Riemannian metrics on the moduli space of GHMC anti-de Sitter structures

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Abstract
We first extend the construction of the pressure metric to the deformation space of globally hyperbolic maximal Cauchy-compact anti-de Sitter structures. We show that, in contrast with the case of the Hitchin components, the pressure metric is degenerate and we characterize its degenerate locus. We then introduce a nowhere degenerate Riemannian metric adapting the work of Qiongling Li on the $\text{SL}(3, \mathbb{R})$-Hitchin component to this moduli space. We prove that the Fuchsian locus is a totally geodesic copy of Teichmüller space endowed with a multiple of the Weil–Petersson metric.

Keywords Anti-de Sitter geometry · (higher) Teichmüller theory · Weil–Petersson metric · Pressure metric.

Mathematics Subject Classification 53C50 · 58D27

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Introduction

Let $S$ be a closed, connected, oriented surface of negative Euler characteristic. The aim of this short note is to introduce two Riemannian metrics on the deformation space $\mathcal{GH}(S)$ of convex co-compact anti-de Sitter structures on $S \times \mathbb{R}$. These are the geometric structures relevant for the study of pairs of conjugacy classes of representations $\rho_{L,R} : \pi_1(S) \to \mathbb{P}\text{SL}(2, \mathbb{R})$ that are faithful and discrete.

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In recent years, much work has been done in order to understand the geometry of $\mathcal{GH}(S)$ [21,26–30]. It turns out that many of the phenomena described in the aforementioned papers have analogous counterparts in the theory of Hitchin representations in $\text{SL}(3, \mathbb{R})$ [9,14–17,20,31]. Pushing this correspondence even further, we explain in this paper how to construct two Riemannian metrics in $\mathcal{GH}(S)$ following analogous constructions known for the $\text{SL}(3, \mathbb{R})$-Hitchin component.

The first Riemannian metric we define is the pressure metric introduced by Brigedeman et al. [2,3] for the Hitchin components, and inspired by previous work of Bridgeman [6] on quasi-Fuchsian representations and McMullen’s thermodynamic interpretation [18] of the Weil–Petersson metric on Teichmüller space. Although the construction of the pressure metric in $\mathcal{GH}(S)$ can be carried out analogously, we show that, unlike in the Hitchin components, the pressure metric is degenerate and we characterize its degenerate locus:

**Theorem A** The pressure metric on $\mathcal{GH}(S)$ is degenerate only at the Fuchsian locus along pure bending directions.

Here, the Fuchsian locus in $\mathcal{GH}(S)$ consists of pairs of discrete and faithful representations of $\pi_1(S)$ that coincide up to conjugation, and pure bending directions correspond to deformations of representations away from the Fuchsian locus that are analogs of bending deformations for quasi-Fuchsian representations in $\mathbb{P}\text{SL}(2, \mathbb{C})$ [27].

The second Riemannian metric we define follows instead the construction of Li on the $\text{SL}(3, \mathbb{R})$-Hitchin component [13] and it is based on the introduction of a preferred $\rho$-equivariant scalar product in $\mathbb{R}^4$ for a given $\rho \in \mathcal{GH}(S)$. The main result is the following:

**Theorem B** This Riemannian metric is nowhere degenerate in $\mathcal{GH}(S)$ and restricts to a multiple of the Weil–Petersson metric on the Fuchsian locus, which, moreover, is totally geodesic.

1 Pressure metric on $\mathcal{GH}(S)$

In this section we adapt the construction of the pressure metric on the Hitchin component [2,3] to the deformation space of globally hyperbolic maximal Cauchy-compact anti-de Sitter manifolds. We will show that the pressure metric is degenerate at the Fuchsian locus along “pure bending” directions.

1.1 Background on anti-de Sitter geometry

We briefly recall some notions of anti-de Sitter geometry that will be used in the sequel.

The 3-dimensional anti-de Sitter space AdS$_3$ is the local model of Lorentzian manifolds of constant sectional curvature $-1$ and can be defined as the set of projective classes of time-like vectors of $\mathbb{R}^4$ endowed with a bilinear form of signature $(2, 2)$.

We are interested in a special class of spacetimes locally modelled on AdS$_3$, introduced by Mess [19], called Globally Hyperbolic Maximal Cauchy-compact (GHMC). This terminology comes from physics and indicates that these spacetimes contain an embedded space-like surface that intersects any inextensible causal curve in exactly one point. From a modern mathematical point of view [8], we can describe these manifolds as being convex co-compact anti-de Sitter manifolds diffeomorphic to $S \times \mathbb{R}$, where $S$ is a closed surface of genus at least 2. This means that, identifying the fundamental group of $S$ with a discrete
subgroup $\Gamma$ of $\text{Isom}_0(\text{AdS}_3) \cong \mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R})$ via the holonomy representation $\text{hol} : \pi_1(S) \to \mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R})$, the group $\Gamma$ acts properly discontinuously and co-compactly on a convex domain in $\text{AdS}_3$.

We denote by $\mathcal{GH}(S)$ the deformation space of globally hyperbolic maximal Cauchy-compact anti-de Sitter structures on $S \times \mathbb{R}$. It turns out that the holonomy of a GHMC anti-de Sitter manifold into $\mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R})$ is faithful and discrete in each factor. Moreover, we have a homeomorphism between $\mathcal{GH}(S)$ and the product $T(S) \times T(S)$ of two copies of the Teichmüller space of $S$ [19]. In particular, each simple closed curve $\gamma \in \pi_1(S)$ is sent by the holonomy representation $\rho = (\rho_L, \rho_R)$ to a pair of hyperbolic isometries of $\mathbb{H}^2$, which preserves a space-like geodesic in the convex domain of discontinuity of $\rho$ in $\text{AdS}_3$, on which $\rho(\gamma)$ acts by translation by

$$\ell_{\rho}(\gamma) = \frac{1}{2}(\ell_{\rho_L}(\gamma) + \ell_{\rho_R}(\gamma)).$$

We will refer to $\ell_{\rho}(\gamma)$ as the translation length of the isometry $\rho(\gamma)$ (see [27]).

We will say that the holonomy $\rho : \pi_1(S) \to \mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R})$ of a GHMC anti-de Sitter structure is Fuchsian if, up to conjugation, its left and right projections coincide.

### 1.2 Background on thermodynamical formalism

Let $X$ be a Riemannian manifold. A smooth flow $\phi = (\phi_t)_{t \in \mathbb{R}}$ is Anosov if there is a flow-invariant splitting $TX = E^s \oplus E_0 \oplus E^u$, where $E_0$ is the bundle parallel to the flow and, for $t \geq 0$, the differential $d\phi_t$ exponentially contracts $E^s$ and exponentially expands $E^u$. We say that $\phi$ is topologically transitive if it has a dense orbit.

Given a periodic orbit $a$ for the flow $\phi$, we denote by $\ell(a)$ its period. Let $f : X \to \mathbb{R}$ be a positive Hölder function. It is possible [2] to reparametrize the flow $\phi$ and obtain a new flow $\phi^f$ with the property that each closed orbit $a$ has period

$$\ell_f(a) := \int_0^{\ell(a)} f(\phi_s(x))ds \quad x \in a.$$

We define

- the topological entropy [4] of $f$ as
  $$h(f) = \limsup_{T \to +\infty} \frac{\log(|R_T(f)|)}{T}$$
  where $R_T = \{ a \text{ closed orbit of } \phi \mid \ell_f(a) \leq T \}$;
- the topological pressure [5] of a Hölder function $g$ (not necessarily positive) as
  $$P(g) = \limsup_{T \to +\infty} \frac{1}{T} \log \left( \sum_{a \in R_T} e^{\ell_f(a)} \right).$$

These two notions are related by the following result:

**Lemma 1.1** [25]. Let $\phi$ be a topologically transitive Anosov flow on $X$ and let $f : X \to \mathbb{R}$ be a positive Hölder function. Then $P(-hf) = 0$ if and only if $h = h(f)$.

