Few-qubit lasing in circuit QED

Stephan André$^{1,2}$, Valentina Brosco$^3$, Michael Marthaler$^{1,2}$, Alexander Shnirman$^{2,4}$ and Gerd Schönh$^{1,2}$

$^1$ Institut für Theoretische Festkörperphysik, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
$^2$ DFG Center for Functional Nanostructures (CFN), Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
$^3$ Dipartimento di Fisica, Università ‘La Sapienza’, Ple A Moro 2, 00185 Roma, Italy
$^4$ Institut für Theorie der Kondensierten Materie, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany

E-mail: schoen@kit.edu

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Abstract
Motivated by recent experiments, which demonstrated lasing and cooling of the electromagnetic modes in a resonator coupled to a superconducting qubit, we describe the specific mechanisms creating the population inversion, and we study the spectral properties of these systems in the lasing state. Different levels of the theoretical description, i.e. the semi-classical and the semi-quantum approximation, as well as an analysis based on the full Liouville equation are compared. We extend the usual quantum optics description to account for strong qubit–resonator coupling and include the effects of low-frequency noise. Beyond the lasing transition we find for a single- or few-qubit system the phase diffusion strength to grow with the coupling strength, which in turn deteriorates the lasing state.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The search for efficient coupling and read-out architectures of scalable solid-state quantum computing systems has opened a new field, called ‘circuit QED’ [1]. It is the on-chip analogue of quantum optics ‘cavity QED’, with superconducting qubits playing the role of (artificial) atoms and an electromagnetic resonator replacing the cavity. The resonators can be used to read out the qubit state [2–5] or to couple qubits to perform single- and two-qubit gates [6–9].

Apart from these applications for quantum information processing, circuit QED offers the possibility to study effects known from quantum optics in electrical circuits: Fock states of the electromagnetic field were created and detected [10], and the coherent control of photon propagation via electromagnetically induced transparency was shown [11]. In addition, lasing and cooling of the electromagnetic field in the resonator has been demonstrated: By creating a population inversion in a driven superconducting single-electron transistor (SSET) coupled capacitively to a microstripline resonator, Astafiev et al [12] could excite a lasing peak in the spectrum. In another experiment, Grajcar et al [13] coupled a driven flux qubit to a low-frequency LC resonator and observed both cooling and a tendency towards lasing via the so-called Sisyphus mechanism.

In contrast to conventional lasers where many atoms are coupled weakly to the light field in a Fabry–Perot cavity, in the micromasers realized, e.g., in [12, 13] a single superconducting qubit is coupled strongly to the microwave field in the resonator. Compared to conventional lasers one expects for single-atom lasers a lower intensity of the radiation but stronger fluctuation effects. Specifically, the quantum fluctuations of the photon number associated with spontaneous emission, which are known to lead to the phase diffusion of the laser field [14], have more pronounced consequences. As a result, even in the lasing state, phase coherence is lost after a characteristic time $\tau_d$, which sets a limit on the linewidth of the laser radiation and thus on the visibility of the lasing signal. Phase diffusion was observed...
experimentally in a single-qubit maser in [12]; the dependence of the phase diffusion on the coupling strength was analyzed theoretically by the present authors in [15].

In the present work, we analyze static and spectral properties of single- and few-qubit lasers, focusing on the regime of strong qubit–resonator coupling realized in most circuit QED experiments. Using a master equation approach, we analyze the consequences of qubit-field correlations. In the strong coupling regime, they have significant quantitative effects on the laser linewidth. In addition for the case of few-qubit lasing, we evaluate the corrections due to qubit–qubit correlations. They are strongest at the transition to the lasing regime but yield only small corrections to the power spectrum. In the frame of the so-called ‘semi-quantum’ approximation [16], we describe the qualitative differences between multi-atom lasers and superconducting micromasers; specifically we analyze the scaling of the lasing transition and diffusion constant with the number of atoms.

The paper is organized as follows. We start discussing in the following section the two experimental realizations of superconducting micromasers reported in [12] and [2, 13], respectively. In particular, we describe how the population inversion in the qubit is created in the two examples. In section 3, we formulate the theoretical model and derive the dynamical equations for the micromaser using a master equation approach. In section 4, we review the theory of lasing, paying attention to effects which are usually ignored for conventional lasers but are prominent in single- or few-atom lasers in the strong coupling regime. We introduce the different approximation schemes. Static properties of single-qubit lasers are presented, such as the average photon number and the qubit–field and qubit–qubit correlations. We show explicitly that due to spontaneous emission in few-qubit lasers the sharp lasing threshold is replaced by a smooth, but still well-localized transition to the lasing regime.

Next, in section 5, we analyze the spectral properties of superconducting micromasers. We discuss the effects of correlations between qubit and resonator on the phase diffusion process and we show that, beyond the lasing threshold, the linewidth grows with increasing coupling strength, thus deteriorating the lasing state. Finally, in section 6, we analyze the scaling of the photon number and of the diffusion constant with the number of atoms. In addition, we demonstrate how low-frequency noise leads to inhomogeneous broadening of the lasing peak.

2. Inversion mechanisms in superconducting micromasers

2.1. The SSET laser

The ‘SSET laser’ realized by Astafiev et al [12] consists of a SSET coupled capacitively to a microstripline resonator, as shown in figure 1(a). The properties of the coupled system and the specific form of the Hamiltonian will be analyzed further in later sections and in the appendix. In the present section, we describe how a population inversion is created in a suitably biased SSET.

