Introducing the trapezoidal pendulum: dynamics of coupled pendula suitable for distance teaching

J J Bissell and S K Bhamidimarri

Department of Electronic Engineering, University of York, York, YO10 5DD, United Kingdom

E-mail: john.bissell@york.ac.uk

Abstract

Two pendula of length $l$, suspended from points separated by a horizontal distance $a$, can be joined in a trapezium configuration by linking the bobs with a light inextensible string of length $b < a$. Here we introduce this ‘trapezoidal pendulum’ as a charming and novel oscillatory system suitable for introductory physics laboratories. In particular, we show that the time period $\tau$ is given by a scale-invariant equation of the form $(\tau/\tau_s) = P(a, b, l)$, where $\tau_s(l) = 2\pi \sqrt{l/g}$ (with $g$ gravity), and $P(a, b, l)$ is a dimensionless function of $a$, $b$, and $l$. The trapezoidal pendulum exhibits richer behaviour than the canonical simple pendulum, but is nevertheless easy to construct from basic household items, making it an ideal system for home experimentation. Indeed, here we verify the $\tau$ dependencies empirically using modest, improvised apparatus. These investigations suggest exciting possibilities for distance teaching activities, such as collaborative studies of the dimensionless scalings, using data obtained by students working independently, and on apparatus of different sizes.

Keywords: simple harmonic motion, distance teaching, coupled oscillator, trapezoidal pendulum, home experimentation

1. Introduction

Two simple pendula—each of length $l$, and bob mass $m$—suspended from fixed points separated by a horizontal distance $a$ can be joined by a light inextensible string of length $b \leq a$ connecting their bobs. If $a = b$, then the equilibrium configuration of this combined system traces out a rectangle (cf figure 1), and the pendula can be made to oscillate in parallel without exchanging energy. In this case the time period $\tau$ of oscillations is equal to the period of the simple pendulum
It has come to our attention when preparing the proofs of this manuscript that a related system involving rigid pendula has been considered theoretically by Ramachandra et al [6]; however, their system is limited to the case where $b < a$, and is solved via a linearised $5 \times 5$ matrix eigenvalue problem derived using Lagrangian mechanics. Our system is thus more general, and our theory far simpler (with the further advantage that it is supported by experimental data).

\[ \tau_a, \text{ that is,} \]
\[ \tau = \tau_a(l) = 2\pi \sqrt{\frac{l}{g}}, \]
\[ \tau(a, b, l) = 2\pi \sqrt{\frac{l}{g} P(a, b, l)}, \]

where $g$ is the acceleration due to gravity. Alternatively, if $b < a$, then the equilibrium configuration traces out an isosceles trapezium (see figure 1), and the pendula can be made to oscillate in the equilibrium plane according to a rocking motion [1]. In this case the pendula exchange energy throughout the motion, meaning that the oscillation time period becomes a more complicated function of $a$, $b$, and $l$. Such energy exchange motivates describing the combined system as a coupled oscillator [2–5, 8]; however, because the motion of one pendulum is entailed entirely by the motion of the other, here we shall refer to the system collectively as a single 'trapezoidal pendulum'.

In this article we introduce the trapezoidal pendulum as a charming and novel oscillatory system suitable for use in introductory degree-level physics laboratories on simple harmonic motion. The idea of a trapezoidal pendulum appears to have been somewhat overlooked from the standard undergraduate repertoire, and we therefore consider the problem it poses in some detail\(^1\).

Beginning with a description of the basic model (section 2), we propose a linear theory for the time period $\tau$ that yields an expression of the form

\[ \tau(a, b, l) = 2\pi \sqrt{\frac{l}{g} P(a, b, l)}, \]

where $P(a, b, l)$ is a scale invariant function of $a$, $b$, and $l$ (section 3). This expression is then verified empirically in section 4, where we describe a set of measurements designed to assess the impact of each of the key parameters. Our experience shows that the trapezoidal pendulum is easy to construct from common household items, making it an ideal system for home experimentation, and distance teaching activities (section 5). We discuss some innovative possibilities for these, such as mass-participatory collaborative work on dimensionless scaling, in sections 5 and 6.

