ANALYTICAL STUDY OF RESONANCE REGIONS FOR SECOND KIND COMMENSURATE FRACTIONAL SYSTEMS

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Abstract. The aim of this paper is to determine analytically the resonance limits for second kind commensurate fractional systems in terms of the pseudo damping factor $\xi$ and the commensurate order $v$ and in addition specify the different resonance regions. In the literature, these limits and regions have never been discussed mathematically, they are determined numerically. Second kind commensurate fractional systems are resonant if the equation:

$$\Omega^{2v} + 3\xi \cos(\frac{v\pi}{2})\Omega^{2v} + (2\xi^2 + \cos(v\pi))\Omega^v + \xi \cos(v\pi/2) = 0,$$

obtained by setting the first derivative of the amplitude-frequency response equal to zero, has at last one strictly positive root. As in the conventional case, resonance limits correspond to zero discriminant of the last equation. This discriminant is a cubic equation in $\xi^2$ whose coefficients change depending on $v$. To resolve this equation, the tangent trigonometric solving method is used and the relationship between $\xi$ and $v$ is established, which represents the resonance limits expression. To search resonance regions, a mathematical study is conducted on the first equation to find the positive roots number for each $(v, \xi)$ combination. Compared to works already achieved, a new region appeared in the region of single resonant frequency with an anti-resonant one. The results are tested through numerical examples and applied to a fractional filter.

1. Introduction. Fractional Order Calculus (FOC) is the field of mathematics which is concerned with the investigation and application of derivatives and integrals of arbitrary order (real or complex). FOC is a powerful tool for modelling real-world phenomena; it can provide more accurate and compact mathematical descriptions for some complicated systems.

Recently, Fractional Order Systems (FOS) have received increasing attentions in many engineering applications [9] such as mechanics, electricity, chemistry, biology, and so on. FOS raise exciting challenges to develop new methodologies for modelling and identification [15, 24, 1, 32, 34, 31, 8], control [22, 23, 6, 21, 29, 14] and diagnosis [2, 3, 4, 33] by involving fractional order dynamics to physical systems.

1.1. Mathematical background. Motivated by the need of time-domain and frequency-domain analysis, several researchers have studied FOS in different contexts (linear and non linear, commensurate and non-commensurate, ...).

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These systems can be described by transfer function form

\[ F(s) = \frac{\sum_{i=0}^{m} b_i s^{\alpha_i}}{1 + \sum_{j=1}^{n} a_j s^{\beta_j}} \]  

(1)

where \((a_j, b_i) \in \mathbb{R}^2\), differentiation orders \(\alpha_1 < \alpha_2 < \ldots < \alpha_m\) and \(\beta_1 < \beta_2 < \ldots < \beta_n\) are allowed to be non-integer positive numbers.

When differentiation orders are arbitrary, equation (1) is called non-commensurate fractional transfer function. In this case, viewed the complexity of the algebraic methods as Routh and Jury criteria \[11\], stability has been essentially investigated using graphical methods as in \[30, 20, 28\]. However, very recently, \[5, 12\] establish numerically the stability and resonance limits of uncommensurate systems.

In the case of commensurate order, fractional systems can be represented as rational transfer functions in \(s^v\)

\[ F(s) = \frac{\sum_{i=0}^{m} b_i (s^v)^i}{1 + \sum_{j=1}^{n} a_j (s^v)^j} \]  

(2)

where the commensurate order \(v\) is a strictly positive real number. The roots in \(s^v\) of respectively numerator and denominator are called \(s^v\)-zeros and \(s^v\)-poles of \(F(s)\).

For commensurate fractional order systems, some powerful criteria have been proposed. The most well-known one is Matignon’s stability theorem \[19\]. It permits us to check system stability straightforwardly through the location of its \(s^v\)-poles instead of computing the \(s\)-poles. Matignon’s theorem is the starting point of several results in the field.

**Theorem 1.1.** \[19\] Extended. A commensurate transfer function with an order \(v\), as in (2), is stable iff

\[ 0 < v < 2 \text{ and } |\arg(p_k)| > \frac{v\pi}{2}, \quad k = 1, \ldots, n \]  

(3)

where \(p_k\) represent \(s^v\)-poles of \(F\).

