INDEPENDENCE BY RANDOM SCALING

Lancelot F. James and Peter Orbanz

HKUST and Columbia University

We give conditions under which a scalar random variable $T$ can be coupled to a random scaling factor $\xi$ such that $T$ and $\xi T$ are rendered stochastically independent. A similar result is obtained for random measures. One consequence is a generalization of a result by Pitman and Yor on the Poisson-Dirichlet distribution to its negative parameter range. Another application are diffusion excursions straddling an exponential random time.

1. Introduction and main results. Distributional identities involving elementary random variables play an important role in probability and related fields. Such identities arise, for instance, in the study of path properties of stochastic processes [13, 12, 10], and in applications of Stein’s method [15]. A fundamental example is the following: For $a > 0$, generically denote by $G_a$ a Gamma($a, 1$) variable. If $G_a$ and $G_b$ are independent, then $(G_a + G_b) \perp \perp G_a/(G_a + G_b)$. Lukacs [11] has shown this property is exclusive to gamma variables, and hence characterizes the gamma distribution. This result and its ramifications are collectively known as the beta-gamma algebra. Its relevance to path properties of Brownian motion and related phenomena is highlighted by Revuz and Yor [22].

The distributional properties studied in the following are of the form $T \perp \perp \xi T$ for positive random variables $\xi$ and $T$. (1)

Lukacs’ characterization shows such variables exist—take $T = G_a + G_b$ and $\xi = 1/G_a$—but also implies $T$ is a sum of independent variables only if these variables are gamma. Pitman and Yor [21] have identified another case: Fix $\alpha \in (0, 1)$ and $\theta > 0$, and abbreviate $\zeta := G_{\theta/\alpha}$. Let $f_{\alpha}$ be an $\alpha$-stable density, $S_{\alpha, \theta}$ a variable with density proportional to $t^{-\theta}f_{\alpha}$, and denote by $(\tau_{\alpha}(y))_{y \geq 0}$ a generalized gamma subordinator, i.e. a non-decreasing Lévy process on $(0, \infty)$ with Lévy density $t \mapsto \alpha t^{-\alpha-1}e^{-t}/\Gamma(1-\alpha)$. Then

\begin{equation}
\begin{aligned}
& (i) \quad \frac{\tau_{\alpha}(\zeta)}{\zeta^{1/\alpha}} \perp \tau_{\alpha}(\zeta) \\
& (ii) \quad \tau_{\alpha}(\zeta) \overset{d}{=} G_{\theta} \\
& (iii) \quad \frac{\tau_{\alpha}(\zeta)}{\zeta^{1/\alpha}} \overset{d}{=} S_{\alpha, \theta},
\end{aligned}
\end{equation}

which follows from the proof of [21, Proposition 21]. Rescaling to $\tilde{\tau}_{\alpha}(y) := \tau_{\alpha}(y)/y^{1/\alpha}$ gives

\begin{equation}
\begin{aligned}
& (i) \quad \tilde{\tau}_{\alpha}(\zeta) \perp \tilde{\tau}_{\alpha}(\zeta) \cdot \zeta^{1/\alpha} \\
& (ii) \quad \tilde{\tau}_{\alpha}(\zeta) \cdot \zeta^{1/\alpha} \overset{d}{=} G_{\theta} \\
& (iii) \quad \tilde{\tau}_{\alpha}(\zeta) \overset{d}{=} S_{\alpha, \theta}.
\end{aligned}
\end{equation}

Clearly, (2i) is an instance of (1).

Since both gamma and stable variables are distinguished by their scaling behavior, it is natural to ask in how far scaling properties are intrinsic to (1). Our first result shows that
the relevant property is not scaling per se, but rather a form of exponential tilting. In the case of the stable, this exponential tilt manifests as a scaling operation; at close inspection, the relationship is visible already in [20]. Let \( T_0 \) be a non-negative random variable with density \( f_{T_0} \) and cumulant function \( \psi(s) := -\log \mathbb{E}[e^{-sT_0}] \). For any such random variable, the exponentially tilted variable \( T^{(s)}_0 \) and the polynomially tilted variable \( T^{(\nu)}_0 \) are given by the densities

\[
P(T^{(s)}_0 \in dt) = e^{-st+\psi(s)} f_{T_0}(t) dt \quad \text{and} \quad P(T^{(\nu)}_0 \in dt) = \frac{t^{-\nu}}{\mathbb{E}[T_0^{-\nu}]} f_{T_0}(t) dt,
\]

for \( s \geq 0 \), and for \( \nu > 0 \) chosen such that \( \mathbb{E}[T_0^{-\nu}] < \infty \).

