On Stefan-Boltzmann law and the Casimir effect at finite temperature in the
Schwarzschild spacetime

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Abstract

This paper deals with quantum field theory in curved space-time using the Thermo Field Dynamics. The scalar field is coupled to the Schwarzschild space time and then thermalised. The Stefan-Boltzmann law is established at finite temperature and the entropy of the field is calculated. Then the Casimir energy and pressure are obtained at zero and finite temperature.
I. INTRODUCTION

Quantum Field Theory (QFT) has been a very successful theory in physics [1]. Together with general relativity it constitutes a great pillar of modern physics. Nevertheless there is no widely recognized theory that unifies a diverse areas of physics even though there are numerous attempts to establish a quantum theory of gravitation [2–4]. There are attempts to unify through a Quantum Field Theory in curved space-time. This strand has been widely investigated in understanding the fundamental characteristics of black holes. Particuarly the no-hair theorem applied to the Kerr black hole has been considered [5] in which a scalar field was coupled to the black hole physics and the field equations is solved by using numerical technique.

On the other hand QFT in curved space leads to the well known Hawking effect, that is, the gravitational field creates particles near the event horizon of a black hole, which in turn emits a spectrum of thermal radiation. This effect is not experimentally verified but systems analogous to black holes in condensed matter reveal that this effect may be real [6]. It is important to note that the thermalisation of a field in a curved space-time is not unique, but its importance is fundamental from the point of view of physical reality and everything happens at finite temperature. There are different ways to introduce temperature under such conditions. The most widespread form for this purpose is to associate time with temperature by a Wick rotation [7]. However this procedure has an undesirable characteristic under certain conditions, the information about the field dynamics, that is due to the temporal coordinate is lost. However in this article, a different approach is based on Thermo Field Dynamics (TFD), temperature is introduced by doubling the Fock space [8–12]. The main advantage of TFD is that the duplicate space topology allows a study of the Stefan-Boltzmann law and the Casimir effect. Both effects are manifestations of the invariance of Bogoliubov transformation in the Fock space. The Casimir effect is well known in the literature [13], as well as its interaction with the scalar field [14], even at finite temperature [15]. On the other hand, the investigation of this physical system in different geometries different from the flat space has revealed that Casimir force plays an important role [16]. Over the years TFD has been applied in various fields such as the scalar field and Dirac field to calculate both Stefan-Boltzmann law and Casimir effect at finite temperature [17–20]. In this sense, we propose to advance the investigation of TFD in curved spaces. Schwarzschild space-time is chosen for this purpose fundamentally for two reasons. The first is of an experimental nature, that is, given the advance of experimental techniques in recent years, it is hoped that some measure of the Casimir effect in Schwarzschild space can be verified soon, for instance in the vicinity of the Sun. The second reason is that gravitational
thermodynamics was initially developed in such a curved space. Thus, from a theoretical point of view, we hope that thermalization by TFD will lead to an understanding of the entropy of this space, different from that normally accepted.

This article is divided as follows. In section II, the scalar field is described in a curved space. The energy-momentum tensor is considered in Schwarzschild space-time. In section III, TFD formalism is introduced. And the generalized Bogoliubov transformation is defined. In section IV, Stefan-Boltzmann law in Schwarzschild space-time is defined and the entropy of the scalar field is calculated. In section V, the Casimir effect at zero temperature in the curved space is established. In section VI, the Casimir effect in Schwarzschild space is calculated at finite temperature. Finally section VII, some brief conclusions are presented.

