Topological phase transition in complex networks

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Preferential attachment is a central paradigm in the theory of complex networks. In this contribution we consider various generalizations of preferential attachment including for example node removal and edge rewiring. We demonstrate that generalized preferential attachment networks can undergo a topological phase transition. This transition separates networks having a power-law tail degree distribution from those with an exponential tail. The appearance of the phase transition is closely related to the breakdown of the continuous variable description of the network dynamics.
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Preferential attachment is a central paradigm in the theory of complex networks. In this contribution we consider various generalizations of preferential attachment including for example node removal and edge rewiring. We demonstrate that generalized preferential attachment networks can undergo a topological phase transition. This transition separates networks having a power-law tail degree distribution from those with an exponential tail. The appearance of the phase transition is closely related to the breakdown of the continuous variable description of the network dynamics.

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Complex networks are found in nature, socio-economic systems, technical infrastructures, and countless other fields. The macroscopic properties of such networks emerge from the microscopic interaction of many individual constituents. Various models of complex networks have been proposed and statistical mechanics of complex networks has become an established branch of statistical physics. Since complex networks are non-equilibrium systems they do not have to obey detailed balance and may show many fascinating features not found in equilibrium systems.

In order to explain the power-law degree distribution of many complex networks Barabási and Albert [3] introduced a model that combines network growth and preferential attachment. Preferential attachment means that new nodes enter a network by attaching preferably to nodes having a high degree. The probability that a new node establishes an edge to an old node \( v \) with degree \( k_v \) is determined by a node attractiveness function \( A_k \); it is given by \( p_{\text{add}}(v) = A_{k_v}/\sum A_{k_u} \). In the original Barabási-Albert model [3] the attractiveness of a node is strictly-proportional to its degree, viz. \( A_k = k \).

The linear preferential attachment function of the Barabási-Albert model was introduced as an ad hoc ansatz without any fundamental justification and many generalizations are conceivable, including attractiveness functions with small perturbations,

\[
A_k = k + \varepsilon(k, k^*) \quad \text{with} \quad \varepsilon(k, k^*) \in \mathcal{O}(1), \quad (1)
\]

random node removal, rewiring, etc. Here we will consider generalized preferential attachment models that include tunable parameters that allow to adjust the relative importance of node removal, rewiring, etc. These parameters define the phase space of a model.

We will show that generalizations of the Barabási-Albert model as considered here can dramatically affect the degree distribution; it falls into one of two possible universality classes. At each point of the phase space the universality class is uniquely determined by a parameter \( \beta \). For \( \beta > 0 \) the tail of the degree distribution is given by a power-law, while if \( \beta < 0 \) it is an exponential. The two universality classes are separated by a topological phase transition. The behavior at the phase transition (\( \beta = 0 \)) depends on the specific model.

We claim that our results are valid for a large class of generalized preferential attachment models, but let us consider two specific examples for illustrative purposes.

**Model I** At each time step with probability \( \varepsilon \) a randomly chosen node is removed from the network and with probability \( 1 - \varepsilon \) a new node is added to the network by establishing \( m \) links to \( m \) old nodes. These nodes are chosen randomly by preferential attachment with the attractiveness function \( A_k \). The perturbation function may be any bounded function \( \varepsilon(k, k^*) \in \mathcal{O}(1) \) such that \( A_k \geq 0 \), e.g.,

\[
\varepsilon(k, k^*) = \begin{cases} k^*/m & \text{if } k \leq m \\ k^* & \text{else.} \end{cases} \quad (2)
\]

For \( k \geq m \) function (2) implements a constant shift of size \( k^* \). The network starts to grow at time \( t = 0 \) with \( N_0 \) nodes. We are interested in behaviour where the number of nodes is much larger than unity. For \( \varepsilon = 1 \) the number of nodes is conserved so \( N_0 \gg 1 \), but for \( \varepsilon < 1 \) the average number of nodes grows with time and one may start with \( N_0 = \mathcal{O}(1) \).

**Model II** We start from a small initially given network. At each time step one of three different operations is performed with probability \( p \), \( q \), or \( r \), respectively, such that \( p + q + r = 1 \).

**Edge insertion (probability \( p \))** There are \( m \) new edges inserted. For each edge one end \( u \) is chosen randomly from the set of nodes without any preference, while the other end \( v \) is selected preferentially with probability proportional to \( A_{k_u} = k_u + 1 \).

**Rewiring (probability \( q \))** \( m \) links are rewired by selecting a node \( u \) (with non-zero degree) and one of its neighbors \( v \). The edge between \( u \) and \( v \) is removed and a new edge from \( u \) to \( v' \) is added, where \( v' \) is chosen with probability proportional to \( A_{k_{v'}} = k_{v'} + 1 \).

**Node addition (probability \( r \))** A new node is added to the network by establishing \( m \) links to \( m \) old nodes \( v \) which are chosen with probability proportional to \( A_{k_v} = k_v + 1 \).