An SSET consists of two superconducting leads coupled by tunnel junctions to a superconducting island. A gate voltage $U$ shifts the electrostatic energy of the island and controls, together with the bias voltage $V$, the current through the device. The Josephson coupling, $E_J$, allowing for coherent Cooper pair tunneling through the junctions, is weak compared to the superconducting energy gap $\Delta_{sc}$ and to the charging energy of the island, $E_C = e^2/2C$, $C$ being the total island’s capacitance. In addition, quasi-particles can tunnel incoherently (with rate $\propto V/eR$, where $R$ is the resistance of the tunnel junction) when the energy difference between initial and final states is sufficient to create a quasiparticle excitation, i.e. when it exceeds twice the gap, $|\Delta E| \geq 2\Delta_{sc}$. We denote by $N$ the number of excess charges on the island; it changes by $\pm 1$ in a single-electron tunneling process and by $\pm 2$ in a Cooper pair tunneling event. At low temperatures for the conditions realized experimentally the number of accessible charge states of the island is strongly reduced. For the further calculations we can restrict our analysis to $N = 0, 1, 2$.

We assume the SSET to be tuned close to the Josephson quasi-particle (JQP) cycle, where the current is transported by a combination of Cooper pair tunneling through one junction and two consecutive quasi-particle tunneling events through the other junction. The parameters of the junctions are chosen asymmetrically. By changing the transport voltage $V$ and the gate voltage $U$, we can tune to a situation, where resonant Cooper pair tunneling is strong across, say, the lower
the energy of the island.

In the setup of [2] an externally driven three-junction flux qubit is coupled inductively to an LC oscillator.

\[ H = -\frac{1}{2} \Delta E \sigma_z + h \Omega_{R_0} \cos (\omega_d t) \sigma_z - \frac{1}{2} \left( b_+ \sigma_z + b_\sigma^\dagger \sigma_z + b_\sigma^\dagger \sigma_z + b_\sigma \right) \hat{X} + H_{\text{bath}}. \]  

In the absence of driving, \( \Omega_{R_0} = 0 \), and for regular (i.e. smooth as a function of the frequency) power spectra of the fluctuating bath observables we can proceed using Golden rule type arguments [18, 19]. The transverse noise, coupling to \( \sigma_x \) and \( \sigma_y \), is responsible for relaxation and excitation processes with rates

\[ \Gamma_\downarrow = \frac{|b|^2}{4\hbar^2} (\hat{X}^2)_{\omega=\Delta E}, \]
\[ \Gamma_\uparrow = \frac{|b|^2}{4\hbar^2} (\hat{X}^2)_{\omega=\Delta E}, \]

while longitudinal noise, coupling to \( \sigma_z \), produces a pure dephasing with rate

\[ \Gamma_\varphi = \frac{|b|^2}{2\hbar^2} S_X(\omega = 0). \]

Here \( b_\perp \equiv b_x + i b_y \), and we introduced the ordered correlation function \( \langle \hat{X}^2 \rangle_{\omega} = \int dt \ e^{i\omega t} \langle \hat{X}(t) \hat{X}(0) \rangle \), as well as the power spectrum, i.e. the symmetrized correlation function, \( S_X(\omega) = \langle (\hat{X}^2)_{\omega} + (\hat{X}^2)_{-\omega} \rangle / 2 \). The rates (5) and (6) also define the relaxation rate \( 1/T_1 = \Gamma_\downarrow = \Gamma_\uparrow + \Gamma_\varphi \) and the total dephasing rate \( 1/T_2 = \Gamma_\varphi = \Gamma_1 / 2 + \Gamma_\varphi \), which appear in the Bloch equations for the qubit.

To account for the driving with frequency \( \omega_d \) it is convenient to transform to the rotating frame via a unitary transformation

\[ U_t = \exp(-i \omega_d t \sigma_z / 2). \]

Within the rotating wave approximation (RWA) the transformed Hamiltonian reduces to

\[ \tilde{H} = \frac{1}{2} h \Omega_{R_0} \sigma_z + \frac{1}{2} h \delta \omega \sigma_z - \frac{1}{2} [b_\sigma^\dagger \sigma_z + b_\sigma \sigma_z^\dagger] \sigma_z + b_\sigma^\dagger e^{-i\omega_d t} \sigma_z, \]

with detuning \( \delta \omega \equiv \omega_d - \Delta E / h \). The RWA cannot be used in the second line of (7) since the fluctuations \( \hat{X} \) contain potentially frequencies close to \( \pm \omega_d \), which can compensate fast oscillations. Diagonalizing the first two terms of (7) one obtains

\[ \tilde{H} = \frac{1}{2} h \Omega_{R_0} \sigma_z + H_{\text{bath}} - \left[ \frac{\sin \beta}{2} b_\sigma^\dagger \right] \]
\[ + \frac{\cos \beta}{4} \left( b_\sigma^\dagger e^{-i\omega_d t} + b_\sigma e^{i\omega_d t} \right) \sigma_z \hat{X} - \left[ \frac{\sin \beta + 1}{4} b_\sigma \right] e^{-i\omega_d t} \]
\[ + \frac{\sin \beta - 1}{4} b_\sigma e^{i\omega_d t} - \frac{\cos \beta}{2} b_\sigma \sigma_z \hat{X} + \text{h.c.}, \]

where the full Rabi frequency is \( \Omega_R = \sqrt{\Omega_{R_0}^2 + \delta \omega^2} \), and the detuning determines the parameter \( \beta \) via

\[ \tan \beta = \delta \omega / \Omega_{R_0}. \]
From here Golden-rule arguments lead to the relaxation and excitation rates in the rotating frame as well as the 'pure' dephasing rate [21]

