Higher Equations of Motion in $N = 1$ SUSY Liouville Field Theory

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Abstract

Similarly to the ordinary bosonic Liouville field theory, in its $N = 1$ supersymmetric version an infinite set of operator valued relations, the “higher equations of motions”, holds. Equations are in one to one correspondence with the singular representations of the super Virasoro algebra and enumerated by a couple of natural numbers $(m,n)$. We demonstrate explicitly these equations in the classical case, where the equations of type $(1,n)$ survive and can be interpreted directly as relations for classical fields. General form of the higher equations of motion is established in the quantum case, both for the Neveu-Schwarz and Ramond series.

Motivations. In ref. [1] it has been shown that in the Liouville field theory (LFT) an infinite set of relations holds for quantum operators. They are parameterized by pairs of positive integers $(m,n)$ and called conventionally the “higher equations of motion” (HEM), because the first one $(1,1)$ coincides with the usual Liouville equation of motion. These equations relate different basic LFT primary fields $V_a(x)$ (operators $V_a$ can be thought of as

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regularized version of the exponential \(\exp(2a\phi)\) of the basic Liouville field \(\phi\). The equations are “derived” on the basis of two conjectures. First one is the vanishing of all singular vectors in the representations built on the exponential fields. This is a natural continuation of the easily verified relations in the classical LFT, and also a mandatory requirement imposed on LFT by the unitarity. The second conjecture is much less justified and states basically that the set of exponential fields \(\{V_a\}\) (with complex \(a\) allowed) covers in some sense the whole variety of primary fields in LFT. One of the open problems here is the concept of the space of local fields and its completeness. In LFT, unlike the more familiar rational conformal field theories [2], the operator state correspondence doesn’t hold literally, thus making difficult a straightforward inheritance of the completeness from the unitary space of physical states. Because of this conceptual problem the status of the second conjecture in unclear and waits for a mathematically correct formulation. For the moment we simply take for granted that the space of primary local fields, the norms and completeness regardless, is spanned by an appropriate subset of \(\{V_a\}\).

Similar operator valued relations have been observed also in \(SL(2,R)_k\) WZNW model [3]. Further study of these and the Liouville related HEM’s can be found in ref. [4].

Higher equations turned to be useful in practical calculations. In particular, in [5] they were used to derive general four-point correlation function in the minimal Liouville gravity (see [6] for the dictionary) with one degenerate matter field. It’s very likely that HEM’s are potentially important in the general program of explicit construction of the complete set of correlation functions in the minimal Liouville gravity.

It is the purpose of this note to reveal a similar set of higher equations in the supersymmetric Liouville field theory (SLFT). The \(N = 1\) SUSY version of LFT [7] is known to be closest partner of the bosonic theory, having very similar properties. In particular, the existence of a supersymmetric version of the HEM’s is naturally expected.

**Higher equations in classical SLFT.** We begin with the classical equations of motion in the \(N = 1\) SLFT [8,9]

\[
\begin{align*}
\bar{\partial}\psi_c &= iM\bar{\psi}_ce^\varphi \\
\partial\bar{\psi}_c &= -iM\psi_ce^\varphi \\
\partial\bar{\partial}\varphi &= iM\bar{\psi}_c\psi_ce^\varphi + M^2e^{2\varphi}
\end{align*}
\]

(1)

where \(\varphi\) is the boson and \((\psi_c, \bar{\psi}_c)\) the Majorana fermion components\(^2\). As in [1] we use complex 2D coordinates \(z = x + iy\) and \(\bar{z} = x - iy\) (\(\partial = \partial_z\) and \(\bar{\partial} = \partial_{\bar{z}}\)) and introduce a redundant parameter \(M\) for the sake of convenience. Equations (1) can be obtained as the extremum conditions for the following classical Lagrangian density

\[
\mathcal{L}_{\text{cl}} = \partial\varphi\bar{\partial}\varphi + \psi_c\bar{\partial}\psi_c + \bar{\psi}_c\partial\bar{\psi}_c + 2iM\bar{\psi}_c\psi_ce^\varphi + M^2e^{2\varphi}.
\]

(2)

The classical (as well as the quantum) SLFT has been introduced and studied in [8–10] shortly after it appeared in the string context in [7]. Here we recapitulate only those properties of the classical theory which will be of further use.

\(^2\)Index “c” is attached to the fields \(\psi_c\), and also to the classical supercurrent \(S_c\) and to the stress tensor \(T_c\), to distinguish them from differently normalized quantum fields, which appear in the subsequent sections.
The superconformal invariance at the classical level is equivalent to the statement that the components of the classical supercurrent

\[ S_c = -i\bar{\psi}_c \partial \varphi + i\partial \psi_c \]

\[ \bar{S}_c = -i\bar{\psi}_c \partial \varphi + i\partial \psi_c \]

are holomorphic \( \bar{\partial}S_c = 0 \) and antiholomorphic \( \partial \bar{S}_c = 0 \) respectively. These relations, as well as similar statements \( \bar{\partial}T_c = \partial \bar{T}_c = 0 \) about the stress tensor components

\[ T_c = -\frac{1}{2}(\partial \varphi)^2 + \frac{1}{2}\partial^2 \varphi + \frac{1}{2}\partial \psi_c \psi_c \]

\[ \bar{T}_c = -\frac{1}{2}(\bar{\partial} \varphi)^2 + \frac{1}{2}\bar{\partial}^2 \varphi + \frac{1}{2}\bar{\partial} \bar{\psi}_c \bar{\psi}_c \]

are easily verified to be consequences of the equations (1). To fully describe the supersymmetry we need also classical generators \( G \) and \( \bar{G} \), the right and left supercharges. These operators act on the classical fields and satisfy the standard relations

\[ G^2 = \partial ; \quad \bar{G}^2 = \bar{\partial} ; \quad \{G, \bar{G}\} = 0 . \]

