GENERALIZED FORMS OF AN OVERCONSTRAINED SLIDING MECHANISM CONSISTING OF TWO CONGRUENT TETRAHEDRA

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2020 Mathematics Subject Classification: 51M99
Keywords and phrases: Sliding mechanisms, tetrahedra, bar structures

*Research (partially) supported by Hungarian National Foundation for Scientific Research, grant nos. K68398, T046846.
**Research (partially) supported by Hungarian National Foundation for Scientific Research, grant no. T046846 and NKFI, grant no. K138615.

ABSTRACT. We investigate the motions of a bar structure consisting of two congruent tetrahedra, whose edges in their basic position form the face diagonals of a rectangular parallelepiped. The constraint of the motion is that the originally intersecting edges should remain coplanar. We determine all finite motions of our bar structure. This generalizes our earlier work, where we did the same for the case when the rectangular parallelepiped was a cube. At the end of the paper we point out three further possibilities to generalize the question about the cube, and give for them examples of finite motions.

1. INTRODUCTION

Drawing all the diagonals of all faces of a cube, we obtain the edges of two congruent regular tetrahedra. This position of these tetrahedra is called their basic position.

We keep one of the tetrahedra fixed. We move the other one under the following condition. Each pair of edges of the two tetrahedra, which were originally diagonals of some face of the cube, should remain coplanar.

The structure consisting of the above described two tetrahedra has been invented in 1982 by L. Tompos, Jr. He was then an undergraduate of the Hungarian Academy of Craft and Design. He has built a physical model of the bar (i.e., edge) structure of these tetrahedra, as follows. The bars of one of the tetrahedra touched those of the other tetrahedron from inside (Fig. 1). He has observed

FIGURE 1 ABOUT HERE
that this structure admits continuous motions. We note that [2], p. 7 contains a figure of these tetrahedra, but their mobility is not investigated there. We have found the same figure as a decoration of the dining room of Hotel Arcas de Agua in Spain, in village Arcas, Cuenca (Fig. 2).

By a \textit{motion} (sometimes we will say a \textit{finite motion}) we will not mean a continuous motion from the basic position, always satisfying the constraints, but \textit{any position of}
our structure that satisfies the constraints. (Possibly this position is not the result of a continuous motion, always satisfying the constraints.) We mean by this the following. One of the tetrahedra is fixed. The other one is obtained from the basic position of itself by the application of an isometry (i.e., congruence) of the space of determinant $+1$. Additionally, the coplanarity conditions are satisfied. An isometry of determinant $+1$ will be written in the form $\Phi(x) = Ax + b$. Here $A$ is a $3 \times 3$ orthogonal matrix of determinant $+1$, and $b$ is a vector in $\mathbb{R}^3$. Geometrically, $A$ is a rotation about some straight line containing the origin.

[10] and [12] determined all motions of this pair of tetrahedra. We give a brief description of them, with the names used for them in [12]. Note that the bars of the model have non-zero width. Hence not all motions can be realized by the physical model. We can realize only such motions for which each respective pair of edges (which have to be coplanar) actually has a common point. A discussion of the question which motions are physically admissible, i.e., satisfy this more restrictive condition, is contained in [1], Ch. 4. Further in this paper we will make no distinction between physically admissible and inadmissible motions.

We suppose that the vertices of the tetrahedra in the basic position are the points $(\pm 1, \pm 1, \pm 1)$. More exactly, the vertices of the fixed tetrahedron are $P_1^0(1, -1, -1)$, $P_2^0(-1, 1, -1)$, $P_3^0(-1, -1, 1)$, $P_4^0(1, 1, 1)$. The vertices of the moving tetrahedron are $Q_1^0, \ldots, Q_4^0$. Here, in the basic position, $Q_i^0$ is the mirror image of $P_i^0$ w.r.t. the origin (i.e., the centre of the cube with vertices $(\pm 1, \pm 1, \pm 1)$), see Fig. 3.

FIGURE 3 ABOUT HERE

There exist motions $\Phi(x) = Ax + b$ of Tompos’s tetrahedra, where $A$ is a rotation about an axis $0e_i$, $0(e_i \pm e_j)$ or $0(e_i \pm e_j \pm e_k)$, resp. Here $e_1, e_2, e_3$ are the basic unit vectors in the space, and $i, j, k$ are different. These motions $\Phi(x)$ are called the motions of the first, second and third kind, resp. In the first case, for angle of rotation $\pi$, $b$ is not
unique, but we count only the case $b = 0$ to the motion of the first kind. The angle of rotation of $A$ in the first case is arbitrary, in the second case it is arbitrary, except $\pi$, in the third case it is arbitrary, except $\pm \pi/2$. (The angle of rotation is positive if, looking from the axis vector, e.g., $0e_i$, backwards, it is positive.)

It turned out that for each above rotation $A$, except in the first case the rotation through the angle $\pi$, we had the following. There existed a unique translation $b$ such that $\Phi(x) = Ax + b$ was a motion of Tompos’s tetrahedra. In the first case, for an angle of rotation different from $\pi$, we had $b = 0$.

There also exist motions $\Phi(x) = Ax + b$ of Tompos’s tetrahedra, with $A$ a rotation about an axis $0(C_1e_i + C_2e_j)$, where $i \neq j$, and $C_1, C_2$ are real, not both 0. If $C_1C_2 \neq 0$, and so this is not a motion of the first kind, then the angle of the rotation $A$ is arbitrary, except $\pi$. Moreover, for each such rotation $A$ the translation $b$ is uniquely determined. These motions, for $C_1C_2 \neq 0$, together with the motions of the first kind (where $C_1C_2 = 0$), both with angle of rotation different from $\pi$, are called the motions of the intermediate kind. These motions also contain the motions of the second kind as a special case.

Now let $A$ be a rotation about an axis $0e_i$, through the angle $\pi$. Then let $b$ be any vector of the form $Ce_j$, or $C_1e_j + C_2e_k$, resp., where $C, C_1, C_2$ are real and $i, j, k$ are different. Then $\Phi(x) = Ax + b$ is a motion of Tompos’s tetrahedra, which is called the motion of the fourth, or fifth kind, resp. The motions of the fifth kind contain the motions of the fourth kind as a special case. The motions of the first, second, third, fourth and fifth kinds are drawn in Figs. 4a, 4b, 4c, 4d, 4e.

**FIGURES 4a, 4b, 4c, 4d, 4e ABOUT HERE**

![Figure 4a. The motion of the first kind.](image)
Figure 4b. The motion of the second kind.

Figure 4c. The motion of the third kind.

Figure 4d. The motion of the fourth kind.
It turned out that the motions of each kind, for given \( e_i, e_i \pm e_j, e_i \pm e_j \pm e_k, \{e_i, e_j\}, e_j \) and \( \{e_j, e_k\} \) (in the above order), resp., constituted a smooth manifold in the six-dimensional manifold of all motions (i.e., isometries of determinant +1) of the space. They had dimensions 1, 1, 1, 2, 1, 2, resp. (in the above order). The motions of the third kind formed a manifold of two connected components, cf. [13], p. 141.

These manifolds show certain bifurcation phenomena, which have been analyzed in [12]. Moreover, as shown by [10] and [12], the above enumerated motions are the only motions of Tompos’s tetrahedra. We note that [10] also described the trajectories of the vertices during the physically admissible motions. Moreover, in [4, 5, 6] (of which only [6] has been available to the authors) and in [1] the motions of Tompos’s tetrahedra have been further investigated. They have pointed out some possible mechanical engineering applications.

In our paper we generalize the above investigations. We start not with a cube, but with a general rectangular parallelepiped. All diagonals of all of its faces constitute the edges of two congruent tetrahedra. This position of these two tetrahedra is called their basic position.

We keep one of the tetrahedra fixed. We move the other one (i.e., apply to it an isometry of the space, of determinant +1) under the following condition. Each pair of edges of the two tetrahedra, which were originally diagonals of some face of the rectangular parallelepiped, should remain coplanar.

(A)

In the physical model, the bars (edges) of one of the tetrahedra touch those of the other tetrahedron from inside, as in Fig. 1. First we give the description of all such motions, and then we give a mathematical proof that this list of motions is complete. The results are rather analogous to the case of the cube. Only in a special case there is a motion of the sixth kind, which constitutes a 1-manifold (one-dimensional manifold). These results, more exactly, each of Theorems 1, 2 and 3 of this paper, have been announced in [8]. We also analyze the bifurcation properties of the solution manifolds.

Second, we give three further generalizations of our pair of tetrahedra. For these we
will not be able to determine all finite motions, but we will be able to give certain finite motions. These generalizations are the following. (1) A pair of tetrahedra derived from a general parallelepiped. (2) A pair of regular \( n \)-gonal pyramidal frames (the bases are allowed to change their shape, and also to become non-planar). (3) A pair of regular tetrahedra with congruent circular arc edges. In case (2) we will present some numerical evidence that all continuous finite motions from the basic position, always satisfying the constraints, might form the one-parameter family of finite motions which we have found. Some of these results, namely concerning (1) and (2), also have been announced in [8].

Related results on pairs of polyhedra or polyhedral frames moving with sliding constraints cf. in [7] and [9].

2. THE MOTIONS OF THE TWO TETRAHEDRA DERIVED FROM A RECTANGULAR PARALLELEPIPED

2.1. We will use analogous notations as in the case of the cube. Let the vertices of the rectangular parallelepiped be \((\pm d_1, \pm d_2, \pm d_3)\), where \(d_1, d_2, d_3 > 0\) are constant. The fixed vertices are \(P_1(d_1, -d_2, -d_3), P_2(-d_1, d_2, -d_3), P_3(-d_1, -d_2, d_3), P_4(d_1, d_2, d_3)\). The moving vertices are \(Q_1, Q_2, Q_3, Q_4\), where in the basic position \(Q_i\) is the mirror image of \(P_i\) w.r.t. the origin. Thus \(P_1P_2P_3P_4\) is the fixed tetrahedron, and \(Q_1Q_2Q_3Q_4\) is the moving tetrahedron.

We move the moving tetrahedron under condition (A), which is the same as in the case of the cube. First we describe the motions \(\Phi(x) = Ax + b\) of this moving tetrahedron.

First of all, the motions of the fourth and fifth kinds are defined word for word as in the case of the cube. They evidently exist (i.e., the originally intersecting edges remain coplanar).

The motion of the first kind, also defined in the same way as for the cube, exists for any value of the rotation angle. Moreover, we have for this motion \(b = 0\). These are shown as in [12], p. 428. Namely, let the axis of rotation be, e.g., \(0e_3\). Then the edges of the moving tetrahedron, originally lying on some horizontal face of the rectangular parallelepiped, contain \(\pm d_3e_3\). Hence they remain coplanar. Now consider the straight lines spanned by the edges of the moving tetrahedron, originally lying on some two opposite vertical faces of the rectangular parallelepiped. Their rotated copies, through all angles, constitute a ruling of a (doubly) ruled surface, a one-sheet hyperboloid of revolution. The other diagonals of these two faces belong to the other ruling of this hyperboloid. However, any two lines from different rulings of this hyperboloid are coplanar. Hence this motion exists for all \(\varphi\).

The motion of the intermediate kind is also defined in the same way as for the cube. (Now we do not need to treat separately any analogue of the motion of the second kind.) The motion of the intermediate kind exists for any value of the rotation angle, different from \(\pi\). (But sometimes it may exist also for rotation angle \(\pi\), cf. below.) We apply the analogues of the arguments in [12], p. 429. Let the axis of rotation be \(0u\), where \(u = C_1e_1 + C_2e_2\) has length 1.

