Parametric Resonance Phenomena in Bose-Einstein Condensates: 
Enhanced Quantum Tunneling of Coherent Matter Pulses

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Abstract

We investigate the quantum tunneling of a Bose-Einstein condensate confined in an optical trap. We show that periodic pulses of coherent matter are emitted from the trap by using an oscillating energy barrier. Moreover, the emitted fraction of condensed atoms strongly increases if the period of oscillation of the height of the energy barrier is in parametric resonance with the period of oscillation of the center of mass of the condensate inside the potential well. Our model is analyzed by numerically solving the nonpolynomial Schrödinger equation (NPSE), an effective one-dimensional equation which describes the macroscopic wavefunction of Bose condensates under transverse harmonic confinement. The range of validity of NPSE is discussed and compared with that of Gross-Pitaevskii equation.
I. INTRODUCTION

Nowadays dynamical properties and resonant phenomena of Bose-Einstein condensates are the subject of many experimental and theoretical investigations. Recently, the parametric excitation of cold trapped atoms in far-off-resonance optical lattices has been experimentally obtained by modulating the potential depth [1,2]. Another kind of parametric resonance, namely the parametric resonance between the collective oscillations of a trapped Bose-Einstein condensate and the oscillations of the confining harmonic trapping potential has been demonstrated with numerical simulations [3]. Moreover, by imposing particular values of anisotropy to the trapping harmonic potential, it has been shown that different collective modes of the condensate are in resonance and the oscillations can be chaotic [4,5].

In this paper we discuss a mechanism to generate coherent matter pulses by means of macroscopic quantum tunneling of a Bose condensate through the barrier of a potential well [6]. In our model the Bose-Einstein condensate is confined in the vertical axial direction by two Gaussian optical barriers and in the transverse direction by a magnetic or optical harmonic potential. The zero-temperature dynamics of Bose-Einstein condensate under transverse harmonic confinement is well described by the one-dimensional nonpolynomial Schrodinger equation (NPSE) we have recently derived from the three-dimensional Gross-Pitaevskii equation (3D GPE) [7,8]. In the first part of the paper we discuss the range of validity of the 3D GPE and that of the NPSE. In the second part of the paper by the numerical integration of NPSE we show that it is possible to generate periodic waves of coherent matter by periodically changing the height of the lower-lying Gaussian barrier. Moreover, the emission probability is strongly enhanced when the oscillation of the height of the energy barrier is in parametric resonance with the oscillation of the center of mass of the condensate inside the potential well.
II. CONDITIONS ON THE GROSS-PITAEVSKII EQUATION

The time-dependent three-dimensional Gross-Pitaevskii equation (3D GPE) [9], that describes the macroscopic wavefunction (or order parameter) $\psi(\mathbf{r}, t)$ of the Bose condensate, is given by

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + gN|\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t),$$

where $U(\mathbf{r})$ is the external trapping potential and $g = 4\pi\hbar^2a_s/m$ is the scattering amplitude and $a_s$ the s-wave scattering length. $N$ is the number of condensed bosons and the wavefunction is normalized to one. Note that the 3D GPE is a nonlinear Schrödinger (Hartree) equation and the nonlinear term is due to the interatomic interaction described by the two-body pseudo-potential $V(\mathbf{r}, \mathbf{r}') = g\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ [10].

The 3D GPE is accurate to describe a condensate of dilute gas of bosons only near zero temperature, where thermal excitations can be neglected [10]. The dilute gas condition is given by $a_s^3 \rho \ll 1$, where $a_s$ is the s-wave scattering length of the interatomic potential and $\rho$ is the density of the Bose gas. This diluteness condition means that the range ($a_s$) of the interatomic potential is much smaller than the average distance ($\rho^{-1/3}$) between atoms. By introducing the healing length

$$\xi = \frac{1}{\sqrt{8\pi a_s \rho}},$$

the dilute gas condition can be written as $a_s \ll \xi$. Note that the condensate can be dilute but strongly interacting if its interatomic energy ($4\pi\hbar^2a_s\rho/m$) is much larger than its kinetic energy. In the case of an axially symmetric harmonic confinement by using the healing length $\xi$ the dilute and strongly-interacting gas condition can be written as $a_s \ll \xi \ll a_H$, where $a_H = \sqrt{\hbar/m\omega_H}$ is the characteristic length of the harmonic confinement with $\omega_H = (\omega_\perp \omega_z)^{1/3}$.

