Normal Covering Spaces with Maximal Bottom of Spectrum

Panagiotis Polymerakis\textsuperscript{1,2}

Received: 23 September 2022 / Accepted: 26 May 2023 / Published online: 14 June 2023
© The Author(s) 2023

Abstract
We study the property of spectral-tightness of Riemannian manifolds, which means that the bottom of the spectrum of the Laplacian separates the universal covering space from any other normal covering space of a Riemannian manifold. We prove that spectral-tightness of a closed Riemannian manifold is a topological property characterized by its fundamental group. As an application, we show that a non-positively curved, closed Riemannian manifold is spectrally-tight if and only if the dimension of its Euclidean local de Rham factor is zero. In their general form, our results extend the state of the art results on the bottom of the spectrum under Riemannian coverings.

Keywords Bottom of spectrum \cdot Schrödinger operator \cdot Riemannian covering \cdot Amenable covering \cdot Spectral-tightness

Mathematics Subject Classification 58J50 \cdot 35P15 \cdot 53C99

1 Introduction
The spectrum of the Laplacian of a Riemannian manifold is an interesting isometric invariant, which attracted much attention over the last years. Aiming to a better comprehension of its relations with the geometry of the underlying manifold, its behavior under maps between Riemannian manifolds that respect the geometry of the manifolds to some extent has been studied. In particular, there are various results on the behavior of the spectrum under Riemannian coverings and open questions arising from them.

To be more precise, let $p: M_2 \to M_1$ be a Riemannian covering. Then the bottoms of the spectra of the Laplacians satisfy $\lambda_0(M_1) \leq \lambda_0(M_2)$. Brooks was the first one to investigate when the equality holds, and this is closely related to the notion of

\textsuperscript{1} Département de Mathématiques et de Statistique, Pavillon Alexandre-Vachon, Université Laval, Quebec, QC G1V 0A6, Canada
\textsuperscript{2} Max Planck Institute for Mathematics, Vivatsgasse, 7, 53111 Bonn, Germany
amenability. The covering \( p \) is called amenable if the monodromy action of \( \pi_1(M_1) \) on the fiber of \( p \) is amenable. It is noteworthy that a normal covering is amenable if and only if its deck transformation group is amenable. Brooks proved in [10] that the universal covering of a closed (that is, compact and without boundary) manifold preserves the bottom of the spectrum if and only if it is amenable. This result has been generalized in various ways over the last years (cf. for instance, the survey [4]). In [3], we showed that if \( p \) is amenable, then \( \lambda_0(M_1) = \lambda_0(M_2) \), without imposing any assumptions on the geometry or the topology of the manifolds. The converse implication is not true in general, but holds under a natural condition involving the bottom \( \lambda^{\text{ess}}_0(M_1) \) of the essential spectrum of the Laplacian. More specifically, according to [20, Theorem 1.2], if \( \lambda_0(M_2) = \lambda_0(M_1) < \lambda^{\text{ess}}_0(M_1) \), then \( p \) is amenable. The situation is very unclear in the case where \( \lambda_0(M_1) = \lambda^{\text{ess}}_0(M_1) \), as pointed out in [4, Question 1.5].

Given a normal Riemannian covering \( p : M_2 \to M_1 \), besides the aforementioned inequality, we have that \( \lambda_0(M_2) \leq \lambda_0(M) \), where \( \tilde{M} \) is the universal covering space of \( M_1 \). One of the purposes of this paper is to examine the validity of the equality. This fits into the study of manifolds with maximal bottom of spectrum under some constraint. Examples of remarkable works on this problem are [19] and [18], which focus on complete manifolds with Ricci curvature bounded from below and quotients of symmetric spaces of non-compact type, respectively. This setting, where \( M_2 \) is a normal covering space of \( M_1 \), may seem more restrictive, but our goal is actually different. In addition to obtaining information about the maximizer \( M_2 \), we want to characterize the existence of a maximizer (different from the universal covering space) in terms of properties of \( M_1 \).

Our motivation is to investigate to what extent the bottom of the spectrum of the Laplacian separates the universal covering space from the rest of the normal covering spaces of a Riemannian manifold. The corresponding question about the exponential volume growth has been addressed in [23]. To be more precise, if \( M_2 \) is a normal covering space of \( M_1 \), then the exponential volume growths satisfy \( \mu(M_1) \leq \mu(M_2) \). Hence, the universal covering space \( \tilde{M} \) of \( M_1 \) has maximal bottom of spectrum and maximal exponential volume growth among all the normal covering spaces of \( M_1 \). A Riemannian manifold \( M_1 \) is called spectrally-tight or growth tight if the universal covering space of \( M_1 \) is the unique normal covering space of \( M_1 \) with maximal bottom of spectrum or maximal exponential volume growth, respectively. Sambusetti proved in [23] that negatively curved, closed manifolds are growth tight, which yields that negatively curved, closed, locally symmetric spaces are spectrally-tight. One of the aims of this paper is to study the notion of spectral-tightness, establish that negatively curved, closed manifolds enjoy this property, and more generally, characterize this property for non-positively curved, closed Riemannian manifolds.

To set the stage, we consider Riemannian coverings \( p : M_2 \to M_1, \ q : M_1 \to M_0 \), with \( q \) normal, a Schrödinger operator \( S_0 \) on \( M_0 \) and its lifts \( S_1, S_2 \) on \( M_1, M_2 \), respectively. It turns out that the validity of \( \lambda_0(S_2) = \lambda_0(S_1) \) is intertwined with a property similar, but weaker than the amenability of the covering.

The covering \( p \) is called relatively amenable with respect to \( q \), or for short, \( q \)-amenable if the monodromy action of \( q_*\pi_1(M_1) \) on the fiber of \( q \circ p \) is amenable. It is evident that a covering is amenable if and only if it is relatively amenable with respect
to the identity. The notion of relative amenability is naturally related to the amenability of the composition. More specifically, the composition $q \circ p$ is amenable if and only if $q$ is amenable and $p$ is $q$-amenable. In general, amenable coverings are relatively amenable, but there exist relatively amenable coverings that are not amenable, as we will show by an example. It is worth to point out that if $q$ is finite sheeted or more importantly, if $q \circ p$ is normal, then $p$ is $q$-amenable if and only if it is amenable. Our first result illustrates the role of this notion in the behavior of the bottom of the spectrum, extending [3, Theorem 1.2] and [20, Theorem 1.2].

**Theorem 1.1** Let $p : M_2 \to M_1$ and $q : M_1 \to M_0$ be Riemannian coverings, with $q$ normal. Consider a Schrödinger operator $S_0$ on $M_0$ and denote by $S_1$, $S_2$ its lift on $M_1$, $M_2$, respectively. Then:

(i) If $p$ is $q$-amenable, then $\lambda_0(S_2) = \lambda_0(S_1)$.

(ii) Conversely, if $\lambda_0(S_2) = \lambda_0(S_1) < \lambda_{\text{ess}}^0(S_0)$, then $p$ is $q$-amenable.

It should be noticed that if $q$ is finite sheeted, then $\lambda_0(S_1) = \lambda_{\text{ess}}^0(S_1)$ (cf. for example [21, Corollary 1.3]). Therefore, the setting of Theorem 1.1 fits into the context of [4, Question 1.5]. Conceptually, it seems interesting that even in this special case, a property different from amenability has such an effect on the behavior of the bottom of the spectrum.

In view of Theorem 1.1, the study of spectral-tightness becomes quite easier, bearing in mind that in this theorem, if $M_2$ is simply connected, then $p$ is $q$-amenable if and only if it is amenable. More precisely, it follows that spectral-tightness of a closed manifold is a topological property determined by its fundamental group as follows:

**Theorem 1.2** A closed Riemannian manifold is spectrally-tight if and only if the unique normal, amenable subgroup of its fundamental group is the trivial one.

