Induced cycles in triangle graphs *

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Abstract

The triangle graph of a graph $G$, denoted by $\mathcal{T}(G)$, is the graph whose vertices represent the triangles ($K_3$ subgraphs) of $G$, and two vertices of $\mathcal{T}(G)$ are adjacent if and only if the corresponding triangles of $G$ share an edge. In this paper, we characterize graphs $G$ whose triangle graph is a cycle and then extend the result to obtain a characterization of $C_n$-free triangle graphs. As a consequence, we give a forbidden subgraph characterization for graphs $G$ whose $\mathcal{T}(G)$ is a tree, a chordal graph, or a perfect graph. For the class of graphs whose $\mathcal{T}(G)$ is perfect, we also prove a conjecture of the third author concerning the packing and covering of triangles.

Keywords: Triangle graph, $F$-free graph, perfect graph, Tuza’s Conjecture

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1 Introduction

In a simple undirected graph $G$, a triangle is a complete subgraph on three vertices. The triangle graph of $G$, denoted by $T(G)$, is the graph whose vertices represent the triangles of $G$, and two vertices of $T(G)$ are adjacent if and only if the corresponding triangles of $G$ share an edge. This notion was introduced independently several times under different names and in different contexts [15, 21, 7, 4]. One fundamental motivation is its obvious relation to the important class of line graphs.

In a more general setting, for a $k \geq 1$, the $k$-line graph $L_k(G)$ of $G$ is a graph which has vertices corresponding to the $K_k$ subgraphs of $G$, and two vertices are adjacent in $L_k(G)$ if the represented $K_k$ subgraphs of $G$ have $k - 1$ vertices in common. Hence, 2-line graph means line graph in the usual sense, whilst 3-line graph is just the triangle graph, which is our current subject.

Based on the classic result of Beineke [5], 2-line graphs can be recognized in polynomial time. In contrast to this, as proved very recently in [2], the recognition problem of triangle graphs (and also, that of $k$-line graphs for each $k \geq 3$) is NP-complete. In the same manuscript [2], a necessary and sufficient condition is given for nontrivial connected graphs $G$ and $F$ to ensure that their Cartesian product is a triangle graph.

Further related results are achieved in the recently published paper [10] of Laskar, Mulder and Novick. They prove that for an ‘edge-triangulated’ and ‘path-neighborhood’ graph $G$ (that is when the open neighborhood of $v$ induces a non-trivial path for each vertex $v \in V(G)$), $T(G)$ is a tree if and only if $G$ is maximal outerplanar. Also, they raise the characterization problem for a path-neighborhood graph $G$ whose $T(G)$ is a cycle ([10, Problem 3]). As an immediate consequence of our Theorem 3, we will answer this question; moreover we will give a forbidden subgraph characterization for graphs whose triangle graph is a tree.

Triangle graphs were studied from several further aspects, too; see e.g. [3, 4, 7, 11, 12, 13, 16, 17, 18].
1.1 Standard definitions

Given a graph $F$, a graph $G$ is called $F$-free if no induced subgraph of $G$ is isomorphic to $F$. When $\mathcal{F}$ is a set of graphs, $G$ is $\mathcal{F}$-free if it is $F$-free for all $F \in \mathcal{F}$. On the other hand, when we say that an $F$ is a forbidden subgraph for a class $\mathcal{G}$ of graphs, it means that no $G \in \mathcal{G}$ may contain any subgraph isomorphic to $F$.

As usual, the complement of a graph $G$ is denoted by $\overline{G}$. The $n$th power of a graph $G$ is the graph $G^n$ whose vertex set is $V(G^n) = V(G)$ and two vertices are adjacent in $G^n$ if and only if their distance is at most $n$ in $G$. Moreover, given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we use the notation $G_1 \lor G_2$ for the join of $G_1$ and $G_2$, that is a graph with one copy of $G_1$ and $G_2$ each, being vertex-disjoint, and all the vertices of $G_1$ are made adjacent with all the vertices of $G_2$. In particular, the $n$-wheel $W_n$ ($n \geq 3$) is a graph $K_1 \lor C_n$ (where, as usual, $K_n$ and $C_n$ denote the $n$-vertex complete graph and the $n$-cycle, respectively). An odd wheel is a graph $W_n$ where $n \geq 3$ is odd; and an odd hole in a graph is an induced $n$-cycle of odd length $n \geq 5$, whereas an odd anti-hole is the complement of an odd hole.

While an acyclic graph does not contain any cycles, a chordal graph is a graph which does not contain induced $n$-cycles for $n \geq 4$. The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colors required to color the vertices of $G$ in such a way that no two adjacent vertices receive the same color. A set of vertices is independent if all pairs of its vertices are non-adjacent. The independence number $\alpha(G)$ of $G$ is the maximum cardinality of an independent vertex set in $G$. A clique is a complete subgraph maximal under inclusion (i.e., in our terminology different cliques in the same graph may have different size). The clique number $\omega(G)$ is the maximum number of vertices of a clique in $G$. The clique covering number $\theta(G)$ is the minimum cardinality of a set of cliques that covers all vertices of $G$. A graph $G$ is perfect if $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$.

As usual, the open neighborhood $N(v)$ of $v$ is the set of neighbors of $v$, whilst its closed neighborhood is $N[v] = N(v) \cup \{v\}$. In a less usual way, we also refer to the subgraphs induced by them as $N(v)$ and $N[v]$, respectively.

1.2 New definitions and terminology

In this paper, we apply the following special terminology for some types of graphs.

- The elementary types are:
(a) the wheel $W_4$,
(b) the square $C_n^2$ of a cycle of length $n \geq 7$.

• The supplementary types are the following ones. (For illustration, see Fig. [1])

(A) $S_A = (V_A, E_A)$, where $V_A = \{v_i, u_i \mid 1 \leq i \leq 4\}$ and

$$E_A = \{v_i v_{i+1} \mid 1 \leq i \leq 4\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 4\}$$

(subscript addition taken modulo 4).

(B) $S_B = (V_B, E_B)$, where $V_B = \{v_i \mid 1 \leq i \leq 5\} \cup \{u_1, u_2, u_3\}$ and

$$E_B = \{v_i v_{i+1} \mid 1 \leq i \leq 5\} \cup \{v_3 v_5, v_4 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 3\}$$

(subscript addition taken modulo 5).

(C) $S_C = (V_C, E_C)$, where $V_C = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_2\}$ and

$$E_C = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_2 v_4, v_3 v_5, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 2\}$$

(subscript addition taken modulo 6).

