Reduction and submanifolds of generalized complex manifolds

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ABSTRACT. We recall the presentation of the generalized, complex structures by classical tensor fields, while noticing that one has a similar presentation and the same integrability conditions for generalized, paracomplex and subtangent structures. This presentation shows that the generalized, complex, paracomplex and subtangent structures belong to the realm of Poisson geometry. Then, we prove geometric reduction theorems of Marsden-Ratiu and Marsden-Weinstein type for the mentioned generalized structures and give the characterization of the submanifolds that inherit an induced structure via the corresponding classical tensor fields.

The study of generalized, complex structures is a recent subject that was started by N. J. Hitchin [10] and M. Gualtieri [9] and was continued by several authors [1, 2, 5, 11, 13, 16, 28]. The subject is motivated by the fact that generalized complex manifolds appear as target manifolds of \( \sigma \)-models [16].

The framework of the present paper is the \( C^\infty \)-category and \( M \) is a differentiable manifold. The generalized complex structures are defined like the classical complex structures but, with the tangent bundle \( TM \) replaced by \( T^{\text{big}}M = TM \oplus T^*M \) and with the Lie bracket of vector fields replaced by the Courant bracket \( [\cdot,\cdot] \) [7, 8]. Lindström-Minasian-Tomasiello-Zabzine [16] and

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Crainic [5] gave the full interpretation of a generalized, complex structure by means of classical tensor fields. Particularly, among these tensor fields there is a Poisson bivector field, which was also discovered in [9] and [1].

The fact that a generalized, complex manifold has an underlying Poisson structure justifies the study of the Poisson geometry of the generalized, complex manifolds. In particular, in [5] the subject is integrability of a generalized, complex manifold to a certain type of symplectic groupoid. In the present paper we will discuss reduction and submanifolds from the Poisson point of view. In brief, the content of the paper is as follows.

In section 1, we present the basics of the theory of the generalized, complex structures using the corresponding classical tensor fields. The content of this section is not original, and its length is justified by the fact that the whole theory is pretty new and, presumably, not very popular yet. However, the section also contains some novelties: the results are formulated for three kinds of generalized structures, complex, paracomplex and subtangent; we show the connection between the generalized structures and Poisson-Nijenhuis structures\(^1\); we indicate the algebraic expression of a generalized structure along a symplectic leaf of its Poisson structure; finally, we refer to the possible Lie groups with a compatible generalized structure.

In Section 2 we discuss reduction of generalized structures. We start with the Marsden-Ratiu definition of the geometric reduction of a Poisson structure via a submanifold and a control vector bundle [19] and prove a geometric reduction theorem for generalized structures. Then, we particularize the theorem for interesting special control bundles. In particular, we obtain a corollary which is a Marsden-Weinstein reduction theorem for generalized structures.

Finally, in Section 3 we discuss the notion of a submanifold defined by Ben-Bassat-Boyarchenko in [2]. These submanifolds inherit an induced generalized structure. We give the characterization of the submanifolds in the sense of [2] by means of the classical tensor fields of the structure. In particular, the submanifolds under consideration have to be Poisson-Dirac submanifolds in the sense of [6]. The same characterization may also be used as a good definition of Poisson-Nijenhuis submanifolds of a Poisson-Nijenhuis manifold, such that the submanifold inherits an induced hierarchy of Poisson structures.

\(^1\)Recently, I learned from P. Xu that he has also indicated such a connection in his lecture at the conference in Trieste, Italy, July 2005.
While work on this paper was in progress, several papers on reduction of generalized complex structures were posted on the web [12, 14, 22, 3, 15]. These papers provide various ways of extending the notion of a Hamiltonian Lie group action with an equivariant momentum map and the Marsden-Weinstein reduction theorem to generalized complex structures. Instead, in the present paper we extend the Marsden-Ratiu reduction theorem of [19].

1 Generalized structures in classical terms

We begin by recalling the Courant bracket \([\cdot, \cdot] : \Gamma T^{big}M \times \Gamma T^{big}M \rightarrow \Gamma T^{big}M\) (\(\Gamma\) denotes spaces of cross sections of vector bundles), which is defined by [7, 8]

\[
[(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)),
\]

where \(X, Y \in \chi^1(N), \alpha, \beta \in \Omega^1(N)\) (we denote by \(\chi^k(M)\) the space of \(k\)-vector fields and by \(\Omega^k(M)\) the space of differential \(k\)-forms on \(M\)). We also recall that \(T^{big}M\) has the neutral metric

\[
g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X))
\]

and the non degenerate 2-form

\[
\omega((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) - \beta(X)).
\]

A maximal, \(g\)-isotropic subbundle \(L \subseteq T^{big}M\) is called an almost Dirac structure of \(M\). If \(L\) is also closed by Courant brackets it is called a Dirac structure.

**Remark 1.1.** The motivation for the study of Dirac structures comes from the theory of constrained mechanical systems as shown by the following facts [7, 8]. An almost Dirac structure \(L\) defines a generalized distribution \(D = pr_{TM}L\) (the projection \(pr_{TM}\) is given by the direct sum structure of \(T^{big}M\) endowed with a 2-form \(\vartheta\) induced by \(\omega\). The structure \(L\) may be reconstructed from the pair \((D, \vartheta)\) by the formula

\[
L = \{(X, \alpha) / X \in D, \ a|_{L^\perp} = b_\vartheta X\}.
\]
Furthermore, $L$ is Dirac iff $D$ is integrable and $\vartheta$ is closed along the leaves; one says that $D$ is the \textit{presymplectic foliation} of $L$. Thus, a Dirac structure is equivalent with a generalized, presymplectic foliation with a presymplectic form that is differentiable on $M$.

The definition of generalized, complex structures uses a generalization of the Nijenhuis tensor. Namely, if $\Phi \in \Gamma(End T^{\text{big}}M)$ one defines the \textit{Courant-Nijenhuis torsion} of $\Phi$ by
\begin{equation}
N_\Phi(X,Y) = [\Phi X, \Phi Y] - \Phi[X, \Phi Y] - \Phi[\Phi X, Y] + \Phi^2[X, Y],
\end{equation}
where $X = (X,\alpha), Y = (Y,\beta) \in \Gamma T^{\text{big}}M$ and the brackets are Courant brackets. In the general case, the Courant-Nijenhuis torsion is not $C^\infty(M)$-bilinear since, in view of the properties of the Courant bracket $[8]$, $N_\Phi(X,fY)$ ($f \in C^\infty(M)$) includes the terms
\begin{equation}
[\Phi^2 g(X,Y) + g(\Phi X,\Phi Y)] \Phi(0,df) - [g(X,Y)\Phi^2(0,df) + g(\Phi X,\Phi Y)(0,df)].
\end{equation}
But, if $\Phi$ is $g$-skew-symmetric, i.e.,
\begin{equation}
g(X,\Phi Y) + g(\Phi X,Y) = 0,
\end{equation}
and $\epsilon$-potent, i.e.,
\begin{equation}
\epsilon^2 = \epsilon \text{Id}, \quad \epsilon = \pm 1, 0,
\end{equation}
$\Phi$ also satisfies the condition
\begin{equation}
g(\Phi X, \Phi Y) + \epsilon g(X,Y) = 0,
\end{equation}
and the terms (1.5) vanish.

If $\Phi$ satisfies (1.7) with $\epsilon = -1$ and (1.6) $\Phi$ is equivalent with a decomposition of the complexification $T^{\text{big}}_c M = T^{\text{big}}M \otimes \mathbb{C}$ into a Whitney sum of conjugated complex, almost Dirac structures, the $\pm i$-eigenbundles $L_{\pm}$ of $\Phi$, and $\Phi$ is called a \textit{generalized, almost complex structure} of $M$. If $\Phi$ satisfies (1.7) with $\epsilon = 1$ and (1.6) $\Phi$ is equivalent with a decomposition of $T^{\text{big}}M$ into a Whitney sum of two maximally $g$-isotropic subbundles, the $\pm 1$-eigenbundles $E_{\pm}$ of $\Phi$ and $\Phi$ is called a \textit{generalized, almost paracomplex structure} of $M$. If $\Phi$ satisfies (1.7) with $\epsilon = 0$ and (1.6) we will say that $\Phi$ is a \textit{generalized, almost subtangent structure} and im$\Phi$, the 0-eigenspaces field.
$S$ of $\Phi$ is $g$-isotropic in $T^{big}M$. The name of this kind of structures comes from the fact that a structure defined by a tensor field $\Phi \in \Gamma \text{End}(TM)$ such that $\Phi^2 = 0$ and

\begin{equation}
\text{im} \Phi = \ker \Phi
\end{equation}

is called an almost tangent structure on $M$. If the generalized, almost sub-tangent structure $\Phi$ satisfies (1.9) $\Phi$ is a generalized, almost tangent structure of $M$ and $S = \text{im} \Phi$ is an almost Dirac structure. In all the cases mentioned above (i.e., $\epsilon = \pm 1, 0$), if $N_\Phi = 0$ the adjective “almost” is dropped and $\Phi$ is said to be integrable.

The following proposition gives an alternative characterization of the generalized complex and paracomplex structures (but does not provide a sufficient condition for the integrability of a generalized, almost sub-tangent structure).

**Proposition 1.1.** [9, 28] A generalized, almost complex or almost paracomplex structure $\Phi$ is integrable iff the eigenbundles of $\Phi$ are Dirac structures. If a generalized, almost sub-tangent structure $\Phi$ is integrable $\text{im} \Phi$ is closed by Courant brackets, and it is a Dirac structure in the tangent case; if $\text{im} \Phi$ is Dirac one has $\Phi \circ N_\Phi = 0$.

**Proof.** Compute the values of the Courant-Nijenhuis torsion $N_\Phi$ on eigenvectors. For the tangent case, look at (1.4) and (1.8).

As indicated by the title, we are interested in the generalized, complex manifolds. However, at almost no extra cost, we get results for all the structures mentioned above. Accordingly, we will use the term *generalized (almost) c.p.s. structure*, where the letters c,p,s stand for complex, paracomplex and sub-tangent, respectively.

**Remark 1.2.** The above definitions may be applied to vector bundles and Courant algebroids [17, 3]. For instance [17], if $\pi$ is a Poisson bivector field on $M$, $T^{big}M$ also has the Courant algebroid structure with anchor $Id + \frac{\pi}{2}$ and bracket

\begin{align}
[(X, \alpha), (Y, \beta)]_\pi &= ([X, Y] + L_\alpha Y - L_\beta X - \frac{1}{2}\sigma(\alpha(Y) - \beta(X)), \\
\{\alpha, \beta\}_\pi + L_X \beta - L_Y \alpha + \frac{1}{2}d(\alpha(Y) - \beta(X)))
\end{align}
\begin{align*}
&= ([X, Y] + i(\beta)L_X \pi - i(\alpha)L_Y \pi - \frac{1}{2} \pi d(\alpha(Y) - \beta(X)), \\
\{\alpha, \beta\}_\pi + L_X \beta - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X))).
\end{align*}

The notation is that of [23], \(\sigma\) is the Lichnerowicz-Poisson differential and

\begin{equation}
\{\alpha, \beta\}_\pi = L^{\sharp}_\pi \alpha \beta - L^{\sharp}_\pi \beta \alpha - d(\pi(\alpha, \beta)).
\end{equation}

This example is not very interesting because the mapping

\begin{equation}
(X, \alpha) \mapsto (X + \sharp_p \alpha, \alpha)
\end{equation}

yields an isomorphism from the new Courant algebroid to the classical Courant algebroid, which sends the bracket \((\ref{eq:bracket})\) to \((\ref{eq:classical-bracket})\) and commutes with the Nijenhuis torsion, therefore, it sends generalized c.p.s. structures with respect to \((\ref{eq:bracket})\) to generalized c.p.s. structures with respect to \((\ref{eq:classical-bracket})\).

