Intersecting families, cross-intersecting families, and a proof of a conjecture of Feghali, Johnson and Thomas

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Abstract

A family \( A \) of sets is said to be intersecting if every two sets in \( A \) intersect. Two families \( A \) and \( B \) are said to be cross-intersecting if each set in \( A \) intersects each set in \( B \).

For a positive integer \( n \), let \([n] = \{1, \ldots, n\} \) and \( S_n = \{ A \subseteq [n] : 1 \in A \} \). In this note, we extend the Erdős–Ko–Rado Theorem by showing that if \( A \) and \( B \) are non-empty cross-intersecting families of subsets of \([n]\), \( A \) is intersecting, and \( a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \) are non-negative real numbers such that \( a_i + b_i \geq a_{n-i} + b_{n-i} \) and \( a_{n-i} \geq b_i \) for each \( i \leq n/2 \), then

\[
\sum_{A \in A} a_{|A|} + \sum_{B \in B} b_{|B|} \leq \sum_{A \in S_n} a_{|A|} + \sum_{B \in S_n} b_{|B|}.
\]

For a graph \( G \) and an integer \( r \), let \( \mathcal{I}_G^{(r)} \) denote the family of \( r \)-element independent sets of \( G \). Inspired by a problem of Holroyd and Talbot, Feghali, Johnson and Thomas conjectured that if \( r < n \) and \( G \) is a depth-two claw with \( n \) leaves, then \( G \) has a vertex \( v \) such that \( \{ A \in \mathcal{I}_G^{(r)} : v \in A \} \) is a largest intersecting subfamily of \( \mathcal{I}_G^{(r)} \). They proved this for \( r \leq \frac{n+1}{2} \). We use the result above to prove the full conjecture.

1 Introduction

Unless otherwise stated, we shall use small letters such as \( x \) to denote non-negative integers or elements of a set, capital letters such as \( X \) to denote sets, and calligraphic letters such as \( \mathcal{F} \) to denote families (that is, sets whose members are sets themselves). It is to be assumed that arbitrary sets and families are finite. We call a set \( A \) an \( r \)-element set if its size \( |A| \) is \( r \), that is, if it contains exactly \( r \) elements (also called members).

The set \( \{1, 2, \ldots\} \) of positive integers is denoted by \( \mathbb{N} \). For any integer \( n \geq 0 \), the set \( \{ i \in \mathbb{N} : i \leq n \} \) is denoted by \([n] \). Note that \([0] \) is the empty set \( \emptyset \). For a set \( X \), the power set of \( X \) (that is, \( \{ A : A \subseteq X \} \)) is denoted by \( 2^X \). The family of \( r \)-element subsets of \( X \) is denoted by \( \binom{X}{r} \). The family of \( r \)-element sets in a family \( \mathcal{F} \) is denoted by \( \mathcal{F}^{(r)} \). If \( \mathcal{F} \subseteq 2^X \)
and $x \in X$, then we denote the family $\{F \in \mathcal{F} : x \in F\}$ by $\mathcal{F}(x)$. We call $\mathcal{F}(x)$ a star of $\mathcal{F}$ if $\mathcal{F}(x) \neq \emptyset$.

We say that a set $A$ intersects a set $B$ if $A$ and $B$ have at least one common element (that is, $A \cap B \neq \emptyset$). A family $\mathcal{A}$ is said to be intersecting if for every $A, B \in \mathcal{A}$, $A$ and $B$ intersect. The stars of a family $\mathcal{F}$ (with $|\bigcup_{F \in \mathcal{F}} F| \geq 1$) are the simplest intersecting subfamilies of $\mathcal{F}$. We say that $\mathcal{F}$ has the star property if at least one of the largest intersecting subfamilies of $\mathcal{F}$ is a star of $\mathcal{F}$.

One of the most popular endeavours in extremal set theory is that of determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$. This started in $[12]$, which features the following classical result, known as the Erdős-Ko-Rado (EKR) Theorem.

