CERTAIN FLOOR FUNCTION SUMS OVER POWERS

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Abstract

We study the sums
\[ S_f(x) = \sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \]
when \( f \) is supported on \( r \)th powers with \( r \geq 2 \). This restriction allows us to give nontrivial estimates for one of the error terms in the asymptotic expansion of \( S_f(x) \). We also state several conjectures related to our results.

1. Introduction and Statement of Results

Since their recent introduction by Bordellès et. al. [1], there has been a fair amount of research into the sums
\[ S_f(x) = \sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right), \]
where \( f \) is an arithmetic function and \( \lfloor \cdot \rfloor \) denotes the integer part function. Under mild assumptions on the growth of \( f \), one can establish decent asymptotic estimates for such sums. For instance, Wu [8] and Zhai [9] showed independently that if \( f(n) \ll n^\alpha \) for some \( 0 \leq \alpha < 1 \), then
\[ S_f(x) = C_f x + O \left( x^{1 + \alpha/2} \right), \tag{1.1} \]
where here and throughout we denote
\[ C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}. \tag{1.2} \]
This estimate has been improved by a number of authors for functions satisfying various additional properties. For instance, if one has a mean square estimate
\[ \sum_{n \leq x} |f(n)|^2 \ll x^\alpha \]
for some \( \alpha \in [1, 2) \), then Wu and Zhao [11] have shown that
\[ C_f x + O \left( x^{2 + 3\alpha/8} \right). \]

In a different direction, one can use exponential sum estimates to improve (1.1) when \( f \) has a convolution structure, say \( f = 1 * g \), or an analogous structure such as that given by Vaughan’s identity in the case \( f = \Lambda \) (see, e.g. [4, 6, 7, 10]).

In this article, we introduce another property that \( f \) may possess that allows us to improve (1.1). Specifically, we consider functions of the form
\[ f(n) = \begin{cases} h(d) & \text{if } n = dr, \\ 0 & \text{otherwise}. \end{cases} \tag{1.3} \]
The reason we consider such functions is that when \( r \geq 2 \), we can give a nontrivial estimate for one of the error terms arising in the asymptotic formula for \( S_f \), namely
\[
S_f^\dagger(x; A) = \sum_{n < A} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right).
\]  
(1.4)

In all previous works, this quantity is estimated trivially using a point-wise upper bound for \( f \).

For functions supported on \( r \)-th powers, we are able to improve this estimate by understanding the spacing between the integers \( n \) for which there exists an integer \( d \) satisfying
\[
\left\lfloor \frac{x}{n} \right\rfloor = d^r.
\]

We do so by following the method of Roth and Halberstam (see, e.g., [2] for a nice survey of these ideas). As an example of our results, if \( h(n) \ll n^\alpha \) for some \( \alpha \in [0, \frac{1}{2} - \frac{1}{r-1}] \), then for \( A \ll x^{\frac{1}{r+1}} \),
\[
S_f^\dagger(x; A) \ll \left( \frac{x}{A} \right)^{\frac{\alpha}{r}} A^{\frac{1}{2(r-1)}},
\]  
(1.5)

which is better than the trivial estimate
\[
S_f^\dagger(x; A) \ll A \left( \frac{x}{A} \right)^{\frac{\alpha}{r}}
\]  
(1.6)

so long as \( A \geq x^{\frac{1}{r+1}} \). Using Lemma 2.2, we will prove the following generalization of (1.1).

**Theorem 1.1.** Let \( r \geq 1 \) be an integer and let
\[
\eta = \begin{cases} 
1 & \text{if } r \text{ is odd}, \\
0 & \text{if } r \text{ is even}. 
\end{cases}
\]  
(1.7)

If \( f \) is given by (1.3) with \( h \) satisfying \( |h(d)| \ll d^\alpha \) for some \( \alpha \in [0, \frac{r+2-\eta}{2r-2+2\eta}] \), then
\[
S_f(x) = C_f x + O \left( x^{\frac{2(1+\alpha)}{3r+\eta}} \right),
\]  
(1.8)

where \( C_f \) is given by (1.2).

We note that the trivial estimate (1.6) leads to
\[
S_f(x) = C_f x + O \left( x^{\frac{1+\alpha}{r+1}} \right),
\]
which is weaker than (1.8) when \( r \geq 3 \). The form of Theorem 1.1 is someone unnatural, owing primarily to the restriction on the range of \( A \) in Lemma 2.2. It seems reasonable that the following simpler and stronger result should hold for a more restricted range of \( \alpha \).