We intend to use the interpretation of the generalized structures in terms of classical tensor fields on \(M\). For this purpose we represent \(\Phi\) in the following matrix form \[9\]

\begin{equation}
\Phi \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \pi \pi \\
\sigma & B \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}
\end{equation}

where \((X, \alpha) \in T^\text{big}M\) and, if we denote by \(pr\) the natural projections and by \(\iota\) the natural embeddings, we have

\[
A = pr_{TM} \circ \Phi \circ \iota_{TM} : TM \to TM, \quad \pi \pi = pr_{TM} \circ \Phi \circ \iota_{T^*M} : T^*M \to TM,
\]

\[
\sigma = pr_{T^*M} \circ \Phi \circ \iota_{TM} : TM \to T^*M, \quad B = pr_{T^*M} \circ \Phi \circ \iota_{T^*M} : T^*M \to T^*M.
\]

With this notation, condition \((\ref{eq:condition})\) is equivalent with the following three facts:

i) \(\pi \pi\) is defined by a bivector \(\pi\) by \(\pi \pi \alpha = i(\alpha)\pi\),

ii) \(\sigma\) is defined by a 2-form \(\sigma\) by \(\sigma X = i(X)\sigma\),

iii) \(B = -tA\), where \(t\) denotes transposition, i.e., \(B\alpha = -\alpha \circ A\),

and we have

\begin{equation}
\Phi(X, \alpha) = (AX + \pi \pi \alpha, \sigma X - \alpha \circ A).
\end{equation}
Furthermore, condition (1.15) is equivalent to
\[(1.15) \quad A^2 = \epsilon Id - \sharp_{\pi} \circ \flat_{\sigma}, \quad \pi(\alpha \circ A, \beta) = \pi(\alpha, \beta \circ A), \quad \sigma(AX, Y) = \sigma(X, AY).\]

If the second, respectively the third, condition (1.15), holds, \(\pi\), respectively \(\sigma\), is said to be compatible with \(A\).

**Remark 1.3.** As a consequence of (1.15), it follows that if a manifold \(M\) has a generalized, almost complex structure, the dimension of \(M\) is even [9, 2]. Indeed, (1.15) implies that \((A|_{\ker \flat_{\sigma}})^2 = \epsilon Id\). Hence, for \(\epsilon = -1\), \(\dim(\ker \flat_{\sigma})\) is even. Since \(\dim(\text{im} \flat_{\sigma})\) is even too, \(\dim M\) is even. The same is true for generalized almost tangent manifolds \(M\) but, the argumentation is different. If \(\Phi\) is the generalized almost tangent structure, there exist decompositions \(T^{big}M = \text{im} \Phi \oplus D\), where the terms are maximal, \(g\)-isotropic and \(\Phi|_D : D \to \text{im} \Phi\) is an isomorphism. Hence, there exists a non-degenerate 2-form on \(D\) given by \(\varpi(Z_1, Z_2) = g(\Phi Z_1, Z_2)\) \((Z_1, Z_2 \in D)\), and \(\dim D = \dim M\) must be even. On the other hand, on any manifold \(M\) the decomposition \(T^{big}M = TM \oplus T^* M\) is a generalized, almost paracomplex structure while \(M\) may also be odd-dimensional.

For \(\Phi^2 = \epsilon Id, \epsilon = -1\), the invariant computation of the Nijenhuis torsion \(\mathcal{N}_\Phi\) with \(\Phi\) given by (1.13) was done by Crainic [5]. (The corresponding computation in local coordinates appeared in [10].) With minor adjustments, Crainic’s computation also holds for \(\epsilon = 1, 0\) and the result is

**Theorem 1.1.** [5] The almost c.p.s. structure \(\Phi\) given by (1.13) is integrable iff the following conditions hold:

i) the bivector field \(\pi\) defines a Poisson structure on \(M\);

ii) the bracket \(\{\alpha, \beta\}_\pi\) defined by (1.11) satisfies the condition
\[(1.16) \quad \{\alpha, \beta\}_\pi \circ A = L_{\sharp_{\pi} \alpha}(\beta \circ A) - L_{\sharp_{\pi} \beta}(\alpha \circ A) - d(\pi(\alpha \circ A, \beta));\]

iii) the Nijenhuis tensor of \(A\) satisfies the condition
\[(1.17) \quad \mathcal{N}_A(X, Y) = \sharp_{\pi}[i(Y)i(X)d\sigma];\]

iv) the associated form
\[(1.18) \quad \sigma_A(X, Y) = \sigma(AX, Y)\]

satisfies the condition
\[(1.19) \quad d\sigma_A(X, Y, Z) = \sum_{Cycl(X,Y,Z)} d\sigma(AX, Y, Z).\]
It is interesting to notice the following interpretation of condition ii). A pair of tensor fields $\pi \in \chi^2(M), A \in \Gamma(\text{End}TM)$ defines the Schouten concomitant \cite{18}

\[ R(\pi, A)(\alpha, X) = \sharp_\pi(L_X(\alpha \circ A)) - (L_{\sharp_\pi \alpha}A)(X) - \sharp_\pi(L_{AX}\alpha) \]

(note that our sign convention for $\sharp_\pi$ and $\flat_\sigma$ is opposite to that of \cite{18}), which is equivalent to the $T^*M$-valued bivector field \cite{24}

\[ C(\pi, A)(\alpha, \beta) = \beta \circ L_{\sharp_\pi \alpha}A - \alpha \circ L_{\sharp_\pi \beta}A + d(\pi(\alpha, \beta)) \circ A - d(\pi(\alpha \circ A, \beta)) \]

in the sense that

\[ < R(\pi, A)(\alpha, X), \beta > = - < C(\pi, A)(\alpha, \beta), X > . \]

If the expression (1.11) of the bracket of 1-forms is inserted in (1.16), it follows that condition ii) may be reformulated as

\[ ii') \text{ the Schouten concomitant } C(\pi, A) \text{ vanishes.} \]

This interpretation of condition ii) has interesting consequences.

**Proposition 1.2.** If the 2-form $\sigma$ of (1.13) is symplectic, the structure $\Phi$ is integrable iff the pair $(A, \sigma)$ is a symplectic-Nijenhuis structure.

**Proof.** We recall that a pair $(w \in \chi^2(M), A \in \text{End}TM)$ is a Poisson-Nijenhuis structure if $A$ and $w$ are compatible, the Schouten-Nijenhuis bracket $[w, w] = 0$, the Nijenhuis tensor $N_A = 0$ and the Schouten concomitant $R(w, A) = 0$ \cite{18, 24}. In particular, if $w$ comes from a symplectic form $\sigma$, i.e., $\sharp_w \circ \flat_\sigma = -\text{Id}$, $(\sigma, A)$ is called a symplectic-Nijenhuis structure. One can prove that this happens iff the associated 2-form $\sigma_A$ is also closed \cite{18, 26}.

The fundamental property of a Poisson-Nijenhuis structure $(w, A)$ is the existence of a corresponding family of pairwise compatible Poisson-Nijenhuis structures $(W, P_1(A))$, where $\sharp_w = P_2(A) \circ \sharp_w$ and $P_{1,2}(A)$ are either polynomials or convergent power series with constant coefficients in the argument $A$, called the Poisson hierarchy \cite{18, 24}. (The compatibility of two Poisson structures $(w, W)$ means that the Schouten-Nijenhuis bracket $[w, W] = 0$, equivalently, that $w + cW \ (c = \text{const.})$ is again a Poisson bivector field.)

Now, we notice that, for a non degenerate 2-form $\sigma$, conditions (1.15) are equivalent with

\[ \sharp_\pi = (A^2 - \epsilon \text{Id}) \circ \sharp_w, \ \sigma(AX, Y) = \sigma(X, AY) \]
(the condition for $\pi$ in (1.15) is a consequence of (1.22)) and we have

$$
\Phi = \begin{pmatrix}
A & (A^2 - \epsilon Id) \circ \sharp w \\
\flat \sigma & -t A
\end{pmatrix}.
$$

Then, if $d\sigma = 0$ and $\Phi$ of (1.23) is integrable, the integrability conditions ii'), iii), iv) show that $(\sigma, A)$ is a symplectic-Nijenhuis structure.

Conversely, if $(\sigma, A)$ is a symplectic-Nijenhuis structure conditions ii'), iii), iv) for $\Phi$ hold and the integrability condition i) follows from the Poisson hierarchy theorem.

Thus, the generalized, c.p.s. structures with a symplectic form $\sigma$ are equivalent with the symplectic-Nijenhuis structures with the same form $\sigma$. Moreover, a symplectic-Nijenhuis manifold $(M, \sigma, A)$ is endowed with families of generalized, c.p.s. structures defined by replacing $A$ by $P_1(A)$, where $P_1(A)$ is either a polynomial or a convergent power series with constant coefficients, in formula (1.23). Formula (1.23) also shows that the Poisson structure $\pi$ of $\Phi$ is compatible with the Poisson structure $w$ defined by the symplectic form $\sigma$.

Furthermore, it is known that the symplectic-Nijenhuis structures $(\sigma, A)$ of a manifold $M$ are in a bijective correspondence with the compatible pairs of Poisson structures $(w, W)$ \cite{15, 26}. This correspondence sends $(\sigma, A)$ to $(w, W)$ where $\sharp w = A \circ \sharp W$, i.e., $W$ is the first new Poisson structure of the Poisson hierarchy of $(\sigma, A)$. Conversely, the pair $(w, W)$ is sent to $(\sigma, -\sharp W \circ \flat \sigma)$.

This proves the following result

**Proposition 1.3.** On a symplectic manifold $(M, \sigma)$ there exists a bijective correspondence between the Poisson structures $W$ on $M$ that are compatible with $w$ and the generalized, c.p.s. structures of $M$ which have the form $\sigma$ in their matrix representation. This correspondence is given by

$$
W \mapsto \Phi_W = \begin{pmatrix}
B & \sharp w \\
\flat \sigma & -t B
\end{pmatrix} = \begin{pmatrix}
-\sharp w \circ \flat \sigma & -\epsilon \sharp w - \sharp w \circ \flat \sigma \circ \sharp w \\
\flat \sigma & \flat \sigma \circ \sharp w
\end{pmatrix}.
$$

**Remark 1.4.** In the complex case, the $\pm i$-eigenbundles of $\Phi_W$ are $L, \bar{L}$ where

$$
L = \text{graph}(\sharp_{-(W+iw)}: T^* cM \to T cM).
$$
and \( \bar{L} \) is the complex conjugate bundle of \( L \). Indeed, the conditions \([W, W] = 0, [W, w] = 0\) imply that \( W + iw \) is a complex-valued Poisson bivector field on \( M \), hence \( L \) defined by (1.25) is a complex Dirac structure. Furthermore, since \( \sigma \) is non degenerate, \( L \cap \bar{L} = \{0\} \) and there exists a unique, generalized, complex structure with the \( \pm i \)-eigenbundles \( L, \bar{L} \). In order to show that this structure is \( \Phi_W \) it suffice to compute the \( i \)-eigencomponent of \((X, \alpha) \in T^{b\theta}M\) with respect to \( \Phi_W \):

\[
\frac{1}{2}(Id - i\Phi_W)(X, \alpha) = \frac{1}{2}(X - i(BX + i_\sigma \alpha), \alpha - i(\beta_\sigma X - \alpha \circ B))
\]

\[
= \frac{1}{2}(\sharp - (W + iw)(\alpha - i(\beta_\sigma X - \alpha \circ B), \alpha - i(\beta_\sigma X - \alpha \circ B)).
\]

Similar computations of eigenbundles may be done in the paracomplex and subtangent cases.