(D) $S_D = (V_D, E_D)$, where $V_D = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_4\}$ and

$$E_D = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_1 v_3, v_2 v_4, v_3 v_5, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 4\}$$

(subscript addition taken modulo 6).

We also define two operations as follows.

• Suppose that $e = xy$ is an edge contained in exactly one triangle $xyz$, whilst $xz$ and $zy$ belong to more than one triangle. An edge splitting of $e$ means replacing $e$ with the 3-path $xwy$ (where $w$ is a new vertex) and inserting the further edge $wz$.

• Let $u$ and $v$ be two vertices at distance at least 4 apart. The vertex sticking of $u$ and $v$ means removing $u$ and $v$ and introducing a new vertex $w$ adjacent to the entire $N(u) \cup N(v)$.

The inverses of these operations can also be introduced in a natural way.

\[\text{1. 'Vertex sticking' and its inverse operator were also introduced in [10].}\]
Suppose that $xwz$ and $ywz$ are two triangles in the following position: $w$ has degree 3, $z$ is the unique common neighbor of $w$ and $x$ and also of $w$ and $y$ (in particular, $x$ and $y$ are not adjacent), and $w$ and $z$ are the only common neighbors of $x$ and $y$. The inverse edge splitting at $w$ means deleting $w$ and its three incident edges, and inserting the new edge $xy$.

Let $w$ be a vertex whose neighborhood $N(w)$ is disconnected. The inverse vertex sticking at $w$ means deleting $w$ and its incident edges, and inserting two new vertices $u$ and $v$ in such a way that $N(u) \cup N(v) = N(w)$, $N(u) \cap N(v) = \emptyset$, and each connected component of $N(w)$ is either inside $N(u)$ or inside $N(v)$.

Figure 1: The four graphs of supplementary type
1.3 Our results

In Section 2, we characterize the graphs whose triangle graph is a cycle (Theorem 3) and then conclude a characterization for path-neighborhood graphs \( G \) with \( T(G) \cong C_n \) for some \( n \). The latter one (Corollary 4) solves a problem raised in [10].

In Section 3, we prove a forbidden subgraph characterization for graphs \( G \) with \( C_n \)-free triangle graphs for any specified \( n \geq 3 \) (Theorem 11). Applying this result, we give necessary and sufficient conditions for graphs \( G \) whose \( T(G) \) is a tree, a chordal graph, and a perfect graph, respectively. In a sense these results form a hierarchy since every tree is chordal (obviously), and every chordal graph is perfect (as it is well known).

In the last section, we prove an old conjecture of the third author regarding packing and covering the triangles of a graph, for the class of graphs whose triangle graph is perfect (Theorem 16).

2 Graphs whose triangle graph is a cycle

In this section, we give characterization for graphs whose triangle graph is a cycle. We assume that every edge of \( G = (V, E) \) is contained in a triangle, and there are no isolated vertices. Before stating the theorem, let us prove that the required property is invariant under the two operations introduced in Section 1.2.

**Lemma 1** For a graph \( G \), let \( G' \) be a graph obtained from \( G \) by splitting an edge or sticking two vertices. Then \( T(G') \) is a cycle if and only if so is \( T(G) \).

**Proof.** Let first \( G' \) be the graph obtained from \( G \) by splitting an edge \( e = uv \) to the path \( uwx \), where \( e \) belongs to exactly one triangle \( uwx \). Let \( t_1, t_2, \ldots, t_n \) be the triangles in \( G \). Without loss of generality we may assume that \( t_1 = uwx, t_2 \) contains the edge \( vx \) and \( t_k \) contains the edge \( ux \). Hence, by the edge splitting, two neighboring triangles \( uwx \) and \( wvx \) arose, where \( uwx \) and \( t_k \), moreover \( wvx \) and \( t_2 \) also have a common edge. Then, clearly, \( t_1, t_2, \ldots, t_n \) is an \( n \)-cycle in \( T(G) \) if and only if the triangles \( uwx, wvx, t_2, \ldots, t_n \) of \( G' \) form an \( (n + 1) \)-cycle in \( T(G') \) in this cyclic order. Therefore, \( T(G) \cong C_n \) if and only if \( T(G') \cong C_{n+1} \).

Let now \( G' \) be the graph obtained from \( G \) by sticking two vertices \( u \) and \( v \) to a new vertex \( w \), where the distance between \( u \) and \( v \) is at least four. If a triangle \( t \) of \( G \) contains neither \( u \) nor \( v \), then \( t \) is a triangle in \( G' \), too. If \( t \) is a triangle which contains \( u \), say \( t = wx_1x_2 \), then \( wx_1x_2 \) is a triangle in \( G' \). The same holds for triangles containing \( v \). Since \( u \) and \( v \) are at distance at least 4 apart, no triangle of \( G \) is damaged and no new triangle can arise when \( u \) and \( v \) are stuck. Moreover,
two triangles share an edge in $G$ if and only if the corresponding triangles have a common edge in $G'$. Therefore, $\mathcal{T}(G) \cong \mathcal{T}(G')$. □

Observe that splitting an edge increases the number of triangles by exactly one, whilst sticking two vertices far enough does not change the number of triangles and the number of edges in a graph but increases edge density. These observations have the following simple but important consequence.

**Corollary 2 (Finite Reduction Lemma)** For every fixed $n$ there is a finite $s_n$ such that, starting from any graph whose triangle graph is a cycle, after $s_n$ applications of edge splitting and vertex sticking in any feasible order, the length of the cycle $\mathcal{T}(G)$ of the graph $G$ obtained surely exceeds $n$. Equivalently, starting from any $G$ whose triangle graph is a cycle, inverse edge splitting and inverse vertex sticking can be applied only finitely many times.

If an edge $e$ belongs to exactly one triangle, we call $e$ a private edge (of that triangle); and if $e$ is contained in exactly two triangles, it is a doubly covered edge.

**Theorem 3** Let $G$ be a graph which contains no isolated vertices and whose every edge is contained in at least one triangle. Then, $\mathcal{T}(G) \cong C_n$ for some $n \geq 3$ if and only if

(i) $G \cong K_5 - K_3$, or

(ii) $G$ is one of the elementary types or supplementary types, or

(iii) $G$ can be obtained from one of the elementary types or supplementary types by a sequence of edge splittings and vertex stickings.

Moreover, graphs whose triangle graph is an odd hole are characterized by (ii) and (iii), with properly chosen parity of the number of edge splittings.

