LEAST UPPER BOUND OF THE EXACT FORMULA FOR OPTIMAL QUANTIZATION OF SOME UNIFORM CANTOR DISTRIBUTIONS

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Abstract. The quantization scheme in probability theory deals with finding a best approximation of a given probability distribution by a probability distribution that is supported on finitely many points. Let $P$ be a Borel probability measure on $\mathbb{R}$ such that $P = \frac{1}{2} P \circ S_1^{-1} + \frac{1}{2} P \circ S_2^{-1}$, where $S_1$ and $S_2$ are two contractive similarity mappings given by $S_1(x) = rx$ and $S_2(x) = rx + 1 - r$ for $0 < r < \frac{1}{2}$ and $x \in \mathbb{R}$. Then, $P$ is supported on the Cantor set generated by $S_1$ and $S_2$. The case $r = \frac{1}{3}$ was treated by Graf and Luschgy who gave an exact formula for the unique optimal quantization of the Cantor distribution $P$ (Math. Nachr., 183 (1997), 113-133). In this paper, we compute the precise range of $r$-values to which Graf-Luschgy formula extends.

1. Introduction. The most common form of quantization is rounding-off. Its purpose is to reduce the cardinality of the representation space, in particular, when the input data is real-valued. It has broad applications in communications, information theory, signal processing and data compression (see [4, 9, 6, 8, 11, 14, 15]). Let $\mathbb{R}^d$ denote the $d$-dimensional Euclidean space equipped with the Euclidean norm $\| \cdot \|$, and let $P$ be a Borel probability measure on $\mathbb{R}^d$. Then, the $n$th quantization error for $P$, with respect to the squared Euclidean distance, is defined by

$$V_n := V_n(P) = \inf \left\{ V(P, \alpha) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where $V(P, \alpha) = \int \min_{a \in \alpha} \| x - a \|^2 dP(x)$ represents the distortion error due to the set $\alpha$ with respect to the probability distribution $P$. A set $\alpha \subset \mathbb{R}^d$ is called an optimal set of $n$-means for $P$ if $V_n(P) = V(P, \alpha)$. For a finite set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$, by $M(a|\alpha)$ we denote the set of all elements in $\mathbb{R}^d$ which are nearest to $a$ among all the elements in $\alpha$, i.e.,

$$M(a|\alpha) = \{ x \in \mathbb{R}^d : \| x - a \| = \min_{b \in \alpha} \| x - b \| \}.$$
$M(a|\alpha)$ is called the Voronoi region generated by $a \in \alpha$. On the other hand, the set $\{M(a|\alpha) : a \in \alpha\}$ is called the Voronoi diagram or Voronoi tessellation of $\mathbb{R}^d$ with respect to the set $\alpha$.

**Definition 1.1.** A set $\alpha \subset \mathbb{R}^d$ is called a centroidal Voronoi tessellation (CVT) with respect to a probability distribution $P$ on $\mathbb{R}^d$, if it satisfies the following two conditions:

(i) $P(M(a|\alpha) \cap M(b|\alpha)) = 0$ for $a, b \in \alpha$, and $a \neq b$;

(ii) $E(X : X \in M(a|\alpha)) = a$ for all $a \in \alpha$,

where $X$ is a random variable with distribution $P$, and $E(X : X \in M(a|\alpha))$ represents the conditional expectation of the random variable $X$ given that $X$ takes values in $M(a|\alpha)$.

A Borel measurable partition $\{A_a : a \in \alpha\}$ is called a Voronoi partition of $\mathbb{R}^d$ with respect to the probability distribution $P$, if $P$-almost surely $A_a \subset M(a|\alpha)$ for all $a \in \alpha$. Let us now state the following proposition (see [4, 6]).

**Proposition 1.2.** Let $\alpha$ be an optimal set of $n$-means with respect to a probability distribution $P$, $a \in \alpha$, and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$. Then, for every $a \in \alpha$,

(i) $P(M(a|\alpha)) > 0$, (ii) $P(M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$, and (iv) $P$-almost surely the set $\{M(a|\alpha) : a \in \alpha\}$ forms a Voronoi partition of $\mathbb{R}^d$.

If $\alpha$ is an optimal set of $n$-means and $a \in \alpha$, then by Proposition 1.2, we see that $a$ is the centroid of the Voronoi region $M(a|\alpha)$ associated with the probability measure $P$, i.e., for a Borel probability measure $P$ on $\mathbb{R}^d$, an optimal set of $n$-means forms a CVT of $\mathbb{R}^d$; however, the converse is not true in general (see [3, 2, 12, 13]).

Let $S_1$ and $S_2$ be two contractive similarity mappings on $\mathbb{R}$ such that $S_1(x) = rx$ and $S_2(x) = rx + 1 - r$ for $0 < r < \frac{1}{2}$, and $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$, where $P \circ S_i^{-1}$ denotes the image measure of $P$ with respect to $S_i$ for $i = 1, 2$ (see [10]). Then, $P$ is a unique Borel probability measure on $\mathbb{R}$ which has support the limit set generated by $S_1$ and $S_2$. By a word $\sigma$ of length $k$, where $k \geq 1$, over the alphabet $\{1, 2\}$, it is meant that $\sigma := \sigma_1 \sigma_2 \cdots \sigma_k \in \{1, 2\}^k$, and write $S_\sigma := S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_k}$. For $\sigma := \sigma_1 \sigma_2 \cdots \sigma_k \in \{1, 2\}^k$ and $\tau := \tau_1 \tau_2 \cdots \tau_\ell$ in $\{1, 2\}^\ell$, $k, \ell \geq 1$, by $\sigma \tau := \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_\ell$ we mean the word obtained from the concatenation of the words $\sigma$ and $\tau$. A word of length zero is called the empty word and is denoted by $\emptyset$. For the empty word $\emptyset$, by $S_\emptyset$ we mean the identity mapping on $\mathbb{R}$, and write $J := J_\emptyset = S_{\emptyset}([0, 1]) = [0, 1]$. Then, the set $C := \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} J_\sigma$ is known as the Cantor set generated by the two mappings $S_1$ and $S_2$, and equals the support of the probability measure $P$ given by $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$, where $J_\sigma := S_\sigma(J)$. For any $\sigma \in \{1, 2\}^k$, $k \geq 1$, the intervals $J_{\sigma_1}$ and $J_{\sigma_2}$ into which $J_\sigma$ is split up at the $(k + 1)$th level are called the basic intervals of $J_\sigma$.