**Theorem 1.1 (EKR Theorem $[12]$)** If $r \leq n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

This means that $\binom{[n]}{r}$ has the star property. There are various proofs of the EKR Theorem (see $[10, 23, 25]$), two of which are particularly short and beautiful: Katona’s $[23]$, which introduced the elegant cycle method, and Daykin’s $[10]$, using the fundamental Kruskal–Katona Theorem $[24, 26]$. The EKR Theorem gave rise to some of the highlights in extremal set theory $[1, 15, 25, 28]$ and inspired many results that establish how large a system of sets can be under certain intersection conditions; see $[4, 11, 14, 16, 17, 20, 21]$.

If $\mathcal{A}$ and $\mathcal{B}$ are families such that each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are said to be cross-intersecting.

For intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. A natural variant of this intersection problem is the problem of maximizing the sum or the product of sizes of cross-intersecting subfamilies (not necessarily distinct or non-empty) of $\mathcal{F}$. This has recently attracted much attention. The relation between the original intersection problem, the sum problem and the product problem is studied in $[6]$. Solutions have been obtained for various families; most of the known results are referenced in $[2, 7]$, which treat the product problem for families of subsets of $[n]$ of size at most $r$.

Here we consider the sum problem for the case where at least one of two cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $[n]$ is an intersecting family. We actually consider a more general setting of weighted sets, where each set of size $i$ is assigned two non-negative integers $a_i$ and $b_i$, and the objective is to maximize $\sum_{A \in \mathcal{A}} a_{|A|} + \sum_{B \in \mathcal{B}} b_{|B|}$. Let $\mathcal{S}_n$ denote the star $\{A \subseteq [n] : 1 \in A\}$ of $2^{[n]}$. In Section 2 we prove the following extension of the EKR Theorem.

**Theorem 1.2** If $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting families of subsets of $[n]$, $\mathcal{A}$ is intersecting, and $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are non-negative real numbers such that $a_i + b_i \geq a_{n-i} + b_{n-i}$ and $a_{n-i} \geq b_i$ for each $i \leq n/2$, then

$$\sum_{A \in \mathcal{A}} a_{|A|} + \sum_{B \in \mathcal{B}} b_{|B|} \leq \sum_{A \in \mathcal{S}_n} a_{|A|} + \sum_{B \in \mathcal{S}_n} b_{|B|}.$$

The EKR Theorem is obtained by taking $r \leq n/2$, $\mathcal{B} = \mathcal{A} \subseteq \binom{[n]}{r}$, and $b_i = 0 = a_i - 1$ for each $i \in \{0\} \cup [n]$.

We use Theorem 1.2 to prove a conjecture of Feghali, Johnson and Thomas $[13]$, Conjecture 2.1. Before stating the conjecture, we need some further definitions and notation.
A graph $G$ is a pair $(X, Y)$, where $X$ is a set, called the vertex set of $G$, and $Y$ is a subset of $\binom{X}{2}$ and is called the edge set of $G$. The vertex set of $G$ and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. An element of $V(G)$ is called a vertex of $G$, and an element of $E(G)$ is called an edge of $G$. We may represent an edge $\{v, w\}$ by $vw$. If $vw$ is an edge of $G$, then we say that $v$ is adjacent to $w$ (in $G$). A subset $I$ of $V(G)$ is an independent set of $G$ if $vw \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G$ denote the family of independent sets of $G$. An independent set $J$ of $G$ is maximal if $J \notin I$ for each independent set $I$ of $G$ such that $I \neq J$. The size of a smallest maximal independent set of $G$ is denoted by $\mu(G)$.

Holroyd and Talbot introduced the problem of determining whether $\mathcal{I}_G^{(r)}$ has the star property for a given graph $G$ and an integer $r \geq 1$. The Holroyd–Talbot (HT) Conjecture [21] Conjecture 7] claims that $\mathcal{I}_G^{(r)}$ has the star property if $\mu(G) \geq 2r$. It is proved in [3] that the conjecture is true if $\mu(G)$ is sufficiently large depending on $r$. By the EKR Theorem, it is true if $G$ has no edges. The HT Conjecture has been verified for several classes of graphs [8, 9, 18, 20, 21, 22, 27, 29]. As demonstrated in [9], for $r > \mu(G)/2$, whether $\mathcal{I}_G^{(r)}$ has the star property or not depends on $G$ and $r$ (both cases are possible).

