Adomian Decomposition Method for the solitary wave solution to the modified Korteweg-de Vries equation

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Abstract. The Adomian decomposition method (ADM), is the one of the semi-analytical method, will be applied to study the time evolution of the soliton solution to the modified Korteweg-de Vries equation. We also provide the conservation laws to verify the dark solution obtained from this method. The ADM gives a good result for the approximation solution of the weakly nonlinear wave equation.

1. Introduction
Solitary waves are well known as the solution of the nonlinear wave equations. These kind of waves retain their shape as a result of the balance between nonlinearity and dispersion. The most familiar form of nonlinear wave equation to produce the solitary wave, is called Korteweg-de Vries (KdV) equation,

\[ \phi_t + \phi \phi_x + \phi_{xxx} = 0, \]

where \( \phi \) subscripts \( x \) and \( t \) denote differentiation with respect to space and time, respectively, which originally derived for water, (see e.g. [1] for further details). \( \phi \) represents the amplitude of the wave. Although originally derived for water, the KdV equation occurs in many other contexts. It governs weakly nonlinear ion-acoustic waves in plasma when the electrons have a Maxwellian distribution [2]. In this case, \( \phi \) is the electrostatic potential. Schamel proposed allowing for the trapping of some of the electrons on ion-acoustic waves [3]. The free and trapped electrons have different temperatures, although both still have Maxwellian distributions. This leads a modified equation for ion-acoustic waves,

\[ \phi_t + \phi^{1/2} \phi_x + \phi_{xxx} = 0, \] (1)

The traveling wave solution of (1) is not difficult to find [4], but to express both of spatial and temporal coordinates might be more difficult. As well as, (1) is not an integrable equation [5], the inverse scattering method [6] can not be applied to obtain the solution. To obtain the soliton solution of (1), we then consider method known as the Adomian decomposition method (ADM) [7, 8, 9]. The initial condition for (1) will be used as,

\[ \phi_0(x) = (30\eta^2)^2 \text{sech}^4 \eta(x-x_0), \] (2)
η is a constant and $x_0$ denotes the initial position of the solitary wave. We also derive the conservation laws for the modified KdV equation where the solution must be satisfied to confirm the validity of the solution.

2. Adomian decomposition method for the modified KdV equation

The ADM has been extensively used for semi-analytical solution of the nonlinear wave equations such as nonlinear Schrödinger equation and its modified form [10, 11, 12]. The idea of this method is to rewrite the nonlinear terms into the Adomian polynomials. We now consider the nonlinear evolution equation as follows

$$\phi_t + \hat{L}\phi + \hat{N}\phi = 0,$$

where $\hat{L}$ is the linear differential operator and $\hat{N}$ denotes the nonlinear differential operator. The solution will be written in the form of a power series for $N$ terms, namely,

$$\phi(x, t) = \sum_{n=0}^{N} u_n(x, t),$$

(3)

where $u_n(x, t)$ represents the unknown function. For the nonlinear operator, the expression for the power series will be shown as

$$\hat{N}\phi = \sum_{n=0}^{N} A_n$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \hat{N} \left( \sum_{i=0}^{\infty} \lambda^i u_i(x, t) \right) \right]_{\lambda \rightarrow 0}.$$  

The solution of (3) can be then determined as

$$\sum_{n=0}^{N} u_n(x, t) = L_t^{-1} \left[ \hat{L} \left( \sum_{n=0}^{N} u_n \right) \right] - L_t^{-1} \sum_{n=0}^{N} A_n$$

We will apply this method to find the solitary wave solution.

3. Conservation laws for modified KdV equation

The conserved quantities will be derived which they do not vary with time. A conservation law can be expressed as

$$\frac{dQ}{dt} = 0,$$

where $Q$ is a conserved quantity. Rearrange (1) to give

$$\phi_t = -\phi^{1/2}\phi_x - \phi_{xxx},$$

and integrate with respect to $x$, we then obtain

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \phi dx = -\frac{2\phi^{3/2}}{3} \bigg|_{-\infty}^{\infty} = 0,$$

$\phi$, $\phi_x$ and $\phi_{xx}$ have the same value as $x \rightarrow \pm \infty$. It is sometimes referred as the 'Mass'

$$\int_{-\infty}^{\infty} \phi dx = M.$$  

(4)
Another conserved quantity is easily to obtain by multiplying both sides with $\phi$ before integrating. This leads to

