Kummer quartic double solids

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Abstract
We study equivariant birational geometry of (rational) quartic double solids ramified over (singular) Kummer surfaces.

Keywords Quadratic double solid · Kummer surface · Equivariant birational rigidity · Fano variety

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A Kummer quartic surface is an irreducible normal surface in $\mathbb{P}^3$ of degree 4 that has the maximal possible number of 16 singular points, which are ordinary double singularities. Any such surface is the Kummer variety of the Jacobian surface of a smooth genus 2 curve. Vice versa, the Jacobian surface of a smooth genus 2 curve admits a natural involution such that the quotient surface is a Kummer quartic surface in $\mathbb{P}^3$ (Fig. 1). Let $\mathcal{S}$ be a Kummer surface in $\mathbb{P}^3$, and let $\mathcal{C}$ be the smooth genus 2 curve such that

\[ \mathcal{S} \cong J(\mathcal{C}) / \langle \tau \rangle, \]

where $\tau$ is the involution of the Jacobian $J(\mathcal{C})$ that sends a point $P$ to the point $-P$. Recall from [22, 23, 30, 35, 36, 42, 48] that the surface $\mathcal{S}$ can be given by the equation

\[
\begin{align*}
& a(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 2b(x_0^2x_1^2 + x_2^2x_3^2) + 2c(x_0^2x_2^2 + x_1^2x_3^2) + 2d(x_0^2x_3^2 + x_1^2x_2^2) + 4e x_0x_1x_2x_3 = 0 \tag{2}
\end{align*}
\]

for some $[a : b : c : d : e] \in \mathbb{P}^4$ such that

\[ a(a^2 + e^2 - b^2 - c^2 - d^2) + 2bcd = 0. \tag{3} \]

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

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Note that the curve $C$ is hyperelliptic, and equation (3) defines a cubic threefold in $\mathbb{P}^4$, which is projectively equivalent to the Segre cubic threefold \cite{22,48}. Using a formula from the book \cite{11} implemented in Magma \cite{7}, we can easily extract an equation of the surface $\mathcal{S}$ from the curve $C$. However, the resulting equation may differ from (2). For instance, if $C$ is the unique genus 2 curve such that $\text{Aut}(C) \cong \mu_2 \times \mathbb{Z}_4$, then $C$ is isomorphic to the curve

$$\left\{ z^2 = xy(x^4 - y^4) \right\} \subset \mathbb{P}(1, 1, 3)$$

where $x, y, z$ are homogeneous coordinates on $\mathbb{P}(1, 1, 3)$ of weights 1, 1, 2, respectively. In this case, Magma produces the following Kummer quartic surface:

$$\left\{ x_0^4 + 2x_0^2x_2x_3 - 2x_0^2x_2^2 + 4x_0x_2^2x_2 - 4x_0x_2x_3^2 + x_2^2x_3^2 - 2x_2^2x_3x_3 + x_2^4 = 0 \right\} \subset \mathbb{P}^3,$$

which is projectively equivalent to the surface given by (2) with parameters $a = b = 1$, $c = d = -1$, $e = -4$ that do not satisfy (3). But this surface is projectively equivalent to

$$\left\{ x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4ix_0x_1x_2x_3 = 0 \right\} \subset \mathbb{P}^3,$$

which is given by (2) with parameters $a = 1$, $b = c = d = 0$, $e = -i$ that do satisfy (3). Here, we use the following Magma code provided to us by Michela Artebani:

\begin{verbatim}
\texttt{x^4 + y^4 + z^4 - 5 ( x^2 y^2 + y^2 z^2 + z^2 x^2 ) + 56 \times y \times z - 20 (x^2 + y^2 + z^2) + 16 = 0}
\end{verbatim}
R<x>:=PolynomialRing(Rationals());
C:=HyperellipticCurve(x^5-x);
GroupName(GeometricAutomorphismGroup(C));
KummerSurfaceScheme(C);

It is not very difficult to recover the hyperelliptic curve $C$ from the quartic surface $S$. Indeed, $\mathbb{P}^3$ contains 16 planes $\Pi_1, \ldots, \Pi_{16}$, called tropes, such that $S|_{\Pi_i} = 2C$ for each $i$, where $C_i$ is a smooth conic. One can show (see, for example, [31]) that

- each trope $\Pi_i$ contains exactly six singular points of the surface $S$,
- each singular point of the surface $S$ is contained in six planes among $\Pi_1, \ldots, \Pi_{16}$.

Moreover, for every conic $C_i$, there exists a double cover $C \to C_i$ which is ramified over the six points $C_i \cap \text{Sing}(S)$. This gives us an algorithm how to recover $C$ from $S$.

**Example 5** Suppose that the surface $S$ is given by the equation (2) with

$$\begin{cases}
a = 2, \\
b = -t^2 - 1, \\
c = -t^2 - 1, \\
d = -t^2 - 1, \\
e = t^3 + 3t,
\end{cases}$$

where $t \in \mathbb{C} \setminus \{\pm 1, \pm \sqrt{3}i\}$. Then the surface $S$ is given by the following equation:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + 2(t^3 + 3t)x_0x_1x_2x_3 = (t^2 + 1)(x_0^2x_1^2 + x_2^2x_3^2) + x_0^2x_2^2 + x_1^2x_3^2.$$  \hfill (6)

Its singular locus $\text{Sing}(S)$ consists of the following 16 points:

$$[1 : 1 : 1 : t], [1 : 1 : t : -1], [1 : t : 1 : -1], [t : 1 : 1 : -1], [1 : -1 : -1 : t], [1 : -1 : 1 : t], [1 : 1 : t : -1],$$

$$[t : 1 : 1 : 1], [t : 1 : -1 : -1], [t : -1 : 1 : -1], [t : -1 : -1 : 1], [1 : 1 : t : -1], [-1 : 1 : -1 : 1], [-1 : -1 : t : 1].$$

Moreover, the tropes $\Pi_1, \ldots, \Pi_{16}$ are listed in the following table:

\[
\begin{array}{ll}
\Pi_1 = \{ x_0 + x_1 + x_2 + tx_3 = 0 \} & \Pi_2 = \{ x_0 - x_1 + x_2 + tx_3 = 0 \} \\
\Pi_3 = \{ x_0 + x_1 - x_2 + tx_3 = 0 \} & \Pi_4 = \{ x_0 - x_1 - x_2 + tx_3 = 0 \} \\
\Pi_5 = \{ x_0 + x_1 + tx_2 + x_3 = 0 \} & \Pi_6 = \{ x_0 - x_1 + tx_2 - x_3 = 0 \} \\
\Pi_7 = \{ x_0 - tx_2 + x_1 - x_3 = 0 \} & \Pi_8 = \{ x_0 - x_1 - tx_2 + x_3 = 0 \} \\
\end{array}
\]
Then $C_1$ is the smooth conic

$$\{ x_0 + x_1 + x_2 + t x_3 = t x_1 x_3 + t x_2 x_3 + x_1^2 + x_1 x_2 + x_2^2 - x_3^2 = 0 \} \subset \mathbb{P}^3.$$

This conic contains the following six singular points of our surface:

\[
\begin{align*}
[1 : -1 : t : -1], & \quad [-1 : 1 : t : -1], \quad [t : -1 : 1 : -1], \\
[t : 1 : -1 : -1], & \quad [1 : t : -1 : -1], \quad [-1 : t : 1 : -1].
\end{align*}
\]

Projecting from $[t : 1 : -1 : -1]$, we get an isomorphism $C_1 \cong \mathbb{P}^1$ that maps these points to

\[
[t + 1 : -2], [1 : 0], [-1 : 1], [1 - t : 1 + t], [0 : 1], [t - 1 : 2].
\]

Therefore, the hyperelliptic curve $\mathcal{C}$ is isomorphic to the curve

$$\{ z^2 = xy(x - y)(t - 1)x + 2y)(2x - t + 1)y((t + 1)x - (t - 1)y) \} \subset \mathbb{P}(1, 1, 3).$$

In particular, it follows from [9] or Magma computations that

\[
\text{Aut}(\mathcal{C}) \cong \begin{cases} 
\mu_2 \times \mathbb{Z}_4 & \text{if } t \in \{0, \pm i, 1 \pm 2i, -1 \pm 2i\}, \\
\mu_2 \times D_{12} & \text{if } t \in \{0, \pm 3\}, \\
\mu_2 \times \mathbb{Z}_3 & \text{if } t \text{ is general.}
\end{cases}
\]

For instance, to identify \text{Aut}(\mathcal{C}) in the case when $t = i$, one can use the following script:

\[
\begin{align*}
K:=&\text{CyclotomicField}(4); \\
R<x>&:=\text{PolynomialRing}(K); \\
i:=&\text{Roots}(x^2+1,K)[1,1]; \\
t:=&i; \\
f:=&x*(x-1)*((t-1)*x+2)*(2*x-(t+1))*((t+1)*x-(t-1)); \\
C:=&\text{HyperellipticCurve}(f); \\
\text{GroupName}(&\text{GeometricAutomorphismGroup}(C));
\end{align*}
\]

In this example, we assume that $t \notin \{ \pm 1, \pm \sqrt{3}i \}$, because

- if $t = \pm 1$ or $t = \infty$, then the equation (6) defines a union of 4 planes,
- if $t = \pm \sqrt{3}i$, the equation (6) defines a double quadric.

These are semistable degenerations with minimal $\text{PGL}_4(\mathbb{C})$-orbits [53, Theorem 2.4].

Let $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ be the subgroup in $\text{PGL}_4(\mathbb{C})$ consisting of projective transformations that leave $\mathcal{S}$ invariant. Then $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ contains a subgroup $\mathbb{H} \cong \mu_2^2$ generated by

\[ \mathbb{H} \]

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Example 8 Let us use assumptions and notations of Example 5. For \( t \in \mathbb{C} \setminus \{ \pm 1, \pm \sqrt{3}i \} \), the group \( \text{Aut}(\mathbb{P}^3, \mathcal{S}) \) contains the subgroup isomorphic to \( \mu_2^4 \rtimes \mathbb{S}_3 \) generated by

\[
A_1, A_2, A_3, A_4, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

In fact, this is the whole group \( \text{Aut}(\mathbb{P}^3, \mathcal{S}) \) if \( t \) is general. On the other hand, if \( t = 0 \), then it follows from [12, 25] that \( \text{Aut}(\mathbb{P}^3, \mathcal{S}) \cong \mu_2^4 \rtimes D_12 \), and this group is generated by

\[
A_1, A_2, A_3, A_4, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

If \( t = \pm i \), then \( \mathcal{S} \) is the surface (4), and \( \text{Aut}(\mathbb{P}^3, \mathcal{S}) \cong \mu_2^4 \rtimes \mathbb{S}_4 \) is generated by

\[
A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Example 9 Suppose that \( \mathcal{S} \) is given by the equation (2) with

\[
\begin{align*}
a & = 2\zeta_5^3 + 2\zeta_5^2 + 6\zeta_5 - 1, \\
b & = 4\zeta_5^3 + 4\zeta_5^2 - 10\zeta_5 + 9, \\
c & = -6\zeta_5^3 - 6\zeta_5^2 + 4\zeta_5 + 3, \\
d & = 11, \\
e & = -20\zeta_5^3 + 24\zeta_5^2 - 16\zeta_5 + 10.
\end{align*}
\]

Then \( \text{Aut}(\mathcal{S}) \cong \mu_2 \times \mu_5 \) and \( \text{Aut}(\mathbb{P}^3, \mathcal{S}) \cong \mu_2^4 \rtimes \mu_5 \), which is generated by
Looking at Examples 5, 8 and 9, we can see a relation between $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ and $\text{Aut}(\mathcal{C})$. In fact, this relation holds for all Kummer surfaces in $\mathbb{P}^3$ by the following well-known result, about which we learned from Igor Dolgachev.

**Lemma 10** Let $\iota \in \text{Aut}(\mathcal{C})$ be the hyperelliptic involution of the curve $\mathcal{C}$. Then

$$\text{Aut}(\mathbb{P}^3, \mathcal{S}) \cong \mu_2^4 \rtimes (\text{Aut}(\mathcal{C})/\langle \iota \rangle).$$

**Proof** Let us identify $\mathcal{C}$ with the theta divisor in $J(\mathcal{C})$ via the Abel–Jacobi map whose base point is one of the fixed points of the involution $\iota$ (one of the six Weierstrass points). Then the linear system $|2\mathcal{C}|$ gives a morphism $J(\mathcal{C}) \to \mathbb{P}^3$ whose image is the surface $\mathcal{S}$. Taking the Stein factorization of the morphism $J(\mathcal{C}) \to \mathcal{S}$, we get the isomorphism (1).

On the other hand, elements in $\text{Aut}(\mathcal{C})$ give automorphisms in $\text{Aut}(J(\mathcal{C}))$ that leave the linear system $|2\mathcal{C}|$ invariant. This gives us a homomorphism $\text{Aut}(\mathcal{C}) \to \text{Aut}(\mathbb{P}^3, \mathcal{S})$, whose kernel is the hyperelliptic involution $\iota$, since $\iota$ induces the involution $\tau \in \text{Aut}(J(\mathcal{C}))$.

The image of the group $\text{Aut}(\mathcal{C})$ in $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ normalizes the subgroup $\mathbb{H}$, because elements in $\mathbb{H}$ are induced by the translations of the Jacobian $J(\mathcal{C})$ by two-torsion points. This gives a monomorphism $\vartheta : \mu_2^4 \rtimes (\text{Aut}(\mathcal{C})/\langle \iota \rangle) \to \text{Aut}(\mathbb{P}^3, \mathcal{S})$.