2. Model

The basic configuration of the trapezoidal pendulum is shown in figure 1: two identical simple pendula, each of length $l$, are suspended from fixed points separated by a horizontal distance $a$, and their bobs joined by a light inextensible string of length $b \leq a$. These constituent pendula define angles $\theta$ and $\phi$ with the vertical, where $\theta$ is the angle of the first pendulum, and $\phi$ is the angle of the second. In this way we may define static equilibrium angles $\theta_0$ and $\phi_0$ such that

\[ \theta = \theta_0 = \phi_0 = \phi. \]

Thus, the equilibrium configuration traces out a rectangle when $b = a$ (in which case $\theta_0 = \phi_0 = 0$), and an equilateral trapezium when $b < a$, as in figure 1(a).

\[ (l \cos \theta - l \cos \phi)^2 + (a - l \sin \theta - l \sin \phi)^2 = b^2. \]

According to this equation, the equilibrium angle $\theta = \theta_0 = \phi_0 = \phi$ satisfies

\[ \sin \theta_0 = \frac{(a - b)}{2l}. \]

\[ \text{Figure 1. Schematics of the trapezoidal pendulum when: (a) in the equilibrium configuration, with } \theta = \theta_0 = \phi_0 = \phi; \text{ and (b) displaced from equilibrium, with } \theta \neq \phi. \]
Introducing the trapezoidal pendulum: dynamics of coupled pendula suitable for distance teaching

Notice that the equilibrium configuration also defines a height $h$ for the trapezium,

$$h = l \cos \theta_0 = l(1 - \sin^2 \theta_0)^{1/2}$$

$$= l \left(1 - \frac{(a-b)^2}{4l^2}\right)^{1/2}, \quad (6)$$

where $h = h(a, b, l)$ is specified completely by $a$, $b$, and $l$, as shown in figure 1(a).

3. Theory

By adopting an energy method, an equation describing simple harmonic motion of the trapezoidal pendulum may be found using mathematics no more complicated than differentiation. In this way the theory can be considered analytically tractable for first-year undergraduates; however, students can easily experiment on the pendulum without deriving the time period from scratch, and should be encouraged to do so. Our approach is to determine the pendulum’s total energy (section 3.1), and then use the theory of small oscillations (section 3.2) to obtain an expression for the time period (sections 3.3 and 3.4).

3.1. Total energy

Recall that the trapezoidal pendulum comprises two simple pendula: one that makes an angle $\theta$ with the vertical, and another that makes an angle $\phi$, as depicted in figure 1(b). The total energy $E_\theta$ of the first of these pendula is thus

$$E_\theta = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl \cos \theta,$$  \quad (7)

where we adopt the ‘dot’ notation for time derivatives throughout, i.e.

$$\dot{\theta}(t) = \frac{d\theta}{dt}, \quad \text{and} \quad \ddot{\theta}(t) = \frac{d^2\theta}{dt^2}. \quad (8)$$

Similarly, the second pendulum has total energy $E_\phi$ given by

$$E_\phi = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl \cos \phi.$$  \quad (9)

Assuming a frictionless system, therefore, the total energy $E$ of the trapezoidal pendulum is

$$E = E_\theta + E_\phi = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \dot{\phi}^2\right) - mgl(\cos \theta + \cos \phi),$$  \quad (10)

where this energy is constant ($\dot{E} = 0$) because the pendulum does no work.

Now the equilibrium angle $\theta_0 = \phi_0$ is fixed, meaning that any time-dependence in $\theta$ and $\phi$ can be described by perturbations $\theta_1(t)$ and $\phi_1(t)$ defined such that

$$\theta(t) = \theta_0 + \theta_1(t) \quad \text{and} \quad \phi(t) = \phi_0 + \phi_1(t). \quad (11)$$

Indeed, in this way we have

$$\dot{\theta}(t) = \dot{\theta}_1(t) \quad \text{and} \quad \dot{\phi}(t) = \dot{\phi}_1(t), \quad (12)$$

and equation (10) can be written in the form

$$E = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\phi}_1^2\right) - mgl\left[\cos(\theta_0 + \theta_1) + \cos(\theta_0 + \phi_1)\right],$$  \quad (13)

where we noted that $\phi_0 = \theta_0$. Hence, by appealing to the compound angle formula, i.e. $\cos(\theta_0 + \theta_1) = \cos \theta_0 \cos \theta_1 - \sin \theta_0 \sin \theta_1$ (and likewise for $\phi$) [7], we obtain

$$\frac{E}{ml^2} = \frac{1}{2} \left(\dot{\theta}_1^2 + \dot{\phi}_1^2\right) - \frac{g}{l} \left[h \left(\cos \theta_1 + \cos \phi_1\right) - \frac{(a-b)}{2l} \left(\sin \theta_1 + \sin \phi_1\right)\right],$$  \quad (14)

where we substituted for $\sin \theta_0$ and $\cos \theta_0$ using equations (5) and (6). This equation gives the total energy of the trapezoidal pendulum for arbitrary perturbations $\theta_1$ and $\phi_1$.