**Theorem 1.** Fix \( \nu > 0 \), and let \( T_0 \) be a positive random variable with cumulant function \( \psi \) and \( \mathbb{E}[T_0^{-\nu}] < \infty \). Let \( \xi \) and \( T \) be positive, absolutely continuous random variables. Then

\[
(i) \quad T \perp \xi T \quad \quad (ii) \quad \xi T \overset{d}{=} G_\nu \quad \quad (iii) \quad T \overset{d}{=} T^{(\nu)}_0
\]

holds if and only if the pair \((T, \xi)\) satisfies

\[
P_{\psi, \nu}(\xi \in ds) = \frac{-\psi(s) s^{\nu-1}}{\mathbb{E}[T_0^{-\nu}] \Gamma(\nu)} ds \quad \text{and} \quad T|\{\xi = s\} \overset{d}{=} T^{(\nu)}_0.
\]

Conditional tilting thus yields a large class of random variables satisfying (1), and the scaled variable is always gamma.

The variables in (2) take scalar values. It is shown in [21], however, that the property extends to the entire path of the process \( \tau_\alpha \): For any \( y \in [0, 1] \),

\[
(i) \quad \frac{\tau_\alpha(y \xi)}{\xi^{1/\alpha}} \perp \tau(\xi) \quad \quad (ii) \quad \tau_\alpha(\xi) \overset{d}{=} G_\theta \quad \quad (iii) \quad \frac{\tau_\alpha(\xi)}{\xi^{1/\alpha}} \overset{d}{=} \tau_\alpha(1)^{[\nu]}.
\]

Combined with Theorem 1, this suggests an analogous result for general subordinators, which we state in terms of random measures: Let \( \Omega \) be a Polish space, \( \mu \) a probability measure on \( \Omega \), and \( \lambda \) a Lévy density on \((0, \infty)\). We assume \( \lambda \) is strictly positive and continuous, and \( f_0^\infty \min \{1, t\} \lambda(t) dt < \infty \). Let \((J_n, \omega_n)\) be the points of a Poisson process on \((0, \infty) \times \Omega\) with mean measure \( \lambda(t) dt \mu(d\omega) \). Then \( N := \sum_n J_n \delta_{\omega_n} \) is a random measure on \( \Omega \), with \( N(\Omega) < \infty \) a.s. If \( h \) is a non-negative function with \( \mathbb{E}[h(N(\Omega))] = 1 \), the random measure \( M \) specified by

\[
P(M \in dm) = h(m(\Omega))P(N \in dm)
\]

again satisfies \( M(\Omega) < \infty \). If \( \psi \) is the cumulant function of the scalar variable \( M(\Omega) \), one can define an exponential tilt \( M^{(s)} \) of \( M \) as \( P(M^{(s)} \in dm) = e^{\psi(s)-s\mu}P(M \in dm) \).

**Theorem 2.** Let \( M_0 \) and \( M \) be random measures of the general form (6), let \( \psi \) be the cumulant function of \( M_0(\Omega) \), and fix \( \nu > 0 \) such that \( \mathbb{E}[M_0(\Omega)^{-\nu}] < \infty \). Then

\[
(i) \quad M \perp \xi M(\Omega) \quad \quad (ii) \quad \xi M(\Omega) \overset{d}{=} G_\nu \quad \quad (iii) \quad M(\Omega) \overset{d}{=} M_0(\Omega)^{[\nu]} \quad (7)
\]

if and only if \( \xi \sim P_{\psi, \nu} \) and \( \xi(\xi = s) = M_0^{(s)} \).
The results are related through the total masses of the random measures: If $M_0$ and $M$ satisfy Theorem 2, their total masses $T_0 := M_0(\Omega)$ and $T := M(\Omega)$ satisfy Theorem 1.

Theorem 2 can be applied to normalized random measures: As $T = M(\Omega)$ is almost surely finite, $P := M/T$ is a random discrete probability measure [8]. Since $M$ in (7i) is independent of $\xi T$, and $T$ is a functional of $M$, it follows that

$$P \perp \xi T \quad \text{where} \quad \xi T \overset{d}{=} G_{\nu} . \quad (8)$$

The conditions above imply $P$ is of the form

$$P \overset{d}{=} \sum_{n \in \mathbb{N}} P_n \delta_{\omega_n} \quad \text{where} \quad (P_n) \perp (\omega_n) \quad \text{and} \quad \omega_1, \omega_2 \sim_{\text{iid}} \mu ,$$