We claim that $\vartheta$ is an epimorphism. Indeed, the action of an element $g \in \text{Aut}(\mathbb{P}^3, \mathcal{S})$ on the surface $\mathcal{S}$ lifts to its action on the Jacobian $J(\mathcal{C})$ that leaves $|2\mathcal{C}|$ invariant, so composing $g$ with some $h \in \mathbb{H}$, we obtain an element $goh$ that preserves the class $[\mathcal{C}]$. Thus, since $[\mathcal{C}]$ is a principal polarization, the composition $goh$ preserves $\mathcal{C}$, and it acts faithfully on $\mathcal{C}$, since $\mathcal{C}$ generates $J(\mathcal{C})$. This gives $goh \in \text{im}(\vartheta)$, so $\vartheta$ is surjective. $\square$

Since $\text{Aut}(\mathcal{C})$ is isomorphic to a group among $\mu_2, \mu_2^2, D_8, D_{12}, \mu_2D_{12}, \mu_2\mathbb{S}_4$, $\mu_2 \rtimes \mu_5$, we conclude that $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ is isomorphic to one of the following groups:

$$\mu_2^4 \rtimes \mu_2, \mu_2 \rtimes \mu_2^4 \rtimes \mathbb{S}_5, \mu_2^4 \rtimes D_{12}, \mu_2^3 \rtimes \mathbb{S}_4, \mu_2^2 \rtimes \mu_5.$$

Note that the group $\text{Aut}(\mathcal{S})$ is always larger that $\text{Aut}(\mathbb{P}^3, \mathcal{S})$ [37, 40].

**Remark 11** ([5, 30, 48]) Let $\mathcal{R}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\text{PGL}_4(\mathbb{C})$. Then $\text{Aut}(\mathbb{P}^3, \mathcal{C}) \subset \mathcal{R}$, and there exists an exact sequence $1 \to \mathbb{H} \to \mathcal{R} \to \mathbb{S}_6 \to 1$, which can be described as follows. Let

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} -i & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -i & 0 & 1 \end{pmatrix}.$$
\[
S_1 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6(x_0^2 x_1^2 + x_0^2 x_2^2) - 6(x_0 x_1 x_2 x_3) = 0 \},
S_2 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6(x_0^2 x_1^2 + x_0^2 x_2^2) + 6(x_0^2 x_1^2 + x_1^2 x_3) + 6(x_0^2 x_2^2 + x_1^2 x_3) = 0 \},
S_3 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6(x_0^2 x_1^2 + x_0^2 x_2^2) - 6(x_0^2 x_1^2 + x_1^2 x_3) + 6(x_0^2 x_2^2 + x_1^2 x_3) = 0 \},
S_4 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6(x_0^2 x_1^2 + x_0^2 x_2^2) + 6(x_0^2 x_1^2 + x_1^2 x_3) + 6(x_0^2 x_2^2 + x_1^2 x_3) = 0 \},
S_5 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 - 12x_0 x_1 x_2 x_3 = 0 \},
S_6 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0 x_1 x_2 x_3 = 0 \}.
\]

Then \( S_1, S_2, S_3, S_4, S_5, S_6 \) are \( \mathcal{H} \)-invariant surfaces, and the quotient \( \mathcal{N}/\mathcal{H} \) permutes them. For instance, the transformation \( w \) and \( x \) is a homogeneous coordinate on \( \mathbb{P}^\mathcal{H} = \mathbb{P}^4 \). The quotient \( \mathcal{N}/\mathcal{H} \) acts on the set \( \{ S_1, S_2, S_3, S_4, S_5, S_6 \} \) as \( (1 \ 2 \ 3 \ 4) \) \( \mathcal{N}/\mathcal{H} \). This gives an explicit isomorphism \( \mathcal{N}/\mathcal{H} \cong \mathcal{S}_6 \).

**Remark 12** The quotient \( \text{Aut}(\mathbb{P}^3, \mathcal{S})/\mathcal{H} \) naturally acts on the threefold (3) fixing the point \([a : b : c : d : e] \) that corresponds to \( \mathcal{S} \). Projecting the threefold from this point, we obtain a (rational) double cover \( \mathbb{P}^3 \) that is branched along the surface \( \mathcal{S} \).

Let \( \pi : X \to \mathbb{P}^3 \) be the double cover branched along the surface \( \mathcal{S} \). Set \( H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1)) \). Then Pic\((X) = \mathbb{Z}[H] \), \( H^3 = 2 \) and \( -K_X \sim 2H \), so \( X \) is a del Pezzo threefold of degree 2, which has 16 ordinary double points. We say that \( X \) is a *Kummer quartic double solid* [57].

The threefold \( X \) is a hypersurface in \( \mathbb{P}(1, 1, 1, 1, 2) \) given by

\[
w^2 = a(x_0^3 + x_1^3 + x_2^3 + x_3^3) + 2b(x_0^2 x_1^2 + x_0^2 x_2^2 + x_0^2 x_3^2) + 2c(x_0^2 x_1^2 + x_0^2 x_2^2) + 2d(x_0^2 x_1^2 + x_0^2 x_3^2) + 4e x_0 x_1 x_2 x_3,
\]

where we consider \( x_0, x_1, x_2, x_3 \) as homogeneous coordinates on \( \mathbb{P}(1, 1, 1, 1, 2) \) of weight 1, and \( w \) is a homogeneous coordinate on \( \mathbb{P}(1, 1, 1, 1, 2) \) of weight 2.

It is well-known that the threefold \( X \) is rational \([15, 50, 52, 57]\), see also Remark 12. Let us describe one birational map \( \mathbb{P}^3 \to X \) following [50, 52]. To do this, fix a twisted cubic curve \( C_3 \subset \mathbb{P}^3 \). Then \( C_3 \) contains six distinct points \( P_1, \ldots, P_6 \) such that there exists a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^3 & \stackrel{\chi}{\longrightarrow} & X \\
\downarrow & \searrow & \downarrow \pi \\
\mathbb{P}^3 & - & - & - & - \to & \mathbb{P}^3
\end{array}
\]

where \( \eta \) is a blow up of the points \( P_1, \ldots, P_6 \), the morphism \( \varphi \) is a contraction of the proper transform of the curve \( C_3 \) and proper transforms of 15 lines in \( \mathbb{P}^3 \) that pass through two points among \( P_1, \ldots, P_6 \), and \( \chi \) is a rational map given by the linear system of quadric surfaces that pass through the points \( P_1, \ldots, P_6 \).

**Corollary 15** ([19, 27]) One has \( \text{Cl}(X) \cong \mathbb{Z}^7 \).

**Remark 16** The vertices of the quadric cones in \( \mathbb{P}^3 \) that pass through \( P_1, \ldots, P_6 \) span an irreducible singular quartic surface \( \mathcal{S} \) which is known as the *Weddle surface* [35, 57].
This surface is singular at the points $P_1, \ldots, P_6$, and $\chi$ induces a birational map $\mathcal{G} \to \mathcal{J}$. On the other hand, the double cover of $\mathbb{P}^3$ branched along $\mathcal{G}$ is irrational [15, 57].

Let $\sigma \in \text{Aut}(X)$ be the Galois involution of the double cover $\pi$. Then $\sigma$ is contained in the center of the group $\text{Aut}(X)$. Moreover, since $\pi$ is $\text{Aut}(X)$-equivariant, it induces a homomorphism $\nu : \text{Aut}(X) \to \text{Aut}(\mathbb{P}^3, \mathcal{J})$ with $\ker(\nu) = \langle \sigma \rangle$, so we have exact sequence

\[
1 \longrightarrow \langle \sigma \rangle \longrightarrow \text{Aut}(X) \xrightarrow{\nu} \text{Aut}(\mathbb{P}^3, \mathcal{J}) \longrightarrow 1.
\]

The main result of this paper is the following theorem (cf. [3, 4, 14]).

**Theorem 17** Let $G$ be any subgroup in $\text{Aut}(X)$ such that $\text{Cl}^G(X) \cong \mathbb{Z}$ and $\mathbb{H} \subseteq \nu(G)$. Then the Fano threefold $X$ is $G$-birationally super-rigid.

**Corollary 18** Let $G$ be any subgroup in $\text{Aut}(X)$ such that $G$ contains $\sigma$ and $\mathbb{H} \subseteq \nu(G)$. Then $X$ is $G$-birationally super-rigid.

The condition $\text{Cl}^G(X) \cong \mathbb{Z}$ in Theorem 17 simply means that $X$ is a $G$-Mori fibre space, which is required by the definition of $G$-birational super-rigidity (see [17, Definition 3.1.1]). The condition $\mathbb{H} \subseteq \nu(G)$ does not imply that $\text{Cl}^G(X) \cong \mathbb{Z}$, see Examples 28 and 29 below. The following example shows that we cannot remove the condition $\mathbb{H} \subseteq \nu(G)$.

**Example 19** Observe that $\text{Cl}^{\langle \sigma \rangle}(X) \cong \mathbb{Z}$. Let $S_1$ and $S_2$ be two general surfaces in $|H|$, and let $C = S_1 \cap S_2$. Then $C$ is a smooth irreducible $\langle \sigma \rangle$-invariant curve, $\pi(C)$ is a line, and there exists $\langle \sigma \rangle$-commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & X \\
\downarrow{\beta} & & \downarrow{\text{blow up of the curve } C} \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
\]

where $\alpha$ is the blow up of the curve $C$, the dashed arrow $\dashrightarrow$ is given by the pencil generated by the surfaces $S_1$ and $S_2$, and $\beta$ is a fibration into del Pezzo surfaces of degree 2. Therefore, the threefold $X$ is not $\langle \sigma \rangle$-birationally rigid.

Let $G$ be a subgroup in $\text{Aut}(X)$ such that $\nu(G)$ contains $\mathbb{H}$. Before proving Theorem 17, let us explain how to check the condition $\text{Cl}^G(X) \cong \mathbb{Z}$. For a homomorphism $\rho : \mathbb{H} \to \mathbb{M}_2$, consider the action of the group $\mathbb{H}$ on the threefold $X$ given by

- $A_1 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_1)w]$,
- $A_2 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [-x_0 : x_1 : -x_2 : x_3 : \rho(A_2)w]$,
- $A_3 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_1 : x_2 : x_3 : x_2 : \rho(A_3)w]$,
- $A_4 : [x_0 : x_1 : x_2 : x_3 : w] \mapsto [x_3 : x_2 : x_1 : x_0 : \rho(A_4)w]$. 

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This gives a lift of the subgroup $\mathbb{H}$ to $\text{Aut}(X)$. Let $\mathbb{H}^\rho$ be the resulting subgroup in $\text{Aut}(X)$. Since $\mathbb{H} \subset \sigma(G)$, we may assume that $\mathbb{H}^\rho \subset G$. If $\rho$ is trivial, we let $\mathbb{H} = \mathbb{H}^\rho$ for simplicity.

For every plane $\Pi_i$, one has $\pi^*(\Pi_i) = \Pi_i^+ + \Pi_i^-$, where $\Pi_i^+$ and $\Pi_i^-$ are two irreducible surfaces such that $\Pi_i^+ \neq \Pi_i^-$ and $\sigma(\Pi_i^+) = \Pi_i^-$. Note that we do not have a canonical way to distinguish between the surfaces $\Pi_i^+$ and $\Pi_i^-$. Namely, if $\pi^*(\Pi_i)$ is given by

$$
\begin{cases}
  h_i(x_0, x_1, x_2, x_3) = 0, \\
  w^2 = g_i^2(x_0, x_1, x_2, x_3),
\end{cases}
$$

where $h_i$ is a linear polynomial such that $\Pi_i = \{ h_i = 0 \} \subset \mathbb{P}^3$, and $g_i$ is a quadratic polynomial such that the conic $C_i$ is given by $h_i = g_i = 0$, then

$$
\Pi_i^\pm = \{ w \pm g_i(x_0, x_1, x_2, x_3) = h_i(x_0, x_1, x_2, x_3) = 0 \} \subset \mathbb{P}(1, 1, 1, 2).
$$

But the choice of $\pm$ here is not uniquely defined, because we can always swap $g_i$ with $-g_i$.

On the other hand, since $\mathbb{H}$ acts transitively on the set (7), the set

$$
\left\{ \Pi_i^+, \Pi_i^-, \Pi_i^+, \Pi_i^- \right\}
$$

splits into two $\mathbb{H}^\rho$-orbits consisting of 16 surfaces such that each of them contains exactly one surface among $\Pi_i^+$ and $\Pi_i^-$ for every $i$. Hence, we may assume that these $\mathbb{H}^\rho$-orbits are

$$
\left\{ \Pi_1^+, \Pi_2^+, \Pi_3^+, \Pi_4^+, \Pi_5^+, \Pi_6^+, \Pi_7^+, \Pi_8^+, \Pi_9^+, \Pi_{10}^+, \Pi_{11}^+, \Pi_{12}^+, \Pi_{13}^+, \Pi_{14}^+, \Pi_{15}^+, \Pi_{16}^+ \right\}
$$

and

$$
\left\{ \Pi_1^-, \Pi_2^-, \Pi_3^-, \Pi_4^-, \Pi_5^-, \Pi_6^-, \Pi_7^-, \Pi_8^-, \Pi_9^-, \Pi_{10}^-, \Pi_{11}^-, \Pi_{12}^-, \Pi_{13}^-, \Pi_{14}^-, \Pi_{15}^-, \Pi_{16}^- \right\}.
$$

**Corollary 20** The surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ generate the group $\text{Cl}(X)$.

**Corollary 21** Either $\text{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}$ or $\text{Cl}^{\mathbb{H}^\rho}(X) \cong \mathbb{Z}^2$.

The surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ are called *planes* [52]. They are not Cartier divisors. To describe their strict transforms on the threefold $\hat{X}$ from the commutative diagram (14), let us introduce the following notations:

(a) let $E_i$ be the $\eta$-exceptional surfaces such that $\eta(E_i) = P_i$, where $1 \leq i \leq 6$;
(b) let $\hat{\Pi}_{i,j,k}$ be the strict transform on the threefold $\hat{X}$ of the plane in $\mathbb{P}^3$ that passes through the three distinct points $P_j$, $P_j$, $P_k$, where $1 \leq i < j < k \leq 6$;
(c) let $\hat{Q}_i$ be the strict transform of the irreducible quadric cone in $\mathbb{P}^3$ that contains all points $P_1, \ldots, P_6$ and is singular at the point $P_i$, where $1 \leq i \leq 6$.

Then strict transforms of the surfaces $\Pi_1^+, \Pi_1^-, \ldots, \Pi_{16}^+, \Pi_{16}^-$ on the threefold $\hat{X}$ are

$$
E_1, E_2, \ldots, E_6, \hat{\Pi}_{1,2,3}, \hat{\Pi}_{1,2,4}, \ldots, \hat{\Pi}_{3,5,6}, \hat{\Pi}_{4,5,6}, \hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_6.
$$

The involution $\sigma$ acts birationally on $\hat{X}$ as a composition of flops of $\varphi$-contracted curves, it swaps $E_i \leftrightarrow \hat{Q}_i$, and it swaps $\hat{\Pi}_{i,j,k} \leftrightarrow \hat{\Pi}_{r,s,t}$ such that $(i,j,k,r,s,t) = \{1, 2, 3, 4, 5, 6\}$. For
a non-trivial element $g \in \langle \tau, \mathbb{H}^p \rangle$, the involution $\varphi^{-1} \circ g \circ \varphi$ is a composition of flops, and either $\eta \circ \varphi^{-1} \circ g \circ \varphi \circ \eta^{-1}$ or $\eta \circ \varphi^{-1} \circ \sigma \circ \varphi \circ \eta^{-1}$ is a Cremona involution whose fundamental points are four points among $P_1, \ldots, P_6$ that swaps the remaining two points.

Now, let us give a criterion for $\text{Cl}^G(X) \cong \mathbb{Z}$. To do this, we set

$$\Pi^\pm = \sum_{i=1}^{16} \Pi^\pm_i.$$ Then $\Pi^+$ and $\Pi^-$ are $\mathbb{H}^p$-invariant divisors, $\sigma(\Pi^+) = \Pi^-$ and $\Pi^+ + \Pi^- \sim 16H$.

**Lemma 22** One has $\text{Cl}^G(X) \cong \mathbb{Z}$ is at least one of the following conditions is satisfied:

1. The group $G$ swaps $\Pi^+$ and $\Pi^-$;
2. The divisor $\Pi^+$ is Cartier;
3. The divisor $\Pi^-$ is Cartier;
4. The surfaces $\Pi^+_1, \ldots, \Pi^+_6$ generate the group $\text{Cl}(X)$;
5. The surfaces $\Pi^-_1, \ldots, \Pi^-_{16}$ generate the group $\text{Cl}(X)$.

**Proof** The assertion follows from Corollary 21, since we assume that $\mathbb{H}^p \subset G$. \hfill \Box

This lemma is easy to apply if we fix $\mathcal{S}$ and the group $G \subset \text{Aut}(X)$ such that $\mathbb{H} \subset \nu(G)$. For instance, to check whether the surfaces $\Pi^+_1, \ldots, \Pi^+_6$ generate the group $\text{Cl}(X)$ or not, we can use the fact that $\text{Cl}(X) \cong \mathbb{Z}'$ is naturally equipped with an intersection form [50]. Namely, fix a smooth del Pezzo surface $S \in |H|$, and let

$$D_1 \cdot D_2 = D_1|_S \cdot D_2|_S \in \mathbb{Z}$$

for any two Weil divisors $D_1$ and $D_2$ in $\text{Cl}(X)$. Then

$$\Pi^\pm_i \cdot \Pi^\pm_j = \begin{cases} 0 & \text{if } i \neq j \text{ and } \Pi^\pm_i \cap \Pi^\pm_j \text{ does not contain curves,} \\ 1 & \text{if } i \neq j \text{ and } \Pi^\pm_i \cap \Pi^\pm_j \text{ contains a curve,} \\ -1 & \text{if } i = j \text{ and } \Pi^\pm_i = \Pi^\pm_j, \\ 2 & \text{if } i = j \text{ and } \Pi^\pm_i \neq \Pi^\pm_j, \end{cases}$$

where two $\pm$ in $\Pi^\pm_i$ and $\Pi^\pm_j$ are independent.

**Remark 23** Let $\Lambda$ be the sublattice in $\text{Cl}(X)$ consisting of divisors $D$ such that $D \cdot H = 0$. Then $\Lambda$ is isomorphic to a root lattice of type $D_6$ by [50, Theorem 1.7], and the natural homomorphism $\text{Aut}(X) \to \text{Aut}(\Lambda)$ is injective [50], where $\text{Aut}(\Lambda) \cong (\mu_2 \rtimes \mathbb{S}_6) \rtimes \mu_2$.

Applying Lemma 22, we get

**Corollary 24** If $\text{rank}(\Pi^+_1 \cdot \Pi^+_7) = 7$ or $\text{rank}(\Pi^-_1 \cdot \Pi^-_7) = 7$, then $\text{Cl}^{\mathbb{H}^p}(X) \cong \mathbb{Z}$.

Let us show how to apply Corollary 24.
\[
\begin{align*}
\Pi_1^+ &= x_0 + x_1 + x_2 + \frac{24}{s^4 + 1} x_3 = 0 \\
\Pi_2^+ &= s^2 - (s + 1) x_0^2 + (s^2 + 1) x_1 x_2 + 2 s x_1 x_3 + (s^2 + 1) x_4^2 + 2 s x_2 x_3 - (s^2 + 1) x_5^2 \\
\Pi_3^+ &= x_0 - x_1 + x_2 + \frac{24}{s^4 + 1} x_3 = 0 \\
\Pi_4^+ &= s^2 - (s + 1) x_0^2 + (s^2 + 1) x_1 x_2 - 2 s x_1 x_3 + (s^2 + 1) x_4^2 + 2 s x_2 x_3 - (s^2 + 1) x_5^2 = 0 \\
\Pi_5^+ &= x_0 - x_1 - x_2 + \frac{24}{s^4 + 1} x_3 = 0 \\
\Pi_6^+ &= s^2 - (s + 1) x_0^2 + 2 s x_2 x_0 + (s^2 + 1) x_3 x_5 - (s^2 + 1) x_4^2 + 2 s x_2 x_3 + (s^2 + 1) x_5^2 = 0 \\
\Pi_7^+ &= s^2 - (s + 1) x_0^2 - 2 s x_2 x_0 - (s^2 + 1) x_3 x_5 - (s^2 + 1) x_4^2 + 2 s x_2 x_3 + (s^2 + 1) x_5^2 = 0 \\
\Pi_8^+ &= x_0 - x_1 - \frac{24}{s^4 + 1} x_3 = 0 \\
\Pi_9^+ &= s^2 - (s + 1) x_0^2 - 2 s x_2 x_0 + (s^2 + 1) x_3 x_5 - (s^2 + 1) x_4^2 + 2 s x_2 x_3 + (s^2 + 1) x_5^2 = 0 \\
\Pi_{10}^+ &= x_0 - \frac{24}{s^4 + 1} x_1 + x_2 + x_3 = 0 \\
\Pi_{11}^+ &= s^2 - (s + 1) x_0^2 - 2 s x_1 x_0 + (s^2 + 1) x_3 x_5 - (s^2 + 1) x_4^2 + 2 s x_1 x_3 + (s^2 + 1) x_5^2 = 0 \\
\Pi_{12}^+ &= x_0 - \frac{24}{s^4 + 1} x_1 + x_2 - x_3 = 0 \\
\Pi_{13}^+ &= s^2 - (s + 1) x_0^2 + 2 s x_1 x_0 - (s^2 + 1) x_3 x_5 - (s^2 + 1) x_4^2 + 2 s x_1 x_3 + (s^2 + 1) x_5^2 = 0 \\
\Pi_{14}^+ &= x_0 + \frac{24}{s^4 + 1} x_1 - x_2 + x_3 = 0 \\
\Pi_{15}^+ &= s^2 - (s + 1) x_0^2 + 2 s x_1 x_0 - (s^2 + 1) x_3 x_5 + (s^2 + 1) x_4^2 + 2 s x_1 x_3 - (s^2 + 1) x_5^2 = 0 \\
\Pi_{16}^+ &= x_0 + \frac{24}{s^4 + 1} x_1 + x_2 - x_3 = 0
\end{align*}
\]

Fig. 2 Defining equations of the surfaces $\Pi_1^+, \ldots, \Pi_{16}^+$ in Example 25
Example 25 Let us use assumptions and notations of Example 5. Suppose, in addition, that \( \rho : \mathbb{H} \to \mu_3 \) is the trivial homomorphism. Therefore, we have \( \mathbb{H}^\rho = \mathbb{H} \). Set \( t = \frac{2i}{s^2+1} \).

Observe that \( \pi^*(\Pi_1) \) is given by \( \mathbb{P}(1, 1, 1, 1, 2) \) by the following equations:

\[
\begin{align*}
x_0 + x_1 + x_2 + \frac{2s}{s^2 + 1} x_3 &= 0, \\
w^2 &= \frac{(s^2 - 1)^2}{(s^2 + 1)^4} ((s^2 + 1)x_1^2 + (s^2 + 1)x_1x_2 + 2sx_1x_3 + (s^2 + 1)x_2^2 + 2sx_2x_3 - (s^2 + 1)x_3^2)^2. 
\end{align*}
\]

Thus, without loss of generality, we may assume that the surface \( \Pi_1^+ \) is given by

\[
\begin{align*}
x_0 + x_1 + x_2 + \frac{2s}{s^2 + 1} x_3 &= 0, \\
w &= \frac{s^2 - 1}{(s^2 + 1)^2} ((s^2 + 1)x_1^2 + (s^2 + 1)x_1x_2 + 2sx_1x_3 + (s^2 + 1)x_2^2 + 2sx_2x_3 - (s^2 + 1)x_3^2). 
\end{align*}
\]

Then the defining equations of the remaining surfaces \( \Pi_2^+, \ldots, \Pi_{16}^+ \) are listed in Figure 2.

Now, the intersection matrix \( (\Pi_i^+ \cdot \Pi_j^+) \) can be computed as follows:

\[
\begin{pmatrix}
-1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

The rank of this matrix is 7. Therefore, we conclude that \( \text{Cl}_\mathbb{H}(X) \cong \mathbb{Z} \) by Corollary 24. Note that we can also prove this using Lemma 22(ii). To do this, it is enough to show that the divisor \( \Pi^+ \) is a Cartier divisor, which can be done locally at any point in \( \text{Sing}(X) \).

Example 26 Let us use assumptions and notations of Example 5. Then \( \text{Aut}(X) \) contains a unique subgroup \( G \) such that \( G \cong \nu(G) \cong \mu_2 \rtimes \mu_3 \), and \( \nu(G) \) is generated by
One can check that $G$ contains the subgroup $\mathbb{H} = \mathbb{H}^\rho$, where $\rho$ is a trivial homomorphism. Therefore, it follows from Example 25 that $\text{Cl}^G(X) \cong \mathbb{Z}$.

If $\text{Cl}^G(X) \not\cong \mathbb{Z}$, we have the following result.

**Lemma 27** Let $G$ be any subgroup in $\text{Aut}(X)$ such that $\text{Cl}^G(X) \not\cong \mathbb{Z}$ and $\mathbb{H} \subseteq \nu(G)$. Then $\text{Cl}^G(X) \cong \mathbb{Z}^2$, and exists a $G$-Sarkisov link

$$A_1, A_2, A_3, A_4, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

where $\varpi$ is a $G$-equivariant small resolution, $\zeta$ is a birational involution that flops sixteen $\varpi$-contracted curves, $\phi$ is a $\mathbb{P}^1$-bundle, and $Z$ is a smooth del Pezzo surface of degree 4.

**Proof** By Corollary 21, one has $\text{Cl}^G(X) \cong \mathbb{Z}^2$, but $\text{Pic}(X) \cong \mathbb{Z}$. Using this and the fact that $G$ acts transitively on the set $\text{Sing}(X)$, we see that there exists a $G$-equivariant small resolution of singularities $\varpi : V \to X$. Then $-K_V \sim \varpi^*(2H)$.

Now, applying $G$-equivariant Minimal Model Program to $V$, we obtain a $G$-equivariant morphism $\phi : V \to Z$ such that $\phi_* (\mathcal{O}_V) = \mathcal{O}_Z$, and one of the following possibilities hold:

1. $Z$ is a possibly singular Fano threefold, and $\phi$ is birational;
2. $Z$ is a normal surface, and $\phi$ is a conic bundle;
3. $Z \cong \mathbb{P}^1$, and $\phi$ is a del Pezzo fibration.

Note that $\text{rk} \text{Cl}^G(V/Z) = 1$. In particular, the divisor $-K_V$ is $\phi$-ample.

