Combinatorics of Nahm sums, quiver resultants and the K-theoretic condition

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Abstract: Algebraic Nahm equations, considered in the paper, are polynomial equations, governing the $q \to 1$ limit of the $q$-hypergeometric Nahm sums. They make an appearance in various fields: hyperbolic geometry, knot theory, quiver representation theory, topological strings and conformal field theory. In this paper we focus primarily on Nahm sums and Nahm equations that arise in relation with quivers. For a large class of symmetric quivers, we prove that quiver A-polynomials, that is, specialized resultants of the Nahm equations, are tempered (the so-called K-theoretic condition). This implies that they are quantizable. Moreover, we find that their face polynomials obey a remarkable combinatorial pattern, reminiscent of the permutohedron. We use the machinery of initial forms and mixed polyhedral decompositions to investigate the edges of the Newton polytope. We work out all diagonal quivers with adjacency matrix $C = \text{diag}(\alpha, \alpha, \ldots, \alpha)$, $\alpha \geq 2$, and give a sketch when the diagonal entries are all distinct. We also conjecture that it holds for all symmetric quivers.
1. Introduction

Algebraic Nahm equations\footnote{Not to be confused with Nahm equations in gauge theory and differential geometry} govern the $q \to 1$ limit of the $q$-hypergeometric Nahm sums, which arise in various fields: conformal field theory \cite{1}, quiver representation theory \cite{2–4}, hyperbolic geometry and ideal triangulations of 3-manifolds \cite{5, 6}, knots-quivers correspondence \cite{7–11} and topological strings \cite{9, 10, 12}. In the realm of quivers, the Nahm sums incarnate as the motivic Donaldson-Thomas (DT) generating series \cite{2–4, 13, 14}:

\[
P_C(x_1, \ldots, x_m) = \sum_{(d_1, \ldots, d_m) \geq 0} \frac{(-q^{1/2}\sum_{i,j=1}^{m} C_{ij} d_i d_j}{(q; q)_{d_1} \cdots (q; q)_{d_m}} x_1^{d_1} \cdots x_m^{d_m},
\]

where $C$ is symmetric matrix with integer entries, $q \in \mathbb{C}$ and $x_i$ are formal variables which commute with each other, and $(a; q)_n := \prod_{k=0}^{n-1}(1 - aq^k)$ is the $q$-Pochhammer symbol. If $C_{i,j}$ are non-negative, the matrix $C$ is the adjacency matrix for some symmetric quiver. Otherwise we can apply the framing transformation $C \mapsto C + [f]$, where $[f]$ is a matrix with all values equal to $f \in \mathbb{Z}$, in order to get rid of the negative entries. It transforms the quiver series \eqref{eq:1.1} in a simple way \cite{15}. For a curious reader, we sketch the derivation of \eqref{eq:1.1} from
the quiver representation theory in Section 2. The crucial property of (1.1) is the following factorization
\[ P_C(x_1, \ldots, x_m) = \prod_{(d_1, \ldots, d_m) \neq 0} \prod_{k \geq 0} (1 - q^{k+(j-1)/2} x_1^{d_1} \ldots x_m^{d_m})^{\Omega_{d_1, \ldots, d_m,j}} \] (1.2)

The exponents \( \Omega_{d_1, \ldots, d_m,j} \) are called the motivic DT invariants (or refined BPS invariants in physics), and were shown to be integers in [14]. Consider the Laurent expansion at \( q \to 1 \) of the natural logarithm of (1.1):
\[ \log P_c(x_1, \ldots, x_m) \big|_{q=e^{\hbar} \to 1} = \frac{1}{\hbar} S_0 + S_1 + \hbar S_2 + O(\hbar^2), \] (1.3)

where \( S_i = S_i(x_1, \ldots, x_m, z_1, \ldots, z_m) \) with \( z_i := q^{d_i} \). The (algebraic) Nahm equations are extracted from critical points of the leading term (superpotential) in (1.3):
\[ \frac{\partial S_0}{\partial z_i} = 0, \quad i = 1 \ldots m, \] implies
\[ F_i := z_i - 1 + (-1)^{C_i,i} x_i \prod_{j=1}^{m} z_j^{C_{i,j}} = 0, \quad i = 1 \ldots m \] (1.4)
(see [12,15] for the details). We add one extra equation, which gives \( y \)-dependence:
\[ F_0 := y - z_1 \ldots z_m = 0 \] (1.5)
and introduce the quiver resultant \( R := \text{res}_{z_1, \ldots, z_m}(F_0, F_1, \ldots, F_m) \). Informally, it is a unique (up to a sign) irreducible polynomial in the coefficients of (1.4), which vanishes whenever all \( F_i \) have a common root with respect to \( z_i, \quad i = 1 \ldots m \) [16]. We will give a slightly refined definition in Section 4.

Recall that the quiver A-polynomial is \( A(x,y) = R(a_1 x, \ldots, a_m x, y) \), where \( a_i \) are non-zero complex parameters (moduli), \( x = e^u, y = e^v, \quad (u,v) \in \mathbb{C} \times \mathbb{C} \). It has been introduced in [11] and further studied in [8] and [15]. Ultimately, it is a polynomial invariant of symmetric quivers. Under a suitable choice of the quiver matrix \( C \) and \( a_i \), it can be related to augmentation variety or geometric A-polynomial for a knot [8], where \( u \) and \( v \) would represent the holonomies around the meridian and longitude of a knot tubular neighbourhood. Also, from the mirror symmetry perspective quiver A-polynomials are the mirror curves (B-model) for some Calabi-Yau 3-folds. The case of strip geometries was studied in [12], whereas the relation to Ooguri-Vafa large \( N \) duality in [9] and [10].

We wish to study the Newton polytope \( N(R) \), that is, the convex hull of its monomials. It has the dimension \((m-1)\), where \( m \) is the number of vertices of a quiver. Our focus is on the diagonal quivers
\[ C = \text{diag}(\alpha_1, \ldots, \alpha_m), \quad \alpha_i \geq 2, \quad i = 1 \ldots m \]
The reason for that is when diagonal, each convex hull of monomials in \( F_i \) is one-dimensional, which greatly simplifies (and in some sense is more fundamental, upon which we can later

\[ \text{This formula is the cornerstone in [2], which led to the mathematical theory of BPS invariants in 3d } \mathcal{N} = 2 \text{ theories, using quivers and their representations} \]
generalize) calculation of its Newton polytope. The Minkowski sum of supports of \( F_i \) in this case would be a zonotope (projection of a hypercube onto \( \mathbb{R}^m \)).

The ultimate goal is to prove certain statements about the resultant \( R \) and quiver A-polynomial \( A(x,y) \). We have:

\[
R \mapsto A(x,y) \implies N(R) \mapsto N(A)
\]

where the arrow “\( \mapsto \)” stands for the projection onto \((x,y)\)-plane.

We conjecture that \( A(x,y) \) is tempered, i.e. all its face polynomials have roots only on the unit circle. By a face polynomial we simply mean the sum all monomials in \( A(x,y) \), which lie on a particular face of a Newton polygon \( N(A) \). This is called the \textit{K-theoretic condition}, because of an elegant interpretation in terms of the group \( K_2 \) for a compact Riemann surface (in our case it is given by \( A(x,y) = 0 \)) [17–19]. It turns out that this condition relates to quantization, modularity and integrality properties for \( A(x,y) \). It is confirmed true for all knot A-polynomials [20], but, to our knowledge, has not been studied for quivers so far. We give a brief review of this topic in Section 3.

Our main result is:

For a diagonal quiver with adjacency matrix \( C = \text{diag}(\alpha,\alpha,\ldots,\alpha) \), \( m \geq 1, \alpha \geq 2 \), its quiver A-polynomial \( A(x,y) \) is tempered. Moreover, all face polynomials of \( A(x,y) \) factorize to binomials, forming a remarkable combinatorial pattern.

This is the content of Section 7, and Theorem 7.1 in particular. The beautiful combinatorial pattern is given in Proposition 7.1. It involves permutations of rows and columns of diagrams, representing the sub-resultants. One can think of it as a “cellular automation” acting on the faces of the Newton polytope. To understand the mechanism better (and also for a nicer presentation), we study the low-dimensional cases \( m = 2 \) and \( m = 3 \) separately in Sections 5 and 6. E.g., for \( \text{diag}(2,2) \) there are four face polynomials: \( \tau + 1, \tau - 1, (\tau + 1)^2 \) and \( (\tau - 1)^2 \), and all their roots are equal to \( \pm 1 \), as shown on Figure 1.

The key point is that we don’t have to compute the resultant explicitly. Instead, we use the machinery of initial forms [21,22] and mixed polyhedral decompositions, developed in [21] and [23]. These guys generalize extremal A-polynomials from knot theory [7,8,24,25], and, under certain assumption, are in bijection with the faces of \( N(R) \). As a consequence, we obtain the “extremalization” of quiver A-polynomials, provided by a particular face of \( N(R) \).

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2. Quiver representations and motivic DT series

In this section we give a simplified derivation of the motivic DT series for symmetric quivers, based on quiver representation theory [2, 4, 13, 26, 27].

We start with the definition of a quiver: it is a directed finite graph $Q = (Q_0, Q_1, h, t)$, where $Q_0$ and $Q_1$ are the sets of vertices and arrows, and $h, t$ are maps from $Q_1$ to $Q_0$, picking up a head or a tail vertex for a given arrow. The term “quiver” is used instead of “graph”, since one considers an additional structure, called a quiver representation: to every vertex $v_i \in Q_0$ it associates a finite-dimensional vector space $V_i$, and to every arrow $a_i \in Q_1$ – a linear map between these vector spaces: $f : V_{h(a)} \to V_{t(a)}$ (here we order the vertices in some way). By abuse of notation, we will write $V(a)$ instead of $f(a)$. Therefore, a dimension vector $d = (d_1, \ldots, d_m) \in \mathbb{Z}^m$, $d_i = \dim V_i$ encodes all the dimensions for a given representation of $Q$ with $m$ vertices.

We are not interested in particular representations, but rather in the moduli space of representations with a fixed dimension vector. There are two related notions: the representation space and the category of quiver representations. It will be useful to consider both of them.

The representation space $\text{Rep}_d(Q) := \prod_{a \in Q_1} \text{Mat}_{d(h(a)), d(t(a))}$, that is, the space of matrices representing the arrows of $Q$, where the dimension vector $d$ is fixed. This can be seen as the moduli space of representations. On another hand, the category $\text{Rep}(Q)$ by definition consists of representations of any dimension vector. Its objects are, of course, quiver representations, while arrows are morphisms between them. A morphism between two representations $V, W$
of \( Q \) can be drawn as a commutative diagram:

\[
\begin{array}{ccc}
V(ta) & \xrightarrow{V(a)} & V(ha) \\
\phi(ta) \downarrow & & \downarrow \phi(ha) \\
W(ta) & \xrightarrow{W(a)} & W(ha)
\end{array}
\] (2.1)

The category \( \text{Rep}(Q) \) has zero object: the representation which associates zero vector space \( \{0\} \) to all vertices, and every linear map is of the form \( \{0\} \to \{0\} \). It has a distinguished property: every other object in \( \text{Rep}(Q) \) has a unique arrow going to, and a unique arrow from the zero object. It can be shown that \( \text{Rep}(Q) \) is \( k \)-linear, abelian category (kernels and cokernels are well-defined).

We would also need the notion of a subrepresentation. A representation \( W \) is called a subrepresentation of \( V \), if \( W(x) \) is a subspace of \( V(x) \) for all \( x \in Q_0 \) and \( V(a) : V(ta) \to V(ha) \) restricts to \( W(a) : W(ta) \to W(ha) \) for all \( a \in Q_1 \). Suppose that \( W \) is a subrepresentation of \( V \). For every \( x \in Q_0 \), let \( \iota(x) : W(x) \to V(x) \) be the inclusion map. Then \( \iota = (\iota(x), x \in Q_0) \) is an injective morphism.

Let \( \phi : V \to W \) be a morphism of representations of \( Q = (Q_0, Q_1, h, t) \). For every \( x \in Q_0 \), define \( K(x) \) and \( C(x) \) to be the kernel and cokernel of \( \phi(x) : V(x) \to W(x) \). Also, let \( \iota(x) : K(x) \to V(x) \) be an inclusion and \( \pi(x) : W(x) \to C(x) \) the projection maps. Keeping in mind (2.1), for every arrow \( a \in Q_1 \) we get:

\[
(\phi(ha) \circ V(a))(K(ta)) = (W(a) \circ \phi(ta))(K(ta)) = 0,
\] (2.2)

which implies that

\[
V(a)(K(ta)) \subseteq K(ha).
\] (2.3)

This means that we can restrict \( V(a) \) to a linear map \( K(a) : K(ta) \to K(ha) \), so \( K \) is a subrepresentation of \( V \), which is called the kernel of \( \phi \) (the cokernel is defined analogously).