As in the conventional case, commensurate fractional transfer functions have two elementary functions.

- Transfer functions of first kind

\[ F(s) = \frac{1}{1 + \left(\frac{s}{w_n}\right)^v} \]  

(4)

- Transfer functions of second kind

\[ F(s) = \frac{1}{1 + 2\xi\left(\frac{s}{w_n}\right)^v + \left(\frac{s}{w_n}\right)^{2v}} \]  

(5)

where \(w_n\), \(\xi\) and \(v\) represent natural frequency, pseudo-damping factor and commensurable order respectively \((w_n \in \mathbb{R}^*_+, \xi \in \mathbb{R} \text{ and } v \in \mathbb{R}^*_+)\).

In the literature, several works have studied commensurate fractional order systems as in \[10\] where the authors consider the time response of first kind elementary transfer functions. Moreover, \[27\] examined the time-domain performances (maximum overshoot and setting time) of elementary transfer functions (4) and (5) in terms of the differentiation order \(v\) and, for second kind transfer functions, the pseudo-damping factor \(\xi\) (denoted \(\cos(\theta)\)). In \[26, 25\], the researchers have generalized 1st and 2nd order filters to the fractional domain showing some advantages.
Traditional analog filters are of integer order. However, using fractional order capacitors and inductors, filters may be represented by the more general fractional order differential equations in which integer order filters are only a tight subset of fractional order filters. The design of the conventional integer-order filters is limited to 1st, 2nd, ... orders, whereas the design of fractional order filters enables the designers to implement any arbitrary filter order such as 0.6 or 2.7. These fractional order filters offer better performance and flexibility compared to their integer-order counterparts [9]. The study conducted in [17] computes the $H_2$-norm of fractional systems of the first and second kind. Additionally, [18] deduces, from the Matignon’s theorem, the stability of such transfer function. The resonance limits are computed analytically for systems of the first kind and numerically for systems of the second kind (Note that no analytical study has carried out so far for this type of system hence the interest of this work). Moreover, [16] determines finiteness conditions of $L_p$-norms of the impulse response for such system.

1.2. Problem formulation. In this work, we consider the second kind commensurate fractional system (5). This system has been studied in [18] in which the stability condition is deduced from the Matignon’s theorem [19] where the resonance limits and resonance regions are determined numerically (no mathematical study).

A) Stability of second kind fractional transfer functions [18]

Theorem 1.2. The transfer function (5) is stable iff

$$0 < v < 2 \text{ and } \xi > -\cos\left(\frac{v\pi}{2}\right)$$

(6)

B) Resonance limits of second kind fractional transfer functions [18]

The frequency response of (5) is given by

$$F(j\Omega) = \frac{1}{1 + 2\xi(j\Omega)^v + (j\Omega)^{2v}}$$

(7)

with $\Omega = \frac{w}{w_n}$ named the normalized frequency.

Transfer functions described by (5) are resonant if

$$\frac{d}{d\Omega} |F(j\Omega)| = 0 \Rightarrow \Omega^{3v} + 3\xi \cos\left(\frac{v\pi}{2}\right)\Omega^{2v} + (2\xi^2 + \cos(v\pi))\Omega^v + \xi \cos(v\frac{\pi}{2}) = 0$$

(8)

has at least one real and strictly positive solution corresponding to the maxima of $|F(j\Omega)|$. The last cubic equation in $\Omega^v$ can have, in $\mathbb{R}$, one (simple or triple), two (which one is a double root) or three (distinct) roots. The number of resonant frequencies of $F$ depends on the number of positive real roots of (8). In [18, 13], these roots are computed numerically then the $v\xi$-plane is divided into resonance regions according to the number of positive real roots. We redesigned the found result in Figure 1. Such region represents combinations of $v$ and $\xi$ and reflects the resonance of systems; the gray region produces no resonant systems, the yellow and green regions produce resonant systems with respectively one and two resonant frequencies. The red region represents unstable systems.

The upper contour of yellow region in the interval $v \in [0.5, 1]$ and the lower contour of green region, denoted respectively $\xi_0$ and $\xi_1$ in [18, 13], are computed numerically. Furthermore, the beginning of green region according $v$, denoted $v_1$ in [13], is not determined (Figure 1).

