Economic optimization of acceptance interval in conformity assessment: 2. Process with unknown systematic effect

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Abstract
In this paper, optimization of acceptance interval in conformity assessment using the expression of the revenue that we showed in part 1 of this study is proposed under the condition of a systematic effect component being present in the measurement uncertainty. Systematic effect components are caused by unknown biases. Using the distribution based on the uncertainty information of the bias, we can develop the distribution of the revenue. Our idea is to set a percentile of the revenue distribution as the target of the maximization to reflect the systematic effect component in the optimization of the acceptance interval. We provide an equation for the optimum acceptance interval that can be numerically solved with little burden. We found that the choice of the probability for the percentile is an essential task in the optimization. Our proposed method incorporating the systematic effect seems to be sufficiently practical to be applied to actual processes.

Keywords: conformity assessment, uncertainty, global consumer’s risk, acceptance interval, guard band

Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

Conformity assessment is implemented to determine the quality of products and services in various fields. In production industries, inspection is an essential process in conformity assessment. It should be noted that an inspection usually has measurement uncertainty, which can lead to an incorrect decision being made in the conformity assessment. Since measurement uncertainty cannot be completely eliminated, methods are needed to control the risks caused by it. In our previous report [1], we showed a method to determine the optimum acceptance interval focusing on the economic perspective. In the present study, we extend this method to enhance its practical applicability.

In this study, a quality characteristic is defined as a true physical quantity that we wish to check by means of the inspection, and the measured value of the quality characteristic in the inspection process is referred to as the measurement value. An item is defined as conforming when its quality characteristic is within a range called the tolerance interval. Since we cannot directly know the value of a quality characteristic in an actual inspection, we must make decisions based on the
measurement values. An item is accepted and made available in the market when the measured value is within a range called the acceptance interval. On the contrary, an item is rejected and discarded without being made available in the market when the value is outside the acceptance interval.

Because of the measurement uncertainty, it can happen that a non-conforming item is accepted (false acceptance) or a conforming item is rejected (false rejection). The term of risk is used in this study to mean the probability of an undesirable event’s occurring, and we wish to reduce the risks of false acceptance and rejection in quality control. Some previous studies have reported on the quantification of such risks [2–7] and analysis of the costs incurred by them [8–10]. Pendrill reported several reviews [11–13] on this issue, and showed some practical applications focusing on optimization of the measurement uncertainty [14–17] by himself. JCGM 106 [18] is a guideline for applying measurement uncertainty in conformity assessment in which the results of these studies have been incorporated.

The scope of our study is mass production processes, for which JCGM 106 gives the definition of the global consumer’s risk (gCR). In our previous report [1], we redefined the gCR, gave a price model with it and a production cost model, and provided a method to optimize the magnitude of the acceptance interval. However, the discussion there was limited to cases with no systematic effect component of the measurement uncertainty. In the present study we consider the systematic effect component of the measurement uncertainty, which is caused by an error that cannot be reduced through repetitive measurements, because it is sometimes not negligible in realistic processes. One of the systematic effects that it is difficult to remove in actual practice is that caused through the calibration process, which is essential to establish metrological traceability.

In principle, any known systematic effects are compensated before the uncertainty evaluation by GUM (JCGM 100) [19]. Since we assume the compensation in this study, the systematic effect component of the measurement uncertainty is only influenced by unknown systematic effects. In GUM, the unknown systematic effect component of the measurement uncertainty is not explicitly discriminated from the random one in assessing the uncertainty associated with a specific measurement value. However, the influence of the systematic effect component is essentially different from that of the random effect component when considering not a specific measurement but an inspection in a process. It should be noted that all inspections are influenced by only one error caused by a systematic effect component, and the influence cannot be averaged over a large number of inspections.

Forbes [10] reported a study focusing on the systematic effect in inspection processes. His study showed how the measurement uncertainty affects decision making in conformity assessment considering the costs of false acceptance and rejection. In his study, the effects of the random and systematic effects were separately considered. It was concluded that while the expectation of the total cost depends mainly on the combined uncertainty of the two components, the variance of the total cost depends on the balance between them. It was noted that ‘the presence of systematic effects could mean that the actual cost could be significantly greater than that expected from a system with only random effects’.

It may be emphasized that because of the systematic effect, we cannot know the revenue precisely. In our previous report [1], we assessed the revenue based on the distributions of the item’s quality characteristics and measurement values. Since these distributions are affected by the unknown bias, we cannot evaluate the gCR precisely so we can never compute the revenue. Optimization of the acceptance interval should therefore be implemented based on a suggested approach that has not been considered so far in the discipline of metrology. The approach involves decision making based on a stochastic phenomenon. Such decision making is conducted not only in production industries but also in various fields including those where financial engineering is applied. Our idea in this study is to apply an index used in financial engineering to conformity assessment.

