On the Composition of Two-Prover Commitments, and Applications to Multi-Round Relativistic Commitments*

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Abstract. We consider the related notions of two-prover and of relativistic commitment schemes. In recent work, Lunghi et al. proposed a new relativistic commitment scheme with a multi-round sustain phase that keeps the binding property alive as long as the sustain phase is running. They prove security of their scheme against classical attacks; however, the proven bound on the error parameter is very weak: It blows up double exponentially in the number of rounds. In this work, we give a new analysis of the multi-round scheme of Lunghi et al., and we show a linear growth of the error parameter instead (also considering classical attacks only). Our analysis is based on a new composition theorem for two-prover commitment schemes. The proof of our composition theorem is based on a better understanding of the binding property of two-prover commitments that we provide in the form of new definitions and relations among them. These new insights are certainly of independent interest and are likely to be useful in other contexts as well. Finally, our work gives rise to several interesting open problems, for instance extending our results to the quantum setting, where the dishonest provers are allowed to perform measurements on an entangled quantum state in order to try to break the binding property.

* This paper is an extended version of our EUROCRPYT 2016 paper. The eprint version is available at https://eprint.iacr.org/2016/113.
** Supported by the NWO Free Competition grant 617.001.203.
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1 Introduction

TWO-PROVER COMMITMENT SCHEMES. We consider the notion of 2-prover commitment schemes, as originally introduced by Ben-Or, Goldwasser, Kilian and Wigderson in their seminal paper [BGKW88]. In a 2-prover commitment scheme, the prover (i.e., the entity that is responsible for preparing and opening the commitment) consists of two agents, $P$ and $Q$, and it is assumed that these two agents cannot communicate with each other during the execution of the protocol. With this approach, the classical and quantum impossibility results for unconditionally secure commitment schemes [May97,LC97] can be circumvented.

A simple 2-prover bit commitment scheme is the scheme proposed by Crépeau et al. [CSST11], which works as follows. The verifier $V$ chooses a uniformly random $a \in \{0,1\}^n$ and sends it to $P$, who replies with $x := y + a \cdot b$, where $b$ is the bit to commit to, and $y \in \{0,1\}^n$ is a uniformly random string (only) to $P$ and $Q$. Furthermore, “$+$” is bit-wise XOR, and “$\cdot$” is scalar multiplication (of the scalar $b$ with the vector $a$). In order to open the commitment (to $b$), $Q$ sends $y$ to $V$, and $V$ checks if $x + y = a \cdot b$. It is clear that this scheme is hiding: The commitment $x = y + a \cdot b$ is uniformly random and independent of $a$ no matter what $b$ is. On the other hand, the binding property follows from the observation that in order to open the commitment to $b = 0$, $Q$ needs to announce $y = x$, and in order to open to $b = 1$, he needs to announce $y = x + a$. Thus, in order to open to both, he must know $x$ and $x + a$, and thus $a$, which is a contradiction to the no-communication assumption, because $a$ was sent to $P$ only.

In the quantum setting, where the dishonest provers are allowed to share an entangled quantum state and can produce $x$ and $y$ by means of performing measurements on their respective parts of the state, the above reasoning for the binding property does not work anymore. Nevertheless, as shown in [CSST11], the binding property still holds (though with a weaker parameter).

RELATIVISTIC COMMITMENT SCHEMES. The idea of relativistic commitment schemes, as introduced by Kent [Ken99], is to take a 2-prover commitment scheme as above and enforce the no-communication assumption by means of relativistic effects: Place $P$ and $Q$ spatially far apart, and execute the scheme fast enough, so that there is not enough time for them to communicate. The obvious downside of such a relativistic commitment scheme is that the binding property stays alive only for a very short time: The opening has to take place almost immediately after the committing, before the provers have the chance to exchange information. This limitation can be circumvented by considering multi-round schemes, where after the actual commit phase there is a sustain phase, during which the provers and the verifier keep exchanging messages, and as long as this sustain phase is running, the commitment stays binding (and hiding), until the commitment is finally opened. Such schemes were proposed in [Ken99] and [Ken05], but they are rather inefficient, and the security analyses are somewhat informal (e.g., with no formal security definitions) and of asymptotic nature. Schemes that require quantum communication were also considered and studied [Ken12,KT13,LKB13], but those were all without sustain phase.

More recently, Lunghi et al. [LKB13] proposed a new and simple multi-round relativistic commitment scheme, and provided a rigorous security analysis. Their scheme works as follows. The actual commit protocol is the commit protocol from the Crépeau et al. scheme: $V$ sends a uniformly random string $a_0 \in \{0,1\}^n$ to $P$, who returns $x_0 := y_0 + a_0 \cdot b$. Then, to sustain the commitment, before $P$ has the chance to tell $a_0$ to $Q$, $V$ sends a new uniformly random string $a_1 \in \{0,1\}^n$ to $Q$ who replies with $x_1 := y_1 + a_1 \cdot y_0$, where $y_1 \in \{0,1\}^n$ is another random string shared between $P$ and $Q$, and the multiplication $a_1 \cdot y_0$ is in a suitable finite field. Then, to further sustain the commitment, $V$ sends a new uniformly random string $a_2 \in \{0,1\}^n$ to $P$, who replies with $x_2 := y_2 + a_2 \cdot y_1$, etc. Finally, after the last sustain round where $x_m := y_m + a_m \cdot y_{m-1}$ has been sent to $V$, in order to finally open the commitment, $y_m$ is sent to $V$ (by the other prover). See Figure 1. In order to verify the opening, $V$ computes $y_{m-1}, y_{m-2}, \ldots, y_0$ inductively in the obvious way, and checks if $x_0 + y_0 = a_0 \cdot b$.

What is crucial is that in round $i$ (say for odd $i$), when preparing $x_i$, the prover $Q$ must not know $a_{i-1}$, but he is allowed to know $a_1, \ldots, a_{i-2}$. Thus, the execution must be timed in such a way that between subsequent rounds there is not enough time for the provers to communicate, but they may communicate over multiple rounds.

As for the security of this scheme, it is obvious that the hiding property stays satisfied up to the open phase: Every single message $V$ receives is one-time-pad encrypted. As for the binding property, Lunghi et al. prove that the scheme with a $m$-round sustain phase is $\varepsilon_m$-binding against classical attacks, where $\varepsilon_m$ satisfies $\varepsilon_0 = 2^{-n}$ (this is just the standard Crépeau et al. scheme) and $\varepsilon_m \leq 2^{-n-1} + \sqrt{\varepsilon_{m-1}}$ for
Fig. 1. The Lunghi et al. multi-round scheme (for \(m = 3\)).

\[ P \quad V \quad Q \]

| commit: | | | |
|---------|---------|---------|
| \(x_0 := y_0 + a_0 \cdot b\) | | | \(a_0\) |

| sustain: | | | |
|---------|---------|---------|
| \(a_1\) | | | \(x_1 := y_1 + a_1 \cdot y_0\) |
| \(x_2 := y_2 + a_2 \cdot y_1\) | | | \(a_2\) |
| \(a_3\) | | | \(x_3 := y_3 + a_3 \cdot y_2\) |

| open: | | |
| \(y_3\) | | |

\(m \geq 1\). Thus, even when reading this recursive formula liberally by ignoring the \(2^{-n-1}\) term, we obtain

\[ \varepsilon_m \lesssim 2 \cdot 2^{-\frac{2m}{3}} \]

i.e., the error parameter blows up double exponentially in \(m\). In other words, in order to have a non-trivial \(\varepsilon_m\) we need that \(n\), the size of the strings that are communicated, is exponential in \(m\). This means that Lunghi et al. can only afford a very small number of rounds. For instance, in their implementation where they can manage \(n = 512\) (beyond that, the local computation takes too long), asking for an error parameter \(\varepsilon_m\) of approximately \(2^{-32}\), they can do \(m = 4\) rounds.\(^2\) This allows them to keep a commitment alive for 2 ms.

**OUR RESULTS.** Our main goal is to improve the bound on the binding parameter of the above multi-round scheme. Indeed, our results show that the binding parameter blows up only linearly in \(m\), rather than double exponentially. Explicitly, our results show that (for classical attacks)

\[ \varepsilon_m \leq (m + 1) \cdot 2^{-\frac{m}{2} + 2} \]

Using the same \(n\) and error parameter as in the implementation of Lunghi et al., we can now afford approximately \(m = 2^{224}\) rounds. Scaling up the 2 ms from the Lunghi et al. experiment for 4 rounds gives us a time that is in the order of 10\(^{56}\) years. We also show tightness of our bound up to a small constant factor (for even \(n\)).

We use the following strategy to obtain our improved bound on \(\varepsilon_m\). We observe that the first sustain round can be understood as committing on the opening information \(y_0\) of the actual commitment, using an extended version of the Crépeau et al. scheme that commits to a string rather than to a bit. Similarly, the second sustain round can be understood as committing on the opening information \(y_1\) of that commitment from the first sustain round, etc. Thus, thinking of the \(m = 1\) version of the scheme, what we have to prove is that if we have two commitment schemes \(S\) and \(S'\), and we modify the opening phase of \(S\) in that we first commit to the opening information (using \(S'\)) and then open that commitment, then the resulting commitment scheme is still binding; note that, intuitively, this is what one would indeed expect. Given such a composition theorem, we can then apply it inductively and conclude security (i.e. the binding property) of the Lunghi et al. multi-round scheme.

Our main result is such a general composition theorem, which shows that if \(S\) and \(S'\) are respectively \(\varepsilon\)- and \(\delta\)-binding (against classical attacks) then the composed scheme is \((\varepsilon + \delta)\)-binding (against classical attacks), under some mild assumptions on \(S\) and \(S'\). Hence, the error parameters simply add up; this is what gives us the linear growth. The proof of our composition theorem crucially relies on new definitions

\(^1\) Lunghi et al. also provide a more complicated recursive formula for \(\varepsilon_m\) that is slightly better, but the resulting blow-up is still double exponential.

\(^2\) Note that [LKB+15] mentions \(\varepsilon_m \approx 10^{-5} \approx 2^{-16}\), but this is an error, as communicated to us by the authors, and as can easily be verified. Also, [LKB+15] mentions \(m = 5\) rounds, but this is because they include the commit round in their counting, and we do not.
of the binding property of 2-prover commitment schemes, which seem to be handier to work with than the $p_0 + p_1 \leq 1 + \epsilon$ definition as for instance used by Lunghi et al. Our definitions formalize the following intuitive requirement: After the commit phase, even if the provers are dishonest, there should exist some bit $\hat{b}$ such that opening the commitment to any other bit fails (with high probability). We show that one of our new definitions is equivalent to the $p_0 + p_1$-definition, while the other one is strictly stronger. Our result holds for both definitions, so we not only obtain a better parameter than Lunghi et al. but also with respect to a stronger definition, and thus we improve the result also in that direction.

One subtle issue is that the extended version of the Crépeau et al. scheme to strings, as it is used in the sustain phase, is not a fully secure string commitment scheme. The reason is that for any $y$ that may be announced in the opening phase, there exists a string $s$ such that $x \neq y = a \cdot s$; as such, the provers can commit to some fixed string, and then can still decide to either open the commitment to that string (by running the opening phase honestly), or to open it to a random string that is out of their control (by announcing a random $y$). We deal with this by also introducing a relaxed version (which we call fairly-binding) of the binding property, which captures this limited freedom for the provers, and we show that it is satisfied by the (extended version of the) Crépeau et al. scheme and that our composition theorem holds for this relaxed version; finally, we observe that the composed fairly-binding string commitment scheme is a binding bit commitment scheme when restricting the domain to a bit.

As such, we feel that our techniques and insights not only give rise to an improved analysis of the Lunghi et al. multi-round scheme, but they significantly improve our understanding of the security of 2-prover commitment schemes, and as such are likely to find further applications.

Open Problems. Our work gives rise to a list of interesting and challenging open problems. For instance, our composition theorem only applies to pairs $\mathcal{S}, \mathcal{S}'$ of commitment schemes of a certain restricted form, e.g., only one prover should be involved in the commit phase (as it is the case in the Crépeau et al. scheme). Our proof crucially relies on this, but there seems to be no fundamental reason for such a restriction. Thus, we wonder if it is possible to generalize our composition theorem to a larger class of pairs of schemes, or, ultimately, to all pairs of schemes (that “fit together”).

In another direction, some of our observations and results generalize immediately to the quantum setting, where the two dishonest provers are allowed to compute their messages by performing measurements on an entangled quantum state, but in particular our main result, the composition theorem, does not generalize. Also here, there seems to be no fundamental reason, and thus, generalizing our composition theorem to the quantum setting is an interesting open problem. Finally, in order to obtain security of the Lunghi et al. multi-round scheme against quantum attacks, beyond a quantum version of the composition theorem, one also needs to prove security against quantum attacks of the (extended version of the) original Crépeau et al. scheme as a (fairly-binding) string commitment scheme.

Concurrent Work. In independent and concurrent work, Chakraborty, Chailloux and Leverrier [CCL15] showed (almost) the same linear bound for the Lunghi et al. scheme, but with respect to the original—and thus weaker—notion of security. Their approach is more direct and tailored to the specific scheme; our approach is more abstract and provides more insight, and our result applies much more generally.

