IRREDUCIBLE COMPONENTS OF THE SPACE OF FOLIATIONS BY SURFACES

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Abstract. Let \( \mathcal{F} \) be written as \( f^*(\mathcal{G}) \), where \( \mathcal{G} \) is a 1-dimensional foliation on \( \mathbb{P}^{n-1} \) and \( f: \mathbb{P}^n \to \mathbb{P}^{n-1} \) a non-linear generic rational map. We use local stability results of singular holomorphic foliations, to prove that: if \( n \geq 4 \), a foliation \( \mathcal{F} \) by complex surfaces on \( \mathbb{P}^n \) is globally stable under holomorphic deformations. As a consequence, we obtain irreducible components for the space of two-dimensional foliations in \( \mathbb{P}^n \). We present also a result which characterizes holomorphic foliations on \( \mathbb{P}^n, n \geq 4 \) which can be obtained as a pull back of 1-foliations in \( \mathbb{P}^{n-1} \) of degree \( d \geq 2 \).

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1. Introduction
A two singular foliation \( \mathcal{F} \) of a holomorphic manifold \( M, \dim_{\mathbb{C}} \geq 3 \), may be defined by:

1. A covering \( \mathcal{U} = (U_\alpha)_{\alpha \in A} \) of \( M \) by open sets.
2. A collection \( (\eta_\alpha)_{\alpha \in A} \) of integrable \((n-2)\)-forms, \( \eta_\alpha \in \Omega^{n-2}(U_\alpha) \), where \( \eta_\alpha \neq 0 \) and defines a 2-dimensional foliation in \( U_\alpha \).
3. A multiplicative cocycle \( G := (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset} \) such that \( \eta_\alpha = g_{\alpha\beta} \eta_\beta \).

If \( N_\mathcal{F} \) denotes the holomorphic line bundle represented by the cocycle \( G \), the family \( (\eta_\alpha)_{\alpha \in A} \) defines a holomorphic section of the vector bundle \( \Omega^{n-2}(M) \otimes N_\mathcal{F} \) i.e. an element \( \eta \) of the cohomology vector space \( H^0(M, \Omega^{n-2}(M) \otimes N_\mathcal{F}) \). The analytic subset \( \text{Sing}(\eta) := \{ p \in M | \eta(p) = 0 \} \) is the singular set of \( \mathcal{F} \). In the case of \( M = \mathbb{P}^n \), n-dimensional complex projective space, we have a theorem of Chow-type. Denote by \( \pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) the natural projection, and consider \( \pi^* \mathcal{F} \) of the foliation \( \mathcal{F} \) by \( \pi \); with the previous notations, \( \pi^* \mathcal{F} \) is defined by \((n-2)\)-forms, \( \pi^* \eta_\alpha \in \Omega^{n-2}[\pi^{-1}(U_\alpha)] \). Recall that for \( n \geq 2 \) we have \( H^1(\mathbb{C}^{n+1} \setminus \{0\}, \mathcal{O}^*) = \{ 1 \} \); it is a result from Cartan. As a consequence, there exists a global holomorphic \((n-2)\)-form \( \eta \) on \( \mathbb{C}^{n+1} \setminus \{0\} \) which defines \( \pi^* \mathcal{F} \) on \( \mathbb{C}^{n+1} \setminus \{0\} \). By Hartog’s extension
Theorem, \( \eta \) can be extended holomorphically at 0. By construction we have \( i_R \eta = 0 \), where \( R \) is the radial vector field. This fact and the integrability condition imply that each coefficient of \( \eta \) is a homogeneous polynomial of degree \( \text{deg}(F) + 1 \). Moreover, if we take a section by a generic immersion of hyperplane \( H := (i : \mathbb{P}^{n-1} \to \mathbb{P}^n) \), this procedure gives a foliation by curves \( \iota^*(F) \) on \( \mathbb{P}^{n-1} \). We then define the degree of \( F \), for short \( \text{deg}(F) \), as the degree of a generic section as before. From now on we will always assume that the singular set of \( F \) has codimension greater or equal than two.

The projectivisation of the set of \( n - 2 \)-forms which satisfies the previous conditions will be denoted by \( \mathfrak{F}(d; 2, n) \), the space of \( 2 \)-dimensional foliations on \( \mathbb{P}^n \) of degree \( d \). Note that \( \mathfrak{F}(d; 2, n) \) can be considered as a quasi projective algebraic subset of \( \mathbb{P}H^0(\mathbb{P}^n, \Omega^{n-2}(\mathbb{P}^n) \otimes O_{\mathbb{P}^n}(d+n-1)) \). In this scenario we have the following:

**Problem:** Describe and classify the irreducible components of \( \mathfrak{F}(d; 2, n) \) on \( \mathbb{P}^n \), such that \( n \geq 3 \).

We observe that the classification of the irreducible components of \( \mathfrak{F}(0; 2, n) \) is given in [2] Th. 3.8 p. 46 and that the classification of the irreducible components of \( \mathfrak{F}(1; 2, n) \) is given in [17] Th. 6.2 and Cor. 6.3 p. 935-936. We refer the reader to [2] and [17] and references therein for a detailed description of them. In the case of foliations of codimension 1, the definitions of foliation and degree are analogous and we denote by \( \mathfrak{F}(k, n) \) the space of codimension 1 foliations of degree \( k \) on \( \mathbb{P}^n \), such that \( n \geq 3 \). The study of irreducible components of these spaces has been initiated by Jouanolou in [10], where the irreducible components of \( \mathfrak{F}(k, n) \) for \( k = 0 \) and \( k = 1 \) are described. In the case of codimension one foliations one can exhibit some kind of list of irreducible components in every degree, but this list is incomplete. In the paper [3], the authors proved that \( \mathfrak{F}(2, n) \) has six irreducible components, which can be described by geometric and dynamic properties of a generic element. We refer the reader to [3] and [11] for a detailed description of them. There are known families of irreducible components in which the typical element is a pull-back of a foliation on \( \mathbb{P}^2 \) by a rational map. Given a generic rational map \( f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2 \) of degree \( \nu \geq 1 \), it can be written in homogeneous coordinates as \( f = (F_0, F_1, F_2) \) where \( F_0, F_1 \) and \( F_2 \) are homogeneous polynomials of degree \( \nu \). Now consider a foliation \( \mathcal{G} \) on \( \mathbb{P}^2 \) of degree \( \geq 2 \). We can associate to the pair \( (f, \mathcal{G}) \) the pull-back foliation \( F = f^* \mathcal{G} \). The degree of the foliation \( F \) is \( \nu(d + 2) - 2 \) as proved in [4]. Denote by \( \mathcal{PB}(d, \nu; n) \) the closure in \( \mathfrak{F}(\nu(d + 2) - 2, n) \), \( n \geq 3 \) of the set of foliations \( \mathcal{F} \) of the form \( f^* \mathcal{G} \). Since \( (f, \mathcal{G}) \to f^* \mathcal{G} \) is an algebraic parametrisation of \( \mathcal{PB}(d, \nu; n) \) it follows that \( \mathcal{PB}(d, \nu; n) \) is an unirational irreducible algebraic subset of \( \mathfrak{F}(\nu(d + 2) - 2, n) \), \( n \geq 3 \). We have the following result:

**Theorem 1.1.** \( \mathcal{PB}(d, \nu; n) \) is a unirational irreducible component of \( \mathfrak{F}(\nu(d + 2) - 2, n) \); \( n \geq 3, \nu \geq 1 \) and \( d \geq 2 \).

