On Dris conjecture about odd perfect numbers

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Abstract: The Euler’s form of odd perfect numbers, if any, is \( n = \pi^\alpha N^2 \), where \( \pi \) is prime, \((\pi, N) = 1\) and \( \pi \equiv \alpha \equiv 1 \pmod{4}\). Dris conjecture states that \( N > \pi^\alpha \). We find that \( N^2 > \frac{1}{2} \pi^\gamma \), with \( \gamma = \max\{\omega(n) - 1, \alpha\}; \omega(n) \geq 10 \) is the number of distinct prime factors of \( n \).

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1 Introduction

Without explicit definitions all the numbers considered in what follows must be taken as strictly positive integers. Let \( \sigma(n) \) be the sum of the divisors of a number \( n \); \( n \) is said to be perfect if and only if \( \sigma(n) = 2n \). The multiplicative structure of odd perfect numbers, if any, is

\[ n = \pi^\alpha N^2, \]

where \( \pi \) is prime, \( \pi \equiv \alpha \equiv 1 \pmod{4} \) and \((\pi, N) = 1\) (Euler, cited in [3, p. 19]); \( \pi^\alpha \) is called the Euler’s factor. From equation (1) and from the fact that the \( \sigma \) is multiplicative, it results also

\[ n = \frac{\sigma(\pi^\alpha)}{2} \sigma(N^2), \]

where \( \sigma(N^2) \) is odd and \( 2\|\sigma(\pi^\alpha) \). Many details concerning the Euler’s factor and \( N^2 \) are given, for example, in [2, 5, 8, 9, 10].

Regarding the relation between the magnitude of \( N^2 \) and \( \pi^\alpha \) it has been conjectured by Dris that \( N > \pi^\alpha \) [4]. The result obtained in this paper is a necessary condition for odd perfection (Theorem 2.1) which provides an indication about Dris conjecture.
Indicating with \( \omega(n) \) the number of distinct prime factors of \( n \), we prove that (Corollary 2.3):

\[
(i) \quad N^2 > \frac{1}{2} \pi^\gamma, \text{ where } \gamma = \max\{\omega(n) - 1, \alpha\}.
\]

Since \( \omega(n) \geq 10 \) (Nielsen, [6]), it follows:

\[
(i_1) \quad N^2 > \frac{1}{2} \pi^0; \text{ this improves the result } N > \pi \text{ claimed in [1] by Brown in his approach to Dris conjecture.}
\]

Besides

\[
(i_2) \quad \text{If } \omega(n) - 1 > 2\alpha, \text{ then } N > \pi^\alpha,
\]

so that

\[
(i_3) \quad \text{If } \omega(n) - 1 > 2\alpha \text{ for each odd perfect number } n, \text{ then Dris conjecture is true.}
\]

Now, some questions arise: \( \omega(n) \) depends on \( \alpha \)? Is there a maximum value of \( \alpha \)? The minimum value of \( \alpha \) is 1? The only possible value of \( \alpha \) is 1 (Sorli, [7, conjecture 2]) so that Dris conjecture is true? Without ever forgetting the main question: do odd perfect numbers exist?

## 2 The proof

Referring to an odd perfect number \( n \) with the symbols used in equation (1), we obtain:

**Lemma 2.1.** If \( n \) is an odd perfect number, then

\[
N^2 = A \frac{\sigma(\pi^\alpha)}{2} \quad \text{and} \quad \sigma(N^2) = A\pi^\alpha.
\]

**Proof.** From equation (2) and from the fact that \((\sigma(\pi^\alpha), \pi^\alpha) = 1\), it follows

\[
N^2 = A \frac{\sigma(\pi^\alpha)}{2},
\]

where \( A \) is an odd positive integer given by

\[
A = \frac{\sigma(N^2)}{\pi^\alpha}.
\]

In relation to the odd parameter \( A \) in Lemma 2.1, we give two further lemmas:

**Lemma 2.2.** If \( A = 1 \), then \( \alpha \geq \omega(n) - 1 \) and \( N^2 > \frac{1}{2} \pi^\alpha \).

**Proof.** Let \( q_k, k = 1, 2, ..., \omega(N) = \omega(N^2) \), are the prime factors of \( N^2 \); from hypothesis and from (4) we have
\[ \pi^\alpha = \sigma(N^2) = \sigma(\prod_{k=1}^{\omega(N)} 2^{\beta_k}) = \prod_{k=1}^{\omega(N)} q_k^{2\beta_k} = \prod_{k=1}^{\omega(N)} \sigma(\delta_k) \]

in which \( \alpha = \sum_{k=1}^{\omega(N)} \delta_k \geq \sum_{k=1}^{\omega(N)} 1_k = \omega(N) \).

