A REGULARITY THEORY FOR QUASI-LINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN WEIGHTED SOBOLEV SPACES

ILDOO KIM AND KYEONG-HUN KIM

Abstract. We study the second-order quasi-linear stochastic partial differential equations (SPDEs) defined on $C^1$ domains. The coefficients are random functions depending on $t$, $x$ and the unknown solutions. We prove the uniqueness and existence of solutions in appropriate Sobolev spaces, and in addition, we obtain $L^p$ and Hölder estimates of both the solution and its gradient.

1. INTRODUCTION

In this article we present a weighted Sobolev space theory of the following stochastic partial differential equation (SPDE):

$$
\begin{align*}
\frac{du}{dt} & = \left[ D_i \left( a^{ij}(t, x, u) u_x^j + b^i(t, x, u) u + f^i \right) + \tilde{b}^i(t, x, u) u_x^i + c(t, x, u) u + f \right] dt \\
& \quad + (\nu^k(t, x) u + g^k) dW^k_t, \quad t \leq \tau, \quad x \in \mathcal{O}; \quad u(0, \cdot) = u_0.
\end{align*}
$$

(1.1)

Here $\tau$ is an arbitrary bounded stopping time, $\mathcal{O}$ is a bounded $C^1$-domain in $\mathbb{R}^d$, and $W^k_t (k = 1, 2, \cdots)$ are independent one-dimensional Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$. The indices $i$ and $j$ move from 1 to $d$, and $k$ runs through $\{1, 2, 3, \cdots \}$. The Einstein’s summation convention with respect to $i$ and $j$ is assumed throughout the article. All the coefficients are random, the coefficients $a^{ij}, \tilde{b}^i, \nu^k, c$ depend also on $t, x$ and the unknown $u$, and the coefficients $\nu^k (k = 1, 2, \cdots)$ depend on $\omega, t$, and $x$. We assume that the coefficients are only measurable with respect to $(\omega, t)$, Hölder continuous with respect to $x$, and Lipschitz continuous with respect to the unknown $u$.

Let $u(t, x)$ denote the density of diffusing particles at the time $t$ and the location $x$. Typically, the flux density $F(t, x)$ is proportional to $-\nabla u$ or more generally to $-\sum_j a^{ij} u_x^j$, and the classical heat equation $u_t = D_i (a^{ij} u_x^j)$ is a consequence of the relation $u_t = -\text{div} F$. Then motivation of studying equation (1.1) is obvious since the diffusion coefficients $a^{ij}$ related to the flux density $F(t, x)$ can depend also on their point density $u(t, x)$. Our equation is this type general equation with noises and random external forces. The external forces $f^i, f,$ and $g$ are contained

2000 Mathematics Subject Classification. 60H15, 35R60.

Key words and phrases. Quasilinear stochastic partial differential equations, Equations of divergence type, Weighted Sobolev space.

The research of the first author was supported by the TJ Park Science Fellowship of POSCO TJ Park Foundation.

The research of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (20120005168).
in a weighted Sobolev space. More precisely,
\[ f^i \in H^\gamma_{p,d}(O, \tau) := L_p(O) \times (0, \tau), \]
where \( p > d + 2 \) and \( \gamma \in ((d + 2)/p, 1) \). The spaces \( H^\gamma_{p,d}(O, \tau) \) and \( H^\gamma_{p,d}(O, \tau) \) are introduced in Section 2. We only remark that if \( \gamma \) is a natural number then \( u \in H^\gamma_{p,d}(O, \tau) \)
\[ |\rho^{\gamma - \varepsilon} u|^C_{\gamma - \frac{1}{p} \times \gamma - \frac{2 \kappa - d}{p} \times \gamma - \frac{d}{p} \times \gamma - \frac{d - \kappa}{p}} < \infty, \quad \forall \varepsilon \in [0, 1 - 2\kappa - d/p]. \]
(1.2)
\[ |\rho^{\gamma - \varepsilon'} u_x|^C_{\gamma - \frac{1}{p} \times \gamma - \frac{2 \kappa - d}{p} \times \gamma - \frac{d}{p} \times \gamma - \frac{d - \kappa}{p}} < \infty, \quad \forall \varepsilon' \in [0, 1 - 2\beta - d/p]. \]
(1.3)
Note that (1.2) is a much better result than the one without \( \rho \). For instance, (1.2) with \( \varepsilon = 1 - 2\kappa - d/p \) implies \( \sup_{s \leq t} |u(s, x)| = O(\rho^{1 - d/p - 2\kappa}(x)) \). This shows how fast \( u(t, x) \to 0 \) as \( \rho(x) \to 0 \). By taking \( p \to \infty \) one can make \( 1 - 2\kappa - d/p \) (the Hölder exponent of \( u \) in space variable) as close to 1 as one wishes, and similarly \( \kappa_1 - 1/p \) (the Hölder exponent of \( u \) in time variable) can be any number in \((0, 1/2)\).

Below we introduce some related results handling divergence or non-divergence type SPDEs whose leading coefficients depend also on the solution \( u \). The 1-dimensional non-divergence type equation
\[ du = a(t, x, u) u'' dt + (b(t, x) u' + h(t, x) u) dW_t, \quad t \leq \tau, \quad x \in \mathbb{R}. \]
is studied \([1]\) under the assumption that coefficients \( a, b, h \) are infinitely differentiable with bounded derivatives. A similar equation
\[ du = [a(t, x, u) u'' + f(t, x)] dt + g(t, x) dW^k_t. \]
is studied in \([18]\). Compared to \([1]\), the condition on \( a \) is much weaker in \([18]\). Here the diffusion coefficient \( a(t, x, u) \) is Hölder continuous in \( t \), differentiable in \( x \), and twice continuously differentiable in \( u \). However both \([1]\) and \([18]\) considered only one-dimensional equation. In \([4]\), we obtained \( L_p \) and Hölder estimates for the divergence type equation
\[ du = \left[ D_i \left( a^{ij}(t, x, u) u_{x^i} + f^i \right) \right] dt + g^k(t, x) dW^k_t, \quad t \leq \tau, \quad x \in \mathcal{O}, \]
where \( a^{ij}(t, x, u) \) are Hölder continuous in \( x \) and twice continuously differentiable in \( u \). The present article is a generalization of \([4]\). Firstly, we generalize the equation. We have multiplicative noises in the stochastic part of \([11]\) together with nonlinear lower order terms of solutions in the deterministic part. Secondly, our smoothness conditions on the coefficients are weaker than those in \([4]\). We only impose the
Lipschitz continuity to \( a^{ij}(t, x, u) \) with respect to \( u \). Thirdly, our \( L_p \) and Hölder theory work for any \( p > d + 2 + \gamma_0 \in ((d + 2)/p, 1) \), whereas in [4], \( p \) and \( \gamma_0 \) are some constants (hard to know exactly) coming from the deterministic theory.

Our approach is based on a weighted Sobolev space theory for divergence type linear SPDEs. It might be possible to study equation \( (1.1) \) using an infinite dimensional SDE theory so-called variational approach. See, for instance, [12, 14] and references therein. The monotonicity or local monotonicity condition is crucial in such theory. It is easy to check that the operator \( D_i (a^{ij}(t, x, u) u_{x_i}) \) does not satisfy the monotonicity condition, but it is not clear to us if local monotonicity condition holds for this operator. Regardless of the possibility of using the variational approach, our approach has many advantages. In particular, it provides \( L_p \) and Hölder estimates of both the solution and its gradient. Furthermore, as can be seen in [1.2] and [1.3], it provides very delicate behaviors of the solution and its derivatives near the boundary.

This paper is organized as follows. We introduce our main results and related function spaces in Section 2. In Section 3, we collect some auxiliary results related to linear SPDEs. The (time) local well-posedness of equation \( (1.1) \) is given in Section 4. Finally, the proof of the main theorem is presented in Section 5.

We finish the introduction with notation used in the article. \( N \) and \( Z \) denote the natural number system and the integer number system, respectively. As usual, \( \mathbf{R}^d \) stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \), \( \mathbf{R}_+^d := \{ x = (x^1, \ldots, x^d) \in \mathbf{R}^d : x^1 > 0 \} \) and \( B_r(x) := \{ y \in \mathbf{R}^d : |x - y| < r \} \). For \( i = 1, \ldots, d \), multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i \in \{0, 1, 2, \ldots \} \), and functions \( u(x) \) we set

\[
\frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_{\alpha_1} u \cdots D_{\alpha_d} u, \quad \nabla u = (u_{x^1}, u_{x^2}, \ldots, u_{x^d}).
\]

We also use the notation \( D^m \) for a partial derivative of order \( m \) with respect to \( x \).

For \( p \in [1, \infty) \), a normed space \( F \) and a measure space \( (X, \mathcal{M}, \mu) \), \( L_p(X, \mathcal{M}, \mu; F) \) denotes the space of all \( F \)-valued \( \mathcal{M}^\mu \)-measurable functions \( u \) so that

\[
\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|^p_F \mu(dx) \right)^{1/p} < \infty,
\]

where \( \mathcal{M}^\mu \) denotes the completion of \( \mathcal{M} \) with respect to the measure \( \mu \). For \( p = \infty \), we write \( u \in L_\infty(X, \mathcal{M}, \mu; F) \) iff

\[
\sup_X |u(x)| := \|u\|_{L_\infty(X, \mathcal{M}, \mu; F)} := \inf \{ \nu \geq 0 : \mu(\{x: \|u(x)\|_F > \nu\}) = 0 \} < \infty.
\]

If there is no confusion for the given measure and \( \sigma \)-algebra, we usually omit the measure and the \( \sigma \)-algebra. If we write \( N = N(a, b, \cdots) \), this means that the constant \( N \) depends only on \( a, b, \cdots \). For functions depending on \( \omega, t, x \), the argument \( \omega \in \Omega \) will be usually omitted. We say that a stopping time \( \tau \) is nonzero iff \( P(\{\tau \neq 0\}) > 0 \).