The case \( \nu = 1 \), of linear pull-backs, was proven in [1], whereas the case \( \nu > 1 \), of nonlinear pull-backs, was proved in [4]. The search for new components of pull-back type for the space of codimension 1 foliations was considered in the Ph.D thesis of the author [6] and after in [7]. There we investigated branched rational maps and foliations with algebraic invariant sets of positive dimensions.

Recently, A.Lins Neto in [14] generalized the results contained in [12] about singularities of integrable 1-forms for the 2-dimensional case and he has obtained as a corollary components of linear pull-back type for the case of 2-dimensional
foliations on $\mathbb{P}^n$. In the present work we will explore the result contained in [14] and some ideas contained in [4] to show that, in fact, there exist families of irreducible components of non-linear pull-back type for the 2-dimensional case. We would like to mention that in [4] the authors have shown that linear pull-back components exist in all codimension. However, the techniques that they use to prove this fact can not be applied to the non-linear case.

1.1. The present work. Let us describe, briefly, the type of pull-back foliation that we shall consider.

Let us fix some homogeneous coordinates $Z = (z_0, \ldots, z_n)$ on $\mathbb{C}^{n+1}$ and $X = (x_0, \ldots, x_{n-1})$ on $\mathbb{C}^n$. Let $f : \mathbb{P}^n \to \mathbb{P}^{n-1}$ be a rational map represented in the coordinates $Z \in \mathbb{C}^{n+1}$ and $X \in \mathbb{C}^n$ by $f = (F_0, F_1, \ldots, F_{n-1})$ where $F_i \in \mathbb{C}[X]$ are homogeneous polynomials without common factors of degree $\nu$. Let $\mathcal{G}$ be a foliation by curves on $\mathbb{P}^{n-1}$. This foliation can be represented in these coordinates by a homogenous polynomial $(n-2)$-form of the type

$$\Omega = (-1)^{i+k+1} \sum_{i,k} x_k F_i dx_0 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_k \wedge \ldots \wedge dx_{n-1},$$

for all $i, k \in \{0, \ldots, n-1\}$ where each $F_i$ is a homogeneous polynomial of degree $d$.

The pull back foliation $f^*(\mathcal{G})$ is then defined in homogeneous coordinates by the $(n-2)$-form

$$\tilde{\eta}_{[f, \mathcal{G}]}(Z) = \left[ (-1)^{i+k+1} \sum_{i,k} F_k (P_i \circ f) dF_0 \wedge \ldots \wedge \widehat{dF_i} \wedge \ldots \wedge dF_k \wedge \ldots \wedge dF_{n-1} \right],$$

for all $i, k \in \{0, \ldots, n-1\}$ where each coefficient of $\tilde{\eta}_{[f, \mathcal{G}]}(W)$ has degree $\Theta(\nu, d, n) + 1 = [(d+n-1)\nu - 2]$. Let $PB(\Theta; 2, n)$ be the closure in $\text{Fol}(\Theta; 2, n)$ of the set $\{ \tilde{\eta}_{[f, \mathcal{G}]} \}$, where $\tilde{\eta}_{[f, \mathcal{G}]}$ is as before. The pull-back foliation’s degree is $\Theta(\nu, d, n) = [(d+n-1)\nu - 3]$ and for simplicity we will denote it by $\Theta$. Let us state the main result of this work.

**Theorem A.** $PB(\Theta; 2, n)$ is a unirational irreducible component of $\text{Fol}(\Theta; 2, n)$ for all $n \geq 4$, $\nu \geq 2$ and $d \geq 2$.

It is worth pointing out that the case $n = 3$ is also true and it is contained in theorem [14]. So we can think this result as the $n \geq 4$-dimensional generalization of [4] for 2-dimiosal foliations in $\mathbb{P}^n$.

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## 2. 1-DIMENSIONAL FOLIATIONS ON $\mathbb{P}^{n-1}$

2.1. Basic facts. We recall the basic definitions and properties of foliations by curves on $\mathbb{P}^{n-1}$ that we will use in this work. Proofs and details can be found in [16] and [14].

Let $R = \sum_{i=0}^{n-1} x_i \frac{\partial}{\partial x_i}$ be the radial vector field in $\mathbb{C}^n$. Denote by $\Sigma(R, d-1) = \{ Z \mid [R, Z] = (d-1)Z \}$, where $[R, Z]$ stands for the Lie’s bracket between the two vector fields $R$ and $Z$. Observe that $\Sigma(R, d-1)$ is a finite dimensional vector space whose elements are homogeneous polynomials of degree $d$. Let us write $X = \ldots = X_{n-1} = 0$. Let $x_{n-1} = ... = x_0$ be homogeneous coordinates on $\mathbb{P}^{n-1}$.
Let $\mathcal{X}_0, \ldots, \mathcal{X}_n$ and $\nabla \mathcal{X} = \sum_{i=0}^{n-1} \frac{\partial \mathcal{X}}{\partial x_i}$. Let $\mathcal{E}(R, d-1) = \{ \mathcal{X} \in \Sigma(R, d-1) | \nabla \mathcal{X} = 0 \}$, and $\mathcal{K}(R, d-1) = \{ \mathcal{X} \in \mathcal{E}(R, d-1) \}$. It can be verified that $\mathcal{K}(R, d-1)$ is a Zariski open and dense subset of $\Sigma(R, d-1)$.

Observe that the restriction $f$ can be verified that $\mathcal{K}(R, d-1)$ is a Zariski open and dense subset of $\Sigma(R, d-1)$ and for each $\mathcal{X} \in \mathcal{K}(R, d-1)$ then the $(n-2)$-form

$$\Omega = i_{H} d\sigma = (-1)^{i+k+1} \sum_{i,k} x_k P_i dx_0 \wedge \ldots \wedge \hat{d}x_i \wedge \ldots \wedge \hat{d}x_k \wedge \ldots \wedge dx_{n-1},$$

where $d\sigma = dx_0 \wedge \ldots \wedge dx_1 \wedge \ldots \wedge dx_k \wedge \ldots \wedge dx_{n-1}$ is the volume form in $\mathbb{C}^n$, $\mathcal{X} = \sum P_i \frac{\partial}{\partial x_i}$ and $0 \in \text{Sing}(\Omega)$ is a n.g.K singularity, with rotational $(d+n)\mathcal{X}$ (see section 5.2 and [14] for more details. Observe that if $\text{cod} \text{Sing}(\Omega) \geq 2$ then $\Omega$ defines a 1-dimensional foliation $\mathcal{G}$ on $\mathbb{P}^{n-1}$ of degree $d$.

**Definition 2.1.** Let us denote by $\mathcal{Fol}(d; 1, n-1)$ the set of 1-dimensional foliations on $\mathbb{P}^{n-1}$.

**Theorem 2.2.** [16] Given, $n \geq 3$, and $d \geq 2$ there exists a Zariski open subset $\mathcal{M}(d)$ of $\mathcal{Fol}(d; 1, n-1)$ such that any $\mathcal{G}$ satisfies:

1. $\mathcal{G}$ has exactly $N = \frac{d^2-1}{d-1}$ hyperbolic singularities and is regular on the complement.
2. $\mathcal{G}$ has no invariant algebraic curve.