1This figure is obtained by simulation using Matlab.
According to [18, 13], solving (8) analytically is not an easy task. In this context, the present paper complements the existing study of second kind commensurate fractional systems. Based on a detailed mathematical study of equation (8), this work provides two main contributions: determine analytically resonance limits $\xi_0$ and $\xi_1$ (also $v_1$) and divide $v\xi$-plane into resonance regions according to positive roots number.

The remainder of this paper is organized as follows: Section 2 presents firstly the required mathematical formulation of resonance limits problem which conducted to a cubic equation. Next, the solution of this equation is discussed to lead finally to the analytical expression of resonance limits. In addition, in order to give a mathematical proof for resonance regions division, Descartes Rule of Signs is applied to (8). In section 3, the $v\xi$-plane is divided into resonance regions and four numerical examples validate our results. Conclusions and further work are presented in section 4.

2. Analytical study of resonance regions. The first problem encountered when searching the expression of resonance limits is: How to formulate mathematically these limits and what does represent these limits for (8)?

To answer to these questions, the idea was to return to conventional systems then inspire from their procedure.

Let’s remind the resonance limit for conventional second order systems. These systems represent a subset of second kind fractional commensurate systems (5). The substitution ($v=1$) in the last equation gives

$$F(s) = \frac{1}{1 + 2\xi(\frac{\omega}{\omega_n}) + (\frac{\omega}{\omega_n})^2}$$

Second order systems are resonant if (8) with $v=1$

$$\Omega(\Omega^2 + 2\xi^2 - 1) = 0$$
has at least one strictly positive root. The known resonance limit $\xi_{rl} = \sqrt{2}$, which separates two regions (resonant and no resonant), corresponds to zero discriminant of (10) (i.e. (8)).

In the similar manner, resonance limits $\xi_0$ and $\xi_1$ for second kind fractional commensurate systems correspond to zero discriminant $D$ of the equation (8). After calculus, $D$ is given by

$$D = 4(9\cos^2(v\pi/2) - 8)\xi^6 + 12(-3\cos^4(v\pi/2) - 2\cos^2(v\pi/2) + 4)\xi^4$$

$$+ 12(3\cos^6(v\pi/2) - 2\cos^4(v\pi/2) + 2\cos^2(v\pi/2) - 2)\xi^2 - 4\cos^3(v\pi)$$

(11)

The latter is a cubic equation in $\xi^2$. So to resolve it, we define a new variable $y = \xi^2$

(12)

Consequently, we obtain a third order equation in $y$

$$D = 0 \Rightarrow ay^3 + by^2 + cy + d = 0$$

(13)

with

$$\begin{align*}
a &= -9\cos^2(v\pi/2) + 8 \\
b &= 3(3\cos^4(v\pi/2) - 2) \\
c &= 3(-3\cos^6(v\pi/2) + 2\cos^4(v\pi/2) - 2\cos^2(v\pi/2) + 2) \\
d &= \cos^3(v\pi)
\end{align*}$$

(14)

Next, the tangent trigonometric solving method is applied. Starting by substitution

$$y = z - T$$

(15)

with

$$T = \frac{bc - 9ad}{2(b^2 - 3ac)}$$

(16)

After calculus, $T$ is reduced to

$$T = (\cos(v\pi) - 9)/8$$

(17)

Consequently, substituting (15) in (13) yields

$$rz^3 + sz^2 + pz + q = 0$$

(18)

with

$$\begin{align*}
r &= 8 - 9\cos^2(v\pi/2) \\
s &= \frac{9}{32}(8 - 7\cos^4(v\pi/2))\sin^2(v\pi/2) \\
p &= \frac{9}{16}(8 - 9\cos^2(v\pi/2))\sin^4(v\pi/2) \\
q &= \frac{9}{64}(8 - 7\cos^2(v\pi/2))\sin^6(v\pi/2)
\end{align*}$$

(19)