**Definition 1.3.** For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n)$ be the unique natural number with $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. For $I \subset \{1, 2\}^{\ell(n)}$ with $\text{card}(I) = n - 2^{\ell(n)}$ let $\beta_n(I)$ be the set consisting of all midpoints $a_\sigma$ of intervals $J_\sigma$ with $\sigma \in \{1, 2\}^{\ell(n)} \setminus I$ and all midpoints $a_{\sigma_1}, a_{\sigma_2}$ of the basic intervals of $J_\sigma$ with $\sigma \in I$. Formally, $\beta_n(I) = \{a_\sigma : \sigma \in \{1, 2\}^{\ell(n)} \setminus I\} \cup \{a_{\sigma_1} : \sigma \in I\} \cup \{a_{\sigma_2} : \sigma \in I\}$. Moreover, 

$$\int_{a \in \beta_n(I)} \|x - a\|^2 dP = \frac{1}{2^{\ell(n)}} r^{2\ell(n)} V \left(2^{\ell(n)+1} n + r^2 (n - 2^{\ell(n)})\right),$$

where $V$ is the variance.
Remark 1.4. In the sequel, there are some ten digit decimal numbers. They are all rational approximations of some real numbers.

In [7], Graf and Luschgy showed that $\beta_n(I)$ forms an optimal set of $n$-means for the probability distribution $P$ when $r = \frac{1}{2}$, and the $n$th quantization error is given by

$$V(\beta_n(I)) = \frac{1}{18 \ell(n)} \cdot \frac{1}{8} \left( 2^{\ell(n)+1} - n + \frac{1}{9} (n - 2^{\ell(n)}) \right).$$

Notice that $\beta_n(I)$ forms a CVT of the Cantor set generated by the two mappings $S_1(x) = rx$ and $S_2(x) = rx + (1 - r)$ for $0 < r \leq \frac{5 - \sqrt{17}}{2}$, i.e., if $0 < r \leq 0.4384471872$ (up to ten significant digits). In [13], we have shown that if $0.4371985206 < r \leq 0.4384471872$ and $n$ is not of the form $2^\ell(n)$ for any positive integer $\ell(n)$, then there exists a CVT for the Cantor set for which the distortion error is smaller than the CVT given by $\beta_n(I)$ implying the fact that $\beta_n(I)$ does not form an optimal set of $n$-means for all $0 < r \leq \frac{5 - \sqrt{17}}{2}$. It was still not known what is the least upper bound of $r$ for which $\beta_n(I)$ forms an optimal set of $n$-means for all $n \geq 2$. In the following theorem, which is the main theorem of the paper, we give the answer of it.

Theorem 1.5. Let $\beta_n(I)$ be the set defined by Definition 1.3. Let $r \in (0, \frac{1}{2})$ be the unique real number such that

$$(r - 1)(r^4 + r^2) = \frac{r^7 + r^6 + 4r^5 - 2r^4 - 2r^3 - 8r^2 + 9r - 3}{6}.$$  

Then, $r_0 = r \approx 0.4350411707$ (up to ten significant digits) gives the least upper bound of $r$ for which the set $\beta_n(I)$ forms an optimal set of $n$-means for the uniform Cantor distribution $P$.

In the sequel instead of writing $r \approx 0.4350411707$, we will write $r = 0.4350411707$. The arrangement of the paper is as follows: In Definition 2.7, we have constructed a set $\gamma_n(I)$, and in Proposition 2.8, we have shown that $\gamma_n(I)$ forms a CVT for the Cantor distribution $P$ if $0.3613249509 \leq r \leq 0.4376259168$ (written up to ten decimal places). In Theorem 3.1, we have proved that the set $\beta_n(I)$ forms an optimal set of $n$-means for $r = 0.4350411707$. In Proposition 4.1, we have shown that $V(P, \beta_n(I)) = V(P, \gamma_n(I))$ if $r = 0.4350411707$, and $V(P, \beta_n(I)) > V(P, \gamma_n(I))$ if $0.4350411707 < r \leq 0.4376259168 < \frac{5 - \sqrt{17}}{2}$. In Definition 4.2, we have constructed a set $\delta_n(I)$. In Proposition 4.3, we have shown that if $0.4371985206 < r \leq \frac{5 - \sqrt{17}}{2}$, then $V(P, \beta_n(I)) > V(P, \delta_n(I))$. Hence, if $0.4350411707 < r \leq \frac{5 - \sqrt{17}}{2}$, then the set $\beta_n(I)$ forms a CVT but does not form an optimal set of $n$-means implying the fact that the least upper bound of $r$ for which $\beta_n(I)$ forms an optimal set of $n$-means is given by $r_0 = r \approx 0.4350411707$ (up to ten significant digits) which is Theorem 1.5. Notice that the optimal sets of $n$-means and the $n$th quantization errors are not known for all Cantor distributions $P$ given by $P := \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$, where $S_1(x) = rx$ and $S_2(x) = rx + 1 - r$ for $0 < r < \frac{1}{2}$. Thus, it is worthwhile to investigate the least upper bound of $r$ for which the exact formula to determine the optimal quantization given by Graf-Luschgy works.

2. Preliminaries. As defined in the previous section, let $S_1$ and $S_2$ be the two similarity mappings on $\mathbb{R}$ given by $S_1(x) = rx$ and $S_2(x) = rx + 1 - r$, where $0 < r < \frac{1}{2}$, and $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$ be the probability distribution on $\mathbb{R}$ supported on the Cantor set generated by $S_1$ and $S_2$. Write $p_1 = p_2 = \frac{1}{2}$, and
Corollary 2.2. Let \( s_1 = s_2 = r \). By \( I^* \) we denote the set of all words over the alphabet \( I := \{1, 2\} \) including the empty word \( \emptyset \). For \( \omega \in I^* \), by \( s_\omega \) we represent the similarity ratio of the composition mapping \( S_\omega \). Notice that the identity mapping has the similarity ratio one. Thus, if \( \omega := \omega_1 \omega_2 \cdots \omega_k \), then we have \( s_\omega = r^k \). Let \( X \) be a random variable with probability distribution \( P \). By \( E(X) \) and \( V := V(X) \) we mean the expectation and the variance of the random variable \( X \). For words \( \beta, \gamma, \cdots, \delta \) in \( \{1, 2\}^* \), by \( a(\beta, \gamma, \cdots, \delta) \) we mean the conditional expectation of the random variable \( X \) given \( J_\beta \cup J_\gamma \cup \cdots \cup J_\delta \), i.e.,

\[
a(\beta, \gamma, \cdots, \delta) = E(X | X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta \cup \cdots \cup J_\delta} x dP(x).
\]

We now give the following lemma.

