Some coset actions in $G_2(q)$ and distance-transitive graphs

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Abstract

This paper studies whether there are distance-transitive graphs arising from the coset actions of $G_2(q)$ on the subfield subgroup $G_2(\sqrt{q})$ or $G_2(q)$ on the Ree subgroup $^2G_2(q)$. It is found that there are no such graphs, even if the groups are extended by outer automorphisms of $G_2(q)$.

1 Introduction

In this paper, a graph means a finite connected undirected graph without loops or multiple edges. A graph $\Gamma$ is called distance-transitive if for every two pairs $(w, x)$ and $(y, z)$ of the graph such that the distance from $w$ to $x$ is the same as the distance from $y$ to $z$, there is an automorphism of the graph that carries $w$ to $y$ and $x$ to $z$.

By a result of D.H. Smith\[1\], distance-transitive graphs can be classified into either primitive or non-primitive, and non-primitive distance-transitive graphs can be derived from primitive distance-transitive graphs, so the classification of primitive distance-transitive graphs serves as a step to classify all distance-transitive graphs.

Praeger, Saxl and Yokoyama\[2\] showed that a primitive distance-transitive graph is either a Hamming graph, has an automorphism group of affine type, or has an almost-simple automorphism group (a group $G$ is almost simple if there is a nonabelian simple group $H$ such that $H \triangleleft G \triangleleft \text{Aut}(H)$; here $H$ is the socle of $G$). The primitive distance-transitive graphs having automorphism groups of affine type are classified by John van Bon\[3\].

The vertex stabilizer in a primitive distance-transitive graph $\Gamma$ is a maximal subgroup of $\text{Aut}\Gamma$. Thanks to the classification of all finite simple groups and their maximal subgroups, the classification of primitive distance-transitive graphs with almost-simple automorphism groups can be dealt with on a case-by-case basis. A lot of work has already been done; for example, the graphs $\Gamma$ where $\text{Aut}\Gamma$ have alternating, sporadic or linear socle are completely determined in [4], [5] and [6] respectively.

There is a combinatorial generalization of distance-transitive graphs called distance-regular graphs: a distance-regular graph is a regular graph such that for any two vertices $x$ and $y$, the number of vertices at distance $i$ from $x$ and at distance $j$ from $y$ is independent from $x$ and $y$, and only dependent upon $i$, $j$ and the distance between $x$ and $y$. It is evident from the definition that every distance-transitive graph is distance-regular, because every ordered pair $(x, y)$ of vertices with the same distance between $x$ and $y$ are equivalent. Distance-regular graphs are usually characterized by intersection arrays.

In this paper we investigate two open cases where $\text{Aut}\Gamma$ has socle $G_2(q)$: they correspond to the vertex stabilizer having type $G_2(\sqrt{q})$ and $^2G_2(q)$. More formally, we assume that $\text{Aut}\Gamma = G_2(q):X$ where $X$ is a subgroup of the outer automorphism group of $G_2(q)$. The vertex stabilizer of $\Gamma$ is $G_2(\sqrt{q}):X$ and...
$G_2(q):X$, respectively. So the vertices of $\Gamma$ can be identified with the cosets $(G_2(q):X)/(G_2(\sqrt{q}):X)$ and $(G_2(q):X)/(2G_2(q):X)$, respectively.

2 $G_2(q)$ on $G_2(\sqrt{q})$

Let $G$ be the finite simple group $G_2(q)$, where $q = p^{2n}$, and $H$ the subgroup of $G$ isomorphic to the subfield subgroup $G_2(\sqrt{q})$; all such subgroups are pairwise conjugate in $G$. Let $r = \sqrt{q}$. Let $\sigma$ be the Frobenius isomorphism $\sigma : x \to x^r$.

The first thing to note is that $q$ is a power of 3. If $\Gamma$ is distance-transitive, the permutation character of $\text{Aut}\Gamma$ acting on the vertices of $\Gamma$ is multiplicity-free. The multiplicity-free actions of $(G:X)/(H:X)$ are determined in [9]: The action is multiplicity-free iff $q$ is a power of 3 and $X$ contains the graph automorphism of $G_2(q)$. So $q$ is a power of 3.