If $\phi$ is a del Pezzo fibration, then its general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$ by the adjunction formula, because $-K_V \sim \varpi^*(2H)$. Moreover, in this case all fibers of the morphism $\phi$ are reduced and irreducible [29], which gives $\text{rk} \text{Cl}(V) \leq 3$, which is impossible since $\text{Cl}(V) \cong \mathbb{Z}'$. Therefore, either $\phi$ is birational, or $\phi$ is a conic bundle.

Now, applying relative Minimal Model Program to the threefold $V$ over $Z$, we obtain a commutative diagram

$$V^\prime \xrightarrow{\alpha} V \xrightarrow{\zeta} V^\prime$$

$$Z \xrightarrow{\phi} V \xrightarrow{\zeta} V \xrightarrow{\phi} Z$$

where $\alpha$ is an extremal contraction, and $\beta$ is a morphism with connected fibers.
Suppose that \( \alpha \) is birational. Let \( E \) be the \( \alpha \)-exceptional surface. Then \( U \) is a smooth threefold, and \( \alpha \) is a usual blow up of a point in \( U \) (see [10, Lemma 2.8] or [21, 46]). Then \( E \cong \mathbb{P}^2 \) and \( E|_E \cong \mathcal{O}_{\mathbb{P}^2}(-1) \). So, the adjunction formula gives \( \nu^*(H)|_E \cong \mathcal{O}_{\mathbb{P}^2}(1) \). Therefore, we conclude that \( \nu \circ \nu^*(E) \cong \nu^*(E) \cong \mathbb{P}^2 \) and \( \nu \circ \nu^*(E) \) is a plane in \( \mathbb{P}^3 \), which implies that \( \nu(E) = \Pi^+_i \) or \( \nu(E) = \Pi^-_i \) for some \( i \in \{1, \ldots, 16\} \).

On the other hand, for every \( g \in G \), we have
\[
g(E) \cap E = \emptyset,
\]
because all curves in \( E \) are contained in one extremal ray of the Mori cone \( \overline{\text{NE}}(V) \). Moreover, since \( H \subseteq \nu(G) \), the \( G \)-orbit of the surface \( E \) consists of at least 16 surfaces. This implies that \( \text{rk} \text{Cl}(V) \geq 17 \), which is impossible, since \( \text{Cl}(V) \cong \text{Cl}(X) \cong \mathbb{Z}^7 \).

Thus, we conclude that \( \alpha \) is not birational, so \( \phi \) is not birational either. Hence, we see that \( \phi \) is a conic bundle. Then \( \alpha \) is also a conic bundle, because otherwise \( \alpha \) would be birational. Since general fibers of both conic bundles \( \alpha \) and \( \beta \) are the same, they generate one extremal ray of the Mori cone \( \overline{\text{NE}}(V) \). This implies that \( \beta \) must be an isomorphism. Therefore, we may assume that \( \alpha = \phi \) and \( \beta = \text{Id}_U \).

Now, it follows from [10, Lemma 2.5] that \( Z \) is smooth, it is a weak del Pezzo surface, and \( V \cong \mathbb{P}(\mathcal{E}) \) for some rank two vector bundle \( \mathcal{E} \) on the surface \( Z \).

To complete the proof, it is enough to show that \( Z \) is a del Pezzo surface of degree 4. First, we observe that \( \text{Cl}(Z) \cong \mathbb{Z}^6 \), since \( \text{Cl}(V) \cong \mathbb{Z}^7 \). This gives \( K_Z^2 = 4 \).

Let \( C \) be a general fiber of the \( \mathbb{P}^1 \)-bundle \( \phi \). Then we have \( 2H \cdot \nu^*(C) = -K_U \cdot C = 2 \), which implies that \( \nu \circ \phi(C) \) is a line in \( \mathbb{P}^3 \) that is tangent to the surface \( \mathcal{S} \) at two points. Now, for \( i \in \{1, \ldots, 16\} \), let \( \tilde{\Pi}^+_i \) be the proper transform on \( V \) of the surface \( \Pi^+_i \). Then
\[
1 = \nu^*(H) \cdot \phi = (\tilde{\Pi}^+_i + \tilde{\Pi}^-_i) \cdot S = \tilde{\Pi}^+_i \cdot C + \tilde{\Pi}^-_i \cdot C.
\]
Thus, without loss of generality, we may assume that \( \tilde{\Pi}^+_i \cdot C = 1 \) and \( \tilde{\Pi}^-_i \cdot C = 0 \) for every \( i \). Then \( \tilde{\Pi}^-_i \) is mapped to a curve in \( Z \), and \( \phi \) induces a birational morphism \( \tilde{\Pi}^+_i \to Z \).

Recall that \( \nu \) induces a birational morphism \( \tilde{\Pi}^+_i \to \Pi^+_i \) that is a blow up of \( k \geq 0 \) points in the intersection \( \Pi^+_i \cap \text{Sing}(X) \), where \( k \leq 6 \), since \( \Pi^+_i \cap \text{Sing}(X) \) consists of six points. Moreover, the morphism \( \nu \) also induces a birational morphism \( \tilde{\Pi}^-_i \to \Pi^-_i \) that blows up the remaining \( 6 - k \) points in \( \Pi^+_i \cap \text{Sing}(X) = \Pi^-_i \cap \text{Sing}(X) \). We also know that the intersection \( \Pi^+_i \cap \text{Sing}(X) \) is contained in an irreducible conic in \( \Pi^+_i \cong \mathbb{P}^2 \). Then

- either \( k \leq 5 \) and \( \tilde{\Pi}^+_i \) is a smooth del Pezzo surface of degree \( 9 - k \),
- or \( k = 6 \) and \( \tilde{\Pi}^+_i \) is a smooth weak del Pezzo surface of degree 3.

If \( k = 6 \), then \( \tilde{\Pi}^-_i \cong \Pi^-_i \cong \mathbb{P}^2 \), and the surface \( \tilde{\Pi}^-_i \) cannot be mapped to a curve by \( \phi \).

Thus, we conclude that \( k \leq 5 \), and \( \tilde{\Pi}^+_i \) is a smooth del Pezzo surface of degree \( 9 - k \geq 4 \). Since \( \phi \) induces a birational morphism \( \tilde{\Pi}^+_i \to Z \) and \( K_Z^2 = 4 \), we get \( k = 5 \) and \( \tilde{\Pi}^+_i \cong Z \), which implies that \( Z \) is a del Pezzo surface of degree 4, and \( \phi(\tilde{\Pi}^+_i) \) is a \((-1)\)-curve.

Before we proceed, let us make few remarks:

1. In the proof of Lemma 27, we show that general fibers of the \( \mathbb{P}^1 \)-bundle \( \phi \) are mapped by the morphism \( \pi \circ \nu \) to the lines in \( \mathbb{P}^3 \) tangent to \( \mathcal{S} \) at two points. These lines form a classical congruence in \( \mathbb{P}^3 \) of degree (2, 2), which is related to the quartic surface \( \mathcal{S} \), see [8, 23, 36, 43, 44, 56, 2, Example 5.5], [49, Example 2.3]. This implies that the weak Fano threefold \( V \) in Lemma 27 is isomorphic to \( \mathbb{P}(\mathcal{E}) \), where \( \mathcal{E} \) is a rank two vector
bundle on the quartic del Pezzo surface $Z \subset \text{Gr}(2, 4)$, which is obtained by restricting to $Z$ the universal quotient bundle from $\text{Gr}(2, 4)$.

(2) It was pointed to us by Alexander Kuznetsov that the diagram in Lemma 27 can be constructed as follows. We know from [55] the existence of the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{N}) & \cong & \mathbb{P}(\mathcal{E}) \\
\pi_{\mathbb{P}^3} & & \pi_Q \\
\mathbb{P}^3 & \rightarrow & Q
\end{array}
\]

$\mathcal{N}$ is the rank two null-correlation vector bundle on $\mathbb{P}^3$, $Q$ is a smooth quadric threefold, $\mathcal{E}$ is a rank two vector bundle on the quadric $Q \subset \text{Gr}(2, 4)$ obtained by restricting to $Q$ the universal quotient bundle from the Grassmannian, and $\pi_{\mathbb{P}^3}$ and $\pi_Q$ are projections. Thus, since $Z$ is cut out on $Q$ by another quadric, we can identify $V$ with a preimage of the del Pezzo surface $Z$ in the fourfold $\mathbb{P}(\mathcal{E})$. Then $\pi_{\mathbb{P}^3}$ induces a morphism $V \rightarrow \mathbb{P}^3$, which is the composition $\pi \circ \sigma$.

(3) We can also describe the $\mathbb{P}^1$-bundle $\phi$ in Lemma 27 using the diagram (14). Namely, a general fiber of this $\mathbb{P}^1$-bundle is mapped by $\eta \circ \phi^{-1} \circ \sigma$ to

- either a line that passes through one of the points $P_1, \ldots, P_6$,
- or a twisted cubic curve that passes through five points among $P_1, \ldots, P_6$.

(4) If $\mathcal{T}$ is the preimage of the quartic surface $\mathcal{S}$ on the threefold $V$ in Lemma 27, then $\mathcal{T}$ is smooth, the morphism $\pi \circ \sigma$ induces a minimal resolution $\mathcal{T} \rightarrow \mathcal{S}$, and the $\mathbb{P}^1$-bundle $\phi$ induces a $G$-equivariant double cover $\mathcal{T} \rightarrow Z$, cf. [54].

(5) Using Lemma 27 one can relate the subgroup $G \subset \text{Aut}(X)$ such that $\text{Cl}^G(X) \cong \mathbb{Z}^2$ with a corresponding subgroup in the group $\text{Aut}(Z)$, see [24, 33].

Note that $\text{Cl}^G(X) \cong \mathbb{Z}^2$ is indeed possible. Let us give two (related) examples.

**Example 28** Let us use all assumptions and notations of Example 25, and let $G = \mathbb{H}^p$, where the homomorphism $\rho$ is defined by $\rho(A_1) = -1$, $\rho(A_2) = 1$, $\rho(A_3) = -1$, $\rho(A_4) = 1$. Then, arguing as in Example 25, we compute $\text{Cl}^G(X) \cong \mathbb{Z}^2$.

**Example 29** Let us use all assumptions and notations of Example 9. Then

$$\text{Aut}(X) \cong \mu_2 \times \text{Aut}(\mathbb{P}^3, \mathcal{J}) \cong \mu_2 \times \left( \mu_4^5 \rtimes \mu_5 \right),$$

and the group $\text{Aut}(X)$ contains a unique subgroup isomorphic to $\text{Aut}(\mathbb{P}^3, \mathcal{J}) \cong \mu_2^4 \rtimes \mu_5$. Suppose that $G$ is this subgroup. It follows from Remark 23 that $\text{Cl}(X) \otimes \mathbb{Q}$ is a faithful seven-dimensional $G$-representation. Using this, it is easy to see that $\text{Cl}(X) \otimes \mathbb{Q}$ splits as a sum of an irreducible five-dimensional representation and two trivial one-dimensional representations. Hence, we conclude that $\text{Cl}^G(X) \cong \mathbb{Z}^2$.

Before proving Theorem 17, let us prove its two baby cases, which follow from [16, 18].

**Proposition 30** Suppose $G = \text{Aut}(X)$, and $\mathcal{I}$ is the quartic surface from Example 9. Then $\text{Cl}^G(X) \cong \mathbb{Z}$ and $X$ is $G$-birationally super-rigid.
Proof Since $\sigma \in G$, we get $\text{Cl}^G(X) \cong \mathbb{Z}$. Let us show that $X$ is $G$-birationally super-rigid.

Note that the $\nu(G)$-equivariant birational geometry of the projective space $\mathbb{P}^3$ has been studied in [18]. In particular, we know from [18, Corollary 4.7] and [18, Theorem 4.16] that

- $\mathbb{P}^3$ does not contain $\nu(G)$-orbits of length less than 16,
- $\mathbb{P}^3$ does not contain $\nu(G)$-invariant curves of degree less than 8.

Let $\mathcal{M}$ be a $G$-invariant linear system on $X$ such that $\mathcal{M}$ has no fixed components. Choose a positive integer $n$ such that $\mathcal{M} \subset |nH|$. Then, by [17, Corollary 3.3.3], to prove that the threefold $X$ is $G$-birationally super-rigid it is enough to show that $(X, \frac{2}{n}\mathcal{M})$ has canonical singularities. Suppose that the singularities of this log pair are not canonical.

Let $Z$ be a center of non-canonical singularities of the pair $(X, \frac{2}{n}\mathcal{M})$ that has the largest dimension. Since the linear system $\mathcal{M}$ does not have fixed components, we conclude that either $Z$ is an irreducible curve, or $Z$ is a point. In both cases, we have

$$\text{mult}_Z(\mathcal{M}) > \frac{n}{2}$$

by [39, Theorem 4.5].

Let $M_1$ and $M_2$ be general surfaces in $\mathcal{M}$. If $Z$ is a curve, then

$$M_1 \cdot M_2 = (M_1 \cdot M_2)_Z \mathcal{Z} + \Delta$$

where $\mathcal{Z}$ is the $G$-irreducible curve in $X$ whose irreducible component is the curve $Z$, and $\Delta$ is an effective one-cycle whose support does not contain $\mathcal{Z}$, which gives

$$2n^2 = n^2H^2 = H \cdot M_1 \cdot M_2 = (M_1 \cdot M_2)_Z \mathcal{Z} + \Delta$$

$$= (M_1 \cdot M_2)_Z (H \cdot \mathcal{Z}) + H \cdot \Delta \geqslant (M_1 \cdot M_2)_Z (H \cdot \mathcal{Z}) \geqslant \text{mult}_Z(\mathcal{M}) (H \cdot \mathcal{Z}) > \frac{n^2}{4} (H \cdot \mathcal{Z}) \geqslant \frac{n^2}{4} \deg(\pi(\mathcal{Z})),$$

so $\pi(\mathcal{Z})$ is a $\nu(G)$-invariant curve of degree $\leqslant 7$, which contradicts [18, Theorem 4.16].