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{\phi^2}{2} dx = -\frac{2}{5} \left( \phi^{3/2} \bigg|_{-\infty}^{\infty} \right) + \left( \frac{\phi_x^2}{2} \bigg|_{-\infty}^{\infty} \right) = 0. $$

This is referred as the 'Momentum',

$$\int_{-\infty}^{\infty} \frac{\phi^2}{2} dx = P. \quad (5)$$

The last conserved quantity, we first differentiate (1) with respect to $x$ and multiply both sides with $\phi_x$ and then integrate both sides with respect to $x$,

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{\phi^2}{2} dx = \int_{-\infty}^{\infty} \phi^{1/2} \phi_x \phi_{xx} dx \quad (6)$$

To determine the right-hand side of (6), we then multiply (1) with $\phi^{3/2}$ and integrate with respect to $x$,

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{2\phi^{5/2}}{5} dx = \frac{3}{2} \int_{-\infty}^{\infty} \phi^{1/2} \phi_x \phi_{xx} dx \quad (7)$$

From (6) and (7), we have the conserved quantity called 'Energy',

$$\int_{-\infty}^{\infty} \frac{3\phi^2}{4} - \frac{2\phi^{5/2}}{5} dx = E. \quad (8)$$

To examine some results, we consider $\eta = 0.15$, and the initial condition at the origin ($\xi_0 = 0$), can be shown in Figure 1

**Figure 1.** shows the initial profile

We corrected up to 3 terms, ($N = 3$), where the initial condition is written as

$$u_0 = 0.455 \text{sech}^4(0.15x),$$
while $u_1$, $u_2$ and $u_3$ are determined as follows,

$$u_1 = t \tanh(0.15x)(0.098 \text{sech}^6(0.15x) + 0.098 \text{sech}^4(0.15x) \tanh^3(0.15x))$$

$$u_2 = t^2(-0.027 \text{sech}^{10}(0.15x) + 0.0053 \text{sech}^8(0.15x) \tanh^2(0.15x) + 0.0186 \text{sech}^6(0.15x) \tanh^4(0.15x) + 0.106 \text{sech}^4(0.15x) \tanh^6(0.15x))$$

$$u_3 = t^3(\text{csch}^7(0.3x) \sinh^8(0.15x)(\text{csch}^4(0.3x) \sinh^4(0.15x)))^{3/2} \left(-14.693 - 2.938 \sinh^2(0.15x) + 38.202 \sinh^4(0.15x) + 26.447 \sinh^6(0.15x)\right) + \text{sech}^4(0.15x) \tanh(0.15x)(0.001124 \text{sech}^8(0.15x) - 0.000884 \text{sech}^6(0.15x) \tanh^2(0.15x) - 0.00437 \text{sech}^4(0.15x) \tanh^4(0.15x) - 0.0016 \text{sech}^2(0.15x) \tanh^6(0.15x) + 0.000765 \tanh^8(0.15x))$$

The time evolution of the solution is shown in Figure 2. The structure of the solution is nearly coherent or traveling without changing its shape.

![Figure 2](image_url)

**Figure 2.** Time evolution of (1) with $0 \leq N \leq 3$ and $t \in [0, 5]$.

However, we can provide the Mathematica file by direct contact the author (SC).

The conservation laws, moreover, are given in Table 1, where these parameters are conserved during the time evolution.

These results might be improved by adding a few terms for the approximation solution ($N > 3$) but this also increases a computational time.

4. **Conclusion**

The ADM has been quite successfully applied to calculate the solitary wave solution for the modified KdV equation. The good thing about this method is directly obtain the solution without using any assumption to the equation. We are strongly believed to the solitary solution by considering the conservation laws. However, if the nonlinear term contains a strong nonlinearity or some messy forms to the nonlinear operator, ADM will take some computational time to process.
time & Mass (4) & Momentum (5) & Energy (8) \\ 
0 & 4.05 & 1.26534 & -5.69401 \\ 
1 & 4.05 & 1.26533 & -5.69394 \\ 
2 & 4.05 & 1.26519 & -5.69299 \\ 
3 & 4.05 & 1.26465 & -5.68917 \\ 
4 & 4.05 & 1.26335 & -5.68004 \\ 
5 & 4.05 & 1.26102 & -5.66397 \\ 

Table 1. Computed quantities M, P and E for modified KdV which are satisfied the ADM

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