A Two-loop Test of Buscher’s T-duality I

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Abstract

We study the two loop quantum equivalence of sigma models related by Buscher’s T-duality transformation. The computation of the two loop perturbative free energy density is performed in the case of a certain deformation of the $SU(2)$ principal sigma model, and its T-dual, using dimensional regularization and the geometric sigma model perturbation theory. We obtain agreement between the free energy density expressions of the two models.

PACS codes: 02.40-k, 03.50.Kk, 03.70, 11.10.L, 11.10.Kk
key words: sigma models, duality, quantum corrections

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1 Introduction

Among the wealth of different dualities relating the perturbative string theories and M-theory, the most ‘ancient’ one is T-duality. At the same time T-duality was the starting point in the discovery of D-branes \[1\]. In string theory T-duality can be proven in arbitrary order of the string perturbation theory \[2, 3\], as long as the vacuum preserves conformal invariance \[4\]. Generalizing the \(R \rightarrow \frac{1}{R}\) symmetry of the toroidally compactified strings, Buscher \[5\] gave a set of equations that describe the transformation of the Neveu-Schwarz background fields of the sigma model action, and works whenever there is an isometry. Denoting by \(g_{\mu\nu}\) and \(b_{\mu\nu}\) the background metric and the antisymmetric tensor field, the explicit form of the Buscher transformation is (since we work on a flat world sheet, we neglect the dilaton):

\[
\tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0\alpha} = \frac{b_{0\alpha}}{g_{00}}, \quad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}},
\]

\[
\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta} - b_{0\alpha} b_{0\beta}}{g_{00}}, \quad \tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \frac{g_{0\alpha} b_{0\beta} - g_{0\beta} b_{0\alpha}}{g_{00}},
\]

(1)

where \(\tilde{g}_{\mu\nu}\) and \(\tilde{b}_{\mu\nu}\) denote the new background fields.

To derive this transformation Buscher used functional integral arguments, that in the meantime have become widely known, and applied in many context (see e.g. \[6\]). The idea is to gauge the aforementioned isometry of the sigma model action and impose a constraint using a Lagrange multiplier. Integrating the multiplier one recovers the original theory to start with, while after a gauge fixing and integrating over one of the original fields gives the dual theory.

It was also shown \[7\], that, at the classical level, the duality transformation rule can be recovered in an elegant way by performing a canonical transformation \[8\]. This clearly shows that the models connected by the Buscher transformations are equivalent classically. At the quantum level the only loophole of the path integral argument mentioned above is that it neglects the effect of the regularization and renormalization. Though it was shown in \[4\] that for conformal invariant models one has full quantum equivalence, in other words the path integral argument holds, this is not the case in the non-conformal setting. It was shown in \[4\] for a deformed \(SU(2)\) principal sigma model, and further clarified in \[10\] for several different deformed \(SU(3)\) principal sigma models, that Buscher’s formula – as applied to renormalized quantities – has to be modified to give two loop quantum equivalence.

Later it was shown that the one loop beta functions of the original and dual models always agree \[11\]. Work has been done to establish the corrections to Buscher’s formulae \[12, 13, 14\]. Advances were made from a different point of view. It is widely known that the low energy degrees of freedom of the sigma model can be described using an effective action, that contains gravity in target space. This fact constrains the possible leading terms of the low energy effective action to a computable form, that is known. It was shown \[15\] that the low energy effective action consistent with the two loop sigma model beta function equations is not invariant under Buscher’s transformation. The leading part of the above action, the one loop part, nevertheless is invariant,
in accordance with the one loop findings of [1]. Using the non-invariance of
the two loop action it was found how to modify Buscher's transformation, with
order $\alpha'$ terms, such that the two loop action would remain invariant under
the modified transformations. What is not clear after all is how to pull back the
modification of the Buscher's formulae found in the low energy action to the
sigma model. Related work in the supersymmetric case, concentrating on the
absence of the above mentioned corrections, was done in [18, 19].

Any satisfactory criterion of the quantum equivalence among dually related
sigma models should be based on the comparison of physical quantities as opposed
to just considering beta functions. If there are global symmetries in the
model then their associated conserved quantities (Noether currents) may be
considered physical. The definition of physical quantities, however, is not very
clear in diffeomorphic invariant sigma models without a sufficient number of
isometries. To circumvent this problem the study of Weyl anomaly coefficients
was suggested in [13]. In the present paper – as an alternative – we study a
thermodynamic quantity namely the free energy density in the presence of a
chemical potential in the dually related sigma models. This quantity surely
qualifies as physical, thus its equality in the two models gives a non trivial
check on their quantum equivalence. Furthermore the free energy density can
be computed perturbatively – at least in asymptotically free models, thus one
can compare the two free energy densities using the first few orders of pertur-
bation theory. The aim of this paper is to carry out this comparison in the two
loop order, where the first really 'quantum' effects appear, thus improving the
almost 'classical' one loop case studied earlier in [16], [17].

The paper is organized as follows: in Section 2 we give a brief review of the
pertinent facts that we need from the renormalization of the $SU(2)$ principal
$\sigma$ model and its dual and develop a Lagrangian with more parameters, that
can accommodate both Lagrangians as special cases. In Section 3 we investi-
gate the conserved charges that can be coupled to the models, calculate the
Hamiltonian, find the ground states in the presence of the external field, and
make the Lagrangians suitable for a perturbative computation. Section 4 deals
with the definition of the perturbative free energy density, and its computation,
using dimensional regularization. First we define the perturbative free energy
density, and set up a scheme to compute it systematically to any order. The
rest of Section 4 deals with the actual computation of the bare free energy up
to two loops. In Section 5 we deal with the issue of renormalization and obtain
the one and two loop renormalized free energies of the original and dual models,
compare them, and improve them using the renormalization group. We close
Section 5 with the analysis of the composite operator renormalization of the
relevant operators. We make our conclusions in Section 6.

2 Lagrangians and T-duality

2.1 The deformed $SU(2)$ principal $\sigma$ model

This section has a twofold role. Primarily it is intended to give an overview of
the results that we need in the rest of the paper and at the same time fix the
notations. Secondarily it extends some of the earlier results in a way suitable for our applications.

In [9] the following one parameter deformation of the \( SU(2) \) principal \( \sigma \)-model was considered:

\[
\mathcal{L}_O = -\frac{1}{2\lambda} \left( \sum_{a=1}^{3} J^a_{\mu} J^{\mu a} + g J^3_{\mu} J^{\mu 3} \right),
\]

(2)

where \( J_{\mu} = G^{-1} \partial_{\mu} G = J^a_{\mu} \tau^a \), and \( \tau^a = \sigma^a / 2 \) with \( \sigma^a \) being the standard Pauli matrices. Thus \( G \) is an element of \( SU(2) \) and \( g \) is the parameter of the deformation. From the Lagrangian (2) it is clear that the global \( SU(2)_L \times SU(2)_R \) symmetry of the undeformed principal \( \sigma \)-model is broken to \( SU(2)_L \times U(1)_R \) by the \( J^3_{\mu} J^{\mu 3} \) term. Setting \( g = 0 \) corresponds to the principal \( \sigma \)-model, while for \( g = -1 \) the \( O(3) \) \( \sigma \)-model is obtained as it can be seen from eq. (4) below.

The authors of [9] investigated the renormalization of \( \lambda \) and \( g \) in the two-loop order of perturbation theory, treating \( \lambda \) as the coupling constant and \( g \) as a parameter. Using the Euler angles \( (\phi, \theta, \psi) \) to parameterize the elements of \( SU(2) \), \( G \) is written as

\[
G = e^{i\phi \tau^3} e^{i\theta \tau^1} e^{i\psi \tau^3}.
\]

(3)

Using this converts the Lagrangian of the deformed \( \sigma \) model, which for the time being we shall call 'the original model', into the following form:

\[
\mathcal{L}_O = \frac{1}{2\lambda} \left\{ (\partial_{\mu} \phi)^2 + (\partial_{\mu} \psi)^2 (1 + g \cos^2 \theta) + (1 + g)(\partial_{\mu} \psi)^2 + 2(1 + g)\partial_{\mu} \phi \partial^{\mu} \psi \cos \theta \right\}.
\]

(4)

Using the Killing vectors of the \( SU(2)_L \times U(1)_R \) symmetry and exploiting the manifest target space covariance of the background field method it was proved in [9] that the model is renormalizable in the ordinary sense: there is no wave function renormalization for \( \theta, \phi \) and \( \psi \), while the coupling constant and the parameter got renormalized according to:

\[
\lambda_0 = \mu^{-\epsilon} Z_{\lambda}(\lambda, g) \lambda, \quad g_0 = Z_{g}(\lambda, g) g.
\]

(5)

Both in the one and in the two loop orders the residues of the single poles in \( Z_{\lambda}(\lambda, g) = 1 - y_{\lambda}(\lambda, g) / \epsilon + \ldots \) and \( Z_{g}(\lambda, g) = 1 - y_{g}(\lambda, g) / \epsilon + \ldots \) were determined:

\[
y_{\lambda} = -\frac{\lambda}{4\pi} \left( 1 - g + \frac{\lambda}{16\pi} (1 - 2g + 5g^2) \right),
\]

\[
y_{g} = \frac{\lambda}{2\pi} (1 + g) \left( 1 + \frac{\lambda}{8\pi} (1 - g) \right).
\]

(6)

Note the sign difference between our formulas (5) and (6), and the corresponding ones in [9]. It is consequence of the fact that in our notation notation \( n = 2 + \epsilon \) rather than \( n = 2 - \epsilon \) as used in [9].
The standard definition of the $\beta$ functions: $\beta_\alpha = \mu \frac{d\alpha}{d\mu}$, $\beta_\gamma = \mu \frac{d\gamma}{d\mu}$, lead to the following two-loop $\beta$ functions (eq. (20) in [9]):

$$\begin{align*}
\beta_\lambda &= -\frac{\lambda^2}{4\pi}(1 - g + \frac{\lambda}{8\pi}(1 - 2g + 5g^2)), \\
\beta_g &= \frac{\lambda}{2\pi}g(1 + g)(1 + \frac{\lambda}{4\pi}(1 - g)).
\end{align*}$$ (7)

It is easy to see, that the $g = 0$ resp. the $g = -1$ lines are fixed lines under the renormalization group flow, and $\beta_\lambda$ reduces to the $\beta$ function of the principal $\sigma$-model, resp. of the $O(3)$ $\sigma$-model on them. In the $(\lambda, g) \geq 0, g < 0$ quarter of the $(\lambda, g)$ plane the renormalization group trajectories run into $\lambda = 0, g = -1$; while for $g > 0$ they run to infinity. This implies that the $g = 0$ fixed line corresponding to the principal $\sigma$-model is 'unstable' under the deformation.