Other connections with Poisson-Nijenhuis structures are given by

**Proposition 1.4.** a) If \( \Phi \) is integrable and \( \sigma \) is closed, \((\pi, A)\) is a Poisson-Nijenhuis structure on \( M \). b) If \((\pi, A)\) is a symplectic-Nijenhuis structure, \( \Phi \) is integrable iff the forms \( \sigma \) and \( \sigma_A \) are closed. c) Let \( \Phi \) be a generalized, almost c.p.s. structure on \( M \) such that \( \pi \) is a Poisson bivector field, \( A \) is a Nijenhuis tensor field, and the 2-forms \( \sigma, \sigma_A \) are closed. Then \( \Phi \) is integrable.

**Proof.** Assertions a) and b) are trivial. For c), the only condition we still have to check is \( C_{(\pi, A)} = 0 \). For this purpose, we write down the following formula, which holds for any tensor fields \( \pi \in \chi^2(M), \sigma \in \Omega^2(M) \) and is equivalent with formula (B.3.9) of [18],

\[
C_{(\pi, \sharp_\pi \circ \beta_\sigma)}(\alpha, \beta) = i(\sharp_\pi \beta)i(\sharp_\pi \alpha)d\sigma - (i(\beta)i(\alpha)[\pi, \pi]) \circ b_\sigma,
\]

where \([\pi, \pi]\) is the Schouten-Nijenhuis bracket. For a generalized, almost c.p.s. structure \( \Phi \), the first condition (1.26) changes (1.27) to

\[
-C_{(\pi, A^2)} = i(\sharp_\pi \beta)i(\sharp_\pi \alpha)d\sigma - (i(\beta)i(\alpha)[\pi, \pi]) \circ b_\sigma.
\]

Under the hypotheses of iii), (1.28) gives \( C_{(\pi, A^2)} = 0 \), and \((\pi, A^2)\) is a Poisson-Nijenhuis structure. Then, by the hierarchy theorem for Poisson-Nijenhuis structures [18] [24], \((\pi, A)\) is also a Poisson-Nijenhuis structure and \( C_{(\pi, A)} = 0 \).
Remark 1.5. In [16] it is shown that supersymmetry is also related with generalized, almost complex structures that are integrable (i.e., have a vanishing tensor (1.4)) with respect to the Ševera-Weinstein Courant bracket [21]

\[(1.29) \quad [(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha)\]

where \(\Lambda\) is a closed 3-form on \(M\), and these new integrability conditions are expressed in local coordinates. The computations done to prove Theorem 1.1 may be easily extended to (1.29), and it follows that \(\mathcal{N}_\Phi\) with brackets (1.29) is zero iff

i) the bivector field \(\pi\) defines a Poisson structure on \(M\);

ii) the Schouten concomitant of the pair \((\pi, A)\) satisfies the condition

\[(1.30) \quad C_{(\pi, A)}(\alpha, \beta) = i(\sharp_{\pi} \beta)i(\sharp_{\pi} \alpha)\Lambda;\]

iii) the Nijenhuis tensor of \(A\) is given by the formula

\[(1.31) \quad \mathcal{N}_A(X, Y) = \sharp_{\pi} [i(Y)i(X)d\sigma + i(AY)i(X)\Lambda - i(AX)i(Y)\Lambda];\]

iv) the exterior differential of the associated form \(\sigma_A\) satisfies the equality

\[(1.32) \quad d\sigma_A(X, Y, Z) = \epsilon \Lambda(X, Y, Z)\]

\[= \sum_{Cycl(X, Y, Z)} [d\sigma(AX, Y, Z) + \Lambda(AX, AY, Z)].\]

Following Crainic [5], we will say that a generalized, almost c.p.s. structure \(\Phi\) is non degenerate if its bivector field \(\pi\) is non degenerate. Then, we will denote by \(\varpi\) the non degenerate 2-form defined by \(b_\varpi \circ \sharp_{\pi} = -Id\), and (1.15) implies

\[(1.33) \quad b_{\varpi} = b_\varpi \circ A^2 - \epsilon b_\varpi.\]

Theorem 1.2. [5] Let \(\Phi\) be a non degenerate, generalized, almost c.p.s. structure. Then, \(\Phi\) is integrable iff \(\pi\) is Poisson and the 2-form \(\varpi_A\) is closed.
A pair \((\varpi, A)\) where \(\varpi\) is a symplectic form and \(A\) is a compatible \((1, 1)\)-tensor field is called a Hitchin pair if the associated 2-form \(\varpi_A\) is closed \([5]\).

The previous theorem may be used to show the existence of a \(1 - 1\) correspondence between each of the three classes of non degenerate, integrable, almost c.p.s. structures (separately) and Hitchin pairs, which is defined by the formula \([5]\),

\[
(\varpi, A) \mapsto \begin{pmatrix} A & \# \pi \\ \flat_{\varpi, A^2} - \epsilon \flat \varpi & -\flat A \end{pmatrix}.
\]

Accordingly, Crainic’s results on Lie groupoids and algebroids connected with generalized complex structures have corresponding variants for generalized paracomplex and tangent structures.

We continue the presentation of the basic results on generalized c.p.s. structures by indicating some examples.

**Example 1.1.** \([9]\) For any classical c.p.s. structure \(A\) on \(TM\), the matrix

\[
\begin{pmatrix} A & 0 \\ 0 & -\flat A \end{pmatrix}
\]

is a generalized c.p.s. structure, respectively.

**Example 1.2.** \([9, 5]\) Any symplectic form \(\varpi\) produces the c.p.s. structures

\[
\begin{pmatrix} 0 & \# \pi \\ -\epsilon \flat \varpi & 0 \end{pmatrix}, \quad \begin{pmatrix} \text{Id} & \# \pi \\ (1 - \epsilon) \flat \varpi & -\text{Id} \end{pmatrix},
\]

where \(\flat \varpi \circ \# \pi = -\text{Id}\), associated with the Hitchin pairs \((\varpi, 0), (\varpi, \text{Id})\), respectively.

**Example 1.3.** \([2]\) If \((M, F)\) is a locally product manifold with structural foliations \(F_1, F_2\) (i.e., \(TM = T F_1 \oplus T F_2\)), and if these foliations have generalized c.p.s. structures \(\Phi_1, \Phi_2\) along the leaves, which are differentiable on \(M\), then \(\Phi = \Phi_1 \oplus \Phi_2\) is a generalized c.p.s. structure on \(M\). Sometimes, it is interesting to change \(\Phi_1\) by its opposite structure \(\Phi_1^\circ\), which is defined by changing the sign of the tensor fields \(\pi, \sigma\) in the matrix \((1.13)\) of \(\Phi_1\), and use the twisted direct sum \(\Phi_1^\circ \oplus \Phi_2\). Theorem \([1.14]\) shows that \(\Phi\) and \(\Phi_1^\circ \oplus \Phi_2\) are simultaneously integrable.
Example 1.4. Let $M$ be a complex analytic manifold. Then we may define the notion of a **holomorphic Dirac structure** in the same way as a real Dirac structure, using the holomorphic tangent and cotangent bundles of $M$. If $L_1, L_2$ are two holomorphic Dirac structures on $M$, $L_1 \oplus \bar{L}_2$ (where the bar denotes complex conjugation) obviously is the $\sqrt{-1}$-eigenbundle of a generalized, complex structure of $M$. (Hitchin’s example of the generalized, complex structure associated with a holomorphic Poisson structure of $M$ is a particular case of the previous construction.)

Another basic notion of the theory is that of gauge equivalence. This notion is based on Hitchin’s remark that, for any closed 2-form $B$ on $M$, the mapping

$$ (X, \alpha) \mapsto (X, \alpha + i(X)B), $$

(1.37)

called a $B$-field or gauge transformation (equivalence) is a bundle automorphism $\mathcal{B}$ of $T^{\text{big}}M$ which is compatible with the metric $g$ and the Courant bracket. The matrix of the gauge transformation (1.37) is

$$ \mathcal{B} = \begin{pmatrix} \text{Id} & 0 \\ b_B & \text{Id} \end{pmatrix}, $$

(1.38)

and $\mathcal{B}$ acts on generalized, almost c.p.s. structures by the invertible mapping $\Phi \mapsto \mathcal{B}^{-1}\Phi \mathcal{B}$. By Proposition in the complex and paracomplex cases $\mathcal{B}$ also preserves integrability.

From the algebraic point of view, $B$-field transformations may be defined in the same way for $E \oplus E^*$, where $E$ is an arbitrary vector bundle with a generalized, almost c.p.s. structure and $B \in \Gamma \wedge^2 E^*$ but, of course, no properties of any bracket will be involved.

The action of $\mathcal{B}$ sends the matrix (1.13) of $\Phi$ to the matrix

$$ \begin{pmatrix} A + \frac{\pi}{\pi} \circ b_B & \frac{\pi}{\pi} \\ b_{\sigma} - b_B \circ \frac{\pi}{\pi} \circ b_B - b_B \circ A - tA \circ b_B & -tA - b_B \circ \frac{\pi}{\pi} \end{pmatrix}. $$

Thus, the Poisson bivector field of $\Phi$ is preserved, the tensor field $A$ goes to $A + \frac{\pi}{\pi} \circ b_B$, and the 2-form $\sigma$ is changed to

$$ \sigma'(X, Y) = \sigma(X, Y) + \pi(b_B X, b_B Y) - B(AX, Y) - B(X, AY). $$

(1.40)
Example 1.5. Any non-degenerate c.p.s. structure $\Phi$ represented by the right hand side of the mapping (1.34) is gauge equivalent with the symplectic structure $\varpi$ seen as the first matrix (1.36), by the field $B = \varpi_A$, and seen as the second matrix (1.36), by the field $B = \varpi - \varpi$.

Example 1.6. The $B$-transform of the structure (1.35) is of the form

$$\begin{pmatrix} A & 0 \\ b_\sigma & -^tA \end{pmatrix},$$

where

$$\sigma(X,Y) = -B(AX,Y) - B(X,AY).$$

Conversely, if the Poisson structure of a generalized c.p.s. structure $\Phi$ is zero, the structure is gauge equivalent with a classical c.p.s. structure, seen as (1.35), iff there exists a closed 2-form $B$ such that

$$\sigma(X,Y) = B(AX,Y) + B(X,AY).$$

For $\epsilon = \pm 1$, the general algebraic solution of (1.42) is

$$B(X,Y) = \frac{\epsilon}{2} \sigma(AX,Y) + B'(X,Y),$$

where $B'(AX,Y) + B'(X,AY) = 0$, and we see that $\Phi$ defined by (1.41), where $\sigma_A$ is closed, is gauge equivalent with a classical structure.

The importance of gauge equivalence is shown by the local structure theorems of Gualtieri [9] and Abouzaid-Boyarchenko [1], which show that any generalized, complex manifold is gauge equivalent with the direct sum of a symplectic and a classical complex structure in a neighborhood of a point. One also has the algebraic result that any generalized, complex structure of a vector space is gauge equivalent with such a direct sum [2]. We extend this algebraic result in the following proposition.

**Proposition 1.5.** Let $(M, \Phi)$ be a generalized, almost c.p.s. manifold, where $\Phi$ is defined by the matrix (1.13) and the bivector field $\pi$ is Poisson. Let $S$ be a symplectic leaf of $\pi$. Let $\nu S$ be a normal bundle of $S$, i.e., a subbundle of $T_S M$ such that

$$T_S M = TS \oplus \nu S.$$  

Then, the restriction of $\Phi$ to the bundle $T^\text{big}_S M$ is algebraically gauge equivalent with a direct sum of a symplectic structure on $T^\text{big} S$ and a c.p.s. structure on $\nu^\text{big} S = \nu S \oplus \nu^* S$. 

