GLOBAL SOLUTIONS TO THE LAGRANGIAN AVERAGED NAVIER-STOKES EQUATION IN LOW REGULARITY BESOV SPACES

NATHAN PENNINGTON

ABSTRACT. The Lagrangian Averaged Navier-Stokes (LANS) equations are a recently derived approximation to the Navier-Stokes equations. Existence of global solutions for the LANS equation has been proven for initial data in the Sobolev space $H^{3/4,2}(\mathbb{R}^3)$ and in the Besov space $B_{2,2}^{3/2}(\mathbb{R}^3)$. In this paper, we use an interpolation based method to prove the existence of global solutions to the LANS equation with initial data in $B_{p,q}^{3/p}(\mathbb{R}^3)$ for any $p > 3$.

1. Introduction and Main Results

The LANS equation is a recently derived approximation to the Navier-Stokes equation and is derived by averaging at the Lagrangian level. For an exhaustive treatment of this process, see [12], [13], [6] and [8]. In [9] and [3], the authors discuss the numerical improvements that use of the LANS equation provides over more common approximation techniques of the Navier-Stokes equation.

On $\mathbb{R}^n$, the isotropic, incompressible form of the LANS equation is given by

$$
\partial_t w + (w \cdot \nabla) w + \text{div} \tau^\alpha(w, w) = -(1 - \alpha^2 \Delta)^{-1} \text{grad} p + \nu \Delta w
$$

$$
w : [0, T) \times \mathbb{R}^n \to \mathbb{R}^n, w(0, x) = w_0(x), \text{ div } w = \text{ div } w_0 = 0,
$$

where all the differential operators (except $\partial_t$) are spatial differential operators, $\alpha > 0$ is a constant, $\nu > 0$ is the viscosity of the fluid, $p$ denotes the fluid pressure, and $w_0$ is the initial data. The Reynolds stress $\tau^\alpha(w, w)$ is given by

$$
\tau^\alpha(f, g) = \frac{\alpha^2}{2} (1 - \alpha^2 \Delta)^{-1} [\text{Def}(f) \cdot \text{Rot}(g) + \text{Def}(f) \cdot \text{Rot}(g)]
$$

where $\text{Rot}(f) = (\nabla f - \nabla f^T)/2$ and $\text{Def}(f) = (\nabla f + \nabla f^T)/2$. Abusing notation, we set $\tau^\alpha(f, f) = \tau^0(f)$. We note that setting $\alpha = 0$ returns the Navier-Stokes equation.

The difference between the LANS equation and the Navier-Stokes equation is the additional nonlinear term $\tau^\alpha$. This additional term complicates local existence theory, but makes it easier to control the long time behavior of local solutions. Local existence results for the LANS equation in various settings can be found in [12], [6], [7], [10] and [11]. In [7], Marsden and Shkoller proved the existence of a global solution to the LANS equation with initial data in the Sobolev space $H^{3,2}(\mathbb{R}^3)$. In [10], this result was improved, achieving global existence for data in the space $H^{3/4,2}(\mathbb{R}^3)$. In [11],
existence of local solutions was proven for initial data in Besov spaces, and the local solution is extended to a global solution for initial data in $B^s_{2,q}(\mathbb{R}^3)$ for $s > 3/4$.

In this article we prove new global existence results to the LANS equation, guided by the method used by Gallagher and Planchon in [1] (which has its origins in [1]) for the Navier-Stokes equation, which will be outlined below. We now state the main result of this article.

**Theorem 1.** Let $w_0 \in B^{3/p}_{p,q}(\mathbb{R}^3)$ be divergence free. Then there exists a unique global solution to the LANS equation $w \in C([0, \infty) : B^{3/p}_{p,q}(\mathbb{R}^3))$ with $w(0) = w_0$, provided $p > 2$.

This result expands on the global existence result from [11], which only held in the case $p = 2$. The primary emphasis here is the large $p$ case, where we obtain global existence for data with regularity close to zero.

The rest of this section is devoted to proving Theorem 1 up to Theorem 2, the proof of which is the focus of the rest of the article. We start with our solution space $W = B^{3/p}_{p,q}(\mathbb{R}^3)$, and define $U = B^{3/2}_{2,q}(\mathbb{R}^3)$ and $V = B^{3/p}_{p,q}(\mathbb{R}^3)$, where $\tilde{p} > p$. Then, choosing $\theta$ such that $3/p = 3\theta/2 + 3(1 - \theta)/\tilde{p}$, we have that

$$W = [U, V]_{\theta,q}.$$  

For our given $w_0 \in W$, this means there exists $u_0 \in U$ and $v_0 \in V$ such that $w_0 = u_0 + v_0$. We can also choose $\|v_0\|_V$ to be arbitrarily small. By one of the results of [11] (recalled in Section 2 below as Theorem 3) there exists a unique global solution $v(t) \in V$ to the LANS equation with initial data $v_0$. This result also provides a unique local solution $w$ to the LANS equation such that $w(t) \in W$ and $w(0) = w_0$.

With this global solution $v$ to the LANS equation, the next step is to derive the following modified version of the LANS equation:

$$\partial_t u - \triangle u + \text{div} (u \otimes u + u \otimes v + v \otimes u) + \text{div} (\tau^\alpha(u, u) + 2\tau^\alpha(u, v)),
\quad u(0) = u_0, \text{ div } u = \text{ div } u_0 = 0,$$

where we recall that $\tau^\alpha$ is defined in equation (1.2). We will refer to this as the mLANS equation, and it is derived by replacing $u$ in (1.1) with $u + v$. This process is explicitly detailed in the beginning of Section 3.

Now that the mLANS equation has been defined, we require the following result.

**Theorem 2.** For any $u_0 \in U$, there exists a unique global solution $u \in C([0, \infty) : U)$ to the mLANS equation.

Proving Theorem 2 will be the primary task of the rest of the article. For now, assuming Theorem 2 we proceed with the proof of Theorem 1. By the construction of the mLANS equation, because $u$ is a global solution to the mLANS equation, we have that $u + v$ is a global solution to the LANS equation, and that $u(0) + v(0) = u_0 + v_0 = w_0$. We also have a unique local solution $w$ to the LANS equation with $w(0) = w_0$ and $w(t) \in W$. By uniqueness, if $u(t) + v(t) \in W$ for all $t$, then $u(t) + v(t) = w(t)$ for all $t$, and the proof of Theorem 1 will be complete.

So our last remaining task is to show that $u(t) + v(t) \in W$ for all $t$, and this is a special case of a general interpolation result found in [3] which will be presented at
the end of Section 5. The key requirement for this result is that $U \hookrightarrow W \hookrightarrow V$. Using Besov embedding (see equation (2.6)), this holds for $U = B_{3/2}^{3/2}(\mathbb{R}^3)$, $V = B_{3/4}^{3/4}(\mathbb{R}^3)$, and $W = B_{3/4}^{3/4}(\mathbb{R}^3)$. Satisfying this embedding relation is the reason we do not use the optimal existence results from [11] for the interpolation, since $B_{3/4+\varepsilon}^{3/4+\varepsilon}(\mathbb{R}^3)$ does not inject into $B_{3/2+\varepsilon}^{3/2+\varepsilon}(\mathbb{R}^3)$.

This completes the proof of Theorem 1, up to proving Theorem 2, which is the focus of the remainder of the article. In Section 2 we recall the basic construction of Besov spaces, some standard Besov space estimates, and local and global existence theorems from [11]. The mLANS equation is derived and local solutions for the mLANS equation are constructed in Section 3, and the extension to a global result is the focus of Section 4 and Section 5.

2. Besov Spaces

We begin by defining the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$. Let $\psi_0 \in \mathcal{S}$ be an even, radial function with Fourier transform $\hat{\psi}_0$ that has the following properties:

$$
\hat{\psi}_0(x) \geq 0
$$

support $\hat{\psi}_0 \subset A_0 := \{\xi \in \mathbb{R}^n : 2^{-1} < |\xi| < 2\}$

$$
\sum_{j \in \mathbb{Z}} \hat{\psi}_0(2^{-j} \xi) = 1, \text{ for all } \xi \neq 0.
$$

We then define $\hat{\psi}_j(\xi) = \hat{\psi}_0(2^{-j} \xi)$ (from Fourier inversion, this also means $\psi_j(x) = 2^{jn} \psi_0(2^j x)$), and remark that $\hat{\psi}_j$ is supported in $A_j := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. We also define $\Psi$ by

$$
\hat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \hat{\psi}_k(\xi).
$$

We define the Littlewood Paley operators $\triangle_j$ and $S_j$ by

$$
\triangle_j f = \psi_j * f, \quad S_j f = \sum_{k = -\infty}^{j} \triangle_k f,
$$

and record some properties of these operators. Applying the Fourier Transform and recalling that $\hat{\psi}_j$ is supported on $2^{j-1} \leq |\xi| \leq 2^{j+1}$, it follows that

$$
\triangle_j \triangle_k f = 0, \quad |j - k| \geq 2
$$

$$
\triangle_j (S_{k-3} f \triangle_k g) = 0 \quad |j - k| \geq 4,
$$

and, if $|i - k| \leq 2$, then

$$
\triangle_j (\triangle_k f \triangle_l g) = 0 \quad j > k + 4.
$$
For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ we define the space $\tilde{B}^{s}_{p,q}(\mathbb{R}^n)$ to be the set of distributions such that

$$\|u\|_{\tilde{B}^{s}_{p,q}} = \left( \sum_{j=0}^{\infty} (2^{js} \|\triangle_j u\|_{L^p})^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$. Finally, we define the Besov spaces $B^{s}_{p,q}(\mathbb{R}^n)$ by the norm

$$\|f\|_{B^{s}_{p,q}} = \|\Psi*f\|_{p} + \|f\|_{\tilde{B}^{s}_{p,q}},$$

for $s > 0$. For $s > 0$, we define $B^{-s}_{p',q'}$ to be the dual of the space $B^{s}_{p,q}$, where $p', q'$ are the Holder-conjugates to $p, q$.