Drawing heavily from the work of Eberlein [16], as an application of Theorem 1.2, we characterize the property of spectral-tightness for non-positively curved, closed Riemannian manifolds.

**Theorem 1.3** A non-positively curved, closed Riemannian manifold is not spectrally-tight if and only if its universal covering space splits as the Riemannian product of a Euclidean space with another manifold.

The notion of spectral-tightness seems more complicated on non-compact Riemannian manifolds. However, it is worth to mention that we prove the characterization of Theorem 1.2 for any Riemannian manifold $M$ with $\lambda_{\text{ess}}^0(M) > \lambda_0(\tilde{M})$, where $\tilde{M}$ is the universal covering space of $M$. Exploiting this, we present examples demonstrating that spectral-tightness of a non-compact Riemannian manifold depends on its Riemannian metric.

As another application of this characterization, we establish the spectral-tightness of certain geometrically finite manifolds, in the sense of Bowditch [9]. Let $M$ be a complete Riemannian manifold of bounded sectional curvature $-b^2 \leq K \leq -a^2 < 0$. Then its universal covering space $\tilde{M}$ is a Hadamard manifold, and $M$ is written as a quotient $\tilde{M} / \Gamma$. Denoting by $M_c = \tilde{M} \cup \tilde{M}_t$ the geometric compactification of $\tilde{M}$, let
\( \Lambda \subset \hat{M} \) be the limit set of \( \Gamma \). Then \( \Gamma \) acts properly discontinuously on \( \hat{M} \setminus \Lambda \). The manifold \( M \) is called geometrically finite if the topological manifold \( (\hat{M} \setminus \Lambda) / \Gamma \) with possibly empty boundary has finitely many ends and each of them is parabolic. It is worth to mention that if \( M \) is geometrically finite, then \( \lambda_0^{\text{ess}}(M) \geq \lambda_0(\hat{M}) \), according to [6, Theorem B].

**Corollary 1.4** Let \( M \) be a non-simply connected, geometrically finite manifold, and denote by \( \hat{M} \) its universal covering space. If \( \lambda_0^{\text{ess}}(M) > \lambda_0(\hat{M}) \), then \( M \) is not spectrally-tight if and only if \( \pi_1(M) \) is elementary.

The paper is organized as follows: In Sect. 2, we give some preliminaries on Schrödinger operators, amenable actions and coverings. In Sect. 3, we introduce the notion of relatively amenable coverings and discuss some basic properties. Section 4 is devoted to Theorem 1.1 and some applications. In Sect. 5, we focus on the notion of spectral-tightness and establish Theorems 1.2, 1.3, and Corollary 1.4.

**2 Preliminaries**

Throughout this paper, manifolds are assumed to be connected and without boundary, unless otherwise stated. Moreover, non-connected manifolds are assumed to have at most countably many connected components.

A *Schrödinger operator* \( S \) on a possibly non-connected Riemannian manifold \( M \) is an operator of the form \( S = \Delta + V \), where \( \Delta \) is the Laplacian and \( V \in C^\infty(M) \), such that there exists \( c \in \mathbb{R} \) satisfying

\[
\langle Sf, f \rangle_{L^2(M)} \geq c\|f\|_{L^2(M)}^2
\]

for any \( f \in C^\infty_c(M) \). Then the linear operator

\[
S: C^\infty_c(M) \subset L^2(M) \rightarrow L^2(M)
\]

is densely defined, symmetric and bounded from below. Hence, it admits Friedrichs extension. Denote by \( \tilde{S} \) this extension and by \( D(\tilde{S}) \) its domain of definition. Recall that the *spectrum* of \( S \) is defined as

\[
\sigma(S) = \{ \lambda \in \mathbb{R} : \tilde{S} - \lambda : D(\tilde{S}) \rightarrow L^2(M) \text{ is not bijective} \},
\]

and is decomposed into the *essential spectrum* of \( S \), which is given as

\[
\sigma^{\text{ess}}(S) = \{ \lambda \in \mathbb{R} : \tilde{S} - \lambda : D(\tilde{S}) \rightarrow L^2(M) \text{ is not Fredholm} \},
\]

and into the *discrete spectrum* \( \sigma_d(S) = \sigma(S) \setminus \sigma^{\text{ess}}(S) \) of \( S \). It is well-known that the discrete spectrum of \( S \) consists of isolated points of the spectrum which are eigenvalues of \( \tilde{S} \) of finite multiplicity.

The bottoms of (that is, the infimums) of the spectrum and the essential spectrum of \( S \) are denoted by \( \lambda_0(S) \) and \( \lambda_0^{\text{ess}}(S) \), respectively. In the case of the Laplacian (that is,
$V = 0)$, these sets and quantities are denoted by $\sigma(M)$, $\sigma_{\text{ess}}(M)$ and $\lambda_0(M)$, $\lambda_{0\text{ ess}}(M)$, respectively. We have by definition that $\lambda_{0\text{ ess}}(S) = +\infty$ if $\sigma_{\text{ess}}(S)$ is empty, and we then say that $S$ has discrete spectrum.

The Rayleigh quotient of a non-zero $f \in \text{Lip}_c(M)$ with respect to $S$ is defined by

$$R_S(f) = \frac{\int_M (\|\text{grad} f\|^2 + Vf^2)}{\int_M f^2}.$$ 

(1)

The Rayleigh quotient of $f$ with respect to the Laplacian is denoted by $R(f)$. According to the following proposition, the bottom of the spectrum of $S$ is expressed as an infimum of Rayleigh quotients. This is a straightforward consequence of Rayleigh’s theorem and standard approximations, and may be found for instance in [20, Proposition 3.2].

**Proposition 2.1** Let $S$ be a Schrödinger operator on a possibly non-connected Riemannian manifold $M$. Then the bottom of the spectrum of $S$ is given by

$$\lambda_0(S) = \inf_f R_S(f),$$

where the infimum is taken over all $f \in C^\infty_c(M) \setminus \{0\}$, or over all $f \in \text{Lip}_c(M) \setminus \{0\}$.

In the case where $M$ is connected, the bottom of the spectrum of $S$ is characterized as the maximum of the positive spectrum of $S$ (cf. for instance [4, Theorem 3.1] and the references therein).

**Proposition 2.2** Let $S$ be a Schrödinger operator on a Riemannian manifold $M$. Then $\lambda_0(S)$ is the maximum of all $\lambda \in \mathbb{R}$ such that there exists a positive $\varphi \in C^\infty(M)$ satisfying $S\varphi = \lambda \varphi$.

It should be emphasized that the positive, smooth functions involved in this proposition are not required to be square-integrable.

We now focus on the essential spectrum of a Schrödinger operator $S$ on a (connected) Riemannian manifold $M$. The decomposition principle asserts that

$$\sigma_{\text{ess}}(S) = \sigma_{\text{ess}}(S, M \setminus K)$$

for any smoothly bounded, compact domain $K$ of $M$. This is well known in the case where $M$ is complete (compare with [15, Proposition 2.1]), but also holds if $M$ is non-complete, as explained for example in [4, Theorem A.17]. This yields the following expression for the bottom of the essential spectrum of $S$ (cf. for instance [8, Proposition 3.2] for complete Riemannian manifolds. The proof for non-complete manifolds is identical, since the decomposition principle holds).