**Proof.** Clearly, $\mathcal{T}(K_5 - K_3) \cong C_3$, $\mathcal{T}(W_4) \cong C_4$, and $\mathcal{T}(C_n^2) \cong C_n$ if $n \geq 7$, moreover the triangle graphs of $S_A$, $S_B$, $S_C$ and $S_D$ are isomorphic to $C_8$. By Lemma 1 also the triangle graphs of graphs satisfying (iii) are cycles.

Now, assume that for a graph $G = (V, E)$, fulfilling the conditions of the theorem, its triangle graph $\mathcal{T}(G)$ is a cycle $C_n$. If an edge $e \in E$ is contained in more than two triangles, then those triangles induce a complete subgraph of order at least 3 in the triangle graph. Hence, if the latter is a cycle, then there cannot be more than three triangles, thus $\mathcal{T}(G) = C_3$ and $G \cong K_5 - K_3$. 

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On the other hand, if a triangle has no private edge then the degree of the corresponding vertex in \(T(G)\) will be at least three, which contradicts the assumption that \(T(G)\) is a cycle.

From now on, assume that \(G \not\cong K_5 - K_3\) and consequently, each triangle has precisely one private edge and exactly two doubly covered edges. Moreover, we will suppose that the inverse operations of edge splitting and vertex sticking cannot be applied to \(G\). (Due to the Finite Reduction Lemma above, we may assume this without loss of generality.) It will suffice to prove that such non-reducible graphs necessarily belong to an elementary or a supplementary type.

**Claim 1.** For every vertex \(v \in V\) the neighborhood \(N(v)\) is a path or a cycle; and in the latter case we have \(G \cong W_4\).

**Proof.** First, assume for a contradiction that \(v\) is a vertex whose \(N(v)\) is not connected. Let \(N_1\) be a component of \(N(v)\), and set \(N_2 = N(v) - N_1\). Let \(G'\) be the graph obtained from \(G\) by deleting \(v\) and introducing two new vertices \(v_1\) and \(v_2\) adjacent to the vertices in \(N_1\) and \(N_2\), respectively. In \(G'\) the distance between \(v_1\) and \(v_2\) is at least 4. Therefore, the inverse operation of vertex splitting can be applied to \(G\), which is a contradiction. Consequently, each \(N(v)\) induces a connected graph.

If a vertex \(u \in N(v)\) had more than two neighbors in \(N(v)\), then the edge \(vu\) would belong to more than two triangles, which contradicts our present assumption. Hence, for every \(v \in V\) and every \(u \in N(v)\), the degree \(\deg_{N(v)}(u)\) is at most two. Since \(N(v)\) is connected, this implies that \(N(v)\) must be a path or a cycle. If \(N(v) \cong C_n\), then \(N[v] \cong W_n\) and \(T(W_n) \cong C_n\). Therefore, \(G \cong W_n\), and since it is assumed that \(G\) cannot be reduced by inverse edge splitting, \(G \cong W_4\) must be valid.

Therefore, consider only the case where \(G \not\cong W_4\) and \(N(v)\) is a path for every \(v \in V\). Partition the edge set of \(G\) as \(E = F \cup H\) where \(F\) is the set of doubly covered edges and \(H\) is the set of private edges of triangles.

If \(t_1, \ldots, t_n\) are the triangles in the assumed cyclic order, then we use the notation \(F = \{f_1, \ldots, f_n\}\) where \(f_i = E(t_i) \cap E(t_{i+1})\), and denote by \(h_i\) the private edge of \(t_i\). Hence \(E(t_i) = \{f_{i-1}, f_i, h_i\}\) (subscript addition is considered modulo \(n\)). The graphs \(G_F = (V, F)\) and \(G_H = (V, H)\) contain no triangles. It is worth noting that the original graph \(G\) can be obtained from \(G_F\) if, for every \(1 \leq i \leq n\), the non-common ends of \(f_i\) and \(f_{i+1}\) are connected by an edge (which is just the private edge \(h_{i+1}\) in \(G\)).

**Claim 2.** The graph \(G_F\) of doubly covered edges is a hairy cycle (a cycle with any number of pendant vertices attached to its vertices).
Proof. Let every vertex \( v \in V \) be associated with the set \( I(v) \) of indices of doubly covered edges incident to \( v \):

\[
i \in I(v) \iff f_i \text{ is incident to } v.
\]

By definition, every index \( 1 \leq i \leq n \) is contained in exactly two sets \( I(v) \). Since \( F \) contains exactly two edges from each triangle, every vertex of \( G \) is incident with at least one edge of \( F \) and hence, no \( I(v) \) is empty. The fact that \( f_i \) and \( f_{i+1} \) share a vertex for every \( 1 \leq i \leq n \), implies that for every \( i \) there exists a vertex \( v \) such that \( \{i, i+1\} \subseteq I(v) \). The connectivity of \( G_F \) also follows.

Now, consider a vertex \( v \in V \), say of degree \( d \). By Claim 1, its neighborhood \( N(v) \) induces a path \( P_d = u_1u_2\ldots u_d \) in \( G \). Any two consecutive vertices of \( P_d \) together with \( v \) form a triangle. Hence, the doubly covered edges incident to \( v \) are exactly \( vu_2, vu_3, \ldots, vu_{d-1} \), and the \( d-1 \) triangles just mentioned correspond to consecutive vertices in the cycle \( T(G) \cong C_n \). It also follows that the index set \( I(v) \) of \( v \) contains \( d-2 \) consecutive integers (viewing 1 as the successor of \( n \)).

By these facts we obtain that the set of vertices which have at least two incident doubly covered edges induce a cycle in \( G_F \). This cycle will be referred to as \( C_k = v_1v_2\ldots v_k \), where the vertices are indexed according to the cyclic order. It contains exactly those vertices \( v_i \) for which \( |I(v_i)| \geq 2 \) and the two edges of \( C_k \) incident to \( v_i \) are exactly the doubly covered edges with smallest and largest indices from \( I(v) \) (where ‘smallest’ and ‘largest’ are meant along a fixed cyclic order of \( 1, 2, \ldots, n \)).

If a vertex \( v_i \) from \( C_k \) is incident to an edge \( f_t = v_iu \) which does not belong to \( C_k \), then \( u \) cannot be incident to any other edge of \( G_F \), since both \( f_{t-1} \) and \( f_{t+1} \) are also incident to \( v_i \), and vertex \( u \), too, must be incident to edges with consecutive indices without a gap. Thus, all vertices and edges not contained in the cycle \( C_k \) are pendant vertices and edges in \( G_F \). Consequently, \( G_F \) is a hairy cycle.