3.2. Small oscillations

For small amplitude perturbations ($\theta_1 \ll 1$) we make the approximations

$$\sin \theta_1 \approx \theta_1 \quad \text{and} \quad \cos \theta_1 \approx 1 - \frac{\theta_1^2}{2},$$  \quad (15)

(and likewise for $\phi_1$) such that equation (14) becomes

$$\frac{1}{2} \left(\dot{\theta}_1^2 + \dot{\phi}_1^2\right) - \frac{g}{l} \left[h \left(2 - \frac{\theta_1^2}{2} - \frac{\phi_1^2}{2}\right)\right].$$
\[ -\frac{(a-b)}{2l}(\theta_1 + \phi_1) \approx \frac{E}{ml^2}. \]  

(16)

It may be shown that the perturbations \( \theta_1 \) and \( \phi_1 \) are related by

\[ \phi_1^2 \approx \theta_1^2, \quad (\phi_1 + \theta_1) \approx \frac{a(a-b)}{2bh} \theta_1^2, \quad \text{and} \quad \phi_1^2 \approx \dot{\theta}_1^2 \]

(17)

(see appendix). Crucially, therefore, by putting these approximations into equation (16) we obtain an expression for the total energy in terms of \( \theta_1 \) only, viz

\[ \ddot{\theta}_1 + \left( \frac{g}{l} + \frac{a(a-b)^2}{4bh^2} \right) \dot{\theta}_1 \approx \frac{E + 2mgh}{ml^2}. \]

(18)

Thus, differentiating with respect to time, for dynamic solutions \( (\dot{\theta}_1 \neq 0) \) we require

\[ \ddot{\theta}_1(t) + \omega^2 \theta_1(t) \approx 0 \quad \text{with} \]

\[ \omega = \left[ \frac{g}{l} + \frac{a(a-b)^2}{4bh^2} \right]^{1/2}. \]

(19)

This expression is the equation for simple harmonic motion, and has general solution

\[ \theta_1(t) = \theta_1(0)e^{i\omega t}, \]

(20)

where \( \omega \) is the oscillation frequency [2].

3.3. Time period

It follows from equations (19) and (20) that small amplitude perturbations \( (\dot{\theta}_1, \phi_1 \ll 1) \) of the trapezoidal pendulum are oscillatory, with time period \( \tau \) given by

\[ \tau(a,b,l) = \frac{2\pi}{\omega} = \tau_0(l)P(a,b,l), \]

(21)

where \( \tau_0(l) = 2\pi\sqrt{l/g}, \) and

\[ P(a,b,l) = \left[ \frac{h}{l} + \frac{a(a-b)^2}{4bh^2} \right]^{-1/2} \]

(22)

is a dimensionless function of \( a, b, \) and \( l \) only, since

\[ h(a,b,l) = l\left(1 - \frac{(a-b)^2}{4l^2}\right)^{1/2}. \]

(23)

To consider the behaviour of the time period \( \tau \), let us suppose that \( l \) and \( a \) are fixed, but that \( (a-b) \geq 0 \) is varied by changing \( b \). If \( b = a \), then \( P(a,b,l) = 1 \), and we find as expected that the period is equal to that of the simple pendulum, i.e. \( \tau = \tau_0 \) (see figure 2). However, as \( b \) is decreased \( (b < a) \), the value of \( P(a,b,l) \) also decreases, and the time period becomes an ever smaller fraction of \( \tau_0 \) (see figure 2). Indeed, equation (21) implies that the time period vanishes in the limits that either \( b \to 0 \) or \( h \to 0 \), i.e.

\[ \tau \to 0 \quad \text{as either} \quad b \to 0 \quad \text{or} \quad b \to (a-2l), \]

(24)

where the latter limit requires a geometry with \( a > 2l \). We expect the approximations used to derive equation (19) to become invalid before these limits are reached. An alternative extreme limit may be taken when \( l \gg a, b \), in which case \( h \approx l \) and \( \tau \approx \tau_0 \).

Note that although our analysis requires the perturbations \( \dot{\theta}_1 \) and \( \phi_1 \) to be small, the equilibrium angle \( \theta_0 = \phi_0 \) need not be. In this way the trapezoidal pendulum can be used to emphasise the important conceptual distinction between small angle perturbations from an equilibrium, and small angles per se.