\[
\Gamma_\downarrow \approx \frac{b^2}{4h} \cos^2 \beta \langle \hat{X}^2 \rangle_{\Omega_R} + \left| \frac{b_\perp^2}{16h^2} \right| (1 - \sin \beta)^2 \langle \hat{X}^2 \rangle_{\omega_{\downarrow} + \Omega_R} + (1 + \sin \beta)^2 \langle \hat{X}^2 \rangle_{\omega_{\downarrow} - \Omega_R},
\]

\[
\Gamma_\uparrow \approx \frac{b^2}{4h} \cos^2 \beta \langle \hat{X}^2 \rangle_{\omega_{\uparrow} - \Omega_R} + \left| \frac{b_\perp^2}{16h^2} \right| (1 - \sin \beta)^2 \langle \hat{X}^2 \rangle_{\omega_{\uparrow} + \Omega_R} + (1 + \sin \beta)^2 \langle \hat{X}^2 \rangle_{\omega_{\uparrow} - \Omega_R},
\]

\[
\Gamma_\psi^* \approx \frac{b^2}{4h^2} \sin^2 \beta \left( S_X(\omega = 0) + \left| \frac{b_\perp^2}{16h^2} \right| \cos \beta \right) S_X(\omega_R).
\]

We note the effect of the frequency mixing. In addition, due to the diagonalization the effects of longitudinal and transverse noise on relaxation and decoherence get mixed. We further note that the rates also depend on the fluctuations' power spectrum at the Rabi frequency, \(\langle \hat{X}^2 \rangle_{\pm \Omega_R}\).

For a sufficiently regular power spectrum of the fluctuations at frequencies \(\omega \approx \pm \Delta E / h\), we can ignore the effect of detuning and the small shifts by \(\pm \delta k\) as compared to the high frequency \(\omega_R \approx \Delta E / h\). We further assume that \(\Omega_R \ll kT / h\). In this case, we find the simple relations

\[
\Gamma_\uparrow = \frac{(1 + \sin \beta)^2}{4} \Gamma_\downarrow + \frac{(1 - \sin \beta)^2}{4} \Gamma_\psi + \frac{1}{2} \cos^2 \beta \Gamma_\psi,
\]

\[
\Gamma_\downarrow = \frac{(1 - \sin \beta)^2}{4} \Gamma_\downarrow + \frac{(1 + \sin \beta)^2}{4} \Gamma_\psi + \frac{1}{2} \cos^2 \beta \Gamma_\psi,
\]

\[
\Gamma_\psi^* = \frac{\sin^2 \beta}{4} \Gamma_\psi + \frac{\cos^2 \beta}{2} (\Gamma_\downarrow + \Gamma_\uparrow),
\]

where the rates in the lab frame are given by equations (5) and (6) and the new rate

\[
\Gamma_\psi = \frac{1}{2h^2} b^2 \Gamma_\psi^* S_X(\Omega_R)
\]

depends on the power spectrum at the Rabi frequency.

To proceed, we concentrate on the most relevant regime. At low temperatures, \(k_B T \ll \Delta E \approx h\omega_R\), we can neglect \(\Gamma_\uparrow\) as it is exponentially small. We also assume that \(\Gamma_\psi\) can be neglected as compared to \(\Gamma_\downarrow\), which is justified, e.g., when the qubit is tuned close to the symmetry point where \(b_\perp \ll |b_\perp|\).

Since the rate \(\Gamma_\psi\) depends on the noise power spectrum at the frequency \(\Omega_R\), which is usually higher than the frequency range of the \(1/f\) noise, the latter does not change the situation. Thus, we neglect \(\Gamma_\psi\), and we are left with

\[
\Gamma_\downarrow \approx \frac{(1 + \sin \beta)^2}{4} \Gamma_\downarrow, \quad \Gamma_\psi^* \approx \frac{\cos^2 \beta}{2} \Gamma_\downarrow.
\]

The ratio of up- and down-transitions depends on the detuning and can be expressed by an effective temperature. Right on resonance, where \(\beta = 0\), we have \(\Gamma_\uparrow = \Gamma_\downarrow\), corresponding to infinite temperature or a classical drive. For ‘blue’ detuning, \(\beta > 0\), we find \(\Gamma_\uparrow > \Gamma_\downarrow\), i.e. negative temperature. This leads to a population inversion of the qubit, which is the basis for the lasing behavior which will be described below.

In a more careful analysis, paying attention to the small frequency shifts by \(\pm \delta k\), we obtain for \(\beta = 0\)

\[
\Gamma_\psi^* = \frac{(1 + \sin \beta)^2}{4} \Gamma_\psi + \frac{\cos^2 \beta}{2} \Gamma_\psi.
\]

For example, for Ohmic noise and low bath temperature this reduces to \(\Gamma_\psi^* / \Gamma_\downarrow \approx 1 + 2\Delta E / h\omega_R\), which corresponds to an effective temperature of order \(\Omega_R^2 / k_B \approx 2\Delta E / k_B\), which by assumption is high but finite. The infinite temperature threshold is crossed toward negative temperatures at weak blue detuning when the condition

\[
\left( \frac{1 + \sin \beta)^2}{4} \left( \frac{1 - \sin \beta)^2}{4} \right) = 1 + 2\Delta E / h\omega_R
\]

is satisfied. We note that all qualitative features are well reproduced by the approximation (13).

To illustrate the calculations outlined above and the mechanism creating the population inversion for blue detuning we show in figure 3 the level structure, i.e., the formation of dressed states, of a near-resonantly driven qubit. For the purpose of the present discussion, we assume that also the driving field is quantized. This level structure was described first by Mollow [17]. The picture also illustrates how for blue detuning a pure relaxation process, \(\Gamma_\downarrow\), in the laboratory frame predominantly leads to an excitation process,
\( \Gamma_\uparrow \), in the rotating frame, thus creating a population inversion on the basis of ‘dressed states’.