**Conjecture 1.2.** Let \( r \geq 2 \) be an integer. If \( f \) is given by (1.3) with \( h \) satisfying \( |h(d)| \ll d^\alpha \) for some \( \alpha \in [0, \frac{r+2-\eta}{2r-2+2\eta}] \), then
\[
S_f(x) = C_f x + O \left( x^{\frac{1+\alpha}{2r}} \right),
\]
where \( C_f \) is given by (1.2).
It is possible that Conjecture 1.2 holds for a larger range of $\alpha$. The reason to consider only the range given in Conjecture 1.2 is that if we take $A$ maximally in (1.5), then
\[ S_f^+ (x; x^{1+\varepsilon}) \ll x^{\frac{1}{2} + \varepsilon} x^{1+\varepsilon} \ll x^{1+\alpha} \]
precisely when $\alpha \leq \frac{1}{2}$. Conjecture 1.2 then follows from the estimate
\[ \left| \sum_{n \leq x^{1+\varepsilon}} \psi \left( \frac{x}{n^r + \delta} \right) \right| \ll x^{\frac{1}{2} + \varepsilon} (x^{1+\varepsilon}), \]
where $\delta = 0$ or 1 and $\psi(t) = t - \lfloor t \rfloor - \frac{1}{2}$. Using estimates for exponential sums, we are able to prove this in the case $r = 2$, as well as an estimate for general $r$ assuming the existence of certain exponent pairs.

**Theorem 1.3.** Let $r \geq 2$ and let $f$ be given by (1.3) with $h$ satisfying $|h(d)| \ll d^\alpha$ for some $\alpha \in [0, \frac{1}{2}]$. Suppose there exists an exponent pair $(k, \ell)$ such that $\ell = rk$. Then
\[ S_f(x) = C_f x + O \left( x^{\frac{1}{3} + \frac{2}{3} \log x} + x^{\frac{1+\alpha}{2}} \right), \]
where the factor $\log x$ is present only when $\alpha = 0$.

This implies the following improvements to Theorem 1.1 in the cases $r = 2, 3, 4$.

**Corollary 1.4.**
1. Let $r = 2$ and let $f$ be given by (1.3) with $h$ satisfying $|h(d)| \ll d^\alpha$ for some $\alpha \in [0, \frac{1}{3}]$. Then
\[ S_f(x) = C_f x + O \left( x^{\frac{1}{2} + \frac{2}{3}} \log x} + x^{\frac{1+\alpha}{2}} \right), \]
2. Let $r = 3$ and let $f$ be given by (1.3) with $h$ satisfying $|h(d)| \ll d^\alpha$ for some $\alpha \in [0, \frac{1}{3}]$. Then
\[ S_f(x) = C_f x + O \left( x^{\frac{4}{3} + \frac{2}{3}} \log x} \right), \]
3. Let $r = 4$ and let $f$ be given by (1.3) with $h$ satisfying $|h(d)| \ll d^\alpha$ for some $\alpha \in [0, \frac{1}{3}]$. Then
\[ S_f(x) = C_f x + O \left( x^{\frac{5}{2} + \frac{1}{2}} \log x} + x^{\frac{1+\alpha}{2}} \right). \]
In each case, $C_f$ is given by (1.2) and the factor $\log x$ is present only when $\alpha = 0$. In particular, Conjecture 1.4 holds in the case $r = 2$.

In light of Theorem 1.3, we also make the following general conjecture regarding exponent pairs, which may be of independent interest.

**Conjecture 1.5.** For any real $\varepsilon > 0$ and $r \geq 1$, there exists an exponent pair $(k, \ell)$ such that
\[ \left| \frac{\ell}{k} - r \right| < \varepsilon. \]

**Notation.** The Vinogradov-Landau symbols $O, o, \ll, \gg$ have their usual meanings, and we always allow the constants implied by these symbols to depend on the parameter $r$ and the function $f$. We use $n \sim N$ to denote the condition $N < n \leq 2N$ and use $n \asymp N$ to denote the more general condition $c_1N \leq n \leq c_2N$ for some positive constants $c_1 < c_2$. A sum $\sum^D$ denotes a dyadic sum over powers $2^k$. Throughout, $\eta$ is given by (1.7), $e(x) = e^{2\pi ix}$, and $\psi(x) = x - |x| - \frac{1}{2}$. 
2. A Spacing Lemma

In this section, we prove Lemma 2.2 which is the main input needed for Theorem 1.1. To do so, we require the following lemma concerning certain polynomial identities. It is a special case of Lemma 2 of Huxley and Nair [4].

