PRODUCT SYSTEMS OF GRAPHS AND
THE TOEPLITZ ALGEBRAS OF HIGHER-RANK GRAPHS

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ABSTRACT. There has recently been much interest in the $C^*$-algebras of directed graphs. Here we consider product systems $E$ of directed graphs over semigroups and associated $C^*$-algebras $C^*(E)$ and $\mathcal{T}C^*(E)$ which generalise the higher-rank graph algebras of Kumjian-Pask and their Toeplitz analogues. We study these algebras by constructing from $E$ a product system $X(E)$ of Hilbert bimodules, and applying recent results of Fowler about the Toeplitz algebras of such systems. Fowler’s hypotheses turn out to be very interesting graph-theoretically, and indicate new relations which will have to be added to the usual Cuntz-Krieger relations to obtain a satisfactory theory of Cuntz-Krieger algebras for product systems of graphs; our algebras $C^*(E)$ and $\mathcal{T}C^*(E)$ are universal for families of partial isometries satisfying these relations.

Our main result is a uniqueness theorem for $\mathcal{T}C^*(E)$ which has particularly interesting implications for the $C^*$-algebras of non-row-finite higher-rank graphs. This theorem is apparently beyond the reach of Fowler’s theory, and our proof requires a detailed analysis of the expectation onto the diagonal in $\mathcal{T}C^*(E)$.

1. Introduction

The $C^*$-algebras $C^*(E)$ of infinite directed graphs $E$ are generalisations of the Cuntz-Krieger algebras which include many interesting $C^*$-algebras and provide a rich supply of models for simple purely infinite algebras (see, for example, [13, 3, 9, 19]). In the first papers, it was assumed for technical reasons that the graphs were locally finite. However, after $C^*(E)$ had been realised as the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ of a Hilbert bimodule $X(E)$ in [7], it was noticed that $\mathcal{O}_{X(E)}$ made sense for arbitrary infinite graphs. The analysis in [7] applied to the Toeplitz algebra $\mathcal{T}X(E)$ rather than $\mathcal{O}_{X(E)}$, but the two coincide for some infinite graphs $E$, and hence the results of [7] gave information about $\mathcal{O}_{X(E)}$ for these graphs. The results of [12] therefore suggested an appropriate definition of $C^*(E)$ for arbitrary $E$, which was implemented in [9].

Higher-rank analogues of Cuntz-Krieger algebras and of the $C^*$-algebras of row-finite graphs have been studied by Robertson-Steger [18] and Kumjian-Pask [11], respectively. It was observed in [8] that the higher-rank graphs of Kumjian and Pask could be viewed as product systems of graphs over the semigroup $\mathbb{N}^k$. The main object of this paper is to extend the construction $E \mapsto X(E)$ to product systems of graphs over $\mathbb{N}^k$ and other semigroups, to apply the results of [7] to the resulting product systems of Hilbert bimodules, and to see what insight might be gained into the $C^*$-algebras of arbitrary higher-rank graphs.

It is relatively easy to extend the construction of $X(E)$ to product systems, and to identify Toeplitz $E$-families which correspond to the Toeplitz representations of $X(E)$.

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studied in \[5\]. The story becomes interesting when we investigate the conditions on \(E\) and on Toeplitz \(E\)-families which ensure that we can apply \[5\] Theorem 7.2] to the corresponding representation of \(X(E)\). To understand the issues, we digress briefly.

The isometric representation theory of semigroups suggests that in general \(\mathcal{T}_{X(E)}\) will be too big to behave like a Cuntz-Krieger algebra, and that we should restrict attention to the Nica-covariant representations of \[15, 14, 4, 5\]. However, Nica covariance is in general a spatial phenomenon, and to talk about the universal \(C^\ast\)-algebra \(\mathcal{T}_{\text{cov}}(X)\) generated by a Nica-covariant Toeplitz representation of a product system \(X\) of bimodules, we need to assume that \(X\) is compactly aligned in the sense of \[4, 5\].

We identify the \textit{finitely aligned} product systems \(E\) of graphs for which \(X(E)\) is compactly aligned, and the \textit{Toeplitz-Cuntz-Krieger} \(E\)-families \(\{S_\lambda\}\) which correspond to Nica-covariant Toeplitz representations of \(X(E)\). The \(C^\ast\)-algebra generated by \(\{S_\lambda\}\) is then spanned by the products \(S_\lambda S_\mu^\ast\), as Cuntz-Krieger algebras and their Toeplitz analogues are. We therefore define the Toeplitz algebra \(\mathcal{T}C^\ast(E)\) of a finitely aligned product system \(E\) to be the universal \(C^\ast\)-algebra generated by a Toeplitz-Cuntz-Krieger \(E\)-family; for technical reasons, we only define the Cuntz-Krieger algebra \(C^\ast(E)\) to be the appropriate quotient of \(\mathcal{T}C^\ast(E)\) when \(E\) has no sinks.

Fowler’s \[3\] Theorem 7.2] gives a spatial condition under which a Nica-covariant Toeplitz representation of a compactly aligned product system \(X\) of Hilbert bimodules is faithful on \(\mathcal{T}_{\text{cov}}(X)\). Since \(\mathcal{T}C^\ast(E)\) has essentially the same representation theory as \(\mathcal{T}_{\text{cov}}(X(E))\), Fowler’s theorem describes some faithful representations of \(\mathcal{T}C^\ast(E)\). However, the resulting theorem about Toeplitz-Cuntz-Krieger \(E\)-families is not as sharp as we would like, for the same reasons that \[7, Theorem 2.1\] is not applied to the single graph \(E\) with \(\mathcal{T}C^\ast(E) = \mathcal{O}_\infty\), it says that isometries \(\{S_i\}\) satisfying \(1 > \sum_{i=1}^{\infty} S_i S_i^\ast\) generate an isomorphic copy of \(\mathcal{O}_\infty\), whereas we know from \[1\] that \(1 \geq \sum_{i=1}^{\infty} S_i S_i^\ast\) suffices. Our main theorem is sharp in this sense: it is an analogue of \[7, Theorem 3.1\] rather than \[7, Theorem 2.1\]. It suggests an appropriate set of Cuntz-Krieger relations for product systems of not-necessarily-row-finite graphs, and gives a uniqueness theorem of Cuntz-Krieger type for \(k\)-graphs in which each vertex receives infinitely many edges of each degree.

We start with a short review of the basic facts about graphs and the Cuntz-Krieger bimodule \(X(E)\) of a single graph \(E\). In \[3\] we associate to each product system \(E\) of graphs a product system \(X(E)\) of Cuntz-Krieger bimodules (Proposition 3.2]. In \[1\] we define Toeplitz \(E\)-families, and show that there is a one-to-one correspondence between such families and Toeplitz representations of \(X(E)\) (Theorem 4.2]. We then restrict attention to product systems over the quasi-lattice ordered semigroups of Nica, and identify the finitely aligned product systems \(E\) of graphs for which \(X(E)\) is compactly aligned (Theorem 5.4]. In \[4\] we discuss Nica covariance, and show that for finitely aligned systems, it becomes a familiar relation which is automatically satisfied by Cuntz-Krieger families of a single graph. By adding this relation to those of a Toeplitz family, we obtain an appropriate definition of Toeplitz-Cuntz-Krieger \(E\)-families for more general \(E\), and then \(\mathcal{T}C^\ast(E)\) is the universal \(C^\ast\)-algebra generated by such a family. We can now apply Fowler’s theorem to \(X(E)\) (Proposition 6.6], and deduce that the Fock representation of \(\mathcal{T}C^\ast(E)\) is faithful (Corollary 7.7].
Our main Theorem 8.1 is a $C^*$-algebraic uniqueness theorem. It does not appear to follow from Fowler’s results: its proof requires a detailed analysis of the expectation onto the diagonal in $T_{C^*}(E)$ and its spatial implementation, as well as an application of Corollary 7.7. In the last section, we apply Theorem 8.1 to the $k$-graphs of [11]. Our results are all interesting in this case, and those interested primarily in $k$-graphs could assume $P = N^k$ throughout the paper without losing the main points.

2. Preliminaries

2.1. Graphs and Cuntz-Krieger families. A directed graph $E = (E_0, E_1, r, s)$ consists of a countable vertex set $E_0$, a countable edge set $E_1$, and range and source maps $r, s : E_1 \to E_0$. All graphs in this paper are directed.

A Toeplitz-Cuntz-Krieger $E$-family in a $C^*$-algebra $B$ consists of mutually orthogonal projections $\{ p_v : v \in E_0 \}$ in $B$ and partial isometries $\{ s_\lambda : \lambda \in E_1 \}$ in $B$ satisfying $s_\lambda^* s_\lambda = p_{r(\lambda)}$ for $\lambda \in E_1$ and

\[ p_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^* \quad \text{for every } v \in E_0 \text{ and every finite set } F \subset s^{-1}(v). \]

It is a Cuntz-Krieger $E$-family if

\[ p_v = \sum_{\lambda \in s^{-1}(v)} s_\lambda s_\lambda^* \quad \text{whenever } s^{-1}(v) \text{ is finite and nonempty.} \]

2.2. Hilbert bimodules. Let $A$ be a $C^*$-algebra. A right-Hilbert $A-A$ bimodule (or Hilbert bimodule over $A$) is a right Hilbert $A$-module $X$ together with a left action $(a, x) \mapsto a \cdot x$ of $A$ by adjointable operators on $X$; we denote by $\phi$ the homomorphism of $A$ into $L(X)$ given by the left action. We say $X$ is essential if

\[ \text{span}\{a \cdot x : a \in A, x \in X\} = X. \]

A Toeplitz representation $(\psi, \pi)$ of a Hilbert bimodule $X$ in a $C^*$-algebra $B$ consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

\[ \psi(a \cdot x) = \psi(x)\pi(a), \quad \psi(a \cdot x) = \pi(a)\psi(x), \quad \text{and } \psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A) \]

for $x, y \in X$ and $a \in A$. There is then a unique homomorphism $\psi^{(1)} : K(X) \to B$ such that

\[ \psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for } x, y \in X; \]

see [16] page 202], [10] Lemma 2.2], or [7] Remark 1.7] for details. The representation $(\psi, \pi)$ is Cuntz-Pimsner covariant if

\[ \psi^{(1)}(\phi(a)) = \pi(a) \quad \text{whenever } \phi(a) \in K(X). \]

Pimsner associated to each Hilbert bimodule $X$ a $C^*$-algebra $T_X$ which is universal for Toeplitz representations of $X$, and a quotient $O_X$ which is universal for Cuntz-Pimsner covariant Toeplitz representations of $X$ ([16]; see also [7] §1]).
2.3. Cuntz-Krieger bimodules. The Cuntz-Krieger bimodule \( X(E) \) of a graph \( E \), as in [7, Example 1.2], consists of the functions \( x : E^1 \to \mathbb{C} \) such that
\[
\rho_x : v \mapsto \sum_{\lambda \in E^1, r(\lambda) = v} |x(\lambda)|^2
\]
vanishes at infinity on \( E^0 \). With
\[
(a \cdot x)(\lambda) := x(\lambda)a(r(\lambda)) \quad \text{and} \quad (a \cdot x)(\lambda) := a(s(\lambda))x(\lambda) \quad \text{for} \ \lambda \in E^1,
\]
\(
(\langle x, y \rangle_{C_0(E^0)}(v) := \sum_{\lambda \in E^1, r(\lambda) = v} \overline{x(\lambda)}y(\lambda) \quad \text{for} \ v \in E^0
\)
\( X(E) \) is a Hilbert bimodule over \( C_0(E^0) \). The Toeplitz representations of \( X(E) \) are in one-to-one correspondence with the Toeplitz-Cuntz-Krieger \( E \)-families via \( (\psi, \pi) \leftrightarrow \{\psi(\delta_\lambda), \pi(\delta_v)\} \) [7, Example 1.2]. Hence \( T_X(E) \) is universal for Toeplitz-Cuntz-Krieger \( E \)-families. When \( E \) has no sinks, the left action of \( C_0(E^0) \) on \( X(E) \) is faithful, the Cuntz-Pimsner covariant representations correspond to Cuntz-Krieger \( E \)-families, and the quotient \( O_X(E) \) is the usual graph \( C^* \)-algebra \( C^*(E) \).

Because of the correspondence \( (\psi, \pi) \leftrightarrow \{\psi(\delta_\lambda), \pi(\delta_v)\} \), it is convenient in calculations to work with the point masses \( \delta_\lambda \in X(E) \). The following lemma explains why this suffices.

**Lemma 2.1.** The space \( X_c(E) := C_c(E^1) \) is a dense submodule of \( X(E) \), and the point masses \( \{\delta_\lambda : \lambda \in E^1\} \) are a vector-space basis for \( X_c(E^1) \).

**Proof.** As a Banach space, \( X(E) \) is the \( c_0 \)-direct sum \( \bigoplus_{v \in E^0} \ell^2(\mathbb{Z}(r^{-1}(v))) \), and \( X_c(E) \) is the algebraic direct sum of the subspaces \( C_c(\ell^2(\mathbb{Z}(r^{-1}(v)))) \). So it is standard that \( X_c(E) \) is dense. For \( x \in X_c(E) \), we have \( x = \sum_{\lambda \in E^1} x(\lambda) \delta_\lambda \). \( \square \)

3. Product systems of graphs and of Hilbert bimodules

Throughout the next two sections, \( P \) denotes an arbitrary countable semigroup with identity \( e \). If \( E = (E^0, E^1, r_E, s_E) \) and \( F = (F^0, F^1, r_F, s_F) \) are two graphs with the same vertex set \( E^0 \), then \( E \times F \) denotes the graph with \( (E \times F)^0 := E^0 \), \( E \times F^1 := \{ \langle \lambda, \mu \rangle : \lambda \in E^1, \mu \in F^1, r_E(\lambda) = s_F(\mu) \} \), and \( s(\lambda, \mu) := s_E(\lambda), r(\lambda, \mu) := r_F(\mu) \).