II. GRAVITY COUPLED TO A SCALAR FIELD

The action of the mass zero scalar field in curved background is given by

\[ S = \frac{1}{2} \int d^4 x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \xi R \phi(x)^2 \right), \tag{1} \]

where \( g \) is the metric determinant and \( \xi \) is the coupling parameter for the scalar curvature, \( R \). Then the field equation in curved space-time is

\[ (\partial_\mu \partial^\mu + \xi R) \phi(x) = 0. \tag{2} \]

The energy-momentum tensor, which is defined as

\[ T_{\gamma\rho} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\gamma\rho}}, \tag{3} \]

for a scalar field that leads to

\[ T_{\gamma\rho} = \frac{1}{2} g_{\gamma\rho} \partial^\mu \phi(x) \partial_\mu \phi(x) - \partial_\gamma \phi(x) \partial_\rho \phi(x) + \xi \left( R_{\gamma\rho} - \frac{1}{2} g_{\gamma\rho} R + g_{\gamma\rho} \square - \partial_\gamma \partial_\rho \right) \phi(x)^2. \tag{4} \]

Now the energy-momentum tensor is used to calculate the Stefan-Boltzmann law and the Casimir effect at finite temperature. In order to avoid divergences, the energy-momentum tensor is written at different space-time points leading to

\[ T_{\gamma\rho}(x) = \lim_{x' \rightarrow x} \tau \left[ \frac{1}{2} g_{\gamma\rho} \partial^\mu \phi(x) \partial_\mu \phi(x') - \partial_\gamma \phi(x) \partial_\rho \phi(x') + \xi \left( R_{\gamma\rho} - \frac{1}{2} g_{\gamma\rho} R + g_{\gamma\rho} \square - \partial_\gamma \partial_\rho \right) \phi(x) \phi(x') \right], \tag{5} \]
where $\tau$ is the time ordering operator.

Considering the canonical quantization, the commutation relation is

$$[\phi(x), \partial^\mu \phi(x')] = i n_0^\mu \delta(\bar{x} - \bar{x}'),$$

(6)

where $n_0^\mu = (1, 0, 0, 0)$ is a time-like vector and

$$\partial^\rho \theta(x_0 - x'_0) = n_0^\rho \delta(x_0 - x'_0),$$

(7)

where $\theta(x_0 - x'_0)$ is the step function. The energy-momentum tensor becomes

$$T_{\gamma\rho}(x) = \lim_{x' \to x} \left\{ \Gamma_{\gamma\rho} \tau \left[ \phi(x)\phi(x') \right] - I_{\gamma\rho} \delta(x - x') \right\},$$

(8)

with

$$\Gamma_{\gamma\rho} = \frac{1}{2} g_{\gamma\rho} \partial^\mu \partial_\mu - \partial_\gamma \partial_\rho + \xi \left( R_{\gamma\rho} - \frac{1}{2} g_{\gamma\rho} R + g_{\gamma\rho} \Box - \partial_\gamma \partial_\rho \right)$$

(9)

and

$$I_{\gamma\rho} = - \frac{i}{2} g_{\gamma\rho} n_0^\mu n_{0\mu} + i n_{0\gamma} n_{0\rho}.$$  

(10)

In order to calculate the Stefan-Boltzmann law and the Casimir effect associated with the scalar field in the curved space-time, the average momentum of the energy-momentum tensor is determined. Then

$$\langle T_{\gamma\rho}(x) \rangle = \lim_{x' \to x} \left\{ i \Gamma_{\gamma\rho} G_0(x - x') - I_{\gamma\rho} \delta(x - x') \right\},$$

(11)

where

$$i G_0(x - x') = \langle 0 | \tau [\phi(x)\phi(x')] | 0 \rangle,$$

(12)

is the scalar field propagator.

### A. The Schwarzschild metric

The Schwarzschild metric as a cosmological background is considered. This metric provides the solution to the Einstein field equations that describe the gravitational field outside a spherical mass. In spherical coordinates $\{t, r, \theta, \phi\}$ this solution is given by

$$ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2,$$

(13)
where $M$ is mass of the black hole and
\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \] (14)

Since the interest is in the solution outside a spherical body (the black hole) the Einstein equation in vacuum is given as
\[ R_{\gamma\rho} = 0. \] (15)

Using this equation in vacuum, the quantity $\Gamma_{\gamma\rho}$ defined in eq. (9) becomes
\[ \Gamma_{\gamma\rho} = \frac{1}{2} g_{\gamma\rho} \partial^\mu \partial_\mu - \partial_{\gamma} \partial_{\rho} + \xi (g_{\gamma\rho} \Box - \partial_{\gamma} \partial_{\rho}). \] (16)