2 Preliminaries

2.1 Basic Notation

Probability Distributions. For the purpose of this work, a (probability) distribution is a function $p : \mathcal{X} \rightarrow [0, 1]$, $x \mapsto p(x)$, where $\mathcal{X}$ is a finite non-empty set, with the property that $\sum_{x \in \mathcal{X}} p(x) = 1$. For specific choices $x_0 \in \mathcal{X}$, we tend to write $p(x = x_0)$ instead of $p(x_0)$. For any subset $A \subset \mathcal{X}$, called an event, the probability $p(A)$ is naturally defined as $p(A) = \sum_{x \in A} p(x)$, and it holds that

$$p(A) + p(\Gamma) = p(A \cup \Gamma) + p(A \cap \Gamma) \leq 1 + p(A \cap \Gamma)$$

(1)

for all $A, \Gamma \subset \mathcal{X}$, and, more generally, that

$$\sum_{i=1}^k p(A_i) \leq p(A_1 \cup \ldots \cup A_k) + \sum_{i<j} p(A_i \cap A_j) \leq 1 + \sum_{i<j} p(A_i \cap A_j)$$

(2)
Remark 2.3. For a distribution $p : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ on two (or more) variables, probabilities like $p(x = y)$, $p(x = f(y))$, $p(x \neq y)$ etc. are naturally understood as

$$p(x = y) = p\{ (x, y) \in \mathcal{X} \times \mathcal{Y} \mid x = y \} = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y)$$

eq \quad \text{etc., and the marginals } p(x) \text{ and } p(y) \text{ are given by } p(x) = \sum_y p(x, y) \text{ and } p(y) = \sum_x p(x, y) \text{, respectively.}

Vice versa, given two distributions $p(x)$ and $p(y)$, we say that a distribution $p(x, y)$ on two variables is a consistent joint distribution if the two marginals $p(x, y)$ coincide with $p(x)$ and $p(y)$, respectively. We will make use of the following property on the existence of a consistent joint distribution that maximizes the probability that $x = y$; the proof is given in the appendix.

Lemma 2.1. Let $p(x)$ and $p(y)$ be two distributions on a common set $\mathcal{X}$. Then there exists a consistent joint distribution $p(x, y)$ such that $p(x = y = x_0) = \min\{ p(x = x_0), p(y = x_0) \}$ for all choices of $x_0 \in \mathcal{X}$. Additionally, $p(x, y)$ satisfies $p(x, y|x \neq y) = p(x|x \neq y) \cdot p(y|x \neq y)$.

Protocols. In this work, we will consider $3$-party (interactive) protocols, where the parties are named $P$, $Q$ and $V$ (the two “provers” and the “verifier”). Such a protocol $\text{prot}_{PQV}$ consists of a triple $(\text{prot}_P, \text{prot}_Q, \text{prot}_V)$ of $L$-round interactive algorithms for some $L \in \mathbb{N}$. Each interactive algorithm takes an input, and for every round $\ell \leq L$ computes the messages to be sent to the other algorithms/parties in that round as deterministic functions of its input, the messages received in the previous rounds, and the local randomness. In the same way, the algorithms produce their respective outputs after the last round.

We write $(\text{out}_P, \text{out}_Q, \text{out}_V) \leftarrow (\text{prot}_P(\text{in}_P), \text{prot}_Q(\text{in}_Q), \text{prot}_V(\text{in}_V))$ to denote the execution of the protocol $\text{prot}_{PQV}$ on the respective inputs $\text{in}_P$, $\text{in}_Q$, $\text{in}_V$, and that the respective outputs $\text{out}_P$, $\text{out}_Q$ and $\text{out}_V$ are produced. Clearly, for any protocol $\text{prot}_{PQV}$ and any input $\text{in}_P$, $\text{in}_Q$, $\text{in}_V$, the probability distribution $p(\text{out}_P, \text{out}_Q, \text{out}_V)$ of the output is naturally well defined.

If we want to make the local randomness explicit, we write $\text{prot}_P[\xi_P](\text{in}_P)$ etc., and understand that $\xi_P$ is correctly sampled — without loss of generality, we may assume it to be a uniformly random bit string of sufficient length. Furthermore, we write $\text{prot}_P[\xi_P](\text{in}_P)$ and $\text{prot}_Q[\xi_Q](\text{in}_Q)$ to express that $\text{prot}_P$ and $\text{prot}_Q$ use the same randomness, in which case we speak of joint randomness.

We can compose two interactive algorithms $\text{prot}_P$ and $\text{prot}_P'$, in the obvious way, by applying $\text{prot}_P'$ to the output of $\text{prot}_P$. The resulting interactive algorithm is denoted as $\text{prot}_P' \circ \text{prot}_P$. Composing the respective algorithms of two protocols $\text{prot}_{PQV} = (\text{prot}_P, \text{prot}_Q, \text{prot}_V)$ and $\text{prot}'_{PQV} = (\text{prot}_P', \text{prot}_Q', \text{prot}_V')$ results in the composed protocol $\text{prot}'_{PQV} \circ \text{prot}_{PQV}$. If $\text{prot}_P$ is a non-interactive algorithm, then $\text{prot}_P' \circ \text{prot}_P$ is naturally understood as the protocol $\text{prot}_P' \circ \text{prot}_P = (\text{prot}_P' \circ \text{prot}_P, \text{prot}_Q', \text{prot}_V')$, and similarly $\text{prot}_P' \circ \text{prot}_{PQV}$ in case $\text{prot}_{PQV}$ is a protocol among $Q$ and $V$ only.

2.2 2-Prover Commitment Schemes

Definition 2.2. A 2-prover (string) commitment scheme $S$ consists of two interactive protocols $\text{com}_{PQV} = (\text{com}_P, \text{com}_Q, \text{com}_V)$ and $\text{open}_{PQV} = (\text{open}_P, \text{open}_Q, \text{open}_V)$ between the two provers $P$ and $Q$ and the verifier $V$, with the following syntaxs. The commit protocol $\text{com}_{PQV}$ uses joint randomness $\xi_{PQ}$ for $P$ and $Q$ and takes a string $s \in \{0,1\}^n$ as input for $P$ and $Q$ (and independent randomness and no input for $V$), and it outputs a commitment $\text{com}$ to $V$ and some state information to $P$ and $Q$:

$$(\text{state}_P, \text{state}_Q)(c) \leftarrow (\text{com}_P[\xi_{PQ}](s))|\text{com}_Q[\xi_{PQ}](s))|\text{com}_V(c).$$

The opening protocol $\text{open}_{PQV}$ uses joint randomness $\eta_{PQ}$ and outputs a string or a rejection symbol to $V$, and nothing to $P$ and $Q$:

$$(\emptyset|\emptyset)(s) \leftarrow (\text{open}_P[\eta_{PQ}](\text{state}_P))|\text{open}_Q[\eta_{PQ}](\text{state}_Q))|\text{open}_V(c)$$

with $s \in \{0,1\}^n \cup \{\bot\}$. The set $\{0,1\}^n$ is called the domain of $S$; if $n = 1$ then we refer to $S$ as a bit commitment scheme instead, and we tend to use $b$ rather than $s$ to denote the committed bit.

Remark 2.3. By convention, we assume throughout the paper that the commitment $c$ output by $V$ equals the communication that takes place between $V$ and the provers during the commit phase. This is without loss of generality since, in general, $c$ is computed as a (possibly randomized) function of the communication, which $V$ as just as well can apply in the opening phase.
Definition 2.5. A 2-prover commitment scheme is γ-complete if in an honest execution \(V\)'s output \(s\) of \(\text{open}_{PQ}V\) equals \(P\) and \(Q\)'s input \(s\) to \(\text{com}_{PQ}V\) except with probability \(\eta\), for any choice of \(P\) and \(Q\)'s input \(s \in \{0,1\}^n\).

The standard definition for the hiding property is as follows:

Definition 2.6. A 2-prover commitment scheme is δ-hiding if for any commit strategy \(\text{com}_V\) and any two strings \(s_0\) and \(s_1\), the distribution of the commitments \(c_0, c_1\), produced as

\[(\text{state}_P||\text{state}_Q||c_b) \leftarrow (\text{com}_P[\xi_{PQ}](s_b)||\text{com}_Q[\xi_{PQ}](s_b)||\text{com}_V(\emptyset)), b = 0, 1\]

have statistical distance at most \(\delta\). A 0-hiding scheme is also called perfectly hiding.

Defining the binding property is more subtle. First, note that an attack against the binding property consists of an “allowed” commit strategy \(\text{com}_{PQ}\) and an “allowed” opening strategy \(\text{open}_{PQ} = (\text{open}_P, \text{open}_Q)\) for \(P\) and \(Q\). Any such attack fixes \(p(s)\), the distribution of \(s \in \{0,1\}^n \cup \{\bot\}\) that is output by \(V\) after the opening phase, in the obvious way.

What exactly “allowed” means may depend on the scheme and needs to be specified. Typically, in the 2-prover setting, we only allow strategies \(\text{com}_{PQ}\) and \(\text{open}_{PQ}\) with no communication at all between the two provers, but we may also be more liberal and allow some well-controlled communication, as in the Lunghi et al. multi-round scheme. Furthermore, in this work, we focus on classical attacks, where \(\text{com}_{PQ}, \text{open}_{PQ}\) and \(\text{open}_Q\) are classical interactive algorithms as specified in the previous section, with access to joint randomness. But one could also consider quantum attacks, in which the provers can perform measurements on an entangled quantum state. Our main result holds for classical attacks only, and so the unfamiliar reader can safely ignore the possibility of quantum attacks, but some of our insights also apply to quantum attacks.

A somewhat accepted definition for the binding property of a 2-prover bit commitment scheme, as it is for instance used in [CSS11][LKB+15][TT15] (up to the factor 2 in the error parameter), is as follows. Here, we assume it has been specified which attacks are allowed, e.g., those where \(P\) and \(Q\) do not communicate during the course of the scheme.

Definition 2.7. A 2-prover bit commitment scheme is \(\varepsilon\)-binding in the sense of \(p_0 + p_1 \leq 1 + 2\varepsilon\) if for every allowed commit strategy \(\text{com}_{PQ}\), and for every pair of allowed opening strategies \(\text{open}_{PQ}^0\) and \(\text{open}_{PQ}^1\), which fix distributions \(p(b_0)\) and \(p(b_1)\) for \(V\)'s respective outputs, it holds that

\[p(b_0=0) + p(b_1=1) \leq 1 + 2\varepsilon\]

In the literature (see e.g. [CSS11] or [LKB+15]), the two probabilities \(p(b_0=0)\) and \(p(b_1=1)\) above are usually referred to as \(p_0\) and \(p_1\), respectively.

2.3 The CHSH\(^n\) Scheme

Our main example is the bit commitment scheme by Crépeau et al. [CSS11] we mentioned in the introduction, and which works as follows. The commit phase \(\text{com}_{PQV}\) instructs \(V\) to sample and send to \(P\) a uniformly random \(a \in \{0,1\}^n\), and it instructs \(P\) to return \(x := r + a \cdot b\) to \(V\), where \(r\) is the joint randomness, uniformly distributed in \(\{0,1\}^n\), and \(b\) is the bit to commit to, and the opening phase \(\text{open}_{PQV}\) instructs \(Q\) to send \(y := r\) to \(V\), and \(V\) outputs the (smaller) bit \(b\) that satisfies \(x + y = a \cdot b\), and \(b := \bot\) in case no such bit exists. Note that the provers in this scheme use the same randomness in the commit and opening phase; thus, formally, \(Q\) needs to output the shared randomness \(r \leftarrow \xi_{PQ}\) as \(\text{state}_Q\). The opening phase uses no fresh randomness.
It is easy to see that this scheme is $2^{-n}$-complete and perfectly hiding (completeness fails in case \( a = 0 \)). For classical provers that do not communicate at all, the scheme is $2^{-n-1}$-binding in the sense of \( p_0 + p_1 \leq 1 + 2^{-n} \), i.e. according to Definition \ref{27}. As for quantum provers, Crépeau et al. showed that the scheme is $2^{-n/2}$-binding; this was recently minorly improved to $2^{-n+1/2}$ by Sikora, Chailloux and Kerenidis \cite{SCK}.

We also want to consider an extended version of the scheme, where the bit \( b \) is replaced by a string \( s \in \{0, 1\}^n \) in the obvious way (where the multiplication \( a \cdot s \) is then understood in a suitable finite field), and we want to appreciate this extension as a 2-prover string commitment scheme. However, it is a priori not clear what is a suitable definition for the binding property, especially because for this particular scheme, the dishonest provers can always honestly commit to a string \( s \), and can then decide to correctly open the commitment to \( s \) by announcing \( y := r \), or open to a random string by announcing a randomly chosen \( y \) — any \( y \) satisfies \( x + y = a \cdot s \) for some \( s \) (unless \( a = 0 \), which almost never happens).

Due to its close relation to the CHSH game \cite{CHSH69}, in particular to the arbitrary-finite-field version considered in \cite{BSL15}, we will refer to this string commitment scheme as CHSH\(^n\).

### 3 On the Binding Property of 2-Prover Commitment Schemes

We introduce new definitions for the binding property of 2-prover commitment schemes. In the case of \( \text{bit} \) commitment schemes, they imply Definition \ref{27} as we will show. Although not necessarily simpler, we feel that our definitions are closer to the intuition of what is expected from a commitment scheme, and as such easier to work with. Indeed, the proofs of our composition results are heavily based on our new definitions. Also, our new notions are more flexible in terms of tweaking it; for instance, we modify them to obtain a \( \text{relaxed} \) notion for the binding property, which captures the binding property that is satisfied by the string commitment scheme CHSH\(^n\).

Throughout this section, when quantifying over attacks against (the binding property of) a scheme, it is always understood that there is a notion of \( \text{allowed} \) attacks for that scheme (e.g., all attacks for which \( P \) and \( Q \) do not communicate), and that the quantification is over all such allowed attacks. Also, even though our focus is on classical attacks, Proposition \ref{310} and Theorem \ref{311} also apply to quantum attacks.