Consider then the space

$$\mathcal{P}(X) = \{ f : X \to \mathbb{R} \mid f \text{ Hölder}, \ P(f) = 0 \}$$
and its quotient $\mathcal{H}(X)$ by the equivalence relation that identifies Hölder functions with the same periods. The analytic regularity of the pressure [22,24] allows to define the pressure metric on $T_f \mathcal{P}(X)$ as

$$
\|g\|^2_P = -\frac{d^2}{dt^2} P(f + tg)_{|t=0} - \frac{d}{dt} P(f + tg)_{|t=0}.
$$

**Theorem 1.2** [22,24]. Let $X$ be a Riemannian manifold endowed with a topologically transitive Anosov flow. Then the pressure metric on $\mathcal{H}(X)$ is positive definite.

In particular, given a one parameter family of positive Hölder functions $f_t : X \to \mathbb{R}$, the functions $\Phi(t) = -h(f_t)f_t$ describe a path in $\mathcal{P}(X)$ by Lemma 1.1 and $\|\hat{\Phi}\|_P = 0$ if and only if $\Phi$ has vanishing periods, hence if and only if $\frac{d}{dt} |_{t=0} h(f_t)\ell_{f_t}(a) = 0$ for every closed orbit $a$ for $\phi$.

### 1.3 Pressure metric on $\mathcal{GH}(S)$

We apply the above theory to the unit tangent bundle $X = T^1 S$ of a hyperbolic surface $(S, \rho_0)$ endowed with its geodesic flow $\phi = \phi^{\rho_0}$. Here, $\rho_0 : \pi_1(S) \to \mathbb{PSL}(2, \mathbb{R})$ is a fixed Fuchsian representation that defines a marked hyperbolic metric on $S$. We also fix an identification of the universal cover $\tilde{S}$ with $\mathbb{H}^2$, and, consequently, of the Gromov boundary $\partial_{\infty}\pi_1(S)$ of the fundamental group with $S^1$. The following facts are well-known from hyperbolic geometry:

**Proposition 1.3** [3]. If $\rho, \eta : \pi_1(S) \to \mathbb{PSL}(2, \mathbb{R})$ are two Fuchsian representations, then there is a unique $(\rho, \eta)$-equivariant Hölder homeomorphism $\xi_{\rho, \eta} : \partial_{\infty}\mathbb{H}^2 \to \partial_{\infty}\mathbb{H}^2$ that varies analytically in $\eta$.

**Proposition 1.4** [3]. For every Fuchsian representation $\eta$, there is a positive Hölder function $f_\eta : X \to \mathbb{R}$ with period $\ell_{f_\eta}(\gamma)$ coinciding with the hyperbolic length $\ell_\eta(\gamma)$ of the closed geodesic $\gamma$ for the hyperbolic metric induced by $\eta$. Moreover, $f_\eta$ varies analytically in $\eta$.

**Corollary 1.5** For every $\rho = (\eta_L, \eta_R) \in \mathcal{GH}(S)$, there exists a positive Hölder function $f_\rho : X \to \mathbb{R}$ such that $\ell_{f_\rho}(\gamma) = \ell_\rho(\gamma)$ for every simple closed curve $\gamma \in \pi_1(S)$.

**Proof** Recall that $\ell_\rho(\gamma) = \frac{1}{2}(\ell_{\eta_L}(\gamma) + \ell_{\eta_R}(\gamma))$, thus it is sufficient to choose $f_\rho = \frac{1}{2}(f_{\eta_L} + f_{\eta_R})$. Moreover, $f_\rho$ varies analytically in $\rho$ by Proposition 1.4.

We can then introduce the thermodynamic mapping:

$$
\Phi : \mathcal{GH}(S) \to \mathcal{P}(X)
$$

$$
\rho \mapsto -h(f_\rho)f_\rho.
$$

By pulling-back the pressure metric via $\Phi$, we obtain a semi-definite metric on $\mathcal{GH}(S)$, which we still call pressure metric.

**Proposition 1.6** The restriction of the pressure metric to the Fuchsian locus in $\mathcal{GH}(S)$ is a constant multiple of the Weil–Petersson metric.

**Proof** Let $\rho_t = (\eta_t, \eta_t)$ be a path on the Fuchsian locus. Then

$$
\Phi(\rho_t) = -h(f_{\rho_t})f_{\rho_t} = -h(f_{\rho_t})f_{\eta_t} = -f_{\eta_t},
$$

where in the last step we used the fact that the entropy of a Fuchsian representation is 1 [10,29]. Therefore, $d\Phi(\dot{\rho}_0) = -\dot{f}_{\eta_0}$ and the result follows from [18].
Lemma 1.8  The pressure metric on $\mathcal{G}\mathcal{H}(S)$ is degenerate on the Fuchsian locus along pure bending directions.

**Proof**  Let $\rho_t = (\eta_{L,t}, \eta_{R,t})$ be a path in $\mathcal{G}\mathcal{H}(S)$ such that $\rho_0$ is Fuchsian and $\dot{\rho}_0 = \frac{d}{dt}_{t=0} \rho_t = (v, -v)$ for some $v \in T_{\eta}\mathcal{T}(S)$. By definition of the pressure metric and Theorem 1.2, we have $\|d\Phi(\rho_0)\| = 0$ if and only if $\frac{d}{dt}_{t=0} h(\rho_t) \ell_{\rho_t}(\gamma) = 0$ for every closed geodesic $\gamma$ on $S$. By the product rule and the fact that the entropy is maximal and equal to 1 at the Fuchsian locus [7,29], we get

$$\frac{d}{dt}_{t=0} h(\rho_t) \ell_{\rho_t}(\gamma) = \frac{d}{dt}_{t=0} \ell_{\rho_t}(\gamma) = \frac{1}{2} \left( \frac{d}{dt}_{t=0} \ell_{\eta_{L,t}}(\gamma) + \frac{d}{dt}_{t=0} \ell_{\eta_{R,t}}(\gamma) \right) = \frac{1}{2} (d\ell_{\eta_0}(v) + d\ell_{\eta_0}(-v)) = 0.$$  

□

Remark 1.9  As remarked in [6], we note that, along a general path $\rho_t \in \mathcal{G}\mathcal{H}(S)$, the condition $\frac{d}{dt}_{t=0} h(\rho_t) \ell_{\rho_t}(\gamma) = 0$ for every closed geodesic $\gamma$ is equivalent to the existence of a constant $k \in \mathbb{R}$ such that

$$\frac{d}{dt}_{t=0} \ell_{\rho_t}(\gamma) = k \ell_{\rho_0}(\gamma).$$

In fact, $k = -\frac{1}{h(\rho_0)} \frac{d}{dt}_{t=0} h(\rho_t)$.