As in [12], for convenience, rather than taking one tetrahedron as fixed, the other one as moving, we make the following. We rotate both tetrahedra about the rotation axis, through angles \(\pm \varphi/2\), in such a way that their symmetry w.r.t. the \(xy\)-plane is preserved.
(thus they are rotated in opposite senses). Then the edges of the two tetrahedra, originally lying on one vertical face of the rectangular parallelepiped, remain symmetric w.r.t. the \( xy \)-plane. Hence they remain coplanar. This symmetry, and consequently coplanarity property remains unchanged if we still translate vertically both of these tetrahedra, in a way symmetric w.r.t. the \( xy \)-plane.

Consider the edges of the already rotated (but not yet translated) two tetrahedra, originally lying on one horizontal face of the rectangular parallelepiped. They may have projections to the \( xy \)-plane, spanning properly intersecting lines. In this case, some unique symmetric vertical translations of the already rotated tetrahedra will make the lines spanned by these two already rotated, and translated edges intersecting. Moreover, this happens simultaneously for the pairs of edges, originally lying on both horizontal faces.

Suppose \(-\pi \leq \varphi \leq \pi\).

First we investigate the case \(-\pi < \varphi < \pi\). We are going to show that in this case we have the following. The edges of the already rotated (but not yet translated) two tetrahedra, originally lying on one horizontal face of the rectangular parallelepiped, satisfy the following. They have projections to the \( xy \)-plane, which span properly intersecting lines.

The lines spanned by the diagonals of the upper horizontal face of the rectangular parallelepiped divide the plane spanned by this face to four angular domains. We translate these angular domains to the \( xy \)-plane, so that their vertices are translated to 0. We may suppose that \( \mathbf{u} \) lies, e.g., in the closed angular domain \((Q_2 - Q_1)0(P_4 - P_3)\) (in the unrotated position). We investigate the projections of the rotated diagonals \( Q_1Q_2 \) and \( P_3P_4 \) to the \( xy \)-plane. Let us replace these rotated diagonals by their translated copies \( 0(Q_2 - Q_1) \) and \( 0(P_4 - P_3) \). Then the projections of these translated copies of the rotated diagonals \( Q_1Q_2 \) and \( P_3P_4 \), to the \( xy \)-plane, are translates of the projections of the rotated diagonals \( Q_1Q_2 \) and \( P_3P_4 \), to the \( xy \)-plane.

Now we investigate the respective rotations of the segments \( 0(Q_2 - Q_1) \) and \( 0(P_4 - P_3) \), about the axis \( 0\mathbf{u} \). Their points 0 remain fixed by these rotations. Moreover, these segments move on directrices of two semi-infinite circular cones, with apex 0, and rotation axis \( Ru \), and with sum of semiapertures less than \( \pi \). (One of these cones, but not both, may degenerate to a half-line. This occurs exactly if \( \mathbf{u} \) lies on a side of the angular domain \((Q_2 - Q_1)0(P_4 - P_3)\), in the unrotated position.) This rotation axis \( Ru \) divides the \( xy \)-plane to two half-planes. Then the projections of the respective rotated copies of the segments \( 0(Q_2 - Q_1) \) and \( 0(P_4 - P_3) \) to the \( xy \)-plane satisfy the following. They lie, except for 0, in one open half-plane, and in the complementary closed half-plane, bounded by this rotation axis, resp. Hence, by the semiaperture condition, the lines spanned by them in fact properly intersect, as stated. Therefore this motion exists for all \( \varphi \in (-\pi, \pi) \).

There remained to investigate the case \( \varphi = \pm \pi \). Then the edges of the two tetrahedra, originally lying on a horizontal face of the rectangular parallelepiped, are rotated to positions lying on two parallel vertical planes. These planes have as distance the height \( 2d_3 \) of our rectangular parallelepiped. So no vertical translations can make these rotated edges of the two tetrahedra intersecting. However, these vertical translates still can be coplanar, namely if they are parallel. To check their possible parallelity, it is sufficient to
replace them by their already rotated and translated copies $0(Q_2 - Q_1)$ and $0(P_4 - P_3)$, and check their parallellity. Since both of them contain $0$, therefore their parallellity means their coincidence. They lie in a vertical plane containing $Ru$. They coincide exactly if $u$ lies on the bisector of the angular domain $(Q_2 - Q_1)0(P_4 - P_3)$, in the unrotated position. We have the analogous statements for any of the above four angular domains. Since the horizontal face of the rectangular parallelepiped is a rectangle, this means that $u$ (of length 1) equals $\pm e_1$ or $\pm e_2$. Thus we have a motion of the first kind, which exists for all $\varphi$, so, in particular, for $\varphi = \pi$.

For a motion of the intermediate kind (in particular, for a motion of the first kind, with $\varphi \neq \pi$), for any given rotation $A$, the translation $b$ is uniquely determined. This is also shown in the same way as in [12], p. 429 (or cf. above, at the “properly intersecting lines”). However, possibly for such an $A$ there exist several other $b$’s yielding a motion of another kind, see the next paragraph.

We continue with describing the novel motion of the sixth kind. Let us suppose that our rectangular parallelepiped satisfies $d_k = d_i d_j / (d_i^2 + d_j^2)^{1/2}$, where $i, j, k$ are different. Here $d_k$ is half the length of the altitude belonging to the hypotenuse of the right triangle bounded by two sides and a diagonal of a face perpendicular to $e_k$. Let, e.g., $d_3 = d_1 d_2 / (d_1^2 + d_2^2)^{1/2}$. Let us take an axis of rotation passing through $0$ and parallel to one of the diagonals of a horizontal face, say, to $P_1P_2$, with axis vector $0(P_2 - P_1)$. Let us consider a rotation about this axis through the angle $\pm \pi/2$.

For convenience, rather than taking one tetrahedron as fixed, the other one as moving, we argue as at the motion of the intermediate kind. We rotate both tetrahedra about this axis, through an angle $\pm \pi/4$, in a way symmetric w.r.t. the $xy$-plane. Then we translate vertically both of these tetrahedra, in a way symmetric w.r.t. the $xy$-plane, through a suitable distance. Thus we achieve that all the pairs of the edges of the tetrahedra, which originally lay on the same face of the parallelepiped, will be, simultaneously, coplanar. Thus we obtain a position corresponding to a motion of the intermediate kind. This is already known to exist, even for all $\varphi \neq \pi$, so, in particular, for $\varphi = \pm \pi/2$. See Fig. 5.
where the rotated and not yet translated tetrahedra are denoted by $P'_1P'_2P'_3P'_4$ and $Q'_1Q'_2Q'_3Q'_4$. Here $P'_i$ and $Q'_i$ are the rotated copies of $P_i$ and $Q_i$, resp. We assume that the sense of rotation is as drawn in Fig. 5 (the other case is similar, only the role of the indices is changed).

The figure shows the orthogonal projection of the rectangular parallelepiped along the axis of rotation. By $d_3 = d_1 d_2 / (d_1^2 + d_2^2)^{1/2}$ this projection is a rectangle, whose horizontal side is twice as long as its vertical side. Note that this implies that the edges $P'_1P'_3$, $P'_2P'_3$ and their mirror images $Q'_2Q'_4$, $Q'_1Q'_4$ w.r.t. the $xy$-plane are horizontal. Hence $P'_1P'_3$, $Q'_2Q'_4$, as well as $P'_2P'_3$, $Q'_1Q'_4$, are parallel, resp. Therefore, their any translated copies remain coplanar, resp. Further the edges $P'_1P'_4$, $P'_2P'_4$, and their mirror images $Q'_2Q'_3$, $Q'_1Q'_3$ w.r.t. the $xy$-plane lie in a vertical plane (whose projection is a vertical line in the figure). Hence any translation in this vertical plane will leave them coplanar. Last, the edges $P'_1P'_2$ (whose projection in the figure is a point) and $Q'_3Q'_4$ will intersect, thus will be coplanar, after some vertical translations of the two tetrahedra, in a way symmetric w.r.t. the $xy$-plane. Then, by symmetry, also $Q'_2Q'_4$ and $P'_4P'_3$ will intersect. However, then any subsequent translation of the moving tetrahedron in the direction of $P'_1P'_2$ (= the direction of $Q'_3Q'_4$ = the direction of the rotation axis) leaves both of these last two pairs of edges coplanar.

Summing up: each pair of originally intersecting edges remains coplanar if we make the following operations. First we make two rotations — symmetric w.r.t. the $xy$-plane — about the above described axis of rotation, through angles $\pm \pi/4$. Second we make two vertical translations — symmetric w.r.t. the $xy$-plane — making $P'_1P'_2$ and $Q'_3Q'_4$ intersecting. Third we translate the moving tetrahedron in the direction of the rotation axis, through an arbitrary distance. Hence, this is an affine 1-manifold of solutions for $b$. 

Figure 5. The motion of the sixth kind.
(An affine manifold is a translate of a linear subspace, or the empty set.) We call this, for any choice of the permutation $(ijk)$ (for which $d_k = d_i d_j/(d_i^2 + d_j^2)^{1/2}$ holds) a motion of the sixth kind. (Observe that then $d_k < d_i, d_j$, hence if this condition holds, then it can hold only for one $k$.)

Last, we turn to the motion of the third kind, which we are able to give in analytic form only. Let $u = [u_1 \ u_2 \ u_3]^T$ and $u_1^2 + u_2^2 + u_3^2 = 1$. Let $D_i = d_i^{-2}$ and let $\Phi(x) = A x + b$ be a motion, where $A$ is a rotation about the axis $0 u$ through an angle $\varphi$. Note that $(u, \varphi)$ and $(-u, -\varphi)$ represent the same rotation. Suppose $0 < \varphi < 2\pi$, with $\varphi \neq \pi/2, 3\pi/2$, and let $s := \cot(\varphi/2) (\neq \pm 1)$. Then, for $i = 1, 2, 3$, let

$$u_i^2 := \frac{(s^4 + 3s^2 + 1) - (3s^4 + 7s^2 + 1)D_i/(D_1 + D_2 + D_3)}{2(s^2 + 1)},$$

provided

$$\frac{s^4 + 3s^2 + 1}{3s^4 + 7s^2 + 1} > \frac{\max D_i}{D_1 + D_2 + D_3}$$

(then for each $i$ we have $u_i^2 > 0$). This determines the rotation part $A$ of the motion $\Phi(x)$, and the translation part $b$ is then uniquely determined among all admitted motions. For given signs of the $u_i$’s, this is a 1-manifold of solutions. (But note that actually $\pm(u, \varphi)$ give the same rotation.)

We exclude the case $D_1 = D_2 = D_3$, since it has been settled in [10] and [12]. First we restrict our attention to the case $u_1, u_2, u_3 > 0$. Then, for max $D_i/(D_1 + D_2 + D_3) \geq 5/11$, this solution manifold is connected. On the other hand, for max $D_i/(D_1 + D_2 + D_3) < 5/11$, this solution manifold consists of three connected components, one for $s < -1$ (with $s$ bounded), one for $-1 < s < 1$, and one for $s > 1$ (with $s$ bounded).

Suppose $D_i \geq D_j, D_k$ (thus earlier max $D_i$ will be replaced by $D_i$). Here and later till Theorem 1, $(ijk)$ is a permutation of $\{1, 2, 3\}$. For $D_i/(D_1 + D_2 + D_3) \geq 5/11$ this solution manifold ends (at the infimum or supremum of $s$) at two points satisfying $u_i = 0$ and $|s| \in (0, 1]$ (with $|s| = 1$ exactly for $D_i/(D_1 + D_2 + D_3) = 5/11$). For $D_i/(D_1 + D_2 + D_3) < 5/11$ the component for $s < -1$ begins (at the infimum of $s$) and the component for $s > 1$ ends (at the supremum of $s$) at two points satisfying $u_i = 0$ and $|s| > 1$. Moreover, there are no endpoints at $s = \pm 1$, the manifold components going to infinity there. For any of the above two cases, these two endpoints are not considered as lying on this solution manifold. They correspond to two motions of the intermediate kind.