Let $X$ be a germ of vector field at $0 \in \mathbb{C}^n$ with an isolated singularity at 0 and denote by $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}$ the spectrum of its linear part. We say that $X$ is hyperbolic at 0 if none of the quotients $\frac{\lambda_k}{\lambda_i}$ are real. We have the following proposition:

**Proposition 2.3.** Let $Q$ be a germ of vector field with a hyperbolic singularity at $0 \in \mathbb{C}^{n-1}$ and denote by $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}$ its spectrum. Then, there are exactly $n-1$ germs of irreducible invariant analytic invariant curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$ at 0 where each $\Gamma_i$ is smooth and tangent to the eigendirection corresponding to $\lambda_i$.

In a local coordinate system near the singularity for instance, $0 \in (\mathbb{C}^{n-1}, u)$ where $u = (u_1, \ldots, u_{n-1})$ the foliation can be written as

$$Q(u) = (\lambda_1 u_1) \frac{\partial}{\partial u_1} + \cdots + (\lambda_{n-1} u_{n-1}) \frac{\partial}{\partial u_{n-1}} + h.o.t,$$

where $h.o.t$ stands for higher order terms. Let us denote by $\mathcal{F}(R, d-1) = L(d)$ and let $A(d) := \mathcal{M}(d) \cap L(d)$ be their intersection. An element of the open and dense subset $A(d) \subset \mathcal{Fol}(d; 1, n-1)$ is the well-known generalized Joanoulou’s example, see [10] and [14].

3. **Rational maps**

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be a rational map and $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ its natural lifting in homogeneous coordinates. We are considering the same homogeneous coordinates used in the introduction.

The indeterminacy locus of $f$ is, by definition, the set $I(f) = \Pi_n \left( \tilde{f}^{-1}(0) \right)$. Observe that the restriction $f|_{\mathbb{P}^n \setminus I(f)}$ is holomorphic. We characterize the set of rational maps used throughout this text as follows:

**Definition 3.1.** We denote by $\text{RM} (n, n-1, \nu)$ the set of maps $\{ f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1} \}$ of degree $\nu$ given by $f = (F_0 : F_1 : \ldots : F_{n-1})$ where the $F_j$, are homogeneous polynomials without common factors, with the same degree.
Let us note that the indeterminacy locus \( I(f) \) is the intersection of the hypersurfaces \( \Pi_n(F_i = 0) \) and \( \Pi_n(F_j = 0) \) for \( i \neq j \).

**Definition 3.2.** We say that \( f \in \text{RM} \,(n,n-1,\nu) \) is *generic* if for all \( p \in \tilde{f}^{-1}(0) \setminus \{0\} \) we have \( dF_0(p) \wedge dF_1(p) \wedge ... \wedge dF_{n-1}(p) \neq 0 \).

This is equivalent to saying that \( f \in \text{RM} \,(n,n-1,\nu) \) is *generic* if \( I(f) \) is the transverse intersection of the \( n \) hypersurfaces \( \Pi_n(F_i = 0) \) for \( i = 0,...,n-1 \).

Moreover if \( f \) is generic and \( \text{deg}(f) = \nu \), then by Bezout’s theorem \( I(f) \) consists of \( \nu^n \) distinct points.

Now let \( V(f) = \mathbb{P}^n \setminus I(f) \), \( P(f) \) the set of critical points of \( f \) in \( V(f) \) and \( C(f) = f(P(f)) \) the set of the critical values of \( f \). If \( f \) is generic, then \( \overline{P(f)} \cap I(f) = \emptyset \), so that \( \overline{P(f)} = P(f) \subset V(f) \) (where \( \overline{A} \) denotes the closure of \( A \subset \mathbb{P}^n \) in the usual topology). Since \( P(f) = \{ p \in V(f): \text{rank}(df(p) \leq (n-2)) \} \), it follows from Sard’s theorem that \( C(f) = f(P(f)) \) is a subset of Lebesgue’s measure 0 in \( \mathbb{P}^{n-1} \), in fact \( C(f) \) is an algebraic curve.

The set of generic maps will be denoted by \( \text{Gen} \,(n,n-1,\nu) \). We state the following result, whose proof is standard in algebraic geometry:

**Proposition 3.3.** \( \text{Gen} \,(n,n-1,\nu) \) is a Zariski dense subset of \( \text{RM} \,(n,n-1,\nu) \).

4. **Generic pull-back components - Generic conditions**

**Definition 4.1.** Let \( f \in \text{Gen} \,(n,n-1,\nu) \). We say that \( \mathcal{G} \in A(d) \) is in generic position with respect to \( f \) if \( \text{Sing}(\mathcal{G}) \cap C(f) = \emptyset \).

In this case we say that \( (f,\mathcal{G}) \) is a generic pair. In particular, when we fix a map \( f \in \text{Gen} \,(n,n-1,\nu) \) the set \( A = \{ \mathcal{G} \in A(d) | \text{Sing}(\mathcal{G}) \cap C(f) = \emptyset \} \) is an open and dense subset in \( A(d) \) \([15]\), since \( C(f) \) is an algebraic curve in \( \mathbb{P}^{n-1} \). The set \( U_1 := \{ (f,\mathcal{G}) \in \text{Gen} \,(n,n-1,\nu) \times A(d) | \text{Sing}(\mathcal{G}) \cap C(f) = \emptyset \} \) is an open and dense subset of \( \text{Gen} \,(n,n-1,\nu) \times A(d) \). Hence the set \( \mathcal{W} := \{ \tilde{\eta}_{f,\mathcal{G}} \circ f | (f,\mathcal{G}) \in U_1 \} \) is an open and dense subset of \( \text{PB(} \Theta; 2, n) \).

Consider the set of foliations \( \text{Fol}(d;1,n-1), d \geq 2 \), and the following map:

\[
\Phi : \text{RM} \,(n,n-1,\nu) \times \text{Fol}(d;1,n-1) \rightarrow \text{Fol}(\Theta;2,n) \quad (f,\mathcal{G}) \mapsto f^*(\mathcal{G}) = \Phi (f,\mathcal{G}).
\]

The image of \( \Phi \) can be written as:

\[
(-1)^{i+k+1} \sum_{i,k} F_k(P_i \circ \tilde{f}) dF_0 \wedge ... \wedge d\overline{F}_i \wedge ... \wedge d\overline{F}_k \wedge ... \wedge dF_{n-1} \]

\( i, k \in \{0,...,n-1\} \). Recall that \( \Phi (f,\mathcal{G}) = \tilde{\eta}_{f,\mathcal{G}} \). More precisely, let \( \text{PB(} \Theta; 2, n) \) be the closure in \( \text{Fol}(\Theta;2,n) \) of the set of foliations \( \mathcal{F} \) of the form \( f^*(\mathcal{G}) \), where \( f \in \text{RM} \,(n,n-1,\nu) \) and \( \mathcal{G} \in \text{Fol}(d;1,n-1) \). Since \( \text{RM} \,(n,n-1,\nu) \) and \( \text{Fol}(d;1,n-1) \) are irreducible algebraic sets and the map \( (f,\mathcal{G}) \rightarrow f^*(\mathcal{G}) \in \text{Fol}(\Theta;2,n) \) is an algebraic parametrization of \( \text{PB(} \Theta; 2, n) \), we have that \( \text{PB(} \Theta; 2, n) \) is an irreducible algebraic subset of \( \text{Fol}(\Theta;2,n) \). Moreover, the set of generic pull-back foliations \( \{ \mathcal{F}; \mathcal{F} = f^*(\mathcal{G}) \}, \) where \( (f,\mathcal{G}) \) is a generic pair \( \} \) is an open (not Zariski) and dense subset of \( \text{PB(} \Theta; 2, n) \) for \( \nu \geq 2, d \geq 2 \).