Let \( p_k \) denote the degree distribution of a generalized preferential attachment model, i.e., the relative number of nodes
having degree $k$ averaged over many network instances. This distribution can be calculated by solving a master equation with the boundary values $p_{k-1} = 0$ and $p_{k_{\text{max}}+1} = 0$, where $k_{\text{max}}$ denotes the maximal possible degree in the network. The thermodynamic limit corresponds to $k_{\text{max}} \to \infty$. For model I the master equation reads

$$\delta_{k,m} = -m A_{k-1} A p_{k-1} + \left( m A_k A + rk + 1 \right) p_k - r(k+1)p_{k+1},$$

which contains the mean attractiveness

$$A = \sum_k A_k p_k.$$  

The simultaneous numerical solution of equations (3) and (4) gives either a degree distribution with a power-law tail or an exponential tail, as shown in Figure 1. For model II we can write down a similar master equation and its solution obeys either a power-law or an exponential tail, too, depending on the parameters $p, q, r,$ and $m.$

Analysis of generalized preferential network models in terms of a continuous variable approximation [7] sheds light on the appearance of the phase transition. We will outline the generic mechanism that leads to the topological phase transition first and we will consider specific models later on.

In a continuous variable approximation degrees are treated as continuous variables and random quantities are represented by their means. The degree distribution $p_k$ is given by

$$p_k = \frac{D(i_k,t)}{N(t)} \left[ \beta k(i,t) \right]^{-1}.$$  \ (5a)

Here $N(t)$ denotes the size of the network at time $t$. Node addition and removal lead to an effective network model-dependent growth rate $\alpha$, i.e., $N(t) = \alpha t + N_0$. The quantity $D(i,t)$ takes into account random node removal at rate $p$. It is the probability that a node that entered the network at time $i$ is still present at time $t \geq i$ and is given by the initial condition $D(i,i) = 1$ and the equation

$$\frac{\partial D(i,t)}{\partial t} = -p D(i,t) \frac{N(t)}{N(t)}.$$  \ (6a)

The function $k(i,t)$ denotes the mean degree of a node at time $t \geq i$ that entered the network at time $i$ and has not yet disappeared from the network. The time $i_k \leq t$ is given by the solution of the equation $k(i_k,t) = k$. The quantity $|\partial k(i,t)/\partial t|_{i=i_k}$ can be interpreted as the number of nodes with degree $k$ at time $t$ (neglecting node deletion).

For generalized preferential attachment networks the evolution of $k(i,t)$ is given by an equation which has the generic form

$$\frac{\partial k(i,t)}{\partial t} = \beta \frac{k(i,t)}{N(t)} + f[k(i,t)] \frac{N(t)}{N(t)}.$$  \ (7a)

The term $\beta k(i,t)/N(t)$ is the hallmark of preferential attachment [5], while $f[k(i,t)]/N(t)$ with $f(x) \in O(1)$ appears as a consequence of perturbations of the strictly linear attractiveness function, edge rewiring, node deletion, or other processes.

Before analyzing the equations (5a), (6a), and (7a) in detail it is useful to introduce a new time scale $t'$. For $\alpha > 0$ we set $t = \exp(\alpha t')$ and in the case $\alpha = 0$ we set $t = t' N_0$. With this new time scale the number of node additions per unit-time is proportional to the network size. Now equations (5a), (6a), and (7a) read (omitting primes)

$$p_k = \frac{D(i_k,t)}{\exp(\alpha(t-i_k))} \left[ \beta k(i,t) \right]^{-1}.$$  \ (5b)

$$\frac{\partial D(i,t)}{\partial t} = -p D(i,t),$$  \ (6b)

$$\frac{\partial k(i,t)}{\partial t} = \beta k(i,t) + f[k(i,t)].$$  \ (7b)

The explicit dependence on the network size has dropped out of equations (5b), (6b), and (7b) and we need not to distinguish between growing ($\alpha > 0$) and non-growing ($\alpha = 0$) networks.
The power-law exponent (11) is determined entirely by three second term corresponds to random node deletion. After solution of (7b) is given by

\[ k(i, t) \sim \exp[\beta(t - i)] \]  

and there follows

\[ i_k = t - \ln \frac{k}{\beta}. \]  

Plugging (8), (9), and the solution of (6b),

\[ D(i, t) = \exp[-\rho(t - i)], \]  

into (5) we find that the combination of exponentials with different exponents (9) yields a power-law degree distribution

\[ p_k \sim k^{-\gamma} \]  

with exponent

\[ \gamma = \frac{\alpha + \beta}{\beta} + 1. \]  

The power-law exponent (11) is determined entirely by three rates, the network growth rate \( \alpha \), the node degree growth rate \( \beta \), and the node deletion rate \( \rho \). It diverges as \( \beta \to 0 \).