Lemma 2.1. Let \( r \geq 2 \) be a positive integer. For each integer \( l \) with \( 1 \leq l \leq r \), there exist polynomials \( P_l \) and \( Q_l \) such that, as \( x \to 0 \),

1. \( |P_l(x)(1-x)^r - Q_l(x)| \ll |x|^{2l-1} \);
2. \( P_l \) and \( Q_l \) have degree \( l-1 \);
3. the coefficients of \( P_l \) and \( Q_l \) are nonzero integers.

Lemma 2.2. Let \( r \geq 1 \) and let \( f \) be given by (1.3). Suppose \( h(d) \leq C d^\alpha \) for some \( \alpha, C \geq 0 \). Then for each integer \( l \) with \( 1 \leq l \leq r \), there exists a constant \( C \) depending only on \( r \) such that for \( A \leq C x^{2l+1} \), we have

\[
\sum_{n < A} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \ll x^{\frac{2l+1}{r} + \alpha} \left( A^{\frac{2l+1}{r}} - \frac{1}{r} + 1 \right). \tag{2.1}
\]

Proof. As stated in the introduction, we follow the method described in Section 1 of [2]. Trivially

\[
\sum_{n < A} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \ll \sum_{D} D^\alpha T(D), \tag{2.2}
\]

where

\[
T(D) = \left| \{ d \sim D : d^r = \left\lfloor \frac{x}{n} \right\rfloor \text{ for some } n \} \right|.
\]

Suppose \( d,d-a \in T(D) \) so that

\[
d^r = \left\lfloor \frac{x}{n_1} \right\rfloor, \quad (d-a)^r = \left\lfloor \frac{x}{n_2} \right\rfloor. \tag{2.3}
\]

Removing the floor brackets and rearranging, we have

\[
n_1 = \frac{x}{d^r} + O \left( \frac{x}{D^{2r}} \right), \quad n_2 = \frac{x}{(d-a)^r} + O \left( \frac{x}{D^{2r}} \right). \tag{2.4}
\]

Using Lemma 2.1 there exist polynomials \( P_l, Q_l \) such that for \( a = o(D) \), we have

\[
\left| P_l \left( \frac{a}{d} \right) \left( \frac{1-a}{d} \right)^r - Q_l \left( \frac{a}{d} \right) \right| \ll \left( \frac{a}{D} \right)^{2l-1}. \]

Multiplying through by \( d^{r+l-1} \), we have

\[
|P(a,d)(d-a)^r - Q(a,d)d^r| \ll a^{2l-1} D^{r-l} \tag{2.5}
\]

for some homogeneous polynomials \( P_0(a,d), Q_0(a,d) \) of degree \( l-1 \) with integer coefficients. Consider now the modified difference

\[
D = P_0(a,d)n_1 - Q_0(a,d)n_2.
\]
By (2.4) and (2.5),
\[ |D| \leq \frac{x^{2^{-1}}}{D^{l+\tau}} + \frac{x}{D^{2r+1-l}}. \]

The inequality \( A \leq C_r x^{\frac{r+1-l}{2r+1-l}} \) implies
\[ D \gg \left( \frac{x}{A} \right)^{\frac{1}{2}} \geq C_r' x^{\frac{1}{2r+1-l}} \tag{2.6} \]
for some constant \( C_r' \). Taking \( C_r \) sufficiently small, we see that there is a constant \( C''_r \) such that
\[ |D| \leq C''_r x^{2^{-1}} \frac{x}{D^{l+\tau}} + \frac{1}{2}. \tag{2.7} \]

Since \( D \) is an integer, it follows that if \( a \leq C''_r D^{\frac{l+\tau}{2r+1-l}} x^{-\frac{1}{2r+1-l}} = L \), say, for some sufficiently small constant \( C''''_r \), then \( |D| < 1 \), so \( D = 0 \).