and $(P_n)$ is a random sequence $P_1 \geq P_2 \geq \ldots$ with $\sum_n P_n = 1$ almost surely. It is hence no loss of generality to assume $\mu$ is the uniform law on $[0,1]$, or to neglect the atoms $\omega_n$ altogether. Throughout, we treat random probability measures and random sequences $(P_n)$ interchangeably. If the total mass $N(\Omega)$ of the random measure in (6) has density $f$, then $T = M(\Omega)$ has density $hf$. This density, and the Lévy density $\lambda$ of $N$, completely determine the law of $(P_n)$, which is called a Poisson-Kingman distribution [18], and denoted $PK(\lambda, hf)$. A distinguished example within the Poisson-Kingman family are the two-parameter Poisson-Dirichlet distributions $PD(\alpha, \theta)$, with parameters $\alpha \in [0,1]$ and $\theta > -\alpha$ [8, 21]. For $\theta > 0$, they are known to satisfy (8): There is a random measure $M$ with total mass $T$ such that $P = M/T$ satisfies

$$P \perp \xi T \quad \text{and} \quad \xi T \overset{d}{=} G_{\theta+\alpha} \quad \text{if} \quad (P_n) \sim \text{PD}(\alpha, \theta) .$$

If $\alpha > 0$, this is once again Proposition 21 of [21], and can be derived from (5) by choosing $M[0,y] := \tau(y\zeta)/\zeta^{1/\alpha}$, in which case $(P_n)$ has law $\text{PD}(\alpha, \theta)$. If $\alpha = 0$, choose $M[0,y] = \tau(y\theta)$ for a gamma subordinator $\tau$ instead; then $(P_n) \sim \text{PD}(0, \theta)$, and the result follows from Lukacs’ characterization.

Relative to Proposition 21 of Pitman and Yor [21], our results imply an extension to the case $\theta \leq 0$: Start with a generalized gamma subordinator $\tau_n(y)$ and $b > 0$. Size-biasing the process $\tau_n(yb^\alpha)/b$ turns it into a bridge

$$\frac{\tau_n(yb^\alpha)}{b} + \frac{G_{1-\alpha}}{b} \mathbb{1}\{U \leq y\} \quad \text{for} \quad U \sim \text{Uniform}[0,1] \quad \text{independently.}$$

In Section 2.3, we construct scalar random variables $H_{\alpha, \theta}$ and $\xi_{H_{\alpha, \theta}}$ such that randomizing $b$ by $\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta}$ defines a random measure

$$M[0,y] := \frac{\tau_\alpha(y\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta})^\alpha}{\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta}} + \frac{G_{1-\alpha}}{\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta}} \mathbb{1}\{U \leq y\} \quad (9)$$

for which the weights $(P_n)$ of $P = M/T$ have law $\text{PD}(\alpha, \theta)$.

**Proposition 3.** For any $\alpha \in [0,1]$ and $\theta > -\alpha$, the ranked weights $(P_n)$ derived from (9) satisfy $P \sim \text{PD}(\alpha, \theta)$, independently of

$$(\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta})T \overset{d}{=} G_{1-\theta} \quad \text{and of} \quad \xi_{H_{\alpha, \theta}} T \overset{d}{=} G_{\theta+\alpha} .$$
The remainder of this article describes applications and examples; proofs are collected in the appendix. While the results apply to quite general processes, our examples emphasize the stable subordinator, which leads to interesting extensions of Proposition 21 of [21].

2. Application to generalized gamma subordinators. In this section, we consider the scaled and time-changed process $\tau_\alpha(yb^\alpha)/b$ that already arose above. This process can be equivalently represented by exponentially tilting a stable subordinator [21]: Let $f_\alpha$ denote the density of an $\alpha$-stable random variable, and $\lambda_\alpha$ an $\alpha$-stable Lévy density. If $\sigma_\alpha^{(b)}$ is a subordinator with Lévy density $e^{-bt}\lambda_\alpha(t)$, then

$$\sigma_\alpha^{(b)}(y) \overset{d}{=} \frac{\tau_\alpha(yb^\alpha)}{b},$$

where the left-hand side is well-defined even if $b = 0$. The variable

$$X_{\alpha,b} := \frac{\tau_\alpha(yb^\alpha)}{b} \text{ hence has density } t \mapsto e^{-bt+b^\alpha f_\alpha(t)}.$$

Exponentially tilted Lévy densities as the one above define a class of Poisson-Kingman distributions for which our results take a special form: For a Lévy density $\lambda$ and $\nu > 0$, let $T_0$ be the total mass of a random measure defined by $\lambda$. The Poisson-Kingman distribution $\text{PK}(\lambda, \mathcal{L}(T_0^{[\nu]}))$ can be embedded in a one-parameter family $\text{PK}(e^{-bt}\lambda(t), \mathcal{L}(T_0^{[\nu]}))$, for $b \geq 0$, where $T_b$ is the total mass of a random measure defined by $e^{-bt}\lambda(t)$. The conditioning operation in Theorem 2 then takes the form of a parameter shift: A random probability measure $P$ with law $\text{PK}(e^{-bt}\lambda(t), \mathcal{L}(T_b^{[\nu]}))$ satisfies (8) if and only if

$$T|\xi = s \overset{d}{=} T_{b+s} \quad \text{and} \quad \xi \sim \mathbb{P}_{\psi,\nu}$$

where $\psi$ is the cumulant function of $T_{b=0}$.