We observe that

$$pr > 0$$

(20)

which means that (13) has one real solution. Since

$$p^2 - 3sq = -\frac{729}{8}\sin^4(v\pi/2)\cos^2(v\pi/2)$$

(21)

is negative. So, the solution is expressed in hyperbolic cotangent as follow

$$z = \sqrt{\frac{P}{3r}}\coth(\theta)$$

(22)

where

$$\coth(3\theta) = -\frac{3q}{p} \sqrt{\frac{3r}{p}}$$

(23)
According to \((19)\) and \((23)\), we deduce
\[
\theta = -\frac{1}{6} \ln(8\tan^2(v\frac{\pi}{2})) \quad (24)
\]
Consequently, substituting \((19)\) and \((24)\) in \((22)\) yields
\[
z = 3\frac{\sin^2(v\frac{\pi}{2})}{4} \left(1 + 2\tan^2(v\frac{\pi}{2}) \right) \quad (25)
\]
Then, \((25)\) and \((17)\) in \((15)\) and after rearrangement, the solution of \((13)\) is
\[
y = \frac{\cos^2(v\frac{\pi}{2})(1 + \tan^2(v\frac{\pi}{2}))(1 - \tan^2(v\frac{\pi}{2}))^3}{1 - 2\tan^2(v\frac{\pi}{2})} \quad (26)
\]
This solution is plotted in Figure 2.

Note that the graph has two vertical asymptotes. Their equations are deduced from
\[
1 - 2\tan^2(v\frac{\pi}{2}) = 0 \quad (27)
\]
Hence
\[
v = \frac{2}{\pi} (\pm \text{argtg}(1/2^{3/2}) + k\pi) \quad k \in \mathbb{Z} \quad (28)
\]
Since \(v\) is between 0 and 2 (stable system condition), so \(v\) has two values
\[
\begin{cases} 
\text{asym1} = 0.2163 \\
\text{asym2} = 1.7837 
\end{cases} \quad (29)
\]
Recall that \(y = \xi^2\), so we keep only the positive part of \(y\). To do this, lets search where \(y\) crosses the \(v\)-axis. That means
\[
y = 0 \Rightarrow 1 - \tan^2(v\frac{\pi}{2}) = 0 \quad (30)
\]
Hence
\[
v = \pm 0.5 + 2k, \quad k \in \mathbb{Z} \quad (31)
\]
So \(v\) has two values
\[
\begin{cases} 
\text{cr1} = 0.5 \\
\text{cr2} = 1.5 
\end{cases} \quad (32)
\]
As depicted in Figure 2, the graph of $y$ intersects the $v$-axis at two points ($v_{cr1} = 0.5$ and $v_{cr2} = 1.5$). Now, $\xi$ is deduced

$$\xi = \pm \cos\left(v\frac{\pi}{2}\right) \left(1 - \tan^{2}\left(v\frac{\pi}{2}\right)\right)^{\frac{1}{2}} \sqrt{1 - 2\tan^{2}\left(v\frac{\pi}{2}\right)}$$

with

$$v \in [0; 0.2163] \cup [0.5; 1.5] \cup [1.7837; 2]$$

Figure 3 represents the solution of (13) in $\xi$. On the same figure, the $v\xi$-plane is divided according to the discriminant signs. Curved colored lines (red, blue, purple and green) correspond to $D = 0$.

The discriminant sign of a cubic equation determines the number of real roots. There are three cases:

- If $D < 0$, the equation has one root;
- If $D = 0$, the equation has one root (triple) or two roots (which one is double);
- If $D > 0$, the equation has three distinct roots.

To determine resonance regions, the positive roots number of (8) must be determined. To do it, Descartes’ Rule of Signs [7] is used.

**Theorem 2.1. Descartes Rule of Signs**

- The number of positive roots of $f(x) = 0$ is either equal to the number of variations in sign of $f(x)$, or less than that by an even number.
- The number of negative roots of $f(x) = 0$ is either equal to the number of variations in sign of $f(-x)$, or less than that by an even number.