A depth-two claw is a graph consisting of $n$ pairwise disjoint edges $x_1y_1,\ldots,x_ny_n$ together with a vertex $x_0 \notin \{x_1,\ldots,x_n,y_1,\ldots,y_n\}$ that is adjacent to each of $y_1,\ldots,y_n$. This graph will be denoted by $T_n$. Thus, $T_n = (\{x_0,x_1,\ldots,x_n,y_1,\ldots,y_n\},\{x_0y_1,\ldots,x_0y_n,x_1y_1,\ldots,x_ny_n\})$. For each $i \in [n]$, we may take $x_i$ and $y_i$ to be $(i,1)$ and $(i,2)$, respectively. Let $X_n = \{x_i: i \in [n]\}$ and

$$\mathcal{L}_{n,k} = \left\{ (i_1,j_1),\ldots,(i_r,j_r) : r \in [n], \{i_1,\ldots,i_r\} \in \binom{[n]}{r}, j_1,\ldots,j_r \in [k] \right\}.$$ 

Note that

$$\mathcal{I}_{T_n}^{(r)} = \mathcal{L}_{n,2}^{(r)} \cup \left\{ A \cup \{x_0\} : A \in \binom{[n]}{r-1} \right\}. \quad (1)$$

The family $\mathcal{I}_{T_n}^{(r)}$ is empty for $r > n + 1$, and consists only of the set $\{x_0,x_1,\ldots,x_n\}$ for $r = n + 1$. In [13], Feghali, Johnson and Thomas showed that $\mathcal{I}_{T_n}^{(r)}$ does not have the star property for $r = n$, and they made the following conjecture.

**Conjecture 1.3 [13]** If $r \leq n - 1$, then $\mathcal{I}_{T_n}^{(r)}$ has the star property.

They proved the conjecture for $r \leq \frac{n+1}{2}$.

**Theorem 1.4 [13]** If $n \geq 2r - 1$, then $\mathcal{I}_{T_n}^{(r)}$ has the star property.

In the next section, we settle the full conjecture using Theorem 1.2 for $r > \frac{n+1}{2}$.

**Theorem 1.5** Conjecture 1.3 is true.

Our proof for $r \leq \frac{n}{2} + 1$ provides an alternative proof of Theorem 1.4.
2 Proofs

In this section, we prove Theorems 1.2 and 1.5.

Proof of Theorem 1.2. For each $i \in \{0\} \cup [n]$, let $c_i = |A^{(i)}|a_i + |B^{(i)}|b_i$. Since $\mathcal{A}$ and $\mathcal{B}$ are non-empty and cross-intersecting, we have $\emptyset \notin \mathcal{A}$ and $\emptyset \notin \mathcal{B}$, so $\mathcal{A}^{(0)} = \mathcal{B}^{(0)} = \emptyset$. Thus, $c_0 = 0$. The result is immediate if $n = 1$. Suppose $n > 1$.

Consider any positive integer $r \leq n/2$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting and $\mathcal{A}$ is intersecting, $[n] \setminus \mathcal{A} \notin \mathcal{A}^{(n-r)}$ for each $\mathcal{A} \in \mathcal{A}^{(r)} \cup \mathcal{B}^{(r)}$, so

$$|\mathcal{A}^{(n-r)}| \leq \binom{n}{n-r} - |\{ [n] \setminus \mathcal{A} : \mathcal{A} \in \mathcal{A}^{(r)} \cup \mathcal{B}^{(r)} \}| = \binom{n}{r} - |\mathcal{A}^{(r)} \cup \mathcal{B}^{(r)}| = \binom{n}{r} - |\mathcal{A}^{(r)}| - |\mathcal{B}^{(r)}\setminus \mathcal{A}^{(r)}|.$$