Their action on the fundamental components \( \varphi \) and \( (\psi_c, \bar{\psi}_c) \) is

\[ G\varphi = i\psi_c ; \quad \bar{G}\varphi = i\bar{\psi}_c . \]

The action of the supercharges on a general exponential field is a direct consequence of (6)

\[ Ge^{\sigma \varphi} = i\sigma \psi_c e^\varphi ; \quad \bar{G}e^{\sigma \varphi} = i\sigma \bar{\psi}_c e^\varphi . \]

All three equations (1) follow from the statement

\[ GG\varphi = iMe^\varphi \]

and the algebra (5). This gives, after a simple reckoning

\[ \bar{G}Ge^{\sigma \varphi} = iM\sigma e^{(1+\sigma)\varphi} - \sigma^2 \bar{\psi}_c \psi_c e^{\sigma \varphi} , \]

a relation useful in the subsequent calculations.

The classical \( D \)-operators form an infinite series \( D_{2k-1}^{(c)} \), \( k = 1, 2, \ldots \) Few first representatives read

\[ D_1^{(c)} = G \]
\[ D_3^{(c)} = G\partial + S_c \]
\[ D_5^{(c)} = G\partial^2 + 2T_c G + 3S_c \partial + 2\partial S_c \]
\[ D_7^{(c)} = G\partial^3 + 8T_c G\partial + 4\partial T_c G + 18T_c S_c + 6S_c \partial^2 + 8\partial S_c \partial + 3\partial^2 S_c \]
\[ D_9^{(c)} = G\partial^4 + 20T_c G\partial^2 + 20\partial T_c G\partial + 6\partial^2 T_c G + 36T_c^2 G + 110T_c S_c \partial \]
\[ + 56\partial T_c S_c + 72T_c \partial S_c + 10S_c \partial^3 + 20\partial S_c \partial^2 + 15\partial^2 S_c \partial + 4\partial^3 S_c + \partial S_c S_c G \]
\[ \ldots \]
There is, of course, the identical series of the “left” operators $\bar{D}_{2k-1}^{(c)}$. One only needs to combine the SUSY algebra (5) with the definitions of the supercurrent (3) and the stress tensor (4) to verify the identities

$$D_{2k-1}^{(c)} e^{-(k-1)\varphi} = \bar{D}_{2k-1}^{(c)} e^{-(k-1)\varphi} = 0$$  \hspace{1cm} (11)

Direct calculation with a help of the equations of motion (1) gives

$$\bar{D}_1^{(c)} D_1^{(c)} \varphi = i M e^\varphi$$
$$\bar{D}_3^{(c)} D_3^{(c)} \varphi e^{-\varphi} = -i M^3 e^{2\varphi}$$
$$\bar{D}_5^{(c)} D_5^{(c)} \varphi e^{-2\varphi} = 4i M^5 e^{3\varphi}$$
$$\bar{D}_7^{(c)} D_7^{(c)} \varphi e^{-3\varphi} = -36i M^7 e^{4\varphi}$$
$$\bar{D}_7^{(c)} D_7^{(c)} \varphi e^{-3\varphi} = 576i M^9 e^{5\varphi}.$$  \hspace{1cm} (12)

This allows to conjecture that for general $k = 1, 2, \ldots$

$$\bar{D}_{2k-1}^{(c)} D_{2k-1}^{(c)} e^{(1-k)\varphi} = i (-)^{k-1} [(k - 1)!]^2 M^{2k-1} e^{k\varphi}$$  \hspace{1cm} (13)

We will show in the subsequent sections that this is a classical limit of a (subset of) more general set of relations, the HEM’s of the quantum SLFT.

Quantum SLFT. We remind very briefly the essence of the quantum SLFT (see [11–13]).

The Lagrangian density is

$$L_{\text{SLFT}} = \frac{1}{8\pi} (\partial_a \phi)^2 + \frac{1}{2\pi} \left( \bar{\psi} \hat{D} \psi + \bar{\psi} \hats D \psi \right) + 2i \mu b^2 \bar{\psi} \psi e^{b\phi} + 2\pi \mu^2 \mu^2 e^{2b\phi}$$  \hspace{1cm} (14)

Here $\mu$ is a scale coupling called conventionally the cosmological constant while $b$ is a quantum parameter, the classical limit corresponding to $b \to 0$. A convenient combination

$$Q = b^{-1} + b$$  \hspace{1cm} (15)

is traditionally called the background charge. In the form (14) the action is explicitly supersymmetric, the classical one (2) being achieved straightforwardly in the limit $b \to 0$ with $b\phi \to \varphi$, $b\psi \to \psi_c$, $2\pi \mu^2 b^2 \to M$ and

$$\int L_{\text{SLFT}} d^2 x \to \frac{1}{2\pi b^2} S_{\text{cl}}$$  \hspace{1cm} (16)

For the “perturbed CFT” interpretation the last two terms in (14) are better understood through the field

$$- \bar{G} G e^{b\phi} = b^2 \bar{\psi} \psi e^{b\phi} - 2i \pi \mu b^2 e^{2b\phi},$$  \hspace{1cm} (17)
a “top” component of an appropriate supermultiplet.

