Abstract

This paper establishes some properties for stable solutions of a semilinear elliptic equation with homogeneous Neumann boundary conditions in unbounded domains. A seminal result of Casten, Holland [16] and Matano [23] states that, in convex bounded domains, such solutions must be constant. This paper examines if this property extends to unbounded convex domains. We give a positive answer for stable non-degenerate solutions, and for stable solutions if the domain \( \Omega \) further satisfies \( \Omega \cap \{|x| \leq R\} = O(R^2) \), when \( R \to +\infty \). If the domain is a straight cylinder, a natural additional assumption is needed. We also derive some symmetry properties. Our results can be seen as an extension to more general domains of some results on the De Giorgi’s conjecture.

Keywords: Semilinear elliptic equations Stability Symmetry Neumann boundary conditions De Giorgi’s conjecture Liouville property Convex domains Unbounded domains Generalized principal eigenvalue

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1 Introduction

1.1 General Framework

Consider the following semilinear elliptic equation, with homogeneous Neumann boundary conditions

\[
\begin{cases}
-\Delta u(x) = f(u(x)) & \forall x \in \Omega, \\
\partial_{\nu} u(x) = 0 & \forall x \in \partial\Omega, \\
u \in C^{2,1}(\overline{\Omega}) \cap L^{\infty}(\Omega),
\end{cases}
\]

where \(\partial_{\nu}\) denotes the outer normal derivative, \(f\) is a \(C^1\) function and \(\Omega \subset \mathbb{R}^n\) is a uniformly \(C^{2,1}\) domain. In two seminal papers, Casten, Holland [16] and Matano [23] established the following: if the domain \(\Omega\) is bounded and convex, then any stable solution of \((E)\) in \(\Omega\) is constant. \((P_\Omega)\)

In this work, we investigate if property \((P_\Omega)\) extends to unbounded convex domains. We also state some symmetry results, namely that stable solutions of \((E)\) inherit symmetries from the domain’s invariances by translations or planar rotations.

Note that we only consider bounded solutions and that our results do not hold for unbounded solutions. For example \(u(x) := e^x\) is a nonconstant solution of \(-u'' = -u\) in \(\mathbb{R}\) which is stable non-degenerate. Note also that we need no other assumptions on \(f\) but smoothness. We choose to stick to this general context even if additional assumptions could lead to some stronger results. For example, Casten and Holland proved in [16] that if \(f\) is convex or concave, then \((P_\Omega)\) holds in any bounded domain, possibly not convex.

We point out that the question we address, in the case \(\Omega = \mathbb{R}^n\), is strongly related to the De Giorgi’s conjecture, to which an extensive literature is devoted (see [2, 3, 21, 25, 26] and references therein). This conjecture claims that any solution of the Allen-Cahn equation in \(\mathbb{R}^n, n \leq 8\) which is monotonic in one variable must be planar. As monotonicity implies stability (see e.g Corollary 4.3 of [2]), the question of classifying stable solutions in \(\mathbb{R}^n\) is crucial in this context and has been considered by many authors. To that extent, our results in the particular case of the whole space are already contained in several papers, see for example [17, 19, 21]. See also [15] for radial solutions. Moreover, the method we use is inspired by this literature, in particular, Theorem 1.7 from [8], of which a refined version is presented in Lemma 4.3. This work can thus be seen as an extension to more general domains of some results on the De Giorgi’s conjecture.

Outline. The paper is organized as follows. We introduce the results in section 2 and give a brief discussion on the available counterexamples. In section 3, we recall the classical proof of the Casten, Holland and Matano result and prove Theorem 2.1. In section 4 we state some general properties of the generalized
1.2 Definition of stability

We define stability through a linearization at equilibrium, as follows.

**Definition 1.1** For \( u \) a solution of (\( E \)), define

\[
\lambda_1(u, \Omega) := \inf_{\psi \in H^1, \|\psi\|_{L^2} = 1} \mathcal{F}_{(u, \Omega)}(\psi), \quad \text{where} \quad \mathcal{F}_{(u, \Omega)}(\psi) := \int_{\Omega} |\nabla \psi|^2 - f'(u)\psi.
\]

The solution \( u \) is said to be stable if \( \lambda_1(u, \Omega) \geq 0 \) and stable non-degenerate if \( \lambda_1(u, \Omega) > 0 \).

In the sequel, we often omit to mention the dependance on \((u, \Omega)\) and simply write \( \lambda_1 \) or \( \mathcal{F} \). It is important to note that, if the domain is bounded, \( \lambda_1 \) coincides with the classical principal eigenvalue of the linearized operator with Neumann boundary conditions. It formally corresponds to the lowest eigenvalue of the second variation of the energy associated with (\( E \)), thus \( \lambda_1 \geq 0 \) implies that the solution is a local minimum of the energy. The properties of \( \lambda_1 \) in unbounded domains have been studied in a general context (see [13] and references therein). In the appendix, we give a brief discussion on the link between this definition of stability and the usual dynamical point of view.

2 The results

2.1 Convex domains

The first result states that, considering stable non-degenerate solutions, property (\( P_{\Omega} \)) fully extends to unbounded convex domain.

**Theorem 2.1** Let \( \Omega \) be a convex domain. If \( u \) is a stable non-degenerate solution of (\( E \)), then \( u \) is constant.

Now, let us focus on the classification of stable possibly degenerate solutions. We need a further assumption on the size of the domain \( \Omega \) at infinity, namely

\[
|\Omega \cap \{|x| \leq R\}| = O(R^2), \quad \text{when} \quad R \to +\infty.
\]

(1)

This condition comes from Lemma 4.3, introduced below. Note that under this assumption, we allow for instance convex domains that are subdomains of \( \mathbb{R}^2 \), or of the form \( \mathbb{R}^i \times \omega \) with \( \omega \) bounded and \( i \in \{1, 2\} \), or of the form

\[
\Omega = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x' \in \omega(x_1)\},
\]
where for all \( x_1 \in \mathbb{R}, \omega(x_1) \subset \mathbb{R}^{n-1} \) with \( |\omega(x_1)| = O(|x_1|) \) when \( |x_1| \to +\infty \) \( \omega(x_1) \) can be empty for some ranges of values of \( x_1 \).

The case of a convex straight cylinder, namely \( \Omega = \mathbb{R} \times \omega \) with \( \omega \subset \mathbb{R}^n \) convex, turns out to be a specific case in our results. Note that it corresponds to domains for which convexity is degenerate in one direction. Indeed, some non-constant stable solutions (consisting of planar waves) may exist in such domains. For example, the Allen-Cahn equation in \( \mathbb{R}^n \)

\[-u'' = u(1 - u)(u - 1/2)\]

admits the explicit solution \( u : x \mapsto \tanh(x) + 1/2 \), which is stable (degenerate). Note that, in this example, the nonlinearity is balanced, that is, \( \int_0^1 f = 0 \). This leads up to the following assumption:

\[ \forall (z_1, z_2) \in \mathcal{Z}, z_1 \neq z_2 \int_{z_1}^{z_2} f \neq 0, \quad (2) \]

where

\[ \mathcal{Z} := \{ z \in \mathbb{R} \mid f(z) = 0 \text{ and } f'(z) \leq 0 \} \]

is the set of the stable zeros of \( f \). Formally, the sign of \( \int_{z_1}^{z_2} f \) corresponds to that of the speed \( c \) of a possible traveling wave connecting \( z_1 \) and \( z_2 \). Thus, assumption (2) prevents the existence of a stationary wave (with speed \( c = 0 \)) connecting two stable states. Note that, if the domain is not a straight cylinder, planar waves do not exist anyway and this assumption is not needed.