**Proposition 2.3** Let $S$ be a Schrödinger operator on a Riemannian manifold $M$, and $(K_n)_{n \in \mathbb{N}}$ an exhausting sequence of $M$ consisting of compact domains of $M$. Then the bottom of the essential spectrum of $S$ is given by

$$\lambda_{0\text{ ess}}(S) = \lim_n \lambda_0(S, M \setminus K_n).$$
Consider a positive $\varphi \in C^\infty(M)$ with $S\varphi = \lambda \varphi$ for some $\lambda \in \mathbb{R}$. Denote by $L_\varphi^2(M)$ the $L^2$-space of $M$ with respect to the measure $\varphi^2 \mathfrak{d}v$, where $\mathfrak{d}v$ stands for the volume element of $M$ induced from its Riemannian metric. It is immediate to verify that the map $m_\varphi: L_\varphi^2(M) \to L^2(M)$ defined by $m_\varphi(f) = f \varphi$, is an isometric isomorphism. It is easily checked that $m_\varphi$ intertwines $S - \lambda$ with the diffusion operator $L = m_\varphi^{-1} \circ (S - \lambda) \circ m_\varphi = \Delta - 2 \text{grad} \ln \varphi$.

The operator $L$ is called the renormalization of $S$ with respect to $\varphi$. The Rayleigh quotient of a non-zero $f \in C^\infty_c(M)$ with respect to $L$ is defined as

$$\mathcal{R}_L(f) = \frac{\langle Lf, f \rangle_{L_\varphi^2(M)}}{\|f\|_{L_\varphi^2(M)}^2} = \frac{\int_M \| \text{grad} f \|^2 \varphi^2}{\int_M f^2 \varphi^2}.$$ 

**Proposition 2.4** The Rayleigh quotients of any non-zero $f \in C^\infty_c(M)$ are related by $\mathcal{R}_L(f) = \mathcal{R}_S(f \varphi) - \lambda$. In particular, we have that

$$\lambda_0(S) - \lambda = \inf_f \mathcal{R}_L(f),$$

where the infimum is taken over all non-zero $f \in C^\infty_c(M)$.

**Proof** The first equality follows from a straightforward computation, using the definition of $L$ and that $m_\varphi$ is an isometric isomorphism. This, together with Proposition 2.1, implies the second statement. \qed

Even though our main results involve manifolds without boundary, it is quite important to consider manifolds with boundary in intermediate steps. Let $M$ be a possibly non-connected Riemannian manifold with smooth boundary, and denote by $\nu$ the outward pointing, unit normal to the boundary. Then the Laplacian on $M$ regarded as

$$\Delta: \{ f \in C^\infty_c(M) : \nu(f) = 0 \text{ on } \partial M \} \subset L^2(M) \to L^2(M)$$

admits Friedrichs extension, being densely defined, symmetric and bounded from below. The spectrum of the Friedrichs extension of this operator is called the Neumann spectrum of $M$, and its bottom is denoted by $\lambda_0^N(M)$. We recall the following expression for the bottom of the Neumann spectrum, where $\mathcal{R}(f)$ is defined as in (1) with $V = 0$. This may be found for instance in [20, Proposition 3.2].

**Proposition 2.5** Let $M$ be a possibly non-connected Riemannian manifold with smooth boundary. Then the bottom of the Neumann spectrum of $M$ is given by

$$\lambda_0^N(M) = \inf_f \mathcal{R}(f),$$

where the infimum is taken over all non-zero $f \in C^\infty_c(M)$.

It should be noticed that in this proposition, the test functions $f \in C^\infty_c(M)$ do not have to satisfy any boundary condition.
2.1 Amenable Actions and Coverings

Let $X$ be a countable set and consider a right action of a discrete, countable group $\Gamma$ on $X$. This action is called \textit{amenable} if there exists an \textit{invariant mean} on $\ell^\infty(X)$; that is, a linear functional $\mu: \ell^\infty(X) \to \mathbb{R}$ such that
\[
\inf f \leq \mu(f) \leq \sup f \quad \text{and} \quad \mu(g^* f) = \mu(f)
\]
for any $f \in \ell^\infty(X)$ and $g \in \Gamma$, where $g^* f(x) := f(xg)$ for any $x \in X$. It should be observed that if the action of $\Gamma$ on the orbit of some $x \in X$ is amenable, then the action of $\Gamma$ on $X$ is amenable.

A group $\Gamma$ is called \textit{amenable} if the right action of $\Gamma$ on itself is amenable. Standard examples of amenable groups are solvable groups and finitely generated groups of subexponential growth. It is worth to mention that the free group in two generators, as well as any group containing it, is non-amenable. It is not difficult to see that if $\Gamma$ is an amenable group, then any action of $\Gamma$ is amenable.

The following characterization of amenability is due to Følner in the case of groups [17, Main Theorem and Remark], and extended to actions by Rosenblatt [22, Theorems 4.4 and 4.9].

\textbf{Proposition 2.6} The right action of $\Gamma$ on $X$ is amenable if and only if for any $\varepsilon > 0$ and any finite subset $G$ of $\Gamma$ there exists a finite subset $F$ of $X$ such that $|Fg \setminus F| < \varepsilon|F|$ for any $g \in G$.

In particular, it follows that the right action of $\Gamma$ on $X$ is amenable if and only if the right action of any finitely generated subgroup of $\Gamma$ on $X$ is amenable. Moreover, if the action of $\Gamma$ on $X$ has finitely many orbits $X_i$, $1 \leq i \leq n$, then the action of $\Gamma$ on $X$ is amenable if and only if the action of $\Gamma$ on $X_i$ is amenable for some $1 \leq i \leq n$.

Let $p: M_2 \to M_1$ be a smooth covering, where $M_1$ has possibly empty, smooth boundary and $M_2$ is possibly non-connected. Fix a point $x \in M_1$ and consider the fundamental group $\pi_1(M_1)$ with base point $x$. For $g \in \pi_1(M_1)$, let $\gamma_g$ be a representative loop of $g$ based at $x$. Given $y \in p^{-1}(x)$, let $\tilde{\gamma}_g$ be the lift of $\gamma_g$ starting at $y$ and denote its endpoint by $yg$. In this way, we obtain a right action of $\pi_1(M_1)$ on $p^{-1}(x)$, which is called the \textit{monodromy action} of the covering. The covering $p$ is called \textit{amenable} if its monodromy action is amenable. It is easy to see that if $M_2$ is connected and $p$ is normal, then $p$ is amenable if and only if its deck transformation group is amenable.

\textbf{Example 2.7} For any smooth covering $p: M_2 \to M_1$, the covering
\[
p \sqcup \text{Id}: M_2 \sqcup M_1 \to M_1
\]
is amenable.

Recall that Følner’s condition characterizes the amenability of an action in terms of the action of finitely generated subgroups. In the context of coverings, this yields the following characterization in terms of smoothly bounded, compact domains.
Proposition 2.8 (4, Proposition 2.14) Let \( p : M_2 \to M_1 \) be a smooth covering, where \( M_2 \) is possibly non-connected, and \( (K_n)_{n \in \mathbb{N}} \) an exhausting sequence of \( M_1 \) consisting of smoothly bounded, compact domains. Then \( p \) is amenable if and only the restriction \( p : p^{-1}(K_n) \to K_n \) is amenable for any \( n \in \mathbb{N} \).

This demonstrates the importance of considering non-connected covering spaces, since \( p^{-1}(K) \) does not have to be connected even if \( M_2 \) is connected.

Finally, we briefly recall some results on the bottom of the spectrum under Riemannian coverings, which will be used in the sequel. Let \( p : M_2 \to M_1 \) be a Riemannian covering, with \( M_2 \) possibly non-connected, \( S_1 \) a Schrödinger operator on \( M_1 \) and \( S_2 \) its lift on \( M_2 \). From Proposition 2.2 it is not hard to see that the bottoms of the spectra satisfy

\[
\lambda_0(S_1) \leq \lambda_0(S_2). \tag{2}
\]

The validity of the equality is closely related to the amenability of the covering.

Theorem 2.9 Let \( p : M_2 \to M_1 \) be an amenable Riemannian covering, with \( M_2 \) possibly non-connected. Consider a Schrödinger operator \( S_1 \) on \( M_1 \) and its lift \( S_2 \) on \( M_2 \). Then \( \lambda_0(S_2) = \lambda_0(S_1) \).