We recall from [8] that a *product system* \( (E, \phi) \) of graphs over \( P \) consists of graphs \( \{(E^0_p, E^1_p, r_p, s_p) : p \in P\} \) with common vertex set \( E^0 \) and disjoint edge sets \( E^1_p \), and isomorphisms \( \phi_{p,q} : E_p \times E^0 E_q \to E_{pq} \) for \( p, q \in P \) satisfying the associativity condition
\[
\phi_{pq,r}(\phi_{p,q}(\lambda, \mu)), \nu) = \phi_{p,q,r}(\lambda, \phi_{q,r}(\mu, \nu))
\]
for all \( p, q, r \in P, (\lambda, \mu) \in (E_p \times E^0 E_q)^1 \), and \( (\mu, \nu) \in (E_q \times E^0 E_r)^1 \); we require that
\[
E_c = (E^0, E^0, \text{id}_{E^0}, \text{id}_{E^0}, E^1_c, E^1_c, \text{id}_{E^1_c}, \text{id}_{E^1_c})
\]
We write \( d(\lambda) = p \) to mean \( \lambda \in E^1_p \), because the \( E^1_p \) are disjoint, this gives a well-defined *degree map* \( d : E^1 := \bigcup_{p \in P} E^1_p \to P \), which gives the vertices \( E^0 = E^0_c \) degree \( e \). The range and source maps combine to give maps \( r, s : E^1 \to E^0 \).

The isomorphisms \( \phi_{p,q} \) in a product system \( (E, \phi) \) combine to give a partial multiplication on \( E^1 \): for \( (\lambda, \mu) \in E^1_p \times E^0 E^1_q \), we define \( \lambda \mu = \phi_{p,q}(\lambda, \mu) \in E^1_{pq} \). This multiplication
is associative by (3.1). Since each \( \varphi_{p,q} \) is an isomorphism, the multiplication has the following factorisation property: for each \( \gamma \in E^1_{pq} \), there is a unique \((\lambda, \mu) \in (E_p \times_{E^0} E_q)^1 \) such that \( \gamma = \lambda \mu \). It follows that if \( \lambda \in E^1_{pq} \), then there is a unique \( \lambda(p, pq) \in E^1_q \) such that \( \lambda = \lambda \lambda(p, pq) \lambda'' \) with \( d(\lambda') = p \) and \( d(\lambda'') = r \). By (3.1) and the factorisation property, \( s(\lambda) = \lambda = \lambda r(\lambda) \) for all \( \lambda \).

A single graph \( E \) gives a product system over \( \mathbb{N} \) in which \( E^1_n \) consists of the paths of length \( n \) in \( E \). More generally:

**Example 3.1 (k-graphs).** It is shown in [8] Examples 1.5, (4)] that the product systems of graphs over \( \mathbb{N}^k \) are essentially the same as the k-graphs of [11 Definitions 1.1]:

- Given a product system \((E, \varphi)\) of graphs over \( \mathbb{N}^k \), let \( \Lambda_E \) be the category with objects \( E^0 \) and morphisms \( E^1 \), with \( \text{dom}(\lambda) := r(\lambda) \) and \( \text{cod}(\lambda) := s(\lambda) \). The degree map is that of \( E \), the morphism \( \lambda \circ \mu \) is by definition the morphism associated to the edge \( \lambda \mu \), and the factorisation property for \( \Lambda_E \) reduces to that of \( E \).
- Given a k-graph \((\Lambda, d)\), let \((E_\Lambda)^0 := \Lambda^0, (E_\Lambda)^1 := \Lambda^n \) for \( n \in \mathbb{N}^k \), \( \lambda \mu := \lambda \circ \mu \in \Lambda^n \) whenever \((\lambda, \mu) \in (E_m \times_{E^0} E_n)^1 \), and define \( r := \text{dom} \) and \( s := \text{cod} \).

The direction of the edges is reversed in going from \((\Lambda, d)\) to \((E_\Lambda, \varphi_\Lambda)\) to ensure that the representations of the two coincide (compare Definition 2.1 with [11 Definitions 1.5]).

**Proposition 3.2.** If \((E, \varphi)\) is a product system of graphs over \( P \), then there is a unique associative multiplication on \( X(E) := \bigcup_{p \in P} X(E_p) \) such that

\[
\delta_\lambda \delta_\mu := \begin{cases} 
\delta_{\lambda \mu} & \text{if } (\lambda, \mu) \in (E_{d(\lambda)} \times_{E^0} E_{d(\mu)})^1 \\
0 & \text{otherwise},
\end{cases}
\]

and \( X(E) \) thus becomes a product system of Hilbert bimodules over \( C_0(E^0) \) as in [5] Definition 2.1].

**Remark 3.3.** We have described the multiplication using point masses because we want to use them in calculations. However, we also write it out explicitly in Corollary 3.4.

**Proof of Proposition 3.2.** It follows from Lemma 2.1 that the elements \( \delta_\lambda \otimes \delta_\mu \) are a basis for the algebraic tensor product \( X_c(E_p) \otimes X_c(E_q) \), and hence there is a well-defined linear map \( \pi : X_c(E_p) \otimes X_c(E_q) \to X_c(E_{pq}) \) such that

\[
\pi(\delta_\lambda \otimes \delta_\mu) = \begin{cases} 
\delta_{\lambda \mu} & \text{if } (\lambda, \mu) \in (E_{d(\lambda)} \times_{E^0} E_{d(\mu)})^1 \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \lambda, \mu, \eta, \xi \in E^1 \). Then

\[
\left\langle \delta_\lambda \otimes \delta_\mu, \delta_\eta \otimes \delta_\xi \right\rangle_{C_0(E^0)}(v) = \left\langle \delta_\eta, \delta_\lambda \right\rangle_{C_0(E^0)} \circ \delta_\mu \delta_\xi \right\rangle_{C_0(E^0)}(v)
\]

\[
= \begin{cases} 
1 & \text{if } \eta = \lambda, \xi = \mu, r(\lambda) = s(\mu) \text{ and } r(\mu) = v \\
0 & \text{otherwise}.
\end{cases}
\]

(3.3)
On the other hand,

\[
\langle \pi(\delta_\lambda \otimes \delta_\mu), \pi(\delta_\eta \otimes \delta_\xi) \rangle_{C_0(E^0)}(v)
\]

\[
= \begin{cases} 
\langle \delta_{\lambda \mu}, \delta_{\eta \xi} \rangle_{C_0(E^0)}(v) & \text{if } r(\lambda) = s(\mu) \text{ and } r(\eta) = s(\xi) \\
0 & \text{otherwise} \\
1 & \text{if } r(\lambda) = s(\mu), r(\eta) = s(\xi), \lambda \mu = \eta \xi \text{ and } r(\mu) = v \\
0 & \text{otherwise,}
\end{cases}
\]

which by the factorisation property is (3.3). Since \(X_c(E_p)\) is dense in \(X(E_p)\) (Lemma 2.1), it follows that \(\pi\) extends to an isometric linear isomorphism of \(X(E_p) \otimes_{C_0(E^0)} X(E_q)\) onto \(X(E_{pq})\). It is easy to check on dense subspaces \(X_c(E_p)\) and \(\text{span}\{\delta_v\} \subset C_0(E^0)\) that \(\pi\) is an isomorphism of Hilbert \(C_0(E^0)\)-bimodules. We now define \(xy := \pi(x \otimes y)\), and associativity of this multiplication follows from (3.3). More calculations on dense subspaces show that \(xa = x \cdot a\) and \(ax = a \cdot x\) for \(a \in C_0(E^0) = X(E_c)\) and \(x \in X(E_p)\).

\[\Box\]

**Corollary 3.4.** For \(x \in X(E_p)\) and \(y \in X(E_q)\), we have

\[
(xy)(\lambda \mu) = x(\lambda) y(\mu) \quad \text{for } (\lambda, \mu) \in (E_p \times E^0 E_q)^1.
\]

**Proof.** The multiplication extends to an isomorphism of \(X(E_p) \otimes_{C_0(E^0)} X(E_q)\) onto \(X(E_{pq})\), \((x, y) \mapsto x \otimes y\) is continuous, and the various evaluation maps \(z \mapsto z(\lambda)\) are continuous, so Lemma 2.1 implies that it is enough to prove (3.4) for \(x \in X_c(E_p)\) and \(y \in X_c(E_q)\). For such \(x, y\) we have

\[
(xy)(\lambda \mu) = \sum_{\alpha \in E^1_p, \beta \in E^1_q} x(\alpha) y(\beta) (\delta_\alpha \delta_\beta)(\lambda \mu),
\]

which collapses to \(x(\lambda) y(\mu)\) by the factorisation property. \[\Box\]

### 4. Representations of product systems

Throughout this section, \((E, \varphi)\) is a product system of graphs over \(P\).

**Definition 4.1.** Partial isometries \(\{s_\lambda : \lambda \in E^1\}\) in a \(C^*\)-algebra \(B\) form a **Toeplitz \(E\)-family** if:

1. \(\{s_v : v \in E^0\}\) are mutually orthogonal projections,
2. \(s_\lambda s_\mu = s_{\lambda \mu}\) for all \(\lambda, \mu \in E^1\) such that \(r(\lambda) = s(\mu)\),
3. \(s_\lambda^* s_\lambda = s_{r(\lambda)}\) for all \(\lambda \in E^1\), and
4. for all \(p \in P \setminus \{e\}, v \in E^0\) and every finite \(F \subset s_p^{-1}(v), s_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^*\).

We recall from [5] that a Toeplitz representation \(\psi\) of a product system \(X\) of bimodules consists of linear maps \(\psi_p : X_p \to B\) such that each \((\psi_p, \psi_e)\) is a Toeplitz representation of \(X_p\), and \(\psi_p(x) \psi_q(y) = \psi_{pq}(xy)\). It is Cuntz-Pimsner covariant if each \((\psi_p, \psi_e)\) is Cuntz-Pimsner covariant. Fowler proves that there is a \(C^*\)-algebra \(\mathcal{T}_X\) generated by a universal Toeplitz representation \(i_X\), and a quotient \(\mathcal{O}_X\) generated by a universal Cuntz-Pimsner covariant representation \(j_X\) [5 §2].
Theorem 4.2. Let \((E, \phi)\) be a product system of graphs over a semigroup \(P\), and let \(X(E)\) be the corresponding product system of Cuntz-Krieger bimodules. If \(\psi\) is a Toeplitz representation of \(X(E)\), then

\[
\{s_\lambda := \psi_{d(\lambda)}(\delta_\lambda) : \lambda \in E^1\}
\]

is a Toeplitz \(E\)-family; conversely, if \(\{s_\lambda : \lambda \in E^1\}\) is a Toeplitz \(E\)-family, then the map

\[
x \in C_c(E_p^1) \mapsto \sum_{\lambda \in E_p^1} x(\lambda)s_\lambda
\]

extends to a Toeplitz representation of \(X(E)\) from which we can recover \(s_\lambda = \psi_{d(\lambda)}(\delta_\lambda)\).

The representation \(\psi\) is Cuntz-Pimsner covariant if and only if \(\{s_\lambda\}\) satisfies

\[
s_v = \sum_{\lambda \in s_p^{-1}(v)} s_\lambda s_\lambda^* \text{ whenever } s_p^{-1}(v) \text{ is finite (possibly empty).}
\]

Proof. If \(\psi\) is a Toeplitz representation of \(X(E)\), then [7, Example 1.2] shows that

\[
\{\psi_c(\delta_v), \psi_p(\delta_\lambda) : v \in E^0, \lambda \in E_p^1\}
\]

is a Toeplitz-Cuntz-Krieger family for \(E_p\) as in [7], and this gives (1), (3), and (4) of Definition 4.2. Definition 4.2(2) follows from (3.2) because \(\psi\) is a homomorphism.

Now suppose that \(\psi\) is Cuntz-Pimsner covariant and \(s_p^{-1}(v)\) is finite. Write \(\phi_p : C_0(E^0) \to \mathcal{L}(X_p)\) for the homomorphism that implements the left action on \(X_p\). Then

\[
\sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^* = \sum_{\lambda \in s_p^{-1}(v)} \psi_p(1)(\Theta_{\delta_\lambda, \delta_\lambda}) = \psi_p(1)\left(\sum_{\lambda \in s_p^{-1}(v)} \Theta_{\delta_\lambda, \delta_\lambda}\right).
\]

For \(x \in X_p, w \in E^0\) and \(\mu \in E_p^1\),

\[
\left(\sum_{\lambda \in s_p^{-1}(v)} \Theta_{\delta_\lambda, \delta_\lambda}(x)\right)(\mu) = \begin{cases} x(\mu) & \text{if } \mu \in s_p^{-1}(w) \\ 0 & \text{otherwise} \end{cases} = (\delta_w \cdot x)(\mu).
\]

Hence the right hand side of (4.4) is just \(\psi_p(1)(\phi_p(\delta_v))\). Since \(\phi_p(\delta_v)\) belongs to \(\mathcal{K}(X_p)\) [7, Proposition 4.4], Cuntz-Pimsner covariance gives \(\psi_p(1)(\phi_p(\delta_v)) = \psi_c(\delta_v)\). Thus

\[
\sum_{\lambda \in s_p^{-1}(v)} s_\lambda s_\lambda^* = \sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^* = \psi_c(\delta_v) = s_v.
\]

If \(\{s_\lambda : \lambda \in E^1\}\) is a Toeplitz \(E\)-family, [7, Example 1.2] implies that \(\psi_p(\delta_\lambda) := s_\lambda\) extend to Toeplitz representations \((\psi_p, \psi_c)\) of \(X_p\) for \(p \in P\); since

\[
\psi_{pq}(\delta_\lambda \delta_\mu) = \psi_{pq}(\delta_\lambda \delta_\mu) = s_\lambda s_\mu = \psi_p(\delta_\lambda)\psi_q(\delta_\mu),
\]

it follows that \(\psi\) is a Toeplitz representation of \(X(E)\). We trivially have \(s_\lambda = \psi_{d(\lambda)}(\delta_\lambda)\).