Computation is important for the practical application of this approach. We show a simple but precise computation method to optimize the acceptance interval. The optimized acceptance interval is consequently of a sufficiently realistic magnitude to be employed in actual processes.

This paper is organized as follows: section 2 presents the mathematical expressions of the gCR, price, and revenue with an unknown bias. Section 3 explains the main concept of this study to optimize the acceptance interval. Section 4 provides the computational procedures to realize the optimization. In section 5, the characterization of the proposed method is shown through simulations. Section 6 gives a summary of this study. Online appendices are also provided, mainly for the derivation of some of the mathematical expressions.

2. Expression of the global consumer’s risk, price, and revenue with an unknown bias

2.1. Global consumer’s risk with an unknown bias

We consider cases where every manufactured item is checked by means of the inspection process. It is assumed that rejected items are never measured twice or more but simply discarded. The general expression of the gCR in this study independent of the distribution is given in our previous report [1]. In the present report, the normal distributions are used for the quality characteristics and the measurement values. The details of the distribution and expression of the gCR are described in this subsection.

The target value for the quality characteristics of the manufactured items is assumed to be zero. We can readily expand the conclusions in this study to general cases by understanding the quality characteristic of an item as the difference between the quantity intended to be measured on the item and the target value.

In this study, we handle only cases where the manufacturing process is well investigated before the running to adjust the population mean of the measured values on the quality characteristic to be zero. The investigation is implemented by the
measurement system to be used in the manufacturing process. Even though any known biases are corrected, the adjustment cannot be perfect because of the unknown bias for the measurement system. The unknown bias in the measurement value is defined as \( \delta \), \( \xi \) and \( x \) denote the quality characteristic of an item and the measured value of it, respectively. Thus, we assume that:

(a) Any known biases are compensated for,
(b) The population mean of the measured value \( x \) given \( \xi \) is always smaller by \( \delta \) than its true value, that is, \( \xi \), meaning that the population mean of \( x \) given \( \xi \) and \( \delta \) is \( \xi - \delta \).
(c) The adjustment is conducted so that the population mean of the unconditional distribution of \( x \) with respect to \( \xi \) is zero, then.
(d) The population mean of \( \xi - \delta \) is consequently given to be zero when \( \xi \) is regarded as a random variable, meaning that the population mean of \( \xi \) is \( \delta \).

Furthermore, the standard uncertainty of \( \delta \) is given as \( u_\delta \). The standard deviation of the quality characteristics of items is given as \( \sigma \), and the standard uncertainty of the random effect component in the measurement is given as \( u_r \).

Since the population mean of \( \xi \) is given as \( \delta \), the distribution of \( \xi \) is given as follows:
\[
\xi|\delta \sim \mathcal{N}(\delta, \sigma^2). \tag{1}
\]
The distribution of \( \xi \) thus depends on \( \delta \). As mentioned, the population mean of the measured value \( x \) given \( \xi \) and \( \delta \) is not \( \xi \) but \( \xi - \delta \). The distribution of \( x \) is given as
\[
x|\xi, \delta \sim \mathcal{N}(\xi - \delta, u_r^2). \tag{2}
\]

Figure 1 shows a schematic explanation of false acceptance based on the models given by expressions (1) and (2).

To obtain the unconditional distribution of \( x \) with respect to \( \xi \), we integrate \( \xi \) over expression (2) using expression (1), as follows:
\[
x|\delta \sim \mathcal{N}(0, \sigma^2 + u_r^2). \tag{3}
\]
where
\[
\sigma_r = \sqrt{\sigma^2 + u_r^2}. \tag{4}
\]
The distribution of \( x \) is thus independent of \( \delta \) after integrating over \( \xi \), because of the adjustment explained above.

We define \( A \) as the acceptable difference with which the item is accepted when \(-A \leq x \leq A \) and rejected otherwise. The acceptance interval as described in section 1 denotes the range of \{\( x \) | \(-A \leq x \leq A \)\}. Furthermore, we define \( T \) as the permissible difference for the tolerance with which the item is conforming when \(-T \leq \xi \leq T \) and non-conforming otherwise. The tolerance interval denotes the range of \{\( \xi \) | \(-T \leq \xi \leq T \)\}. (Note that in JCGM 106 [18], the symbols \( T \) and \( A \) are differently employed and indicate the magnitudes of the tolerance and the acceptance intervals, respectively.)