From now on, when the inverse operation of edge splitting is applied in \( G \) to a vertex \( w \) which is pendant in \( G_F \), we say that \( w \) is eliminated. Let us emphasize that we excluded this situation by assumption at the very beginning; hence, several proofs below will apply the fact that it is impossible to identify a vertex which can be eliminated.

Claim 3. In \( G_F \) each vertex is incident with at most one pendant edge.

Proof. Assume for a contradiction that a vertex \( v_i \) has at least two pendant neighbors. Then, there exist two pendant neighbors \( u \) and \( w \) such that \( f_j = v_iu \) and \( f_{j+1} = v_iw \) for some \( j \). Also, \( f_{j-1} = v_iv' \) and \( f_{j+2} = v_iv'' \) have to be
incident to \( v_i \) (otherwise \( u \) or \( w \) would be a vertex from the cycle). Under these assumptions vertex \( w \) could be eliminated, since \( v''w \) and \( uw \) are private edges, moreover the only possible common neighbor, \( x \neq v_i \), of \( v'' \) and \( u \) in \( G \) might be \( v' \), but since \( G \not\cong W_4 \), it cannot be. This contradiction proves the claim. 

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Claim 4. If there exists a vertex \( v_i \) incident to a pendant edge \( f_j = v_iu \) in \( G_F \) then the cycle \( C_k \) of \( G_F \) is of length at most 6.

Proof. The graph \( G_F \) contains no triangle, hence \( v_{i-1}v_{i+1} \notin F \). Moreover, since \( f_{j-1} = v_{i-1}v_i \) and \( f_{j+1} = v_iv_{i+1} \) are not consecutive doubly covered edges, \( v_{i-1}v_{i+1} \) cannot be a private edge of \( G \) and so, \( v_{i-1}v_{i+1} \notin E \). If the only common neighbor of \( v_{i-1} \) and \( v_{i+1} \) were \( v_i \), then \( u \) could be eliminated and replaced by the edge \( v_{i-1}v_{i+1} \). As \( u \) cannot be eliminated, vertices \( v_{i-1} \) and \( v_{i+1} \) have some common neighbor \( x \neq v_i \) in \( G \). Now, assume that the cycle of doubly covered edges is of length \( k > 6 \). Then, every vertex of \( C_k \) is at distance greater than two apart from at least one of \( v_{i-1} \) and \( v_{i+1} \) in \( G_F \), and the same is true for the possible pendant vertices of \( G_F \). Thus, no doubly covered edges and no private edges form a second triangle with \( v_{i-1}v_{i+1} \) in \( G \), and \( u \) can be eliminated if \( k > 6 \), which contradicts the present conditions. Thus, \( k \leq 6 \) follows. 

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If there exist no pendant edges in \( G_F \) and \( k \geq 7 \), then \( G \) belongs to the elementary type \((b)\). Assume that this is not the case and \( k \leq 6 \). Since \( G_F \) contains no triangle, the length of \( C_k \) is either 4 or 5 or 6.

Claim 5. If \( k = 4 \) then \( G \cong S_A \).

Proof. No chord of the four-cycle belongs to the graph \( G \), since it would be a doubly covered edge and \( G_F \) cannot contain triangles. Avoiding such a case, \( G_F \) has to contain precisely one pendant edge on each vertex of \( C_4 \). Supplementing \( G_F \) with the private edges between the non-common ends of consecutive double edges, the graph \( G \cong S_A \) is obtained. 

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Claim 6. If \( k = 5 \) then \( G \cong S_B \).

Proof. Let the five-cycle be \( v_1v_2v_3v_4v_5 \). If one of these vertices, say \( v_i \) has no pendant edge in \( G_F \), then \( v_{i-1}v_{i+1} \in H \). This edge cannot belong to any other triangle in \( G \); hence, neither \( v_{i-2}v_{i+1} \) nor \( v_{i-1}v_{i+2} \) can be an edge in \( G \). This means that both \( v_{i+2} \) and \( v_{i-2} \) must have pendant edges in \( G_F \). Similarly, if there is no pendant edge on \( v_{i+1} \), then there is one on vertex \( v_{i-1} \). This proves that if \( G \) is complying with our assumption, there exist three consecutive vertices on the five-cycle of \( G_F \) such that each of them has a pendant edge.
On the other hand, if four consecutive vertices, say \( v_i, v_{i+1}, v_{i+2} \) and \( v_{i+3} \) have pendant edges, the pendant vertex adjacent to \( v_i \) can be eliminated. Consequently, if \( G_F \) has a cycle of length 5, then exactly three of its consecutive vertices have pendant edges, and \( G \cong S_B \) holds.

**Claim 7.** If \( k = 6 \) then \( G \cong S_C \) or \( G \cong S_D \).

**Proof.** Consider a six-cycle \( v_1v_2v_3v_4v_5v_6 \). If none of the vertices \( v_1, v_3 \) and \( v_5 \) (or \( v_2, v_4 \) and \( v_6 \)) has a pendant edge, then a forbidden triangle \( v_2v_4v_6 \) \( (v_1v_3v_5) \) would arise in \( H \). Hence, two consecutive or two opposite vertices of the six-cycle surely have pendant edges. On the other hand, if two of the vertices \( v_1, v_3 \) and \( v_5 \) (or two of \( v_2, v_4 \) and \( v_6 \)) have pendant edges, then one of the corresponding pendant vertices can be eliminated. Hence, under the given conditions, either two consecutive or two opposite vertices have pendant edges and \( G \cong S_C \) or \( G \cong S_D \) is obtained, respectively.

The above cases cover all possibilities, therefore the theorem follows. \( \Box \)

Note that \( T(G) \) is an odd hole if and only if

- \( G \) can be obtained from \( W_4 \), or from one of the four supplementary types, or from \( C_n^2 \) where \( n \geq 8 \) is even, by a sequence of an odd number of edge splittings and an arbitrarily number of vertex stickings; or

- \( G \) can be obtained from \( C_n^2 \) where \( n \geq 7 \) is odd, by a sequence of an even number of edge splittings and an arbitrarily number of vertex stickings.

As a vertex sticking clearly creates a graph which is not path-neighborhood and then, applying any further vertex stickings or edge splittings, this property does not change, we infer the following characterization for path-neighborhood graphs, which solves an open problem raised in [10].

**Corollary 4** Let \( G \) be a graph and assume that for every vertex \( v \in V(G) \) the open neighborhood \( N(v) \) induces a path on at least two vertices. Then, \( T(G) \cong C_n \) for some \( n \geq 3 \) if and only if

(i) \( G \cong C_k^2 \) for a \( k \geq 7 \); or

(ii) \( G \cong S_A \) or \( S_B \) or \( S_C \) or \( S_D \); or

(iii) \( G \) can be obtained by a sequence of edge splittings from a graph which satisfies (i) or (ii).
Remark 5  It is worth investigating the status of graphs $C_n^2$ for $n \leq 6$.