We see that $Z$ is a point, and $(X, \frac{2}{n}\mathcal{M})$ is canonical away from finitely many points.

We claim that $Z \notin \text{Sing}(X)$. Indeed, suppose $Z$ is a singular point of the threefold $X$. Let $h : \overline{X} \rightarrow X$ be the blow up of the locus $\text{Sing}(X)$, let $E_1, \ldots, E_{16}$ be the $h$-exceptional surfaces, let $\overline{M}_1$ and $\overline{M}_2$ be the proper transforms on $\overline{X}$ of the surfaces $M_1$ and $M_2$, respectively. Write $E = E_1 + \cdots + E_{16}$. Since $\text{Sing}(X)$ is a $G$-orbit, we have

$$\overline{M}_1 \sim \overline{M}_2 \sim h^*(H) - eE$$

for some integer $e \geqslant 0$. Using [20, Theorem 3.10] or [13, Theorem 1.7.20], we get $e > \frac{n}{2}$. On the other hand, the linear system $|h^*(3H) - E|$ is not empty and does not have base curves away from the locus $E_1 \cup E_2 \cup \cdots \cup E_{16}$, because $\text{Sing}(\mathcal{Z})$ is cut out by cubic surfaces in $\mathbb{P}^3$. In particular, the divisor $h^*(3H) - E$ is nef, so

$$0 \leqslant (h^*(3H) - E) \cdot \overline{M}_1 \cdot \overline{M}_2 = (h^*(3H) - E) \cdot (h^*(3nH) - eE)^2 = 6n^2 - 32e^2,$$

which is impossible, since $e > \frac{n}{2}$. So, we see that $Z$ is a smooth point of the threefold $X$.

Then the pair $(X, \frac{2}{n}\mathcal{M})$ is not log canonical at $Z$. Let $\mu$ be the largest rational number such that the log pair $(X, \mu\mathcal{M})$ is log canonical. Then $\mu < \frac{2}{n}$ and
Orb$_G(Z) \subseteq \text{Nklt}(X, \mu \mathcal{M})$.

Observe that Nklt$(X, \mu \mathcal{M})$ is at most one-dimensional, since $\mathcal{M}$ has no fixed components. Moreover, this locus is $G$-invariant, because $\mathcal{M}$ is $G$-invariant.

We claim that Nklt$(X, \mu \mathcal{M})$ does not contain curves. Indeed, suppose this is not true. Then Nklt$(X, \mu \mathcal{M})$ contains a $G$-irreducible curve $C$. We write $M_1 \cdot M_2 = mC + \Omega$, where $m$ is a non-negative integer, and $\Omega$ is an effective one-cycle whose support does not contain the curve $C$. Then it follows from [20, Theorem 3.1] that

$$m \geq \frac{4}{\mu^2} > \frac{4n^2}{9}.$$

Therefore, we have

$$2n^2 = n^2H^3 = H \cdot M_1 \cdot M_2 = m(H \cdot C) + H \cdot \Omega \geq m(H \cdot C) > \frac{4n^2}{9}(H \cdot C),$$

which implies that $H \cdot C \leq 4$. Then $\pi(C)$ is a $\nu(G)$-invariant curve in $\mathbb{P}^3$ of degree $\leq 4$, which contradicts [18, Theorem 4.16]. Thus, the locus Nklt$(X, \mu \mathcal{M})$ contains no curves.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X, \mu \mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in $X$. Then $\mathcal{L}$ is a zero-dimensional (reduced) subscheme such that

$$\text{Orb}_G(Z) \subseteq \text{Supp}(\mathcal{L}) = \text{Nklt}(X, \mu \mathcal{M}).$$

Applying Nadel’s vanishing [45, Theorem 9.4.8], we get $h^1(X, \mathcal{I} \otimes \mathcal{O}_X(H)) = 0$. This gives

$$4 = h^0(X, \mathcal{O}_X(H)) \geq h^0(\mathcal{O}_C \otimes \mathcal{O}_X(H)) = h^0(\mathcal{O}_C) \geq |\text{Orb}_G(Z)|.$$

In particular, we conclude that the length of the $\nu(G)$-orbit of the point $\pi(Z)$ is at most 4, which is impossible by [18, Corollary 4.7].

**Proposition 31** Suppose that $\mathcal{I}$ is the surface from Example 5, and $G$ is the subgroup described in Example 26. Then $\text{Cl}^G(X) \cong \mathbb{Z}$ and $X$ is $G$-birationally super-rigid.

**Proof** Recall from Example 26 that $G \cong \mu_4 \rtimes \mu_3$ and $\text{Cl}^G(X) \cong \mathbb{Z}$.

The $\nu(G)$-equivariant geometry of the projective space $\mathbb{P}^3$ has been studied in [16]. In particular, we know from [16] that $\mathbb{P}^3$ does not contain $\nu(G)$-orbits of length 1, 2 or 3, and the only $\nu(G)$-orbits in $\mathbb{P}^3$ of length 4 are

$$\Sigma_4 = \{[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]\},$$

$$\Sigma_4' = \{[1 : 1 : 1 : -1], [1 : 1 : -1 : 1], [1 : -1 : 1 : 1], [-1 : 1 : 1 : 1]\},$$

$$\Sigma_4'' = \{[1 : 1 : 1 : 1], [1 : 1 : -1 : -1], [1 : -1 : -1 : 1], [-1 : -1 : 1 : 1]\}.$$  

We also know from [16] the classification of $\nu(G)$-invariant curves in $\mathbb{P}^3$ of degree at most 7. Namely, let $\mathcal{L}_4$, $\mathcal{L}_4'$, $\mathcal{L}_4''$, $\mathcal{L}_4'''$, $\mathcal{L}_6$, $\mathcal{L}_6'$, $\mathcal{L}_6''$, $\mathcal{L}_6'''$ be $\nu(G)$-irreducible curves in $\mathbb{P}^3$ whose irreducible components are the lines
\begin{align*}
\{ 2x_0 + (1 + \sqrt{3})x_2 - (1 - \sqrt{3})x_3 = 2x_1 + (1 - \sqrt{3})x_2 + (1 + \sqrt{3})x_3 = 0 \}, \\
\{ 2x_0 + (1 - \sqrt{3})x_2 - (1 + \sqrt{3})x_3 = 2x_1 + (1 + \sqrt{3})x_2 + (1 - \sqrt{3})x_3 = 0 \}, \\
\{ 2x_0 - (1 - \sqrt{3})x_2 + (1 + \sqrt{3})x_3 = 2x_1 + (1 + \sqrt{3})x_2 + (1 - \sqrt{3})x_3 = 0 \}, \\
\{ 2x_0 - (1 + \sqrt{3})x_2 + (1 - \sqrt{3})x_3 = 2x_1 + (1 - \sqrt{3})x_2 + (1 + \sqrt{3})x_3 = 0 \}, \\
\{ x_0 = x_1 = 0 \}, \\
\{ x_0 + x_1 = x_2 - x_3 = 0 \}, \\
\{ x_0 + x_1 = x_2 + x_3 = 0 \}, \\
\{ x_0 + ix_2 = x_1 + ix_3 = 0 \}, \\
\{ x_0 + ix_3 = x_1 + ix_2 = 0 \},
\end{align*}

respectively. Then the curves \(L_4, L_4', L_4'', L_4''', L_6, L_6'', L_6''', L_6'''\) are unions of 4, 4, 4, 4, 6, 6, 6, 6, 6 lines, respectively. Moreover, it follows from [16] that \(L_4, L_4', L_4'', L_4''', L_6, L_6'', L_6''', L_6''''\) are the only \(v(G)\)-invariant curves in \(\mathbb{P}^3\) of degree at most 7.

Now, using the defining equation of the surface \(\mathcal{S}\), one can check that any irreducible component of any curve among \(L_4, L_4', L_4'', L_4''', L_6, L_6', L_6'', L_6''', L_6'''\) intersects the quartic surface \(\mathcal{S}\) transversally by 4 distinct points, so that its preimage in \(X\) via the double cover \(\pi\) is a smooth elliptic curve. Thus, if \(C\) is a \(G\)-invariant curve in \(X\), then \(H \cdot C \geq 8\).

Suppose that \(X\) is not \(G\)-birationally super-rigid. It follows from [17, Corollary 3.3.3] that there are a positive integer \(n\) and a \(G\)-invariant linear subsystem \(\mathcal{M} \subset \mathcal{O}_H[n]\) such that \(\mathcal{M}\) does not have fixed components, but the log pair \((X, \frac{2}{n} \mathcal{M})\) is not canonical.

Arguing as in the proof of Proposition 30, we see that the log pair \((X, \frac{2}{n} \mathcal{M})\) is canonical away from finitely many points. Let \(P\) be a point in \(X\) that is a center of non-canonical singularities of the log pair \((X, \frac{2}{n} \mathcal{M})\). Now, arguing as in the proof of Proposition 30 again, we see that \(P\) is a smooth point of the threefold \(X\).

Then the log pair \((X, \frac{2}{n} \mathcal{M})\) is not log canonical at \(P\). Let \(\mu\) be the largest rational number such that \((X, \mu \mathcal{M})\) is log canonical. Then \(\mu < \frac{3}{n}\) and

\[\text{Orb}_G(P) \subseteq \text{Nklt}(X, \mu \mathcal{M}).\]

Observe that \(\text{Nklt}(X, \mu \mathcal{M})\) is at most one-dimensional, since \(\mathcal{M}\) has no fixed components. Moreover, this locus is \(G\)-invariant, because \(\mathcal{M}\) is \(G\)-invariant. Furthermore, arguing as in the proof of Proposition 30, we see that

\[\dim \left( \text{Nklt}(X, \mu \mathcal{M}) \right) = 0.\]

Let \(\mathcal{I}\) be the multiplier ideal sheaf of the pair \((X, \mu \mathcal{M})\), and let \(\mathcal{L}\) be the corresponding subscheme in \(X\). Then \(\mathcal{L}\) is a zero-dimensional (reduced) subscheme such that

\[\text{Orb}_G(P) \subseteq \text{Supp}(\mathcal{L}) = \text{Nklt}(X, \mu \mathcal{M}).\]

On the other hand, applying Nadel’s vanishing theorem [45, Theorem 9.4.8], we get

\[h^1(X, \mathcal{I} \otimes \mathcal{O}_X(H)) = 0.\]

This gives
4 = h^0(X, \mathcal{O}_X(H)) \geq h^0(\mathcal{O}_X \otimes \mathcal{O}_X(H)) = h^0(\mathcal{O}_X) \geq |\text{Orb}_G(P)|.

Thus, we conclude that $|\text{Orb}_G(P)| = 4$ and $\pi(P) \in \Sigma_4 \cup \Sigma_5' \cup \Sigma_5''$.

Let $M_1$ and $M_2$ be two general surfaces in $\mathcal{M}$. Using [51] or [20, Corollary 3.4], we get

$$\{M_1 \cdot M_2\}_p > n^2.$$  \hspace{1cm} (32)

Let $\mathcal{S}$ be a linear subsystem in $|3H|$ that consists of all surfaces that are singular at every point of the $G$-orbit $\text{Orb}_G(P)$. Then its base locus does not contain curves, which implies that there is a surface $S \in \mathcal{S}$ that does not contain components of the cycle $M_1 \cdot M_2$. Thus, using (32) and $	ext{mult}_p(S) \geq 2$, we get

$$6n^2 = S \cdot M_1 \cdot M_2 \geq \sum_{O \in \text{Orb}_G(P)} 2(M_1 \cdot M_2)_O = 2|\text{Orb}_G(P)|\{M_1 \cdot M_2\}_p$$

$$= 8(M_1 \cdot M_2)_p > 8n^2,$$

which is absurd. This completes the proof of Proposition 31. \hfill \Box

In the remaining part of the paper, we prove Theorem 17, and consider one application. Let us recall from [1, 26, 30, 35, 48] basic facts about the $\mathbb{H}$-equivariant geometry of $\mathbb{P}^3$. Set

$$Q_1 = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\},$$

$$Q_2 = \{x_0^2 + x_1^2 = x_2^2 + x_3^2\},$$

$$Q_3 = \{x_0^2 - x_1^2 = x_2^2 - x_3^2\},$$

$$Q_4 = \{x_0^2 - x_1^2 = x_3^2 - x_2^2\},$$

$$Q_5 = \{x_0 x_2 + x_1 x_3 = 0\},$$

$$Q_6 = \{x_0 x_3 + x_1 x_2 = 0\},$$

$$Q_7 = \{x_0 x_1 + x_2 x_3 = 0\},$$

$$Q_8 = \{x_0 x_2 = x_1 x_3\},$$

$$Q_9 = \{x_0 x_3 = x_1 x_2\},$$

$$Q_{10} = \{x_0 x_1 = x_2 x_3\}.$$ Then $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}$ are all $\mathbb{H}$-invariant quadric surfaces in $\mathbb{P}^3$. These quadrics are smooth, and $\mathbb{H} \cong \mu^4$ acts naturally on each quadric $Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$.