We define the rejection rate \( R \) as the fraction of items discarded without being made available in the market among all of the manufactured items. \( R \) can be expressed mathematically as
\[
R = \Phi\left(-\frac{A}{\sigma_r}\right) + \left[1 - \Phi\left(\frac{A}{\sigma_r}\right)\right] = 2 \times \Phi\left(-\frac{A}{\sigma_r}\right), \tag{5}
\]
where \( \Phi(y) \) denotes the cumulative probability function of the standardized normal distribution for variable \( y \). The gCR in this study is given as follows:
\[
\theta = \frac{H(-\infty, -T) + H(T, +\infty)}{1 - R}, \tag{6}
\]
where
\[
H(a, b) = \int_a^b \left[\phi\left(\frac{A - (\xi - \delta)}{u_r}\right) - \Phi\left(-\frac{A - (\xi - \delta)}{u_r}\right)\right] \times \frac{1}{\sigma} \phi\left(\frac{\xi - \delta}{\sigma}\right) \, d\xi, \tag{7}
\]
and \( \phi(y) \) denotes the probability density function of the standardized normal distribution for variable \( y \). The mathematical form of \( H(a, b) \) in expression (7) is different from that in our previous report [1] because of the existence of the bias \( \delta \).

The gCR in expression (6) shows the quality based on the market-available items. An important point to be noted is that the gCR \( \theta \) is given as a function of \( \delta \). We consider the distribution of \( \delta \) in section 3.

2.2. Model for the price

In accordance with our previous report [1], the price of an item is given as a function of the gCR \( \theta \) as follows:
\[
P = P_0(1 - \beta \theta). \tag{8}
\]
The right side of expression (8) has the two parameters, \( P_0 \) and \( \beta \). Since \( P = P_0 \) when \( \beta = 0 \), \( P_0 \) means the price of a conforming item. Similarly, \( P_0(1 - \beta) \) is the price of a non-conforming item, where we can consider negative price as the expression of an economic loss. The difference between these
two prices \( P_0, \beta \) can hence be interpreted as the loss caused by a non-conforming item, and \( \beta \) is the ratio of the loss to the price of a conforming item. We refer to \( \beta \) as the ‘degree of loss’ in this study. The determination of \( \beta \) is an important task, for which some information is given in our previous report.

A reference condition is assumed to exist in which positive revenue is given, and \( P_{\text{ref}} \) is defined as the price of an item produced under that condition. It should be noted that not only the acceptable difference under the reference condition, \( A_{\text{ref}} \), but also the bias on it, \( \delta_{\text{ref}} \), is required to determine \( P_{\text{ref}} \). While we cannot know the magnitude of the bias in an actual process, \( \delta_{\text{ref}} \) can be virtually given because the reference condition does not need to be an actual condition but a condition under which we evaluate \( P_{\text{ref}} \).

Once we determine \( P_{\text{ref}} \) for \( A = A_{\text{ref}} \) and \( \delta = \delta_{\text{ref}} \), the gCR under the reference condition \( \theta_{\text{ref}} \) is computed. \( P_0 \) can be determined as follows:

\[
P_0 = \frac{P_{\text{ref}}}{1 - \beta \theta_{\text{ref}}}. \tag{9}
\]

### 2.3. Model of the revenue

In this subsection, the model for the revenue is given. Since the content is the same as that in our previous report [1], the details are omitted here. Please refer to our previous report if more information is required.

In this study, we discriminate between manufacturing and production considering the steps of inspection, discarding, and selling in a process. The production cost consists of the manufacturing, discarding, and selling costs, \( c_1, c_D, \) and \( c_S \) denote the variable costs for the inspection of an item, the discarding of a rejected item, and the selling of an accepted item, respectively. The non-manufacturing variable cost of an accepted item, \( \eta \), is given as follows:

\[
\eta = \frac{1}{1 - R} \left[ c_1 + c_D R + c_S (1 - R) \right]. \tag{10}
\]

It is assumed that we know the acceptable difference \( A_{\text{ref}} \) and the bias \( \delta_{\text{ref}} \) to specify the reference condition under which we can determine the price for a market-available item \( P_{\text{ref}} \). \( R_{\text{ref}} \) and \( \eta_{\text{ref}} \) denote the rejection rate and the non-manufacturing cost for an accepted item under the reference condition. Since the revenue for the reference condition is assumed to be positive, \( \eta_{\text{ref}} \) must be less than \( P_{\text{ref}} \). Furthermore, the elasticity of supply under the reference condition \( E_{\text{ref}} \) is also given as a positive value.