Similarly, since $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, $|\mathcal{B}^{(n-r)}| \leq \binom{n}{r} - |\mathcal{A}^{(r)}|$. Thus,

$$c_r + c_{n-r} \leq |\mathcal{A}^{(r)}|a_r + |\mathcal{A}^{(r)}|b_r + |\mathcal{B}^{(r)}\setminus \mathcal{A}^{(r)}|b_r + \binom{n}{r} - |\mathcal{A}^{(r)}| - |\mathcal{B}^{(r)}\setminus \mathcal{A}^{(r)}|a_{n-r} + \binom{n}{r} - |\mathcal{A}^{(r)}|b_{n-r}

= |\mathcal{A}^{(r)}|a_r + b_r - a_{n-r} - b_{n-r} - |\mathcal{B}^{(r)}\setminus \mathcal{A}^{(r)}|(a_{n-r} + b_r) + \binom{n}{r}(a_{n-r} + b_{n-r})

\leq \binom{n-1}{r-1}(a_r + b_r - a_{n-r} - b_{n-r}) + \binom{n}{r}(a_{n-r} + b_{n-r})$$

by Theorem 1.1 and the given conditions $a_r + b_r \geq a_{n-r} + b_{n-r}$ and $a_{n-r} \geq b_r$. Therefore, $c_r + c_{n-r} \leq \binom{n-1}{r-1}(a_r + b_r) + \binom{n-1}{r-1}(a_{n-r} + b_{n-r}) = \binom{n-1}{r-1}(a_r + b_r) + \binom{n-1}{r-1}(a_{n-r} + b_{n-r})$. Note that if $r = n/2$, then we have $c_{n/2} + c_{n/2} \leq \binom{n-1}{n/2-1}(a_{n/2} + b_{n/2}) + \binom{n-1}{n/2-1}(a_{n/2} + b_{n/2})$, and hence $c_{n/2} \leq \binom{n-1}{n/2-1}(a_{n/2} + b_{n/2})$.

If $n$ is odd, then

$$\sum_{\mathcal{A} \in \mathcal{A}} a_{|\mathcal{A}|} + \sum_{\mathcal{B} \in \mathcal{B}} b_{|\mathcal{B}|} = \sum_{i=0}^{n-1} (|\mathcal{A}^{(i)}|a_i + |\mathcal{A}^{(n-i)}|a_{n-i} + |\mathcal{B}^{(i)}|b_i + |\mathcal{B}^{(n-i)}|b_{n-i})$$

$$= c_0 + c_n + \sum_{i=1}^{n-1} (c_i + c_{n-i})$$

$$\leq a_n + b_n + \sum_{i=1}^{n-1} \left( \binom{n-1}{i-1}(a_i + b_i) + \binom{n-1}{n-i-1}(a_{n-i} + b_{n-i}) \right)$$

$$= \sum_{\mathcal{A} \in \mathcal{S}_n} a_{|\mathcal{A}|} + \sum_{\mathcal{B} \in \mathcal{S}_n} b_{|\mathcal{B}|}.$$
Similarly, if \( n \) is even, then

\[
\sum_{A \in \mathcal{A}} a_{|A|} + \sum_{B \in \mathcal{B}} b_{|B|} = c_0 + c_n + c_{n/2} + \sum_{i=1}^{n-1}(c_i + c_{n-i})
\]

\[
\leq a_n + b_n + \left(\frac{n-1}{n/2 - 1}\right)(a_{n/2} + b_{n/2})
\]

\[
+ \sum_{i=1}^{n-1} \left(\frac{n-1}{i-1}\right)(a_i + b_i) + \left(\frac{n-1}{n-i-1}\right)(a_{n-i} + b_{n-i})
\]

\[
= \sum_{A \in \mathcal{A}} a_{|A|} + \sum_{B \in \mathcal{B}} b_{|B|},
\]

as required. \( \square \)