These Littlewood-Paley operators are also used to define Bony’s paraproduct. We have

$$fg = \sum_{k} S_{k-3}f \triangle_k g + \sum_{k} S_{k-3}g \triangle_k f + \sum_{k} \triangle_k f \sum_{l=-2}^{2} \triangle_{k+l} g.$$  

The estimates (2.2) and (2.3) imply that

$$\triangle_j(fg) \leq \sum_{k=-3}^{3} \triangle_j (S_{j+k-3}f \triangle_{j+k} g) + \sum_{k=-3}^{3} \triangle_j (S_{j+k-3}g \triangle_{j+k} f)
\quad + \sum_{k>j-4} \triangle_j \left( \triangle_k f \sum_{l=-2}^{2} \triangle_{k+l} g \right).$$

This calculation will be very useful in Section 7.

Now we turn our attention to establishing some basic Besov space estimates. First, we let $1 \leq q_1 \leq q_2 \leq \infty$, $\beta_1 \leq \beta_2$, $1 \leq p_1 \leq p_2 \leq \infty$, $\gamma_1 = \gamma_2 + n(1/p_1 - 1/p_2)$, and $r > s > 0$. Then we have the following:

$$\|f\|_{B^{\beta_1}_{p_1,q_1}} \leq C\|f\|_{B^{\beta_2}_{p_2,q_2}},$$

$$\|f\|_{B^{\beta_2}_{p_2,q_2}} \leq C\|f\|_{B^{\beta_1}_{p_1,q_1}},$$

$$\|f\|_{H^{s,p}} \leq \|f\|_{B^{s}_{p,q}},$$

$$\|f\|_{H^{s,2}} = \|f\|_{B^{s}_{2,2}} \leq \|f\|_{B^{s}_{2,q}}.$$  

These will be referred to as the Besov embedding results. Next, we record a Leibnitz-rule type estimate. This can be found in [2], and for the reader’s convenience, the proof can be found in Section 7.

**Proposition 1.** Let $f \in B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ and let $g \in B^{s_2}_{p_2,q}(\mathbb{R}^n)$. Then, for any $p$ such that $1/p \leq 1/p_1 + 1/p_2$ and with $s = s_1 + s_2 - n(1/p_1 + 1/p_2 - 1/p)$, we have

$$\|fg\|_{B^{s}_{p,q}} \leq \|f\|_{B^{s_1}_{p_1,q_1}} \|g\|_{B^{s_2}_{p_2,q}},$$

provided $s_1 < n/p_1$, $s_2 < n/p_2$, and $s_1 + s_2 > 0$. 

Our third result is the Bernstein inequalities (see Appendix A in [14]). We let $A = (-\Delta)$, $n \geq 0$, and $1 \leq p \leq q \leq \infty$. If $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^j K \}$ and $\text{supp} \hat{g} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^j K_2 \}$ for some $K, K_1, K_2 > 0$ and some integer $j$, then

$$\hat{A} = \sum_{j=0}^\infty \hat{A}_j = \sum_{j=0}^\infty \sum_{L \in I_j} \hat{A}_L$$

where

$$\hat{A}_L \hat{t} = \hat{A}_L \hat{t} \hat{f} = \sum_{\ell \in I_j} \left( \hat{A}_{L_\ell} \hat{t} \right) \hat{f}$$

and $I_j$ is a finite set of integers.

Proposition 1. Let $1 \leq p_1 \leq p_2 < \infty$, $\alpha \geq 0$, and $n \geq 1$. If $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^j K \}$ and $\text{supp} \hat{g} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^j K_2 \}$ for some $K, K_1, K_2 > 0$ and some integer $j$, then

$$\hat{A} \hat{f} \hat{g} = \sum_{j=0}^\infty \hat{A}_j \hat{f} \hat{g}$$

where

$$\hat{A}_j \hat{f} \hat{g} = \sum_{L \in I_j} \hat{A}_L \hat{f} \hat{g}$$

and $I_j$ is a finite set of integers.

The Bernstein inequalities (see Appendix A in [14]) state that

$$\hat{A} \hat{f} \hat{g} = \sum_{j=0}^\infty \sum_{L \in I_j} \hat{A}_L \hat{f} \hat{g}$$

where $I_j$ is a finite set of integers.

Our last Besov space estimate governs the behavior of the heat kernel on Besov spaces.

Proposition 2. Let $1 \leq p_1 \leq p_2 < \infty$, $-\infty < s_1 \leq s_2 < \infty$, and let $0 < q < \infty$. Then

$$\|e^{t \alpha} f\|_{B_{p_2,q}^{s_2}} \leq C t^{-(s_2-s_1+n/p_1-n/p_2)/2} \|f\|_{B_{p_1,q}^{s_1}},$$

provided $0 < t < 1$.
We let $\dot{C}_{a,s,p,q}^T$ denote the subspace of $C_{a,s,p,q}^T$ consisting of $f$ such that
\[ \lim_{t \to 0^+} \dot{t}sf(t) = 0 \text{ (in } B_{p,q}^s(\mathbb{R}^n)). \]

Note that while the norm $\| \cdot \|_{a,s,p,q}$ lacks an explicit reference to $T$, there is an implicit $T$ dependence. Finally, we state a local existence theorem for the LANS equation. This result is a special case of Theorem 4 in [11].

**Theorem 3.** Let $v_0 \in B_{p,q}^{n/p}(\mathbb{R}^n)$ be divergence free, where $p > n$, and let $r$ satisfy $n/p < r < n/p + 1$. Then there exists a time $T$ and a unique solution $v$ to the LANS equation (1.1) such that
\[ v \in BC([0,T) : B_{p,q}^{n/p}(\mathbb{R}^n)) \cap \dot{C}_{\frac{n}{p},r,p,q}^T, \]
with $v(0) = v_0$. We remark that the time $T$ depends only on $\|v_0\|_{B_{p,q}^{n/p}}$, and for sufficiently small $\|v_0\|_{B_{p,q}^{n/p}}$, $T = \infty$. Furthermore, for a given $T^* < \infty$ and a given real number $\varepsilon$, if $\|v_0\|_{B_{p,q}^{n/p}}$ is sufficiently small, then $\sup_{0 \leq t < T^*} \|v(t)\|_{B_{p,q}^{n/p}} < \varepsilon$.

The result can be extended in the following fashion.

**Corollary 1.** Let $v_0 \in B_{p,q}^{n/p}(\mathbb{R}^n)$ be divergence free, and let $v$ be the solution given in the above theorem. Then the requirement that $r < n/p + 1$ can be removed.

The proof of a similar extension for solutions to the mLANS equation can be found in Lemma 4 in Section 6. A more complete discussion of this type of result can also be found there.

### 3. Derivation of and Local Solutions to the mLANS equation

We let $v(t)$ denote the solution to the LANS equation with $v(0) = v_0$ given by Theorem 3. We seek a $u$ such that, defining $w(t)$ by $w(t) = u(t) + v(t)$, $w$ will solve the LANS equation. This means
\[ \partial_t(u + v) - \Delta(u + v) + \text{div}(u + v) + \tau^\alpha(u + v, u + v). \]

Using the fact that $v$ satisfies the LANS equation, and requiring that $u(0) = u_0$ and $\text{div } u = \text{div } u_0 = 0$, this (essentially) simplifies to
\[ \partial_t u - \Delta u + \text{div}(u \otimes u + u \otimes v + v \otimes v) + \text{div}(1 - \alpha^2 \Delta)^{-1}(\nabla u \nabla u + \nabla v \nabla v), \]
with $u(0) = u_0$, $\text{div } u = \text{div } u_0 = 0$.

This is not exact because of the second non-linear term. There are actually several more terms involving products of $\nabla u$, $(\nabla u)^T$, $\nabla v$, and $(\nabla v)^T$ (but no terms involving only products of $\nabla v$ and $(\nabla v)^T$). In most of the following calculations, the additional terms have no effect on our argument, and so will often be omitted. We call equation (3.1) the mLANS equation.

Throughout the remainder of the article, we set the $v$ in the mLANS equation to be the small initial data solution to the LANS equation given by Theorem 3 with $p >> n$, which means $v$ is divergence-free and
\[ v \in \tilde{E} = BC([0,T) : B_{p,q}^{n/p}(\mathbb{R}^n)) \cap \dot{C}_{\frac{n}{p},r,p,q}^T, \]
for any \( r > n/p \) and any \( T \), where \( b = (r - n/p)/2 \).

**Theorem 4.** Let \( u_0 \in B_{2,q}^{n/2}(\mathbb{R}^n) \) be divergence free. Then there exists a local solution \( u \) to the mLANS equation (3.1) such that

\[
u \in BC([0,T) : B_{2,q}^{n/2}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s,2,q},
\]

where \( a = (s - n/2)/2, \) \( 0 < s - n/2 < 1 \), and \( T \) depends only on \( \|u_0\|_{n/2,2,q} \).

In the next section, we will extend this local solution to a global solution. The following Corollary is instrumental in this task.

**Corollary 2.** The requirement in Theorem 4 that \( s - n/2 < 1 \) can be removed.

The proof of the corollary follows from Lemma 4 in Section 6. This result (and its proof) is similar to Proposition 8 in [10], which has its origins in an induction argument from [5].

The proof of Theorem 4 will follow from the standard contraction mapping method and heavy use of the results from Section 2. We begin by defining the nonlinear operator \( \Phi \) by

\[
\Phi(u) = e^{t\nabla}u_0 + \tilde{\Phi}(u) + \Psi(u,v),
\]

where

\[
\tilde{\Phi}(u) = \int_0^t e^{(t-s)\nabla}V(u)ds,
\]

\[
\Psi(u,v) = \int_0^t e^{(t-s)\nabla}W(u,v)ds,
\]

with \( V \) and \( W \) (essentially) given by

\[
V(u) = \text{div} \, (u \otimes u) + \text{div} \, (1 - \triangle)^{-1}(\nabla u \nabla u),
\]

\[
W(u,v) = \text{div} \, (u \otimes v) + \text{div} \, (1 - \triangle)^{-1}(\nabla u \nabla v),
\]

where, as above, the full definitions of \( V \) and \( W \) involve additional terms whose behavior is controlled by the terms shown.