In order to obtain the Stefan-Boltzmann law and the Casimir effect for the scalar field coupled to gravity, in a context where the Schwarzschild space-time is the geometric background, two components of eq. (16) are used
\[ \Gamma_{00} = \frac{1}{2} \partial_0 \partial'_0 - \left( \frac{1}{2} + \xi \right) \left( 1 - \frac{2M}{r} \right) \left[ \partial_1 \partial'_1 + \frac{1}{r^2} \partial_2 \partial'_2 + \frac{1}{r^2 \sin^2 \theta} \partial_3 \partial'_3 \right] \] (17)
and
\[ \Gamma_{11} = \frac{1}{2} \partial_1 \partial'_1 + \left( \frac{1}{2} + \xi \right) \left( 1 - \frac{2M}{r} \right)^{-1} \left[ \partial_0 \partial'_0 + \frac{1}{r^2} \partial_2 \partial'_2 + \frac{1}{r^2 \sin^2 \theta} \partial_3 \partial'_3 \right]. \] (18)

Another ingredient that will be considered in this cosmological background is the temperature. Finite temperature is introduced in order to consider the cosmological background using the TFD formalism.

III. THERMO FIELD DYNAMICS (TFD) FORMALISM

Thermal quantum field theory with a thermal vacuum, $|0(\beta)\rangle$, is considered, with $\beta = \frac{1}{k_B T}$, where $T$ is the temperature and $k_B$ is the Boltzmann constant. The main objective is to interpret the statistical average of an arbitrary operator $\mathcal{O}$, as the expectation value in a thermal vacuum, i.e., $\langle \mathcal{O} \rangle = \langle 0(\beta) | \mathcal{O} | 0(\beta) \rangle$. This interpretation leads to two fundamental conditions: (i) the Hilbert space $\mathcal{S}$ is doubled and (ii) the Bogoliubov transformation is used. The doubled space consists of a thermal space that is defined as $\mathcal{S}_T = \mathcal{S} \otimes \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ is the dual (tilde) Hilbert space. The Bogoliubov transformation introduces temperature effects through a rotation between tilde ($\tilde{\mathcal{S}}$) and non-tilde ($\mathcal{S}$) operators.
For arbitrary operators $O$ and $\hat{O}$ in Hilbert space $S$ and tilde space $\tilde{S}$ respectively, the Bogoliubov transformation is given as

$$
\begin{pmatrix}
O(k, \alpha) \\
\eta\hat{O}^\dagger(k, \alpha)
\end{pmatrix} = B(\alpha) \begin{pmatrix}
O(k) \\
\eta\hat{O}^\dagger(k)
\end{pmatrix},
$$

(19)

where $k$ is the 4-momentum and $\eta = -1(+1)$ for bosons (fermions) and $B(\alpha)$ is defined as

$$
B(\alpha) = \begin{pmatrix}
u(\alpha) & -v(\alpha) \\
\eta v(\alpha) & u(\alpha)
\end{pmatrix},
$$

(20)

with $u^2(\alpha) + \eta v^2(\alpha) = 1$. The $\alpha$ parameter is assumed to be the compactification parameter defined by $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{D-1})$, with $D$ being the space-time dimension. The effect of temperature is described by the choice $\alpha_0 \equiv \beta$ and $\alpha_1, \cdots, \alpha_{D-1} = 0$. In general, $\alpha$ may be associated with any physical quantity. The functions $u(\alpha)$ and $v(\alpha)$, are related to the Bose distribution, and are given as

$$
v^2(\alpha) = (e^{\alpha \omega_k} - 1)^{-1}, \quad u^2(\alpha) = 1 + v^2(\alpha),
$$

(21)

with $\omega_k = k_0$ being the energy.