2.1. The basic case. Suppose the random measure $M_0$ in Theorem 2 is defined as $M_0[0,y] := \tau_\alpha(yb^\alpha)/b$ for all $y \in [0,1]$. The total mass $M_0[0,1] \overset{d}{=} X_{\alpha,b}$ then has cumulant function $\psi(s) = (b+s)^\alpha - b^\alpha$, and substituting into the theorem yields

$$\mathbb{P}_{\psi,\nu}(\xi \in ds) = \frac{e^{-(b+s)^\alpha+b^\alpha}s^{\nu-1}}{E[X_{\alpha,b}^{\nu}]}ds \quad \text{and} \quad M[0,y]|(\xi = s) = \frac{\tau_\alpha(y(b+s)^\alpha)}{b+s}.$$}

Consequently, the random probability measure

$$P[0,y] := \frac{\tau_\alpha(y(b+\xi)^\alpha)}{\tau_\alpha((b+\xi)^\alpha)} \quad \text{satisfies} \quad P \overset{d}{=} \mathbb{P}(T \in dt \propto t^{-\nu}e^{-bt+b^\alpha f_\alpha(t)}dt). \quad (10)$$

The variables $T = M[0,1]$ and $T_0 = M_0[0,1]$ satisfy

$$T = T^{[\nu]}_0 \overset{d}{=} \frac{\tau_\alpha((b+\xi)^\alpha)}{b+\xi} \quad \text{and hence} \quad \mathbb{P}(T \in dt \propto t^{-\nu}e^{-bt+b^\alpha f_\alpha(t)}dt). \quad (11)$$

The resulting law of $(P_n)$ is $\text{PK}(\lambda_\alpha, \mathcal{L}(T))$. For $b := 0$ and $\nu > 0$, this law is specifically $\text{PD}(\alpha,\nu)$, which recovers Proposition 21 of Pitman and Yor [21]. Both the independence property in (10) and equality in distribution to $G_\nu$ remain true if $b$ is randomized by mixing against any positive random variable.
2.2. Size-biasing. If \( Y \) is any positive random variable with density \( f_Y \), we denote by \( Y^\ast \) the size-biased variable with density \( yf_Y(y)/E[Y] \). For an independent uniform variable \( U \) on \([0,1]\), the process

\[
\tau_{\alpha,b}^\ast(y) := \frac{\tau_{\alpha}(yb^\alpha)}{b} + \frac{G_{1-\alpha}}{b} 1\{U \leq y\}
\]

hence \( \tau_{\alpha,b}^\ast(1) \overset{d}{=} \left( \frac{\tau_{\alpha}(b^\alpha)}{b} \right)^{+} \),

and can be regarded as a size-biased form of \( \tau_{\alpha}(yb^\alpha)/b \) [14, 16]. Since the summands are independent, their cumulant functions are additive, and the cumulant function of \( \tau_{\alpha,b}^\ast(1) \) is

\[
\psi(s) = -(\alpha - 1) \log(1 + \frac{s}{b}) + (b + s)^\alpha - b^\alpha.
\]

For the random measure defined on the interval by \( M_\alpha[0,y] := \tau_{\alpha,b}^\ast(y) \), the distributions in Theorem 2 then take the form

\[
\mathbb{P}_{\psi,\nu}(\xi \in ds) = \frac{\alpha(b+s)^{\alpha-1}e^{-(b+s)^\alpha+b^\alpha}s^\nu-1}{\Gamma(\nu)E[\tau_{\alpha,b}^\ast(1)^{\nu+1}]}ds \quad \text{and} \quad M_\psi(\xi = s) = \tau_{\alpha,b}^\ast(s).
\]

As the variable \( \tau_{\alpha,b}^\ast(1) \) can be defined by tilting and size-biasing a stable variable, its density is \( g_{\alpha,b}(t) := b^{1-\alpha}e^{-bt+b^\alpha}tf_{\alpha}(t)/\alpha \). The marginal law of \( T = M[0,1] \) is then

\[
\mathbb{P}(T \in dt) = \frac{t^{-\nu}g_{\alpha,b}(t)}{\int s^{-\nu}g_{\alpha,b}(s)ds}dt = \mathbb{P}(X_{\alpha,b}^{[\nu-1]} \in dt),
\]

and we obtain:

**Proposition 4.** Let \( f_{\alpha} \) be the \( \alpha \)-stable density. If the weights of a random probability measure \( P \) have law \( \mathbb{P}(f_{\alpha}, \mathcal{L}(X_{\alpha,b}^{[\nu]})) \), it can be represented as \( P = M/T \) for

\[
M[0,y] = \frac{\tau_{\alpha}(y(b+\xi)^\alpha)}{b+\xi} + \frac{G_{1-\alpha}}{b+\xi} 1\{U \leq y\} = \frac{\tau_{\alpha}(y((b+\xi)^\alpha + G_{1-\alpha}/\alpha 1\{U \leq y\}))}{b+\xi}
\]

and satisfies \( P \bot \xi T \) and \( \xi T \overset{d}{=} G_{\nu} \) and \( T \overset{d}{=} X_{\alpha,b}^{[\nu-1]} \).

For example, choose \( \nu = 1 \), and abbreviate \( Z := (G_1 + b^\alpha)^{1/\alpha} \). Then, for any \( b > 0 \),

\[
\frac{\tau_{\alpha}(Z^\alpha + G_{1-\alpha})}{Z} \quad \text{and} \quad \frac{\tau_{\alpha}(Z^\alpha + G_{1-\alpha})}{Z} \overset{d}{=} G_1,
\]

i.e. the value of the process \( \tau_{\alpha} \), taken at a suitable random time, decouples from itself by random scaling. This also shows the unbiased case in Section 2.1 can be recovered from the size-biased one by choosing \( \nu = 1 \): Observe the term on the left is distributed as \( \tau_{\alpha}(Z^\alpha + G_{1-\alpha})/Z \approx \tau_{\alpha}(b^\alpha)/b \), for any \( b > 0 \). For \( b' \geq 0 \) and \( \nu' > 0 \), we may substitute \( b = b' + \xi' \), where \( \xi' \) has density proportional to \( e^{-(b'+s)^\alpha+b^\alpha s^\nu-1} \) as in Section 2.1 above. Then

\[
T \overset{d}{=} \tau_{\alpha}(b'+\xi')^{\alpha}/(b'+\xi') \quad \text{and hence} \quad \mathbb{P}(T \in dt) \propto t^{-\nu}e^{-b't+b^\alpha},
\]

which recovers all cases in Section 2.1.
2.3. Poisson-Dirichlet models. A PD($\alpha, \theta$) random measure $P$ can be represented as

$$P[0, y] \doteq \frac{\tau_\alpha(G(\alpha+\theta)/\alpha y + G(1-\alpha)/\alpha I\{U \leq y\})}{\tau_\alpha(G(\alpha+\theta)/\alpha + G(1-\alpha)/\alpha)}$$

for any $\theta > -\alpha$. (13)

This can be read from Dong, Goldschmidt, and Martin [2], or indeed from Pitman and Yor [20]. Define

$$H_{\alpha, \theta} := G^{1/\alpha}_{\alpha+\theta} B_{1-\alpha, \theta+\alpha}$$

Now index the random variable $\xi$ in (12) explicitly by the value of $b$ as $\xi_b$, and let $\xi_{H_{\alpha, \theta}}$ denote the variable obtained by mixing $b$ against $H_{\alpha, \theta}$.

**Lemma 5.** Let $\mathbb{P}_{\psi, \nu}$ be defined as in (12). For each $b > 0$, let $\xi_b \sim \mathbb{P}_{\psi, \theta+\alpha}$. Then

(i) $$(\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta})^\alpha \doteq G(\alpha+\theta)/\alpha$$

(ii) $$(\xi_{H_{\alpha, \theta}}, H_{\alpha, \theta}) \doteq G^{1/\alpha}_{\alpha+\theta}(B_{\theta+1, 1-\alpha}, 1 - B_{\theta+1, 1-\alpha})$$

(14)

We have hence established the result stated in the introduction:

**Proof of Proposition 3.** Substitution of $\xi_{H_{\alpha, \theta}} + H_{\alpha, \theta}$ for $b$ in Proposition 4 yields the random measure defined in (9). By (13), it normalizes to a measure with weights $(P_n) \sim \text{PD}(\alpha, \theta)$, and the claim follows from Proposition 4.

2.4. Implications for $\alpha$-diversities. For $(P_n) \sim \text{PK}(\lambda_\alpha, h_{\alpha})$, it is known [19] that

$$\Gamma(1-\alpha)^{-1} \lim_{\varepsilon \to \infty} \varepsilon^\alpha |\{n|P_n \geq \varepsilon\}| = T^{-\alpha} \quad \text{a.s.}$$

The random variable $T^{-\alpha}$ can be interpreted in terms of a local time, and is also known as the $\alpha$-diversity of the exchangeable random partition of $N$ defined by $P$: If $K_n$ is the number of distinct blocks in the restriction of this partition to the subset $[n]$, then $n^{-\alpha} K_n \to T^{-\alpha}$ almost surely as $n \to \infty$. The case $T = S_{\alpha, \theta}$ arises in Bayesian statistics, stochastic processes, and models for random trees and graphs [e.g. 3, 5, 6, 15, 24].