The Lagrangian of the deformed $\sigma$-model, eq. (4), exhibits two obvious Abelian isometries that can be used to construct two different (Abelian) duals: namely the translations in the $\phi$ and $\psi$ fields. We call the models obtained this way the 'φ dual' respectively the 'ψ dual' of the deformed $\sigma$ model (4).

In [9] it was found that for the 'ψ dual' model, as summarized below, the renormalization of the coupling and the parameter are equivalent to that of the original model. Therefore, in the present context we deal with the original, deformed $SU(2)$ principal $\sigma$-model and it's 'ψ dual', which we shall simply call 'the dual model'.

2.2 The 'ψ dual' model

For the Lagrangian of the 'ψ-dual' model, using Buscher's formulae [5, 9], one has an expression analogous to eq. (4):

$$L_D = \frac{1}{2\lambda}((\partial_\mu \theta)^2 + (\partial_\mu \phi)^2 \sin^2 \theta + (\partial_\mu \chi)^2 + 2a \cos \theta \epsilon^{\mu\nu} \partial_\mu \chi \partial_\nu \phi).$$ (8)

Here $\chi$ denotes the (appropriately scaled) variable dual to $\psi$, and $(\tilde{\lambda}, \tilde{g})$ stands for the couplings of the dual model. One can show that $L_D$ exhibits the expected $SU(2) \times U(1)$ symmetry for all values of the parameter $a$.

The couplings of the original, (4), and of the dual models, (8), are related (at the classical level) as a direct consequence of T-duality as

$$\tilde{\lambda} = \lambda, \quad a = \sqrt{1 + \tilde{g}}, \quad \tilde{g} = g.$$ (9)

These relations can be maintained at the two-loop level if one performs the renormalization and, in addition, in both theories the couplings $(g, \tilde{g})$ are expressed in terms of the corresponding renormalization group invariant quantities, which at the end are set equal to each other [9]. In addition one also has to take into account the freedom in the choice of the renormalization group invariant, due to the scheme dependence of the two-loop $\beta$ function.

The two-loop renormalization invariants, that characterize the flows under the corresponding sets of $\beta$ functions, for the original and dual models can be
easily computed:
\[
M_O = -\frac{g}{(1+g)^2}\lambda^2 - \frac{1}{4\pi}\frac{g}{1+g}\lambda^3 + o(\lambda^4),
\]
\[
M_D = -\frac{a^2 - 1}{a^4}\tilde{\lambda}^2 - \frac{1}{4\pi}\frac{1}{a^2}\tilde{\lambda}^3 + o(\tilde{\lambda}^4).
\] (10)

The next step is to express \(g\) (resp. \(a\)) in terms of the renormalization group invariant. This can be done by inverting eq. (10) perturbatively. In both cases at leading order one has to solve quadratic equations and the next to leading order terms correct the results including the two-loop effects. Assuming that neither \(M_O\) nor \(M_D\) vanishes, one obtains (the leading terms were obtained in [9], but we shall need the whole expression):
\[
g(\lambda, M_O) = -1 \pm \frac{1}{\sqrt{M_O}}\lambda - \frac{1}{2M_O}\lambda^2 + o(\lambda^3),
\]
\[
a^2(\tilde{\lambda}, M_D) = \pm \frac{1}{\sqrt{M_D}}\tilde{\lambda} - \frac{1}{2M_D}\tilde{\lambda}^2 + o(\tilde{\lambda}^3).
\] (11)

The sign ambiguity can be removed by studying the renormalization group flows, as it was briefly mentioned in connection with eq. (7). In the original model, it turns out that the interesting region is the vicinity of \(g = -1\) with \(g \geq -1\). In the dual model, since \(a = \sqrt{1 + g}\), choosing \(g \geq -1\) one is uniquely led to the solution with the plus sign. Thus in both cases one has to consider the solution with the plus sign. Moreover if one sets \(M_O\) and \(M_D\) equal, \(M_O = M_D = M\), then the classical \(a = \sqrt{1 + g}\) relation can be maintained. We note (though we didn’t display it in eq. (11)) that the two expressions differ already at the order of \(\lambda^3\), as expected, since \(M\) is renormalization group invariant only up to two loop order.

Using \(g(\lambda, M_O)\) (\(a^2(\tilde{\lambda}, M_D)\)) in the two loop beta functions of the coupling constants of the original and dual models yields a universal expression [9]:
\[
\beta_\lambda = -\frac{\lambda^2}{4\pi}\left(2 + \lambda\left(\frac{1}{\pi} - \frac{1}{\sqrt{M}}\right)\right).
\] (12)

As far as the coupling constant renormalization is concerned, this universal beta function shows that the two models are equivalent, both are asymptotically free, and the actual value of \(M\) effects only the two loop coefficient.

According to eq. (11) one can express \(a\) in terms of the common renormalization group (RG) invariant. For the sake of simplicity it is useful to introduce the following notations:
\[
A = \sqrt{\lambda}, \quad \alpha_1 = \frac{1}{\sqrt{M}}, \quad \alpha_2 = \frac{1}{4\sqrt{M^3}}.
\] (13)

It will turn out that \(A = \sqrt{\lambda}\) is the proper coupling of the two models. In terms of these
\[
a = \alpha_1 A - \alpha_2 A^3 + o(A^5), \quad g = -1 + \alpha_1^2 A^2 - \frac{1}{2}\alpha_1^4 A^4 + o(A^6).
\] (14)
2.3 Unified description

In order to make the computations more general it is useful to express both theories as particular cases of a generalized \( \sigma \) model. To achieve this we rescale the field \( \psi \rightarrow \sqrt{1+g} \psi \) and introduce \( a = \sqrt{1+g} \) in place of \( g \) in the original model. After some obvious changes of symbols, \( \mathcal{L}_O \) and \( \mathcal{L}_D \) can be described by the following unified \( \sigma \) model:

\[
\mathcal{L} = \frac{1}{2\lambda} \left\{ (\partial_\mu \theta)^2 + (\partial_\mu \phi)^2 (1 + r \cos^2 \theta) + (\partial_\mu \psi)^2 + 2a \omega_{\mu\nu} \cos \theta \partial_\mu \phi \partial_\nu \psi \right\} .
\] (15)

The previous models can be recovered by the special choices: \( r = g \) and \( \omega_{\mu\nu} = \eta_{\mu\nu} \) for the original model, and \( r = -1 \) and \( \omega_{\mu\nu} = -\epsilon_{\mu\nu} \) for the dual model. Expressing \( r \) in terms of the RG invariant parameter we have:

\[
r = -1 + 2\beta A^2 + \gamma A^4 + o(A^6) ,
\] (16)

where \( \beta = \alpha_1^2/2 \) and \( \gamma = -1/4 \alpha_1^4 \) in the original resp. \( \beta = 0 \gamma = 0 \) in the dual model. Observe that both \( \beta \) and \( \gamma \) are renormalization group (RG) invariant.

This unified Lagrangian (15) is more than it might appear at first sight. As it was shown in [9] and argued above, the deformed principal \( \sigma \) model and its dual can be viewed as being quantum equivalent from the point of view of the two loop beta functions. The unifying Lagrangian (15) can be viewed as a genuine quantum generalization of the deformed principal \( \sigma \) model and its dual. It reduces to the latter ones at special values of the parameters \( \beta, \gamma \) and \( \omega_{\mu\nu} \), and at the same time gives the corresponding beta functions. Thus the renormalization properties of the two models are encompassed in this generalized Lagrangian. Based on this we will be able to renormalize the free energy in both theories at the same time, shortening the computations and obtaining better control on the different contributions.. Checking the quantum equivalence will amount to compare the renormalized free energies computed at the special values of the parameters \( \beta, \gamma \) and \( \omega_{\mu\nu} \) corresponding to the two models.

3 The ground state and perturbative Lagrangian

3.1 Outline of the method - Noether currents

So far testing the quantum equivalence of the dual models was mostly reduced to comparisons of the corresponding beta-functions [1, 10, 11]. Of course there are more to test before one can ascertain about an equivalence. In this paper to test the physical equivalence between the original and dual theories we couple both of them to some particular conserved charge \( Q \). This is accomplished by modifying the respective Hamiltonians \( H_O \) \( (H_D) \) to \( H_O - hQ \) \( (H_D - hQ) \), where \( h \) is an external field (chemical potential type of parameter), having mass dimension one. The corresponding changes in the ground state energy densities (i.e. in the free energy densities, that we shall call for simplicity free energies) can be computed, at least in principle, to any order in perturbation theory. The comparison of the free energy densities, as functions of \( h \), up to
a certain order of perturbation theory, in the original and dual models, then provides a useful check whether the two models do really, physically correspond to each other. For a comparison of this type to make sense both theories must be asymptotically free (to guarantee that perturbation theory applies), and of course we have to choose \( Q \) to be really the same.

As the global symmetry group of both the original and the dual models is \( SU(2) \times U(1) \) we may think naively that any linear combination of an \( SU(2) \) Noether charge and the \( U(1) \) Noether charge, \( x^a Q^a + y Q_{U(1)} \), can be used as the charge above \( Q \) to couple to the Hamiltonians. However for our test we need the same \( Q \) coupled to \( H_O \) and \( H_D \), thus we can choose only such charges that are mapped to themselves under the canonical transformation connecting the original and dual models. This, of course, implies that the charge \( Q \) must stay local under the canonical transformation.