In the case where \( M_2 \) is connected, this coincides with [3, Theorem 1.2]. With very slight modifications, its proof extends [3, Theorem 1.2] to the case where \( M_2 \) is possibly non-connected. This may be found also in [5, Theorem A], which involves Riemannian coverings of orbifolds, where the covering space may be non-connected.

Amenability of a Riemannian covering of a compact manifold with smooth boundary is characterized in terms of the Neumann spectrum, according to the following analogue of Brooks’ result [10].

Theorem 2.10 Let \( p : M_2 \to M_1 \) be a Riemannian covering, where \( M_1 \) is compact with smooth boundary and \( M_2 \) is possibly non-connected. Then \( p \) is amenable if and only if \( \lambda_0^N(M_2) = 0 \).

Proof The converse implication is known by [20, Theorem 4.1]. For the other direction, consider a Riemannian metric on \( M_1 \) such that its boundary has a neighborhood isometric to a cylinder \( \partial M_1 \times [0, \varepsilon) \), and endow \( M_2 \) with the lifted metric. Since this metric is uniformly equivalent to the original, it suffices to show that \( \lambda_0^N(M_2) = 0 \) with respect to this metric. Denote by \( 2M_i \) the Riemannian manifold obtained by gluing two copies of \( M_i \) along their boundaries, \( i = 1, 2 \). Then \( p : M_2 \to M_1 \) extends to a Riemannian covering \( 2p : 2M_2 \to 2M_1 \). Choose \( x \in \partial M_1 \subset 2M_1 \) as base point for \( \pi_1(2M_1) \), and observe that any loop \( c \) based at \( x \) is written as \( c = c_2n * \ldots * c_1 \) for some paths (not necessarily loops, since \( \partial M_1 \) may be non-connected) \( c_i \), with the image of \( c_2i-1 \) contained in \( M_1 \), and the image of \( c_2i \) in \( 2M_1 \setminus M_1 \), \( 1 \leq i \leq n \). Denote by \( c_i' \) the reflection of \( c_i \) along \( \partial M_1 \), and observe that the lifts of \( c_i \) and \( c_i' \) starting from the same point also have the same endpoint. It is now apparent that given \( y \in (2p)^{-1}(x) = p^{-1}(x) \), the lifts of \( c \) and \( c_2n * c_2n-1 * \ldots * c_i' * c_1 \) starting at \( y \) have the same endpoint. Since the image of the latter loop is contained in \( M_1 \), it is now easy...
to verify that $2p$ is amenable, $p$ being amenable. Since $2M_1$ is closed, we derive from
Theorem 2.9 that $\lambda_0(2M_2) = 0$. In view of Proposition 2.1, this means that for any
$\varepsilon > 0$ there exists $f \in C^\infty_c(2M_2) \setminus \{0\}$ with $R(f) < \varepsilon$. Without loss of generality,
we may assume that $f$ is not identically zero neither on $M_2$ nor on $2M_2 \setminus M_2$. Indeed,
otherwise one may extend $f$ beyond $\partial M_2$ to obtain a function invariant under reflection
along $\partial M_2$, with the same Rayleigh quotient and the aforementioned property. Then
we readily see from Proposition 2.5 that

$$
\varepsilon > R(f) \geq \min\{R(f|_{M_2}), R(f|_{2M_2 \setminus M_2})\} \geq \min\{\lambda_0^N(M_2), \lambda_0^N(2M_2 \setminus M_2)\}.
$$

Because $M_2$ and $2M_2 \setminus M_2$ are isometric, we conclude that $\lambda_0^N(M_2) = 0$.  

By virtue of the preceding theorem, we may reformulate Proposition 2.8 as follows.

**Corollary 2.11** Let $p : M_2 \to M_1$ be Riemannian covering, with $M_2$ possibly non-
connected, and $(K_n)_{n \in \mathbb{N}}$ an exhausting sequence of $M_1$ consisting of smoothly
bounded, compact domains. Then $p$ is amenable if and only if $\lambda_0^N(p^{-1}(K_n)) = 0$
for any $n \in \mathbb{N}$.

### 3 Relatively Amenable Coverings

In this section, we introduce the notion of relatively amenable coverings and present
some of their properties.

**Definition 3.1** Let $p : M_2 \to M_1$ and $q : M_1 \to M_0$ be smooth coverings, where $q$
is normal. Fix a base point $x \in M_1$ and set $x_0 = q(x)$. The covering $p$ is called
**relatively amenable with respect to $q$**, or for short, $q$-**amenable** if the monodromy
action of $q_*\pi_1(M_1)$ on $(q \circ p)^{-1}(x_0)$ is amenable.

In the setting of this definition, to provide another description of this action, denote
by $\Gamma$ the deck transformation group of $q$. Let $s \in \pi_1(M_1)$ and $\gamma_s$ a representative loop
of $s$ based at $x$. It is clear that for any $z \in (q \circ p)^{-1}(x_0)$, there exists a unique $g \in \Gamma$,such that $z \in p^{-1}(gx)$. Then $x_0$ is the endpoint of the lift of $g \circ \gamma_s$ starting at $z$.

For $g \in \Gamma$, consider the covering $p_g : M_2 \to M_1$ defined by $p_g = g^{-1} \circ p$, and
denote by $\hat{p} : \hat{M} \to M_1$ the induced covering

$$
\cup_{g \in \Gamma} p_g : \cup_{g \in \Gamma} M_2 \to M_1.
$$

From the above discussion, we arrive at the following characterization of relatively
amenable coverings.

**Lemma 3.2** The covering $p$ is $q$-amenable if and only if the induced covering $\hat{p}$ is
amenable.

**Proof** It is easily checked that the monodromy action of $\hat{p}$ coincides with the monodromy action of $q_*\pi_1(M_1)$ on $(q \circ p)^{-1}(x_0)$.  

$\square$
In view of the preceding lemma, it is evident that amenable coverings are relatively amenable. Furthermore, we readily see that if \( q \) is finite sheeted, then \( p \) is \( q \)-amenable if and only if it is amenable. We now discuss an analogue of Corollary 2.11.

**Proposition 3.3** Let \( p : M_2 \to M_1 \) and \( q : M_1 \to M_0 \) be Riemannian coverings, where \( q \) is normal with deck transformation group \( \Gamma \), and fix an exhausting sequence \((K_n)_{n \in \mathbb{N}}\) of \( M_1 \) consisting of smoothly bounded, compact domains. Then \( p \) is \( q \)-amenable if and only if

\[
\inf_{g \in \Gamma} \lambda^N_0(p^{-1}(gK_n)) = 0
\]

for any \( n \in \mathbb{N} \).

**Proof** It is obvious that \( \hat{p}^{-1}(K) \) is isometric to the disjoint union of \( p^{-1}(gK) \) with \( g \in \Gamma \), for any smoothly bounded, compact domain \( K \) of \( M_1 \). In particular, the bottoms of the Neumann spectra are related by

\[
\lambda^N_0(\hat{p}^{-1}(K)) = \inf_{g \in \Gamma} \lambda^N_0(p^{-1}(gK)).
\]

The proof is completed by Corollary 2.11 and Lemma 3.2. \( \square \)

**Corollary 3.4** Let \( p : M_2 \to M_1 \) and \( q : M_1 \to M_0 \) be smooth coverings, with \( q \) and \( q \circ p \) normal. Then \( p \) is \( q \)-amenable if and only if it is amenable.