If \(\{s_\lambda : \lambda \in E^1\}\) satisfies (4.3), then for \(p \in P\) and \(v \in E^0\) with \(s_p^{-1}(v)\) finite,

\[
\psi_p(1)(\phi_p(\delta_v)) = \psi_p(1)\left(\sum_{\lambda \in s_p^{-1}(v)} \Theta_{\delta_\lambda, \delta_\lambda}\right) = \sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^*,
\]

which is \(\psi_c(\delta_v)\) by (4.3). Proposition 4.4 of [7] ensures that \(\{\delta_v : |s_p^{-1}(v)| < \infty\}\) spans a dense subspace of \(\{a \in C_0(E^0) : \phi(a) \in \mathcal{K}(X_p)\}\), so \(\psi\) is Cuntz-Pimsner covariant. □
Corollary 4.4. Let \((E, \varphi)\) be a product system of graphs over a semigroup \(P\). Then \((\mathcal{T}_{X(E)}, i_{X(E)})\) is universal for Toeplitz \(E\)-families in the sense that

1. \(\{s_\lambda\} := \{i_{X(E)}(\delta_\lambda)\}\) is a Toeplitz \(E\)-family which generates \(\mathcal{T}_{X(E)}\); and
2. for every Toeplitz \(E\)-family \(\{s_\lambda\}\), there is a representation \(\psi_*\) of \(\mathcal{T}_{X(E)}\) such that

\[(\psi_* \circ i_{X(E)})(\delta_\lambda) = s_\lambda \text{ for every } \lambda \in E^1.\]

Similarly, \((\mathcal{O}_{X(E)}, j_{X(E)})\) is universal for Toeplitz \(E\)-families satisfying (1.3).

Proof. This follows from Theorem 4.2 and the universal properties of \(\mathcal{T}_{X(E)}\) and \(\mathcal{O}_{X(E)}\) described in \([5, \text{Propositions 2.8 and 2.9}]\).

If \((E, \varphi)\) is a product system of row-finite graphs without sinks over \(\mathbb{N}^k\), then \(\Lambda_E\) is row-finite and has no sources as in \([11]\), and the Toeplitz \(E\)-families which satisfy (1.3) are precisely the \(*\)-representations of \(\Lambda_E\). Hence:

Corollary 4.4. Let \(\Lambda\) be a row-finite \(k\)-graph with no sources as in \([11]\), define \(E_\Lambda\) as in Example \([3.7]\), and let \(X = X(E_\Lambda)\). Then there is an isomorphism of \(C^*(\Lambda)\) onto \(\mathcal{O}_X\) carrying \(s_\lambda\) to \(i_X(\delta_\lambda)\).

Remark 4.5. If there are vertices which are sinks in one or more \(E_p\), then some subtle issues arise, and the Toeplitz \(E\)-families satisfying (1.3) are not necessarily the Cuntz-Krieger \(\Lambda_E\)-families studied in \([17]\). Here, though, we care primarily about Toeplitz families, and the presence of sinks does not cause problems.

5. Compactly aligned product systems of Cuntz-Krieger bimodules

The compactly aligned product systems are a large class of product systems whose Toeplitz algebras have been analysed in \([4, 5]\) and \([11]\). To apply the results of \([3]\), we need to identify the product systems \(E\) of graphs for which \(X(E)\) is compactly aligned.

In compactly aligned product systems, the underlying semigroup \(P\) has to be quasi-lattice ordered in the sense of Nica \([15, 14]\). Suppose \(P\) is a subsemigroup of a group \(G\) such that \(P \cap P^{-1} = \{e\}\). Then \(g \leq h \iff g^{-1}h \in P\) defines a partial order on \(G\), and \(P\) is quasi-lattice ordered if every finite subset of \(G\) with an upper bound in \(P\) has a least upper bound in \(P\). (Strictly speaking, it is the pair \((G, P)\) which is quasi-lattice ordered.) If two elements \(p\) and \(q\) have a common upper bound in \(P\), \(p \vee q\) denotes their least upper bound; otherwise, we write \(p \vee q = \infty\).

Totally ordered groups, free groups, and products of these groups are all quasi-lattice ordered. The main example of interest to us is \((G, P) = (\mathbb{Z}^k, \mathbb{N}^k)\), which is actually lattice-ordered: each pair \(m, n \in \mathbb{N}^k\) has a least upper bound \(m \vee n\) with \(i\)th coordinate \((m \vee n)_i := \max\{m_i, n_i\}\).

Let \(X\) be a product system of bimodules over a quasi-lattice ordered semigroup \(P\), and suppose \(p, q \in P\) have \(p \vee q < \infty\). Since \(S \in \mathcal{L}(X_p)\) acts as an adjointable operator \(S \otimes 1\) on \(X_p \otimes_A X_{p^{-1}(p \vee q)}\), the isomorphism of \(X_p \otimes A X_{p^{-1}(p \vee q)}\) onto \(X_{p \vee q}\) induced by the multiplication gives an action of \(\mathcal{L}(X_p)\) on \(X_{p \vee q}\); we write \(S_p^{p \vee q}\) for the image of \(S \in \mathcal{L}(X_p)\), so that \(S_p^{p \vee q}\) is characterised by

\[(5.1)\]

\[S_p^{p \vee q}(xy) := (Sx)y \quad \text{for } x \in X_p, y \in X_{p^{-1}(p \vee q)}.\]

The product system \(X\) is compactly aligned \([3, \text{Definition 5.7}]\) if

\[S \in \mathcal{K}(X_p)\] and \(T \in \mathcal{K}(X_q)\) imply \((S_p^{p \vee q})(T_p^{p \vee q}) \in \mathcal{K}(X_{p \vee q})\).
When \( X = X(E) \) is a product system of Cuntz-Krieger bimodules, Lemma \( \ref{lem:compactness} \) implies that the point masses span dense subspaces of \( X(E_p) \), and the rank-one operators \( \Theta_{x,y} \) span dense subspaces of \( \mathcal{K}(X) \); thus to prove that \( X(E) \) is compactly aligned, it suffices to check that every
\[
(\Theta_{\delta_{\mu_1},\delta_{\mu_2}})^{p^\nu q}(\Theta_{\delta_{\nu_1},\delta_{\nu_2}})^{p^\nu q} \text{ belongs to } \mathcal{K}(X(E_{p^\nu q})).
\]

To prove that a given \( X(E) \) is not compactly aligned, we need to be able to recognise non-compact operators on \( X(E) \).

**Lemma 5.1.** Let \( X(E) \) be the Cuntz-Krieger bimodule of a graph, and let \( S \in \mathcal{K}(X(E)) \). Then the function \( x_S : E^1 \to \mathbb{R} \) defined by \( x_S(\lambda) := \|S(\delta_\lambda)\|_{C_0(E^0)} \) vanishes at infinity on \( E^1 \).

**Proof.** First suppose \( S = \Theta_{x,y} \) for some \( x, y \in X(E) \). Then for \( \lambda \in E^1 \), we have
\[
\|\Theta_{x,y}(\delta_\lambda)\|^2 = \sum_{r(\mu) = r(\lambda)} |x(\mu)y(\lambda)|^2 \leq |y(\lambda)|\|x\|^2;
\]
since \( y \in X(E) \subseteq C_0(E^1) \), so is \( \lambda \mapsto \|\Theta_{x,y}(\delta_\lambda)\| \). Easy calculations show that \( |x_w s + z T(\lambda)| \leq |w| |x_S(\lambda)| + |z| |x_T(\lambda)| \) and \( |x_S(\lambda)| \leq \|S\|_{\mathcal{L}(X(E))} \), so the result for arbitrary \( S \in \mathcal{K}(X(E)) \) follows by linearity and continuity. \( \square \)

**Example 5.2.** (A Cuntz-Krieger bimodule which is not compactly aligned.) Let \( (G, P) = (\mathbb{Z}^2, \mathbb{N}^2) \). Let \( E^0 := \{(0,0), (0,1), (1,0), (1,1)\} \),
\[
E_{(1,0)}^1 := \{\lambda\} \cup \{\alpha_i : i \in \mathbb{N}\}, \quad E_{(0,1)}^1 := \{\mu_i : i \in \mathbb{N}\} \cup \{\beta\},
\]

and define
\[
r(\lambda) = (1,0), \quad s(\lambda) = (0,0), \quad r(\alpha_i) = (1,1), \quad s(\alpha_i) = (0,1), \quad \text{and}
\]
\[
r(\mu_i) = (1,1), \quad s(\mu_i) = (1,0), \quad r(\beta) = (0,1), \quad s(\beta) = (0,0).
\]

By [\( \square \) Theorem 2.1], there is a unique product system \( E \) over \( \mathbb{N}^2 \) in which \( \beta \alpha_i = \lambda \mu_i \).

In pictures:
\[
E_{(1,0)} = (0,1) \xrightarrow{\alpha_1} (1,1) \quad E_{(0,1)} = (0,1) \xrightarrow{\beta} (1,1) \quad E_{(1,1)} = (0,1) \xrightarrow{\beta \alpha_i = \lambda \mu_i} (1,1)
\]

For \( S := \Theta_{\delta_{\lambda},\delta_{\lambda}} \) and \( T := \Theta_{\delta_{\beta},\delta_{\beta}} \), we can compute \( S_{(1,0)}^{(1,1)} \circ T_{(0,1)}^{(1,1)}(\delta_{\lambda \mu_i}) \) using (5.1). To evaluate \( T_{(0,1)}^{(1,1)}(\delta_{\lambda \mu_i}) \) we need to factor \( \lambda \mu_i \) as \( \beta \alpha_i \), so that \( \delta_{\lambda \mu_i} = \delta_{\beta} \delta_{\alpha_i} \). Then
\[
\begin{align*}
S_{(1,0)}^{(1,1)} \circ T_{(0,1)}^{(1,1)}(\delta_{\lambda \mu_i}) &= S_{(1,0)}^{(1,1)}(T(\delta_{\beta}) \delta_{\alpha_i}) = S_{(1,0)}^{(1,1)}(\delta_{\beta} \delta_{\alpha_i}) \\
&= S_{(1,0)}^{(1,1)}(\delta_{\lambda} \delta_{\mu_i}) = S(\delta_{\lambda}) \delta_{\mu_i} = \delta_{\lambda \mu_i}.
\end{align*}
\]
Thus \( \lambda \mu_i \mapsto \| S_{(1,0)}^{(1,1)} \circ T_{(0,1)}^{(1,1)}(\delta_{\lambda \mu_i}) \| \) does not vanish at infinity on \( E_1^{1,1} \). Lemma 5.1 therefore implies that \( S_{(1,0)}^{(1,1)} \circ T_{(0,1)}^{(1,1)} \) is not compact, and \( E \) is not compactly aligned.

To identify the \( E \) for which \( X(E) \) is compactly aligned, we legislate out the behaviour which makes Example 5.2 work. More precisely:

**Definition 5.3.** Suppose \((E, \varphi)\) is a product system of graphs over a quasi-lattice ordered semigroup \( P \), and let \( \mu \in E_p^1 \) and \( \nu \in E_q^1 \). A common extension of \( \mu \) and \( \nu \) is a path \( \gamma \) such that \( \gamma(0, p) = \mu \) and \( \gamma(0, q) = \nu \). Notice that \( d(\gamma) \) is then an upper bound for \( p \) and \( q \), so \( p \vee q < \infty \); we say that \( \gamma \) is a minimal common extension if \( d(\gamma) = p \vee q \).

We denote by \( \text{MCE}(\mu, \nu) \) the set of minimal common extensions of \( \mu \) and \( \nu \), and say that \((E, \varphi)\) is finitely aligned if \( \text{MCE}(\mu, \nu) \) is finite (possibly empty) for all \( \mu, \nu \in E^1 \).

**Theorem 5.4.** Let \((E, \varphi)\) be a product system of graphs over a quasi-lattice ordered semigroup \( P \). Then \( X(E) \) is compactly aligned if and only if \((E, \varphi)\) is finitely aligned.

**Proof.** If \( \text{MCE}(\lambda, \beta) \) is infinite for some \( \alpha \) and \( \beta \), there are infinitely many paths \( \mu_i \) and \( \alpha_i \) such that \( \lambda \mu_i = \beta \alpha_i \), and the argument of Example 5.2 shows that \( X(E) \) is not compactly aligned. Suppose that \((E, \varphi)\) is finitely aligned, \( p, q \in P \) satisfy \( p \vee q < \infty \), and \( \mu_1, \mu_2 \in E_p^1, \nu_1, \nu_2 \in E_q^1 \). Then computations like \( 5.3 \) show that \( (\Theta_{\delta_{\mu_1, \mu_2}})^{p \vee q}(\Theta_{\delta_{\nu_1, \nu_2}})^{p \vee q}(\delta_{\lambda}) = 0 \) unless \( \lambda(e, q) = \nu_2 \), and then with \( \sigma := \nu_1 \lambda(q, p \vee q) \) we have

\[
(\Theta_{\delta_{\mu_1, \mu_2}})^{p \vee q}(\Theta_{\delta_{\nu_1, \nu_2}})^{p \vee q}(\delta_{\lambda}) = \delta_{\nu_2}(\lambda(0, q))\delta_{\mu_2}(\sigma(0, p))\delta_{\mu_1 \sigma(p, p \vee q)} = \begin{cases} \delta_{\mu_1 \sigma(p, p \vee q)} & \text{if } \sigma(0, p) = \mu_2 \\ 0 & \text{otherwise.} \end{cases}
\]

Thus

\[
(\Theta_{\delta_{\mu_1, \mu_2}})^{p \vee q}(\Theta_{\delta_{\nu_1, \nu_2}})^{p \vee q} = \sum_{\sigma \in \text{MCE}(\mu_2, \nu_1)} \Theta_{\delta_{\mu_1 \sigma(p, p \vee q), \nu_2 \sigma(q, p \vee q)}}
\]

which belongs to \( \mathcal{K}(X(E)) \) because \( \text{MCE}(\mu_2, \nu_1) \) is finite.

