On a Conjecture on the Representation of Positive Integers as the Sum of Three Terms of the Sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$

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Abstract

We prove some cases of a conjecture by Farhi on the representation of every positive integer as the sum of three terms of the sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$. This is done by generalizing a method used by Farhi in his original paper.

1 Introduction

In the following we let $\mathbb{N}$ denote the set of non-negative integers, $\lfloor \cdot \rfloor$ is the integer part function, and $\langle \cdot \rangle$ is the fractional part function.

A classical result by Legendre \cite{3} states that every natural number not of the form $4^s(8t + 7)$, $s, t \in \mathbb{N}$ can be written as the sum of three squares.

In relation to this Farhi recently conjectured the following:

Conjecture 1 (Farhi \cite{2}). Let $a \geq 3$ be an integer. Then every natural number can be represented as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor\right)_{n \in \mathbb{N}}$. 

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The conjecture was confirmed by Farhi [1] and Mezroui, Azizi, and Ziane [4] for \( a \in \{3, 4, 8\} \).

In this paper we generalize the method used by Farhi for \( a = 4 \), and partially for \( a = 3 \), to prove that the conjecture holds for \( a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\} \). The method uses Legendre’s three-square theorem and properties of quadratic residues.

We also note that the set of integers, \( a \), such that Conjecture [1] holds is closed under multiplication by a square.

\section{Method and results}

We start by introducing the following sets:

\textbf{Definition 2.} For any nonzero \( a \in \mathbb{N} \) we define

\[ Q_a = \{ 0 < \varphi < a \mid \exists x \in \mathbb{Z}: \varphi \equiv x^2 \pmod{a} \}. \]

Therefore, \( Q_a \) is the set of quadratic residues modulo \( a \).

\textbf{Definition 3.} For any nonzero \( a \in \mathbb{N} \) we define

\[ A_a = \{ \varphi \in \mathbb{N} \mid \exists x, y, z \in Q_a \cup \{0\}: \varphi = x + y + z \}. \]

Thus, \( A_a \) is the set of integers that can be written as the sum of three elements of \( Q_a \cup \{0\} \).

\textbf{Definition 4.} For any nonzero \( a \in \mathbb{N} \) we define

\[ R_a = \{ \varphi \in A_a \mid \forall \psi \in A_a: \varphi \equiv \psi \pmod{a} \Rightarrow \varphi = \psi \}. \]

So, \( R_a \) is the set of integers that can be written as the sum of three elements of \( Q_a \cup \{0\} \), and such that no other integer in the same residue class modulo \( a \) has this property.

Now, we are ready to formulate the main result.

\textbf{Theorem 5.} Let \( a \in \mathbb{N} \) be nonzero and assume that for every \( k \in \mathbb{N} \) there exists an \( r \in R_a \) such that \( ak + r \neq 4^s(8t + 7) \) for any \( s, t \in \mathbb{N} \). Then every \( N \in \mathbb{N} \) can be written as the sum of three terms of the sequence \( \left( \left\lfloor \frac{n^2}{a} \right\rfloor \right)_{n \in \mathbb{N}} \).

\textbf{Proof.} Let \( N \in \mathbb{N} \) be fixed. By assumption we can choose \( r \in R_a \) such that \( aN + r \neq 4^s(8t + 7) \) for any \( s, t \in \mathbb{N} \). By Legendre’s theorem it follows that \( aN + r \) can be written of the form

\[ aN + r = A^2 + B^2 + C^2 \quad (1) \]
for some $A, B, C \in \mathbb{N}$. Now we have

$$r \equiv A^2 + B^2 + C^2 \pmod{a},$$

so

$$r = (A^2 \mod a) + (B^2 \mod a) + (C^2 \mod a), \tag{2}$$

since $r \in \mathcal{R}_a$. Dividing by $a$ and separating the integer and fractional parts of the right hand side in (1), we get

$$N + \frac{r}{a} = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor + \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

and from (2) we have

$$\frac{r}{a} = \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

so

$$N = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor.$$

Since we can find the sets $\mathcal{R}_a$ by computation, we can now apply the main theorem to get the following corollary.