Lemma 2.1. Let \( P \) be the Cantor distribution and \( i = 1, 2 \). A set \( \alpha \subset \mathbb{R} \) is a CVT for \( P \) if and only if \( S_i(\alpha) \) is a CVT for the image measure \( P \circ S_i^{-1} \).

Proof. Notice that \( a, b \in \alpha \) if and only if \( S_i(a), S_i(b) \in S_i(\alpha) \). Moreover,

\[
(P \circ S_i^{-1}) (M(S_i(a)|S_i(\alpha)) \cap M(S_i(b)|S_i(\alpha))) = P(M(a|\alpha) \cap M(b|\alpha)).
\]

The last equation is true, since for any \( c \in \alpha \),

\[
S_i^{-1}(M(S_i(c)|S_i(\alpha))) = \{ S_i^{-1}(x) \in \mathbb{R} : \| x - S_i(c) \| = \min_{b \in S_i(\alpha)} \| x - b \| \}
\]

\[
= \{ y \in \mathbb{R} : \| S_i(y) - S_i(c) \| = \min_{b \in \alpha} \| S_i(y) - S_i(b) \| \}
\]

\[
= \{ y \in \mathbb{R} : \| y - c \| = \min_{b \in \alpha} \| y - b \| \}
\]

\[
= M(c|\alpha).
\]

Hence, by Definition 1.1, the lemma follows.

From Lemma 2.1 the following corollary follows.

Corollary 2.2. Let \( i = 1, 2 \), and let \( \beta \) form a CVT for the image measure \( P \circ S_i^{-1} \). Then, \( S_i^{-1}(\beta) \) forms a CVT for the probability measure \( P \).

The following two lemmas are well-known and easy to prove (see [7, 13]).

Lemma 2.3. Let \( f : \mathbb{R} \to \mathbb{R}^+ \) be Borel measurable and \( k \in \mathbb{N} \), and \( P \) be the probability measure on \( \mathbb{R} \) given by \( P = \frac{1}{2} P \circ S_1^{-1} + \frac{1}{2} P \circ S_2^{-1} \). Then

\[
\int f(x) dP(x) = \sum_{\sigma \in \{1, 2\}^*} \frac{1}{2^k} \int f \circ S_\sigma(x) dP(x).
\]

Lemma 2.4. Let \( X \) be a random variable with the probability distribution \( P \). Then, \( E(X) = \frac{1}{2} \) and \( V := V(X) = \frac{1}{4(1 + r)} \), and \( \int (x - x_0)^2 dP(x) = V(X) + (x_0 - \frac{1}{2})^2 \), where \( x_0 \in \mathbb{R} \).

We now give the following corollary.

Corollary 2.5. Let \( \sigma \in \{1, 2\}^k \) for \( k \geq 1 \), and \( x_0 \in \mathbb{R} \). Then,

\[
\int_{J_{\sigma}} (x - x_0)^2 dP(x) = \frac{1}{2^k} \left( r^{2k} V + \left( S_\sigma(\frac{1}{2}) - x_0 \right)^2 \right).
\]
Note 2.6. Corollary 2.5 is useful to obtain the distortion error. By Lemma 2.4, it follows that the optimal set of one-mean is the expected value and the corresponding quantization error is the variance $V$ of the random variable $X$. For $\sigma \in \{1, 2\}^k$, $k \geq 1$, since $a(\sigma) = E(X : X \in J_\sigma)$, using Lemma 2.3, we have

$$a(\sigma) = \frac{1}{P(J_\sigma)} \int_{J_\sigma} x \, dP(x) = \int_{J_\sigma} x \, d(P \circ S_\sigma)^{-1}(x) = \int S_\sigma(x) \, dP(x) = E(S_\sigma(X)).$$

Since $S_1$ and $S_2$ are similarity mappings, it is easy to see that $E(S_j(X)) = S_j(E(X))$ for $j = 1, 2$ and so by induction, $a(\sigma) = E(S_\sigma(X)) = S_\sigma(E(X)) = S_\sigma(\frac{1}{2})$ for $\sigma \in \{1, 2\}^k$, $k \geq 1$.

Definition 2.7. For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n)$ be the unique natural number with $2^\ell(n) \leq n < 2^{\ell(n)+1}$. Write

$$\gamma_2 := \{a(1), a(2)\} \text{ and } \gamma_3 := \{a(11,121), a(122, 211), a(212, 22)\}.$$

For $n \geq 4$, define $\gamma_n := \gamma_n(I)$ as follows:

$$\gamma_n(I) = \begin{cases} \bigcup_{\omega \in I} S_\omega(\gamma_3) \bigcup_{\omega \in \{1, 2\}^{\ell(n)-1} \setminus I} S_\omega(\gamma_2) & \text{if } 2^\ell(n) \leq n \leq 3 \cdot 2^\ell(n)-1, \\ \bigcup_{\omega \in \{1, 2\}^{\ell(n)-1} \setminus I} S_\omega(\gamma_3) \bigcup_{\omega \in I} S_\omega(\gamma_4) & \text{if } 3 \cdot 2^\ell(n)-1 < n < 2^\ell(n)+1, \end{cases}$$

where $I \subset \{1, 2\}^{\ell(n)-1}$ with $\text{card}(I) = n - 2^\ell(n)$ if $2^\ell(n) \leq n \leq 3 \cdot 2^\ell(n)-1$, and $\text{card}(I) = n - 3 \cdot 2^\ell(n)-1$ if $3 \cdot 2^\ell(n)-1 < n < 2^\ell(n)+1$.

Proposition 2.8. Let $\gamma_n := \gamma_n(I)$ be the set defined by Definition 2.7. Then, $\gamma_n(I)$ forms a CVT for the Cantor distribution $P$ if $0.3613249509 \leq r \leq 0.4376259168$ (written up to ten decimal places).

