Quantile hedging for multiple assets derivatives

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Abstract
The problem of quantile hedging for multiple assets derivatives in the Black-Scholes model with correlation is considered. Explicit formulas for the probability maximizing function and the cost reduction function are derived. Applicability of the results for the widely traded derivatives as digital, quantos, outperformance and spread options is shown.

Key words: quantile hedging, multiple assets options, correlated assets.
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1 Introduction

As recent events on the market have shown the risk appearing in pricing of financial contracts should be more thoroughly surveyed. Although the problem of minimizing risk is widely studied in the literature, the great majority of the results do not meet the expectations of practitioners who are interested in straightforward applications. This paper is concerned with the issue of risk analysis for the multiple assets derivatives and provides explicit computing methods for the risk parameters.

The risk is measured by the possibility of a partial hedging of the payoff. Thus our approach is based on the idea of quantile hedging which was introduced in [6] and later developed in various directions, see for instance [3], [10], [2], [1]. Let us briefly sketch a general concept. Denote by $H$ a contingent claim and assume that the arbitrage free pricing method indicates its price $p(H)$. This means that if the investor has an initial endowment $x \geq p(H)$ then he is able to follow some trading strategy such that his portfolio hedges $H$ with probability 1. If this is the case, then $x$ carries no risk and the probability maximizing function $\Phi_1$ equals 1, i.e. $\Phi_1(x) = 1$. On the other hand, if $x < p(H)$ then the shortfall probability is strictly greater than zero for each trading strategy and then $\Phi_1(x) < 1$. The greater the probability of shortfall is the smaller the value $\Phi_1(x)$ is. Thus the function $\Phi_1$ can be viewed as a measure of the risk sensitivity to the price reduction of the option. There is also another aspect of the problem. Assume that the hedger is willing to accept some risk measured by the shortfall probability in order to reduce initial cost. He chooses a number $\alpha \in [0,1]$ and searches for a minimal initial capital $\Phi_2(\alpha)$ which allows to find a strategy such that the probability of the shortfall is smaller then $1 - \alpha$. Thus if the hedger accepts no risk, i.e. $\alpha = 0$, then the minimal cost required to replicate $H$ is just $p(H)$. In this case the cost reduction function satisfies $\Phi_2(0) = p(H)$. However, if $\alpha > 0$ then
\( \Phi_2(\alpha) < p(H) \) and the function \( \Phi_2 \) enables us to view the effect how the risk acceptance affects the cost reduction of the option. Recall the numerical example from [6] p. 261 which shows that \( \Phi_2(0.05) = 0.59 \cdot p(H) \) for a call option with certain parameters. This means that the acceptance of a 5% margin of risk reduces the hedging cost by 41%. This shows that quantile hedging is an attractive tool for the risk analysis and should be taken into account by traders.

The basic problem, however, is to determine functions \( \Phi_1 \) and \( \Phi_2 \) for specific derivatives. There are only a few examples in the literature where they are explicitly found. In [6] explicit formulas are given for the most important case of a call option in a classical Black-Scholes model. The method can be mimicked to obtain formulas for the put option. The idea is based on reducing the original dynamic problem to the static one which can be solved with methods used in the theory of statistical tests. Since the market was complete the solution of the static problem could be obtained, via Neyman-Pearson lemma, by indicating a non-randomized test for the appropriate probability measures. The Neyman-Pearson lemma can be generalized for the case of composite hypotheses, i.e. when measures are replaced by the families of measures, see [4] where the solution in the abstract form is presented. However, straightforward applicability of this result towards incomplete markets seems to be questionable. This paper is devoted to determining functions \( \Phi_1 \) and \( \Phi_2 \) for the multiple assets derivatives in the Black-Scholes framework with correlation. As the market is complete, we follow the same general method as in [6], but we find the solutions explicitly using specific features of the model. More precisely, we show that the original problem can be reduced to that of finding another two deterministic functions \( \Psi_1, \Psi_2 \) depending on \( H \), which turned out to be regular, i.e. continuous and strictly monotone if \( H \) is of a reasonable form, see Proposition 3.4 and Proposition 3.5.

Then, roughly speaking, \( \Phi_1 = \Psi_1 \circ \Psi_2^{-1} \) and \( \Phi_2 = \Psi_2 \circ \Psi_1^{-1} \); for a precise formulation see Theorem 3.6. In the one dimensional case when \( H \) is a call option the result covers the above mentioned example from [6]. We also determine explicit forms of \( \Psi_1 \) and \( \Psi_2 \) for commonly traded derivatives, see Section 4 and its subsections. As \( \Psi_1, \Psi_2 \) are rather of a complicated form, the inverse functions can not be given by analytic formulas but can be determined with the use of numerical methods. Thus a great advantage of our results is that they can be used in practice.

The paper is organized as follows. In Section 2 we briefly recall the multidimensional Black-Scholes model and formulate the problem strictly. Section 3 contains the main result - Theorem 3.6 which is proceeded by a general discussion on the results from [6] and the Neyman-Pearson technique. The method established in Theorem 3.6 is used in Section 4 for calculating the functions \( \Psi_1, \Psi_2 \) for two assets derivatives which are widely traded. At the beginning of Section 4 we adopt general setting to a two dimensional model and recall basic properties of a multidimensional normal distribution. In the sequel explicit formulas for \( \Psi_1 \) and \( \Psi_2 \) are presented in the case of digital option, quanto domestic, quanto foreign, outperformance and spread options.
2 The model

Let \((\Omega, \mathcal{F}, t \in [0, T], P)\) be a fixed probability space with filtration. The prices of \(d\) shares are given by the Black-Scholes equations

\[
    dS^i_t = S^i_t(\alpha^i_t dt + \sigma^i_t dW^i_t), \quad i = 1, 2, ..., d, \quad t \in [0, T],
\]

where \(\alpha^i \in \mathbb{R}, \sigma^i > 0, i = 1, 2, ..., d\) and \(W_t = (W^1_t, W^2_t, ..., W^d_t), t \in [0, T]\), is a sequence of standard Wiener processes adapted to \(\mathcal{F}_t; t \in [0, T]\) with the correlation matrix \(Q\) of the form

\[
    Q = \begin{bmatrix}
        1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,d} \\
        \rho_{2,1} & 1 & \rho_{2,3} & \cdots & \rho_{2,d} \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        \rho_{d,1} & \rho_{d,2} & \rho_{d,3} & \cdots & 1
    \end{bmatrix},
\]

where

\[
    \rho_{i,j} = \text{cor}\{W^i_t, W^j_t\}, \quad i, j = 1, 2, ..., d.
\]

We assume that \(Q\) is positive definite. The process given above will be called a \(Q\)-Wiener process. The trader can invest his money in stocks as well as put it on a savings account which dynamics is given by

\[
    dB_t = rB_t dt, \quad t \in [0, T],
\]

with \(r\) standing for a constant short rate.

**Remark 2.1** The most common approach for the description of the market is based on a sequence of independent Wiener processes, see for instance a classical textbook \([9]\). It can be shown that the model described above is equivalent to the model with \(d\) independent Wiener processes and the \(d \times d\) diffusion matrix with constant coefficients. We work with a correlated Wiener process because it is more convenient for later calculations. Let us also mention that parameters in such model can be easily estimated from data, see \([7]\) p.104.

Let us now briefly characterize a martingale measure of the model, i.e. a measure \(\tilde{P}\) which is equivalent to \(P\) such that the discounted price processes \(\tilde{S}^i_t := e^{-rt}S^i_t, i = 1, 2, ..., d\) are local martingales. The following is a version of Theorem 10.14 in \([5]\) adapted to our finite dimensional setting.

**Theorem 2.2** Let \(\varphi\) be a predictable process taking values in \(\mathbb{R}^d\) satisfying

\[
    \mathbb{E}\left(e^{\int_0^T (Q^{-\frac{1}{2}}\varphi_t dW_t) - \frac{1}{2} \int_0^T |\varphi_t|^2 dt}\right) = 1.
\]

Then the process

\[
    \tilde{W}_t = W_t - \int_0^t Q^{-\frac{1}{2}}\varphi_s ds, \quad t \in [0, T],
\]

is a \(Q\)-Wiener process with respect to the measure \(\tilde{P}\) with a density

\[
    \frac{d\tilde{P}}{dP} = e^{\int_0^T (Q^{-\frac{1}{2}}\varphi_t dW_t) - \frac{1}{2} \int_0^T |\varphi_t|^2 dt}.
\]
It can be shown that each measure equivalent to $P$ can be characterized by a density process

$$Z_t := e^{\int_0^t (\alpha - r 1_t) d\tilde{W}_s - \frac{1}{2} \int_0^t |\varphi_s|^2 ds}, \quad t \in [0, T],$$

(2.1)

for some predictable $\mathbb{R}^d$-valued process $\varphi$. The process $\tilde{S}^i$ is a $\tilde{P}$ local martingale if and only if $\tilde{S}^i Z$ is a $P$ local martingale. Thus the measure $\tilde{P}$ can be determined by finding a process $\varphi$ in (2.1) such that $\tilde{S}^i Z, i = 1, 2, ..., d$ are $P$ local martingales. Simple calculations based on the Itô formula yield

$$\varphi_t = -Q^{-\frac{1}{2}} \left[ \frac{\alpha - r 1_d}{\sigma} \right] := -Q^{-\frac{1}{2}} \begin{bmatrix} \frac{\varphi_1 - r}{\sigma_1} \\ \frac{\varphi_2 - r}{\sigma_2} \\ \vdots \\ \frac{\varphi_d - r}{\sigma_d} \end{bmatrix}, \quad t \in [0, T].$$