If $D_i > D_j, D_k$, then the above mentioned two endpoints (for any of the above two cases) do not correspond to motions of the first kind. If $D_i = D_j > D_k$, then the above mentioned two endpoints (for any of the above two cases) correspond to motions of the first kind.

This motion of the third kind never passes through the basic position, unless $D_1 = D_2 = D_3$, when it does (for $s^2 = \infty$).

These formulas in fact describe a motion of our tetrahedra. This fact, and the mentioned properties of this motion will follow from the proof of Theorem 1.

For min $u_i^2 > 0$ this 1-manifold of motions of the third kind is smooth. However, unless $D_1 = D_2 = D_3$, this 1-manifold, parametrized by $\varphi$, or by $s$, is in general only
topologically a manifold with boundary (i.e., its above two endpoints). Namely, at its endpoints it is in general not differentiable. For $D_i \geq D_j, D_k$, and for $s$ yielding $u_i = 0$, we have $du_i/ds = (du_i^2/ds)/(2u_i) = \pm \infty$, provided that $du_i^2/ds \neq 0$. Here $du_i^2/ds$ exists, and is finite. E.g., for $D_i = D_j = 5/11$ and $D_3 = 1/11$, an easy calculation shows that for $s$ yielding $u_i = 0$ we have $du_i^2/ds \neq 0$. Therefore in general, for this $s$, we have $du_i^2/ds \neq 0$. Hence this 1-manifold with boundary is, in general, not differentiable at its endpoints.

Even for $D_i = D_j > D_k$, in general, for $s$ yielding $u_i = u_j = 0$, we have $du_i^2/ds = du_j^2/ds \neq 0$. This is shown by the same example. Similarly, for $D_i \geq D_j, D_k$, we can specialize to $D_i/(D_1 + D_2 + D_3) = 5/11$, or to $D_i/(D_1 + D_2 + D_3) = D_j/(D_1 + D_2 + D_3) = 5/11$. Then we see that also in these two special cases, in general, for $s$ yielding $u_i = 0$, we have $du_i^2/ds \neq 0$. So, even in each of these three special cases, we have in general non-differentiability of this 1-manifold with boundary, at its endpoints.

Now we allow any signs of the $u_i$’s. The number of 1-manifolds of motions of the third kind is $2^3/2 = 4$. Namely the number of sign combinations of the $u_i$’s, when they are non-zero, is $2^3$, but $\pm(u, \varphi)$ give the same rotation. Then for $D_i > D_j, D_k$ these four 1-manifolds by twos have both endpoints in common. However, for different twos these both endpoints are disjoint pairs. These endpoints correspond to motions of the intermediate kind (but not of the first kind). For $D_i = D_j > D_k$ all four of these 1-manifolds have both endpoints in common. These endpoints correspond to motions of the first kind.

At both endpoints of this motion (at the infimum or supremum of $s$) we have bifurcations. First suppose $D_i > D_j, D_k$. Then at an endpoint the solution set locally consists of two 1-manifolds with boundary (this endpoint), of motions of the third kind. Moreover, of one 2-manifold, of motions of the intermediate kind. Second suppose $D_i = D_j > D_k$. Then at an endpoint the solution set locally consists of all four 1-manifolds with boundary (this endpoint), of the third kind. Moreover, of one 1-manifold of the motions of the first kind, and of two 2-manifolds of the motions of the intermediate kind.

Observe that the motions of the fifth and sixth kinds exist only for $\varphi = \pi$, i.e., $s = 0$, and for $\varphi = \pi/2, 3\pi/2$, i.e., $s = \pm 1$, resp. However, $s = 0$ does not yield an endpoint of the 1-manifold of motions of the third kind. Hence the motion of the fifth kind cannot occur locally at the endpoints of the 1-manifold of motions of the third kind. Further, $s = \pm 1$ yield endpoints of the 1-manifold of motions of the third kind if and only if $\max D_i/(D_1 + D_2 + D_3) = 5/11$. The motion of the sixth kind exists only for $D_i = D_j + D_k$ (where $(ijk)$ is a permutation of $\{1, 2, 3\}$; cf. (7) of the proof of Theorem 1). If an endpoint of the 1-manifold of motions of the third kind, and a motion of the sixth kind coincided, then we had $1/2 = D_i/(D_i + D_j + D_k) = 5/11$, a contradiction. Hence the motion of the sixth kind cannot occur locally at the endpoints of the 1-manifold of the motions of the third kind.

Also at each counterbasic position, i.e., a motion of the first kind with rotation angle $\varphi = \pi$ (i.e., $s = 0$), we have bifurcations. We may suppose $u = (0, 0, 1)$. At a counterbasic position there locally occur one 1-manifold of the motions of the first kind, two 2-manifolds of the motions of the intermediate kind, and one 2-manifold of the motions of the fifth kind. However, manifolds of the motions of the third kind cannot occur at a counterbasic position. Namely, for $s = 0$ the formula for $u_i^2$ (at the introduction of the motion of the
third kind) simplifies to \( u_i^2 = (1 - D_i/(D_1 + D_2 + D_3))/2 \), for \( i = 1, 2, 3 \). Then \( 0 = u_i^2 = (1 - D_1/(D_1 + D_2 + D_3))/2 > 0 \), a contradiction. Also manifolds of the motions of the sixth kind cannot occur at a counterbasic position, because for them we have \( \varphi = \pm \pi/2 \).

We recall that we mean by a motion an isometry (congruence) of the space, of determinant +1.

**Theorem 1.** Consider the two tetrahedra \( P_1P_2P_3P_4 \) and \( Q_1Q_2Q_3Q_4 \), derived above from the rectangular parallelepiped of vertices \((\pm d_1, \pm d_2, \pm d_3)\). The only finite motions admitted by these tetrahedra — i.e., all positions of the moving tetrahedron, satisfying (A) — are the following: The motions of the first, intermediate, third, fifth kinds and, provided \( \Phi \) to be considered.

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

... is the basic position of the \( Q_i \)'s from the beginning of 2.1,

\[
\begin{align*}
Q_1(x_1, y_1, z_1) &= A[-d_1 \quad d_2 \quad d_3]^T + b, \\
Q_2(x_2, y_2, z_2) &= A[d_1 \quad -d_2 \quad d_3]^T + b, \\
Q_3(x_3, y_3, z_3) &= A[d_1 \quad d_2 \quad -d_3]^T + b, \\
Q_4(x_4, y_4, z_4) &= A[-d_1 \quad -d_2 \quad -d_3]^T + b.
\end{align*}
\]

Let us denote

\[
D = \begin{bmatrix} d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3 \end{bmatrix}.
\]

Using the notations \( P_i^0, Q_i^0 \) from §1 (e.g., Fig. 3), and \( P_i \) from the beginning of 2.1, we have \( P_i = DP_i^0 \) and \( P_2 = DP_2^0 \), and of the moving vertices \( Q_3 = \Phi(Q_3) = \Phi(D(Q_3^0)) \) and \( Q_4 = \Phi(Q_4) = \Phi(D(Q_4^0)) \), is equivalent to the following. The points \( P_1^0, P_2^0 \), and \( D^{-1}(\Phi(D(Q_3^0))) = D^{-1}AD(Q_3^0) + D^{-1}b \) and \( D^{-1}(\Phi(D(Q_4^0))) = D^{-1}AD(Q_4^0) + D^{-1}b \) are coplanar. We have analogous equivalent conditions for the coplanarity of the other quadruples of vertices to be considered.

So \( \Phi(x) = Ax + b \) represents a motion of our tetrahedra if and only if the following holds. The transformation \( D^{-1}AD(x) + D^{-1}b \) of the vertices \( Q_1^0, \ldots, Q_4^0 \) preserves coplanarity of the four vertices of any face of the cube with vertices \((\pm 1, \pm 1, \pm 1)\). Simultaneously, two of these vertices, namely the vertices \( P_i^0 \), are fixed, and only two
of them, namely the vertices \( Q_i^0 \), are transformed by this transformation. We write 
\[
[u_{ij}^0] := D^{-1}AD = [d_i^{-1}a_{ij}d_j],
\]
and [\( b_1^0 \)  \( b_2^0 \)  \( b_3^0 \)] := \( D^{-1}b = [d_1^{-1}b_1 \  d_2^{-1}b_2 \  d_3^{-1}b_3]^T \). Then, like in [12], p. 435, by these coplanarities we have
\[
-(a_{22}^0 + a_{33}^0)b_1^0 + a_{12}^0b_2^0 + a_{13}^0b_3^0 = a_{21}^0a_{13} + a_{31}^0a_{12} + (a_{23}^0 + a_{32}^0)(1 - a_{11}^0),
\]
\text{(I/1)}
\[
-(a_{23}^0 + a_{32}^0)b_1^0 + a_{13}^0b_2^0 + a_{12}^0b_3^0 = a_{12}^0a_{21} + a_{31}^0a_{13} + (a_{22}^0 + a_{33}^0)(1 - a_{11}^0).
\]
\text{(II/1)}

There hold the analogous equations obtained from these ones by the permutation of the indices \( 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1 \). These equations will be denoted by \( (I/2) \) and \( (II/2) \), resp. Similarly, using the permutation \( 1 \leftrightarrow 3 \leftrightarrow 2 \leftrightarrow 1 \) we get equations \( (I/3) \) and \( (II/3) \). Thus we obtain a system of six linear equations for \( b_1, b_2 \) and \( b_3 \). This system of equations expresses the coplanarity of the respective six quadruples from the fixed points \( P_i \) and the moved points \( Q_i \), cf. [12], p. 435.

(2) In these six equations \( (I/i) \), \( (II/i) \), where \( i = 1, 2, 3 \), we replace \( a_{ij}^0 \) by \( d_i^{-1}a_{ij}d_j \), and \( b_i^0 \) by \( d_i^{-1}b_i \). Then we express \( a_{ij} \) by \( u_1, u_2, u_3 \) and \( \varphi \), by the well-known formula
\[
Ax = \langle x, u \rangle u + \cos \varphi \cdot (x - \langle x, u \rangle u) + \sin \varphi \cdot (x \times u)
\]
(cf., e.g., [12], p. 436).

Then let us consider the \( 6 \times 4 \) matrix \( (B') \) formed by the coefficients of \( b_1, b_2, b_3 \), and the right-hand sides of these six equations, rewritten as indicated above. (Its entries depend on \( u, \varphi \), and the constants \( d_i \).)