Proof. $\gamma_2$ forms a CVT for any $0 < r < \frac{1}{2}$. Using the similar arguments as [13, Lemma 3.13], we can show that $\gamma_3$ forms a CVT for $P$ if

$$0.3613249509 < r < 0.4376259168. \quad (1)$$

We now prove the proposition for $n \geq 4$. Let $\ell(n)$ be the unique natural number with $2^\ell(n) \leq n < 2^{\ell(n)+1}$. Thus, for $n \geq 4$, we have $\ell(n) \geq 2$. Notice that the two similarity mappings $S_1$ and $S_2$ are increasing mappings in the sense that $S_i(x) < S_i(y)$ for all $x, y \in \mathbb{R}$ with $x < y$, where $i = 1, 2$. This induces an order relation $\prec$ on $I^*$ as follows: for $\omega, \tau \in I^*$, we write $\omega \prec \tau$ if $S_\omega(x) < S_\tau(y)$ for $x, y \in \mathbb{R}$ with $x < y$. Let $\omega^{(1)} \prec \omega^{(2)} \prec \omega^{(3)} \prec \cdots \prec \omega^{(2^{\ell(n)-1})}$ be the order of the $2^{\ell(n)-1}$ elements in the set $I^{\ell(n)-1}$. For $1 \leq i < 2^{\ell(n)-1}$ and $\omega^{(i)} \in I^{\ell(n)-1}$, $a(\omega^{(i)}2)$ and $a(\omega^{(i+1)}1)$ are, respectively, the midpoints of the basic intervals of $J_{\omega^{(i)}}2$ and $J_{\omega^{(i+1)}}1$ yielding

$$S_{\omega^{(i)}}2(1) < \frac{1}{2}(a(\omega^{(i)}2) + a(\omega^{(i+1)}1)) < S_{\omega^{(i+1)}}2(0),$$

and so, for $n = 2^\ell(n)$, the set $\gamma_n$ forms a CVT for $P$. Let us now assume that $2^\ell(n) < n < 2^{\ell(n)+1}$. Then, the set $\gamma_n$ will form a CVT for $P$ if for $1 \leq i < 2^{\ell(n)-1}$
we can show that the following inequalities are true:

\[ S_{\omega(i)22}(1) < \frac{1}{2}(a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11, \omega(i+1)121)) < S_{\omega(i+1)11}(0), \]

\[ S_{\omega(i)22}(1) < \frac{1}{2}(a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11)) < S_{\omega(i+1)11}(0), \]

\[ S_{\omega(i)22}(1) < \frac{1}{2}(a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11)) < S_{\omega(i+1)11}(0). \]

We call \( \tau \) a predecessor of a word \( \omega \in \mathcal{I} \), if \( \omega = \tau \delta \) for some \( \delta \in \mathcal{I} \). If \( \omega(i) \) and \( \omega(i+1) \) have a common predecessor \( \tau \), then notice that the points \( a(\omega(i)212, \omega(i)22) \) and \( a(\omega(i+1)11, \omega(i+1)121) \) are reflections of each other about the point \( S_r(\frac{1}{2}) \), i.e.,

\[ \frac{1}{2}(a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11, \omega(i+1)121)) = S_r(\frac{1}{2}) \]

and \( S_{\omega(i)22}(1) \) and \( S_{\omega(i+1)11}(0) \) are in opposite sides of \( S_r(\frac{1}{2}) \), and so, the inequalities in (3) are true. If \( \omega(i) \) and \( \omega(i+1) \) have no common predecessor, i.e., the predecessor is the empty word \( \emptyset \), then the two points \( a(\omega(i)212, \omega(i)22) \) and \( a(\omega(i+1)11, \omega(i+1)121) \) are reflections of each other about the point \( \frac{1}{2} \), i.e.,

\[ \frac{1}{2}(a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11, \omega(i+1)121)) = \frac{1}{2} \]

and so, the inequalities in (3) are true. We now prove the inequalities in (4) and (5). To prove the inequalities in the following, by \( \omega(i) \wedge \omega(i+1) \), we denote the common predecessor of the words \( \omega(i) \) and \( \omega(i+1) \). Notice that \( s_{\omega(i+1)} = s_{\omega(i)} \) and \( s_{\omega(i+1)1} = s_{\omega(i)} r \) for all \( \omega(i), \omega(i+1) \in \mathcal{I}^{1(n)-1} \). We have

\[ a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11) - 2S_{\omega(i)22}(1) \]

\[ = \frac{1}{2} \left( \frac{1}{2}S_{\omega(i)212}(\frac{1}{2}) + \frac{1}{22}S_{\omega(i)22}(\frac{1}{2}) \right) + S_{\omega(i+1)1}(\frac{1}{2}) - 2S_{\omega(i)22}(1) \]

\[ = \frac{1}{3} \left( S_{\omega(i)212}(\frac{1}{2}) - S_{\omega(i)22}(1) \right) + \frac{2}{3} \left( S_{\omega(i)22}(\frac{1}{2}) - S_{\omega(i)22}(1) \right) \]

\[ + \left( S_{\omega(i+1)1}(\frac{1}{2}) - S_{\omega(i)22}(1) \right) \]

\[ = s_{\omega(i)} \left( \frac{1}{3} (S_{212}(\frac{1}{2}) - S_{22}(1)) + \frac{2}{3} (S_{22}(\frac{1}{2}) - S_{22}(1)) \right) \]

\[ + (1 - 2r)s_{\omega(i)\wedge\omega(i+1)} + \frac{1}{2}s_{\omega(i+1)1} \]

\[ = r^{\ell(n)-1} \left( \frac{1}{3} (S_{212}(\frac{1}{2}) - S_{22}(1)) + \frac{2}{3} (S_{22}(\frac{1}{2}) - S_{22}(1)) \right) \]

\[ + (1 - 2r)s_{\omega(i)\wedge\omega(i+1)} + \frac{1}{2} r^{\ell(n)}. \]

If \( n = 5 \), i.e., when \( \ell(n) = 2 \), then (6) reduces to

\[ a(1212, 122) + a(21) - 2S_{122}(1) \]

\[ = r \left( \frac{1}{3} (S_{212}(\frac{1}{2}) - S_{22}(1)) + \frac{2}{3} (S_{22}(\frac{1}{2}) - S_{22}(1)) \right) + (1 - 2r) + \frac{1}{2} r^2. \]

Let \( \tau \) be the predecessor with the maximum length among all the predecessors of any two consecutive words \( \omega(i) \) and \( \omega(i+1) \) as defined in (2). Then, by (6) and (7), we have

\[ a(\tau 1212, \tau 122) + a(\tau 21) - 2S_{\tau 122}(1) \leq a(\omega(i)212, \omega(i)22) + a(\omega(i+1)11) - 2S_{\omega(i)22}(1) \]

\[ \leq a(1212, 122) + a(21) - 2S_{122}(1). \]
Similarly, we can prove that
\[ 2S_{r21}(0) - a(\tau 1212, \tau 122) - a(\tau 21) \leq 2S_{\omega, \gamma}(0) - a(\omega^{(i)} 212, \omega^{(i)} 22) - a(\omega^{(i+1)} 1) \]
\[ \leq 2S_{21}(0) - a(1212, 122) - a(21). \]