To apply Descartes’ theorem, let’s consider the first polynomial function of (8)

$$f(x) = x^3 + 3\xi\cos(v\pi/2)x^2 + (2\xi^2 + \cos(v\pi))x + \xi\cos(v\pi/2)$$

with $x = \Omega^v$, then the second polynomial function

$$f(-x) = -x^3 + 3\xi\cos(v\pi/2)x^2 - (2\xi^2 + \cos(v\pi))x + \xi\cos(v\pi/2)$$
Firstly, covering the \( v\xi \)-plane with the coefficients signs of \( f(x) \) and \( f(-x) \) as shown in Figure 4. We notice that the elliptical shape corresponds to the third coefficient equal to zero.

Figure 4. Coefficients signs of \( f(x) \) and \( f(-x) \) in the \( v\xi \)-plane.

Then, let’s retake Figures 3 and 4 to make a new division of the space by covering all combinations of discriminant signs and coefficients signs of \( f(x) \) and \( f(-x) \) (Figure 5).

Figure 5. Division of the \( v\xi \)-plane according to \( D \) signs and coefficients signs of \( f(x) \) and \( f(-x) \).

Finally, applying Descartes Rule of Signs to find positive roots of (8). Table 1 reveals the different regions for all possible combinations \((v, \xi)\) according to Figure 5 division. For each region, the discriminant sign (Figure 3) and coefficients signs of \( f(x) \) and \( f(-x) \) (Figure 4) are determined. Consequently, the roots number of (8), the positive and negative roots number possibilities are determined. By analysis of obtained numbers, the positive roots number of (8) can be deduced.
Table 1. Application of Descartes Rule of Signs according to Figure 5 division.

| Regions | 1 | 2 | 31 | 32 | 41 | 42 | 5 | 6 |
|---------|---|---|----|----|----|----|---|---|
| $D$ sign | + | + | - | - | - | - | + | + |
| $R$ Nbr | 3 | 3 | 1 | 1 | 1 | 1 | 3 | 3 |
| $f(x)$ coefs signs | + + + + - + + + + + + + + + + + + + - - - - - - - |
| Sign changes Nbr | 0 | 3 | 0 | 2 | 3 | 1 | 2 | 1 |
| $R^+$ Nbr Possibility | 0 | 3 or 1 | 0 | 2 or 0 | 3 or 1 | 1 | 2 or 0 | 1 |
| $f(-x)$ coefs signs | - + + - - - - - - - - - - - + + + - + + + |
| Sign changes Nbr | 3 | 0 | 3 | 1 | 0 | 2 | 1 | 2 |
| $R^-$ Nbr Possibility | 3 or 1 | 0 | 3 or 1 | 1 | 0 | 2 or 0 | 1 | 2 or 0 |
| $R^+$ Nbr | 0 | 3 | 0 | 1 | 1 | 2 | 1 |
| $R^-$ Nbr | 3 | 0 | 1 | 1 | 0 | 0 | 1 | 2 |

Table 1 can be reduced to Table 2 associating every positive roots number with corresponding regions.

Table 2. Positive roots number of (8) and corresponding regions.

| $R^+$ Nbr | 0 | 1 | 2 | 3 |
|-----------|---|---|---|---|
| Regions   | 1 and 3 (31 and 32) | 4 (41 and 42) and 6 | 5 | 2 |

The roots of (8) represent the intersection of the $f$-graph with the $x$-axis. To illustrate graphically these roots, refering to Figure 3, we choose firstly an arbitrary value of $(v, \xi)$ for each region from 1 to 6, also for each colored curved lines (red, blue, purple and green) and colored straight lines (black and coral). Then, the chosen combination $(v, \xi)$ is replaced in (34). Finally, the obtained $f(x)$ is plotted as shown in Figure 6.

Figure 7 presents a new division of the $v\xi$-plane according to the number of strictly positive roots as shown in Table 2. We can distinguish four regions denoted $R_0^+, R_1^+, R_2^+$ and $R_3^+$ with respectively zero, one, two and three strictly positive real roots. In comparison with Figure 3, we observe that the curved colored lines (red and green) separating respectively the regions (1, 3) and (4, 6) disappeared because each pair of regions has the same positive roots number.
Figure 6. Roots of (8) for different regions of the $v_\xi$-plane according to Figure 3 division.
Applying stability condition (6), Figure 7 is reduced to Figure 8.