Ross Lawther has computed the suborbit lengths of $G$ acting on $G/H$ when $q$ is a power of 3, shown in Table 1, where $z$ is the $G_2(\overline{F}_q)$-class representative of $x^{-1}\sigma(x)$ in $G_2(q)$ listed in [11].

The elements of the form $x_{na+nb}(1)$ are unipotent elements of the algebraic group $G_2(\overline{F}_q)$, so they have order 3. Note that $x_{2a+b}(1)$ and $x_{3a+2b}(1)$ commute, so $x_{2a+b}(1)x_{3a+2b}(1)$ also has order 3.

Let $T$ be a Cartan subgroup of the algebraic group $G_2(\overline{F}_q)$. The Dynkin diagram with respect to $T$ is of type $G_2$. Denote its short simple root as $a$ and long simple root as $b$ under a choice of simple roots. Then $z_1 = a + b$, $z_2 = a$, $z_3 = 2a - b$ are three short roots of the root diagram. The elements $h(x_1, x_2, x_3 | x_1, x_2, x_3 \in \overline{F}_q^*)$ are elements in $T$, mapping $z_i$ to $x_i$, $i = 1, 2, 3$.

The letters $\sigma, \tau, \theta, \eta$ and $\gamma$ in Table 1 represents certain elements of the multiplicative group $\overline{F}_q^*$. Let $\kappa$ be a generator of the multiplicative group $\overline{F}_q^*$, and

\[
\sigma = \kappa^{(r+1)(r^3-1)}, \quad \tau = \kappa^{(r-1)(r^3+1)}, \quad \theta = \kappa^{q^2+q+1},
\eta = \theta^{r-1}, \quad \gamma = \theta^{r+1}.
\]

The shorthand $h_*(x_1, x_2, x_3)$ represents $h(*^{x_1}, *^{x_2}, *^{x_3})$ where $*$ is any letter.
Table 1: The suborbits of the coset action $G/H$ when $q$ is a power of 3.

| $z$                                    | Size of suborbits                                      | Number of classes |
|----------------------------------------|-------------------------------------------------------|-------------------|
| $x_{3a+2b}(1)$                         | 1                                                     | 1                 |
| $x_{2a+b}(1)$                          | $(r^6 - 1)$                                           | 1                 |
| $x_{2a+b}(1)x_{3a+2b}(1)$              | $(r^6 - 1)(r^2 - 1)$                                  | 1                 |
| $x_{a+b}(1)x_{3a+b}(1)$                | $r^2(r^6 - 1)(r^2 - 1)/2$                             | 1                 |
| $x_a(1)x_b(1)$                         | $r^4(r^6 - 1)(r^2 - 1)/2$                             | 1                 |
| $h(-1,-1,1)$                           | $r^4(r^4 + r^2 + 1)$                                  | 1                 |
| $h(-1,-1,1)x_b(1)$                    | $r^4(r^6 - 1)$                                        | 1                 |
| $h(-1,-1,1)x_{2a+b}(1)$               | $r^4(r^6 - 1)$                                        | 1                 |
| $h(-1,-1,1)x_b(1)x_{2a+b}(1)$         | $r^4(r^6 - 1)(r^2 - 1)/2$                             | 1                 |
| $h_x(i,-2t,i)$                        | $r^5(r^3 - 1)(r^2 - r + 1)$                           | $(r - 3)/2$       |
| $h_x(i,-2t,i)x_{3a+2b}(1)$            | $r^5(r^6 - 1)(r - 1)$                                 | $(r - 3)/2$       |
| $h_x(i,-i,0)$                          | $r^5(r^3 - 1)(r^2 - r + 1)$                           | $(r - 3)/2$       |
| $h_x(i,-i,0)x_{2a+b}(1)$              | $r^5(r^6 - 1)(r - 1)$                                 | $(r - 3)/2$       |
| $h_x(i,j,-i-j)$                       | $r^6(r^3 - 1)(r^2 - r + 1)(r - 1)$                    | $(r^2 - 8r + 15)/12$ |
| $h_y(i,-2t,i)$                        | $r^5(r^3 + 1)(r^2 + r + 1)$                           | $(r - 1)/2$       |
| $h_y(i,-2t,i)x_{3a+2b}(1)$            | $r^5(r^6 - 1)(r + 1)$                                 | $(r - 1)/2$       |
| $h_y(i,-i,0)$                         | $r^5(r^3 + 1)(r^2 + r + 1)$                           | $(r - 1)/2$       |
| $h_y(i,-i,0)x_{2a+b}(1)$              | $r^5(r^6 - 1)(r + 1)$                                 | $(r - 1)/2$       |
| $h_y(i,j,-i-j)$                       | $r^6(r^3 + 1)(r^2 + r + 1)(r + 1)$                    | $(r^2 - 4r + 3)/12$ |
| $h_y(i,(r-1)i,-ri)$                   | $r^6(r^6 - 1)$                                        | $(r - 1)^2/4$     |
| $h_y(i,ri,-(r+1)i)$                   | $r^6(r^6 - 1)$                                        | $(r - 1)^2/4$     |
| $h_y(i,ri,r^2i)$                      | $r^6(r^3 - 1)(r^2 - 1)(r + 1)$                        | $r(r + 1)/6$      |
| $h_y(i,-ri,r^2i)$                     | $r^6(r^3 + 1)(r^2 - 1)(r - 1)$                        | $r(r - 1)/6$      |