Since \( \omega(n) = \omega(N) + 1 \), it results in
\[ \alpha \geq \omega(n) - 1. \]

Besides, from Equation (3) it follows
\[ N^2 = \frac{1}{2} \sigma(\pi^\alpha) > \frac{1}{2} \pi^\alpha. \]

**Lemma 2.3.** If \( A > 1 \), then \( N^2 > \frac{3}{2} \pi^\alpha \).

**Proof.** From Equation (3) it results \( A \geq 3 \). Thus
\[ N^2 \geq \frac{3}{2} \sigma(\pi^\alpha) > \frac{3}{2} \pi^\alpha. \]

The following theorem summarizes a necessary condition for odd perfection.

**Theorem 2.1.** If \( n \) is an odd perfect number, then
\[ (\neg a \land d) \lor (a \land b \land c) \lor (b \land c \land d), \]
where: \( a \cong (A = 1), \neg a \cong (A > 1), b \cong (\alpha \geq \omega(n) - 1), c \cong (N^2 > \frac{1}{2} \pi^\alpha), d \cong (N^2 > \frac{3}{2} \pi^\alpha). \)

**Proof.** We combine Lemmas 2.2 and 2.3, setting
\[ \left\{ \begin{array}{l}
\text{lemma 2.2: } (a \implies b \land c) \\
\text{lemma 2.3: } (\neg a \implies d)
\end{array} \right. \]  \tag{5}

where, since it cannot be \( A < 1 \), it is \( (a) \cong (A = 1) \) and \( (\neg a) \cong (A > 1) \). One obtains from (5)
\[ (\neg a \lor (b \land c)) \land (a \lor d), \]
which is equivalent to
\[ (\neg a \land d) \lor (a \land b \land c) \lor (b \land c \land d). \]  \tag{6}

Considering cases in which the necessary condition for odd perfection (6) is false, we obtain the following corollaries:

**Corollary 2.1.** If \( n \) is an odd perfect number, then \( N^2 > \frac{1}{2} \pi^\alpha \).

**Proof.** We have
\[ (\neg c \land \neg d)(\cong N^2 < \frac{1}{2} \pi^\alpha) \implies n \text{ is not an odd perfect number}. \]  \tag{7}

From the contrapositive formulation of (7) it follows the proof.
Corollary 2.2. If \( n \) is an odd perfect number, then

\[
N^2 > \frac{3}{2} \pi^{\omega(n)-1} > \frac{1}{2} \pi^{\omega(n)-1}.
\]

Proof. We have

\[
(\neg b \land \neg d) \Leftarrow N^2 < \frac{3}{2} \pi^{\omega(n)-1} \implies n \text{ is not an odd perfect number.}
\]

From the contrapositive formulation of (8) it follows the proof.

Combining these two corollaries, we have

Corollary 2.3. If \( n \) is an odd perfect number, then

\[
N^2 > \frac{1}{2} \pi^\gamma, \text{ where } \gamma = \max\{\omega(n) - 1, \alpha\}.
\]

Proof. Immediate.

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References

[1] Brown, P. (2016) A partial proof of a conjecture of Dris, http://arxiv.org/abs/1602.01591v1.

[2] Chen, S. C., & Luo, H. (2011) Odd multiperfect numbers, http://arxiv.org/abs/1102.4396.

[3] Dickson, L. E. (2005) History of the Theory of Numbers, Vol. 1, Dover, New York.

[4] Dris, J. A. B. (2008), Solving the odd perfect number problem: some old and new approaches, M.Sc. thesis, De La Salle University, Manila, http://arxiv.org/abs/1204.1450.

[5] MacDaniel, W. L., & Hagis, P. (1975) Some results concerning the non-existence of odd perfect numbers of the form \( \pi^\alpha M^{2\beta} \), Fibonacci Quart., 131, 25–28.

[6] Nielsen, P. P. (2015) Odd perfect numbers, Diophantine equations, and upper bounds, Math. Comp., 84, 2549–2567.

[7] Sorli, R. M. (2003) Algorithms in the study of multiperfect and odd perfect numbers, Ph.D. thesis, University of Technology, Sydney, http://epress.lib.uts.edu.au/research/handle/10453/20034.
[8] Starni, P. (1991) On the Euler’s factor of an odd perfect number, *J. Number Theory*, 37, 366–369.

[9] Starni, P. (1993) Odd perfect numbers: a divisor related to the Euler’s factor, *J. Number Theory*, 44, 58–59.

[10] Starni, P. (2006) On some properties of the Euler’s factor of certain odd perfect numbers, *J. Number Theory*, 116, 483–486.