**Theorem 2.2** Let \( \Omega \subset \mathbb{R}^n \) be a convex domain which satisfies (1) and \( u \) be a stable solution of (E).

1. If \( \Omega \) is not a straight cylinder, then \( u \) is constant.
2. If \( \Omega \) is a straight cylinder, then \( u \) is either constant, or a planar monotonic stationary wave connecting two stable roots \((z^-, z^+) \in \mathbb{Z}^2\), such that \( \int_{z^+}^{z^-} f = 0 \). As a consequence, if we further assume (2) then \( u \) is constant.

**2.2 Symmetry properties**

The following result deals with straight cylinders, possibly not convex.

**Theorem 2.3** Let \( \Omega = \mathbb{R}^n \times \omega \) with \( \omega \subset \mathbb{R}^m \) bounded and let \( u \) be a stable solution of (E). For \( x \in \Omega \), we generically denote \( x = (x_1, \ldots, x_n, x_1', \ldots, x_m') \).

1. If \( u \) is stable non-degenerate, then \( u \) does not depend on \((x_1, \ldots, x_n)\).
2. If \( u \) is stable degenerate and \( n = 1 \), then \( u \) is monotonic with respect to \( x_1 \).
3. If \( u \) is stable degenerate and \( n = 2 \), the dependance of \( u \) with respect to \((x_1, x_2)\) is only through a single scalar variable \( x_0 \in \mathbb{R} \). Moreover, \( u \) is monotonic with respect to \( x_0 \).
We are now interested in cylinders which are invariant with respect to a planar rotation.

**Definition 2.4** A domain \( \Omega \subset \mathbb{R}^{n+2} \) is said to be \( \theta \)-invariant if \( \Omega = \Omega' \times [0, 2\pi) \), where \( \Omega' \subset \mathbb{R}^n \times \mathbb{R}^+ \) in some cylindrical coordinates \( (x, r, \theta) \in \mathbb{R}^n \times \mathbb{R}^+ \times [0, 2\pi) \).

When considering a \( \theta \)-invariant domain \( \Omega \), we further assume that the “radial section” is uniformly bounded:

\[
\sup_{(x, r) \in \Omega'} r < +\infty.
\]

(3)

In particular, it guarantees that if \( u \) is a solution of \((\mathcal{E})\) in \( \Omega \), then \( \partial_\theta u \) is bounded.

**Theorem 2.5** Let \( \Omega \) be a \( \theta \)-invariant domain which satisfies (3) and \( u \) be a stable solution of \((\mathcal{E})\). Assume (1) or that \( u \) is stable non-degenerate. Then \( \partial_\theta u = 0 \).

**Corollary 2.6** Under the same conditions, if we further assume that \( \Omega' \) is convex, then Theorem 2.1 and Theorem 2.2 apply.

As a consequence, property \((P_\Omega)\) holds in some nonconvex domains, such as cylinders whose section is a torus or an annulus.

Theorem 2.3 and Theorem 2.5 deal with domains which are invariant with respect to a translation or a planar rotation. More generally, one can consider domains which are invariant with respect to a vector field \( X(\cdot) \in C^1(\overline{\Omega}, \mathbb{R}^n) \) and ask whether stable solutions inherit the same symmetry, that is to say \( v := \partial_X u = \nabla u \cdot X \) is zero. Based on the following observation, it is reasonable to think that the answer is negative in general. To fix ideas, let \( \Omega \subset \mathbb{R}^2 \) and let \( u \) be a stable solution of \((\mathcal{E})\). Our method requires \( v \) to satisfy the linearized equation

\[
-\Delta v - f'(u)v = 0 \quad \text{in } \Omega.
\]

This essentially means that \( \partial_X \) and \( \Delta \) commute, which implies \( \partial_1 X_1 = \partial_2 X_2 = 0 \) and \( \partial_1 X_2 = -\partial_2 X_1 \). Thus, the line integrals of \( X \) are parallel lines or concentric circles, i.e the domain is invariant with respect to a translation or a planar rotation, which is already covered by Theorem 2.3 and Theorem 2.5.

### 2.3 Counterexamples

The existence of a counterexample to, say, Theorem 2.2 can be investigated when relaxing the assumptions either on the convexity of the domain, or assumption (1).

The latter case amounts to the existence of a counterexample to Lemma 4.3. This problem, which is related to the De Giorgi’s conjecture, turns out to be intricate. In [5], it is proved that Lemma 4.3 does not hold in \( \mathbb{R}^n \), \( n \geq 3 \).
this reason, it is reasonable to think that Theorem 2.2 does not hold in \( \mathbb{R}^n \), \( n \geq 3 \) either. However, this seems to be an open question in low dimensions (See [25] for \( n \geq 9 \), or [15] for \( n \geq 11 \)). See section 4.2 for more details.

Regarding counterexamples in bounded nonconvex domains, Matano constructs in [23] a “dumbbell domain” (consisting in two balls connected by a narrow passage) that admits a nonconstant stable solution, for a general class of bistable nonlinearities. Similar constructions can be found in [9, 7].

Theorem 2.2 along with a blow-up argument leads to the following remarkable corollary. We further assume

the zeros of \( f \) are simple and isolated,

\[
\exists (M, m) \in \mathbb{R}^2 \text{ s.t. } \begin{cases} f < 0 & \text{on } (-\infty, m), \\ f > 0 & \text{on } (M, +\infty). \end{cases}
\]  

(4)

**Corollary 2.7** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth domain. We assume without loss that \( 0 \in \Omega \). For \( \mu > 0 \) we define the dilated domain \( \Omega_{\mu} := \{ \mu x, x \in \Omega \} \).

If \( f \) satisfies (2) and (4), then there exists \( \mu^* > 0 \) such that \( (P_{\Omega}) \) holds in \( \Omega_{\mu} \) for all \( \mu \geq \mu^* \).

**Proof** A complete proof of Corollary 2.7 can be adapted from the proof of Theorem 5 in [17]. We only give a sketch of the proof. By contradiction, assume there exists a sequence \( u_n \) of nonconstant stable solutions of \( (E) \) in \( \Omega_{\mu_n} \), for \( \mu_n \to +\infty \). From assumption (4), we infer a uniform \( L^\infty \) bound for \( u_n \), then a uniform \( C^{2,\alpha} \) bound by elliptic estimates. Thus we can extract a subsequence that converges to some \( u \), which is a solution of \( (E) \) in the whole space \( \mathbb{R}^2 \). Using Corollary 4.2 (below), we can prove that \( u \) is stable. Now, from assumption (4) and the fact that \( u_n \) is not constant, we can show that \( u \) is not constant, which contradicts Theorem 2.2.