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Proof. In the conclusion, by a symplectic structure we mean a structure defined by the first matrix (1.36) and by a c.p.s. structure we mean a matrix (1.35) which satisfies the algebraic conditions of a generalized c.p.s. structure.

First, we show the existence of \( B \in \Gamma \wedge^2 T^*_S M \) such that the \( B \)-field equivalent structure

\[
\Phi' = \begin{pmatrix}
A' & \sharp \pi \\
\flat \sigma' & -tA'
\end{pmatrix}
\]

of \( \Phi \) has the property that \( \nu S \) invariant by \( A' \). Indeed, by (1.39), this means that we have to choose \( B \) such that

\[
(1.45) \quad \sharp \pi \flat B(V) = -\text{pr}_{TS} A(V), \quad \forall V \in \nu S,
\]

where the projection is defined by the decomposition (1.44). Since \( S \) is a symplectic leaf of \( \pi \), there exists a unique \( \lambda \in T^*S \) such that \( \text{pr}_{TS} \circ A(V) = \sharp \pi \lambda \), and \( V \mapsto -\lambda \) yields a well defined mapping \( \varphi : \nu S \to T^*S \) such that (1.45) is satisfied if \( \flat B|_{\nu S} = \varphi \). This mapping extends to a mapping \( \flat B : T S \oplus \nu S \to T^*S \oplus \nu^*S^* \) defined in matrix form by

\[
\flat B = \begin{pmatrix}
0 & \varphi \\
-t \varphi & 0
\end{pmatrix},
\]

which is associated with a 2-form \( B \). (Above and hereafter \( T^*S, \nu^*S \) are seen as the terms of the decomposition \( T^*_S M = T^*S \oplus \nu^*S \) induced by (1.44).)

Furthermore, we shall see that \( T S \oplus T^*S \) and \( \nu S \oplus \nu^*S \) are invariant by \( \Phi' \) and the latter is the direct sum of its restrictions to these invariant subbundles, which are of the form

\[
(1.46) \quad \Phi'|_{TS \oplus T^*S} = \begin{pmatrix}
A' & \sharp \pi \\
\flat \sigma' & -tA'
\end{pmatrix}, \quad \Phi'|_{\nu S \oplus \nu^*S} = \begin{pmatrix}
A' & 0 \\
\flat \sigma' & -tA'
\end{pmatrix}.
\]

Indeed, we obviously have \( A' = A'|_{TS} + A'|_{\nu S} \) and \( \sharp \pi = (\sharp \pi)|_{T^*S} \oplus 0 \) (remember that \( TS = \text{im} \sharp \pi \) and \( \nu^*S = \text{ann} TS \)). In order to see that \( \flat \sigma' = (\flat \sigma')|_{TS} \oplus (\flat \sigma')|_{\nu S} \) we have to check the \( \sigma' \)-orthogonality of \( TS \) and \( \nu S \), which is seen as follows. For \( X = \sharp \pi \xi \in TS \) (\( \xi \in T^*S \)) and \( V \in \nu S \), (1.45) for \( \Phi' \) implies

\[
\sigma'(V, X) = < \flat \sigma'V, \sharp \pi \xi > = - < \sharp \pi \flat \sigma'V, \xi > = - < \varepsilon V - A^2 V, \xi > = 0.
\]

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Now, since $\Phi'|_{T^*S}$ is non degenerate, Example 1.5 tells us that, algebraically, this component of $\Phi'$ is gauge equivalent with the symplectic structure defined by the first matrix (1.36) associated to $\#_\pi|_{T^*S}$.

Then, in view of (1.42), a gauge transformation that sends $\Phi'|_{\nu S \oplus \nu^*S}$ to a structure with a matrix form (1.35) is defined by a new $B \in \Gamma \wedge^2 \nu^*S$ such that

$$\sigma'(V_1, V_2) = B(A'V_1, V_2) + B(V_1, A'V_2)$$

$(V_1, V_2 \in \nu S)$. This condition holds if we ask

(1.47) $$B(A'V_1, V_2) = \frac{1}{2} \sigma'(V_1, V_2).$$

Such a form $B$ exists: for $\epsilon = \pm 1$ $A'|_{\nu S}$ is non degenerate, and (1.47) fully defines $B$; for $\epsilon = 0$, (1.47) defines $B$ on $im A'|_{\nu S}$ and we may use an arbitrary extension to $\nu S$.

The composition of all the algebraic gauge transformations described above yields the required conclusion. \(\square\)

**Remark 1.6.** From the algebraic point of view again, it is also interesting to refer to a dual notion of $\beta$-field transformation [9, 2]

(1.48) $$\Phi \mapsto \mathcal{B}'^{-1} \Phi \mathcal{B}'$$

where

$$\mathcal{B}' = \left( \begin{array}{cc} Id & \#_\beta \\ 0 & Id \end{array} \right) \quad (\beta \in \chi^2(M)).$$

Preservation of integrability by a $\beta$-field transformation is rare. An example is given by formula (1.24) where $\Phi_W$ is the result of the $W$-field transformation of the symplectic structure $\sigma$ seen as in the first matrix (1.36) and $\Phi_W$ is integrable.

We finish this section by referring to a notion of generalized c.p.s. mapping (generalized, holomorphic mapping in the complex case). This is not simple because both contravariant and covariant tensor fields are involved. The most appropriate definition seems to be that of Crainic [5], even though it is very restrictive. We justify Crainic’s definition as follows.

A mapping $f: (M_1, \Phi_1) \rightarrow (M_2, \Phi_2)$, where $\Phi_1, \Phi_2$ are generalized c.p.s. structures, produces relations

(1.49) $$f^{rel}_x = \{((X, f^*\alpha), (f_*X, \alpha)) / X \in T_xM_1, \alpha \in T_x^*M_2\}$$
\[ \subseteq T_{x}^{\text{big}}M_{1} \times T_{\text{f}(x)}^{\text{big}}M_{2} \]

defined \( \forall x \in M_{1} \). The mapping \( f \) will be called a generalized c.p.s. mapping if, \( \forall x \in M_{1} \), \( \forall ((X, f^{*}\alpha), (f_{*}X, \alpha)) \in f_{x}^{rel} \) one has \((\Phi_{1}(X, f^{*}\alpha), \Phi_{2}(f_{*}X, \alpha)) \in f_{x}^{rel} \).

Accordingly, if the matrices of \( \Phi_{1}, \Phi_{2} \) are as in (1.13) with indices 1, 2, respectively, \( f \) is generalized c.p.s. iff, \( \forall x \in M_{1} \) and \( \forall X \in T_{x}M_{1} \), \( \forall \alpha \in T \ast_{f}(x)M_{2} \) one has

\[ (1.50) \]

\[
A_{2}(f_{*}X) + \sharp\pi_{2}\alpha = f_{*}(A_{1}X + \sharp\pi_{1}f^{*}\alpha),
\]

\[
\flat_{\sigma_{1}}X - (f^{*}\alpha) \circ A_{1} = f^{*}(\flat_{\sigma_{2}}(f_{*}X) - \alpha \circ A_{2}).
\]

Furthermore, if we look at (1.50) for either \( X = 0 \) or \( \alpha = 0 \), we see that \( f \) is generalized c.p.s. iff the following three conditions required in [5] hold

\[ (1.51) \]

\[
\pi_{2} = f_{*}\pi_{1}, \quad \sigma_{1} = f^{*}\sigma_{2}, \quad A_{2} \circ f_{*} = f_{*} \circ A_{1}.
\]

Following is an example of utilization of the notion of a c.p.s. mapping which shows its restrictive character.

We define a generalized c.p.s. Lie group to be a real Lie group \( G \) endowed with a generalized c.p.s. structure \( \Phi \) such that the multiplication mapping \( \mu(g_{1}, g_{2}) = g_{1}g_{2} \) is a generalized c.p.s. mapping \( \mu : (G \times G, \Phi \oplus \Phi) \to (G, \Phi) \).

If the matrix of \( \Phi \) is that of formula (1.13), the first condition (1.51) tells that the pair \((G, \pi)\) is a Poisson-Lie group [23]. The second condition (1.51) gives

\[ (1.52) \]

\[
\sigma_{g_{1}}(X, X') + \sigma_{g_{2}}(Y, Y') = \sigma_{g_{1}g_{2}}(L_{g_{1}}Y + R_{g_{2}g_{1}}X, L_{g_{1}}Y' + R_{g_{2}g_{1}}X'),
\]

\[ \forall g_{1}, g_{2} \in G, X, X' \in T_{g_{1}}G, Y, Y' \in T_{g_{2}}G \text{ and where } L, R \text{ denote left and right translations, respectively. Indeed, it is easy to see (e.g., [23]) that}
\]

\[ \mu_{*}(X, Y) = L_{g_{1}}Y + R_{g_{2}g_{1}}X. \]

For \( X = X' = 0 \), respectively, \( Y = Y' = 0 \) condition (1.52) shows that \( \sigma \) is left-invariant and right-invariant, respectively. Then, the case \( X' = Y = 0 \) shows that the only possibility is \( \sigma = 0 \). Finally, in a similar way, we see that the third condition (1.51) simply means that \( A \) is a bi-invariant tensor field on \( G \). Moreover, in view of \( \sigma = 0 \) and of (1.13), in the complex and paracomplex cases \( A \) is an almost c.p. structure, respectively.

Now, if we also look at the integrability conditions given by Theorem 1.1 we get
Proposition 1.6. A generalized complex, respectively paracomplex, Lie group is a classical complex, respectively paracomplex Lie group \((G, A)\) endowed with a multiplicative Poisson bivector field \(\pi\) such that the pair \((\pi, A)\) is a Poisson-Nijenhuis structure.

2 Reduction of generalized c.p.s. structures

Reduction theory is an important chapter of symplectic and Poisson geometry. Geometric reduction leads to a symplectic, respectively Poisson, structure on a quotient of a submanifold of a given symplectic or Poisson manifold. If the submanifold is obtained as a non critical level set of a momentum map of a Hamiltonian group action, one has the Marsden-Weinstein reduction, which has many applications in mechanics and physics.

The general, Poisson framework of geometric reduction was given by Marsden and Ratiu [19] and we briefly describe it as follows. Let \((M, \pi)\) be a Poisson manifold, and \(\iota : N \hookrightarrow M\) a submanifold. A subbundle \(E \subseteq T_N M\) is a reduction-control bundle on \(N\) if

a) \(E \cap TN = TF\) where \(F\) is a foliation of \(N\) by the fibers of a submersion \(s : N \to Q\),

b) \(\forall \varphi, \psi \in \mathcal{C}^\infty(M)\) such that \(d\varphi|_N, d\psi|_N \in \text{ann} E\) the Poisson bracket \(\{\varphi, \psi\}\) satisfies the same condition \(d\{\varphi, \psi\}|_N \in \text{ann} E\) (\(\text{ann}\) denotes the annihilator of the bundle \(E\)),

c) \(\sharp_\pi(\text{ann} E) \subseteq TN + E\).