We seek a fixed point of \( \Phi \) in the space

\[
E = \{ f \in BC((0,T) : B_{2,q}^{n/2}(\mathbb{R}^n)) \cap \dot{C}^T_{s-n/2,2,q} : \sup_t \| f - e^{t\nabla}u_0 \|_{n/2,2,q} + \| f \|_{(s-n/2)/2,2,q} < M \}.
\]

We first show that \( \Phi : E \to E \), and we begin by showing that \( \Psi : E \to E \), which requires estimating

\[
I = I_1 + I_2 = \sup_{0 \leq t < T} \left\| \int_0^t e^{(t-\tau)\nabla} \text{div} \, (u \otimes v) d\tau \right\|_{B_{2,q}^{n/2}}
\]

\[
+ \sup_{0 < t < T} t^a \left\| \int_0^t e^{(t-s)\nabla} \text{div} \, (u \otimes v) d\tau \right\|_{B_{2,q}^s}.
\]

(3.3)
\[ J = J_1 + J_2 = \sup_{0 \leq t < T} \left\| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( (1 - \Delta)^{-1}(\nabla u \nabla v) d\tau \right) \right\|_{B^{n/2}_{2,q}} \]

\[ + \sup_{0 \leq t < T} t^a \left\| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( (1 - \Delta)^{-1}(\nabla u \nabla v) d\tau \right) \right\|_{B^{n/2}_{2,q}} \]

(3.4)

3.1. Estimating \( I \). To bound \( I_1 \), we start by setting \( \alpha = \alpha_1 + \alpha_2 \), where \( \alpha_1 = n/2 - \varepsilon \) and \( \alpha_2 = n/p - \varepsilon \). Then we have

\[ \left\| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( u \otimes v \right) d\tau \right\|_{B^{n/2}_{2,q}} \leq \int_0^t \left| t - \tau \right|^{-(n/2 - (\alpha_1 + n/p - n)/2)} \left\| u \otimes v \right\|_{B^{n/2}_{p,q}} d\tau, \]

where \( n/p = n/2 + n/p \) and we used that \( \alpha - 1 = n/2 + n/p - 2\varepsilon - 1 \leq n/2 \) for \( p > n \). Using Proposition 1, we have

\[ \left\| u \otimes v \right\|_{B^{n/2}_{p,q}} \leq \left\| u \right\|_{B^{n/2}_{2,q}} \left\| v \right\|_{B^{n/2}_{p,q}} \leq \left\| u \right\|_{B^{n/2}_{2,q}} \left\| v \right\|_{B^{n/2}_{p,q}}. \]

Returning to the integral, we have

\[ \int_0^t \left| t - \tau \right|^{-(n/2 - (\alpha_1 + n/p - n)/2)} \left\| u(\tau) \otimes v(\tau) \right\|_{B^{n/2}_{p,q}} d\tau \]

\[ \leq C \int_0^t \left| t - \tau \right|^{-1/2} \left\| u(\tau) \right\|_{B^{n/2}_{0,q}} \left\| v(\tau) \right\|_{B^{n/2}_{0,q}} d\tau \]

\[ \leq C \left\| u \right\|_{0,n/2,q} \left\| v \right\|_{0,n/2,q} \int_0^t \left| t - \tau \right|^{-1/2} d\tau \]

\[ \leq C \left\| u \right\|_{0,n/2,q} \left\| v \right\|_{0,n/2,q} t^{-1/2 + 1}, \]

provided \( 1 + 2\varepsilon < 2 \), which is easily satisfied for small \( \varepsilon \). From (3.2), we know that \( \left\| v \right\|_{0,n/2,p,q} \) is finite, so

\[ I_1 \leq \sup_{0 \leq t < T} C \left\| u \right\|_{0,n/2,q} \left\| v \right\|_{0,n/2,p,q} t^{-1/2 + 1} \]

(3.5)

\[ \leq CMT^{1/2 - \varepsilon}. \]

For \( I_2 \), recalling that \( a = (s - n/2)/2 \), a similar argument gives

\[ I_2 \leq \sup_{0 \leq t < T} t^a \left\| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( u(\tau) \otimes v(\tau) \right) d\tau \right\|_{B^{n/2}_{2,q}} \]

\[ \leq \sup_{0 \leq t < T} t^a \int_0^t \left| t - \tau \right|^{-1/2 - (s - \alpha)/2} \left\| u(\tau) \otimes v(\tau) \right\|_{B^{n/2}_{p,q}} d\tau \]

(3.6)

\[ \leq \sup_{0 \leq t < T} C t^a \left\| u \right\|_{0,n/2,q} \left\| v \right\|_{0,n/2,q} \int_0^t \left| t - \tau \right|^{-1/2 - (s - \alpha)/2} d\tau \]

\[ \leq \sup_{0 \leq t < T} C \left\| u \right\|_{0,n/2,q} \left\| v \right\|_{0,n/2,q} t^{-1/2 - (s - \alpha)/2} \]

\[ \leq CMT^{(1-n/p - n/2 + a)/2} \leq CMT^{1/2 - \varepsilon}, \]
provided \( s - \alpha + n/p < 1 \). So we have that

\[
I = I_1 + I_2 < CMT^{1/2 - \varepsilon},
\]

provided

\[
1 > s - \alpha + n/p = s - n/2 + 2\varepsilon
\]

\[
1 \geq n/2 - \alpha + n/p = 2\varepsilon.
\]

The first requirement is equivalent to \( s - n/2 < 1 \) and the second is vacuously satisfied.

3.2. Estimating \( J \). For \( J_1 \), we have

\[
\left\| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( (1 - \Delta)^{-1}(\nabla u(\tau) \nabla v(\tau)) \right) d\tau \right\|_{B_{q/2}^n}
\]

\[
\leq \int_0^t |t - \tau|^{-n/(p-n/2)/2} \| \text{div} \left( (1 - \Delta)^{-1}(\nabla u(\tau) \nabla v(\tau)) \right) \|_{B_{p,q}^{n/2}} d\tau
\]

\[
\leq \int_0^t |t - \tau|^{-n/(p-n/2)/2} \| \nabla u(\tau) \nabla v(\tau) \|_{B_{p,q}^{n/2-1}} d\tau
\]

where \( n/p = n/2 + n/p \). Setting \( n/2 - 1 = \beta_1 + \beta_2 \), where \( \beta_1 < n/2 \) and \( \beta_2 < n/p \), and again using Proposition \( \square \) we have

\[
\| \nabla u \nabla v \|_{B_{p,q}^{n/2-1}} \leq \| u \|_{B_{p,q}^{\beta_1 + 1}} \| v \|_{B_{p,q}^{\beta_2 + 1}} \leq \| u \|_{B_{p,q}^{n/2}} \| v \|_{B_{p,q}^{r}},
\]

where \( r \geq \beta_2 + 1 \).

Recalling that \( b = (r - n/p)/2 \), we get that \( J_1 \) is bounded by

\[
J_1 \leq \sup_{0 \leq t < T} \int_0^t |t - \tau|^{-n/(p-n/2)/2} \| \text{div} \left( (1 - \Delta)^{-1}(\nabla u(\tau) \nabla v(\tau)) \right) \|_{B_{p,q}^{n/2}} d\tau
\]

\[
\leq \sup_{0 \leq t < T} \int_0^t |t - \tau|^{-n/2p} \| u(\tau) \|_{B_{p/q}^{n/2}} \| v(\tau) \|_{B_{p/q}^{r}} d\tau
\]

\[
\leq \sup_{0 \leq t < T} C \| u \|_{0:n/2,2,q} \| v \|_{b:r,p,q} \int_0^t |t - \tau|^{-n/2p} \| T^{-b} \| d\tau
\]

\[
\leq CMT^{-n/2p-b+1} = CMT^{1-r/2},
\]

provided \( b < 1 \) (recall that, by equation \( (3.2) \), \( \| v \|_{b:r,p,q} \) is bounded).

For \( J_2 \), we have

\[
J_2 \leq \sup_{0 < t < T} t^a \| \int_0^t e^{(t-\tau)\Delta} \text{div} \left( (1 - \Delta)^{-1}(\nabla u(\tau) \nabla v(\tau)) \right) d\tau \|_{B_{q/2}^n}
\]

\[
\leq \sup_{0 < t < T} t^a \int_0^t |t - \tau|^{-s-n/2+n/p-n/2)/2} \| \text{div} \left( (1 - \Delta)^{-1}(\nabla u(\tau) \nabla v(\tau)) \right) \|_{B_{p,q}^{n/2}} d\tau
\]

\[
\leq C \sup_{0 < t < T} t^a \| u \|_{0:n/2,2,q} \| v \|_{b:r,p,q} \int_0^t |t - \tau|^{-s-n/2+n/p)/2} \| T^{-b} \| d\tau
\]

\[
\leq CMT^{1-r/2},
\]

provided \( s - n/2 + n/p < 2 \) and \( b < 1 \).
Combining the restrictions, we get that
\[ J = J_1 + J_2 \leq CMT^{1-r/2}, \]
provided
\[ \frac{n}{2} > \beta_1 + 1 \]
\[ \beta_2 < n/p \]
\[ r \geq \beta_2 + 1 \]
\[ 2 > s - n/2 + n/p \]
\[ 2 \geq r. \]

Setting \( \beta_1 = \frac{n}{2} - 1 - \frac{n}{2p} \), \( \beta_2 = \frac{n}{2p} \), and \( r = \beta_2 + 1 \), we get
\[ J = J_1 + J_2 \leq CMT^{1/2-n/4p} \]
provided \( s - n/2 < 1 \).

3.3. **Finishing Theorem** \( \text{[4]} \). From equations \( (3.7) \) and \( (3.8) \), we have that
\[ \| \Psi \|_E \leq I + J < CM(T^{1/2-n/4p} + T^{1/2-\varepsilon}) < CMT^{1/2-n/4p}, \]
for \( p \gg n \). For \( \tilde{\Phi} \), similar calculations yield
\[ \| \tilde{\Phi} \|_E \leq CM^2. \]
Thus
\[ \| \tilde{\Phi} \|_E + \| \Psi \|_E \leq M/2, \]
provided \( M \) and \( T \) are sufficiently small. We remark that the size of \( T \) required here depends only on the parameters and on constants, not on \( u \) or \( M \). For the linear term \( e^{t\Delta}u_0 \), Proposition \( \text{[2]} \) and equation \( (2.8) \) give that
\[ \| e^{t\Delta}u_0 \|_E = \| e^{t\Delta}u_0 \|_{a,s,2,q} < M/2, \]
provided \( T \) is sufficiently small. We note that the desired \( T \) depends only on \( M \) and \( \| u_0 \|_{B^{s/2}_{2,q}} \). So we have that \( \Phi : E \to E \) provided \( M \) and \( T \) are sufficiently small, and \( T \) can be taken taken as a function of \( \| u_0 \|_{B^{s/2}_{2,q}} \). The proof that \( \Phi \) is a contraction follows from the standard contraction mapping argument and will be omitted.