We call its rows \( I/i \) and \( II/i \), according to the equation they correspond to. We will not write out \( (B') \) explicitly, but rather we will make some simplifications in it. In matrix \( (B') \) we multiply rows \( I/i \), \( II/i \) by \( d_i \), then divide the fourth column by \( 2d_1d_2d_3 \), and then divide row \( II/i \) by \( d_1d_2d_3d_i^{-1} \). Like in 2.1, before Theorem 1, denote \( D_i = d_i^{-2} \). Note that for \( \varphi = 0 \), thus \( A = I \), we have the unique motion with \( b = 0 \). Henceforward we will assume \( 0 < \varphi < 2\pi \). Letting \( s := \cot(\varphi/2) \), as in 2.1, before Theorem 1, we still multiply each entry of the matrix obtained last time from \( (B') \) by \( (s^2 + 1)/2 \). Thus we obtain the matrix
\[
\begin{bmatrix}
  u_1^2 - s^2 & u_1u_2 + su_3 & u_3u_1 - su_2 & u_2u_3(D_2 + D_3) \\
  u_1u_2 - su_3 & u_2^2 - s^2 & u_2u_3 + su_1 & u_3u_1(D_3 + D_1) \\
  u_3u_1 + su_2 & u_2u_3 - su_1 & u_3^2 - s^2 & u_1u_2(D_1 + D_2) \\
  -u_2u_3(D_2 + D_3) + & (u_3u_1 - su_2)D_2 & (u_1u_2 + su_3)D_3 & 0 \\
  +su_1(D_3 - D_2) & -u_3u_1(D_3 + D_1) + & (u_1u_2 - su_3)D_3 & 0 \\
  (u_2u_3 + su_1)D_1 & (u_3u_1 + su_2)D_2 & -u_1u_2(D_1 + D_2) + & 0 \\
  (u_2u_3 - su_1)D_1 & (u_3u_1 + su_2)D_2 & +su_3(D_2 - D_1) & 0
\end{bmatrix}.
\]

(Note that for \( D_1 = D_2 = D_3 = 1 \) this reduces to \( (B) \) in [12], p. 437, up to a factor 1/2 in the fourth column.) The rows of matrix \( (B) \) corresponding to equations \( (I/i) \), \( (II/i) \) will be called rows \( I/i, II/i \) of \( (B) \). Later, unless stated otherwise, we will consider only rows \( I/i, II/i \) of matrix \( (B) \), and not of the original matrix \( (B') \). Now suppose that \( u \) and \( s \) are
fixed. Then the solvability of the system of equations corresponding to this new matrix (B), for \( b \), is equivalent to the solvability of the system of equations corresponding to the original matrix (B’). (Thus of our original system of equations (I/i), (II/i), for \( i = 1, 2, 3 \), for \( b \). More exactly, \( u, s, b \) is a solution of the system of equations corresponding to (B’), if and only if \( u, s, b/(2d_1d_2d_3) \) is a solution of the system of equations corresponding to (B). Hence, for \( u \) and \( s \) fixed, the dimensions of the solution manifolds, for \( b \), of the two systems of equations (if they are not empty) are the same. Moreover, they can be obtained from each other by multiplication with a non-zero constant. Later we will not be interested in formulas for \( b \), therefore, unless stated otherwise, we will use the system of equations corresponding to (B).

(3) The upper left 3 \times 3 submatrix of (B) is independent of \( D_i \), hence it is singular in the same case when it is singular for \( D_1 = D_2 = D_3 = 1 \). Observe that the left-hand side of equation (I/1) equals \( d_1^{-1}[-(a_{22} + a_{33})b_1 + a_{12}b_2 + a_{13}b_3] \), and similarly for (I/2), (I/3). Hence the determinant of the considered 3 \times 3 submatrix of (B) is a non-zero number times the determinant \( |a_{ij} - \delta_{ij}(a_{11} + a_{22} + a_{33})| \). By [11], p. 270 or [12], p. 438, this determinant is 0 if and only if \( \varphi = \pm \pi/2 \) or \( \varphi = \pi \). (We do not distinguish between angles differing by multiples of \( 2\pi \).) Hence our equations (for matrix (B)) can have a non-unique solution for \( b \) only for \( \varphi = \pm \pi/2 \) and \( \varphi = \pi \), i.e., for \( s = \pm 1 \) and \( s = 0 \).

The last three rows of matrix (B) are linearly dependent. Namely, multiplying row II/1 by \( D_1 \) and then summing them, we obtain the zero row.

Multiplying row I/1 of (B) by \( u_i \), and then summing them, we obtain

\[
[u_1(1 - s^2) \quad u_2(1 - s^2) \quad u_3(1 - s^2) \quad u_1u_2u_32(D_1 + D_2 + D_3)].
\]

The corresponding equation implies that for \( s^2 = 1 \), i.e., for \( \varphi = \pm \pi/2 \), for any solution of our equations (for matrix (B)) we have \( u_1u_2u_3 = 0 \).

(4) The determinant of the submatrix of (B) formed by its rows I/1, I/2, I/3, II/1 is a homogeneous eighth degree polynomial of \( u_1, u_2, u_3 \) and \( s \), with coefficients polynomials of the \( D_i \)'s. A straightforward but somewhat lengthy calculation gives that it equals

\[
su_1u_2u_3(D_1 + D_2 + D_3)\{(u_1^2 + u_2^2 + u_3^2) \times [u_1^2(D_3 - D_2) + u_2^2(-D_2 - D_3) + u_3^2(D_2 + D_3)] + s^2[u_1^2(3D_3 - 3D_2) + u_2^2(-3D_2 + 3D_3) + u_3^2(-D_2 + 3D_3)] + s^4(D_3 - D_2)\}, \tag{C}
\]

which equals 0. Now suppose \( su_1u_2u_3 \neq 0 \). Then the factor of (C) in braces is 0. Moreover, two more analogous expressions are equal to 0, which are obtained from this expression by cyclic permutations of the indices. (These arise analogously from the determinants of the submatrices formed by rows I/1, I/2, I/3, II/2, and I/1, I/2, I/3, II/3 of (B), resp.) These three equations are homogeneous linear in \( D_2, D_3 \), in \( D_3, D_1 \), and in \( D_1, D_2 \), resp.,
and can be written as
\[
\begin{align*}
(u_1^2 + u_2^2 + u_3^2)(-u_1^2 + u_2^2 + u_3^2) + s^2(u_1^2 + 3u_2^2 + 3u_3^2) + s^4 &= D_1 \\
= (u_1^2 + u_2^2 + u_3^2)(u_1^2 - u_2^2 + u_3^2) + s^2(3u_1^2 + u_2^2 + 3u_3^2) + s^4 &= D_2 \\
= (u_1^2 + u_2^2 + u_3^2)(u_1^2 + u_2^2 - u_3^2) + s^2(3u_1^2 + 3u_2^2 + u_3^2) + s^4 &= D_3
\end{align*}
\]
(5) First we discuss the case when the first factor of (C), i.e., \(s\), equals 0. Consider the \(4 \times 4\) matrix formed by rows I/1, I/2, II/1, and the sum of \((D_1 + D_2)\) times row II/2 and \(D_3\) times row II/3 of our matrix (B). Its determinant is
\[
-u_1 u_2 u_3 D_1 D_3 (D_1 + D_2 + D_3) [u_2^2 (D_2 + D_3) - u_1^2 (D_3 + D_1)] (u_1^2 + u_2^2 + u_3^2),
\]
which equals 0. By cyclic permutation of rows I/i and II/i we get two more similar equations. Namely the expressions, obtained from the last expression by cyclic permutations of the indices, are equal to 0.

These three equations together imply \(u_1 u_2 u_3 = 0\), or \(u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)\).

First let us suppose, e.g., \(u_1 = 0\). Then the equation corresponding to row I/1 becomes 0 = \(u_2 u_3 (D_2 + D_3)\), thus \(u_2 u_3 = 0\). Let, e.g., \(u_1 = u_2 = 0\) (thus \(u = \pm e_3\) by \(u_1^2 + u_2^2 + u_3^2 = 1\)). By \(s = 0\) we have \(\varphi = \pi\). Then the pairs of the edges of the two tetrahedra, originally lying on some vertical face of the rectangular parallelepiped, are parallel. Thus they remain coplanar after any translation of the moving tetrahedron.

The pairs of the edges of the two tetrahedra, originally lying on some horizontal face of the rectangular parallelepiped, are properly intersecting and horizontal. Thus they remain coplanar after a translation of the moving tetrahedron, through a vector \([b_1 \ b_2 \ b_3]^T\), exactly when \(b_3 = 0\). Therefore the set of solution vectors \([b_1 \ b_2 \ b_3]^T\) of our equations (for matrix (B)) is given by all vectors with \(b_3 = 0\). This gives the motion of the fifth kind.

Second let us suppose \(u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)\). Then by \(u_1^2 + u_2^2 + u_3^2 = 1\) we have \(u_1 u_2 u_3 \neq 0\). Observe that in this case the double equality (D) is satisfied (for \(s = 0\)). We will investigate this case further together with the investigation of (D), in (10) (where both cases \(s = 0\) and \(s \neq 0\) will be allowed). Now we only show that in this case there is a unique solution \(b\) of our equations (for matrix (B)). (The existence of the solution for \(b\) will be showed in (10).) For this consider the \(4 \times 4\) matrix from the beginning of (5). Then take its \(3 \times 3\) submatrix consisting of the first three entries of its first, third and fourth rows. Its determinant is a homogeneous polynomial of third degree of the \(D_i\)’s, with coefficients homogeneous sixth degree polynomials of the \(u_i\)’s. To test if it is zero or not, it suffices to substitute \(D_2 + D_3 := u_1^2\), and \(D_3 + D_1 := u_2^2\), and \(D_1 + D_2 := u_3^2\). Thus this determinant becomes
\[
u_1^2 u_2 u_3 (u_1^2 + u_2^2 + u_3^2) (u_1^2 + u_2^2 - u_3^2) (-u_1^2 + u_2^2 + u_3^2) / 8.
\]
Here $u_2^2 u_3^2 u_1 \neq 0$, as shown above, and each other factor is positive. E.g., $(u_1^2 + u_2^3 - u_3^3)/(u_2^2 + u_2^2 + u_2^3) = D_3/(D_1 + D_2 + D_3) > 0$. Hence the considered determinant is non-zero, showing unicity of the solution for $b$ (for matrix (B)).

(6) Second we discuss the case when the factor $u_1 u_2 u_3$ of (C) equals 0. Let, e.g., $u_1 = 0$. In this case we have the motions of the first, intermediate, fifth and sixth kinds. We have to show for $u_2 u_3 \neq 0$, that if the motion of the intermediate kind does not exist — that is $\varphi = \pi$, i.e., $s = 0$ — then we do not have any solution. (Recall the following. The motion of the intermediate kind exists only for $\varphi \neq \pi$. The motion of the fifth kind — for $u_1 = 0$ — exists only for $u_2 u_3 = 0$. The motion of the sixth kind exists only for $\varphi = \pm \pi/2$.) However, in (5) it has been shown that $s = 0$ and $u_1 = 0$ imply $u_2 u_3 = 0$.

There remains the question of the unicity of the translation part $b$ of the motion. However, in (3) it has been shown that our equations (for matrix (B)) can have a non-unique solution for $b$ only for $s = 0$ and $s = \pm 1$. The case $s = 0$ has been dealt with in (5), and the question of unicity has been completely settled there. We will deal with $s = \pm 1$ in (7).

(7) We turn to discuss unicity of $b$ for $s = \pm 1$, i.e., $\varphi = \pm \pi/2$. Replacing $u$ by $-u$ if necessary, we may assume $s = 1$. In (3) it has been shown that $s^2 - 1$ implies $u_1 u_2 u_3 = 0$. Let, e.g., $u_3 = 0$. Then our matrix (B) becomes a function only of $u_1$, $u_2$ and $D_1$, $D_2$, $D_3$. Loosing homogeneity, we will use $u_1^2 + u_2^2 = 1$. Thus we see that rows I/1 and I/2 are proportional, they are $-u_2, s = 0$, and $u_1$ times $[u_2 \quad u_1 \quad 0]$, resp. Since $u_1$, $u_2$ are not both zero, we may replace rows I/1 and I/2 by one row $[u_2 \quad u_1 \quad 0]$. Row I/3 is retained. As mentioned in (3), rows II/1, II/2 and II/3 are linearly dependent, with non-zero coefficients. Hence we may omit from among them row II/3. Thus we obtain a $4 \times 4$ matrix $M$. The question of the dimension of the solution manifold, for $b$, of the equations corresponding to $M$, at the considered rotation part of the motion, is equivalent to the same question regarding matrix (B). By $u_3 = 0$ and $\varphi = \pm \pi/2 \neq \pi$ one solution always exists, namely a motion of the intermediate kind. Thus the dimension of this solution manifold is $3 - r$, where $r$ is the rank of the matrix

$$
N = \begin{pmatrix}
    u_2 & -u_1 & 1 \\
    u_2 & -u_1 & -1 \\
    u_1(D_3 - D_2) & -u_2 D_2 & u_1 u_2 D_3 \\
    u_1 D_1 & u_2(D_1 - D_3) & u_1 u_2 D_3
\end{pmatrix},
$$

obtained by omitting the last column from matrix $M$.