SLFT is a superconformal field theory (SCFT), the symmetry being generated by the holomorphic and antiholomorphic components of the supercurrent

\[ S(z) = -i\phi \partial \psi + iQ \partial \bar{\psi} \; ; \; \bar{S}(\bar{z}) = -i\phi \bar{\partial} \bar{\psi} + iQ \partial \psi \]

In the classical limit they apparently turn to the fields \( S \rightarrow b^{-2}S_{c} \), \( \bar{S} \rightarrow b^{-2}\bar{S}_{c} \). In the same way the classical holomorphic and antiholomorphic stress tensors are the limits

\[ T(z) = -\frac{1}{2}(\partial \phi)^2 + \frac{Q}{2} \partial^2 \phi + \frac{1}{2} \partial \psi \bar{\psi} \]

\[ \bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\partial} \bar{\phi})^2 + \frac{Q}{2} \bar{\partial}^2 \bar{\phi} + \frac{1}{2} \bar{\partial} \bar{\psi} \bar{\bar{\psi}} \]

Together with the supercurrent \( (18) \) they form the superconformal algebra of operator product expansions

\[ S(z)S(z') = \frac{\hat{c}}{(z - z')^3} + \frac{T(z')}{z - z'} + \text{reg.} \]

\[ T(z)S(z') = \frac{3S(z')}{2(z - z')^2} + \frac{\partial S(z')}{z - z'} + \text{reg.} \]

\[ T(z)T(z') = \frac{3\hat{c}}{4(z - z')^4} + \frac{2T(z')}{(z - z')^2} + \frac{\partial T(z')}{z - z'} + \text{reg.} \]

where the central charge is related to \( b \) as

\[ \hat{c} = 1 + 2Q^2 \]

In terms of the Laurent components

\[ T(z) = \sum_{n} L_n z^{-n-2} ; \; S(z) = \sum_{k} G_k z^{-k-3/2} \]

(index \( k \in Z + 1/2 \) in the Neveu-Schwarz (NS) representations and \( k \in Z \) in the Ramond (R) ones) the algebra takes the conventional form of super Virasoro algebra (SV)

\[ \{G_k, G_l\} = 2L_{k+l} + \frac{\hat{c}}{2} \left( k^2 - \frac{1}{4} \right) \delta_{k+l} \]

\[ [L_n, G_k] = \left( \frac{n}{2} - k \right) G_{n+k} \]

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m+n} \]

An identical, “left” algebra, is generated by the antiholomorphic components \( \bar{S}(\bar{z}) \) and \( \bar{T}(\bar{z}) \) and their Laurent components \( \bar{G}_k \) and \( \bar{L}_n \).
Local fields form the highest weight representations of the right and left algebras $SV \otimes \overline{SV}$, both either Neveu-Schwarz or Ramond ones. The corresponding highest weight vectors are the SCFT primary fields, denoted $V_a$ for the NS representations and $R^\pm_a$ for the R representations. Their dimensions depend on the parameter $a$ as

$$\Delta_{\text{NS}}(a) = \frac{a(Q-a)}{2}; \quad \Delta_{\text{R}}(a) = \Delta_{\text{NS}}(a) + \frac{1}{16},$$

(differently for the NS and R sectors. It is not a bad idea to compare the basic SLFT fields $\phi$ and $(\psi, \bar{\psi})$ with free massless boson and Majorana fermion. In this dictionary at $\text{Re} \, a < Q/2$ the primary field $V_a$ corresponds to the normal ordered exponential : $\exp(a\phi)$ :. Another convenient parameter $\lambda = Q/2 - a$ is often used instead of $a$. Correspondingly

$$\Delta_{\text{NS}}(a) = \frac{(-1)}{16} - \frac{\lambda^2}{2}; \quad \Delta_{\text{R}}(a) = \frac{\hat{c}}{16} - \frac{\lambda^2}{2}. \quad (25)$$

In the last Ramond case there are two equally good highest weight vectors $R^\pm_a$, forming a two dimensional representation

$$G_0(R^+_a) = \begin{pmatrix} 0 & \frac{(-1+i)\lambda}{2} \\ (1+i)\lambda & 2 \\ \frac{(-1+i)\lambda}{2} & 0 \end{pmatrix} \begin{pmatrix} R^+_a \\ R^-_a \end{pmatrix}$$

and

$$G_0(R^-_a) = \begin{pmatrix} 0 & \frac{(1+i)\lambda}{2} \\ (-1+i)\lambda & 2 \\ \frac{(1+i)\lambda}{2} & 0 \end{pmatrix} \begin{pmatrix} R^+_a \\ R^-_a \end{pmatrix} \quad (26)$$

of the Cartan subalgebra $G^2_0 = G^2_0 = L_0 - \hat{c}/16$ and $\{G_0, \bar{G}_0\} = 0$. In the free field language $R^+_a$ and $R^-_a$ can be related respectively to the fields $\sigma : \exp(a\phi)$ : and $\mu : \exp(a\phi)$ ::, where $\sigma$ and $\mu$ are the standard order and disorder spin fields with respect to the free fermion, familiar from the Ising model [14]. In our further development this doubling of the Ramond primary fields is not very important and we will often omit the index $\pm$ near $R^\pm_a$, keeping however in mind this feature.