2. MAIN RESULT

Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( \{\mathcal{F}_t, t \geq 0\} \) be an increasing filtration of \( \sigma \)-fields on \( \Omega \) satisfying the usual condition, i.e. \( \mathcal{F}_t \subset \mathcal{F} \) contains all \((\mathcal{F}, P)\)-null sets and \( \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s \). By \( \mathcal{P} \) we denote the predictable \( \sigma \)-field, that is \( \mathcal{P} \) is the smallest \( \sigma \)-field containing the collection of all sets \( A \times (s, t] \), where \( 0 \leq s < t < \infty \) and \( A \in \mathcal{F}_s \). The processes \( W^1_t, W^2_t, \cdots \) are independent one-dimensional Wiener processes defined on \( \Omega \), each of which is a Wiener process relative to \( \{\mathcal{F}_t, t \geq 0\} \).
For $p > 1$ and $\gamma \in \mathbb{R}$, let $H_p^\gamma = H_p^\gamma(\mathbb{R}^d)$ denote the class of all (tempered) distributions $u$ on $\mathbb{R}^d$ such that
\[ ||u||_{H_p^\gamma} := ||(1 - \Delta)^{\gamma/2}u||_{L_p} < \infty, \] (2.1)
where
\[ (1 - \Delta)^{\gamma/2}u = \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u) \right). \]
Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier and inverse Fourier transforms respectively. It is well-known that if $\gamma = 1, 2, \cdots$, then
\[ H_p^\gamma = W_p^\gamma := \{ u : D_x^\alpha u \in L_p(\mathbb{R}^d), \ |\alpha| \leq \gamma \}, \quad H_p^{-\gamma} = \left( H_p^{\gamma/(p-1)} \right)^*, \]
where $\left( H_p^{\gamma/(p-1)} \right)^*$ denotes the dual space of $H_p^{\gamma/(p-1)}$. For a tempered distribution $u \in H_p^\gamma$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, the action of $u$ on $\phi$ (or the image of $\phi$ under $u$) is defined as
\[ (u, \phi) = \left( (1 - \Delta)^{\gamma/2}u, (1 - \Delta)^{-\gamma/2}\phi \right) = \int_{\mathbb{R}^d} (1 - \Delta)^{\gamma/2}u(x) \cdot (1 - \Delta)^{-\gamma/2}\phi(x) \, dx. \]

Let $l_2$ denote the set of all sequences $a = (a_1, a_2, \cdots)$ such that
\[ |a|_{l_2} := \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} < \infty. \]
By $H_p^\gamma(l_2) = H_p^\gamma(\mathbb{R}^d; l_2)$ we denote the class of all $l_2$-valued (tempered) distributions $v = (v_1, v_2, \cdots)$ on $\mathbb{R}^d$ such that
\[ ||v||_{H_p^\gamma(l_2)} := ||(1 - \Delta)^{\gamma/2}v||_{L_p} < \infty. \]

Next we introduce weighted Sobolev spaces $H_p^\gamma,\theta(\mathcal{O})$ defined on domains, where $\gamma, \theta \in \mathbb{R}$. Let $\mathcal{O}$ be a bounded $C^1$ domain in $\mathbb{R}^d$ and denote $\rho(x) := \text{dist}(x, \partial \mathcal{O})$. Then one can choose a smooth function $\psi$ defined on $\mathcal{O}$ satisfying the followings (see, e.g. [2, 8]):
- $\psi$ is comparable to $\rho$, that is there is a constant $N = N(\mathcal{O})$ so that $N^{-1} \rho(x) \leq \psi(x) \leq N \rho(x), \ \forall x \in \mathcal{O}$.
- $\psi$ is infinitely differentiable in $\mathcal{O}$ (not up to the boundary), and for any multi-index $\alpha$,
\[ \sup_{\mathcal{O}} \psi^{(|\alpha|)}(x)|D^\alpha \psi(x)| < \infty. \]

Fix a nonnegative function $\zeta(x) = \zeta(x^1) \in C_0^\infty(\mathbb{R}_+)$ such that
\[ \sum_{n=-\infty}^{\infty} \zeta^p(e^n x^1) > c > 0, \ \forall x^1 \in \mathbb{R}_+, \] (2.2)
where $c$ is a constant. Note that any nonnegative function $\zeta$ with $\zeta > 0$ on $[1, e]$ satisfies $\mathcal{L}_2$. For $x \in \mathcal{O}$ and $n \in \mathbb{Z} = \{0, \pm 1, \cdots\}$ define
\[ \zeta_n(x) = \zeta(e^n \psi(x)). \]
Then we have $\sum_n \zeta_n \geq c > 0$ in $\mathcal{O}$ and
\[ \zeta_n \in C_0^\infty(\mathcal{O}), \quad |D^m \zeta_n(x)| \leq N(m)e^{mn}. \]
For $\theta, \gamma \in \mathbb{R}$, let $H^{\gamma}_{p,\theta}(\mathcal{O})$ be the set of all distributions $u$ on $\mathcal{O}$ such that
\[
\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} := \sum_{n \in \mathbb{Z}} e^{\alpha n} \|\zeta_{-n}(e^{\alpha n})u(e^{\alpha n})\|_{L^p} < \infty.
\]
Similarly, for $l_2$-valued functions $g = (g^1, g^2, \cdots)$ we define
\[
\|g\|_{H^{\gamma}_{p,\theta}(\mathcal{O}, l_2)} := \sum_{n \in \mathbb{Z}} e^{\alpha n} \|\zeta_{-n}(e^{\alpha n})g(e^{\alpha n})\|_{H^{\gamma}_{p,\theta}(l_2)}.
\]
It is known (see [9, 14]) that up to equivalent norms the space $H^{\gamma}_{p,\theta}(\mathcal{O})$ is independent of the choice of $\zeta$ and $\psi$. Moreover if $\gamma$ is a non-negative integer then

\[H^{\gamma}_{p,\theta} = \{ u : \rho^{\alpha} D^\alpha u \in L_p(\mathcal{O}, \rho^\theta - d x), |\alpha| \leq \gamma \},\]

and
\[
\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \sim \sum_{|\alpha| \leq \gamma} \int_{\mathcal{O}} |\rho^{\alpha} D^\alpha u(x)|^p \rho^{\theta - d}(x) \, d x.
\]

To state our assumptions on the coefficients, we take some notation from [3, 8]. Denote $\rho(x,y) = \rho(x) \wedge \rho(y)$. For $\alpha \in (0,1]$, and $k = 0,1,2,\ldots$, we define interior Hölder norm $\cdot \cdot_{k+\alpha}$ as follows.
\[
[f]_{k+\alpha}^{(0)} = \sum_{|\beta| \leq k} \sup_{x,y \in \mathcal{O}} \rho^{|\beta|}(x) |D^\beta f(x)|,
\]
\[
[f]_{k+\alpha}^{(0)} = \sup_{x,y \in \mathcal{O}} \rho^{k+\alpha}(x,y) \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^{\alpha}},
\]
\[
[f]_{k+\alpha}^{(0)} = [f]_k^{(0)} + [f]_{k+\alpha}^{(0)}.
\]

For $l_2$-valued functions $f = (f^1, f^2, \cdots)$ we define $[f]_{k+\alpha}^{(0)}$ by using $|D^\beta f(x)|$ in place of $|D^\beta f(x) - D^\beta f(y)|$ in place of $|D^\beta f(x) - D^\beta f(y)|$, respectively. One can easily check that there exists a constant $N > 0$ such that for any $\gamma \in [0,1],$
\[
[f]_{\gamma}^{(0)} \leq N \left( \|f\|_{C(\mathcal{O})} + |\psi D^\gamma f|_{C(\mathcal{O})} \right),\tag{2.3}
\]
where $N$ is independent of $\gamma$ and $f$.

Below we collect some well-known properties of the space $H^{\gamma}_{p,\theta}(\mathcal{O})$. For $\alpha \in \mathbb{R}$, we write $f \in \psi^{\alpha} H^{\gamma}_{p,\theta}(\mathcal{O})$ if and only if $\psi^{-\alpha} f \in H^{\gamma}_{p,\theta}(\mathcal{O})$.

**Lemma 2.1.** (i) For any $\gamma, \theta \in \mathbb{R}$, $C^\infty(\mathcal{O})$ is dense in $H^{\gamma}_{p,\theta}(\mathcal{O})$.

(ii) Assume that $\gamma - d/p = m + \nu$ for some $m = 0,1,\cdots$ and $\nu \in (0,1]$. Then for any $u \in H^{\gamma}_{p,\theta}(\mathcal{O})$ and $i \in \{0,1,\cdots, m\}$, we have
\[
|\psi^{i+\theta/p} D^i u|_{C(\mathcal{O})} + |\psi^{m+\nu+\theta/p} D^m u|_{C^{\nu}} \leq N \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.
\]

(iii) Let $\alpha \in \mathbb{R}$, then $\psi^{\alpha} H^{\gamma}_{p,\theta+\alpha}(\mathcal{O}) = H^{\gamma}_{p,\theta}(\mathcal{O}),$
\[
\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \leq N \|\psi^{-\alpha} u\|_{H^{\gamma}_{p,\theta+\alpha}(\mathcal{O})} \leq N \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.
\]

(iv) Let $\varepsilon = 0$ if $\gamma$ is an integer, and $\varepsilon > 0$ otherwise. Then
\[
\|au\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \leq N |a|^{(0)}_{\gamma+\varepsilon} \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.
\]

(v) $\psi D_1, D_2 \psi : H^{\gamma}_{p,\theta}(\mathcal{O}) \to H^{\gamma-1}_{p,\theta}(\mathcal{O})$ are bounded linear operators, and
\[
\|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})} \leq N \|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} + N \|\psi \nabla u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} \leq N \|u\|_{H^{\gamma}_{p,\theta}(\mathcal{O})}.
\]
\[\|u\|_{H^\gamma_p(\mathcal{O})} \leq N\|u\|_{H^{-1}_p(\mathcal{O})} + N\|\nabla(\psi u)\|_{H^{-1}_p(\mathcal{O})} \leq N\|u\|_{H^\gamma_p(\mathcal{O})}.\]

Now we introduce stochastic Banach spaces. For a stopping time \(\tau\), denote \((0, \tau) = \{(\omega, t) : 0 < t \leq \tau(\omega)\},\)
\[H^\gamma_{p, \theta}(\mathcal{O}, \tau) = L_p((0, \tau], \mathcal{P}; H^\gamma_{p, \theta}(\mathcal{O})), \quad \mathbb{H}^\gamma_{p, \theta}(\mathcal{O}, \tau, l_2) = L_p((0, \tau], \mathcal{P}; H^\gamma_{p, \theta}(\mathcal{O}, l_2)),\]
\[L^0_{p, \theta}(\mathcal{O}, \tau) = \mathbb{H}^0_{p, \theta}(\mathcal{O}, \tau), \quad U^\gamma_{p, \theta}(\mathcal{O}) = \psi^{1-2/p}L_0(\Omega, \mathcal{F}_0; H^{\gamma-2/p}(\mathcal{O})).\]

where
\[\|u\|_{H^\gamma_p(\mathcal{O}, \tau)}^p = \mathbb{E} \int_0^\tau \|u\|_{H^\gamma_p(\mathcal{O})}^p dt, \quad \|f\|_{U^\gamma_p(\mathcal{O})}^p := \mathbb{E}\|f\|^{q/2/p-1} \|\cdot\|_{H^{\gamma-2/p}(\mathcal{O})}^p.\]

For instance, \(u \in H^\gamma_{p, \theta}(\mathcal{O}, \tau)\) if \(u\) has an \(H^\gamma_{p, \theta}(\mathcal{O})\)-valued \(\mathcal{P}\)-measurable version \(v\) defined on \((0, \tau]\) (i.e. \(u = v\) a.e. in \((0, \tau]\)) and \(\|u\|_{H^\gamma_{p, \theta}(\mathcal{O}, \tau)} < \infty\).

**Definition 2.2.** We write \(u \in \mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)\) if \(u \in \mathbb{H}^\gamma_{p, \theta}(\mathcal{O}, \tau), u_0 \in U^\gamma_{p, \theta}(\mathcal{O})\) and for some \(h \in \psi^{-1}H^\gamma_{p, \theta}(\mathcal{O}, \tau)\) and \(g = (g^1, g^2, \cdots) \in \mathbb{H}^\gamma_{p, \theta}(\mathcal{O}, \tau, l_2)\), it holds that
\[du = h dt + g^k dw^k_t, \quad t \leq \tau; \quad u(0, \cdot) = u_0\]

in the sense of distributions, that is for any \(\phi \in C_c(\mathcal{O})\), the equality
\[(u(t), \phi) = (u_0, \phi) + \int_0^t (h(s), \phi) ds + \sum_k \int_0^t (g^k(s), \phi) dw^k_s\]
holds for all \(t \leq \tau\) (a.s.). In this case we write
\[h = \mathcal{D}u, \quad g = \mathcal{S}u.\]

Especially, we say that \(u\) is a solution to equation \((1.1)\) if
\[Du = D_1(a^{ij}(u)u_{x^i}) + b_i(u)u + \hat{b}(u)u_{x^i} + c(u)u + f^i + f, \quad \mathcal{S}u = \nu u + g.\]

The norm in \(\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)\) is given by
\[\|u\|_{\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)} = \|\psi^{-1}u\|_{\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)} + \|\mathcal{D}u\|_{\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)} + \|\mathcal{S}u\|_{\mathcal{H}^\gamma_{p, \theta}(\mathcal{O}, \tau, l_2)} + \|u(0, \cdot)\|_{U^\gamma_{p, \theta}(\mathcal{O})}.\]

Finally, we write \(u \in \mathcal{S}_{p, \theta, \text{loc}}^{\gamma+1}(\mathcal{O}, \tau)\) if there exists a sequence of stopping times \(\tau_n \uparrow \tau\) so that \(u \in \mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau_n)\).