Subtracting the second row of $N$ from the first one, the first row becomes $[0 \quad 0 \quad 2]$. Hence $r = 1 + r'$, where $r'$ is the rank of the $3 \times 2$ submatrix $N'$ of $N$ at the lower left corner. If $u_1$ or $u_2$ is 0, then we have $r' = 2$, thus $r = 3$. Then we have a unique solution for $b$ (for matrix $M$). Now let $u_1 u_2 \neq 0$. The determinants of the $2 \times 2$ submatrices of $N'$, obtained by omitting its first, second or third row, resp., are the following: $u_1 u_2 D_3(D_1 + D_2 - D_3)$, and $u_2^2(D_1 - D_3) + u_2^2 D_1$, and $-u_2^2 D_2 + u_1^2(D_3 - D_2)$. If any of these expressions is not 0, then we have $r' = 2$, thus $r = 3$. Then again there is a unique solution of our equations for $b$ (for matrix $M$). If all these above expressions are equal to 0, then we have (equivalently)
$D_3 = D_1 + D_2$ and $u_1^2D_1 = u_2^2D_2$. Hence, by $u_3 = 0$, we have

$$u_1^2 = D_2/(D_1 + D_2) = d_1^2/(d_1^2 + d_2^2)$$

$$u_2^2 = D_1/(D_1 + D_2) = d_2^2/(d_1^2 + d_2^2).$$

In this case $r' = 1$, thus $r = 2$, and then the dimension of the affine manifold of solutions, for $b$ (for matrix $M$, or matrix $(B)$), is $1$ (for this rotation part of the motion).

It remains to show that geometrically this is the motion of the sixth kind. Since $D_i = d_i^{-2}$, therefore $D_3 = D_1 + D_2$ means $d_3 = d_1d_2/(d_1^2 + d_2^2)^{1/2}$. We have from above $[u_1 \ u_2 \ u_3]^T = [\pm d_1/(d_1^2 + d_2^2)^{1/2} \ \pm d_2/(d_1^2 + d_2^2)^{1/2} \ 0]^T$ (the $\pm$ signs being independent). Hence the axis of rotation of the rotation part $A$ of the motion is parallel to a diagonal of a horizontal face of our rectangular parallelepiped in its basic position. Further, the angle of rotation is $\pm \pi/2$. This is just the rotation part of the motion of the sixth kind (for $k = 3$, cf. the introduction of the motion of the sixth kind, in 2.1, before Theorem 1). At describing the motion of the sixth kind, we have exhibited an affine 1-manifold $A_1$ of solutions for $b$, with the above $A$ (for matrix $(B')$). This is therefore a subset of the entire solution manifold $A_2$, for $b$, with this $A$ (for matrix $(B')$). Now we have shown that the entire affine manifold of solutions $b$, with this $A$ (for matrix $(B)$), is exactly 1-dimensional. Recall that the solution manifolds, for $b$, for matrices $(B')$ and $(B)$, are obtained from each other by multiplication with a non-zero constant, cf. (2). Hence also $A_2$ is an affine 1-manifold, containing the affine 1-manifold $A_1$. Hence $A_2 = A_1$, and this is the manifold of the motions of the sixth kind.

(8) Recall that a non-unique solution for $b$ (for matrix $(B)$) is possible only for $s = 0$, $\pm 1$ (cf. (3)). These non-unique solutions have been discussed in (5) and (7), resp.

For the existence of solutions we have derived in (4) the equation that $(C)$ equals $0$, and some of its consequences. The case when the first factor of $(C)$, i.e., $s$, equals $0$, has been settled in (5), except the case when $u_1u_2u_3 \neq 0$ and (D) is satisfied. The case when the factor $u_1u_2u_3$ of $(C)$ equals $0$, has been settled in (6), except the case of unicity at $s = \pm 1$. This in turn has been settled in (7). If $su_1u_2u_3 \neq 0$, then we have derived in (4) equations (D).

Therefore all that remains is the following. We have to solve equations (D), where $s$ can be $0$ or any non-zero number, and $u_1u_2u_3 \neq 0$. Moreover, we have to verify if they are solutions of our equations (for matrix $(B)$). Recall that by (3) $s^2 = 1$ implies $u_1u_2u_3 = 0$, hence we will suppose $s \neq \pm 1$.

(9) Now we show that for $u_1u_2u_3 \neq 0$ and $s \neq \pm 1$ any solution of equations (D) is a solution of our equations for matrix $(B)$. This is necessary since equations (D) are only consequences of the equations for matrix $(B)$, They have not been gained from these equations by equivalent transformations. Moreover, we do not have a geometrical description of the motion of the third kind, making its existence evident.

First we show that, for $u_1u_2u_3 \neq 0$ and $s \neq 0, \pm 1$, any solution of equations (D) is a solution of our equations for matrix $(B)$. In fact, equations (D) have been derived for $su_1u_2u_3 \neq 0$ from the equation that expression $(C)$ equals $0$, and from two other analogous equations. These express linear dependence of rows $I/1, I/2, I/3$ and $II/i$ $(i = 1, 2, 3)$ of matrix $(B)$, resp. For $s \neq 0, \pm 1$ the determinant of the matrix formed by the first three
entries of rows I/1, I/2 and I/3 is not 0, cf. (3). Hence (D) expresses linear dependence of rows II/1, II/2 and II/3 on the linearly independent rows I/1, I/2 and I/3. (Observe that already their first three entries form linearly independent row vectors.) Hence (D) implies that the rank of (B) is at most 3, thus that the four column vectors of (B) are linearly dependent. However, at this linear dependence the fourth column vector must have a non-zero coefficient. Namely the first three column vectors are linearly independent. (Their first three entries already form linearly independent column vectors.) This just means that our equations have a solution for \( \mathbf{b} \) (for matrix (B)).

Now let \( u_1u_2u_3 \neq 0 \) and \( s = 0 \). (Recall that \( s = \pm 1 \) has been excluded, cf. (8).) We show that also now any solution of equations (D) is a solution of our equations for matrix (B). For \( s = 0 \) (D) gives \((-u_1^2 + u_2^2 + u_3^2)/D_1 = (u_1^2 - u_2^2 + u_3^2)/D_2 = (u_1^2 + u_2^2 - u_3^2)/D_3\). Let their common value be \( \lambda \), say. Then \( u_i^2 = (\lambda D_j + \lambda D_k)/2, \) for \( i, j, k \) different (hence \( \lambda \neq 0 \) by \( u_1^2 + u_2^2 + u_3^2 = 1 \)), thus \( u_i^2 : u_2^2 = u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2) \). Then rows I/1, I/2 and I/3 of matrix (B) are all proportional to \([u_1 \quad u_2 \quad u_3 \quad 2u_1u_2u_3/\lambda]\). Further, rows II/1, II/2 and II/3 are linearly dependent by (3). Hence (D) implies that the rank of (B) is at most 3. Thus after some row manipulations some \( 3 \times 3 \) submatrix, contained in the first three columns, has a non-zero determinant by the last paragraph of (5) (for \( s = 0 \)). Therefore we have, like at the case \( s \neq 0, \pm 1 \), that also in this case our equations have a solution for \( \mathbf{b} \) (for matrix (B)). Cf. the italicized text in the previous paragraph.

(10) By the last paragraph of (8), it remained to solve equations (D) for \( u_1u_2u_3 \neq 0 \), where \( s \) can be any real number different from \( \pm 1 \).

Using \( u_1^2 + u_2^2 + u_3^2 = 1 \), equations (D) become

\[
\frac{(1 - 2u_i^2) + s^2(3 - 2u_i^2) + s^4}{D_i} = \lambda, \text{ where } \lambda \text{ is independent of } i \ (i = 1, 2, 3). \quad (E)
\]

Solving this for \( u_i^2 \), we obtain

\[
u_i^2 = \frac{1}{2} \left( s^2 + 2 - \frac{\lambda D_i + 1}{s^2 + 1} \right). \quad (F)
\]

Summing these for \( i = 1, 2, 3 \), we obtain

\[
1 = \frac{1}{2} \left( 3s^2 + 6 - \frac{\lambda(D_1 + D_2 + D_3) + 3}{s^2 + 1} \right),
\]

from which we express \( \lambda \) and put it into (F). Thus we obtain

\[
u_i^2 = \frac{(s^4 + 3s^2 + 1) - (3s^4 + 7s^2 + 1)D_i/(D_1 + D_2 + D_3)}{2(s^2 + 1)}, \quad (G)
\]

provided of course that all these expressions are non-negative. Actually, by \( u_1u_2u_3 \neq 0 \), all these expressions have to be positive. It is easily seen that these expressions actually satisfy (E) and have sum 1, thus we have made equivalent transformations.
Using (G), the condition \( \min u_i^2 > 0 \) is equivalent to \( f(s^2) := (s^4 + 3s^2 + 1)/(3s^4 + 7s^2 + 1) > \max D_i/(D_1 + D_2 + D_3) \). Here \( s^2 \mapsto f(s^2) \) strictly decreases in \([0, \infty)\), with image \((1/3, 1]\). Hence, except the case \( D_1 = D_2 = D_3 \), when this inequality is satisfied for all \( s \in \mathbb{R} \), we have that this inequality is satisfied for \( s^2 < f^{-1}[\max D_i/(D_1 + D_2 + D_3)] < \infty \). (Thus in this case this solution set is far from the basic position, which is characterized by \( s^2 = \infty \).) Here \( f^{-1} \), defined on \((1/3, 1]\), and strictly decreasing there, with image \([0, \infty)\), is the inverse of \( f \), defined on \([0, \infty)\). We have \( f(1) = 5/11 \).

Exclude further the case \( D_1 = D_2 = D_3 \), which has been completely settled by [10] and [12]. We may restrict our attention to the case \( u_1, u_2, u_3 \geq 0 \). We write \( s_0 := [f^{-1}(\max D_i/(D_1 + D_2 + D_3))]^{1/2} \). By \( f(0) = 1 > \max D_i/(D_1 + D_2 + D_3) \), we have \( s_0 > 0 \). By positivity of (G), we have \( s \in (-s_0, s_0) \).

First suppose \( s_0 < 1 \) (i.e., \( \max D_i/(D_1 + D_2 + D_3) > 5/11 \)). We claim that \( s \) varies in \( I := (-s_0, s_0) \). At \( \pm s_0 \) the solution of our equations for \( b \) (for matrix (B)) is unique, cf. (3). For \( s \in (-s_0, s_0) \) it also is continuous in \( s \) and \( u \), hence in \( s \) (since now, by (G), \( u \) is continuous in \( s \)). Recall from (9) that for \( u_1u_2u_3 \neq 0 \) and \( s \neq \pm 1 \) any solution of equations (D) is a solution for our equations (for matrix (B)). However, as we have seen in (G), also using the equivalence of (G) and (E), each \( s \in (-s_0, s_0) \not\in \pm 1 \) can occur for a solution for our equations (for matrix (B)). Moreover, for \( s \in (-s_0, s_0) \) we have \( u_i^2 > 0 \). Summing up: in this case we have a connected manifold of solutions. Moreover, adding to it its two endpoints, at \( s = \pm s_0 \), we obtain a topological 1-manifold with boundary (these two points), contained in the solution set.