(i) The square $C_4^2$ of the four-cycle is $K_4$, and also its triangle graph is $K_4$. But if we double the two diagonals added to $C_4$ for $C_4^2$, and apply four edge splittings on these two pairs of parallel edges, we obtain $S_A$ from the supplementary type (A). In fact, this does not correspond precisely to the definition of edge splitting, but we can view the case as if one of the edges $v_1v_3$ belonged to triangle $v_1v_2v_3$, and its ‘twin’ edge to $v_1v_4v_3$. The two edges $v_2v_4$ can be treated similarly.

(ii) For the five-cycle $C_5 = v_1 \ldots v_5$, the square graph $C_5^2$ is a complete $K_5$, and its triangle graph is the complement of the Petersen graph, hence not at all a cycle. But if we consider only the five triangles of the form $v_iv_{i+1}v_{i+2}$ (where $1 \leq i \leq 5$), and the remaining triangles of the form $v_iv_{i+1}v_{i+3}$ are “damaged” by properly chosen edge splittings, we obtain a graph $G$ whose triangle graph is a cycle. If such a $G$ cannot be reduced by the inverse of edge splitting, then it is isomorphic to $S_B$.

(iii) A similar observation can be made for $n = 6$, where at least two edge splittings have to be taken in $C_6^2$ to achieve that the triangle graph becomes a cycle. In the minimal (non-reducible) configurations exactly two edge splittings are needed, either neighboring or opposite. This yields a graph isomorphic either to $S_C$ or to $S_D$.

3 Forbidden subgraph characterizations

First we prove the main result of this section, which is a forbidden subgraph characterization for graphs $G$ whose triangle graph does not contain an induced cycle of a given length $n \geq 3$. This problem is in close connection with the problem we solved in Theorem 3. Clearly, if the triangle graph $T(G)$ is $C_n$-free, then $G$ contains no subgraph $H$ with $T(H) \cong C_n$. But excluding these subgraphs is not sufficient for making $T(G)$ to be $C_n$-free. For instance none of the subgraphs of $C_6^2$ has a triangle graph isomorphic to the 6-cycle, but $T(C_6^2)$ contains an induced $C_6$; moreover, no subgraph of $K_4$ has a triangle graph isomorphic to a cycle, but $T(K_4) \cong K_4$ contains 3-cycles.

In the second subsection we establish some immediate consequences of the main result. This contains a forbidden subgraph characterization for graphs whose triangle graph is a tree, a chordal graph, or a perfect graph. Further, in the third subsection we determine a graph class on which the longstanding
conjecture of the third author concerning the maximum packing and minimum covering of triangles of a graph holds.

Let us recall that $C_n$-free triangle graph means that no induced subgraph of $\mathcal{T}(G)$ is isomorphic to $C_n$; whilst forbidden subgraphs given for $G$ are meant that they must not occur as non-induced subgraphs either.

3.1 Graphs with $C_n$-free triangle graphs

In this subsection “minimal forbidden graph for $C_n$” is meant as a graph $G$ whose triangle graph contains an induced cycle $C_n$ but this does not hold for any proper subgraph of $G$. If $G$ is a minimal forbidden graph for $C_n$, the triangles $T_1, \ldots, T_n$ in $G$, which correspond to the vertices $t_1, \ldots, t_n$ inducing a specified cycle in $\mathcal{T}(G)$, are called cycle-triangles, while the other triangles of $G$ are called additional triangles. By definition, in a minimal forbidden graph, every edge is contained in at least one cycle-triangle. The edge $e \in E(G)$ will be called private edge if it belongs to exactly one cycle-triangle, and those contained in exactly two cycle-triangles will be called doubly covered. That is, when an edge is declared to be private or doubly covered, additional triangles are disregarded.

**Proposition 6** Let $n \geq 3$ be an integer. Assume that $G$ is a minimal forbidden graph for $C_n$. Then, either $G$ is isomorphic to $K_5 - K_3$, or else no edge of $G$ is covered by more than two cycle-triangles and furthermore $G$ contains exactly $2n$ edges, from which $n$ edges are private and $n$ edges are doubly covered.

**Proof.** Three cycle-triangles sharing a common edge correspond to a 3-cycle in the triangle graph. Hence, if an edge of $G$ is covered by three cycle-triangles, then $n = 3$. Moreover, by the minimality assumption, $G$ has no additional triangles and $G \cong K_5 - K_3$ must hold.

Now, assume that $G \not\cong K_5 - K_3$ and $G$ is minimal for $C_n$. Fixing an induced $n$-cycle in $\mathcal{T}(G)$, each cycle-triangle in $G$ has exactly two neighboring cycle-triangles, and hence exactly two doubly covered edges. By minimality, each edge of $G$ belongs to at least one cycle-triangle. Therefore, $E(G)$ consists of exactly $n$ private and $n$ doubly covered edges. \(\square\)

**Corollary 7** $G$ is a minimal forbidden graph for $C_n$ for an $n \geq 4$ if and only if $\mathcal{T}(G)$ contains an induced $n$-cycle and $G$ has exactly $2n$ edges.

**Proof.** If $\mathcal{T}(G)$ contains an induced $n$-cycle, then either $G$ or some proper subgraph of it must be minimal forbidden for $C_n$. Since each proper subgraph
has fewer than $2n$ edges, Proposition 6 implies that $G$ itself is a minimal forbidden graph. The other direction follows immediately from Proposition 6.

\[ \square \]

**Corollary 8** There exists no graph which is minimal forbidden for both $C_n$ and $C_m$ if $n \neq m$.

The operators edge splitting and vertex sticking will be meant as introduced in the previous section, but the conditions of their applicability are modified as described next. Recall that throughout this section the position of an $n$-cycle in $T(G)$ is assumed to be specified, in order to distinguish between cycle-triangles and additional triangles.

- **Weak edge splitting** can be applied for any private edge $e$. If this edge $e = uv$ belongs to the cycle-triangle $ux$, we introduce a new vertex $w$ and change the edge set from $E$ to $E \setminus \{uv\} \cup \{uw, wu, wx\}$. This damages the additional triangles (if any) incident with $e$. The two edges $uw$ and $wv$ originated from $e$ will be two incident private edges in the graph obtained. **Inverse weak edge splitting** can be applied for any two incident private edges $uw$ and $wv$ subject to that $uv$ is not an edge in the graph. This inverse operation may create new additional triangles. If no additional triangles are incident with $e$, a weak edge splitting applied to $e$ is also called **strong edge splitting**. The **inverse strong edge splitting** can be applied for the private edges $uw$ and $wv$ if there is no edge $uv$ and if $w$ is the only common neighbor of $u$ and $v$. The inverse strong edge splitting creates no new additional triangles.