3.4. Scale invariance

Observe that the overall size of the pendulum is defined by the length \( l \), while the other length-scales \( a \) and \( b \) are fixed by whatever values are
chosen for the ratios \( a/l \) and \( b/l = (a/l - 2\sin\theta_0) \). Thus, the function \( P(a, b, l) \) describes the proportions of the trapezoidal pendulum when it is in equilibrium. It is in this sense that \( P(a, b, l) \) can be considered scale-invariant: given two trapezoidal pendula of identical proportions, but with one larger than the other by the factor \( r \), we have by equation (22) that

\[
P(a, b, l) = P(ra, rb, rl).
\]  
(25)

Hence, increasing the size of a trapezoidal pendulum by the factor \( r \) increases its time period by the factor \( \sqrt{r} \), viz.

\[
\frac{\tau(ra, rb, rl)}{\tau(a, b, l)} = \frac{\tau_s(rl)}{\tau_s(l)} = \sqrt{r}
\]

(26)

(cf the simple pendulum). The trapezoidal pendulum therefore presents interesting possibilities for investigating scaling laws (see section 5).

4. Experiments

Because the trapezoidal pendulum’s time period \( \tau = \tau_s(l)P(a, b, l) \) depends on three parameters \( a, b, \) and \( l \), it exhibits richer behaviour than the classical simple pendulum, with more possibilities for experimentation. Like the simple pendulum, however, the trapezoidal pendulum can be constructed easily from non-specialist equipment. Thus, while the trapezoidal pendulum is perfectly suitable for use in traditional teaching contexts, it is also an ideal system for home experimentation, and remote learning.

In this section we describe a set of empirical investigations on the time period \( \tau \) conducted during the first lockdown period of the 2019–2020 coronavirus pandemic\(^2\). These experiments demonstrate the ease with which the trapezoidal pendulum can be studied in a distance learning context, and reveal practical tips on assembling safe and reliable testing apparatus from common household items (sections 4.1 and 4.2). Ultimately, of course, they also serve to verify our theoretical predictions (section 4.3).

---

\(^2\) The 2019–2020 coronavirus (COVID-19) outbreak was declared a pandemic by the World Health Organisation on 11th March 2020 [9].
Table 1. Five selected values for $l$ and $a$, and (essentially) four different ratios $a/l$. Measuring these lengths with a tape-measure limited our precision to about $\pm 0.5$ cm.

| $l$/cm | $a$/cm | $a/l$ |
|--------|--------|-------|
| 68.5 $\pm$ 0.5 | 132.5 $\pm$ 0.5 | 1.934 $\pm$ 0.016 |
| 68.5 $\pm$ 0.5 | 155.0 $\pm$ 0.5 | 2.263 $\pm$ 0.018 |
| 77.5 $\pm$ 0.5 | 100.0 $\pm$ 0.5 | 1.290 $\pm$ 0.011 |
| 77.5 $\pm$ 0.5 | 119.5 $\pm$ 0.5 | 1.542 $\pm$ 0.012 |
| 77.5 $\pm$ 0.5 | 150.0 $\pm$ 0.5 | 1.935 $\pm$ 0.014 |

4.2. Measurements

The dependence of $\tau(a, b, l)$ on the length $l$ is to some extent intuitive, and for this reason our experiments focused on the proportions of the trapezoidal pendulum (i.e. the ratios $a/l$ and $b/l$), rather than its absolute size. To this end we selected two values for $l$, and five values for $a$, as listed in table 1 (cf figure 4). These pairs of values were chosen such that two of the pairs yielded the ratio $a/l \approx 1.94$ (see section 4.3).

Given fixed values of $a$ and $l$, the procedure for testing the dependence of $\tau(a, b, l)$ on $b$ can be understood as follows. For the first measurement of $\tau$, the trapezoidal pendulum is set up with $b$ just less than $a$, and the time period inferred by measuring the total time $t_f$ for some number of oscillations $n$, and computing $\tau = t_f/n$. (Time measurements can be taken to a reasonable level of precision ($\sim 0.5 \text{ s}$) using the stop-clock on a smartphone.) For subsequent measurements, $b$ is reduced incrementally by shortening the string connecting the bobs. In this way one obtains a set of $\tau(a, b, l)$ measurements with $(a - b)/2l$ increasing.