The martingale measure $\tilde{P}$ is thus unique and given by the density process

$$\tilde{Z}_t := e^{-\left( Q^{-\frac{1}{2}} |\alpha - r 1_t|_{\tilde{W}_t} - \frac{1}{2} Q^{-\frac{1}{2}} |\alpha - r 1_t|^2 t \right)}, \quad t \in [0, T].$$

(2.2)

Moreover, it follows from Theorem 2.2 that the process

$$\tilde{W}_t := W_t + \frac{\alpha - r 1_d}{\sigma} t, \quad t \in [0, T],$$

is a $Q$-Wiener process under $\tilde{P}$. The dynamics of the prices under the measure $\tilde{P}$ can be written as

$$dS^i_t = S^i_t (rdt + \sigma_t d\tilde{W}_t^i), \quad i = 1, 2, ..., d.$$

The wealth process with the initial endowment $x$ and the trading strategy $\pi$ is defined by

$$X^{x,\pi}_t := \pi^0_t B_t + \sum_{i=1}^d \pi^i_t S^i_t, \quad t \in [0, T].$$

and assumed to satisfy $X^{x,\pi}_0 = x$. All the strategies are assumed to be admissible, i.e. $X^{x,\pi}_t \geq 0$ for each $t \in [0, T]$ almost surely and self-financing, i.e.

$$dX^{x,\pi}_t = \pi^0_t dB_t + \sum_{i=1}^d \pi^i_t dS^i_t, \quad t \in [0, T].$$

A contingent claim, representing future random payoff, is a random variable $H \geq 0$ measurable wrt. $F_T$. A hedging strategy against $H$ is a pair $(x, \pi)$ such that

$$P(X^{x,\pi}_T \geq H) = 1.$$

A replicating strategy is a pair $(x, \pi)$ such that

$$P(X^{x,\pi}_T = H) = 1.$$

A price of $H$ is defined by

$$p(H) := \inf \{ x : \exists \pi \text{ s.t. } P(X^{x,\pi}_T \geq H) = 1 \}$$
and, due to the fact that the market is complete, it follows from the general theory that 
\[ p(H) = \mathbb{E}[e^{-rT}H], \]
where the expectation is calculated under the measure \( \tilde{P} \).

If \( x < p(H) \) then \( P(X_T^{x,\pi} \geq H) < 1 \) for all \( \pi \) and the question under consideration is to find a strategy maximizing the probability of successful hedge, i.e.
\[ P(X_T^{x,\pi} \geq H) \rightarrow \max_{\pi}. \quad (2.3) \]

We will refer the corresponding function \( \Phi_1 : [0, +\infty) \rightarrow [0, 1] \) given by
\[ \Phi_1(x) := \max_{\pi} P(X_T^{x,\pi} \geq H), \]
as the maximal probability function. The strategy \( \hat{\pi} \) such that \( P(X_T^{x,\hat{\pi}} \geq H) = \Phi_1(x) \) will be called the probability maximizing strategy for \( x \).

We also consider the problem of cost reduction. Let \( \alpha \in [0, 1] \) be a fixed number describing the level of shortfall risk accepted by the trader. Then we are searching for a minimal initial cost such that there exists a strategy with the probability of successful hedge exceeding \( 1 - \alpha \), i.e.
\[ x \rightarrow \min; \exists \pi \text{ s.t. } P(X_T^{x,\pi} \geq H) \geq 1 - \alpha. \quad (2.4) \]

The cost reduction function \( \Phi_2 : [0, 1] \rightarrow [0, p(H)] \) is thus defined by
\[ \Phi_2(\alpha) := \min \{ x : \exists \pi \text{ s.t. } P(X_T^{x,\pi} \geq H) \geq 1 - \alpha \}. \]

The strategy \( \hat{\pi} \) such that \( P(X_T^{\Phi_2(\alpha),\hat{\pi}} \geq H) \geq 1 - \alpha \) will be called the cost minimizing strategy for \( \alpha \).

In the sequel we study the problem of determining the functions \( \Phi_1 \) and \( \Phi_2 \) for the contingent claim \( H \) of a general form. Then in Section 4 specific payoffs are examined.

3 Main results

In this section we present a general method of determining functions \( \Phi_1 \) and \( \Phi_2 \). Let us start with the auxiliary problems which can be solved via the Neyman-Pearson lemma.

Assume that we are given two probability measures \( P_1, P_2 \) with strictly positive density \( \frac{dP_1}{dP_2} \) and consider two types of optimizing problems
\[
\left\{ \begin{array}{l}
P_1[A] \rightarrow \max, \\
P_2[A] \leq x,
\end{array} \right. \quad (3.5)
\]
\[
\left\{ \begin{array}{l}
P_1[B] \geq 1 - \alpha \\
P_2[B] \rightarrow \min,
\end{array} \right. \quad (3.6)
\]
where \( \alpha, x \in [0, 1] \) are fixed constants. Problem (3.5) is a classical one appearing in the statistical hypotheses testing. Recall, that if there exists a constant \( c \geq 0 \) such that \( P_2(\frac{dP_1}{dP_2} \geq c) = x \) then the set
\[ \tilde{A} := \left\{ \frac{dP_1}{dP_2} \geq c \right\} \]
is a solution of (3.5). It is not surprising that the solution of the problem (3.6) is of a similar form. For the convenience of the reader we prove the following.
Proposition 3.1 If there exists a constant $c \geq 0$ satisfying $P_1(\frac{dP_2}{dP_1} \leq c) = 1 - \alpha$ then the set

$\tilde{B} := \left\{ \frac{dP_2}{dP_1} \leq c \right\}$

is a solution of the problem (3.6).

Proof: Let $B$ be an arbitrary set satisfying $P_1(B) \geq 1 - \alpha$. We will show that $P_2(B) \geq P_2(\tilde{B})$. The following estimation holds.

$$P_2(B) - P_2(\tilde{B}) = \int_{\Omega} (1_B - 1_{\tilde{B}})dP_2 = \int_{\{\frac{dP_2}{dP_1} \leq c\}} (1_B - 1_{\tilde{B}})dP_2 + \int_{\{\frac{dP_2}{dP_1} > c\}} (1_B - 1_{\tilde{B}})dP_2$$

$$\geq c \int_{\{\frac{dP_2}{dP_1} \leq c\}} (1_B - 1_{\tilde{B}})dP_1 + c \int_{\{\frac{dP_2}{dP_1} > c\}} 1_BdP_1$$

$$= c \left( \int_{\Omega} 1_BdP_1 - \int_{\Omega} 1_{\tilde{B}}dP_1 \right) = c(P_1(B) - P_1(\tilde{B}))$$

$$\geq c \left( P_1(B) - (1 - \alpha) \right) \geq 0.$$

Let us notice that both optimal sets $\tilde{A}$, $\tilde{B}$ have a similar form

$$\left\{ \frac{dP_1}{dP_2} \geq c \right\},$$

with suitable constants $c \geq 0$. More precisely, for $\tilde{A}$ the constant $c$ is s.t.

$$P_2 \left( \frac{dP_1}{dP_2} \geq c \right) = x$$

and for $\tilde{B}$ is s.t.

$$P_1 \left( \frac{dP_1}{dP_2} \geq c \right) = 1 - \alpha.$$

Now, come back to the initial problem of determining functions $\Phi_1$, $\Phi_2$. Let us start with presenting two auxiliary results which are nonrandomized versions of Theorems 2.34 and 2.42 in [6].

Theorem 3.2 Let $x \geq 0$. If $\tilde{A}$ is a set solving the problem

$$\left\{ \begin{array}{l} P[A] \rightarrow \text{max,} \\ \mathbb{E}[e^{-rT}H1_A] \leq x, \end{array} \right.$$  \hspace{1cm} (3.10)

then $\Phi_1(x) = P(\tilde{A})$ and the probability maximizing strategy for $x$ is that one replicating the payoff $H1_{\tilde{A}}$.

Let us notice that if $x \geq p(H)$ then $\tilde{A} = \Omega$ and thus $\Phi_1(x) = 1$. Moreover, if (3.10) has a solution for every $x \geq 0$, then the function $\Phi_1$ is increasing.
Theorem 3.3 Let $\alpha \in [0, 1]$ be a fixed number. If $\tilde{B}$ is a set solving the problem

$$
\begin{align*}
P[B] &\geq 1 - \alpha, \\
\tilde{E}[e^{-rT}H1_\tilde{B}] &\longrightarrow \min,
\end{align*}
$$

(3.11)

then $\Phi_2(\alpha) = \tilde{E}[e^{-rT}H1_\tilde{B}]$ and the cost minimizing strategy for $\alpha$ is that one replicating the payoff $H1_\tilde{B}$.

Notice that $\Phi_2(0) = p(H)$ and if (3.11) has a solution for each $\alpha \in [0, 1]$ then $\Phi_2$ is decreasing.