Lemma 1.10  Let $v \in T_{\rho}\mathcal{G}\mathcal{H}(S)$ be a non-zero vector. If there exists $k \in \mathbb{R}$ such that

$$\frac{d}{dt}_{t=0} \ell_{\rho_t}(\gamma) = k \ell_{\rho_0}(\gamma) \quad (1.1)$$

for every closed geodesic $\gamma$, then $k = 0$ or $\rho$ is Fuchsian.

**Proof**  We show that if $\rho$ is not Fuchsian, then $k$ is necessarily 0. The proof follows the line of [6, Lemma 7.4]. Let $v = (v_1, v_2)$ and $\rho_t = (\rho_{1,t}, \rho_{2,t})$. Choose simple closed curves $\alpha$ and $\beta$ in $S$. Up to conjugation we can assume that

$$A_i(t) = \rho_{i,t}(\alpha) = \begin{pmatrix} \lambda_i(t) & 0 \\ 0 & \lambda_i(t)^{-1} \end{pmatrix},$$

where we denoted by $\lambda_i(t)$ the largest eigenvalue of the hyperbolic isometry $\rho_{i,t}(\alpha)$. Let

$$B_i(t) = \rho_{i,t}(\beta) = \begin{pmatrix} a_i(t) & b_i(t) \\ c_i(t) & d_i(t) \end{pmatrix}$$

such that $\det(B_i(t)) = 1$ and $\text{tr}(B_i(t)) > 2$. Notice that $b_i(t)c_i(t) \neq 0$ because $B_i(t)$ is hyperbolic and $A_i(t)$ and $B_i(t)$ have different axes. For every $n \geq 0$, we consider the matrices

$$C_{i,n}(t) = A_i^n(t)B_i(t) = \rho_{i,t}(\gamma_n) = \begin{pmatrix} \lambda_i(t)^n a_i(t) & \lambda_i(t)^n b_i(t) \\ \lambda_i(t)^{-n} c_i(t) & \lambda_i(t)^{-n} d_i(t) \end{pmatrix}$$
associated to some closed curves $\gamma_n$ on $S$. The eigenvalues $\mu_{i,n}$ of $C_{i,n}(t)$ satisfy
\[
\log(\mu_{i,n}(t)) = n \log(\lambda_i(t)) + \log(a_i(t)) + \lambda_i(t)^{-2n} \left( \frac{a_i(t)d_i(t) - 1}{a_i(t)^2} \right) + O(\lambda_i(t)^{-4n})
\]
as $n \to +\infty$. Applying Eq. (1.1) to the curves $\gamma_n$, we obtain
\[
0 = \frac{d}{dt} \bigg|_{t=0} \ell_{\rho_i}(\gamma_n) - k \ell_{\rho_i}(\gamma_n) = n \log(\lambda_1 \lambda_2)' - kn \log(\lambda_1 \lambda_2) + \log(a_1a_2)' - k \log(a_1a_2)
\]
\[
-2n \left[ \lambda_1^{-2n-1} \lambda_1' \left( \frac{a_1d_1 - 1}{a_1^2} \right) + \lambda_2^{-2n-1} \lambda_2' \left( \frac{a_2d_2 - 1}{a_2^2} \right) \right] + n \lambda_1^{-2n} \left[ \left( \frac{a_1d_1 - 1}{a_1^2} \right)' - k \left( \frac{a_1d_1 - 1}{a_1^2} \right) \right] + n \lambda_2^{-2n} \left[ \left( \frac{a_2d_2 - 1}{a_2^2} \right)' - k \left( \frac{a_2d_2 - 1}{a_2^2} \right) \right] + o(\lambda_i^{-2n})
\]
where all derivatives and all functions are intended to be taken and evaluated at $t = 0$. The term $n \log(\lambda_1 \lambda_2)' - kn \log(\lambda_1 \lambda_2)$ vanishes by assumption because
\[
\ell_{\rho_i}(\alpha) = \frac{1}{2} \left( \ell_{\rho_{i,t}}(\alpha) + \ell_{\rho_{2,t}}(\alpha) \right) = \log(\lambda_i(t)\lambda_2(t))
\]
and Eq. (1.1) holds for the curves $\alpha$. Taking the limit of the above expression as $n \to +\infty$, we deduce that $\log(a_1a_2)' - k \log(a_1a_2) = 0$. Because $\rho$ is not Fuchsian, we can assume to have chosen $\alpha$ and $\beta$ so that $\lambda_1(0) > \lambda_2(0)$. Then if we multiply the equation above by $\frac{\lambda_1' n}{2}$ and take the limit as $n \to +\infty$, we deduce that $\lambda_1' = 0$. Similarly, multiplying by $\frac{\lambda_2' n}{2}$, we find that $\lambda_2' = 0$. Therefore,
\[
\frac{d}{dt} \bigg|_{t=0} \ell_{\rho_i}(\alpha) = \frac{d}{dt} \bigg|_{t=0} \log(\lambda_i(t)\lambda_2(t)) = 0,
\]
hence $k = 0$. 

**Theorem 1.11** Let $v = (v_L, v_R) \in T_pG\mathcal{H}(S)$ be a non-zero tangent vector such that $\|d\Phi(v)\| = 0$. Then $\rho$ is Fuchsian and $v$ is a pure bending direction.

**Proof** Let $\rho_t$ be a path in $G\mathcal{H}(S)$ such that $\rho_0 = \rho$ and $\rho_t$ is tangent to $v$. If $\rho = (\eta, \eta)$ is Fuchsian, then, combining Remark 1.9 with the fact that the entropy is maximal and equal to 1 at the Fuchsian locus, we get
\[
0 = \frac{d}{dt} \bigg|_{t=0} h(\rho_t)\ell_{\rho}(\gamma) = \frac{d}{dt} \bigg|_{t=0} \ell_{\rho_i}(\gamma)
\]
for every simple closed geodesic $\gamma$ in $S$. Therefore,
\[
0 = \frac{d}{dt} \bigg|_{t=0} \ell_{\rho_i}(\gamma) = \frac{1}{2} (d\ell_{\eta}(\gamma)(v_L) + d\ell_{\eta}(\gamma)(v_R))
\]
from which we deduce that $v_R = -v_L$, because $\{d\ell_{\eta}(\gamma)\}_{\gamma}$ generates $T_{\eta}^*\mathcal{T}(S)$. Hence, $v$ is a pure-bending direction.
We are thus left to show that $\rho$ is necessarily Fuchsian. Suppose it is not and denote with $\rho_L \neq \rho_R$ the projections of $\rho$. By the previous lemma

$$\ell'_\rho(\gamma) = \frac{d}{dt}_{|t=0} \ell_\rho(t) = 0$$

for every simple closed geodesic $\gamma$ in $S$. Moreover, we have shown in the proof of Lemma 1.10 that if $\ell_{\rho_L}(\gamma) \neq \ell_{\rho_R}(\gamma)$ then $\ell'_{\rho_L}(\gamma) = \ell'_{\rho_R}(\gamma) = 0$. Otherwise, $\ell'_{\rho_L}(\gamma) = -\ell'_{\rho_R}(\gamma)$.