The law considered above is $(P_n) \sim \text{PK}(\lambda_\alpha, \mathcal{L}(T))$, where $T = \tau_\alpha((b + \xi)^\alpha)/(b + \xi)$ as in (11). For $b = 0$, the variable $T^{-\alpha}$ is the $\alpha$-diversity of the two-parameter Chinese restaurant process [19]. More generally, for any value $b \geq 0$, the resulting partition is of Gibbs type [19, 19], since $\lambda_\alpha$ defines a stable subordinator. There hence exists a subclass of Gibbs-type measures that is strictly larger than the Poisson-Dirichlet family, and whose $\alpha$-diversity exhibits a similar independence property $\xi^{\alpha} T^{-\alpha} \perp \perp P$.

3. Application to excursions straddling an exponential random time. This section considers applications to a type of distributions and processes that arise in a range of contexts, including passage times of Lévy processes, excursions of regular linear diffusions, interval partitions generated by a subordinator, and also in applications in statistics and finance [e.g. 1, 4, 7, 9, 21, 17, 24].
3.1. Independence of scaled excursion durations. Let \( \tau \) again be a subordinator, with Lévy density \( \lambda \) and \( \tau(1) < \infty \) a.s., and denote its Lévy exponent \( \Psi(s) := \int_0^\infty (1 - e^{-st}) \lambda(t) dt \).

Following Winkel [24] and the exposition in [1], define the local time process \( L \), overshoot process \( O \), and undershoot process \( U \) as

\[
L_t := \inf \{ s \mid \tau(s) > t \} \quad O(t, \tau) := \tau(L_t) - t \quad U(t, \tau) := t - \tau(L_t -)
\]

where \( \tau(L_t-) \) is the prepassage height, i.e. the left-hand limit \( \lim_{t \searrow L_t} \tau(t) \). For an independent exponential time \( G_1 \), abbreviate

\[
O(\tau) := O(G_1, \tau) \quad \text{and} \quad U(\tau) := U(G_1, \tau) \quad \text{and define} \quad \Delta(\tau) := O(\tau) + U(\tau).
\]

The variable \( \Delta(\tau) \) can be interpreted as the duration of the excursion from 0 to 0 of a strongly recurrent linear diffusion that straddles the random time \( G_1 \), and whose inverse local time is \( \tau \) [23]. The density \( f_\lambda \) of \( \Delta(\tau) \) and the joint density \( h_\lambda \) of \( (O(\tau), U(\tau)) \) are known to be

\[
\lambda^{(b)}(s) = e^{-bs} \lambda(s) \quad \text{which has Lévy exponent} \quad \Psi(s + b) - \Psi(b).
\]

An additional polynomial tilt yields the subordinator \( \tau^{(b)}_\nu \) with Lévy density

\[
\lambda^{(b)}_\nu(s) = s^{\nu} e^{-bs} \lambda(s) \quad \text{with Lévy exponent} \quad \Psi^{(b)}(s) := \int_0^\infty (1 - e^{-st}) \lambda^{(b)}(s) dt.
\]

If the scalar variable \( T_0 \) in Theorem 1 is chosen as \( T_0 := \Delta(\tau^{(b)}_\nu) \), the resulting law of \( \xi \) is

\[
\mathbb{P}_{\psi, \nu}(\xi \in ds) = \frac{\Psi^{(b)}(b + s) s^{\nu - 1}}{\Gamma(\nu)(\Psi^{(b + 1)} - \Psi^{(b)}(b))} ds. \tag{15}
\]

The variable \( T \) in the theorem is then \( T = \Delta(\tau^{(b+\xi)}_\nu) \).

**Proposition 6.** Fix \( \nu > 0 \) and \( b \geq 0 \), let \( \xi \) be a random variable with law (15). Then the conditional density of \( L(O(\tau^{(b+\xi)}_\nu), U(\tau^{(b+\xi)}_\nu)|\xi = s) \) is \( h_{\lambda^{(b+s)}_\nu} \), and

\[
(O(\tau^{(b+\xi)}_\nu), U(\tau^{(b+\xi)}_\nu)) \overset{\text{d}}{=} (O(\tau^{(b)}_\nu), U(\tau^{(b)}_\nu)) \quad \text{independently of} \quad \xi \Delta(\tau^{(b+\xi)}_\nu) \overset{\text{d}}{=} G_\nu.
\]

The process \( \tau^{(b+\xi)}_\nu|\xi = s \) is compound Poisson with rate and jump density

\[
r_{b, \nu} := \int_0^\infty e^{-(b+s)t^{\nu}} \lambda(t) dt \quad \text{and} \quad s \mapsto e^{-(b+s)t^{\nu}} \lambda(t)/r_{b, \nu}
\]

whenever \( r_{b, \nu} < \infty \), in particular for \( \nu \geq 1 \).