The manufacturing cost is assumed to be linear to the manufactured amount, which includes the amount of rejected and discarded items, with the elasticity of supply under the reference condition \( E_{\text{ref}} \). However, to prevent the marginal manufacturing cost from being negative, it is given as zero for a small manufactured amount, as explained in our previous report. It is assumed that the producer adjusts the output amount to maximize the revenue in response to the price of an item in the market.

Based on our previous report, we introduce the standardized revenue \( m \) which shows the difference of the revenue from that under the reference condition divided by \( g_{\text{ref}} W_{\text{ref}} \), where \( g_{\text{ref}} \) is the marginal manufacturing cost under the reference condition and is given as

\[
g_{\text{ref}} = (1 - R_{\text{ref}}) (P_{\text{ref}} - \eta_{\text{ref}}), \tag{11}
\]

and \( W_{\text{ref}} \) is the manufactured amount. Specifically, the standardized revenue \( m \) is given as follows:

\[
m = \begin{cases} 
\frac{1}{2} \chi_{\text{ref}} & \text{for } \chi_{\text{ref}} \geq 1 \text{ and } K \leq 1 - \frac{1}{\chi_{\text{ref}}}, \\
\chi_{\text{ref}} - 1 & \text{for } \chi_{\text{ref}} < 1 \text{ and } K \leq 0
\end{cases} \tag{12}
\]

\[
(K - 1) + \frac{\chi_{\text{ref}}}{2} (K - 1)^2 \quad \text{Otherwise,} \tag{14}
\]

where

\[
K = \frac{(1 - R) (P - \eta)}{(1 - R_{\text{ref}}) (P_{\text{ref}} - \eta_{\text{ref}})}, \tag{15}
\]

\[
\chi_{\text{ref}} = E_{\text{ref}} \left( 1 - \frac{\eta_{\text{ref}}}{P_{\text{ref}}} \right). \tag{16}
\]

\( \chi_{\text{ref}} \) is positive, since \( E_{\text{ref}} > 0 \) and \( P_{\text{ref}} > \eta_{\text{ref}} \).

We believe that the larger \( m \) is, the better the process control is. \( A_{\text{opt}} \) denotes the optimized acceptable difference. In our previous report, \( A_{\text{opt}} \) is evaluated under the condition that the bias is fixed as \( \delta = 0 \). It is assumed that \( \delta \) is, however, not zero but an unknown value with the standard uncertainty \( \delta_u \) in the present study. The approach in our previous report [1] cannot therefore be simply applied.

### 3. Optimization of the acceptance interval with the concept of the value at risk

We cannot regard \( \delta \) as a specific value in practice, but as a random variable. The standard uncertainty of \( \delta \) is given as \( \delta_u \), and the following distribution for \( \delta \) is considered:

\[
\delta \sim N(0, \delta_u^2). \tag{17}
\]

Since the standardized revenue \( m \) given in subsection 2.3 is a function of \( \delta \), \( m \) can also be regarded as a random variable. We consider the distribution of \( m \) for the case where

\[
T = 2.00, \quad \sigma = 1.0, \quad \delta_u = 0.25, \quad \delta_u = 0.1, \quad A_{\text{ref}} = 2.00, \quad \delta_{\text{ref}} = 0, \quad P_{\text{ref}} = 1000, \tag{18}
\]

\[
c_1 = 10, \quad c_D = 100, \quad c_S = 200.
\]

Although the units are omitted for simplicity, an identical physical unit is associated with \( T, \sigma, \delta_u, A_{\text{opt}}, \) and \( \delta_{\text{ref}} \), and an identical financial unit with \( P_{\text{ref}}, c_1, c_D, \) and \( c_S \). We reported the optimization of the acceptance interval in section 6 of our previous report [1] using the same parameters as those in
expression (18) for the case in which $\delta = 0$, and $m$ is maximized to be 0.93 with the optimum acceptable difference of $A = 1.59$.