This result may be put in perspective with the aforementioned Matano’s counterexample. Corollary 2.7 somehow states that such a counterexample could not be achieved in a domain with no “narrow passage”, regardless of its nonconvexity, at least for \( n = 2 \). In the same spirit, it is proved in [7] that considering a generalized traveling front, there is a complete invasion if \( \Omega \) is a cylinder containing a sufficiently wide straight cylinder whose section \( \omega(x_1) \) is star-shaped. This geometrical assumption somehow formulates the fact that the domain has no narrow passage.

### 3 Preliminaries

#### 3.1 The classical case of bounded convex domains

To give a grasp of the method and the difficulties arising when the domain is unbounded, we recall the proof of \( (P_{\Omega}) \) for bounded convex domains. Let \( \Omega \) be a convex bounded domain, \( u \) a stable solution of \( (E) \) and set \( v_i := \partial_{x_i} u, \) for all
\( i \in \{1, \ldots, n\} \).

**Step 1.** One the one hand, differentiating \((E)\) with respect to \(x_i\), we find that 
\[ v_i := \partial_{x_i} u \] satisfies the linearized equation
\[ -\Delta v_i - f'(u)v_i = 0 \quad \text{in} \quad \Omega. \] (5)

From an integration by part we have
\[ F(v_i) = \int_{\partial \Omega} v_i \partial_\nu v_i = \frac{1}{2} \int_{\partial \Omega} \partial_\nu [v_i^2], \]
with \( F = F_{(u, \Omega)} \) from Definition 1.1. On the other hand, as \( u \) is stable, we have 
\( F(\cdot) \geq 0 \) and
\[ 0 \leq F(v_i) \leq \sum_{k=1}^n F(v_k) = \frac{1}{2} \int_{\partial \Omega} \partial_\nu [|\nabla u|^2]. \]

**Step 2.** When the domain is convex, the above integral turns out to be non-positive, as stated in the following key lemma. This is where the *convexity* of the domain comes into play. It can be found in [16, 23], but a simple proof is presented at the end of the section for completeness. Note that the lemma still holds when the domain is unbounded.

**Lemma 3.1 ([16, 23])** Let \( \Omega \) be a smooth convex domain. If \( u \) is a \( C^2 \) function such that
\[ \partial_\nu u = 0 \quad \text{on} \quad \partial \Omega, \] (6)
then
\[ \partial_\nu |\nabla u|^2 \leq 0 \quad \text{on} \quad \partial \Omega. \]

We then conclude that for all \( i \in \{1, \ldots, n\} \), we have \( F(v_i) = 0 \) thus \( v_i \) minimizes \( F \). Note that, at this step, if we assume that \( u \) is not constant, i.e \( v_i \not\equiv 0 \) for some \( i \), then we deduce \( \lambda_1 = 0 \), i.e \( u \) is *stable degenerate*.

Note also that, if \( \Omega \) is unbounded, the computations are not licit and need to be adapted. This is done in section 3.2.

**Step 3.** Owing to the above conclusion, we deduce as a classical fact that for all \( i \in \{1, \ldots, n\} \), \( v_i \) is a multiple of the principal eigenvalue associated to (5), which is denoted \( \varphi \) and is positive in \( \Omega \).

Note that if \( \Omega \) is unbounded, this step also needs to be adapted. This is done in section 4.

**Step 4.** From \( \partial_\nu u = 0 \) on the closed surface \( \partial \Omega \), we deduce that \( v_i \) vanishes on some point of the boundary. But as \( v_i \) is colinear to \( \varphi \), we conclude \( v_i \equiv 0 \), which completes the proof.

Note that if \( \Omega \) is a straight cylinder, the above conclusion fails and \( v_i \) may be a nonzero multiple of \( \varphi \).
Before proving Lemma 3.1, we need the following definition.

**Definition 3.2** Let $\Omega \subset \mathbb{R}^n$. A “representation of the boundary” is a pair $(\rho, U)$ where $\rho$ is a $C^2$ function defined on $U$ a neighborhood of $\partial \Omega$ such that

$$
\rho(x) = \begin{cases} 
< 0 & \text{if } x \in \Omega \cap U \\
0 & \text{if } x \in \partial \Omega \\
> 0 & \text{if } x \in U \setminus \overline{\Omega}
\end{cases}
$$

and $\nabla \rho(x) = \nu(x)$ $\forall x \in \partial \Omega$, where $\nu(x)$ is the outer normal unit vector of $\partial \Omega$ at $x$.

It is classical that such a representation of the boundary always exists for $C^{2,1}$ domains, see e.g section 6.2 of [22].

**Proof (of Lemma 3.1)** Let us consider $(\rho, U)$ a representation of the boundary for $\Omega$. Equation (6) becomes

$$\nabla u \cdot \nabla \rho = 0$$

on $\partial \Omega$.

As $\nabla u$ is tangential to $\partial \Omega$, we can differentiate the above equality with respect to the vector field $\nabla u$. It gives, on $\partial \Omega$,

$$0 = \nabla (\nabla u \cdot \nabla \rho) \cdot \nabla u = \nabla u \cdot \nabla^2 u \cdot \nabla \rho + \nabla u \cdot \nabla^2 \rho \cdot \nabla u.$$

From this, we infer

$$\partial_u |\nabla u|^2 = \nabla (|\nabla u|^2) \cdot \nabla \rho = 2 \nabla u \cdot \nabla^2 u \cdot \nabla \rho = -2 \nabla u \cdot \nabla^2 \rho \cdot \nabla u,$$

Since $\Omega$ is convex, for all $x_0 \in \partial \Omega$, we have that $\nabla^2 \rho(x_0)$ is a nonnegative quadratic form in the tangent space of $\partial \Omega$ at $x_0$. As $\nabla u$ is tangential to $\partial \Omega$, we deduce from the above equation that $\partial_u |\nabla u|^2$ is nonpositive. ■

In [16], the authors give the following remarkable geometrical interpretation of the above lemma. Consider a bounded convex domain $\Omega \subset \mathbb{R}^2$. As $u$ satisfies Neumann boundary conditions, its level set cross the border $\partial \Omega$ orthogonally.

Since the domain is convex, these level sets go apart one from each other as we move outward $\partial \Omega$. As $|\nabla u|$ corresponds to the inverse of the distance of two level sets, it implies that $|\nabla u|$ decreases as we move outward $\Omega$, hence the result.

### 3.2 Non-degenerate stable solutions - proof of Theorem 2.1

The following proof is adapted from the first two steps of section 3.1. The method is inspired from [21]. As $\Omega$ is unbounded, the computations which lead to "$\mathcal{F}(v_i) = 0$" are not licit. We shall instead perform the computations on a truncated function $\chi_R v_i$. For $R > 0$, we set

$$\chi_R(x) := \chi \left( \frac{|x|}{R} \right), \quad \forall x \in \mathbb{R}^n,$$

for $\chi$ a smooth nonnegative function such that

$$\chi(z) = \begin{cases} 
1 & \text{if } 0 \leq z \leq 1, \\
0 & \text{if } z \geq 2, \quad |\chi'| \leq 2.
\end{cases}$$
Lemma 3.3 Let \( v \in C^2(\Omega) \) be bounded and satisfy
\[
v(\Delta v + f'(u)v) \geq 0 \quad \text{in } \Omega, \tag{8}
\]
and
\[
\int_{\partial \Omega} \chi_R^2 v \partial_\nu v \leq 0, \quad \forall R \gg 1. \tag{9}
\]
If \( u \) is stable non-degenerate, then \( v \equiv 0 \).