Our aim is to define the extensions of quiver representations. We need one more ingredient – a direct sum of two representations \( V, W \):

\[
\begin{align*}
\text{for } x \in Q_0, & \text{ define } (V \oplus W)(x) := V(x) \oplus W(x), \\
\text{for } a \in Q_1, & \text{ define } (V \oplus W)(a) := V(ta) \oplus W(ta) \to V(ha) \oplus W(ha),
\end{align*}
\] (2.4)

where the latter arrow is given by the matrix: \[
\begin{pmatrix}
V(a) & 0 \\
0 & W(a)
\end{pmatrix}.
\]

Then there are two natural inclusions: \( \iota_1 : V \hookrightarrow V \oplus W, \iota_2 : W \hookrightarrow V \oplus W \), such that \( \iota_1 = \begin{pmatrix} 1_{V(x)} \\ 0 \end{pmatrix} \), and \( \iota_2 = \begin{pmatrix} 0 \\ 1_{W(x)} \end{pmatrix} \). It turns out that for a pair of representations \( V, W \), a tuple \( \phi = (\phi(a))_{a \in Q_1} \in \]
\[ \prod_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \] also determines a representation of \( Q \):

\[
e(V, W, \phi) := \left( (W(x) \oplus V(x))_{x \in Q_0}, \begin{pmatrix} W(a) & \phi(a) \\ V(a) & 0 \end{pmatrix}_{a \in Q_1} \right), \tag{2.5}
\]

which fits into the exact sequence:

\[
0 \xrightarrow{\partial_1} W \xrightarrow{\partial_2} e(V, W, \phi) \xrightarrow{\partial_3} V \xrightarrow{\partial_4} 0, \tag{2.6}
\]

\( \partial_1 \) and \( \partial_2 \) being the inclusion and projection maps, and the term “exact” stands for the property: \( \text{im}(\partial_k) = \ker(\partial_{k+1}) \), \( k = 0 \ldots 2 \). For example:

\[
\begin{pmatrix} W(a) & \phi(a) \\ 0 & V(a) \end{pmatrix} \begin{pmatrix} W(ta) \\ V(ta) \end{pmatrix} = \begin{pmatrix} W(ha) + \phi(a)V(ta) \\ V(ha) \end{pmatrix} \tag{2.7}
\]

Such exact sequence (2.6) is of course not generic at all, since the middle term \( e(V, W, \phi) \) has a very special structure. We call (2.6) the extension of \( V \) by \( W \).

**Remark:** Every exact sequence of finite-dimensional representations \( V, V', V'' \), of the form

\[
0 \to V' \to V \to V'' \to 0 \tag{2.8}
\]

satisfies \( d(V) = d(V') + d(V'') \), where \( d(\ast) \) are the corresponding dimension vectors. Also, any two isomorphic finite-dimensional representations have necessarily the same dimension vector.

Now that we have defined all the basic notions, it’s time to count some objects! Assume our representations are now vector spaces over a finite field \( \mathbb{F}_q \) (and of course maps between them), where \( q = p^r \) and \( p \) is prime. These are integers modulo \( q \), i.e. \( \mathbb{F}_q = \{0, 1, 2, \ldots, q-1\} \) with modular multiplication. For a fixed dimension vector \( d = (d_1, \ldots, d_m) \), define

\[
s_d := \sum_{[V]|\dim V = d} \frac{1}{|\text{Aut}(V)|}, \tag{2.9}
\]

where the summation is over all isomorphism classes of representations \( V \) with dimension vector \( d \). \( |\text{Aut}(V)| \) is the size of the automorphism group of \( V \), which is the orbit of \( G := \prod_{i=1}^m \text{GL}_{d_i} (\mathbb{F}_q) \), acting on each linear map \( V(a) \) in \( V \) as follows:

\[
(g_i)(A_a) = (g_j V(a) g_i^{-1})_{(a; x_i \to x_j)}, \quad \forall a \in Q_1 \tag{2.10}
\]

This group is of course also finite, since we are dealing with finite fields! Also notice that each isomorphism class \([V]\) is exactly the orbit of \( V \) under the action of \( G \). It follows that the total number of representations of dimension vector \( d \) is \( q^{\sum_{a \in Q_1} d_i d_j} = q^{\sum_{i,j=1}^m C_{ij} d_i d_j} \), and the number of points in the orbit of \( V \) is \( \frac{|G|}{|\text{Aut}(V)|} \). Thus,

\[
q^{\sum_{i,j=1}^m C_{ij} d_i d_j} = \sum_{[V]|\dim V = d} \frac{|G|}{|\text{Aut}(V)|}, \tag{2.11}
\]
from which it follows that
\[ s_d = \frac{q^{\sum_{i,j=1}^m C_{ij} d_i d_j}}{\prod_{i=1}^m |GL_{d_i}(\mathbb{F}_q)|} \]  
(2.12)

where \( C_{ij} \) are elements of the adjacency matrix of \( Q \), and \( |GL_{d_i}(\mathbb{F}_q)| = (q^m - 1)(q^m - q) \cdots (q^m - q^{m-1}) \). The latter equality comes from the counting of admissible columns of a matrix in \( GL_{d_i}(\mathbb{F}_q) \). The first one can be only zero vector, hence \((q^m - 1)\) factor, the second one can be anything but the first one, hence \((q^m - q)\), and so on.

The formula (2.12) gives the Poincare polynomial of the ordinary cohomology of the moduli space of quiver representations.

**Definition 2.1** Motivic DT series for symmetric quivers:

\[ P_C(x_1, \ldots, x_m) := \sum_{(d_1, \ldots, d_m) \geq 0} s_d x_1^{d_1} \cdots x_m^{d_m} \sim \sum_{(d_1, \ldots, d_m) \geq 0} \frac{(-q^{1/2})^{\sum_{i,j=1}^m C_{ij} d_i d_j}}{(q; q)_{d_1} \cdots (q; q)_{d_m}} x_1^{d_1} \cdots x_m^{d_m} \]
(2.13)

where \( x_i \) are commutative variables, and \( C \) is the adjacency matrix of \( Q \), up to framing transformation \( C \mapsto C + [f] \) with \([f]_{ij} = f \in \mathbb{Z}\).

**Remark:** In the definition above, \( q \) is considered to be an arbitrary complex number (\( \equiv \) analytic continuation of (2.12)), and one may notice that the coefficients \( s_d \) have poles at \( q = \) root of unity, due to the \( q \)-Pochhammer symbols in the denominator. It is of our interest to consider the perturbative expansion of (2.13) at \( q = 1 \), which will eventually lead to a system of algebraic equations (1.4). Also, in [2, 3, 13] the definition of quiver series is given for an arbitrary (not necessarily symmetric) quiver, but for our purpose we deal strictly with symmetric case.

3. Algebraic K-theory and tempered polynomials

What is the algebraic K-theory? Roughly speaking, it’s all about a study of the family of functors \( K_n : \text{Rings} \to \text{Abelian groups} \) (it was invented to produce nice invariants of rings). \( K_0, K_1 \) and \( K_2 \) are classically known from the sixties. Higher \( K \)-groups, as well as those with the negative index, were defined in the following decades. However, our main character is the group \( K_2(F) \), where \( F \) is a field. The exposition here is mostly borrowed from Milnor’s classical book [28]. We start with a rather informal definition:

\[ K_2(F) := \text{a group of non-trivial relations satisfied by elementary matrices of any size with entries in } F \]
(3.1)

Recall that elementary matrix is a matrix \( e^{\lambda}_{ij} \in GL_n(F) \), which differs from the identity matrix of size \( n \) by a single element \( \lambda \) in the \((i, j)\)-th position, \( i, j = 1 \ldots n \), or a matrix obtained from such by elementary row operations. In other words, we can say that \( e^{\lambda}_{ij} \) generate the
subgroup of elementary matrices, sitting in $GL_n(F)$. If $e^\lambda_{ij}, e^\mu_{kl}$ are elementary matrices, their commutator is

$$[e^\lambda_{ij}, e^\mu_{kl}] = \begin{cases} 1; & j \neq k, i \neq l \\ e^\lambda_{ii} & j = k, i \neq l \\ e^{-\mu}_{kj} & j \neq k, i = l \end{cases} \tag{3.2}$$

We can forget for a moment that we are dealing with matrices, and consider an abstract group generated by the relations:

(a) $x^\lambda_{ij} x^\mu_{ij} = x^{\lambda+\mu}_{ij}$

(b) $[x^\lambda_{ij}, x^\mu_{jl}] = x^{\lambda \mu}_{il}; i \neq l$

(c) $[x^\lambda_{ij}, x^\mu_{kl}] = 1; j \neq k, i \neq l$

These relations define Steinberg group, denoted by $St(n, F)$ for $n \geq 3$ (for $n < 3$ the relations will degenerate). $x^\lambda_{ij}, x^\mu_{kl}$ are group elements.

For each $n \geq 3$ we have a homomorphism of groups:

$$St(n, F) \to GL_n(F), \tag{3.3}$$

which associates an elementary matrix of size $n$ to each element of $St(n, F)$: $\varphi(x^\lambda_{ij}) = e^\lambda_{ij}$. Now we can pass through the direct limit of a sequence of groups when $n \to \infty$, denoting it $GL(F)$, which is understood as follows:

$$GL_1(F) \subset GL_2(F) \subset GL_3(F) \subset \ldots, \tag{3.4}$$

and each $GL_n(F)$ is injected into $GL_{n+1}(F)$ by the map:

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5}$$

Therefore, $GL(F)$ is determined by taking the union of all elements in the infinite sequence (3.4). Analogously, one can define $St(F)$. In what follows is the formal definition of $K_2(F)$:

$$K_2(F) := \text{Kernel of the map } \phi : St(F) \to GL(F), \tag{3.6}$$

where the kernel elements are mapped to an identity matrix in $GL(F)$. Let’s show this by example: pick up a rotation by 90 degrees matrix, which is elementary:

$$e_1^{12} e_2^{-1} e_1^{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.7}$$

and is decomposed as a product of the generators $e^\lambda_{ij}$. This matrix has period 4:

$$(e_1^{12} e_2^{-1} e_1^{12})^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.8}$$
Therefore, the relation (3.8) is a non-trivial relation between elementary matrices, since the identity matrix is of course also elementary. If we associate to the left hand side of (3.8) the element in St, that is, the preimage of $\phi$, it will belong to the kernel of $\phi$, and thus giving an element in $K_2(\mathbb{R})$:

$$\left(x_{12}^{-1}x_{21}^{-1}x_{12}\right)^4 \in \ker \phi, \quad \phi \left(\left(x_{12}^{-1}x_{21}^{-1}x_{12}\right)^4\right) = \left(e_{12}^{-1}e_{21}^{-1}e_{12}\right)^4 \quad (3.9)$$

since it evaluates as an identity matrix, which means that “the relation holds”. In general, such identities are of the form:

$$e_{\lambda_1}^{i_{1j_1}} e_{\lambda_2}^{i_{2j_2}} \cdots e_{i_{rj_r}}^{\lambda_r} = \text{Id} \quad \iff \quad x_{\lambda_1}^{i_{1j_1}} x_{\lambda_2}^{i_{2j_2}} \cdots x_{i_{rj_r}}^{\lambda_r} \quad (3.10)$$

Following [18], we restrict ourselves to $F = \mathbb{Q}(C)$ – the field of rational functions on a compact Riemann surface $C$. Choose a pair $(x,y)$ of such functions. Since $C$ is compact, there is always a unique minimal irreducible polynomial $A(x,y)$ defining it. For example, if $C$ is homeomorphic to a sphere, $x = x(t), y = y(t)$ give a rational parametrization of $A(x,y)$. For higher genus, however, we would need more parameters, in order to make a proper parametrization (see some examples in [17]).

Now take a pair of elementary matrices:

$$D_x = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D'_y = \begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^{-1} \end{pmatrix}, \quad (3.11)$$

and define

$$\{x,y\} := wvu^{-1}v^{-1} \quad (3.12)$$

with $u = \phi^{-1}(D_x), v = \phi^{-1}(D'_y)$. We call this bracket the universal symbol of $(x,y)$. The commutator is always identity matrix, therefore $\{x,y\} \in K_2(\mathbb{Q}(C))$. It turns out that $K_2(F)$ is generated by the symbols $\{x,y\}$ ([28], Corollary 9.13 p. 78), and it holds exactly when $F$ is a field.

