Hyperbolic Chaos of Turing Patterns

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We consider time evolution of Turing patterns in an extended system governed by an equation of the Swift-Hohenberg type, where due to an external periodic parameter modulation long-wave and short-wave patterns with length scales related as 1:3 emerge in succession. We show theoretically and demonstrate numerically that the spatial phases of the patterns, being observed stroboscopically, are governed by an expanding circle map, so that the corresponding chaos of Turing patterns is hyperbolic, associated with a strange attractor of the Smale-Williams solenoid type. This chaos is shown to be robust with respect to variations of parameters and boundary conditions.

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In nonlinear dynamics the notion of structural stability, or robustness, is one of the key tools allowing one to specify systems and effects that are really significant for theoretical and numerical researches, and especially for practical applications [1, 2]. Among chaotic attractors, structural stability is intrinsic to those possessing the uniform hyperbolicity (“the systems with axiom A”), mathematical examples of which were advanced already since 60’s – 70’s [3–6]. That time, such attractors were mathematical examples of which were advanced already since 60’s – 70’s [3–6]. That time, such attractors were introduced on the one-dimensional Swift-Hohenberg equation [12]. Consider its following modification:

\[ \partial_t u + [1 + \kappa^2(t) \partial_x^2] u = [A + B \chi(x)] u - u^3. \] (1)

Here \( A \) is a positive parameter controlling the Turing instability. An instant value of \( \kappa \) determines the wave number of the unstable Turing mode. In our case \( \kappa(t) \) is assumed to be a periodic function: \( \kappa(t) = 1 \) for \( nT < t < (n+1/2)T \), and \( \kappa(t) = 1/3 \) for \( (n+1/2)T < t < (n+1)T \). This switching provides the excitation of two distinct alternating in time Turing patterns characterized by the dominating wave numbers, \( k = 1 \) and \( k = 3 \), respectively. The time interval \( T \) between the switchings is supposed to exceed the characteristic time duration of the formation or decay of the Turing patterns. A nonlinear cubic term in the equation is responsible for saturation of the instability. Also, the coefficient at the linear term in the equation is assumed to depend on the spatial coordinate that corresponds to the presence of a spatial non-uniformity characterized by a function \( \chi(x) \); its role will be clarified below. Assuming the ring geometry and periodic boundary conditions \( u(x, t) \equiv u(x+L, t) \) (PBC), it is natural to set the length of the system as \( L = 2\pi \ell \), with integer \( \ell \), to get the geometry supporting the Turing patterns of both the wave numbers \( k = 1 \) and 3.

The system operates as follows. In each time interval with \( \kappa(t) = 1 \) the Turing pattern with the dominating wave number \( k = 1 \) arises, which is characterized by some spatial phase \( \varphi \): \( u \sim U_1 \cos(x + \varphi) + U_3 \cos(3x + 3\varphi) \), where \( U_3 \ll U_1 \), and \( U_1 \) is of the order of \( \sqrt{A} \). (The
third harmonic appears naturally due to the cubic nonlinear term in the equation.) After the switch to $\kappa = 1/3$ the system becomes unstable in respect to the harmonic component with $k = 3$, while that with $k = 1$ starts to decay. The initial stimulation of the short-wave pattern is provided by the component $U_3$; so, it accepts the spatial phase $3\varphi$. At the end of the considered time interval the first harmonic component practically disappears, and we have $u \sim U_3 \cos(3x + 3\varphi)$, with $U_3$ of the order of $\sqrt{A}$. After the next switch, when $\kappa = 1$ again, the third harmonic decays, but the first harmonic becomes unstable and starts to grow. A germ for this growth is provided by a component at the wave number $k = 1$ arising from the combination of the decaying short-wave pattern and the spatially dependent coefficient $\chi(x)$. If the Fourier expansion of $\chi(x)$ contains a dominating second harmonic $k = 2$, the long-wave mode will arise with the phase $3\varphi$, due to the term proportional to $\cos 2x \cos(3x + 3\varphi) = (1/2) \cos(x + 3\varphi) + \ldots$. Thus, on each complete period of modulation $T$ the phase of the spatial pattern undergoes the tripling (up to a constant phase shift): $\varphi_{n+1} = 3\varphi_n + \text{const}$. This is an expanding circle map with chaotic behavior characterized by the positive Lyapunov exponent $\Lambda = \ln 3 \approx 1.0986$ [13]. Since the phase map is uniformly expanding, the stroboscopic map corresponding to the transformation of the states $u_n(x) \equiv u(x, t_n)$ from $t_n = nT + \text{const}$ to $t_{n+1}$ is expected to be hyperbolic.

Of course, this mode of operation occurs under the proper choice of the parameters. A value of $A$ is selected to get an instability at $k = 1$ with a decay at $k = 3$, or vice versa, at successive half-periods of parameter modulation. The term $BY(x)$ must be small (comparing to the fully developed pattern amplitude) to contribute only as a germ for the formation of the long-wave pattern, although this germ should be of a sufficient level to start the process with saturation on the time scale $T$. In fact, the requirements are not very strict: the described type of behavior occurs in a fairly wide parameter range.