Exploiting the isomorphism $\mathbb{P}SL(2, \mathbb{R}) \times \mathbb{P}SL(2, \mathbb{R}) \cong SO_0(2, 2)$, we find that the matrix $\rho_t(\gamma)$ is conjugated to

$$\exp\left(\frac{1}{2} \text{diag}(\ell_{\rho_L}, \ell_{\rho_R}, \ell_{\rho_L}, -\ell_{\rho_R}, -\ell_{\rho_L}, -\ell_{\rho_R})\right),$$

thus

$$d\text{tr}(\rho(\gamma))(v) = \frac{d}{dt}_{|t=0} \text{tr}(\rho_t(\gamma)) = 0$$

for every simple closed geodesic $\gamma$. Because $\rho$ is generic in the sense of [2, Proposition 10.3], i.e. there exists an element $\gamma_0 \in \pi_1(S)$ such that $\rho(\gamma_0)$ is diagonalizable with different eigenvalues, the differentials of traces $\{d\text{tr}(\rho(\gamma))\}_\gamma$ generate $T^*_\rho \mathcal{H}(S)$ and we must have $v = 0$.

2 A non-degenerate Riemannian metric on $\mathcal{H}(S)$

In this section we define a non-degenerate Riemannian metric on $\mathcal{H}(S)$ following Li’s construction [13] for the $SL(3, \mathbb{R})$-Hitchin component.

2.1 Preliminaries

In this section we identify $\mathcal{H}(S)$ with a connected component of the space of representations $\text{Hom}(\pi_1(S), SO_0(2, 2))/SO_0(2, 2)$ via the holonomy map. Recall that by Mess’ parametrization [19], this component is smooth and diffeomorphic to $T(S) \times T(S)$. This allows to identify the tangent space $T_\rho \mathcal{H}(S)$ at $\rho \in \mathcal{H}(S)$ with the cohomology group $H^1(S, \mathfrak{s}_0(2, 2)_{\text{Ad}_\rho})$, where $\mathfrak{s}_0(2, 2)_{\text{Ad}_\rho}$ denotes the flat $\mathfrak{s}_0(2, 2)$ bundle over $S$ with holonomy $\text{Ad}_\rho$. Explicitly,

$$\mathfrak{s}_0(2, 2)_{\text{Ad}_\rho} = (\tilde{\mathfrak{s}} \times \mathfrak{s}_0(2, 2))/\sim$$

where $(\tilde{x}, v) \sim (\gamma \tilde{x}, \text{Ad}_\rho(\gamma)(v))$ for any $\gamma \in \pi_1(S)$, $x \in \tilde{\mathfrak{s}}$ and $v \in \mathfrak{s}_0(2, 2)$.

In order to define a Riemannian metric on $\mathcal{H}(S)$ it is thus sufficient to introduce a non-degenerate scalar product on $H^1(S, \mathfrak{s}_0(2, 2)_{\text{Ad}_\rho})$. Let us assume for the moment that we have chosen an inner product $\iota$ on the bundle $\mathfrak{s}_0(2, 2)_{\text{Ad}_\rho}$ and a Riemannian metric $h$ on $S$. A Riemannian metric in cohomology follows then by standard Hodge theory that we recall briefly here. The Riemannian metric $h$ and the orientation on $S$ induce a scalar product $\langle \cdot, \cdot \rangle$ on the space $\mathcal{A}^p(S)$ of $p$-forms on $S$, which allows to define a Hodge star operator

$$\star : \mathcal{A}^p(S) \to \mathcal{A}^{2-p}(S)$$

by setting

$$\alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle_h dA_h.$$
This data gives a bi-linear pairing $\tilde{g}$ in the space of $so_0(2,2)_{Ad_\rho}$-valued 1-forms as follows:

$$\tilde{g}(\sigma \otimes \phi, \sigma' \otimes \phi') = \int_S \iota(\phi, \phi')_{\sigma} \wedge (\ast \sigma'),$$

where $\sigma, \sigma' \in \mathcal{A}^1(S)$ and $\phi, \phi'$ are sections of $so_0(2,2)_{Ad_\rho}$.

Given $\rho \in \mathcal{G}\mathcal{H}(S)$, we denote by $\rho^*$ the contragradient representation (still into $SO_0(2,2)$) defined by $(\rho^*(\gamma)L)(v) = L(\rho^{-1}(\gamma)v)$ for every $v \in \mathbb{R}^4$ and $L \in \mathbb{R}^{4*} = \text{Hom}(\mathbb{R}^4, \mathbb{R})$. The flat bundle $so_0(2,2)_{Ad_\rho^*}$ is dual to $so_0(2,2)_{Ad_\rho}$ and the inner product $\iota$ induces an isomorphism [23]

$$\#: so_0(2,2)_{Ad_\rho} \rightarrow so_0(2,2)_{Ad_\rho^*}$$

defined by setting

$$(#A)(B) = \iota(A, B)$$

for $A, B \in so_0(2,2)$. This extends naturally to an isomorphism

$$\#: \mathcal{A}^P(S, so_0(2,2)_{Ad_\rho}) \rightarrow \mathcal{A}^P(S, so_0(2,2)_{Ad_\rho^*}).$$

Consequently, we can introduce a coboundary map

$$\delta : \mathcal{A}^P(S, so_0(2,2)_{Ad_\rho}) \rightarrow \mathcal{A}^{P-1}(S, so_0(2,2)_{Ad_\rho})$$

by setting $\delta = -(#)^{-1} \ast^{-1} d \ast #$, and then a Laplacian operator

$$\Delta : \mathcal{A}^P(S, so_0(2,2)_{Ad_\rho}) \rightarrow \mathcal{A}^P(S, so_0(2,2)_{Ad_\rho})$$

given by $\Delta = d\delta + \delta d$. A 1-form $\xi$ is said to be harmonic if $\Delta \xi = 0$, or, equivalently, if $d\xi = \delta \xi = 0$. We have an orthogonal decomposition

$$\mathcal{A}^1(S, so_0(2,2)_{Ad_\rho}) = \text{Ker}(\Delta) \oplus \text{Im}(d) \oplus \text{Im}(\delta)$$

and by the non-abelian Hodge theory [23] every cohomology class contains a unique harmonic representative. Therefore, the bi-linear pairing $\tilde{g}$ induces a scalar product in cohomology by setting

$$g : H^1(S, so_0(2,2)_{Ad_\rho}) \times H^1(S, so_0(2,2)_{Ad_\rho}) \rightarrow \mathbb{R}$$

$$([\alpha], [\beta]) \mapsto \tilde{g}(\alpha_{\text{harm}}, \beta_{\text{harm}}),$$

where $\alpha_{\text{harm}}$ and $\beta_{\text{harm}}$ are the harmonic representatives of $\alpha$ and $\beta$.

### 2.2 Definition of the metric

As explained before, in order to define a Riemannian metric on $\mathcal{G}\mathcal{H}(S)$ it is sufficient to define a Riemannian metric $h$ on $S$ and a scalar product $\iota$ on $so_0(2,2)_{Ad_\rho}$.

Let us begin with the metric $h$ on $S$. Given $\rho \in \mathcal{G}\mathcal{H}(S)$, we denote by $M_\rho$ the unique GHMC anti-de Sitter manifold with holonomy $\rho$, up to isotopy. It is well-known that $M_\rho$ contains a unique embedded maximal (i.e. with vanishing mean curvature) surface $\Sigma_\rho$ [1]. A natural choice for $h$ is thus the induced metric on $\Sigma_\rho$.