\( \square \)

6. Nica covariance

In this section, we show that when \( X = X(E) \), Fowler’s Nica-covariance condition reduces to an extra relation for Toeplitz \( E \)-families, which will look familiar to anyone who has studied any generalisation of Cuntz-Krieger algebras. This relation automatically holds for Toeplitz-Cuntz-Krieger families of single graphs, but is not automatic for the Toeplitz families of product systems.

Suppose \( X \) is a product system of \( A \longrightarrow A \) bimodules over a quasi-lattice ordered semigroup \( P \), and \( \psi \) is a nondegenerate Toeplitz representation of \( X \) on \( \mathcal{H} \). Fowler shows in [5, Proposition 4.1] that there is an action \( \alpha^{\psi} : P \to \text{End} \psi_e(A)' \) such that

\[
(6.1) \quad \alpha_p^{\psi}(T)\psi_p(x) = \psi_p(x)T \quad \text{for } T \in \psi_e(A)' \quad \text{and} \quad \alpha_p^{\psi}(1)h = 0 \quad \text{for } h \in \psi_p(X_p)'.
\]

The representation \( \psi \) is \text{Nica covariant} if

\[
(6.2) \quad \alpha_p^{\psi}(1_p)\alpha_q^{\psi}(1_q) = \begin{cases} \alpha_{p \vee q}^{\psi}(1_{p \vee q}) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}
\]
We denote by \((\mathcal{T}_{\text{cov}}^0(X), i_X)\) the pair which is universal for Nica-covariant Toeplitz representations of \(X\) in the sense of [5, Theorem 6.3]. When \(X\) is compactly aligned, it follows from [5, Lemma 5.5 and Proposition 5.6] that the Nica covariance condition (6.2) makes sense for a representation taking values in a \(C^*\)-algebra, and then \((\mathcal{T}_{\text{cov}}^0(X), i_X)\) is universal in the usual sense of the word.

When \(P\) is the positive cone in a totally ordered group, \(p \vee q\) is either \(p\) or \(q\), and Nica covariance is automatic. Thus Toeplitz representations of a single Cuntz-Krieger bimodule \(X(E)\) are always Nica covariant. For product systems of row-finite graphs over lattice-ordered semigroups such as \(\mathbb{N}^k\), Nica covariance is a consequence of Cuntz-Pimsner covariance:

**Lemma 6.1.** Let \((E, \varphi)\) be a product system of graphs over a lattice-ordered semigroup \(P\). If every \(E_p\) is row-finite, then every Toeplitz representation of \(X(E)\) which is Cuntz-Pimsner covariant is also Nica covariant. In particular, if \(\Lambda\) is a row-finite \(k\)-graph, every Cuntz-Pimsner covariant representation of \(X(E_{\Lambda})\) is Nica covariant.

**Proof.** Since each \(E_p\) is row-finite, \(C_0(E^0)\) acts by compact operators on the left of each \(X(E_p)\) [7, Proposition 4.4], and the result follows from [5, Proposition 5.4]. \(\square\)

**Corollary 6.2.** Let \((E, \varphi)\) be a product system of row-finite graphs over a lattice-ordered semigroup \(P\). Then \(\mathcal{O}_{X(E)}\) is isomorphic to a quotient of \(\mathcal{T}_{\text{cov}}^0(X(E))\).

**Proposition 6.3.** Let \((E, \varphi)\) be a product system of graphs over a quasi-lattice ordered semigroup \(P\), and let \(\psi\) be a nondegenerate Toeplitz representation of \(X(E)\) on \(\mathcal{H}\). For \(p \in P, T \in B(\mathcal{H})\) and \(h \in \mathcal{H}\), the sum

\[
\sum_{\lambda \in E^1_p} \psi_p(\delta_{\lambda}) T \psi_p(\delta_{\lambda})^* h
\]

converges in \(\mathcal{H}\); if \(T \in \psi_{\varepsilon}(C_0(E^0))'\), it converges to \(\alpha_p^\psi(T) h\).

**Proof.** By [3, Proposition 4.1(1)], it suffices to work with a representation \((\psi, \pi)\) of a single graph \(E\), and show

1. that the sum \(\alpha(T) h := \sum_{\lambda \in E^1} \psi(\delta_{\lambda}) T \psi(\delta_{\lambda})^* h\) converges for all \(h \in \mathcal{H}\);
2. that \(\alpha(T) \in B(\mathcal{H})\) for each \(T \in B(\mathcal{H})\);
3. that \(\alpha\) is an endomorphism of \(\pi(C_0(E^0))'\); and
4. that \(\alpha\) satisfies \(\alpha(T) \psi(x) = \psi(T x)\) for \(T \in \psi_{\varepsilon}(C_0(E^0))'\), and \(\alpha(1)\mid_{(\psi(\mathcal{H}))^\perp} = 0\).

Because the \(\psi(\delta_{\lambda})\) are partial isometries with orthogonal ranges, we have

\[
\sum_{\lambda \in E^1} \|\psi(\delta_{\lambda}) T \psi(\delta_{\lambda})^* h\|^2 \leq \sum_{\lambda \in E^1} \|T\|^2 \|\psi(\delta_{\lambda})^* h\|^2 \leq \|T\|^2 \|h\|^2.
\]

Thus \(\sum_{\lambda \in E^1} \psi(\delta_{\lambda}) T \psi(\delta_{\lambda})^* h\) is a sum of orthogonal vectors which converges in \(\mathcal{H}\), and the sum satisfies

\[
\|\alpha(T) h\|^2 = \left\| \sum_{\lambda \in E^1} \psi(\delta_{\lambda}) T \psi(\delta_{\lambda})^* h \right\|^2 = \sum_{\lambda \in E^1} \|\psi(\delta_{\lambda}) T \psi(\delta_{\lambda})^* h\|^2 \leq \|T\|^2 \|h\|^2.
\]

This gives (1) and (2).
Multiplying \( \psi(\delta_\lambda)T\psi(\delta_\lambda)^* \) on either side by \( \psi(\delta_\lambda) \) gives 0 unless \( \nu = s(\lambda) \), and leaves it alone if \( \nu = s(\lambda) \). Thus each \( \psi(\delta_\lambda)T\psi(\delta_\lambda)^* \) belongs to \( \pi(C_0(E^0))' \), and so does the strong sum \( \alpha(T) \). If \( S \) and \( T \) belong to \( \pi(C_0(E^0))' \), then
\[
\psi(\delta_\lambda)S\psi(\delta_\lambda)^*T\psi(\delta_\mu)^* = \begin{cases} 
\psi(\delta_\lambda)ST\psi((\delta_\lambda, \delta_\mu)C_0(E^0))T\psi(\delta_\mu)^* & \text{if } \mu = \lambda \\
0 & \text{otherwise}
\end{cases}
\]
and it follows by taking sums and limits that \( \alpha \) is multiplicative on \( \pi(C_0(E^0))' \). It is clearly \( * \)-preserving.

For (4), we let \( T \in \psi_e(C_0(E^0))' \) and calculate:
\[
\alpha(T)\psi(\delta_\lambda) = \sum_{\mu \in E^1} \psi(\delta_\mu)T\psi(\delta_\mu)^*\psi(\delta_\lambda) = \psi(\delta_\lambda)T\pi(\delta(\lambda)) = \psi(\delta_\lambda)\pi(\delta(\lambda))T = \psi(\delta_\lambda)T.
\]
Extending by linearity gives \( \alpha(T)\psi(x) = \psi(x)T \) for \( x \in X_e(E) \), which suffices by continuity. If \( h \perp \psi(X)H \), then \( \psi(\delta_\lambda)^*h = 0 \) for all \( \lambda \), and \( \alpha(T)h = 0 \). \( \square \)

Suppose that \( \{S_\lambda\} \subset B(H) \) is a Toeplitz \( E \)-family for a product system \((E, \varphi)\) of graphs over a quasi-lattice ordered semigroup \( P \). Proposition 6.3 implies that the corresponding Toeplitz representation \( \psi \) of \( X(E) \) is Nica covariant if and only if
\[
\left( \sum_{\mu \in E^1_p} S_\mu S^*_\mu \right) \left( \sum_{\nu \in E^1_q} S_\nu S^*_\nu \right) = \begin{cases} 
\sum_{\lambda \in E^1_{p\lor q}} S_\lambda S^*_\lambda & \text{if } p \lor q < \infty \\
0 & \text{otherwise}
\end{cases}
\]
The sums in (6.3) may be infinite, and then only converge in the strong operator topology, so this is a spatial criterion rather than a \( C^* \)-algebraic one. When \( E \) is finitely aligned, however, there is an equivalent condition which only uses finite sums.

**Proposition 6.4.** Let \((E, \varphi)\) be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup \( P \), and let \( \{S_\lambda\} \subset B(H) \) be a Toeplitz \( E \)-family. The corresponding Toeplitz representation \( \psi \) of \( X(E) \) is Nica covariant if and only if, for all \( p, q \in P, \mu \in E^1_p \) and \( \nu \in E^1_q \), we have
\[
S^*_\mu S_\nu = \sum_{\alpha = \nu \beta \in MCE(\mu, \nu)} S_\alpha S^*_\beta \quad (\text{which is 0 if } p \lor q = \infty).
\]

**Proof.** First suppose \( \psi \) is Nica covariant, and let \( \mu \in E^1_p \) and \( \nu \in E^1_q \). Then because the \( S_\lambda \) corresponding to \( \lambda \) of the same degree have mutually orthogonal ranges, we have
\[
S^*_\mu S_\nu = S^*_\mu \left( \sum_{\gamma \in E^1_p} S_\gamma S^*_\gamma \right) \left( \sum_{\sigma \in E^1_q} S_\sigma S^*_\sigma \right) S_\nu
= \begin{cases} 
S^*_\mu \left( \sum_{\lambda \in E^1_{p\lor q}} S_\lambda S^*_\lambda \right) S_\nu & \text{if } p \lor q < \infty \\
0 & \text{if } p \lor q = \infty
\end{cases}
= \sum_{\alpha = \nu \beta \in MCE(\mu, \nu)} S_\alpha S^*_\beta,
\]
because \((S^*_\mu S_\lambda)(S^*_\nu S_\mu) = 0\) unless \(\lambda = \mu \alpha = \nu \beta\), and MCE(\(\mu, \nu\)) is empty if \(p \lor q = \infty\).

On the other hand, let \(p, q \in \mathcal{P}\) and suppose that (6.4) holds. Then

\[
\left( \sum_{\mu \in E^1_\mu} S_\mu^* S_\mu \right) \left( \sum_{\nu \in E^1_\nu} S_\nu^* S_\nu \right) = \sum_{\mu \in E^1_\mu, \nu \in E^1_\nu} S_\mu \left( \sum_{\mu\alpha = \nu\beta \in \text{MCE}(\mu, \nu)} S_\alpha S_\beta^* \right) S_\nu^*
\]

which is \(\sum \{ S_\lambda^* S_\lambda^* : \lambda \in E^1_{p \lor q} \}\) if \(p \lor q < \infty\) because the factorisation property implies that each \(\lambda\) appears exactly once as a \(\mu \alpha\) and as a \(\nu \beta\), and 0 if \(p \lor q = \infty\) because then each MCE(\(\mu, \nu\)) is empty. \(\square\)

7. Toeplitz-Cuntz-Krieger families

Relation (6.4) is familiar: some version of it is used in every theory of Cuntz-Krieger algebras to ensure that \(\text{span} \{ S_\mu^* S_\nu \}\) is a dense *-subalgebra of \(C^*\{(S_\mu)\}\) (see, for example, [2 Lemma 2.2], [12 Lemma 1.1], [17 Proposition 3.5]). As Lemma 6.1 shows, it is often automatic when the graphs are row-finite, but otherwise it will have to be assumed if we want \(C^*\{(S_\mu)\}\) to behave like a Cuntz-Krieger algebra.

We therefore make the following definition:

**Definition 7.1.** Let \(E\) be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup \(\mathcal{P}\). Partial isometries \(\{s_\lambda : \lambda \in E^1\}\) in a \(C^*\)-algebra \(B\) form a **Toeplitz-Cuntz-Krieger \(E\)-family** if:

1. \(\{s_v : v \in E^0\}\) are mutually orthogonal projections,
2. \(s_\lambda s_\mu = s_\lambda s_\mu\) for all \(\lambda, \mu \in E^1\) such that \(r(\lambda) = s(\mu)\),
3. \(s_\lambda^* s_\lambda = s_{r(\lambda)}\) for all \(\lambda \in E^1\),
4. for all \(p \in \mathcal{P} \setminus \{e\}, v \in E^0\) and every finite \(F \subset s_p^{-1}(v)\), \(s_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^*\),
5. \(s_\mu^* s_\nu = \sum_{\mu\alpha = \nu\beta \in \text{MCE}(\mu, \nu)} s_\alpha s_\beta^*\) for all \(\mu, \nu \in E^1\).

They form a **Cuntz-Pimsner \(E\)-family** if they also satisfy

6. \(s_v = \sum_{\lambda \in s_p^{-1}(v)} s_\lambda s_\lambda^*\) whenever \(s_p^{-1}(v)\) is finite.

**Remark 7.2.** Multiplying both sides of (5) on the left by \(s_\mu\) and on the right by \(s_\nu^*\) gives

\[
(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \sum_{\gamma \in \text{MCE}(\mu, \nu)} s_\gamma s_\gamma^*,
\]

and this is equivalent to (5) because we can get back by multiplying on the left by \(s_\mu^*\) and on the right by \(s_\nu\).