For \((i, j) \in [n] \times [2, k]\), let \( \delta_{i,j} : \mathcal{L}_{n,k} \rightarrow \mathcal{L}_{n,k} \) be defined by

\[
\delta_{i,j}(A) = \begin{cases} 
(A \setminus \{(i, j)\}) \cup \{(i, 1)\} & \text{if } (i, j) \in A; \\
A & \text{otherwise},
\end{cases}
\]

and let \( \Delta_{i,j} : 2^{\mathcal{L}_{n,k}} \rightarrow 2^{\mathcal{L}_{n,k}} \) be the compression operation defined by

\[
\Delta_{i,j}(\mathcal{A}) = \{ \delta_{i,j}(A) : A \in \mathcal{A} \} \cup \{ A \in \mathcal{A} : \delta_{i,j}(A) \in \mathcal{A} \}.
\]

It is well known that \( |\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}| \) and that, if \( \mathcal{A} \) is intersecting, then \( \Delta_{i,j}(\mathcal{A}) \) is intersecting. Moreover, we have the following special case of \[5\] Corollary 3.2.

**Lemma 2.1** If \( \mathcal{A} \) is an intersecting subfamily of \( \mathcal{L}_{n,k} \) and

\[
\mathcal{A}^* = \Delta_{n,k} \circ \cdots \circ \Delta_{n,2} \circ \cdots \circ \Delta_{1,k} \circ \cdots \circ \Delta_{1,2}(\mathcal{A}),
\]

then \( |A \cap B \cap X_n| \geq 1 \) for any \( A, B \in \mathcal{A}^* \).

**Proof of Theorem 1.5.** The result is trivial for \( 0 \leq r \leq 1 \), so consider \( 2 \leq r \leq n - 1 \). Let \( \mathcal{R} = \mathcal{I}_{T_n}(r) \). Let \( \mathcal{E} \) be an intersecting subfamily of \( \mathcal{R} \). Let \( \mathcal{F} = \{ A \in \mathcal{R} : x_1 \in A \} \). For any \( \mathcal{H} \subseteq \mathcal{R} \), let \( \mathcal{H}_0 = \{ H \in \mathcal{H} : x_0 \notin H \} \), \( \mathcal{H}_1 = \{ H \in \mathcal{H} : x_0 \in H \} \) and \( \mathcal{H}' = \{ H \setminus \{x_0\} : H \in \mathcal{H}_1 \} \); by \[11\], \( \mathcal{H}_0 \subseteq \mathcal{L}_{n,2}(r) \) and \( \mathcal{H}' \subseteq \binom{X_n}{r-1} \).

**Case 1:** \( n \geq 2r - 2 \). Then \( r \leq \frac{n}{2} + 1 \). Let \( \gamma : \mathcal{R} \rightarrow \mathcal{R} \) be defined by

\[
\gamma(A) := \begin{cases} 
(A \setminus \{x_0\}) \cup \{y_1\} & \text{if } x_0 \in A, y_1 \notin A \text{ and } (A \setminus \{x_0\}) \cup \{y_1\} \in \mathcal{R}; \\
A & \text{otherwise},
\end{cases}
\]

and let \( \Gamma : 2^{\mathcal{R}} \rightarrow 2^{\mathcal{R}} \) be the compression operation defined by

\[
\Gamma(\mathcal{A}) := \{ \gamma(A) : A \in \mathcal{A} \} \cup \{ A \in \mathcal{A} : \gamma(A) \in \mathcal{A} \}.
\]

Note that \( \gamma(A) \neq A \) if and only if \( x_0 \in A \) and \( x_1, y_1 \notin A \). Let \( \mathcal{G} = \Gamma(\mathcal{E}) \). Then \( \mathcal{G} \subseteq \mathcal{R} \) and \( |\mathcal{G}| = |\mathcal{E}| \). By \[9\] Lemma 2.1, \( \mathcal{G}_0 \) and \( \mathcal{G}' \) are intersecting. Since \( r - 1 \leq \frac{n}{2} \) and \( \mathcal{G}' \subseteq \binom{X_n}{r-1} \),
Theorem 11 gives us $|G'_1| \leq |\{A \in (X_n)^{(r)}: x_1 \in A\}| = |F'_1|$. It is well known that, for any $k \geq 2$, $L_{n,k}^{(r)}$ has the star property [11, Theorem 5.2]. Thus, $|G_0| \leq |\{A \in L_{n,2}^{(r)}: x_1 \in A\}| = |F_0|$. Therefore, we have $|E| = |G| = |G_0| + |G'_1| \leq |F_0| + |F'_1| = |F|$.