We first point out the relation between the appropriate Noether currents and charges of the global symmetries of \( L_O \) and \( L_D \). An exhaustive treatment will be given elsewhere [20]. Let's start remarking that under Abelian duality transformations the image of the \( U(1) \) current, that belongs to the distinguished isometry used in duality, is a topological current built from the dual field. Thus the image of the \( U(1)_R \) current of the deformed sigma model is the topological current of the \( \chi \) field, \( \epsilon_{\mu\nu} \partial^\nu \chi \), and as such, its charge should vanish on a topologically trivial 2d space-time. Therefore \( Q_{U(1)} \) cannot be present in both \( H_O \) and \( H_D \).

The next simplest possibility is to use the \( U(1) \) charges corresponding to the \( \phi \)-translations in both the original and dual theories: \( N_0^3 \) and \( \tilde{N}_0^3 \). It can be shown (see [20] for details) that the canonical transformation which implements Abelian duality, effectively exchanges only \( p_\psi \) and \( \chi' (p_\chi' \text{ and } \psi') \) and leaves \( p_\phi \), \( p_\theta \) unchanged. It is obvious that \( \tilde{N}_0^3 \) is 'identical' under duality to \( N_0^3 \), but it is not entirely trivial (though it is true) that the space component of \( N_\mu^3 \) really becomes the space component of \( \tilde{N}_\mu^3 \).

### 3.2 From the Hamiltonian to the perturbative Lagrangian

In conclusion, to test the physical equivalence of the original and dual models, in both of them we introduce the coupled Hamiltonian densities \( \hat{H}_{O,D} = \hat{H}_{O,D} - h N_0^3 \). Performing the inverse Legendre transformation on these quantities one obtains the \( \hat{L}_{O,D} \) Lagrange densities of the coupled models. By explicit computation one can show that this procedure of obtaining \( \hat{L}_{O,D} \) is equivalent to the following formal gauging \((\partial_\mu \xi^i \rightarrow D_\mu \xi^i)\) of the original Lagrangian \( \hat{L}_{O,D} \):

\[
D_\mu \psi = \partial_\mu \psi, \quad D_\mu \phi = \partial_\mu \phi + h \delta_0^\mu, \quad D_\mu \theta = \partial_\mu \theta.
\]

This gauging, of course, can be done on the universal, common Lagrangian \((\xi^i)\); the outcome being denoted by \( \hat{L} \). As it can be seen in [17], the coupling of the chemical potential explicitly breaks Lorentz invariance. This will play an important role in our analysis of the quantum equivalence.

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1 One can also investigate different linear combinations of the Noether charges [20].
To perform the perturbative analysis first one has to determine the classical ground state of the system, in other words the minima of the Hamiltonian \( \hat{\mathcal{H}} = \hat{\mathcal{H}}(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi) \), finding the critical points and checking the Hessian’s positive definiteness. Using the actual (perturbative) expression for \( r \), eq. (16), it is straightforward to show that in the perturbative region around \( r = -1 \) (which in the original model corresponds to \( g = -1 \) as opposed to the dual model where it is exactly -1) there is a two-parameter family of local minima, all of them being physically equivalent, given by: \( (p_\theta = p_\psi = 0, p_\phi = \frac{h}{A}, \theta = \frac{\pi}{2}, \phi = \text{const.}, \psi = \text{const.}) \). Since \( \phi \) and \( \psi \) does not appear explicitly in the Hamiltonian, we can choose for convenience the stable classical ground state, common to both models, to be given by \( \theta = \frac{\pi}{2}, \phi = 0, \psi = 0 \). We shall expand our fields around this background.

The Lagrangian, \( \hat{\mathcal{L}} \), as emerging from eq. (15) is not suitable for a direct perturbative computation on account of the overall \( \frac{1}{\lambda} \) factor. However this factor can be removed by an appropriate rescaling of the fields:

\[
\tilde{\theta} = \frac{\theta}{A}, \quad \tilde{\phi} = \frac{\phi}{A}, \quad \tilde{\psi} = \frac{\psi}{A},
\]

where \( A = \sqrt{\lambda} \) as it was given in (13). We note here that originally the fields had no renormalization in either the original or in the dual model, consistent with the fact that the Euler angles are compact variables. The rescaling nevertheless introduces nontrivial renormalization, but this can be deduced from the renormalization of \( \lambda \).

Deleting the tilde from the rescaled fields, we get the following result:

\[
\hat{\mathcal{L}} = \frac{1}{2} \left\{ (\partial_\mu \theta)^2 + (1 + r \cos^2(A\theta))(\partial_\mu \phi + \frac{h}{A}\delta_{0\mu})^2 + (\partial_\mu \psi)^2 + 2a \sin(A\theta)(\omega_{\mu\nu} \partial_\mu \phi \partial_\nu \psi + \frac{h}{A}\Omega_{0\nu}) \right\},
\]

(19)

According to the general prescription of Wick rotation, we define the Euclidian continuation of our model by \( \hat{\mathcal{L}}_E = -\hat{\mathcal{L}}_M(\partial_0 \rightarrow i\partial_1, \partial_1 \rightarrow \partial_0) \). Note that a similar continuation of \( h \rightarrow ih \) would lead to inconsistencies (like wrong signs in the propagators: \( \frac{1}{-p_0^2 + \hbar^2} \) instead of \( \frac{1}{p_0^2 + \hbar^2} \)).

Before starting the perturbative expansion of the trigonometric functions we make another field transformation (redefinition): \( \theta \rightarrow \frac{\theta}{A} + \theta \), guaranteeing that the minimum we expand around is: \( (\theta = 0, \phi = 0, \psi = 0) \). Thus the Euclidian Lagrangian we use has the form:

\[
\hat{\mathcal{L}} = \frac{1}{2} \left\{ (\partial_\mu \theta)^2 + (1 + r \sin^2(A\theta))(\partial_\mu \phi + \frac{h}{A}\delta_{0\mu})^2 + (\partial_\mu \psi)^2 + 2a \sin(A\theta)(\omega_{\mu\nu} \partial_\mu \phi \partial_\nu \psi + \frac{h}{A}\Omega_{0\nu}) \right\},
\]

(20)

where \( r \) and \( a \) are given by (16) resp. (14), and the new parameters \( \omega \) and \( \Omega \), which bear the model dependence, are: \( \omega_{\mu\nu} = -\delta_{\mu\nu}, \Omega_\nu = i\delta_{0\nu} \) in the original and \( \omega_{\mu\nu} = -i\epsilon_{\mu\nu}, \Omega_\nu = -\epsilon_{0\nu} \) respectively in the dual model.

Next we expand the Lagrangian \( \hat{\mathcal{L}} \) around the classical ground state: \( (\theta = 0, \phi = 0, \psi = 0) \), with \( A = \sqrt{\lambda} \) being the relevant coupling constant. We note
at this point that one could follow a different route and expand the parameters $r$, $g$ and $a$ in terms of the RG invariant. The motivation for this would be that at the end of the computation this has to be done anyway. This possibility and the complications that arise will be investigated elsewhere [20].

After some algebra the result is as follows (for the sake of simplicity we denote $\bar{\mathcal{L}}$ by $\mathcal{L}$):

$$
\mathcal{L} = \mathcal{L}_{-2} A^{-2} + \mathcal{L}_{-1} A^{-1} + \mathcal{L}_0 + \mathcal{L}_1 A + \mathcal{L}_2 A^2 + o(A^3),
$$

(21)

where

$$
\mathcal{L}_{-2} = -\frac{1}{2} h^2, \quad \mathcal{L}_{-1} = -ih \partial_0 \phi,
$$

(22)

$$
\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \theta)^2 - r \frac{1}{2} h^2 \theta^2 + ah \Omega_\nu \theta \partial_\nu \psi,
$$

$$
\mathcal{L}_1 = -ir h \theta^2 \partial_0 \phi + aw \mu \nu \theta \partial_\mu \phi \partial_\nu \psi,
$$

$$
\mathcal{L}_2 = \frac{1}{2} r \theta^2 (\partial_\mu \phi)^2 + \frac{1}{6} r h^2 \theta^4 - \frac{1}{6} ah \Omega_\nu \theta^3 \partial_\nu \psi.
$$

Notice that $\mathcal{L}_{-2}$ is a constant (i.e., it is independent of the fields), while $\mathcal{L}_{-1}(h, \varphi) = -ih \partial_0 \varphi$ is a total derivative, thus it can be discarded in this non-topological sector of the theory. From $\mathcal{L}_0$ we see that $\phi$ is a massless scalar field, while $\theta$ and $\psi$ are mixed, apart from the mixing the former is a massive scalar field with mass $\sqrt{-r} h$, the latter is massless. The interaction of the different fields is highly non-trivial, as can be seen above, and contains infinitely many vertices. Nevertheless, these vertices are naturally separated in the weak coupling regime. We emphasize that only the first two terms, $\mathcal{L}_{-2}$ and $\mathcal{L}_{-1}$, are common to both models, as the model dependent parameters $\alpha_i$, $\beta$, $\omega$ and $\Omega$ appear in $\mathcal{L}_j$ for all $j \geq 0$.

## 4 The free energy

Our goal is to define and compute the free energy density in perturbation theory. After setting the stage we do the explicit computations.

### 4.1 Definition of the free energy

At this point our aim is to define the free energy density perturbatively. Denoting the fields collectively by $\varphi = (\theta, \phi, \psi)$, the free energy (density) reads:

$$
e^{-F(h)} V = \frac{\int \mathcal{D} \varphi e^{-S[h, \varphi]} d^2 x}{\int \mathcal{D} \varphi e^{-S[h=0, \varphi]}},
$$

(23)

where $V$ is the volume of the system, $S[h, \varphi] = \int d^2 x \mathcal{L}(h, \varphi(x))$, and $\int \mathcal{D} \varphi$ denotes the functional integration over the field configurations $\varphi = (\theta, \phi, \psi)$: $\int \mathcal{D} \varphi = \int \mathcal{D} \theta \mathcal{D} \psi \mathcal{D} \phi$. The role of the denominator in (23) is to insure the correct normalization: $F(h=0) = 0$. From dimensional arguments one expects the following functional dependence: $F(h) = h^2 \Psi(h)$, where $\Psi = \Psi(h)$ is a dimensionless function.
Let us note the similarity between the free energy defined above and quantum effective action (the generator of the 1PI graphs). The role of the external field is played by $h$, that couples to a conserved charge (composite operator) rather than an elementary field. This similarity will play a structurally simplifying role when we discuss the renormalization of the model.