If this effective inverted two-state system is coupled to an oscillator, a lasing state is induced. In [20], it was proposed to couple the oscillator to the dressed states belonging to the neighboring doublet (see figure 3). Then, to be in resonance with the pair of dressed states with population inversion, the oscillator frequency should satisfy \( \omega_0 = \omega_\uparrow + \Omega_R \).

As \( \omega_\uparrow \approx \Delta \omega \) and \( \Omega_R \ll \Delta \omega \) this can work for a high-frequency resonator approximately in resonance with the qubit, \( \omega_0 \approx \Delta \omega \). In contrast, in [22] a different situation was considered where the oscillator was coupled to the dressed states belonging to the same doublet. The resonance condition then reads \( \omega_0 = \Omega_R \), and the lasing can be reached for an oscillator much slower than the qubit, \( \omega_0 \ll \Delta \omega \), which is the situation realized in [2]. An additional complication arises at the symmetry point of the qubit, since there the single-photon coupling between the oscillator and the doublet of the dressed states vanishes. Then, two-photon processes become relevant with the resonance condition \( 2\omega_0 = \Omega_R \) [22].

3. Modeling the single- or few-qubit laser

We consider a single-mode quantum resonator coupled to \( N_d \) qubits (labeled by \( \mu \)). In the absence of dissipation, in the rotating wave approximation, the dynamics of the system is described by the Tavis–Cummings Hamiltonian [23]:

\[
H_{TC} = \hbar \omega_0 a^\dagger a + \frac{1}{2} \hbar \kappa \sum_\mu \sigma_\mu^z + \hbar g \sum_\mu (\sigma_\mu^x a + \sigma_\mu^x a^\dagger). \tag{16}
\]

Here we introduced, apart from the photon annihilation and creation operators, \( a \) and \( a^\dagger \), the Pauli matrices acting on the single-qubit eigenstates \( \sigma_\mu^z = |\uparrow_\mu\rangle \langle \uparrow_\mu| - |\downarrow_\mu\rangle \langle \downarrow_\mu| \), \( \sigma_\mu^x = |\uparrow_\mu\rangle \langle \downarrow_\mu| + |\downarrow_\mu\rangle \langle \uparrow_\mu| \). Including both resonator and qubit dissipation, the total Hamiltonian becomes

\[
H = H_{TC} + (a + a^\dagger)X_a + \sum_\mu \left( X_\mu^x \sigma_\mu^z + X_\mu^y \sigma_\mu^x + X_\mu^z \sigma_\mu^y \right) + H_{bath}. \tag{17}
\]

Dissipation is modeled by assuming that the oscillator and the qubits interact with noise operators, \( X_a \) and \( X_\mu^x \), \( X_\mu^y \), \( X_\mu^z \), belonging to independent baths with Hamiltonian \( H_{bath} \) in thermal equilibrium [24]. The noise coupling longitudinally to the qubits, \( X_\mu^\dagger \sigma_\mu^x \), is responsible for the qubits’ pure dephasing.

In this section and beyond, we do not describe anymore the detailed mechanism creating the population inversion in the qubits, which is necessary to obtain lasing. Rather we introduce it by assuming that the effective temperature fixing the ratio of excitation and relaxation rates of the qubits is negative. (In the same spirit the transition rates \( \Gamma \) appearing below are those of the effective two-level system, even if they refer to transition between dressed states, for which the rates were denoted above by \( \Gamma \).) Possible deviations from the Tavis–Cummings oscillator–qubit coupling used in equation (17) are discussed in the appendix.

The dynamics of a single- or few-qubit laser can be analyzed in the frame of a master equation approach, as discussed by several authors [22, 26–29]. In the Schrödinger picture, the master equation for the reduced density matrix \( \rho \) of the qubits and the oscillator reads

\[
\dot{\rho} = -\frac{i}{\hbar} [H_{TC}, \rho] + L_Q \rho + L_R \rho. \tag{18}
\]

The Liouville operators \( L_R \) and \( L_Q \) describe the resonator’s and qubits’ dissipative processes. For Markovian processes, it is sufficient to approximate them by Lindblad forms,

\[
L_R \rho = \frac{\kappa}{2} \left[ (N_{bh} + 1) (2a^\dagger a^\dagger a^\dagger a - a^\dagger a^\dagger a^\dagger a) + N_{bh} (2a^\dagger a^\dagger a^\dagger a - a^\dagger a^\dagger a^\dagger a) \right] \tag{19}
\]

and

\[
L_Q \rho = \sum_\mu \left[ \frac{\Gamma_\uparrow}{2} (\sigma_\mu^x \rho \sigma_\mu^x - \rho) + \frac{\Gamma_\downarrow}{2} (2\sigma_\mu^x \rho \sigma_\mu^x - \rho \sigma_\mu^x \rho - \sigma_\mu^x \rho \sigma_\mu^x \rho) \right]. \tag{20}
\]

The dissipative evolution of the system depends on the excitation, relaxation and pure dephasing rates of the qubits, \( \Gamma_\uparrow \), \( \Gamma_\downarrow \), and \( \Gamma_\psi \), as well as on the bare damping rate of the resonator, \( \kappa \), and on the thermal photon number \( N_{bh} \).