#### 3.1 Defining The Binding Property

Intuitively, we say that a scheme is binding if after the commit phase there exists a string \( \hat{s} \) so that no matter what the provers do in the opening phase, the verifier will output either \( s = \hat{s} \) or \( s = \bot \) (except with small probability). We consider two definitions of the binding property which interpret this intuitive requirement in two different ways. In the first definition, which we introduce in this section, \( \hat{s} \) is a function of the provers’ (combined) view immediately after the commit phase. In the second one, which we introduce in Section \ref{33}, \( \hat{s} \) is specified by its distribution only. Both of these definitions admit a composition theorem.

**Definition 3.1 (Binding property).** A 2-prover commitment scheme \( S \) is \( \varepsilon \)-binding if for every commit strategy \( \text{com}_{PQ}[\xi_{PQ}] \) there exists a function \( \hat{s}(\xi_{PQ}, c) \) of the joint randomness \( \xi_{PQ} \) and the commitment \( c \) such that for every opening strategy \( \text{open}_{PQ} \) it holds that \( p(s \neq \hat{s}(\xi_{PQ}, c) \land s \neq \bot) \leq \varepsilon \). In short:

\[
\forall \text{com}_{PQ} \exists \hat{s}(\xi_{PQ}, c) \forall \text{open}_{PQ} : p(s \neq \hat{s} \land s \neq \bot) \leq \varepsilon .
\]

The string commitment scheme CHSH\(^n\) does \( \text{not} \) satisfy this definition (the bit commitment version does, as we will show): After the commit phase, the provers can still decide to open the commitment to a \( \text{fixed} \) string, chosen before the commit phase, or to a \( \text{random} \) string that is out of their control. We capture this by the following relaxed version of the binding property: We allow \( V \)’s output \( s \) to be different from \( \hat{s} \) and \( \bot \), but in this case the provers should have little control over \( s \): For any fixed target string \( s_\circ \), it should be unlikely that \( s = s_\circ \). Formally, this is captured as follows; we will show in Section \ref{33} that CHSH\(^n\) is \( \text{fairly-binding} \) in this sense.

\[\text{This could easily be prevented by requiring } Q \text{ to announce } s \text{ (rather than letting } V \text{ compute it), but we want the information announced during the opening phase to fit into the domain of the commitment scheme.}\]

\[\text{Recall that by convention (Remark 2.3), } c \text{ equals the communication between } V \text{ and the provers during the commit phase.}\]
Definition 3.2 (Fairly binding property). A 2-prover commitment scheme $S$ is $\varepsilon$-fairly-binding if for every commit strategy $\xi_{PQ}$ there exists a function $\hat{s}(\xi_{PQ}, c)$ such that for every opening strategy $\eta_{PQ}$ and all functions $s_0(\xi_{PQ}, \eta_{PQ})$ it holds that $p(s \neq \hat{s}(\xi_{PQ}, c) \land s = s_0(\xi_{PQ}, \eta_{PQ})) \leq \varepsilon$. In short:

$$\forall \xi_{PQ} \exists \hat{s}(\xi_{PQ}, c) \forall \eta_{PQ} \forall s_0(\xi_{PQ}, \eta_{PQ}): p(s \neq \hat{s} \land s = s_0) \leq \varepsilon.$$  

(4)

Remark 3.3. By means of standard techniques, one can easily show that it is sufficient for the (fairly) binding property to consider deterministic provers. In this case, $\hat{s}$ is a function of $c$ only, and, in the case of fairly-binding, $s_0$ runs over all fixed strings.

Remark 3.4. Clearly, the binding property implies the fairly binding property. Furthermore, in the case of bit commitment schemes it obviously holds that $p(\hat{b} \neq \bar{b}) = p(b \neq b \land b = 0) + p(b \neq b \land b = 1)$, and thus the fairly-binding property implies the binding property with a factor-2 loss in the parameter. Furthermore, every fairly-binding string commitment scheme gives rise to a binding bit commitment scheme in a natural way, as shown by the following proposition.

Proposition 3.5. Let $S$ be a $\varepsilon$-fairly-binding string commitment scheme. Fix any two distinct strings $s_0, s_1 \in \{0,1\}^n$ and consider the bit-commitment scheme $S'$ obtained as follows. To commit to $b \in \{0,1\}$, the provers commit to $s_0$ using $S$, and in the opening phase $V$ checks if $s = s_0$ for some bit $b \in \{0,1\}$ and outputs this bit if it exists and else outputs $b = \perp$. Then, $S'$ is a $2\varepsilon$-binding bit commitment scheme.

Proof. Fix some commit strategy $\xi_{PQ}$ for $S'$ and note that it can also be used to attack $S$. Thus, there exists a function $\hat{s}(\xi_{PQ}, c)$ as in Definition 3.2. We define

$$\hat{b}(\xi_{PQ}, c) = \begin{cases} 
0 & \text{if } \hat{s}(\xi_{PQ}, c) = s_0 \\
1 & \text{otherwise}
\end{cases}$$

Now fix an opening strategy $\eta_{PQ}$ for $S'$, which again is also a strategy against $S$. Thus, we have $p(\hat{s} \neq s = s_0) \leq \varepsilon$ for any $s_0$ (and in particular $s_0 = s_0$ or $s_1$). This gives us

$$p(\hat{b} \neq \bar{b}) = p(\hat{b} = 1 \land b = 0) + p(\hat{b} = 0 \land b = 1)$$

$$= p(\hat{s} \neq s_0 \land s = s_0) + p(\hat{s} = s_0 \land s = s_1)$$

$$\leq p(\hat{s} \neq s_0 \land s = s_0) + p(\hat{s} \neq s_1 \land s = s_1)$$

$$\leq 2\varepsilon$$

and thus $S'$ is a $2\varepsilon$-binding bit-commitment scheme. \hfill \Box

Remark 3.6. The proof of Proposition 3.5 generalizes in a straightforward way to $k$-bit string commitment schemes: Given a $\varepsilon$-fairly-binding $n$-bit string commitment scheme $S$, for $k < n$, we define a $k$-bit string commitment scheme $S_k$ as follows: To commit to a $k$-bit string, the provers pad the string with $n - k$ zeros and then commit to the padded string using $S$. In the opening phase, the verifier outputs the first $k$ bits of $s$ if the remaining bits in $s$ are all zeros, and $\perp$ otherwise. Then, $S'$ is $2^k\varepsilon$-binding.

3.2 The Weak Binding Property

Here, we introduce yet another definition for the binding property. It is similar in spirit to Definition 3.2 but weaker. One advantage of this weaker notion is that it is also meaningful when considering quantum attacks, whereas Definition 3.2 is not. In the subsequent section, we will see that for bit commitment schemes, this weaker notion of the binding property is equivalent to Definition 3.2.

Definition 3.7 (Weak binding property). A 2-prover commitment scheme $S$ is $\varepsilon$-weak-binding if for all commit strategies $\xi_{PQ}$ there exists a distribution $p(\hat{s})$ such that for every opening strategy $\eta_{PQ}$ (which then fixes the distribution $p(s)$ of $V$’s output $s$) there is a consistent joint distribution $p(\hat{s}, s)$ such that $p(s \neq \hat{s} \land s \neq \perp) \leq \varepsilon$. In short:

$$\forall \xi_{PQ} \exists p(\hat{s}) \forall \eta_{PQ} \exists p(\hat{s}, s): p(s \neq \hat{s} \land s \neq \perp) \leq \varepsilon.$$  

(5)

We also consider a related, i.e., “fairly”, version of this binding property, similar to Definition 3.2.

Definition 3.8 (Fairly weak binding property). A 2-prover commitment scheme $\mathcal{S}$ is $\varepsilon$-fairly-weak-binding if for all open strategies $\mathcal{C}_{\mathcal{PQ}}$ there exists a distribution $p(\hat{s})$ such that for every opening strategy $\mathcal{O}_{\mathcal{PQ}}$ (which then fixes the distribution $p(s)$ of $V$’s output $s$) there is a consistent joint distribution $p(\hat{s}, s)$ so that for all $s_0 \in \{0, 1\}^n$ it holds that $p(s \neq \hat{s} \land s = s_0) \leq \varepsilon$. In short:

$$\forall \mathcal{C}_{\mathcal{PQ}} \exists p(\hat{s}) \forall \mathcal{O}_{\mathcal{PQ}} \exists p(\hat{s}, s) \forall s_0 : p(s \neq \hat{s} \land s = s_0) \leq \varepsilon \ .$$

(6)

Remark 3.9. Remarks 3.3 and 3.4 also hold for the weak binding properties. Furthermore, it is easy to see that the binding and fairly-binding properties imply their weak counterparts.

Proposition 3.10. Let $\mathcal{S}$ be a $\varepsilon$-fairly-weak-binding string commitment scheme and define $\mathcal{S}'$ as in Proposition 3.5. Then, $\mathcal{S}'$ is a $2\varepsilon$-weak-binding bit commitment scheme.

Proof. The proof of Proposition 3.5 can be easily adapted: Let $p(\hat{s})$ be as required by Definition 3.8. We define $p(\hat{b})$ by taking the marginal of $p(\hat{s}, \hat{b})$ where $\hat{b} = 0$ if $\hat{s} = s_0$, and $\hat{b} = 1$ otherwise. An opening strategy $\mathcal{O}_{\mathcal{PQ}}$ for $\mathcal{S}'$ can also be viewed as a strategy for $\mathcal{S}$. As such, there is a joint distribution $p(\hat{s}, \hat{b}, s, b)$ as required by Definition 3.7 which we can extend to $p(\hat{s}, s, b)$ by setting $b = 0$ if $s = s_0$, $b = 1$ if $s = s_1$, and $b = \perp$ otherwise. We define $p(\hat{b}, b) := \sum_{\hat{s}} p(\hat{b}, \hat{s}) \cdot p(s, b|\hat{s})$. As in the proof of Proposition 3.5, one can easily check that $p(\hat{b} \neq b \neq \perp) \leq 2\varepsilon$ holds.

3.3 Relations Between The Definitions

Here, we show that in case of bit commitment schemes, the weak binding property as introduced in Definition 3.7 above is actually equivalent to the $(p_0 + p_1)$-definition. Even though our focus is on classical attacks, the proof immediately carries over to quantum attacks as well.

Theorem 3.11. A 2-prover bit-commitment scheme is $\varepsilon$-binding in the sense of $p_0 + p_1 \leq 1 + 2\varepsilon$ if and only if it is $\varepsilon$-weak-binding.

Proof. First, consider a scheme that is $\varepsilon$-binding according to Definition 2.7. Fix a commit strategy $\mathcal{C}_{\mathcal{PQ}}$ and opening strategies $\mathcal{O}_{\mathcal{PQ}}$ and $\mathcal{O}'_{\mathcal{PQ}}$ so that $p_0 = p(b_0 = 0)$ and $p_1 = p(b_1 = 1)$ are maximized, where $b_i \in \{0, 1, \perp\}$ is $V$’s output when the dishonest provers use opening strategy $\mathcal{O}_{\mathcal{PQ}}$. Let $p_0 + p_1 = 1 + 2\varepsilon'$. Since the scheme is $\varepsilon$-binding, we have $\varepsilon' \leq \varepsilon$. We define the distribution $p(\hat{b})$ as $p(\hat{b} = 0) := p_0 - \varepsilon'$ and $p(\hat{b} = 1) := p_1 - \varepsilon'$. To see that this is indeed a probability distribution, note that $p_0, p_1 \geq 2\varepsilon'$ (otherwise, we would have $p_0 > 1$ or $p_1 > 1$) and that $p(\hat{b} = 0) + p(\hat{b} = 1) = p_0 + p_1 - 2\varepsilon' = 1$.

Now we consider an arbitrary opening strategy $\mathcal{O}_{\mathcal{PQ}}$ which fixes a distribution $p(\hat{b})$. By definition of $p_0$ and $p_1$, we have $p(\hat{b} = i) \leq p_i$ and thus $p(\hat{b} = i) \leq p(\hat{b} = i) + \varepsilon' \leq p(\hat{b} = i) + \varepsilon$. By Lemma 2.1 there exists a consistent joint distribution $p(\hat{b}, b)$ with the property that $p(\hat{b} = b = i) = \min\{p(\hat{b} = i), p(\hat{b} = i)\}$. We wish to bound $p(\hat{b} \neq b \neq \perp) = p(\hat{b} = 0 \lor b = 1) + p(\hat{b} = 1 \land b = 0)$. For $i \in \{0, 1\}$, it holds that

$$p(\hat{b} = 1 - i \land b = i) = p(\hat{b} = i) - p(\hat{b} = b = i)$$

$$= p(\hat{b} = i) - \min\{p(\hat{b} = i), p(\hat{b} = i)\}$$

$$= \max\{0, p(\hat{b} = i) - p(\hat{b} = i)\}$$

$$\leq \varepsilon$$

and furthermore, there is at most one $i \in \{0, 1\}$ such that $p(\hat{b} = i) > p(\hat{b} = i)$, for if $p(\hat{b} = i) > p(\hat{b} = i)$ for both $i = 0$ and $i = 1$, then $p(\hat{b} = 0) + p(\hat{b} = 1) > p(\hat{b} = 0) + p(\hat{b} = 1) = 1$ which is a contradiction. Thus, we have $p(\hat{b} \neq b \neq \perp) \leq \varepsilon$. This proves one direction of our claim.