For the perturbative expansion, in view of eqs. (21-23), it proves useful to introduce

$$S[h, \varphi] = \sum_{i=-2}^{\infty} S_i[h, \varphi] A^i,$$  \hspace{1cm} (24)

with $S_i[h, \varphi] = \int d^2x \mathcal{L}_i(h, \varphi(x))$, where $i \geq -2$. Using this in eq. (24) we obtain a similar expression for the free energy:

$$\mathcal{F}(h) = \sum_{i=-2}^{\infty} \mathcal{F}_i(h) A^i.$$  \hspace{1cm} (25)

Our task will be to determine the first few terms in this expansion, in both models, and compare them. More precisely we determine the first six terms of $\mathcal{F}(h)$ and check whether they are equal.

As $\mathcal{L}_{-2}$ is independent of the fields $\varphi$, it results that $S_{-2}[h, \varphi] = -\frac{1}{2} h^2 V$, and $\exp(-S_{-2}[h, \varphi])$ factorizes (we will come back to this) in the functional integral (23). This way one readily obtains that the first term of (25) is

$$\mathcal{F}_{-2}(h) = -\frac{1}{2} h^2.$$  \hspace{1cm} (26)

Of course this is valid both in the original and dual model, implying that at leading order the perturbative free energy densities coincide. In addition, since $\mathcal{L}_{-1}$ is a total derivative, it implies that $S_{-1}[h, \varphi] = 0$, thus

$$\mathcal{F}_{-1}(h) = 0;$$  \hspace{1cm} (27)

again a model independent statement.

Thus all what remains to be dealt with is the reduced action

$$\bar{S}[h, \varphi] = \sum_{i=0}^{\infty} S_i[h, \varphi] A^i,$$  \hspace{1cm} (28)

and the reduced free energy

$$\bar{\mathcal{F}}(h) = \sum_{i=0}^{\infty} \mathcal{F}_i(h) A^i, \hspace{0.5cm} \mathcal{F}(h) = \mathcal{F}_{-2}(h) + \bar{\mathcal{F}}(h).$$  \hspace{1cm} (29)

Eq. (27) might also suggest that all $\mathcal{F}_i$ with $i$ odd vanishes, this would comply with the fact that the coupling $A = \sqrt{\lambda}$ is just an artifact of our perturbation theory, as it was $\lambda$ that appeared in the original Lagrangian. The vanishing of all odd power contributions in $A$ would indeed imply that the true coupling is in fact $\lambda$. 

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Introducing the $Z(h) = \int \mathcal{D}\varphi e^{-S[h,\varphi]}$ auxiliary function then we can rewrite eq. (23) as $e^{-\mathcal{F}(h)V} = Z(h)/Z(h=0)$. Moreover, if $M$ is an operator we define the following 'expectation value':

$$\langle M \rangle = \frac{\int \mathcal{D}\varphi e^{-S_0[h,\varphi]} M}{\int \mathcal{D}\varphi e^{-S_0[h,\varphi]}}.$$  \hspace{1cm} (30)

Expanding $Z(h)$ as a power series in $A$ we have:

$$Z(h) = [1 - \langle S_1 \rangle A - \langle S_2 - \frac{1}{2}S_1^2 \rangle A^2 - \langle S_3 - S_1S_2 - \frac{1}{6}S_1^3 \rangle A^3 + \text{o}(A^4)] \int \mathcal{D}\varphi e^{-S[h,\varphi]}.$$

For simplicity we have omitted to write the functional dependence of the $\tilde{S}_i[h,\varphi]$-s. Using the identity

$$1 + y_1 A + y_2 A^2 + y_3 A^3 + \text{o}(A^4) = e^{x_1 A + x_2 A^2 + x_3 A^3 + \text{o}(A^4)},$$

we can read off the various components of the reduced free energy density

$$e^{-\mathcal{F}_0(h)V} = \frac{\int \mathcal{D}\varphi e^{-S_0[h,\varphi]}}{\int \mathcal{D}\varphi e^{-S_0[h=0,\varphi]}},$$

$$\mathcal{F}_1(h) = \frac{1}{V} \langle S_1 \rangle,$$

$$\mathcal{F}_2(h) = \frac{1}{V} \left[ \langle S_2 - \frac{1}{2}S_1^2 \rangle + \frac{1}{2} \langle S_1 \rangle^2 \right].$$

We emphasize that in the above formulas we kept only those terms which depend on $h$. In other words we discarded the contribution of $Z(h=0)$, which in fact is a divergent quantity. This is consistent with the fact the that only the derivatives of the free energy are observable thermodynamical quantities.

### 4.2 Propagators

To compute the various vacuum expectation values (or correlators since we are in Euclidean space) determining $\mathcal{F}_i(h)$ we use dimensional regularization (with $n = 2 + \epsilon$). Since

$$S_0(h) = \int d^2 x \mathcal{L}_0(h) = \frac{1}{2} \int d^2 x \varphi^4(x) M(x)\varphi(x)$$

where

$$\varphi^i(x) = (\theta(x), \phi(x), \psi(x)),
\quad
M(x) = \begin{pmatrix}
-\partial^2 - r h^2 & 0 & a h \Omega_\nu \partial_\nu \\
0 & -\partial^2 & 0 \\
-a h \Omega_\nu \partial_\nu & 0 & -\partial^2
\end{pmatrix},
\quad
\partial^2 = \partial_\mu \partial_\mu.$$
in order to determine the propagators of the various fields we have to invert the matrix operator \( M(x) \). This is easily done in momentum space resulting:

\[
G_\theta(x) = \int \frac{d^n p}{(2\pi)^n} e^{-ipx} g(p), \quad G_\psi(x) = \int \frac{d^n p}{(2\pi)^n} e^{-ipx} (p^2 - r h^2) g(p),
\]

\[
G_{\theta\psi}(x) = i a h \int \frac{d^n p}{(2\pi)^n} e^{-ipx} \Omega \cdot p g(p), \quad G_{\phi\phi}(x) = \int \frac{d^n p}{(2\pi)^n} e^{-ipx} \frac{1}{p^2}, \quad G_{\psi\phi}(x) = 0,
\]

where

\[
g(p) = \frac{1}{p^4 - h^2(rp^2 + (a\Omega \cdot p)^2)}, \quad \Omega \cdot p \equiv \Omega_\nu p_\nu. \tag{40}
\]

We have used commonly the notation \( G_{\varphi,\varphi'}(x - y) = \langle \varphi(x) \varphi'(y) \rangle \), with \( G_\varphi \equiv G_{\varphi,\varphi} \). Just note in passing a few simple properties: \( G_{\theta\theta}(x) \) is even, while \( G_{\theta\psi}(x) \) and \( G_{\psi\phi}(x) \) are odd, and \( G_{\psi\theta}(x) = -G_{\theta\psi}(x) \). It is also obvious that \( \phi \) behaves like a massless scalar.

### 4.3 Computation of \( \mathcal{F}_0(h) \)

The computation of \( \mathcal{F}_0(h) \) involves in fact the evaluation of a functional determinant, similarly to the case of the quantum effective action. Evaluating the Gaussian integral from eq. (34), using the identity \( \det X = e^{Tr \ln X} \), results:

\[
\mathcal{F}_0(h) = \frac{1}{2} \int \frac{d^n p}{(2\pi)^n} \ln \left( 1 - \frac{h^2(rp^2 + (a\Omega \cdot p)^2)}{p^4} \right). \tag{41}
\]

A proper way to compute this expression is to take its derivative with respect to \( h \) and solve the following initial value problem:

\[
\frac{d \mathcal{F}_0(h)}{dh} = -h \int \frac{d^n p}{(2\pi)^n} \frac{rp^2 + (a\Omega \cdot p)^2}{p^4 - h^2(rp^2 + (a\Omega \cdot p)^2)}, \quad \mathcal{F}_0(0) = 0. \tag{42}
\]

Rescaling \( p \to hp \), the \( h \) dependence factorizes and we get:

\[
\mathcal{F}_0(h) = -\frac{h^n}{n} \int \frac{d^n p}{(2\pi)^n} \frac{r_0 p^2 + (a_0 \Omega \cdot p)^2}{p^4 - h^2[r_0 p^2 + (a_0 \Omega \cdot p)^2]}.
\]

Above we have made it explicit that the integral is computed in terms of the bare quantities \( r_0 \) and \( a_0 \), rather than the renormalized ones, \( r \) and \( a \). From now on we are going to make this distinction clear in all subsequent formulas. This same expression was obtained in [13], though the initial Lagrangian differed from the one used here by a certain rescaling of the fields. A rescaling usually cannot cause major discrepancies, at least at low orders of perturbation theory, as this example also reflects.

The model dependence is manifest in (43). The integral is divergent in two dimensions more precisely it has a first order pole in \( \epsilon \). The analysis of [17] has computed the pole term and the constant term in the \( \epsilon \) expansion of (39) in closed form, in terms of generalized hyper-geometric functions. After renormalization, the two expressions – as functions of the original (respectively dual) coupling and parameter – were not equal, but, the difference could be accounted for by the scheme dependence of the two loop beta functions. Indeed
it was pointed out in [9] that the equivalence of the original and dual $\beta$ functions corresponds to a perturbative redefinition of the coupling constants in the dual model:

\[
\tilde{\lambda} = \lambda + \frac{\lambda^2}{4\pi}(1 + g), \quad \tilde{g} = g + \frac{\lambda}{4\pi}(1 + g)^2.
\] (44)

Implementing this redefinition in the expressions of the renormalized one loop free energy densities revealed their equality.