**Remark 4.2.** We observe that if \( \nu = 1 \) the theorem is also true and, in this case, we re-obtain the result \([14]\ Cor. 1 p.7] and \([4]\ Cor. 5.1 p. 426] for the case of bi-dimensional foliations.
Remark 4.3. To visualize that the degree of a generic pull-back foliation is indeed
\(\Theta(v, d, n) = [(d + n - 1)v - 3]\), do the pull-back of a generic map of the Joanoulou’s
foliation on \(\mathbb{P}^{n-1}\) to obtain that the degree of this generic element coincides with
this number.

5. Description of generic pull-back foliations on \(\mathbb{P}^n\)

5.1. The Kupka set of \(\mathcal{F} = f^*(\mathcal{G})\). Let \(q_i\) be a singularity of \(\mathcal{G}\) and \(V_{q_i} = f^{-1}(q_i)\).
If \((f, \mathcal{G})\) is a generic pair then \(V_{q_i} \cap I(f)\) is contained in the Kupka set of \(\mathcal{F}\).

Fix \(p \in V_{q_i} \setminus I(f)\). Since \(f\) is a submersion at \(p\), there exist local analytic co-
ordinate systems \((U, y, t), y : U \to \mathbb{C}^{n-1}, t : U \to \mathbb{C}\), and \((V, u), u : V \to \mathbb{C}^{n-1},\)
at \(p\) and \(q_i = f(p)\) respectively, such that \(f(y_1, y_2, ..., y_{n-1}, t) = (y_1, y_2, ..., y_{n-1}), u(q_i) = 0\).
Suppose that \(\mathcal{G}\) is represented by the vector field \(Q = \sum_{j=1}^{n-1} Q_j(u) \frac{\partial}{\partial u_j}\)
in a neighborhood of \(q_i\). Then \(\mathcal{F}\) is represented by \(Y = \sum_{j=1}^{n-1} Y_j(y) \frac{\partial}{\partial y_j}\). It follows
that in \(U\), the foliation \(\mathcal{F}\) is equivalent to the product of two foliations of dimension
one: the singular foliation induced by the vector field \(Y\) in \((\mathbb{C}^{n-1}, 0)\) and the regular
foliation of dimension one given by the fibers of the first projection \(F(y, t) = y\).

Remark 5.1. Note that, \(\mathcal{F}\) has other singularities which are contained in
\(f^{-1}(\mathcal{G}(f))\). We remark that \(\mathcal{F}\) has local holomorphic first integral in a neighbor-
hood of each singularity of this type. In fact, this is the obstruction to try to
apply the results contained in [4] to prove theorem A since these pull-back foliations
do not have totally decomposable tangent sheaf.

Since \(\mathcal{G}\) has degree \(d\) and all of its singularities are non degenerate it has \(N = \frac{d^{n-1}}{(n-1)!}\)
singularities, say, \(q_1, ..., q_N\). We will denote the curves \(f^{-1}(q_1), ..., f^{-1}(q_N)\)
by \(V_{q_1}, ..., V_{q_N}\) respectively. We have the following:

Proposition 5.2. For each \(\{j = 1, ..., N\}, V_{q_j}\) is a complete intersection of \((n - 1)\)
transversal algebraic hypersurfaces. Furthermore, \(V_{q_j} \cap I(f)\) is contained in the
Kupka set of \(\mathcal{F} = f^*\mathcal{G}\).

5.2. Generalized Kupka singularities for 2-dimensional foliations. In this
section we will recall the generalized Kupka singularities of an integrable holomor-
phic \((n - 2)\)-form, for more detail we refer the reader to [14]. They appear in the
indeterminacy set of \(f\) and play a central role in great part of the proof of the main
theorem. Let \(\Omega\) be an holomorphic integrable \((n - 2)\)-form defined in a neighbor-
hood of \(p \in \mathbb{C}^n\). In particular, since \(d\Omega\) is a \((n - 1)\)-form, there exists a holomorphic
vector field \(Z\) defined in a neighborhood of \(p\) such that:
\[d\Omega = i_Z dw_0 \wedge \cdots \wedge dw_{n-1}\]

Definition 5.3. We say that \(p\) is a singularity of generalized Kupka type of \(\Omega\) if
\(Z(p) = 0\) and \(p\) is an isolated zero of \(Z\).

Definition 5.4. We say that \(p\) is a nilpotent generalized singularity, for short n.g.k
singularity, if the linear part of \(Z\), \(DZ(p)\) is nilpotent.

This definition is justified by the following result (that can be found in [14]).
Theorem 5.5. Assume that $0 \in \mathbb{C}^n$ is a n.g.k singularity of $\Omega$. Then there exists two holomorphic vector fields $S$ and $Z$ and a holomorphic coordinate system $x = (x_0, ..., x_{n-1})$ around $0 \in \mathbb{C}^n$ where $\Omega$ has polynomial coefficients. More precisely, there exists two polynomial vector fields $X$ and $Y$ in $\mathbb{C}^n$ such that:

(a) $Y = S + N$, where $S = \sum_{j=0}^{n-1} k_j w_j \frac{\partial}{\partial x_j}$ is linear semi-simple with eigenvalues $k_0, ..., k_{n-1} \in \mathbb{N}$, $DN(0)$ is linear nilpotent and $[S, N] = 0$.

(b) $[N, X] = 0$ and $[S, X] = kX$, where $k \in \mathbb{N}$. In other words, $X$ is quasi-homogeneous with respect to $S$ with weight $k$.

(c) In this coordinate system we have $\Omega = iv_{X}dx_0 \wedge \cdots \wedge dx_{n-1}$ and $L_Y(\Omega) = (k + tr(S))\Omega$.

In particular, the foliation given by $\Omega = 0$ can be defined by a local action of the affine group.

Definition 5.6. In the situation of the theorem 5.5, $S = \sum_{j=0}^{n-1} k_j x_j \frac{\partial}{\partial x_j}$ and $L_S(X) = kX$, we say that the n.g.K is of type $(k_0, ..., k_{n-1}; k)$.

Remark 5.7. We would like to observe that in many cases it can be proved that the vector field $N$ of the statement of theorem 2 vanishes. In order to discuss this assertion it is convenient to introduce some objects. Given two germs of vector fields $Z$ and $W$ set $L_Z(W) := [Z, W]$. Recall that $\Sigma(S, \ell) = \{Z \in X | L_S(Z) = \ell Z\}$. Let $X$ and $Y = S + N$ be as in theorem 5.5. Observe that:

- Jacobi’s identity implies that if $W \in \Sigma(S, k)$ and $Z \in \Sigma(S, \ell)$ then $[W, Z] \in \Sigma(S, k + \ell)$.
- For all $k \in \mathbb{Z}$ we have $\dim_{\mathbb{C}}(\Sigma(S, k)) < \infty$ (because $k_0, ..., k_{n-1} \in \mathbb{N}$).
- $N \in \Sigma(S(0), X) \in \Sigma(S, \ell)$ and $L_X(N) = 0$, so that $N \in ker(L_X^0)$, where $L_X^0 := LX : \Sigma(S, 0) \rightarrow \Sigma(S, \ell)$. In particular, the vector field $N \in \Sigma(S, 0)$ of theorem 5.5 necessarily vanishes $\iff ker(L_X^0) = \{0\}$.