For later purposes, we also introduce the rate \( \Gamma_0 = \Gamma_\uparrow + \Gamma_\downarrow \), which is the sum of excitation and relaxation rates, and \( \Gamma_\psi = \Gamma_\uparrow/2 + \Gamma_\downarrow \), the total dephasing rate, including the ‘pure dephasing’ due to longitudinal noise described by \( \Gamma_\psi \). In contrast to relaxation and excitation processes, pure dephasing arises due to processes with no energy exchange between qubit and environment and thus does not affect the populations of the two qubit states. The parameter \( D_0 = (\Gamma_\uparrow - \Gamma_\downarrow)/\Gamma_\uparrow \) denotes the stationary qubit polarization in the absence of the resonator. In the present case, since we assume a negative temperature of the qubit baths and a population inversion we have \( D_0 > 0 \).

The master equation (18) allows us to determine completely the quantum state of the system. However, its full solution is numerically demanding in the experimental regime of parameters due to the high number of photons in the resonator (of the order of \( 10^5 \) or higher for a single-qubit laser). For this reason, we will use, whenever possible, different approximation schemes to calculate the physically relevant quantities.

4. Approximations and static properties

To describe the single- or few-qubit laser in the strong coupling regime, we start from the master equation (18) for the density matrix. In some cases, we find that approximate analytical results, which are presented in this section, are sufficient. In general, however, we rely on a numerical solution.

From equation (18) we obtain the following equations for the average photon number \( \langle n \rangle \), the qubit polarization \( \langle \sigma_\mu^x \rangle \), and the resonator’s occupation \( n_\psi \).

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shows the qubit–qubit we also plot the quantum optics as ‘semi-quantum model’ \( \text{\cite{48}} \) which depends on the parameter \( \tilde{\alpha} \), the average photon number (per qubit) \( \langle \sigma_z^0 \rangle \) and the total dephasing rate \( \gamma = \Gamma_{\phi} + \frac{1}{2} \).

In the stationary limit, after isolating the correlations between different qubits by writing \( \langle \sigma_i^0 \sigma_j^\mu \rangle = \delta_{ij}(1 + \langle \sigma_i^0 \sigma_i^\mu \rangle) + (1 - \delta_{ij})\langle \sigma_i^0 \sigma_j^\mu \rangle \), we derive from the previous equations the following two exact relations between four quantities: the average qubit polarization \( \langle S_z(t) \rangle \) with \( S_z = \frac{1}{N_0} \sum_n \sigma_z^0 \), the photon number \( \langle n(t) \rangle \), and the correlators \( \langle n_S \rangle \) and \( C_{\langle Q Q \rangle} = \frac{1}{N_0} \sum_{n_0,n} \langle \sigma_z^0 \sigma_z^n \rangle \).

\[
(n) = N_{th} + \frac{2g^2N_a}{\kappa} \frac{\gamma}{\gamma^2 + \Delta^2} \left[ \langle S_n \rangle + \frac{1}{2} \langle \langle S_1 \rangle + 1 \rangle + C_{\langle Q Q \rangle} \right],
\]

\[
\langle S_z \rangle = D_0 - \frac{4g^2}{\Gamma_1} \frac{\gamma}{\gamma^2 + \Delta^2} \left[ \langle S_n \rangle + \frac{1}{2} \langle \langle S_1 \rangle + 1 \rangle + C_{\langle Q Q \rangle} \right].
\]

If two of them are known, e.g., from a numerical solution of the master equation, the other two can be determined.

4.1. Semi-quantum model

Factorizing the correlator, \( \langle S_n \rangle \sim \langle S_z \rangle \langle n \rangle \), on the right-hand side of equations (22) and neglecting the qubit–qubit correlations, \( C_{\langle Q Q \rangle} \approx 0 \), we reproduce results known in quantum optics as ‘semi-quantum model’ \( \text{\cite{16}} \). This approximation yields a quadratic equation for the scaled average photon number (per qubit) \( \bar{n} = \langle n \rangle / N_a \),

\[
\bar{n}^2 + \left( \bar{n}_0 - \Gamma_{\phi}D_0 \frac{N_{th}}{N_a} \frac{\kappa}{2N_a} + \frac{1}{2N_a} \right) \bar{n} - \left( \bar{n}_0 \frac{N_{th}}{N_a} + \frac{N_{th}}{2N_a} + \frac{\Gamma_{\phi}D_0}{4\kappa} \right) = 0,
\]

which depends on the parameter \( \bar{n}_0 = \frac{\Gamma_{\phi}D_0}{g^2/N_a}(1 + \frac{\gamma}{\gamma^2}) \). This equation has always one positive solution \( \bar{n} > 0 \).

4.2. Semiclassical approach

Before continuing the analysis of the properties of the semi-quantum solution, for the sake of comparison, we recall the standard semiclassical results. In this approximation, the operator \( \alpha \) is treated as a classical stochastic variable, \( \alpha \). After adiabatic elimination of the qubits’ degrees of freedom, i.e., assuming \( \Gamma_{\phi} \gg \kappa / 2 \), one obtains a classical Langevin equation for \( \alpha \),

\[
\dot{\alpha} = -\frac{\kappa}{2} \frac{g^2 N_a}{\Gamma_{\phi} - i \Delta} \alpha + \xi(t).
\]

Here \( \xi(t) \) is a classical Langevin force due to thermal noise, \( \langle \xi(t)\xi^*(t') \rangle = \kappa N_{th} \delta(t - t') \), and \( i \Delta = D_0 / (1 + |\alpha|^2 / (\bar{n}_0 N_a)) \) denotes the stationary qubits’ polarization.