The second thing to note is that the subgroup $G_2(\sqrt{q})$ is the fixed subgroup of the Frobenius isomorphism $\sigma: x \rightarrow x^q$. If $\sigma \in X$, then the subgroup $H:X$ is an involution centralizer in $G:X$, and the action of $G:X$ on cosets $(G:X)/(H:X)$ can be identified with the action of $G:X$ on the conjugacy class of $\sigma$ by conjugation. Then it would be possible to apply the methodology of [12], summarized in the following theorem:

**Theorem 2.1** Let $\Gamma$ be a distance-transitive graph with distance-transitive group $G$. Suppose that the vertex set $V\Gamma$ of $\Gamma$ is a conjugacy class of involutions in $G$, that $G$ acts on $\Gamma$ by conjugation and that there are elements in $V\Gamma$ which commute in $G$. Take $x, y \in \Gamma$ with $x$ adjacent to $y$. Then at least one of the following statements holds.

- $\Gamma$ is a polygon or an antipodal 2-cover of a complete graph.
- $G$ is a 2-group.
- The order of $xy$ is an odd prime, if $a, b \in \Gamma$ with $ab$ of order 2, then $a$ and $b$ have maximal distance in $\Gamma$, and if $a, b \in \Gamma$ the order of $ab$ is not 4.
The elements $x$ and $y$ commute, and if $z \in O_2(x)$ then $xz$ has order 2, 4 or an odd prime. Moreover either $O_2(C_G(x)) = \langle x \rangle$ or $C_G(x)$ contains a normal subgroup generated by $p$-transpositions.

By the following lemma of [12], we can assume $\sigma \in X$:

**Lemma 2.1** Let $\Gamma$ be a graph on which $G$ acts primitively distance-transitively, and denote by $H$ the stabilizer in $G$ of a vertex of $\Gamma$. Suppose $\sigma$ is an automorphism of $G$.

- If $\sigma$ centralizes $H$ and $\text{diam} \, \Gamma \geq 3$, then $\sigma \in \text{Aut}(\Gamma)$;
- If $\sigma$ normalizes $H$ and $\text{diam} \, \Gamma \geq 5$, then the same conclusion holds.

The suborbit lengths shown in Table 1 indicates that a distance-transitive $\Gamma$ cannot have diameter 2 (because there are at least 3 different nontrivial suborbit lengths in $G$ acting on $G/H$, and the outer automorphism group can only fuse together suborbits of the same length). So we may assume $\Gamma$ has diameter $\geq 3$ and thus $\sigma \in X$.