Now apply the method of solving the problems (3.5) and (3.6) to (3.10) and (3.11). Notice that (3.10) and (3.11) can be reformulated to the following form

$$
\begin{align*}
P[A] &\longrightarrow \max, \\
P^*(A) &\leq \frac{x}{\tilde{E}[e^{-rT}H]},
\end{align*}
$$

(3.12)

and

$$
\begin{align*}
P[B] &\geq 1 - \alpha \\
P^*(B) &\longrightarrow \min,
\end{align*}
$$

(3.13)

where $P^*$ is a probability measure given by the density

$$
\frac{dP^*}{dP} = \frac{H}{\tilde{E}[e^{-rT}H]}.
$$

In view of (3.7) we are searching for the solutions $\tilde{A}$, $\tilde{B}$ to (3.12), (3.13) in the family of sets

$$
\left\{ \frac{dP}{dP^*} \geq c \right\} = \left\{ \frac{dP}{dP} \frac{dP}{dP^*} \geq c \right\} = \left\{ \tilde{Z}_T^{-1} \geq \frac{c}{\tilde{E}[e^{-rT}H]}H \right\}; \quad c \geq 0,
$$

where $\tilde{Z}_T$ is given by (2.2). Denoting, for the sake of simplicity, the constant $c\tilde{E}[e^{-rT}H]$ by $c$ we see that the optimal sets $\tilde{A}$, $\tilde{B}$ are of the form

$$
A_c := \left\{ \tilde{Z}_T^{-1} \geq cH \right\},
$$

(3.14)

where, by (3.8) and (3.9), $c$ is s.t.

$$
P^*(A_c) = \frac{x}{\tilde{E}[e^{-rT}H]} \quad \text{for } \tilde{A},
$$

(3.15)

and

$$
P(A_c) = 1 - \alpha \quad \text{for } \tilde{B}.
$$

(3.16)

Now define two functions $\Psi_1 : [0, +\infty) \longrightarrow [0, 1]$, $\Psi_2 : [0, +\infty) \longrightarrow [0, p(H)]$ by

$$
\begin{align*}
\Psi_1(c) &:= P(A_c), \\
\Psi_2(c) &:= P^*(A_c) \cdot \tilde{E}[e^{-rT}H] = \tilde{E}[e^{-rT}H1_{A_c}].
\end{align*}
$$

(3.17)

(3.18)

Let us notice that both functions $\Psi_1, \Psi_2$ are decreasing and $\Psi_1(0) = 1$, $\Psi_2(0) = p(H)$. Thus $\Psi_2(0)$ provides the arbitrage free price of the continent claim $H$. Below we list some properties of functions $\Psi_1, \Psi_2$ needed in the sequel. First let us introduce two conditions concerning the real function $f : \mathbb{R}^d \longrightarrow [0, +\infty)$:
(C1) $\lambda_d(\{ z : f(z) = c \}) = 0$ for each $c > 0$,

(C2) $\lambda_d(\{ z : f(z) \in (a, b] \}) > 0$ for each $0 < a < b$.

Above $\lambda_d$ stands for the Lebesgue measure on $\mathbb{R}^d$.

**Proposition 3.4**

a) The function $\Psi_1$ is left continuous with right hand side limits in each point of the domain.

b) The following holds

$$\lim_{c \to +\infty} \Psi_1(c) = P(H = 0).$$

Assume that $Z_TH = f(W_T)$ where $f : \mathbb{R}^d \to [0, +\infty)$. Then $\Psi_1$ is

c) continuous if and only if (C1) is satisfied,

d) strictly decreasing if and only if (C2) is satisfied.

**Proof:**

a) The function $\Psi_1$ can be written in the form

$$\Psi_1(c) = P\left(\tilde{Z}_TH \leq \frac{1}{c}\right) = F_{\tilde{Z}_TH}\left(\frac{1}{c}\right), \quad c > 0,$$

where $F_{\tilde{Z}_TH}$ stands for the distribution function of the random variable $\tilde{Z}_TH$. Thus $\Psi_1$ has one sided limits for any $c > 0$ and the left continuity follows from the right continuity of $F_{\tilde{Z}_TH}$ for any $c > 0$. Left continuity at $c = 0$ follows from monotonicity.

b) The assertion follows from the formula

$$\Psi_1(c) = P(\tilde{Z}_T^{-1} \geq cH \mid H > 0)P(H > 0) + P(\tilde{Z}_T^{-1} \geq cH \mid H = 0)P(H = 0)$$

and the following

$$\lim_{c \to +\infty} P(\tilde{Z}_T^{-1} \geq cH \mid H > 0) = 0.$$

c) First show continuity at zero. If $c_n \downarrow 0$ then $\{ \tilde{Z}_T^{-1} \geq c_nH \}_n$ is an increasing family of sets and by the continuity of probability we have

$$\lim_{n \to +\infty} \Psi_1(c_n) = \lim_{n \to +\infty} P(\tilde{Z}_T^{-1} \geq c_nH) = P\left(\bigcup_n \{ \tilde{Z}_T^{-1} \geq c_nH \}\right) = P(\tilde{Z}_T^{-1} > 0) = 1 = \Psi_1(0).$$

Taking into account (3.19) we see that $\Psi_1$ is continuous for each $c > 0$ if and only if the random variable $\tilde{Z}_TH = f(W_T)$ has no positive atoms. In view of the equality

$$P(\tilde{Z}_TH = c) = P(f(W_T) = c) = \mathcal{L}_{W_T}\left(\{ z : f(z) = c \}\right), \quad c > 0,$$

and the fact that the distribution of $W_T$ is nondegenerate we see that the continuity of $\Psi_1$ is equivalent to (C1). $\mathcal{L}_{W_T}$ above stands for the distribution of $W_T$.

d) For $0 < c_1 < c_2$ we have

$$\Psi_1(c_1) - \Psi_1(c_2) = P\left(\tilde{Z}_TH \leq \frac{1}{c_1}\right) - P\left(\tilde{Z}_TH \leq \frac{1}{c_2}\right) = P\left( f(W_T) \in \left(\frac{1}{c_2}, \frac{1}{c_1}\right) \right)$$

$$= \mathcal{L}_{W_T}\left(\{ z : f(z) \in \left(\frac{1}{c_2}, \frac{1}{c_1}\right) \}\right),$$

and it follows from the nondegeneracy of the distribution of $W_T$ that the strict monotonicity of $\Psi_1$ is equivalent to (C2). □
Proposition 3.5  

a) The function $\Psi_2$ is left continuous with right hand side limits in each point of the domain.

b) The following holds

$$\lim_{c \to +\infty} \Psi_2(c) = 0.$$  

Assume that $Z_T H = f(W_T)$ where $f : \mathbb{R}^d \to [0, +\infty)$. Then $\Psi_2$ is

c) continuous if and only if (C1) is satisfied,

d) strictly decreasing if and only if (C2) is satisfied.

Proof: a) It follows from monotonicity that one sided limits exist. We show left continuity for any $c > 0$. For $c_n \uparrow c$ the family

$$\{\tilde{Z}_T^{-1} \geq c_n H\}_n$$

is decreasing and

$$\bigcap_n \{\tilde{Z}_T^{-1} \geq c_n H\} = \{H = 0\} \cup \{\tilde{Z}_T^{-1} \geq cH\} = \{\tilde{Z}_T^{-1} \geq cH\}.$$  

Thus by the dominated convergence we have

$$\lim_{n \to +\infty} \Psi_2(c_n) = \lim_{n \to +\infty} \mathbb{E}[e^{-rTH}_1(\tilde{Z}_T^{-1} \geq c_n H)] = \mathbb{E}[e^{-rTH}_1(\tilde{Z}_T^{-1} \geq cH)] = \Psi_2(c).$$

b) For $c_n \uparrow +\infty$ we have

$$\{\tilde{Z}_T^{-1} \geq c_n H\}_n \downarrow \bigcap_n \{\tilde{Z}_T^{-1} \geq c_n H\} = \{H = 0\} \cup \{\tilde{Z}_T^{-1} = +\infty\} = \{H = 0\},$$

and thus

$$\lim_{n \to +\infty} \Psi_2(c_n) = \lim_{n \to +\infty} \mathbb{E}[e^{-rTH}_1(\tilde{Z}_T^{-1} \geq c_n H)] = \mathbb{E}[e^{-rTH}_1(H = 0)] = 0.$$  

c) We show that the right continuity of $\Psi_2$ is equivalent to (C1). Then continuity follows from (a). For $c_n \downarrow c \geq 0$ we have

$$\{\tilde{Z}_T^{-1} \geq c_n H\} \uparrow \bigcup_n \{\tilde{Z}_T^{-1} \geq c_n H\} = \{H = 0\} \cup \{H > 0, \tilde{Z}_T^{-1} > cH\}$$

$$= \{\tilde{Z}_T^{-1} > cH\} = \{1 > cf(W_T)\},$$

and thus

$$\lim_{n \to +\infty} \Psi_2(c_n) = \mathbb{E}[e^{-rTH}_1(1 > cf(W_T))].$$

The condition $\lim_{n \to +\infty} \Psi_2(c) = \Psi_2(c)$ holds if and only if $\tilde{P}(1 \geq cf(W_T)) = \tilde{P}(1 > cf(W_T))$. The last condition holds for $c = 0$ and for $c > 0$ it is equivalent to (C1).

d) Fix $0 < c_1 < c_2$. The inequality

$$\Psi_2(c_1) - \Psi_2(c_2) = \mathbb{E}[e^{-rTH}_1(\frac{1}{c_1} < f(W_T) \leq \frac{1}{c_2})] > 0$$

holds if and only if $\tilde{P}(\frac{1}{c_1} < f(W_T) \leq \frac{1}{c_2}) > 0$. The last condition is equivalent to (C2). □
Now assume that \( \tilde{Z}_TH = f(W_T) \) for some \( f: \mathbb{R}^d \rightarrow [0, +\infty) \). Let us fix \( \alpha \in [0, 1] \), \( x > 0 \) and consider the problem of existence of solutions to the equation

\[
\Psi_1(c) = 1 - \alpha, \tag{3.20}
\]
as well as to

\[
\Psi_2(c) = x. \tag{3.21}
\]

In view of Propositions 3.4 and 3.5 it follows that if (C1) is satisfied then \( \Psi_1, \Psi_2 \) are continuous decreasing functions with images \( (P(H = 0), 1) \) and \( (0, p(H)) \) respectively. Thus for \( \alpha \in [0, P(H \neq 0)) \) and \( x \in (0, p(H)) \) the equations (3.20) and (3.21) do have solutions. Moreover, if (C2) is satisfied then the solutions are unique.