Suppose now that \( I \) is a subinterval of \( (D, 2D] \) of length at most \( L \). Using the above analysis, we will show that \( I \) contains at most \( 2l \) elements of \( T(D) \). Suppose that \( d, d-a, d-a-b \in T(D) \), so
\[ d'' = \left\lfloor \frac{x}{n_1} \right\rfloor, \quad (d-a)^r = \left\lfloor \frac{x}{n_2} \right\rfloor, \quad (d-a-b)^r = \left\lfloor \frac{x}{n_3} \right\rfloor. \]

Then since \( |I| \leq L \), it follows that
\[ P_0(a, d)n_2 - Q_0(a, d)n_1 = 0 \]
\[ P_0(a + b, d)n_3 - Q_0(a + b, d)n_1 = 0 \]
\[ P_0(b, d + a)n_3 - Q_0(b, d + a)n_2 = 0 \]

We multiply the second of these by \( P_0(b, d + a) \) and the third by \( P_0(a + b, d) \), then subtract to see that
\[ P_0(a + b, d)Q_0(b, d + a)n_2 - P_0(b, d + a)Q_0(a + b, d)n_1 = 0. \tag{2.8} \]

We view \( d \) and \( a \) as fixed and view the left hand side as a polynomial in \( b \). Note that, by construction, \( P_0(a + b, d)Q_0(b, d + a) \) and \( P_0(b, d + a)Q_0(a + b, d) \) have the same leading coefficient as polynomials in \( b \), namely \( v_P v_Q \), where \( v_P \) and \( v_Q \) are the leading coefficients of \( P_l \) and \( Q_l \), respectively. Clearly we cannot have \( n_1 = n_2 \), and therefore the left side of (2.8) is a polynomial of degree \( 2l - 2 \). Thus (2.8) has at most \( 2l-2 \) solutions in \( b \), and it follows that any interval \( I \) of length at most \( L \) contains at most \( 2l \) elements of \( T(D) \). Dividing the interval \( (D, 2D] \) into \( O(DL^{-1}) \) intervals of length at most \( L \), we deduce that
\[ T(D) \ll \frac{L}{D} + 1 \ll \left( \frac{x}{D^{r-l+1}} \right)^{\frac{1}{3r-1}}. \]

We conclude the proof of Lemma 2.2 by substituting this into (2.2) and executing the dyadic sum in \( D \).

\[ \square \]

3. Proofs of Theorems 1.1 and 1.3

3.1. Main and Error Terms

Similar to previous works, we decompose \( S_f(x) \) via
\[ S_f(x) = S_f^h(x; B) + S_f^\theta(x; A; B) + S_f^\tau(x; A), \]
where
\[ S_{f}^{1}(x; B) = \sum_{n \leq B} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right), \]
\[ S_{f}^{2}(x; A; B) = \sum_{(\frac{x}{A})^{\frac{1}{r}} < d < (\frac{x}{A})^{\frac{1}{r}} + \delta} h(d) \left( \left\lfloor \frac{x}{d^{r}} \right\rfloor - \left\lfloor \frac{x}{d^{r} + 1} \right\rfloor \right), \]
\[ S_{f}^{3}(x; A) = \sum_{d \leq (\frac{x}{A})^{\frac{1}{\alpha}}} h(d) \left( \left\lfloor \frac{x}{d^{r}} \right\rfloor - \left\lfloor \frac{x}{d^{r} + 1} \right\rfloor \right), \]
and \(1 \leq B \leq A \leq \sqrt{x}\) are parameters to be chosen. We treat these sums as follows.

For \(S_{f}^{1}(x; B)\), we apply Lemma 2.2. In the range
\[ 1 \leq l \leq r, \quad \alpha < \frac{r + 1 - l}{2l - 1}, \quad B \leq C_{v} x^{\frac{r + 1 - l}{2l - 1}}, \]
we have
\[ S_{f}^{1}(x; B) \ll \left( \frac{x}{B} \right)^{\frac{\alpha}{r}} x^{\frac{1}{r(d + 1)}} B^{r(\frac{1}{d} - \frac{1}{q})}. \]