Here we take a different route from the one described above. Our strategy is to express the parameters that bear the model dependence in terms of the RG invariant in both models, then set these two RG invariants equal and compare the results. Evaluating

\[
-p_0^2 - \left(a_0 \Omega \cdot p\right)^2
\]

we get

\[
-p_0^2 - h_0^2 \left(\frac{\partial^2}{\partial x^2}\right)^2
\]
in the original model and

\[
-\tilde{p}_0^2 - \tilde{h}_0^2 \left(\frac{\partial^2}{\partial \tilde{x}^2}\right)^2
\]

respectively in the dual model, where $\tilde{p}_{\mu} = \epsilon_{\mu\nu} p_{\nu}$. Due to [26] we have $\tilde{p}_0 = p_0$, and one can perform a change of variables from $p$ to $\tilde{p}$. This way, changing also $p_1 \leftrightarrow p_0$, one can obtain formally identical expressions in the two cases. At this point we only remark that the role of $p_0$ and $p_1$ in dimensional regularization is different.

### 4.4 Computation of $F_1(h)$

According to (35) and (22) the computation of $F_1(h)$ involves the following correlation functions: $\langle \theta(x) \partial_\mu \psi(x) \rangle$ and $\langle \theta^2(x) \partial_0 \phi(x) \rangle$. Using Wick’s theorem we obtain:

\[
F_1(h) = 0.
\] (45)

Eq. (45) has twofold meaning. It shows once again the model independence of the free energy, though we have already encountered explicit model dependence. On the other hand supports our earlier statement about the vanishing of the non-analytic corrections in $\lambda$.

### 4.5 Computation of $F_2(h)$

In the case of $F_2(h)$ we will not be able to obtain the result in a closed form, nevertheless what we can actually compute will suffice to achieve our goals. According to eq. (36) we have to compute $\langle S_2(h) \rangle$, $\langle S_1(h)^2 \rangle$ and $\langle S_1(h) \rangle^2$. We can immediately quote eq. (45) and $\langle S_1(h) \rangle = 0$.

#### 4.5.1 Computation of $\langle S_2(h) \rangle$

From eq. (22) we see that

\[
\langle S_2(h) \rangle = \int d^n x \left[ \frac{1}{2} r_0 \langle \theta^2(x) \partial_\mu \phi \rangle^2(x) \right] + \frac{1}{6} r_0 h^2 \langle \theta^4(x) \rangle - \frac{1}{6} a_0 h \Omega_{\nu} \langle \theta^3 \partial_\nu \psi(x) \rangle.
\] (46)

At first sight one might want to discard the terms that are not coupled to $h$. Nevertheless these terms acquire $h$ dependence through the $h$-dependent propagators.

Based on Wick’s theorem for the first term we have $\langle \theta^2(x) \partial_\mu \phi \rangle^2(x) = \langle \theta^2(x) \rangle \langle (\partial_\mu \phi)^2(x) \rangle$. Since the $\phi$ propagator is in fact a Green’s function, or in
The computation of \( \langle (\partial_\mu \phi)^2 (x) \rangle = \delta^{(n)}(0) \) (modulo equal time commutator terms), where \( \delta^{(n)}(0) \) is the Dirac delta distribution 'evaluated at 0'. But in dimensional regularization the latter is set to zero, yielding no contribution.

The next term in eq. (46) is also readily evaluated: \( \langle \theta^4(x) \rangle = 3(\theta^2(x))^2 = 3 G_\theta(0)^2 \). Evaluating the third term gives (recall that the system in finite volume: \( \int d^n x = V \)):

\[
\langle S_2(h) \rangle = \frac{1}{2} h^2 V \left[ r_0 + \frac{1}{n} a^2_0 \Omega \Omega \right] G_\theta(0)^2. \tag{47}
\]

### 4.5.2 Computation of \( \langle S_1(h)^2 \rangle \)

The computation of \( \langle S_1(h)^2 \rangle \) involves in fact the evaluation of double integral \( \int d^n x d^n y \langle L(x)L(y) \rangle \). From eq. (22) it turns out that \( \langle S_1(h)^2 \rangle \) equals \( \int d^n x \) times

\[
\begin{align*}
r_0^2 h^2 & \partial_0^2 G_\phi(x)[G_\theta(0)^2 + 2G_\theta(x)^2] - 4i r_0 h a_0 \omega_\mu \partial_0 \partial_\mu G_\phi(x)G_\theta(x)\partial_\nu G_\theta(x) + \\
+ & a^2_0 \omega_\mu \omega_\lambda \partial_\mu \partial_\lambda G_\phi(x)[-\partial_\nu G_\theta(x)\partial_\rho G_\theta(x) - G_\theta(x)\partial_\nu \partial_\rho G_\psi(x)] + \\
+ & \partial_\nu G_\theta(0)\partial_\rho G_\theta(0)].
\end{align*} \tag{48}
\]

A priori it is not clear at all how to obtain the overall \( h^2 \) factor required by dimensional analysis. Moreover it is also puzzling how the explicit factors of \( i \) disappear during the computation. As we shall see shortly it is the form of the 'tensor' parameters \( \Omega, \omega_\mu \) and propagators that is responsible for the correct answers.

The first term in (48) results

\[
- 2r_0^2 h^2 V \frac{1}{n} G_\theta(0)^2 \delta_{00}. \tag{49}
\]

The only non-trivial fact that one has to use an IR regularization of the field \( \phi \), the regulator’s mass we denote by \( m \). Then the first term in this parenthesis will be proportional to

\[
\int d^n x \, \partial_0^2 G_\phi(x) \approx \int d^n k \frac{k_0^2}{k^2 + m^2} \delta^{(n)}(k) = 0. \tag{50}
\]

The second term in (48) results

\[
4r_0 a_0^2 h^2 V i \omega_\mu \Omega \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(k_1 + k_2)_0 (k_1 + k_2)_\mu k_1^2 g(k_1) k_2^\rho g(k_2)}{(k_1 + k_2)^2}, \tag{51}
\]

while the last terms of (48) give

\[
a^2_0 \omega_\mu \omega_\lambda \Omega \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(k_1 + k_2)_\lambda (k_1 + k_2)_\mu g(k_1) g(k_2) k_2^\rho}{(k_1 + k_2)^2} [a_0^2 h^2 k_1^\nu (k_1 \cdot \Omega)(k_2 \cdot \Omega) + k_1^2 k_2^\nu (k_2^2 - r_0 h^2)]. \tag{52}
\]
The general structure of the integrals that appear (with one exception) is of the following form:

\[
\int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(k_1 + k_2)_{\mu_1} (k_1 + k_2)_{\mu_2}}{(k_1 + k_2)^2} k_{1 \mu_3} k_{1 \mu_4} k_{2 \mu_5} k_{2 \mu_6} g(k_1) g(k_2).
\]

(53)

The only integral that cannot be brought to this form has \(k_2^2 g(k_2)\) instead of \(g(k_2)\). If \(g(k)\) were a covariant expression in \(k\), then the value of the integral would be given completely by the index structure. More precisely covariance would require the result to be the sum of triple products of the Kronecker delta functions. The number of independent possibilities is \(6!(2!)^3/3! = 15\).

Exploiting the obvious symmetries under the exchange of \(\mu_1 \leftrightarrow \mu_2, \mu_3 \leftrightarrow \mu_4, \mu_5 \leftrightarrow \mu_6\), and the less obvious exchange \((\mu_3, \mu_4) \leftrightarrow (\mu_5, \mu_6)\), which can be seen by a change of integration variables, the tensor structure reduces significantly to only four independent terms. Denoting \(\delta_{\mu_i \mu_j}\) by \((ij)\), we get for (53)

\[
(12)(34)(56) I_1 + (12)[(35)(46) + (36)(45)] I_2 + (34)[(15)(26) + (16)(25)] I_3 + \ldots
\]

(54)

Using the standard methodology one can compute the unknowns \(I_1\) through \(I_4\), in a straightforward manner. Unfortunately \(g(k)\) is not covariant with respect to the full \(SO(n)\). As we pointed out, covariance was already broken at the level of the Lagrangian. If we analytically continue \(p_0\) to \(n_0\) dimensions and \(p_1\) to \(n_1\) dimensions, with \(n_0 + n_1 = n\), then instead of the full \(SO(n)\) group we get \(SO(n_0) \times SO(n_1)\). In other words, among the \(p_0\)-s and \(p_1\)-s standard covariance arguments remain valid, and the computation sketched above makes sense. Thus if all the indices are of \(p_0\) or \(p_1\) type, then we can reliably compute the integrals.

In the expression of the two-loop free energy (51, 52) the free indices we have dealt above are contracted with different tensor structures. In the original model \(\Omega_\nu\) and \(\omega_{\mu \nu}\) involve only delta functions, and this way what we said applies. On the other hand for the dual model \(\tilde{\Omega}_\nu\) and \(\tilde{\omega}_{\mu \nu}\) are epsilon tensors, mixing the indices, and the arguments presented break down. The same is true for the sum of the terms involved, that we eventually want to compute.

There is an independent argument that shows that even if we were able to do the integrals, the result would not be reliable, due to the behavior of the \(\epsilon\) tensor in dimensional regularization. More precisely, even if the integrals in (51) and (52) were covariant, based on the definition of \(\omega_{\mu \nu}\) and on (52), we could conclude that in the dual model \(\langle S_1(h)^2 \rangle\), and as a result \(\mathcal{F}_2(h)\), contains terms proportional to the product of two \(\epsilon\) tensors, with uncontracted indices. The broken covariance by the external field \(h\), invalidates the above argument, but is highly likely that it complicates matters, rather than simplifies, and as a result we would still end up with \(\epsilon_{\mu \nu} \epsilon_{\alpha \beta}\) terms. As it is well known, there is no consistent way defining such an object in dimensional regularization (2).
in order to compare the results, at the very end of the computation we want to express them in terms of the two RG invariants, which are finally set equal. In other words, based on (11) and (16) we trade the renormalized quantities \( g \) and \( \beta \) for the RG invariant \( M \), implicitly meaning that by this we have also set the RG invariants equal. Since it is the bare \( a_0 \) that appears in (51) and (52), and also in (43), let’s investigate more closely what happens to this term during renormalization and expansion in terms of the RG invariant.

At the level of the bare quantities we aim to have \( a_0 = \sqrt{1 + g_0} \). Based on (9) this is certainly true at two loop level in the original model, while in the dual one this is more subtle. As it was shown in (9) naively the relation \( a_0 = \sqrt{1 + g_0} \) cannot be maintained at two loop level in the dual model. Nevertheless, taking into account the redefinition of the dual model’s coupling and parameter, eq. (44), the above relation can be maintained at one loop order.