In order to obtain an expression for the average photon number \( \langle n \rangle = \langle |\alpha|^2 \rangle \), we rewrite the Langevin equation as \( \dot{\alpha} = -f(|\alpha|^2) + \xi(t) \) and approximate \( f(|\alpha|^2) \approx f(\langle n \rangle) \cdot \langle n \rangle \). Thus, we arrive at the equation \( \frac{d}{dt} \langle n \rangle = -2\text{Re} \langle f(\langle n \rangle) \rangle + \kappa N_{th} \), from which we obtain in the steady state a quadratic equation for the scaled photon number \( \bar{n} = \langle n \rangle / N_a \),

\[
\bar{n}^2 + \left( \bar{n}_0 - \Gamma_{\phi}D_0 \frac{N_{th}}{2\kappa} \right) \bar{n} - \frac{\bar{n}_0 \bar{n}_0}{N_a} = 0.
\]

In the low-temperature limit, \( N_{th} \sim 0 \), the semiclassical results can be rewritten in the simple form, \( \bar{n}^2 + \left( \bar{n}_0 - \Gamma_{\phi}D_0 / 2\kappa \right) \bar{n} \approx 0 \), with \( \bar{n}_0 = \frac{\Gamma_{\phi}D_0}{g^2/N_a}(1 + \frac{\gamma}{\gamma^2}) \). This gives the well-known threshold condition \( D_0 > \kappa \Gamma_{\phi} / (2g^2 N_a) \) for the lasing state.

4.3. Comparison of the different approaches

Different from the semiclassical picture, the semi-quantum model includes the effects of spontaneous emission processes, described by the term proportional to \( \langle \langle S_z \rangle + 1 \rangle \) in equations (22). Spontaneous emission is responsible for the linewidth of the lasers, and, as noticed in \( \text{\cite{27}} \), due to the low photon number, spontaneous emission is especially relevant for the dynamics of single-atom lasers.

To illustrate the effect of spontaneous emission on the lasing transition and at the same time the quality of the semi-quantum approximation, we plot the photon number as a function of the coupling strength \( g \) for \( N_a = 1 \) (figure 4, left panel) and \( N_a = 2 \) (figure 4, right panel). The plots show the semi-quantum, semiclassical and master equation results. We note that the semi-quantum approximation gives results in very good agreement with the master equation. Moreover, both the semi-quantum and the master equation solutions show a smooth crossover between the normal and the lasing regimes, an effect that is due to spontaneous emission. While we cannot define a sharp threshold condition, we can still identify, even for a single-atom laser, a well-localized transition region centered at the threshold coupling predicted by the semiclassical approximation.

In the left panel of figure 4 we also plot the qubit–oscillator correlator, \( \langle S_z \rangle \), and the factorized approximation, \( \langle S_z \rangle \langle n \rangle \). For strong coupling, they differ significantly. However, as the good agreement between the semi-quantum approximation and the numerical solution of the master equation demonstrates, the qubit–field correlations have only a weak effect on the average photon number. On the other hand, as we will see in the following section, the qubit–field correlations have an important effect on the spectral properties of the single-qubit laser.

The right panel of figure 4 shows the qubit–qubit correlations \( C_{\langle Q Q \rangle} \) for a two-qubit-laser. Similar to the qubit–field correlations, they are neglected in the semi-quantum approximation. Both correlations are maximum at the lasing transition, but decay away from this point. The reason is that qubit–field and qubit–qubit correlations scale as \( g^2 / \Gamma_{\phi}^2 \), thus they are small for weak coupling. On the other hand, they are proportional to the qubit inversion \( \langle S_z \rangle \) and hence vanish rapidly above the transition.
To investigate similarly to equations (5.1. Spectral properties in the semi-quantum theory) for single-qubit lasers. The emission spectrum, \( \langle n \rangle \) in the resonator and qubit-field correlations for a single-qubit-laser. The photon number is calculated using the master equation (ME, solid black line), the semi-quantum (SQ, solid orange (light grey) line), and the semiclassical approximation (SC, dotted line). The dot-dashed and dashed lines show the average values \( (\sigma z | n \rangle \) and \( | \sigma z \rangle | n \rangle \). Right panel: average photon number \( \langle n \rangle \) in the resonator and qubit-field correlations \( C_{\sigma z} \) (dashed line) for a two-qubit-laser. The photon number was calculated using the master equation (solid black line), the semi-quantum (solid orange (light grey) line), and the semiclassical approximation (dotted line). The parameters are \( \epsilon = \omega_0, \Gamma_1/\omega_0 = 0.016, \Gamma_2/\omega_0 = 0.004, D_N = 0.975, \kappa/\omega_0 = 3 \times 10^{-4} \) and \( N_q = 0 \).

5. Spectral properties

In this section, we will study the spectral properties of single-qubit lasers. The emission spectrum \( \hat{O}(\omega) \) is given by the Fourier transform of the correlation function \( \langle O(\tau) \rangle = \lim_{\tau \to \infty} \langle a^\dagger(t+\tau)a(t) \rangle \). As we will see, for typical circuit QED parameters, i.e., for strong coupling \( g \), the semi-quantum approximation, in spite of giving a sufficient estimate of the stationary photon number, cannot be used for a quantitative study of spectral functions. We evaluate the correlation function by performing a time-dependent simulation of the master equation (18) using the method described in [28]. This method is numerically demanding, especially when we consider lasing with more than one qubit, \( N_q > 1 \). We will show that at resonance, \( \epsilon = \omega_0 \), the semi-quantum theory catches the most qualitative features, both below and above the transition to the lasing regime. We will use this method later in section 6.1 to investigate the scaling of the spectral properties with the number of qubits \( N_q \).