- **Weak vertex sticking** can be applied for any two vertices at distance at least 3 apart. If this distance is at least 4, the triangle graph remains unchanged and the operation is also a **strong vertex sticking**, and corresponds to ‘vertex sticking’ introduced in Section 1.2. A weak vertex sticking, when applied for vertices at distance 3, creates some new additional triangle, but a strong vertex sticking cannot cause change in the triangle graph. For the inverse operators, **inverse strong vertex sticking** can be applied for a vertex $v$ if its neighborhood is disconnected, whilst **inverse weak vertex sticking** can be applied if the neighborhood of $v$ restricted to the cycle-triangles is disconnected. The former inverse transformation always keeps the triangle graph the same, the latter one can damage some of the additional triangles.
Proposition 9

(i) If \( G' \) is obtained from \( G \) by a weak edge splitting (or, equivalently, if \( G \) is obtained from \( G' \) by an inverse weak edge splitting), then \( G' \) is a minimal forbidden graph for \( C_{n+1} \) if and only if \( G \) is minimal forbidden for \( C_n \).

(ii) If \( G'' \) is obtained from \( G \) by a weak vertex sticking (or, equivalently, if \( G \) is obtained from \( G'' \) by an inverse weak vertex sticking), then \( G'' \) is a minimal forbidden graph for \( C_n \) if and only if \( G \) is minimal forbidden for \( C_n \).

Proof. (i) It is clear from the definition of weak edge splitting that the fixed \( n \)-cycle of \( T(G) \) is transformed into an induced \((n+1)\)-cycle of \( T(G') \) and vice versa. Moreover, \( |E(G)| = 2n \) if and only if \( |E(G')| = 2n + 2 \). Hence, statement follows by Proposition 7.

(ii) If \( G'' \) is obtained from \( G \) by a weak vertex splitting, \( T(G) \) contains an induced \( n \)-cycle if and only if \( T(G'') \) contains an induced \( n \)-cycle. Additionally, \( |E(G)| = 2n \) holds if and only if \( |E(G'')| = 2n \). Similarly to the previous case, Proposition 7 implies the statement. \( \square \)

Concerning the order in which these transformations can be applied, we prove the following property. (Although the third part could also be made more detailed, by performing strong vertex stickings before non-strong ones, we do not need this fact in the current context.)

Proposition 10 Assume that graph \( F \) can be obtained from \( G \) by a sequence of weak edge splittings and weak vertex stickings. Then, \( F \) can also be obtained from \( G \) by performing the operators in the following order:

1. some (maybe zero) weak but not strong edge splittings,
2. some (maybe zero) strong edge splittings,
3. some (maybe zero) weak vertex stickings.

Consequently, \( G \) can be obtained from \( F \) in the reverse order of the corresponding inverse operators.

Proof. First, observe that if the \( i \)th transformation \( O_i \) is a weak sticking of \( x \) and \( y \), moreover the \( (i+1) \)st transformation \( O_{i+1} \) is the weak splitting of the edge \( e \), then they can also be applied in the order \( O_{i+1}, O_i \). Indeed, a vertex sticking does not create a new private edge, hence \( O_{i+1} \) can be performed before \( O_i \). On the
other hand, an edge splitting cannot decrease the distance of \( x \) and \( y \) and cannot create a new edge between two vertices which were present previously. Hence, the order \( O_{i+1}, O_i \) is feasible and gives the same result as \( O_i, O_{i+1} \). Therefore, we can re-order the transformations in such a way that all edge splittings precede all vertex stickings.

A sequence of edge splittings can be unambiguously described by assigning a nonnegative integer \( s(e) \) to each edge \( e \in E(G) \), where \( s(e) \) is the number of edge splittings applied to \( e \) and to the edges originated from \( e \). Equivalently, this is the number of subdivision vertices we have on \( e \) at the end. This also shows that edge splittings can be performed in any order. If we want to start with weak but not strong edge splittings, we just take an edge with \( s(e) \geq 1 \) which is incident with an additional triangle and apply an edge splitting as long as such an edge exists. □

**Theorem 11.** Let \( n \geq 3 \) be a given integer. The triangle graph \( T(G) \) of a graph \( G \) does not contain an induced cycle of length \( n \) if and only if \( G \) has no subgraph which is isomorphic to any of the following forbidden ones.

(a) If \( n = 3 \), the forbidden subgraphs are \( K_4 \) and \( K_5 - K_3 \).

(b) If \( n = 4 \), the only forbidden subgraph is \( W_4 \).

(c) If \( n = 5 \), the forbidden subgraphs are \( W_5 \) and \( C_5^2 \cong K_5 \).

(d) If \( n = 6 \), the forbidden subgraphs are \( W_6, C_6^2, K_6 - K_3, K_6 - P_4 \), and the graph obtained from \( C_5^2 \cong K_5 \) by a weak edge splitting.

(e) If \( n \geq 7 \), the forbidden subgraphs are

(i) \( W_n \);

(ii) graphs obtained from \( C_m^2 \) by \( n - m \) weak edge splittings \((5 \leq m \leq n)\);

(iii) graphs obtained from \( K_6 - K_3 \) by \( n - 6 \) weak edge splittings;

(iv) graphs obtained from \( K_6 - P_4 \) by \( n - 6 \) weak edge splittings;

(v) graphs obtained from any graphs described in (ii)–(iv) by any number of weak vertex stickings.

**Proof.** First, observe that three triangles, any two of which share an edge, are either three triangles having a fixed common edge, or they belong to a common \( K_4 \) subgraph. Hence, a \( C_3 \) in \( T(G) \) originates either from a \( K_5 - K_3 \) or from a \( K_4 \). Consequently, by minimality, if \( n = 3 \) then either \( G \cong K_5 - K_3 \) or \( G \cong K_4 \) holds.
From now on, we assume that cycle-triangles do not form a $K_5 - K_3$ subgraph and consider a graph $G$ which is minimal forbidden for an $n \geq 4$. By Proposition 6, there are exactly $n$ private edges and $n$ doubly covered edges in $G$.

By Proposition 9, weak edge splittings, vertex stickings and their inverse operators do not change the status of a graph being minimal forbidden for at least one cycle $C_n$. Hence, we assume further that $G$ is a minimal forbidden graph for which inverse weak vertex sticking and inverse weak edge splitting cannot be applied.