4. **Extension to Global existence**

In this section, we prove the following Theorem.

**Theorem 5.** The solution \( u \) to the mLANS equation given by Theorem \( \text{[4]} \) with initial data \( u_0 \in B^{3/2}_{2,q}(\mathbb{R}^3) \) can be extended to a global solution.

The proof follows from a bootstrapping argument and a priori estimates proven in the next section. We begin here by setting up the bootstrap, and start by assuming the unique local solution \( u \) with \( u(0) = u_0 \in B^{3/2}_{2,q}(\mathbb{R}^3) \) given by Theorem \( \text{[4]} \) satisfies \( u \in BC([0,T_0) : B^{3/2}_{2,q}(\mathbb{R}^3)) \) for some \( T_0 < \infty \). By definition (equation \( (2.9) \)), this means
\[ \sup_{0 \leq t < T_0} \| u(t) \|_{B^{3/2}_{2,q}} = M < \infty. \]
For any \( t \in [0, T_0) \), define \( v_t(0) = u(t) \). Then by Theorem 4 there is a unique solution to the mLANS equation \( v_t \in BC([0, T(\|v_t(0)\|_{B^{3/2}_{2,q}})) : B^{3/2}_{2,q}(\mathbb{R}^3) \) with initial data \( v_t(0) = u(t) \), where \( T(\|v_t(0)\|_{B^{3/2}_{2,q}}) \) indicates that the time interval of the solution depends only on \( \|v_t(0)\|_{B^{3/2}_{2,q}} \). The key fact here is that, since \( \|u(t)\|_{B^{3/2}_{2,q}} = \|v_t(0)\|_{B^{3/2}_{2,q}} \leq M \) for any \( t \), there exists a \( \tilde{T} \) such that \( T(\|v_t(0)\|_{B^{3/2}_{2,q}}) \geq \tilde{T} \), and thus, for any \( t \), \( v_t \) exists on a time interval of at least length \( \tilde{T} \).

By uniqueness, we also have that \( v_t(s) = u(t + s) \), for \( s \in [0, \tilde{T}) \), which means \( u \) exists on the interval \([t, t + \tilde{T})\) for any \( t \in [0, T) \). By choosing \( t^* = T_0 - \tilde{T}/2 \), the original solution \( u \) is extended to \( u \in BC([0, T_1) : B^{3/2}_{2,q}(\mathbb{R}^n)) \), where \( T_1 = T_0 + \tilde{T}/2 \). This completes the bootstrap.

By this bootstrapping argument, given that

\[
(4.1) \quad \sup_{0 \leq t < T} \|u(t)\|_{B^{3/2}_{2,q}} = M < \infty,
\]

the local solution \( u \) can be extended to a time interval \([0, T_1)\), where \( T < T_1 \).

To prove Theorem 5 we will assume for contradiction that our solution \( u \) is not a global solution. This means there exists a \( T^* < \infty \) such that \( u \in BC([0, T) : B^{3/2}_{2,q}(\mathbb{R}^3)) \) for any \( T < T^* \), but \( u \notin BC([0, T^*) : B^{3/2}_{2,q}(\mathbb{R}^3)) \).

To contradict this assumption, we use the following \textit{a priori} results. We first recall, by Corollary 1 that \( u(t) \in B^r_{2,q}(\mathbb{R}^3) \) for any real \( r \). Then by Besov embedding (equation (2.6)), we have

\[
(4.2) \quad \|u(t)\|_{B^{3/2}_{2,q}} \leq \|u(t)\|_{B^2_{2,2}} = \|u(t)\|_{H^{2,2}},
\]

\[
\|u(t)\|_{H^{3,2}} \leq \|u(t)\|_{B^{3+\varepsilon}_{2,q}},
\]

for any \( q \in [1, \infty) \). This means \( u \) satisfies the hypothesis of Theorem 6 (proven in Section 5 below), so by Theorem 6 for any \( a \in (0, T^*) \),

\[
(4.3) \quad \sup_{a \leq t < T^*} \|u(t)\|_{B^{3/2}_{2,q}} = K < \infty.
\]

By assumption, since \( a < T \), we have that \( u \in BC([0, a) : B^{3/2}_{2,q}(\mathbb{R}^3)) \), so we have finally proven that \( u \in BC([0, T^*) : B^{3/2}_{2,q}(\mathbb{R}^3)) \), which provides the desired contradiction, and finishes the proof of Theorem 5 up to the proof of Theorem 6, which is the main result of the next section.

5. Sobolev space \textit{a priori} estimates

As mentioned in the introduction, it is easier to control the long time behavior of solutions to the LANS equation than solutions to the Navier-Stokes equation. More specifically, cancellation in the non-linear terms leads to uniform-in-time bounds on the Sobolev space norms of the solution, which, combined with standard bootstrapping arguments, can extend local solutions to global solutions. The first goal of this section is to prove Theorem 6 an analogous \textit{a priori} bound for the mLANS equation, which completes the proof of Theorem 5. At the end of this section, we address the
abstract interpolation result referenced at the end of the introduction. These results complete the proof of Theorem 2.

Before stating the a priori results, we recall some notation from the previous section and some properties of our small-data global solution $v$ that will be used throughout this section. For notational convenience, we set $(-\Delta) = A$. We let $T^*$ be as in the previous section, and we let $a \in (0, T^*)$. From Theorem 3, Corollary 1, and the Besov space embedding results (equation (2.6)), we have that

\[
(5.1) \quad \sup_{a \leq t < T^*} \|v(t)\|_{H^{r,p}} \leq N < \infty,
\]

and that, for any $\varepsilon > 0$,

\[
(5.2) \quad \sup_{0 \leq t < T^*} \|v(t)\|_{B^{n/p}_{p,q}} < \varepsilon,
\]

provided $\|v_0\|_{B^{n/p}_{p,q}}$ is small enough.

The first a priori result provides a bound for the $H^{1,2}(\mathbb{R}^n)$ norm of a solution $u$ to the mLANS equation.

**Lemma 1.** Let $u$ be a solution to the mLANS equation, with $v$ as described above. Then

\[
\sup_{a \leq t < T^*} \|u(t)\|_{L^2}^2 + \alpha^2 \|u(t)\|_{H^{1,2}}^2 \leq C(N)(\|u(a)\|_{L^2}^2 + \alpha^2 \|u(a)\|_{H^{1,2}}^2),
\]

where $\| \cdot \|_{H^{r,p}}$ denotes the homogeneous Sobolev space norm.

Note that, if $\alpha = 0$, this only provides an $L^2$ bound, which is not sufficient to extend the local solutions to global solutions. The second lemma provides a bound for the $\dot{H}^{2,2}(\mathbb{R}^3)$ norm.

**Lemma 2.** Let $u$ be a solution to the mLANS equation, with $v$ as specified in the beginning of the section. We assume $u(t) \in \dot{H}^{3,2}(\mathbb{R}^3)$ for any $t \in [a, T^*)$. Then

\[
(5.3) \quad \sup_{t \in [a, T^*)} \|u(t)\|_{H^{1,2}} = K < \infty
\]

for some real number $K$.

The combination of these two Lemma’s and the Besov embeddings in equation (2.6) proves the following Theorem.

**Theorem 6.** Let $u$ be a solution to the mLANS equation, with $v$ as specified in the beginning of the section. We assume $u(t) \in H^{3,2}(\mathbb{R}^3)$ for any $t \in [a, T^*)$. Then

\[
(5.4) \quad \sup_{t \in [a, T^*)} \|u(t)\|_{B^{3/2}_{2,q}} = K < \infty
\]

for some real number $K$. 

5.1. **Proof of Lemma 1.** We begin the proof of the Lemma by stating the following equivalent form of the mLANS equation (see Section 3 of [17]):

\[
\begin{align*}
\partial_t (1 + A\alpha^2) u(t) + (1 + A\alpha^2) Au(t) &= -\nabla p - \alpha^2 (\nabla u(t))^T \cdot Au(t) \\
- \nabla u_0 \cdot [(1 + A\alpha^2) u(t)] - (1 + A\alpha^2) (\text{div} (u(t) \otimes v(t))) - \text{div} (\nabla u(t) \nabla v(t))
\end{align*}
\]

Taking the \(L^2\) product of the equation with \(u(t)\), we get

\[
\partial_t (\|u(t)\|_{L^2}^2 + \alpha^2 \|A^{1/2} u(t)\|_{L^2}^2) + \|A^{1/2} u(t)\|_{L^2}^2 + \alpha^2 \|Au(t)\|_{L^2}^2 \\
\leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]

where

- \(I_1 = (\nabla u(t), u(t))\),
- \(I_2 = \alpha^2 \left((\nabla u(t) \Delta u(t), u(t)) + ((\nabla u(t))^T \cdot Au(t), u(t))\right)\),
- \(I_3 = (\nabla p, u(t))\),
- \(I_4 = (\text{div} (u(t) \otimes v(t)), u(t))\),
- \(I_5 = \alpha^2 (\text{div} (u(t) \otimes v(t)), u(t))\),
- \(I_6 = (\text{div} (\nabla u(t) \nabla v(t)), u(t))\).