**Proof** It is clear that if \( p \) is amenable, then it is \( q \)-amenable. For the converse implication, endow \( M_0 \) with a Riemannian metric and \( M_1, M_2 \) with the lifted metrics. Denote by \( \Gamma \) the deck transformation group of \( q \) and let \((K_n)_{n \in \mathbb{N}}\) be an exhausting sequence of \( M_1 \) consisting of smoothly bounded, compact domains. Then Proposition 3.3 states that

\[
\inf_{g \in \Gamma} \lambda^N_0(p^{-1}(gK_n)) = 0
\]

for any \( n \in \mathbb{N} \). Since \( q \circ p \) is normal, we deduce that any \( g \in \Gamma \) can be lifted to an isometry of \( M_2 \). Indeed, given \( x \in M_2 \), write \( x_0 = (q \circ p)(x) \) and observe that \((q_s \circ g_s \circ p_s)(\pi_1(M_2, x)) = (q_s \circ p_s)(\pi_1(M_2, x)) \). Therefore, we derive that \((q_s \circ g_s \circ p_s)(\pi_1(M_2, x)) = (q_s \circ p_s)(\pi_1(M_2, y)) \) for any \( y \in p^{-1}(gp(x)) \), \( q \circ p \) being normal. Keeping in mind that \( q_s : \pi_1(M_1, gp(x)) \to \pi_1(M_0, x_0) \) is injective, this implies that \((g_s \circ p_s)(\pi_1(M_2, x)) = p_s(\pi_1(M_2, y)) \). It now follows from the lifting theorem that \( g \) can be lifted to a local isometry of \( M_2 \) mapping \( x \) to \( y \). Since \( x \in M_2 \), \( g \in \Gamma \) and \( y \in p^{-1}(gp(x)) \) are arbitrary, we readily see that the lift is invertible, and hence, an isometry. In particular, we derive that \( p^{-1}(gK_n) \) is isometric to \( p^{-1}(K_n) \) for any \( g \in \Gamma \) and \( n \in \mathbb{N} \), which yields that

\[
\lambda^N_0(p^{-1}(K_n)) = \inf_{g \in \Gamma} \lambda^N_0(p^{-1}(gK_n)) = 0
\]

for any \( n \in \mathbb{N} \). We conclude from Corollary 2.11 that \( p \) is amenable. \( \square \)
4 Spectrum Under Relatively Amenable Coverings

In this section we study the behavior of the bottom of the spectrum under relatively amenable coverings, and give some applications and examples.

Proof of Theorem 1.1  Suppose first that \( p \) is \( q \)-amenable. Then the induced covering \( \hat{p} \) is amenable, from Lemma 3.2. Denoting by \( \hat{S} \) the lift of \( S_1 \) on \( \hat{M} \), we derive from Theorem 2.9 that \( \lambda_0(\hat{S}) = \lambda_0(S_1) \). Moreover, bearing in mind that \( S_1 \) is the lift of \( S_0 \), we readily see that \( S_1 \) is invariant under deck transformations of \( q \). Therefore, any connected component of \( \hat{M} \) is isometric to \( M_2 \) via an isometry that identifies \( \hat{S} \) with \( S_2 \). This yields that \( \lambda_0(\hat{S}) = \lambda_0(S_2) \), which establishes the asserted equality.

To prove the second assertion, notice that the assumption that \( \lambda_{\text{ess}}(S_0) > \lambda_0(S_1) \), together with Proposition 2.3, implies that there exists a compact domain \( D_0 \) of \( M_0 \) such that
\[
\lambda_0(S_0, M_0 \setminus D_0) > \lambda_0(S_1).
\]
Let \((K_m)_{m \in \mathbb{N}}\) be an exhausting sequence of \( M_1 \) consisting of smoothly bounded, compact domains, such that \( D_0 \) is contained in the interior of \( q(K_1) \).

Assume to the contrary that \( p \) is not \( q \)-amenable, and denote by \( \Gamma \) the deck transformation group of \( q \). By virtue of Proposition 3.3, there exists \( m \in \mathbb{N} \) and \( c > 0 \), such that
\[
\lambda_0^N(p^{-1}(gK_m)) \geq c
\]
for any \( g \in \Gamma \). Set \( K = K_m \) and \( D = q(K_m) \).

We know from Proposition 2.2 that there exists a positive \( \varphi_1 \in C^\infty(M_1) \) satisfying \( S_1 \varphi_1 = \lambda_0(S_1) \varphi_1 \). Denote by \( \varphi_2 \) the lift of \( \varphi_1 \) on \( M_2 \) and by \( L \) the renormalization of \( S_2 \) with respect to \( \varphi_2 \). Since \( \lambda_0(S_2) = \lambda_0(S_1) \), we derive from Proposition 2.4 that there exists \((f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2)\) with \( \|f_n\|_{L^2_c(M_2)} = 1 \) and \( R_L(f_n) \rightarrow 0 \).

According to the gradient estimate \([14, \text{Theorem 6}]\), there exists \( C > 0 \) such that
\[
\max_K \psi \leq C \min_K \psi
\]
for any positive \( \psi \in C^\infty(M_1) \) with \( S_1 \psi = \lambda_0(S_1) \psi \). For \( \psi = \varphi_1 \circ g \), this means that
\[
\max_g \varphi_2 \leq \max_g \varphi_1 \leq C \min_g \varphi_1 = C \min_g \varphi_2
\]
for any \( g \in \Gamma \). Using this and Proposition 2.5, we compute
\[ \int_{p^{-1}(gK)} \| \text{grad } f_n \|^2 \varphi_2^2 \geq \left( \min_{p^{-1}(gK)} \varphi_2^2 \right) \int_{p^{-1}(gK)} \| \text{grad } f_n \|^2 \]
\[ \geq \frac{1}{C^2} \left( \max_{p^{-1}(gK)} \varphi_2^2 \right) \lambda_0^N (p^{-1}(gK)) \int_{p^{-1}(gK)} f_n^2 \]
\[ \geq \frac{c}{C^2} \int_{p^{-1}(gK)} f_n^2 \varphi_2^2 \]

(3)

for any \( n \in \mathbb{N} \) and \( g \in \Gamma \).

Since \( K \) is compact, it is clear that there exists \( k \in \mathbb{N} \) such that any point of \( M_1 \) belongs to at most \( k \) different translates \( gK \) of \( K \), with \( g \in \Gamma \). Thus, any point of \( M_2 \) belongs to at most \( k \) different \( p^{-1}(gK) \), with \( g \in \Gamma \). This, together with (3), gives the estimate

\[ \int_{(q \circ p)^{-1}(D)} \| \text{grad } f_n \|^2 \varphi_2^2 \geq \frac{1}{k} \sum_{g \in \Gamma} \int_{p^{-1}(gK)} \| \text{grad } f_n \|^2 \varphi_2^2 \geq \frac{c}{kC^2} \int_{(q \circ p)^{-1}(D)} f_n^2 \varphi_2^2, \]

where we used that \( (q \circ p)^{-1}(D) \) is the union of \( p^{-1}(gK) \) with \( g \in \Gamma \). Bearing in mind that \( \| f_n \|_{L^2_{\varphi_2}(M_2)} = 1 \) and \( \mathcal{R}_L (f_n) \rightarrow 0 \), this yields that

\[ \int_{(q \circ p)^{-1}(D)} f_n^2 \varphi_2^2 \rightarrow 0 \quad \text{and} \quad \int_{M_2 \setminus (q \circ p)^{-1}(D)} f_n^2 \varphi_2^2 \rightarrow 1. \]

(4)