As \( a_0 = \sqrt{1 + g_0} \) is doubtlessly valid in the leading order, the renormalization (5) amounts to \( a_0 = \sqrt{1 + Z_g(\lambda, g)g} \). But (1) shows that \( Z_g(\lambda, g) = 1 + o(\lambda) \), thus to leading order we have \( a_0 = \sqrt{1 + g} \), with \( g \) the renormalized coupling. On the other hand at this point we can use (11) and conclude that, after renormalization and expanding in the RG invariant, \( a_0 \) becomes proportional to \( \sqrt{\lambda} \).

The good news is that all the terms we were unable to compute (51) and (52) are proportional to \( a_0^2 \), hence are of order \( \lambda^2 \). Since we have not even attempted to compute the \( F_4(h) \) term that is of the same order, we neglect them for the time being. Having in mind the insertion of the RG expressions for the parameters, we simply get:

\[
F_2(h) = -h^2 \frac{n-2}{2n} G_\theta(0)^2.
\]  

(55)

Naturally this result is to be interpreted as modulo terms that will be of higher order after renormalization and expansion in the RG invariant.

5 Renormalization

The results from the previous section are divergent, and we used dimensional regularization to compute them. As advertised we make use of the similarities between the free energy and the quantum effective action to discuss the renormalization of the former, modeled by the renormalization of the latter (see e.g. [21]). In this section we follow a more or less naive renormalization procedure, and compute the renormalization group improved perturbative two-loop free energy. It will be the role of the next section to tighten the loose end, and prove that what we did is indeed correct.

Since we are at second order of perturbation theory, we have both first and second order poles in dimensional regularization, as can be seen in eq. (55). Since the free energy is a physical quantity, it has to be well defined after renormalization. The recipe of this section is to use the renormalization of the deformed principal \( \sigma \) model (8), (9), and the renormalization of its dual, to renormalize the free energy. The above renormalizations were performed using the geometric method of [23, 25].
This procedure can be immediately objected since the free energy is computed in a theory that has additional terms in the Lagrangian (19), compared to the deformed principal $\sigma$ model (4) and its dual (8). A priori there is no reason to expect that the wavefunction and coupling constant renormalization functions are the same, though in fact they are. In the following we present a simple argument in favor of the above statement. We shall give a complete proof in section 5.5 below.

Coupling the external field amounts to the appearance of terms proportional to $h$. Setting $h = 0$ we get the original models (deformed principal $\sigma$ model and its dual). Thus the wavefunction and coupling constant renormalization functions can at most differ from the ones of the original models by terms proportional to $h$. But $h$ is a dimensionful quantity, having mass dimension +1 (it is a super-renormalizable coupling or relevant perturbation). On the other hand our theories have no other dimensionful quantities, and the wavefunction and coupling constant renormalization functions are dimensionless. We conclude that $h$ cannot appear in the latter ones, proving the assertion.

In addition since $h$ couples to a conserved charge, it is not renormalized. The only issue that remains will be to deal with the renormalization of the operators that couple to $h$, viewed as composite operators. We postpone this to section 5.5.

In order to cancel the second order poles of the regularized free energy (55) we have to go beyond the computation of [9]. Let us review what we know at this stage about the renormalization of the deformed principal $\sigma$ model and its dual. In (15) we have introduced a generalized Lagrangian, and argued in the last paragraph of the section that it encompasses the renormalization properties of both models. Thus we can translate the renormalization properties of the original models to those of the generalized Lagrangian. The Euler angles ($\phi, \theta, \psi$) are compact and have no wavefunction renormalization. The renormalization of $\tau, \lambda$ and $a$ in the unified model follows from those of $g$ (resp. $a$) and $\lambda$ in the original models.

In (18) we rescaled the fields ($\phi, \theta, \psi$), this way in (22) they have the corresponding nontrivial wavefunction renormalization. Thus $Z_\lambda(\lambda, M)$, the coupling constant renormalization of $\lambda$, is the central object for their renormalization. (Recall that $M$ is the renormalization group invariant that appeared in (10)). $Z_\lambda(\lambda, M)$ was computed at two loop order in perturbation theory in [9]. The interest in [9] was restricted to the single pole terms. On the other hand we are constrained to deal with the second order poles too. First we compute the residue of the second order pole in $Z_\lambda(\lambda, M)$.

5.1 $\lambda$ at two loop

The goal is to determine the terms $y_\lambda(\lambda, M)$ and $\bar{y}_\lambda(\lambda, M)$ in the following expansion:

$$Z_\lambda(\lambda, M) = 1 + \frac{1}{\epsilon} y_\lambda(\lambda, M) + \frac{1}{\epsilon^2} \bar{y}_\lambda(\lambda, M) + o(\frac{1}{\epsilon^3}).$$

The first term was basically determined in [9]. All we need is to use [3] for $Z_\lambda(\lambda, g)$ and [14] for $g$, in terms of the renormalization group invariant $M$, to
obtain:

\[ y_\lambda(\lambda, M) = -\frac{1}{2\pi} \lambda + \frac{1}{8\pi} (2\alpha_1^2 - \frac{1}{\pi}) \lambda^2. \]  
(57)

Our task of computing \( \bar{y}_\lambda(\lambda, M) \) is highly simplified by the special properties of the non-linear \( \sigma \) models. Following [22] it was shown in [23] that the generalized renormalization theory [24] of non-linear \( \sigma \) models lead to generalized renormalization group equations, that allow one to determine the residues of the higher order poles in a given coupling constant renormalization function like \( Z_\lambda(\lambda, M) \), without extra diagrammatic computations.

More precisely it is shown that in a theory (like ours) with a single dimensionless coupling constant \( \lambda \), with mass scale parameter \( \mu \), defined in \( n \) dimensions (dimensionally regularized and minimally subtracted), having an expansion of the bare coupling \( \lambda_0 \) in terms of the renormalized coupling \( \lambda \) of the form

\[ \lambda_0 = \mu^{2-n} \left( \lambda + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda)}{(n-2)^\nu} \right), \]  
(58)

the pole residues \( a_\nu(\lambda) \) satisfy the recursive pole equations:

\[ (1 - \lambda \frac{\partial}{\partial \lambda}) a_{\nu+1}(\lambda) = (1 - \lambda \frac{\partial}{\partial \lambda}) a_1(\lambda) \frac{\partial}{\partial \lambda} a_\nu(\lambda). \]  
(59)

We can apply this directly to compute \( \bar{y}_\lambda(\lambda, M) \). From the expressions following (5) we see that in our case \( a_1(\lambda) = \lambda y_\lambda(\lambda, M) \), with \( y_\lambda(\lambda, M) \) given in (57). Using (59) for \( \nu = 1 \) we can determine the leading term in the expansion of \( a_2(\lambda) \). A simple calculation shows that \( a_2(\lambda) = \frac{1}{4\pi^2} \lambda^3 + \ldots \), implying

\[ \bar{y}_\lambda(\lambda, M) = \frac{1}{4\pi^2} \lambda^2 + \ldots. \]  
(60)

5.2 One loop free energy

By now it is a computation on the back of an envelope to obtain the renormalized one loop free energy density. We have to use (26) for the leading term (with \( \lambda \) as the bare coupling \( \lambda_0 \)) and (43) for the one loop regularized result. The integral itself, as it was pointed out in [17], is hard to deal with exactly. Nevertheless, with the parameters traded for the RG invariants, and keeping only the leading contribution in \( \lambda \) (as explained in the last paragraph of Section 4), and renormalizing the expression using (5), (56) and (57), we obtain:

\[ \mathcal{F}^{1\text{-loop}}(\mu) = h^2 \left( -\frac{1}{2\lambda} - \frac{1}{8\pi} \left( \ln \left( \frac{h}{\mu} \right)^2 + \gamma - 1 - \ln(4\pi) \right) \right). \]  
(61)

Some of the terms that are higher order in \( \lambda \), and are not yet displayed, will be used for the computation of the \( \mathcal{F}_2 \).

We note two things here. Firstly, the one loop equivalence might be argued to be not surprising based on the experience gained in [10] and the general one-loop beta function result of [11]. Secondly, the above expression correctly reproduces the known one-loop free energy of the \( O(3) \) sigma model, that arises in the \( \alpha_1 = 0 \) limit.
As opposed to the corresponding computation in [17], the procedure that leads to (61) does not require the splitting of the momentum integration in the 0-th direction and the rest. The reason is simply that due to the expansion in the RG invariant, the order \( \lambda \) term becomes totally covariant. The non-covariance is shifted to the next order.

5.3 Two loop free energy

For the computation of the two loop free energy we need all the results developed. The \( o(\lambda) \) corrections arising from the leading bare term (26) are obvious in the light of (3), (56), (57) and (60). The \( o(\lambda) \) contributions from the bare next to leading term of the one loop free energy (43) can be computed as an expansion in the RG invariant, using (14) and (16). The results are:

\[
\lambda h^2 \alpha^2 \left[ \frac{1}{4\pi\epsilon} - \frac{1}{8\pi} \left( \gamma - \ln(4\pi) + 2 \ln(h) - 1 \right) \right] M, \tag{62}
\]

where \( M \) equals \( \delta_{11} \) in the original model, and \( \epsilon_0 \epsilon_0 \) in the dual one. Although, as argued above, the product of two epsilon tensors with uncontracted indices is ambiguous in dimensional regularization, the contraction of one index gives a meaningful expression. Following [26], we assume that \( \epsilon_{\mu\alpha} \epsilon_{\nu\alpha} \) has a consistent continuation, namely:

\[
\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \quad \epsilon_{\mu\alpha} \epsilon_{\nu\alpha} = \delta_{\mu\nu}. \tag{63}
\]

These two expressions might differ depending on the regularization schemes chosen. As we have pointed out in connection with (54), we can a priori continue the 0th direction into \( n_0 \) dimensions and the 1st direction into \( n_1 \) dimensions, provided \( n_0 + n_1 = n \). We checked that the final result is consistent with dimensional analysis in either case. While we know [17] that the choice \( n_0 = 1 \) is a consistent scheme with the continuation of the \( \epsilon \) tensor, we cannot claim the same about the general case.