For the other direction, consider a scheme that is $\varepsilon$-binding. Fix $\mathcal{C}_{\mathcal{PQ}}$ and let $p(\hat{b})$ be a distribution such that for every opening strategy $\mathcal{O}_{\mathcal{PQ}}$, there is a joint distribution $p(\hat{b}, b)$ with $p(\hat{b} \neq b \neq \perp) \leq \varepsilon$. Now consider two opening strategies $\mathcal{O}_{\mathcal{PQ}}$ and $\mathcal{O}'_{\mathcal{PQ}}$ which give distributions $p(b_0)$ and $p(b_1)$. We need to bound $p(b_0 = 0) + p(b_1 = 1)$. There is a joint distribution $p(\hat{b}, b_0)$ such that $p(\hat{b} \neq b_0 \neq \perp) \leq \varepsilon$ and likewise for $b_1$. Thus,

$$p(b_0 = 0) + p(b_1 = 1) = p(\hat{b} = 0, b_0 = 0) + p(\hat{b} = 1, b_0 = 0) + p(\hat{b} = 0, b_1 = 1) + p(\hat{b} = 1, b_1 = 1)$$

$$\leq p(\hat{b} = 0) + p(\hat{b} = 1) + p(\hat{b} \neq b_0 \neq \perp) + p(\hat{b} \neq b_1 \neq \perp)$$

$$\leq 1 + 2\varepsilon$$

which proves the other direction. □
Remark 3.12. By Remark 8.9, it follows that Definition 3.1 also implies the \( p_0 + p_1 \)-definition. In fact, Definition 3.1 is strictly stronger (and hence, also strictly stronger than the weak-binding definition). Consider the following (artificial and very non-complete) scheme: In the commit phase, \( V \) chooses a uniformly random bit and sends it to the provers, and then accepts everything or rejects everything during the opening phase, depending on that bit. Then, \( p_0 + p_1 = 1 \), yet a commitment can be opened to \( 1 - \hat{b} \) (no matter how \( \hat{b} \) is defined) with probability \( \frac{1}{2} \).

Since a non-complete separation example may not be fully satisfying, we note that it can be converted into a complete (but even more artificial) scheme. Fix a “good” (i.e., complete, hiding and binding with low parameters) scheme and call our example scheme above the “bad” scheme. We define a combined scheme as follows: At the start, the first prover can request either the “good” or “bad” scheme to be used. The honest prover is instructed to choose the former, guaranteeing completeness. The dishonest prover may choose the latter, so the combined scheme inherits the binding properties of the “bad” scheme: It is binding according to the \( (p_0 + p_1) \)-definition, but not according to Definition 3.1.

3.4 Security of \( CHSH^n \)

In this section, we show that \( CHSH^n \) is a fairly-binding string commitment scheme. To this end, we introduce yet another version of the binding property and show that \( CHSH^n \) satisfies this property. Then we show that this version of the binding property implies the fairly-binding property (up to some loss in the parameter, and some mild restrictions on the scheme).

This new binding property is based on the intuition that it should not be possible to open a commitment to two different values simultaneously (except with small probability). For this, we observe that (for classical attacks), when considering a commit strategy \( \text{com}_{PQ} \), as well as two opening strategies \( \text{open}_{PQ} \) and \( \text{open}'_{PQ} \), we can run both opening strategies simultaneously on the produced commitment with two (independent) copies of \( \text{open}_V \); by applying \( \text{open}_{PQ} \) and \( \text{open}'_{PQ} \) to two copies of the respective internal states of \( P \) and \( Q \). This gives rise to a joint distribution \( p(s,s') \) of the respective outputs \( s \) and \( s' \) of the two copies of \( \text{open}_V \).

Definition 3.13 (Simultaneous opening). A 2-prover commitment scheme \( S \) is \( \varepsilon \)-fairly-binding in the sense of simultaneous opening if for all \( \text{com}_{PQ} \), all pairs of opening strategies \( \text{open}_{PQ} \) and \( \text{open}'_{PQ} \), and all pairs \( s,s' \) of distinct strings, we have \( p(s = s') \leq \varepsilon \).

Remark 3.14. Also for this notion of fairly-binding, it is sufficient to consider deterministic strategies, as can easily be seen.

Proposition 3.15. The string commitment scheme \( CHSH^n \) is \( 2^{-n} \)-fairly-binding in the sense of simultaneous opening.

Proof. By Remark 3.14, it suffices to consider deterministic attack strategies. Fix a deterministic strategy \( \text{com}_{PQ} \) and two deterministic opening strategies \( \text{open}_{PQ} \) and \( \text{open}'_{PQ} \). The strategy \( \text{com}_{PQ} \) specifies \( P \)'s output \( x \) as a function \( f(a) \) of the verifier’s message \( a \). The opening strategies are described by constants \( y \) and \( y' \). By definition of \( CHSH^n \), \( s = s_o \) implies \( f(a) + y = a \cdot s_o \) and likewise, \( s' = s'_o \) implies \( f(a) + y' = a \cdot s'_o \). Therefore, \( s = s_o \land s' = s'_o \) implies \( a = (y - y')/(s_o - s'_o) \). It thus holds that \( p(s = s') \leq p(a = (y - y')/(s_o - s'_o)) \leq \frac{1}{2n} \), which proves our claim.

Remark 3.16. It follows directly from [1] that every bit commitment scheme that is \( \varepsilon \)-fairly-binding in the sense of simultaneous opening (against classical attacks) is \( \varepsilon/2 \)-binding in the sense of \( p_0 + p_1 \leq 1 + \varepsilon \) (and thus also according to Definitions 3.7). The converse is not true though: The schemes from Remark 3.12 again serve as counterexamples.

Theorem 3.17. Let \( S = (\text{com}_{PQV}, \text{open}_{PQV}) \) be a 2-prover commitment scheme. If \( S \) is \( \varepsilon \)-fairly-binding in the sense of simultaneous opening and \( \text{open}_V \) is deterministic, then \( S \) is \( 2\sqrt{\varepsilon} \)-fairly-binding.

Proof. By Remark 3.3, it suffices to consider deterministic strategies for the provers. We fix some deterministic commit strategy \( \text{com}_{PQ} \) and an enumeration \( \{\text{open}_{PQ}\}_{i=1}^N \) of all deterministic opening strategies. Since we assume that \( \text{open}_V \) is deterministic, for any fixed opening strategy for the provers, the verifier’s\(^5\) It is understood that the allowed attacks against \( CHSH^n \) are those where the provers do not communicate.\(^6\) We use “fairly” here to distinguish the notion from a “non-fairly” version with \( p(\perp \neq s \neq s' \neq \perp) \leq \varepsilon \); however, we do not consider this latter version any further here.
output $s$ is a function of the commitment $c$. Thus, for each opening strategy $p_{\text{PO}}$ there is a function $f_i$ such that the verifier’s output is $s = f_i(c)$. We will now define the function $\hat{s}(c)$ that satisfies the properties required by Definition 3.2. Our definition depends on a parameter $\alpha > 0$ which we fix later. To define $\hat{s}$, we partition the set $C$ of all possible commitments into disjoint sets $C = R \cup \bigcup_{s,i} C_{s,i}$ that satisfy the following three properties for every $i$ and every $s$:

$$C_{s,i} \subseteq f_i^{-1}(\{s\}), \quad (c \in C_{s,i}) \implies \alpha \geq C_{s,i} = \emptyset, \quad \text{and} \quad (c \in R \cap f_i(c) = s) < \alpha.$$  

The second property implies that there are at most $\alpha^{-1}$ non-empty sets $C_{s,i}$. It is easy to see that such a partitioning exists: Start with $R = C$ and while there exist $s$ and $i$ with $p(c \in R \cap f_i(c) = s) > \alpha$, let $C_{s,i} = \{c \in R | f_i(c) = s\}$ and remove the elements of $C_{s,i}$ from $R$. For any $c \in C$, we now define $\hat{s}(c)$ as follows. We set $\hat{s}(c) = s$ for $c \in C_{s,i}$ and $\hat{s}(c) = 0$ for $c \in R$.

Now fix some opening strategy $p_{\text{PO}}$ and a string $s_i$, and write $s_i$ for the verifier’s output. Using $C_{s,i}$ as a shorthand for $\bigcup_{s \neq s_i} \bigcup_{j} C_{s,j}$, we note that if $\hat{s}(c) \neq s_i$ then $c \in R \cup C_{\neq s_i}$. Thus, it follows that

$$p(s_i \neq \hat{s}(c) \land s_i = s_0) = p(\hat{s}(c) \neq s_0 \land s_i = s_0) \leq p(c \in (R \cup C_{\neq s_i}) \land f_i(c) = s_0) = p(c \in R \land f_i(c) = s_0) + \sum_{s \neq s_i,j} p(c \in C_{s,j} \land f_i(c) = s_0)$$

$$\leq p(c \in R \land f_i(c) = s_0) + \sum_{s \neq s_i,j} p(f_j(c) = s \land f_i(c) = s_0)$$

$$< \alpha + \alpha^{-1} \cdot \varepsilon$$

where the final inequality holds because $p(c \in R \land f_i(c) = s_0) < \alpha$ by the choice of $R$, because $p(f_j(c) = s \land f_i(c) = s_0) \leq \varepsilon$ by the assumed binding property, and because the number of non-empty $C_{s,i}$ is at most $1/\alpha$. It is easy to see that the upper bound $\alpha + \alpha^{-1} \cdot \varepsilon$ is minimized by setting $\alpha = \sqrt{\varepsilon}$. We conclude that $p(s_i \neq \hat{s}(c) \land s_i = s_0) < 2\sqrt{\varepsilon}$.

Combining Proposition 3.15 and Theorem 3.17 we obtain the following statement for the (fairly-)binding property of the 2-prover string commitment scheme $CHSH^\alpha$.

**Corollary 3.18.** $CHSH^\alpha$ is $2^{-\frac{\varepsilon}{2} + 1}$-fairly-binding.

For the fairly-weak-binding property, we can get a slightly better parameter. Note that we do not require open$_V$ to be deterministic here. The proof of the theorem below is given in Appendix C.

**Theorem 3.19.** Every 2-prover commitment scheme $\mathcal{S}$ that is $\varepsilon$-fairly-binding in the sense of simultaneous opening (against classical attacks) is $\sqrt{2\varepsilon}$-fairly-weak-binding (against classical attacks).

**Corollary 3.20.** $CHSH^\alpha$ is $2^{-\frac{\varepsilon}{2}}$-fairly-weak-binding.

**Remark 3.21.** It is not too hard to see that Corollary 3.20 above implies an upper bound on the classical value $\omega$ of the game $CHSH_2$ considered in [BS15] of $\omega(CHSH_{2^n}) \leq 2^{-\frac{\varepsilon}{2}} + 2^{-n}$. As such, Theorem 1.3 in [BS15] implies that the above $\varepsilon$ is asymptotically optimal for odd $n$, i.e., the square root loss to the binding property of the bit commitment version is unavoidable (for odd $n$).

As for security against quantum attacks, we point out that [BS15, RAM15] provide an upper bound on the quantum value $\omega^*(CHSH)$ of general finite-field CHSH; however, this does not directly imply security against quantum attacks of $CHSH^\alpha$ as a (fairly-weak-binding) string commitment scheme.

## 4 Composing Commitment Schemes

### 4.1 The Composition Operation

We consider two 2-prover commitment schemes $\mathcal{S}$ and $\mathcal{S}'$ of a restricted form, and we compose them to a new 2-prover commitment scheme $\mathcal{S}'' = \mathcal{S} \circ \mathcal{S}'$ in a well-defined way; our composition theorem then shows that $\mathcal{S}''$ is secure (against classical attacks) if $\mathcal{S}$ and $\mathcal{S}'$ are. We start by specifying the restriction to $\mathcal{S}$ and $\mathcal{S}'$ that we impose.
Definition 4.1. Let $S$ and $S'$ be two 2-prover string commitment schemes. We call the pair $(S, S')$ eligible if the following three properties hold, or they hold with the roles of $P$ and $Q$ exchanged.

1. The commit phase of $S$ is a protocol $\text{com}_{PV} = (\text{com}_P, \text{com}_V)$ between $P$ and $V$ only, and the opening phase of $S$ is a protocol $\text{open}_{QV} = (\text{open}_Q, \text{open}_V)$ between $Q$ and $V$ only. In other words, $\text{com}_Q$ and $\text{open}_P$ are both trivial and do nothing. Similarly, the commit phase of $S'$ is a protocol $\text{com}'_{QV}$ between $Q$ and $V$ only (but both provers may be active in the opening phase).

2. The opening phase $\text{open}_{QV}$ of $S$ is of the following simple form: $Q$ sends a bit string $y \in \{0, 1\}^m$ to $V$, and $V$ computes $s$ deterministically as $s = \text{Extr}(y, c)$, where $c$ is the commitment.

3. The domain of $S'$ contains (or equals) $\{0, 1\}^m$.

Furthermore, we specify that the allowed attacks on $S$ are so that $P$ and $Q$ do not communicate during the course of the entire scheme, and the allowed attacks on $S'$ are so that $P$ and $Q$ do not communicate during the course of the commit phase but there may be limited communication during the opening phase.

An example of an eligible pair of 2-prover commitments is the pair $(\text{CHSH}^n, \text{XCHSH}^n)$, where $\text{XCHSH}^n$ coincides with scheme $\text{CHSH}^n$ except that the roles of $P$ and $Q$ are exchanged.

Remark 4.2. For an eligible pair $(S, S')$, it will be convenient to understand $\text{open}_Q$ and $\text{open}_V$ as non-interactive algorithms, where $\text{open}_Q$ produces $y$ as its output, and $\text{open}_V$ takes $y$ as additional input (rather than viewing the pair as a protocol with a single one-way communication round).