4.3. Results

Data from our experiments are shown in figure 4, with $\tau$ normalised to $\tau_s(l)$, and exhibit good agreement with the theoretical prediction $\tau/\tau_s = P(a, b, l)$ of equation (21). Notice that although five pairs of values were used for $a$ and $l$, two of these pairs yielded the ratio $a/l \approx 1.94$, and therefore sit on the same theory curve (see figure 4). In this way our data supports both: (i) the overall functional dependence of $\tau$ with $a$, $b$, and $l$, as described in section 3.3; and (ii) the dimensionless scaling of $\tau/\tau_s = P(a, b, l)$ described in section 3.4.

Overall we found that experimental uncertainty could be kept small, despite the use of relatively crude apparatus. In particular, the fractional error in $(a - b)/2l$ was typically around 1–2%, only peaking to ~5–15% for the very smallest values, i.e. when $(a - b)/2l \leq 0.1$. Similarly, the errors in $\tau/\tau_s(l)$ all lay within the range ~1–3%, and were largest when $\tau/\tau_s(l)$ was small. These uncertainties are either less than, or similar to the magnitude of the marker size used in figure 4, rendering error bars unnecessary.

5. Discussion

Our empirical results lend compelling support to the theory described in section 3, but they also demonstrate that this theory can be investigated effectively using modest, improvised apparatus. From a teaching perspective, therefore, the trapezoidal pendulum neatly illustrates how scientific thinking can be applied effectively without the need for specialist equipment.

One of the most exciting possibilities raised by the trapezoidal pendulum, however, is the potential for coordinating mass-participatory investigations on the dimensionless scaling of $P(a, b, l)$. In particular, the dependence of $P(a, b, l)$

---

**Figure 4.** Experimental variation of the time period $\tau/\tau_s$ with $(a - b)/2l$ (markers), compared to theory (curves). Five pairs of values have been chosen for $a$ and $l$; two of these pairs yield the ratio $a/l \approx 1.94$ (circles), and thus sit on the same theory curve (cf figure 2). Error bars are similar to marker size, and have been omitted for clarity. We take gravity as $g = 9.81 \text{ms}^{-2}$ throughout.
Introducing the trapezoidal pendulum: dynamics of coupled pendula suitable for distance teaching

on \((a - b)/2l\) can be verified by collecting data obtained by students working independently, and on apparatus of different sizes: provided each student uses the same proportion for \(a/l\), collectively the data should all lie on the same theory curve (cf figure 4).

Such investigations can be conducted within the familiar setting of a conventional undergraduate laboratory, but they can also be undertaken by students working remotely on homemade apparatus. Indeed, the trapezoidal pendulum is an ideal system for use in distance teaching activities: it is easy to construct from household items; it requires no specialist measuring equipment; and it yields good quality data. With these possibilities in mind, however, students tasked with devising their own experiments on the trapezoidal pendulum should be reminded of the following points:

- The supporting structure should be rigid.
- The suspension points should be on the same horizontal level.
- The string used to suspend the bobs should be inelastic.
- The pendulum should be long compared to the size of the bobs.
- The masses used for the bobs should be identical.

Crucially, of course, the process of designing apparatus offers a valuable, engaging, and liberating educational experience for students its own right.

6. Conclusion

Two simple pendula can be coupled in a trapezium configuration by linking their bobs with a light inextensible string. Here we have introduced this ‘trapezoidal pendulum’ as a novel oscillatory system, and considered its applications to introductory degree-level physics laboratories, and remote learning (section 2). In so doing we have proposed a basic linear theory for the pendulum, and derived an equation for its time period \(\tau\) of the form

\[
\tau = \tau(l)P(a, b, l),
\]

where \(P(a, b, l)\) is a dimensionless function of the pendulum’s length-scales \(a, b\) and \(l\), and \(\tau(l) = 2\pi\sqrt{l/g}\) is the period of the simple pendulum (section 3).

From a modelling perspective, the trapezoidal pendulum illustrates how energy arguments can be combined with the method of small angles to study the way in which simple harmonic motion arises in systems perturbed from equilibrium. Although there is no suggestion that students should be able to derive this theory from scratch, it represents an accessible context in which the method of small oscillations can be considered analytically tractable for first-year undergraduates [10].