An application of integration by parts and recalling that \(\text{div} u(t) = 0\) gives that \(I_1 = I_3 = 0\). For \(I_2\), writing it in coordinates (and temporarily suppressing the time dependence), we see that

\[
I_2 = \sum_{i,j=1}^{3} \alpha^2 \int u_i (\partial_x \Delta u_j) u_j + (\Delta u_i) (\partial_x u_i) u_j \\
= \sum_{i,j=1}^{3} \alpha^2 \int -(u_i (\Delta u_j) (\partial_x u_j)) + (\Delta u_i) (\partial_x u_i) u_j = 0,
\]

where we again used integration by parts and exploited the divergence free condition. We remark here that it is these cancellations which make it easier to control the long time behavior of the LANS equations. For \(I_4\), using Holder’s inequality and the Sobolev embedding theorem (and recalling that \(\|\cdot\|_{H^{s,p}}\) denotes the homogeneous Sobolev space norm), we have

\[
I_4 \leq \|\nabla u(t)\|_{L^2} \|u(t) \otimes v(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^p} \|v(t)\|_{L^p} \\
\leq \|\nabla u(t)\|_{L^2} \|u(t)\|_{H^{s,2}} \|v(t)\|_{L^p},
\]

where Holder’s inequality requires \(1/2 = 1/\tilde{p} + 1/p\) and the Sobolev embedding theorem requires \(1/\tilde{p} = 1/2 - s/3\), with \(2s < 3\). Solving the system for \(s\), we get that \(s = 3/p\), and for \(p > 3\), we have that \(s < 1\), so we finally bound \(I_4\) by

\[
I_4 \leq C \alpha^{-2} (\|u(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t)\|_{L^2}^2) \|v(t)\|_{L^p}.
\]
To bound $I_5$, we use integration by parts, the Leibnitz rule, and then Holder’s inequality and Sobolev embeddings as in the estimate of $I_4$ to get

$$I_5 \leq \alpha^2 (\langle Au(t)\rangle v(t), \nabla u(t)) + (\nabla u(t)\nabla v(t), \nabla u(t)) + (u(t)(\Delta v(t)), \nabla u(t))$$

$$\leq \alpha^2 \|u(t)\|_{H^{2,2}} \|v(t)\|_{L^2} + \|\nabla u(t)\|_{L^2} \|\nabla v(t)\|_{L^2} + \|\nabla u(t)\|_{L^2} \|u(t)Au(t)\|_{L^2}$$

$$\leq C\alpha^2 (\|u(t)\|_{H^{2,2}} \|v(t)\|_{L^p} + \|\nabla u(t)\|_{L^2} \|v(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^2} \|Av(t)\|_{L^p})$$

$$\leq C\alpha^2 (\|u(t)\|_{H^{2,2}} \|v(t)\|_{L^p}) + (\|u(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t)\|_{L^2}^2) \|v(t)\|_{H^{2,p}}.$$

For $I_6$, the same type of argument gives

$$I_6 \leq ((\nabla u(t)\nabla v(t)), \nabla u(t)) \leq \|\nabla u(t)\|_{L^2}^2 \|v(t)\|_{H^{1+n/p+\epsilon,p}}$$

$$\leq C\alpha^{-2}(\|u(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t)\|_{L^2}^2) \|v(t)\|_{H^{2,p}}.$$

Plugging the estimates for $I_1$ through $I_6$ back into equation (5.1), we get

$$(5.7) \quad \partial_t (\|u(t)\|_{L^2}^2 + \alpha^2 \|u(t)\|_{L^2}^2) \leq J_1 + J_2 + J_3,$$

where

$$J_1 = C\alpha^{-2}(\|u(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t)\|_{L^2}^2)(\|v(t)\|_{L^p} + \|v(t)\|_{H^{2,p}}),$$

$$J_2 = \alpha^2 C\|u(t)\|_{H^{2,2}} (C\|v(t)\|_{L^p} - 1),$$

$$J_3 = -\|\nabla u(t)\|_{L^2}^2.$$

From equation (5.2), for sufficiently small $\|v_0\|_{B^s_{p,q}}$, we have that $C\|v(t)\|_{L^p} - 1 < 0$ for all $t$. This makes $J_2$ and $J_3$ negative, so (5.7) becomes

$$\partial_t (\|u(t)\|_{L^2}^2 + \alpha^2 \|u(t)\|_{L^2}^2) \leq C\alpha^{-2}(\|u(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t)\|_{L^2}^2)(\|v(t)\|_{L^p} + \|v(t)\|_{H^{2,p}}).$$

Applying Gronwall’s inequality gives

$$\|u(t)\|_{L^2}^2 + \alpha^2 \|u(t)\|_{L^2}^2$$

$$\leq (\|u(a)\|_{L^2}^2 + \alpha^2 \|u(a)\|_{L^2}^2) \exp\{C\alpha^{-2} \int_a^T \|v(s)\|_{H^{2,p}} ds\},$$

and an application of equation (5.1) completes the Lemma. We observe that this result could be extended to higher dimensions by taking more care with the Sobolev embeddings.

5.2. **Proof of Lemma 2.** We first observe that, to prove Lemma 2, it is sufficient to take the supremum over all $t$ such that $\|u(t)\|_{H^{2,2}}$ and $\|u(t)\|_{H^{3,2}}$ are greater than one. For the proof, we start with the standard form of the mLANS equation, apply $A$ to both sides and take the $L^2$ product with $Au$ to get

$$(5.8) \quad (\partial_t Au(t), Au(t)) + (A^2 u(t), Au(t)) = I + J,$$

where

$$I = -(AP^\alpha (\nabla u(t)^2) + \div (1 + A\alpha^2)^{-1}(\nabla u(t)^2 \nabla u(t))), Au(t),$$

$$J = -(AP^\alpha (\div (u(t) \otimes v) + \div (1 + A\alpha^2)^{-1}(\nabla u(t) \nabla v(t))), Au(t)).$$
For the left hand side, we have
\[
(\partial_t Au(t), Au(t)) = \frac{1}{2} \partial_t \|u(t)\|_{H^{2,2}}^2
\]
and
\[
(A^2 u(t), Au(t)) = (A^{3/2} u(t), A^{3/2} u(t)) = \|u(t)\|_{H^{3,2}}^2.
\]

Estimating $I$ and $J$ is significantly harder, and is the subject of the next two subsections.

5.2.1. Estimating $I$. We start by re-writing $I$ as $I = K_1 + K_2$, where
\[
K_1 = -(AP^\alpha(\nabla u(t)Au(t)), Au(t)),
\]
\[
K_2 = -(AP^\alpha \nabla \tau^\alpha u(t), Au(t)).
\]

We will make heavy use of the following Ladyzhenskaya inequality ((5.3) in [7]) which holds in $\mathbb{R}^3$:
\[
\|f\|_{H^{r_1,2}} \leq C\|f\|_{L^1}^{1-r_1/r_2}\|f\|_{L^2}^{r_1/r_2},
\]
Starting with $K_1$, making liberal use of integration by parts, the product rule, and Holder's inequality, we have
\[
|K_1| \leq (A^{1/2}(\nabla u(t)Au(t)), A^{3/2} u(t))
\]
\[
\leq C\|A^{3/2} u(t)\|_{L^2}(\|A^{1/2} \nabla u(t)\|_{L^2} + \|A^{1/2} u(t)\| \nabla u(t)\|_{L^2})
\]
\[
\leq C\|u(t)\|_{H^{3,2}}\|u(t)\|_{L^\infty}\|u(t)\|_{H^{2,2}} + \|A^{1/2} u(t)\|_{L^\infty}\|u(t)\|_{H^{1,2}}.
\]

By Sobolev embedding, we have
\[
\|u\|_{L^\infty} \leq \|u(t)\|_{L^2} + \|u(t)\|_{H^{1,2}},
\]
\[
\|A^{1/2} u(t)\|_{L^\infty} \leq \|\nabla u(t)\|_{L^2} + \|u(t)\|_{H^{2,2}},
\]
provided $k_1 = 3/2 + \varepsilon$ and $k_2 = 5/2 + \varepsilon$ for positive $\varepsilon$. Recalling that Lemma [1] provides a uniform bound of $M$ on $\|u(t)\|_{H^{1,2}}$, we can now bound $K_1$ by
\[
|K_1| \leq C\|u(t)\|_{H^{3,2}}\|u(t)\|_{H^{2,2}}(M + \|u(t)\|_{H^{1,2}})
\]
\[
+ CM\|u(t)\|_{H^{3,2}}(M + \|u(t)\|_{H^{2,2}}).
\]

By (5.11), we have
\[
\|u(t)\|_{H^{2,2}} = \|\nabla u(t)\|_{H^{1,2}} \leq C\|u(t)\|_{H^{1,2}}^{1/2}\|u(t)\|_{H^{3,2}}^{1/2}
\]
\[
\|u(t)\|_{H^{k_1,2}} = \|\nabla u(t)\|_{H^{k_1-1,2}} \leq C\|u(t)\|_{H^{1,2}}^{1-(k_1-1)/2}\|u(t)\|_{H^{3,2}}^{(k_1-1)/2}
\]
\[
\|u(t)\|_{H^{k_2,2}} = \|\nabla u(t)\|_{H^{k_2-1,2}} \leq C\|u(t)\|_{H^{1,2}}^{1-(k_2-1)/2}\|u(t)\|_{H^{3,2}}^{(k_2-1)/2}.
\]

Applying (5.15) to (5.14) and recalling that we have assumed both $\|u(t)\|_{H^{2,2}}$ and $\|u(t)\|_{H^{3,2}}$ are no less than 1, we have
\[
|K_1| \leq C(M)(\|u(t)\|_{H^{3,2}}^{3/2} + \|u(t)\|_{H^{3,2}}^{1+k_1/2} + \|u(t)\|_{H^{3,2}}^{(k_2+1)/2}),
\]
where $C(M)$ indicates that $C$ is a function only of $M$. Choosing $\varepsilon = 1/4$, we get
\[
|K_1| \leq C(M)\|u(t)\|_{H^{3,2}}^{15/8}
\]
which finishes our $K_1$ estimate.

For $K_2$, using Holder’s inequality, we have

$$|K_2| \leq \|u(t)\|_{\dot{H}^{2,2}}^{1/2}(\nabla u(t)\nabla u(t))\|_{L^2}.$$

Using (5.13) and (5.15) gives

$$|K_2| \leq C\|u(t)\|_{\dot{H}^{2,2}}^2(M + \|u(t)\|_{\dot{H}^{4,2}})$$

where the last inequality is due to equation (5.1). Applying this to equation (5.23),

$$|I| \leq C(M)\|u(t)\|_{\dot{H}^{3}}^{15/8}.$$

Applying Young’s multiplicative inequality with $q = 16/15$, we get

$$I \leq \varepsilon(\|u(t)\|_{\dot{H}^3}^{15/8})^{16/15} + C(M)(\varepsilon)^{-1}.$$

Choosing $\varepsilon = 1/4$, our final bound for $I$ is

$$I \leq \frac{1}{4}\|u(t)\|_{\dot{H}^3}^2 + C(M).$$

Now we turn our attention to $J$.