Proof (of Lemma 3.3) By contradiction, assume \( v \not\equiv 0 \). By a standard elliptic argument, \( v \) cannot be identically zero on any subset \( \Omega' \subset \Omega \). For \( R > 0 \), multiplying (8) by \( \chi_R^2 \), integrating on \( \Omega \), using the divergence theorem and (9) we find
\[
F\left( \frac{\chi_R^2 v}{\|\chi_R^2 v\|_{L^2}} \right) = \frac{\int_{\partial \Omega} \chi_R^2 v \partial_\nu v + \int_{\Omega} |\nabla \chi_R|^2 v^2}{\int_{\Omega} \chi_R^2 v^2} \leq \frac{\int_{\Omega} |\nabla \chi_R|^2 v^2}{\int_{\Omega} v^2} \leq \frac{4}{R^2} \int_{\Omega_R} v^2 \leq 4 \alpha_R,
\]
denoting,
\[
\Omega_R := \Omega \cap \{|x| \leq R\}, \quad C(R) := \int_{\Omega_R} v^2, \quad \alpha_R := \frac{C(2R) - C(R)}{R^2 C(R)}.
\]
Now, let us show
\[
\liminf_{R \to +\infty} \alpha_R \leq 0. \tag{10}
\]
If (10) holds, then \( \lambda_1 \leq 0 \), which contradicts the fact that \( u \) is stable non-degenerate and thereby completes the proof. By contradiction, let us assume \( \alpha_R \geq \delta > 0 \). We have \( C(2R) \geq \delta R^2 C(R) \). Iterating, we find, for \( R \) large enough
\[
C(2^j R) \geq K (\delta R^2)^j \quad \text{for all integer } j \geq 1,
\]
where positive constants are generically denoted \( K \). In addition, \( v \) is bounded, hence \( C(R) \leq KR^n \). We have
\[
K (2^j R)^n \geq (\delta R^2)^j.
\]
Fixing \( R \) large enough, we reach a contradiction as \( j \) goes to \( +\infty \). Thereby, we have proved (10) and the proof is complete.

We denote \( v_i := \partial_{x_i} u \) for \( i \in \{1, \ldots, n\} \). As \( \Omega \) is convex, Lemma 3.1 implies
\[
\sum_{i=1}^n \int_{\partial \Omega} \chi_R^2 v_i \partial_\nu v_i = \frac{1}{2} \int_{\partial \Omega} \chi_R^2 \partial_\nu \|\nabla u\|^2 \leq 0. \tag{11}
\]
From a differentiation of (E), we find that all the \( v_i \) satisfy (5). Moreover, since \( u \) is bounded, classical global Schauder estimates (see e.g Theorem 6.30 in [22]) ensure that all the \( v_i \) are bounded. Then, Lemma 3.3 implies in particular \( \int_{\partial \Omega} \chi_R^2 v_i \partial_\nu v_i \geq 0 \) for all \( i \), i.e all the terms of the sum in (11) are nonnegative. As the sum is nonpositive, all the terms must be zero. From Lemma 3.3, we find \( v_i \equiv 0 \) for all \( i \), i.e \( u \) is constant, which completes the proof.
4 Properties of $\lambda_1$

4.1 Existence of a positive eigenfunction

If the domain is bounded, it is classical that there exists a positive eigenfunction associated to $\lambda_1$. This property extends to unbounded domains, as stated in the following proposition.

**Proposition 4.1** Let $\Omega \subset \mathbb{R}^n$ and $u$ be a solution of $(\mathcal{E})$. There exists $\varphi \in W^{2,p}_{loc}(\Omega)$, $\forall p \geq 1$, which is positive on $\overline{\Omega}$ and satisfies

$$
\begin{cases}
-\Delta \varphi - f'(u)\varphi = \lambda_1 \varphi & \text{in } \Omega, \\
\partial_\nu \varphi = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(12)

$\varphi$ is referred as a principal eigenfunction of the linearized operator.

This statement is adapted from the results of [13]. However, presenting a full proof would be too technical and slightly off topic. See the appendix for more details.

As a corollary, we can prove that $\lambda_1 \geq 0$ is equivalent to the existence of a positive supersolution. This result is essentially classical in the theory of linear elliptic equations. The proof is postponed to the appendix.

**Corollary 4.2** Let $\Omega \subset \mathbb{R}^n$ and $u$ be a solution of $(\mathcal{E})$. The two following statements are equivalent:

1. $u$ is stable
2. There exists $\phi \in W^{2,p}_{loc}(\Omega)$, $\forall p \geq 1$, which is positive on $\overline{\Omega}$ and satisfies

$$
\begin{cases}
-\Delta \phi - f'(u)\phi \geq 0 & \text{in } \Omega, \\
\partial_\nu \phi \geq 0 & \text{on } \partial \Omega.
\end{cases}
$$

(13)

In the sequel, $\varphi$ could be replaced by any $\phi$ satisfying (13) for our purposes.

4.2 A Liouville result, or the simplicity of $\lambda_1$

When the domain is bounded, it is classical that the only minimizers of $\mathcal{F}_{(u,\Omega)}$ are multiples of $\varphi$. The following lemma claims that, if the minimizer is a bounded function, the above conclusion extends to unbounded domains which satisfy (1). It is a refinement of Theorem 1.7 from [8].

**Lemma 4.3** Let $\Omega \subset \mathbb{R}^n$ satisfy (1) and let $u$ be a stable solution of $(\mathcal{E})$. If $v$ is smooth, bounded and satisfies (8)-(9), then $v \equiv C\varphi$ for some constant $C$, where $\varphi$ is defined in Proposition 4.1.
Proof (of Lemma 4.3) We follow the method of [8]. Let us set $\sigma = \frac{\varphi}{\varphi}$ and show that $\sigma$ is constant. From (8), we deduce

$$\sigma \varphi (\varphi \Delta \sigma + 2 \nabla \varphi \cdot \nabla \sigma + \sigma (\Delta \varphi + f'(u) \varphi)) \geq 0.$$  

From (12) and $\lambda_1 \geq 0$, we obtain

$$\sigma \nabla \cdot (\varphi^2 \nabla \sigma) \geq 0.$$  

Multiplying by $\chi^2_R$ (defined in (7)), integrating on $\Omega$ and using the divergence theorem, we find

$$0 \leq \int_{\partial \Omega} \chi^2_R \varphi^2 \partial_\nu \sigma - \int_{\Omega} \varphi^2 \nabla (\chi^2_R \sigma) \cdot \nabla \sigma$$
$$= \int_{\partial \Omega} \chi^2_R \varphi^2 \partial_\nu \sigma - \int_{\Omega} \varphi^2 \chi^2_R |\nabla \sigma|^2 - 2 \int_{\Omega} \varphi^2 \chi_R \nabla \chi_R \cdot \nabla \sigma.$$  

As $\partial_\nu \varphi = 0$ on $\partial \Omega$, the boundary term reads as $\int_{\partial \Omega} \chi^2_R \varphi \partial_\nu v$, which is nonpositive from (9). Using the Cauchy-Schwarz inequality, we deduce

$$\int_{\Omega} \chi^2_R \varphi^2 |\nabla \sigma|^2 \leq 2 \sqrt{\int_{\Omega_{2R} \setminus \Omega_R} \chi^2_R \varphi^2 |\nabla \sigma|^2} \sqrt{\int_{\Omega} \varphi^2 |\nabla \chi_R|^2},$$  

(14)

where $\Omega_R = \Omega \cap \{|x| \leq R\}$.