The Marsden-Ratiu reduction theorem says that if \(E\) is a reduction-control bundle there exists a unique Poisson structure \(\pi_Q\) on \(Q\) such that \(\forall \lambda \in T^*Q\) one has

\[
\sharp_{\pi_Q} \lambda = s_*(pr_{TN} \sharp_\pi s^* \lambda),
\]

where \(s^* \lambda\) is an extension of \(s^* \lambda\) to \(T_N M\) such that \(s^* \lambda|_N \in \text{ann} E\) (an \(E\)-controlled extension). Formula (2.1) holds \(\forall x \in N, y = s(x) \in Q\). The projection \(pr_{TN}\) is defined by the decomposition of condition c) for \(E\); it may not be uniquely defined but, any two values of this projection differ by a vector in \(TF\) and the projection by \(s_\pi\) is well defined. \(E\)-controlled extensions of \(s^* \lambda\) may be obtained by asking them to vanish on a normal bundle \(\nu N\) of \(N\) in \(M\) (i.e., \(T_N M = TN \oplus \nu N\)) which is of the form \(\nu N = E' \oplus C\), where \(E'\) is a complement of \(TF\) in \(E\) and \(C\) is a complement of \(E\) in \(T_N M\).

The independence of \(\pi_Q\) on the choice of the controlled extensions is part of the proof of the reduction theorem [19]. The Poisson structure \(\pi_Q\) is the reduction of \(\pi\) via \((N, E)\).
Now, let us consider a generalized c.p.s. manifold \((M, \Phi)\), where \(\Phi\) has the matrix (1.13), and a submanifold \(\iota : N \hookrightarrow M\) with a \(\pi\)-reduction-control bundle \(E\) and with the reduced structure \(\pi_Q\) of the quotient manifold \(Q = M/F\) \((TF = E \cap TN)\). We would like to be able to reduce the whole structure \(\Phi\) to \(Q\), and we will prove a theorem which shows that, if hypotheses that ensure the reducibility of \(\Phi\) to a generalized, almost c.p.s. structure are satisfied, the reduced structure is integrable.

**Theorem 2.1.** Assume that the configuration \((M, \Phi, N, E, Q, \pi_Q)\) considered above satisfies the following hypotheses: 1) \(\sharp \pi(\text{ann} E) \subseteq TN\); 2) \(A(TN) \subseteq TN, A(E) \subseteq E\) and \(A|_{TN}\) sends \(\mathcal{F}\)-projectable vector fields \(X \in \Gamma TN\) to \(\mathcal{F}\)-projectable vector fields \(AX \in \Gamma TN\); 3) \(\forall Z \in \Gamma E\) one has \(\iota^*(i(Z)\sigma) = 0, L_Z(\iota^*\sigma) = 0\). Then \(Q\) has unique tensor fields \(A_Q, \sigma_Q\) which, together with the reduced Poisson structure \(\pi_Q\), define a generalized c.p.s. structure \(\Phi_Q\) on \(Q\).

**Proof.** We know that \(\pi_Q\) exists and is Poisson from the Marsden-Ratiu theorem; hypothesis 1) means that \(\text{ann} E\) and \(\text{ann} TN\) are \(\pi\)-orthogonal, it is stronger than property c) of \(E\) and is required for the continuation of the present proof. The existence of a projection \(A_Q\) of \(A\) is obvious from hypothesis 2). Hypothesis 3) ensures that \(\iota^*\sigma = s^*\sigma_Q\) for a well defined 2-form \(\sigma_Q\); indeed, in foliation theory it is known that the existence of \(\sigma_Q\) is ensured by the conditions \(i(Z)(\iota^*\sigma) = 0, L_Z(\iota^*\sigma) = 0\), \(\forall Z \in \Gamma TF\) and these conditions are implied by hypothesis 3). Thus, the theorem will be proven if we check the algebraic and integrability conditions of a generalized c.p.s. structure, which is done as follows.

1. \(A_Q^2[X]_{T_xN} = [A^2 X]_{T_xN} = [\epsilon X - \sharp \pi \circ b_\sigma X]_{T_xN} = \epsilon[X]_{T_xN} - \sharp \pi_Q \circ b_\sigma [X]_{T_xN}\), where \(x \in N\), we have identified a vector in \(T_{s(x)}Q\) with an equivalence class \([X]_{T_xN}\) modulo \(T_xN\) on \(N\) and
   \[
   \sharp \pi_Q \circ b_\sigma [X]_{T_xN} = [\sharp \pi \circ b_\sigma X]_{T_xN}
   \]
   in view of (2.1) and because, by hypotheses 3), \(b_\sigma \tilde{X}\) is a controlled extension of \(s^*(b_\sigma [X]_{T_xN})\).

2. The compatibility of \(\sigma_Q\) with \(A_Q\) is trivial and the compatibility of \(\pi_Q\) with \(A_Q\) is a consequence of (2.1) and of the fact that, if \(\lambda \in \Omega^1(Q)\) and \(\tilde{\lambda}\) is a controlled extension of \(s^*\lambda, \tilde{\lambda} \circ A\) is a controlled extension of \(s^*(\lambda \circ A_Q)\).
3. The integrability condition i), of Theorem 1.1 holds, and checking condition iv) of Theorem 1.1 is trivial because we have \( s^*(\sigma_{AQ}) = \iota^* \sigma_A \), and \( s^* \) is injective.

4. Condition iii) of Theorem 1.1 holds since we have

\[
N_{AQ}([X]_{T_xF}, [Y]_{T_xF}) = [N_A(X,Y)]_{T_xF}
\]

\[
= [\sharp_{\pi}(i(Y)i(X)d\sigma)]_{T_xF} = \sharp_{\pi_Q}(i([Y]_{T_xF})i([X]_{T_xF})d\sigma_Q);
\]

the last equality holds because hypothesis 3) implies that \( i(Y)i(X)d\sigma \) is a controlled extension of \( s^*(i([Y]_{T_xF})i([X]_{T_xF})d\sigma_Q) \).

5. The proof of the fact that the Schouten concomitant \( C_{(\pi_Q, AQ)} \) vanishes appears in the proof of the reduction theorem for Poisson-Nijenhuis structures \([25]\), page 92-93.

\[\square\]

Remark 2.1. From facts included in the proof of Theorem 2.1 we get the following explicit formula for the reduced structure \( \Phi_Q \)

\[(2.2)\]

\[
\Phi_Q([X]_{T_xF}, \lambda) = (s_*pr_{TN}(pr_{TM}\Phi(X, s^*\lambda)), \iota^*pr_{T^*M}\Phi(X, s^*\lambda)),
\]

where \( X \in T_xN, \lambda \in T^*_s(x)Q, x \in N \) and \( s^*\lambda \) is a controlled extension of \( s\lambda \).

For any submanifold \( \iota : N \hookrightarrow (M, \Phi) \), where \( \Phi \) is a generalized, almost c.p.s. structure, one has the following differentiable field of planes along \( N \)

\[(2.3)\]

\[
\nu^{ei}N = pr_{TM}(\Phi(TN \oplus ann\, TN))
\]

\[
= \{AX + \sharp_{\pi}\alpha / X \in TN, \alpha \in ann\, TN\} = A(TN) + \sharp_{\pi}(ann\, TN).
\]

The field \( \nu^{ei}N \), which may not have a constant dimension, will be called the enlarged image field of \( N \).

We assume that \( \Phi \) is integrable and that \( \nu^{ei}N \) is a vector bundle over \( N \) (i.e., all its planes are of the same dimension). Then, we shall discuss conditions ensuring that \( \Phi \) can be reduced via \( (N, \nu^{ei}N) \).

From the last equality (2.3) it follows that

\[(2.4)\]

\[
ann(\nu^{ei}N) = ann(A(TN)) \cap (ann\, TN)^{\perp_{\pi}}
\]

and, as a consequence of (2.4), we get \( \sharp_{\pi}ann(\nu^{ei}N) \subseteq TN \), which is condition 1) of Theorem 2.1. Moreover, hypothesis \( A(E) \subseteq E \) of condition 2) of
Theorem 2.1 is implied by $A(TN) \subseteq TN$. Indeed, if we ask the latter, we also have $A(\text{ann} TN) \subseteq \text{ann} TN$ and (2.3) yields $A(\nu^{ei} N) \subseteq \nu^{ei} N$.

Condition b) of reduction-control is equivalent with (2.5) $$(L_{\tilde{Z}} \pi)|_{\text{ann} E} = 0,$$
where $\tilde{Z}$ is an extension of $Z \in \Gamma E$ to $M$ (evaluate the Lie derivative (2.4) on the arguments $d\varphi|_{N}, d\psi|_{N}$ of condition b)) and for $E = \nu^{ei} N$ (2.3) is equivalent to

(2.6) $$(L_{AX} \pi)|_{\text{ann}(\nu^{ei} N)} = 0, (L_{\tilde{\alpha}} \pi)|_{\text{ann}(\nu^{ei} N)} = 0,$$
where $\tilde{X}$ extends $X \in \Gamma TN$ and $\tilde{\alpha}$ extends $\alpha \in \Gamma(\text{ann} TN)$. The second condition (2.6) always holds. To see this, we use the characterization of Poisson structures via the Schouten-Nijenhuis bracket, $[\pi, \pi] = 0$, and the fact that $-[\pi,.]$ is the Lichnerowicz coboundary $\sigma$, which is the contravariant, exterior differential on the Poisson manifold $(M, \pi)$ [23]. Since $\forall \lambda \in \text{ann}(\nu^{ei} N)$ we have $\sharp_{\pi} \lambda \in TN$, the usual formulas of the Lie derivative and of the contravariant, exterior differential (e.g., [23], formula (4.8)) yield

$$(L_{\sharp_{\pi} \alpha} \pi)(\lambda_1, \lambda_2) = -\sigma(\pi) = [\pi, \pi](\alpha, \lambda_1, \lambda_2) = 0.$$ 

In what follows, invariant always means $A$-invariant, and we assume that this condition holds. Then

(2.7) $$\nu^{ei} N \cap TN = \{AX + \sharp_{\pi} \xi / X \in TN, \xi \in \text{ann} TN, \sharp_{\pi} \xi \in TN\}$$
$$= A(TN) + \sharp_{\pi}((\text{ann} TN) \cap (\text{ann} TN)^{\perp_\pi}),$$
and we shall compute the Lie bracket of vector fields (2.7) with differentiable $X$ and $\xi$ (if any). Using integrability conditions i) (which implies $[\sharp_{\pi} \xi, \sharp_{\pi} \eta] = \sharp_{\pi}\{\xi, \eta\}_{\pi}$, ii) (under the form $R(\pi, A) = 0$, where $R$ is defined by (1.20)) and iii) of Theorem 1.1 we get

(2.8) $$[AX + \sharp_{\pi} \xi, AY + \sharp_{\pi} \eta] = A([AX, Y] + [X, AY])$$
$$-A[X, Y] + [\sharp_{\pi} \xi, Y] - [\sharp_{\pi} \eta, X]) + \sharp_{\pi}(i(Y)i(X)d\sigma + \{\xi, \eta\}_{\pi}$$
$$-L_{AX} \xi + L_{Y}(\xi \circ A) + L_{AX} \eta - L_{X}(\eta \circ A)).$$

The first term of (2.8) is of the form required by (2.7). Since the left hand side and the first term of the right hand side of (2.8) are in $TN$, so is
the second term of the right hand side. Thus, the only condition required in order to ensure that the bracket (2.8) belongs to $\nu^e N \cap TN$ is

\[
(2.9) \quad i(Y)i(X)d\sigma + \{\xi, \eta\}_\pi - L_{AY}\xi + L_Y(\xi \circ A) + L_{AY}\eta - L_X(\eta \circ A) \in \text{ann} TN.
\]

Using (1.11) to evaluate $\{\xi, \eta\}_\pi$ on $V \in TN$ we get 0 (Lie derivatives are to be computed using extensions of vector fields and forms from $N$ to $M$ and the result does not depend on the choice of the extension), hence, $\{\xi, \eta\}_\pi \in \text{ann} TN$. Similarly, the evaluation of the last four terms of (2.9) on $V \in TN$ shows that each of them belongs to $\text{ann} TN$. Therefore, (2.9) reduces to

\[
(2.10) \quad \iota^* d\sigma = 0.
\]

Thus, if (2.10) holds and if $\nu^e N \cap TN$ is a distribution of planes on $N$ that has a constant dimension and local generators $AX + \sharp\pi \xi$ ($X \in TN, \xi \in (\text{ann} TN) \cap (\text{ann} TN)^{\perp\pi}$) where $X, \xi$ are differentiable then $\nu^e N \cap TN$ is a foliation $\mathcal{F}_N$ of the submanifold $N$. By (2.7), the existence of the required local generators is ensured if we ask $(\text{ann} TN) \cap (\text{ann} TN)^{\perp\pi}$ to have a constant dimension or, equivalently, if we ask that $\dim(TN + \sharp\pi \text{ann} TN) = \text{const}$.