5.2.2. Estimating $J$. As in the preceding subsection, we begin by writing $J$ as $J = L_1 + L_2$, where

$$L_1 = -(AP^\alpha(div (u(t) \otimes v(t))), Au(t)),$$

$$L_2 = -(AP^\alpha(div (1 - \alpha^2\Delta)^{-1}(\nabla u(t)\nabla v(t))), Au(t)).$$

Starting with $L_1$, making liberal use of integration by parts, the product rule, and Holder’s inequality, we have

$$\|L_1\| \leq (A^{1/2}(\nabla u(t)\nabla v(t))), A^{3/2}u(t))$$

Using (5.13) and (5.15) gives

where $\bar{p}$ is as in equation (5.6). By Sobolev embedding (and recalling that $p > 3$), we have

$$\|v(t)\|_{L^\infty} + A^{1/2}(\nabla v(t))\|_{L^p} + \|Av(t)\|_{L^p} \leq \|v(t)\|_{H^{2,p}} < N,$$

where the last inequality is due to equation (5.1). Applying this to equation (5.23), we get

$$|L_1| \leq C(N)\|u(t)\|_{\dot{H}^{3,2}(\|Au(t)\|_{L^2} + A^{1/2}u(t))L^2).$$
As in the estimate for $K_1$, we use (5.15) to get

\begin{align}
|L_1| &\leq C(N)\|u(t)\|_{H^{3,2}}(\|u(t)\|_{H^{1,2}}^{1/2} \|u(t)\|_{H^{3,2}}^{1/2} + \|u(t)\|_{H^{1,2}}) \\
&\leq C(N, M)\|u(t)\|_{H^{3,2}}^{3/2},
\end{align}

where we recall that $\|u(t)\|_{H^{1,2}} \leq M$. This finishes our $L_1$ estimate. For $L_2$, using Holder’s inequality, we have

\begin{align}
|L_2| &\leq \|u(t)\|_{H^{2,2}}^2 A^{1/2}(\nabla u(t) \nabla v(t))\|_{L^2} \\
&\leq C\|u(t)\|_{H^{2,2}}^2 \|\nabla v(t)\|_{L^\infty} \leq C(N)\|u(t)\|_{H^{2,2}}^2.
\end{align}

Using (5.15) gives

\begin{align}
|L_2| &\leq C(N)\|u(t)\|_{H^{3,2}}\|u(t)\|_{H^{1,2}} \leq C(N, M)\|u(t)\|_{H^{3,2}},
\end{align}

and this finishes our work on $L_2$. Combining equations (5.24) and (5.25), we bound $J$ by

\begin{align}
|J| &\leq C(N, M)\|u(t)\|_{H^{3,2}}^{3/2}.
\end{align}

Applying Young’s inequality for products (and choosing $\varepsilon = 1/4$), we get

\begin{align}
|J| &\leq \frac{1}{4}(\|u(t)\|_{H^{3,2}}^{3/2})^{4/3} + C(N, M) = \frac{1}{4}\|u(t)\|_{H^{3,2}}^2 + C(N, M).
\end{align}

We conclude this section by combining equations (5.22) and (5.27) to get

\begin{align}
|I| + |J| &\leq \frac{1}{2}\|u(t)\|_{H^{3,2}}^2 + C(N, M).
\end{align}

5.2.3. Prove of equation (5.3). Returning to equation (5.8), and using (5.9), (5.10), and (5.28), we get

\begin{align}
\partial_t\|u(t)\|_{H^{2,2}}^2 + \|u(t)\|_{H^{3,2}}^2 &\leq \frac{1}{2}\|u(t)\|_{H^{3,2}}^2 + C(N, M).
\end{align}

Subtracting $\|u(t)\|_{H^{3,2}}^2$ from both sides, we finally get

\begin{align}
\partial_t\|u(t)\|_{H^{2,2}} &\leq -\frac{1}{2}\|u(t)\|_{H^{1,2}}^2 + C(N, M) \leq C(N, M).
\end{align}

Integrating from $a$ to $t$, we get

\begin{align}
\|u(t)\|_{H^{2,2}}^2 &\leq \|u(a)\|_{H^{2,2}}^2 + \int_a^T C(N, M)ds \leq K.
\end{align}

Taking the supremum over $t \in [a, T^*)$ gives equation (5.3) and completes the proof of Lemma 2.
5.3. An abstract interpolation result. In this subsection we address the following result.

Lemma 3. Let \( v \) be as in Theorem 5 and let \( u \) be the global solution to the mLANS equation given by Theorem 5. Also assume \( u_0 + v_0 \in [B^{3/p}_{p,q} (\mathbb{R}^3), B^{3/2}_{2,q} (\mathbb{R}^3)] \) for some \( \theta \in (0, 1) \). Then, for all \( t \), \( u(t) + v(t) \in [B^{3/p}_{p,q} (\mathbb{R}^3), B^{3/2}_{2,q} (\mathbb{R}^3)] \) and \( \theta,q \).

This is a specific case of the result proven in Section 4.4 in [1]. As stated in equations (4.11) and (4.12) there, the two key requirements, adapted to this case, are that

\[
B^{3/p}_{p,q} (\mathbb{R}^3) \hookrightarrow [B^{3/p}_{p,q} (\mathbb{R}^3), B^{3/2}_{2,q} (\mathbb{R}^3)] \hookrightarrow B^{3/2}_{2,q} (\mathbb{R}^3),
\]

and that

\[
\| v(t) \|_{B^{3/p}_{p,q}} \leq C \| v_0 \|_{B^{3/p}_{p,q}},
\]

\[
\| u(t) \|_{B^{3/2}_{2,q}} \leq C (\| v_0 \|_{B^{3/2}_{2,q}}) \| u_0 \|_{B^{3/2}_{2,q}}.
\]

The first requirement follows directly from the Besov embeddings in equation (2.6). For the second requirement, the first part follows from Theorem 5. The second part follows from the fact that, by Theorem 5, \( u \in BC([0,T) : B^{3/2}_{2,q} (\mathbb{R}^3) \) for any \( T > 0 \). This result, combined with Theorem 5, completes the proof of Theorem 1.

6. Higher regularity for the local existence result

Here we prove Corollary 2. The proof is an induction argument, similar to the one in [10] applied to the LANS equation (which was in turn inspired by the argument in [5] for the Navier-Stokes equation).

As usual, before stating the theorem, we construct a solution to the LANS equation \( v \). Here, we pick \( p > n \), and let \( v_0 \in B^{n/p}_{p,q} (\mathbb{R}^n) \) with \( \| v_0 \|_{B^{n/p}_{p,q}} \) arbitrarily small, so by Theorem 5 and Corollary 1 we have a global solution \( v \) to the LANS equation where \( v \in BC([0,T) : B^{n/p}_{p,q} (\mathbb{R}^n) \cap \hat{C}^T_{\alpha,r,p,q}) \), with \( \alpha = (r - n/p)/2 \) for any real \( r > n/p \).

Lemma 4. With \( v \) as in the preceding paragraph, let \( u_0 \in B^{n/2}_{2,q} (\mathbb{R}^n) \) and let \( u \) be the associated unique solution to the mLANS equation with initial data \( u_0 \) such that

\[
u \in BC([0,T) : B^{n/2}_{2,q} (\mathbb{R}^n) \cap \hat{C}^T_{(s-n/p)/2:s,2,q},
\]

where \( 0 < s - n/2 < 1 \). Then for all \( k \geq s \), we have that \( u \in \hat{C}^T_{(k-n/2)/2:k,2,q} \).

Proof. We start with the solution to the mLANS equation \( u \). Then let \( \delta > 0 \) be arbitrary, and let \( w = t^\delta u \). We note that \( w(0) = 0 \). Then

\[
\partial_t w = \delta t^{\delta-1} u + t^\delta \partial_t u
\]

\[
= \delta t^{-1} w + t^\delta (\Delta u - \text{div} (u \otimes u + \tau^\alpha(u,u)) - \text{div} (u \otimes v + \tau^\alpha(u,v)))
\]

\[
= \delta t^{-1} w + \Delta w - t^{-\delta} \text{div} (w \otimes w + \tau^\alpha(w,w)) - \text{div} (w \otimes v + \tau^\alpha(w,v)).
\]
Applying Duhamel’s principle, we get
\[ w = e^{t\Delta}w_0 + \int_0^t e^{(t-s)\Delta}s^{-1}w(s)ds + \int_0^t e^{(t-s)\Delta}s^{-\delta}(\text{div}(w(s) \otimes w(s) + \tau^\alpha(w(s), w(s))))ds \\
+ \int_0^t e^{(t-s)\Delta}(\text{div}(w(s) \otimes v(s) + \tau^\alpha(w(s), v(s))))ds. \]

Recalling that \( w(0) = w_0 = 0 \), and substituting \( w = t^\delta u \), we get
\[ u = t^{-\delta} \int_0^t e^{(t-s)\Delta}s^{\delta-1}u(s)ds + t^{-\delta} \int_0^t e^{(t-s)\Delta}s^{\delta}(\text{div}(u(s) \otimes u(s) + \tau^\alpha(u(s), u(s))))ds \\
+ t^{-\delta} \int_0^t e^{(t-s)\Delta}s^{\delta}(\text{div}(u(s) \otimes v(s) + \tau^\alpha(u(s), v(s))))ds. \]

Now we are ready to apply the induction. We have by assumption that \( u \) is in \( \hat{C}^T_{(s-n/2),2;2,q} \), where \( s > 1 \). For induction, we assume this solution \( u \) is also in \( \hat{C}^T_{(k-n/2),2;2,q} \), and seek to show that \( u \) is in \( \hat{C}^T_{(k+h-n/2),2;2,q} \), where \( 0 < h < 1 \) is fixed and will be chosen later. We have
\[ \|u\|_{B^{k+h}_{2,q}} \leq I + J_1 + J_2 + K_1 + K_2, \]

with \( I, J_1, J_2, K_1 \), and \( K_2 \) defined by
\[ I = t^{-\delta} \int_0^t \|e^{(t-s)\Delta}s^{\delta-1}u(s)\|_{B^{k,h}_{2,q}}ds \\
J_1 = t^{-\delta} \int_0^t \|e^{(t-s)\delta}s^{\delta}(\text{div}(1 - \alpha^2\Delta)^{-1}(\nabla u(s)\nabla u(s)))\|_{B^{k,h}_{2,q}}ds \\
J_2 = t^{-\delta} \int_0^t \|e^{(t-s)\delta}s^{\delta}(\text{div}(u(s) \otimes u(s)))\|_{B^{k,h}_{2,q}}ds \\
K_1 = t^{-\delta} \int_0^t \|e^{(t-s)\delta}s^{\delta}(\text{div}(1 - \alpha^2\Delta)^{-1}(\nabla u(s)\nabla v(s)))\|_{B^{k,h}_{2,q}}ds \\
K_2 = t^{-\delta} \int_0^t \|e^{(t-s)\delta}s^{\delta}(\text{div}(u(s) \otimes v(s)))\|_{B^{k,h}_{2,q}}ds, \]

where, as usual, we have suppressed terms from \( \tau^\alpha \) that are controlled by the terms we included. The \( I, J_1, \) and \( J_2 \) terms are the terms from the LANS equation, while \( K_1 \) and \( K_2 \) are the terms resulting from the modification of the LANS equation. We address \( I, J_1, \) and \( J_2 \) first.