Thus, the inequalities in (4) will be true if we can prove that
\[ S_{r212}(1) < \frac{1}{2} (a(\tau 1212, \tau 122) + a(\tau 21)) < S_{r21}(0). \tag{8} \]

Proceeding in the similar way, we can prove that the inequalities in (5) will be true if we can prove that
\[ S_{r212}(1) < \frac{1}{2} (a(\tau 1212, \tau 122) + a(\tau 211)) < S_{r211}(0). \tag{9} \]

Using Lemma 2.1, we can say that the inequalities in (8) and (9) will be true if we can prove that
\[ S_{122}(1) < \frac{1}{2} (a(1212, 122) + a(21)) < S_{21}(0), \tag{10} \]
\[ S_{122}(1) < \frac{1}{2} (a(1212, 122) + a(211)) < S_{211}(0). \tag{11} \]

The inequalities in (10) are true if \(0 < r < 0.4850084548\), and the inequalities in (11) are true if \(0 < r < 0.4376259168\) (written up to ten decimal places). Thus, the proof of the proposition is complete. \(\square\)

**Proposition 2.9.** For \(n \geq 4\) let \(\gamma_n(I)\) be the set defined by Definition 2.7. Then,

\[
\int \min_{a \in \gamma_n(I)} (x - a)^2 dP = \begin{cases} 
12^{(n)} r (n - 2^{(n)}) & \text{if } n = 2^{(n)}, \\
\frac{1}{2^{(n)}} r 2^{(n)-1} V_3 (n - 2^{(n)}) + V_2 (3 \cdot 2^{(n)-1} - n) & \text{if } 2^{(n)} < n \leq 3 \cdot 2^{(n)-1}, \\
\frac{1}{2^{(n)}} r 2^{(n)-1} V_3 (2^{(n)+1} - n) + V_4 (n - 3 \cdot 2^{(n)-1}) & \text{if } 3 \cdot 2^{(n)-1} < n < 2^{(n)+1},
\end{cases}
\]

where \(V_2 := V(P, \gamma_2)\) and \(V_3 := V(P, \gamma_3)\), respectively, denote the distortion errors for the CVTs \(\gamma_2(I)\) and \(\gamma_3(I)\).

**Proof.** For \(n = 2^{(n)}\), we have

\[
\sum_{\omega \in \{1, 2\}^{(n)} \mathcal{J}_\omega} \int (x - a(\omega))^2 dP = \frac{1}{2^{(n)}} \sum_{\omega \in \{1, 2\}^{(n)}} \int (x - a(\omega))^2 dP (P \circ S_\omega^{-1}) = r 2^{(n)} V.
\]

For \(2^{(n)} < n \leq 3 \cdot 2^{(n)-1}, \)

\[
\int \min_{a \in \gamma_n(I)} (x - a)^2 dP = \sum_{\omega \in \mathcal{J}_\omega} \int \min_{a \in S_{\omega}(\gamma_3)} (x - a)^2 dP + \sum_{\omega \in \{1, 2\}^{(n)-1} \mathcal{J}_\omega} \int \min_{a \in S_{\omega}(\gamma_2)} (x - a)^2 dP
\]

\[
= \sum_{\omega \in \mathcal{J}_\omega} \frac{1}{2^{(n)-1}} \int \min_{a \in S_{\omega}(\gamma_3)} (x - a)^2 d(P \circ S_\omega^{-1})
\]
Thus, the proof of the proposition is complete. \[ \square \]

3. **Optimal sets of \( n \)-means for \( r = 0.4350411707 \) and \( n \geq 2 \).** Recall that \( \beta_n(I) \) forms a CVT if \( r = 0.4350411707 \). In this section, we state and prove the following theorem.

**Theorem 3.1.** Let \( n \geq 2 \), and let \( \beta_n(I) \) be the set given by Definition 1.3. Then, \( \beta_n(I) \) forms an optimal set of \( n \)-means for \( r = 0.4350411707 \).

To prove the theorem, we need some basic lemmas and propositions.

The following two lemmas are true. Due to technicality the proofs of them are not shown in the paper.

**Lemma 3.2.** Let \( \alpha := \{a_1, a_2\} \) be an optimal set of two-means, \( a_1 < a_2 \). Then, \( a_1 = a(1) = S_1(\frac{1}{2}) = 0.2175 \), \( a_2 = a(2) = S_2(\frac{1}{2}) = 0.7825 \), and the corresponding quantization error is \( V_2 = r^2V = 0.0186274 \).

**Lemma 3.3.** The sets \( \{a(1), a(21), a(22)\} \) and \( \{a(11), a(12), a(2)\} \) form two optimal sets of three-means with quantization error \( V_3 = 0.0110764 \).

We now prove the following lemma.

**Lemma 3.4.** Let \( \alpha_n \) be an optimal set of \( n \)-means for \( n \geq 2 \). Then, \( \alpha_n \cap [0, r) \neq \emptyset \) and \( \alpha_n \cap (1 - r, 1] \neq \emptyset \).

**Proof.** For \( n = 2 \) and \( n = 3 \), the statement of the lemma follows from Lemma 3.2 and Lemma 3.3. Let us now prove that the lemma is true for \( n \geq 4 \). Consider the
set of four points \( \beta \) given by \( \beta := \{a(\sigma) : \sigma \in \{1, 2\}^2\} \). Then,
\[
\int \min_{a \in \beta} (x - a)^2 dP = \sum_{\sigma \in \{1, 2\}^2} \int_{J_\sigma} (x - a(\sigma))^2 dP = 0.00352544.
\]
Since \( V_n \) is the \( n \)th quantization error for \( n \geq 4 \), we have \( V_n \leq V_4 \leq 0.00352544 \).

Let \( \alpha_n \) be an optimal set of \( n \)-means. Write \( \alpha_n := \{a_1, a_2, \ldots, a_n\} \), where \( 0 < a_1 < a_2 < \cdots < a_n < 1 \). If \( a_1 \geq r \), using Corollary 2.5, we have
\[
V_n \geq \int_{J_1} (x - a_1)^2 dP \geq \int_{J_1} (x - r)^2 dP
\]

\[
= \frac{1}{2} (r^2 V + (a(1) - r)^2) = 0.0329713 > V_4 \geq V_n,
\]
which is a contradiction. Thus, we can assume that \( a_1 < r \). Similarly, we can show that \( a_n > (1 - r) \). Thus, we see that if \( \alpha_n \) is an optimal set of \( n \)-means with \( n \geq 2 \), then \( \alpha_n \cap [0, r) \neq \emptyset \) and \( \alpha_n \cap (1 - r, 1) \neq \emptyset \). Thus, the lemma is yielded. \( \Box \)

The following lemma is a modified version of Lemma 4.5 in [7], and the proof follows similarly.