In addition to demonstrating that the trapezoidal pendulum oscillates harmonically, the main purpose of the theory is to introduce the concept of scale invariance. Indeed, the trapezoidal pendulum is a fundamentally geometric system, making it an ideal context in which to explore this idea in a way that is engaging, and intuitive (section 3.4). Here we have succeeded in verifying both the functional dependence of \(\tau\), and the dimensionless scaling of \(P(a, b, l)\) using apparatus improvised from inexpensive household items (section 4). In this respect an especially exciting possibility for the trapezoidal pendulum is as a system for home experimentation, and distance teaching activities. For example, by coordinating a mass-participation study, the scale invariance of \(P(a, b, l)\) may be investigated using data supplied by students working independently, on homemade apparatus (section 5). Ultimately, however, the trapezoidal pendulum is simply a beautiful system, and complements the standard undergraduate curriculum by offering a fresh context in which to study oscillatory motion.

Appendix

The approximations listed in equation (17) may be derived using a Taylor expansion as follows. First observe that \(\phi\) and \(\theta\) are implicit functions of each other, say \(\phi(\theta)\) and \(\theta(\phi)\), according to equation (4), viz.

\[
(\phi - \phi(\theta) - \theta(\phi)) = 0.\]  

(A1)

Although it does not look easy to solve this equation, by Taylor expanding \(\phi(\theta)\) about \(\theta_0\) we obtain a relationship between \(\phi\) and \(\theta\) of the form...
\[ \phi(\theta_0 + \theta_1) = \phi(\theta_0) + \frac{d\phi}{d\theta} \bigg|_{\theta_0} \theta_1 + \frac{d^2\phi}{d\theta^2} \bigg|_{\theta_0} \frac{\theta_1^2}{2} + O(\theta_1^3), \quad (A2) \]

where \(O(\theta_1^3)\) means ‘of magnitude \(\theta_1^3\)' [7], and the derivatives may be found by differentiating equation (A1) implicitly to give

\[ \frac{d\phi}{d\theta} \bigg|_{\theta_0} = -1 \quad \text{and} \quad \frac{d^2\phi}{d\theta^2} \bigg|_{\theta_0} = \frac{a(a - b)}{bh}. \quad (A3) \]

Since \(\phi(\theta_0 + \theta_1) = \phi_0 + \phi_1\), with \(\phi_0 = \phi(\theta_0)\), the expansion in equation (A2) may then be written

\[ \phi_1 = -\theta_1 + \frac{a(a - b)}{2bh} \theta_1^2 + O(\theta_1^3). \quad (A4) \]

Hence, for very small perturbations \(\theta_1 \ll 1\) we have

\[ \phi_1 \approx -\theta_1 \quad \text{and} \quad \phi_1^2 \approx \theta_1^2, \quad (A5) \]

alongside the relationship

\[ (\phi_1 + \theta_1) \approx \frac{a(a - b)}{2bh} \theta_1^2. \quad (A6) \]

Similarly, by differentiating the Taylor expansion (A4) with respect to time, we obtain

\[ \dot{\phi}_1 \approx -\dot{\theta}_1 \quad \text{and} \quad \dot{\phi}_1^2 \approx \dot{\theta}_1^2. \quad (A7) \]

Equations (A5)–(A7) are the approximations listed in equation (17) as required.

References

[1] Bissell J J, and Bhamidimarri S K 2020 The Trapezoidal Pendulum (https://www.youtube.com/watch?v=GZAkr-OyTbM&feature=youtu.be)
[2] Kibble T W B 1966 Classical Mechanics (New York: McGraw-Hill)
[3] Kodejška Č, Lepil O and Sedláčková H 2018 Coupled oscillators: interesting experiments for high school students Phys. Educ. 53 045021
[4] Picciarelli V and Stella R 2010 Coupled pendulums: a physical system for laboratory investigations at upper secondary school Phys. Educ. 45 402–8
[5] Wang H 2002 Two pendulums on a string Phys. Educ. 37 347
[6] Ramachandran P, Krishna S G and Ram Y M 2011 Instability of a constrained pendulum system Am. J. Phys. 79 395–400
[7] Riley K F, Hobson M P and Bence S J 2006 Mathematical Methods for Physics and Engineering 3rd edn (Cambridge: Cambridge University Press)
[8] Oliveira V 2020 Measuring g with a classroom pendulum using changes in the pendulum string length Phys. Educ. 51 063007
[9] World Health Organisation 2020 Coronavirus disease 2019 (COVID-19) Situation Report vol 52
[10] Troy T, Reiner M, Haugen A J and Moore N T 2017 Small oscillations via conservation of energy Phys. Educ. 52 065009