Figure 1 illustrates the spatio-temporal behavior of the system observed for the case of PBC. The 3D-plot $u(x, t)$ is obtained using computations on a spatial grid with the node separation $\Delta x = L/N$, where $N$ is a number of the nodes. One can observe the alternating evolution of the Turing structures: a long-wave pattern first appears, then decays, and is replaced by a short-wave one. After the period $T$, the long-wave pattern reappears but with a different spatial phase (shift along $x$-axis), and the process repeats. As we show in Fig. 2(a), the spatial phases recorded stroboscopically follow a chaotic map of the expected type. To obtain this diagram, we determine the spatial phases at $t_n = (n + 1/4)T$ as $\varphi_n = \arg[u(L/2, t_n) + i\partial_x u(L/2, t_n)]$, where the spatial derivative $\partial_x u$ is estimated by the numerical differentiation, and the results are plotted in coordinates $\varphi_{n+1}$ versus $\varphi_n$. This empirical map is of the expected topological type: one revolution for the pre-image corresponds to three revolutions for the image.

To characterize chaos quantitatively and demonstrate its robustness, we calculate the Lyapunov exponents. Figure 2(a) shows the first five Lyapunov exponents for the stroboscopic map as functions of the parameter $A$. The chaotic mode of operation occurs above some threshold around $A \approx 0.38$. In the chaotic regime there is one positive Lyapunov exponent, which remains almost constant in a wide parameter range. In particular, at $A = 0.6$ the Lyapunov exponents are $\Lambda = \{1.018, -9.34, -9.34, -11.42, -18.64, \ldots\}$. As expected, the largest exponent is close to $\ln 3$. As seen from the diagram, all the exponents depend on the parameter smoothly, without sharp spikes or dips. This is a manifestation of robustness of the hyperbolic chaos [9, 10]. The Kaplan-Yorke dimension of the attractor varies slightly, see the solid line in Fig. 2(b); in particular $D_{\text{KY}} \approx 1.11$ at $A = 0.6$.

To confirm the validity of the used spatial discretization, in Figure 2(c) we show the sixteen largest Lyapunov exponents obtained at a fixed length $L$ with different sizes of the numerical mesh $N$. The decrease of
**Figure 3.** (Color online) (a) Five largest Lyapunov exponents vs. $A$ for the stroboscopic map of the system (1) at $t_n = (n + 1/4)T$, PBC. (b) Kaplan-Yorke dimension for PBC (solid line) and ZBC (dotted line). (c) PBC, first sixteen exponents for different $N$: pluses, crosses, stars and squares refer to $N = 64, 128, 256$, and $512$, respectively. (d) ZBC, first five exponents. Other parameters are the same as in Fig. 4 for PBC and $B = 0.03$, $T = 25$, $L = 8\pi$, $N = 128$ for ZBC.

$\Delta x = L/N$ corresponds obviously to approaching the continuous limit. The left-hand parts of the curves overlap perfectly; so, the larger exponents are in good correspondence for all tested step sizes. The discrepancy visible in the right-hand part of the plot for large negative exponents decreases with the growth of $N$. Hence, we can be sure that the properties revealed in the computations with the finite discretization size are valid for the continuous system as well.

Next, we perform a direct test of the hyperbolicity. The hyperbolicity implies that there are no tangencies between the stable and unstable manifolds of orbits belonging to the attractor. Occurrence of a tangency is determined by the zero angle between the expanding and contracting tangent subspaces spanned by the corresponding covariant Lyapunov vectors (14). Following the method for testing hyperbolicity described in (15), we examine the distribution of these angles by considering the orthogonal complement to the contracting subspace, which is normally much less dimensional than the contracting subspace itself. If there are $K$ expanding directions, it is sufficient to calculate $K$ orthogonal backward and forward Lyapunov vectors, to construct a $K \times K$ matrix $P$ of their scalar products, and to check how close to zero is the normalized characteristic number

$$d_K = | \text{det}(P) |. \quad (2)$$

By the definition, $0 \leq d_K \leq 1$. The procedure is applied at a representative set of points on a trajectory on the attractor. The distribution of $d_K$ separated well from zero means that the chaos is detected as hyperbolic: the tested trajectory does not contain any points with tangencies of the expanding and contracting Lyapunov vectors.

In application to the stroboscopic map of the system (1) the calculations are simple because $K = 1$. For the parameters used in Fig. 4 we processed $10^5$ points and observed that $(1 - 5 \times 10^{-5}) < d_1 \leq 1$. It means that the expanding direction is always almost orthogonal to the contracting subspace. Thus, the conjecture that the attractor is uniformly hyperbolic is confirmed, but, of course, a rigorous mathematical proof of the hyperbolicity would be desirable anyway.