**Remark 7.3.** We have called families satisfying (6) Cuntz-Pimsner families rather than Cuntz-Krieger families because of the problems with sinks mentioned in Remark 4.5: if \(v\) is a sink in a single graph \(E\), then (6) implies that \(s_v = 0\), whereas the generally accepted Cuntz-Krieger relations impose no relation at \(v\). The Cuntz-Pimsner families are the ones which correspond to Cuntz-Pimsner covariant representations of \(X(E)\).

**Example 7.4 (The Fock representation).** For \(\lambda \in E^1\), let \(S_\lambda\) be the partial isometry on \(\ell^2(E^1)\) such that

\[
S_\lambda e_\mu := \begin{cases} e_{\lambda \mu} & \text{if } r(\lambda) = s(\mu) \\ 0 & \text{otherwise}. \end{cases}
\]
We claim that \( \{ S_\lambda : \lambda \in E^1 \} \) is a Toeplitz-Cuntz-Krieger \( E \)-family. Conditions (1)--(3) of Definition 7.1 are obvious, and (4) holds because
\[
S_v = \sum_{\lambda \in s_p^{-1}(v)} S_\lambda S_\lambda^* = e_v
\]
for all \( v \in E^0 \) and \( p \in P \setminus \{ e \} \). To verify (5), we compute on the one hand
\[
(S_\lambda S_\mu e_\nu | e_\sigma) = (S_\mu e_\nu | S_\lambda e_\sigma) = \begin{cases} 1 & \text{if } \mu \nu = \lambda \sigma \\ 0 & \text{otherwise,} \end{cases}
\]
and on the other hand,
\[
\left( \sum_{\lambda = \mu \beta \in \text{MCE}(\lambda, \mu)} S_\alpha S_\beta^* e_\nu | e_\sigma \right) = \sum_{\lambda = \mu \beta \in \text{MCE}(\lambda, \mu)} (S_\beta^* e_\nu | S_\alpha^* e_\sigma) = \sum_{\lambda = \mu \beta \in \text{MCE}(\lambda, \mu)} \begin{cases} 1 & \text{if } \nu = \beta \tau \text{ and } \sigma = \alpha \tau \text{ for some } \tau \\ 0 & \text{otherwise.} \end{cases}
\]

By the factorisation property, at most one term in this last sum can be nonzero, and there is one precisely when \( \lambda \alpha \tau = \mu \beta \tau \) for some \( \lambda \alpha = \mu \beta \in \text{MCE}(\lambda, \mu) \), giving (5).

If there is a vertex \( v \) which emits just finitely many edges in some \( E_p \), then (7.2) implies that (6) does not hold, and hence \( \{ S_\lambda \} \) is not a Cuntz-Pimsner family.

If \( (E, \varphi) \) is finitely aligned, then Theorem 4.2 and Proposition 6.2 imply that the Toeplitz \( E \)-family \( \{ i_{X(E)}(\delta_\lambda) : \lambda \in E^1 \} \) in \( \mathcal{T}_{\text{cov}}(X(E)) \) is a Toeplitz-Cuntz-Krieger \( E \)-family. It then follows from Lemma 6.1 that \( \mathcal{T}_{\text{cov}}(X(E)) \) is generated by \( \{ i_{X(E)}(\delta_\lambda) \} \). We can now apply the other direction of Theorem 4.2 to see that \( \mathcal{T}_{\text{cov}}(X(E)) \) is universal for Toeplitz-Cuntz-Krieger \( E \)-families. Thus:

**Corollary 7.5.** Let \( (E, \varphi) \) be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup \( P \). Then \( (\mathcal{T}_{\text{cov}}(X(E)), \{ i_{X(E)}(\delta_\lambda) \}) \) is universal for Toeplitz-Cuntz-Krieger \( E \)-families.

In view of Corollary 7.5, we define \( \mathcal{T}C^*(E) \) to be the universal algebra \( \mathcal{T}_{\text{cov}}(X(E)) \). If there are no sinks, we define \( C^*(E) \) to be the quotient of \( \mathcal{T}C^*(E) \) which is universal for Cuntz-Pimsner \( E \)-families. If \( \Lambda \) is a row-finite \( k \)-graph with no sources, it follows from Lemma 6.1 that \( C^*(E_\Lambda) \) is the \( C^* \)-algebra \( C^*(\Lambda) \) studied in [11].

From now on, we denote by \( \{ s_\lambda : \lambda \in E^1 \} \) the canonical generating family in \( \mathcal{T}C^*(E) \), and if \( \{ t_\lambda : \lambda \in E^1 \} \) is a Toeplitz-Cuntz-Krieger \( E \)-family in a \( C^* \)-algebra \( B \), then we write \( \pi_t \) for the homomorphism of \( \mathcal{T}C^*(E) \) into \( B \) such that \( \pi_t(s_\lambda) = t_\lambda \).

We now see what Fowler’s theory tells us about faithful representations.

**Proposition 7.6.** Let \( (G, P) \) be quasi-lattice ordered with \( G \) amenable, and let \( (E, \varphi) \) be a finitely aligned product system of graphs over \( P \). Let \( \{ S_\lambda : \lambda \in E^1 \} \) be a Toeplitz-Cuntz-Krieger \( E \)-family in \( B(\mathcal{H}) \), and suppose that for every finite subset \( R \) of \( P \setminus \{ e \} \) and every \( v \in E^0 \), we have
\[
\prod_{p \in R} \left( S_v - \sum_{\lambda \in s_p^{-1}(v)} S_\lambda S_\lambda^* \right) > 0.
\]
Then the corresponding representation \( \pi_S : \mathcal{T}C^*(E) \to B(\mathcal{H}) \) is faithful.
Proof. We consider the representation $\psi$ of $X(E)$ associated to $\{s_\lambda\}$. Theorem 5.4 says that $X(E)$ is compactly aligned, and Proposition 6.4 that $\psi$ is Nica covariant. Since the $\delta_i$ span a dense subspace of $\mathcal{C}_0(E^0)$ and the $\psi_i(\delta_i) = S_v$ are mutually orthogonal, Proposition 6.3 implies that (7.3) is equivalent to the displayed hypothesis in [5, Theorem 7.2]. Thus [5, Theorem 7.2] implies that $\psi_*$ is faithful on $\mathcal{T}_{\text{cov}}(X(E))$. But $\pi_S$ is by definition the representation $\psi_*$ of $\mathcal{T}C^*(E) := \mathcal{T}_{\text{cov}}(X(E))$. \hfill $\Box$

Corollary 7.7. Let $(G, P)$ be a quasi-lattice ordered group such that $G$ is amenable, and let $(E, \varphi)$ be a finitely aligned product system of graphs over $P$. Then the representation $\pi_S$ of $\mathcal{T}C^*(E)$ associated to the Fock representation of Example 7.4 is faithful.

Proof. Equation (7.3) follows from (7.2). \hfill $\Box$

8. A $C^*$-ALGEBRAIC UNIQUENESS THEOREM

Theorem 8.1. Let $(G, P)$ be a quasi-lattice ordered group such that $G$ is amenable, and let $(E, \varphi)$ be a finitely aligned product system of graphs over $P$. Let $\{t_\lambda : \lambda \in E^1\}$ be a Toeplitz-Cuntz-Krieger $E$-family in a $C^*$-algebra $B$. Suppose that for every finite subset $R$ of $P \setminus \{e\}$, every $v \in E^0$, and every collection of finite sets $F_p \subset s_p^{-1}(v)$, we have

$$\prod_{p \in R} \left( t_v - \sum_{\lambda \in F_p} t_\lambda t_\lambda^* \right) > 0. \tag{8.1}$$

Then the associated homomorphism $\pi_t : \mathcal{T}C^*(E) \rightarrow B$ is injective.

To prove Theorem 8.1 we first establish that there is a linear map $\Phi^E$ onto the diagonal in $\mathcal{T}C^*(E)$ which is faithful on positive elements, and show that there is a norm-decreasing linear map $\Phi^B$ on $\pi_t(\mathcal{T}C^*(E))$ such that $\pi_t \circ \Phi^E = \Phi^B \circ \pi_t$.

Proposition 8.2. There is a linear map $\Phi^E : \mathcal{T}C^*(E) \rightarrow \mathcal{T}C^*(E)$ such that

$$\Phi^E(s_\lambda s_\mu^*) = \begin{cases} s_\lambda s_\mu^* & \text{if } \lambda = \mu \\ 0 & \text{otherwise,} \end{cases}$$

and $\Phi^E$ is faithful on positive elements.

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}$, and for $i \in I$, let $P_i$ be the projection onto $\mathcal{C}e_i$. Then for $T \in B(\mathcal{H})$, $\sum_{i \in I} P_i TP_i$ converges in the strong operator topology, and $T \mapsto \sum_{i \in I} P_i TP_i$ is the diagonal map on $B(\mathcal{H})$ which takes the rank-one operator $\Theta_{e_i,e_j}$ to $\Theta_{e_i,e_j}$ if $i = j$ and to 0 otherwise. It follows that this diagonal map is linear and norm-decreasing, and it is faithful on positive elements: $\Phi(T^*T) = 0$ implies $(T^*T)e_i|e_i) = 0$ for all $i$, and hence $T = 0$.

Let $\mathcal{H} := \ell^2(E^1)$ and let $\{S_\lambda : \lambda \in E^1\}$ be the Toeplitz-Cuntz-Krieger family of Example 7.4. Then a calculation using the basis elements $\{e_\nu : \nu \in E^1\}$ shows that

$$P_\gamma S_\lambda S_\mu^* P_\gamma = \begin{cases} P_\gamma & \text{if } \lambda = \mu = \gamma(e,d(\mu)) \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $\Phi$ denotes the diagonal map on $\ell^2(E^1)$, then

$$\Phi(S_\lambda S_\mu^*) = P_{\gamma_{\text{pair}}(e_i,\lambda=\mu=\gamma(e,d(\mu)))} S_\lambda S_\mu^* = \begin{cases} S_\lambda S_\mu^* & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$
because the representation $\pi_S$ associated to the Fock representation is faithful by Corollary \ref{cor:MCE} and because $\Phi$ has the required properties, we can pull $\Phi$ back to $\mathcal{TC}^*(E)$ to get the required map $\Phi^E$. \hfill \Box

We must now establish the existence of $\Phi^B: \pi_t(\mathcal{TC}^*(E)) \to \pi_t(\mathcal{TC}^*(E))$ and show that $\pi_t$ is faithful on $\Phi^E(\mathcal{TC}^*(E))$. To do this, we analyse the structure of the diagonal $\Phi^E(\mathcal{TC}^*(E))$. Since $\mathcal{TC}^*(E)$ is spanned by elements of the form $s_\lambda s_\mu^*$, we consider the image of $\text{span}\{s_\lambda s_\mu^*: \lambda, \mu \in E^1\}$ in the diagonal. We show that for a finite subset $F$ of $E^1$, $C^*(\{t_\lambda t_\lambda^*: \lambda \in F\})$ sits inside a finite-dimensional diagonal subalgebra of $B$, and use the matrix units in this diagonal subalgebra to show that $\Phi^B$ exists and is norm-decreasing. We can then show that $\pi_t$ is faithful on $\text{span}\{s_\lambda s_\mu^*: \lambda \in E^1\}$ just by checking that the matrix units are nonzero.

Condition (5) of Definition \ref{def:MCE} shows that $C^*(\{t_\lambda t_\lambda^*: \lambda \in F\})$ is typically bigger than $\text{span}\{t_\lambda t_\lambda^*: \lambda \in F\}$; the two can only be equal if $\lambda, \mu \in F$ implies $\text{MCE}(\lambda, \mu) \subset H$. Thus we need to pass to a larger finite set $H$ such that $\lambda, \mu \in H$ imply $\text{MCE}(\lambda, \mu) \subset H$.

**Definition 8.3.** For each finite subset $F$ of $E^1$, let

$$\text{MCE}(F) := \{\lambda \in E^1: d(\lambda) = \bigvee_{\alpha \in F} d(\alpha) \text{ and } \lambda(e, d(\alpha)) = \alpha \text{ for all } \alpha \in F\},$$

and let $\forall F := \bigcup_{G \subset F} \text{MCE}(G)$.

Definition \ref{def:MCE} is consistent with Definition \ref{def:MCE} since $\text{MCE}\{\lambda, \mu\} = \text{MCE}(\lambda, \mu)$.

**Lemma 8.4.** Let $F$ be a finite subset of $E^1$. Then

1. $F \subset \forall F$;
2. $\forall F$ is the union of the disjoint sets $\forall \{\lambda \in F: s(\lambda) = v\}$ over $v \in s(F)$;
3. $\forall F$ is finite; and
4. $G \subset \forall F$ implies $\text{MCE}(G) \subset \forall F$.

**Proof.**

1. For $\lambda \in F$, $\{\lambda\} \subset F$ and $\lambda \in \text{MCE}\{\lambda\}$.

2. If $\lambda, \mu \in G$ and $s(\lambda) \neq s(\mu)$, then $\text{MCE}(G)$ is empty.

3. It suffices to show that if $F \subset E^1$ is finite, then $\text{MCE}(F)$ is finite. When $|F| = 1$, this assertion is trivial. Suppose as an inductive hypothesis that $\text{MCE}(F)$ is finite whenever $|F| \leq k$ for some $k \geq 1$, and suppose that $|F| = k + 1$. Let $\lambda \in F$, and let $F' := F \setminus \{\lambda\}$. Suppose that $\gamma \in \text{MCE}(F)$. Since $\gamma(e, \bigvee_{\alpha \in F'} d(\alpha)) \in \text{MCE}(F')$, we have $\gamma \in \text{MCE}(\lambda, \mu)$ for some $\mu \in \text{MCE}(F')$. Hence $|\text{MCE}(F)| \leq \sum_{\mu \in \text{MCE}(F')} |\text{MCE}(\lambda, \mu)|$. Each term in this sum is finite because $(E, \phi)$ is finitely aligned, and the sum has only finitely many terms by the inductive hypothesis. Hence $\text{MCE}(F)$ is finite.