**Theorem 2.3.** (i) For any \(\gamma, \theta \in \mathbb{R}\) and \(p \geq 2\), \(\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)\) is a Banach space.

(ii) If \(\tau \leq T\), \(p > 2\) and \(1/2 > \beta > \alpha > 1/p\), then for any \(u \in \mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)\), it holds that \(u \in C^{0, -1/p}([0, \tau], H^{\gamma+1-2\beta}_{p, \theta}(\mathcal{O}))\) (a.s.) and
\[\mathbb{E}\|\psi^{2\beta-1}u\|_{C^{0, -1/p}([0, \tau], H^{\gamma+1-2\beta}_{p, \theta}(\mathcal{O}))} \leq N(d, p, \alpha, \beta, T)\|u\|_{\mathcal{S}_{p, \theta}^{\gamma+1}(\mathcal{O}, \tau)} \quad (2.4)\]

(iii) If \(p = 2\) then \((2.4)\) holds with \(\beta = 1/2\) and \(\alpha = 1/p = 1/2\). That is, if \(u \in \mathcal{S}_{2, \theta}^{\gamma+1}(\mathcal{O}, \tau)\), then \(u \in C([0, \tau]; H^\gamma_{2, \theta}(\mathcal{O}))\) (a.s.), and
\[\mathbb{E}\sup_{t \leq T}\|u\|_{H^\gamma_{2, \theta}(\mathcal{O})}^2 \leq N\|u\|_{\mathcal{S}_{2, \theta}^{\gamma+1}(\mathcal{O}, \tau)}^2.\]

**Proof.** This theorem is proved by Krylov in [10] if \(\mathcal{O} = \mathbb{R}^d\); see [10] Theorem 4.1] for (ii) and [10] Remark 4.5] for (iii). For bounded \(C^1\) domains, see e.g. Lemma 3.1 of [5] and (2.21) of [16].
In this article we mostly use the above theorem when $\theta = d$, and thus we only consider the case $\theta = d$ in the following corollary.

**Corollary 2.4.** (i) Let $\hat{\alpha} := 1 - d/p - 2/p > 0$. Then for any $\kappa \in (0, \hat{\alpha})$,
\[
E|u|^p_{C([0,\tau];(0,|\kappa|)C^\infty(\Omega))} + E|u|^p_{C^{\infty}(\Omega)} \leq N\|u\|^p_{H^1_p,d(\Omega)}, \tag{2.5}
\]

(ii) Let the constants $\kappa, \kappa_1, \gamma, \beta$ and $\beta_1$ satsify
\[
1/p < \kappa_1 < \kappa < 1/2, \quad 1 - 2\kappa - d/p > 0,
\]
\[
1/p < \beta_1 < \beta < 1/2, \quad \gamma - 2\beta - d/p > 0.
\]
Then for any $\varepsilon$ and $\varepsilon'$ satisfying
\[
0 \leq \varepsilon \leq 1 - 2\kappa - d/p, \quad 0 \leq \varepsilon' \leq \gamma - 2\beta - d/p,
\]
we have
\[
E|\psi^{-\varepsilon} u|^p_{C^{\alpha}(\Omega)} + E|\psi^{\varepsilon'} Du|^p_{C^{\alpha}(\Omega)} \leq N\|u\|^p_{\delta^{1+\gamma}_p,d(\Omega)}, \tag{2.6}
\]

In particular, we have
\[
E|u|^p_{C([0,\tau];\Omega)})} + E|\psi^{\varepsilon} Du|^p_{C([0,\tau];\Omega)}} \leq N\|u\|^p_{\delta^{1+\gamma}_p,d(\Omega)}, \tag{2.7}
\]

**Proof.** (i) We only consider the first term of (2.5). The second one can be treated similarly. Take $1/p < \alpha < \beta < 1/2$ such that
\[
1 - 2\beta - d/p = \kappa.
\]
To apply Lemma 2.4(ii) we take $\gamma = 1 - 2\beta, \theta = d/p + (2\beta - 1)$ and $\nu = \gamma - d/p = \kappa$, and get
\[
|u|^p_{C^{\infty}(\Omega)} \leq N\|u\|^p_{H^{1-2\beta}_p,d(\Omega)} \leq N\|\psi^{\beta} u\|^p_{H^{1-2\beta}_p,d(\Omega)},
\]
where the second inequality is due to Lemma 2.4(iii). Therefore the claim follows from 2.4.

(ii) Since, $u \in \delta^{1+\gamma}_p,d(\Omega)$, by Theorem 2.3, we get
\[
E|\psi^{-1/2\kappa} u|^{p}_{C^{\infty}(\Omega)} + E|\psi^{1-2\kappa} u|^{p}_{C^{\infty}(\Omega)} \leq N\|\psi^{-1/2\kappa} u\|^p_{H^{1-2\kappa}_p,d(\Omega)} \leq N\|\psi^{1-2\kappa} u\|^p_{H^{1-2\kappa}_p,d(\Omega)}.
\]

Lemma 2.4(ii) with $\gamma = 1 - 2\kappa$ and $\nu = 1 - 2\kappa - d/p - \varepsilon$ to get
\[
|\psi^{-\varepsilon} u|^p_{C^{\gamma-p-2\kappa} - \varepsilon}(\Omega) \leq N\|\psi^{-1/2\kappa} u\|^p_{H^{1-2\kappa}_p,d(\Omega)} \leq N\|\psi^{1-2\kappa} u\|^p_{H^{1-2\kappa}_p,d(\Omega)}.
\]
Hence the first term of (2.6) is handled. The second term is treated similarly using $u \in \delta^{1+\gamma}_p,d(\Omega)$, instead of $u \in \delta^{1+\gamma}_p,d(\Omega)$. \hfill \Box

Below are our assumptions on the coefficients.

**Assumption 2.5 (Measurability).** The coefficients $a^ij(t, x, u), b^j(t, x, u), \tilde{b}^i(t, x, u), c(t, x, u)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R})$-measurable and $\nu^k(t, x)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable.
Assumption 2.6 (Ellipticity and Boundedness). There exist constants $\delta_0 \in (0, 1]$ and $K_1 > 0$ such that
\[ |\alpha^j(t, x, u)|^2 \leq a^j(t, x, u)\xi^i\xi^j \leq \delta_0^{-1}|\xi|^2 \]
and
\[ |a^j(t, x, u) + |b^j(t, x, u)| + |\bar{b}^j(t, x, u)| + |c(t, x, u)| + |\nu^j(t, x)||_2 \leq K_1 \]
for all $\omega, t, x, u$, and $\xi \in \mathbb{R}^d$.

Assumption 2.7 (Interior Hölder continuity in $x$). $p > d + 2, \gamma_0 \in \left( \frac{d+2}{p}, 1 \right)$, and there exists a $K_2$ such that for any $\omega, t, u$,
\[ |a^{ij}(t, \cdot, u)|^{(0)} + |b^{i}(t, \cdot, u)|^{(0)} + |\bar{b}^{j}(t, \cdot, u)|^{(0)} + |c(t, \cdot, u)|^{(0)} + |\nu(t, \cdot)|^{(0)} \leq K_2. \]

Assumption 2.8 (Lipschitz continuity with respect to the unknown). There exists a constant $K_3$ such that for any $\omega, t, x, v, i, j$,
\[ |a^{ij}(t, x, u) - a^{ij}(t, x, v)| + \psi(x)|b^{i}(t, x, u) - b^{i}(t, x, v)| + \psi(x)|\bar{b}^{j}(t, x, u) - \bar{b}^{j}(t, x, v)| + \psi^2(x)|c(t, x, u) - c(t, x, v)| \leq K_3|u - v|. \]

Here is the main result of this article.

Theorem 2.9. Let $\tau \leq T$ be a stopping time. Suppose Assumptions 2.6, 2.7, and 2.8 hold. Then for any given $f^i \in \mathbb{H}^{\gamma_0}_{p,d}(\mathcal{O}, \tau)$, $f \in \psi^{-1}\mathbb{H}^{\gamma_0+1}_{p,d}(\mathcal{O}, \tau), g \in \mathbb{H}^{\gamma_0}_{p,d}(\mathcal{O}, \tau, l_2)$, and $u_0 \in U_{p,d}^{\gamma_0+1}(\mathcal{O})$, equation (1.1) has a unique solution $u$ in $\mathcal{D}_{2,d}(\mathcal{O}, \tau)$, and for this solution $u$, we have
\[ \|u\|_{\mathcal{D}_{2,d}(\mathcal{O}, \tau)} \leq N(\|f^i\|_{L_{2,d}(\mathcal{O}, \tau)} + \|\psi f\|_{\mathbb{H}^{\gamma_0+1}(\mathcal{O}, \tau)} + \|g\|_{L_{2,d}(\mathcal{O}, \tau, l_2)} + \|u_0\|_{U_{p,d}^{\gamma_0+1}(\mathcal{O})}), \tag{2.8} \]
where $N$ depends only on $d, p, \delta_0, K_1, T, \text{ and } \mathcal{O}$. Furthermore,
\[ u \in \mathcal{D}_{p,d,\text{loc}}^{1 + \gamma}(\mathcal{O}, \tau), \quad \forall \gamma < \gamma_0, \]
and for any constants $\kappa, \kappa_1, \gamma, \beta, \beta_1, \varepsilon$ and $\varepsilon'$ satisfying
\[ 1/p < \kappa < 1 < 1/2, \quad 1 - 2\kappa - d/p > 0, \]
\[ 1/p < \beta_1 < \beta < 1/2, \quad \gamma - 2\beta - d/p > 0, \]
\[ \varepsilon \in [0, 1 - 2\kappa - d/p], \quad \varepsilon' \in [0, \gamma - 2\beta - d/p], \]
\[ \text{it holds that for all } t < \tau \text{ (a.s.)} \]
\[ |\rho^{-t} u|_{C^{\kappa_1-1/p}([0,t],C^{1-2\kappa-d/p-\varepsilon}(\mathcal{O}))} + |\rho^{-t} u|_{C^{\beta_1-1/p}([0,t],C^{1-2\beta-d/p-\varepsilon'}(\mathcal{O}))} < \infty. \tag{2.9} \]

Remark 2.10. (i) Taking $\varepsilon = 0$ in (2.9), we find that $u$ is Hölder continuous in $t$ with exponent $\kappa_1 - 1/p$ (which can be very close to $1/2$ if $p$ is large) and Hölder continuous in $x$ with exponent $1 - 2\kappa - d/p$ (this can be very close to 1).

(ii) Take $\varepsilon = 1 - 2\kappa - d/p$, then we get
\[ \sup_{s < t} |u(s, x)| \leq N \rho^{\gamma}(x) \to 0 \]
substantially fast as $\rho(x) \to 0$. Hence $u$ vanishes on the boundary, and this is a reason we do not need to explicitly impose the zero boundary condition to the equation.

(iii) Since $\gamma - \varepsilon' > 0$, from (2.9) it only follows that $u_x$ is Hölder continuous in compact subsets of $\mathcal{O}$. 