We now define the composition operation. Informally, committing is done by means of committing using $S$, and to open the commitment, $Q$ uses $\text{open}_Q$ to locally compute the opening information $y$ and he commits to $y$ with respect to the scheme $S'$, and then this commitment is opened (to $y$), and $V$ computes and outputs $s = \text{Extr}(y, c)$. Formally, this is captured as follows (see also Figure 2).

Definition 4.3. Let $S = (\text{com}_{PV}, \text{open}_{QV})$ and $S' = (\text{com}'_{QV}, \text{open}'_{PVQ})$ be an eligible pair of 2-prover commitment schemes. Then, their composition $S \star S'$ is defined as the 2-prover commitment scheme consisting of $\text{com}_{PV} = (\text{com}_P \circ \text{com}'_Q, \text{com}_V)$ and

$$\text{open}'_{PVQ} = (\text{open}'_P, \text{open}'_Q \circ \text{com}'_Q \circ \text{open}_Q, \text{open}_V \circ \text{open}'_V \circ \text{com}_V),$$

where we make it explicit that $\text{com}_P$ and $\text{open}_Q$ use joint randomness, and so do $\text{com}'_Q$ and $\text{open}'_P$.

When considering attacks against the binding property of the composed scheme $S \star S'$, we declare that the allowed deterministic attacks are those of the form $(\text{com}_P, \text{open}'_{PVQ} \circ \text{ptoc}_{PVQ} \circ \text{com}_Q)$, where $\text{com}_P$ is an allowed deterministic commit strategy for $S$, $\text{com}_Q$ and $\text{open}'_P$ are allowed deterministic commit and opening strategies for $S'$, and $\text{ptoc}_{PVQ}$ is the one-way communication protocol that communicates $P$’s input to $Q$ (see also Figure 3).

Remark 4.4. It is immediate that $S \star S'$ is a commitment scheme in the sense of Definition 2.2 and that it is complete if $S$ and $S'$ are, with the error parameters adding up. Also, the hiding property is obviously inherited from $S$; however, the point of the composition is to keep the hiding property alive for longer, namely up to before the last round of the opening phase — recall that, using the terminology used in context of relativistic commitments, these rounds of the opening phase up to before the last would then be referred to as the sustain phase. We show in Appendix D that $S \star S'$ is hiding up to before the last round, with the error parameters adding up.

It is intuitively clear that $S \star S'$ should be binding if $S$ and $S'$ are: Committing to the opening information $y$ and then opening the commitment allows the provers to delay the announcement of $y$ (which is the whole point of the exercise), but it does not allow them to change $y$, by the binding property of $S'$; thus, $S \star S'$ should be (almost) as binding as $S$. This intuition is confirmed by our composition theorem below.

---

7 Except that $\text{com}_Q$ may output state information to the opening protocol $\text{open}_Q$, e.g., in order to pass on the commit phase randomness.

8 Our composition theorem also works for a randomized Extr, but for simplicity, we restrict to the deterministic case.

9 The allowed randomized attacks are then naturally given as those that pick one of the deterministic attacks according to some distribution.

10 This one-way communication models that in the relativistic setting, sufficient time has passed at this point for $P$ to inform $Q$ about what happened during $\text{com}_P$. 
Fig. 2. The composition of $S$ and $S'$ (assuming single-round commit phases). The dotted arrows indicate communication allowed to the dishonest provers.

Remark 4.5. We point out that the composition $S \times S'$ can be naturally defined for a larger class of pairs of schemes (e.g., where both provers are active in the commit phase of both schemes), and the above intuition still holds. However, our proof only works for this restricted class of (pairs of) schemes. Extending the composition result in that direction is an open problem.

Remark 4.6. We observe that if $(S, XS)$ is an eligible pair, where $XS$ coincides with $S$ except that the roles of $P$ and $Q$ are exchanged, then so is $(XS, S \times XS)$. As such, we can then compose $XS$ with $S \times XS$, and obtain yet another eligible pair $(S, XS \times S \times XS)$, etc. We write $S_m$ for the $m$-fold composition of $S$ with itself, i.e., $S_m = S \times XS \times S \times \ldots$ for $m$ terms. Applying this to the schemes $S = CHSH^n$, we obtain the multi-round scheme from Lunghi et al. [LKB+15]. As such, our composition theorem below implies security of their scheme—with a linear blow-up of the error term (instead of double exponential).

We point out that formally we obtain security of the Lunghi et al. scheme as a 2-prover commitment scheme under an abstract restriction on the provers’ communication: In every round, the active prover cannot access the message that the other prover received in the previous round. As such, when the rounds of the protocol are executed fast enough so that it is ensured that there is no time for the provers to communicate between subsequent rounds, then security as a relativistic commitment scheme follows immediately.

Before stating and proving the composition theorem, we need to single out one more relevant parameter.

**Definition 4.7.** Let $(S, S')$ be an eligible pair, which in particular means that $V$’s action in the opening phase of $S$ is determined by a function $\text{Extr}$. We define $k(S) := \max_s |\{y | \text{Extr}(y, c) = s\}|$.

I.e., $k(S)$ counts the number of $y$’s that are consistent with a given string $s$ (in the worst case). Note that $k(CHSH^n) = 1$: For every $a, x, s \in \{0, 1\}^n$ there is at most one $y \in \{0, 1\}^n$ such that $x + y = a \cdot s$.

4.2 The Composition Theorems

In the following composition theorems, we take it as understood that the assumed respective binding properties of $S$ and $S'$ hold with respect to a well-defined respective classes of allowed attacks. We start with the composition theorem for the fairly-binding property, which is easier to prove than the one for the fairly-weak-binding property.

**Theorem 4.8.** Let $(S, S')$ be an eligible pair of 2-prover commitment schemes, and assume that $S$ and $S'$ are respectively $\varepsilon$-fairly-binding and $\delta$-fairly-binding. Then, their composition $S'' := S \times S'$ is $(\varepsilon + k(S) \cdot \delta)$-fairly-binding.
Proof. We first consider the case \( k(S) = 1 \). We fix an attack \((\text{com}_P, \text{open}'_{PQ})\) against \( S'\). Without loss of generality, the attack is deterministic, so \( \text{open}'_{PQ} \) is of the form \( \text{open}'_{PQ} = \text{open}'_{PQ} \circ \text{ptoq}_{PQ} \circ \text{com}'_{Q} \).

Note that \( \text{com}_P \) is also a commit strategy for \( S \). As such, by the fairly-binding property of \( S \), there exists a function \( \tilde{s}(c) \), only depending on \( \text{com}_P \), so that the property specified in Definition 3.2 is satisfied for every opening strategy \( \text{open}'_{PQ} \) for \( S \). We will show that it is also satisfied for the (arbitrary) opening strategy \( \text{open}'_{PQ} \) for \( S' \), except for a small increase in \( \varepsilon \). We will show that \( p(\tilde{s}(c) \neq s \wedge s = s_o) \leq \varepsilon + \delta \) for every fixed target string \( s_o \). This then proves the claim.

To show this property on \( \tilde{s}(c) \), we “decompose and reassemble” the attack strategy \((\text{com}_P, \text{open}'_{PQ} \circ \text{ptoq}_{PQ} \circ \text{com}'_{Q})\) for \( S' \) into an attack strategy \((\text{com}_Q, \text{newopen}'_{PQ})\) for \( S' \) with \( \text{newopen}'_{PQ} \) formally defined as

\[
\text{newopen}'_{PQ}[c](\text{state}_P(c) := \text{open}'_{PQ}(\text{state}_P(c)||\text{state}(P,c,\text{state}_Q))
\]

where

\[
(\text{state}_P(c)||c) \leftarrow (\text{com}_P||\text{com}_V).
\]

Informally, this means that ahead of time, \( P \) and \( Q \) simulate an execution of \((\text{com}_P(\emptyset)||\text{com}_V(\emptyset))\) and take the resulting communication/commitment \( c \) as shared randomness, and then \( \text{newopen}'_{PQ} \) computes \( \text{state}_P \) from \( c \) as does \( \text{com}_P \), and runs \( \text{open}'_{PQ} \) (see Figure 3). It follows from the fairly-binding property that there is a function \( \tilde{y}(c') \) of the commitment \( c' \) so that \( p(\tilde{y}(c') \neq y \wedge y = y_o(c)) \leq \delta \) for every function \( y_o(c) \).

Fig. 3. Constructing the opening strategy \( \text{newopen}'_{PQ} \) against \( S' \).

The existence of \( \tilde{y} \) now gives rise to an opening strategy \( \text{open}'_{PQ} \) for \( S \); namely, simulate the commit phase of \( S' \) to obtain the commitment \( c' \), and output \( \tilde{y}(c') \). By Definition 3.2 for \( s := \text{Extr}(\tilde{y}(c'),c) \) and every \( s_o \), \( p(\tilde{s}(c) \neq \tilde{s} \wedge \tilde{s} = s_o) \leq \varepsilon \).

We are now ready to put things together. Fix an arbitrary target string \( s_o \). For any \( c \) we let \( y_o(c) \) be the unique string such that \( \text{Extr}(y_o(c),c) = s_o \) (and some default string if no such string exists); recall, we assume for the moment that \( k(S) = 1 \). Omitting the arguments in \( \tilde{s}(c), \tilde{y}(c') \) and \( y_o(c) \), it follows that

\[
p(\tilde{s} \neq s \wedge s = s_o) \leq p(\tilde{s} \neq s \wedge s = s_o \wedge s \neq \tilde{s}) + p(s = s_o \wedge s \neq \tilde{s})
\]
\[
\leq p(\tilde{s} \neq \tilde{s} \wedge \tilde{s} = s_o) + p(\text{Extr}(y,c) \neq \text{Extr}(\tilde{y},c) \wedge \text{Extr}(y,c) = s_o)
\]
\[
\leq p(\tilde{s} \neq \tilde{s} \wedge \tilde{s} = s_o) + p(y \neq \tilde{y} \wedge y = y_o)
\]
\[
\leq \varepsilon + \delta.
\]

Thus, \( \tilde{s} \) is as required.

For the general case where \( k(S) > 1 \), we can reason similarly, except that we then list the \( k \leq k(S) \) possibilities \( y^{(1)}_o(c), \ldots, y^{(k)}_o(c) \) for \( y_o(c) \), and conclude that \( p(s \neq \tilde{s} \wedge s = s_o) \leq \sum_i p(y \neq \tilde{y} \wedge y = y^{(i)}_o) \leq k(S) \cdot \delta \), which then results in the claimed bound. \(\square\)

\(^{11}\) Recall that by convention (Remark 2.3), the commitment \( c \) equals the communication between \( V \) and, here, \( P \).

\(^{12}\) We are using here that \( Q \) is inactive during \( \text{com}_P \) and \( P \) during \( \text{com}'_{Q} \), and thus the two “commute.”
Definition 3.8 is satisfied for every opening strategy.

Remark 4.9. Putting things together, we can now conclude the security (i.e., the binding property) of the Lunghi et al. multi-round commitment scheme. Corollary 3.18 ensures the fairly-binding property of CHSH, i.e., the Crépeau et al. scheme as a string commitment scheme, with parameter \(2^{-n/2+1}\). The composition theorem (Theorem 3.18) then guarantees the fairly-binding property of the \(m\)-fold composition as a string commitment scheme, with parameter \((m + 1) \cdot 2^{-n/2+1}\). Finally, Proposition 3.5 implies that the \(m\)-fold composition of CHSH with itself is a \(\varepsilon_m\)-binding bit commitment scheme with error parameter \(\varepsilon_m = (m + 1) \cdot 2^{-n/2+2}\) as claimed in the introduction, or, more generally, and by taking Remark 3.6 into account, a \((m + 1) \cdot 2^{-n/2+k+1}\)-binding \(k\)-bit-string commitment scheme.

For completeness, we also show the composition theorem for the weak version of the binding property. Since this notion makes sense also against quantum attacks, we emphasize the restriction to classical attacks—extending the theorem to quantum attacks is an open problem.

**Theorem 4.10.** Let \((S, S')\) be an eligible pair of 2-prover commitment schemes, and assume that \(S\) and \(S'\) are respectively \(\varepsilon\)-fairly-weak-binding and \(\delta\)-fairly-weak-binding against classical attacks. Then, their composition \(S'' = S \star S'\) is a \((\varepsilon + k(S) \cdot \delta)\)-fairly-weak-binding 2-prover commitment scheme against classical attacks.

**Proof.** We first consider the case \(k(S) = 1\). We fix an arbitrary deterministic attack \((\text{comp}, \text{open}_{PQ})\) against \(S''\), where \(\text{open}_{PQ}\) is of the form \(\text{open}_{PQ} = \text{open}_{PQ} \circ \text{ptoq}_{PQ} \circ \text{comp}'\). Let \(a\) be \(V\)'s randomness in \(\text{comp}\). Then, \(c\) is a function \(c(a)\) of \(a\), and the distribution \(p(a, y)\) is well defined. Since \(\text{comp}\) is also an attack strategy against \(S\), there exists a distribution \(\hat{p}(\hat{s})\) (only depending on \(\text{comp}\)) such that Definition 3.8 is satisfied for every opening strategy \(\text{open}_{PQ}\) for \(S\).