As for the scalar product $\iota$, we first introduce a scalar product in $\mathbb{R}^4$ that is closely related to the maximal surface and its induced metric. Lifting the surface $\Sigma_\rho$ to the universal cover, we can find a $\rho$-equivariant maximal embedding $\tilde{\sigma} : \tilde{S} \rightarrow \tilde{\text{AdS}}_3 \subset \mathbb{R}^4$, where $\tilde{\text{AdS}}_3$ denotes the double cover of $\text{AdS}_3$ consisting of unit time-like vectors in $\mathbb{R}^4$ endowed with a bi-linear
form of vectors $u_1(\tilde{x})$ and $u_2(\tilde{x})$ to the surface at $\tilde{\sigma}(\tilde{x})$, the time-like unit normal vector $N(\tilde{x})$ at $\tilde{\sigma}(\tilde{x})$ and the position vector $\tilde{\sigma}(x)$. We can define a scalar product $\iota_{\tilde{x}}$ on $\mathbb{R}^4$ depending on the point $\tilde{x} \in \hat{S}$ by declaring the frame $\{u_1(\tilde{x}), u_2(\tilde{x}), \tilde{\sigma}(x), N(\tilde{x})\}$ to be orthonormal for $\iota_{\tilde{x}}$. 

Because $so_0(2, 2) \subset gl(4, \mathbb{R}) \cong \mathbb{R}^4 \times \mathbb{R}^4$, the inner product $\iota_{\tilde{x}}$ induces an inner product on $so_0(2, 2)$ and, consequently, on the trivial bundle $\hat{S} \times so_0(2, 2)$ over $\hat{S}$. This descends to a metric $\iota$ on $so_0(2, 2)_{Ad\rho}$ by setting

$$\iota_p(\phi, \phi') := \iota_{\tilde{x}}(\tilde{\phi} \tilde{x}, \tilde{\phi}' \tilde{x})$$

where $p \in \bar{S}$, $\pi : \hat{S} \rightarrow S$ is the natural projection and $\tilde{\phi}, \tilde{\phi}'$ are lifts of $\phi, \phi'$ to the trivial bundle $\hat{S} \times so_0(2, 2)$ evaluated at $\tilde{x}$. Because $\iota_{\tilde{x}}$ is $\rho$-equivariant, it is easy to check (see [13]) that $\iota_p$ does not depend on the choice of $\tilde{x} \in \pi^{-1}(p)$ and thus $\iota$ is a well-defined metric on the flat bundle $so_0(2, 2)_{Ad\rho}$.

The following lemma is useful for computations with this metric:

**Lemma 2.1** [13]. Assume that we have a matrix representation $H$ of the inner product $\iota_{\tilde{x}}$ at a point $\tilde{x} \in \pi^{-1}(p)$ with respect to the canonical basis of $\mathbb{R}^4$. Then

$$\iota_p(A, B) = \text{tr}(A^t H^{-1} B H) \quad \text{for } A, B \in so_0(2, 2).$$

### 2.3 Restriction to the Fuchsian locus

In order to compute the restriction of the metric $g$ to the Fuchsian locus, we need to understand the induced metric on the equivariant maximal surface and find a matrix representation of the inner product $\iota$.

If $\rho \in \mathcal{H}(S)$ is Fuchsian, the representation preserves a totally geodesic space-like plane in $AdS_3$. Realizing explicitly (the double cover of) anti-de Sitter space as

$$\widehat{AdS}_3 = \{ x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1 \},$$

we can assume, up to post-composition by an isometry, that $\rho$ preserves the hyperboloid

$$\mathcal{H} = \{ x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 - x_3^2 = 1, x_4 = 0 \},$$

which is isometric to the hyperbolic plane $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$: an explicit isometry [13] being

$$f : \mathbb{H}^2 \rightarrow \mathcal{H} \subset \mathbb{R}^4 \quad \quad (x, y) \mapsto \left( \frac{x}{y} \cdot \frac{x^2 + y^2 - 1}{2y}, \frac{x^2 + y^2}{2y}, 0 \right).$$

The representation $\rho : \pi_1(S) \rightarrow \text{SO}_0(2, 2)$ factors then through the standard copy of $\text{SO}_0(2, 1)$ inside $\text{SO}_0(2, 2)$, which is isomorphic to $\mathbb{P}SL(2, \mathbb{R})$ via the map [12]

$$\Phi : \mathbb{P}SL(2, \mathbb{R}) \rightarrow \text{SO}_0(2, 1) \subset \text{SO}_0(2, 2) \quad \quad (a \ b) \mapsto \left( \begin{array}{ccc} ad + bc & ac - bd & 0 \\ ad - cd & a^2 - b^2 - c^2 + d^2 & a^2 + b^2 - c^2 - d^2 \\ ab + cd & a^2 - b^2 + c^2 - d^2 & a^2 + b^2 + c^2 + d^2 \end{array} \right).$$
The map $\Phi$ induces a Lie algebra homomorphism, still denoted by $\Phi$, given by
\[
\Phi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{so}_0(2, 2)
\]
by declaring the frame
\[
\begin{pmatrix}
a & b \\ c & -a
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & c - b & c + b & 0 \\ b - c & 0 & 2a & 0 \\ b + c & 2a & 0 & 0 \\ 0 & 0 & 0 & 0
\end{pmatrix}
\]
(2.5)

It follows that if $\rho(\pi_1(S)) = \Gamma < SO_0(2, 2)$, then the maximal surface $\Sigma_\rho$ is realized by $\mathcal{H}/\Gamma$ and is isometric to the hyperbolic surface $\mathbb{H}^2/\Phi^{-1}(\Gamma)$.

Let us now turn our attention to the scalar product $\iota$ on $\mathfrak{so}_0(2, 2)_{\text{Ad} \rho}$. Recall that $\iota$ is determined by a family of inner products $\iota_\Sigma$ on $\mathbb{R}^4$ depending on $\Sigma \in \hat{S}$, which is obtained by declaring the frame $\{u_1(\Sigma), u_2(\Sigma), v(\Sigma), N(\Sigma)\}$ orthonormal. If we identify the universal cover of $\hat{S}$ with $\mathbb{H}^2$, the map $f$ gives an explicit $\rho$-equivariant maximal embedding of $\hat{S}$ into $\text{Ad}_S^3$. Therefore, the coordinates of the vectors tangent and normal to the embedding with respect to the canonical basis of $\mathbb{R}^4$ can be explicitly computed and the following matrix representation $H$ of $\iota_\Sigma$ can be obtained for any $z \in \mathbb{H}^2$ [13, Corollary 6.5]:
\[
H = \begin{pmatrix}
\frac{2x^2}{y^2} + 1 & \frac{x(x^2+y^2-1)}{y^2} & \frac{-x(x^2+y^2+1)}{y^2} & 0 \\
-x(x^2+y^2-1)(x^2+y^2+1) & \frac{2x^2}{y^2} + 1 & \frac{x^2+y^2-1)(x^2+y^2+1)}{2y^2} & 0 \\
-x^2 & \frac{2x^2}{y^2} & \frac{x^2+y^2-1)(x^2+y^2+1)}{2y^2} & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with
\[
H^{-1} = \begin{pmatrix}
\frac{2x^2}{y^2} + 1 & \frac{x(x^2+y^2-1)}{y^2} & \frac{x(x^2+y^2+1)}{y^2} & 0 \\
-x(x^2+y^2-1)(x^2+y^2+1) & \frac{2x^2}{y^2} + 1 & \frac{x^2+y^2-1)(x^2+y^2+1)}{2y^2} & 0 \\
-x^2 & \frac{2x^2}{y^2} & \frac{x^2+y^2-1)(x^2+y^2+1)}{2y^2} & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Together with Lemma 2.1 we obtain the following:

**Corollary 2.2** [13]. For any $z \in \mathbb{H}^2$, after extending the definition of Lemma 2.1 to $A, B \in \mathfrak{so}(4, \mathbb{C})$ by $\iota_\Sigma(A, B) = \text{tr}(A^t H^{-1} B H)$, we have
\[
\iota_\Sigma \left( \Phi \begin{pmatrix} -z \\ -1 \\ z^2 \\ z \end{pmatrix} , \Phi \begin{pmatrix} -z \\ -1 \\ z^2 \\ z \end{pmatrix} \right) = 16y^2 .
\]

The last ingredient we need in order to describe the restriction of $g$ to the Fuchsian locus is an explicit realization of the tangent space to the Fuchsian locus inside $T \mathcal{G} \mathcal{H}(S)$.