4. Let $G \subset \forall F$ and for $\alpha \in G$ choose $G_\alpha \subset F$ such that $\alpha \in \text{MCE}(G_\alpha)$. Let $H := \bigcup_{\alpha \in G} G_\alpha$. We will show that $\text{MCE}(G) \subset \text{MCE}(H) \subset \forall F$. Suppose $\lambda \in \text{MCE}(G)$. Then $d(\lambda) = \bigvee_{\alpha \in G} d(\alpha) = \bigvee_{\alpha \in G} \left(\bigvee_{\beta \in H} d(\beta)\right) = \bigvee_{\beta \in H} d(\beta)$. For $\beta \in H$, choose $\alpha \in G$ such that $\beta \in G_\alpha$. Then $\lambda(e, d(\beta)) = \alpha(e, d(\beta)) = \beta$. Thus $\lambda \in \text{MCE}(H)$. \hfill \Box

It follows from Lemma \ref{lem:MCE}(4) that $\lambda, \mu \in \forall F$ implies that $\text{MCE}(\lambda, \mu) \subset \forall F$. Consequently, Lemma \ref{lem:MCE}(1) and \ref{lem:MCE} imply that

$$C^*(\{t_\lambda t_\lambda^*: \lambda \in F\}) \subset C^*(\{t_\lambda t_\lambda^*: \lambda \in \forall F\}) = \text{span}\{t_\lambda t_\lambda^*: \lambda \in \forall F\}.$$

To write this as a diagonal matrix algebra, we need to be able to orthogonalise the range projections associated to the edges in $\forall F$. 
Lemma 8.5. Let $\lambda \in E^1$. If $F \subset s^{-1}(r(\lambda))$ is finite and $r(\lambda) \not\in F$, then
\[ t_{\lambda}t_{\lambda}^{*} \left( \prod_{\mu \in F} (t_{\mu}(\lambda) - t_{\lambda \mu}t_{\lambda \mu}^{*}) \right) > 0. \]

Proof. We have
\[ \left\| t_{\lambda}t_{\lambda}^{*} \left( \prod_{\mu \in F} (t_{\mu}(\lambda) - t_{\lambda \mu}t_{\lambda \mu}^{*}) \right) \right\| = \left\| \prod_{\mu \in F} (t_{\lambda} - t_{\lambda \mu}t_{\lambda \mu}^{*}) \right\| = \left\| t_{\lambda} \left( \prod_{\mu \in F} (t_{\mu}(\lambda) - t_{\lambda \mu}t_{\lambda \mu}^{*}) \right) t_{\lambda}^{*} \right\|, \]
which is nonzero by (8.1).

We now define our matrix units. First note that (2.1) for the Toeplitz-Cuntz-Krieger family $\{t_{\lambda}\}$ implies that the range projections $t_{\lambda}t_{\lambda}^{*}$ commute with each other. Thus for every finite subset $F$ of $E^1$ and every $\lambda \in \vee F$, the operator $Q_{\lambda}^{\vee F}$ defined by
\[ Q_{\lambda}^{\vee F} := t_{\lambda}t_{\lambda}^{*} \left( \prod_{\lambda \mu \in \vee F, d(\lambda) \neq e} (t_{\lambda}(\mu) - t_{\lambda \mu}t_{\lambda \mu}^{*}) \right) \]
is a projection which commutes with every $t_{\mu}t_{\mu}^{*}$.

Proposition 8.6. Let $F$ be a finite subset of $E^1$ such that $\lambda \in F$ implies $s(\lambda) \in F$. Then $\{Q_{\lambda}^{\vee F} : \lambda \in \vee F\}$ is a collection of nonzero mutually orthogonal projections in $B$ such that $\text{span}\{Q_{\lambda}^{\vee F} : \lambda \in \vee F\} = \text{span}\{t_{\lambda}t_{\lambda}^{*} : \lambda \in \vee F\}$. In particular,
\[ \sum_{\lambda \in \vee F} Q_{\lambda}^{\vee F} = \sum_{v \in s(F)} t_{v}. \]

The key to proving Proposition 8.6 is establishing (8.2), which we do by induction on $|F|$. This requires two technical lemmas.

Lemma 8.7. Let $F$ be as in Proposition 8.6, suppose $\lambda \in F \setminus E^0$ and let $G := F \setminus \{\lambda\}$. Then for every $\gamma \in \vee F \setminus \vee G$ there is a unique $\mu_{\gamma} \in \vee G$ such that
\[ (8.3) \quad \text{if } \mu \in \vee G \text{ and } \gamma(e, d(\mu)) = \mu \text{ then } d(\mu) \leq d(\mu_{\gamma}). \]
We then have $\gamma \in \text{MCE}(\mu_{\gamma}, \lambda)$; in particular, $d(\gamma) = d(\mu_{\gamma}) \lor d(\lambda)$.

Proof. For $\gamma \in \vee F \setminus \vee G$, let $\text{v}(G)_{\gamma} := \{ \mu \in \vee G : \gamma(e, d(\mu)) = \mu \}$, which is nonempty because $s(\gamma) \in (\vee G)_{\gamma}$. For every $\mu \in (\vee G)_{\gamma}$, $d(\mu) \leq d(\gamma)$, so $d := \bigvee_{\mu \in (\vee G)_{\gamma}} d(\mu)$ satisfies $d \leq d(\gamma)$. Lemma 8.4 shows that $\gamma(e, d) \in \vee G$, and then $\mu_{\gamma} := \gamma(e, d)$ has the required property. To see that $\gamma \in \text{MCE}(\mu_{\gamma}, \lambda)$, notice that $\gamma \in \vee F \setminus \vee G$ implies $\gamma \in \text{MCE}(\mu, \lambda)$ for some $\mu \in \vee G$. Thus $\mu \in (\vee G)_{\gamma}$, $d(\mu) \leq d(\mu_{\gamma})$, and
\[ d(\gamma) = d(\mu) \lor d(\lambda) \leq d(\mu_{\gamma}) \lor d(\lambda). \]
On the other hand, we have $d(\gamma) \geq d(\mu_{\gamma})$ by definition, and $d(\gamma) \geq d(\lambda)$ since $\gamma \in \text{MCE}(\lambda, \mu)$. Hence $d(\gamma) = d(\mu_{\gamma}) \lor d(\lambda)$, and $\gamma \in \text{MCE}(\mu_{\gamma}, \lambda)$.

Lemma 8.8. Let $F$ be as in Proposition 8.6, suppose $\lambda \in F \setminus E^0$ and let $G := F \setminus \{\lambda\}$. Then for each $\delta \in \vee F \setminus \vee G$,
\[ (8.4) \quad Q_{\delta}^{\vee F} = Q_{\mu_{\delta}}^{\vee G} Q_{\delta}^{\vee F}. \]

Proof. We shall show that
(1) $Q_{\delta}^{\vee F} = Q_{\mu_{\delta}}^{\vee G} Q_{\delta}^{\vee F}$, and
\[(2) \ Q_{\mu_5}^{\nu} t_{\delta \varepsilon} t_{\delta \varepsilon} = 0 \text{ whenever } \delta \varepsilon \in \nu F \text{ and } d(\varepsilon) \neq e,\]

and then use these to prove \((8.3)\).

To prove (1), let \(\delta \in \nu F \setminus \nu G\). Since \(t_{\mu_3} t_{\mu_3}^* \geq t_{\delta} t_{\delta}^*\),

\[Q_{\mu_5}^{\nu} Q_{\delta}^{\nu} = t_{\delta} t_{\delta}^* \left( \prod_{\mu_3 \nu \in \nu G, \ d(\nu) \neq e} (t_{s(\delta)} - t_{\mu_3} t_{\mu_3}) \right) Q_{\delta}^{\nu}.\]

Suppose \(\mu_3 \nu \in \nu G\) and \(d(\nu) \neq e\). Then

\[t_{\delta} t_{\delta}^* (t_{s(\delta)} - t_{\mu_3} t_{\mu_3}) = t_{\delta} t_{\delta}^* - \sum_{\gamma \in \text{MCE}(\delta, \mu_3 \nu)} t_{\gamma} t_{\gamma}^* \text{ by (2)}.\]

Now suppose \(\gamma \in \text{MCE}(\delta, \mu_3 \nu)\). Then \(d(\mu_3) \geq d(\mu_3 \nu)\) because \(\mu_3 \nu \in \nu G\), and \(d(\mu_3 \nu) > d(\mu_3)\) because \(d(\nu) \neq e\). In particular \(\gamma \neq \delta\). But \(\gamma(e, \nu) = \delta\) because \(\gamma \in \text{MCE}(\delta, \mu_3 \nu)\). Hence there exists \(\varepsilon \in E^1\) such that \(d(\varepsilon) \neq e\) and \(\gamma = \delta \varepsilon\). Since \(\delta\) and \(\mu_3 \nu\) are in \(\nu F\), Lemma \((8.4)\) ensures that \(\gamma \in \nu F\), so \(t_{s(\delta)} - t_{\gamma} t_{\gamma}^*\) is a factor in \(Q_{\delta}^{\nu}\), and \(t_{\gamma} t_{\gamma}^* Q_{\delta}^{\nu} = 0\). Thus

\[t_{\delta} t_{\delta}^* (t_{s(\delta)} - t_{\mu_3} t_{\mu_3}) = t_{\delta} t_{\delta}^* Q_{\delta}^{\nu} - \left( \sum_{\gamma \in \text{MCE}(\delta, \mu_3 \nu)} t_{\gamma} t_{\gamma}^*\right) Q_{\delta}^{\nu} = Q_{\delta}^{\nu}.\]

Applying this equation to each \(\mu_3 \nu \in \nu G\) with \(d(\nu) \neq e\) establishes (1).

To prove (2), suppose that \(\delta \varepsilon \in \nu F\) with \(d(\varepsilon) \neq e\). Then \(\mu_3 \varepsilon \in \nu G\), and \(\mu_3 \varepsilon \neq \mu_3\): if \(\mu_3 \varepsilon = \mu_3\), then \(d(\delta \varepsilon) = d(\lambda) \lor d(\mu_3 \varepsilon) = d(\lambda) \lor d(\mu_3) = d(\delta)\), contradicting \(d(\varepsilon) \neq e\). However, \((\delta \varepsilon)(e, \mu_3) = d(\varepsilon) \neq e\), so Lemma \((8.4)\) implies that \(d(\mu_3) < d(\mu_3 \varepsilon)\), and \(\mu_3 = \mu_3 \alpha\) for some \(\alpha\) with \(d(\alpha) \neq e\). Since \(\mu_3 \varepsilon \in \nu G\), it follows that

\[Q_{\mu_3}^{\nu} t_{\delta \varepsilon} t_{\delta \varepsilon}^* \leq (t_{s(\mu_3)} - t_{\mu_3} t_{\mu_3}^*) t_{\delta \varepsilon} t_{\delta \varepsilon}^*,\]

which vanishes because \(\mu_3 \alpha = (\delta \varepsilon)(e, d(\mu_3 \varepsilon))\). This gives (2).

To finish off, we compute:

\[Q_{\delta}^{\nu} = Q_{\mu_5}^{\nu} Q_{\delta}^{\nu} \text{ by (1)} = Q_{\mu_5}^{\nu} \left( \prod_{\delta \in \nu F, d(\varepsilon) \neq e} (t_{s(\mu_3)} - t_{\delta \varepsilon} t_{\delta \varepsilon}^*) \right) t_{\delta} t_{\delta}^* = Q_{\mu_5}^{\nu} t_{\delta} t_{\delta}^* \text{ by (2)}.\]

**Proof of Proposition** \((8.4)\). The \(Q_{\lambda}^{\nu}\) are nonzero by Lemma \((8.3)\). To see that the \(Q_{\lambda}^{\nu}\) are orthogonal, suppose that \(\lambda \neq \mu \in \nu F\). If \(d(\lambda) = d(\mu)\) then \(Q_{\lambda}^{\nu} Q_{\mu}^{\nu} \leq t_{\lambda} t_{\mu}^* t_{\mu} t_{\mu}^* = 0\) by (4) of Definition \((1)\). So suppose that \(d(\lambda) \neq d(\mu)\). We can assume without loss of generality that \(d(\lambda) \lor d(\mu) > d(\lambda)\). Then \(\gamma \in \text{MCE}(\lambda, \mu)\) implies \(\gamma = \lambda \alpha\) where \(d(\alpha) \neq e\), and \(\gamma \in \nu F\) by Lemma \((8.4)\). Thus \((1)\) shows that

\[Q_{\lambda}^{\nu} Q_{\mu}^{\nu} \leq \left( \sum_{\gamma \in \text{MCE}(\lambda, \mu)} t_{\gamma} t_{\gamma}^*\right) Q_{\lambda}^{\nu} = 0.\]
Assuming that (8.2) has been established, let \( \lambda \in \forall F \) and calculate:

\[
t_{\lambda}t_{\lambda}^* = t_{\lambda}t_{\lambda}^* \left( \sum_{\mu \in \forall F} Q_{\mu}^{V_F} \right) \quad \text{by (8.2)}
\]

\[
= \sum_{\mu \in \forall F} \left( t_{\lambda}t_{\lambda}^* t_{\mu}^* \left( \prod_{\mu \alpha \in \forall F, d(\alpha) \neq e} (t_{s(\mu)} - t_{\mu \alpha}t_{\mu \alpha}^*) \right) \right)
\]

(8.5)

\[
= \sum_{\mu \in \forall F} \left( \left( \sum_{\gamma \in \text{MCE}(\lambda, \mu)} t_{\gamma}t_{\gamma}^* \right) \left( \prod_{\mu \alpha \in \forall F, d(\alpha) \neq e} (t_{s(\mu)} - t_{\mu \alpha}t_{\mu \alpha}^*) \right) \right).
\]

Suppose \( \mu \in \forall F \) and \( \mu \neq \lambda \lambda' \) for any path \( \lambda' \), and that \( \gamma \in \text{MCE}(\lambda, \mu) \). Lemma 8.4) ensures that \( \gamma \in \forall F \), and \( \gamma \neq \mu \) because \( \mu \neq \lambda \lambda' \). Thus \( \gamma = \mu \alpha \) for some path \( \alpha \) such that \( d(\alpha) \neq e \). Hence the product in (8.5) vanishes for such \( \mu \), and (8.5) collapses to

\[
t_{\lambda}t_{\lambda}^* = \sum_{\lambda' \in \forall F} Q_{\lambda \lambda'}^{V_F}.
\]

It therefore suffices to establish (8.2). Indeed, \( Q_{\lambda}^{V_F} \leq s(\lambda) \) for all \( \lambda \), so Lemma 8.4(2) shows that it suffices to establish (8.2) when \( F \subseteq s^{-1}(v) \) for some \( v \in E^0 \). We do this by induction on \( |F| \). Recall that \( \lambda \in F \) implies \( s(\lambda) \in F \), so if \( |F| = 1 \) then \( F = \forall F = \{v\} \) and \( Q_v^{V_F} = t_v \).