For a non-trivial element $g \in \mathbb{H}$, the locus of its fixed points in $\mathbb{P}^3$ consists of two skew lines, which we will denote by $L_g$ and $L'_g$. For two non-trivial elements $g \neq h$ in $\mathbb{H}$, one has

$$\{L_g, L'_g\} \cap \{L_h, L'_h\} = \emptyset.$$ In total, this gives 30 lines $\ell_1, \ldots, \ell_{30}$, whose equations are listed in the following table:

| $\ell_1 = \{x_0 = x_1 = 0\}$ | $\ell_2 = \{x_2 = x_3 = 0\}$ |
| $\ell_3 = \{x_0 = x_2 = 0\}$ | $\ell_4 = \{x_1 = x_3 = 0\}$ |
| $\ell_5 = \{x_0 = x_3 = 0\}$ | $\ell_6 = \{x_1 = x_2 = 0\}$ |
| Q | Q_1 | Q_2 | Q_3 | Q_4 | Q_5 | Q_6 | Q_7 | Q_8 | Q_9 | Q_10 |
|---|---|---|---|---|---|---|---|---|---|---|
| ℓ₁ | - | - | - | - | + | + | - | + | + | - |
| ℓ₂ | - | - | - | - | + | + | - | + | + | - |
| ℓ₃ | - | - | - | - | - | + | + | - | + | + |
| ℓ₄ | - | - | - | - | - | + | + | - | + | + |
| ℓ₅ | - | - | - | - | + | - | + | + | - | + |
| ℓ₆ | - | - | - | - | + | + | - | + | - | + |
| ℓ₇ | - | + | + | + | - | - | + | + | - | + |
| ℓ₈ | - | + | + | + | - | - | + | + | - | + |
| ℓ₉ | - | + | - | - | - | - | + | + | - | + |
| ℓ₁₀ | - | + | + | - | - | - | - | + | + | - |
| ℓ₁₁ | - | + | - | + | - | - | + | + | - | + |
| ℓ₁₂ | - | + | - | + | + | - | - | + | + | - |
| ℓ₁₃ | - | - | + | + | + | + | - | + | - | - |
| ℓ₁₄ | - | - | + | + | + | + | - | + | - | - |
| ℓ₁₅ | - | + | + | - | - | - | + | + | - | - |
| ℓ₁₆ | - | + | + | - | - | + | - | + | - | - |
| ℓ₁₇ | - | + | - | + | - | + | - | - | - | - |
| ℓ₁₈ | - | + | - | + | + | - | + | - | - | - |
| ℓ₁₉ | + | + | - | - | + | - | - | + | - | + |
| ℓ₂₀ | + | + | - | - | + | - | - | - | + | - |
| ℓ₂₁ | + | - | - | + | - | - | - | + | - | + |
| ℓ₂₂ | + | - | - | + | - | - | - | + | - | + |
| ℓ₂₃ | + | + | + | - | - | + | - | + | - | - |
| ℓ₂₄ | + | - | + | - | - | - | - | + | - | - |
| ℓ₂₅ | + | + | - | - | - | + | - | + | - | - |
| ℓ₂₆ | + | + | - | - | + | - | - | - | + | - |
| ℓ₂₇ | + | + | - | + | - | - | - | - | + | - |
| ℓ₂₈ | + | - | - | - | + | + | - | - | - | + |
| ℓ₂₉ | + | - | + | - | - | + | - | - | - | + |
| ℓ₃₀ | + | - | + | - | + | - | - | - | - | + |

**Fig. 3** Ten ℍ-invariant quadrics in ℙ¹ and thirty lines in them

\begin{align*}
ℓ₇ &= \{x_0 + x_1 = x_2 + x_3 = 0\} \\
ℓ₈ &= \{x_0 - x_1 = x_2 - x_3 = 0\} \\
ℓ₉ &= \{x_0 + x_2 = x_1 + x_3 = 0\} \\
ℓ₁₀ &= \{x_0 - x_2 = x_1 - x_3 = 0\} \\
ℓ₁₁ &= \{x_0 + x_3 = x_1 + x_2 = 0\} \\
ℓ₁₂ &= \{x_0 - x_3 = x_1 - x_2 = 0\} \\
ℓ₁₃ &= \{x_0 + x_1 = x_2 - x_3 = 0\} \\
ℓ₁₄ &= \{x_0 - x_1 = x_2 + x_3 = 0\} \\
ℓ₁₅ &= \{x_0 + x_2 = x_1 - x_3 = 0\} \\
ℓ₁₆ &= \{x_0 - x_2 = x_1 + x_3 = 0\}
\end{align*}
\[\ell_{17} = \{x_0 + x_3 = x_1 - x_2 = 0\}\]
\[\ell_{18} = \{x_0 - x_3 = x_1 + x_2 = 0\}\]
\[\ell_{19} = \{x_0 + ix_1 = x_2 + ix_3 = 0\}\]
\[\ell_{20} = \{x_0 - ix_1 = x_2 - ix_3 = 0\}\]
\[\ell_{21} = \{x_0 + ix_2 = x_1 + ix_3 = 0\}\]
\[\ell_{22} = \{x_0 - ix_2 = x_1 - ix_3 = 0\}\]
\[\ell_{23} = \{x_0 + ix_3 = x_1 + ix_2 = 0\}\]
\[\ell_{24} = \{x_0 - ix_3 = x_1 - ix_2 = 0\}\]
\[\ell_{25} = \{x_0 - ix_1 = x_2 + ix_3 = 0\}\]
\[\ell_{26} = \{x_0 + ix_1 = x_2 - ix_3 = 0\}\]
\[\ell_{27} = \{x_0 + ix_2 = x_1 - ix_3 = 0\}\]
\[\ell_{28} = \{x_0 - ix_2 = x_1 + ix_3 = 0\}\]
\[\ell_{29} = \{x_0 + ix_3 = x_1 - ix_2 = 0\}\]
\[\ell_{30} = \{x_0 - ix_3 = x_1 + ix_2 = 0\}\]

Note that \(\ell_1, \ldots, \ell_{30}\) are irreducible components of the curves \(\ell_6', \ell_6'', \ell_6''', \ell_6''''\) which have been introduced in the proof of Proposition 31. One can check that

- for every \(k \in \{1, \ldots, 15\}\), the curve \(\ell_{2k-1} + \ell_{2k}\) is \(\mathbb{P}\)-irreducible,
- each line among \(\ell_1, \ldots, \ell_{30}\) is contained in 4 quadrics among \(Q_1, \ldots, Q_{10}\),
- each quadric among \(Q_1, \ldots, Q_{10}\) contains 12 lines among \(\ell_1, \ldots, \ell_{30}\),
- every two quadrics among \(Q_1, \ldots, Q_{10}\) intersect by 4 lines among \(\ell_1, \ldots, \ell_{30}\).

The incidence relation between \(\ell_1, \ldots, \ell_{30}\) and \(Q_1, \ldots, Q_{10}\) is presented in Figure 3.

Now, let us describe the intersection points of the lines \(\ell_1, \ldots, \ell_{30}\). To do this, we set

\[\Sigma_4^1 = \{[1 : 0 : 0 : 0],[0 : 1 : 0 : 0],[0 : 0 : 1 : 0],[0 : 0 : 0 : 1]\},\]
\[\Sigma_4^2 = \{[1 : 1 : 1 : 1],[1 : 1 : -1 : 1],[1 : -1 : 1 : 1],[1 : -1 : -1 : 1]\},\]
\[\Sigma_4^3 = \{[0 : 0 : 1 : 1],[0 : 0 : 0 : 1],[0 : 0 : 1 : 0],[0 : 0 : -1 : 0]\},\]
\[\Sigma_4^4 = \{[1 : 0 : 1 : 0],[1 : 0 : 0 : 1],[0 : -1 : 1 : 0],[0 : -1 : 0 : 1]\},\]
\[\Sigma_4^5 = \{[i : 0 : 1 : 1],[0 : i : 1 : 0],[0 : 0 : -1 : i],[0 : -1 : 0 : i]\},\]
\[\Sigma_4^6 = \{[i : 0 : -1 : 1],[0 : 0 : 1 : i],[0 : 0 : 1 : -i],[0 : 0 : -1 : i]\},\]
\[\Sigma_4^7 = \{[i : i : 1 : 1],[0 : 0 : 0 : 1],[0 : 0 : 1 : i],[0 : 0 : 1 : -i]\},\]
\[\Sigma_4^8 = \{[i : 1 : 0 : 1],[0 : 0 : 0 : 1],[0 : 0 : 1 : 1],[0 : 0 : -1 : 1]\},\]
\[\Sigma_4^9 = \{[-i : -i : 1 : 1],[0 : -i : 0 : 1],[0 : -i : 0 : -1],[0 : -i : 0 : 1]\},\]
\[\Sigma_4^{10} = \{[-i : -i : 1 : 1],[0 : -i : 1 : 0],[0 : -i : 1 : 0],[0 : -i : -1 : 1]\},\]
\[\Sigma_4^{11} = \{[i : i : 1 : 1],[i : -i : -1 : 1],[1 : 1 : -i : 1],[1 : -1 : i : 1]\},\]
\[\Sigma_4^{12} = \{[i : i : -i : 1],[1 : 1 : i : 1],[1 : -i : 1 : 1],[1 : -1 : -i : 1]\},\]
\[\Sigma_4^{13} = \{[i : 1 : 1 : 1],[1 : -i : -1 : 1],[1 : i : 1 : 1],[1 : i : -1 : 1]\},\]
\[\Sigma_4^{14} = \{[i : 1 : i : 1],[1 : 1 : i : 1],[1 : -i : -1 : 1],[1 : i : 1 : 1]\},\]
\[\Sigma_4^{15} = \{[i : -i : -1 : 1],[i : 1 : i : 1],[i : -i : -1 : 1],[i : 1 : i : 1]\}.

Then the subsets \(\Sigma_4^1, \ldots, \Sigma_{4}^{15}\) are \(\mathbb{P}\)-orbits of length 4. Moreover, one has

\[\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15} = \text{Sing}(\ell_1 + \ell_2 + \cdots + \ell_{30}).\]

So, for every \(\ell_i\) and \(\ell_j\) such that \(\ell_i \neq \ell_j\) and \(\ell_i \cap \ell_j \neq \emptyset\), one has \(\ell_i \cap \ell_j \in \Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}\). Furthermore, one can also check that

- every line among \(\ell_1, \ldots, \ell_{30}\) contains 6 points in \(\Sigma_4^1 \cup \Sigma_4^2 \cup \cdots \cup \Sigma_4^{15}\),

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• every point in $\Sigma_1^4 \cup \Sigma_2^4 \cup \cdots \cup \Sigma_{15}^4$ is contained in 3 lines among $\ell_1, \ldots, \ell_{30}$.

As in Remark 11, let $\mathcal{N}$ be the normalizer of the subgroup $\mathbb{H}$ in the group $\text{PGL}_4(\mathbb{C})$. Then $\text{Aut}(\mathbb{P}^3, \mathcal{E}) \subset \mathcal{N}$ and $\mathcal{N} \cong \mathbb{H}. \mathbb{S}_6$ by Remark 11. Moreover, one can show that

• the group $\mathcal{N}$ acts transitively on the set $\{Q_1, \ldots, Q_{10}\}$,
• the group $\mathcal{N}$ acts transitively on the set $\{\ell_1, \ldots, \ell_{30}\}$,
• the group $\mathcal{N}$ acts transitively on the set $\{\Sigma_1^4, \ldots, \Sigma_{15}^4\}$.

Now, we are ready to describe $\mathbb{H}$-orbits in $\mathbb{P}^3$. They can be described as follows:

1. $\Sigma_1^4, \ldots, \Sigma_{15}^4$ are $\mathbb{H}$-orbits of length 4;
2. $\mathbb{H}$-orbit of every point in $(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}) \setminus (\Sigma_1^4 \cup \Sigma_2^4 \cup \cdots \cup \Sigma_{15}^4)$ has length 8;
3. $\mathbb{H}$-orbit of every point in $\mathbb{P}^3 \setminus (\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30})$ has length 16.

Lemma 33 The surface $\mathcal{S}$ does not contain $\mathbb{H}$-orbits of length 4.

Proof The assertion follows from [58, Theorem 3], because the $\mathbb{H}$-action on the minimal resolution of the surface $\mathcal{S}$ is symplectic [32, 47]. Instead, we can check this explicitly. Indeed, it is enough to check that $\mathcal{S}$ does not contain $\Sigma_1^4$, since the group $\mathcal{N}$ transversely permutes the orbits $\Sigma_1^4, \ldots, \Sigma_{15}^4$. If $\Sigma_1^4 \subset \mathcal{S}$, then $\mathcal{S}$ is given by (2) with $a = bcd = 0$, which implies that $\mathcal{S}$ has non-isolated singularities.

Corollary 34 Every line among $\ell_1, \ldots, \ell_{30}$ intersects $\mathcal{S}$ transversally by 4 points.

Proof Fix $k \in \{1, \ldots, 15\}$. If $|\ell_{2k-1} \cap \mathcal{S}| < 4$, then the subset $(\ell_{2k-1} \cup \ell_{2k}) \cap \mathcal{S}$ contains an $\mathbb{H}$-orbit of length 4, which contradicts Lemma 33. Therefore, we have $|\ell_{2k-1} \cap \mathcal{S}| = 4$. Similarly, we see that $|\ell_{2k} \cap \mathcal{S}| = 4$.

Now, let us prove one result that plays a crucial role in the proof of Theorem 17.

Lemma 35 Let $C$ be a possibly reducible $\mathbb{H}$-irreducible curve in $\mathbb{P}^3$ such that $\deg(C) < 8$. Then one of the following two possibilities hold:

(a) either $C = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$;
(b) or $C$ is a union of 4 disjoint lines and $C \subset Q_i$ for some $i \in \{1, \ldots, 10\}$.