**Lemma 3.5.** Let \( n \geq 2 \), and let \( \alpha_n \) be an optimal set of \( n \)-means such that \( \alpha_n \cap J_1 \neq \emptyset \), \( \alpha_n \cap J_2 \neq \emptyset \), and \( \alpha_n \cap (r, 1 - r) = \emptyset \). Further assume that the Voronoi region of any point in \( \alpha_n \cap J_1 \) does not contain any point from \( J_2 \), and the Voronoi region of any point in \( \alpha_n \cap J_2 \) does not contain any point from \( J_1 \). Set \( \alpha_1 := \alpha_n \cap J_1 \) and \( \alpha_2 := \alpha_n \cap J_2 \), and \( j := \text{card}(\alpha_1) \). Then, \( S_j^{-1}(\alpha_1) \) is an optimal set of \( j \)-means and \( S_{2-j}^{-1}(\alpha_2) \) is an optimal set of \((n - j)\)-means. Moreover,
\[
V_n = \frac{1}{2} r^2 (V_j + V_{n-j}).
\]

**Remark 3.6.** Lemma 4.5 in [7] does not work for all \( 0 < r < \frac{1}{2} \). Due to that we have added an extra condition to Lemma 4.5 in [7] to work for all \( 0 < r < \frac{1}{2} \).

**Lemma 3.7.** Let \( \alpha_4 \) be an optimal set of four-means. Then,
\[
\alpha_4 := \{a(11), a(12), a(21), a(22)\},
\]
and the quantization error is \( V_4 = 0.00352544 \).

**Proof.** Consider the four-point set \( \beta \) given by \( \beta := \{a(\sigma) : \sigma \in \{1, 2\}^2\} \). Then,
\[
\int \min_{a \in \beta} (x - a)^2 dP = \sum_{\sigma \in \{1, 2\}^2} \int_{J_\sigma} (x - a(\sigma))^2 dP = 0.00352544.
\]
Since \( V_4 \) is the \( n \)th quantization error for \( n = 4 \), we have \( V_4 \leq 0.00352544 \).

Let \( \alpha_4 := \{a_1, a_2, a_3, a_4\} \), where \( 0 < a_1 < a_2 < a_3 < a_4 < 1 \), be an optimal set of four-means. If \( a_1 > 0.20 > 0.189261 = S_{11}(1) \), using Corollary 2.5, we have
\[
V_4 \geq \int_{J_{a_1}} (x - a_1)^2 dP \geq \int_{J_{a_1}} (x - 0.20)^2 dP = 0.00365705 > V_4,
\]
which is a contradiction. So, we can assume that \( a_1 \leq 0.20 \). Similarly, \( a_4 \geq 0.80 \). We now show that \( \alpha_4 \) does not contain any point from \((r, 1 - r)\). Suppose that \( \alpha_4 \) contains a point from \((r, 1 - r)\). Then, due to Lemma 3.4, without any loss of generality, we can assume that \( a_2 \in (r, 1 - r) \), and \( 1 - r \leq a_3 < a_4 \). Two cases can arise:

**Case 1.** \( a_2 \in \left[\frac{1}{2}, 1 - r\right) \).
Then, \( a_1 \leq 0.20 < S_{121}(0) = S_{121}(1) = 0.328117 < \frac{1}{7}(0.20 + \frac{1}{2}) = 0.35 < 0.352705 = S_{122}(0) \). Notice that \( a(11, 121) = 0.158736 < 0.20 \), and thus,

\[
V_4 \geq \int_{J_{11} \cup J_{121}} (x - a(11, 121))^2 dP = 0.00404695 > V_4,
\]

which is a contradiction.

**Case 2.** \( a_2 \in (r, \frac{1}{2}] \).

Then, \( S_{1211}(1) = 0.2816 < 0.281742 = \frac{1}{2}(a(11, 1211) + r) < 0.292297 = S_{1212}(0) \) implying the fact that \( J_{11} \cup J_{1211} \subset M(a(11, 1211)|\alpha_4) \) and \( J_{122} \subset M(r|\alpha_4) \). Again,

\[
\int_{J_{2}} \min_{a \in \{a_2, a_3, a_4\}} (x - a)^2 dP \\
\geq \int_{J_{2}} \min_{a \in S_2(\alpha_4)} (x - a)^2 dP = \frac{1}{2} \int_{J_{2}} \min_{a \in S_2(\alpha_4)} (x - a)^2 d(P \circ S_a^{-1}) \]

\[
= \frac{1}{2} \int_{a \in S_2(\alpha_3)} (S_2(x) - a)^2 dP = \frac{1}{2} \int_{a \in \alpha_3} \min(S_2(x) - S_2(a))^2 dP = \frac{1}{2} r^2 V_3,
\]

where \( \alpha_3 \) is an optimal set of three-means as given by Lemma 3.3. Thus, we obtain

\[
V_4 \geq \int_{J_{11} \cup J_{1211}} (x - a(11, 1211))^2 dP + \int_{J_{122}} (x - r)^2 dP + \frac{1}{2} r^2 V_3 = 0.00366173 > V_4
\]

which gives a contradiction.

Hence, we can assume that \( \alpha_4 \) does not contain any point from the open interval \((r, 1 - r)\). We now show that \( \text{card}(\alpha_4 \cap J_1) = \text{card}(\alpha_4 \cap J_2) = 2 \). For the sake of assumption, assume that \( a_1 \in J_1 \) and \( \{a_2, a_3, a_4\} \subset J_2 \). If the Voronoi region of \( a_2 \) does not contain any point from \( J_1 \), then

\[
V_4 \geq \int_{J_1} (x - a(1))^2 dP = 0.00931372 > V_4,
\]

which leads to a contradiction. So, we can assume that the Voronoi region of \( a_2 \) contains points from \( J_1 \). Then, \( \frac{1}{2}(a_1 + a_2) < r \) implying \( a_1 < 2r - a_2 \leq 2r - (1 - r) = 3r - 1 = 0.305124 < 0.312534 = S_{12122}(0) \). Notice that \( a(11, 1211, 12121) = 0.144047 < 0.305124 \), and \( S_{12121}(1) = 0.30788 < 0.354503 = \frac{1}{2}(a(11, 1211, 12121) + (1 - r)) \). Then, using (12), we have