Now the K-theoretic condition for $A(x,y)$ would be stated as follows ([18], also [17] and [19] give slightly different at the first sight, but in fact equivalent formulations):

$$\{x,y\}^N \in K_{2,0} \text{ for some } N \in \mathbb{N} \iff A(x,y) \text{ is tempered} \quad (3.13)$$

where “tempered” means that the face polynomials of $A(x,y)$ have roots only on the unit circle (are products of cyclotomic polynomials), and $K_{2,0}$ is the set of “trivial” elements in $K_2(\mathbb{Q}(C))$:

$$K_{2,0} := \bigcap_w \ker \lambda_w \subset K_2(\mathbb{Q}(C)), \quad (3.14)$$

where $w \in C$, and $\lambda_w : K_2 \to \mathbb{C}^*$ corresponds to the tame symbol:

$$\{x,y\}_w := (-1)^{w(x)w(y)} \frac{x^{w(y)}}{y^{w(x)}} \bigg|_w \quad (3.15)$$
Here the point \( w \in C \) induces a functional \( w(\cdot) \) on \( \mathbb{Q}(C) \), called the valuation. Such that \( w(x(t)) \) or \( w(y(t)) \) equals to the degree of a leading term of \( x(t) \) (or \( y(t) \)) around \( t = w \), where \( x(t), y(t) \) are the Puiseaux parametrizations of a local branch.

**Remark:** As the reader may notice, the tame symbol is a map \( F^* \times F^* \to \mathbb{C}^* \), where \( F^* := F \setminus \{0, 1\} \). Where does then \( \lambda_w \) come from? In fact, every symbol on \( F \), that is, a map

\[
F^* \times F^* \to A, \tag{3.16}
\]

where \( A \) is any abelian group, gives rise to a unique homomorphism \( K_2(F) \to A \). This is the content of the theorem by Matsumoto [18], which states that \( K_2(F) \) is the universal target of all symbols on \( F \). So in the case of the tame symbol, we simply denote this homomorphism by \( \lambda_w \). Its kernel consists of all elements in \( K_2(F) \), which are mapped to 1 \( \in \mathbb{C}^* \). Rephrasing, we require that all tame symbols for any \( w \in C \) are roots of unity.

It turns out that this criterion has many exciting implications: relation to modular forms and special values of Zeta function [18], Chern-Simons geometric quantization [19], knot theory [20], modularity properties of the Mahler measure [17, 18], etc. The proof of (3.13) is due to the fact that for each slope \( p/q \) of \( N(A) \), there is a valuation \( v \) such that \( \frac{p}{q} = -\frac{v(x)}{v(y)} \). Moreover, the value of the tame symbol \( (x, y)_w \) equals to the root of the corresponding face polynomial with this slope (details in [20]).

In other words, by choosing \( (x, y) \), we have to evaluate tame symbols \( (x, y)_w \) for each \( w \in S \), where \( S \) is the set of zeroes and poles of \( x \) and \( y \) on \( C \), and thus must be sure to get the roots of unity. It holds if and only if the polynomial \( A(x, y) \) is tempered.

**(An) example**

Take the genus zero curve \( A(x, y) = 0 \), also studied in [17], with

\[
A(x, y) = x^2 - 2xy + y^2 - 2x - y + 1 \tag{3.17}
\]

Its Newton polygon \( N(A) \) is a triangle with vertices \((0, 0), (0, 2), (2, 0)\). Notice that \( A(x, y) \) is not self-reciprocal, since

\[
A(x^{-1}, y^{-1}) \neq \pm x^py^qA(x, y), \quad \text{for some integers } p, q
\tag{3.18}
\]

which means it cannot be realized as a geometric A-polynomial for some hyperbolic 3-manifold. The slopes are \( 0, \infty, -1 \). The face polynomials are \((\tau - 1)^2, (\tau - 1)^2, \tau^2 - \tau + 1\), where the variable \( \tau \) decorates the monomials on a given edge of \( N(A) \). Here’s an explanation:

write \( A(x, y) = \sum c_{i,j}x^iy^j \). To get a face polynomial \( f_e \), label all the monomials on an edge \( e \) consequently from one vertex to another by \( E = \{1, 2, \ldots, |E|\} \) and sum them up, replacing \( x^iy^j \) by some power of \( \tau \) (keeping the ordering): \( f_e := \sum_{s \in E} c_{i,j}(s)\tau^s, \quad E = \{1, 2, \ldots, |E|\} \). In this way, starting from a vertex and going through all edge monomials consequently, we end up in the opposite vertex, and get:

\[
x^2 - 2x + 1 \leftrightarrow (\tau - 1)^2
\]
\[
y^2 - y + 1 \leftrightarrow \tau^2 - \tau + 1
\]
\[
x^2 - 2xy + y^2 \leftrightarrow (\tau - 1)^2
\tag{3.19}
All of them are obviously cyclotomic. Choose the rational parametrization, e.g.:

\[
x(t) = \frac{t^2 + t + 1}{(t - 1)^2}, \quad y(t) = \frac{3t^2}{(t - 1)^2}
\]  

(3.20)

Now compute the tame symbols at \( w \in S \) for this parametrization. In our case the set of zeroes and poles \( S \) of \( x(t) \) and \( y(t) \) is

\[
S = \{0, 1, \zeta_3^{(1)}, \zeta_3^{(2)}\},
\]  

(3.21)

where \( \zeta_3^{(1)}, \zeta_3^{(2)} \) are two complex-conjugated cubic roots of unity. We get:

\[
\begin{align*}
\text{horizontal} & : (x, y)_0 = 1, \quad \text{slope -1: } (x, y)_1 = 1, \quad \text{vertical: } (x, y)_{\zeta_3^{(i)}} = \zeta_3^{(i)}
\end{align*}
\]  

(3.22)

For instance,

\[
(x, y)_0 = (-1)^{0} x(t)^2 \bigg|_{t=0} = 1,
\]  

(3.23)

since \( x(t) = 1 + 3t + 6t^2 + O(t^3) \), and \( y(t) = 3t^2 + O(t^3) \) around \( w = 0 \), this gives \( w(x) = 0 \) and \( w(y) = 2 \). As we see, each of the values \((x, y)_0, (x, y)_1, (x, y)_{\zeta_3^{(i)}}\) corresponds to a root of some face polynomial. All of them are roots of unity, which shows that \( A(x, y) \) (3.17) is tempered, i.e. the K-theoretic property holds for the underlying curve. Also, by computing the tame symbols we indeed see the surjection, but not the bijection between valuations and slopes (of course in this example one of the face polynomials has degree two and is irreducible, thus giving the two distinct roots with the same slope).

4. The main conjecture and sketch of the proof

Here comes the main conjecture of the paper:

**Conjecture 4.1** Quiver A-polynomials are tempered, for every choice of the adjacency matrix.

If true, it means that all quiver A-polynomials are quantizable, according to [19]. We focus on the diagonal case \( C = \text{diag}(\alpha_1, \ldots, \alpha_m) \), because it is the simplest one from the combinatorial perspective. We will rely on the notion of initial form, which intuitively is the extremal counterpart of a quiver resultant, supported at a particular face of \( N(R) \). The strategy is:

- compute the initial forms, supported at the edges of \( N(R) \)
- study their projection onto \( (x, y) \)-plane, given by the principal specialization
- binomiality of all face polynomials of \( A(x, y) \) would follow from factorization of those initial forms, which project onto the edges of \( N(A) \)
- it would imply that \( A(x, y) \) is tempered, since these binomials have all roots = ±1
We begin by defining the sparse mixed resultant [16, 21, 29]. Let $A = \{A_0, \ldots, A_m\}$ be a collection of finite subsets $A_i \subset \mathbb{Z}^m$, $n_i = |A_i|$ and $Q_i = \text{conv}(A_i)$ – their convex hulls. Denote $\prod \mathbb{C}^A$ the space of all Laurent polynomials with supports equal to $A$, and take an $(m+1)$-tuple of its generic representatives:

$$f_i = \sum_{a \in A_i} c_{i,a} a^a, \ i = 0, \ldots, m \quad (4.1)$$

The coefficients $c_{i,a}$ are assumed to be non-zero complex parameters. Let $Z$ be the subvariety in $\prod \mathbb{C}^A$ consisting of those tuples of the form (4.1), which have a common solution $z'(z') = 0 \ i = 0, \ldots, m$ in $(\mathbb{C}^*)^m$, and let $\overline{Z}$ be its Zariski closure. Then, $\overline{Z}$ is an irreducible hypersurface in $\prod \mathbb{C}^A$ [16].

**Definition 4.1** Sparse mixed resultant $R$ is the unique (up to a sign) irreducible polynomial in $\{c_{i,a}\}$ with integral coefficients, which vanishes on $\overline{Z}$ if $\text{codim}(\overline{Z}) = 1$, and $R := 1$ if $\text{codim}(\overline{Z}) \geq 2$.

Now we return to Nahm equations and quivers. In order to study Newton polytopes and initial forms, we do have to add generic coefficients as extra variables in front of each monomial in Nahm equations (1.4), thus slightly generalizing them:

$$F_0 := a_0 + a_1 z_1 \ldots z_m$$
$$F_i := b_{i,0} + b_{i,1} z_i + b_{i,2} \prod_{j=1}^m z_j^{C_{i,j}} \quad (4.2)$$

(or we can say that we do not care about the coefficients, thus leaving them to be generic, and only the powers of monomials are important). We will shorthand $b := \{b_{i,j}\}$ for $i = 0 \ldots m, \ j = 1, 2$.

**Definition 4.2** Quiver resultant $R(a_0, a_1, b)$ is the sparse mixed resultant from the supports of $F_i$, $i = 0 \ldots m$ (4.2).

**Remark:** this philosophy naturally leads to connection with toric geometry [16, 23]. Given a set of supports $\{A_0, \ldots, A_m\}$, $A_i = \text{supp}(F_i)$, we can form a (projective) toric variety $X_{A_0, \ldots, A_m}$, associated to Nahm equations. Then, the zero locus of $R$ defines the hypersurface which is rulled out by intersecting $X$ with hyperplanes in the projective space. Initial forms would be then defined as algebraic cycles arising in a similar way [23].

Therefore, quiver $A$-polynomials are specialized quiver resultants:

$$A(x, y) = R(y, -1 | 1, -1, (-1)^{C_{1,1}} a_1 x | \ldots | 1, -1, (-1)^{C_{m,m}} a_m x) \quad (4.3)$$

The formula (4.3) relates (4.2) to (1.4), and then sends $x_i = a_i x$, giving the principal specialization. Denote $N(R)$ the Newton polytope of $R$, i.e. the convex hull of its support. Due to the results of [21], dimension of $N(R)$ is equal to $(m - 1)$, and the total degree is the mixed
volume of the Minkowski sum $Q = Q_0 + \cdots + Q_m$. To work out higher dimensional resultant polytopes, we use the language of perfograms. Each perfogram is just a pictorial presentation of a sub-resultant, for example:

\[
\begin{align*}
  b_0 + b_1 z_1 + b_2 z_2^{c_{1,2}} \cdots z_m^{c_{1,m}} & \iff \bullet \quad \bullet \\
  c_0 + c_1 z_1 + c_2 z_2^{c_{2,1}} \cdots z_m^{c_{2,m}} & \iff \bullet \quad \bullet \\
  d_0 + d_1 z_3 + d_2 z_1^{c_{3,1}} \cdots z_m^{c_{3,m}} & \iff \bullet \quad \bullet
\end{align*}
\]

(4.4)

Or, with $F_0$ included:

\[
\begin{align*}
  F_0 &= a_0 + a_1 z_1 \cdots z_m & \iff \bullet \quad \bullet \\
  F_1 &= b_0 + b_1 z_1 + b_2 z_2^{c_{1,2}} \cdots z_m^{c_{1,m}} & \iff \bullet \quad \bullet \\
  F_2 &= c_0 + c_1 z_1 + c_2 z_2^{c_{2,1}} \cdots z_m^{c_{2,m}} & \iff \bullet \quad \bullet \\
  F_3 &= d_0 + d_1 z_3 + d_2 z_1^{c_{3,1}} \cdots z_m^{c_{3,m}} & \iff \bullet \quad \bullet
\end{align*}
\]

(4.5)

Now we shall review the combinatorics of mixed decompositions. Given the Minkowski sum $Q = Q_1 + \cdots + Q_m$ which is the sum of convex hulls of supports of $F_i$, we may construct its mixed decomposition (or subdivision\(^3\)) $\text{MD}_\phi$ as follows: each cell of the decomposition is a Minkowski sum of sub-supports, computed for subsets $A'_0 \subset A_0, \ldots, A'_m \subset A_m$. Each choice of them correspond to a particular sub-resultant. Then, another cell would be given by yet another subsets $A''_0 \subset A_0, \ldots, A''_m \subset A_m$, and so on, which yields a partition of $Q$ into non-overlapping cells (if the sub-resultants are chosen properly). Example of such mixed decomposition for the Nahm equations with $m = 2$ is on the figure (2).

\[\text{Figure 2: An example of mixed decomposition for } m = 2: \text{ the Minkowski sum } Q = Q_0 + Q_1 + Q_2 \text{ (left) is decomposed into 4 non-overlapping cells (right)}\]

Since perfograms represent sub-resultants, each initial form is given by a collection of perfograms [21, 29]:

**Definition 4.3** Initial form $\text{init}_\phi$, supported on the face $\phi$ of $N(\mathcal{R})$, is the product of certain sub-resultants $\overline{R}_k^l$ times a monomial $\mu_\phi$:

\[
\text{init}_\phi := \mu_\phi \prod_{i \in \text{MD}_\phi} \overline{R}_k^l,
\]

(4.6)

\(^3\)Both terms are used in the literature.
where the product is taken over all cells in \( \text{MD}_\phi \), and the exponents \( k_i \) are chosen such that the volume of \( i \) equals to the total degree of \( \tilde{R}_i^{k_i} \), for every cell \( i \).