Let \( \chi_0 \in C_c^{\infty}(M_0) \) with \( \chi_0 = 1 \) in a neighborhood of \( D_0 \) and \( \text{supp } \chi_0 \) contained in the interior of \( D \). Set \( \chi_2 = \chi_0 \circ q \circ p \) and set \( h_n = (1 - \chi_2) f_n \in C_c^{\infty}(M_2) \). In view of (4), it is immediate to verify that

\[ \int_{M_2} h_n^2 \varphi_2^2 \rightarrow 1 \quad \text{and} \quad \int_{M_2} \| \text{grad } h_n \|^2 \varphi_2^2 \rightarrow 0, \]

which shows that that \( \mathcal{R}_L (h_n) \rightarrow 0 \), and thus, \( \mathcal{R}_{S_2} (h_n \varphi_2) \rightarrow \lambda_0 (S_1) \), by Proposition 2.4. Since \( h_n \varphi_2 \) is compactly supported in \( M_2 \setminus (q \circ p)(D_0) \), Proposition 2.1 implies that

\[ \mathcal{R}_{S_2} (h_n \varphi_2) \geq \lambda_0 (S_2, M_2 \setminus (q \circ p)^{-1}(D_0)) \geq \lambda_0 (S_0, M_0 \setminus D_0) > \lambda_0 (S_1), \]

where the intermediate inequality follows from (2) applied to the restriction of \( q \circ p \) over any connected component of \( M_0 \setminus D_0 \). This is a contradiction, which completes the proof. \( \square \)

The next example illustrates that in Theorem 1.1(ii), the covering \( p \) is \( q \)-amenable, but not necessarily amenable, and in particular, that relative amenability is indeed a weaker property than amenability. This example also demonstrates that in Corollary 3.4, the assumption that \( q \circ p \) is normal cannot be replaced with \( p \) being normal.
Example 4.1 Let $N$ be a closed Riemannian manifold of dimension $n \geq 3$ with non-amenable fundamental group. Fix two sufficiently small open balls $B_i$ with disjoint closures and $\partial B_i$ diffeomorphic to the sphere $S^{n-1}$, $i = 1, 2$. Denote by $M_0$ the closed manifold obtained by gluing a cylinder $S^{n-1} \times [0, 1]$ along the boundary of $N \smallsetminus (B_1 \cup B_2)$, so that $S^{n-1} \times \{0\}$ gets identified with $\partial B_1$, and $S^{n-1} \times \{1\}$ with $\partial B_2$. Endow $M_0$ with a Riemannian metric.

Consider now the disjoint union of copies $N_k$ of $N \smallsetminus (B_1 \cup B_2)$, with $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, glue a cylinder $S^{n-1} \times [0, 1]$ along the boundary of this disjoint union, so that $S^{n-1} \times \{0\}$ gets identified with $\partial B_1$ in $N_k$, and $S^{n-1} \times \{1\}$ with $\partial B_2$ in $N_{k+1}$. In this way, we obtain a manifold $M_1$ on which $\mathbb{Z}$ acts via diffeomorphisms and the quotient is diffeomorphic to $M_0$. That is, we have a covering $p : M_1 \to M_0$ with deck transformation group $\mathbb{Z}$. We endow $M_1$ with the lift of the Riemannian metric of $M_0$.

Let $F$ be a fundamental domain of $q$ which is diffeomorphic to $N_0$ with two cylinders $S^{n-1} \times [0, 1/2]$ attached along its boundary. Then $\pi_1(F)$ is non-amenable and $\partial F$ has two connected components $C_1$ and $C_2$, which are diffeomorphic to $S^{n-1}$. Hence, the universal covering $p : \hat{F} \to F$ of $F$ is non-amenable. It should be noticed that the restriction of $p$ on any connected component of $\partial \hat{F}$ is a covering over a connected component of $\partial F$. Since the connected components of $\partial F$ are diffeomorphic to $S^{n-1}$, where $n \geq 3$, we derive that $p$ restricted to any connected component of $\partial \hat{F}$ is an isometry. This means that $p^{-1}(C_i)$ is a disjoint union of copies of $C_i$, $i = 1, 2$. Write $M_1 \setminus F^\circ = D_1 \sqcup D_2$, with $C_i$ contained in $D_i$, $i = 1, 2$. Denote by $M_2$ the manifold obtained by gluing a copy of $D_i$ to $\hat{F}$ along any connected component of $p^{-1}(C_i)$, $i = 1, 2$. We readily see that the covering $p : \hat{F} \to F$ is extended to a normal covering $p : M_2 \to M_1$ with the same deck transformation group. Therefore, $p : M_2 \to M_1$ is non-amenable.

It remains to establish that $p : M_2 \to M_1$ is $q$-amenable. To this end, we endow $M_2$ with the lift of the Riemannian metric of $M_1$, and by virtue of Theorem 1.1, it suffices to prove that $\lambda_0(M_2) = 0$. Since $q$ is amenable, Theorem 2.9 states that $\lambda_0(M_1) = 0$. From Proposition 2.1, for any $\varepsilon > 0$ there exists a non-zero $f \in C^\infty_c(M_1)$ with $\mathcal{R}(f) < \varepsilon$. Then there exists a deck transformation $g$ of $q$, such that $\text{supp}(f \circ g)$ is contained in the interior of $D_1$, $\text{supp} f$ being compact. It is evident that $\mathcal{R}(f \circ g) < \varepsilon$, and since $D_1$ is isometric to a domain of $M_2$, the corresponding $f_2 \in C^\infty_c(M_2)$ satisfies $\mathcal{R}(f_2) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude from Proposition 2.1 that $\lambda_0(M_2) = 0$, as we wished.

Based on the preceding example, we now present examples of amenable smooth coverings $p : M_2 \to M_1$ with $M_2$ non-connected, such that the restriction of $p$ on any connected component of $M_2$ is non-amenable.

Example 4.2 Let $p : M_2 \to M_1$ and $q : M_1 \to M_0$ be smooth coverings, where $q$ is normal with deck transformation group $\Gamma$. Suppose that $p$ is $q$-amenable, but not amenable. Then the induced covering $\hat{p} : \hat{M} \to M_1$ is amenable, from Lemma 3.2. It is evident that the restriction of $\hat{p}$ on any connected component of $\hat{M}$ is of the form $p_g = g^{-1} \circ p : M_2 \to M_1$ for some $g \in \Gamma$. It follows from Corollary 2.11 that $p_g$ is non-amenable for any $g \in \Gamma$, $p$ being non-amenable.

We end this section with a characterization of the amenability of a composition of coverings.
Corollary 4.3 Let \( p : M_2 \to M_1 \) and \( q : M_1 \to M_0 \) be smooth coverings, with \( q \) normal. Then \( q \circ p \) is amenable if and only if \( q \) is amenable and \( p \) is \( q \)-amenable.

**Proof** Endow \( M_0 \) with a Riemannian metric and \( M_1, M_2 \) with the lifted metrics. Let \( S_0 \) be a Schrödinger operator on \( M_0 \) with discrete spectrum (for instance, \( S_0 = \Delta + V \), where \( V \) has compact sublevels), and denote by \( S_1, S_2 \) its lift on \( M_1, M_2 \), respectively.

If \( q \circ p \) is amenable, then \( \lambda_0(S_0) = \lambda_0(S_2) \), from Theorem 2.9. Furthermore, we know from (2) that \( \lambda_0(S_0) \leq \lambda_0(S_1) \leq \lambda_0(S_2) \). As a consequence, we obtain that \( \lambda_0(S_1) = \lambda_0(S_0) < \lambda_0^{ss}(S_0) \), which yields that \( q \) is amenable, from [20, Theorem 1.2]. Since \( \lambda_0(S_2) = \lambda_0(S_1) < \lambda_0^{ss}(S_0) \), Theorem 1.1 implies that \( p \) is \( q \)-amenable.

Conversely, if \( q \) is amenable and \( p \) is \( q \)-amenable, then \( \lambda_0(S_2) = \lambda_0(S_1) = \lambda_0(S_0) \), where we used Theorem 1.1 for the first equality, and Theorem 2.9 for the second one. Since \( S_0 \) has discrete spectrum, [20, Theorem 1.2] states that \( q \circ p \) is amenable. \(\square\)

5 Spectral-Tightness of Riemannian Manifolds

In this section we study the notion of spectral-tightness of Riemannian manifolds. Recall that a Riemannian manifold \( M \) is called spectral-tight if \( \lambda_0(M') < \lambda_0(M) \) for any non-simply connected, normal covering space \( M' \) of \( M \), where \( \tilde M \) is the universal covering space of \( M \). We begin with a straightforward consequence of Theorem 1.1.