The description of functions \( \Phi_1 \) and \( \Phi_2 \) is provided by the following theorem, which is the main result of the paper.

**Theorem 3.6** Assume that \( \tilde{Z}_TH = f(W_T) \) for some \( f: \mathbb{R}^d \rightarrow [0, +\infty) \) satisfying (C1).

a) Let \( c = c(x) \in [0, +\infty) \) be a solution of the equation

\[
\Psi_2(c) = x, \quad x \in (0, p(H)). \tag{3.22}
\]

Then the maximal probability function is given by

\[
\Phi_1(x) = \begin{cases} 
P(H = 0) & \text{for } x = 0, \\
\Psi_1(c(x)) & \text{for } x \in (0, p(H)), \\
1 & \text{for } x \geq p(H)
\end{cases}
\]

Moreover, for any \( x \in (0, p(H)) \) the probability maximizing strategy for \( x \) is that one replicating the payoff \( H1_{A_c(x)} \).

b) Let \( c = c(\alpha) \in [0, +\infty) \) be a solution of the equation

\[
\Psi_1(c) = 1 - \alpha, \quad \alpha \in [0, P(H \neq 0)). \tag{3.23}
\]

Then the cost reduction function is given by

\[
\Phi_2(\alpha) = \begin{cases} 
\Psi_2(c(\alpha)) & \text{for } \alpha \in [0, P(H \neq 0)), \\
0 & \text{for } \alpha \in [P(H \neq 0), 1].
\end{cases}
\]

Moreover, for any \( \alpha \in [0, P(H \neq 0)) \) the cost reduction strategy for \( \alpha \) is that one replicating the payoff \( H1_{A_c(\alpha)} \).

**Proof:** The proof is based on the consideration proceeding the formulation of the Theorem.

a) If \( x \geq p(H) \) then the hedging strategy is the probability maximizing strategy and then clearly \( \Phi_1(x) = 1 \). Consider the case \( x \in (0, p(H)) \). By Theorem 3.2 we know that \( \Phi_1(x) = P(\hat{A}) \), where \( \hat{A} \) is a solution of (3.10). The solution of (3.12), which is equivalent to (3.10), is of the form (3.14) with \( c \) satisfying (3.15). But (3.15) is equivalent to (3.22). Thus we have

\[
\Phi_1(x) = P(A_c) = \Psi_1(c),
\]
where $c$ is given by the condition $\Psi_2(c) = x$. For $x = 0$ consider the trivial strategy $\pi = 0$. Then $P(X_T^{x,\pi} \geq H) = P(H = 0)$. On the other hand, due to the monotonicity of $\Phi_1$, we have $\Phi_1(0) \leq \lim_{x \to 0} \Phi_1(x) = \lim_{x \to 0} \Psi_1(c(x)) = \lim_{z \uparrow +\infty} \Psi_1(z) = P(H = 0)$. As a consequence we obtain $\Phi_1(0) = P(H = 0)$. The second part of the assertion follows from Theorem 3.2.

b) If $\alpha \in [P(H \neq 0), 1]$ then consider a trivial strategy $\pi = 0$ with zero initial endowment $x = 0$. Then $X_T^{x,\pi} = 0$ and thus $P(X_T^{x,\pi} \geq H) = P(H = 0) \geq 1 - \alpha$. As a consequence we have $\Phi_2(\alpha) = 0$. Now consider the case $\alpha \in [0, P(H \neq 0)]$. It follows from Theorem 3.3 that $\Phi_2(\alpha) = \tilde{E}[e^{-rT}H1_{B}]$, where $\tilde{B}$ is a solution to (3.11). The optimal solution of (3.11) is the same as for (3.13) and has the form (3.14) with $c$ satisfying (3.16). The condition (3.10) can be written as $\Psi_1(c) = 1 - \alpha$. Thus we have

$$\Phi_2(\alpha) = \tilde{E}[e^{-rT}H1_{A_c}] = \Psi_2(c).$$

The second part of the assertion follows from Theorem 3.3.

In virtue of Theorem 3.6 the only problem to determine functions $\Phi_1$, $\Phi_2$ is to find functions $\Psi_1$, $\Psi_2$ and to solve the equations (3.22), (3.23). In general, due to the fact that $\Psi_1$, $\Psi_2$ are rather of a sophisticated form, one should not expect to find analytic formulas for the constants in (3.22), (3.23). However, the equations (3.22), (3.23) can be solved with the use of numerical methods. In the sequel we solve the problem of determining functions $\Psi_1$, $\Psi_2$ for the most common multiple assets derivatives.

4 Quantile hedging in two dimensional model

In this section we determine explicit formulas for the functions $\Psi_1$, $\Psi_2$ for a few examples of popular options. Since our derivatives depend on two underlying assets we simplify at the beginning general formulas from Section 3 to the case $d = 2$. In the calculations we base on properties of the multidimensional normal distribution which are recalled in the sequel.

For the case $d = 2$ we denote the correlation matrix by

$$Q = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$ 

Consequently, we have

$$Q^{-1} = \frac{1}{\rho^2 - 1} \begin{bmatrix} -1 & \rho \\ \rho & -1 \end{bmatrix}, \quad Q^{-\frac{1}{2}} = \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \end{bmatrix}.$$ 

Hence the density of the martingale measure (2.2) can be written as

$$\tilde{Z}_T = e^{-A_1W_1^x - A_2W_2^{y} - BT}, \quad (4.24)$$
where
\[ A_1 := \frac{1}{\rho^2 - 1} \left( -\frac{\alpha_1 - r}{\sigma_1} + \frac{\rho \alpha_2 - r}{\sigma_2} \right) \]
\[ A_2 := \frac{1}{\rho^2 - 1} \left( \frac{\rho \alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right) \]
\[ B := \frac{1}{8} \left( \left( \frac{1}{\sqrt{1 + \rho}} + \frac{1}{\sqrt{1 - \rho}} \right) \frac{\alpha_1 - r}{\sigma_1} + \left( \frac{1}{\sqrt{1 + \rho}} - \frac{1}{\sqrt{1 - \rho}} \right) \frac{\alpha_2 - r}{\sigma_2} \right)^2 \]
\[ + \left( \left( \frac{1}{\sqrt{1 + \rho}} - \frac{1}{\sqrt{1 - \rho}} \right) \frac{\alpha_1 - r}{\sigma_1} + \left( \frac{1}{\sqrt{1 + \rho}} + \frac{1}{\sqrt{1 - \rho}} \right) \frac{\alpha_2 - r}{\sigma_2} \right)^2 \].

The formula (3.14) for the set \( A_c \) simplifies to the form
\[ A_c = \{ \tilde{Z}^{-1}_T \geq cH \} = \{ e^{A_1 W_1^1 + A_2 W_2^2 + BT} \geq cH \}, \]
and consequently formulas (3.17), (3.18) become
\[ \Psi_1(c) = P(e^{A_1 W_1^1 + A_2 W_2^2 + BT} \geq cH), \]
\[ \Psi_2(c) = \tilde{E}[e^{-rT} I_{\{ e^{A_1 W_1^1 + A_2 W_2^2 + BT} \geq cH \}}]. \]

Now set the notation concerning the multidimensional normal distribution and recall its basic properties, which can be found in standard textbooks on probability theory or statistics, see for instance [8]. A random vector \( X \) taking values in \( \mathbb{R}^d \) has a multidimensional normal distribution if its density is of the form
\[ f_X(x) = \frac{1}{(2\pi)^{d/2} \det(\Sigma)^{1/2}} \cdot e^{-\frac{1}{2}(x-m)^T \Sigma^{-1} (x-m)}, \quad x \in \mathbb{R}^d, \quad (4.25) \]
where \( m \in \mathbb{R}^d \) is a mean of \( X \) and \( \Sigma \) is a symmetric positive definite \( d \times d \) covariance matrix of \( X \). The fact that \( X \) has a density \( (4.25) \) will be denoted by \( X \sim N_d(m, \Sigma) \) or \( \mathcal{L}(X) = N_d(m, \Sigma) \). If \( d = 1 \) then the subscript is omitted and \( N(m, \sigma) \) denotes the normal distribution with mean \( m \) and variance \( \sigma \). If \( X \sim N_d(m, \Sigma) \) and \( A \) is a \( k \times d \) matrix then,
\[ AX \sim N_k(AM, A\Sigma A^T); \quad (4.26) \]
in particular if \( a \in \mathbb{R}^d \) then
\[ a^T X \sim N(a^T m, a^T \Sigma a). \quad (4.27) \]