ILDOO KIM AND KYEONG-HUN KIM
3. Some auxiliary results related to linear equations

In this section, we collect a few results related to the following linear equation:

\[ du = \left[ D_i \left( a^{ij}(t, x) u_{x_j} + b^i(t, x) u + f^i \right) + \tilde{b}^i(t, x) u_{x_i} + c(t, x) u + f \right] dt \\
+ \left[ \nu^k(t, x) u + g^k \right] dW^k_t, \quad t \leq \tau; \quad u(0, \cdot) = u_0. \]  \hspace{1cm} (3.1)

**Assumption 3.1.** (i) The coefficients \( a^{ij}(t, x), b^i(t, x), \tilde{b}^i(t, x), c(t, x), \) and \( \nu^k(t, x) \)
are \( \mathcal{P} \times \mathcal{B}(\Omega) \)-measurable functions.

(ii) There exists a constant \( \delta_0 > 0 \) such that

\[ \delta_0 |\xi|^2 \leq a^{ij} \xi^i \xi^j \leq \delta_0^{-1} |\xi|^2, \]  \hspace{1cm} (3.2)
for all \( \omega, t, x \) and \( \xi \in \mathbb{R}^d \).

**Assumption 3.2.** For all \( \omega, t, x, \)

\[ \rho(x) |b^i(t, x)| + |\tilde{b}^i(t, x)| + |\nu(t, x)|_{L^2} + \rho^2(x) |c(t, x)| \leq \delta_0^{-1}, \]

and there is a control on the blow up of the coefficients near the boundary:

\[ \rho(x)|b^i(t, x)| + \rho(x)|\tilde{b}^i(t, x)| + \rho^2(x)|c(t, x)| + \rho(x)|\nu(t, x)|_{L^2} \rightarrow 0 \]
as \( \rho(x) \rightarrow 0 \). In other words, there exists a nondecreasing function \( \pi_0 : [0, \infty) \mapsto [0, \infty) \) such that \( \pi_0(t) \downarrow 0 \) as \( t \downarrow 0 \) and

\[ \rho(x)|b^i(t, x)| + \rho(x)|\tilde{b}^i(t, x)| + \rho^2(x)|c(t, x)| + \rho(x)|\nu(t, x)|_{L^2} \leq \pi_0(\rho(x)). \]

**Remark 3.3.** Obviously Assumption 3.2 holds if the coefficients are bounded. It also holds if

\[ |b^i(t, x)| + |\tilde{b}^i(t, x)| + |\nu(t, x)|_{L^2} \leq N \rho^{-1+\varepsilon}(x), \quad |c(t, x)| \leq N \rho^{-2+\varepsilon}(x), \]  \hspace{1cm} (3.3)
for some \( \varepsilon, N > 0 \). Note that (3.3) allows the coefficients to blow up substantially fast near the boundary.

**Assumption 3.4.** \( a^{ij} \) are uniformly continuous in \( x \), that is, there exists a nondecreasing function \( \pi_0 : [0, \infty) \rightarrow [0, \infty) \) such that

\[ |a^{ij}(t, x) - a^{ij}(t, y)| \leq \pi_0(|x - y|), \quad \forall \omega, t, \]
and \( \pi_0(\lambda) \rightarrow 0 \) as \( \lambda \rightarrow 0 \).

Fix \( \kappa_0 \in (0, 1) \), and for \( \gamma \geq 0 \), denote \( \gamma_+ = \gamma \) if \( \gamma \) is integer and otherwise \( \gamma_+ = \gamma + \kappa_0 \).

**Theorem 3.5.** Let \( p \geq 2, \tau \leq T \) be a stopping time, and

\[ d - 1 + p < \theta < d - 1 + p. \]  \hspace{1cm} (3.4)

Suppose that Assumptions 3.1, 3.2, and 3.4 hold, and there exists a constant \( \bar{K} \)
such that

\[ |a^{ij}(t, \cdot)|_{\gamma_+}^{(0)} + |\psi b^i(t, \cdot)|_{\gamma_+}^{(0)} + |\psi c(t, \cdot)|_{\gamma_+}^{(0)} + |\psi^2 c(t, \cdot)|_{\gamma_+}^{(0)} \leq \bar{K} \quad \forall \omega, t. \]  \hspace{1cm} (3.5)
Then for any $f^i \in \mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O}, \tau)$, $f \in \psi^{-1}\mathbb{H}^{-1}_{p,\theta}(\mathcal{O}, \tau)$, $g \in \mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O}, \tau, l_2)$, and $u_0 \in \mathcal{U}^{\gamma+1}_{p,\theta}(\mathcal{O})$ equation (3.7) with initial data $u_0$ has a unique solution $u$ in $\mathcal{D}^{\gamma+1}_{p,\theta}(\mathcal{O}, \tau)$ and

$$\|u\|_{\mathcal{D}^{\gamma+1}_{p,\theta}(\mathcal{O}, \tau)} \leq N(\|f^i\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O}, \tau)} + \|\psi f\|_{\mathbb{H}^{-1}_{p,\theta}(\mathcal{O}, \tau)} + \|g\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O}, \tau, l_2)} + \|u_0\|_{\mathcal{U}^{\gamma+1}_{p,\theta}(\mathcal{O})}),$$

(3.6)

where $N$ depends only on $d$, $p$, $\gamma$, $\delta_0$, $T$, $\mathcal{O}$, and the function $\pi_0(t)$.

Proof. See [4, Theorem 3.13]. We only mention that the result of this theorem was first proved by Krylov and Lototsky [11] when $\mathcal{O} = \mathbb{R}^d_+$ and the coefficients are independent of $x$. Then the result was extended to general $C^1$-domains in [7] based on localization and flattening the boundary arguments. In [7], the coefficients $a^{ij}$ are continuous in $x$ and consequently we only have $u \in \mathcal{D}^1_{p,d}(\mathcal{O}, \tau)$. Finally better regularity of the solution is obtained in [4] under Hölder continuity (3.5). \[\square\]

**Corollary 3.6** ($\theta = d$ with no stochastic term). Assume $\nu^k = g^k = 0$ for each $k$ and let $u$ be the solution in Theorem 3.7 corresponding to the case $\theta = d$, that is, $u$ is the solution to

$$du = \left[ D_i \left( a^{ij}(t, x) u_{x^j} + b^i(t, x) u + f^i \right) + b^i(t, x) u_{x^i} + c(t, x) u + f \right] dt, \quad t \leq \tau, \quad u(0, \cdot) = u_0.$$

Then (a.s.)

$$\|\psi^{-1}u\|_{L^p([0,\tau]; H^{1+\gamma}_{p,\theta}(\mathcal{O}, \tau))} + \|\psi Du\|_{L^p([0,\tau]; H^{-1}_{p,\theta}(\mathcal{O}, \tau))} \leq N \left( \|f^i\|_{L^p([0,\tau]; H^{\gamma+1}_{p,\theta}(\mathcal{O}, \tau))} + \|\psi f\|_{L^p([0,\tau]; H^{-1}_{p,\theta}(\mathcal{O}, \tau))} + \|u_0\|_{H^{\gamma+1}_{p,\theta}(\mathcal{O})} \right),$$

where $N = N(d, p, \gamma, \delta_0, T, \pi_0, \mathcal{O})$ is independent of $\omega$.

Proof. It is enough to fix $\omega$, and then apply Theorem 3.7 to the corresponding deterministic equation. \[\square\]

**Theorem 3.7** ($p = 2$ and $\theta = d$ with only measurable coefficients). Let $\tau \leq T$ and Assumptions 3.7 and 3.2 hold. Then for any $f^i \in \mathbb{L}^2_d(\mathcal{O}, \tau)$, $f \in \psi^{-1}\mathbb{H}^{-1}_{2,\theta}(\mathcal{O}, \tau)$, $g \in \mathbb{L}^2_d(\mathcal{O}, \tau, l_2)$, and $u_0 \in \mathcal{U}^1_{2,d}(\mathcal{O})$ equation (3.7) with initial data $u_0$ has a unique solution $u \in \mathcal{D}^1_{2,d}(\mathcal{O}, \tau)$, and for this solution

$$\|u\|_{\mathcal{D}^1_{2,d}(\mathcal{O}, \tau)} \leq N(\|f^i\|_{\mathbb{L}^2_d(\mathcal{O}, \tau)} + \|\psi f\|_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O}, \tau)} + \|g\|_{\mathbb{L}^2_d(\mathcal{O}, \tau, l_2)} + \|u_0\|_{\mathcal{U}^1_{2,d}(\mathcal{O})}),$$

where $N$ depends only on $\delta_0$, $T$, $\mathcal{O}$, and the function $\pi_0$.

Proof. This is a very classical result (see e.g. [17]) if the coefficients are bounded. See [6, Theorem 2.19] for the general case. We remark that in our main theorem, Theorem 2.9, the coefficients are assumed to be bounded. Hence, the classical result of [17] is enough for our need. \[\square\]
\textbf{Theorem 3.8.} Let \( u \in \mathcal{S}^1_{p,d}(\mathcal{O}, \tau) \) be the solution taken from Theorem 3.7 and assume \( v^k = 0 \) for each \( k \). Assume
\[ |b^i(t, x) + \tilde{b}(t, x)| + |c(t, x)| \leq K \quad \forall \omega, t, x, \quad (3.7) \]
for some \( p > d + 2 \). Then there exists a constant \( \bar{\alpha} > 0 \) so that if \( \alpha < \bar{\alpha} \) and \( u_0 \in L_p(\bar{\Omega}, C^{\alpha}(\mathcal{O})) \) then
\[ \mathbb{E}|u|^p_{C^{\alpha}([0, \tau] \times \mathcal{O})} \leq N \| f \|^p_{L_p(\mathcal{O})} + N \| \psi f \|^p_{H^{-1,p}(\mathcal{O})} + N \| g \|^p_{L_p(\mathcal{O}, \tau, t_2)} \]
where \( N \) depends only on \( d, p, \alpha, \delta_0, K, T, \) and \( \mathcal{O} \).

\textit{Proof.} See [4, Theorem 2.9] or [5, Theorem 2.4] for detailed proof. Below we only give a sketch of the proof. For simplicity assume \( b^i = \tilde{b} = c = u_0 = 0 \). By Theorem 3.6 there is a unique solution \( v \in \mathcal{S}^1_{p,d}(\mathcal{O}, \tau) \) to
\[ dv = (\Delta v + f) dt + g^k dW^k, \quad v(0) = 0. \]
By (3.0) and Corollary 3.6 with \( \gamma = 0 \), there is \( \alpha_1 > 0 \) so that
\[ \mathbb{E}\|v\|^p_{C^{\alpha_1}([0, \tau] \times \mathcal{O})} + \mathbb{E}\|v\|^p_{C^{\alpha_2}([0, \tau] \times \mathcal{O})} \leq N (\|f\|^p_{L_p(\mathcal{O})} + \|g\|^p_{L_p(\mathcal{O}, \tau, t_2)}). \]
Note that, for each fixed \( \omega \), the function \( \tilde{u} := u - v \) satisfies the deterministic equation
\[ \frac{d\tilde{u}}{dt} = D_i(a^{ij}(\tilde{u})_{x^j} + \tilde{f}^i), \quad \tilde{u}(0, \cdot) = 0, \]
where \( \tilde{f}^i = (\delta^{ij} - \delta^{ij})v_{x^j} + f^i \). Here \( \delta^{ij} \) is the Kronecker delta, i.e. \( \delta^{ij} = 1 \) if \( i = j \) and otherwise \( \delta^{ij} = 0 \). Then using a classical result for the deterministic equation (e.g. [13]), for some \( \alpha_2 > 0 \) we have
\[ \|\tilde{u}\|^p_{C^{\alpha_2}([0, \tau] \times \mathcal{O})} \leq N \|\tilde{f}\|^p_{L_p([0, \tau] \times \mathcal{O})}, \]
where \( N \) is independent of \( \omega \in \bar{\Omega} \). Combining above two estimates we get (3.8) with \( \bar{\alpha} = \alpha_1 \land \alpha_2 \).