\[
V_4 \geq \int_{J_{11} \cup J_{1211} \cup J_{1212}} (x - a(11, 1211, 12121))^2 dP \\
+ \int_{J_{1222} \cup J_{122}} (x - 0.305124)^2 dP + \frac{1}{2} r^2 V_3 = 0.00528016 > V_4,
\]

which is a contradiction. Thus, \( \text{card}(\alpha_4 \cap J_1) = 1 \) and \( \text{card}(\alpha_4 \cap J_2) = 3 \) give a contradiction. Since \( \text{card}(\alpha_4 \cap J_1) = 3 \) and \( \text{card}(\alpha_4 \cap J_2) = 1 \) is a reflection of the case \( \text{card}(\alpha_4 \cap J_1) = 1 \) and \( \text{card}(\alpha_4 \cap J_2) = 3 \) about the point \( \frac{1}{2} \), we can say that \( \text{card}(\alpha_4 \cap J_1) = 3 \) and \( \text{card}(\alpha_4 \cap J_2) = 1 \) also yield a contradiction. Again, we have seen that \( \alpha_4 \cap J_1 \neq \emptyset \) and \( \alpha_4 \cup J_2 \neq \emptyset \). Thus, we have \( \text{card}(\alpha_4 \cap J_1) = \text{card}(\alpha_4 \cap J_2) = 2 \). Since \( P \) has symmetry about the point \( \frac{1}{2} \), i.e., if two intervals of equal lengths are equidistant from the point \( \frac{1}{2} \) then they have the same probability, and \( \text{card}(\alpha_4 \cap J_1) = \text{card}(\alpha_4 \cap J_2) = 2 \), we can assume that the boundary of the Voronoi regions of \( a_2 \) and \( a_3 \) passes through the point \( \frac{1}{2} \), i.e., the Voronoi region of any point in \( \alpha_4 \cap J_1 \) does not contain any point from \( J_2 \), and the Voronoi region of any point in \( \alpha_4 \cap J_2 \) does not contain any point from \( J_1 \). Hence, By Lemma 3.5, both \( S^{-1}_1(\alpha_4 \cap J_1) \) and \( S^{-1}_2(\alpha_4 \cap J_2) \) are optimal sets of two-means,
i.e., $S^1_-(\alpha_4 \cap J_1) = S^1_-(\alpha_4 \cap J_2) = \{a(1), a(2)\}$ yielding $\alpha_4 \cap J_1 = \{a(11), a(12)\}$ and $\alpha_4 \cap J_2 = \{a(21), a(22)\}$. Thus, we have $\alpha_4 = \{a(11), a(12), a(21), a(22)\}$, and the corresponding quantization error is

$$V_4 = \frac{1}{2}r^2(V_2 + V_2) = r^2V_2 = 0.00352544,$$

which is the lemma. \qed

**Proposition 3.8.** Let $n \geq 2$, and $\alpha_n$ be an optimal set of $n$-means. Then, $\alpha_n$ does not contain any point from the open interval $(r, 1-r)$, i.e., $\alpha_n \cap (r, 1-r) = \emptyset$.

**Proof.** By Lemma 3.2, Lemma 3.3, and Lemma 3.7, the proposition is true for $n = 2, 3, 4$. We now prove that the proposition is true for $n = 5$. Let $\alpha_5 := \{a_1, a_2, a_3, a_4, a_5\}$ be an optimal set of five-means, such that $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 1$. Consider the set of five points $\beta$ given by

$$\beta := \{a(11), (12), a(21), a(221), a(222)\}.$$

The distortion error due to the set $\beta$ is given by

$$\int \min_{b \in \beta} (x - b)^2 dP = 3 \int_{J_{11}} (x - a(11))^2 dP + 2 \int_{J_{221}} (x - a(221))^2 dP = 0.00281089.$$

Since $V_5$ is the quantization error for five-means, we have $V_5 \leq 0.00281089$. If $V_5 = S_{11}(1) < a_1$, then

$$V_5 \geq \int_{J_{11}} (x - S_{11}(1))^2 dP = 0.00312009 > V_5,$$

which gives a contradiction. Hence, we can assume that $a_1 < S_{11}(1) = 0.189261$. Similarly, $S_{22}(0) < a_5$. For the sake of contradiction, assume that $\alpha_5$ contains a point from $(r, 1-r)$. Notice that due to Proposition 1.2, if $\alpha_5$ contains a point from $(r, 1-r)$, then it cannot contain more than one point from $(r, 1-r)$. Suppose that $a_2 \in (r, 1-r)$. Two cases can arise:

**Case 1.** $a_2 \in [0, 1-r]$.

Then, $a_1 \leq 0.189261 = S_{11}(1) < S_{121}(0) < S_{212}(1) = 0.328117 < 0.344631 = \frac{1}{2}(0.189261 + \frac{1}{2}) < 0.352705 = S_{22}(0)$, and so,

$$V_5 \geq \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{121}} (x - S_{11}(1))^2 dP + \int_{J_{122}} (x - \frac{1}{2})^2 dP = 0.00364889 > V_5,$$

which is a contradiction.

**Case 2.** $a_2 \in (r, 1-r)$.

Then, $S_{122}(1) = 0.2816 < 0.281742 = \frac{1}{2}(a(11), 1211) + r < 0.292297 = S_{212}(0)$ implying the fact that $J_{11} \cup J_{121} \subset M(a(11), 1211) / \alpha_5$ and $J_{122} \subset M(r / \alpha_5)$. Again, we have

$$\int_{J_2} \min_{a \in \{a_2, a_3, a_4, a_5\}} (x - a)^2 dP \geq \int_{J_2} \min_{a \in S_2(\alpha_4)} (x - a)^2 dP$$

$$= \frac{1}{2} \int_{J_2} \min_{a \in S_2(\alpha_4)} (x - a)^2 d(P \circ S_2^{-1}) = \frac{1}{2} \int \min_{a \in S_2(\alpha_4)} (S_2(x) - a)^2 dP$$

$$= \frac{1}{2} \int \min_{a \in \alpha_4} (S_2(x) - S_2(a))^2 dP = \frac{1}{2}r^2V_5.$$
where \( \alpha_4 \) is an optimal set of four-means. Thus, we obtain

\[
V_5 \geq \int_{J_1 \cup J_{1211}} (x - a(11, 1211))^2 dP + \int_{J_{122}} (x - r)^2 dP + \frac{1}{2} r^6 V = 0.00294718 > V_5
\]

which gives a contradiction.