Therefore, \( \text{init}_\phi \) is a polynomial in \( (a_0, a_1, b) \), characterized by the property:

\[
\phi = \text{conv}(\text{supp}(\text{init}_\phi))
\] (4.7)

For example, the initial form corresponding to (2, right) is:

\[
a_0 a_1^{(\alpha - 1)(\beta - 1)} (a_0^{\alpha - 1} b_2 c_1^{\alpha - 1} + a_1^{\alpha - 1} b_1 c_0^{\alpha - 1}) (a_0^{\beta - 1} c_2 b_1^{\beta - 1} + a_1^{\beta - 1} c_1 b_0^{\beta - 1})
\] (4.8)

Each cell is related to a particular sub-resultant: hexagons are the two binomial factors, while rectangles contribute to the monomial in (4.8).

Sometimes we will write \( \phi \) instead of \( \text{init}_\phi \), which will be clear from the context.

**Definition 4.4** We call the initial form \( \phi \) simple, if \( \text{supp}(\phi) \) lies on the one-dimensional skeleton of the corresponding face \( \phi \) of \( N(\mathcal{R}) \). Otherwise, if there are monomials which do not fit there, we refer to them as “interior”.

**Remark:** In general, the correspondence between faces and initial forms is surjective, but not bijective. For a given face, there is always more than one (but finitely many) initial forms touching all its vertices. It depends on whether we want to include the interior monomials or not, and which ones (by switching the intermediate bullets im each row of a perfogram).

However, we may get better results with simple initial forms:

**Proposition 4.1** The initial form \( \phi \) is simple if and only if all its perfograms, corresponding to \( \tilde{R}_i \), do not have intermediate bullets in each of its row. Moreover, there is a bijection between the set of all simple initial forms and the set of faces of \( N(\mathcal{R}) \).

Here’s an example:

- \( \begin{array}{ccc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array} \) is simple, whereas \( \begin{array}{ccc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array} \) is not.

(4.9)

Proof. Start with a simple initial form \( \phi \). By definition, \( \text{supp}(\text{init}_\phi) \subset \text{skel}_d (\phi) \), i.e. the support of \( \phi \) does not have interior monomials. We can ignore the monomial prefactor \( \mu_\phi \), since it simply rescales the support lattice, and the resulting polytopes are affinely isomorphic. Then, from the product formula (4.3) we deduce that the \( N(\prod \tilde{R}_i) \) decomposes as a Minkowski sum of \( N(\tilde{R}_i) \), for \( i = 1 \ldots |\text{MD}_\phi| \). But since \( \phi \) is simple, each \( \tilde{R}_i \) should be simple as well, i.e. not containing any interior monomials (otherwise it would hold also for their Minkowski sum). Another way around is immediate: since all \( \tilde{R}_i \) are simple, their Minkowski sum does not have interior monomials, which implies \( \phi \) is simple. Finally, the bijection is provided by:

\[
\text{vertices}(\phi) = \text{vertices}(\text{conv}(\text{supp}(\text{init}_\phi)))) \subseteq \text{supp}(\text{init}_\phi),
\] (4.10)

and the set \( \text{supp}(\text{init}_\phi) \setminus \text{vertices}(\phi) \) is fixed uniquely, by requiring all the perfograms to have no intermediate bullets “\( \bullet \)” in each of its row. \( \square \)
Proposition 4.2 If $\text{init}(\phi)$ is simple, then $\text{dim}(\phi)$ is equal to the number of its distinct binomial factors.

Proof. The case $\text{dim}(\phi) = 1$ is trivial, since if $\phi$ is an edge and $\text{init}(\phi)$ is simple, then it cannot be anything but just a single binomial (one vertex + another vertex, and if there are intermediate monomials, it factorizes into a power of this binomial). When $\text{dim}(\phi) = 2$, $\text{init}(\phi)$ would have two distinct binomial factors. Conversely, for any initial form with two binomial factors, these factors cannot belong to the same edge – the initial form is said to be simple. The only monomials are vertices of its convex hull (in the opposite situation we would encounter some monomials which are not the vertices – a contradiction).

The same argument is applied by induction to any number of binomial factors. Namely, assume we have a product of $n$ binomials, which defines a face of dimension $n$. If we join to them one more binomial, the dimension will increase to $n + 1$, due to convexity and the fact that the faces $\phi_n$ and $\phi_{n+1}$ are both simple (so it will never happen that the extra face $\phi_{n+1}$ will be linearly dependent with any of sub-faces of $\phi_n$. Since if it would, then we will unavoidable loose some of its edges by taking the convex hull, which contradicts the simplicity property, and also the fact that $\phi_n$ is actually a face of $\phi_{n+1}$), see the figure (3). □

Figure 3: Simple initial form with two distinct binomial factors corresponds to a 2d face $\phi_2$. Joining an extra edge to $\phi_2$ will lead to $\phi_3$. Since the simplicity relation is preserved, it increases the dimension by one. The configuration in the middle does not preserve this relation, therefore is not simple. For the middle picture, the bold edge of $\phi_2$ is not an edge of the resulting convex hull.

It’s also important to mention that each initial form is a summand of $\mathcal{R}$, and $\mathcal{R}$ itself corresponds to the “filled” diagram:

\[
\mathcal{R} \simeq \begin{array}{c|c|c|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]  \quad (4.11)

In what follows, we will consider only simple initial forms.
5. Two-dimensional case

Our first result concerns the $m = 2$ case:

\[
\begin{align*}
F_0 &= a_0 + a_1 z_1 z_2 \\
F_1 &= b_0 + b_1 z_1 + b_2 z_1^2 \\
F_2 &= c_0 + c_1 z_2 + c_2 z_2^2
\end{align*}
\] (5.1)

Without loss of generality, we assume $\alpha, \beta \geq 2$ (for $\alpha, \beta \in \{0, 1\}$ the Newton polytope degenerates; for the negative values, after multiplying each $F_i$ by a suitable monomial, we end up with an isotopy of the polytopes). We have subtracted the anti-diagonal, since it simply amounts to framing. Therefore, $C = \text{diag}(\alpha, \beta)$ is of our main interest. In this case $N(\mathcal{R}) = N_{2,2}$, the famous Gelfand-Kapranov-Zelevinski polytope of the classical resultant [16]. However, our novelty is that we equip this polytope with some additional structure, carried out by initial forms at its edges. Therefore, they will keep track of all the monomials on the 1-dimensional skeleton of $N(\mathcal{R})$. We are aiming to get a universal formula with $(\alpha, \beta)$ as parameters. Afterwards, we will see that the $(x, y)$-projection of this polytope is well-behaved, which means that $A(x, y)$ is tempered.

Warm-up example: $(\alpha, \beta) = (2, 2)$

At the beginning we give a detailed exposition of the first non-trivial case $C = \text{diag}(2, 2)$ (4).

We eliminate $z_1, z_2$ from the Nahm equations (1.4), to get the specialized quiver resultant:

\[
\mathcal{R}_{\text{diag}(2,2)}(x_1, x_2, y) = x_1^2 x_2^2 y^4 + x_1 x_2 y^3 - 2x_1 x_2 y^2 + x_1 y^2 + x_2 y^2 + y + 1.
\] (5.2)

The Newton polytope $N(\mathcal{R})$ coincides with the Gelfand-Kapranov-Zelevinsky (GKZ) polytope $N_{2,2}$ (5), extensively studied in [16, 30]. Its vertices $(i, j, k)$ encode the powers of monomials $x_1^i x_2^j y^k$ in (5.2):

\[(2, 2, 4), (1, 1, 3), (1, 0, 2), (0, 1, 2), (0, 0, 1), (0, 0, 0)\] (5.3)

In this case we have only one monomial $-2x_1 x_2 y^2$, which is not a vertex of $N(\mathcal{R})$. Instead, it divides the bottom edge into two equal intervals. We can combine some monomials in (5.2) and write

\[
\mathcal{R}_{\text{diag}(2,2)}(x_1, x_2, y) = (x_1 x_2 y^2 - 1)^2 - y (x_1 y + 1)(x_2 y + 1)
\] (5.4)

Recall that Gelfand-Kapranov-Zelevinsky polytope $N_{m,n}$ is the Newton polytope for classical resultant of two polynomials of degrees $m$ and $n$. 

---
Figure 5: GKZ polytope $N_{2,2}$, along with the monomials (blue nodes) of $R_{\text{diag}(2,2)}$, compare with the $(x,y)$-projection (1)

It turns out that the two binomial summands are supported on the distinct faces of $N(R)$: the convex hull of $(x_1x_2y^2 - 1)^2$ gives the “bottom” edge $\phi_{0,0}$, and $-y(x_1y + 1)(x_2y + 1)$ gives the 2-dimensional quadrangular face $\phi_{1,1}$. The latter belongs to the plane defined by equation:

$$1 + x_1 + x_2 - y = 0$$

(5.5)

Its normal vector is $\omega = (1, 1, -1)$, up to translation and multiplication by a scalar. Rescale the variables $x_1, x_2, y$ with respect to this vector:

$$R_{\text{diag}(2,2)}(c^1 x_1, c^1 x_2, c^{-1} y) = (x_1 x_2 y^2 - 1)^2 - c^{-1} y(x_1y + 1)(x_2y + 1), \quad c \in \mathbb{C}$$

(5.6)

Therefore, the parameter $c$ separates the faces of $N(R)$ as the two summands in (5.4). So, we get the two distinguished initial forms:

$$\text{init}_{\phi_{0,0}} = \lim_{c \to \infty} R_{\text{diag}(2,2)}(c^{\omega_1} x_1, c^{\omega_2} x_2, c^{\omega_3} y) = (x_1 x_2 y^2 - 1)^2$$

$$\text{init}_{\phi_{1,1}} = \lim_{c \to 0} (c \cdot R_{\text{diag}(2,2)}(c^{\omega_1} x_1, c^{\omega_2} x_2, c^{\omega_3} y)) = y(x_1y + 1)(x_2y + 1)$$

(5.7)

Let us move to the unspecialized case (4.2):

$$F_0 = a_0 + a_1 z_1 z_2$$
$$F_1 = b_0 + b_1 z_1 + b_2 z_1^2$$
$$F_2 = c_0 + c_1 z_2 + c_2 z_2^2$$

(5.8)

The sparse mixed resultant from the supports of (5.8) reads:

$$R(a_0, a_1, b) = (a_0^2 b_2 c_2 - a_1^2 b_0 c_0)^2 + a_0 a_1 (a_0 b_2 c_1 + a_1 b_1 c_0)(a_0 b_1 c_2 + a_1 b_0 c_1).$$

(5.9)

We have: $R(y, -1, -1, 1, x_1, -1, 1, x_2) = R_{\text{diag}(2,2)}(x_1, x_2, y).$
On another hand, we can use mixed decompositions of the Minkowski sum $Q = Q_0 + Q_1 + Q_2$, where $Q_i = \text{conv}(F_i)$. Consider the first initial form: $(a_0^2b_2c_2 - a_1^2b_0c_0)^2$. It is attached to the bottom edge of $N(R)$. The square comes from the areal factor of the grey hexagon (6, left). The total degree of a given binomial equals to the euclidean volume of the corresponding cell of a mixed decomposition, which is equal to 6 in our case. Notice that the binomial $a_0^2b_2c_2 - a_1^2b_0c_0$ is the sub-resultant for $b_1 = c_1 = 0$.