Proof Intersecting $C$ with quadric surfaces $Q_1, \ldots, Q_{10}$, we conclude that $\deg(C)$ is even. This gives $\deg(C) \in \{2, 4, 6\}$. Suppose that $C$ is reducible. Since $|\mathcal{N}| = 16$, we have the following possibilities:

(i) $C$ is a union of 2 lines,
(ii) $C$ is a union of 4 lines,
(iii) $C$ is a union of 2 irreducible conics,
(iv) $C$ is a union of 3 irreducible conics.
(v) $C$ is a union of 2 irreducible plane cubics,
(vi) $C$ is a union of 2 twisted cubics,

Since $\mathbb{P}^3$ does not have $\mathbb{H}$-orbits of length 2 and 3, cases (iii), (iv) and (v) are impossible. Similarly, case (vi) is also impossible, because $\mu_2^3$ cannot faithfully act on a rational curve. Thus, either $C$ is a union of 2 lines, or $C$ is a union of 4 lines.

Suppose that $C = L_1 + L_2$, where $L_1$ and $L_2$ are lines. Then $\text{Stab}_{\mathbb{H}}(L_1) \cong \mu_2^3$, and this group cannot act faithfully on $L_1 \cong \mathbb{P}^1$. Therefore, there exists a non-trivial $g \in \text{Stab}_{\mathbb{H}}(L_1)$ such that $g$ pointwise fixes the line $L_1$. But this means that $L_1$ is one of the lines $\ell', \ldots, \ell_{30}$, so we have $C = \ell_{2k-1} + \ell_{2k}$ for some $k \in \{1, \ldots, 15\}$ as required.

Suppose $C = L_1 + L_2 + L_3 + L_4$, where $L_1, L_2, L_3, L_4$ are lines. Then $\text{Stab}_{\mathbb{H}}(L_1) \cong \mu_2^3$. Note that $\text{Stab}_{\mathbb{H}}(L_1)$ must act faithfully on $L_1$, because $L_1$ is not one of the lines $\ell', \ldots, \ell_{30}$. This implies that $L_1$ does not have $\text{Stab}_{\mathbb{H}}(L_1)$-fixed points, which implies that $\mathbb{P}^3$ also does not have $\text{Stab}_{\mathbb{H}}(L_1)$-fixed points. All subgroups in $\mathbb{H}$ isomorphic to $\mu^2$ with these property are conjugated by the action of the group $\mathcal{R}$. Thus, we may assume that

$$\text{Stab}_{\mathbb{H}}(L_1) = \langle A_1, A_2, A_3 \rangle.$$  

This subgroup leaves invariant rulings of the quadric surface $Q_8 \cong \mathbb{P}^1 \times \mathbb{P}^1$. To be precise, for every $[\lambda : \mu] \in \mathbb{P}^1$, the group $\langle A_1, A_2, A_3 \rangle$ leaves invariant the line

$$\{ \lambda x_0 + \mu x_3 = \lambda x_1 + \mu x_2 = 0 \} \subset Q_8,$$

and these are all $\langle A_1, A_2, A_3 \rangle$-invariant lines in $\mathbb{P}^3$. So, the lines $L_1, L_2, L_3, L_4$ are disjoint, and all of them are contained in the quadric $Q_8$. Thus, we are done in this case.

Therefore, to complete the proof of the lemma, we may assume that $C$ is irreducible. Observe that the curve $C$ is not planar, because $\mathbb{P}^3$ does not contain $\mathbb{H}$-invariant planes. Moreover, the curve $C$ is singular: otherwise its genus is $\leq 4$ by the Castelnuovo bound, but $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [16, Lemma 3.2]. Therefore, we conclude that $\deg(C) = 6$, since otherwise the curve $C$ would be planar.

We claim that the curve $C$ does not contain $\mathbb{H}$-orbits of length 4. Suppose that it does. Since $\mathcal{R}$ transitively permutes the orbits $\Sigma_4^1, \ldots, \Sigma_4^{15}$, we may assume that $\Sigma_4^1 \subset C$. Then

$$\Sigma_4^1 \subset \text{Sing}(C),$$

because the stabilizer in $\mathbb{H}$ of a smooth point in $C$ must be a cyclic group [28, Lemma 2.7]. Let $\iota : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the standard Cremona involution, which is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_1 x_2 x_3 : x_0 x_2 x_3 : x_0 x_1 x_3 : x_0 x_1 x_2].$$

Then $\iota$ centralizes $\mathbb{H}$. On the other hand, the curve $n(C)$ is a conic, because $\deg(C) = 6$, and $C$ is singular at every point of the $\mathbb{H}$-orbit $\Sigma_4^1$. But $\mathbb{P}^3$ contains no $\mathbb{H}$-invariant conics, because it contains no $\mathbb{H}$-invariant planes. Thus, $C$ contains no $\mathbb{H}$-orbits of length 4.

Note that $Q_1 \cap Q_2 \cap \cdots \cap Q_{10} = \emptyset$. So, at least one quadric among $Q_1, \ldots, Q_{10}$ does not contain the curve $C$. Without loss of generality, we may assume that $C \not\subset Q_1$. Then

$$12 = Q_1 \cdot C \geq |Q_1 \cap C|,$$

which implies that the intersection $Q_1 \cap C$ is an $\mathbb{H}$-orbit of length 8, because we already proved that $C$ does not contain $\mathbb{H}$-orbits of length 4. For a point $P \in Q_1 \cap C$, we have...
Corollary 36. Let $Q$ be an $\mathbb{H}$-invariant quadric among $Q_1, \ldots, Q_{10}$, and let $C = \mathcal{A}_Q$. Then one of the following possibilities holds:

- $C$ is a smooth irreducible curve of degree 8 and genus 9;
- $C$ is an $\mathbb{H}$-irreducible curve which is a union of two irreducible smooth quartic elliptic curves that intersect each other transversally by an $\mathbb{H}$-orbit of length 8;
- $C = \mathcal{L}_4 + \mathcal{L}'_4$ for $\mathbb{H}$-irreducible curves $\mathcal{L}_4$ and $\mathcal{L}'_4$ consisting of 4 disjoint lines such that the intersection $\mathcal{L}_4 \cap \mathcal{L}'_4$ is an $\mathbb{H}$-orbit of length 16.

**Proof.** If $C$ is not $\mathbb{H}$-irreducible, then Lemma 35 and Corollary 34 imply the assertion. Thus, we may assume that $C$ is $\mathbb{H}$-irreducible.

Suppose that $C$ is reducible. By Lemma 33, $C$ does not contain $\mathbb{H}$-orbits of length 4. Observe also that $\text{Pic}^H(Q) \cong \mathbb{Z}^2$. Thus, since $C$ is $\mathbb{H}$-irreducible, we conclude that

(i) either $C$ is a union of 4 irreducible conics,
(ii) or $C$ is an $\mathbb{H}$-irreducible curve which is a union of two irreducible smooth quartic elliptic curves that intersect each other transversally by an $\mathbb{H}$-orbit of length 8.

In the latter case, we are done. In the former case, we have $|\text{Sing}(C)| \leq 12$, which implies that $\text{Sing}(C)$ is an $\mathbb{H}$-orbit of length 8, which immediately leads to a contradiction.

Thus, we may assume that $C$ is irreducible. The arithmetic genus of the curve $C$ is 9. If $C$ is smooth, we are done. If $C$ is singular, then the genus of it normalization is $\leq 1$, because $C$ does not contain $\mathbb{H}$-orbits of length 4 by Lemma 33. But $\mathbb{H}$ cannot faithfully act on a smooth curve of genus less than 5 by [16, Lemma 3.2].

Now, we are ready to prove Theorem 17.

**Proof of Theorem 17.** Let $G$ be a subgroup in $\text{Aut}(X)$ such that $\text{Cl}^G(X) \cong \mathbb{Z}$ and

$$\mathbb{H} \subseteq \nu(G),$$

so $G$ contains a subgroup $\mathbb{H}^\rho$ for some homomorphism $\rho : \mathbb{H} \to \mu_2$. We must prove that the threefold $X$ is $G$-birationally super-rigid. Suppose it is not $G$-birationally super-rigid. Then there are a positive integer $n$ and a $G$-invariant linear subsystem $\mathcal{M} \subset [nH]$ such that the linear system $\mathcal{M}$ does not have fixed components, but $(X, \frac{2}{n}\mathcal{M})$ is not canonical.

Starting from this moment, we are going to forget about the group $G$. In the following, we will work only with its subgroup $\mathbb{H}^\rho$. Recall that $\nu(\mathbb{H}^\rho) = \mathbb{H}$.

Let $Z$ be the center of non-canonical singularities of the log pair $(X, \frac{2}{n}\mathcal{M})$ that has maximal dimension. We claim that $Z$ must be a point. Indeed, suppose that $Z$ is a curve. Let $M$ be sufficiently general surface in the linear system $\mathcal{M}$. Then

$$\text{mult}_Z(M) > \frac{n}{2}$$

(37)
by [39, Theorem 4.5]. Let us seek for a contradiction.

Let \( \mathcal{Z} \) be an \( \mathbb{H}^e \)-irreducible curve in \( X \) whose irreducible components is the curve \( Z \).
Then, arguing as in the proof of Proposition 30, we see that
\[
H \cdot \mathcal{Z} \leq 7.
\]
In particular, we conclude that \( \pi(\mathcal{Z}) \) is a \( \mathbb{H} \)-invariant curve of degree \( \leq 7 \). By Lemma 35, the curve \( \pi(Z) \) is a line, and one of the following two possibilities hold:

(a) either \( \pi(\mathcal{Z}) = \ell_{2k-1} + \ell_{2k} \) for some \( k \in \{1, \ldots, 15\} \);
(b) or \( \pi(\mathcal{Z}) \) is a union of 4 disjoint lines and \( \pi(\mathcal{Z}) \subset \mathcal{Q}_i \) for some \( i \in \{1, \ldots, 10\} \).

Let us deal with these two cases separately.

Suppose we are in case (a). Without loss of generality, we may assume \( \pi(\mathcal{Z}) = \ell_1 + \ell_2 \).
Let \( C_1 \) and \( C_2 \) be the preimages on the threefold \( X \) of the lines \( \ell_1 \) and \( \ell_2 \), respectively. Then it follows from Corollary 34 that \( C_1 \) and \( C_2 \) are smooth irreducible elliptic curves. In particular, the curves \( C_1 \) and \( C_2 \) are disjoint and
\[
\mathcal{Z} = C_1 + C_2.
\]
Let \( f : \tilde{X} \to X \) be the blow up of the curves \( C_1 \) and \( C_2 \), let \( E_1 \) and \( E_2 \) be the \( f \)-exceptional surfaces such that \( f(E_1) = C_1 \) and \( f(E_2) = C_2 \), and let \( \tilde{M} \) be the proper transform on the threefold \( \tilde{X} \) of the surface \( M \). Then \( |f^*(2H) - E_1 - E_2| \) is base point free, so
\[
0 \leq (f^*(2H) - E_1 - E_2)^2 \cdot \tilde{M} = (f^*(2H) - E_1 - E_2)^2 \cdot (f^*(nH) - \text{mult}_Z(M)(E_1 + E_2)) = 4n - 8\text{mult}_Z(M),
\]
which contradicts (37). This shows that case (a) is impossible.

Suppose we are in case (b). Without loss of generality, we may assume that \( \pi(\mathcal{Z}) \subset \mathcal{Q}_1 \).
Let \( S \) be the preimage of the quadric surface \( \mathcal{Q}_1 \) via the double cover \( \pi \). Then it follows from Corollary 36 that \( S \) is an irreducible normal surface such that

(i) either \( S \) is a smooth K3 surface,
(ii) or \( S \) is a singular K3 surface that has 8 or 16 ordinary double points.

Note that \( \mathcal{Z} \subset S \) by construction. Let \( \mathcal{C} \) be the preimage in \( X \) of a sufficiently general line in the quadric \( \mathcal{Q}_1 \) that intersect the line \( \pi(Z) \). Then \( \mathcal{C} \) is a smooth irreducible elliptic curve, which is contained in the smooth locus of the K3 surface \( S \). Observe that \( H \cdot \mathcal{C} = 2 \). Moreover, we also have \( |\mathcal{C} \cap \mathcal{Z}| \geq 4 \). Thus, since \( \mathcal{C} \not\subset \text{Supp}(M) \), we get
\[
2n = nH \cdot \mathcal{C} = M \cdot \mathcal{C} \geq \sum_{0 \in \mathcal{C} \cap \mathcal{Z}} \text{mult}_O(M) \geq \text{mult}_Z(M)|\mathcal{C} \cap \mathcal{Z}| \geq 4\text{mult}_Z(M),
\]
which contradicts (37). This shows that case (b) is also impossible.

Hence, we see that \( Z \) is a point. In particular, the pair \( (X, \frac{1}{2} \mathcal{M}) \) is canonical away from finitely many points. Now, arguing as in the proof of Proposition 30, we get
\[
Z \not\in \text{Sing}(X).
\]
Let \( M_1 \) and \( M_2 \) be two general surfaces in \( \mathcal{M} \). Using [51] or [20, Corollary 3.4], we get
\[(M_1 \cdot M_2)_Z > n^2. \quad (38)\]

Let \(P = \pi(Z)\). Then, arguing as in the proof of Proposition 31, we get \(|\text{Orb}_{\mathcal{H}}(P)| \neq 4\).