**Degenerate primaries.** At certain special values of the parameter $a$ the $SV$ representations are singular. This happens at [15] $a = a_{m,n}$ (or, equivalently, at $a = Q - a_{m,n}$), where $a_{m,n} = -\lambda_{m-1,n-1}$ and $(m, n)$ is a pair of positive integers. We introduced a convenient notation

$$\lambda_{m,n} = \frac{mb^{-1} + nb}{2} \quad (27)$$

In general at $a = a_{m,n}$ the corresponding dimensions are

$$\Delta_{m,n}^{(\text{NS})} = \Delta_{\text{NS}}(a)|_{a=a_{m,n}} \quad \text{or} \quad \Delta_{m,n}^{(\text{R})} = \Delta_{\text{R}}(a)|_{a=a_{m,n}} \quad (28)$$

and one singular vector appears at level $mn/2$ in the Verma module, over $V_{a_{m,n}} = V_{m,n}$ (at $m - n \in 2\mathbb{Z}$) or over $R_{a_{m,n}} = R_{m,n}$ ($m - n \in 2\mathbb{Z} + 1$) respectively. It is convenient to
introduce for each pair \((m,n)\) a “singular vector creation operator” \(D_{m,n}\), which is graded polynomial in \(G_{-k}\) and \(L_{-k}\) of level \(mn/2\) with coefficients functions of the central charge parameter \(b^2\), such that the singular vector appears when \(D_{m,n}\) is applied to \(V_{m,n}\) or \(R_{m,n}\), whichever appropriate. In the NS case the normalization is unambiguously fixed through the coefficient near the highest order term \(D_{m,n} = G_{-1/2}^{mn} + \ldots\). Apparently the fermion parity of this operator is that of the product \(mn\). In the R case \(mn\) is always even, while \(G_0\) allows in any case to choose \(D_{m,n}\) bosonic. Let us agree to put all the fermion operators \(G_{-k}\) to the right from the bosonic ones \(L_{-k}\), arranging each group in the order of increasing index \(-k\). Then an unambiguous normalization \(D_{m,n} = L_{-1}^{mn/2} + \ldots\) is prescribed by the coefficient near \(L_{-1}^{mn/2}\).

At moderate \((m, n)\) the polynomial \(D_{m,n}\) can be carried out manually. Here we list few first examples.

- **Level 1/2**: A singular module is over \(V_{1,1} = V_0\) with the singular vector created by

\[ D_{1,1} = G_{-1/2} \]  

(29)

- **Level 1**: A singular vector in the module over \(R_{1,2}\) with

\[ D_{1,2} = L_{-1} - \frac{2b^2}{1 + 2b^2} G_{-1} G_0 \]  

(30)

appears at \(a = a_{1,2} = -b/2\). There is similar singular module over \(R_{2,1}\). It is however needless to write down \(D_{2,1}\) separately, as it is obtained from \(D_{1,2}\) through the symmetry \(m \leftrightarrow n, b \rightarrow b^{-1}\). Henceforth we will systematically omit such mirror images without any special warnings.

- **Level 3/2**: There is an NS singular vector over \(V_{1,3}\) with

\[ D_{1,3} = L_{-1} G_{-1/2} + b^2 G_{-3/2} \]  

(31)

- **Level 2**: A singular representation over \(R_{1,4}\) with the creation operator

\[ D_{1,4} = L_{-1}^2 + \frac{3b^2}{2} L_{-2} + \frac{b^2 (1 - 6b^2)}{1 + 4b^2} G_{-2} G_0 - \frac{4b^2}{1 + 4b^2} L_{-1} G_{-1} G_0, \]  

(32)

and yet another degenerate representation of NS type, where

\[ D_{2,2} = L_{-1}^2 + \frac{(1 + b^2)^2}{2b^2} L_{-2} - G_{-3/2} G_{-1/2} \]  

(33)

- **Level 5/2**: An NS degenerate field \(V_{1,5}\) with

\[ D_{1,5} = L_{-1}^2 G_{-1/2} + 2b^2 (1 + 3b^2) G_{-5/2} + 3b^2 G_{-3/2} L_{-1} + 2b^2 L_{-2} G_{-1/2} \]  

(34)
• Level 3: Here we have two Ramond degenerate moduli over $R_{1,6}$ and $R_{2,3}$, the corresponding creation operators being

$$D_{1,6} = L_{-1}^3 + \frac{13b^2}{2} L_{-2} L_{-1} + (3b^2 + 10b^4) L_{-3}$$

$$- \frac{6b^2}{1 + 6b^2} L_{-2} G_{-1} G_0 - \frac{15b^4}{1 + 6b^2} L_{-2} L_{-1} G_0$$

$$+ \frac{3b^2 (1 - 8b^2)}{1 + 6b^2} L_{-1} G_{-2} G_0 + \frac{b^2}{4} G_{-2} G_{-1} - \frac{3 (b^2 - 12b^4 + 40b^6)}{2 (1 + 6b^2)} G_{-3} G_0$$

and

$$D_{2,3} = L_{-1}^3 + \frac{1 + 4b^4}{2 b^2} L_{-2} L_{-1} + (1 + 3b^2 + b^4) L_{-3}$$

$$- \frac{1 - 4b^4}{b^2 (2 + 3b^2)} L_{-2} G_{-1} G_0 - \frac{2}{2 + 3b^2} L_{-1}^2 G_{-1} G_0$$

$$+ \frac{1 - 8b^4}{2 + 3b^2} L_{-1} G_{-2} G_0 + \frac{1 - 8b^4}{4b^2} G_{-2} G_{-1} - \frac{5 - 12b^2 + 4b^4}{2 (2 + 3b^2)} G_{-3} G_0$$