Finally, resonance limits $\xi_{rl}$ (purple and blue curved lines in Figure 8), which correspond to two roots (one is double) of (8), are then expressed by

$$\xi_{rl} = \cos(v_\frac{\pi}{2})(\tan^2(v_\frac{\pi}{2}) - 1) \sqrt{\frac{1 - \tan^2(v_\frac{\pi}{2})}{1 - 2\tan^2(v_\frac{\pi}{2})}}$$

with

$$v \in [0.5 ; 1] \cup \{1.7837 ; 2\}$$

This result is in accordance with the rational case. Indeed, for the order $v = 1$, the known resonance limit for rational transfer function of second order can be
found from (36). After removing the indeterminate case, $\xi_{rl} = \frac{\sqrt{2}}{2}$. Note that the resonance limits, denoted by $\xi_0$ and $\xi_1$ in [18], have the same expression (36) (for two different intervals of $v$) because they belong to the same curve (Solution of $D = 0$). While $v_1$ is the second vertical asymptote equation ($v_1 = v_{asy2} = 1.7837$).

3. Resonance regions and numerical examples. The Figure 9 shows resonance regions for stable commensurate fractional systems of the second kind. Four regions can be distinguished:

- **Gray region**: no-resonant region;
- **Yellow region**: resonant region with one resonant frequency;
- **Brown region**: resonant region with one resonant and one anti-resonant frequencies;
- **Green region**: resonant region with two resonant frequencies and one anti-resonant frequency.

![Figure 9. New division of resonance regions of the second kind commensurate fractional systems in the $v\xi$-plane.](image)

Referring to Figure 1, Figure 9 presents a new region in brown color. We summarise the different regions in Table 3 in which for each one an example is given. The Figure 10 illustrates the frequency response for each region.

Fractional order Low-Pass Filter (FLPF) constitutes a typical example for second fractional order systems (5). They were firstly proposed by Radwan and al [26], where classical first-order filter networks were generalized to be fractional order. The design of low-pass, high-pass, band-pass and all-pass filters were investigated using a single fractional-order capacitor with impedance $Z = \frac{1}{s^\alpha}$. Then, the same authors [25] were presented second fractional order filters with two fractional-order capacitors of the same order $\alpha$. They used, as design examples, popular Sallen-Key and KHN filters. In their study, due to the complexity of the mathematical analysis, the frequency responses obtained are based on an arbitrary choice of the transfer function parameters. Hence the interest of this study; the mathematical analysis done above provides a detailed solution according to transfer function parameters.
Table 3. Resonance regions of the second kind commensurate fractional systems.