For \(S_{f}^{2}(x; A; B)\), we have
\[ \left| S_{f}^{2}(x; A; B) \right| \leq \sum_{(\frac{x}{A})^{\frac{1}{r}} < d < (\frac{x}{A})^{\frac{1}{r}} + \delta} |h(d)| \left( \left\lfloor \frac{x}{d^{r}} \right\rfloor - \left\lfloor \frac{x}{d^{r} + 1} \right\rfloor \right) \]
\[ \ll \sum_{(\frac{x}{A})^{\frac{1}{r}} < d < (\frac{x}{A})^{\frac{1}{r}} + \delta} d^{\alpha} \left( \left\lfloor \frac{x}{d^{r}} \right\rfloor - \left\lfloor \frac{x}{d^{r} + 1} \right\rfloor \right) \]
\[ \leq x \sum_{(\frac{x}{A})^{\frac{1}{r}} < d < (\frac{x}{A})^{\frac{1}{r}} + \delta} \frac{d^{\alpha}}{d^{r}(d^{r} + 1)} \left| \mathcal{E}_{0}^{\alpha}(x; A, B) \right| + \left| \mathcal{E}_{1}^{\alpha}(x; A, B) \right|, \]
where
\[ \mathcal{E}_{0}^{\alpha}(x; A, B) = \sum_{(\frac{x}{A})^{\frac{1}{r}} < d < (\frac{x}{A})^{\frac{1}{r}} + \delta} d^{\alpha} \psi \left( \frac{x}{d^{r} + \delta} \right). \]

Since \(r \geq 2\), we have \(\alpha < 2r - 1\) by (3.1). Thus
\[ S_{f}^{2}(x; A; B) \ll \frac{A^{2}}{x} \left( \frac{x}{A} \right)^{\frac{1 + \alpha}{r}} + \left| \mathcal{E}_{0}^{\alpha}(x; A, B) \right| + \left| \mathcal{E}_{1}^{\alpha}(x; A, B) \right|. \]

Finally, for \(S_{f}^{3}(x; A)\), we remove the floor brackets to get
\[ S_{f}^{3}(x; A) = x \sum_{d \leq (\frac{x}{A})^{\frac{1}{r}} + \delta} h(d) \left( \psi \left( \frac{x}{d^{r} + \delta} \right) - \psi \left( \frac{x}{d^{r}} \right) \right). \]
The first sum is
\[ x \sum_{d \leq (\frac{x}{A})^{\frac{1}{r}} + \delta} h(d) \frac{d^{\alpha}}{d^{r}(d^{r} + 1)} = C_{f} x + O \left( \frac{A^{2}}{x} \left( \frac{x}{A} \right)^{\frac{1 + \alpha}{r}} \right). \]
and thus
\[
S^x_f(x; A) = C_f x + O\left(\frac{A^2}{x} \left(\frac{x}{A}\right)^{\frac{1}{r} + \alpha} + \left|\mathcal{E}^x_0(x; A)\right| + \left|\mathcal{E}^x_1(x; A)\right|\right),
\]

(3.5)

where
\[
\mathcal{E}^x_\delta(x; A) = \sum_{d \leq \left(\frac{x}{A}\right)^{\frac{1}{r}}} h(d) \psi \left(\frac{x}{d^r + \delta}\right).
\]

(3.6)

Combining (3.2), (3.4), and (3.5), we have, in the range (3.1),
\[
S_f(x) = C_f x + O\left(\frac{A^2}{x} \left(\frac{x}{A}\right)^{\frac{1}{r} + \alpha} + \left|\mathcal{E}^x_0(x; A)\right| + \left|\mathcal{E}^x_1(x; A)\right| + \mathcal{E}(x; A, B)\right),
\]

(3.7)

where
\[
\mathcal{E}(x; A, B) = \left|\mathcal{E}^x_0(x; A, B)\right| + \left|\mathcal{E}^x_1(x; A, B)\right| + \left|\mathcal{E}^x_0(x; A)\right| + \left|\mathcal{E}^x_1(x; A)\right|
\]

(3.8)

and \(\mathcal{E}^x_\delta\) and \(\mathcal{E}^x_\beta\) are given by (3.3) and (3.6), respectively. We note the trivial bound
\[
\mathcal{E}(x; A, B) \ll \left(\frac{x}{B}\right)^{\frac{1}{r}}.
\]

(3.9)

### 3.2. Proof of Theorem 1.1

Using the trivial estimate (3.9) with \(B = A\) and recalling that we will choose \(A \leq \sqrt{x}\), we have, in the range (3.1)
\[
S_f(x) = C_f x + O\left(\frac{x^{\frac{1}{r} + \alpha}}{A} + \left(\frac{x}{A}\right)^{\frac{1}{r} + \alpha} + \mathcal{E}(x; A, B)\right).
\]

The optimal choice \(A = x^{\frac{1}{r+\eta}}\) gives the estimate
\[
S_f(x) = C_f x + O\left(x^{\frac{1}{r+\eta}}\right).
\]

However, we must choose \(A\) subject to (3.1), and this requires
\[
l \leq \frac{r + 1}{2}.
\]

We conclude the proof of Theorem 1.1 by choosing \(l = \frac{r + \eta}{2}\).