• Level 7/2: A singular vector in the seven dimensional space of this level in the module over $V_{1,7}$ is created by

$$D_{1,7} = L_{-1}^3 G_{-1/2} + 8b^2 L_{-2} L_{-1} G_{-1/2} + (4b^2 + 15b^4) L_{-3} G_{-1/2} + 6b^2 L_{-1}^2 G_{-3/2}$$

$$+ 18b^4 L_{-2} G_{-3/2} - 2 (2b^2 - 15b^4) L_{-1} G_{-5/2} + (2b^2 - 27b^4 + 90b^6) G_{-7/2}$$

• Level 4: In the Ramond module $R_{1,8}$ the singular vector is created by the operator

$$D_{1,8} = L_{-1}^4 + 17b^2 L_{-2} L_{-1}^2 + 2 (8b^2 + 33b^4) L_{-3} L_{-1} + \frac{105b^4}{4} L_{-2}^2$$

$$+ \frac{6b^2 (1 - 10b^2)}{1 + 8b^2} L_{-1}^2 G_{-2} G_0 + \frac{71b^2 + 492b^4 + 1260b^6}{8} L_{-4}$$

$$- \frac{8b^2}{1 + 8b^2} L_{-1}^2 G_{-1} G_0 - \frac{76b^4}{1 + 8b^2} L_{-2} L_{-1} G_{-1} G_0 - \frac{2 (19b^4 + 84b^6)}{1 + 8b^2} L_{-3} G_{-1} G_0$$

$$+ \frac{19b^4 - 210b^6}{1 + 8b^2} L_{-2} G_{-2} G_0 + b^2 L_{-1} G_{-2} G_{-1} - \frac{6 (b^2 - 15b^4 + 60b^6)}{1 + 8b^2} L_{-1} G_{-3} G_0$$

$$- \frac{3b^2 (1 - 6b^2)}{4} G_{-3} G_{-1} + \frac{15b^2 - 292b^4 + 2052b^6 - 5040b^8}{4 (1 + 8b^2)} G_{-4} G_0$$
There is another, NS module over $V_{2,4}$, singular at this level with

$$D_{2,4} = L_1^4 + \frac{1 + 2b^2 + 5b^4}{b^2} L_2 L_1^2 + \frac{(1 + 2b^2)(1 + 5b^2 + 3b^4)}{b^2} L_3 L_1\ + \frac{(1 - b^2)^2 (1 + 3b^2)^2}{4b^4} L_2^2 + \frac{(1 + 3b^2) (2 + 8b^2 + 11b^4 + 3b^6)}{2b^2} L_4\ - 2L_1^2 G_{-3/2} G_{-1/2} + 2 \left(1 - 4b^2\right) L_1 G_{-5/2} G_{-1/2} - \frac{1 - 3b^4}{b^2} L_2 G_{-3/2} G_{-1/2}\ + \frac{(1 + 3b^2) (1 + 3b^2 - 12b^4)}{4b^2} G_{-5/2} G_{-3/2} + \frac{1 - 14b^2 + 29b^4 - 12b^6}{4b^2} G_{-7/2} G_{-1/2}\$$

(39)

- **Level 9/2**: At this level there are two NS highest weight vectors, $V_{1,9}$ and $V_{3,3}$, the corresponding $D$-operators reading

$$D_{1,9} = L_1^4 G_{-1/2} + 20b^2 L_2 L_1^2 G_{-1/2} + b^2 \left(20 + 91b^2\right) L_3 L_1 G_{-1/2} + \ + 6b^2 \left(2 + 15b^2 + 42b^4\right) L_4 G_{-1/2} + 36b^4 L_2^2 G_{-1/2} + 10b^2 L_3^3 G_{-3/2}\ + 110b^4 L_2 L_1 G_{-3/2} + 56 \left(b^4 + 5b^8\right) L_3 G_{-3/2} - 10b^2 \left(1 - 9b^2\right) L_1^2 G_{-5/2}\ - (38b^4 - 360b^6) L_2 G_{-5/2} + 2b^2 \left(5 - 81b^2 + 315b^4\right) L_1 G_{-7/2}\ - 6b^2 \left(1 - 23b^2 + 171b^4 - 420b^6\right) G_{-9/2} + b^4 G_{-5/2} G_{-3/2} G_{-1/2}\$$

and

$$D_{3,3} = L_1^4 G_{-1/2} + \frac{2(1 + b^4)}{b^2} L_2 L_1^2 G_{-1/2} + \frac{1 + 2b^2 + b^4 + 2b^6 + b^8}{b^4} L_3 L_1 G_{-1/2}\ + \frac{2(2 + 5b^2 + 2b^4)}{b^2} L_4 G_{-1/2} + 4L_2^2 G_{-1/2} + \frac{2(1 + 3b^4 + b^8)}{b^4} L_2 L_1 G_{-3/2}\ + \frac{1 + b^4}{b^2} L_3^3 G_{-3/2} + \frac{(1 + b^2)^2 (1 - 6b^4 + b^8)}{b^6} L_3 G_{-3/2} - \frac{1 - 10b^2 + b^4}{b^2} L_2^2 G_{-5/2}\ + \frac{2(4 - 9b^2 + 4b^4)}{b^2} L_1 G_{-7/2} + \frac{2\left(1 - 2b^2 - 7b^4 - 2b^6 + b^8\right)}{b^4} L_2 G_{-5/2}\ + \frac{2(2 - 6b^2 + 5b^4 - 6b^6 + 2b^8)}{b^4} G_{-9/2} + \frac{1 - 9b^4 + b^8}{b^4} G_{-5/2} G_{-3/2} G_{-1/2}\$$

