrum by using optimization technique. J. of the Korean Society for Noise and Vibration Engineering 12 (10) 792–98
[12] Ahn S J, Jeong W B and Yoo W S 2004 An estimation of error-free frequency response function from impact hammer testing. JSME Int. J. Series C Mechanical Systems, Machine Elements and Manufacturing 47(3) 852–57
[13] Fladungand W and Rost R 1997 Application and correction of the exponential window for frequency response functions. Mechanical systems and signal processing 11(1) 23-36
[14] Peeters B 2005 PolyMAX: A revolution in operational modal analysis. 1st. Int. Operation Modal Analysis Conf. Copenhagen, Denmark, 26-27
[15] Ahn S J, Jeong W B and Yoo W S 2004 Unbiased expression of FRF with exponential window function in impact hammer testing. J. of Sound and Vibration 277(4–5) 931–41
[16] Kawamura S, Kato Y, Harada M and Minamoto H 2013 Estimation of dynamic properties of a lightly damped element Proc. of Dynamics and Design Conf. 2013 322 (in Japanese)
[17] Kawamura S, Kita M and Matsubara M 2015 Identification of modal properties of a lightly damped element DVD. Proceedings of the 2015 Annual meeting G1000306 (in Japanese)

[18] Kitahara A and Yoshimura T 2015 Modal identification of cylindrical shell using circumference reduction method. Transactions of the JSME 81(822) DOI: 10.1299/transjsme.14-00461 (in Japanese)

[19] Kawamura S, Kita M, Matsubara M and Ise T 2016 Study of the effect of specimen size and frequency on the structural damping property of beam Mechanical Engineering Journal 3(6) DOI: 10.1299/mej.16-00446

[20] Matsubara M, Ise T and Kawamura S 2018 Application of modal properties identification to multi-degree-of-freedom system using simultaneous equations of the real and imaginary parts of frequency response function Transactions of the JSME 84(860) DOI: 10.1299/transjsme.17-00540 (in Japanese)