We have the following two cases concerning the additional triangles of $G$.

**Case 1.** Each additional triangle contains at least one private edge (or there is no additional triangle).

Then, applying a weak edge splitting for at least one private edge from each additional triangle, all the additional triangles are destroyed and we have a graph $G'$ with $T(G') \cong C_n$. This $G'$ is also a minimal forbidden graph for $C_n$. Next, while it is possible, we apply inverse strong vertex sticking and inverse strong edge splitting for $G'$, and then we obtain $G''$. Due to Theorem 3, $G''$ is isomorphic either to $W_4$, or to $C_n^2$ with an $n \geq 7$, or to one of the supplementary types $S_A$, $S_B$, $S_C$, $S_D$. By Proposition 10, $G$ can be reconstructed from $G''$ by applying inverse weak edge splittings while it is possible.

Checking all items from our list, we can observe the following.

- $W_4$ yields $K_4 \cong W_3$.
- For $C_n^2$ (with $n \geq 7$) inverse weak edge splitting cannot be applied, as any two neighboring triangles have disjoint private edges.
- In $S_A$ we can choose from four possible inverse weak edge splittings at the first step. But after one is performed, only two further possibilities remain. In any case, finally we obtain $K_6 - P_4$.
- For $S_B$ three inverse weak edge splittings can be applied. They can be performed in any order, the result will be $C_5^2 \cong K_5$.
- For $S_C$ and $S_D$ we can apply inverse weak edge splitting twice. After performing them we have a $C_6^2$.

As we assumed, no inverse weak edge splittings and vertex stickings can be applied for $G$. Hence, in this case, $G$ must be isomorphic either to $K_4$, or to $C_n^2$ with $n \geq 5$, or to $K_6 - P_4$. 

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Case 2. There is at least one additional triangle $uvw$ in $G$ such that each of the edges $uv$, $vw$, $uw$ is doubly covered.

By our present assumption, no inverse weak vertex sticking can be applied for $G$. Hence, every vertex $x$ has a connected neighborhood even if only the cycle-triangles incident with $x$ are considered. Since every triangle $T_i$ has exactly one private edge and the two doubly covered edges correspond to $T_i - 1 \cap T_i$ and $T_i \cap T_{i+1}$, the triangles incident with $x$ are consecutive triangles along the triangle cycle. Let us refer to this property as ‘continuity’.

Since $uv$, $vw$ and $uw$ are doubly covered, the incident triangles can be given with their vertex sets in the form $T_1 = \{uva_1\}$, $T_2 = \{uva_2\}$, $T_i = \{vwb_1\}$, $T_{i+1} = \{vwb_2\}$, $T_j = \{wuc_1\}$ and $T_{j+1} = \{wuc_2\}$, where these six triangles are not assumed to be consecutive, but they are given in a cyclic order.

If $a_2 = b_1$, $b_2 = c_1$ and $c_2 = a_1$, we get the desired result $G \cong K_6 - K_3$. Now, suppose that $a_2 \neq b_1$, which is equivalent to $i \neq 3$. Since $T_2$ and $T_i$ are incident with vertex $v$ (but $T_j$ is not), the continuity of triangles at $v$ implies that $T_3$ has vertices $va_2x$, moreover $a_2x$ must be the private edge. Now, $ua_2$ and $ax$ are two incident private edges of consecutive triangles. The inverse weak vertex sticking could not be applied for them only if $ux$ is an edge in $G$. Then, the triangle $T_i$ having this edge $ux$ belongs to triangles incident with $u$. Since $T_1$, $T_2$ are incident with $u$ but $T_3$, $T_i$, $T_{i+1}$ are not, $i + 2 \leq \ell$ follows. But in this case, the triangles incident with $x$ cannot satisfy the continuity (since $x$ belongs to $T_3$, might belong to $T_i$, but it is surely not contained in $T_{i+1}$ and $T_1$). This contradiction proves that $a_2 = b_1$ and similarly, $b_2 = c_1$ and $c_2 = a_1$ must be valid, as well. Thus, $G \cong K_6 - K_3$ holds.

These cases cover all possibilities, therefore the statement is proved. □

3.2 Trees, chordal graphs and perfect graphs

Lemma 12 $T(G)$ is connected if and only if there does not exist a partition of $E(G)$ into two sets $A$ and $B$ such that each of $A$ and $B$ contains at least three edges which induce a triangle and each triangle in $G$ is either in $A$ or in $B$.

Proof. Let $A \cup B = E(G)$ be an edge partition such that each triangle of $G$ is contained in either $A$ or $B$. Then in $T(G)$ there cannot be any edges from the vertices representing the triangles inside $A$ to those representing the triangles inside $B$. Thus, if there exist two triangles $T_A \subset A$ and $T_B \subset B$ in $G$, then $T(G)$ has at least two components.

Conversely, assume that $T(G)$ is disconnected. Let $A$ be the collection of all edges of $G$ corresponding to the triangles in one of the components of $T(G)$, and
let \( B = E(G) - A \). This \( B \) also contains at least one triangle, since \( T(G) \) is disconnected. Now, \( \{A, B\} \) is a partition of \( E(G) \) such that each of \( A \) and \( B \) contains at least one triangle and each triangle is either in \( A \) or in \( B \).

Let us say that graph \( G \) is triangle-connected if \( T(G) \) is connected.

Now, the characterization of graphs whose triangle graph is a tree or a chordal graph follows immediately from Theorem 11.

**Corollary 13** For a graph \( G \), its triangle graph \( T(G) \) is a tree if and only if \( G \) is triangle-connected and does not contain a subgraph which is isomorphic to one of the following graphs.

1. \( W_n \), for \( n \geq 3 \);
2. \( K_5 - K_3 \);
3. \( C_n^2 \), for \( n \geq 5 \);
4. \( K_6 - K_3 \);
5. \( K_6 - P_4 \);
6. graphs obtained from any of the graphs described in (c) – (e) by any number of weak edge splittings and weak vertex stickings.

**Corollary 14** For a graph \( G \), its triangle graph \( T(G) \) is chordal if and only if \( G \) does not contain a subgraph which is isomorphic to any of the following graphs:

1. \( W_n \), for \( n \geq 4 \);
2. \( C_n^2 \), for \( n \geq 5 \);
3. \( K_6 - K_3 \);
4. \( K_6 - P_4 \);
5. graphs obtained from any graphs described in (b) – (d) by any number of weak edge splittings and weak vertex stickings.