Suppose that \( |F| = k + 1 \geq 2 \), and that the proposition holds for all subsets of \( s^{-1}(v) \) containing \( v \) and having at most \( k \) elements. Since \( |F| > 1 \) there exists \( \lambda \neq v \) in \( F \). Let \( G := F \setminus \{\lambda\} \). For \( \mu \in \forall G \), we have

\[
Q_{\mu}^{V_F} = t_{\mu}t_{\mu}^* \left( \prod_{\mu \alpha \in \forall G, d(\alpha) \neq e} (t_v - t_{\mu \alpha}t_{\mu \alpha}^*) \right) \left( \prod_{\gamma = \mu \beta \in \forall F \setminus V_G} (t_v - t_{\gamma}t_{\gamma}^*) \right).
\]

Suppose that \( t_v - t_{\gamma}t_{\gamma}^* \) is a factor in the second product and \( \mu \gamma \neq \mu \). Then \( \mu \gamma = \mu \alpha \) for some \( \alpha \) such that \( d(\alpha) \neq e \) because \( \mu \gamma \) is the maximal subpath of \( \gamma \) in \( \forall G \). Thus \( t_v - t_{\gamma}t_{\gamma}^* \) is larger than the factor \( t_v - t_{\mu \alpha}t_{\mu \alpha}^* \) from the first product. So such terms in the second product can be deleted without changing the product, and we have

\[
Q_{\mu}^{V_F} = Q_{\mu}^{V_G} \left( \prod_{\gamma \in V_F \setminus V_G, \mu \gamma = \mu} (t_v - t_{\gamma}t_{\gamma}^*) \right).
\]

Thus

\[
\sum_{\lambda \in \forall F} Q_{\lambda}^{V_F} = \sum_{\mu \in \forall G} Q_{\mu}^{V_G} \left( \prod_{\gamma \in V_F \setminus V_G, \mu \gamma = \mu} (t_v - t_{\gamma}t_{\gamma}^*) \right) + \sum_{\delta \in \forall F \setminus V_G} Q_{\delta}^{V_F}
\]

\[
= \sum_{\mu \in \forall G} \left( Q_{\mu}^{V_G} \left( \prod_{\gamma \in V_F \setminus V_G, \mu \gamma = \mu} (t_v - t_{\gamma}t_{\gamma}^*) \right) + \sum_{\delta \in \forall F \setminus V_G, \mu \delta = \mu} Q_{\delta}^{V_F} \right)
\]

by Lemma 8.7 and Lemma 8.8 gives

\[
\sum_{\lambda \in \forall F} Q_{\lambda}^{V_F} = \sum_{\mu \in \forall G} \left( Q_{\mu}^{V_G} \left( \prod_{\gamma \in V_F \setminus V_G, \mu \gamma = \mu} (t_v - t_{\gamma}t_{\gamma}^*) \right) + \sum_{\delta \in \forall F \setminus V_G, \mu \delta = \mu} Q_{\mu}^{V_G} t_{\delta}t_{\delta}^* \right)
\]

(8.6)

\[
= \sum_{\mu \in \forall G} Q_{\mu}^{V_G} \left( \prod_{\gamma \in V_F \setminus V_G, \mu \gamma = \mu} (t_v - t_{\gamma}t_{\gamma}^*) \right) + \sum_{\delta \in \forall F \setminus V_G, \mu \delta = \mu} t_{\delta}t_{\delta}^*.
\]
If \( \mu \in \forall G \) and \( \delta \in \forall F \setminus \forall G \) satisfies \( \mu_\delta = \mu \), then Lemma 8.6 implies that \( d(\delta) = d(\mu) \vee d(\lambda) \). Thus \( \{ t_\delta t_\delta^* : \mu_\delta = \mu \} \) are mutually orthogonal, and (8.6) is just \( \sum_{\mu \in \forall G} Q^G_\mu \).

Applying the inductive hypothesis to \( G \) now establishes (8.2) for the given \( F \). \( \square \)

**Proposition 8.9.** There is a norm-decreasing linear map

\[
\Phi^F : C^*(\{ t_\lambda : \lambda \in E^1 \}) \to \overline{\text{span}}\{ t_\lambda t_\lambda^* : \lambda \in E^1 \}
\]

such that \( \Phi^F \circ \pi_t = \pi_t \circ \Phi^F \).

**Proof.** It suffices to show that if \( F \subset E^1 \) is finite and \( \{ \alpha_{\lambda,\mu} : \lambda, \mu \in F \} \subset \mathbb{C} \), then

\[
\| \sum_{\lambda,\mu \in F} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \| \geq \| \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \|.
\]

Since \( \sum_{\gamma \in F} Q^F_\gamma = \sum_{v \in s(F)} t_v \) and the \( Q^F_\gamma \) commute with the \( t_\lambda t_\lambda^* \), there exists \( \gamma \in \forall F \) such that

\[
\| Q^F_\gamma \left( \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \right) \| = \| \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \|.
\]

If \( \lambda \in F \) and \( \gamma \neq \lambda \beta \) for any \( \beta \), then \( \delta \in \text{MCE}(\lambda, \gamma) \) implies \( d(\delta) > d(\gamma) \), giving

\[
Q^F_\gamma t_\lambda = Q^F_\gamma t_\lambda t_\lambda^* t_\lambda = \left( \prod_{\gamma \beta \in \forall F, d(\beta) \neq e} \left( t_\beta t_\beta^* - t_\gamma \beta^* t_\gamma^* \right) \right) \left( \sum_{\delta \in \text{MCE}(\gamma, \lambda)} t_\delta t_\delta^* \right) t_\lambda = 0.
\]

Thus

\[
Q^F_\gamma \left( \sum_{\lambda,\mu \in F} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \right) Q^F_\gamma = Q^F_\gamma \left( \sum_{\lambda,\mu \in F} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \right) Q^F_\gamma.
\]

In particular, notice that for \( \lambda \in \forall F \),

\[
Q^F_\gamma t_\lambda t_\lambda^* = \begin{cases} Q^F_\gamma & \text{if } d(\gamma) \geq d(\lambda) \text{ and } \gamma(e, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases}
\]

We will replace \( Q^F_\gamma \) with a smaller nonzero projection \( Q_\gamma \) so that the remaining off-diagonal terms are eliminated. Since \( 0 < Q_\gamma \leq Q^F_\gamma \), we will then have

\[
Q_\gamma t_\lambda t_\lambda^* = \begin{cases} Q_\gamma & \text{if } d(\gamma) \geq d(\lambda) \text{ and } \gamma(e, d(\lambda)) = \lambda \\ 0 & \text{otherwise,} \end{cases}
\]

which, in conjunction with (8.8), will imply that

\[
\| Q_\gamma \left( \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \right) \| = \| \sum_{\lambda \in F, d(\lambda) \leq d(\gamma), \gamma(e, d(\lambda)) = \lambda} \alpha_{\lambda,\lambda} \| = \| Q^F_\gamma \left( \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \right) \|.
\]

To produce \( Q_\gamma \), we consider pairs \( \lambda, \mu \in \forall F \) such that \( \gamma(e, d(\lambda)) = \lambda \) and \( \gamma(e, d(\mu)) = \mu \). For each such \( (\lambda, \mu) \), factorise \( \gamma \) as \( \lambda \lambda' = \gamma = \mu \mu' \), and define

\[
d_\gamma(\lambda, \mu) := \{ \sigma : \sigma = d(\lambda', d(\delta)) \text{ or } \sigma = d(\mu', d(\delta)) \text{ for some } \delta \in \text{MCE}(\lambda', \mu') \}.
\]

Now \( \lambda' \) and \( \mu' \) are uniquely determined by \( \lambda, \mu \) and \( \gamma \), each \( \text{MCE}(\lambda', \mu') \) is finite, and \( d(\lambda', d(\delta)) \) and \( d(\mu', d(\delta)) \) are uniquely determined by \( \delta \in \text{MCE}(\lambda', \mu') \), so each
\( d_\gamma(\lambda, \mu) \) is finite. Let
\[
Q_\gamma := Q_\gamma^F \prod_{\lambda \neq \mu \in \forall F, (\gamma, \delta(\lambda)) = \lambda, (\gamma, \delta(\mu)) = \mu, \sigma \in d_\gamma(\lambda, \mu)} (t_\gamma t_\sigma^* - t_\gamma \sigma t_{\gamma \sigma}^*).
\]

Lemma 8.5 implies \( Q_\gamma > 0 \), and \( Q_\gamma \leq Q_\gamma^F \) by definition, so we have (8.9) and (8.10).

For \( \lambda, \mu \in \forall F \) with \( \lambda \lambda = \gamma = \mu \mu \) and \( \lambda \neq \mu \), we calculate:
\[
Q_\gamma t_\lambda t_\mu^* Q_\gamma = Q_\gamma (t_\lambda(t_\lambda^* t_\mu t_\mu^*) t_\mu^*) Q_\gamma
\]
\[
= Q_\gamma \left( t_\lambda \left( \sum_{\nu \in MCE(\lambda', \mu')} t_\nu^* \right) t_\mu^* \right) Q_\gamma,
\]
which vanishes because \( \nu \in MCE(\lambda', \mu') \) implies that \( \lambda \nu = \gamma \sigma \) for some \( \sigma \in d_\gamma(\lambda, \mu) \).

Thus
\[
\left\| \sum_{\lambda, \mu \in F} \alpha_{\lambda, \mu} t_\lambda^* t_\mu \right\| \geq \left\| Q_\gamma \left( \sum_{\lambda, \mu \in F} \alpha_{\lambda, \mu} t_\lambda^* t_\mu \right) Q_\gamma \right\|
\]
\[
= \left\| Q_\gamma \left( \sum_{\lambda \in F} \alpha_{\lambda, \gamma} t_\lambda^* \right) Q_\gamma \right\|
\]
\[
= \left\| Q_\gamma^F \left( \sum_{\lambda \in F} \alpha_{\lambda, \gamma} t_\lambda^* \right) \right\| \quad \text{by (8.11)}
\]
\[
= \left\| \sum_{\lambda \in F} \alpha_{\lambda, \gamma} t_\lambda^* \right\| \quad \text{by (8.7)}. \quad \square
\]

**Proof of Theorem 8.7** It suffices to show that if \( F \) is a finite subset of \( E^1 \) and
\[
a = \sum_{\lambda, \mu \in F} \alpha_{\lambda, \mu} s_\lambda s_\mu^*,
\]
then \( \pi_1(a) = 0 \) implies \( a = 0 \). Suppose \( \pi_1(a) = 0 \). Then \( \pi_1(a^* a) = 0, \Phi^B(\pi_1(a^* a)) = 0, \) and Proposition 8.9 implies that \( \pi_1(\Phi^E(a^* a)) = 0 \). Now \( \Phi^E(a^* a) \) belongs to \( D := \text{span}\{s_\lambda s_\lambda^* : \lambda \in \forall F\} \), and applying Proposition 8.6 to the universal Toeplitz-Cuntz-Krieger \( E \)-family \( \{s_\lambda\} \) shows that \( D \) is a finite-dimensional diagonal matrix algebra with matrix units
\[
\{e_{\lambda, \lambda} := s_\lambda s_\lambda^* \prod_{\lambda \sigma \in \forall F, \delta(\lambda) \neq \delta(\sigma)} (s_\lambda - s_\lambda s_\sigma^*) : \lambda \in \forall F\}.
\]

Lemma 8.5 implies that \( \pi_1(e_{\lambda, \lambda}) \neq 0 \) for \( \lambda \in \forall F \), so \( \pi_1 \) is faithful on \( D \). In particular \( \|\Phi^E(a^* a)\| = \|\pi_1(\Phi^E(a^* a))\| = 0 \). Proposition 8.2 now shows that \( a^* a = 0 \), and hence \( a = 0 \). \quad \square

9. The \( C^* \)-algebra of an infinite \( k \)-graph

We show how the finitely-aligned hypothesis, relation (5) of Definition 7.1 and the hypothesis (8.1) in Theorem 8.1 all simplify when the underlying semigroup is \( \mathbb{N}^k \). We then prove a uniqueness theorem for the \( C^* \)-algebras of \( k \)-graphs in which every vertex receives infinitely many paths of every degree.
9.1. Product systems of graphs over $\mathbb{N}^k$.