Now, assumption (1) implies

$$\int_{\Omega} \varphi^2 |\nabla \chi_R|^2$$  

is bounded, uniformly in $R \geq 1$.  

(15)

From (14), we have that $\int_{\Omega} \chi^2_R \varphi^2 |\nabla \sigma|^2$ is uniformly bounded. Using (14) again, we infer that it converges to 0 as $R \to \infty$. At the limit, we find $\int_{\Omega} \varphi^2 |\nabla \sigma|^2 \leq 0$. Hence $\nabla \sigma = 0$, which ends the proof. □

The cornerstone of the proof is that $\sigma \nabla \cdot (\varphi^2 \nabla \sigma) \geq 0$ implies $\nabla \sigma = 0$, where $\sigma := \frac{\varphi}{\varphi}$. The literature refers to this property as the Liouville property. Originally introduced in [8], it has been extensively discussed (see [4, 6, 20, 21, 24]) and used to derive numerous results (e.g. [10, 15, 17, 19]), in particular to prove the De Giorgi’s conjecture in low dimension (see [2, 3, 14, 21]). Lemma 4.3 is a refinement of this property for domains with a boundary, instead of $\Omega = \mathbb{R}^n$. This is why we need a boundary condition (9).

This is the only step where (1) is needed, it is thus a natural question to ask if this assumption can be relaxed. In the proof, (1) is used to derive (15), thus the choice of $\chi_R$ seems crucial. However, in [21], the authors consider the optimal $\chi_R$ by taking a solution of the minimization problem

$$\inf_{\chi \in H^1(\mathbb{R}^2)} \left[ \int_{R \leq |x| \leq R'} |\nabla \chi(x)|^2 \, dx, \, \xi(x) = \begin{cases} 1 & \text{if } |x| \leq R' \\ 0 & \text{if } |x| \geq R' \end{cases} \right].$$  

(16)
That, in fact, does not allow to substantially relax condition (1). In [5], Bar-low uses a probabilistic approach to establish that the aforementioned Liouville property (and consequently Lemma 4.3) does not hold in \( \Omega = \mathbb{R}^n, n \geq 3 \). It is thus reasonable to think that condition (1) cannot be relaxed, yet this is an open question. We also cite [20], in which (1) is proved to be sharp, however we point out that, there, the condition \( v \in L^\infty \) is not satisfied.

Note also that, in this work, we only apply Lemma 4.3 to functions \( v \) which are derivatives of \( u \), which is a stronger condition than (8). In this context, not much is known about whether (1) could be relaxed. Indeed, up to the author’s knowledge, the only available counterexamples are for \( \Omega = \mathbb{R}^n, n \leq 9 \), as a consequence of [25] in which the authors construct counterexamples to the De Giorgi’s conjecture for \( n \leq 9 \).

However, we can sometimes relax (1) under further assumptions, either on \( \Omega, f, \) or \( v \). From a remark in [19], if \( f \geq 0 \), we can relax (1) to \( |\Omega \cap \{|x| \leq R\}| = O(R^4) \). Note also that, if \( \Omega = \mathbb{R}^n \), then (1) can be replaced by \( v \in H^1(\Omega) \), or \( v = o(|x|^{1-\frac{n}{2}}) \), see [8, 15]. In addition, we can show that Lemma 4.3 holds for a large class of domains satisfying

\[
|\Omega \cap \{|x| \leq R\}| = O(R^2 \ln(R)) \quad \text{when } R \to +\infty.
\]

More precisely, let \( \Omega \) be of the form

\[
\Omega := \{(x, x') \in \mathbb{R}^2 \times \mathbb{R}^n : x' \in \omega(x)\},
\]

where \( \forall x \in \mathbb{R}^n, \omega(x) \subset \mathbb{R}^n \) is bounded and

\[
|\omega(x)| = O(\ln(|x|)) \quad \text{when } |x| \to +\infty.
\]

Then, to show (15) we use the cut-off

\[
\chi_R(x) = \begin{cases} 
1 & \text{if } |x| \leq R, \\
\frac{\ln R^2 - \ln |x|}{\ln R^2 - \ln R} & \text{if } R \leq |x| \leq R^2, \\
0 & \text{if } |x| \geq R^2.
\end{cases}
\]

This cut-off was first introduced in [21] as a solution of (16) for \( n = 2 \).

5 Proof of the symmetry properties

5.1 Proof of Theorem 2.3

We set \( v_i := \partial_{x_i} u \) for \( i \in \{1, \ldots, n\} \), which is bounded and satisfies (8). Since \( \Omega \) is straight in the directions \( x_i \), we have \( \partial_{x_i} v_i = 0 \) on \( \partial \Omega \), therefore \( v_i \) satisfies (9). Thus, the first assertion follows from Lemma 3.3. Next, the second assertion follows from Lemma 4.3 and the fact that \( \varphi > 0 \).

To show the second assertion (we assume \( n = 2 \)), we follow an idea from [8, 21] and apply Lemma 4.3 to \( \partial_{\xi} u = (\partial_{x_1} u, \partial_{x_2} u) \cdot \xi \), for \( \xi \) being any unit
vector of $\mathbb{R}^2$. We infer that there exists a constant $C_\xi$ such that $\partial_\xi u = C_\xi \varphi$. Since $C_\xi$ depends continuously on $\xi$, it must vanish for some $\xi_0$ when $\xi$ moves on the sphere from direction $x_1$ to $-x_1$. Using a change of coordinates, we may assume $\xi_0 = x_2$. Hence $\partial_{x_2} u \equiv 0$ and $u$ depends on one coordinate only. The monotonicity is deduced from the second assertion.

5.2 Proof of Theorem 2.5 and Corollary 2.6

We set $v := \partial_\theta u$. Note that in a Cartesian system of coordinates

$$z = (x_1, \ldots, x_n, y_1, y_2) \in \mathbb{R}^{n+2},$$

we have $v(z) = y_2 \partial_{y_1} u(z) - y_1 \partial_{y_2} u(z)$. A direct computation shows that $v$ satisfies (5). We also know that $v$ is bounded from (3) and classical elliptic estimates. We use the following lemma.

**Lemma 5.1** If $\Omega$ is $\theta$-invariant and $u$ is a smooth function satisfying

$$\partial_\nu u = 0 \quad \text{on } \partial \Omega, \quad (17)$$

then

$$\partial_\nu \partial_\theta u = 0 \quad \text{on } \partial \Omega.$$ 

**Proof (of Lemma 5.1)** Let us consider $(\rho, U)$ a representation of the boundary for $\Omega$ (see Definition 3.2). Since $\Omega$ is $\theta$-invariant, we can choose $\rho$ to be $\theta$-invariant, i.e $\partial_\theta \rho = 0$. It implies that $\partial_\theta$ and $\partial_\nu$ commute. We then differentiate (17) with respect to $\theta$ to prove the claim. ■

As a consequence, $v$ is bounded and satisfies (8)-(9). If $u$ is stable non-degenerate, we conclude with Lemma 3.3. Assume (1). From Lemma 4.3, we deduce $v \equiv C \varphi$ for some constant $C$. Thus, $v$ is of constant sign. For $(x, x', r, \theta) \in \Omega$ we have

$$\int_0^{2\pi} v(x, x', r, \theta)d\theta = 0.$$ 

Therefore $v \equiv 0$, which completes the proof of Theorem 2.5.