Moreover, we can also prove that $A|_N$ sends $\mathcal{F}_N$-foliated vector fields to $\mathcal{F}_N$-foliated vector fields. Let $X \in \chi^1(N)$ be $\mathcal{F}_N$-foliated and take $Y \in \Gamma TF_N$. We have to check that $[Y, AX] \in TF_N$. If $Y = AV$ with $V \in \chi^1(N), A$-invariance, condition (2.10), and the integrability condition iii) of Theorem 1.1 yield

\[
[A, AX] = A[AV, X] + A[V, AX] - A^2[V, X] + \sharp\pi (i(X)i(V)d\sigma) \in TF_N.
\]

If $Y = \sharp\pi \xi \in \Gamma TN$ where $\xi \in \text{ann} TN$, the integrability condition ii') of Theorem 1.1 and the expression (1.20) of the Schouten concomitant $R(\pi, A)$ yield

\[
[\sharp\pi \xi, AX] = (L_{\sharp\pi \xi} A)(X) + A[\sharp\pi \xi, X] = \sharp\pi (L_X(\xi \circ A) - L_{AX}\xi) + A[\sharp\pi \xi, X] \in TF_N.
\]

Continuing to keep the $A$-invariance condition enforced, let us see the meaning of hypothesis 3) of Theorem 2.1, where we look at arguments $Z = AX$ ($X \in TN$) and $Z = \sharp\pi \alpha$ ($\alpha \in \text{ann} TN$). The conditions for $\sigma$ are

\[
(2.11) \quad \sigma(AX, Y) = 0, \sigma(\sharp\pi \alpha, Y) = 0, \quad Y \in TN, \alpha \in \text{ann} TN.
\]
The second condition (2.11) is equivalent with the invariance of $TN$ by $\sharp_{\pi} \circ \flat_{\sigma}$ and, in view of (1.15), this is ensured by the $A$-invariance of $N$.

The condition 3) for $d\sigma$ means

\[(2.12)\quad d\sigma(AX, Y_1, Y_2) = 0, \quad d\sigma(\sharp_{\pi} \alpha, Y_1, Y_2) = 0,\]

where $X, Y_1, Y_2 \in TN, \alpha \in \text{ann} TN$. The first condition (2.12) is implied by (2.10) and the second condition is a consequence of integrability condition iii), Theorem 1.1.

Accordingly, we get the following reduction theorem.

**Theorem 2.2.** Let $(M, \Phi)$ be a generalized c.p.s. manifold and $\iota : N \to M$ an $A$-invariant submanifold. Assume that the following hypotheses are satisfied: 1) $\dim \nu^{\iota^\ast} N = \text{const.}$, $\dim(\nu^{\iota^\ast} N \cap TN) = \text{const.}$ and $\dim(TN + \sharp_{\nu} \text{ann} TN) = \text{const.}$, 2) the 2-form $\sigma$ satisfies the conditions $\iota^\ast \sigma_A = 0, \iota^\ast d\sigma = 0$, 3) the foliation $F_N$, which exists because of 1) and 2), consists of the fibers of a submersion $s : N \to Q$, 4) the underlying Poisson structure satisfies the first condition (2.6). Then, $Q$ has a reduced generalized c.p.s. structure $\Phi_Q$.

**Proof.** The hypotheses and the previous analysis show that $N$ and $E = \nu^{\iota^\ast} N$ satisfy all the hypotheses of Theorem 2.1. □

**Remark 2.2.** If the Poisson structure $\pi$ of the generalized c.p.s. structure $\Phi$ is zero (e.g., $\Phi$ is a classical c.p.s. structure) and $N$ is an invariant submanifold then $(A|_N, 0, \iota^\ast \sigma)$ is a generalized c.p.s. structure $\Phi_N$ of $N$ and reductions provided by Theorem 2.1 are just the projection of $\Phi_N$ to a space of leaves.

Another interesting field of planes along $N$ is the pseudo-normal field of the submanifold $N$ with respect to the Poisson structure $\pi$, $\nu_\pi N = \sharp_{\pi} (\text{ann} TN)$ (the name pseudo-normal was introduced in [27]). It follows immediately that

\[(2.13)\quad \text{ann}(\nu_\pi N) = \{\lambda \in T_N^* M / \sharp_{\pi} \lambda \in TN\},\]

therefore, $\sharp_{\pi} (\text{ann} \nu_\pi N) \subseteq TN$. Furthermore, for two vector fields in $\nu_\pi N \cap TN$ that are of the form $\sharp_{\pi} \lambda_1, \sharp_{\pi} \lambda_2$ where $\lambda_1, \lambda_2$ are differentiable and belong to $(\text{ann} TN) \cap (\nu_\pi N)$, the bracket necessarily belongs to $TN$ and it is given by

\[(2.14)\quad [\sharp_{\pi} \lambda_1, \sharp_{\pi} \lambda_2] = \sharp_{\pi} \{\lambda_1, \lambda_2\}_\pi.\]
It is easy to check that \( \{ \lambda_1, \lambda_2 \}_\pi \in \text{ann } TN \), hence, the bracket also belongs to \( \nu_\pi N \). Thus, if the field \( \nu_\pi N \cap TN \) consists of planes of the same dimension and is locally spanned by vector fields \( \sharp_\pi \lambda \) with differentiable 1-forms \( \lambda \in (\text{ann } TN) \cap (\text{ann } \nu_\pi N) \) then this field is an involutive distribution and we have a foliation \( \mathcal{C}(N) \) of \( N \) such that \( TC = \nu_\pi N \cap TN \) (see also [23], p.104).

This situation leads to one more reduction theorem:

**Theorem 2.3.** Let \( N \) be an \( A \)-invariant submanifold of a generalized c.p.s. manifold \( (M, \Phi) \). If \( \dim \nu_\pi N = \text{const.} \) and \( \dim(\nu_\pi N \cap TN) = \text{const.} \), \( N \) has a foliation \( \mathcal{C}(N) \) with tangent bundle \( TC = \nu_\pi N \cap TN \), and in case the leaves of \( \mathcal{C}(N) \) are the fibers of a submersion \( s : N \to Q \), \( Q \) has a reduced generalized c.p.s. structure \( \Phi_Q \) of \( \Phi \) via \( (N, \nu_\pi N) \).

**Proof.** The hypotheses of the corollary imply conditions 1), 2) and 3) of Theorem 2.1. In particular, \( \nu_\pi N \cap TN \) is spanned by vector fields \( \sharp_\pi \lambda \) with differentiable 1-forms \( \lambda \in (\text{ann } TN) \cap (\text{ann } \nu_\pi N) \) because the constancy of the dimensions of \( \nu_\pi N \), \( \nu_\pi N \cap TN \) implies \( \dim(TN + \nu_\pi N) = \text{const.} \), therefore \( \dim(\text{ann } TN) \cap (\text{ann } \nu_\pi N) = \text{const.} \). The projectability of \( A \), the existence of the reduced Poisson structure \( \pi_Q \), and the conditions of hypothesis 3) were proven during the proof of Theorem 2.2 (the difference between the present situation and the one in Theorem 2.2 is that the vectors \( \sharp_\pi \xi \) with \( \xi \in (\text{ann } TN) \cap (\text{ann } TN)^{\perp_\pi} \) suffice to span \( TC(N) \)).

**Corollary 2.1.** Let \( (M, \Phi) \) be a non degenerate, generalized c.p.s. manifold where \( \Phi \) is associated to the Hitchin pair \( (\varpi, A) \). Let \( \iota : N \hookrightarrow (M, \Phi) \) be an \( A \)-invariant submanifold such that: 1) \( \text{rank } \iota^* \varpi = \text{const.} \), 2) the leaves of the foliation \( \mathcal{C}(N) \) \( (TC = \nu_\pi N \cap TN) \) are the fibers of a submersion \( s : N \to Q \). Then \( Q \) has the reduced generalized c.p.s. structure \( \Phi_Q \) of \( \Phi \) via \( (N, \nu_\pi N) \) and \( \Phi_Q \) is the non degenerate, generalized c.p.s. structure associated to the Hitchin pair \( (\varpi_Q, A_Q) \), where \( \varpi_Q \) is the reduction of \( \varpi \) and \( A_Q \) is the projection of \( A_N = A|_{TN} \).

**Proof.** Under the hypotheses, \( \nu_\pi N = (TN)^{\perp_\varpi} \) and the existence of the reduced symplectic form \( \varpi_Q \) is well known (e.g., [23], p. 103). Then, the assertion of the present corollary clearly follows from Theorem 2.3. We may also notice that it is easy to justify the assertion of the corollary straightforwardly. Indeed all we still need is the fact that \( A_N \) sends a \( \mathcal{C}(N) \)-foliated vector field \( X \in \chi^1(N) \) to a \( \mathcal{C}(N) \)-foliated vector field \( AX \). In view of the
definition of $C(N)$, this is equivalent with

$$\varpi([Y,AX],X') = 0, \ \forall X, X' \in \chi^1(N), \ \forall Y \in \Gamma(TN \cap T^\perp)$$

where $X, X'$ are $C(N)$-foliated vector fields, which follows from

$$d\varpi_A(Y,X,X') = 0, \ d\varpi(Y,AX,X') = 0.$$

Finally, the following result is a straightforward consequence of Theorem 2.3 and may be seen as a Marsden-Weinstein reduction theorem for generalized c.p.s. manifolds.

**Theorem 2.4.** Let $(M, \Phi)$ be a generalized c.p.s. manifold with the Poisson structure $\pi$ and the tensor fields $A, \sigma$. Assume that one has a $\pi$-Hamiltonian action of the Lie group $G$ on $M$ with an equivariant momentum map $J : M \to G^*$ ($G$ is the Lie algebra of $G$) such that $J_* \circ A = J_*$. Let $\gamma \in G^*$ be a common regular value of all the restrictions of $J$ to the symplectic leaves of $\pi$ with isotropy group $G_\gamma$. Assume that the level set $M_\gamma = J^{-1}(\gamma)$ is non void and the foliation of $M_\gamma$ by the orbits of $G_\gamma$ is by the fibers of a submersion $s : M \to Q$. Then the structure $\Phi$ reduces to a generalized c.p.s. structure $\Phi_Q$ of $Q$.

**Proof.** For all the notions involved in Theorem 2.4 and for the existence of the reduced Poisson structure $\pi_Q$ we refer the reader to [23], pp. 110-113. The hypothesis $J_* \circ A = J_*$ ensures that $M_\gamma$ is $A$-invariant, and the conclusion follows from Theorem 2.3.

**Remark 2.3.** Notice that in Theorem 2.4 we didn’t have to ask the action to be by generalized c.p.s. mappings, which would have been more restrictive. On the other hand, if $\pi = 0$ the action of $G$ must be trivial, $Q = M$, and we do not get a true reduction.