Let \( X \) be a random vector taking values in \( \mathbb{R}^d \) and fix an integer \( 0 < k < d \). Let us divide \( X \) into two vectors \( X^{(1)} \) and \( X^{(2)} \) with lengths \( k, d - k \) respectively, i.e.
\[ X^{(1)} = (X_1, X_2, \ldots, X_k)^T, \quad X^{(2)} = (X_{k+1}, X_{k+2}, \ldots, X_d)^T. \]

Analogously, divide the mean vector \( m \) and the covariance matrix \( \Sigma \)
\[ m = \begin{pmatrix} m^{(1)} \\ m^{(2)} \end{pmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma^{(11)} & \Sigma^{(12)} \\ \Sigma^{(21)} & \Sigma^{(22)} \end{bmatrix}, \]
where
\[ \Sigma^{(11)} = \begin{pmatrix} \Sigma_{11} \end{pmatrix}; \quad \Sigma^{(22)} = \begin{pmatrix} \Sigma_{22} \end{pmatrix}. \]
so that $EX^{(1)} = m^{(1)}$, $EX^{(2)} = m^{(2)}$, $CovX^{(1)} = \Sigma^{(11)}$, $CovX^{(2)} = \Sigma^{(22)}$, $Cov(X^{(1)}, X^{(2)}) = \Sigma^{(12)} = \Sigma^{(21)}^T$. Denote by $L \left( X^{(1)} \mid X^{(2)} = x^{(2)} \right)$ the conditional distribution of $X^{(1)}$ given $X^{(2)} = x^{(2)} \in \mathbb{R}^{d-k}$. If $\Sigma^{(22)}$ is nonsingular then

$$L \left( X^{(1)} \mid X^{(2)} = x^{(2)} \right) = N_k(m^{(1)}(x^{(2)}), \Sigma^{(11)}(x^{(2)})), \quad (4.28)$$

where

$$m^{(1)}(x^{(2)}) = m^{(1)} + \Sigma^{(12)}\Sigma^{(22)}^{-1}(x^{(2)} - m^{(2)}),$$

$$\Sigma^{(11)}(x^{(2)}) = \Sigma^{(11)} - \Sigma^{(12)}\Sigma^{(22)}^{-1}\Sigma^{(21)}. \quad (4.29)$$

Actually the conditional variance $\Sigma^{(11)}(x^{(2)})$ does not depend on $x^{(2)}$ but we keep the notation for the sake of consistency. The conditional density will be denoted by $f_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$, where $x^{(1)} \in \mathbb{R}^k$. In particular if $(X, Y)$ is a two dimensional normal vector with parameters

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

then

$$L(X \mid Y = y) = N(m_1(y), \sigma_1(y)), \quad (4.30)$$

If $X$ is a random vector then its distribution wrt. the measure $\tilde{P}$ will be denoted by $\tilde{L}(X)$ and its density by $\tilde{f}_X$. Analogously, $f_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$ stands for the conditional density with respect to the measure $\tilde{P}$.

In the following subsections we will use the universal constants: $A_1, A_2, B$ defined in (4.21) as well as $a_1, a_2, b, \tilde{a}_1, \tilde{a}_2, \tilde{b}$ introduced below.

Fix a number $K > 0$. One can check the following

$$\{ S^1_T \geq K \} = \{ W^1_T \geq a_1 \} = \left\{ \tilde{W}^1_T \geq \tilde{a}_1 \right\}, \quad (4.31)$$

$$\{ S^2_T \geq K \} = \{ W^2_T \geq a_2 \} = \left\{ \tilde{W}^2_T \geq \tilde{a}_2 \right\}, \quad (4.32)$$

$$\{ S^1_T \geq S^2_T \} = \{ \sigma_1 W^1_T - \sigma_2 W^2_T \geq b \} = \left\{ \sigma_1 \tilde{W}^1_T - \sigma_2 \tilde{W}^2_T \geq \tilde{b} \right\}, \quad (4.33)$$

where

$$a_1 := \frac{1}{\sigma_1} \left( \ln \frac{K}{S^0_0} - (\alpha_1 - \frac{1}{2} \sigma_1^2)T \right), \quad \tilde{a}_1 := \frac{1}{\sigma_1} \left( \ln \frac{K}{S^0_0} - (r - \frac{1}{2} \tilde{\sigma}_1^2)T \right),$$

$$a_2 := \frac{1}{\sigma_2} \left( \ln \frac{K}{S^0_0} - (\alpha_2 - \frac{1}{2} \sigma_2^2)T \right), \quad \tilde{a}_2 := \frac{1}{\sigma_2} \left( \ln \frac{K}{S^0_0} - (r - \frac{1}{2} \tilde{\sigma}_2^2)T \right)$$

$$b := \ln \left( \frac{S^2_0}{S^0_0} \right) + (\alpha_2 - \alpha_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2))T, \quad \tilde{b} := \ln \left( \frac{S^2_0}{S^0_0} \right) - \frac{1}{2}(\tilde{\sigma}_2^2 - \tilde{\sigma}_1^2)T.$$
4.1 Digital option

In this section we determine Ψ₁, Ψ₂ for the payoff

\[ H = K \cdot 1_{(S^1_T \geq S^2_T)}, \quad \text{where} \quad K > 0. \]  (4.34)

By (4.33) we have

\[ \Psi_1(c) = P(A_c) = P \left( e^{A_1W^1_T + A_2W^2_T + BT} \geq cK1_{(S^1_T \geq S^2_T)} \right) \]

\[ = P \left( A_1W^1_T + A_2W^2_T + BT \geq \ln(cK), S^1_T \geq S^2_T \right) + P(e^{A_1W^1_T + A_2W^2_T + BT} \geq 0, S^1_T < S^2_T) \]

\[ = P \left( A_1W^1_T + A_2W^2_T + BT \geq \ln(cK) \mid \sigma_1W^1_T - \sigma_2W^2_T \geq b \right) P(\sigma_1W^1_T - \sigma_2W^2_T \geq b) \]

\[ + P(\sigma_1W^1_T - \sigma_2W^2_T < b). \]  (4.35)

Let us notice that

\[ X := \left[ \begin{array}{c} A_1W^1_T + A_2W^2_T \\ \sigma_1W^1_T - \sigma_2W^2_T \end{array} \right] = \left[ \begin{array}{cc} A_1 & A_2 \\ \sigma_1 & -\sigma_2 \end{array} \right] \left[ \begin{array}{c} W^1_T \\ W^2_T \end{array} \right], \]

so in view of (4.20) we have \( X \sim N(0, \Sigma), \) where

\[ \Sigma = \left[ \begin{array}{cc} (A_1 + A_2r)TA_1 + (A_1r + A_2)TA_2 & (\sigma_1 - \sigma_2r)TA_1 + (\sigma_1r - \sigma_2)TA_2 \\ (\sigma_1 - \sigma_2r)TA_1 + (\sigma_1r - \sigma_2)TA_2 & (\sigma_1 - \sigma_2r)TA_1 - (\sigma_1r - \sigma_2)TA_2 \end{array} \right]. \]

In virtue of (4.30) we have

\[ \mathcal{L}(A_1W^1_T + A_2W^2_T \mid \sigma_1W^1_T - \sigma_2W^2_T = y) = N(m(y), \sigma(y)), \]

where

\[ m(y) = y \frac{(\sigma_1 - \sigma_2r)A_1 + (\sigma_1r - \sigma_2)A_2}{(\sigma_1 - \sigma_2r)\sigma_1 - (\sigma_1r - \sigma_2)\sigma_2}, \quad \sigma(y) = \frac{T(A_1\sigma_2 + A_2\sigma_1)^2(\rho^2 - 1)}{-\sigma_1^2 + 2\rho\sigma_1\sigma_2 - \sigma_2^2}. \]

By (4.27) we have: \( \sigma_1W^1_T - \sigma_2W^2_T \sim N(0, T(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)). \) Going back to (4.35) we have

\[ \Psi_1(c) = \int_b^{+\infty} P(A_1W^1_T + A_2W^2_T \geq \ln(cK) - BT \mid \sigma_1W^1_T - \sigma_2W^2_T = y) f_{\sigma_1W^1_T - \sigma_2W^2_T}(y) dy \]

\[ + P(\sigma_1W^1_T - \sigma_2W^2_T < b) \]

\[ = \int_b^{+\infty} \Phi \left( \frac{m(y) + BT - \ln(cK)}{\sqrt{\sigma(y)}} \right) f_{\sigma_1W^1_T - \sigma_2W^2_T}(y) dy + \Phi \left( \frac{b}{\sqrt{T(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}} \right). \]