5.1. Spectral properties in the semi-quantum theory

Similarly to equations (21) for the average values, we can derive equations for the laser and cross correlation functions, \( \hat{O}(\tau) \) and \( G(\tau) = \lim_{\tau \to \infty} \langle \sigma_z(t+\tau)a(t) \rangle \). Assuming that the oscillator damping is much weaker than the qubits’ dephasing, \( \kappa/2 \ll \Gamma_\psi \), which is usually satisfied in single-qubit lasing experiments, we obtain a single equation for the oscillator correlation function: \( \frac{dt}{d\tau} \langle a^\dagger(t+\tau)a(t) \rangle = (i\omega_0 - \kappa/2)\langle a^\dagger(t+\tau)a(t) \rangle + g^2/\Gamma_\psi \langle \sigma_z \rangle \langle a^\dagger(t+\tau)a(t) \rangle \). Thus, the semi-quantum theory predicts an exponential decay of the correlation function \( O(\tau) \), which corresponds to a Lorentzian shape of the emission spectrum,

\[
\hat{O}(\omega) = \frac{2\kappa_4 \langle n \rangle}{(\omega - \omega_0)^2 + \kappa_4^2},
\]

where the width of the spectrum is given by the expression

\[
\kappa_4 = \frac{\kappa N_q}{2 \langle n \rangle} \frac{g^2 N_q ((S_z) + 1)}{\Gamma_\psi 2\langle n \rangle}.
\]

5.2. Numerical investigation of the spectral properties

For the following discussion, we focus on the case of a single qubit, \( N_q = 1 \). In the left panel of figure 5, we plot the diffusion constant \( \kappa_4 \), as a function of the coupling strength \( g \), covering the whole range from below to above the transition, and compare it to the diffusion constant \( \kappa_4^{SC} \), obtained from the semi-quantum theory, i.e. by neglecting the qubit-field correlations \( (\sigma_z | n \rangle - | \sigma_z \rangle | n \rangle \).

Upon approaching the broadened lasing threshold from the weak coupling side, we observe the linewidth narrowing characteristic for the lasing transition. However, above the transition, the linewidth increases again with growing coupling strength, thus deteriorating the lasing state. By comparing the semi-quantum approximation with the full solution of the master equation, we observe that qubit–oscillator correlations have a significant quantitative effect on the phase diffusion, leading to a reduction of the linewidth by roughly a factor of 1/2, but they do not change the qualitative conclusions.

Also in figure 5, we note that in the transition region there is an ‘optimal’ value of the qubit–oscillator coupling where the height of the spectral line, which is given by the ratio \( \langle n \rangle/\kappa_4 \) of the photon number and the linewidth, is maximum. This interesting feature is due to the fact that in single- and few-qubit lasers far above the lasing transition a increase of the coupling has little effect on the saturated photon number, but leads to an increase of the incoherent photon emission rate and the linewidth.

When the qubit and the resonator are not in resonance, \( \Delta \neq 0 \), the emission spectrum is shifted with respect to the natural frequency \( \omega_0 \) of the resonator. This is shown in the right panel of figure 5, where we plot the average photon number, the linewidth and the frequency shift \( \delta \omega_0 \) as functions of the detuning \( \Delta \) in the strong coupling regime.

The numerical results for the linewidth \( \kappa_4 \) shown in this plot differ qualitatively from the results presented in our previous paper [15], where we use a factorization scheme to obtain analytical expressions for the linewidth. Specifically, moving away from the resonance the linewidth...
6. Discussion

6.1. Scaling in the semi-quantum approximation

As discussed in a quantum optics context in [27, 28], various properties of single qubit masers are due to the fact that in these systems only one artificial atom (a microscopic system from a thermodynamical point of view) interacts with the electromagnetic radiation. To clarify the main differences between single qubit masers and conventional (many atom) lasers, we use the semi-quantum approximation to study the scaling of the average photon number and of the phase diffusion with the number of atoms.

In figure 6, we plot the scaled photon number $\langle n \rangle / N_a$ in the transition region and $\kappa_d$ versus the scaled coupling, $g \sqrt{N_a}$, for different values of the number of qubits $N_a$. Plotted in these scaled forms, all curves have the same asymptotic behavior, and the transition occurs at the same position. As expected, in figure 6 (left panel), we observe that for low values of $N_a$, there is a smoothing of the lasing transition, which is due to spontaneous emission processes and disappears in the large $N_a$ limit. In figure 6 (right panel), we show the scaling of the phase diffusion constant. Here the qubits’ relaxation processes are responsible for the increase of the phase diffusion rate in the case of strong qubit–oscillator coupling for small $N_a$.

6.2. Effect of the low-frequency noise

The linewidth of the order of 0.3 MHz observed in [12] is about one order of magnitude larger than what follows from our results (of the order of the Schawlow–Townes linewidth). Moreover, in the experiment, the emission spectrum shows a Gaussian rather than a Lorentzian shape. Both discrepancies can be explained if we note that the qubits’ dephasing is mostly due to low-frequency charge noise, which cannot be treated within the Markov approximation used in the present analysis.

However, low-frequency (quasi-static) noise can be taken into account by averaging the Lorentzian spectral line over different values of the energy splitting $\epsilon$ of the qubit [30], or equivalently, over different values of the detuning $\Delta$. 

is increasing, while the factorization predicted a decrease. On the other hand, the approximation based on the factorization yields results very similar to the numerical ones right on resonance, as well as in the far off-resonant situation and the weak-coupling regime.

Figure 5. Left panel: phase diffusion constant and average photon number (solid line) as a function of the coupling strength $g$ for a single-qubit laser. The phase diffusion constant is calculated using the numerical simulation (dashed line) and the semi-quantum approximation (dotted line). Right panel: phase diffusion constant (dashed line), frequency shift (dotted line) and photon number (solid line) as a function of the detuning for a fixed coupling strength $g/\omega_0 = 0.005$. In both plots, we used $\kappa/\omega_0 = 5 \times 10^{-4}$ for the bare damping rate of the resonator, other parameters as in figure 4.