Another interesting condition is the one-dimensional gas condition. One has the 1D regime when the potential energy of the transverse confinement is much larger than the interatomic energy of the Bose condensate. In the case of a transverse harmonic confinement,
again using the healing length $\xi$, the 1D gas condition for a Bose condensate can be written as $a_{\perp} \ll \xi$, where $a_{\perp} = \sqrt{\hbar / m\omega_{\perp}}$ is the characteristic length of the transverse harmonic confinement. It follows that, if the inequalities $a_s \ll a_H < a_{\perp} \ll \xi$ are satisfied, then the Bose condensate is dilute, cigar-shaped and one-dimensional.

It is important to observe that for $\xi \gg a_{1D}$, where $a_{1D}$ is the 1D scattering length given by $a_{1D} = a_{\perp}^2 / a_s$ [10], the Bose gas enters in the Tonks-Girardeau regime: a one-dimensional and very dilute gas of impenetrable Bosons for which the Bose-Einstein condensation is absent [11]. Thus in the Tonks-Girardeau regime the 3D GPE cannot be applied. In conclusion, the 3D GPE is valid if the healing length $\xi$ of the Bose condensate satisfies the condition

$$a_s \ll \xi < a_{\perp}^2 / a_s ,$$

and it describes a one-dimensional Bose condensate if $a_s \ll a_{\perp} \ll \xi < a_{\perp}^2 / a_s$.

III. NONPOLYNOMIAL SCHRÖDINGER EQUATION

In many experiments with Bose-Einstein condensates the external trapping potential can be described by a harmonic potential in the transverse direction and a generic potential in the axial direction:

$$U(r) = \frac{1}{2} m\omega_{\perp}^2 (x^2 + y^2) + V(z) .$$

This external potential suggests to map the 3D GPE into an effective 1D equation, which simplifies greatly the solution of the 3D GPE. This problem is not trivial due to the nonlinearity of the GPE.

The 3D GPE is the Euler-Lagrange equation of the following Lagrangian density

$$L = \psi^* (r, t) \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(r) - \frac{1}{2} gN |\psi(r, t)|^2 \right] \psi(r, t) .$$

For the wavefunction we choose the following variational ansatz

$$\psi(r, t) = \phi(x, y, t; \sigma(z, t)) f(z, t) ,$$
where both $\phi$ and $f$ are normalized to one and $\phi$ is represented by a Gaussian:

$$\phi(x, y, t; \sigma(z, t)) = \frac{e^{-\frac{x^2+y^2}{2\sigma(z,t)^2}}}{\pi^{1/2}\sigma(z,t)}.$$  (7)

Moreover we assume that the transverse wavefunction $\phi$ is slowly varying along the axial direction with respect to the transverse direction, i.e. $\nabla^2 \phi \simeq \nabla^2_\perp \phi$ where $\nabla^2_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

By inserting the trial wave-function in the Lagrangian density and after spatial integration along $x$ and $y$ variables the Lagrangian density becomes

$$L = f^* \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} - V - \frac{1}{2} gN \frac{\sigma^{-2}}{2\pi} |f|^2 - \frac{\hbar^2}{2m} \frac{\sigma^{-2}}{2} - \frac{m\omega_\perp^2}{2} \right] f.$$  (8)

The Euler-Lagrange equations with respect to $f^*$ and $\sigma$ read

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + gN \frac{\sigma^{-2}}{2\pi} |f|^2 + \left( \frac{\hbar^2}{2m} \frac{\sigma^{-2}}{2} + \frac{m\omega_\perp^2}{2} \right) \right] f,$$  (9)

$$\frac{\hbar^2}{2m} \sigma^{-3} - \frac{1}{2} m\omega_\perp^2 \sigma + \frac{1}{2} gN \frac{\sigma^{-3}}{2\pi} |f|^2 = 0.$$  (10)