To prove Corollary 2.6, it suffices to note that, since $\partial_\theta u = 0$, $w := \partial_r u$ satisfies

$$w(\Delta w + f'(u)w) = \frac{1}{r^2} w^2 \quad \text{a.e in } \Omega,$$

thus $w$ satisfies (8), hence the conclusion.

6 Proof of Theorem 2.2

We consider $(e_1, \ldots, e_n)$ an orthonormal basis of $\mathbb{R}^n$, $(x_1, \ldots, x_n)$ the associated variables, and we set $v_i := \partial_{x_i} u$ for $i \in \{1, \ldots, n\}$. As a consequence of Lemma 4.3 and Lemma 3.1, we have the following result.
**Lemma 6.1** For all \(i \in \{1, \ldots, n\} \), \(v_i \equiv C_i \varphi \) for some constant \(C_i\).

**Proof** (of Lemma 6.1) Differentiating \((E)\), we find that \(v_i\) satisfies (8). Moreover, since \(u\) is bounded, classical global Schauder’s estimates guarantee that all the \(v_i\) are bounded.

Now, we show that all the \(v_i\) satisfy (9). On the one hand, as \(\Omega\) is convex, Lemma 3.1 implies

\[
\sum_{i=1}^{n} \int_{\partial \Omega} \xi_R v_i \partial_v v_i = \frac{1}{2} \int_{\partial \Omega} \xi_R \partial_v |\nabla u|^2 \leq 0.
\]

On the other hand, since \(\partial_v \varphi = 0\) on \(\partial \Omega\), Lemma 4.3 implies, in particular, that \(\int_{\partial \Omega} \lambda_R^2 v_i \partial_v v_i \geq 0\) for all \(i\), i.e. all the terms of the above sum are nonnegative.

As the sum is nonpositive, all the terms must be zero.

Then, we apply Lemma 4.3 to conclude. \(\blacksquare\)

### 6.1 First case: \(\Omega\) is not a straight cylinder

We have to prove that \(v_i \equiv 0\), for all \(i \in \{1, \ldots, n\} \). We proceed by induction: given \(k \in \{0, \ldots, n\} \), we show \(v_i \equiv 0\) for all \(i \in \{1, \ldots, k\} \). If \(k = 0\), the claim is trivial. Let us assume the claim for a fixed \(k \in \{0, n-1\} \). We denote \(S = \text{span} \{e_1, \ldots, e_k\} \). As \(\Omega\) is not straight in any direction, there exists \(\bar{x} \in \partial \Omega\) such that \(\nu(\bar{x}) \not\in S\). Up to an isometric transformation of the orthonormal basis \(\{e_k+1, \ldots, e_n\}\), we can assume without loss that \(\nu(\bar{x}) \in S^\perp \mathbb{R} e_{k+1}\). From \(\partial_v u(\bar{x}) = 0\) and the induction assumption, we find \(v_{k+1}(\bar{x}) = 0\). Since \(\varphi > 0\) on \(\Omega\), we have \(v_{k+1} \equiv 0\), which ends the proof.

### 6.2 Second case: \(\Omega\) is a straight cylinder

If \(\Omega\) is a straight cylinder, since it is convex and satisfies (1), it is either of the form \(\mathbb{R} \times \omega\) or \(\mathbb{R}^2 \times \omega\), with \(\omega\) a bounded domain. From the previous case and Theorem 2.3, we infer that \(u\) depends on only one variable. Thus, we can assume without loss of generality that \(v_i \equiv 0\), \(\forall i \in \{2, \ldots, n\} \). Moreover, as \(v_1 \equiv C_1 \varphi\), it is of constant sign, hence \(u\) is monotonic. Since \(u\) is bounded, it has a limit \(z^+\) when \(x_1 \to +\infty\). Setting \(u_0(x_1) = u(x_1 + n)\) and using classical elliptic estimates, we can extract a subsequence that converges in \(C^{2+\alpha}\) to a stable solution \(u_\infty\) of \((E)\) (note that \(\Omega\) is invariant under translation in the \(x_1\) direction). From \(u_\infty \equiv z^+\), we deduce that \(z^+\) must be a stable root of \(f\). Identically, \(u\) has a constant limit \(z^- \in \mathcal{Z}\) as \(x_1 \to -\infty\).

If \(z^+ = z^-\), then \(u\) is constant. Let us assume \(z^- \neq z^+\), and fix \(M > 0\). Integrating (5) on \(x_1 \in [-M, M]\) gives

\[
|u'(M) - u'(-M)| = \left| \int_{-M}^{M} f(u(x)) \, dx \right| = \left| \int_{u(-M)}^{u(M)} f(x) \, dx \right| = K \left| \int_{u(-M)}^{u(M)} f(x) \, dx \right|,
\]
where $K$ is a positive constant. As $u'(\pm \infty) = 0$ (indeed, $u'$ is integrable and $u''$ is bounded), when $M$ goes to $+\infty$ we obtain $\int_{-\infty}^{+\infty} f = 0$. The proof is thereby complete.

**Remark 6.2** Note that if a stationary traveling wave exists, it is always a stable (degenerate) solution of $(E)$. This can be shown using Corollary 4.2 with $\varphi := |\partial_x u|$.

### A Generalized principal eigenvalue

Given an elliptic operator $L$ and a smooth bounded domain $\Omega$ along with some proper boundary conditions, the classical Krein-Rutman theory provides a minimal eigenvalue, referred as the **principal eigenvalue**. This notion has been extended to non smooth and non bounded domains, under Dirichlet boundary conditions, see [13, 11]. Considering smooth unbounded domains, the approach of [13] can be adapted to Neumann boundary conditions, by substituting the functional space $W^{2,p}_{loc}(\Omega)$ with $\tilde{W}^{2,p}_{loc}(\Omega) := \{ \psi \in W^{2,p}_{loc}(\Omega) : \partial_\nu \psi = 0 \text{ on } \partial \Omega \}$.

Indeed, in [12] the authors define

$$\lambda_{1,N} := \sup \left\{ \lambda \in \mathbb{R} : \exists \psi \in \tilde{W}^{2,p}_{loc}(\Omega), \psi > 0, (\Delta + f'(u) + \lambda)\psi \leq 0 \text{ a.e in } \Omega \right\}$$

and prove the existence of an associated positive eigenfunction, namely

**Proposition A.1 (Theorem 3.1 and Proposition 1 in [12])** There exists $\varphi \in W^{2,p}_{loc}, \forall p > 1$, which is positive on $\Omega$ and satisfies

$$\begin{cases} -\Delta \varphi - f'(u) \varphi = \lambda_{1,N} \varphi & \text{a.e in } \Omega, \\ \partial_\nu \varphi = 0 & \text{a.e on } \partial \Omega. \end{cases}$$

In fact, the quantities $\lambda_{1,N}$ and $\lambda_1$ (from Definition 1.1) are equal, as stated in the following lemma. Note however that the definition of $\lambda_1$ relies on the fact that the operator $-\Delta - f'(u)$ is self-adjoint, whereas $\lambda_{1,N}$ can be defined for more general operators.