Now let us determine Ψ₂. In virtue of (4.33) we have

\[ \Psi_2(c) = e^{-rT}E[H1_{A_c}] = e^{-rT}E[K1_{(S^1_T \geq S^2_T)} \cdot 1_{(Z^1_T \geq cK1_{(S^1_T \geq S^2_T)})}] = e^{-rT}K \tilde{P} \left( S^1_T \geq S^2_T, \tilde{Z}^{-1}_T \geq cK1_{(S^1_T \geq S^2_T)} \right) \]

\[ = e^{-rT}K \tilde{P} \left( \tilde{Z}^{-1}_T \geq cK \mid S^1_T \geq S^2_T \right) \tilde{P}(S^1_T \geq S^2_T) \]

\[ = e^{-rT}K \tilde{P}(e^{A_1W^1_T + A_2W^2_T + BT} > cK \mid \sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T \geq \tilde{b}) \tilde{P}(\sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T \geq \tilde{b}) \]

\[ = e^{-rT}K \int_b^{+\infty} \tilde{P}(e^{A_1W^1_T + A_2W^2_T + BT} > cK \mid \sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T = \tilde{y}) f_{\sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T}(y) dy \]

\[ = e^{-rT}K \cdot \int_b^{+\infty} \tilde{P}(A_1\tilde{W}^1_T + A_2\tilde{W}^2_T > \ln(cK) + A_1\frac{\alpha_1 - r}{\sigma_1}T + A_2\frac{\alpha_2 - r}{\sigma_2}T + BT \mid \sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T = \tilde{y}) \]

\[ \cdot f_{\sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T}(y) dy \]

\[ = e^{-rT}K \int_b^{+\infty} \Phi \left( \frac{(m(y) - \ln(cK) - A_1\frac{\alpha_1 - r}{\sigma_1}T - A_2\frac{\alpha_2 - r}{\sigma_2}T + BT)}{\sqrt{\sigma(y)}} \right) f_{\sigma_1\tilde{W}^1_T - \sigma_2\tilde{W}^2_T}(y) dy. \]
4.2 Quantos

4.2.1 Quanto domestic

The contingent claim is of the form

$$H = S_T^2(S_T^1 - K)^+, \quad K > 0.$$  \hspace{1cm} (4.36)

At the beginning let us notice that

$$\left\{ e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K) \right\} = \left\{ (A_2 - \sigma_2) W_T^2 \geq v(c, W_T^1) \right\} = \left\{ (A_2 - \sigma_2) W_T^2 \geq w(c, W_T^1) \right\},$$

where

$$v(c, x) := \ln \left( c S_0^2 e^{(\alpha_2 - \frac{1}{2} \sigma_2^2)T - A_1 x} (S_0^1 e^{(\alpha_1 - \frac{1}{2} \sigma_1^2)T + \sigma_1 x} - K) \right),$$

$$w(c, x) := \ln \left[ c S_0^2 e^{(r - \frac{1}{2} \sigma_2^2 - B + A_1 \frac{\alpha_1 - r}{\sigma_1} + A_2 \frac{\alpha_2 - r}{\sigma_2})T - A_1 x} (S_0^1 e^{(r - \frac{1}{2} \sigma_1^2)T + \sigma_1 x} - K) \right].$$

By (4.31) and (4.37) we have

$$\Psi_1(c) = P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K)^+ \right)$$

$$= P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K) \mid S_T^1 \geq K \right) P(S_T^1 \geq K)$$

$$+ P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0 \mid S_T^1 < K \right) P(S_T^1 < K)$$

$$= P \left( (A_2 - \sigma_2) W_T^2 \geq \ln \left( c S_0^2 e^{(\alpha_2 - \frac{1}{2} \sigma_2^2)T - A_1 x} (S_0^1 e^{(\alpha_1 - \frac{1}{2} \sigma_1^2)T + \sigma_1 x} - K) \right) \mid W_T^1 \geq a_1 \right) P(W_T^1 \geq a_1) + P(W_T^1 < a_1)$$

$$= \int_{a_1}^{+\infty} P((A_2 - \sigma_2) W_T^2 \geq v(c, W_T^1) \mid W_T^1 = x) f_{W_T^1}(x) dx + \Phi \left( \frac{a_1}{\sqrt{T}} \right).$$

The conditional distribution is given by

$$\mathcal{L}((A_2 - \sigma_2) W_T^2 \mid W_T^1 = x) \sim N(m(x), \sigma(x)),$$

where $m(x), \sigma(x)$ are given by (4.29). Hence we have

$$\Psi_1(c) = \int_{a_1}^{+\infty} \Phi \left( \frac{m(x) - v(c, x)}{\sqrt{\sigma(x)}} \right) f_{W_T^1}(x) dx + \Phi \left( \frac{a_1}{\sqrt{T}} \right).$$

To avoid technicalities assume that $A_2 \neq \sigma_2$. We have

$$\Psi_2(c) = e^{-rT} \mathbb{E} \left[ S_T^2(S_T^1 - K)^+ \mathbf{1}_{\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K)^+\}} \right]$$

$$= e^{-rT} \mathbb{E} \left[ S_T^2(S_T^1 - K)^+ \mathbf{1}_{\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K)^+\}} \mid S_T^1 \leq K \right] \tilde{P}(S_T^1 \leq K)$$

$$+ e^{-rT} \mathbb{E} \left[ S_T^2(S_T^1 - K)^+ \mathbf{1}_{\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K)^+\}} \mid S_T^1 > K \right] \tilde{P}(S_T^1 > K)$$

$$= e^{-rT} \mathbb{E} \left[ S_T^2(S_T^1 - K) \mathbf{1}_{\{e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c S_T^2(S_T^1 - K)\}} \mid S_T^1 > K \right] \tilde{P}(S_T^1 > K).$$
By (4.31) and (4.37) we have

\[ \Psi_2(c) = e^{-rT} \mathbb{E} \left[ S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 \tilde{W}_T^2} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 \tilde{W}_T^1} - K) \mathbb{1}_{\{s \geq u(c, \tilde{W}_T^1)\}} \tilde{P}(\tilde{W}_T^1 > \tilde{a}_1) \right] = e^{-rT} \int_{\tilde{a}_1}^{+\infty} \mathbb{E} \left[ S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 \tilde{W}_T^2} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 \tilde{W}_T^1} - K) \mathbb{1}_{\{s \geq u(c, \tilde{W}_T^1)\}} \right] \tilde{f}_{\tilde{W}_T^1}(x) dx \]

By (4.31) and (4.37) we have

\[ \Psi_2(c) = e^{-rT} \mathbb{E} \left[ S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 \tilde{W}_T^2} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 \tilde{W}_T^1} - K) \mathbb{1}_{\{s \geq u(c, \tilde{W}_T^1)\}} \tilde{P}(\tilde{W}_T^1 > \tilde{a}_1) \right] = e^{-rT} \int_{\tilde{a}_1}^{+\infty} \mathbb{E} \left[ S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 \tilde{W}_T^2} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 \tilde{W}_T^1} - K) \mathbb{1}_{\{s \geq u(c, \tilde{W}_T^1)\}} \right] \tilde{f}_{\tilde{W}_T^1}(x) dx \]

It follows from (4.30) that \( \hat{L}((A_2-\sigma_2)\tilde{W}_T^2 \mid \tilde{W}_T^1 = x) = N((A_2-\sigma_2)x, (1-\rho^2)(A_2-\sigma_2)^2) \) and hence

\[ \Psi_2(c) = e^{-rT} S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T} \frac{1}{\sqrt{2\pi T(1-\rho^2)(A_2-\sigma_2)^2}} \int_{\tilde{a}_1}^{+\infty} \mathbb{E} \left[ S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 \tilde{W}_T^2} \right] \tilde{f}_{\tilde{W}_T^1}(x) dx \]

4.2.2 Quanto foreign

The payoff is of the form

\[ H = \left( S_T^1 - \frac{K}{S_T^2} \right)^+, \quad K > 0. \]

First let us notice that

\[ \left\{ S_T^1 - \frac{K}{S_T^2} \geq 0 \right\} = \left\{ \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d \right\} = \left\{ \sigma_1 \tilde{W}_T^1 + \sigma_2 \tilde{W}_T^2 \geq \tilde{d} \right\}, \quad (4.38) \]

where

\[ d := \ln \frac{K}{S_0^1 S_0^2} - (\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)) T, \quad \tilde{d} := \ln \frac{K}{S_0^1 S_0^2} - \left( 2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \right) T, \]

and

\[ \left\{ e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c \left( S_T^1 - \frac{K}{S_T^2} \right) \right\} = \left\{ A_1 W_T^1 + (A_2 + \sigma_2) W_T^2 \geq v(c, \sigma_1 W_T^1 + \sigma_2 W_T^2) \right\} = \left\{ A_1 \tilde{W}_T^1 + (A_2 + \sigma_2) \tilde{W}_T^2 \geq w(c, \sigma_1 \tilde{W}_T^1 + \sigma_2 \tilde{W}_T^2) \right\}, \quad (4.39) \]

where

\[ v(c, z) := \ln \left( \frac{c}{S_0^1 S_0^2 e^{(\frac{1}{2}\sigma_2^2-\frac{1}{2}(\sigma_1^2+\sigma_2^2)T+z-K)}} \right), \]

\[ w(c, z) := \ln \left( \frac{c}{S_0^1 S_0^2 e^{(2r-\frac{1}{2}(\sigma_1^2+\sigma_2^2))T+z-K}} \right). \]
By \(4.38\) we have
\[
\Psi_1(c) = P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c \left( S_T^1 - \frac{K}{S_T^2} \right)^+ \right)
= P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c \left( S_T^1 - \frac{K}{S_T^2} \right) \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d \right) P(\sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d)
+ P \left( e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq 0 \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 < d \right) P(\sigma_1 W_T^1 + \sigma_2 W_T^2 < d)
= P \left( S_T^2 e^{A_1 W_T^1 + A_2 W_T^2 + BT} \geq c \left( S_T^1 - K \right) \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d \right) P(\sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d)
+ P(\sigma_1 W_T^1 + \sigma_2 W_T^2 < d).
\]