We claim that \(|\text{Orb}_{\mathcal{H}}(P)| \neq 8\). Indeed, suppose \(|\text{Orb}_{\mathcal{H}}(P)| = 8\). Then
\[P \in \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}.
\]

Without loss of generality, we may assume that \(P \in \ell_1\). Let \(C_1\) and \(C_2\) be the preimages on the threefold \(X\) of the lines \(\ell_1\) and \(\ell_2\), respectively. Recall that \(C_1\) and \(C_2\) are smooth irreducible elliptic curves, and the curve \(C_1 + C_2\) is \(\mathbb{H}^\ell\)-irreducible. Write
\[M_1 \cdot M_2 = m(C_1 + C_2) + \Delta,
\]
where \(m\) is a non-negative integer, and \(\Delta\) is an effective one-cycle whose support does not contain the curves \(C_1\) and \(C_2\). Then \(m \leq \frac{n^2}{2}\), because
\[2n^2 = H \cdot M_1 \cdot M_2 = mH \cdot (C_1 + C_2) + H \cdot \Delta \leq mH \cdot (C_1 + C_2) = 4m.
\]
On the other hand, since \(C_1\) and \(C_2\) are smooth curves, it follows from (38) that
\[\text{mult}_O(\Delta) > n^2 - m \quad (39)
\]
for every point \(O \in \text{Orb}_{\mathbb{H}^\ell}(Z)\). Note also that \(Z \in C_1\) and \(|\text{Orb}_{\mathbb{H}^\ell}(Z)| \geq 8\).

Let \(D\) be the linear subsystem in \(|2H|\) that consists of surfaces passing through \(C_1 \cup C_2\). Then, as we already implicitly mentioned, the linear system \(D\) does not have base curves except for \(C_1\) and \(C_2\). Therefore, if \(D\) is a general surface in \(D\), then \(D\) does not contain irreducible components of the one-cycle \(\Delta\), so (39) gives
\[4n^2 - 8m = D \cdot \Delta \geq \sum_{O \in \text{Orb}_{\mathbb{H}^\ell}(Z)} \text{mult}_O(\Delta) = |\text{Orb}_{\mathbb{H}^\ell}(Z)| \text{mult}_Z(\Delta) > |\text{Orb}_{\mathbb{H}^\ell}(Z)| (n^2 - m) \geq 8(n^2 - m),
\]
which is absurd. This shows that \(|\text{Orb}_{\mathbb{H}^\ell}(P)| \neq 8\).

In particular, we see that \(|\text{Orb}_{\mathbb{H}^\ell}(P)| = 16\) and \(P \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30}\).

We claim that \(P \notin Q_1 \cup Q_2 \cup \cdots \cup Q_{10}\). Indeed, suppose that \(P \in Q_1 \cup Q_2 \cup \cdots \cup Q_{10}\). Without loss of generality, we may assume that
\[\pi(Z) = P \in Q_1.
\]

As above, denote by \(S\) the preimage of the quadric surface \(Q_1\) via the double cover \(\pi\). Then \(S\) is a K3 surface with at most ordinary double singularities, and it follows from the inversion of adjunction [39, Theorem 5.50] that \((S, S^2_\mathcal{M}|_S)\) is not log canonical at \(Z\). Let \(\lambda\) be the largest rational number such that \((S, \lambda S^2_\mathcal{M}|_S)\) is log canonical at \(Z\). Then
\[\text{Orb}_{\mathbb{H}^\ell}(Z) \subseteq \text{Klt}(S, \lambda S^2_\mathcal{M}|_S).
\]

Note that the locus \(\text{Klt}(S, \lambda S^2_\mathcal{M}|_S)\) is \(\mathbb{H}^\ell\)-invariant, because \(\mathcal{M}\) and \(S\) are \(\mathbb{H}^\ell\)-invariant.

Suppose \(\text{Klt}(S, \lambda S^2_\mathcal{M}|_S)\) contains an \(\mathbb{H}^\ell\)-irreducible curve \(C\) that passes through \(Z\). This means that \(\lambda S^2_\mathcal{M}|_S = C + \Omega\), where \(\Omega\) is an effective \(\mathbb{Q}\)-linear system on \(S\). Then
\[H \cdot C \leq H \cdot (C + \Omega) = 4n\lambda < 8,
\]
hence \( \pi(C) \) is a union of 4 disjoint lines in \( Q_4 \) by Lemma 35, since \( P \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30} \). Let \( C \) be the preimage in \( X \) of a general line in \( Q_4 \) that intersect \( \pi(C) \). Then

\[
4 \leq C \cdot C \leq C \cdot (C + \Omega) = \lambda n (H \cdot C) = 2\lambda n < 4,
\]

which is absurd. So, the locus \( \text{Nklt}(S, \lambda M|_{\mathcal{S}}) \) contains no curves that pass through \( Z \).

Let \( \mathcal{I}_S \) be the multiplier ideal sheaf of the pair \((S, \lambda M|_{\mathcal{S}})\), let \( \mathcal{L}_S \) be the corresponding subscheme in \( S \). Then

\[
\text{Supp}(\mathcal{L}_S) = \text{Nklt}(S, \lambda M|_{\mathcal{S}}).
\]

Now, applying Nadel’s vanishing theorem [45, Theorem 9.4.8], we get

\[
h^1(S, \mathcal{I}_S \otimes \mathcal{O}_S(2H|_{\mathcal{S}})) = 0.
\]

Now, using the Riemann–Roch theorem and Serre’s vanishing, we obtain

\[
10 = h^0(S, \mathcal{O}_S(2H|_{\mathcal{S}})) \geq h^0(\mathcal{O}_{\mathcal{L}_S} \otimes \mathcal{O}_S(2H|_{\mathcal{S}})) \geq |\text{Orb}_{\mathcal{I}_S}(Z)|.
\]

because \( \mathcal{L}_S \) has at least \( |\text{Orb}_{\mathcal{I}_S}(Z)| \) disjoint zero-dimensional components, whose supports are points in \( \text{Orb}_{\mathcal{I}_S}(Z) \), because \( \text{Orb}_{\mathcal{I}_S}(Z) \subseteq \text{Nklt}(S, \lambda M|_{\mathcal{S}}) \), and \( \text{Nklt}(S, \lambda M|_{\mathcal{S}}) \) does not contain curves that are not disjoint from \( \text{Orb}_{\mathcal{I}_S}(Z) \). Hence, we see that \( |\text{Orb}_{\mathcal{I}_S}(Z)| \leq 10 \), which is impossible, since \( |\text{Orb}_{\mathcal{I}_S}(Z)| = 16 \). This shows that

\[
\pi(Z) = P \notin Q_1 \cup Q_2 \cup \cdots \cup Q_{10}.
\]

Let us summarize what we proved so far. Recall that \( M \) is a mobile \( \mathbb{H}/\mathbb{I} \)-invariant linear subsystem in \( \mathbb{H} \mathcal{H} \), the log pair \((X, \frac{2}{n} M)\) is canonical away from finitely many points, but the singularities of the pair \((X, \frac{2}{n} M)\) are not canonical at the point \( Z \in X \) such that

- \( Z \notin \text{Sing}(X) \),
- \( \pi(Z) \notin \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{30} \),
- \( \pi(Z) \notin Q_1 \cup Q_2 \cup \cdots \cup Q_{10} \),
- \( |\text{Orb}_{\mathcal{I}_S}(Z)| = 16 \),
- \( |\text{Orb}_{\mathcal{I}_S}(\pi(Z))| = 16 \).

By Lemma 35, \( \pi(Z) \) is not contained in any \( \mathbb{H}/\mathbb{I} \)-invariant curve whose degree is at most 7. Let us use this and Nadel’s vanishing [45, Theorem 9.4.8] to derive a contradiction.

As in the proofs of Propositions 30 and 31, we observe that \((X, \frac{2}{n} M)\) is not log canonical at the point \( Z \), because \( X \) is smooth at \( Z \). Let \( \mu \) be the largest rational number such that the log pair \((X, \mu M)\) is log canonical at \( Z \). Then \( \mu < \frac{3}{n} \) and

\[
\text{Orb}_{\mathcal{I}_S}(Z) \subseteq \text{Nklt}(X, \mu M).
\]

Moreover, if the locus \( \text{Nklt}(X, \mu M) \) contains an \( \mathbb{H}/\mathbb{I} \)-irreducible curve \( C \), then arguing as in the proof of Proposition 30, we see that

\[
\deg(\pi(C)) \leq H \cdot C \leq 4,
\]

which implies that the curve \( C \) does not pass through \( Z \). Hence, we conclude that every point of the orbit \( \text{Orb}_{\mathcal{I}_S}(Z) \) is an isolated irreducible component of the locus \( \text{Nklt}(X, \mu M) \).
Let $\mathcal{I}$ be the multiplier ideal sheaf of the pair $(X, \mu\mathcal{M})$, and let $\mathcal{L}$ be the corresponding subscheme in $X$. Then

$$\text{Supp} (\mathcal{L}) = \text{Nklt} (X, \mu\mathcal{M}),$$

so the subscheme $\mathcal{L}$ contains at least $|\text{Orb}_{\mathbb{P}^2}(Z)| = 16$ zero-dimensional components whose supports are points in the orbit $\text{Orb}_{\mathbb{P}^2}(Z)$. On the other hand, we have

$$h^1 (X, \mathcal{I} \otimes \mathcal{O}_X (H)) = 0$$

by Nadel’s vanishing theorem [45, Theorem 9.4.8]. This gives

$$4 = h^0 (X, \mathcal{O}_X (H)) \geq h^0 (\mathcal{O}_\mathcal{L} \otimes \mathcal{O}_X (H)) \geq |\text{Orb}_{\mathbb{P}^2}(Z)| = 16,$$

which is absurd. The obtained contradiction completes the proof of Theorem 17.

Let us conclude this paper with one application of Theorem 17, which was the initial motivation for this paper—we were looking for various embeddings $\mu_2^4 \rtimes \mu_3 \rhd \text{Bir}(\mathbb{P}^3)$.

**Example 40** (cf. Examples 5, 8, 26) Let $G_{48,50}$ be the subgroup in $\text{PGL}_4 (\mathbb{C})$ generated by

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Then one can check that $G_{48,50} \cong \mu_2^4 \rtimes \mu_3$ and the GAP ID of the group $G_{48,50}$ is [48,50]. For every $t \in \mathbb{C} \setminus \{ \pm 1, \pm \sqrt{3} \}$, let $S_t$ be the quartic surface in $\mathbb{P}^3$ given by the equation (6), i.e. the surface $S_t$ is the quartic surface in $\mathbb{P}^3$ given by

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 - (t^2 + 1)(x_0^2x_1^2 + x_2^2x_3^2)
+ x_0^2x_2^2 + x_1^2x_3^2 + x_0^2x_3^2 + x_1^2x_2^2 + 2(t^3 + 3t)x_0x_1x_2x_3 = 0.$$ 

Then $S_t$ is $G_{48,50}$-invariant, and $S_t$ has 16 ordinary double singularities (see Example 5). Now, let $X_t$ be the hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ that is given by

$$w^2 = x_0^4 + x_1^4 + x_2^4 + x_3^4 - (t^2 + 1)(x_0^2x_1^2 + x_2^2x_3^2)
+ x_0^2x_2^2 + x_1^2x_3^2 + x_0^2x_3^2 + x_1^2x_2^2 + 2(t^3 + 3t)x_0x_1x_2x_3,$$

where we consider $x_0, x_1, x_2, x_3$ as homogeneous coordinates on $\mathbb{P}(1, 1, 1, 1, 2)$ of weight 1, and $w$ is a coordinate of weight 2. Consider the faithful action $G_{48,50} \rhd X_t$ given by
Since the threefold $X_t$ is $G_{48,50}$-invariant, this gives an embedding $G_{48,50} \hookrightarrow \text{Aut}(X_t)$. Then it follows from Theorem 17 that the threefold $X_t$ is $G_{48,50}$-birationally super-rigid. In particular, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{ \pm 1, \pm \sqrt{3}i \}$, the following conditions are equivalent:

- the threefolds $X_{t_1}$ and $X_{t_2}$ are $G_{48,50}$-birational;
- the surfaces $S_{t_1}$ and $S_{t_2}$ are projectively equivalent.

Recall that $X_t$ is rational. For $t \in \mathbb{C} \setminus \{ \pm 1, \pm \sqrt{3}i \}$, fix a birational map $X_t : \mathbb{P}^3 \rightarrow X_t$, and consider the monomorphism $\eta_t : G_{48,50} \hookrightarrow \text{Bir}(\mathbb{P}^3)$ that is given by $g \mapsto X_t^{-1} \circ g \circ X_t$. Then, for any $t_1 \neq t_2$ in $\mathbb{C} \setminus \{ \pm 1, \pm \sqrt{3}i \}$, we have the following assertion:

$$\eta_{t_1}(G_{48,50}) \text{ and } \eta_{t_2}(G_{48,50}) \text{ are conjugate in } \text{Bir}(\mathbb{P}^3) \iff X_{t_1} \text{ and } X_{t_2} \text{ are } G_{48,50} - \text{birational}.$$

Thus, if $t_1 \neq t_2$ are general, then $\eta_{t_1}(G_{48,50})$ and $\eta_{t_2}(G_{48,50})$ are not conjugate in $\text{Bir}(\mathbb{P}^3)$. Similarly, we see that $\eta_t(G_{48,50})$ is not conjugate in $\text{Bir}(\mathbb{P}^3)$ to the group $G_{48,50} \subset \text{PGL}_4(\mathbb{C})$, which also follows from [16]. Can we show this using other obstructions [6, 34, 41]?

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