Corollary 5.1 Let \( M \) be a Riemannian manifold with \( \lambda_0(\hat M) < \lambda_0^{ss}(M) \), where \( \hat M \) is the universal covering space of \( M \). Then \( M \) is spectrally-tight if and only if the unique normal, amenable subgroup of \( \pi_1(M) \) is the trivial one.

**Proof** If \( M \) is not spectrally-tight, then there exists a normal covering \( q : M' \to M \) with \( \lambda_0(M') = \lambda_0(\hat M) \) and \( M' \) non-simply connected. In view of Theorem 1.1, the universal covering \( p : \tilde M \to M' \) is \( q \)-amenable. Since \( q \circ p \) is normal, Corollary 3.4 asserts that \( p \) is amenable, or equivalently, that \( \pi_1(M') \) is amenable. It is clear that \( \pi_1(M') \) is a non-trivial, normal subgroup of \( \pi_1(M) \).

Conversely, if \( \pi_1(M) \) has a non-trivial, amenable, normal subgroup \( \Gamma \), then the action of \( \Gamma \) on \( \tilde M \) gives rise to an amenable Riemannian covering \( p : \tilde M \to \tilde M / \Gamma \). We deduce from Theorem 2.9 that \( \lambda_0(\hat M) = \lambda_0(\tilde M / \Gamma) \), while \( \tilde M / \Gamma \) is a non-simply connected, normal covering space of \( M \). \(\square\)

**Proof of Theorem 1.2** It is an immediate consequence of Corollary 5.1, since the spectrum of the Laplacian on a closed Riemannian manifold is discrete. \(\square\)

Before proceeding to the proof of Theorem 1.3, we recall some terminology from [16]. Let \( M \) be a non-positively curved, closed Riemannian manifold. The Euclidean local de Rham factor of \( M \) is the maximum of all \( n \in \mathbb{N} \cup \{0\} \) such that the universal covering space \( \hat M \) of \( M \) splits as the Riemannian product \( \mathbb{R}^n \times N \) for some Riemannian manifold \( N \) (under the convention that if \( \hat M \) cannot be written as \( \mathbb{R} \times N \), then this number is zero). According to [16, Theorem], the Euclidean local de Rham factor of \( M \) coincides with the rank of the unique maximal normal abelian subgroup of \( \pi_1(M) \).

Finally, we recall the well-known result of Bieberbach. An isometry of the Euclidean space \( \mathbb{R}^n \) is of the form \( \varphi(x) = Ax + v \), where \( A \) is an orthogonal transformation.
and \(v \in \mathbb{R}^n\). An isometry is called translation if it is of the form \(\varphi(x) = x + v\). A Bieberbach group \(\Gamma\) is a discrete group of isometries of \(\mathbb{R}^n\) such that \(\mathbb{R}^n / \Gamma\) is compact. According to Bieberbach’s result, the subgroup \(G\) consisting of the translations in \(\Gamma\) is the unique maximal normal abelian subgroup of \(\Gamma\) (cf. for instance [1, § 9]).

**Proof of Theorem 1.3** Let \(M\) be a non-positively curved, closed Riemannian manifold. Suppose first that the dimension of the Euclidean local de Rham factor of \(M\) is non-zero. We derive from [16, Theorem] that there exists a non-trivial, normal abelian subgroup of \(\pi_1(M)\). By virtue of Theorem 1.2, this shows that \(M\) is not spectrally-tight.

Conversely, if \(M\) is not spectrally-tight, then there exists a non-trivial, normal amenable subgroup \(\Gamma\) of \(\pi_1(M)\), according to Theorem 1.2. We obtain from [12, Corollary 2] that \(\Gamma\) is a Bieberbach group. Denote by \(G\) the subgroup of translations in \(\Gamma\). It should be noticed that \(G\) is non-trivial, since otherwise, \(\Gamma\) is finite and in particular, there exist non-trivial elements of finite order in \(\pi_1(M)\), which is a contradiction. Furthermore, it is known that \(G\) is the unique maximal normal abelian subgroup of \(\Gamma\), and thus, a characteristic subgroup of \(\Gamma\). Since \(\Gamma\) is a normal subgroup of \(\pi_1(M)\), we readily see that so is \(G\). Since \(G\) is isomorphic to \(\mathbb{Z}^n\) for some \(n \in \mathbb{N}\), we conclude that the rank of the unique normal maximal abelian subgroup of \(\pi_1(M)\) is non-zero and the proof is completed by [16, Theorem]. \(\square\)

We now recall some notions on manifolds of non-positive curvature, which may be found in [1]. Let \(g\) be an isometry of a Hadamard manifold \(M\). The **displacement function** \(d_g : M \to \mathbb{R}\) of \(g\) is defined as \(d_g(x) = d(x, gx)\). Then \(g\) is called:

- **elliptic** if \(\min_M d_g = 0\),
- **axial** if \(\min_M d_g > 0\),
- **parabolic** if \(d_g\) does not achieve its infimum.

It is evident that \(g\) is elliptic if and only if it fixes a point of \(M\). If \(g\) is axial, then it acts as translation by \(\min_M d_g\) along a geodesic of \(M\) (cf. [1, Lemma 6.5]). Finally, if \(g\) is parabolic, then it fixes a point of the ideal boundary \(M_i\) and all horospheres (as sets) centered at that point (see [1, Lemma 6.6.]).

A discrete group \(G\) of isometries of \(M\) is called:

- **elliptic** if \(G\) fixes a point of \(M\),
- **axial** if \(G\) fixes a geodesic of \(M\) (as a set) but does not fix any point of \(M\),
- **parabolic**, if \(G\) fixes a point \(x \in M_i\) and horospheres centered at \(x\) (as sets), but does not fix any point of \(M \cup M_i \setminus \{x\}\),
- **elementary** if \(G\) is elliptic, axial or parabolic.

Suppose that \(G\) acts freely on \(M\). Then \(G\) is elliptic if and only if it is the trivial group. If \(G\) is axial, taking into account that any element of \(G\) acts by translation along the same geodesic, it is not hard to see that \(G\) is abelian. According to [9, Proposition 4.1], parabolic groups are virtually nilpotent. It is worth to point out that these groups are amenable, being of subexponential growth.

Finally, we recall another definition (equivalent to the one stated in the Introduction) of geometrically finite manifolds, in the sense of Bowditch [9]. Let \(M\) be a complete Riemannian manifold of bounded sectional curvature \(-b^2 \leq K \leq -a^2 < 0\). Then
the universal covering space $\tilde{M}$ of $M$ is a Hadamard manifold, and $M$ is the quotient $M/\Gamma$, for some discrete group $\Gamma$ of isometries of $\tilde{M}$. Let $M_t$ be the ideal boundary of $M$ and consider the compactification $\tilde{M}_t = M \cup M_t$. Denote by $\Lambda \subset \tilde{M}_t$ the limit set of $\Gamma$ and by $H$ the convex hull of $\Gamma$. The set $C = (H \cap \tilde{M})/\Gamma$ is called the convex core of $M$, and $M$ is called geometrically finite if some/any tubular neighborhood of $C$ has finite volume. The reader may consult the seminal paper of Bowditch [9] for more details on geometrically finite manifolds, or [6, Section 3] for a brief exposition.

Before proceeding to the proof of Corollary 1.4, we need the following observation, which will be also exploited in the examples in the sequel.

**Lemma 5.2** Let $M$ be a geometrically finite manifold. If $\pi_1(M)$ contains a non-trivial, amenable normal subgroup, then $\pi_1(M)$ is elementary.