It turns out that the exponential growth of a node’s degree (8) is the key ingredient for the emergence of a power-law degree distribution. For \( \beta = 0 \) this ingredient is lost and the phase transition appears. The equation \( \beta = 0 \) defines the critical phase line that separates the two universality classes. At the phase transition term I in equation (7b) vanishes and as a consequence the qualitative behavior of the solution of (7b) is determined by term II and not universal. Because \( f(x) \in \mathcal{O}(1) \) function \( k(i, t) \) must grow sub-exponentially. The solution of (7b) is given by the implicit equation

\[ \int_m^{k(i, t)} \frac{1}{f(k')} \, dk' = t - i. \]  

For \( \beta < 0 \) term I dominates equation (7b) again. But now the continuous variable approximation breaks down. In fact, the time \( t \) becomes greater than \( t \); that means it is in the future. The continuous variable approximation is not able to predict the correct degree distribution for \( \beta < 0 \).

Let us see how our generic continuous variable analysis translates to specific models. In model I in each time step a node is added to the network and with probability \( r \) a node is removed from the network, as a consequence its size grows at rate \( \alpha = (1 - r) \). The function \( k(i, t) \) is given by

\[ \frac{dk(i, t)}{dt} = m - k(i, t) + e[k(i, t), k^*] \frac{k(i, t)}{AN(t)} - \frac{k(i, t)}{N(t)}. \]  

The first term takes into account node addition via preferential attachment with attractiveness function (1) and the second term corresponds to random node deletion. After reordering equation (13) and comparing with (10) we find

\[ \beta = \frac{m}{(rA) - 1} \]  

for model I networks with perturbation function (2) and \( m = 3 \). Symbols indicate locations in the phase space for data shown in Figure 3. Direct simulations of the network dynamics show that in the universality class of power-law degree distributions \( k(i, t) \) grows indeed exponentially, as predicted by (8), while in the exponent.
with exponential class \(k(i,t)\) grows sub-exponentially, see Figure\textsuperscript{4}. Moreover, our findings for model I put some known results for specific variations of the Barabási-Albert model into a unifying framework. For a vanishing node deletion rate \(r = 0\) and \(\varepsilon(k, k') = k'\) the mean attractiveness equals \(A = 2m + k'\) and for any \(k'\) the degree distribution has a power-law tail with \(\alpha = 3 + k'/m\). In the special case \(\varepsilon(k, k') = 0\) with node deletion rate \(0 < r < 1\) the mean attractiveness \(A\) equals the mean degree of the network, which is given by \(2m/(1 + r)\). It follows that the degree distribution is in the power-law class with \(\alpha = (3 - r)/(1 - r)\).

Model II can be analyzed in a similar way. In this model the network size grows at rate \(\alpha = r = 1 - p - q\), and the evolution equation for \(k(i,t)\) is given by

\[
\frac{\partial k(i,t)}{\partial t} = \frac{mp(k(i,t) + 1)}{N(t)(k + 1)} - \frac{mqk(i,t)}{N(t)k} + \frac{mq[k(i,t) + 1]}{N(t)(k + 1)} + \frac{mr[k(i,t) + 1]}{N(t)(k + 1)},
\]

where \(\bar{k} = 2m(1 - q)/r\) denotes the mean degree of the network. After some rearranging of equation\textsuperscript{14} and comparing with \textsuperscript{18} we identify

\[
\beta = \frac{1}{2} \frac{(1 - p - q)[-(2m + 1)q^2 + (4m + 1 - p)q - 2m]}{(1 - q)[(2m + 1)q - (2m + 1) + p]}. \tag{14}
\]

So, for \(\beta > 0\) the degree distribution will follow a power-law with exponent

\[
\gamma = 2\frac{(1 - q)[(2m + 1)q - (2m + 1) + p]}{-(2m + 1)q^2 + (4m + 1 - p)q - 2m} + 1
\]

and the critical phase line is determined by the equation \((2m + 1)q^2 - (4m + 1 - p)q + 2m = 0\), as shown in Figure\textsuperscript{4}.

To summarize, we have investigated a broad class of generalized preferential attachment models and shown that small perturbations of the attractiveness function \(\varepsilon\), node rewiring, or edge rewiring cannot only change the power-law exponent of the original Barabási-Albert model \(\gamma\). These variations can even lead to a qualitative change of the degree distribution from a power-law to an exponential.

It had been argued \(\gamma\) “that net growth is one of the fundamental requirements for the generation of power-law degree distributions” of preferential attachment models. However, we have shown, that it is not the growth of the network itself that leads to a power-law degree distribution. The mean degree of individual nodes has to grow sufficiently strongly, i.e. \(\beta > 0\), to generate a power-law degree distribution. We also demonstrated that negative perturbations of the linear attractiveness function as in model I have a tendency to stabilize power-laws. In the light of this fact and the ubiquity of power-laws in nature it would be interesting to know if in real world networks such negative perturbations do exist.

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