The TFD formalism is used to introduce the compactification parameter $\alpha$ in the propagator of the theory. Here the scalar field is considered as an example. The propagator of the scalar field is written as

$$
G_0^{(AB)}(x - x'; \alpha) = i\langle 0(\alpha)|\tau[\phi^A(x)\phi^B(x')]|0(\alpha)\rangle,
$$

$$
= i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} G_0^{(AB)}(k; \alpha),
$$

(22)

where $A, B = 1, 2$ represent the doubled space. Then

$$
G_0^{(AB)}(k; \alpha) = B^{-1}(\alpha) G_0^{(AB)}(k) B(\alpha),
$$

(23)

with

$$
G_0^{(AB)}(k) = \begin{pmatrix}
G_0(k) & 0 \\
0 & \eta G_0^*(k)
\end{pmatrix},
$$

(24)

and

$$
G_0(k) = \frac{1}{k^2 + i\epsilon},
$$

(25)

being the usual massless scalar field propagator. $G_0^*(k)$ is the conjugate complex of $G_0(k)$. 

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Then the Green function (the physical component is given for $A = B = 1$) is

$$G_0^{(11)}(k; \alpha) = G_0(k) + \eta \, v^2(k; \alpha)[G_0^*(k) - G_0(k)],$$

where $v^2(k; \alpha)$ is the generalized Bogoliubov transformation which is given as

$$v^2(k; \alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} \sum_{l_{\sigma_1} \cdots l_{\sigma_s} = 1}^{\infty} (-\eta)^{s+\sum_{r=1}^{s} l_{\sigma_r}} \exp\left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j}\right],$$

with $d$ being the number of compactified dimensions, $\eta = \frac{1}{1}$ for fermions (bosons), $\{\sigma_s\}$ denotes the set of all combinations with $s$ elements.

In the next section, the TFD formalism is used to calculate the Stefan-Boltzmann law and the Casimir effect at temperature $T$ for a mass zero scalar field coupled to gravity in a geometric background described by the Schwarzschild spacetime.

In order to calculate these quantities for the scalar field coupled to gravitational field, a field theory on the topology $\Gamma_D^d = (S^1)^d \times \mathbb{R}^{D-d}$ with $1 \leq d \leq D$ is considered. Here $d$ is the number of compactified dimensions. This establishes a formalism such that any set of dimensions of the manifold $\mathbb{R}^D$ are compactified, where the circumference of the $n$th $S^1$ is specified by $\alpha_n$.

**IV. STEFAN-BOLTZMANN LAW IN THE SCHWARZSCHILD SPACETIME**

In order to calculate the Stefan-Boltzmann law a physical (renormalized) energy-momentum tensor is calculated. Using the Casimir prescription, the finite energy-momentum tensor has the form

$$\mathcal{T}_{\gamma\rho}(x; \alpha) = \left\langle T_{\gamma\rho}^{(AB)}(x; \alpha) \right\rangle - \left\langle T_{\gamma\rho}^{(AB)}(x) \right\rangle,$$

where the duplicate notation of TFD formalism is used with one component dependant on constant ($\alpha$). Then

$$\mathcal{T}_{\gamma\rho}(x; \alpha) = \lim_{x' \to x} \left\{ i \Gamma_{\gamma\rho} \mathcal{G}_0^{(AB)}(x-x'; \alpha) \right\},$$

where

$$\mathcal{G}_0^{(AB)}(x-x'; \alpha) = G_0^{(AB)}(x-x'; \alpha) - G_0^{(AB)}(x-x'),$$

and constant $\Gamma_{\gamma\rho}$ is given in eq. [9].
In order to calculate the Stefan-Boltzmann law in the Schwarzschild spacetime, the topology \( \Gamma_4^1 = S^1 \times \mathbb{R}^3 \), with \( \alpha = (\beta, 0, 0, 0) \) is used. Using the generalized Bogoliubov transformation
\[
v^2(\beta) = \sum_{l_0=1}^{\infty} e^{-\beta l_0},
\]
the Green function becomes
\[
\mathcal{G}_0(x - x'; \beta) = 2 \sum_{l_0=1}^{\infty} G_0(x - x' - i\beta l_0 n_0),
\]
where \( n_0 = (1, 0, 0, 0) \) and \( \mathcal{G}_0(x - x'; \beta) \equiv \mathcal{G}_0^{(11)}(x - x'; \beta) \), that is the physical component. Then the energy-momentum tensor is
\[
T_{\gamma \rho}^{(11)}(x; \beta) = 2i \lim_{x' \to x} \left\{ \Gamma_{\gamma \rho} \sum_{l_0=1}^{\infty} G_0(x - x' - i\beta l_0 n_0) \right\}.
\]