The second Euler-Lagrange equation reduces to an algebraic relation providing a one to one correspondence between $\sigma$ and $f$: $\sigma^2 = a_\perp^2 \sqrt{1 + 2a_s N |f|^2}$, where $a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}$ is the oscillator length in the transverse direction. One sees that $\sigma$ depends implicitly on $z$ and $t$ because of the space and time dependence of $|f|^2$. Inserting this result in the first equation one finally obtains

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + gN \frac{\sigma^{-2}}{2\pi} \frac{|f|^2}{\sqrt{1 + 2a_s N |f|^2}} \right.$$

$$+ \left. \frac{\hbar\omega_\perp}{2} \left( \frac{1}{\sqrt{1 + 2a_s N |f|^2}} + \sqrt{1 + 2a_s N |f|^2} \right) \right] f.$$  (11)

This equation is a time-dependent non-polynomial Schrödinger equation (NPSE).

We observe that under the condition $a_s N |f|^2 \ll 1$ one has $\sigma^2 = a_\perp^2$ and NPSE reduces to

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + \frac{gN}{2\pi a_\perp^2} |f|^2 \right] f,$$  (12)
where the additive constant $\hbar \omega_\perp$ has been omitted because it does not affect the dynamics. This equation is a 1D Gross-Pitaevskii equation (1D GPE). The nonlinear coefficient $g'$ of this 1D GPE can be thus obtained from the nonlinear coefficient $g$ of the 3D GPE by setting $g' = g/(2\pi a_\perp^2)$ such that $V(z, z') = g'\delta(z - z') = (2\hbar^2/ma_{1D})\delta(z - z')$ is the effective 1D interatomic pseudo-potential and the effective 1D scattering length is $a_{1D} = a_\perp^2/a_s$, in agreement with the definition of the previous section. Note that the limit $a_s N |f|^2 \ll 1$ is precisely the condition for the one-dimensional regime where the healing length $\xi$ is larger than $a_\perp$. But, as shown in the previous section, the 3D GPE and consequently the NPSE are no more valid if $\xi \gg a_\perp^2/a_s$. It follows that the 1D GPE is valid only where the condition $a_s^2/a_\perp^2 < a_s N |f|^2 \ll 1$ is satisfied.

In papers [9,10] we have tested the accuracy of the NPSE in the determination of the ground-state and collective oscillations of the condensate with axial harmonic confinement and also in the description of tunneling through a Gaussian barrier. In particular, we have compared NPSE with the the full 3D GPE. The conclusion of these investigations is that NPSE is very accurate in the description of cigar-shaped condensates ($a_z/a_\perp \geq \sqrt{10}$) both in the 3D regime and in the 1D regime ($a_s \ll \xi < a_\perp^2/a_s$).

Note that the NPSE has been recently used by Massignan and Modugno to study the dynamics of a Bose condensate in a optical lattice [12].

**IV. PERIODIC EMISSION OF MATTER WAVES**

In this section we use the NPSE to study a mechanism which produces periodic emission of matter waves by means of the quantum tunneling of a Bose condensate. We consider a cigar-shaped condensate confined by a harmonic potential in the transverse direction and under the action of the gravity potential $mgz$ and two Gaussian functions that model a confining potential well in the vertical axial direction:

$$V(z) = V_1 e^{-(z-z_1)^2/\sigma^2} + V_2 e^{-(z-z_2)^2/\sigma^2} + mgz.$$  (13)
Such a configuration can be experimentally obtained by using two blue-detuned laser beams (perpendicular to the axial direction) which are modelled by the two Gaussian potentials. By varying the intensity of the lower-lying laser beam one controls the height of the energy barrier and therefore the tunneling probability. The periodic height given by:

$$V_1(t) = V_a + V_b \sin \left( \frac{2\pi}{\tau} t \right), \quad (14)$$

where $\tau$ is the period of oscillation.