Figure 2(a) shows the probability density function of the standardized revenue $m$, $p(m)$, with $A = 1.59$ obtained by the Monte Carlo method. ($p(m)$ is computed by dividing the fraction of the counts in the bins by the width of the bins.) $m$ seems to have a finite maximum value, and $p(m)$ is at a maximum around the maximum value of $m$. The distribution is strongly asymmetric and has a tail only on the side with the decreasing value of $m$. Some summary statistics of this distribution are as follows:

$$m_{[0.001]} = 0.14, \quad m_{[0.01]} = 0.50, \quad m_{[0.05]} = 0.69, \quad \overline{m} = 0.86, \quad m_{[0.5]} = 0.87, \quad \max[m] = 0.93,$$

where $m_{[\alpha]}$ means the $100\alpha$ percentile of the distribution of $m$. $\overline{m}$ and $\max[m]$ are the mean and the maximum value of $m$, respectively. $\max[m] = 0.93$ is given, and it is the value when $\delta = 0$. We show that the maximum value of $m$ is always given when $\delta = 0$ in subsection 4.1.

One idea is to take one of the summary statistics of the distribution of $m$ as the index to be maximized or minimized to optimize the acceptance interval. While taking $\overline{m}$ as the index is possible, Forbes [10] pointed out the problem in doing so. Forbes applied a type of cost expression which is different from our suggestion in this study, and proposed the determination of the acceptance interval by minimizing the total cost. In his study, it was suggested that the variance of the costs caused by the systematic effect in the measurement uncertainty is quite large. The results implied that it is not practical to handle the systematic and random effects in the same way. Although the cost in his study and the revenue in the present report are not identical or simply reciprocal, we can find the same tendency.

In financial engineering, rather than a central value such as the mean or the median, a value that appears only with quite a small probability is often used in the control of financial damages. This value is called the value at risk (VaR) [20], and the small probability is often set at 0.1% to 5%. In the VaR concept, the value with the small probability is regarded as the possible minimum value in a practical sense. The VaR may not be the percentile of $m$ with a small probability but a negative value of it, since the focus of concern in the VaR concept is usually not the revenue but the loss. This way of thinking can, however, be applied to the discussion in the present report.

In figure 2(b), the probability density function of $m$ for $A = 1.39$ with the parameters in expression (18) is shown. The summary statistics are assessed as

$$m_{[0.001]} = 0.58, \quad m_{[0.01]} = 0.71, \quad m_{[0.05]} = 0.77, \quad \overline{m} = 0.81, \quad m_{[0.5]} = 0.81, \quad \max[m] = 0.82.$$  

(20)

The difference between $m_{[0.01]}$ and $\overline{m}$, which is 0.36 for $A = 1.59$, is 0.10. The distribution is found to be sharper than that with $A = 1.59$. While $\max[m]$, $m_{[0.05]}$, and $\overline{m}$ with $A = 1.59$ are larger than those with $A = 1.39$, $m_{[0.05]}$, $m_{[0.01]}$, and $m_{[0.001]}$ show the opposite trend. It can be said that by taking $A = 1.39$ instead of $A = 1.59$, the possible maximum revenue is reduced but the fifth or less percentile of the revenue distribution is increased.

It is not an objective but a subjective task to determine which of the possible maximum revenue, the average revenue, the possible minimum revenue, or another factor should serve as the index to be optimized. However, we suggest that it is risky to neglect an event with a realistic possibility. It should be noted that the neglect of systematic small probability events can cause a loss in the revenue over a long period. Maximizing the possible minimum revenue, or another factor should serve as the index to be optimized. However, we suggest that it is risky to neglect an event with a realistic possibility. It should be noted that the neglect of systematic small probability events can cause a loss in the revenue over a long period. Maximizing the possible minimum revenue, or another factor should serve

In this study, we thus propose the determination of the optimum acceptable difference $A_{\text{opt}}$ by solving the problem of

$$A_{\text{opt}} = \arg \max_A m_{[\alpha]}$$

(21)

Determining $A_{\text{opt}}$ in accordance with expression (21) does not mean that the revenue from the process is $m_{[\alpha]}$, but that it is greater than $m_{[\alpha]}$ with a probability of $1 - \alpha$. We refer to $\alpha$ as the ‘level of risk’ in this study.

Figure 3 shows the variation of the maximum and average values and the percentiles of $m$. When $\alpha = 1\%$, $m_{[0.01]}$ is maximized when $A = 1.39$ so $A_{\text{opt}} = 1.39$. When $\alpha = 0.1\%$, $A_{\text{opt}} = 1.32$. We hereafter basically use $\alpha = 1\%$ in this study.
The acceptance interval is optimized to maximize the bias.