Similar to the proof of Theorem 3.18, wereassemble the attack strategy \((\text{comp}, \text{open}_{PQ} \circ \text{ptoq}_{PQ} \circ \text{comp})\) for \(S''\) into an attack strategy \((\text{comp}, \text{newopen}_{PQ,a})\) for \(S'\). Concretely, for every fixed choice of \(a\), we obtain a deterministic opening strategy \(\text{newopen}_{PQ,a}\) given by

\[
\text{newopen}_{PQ,a}(\text{state}'): = \text{open}_{PQ}(\text{state}_P(c(a)))(|\text{state}_P(c(a)), \text{state}_Q'|),
\]

and the distribution of the verifier’s output \(y\) when the provers use \(\text{newopen}_{PQ,a}\) is \(p(y|a)\). It follows from the fairly-binding property of \(S'\) that there exists a distribution \(p(\hat{y})\), only depending on \(\text{comp}_{Q}\), so that for every choice of \(a\) there exists a consistent joint distribution \(p(\hat{y}, y|a)\) such that \(p(\hat{y} \neq y \wedge y = y_a|a) \leq \delta\) for every fixed target string \(y_a\). Note that here, consistency in particular means that \(p(\hat{y}|a) = p(\hat{y})\). This joint conditional distribution \(p(\hat{y}, y|a)\) together with the distribution \(p(a)|a\) then naturally defines the distribution \(p(\hat{a}, y, \hat{y})\), which is consistent with \(p(a, y)|a\) considered above.

The existence of \(p(y)\) now gives rise to an opening strategy \(\text{open}_{PQ}\) for \(S\); namely, sample \(\hat{y}\) according to \(p(y)\) and output \(\hat{y}\). Note that the joint distribution of \(a\) and \(\hat{y}\) in this “experiment” is given by

\[
p(a) \cdot p(y) = p(a) \cdot p(\hat{y}|a) = p(a, \hat{y}),
\]

i.e., is consistent with the distribution \(p(a, \hat{y}, y)|a\). By Definition 3.8, we know there exists a joint distribution \(p(\hat{s}, \hat{y})\), consistent with \(p(\hat{s})\) fixed above and with \(p(\hat{s})\) determined by \(\hat{s} := \text{Extr}(\hat{y}, c(a))\), and such that \(p(\hat{s} \neq \hat{s} \wedge \hat{s} = s_a) \leq \varepsilon\) for every \(s_a\). We can now “glue together” \(p(\hat{s}, \hat{y})\) and \(p(\hat{c}, \hat{y}, \hat{s})\), i.e., find a joint distribution that is consistent with both, by setting

\[
p(a, \hat{y}, y, \hat{s}, \hat{\tilde{s}}) := p(a, \hat{y}, y, \hat{s}) 
\cdot p(\hat{s}|\tilde{s}).
\]

We now fix an arbitrary target string \(s_a\). Furthermore, for any \(a\) we let \(y_a(a)\) be the unique string such that \(\text{Extr}(y_a(a), c(a)) = s_a\) (and to some default string if no such string exists); recall, we assume for the moment that \(k(S) = 1\). With respect to the above joint distribution, it then holds that

\[
p(\hat{s} \neq s \wedge s = 0) = p(\hat{s} \neq s \wedge s = s_0 \wedge s = \hat{s}) + p(s = s_0 \wedge s = s_0 \wedge s \neq \hat{s})
\leq p(\hat{s} \neq s \wedge s = s_0 \wedge s = \hat{s}) + p(s \neq \hat{s} \wedge s = s_0)
\leq p(\hat{s} \neq \hat{s} \wedge s = s_0) + p(\text{Extr}(y, c(a)) \neq \text{Extr}(\hat{y}, c(a)) \wedge \text{Extr}(y, c(a)) = s_a)
\leq p(\hat{s} \neq \hat{s} \wedge \hat{s} = s_a) + p(y \neq \hat{y} \wedge y = y_a|a)
\leq \varepsilon + \delta.
\]
Thus, the distribution \( p(\hat{s}, s) \) is as required.

For the case where \( k(S) > 1 \), we can reason similarly, except that we then list the \( k \leq k(S) \) possibilities \( y_1(a), \ldots, y_k(a) \) for \( y(a) \), and conclude that \( p(s \neq \hat{s} \land s = s_0) \leq \sum_i p(y_i \neq y = y_i(a)) \leq k(S) \cdot \delta \), which then results in the claimed bound. \( \square \)

**Remark 4.11.** Analogously to Remark 4.9 we can conclude from Corollary 3.20 and Theorem 4.10 that \( CHSH^n \) is \( (m + 1) \cdot 2^{-(n-1)/2} \)-fairly-weak-binding. It follows from Proposition 3.10 that \( CHSH^n \) is a \( (m + 1) \cdot 2^{-(n+1)/2} \)-weak-binding bit-commitment scheme. More generally, we can conclude that for any \( k < n \), it is a \( (m + 1) \cdot 2^{-(n-1)/2+k} \)-weak-binding \( k \)-bit string commitment scheme. Below, we show how to avoid the factor 2 introduced by invoking Proposition 3.10.

### 4.3 Variations

In this section, we show two variants of the composition theorems. The first one says that if we compose a weak-binding with a fairly-weak-binding scheme, we obtain a weak-binding scheme. This allows us to slightly improve the parameter in Remark 4.11. The proof crucially relies on the fact that, in the weak definition, there is some freedom in “gluing together” the distributions \( p(s) \) and \( p(\hat{s}) \). The second variant says that composing two binding (or weak-binding) schemes yields a binding (or weak-binding, respectively) scheme.

We start by proving the following two properties for fairly-weak-binding commitment schemes. The first property shows that one may assume the joint distribution \( p(\hat{s}, s) \) to be such that \( s \) and \( \hat{s} \) are independent conditioned on \( s \neq \hat{s} \).

**Lemma 4.12.** Let \( S \) be a \( \varepsilon \)-fairly-weak-binding commitment scheme. Then, for any \( \mathsf{COM}_{PQ} \) and \( \mathsf{OPEN}_{PQ} \), there exists a joint distribution \( p(\hat{s}, s) \) as required by Definition 3.8 but with the additional property that

\[
p(\hat{s}, s|s \neq \hat{s}) = p(\hat{s}|s \neq \hat{s}) \cdot p(s|s \neq \hat{s}).
\]

**Proof.** Since the scheme is \( \varepsilon \)-fairly-weak-binding, it follows that there exists a consistent joint distribution \( p(\hat{s}, s) \) such that \( p(s \neq \hat{s} \land s = s_0) \leq \varepsilon \) for every \( s_0 \). Because of this, we have

\[
p(s = s_0) = p(s = s_0 \land \hat{s} = s_0) + p(s = s_0 \land \hat{s} = \neq s_0) = p(s = s_0 \land s = s_0) + p(s = \neq s \land s = s_0) \leq p(\hat{s} = s_0) + \varepsilon.
\]

We apply Lemma 2.1 to the marginal distributions \( p(\hat{s}) \) and \( p(s) \). The resulting joint distribution \( p(\hat{s}, s) \) satisfies \( \bar{p}(\hat{s} = s_0 \land s = s_0|s = \hat{s}) = \min\{p(s = s_0), p(\hat{s} = s_0)\} \) and \( \bar{p}(\hat{s}, s|s \neq \hat{s}) = \bar{p}(\hat{s}|s \neq \hat{s}) \cdot \bar{p}(s|s \neq \hat{s}) \). It remains to show that \( \bar{p}(s \neq \hat{s} \land s = s_0) \leq \varepsilon \) for all \( s_0 \). Indeed, we have

\[
\bar{p}(s \neq \hat{s} \land s = s_0) = \bar{p}(s = s_0) - \bar{p}(s = s_0 \land s = s_0) \\
= p(s = s_0) - p(s = s_0 \land s = s_0) \\
= \min\{p(s = s_0), p(s = s_0)\} - \min\{p(s = s_0), p(s = s_0)\} \\
\leq p(s = s_0) - \min\{p(s = s_0), p(s = s_0)\} - \varepsilon \\
= \varepsilon
\]

as claimed. \( \square \)

The second property shows that the quantification over all fixed \( s_0 \) in Definition 3.8 of the fairly-weak-binding property can be relaxed to \( s_0 \) that may depend on \( s \), but only on \( \hat{s} \). Note that we can obviously not allow \( s_0 \) to depend (arbitrarily) on \( s \), since then one could choose \( s_0 = s \).

**Proposition 4.13.** Let \( S \) be a \( \varepsilon \)-fairly-weak-binding commitment scheme. Then

\[
\forall \mathsf{COM}_{PQ} \exists p(\hat{s}) \forall \mathsf{OPEN}_{PQ} \exists p(s, \hat{s}) \forall p(s_0|\hat{s}) : p(s \neq \hat{s} \land s = s_0) \leq \varepsilon,
\]

where it is understood that \( p(\hat{s}, s, s_0) := p(\hat{s}, s) \cdot p(s_0|\hat{s}) \). Thus, the joint distribution \( p(\hat{s}, s) \) is such that \( p(s \neq \hat{s} \land s = s_0) \leq \varepsilon \) holds in particular for any function \( s_0 = f(\hat{s}) \) of \( \hat{s} \).

**Proof.** For given \( \mathsf{COM}_{PQ} \) and \( \mathsf{OPEN}_{PQ} \), let \( p(s, \hat{s}) \) be as guaranteed by the fairly-weak-binding property. By Lemma 4.12, we may assume without loss of generality that \( p(\hat{s}, s|s \neq \hat{s}) = p(\hat{s}|s \neq \hat{s}) \cdot p(s|s \neq \hat{s}) \).

Then, by Lemma 3.1, we also have that \( p(s, s_0|s \neq \hat{s}) = p(s|s \neq \hat{s}) \cdot p(s_0|s \neq \hat{s}) \). It follows that

\[
p(s \neq \hat{s} \land s = s_0) = p(s \neq \hat{s}) \cdot p(s = s_0|s \neq \hat{s})
\]
where the inequality follows from the fact that $p(s \neq \hat{s} \wedge s = s'_o) \leq \epsilon$ for every fixed $s'_o$. \hfill \Box

For the rest of the section, we take it as understood that we only consider classical attacks.

**Theorem 4.14.** Let $(S, S')$ be an eligible pair of $2$-prover commitment schemes, where $S$ is $\epsilon$-weak-binding and $S'$ is $\delta$-fairly-weak-binding, and let $\{0,1\}^m$ be the domain of $S$. Then, the composition $S \circ S'$ is a $((\epsilon + (2m - 1) \cdot k(S)) \cdot \delta)$-weak-binding commitment scheme.

In particular, if $S$ is a bit commitment scheme then $S \circ S'$ is a $(\epsilon + k(S)) \cdot \delta$-weak-binding.

**Proof.** We follow the proof of Theorem 4.10 up to when it comes to choosing $y_o$. Let us first consider the case $m = 1$, i.e., $S$ is a bit commitment scheme. In that case, and assuming for the moment that $k(S) = 1$, we let $y_o$ be the unique string that satisfies $\text{Extr}(y_o, c) = s_o$, but where now $s_o := 1 - \hat{s}$. We emphasize that for a fixed $c$, this choice of $y_o$ is not fixed anymore (in contrast to the choice in the proof of Theorem 4.10), namely, it is a function of $\hat{s} = \text{Extr}(\hat{y}, c)$, which in turn is a function of $\hat{y}$. Therefore, by Proposition 4.13 it still holds that $p(y \neq \hat{y} \wedge y = y_o(a)) \leq \delta$, and we can conclude that

$$p(s \neq \hat{s} \wedge s \neq \perp) \leq p(s \neq \hat{s} \wedge s \neq \perp \wedge s = \hat{s}) + p(s \neq \hat{s} \wedge s = 1 - \hat{s})$$

$$= p(s \neq \hat{s} \wedge s \neq \perp \wedge s = \hat{s}) + p(s \neq \hat{s} \wedge s = 1 - \hat{s})$$

$$\leq \epsilon \cdot \sum_a p(a) \left( p(y \neq \hat{y} \wedge y = y_o(a) \right. + p(y \neq \hat{y} \wedge y = y_o(a) \right.$$}

$$\leq \epsilon + \sum_a p(a) \delta$$

$$= \epsilon + \delta.$$

In the case that $k(S) > 1$, we instead randomly select one of the at most $k(S)$ strings $y_o$ that satisfy $\text{Extr}(y_o, c) = s_o = 1 - \hat{s}$. Then, conditioned on $a$, $y_o$ is still independent of $y$ given $\hat{y}$, so that Proposition 4.13 still applies, and we can argue as above, except that we get a factor $k(S)$ blow-up from $p(s \neq \hat{s} \wedge s = 1 - \hat{s}) \leq k(S) \cdot p(y \neq \hat{y} \wedge y = y_o)$. We finally, for the case $m > 1$, we first pick a random $s_o \in \{0,1\}^m \setminus \{\hat{s}\}$, and then choose $y_o$ such that $\text{Extr}(y_o, c) = s_o$, uniquely or at random, depending of $k(S)$. Conditioned on $a$, $y_o$ is still independent of $y$ given $\hat{y}$, and therefore Proposition 4.13 still applies, but now we get an additional factor $(2^m - 1)$ blow-up from $p(s \neq \hat{s} \wedge s \neq \perp) \leq (2^m - 1) \cdot p(s \neq \hat{s} \wedge s = s_o)$. \hfill \Box

**Remark 4.15.** Theorem 4.14 allows us to slightly improve the bound we obtain in Remark 4.11 on the Lunghi et al. multi-round commitment scheme. By Theorem 4.10 we can compose $m$ instances of CHSH$^m$ to obtain a $m \cdot 2^{-((n-1)/2)}$-fairly-weak-binding string commitment scheme. Then, we can compose the Crépeau et al. bit commitment scheme (i.e., the bit commitment version of CHSH$^m$), which is $2^{-(n-1)}$-weak-binding, with this fairly-weak-binding string commitment scheme; by Theorem 4.14 this composition, which is the Lunghi et al. multi-round bit commitment scheme, is $(m \cdot 2^{-(n-1)/2} + 2^{-((n-1)/2)})$-weak-binding.