Since \( \Delta = O + U \), it follows that the excursion duration satisfies

\[
\Delta(\tau^{(b+\xi)}_\nu) \overset{\text{d}}{=} \Delta(\tau^{(b)}_\nu) \quad \text{independently of} \quad \xi \Delta(\tau^{(b+\xi)}_\nu).
\]

The result does not imply independence of \( \xi \Delta(\tau^{(b+\xi)}_\nu) \) and the entire process \( \tau^{(b+\xi)}_\nu \).
3.2. A concrete example. Let $\tau_{\nu}^{(b+\xi)}$ have Lévy density

$$
\lambda_{\nu}^{(b+\xi)}(s) = \frac{\alpha}{\Gamma(1-\alpha)} s^{\nu-\alpha-1} e^{-bs} \quad \text{for } s \in (0, \infty).
$$

(16)

Changing parameters to $\delta := \alpha - \nu$, and comparing to the generalized gamma subordinator $\tau_\alpha$ used in Section 2, shows (16) is up to a constant the Lévy density of the subordinator $\tau_\delta(\alpha b^\delta)/b$. The Lévy exponent $\Psi_\nu$ of $\lambda_{\nu}^{(b)}$, and hence the variable $\xi$ defined by (15), depend on the sign of $\delta$. We must distinguish three cases:

1. $\delta \in (0, \alpha)$ and $b \geq 0$: $\tau_{\nu}^{(b)}$ is a generalized gamma process with infinite activity and parameter $\delta$, with

$$
\Psi_\nu(b) = \frac{\alpha \Gamma(1-\delta)}{\delta \Gamma(1-\alpha)} ((b+1)^\delta - b^\delta) \quad \text{and} \quad \tau_{\nu}^{(b+\xi)}(t) \overset{d}{=} \frac{\tau_\delta(t(b+\xi)\delta^{(1-\delta)/\delta})}{b+\xi},
$$

where $\tau_\delta = \tau_{\alpha=\delta}$ is a generalized gamma subordinator.

2. $\delta = 0$ and $b > 0$: $\tau_{\nu}^{(b)}$ is a gamma process, with

$$
\Psi_\nu(b) = \frac{\alpha}{\Gamma(1-\alpha)} \log(1 + 1/b) \quad \text{and} \quad \tau_{\nu}^{(b+\xi)}(t) \overset{d}{=} \frac{\gamma(t\alpha)}{b+\xi},
$$

where $\gamma$ is a gamma subordinator with Lévy density $s^{-1}e^{-s}$. The weights of the random measure $P = \tau_{\nu}^{(b+\xi)}(y)/\tau_{\nu}^{(b+\xi)}(1)$ have law $PD(0,\delta)$, independently of $\xi$.

3. For $\delta < 0$ and $b > 0$, one obtains a compound Poisson process, with

$$
\Psi_\nu(b) = \frac{\alpha \Gamma(\nu-\alpha)}{\Gamma(1-\alpha)} (b^{\nu-\alpha} - (b+1)^{\nu-\alpha}) \quad \text{and} \quad \tau_{\nu}^{(b+\xi)}(t) \overset{d}{=} \sum_{i=1}^{\tilde{N}(t)} \frac{G_i}{b+\xi},
$$

where $\tilde{N}$ is a Poisson process with rate $(b+\xi)^{\nu-\alpha} \frac{\alpha \Gamma(\nu-\alpha)}{\Gamma(1-\alpha)}$, and the variables $G_1, G_2, \ldots$ are i.i.d. $G_i \overset{d}{=} G_{\nu-\alpha}$.

In each case, the excursion duration is conditionally distributed as

$$
\mathcal{L}(\Delta(\tau_{\nu}^{(b+\xi)})|\xi = s) = \frac{\alpha (1 - e^{-t}) e^{-(b+s)t^{\nu-\alpha-1}}}{\Gamma(1-\alpha) \Phi_\nu(b+s)} dt,
$$

and satisfies $\Delta(\tau_{\nu}^{(b+\xi)}) \overset{d}{=} \Delta(\tau_0^{(b)})$, independently of $\xi \Delta(\tau_{\nu}^{(b+\xi)})$.