**Lemma 2.3** Let $\rho \in \mathcal{G} \mathcal{H}(S)$ be a Fuchsian representation.

(i) The tangent space at $\rho$ to the Fuchsian locus is spanned by the cohomology class of $\phi(z) dz \otimes \Phi \begin{pmatrix} -z \\ -1 \\ z^2 \\ z \end{pmatrix}$, where $\phi(z) dz^2$ is a holomorphic quadratic differential on $\Sigma_\rho$.

(ii) the $\mathfrak{so}_0(2, 2)_{\text{Ad} \rho}$-valued 1-forms $\phi(z) dz \otimes \Phi \begin{pmatrix} -z \\ -1 \\ z^2 \\ z \end{pmatrix}$ are harmonic representatives in their own cohomology class.
Proof (i) Let $\rho' = \Phi^{-1}(\rho)$ be the corresponding Fuchsian representation in $\mathbb{P}SL(2, \mathbb{R})$.

The claim follows from the fact [11] that the tangent space to Teichmüller space is generated by the $\mathfrak{sl}(2, \mathbb{R})_{Ad\rho'}$-valued 1-forms $\phi(z)dz \otimes \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}$ and thus the tangent space to the Fuchsian locus is generated by the inclusion of $H^1(S, \mathfrak{sl}(2, \mathbb{R})_{Ad\rho})$ induced by the map $\Phi$.

(ii) We need to show that $\phi(z)dz \otimes \Phi \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}$ is $d$-closed and $\delta$-closed. The first fact has been proved in [13, Lemma 6.6]. As for $\delta$-closedness, we will follow the lines of the aforementioned lemma. From the definition of $\delta$, it is enough to show that $d \ast (#(\phi(z)dz \otimes \Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})) = 0$. By linearity

$$
#(\phi(z)dz \otimes \Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = z^2 \phi(z)dz \otimes \Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \phi(z)dz \otimes \Phi \begin{pmatrix} 0 & 0 \\ 1/2 & -1/2 \end{pmatrix} - 2z\phi(z)dz \otimes \Phi \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.
$$

We then want to calculate $#\Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $#\Phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $#\Phi \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$.

We choose a basis for $\mathfrak{so}(2, 2)$ given by

$$
E_1 = \Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
E_2 = \Phi \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
E_3 = \Phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
E_4 = \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
E_5 = \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},
E_6 = \Phi \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
$$

The map $# : \mathfrak{so}(2, 2)_{Ad\rho} \to \mathfrak{so}(2, 2)_{Ad\rho^*}$ is defined by setting

$$
(#A)(B) = \iota(A, B)
$$

thus

$$
#A = \sum_{i=1}^{6} \iota(A, E_i) E_i^*,
$$

where $E_i^*$ satisfies

$$
E_i^*(E_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.
$$

Applying Lemma 2.1 to compute $\iota(E_i, E_j)$, we obtain the following

$$
#E_1 = \frac{4}{y^2} (E_1^* - x^2 E_3^* + x E_2^*)
$$
Theorem 2.4 The metric $g$ on $\tilde{\mathcal{M}}$ where here we are extending $\tilde{\mathcal{M}}$ to an isometry of $(\mathcal{N}, g)$.

Proof By Lemma 2.3, it is sufficient to show that $\phi(z)dz \otimes \Phi \left( \begin{matrix} -z \\ z^2 \\ -1 \\ z \end{matrix} \right)$ is $d$-closed and $\delta$-closed, hence it is harmonic.

We can finally prove one of the main results of the section:

Theorem 2.4 The metric $g$ on $\mathcal{G}\mathcal{H}(S)$ restricts on the Fuchsian locus to a constant multiple of the Weil–Petersson metric on Teichmüller space.

Proof By Lemma 2.3, it is sufficient to show that

$$\tilde{g} \left( \phi(z)dz \otimes \Phi \left( \begin{matrix} -z \\ z^2 \\ -1 \\ z \end{matrix} \right), \psi(z)dz \otimes \Phi \left( \begin{matrix} -z \\ z^2 \\ -1 \\ z \end{matrix} \right) \right) = \langle \phi, \psi \rangle_{WP},$$

where here we are extending $\tilde{g}$ to an hermitian metric on the space of $so(4, \mathbb{C})_{Ad,\rho}$-valued 1-forms. From the definition of $\tilde{g}$ and Corollary 2.2 we have

$$\tilde{g} \left( \phi(z)dz \otimes \Phi \left( \begin{matrix} -z \\ z^2 \\ -1 \\ z \end{matrix} \right), \psi(z)dz \otimes \Phi \left( \begin{matrix} -z \\ z^2 \\ -1 \\ z \end{matrix} \right) \right) = \Re \left( \int_S \phi(z)dz \wedge \psi(z)dz \right),$$

$$= \Re \left( \int_S 16i \phi(z)\psi(z) y^2 dz \wedge d\tilde{z} \right),$$

$$= 32 \langle \phi, \psi \rangle_{WP}.$$
of $G\mathcal{H}(S)$ as $T(S) \times T(S)$, there is a natural involution that swaps left and right representations, thus fixing pointwise the Fuchsian locus. Identifying $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ with $\text{SO}_0(2, 2)$, this corresponds to conjugation by $Q = \text{diag}(-1, -1, 1, -1) \in O(2, 2)$. Therefore, we introduce the map

$$q : G\mathcal{H}(S) \to G\mathcal{H}(S)$$

$$\rho \mapsto Q\rho Q^{-1}$$

and show that this is an isometry for the metric $g$.

We first need to compute the induced map in cohomology

$$q_* : H^1(S, \mathfrak{so}_0(2, 2)_{Ad\rho}) \to H^1(S, \mathfrak{so}_0(2, 2)_{Adq(\rho)})$$

It is well-known (see e.g. [11]) that a tangent vector to a path of representations $\rho_t$ is a 1-cocycle, that is a map $u : \pi_1(S) \to \mathfrak{so}_0(2, 2)$ satisfying

$$u(\gamma \gamma') - u(\gamma') = \text{Ad}(\rho(\gamma))u(\gamma')\,.$$

It is then clear that, if $u$ is a 1-cocycle tangent to $\rho$, then $QuQ^{-1}$ is a 1-cocycle tangent to $q(\rho)$. A 1-cocycle represents a cohomology class in $H^1(\pi_1(S), \mathfrak{so}_0(2, 2))$ which is isomorphic to $H^1(S, \mathfrak{so}_0(2, 2)_{Ad\rho})$ via

$$H^1(S, \mathfrak{so}_0(2, 2)_{Ad\rho}) \to H^1(\pi_1(S), \mathfrak{so}_0(2, 2))$$

$$[\sigma \otimes \phi] \mapsto u_{\sigma \otimes \phi} : \gamma \mapsto \int_\gamma \sigma \otimes \phi.$$

**Lemma 2.5** For any $\sigma \in A^1(S)$ and for any section $\phi$ of $\mathfrak{so}_0(2, 2)_{Ad\rho}$, we have

$$q_*[\sigma \otimes \phi] = [\sigma \otimes Q\phi Q^{-1}]$$

**Proof** It is sufficient to show that $u_{\sigma \otimes Q\phi Q^{-1}} = Qu_{\sigma \otimes \phi} Q^{-1}$. This follows because, for any $\gamma \in \pi_1(S)$

$$\int_\gamma \sigma \otimes Q\phi Q^{-1} = Q \left( \int_\gamma \sigma \otimes \phi \right) Q^{-1}.$$

By an abuse of notation, we will still denote by $q_*$ the map induced by $q$ at the level of $\mathfrak{so}_0(2, 2)_{Ad\rho}$-valued 1-forms. Our next step is to show that $q_*$ preserves the metric $\tilde{g}$.