6.1. **Bounding \( I, J_1, \) and \( J_2.** Starting with \( I \), we have
\[ I \leq t^{-\delta} \int_0^t |t-s|^{-h/2}s^{\delta-1}\|u(s)\|_{B^{k,h}_{2,q}}ds \leq t^{-\delta}\|u\|_{(k-n/2),2,k,2,q} \int_0^t |t-s|^{-h/2}s^{\delta-1-(k-n/2)/2}ds \leq C\|u\|_{(k-n/2),2,k,2,q}t^{-\delta}t^{-h/2}t^{\delta-1-(k-n/2)/2+1} \leq C\|u\|_{(k-h-n/2)/2}t^{-(k+h-n/2)/2+1}, \]
provided
\[ 1 > h/2, \]
\[ -1 < \delta - 1 - (k - n/2)/2, \]
which clearly holds for sufficiently large \( \delta \). We observe that, without modifying the PDE to include these \( t^b \) terms, we would need \( (k - n/2)/2 \) to be less than 1, which does not hold for large \( k \).

For \( J_1 \), we choose \( \tilde{r} = n/2 - 1 - \varepsilon \), and with \( n/p = n/2 + \tilde{r} \), we have
\[
J_1 \leq t^{-\delta} \int_0^t |t - s|^{-(h+n/p-n/2)/2} s^{\delta} \| \text{div} (1 - \triangle)^{-1} (\nabla u \nabla u) \|_{B^k_{p,q}} ds
\]
\[
\leq t^{-\delta} \int_0^t |t - s|^{-(h-n/2+\tilde{r})/2} s^{\delta} \| \nabla u \|_{B^k_{p,q}} ds
\]
\[
\leq t^{-\delta} \int_0^t |t - s|^{-(h-n/2+\tilde{r})/2} s^{\delta} \| \nabla u \|_{B^{k-1}_{p,q}} \| \nabla u \|_{B^\varepsilon_{p,q}} ds
\]
\[
(6.2)
\]
\[
\leq t^{-\delta} \int_0^t |t - s|^{-(h-n/2+\tilde{r})/2} s^{\delta} \| u \|_{B^k_{2,q}} \| u \|_{B^{n/2-\varepsilon}_{2,q}} ds
\]
\[
\leq t^{-\delta} \| u \|_{(k-n/2)/2;2,k,2,q} \| u \|_{0,n/2,2,q} \int_0^t |t - s|^{-(h+n/2-\tilde{r})/2} s^{\delta-(k-n/2)/2} ds
\]
\[
\leq t^{-(k-n/2)/2+(1-\varepsilon)/2} \| u \|_{(k-n/2)/2;2,k,2,q} \| u \|_{0,n/2,2,q},
\]
provided
\[ \delta > (k - n/2)/2 + (1 - n/2)/2, \]
\[ 2 > h + n/2, \]
and we again see that this is easily satisfied by choosing \( \delta \) large and \( h \) small. For \( J_2 \), we define \( s = n/2 - \varepsilon \) and \( n/p = n/2 + \tilde{s} \), and have
\[
J_2 \leq t^{-\delta} \int_0^t |t - s|^{-(h+1+n/p-n/2)/2} s^{\delta} \| u \otimes u \|_{B^k_{2,q}} ds
\]
\[
\leq t^{-\delta} \int_0^t |t - s|^{-(h+1+n/2-\tilde{s})/2} s^{\delta} \| u \|_{B^k_{2,q}} ds
\]
\[
\leq t^{-\delta} \| u \|_{(k-n/2)/2;2,k,2,q} \| u \|_{0,n/2,2,q} \int_0^t |t - s|^{-(h+1-\varepsilon)/2} s^{\delta-(k-n/2)/2} ds
\]
\[
\leq t^{-(h+k-n/2)/2+1/2+\varepsilon} \| u \|_{(k-n/2)/2;2,k,2,q} \| u \|_{0,n/2,2,q},
\]
provided
\[ 1 > h - \varepsilon, \]
\[ -1 < \delta - (k - n/2)/2. \]

Combining equations \((6.1)\), \((6.2)\) and \((6.3)\), we have that, for \( h \) small enough and \( \delta \) large enough,
\[
I + J_1 + J_2 \leq Ct^{-(h+k-n/2)/2} \| u \|_{(k-n/2)/2;2,k,2,q} \| u \|_{0,n/2,2,q}.
\]
\[
(6.4)
\]
Now we turn our attention to \( K_1 \) and \( K_2 \).

6.2. **Bounding \( K_1 \) and \( K_2 \).** Starting with \( K_1 \), Defining \( n/\tilde{p} = n/p + n/2 \) and \( a = (r - n/p)/2 \), we have

\[
\begin{align*}
&\quad t^{-\delta} \int_0^t |t - s|^{-(h+n/\tilde{p}-n/2)/2} s^\delta \|\text{div } (1 - \Delta)^{-1} \nabla u \nabla v\|_{B_{p,q}^k} \, ds \\
\leq & t^{-\delta} \int_0^t |t - s|^{-(h+n/p)/2} s^\delta \|\nabla u \nabla v\|_{B_{p,q}^{k-1}} \, ds \\
\leq & t^{-\delta} \int_0^t |t - s|^{-(h+n/p)/2} s^\delta \|\nabla u\|_{B_{p,q}^{k-1}} \|\nabla v\|_{L^p} \, ds \\
\leq & t^{-\delta} \|u\|_{(k-n/2)/2;k,2,q} \|v\|_{a;r,p,q} \int_0^t |t - s|^{-(h+n/p)/2} s^\delta -(k-n/2)/2-a \, ds \\
\leq & t^{-(h+k-n/2)/2+1-r/2} \|u\|_{(k-n/2)/2;k,2,q} \|v\|_{a;r,p,q},
\end{align*}
\]

provided

\[
\begin{align*}
\delta & > (k-n/2)/2 + (1-n/p)/2, \\
r & < 2, \\
2 & > h + n/p,
\end{align*}
\]

all of which are easily satisfied by a sufficiently large choice of \( \delta \) and a sufficiently small choice of \( h \).

For \( K_2 \), we have

\[
\begin{align*}
&\quad t^{-\delta} \int_0^t |t - s|^{-(h+1+n/\tilde{p}-n/2)/2} s^\delta \|u \otimes u\|_{B_{p,q}^k} \, ds \\
\leq & t^{-\delta} \int_0^t |t - s|^{-(h+1+n/p)/2} s^\delta \|u\|_{B_{p,q}^k} \|v\|_{L^p} \, ds \\
\leq & t^{-\delta} \|u\|_{(k-n/2)/2;k,2,q} \|v\|_{0;n/p,q} \int_0^t |t - s|^{-(h+1+n/p)/2} s^\delta -(k-n/2)/2 \, ds \\
\leq & t^{-(h+k-n/2)/2+(1-n/p)/2} \|u\|_{(k-n/2)/2;k,2,q} \|v\|_{0;n/p,q},
\end{align*}
\]

provided

\[
\begin{align*}
1 & > h + n/p, \\
\delta & > (k-n/2)/2.
\end{align*}
\]

So we have that

\[
K_1 + K_2 \leq t^{-(h+k-n/2)/2} \|u\|_{(k-n/2)/2;k,2,q} \|v\|_{(r-n/p)/2;r,p,q}.
\]
6.3. **Finishing the proof of Lemma 4.** Using equations (6.4) and (6.7), we get
\[ \|u\|_{H^{k+h,p}} \leq Ck^{(k+h-n/2)/2} \left\| u \right\|_{(k-n/2)/2;k,2,q}(\|u\|_{0;n/2,2,q} + \|v\|_{(r-n/p)/2;r,p,q}) \]
which immediately gives
\[ \|u\|_{(k+h-n/2)/2;k+h,p} \leq C \left\| u \right\|_{(k-n/2)/2;k,2,q}(\|u\|_{0;n/2,2,q} + \|v\|_{(r-n/p)/2;r,p,q}), \]
which proves the desired result. We remark that \( \delta \) is chosen after beginning the
induction step, while the appropriate value of \( h \) is fixed by the choices of \( n, p, \) and \( n/2. \)

\[ \square \]

7. **Appendix: A Modified Product Estimate**

In this appendix we prove Proposition 1, which can be found in Corollary 1.3.1 in [2]. Before beginning, we establish another result for the Littlewood-Paley operators and make a slight notational change. First, we observe that, by changing variables,
\[ (7.1) \quad \left\| \psi_j \right\|_{L^p} \leq 2^{jn/p'} \left\| \psi_0 \right\|_{L^p} \leq C 2^{jn/p'}, \]
where \( p' \) is the Holder’ conjugate to \( p, \) i.e. \( 1 = 1/p + 1/p'. \)

Next, we make a slight notational change. For \( j > 0, \) we leave \( \psi_j \) as defined in
Section 2. For \( j = 0, \) we set \( \psi_0 = \Psi, \) so \( \hat{\psi}_0 \) is now supported on the ball centered at
the origin of radius \( 1/2 \) and \( \Delta_0 f = \psi_0 \ast f = \Psi \ast f. \) Then the Besov norm can be
defined by
\[ \left\| f \right\|_{B^r_{p,q}} = \left( \sum_{j=0}^{\infty} 2^{rjq} \left\| \Delta_j u \right\|^q_{L^p} \right)^{1/q}. \]
We are now ready to prove Proposition 1.