Second suppose \( s_0 > 1 \) (i.e., \( \max D_i/(D_1 + D_2 + D_3) < 5/11 \)). We claim that \( s \) varies in \( J := (-s_0, s_0) \setminus \{-1, 1\} \). From above, we have \( s \in (-s_0, s_0) \). Like in the case \( s_0 < 1 \), each \( s \in J \) actually can occur for a solution of our equations for \( b \) (for matrix (B)). However, for \( u_1u_2u_3 \neq 0 \), we have that \( s = \pm 1 \) cannot occur for a solution, cf. (3). We are going to show that \( u_1u_2u_3 \neq 0 \) continues to hold also for \( s = \pm 1 \). This will imply that for \( 0, \pm 1 \neq s \rightarrow \pm 1 \) the uniquely existing solutions \( b \) of our equations (for matrix (B)) tend to infinity (recall that non-uniqueness for \( b \) can occur only for \( s = 0, \pm 1 \), cf. (3)). (In the contrary case, a standard compactness argument would give that \( s = \pm 1 \) could occur for a solution.) We have \( f(s_0^2) = \max D_i/(D_1 + D_2 + D_3) \). Thus, applying (G) for \( s = \pm 1 \), we have

\[
\min u_i^2 = \frac{5/11 - \max D_i/(D_1 + D_2 + D_3)}{4/11} = \frac{f(1) - f(s_0^2)}{4/11} > 0.
\]

Then however, \( u_i^2 u_j^2 u_k^2 \geq (\min u_i^2)^3 > 0 \), as asserted. Summing up: in this case we have a solution manifold of three connected components: one for \( s \in (-s_0, -1) \), one for \( s \in (-1, 1) \), and one for \( s \in (1, s_0) \). At \( s = \pm 1 \) the manifold components go to infinity. Like for \( s_0 < 1 \), also now, adding to the solution manifold its two endpoints, at \( s = \pm s_0 \), we obtain a topological 1-manifold with boundary (these two points), contained in the solution set.

Third suppose \( s_0 = 1 \) (i.e., \( \max D_i/(D_1 + D_2 + D_3) = 5/11 \)). Suppose \( D_i \geq D_j, D_k \). Then for \( s = \pm 1 \) the only possibility of non-uniqueness of the solution for \( b \) for our equations (for matrix (B)) is when \( D_i = D_j + D_k \) (cf. (7)). Then however \( 1/2 = D_i/(D_i + D_j + D_k) = 5/11 \), a contradiction. Hence for \( s = \pm 1 \) the solution for \( b \) for our equations (for matrix (B)) is unique. However, for \( s = \pm 1 \) it also exists. Namely, then \( u_i = 0 \), and
$\varphi = \pm \pi/2$, which is a motion of the intermediate kind, which exists even for all $\varphi \neq \pi$, so, in particular, for $\varphi = \pm \pi/2$. Like in the first case above, $s$ varies in $(-s_0, s_0) = (-1, 1)$. By existence and unicity of the solution for $b$ (for matrix (B)), for $s = \pm 1$, we have that the rank of the submatrix of (B), consisting of its first three columns, for $s = \pm 1$, is 3. Otherwise said, for $s = \pm 1$, some $3 \times 3$ submatrix of matrix (B), contained in its first three columns, has a non-zero determinant (this submatrix may depend on $s = \pm 1$).

Then, for $s = \pm 1$, the system of the corresponding three linear equations for $b_1$, $b_2$ and $b_3$ can be uniquely solved by Cramer’s rule, with the denominator being non-zero. Then, in some neighbourhood of an endpoint of this solution manifold, we have the following. The solution vector $b$ for this submatrix depends continuously on the coefficients of these three linear equations, therefore on $u$ and $s$. Hence, in some neighbourhood of an endpoint of this solution manifold, taken in the solution set, we have the following. There is a unique solution for $b$ (for matrix (B)), which is furthermore continuous in $u$ and $s$. Then the statements about the endpoints of this solution manifold follow from this. Summing up: in this case we have a connected manifold of solutions. Moreover, adding to it its two endpoints, at $s = \pm s_0 = \pm 1$, we obtain a topological 1-manifold with boundary (these two points), contained in the solution set.

QED

2.2. In [12], pp. 438–440 a slight generalization of the question of Tompos’s tetrahedra has also been considered. Now we present the corresponding question for the tetrahedra $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$, derived from a rectangular parallelepiped. In the physical model of these tetrahedra, we have the following. The bars (edges) of one tetrahedral frame (of the fixed tetrahedron $P_1P_2P_3P_4$, say) touch the corresponding bars (edges) of the other tetrahedral frame (of the moving tetrahedron $Q_1Q_2Q_3Q_4$) from inside (cf. §1). Thus the actual physical constraint is only that each edge $P_iP_j$ lies “inside $Q_kQ_l$” (here $(ijkl)$ is any permutation of $\{1, 2, 3, 4\}$). This can be defined mathematically as follows (cf. [12], p. 439).

For any permutation $(ijkl)$ of $\{1, 2, 3, 4\}$, the signed volume of the tetrahedron $P_iP_jQ_kQ_l$ is either 0, or has the opposite sign as that of the tetrahedron $P_iP_jR_kR_l$. Here $R_kR_l$ is the translate of the segment $Q_kQ_l$ in the basic position (i.e., of the segment $Q_kQ_l = (-P_k)\cdot(-P_l)$ satisfying the following. The midpoint of $R_kR_l$ is the centre of the rectangular parallelepiped in the basic position. (H)

We take (H) as the definition of a generalized motion of our moving tetrahedron $Q_1Q_2Q_3Q_4$ (while $P_1P_2P_3P_4$ is fixed). We prove

**Theorem 2.** Consider the pair of tetrahedra, derived from a rectangular parallelepiped, considered in Theorem 1. For them the generalized admitted finite motions — i.e., all positions of the moving tetrahedron (obtained from its basic position by applying to it an isometry of the space, of determinant +1), satisfying (H) — are identical with the finite motions admitted by them (described in Theorem 1).

*Proof* is analogous to that of [12], Theorem 2, pp. 439–440. Details cf. there, we only indicate the differences.

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Let $\Phi x = Ax + b$ be a generalized admitted finite motion. For $A = I$ we have $b = 0$. From now on we suppose $0 < \varphi < 2\pi$. Observe that now the constraints are expressed by six inequalities (corresponding to equalities (1g)–(1l) in [12], p. 423). Namely three expressions (the left-hand sides of (1g), (1h), (1i)) are non-negative, three expressions (the left-hand sides of (1j), (1k), (1l)) are non-positive. In [12] the moving vertices $Q_i$ were obtained by the motion $Ax + b$, from the respective points ($\pm 1, \pm 1, \pm 1$). Differing from this, now the images by $D^{-1}$ of the moving vertices $Q_i$ are obtained by the transformation $D^{-1}AD(x) + D^{-1}b$, from the respective points ($\pm 1, \pm 1, \pm 1$), while $P_i^0 = (\pm 1, \pm 1, \pm 1)$ are fixed. Cf. the proof of our Theorem 1, (1).

Subtract from a non-negative above expression a non-positive above expression, corresponding to pairs of edges which were originally diagonals of opposite faces of the rectangular parallelepiped. Then divide this difference by 2. Then like in [12], p. 439, we obtain the following. Rather than our equalities (II/1) in (1) of the proof of Theorem 1, we will have inequalities. Namely the left-hand sides of (II/1) are not less than their right-hand sides, which are equal to 0. However, by (3) of the proof of Theorem 1, a positive linear combination of rows II/i of matrix (B) is 0. Therefore the same holds for matrix (B'). Hence, like in [12], in each of the inequalities, corresponding to equalities (II/1), (II/2) and (II/3), we have equalities.

Now recall that the left hand side of each of the inequalities corresponding to an equality (II/i) was obtained as follows. It was half the difference of a non-negative and a non-positive number. This half difference being 0 implies that both of these non-negative and non-positive numbers are 0. In other words, in all the six original constraint inequalities we have equalities. I.e., each pair of edges $P_iP_j, Q_kQ_l$ (where $(ijkl)$ is any permutation of \{1, 2, 3, 4\}) is coplanar. Thus $\Phi$ is a motion admitted by our pair of tetrahedra. QED

2.3. Let us start, rather than with a rectangular parallelepiped, with a general parallelepiped. All the diagonals of all of its faces constitute the edges of two congruent tetrahedra. This position of the two tetrahedra is called their basic position. We define the admitted motions as in (A), but deleting the word “rectangular”. We have, with the notations from (1) in the proof of Theorem 1, that $P_i = D P_i^0$ and $Q_i = D Q_i^0$. However, now $D = [d_{ij}]$ is a general non-singular matrix. In what follows, we will show (in 3.1) that in certain cases the analogues of the motions for the case of the cube or the rectangular parallelepiped exist. Further we prove the generalization of Theorem 2 to the case of general parallelepipeds. Also we will investigate the unicity of the solutions of our equations for $b$ (in 3.1). In the physical model, the bars (edges) of the fixed tetrahedron touch those of the moving tetrahedron from inside (as in Fig. 1).

Also now we have for $[a_{ij}^0 := D^{-1}AD$ and $[b_1^0, b_2^0, b_3^0]^T := D^{-1}b$ equations (I/i), (II/1), (II/2), (II/3) have sum 0. Their right-hand sides have sum $2\text{Tr}(D^{-1}AD) - 2m_2(D^{-1}AD)$. Here, for any $3 \times 3$ matrix $B$, we write $m_2(B)$ for the sum of the symmetric $2 \times 2$ subdeterminants of $B$. We have $\text{Tr}(D^{-1}AD) = \text{Tr}(A)$. We also have $m_2(D^{-1}AD) = m_2(A)$. Namely these last two numbers are the coefficients of $-\lambda$ in the characteristic polynomials of $D^{-1}AD$ and $A$, resp. However, these polynomials coincide. Hence the sum of the right-hand sides of equations (II/1), (II/2) and (II/3) is the same as for the case $D = I$, i.e. 0. (Cf. [12], p. 22.
Thus the sum of equations (II/1), (II/2) and (II/3) is the equation 0 = 0. Hence, among our six linear equations for \( b_1, b_2 \) and \( b_3 \), there are at most five linearly independent ones.

Defining also for the case of general parallelepipeds the generalized admitted finite motions as in Theorem 2, but in (H) deleting the word “rectangular”, we have

**Theorem 3.** Consider the pair of tetrahedra, derived above from a general parallelepiped. For them the generalized admitted finite motions are identical with the finite motions admitted by them.

**Proof** is the same as for Theorem 2, using that the sum of the linear equations (II/1), (II/2) and (II/3), for \( b_1, b_2, b_3 \), is the equation 0 = 0. \( \text{QED} \)

Further we will deal with the pair of tetrahedra, derived above from a general parallelepiped, in 3.1.

3. THE MOTIONS OF A PAIR OF TETRAHEDRA DERIVED FROM A GENERAL PARALLELEPIPED, OF A PAIR OF REGULAR PYRAMIDAL FRAMES, AND OF A PAIR OF REGULAR TETRAHEDRA WITH CIRCULAR ARC EDGES

### 3.1.

We continue the investigation of the two tetrahedra derived from a general parallelepiped. Like in the beginning of 2.1, in (1) of the proof of Theorem 1, and in 2.3, our parallelepiped is taken as the image of the cube with vertices \((\pm 1, \pm 1, \pm 1)\), by the non-singular matrix \( D \).

Similarly like in (3) of the proof of Theorem 1, we have the following. A non-unique solution of equations (I/\( i \)), (II/\( i \)), \( i = 1, 2, 3 \), for \( b \), can occur only if \( \text{Tr} (D^{-1}AD) \) is an eigenvalue of \( D^{-1}AD \). Equivalently, \( \text{Tr} (A) \) is an eigenvalue of \( A \), i.e., \( \varphi = \pm \pi/2 \) or \( \varphi = \pi \) (cf. [11], p. 270, or [12], p. 438).