The energy associated with the scalar field coupled to gravity in a Schwarzschild background is obtained by choosing \( \gamma = \rho = 0 \). Then
\[
T_{00}^{(11)}(x; \beta) = 2i \lim_{x' \to x} \left\{ \Gamma_{00} \sum_{l_0=1}^{\infty} G_0(x - x' - i\beta l_0 n_0) \right\},
\]
where \( \Gamma_{00} \) is given in eq. (17) and
\[
G_0(x - x' - i\beta l_0 n_0) = -\frac{i}{(2\pi)^2 (x - x' - i\beta l_0 n_0)^2},
\]
with
\[
(x - x' - i\beta l_0 n_0)^2 = -\left(1 - \frac{2M}{r}\right)(t - t' - i\beta l_0)^2 + \left(1 - \frac{2M}{r}\right)^{-1} (r - r')^2 + r^2(\theta - \theta')^2 + r^2 \sin^2 \theta (\phi - \phi')^2.
\]

Using these results, the 00-component of the energy-momentum tensor at finite temperature becomes
\[
T_{00}^{(11)}(T) = \frac{\pi^2(1 + \xi)r}{30(r - 2M)} T^4.
\]
where \( T_{00}^{(11)}(T) \equiv E(T) \) and the Riemann Zeta function,
\[
\zeta(4) = \sum_{l_0=1}^{\infty} \frac{1}{l_0^4} = \frac{\pi^4}{90},
\]
is used. The eq. (37) is the Stefan-Boltzmann law for the black hole. In the limit \( r >> 2M \), the Stefan-Boltzmann law becomes
\[
T_{00}^{(11)}(T) = \frac{\pi^2(1 + \xi)}{30} T^4.
\]
The Stefan-Boltzmann law for the massless scalar field in a flat spacetime is given by \( \mathcal{T}_{00}^{(11)}(T) \), i.e., eq. (39). Then in this limit, the gravitational effect due to the black hole is ignored.

With the pressure given as \( P = \frac{E}{3} \), and the Maxwell relationship given by

\[
\left( \frac{\partial P}{\partial T} \right)_V = \left( \frac{\partial S}{\partial V} \right)_T, \tag{40}
\]

where \( S \) is the entropy leads to

\[
\left( \frac{\partial S}{\partial V} \right)_T = \frac{4\pi^2(1 + \xi)r}{90(r - 2M)} T^3. \tag{41}
\]

Then the entropy associated with the scalar field coupled to gravity in a Schwarzschild spacetime is given as

\[
S = \frac{4\pi^2 (1 + \xi)}{90} T^3 \int \frac{r}{(r - 2M)} r^2 \sin \theta dr d\theta d\phi. \tag{42}
\]

Performing an integration, the gravitational entropy becomes

\[
S = \frac{64M^3\pi^3(1 + \xi)}{45} T^3 \left[ -\ln \left| \frac{2M}{2M - r} \right| + \frac{R}{2M} + \frac{R^2}{8M^2} \right], \tag{43}
\]

where \( R \) is the radius of the integration hypersurface. It is important to note that this expression has two noteworthy regions of singularity. The first is the black hole event horizon, \( r = 2M \). This singularity is not of concern because it stems from the choice of the coordinate system. In addition the causal region is outside the event horizon. The singularity in the \( r \to \infty \) region arises from integration over the whole spacetime. It should be noted, however, that the entropy density remains finite even in the limit of a flat spacetime. Thus its integration with infinite space will also lead to a divergence. It is worth noting that an entropy expression is linked to both macroscopic and microscopic components. But in the case of gravitational interaction we have no information on the latter. Thus the spacetime entropy divergence is an expected result as it would arise from, a priori, an infinite ensemble.