In the rest of this section, we will denote the elements of $G:X$ in external semidirect product notation; that is, an element of $G:X$ is written as $(x,y)$, where $x \in G$ and $y \in X$, and the multiplication rule is $(w,x)(y,z) = (wz(y),xz)$. Thus, the conjugation of $(1,\sigma)$ by $(g,1)$ is $(g^{-1},1)(1,\sigma)(g,1) = (g^{-1},\sigma)(g,1) = (g^{-1}\sigma(g),\sigma)$. Also $(g^{-1}\sigma(g),\sigma)(1,\sigma) = (g^{-1}\sigma(g),1)$. If $(1,\sigma)$ commutes with $(g^{-1}\sigma(g),\sigma)$, we have $g^{-1}\sigma(g) = \sigma(g^{-1}\sigma(g))$. And that means $(g^{-1}\sigma(g))^2 = g^{-1}\sigma(g)g^{-1}\sigma(g) = g^{-1}\sigma(g)\sigma(g^{-1}\sigma(g)) = 1$. The reverse implication also holds, in the sense that $(g^{-1}\sigma(g))^2 = 1$ implies $(1,\sigma)$ commutes with $(g^{-1}\sigma(g),\sigma)$.

There exists elements $g \in G$ such that $g^{-1}\sigma(g)$ is conjugate to $h(-1,-1,1)$. In this case, $g^{-1}\sigma(g)$ has order 2, so $(1,\sigma)$ commutes with $(g^{-1}\sigma(g),\sigma)$. Thus the assumptions of Theorem 2.1 holds.

To find an element with the form $(g^{-1}\sigma(g),\sigma)$ that is connected to $(1,\sigma)$, the following theorem<sup>[3]</sup> is applied:

**Theorem 2.2** Let $G$ be a primitive distance-transitive group of automorphisms of $\Gamma$ (having diameter $d$) and $x \in \eta \Gamma$. Then among the nontrivial $G_x$-orbit lengths, $|\Gamma_1(x)|$ ($\Gamma_1(x)$ means the vertices of $\Gamma$ having distance $i$ to $x$) is among the two smallest. Moreover, if $|\Gamma_1(x)|$ is not the smallest, then $|\Gamma_d(x)|$ is.

Since outer automorphisms can only fuse suborbits with the same length, the smallest suborbits must be one labeled by $x_{3a+2b}(1)$, $x_{2a+b}(1)$ or $x_{2a+b}(1)x_{3a+2b}(1)$. In either case, the element $g^{-1}\sigma(g)$ has order 3, so $(1,\sigma)$ and $(g^{-1}\sigma(g),\sigma)$ does not commute. So the last case of Theorem 2.1 does not hold. As $G:X$ is not a 2-group and $\Gamma$ has degree and diameter at least 3, the first and second case does not hold. It can be concluded that if $\Gamma$ is distance-transitive, then the third case of Theorem 2.1 holds. Specifically, there are no $g \in G:X$ such that $g^{-1}\sigma(g)$ has order 4.

There exists $g \in G$ such that $g^{-1}\sigma(g)$ is conjugate to $h_\gamma(i,-2i,i)$ when $r \geq 9$, and there exists $g \in G$ such that $g^{-1}\sigma(g)$ is conjugate to $h_\eta(i,-2i,i)$. The elements $\gamma$ and $\eta$ has order $r-1$ and $r+1$, respectively. As one of $r-1$ and $r+1$ is divisible by 4, it is possible to choose $i$ such that one of $\gamma^i$ and $\eta^i$ has order 4 in the multiplicative group $\mathbb{F}_q^*$. But this means that $g^{-1}\sigma(g)$ has order 4.

So there is no primitive distance-transitive graph with automorphism group $G_2(q):X$ and vertex stabilizer $G_2(\sqrt{q}):X$. 

4
3 $G_2(q)$ on $^2G_2(q)$

Let $G$ be the finite simple group $G_2(q)$, where $q = 3^{2n+1}$ ($n \geq 0$), and $H$ the subgroup of $G$ isomorphic to the Ree group $^2G_2(q)$; all such subgroups are pairwise conjugate in $G$\cite{7}. Let $X$ be a subgroup of the outer automorphism group of $G$.