Case 2: $n \leq 2r - 3$. Then $r > \frac{n}{2} + 1$. Let $E'_0 = \Delta_{n,2} \circ \cdots \circ \Delta_{1,2}(E_0)$. Thus, $E'_0 \subseteq L_{n,2}^{(r)}$. By Lemma 27

$$E \cap F \cap X_n \neq \emptyset \text{ for any } E, F \in E'_0.$$  

For any $E \in E_0$ and any $F \in E_1$, we have $\emptyset \neq E \cap F \subseteq X_n$ and $E \cap X_n \subseteq \delta_i(E)$ for any $i \in [n]$. It clearly follows that

$$E \cap F \cap X_n \neq \emptyset \text{ for any } E \in E'_0 \text{ and any } F \in E'_1.$$  

Let $A = \{E \cap X_n: E \in E'_0\}$ and $B = E'_1$. By (2), $A$ is intersecting. By (3), $A$ and $B$ are cross-intersecting. Let $a_i = \binom{n-i}{r-i}$ for each $i \in \{0\} \cup [r]$, and let $a_i = 0$ for each $i \in [r + 1, n]$. Since $r \leq n - 1$, we have

$$a_0 > \cdots > a_r > a_{r+1} = \cdots = a_n = 0.$$  

Let $b_{r-1} = 1$ and let $b_i = 0$ for each $i \in \{0\} \cup [n]\} \setminus \{r - 1\}$.

Consider any $i \leq \frac{n}{2}$. Then $i < r - 1$, so $b_i = 0 \leq a_{n-i}$. If $i < n/2$, then $a_i > a_{n-i}$ (by (4) as $i < \frac{n}{2} < r$), so $a_i + b_i \geq a_{n-i} + b_{n-i}$. If $i = n/2$, then $n - i = i$, so $a_i + b_i = a_{n-i} + b_{n-i}$. (Note that if we ignore Case 1 and rely solely on Theorem 1.4, then we need to prove Conjecture 1.3 for $n \leq 2r - 2$, that is, $r \geq \frac{n}{2} + 1$. In this case, we may have $i = r - 1$.) Suppose $i = r - 1$. Since $i \leq \frac{n}{2} \leq r - 1 = i$, $i = \frac{n}{2}$. Thus, $b_i = 1 = b_{n-i}$ and $a_i = a_{n-i} \geq 1$, and hence again $a_i + b_i = a_{n-i} + b_{n-i}$ and $a_{n-i} \geq b_i$.

We have shown that $a_i + b_i \geq a_{n-i} + b_{n-i}$ and $a_{n-i} \geq b_i$ for each $i \leq n/2$. By Theorem 1.2

$$\sum_{A \in A} a_{|A|} + \sum_{B \in B} b_{|B|} \leq \sum_{A \in X_n} a_{|A|} + \sum_{B \in X_n} b_{|B|},$$

where $X_n = \{A \subseteq X_n: x_1 \in A\}$. Now

$$|E_0| = |E'_0| = \sum_{i=0}^{r} \left|\left\{E \in E'_0: E \cap X_n \in A^{(i)}\right\}\right| \leq \sum_{i=0}^{r} \left|\left\{E \in L_{n,2}^{(r)}: E \cap X_n \in A^{(i)}\right\}\right| = \sum_{i=0}^{r} |A^{(i)}|a_i = \sum_{A \in A} a_{|A|}$$

and $|E_1| = |B| = \sum_{B \in B} b_{r-1} = \sum_{B \in B} b_{|B|}$. Clearly, $|F_0| = \sum_{A \in X_n} a_{|A|}$ and $|F_1| = \sum_{B \in X_n} b_{|B|}$. We have

$$|E| = |E_0| + |E_1| \leq \sum_{A \in A} a_{|A|} + \sum_{B \in B} b_{|B|} \leq \sum_{A \in X_n} a_{|A|} + \sum_{B \in X_n} b_{|B|} = |F_0| + |F_1| = |F|.$$  

Hence the result. □
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