As we will see in a moment the only discrepancy in the two-loop free energies of the two models comes from (62). This way we have several choices of different schemes to see the quantum equivalence. With the notation following (62) we can chose in both models the scheme with \( \delta_{00} = \delta_{11} \). The second choice is two different schemes in the two models, but related to each other by: \( \delta_{00}^{D} = \delta_{11}^{O} \). Besides these naive choices we have a highly non-trivial one, with the identical choice in both models: \( \delta_{00} = 1 \) and \( \delta_{11} = n - 1 \). Unlike for the previously mentioned ones, we know the consistency of this scheme, and we are going to work with it in the rest of the paper. As we remarked in connection with (43), it was shown in [17] that in this case the difference in (62) is accountable for a perturbative redefinition of the coupling constant \( \lambda \) in the dual model. Naturally this redefinition has to proceed the expansion in the RG invariant. Unlike in [17], where this gave correction to \( \mathcal{F}^{1-\text{loop}}(h) \), due to the expansion in the RG invariant we get contribution to \( \mathcal{F}^{2-\text{loop}}(h) \). The redefinition of the coupling constant does not effect the genuinely higher order terms as it can be seen from (14).
Thus, as we just noted, we are going to use the last scheme in what follows: \( \delta_{00} = 1 \) and \( \delta_{11} = n - 1 \), and for definiteness we consider the case of the original model first. In the light of what we just said, for the dual model, the results (64), (65) below, must be amended by \( h \frac{\lambda}{16 \pi^2} \), to account for the perturbative redefinition of \( \tilde{\lambda} \), eq. (44). However once this redefinition is taken into account, the two loop results below are identical in the two models. The result for the other schemes will differ from the results to be presented by terms that are some number times \( -\lambda h^2 \frac{\alpha^2}{16 \pi^2} \).

Using the bare result (55) and summing it with the corresponding contributions described, we obtain a cancelation of both the first and second order poles, and obtain a finite expression. The computation is somewhat tedious, but straightforward, the result is simply:

\[
\mathcal{F}^{2\text{-loop}}(h) = -\frac{h^2}{2\lambda} - \frac{h^2}{8\pi} \left[ \ln \left( \frac{h}{\mu} \right)^2 + \frac{\gamma}{2} - \ln(4\pi) \right] - \frac{h^2 \lambda}{16 \pi^2} \left\{ \ln \left( \frac{h}{\mu} \right)^2 + \frac{\gamma}{2} - \ln(4\pi) \right\} - \pi \alpha_1^2 \left[ \ln \left( \frac{h}{\mu} \right)^2 + \frac{\gamma}{2} - \ln(4\pi) \right] + \pi \alpha_1^2 \left[ \ln \left( \frac{h}{\mu} \right)^2 + \frac{\gamma}{2} - \ln(4\pi) \right] \]  

(64)

It simplifies a bit if we use instead of minimal subtraction (MS scheme) the \( \text{M}S \) scheme: \( \ln \mu \to \ln \mu + (\gamma - \ln(4\pi))/2 \). Then the free energy reads:

\[
\mathcal{F}^{2\text{-loop}}(h) = -\frac{h^2}{2\lambda} - \frac{h^2}{8\pi} \left[ \ln \left( \frac{h}{\mu} \right)^2 - 1 \right] - \frac{h^2 \lambda}{16 \pi^2} \left\{ \ln \left( \frac{h}{\mu} \right)^2 - \frac{1}{2} - \ln(4\pi) \right\} - \pi \alpha_1^2 \left[ \ln \left( \frac{h}{\mu} \right)^2 - 1 \right] + \pi \alpha_1^2 \left[ \ln \left( \frac{h}{\mu} \right)^2 - 1 \right] \]  

(65)

### 5.4 Improvement of the perturbation theory

We can take advantage of the asymptotic freedom of our theory and calculate the RG improvement of the perturbative result. This gives the asymptotic expansion of the free energy for large values of the external fields.

Physical quantities depend on the renormalized coupling \( \lambda \), the renormalized parameter \( g \), and the dimensionful scale parameter (or subtraction point) \( \mu \), in such a way that the action of the renormalization group (RG) operator

\[
\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta_\lambda(\lambda, g) \frac{\partial}{\partial \lambda} + \beta_g(\lambda, g) \frac{\partial}{\partial g},
\]

vanishes on them. As the free energy density takes the form \( \mathcal{F}(h) = -h^2 \Psi(\lambda, g, \mu, h) \) and the external field \( h \) is not renormalized, the function \( \Psi \) is renormalization invariant \( \mathcal{D} \Psi = 0 \). Since we are interested in the behaviour of the free energy density for large values of \( h \) we write \( h = h_0 e^x \) where \( h_0 \) is fixed and let \( x \to \infty \). Standard RG considerations then give that

\[
\mathcal{F}(h) = -h^2 \Psi(\lambda(x), g(x), \mu, h_0),
\]

(67)

where \( \lambda(x) \) and \( g(x) \) are the running coupling and parameter. In our final result for the two loop free energy density (65) the parameter \( g \) is eliminated.
in favor of the two loop RG invariant. As a consequence, up to this order, 
\( \Psi = \Psi(\bar{\lambda}(x), \mu, h_0) \) depends only on \( \bar{\lambda}(x) \), which is a solution of

\[
\frac{d}{dx} \bar{\lambda}(x) = \beta_\lambda(\bar{\lambda}(x), \alpha_1), \quad \bar{\lambda}(x = 0) = \lambda,
\]  
(68)

where the beta function has the following expansion:

\[
\beta_\lambda(\lambda) = -b_0 \lambda^2 - b_1 \lambda^3 - b_2 \lambda^4 + \ldots
\]
(69)

with (see eq. (12))

\[
b_0 = \frac{1}{2\pi}, \quad b_1 = \frac{1}{4\pi} \left( \frac{1}{\pi} - \alpha_1^2 \right) = \frac{1}{4\pi^2} \left( 1 - \frac{p}{2} \right), \quad p = \frac{2\pi}{\sqrt{M}}.
\]
(70)

Thus we can go on with the RG analysis as if we had only one coupling constant \( \bar{\lambda}(x) \). The expression of the (RG invariant) \( \Lambda \) parameter is

\[
\frac{\Lambda}{\mu} = e^{-\frac{\alpha}{b_0} \lambda} \frac{1}{b_0} e^{c} \left[ 1 + \left( \frac{b_1}{b_0} - \frac{b_2}{b_0} \right) \lambda + o(\lambda^2) \right],
\]
(71)

or more conveniently

\[
\ln \frac{\Lambda}{\mu} = -\frac{1}{b_0} + \frac{1}{b_0} \ln \lambda + c + \left( \frac{b_1}{b_0} - \frac{b_2}{b_0} \right) \lambda + o(\lambda^2),
\]
(72)

where \( c \) is a constant of integration. We define \( \Lambda \equiv \Lambda_{\text{MS}} \) by choosing \( e^c = \left( \frac{1}{2\pi} \right) \frac{1}{b_0} \) as it simplifies some of the forthcoming expressions.

As the next step of the RG analysis we note, that using expression (72), it is customary to define an effective coupling \( \alpha(h) \) by the following transcendental equation [28]:

\[
b_0 \ln \frac{h}{\Lambda_{\text{MS}}} = \frac{1}{\alpha} + \frac{b_1}{b_0} \ln \alpha.
\]
(73)

The important property of this effective coupling is that it depends on the physical quantity \( s = \ln \left( \frac{h}{\Lambda_{\text{MS}}} \right) \), moreover one can express the running coupling \( \bar{\lambda} \) in terms of the effective coupling \( \alpha \) perturbatively:

\[
\bar{\lambda} = \alpha \left( 1 + \xi_1 \alpha + \xi_2 \alpha^2 + \ldots \right),
\]
(74)

where

\[
\xi_1 = b_0 \left( \ln \frac{h_0}{\mu} - c \right), \quad \xi_2 = \xi_1^2 + \frac{b_1}{b_0} \xi_1 - \frac{b_2}{b_0} + \frac{b_1}{b_0} \xi_1 - \frac{b_2}{b_0}.
\]
(75)

To use the \( o(\lambda) \) part of the free energy density effectively we need the large \( s \) asymptotic expansion of the effective coupling up to \( o(s^{-3}) \):

\[
\alpha = \frac{2\pi}{s} \left( 1 + \frac{A \ln s}{s^2} + \frac{B}{s} + \frac{C \ln^2 s}{s^2} + \frac{D \ln s}{s^2} + \frac{E}{s^2} \right),
\]
(76)

where the coefficients \( A, \ldots E \) can be obtained from (73).
Now improving (65) by the RG, i.e. using eq. (74-76) in (65, 67), results, when the dust settles, the following formula for the asymptotic (large $s$) behaviour of the two loop free energy:

\[
F_{2\text{-loop}}(h) = -\frac{h^2}{4\pi} \left( s + \left(1 - \frac{p}{2}\right) \ln s - \frac{1}{2} + \left(1 - \frac{p}{2}\right)^2 \ln \frac{s}{s} \right) + \frac{1}{s} \left( \left(1 - \frac{p}{2}\right)^2 - 8\pi^2 b_2 + \frac{p - 1}{4} \right) + o\left(\ln^2 \frac{s}{s}\right). \tag{77}
\]

Unfortunately $b_2$ has not yet been computed in the literature, nevertheless we can say a lot about it. Due to the expansion in the RG invariant, it will have terms coming from the lower order beta function coefficients, like in (6), and the genuine three loop coefficient evaluated at $g = -1$. For simplicity we denote this last term $b^{(3)} = b^{(3-\text{loop})}(g = -1)$. But this is exactly the three loop beta function coefficients of the $O(3)$ $\sigma$-model: $b^{(3)} = \frac{\beta_3}{32\pi}$. Carrying one step further the computation of (12) results:

\[
b_2 = \frac{1}{8\pi} \left( \frac{\alpha_1^4}{\alpha_1^2} - \frac{3}{\pi} \alpha_1^2 \right) + b^{(3)} = \frac{1}{16\pi^3} \left( \frac{p^2}{2} - 3p \right) + \frac{5}{32\pi^3}. \tag{78}
\]

Accordingly, as a final result we have:

\[
F_{2\text{-loop}}(h) = -\frac{h^2}{4\pi} \left( s + \left(1 - \frac{p}{2}\right) \ln s - \frac{1}{2} + \left(1 - \frac{p}{2}\right)^2 \ln \frac{s}{s} \right) + \frac{1}{s} \left( \left(1 - \frac{p}{2}\right)^2 - 8\pi^2 b_2 + \frac{p - 1}{4} \right) + o\left(\ln^2 \frac{s}{s}\right). \tag{79}
\]

### 5.5 Renormalization of the composite operators

In this section we give a solid foundation to the results of the previous renormalization procedure. More precisely we consider the renormalization of the composite operators that arise via the coupling of the chemical potential. Our attitude is similar to the standard procedure of mass renormalization in QCD: initially the mass of the light quarks is set to zero, then the bilinear mass operator $\bar{\psi}\psi$ is added, and its effect is accounted by its renormalization as a composite operator. We followed the same ideology so far, neglecting the fact that the renormalization of the coupled theory is different from that of the uncoupled. It is now that we remedy this.