4. Computation of the optimum acceptance interval

4.1. Relation between the revenue and the magnitude of the bias

The acceptance interval is optimized to maximize $m_{[\alpha]}$. In this subsection, it is explained that $m_{[\alpha]}$ is the value of $m$ for $\delta = \delta_{[\alpha/2]}$ or $\delta = \delta_{[1-\alpha/2]}$, where $\delta_{[\alpha]}$ is the 100\% percentile of the distribution of $\delta$. Figure 4 shows the variation of $m$ with the parameters given in expression (18) and $A = 1.59$ and 1.39. This graph implies the following:

(a) $m$ is symmetric around $\delta = 0$, and.
(b) $m$ monotonically decreases when $\delta > 0$.

Online appendices A.3 and A.4 show that propositions (a) and (b) above are true, respectively. Therefore, we can readily understand that the larger $|\delta|$ is, the smaller $m$ is. It should be noted that we assume the normal distribution given in expression (17) for $\delta$. Since the probability that $|\delta|$ is greater than $|\delta_{[\alpha/2]}| = |\delta_{[1-\alpha/2]}|$ is $\alpha$, $m_{[\alpha]}$ is the value of $m$ with $\delta = \delta_{[\alpha/2]}$ or $\delta_{[1-\alpha/2]}$. This fact greatly reduces the computational burden of applying our proposal, because we can compute $m_{[\alpha]}$ with $\delta = \delta_{[1-\alpha/2]}$ without investigating the distribution of $m$.

4.2. Simple numerical determination of the optimized acceptance interval

Here we offer the equation in a simple form to obtain $A_{\text{opt}}$. We assume the following two conditions for the discussion in this subsection:

(a) Expression (14) is applied to at least a part of $A \in \{A|0 \leq A \leq +\infty\}$ with $\delta = \delta_{[1-\alpha/2]}$, and.
(b) $\delta_{[1-\alpha/2]} < T$.

Regarding point (a), it can happen that not expression (14) but expression (12) or (13) is applied throughout the whole range of $0 \leq A \leq +\infty$. In such cases, no items are manufactured in the process for any $A$, and it is impossible to determine $A_{\text{opt}}$ for those cases. The fact that point (a) holds implies there are conditions where manufacturing some items is better in terms of the revenue than no manufacturing. When point(a) is not confirmed in advance, we can understand that $A_{\text{opt}}$ given in this subsection is not the optimum acceptable difference but the sole candidate for it. When expression (12) or (13) is applied to compute $m_{[\alpha]}$ for the obtained value of $A_{\text{opt}}$, we cannot implement the optimization.

We next make remarks regarding point (b). $m_{[\alpha]}$ is a function of the single parameter $A$ with given $T, \sigma, u_r, A_{\text{ref}}, \delta_{\text{ref}}, P_{\text{ref}}, c_l, c_p, c_s$, and $\delta = \delta_{[1-\alpha/2]}$. If we take $K$ as a function of the single parameter $A$ by fixing $\delta = \delta_{[1-\alpha/2]}$, the derivative of $m_{[\alpha]}$ with respect to $A$ is

$$\frac{\partial m_{[\alpha]}}{\partial A} = \frac{dK}{dA}[1 + \chi_{\text{ref}}(K - 1)].$$

(22)

Online appendix A.4 in our previous report [1] shows $[1 + \chi_{\text{ref}}(K - 1)] > 0$ when expression (14) is applied. Thus, for
always holds, as shown in online appendix B.1. Thus, only detailed conditions are described in the manuscript.

Expression (23) shows the equation for $A_{\text{opt}}$ to be solved. Since $K$ is not a function of $E_{\text{ref}}$, the determination of $A_{\text{opt}}$ is independent of $E_{\text{ref}}$. This feature seems important in terms of the application of our proposal because the evaluation of $E_{\text{ref}}$ is sometimes a difficult task. Although

$$
\frac{dK}{dA} \bigg|_{A=A_{\text{opt}}} = 0.
$$

expression (23) is not equivalent to

$$
\frac{dK}{d\theta} \bigg|_{A=A_{\text{opt}}} = 0,
$$

because $d\theta/dA$ can be negative. When $|\delta| < T$, $d\theta/dA \geq 0$ always holds, as shown in online appendix B.1. Thus, only cases of $\delta_{[1-\alpha/2]} < T$, where expressions (21) and (25) are identical to each other as the equation for $A_{\text{opt}}$, are discussed in this subsection. Point (b) is, hence, required. Online appendix B.2 shows that expression (25) can be transformed into the following equation:

$$
\begin{align*}
\phi \left( \frac{T - A_{w} - \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T - A_{w} + \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T + A_{w} - \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T + A_{w} + \delta_{[1-\alpha/2]}}{u_{w}} \right) &= \frac{P_{0} + (c_{D} - c_{S})}{\beta P_{0}},
\end{align*}
$$

where

$$
\frac{u_{w}}{} = \frac{1}{\sqrt{\sigma^{-2} + u_{r}^{-2}}} = \frac{\sigma}{u_{r}}, \quad A_{w} = \frac{u_{w}^{2}}{u_{r}^{2}} A.
$$