The NPSE is integrated by using a finite-difference predictor-corrector method [13,14,15]. The initial wave function of the condensate is found by solving the equation with imaginary time. In Figure 1 we plot the axial density profile $\rho(z)$ of the matter-waves coming out from the potential well with $\tau = 1.5$. The figure shows that, as expected, the emission of pulses is periodic. Moreover, both the period of emission and the tunneling fraction $P_T$ depend on $\tau$. In the inset of Figure 2 the tunneling probability $P_T$ is shown as a function of $\tau$ for a Bose condensate with $N = 10^3$ and $N = 10^4$ atoms. The emission probability has its absolute maximum near $\tau = 0.8$, that is the average period of the oscillations of the center of mass of the condensate inside the potential well.

The enhancement of the emission probability of coherent matter near $\tau = 0.8$ has a classical explanation. We have verified [6] that it corresponds to the condition of resonance between the oscillation of the energy barrier and the oscillation of the center of mass of the condensate. Close to the resonance condition the phenomenon of coherent emission of atoms is no more due to quantum tunneling but to the parametric resonance [16] between the period of oscillation of the center of mass of the condensate and the period of oscillation of the energy barrier. This might explain the different $N$ dependence of $P_T$ and $P_T^0$ as shown in Figure 2: $P_T^0$ is dominated by quantum effects and, as such, is enhanced by interaction, while $P_T$ is essentially a classical effect and is almost $N$ independent, in agreement with the numerical data shown in Figure 2.

Note that in Figure 2 there are other local maxima whose positions follow the text-book [16] resonant condition formula $\tau = (n/2) \tau_{osc}$, where $\tau_{osc}$ is the period of oscillation of the
center of mass in the absence of an oscillating barrier, with \( n \) an integer number.

**CONCLUSIONS**

In this paper we have studied the controlled emission of coherent matter pulses from a trap by changing the period of oscillation of the energy barrier. The experimental investigation of the physical configuration we have considered in this paper may contribute to the realization of novel phenomena in atom lasers. Finally, we observe that the parametric driving of Bose-Einstein condensates can be obtained by current experiments using optical dipole forces with far-detuned laser beams. By varying the intensity of the laser beams one can control the height of the energy barrier that confines the condensed sample.

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FIG. 1. Axial density profile of Bose condensed $^{23}\text{Na}$ atoms tunneling through the potential well (Eq. (13)) with oscillating barrier (Eq. (14)): $V_a = 300$ and $V_b = 200$. Results obtained by solving NPSE. Period of oscillation: $\tau = 1.5$. Effective chemical potential of the initial condensate: $\mu_{\text{eff}} = 57.43$. Number of $^{23}\text{Na}$ atoms: $N = 10^4$. Scattering length: $a_s = 30$ Å. Note that $P_T = 0.26$ with $V_1(t) = V_a - V_b = 100$. Length $z$ in units $a_z = (\hbar/m\omega_z)$, where $\omega_z = \omega_\perp/10$ with $\omega_\perp = 2\pi$ kHz; energy in units $\hbar \omega_z$ and time in units $\omega_z^{-1}$. 
FIG. 2. Tunneling ratio $P_T/P_T^0$ at $t = 5$ as a function of the period $\tau$ of oscillation of the lower-lying Gaussian barrier (Eq. (13,14)), as obtained by solving NPSE, with $V_a = 300$ and $V_b = 200$ (Eq. (6)). The period $\tau_{osc}$ of small oscillations around the minimum of the potential well with $V_a = 300$ and $V_b = 0$ is $\tau_{osc} = 0.84$. $P_T^0$ is the tunneling probability with $V_1 = V_a - V_b = 100$. Effective chemical potential of the initial condensate: $\mu_{eff} = 27.15$ for $N = 10^3$, $\mu_{eff} = 57.43$ for $N = 10^4$. $N$ is the number of $^{23}$Na atoms. Scattering length: $a_s = 30 \text{ Å}$. Inset: tunneling probability $P_T$ at $t=5$ as a function of $\tau$. Units as in Fig. 1.