Finally, consider the second initial form $a_0a_1(a_0b_2c_1 + a_1b_1c_0)(a_0b_1c_2 + a_1b_0c_1)$. It splits into the product of four distinct sub-resultants, which represent four distinct cells of our mixed decomposition on the figure (6, right):

\[
\begin{align*}
b_0 = c_2 &= 0, & a_0b_2c_1 + a_1b_1c_0 \\
b_2 = c_0 &= 0, & a_0b_1c_2 + a_1b_0c_1 \\
a_1 = b_2 = c_2 &= 0, & a_0 \\
a_0 = b_0 = c_0 &= 0, & a_1
\end{align*}
\]

Figure 6: Mixed decompositions in $(z_1, z_2)$-plane: $\varphi_{0,0}$ (left) and $\varphi_{1,1}$ (right)

Therefore, one may start with all possible mixed decompositions and compute the associated initial forms just by looking at the cell arrangement in each such decomposition. This is exactly the formula (4.3).

**General ($\alpha, \beta$)**

Moving to general ($\alpha, \beta$) case, we have to add more definitions.

**Definition 5.1** Given a binomial $\mu^p + \eta^q$, $p = (p_1, \ldots, p_k)$, $q = (q_1, \ldots, q_k)$, define

\[
\text{GCD} (\mu^p + \eta^q) := \left( \frac{\mu}{\text{gcd}(\mu, q)} \right)^{\frac{p}{\text{gcd}(p, q)}} + \left( \frac{\eta}{\text{gcd}(\mu, q)} \right)^{\frac{q}{\text{gcd}(p, q)}} \right)^{\text{gcd}(p, q)}
\]

Also, for a product of binomials for any integer $s \geq 1$:

\[
\text{GCD} \left( \prod_{i=1}^{s} (\mu^{p_i} + \eta^{q_i}) \right) = \prod_{i=1}^{s} \text{GCD} (\mu^{p_i} + \eta^{q_i})
\]

\[\text{(5.12)}\]
This operator implements the rule for computing the exponents \( k_i \) in (4.3). A few examples:

\[
\begin{align*}
\text{GCD} \left( a_0^2 b_2^2 c_2 + a_1^2 b_0^2 c_0 \right) &= a_0^2 b_2^2 c_2 + a_1^2 b_0^2 c_0, \\
\text{GCD} \left( a_4 b_2^2 c_2 + a_1^4 b_0^2 c_0^2 \right) &= (a_0 b_2 c_2 + a_1 b_0 c_0)^2.
\end{align*}
\]  

(5.13)

**Proposition 5.1** The Newton polytope \( N(\mathcal{R}) \) for the system (5.1) supports the following simple initial forms:

\[
\begin{align*}
\text{init}_a &= \text{GCD} \left( a_0^\alpha b_2^\beta c_2 + (-1)^{\alpha+\beta} a_1^\alpha b_0^\beta c_0 \right), \\
\text{init}_b &= a_0 a_1^{(\alpha-1)(\beta-1)} (a_0^\alpha b_2^\beta c_2 + (-1)^{\alpha+\beta} a_1^\alpha b_0^\beta c_0), \\
\text{init}_c &= a_1^{(\alpha-1)\beta} b_0^{\alpha-1} (a_0 a_1^\beta b_2^\alpha c_2 + a_1 b_0^\beta c_2), \\
\text{init}_d &= a_1^{(\alpha-1)\beta} a_0^{\alpha-1} (a_0 a_1^\beta b_2^\alpha c_2 + a_1 b_0^\beta c_2), \\
\text{init}_e &= a_0^\alpha b_2^\beta c_2 \cdot \text{GCD} \left( a_0^{(\alpha-1)\beta} b_2^\beta c_2^\alpha + (-1)^{(\alpha-1)(\beta+1)+\beta} a_1^{(\alpha-1)\beta} b_0^\beta c_0^\alpha \right), \\
\text{init}_f &= a_0^\beta b_2 \cdot \text{GCD} \left( a_0^{(\alpha-1)\beta} b_2^\beta c_2 + (-1)^{(\alpha-1)(\beta+1)+\beta} a_1^{(\alpha-1)\beta} b_0^\beta c_2 \right), \\
\text{init}_g &= a_0^{\alpha+\beta} b_2 c_2 \cdot \text{GCD} \left( a_0^{(\alpha-1)\beta} b_2^\beta c_2^\alpha + (-1)^{(\alpha-1)(\beta-1)+\beta} a_1^{(\alpha-1)\beta} b_0^\beta c_2^\alpha \right), \\
\text{init}_h &= a_1^{\alpha+\beta} b_0^{\beta-1} c_0^{\alpha-1} (a_0 a_1^\beta b_2 + a_1^\beta b_0^\beta c_2).
\end{align*}
\]

where \( a, b, c, d, e, f, g, h \) are the faces of \( N(\mathcal{R}) \) (Figure 7).

Proof. Every binomial factor in \( \text{init}_a \) corresponds to a sub-resultant, which perfogram is given on the right side of each expression \( \text{init}_a = \ldots \). Let’s associate mixed decompositions to these initial forms, as shown on (8). This provides a desired combinatorial interpretation of the faces. Each hexagon in a mixed decomposition gives the distinct binomial factor in the corresponding initial form, and all rectangles together determine the monomial prefactor. The GCD operator (5.1) has the following interpretation: each \( k_i \geq 1 \) in (4.3) is uniquely fixed when \((\alpha, \beta)\) are fixed, so that the total degree of \( \tilde{R}^k \) equals to the area of the \( \iota \)-th cell of a mixed decomposition.

E.g., for \( \text{init}_a \) there is only a single hexagon (the top-left on 8), which is \( Q \) itself – so there is no monomial prefactor. This hexagon gives the sub-resultant \( \tilde{R} = a_0^{\alpha \beta} b_2^\beta c_2 + (-1)^{\alpha \beta+\alpha+ \beta} a_1^{\alpha \beta} b_0^\beta c_0 \). We see that the area of \( Q \) is \( \alpha \beta + \alpha + \beta \), so if \( \alpha \) and \( \beta \) are not co-prime,
Figure 7: The Newton polytope $N_{2,2}$ with the initial forms (5.1): top (left) and bottom (right). Blue faces are those, which do not have other points except the vertices.

Figure 8: Mixed decompositions of $Q$, induced by the faces of the polytope $N_{2,2}$

It would give $k > 1$, hence

$$\text{init}_a = \gcd \left( a_0^\alpha b_2^\beta c_1^\alpha + (-1)^{\alpha\beta + \alpha + \beta} a_1^{\alpha\beta} b_0^\beta c_0^\alpha \right).$$

In the case of $\text{init}_b$ we have two hexagons (giving the two distinct binomial factors) and two quadrangles for the monomial: the bottom square is $a_0$, and the top quadrangle is $a_1^{(\alpha-1)(\beta-1)}$ (also depicted on 2). The rest is carried out analogously.

It turns out that using the mixed decompositions $a, b, c, d, e, f, g, h$ we completely described the bijection between the faces and simple initial forms. $\square$

**Corollary 5.1** Quiver A-polynomial for any two-vertex quiver is tempered, with its face polynomials all being binomials.
It follows directly from factorization formulas for the initial forms \( a, b, c, d, e, f, g, h \). The polytope \( N(\mathcal{R}) \) projects onto \( N(A) \) in such way that the faces of \( N(\mathcal{R}) \) do not collide with each other (it is the property of \( N_{2,2} \), which is depicted on Figure 5). That’s why binomiality of the initial forms implies binomiality of the face polynomials, which further implies that \( A(x, y) \) is tempered. **Remark:** The non-diagonal case is simply a framing transformation \( x \mapsto xy' \), which amounts to affine isotopy of the polytopes, therefore not bringing any substantial changes.

6. Three-dimensional case

\[
\begin{align*}
F_0 &= a_0 + a_1 z_1 z_2 z_3 \\
F_1 &= b_0 + b_1 z_1 + b_2 z_1^2 \\
F_2 &= c_0 + c_1 z_2 + c_2 z_2^2 \\
F_3 &= d_0 + d_1 z_2 + d_2 z_2^2
\end{align*}
\]  

(6.1)

As before, we assume \( \alpha, \beta, \gamma \geq 2 \). In what follows, we study a very special subset of initial forms:

\[
\begin{array}{c|c|c|c}
\phi_{0,0} : & \cdot \cdot & \cdot \cdot & \\
\phi_{3,0} : & \cdot \cdot \cdot & \cdot \cdot \cdot & \\
\phi_{2,1} : & \cdot \cdot \cdot & \times & \cdot \cdot \cdot & \times \cdot \cdot \cdot & \cdot \cdot \cdot \\
\phi_{1,2} : & \cdot \cdot \cdot & \times & \cdot \cdot \cdot & \times & \cdot \cdot \cdot \\
\phi_{0,3} : & \cdot \cdot \cdot & \cdot \cdot \cdot & \\
\end{array}
\]  

(6.2)

Notice that \( \dim \phi_{0,0} = \dim \phi_{3,0} = \dim \phi_{0,3} = 1 \), whereas \( \dim \phi_{2,1} = \dim \phi_{1,2} = 3 \). For curiosity of the reader, we list some extra initial forms which correspond to two-dimensional faces, and together with \( \phi_{0,0}, \phi_{2,1} \) and \( \phi_{1,2} \) they fully describe \( N(\mathcal{R}) \) (but they do not contribute to the edges of \( N(A) \), thus are irrelevant to the K-theoretic property):

\[
\begin{array}{c|c|c|c}
\cdot \cdot \cdot & \times & \cdot \cdot \cdot & \\
\cdot \cdot \cdot & \times & \cdot \cdot \cdot & \\
\cdot \cdot \cdot & \times & \cdot \cdot \cdot & \\
\end{array}
\]  

(6.3)
Returning to (6.2), we compute the corresponding initial forms (but skipping \(\phi_{3,0}\) and \(\phi_{0,3}\), as they are rather trivial and can be recovered from the others):

\[
init(\phi_{0,0}) = \text{GCD} \left( a_0^{\alpha} b_1^{\beta} c_2^{\gamma} d_2^{\alpha} + (-1)^{\sigma+1} a_1^{\alpha} b_0^{\beta} c_0^{\gamma} d_0^{\alpha} \right)
\]

\[
init(\phi_{2,1}) = \mu_{2,1} \cdot \text{GCD} \left( \left( a_0^{\alpha-1} b_2 c_1^{\alpha-1} d_1^{\alpha-1} + (-1)^{\alpha+1} a_1^{\alpha-1} b_1 c_0^{\alpha-1} d_0^{\alpha-1} \right) \times \right.

\left. \left( a_0^{-1} b_1^{\alpha-1} d_1^{\alpha-1} + (-1)^{\alpha+1} a_1^{-1} b_0^{\alpha-1} d_0^{\alpha-1} \right) \times \right.

\left( a_0^{-1} b_1^{\alpha-1} c_1^{\alpha-1} + (-1)^{\alpha+1} a_1^{-1} b_0^{\alpha-1} c_0^{\alpha-1} \right) \right)
\]

\[
init(\phi_{1,2}) = \mu_{1,2} \cdot \text{GCD} \left( \left( a_0^{\beta-1} b_1^{\gamma-1} c_2^{\gamma-1} d_2^{\beta-1} \right) \right.

\left. + \left( -1 \right)^{\beta+\gamma+1} a_1^{\beta-1} b_0^{\gamma-1} c_1^{\gamma-1} d_1^{\beta-1} \right) \times \right.

\left. \left( a_0^{\alpha-1} c_1^{\alpha-1} b_2^{\gamma-1} d_2^{\alpha-1} \right) \times \right.

\left. \left( -1 \right)^{\alpha+\gamma+1} a_1^{\alpha-1} c_0^{\alpha-1} b_1^{\gamma-1} d_1^{\alpha-1} \right) \times \right.

\left. \left( a_0^{\alpha-1} d_1^{\alpha-1} b_2^{\beta-1} c_2^{\alpha-1} \right) \times \right.