As a consequence of \(4.39\) we obtain
\[
\Psi_1(c) = P(\sigma_1 W_T^1 + (A_2 + \sigma_2) W_T^2 \geq v(c, \sigma_1 W_T^1 + \sigma_2 W_T^2) \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d) P(\sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d)
+ P(\sigma_1 W_T^1 + \sigma_2 W_T^2 < d)
= \int_{d}^{+\infty} P(\sigma_1 W_T^1 + (A_2 + \sigma_2) W_T^2 \geq v(c, z) \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 = z) f_{\sigma_1 W_T^1 + \sigma_2 W_T^2}(z) dz
+ P(\sigma_1 W_T^1 + \sigma_2 W_T^2 < d).
\]

By \(4.30\) we have
\[
L(\sigma_1 W_T^1 + (A_2 + \sigma_2) W_T^2 \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 = z) = N(m(z), \sigma(z)),
\]
where
\[
m(z) := \frac{(A_1 + (A_2 + \sigma_2) \rho) \sigma_1 + (A_1 \rho + A_2 + \sigma_2) \sigma_2}{\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2},
\]
\[
\sigma(z) := T \left\{ \frac{(A_1 + (A_2 + \sigma_2) \rho) A_1 + (A_1 \rho + (A_2 + \sigma_2)) (A_2 + \sigma_2)}{\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2} \right\} \left\{ \frac{(A_1 + (A_2 + \sigma_2) \rho) \sigma_1 + (A_1 \rho + (A_2 + \sigma_2) \sigma_2)^2}{\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2} \right\},
\]
and thus
\[
\Psi_1(c) = \int_{d}^{+\infty} \Phi \left( \frac{m(z) - v(c, z)}{\sqrt{\sigma(z)}} \right) f_{\sigma_1 W_T^1 + \sigma_2 W_T^2}(z) dz + \Phi \left( \frac{d}{\sqrt{T(\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2)}} \right).
\]
By (4.38) and (4.39) we have

$$
\Psi_2(c) = e^{-rT} \mathbb{E} \left[ \left( S^1_T - \frac{K}{S^2_T} \right)^{+} 1_{\{ e^{A_1W^1_T + A_2W^2_T + BT} \geq e(S^1_T - \frac{K}{S^2_T}) \}} \right]
$$

By (4.38) and (4.39) we have

$$
= e^{-rT} \mathbb{E} \left[ \left( S^1_T - \frac{K}{S^2_T} \right)^{+} 1_{\{ e^{A_1W^1_T + A_2W^2_T + BT} \geq e(S^1_T - \frac{K}{S^2_T}) \}} \right] \bar{P}(\sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T \geq \bar{d})
$$

$$
= e^{-rT} \mathbb{E} \left[ \left( S^1_T - \frac{K}{S^2_T} \right)^{+} 1_{\{ A_1\tilde{W}^1_T + A_2\tilde{W}^2_T \geq w(c,\sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T) \}} \right] \bar{P}(\sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T \geq \bar{d})
$$

Using (4.28) we find the conditional distributions

$$
\mathcal{L}(\tilde{W}^1_T, A_1\tilde{W}^1_T + A_2\tilde{W}^2_T | c, \sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T = z) = N_2(M^1(z), \Sigma^1(z)),
$$

$$
\mathcal{L}(\tilde{W}^2_T, A_1\tilde{W}^1_T + A_2\tilde{W}^2_T | c, \sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T = z) = N_2(M^2(z), \Sigma^2(z)),
$$

where $M^1(z), M^2(z), \Sigma^1(z), \Sigma^2(z)$ are determined by (4.29). As a consequence we obtain

$$
\Psi_2(c) = e^{-rT} S^1_0 e^{-\frac{1}{2}\sigma^2 T} \int_{d}^{+\infty} \int_{-\infty}^{+\infty} e^{\sigma_1 x} F^1(x, y) dy \int_{w(c,z)}^{+\infty} e^{\sigma_2 x} F^2(x, y) dy \int_{\sigma_1\tilde{W}^1_T + \sigma_2\tilde{W}^2_T(z)}^{+\infty} dz
$$

where $F^1, F^2$ stand for the density functions of the two dimensional normal distributions $N_2(M^1(z), \Sigma^1(z)), N_2(M^2(z), \Sigma^2(z))$ respectively.

### 4.3 Outperformance option

The problem is studied for

$$
H = \left( \max \{ S^1_T, S^2_T \} - K \right)^{+}, \quad K > 0.
$$

Let us notice that

$$
\left\{ e^{A_1W^1_T + A_2W^2_T + BT} \geq e(S^1_T - K) \right\} = \left\{ A_2 W^2_T \geq w_1(c, \tilde{W}^1_T) \right\} = \left\{ A_1 \tilde{W}^1_T \geq w_1(c, \tilde{W}^1_T) \right\} \quad (4.40)
$$

$$
\left\{ e^{A_1W^1_T + A_2W^2_T + BT} \geq e(S^2_T - K) \right\} = \left\{ A_1 W^1_T \geq w_2(c, \tilde{W}^2_T) \right\} = \left\{ A_1 \tilde{W}^1_T \geq w_2(c, \tilde{W}^2_T) \right\} \quad (4.41)
$$

where

$$
w_1(c, x) := \ln \left[ c S^1_0 e^{(\alpha_1 - \frac{1}{2}\sigma^2 T) + \sigma_1 x} \right] - A_1 x - BT, \quad w_2(c, y) := \ln \left[ c S^2_0 e^{(\alpha_2 - \frac{1}{2}\sigma^2 T) + \sigma_2 y} \right] - A_2 y - BT,
$$

$$
w_1(c, x) := \ln \left( c e^{-A_1 x + (A_1 \tilde{W}^1_T - BT)} (S^1_0 e^{(r - \frac{1}{2}\sigma^2 T) + \sigma_1 x} - K) \right) + \frac{\alpha_1 - r}{\sigma_1} T,
$$

$$
w_2(c, y) := \ln \left( c e^{-A_2 y + (A_2 \tilde{W}^2_T - BT)} (S^2_0 e^{(r - \frac{1}{2}\sigma^2 T) + \sigma_2 y} - K) \right) + \frac{\alpha_2 - r}{\sigma_2} T.
$$
By (4.31), (4.32), (4.33) we have
\[ \Psi_1(c) = P \left( e^{A_1 W_1^2 + A_2 W_2^2 + S^2 T} \geq c(S_T^1 \vee S_T^2 - K) \right) \]
\[ = P \left( e^{A_1 W_1^2 + A_2 W_2^2 + S^2 T} \geq c(S_T^1 - K) \mid W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P \left( e^{A_1 W_1^2 + A_2 W_2^2 + S^2 T} \geq 0 \mid W_T^1 < a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^1 < a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P \left( e^{A_1 W_1^2 + A_2 W_2^2 + S^2 T} \geq c(S_T^2 - K) \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P \left( e^{A_1 W_1^2 + A_2 W_2^2 + S^2 T} \geq 0 \mid W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \].

By (4.40), (4.41) we have
\[ \Psi_1(c) = P \left( A_2 W_T^2 \geq v_2(c, W_T^2) \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P(W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P \left( A_1 W_T^1 \geq v_2(c, W_T^2) \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right) P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \]
\[ + P(W_T^2 < a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \],

Let \( m_1(y, z), m_2(x, z), \sigma_1(y, z), \sigma_2(x, z) \) be the means and variances of the conditional distributions
\[ \mathcal{L}(A_1 W_T^1 \mid W_T^2 = y, \sigma_1 W_T^1 - \sigma_2 W_T^2 = z) = N(m_1(y, z), \sigma_1(y, z)), \]
\[ \mathcal{L}(A_2 W_T^2 \mid W_T^1 = x, \sigma_1 W_T^1 - \sigma_2 W_T^2 = z) = N(m_2(x, z), \sigma_2(x, z)), \]
given by (1.29). Then we have
\[ \Psi_1(c) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi \left( \frac{m_2(x, z) - v_1(c, x)}{\sqrt{\sigma_2(x, z)}} \right) f_{W_T^2, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x, z) dz dx \]
\[ + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{W_T^2, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x, z) dz dx \]
\[ + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi \left( \frac{m_1(y, z) - v_2(c, y)}{\sqrt{\sigma_1(y, z)}} \right) f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(y, z) dy dz \]
\[ + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(y, z) dy dz. \]
Let \( m_1(y,z), \sigma_1(y,z) \) and \( m_2(x,z), \sigma_2(x,z) \) denote means and variances of the conditional distributions

\[
\mathcal{L} \left( A_1 \tilde{W}_T^1 \mid \tilde{W}_T^2, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \right) = N \left( m_1(y,z), \sigma_1(y,z) \right),
\]

\[
\mathcal{L} \left( A_2 \tilde{W}_T^2 \mid \tilde{W}_T^1, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \right) = N \left( m_2(x,z), \sigma_2(x,z) \right).
\]