### 3 Generalized c.p.s. submanifolds

In this section we discuss our second subject, submanifolds. The naive definition of a generalized c.p.s. submanifold $N$ of a generalized c.p.s. manifold $(M, \Phi)$ would be by asking the immersion $\iota : N \hookrightarrow M$ to be a generalized...
c.p.s. morphism. Like in Poisson geometry, this condition is very restrictive because it asks $N$ to be a Poisson submanifold of $M$, hence, a union of symplectic leaves of the Poisson structure $\pi$ of $\Phi$. The same situation appears if we try to get the submanifold structure by reducing $\Phi$ via $N$ with control subbundle $E = 0$.

The good notion of a submanifold of a Poisson manifold, which gets an induced Poisson structure, is that of a Poisson-Dirac submanifold [6]. The submanifolds of a generalized, complex manifold with an induced generalized, complex structure were defined by Ben-Bassat-Boyarchenko [2] and, in this section, we discuss the meaning of the Ben-Bassat-Boyarchenko definition in classical terms. (A different notion of submanifold, which does not require an induced structure was studied in [9].)

We begin by recalling the notion of a Poisson-Dirac submanifold. If $f : N \to M$ is a differentiable mapping and $L$ is a Dirac structure on $M$, we obtain a field $f^*(L)$ of maximal isotropic subspaces of the fibers of $\bigotimes N$ by putting

\[(3.1) \ f^*(L)_x = \{(X, f^*\alpha) / X \in T_xN, \alpha \in T^*_xM, (f_\star X, \alpha) \in L_{f(x)} \} \quad (x \in N)\]

(e.g., [4]). The field (3.1) may not be differentiable; if it is, $f$ is called a \textit{backward Dirac map}. If $f$ is the embedding $\iota : N \hookrightarrow (M, L)$ of a submanifold, and if $L_N = \iota^*(L)$ is differentiable, $L_N$ must be integrable [8], $N$ is called a \textit{proper submanifold}, and $L_N$ is the \textit{induced Dirac structure}.

Particularly, since a Poisson structure $\pi$ may be seen as the Dirac structure $\{(\sharp_\pi \alpha, \alpha) / \alpha \in T^*M\}$, one defines [6]

**Definition 3.1.** A proper submanifold $\iota : N \hookrightarrow M$ of a Poisson manifold $(M, \pi)$ such that the induced Dirac structure is Poisson is called a \textit{Poisson-Dirac submanifold}.

It was shown in [6] that the proper submanifold $\iota : N \hookrightarrow (M, \pi)$ is Poisson-Dirac iff

\[(3.2) \quad TN \cap \sharp_\pi(ann TN) = TN \cap \nu_\pi N = 0.\]

An equivalent characterization is obtained by taking the annihilator of (3.2), which yields

\[(3.3) \quad ann(\nu_\pi N) + ann TN = T^*_N M.\]
In [2] one uses a similar procedure for a definition of a notion of generalized, complex submanifold and, in the mean time, we refer to generalized, complex structures only. Let $\iota : N \hookrightarrow (M, \Phi)$ be a submanifold of a generalized, almost complex manifold and let $L \subseteq T_{\text{big}}^c M$ be the $i$-eigenbundle of $\Phi$. Then, $\iota^*(L)$ may be constructed like in the real case and, if $\iota^*(L)$ is differentiable, we will say that the submanifold is proper. If $\Phi$ is integrable and $N$ is proper, $\iota^*(L)$ is closed by Courant brackets (like in the real case [3]). However, we may have $\iota^*L \cap \iota^*\overline{L} \neq 0$, and $\iota^*L$ may not be a generalized, complex structure on $N$. The definition of [2] is

**Definition 3.2.** A submanifold $\iota : N \hookrightarrow M$ of a generalized, almost complex manifold $(M, \Phi)$ is a generalized, almost complex submanifold if $N$ is proper and $\iota^* L$ is a generalized, almost complex structure on $N$, called the induced structure.

The following theorem expresses the conditions of Definition 3.2 in classical terms.

**Theorem 3.1.** Let $\Phi$ be a generalized, almost complex structure of matrix form (1.13) on $M$ and let $N$ be a submanifold of $(M, \Phi)$. Then $N$ is a generalized, almost complex submanifold iff it satisfies the following three conditions: i) $N$ is a Poisson-Dirac submanifold of $(M, \pi)$, ii) $A(TN) \subseteq TN + \text{im} \sharp_\pi = TN \oplus \sharp_\pi(\text{ann} TN)$, iii) $\text{pr}_{TN} \circ A$, where $\text{pr}_{TN}$ is the natural projection of the direct sum of ii) onto its first term, is differentiable.

**Proof.** The equality included in condition ii) of the theorem is an immediate consequence of (3.2), (3.3), which hold if condition i) holds. Now, let us prove the necessity of i). The $i$-eigenbundle of $\Phi$, which is the image of $(1/2)(\text{Id} - i\Phi)$, is given by

\[
(3.4) \quad L = \{(X - i(AX + \sharp_\pi \xi), \xi - i(\flat_\sigma X - \xi \circ A)) / X \in TM, \xi \in T^*M\}.
\]

Denote by $\iota$ the immersion of $N$ in $M$. Using (3.4) and the natural identification between $T^*N$ and $T^*_N M/\text{ann} TN$, which represents the covectors of $N$ as equivalence classes $[\xi]_{\text{ann} TN}$, we see that the pullback of $L$ to $N$ is

\[
(3.5) \quad \iota^*L = \{(X - i(AX + \sharp_\pi \xi), [\xi - i(\flat_\sigma X - \xi \circ A)]_{\text{ann} TN} / X \in TN, AX + \sharp_\pi \xi \in TN\}.
\]
On the other hand, if $N$ is a generalized, almost complex submanifold of $(M, \Phi)$, $\iota^*L$ defines a generalized, almost complex structure $\iota^*\Phi$ on $N$ and must be of the form

\begin{equation}
\iota^*L = \{(Y - i(A'Y + \sharp^{\pi}[\eta]_{ann\,TN}), [\eta]_{ann\,TN} - i(\flat^{\sigma'}Y - [\eta]_{ann\,TN} \circ A'))\},
\end{equation}

where $A', \pi', \sigma'$ are the elements of the matrix representation of $\iota^*\Phi$, and $Y \in TN, \eta \in T_N^*M$. Thus, every pair of the form (3.5) is identifiable with a pair of the form (3.6), and, since the real part of the equal, vector and covector, components of the two pairs must be the same, we must have $X = Y$ and $\xi \sim \eta$ modulo $ann\,TN$. The case $X = Y = 0$ shows that any $\eta \in T_N^*M$ is equivalent modulo $ann\,TN$ with some $\xi \in ann(\nu_\pi N)$, i.e., condition (3.3) must hold and $N$ is a Poisson-Dirac submanifold of $(M, \pi)$ with the induced Poisson structure $\pi'$.

Now, for $Y = X \in TN, \eta \in ann\,TN$, (3.6) is a pair of the form

\begin{equation}
(X - iA'X, -i\flat^{\sigma'}X)
\end{equation}

and the corresponding pair (3.5) must be of the form

\begin{equation}
(X - iAX + \sharp^{\pi}X, -i[\flat^{\sigma}X - \xi \circ A]_{ann\,TN}),
\end{equation}

where $\xi \in ann\,TN$ and $AX + \sharp^{\pi}X \in TN$. The equality of the pairs (3.7), (3.8) yields

\begin{equation}
A'X = AX + \sharp^{\pi}X,
\end{equation}

whence

\begin{equation}
A(TN) \subseteq TN \oplus \sharp^{\pi} (ann\,TN),
\end{equation}

which is condition ii) of the theorem. Furthermore, (3.9) implies

\begin{equation}
A' = pr_{TN} \circ A,
\end{equation}

where the projection is defined by the decomposition (3.10), hence, condition iii) also holds.

For another expression of $A'$ and in order to compute the 2-form $\sigma'$ we denote by $\alpha_X \in ann\,TN$ a 1-form such that $\sharp^{\pi}\alpha_X = pr_{\sharp^{\pi}ann\,TN}AX$. The form $\alpha_X$ is defined up to the addition of a term $\gamma \in ker\,\sharp^{\pi}$, i.e., the equivalence class $[\alpha_X]_{(ann\,TN) \cap (ker\,\sharp^{\pi})}$ is well defined. Thus, for a differentiable vector
field $X \in \chi^1(N)$, the differentiability of $\alpha_X$ is not ensured and may be assumed if we assume $\dim((\text{ann} TN) \cap (\ker \sharp^\pi)) = \text{const.}$ or, equivalently, $\dim(TN + \text{im} \sharp^\pi) = \text{const.}$ Then, $A'$ is given by

$$A'X = AX - \sharp^\pi(\alpha_X).$$

Before going on, we notice the following simple result

**Lemma 3.1.** If conditions i), ii) of Theorem 3.1 hold, and if $\gamma \in (\text{ann} TN) \cap (\ker \sharp^\pi)$ then $\gamma \circ A \in \text{ann} TN$.

**Proof.** \(\forall Z \in TN\) we have

$$\gamma(AZ) = \gamma(\sharp^\pi \alpha_Z) = -\alpha_Z(\sharp^\pi \gamma) = 0.$$

Back to the proof of Theorem 3.1, the definition of $\alpha_X$ shows that $\xi$ of (3.9) is of the form $\xi = -\alpha_X + \gamma$ with $\gamma \in (\text{ann} TN) \cap (\ker \sharp^\pi)$. Accordingly, Lemma 3.1 implies

$$[\flat^\sigma_X - \xi \circ A]_{\text{ann}TN} = [\flat^\sigma_X - \alpha_X \circ A]_{\text{ann}TN},$$

and the equality of the pairs (3.7), (3.8) yields

$$[\flat^\sigma_X - \xi \circ A]_{\text{ann}TN} = [\flat^\sigma_X + \alpha_X \circ A]_{\text{ann}TN},$$

equivalently,

$$\sigma'(X,Y) = (\nu^* \sigma)(X,Y) + \alpha_X(AY)$$

$$= (\nu^* \sigma)(X,Y) - \pi(\alpha_X, \alpha_Y) \quad (X,Y \in TN).$$

Notice that, although $\alpha_X$ is not uniquely defined, the result of (3.14) is well defined in view of Lemma 3.1

Thus, we proved that a generalized, complex submanifold satisfies conditions i), ii), iii) and we computed the classical tensor fields of the induced structure.

For the converse result we first check that i), ii), iii) imply $\nu^* L \cap \overline{\nu^* L} = 0$.

Notice that, if condition i) holds, we have (3.3) and condition ii) is equivalent with (3.10).
Then, a pair of the form (3.5) belongs to $\iota^*L \cap \iota^*L$ iff it also has the form

$$(X + i(AX + \sharp_\pi \zeta), [\zeta + i(b_\sigma X - \zeta \circ A)]_{ann \, TN})$$

with the same vector $X$, i.e., $\exists \zeta \in T^*_N M$ such that

$$\zeta = \xi + \alpha \quad (\alpha \in ann \, TN), \quad 2AX = -\sharp_\pi (\xi + \zeta),$$

$$(\xi + \zeta) \circ A - 2b_\sigma X \in ann \, TN.$$  

Since these conditions imply

$$2(AX + \sharp_\pi \xi) = -\sharp_\pi \alpha,$$

in view of (3.2), we have

$$(3.16) \quad \sharp_\pi \alpha = 0, \quad AX + \sharp_\pi \xi = 0,$$

and, if we apply $A$ to the second condition (3.16) and use (1.15), we get

$$(3.17) \quad X = -\sharp_\pi b_\sigma X + \sharp_\pi (\xi \circ A).$$

Furthermore, using (3.15), (3.16) and Lemma 3.1, we get

$$(3.18) \quad b_\sigma X - \xi \circ A \in ann \, TN.$$  

Accordingly, (3.2) and (3.17) show that $X = 0$ and, then, (3.16) and (3.18) show that the pair we study must be of the form $(0, \xi)_{ann \, TN}$.