Finally, for completeness, we point out that the composition theorem also applies to two ordinary binding or weak-binding commitment schemes.

**Theorem 4.16.** Let $(S, S')$ be an eligible pair of $2$-prover commitment schemes, where $S$ is $\epsilon$-binding and $S'$ is $\delta$-binding. Then, the composition $S \circ S'$ is $(\epsilon + \delta)$-binding. The same holds for the weak-binding property.
Proof. The proof is almost the same as in Theorem 4.8 or Theorem 4.10, respectively, except that now there are no \( s_0 \) and \( y_0 \), and in the end we can simply conclude that
\[
p(s \neq \tilde{s} \land s \neq \perp) \leq p(s \neq \tilde{s} \land s \neq \perp) + p(s \neq \tilde{s} \land s \neq \perp) 
\leq p(s \neq \tilde{s} \land \tilde{s} \neq \perp) + p(y \neq \tilde{y} \land y \neq \perp) 
\leq \varepsilon + \delta,
\]
where the second inequality holds since \( y = \perp \) implies that \( s = \text{Extr}(y, c) = \perp \). \( \square \)

4.4 Tightness

We now show that our composition result is nearly tight for CHSH\(^n\). Let CHSH\(^n\) be the \( m \)-fold composition of CHSH\(^n\) with itself, as defined in Remark 4.4. We show that for even \( n \), this composed scheme can be \( \varepsilon \)-weak-binding as a bit-commitment scheme only if \( \varepsilon \geq 1 \) \( m2^{-n/2} \). A slightly weaker result was proved in [BC16], which shows that \( \varepsilon \geq \frac{1}{8}m2^{-n/2} \) for even \( n \). Furthermore, we show that, as a string commitment scheme, CHSH\(^n\) can be \( \varepsilon \)-fairly-weak-binding only if \( \varepsilon \geq \frac{1}{2}m2^{-n/2} \) (for even \( n \)).

Lemma 4.17. Consider functions \( X_n, Y_n : \mathbb{F}_2^n \times R_n \to \mathbb{F}_{2^n} \). Let
\[
q_n = \max_{X_n, Y_n} p(X_n(a, r) + Y_n(s, r) = a \cdot s)
\]
where \( a, s \) and \( r \) are selected uniformly at random in \( R_n \). It holds that:
1. There are \( X_n \) and \( Y_n \) such that \( p(X_n(a, r) + Y_n(s, r) = a \cdot s) = q_n \) for all \( a, s \in \mathbb{F}_{2^n} \).
2. For even \( n \), we have \( q_n = \Omega(2^{-n/2}) \). For odd \( n \), we have \( q_n = \Omega(2^{-2n/3}) \).

Proof. Fix \( X'_n \) and \( Y'_n \) that achieve the maximum in Equation (7). We show that there also are functions \( X_n \) and \( Y_n \) such that for any \( a \) and \( s \), \( p(X_n(a, r) + Y_n(s, r) = a \cdot s) = q_n \). Without loss of generality, \( X'_n \) and \( Y'_n \) depend only on \( a \) and \( s \), not on \( r \). Intuitively, \( X_n \) and \( Y_n \) do the following: They randomize their inputs \( a \) and \( s \) by adding uniformly random elements \( r_a, r_s \in \mathbb{F}_{2^n} \), then apply \( X'_n \) and \( Y'_n \), and finally remove the random terms again from the output. Formally, we let
\[
X_n(a, (r_a, r_s)) = X'_n(a + r_a) - ar_a - r_ar_s
\]
\[
Y_n(a, (r_a, r_s)) = Y'_n(s + r_s) - rs
\]
For \( r_a \) and \( r_s \) uniformly random, we have \( p(X'_n(a + r_a) + Y'_n(s + r_s) = as + ar_a + r_ar_s + rs) = q_n \). Thus, it is easy to see that \( p(X_n(a, (r_a, r_s)) + Y_n(s, (r_a, r_s)) = as + ar_a + r_ar_s + rs) = q_n \).

The functions \( X_n \) and \( Y_n \) in Equation (7) describe strategies for the CHSH\(^n\) game with classical players and \( q_n \) is the maximal winning probability that classical players can achieve in this game. As shown in [BS14], it holds that \( q_n = \Omega(2^{-n/2}) \) for even \( n \), and \( q_n = \Omega(2^{-2n/3}) \) for odd \( n \). \( \square \)

The following lemma can be seen as a generalization of Theorem 3.11 to string commitment schemes. Intuitively, it bounds the winning probability of the provers in the following game: First, they have to produce a commitment. Then, they receive a uniformly random string \( s_0 \) and, in order to win, they have to open the commitment to \( s_0 \). The winning probability in this game is at most \( \varepsilon + 2^{-n} \), when the scheme is an \( \varepsilon \)-fairly-weak-binding \( n \)-bit string commitment scheme.

Lemma 4.18. Let \( S \) be an \( \varepsilon \)-fairly-weak-binding \( n \)-bit string commitment scheme. Fix an allowed commit strategy \( \text{Com}_{\text{PQ}} \) for \( S \) and, for each \( s_0 \in \mathbb{F}_{2^n} \), an allowed opening strategy \( \text{Open}_{\text{PQ}}(s_0) \). Let \( p(s|s_0) \) be the output distribution of \( S \) if the provers use \( \text{Com}_{\text{PQ}} \) and \( \text{Open}_{\text{PQ}}(s_0) \). Let \( p(s) \) be distributed uniformly over \( \mathbb{F}_{2^n} \). Then, \( p(s = s_0) := \sum_{s_0 \in \mathbb{F}_{2^n}} p(s_0)p(s) = s_0 | s_0 \leq \varepsilon + 2^{-n} \).

Proof. Let \( p(\tilde{s}) \) be a distribution that satisfies Equation (8) for the commit strategy \( \text{Com}_{\text{PQ}} \). Now consider any consistent joint distribution \( p(s, \tilde{s}|s_0) \). Here, consistency also means that \( p(\tilde{s}|s_0) = p(\tilde{s}) \). Thus, for a uniformly random \( s_0 \), \( p(\tilde{s} = s_0) = 2^{-n} \). By the \( \varepsilon \)-fairly-weak-binding property of \( S \), we have
\[
\varepsilon \geq p(s \neq \tilde{s} \land s = s_0) \geq p(s = s_0) - p(\tilde{s} = s_0) = p(s = s_0) - 2^{-n}
\]
and thus our claim follows. \( \square \)

\footnote{The paper states \( \varepsilon \geq \frac{1}{4}m2^{-n/2} \), but their binding definition is \( p_0 + p_1 \leq 1 + \varepsilon \); to convert their bound to our definition (equivalent to \( p_0 + p_1 \leq 1 + 2\varepsilon \)), it must be multiplied by 1/2.}

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With the help of the lemma above, it is easy to see that $q_n$ limits the binding parameter of the one-round scheme $\text{CHSH}^n$. If $P$ sends $X_n(a, r)$ and $Q$ sends $Y_n(s_0, r)$ for uniformly random $r$, then we have $p(s = s_0 | a \neq 0) = q_n$, and thus $p(s = s_0) \geq q_n - 2^{-n}$ for every $s_0$. Thus, by Lemma 4.18 $\text{CHSH}^n$ can be $\varepsilon$-fairly-weak-binding only if $\varepsilon \geq q_n - 2^{-n+1}$. We now show that this bound scales approximately linearly with the number of rounds.

**Theorem 4.19.** Let $q_n$ as in Lemma 4.17. For odd $m$, the $\text{CHSH}_m^n$ commitment scheme can be $\varepsilon$-fairly-weak-binding as a string commitment scheme only if

$$
\varepsilon \geq \frac{(m + 1)q_n}{2} - \frac{(m^2 - 1)q_n^2}{8} - (m + 1)2^{-n}.
$$

If $m = o(q_n^{-1})$, it holds that $\varepsilon \geq \Omega(mq_n)$. If, furthermore, $n$ is even, we have $\varepsilon \geq \Omega(m2^{-n/2})$; if $n$ is odd, $\varepsilon \geq \Omega(m2^{-2n/3})$.

**Proof.** Let $X_n(a, r)$ and $Y_n(b, r)$ be functions as in Lemma 4.17. We define a commit strategy $\text{commit}_{PQ}$ and an opening strategy $\text{open}_{PQ}(s_0)$ for every $s_0$ which aims to open to $s_0$.

We assume that the provers have $m$ uniformly random strings $r_i \in \mathbb{F}_2^m$ and $(m + 1)/2$ uniformly random inputs $r_i', i$ odd, for $X_n$ and $Y_n$ as shared randomness. We write $c_i = (a_i, x_i)$ for the communication between the verifier and the active prover in round $i$, where the $x_i$ are specified below. The dishonest provers exchange their communications as fast as possible, so in round $i + 2$, the active prover knows $c_1, \ldots, c_i$. Let $y_0 = s_0$ and for $i > 0$, let $y_i$ such that $\text{Extr}(y_i, c_i) = y_{i-1}$. Such a $y_i$ exists and is unique if $a_i \neq 0$. We only specify our strategy for the case where the verifier’s messages $a_i$ are all non-zero and assume that the provers fail to open to $s_0$ otherwise. One can compute $y_i$ from $c_1, \ldots, c_i$, so in round $i + 2$, the active prover can compute $y_i$.

If in any round $i$, the commitment is $(a_i, r_i + a_i \cdot y_{i-1})$, the provers can open to $s_0$ simply by following the honest strategy for $\text{CHSH}_m^n$ from that round on. The strategy described below is such that the provers have $(m + 1)/2$ chances to bring about this situation with probability $q_n$.

- **Round 1 (commit):** $P$ produces a “fake commitment” $x_1 = X_n(a_1, r_1')$.
- **Round $i$, $i$ even:** $Q$ computes $y'_{i-1} = Y_n(y_{i-2}, r_{i-1}')$, hoping that $x_{i-1} + y'_{i-1} = a_{i-1} \cdot y_{i-2}$, i.e., $y'_{i-1} = y_{i-1}$. He honestly commits to $y'_{i-1}$ by computing $x_i = a_i \cdot y_{i-1} + r_i$.
- **Round $i + 1$, $i$ even:** $P$ checks if $y_{i-1} = y'_{i-1}$. If yes, both provers proceed honestly from this round on, i.e., they follow the honest strategy for $\text{CHSH}_m^n$ in all subsequent rounds. If not, $P$ again produces a “fake commitment” $x_{i+1} = X_n(a_{i+1}, r_{i+1}')$.
- **Round $m + 1$:** $Q$ sends $y_m' = Y_n(y_{m-1}, r_m')$ to $V$.

By definition, we have $y'_{i-1} = y_{i-1}$ if and only if $X_n(a_{i-1}, r_{i-1}') + Y_n(y_{i-2}, r_{i-1}') = a_{i-1} \cdot y_{i-2}$, which happens with probability $q_n$. In this case, we have $c_i = (a_i, r_i + a_i \cdot y_{i-1})$, so the provers can indeed open to $s_0$ by proceeding honestly (ignoring completeness errors for now).

By definition of $X_n$, $Y_n$, and $q_n$, if the provers use the strategy $\text{commit}_{PQ}(s_0)$, then for

$$q = 1 - (1 - q_n)^{(m+1)/2} \geq \frac{(m + 1)q_n}{2} - \frac{(m + 1)/2}{2} \cdot (m + 1)q_n = \frac{(m + 1)q_n - (m^2 - 1)q_n^2}{8},$$

we have $p(s = s_0 | a_1, \ldots, a_m \neq 0) = q$. Thus, $p(s = s_0) \geq q - m2^{-n}$ for all $s_0$. Applying Lemma 4.18 we conclude that the scheme can be $\varepsilon$-fairly-weak-binding only if

$$\varepsilon \geq q - m2^{-n} \geq \frac{(m + 1)q_n}{2} - \frac{(m^2 - 1)q_n^2}{8} - (m + 1)2^{-n}$$

which is in $\Omega(mq_n)$ if $m = o(q_n^{-1})$. Finally, we have $\Omega(mq_n) = \Omega(m2^{-n/2})$ if $n$ is even and $\Omega(mq_n) = \Omega(m2^{-2n/3})$ if $n$ is odd, by claim 2 of Lemma 4.17. $\square$

From the analysis in the above proof, we can also derive a version of the theorem for the bit-commitment scheme described in Proposition 3.10.

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14 $Q$ can compute $y_{i-1}$ in round $i + 2$ and thus he too knows whether the provers should proceed honestly or not.
Corollary 4.20. For even $m$, the commitment scheme $\text{CHSH}_m^n$ can be $\varepsilon$-binding as a bit-commitment scheme only if

$$\varepsilon \geq \frac{mq_n}{4} - \frac{(m^2 - 2m)q_n^2}{16} - (m + 1)2^{-n}.$$

If $m = o(q_n^{-1})$, it holds that $\varepsilon \geq \Omega(mq_n)$. If $n$ is even, we have $\varepsilon \geq \Omega(m2^{-n/2})$ and if it is odd, $\varepsilon \geq \Omega(m2^{-2n/3})$.

Proof. Let $\text{comp} = \text{comp}(0)$, i.e., $P$ produces an honest commitment to 0. Let $\text{open}_{PQ}(0) = \text{open}_{PQ}$, i.e., the honest opening strategy. Since the provers play honestly, they are successful with probability at least $1 - (m + 1)2^{-n}$.