| Region and defined domain | Exemple ($\omega_n = 1$) |
|---------------------------|--------------------------|
| **Gray region**<br>No resonant frequency<br>$\begin{cases} 0 < v \leq 0.5 \text{ and } 0 \leq \xi \\ 0.5 < v \leq 1 \text{ and } \xi_{rl} \leq \xi \end{cases}$ | $v = 0.5 \text{ and } \xi = 0.5$<br>$\Omega^{1.5} + 1.06\Omega^3 + 0.56\Omega^{0.5} + 0.35 = 0$<br>$\Rightarrow \begin{cases} \Omega_1 = -0.87 \\ \Omega_2 = -0.37 + i0.08 \\ \Omega_3 = -0.37 - i0.08 \end{cases}$ |
| **Yellow region**<br>A single resonant frequency<br>$\begin{cases} 0 < v < 1 \text{ and } \cos(v\pi/2) < \xi < 0 \\ 0.5 < v < 1 \text{ and } \xi = 0 \\ v = 1 \text{ and } 0 < \xi < 0.7071 \\ 1 < v \leq 1.7837 \text{ and } -\cos(v\pi/2) < \xi \\ 1.7837 < v < 2 \text{ and } -\cos(v\pi/2) < \xi \leq \xi_{rl} \end{cases}$ | $v = 1.5 \text{ and } \xi = 0.9$<br>$\Omega^{4.5} - 1.91\Omega^3 + 1.62\Omega^{1.5} - 0.64 = 0$<br>$\Rightarrow \begin{cases} \Omega_1 = 0.93 \\ \Omega_2 = 0.73 + i0.52 \\ \Omega_3 = 0.73 - i0.52 \end{cases}$<br>$\Rightarrow \omega_{\text{res}} = 0.93 \text{rd/sec}$ |
| **Brown region**<br>A single resonant frequency with anti-resonant frequency<br>$0.5 < v < 1 \text{ and } 0 < \xi < \xi_{rl}$ | $v = 0.8 \text{ and } \xi = 0.28$<br>$\Omega^{2.4} + 0.26\Omega^{1.6} - 0.65\Omega^{0.8} + 0.09 = 0$<br>$\Rightarrow \begin{cases} \Omega_1 = 0.09 \\ \Omega_2 = 0.52 \\ \Omega_3 = -1.00 \end{cases}$<br>$\Rightarrow \omega_{\text{anti-res}} = 0.09 \text{rd/sec}$<br>$\omega_{\text{res}} = 0.52 \text{rd/sec}$ |
| **Green region**<br>Two resonant frequencies<br>$1.7837 < v < 2 \text{ and } \xi_{rl} < \xi$ | $v = 1.9 \text{ and } \xi = 2.5$<br>$\Omega^{5.7} - 7.41\Omega^{3.8} + 13.45\Omega^{1.9} - 2.47 = 0$<br>$\Rightarrow \begin{cases} \Omega_1 = 0.44 \\ \Omega_2 = 1.65 \\ \Omega_3 = 2.23 \end{cases}$<br>$\Rightarrow \begin{cases} \omega_{\text{res1}} = 0.44 \text{rd/sec} \\ \omega_{\text{anti-res}} = 1.65 \text{rd/sec} \\ \omega_{\text{res2}} = 2.23 \text{rd/sec} \end{cases}$ |
Figure 10. Magnitude Bode diagram for each region.

Let’s retake the example of Sallen-Key FLPF of $2v$ order constituted of two fractional order capacitors with the same order $v$, one op-amp and resistors (Figure 11).

Figure 11. Sallen-Key FLPF circuit.

The transfer function of this filter can be written as follows

$$F(s) = \frac{V_{out}}{V_{in}} = \frac{G}{R_1 R_2 C_1 C_2 s^{2v} + (R_1 C_1 + R_2 C_1 + R_1 C_2(1 - G)) s^v + 1}$$

(37)

That corresponds to (5) with DC gain $G = 1 + \frac{R_b}{R_a}$ and

$$w_n = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$\xi = \frac{R_1 C_1 + R_2 C_1 + R_1 C_2(1 - G)}{2 \sqrt{R_1 R_2 C_1 C_2}}$$

(38)
According to \( v \) and \( \xi \) values, the frequency response of Sallen-Key FLPF belongs to one of resonant regions (Figure 9). In the special case, where \( R_1 = R_2 \) and \( C_1 = C_2 \), the pseudo damping factor \( \xi \) is reduced to \((3 - G)/2\). It is clear that, in this case, \( \xi \) can not exceed 1. The variation of \( G \), i.e. \( R_b/R_a \), and \( v \) allow to obtain frequency responses (without resonance, with a single resonant frequency or with resonant and anti resonant frequencies). If, for example, \( R_b = 1.5R_a \), which implies \( \xi = 0.25 \), and \( v = 0.8 \), then the frequency response belongs to the brown region. Otherwise, if \( R_1 = R_2 \) and \( C_1 = 4C_2 \), yield \( \xi = (9 - G)/4 \), all resonance regions can be reached by varying \( G \), i.e \( \xi \) and \( v \). For example, if \( G = 3 \), i.e \( \xi = 1.5 \) and \( v = 1.9 \), then frequency response belong to green region.

4. Conclusion. In [18], resonance limits and regions of second kind commensurate fractional systems are determined numerically. In this paper, these limits are expressed analytically (36). To attain this concern, we did a detailed study of cubic equation (8). In addition resonance regions are specified with mathematical proof. Compared to [18], a new resonant region appeared (brown region). Moreover, the beginning of green region according \( v \) is determined analytically \((v_1 = 1.7837)\). The results are tested through numerical examples and applied on the fractional filter (Sallen-Key FLPF). As a further work, we plan to determine analytically the resonance frequencies for each resonant region (yellow, brown and green).

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