\section{Local Solution}

In this section, we construct a nonzero stopping time \( \tau' \leq \tau \) so that the equation
\[ du = \left( D_i\left(a^{ij}(u)_{x^j} + b^i(u) + f^i\right) + \tilde{b}^i(u)_{x^j} + c(u) + f\right) dt + g^k dW^k \]
has a solution for \( t \leq \tau' \).

\textbf{Lemma 4.1.} Suppose that Assumptions 2.5, 2.8, 2.8 hold and assume that \( \tau \) is a nonzero stopping time. Let \( f^i \in L_{p,d}(\mathcal{O}, \tau), \ f \in H^{-1,p}(\mathcal{O}, \tau), \ g \in L_{p,d}(\mathcal{O}, \tau, t_2) \) and \( u_0 \in U^1_{p,d}(\mathcal{O}) \). Then there exists a nonzero stopping time \( \tau' \leq \tau \) such that equation (3.8) with initial data \( u_0 \) has a solution \( u \) in \( \mathcal{S}^1_{p,d}(\mathcal{O}, \tau') \).

\textit{Proof.} \textbf{Step 1.} We prove the lemma if \( K_3 \) is sufficiently close to zero.

Let \( u^1 \in \mathcal{S}^{1+\tau_0}_{p,d}(\mathcal{O}, \tau) \) be the solution of the following equation:
\[ du = \left[ D_i\left(\delta^{ij}u_{x^j} + f^i\right) + f\right] dt + g^k dW^k, \quad t \leq \tau, \quad u(0, \cdot) = u_0. \]
By the assumption \( \gamma_0 \in \left( \frac{d+2}{p}, 1 \right) \), one can choose \( \gamma_1 \) and \( \gamma_2 \) so that
\[
\frac{d + 2}{p} < \gamma_2 < \gamma_1 < \gamma_0 < 1.
\] (4.3)

Since \( p > d + 2 \),
\[
\hat{\alpha} = \hat{\alpha}(p) := 1 - \frac{d}{p} - \frac{2}{p} > 0.
\]

Let \( \alpha \in (0, \hat{\alpha}) \). Then by Theorem 2.6 and Corollary 2.2
\[
\mathbb{E}|u^1|_{(0,\tau]\times C^0(\mathcal{O})}^p + \mathbb{E}|\psi Du^1|_{(0,\tau]\times C(\mathcal{O})}^p \leq N\|u^1\|_{p,d,1+\gamma_0}^p < \infty.
\]

Denote
\[
A_n := \{ \omega \in \Omega : |u^1|_{(0,\tau]\times C^0(\mathcal{O})} + |\psi Du^1|_{(0,\tau]\times C(\mathcal{O})} < n \}.
\]

Then \( P(\cup_{n=1}^{\infty} A_n) = 1 \), and thus we can fix \( n_0 \in \mathbb{N} \) such that
\[
P(\{ \tau \neq 0 \} \cap A_{n_0}) > 0.
\]

Define
\[
\tau'' := \inf \{ t \leq \tau : |u^1|_{(0,\tau]\times C^0(\mathcal{O})} + |\psi Du^1|_{(0,\tau]\times C(\mathcal{O})} \geq n_0 \},
\]
\[
\tau''' := \inf \{ t \leq \tau : \|\psi^{-1}u^1\|_{L_p([0,\tau]; H_{p,d}^{1+\gamma_0}(\mathcal{O}))} > \varepsilon \},
\]
and
\[
\tau' := \tau'' \wedge \tau''',
\] (4.4)

where \( \varepsilon \in (0, 1) \) will be specified later. It is obvious that \( \tau''' > 0 \) (a.s.) and \( \tau' \) is a nonzero stopping time. The latter is because \( \tau'' = \tau \) on \( A_{n_0} \).

Denote
\[
\Phi(\tau') := \left\{ u \in \mathcal{S}_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau') : \|\psi^{-1}(u - u^1)\|_{L_p([0,\tau']; H_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau'))} \leq 1 \text{ (a.s.)}, \right. \\
\left. |u - u^1|_{(0,\tau]\times C(\mathcal{O})} + |\psi Du - u^1|_{(0,\tau]\times C(\mathcal{O})} \leq 1 \text{ (a.s.)}, \quad u(0, \cdot) = u_0 \right\}.
\]

For each \( v \in \Phi(\tau') \), by \( \mathcal{R}v \) we denote the solution in \( \mathcal{S}_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau') \) to the equation
\[
\begin{align*}
du &= \left[ D_i \left( a^{ij}(u^1)u_{x^i} + b^i(u^1)u + [a^{ij}(v) - a^{ij}(u^1)]v_{x^i} + |b^i(v) - b^i(u^1)|v + f^i \right) \\
&\quad + \tilde{b}^i(u^1)u_{x^i} + c(u^1)v + \tilde{b}^i(v) - \tilde{b}^i(u^1)v_{x^i} + |c(v) - c(u^1)|v + f \right] dt \\
&\quad + g^kW^k_t, \quad t \leq \tau'; \quad u(0, \cdot) = u_0.
\end{align*}
\]

The map \( v \to \mathcal{R}v \in \mathcal{S}_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau') \) is well-defined due to Theorem 3.6 since \( a^{ij}(u^1) \) is uniformly continuous in \( x \) (uniformly in \( (\omega, t) \)) and other coefficients are assumed to be bounded. Indeed,
\[
|a^{ij}(t, x, u^1(t, x)) - a^{ij}(t, y, u^1(t, y))| \\
\leq K_2|x - y|^{\gamma_0} + K_3|u^1(t, x) - u^1(t, y)| \leq K_2|x - y|^{\gamma_0} + K_3n_0|x - y|^{\gamma_0}, \quad t \leq \tau'.
\]

To check \( \mathcal{R}v \in \mathcal{S}_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau') \), first note that by (2.3), (3.3), and Assumption 2.7
\[
\begin{align*}
&\sup_{t \leq \tau} |a(t, \cdot, u^1(\cdot))|_{\gamma_1}^{(0)} + \sup_{t \leq \tau} |\psi \tilde{b}(t, \cdot, u^1(\cdot))|_{\gamma_1}^{(0)} \\
&\quad + \sup_{t \leq \tau} |\psi \tilde{c}(t, \cdot, u^1(\cdot))|_{\gamma_1}^{(0)} \leq Nn_0,
\end{align*}
\]
where $N$ is a constant depending only on $d$, $K_1$, and $K_2$. Thus by Theorem 3.3 we conclude $Rv \in \mathcal{S}^{1+\gamma_2}_{p,d}(\mathcal{O}, \tau^{'})$. Here we used the existence result in $\mathcal{S}^{1+\gamma_2}_{p,d}(\mathcal{O}, \tau^{'})$ and the uniqueness result in $\mathcal{S}^{1}_{p,d}(\mathcal{O}, \tau^{'})$.

Next we show $Rv \in \Phi(\tau^{'})$ if $K_3$ is sufficiently small. Note that $(\mathcal{R}v-u^1)(0, \cdot) = 0$ and

$$d(\mathcal{R}v-u^1) = \left[ D_{i} \left( a^{ij}(u^1)(\mathcal{R}v-u^1)_{xj} + \tilde{b}^{ij}(u^1)(\mathcal{R}v-u^1) + \tilde{f}^{ij} \right) + \tilde{b}^{ij}(u^1)(\mathcal{R}v-u^1)_{xj} + c(u^1)(\mathcal{R}v-u^1) + \tilde{f} \right] dt, \quad t \leq \tau^{'},$$

where

$$\tilde{f}^{ij} := [a^{ij}(v) - a^{ij}(u^1)]v_{xj} + [\tilde{b}^{ij}(v) - \tilde{b}^{ij}(u^1)]v + a^{ij}(u^1)_{xj} + b(u^1)u^1 - \delta^{ij} u^1_{xj},$$

$$\tilde{f} := [\tilde{b}^{ij}(v) - \tilde{b}^{ij}(u^1)](v - u^1)_{xj} + [c(v) - c(u^1)](v - u^1) + [\tilde{b}^{ij}(u^1)]u^1_{xj} + [c(v) - c(u^1)]u^1 + [\tilde{b}^{ij}(u^1)]u^1 + c(u^1)u^1.$$ 

Using the deterministic version of (2.7) with $\gamma = \gamma_2$ and Corollary 3.6 we get (a.s.)

$$\|((\mathcal{R}v-u^1)||_{C([0,\tau^{'}, \mathcal{O})} + \|\psi(\mathcal{R}v-u^1)\|_{C([0,\tau^{'}, \mathcal{O})}$$

$$+ \|\psi^{-1}(\mathcal{R}v-u^1)\|_{L_p([0,\tau^{'}, H_{p,d,1}^{1+\gamma_2}))(\mathcal{O}, \tau^{'})}$$

$$\leq N \left( \|\tilde{f}||_{L_p([0,\tau^{'}, H_{p,d,1}^{1+\gamma_2}))(\mathcal{O}, \tau^{'})} + \|\tilde{f}||_{L_p([0,\tau^{'}, H_{p,d,1}^{1+\gamma_2}))(\mathcal{O}, \tau^{'})} \right), \quad (4.5)$$

where $N = N(n_0, d, p, \delta_0, T, K_1, K_2)$.

To estimate $\tilde{f}^{ij}$ and $\tilde{f}$ in (4.5), we show that for any $\delta > 0$,

$$|a^{ij}(v) - a^{ij}(u^1)|_{C([0, \tau^{'}, \mathcal{O})} \leq N(d, \mathcal{O}) \left( K_3(1 + \delta^{-\gamma_1} + n_0) + K_2\delta^{-\gamma_0 - \gamma_1} \right). \quad (4.6)$$

First observe

$$|a^{ij}(v) - a^{ij}(u^1)| \leq K_3 |v - u^1| \leq K_3, \quad t \leq \tau^{'}. $$

If $|x-y| \geq \delta$,

$$\rho^{\gamma_1}(x, y) \frac{|a^{ij}(x, v(x)) - a^{ij}(x, u^1(x)) - (a^{ij}(y, v(y)) - a^{ij}(y, u^1(y)))|}{|x-y|^{\gamma_1}}$$

$$\leq \rho^{\gamma_1}(x, y) \frac{|a^{ij}(x, v(x)) - a^{ij}(x, u^1(x))|}{|x-y|^{\gamma_1}} + \rho^{\gamma_1}(x, y) \frac{|a^{ij}(y, v(y)) - a^{ij}(y, u^1(y))|}{|x-y|^{\gamma_1}}$$

$$\leq 2NK_3 |v - u^1|_{C(\mathcal{O})} \frac{1}{|x-y|^{\gamma_1}} \leq 2NK_3 \delta^{-\gamma_1},$$
and if \( |x - y| \leq \delta \)

\[
\rho^{\tau_1}(x, y) \left| a^{ij}(x, v(x)) - a^{ij}(x, u^1(x)) \right| \leq \rho^{\tau_1}(x, y) \left| a^{ij}(x, v(x)) - a^{ij}(x, y) \right| \left| u^1(x) - y \right|^{\gamma_1} + \rho^{\tau_1}(x, y) \left| a^{ij}(x, y) - a^{ij}(y, v(y)) \right| \left| u^1(y) - v(y) \right|^{\gamma_1} + \rho^{\tau_1}(x, y) \left| a^{ij}(y, v(y)) - a^{ij}(y, y) \right| \left| v(y) - y \right|^{\gamma_1}
\]

\[
\leq K_3([v]^{(0)}_{\gamma_1} + [u^1]^{(0)}_{\gamma_1}) + NK_2 |x - y|^{\gamma_0 - \gamma_1}
\]

\[
\leq K_3([v]^{(0)}_{\gamma_1} + [u^1]^{(0)}_{\gamma_1}) + NK_2 \delta^{\gamma_0 - \gamma_1}.
\]