Ross Lawther has computed the suborbit lengths of $G$ acting on $G/H$\cite{10}:

| Size of suborbits                  | Number of classes |
|------------------------------------|-------------------|
| $1$                                | $1$               |
| $(q^3+1)(q-1)$                     | $1$               |
| $q(q^3+1)(q-1)/2$                  | $1$               |
| $q^2(q^3+1)(q-1)/2$                | $1$               |
| $q^3(q^3+1)(q-1)$                  | $1$               |
| $q^2(q^2-q+1)$                     | $(q-3)/2$         |
| $q^2(q^3+1)(q-1)$                  | $(q-3)/6$         |
| $q^3(q^2-1)(q-3m+1)$               | $(q-3m)/6$        |
| $q^3(q^2-1)(q+3m+1)$               | $(q+3m)/6$        |

Table 2: The suborbits of the coset action $G/H$. Note: $q = 3m^2$

Note that three rows of this table would be absent if $q = 3$; but in this case, the graph would have $|G|/|H| = 4245696/1512 = 2808$ vertices and diameter at least 6 (because the outer automorphism group can only fuse together suborbits of the same length). A graph of this size would be covered in Chapter 14. of [13], but [13] does not contain any intersection arrays of a distance-regular graph with 2808 vertices and diameter at least 6. So we will assume $q \geq 27$ in the rest of this section.

An inequality from [14] states that a distance-regular graph of diameter $d$ and $v$ vertices satisfies $d < 8/3 \log_2(v)$. If there were a distance-regular graph arising from the group action, it would have $q^3(q^3-1)(q+1)$ vertices and diameter at least $(q+6)/|X| = (q+6)/2(2n+1)$, because the outer automorphism group of $G$ is a direct product of the graph automorphism (of order 2) and the field automorphism (of order $2n+1$), and it can never fuse more than $|X|$ suborbits into a suborbit. This inequality is only satisfied when $n \leq 3$.

The main theorem to deal with the cases when $n \leq 3$ is the following theorem from [12]:

**Theorem 3.1** Let $\Gamma$ be a graph of diameter $d$ on which the group $G$ acts distance-transitively as a group of automorphisms. For a vertex $x \in \Gamma$, denote by $G_x^i$ the kernel of the action of the stabilizer of $x$ in $G$ on $\Gamma_i(x)$. If, for some $i \geq 1$, we have $G_x^i \neq 1$, then $G_x^i \subsetneq G_x^{i-1} \subsetneq \ldots \subsetneq G_x^1$ or $G_x^i \subsetneq G_x^{i+1} \subsetneq \ldots \subsetneq G_x^1$.

Since all the suborbit lengths are proper divisors of $|H| = q^3(q^3+1)(q-1)$, the kernel of $H$ acting on any of the suborbits is nontrivial, and so does the extension $H:X$.

The last two rows of Table 2 contain two suborbits. The respective kernel size (the size of $G_x^i$) of one row is divisible by 19 when $n = 1$, 31 when $n = 2$, 43 when $n = 3$, and the other divisible by 37.
when $n = 1$, 271 when $n = 2$, 2269 when $n = 3$. By Theorem 2.2, the suborbit corresponding to $\Gamma_1(x)$ has length $(q^3+1)(q-1)$ or $q^2(q^2 - q + 1)$. Thus $G_1^1$ is not divisible by any of the two kernel sizes, and it is impossible to have $G_1^x \subseteq G_2^x$ for any of the two suborbits. So $G_2^x \nsubseteq G_3^x$ for both suborbits, and $G_2^x$ would have order divisible by the primes $19 \times 37$ when $n = 1$, $31 \times 271$ when $n = 2$, and $43 \times 2269$ when $n = 3$. But no suborbit has such a kernel, even if $X$ is nontrivial (outer automorphisms can only multiply the order of the kernels by some number containing the prime factors 2, 3, 5 and 7.).

So there is no primitive distance-transitive graph with automorphism group $G_2(q):X$ and vertex stabilizer $^2G_2(q):X$.

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