(41)

We quote here these bulky expressions simply to give a general idea of what we’re dealing with. Of course the same expressions with $G_n \rightarrow \bar{G_n}$ and $L_n \rightarrow \bar{L_n}$ give the “left” creation operators $\bar{D}_{m,n}$.

In SLFT, like in the bosonic LFT, all singular vectors “decouple”, i.e., in the sense of quantum operators

$$D_{m,n}(V, R)_{m,n} = \bar{D}_{m,n}(V, R)_{m,n} = 0$$

(42)

where for the primary field stands either NS or Ramond one, dependent in an obvious way on the parity of $m$ and $n$.\footnote{Below we often use this abbreviation $(V, R)$, which stands either for $V$ or for $R$, dependent in an obvious manner on the context. This is the quantum version of the classical relations \( (m, n) \). Precisely
as in the case of LFT, equations (42) can be considered as the basic dynamic principle of SLFT. Even the very first steps in the study of SLFT [8–10], as well as the later achievements in construction of a consistent theory [12, 13], are based on this “decoupling”.

In the next section we state certain algebraic property of the $D_{m,n}$ operators, a supersymmetric generalization of the product formula of [1] for the “norms of logarithmic primaries” (defined more precisely below).

**Norms of logarithmic primaries.** Related to every $D_{m,n}$ of the previous section, define a “conjugate” operator $D^\dagger_{m,n}$ through the prescriptions $G^\dagger_n = G_{-n}$ and $L^\dagger_n = L_{-n}$. Then $D^\dagger_{m,n} D_{m,n}$ obviously acts invariantly at the levels. In particular the highest weight vector $|a\rangle$ (here we use unified notation for the NS and Ramond primary states) of dimension $\Delta$ ($\Delta = \Delta_{NS}(a)$ or $\Delta = \Delta_R(a)$, dependent on the type of the representation) is an eigenvector

$$D^\dagger_{m,n} D_{m,n} |a\rangle = d_{m,n}(a,b) |a\rangle$$

with certain eigenvalue $d_{m,n}(a,b)$. By definition this function is zero at $a = a_{m,n}$ (or $a = Q - a_{m,n}$), where $\Delta_a$ becomes the Kac dimension (28) of singular representation. We’re interested in the quantity $r_{m,n}$, the coefficient in the linear term

$$d_{m,n}(a,b) = r_{m,n}(a - a_{m,n}) + O\left((a - a_{m,n})^2\right)$$

. This coefficient is a function of $b$. “Manual” calculations with the explicit expressions (49) produce the following compact results

$$r_{1,1} = b^{-1}(1 + b^2)$$
$$r_{1,2} = b^{-1}(1 - b^4)$$
$$r_{1,3} = -b^{-1}(1 - b^4)(1 + 3b^2)$$
$$r_{1,4} = -2b^{-1}(1 - b^4)(1 - 9b^4)$$
$$r_{2,2} = b^{-5}(1 - b^4)^2(1 + b^2)^3$$
$$r_{1,5} = 4b^{-1}(1 - b^4)(1 - 9b^4)(1 + 5b^2)$$
$$r_{1,6} = 12b^{-1}(1 - b^4)(1 - 9b^4)(1 - 25b^4)$$
$$r_{2,3} = b^{-7}(1 - b^4)^3(1 - 9b^4)$$
$$r_{1,7} = -36b^{-1}(1 - b^4)(1 - 9b^4)(1 - 25b^4)(1 + 7b^2)$$
$$r_{1,8} = -144b^{-1}(1 - b^4)(1 - 9b^4)(1 - 25b^4)(1 - 49b^4)$$
$$r_{2,4} = -2b^{-9}(1 - b^4)^3(1 - 9b^4)^2(1 + 2b^2)$$
$$r_{1,9} = 5796b^{-1}(1 - b^4)(1 - 9b^4)(1 - 25b^4)(1 - 49b^4)(1 + 9b^2)$$
$$r_{3,3} = -3b^{-13}(1 - b^4)^4(1 + b^2)(9 - b^4)(1 - 9b^2)$$

All of them fall into the “product formula”

$$r_{m,n} = 2^{1-mn} \prod_{(k,l) \in [m,n]} (kb^{-1} + lb)$$

(46)
where symbol \([m, n]\) denotes either \([m, n]_{NS}\), or \([m, n]_R\), dependent on the type of representation. Here

\[
[m, n]_{NS} = \{1 - m : 2 : m - 1, 1 - n : 2 : n - 1\}
\cup \{2 - m : 2 : m, 2 - n : 2 : n\} \setminus \{0, 0\}
\]

\[
[m, n]_R = \{1 - m : 2 : m - 1, 2 - n : 2 : n\}
\cup \{2 - m : 2 : m, 1 - n : 2 : n - 1\} \setminus \{0, 0\}
\]

In these expressions \(a : d : b\) (“from \(a\) to \(b\) step \(d\)”) stands for the “linear” set, i.e., the set of numbers \(a, a + d, a + 2d, \ldots, b\). Symbol \(\{A, B\}\) is for the set of pairs \((k, l)\) with \(k\) and \(l\) running independently the sets \(A\) and \(B\) and \(\{A_1, B_1\} \cup \{A_2, M_2\}\) is the standard union of two sets. Finally, \(\ldots \setminus \{0, 0\}\) means that the pair \((0, 0)\) is excluded.