**Proof** Denote by $\tilde{M}$ the universal covering space of $M$ and write $M = \tilde{M}/\Gamma$. Let $G$ be a non-trivial, amenable normal subgroup of $\Gamma$. It follows from [7, Corollary 1.2] that $G$ is elementary. Since $G$ is non-trivial and acts freely on $\tilde{M}$, this yields that $G$ is either parabolic or axial.

Suppose first that $G$ is parabolic and contained in the stabilizer of a point $x$ of the ideal boundary of $\tilde{M}$. Given $h \in \Gamma$, using that $hgh^{-1} \in G$, we readily see that $G$ fixes $h^{-1}x$. This shows that $h^{-1}x = x$ for any $h \in \Gamma$, and thus, $\Gamma$ is parabolic and fixes $x$.

Assume now that $G$ is axial and contained in the stabilizer of the image of a geodesic $\gamma : \mathbb{R} \to \tilde{M}$ joining two points of the ideal boundary of $\tilde{M}$. Then for any $t \in \mathbb{R}$, there exists $m(t) \in \mathbb{R}$ such that $g(\gamma(t)) = g(t + m(t))$ for any $t \in \mathbb{R}$. Given $h \in \Gamma$, there exists $t' \in \mathbb{R}$ such that $hgh^{-1}(\gamma(t)) = g(t + t')$ for any $t \in \mathbb{R}$, since $hgh^{-1} \in G$. Then $G$ fixes the image of the geodesic $h^{-1} \circ \gamma$, which implies that the images of $h^{-1} \circ \gamma$ and $\gamma$ coincide for any $h \in \Gamma$. We conclude that $\Gamma$ is axial and fixes the image of $\gamma$.

**Proof of Corollary 1.4** If $M$ is not spectrally-tight, we derive from Corollary 5.1 that there exists a non-trivial, amenable normal subgroup of $\pi_1(M)$. Then $\pi_1(M)$ is elementary, by Lemma 5.2. Conversely, if $\pi_1(M)$ is elementary, then $\lambda_0(M) = \lambda_0(\tilde{M})$, in view of Theorem 2.9, $\pi_1(M)$ being amenable. Since $M$ is non-simply connected, this means that $M$ is not spectrally-tight.
respect to this Riemannian metric. Otherwise, we obtain from Theorem 5.1 that there exists a non-trivial, amenable normal subgroup of $\pi_1(M)$. It is known that $M$ admits a complete hyperbolic metric, and with respect to such a metric, $M$ is geometrically finite (cf. for instance [2, Example 1.4]). It now follows from Lemma 5.2 that $\pi_1(M)$ is elementary, and thus, cyclic, $M$ being two-dimensional, which is a contradiction.

Although this gives a quite wide class of examples, it seems reasonable to present a more explicit one.

Example 5.4 Let $M$ be a two-dimensional torus with a cusp attached. Initially, we endow $M$ with a complete Riemannian metric, so that the cusp $D$ is isometric to a domain of a flat cylinder. Then it is clear that $\lambda_0(M) = \lambda_0(D) = \lambda_0^\text{ess}(M) = 0$. Since the fundamental group of $D$ is amenable, it follows from [21, Corollary 1.6] that $\lambda_0(M) = \lambda_0(M)$, where $\tilde{M}$ stands for the universal covering space of $M$. Therefore, $M$ endowed with this Riemannian metric is not spectrally-tight.

We now endow $M$ with a complete Riemannian metric, so that the cusp is isometric to the surface of revolution generated by $e^{-\alpha t^n}$ with $t \geq 1$, for some $\alpha > 1$. Then the spectrum of $M$ is discrete (cf. [11, Theorem 2]) and Corollary 5.1 characterizes spectral-tightness. It should be observed that $\pi_1(M)$ is the free group $F_2$ in two generators. Since the fundamental group of any negatively curved, closed manifold contains a subgroup isomorphic to $F_2$, [13, Theorem 1] shows that any amenable subgroup of $F_2$ is cyclic. It is not difficult to verify that the unique normal, cyclic subgroup of $F_2$ is the trivial one. Hence, $M$ endowed with this Riemannian metric is spectrally-tight, from Corollary 5.1.

Acknowledgements I would like to thank Werner Ballmann for some very helpful comments and remarks. I am also grateful to the Max Planck Institute for Mathematics in Bonn for its support and hospitality. I would also like to thank the referees for their suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Competing Interests The authors have no competing interests to declare that are relevant to the content of this article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of nonpositive curvature. Prog. Math. 61, Birkhäuser Boston, Inc., Boston, MA, vi+263, isbn=0-8176-3181-X (1985)
2. Ballmann, W., Matthiesen, H., Mondal, S.: On the analytic systole of Riemannian surfaces of finite type. Geom. Funct. Anal. 27(5), 1070–1105 (2017)
3. Ballmann, W., Matthiesen, H., Polymerakis, P.: On the bottom of spectra under coverings, Math. Z. 288(3–4), 1029–1036, 0025-5874 (2018)
4. Ballmann, W., Polymerakis, P.: Bottom of spectra and coverings. Surv. Differ. Geom. 23, 1–33 (2020)
5. Ballmann, W., Polymerakis, P.: Bottom of spectra and coverings of orbifolds. Int. J. Math. 32(12), 2140018, 33 (2021)
6. Ballmann, W., Polymerakis, P.: On the essential spectrum of differential operators over geometrically finite orbifolds. J. Differ. Geom. (To Appear)
7. Belegradek, I., Kapovitch, V.: Classification of negatively pinched manifolds with amenable fundamental groups. Acta Math. 196(2), 229–260 (2006)
8. Bessa, G.P., Montenegro, J.F., Piccione, P.: Riemannian submersions with discrete spectrum. J. Geom. Anal. 22(2), 603–620, 1050–6926 (2012)
9. Bowditch, B.H.: Geometrical finiteness with variable negative curvature. Duke Math. J. 77(1), 229–274 (1995)
10. Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. 56(4), 581–598 (1981)
11. Brooks, R.: On the spectrum of non-compact manifolds with finite volume. Math. Z. 187, 425–432 (1984)
12. Burger, M., Schroeder, V.: Amenable groups and stabilizers of measures on the boundary of a Hadamard manifold, Math. Ann. 276(3), 505–514, 0025-5831 (1987)
13. Chen, S.S.: On the fundamental group of a compact negatively curved manifold. Proc. Am. Math. Soc. 71(1), 119–122 (1978)
14. Cheng, S.Y., Yau, S.T.: Differential equations on Riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. 28(3), 333–354 (1975)
15. Donnelly, H., Li, P.: Pure point spectrum and negative curvature for noncompact manifolds. Duke Math. J. 46(3), 497–503 (1979)
16. Eberlein, P.: Euclidean de Rham factor of a lattice of nonpositive curvature. J. Differ. Geom. 18(2), 209–220 (1983)
17. Følner, E.: On groups with full Banach mean value. Math. Scand. 3(243–254), 0025–5521 (1955)
18. Ji, L., Li, P., Wang, J.: Ends of locally symmetric spaces with maximal bottom spectrum. J. Reine Angew. Math. 632(1–35), 0075–4102 (2009)
19. Li, P., Wang, J.: Complete manifolds with positive spectrum. II. J. Differ. Geom. 62(1), 143–162 (2002)
20. Polymerakis, P.: Coverings preserving the bottom of the spectrum. J. Spectr. Theory 12(3), pp. 993–1022 (2022).
21. Polymerakis, P.: On the spectrum of differential operators under Riemannian coverings. J. Geom. Anal. 30(3), 3331–3370 (2020)
22. Rosenblatt, J.M.: A generalization of Følner’s condition. Math. Scand. 33(153–170), 0025–5521 (1973)
23. Sambusetti, A.: Asymptotic properties of coverings in negative curvature. Geom. Topol. 12(1), 617– 637, 1465-3060 (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.