Hence (4.6) is proved. Similarly, for any \( \delta > 0 \),

\[
\left| \psi \mid b^i(v) - b^i(u^1) \right|^{(0)}_{\gamma_1} \leq N \left( K_3 (1 + \delta^{-\gamma_1} + n_0) + K_2 \delta^{\gamma_0 - \gamma_1} \right) =: NI(\delta, K_3).
\]

Therefore by Lemma 2.3 (iv),

\[
\| (a^{ij}(v) - a^{ij}(u^1)) \cdot (v_{x^j} - u^1_{x^j}) \|_{L_p([0, \tau]; H_{p,d}^{\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \| v - u^1 \|_{L_p([0, \tau]; H_{p,d}^{\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \| \psi^{-1}(v - u^1) \|_{L_p([0, \tau]; H_{p,d}^{1+\gamma_0}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \epsilon.
\]

Similarly,

\[
\| (a^{ij}(v) - a^{ij}(u^1)) u^1_{x^j} \|_{L_p([0, \tau]; H_{p,d}^{\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \| u^1 \|_{L_p([0, \tau]; H_{p,d}^{\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \epsilon.
\]

In this way, we get

\[
\| \tilde{f} \|_{L_p([0, \tau]; H_{p,d}^{\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) \epsilon.
\]

Similarly,

\[
\| \psi \tilde{f} \|_{L_p([0, \tau]; H_{p,d}^{1+\gamma_2}(\mathcal{O}, \tau'))} \leq NI(\delta, K_3) + \epsilon.
\]

Therefore taking sufficiently small \( \epsilon \) and \( \delta \), and then assuming \( K_3 \) is very small, we get

\[
R \in \Phi(\tau') \quad \forall \tau \in \Phi(\tau').
\]

Next we claim that the operator \( R \) becomes a contraction mapping on

\[
\Phi(\tau') = \Phi(\tau') \cap \mathcal{S}_{1,d}^1(\mathcal{O}, \tau')
\]
with respect to the norm $\| \cdot \|_{\mathcal{A}^1_{p,d}(\Theta,\tau')}$ if $K_3$ and $\varepsilon$ are small enough. We may assume that $K_3$ and $\varepsilon$ are small so that (4.47) holds. Observe that for each $v, w \in \Phi(\tau')$, $(\mathcal{R}v - \mathcal{R}w)(0,\cdot) = 0$ and

$$
d(\mathcal{R}v - \mathcal{R}w) = \left[ D_i \left( a^{ij}(u^1)(\mathcal{R}v - \mathcal{R}w)_{x^j} + b^i(u^1)(\mathcal{R}v - \mathcal{R}w) + \bar{f}^i \right) + \bar{b}^i(u^1)(\mathcal{R}v - \mathcal{R}w)_{x^i} + c(u^1)(\mathcal{R}v - \mathcal{R}w) + \bar{f} \right] dt, \quad t \leq \tau',
$$

where

$$
\bar{f}^i := [a^{ij}(v) - a^{ij}(u^1)]v_{x^j} + [b^i(v) - b^i(u^1)]v - [a^{ij}(w) - a^{ij}(u^1)]w_{x^j} - [b^i(w) - b^i(u^1)]w = [a^{ij}(v) - a^{ij}(u^1)](v - w)_{x^j} + [b^i(v) - b^i(u^1)](v - w) + [a^{ij}(v) - a^{ij}(w)](w - u^1)_{x^i} + [b^i(v) - b(w)](w - u^1) + [a^{ij}(v) - a^{ij}(w)]u^1_{x^i} + [b^i(v) - b(w)]u^1,
$$

and

$$
\bar{f} := [\bar{b}^i(v) - \bar{b}^i(u^1)]v_{x^i} + [c(v) - c(u^1)]v - [\bar{b}^i(w) - \bar{b}^i(u^1)]w_{x^i} - [c(w) - c(u^1)]w = [\bar{b}^i(v) - \bar{b}^i(u^1)](v - w)_{x^i} + [c(v) - c(u^1)](v - w) + [\bar{b}^i(v) - \bar{b}(w)](w - u^1)_{x^i} + [c(v) - c(w)](w - u^1) + [\bar{b}^i(v) - \bar{b}(w)]u^1_{x^i} + [c(v) - c(w)]u^1.
$$

By Theorem 3.5

$$
\| \mathcal{R}v - \mathcal{R}w \|_{\mathcal{A}^1_{p,d}(\Theta,\tau')} \leq N \left( \| \bar{f}^i \|_{L_{p,d}(\Theta,\tau')} + \| \bar{f} \|_{L_{p,d}(\Theta,\tau')} \right).
$$

Since

$$
\| a^{ij}(v) - a^{ij}(u^1) \| + \| \psi(b^i(v) - b^i(u^1)) \| \leq K_3 |v - u^1| \leq K_3,
$$

it follows that

$$
\| a^{ij}(v) - a^{ij}(u^1) \|_{L_{p,d}(\Theta,\tau')} + \| \psi(b^i(v) - b^i(u^1)) \|_{L_{p,d}(\Theta,\tau')} \leq N K_3 (\| v - w \|_{L_{p,d}(\Theta,\tau')} + \| \psi^{-1}(v - w) \|_{L_{p,d}(\Theta,\tau')}).
$$

Also, using

$$
\sup_x (|w(t,x) - u^1(t,x)| + |\psi(w - u^1)(x,t)| + |u^1(t,x)| + |\psi u^1(t,x)|) \leq 2 + n_0
$$

for $t \leq \tau'$, we get

$$
\| a^{ij}(v) - a^{ij}(w) \|_{L_{p,d}(\Theta,\tau')} \leq 2 \| \psi^{-1}(a^{ij}(v) - a^{ij}(w)) \|_{L_{p,d}(\Theta,\tau')} \leq 2 K_3 \| \psi^{-1}(v - w) \|_{L_{p,d}(\Theta,\tau')},
$$

and similarly

$$
\| b^i(v) - b(w) \|_{L_{p,d}(\Theta,\tau')} + \| [a^{ij}(v) - a^{ij}(w)]u^1_{x^i} \|_{L_{p,d}(\Theta,\tau')} + \| [b^i(v) - b(w)]u^1 \|_{L_{p,d}(\Theta,\tau')} \leq N K_3 |v - w|_{\mathcal{A}^1_{p,d}(\Theta,\tau')}.
$$

Hence,

$$
\| \bar{f}^i \|_{L_{p,d}(\Theta,\tau')} \leq N K_3 |v - w|_{\mathcal{A}^1_{p,d}(\Theta,\tau')}.
$$
where $N$ depends only on $d$, $p$, $\gamma$, $\delta$, $K$, $T$, $\mathcal{O}$, and $n_0$. Furthermore, based on the same computations, we also get

$$\|\psi\tilde{f}\|_{\mathbb{F}^{-1}_{p,d}(O,\tau')} \leq N\|\psi\tilde{f}\|_{\mathbb{F}^{-1}_{p,d}(O,\tau')} \leq NK_3\|v - w\|_{\mathcal{S}^1_{p,d}(O,\tau')}.$$ 

Therefore, taking $K_3$ small enough, we obtain

$$\|\mathcal{R}v - \mathcal{R}w\|_{\mathcal{S}^1_{p,d}(O,\tau')} \leq \frac{1}{2}\|v - w\|_{\mathcal{S}^1_{p,d}(O,\tau')} \quad \forall v, w \in \Phi(\tau').$$

(4.8)

For $n = 2, 3, \ldots$, define $u^{n+1} = \mathcal{R}u^n$ inductively. Then $\{u^n : n = 1, 2, \ldots\}$ becomes a Cauchy sequence in $\mathcal{S}^1_{p,d}(O,\tau')$. Let $u$ be the limit of $u^n$ in $\mathcal{S}^1_{p,d}(O,\tau')$.

Then using the relation

$$du^{n+1} = \left[D_i \left(a^{ij}(u^n)u^{n+1}_{xj} + [a^{ij}(u^n) - a^{ij}(u^n)]u^n_{xj} + b^i(u^n)u + [b^i(u^n) - b^i(u^n)]u^n\right)
+ D_t f^i + \tilde{b}^i(u^n)u^{n+1}_{xt} + \tilde{b}^i(u^n) - \tilde{b}^i(u^n)\right]u^n + c(u^n)u^{n+1}
+ [c(u^n) - c(u^n)]u^n + f \right]dt + g^kDW^k_t, \quad t \leq \tau',$$

and taking $n \to \infty$, we find $u(0, \cdot) = u_0$ and the equality

$$du = \left[D_i \left(\tilde{a}^{ij}(u)u_{xj} + b^i(u)u + f^i\right) + \tilde{b}^i(u)v_{xt} + c(u)v + f \right]dt + g^kDW^k_t$$

holds for almost all $t \leq \tau'$ (a.s.). It follows that the above equality holds for all $t \leq \tau'$ (a.s.) since both sides above are continuous in $t$ due to Theorem 2.3

**Step 2.** We remove the condition that $K_3$ is very small.

For $\delta > 0$, consider the transform $u(t, x) \mapsto v(t, x) := \frac{1}{\delta}u(t, x)$. Then $u$ is a solution of (141) if and only if

$$dv = \left[D_i \left(\tilde{a}^{ij}(v)u_{xj} + \tilde{b}^i(v)v + \tilde{f}^i\right) + \tilde{b}^i(v)v_{xt} + \tilde{c}(v)v + f \right]dt + g^kDW^k_t,$n
\quad t \leq \tau', \quad v(0) = u_0,$n

(4.9)

where

$$\tilde{a}^i(t, x, z) = a^i(t, x, \delta z), \quad \tilde{b}^i(t, x, z) = \tilde{b}^i(t, x, \delta z), \quad \tilde{c}(t, x, \delta z) = \frac{1}{\delta}c(t, x), \quad \tilde{f}^i(t, x) = \frac{1}{\delta}f^i(t, x), \quad \tilde{g}^k(t, x) = \frac{1}{\delta}g^k(t, x).$$

By taking $\delta$ small enough, (4.9) has a solution $v \in \mathcal{S}^1_{p,d}(O, \tau')$ due to Step 1, and therefore (4.1) has a solution $u \in \mathcal{S}^1_{p,d}(O, \tau').$

5. **Proof of Theorem 2.9**

First we prove the uniqueness result.