As in the system only two modes with the wave numbers $k = 1$ and $k = 3$ are basically involved, one can expect that the essential properties of the dynamics can be described with a truncated model. To derive the low-dimensional model we proceed as follows. Accounting for the relevant modes, we use the ansatz $u = a_1(t) \cos x + a_3(t) \cos 3x$, and by substituting to Eq. (3), we multiply the resulting expression by $\cos x$ and $\sin x$, by $\cos 3x$ and $\sin 3x$, and for each case perform the integration over the spatial period $2\pi$. The result is a set of equations for the amplitudes of the modes, which can be compactly expressed in the complex form as

$$\dot{c}_1 = \mu_1 c_1 - \frac{1}{2} |c_3|^2 c_1 - 2Bc_3 + (3c_1^* c_3 - 2B) c_1^* , \quad c_3 = \mu_3 c_3 - \frac{1}{2} |c_3|^2 c_3 - 2Bc_1 + c_1^* , \quad (3)$$

where the asterisk denotes complex conjugation, $c_1 = a_1 + ib_1$, $c_3 = a_3 + ib_3$, $\mu_1 = A - (1 - \kappa^2)^2$, $\mu_3 = A - (1 - 9\kappa^2)^2$, and $\kappa = \kappa(t)$, as before. Notice that the structure of the equations resembles that for the amplitude equations obtained for other models with hyperbolic attractors of Smale-Williams type (16).

Figure 5 illustrates the dynamics of the model (3). Observe the switchings after each next half-period $T/2$. Different heights of the humps of $\text{Re} c_1$ and $\text{Im} c_1$ arise due to the variations of the phases of $c_1$. The phases transform stroboscopically according to the triple expanding circle map; see Fig. 2(b) for the diagram for the phases computed as $\varphi_n = \text{arg}[c_1(t_n)]$. The Lyapunov exponents evaluated for the stroboscopic map of the model (5) at $A = 0.6$, $B = 0.03$, $T = 25$ are
Λ = \{1.083, −12.5, −804.7, −806.5\}, and the Kaplan-Yorke dimension is 1.09. Notice that the first exponent is close to ln 3. The hyperbolicity test described above again shows that the expanding and contracting subspaces are almost perfectly orthogonal.

Moreover, there is a large parameter interval, where the part of the system agrees well with the expected form. However, computations show that such a blocking occurs only in short systems. If the length is large enough, patterns in the middle part of the system still interact in the same way as for PBC, while the parts close to the ends undergo deformations to fit ZBC; see Fig. 2(c) and 2(d) for L = 8π. The map for the spatial phases at the middle part of the system agrees well with the expected form. Moreover, there is a large parameter interval, where the system has a single positive Lyapunov exponent of value almost independent on A. At A = 0.6 the Lyapunov exponents are Λ = \{1.047, −1.59, −3.92, −4.97, −6.16, \ldots\}, and the Kaplan-Yorke dimension is 1.66. The hyperbolicity test shows pronounced separation of d1 from the origin, although the distribution is wider (0.93 < d1 < 1) than for PBC.

Summarizing, in this letter we have shown how the hyperbolic chaotic dynamics can emerge in extended systems due to an interplay of spatial patterns with different wave lengths. In our model system the spatial phases of the patterns evolve in time according to the Bernoulli-type tripling map, and their dynamics is strongly and robustly chaotic, while the amplitudes behave in a rather regular manner. The mechanism of the hyperbolic chaos is similar to that in alternately excited oscillations, studied earlier [4]. In some respects, the chaosization of spatial phases appears to be easier for implementation (there is no necessity to have more than one involved subsystem). We have demonstrated the expected chaotic behavior in the partial differential equation of the Swift-Hohenberg type, and in the truncated model represented by a set of ordinary differential equations. It should be emphasized that the kind of dynamics we consider is not specific for the Swift-Hohenberg equation only. Ingredients needed for the phase multiplication mechanism, namely, the alternation of patterns due to parameter modulation, the non-linearity, and the spatial inhomogeneity can be either found or created in many spatially extended systems. As expected, these results open prospects for the search and constructing for hyperbolic chaos in pattern-formation for systems in fluid dynamics (Faraday ripples, convection rolls) and in reaction-diffusion systems (Turing structures, advection induced patterns) [12]. In the case of microfluidic systems [17, 18], an interesting question for future studies is the effect of hyperbolic chaos on Lagrangian mixing properties.

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[13] Also χ(x) can have the dominating fourth harmonic. In this case \cos 4x \cos (3x + 3ϕ) = (1/2) \cos (x − 3ϕ) + \ldots, so the map for the phase will be ϕ_{n+1} = −3ϕ_n + const. Under the variation of relative weights of the components k = 2 and k = 4 some transitions between the topologically distinct behaviors will occur.

Figure 4. (Color online) Solution of Eq. 13 at A = 0.6, B = 0.03, T = 25. Solid (red) and dotted (blue) lines refer to a1 = Re c1, and b1 = Im c1, respectively.