**Corollary 6.** Conjecture [4] is satisfied for $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$.

**Proof.** Consider the following table:

| $a$ | $\mathcal{R}_a$ |
|-----|-----------------|
| 4   | $\{0, 1, 2, 3\}$ |
| 7   | $\{4, 6\}$      |
| 8   | $\{2, 3, 5, 6\}$ |
| 9   | $\{1, 4, 7, 8\}$ |
| 20  | $\{11, 15, 18, 19\}$ |
| 24  | $\{11, 14, 19, 21, 22\}$ |
| 40  | $\{27, 38\}$    |
| 104 | $\{99\}$        |
| 120 | $\{107\}$       |

Calculating modulo 8 it can be checked fairly easily that for each $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$ and every $k \in \mathbb{N}$ there exists an $r \in \mathcal{R}_a$ such
that $ak + r$ is not of the form $4^s(8t + 7)$, $s, t \in \mathbb{N}$, and thus every natural number can be written as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a}\right\rfloor\right)_{n \in \mathbb{N}}$.

To demonstrate this, we show the case $a = 7$. All the other cases are done in exactly the same way.

For $k \equiv 1, 2, 3, 6$ or 7 (mod 8) we have $7k + 4 \equiv 3, 2, 1, 6$ and 5 (mod 8), respectively, and for $k \equiv 0, 4$ or 5 (mod 8) we have $7k + 6 \equiv 6, 2$ and 1 (mod 8), respectively. Since $4^s(8t + 7) \equiv 0, 4$ or 7 (mod 8), $s, t \in \mathbb{N}$, we conclude that for every $k \in \mathbb{N}$ we can write $7k + r$, for $r \in \mathbb{R}$, $7 = \{4, 6\}$, such that it is not of the form $4^s(8t + 7)$, $s, t \in \mathbb{N}$. The case now follows from Theorem 5.

Further, one should note that the set of integers satisfying Conjecture 1 is closed under multiplication by a square.

**Observation 7.** Let $M$ be the set of integers satisfying Conjecture 1. If $a \in M$, then $ak^2 \in M$ for any integer $k > 0$.

**Proof.** This follows easily since for any $n$ we can find $A, B, C \in \mathbb{N}$ such that

$$n = \left\lfloor \frac{A^2}{a}\right\rfloor + \left\lfloor \frac{B^2}{a}\right\rfloor + \left\lfloor \frac{C^2}{a}\right\rfloor = \left\lfloor \frac{(Ak)^2}{ak^2}\right\rfloor + \left\lfloor \frac{(Bk)^2}{ak^2}\right\rfloor + \left\lfloor \frac{(Ck)^2}{ak^2}\right\rfloor.$$

Knowing this, we see that since Conjecture 1 is satisfied for $a = 3, 9, 4, 8$, it must also hold for $a = 3^k$ for any positive integer $k$ and for $a = 2^k$, $k > 1$.

Finally, using Observation 7, Corollary 6, and the fact 4 that Conjecture 1 holds for $a = 3$, we get that the conjecture holds for the following values up to 120.

$$a \in \{3, 4, 7, 8, 9, 12, 16, 20, 24, 27, 28, 32, 36, 40, 48, 63, 64, 72, 75, 80, 81, 96, 100, 104, 108, 112, 120\}.$$

Unfortunately, it seems that the method deployed in Theorem 5 is not extendable to other cases, since its success relies on $R_a$, and in general $R_a$ does not contain the necessary elements for the condition in the theorem to be satisfied.

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References

[1] B. Farhi, On the representation of the natural numbers as the sum of three terms of the sequence $\left\lfloor \frac{n^2}{3} \right\rfloor$, *J. Integer Seq.*, 16 (2013), [Article 13.6.4].

[2] B. Farhi, An elementary proof that any natural number can be written as the sum of three terms of the sequence $\left\lfloor \frac{n^2}{3} \right\rfloor$, *J. Integer Seq.*, 17 (2014), [Article 14.7.6].

[3] A. M. Legendre, *Théorie des Nombres*, 3rd ed., Vol. 2, 1830.

[4] S. Mezroui, A. Azizi, and M. Ziane, On a conjecture of Farhi, *J. Integer Seq.*, 17 (2014), [Article 14.1.8].

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