In [14], § 3.2 it is shown that under a non-resonance condition, which depends only on $X$, then $ker(L_X^0) = \{0\}$. Let us mention some correlated facts.

(I) If $S$ has no resonances of the type $< \sigma, k > - k_j = 0$, where $< \sigma, k > := \sum_j \sigma_j k_j$, $k = (k_0, ..., k_{n-1})$ and $\sigma = (\sigma_0, ..., \sigma_{n-1}) \in \mathbb{Z}_{>0}^n$, then $ker(L_X^0) = \{0\}$.

(II) When $n = 3$ and $X$ has an isolated singularity at $0 \in \mathbb{C}^3$ then $ker(L_X^0) = \{0\}$ (c.f [12]).

(III) When $N \neq 0$ and $cod_{\mathbb{C}}(sing(N)) = 1$, or $sing(N)$ has an irreducible component of dimension one then it can be proved that $X$ cannot have an isolated singularity at $0 \in \mathbb{C}^n$.

In fact, we think that whenever $X$ has an isolated singularity at $0 \in \mathbb{C}^n$ and $\nabla X = 0$ then $ker(L_X^0) = \{0\}$.

The next result is about the nature of the set $K(S, \ell) := \{X \in \Sigma(S, \ell) | ker(L_X^0) = \{0\} \text{ and } \nabla X = 0\}$.

Proposition 5.8. If $K(S, \ell) \neq \emptyset$ then $K(S, \ell)$ is a Zariski open and dense subset of $E(S, \ell)$. In particular, if there exists $X \in E(S, \ell)$ satisfying the non-resonance condition mentioned in remark 5.7 then $K(S, \ell)$ is a Zariski open and dense in $E(S, \ell)$. Proposition 5.8 is a straightforward consequence of the following facts:
we are considering \( \Omega \in A \)
\( \mathbb{R} \) is a n.g.K singularity of \( \Omega \) of type \((1, \ldots, n)\).

In the case of the radial vector field, the germ of \( \Pi^*_t \) is defined by \( L(X) = L^N_\mathbb{R} \) is linear.

As a consequence, the set \( L^{-1}(\mathcal{N}T) \) is an algebraic subset of \( \mathcal{E}(S, \ell) \).

We leave the details to the reader.

**Remark 5.9.** In the case of the radial vector field, \( R = \sum_{i=0}^{n-1} x_i \frac{\partial}{\partial x_i} \), we have \( K(R, d-1) \neq \emptyset \) for all \( d \geq 2 \). In fact, it is proved in [14], § 3.2 that \( J_d \in (R, d-1) \), where \( J_d \) is the generalized Jouanolou’s vector field.

Consider a holomorphic family of \((n-2)\)-forms, \((\Omega_t)_{t \in U}\), defined on a polydisc \( Q \) of \( \mathbb{C}^n \), where the space of parameters \( U \) is an open set of \( \mathbb{C}^k \) with \( 0 \in U \). Let us assume that:

- For each \( t \in U \) the form \( \Omega_t \) defines a 2-dimensional foliation \( F_t \) on \( Q \).
  
  Let \( (\mathcal{Z}_t)_{t \in U} \) be the family of holomorphic vector fields on \( Q \) such that \( d\Omega_t = i_{\mathcal{Z}_t}dx_0 \wedge \cdots \wedge dx_{n-1} \).
  
- \( F_0 \) has a n.g.K singularity at \( 0 \in Q \).

We can now state the stability result, whose proof can be found in [14]:

**Theorem 5.10.** In the above situation there exists a neighborhood \( 0 \in V \subset U \), a polydisc \( 0 \in P \subset Q \), and a holomorphic map \( P : V \to P \subset \mathbb{C}^n \) such that \( P(0) = 0 \) and for any \( t \in V \) then \( P(t) \) is the unique singularity of \( F_t \) in \( P \). Moreover, \( P(t) \) is the same type as \( P(0) \) in the sense that: If \( 0 \) is a n.g.K singularity of type \((k_0, \ldots, k_{n-1}, k)\) of \( F_0 \) then \( P(t) \) is a n.g.K singularity of type \((k_0, \ldots, k_{n-1}, k)\) of \( F_t, \forall t \in V \).

Let us now describe \( F = f^*(\mathcal{G}) \) in a neighborhood of a point \( p \in I(f) \).

**Proposition 5.11.** If \( p \in I(f) \) then \( p \) is a n.g.K singularity of \( \Omega \) of type \((1, \ldots, 1, n)\).

**Proof.** It is easy to show that there exists a local chart \((U, x = (x_0, \ldots, x_{n-1})) \in \mathbb{C}^n \) around \( p \) such that the lifting \( \tilde{f} \) of \( f \) is of the form \( \tilde{f}|_U = (x_0, \ldots, x_{n-1}) : U \to \mathbb{C}^n \).

In particular \( F|_{U(p)} \) is represented by the homogeneous \((n-2)\)-form

\[
\Omega = (-1)^{i+k+1} \sum_{i,k} x_k P_i dx_0 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge \hat{dx}_k \wedge \cdots \wedge dx_{n-1}.
\]

Observe that \( L_\mathbb{R} \Omega = (d+n) \Omega, X = \sum_i P_i \frac{\partial}{\partial x_i}, Z = (d+n)X, [R, X] = dX \). Since we are considering \( \Omega \in \mathcal{A} \) we have that \( Y = R, N = 0 \) and hence we conclude that \( p \) is a n.g.K singularity of \( \Omega \) of type \((1, \ldots, 1, n)\). In particular the vector field \( S \) as in the Theorem [5.5] is the radial vector field.

It follows from theorem [5.10] that these singularities are stable under deformations. Proposition [5.11] says that the germ \( f^*\mathcal{G} \) of \( f^*\mathcal{G} \) at \( p \in I(f) \) is equivalent to the germ of \( \Pi^*_t \mathcal{G} \) at \( p \in I(f) \).
5.2.1. Deformations of the singular set of $\mathcal{F}_0 = f_0^* (\mathcal{G}_0)$. We have constructed an open and dense subset $\mathcal{W}$ inside $PB(\Theta, 2, n)$ containing the generic pull-back foliations. We will show that for any rational foliation $\mathcal{F}_0 \in \mathcal{W}$ and any germ of a holomorphic family of foliations $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ such that $\mathcal{F}_0 = \mathcal{F}_{t=0}$ we have $\mathcal{F}_t \in PB(\Theta, 2, n)$ for all $t \in (\mathbb{C}, 0)$.

Using the theorem. 5.10, with $\mathcal{V} = (\mathbb{C}, 0)$, it follows that for each $p_j \in I(f_0)$ there exists a deformation $p_j(t)$ of $p_j$ and a deformation of $\mathcal{F}_{t,p_j(t)}$ of $\mathcal{F}_{p_j}$ such that $p_j(t)$ is a n.g.K singularity of $\mathcal{F}_{t,p_j(t)} := \Omega_{p_j(t)}$ of type $(1, \ldots, 1, n)$ and $(\mathcal{F}_{t,p_j(t)})_{t \in (\mathbb{C}, 0)}$ defines a holomorphic family of foliations in $\mathbb{P}^{n-1}$. We will denote by $I(t) = \{p_1(t), \ldots, p_j(t), \ldots, p_{n+1}(t)\}$.