\left. \left( -1 \right)^{\alpha+\beta+1} a_1^{\alpha-1} d_0^{\alpha-1} b_1^{\beta-1} c_1^{\alpha-1} \right) \right)
\]

where \(\sigma = \alpha \beta \gamma + \alpha \beta + \alpha \gamma + \beta \gamma\), and the monomials are:

\[
\mu_{2,1} = a_0 a_1 a_2 b_1 c_0 d_0 \left( -1 \right)^{\alpha+\gamma+1} a_1^{\beta-1} b_0^{\gamma-1} c_1^{\gamma-1} d_1^{\beta-1} \right)
\]

\[
\mu_{1,2} = a_0 a_1 a_2 b_1 c_0 d_0 \left( -1 \right)^{\alpha+\beta+1} a_1^{\alpha-1} b_1^{\gamma-1} c_1^{\gamma-1} d_1^{\alpha-1} \right)
\]

As the reader may have noticed, \(init(\phi_{p,q})\) stands for the initial form, given by the product over permutations of all perfograms with \(p\) rows of the form [● ● ], and \(q\) rows of the form [● ● ] with \(m = p + q\). We will give a general prescription in Section 7.

It turns out that these initial forms are supported on the faces of maximal dimension. Since every row of the form [● ● ] can be permuted only identically, the total number of distinct binomials for initial forms with [● ● ]-type rows is smaller, contrary to those which contain only [● ● ]- and [● ● ]-type rows (recall the Proposition 4.2).

Among other things, we will always include \(\phi_{0,0}\), which is the bottom edge of both \(N(R)\) and \(N(A)\). Its vertices attain the minimal and maximal powers of \(y\), and by assumption the Newton polygon has no horizontal slopes (the non-degeneracy condition).

To calculate (6.4), we perform a cheap trick: write \(a_0 = -a_1 z_1 z_2 z_3\) and raise left and right hand sides of this equation to powers of \(z_i\) consequently, substituting to the right hand side at each step the corresponding expression for \(z_i\).

The monomial part (6.5) is a bit more subtle, since we have to play with “Lego boxes” to form a proper subdivision of \(Q\). Only when all the boxes are aligned properly, we get a mixed decomposition, which amounts to the expressions for \(\mu_{p,q}\) (see Figure (9) as an example).

Now we focus on the case \(\beta = \gamma = \alpha\). We will show that it is well-behaved: the data of (6.2) is sufficient to describe all the edges of \(N(A)\)! The following bullet-points are important to us:
Figure 9: The mixed decomposition induced by $\phi_{1,2}$: there are three red cells, which correspond to the binomials in (6.4), while the union of blue cells give the monomial $\mu_{1,2}$ (6.5). This generalizes (2) to three dimensions.

- every vertex of $N(A)$ has a unique preimage, which is a vertex of $N(\mathcal{R})$

- every edge of $N(A)$ is an image of a unique simple face from the set $\{\phi_{0,0}, \phi_{3,0}, \phi_{2,1}, \phi_{1,2}, \phi_{0,3}\}$

First, let’s put $\alpha = \beta = \gamma$ into the formulas (6.4). For each initial form, pick up the minimal and maximal powers of $a_0$ ($\simeq y$) (notice that they are attained only once for each case):

$$[a_0^{\min} \cdot \text{coeff}(\phi_s, a_0^{\min}), a_0^{\max} \cdot \text{coeff}(\phi_s, a_0^{\max})] =: [\phi_s^{\min}, \phi_s^{\max}]$$

We get for $\phi_{0,0}$, $\phi_{2,1}$ and $\phi_{1,2}$, respectively:

$$\begin{bmatrix}
  a_1 \alpha^3 b_0 \alpha^2 c_0 \alpha^2 d_0 \alpha^2, \\
  a_0 a_1 \alpha^3 - 1 b_0 \alpha^2 - 1 b_1 c_0 \alpha^2 - 1 c_1 d_0 \alpha^2 - 1 d_1, \\
  a_0 3 \alpha - 2 a_1 \alpha^3 - 3 \alpha + 2 b_0 \alpha^2 - 2 \alpha - 1 b_1 2 \alpha - 2 b_2 c_0 \alpha^2 - 2 \alpha - 1 c_1 2 \alpha - 2 c_2 d_0 \alpha^2 - 2 \alpha + 1 d_1 2 \alpha - 2 d_2
\end{bmatrix}$$

(6.6)

Now coming back to $\phi_{3,0}$ and $\phi_{0,3}$: they are simply given by

$$\phi_{3,0} = \phi_{0,0}^{\min} + \phi_{2,1}^{\min}, \quad \phi_{0,3} = \phi_{1,2}^{\max} + \phi_{0,0}^{\max}.$$  

(6.7)

So (6.6) are the monomials, which project onto the vertices $N(A)$. Therefore, each face $\phi_{p,q}$ projects onto one of the edges of $N(A)$, and from (6.6) we can write down all the vertices of
\( N(A) \), starting from the origin \((x, y) = (0, 0)\) and going around clockwise:

\[
(0, 0), \ (0, 1), \ (3, 3\alpha - 2), \ (6\alpha - 3, 3\alpha^2 - 3\alpha + 1), \ (3\alpha^2, \alpha^3). \tag{6.8}
\]

Hence binomiality of the face polynomials for \( \text{diag}(\alpha, \alpha, \alpha) \) follows from binomiality of its preimages in \( \mathcal{R} \): the initial forms (6.2). Unfortunately, the same does not work for generic \( \alpha, \beta, \gamma \). As we will see, we would have to take into account some extra faces lying between \( \phi_{1,2} \) and \( \phi_{2,1} \).

There seems to be the two kind of problems we may encounter:

1. The projection of \( \phi_{1,2} \) captures the endpoints of an edge, but not all its interior points. The simplest example is \((\alpha, \beta, \gamma) = (2, 3, 3)\).

2. The projection of \( \phi_{1,2} \) does not capture one or more endpoints along with some of the interior points of an edge. Examples include \((\alpha, \beta, \gamma) = (2, 3, \gamma) \) for \( \gamma \geq 4 \)

Both situations are fixed by intermediate faces between the projection of \( \phi_{i,j} \). In fact, such faces are present for any \( \alpha, \beta, \gamma \), but the key point is that if \( \alpha = \beta = \gamma \), they project onto the interior of \( N(A) \) and do not contribute to the edges at all. But when \( \alpha, \beta, \gamma \) are generic, we would have to take them into account. We determine one of such faces for \( \alpha \leq \beta \leq \gamma \):

\[
\text{init}_{f_4} = \mu_{f_4} \text{GCD} \left( a_0^{(\beta-1)(\gamma-1)}b_1^{(\beta-1)(\gamma-1)}c_2^{\gamma-1}d_2^{\beta-1} + \right.
\]

\[
(-1)^{\beta+\gamma+1} a_1^{(\beta-1)(\gamma-1)}b_0^{(\beta-1)(\gamma-1)}c_1^{\gamma-1}d_1^{\beta-1}, \tag{6.9}
\]

\[
\mu_{f_4} = a_0^{-1+\beta+\gamma}a_1^\sigma b_1^{-1+\beta+\gamma}c_0^{\gamma-2+\alpha}c_2d_0^{(\alpha-1)(\beta-1)+\alpha-1}
\]

where \( \sigma = (\alpha-1)(\beta-1)(\gamma-1) + \alpha - 1 + (\alpha - 1)(\beta - 1) + (\alpha - 1)(\gamma - 1) \).

This is the extra edge of \( N(\mathcal{R}) \), which is needed to fully describe \( N(A) \) for \( \alpha, \beta, \gamma \) all being distinct. This is a very interesting observation, since \( \text{init}_{f_4} \) contains the binomial factor equal to one of the factors of \( \phi_{1,2} \). On another hand, \( f_4 \) does not lie on the same hyperplane with \( \phi_{1,2} \). That’s why if we take the convex hull of \( f_4 \cup \phi_{1,2} \), we obtain the four-dimensional counterpart of \( N(\mathcal{R}) \), which itself is four-dimensional. We have to emphasise that the two kinds of problems are in question related to the projection, since \( f_4 \) always appears as a face of \( N(\mathcal{R}) \), but not always as an edge of \( N(A) \). Of course, \( f_4 \) has certain “siblings”, given by a different choice of a distinct binomial factor in \( \phi_{1,2} \):

\[
\phi_{1,2} : \quad \{ [F_{1,1}, F_{1,2}, F_{1,3}, F_{1,4}], [F_{2,1}, F_{2,2}, F_{2,3}, F_{2,4}], [F_{3,1}, F_{3,2}, F_{3,3}, F_{3,4}] \}, \tag{6.10}
\]

where \( F_{i,j} \) are the edges of \( \phi_{1,2} \). Therefore, each such quadruple defines a two-dimensional face of \( \phi_{1,2} \). We group them as follows:

\[
F_{1,*} : \quad \bullet \bullet \bullet \quad F_{2,*} : \quad \bullet \bullet \bullet \quad F_{3,*} : \quad \bullet \bullet \bullet \tag{6.11}
\]
To each of the three sets, we add an extra edge \( Q_{*,1} \):

\[
\begin{align*}
Q_{1,1} &= \text{init}_{f_4} \\
Q_{2,1} &= \text{init}_{f_4}(b \leftrightarrow c) \\
Q_{3,1} &= \text{init}_{f_4}(b \leftrightarrow d)
\end{align*}
\] (6.12)

On another hand, there is also a face of mixed type (a part of \( \phi_{2,1} \))

\[
R : \begin{array}{c|c|c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

Its initial form is given by:

\[
\text{init}_R = \mu_R \cdot \text{GCD} \left( \left( a_0^{(\beta-1)(\gamma-1)} b_1^{(\beta-1)(\gamma-1)} c_2^{\gamma-1} d_2^{\beta-1} + \\
(-1)^{\beta+\gamma+1} a_1^{(\beta-1)(\gamma-1)} b_0^{(\beta-1)(\gamma-1)} c_1^{\gamma-1} d_1^{\beta-1} \right) \times \\
\left( a_0^{a-1} b_2 c_1^{a-1} d_1^{a-1} + (-1)^{\alpha+1} a_1^{\alpha-1} b_1 c_0^{\alpha-1} d_0^{\alpha-1} \right) \right),
\]

\[
\mu_R = a_0^{1+\beta+\gamma} a_1^{\sigma} b_1^{\beta-2+\gamma} c_0^{(\alpha-1)(\gamma-1)} d_0^{(\alpha-1)(\beta-1)} d_2
\]

where \( \sigma = (\alpha - 1)(\beta - 1)(\gamma - 1) + (\alpha - 1)(\gamma - 1) + (\alpha - 1)(\beta - 1) \). In what follows, we assume \( 2 \leq \alpha < \beta < \gamma \).

**Proposition 6.1** For \( (\alpha, \beta, \gamma) \), the only contribution to the edges of \( N(A) \) is due to the initial forms \( \{\phi_{i,j}\}_{i+j=3} \), completed with \( Q_{1,1} \) and \( R \) in the following cases:

1. \( (\alpha, \alpha, \alpha) \): \( \phi_{0,0}, \phi_{3,0}, \phi_{2,1}, \phi_{1,2}, \phi_{0,3} \) project onto the edges \( f_0, f_1, f_2, f_3, f_4 \), and every vertex of \( f_i \) has a unique preimage

2. \( (\alpha, \alpha, \beta) \): \( \{\phi_{i,j}\}_{i+j=3} \) are still OK, although some of the monomials of \( \phi_{1,2} \) and \( \phi_{2,1} \) will project onto the interior of \( N(A) \), unlike in the case above. Nevertheless, this does not affect binomiality of the edges of \( N(A) \)

3. \( (\alpha, \beta, \beta) \): we need to include the extra initial form \( \text{init}_R \), since \( \phi_{1,2} \) would have an issue: it will capture the vertices, but not all the intermediate points of the edge. Therefore, \( \text{init}_R \) will fully cover this problematic edge, and since it is a binomial, so is true for the projection.

4. \( (\alpha, \beta, \gamma) \) all distinct: instead of \( R \), we have to take \( Q_{1,1} \)

**Remark:** It is worth to mention that \( Q_{i,j} \) and \( R \) are defined up to permutation of the rows in its perfograms, because such permutations are indistinguishable after projecting onto \((x, y)\)-plane. Therefore, in the proposition above \( Q_{1,1} \) may be safely replaced with \( Q_{2,1} \) or \( Q_{3,1} \).