When $\delta_{[1-\alpha/2]} < T$ and we cannot find the solution for the equation given by expression (25), $A_{\text{opt}} = +\infty$ is the solution. Moreover, we can obtain the following approximation of expression (26) by inserting $c_{D} = c_{S} = 0$:

$$
\begin{align*}
\phi \left( \frac{T - A_{w} - \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T - A_{w} + \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T + A_{w} - \delta_{[1-\alpha/2]}}{u_{w}} \right) + \phi \left( \frac{T + A_{w} + \delta_{[1-\alpha/2]}}{u_{w}} \right) \bigg|_{A=A_{\text{opt}}} \\
= \beta^{-1},
\end{align*}
$$

assuming $c_{D}$ and $c_{S}$ are negligible to $P_{0}$. Expression (28) implies that we do not need to know even the value of $P_{0}$ for this case. The equation of expression (28) can be solved by applying a simple numerical algorithm. Listing 1 shows the program for MATLAB [21]. We can also obtain the solution using other general software.

Figure 5 shows the results of $T - A_{\text{opt}}$ with the parameters $(T, \sigma, u_{t}, u_{r}) = (2 \text{ or } 3, 1, 0.25, 0.1), (c_{1}, c_{D}, c_{S}) = (0, 0, 0)$, and $\beta = 50$ with $\alpha$ varying from 0.001 to 0.9. The magnitude of $T - A$ is often called the guard band, and $T - A_{\text{opt}}$ indicates the optimum guard band. This graph shows the sensitivity of $A_{\text{opt}}$ to the value of $\alpha$. We can conclude that the determination of $\alpha$ influences the variation of $A_{\text{opt}}$.
Listing 1: Matlab code for the optimized acceptable difference: ‘par’, ‘uw’, ‘delta’ and ‘Aopt’ indicate the input vector of $(T, \sigma, ur, \alpha, \beta)^T$, $uw, \delta_{[1, \alpha/2]}$ and $A_{opt}$, respectively.

```
1 target = @(A, T, uw, ur, delta, beta) ...
2   (normcdf((-T-uw^2/ur^2*A-delta)/uw) ...) 
3   + normcdf(((-T-uw^2/ur^2*A+delta)/uw) ...) 
4   + normcdf(((-T+uw^2/ur^2*A-delta)/uw) ...) 
5   + normcdf(((T+uw^2/ur^2*A+delta)/uw)) = 0.5 ...
6   -1/beta;
7 par = [2 1.00 0.25 0.1 0.01 50];
8 uw = sqrt(par(3)^2 * par(2)^2 / (par(3)^2 + par(2)^2));
9 delta = norminv(1-par(5)/2) * par(4);
10 optA = @(A) target(A, par(1), uw, par(3), delta, par(6));
11 Aopt = fzero(optA, par(1))
```

5. Characterization through simulations

We characterize the proposed method through simulations. The simulations are implemented using the parameters shown in expression (18) in section 3. The given reference condition is the same as that in section 6 of our previous report [1]. Under the reference condition, the rejection rate and the gCR for the reference condition are given as $R_{ref} = 5.23\%$ and $\theta_{ref} = 0.84\%$, respectively.

In the simulations in this subsection, fixing $\alpha = 0.01$, we determine $A_{opt}$ to maximize $m_{[0.01]}$. In other words, the revenue for the case with $\delta = \delta_{[0.995]}$ is maximized. Since $\delta_{[0.995]} = 0.26 < T = 2$, we can confirm that one of the conditions exists allowing us to use expression (27) for obtaining $A_{opt}$.

We summarize the simulation conditions and results in table 1. Case 1 is the condition used in section 3. The values of $\beta$ and $E_{ref}$ are different among cases 1 to 3. Specifically, $\beta = 50$ and $E_{ref} = 3$ in case 2, and $\beta = 10$ and $E_{ref} = 1$ in case 3. $P_0$ is computed using these values for each case as shown in table 1. $m_{[0.01]}$ is computed as a function of $A$, and the results are shown in figures 6 and 7 for cases 2 and 3, respectively. For case 1, see figure 3. Not only $m_{[0.01]}$ but also $\overline{m}$ and max[$m$] are shown in figures 6 and 7.