**Lemma A.2** In the definition of $\lambda_1$, it is equivalent to take the infimum on compactly supported smooth test functions, namely

$$\lambda_1(u, \Omega) = \inf_{\psi \in C^1_c(\overline{\Omega})} \frac{\mathcal{F}_{(u,\Omega)}(\psi)}{\|\psi\|_{L^2}^2}$$

where $C^1_c(\overline{\Omega})$ is the space of continuously differentiable functions with compact support in $\overline{\Omega}$. As a consequence, $\lambda_1 = \lambda_{1,N}.$

**Proof** Recalling that $f'(u)$ is bounded, the first statement is deduced from the dominated convergence theorem and classical density results. The remaining can be adapted from the proof of Proposition 2.2 (iv) in [13] (which itself relies on [1, 11]).
Combining Proposition A.1 and Lemma A.2, the proof of Proposition 4.1 follows. We now prove Corollary 4.2.

**Proof (of Corollary 4.2)** If \( u \) is stable, the existence of \( \phi \) is a direct consequence of Proposition 4.1. Conversely, assume that such a \( \phi \) exists. Let \( \psi \) be a test function. Owing to Lemma A.2, \( \psi \) can be chosen in \( C_1^1(\Omega) \). Multiplying (12) by \( \frac{\psi}{\phi} \), integrating and using the divergence theorem, we find

\[
0 \leq \int_\Omega \nabla \phi \cdot \nabla \psi - f'(u)\psi^2
= 2 \int_\Omega \frac{\psi}{\phi} \nabla \phi \cdot \nabla \psi - \frac{\psi^2}{\phi^2} |\nabla \phi|^2 - f'(u)\psi^2
\leq \int_\Omega |\nabla \psi|^2 - f'(u)\psi^2 = F_{(u,\Omega)}(\psi),
\]

using Young’s inequality in the last step. ■

**B Stability**

When considering stability from a dynamical point of view, one can come up with the two following definitions.

**Definition B.1** A solution \( u \) of (\( E \)) is said to be dynamically stable if given any \( \epsilon > 0 \), there exists \( \delta_0 > 0 \) such that for any \( v_0(x) \) with \( \|v_0 - u\|_{L^\infty} \leq \delta_0 \), we have

\[
\|v(t, \cdot) - u(\cdot)\|_{L^\infty} \leq \epsilon, \quad \forall t > 0,
\]

where \( v(t, x) \) is the solution of the evolution problem

\[
\begin{align*}
\partial_t v(t, x) - \Delta v(t, x) &= f(v(t, x)) \quad \forall x \in \Omega, \quad \forall t > 0, \\
\partial_\nu v(t, x) &= 0 \quad \forall x \in \partial \Omega, \quad \forall t > 0, \\
v(t = 0, x) &= v_0(x) \quad \forall x \in \Omega.
\end{align*}
\]

**Definition B.2** A solution \( u \) of (\( E \)) is said to be asymptotically stable if there exists \( \delta_0 > 0 \) such that for any \( v_0(x) \) with \( \|v_0 - u\|_{L^\infty} \leq \delta_0 \), we have

\[
\|v(t, \cdot) - u(\cdot)\|_{L^\infty} \to 0, \quad \text{when } t \to +\infty,
\]

where \( v(t, x) \) is the solution of (18).

The following proposition clarifies the hierarchy of the different definitions of stability.

**Proposition B.3** Let \( u \) be a solution of (\( E \)) and \( \lambda_1 \) from Definition 1.1. The following implications hold

\[
u \text{ asymptotically stable } \Rightarrow \ u \text{ dynamically stable } \Rightarrow \lambda_1 \geq 0.
\]
Proof The first implication is trivial. Let us show the second implication by contradiction: assume $\lambda_1 < 0$ and that $u$ is dynamically stable. From the dominated convergence theorem, there exists a bounded domain $\Omega \subset \tilde{\Omega}$ such that

$$\tilde{\lambda}_1 := \inf_{\psi \in H^1(\tilde{\Omega})} \frac{\int_{\tilde{\Omega}} |\psi|^2 - f'(u)\psi^2}{\|\psi\|_{L_2(\tilde{\Omega})}} < 0.$$  

Note that $\tilde{\Omega}$ could be replaced by a larger subdomain of $\Omega$, therefore we can assume without loss that $\partial\Omega$ and $\partial\tilde{\Omega}$ do not intersect tangentially. From technical but classical arguments (see, e.g. Theorem 3.1 in [12]) we know that there exists a positive function $\tilde{\phi} \in W^{2,p}(\tilde{\Omega}), \forall p > 1$ such that

$$\begin{cases} -\Delta \tilde{\phi} - f'(u)\tilde{\phi} = \tilde{\lambda}_1 \tilde{\phi} & \text{in } \tilde{\Omega}, \\ \partial_\nu \tilde{\phi} = 0 & \text{on } \partial \tilde{\Omega} \cap \partial \Omega, \\ \tilde{\phi} = 0 & \text{on } \partial \tilde{\Omega} \setminus \partial \Omega. \end{cases}$$

We choose the normalization $\|\tilde{\phi}\|_{L^\infty} = \delta$, with $\delta$ small enough such that $0 < \delta < \delta_0$ (where $\delta_0$ is given in Definition B.1) and

$$\eta_{\delta} := \sup_{\bar{u} \in [\inf u, \sup u]} \left| \frac{f'(\bar{u}) - f(\bar{u} + h) - f(\bar{u})}{h} \right| < -\lambda_1. \quad (19)$$

We set $v_0 := u + \tilde{\phi}$. On the one hand, as $\|v_0 - u\|_{L^\infty} \leq \delta$, the stability assumption implies $\|h(t, \cdot)\|_{L^\infty} \leq \delta$ for all time $t \geq 0$, where $h(t, x) := v(t, x) - u(x)$. On the other hand, $h(t, \cdot) > 0$ and satisfies

$$\begin{cases} \partial_t h(t, x) - \Delta h(t, x) \geq (f'(u(x)) - \eta_{\delta}) h(t, x) & \text{in } \tilde{\Omega}, \\ \partial_\nu h = 0 & \text{on } \partial \tilde{\Omega} \cap \partial \Omega, \\ h > 0 & \text{on } \partial \tilde{\Omega} \setminus \partial \Omega. \end{cases}$$

Using the parabolic comparison principle, we infer $h(t, \cdot) \geq \tilde{h}(t, \cdot)$ for all $t \geq 0$, where $\tilde{h}(t, x) := e^{-(\lambda_1 + \eta_{\delta})t} \tilde{\phi}(x)$. From $\lambda_1 + \eta_{\delta} < 0$, we deduce that $\|h(t, \cdot)\|_{L^\infty}$ goes to infinity when $t$ becomes large: contradiction. □

Remark B.4 Note that, in the proof, the perturbation $\tilde{\phi}$ has a compact support and an arbitrarily small $L^\infty$ norm. Thus, if $\lambda_1 < 0$ then (18) drives $u + h$ away from $u$ for any $h$ which is positive or negative on a sufficiently large subdomain $\tilde{\Omega} \subset \Omega$.