To sum up, we established the four cases which produce non-isotopic polytopes, and gave a description of the initial forms, contributing to the edges of \( N(A) \). As a corollary, we obtained that the quiver A-polynomial for \( C = \text{diag}(\alpha, \beta, \gamma) \) is tempered. In the next section we proceed to higher dimensions.
7. Arbitrary dimension

This section contains the main result of the paper. Let \( \mathcal{R} \) be the quiver resultant (4.2) and \( N(\mathcal{R}) \) its Newton polytope. It is the \((m + 1)\)-dimensional convex polytope in \( \mathbb{R}^{2+3m} \), where \( m \) is the quiver size. We take \( m \geq 2 \), since \( m = 1 \) gives \( N(\mathcal{R}) = N(A) \) is just a triangle, and is trivial. Also we fix the quiver matrix \( C = \text{diag}(\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i \geq 2 \). We show the combinatorial structure of the one-dimensional skeleton of \( N(\mathcal{R}) \), which is captured by its initial forms. This picture is the most fundamental (non-diagonal quivers can be build upon the same skeleton, but with additional assumptions, which are outside of the scope of this paper), and it induces binomiality of the face polynomials of the projection polygon \( N(A) \). The latter implies the K-theoretic property (3.13) for \( A(x,y) \), which is equivalent to the fact that it is a tempered polynomial, therefore quantizable. We give a full proof for \( \text{diag}(\alpha, \ldots, \alpha) \) and a sketch when \( \alpha_i \) are all distinct.

7.1 \( \alpha_i \) are all equal

Let \( m = p + q \). We are aiming to define the family of simple initial forms \( \{ \phi_{p,q} \} \), where \( p \) is the number of \([\bullet ]\)-type rows – we indicate them by the index subset \( I = \{ i_1, \ldots, i_p \} \), and \( q \) is the number of \([\bullet \bullet ]\)-type rows \( K = \{ k_1, \ldots, k_q \} \), in each of the perfogram contained in \( \phi_{p,q} \). Therefore, they will differ by a sequence of permutations. Dimension of the corresponding face induced by \( \phi_{p,q} \) is equal to the number of its distinct binomial factors (due to the Proposition 4.2). In what follows, we give a full description of \( \{ \phi_{p,q} \} \) for the diagonal quiver.

Proposition 7.1 Let \( I = \{ i_1, \ldots, i_p \} \), \( K = \{ k_1, \ldots, k_q \} \), \( p + q = m \). Define

\[
\varphi_{I,K} := \left( a_0 \prod_{i \in I} b_{i,1} \right) \prod_{k \in K} \left( b_{k,2} \right)^{\alpha_k - 1} \prod_{k' \in K} \left( b_{k',2} \right)^{\alpha_{k'} - 1} + (-1)^{1+\sum_{k \in K} \prod_{k' \in K, k' \neq k} (\alpha_{k'} - 1)} \left( a_1 \prod_{i \in I} b_{i,0} \right) \prod_{k \in K} \left( b_{k,1} \right)^{\alpha_k - 1} \prod_{k' \in K} \left( b_{k',1} \right)^{\alpha_{k'} - 1} \tag{7.1}
\]

Then \( \phi_{p,q} \) are initial forms:

\[
\phi_{p,q} := \mu_{p,q} \prod_{\text{perm}(I,K)} \text{GCD}(\varphi_{I,K}) \tag{7.2}
\]

where the product is taken over all permutations of \( I, K \) of size \( p \) and \( q \), selected as a partition of \( \{1, 2, \ldots, m\} \), and the monomial \( \mu_{p,q} \) is given by

\[
\mu_{p,q} = a_0^{1+\sum_{|K'|=q-1} \prod_{k' \in K'} (\alpha_{k'} - 1)} \prod_{i \in K} b_{i,0} \prod_{i \notin K'} b_{i,2} \tag{7.3}
\]

where \( \delta(q) = 0 \) if \( q \leq m - 1 \), and \( 1 \) otherwise.
The expression for $\varphi_{I,K}$ is rather easy. Recall that we are dealing with polynomials (4.2), where we take $C = \text{diag}(\alpha_1, \ldots, \alpha_m)$. Write the first equation $F_0 = 0$ as

$$a_0 = -a_1 z_1 \ldots z_m,$$

and then raise it consequently in powers $(\alpha_k - 1)$ where $k \in K$ corresponds to [●●]-type rows, each time plugging $z_k^{\alpha_k - 1} = -\frac{b_{k,1}}{b_{k,2}}$ (do this for all permuted perfograms). The monomial part is just slightly more involved. We have find out that the following permutation moves generate all letters in the monomial $\mu_{p,q}$, where we short-handed (perm) for the set of all permutations of the rows of a given perfogram:

$$a_0 : \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \cdots \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} , \quad \text{unless } |K'| \leq q - 1$$

(7.5)

$$a_1 : \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \cdots \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} , \quad \text{unless } |K'| \geq q + 1$$

(7.6)

$$b_{\star,0} : \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \cdots \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} , \quad \text{unless } |K'| \geq q + 1$$

(7.7)

$$b_{\star,2} : \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} \times \cdots \times \begin{array}{c|c|c|c} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & \bullet & \bullet \end{array} , \quad \text{unless } |K'| \leq q - 1$$

(7.8)

Thus, taking the product over all such perfograms, we get the proper mixed decomposition of the Minkowski sum $Q = \sum_{i=1}^m Q_i$, which is $m$-dimensional. Therefore, $\{\phi_{p,q}\}$ are well-defined initial forms. □
We would also need the “bottom” one-dimensional face of \(N(\mathcal{R})\), which mixed decomposition is just the convex hull of \(Q\) itself, and the initial form is

\[
\phi_{0,0} := \text{GCD} \left( a_0 \prod \alpha_j \prod b_{j,2}^{\sum_j \alpha_j} + (-1)^j \prod \alpha_j + \sum \prod \alpha_j \prod a_i \prod b_{j,0}^{\sum_j \alpha_j} \right) \approx \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\tag{7.9}
\]

In particular, it has \(\mu_{0,0} = 1\).

**Remark:** The GCD rule (5.1) becomes especially important with the increase of quiver size. In fact, it controls how monomials appear (and disappear) on 1d-skeleton of each face of \(N(\mathcal{R})\). The rule turns out to be purely arithmetic (depending on the diagonal values \(\alpha_i\))!

The following result is the most important for us, since it implies that \(A(x, y)\) is tempered for any quiver of the shape diag(\(\alpha, \alpha, \ldots, \alpha\)) of size \(m\):

**Theorem 7.1** For any \(\phi_{p,q}\), denote its extremal monomials (having minimal/maximal powers of \(a_0\)) by \(\phi_{p,q}^{\text{min}}\) and \(\phi_{p,q}^{\text{max}}\). Then the Newton polygon \(N(A)\) is the convex hull of their images under the principal specialization, if and only if \((\alpha_1, \ldots, \alpha_m) = (\alpha, \ldots, \alpha)\). Moreover, it yields a bijection between the initial forms \(\{\phi_{p,q}\}\) and the edges of \(N(A)\) (Figure 10):

\[
\phi_{0,0} \longleftrightarrow f_0, \quad \phi_{m,0} \longleftrightarrow f_1, \quad \phi_{m-1,1} \longleftrightarrow f_2, \quad \ldots
\tag{7.10}
\]

Proof. Let \((\alpha_1, \ldots, \alpha_m) = (\alpha, \ldots, \alpha)\). These are the key observations: a) every vertex of \(N(A)\) has a unique preimage in \(\mathcal{R}\), which is a vertex of some \(\phi_{p,q}\), b) for a given \(\phi_{p,q}\), all its monomials project onto the same line segment, therefore this segment determines a unique edge of \(N(A)\). Going around the faces, we obtain the full polygon \(N(A)\) with all its face polynomials, binomiality (and factorization) properties of which follows immediately from \(\phi_{p,q}\).

The candidate preimages for the vertices of \(N(A)\) are \(\phi_{\ast}^{\text{min}}\) and \(\phi_{\ast}^{\text{max}}\), which are clearly the vertices of \(N(\mathcal{R})\). However, they do not give all the vertices, as we restricted ourselves to a particular subset of the faces of \(N(\mathcal{R})\) (on which \(\phi_{p,q}\) are supported). However, this turns out to be sufficient to fully describe \(N(A)\). We get:

\[
\phi_{0,0}^{\text{min}} = (-1)^j \prod \alpha_j + \sum \prod \alpha_j a_i \prod b_{j,0}^{\sum_j \alpha_j}, \quad \phi_{0,0}^{\text{max}} = a_0 \prod \alpha_j \prod b_{j,2}^{\sum_j \alpha_j} \tag{7.11}
\]

which correspond to the lowest and highest powers of \(y\) in the A-polynomial, that is, the points \((0, 0)\) and \((\sum \prod \alpha_j^j, \prod \alpha_j)\) on the \((x, y)\)-plane. We have to show that there are no points below this slope (in the projection). And after that, we have of course to show that there are no points above the upper part (which is given by the projection of various \(\phi\)'s, as depicted on (10)). We will see below, that it is achieved if and only if \(a_i\) are all equal.
Figure 10: \(N(A)\) for \(\text{diag}(\alpha, \ldots, \alpha)\): each \(\phi_{i,j}\) projects onto the corresponding edge, such that the min/max monomials of \(\phi_{i,j}\) are in bijection with the vertices of \(N(A)\), but the intermediate monomials may mix up on the projection, nevertheless keeping the bijection between the initial forms and the edges.

Let’s write min/max monomials for \(\phi_{p,q}\):

\[
\begin{align*}
\phi_{p,q}^{\text{min}} &= \mu_{p,q} \cdot \prod_{\text{perm}(I,K)} (-1)^{1+\sum_{k \in K} \Pi_{k' \in K}(\alpha_{k'}-1)} \left( a_1 \prod_{i \in I} b_{i,0} \right) \Pi_{k \in K}(\alpha_k-1) \prod_{k \neq k'} b_{k,1} \\
\phi_{p,q}^{\text{max}} &= \mu_{p,q} \cdot \prod_{\text{perm}(I,K)} \left( a_0 \prod_{i \in I} b_{i,1} \right) \Pi_{k \in K}(\alpha_k-1) \prod_{k \neq k'} b_{k,2}
\end{align*}
\]

For example, the first few nodes project onto \((x, y)\)-plane as

\[
\begin{align*}
\phi_{m,0}^{\text{min}} : (0, 0), & \quad \phi_{m,0}^{\text{max}} : (0, 1) \\
\phi_{m-1,1}^{\text{min}} : (0, 1), & \quad \phi_{m-1,1}^{\text{max}} : \left( m, 1 + \sum_{i=1}^{m} (\alpha_i - 1) \right)
\end{align*}
\]

where \((x_i, y_i) = (\deg(\phi_{\text{min}}^{\max}, x), \deg(\phi_{\text{min}}^{\max}, y))\). The first slope, which is vertical, is always presented in \(A(x, y)\), since it encodes the analytic branch of \(y\) as a function of \(x\) (when the leading coefficient in the Puiseaux expansion has non-negative degree), see [15].