Figure 6 shows the result for case 2 ($E_{ref} = 3, \beta = 50$). It is found that $m$ is a constant of about $-0.2$ when $A$ is close to 0. The reason why $m$ is constant is that manufacturing no items is the best decision under these conditions. Since the sales are zero, $m$ is a negative constant with respect not only to $A$ but also to $\alpha$.

When $A$ is more than about 0.5, $m$ becomes large with increasing $A$. $A_{opt} = 1.39$, and $m_{[0.01]}$ is maximized at that point. The maximum value is given as 0.98. Since $A_{opt}$ in case 1 is also given as 1.39, we can confirm the consistency of the values of $A_{opt}$ in cases 1 and 2 where $E_{ref} = 1$ and 3, respectively. However, the maximum value of $m_{[0.01]}$ in case 1 is 0.71, which is smaller than 0.98. While the value of $E_{ref}$ has no effect in the determination of $A_{opt}$, it can influence the maximized value of $m_{[0.01]}$.

$m$ is the difference of the revenue from that under the reference condition divided by $g_{ref}W_{ref}$. However, the fact that $m_{[0.01]} = 0.98$ at $A = A_{opt}$ does not mean that the revenue is $0.98 \times g_{ref}W_{ref}$ greater than that under the reference condition, but that the revenue is beyond that value with the probability of $1 - \alpha = 99\%$. The revenue can increase
For \( A > A_{\text{opt}} \), \( m_{[0.01]} \) decreases with increasing \( A \), because the decrease in the price is dominant with respect to the revenue rather than the increase in the output amount. The smallest value for \( m_{[0.01]} \) is that in the case where no manufacturing is conducted. Thus, \( \max[m] \), \( \overline{m} \), and \( m_{[0.01]} \) have a constant value of about \(-0.2\) when \( A \) is greater than a certain value around \( 2.2 \).

Figure 7 shows the result for case 3 (\( E_{\text{ref}} = 1, \beta = 10 \)). Since \( m \) with no manufacturing is identical to that in case 1 due to the same \( E_{\text{ref}} \), \( m \) is about \(-0.6\) when \( A \) is close to \( 0 \). The qualitative feature of \( m_{[0.01]} \) as a function of \( A \) is the same as those in cases 1 and 2. The optimum acceptable difference is, however, given as \( A_{\text{opt}} = 1.62 \), which is larger than \( A_{\text{opt}} = 1.39 \) in cases 1 and 2. For \( A > A_{\text{opt}} \), the decrease in \( m_{[0.01]} \) with increasing \( A \) is more moderate than in cases 1 and 2.

The maximized value of \( m_{[0.01]} \) is given as \(-0.02\). The negative value of \( m \) does not mean negative revenue but a smaller revenue than that under the reference condition. Defining \( M_{\text{ref}} \) as the revenue under the reference condition, we can make a decision on running the process based on whether \( (M_{\text{ref}} - 0.02 \times g_{\text{ref}} \beta W_{\text{ref}}) \) is positive or negative. In this decision making, \( E_{\text{ref}} \) is necessary, even though it is not required to determine \( A_{\text{opt}} \).

Moreover, assuming \( u_{a} = 0 \), the optimum acceptable difference is given as \( 1.78 \), where \( \max[m] \) is maximized. The gap between that value and \( A_{\text{opt}} = 1.62 \) for \( m_{[0.01]} \) is caused by the systematic effect component in the measurement uncertainty. It can be said that the systematic effect component in the measurement uncertainty has a non-negligible effect on the optimization of the acceptance interval.

6. Summary

In this report, we propose a method for the optimization of acceptance interval in conformity assessment from an economic perspective with the presence of random and systematic effect components in the measurement uncertainty. The systematic effect components are caused by unknown biases. We discuss only cases where the central value of the distribution of the measured quality characteristics is set as the target value, and all manufactured items are inspected. When we consider the distribution of biases, the distribution of the revenue can be given based on the price model and manufacturing model proposed in our previous report. We hence develop the optimization method to maximize the \( 100\alpha \) percentile of the revenue distribution, where \( \alpha \) is referred to as the level of risk in this study. The optimization can be achieved by the solution of a relatively simple equation. We have characterized our proposal and can conclude that the level of risk \( \alpha \) is an important parameter to determine the optimum acceptance interval as well as the degree of loss \( \beta \) suggested our previous report.

By incorporating the systematic effect, we believe that the proposed method is sufficiently practical to be employed in actual industrial settings.
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