**Lemma 9.1.** Let $(E, \varphi)$ be a product system of graphs over $\mathbb{N}^k$. Then $(E, \varphi)$ is finitely aligned if and only if

\[
\text{(9.1)} \quad \text{MCE}(\mu, \nu) \text{ is finite for every pair } \mu \in E^1_{e_i} \text{ and } \nu \in E^1_{e_j} \text{ with } i \neq j.
\]

**Proof.** Every finitely aligned system trivially satisfies (9.1). For the reverse implication, suppose $E$ satisfies (9.1). Then MCE($\mu, \nu$) is finite whenever $|d(\mu) \lor d(\nu)| \leq 2$. Suppose as an inductive hypothesis that MCE($\mu, \nu$) is finite whenever $|d(\mu) \lor d(\nu)| \leq n$, and consider $\mu \in E^1_p$, $\nu \in E^1_q$ with $|p \lor q| = n + 1$.

If the coordinate-wise minimum $p \land q$ of $p$ and $q$ is nonzero, then either $\mu(0, p \land q) \neq \nu(0, p \land q)$, in which case the factorisation property implies MCE($\mu, \nu$) = $\emptyset$, or

\[
\text{MCE}(\mu, \nu) = \{ \mu(0, p \land q) \gamma : \gamma \in \text{MCE}(\mu(p \land q), \nu(p \land q, q)) \}
\]

is finite by the inductive hypothesis. Thus we may assume that $p \land q = 0$, and hence that $p \lor q = p + q$. If $p \geq q$ or $q \geq p$ then MCE($\mu, \nu$) has at most one element. So we may further assume that there exist $i \neq j$ such that $p_i > q_i$ and $q_j > p_j$. Since $p \land q = 0$, this implies that $p_j = q_i = 0$.

Now let $\gamma \in \text{MCE}(\mu, \nu)$. Then $d(\gamma) - e_i = p + q - e_i = (p - e_i) \lor q$ since $q_i = 0$. Thus $\gamma_i := \gamma(0, d(\gamma) - e_i)$ satisfies

\[
\gamma_i(0, p - e_i) = \gamma(0, p - e_i) = \mu(0, p - e_i) \quad \text{and} \quad \gamma_i(0, q) = \gamma(0, q) = \nu,
\]

so $\gamma_i \in \text{MCE}(\mu(0, p - e_i), \nu)$. Similarly, $\gamma_j := \gamma(0, d(\gamma) - e_j) \in \text{MCE}(\mu, \nu(0, q - e_j))$. But now $p \lor q = d(\gamma_i) + e_i = d(\gamma_j) + e_j$, and since $i \neq j$, it follows that $d(\gamma) = d(\gamma_i) \lor d(\gamma_j)$. Furthermore, $\gamma(0, d(\gamma_i)) = \gamma_i$ and $\gamma(0, d(\gamma_j)) = \gamma_j$, so $\gamma \in \text{MCE}(\gamma_i, \gamma_j)$. Hence

\[
|MCE(\mu, \nu)| \leq \sum_{\gamma_i \in \text{MCE}(\mu(0, p - e_i), \nu)} |MCE(\gamma_i, \gamma_j)|.
\]

By the inductive hypothesis, MCE($\mu(0, p - e_i), \nu$) and MCE($\mu, \nu(0, q - e_j)$) are finite, so the sum has only finitely many terms. Thus we take $\gamma_i \in \text{MCE}(\mu(0, p - e_i), \nu)$ and $\gamma_j \in \text{MCE}(\mu, \nu(0, q - e_j))$, and show that MCE($\gamma_i, \gamma_j$) is finite. If it is nonempty, then the initial segments of degree $(p \lor q) - e_i - e_j$ of $\gamma_i$ and $\gamma_j$ are the same; call it $\beta$, and write $\gamma_i = \beta \gamma_i', \gamma_j = \beta \gamma_j'$. Then $d(\gamma_i') = e_i$ and $d(\gamma_j') = e_j$, so $|\text{MCE}(\gamma_i, \gamma_j)| = |\text{MCE}(\gamma_i', \gamma_j')|$ is finite by (9.1). \hfill $\square$

**Lemma 9.2.** Let $(E, \varphi)$ be a finitely aligned product system of graphs over $\mathbb{N}^k$. Then a Toeplitz $E$-family $\{t_\lambda\}$ is a Toeplitz-Cuntz-Krieger $E$-family if and only if

\[
(9.2) \quad t^*_\mu t^*_\nu = \sum_{\mu \alpha = \nu \beta \in \text{MCE}(\mu, \nu)} t^*_\alpha t^*_\beta
\]

for every $\mu \in E^1_{e_i}$ and $\nu \in E^1_{e_j}$ with $s(\mu) = s(\nu)$ and $i \neq j$.

**Proof.** Since (9.2) is a special case of Definition (7.1), we have to show that (9.2) implies Definition (7.1). If $|d(\mu) \lor d(\nu)| \leq 2$, this is trivially true. Suppose as an inductive hypothesis that (6.4) holds whenever $|d(\mu) \lor d(\nu)| \leq n$ for some $n \geq 2$. Suppose $\mu \in E^1_p$ and $\nu \in E^1_q$ where $p$ and $q$ satisfy $|p \lor q| = n + 1$. We give separate arguments for $p \land q \neq 0$ and $p \land q = 0$. 
If $p \land q \neq 0$, then

$$t^*_\mu t^*_\nu = t^*_\mu(p \land q) t^*_\nu(p \land q) = \begin{cases} t^*_\mu(p \land q) t^*_\nu(p \land q) & \text{if } \mu(0, p \land q) = \nu(0, p \land q) \\ 0 & \text{otherwise.} \end{cases}$$

(9.3)

The set $\text{MCE}(\mu, \nu)$ is empty unless $\mu(0, p \land q) = \nu(0, p \land q)$, and if so we have

$$\text{MCE}(\mu, \nu) = \{\mu(0, p \land q) : \gamma \in \text{MCE}(\mu(p \land q), \nu(p \land q))\}.$$ Applying the inductive hypothesis to (9.3) gives Definition (7.5).

Now suppose $p \land q = 0$, or equivalently that $p \lor q = p + q$. Since $|p \lor q| \geq 2$, we can assume that $|q| \geq 2$. If $p \geq q$ then (6.3) is trivial, so we may further assume that there exists $i$ such that $q_i > p_i$, and then $p \land q = 0$ forces $p_i = 0$. In particular, $|p \lor (q - e_i)| = n$, and the inductive hypothesis gives

$$t^*_\mu t^*_\nu = t^*_\mu(0, q - e_i) t^*_\nu(q - e_i) = \left( \sum_{\mu \delta = 0(0, q - e_i) \in \text{MCE}(\mu, \nu(0, q - e_i))} t^*_\mu t^*_\nu \right).$$

Each $\varepsilon$ appearing in this sum has $d(\varepsilon) = p$, so $d(\varepsilon) \lor d(\nu(q - e_i), q) = p + e_i$, which has length at most $n$ because $|q| \geq 2$. Thus we can apply the inductive hypothesis to each summand to get

$$t^*_\mu t^*_\nu = \sum_{\mu \delta = 0(0, q - e_i) \in \text{MCE}(\mu, \nu(0, q - e_i))} t^*_\mu t^*_\nu \circ MCE(\mu, \nu(0, q - e_i)).$$

(9.4)

It remains to show that the pairs $(\delta \sigma, \tau)$ arising in this sum are precisely the pairs $(\alpha, \beta)$ arising in the right-hand side of (6.4). Given $(\delta \sigma, \tau)$, we certainly have

$$\mu \delta \sigma = \nu(0, q - e_i), \varepsilon \sigma = \nu(0, q - e_i), \nu(q - e_i), \tau = \nu \tau,$$

and $d(\delta \sigma) = d(\delta) + d(\sigma) = q - e_i + e_i = q$, so $\mu \delta \sigma \in \text{MCE}(\mu, \nu)$. Conversely, given $(\alpha, \beta)$, we take $\delta := \alpha(0, q - e_i)$, $\sigma := \alpha(q - e_i, q)$ and $\tau := \beta$. \hfill $\Box$

**Lemma 9.3.** Let $E$ be a finitely aligned product system of graphs over $\mathbb{N}^k$. Then a Toeplitz $E$-family $\{t_\lambda\}$ satisfies (8.1) if and only if

$$\prod_{m=1}^k \left( t_\nu - \sum_{\lambda \in G_m} t^*_\lambda \right) > 0$$

(9.5)

for every choice of finite sets $G_m \subset s_\epsilon^{-1}(v)$.

**Proof.** The necessity of (9.5) is obvious. Suppose (9.5) holds, and $R$, $v$ and $F_p$ are as in Theorem 8.1. For $p \in R$, choose $i_p$ such that $p_{i_p} > 0$, and for each $m$, set

$$G_m := \bigcup_{\{p \in R : i_p = m\}} \{\lambda(0, \epsilon) : \lambda \in F_p\}.$$
Then each $G_m$ is a finite subset of $s^{-1}(v)$, and
\[
\prod_{p \in R} \left( t_v - \sum_{\lambda \in F_p} t_{t_{\Lambda}} \right) \geq \prod_{p \in R} \left( t_v - \sum_{\lambda \in F_p} t_{\lambda(0,e_p)} t_{\lambda(0,e_p)^*} \right) = \prod_{m=1}^k \left( t_v - \sum_{\mu \in G_m} t_{\mu t_{\mu}^*} \right),
\]
which is nonzero by (9.5). \hfill \square

9.2. The $C^*$-algebra of an infinite $k$-graph. If $(\Lambda, d)$ is a $k$-graph, and $\lambda, \mu \in \Lambda$, we regard $\text{MCE}(\lambda, \mu) \subset (E_\Lambda)^1$ as a subset of $\Lambda$. In view of Lemma 9.2 we say that $\Lambda$ is finitely aligned if $\text{MCE}(\lambda, \mu)$ is finite whenever $d(\lambda) = e_i$ and $d(\mu) = e_j$. By a Toeplitz-Cuntz-Krieger $\Lambda$-family we mean a Toeplitz-Cuntz-Krieger $E_\Lambda$-family. If $\Lambda$ has no sources, so that the graphs in $E_\Lambda$ have no sinks, then we define a Cuntz-Krieger $\Lambda$-family to be a Cuntz-Pimsner $E_\Lambda$-family. We have only made this last definition for $k$-graphs without sources to avoid clashing with the definitions given for row-finite graphs in \[17\]; for row-finite $k$-graphs without sources, therefore, our $C^*(E_\Lambda)$ coincides with the graph algebra $C^*(\Lambda)$ used in \[11\] and \[17\].

Recall that $\Lambda^e(v) := \{ \lambda \in \Lambda : d(\lambda) = n \text{ and } \text{cod}(\lambda) = v \}$. If $|\Lambda^e(v)| = \infty$ for every $v \in \Lambda^0$, and every $1 \leq i \leq k$, then conditions (6) and (4) of Definition \[11\] are equivalent, so Theorem 8.1 gives a uniqueness theorem for $C^*(\Lambda) = C^*(E_\Lambda)$ is faithful.

Corollary 9.4. Let $(\Lambda, d)$ be a finitely aligned $k$-graph such that $|\Lambda^e(v)| = \infty$ for every $v \in \Lambda^0$ and $1 \leq i \leq k$. Let $\{ t_{\lambda} : \lambda \in \Lambda \}$ be a Cuntz-Krieger $\Lambda$-family such that $t_v \neq 0$ for all $v \in \Lambda^0$. Then the representation $\pi_t$ of $C^*(\Lambda) := C^*(E_\Lambda)$ is faithful.

Proof. That each $|\Lambda^e(v)| = \infty$ implies both that $C^*(E_\Lambda) = \mathcal{T} C^*(E_\Lambda)$, and that $\Lambda$ has no sources, so that $C^*(\Lambda) := C^*(E_\Lambda)$. Lemma 8.1 implies that $(E_\Lambda, \varphi_\Lambda)$ is finitely aligned. To establish (8.1), we fix $v \in \Lambda^0$ and finite sets $G_m \subset \Lambda^e_m(v)$ for $1 \leq m \leq k$. By Lemma 9.3 it suffices to show that
\[
\prod_{m=1}^k \left( t_v - \sum_{\lambda \in G_m} t_{\lambda t_{\lambda}^*} \right) > 0.
\]

We shall construct paths $\mu_m \in \Lambda(v)$ of degree $\sum_{i=1}^m e_i$ for $m \leq k$ such that $\mu_m(0, e_i)$ does not belong to $G_i$ for $1 \leq i \leq m$. We take $\mu_1$ to be any edge of degree $e_1$ which is not in $G_1$. If we have $\mu_m$, then because the set $\Lambda^e_{m+1}(\text{dom}(\mu_m))$ is infinite, there is a path $\mu_{m+1} = \mu_m \alpha$ of degree $\sum_{i=1}^{m+1} e_i$ which is not in the finite set $\bigcup_{\lambda \in \Lambda^e_{m+1}} \text{MCE}(\mu_m, \lambda)$. Then $\mu_{m+1}(0, e_i) = \mu_m(0, e_i)$ is not in $G_i$ for $i \leq m$, and $\mu_{m+1}(0, e_{m+1})$ cannot be in $G_m$ because $\mu_{m+1} \in \text{MCE}(\mu_m, \mu(0, e_{m+1}))$.

Now for $\lambda \in G_i$, we have $\text{MCE}(\lambda, \mu_k) = \emptyset$, and relation (5) of Definition \[7\] in the form (7.1) gives $t_{\lambda t_{\lambda}^* t_{\mu} t_{\mu}^*} = 0$. Thus
\[
\prod_{m=1}^k \left( t_v - \sum_{\lambda \in G_m} t_{\lambda t_{\lambda}^*} \right) t_{\mu_t} t_{\mu_t}^* = t_{\mu_k} t_{\mu_k}^*,
\]
which is nonzero because $t_{\mu_t} t_{\mu_t}^* = t_{s(\mu_t)}$ is nonzero. Since $\mathbb{Z}^k$ is amenable, the result now follows from Theorem 8.1. \hfill \square
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