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**PROOFS**

**Proof of Theorem 1.** If (4) holds, the joint density of $(\xi, T)$ is

$$
\mathbb{P}(\xi \in ds, T \in dt) = \frac{e^{-st}t^{\nu-1}}{\mathbb{E}[T_0^{-\nu}\Gamma(\nu)]} f_0(t)dsdt,
$$

if $f_0$ is the density of $T_0$. It can be disintegrated either into $\mathcal{L}(T|\xi)$ and $\mathcal{L}(\xi)$, which recovers (4), or into $\mathcal{L}(\xi|T)$ and $\mathcal{L}(T)$, in which case

$$
\mathbb{P}(T \in dt) = \frac{t^{-\nu}f_0(t)}{\mathbb{E}[T_0^{-\nu}]} dt = \mathbb{P}(T_0^{-\nu} \in dt),
$$

(17)
which is just (3iii). For any measurable functions $g$ and $h$, a change of variables $y := st$ then yields

$$
E[g(T^*)h(\xi T^*)] = \int g(t)h(ts)\frac{e^{-st}e^{\psi(s)}f_{T}(t)e^{-\psi(s)}s^{-1}}{E[T^{-\nu}]}dt \Psi(s,\nu(h))
$$

so (3i) and (3ii) are also true. Conversely, assume (3) holds, and hence in particular (17). The joint density of $(T, \xi T)$ is then

$$
t^{-\nu}f_{T}(t)dt \frac{s^{-1}e^{-\psi(s)}}{\Gamma(\nu)}h(s) = t^{-\nu}f_{T}(t)dt \frac{s^{-1}e^{-\psi(s)}}{\Gamma(\nu)}h(s) ,
$$

If additionally $h$ is any positive function and $\psi(s) = \log(h(s))$,

$$
s^{-\nu}f_{T}(t)dt \frac{s^{-1}e^{-\psi(s)}}{\Gamma(\nu)}h(s) = s^{-\nu}f_{T}(t)dt .
$$

which is the product of the two terms in (4).

The proof of Theorem 2 is similar.

PROOF OF LEMMA 5. $H_{\alpha,\theta}$ has density $b \mapsto b^{-\alpha}e^{-b}\frac{\mathbb{E}[X^{-\theta\alpha-\alpha}]^\nu/(1-\alpha)\mathbb{E}[S_{\alpha}^{-\theta\alpha}]}{\Gamma(1-\alpha)\mathbb{E}[S_{\alpha}^{-\theta\alpha}]}$. Integrating against the density of $\xi_b$ given in (12) shows $\xi_{H_{\alpha,\theta}}$ has marginal density

$$
s \mapsto Cs^\nu + \alpha - 1 \int_0^1 e^{-b(b+1)}(b+1)^{\alpha-1}b^{-\alpha}db ,
$$

hence $\xi_{H_{\alpha,\theta}} \equiv G_{\nu,\alpha,\theta+\alpha}B_{1-\alpha,\theta+\alpha}$.

Taking Laplace transforms yields (14)(i), which implies $(\xi_{H_{\alpha,\theta}}, \xi_{H_{\alpha,\theta}} + H_{\alpha,\theta})^\nu$ is equal in distribution to $G_{\nu,\alpha,\theta+\alpha}(B_{1-\alpha,\theta+\alpha},1)$, and hence yields (14)(ii).

PROOF OF PROPOSITION 6. (i) holds by construction and Theorem 1. To obtain (ii) and (iii), abbreviate $(O_{\xi}, U_{\xi}) := (O(\tau_{\nu,\theta} + \xi), U(\tau_{\nu,\theta} + \xi))$ and $\Delta_{\xi} := \Delta(\nu, \theta, \nu, \theta)$. The joint density of $(O_{\xi}, U_{\xi}, \xi)$ is then

$$
\frac{e^{-\nu}e^{-\left(\frac{1}{v+w}\right)(v+w)}\lambda(v+w)}{\Psi_v(b+s)} = \frac{s^{-1}e^{-\left(\frac{1}{v+w}\right)(v+w)}\lambda(v+w)s^{-1}}{\Gamma(\nu) (\Psi_v(b+s) - \Psi(b))} .
$$

It follows that, for any measurable function $h$,

$$
\mathbb{E}[h(O_{\xi}, U_{\xi})e^{-\omega \xi_{\Delta_{\xi}}} = \frac{h(v,w)e^{-\nu}e^{-\left(\frac{1}{v+w}\right)(v+w)}e^{-\left(1+\omega\right)(x+v)}\lambda(v+w)s^{-1}}{\Gamma(\nu) (\Psi_v(b+s) - \Psi(b))} ds d\nu dv ,
$$

and integrating out $s$ yields $\mathbb{E}[h(O_{\xi}, U_{\xi})e^{-\omega \xi_{\Delta_{\xi}}} = (1 + \omega)^{-\nu}\mathbb{E}[h(O(\tau_{\nu}, U(\tau_{\nu}))].$