Let’s return to the fact a). Uniqueness of the preimage of each vertex of \(N(A)\) follows from uniqueness of the corresponding mixed decomposition, where the \(a_0\)- and \((\bigcup b_{i,2})\)-cells are fixed. There is no space to vary the other cells, so they would be fixed rigidly and produce a unique extremal monomial. Being projected, each such monomial would give a unique vertex of \(N(A)\).
Next, we verify the validity of b). We use one extra notion: the *detalization map*, which further subdivides a given mixed decomposition, refining its cell structure by dividing cells into smaller cells. For a given face, it corresponds to picking up a particular sub-face. Here is an example (Figure 11), borrowed from Section 5:

\[ \phi_{m-i,i} = \phi_{m-i+1,i+1}, \quad \forall i = 0 \ldots m; \quad m \geq 2 \] (7.14)
Proof. To begin with, let’s see how the detalization map acts on each perfogram, representing a binomial in $\phi_{p,q}$. We study the pattern for $\text{diag}(\alpha_1, \alpha_2)$: $\phi_{2,1}^{\text{max}} = \phi_{1,2}^{\text{min}}$. Then it will follow for any $\phi_{p,q}$ by induction on $m$. First, for $\phi_{2,1}$:

The boxed monomials are those, which remain frozen (always non-zero) when doing detalization, i.e. we do not cross them out. We obtain decompositions of each binomial diagram (of the three in the upper row) into four pieces (forming a column), such that all of them along with $\mu_{2,1}$ define the corresponding extremal monomial $\phi_{2,1}^{\text{max}}$. For the sake of completeness, we also give a formula for $\mu_{2,1}$:

$$(a_0) : \quad (a_1) : \quad (b_{*,0}) : \quad (\text{perm}) $$

$$(7.20)$$
Now we do the same thing for $\phi_{1,2}$:

\begin{align*}
\text{The monomial } \mu_{1,2}: \\
(a_0) & \quad \quad (a_1) & \quad (b_{*,2}) \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array} \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array} \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array} \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array} \\
\end{array} \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array} \\
\end{array} \\
\end{array} \\
\end{array} \\
\end{array} \\
\end{array} \quad \quad (7.21) \\
\end{align*}

We see that the binomial counterparts of both $\phi_{min,2,1}$ and $\phi_{min,2,1}$ have the identical collections of $b_{*,1}$-perfograms. Moreover, it immediately extends to any $\phi_{p,q}$, since detailing any of $b_{*,1}$ corresponds to taking a row of the form $[\bullet \bullet]$ or $[\bullet \bullet]$. So, in order to get the maximum (minimum), we remove the left (right) neighbouring “$\bullet$”, which results in the same perfogram. Next, comparing the $a_0$- and $a_1$-perfograms, we see that those ones, which are in the binomial part of $\phi_{min,2,1}$, coincide with the $\mu$-part in $\phi_{max,2,1}$, and vice versa. This is also true for $b_{*,2}$ counterpart (follows from Proposition (7.1)). Therefore, there is an “exchange relation” between the two collections of perfograms, resulting into identical extremal monomials. Moreover, this rule extends to any $p,q$, hence the claim of the proposition. □

**Proposition 7.3** Send $a_0$ to $y$ and $b_{i,2}$ to $x$, for $i = 1 \ldots m$. For a fixed $(p,q)$ and $\phi_{p,q}$, sort its monomials by $y$-degree, ascending. Then, the $x$- and $y$-degrees of each of monomial grow
linearly, that is, after the principal specialization they project onto the same line segment, if and only if $\alpha_i$ are all equal.

Proof. First, recall that all $\phi_{p,q}$ are simple, meaning that all its monomials are extreme monomials of $N(R)$, and its dimension equals to the number of distinct binomial factors. We already described now to compute its monomials with minimal and maximal powers of $a_0$. Now how we do it for all other monomials? The answer is simple (and was in fact already given in [21]): we have to take all possible combinations of min and max applied to a particular binomial, in such way to obtain its full detalization (much like on the figure 11 for $m = 2$), and the resulting mixed decomposition would give us the extreme monomial of $\phi_{p,q}$, and then changing the min max configuration will give another extremal monomial, and so on.

Consider, for example, $\phi_{2,1}$ versus $\phi_{1,2}$. Each of them contains three distinct binomial factors – denote them as $H_{p,q}^{(s)}$, $(p,q) = (2,1)$ or $(1,2)$. Therefore, their monomials are given by the triples $(\min, \min, \min), \ldots, (\max, \max, \max)$:

| $H_{p,q}^{(1)}$ | $H_{p,q}^{(2)}$ | $H_{p,q}^{(3)}$ |
|-----------------|-----------------|-----------------|
| min | min | min |
| min | min | max |
| min | max | min |
| min | max | max |
| max | min | min |
| max | min | max |
| max | max | min |
| max | max | max |

(7.27)

In total there are 8 monomials for $\phi_{2,1}$ (and for $\phi_{1,2}$). We have the following rule for “min” (“max”): make the $a_1$ ($a_0$) bullet frozen: $\bullet$, along with all the leftmost (rightmost) $b$-type bullets, as shown on (7.15) and (7.21). Then, duplicate the perfogram by removing each non-frozen bullet, to obtain a collection of perfograms corresponding to a single coefficient in a non-negative power. Therefore, the whole collection will now define some extreme monomial.

Recall that the binomial counterpart of $\phi_{2,1}^{\text{min}}$ does not depend on neither $a_0$ nor $b_{1,2}$, which means that the $(x,y)$-coordinates of the minimal monomial are fixed by $\mu_{2,1}$. Moving to the next order gives an increment to both $x$- and $y$-degrees of $\mu_{2,1}$ (which we denote as $\tilde{\mu}_x$ and $\tilde{\mu}_y$).

For the fist increment, we replace a single “min” by “max”, say, in $H_s := H_{p,q}^{(s)}$. This amounts to changing the frozen configuration, so that the $a_0$-degree gets the increment $+\prod_{j \in K_s'}(\alpha_j - 1)$, where $K_s'$ is attached to $H_s$. If we do that again, we modify yet another factor $H_{s'}$, getting the increment: $a_0 \mapsto a_0 + \prod_{j \in K_{s'}'}(\alpha_j - 1)$, and so on, until we reach $\phi_{2,1}^{\text{max}} \simeq (\max, \max, \max)$. Analogously, for $b_{1,2}$ the increment at $H_s$ be like: $+\prod_{j \in K_s' \setminus \{s\}}(\alpha_j - 1)$. The total $x$-degree is given by summing up the latter expression. We obtain the sequence of increments (Figure 12).
Therefore each time, changing min of $H_s$ to max (exactly as shown in (7.27)), we get the increments for the $(x, y)$-coordinates of a monomial on the edge of $N(A)$:

$$x \mapsto x + \sum_{r=1}^m \prod_{j \in K'_s \setminus \{r\}} (\alpha_j - 1), \quad y \mapsto y + \prod_{j \in K'_s} (\alpha_j - 1),$$  \hspace{1cm} (7.28)

We see that the increment, being a function of $(\alpha_1, \ldots, \alpha_m)$, is non-linear. Now it becomes clear by looking at (7.28) that it is linear only when $\alpha_i$ are all equal. Of course the $(x, y)$-degree of the monomial does not depend on permutation (i.e. $(\min, \min, \max)$ and $(\max, \min, \min)$ are indistinguishable on the projected picture). Therefore, eight monomials of $\phi_{2,1}$ ($\phi_{1,2}$) are mapped onto four points on the edge of $N(A)$, see Figure (12). The red nodes are extremal, and in one-to-one correspondence with its preimage, and the green ones correspond to classes of permutations. The picture is drawn when $\alpha$’s are all equal, which guarantees linearity of the increments, therefore a single line segment being a projection slope. Vice versa, linearity of the increment forces all $\alpha_i$ to be equal, since any pair $\alpha_i \neq \alpha_j$ will necessarily create a change of slope on Figure (12) between any of the two nodes.

Finally, we have to clarify the following: if $\alpha_i$ are all equal, there are no intermediate monomials on the edges, except from the projection of $\phi_{p,q}$. This follows from the fact that the minimally allowed $y$-increment is equal to $(\alpha - 1)$. For $m = 2$ this holds trivially, since there are no integer lattice points on each edge of $N(A)$, which are not the monomials of $\phi_{p,q}$ for some $p$ and $q$. Then, for $\alpha > 2$ there are integer points between red - green, and green-green nodes (see the figure 12). But in order to have the corresponding monomials in $A(x, y)$, one has to apply $y$-increment which is smaller than $(\alpha - 1)$, among all the initial forms $\{\phi_{p,q}\}$, which is of course impossible. Those integer lattice points would stay unoccupied, hence the claim.

Figure 12: The $(x, y)$-projection of $\phi_{2,1}$ onto the edge of $N(A)$, and min/max rule
At this point, we have finally completed the proof of the main Theorem (7.1). □

Summing up this section, we achieved the main goal, showing that the A-polynomial for a diagonal quiver with adjacency matrix $C = \text{diag}(\alpha, \ldots, \alpha)$ is tempered, therefore quantizable. It also gives an alluring perspective on the associated quantum curves and modularity properties of the quiver series.

7.2 $\alpha_i$ are all distinct

In this case, we need an additional set of faces of $N(\mathcal{R})$ to fully describe $N(A)$. We deal with the problem for $m = 3$, and conjecture the way to go for $m > 3$ ($m = 2$ has been completely solved in Section 5).

Recall that for $\text{diag}(\alpha, \beta, \gamma)$ we had two distinct initial forms, which we denoted by $\phi_{2,1}$ and $\phi_{1,2}$ (Section 6). They correspond to the 3-dimensional faces of $N(\mathcal{R})$. Now for each of them, we describe its “child” edges, such that their union with $\phi_{p,q}$ gives a polytope of dimension $(\dim \phi_{p,q}) + 1$. In the case of $m = 3$ it is a 4-dimensional counterpart of $N(\mathcal{R})$. It turns out that these edges are exactly the missing ones, in order to fully describe the projection polygon when $\alpha_i$ all are distinct.

In order to get them, we choose $\phi_{p,q}$ and its binomial factor $H_i$. We want to generate an edge of $N(\mathcal{R})$, with a mixed decomposition having the same binomial factor $H_i$ as the chosen sub-face of $\phi_{p,q}$. But its monomial is completely different, so that this edge (denote it by $\tilde{\phi}_{p,q}(H_i)$) is orthogonal (i.e. does not lie on the same 3-plane) to $\phi_{p,q}$ itself! In the case of $\phi_{1,2}$ ($\phi_{2,1}$) there are four 2-dimensional faces with this property. Each such face has four edges.

Using them, we are able to compute $N(A)$ and all its face polynomials explicitly for the case $(\alpha, \beta, \gamma)$, where $2 \leq \alpha < \beta < \gamma$. Its vertices, if going around clockwise, starting from the origin $(0,0)$, are given below:

\[
\begin{align*}
\phi_{3,0} & : [(0,0), (0,1)] \\
\phi_{2,1} & : [(0,1), (1,\gamma)], [(1,\gamma), (2,\beta+\gamma-1)] \\
\tilde{\phi}_{2,1}^+ & : [(2,\beta+\gamma-1), (\beta+\gamma, \beta\gamma)] \\
\tilde{\phi}_{1,2}^- & : [(\beta+\gamma, \beta\gamma), (\beta+\gamma+1, \beta\gamma+\alpha-1)] \\
\phi_{1,2} & : [(\beta+\gamma+1, \beta\gamma+\alpha-1), (\alpha+\beta+2\gamma-1, \alpha\gamma+\beta\gamma-\gamma)], \quad \text{(7.29)} \\
& [(\alpha+\beta+2\gamma-1, \alpha\gamma+\beta\gamma-\gamma), (2(\alpha+\beta+\gamma)-3, \\
& \alpha\beta+\alpha\gamma+\beta\gamma-(\alpha+\beta+\gamma-1))] \\
\phi_{0,3} & : [2(\alpha+\beta+\gamma)-3, \alpha\beta+\alpha\gamma+\beta\gamma-(\alpha+\beta+\gamma-1), \\
& (\alpha\beta+\alpha\gamma+\beta\gamma, \alpha\beta\gamma)]
\end{align*}
\]

We notice that for each edge, the initial form $\phi_*$ captures only its endpoints (the other monomials are projected onto the interior of $N(A)$). The “furuncle” is basically the extreme monomial with coordinates $(\beta + \gamma, \beta\gamma)$. That’s why we had to include it, so that it connects $\phi_{2,1}$ and $\phi_{1,2}$ via their “tilded” partners $\tilde{\phi}_{2,1}^+$ and $\tilde{\phi}_{1,2}^-$. 


The straightforward computation shows: when all $\alpha_i$ are distinct, it is not sufficient to consider just the initial forms $\phi_{p,q}$, defined previously. In what follows, we conjecture how to complete this subset of initial forms, in order to describe all edges of $N(A)$.

**Conjecture 7.1** For any $m \geq 3$ and a pair $(\phi_{i+1,j}, \phi_{i,j+1})$, $i, j \neq 0$, there exists the unique simple initial form

$$\tilde{\phi}_{(i+1,j),(i,j+1)},$$

such that its binomial factors agree with the sources $\phi_{i+1,j}$ and $\phi_{i,j+1}$, following the pattern on Figure 13. Besides that, the monomial $\mu_{(i+1,j),(i,j+1)}$ is defined purely in terms of $\mu_{i+1,j}$ and $\mu_{i,j+1}$.

**Conjecture 7.2** The initial forms $\{\tilde{\phi}_{(p,q),(r,s)}\}$ along with $\{\phi_{p,q}\}$ fully describe $N(A)$ and its face polynomials, implying that the quiver $A$-polynomial $A(x,y)$ is tempered, for a diagonal quiver $\text{diag}(\alpha_1, \ldots, \alpha_m)$ with $\alpha_i < \alpha_j$, $\forall i < j$.

We have confirmed these conjectures by hand for $m = 3$ and using the computer program for $m = 4$ (the Newton polytopes of dimension 4 and 5, correspondingly).
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