V. CASIMIR EFFECT AT ZERO TEMPERATURE

In order to calculate the Casimir effect at zero temperature, a theory with topology \( \Gamma_4^1 = S^1 \times \mathbb{R}^3 \) is considered. Choosing the \( \alpha \) parameter as \( \alpha = (0, 0, 0, i2b) \), where \( 2b \) corresponds to the length of the circumference \( S^1 \). Here the Bogoliubov transformation is

\[
v^2(b) = \sum_{l_3=1}^{\infty} e^{-i2b b^3 l_3} \tag{44}\]
and the Green function is

\[ \mathcal{G}_0(x - x'; b) = 2 \sum_{l_3=1}^{\infty} G_0(x - x' - 2bl_3n_3) \] (45)

with \( n_3 = (0, 0, 0, 1) \). The Casimir energy is calculated for the case \( \gamma = \rho = 0 \). Then the energy-momentum tensor becomes

\[ \mathcal{T}_{00}^{(11)}(x; b) = 2i \lim_{x' \to x} \left\{ \Gamma_{00} \sum_{l_3=1}^{\infty} G_0(x - x' - 2bl_3z) \right\}. \] (46)

Using eq. (17) the Casimir energy \( (E_c \equiv \mathcal{T}_{00}^{(11)}(x; b)) \) at zero temperature associated to the massless scalar field coupled to gravity in a Schwarzschild spacetime is

\[ E_c = \frac{(r - 2M)^2}{1440\pi^2r^4b^4} \left[ \pi^4(1 + \xi)r(2M - r) - 180bM(1 + 2\xi)\zeta(3) \right], \] (47)

where \( \zeta(3) \) is the Riemann Zeta function. Far from the event horizon, i.e. \( r >> 2M \), the Casimir energy has the form

\[ E_c = -\frac{\pi^2(1 + \xi)}{1440b^4}. \] (48)

This is the result found for the case of the flat spacetime, where the gravitational effects due to black hole are ignored.

Similarly the Casimir pressure is calculated by taking \( \gamma = \rho = 1 \). Then the energy-momentum tensor becomes

\[ \mathcal{T}_{11}^{(11)}(x; b) = 2i \lim_{x' \to x} \left\{ \Gamma_{11} \sum_{l_3=1}^{\infty} G_0(x - x' - 2bl_3z) \right\}. \] (49)

Using eq. (18) the Casimir pressure \( (P_c \equiv \mathcal{T}_{11}^{(11)}(x; b)) \) at zero temperature is

\[ P_c = -\frac{60bM\zeta(3) - \pi^4(1 + \xi)r(2M - r)}{480\pi^2r^2b^4}. \] (50)

In the limit \( r >> 2M \) the Casimir pressure is

\[ P_c = -\frac{\pi^2(1 + \xi)}{480b^4}. \] (51)

It is a well-known result in flat spacetime. These results indicate that the gravitational force due to the black hole changes the Casimir energy and pressure for a scalar field at zero temperature.
VI. CASIMIR EFFECT AT FINITE TEMPERATURE

With the topology $\Gamma_4^2 = S^1 \times S^1 \times \mathbb{R}^2$ for $\alpha = (\beta, 0, 0, i2b)$, the Casimir effect at finite temperature is calculated. Then the Bogoliubov transformation becomes

$$v^2(\beta, b) = \sum_{l_0=0}^{\infty} e^{-\beta k^0 l_0} + \sum_{l_3=1}^{\infty} e^{-i2bl_3 l_3} + 2 \sum_{l_0,l_3=1}^{\infty} e^{-\beta k^0 l_0 - i2bl_3 l_3},$$

where the first two terms are associated with the Stefan-Boltzmann law and the Casimir effect at zero temperature and the third term provides the combined effect of temperature and spatial compactification. Indeed, in this case, two compactifications, the time and the other along the $z$-coordinate are explored. Then the Green function is

$$\mathcal{G}_0(x - x'; \beta, b) = 4 \sum_{l_0,l_3=1}^{\infty} G_0(x - x' - i\beta l_0 n_0 - 2bl_3 n_3).$$