Expression (46) is very much like the one obtained in [1] for the similar characteristic related to the singular representations of the usual Virasoro algebra. In that case a line of “physical” arguments has been proposed, based on the consistency of HEM’s with the one point functions in the so called “Poincaré disk geometry” (see [16]). At the same time, it is clear that the product formula is of purely algebraic nature and has nothing to do neither with HEM’s nor with the Poincaré disk. It is desirable therefore to have a direct algebraic derivation of the product formula, as well as of its SUSY version (46). It is plausible that such a derivation can be found studying the structure of moduli embeddings in the non-trivial case of rational \(b^2\) (authors thank B. Feigin for a discussion of this point). Let us mention also an algebraic, although rather complicated proof of a particular case of the product formula, related to the singular representations \((1, n)\) of the Virasoro algebra [18].

In this short note we follow a simplified way, opposite to that of [1]. In the absence of a direct proof, we take eq. (46) for granted and derive the coefficients in HEM’s comparing it with the one point functions on the Poincaré disk [19, 20]. This procedure, unlike the study of multipoint functions in [1], makes the analysis very compact and allows to avoid heavy calculations with the SLFT structure constants.

**Logarithmic degenerate fields and HEM’s.** Now we’re in the position to define, in the spirit of ref. [1], the set of “logarithmic degenerate fields” \(V'_{m,n} (m - n \in 2\mathbb{Z})\) and \(R'_{m,n} (m - n \in 2\mathbb{Z} + 1)\). General logarithmic fields \(V'_a = \partial\gamma_a / \partial a\) and \(R'_a = \partial\gamma_a / \partial a\) are the derivatives in \(a\) (a normal ordered free fields : \(\phi \exp(\alpha\phi)\) : and \(\sigma(\mu) : \phi \exp(\alpha\phi)\) : is what they look like in the \(\phi \to -\infty\) free field limit) of the corresponding primary ones. Let us set

\[
V'_{m,n} = V'_a |_{a = a_{m,n}} \quad m - n \in 2\mathbb{Z}
\]

\[
R'_{m,n} = R'_a |_{a = a_{m,n}} \quad m - n \in 2\mathbb{Z} + 1
\]

Whereas \(V'_{m,n}\) and \(R'_{m,n}\) are logarithmic fields (as well as general \(V'_a\) and \(R'_a\)), holds true the following

**Proposition:**

\[
\bar{D}_{m,n} D_{m,n} (V, R)'_{m,n}
\]

are primary fields.
We will not repeat the proof here, as it follows literally the considerations of ref. [1]. The idea is quite simple. From the algebraic point of view the only difference between the logarithmic fields \((V, R)_a\)' and the primary ones \((V, R)_a\) is the inhomogeneous term in the action of \(L_0\)

\[
L_0(V, R)_a' = \Delta_{(NS,R)}(a)(V, R)_a + \frac{d\Delta_{(NS,R)}(a)}{da}(V, R)_a
\]

and the same for \(\bar{L}_0\). Hence, for every \(k > 0\) the vectors \(L_k D_{m,n}(V, R)_a\)' and \(G_k D_{m,n}(V, R)_a\)' are in the \(SV\) (“right”) module over \((V, R)_a\), and therefore are annihilated by \(\bar{D}_{m,n}\).

Similarly to the Liouville case, in SLFT the primary fields (50) are to be identified with other exponential fields. Comparing dimensions we find

\[
\bar{D}_{m,n} D_{m,n}(V, R)'_{m,n} = \mathcal{B}_{m,n}(\tilde{V}, \tilde{R})_{m,n}
\]

Here the exponential primaries \(\tilde{V}_{m,n} = V_{am,-n}\) and \(\tilde{R}_{m,n} = R_{am,-n}\) have dimensions \(\Delta_{m,n}^{(NS)} + \frac{mn}{2}\) and \(\Delta_{m,n}^{(R)} + \frac{mn}{2}\) respectively.

Equations (52) are our long anticipated SLFT HEM’s. The remaining problem of the numerical coefficients \(\mathcal{B}_{m,n}\) is discussed in the next section. We will do this comparing right and left hand sides of (52) inside correlation functions. The simplest one is the one-point function in the non-compact geometry of the Poincaré disk. In this geometry, unlike sphere or “finite disk” [17], the gauge group \(SL(2, R)\) is an isometry and therefore there is no problem of factoring out its orbits. There are thus all reasons to expect the operator-valued relations to hold already on the one point function level. On the other hand, such one-point functions are relatively simple [19, 20]. Let us turn to their discussion.