**Lemma 2.6** For any $\sigma, \sigma' \in A^1(S)$ and for any sections $\phi$ and $\phi'$ of $\mathfrak{so}_0(2, 2)_{Ad\rho}$, we have

$$\tilde{g}(q_*(\sigma \otimes \phi), q_*(\sigma' \otimes \phi')) = \tilde{g}(\sigma \otimes \phi, \sigma \otimes \phi').$$

**Proof** Given $\rho \in G\mathcal{H}(S)$, we denote by $M_\rho$ the GHMC anti-de Sitter manifold with holonomy $\rho$. Because $M_\rho$ and $M_{Q\rho Q^{-1}}$ are isometric via the map induced in the quotients by $Q : \text{Ad}\hat{S}_3 \to \text{Ad}\hat{S}_3$, the minimal surfaces $\Sigma_\rho$ and $\Sigma_{Q\rho Q^{-1}}$ are isometric as well. In particular, their induced metrics $h$ and $h^Q$ coincide on every $\tilde{x} \in \tilde{S}$. Moreover, if $\tilde{\sigma} : \tilde{S} \to \text{Ad}\hat{S}_3$ is the $\rho$-equivariant maximal embedding, then $Q\tilde{\sigma}$ is $Q\rho Q^{-1}$-equivariant and still maximal. We deduce that if $H$ is a matrix representation of the $\rho$-equivariant inner product $\iota_\tilde{x}$ on $\tilde{S} \times \mathfrak{so}_0(2, 2)$, then $H^Q = Q^t HQ$ is the matrix representation of the $Q\rho Q^{-1}$-equivariant
inner product $\iota^q_{\tilde{\chi}}$. Therefore, noting that $Q = Q' = Q^{-1}$, for any $\phi$ and $\phi'$ sections of $\mathfrak{s}0_0(2, 2)_{\text{Ad}_p}$ and for any $p \in S$, we have

$$
\iota_p(\phi, \phi') = \iota(\phi, \phi') = \text{tr}(A^i H^{-1} B H) \quad \text{by Lemma 2.1}
$$

$$
= \text{tr}(Q'(Q')^{-1} A^i Q Q^{-1} H_1^{-1} Q'(Q')^{-1} Q B Q^{-1} Q(Q')^{-1} Q' H Q Q^{-1})
$$

$$
= \text{tr}((q_*(A))^t (H^q)^{-1} q_*(B) H^q Q^{-1})
$$

$$
= \text{tr}((q_*(A))^t (H^q)^{-1} q_*(B) H^q Q^{-1})
$$

$$
= \iota^q_{\tilde{\chi}}(q_*(A), q_*(B)) = \iota^q_0(q_*(\phi), q_*(\phi')) \quad \text{by Lemma 2.5.}
$$

We can now compute

$$
\tilde{g}((\sigma \otimes \phi, \sigma' \otimes \phi')) = \int_S \iota(\phi, \phi') \sigma \wedge (*\sigma')
$$

$$
= \int_S \iota(\phi, \phi')(\sigma, \sigma')_h dA_h
$$

$$
= \int_S \iota^q(q_*(\phi), q_*(\phi'))(\sigma, \sigma')_h dA_h
$$

$$
= \tilde{g}(q_*(\sigma \otimes \phi), q_*(\sigma' \otimes \phi'))
$$

which shows that $q_*$ is an isometry for the Riemannian metrics on the bundles $A^1(S, \mathfrak{s}0_0(2, 2)_{\text{Ad}_p})$ and $A^1(S, \mathfrak{s}0_0(2, 2)_{\text{Ad}_{q(\rho)}})$.

In order to conclude that $q : \mathcal{G}(\mathbb{H}(S)) \rightarrow \mathcal{G}(\mathbb{H}(S))$ is an isometry for $g$, it is sufficient now to show that the map $q_*$ preserves harmonicity of forms.

**Lemma 2.7** The map $q_* : H^1(S, \mathfrak{s}0_0(2, 2)_{\text{Ad}_p}) \rightarrow H^1(S, \mathfrak{s}0_0(2, 2)_{\text{Ad}_{q(\rho)}})$ sends harmonic forms to harmonic forms.

**Proof** Let $\sum_i \sigma_i \otimes \phi_i$ be the harmonic representative in its cohomology class. This is equivalent to saying that $d(\sum_i \sigma_i \otimes \phi_i) = 0$ and $\delta(\sum_i \sigma_i \otimes \phi_i) = 0$. We need to show that these imply $d(\sum_i \sigma_i \otimes Q\phi_i Q^{-1}) = 0$ and $\delta(\sum_i \sigma_i \otimes Q\phi_i Q^{-1}) = 0$, as well.

The condition $d(\sum_i \sigma_i \otimes Q\phi_i Q^{-1}) = 0$ easily follows by linearity of $d$.

As for $\delta$-closedness, by definition of $\delta$, we have $\delta(\sum_i \sigma_i \otimes \phi_i) = 0$ if and only if $d * \#(\sum_i \sigma_i \otimes \phi_i) = d * (\sum_i \sigma_i \otimes \#\phi_i) = 0$. Let us denote by $\#^q$ the analogous operator defined on $\mathfrak{s}0_0(2, 2)_{\text{Ad}_{q(\rho)}}$-valued 1-forms. Let $\{E_j\}_{j=1}^6$ be the basis of $\mathfrak{s}0_0(2, 2)$ introduced in the proof of Lemma 2.3 and denote by $\{E_j^*\}_{j=1}^6$ its dual. By definition of $\#$ and $\#^q$ we have

$$
\# A = \sum_{j=1}^6 \iota(A, E_j) E_j^* \quad \text{and} \quad \#^q A = \sum_{j=1}^6 \iota^q(A, E_j) E_j^*,
$$

where, as in Lemma 2.6, we denoted by $\iota^q$ the inner product on $\mathfrak{s}0_0(2, 2)_{\text{Ad}_{q(\rho)}}$. Hence, $d * (\sum_i \sigma_i \otimes \#\phi_i) = 0$ if and only if

$$
d * \left(\sum_i \sigma_i \otimes \sum_{j=1}^6 \iota(\phi_i, E_j) E_j^*\right) = 0,
$$

\[\square\]
which implies that
\[ d^{*} \left( \sum_{i} \sigma_{i} \iota(\phi_{i}, E_{j}) \right) = 0 \quad \text{for every } j = 1, \ldots, 6. \quad (2.7) \]