**Lemma 5.1.** Suppose that Assumptions 2.3, 2.6, 2.7, and 2.8 hold. Let $f^i \in \mathbb{H}_{10}^{\gamma} \mathcal{F}^{-1}_{p,d}(O, \tau), g \in \mathbb{F}^{\gamma}_{p,d}(O, \tau, l_2)$, and $u_0 \in U_{p,d}^{\gamma}(O)$. Assume that $u, v \in \mathcal{S}^1_{p,d}(O, \tau)$ are solutions to the equation

$$du = \left[D_i \left(\tilde{a}^{ij}(u)u_{xj} + \tilde{b}^i(u)u + f^i\right) + \tilde{b}^i(u)v_{xt} + c(u)v + f \right]dt + g^kDW^k_t, \quad t \leq \tau$$

$$\quad u(0, \cdot) = u_0.$$

Then $u = v$ in $\mathcal{S}^1_{2,d}(O, \tau)$, and moreover $u \in \mathcal{S}^1_{p,d,loc}(O, \tau)$ for any $\gamma < \gamma_0$. 

\[\square\]
Similarly, by Lemma 2.1(ii),
\[
\mathbb{E}|u_0|_{C^{1+\gamma}_{\mathbb{R}^d\times\mathbb{R}^d}(\Omega)} \leq \mathbb{N}\mathbb{E}|\psi^{2/p-1}u_0|^p_{H^{1+\gamma}_{\mathbb{R}^d\times\mathbb{R}^d}(\Omega)} \leq N\|u_0\|^p_{L^2(\Omega)}.
\]
Thus by Theorem 3.8 there exists a \(\alpha \in (0, 1)\) so that
\[
\mathbb{E}|u|^p_{C^\alpha([0, \tau]\times\Omega)} + \mathbb{E}|v|^p_{C^\alpha([0, \tau]\times\Omega)} < \infty.
\]
Define
\[
\tau_n^{(1)} := \inf\{t \leq \tau : |a^{ij}(t, x, u)|_{C^{\alpha}([0, \tau]\times\Omega)} + |b(t, x, u)|_{C^{\alpha}([0, \tau]\times\Omega)} > n\},
\]
\[
\tau_n^{(2)} := \inf\{t \leq \tau : |a^{ij}(t, x, v)|_{C^{\alpha}([0, \tau]\times\Omega)} + |b(t, x, v)|_{C^{\alpha}([0, \tau]\times\Omega)} > n\},
\]
and \(\tau_n = \tau_n^{(1)} \land \tau_n^{(2)}\). Then \(\tau_n \to \tau\) \(\text{(a.s.)}\) as \(n \to \infty\) and by Theorem 3.5 for each \(n\) we have \(u, v \in \mathcal{H}^{1+\gamma}_{p, \theta}(\tau_n, \Theta)\) for all \(\gamma < \alpha\). Fix \(0 < \gamma < \alpha\). Due to (2.4),
\[
\mathbb{E}|u|^p_{C([0, \tau_n]\times\Omega)} + \mathbb{E}|\psi^{\gamma}Du|^p_{C([0, \tau_n]\times\Omega)} + \mathbb{E}|v|^p_{C([0, \tau_n]\times\Omega)} + \mathbb{E}|\psi^{\gamma}Dv|^p_{C([0, \tau_n]\times\Omega)} < \infty.
\]
Thus there exists a sequence of stopping times \(\tau_{n,m} \leq \tau_n\) such that \(\tau_{n,m}\) converges to \(\tau_n\) \(\text{(a.s.)}\) as \(m \to \infty\), and \(\text{(a.s.)}\)
\[
|u|^p_{C([0, \tau_{n,m}]\times\Omega)} + |v|^p_{C([0, \tau_{n,m}]\times\Omega)} + |\psi^{\gamma}Du|^p_{C([0, \tau_{n,m}]\times\Omega)} + |\psi^{\gamma}Dv|^p_{C([0, \tau_{n,m}]\times\Omega)} \leq m. \hspace{1cm} (5.1)
\]
Due to Assumption 2.8
\[
D_i \left(a^{ij}(u)u_{x^j} - a^{ij}(v)v_{x^j}\right)
\]
\[
= D_i \left(\int_0^1 \frac{d}{d\vartheta} \left( a^{ij}(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v)_{x^j} \right) \right) d\vartheta
\]
\[
= D_i \left(\int_0^1 a^{ij}(\vartheta u + (1 - \vartheta)v) \vartheta (u - v)_{x^j} \right)
\]
\[
+ D_i \left(\int_0^1 a^{ij}_{\vartheta}(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v)_{x^j} \right) du \, (u - v).\]
Similarly,
\[
D_i \left(b(u)v - b^i(v)v\right)
\]
\[
= D_i \left(\int_0^1 \frac{d}{d\vartheta} \left( b(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v) \right) \right) d\vartheta
\]
\[
= D_i \left(\int_0^1 b(\vartheta u + (1 - \vartheta)v) \vartheta (u - v) \right)
\]
\[
+ D_i \left(\int_0^1 b_{\vartheta}(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v) \vartheta (u - v) \right),\]
\[
\bar{b}^i(u)u_{x^j} - \bar{b}^i(v)v_{x^j}
\]
\[
= \int_0^1 \frac{d}{d\vartheta} \left( \bar{b}^i(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v)_{x^j} \right) d\vartheta
\]
\[
= \int_0^1 \bar{b}^i(\vartheta u + (1 - \vartheta)v) \vartheta (u - v)_{x^j}
\]
\[
+ \int_0^1 \bar{b}_{\vartheta}^i(\vartheta u + (1 - \vartheta)v)(\vartheta u - (1 - \vartheta)v)_{x^j} d\vartheta \right) du \, (u - v),\]
and
\[ c(u)v - c(v)v = \int_0^1 \frac{d}{d\theta} (c(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v)) d\theta = \int_0^1 c(\theta u^1 + (1 - \theta)v) d\theta (u - v) + \int_0^1 c_u(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v) d\theta (u - v). \]

Defining
\[ \tilde{a}^{ij} := \int_0^1 a^{ij}(\theta u + (1 - \theta)v) d\theta \]
\[ \tilde{b} := \int_0^1 \tilde{a}^{ij}(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v) x, d\theta + \int_0^1 b(\theta u + (1 - \theta)v) d\theta \]
\[ + \int_0^1 b_u(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v) d\theta \]
\[ \tilde{b}^i := \int_0^1 \tilde{b}(\theta u + (1 - \theta)v) d\theta \]
\[ \tilde{c} := \int_0^1 \tilde{b}_u(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v) x, d\theta + \int_0^1 c(\theta u^1 + (1 - \theta)v) d\theta \]
\[ + \int_0^1 c_u(\theta u + (1 - \theta)v)(\theta u - (1 - \theta)v) d\theta, \]
we have
\[ (u - v)_t := D_t[\tilde{a}^{ij}(u - v)_{x,} + \tilde{b}^i(u - v)] + \tilde{b}^i(u - v) + \tilde{c}(u - v), \quad t \leq \tau_{n,m}. \]

Due to (5.1), (5.3), and the definition of stopping times \( \tau^{(1)}_n, \tau^{(2)}_n \), the coefficients \( \tilde{a}^{ij} \) are uniformly continuous and
\[ \sup_{\omega} \sup_{t \leq \tau_{n,m}} \left( |\rho(\tilde{b} + \tilde{b}^i)| + |\rho^2 c| \right) \rightarrow 0 \]
as \( \rho(x) \rightarrow 0 \). Thus we can apply Theorem 3.5 to conclude \( u = v \) as an element in \( H^{1}_{p,d}(O, \tau_{n,m}) \). To obtain better regularity, recall Assumption 2.7. Due to (5.1), there exists a \( K \) such that for any \( \gamma < \gamma_0 \) and \( (\omega, t) \),
\[ |a^{ij}(t, \cdot, u(\cdot))|_{\gamma} + |\psi b^i(t, \cdot, u(\cdot))|_{\gamma} + |\psi \tilde{b}^i(t, \cdot, u(\cdot))|_{\gamma} + |\psi^2 c(t, \cdot, u(\cdot))|_{\gamma} \leq K. \]

Therefore by Theorem 3.5 again, we conclude \( u \in H^{1+\gamma}_{p,d}(O, \tau_{n,m}) \). The lemma is proved. \( \Box \)

**Proof of Theorem 2.9**

We prove that there is a unique solution \( u \) in the class \( H^{1}_{p,d}(O, \tau) \) and also show this solution belongs to \( H^{1+\gamma}_{p,d, loc}(O, \tau) \) for any \( \gamma < \gamma_0 \). The estimates 2.8 and 2.9 are easy consequences of Theorem 3.7 and Corollary 2.4.

We divide our proof into two steps.
Step 1. We assume \( \nu(t, x) = 0 \).

Due to Lemma 5.1, we only need to prove the existence of a solution in the class \( \mathfrak{H}_{1, d}(O, \tau) \). Define

\[
\Pi := \{ \text{stopping times } \tau_n \leq \tau: \text{equation} \ (1.1) \ \text{has a solution } u \in \mathfrak{H}_{1, d}(O, \tau_n) \}.
\]

Observe that if \( \tau_{a_1}, \tau_{a_2} \in \Pi \), then \( \tau_{a_1} \lor \tau_{a_2} \in \Pi \). Indeed, if \( u^1, u^2 \) are solutions for \( t \leq \tau_{a_1} \) and \( t \leq \tau_{a_2} \), respectively, then \( u^1 = u^2 \) for \( t \leq \tau_{a_1} \land \tau_{a_2} \) by Lemma 5.1. Thus one can find a solution \( u \in \mathfrak{H}_{1, d}(O, \tau_n) \) defining \( u = u^1 \) if \( \tau_{a_1} \geq \tau_{a_2} \), and \( u = u^2 \) otherwise.

Define \( \bar{\tau} := \sup_{\tau_n \in \Pi} \{ E\tau_n \} \). Then there exist nondecreasing stopping times \( \tau_n \) (otherwise one can consider \( \tau_1 \lor \cdots \lor \tau_n \)) such that \( E\tau_n \uparrow \bar{\tau} \). Put \( \tilde{\tau} := \lim_{n \to \infty} \tau_n \). Then \( \tilde{\tau} \) becomes a stopping time since the filtration \( \mathcal{F}_t \) is right continuous and \( E\tilde{\tau} = \bar{\tau} \) by the monotone convergence theorem.

We will show \( \tilde{\tau} \in \Pi \). Since \( \tau_n \) are nondecreasing, using the uniqueness result of Lemma 5.1, we conclude that there exists \( u \in \mathfrak{H}_{1, d, loc} (\tilde{\tau}, O) \) such that \( u(0, \cdot) = u_0 \) and for each \( n \)

\[
du = (D_i [a^{ij}(u)u_{xj} + b(u)u + f] + \tilde{b}(u)u_{xj} + c(u)u + f) + g^k dW^k_t, \quad t \leq \tau_n.
\]

Moreover due to Theorem 3.1,

\[
\|u\|_{\mathfrak{H}_{1, d}(O, \tau_n)} \leq N \left( \sum_{i=1}^d \|f^i\|_{L^2_d(O, \tilde{\tau})} + \|\psi f\|_{H^{-1}_d(O, \tilde{\tau})} + \|g\|_{L^2_d(O, \tilde{\tau}, t_2)} \right)
\]

where \( N \) is independent of \( n \). Thus taking \( n \to \infty \), we have

\[
\|\psi^{-1}u\|_{H^{-1}_d(O, \tilde{\tau})} < \infty.
\]

This implies that the right hand side of \( (5.2) \) is \( \psi^{-1}H^{-1}_d \)-valued continuous function on \( [0, \bar{\tau}] \), and therefore \( u \) is continuously extendible to \( [0, \bar{\tau}] \). Therefore \( (5.2) \) holds for \( t \leq \tilde{\tau} \). This with \( (5.3) \) shows that \( u \in \mathfrak{H}_{1, d}(O, \tilde{\tau}) \). Consequently, \( \tilde{\tau} \in \Pi \), and by Theorem 2.3 \( u \in C([0, \tilde{\tau}); L^2_d(O)) \) (a.s.).