Remark 5.12. Since $I(t)$ is not connected we can not guarantee a priori that $\mathcal{F}_{t,p_i(t)} = \mathcal{F}_{t,p_j(t)}$, if $i \neq j$.

Lemma 5.13. There exist $\epsilon > 0$ and smooth isotopies $\phi_{q_i} : D_\epsilon \times V_{q_i} \rightarrow \mathbb{P}^n, q_i \in \text{Sing}(\mathcal{G}_0)$, such that $V_{q_i}(t) = \phi_t(\{t\} \times V_{q_i})$ satisfies:

(a) $V_{q_i}(t)$ is an algebraic subvariety of dimension 1 of $\mathbb{P}^n$ and $V_{q_i}(0) = V_{q_i}$ for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$.

(b) $I(t) \subset V_{q_i}(t)$ for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. Moreover, if $q_i \neq q_j$ and $q_i, q_j \in \text{Sing}(\mathcal{G}_0)$, we have $V_{q_i}(t) \cap V_{q_j}(t) = I(t)$ for all $t \in D_\epsilon$ and the intersection is transversal.

(c) $V_{q_i}(t) \setminus I(t)$ is contained in the Kapka-set of $\mathcal{F}_t$ for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. In particular, the transversal type of $\mathcal{F}_t$ is constant along $V_{q_i}(t) \setminus I(t)$.

Proof. See [11] lemma 2.3.3, p.83. □

5.3. End of the proof of Theorem [A]. We divide the end of the proof of Theorem [A] in two parts. In the first part we construct a family of rational maps $f_t : \mathbb{P}^n \rightarrow \mathbb{P}^n$, $f_t \in \text{Gen}(n, n-1, \nu)$, such that $(f_t)_{t \in D_\epsilon}$ is a deformation of $f_0$ and the subvarieties $V_{q_i}(t)$ are fibers of $f_t$ for all $t$. In the second part we show that there exists a family of foliations $(\mathcal{G}_t)_{t \in D_\epsilon}, \mathcal{G}_t \in \mathcal{A}$ (see Section [I]) such that $\mathcal{F}_t = f_t^* (\mathcal{G}_t)$ for all $t \in D_\epsilon$.

5.3.1. Part 1. Once $d = \text{deg}(\mathcal{G}_0) \geq 2$, the number of singularities of $\mathcal{G}_0$ is $N = \frac{d^{n+1} - 1}{d - 1} > n$, so we can suppose that the singularities of $\mathcal{G}_0$ are $q_1 = [0 : 0 : \ldots : 1], \ldots, q_{n} = [1 : 0 : \ldots : 0], \ldots, q_N$.

Proposition 5.14. Let $(\mathcal{F}_t)_{t \in D_\epsilon}$ be a deformation of $\mathcal{F}_0 = f_0^* (\mathcal{G}_0)$, where $(f_0, \mathcal{G}_0)$ is a generic pair, with $\mathcal{G}_0 \in \mathcal{A}$. $f_0 \in \text{Gen}(n, n-1, \nu)$ and $\text{deg}(f_0) = \nu \geq 2$. Then there exists a deformation $(f_t)_{t \in D_\epsilon}$ of $f_0$ in $\text{Gen}(n, n-1, \nu)$ such that:

(i) $V_{q_i}(t)$ are fibers of $(f_t)_{t \in D_\epsilon}$.

(ii) $I(t) = I(f_t), \forall t \in D_\epsilon$.

Proof. Let $f_0 = (F_0, \ldots, F_{n-1}) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ be the homogeneous expression of $f_0$. Then $V_{q_1}, V_{q_2}, \ldots, V_{q_n}$ appear as the complete intersections $V_{q_i} = (F_0 = F_1 = \ldots = F_{i-1} = \ldots = 0)$. The remaining fibers, $V_{q_i}$ for $i > n$ are obtained in the same way. With this convention we have that $I(f_0) = V_{q_i} \cap V_{q_j}$ if $i \neq j$. It follows from [19] (see section 4.6 pp 235-236) that each $V_{q_i}(t)$ is a smooth complete intersection generated by polynomials of the same degree. However we can not assure that the set of polynomials which define each $V_{q_i}(t)$ have a correlation among them. In
the next lines we will show this fact. For this let us work firstly with two curves. For instance, let us take \( V_{q_1}(t) \) and \( V_{q_2}(t) \) which are deformations of \( V_q \) and \( V_{q_2} \) respectively. We will see that this two curves are enough to construct the family of deformations \((f_i)_{i \in D_{\nu}}\). After that we will show that the remaining curves \( V_{q_i}(t) \) are also fibers of \((f_i)_{i \in D_{\nu}}\). We can write \( V_{q_1}(t) = (F_1(t) = F_2(t) = \ldots F_{n-1}(t) = 0) \), and \( V_{q_2}(t) = (\tilde{F}_0(t) = F_1(t) = \tilde{F}_2(t) = \ldots = \tilde{F}_{n-1}(t) = 0) \) where \((F_i(t))_{i \in D_{\nu}}\) and \((\tilde{F}_i(t))_{i \in D_{\nu}}\) are deformations of \( F_i \) and \( D_{\nu} \) is a possibly smaller neighborhood of \( 0 \). Observe first that since the \( F_{is}(t) \) and \( \tilde{F}_{is}(t) \) are near \( F_{is} \), they meet as a complete intersection at:

\[
I(f_i) := (F_0(t) = 0) \cap V_i(t)
\]

On the other hand we also have

\[
I(t) = V_{q_1}(t) \cap V_{q_2}(t) = V_{q_1}(t) \cap \{(F_0(t) = 0) \cap \{\tilde{F}_2(t) = \ldots \tilde{F}_{n-1}(t) = 0\}\}.
\]

Let us write \( \{S(t) = 0\} = \{\tilde{F}_2(t) = \ldots \tilde{F}_{n-1}(t) = 0\} \). Hence \( I(f_i) \cap \{S(t) = 0\} = V_{q_1}(t) \cap V_{q_2}(t) = I(t) \), which implies that \( I(t) \subset I(f_i) \). Since \( I(f_i) \) and \( I(t) \) have \( \nu^n \) points, we have that \( I(t) = I(f_i) \) for all \( t \in D_{\nu} \). In particular, we obtain that \( I(t) \subset \{S(t) = 0\} \). We will use the following version of Noether’s Normalization Theorem (see [11] p 86):

**Lemma 5.15.** (Noether’s Theorem) Let \( G_0, \ldots, G_k \in \mathbb{C}[z_1, \ldots, z_m] \) be homogeneous polynomials where \( 0 \leq k \leq m \) and \( m \geq 2 \), and \( X = (G_0, \ldots, G_k) = 0 \). Suppose that the set \( Y := \{p \in X | dG_0(p) \land \ldots \land dG_k(p) = 0\} \) is either \( 0 \) or \( \emptyset \). If \( G \in \mathbb{C}[z_1, \ldots, z_m] \) satisfies \( G_X \equiv 0 \), then \( G \in <G_0, \ldots, G_k> \).