Imposing parity conditions, we also obtain a characterization of graphs whose triangle graph is perfect.

**Theorem 15** For a graph \( G \), its triangle graph \( T(G) \) is perfect if and only if \( G \) does not contain any subgraph which is isomorphic to one of the following graphs:
(a) $W_n$, for an odd integer $n \geq 5$;
(b) graphs obtained from $C_n^2$ by an even number of weak edge splittings for an odd $n \geq 5$;
(c) graphs obtained from $C_n^2$ by an odd number of weak edge splittings for an even $n \geq 6$;
(d) graphs obtained from $K_6 - K_3$ by an odd number of weak edge splittings;
(e) graphs obtained from $K_6 - P_4$ by an odd number of weak edge splittings;
(f) graphs obtained from the graphs described in (b) – (e) by any number of weak vertex stickings.

**Proof.** Since $K_2 \vee P_3$ is forbidden for triangle graphs [12] and $C_n$ contains it as an induced subgraph for all $n \geq 7$, we have that $\mathcal{T}(G)$ is $C_n$-free for $n \geq 7$. Also, $C_5 = C_5$. Therefore, by the Strong Perfect Graph Theorem [6], $\mathcal{T}(G)$ is perfect if and only if has no induced odd hole. Moreover, $\mathcal{T}(G)$ contains an induced odd hole if and only if $G$ has a subgraph from the types described in (a) – (f). This completes the proof. □

### 3.3 Consequences for triangle packing and covering

A family $\mathcal{F}$ of triangles in $G = (V, E)$ is called *independent* if the members of $\mathcal{F}$ are pairwise edge-disjoint. An edge set $E' \subseteq E$ is a $\mathcal{T}$-transversal if every triangle of $G$ contains at least one edge from $E'$. We denote by $\nu_\Delta(G)$ the maximum cardinality of an independent family of triangles in $G$, and by $\tau_\Delta(G)$ the minimum cardinality of a $\mathcal{T}$-transversal in $G$. In 1981 the third author stated the following conjecture.

**Conjecture [19]:** If a graph $G = (V, E)$ does not contain more than $t$ mutually edge-disjoint triangles, then there exists an $E' \subseteq E$ such that $|E'| \leq 2t$ and each triangle of $G$ has at least one edge in $E'$.

The conjecture is proved only for few classes of graphs, namely for planar graphs, some subclasses of chordal graphs, graphs with $n$ vertices and at least $\frac{7}{16} n^2$ edges [20], graphs without a subgraph homeomorphic to $K_{3,3}$ [9], graphs with chromatic number three [8], graphs in which every subgraph has average degree smaller than seven [14], odd-wheel-free graphs, and graphs admitting an edge 3-coloring in which each triangle receives three distinct colors on its edges [1]. (The latter class contains all graphs with chromatic number at most four, and also all
graphs which have a homomorphism into the third power of an even cycle, \( C_{2k}^3 \) with \( k \geq 5 \).

The case of equality \( \tau_\Delta = \nu_\Delta \) has also been studied to some extent (\([22, 1]\)). For instance, it was proved in \([1]\) that \( \tau_\Delta(G) = \nu_\Delta(G) \) is valid for \( K_4 \)-free graphs \( G \) whose triangle graph is odd-hole free. Now, by Theorem 15, this class of graphs is determined in a more direct way by forbidden subgraph characterization. (Some redundancies could be eliminated; e.g., \( K_6 - P_4 \) and \( K_6 - K_3 \) contain \( K_4 \) which is forbidden, too. But after some appropriate edge splittings, the \( K_4 \) subgraphs disappear, hence the graphs listed in parts (d) and (e) of Theorem 15 cannot be totally omitted.)

Here we prove the original conjecture for graphs \( G \) whose triangle graph is perfect, but the \( K_4 \) subgraphs are not excluded from \( G \). The condition that \( T(G) \) is perfect, can be replaced either by assuming that \( T(G) \) is odd-hole-free or by the forbidden subgraph characterization of Theorem 15.

**Theorem 16** If the triangle graph of a graph \( G \) is perfect, then \( \tau_\Delta(G) \leq 2\nu_\Delta(G) \) holds.

**Proof.** As discussed already in \([1]\), the maximum number \( \nu_\Delta(G) \) of independent triangles in \( G \) equals the independence number \( \alpha(T(G)) \) of the triangle graph. Since the triangle graph is supposed to be perfect, its complement is also perfect and we have

\[
\nu_\Delta(G) = \alpha(T(G)) = \omega(T(G)) = \chi(T(G)) = \theta(T(G)),
\]

where \( \theta(T(G)) \) is the minimum number of cliques in the triangle graph which together cover all vertices.

In a triangle graph \( T(G) \) we may have two types of cliques: (A) its vertices correspond to triangles of \( G \) all of which are incident with a fixed edge; (B) its vertices correspond to four triangles of a \( K_4 \) subgraph of \( G \).

Having a minimum clique cover of \( T(G) \) at hand we can construct an edge cover for triangles of \( G \). For every clique \( C_A \) of type A, we put the corresponding edge of \( G \) into the covering set. This edge covers all triangles corresponding to the vertices covered by \( C_A \) in \( T(G) \). Then, for every clique \( C_B \) of type B, we put two independent edges from the corresponding \( K_4 \) subgraph of \( G \) into the covering set. These two edges together cover all the four triangles. Since every vertex \( t \in V(T(G)) \) is covered by a clique in \( T(G) \), every triangle of \( G \) is covered by an edge selected. The covering set contains at most \( 2\theta(T(G)) = 2\nu_\Delta(G) \) edges. Consequently, \( \tau_\Delta(G) \leq 2\nu_\Delta(G) \) is valid. \[\square\]
Note added on 4th November, 2014. After the first appearance of this manuscript, Gregory Puleo kindly informed us that from his results in [14] the inequality $\tau_\Delta(G) \leq 2\nu_\Delta(G)$ follows for a class of graphs which is larger than the one in our Theorem [16]. Namely, more generally than the graphs with perfect $T(G)$, it suffices to assume that $G$ has no subgraph isomorphic to $W_n$ for any odd $n \geq 5$. That is, from the forbidden subgraphs listed in our Theorem [15] already the case (a) is sufficient to derive $\tau_\Delta \leq 2\nu_\Delta$. As Puleo explains in his email, this follows by the properties of the so-called ‘weak König–Egerváry graphs’, proved in Chapter 2 of [14]. This extends Theorem 3 of [1] where the analogous result was proved for graphs without any odd wheels (i.e., excluding $W_3 \cong K_4$, too).

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