Figure 6. Average scaled photon number and phase diffusion constant as a function of the scaled coupling $g \sqrt{N_a}$. Other parameters as in figure 4.
between qubit and oscillator. Assuming that these fluctuations are Gaussian distributed, with mean \( \Delta \) and width \( \sigma \), such that \( \Gamma_1 > \sigma \gg \kappa_d \), we can neglect in the saturated limit the dependence of \( \kappa_d \) and \( \langle n \rangle \) on \( \Delta \) and assume that the frequency shift \( \delta\omega \) depends linearly on the detuning \( \Delta \).

In the strong coupling regime, the shift of the emission spectrum is given, in a good approximation, by \( \delta\omega_0 \simeq \Delta \kappa/(2\Gamma_1) \), which leads to a Gaussian line of width \( \tilde{\sigma} \simeq \sigma \kappa/(2\Gamma_1) \), where we remark that \( \Gamma_1 \) is the total Markovian dephasing rate. In this way, the linewidth observed in the experiment can be reproduced by a reasonable choice of \( \sigma \) of the order of 300 MHz. In the case in which \( \sigma \) is larger than \( \Gamma_1 \), the previous formula overestimates the linewidth since it does not take into account the decay of \( \langle n \rangle \) below the lasing transition. In this case, we can still perform the averaging numerically. In either case, we note that in the presence of low-frequency noise, the linewidth is governed not by \( \kappa_d \), but by \( \delta\omega_0 \).

7. Conclusions

We analyzed in detail the static and spectral properties of single- and few-qubit lasers. Our main conclusions are:

- As compared to a conventional laser setup with many atoms, which has a sharp transition to the lasing state at a threshold value of the coupling strength (or inversion), we find for a single- or few-qubit laser a smearing, but still well-defined transition. Similarly, the decrease of the phase diffusion strength when approaching the transition, i.e., the characteristic linewidth narrowing, is less sharp but still pronounced.
- Above the lasing transition we observe for a single- or few-qubit laser a pronounced increase of the phase diffusion strength, which leads to a deterioration of the lasing state and a reduction of the height of the laser spectrum.
- Low-frequency noise strongly affects the linewidth of the lasing peak, leading to an inhomogeneous broadening. In comparison, the natural laser linewidth due to spontaneous emission is negligible.

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Appendix. Comment on two-photon processes

Here we briefly discuss the validity of the Jaynes–Cummings model introduced in section 3, when applied to describe the SSET laser of Astafiev et al [12]. As discussed in section 2.1, the SSET laser, schematically depicted in figure 1, consists of a biased superconducting island capacitively coupled to a single-mode electrical resonator. Under appropriate conditions only two charge states, corresponding to \( N = 0, 2 \) are relevant to the dynamics of the device. On this basis, the Hamiltonian of the oscillator and the qubit can be written as

\[
H = \frac{1}{2} (\epsilon_{ch} \tau_z + E_j \tau_x) + \hbar \omega_0 a^\dagger a - \hbar g_0 \tau_x (a + a^\dagger),
\]

where the operators \( \tau_x \) and \( \tau_z \) are defined as: \( \tau_x = (|N = 2\rangle\langle N = 2| - |N = 0\rangle\langle N = 0|) \) and \( \tau_z = (|N = 2\rangle\langle N = 0| + |N = 0\rangle\langle N = 2|) \). Rotating to the qubit’s eigenbasis \( \{|\uparrow\rangle, |\downarrow\rangle\} \), defined by equations (1), we can recast the Hamiltonian as follows:

\[
H = \frac{1}{2} \epsilon \sigma_z + \hbar \omega_0 a^\dagger a - \hbar g_0 (\cos \xi \sigma_z - \sin \xi \sigma_x) (a + a^\dagger).
\]

(A.2)

The angle \( \xi \) is defined as in section 2.1, \( \tan \xi = E_j/\epsilon_{ch} \), and the qubit energy splitting, \( \epsilon \), depends on the charging and Josephson energies, \( \epsilon_{ch} \) and \( E_j \) . \( \epsilon = \sqrt{\epsilon_{ch}^2 + E_j^2} \). In order to identify the one- and two-photon coupling strength, we now apply a Schrieffer–Wolff transformation \( U = e^{iS} \) with \( S = i(g_0 \cos \xi/\omega_0) \sigma_z (a - a^\dagger) \) and perform a perturbation expansion in the parameter \( g_0/\omega_0 \). The transformed Hamiltonian, \( \tilde{H} = U^\dagger H U \), thus becomes

\[
\tilde{H} \simeq \frac{1}{2} \epsilon \sigma_z + \hbar \omega_0 a^\dagger a + \hbar g_1 \sigma_x (a + a^\dagger) + \hbar g_2 \sigma_x (a^2 - (a^\dagger)^2).
\]

(A.3)

Here we neglected terms of order \( (g_0/\omega_0)^3 \) and introduced the two coupling constants \( g_1 = -g_0 \sin \xi \) and \( g_2 = (2g_0^2/\omega_0^2) \sin \xi \cos \xi \) for one-photon and two-photon transitions, respectively. For the parameters used in the experiment, the coupling \( g_2 \) is roughly two orders of magnitude smaller than the one-photon coupling and below the semiclassical threshold for the two-photon lasing, \( g_2^{th} = \sqrt{\kappa^2/\Gamma_1 D_0^2} \) [31]. In the parameter regime explored in the experiments, the Hamiltonian used in equation (17) gives thus a good description of the dynamics of the system.

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