One can ask whether

$$\lambda_1 > 0 \Rightarrow u \text{ asymptotically stable}. \quad (20)$$

Note that property (20) is classical in bounded domains (see Proposition 1.4.1 in [18]), but it is not clear whether it extends to unbounded domains.
**Question:** Does (20) hold in any unbounded domain $\Omega$?

A consequence of our results is that (20) holds in unbounded convex domains.

**Proposition B.5** Let $\Omega \subset \mathbb{R}^n$ be a $C^{2,1}$ convex domain (not necessarily bounded) and $u$ be a solution of $(E)$. Then

$$\lambda_1 > 0 \Rightarrow u \text{ asymptotically stable} \Rightarrow u \text{ dynamically stable} \Rightarrow \lambda_1 \geq 0.$$  

**Proof** We only have to show the first implication. Assume $\lambda_1 > 0$. We deduce from Theorem 2.1 that $u$ is constant. Thus $\lambda_1 = -f'(u)$ and $\varphi$ is constant. We choose $\delta_0$ small enough such that $\eta\delta_0 \in (0, \frac{\lambda_1}{2})$, where $\eta$ is defined in (19). Let $v_0$ be as in Definition B.2 (we use the same notations). We set $T := \sup\{t > 0 : \|h(t,\cdot)\|_{L^\infty} \leq \delta_0\}$. By continuity and the choice of $v_0$, we know that $T > 0$. We have

$$\begin{cases}
\partial_t h(t, x) - \Delta h(t, x) \leq (f'(u) + \eta\delta) h(t, x) & \forall t \in (0, T), \ x \in \Omega, \\
\partial_n h(t, x) = 0 & \forall t \in (0, T), \ x \in \partial\Omega.
\end{cases}$$

From $f'(u) + \eta\delta \leq -\frac{\lambda_1}{2}$ and the parabolic comparison principle, we obtain

$$\|h(t,\cdot)\|_{L^\infty} \leq \delta_0 e^{-\frac{\lambda_1}{2}t}, \text{ for all } t \in (0, T).$$

We deduce $T = +\infty$ and $\|h(t,\cdot)\|_{L^\infty} \rightarrow 0$ when $t \rightarrow +\infty$, thus $u$ is asymptotically stable.  ■

**References**

[1] Shmuel Agmon. On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds. *Methods of Functional Analysis and Theory of Elliptic Equations (Naples, 1982)*, pages 19–52, 1983.

[2] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré. On a Long-Standing Conjecture of E. De Giorgi: Symmetry in 3D for General Nonlinearities and a Local Minimality Property. *Acta Applicandae Mathematicae*, 65:9–33, 2001.

[3] Luigi Ambrosio and Xavier Cabré. Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi. *Journal of the American Mathematical Society*, 13(4):725–739, 2000.

[4] Luigi Ambrosio and Xavier Cabré. Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi. *Journal of the American Mathematical Society*, 13(4):725–739, 2000.

[5] Martin T. Barlow. On the Liouville Property for divergence form operators. *Journal of Mathematics*, 50(3):487–496, 1998.

[6] Martin T. Barlow, Richard F. Bass, and Changfeng Gui. The Liouville property and a conjecture of De Giorgi. *Communications on Pure and Applied Mathematics*, 53(8):1007–1038, 2000.
[7] Henri Berestycki, Juliette Bouhours, and Guillemette Chapuisat. Front blocking and propagation in cylinders with varying cross section. *Calculus of Variations and Partial Differential Equations*, 55(3):44, 2016.

[8] Henri Berestycki, Luis Caffarelli, and Louis Nirenberg. Further qualitative properties for elliptic equations in unbounded domains. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, 25(1-2):69–94, 1997.

[9] Henri Berestycki, François Hamel, and Hiroshi Matano. Bistable travelling waves around an obstacle. *Communications in Pure and Applied Mathematics*, 62(6):729–788, 2009.

[10] Henri Berestycki, François Hamel, and Régis Monneau. One-dimensional symmetry of bounded entire solutions of some elliptic equations. *Duke Mathematical Journal*, 103(3):375–396, 2000.

[11] Henri Berestycki, Louis Nirenberg, and S.R.S Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Communications on Pure and Applied Mathematics*, 47(1):47–92, 1994.

[12] Henri Berestycki and Luca Rossi. Reaction-diffusion equations for population dynamics with forced speed II - Cylindrical-type domains. *Discrete and Continuous Dynamical Systems*, 25(1):19–61, 2009.

[13] Henri Berestycki and Luca Rossi. Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *Communications on Pure and Applied Mathematics*, 68(6):1014–1065, 2015.

[14] Xavier Cabré. A Conjecture of De Giorgi on Symmetry for Elliptic Equations in R. *European Congress of Mathematics. Progress in Mathematics (Birkhäuser, Basel)*, 201, 2001.

[15] Xavier Cabré and Antonio Capella. On the stability of radial solutions of semilinear elliptic equations in all of R. *Comptes Rendus Mathematique*, 338:769–774, 2004.

[16] Richard G. Casten and Charles J. Holland. Instability results for reaction diffusion equations with Neumann boundary conditions. *Journal of Differential Equations*, 27(2):266–273, 1978.

[17] E. Norman Dancer. Stable and Finite Morse Index solutions on R^n or on bounded domains with small diffusion. *Transactions of the American Mathematical Society*, 357(304):1225–1243, 2004.

[18] Louis Dupaigne. *Stable solutions of elliptic partial differential equations (Monographs and Surveys in Pure and Applied Mathematics)*. Taylor & Francis, 2011.
[19] Louis Dupaigne and Alberto Farina. stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^n$. ArXiv e-prints, 2008.

[20] Filippo Gazzola. The sharp exponent for a Liouville-type theorem for an elliptic inequality. Rendiconti dell’Istituto di Matematica dell’Università di Trieste, 34(1-2):99–102, 2003.

[21] Nassif Ghoussoub and Changfeng Gui. On a conjecture of De Giorgi and some related problems. Mathematische Annalen, 311(3):481–491, 1998.

[22] David Gilbar and Neil S. Trudinger. Elliptic Partial Differential Equations of Second Order, volume 1542. Springer-Verlag Berlin Heidelberg, 2001.

[23] Hiroshi Matano. Asymptotic behavior and stability of solutions of semilinear diffusion equations. Publications of the Research Institute for Mathematical Sciences, 15:401–454, 1979.

[24] Luisa Moschini. New Liouville theorems for linear second order degenerate elliptic equations in divergence form. Annales de l’Institut Henri Poincare (C) Non Linear Analysis, 22(1):11–23, 2005.

[25] Manuel Del Pino, Michaa L Kowalczyk, and Juncheng Wei. On De Giorgi’s conjecture in dimension $N \geq 9$. Annals of Mathematics, 174:1485–1569, 2011.

[26] Ovidiu Savin. Phase transitions, minimal surfaces and a conjecture of De Giorgi. Annals of Mathematics, 169:41–78, 2009.