Observe that for \( [a_{ij}^0] := D^{-1}AD \) and \( [b_i^0] := D^{-1}b \) the left hand sides of equations (I/1), (I/2) and (I/3) form the vector \([a_{ij}^0 - \delta_{ij}(a_{11}^0 + a_{22}^0 + a_{33}^0)][b_1^0, b_2^0, b_3^0]^T\). Therefore, like in the first paragraph of (3), and in the second paragraph of (9) of the proof of Theorem 1, for \( \varphi \neq \pm \pi/2, \pi \), we have the following. The vanishing of the determinants of the system of equations (I/1), (I/2), (I/3), (II/1), and of the system of equations (I/1), (I/2), (I/3), (II/2), is also a sufficient condition for the solvability of equations (I/\( i \)), (II/\( i \)), for \( i = 1, 2, 3 \), for \( b \). (Recall from 2.3 the linear dependence among equations (II/1), (II/2) and (II/3). Namely, their sum is the equation 0 = 0.)

Because of the linear dependence among our equations, it is to be expected that there is a 1-manifold of solutions. This exists — and is a motion of the third kind — if the parallelepiped has a threefold rotational symmetry about a spatial diagonal. Namely, one tetrahedron remains fixed. Beginning from the basic position, the other one is first rotated about this spatial diagonal. Then it is translated in the direction of this spatial diagonal, till the coplanarity conditions become, simultaneously, satisfied (this position is unique). The angle of rotation can be arbitrary, except \( \pm \pi/2 \).

Now suppose that the mid-plane between two parallel faces contains \( 0 \), and is a plane of symmetry of the parallelepiped. Let these faces be horizontal. Then, as follows from
the considerations in 2.1, we have the motions of the intermediate kind, for any \( \varphi \neq \pi \), and we have the motions of the fifth kind. The axis of rotation \( 0u \) lies in this plane of symmetry, i.e., the \( xy \)-plane (and besides this it can have an arbitrary direction), and is perpendicular to this plane of symmetry, resp. In this symmetric case, suppose that moreover the projection of the parallelepiped along a diagonal of one of the mentioned parallel faces is a rectangle of side ratio 2 : 1. (The other diagonal of this face having a larger projection than the altitude belonging to this face.) Then we have a motion of the sixth kind.

However, in this symmetric case, as follows from the considerations in 2.1, the motion of the intermediate kind may exist also for \( \varphi = \pi \). This happens exactly in the cases when the rotation axis \( 0u \) (supposed to lie in the \( xy \)-plane) is parallel to an angle bisector of the diagonals of a horizontal face. Moreover, then there exists also a new motion, which we call a motion of the seventh kind. Namely, as in 2.1, we rotate both tetrahedra about this rotation axis, in a way symmetric w.r.t. the \( xy \)-plane, through angles \( \pm \pi/2 \). Then we make arbitrary vertical translations of the two tetrahedra, in a way symmetric w.r.t. the \( xy \)-plane. Thus we get positions satisfying our constraints. For the case of a rectangular parallelepiped, this yields a motion of the fourth kind.

For the general case, suppose that the parallelepiped is nearly a cube — more exactly, \( D \) is near to \( I \). Let us choose a point of a solution manifold of the motions of the third kind for the cube (with \( D = I \)). Let it correspond to \( u^0 = (\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3}) \) (the \( \pm \) signs being independent), and to a fixed \( \varphi \in (0, 2\pi) \), with \( \varphi \neq \pi/2, \pi, 3\pi/2 \) (with some unique \( b \)). Then for any fixed \( \varepsilon > 0 \), for a sufficiently small perturbation \( D \) of \( I \), we have the following. There is a solution for the parallelepiped associated to \( D \), with \( u_1^2 + u_2^2 + u_3^2 = 1 \) and \( \| (u_1, u_2, u_3) - u^0 \| < \varepsilon \), and with this fixed value of \( \varphi \) (with some unique \( b \)). (Possibly these points \( (u_1, u_2, u_3) \), for \( \varphi \) varying in a fixed closed subinterval of \( (0, 2\pi) \), avoiding \( \pi/2, \pi, 3\pi/2 \), and for a sufficiently small perturbation \( D \) of \( I \), form a smooth 1-manifold with boundary.)

In fact, our problem is now equivalent to solving the system of the two determinantal equations, mentioned in the third paragraph of 3.1, for \( u_1, u_2, u_3 \) and \( s \). For the case of the cube (with \( D = I \)), these equations say that non-zero multiples of \( u_2 - u_3 \), and of \( u_3 - u_1 \), are 0, cf. [12], p. 437, 3. Thus these multiples change their signs at the curves on \( S^2 \), given by \( u_2 = u_3 \) and \( u_3 = u_1 \), resp. Hence, after a small perturbation of the equations (i.e., for \( D \) near to \( I \)), the zero-sets of the perturbed multiples will be near the above two curves, resp. Therefore, for each fixed \( \varphi \in (0, 2\pi) \), where \( \varphi \neq \pi/2, \pi, 3\pi/2 \), and for a sufficiently small perturbation, the following holds. We have a solution of this perturbed system of our two equations, with \( u_1^2 + u_2^2 + u_3^2 = 1 \) and \( \| (u_1, u_2, u_3) - u^0 \| < \varepsilon \), and with this fixed value of \( \varphi \) (with some unique \( b \)). (Cf. [3], p. 40, Proposition D.)

Of course, for a general non-singular \( D \), the basic position is a solution as well. By the linear dependence of equations (I/\( i \)) and (II/\( i \)) (cf. 2.3), it is to be expected that the basic position lies on a solution manifold of dimension at least 1. Experiences with models, far from the rectangular parallelepipeds (more exactly, with all edge lengths rather different), seem to confirm this, even with dimension exactly 1. (Observe that, for a rectangular parallelepiped, each opposite pair of edges of the tetrahedra have equal lengths.) However,
we cannot identify this (assumed) solution manifold, which we can call the *motion of the eighth kind*.

We have determined the infinitesimal degree of freedom, at the basic position, for several incongruent parallelepipeds having a threefold rotational symmetry about a spatial diagonal. (Details of this calculation will be given in 3.2.) Except for Tompos’s tetrahedra, this infinitesimal degree of freedom always turned out to be 1.

Hence we have the following. The infinitesimal degree of freedom, at a point of a solution manifold, which can be reached from the basic position by a continuous motion, always satisfying the constraints, is probably, in general, not greater than 1. Moreover, in general, the basic position does not lie on a smooth 2-manifold of solutions.

The simplest unsolved case is probably that of a parallelepiped $P$ having a threefold rotational symmetry about a spatial diagonal. Using analogous notations as in Fig. 3, let the vertices be $P_i$ (fixed) and $Q_i$ (moving). We define the admissible motion, obtained from the basic position of the moving tetrahedron $Q_1Q_2Q_3Q_4$, by a rotation about the axis of rotation $P_1Q_4$, say, through an angle $\pm \pi/3$, and a subsequent (unique) translation in the direction of this axis. This is denoted by $\Phi_i^\pm$. Then $\Phi_i^\pm$ degenerates $P$ to a regular double pyramid, with base the triangle $P_1P_2P_3 = \Phi_i^+(Q_1)\Phi_i^+(Q_2)\Phi_i^+(Q_3) = \Phi_i^-(Q_3)\Phi_i^-(Q_2)\Phi_i^-(Q_3)$. Moreover, the plane spanned by this triangle is a plane of symmetry of this double pyramid. Also, $\Phi_i^+$ and $\Phi_i^-$ lie on a connected component of a 1-manifold of solutions, of motions of the third kind (cf. above).

Now let, e.g., $i = 1$. Let $S_1$ be the symmetry w.r.t. the plane spanned by the face $P_2P_3P_4$. Then the symmetric double pyramid $(P_1P_2P_3P_4) \cup S_1(P_1P_2P_3P_4)$ is a degenerate image of $P$. It can be obtained by choosing any of the two orientation-preserving isometries $\Phi_1^+$ and $\Phi_1^-$, which are also admitted motions, and which are defined as follows. We have $\Phi_1^+(Q_1) := P_2$ and $\Phi_1^+(Q_2) := P_3$ and $\Phi_1^+(Q_3) := S_1(P_1)$ and $\Phi_1^+(Q_4) := P_4$. Similarly, we have $\Phi_1^-(Q_1) := P_3$ and $\Phi_1^-(Q_2) := S_1(P_1)$ and $\Phi_1^-(Q_3) := P_2$ and $\Phi_1^-(Q_4) := P_4$. Analogously we define $\Phi_i^+$ and $\Phi_i^-$, for $i = 2, 3$. Then $\Phi_i^+$ and $\Phi_i^-$, for any fixed $i \in \{1, 2, 3\}$, are probably points of assumed three analogues of the 1-manifold of the motions of the third kind, resp. Moreover, a model experiment indicates three connected smooth solution 1-manifolds, containing these three pairs of points, resp.

Concluding: it is to be awaited that the solution manifolds have in general dimension 1, and also the infinitesimal degree of freedom at their points is in general 1. Let $D$ be a small perturbation of $I$. Then probably, in some small neighbourhoods of the four 1-manifolds of the motions of the third kind for the cube (with $D = I$), there are four 1-manifolds of solutions. Possibly these exist even for each non-singular $D$ (as they do for the rectangular parallelepipeds). Moreover, probably there is one 1-manifold of solutions, passing through the basic position. (For the case of a rectangular parallelepiped, this may degenerate to have length 0.)

**3.2.** We consider two congruent right pyramids, with regular $n$-gonal bases ($n \geq 3$). Suppose that their axes of rotation coincide, and the basic edges of one pyramid intersect the lateral edges of the other one, and also conversely, with the vectors from the centres of the bases to the respective apices being opposite. (Without this oppositeness property, the moving pyramid could coincide with the fixed pyramid.) Additionally, we suppose that

(25)
the direction of some basic edge of one pyramid and the direction of some basic edge of the other pyramid enclose an angle $\pi/n$ (Fig. 6). This position is called the *basic position* of this bar structure, consisting of these two pyramids. (Observe that the case $n = 3$ is a special case of 3.1 as well.)

Consider these pyramids as bar structures only. Move each vertex of the bar structure consisting of these two pyramids under the following condition. One triangular face of one pyramid remains fixed, each bar (edge) retains its length, and each pair of originally intersecting edges, one from each pyramid, remains coplanar. (I)

The physical model is built in such a way that the bars (edges) of one pyramidal frame touch those of the other one from inside (as in Fig. 1). Observe that the two pyramids are not supposed a priori to undergo rigid motions (isometries of the space of determinant +1), but the bases are allowed to change their shapes, and also to become non-planar. However, by experimenting with the respective physical models, for positions attainable from the basic position by continuous motions, always satisfying the constraints, the following seems probable. These conditions seem to enforce the rigid motion of the two pyramids, even with their axes of rotations coinciding, and with the vectors from the centres of the bases to the respective apices being opposite (as in Fig. 6). Actually even the weakening of (I), analogously as in (H), seems to enforce this. Also cf. [9].

We make local investigations. Since the motions of Tompos’s tetrahedra have already been described, we further exclude the case that $n = 3$ and the two tetrahedra are regular.

We have a 1-manifold of finite motions, which are conjectured to be the only positions, attainable from the basic position by continuous motions, always satisfying the constraints.

(This is supported by experimenting with the models.) Namely, one pyramid remains fixed, and the other one undergoes a rigid motion, as follows. Its axis of rotation remains fixed, and it undergoes a certain rotation about this axis, followed by a suitable translation. This
translation happens in the direction of the common axis of rotation, through a distance depending on the angle of rotation. We translate the moving pyramid till the coplanarity conditions become, simultaneously, satisfied (this position is unique). This is an analogue of the motion of the third kind for Tompos’s tetrahedra. The angle of rotation can be arbitrary, except $\pm \pi/2$. For $n \geq 4$ (unlike as for $n = 3$, cf. 3.1), we are unaware of any other motions, admitted by our bar structure.