### 3.3. Proof of Theorem 1.3

Recall that
\[
\mathcal{E}^x_\delta(x; A, B) = \sum_{\left(\frac{A}{x}\right)^{\frac{1}{r}} < d \leq \left(\frac{x}{A}\right)^{\frac{1}{r}}} d^\alpha \psi \left(\frac{x}{d^r + \delta}\right).
\]

Subdividing into dyadic intervals and applying partial summation, we have
\[
\mathcal{E}^x_\delta(x; A, B) \ll \sum_{\left(\frac{A}{x}\right)^{\frac{1}{r}} < D \leq \left(\frac{x}{A}\right)^{\frac{1}{r}}} D^\alpha \max_{D \leq D' \leq 2D} \left|\sum_{D < d \leq D'} \psi \left(\frac{x}{d^r + \delta}\right)\right|.
\]

We estimate the sum over \(d\) via the following lemma, which is a special case of Lemma 4.3 of [3].
Lemma 3.1 (Graham and Kolesnik). Let \( g(n) \sim y^{n^{-r}} \) for some \( r > 0 \) and \( g \in C^1([N,2N]) \) with \( N \geq 1 \). Then for any exponent pair \((k,\ell)\) and any \( N' \in [N,2N] \), we have
\[
\sum_{N < n \leq N'} \psi(g(n)) \ll y^{\frac{1}{k+1}} N^{\frac{r+1}{k+1}} + \frac{N'^{r+1}}{y}.
\]

Applying this to \( E^\delta_\psi \) yields
\[
E^\delta_\psi(x;A,B) \ll \sum_{x \leq D < x^\frac{1}{r}} D^\alpha \left( x^{\frac{b}{k+1}} D^{\frac{r+1}{k+1}} + \frac{D^{r+1}}{x} \right).
\]

By hypothesis, there exists an exponent pair \((k,\ell)\) with \( \ell = rk \), and so
\[
E^\delta_\psi(x;A,B) \ll \sum_{x \leq D < x^\frac{1}{r}} D^\alpha \left( x^{\frac{b}{k+1}} + \frac{D^{r+1}}{x} \right) \ll \left( \frac{x}{B} \right)^{\frac{b}{k+1}} \left( x^{\frac{1}{k+1}} \log x + \frac{x^\frac{1}{r}}{B^{1+\frac{1}{r}}} \right),
\]
where the factor \( \log x \) is present only when \( \alpha = 0 \). Estimating \( \mathcal{E}^\delta_\psi(x;A) \) trivially, we find that
\[
\mathcal{E}(x;A,b) \ll \left( \frac{x}{B} \right)^{\frac{b}{k+1}} \left( x^{\frac{1}{k+1}} \log x + \frac{x^\frac{1}{r}}{B^{1+\frac{1}{r}}} \right) + \left( \frac{x}{A} \right)^{1+\alpha}.
\]

We conclude the proof Theorem 1.3 by inserting the above estimate into (3.7) and specifying \( A = \sqrt{x}, \ B = C_r x^{1/r} \) and \( l = r \). Corollary 1.4 then follows from the exponent pairs
\[
\begin{align*}
\left( \frac{2}{7}, \frac{4}{7} \right) & = BA^2 \left( \frac{1}{2}, \frac{1}{2} \right), & \left( \frac{11}{53}, \frac{33}{53} \right) & = BABA^2BA^2 \left( \frac{1}{2}, \frac{1}{2} \right), & \left( \frac{1}{6}, \frac{2}{3} \right) & = A \left( \frac{1}{2}, \frac{1}{2} \right)
\end{align*}
\]
for \( r = 2, 3, 4 \), respectively. Here \( A \) and \( B \) denote the usual \( A \) and \( B \) processes of obtaining new exponent pairs.

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