**One point functions on the Poincaré disk.** In refs. [19, 20] the one point functions were constructed in the so called Poincaré disk geometry. Roughly speaking, this geometry is a quantum version of the “basic” classical solution to the classical SLFT equations of motion (1) inside the unit disk \(|z| < 1\)

\[
e^\varphi = \frac{2}{M^2(1-\bar{z}z)^2}; \quad \bar{\psi} = \bar{\psi} = 0.
\]

The object of study is the one point function of the exponential SLFT fields (the meaning of the index \((m, n)\) near the one point functions, boundary states and amplitudes is explained in [16] and [19, 20])

\[
\langle \langle V, R \rangle_a \rangle_{(1,1)} = \frac{\langle \langle B_{(1,1)} \rangle | (V, R)_a \rangle}{\langle B_{(1,1)} | V_0 \rangle} = U_{(1,1)}^{(NS,R)}(a),
\]

where we denoted as \(\langle B_{(1,1)} \rangle\) the boundary state radiated by the absolute of the Poincaré disk. In this paper we will not repeat the considerations of refs. [19, 20], quoting only the
Substituting equations (52) to these one point functions, we obtain

$$\langle B(1,1) | \bar{D}_{m,n} D_{m,n} (V,R)'_{m,n} \rangle = B_{m,n} \langle B(1,1) | (\tilde{V}, \tilde{R})_{m,n} \rangle$$  \hfill (56)

The boundary state $\langle B(1,1) |$ enjoys superconformal invariance. This means that for all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}, \mathbb{Z} + 1/2$ the following identities hold

$$\langle B(1,1) | \tilde{G}_k = i \langle B(1,1) | G_{-k} \rangle \quad (57)$$

$$\langle B(1,1) | \tilde{L}_n = \langle B(1,1) | L_{-n} \rangle . \quad (58)$$

It is easy to see that these identities entail (operator $D_{m,n}^\dagger$ is defined in sect. 5)

$$\langle B(1,1) | \tilde{D}_{m,n} = \langle B(1,1) | D_{m,n}^\dagger \quad mn \in 2\mathbb{Z} \quad (59)$$

$$\langle B(1,1) | \tilde{D}_{m,n} = i \langle B(1,1) | D_{m,n}^\dagger \quad mn \in 2\mathbb{Z} + 1 \quad (60)$$

In the last case a multiplier $i$ remains because at $mn \in 2\mathbb{Z} + 1$, unlike all other cases, the singular vector creating operator is fermionic (odd in $G$). We find

$$\langle B(1,1) | D_{m,n}^\dagger D_{m,n} (V,R)'_{m,n} \rangle = (-i)^{mn-2[mn/2]} B_{m,n} \langle B(1,1) | (\tilde{V}, \tilde{R})_{m,n} \rangle . \quad (61)$$

In terms on the one point functions eq. (44) has the following interpretation

$$\langle B(1,1) | D_{m,n} D_{m,n} (V,R)'_{m,n} \rangle = r_{m,n} \langle B(1,1) | (V,R)_{m,n} \rangle \quad (62)$$

where numbers $r_{m,n}$ are from the product formulas (46), (47) and (48). Therefore

$$B_{m,n} = i^{mn-2[mn/2]} r_{m,n} U^{(NS,R)}_{(1,1)}(a_{m,n}) U^{(NS,R)}_{(1,1)}(a_{m,-n})$$  \hfill (63)

This equality allows to find easily the coefficients $B_{m,n}$. Two cases should be distinguished
1. NS case \((m \text{ and } n \text{ both either even or odd})\)

\[
B_{m,n} = 2^mn^{m-2n/2}b^{n-m+1} \left[ \pi \mu^2 (bQ/2) \right]^n \gamma \left( \frac{m-nb^2}{2} \right) \prod_{(k,l) \in \langle m,n \rangle_{NS}} \lambda_{k,l} \quad (62)
\]

2. R case \((m \text{ odd and } n \text{ even})\)

\[
B_{m,n} = 2^m b^{n-m} \left[ \pi \mu^2 (bQ/2) \right]^n \gamma \left( \frac{1}{2} + \frac{m-nb^2}{2} \right) \prod_{(k,l) \in \langle m,n \rangle_{R}} \lambda_{k,l} \quad (63)
\]

Symbol \(\lambda_{k,l}\) is defined in (27) while the sets \(\langle m,n \rangle_{NS}\) and \(\langle m,n \rangle_{R}\) include the following pairs of integers \((k,l)\)

\[
\langle m,n \rangle_{NS} = \{1 - m : 2 : m - 1, 1 - n : 2 : n - 1\} \quad (64)
\]

\[
\cup \{2 - m : 2 : m - 2, 2 - n : 2 : n - 2\} \setminus \{0,0\}
\]

\[
\langle m,n \rangle_{R} = \{1 - m : 2 : m - 1, 2 - n : 2 : n - 2\} \quad (65)
\]

\[
\cup \{2 - m : 2 : m - 2, 1 - n : 2 : n - 1\} \setminus \{0,0\}
\]

Only the series \((1,2k-1), k = 1, 2, \ldots\) of HEM’s allows a classical limit. It is easy to check that (62) at \(b \rightarrow 0\) turns to \(B_{1,2k-1} \rightarrow i(-1)^{k-1}b^{-1}(2\pi \mu b^2)^{2k-1}[(k-1)!]^2\), in agreement with the results (12) of the classical calculations.

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