We have considered the basic position of this motion, for $3 \leq n \leq 7$, and for several different values of the quotient of the lengths of the lateral and the basic edges. We have determined for these cases the infinitesimal degrees of freedom of our bar structure, consisting of these two pyramidal frames, as follows.

The number of the free parameters, i.e., of all the three coordinates of all but the fixed three vertices, is $6n - 3$. The constraints are that the lengths of all but the fixed three edges are fixed, and $2n$ pairs of edges are coplanar, i.e., the tetrahedra spanned by their vertices have fixed signed volumes, namely 0. The total number of constraints is also $6n - 3$. Thus we have a function $\mathbb{R}^{6n-3} \rightarrow \mathbb{R}^{6n-3}$. This maps a $(6n - 3)$-tuple of the coordinates of the non-fixed vertices to the vector with coordinates the $6n - 3$ constraints, as functions of the previous $6n - 3$ coordinates. (The constraints are the lengths of the non-fixed edges, and the signed volumes of the above tetrahedra.) Then the number of infinitesimal degrees of freedom of our bar structure is the nullity of the Jacobian $J$ of this map, i.e., $6n - 3 - \text{rank } J$.

Having performed these calculations, like e.g. in [7], §2, we have found the following. This infinitesimal degree of freedom, at the basic position, is in all the cases considered by us equal to 1, except in the case of Tompos’s tetrahedra. This can be considered as numerically supporting the above conjecture.

It would be interesting to clarify even that case, when the two pyramids move as rigid bodies. (Observe that for $n \geq 4$ this yields an overdetermined system of equations, namely we have $2n$ equations about coplanarities, for six unknowns.)

3.3. Another generalization of the pair of tetrahedra, derived in §1 from the cube with vertices $(\pm 1, \pm 1, \pm 1)$, is the following. We replace each edge of both tetrahedra by congruent circular arcs of some fixed radius. These have the same endpoints as the respective edges. Moreover, each of them lies in the plane spanned by the respective edge and the centre of the cube. Further suppose that any congruence of the above cube to itself is also a congruence of this system of circular arcs.

Thus we obtain a figure roughly resembling Fig. 1 or Fig. 3. Thus both tetrahedra become tetrahedron-like frames. The arcs of circles, replacing diagonals of the same face of the cube, intersect. Their point of intersection lies on the straight line, connecting the midpoint of the cube with the midpoint of the considered face of the cube.

Fixing one of these frames, we move the other one in the following way. Each pair of circles, containing the pairs of circular arcs, originally corresponding to diagonals of some face of the cube, continue to have a common point, in the complex projective sense. (J)

Namely, in this sense the condition is to be awaited simpler.
Now suppose that the frames both lie on the surface of the circumsphere of the cube. Then an arbitrary rotation about the centre of the cube, with translation part \( b = 0 \), is an admitted motion. So now we have an at least 3-parameter set of motions.

Again we turn to the general case. We will show that the motions of the intermediate, third and fifth kinds generically exist. (These contain the motions of first, second and fourth kinds as special cases.)

We begin with the analogue of the motion of the fifth kind. At this motion the moving tetrahedron undergoes from the position of first kind — obtained by \( A \) being a rotation through the angle \( \pi \), about the \( z \)-axis, say — a translation, through a vector \( b \). This happens in the following way. An arbitrarily fixed point of the moving circle, containing the circular arc corresponding to the edge \( Q_0Q_2^0 \), in its rotated position, will coincide after translation with an arbitrarily fixed point of the fixed circle, containing the circular arc corresponding to the edge \( P_3^0P_4^0 \) (in analogy with Fig. 4).

This is a two-parameter motion. At this motion the circles, containing the arcs corresponding to the edges \( P_0^1P_2^0 \) and \( Q_0^3Q_4^0 \) (in its rotated position), also intersect. This follows by a simple argument using central symmetry. However, the set of \( b \)'s for this \( A \) does not form an affine 2-manifold — on the contrary, it is bounded. (For Tompos's tetrahedra we had here an affine 2-manifold.) All other pairs of respective circles, which should have common points, in the complex projective sense, lie in respectively parallel or coincident planes. This guarantees that these pairs of circles in fact have common points, in the complex projective sense. (Observe that the complex projective extension of any circular line, lying in a horizontal plane, contains the points \((1, \pm i, 0, 0)\) of the complex projective space.)

However, for motions of the intermediate and third kinds, there is a difference as compared to 2.1.

We turn to the analogue of the motion of the third kind. It will be convenient to rotate our original cube about \( 0 \), so that \( P_4^0 \) becomes \((0, 0, \sqrt{3})\). The fixed circle \( C_{14} \), containing the circular arc replacing the edge \( P_1^0P_2^0 \), should lie in the vertical plane \( y = 0 \). However, the moving circle \( C_{23}' \) (obtained by a rotation, through an angle \( \varphi \), about the \( z \)-axis, from its basic position) containing the circular arc replacing the edge \( Q_2^0Q_3^0 \), will lie in a not vertical plane.

We denote the projection map to the \( xy \)-coordinate plane by \( \pi \). Recalling the definition of the motion of the third kind, we want to find a \( \lambda \), such that \( C_{14} \cap (C_{23}' + \lambda e_3) \neq \emptyset \). (Then, by reason of symmetry, with the evident notations, also \( C_{23} \cap (C_{14}' + \lambda e_3) \neq \emptyset \).)

We have that \( C_{14} \) lies in the \( xz \)-coordinate plane, hence \( \pi(C_{14}) \) is contained in the \( x \)-coordinate axis. The equation system of \( C_{14} \) is \( y = 0 \), and an equation of the form \((x - a)^2 + (z - b)^2 = R^2\). Hence, in the complex case, \( \pi(C_{14}) \) also contains the \( x \)-coordinate axis, hence equals the \( x \)-coordinate axis. Even generically \( xe_1 \) is the projection, by \( \pi \), of two points \((x, 0, z_1(x)), (x, 0, z_2(x)) \in C_{14} \) (\( C_{14} \) meant in the complex sense): namely the two endpoints of a chord of \( C_{14} \), whose difference lies in the \( z \)-coordinate axis.

On the other hand, \( C_{23}' \) (meant in the complex sense) does not lie in a vertical plane, hence the projection from the affine hull of \( C_{23}' \) to the \( xy \)-coordinate plane is a bijection. Hence \( \pi(C_{23}') \) is a non-degenerate ellipse, of an equation of the form \( Ax^2 + Bxy + Cy^2 + \)
\[Dx + Ey + F = 0.\] Its intersection with the \(x\)-axis has an equation \(Ax^2 + Dx + F = 0\), which has generically two zeroes \(x_1, x_2\). Therefore, generically, the number of \((x, 0, z)\)'s, for which \((x, 0, z) \in C_{14} \cap (C_{23} + \lambda e_3)\), for some \(\lambda\) (depending on \(x\) and \(z\)), is four. Namely, \(x_i e_1\) is the projection, by \(\pi\), of one point \((x_i, 0, z'(x_i)) \in C_{23}\), and then we choose \(\lambda_{ij} = z_j(x_i) - z'(x_i)\) (for \(i, j = 1, 2\)). Moreover, these \(\lambda_{ij}\)'s are generically different. This can be seen from the example with radius of the circles \(\sqrt{2}\), and \(\varphi := \pi/2\), where these \(\lambda_{ij}\)'s are all different.

We turn to the analogue of the motion of the intermediate kind. Let the axis of rotation \(0u\) lie in the \(xy\)-plane. Let us apply symmetric rotations, w.r.t. the \(xy\)-plane, through angles \(\pm \varphi/2\), about the rotation axis, to the two tetrahedron-like frames. Then the pairs of the curved edges (circles), corresponding to the two diagonals of any of the originally vertical faces of the cube, remain symmetric images of each other w.r.t. the \(xy\)-plane. The (possibly complex projective) intersection points of the \(xy\)-plane and one of the circles lie also on the other circle.

Moreover, after these symmetric rotations, we have the following. Each pair of the curved edges, corresponding to the two diagonals of an originally horizontal face of the cube, generically have, as projections to the \(xy\)-plane, two elliptical lines. These projections have generically four intersection points, in the complex projective sense. Therefore generically, for any of the four vertical lines, containing some of these four intersection points, the following holds. Suitable vertical translations, symmetric w.r.t. the \(xy\)-plane — in general through different distances for different lines (cf. below) — produce common points of these two curved edges, lying on this particular vertical line. Further, these distances, for the two originally horizontal faces, pairwise coincide. Namely, by reason of symmetry, the distances, associated to the same vertical line, (pairwise) coincide, so that these common points, on any of these, generically four vertical lines are produced, for the two originally horizontal faces, simultaneously.

It remained to show that the four vertical translations are generically different. Again, it suffices to give one example for this. We let the radius of the circles to be \(\sqrt{2}\), and \(u := (1/\sqrt{2}, 1/\sqrt{2}, 0)\), and \(\varphi := \pi\). Then, as at the analogue of the motion of the third kind, the moving circle, containing the arc replacing the diagonal \(Q_0 Q_3\), has as a projection to the \(xy\)-plane, in the complex sense, a fixed straight line. Therefore we can repeat the considerations at the analogue of the motion of the third kind, obtaining that the four \(\lambda_{ij}\)'s are generically different.

This question perhaps could be handled in analogy with [12], Theorem 1. We have six pairs of circles in the space, one fixed and one moving, which pairwise intersect (in the complex projective sense). This for each pair means an equation of degree six for the coefficients of the equations of our circular lines, hence for our parameters \(a_{ij}\) and \(b_i\). Thus we have a system of six equations of degree twelve for our parameters \(u\), \(s\) and \(b\). Unfortunately this is not linear in the \(b_i\)'s. Namely, for the analogues of the intermediate and the third kinds of motions, there are generically four solutions for \(b\) (in the complex projective sense, cf. above). Therefore, rather than calculating determinants, as in [12], one needs to calculate resultants of polynomials (cf. [14]). Possibly some symbolic algebraic calculations, like with Mathematica or Maple, and efficient algorithms from computational algebraic geometry could help.
3.4. In [9] the following general model was considered, which contains the examples in 3.1 and 3.2 as special cases. Let us have two convex polyhedra, which are combinatorially dual. Let one of them have \( f \) faces, \( e \) edges and \( v \) vertices. Then the other one has \( v \) faces, \( e \) edges and \( f \) vertices.

Consider these polyhedra as bar structures only. Move each vertex of the bar structure consisting of these two polyhedra under the following condition. Each bar (edge) retains its length, and each pair of combinatorially corresponding edges is coplanar. (K)

The faces may change their shapes, and may become non-planar, and also convexity may not hold any more. The number of free parameters (all three coordinates of all vertices) is \( 3v + 3f \). The number of constraints (edge lengths and coplanarity conditions) is \( 3e \). By Euler’s theorem, these numbers have a difference 6, i.e., the number of parameters of all rigid motions of the space. This used to indicate that there are not even infinitesimal motions. However, in example 3.2, and sometimes (possibly always) in example 3.1, there are finite motions, so here intuition fails.

An example is a pair of congruent tetrahedra with the same orientation, with the combinatorially corresponding pairs of edges being those induced by a fixed orientation-preserving congruence. Then a rotation of the fixed tetrahedron about any of its altitudes, or a translation of the fixed tetrahedron by any vector, yields a moving tetrahedron satisfying the constraints.

**Problem 1.** Is there some general theorem behind these examples, that under suitable hypotheses, the model described in 3.4 always has a finite motion?

**Problem 2.** Determine the finite motions of the examples in 3.1, 3.2 and 3.3.

**Acknowledgements.** The authors express their thanks to András Lengyel and Ampar López for help with the photography, and special thanks to András Lengyel for producing the line figures of this paper.

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