For $\text{open}_{PQ}(1)$, let $s_0$ such that $\text{Extr}(s_0, c_1) = 1$. The provers then use the strategy in the proof of Theorem 4.19 to produce a fake commitment $c_1$ and open it to $s_0$. Then, we have

$$p(b = 1 | a_1, \ldots, a_m \neq 0) \geq \frac{mq_n}{2} - \frac{(m^2 - 2m)q_n^2}{8} - 2^{-n}$$

and thus,

$$p(b = 1) \geq \frac{mq_n}{2} - \frac{(m^2 - 2m)q_n^2}{8} - (m + 1)2^{-n}.$$

It follows that

$$p(b = 0) + p(b = 1) \geq 1 + \frac{mq_n}{2} - \frac{(m^2 - 2m)q_n^2}{8} - (m + 1)2^{-n+1}$$

and, by Theorem 3.11, the scheme can be $\varepsilon$-weak-binding only if

$$\varepsilon \geq \frac{mq_n}{4} - \frac{(m^2 - 2m)q_n^2}{16} - (m + 1)2^{-n}.$$

Acknowledgments

We would like to thank Jędrzej Kaniewski for helpful discussions regarding [LKB+15], and for commenting on an earlier version of our work.

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A Proof of Lemma 2.1

We first extend the respective probability spaces given by the distributions \( p(x) \) and \( p(y) \) by introducing an event \( \Delta \) and declaring that

\[
p(x = x_o \wedge \Delta) = \min\{p(x = x_o), p(y = x_o)\} = p(y = x_o \wedge \Delta)
\]

for every \( x_o \in \mathcal{X} \). Note that \( p(\Delta) \) is well defined (by summing over all \( x_o \)). As we will see below, \( \Delta \) will become the event \( x = y \). In order to find a consistent joint distribution \( p(x, y) \), it suffices to find a consistent joint distribution \( p(x, y|\Delta) \) for \( p(x|\Delta) \) and \( p(y|\Delta) \), and a consistent joint distribution \( p(x, y|\neg\Delta) \) for \( p(x|\neg\Delta) \) and \( p(y|\neg\Delta) \). The former, we choose as

\[
p(x = x_o \wedge y = x_o|\Delta) := \min\{p(x = x_o), p(y = x_o)\}\ \if p(\Delta)
\]

for all \( x_o \in \mathcal{X} \), and \( p(x = x_o \wedge y = y_o|\Delta) := 0 \) for all \( x_o \neq y_o \in \mathcal{X} \), and the latter we choose as

\[
p(x = x_o \wedge y = y_o|\neg\Delta) := p(x = x_o|\neg\Delta) \cdot p(y = y_o|\neg\Delta)
\]

for all \( x_o, y_o \in \mathcal{X} \). It is straightforward to verify that these are indeed consistent joint distributions, as required, so that \( p(x, y) = p(x, y|\Delta) \cdot p(\Delta) + p(x, y|\neg\Delta) \cdot p(\neg\Delta) \) is also consistent. Furthermore, note that \( p(x = y|\Delta) = 1 \) and \( p(x = y|\neg\Delta) = 0 \); the latter holds because we have \( p(x = x_o \wedge \Delta) = p(x = x_o) \) or \( p(y = x_o \wedge \Delta) = p(y = x_o) \) for each \( x_o \in \mathcal{X} \), and thus \( p(x = x_o \wedge \neg\Delta) = 0 \) or \( p(y = x_o \wedge \neg\Delta) = 0 \). As such, \( \Delta \) is the event \( x = y \), and therefore \( p(x = x_o) = p(x = x_o \wedge \Delta) = \min\{p(x = x_o), p(y = x_o)\} \) for every \( x_o \in \mathcal{X} \) as required. Finally, the claim regarding \( p(x, y|x \neq y) \) holds by construction. \( \square \)

B A Property for Conditionally Independent Random Variables

Let \( p(x, y, z) \) be a distribution, and let \( \Lambda \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \) be an event. Then, we write \( x \rightarrow y \rightarrow z \) to express that \( p(x, z|y) = p(x|y) p(z|y) \), and \( x \rightarrow \Lambda \rightarrow y \rightarrow z \) to express that \( p(x, y|\Lambda) = p(x, y|\Lambda) p(y|\Lambda) \), etc.

**Lemma B.1.** If \( x \rightarrow y \rightarrow z \) and \( x \rightarrow x \neq y \rightarrow y \), then \( x \rightarrow x \neq y \rightarrow z \).

**Proof.** We assume that \( x \rightarrow y \rightarrow z \) and \( x \rightarrow x \neq y \rightarrow y \). We first observe that

\[
p(x, x \neq y, z|y) = p(x, x \neq y|y) p(z|x, y, x \neq y) = p(x, x \neq y|y) p(z|x, y) = p(x, x \neq y|y) p(z|y),
\]

which means that \( (x, x \neq y) \rightarrow y \rightarrow z \), and, by summing over \( x \), implies \( x \neq y \rightarrow y \rightarrow z \). It follows that

\[
p(z|x, y, x \neq y) = p(z|y) = p(z|y, x \neq y),
\]

which actually means that \( x \rightarrow (y, x \neq y) \rightarrow z \). Therefore,

\[
p(x, z|x \neq y) = \sum_y p(x, y, z|x \neq y) = \sum_y p(x, y|z, x \neq y) p(z|x, y, x \neq y)
\]

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By Definition 3.13 and inequality (2), it holds that the output of the verifier when the commit phase. However, the purpose of multi-round schemes is to maintain the commitment over a period of time in the relativistic setting, without disclosing the string $s$ until the very end. In this appendix, we define a hiding property that captures this requirement, and we prove that a composed scheme $S'' = S \ast S'$ is hiding if both $S$ and $S'$ are hiding (with the error parameters adding up).

\[ p(x|x \neq y) = p(x|x \neq y) \sum_{y} p(y|x \neq y) p(z|y, x \neq y) = p(x|x \neq y) \sum_{y} p(y, z|x \neq y) = p(x|x \neq y) p(z|x \neq y), \]

which was to be proven. \qed

\section{Proof of Theorem 3.19}

Fix a commit strategy $\overline{\text{Goal}}_{PQ}$ against $S$. Enumerate all strings in the domain $\{0, 1\}^n$ of $S$ as $s_1, \ldots, s_{2^n}$, and for every $i \in \{1, \ldots, 2^n\}$ let $\overline{\text{Goal}}_{PQ}^i$ be an opening strategy maximizing $p_i := p(s = s_i)$, where $s$ is the output of the verifier when $P$ and $Q$ use this strategy. We assume without loss of generality that the $p_i$s are in descending order. We define $p(\hat{s})$ as follows. Let $N \geq 2$ be an integer which we will fix later. By Definition 3.13 and inequality (2), it holds that

\[ \sum_{i=1}^{N} p_i \leq 1 + \frac{N}{2} \cdot \varepsilon = 1 + \frac{N(N - 1)}{2} \cdot \varepsilon \]

where we let $p_i = 0$ for $i > 2^n$ in case $N > 2^n$. We would like to define $p(\hat{s})$ as $p(\hat{s} = s_i) := p_i - (N - 1)\varepsilon/2$ for all $i \leq N, 2^n$; however, this is not always possible because $p_i - (N - 1)\varepsilon/2$ may be negative. To deal with this, let $N'$ be the largest integer such that $N' \leq N$ and $p_1, \ldots, p_{N'} \geq (N - 1)\varepsilon/2$. (We take $N = 0$ if $p_1 < (N - 1)\varepsilon/2$.) It follows that

\[ \sum_{i=1}^{N'} p_i \leq 1 + \frac{N'(N' - 1)}{2} \cdot \varepsilon \leq 1 + \frac{N'(N - 1)}{2} \cdot \varepsilon \]

and thus \[ \sum_{i=1}^{N'} p_i = 1 + \frac{N'(N - 1)}{2} \cdot \varepsilon \]

for some $\bar{\varepsilon} \leq \varepsilon$. We now set $p(\hat{s})$ to be $p(\hat{s} = s_i) := p_i - (N - 1)\bar{\varepsilon}/2 \geq p_i - (N - 1)\varepsilon/2 \geq 0$ for all $i \leq N'$. Now consider an opening strategy $\overline{\text{Goal}}_{PQ}$ and let $p(s)$ be the resulting output distribution. By definition of the $p_i$, it follows that $p(s = s_i) \leq p_i$ for all $i \leq 2^n$, and $p_i \leq p(\hat{s} = s_i) + (N - 1)\varepsilon/2$ for all $i \leq N'$. By Lemma 2.3, we can conclude that there exists a consistent joint distribution $p(\hat{s}, s)$ with $p(\hat{s} = s = s_i) = \min\{p(s = s_i), p(\hat{s} = s_i)\} \geq p(s = s_i) - (N - 1)\varepsilon/2$ for all $i \leq N'$, and thus $p(\hat{s} \neq s = s_i) = p(s = s_i) - p(\hat{s} = s = s_i) \leq (N - 1)\varepsilon/2$ for all $i \leq N'$. Furthermore, when $N' < i \leq N$, we have $p(\hat{s} \neq s = s_i) = p(s = s_i) \leq p_i < (N - 1)\varepsilon/2$ by definition of $N'$. Since the $p_i$ are sorted in descending order, it follows that for all $i > N$

\[ p(\hat{s} \neq s = s_i) = p(s = s_i) \leq p_i \leq p_N \leq \frac{1}{N} \sum_{i=1}^{N} p_i \leq \frac{1}{N} + \frac{N - 1}{2} \cdot \varepsilon \]

and thus, we have shown for all $s_i \in \{0, 1\}^n$ that

\[ p(\hat{s} \neq s = s) \leq \frac{1}{N} + \frac{N - 1}{2} \cdot \varepsilon. \]

We now select $N$ so that this value is minimized: It is easy to verify that the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto 1/x + (x - 1)\varepsilon/2$ has its global minimum in $\sqrt{2/\varepsilon}$, thus, we pick $N := \lceil \sqrt{2/\varepsilon} \rceil$, which gives us

\[ p(\hat{s} \neq s = s) \leq \frac{1}{N} + \frac{N - 1}{2} \cdot \varepsilon \leq \frac{1}{\sqrt{2/\varepsilon}} + \frac{\sqrt{2/\varepsilon}}{2} \cdot \varepsilon = \sqrt{2/\varepsilon} \]

for any $s_s \in \{0, 1\}^n$, as claimed. \qed

\section{The Hiding Property of Composed Schemes}

We already mentioned that the standard hiding property is not good enough for multi-round bit commitment schemes: The standard definition is not violated if the verifier learns the string $s$ immediately after the commit phase. However, the purpose of multi-round schemes is to maintain the commitment over a longer period of time in the relativistic setting, without disclosing the string $s$ until the very end. In this appendix, we define a hiding property that captures this requirement, and we prove that a composed scheme $S'' = S \ast S'$ is hiding if both $S$ and $S'$ are hiding (with the error parameters adding up).
Definition D.1. Let $S = (\text{com}_{PQV}, \text{open}_{PQV})$ be a commitment scheme. We write $v$ for the verifier's view immediately before the last round of communication in $\text{open}_{PQV}$. We say that a scheme is $\varepsilon$-hiding until the last round if for any (possibly dishonest) verifier $V$ and any two inputs $s_0$ and $s_1$ to the honest provers, we have $d(p(v|s_0), p(v|s_1)) \leq \varepsilon$.

Theorem D.2. Let $S$ be a $\varepsilon$-hiding commitment scheme and $S'$ a scheme that is $\delta$-hiding until the last round. If $(S, S')$ is eligible, then the composed scheme $S'' = S \star S'$ is $(\varepsilon + \delta)$-hiding until the last round.

Proof. Fix a strategy against the hiding-until-the-last-round property of $S''$. We consider the distribution $p(v, y, v'|s)$ where $s$ is the string that the provers commit to, $v$ the verifier's view after $\text{com}_{PQV}$ has been executed, $y$ the opening information to which $Q$ commits using the scheme $S'$, and $v'$ the verifier's view immediately before the last round of communication. We need to show that $d(p(v'|s_0), p(v'|s_1)) \leq \varepsilon + \delta$ for any $s_0$ and $s_1$.

First, note that $p(v'|v, y, s_b) = p(v'|v, y)$ since $v'$ is produced by $P$, $Q$ and $V$ acting on $y$ and $v$ only. From any strategy against $S''$, we can obtain a strategy against $S'$ by fixing $v$. Thus, by the hiding property of $S'$, for any $y_0$ and $y_1$, we have $d(p(v'|v, y = y_0), p(v'|v, y = y_1)) \leq \delta$ and it follows by the convexity of the statistical distance in both arguments that

$$p(v'|v, s_0) = \sum_y p(y|v, s_0)p(v'|v, y) \approx_{\delta} \sum_y p(y|v, s_1)p(v'|v, y) = p(v'|v, s_1)$$

where we use $\approx_{\delta}$ to indicate that the two distributions have statistical distance at most $\delta$. Since we have $d(p(v|s_0), p(v|s_1)) \leq \varepsilon$ by the hiding property of $S$, it follows that

$$p(v'|s_0) = p(v, v'|s_0) = p(v|s_0)p(v'|v, s_0) \approx_{\delta} p(v|s_0)p(v'|v, s_1) \approx_{\varepsilon} p(v|s_1)p(v'|v, s_1) = p(v, v'|s_1) = p(v'|s_1)$$

where the first and last equality hold because $v'$ contains $v$ since $v'$ is the view of $V$ at a later point in time. \qed

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