But, $\xi$ is not arbitrary either. Modulo $X = 0$, (3.16), (3.17) and (3.18) imply

$$\xi \in ker \, \sharp_\pi, \quad \xi \circ A \in (ann \, TN) \cap (ker \, \sharp_\pi),$$

and by Lemma 3.1 $\xi \circ A \in ann \, TN$. Thus, composing again by $A$, we get

$$\xi \circ A^2 = -\xi - b_\sigma \circ \sharp_\pi \xi = -\xi \in ann \, TN,$$

and the considered pair is just $(0, 0)$. In other words, we showed that $\iota^*L \cap \iota^*L = 0$.

The previous conclusion means that, $\forall x \in N$, $\iota^*L_x$ defines a generalized complex structure of $T^{big}_x N$, therefore, $\iota^*L_x$ must be of the form (3.6). Then, $\pi'$ of (3.6) is induced by $\pi$ and it is differentiable because $N$ is a Poisson-Dirac submanifold. Furthermore, $A'$ is differentiable because of condition iii) and $\sigma'$ is differentiable because its values depend only on $\sharp_\pi \alpha_X, \sharp_\pi \alpha_Y$ and the definition of $\alpha_X, \alpha_Y$ shows that the previous vector field are differentiable if $A'$ is differentiable. This justifies the fact that $N$ is proper in $(M, \Phi)$, which finishes the proof of the theorem. ∎
In view of Theorem 3.1, we propose the following general terminology.

**Definition 3.3.** A submanifold \( \iota : N \hookrightarrow M \) of a generalized, almost c.p.s. manifold \((M, \Phi)\) is a quasi-invariant submanifold if i) it is a \(\pi\)-Poisson-Dirac submanifold, ii) \(A(TN) \subseteq TN + im \sharp_\pi = TN \oplus \sharp_\pi(ann TN)\), iii) \(A' = pr_{TN} \circ A\) is differentiable.

We notice that quasi-invariance is preserved by a gauge equivalence. Theorem 3.1 tells that “quasi-invariant submanifold” and “generalized, almost complex submanifold in the sense of [2]” are synonymous terms. In the general c.p.s. case the structure \(\Phi\) induces a differentiable, \(g\)-skew-symmetric endomorphism \(\Phi_N\) of the bundle \(T^{big}N\) defined by the matrix

\[
(3.19) \quad \Phi_N = \begin{pmatrix}
A' & \sharp' \\
\flat' & -^tA'
\end{pmatrix},
\]

where \(\pi'\) is induced by \(\pi\) and \(A', \sigma'\) are given by the formulas (3.12), (3.14).

**Theorem 3.2.** If \(\iota : N \hookrightarrow M\) is a quasi-invariant submanifold of a generalized, almost c.p.s. manifold \((M, \Phi)\), the induced structure \(\Phi_N\) is also almost c.p.s. In the complex and paracomplex cases, if \(\Phi\) is integrable, \(\Phi_N\) is integrable too.

**Proof.** For the algebraic part of the proposition it suffices to work at a fixed point \(x \in N\). It is known from [6] that, since \(N\) is a Poisson-Dirac submanifold of \((M, \pi)\), there exists a normal space \(\nu_xN\) \((T_xM = T_xN \oplus \nu_xN)\) such that

\[
(3.20) \quad \pi = \pi_{\nu_xN} + \pi_{T_xN}, \quad \pi_{\nu_xN} \in \wedge^2\nu_xN, \ \pi_{T_xN} \in \wedge^2T_xN;
\]

and the induced Poisson structure is \(\pi'_x = \pi_{T_xN}\). The compatibility of \((\pi', A')\) is a straightforward consequence of the previous remark on \(\pi'\), of the compatibility of \((\pi, A)\) and of formula (3.11).

In order to check the compatibility of \((\sigma', A')\) we use formula (3.14) and, for \(X, Y \in T_xN\), we get

\[
\sigma'(X, A'Y) = \sigma(X, A'Y) + \alpha_X(AA'Y) = \sigma(X, AY) - \sigma(X, \sharp_\pi\alpha_Y)
\]

\[
+ \alpha_X(A^2Y) - \alpha_X(A^2\sharp_\pi\alpha_Y) - \sigma(X, AY) - \sigma(X, \sharp_\pi\alpha_Y) + \sigma(Y, \sharp_\pi\alpha_X)
\]

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\[ +\pi(\alpha_X \circ A, \alpha_Y). \]

If we change the role of \(X,Y\) we see that \(\sigma'(X, A'Y) = \sigma'(A'X, Y)\) as required.

The first condition (1.15) for the induced structure is checked as follows. We have
\[ \#_{\pi} b_{\sigma'} X = \#_{\pi} [b_{\sigma} X + \alpha_X \circ A]_{\text{ann} T N} \quad (X \in T N), \]
and the right hand side of this equality is computable by a representative form of the equivalence class sent by \(\#_{\pi}\) in \(T N\). By (1.15) for the original structure we have
\[ \#_{\pi} (b_{\sigma} X + \alpha_X \circ A) = \epsilon X - A^2 X + \#_{\pi} (\alpha_X \circ A) \]
\[ = \epsilon X - AA' X = \epsilon X - A^2 X - \#_{\pi} \alpha_{A'} X. \]

Hence,
\[ \#_{\pi} (b_{\sigma} X + \alpha_X \circ A + \alpha_{A'} X) \in T N \]
and, since \(\alpha_{A'} X \in \text{ann} T N\), we deduce that
\[ \#_{\pi} b_{\sigma'} X = \epsilon X - A^2 X, \]
which is the required property.

Now, using (3.11), (3.14), we get
\[ (3.21) \quad \Phi_N(X, [\xi]_{\text{ann} T N}) = (AX - \#_\pi \alpha_X + \#_{\pi} \tilde{\xi}, [b_{\sigma} X + \alpha_X \circ A - \tilde{\xi} \circ A]_{\text{ann} T N}), \]
where \(\tilde{\xi}\) is determined by the decomposition \(\xi = \tilde{\xi} + \xi_0, \tilde{\xi} \in \text{ann}(\nu_{\pi} N), \xi_0 \in \text{ann} T N\). Formula (3.21) allows us to write down the general expression of an element of the \(\pm i\) or \(\pm 1\)-eigenbundles, of \(\Phi_N\), respectively, similar to the pairs (3.6) (which was the case of the eigenvalue \(i\) in the complex situation). The results show that these elements also have a corresponding expression of the form (3.5), where, if the (3.6)-like formula is defined by the pair \((Y = X, [\eta]_{\text{ann} T N} = [\xi]_{\text{ann} T N})\), the corresponding (3.5)-like formula is defined by the pair \((X, \tilde{\xi} - \alpha_X)\).

The conclusion is that the Dirac eigenbundles of the structure \(\Phi_N\) are the \(\iota^*\)-pullbacks of the Dirac bundles of the structure \(\Phi\). Obviously, under the hypotheses of the theorem, these pullbacks are differentiable, therefore, if the Dirac eigenbundles of \(\Phi\) are closed by Courant brackets the same holds for the Dirac eigenbundles of \(\Phi_N\). Thus, the assertion about the integrability of the induced structure follows from Proposition 1.1. \(\square\)
The proof of the integrability part of Theorem 3.2 does not hold in the generalized, almost subtangent case; the 0-eigenbundle of the induced structure $\Phi_N$ is again closed by brackets, if that of $\Phi$ is (same argument as in the proof above), but this is not enough for the integrability of $\Phi_N$. If $\Phi$ is a non degenerate, generalized c.p.s. structure (the subtangent case included), we can justify the integrability of the induced structure as follows. The structure $\Phi$ corresponds by (1.34) to a Hitchin pair $(\varpi, A)$, and the submanifold $N$ must be a symplectic submanifold of $(M, \varpi)$ (a Poisson-Dirac submanifold of a symplectic manifold is a symplectic submanifold [6]). Then, (3.11) yields $\varpi'_A = \iota^* \varpi_A$, which is closed. Therefore, the induced structure $\Phi_N$ is non degenerate, it corresponds to a Hitchin pair, and it is integrable.

In Section 2, we have defined the notion of an invariant submanifold, which was a submanifold that is invariant by $A$. Of course, a Poisson-Dirac, invariant submanifold is quasi-invariant and, by (3.11), (3.14), it has the induced generalized, almost c.p.s. structure defined by the induced Poisson structure $\pi'$ and by

$$A' = A|_{TN}, \sigma' = \iota^* \sigma.$$ (3.22)

In all three c.p.s. cases, if the structure $\Phi$ of $M$ is integrable, and if $N$ is a Poisson-Dirac, invariant submanifold, the induced structure (3.22) is integrable too. Indeed, the only integrability condition of Theorem 1.1 which is a bit less obvious is the annulation of the Schouten concomitant $C(\pi', A')$.

But, it is easy to see that $\forall X \in TN$ one has

$$< C(\pi', A')([\alpha]_{ann TN}, [\beta]_{ann TN}), X > = < C(\pi, A)(\tilde{\alpha}, \tilde{\beta}), X >,$$ (3.23)

where $\tilde{\alpha}, \tilde{\beta}$ are equivalent modulo $ann TN$ with $\alpha, \beta$ and $\#_{\pi} \tilde{\alpha}, \#_{\pi} \tilde{\beta} \in TN$. Hence, $C(\pi, A)$ implies $C(\pi', A') = 0$.

In the terminology of [2] the invariance of a submanifold $N$ means that the submanifold satisfies the *graph condition*. In [2] the authors also define a much stronger property called the *split property*. In “classical terms” a submanifold $\iota: N \hookrightarrow (M, \Phi)$ is a *split submanifold* if it is Poisson-Dirac, invariant and has an $A$-invariant normal bundle $\nu N$ ($T_N M = TN \oplus \nu N$) which is $\sigma$-orthogonal to $TN$. Like any invariant submanifold, a split submanifold has the induced generalized structure $\Phi_N$ defined by (3.22). But, the normal bundle $\nu N$ also has an induced generalized structure $\Phi_\nu$, and $\Phi = \Phi_N \oplus \Phi_\nu$.

Finally, let us also make the following observation. The definitions of quasi-invariance and invariance may also be used for a submanifold $N$ of
a Poisson-Nijenhuis manifold \((M, \pi, A)\) \(((\pi, A)\) is a Poisson-Nijenhuis structure). A Poisson-Dirac, invariant submanifold inherits an induced structure \((\pi', A' = A|_{TN})\) and formula (3.23) shows that the induced structure is a Poisson-Nijenhuis structure too. Moreover, it is easy to see that the Poisson-Nijenhuis hierarchy \((\pi'_k, A'_k)\) \((k, p = 1, 2, ..., \sharp\pi') = A'k \circ \sharp\pi')\) of the induced structure \((\pi', A')\) is induced by the corresponding Poisson-Nijenhuis hierarchy of \((\pi, A)\). For these reasons it is natural to attribute the name of Poisson-Nijenhuis submanifold to an invariant, Poisson-Dirac submanifold \(N\) of a Poisson-Nijenhuis manifold \((M, \pi, A)\). If \(N\) is a quasi-invariant submanifold of \((M, \pi, A)\), \(N\) has the induced Poisson structure \(\pi'\) and a compatible, tensor field \(A'\) defined by (3.11) but \(A'\) may not have a vanishing Nijenhuis tensor.

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