Therefore, using that \( \iota^{q} (QAQ^{-1}, QBQ^{-1}) = \iota(A, B) \) for every \( A, B \in \mathfrak{so}(2, 2) \), we have
\[
\begin{align*}
    d^{*} \left( \sum_{i} \sigma_{i} \otimes \#^{q} Q \phi_{i} Q^{-1} \right) &= d^{*} \left( \sum_{i} \sigma_{i} \otimes \sum_{j=1}^{6} \iota^{q}(Q \phi_{i} Q^{-1}, E_{j} E_{j}^{*}) \right) \\
    &= d^{*} \left( \sum_{i} \sigma_{i} \otimes \sum_{j=1}^{6} \iota^{q}(Q \phi_{i} Q^{-1}, Q Q^{-1} E_{j} Q Q^{-1}) E_{j}^{*} \right) \\
    &= d^{*} \left( \sum_{i} \sigma_{i} \otimes \sum_{j=1}^{6} \iota(\phi_{i}, Q^{-1} E_{j} Q) E_{j}^{*} \right).
\end{align*}
\]

A straightforward computation shows that
\[
Q^{-1} E_{1} Q = -E_{3} \quad Q^{-1} E_{2} Q = -E_{2} \quad Q^{-1} E_{3} Q = -E_{1} \\
Q^{-1} E_{4} Q = E_{4} \quad Q^{-1} E_{5} Q = E_{5} \quad Q^{-1} E_{6} Q = -E_{6}
\]
thus \( d^{*} (\sum_{i} \sigma_{i} \otimes \sum_{j=1}^{6} \iota(\phi_{i}, Q^{-1} E_{j} Q) E_{j}^{*}) = 0 \), because, up to a sign, the coefficients of \( E_{j}^{*} \) coincide with those in Eq. \((2.7)\) for \( j \neq 1, 3 \) and the coefficient of \( E_{1}^{*} \) is swapped with that of \( E_{3}^{*} \) in Eq. \((2.7)\). Hence, \( d^{*} \#(\sum_{i} \sigma_{i} \otimes Q \phi_{i} Q^{-1}) = 0 \), and then \( \delta(\sum_{i} \sigma_{i} \otimes Q \phi_{i} Q^{-1}) = 0 \), as required. \( \square \)

Combining the above result with Lemma 2.6, by definition of the metric \( g \) on \( \mathcal{GH}(S) \) we obtain the following:

**Theorem 2.8** The map \( g : \mathcal{GH}(S) \to \mathcal{GH}(S) \) is an isometry for \( g \). In particular, the Fuchsian locus, which is pointwise fixed by \( g \), is totally geodesic.

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**References**

1. Barbot, T., Béguin, F., Zeghib, A.: Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on \( \text{AdS}_3 \). Geom. Dedicata. **126**, 71–129 (2007)
2. Bridgeman, M., Canary, R., Labourie, F., Sambarino, A.: The pressure metric for Anosov representations. Geom. Funct. Anal. **25**(4), 1089–1179 (2015)
3. Bridgeman, M., Canary, R., Sambarino, A.: An introduction to pressure metrics for higher Teichmüller spaces. Ergod. Theory Dyn. Syst. **38**(6), 2001–2035 (2018)
4. Bowen, R.: Periodic orbits for hyperbolic flows. Am. J. Math. **94**, 1–30 (1972)
5. Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. Invent. Math. **29**(3), 181–202 (1975)
6. Bridgeman, M.: Hausdorff dimension and the Weil–Petersson extension to quasifuchsian space. Geom. Topol. **14**(2), 799–831 (2010)
7. Collier, B., Tholozan, N., Toulisse, J.: The geometry of maximal representations of surface groups into \( \text{SO}_0(2, n) \). Duke Math. J. **168**(15), 2873–2949 (2019)
8. Danciger, J., Guérin, F., Kassel, F.: Convex cocompactness in pseudo-Riemannian hyperbolic spaces. Geom. Dedicata **192**, 87–126 (2018)

\( \odot \) Springer
9. Dumas, D., Wolf, M.: Polynomial cubic differentials and convex polygons in the projective plane. Geom. Funct. Anal. 25(6), 1734–1798 (2015)
10. Glorieux, O., Monclair, D.: Critical exponent and Hausdorff dimension for quasi-fuchsian AdS manifolds (2016). arXiv:1606.05512
11. Goldman, W.M.: The symplectic nature of fundamental groups of surfaces. Adv. Math. 54(2), 200–225 (1984)
12. Kim, I., Zhang, G.: Kähler metric on the space of convex real projective structures on surface. J. Differ. Geom. 106(1), 127–137 (2017)
13. Li, Q.: Teichmüller space is totally geodesic in Goldman space. Asian J. Math. 20(1), 21–46 (2016)
14. Loftin, J.C.: The compactification of the moduli space of convex \( \mathbb{RP}^2 \) surfaces. I. J. Differ. Geom. 68(2), 223–276 (2004)
15. Loftin, J.: Flat metrics, cubic differentials and limits of projective holonomies. Geom. Dedicata 128, 97–106 (2007)
16. Loftin, J.: Convex \( \mathbb{RP}^2 \) structures and cubic differentials under neck separation. J. Differ. Geom. 113(2), 315–383 (2019)
17. Loftin, J., Zhang, T.: Coordinates on the augmented moduli space of convex \( \mathbb{RP}^2 \) structures (2018). arXiv:1812.11389
18. McMullen, C.T.: Thermodynamics, dimension and the Weil–Petersson metric. Invent. Math. 173(2), 365–425 (2008)
19. Mess, G.: Lorentz spacetimes of constant curvature. Geom. Dedicata 126, 3–45 (2007)
20. Ouyang, C., Tamburelli, A.: Limits of Blaschke metrics. Duke Math. J. (2019). arXiv:1911.02119 (to appear)
21. Ouyang, C.: High energy harmonic maps and degeneration of minimal surfaces (2019). arXiv:1910.06999
22. Parry, W., Pollicott, M.: Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque 187(188), 268 (1990)
23. Raghunathan, M.S.: Discrete Subgroups of Lie Groups. Springer, New York (1972). (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68)
24. Ruelle, D.: Thermodynamic Formalism, Volume 5 of Encyclopedia of Mathematics and Its Applications. Addison-Wesley Publishing Co., Reading (1978). (The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota)
25. Sambarino, A.: Quantitative properties of convex representations. Comment. Math. Helv. 89(2), 443–488 (2014)
26. Tamburelli, A.: Degeneration of globally hyperbolic maximal anti-de Sitter structures along pinching sequences. Differ. Geom. Appl. 64, 125–135 (2019)
27. Tamburelli, A.: Fenchel–Nielsen coordinates on the augmented moduli space of anti-de Sitter structures. Math. Z. (2019). arXiv:1906.03715 (to appear)
28. Tamburelli, A.: Polynomial quadratic differentials on the complex plane and light-like polygons in the Einstein universe. Adv. Math. 352, 483–515 (2019)
29. Tamburelli, A.: Degeneration of globally hyperbolic maximal anti-de Sitter structures along rays. Commun. Anal. Geom. (2020) (to appear)
30. Tamburelli, A.: Regular globally hyperbolic maximal anti-de Sitter structures. J. Topol. 13, 416–439 (2020)
31. Tamburelli, A., Wolf, M.: Planar minimal surfaces with polynomial growth in the \( \text{Sp}(4,\mathbb{R}) \)-symmetric space (2020). arXiv:2002.07295

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