**Proposition 1.** We start by taking the \( L^p \) norm of equation (2.5), and get:
\[ \left\| \Delta_j(fg) \right\|_{L^p} \leq \sum_{k=-3}^{3} \left\| \Delta_j(S_{j+k-3}f \Delta_j+k g) \right\|_{L^p} + \sum_{k=-3}^{3} \left\| \Delta_j(S_{j+k-3}g \Delta_j+k f) \right\|_{L^p} \]
\[ + \sum_{k>j-4} \left\| \Delta_j \left( \Delta_k \sum_{l=-2}^{2} \Delta_{k+l}g \right) \right\|_{L^p}. \]
We first observe that, without loss of generality, we can set \( k = l = 0 \) in the finite
sums and replace \( k > j - 4 \) with \( k > j. \) Doing so, we get
\[ \left\| \Delta_j(fg) \right\|_{L^p} \leq \left\| \Delta_j(S_{j-3}f \Delta_jg) \right\|_{L^p} + \left\| \Delta_j(S_{j-3}g \Delta_jf) \right\|_{L^p} \]
\[ + \sum_{k>j} \left\| \Delta_j \left( \Delta_k \Delta_jg \right) \right\|_{L^p}. \]
Starting with the first term, and defining $\tilde{p}$ by $1 + 1/p = 1/\tilde{p} + 1/p_2$, we have
\[ \|\Delta_j(S_{j-3}f\Delta_j g)\|_{L^p} \leq \|\tilde{\psi}_j\|_{L^{\tilde{p}}} \|\Delta_j S_{j-3}g\|_{L^{p_2}} \leq C^2 2^{jn/\tilde{p}} \|\Delta_j g\|_{L^{p_2}} \|S_{j-3}f\|_{L^\infty} \]
\[ \leq C 2^{jn/\tilde{p}} \|\Delta_j g\|_{L^{p_2}} \sum_{m<j-3} \|\Delta_m f\|_{L^\infty} \]
\[ \leq C 2^{jn(1/p_2-1/p)/\tilde{p}} \|\Delta_j g\|_{L^{p_2}} \sum_{m<j-3} 2^{mn/p_1} \|\Delta_m f\|_{L^{p_1}}, \]

where we used Young’s inequality, equation (7.1), Holder’s inequality, and finally Bernstein’s inequality.

A similar calculation for the second term yields
\[ \|\Delta_j(S_{j-3}g\Delta_j f)\|_{L^p} \leq C 2^{jn(1/p_1-1/p)/\tilde{p}} \|\Delta_j f\|_{L^{p_2}} \sum_{m<j-3} 2^{mn/p_2} \|\Delta_m g\|_{L^{p_1}}. \]

For the third term, we have
\[ \sum_{k>j} ||\Delta_j(\Delta_k f \Delta_k g)||_{p_1} \leq \|\tilde{\psi}_j\|_{L^{\tilde{p}}} \sum_{k>j} ||\Delta_k u \Delta_k v||_{L^q} \]
\[ \leq 2^{jn/\tilde{p}} \sum_{k>j} ||\Delta_k f||_{p_1} ||\Delta_k g||_{p_2} \]
\[ \leq 2^{jn(1/p_1-1/p)/\tilde{p}} \sum_{k>j} ||\Delta_k f||_{p_1} ||\Delta_k g||_{p_2}, \]

where $1 + 1/p = 1/\tilde{q} + 1/q$ and $1/q = 1/p_1 + 1/p_2$.

So we have that
\[ \|\Delta_j(fg)\|_{L^p} \leq 2^{jn(1/p_2-1/p)/\tilde{p}} \|\Delta_j g\|_{L^{p_2}} \sum_{m<j-3} 2^{jn/p_1} \|\Delta_m f\|_{L^{p_1}} \]
\[ + 2^{jn(1/p_1-1/p)/\tilde{p}} \|\Delta_j f\|_{L^{p_1}} \sum_{m<j-3} 2^{jn/p_2} \|\Delta_m g\|_{L^{p_2}} \]
\[ + 2^{jn(1/p_1-1/p)/\tilde{p}} \sum_{k>j} ||\Delta_k f||_{p_1} ||\Delta_k g||_{p_2} \]

Multiplying (7.2) by $2^{j(s_1+s_2-n(1/p_2+1/p_1-1/p))}$ and taking the $l^q$ norm in $j$, we get
\[ \|fg\|_{B_{p,q}^n} \leq I + J + K, \]

where
\[ I = \left( \sum_j 2^{(s_1+s_2-n/p_1)j} \|\Delta_j g\|_{L^{p_2}}^{q} \left( \sum_{m<j-3} 2^{mn/p_1} \|\Delta_m f\|_{L^{p_1}}^{q} \right) \right)^{1/q}, \]
\[ J = \left( \sum_j 2^{(s_1+s_2-n/p_2)j} \|\Delta_j f\|_{L^{p_1}}^{q} \left( \sum_{m<j-3} 2^{mn/p_2} \|\Delta_m g\|_{L^{p_2}}^{q} \right) \right)^{1/q}, \]
\[ K = \left( \sum_j (2^{j(s_1+s_2)} \sum_{k>j} ||\Delta_k f||_{p_1} ||\Delta_k g||_{p_2})^{q} \right)^{1/q}. \]
For $I$, we have

\[ I \leq \left( \sum_j 2^{(s_1+s_2-n/p_1)j} \left\| \triangle_j g \right\|_{L^{p_2}}^q \left( \sum_{m<j-3} 2^{jn/p_1} \left\| \triangle_m f \right\|_{L^{p_1}}^q \right) \right)^{1/q} \]

\[ \leq \left( \sum_j (2^{js_2} \left\| \triangle_j g \right\|_{L^{p_2}})^q \left( \sum_{m<j-3} 2^{m(n/p_1+s_1-n/p_1)2(j-m)(s_1-n/p_1)} \left\| \triangle_m f \right\|_{L^{p_1}}^q \right) \right)^{1/q} \]

\[ \leq \left\| f \right\|_{B^{s_1}_{p_1,\infty}} \sum_k 2^{-(s_1-n/p_2)} \left( \sum_j (2^{js_2} \left\| \triangle_j g \right\|_{L^{p_2}}^q) \right)^{1/q} \]

\[ \leq \left\| f \right\|_{B^{s_1}_{p_1,\infty}} \left( \sum_k 2^{-(s_1-n/p_2)} \left( \sum_j 2^{js_2} \left\| \triangle_j g \right\|_{L^{p_2}}^q \right)^{1/q} \right) \]

provided $s_1 < n/p_1$. A similar calculation for $J$ yields

\[ J \leq \left\| f \right\|_{B^{s_1}_{p_1,\infty}} \left\| g \right\|_{B^{s_2}_{p_2,q}}, \]

provided $s_2 < n/p_2$. For $K$, we have, using Young’s inequality for sums,

\[ K = \left( \sum_j \left( \sum_{k>j} 2^{(j-k)(s_1+s_2)} 2^{k s_1} \left\| \triangle_k f \right\|_{L^{p_1}} 2^{k s_2} \left\| \triangle_k g \right\|_{L^{p_2}} \right)^q \right)^{1/q} \]

\[ \leq \left\| g \right\|_{B^{s_2}_{p_2,\infty}} \left( \sum_j \left( \sum_{k>j} 2^{(j-k)(s_1+s_2)} 2^{k s_1} \left\| \triangle_k f \right\|_{L^{p_1}} \right)^q \right)^{1/q} \]

\[ \leq \left\| g \right\|_{B^{s_2}_{p_2,\infty}} \sum_k 2^{-(s_1+s_2)} \left( \sum_k 2^{k s_1} \left\| \triangle_k f \right\|_{L^{p_1}} \right)^q \]

\[ \leq \left\| f \right\|_{B^{s_1}_{p_1,\infty}} \left\| g \right\|_{B^{s_2}_{p_2,q}} + \left\| B^{s_2}_{p_2,q} \right\| \]

provided $s_1 + s_2 > 0$. This finishes the proposition. \hfill \Box

References

1. C. P. Calderón, *Existence of weak solutions for the Navier-Stokes equations with initial data in $L^p$*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 179–200.

2. J.Y. Chemin, *About the navier-stokes system*, Publications du Laboratoire d’analyse numérique (1996).

3. S. Chen, D. D. Holm, L. Margolin, and R. Zhang, *Direct numerical simulations of the Navier-Stokes alpha model*, Phys. D **133** (1999), no. 1-4, 66–83, Predictability: quantifying uncertainty in models of complex phenomena (Los Alamos, NM, 1998).

4. I. Gallagher and F. Planchon, *On global infinite energy solutions to the Navier-Stokes equations in two dimensions*, Arch. Ration. Mech. Anal. **161** (2002), no. 4, 307–337.

5. T. Kato, *The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity*, Differential Integral Equations **7** (1994), no. 3-4, 949–966.

6. J. Marsden, T. Ratiu, and S. Shkoller, *The geometry and analysis of the averaged Euler equations and a new diffeomorphism group*, Geom. Funct. Anal. **10** (2000), no. 3, 582–599.

7. J. Marsden and S. Shkoller, *Global well-posedness for the Lagrangian averaged Navier-Stokes equations on bounded domains*, Phil. Trans. R. Soc. Lond. (2001), no. 359, 1449–1468.
8. [Author], *The Anisotropic Lagrangian Averaged Euler and Navier-Stokes equations*, Arch. Rational Mech. Anal. (2003), no. 166, 27–46.

9. K. Mohseni, B. Kosović, S. Shkoller, and J. Marsden, *Numerical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence*, Phys. Fluids 15 (2003), no. 2, 524–544.

10. N. Pennington, *Lagrangian Averaged Navier-Stokes equation with rough data in Sobolev space*, http://arxiv.org/abs/1011.1856v2.

11. [Author], *Local and global existence for the Lagrangian Averaged Navier-Stokes equation in besov spaces*, http://arxiv.org/abs/1109.1836.

12. S. Shkoller, *On incompressible averaged Lagrangian hydrodynamics*, E-print, (1999), http://xyz.lanl.gov/abs/math.AP/9908109.

13. [Author], *Analysis on groups of diffeomorphisms of manifolds with boundary and the averaged motion of a fluid*, Journal of Differential Geometry (2000), no. 55, 145–191.

14. T. Tao, *Nonlinear Dispersive Equations*, American Mathematical Society, 2006.

NATHAN PENNINGTON, DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, 138 CARDWELL HALL, MANHATTAN, KS-66506, USA.

E-mail address: npenning@math.ksu.edu