Next we claim \( \tilde{\tau} = \tau \) (a.s.). Suppose not. Then \( P(\tilde{\tau} < \tau) > 0 \). By Theorem 2.3 (iii),

\[
\|u(\tilde{\tau}, \cdot)\|_{L^2_d(O)} = E\|u(\tilde{\tau}, \cdot)\|_{L^2_d(O)} < \infty.
\]

Denote \( \tilde{u}_0 := u(\tilde{\tau}, \cdot), \mathcal{F}_t^\tilde{\tau} := \mathcal{F}_{t+\tilde{\tau}}, \) and \( \tilde{w}_k := w_{t+\tilde{\tau}} - w_k^{\tilde{\tau}} \). Then \( \tilde{w}_k \) are independent Wiener processes relative to \( \mathcal{F}_t^\tilde{\tau}, \tilde{u}_0 \) is \( \mathcal{F}_0^\tilde{\tau} \)-measurable, and \( \tilde{\tau} := \tau - \tilde{\tau} \) is a nonzero stopping time with respect to \( \mathcal{F}_t^\tilde{\tau} \). Consider the equation

\[
d\tilde{v} = \left[ D_i \left( a^{ij}(\tilde{\tau} + t, x, v)\tilde{v}_{xj} + b(\tilde{\tau} + t, x, \tilde{v})v + f(\tilde{\tau} + t) \right) \right. \\
\left. + \tilde{b}(\tilde{\tau} + t, x, v)v_{xj} + c(\tilde{\tau} + t, x, v)v + f(\tilde{\tau} + t) \right]dt \\
+ g^k(\tilde{\tau} + t)d\tilde{w}_k^t, \quad t \leq \tilde{\tau}; \quad \tilde{v}(0, \cdot) = \tilde{u}_0.
\]

Then by Lemma 4.1 there exists a nonzero stopping time \( \sigma \leq \tau - \tilde{\tau} \) (with respect to \( \mathcal{F}_t^\tilde{\tau} \)) so that the above equation has an \( \mathcal{F}_t^\tilde{\tau} \)-adapted solution

\[
\tilde{v} \in \mathfrak{H}_{1, d}(O, \sigma) \subset \mathfrak{H}_{1, d}(O, \sigma).
\]
Define \( \tau_0 = \bar{\tau} + \sigma \). It is easy to check that \( \mathbb{E}\tau_0 > r \) and \( \tau_0 \) is a stopping time since \( \mathcal{F}_t \) is right continuous. Define

\[
U(t, x) = \begin{cases} 
  u(t, x) & : t \leq \bar{\tau} \\
  \hat{v}(t - \bar{\tau}, x) & : \bar{\tau} < t \leq \tau_0 
\end{cases}
\]

then \( U \) satisfies (1.1) for \( t \leq \tau_0 \) (a.s.) and \( U \in \mathcal{F}_{\tau_0} \). This is a contradiction since \( \mathbb{E}\tau_0 > r \) and \( \tau_0 \in \Pi \). Therefore, we conclude \( \bar{\tau} = \tau \), and the proof for the case \( \nu = 0 \) is completed.

**Step 2.** (General case) Let \( \gamma < \gamma_1 < \gamma_0 \) and denote

\[
h(t, x) = e^{-\int_0^t \nu^k(s,x) dW^k_s}, \quad \tau_n = \inf\{t \leq \tau : |h(t, \cdot)|^{(0)}_{1+\gamma_1} > n\},
\]

\[
a^{ij}(t, x, z) = a^{ij}(t, x, hz), \quad b^{ij}(t, x, z) = b^{ij}(t, x, hz), \quad \hat{b}^{ij}(t, x, z) = \hat{b}^{ij}(t, x, hz),
\]

\[
\hat{c}(t, x, z) = c(t, x, hz) - \sum_k (\nu^k(t, x))^2, \quad \hat{f}^i = h^{-1} f^i, \quad \hat{g}^k := h^{-1} g^k.
\]

Then Assumptions 2.5, 2.6, 2.7, and 2.8 hold with \( \tilde{a}^{ij}, \tilde{b}^{ij}, \tilde{c}, \tilde{f}^i, \tilde{f}, \) and \( \tilde{g} \) on \([0, \tau_0]\). Therefore by Step 1, there exists a \( u^{(n)} \in \mathcal{F}_{\tau_n} \cap \mathcal{F}^{1+\gamma}_{p,d,loc}(\mathcal{O}, \tau_n) \), \( \gamma < \gamma_0 \), such that

\[
du^{(n)} = \left[ D_i\left( \tilde{a}^{ij}(u^{(n)}) u_x^{(n)} + \tilde{b}^{ij}(u^{(n)}) u_u^{(n)} + \tilde{f}^i \right) + \tilde{b}^{ij}(u^{(n)}) u_x^{(n)} + \tilde{c}(u^{(n)}) u_u^{(n)} + \tilde{f}^j \right] dt,
\]

\[
+ \hat{g}^k dW^k, \quad t \leq \tau_n; \quad u^{(n)}(0, \cdot) = u_0.
\]

Denote

\[
v^{(n)}(t, x) := u^{(n)}(t, x) e^{-\int_0^t \nu^k(s,x) dW^k_s} = u(t, x) h(t, x).
\]

Then

\[
v^{(n)} \in \mathcal{F}_{\tau_n} \cap \mathcal{F}^{1+\gamma}_{p,d,loc}(\mathcal{O}, \tau_n).
\]

Moreover by Itō’s formula

\[
dh = \left( h \nu^k \right)^2 dt - (h \nu^k) dW_t^k
\]

and

\[
dv^{(n)} = u^{(n)}(dh) + (du^{(n)}) h
\]

\[
= \left[ D_i\left( \tilde{a}^{ij}(u^{(n)}) u_x^{(n)} + \tilde{b}^{ij}(u^{(n)}) u_u^{(n)} + \tilde{f}^i \right) + \tilde{b}^{ij}(u^{(n)}) u_x^{(n)} u_u^{(n)} + \tilde{c}(u^{(n)}) u_u^{(n)} + \tilde{f}^j \right] dt
\]

\[
+ \left( u^{(n)} h \nu^k \right)^2 dt - (u^{(n)} h \nu^k + \hat{g}^k h) dW_t^k
\]

\[
= \left[ D_i\left( \tilde{a}^{ij}(v^{(n)}) v_x^{(n)} + \tilde{b}^{ij}(v^{(n)}) v_u^{(n)} + \tilde{f}^i h \right) + \tilde{b}^{ij}(v^{(n)}) v_x^{(n)} u_u^{(n)} + \tilde{c}(v^{(n)}) v_u^{(n)} + \tilde{f}^j \right] dt
\]

\[
+ \left( \nu^k v^{(n)} + \hat{g}^k \right) dW_t^k, \quad t \leq \tau_n; \quad u(0, \cdot) = u_0.
\]

Using the uniqueness result of equation (5.4) (Lemma 5.1), we conclude that \( v^{(n)} \) is also unique solution to (5.5), and

\[
v^{(n)} = v^{(m)} \quad \text{on } [0, \tau_n) \quad \forall n \leq m.
\]
Therefore there exists a \( v \in \tilde{H}^{1}_{2,d,\text{loc}}(O, \tau) \cap \tilde{H}^{1+\gamma}_{p,d,\text{loc}}(O, \tau) \) such that
\[
v^{(n)} = v \quad \text{on} \quad [0, \tau_n) \quad \forall n.
\]

By Theorem 3.7, there exists a \( \tilde{v} \in H^{1/2}_{d}(O, \tau) \) to the equation
\[
d\tilde{v} = \left[ D_{i} \left( a^{ij}(v)\tilde{v}_{x}^{j} + b^{i}(v)\tilde{v} + f^{i}h \right) + \tilde{b}^{i}(v)\tilde{v}_{x}^{i} + c(v)\tilde{v} + f \right] dt + \left( \nu^{k}\tilde{v} + g^{k} \right) dW_{t}^{k}, \quad t \leq \tau, \quad x \in O, \quad u(0, \cdot) = u_{0}.
\]

(5.6)

Due to Lemma 5.1

\[
v = \tilde{v} \quad \text{on} \quad [0, \tau_n) \quad \forall n
\]

and therefore
\[
v = \tilde{v} \in \tilde{H}^{1}_{2,d}(O, \tau) \cap \tilde{H}^{1+\gamma}_{p,d,\text{loc}}(O, \tau).
\]

The theorem is proved. \( \square \)

REFERENCES

[1] G. Da Prato and L. Tubaro. Fully nonlinear stochastic partial differential equations. *SIAM Journal on Mathematical Analysis*, 27(1):40–55, 1996.

[2] D. Gilbarg and L. Hörmander. Intermediate schauder estimates. *Archive for Rational Mechanics and Analysis*, 74(4):297–318, 1980.

[3] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[4] I. Kim and K.-H. Kim. Some \( L_{p} \) and Hölder Estimates for Divergence Type Nonlinear SPDEs on \( C^{1} \)-Domains. *Potential Analysis*, 41(2):583–612, 2014.

[5] K.-H. Kim. \( L_{q}(L_{p}) \) theory and Hölder estimates for parabolic SPDEs. *Stochastic Processes and their Applications*, 114(2):313–330, 2004.

[6] K.-H. Kim. Sobolev space theory of SPDEs with continuous or measurable leading coefficients. *Stochastic Processes and their Applications*, 119(1):16–44, 2009.

[7] K.-H. Kim. On \( L_{p} \) theory of stochastic partial differential equations of divergence form in \( C^{1} \) domains Probab. Theory Related Fields, 130: 473–492, 2004.

[8] K.-H. Kim and N. V. Krylov. On the Sobolev space theory of parabolic and elliptic equations in \( C^{1} \) domains. *SIAM journal on mathematical analysis*, 36(2):618–642, 2004.

[9] N. V. Krylov. Weighted Sobolev spaces and Laplace’s equation and the heat equations in a half space. *Communications in Partial Differential Equations*, 24(9-10):1611–1653, 1999.

[10] N. V. Krylov. Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces. *Journal of Functional Analysis*, 183(1):1–41, 2001.

[11] N.V. Krylov and S.V. Lototsky. A Sobolev space theory of SPDEs with constant coefficients in a half space SIAM. Math. Anal., 32(1): 19–33, 1000.

[12] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Journal of Soviet Mathematics*, 16(4):1233–1277, 1981.

[13] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural’ceva. Linear and Quasi-linear Parabolic Equations Nauka, Moscow, 1967 (in Russian) ; Amer. Math. Soc., Providence, RI, 1968 (in English).

[14] W. Liu and M. Röckner. *Stochastic Partial Differential Equations: An introduction*. Springer, 2015.

[15] S. V. Lototsky. Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations. *Methods and Applications of Analysis*, 7(1):195–204, 2000.

[16] S. V. Lototsky. Linear stochastic parabolic equations, degenerating on the boundary of a domain. *Electronic Journal of Probability*, 6(24):1–14, 2001.

[17] B. L. Rozovskii. Stochastic evolution systems Kluner, Dordrecht, 1990.

[18] H. Yoo. On the unique solvability of some nonlinear stochastic PDEs. *Electronic Journal of Probability*, 3(11):1–22, 1998.
Center for Mathematical Challenges, Korea Institute for Advanced Study (KIAS),
85 Hoegiro Dongdaemun-gu, Seoul 130-722, Republic of Korea
E-mail address: waldo@cias.re.kr

Department of Mathematics, Korea University, 1 Anam-dong Sungbuk-gu, Seoul, South
Korea 136-701
E-mail address: kyeonghun@korea.ac.kr