Take \( k = n - 1 \), \( G_0 = F_0(t) \), \( G_1 = F_1(t) \ldots G_{n-1} = F_{n-1}(t) \). Using Noether’s Theorem with \( Y = 0 \) and the fact that all polynomials involved are homogeneous of the same degree, we have \( \tilde{F}_1(t) \in <F_0(t), F_1(t), \ldots, F_{n-1}(t)> \). More precisely we conclude that each \( \tilde{F}_i(t) = \sum_{j=0}^{n-1} g_{i,j}(t) F_j(t), g_{i,j}(t) \in \mathbb{C} \) and when \( t = 0 \) for each \( i, \tilde{F}_i(0) = F_i(0) = F_i \). On the other hand, if \( V_{q_k}(t) \) is the deformation of another \( V_{q_k} \), then \( V_{q_k}(t) \) is also a complete intersection, say, \( V_{q_k}(t) = (P^k(t) = \ldots = P^k_{n-1} = 0) \) where each \( P^k_i(t) \) for \( i = 1, \ldots, n-1 \) is a homogeneous polynomial of degree \( \nu \).

Since \( I(t) \subset (P^k_i(t) = 0) \) for \( i = 1, \ldots, n-1 \), we have that each \( P^k_i(t) \) is a linear combination of the \( F_{is}(t) \), that is, \( P^k_i(t) \in <F_0(t), F_1(t), \ldots, F_{n-1}(t)> \). This implies that \( V_{q_k}(t) \) is also a fiber of \( f_i \), as the reader can check, say \( V_{q_k}(t) = f_t^{-1}(q_k(t)) \). Since \( q_k(t) = f_t(V_{q_k}(t)) \) and \( f_t \) and \( V_{q_k}(t) \) are deformations of \( f_0 \) and \( V_q \) we get that \( q_k(t) \) is a deformation of \( q_k \), so that for small \( t \), \( q_k(t) \) is a regular value of \( f_t \).

5.3.2. Part 2. Let us now define a family of foliations \((G_i)_{i \in D_{\nu}}\), \( G_i \in \mathcal{A} \) (see Section 3) such that \( \tilde{F}_i = f^*_t(G_i) \) for all \( t \in D_{\nu} \). Let \( M(t) \) be the family of “rational varieties” obtained from \( \mathbb{P}^n \) by blowing-up at the \( \nu^n \) points \( p_1(t), \ldots, p_{\nu^n}(t) \) corresponding to \( I(t) \) of \( \mathcal{F}_i \); and denote by

\[
\pi(t) : M(t) \to \mathbb{P}^n
\]

the blowing-up map. The exceptional divisor of \( \pi(t) \) consists of \( \nu^n \) submanifolds \( E_j(t) = \pi(t)^{-1}(p_j(t)) \), \( 1 \leq j \leq \nu^n \), which are projective spaces \( \mathbb{P}^{n-1} \). More precisely, if we blow-up \( \mathcal{F}_i \) at the point \( p_i(t) \), then the restriction of the strict transform \( \pi^*\mathcal{F}_i \) to the exceptional divisor \( E_j(t) = \mathbb{P}^{n-1} \) is up to a linear automorphism of \( \mathbb{P}^{n-1} \), the homogeneous \((n-2)\)-form that defines \( \mathcal{F}_i \) at the point \( p_j(t) \). With
this process we produce a family of bidimensional holomorphic foliations in \( A \).

This family is the “holomorphic path” of candidates to be a deformation of \( G_0 \).

In fact, since \( A \) is an open set we can suppose that this family is inside \( A \). We fix the exceptional divisor \( E_1(t) \) to work with and we denote by \( \hat{G}_t \) the restriction of \( \pi^*\mathcal{F}_t \) to \( E_1(t) \). Consider the family of mappings \( f_t: \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}, t \in D_\epsilon \), defined in Proposition \[6.14\]. We will consider the family \( (f_t)_{t \in D_\epsilon} \) as a family of rational maps \( f_t: \mathbb{P}^n \rightarrow E_1(t) \); we decrease \( \epsilon \) if necessary. We would like to observe that the mapping \( f_t \circ \pi(t): M(t) \setminus \cup_j E_j(t) \rightarrow \mathbb{P}^{n-1} \) extends as holomorphic mapping \( \hat{f}_t: M(t) \rightarrow \mathbb{P}^{n-1} \) if \( |t| < \epsilon \). This follows from the fact that \( dF_0(t)(p_j(t)) \wedge dF_1(t)(p_j(t)) \wedge \ldots \wedge dF_{n-1}(t)(p_j(t)) \neq 0 \), \( 1 \leq j \leq n^n \), if \( |t| < \epsilon \).

The mapping \( f_t \) can be interpreted as follows. Each fiber of \( f_t \) meets \( p_j(t) \) once, which implies that each fiber of \( \hat{f}_t \) cuts \( E_1(t) \) only one time. Since \( M(t) \setminus \cup_j E_j(t) \) is biholomorphic to \( \mathbb{P}^n \setminus I(t) \), after identifying \( E_1(t) \) with \( \mathbb{P}^{n-1} \), we can imagine that if \( q \in M(t) \setminus \cup_j E_j(t) \) then \( \hat{f}_t(q) \) is the intersection point of the fiber \( \hat{f}_t^{-1}(\hat{f}_t(q)) \) with \( E_1(t) \).

We obtain a mapping

\[
\hat{f}_t: M(t) \rightarrow \mathbb{P}^{n-1}.
\]

With all these ingredients we can define the foliation \( \hat{F}_t = f_t^* (\hat{G}_t) \in \mathbb{P} \Theta(2, n) \).

This foliation is a deformation of \( G_0 \). Based on the previous discussion let us denote \( F_1(t) = \pi(t)^*(\mathcal{F}_t) \) and \( \hat{F}_1(t) = \pi(t)^*(\hat{G}_t) \).

**Lemma 5.16.** If \( \mathcal{F}_1(t) \) and \( \hat{F}_1(t) \) are the foliations defined previously, we have that

\[
\mathcal{F}_1(t)|_{E_1(t) \cap \mathbb{P}^{n-1}} = \hat{G}_t = \hat{F}_1(t)|_{E_1(t) \cap \mathbb{P}^{n-1}}
\]

where \( \hat{G}_t \) is the foliation induced on \( E_1(t) \) by the homogeneous \((n-2)\)-form \( \Omega_{p_1(t)} \).