Let us reconsider the computation of the free energy (23). In (24) we started with the bare Lagrangian. Instead we have to use the renormalized one, with counterterms coming from the wavefunction renormalization of the fields, coupling constant renormalization, and the renormalization of the composite operators, all these coming with a natural grading:

\[
S[h, \varphi] + \Delta S[h, \varphi] = \sum_{i=-2}^{\infty} (S_i[h, \varphi] + \Delta S_i[h, \varphi]) A^i, \tag{80}
\]

with the renormalized quantities in the right hand side. $\Delta S_i[h, \varphi]$ commonly denotes all the counterterms of order $i$. The original Lagrangian that has to be
employed reads:

$$L^0 = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{1}{2}(\partial_\mu \phi)^2 + A a\omega_{\mu\nu} \theta \partial_\mu \phi \partial_\nu \psi + A^2 \frac{1}{2} r \theta^2 (\partial_\mu \phi)^2.$$  (81)

$\Delta S_{-2}$ amounts simply to the multiplicative renormalization we considered in (54). Since multiplied by a factor of $1/\lambda$, it gives rise to a term that is independent of $\lambda$ (already used to renormalize the one loop free energy), one that is $o(\lambda)$ (also used), and higher order terms:

$$\Delta S_{-2} = \Delta S^0_{-2} + \Delta S^1_{-2} \lambda + o(\lambda^2).$$  (82)

$\Delta S_{-1} = 0$ as already $S_{-1} = 0$. A priori $\Delta S_0$ contains terms from the from the wavefunction renormalization of the fields, but these are independent of $h$, and are canceled by the denominator in (23). Thus $\Delta S_0 = \delta_{O^0} O^0 + \delta_{O^1} O^1$, where $O^0 = -r \frac{1}{2} h^2 \theta^2$ and $O^1 = a h \Omega_{\mu} \theta \partial_\mu \psi$. The renormalization of $O^0$ (and the similar operators that appear in the Lagrangian proportionally to $h$) has two contributions: one from the renormalization of $h$ (that is zero as discussed in the previous section) and one from the renormalization of the composite operator $\theta^2$. We expect that $\delta_{O^0} = 1 + o(\lambda^2)$, and this expectation can be confirmed by a short explicit computation.

In order to compute $\delta_{O^0}$ we need a Green’s function involving $O^0$. For simplicity let’s consider the one-point function of $\theta^2$: $\langle \theta^2(x) \rangle$. As a renormalization prescription we normalize the one-point function according to the tree level value, and determine $\delta_{O^2}$ from the condition of preserving the above normalization. At $o(\lambda)$ there are two diagrams that contribute coming from the $o(A)$ and $o(A^2)$ vertices. The latter one has a value proportional to

$$\int d^n y \, G(x-y)^2 \partial^2 G(0) = \int d^n y \, G(x-y)^2 \delta(0) = 0$$  (83)

in dimensional regularization. The same result is obtained if the diagram is evaluated in momentum space, where the masslessness of the fields requires additional IR regularization (as exploited already in a previous section).

The other diagram is readily proportional to $\lambda a^2$. The remaining integral can be computed but we don’t need the result for what follows, because once again, as we express this contribution in terms of the RG invariants we shall have a dependence proportional to $\lambda^2$. Thus we have $\delta_{O^2} = 1 + o(A^3)$ at least. As a consistency check we quote that the same result is obtained by considering for example the $\langle \theta^2(x) \theta(y) \theta(y') \rangle$ Green’s function.

During the computation we employed the $\frac{1}{2} h^2 \theta^2$ operator as a mass term for $\theta$, though it is a composite operator as discussed above. The motivation for this can be given as follows: assume that $\frac{1}{2} h^2 \theta^2$ is a perturbation, and expand it perturbatively with the rest of the terms in (33). The difference is that $\frac{1}{2} h^2 \theta^2$ is independent of $\lambda$, and as such any term in its power series expansion is of the same order, and has to be summed. In other words, any term in the perturbative expansion of $F$ will be multiplied by the full expansion of the exponential of $\frac{1}{2} h^2 \theta^2$, that can be resummed. The resummation on the other hand is equivalent with the corresponding mass term $h$ for the $\theta$ field. From the point of view of the original $h = 0$ theory this is a non-perturbative result.
Next we consider the renormalization of the second operator $O^1$ and the one appearing at the next level $O^2$: $O^2 = \theta^2 \partial_0 \phi$. These operators will have their composite operator renormalization functions: $\delta_{O^1}$ and $\delta_{O^2}$. Since the undeformed ($h = 0$) theory has only interactions proportional to $\lambda$ or to $a\sqrt{\lambda}$, and the latter ones must appear at least twice for a finite contribution, eq. (81), we conclude that the renormalization effects due to these vertices are at least of order $\lambda$ and resp. $\lambda a^2$.

In the spirit outlined above we have to introduce the new terms $z^1 O^1$ and $z^2 O^2$ into the action (80) as terms contributing to $\Delta S[h, \varphi]$, and account for their contribution. As these new terms are proportional to $\lambda$, they contribute to $\Delta S^2[h, \varphi]$. Thus we expect new contribution to $F_2$. Based on (31) these are proportional to $\langle z^1 O^1 \rangle$ resp. $\langle z^2 O^2 \rangle$. But we already know from (45) that these are zero. It is easy to see that all the other combinations of these two operators with the rest of the operators will give higher order contributions to the free energy.

It is obvious that the product of $z^1 O^1$ (resp. $z^2 O^2$) with $O^1$ (resp. $O^2$) will give non-zero contribution to $F_3$, but as far as $F_2$ is concerned there is no deviation from the results we obtained.

At the next level (i.e. at $o(A^2)$) we have the composite operators $O^3 = \theta^4$ and $O^4 = \theta^3 \partial_0 \psi$. Consider first $O^3 = \theta^4$. In order to compute $\delta_{O^3}$ we fix the normalization of the correlation function $\langle \theta^4(x) \rangle$ to the tree level value: $\langle \theta^4(x) \rangle = 3G(0)^2$. We compute $\delta_{O^3}$ by computing the first corrections to the correlation function, and insisting as above that the above normalization remains valid. Using the available interaction terms in (81) we see that using the vertices $\theta \partial_\mu \phi \partial_\nu \psi$ resp. $\theta^2 (\partial_\mu \phi)^2$ give the first nontrivial contribution proportional to $\lambda^4$ resp. $\lambda^2$, while their combination is of order $\lambda^3$. Thus we conclude that $\delta_{O^3} = 1 + o(\lambda^2)$. In the case of $O^4$ after simple considerations it is similarly obtained that $\delta_{O^4} = 1 + o(\lambda^2)$, and we are done.

From the above analysis we conclude that the more or less naive renormalization procedure employed in the first place is completely adequate, and the additional composite operator renormalizations do not disturb the results as far as the $F_2$ is concerned.

6 Conclusions

In this paper we have computed the change in the ground state energy density of a deformed principal $SU(2)$ sigma model, and one of its T-duals, in the presence of an external field. The computations has been carried out at two loop order in perturbation theory. Perfect agreement has been found in the following sense: there were renormalization schemes in the two models that yielded the same expressions for the two loop free energy densities.

We defined the free energy using the conserved charge corresponding to a special symmetry of the model. Care had to be made at the choice of the charge, since we wanted it to exist as a genuine symmetry in both the original and dual model, while it is known that the charge corresponding to the isometry used for the duality becomes topological. We performed the computation for
the simplest choice, but other possibilities are also investigated at least in the one loop order [17], [27]. We computed the relevant diagrams and performed the renormalization using dimensional regularization. Due to asymptotic freedom, we could improve the perturbative result using the renormalization group equations.

This result strengthens the confidence in the findings of [9], that were mainly based on beta function computations. More precisely this way we have given much stronger evidence that in the case of constant $g_{00}$, at least in the case of the deformed principal SU(2) sigma model, the naive Buscher formula that relates the original model and its T-dual, gives a true quantum equivalence. As it was pointed out and exemplified in [10], the constancy of $g_{00}$ is no guarantee for two loop quantum equivalence. Based on this one might expect that a finer test would detect discrepancy even in the case where the beta function arguments show no sign of it. Though the free energy test that we performed does not prove the lack of discrepancies, it hints to the non-existence of these at least at two loop order. It would be interesting to analyze the pertinent SL(3) example of [11] from the free energy point of view, though, at this point it seems to be too big of a computational challenge.

Acknowledgments

We would like to thank J. Balog, P. Forgács for valuable discussions throughout this work. R.L.K. would also like to thank F.P. Esposito, D. Kastler, D. Maison, F. Mansouri, T. Nagy, M.E. Peskin, G. Pócsik, A Rebhan, A. Slavnov, P. Surányi, L.C.R. Wijewardhana, and L. Witten for discussions at different stages of this work. This work was supported in part by the Hungarian National Science Fund (OTKA) under T029802, and by the Ministry of Education under FKFP 0178/1999. R.L.K. was also supported in part by DOE grant DOE-FGO2-84ER40153.

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