Finally we obtain

\[
\Psi(c) = e^{-rT} \int_{a_1}^{+\infty} \int_{b}^{+\infty} \left( S_0^1 e^{(r - \frac{1}{2} \sigma_1^2)T + \sigma_1 x} - K \right) \Phi \left( \frac{m_2(x,z) - w_1(c,x)}{\sqrt{\sigma_2(x,z)}} \right) \tilde{f}_{\tilde{W}_T^1, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2}(x,z)dzdx
\]

\[
+ e^{-rT} \int_{a_2}^{+\infty} \int_{-\infty}^{b} \left( S_0^2 e^{(r - \frac{1}{2} \sigma_2^2)T + \sigma_2 y} - K \right) \Phi \left( \frac{m_1(y,z) - w_2(c,y)}{\sqrt{\sigma_1(y,z)}} \right) \tilde{f}_{\tilde{W}_T^2, \sigma_1 \tilde{W}_T^2 - \sigma_2 \tilde{W}_T^1}(y,z)dzdy.
\]

### 4.4 Spread option

The payoff is of the form

\[ H = (S_T^1 - S_T^2 - K)^+, \quad K > 0. \]

For any \( y \in \mathbb{R} \)

\[ \{ S_T^1 - S_0^2 e^{(\alpha_2 - \frac{1}{2} \sigma_2^2)T + \sigma_2 y} - K \geq 0 \} = \{ W_T^1 \geq e(y) \}, \quad (4.42) \]

where

\[ e(y) := \frac{1}{\sigma_1} \left( \ln \left[ \frac{1}{S_0^2} \left( S_0^2 e^{(\alpha_2 - \frac{1}{2} \sigma_2^2)T + \sigma_2 y} \right) \right] - \left( \alpha_1 - \frac{1}{2} \sigma_1^2 \right) T \right). \]

By conditioning wrt. \( W_T^2 \) we obtain

\[ \Psi(c) = P \left( e^{A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 + BT} \geq c(S_T^1 - S_T^2 - K)^+ \right) \]

\[
= \int_{-\infty}^{+\infty} P \left( e^{A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 + BT} \geq c(S_T^1 - S_T^2 - K)^+ \mid W_T^2 = y \right) f_{W_T^2}(y)dy
\]

\[
= \int_{-\infty}^{+\infty} P \left( e^{A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 + BT} \geq c(S_T^1 - S_T^2 - K), W_T^1 \geq e(y) \mid W_T^2 = y \right) f_{W_T^2}(y)dy
\]

\[
+ \int_{-\infty}^{+\infty} P \left( e^{A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 + BT} \geq 0, W_T^1 < e(y) \mid W_T^2 = y \right) f_{W_T^2}(y)dy
\]

\[
= \int_{-\infty}^{+\infty} P \left( e^{A_1 \tilde{W}_T^1 + BT} e^{A_2 \tilde{W}_T^1 - cS_0^1 e^{(\alpha_1 - \frac{1}{2} \sigma_1^2)T} e^{\sigma_1 \tilde{W}_T^1 - c(S_T^1 + K)}, W_T^1 \geq e(y) \mid W_T^2 = y \right) f_{W_T^2}(y)dy
\]

\[
+ \int_{-\infty}^{+\infty} P \left( W_T^1 < e(y) \mid W_T^2 = y \right) f_{W_T^2}(y)dy.
\]

Let

\[ \mathcal{L}(W_T^1 \mid W_T^2 = y) = N(m(y), \sigma(y)). \]
Then we have

\[
\Psi_1(c) = \int_{-\infty}^{+\infty} \int_{S(c,y) \cap (e(y),+\infty)} f_{W_1}(w_2=y)\,dx\,f_{W_2}(y)\,dy + \int_{-\infty}^{+\infty} \Phi\left(\frac{c(y) - m(y)}{\sqrt{\sigma(y)}}\right) f_{W_2}(y)\,dy,
\]

where \(S(c,y)\) is a set defined \(y \in \mathbb{R}\) by

\[S(c,y) := \left\{ x : e^{A_2 y + BT} e^{A_1 x} - cS_1^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T} e^{\sigma_1 x} \geq -c(S_0^2 e^{(a_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K) \right\}.\]

For the practical applications it is necessary to find a closed form of the set \(S(c,y)\). In the formulation of the next result we will use the solutions of the equation

\[g(x) = -c(S_0^2 e^{(a_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K),\] (4.43)

where \(g(x) := e^{A_2 y + BT} e^{A_1 x} - cS_0^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T} e^{\sigma_1 x}\). These solutions can be found numerically.

**Proposition 4.1** The set \(S(c,y)\) is of the form

a) if \(A_1 > \sigma_1\) and

(i) if \(g(\hat{x}) \geq -c(S_0^2 e^{(a_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K)\) then \(S(c,y) = (-\infty, +\infty)\),

(ii) if \(g(\hat{x}) \geq -c(S_0^2 e^{(a_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K)\) then \(S(c,y) = (-\infty, x_1) \cup (x_2, +\infty)\), where \(x_1 < x_2\) are the unique solutions of (4.43).

Above, \(\hat{x}\) stands for \(\frac{1}{\sigma_1 - A_1} \ln\left(\frac{A_1 e^{A_2 y + BT}}{\sigma_1 S_0^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T}}\right)\).

b) if \(A_1 = \sigma_1\) and

(i) \(e^{A_2 y + BT} \geq cS_0^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T}\) then \(S(c,y) = (-\infty, +\infty)\),

(ii) \(e^{A_2 y + BT} < cS_0^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T}\) then \(S(c,y) = (-\infty, x_0)\), where \(x_0\) is a unique solution of (4.43).

c) if \(A_1 < \sigma_1\) then \(S(c,y) = (-\infty, x_0)\), where \(x_0\) is a unique solution of (4.43).

**Proof:**

a) One can check that \(g\) has a minimum at the point \(\hat{x}\) and is decreasing on \((-\infty, \hat{x})\) and increasing on \((\hat{x}, +\infty)\). Hence (i) and (ii) follow.

b) The formulas for \(S(c,y)\) follows from the simplified form of the function \(g(x) = (e^{A_2 y + BT} - cS_0^1 e^{(a_1 - \frac{1}{2}\sigma_1^2)T}) e^{A_1 x}\).

c) It can be checked that \(g\) is strictly increasing on the set \(\{ x : g(x) < 0 \}\) and that \(\lim_{x \to +\infty} g(x) = -\infty\). Thus (4.43) has a unique solution and the form of the set \(S(c,y)\) follows.

Now let us determine \(\Psi_2\). One can check that for \(y \in \mathbb{R}\)

\[\{ S_1^1 - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \geq 0 \} = \{ W_1^1 \geq f(y) \},\] (4.44)

where

\[f(y) := \frac{1}{\sigma_1} \left( \ln \left[ \frac{1}{S_0^1} (S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y}) \right] - \left( r - \frac{1}{2} \sigma_1^2 \right) T \right) .\]

For \(y \in \mathbb{R}\) define

\[
\tilde{S}(c, y) := \left\{ x : e^{A_1 (x - \frac{a_2 - \frac{1}{2}\sigma_2^2}{\sigma_1}) T} + A_2 (y - \frac{a_2 - \frac{1}{2}\sigma_2^2}{\sigma_1}) T + BT \geq c \left( e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \right) \right\} .
\]

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We have
\[
\Psi_2(c) = e^{-rT} E \left[ (S_T^1 - S_T^2 - K)^+ 1_{\{e^{A_1w_1^1 + A_2w_2^1 + \sigma^1 (+, y) (S_T^1 - S_T^2 - K)^+}\}} \right]
\]
\[
= e^{-rT} \int_{-\infty}^{\infty} E \left[ (S_T^1 - S_T^2 - K)^+ 1_{\{e^{A_1w_1^1 + A_2w_2^1 + \sigma^1 (+, y) (S_T^1 - S_T^2 - K)^+}\}} | \tilde{W}_T^2 = y \right] f_{\tilde{W}_T^2}(y) dy
\]
\[
= e^{-rT} \int_{-\infty}^{\infty} E \left[ (S_T^1 - S_T^2 - K)^+ 1_{\{e^{A_1w_1^1 + A_2w_2^1 + \sigma^1 (+, y) (S_T^1 - S_T^2 - K)^+}\}} | \tilde{W}_T^2 = y \right] f_{\tilde{W}_T^2}(y) dy
\]
\[
+ e^{-rT} \int_{-\infty}^{\infty} E \left[ (S_T^1 - S_T^2 - K)^+ 1_{\{e^{A_1w_1^1 + A_2w_2^1 + \sigma^1 (+, y) (S_T^1 - S_T^2 - K)^+}\}} | \tilde{W}_T^2 = y \right] f_{\tilde{W}_T^2}(y) dy
\]
\[
= e^{-rT} \int_{-\infty}^{\infty} \int_{\tilde{S}(c,y) \cap (f(y), +\infty)} \left( S_0^1 e^{(r-\frac{1}{2} \sigma^2)T + \sigma^1 (+, y) x} - S_0^2 e^{(r-\frac{1}{2} \sigma^2)T + \sigma^2 y} - K \right) f_{\tilde{W}_T^2}(x) dx f_{\tilde{W}_T^2}(y) dy.
\]

The explicit form of the set \( \tilde{S}(c, y) \) can be established in the same way as for \( S(c, y) \) in Proposition 4.1.

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