**Proof.** In a neighborhood of \( p_1(t) \in I(t) \), \( \mathcal{F}_t \) is represented by the homogeneous \((n-2)\)-form \( \Omega_{p_1(t)} \). This \((n-2)\)-form satisfies \( \iota_{R(t)} \Omega_{p_1(t)} = 0 \) and therefore naturally defines a foliation on \( \mathbb{P}^{n-1} \). This proves the first equality. The second equality follows from the geometrical interpretation of the mapping \( \hat{f}_t: M(t) \rightarrow \mathbb{P}^{n-1} \), since \( \hat{F}_1(t) = f_t^*(\hat{G}_t) \). \( \square \)

Let \( q_1(t) \) be a singularity of \( \hat{G}_t \). Since the map \( t \rightarrow q_1(t) \in \mathbb{P}^{n-1} \) is holomorphic, there exists a holomorphic family of automorphisms of \( \mathbb{P}^{n-1}, t \rightarrow H(t) \) such that \( q_1(t) = [0: \ldots : 1] \in E_1(t) \) is kept fixed. Observe that such a singularity has \((n-1)\) non algebraic separatrices at this point. Fix a local analytic coordinate system \((U_t = u_0(t), \ldots, u_n(t))\) at \( q_1(t) \) such that the local separatrices are tangents to \( u_i(t) = 0 \) for each \( i \). Observe that the local smooth hypersurfaces along \( \hat{V}_{q_1(t)} = \hat{f}_t^{-1}(q_1(t)) \) defined by \( \hat{U}_i(t) := (u_i(t) \circ \hat{f}_t = 0) \) are invariant for \( \hat{F}_1(t) \). Furthermore, they meet transversely along \( \hat{V}_{q_1(t)} \). On the other hand, \( \hat{V}_{q_1(t)} \) is also contained in the Kupka set of \( \mathcal{F}_1(t) \). Therefore there are \((n-1)\) local smooth hypersurfaces \( U_i(t) := (u_i(t) \circ \hat{f}_t = 0) \) invariant for \( \mathcal{F}_1(t) \) such that:

1. All the \( U_i(t) \) meet transversely along \( \hat{V}_{q_1(t)} \).
2. \( U_i(t) \cap \pi(t)^{-1}(p_1(t)) = (U_i(t) = 0) = \hat{U}_i(t) \cap \pi(t)^{-1}(p_1(t)) \) (because \( \mathcal{F}_1(t) \) and \( \hat{F}_1(t) \) coincide on \( E_1(t) \simeq \mathbb{P}^{n-1} \)).
3. Each \( U_i(t) \) is a deformation of \( U_i(0) = \hat{U}_i(0) \).
We have proved that the foliations $F_U$ contains a small neighborhood $\tilde{E}$ that is, $F_U$ algebraic surface must have that $\hat{I}$.

Consider the following properties:

- $d \geq 2$
- Theorem B.
- In the conditions above, if properties $P_1, P_2, P_3$ and $P_4$ hold then $F_i$ is a pull back foliation, $F = f^*(G)$, where $G$ is a 1-dimensional foliation of degree $d \geq 2$ on $\mathbb{P}^{n-1}$. 

Proof. Let us consider the projection $\hat{f}_i : M(t) \to \mathbb{P}^{n-1}$ on a neighborhood of the regular fibre $\tilde{V}_{\nu(t)}$, and fix local coordinates $(\tilde{U}_t = u_0(t), \ldots, u_{n-1}(t))$ on $\mathbb{P}^{n-1}$ such that $U_1(t) := (u_1(t) \circ \hat{f}_t = 0)$. For small $\epsilon$, let $H_\epsilon = (u_1(t) \circ \hat{f}_t = \epsilon)$. Thus $\Sigma_\epsilon = \tilde{U}_0(t) \cap H_\epsilon$ are (vertical) compact curves, deformations of $\Sigma_0 = \tilde{V}_{\nu(t)}$. Set $\Sigma_\epsilon = U_0(t) \cap H_\epsilon$. The $\Sigma_\epsilon$'s, as the $\tilde{\Sigma}_\epsilon$'s, are compact curves (for $t$ and $\epsilon$ small), since $U_0(t)$ and $\tilde{U}_0(t)$ are both deformations of the same $U_0$. Thus for small $t$, $U_0(t)$ is close to $\tilde{U}_0(t)$. It follows that $\hat{f}_t(\Sigma_\epsilon)$ is an analytic curve contained in a small neighborhood $\tilde{U}_t$ of $q_1(t)$, for small $\epsilon$. By the maximum principle, we must have that $\hat{f}_t(\Sigma_\epsilon)$ is a point, so that $\hat{f}_t(U_0(t)) = \hat{f}_t(\cup_\epsilon \Sigma_\epsilon)$ is a curve $C \subset \tilde{U}_t$, that is, $U_0(t) = \hat{f}_t(C)$. But $U_0(t)$ and $\tilde{U}_0(t)$ intersect the exceptional divisor $E_1(t) \simeq \mathbb{P}^{n-1}$ along the separatrix $(u_0(t) = 0)$ of $G_t$ through $q_1(t)$. This implies that $U_0(t) = \hat{f}_t^{-1}(C) = \hat{f}_t^{-1}(u_0(t) = 0) = \tilde{U}_0(t)$. 

We have proved that the foliations $F_i$ and $\tilde{F}_i$ have a common local leaf: the leaf that contains $\pi(t)\left(\tilde{U}_0(t)\backslash \tilde{V}_{\nu(t)}\right)$ which is not algebraic. Let $D(t) := Tang(F_i, \tilde{F}_i)$ be the set of tangencies between $F(t)$ and $\tilde{F}(t)$. This set can be defined by $D(t) = \{Z \in \mathbb{C}^{n+1}; \Omega(t) \wedge \Omega(t) = 0\}$, where $\Omega(t)$ and $\Omega(t)$ define $F(t)$ and $\tilde{F}(t)$, respectively. Hence it is an algebraic set. Since this set contains an immersed non-algebraic surface $U_0(t)$, we necessarily have that $D(t) = \mathbb{P}^n$. It follows that $\tilde{F}_i = F_i$.

Recall from Definition 5.12 the concept of a generic map. Let $f \in RM((n, n-1, \nu)$, $I(f)$ its indeterminacy locus and $F$ a foliation by complex surfaces on $\mathbb{P}^n$, $n \geq 4$. Consider the following properties:

- $P_1$ : Any point $p_j \in I(f)$ $F$ has the following local structure: there exists an analytic coordinate system $(U^p_j, x^p_j)$ around $p_j$ such that $x^p_j(p_j) = 0 \in (\mathbb{C}^n, 0)$ and $F|_{U^p_j} = x^p_j$ can be represented by a homogeneous $(n-2)$-form $\Omega_{p_j}$ (as described in the Lemma 5.11) such that
  - (a) $Sing(\mathcal{Z}_{p_j}) = 0$, where $\mathcal{Z}_{p_j}$ is the rotational of $\Omega_{p_j}$.
  - (b) $0$ is a n.g.K singularity of the type $(1, \ldots, 1, n)$

- $P_2$ : There exists a fibre $f^{-1}(q) = V(q)$ such that $V(q) = f^{-1}(q)\backslash I(f)$ is contained in the Kupka-Set of $F$.

- $P_3$ : $V(q)$ has transversal type $Q$, where $Q$ is a germ of vector field on $(\mathbb{C}_{n-1}, 0)$ with at least a non algebraic separatrix and such that the Camacho-Sad index of $G$ with respect to this separatrix is non-real.

- $P_4$ : $F$ has no algebraic hypersurface.

Lemma 5.17 allows us to prove the following result:

Theorem B. In the conditions above, if properties $P_1, P_2, P_3$ and $P_4$ hold then $F$ is a pull back foliation, $F = f^*(G)$, where $G$ is a 1-dimensional foliation of degree $d \geq 2$ on $\mathbb{P}^{n-1}$. 

Choosing $i = 0$ we shall prove that $U_0(t) = \tilde{U}_0(t)$ for small $t$. For our analysis this will be sufficient to finish the proof of Theorem A.
Note that the situation \( n = 3 \) is proved in [4, Th. B p. 709]. So we can think this result as \( n \geq 4 \)-dimensional generalization of [4] for bi-dimensional foliations in \( \mathbb{P}^n \).

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