Using the Green function the energy-momentum becomes

$$\mathcal{T}^{(11)}_{\gamma\rho}(\beta, b) = 4i \lim_{x' \to x} \left\{ \Gamma_{\gamma\rho} \sum_{l_0=1}^{\infty} G_0(x - x' - i\beta l_0 n_0 - 2bl_3 n_3) \right\}.$$ 

This leads to the Casimir energy at finite temperature as

$$\mathcal{T}^{(11)}_{00}(\beta, b) = \frac{-2(2M - r)}{\pi^2 r^2} \sum_{l_0,l_3=1}^{\infty} \frac{1}{[(2bl_3)^2(2M + r) - (\beta l_0)^2(2M - r)]^3} \times \left[ 8(2l_3)^3 M (1 + 2\xi)(4M^2 - r^2) - (2bl_3)^2 r(2M + r) ((1 + \xi)r^2 \\
- 6M^2(1 + 2\xi)) - 2(\beta l_0)^2 bM (1 + 2\xi) l_3 (2M + r) (2M + 3r) \\
+ (\beta l_0)^2 r(2M - r) (M^2 (2 + 4\xi) - 2(1 + \xi)r^2) \right].$$

The Casimir pressure at finite temperature is

$$\mathcal{T}^{(11)}_{11}(\beta, b) = \frac{2}{\pi^2(2M - r)} \sum_{l_0,l_3=1}^{\infty} \frac{1}{[(2bl_3)^2(2M + r) - (\beta l_0)^2(2M - r)]^3} \times \left[ 8(l_0l_3)^3 M (r^2 - 4M^2) + 3(2bl_3)^2 r(2M + r) ((1 + \xi)r^2 \\
- 2M^2) + 2(\beta l_0)^2 bM l_3 (2M - r) (2M + 3r) \\
- (\beta l_0)^2 r(2M - r) (2M^2 - (1 + \xi)r^2) \right].$$

Overall the Casimir effect at finite temperature for the scalar field is modified by the gravitational force, due to the presence of the black hole, and finite temperature.

In the limit $r \gg 2M$ the Casimir energy and pressure at finite temperature become

$$\mathcal{T}^{(11)}_{00}(\beta, b) = -\frac{2}{\pi^2} \sum_{l_0,l_3=1}^{\infty} \frac{(1 + \xi)[2(\beta l_0)^2 - (2bl_3)^2]}{[(2bl_3)^2 + (\beta l_0)^2]^3}.$$
\[ T_{11}^{(1)}(\beta, b) = -\frac{2}{\pi^2} \sum_{l_0, l_3=1}^{\infty} \frac{(1 + \xi)[(\beta l_0)^2 - 3(2bl_3)^2]}{[(2bl_3)^2 + (\beta l_0)^2]^3}. \]  

These results represent the Casimir energy and pressure in a flat space-time, if the gravitational effect at finite temperature due to the black hole is ignored.

VII. CONCLUSIONS

The Stefan-Boltzmann law is obtained by coupling a scalar field to the Schwarzschild space-time. The thermalization of the scalar field is obtained by using TFD. Using the temperature dependent energy, it is possible to calculate the first law of thermodynamics. The entropy should not be confused with Hawking entropy of Schwarzschild black hole since the first is obtained using the quantum field theory in curved space-time while the second is a geometric property of the black hole. This implies that calculated entropy refers to a property of the scalar field and is dependent on the integration surface. Here the energy and the pressure of the Casimir effect generated by the scalar field in the curved space-time and finite temperature are presented. It is important to note that the Casimir entropy at finite temperature does exist even though the integration over a spherical hypersurface is quite intricate. It is possible that the entropy leads to no remnant since the energy limit for \( T \to 0 \) is null.

Acknowledgments

This work by A. F. S. is supported by CNPq projects 308611/2017-9 and 430194/2018-8.

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