Hence, we can assume that \( a_2 \notin (r, 1 - r) \). Likewise, if \( a_3 \in (r, 1 - r) \), we can show that a contradiction arises. Proceeding in the similar fashion, one can show that the proposition is true for all \( 6 \leq n \leq 15 \). We now give the general proof of the proposition for all \( n \geq 16 \). Let \( a_n \) be an optimal set of \( n \)-means for all \( n \geq 16 \), and \( V_n \) is the corresponding quantization error. Consider the set of sixteen points \( \beta \) given by \( \beta := \{a(\sigma) : \sigma \in \{1\}^4\} \). The distortion error due to the set \( \beta \) is given by

\[
\int \min_{b \in \beta} (x - b)^2 dP = r^8 V = 0.00012628.
\]

Since \( V_n \) is the quantization error for \( n \)-means for \( n \geq 16 \), we have \( V_n \leq V_{16} \leq 0.00012628 \). Write \( \alpha_n := \{a_1, a_2, \ldots, a_n\} \), where \( 0 < a_1 < a_2 < \cdots < a_n < 1 \). By Lemma 3.4, we see that \( \alpha_n \cap J_1 \neq \emptyset \) and \( \alpha_n \cap J_2 \neq \emptyset \). Let \( j \) be the largest positive integer such that \( a_j \in J_1 \). Then, \( a_{j+1} > r \). We need to show that \( \alpha_n \cap (r, 1 - r) = \emptyset \). For the sake of contradiction, assume that \( \alpha_n \cap (r, 1 - r) \neq \emptyset \). Proposition 1.2 implies that if \( a_n \) contains a point from the open interval \((r, 1 - r)\), then it cannot contain more than one point from the open interval \((r, 1 - r)\). Thus, we have \( a_j \leq r < a_{j+1} < 1 - r \leq a_{j+2} \). The following two cases can arise:

**Case 1.** \( a_{j+1} \in [\frac{1}{2}, 1 - r) \).

Then, by Proposition 1.2, we have \( \frac{1}{2}(a_j + a_{j+1}) < r \) implying \( a_j < 2r - a_{j+1} \leq 2r - \frac{1}{2} = 0.370082 < S_{12212}(0) = 0.372942 \). Thus,

\[
V_n \geq \int_{J_{12212} \cup J_{1222}} (x - 0.3700816)^2 dP = 0.000150535 > V_{16} \geq V_n,
\]

which is a contradiction.

**Case 2.** \( a_{j+1} \in (r, \frac{1}{2}] \).

Since this case is the reflection of Case 1 with respect to the point \( \frac{1}{2} \), a contradiction arises.

Hence, \( a_n \) does not contain any point from the open interval \((r, 1 - r)\). Thus, the proof of the proposition is complete. \( \square \)

**Proposition 3.9.** Let \( a_n \) be an optimal set of \( n \)-means with \( n \geq 2 \). Then, the Voronoi region of any point in \( \alpha_n \cap J_1 \) does not contain any point from \( J_2 \), and the Voronoi region of any point in \( \alpha_n \cap J_2 \) does not contain any point from \( J_1 \).

**Proof.** Notice that \( \frac{1}{2}(a(1) + a(2)) = \frac{1}{2}, r < \frac{1}{2}(a(1) + a(21)) < 1 - r, r < \frac{1}{2}(a(12) + a(2)) < 1 - r \), and \( r < \frac{1}{2} = \frac{1}{2}(a(12) + a(21)) < 1 - r \). Thus, by Lemma 3.2, Lemma 3.3, and Lemma 3.7, the proposition is true for \( n = 2, 3, 4 \). It can also be shown that the proposition is true for \( 5 \leq n \leq 7 \). Due to lengthy as well as the technicality of the proofs we don’t show them in the paper, and give a general proof of the proposition for all \( n \geq 8 \). Let us consider a set of eight points \( \beta \) given by \( \beta := \{a(\sigma) : \sigma \in \{1, 2\}^3\} \). Then,

\[
\int \min_{a \in \beta} (x - a)^2 dP = \sum_{\sigma \in \{1, 2\}^3} \int_{J_\sigma} (x - a(\sigma))^2 dP = 0.000667229.
\]
Since $V_n$ is the $n$th quantization error for $n \geq 8$, we have $V_n \leq V_8 \leq 0.000667229$. Let $\alpha_n := \{a_1, a_2, \cdots , a_n\}$ be an optimal set of $n$-means for $n \geq 8$ with $0 \leq a_1 < a_2 < \cdots < a_n \leq 1$, and let $j$ be the greatest positive integer such that $a_j \in J_1$. Then, by Proposition 3.8, we have $a_1 < r$ and $1 - r < a_{j+1}$. Suppose that the Voronoi region of $a_{j+1}$ contains points from $J_1$. Then, $\frac{1}{2}(a_j + a_{j+1}) < r$ yielding $a_j < 2r - a_{j+1} \leq 2r - (1 - r) = 3r - 1 = 0.305124 < 0.312534 = S_{12122}(0)$. Hence, by Corollary 2.5,

$$V_n \geq \int_{J_{12122} \cup J_{122}} (x - 0.305124)^2 dP = 0.00107592 > V_8 \geq V_n,$$

which is a contradiction. Thus, we can assume that the Voronoi region of $a_{j+1}$ does not contain any point from $J_1$. Similarly, we can show that the Voronoi region of $a_j$ does not contain any point from $J_2$. Hence, the proposition is true for all $n \geq 8$. Thus, we complete the proof of the proposition.

We are now ready to give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We prove the theorem by induction. For $n \geq 2$ let $\alpha_n$ be an optimal set of $n$-means for $P$. By Lemma 3.2, Lemma 3.3, and Lemma 3.7, the theorem is true for $n = 2, 3, 4$. Suppose that the assertion of the theorem holds for all $m < n$, where $n \geq 2$. Set $\alpha_1 := \alpha_n \cap J_1$ and $\alpha_2 := \alpha_n \cap J_2$, and $j := \text{card}(\alpha_1)$. By Lemma 3.4, Lemma 3.5, Proposition 3.8, and Proposition 3.9, there exists a $j \in \{1, 2, \cdots , n - 1\}$ such that

$$V_n = \frac{1}{2} r^2 (V_j + V_{n-j}),$$

which is same as the expression of $V_n$ given in [7] for $r = \frac{1}{2}$. Without any loss of generality, we can assume that $j \geq n - j$. Then, proceeding similarly, as given in the proof of Theorem 5.2 in [7], we can show that the following inequalities are true:

$$2^{\ell(n)-1} \leq j \leq 2^{\ell(n)} \text{ and } 2^{\ell(n)-1} \leq n - j < 2^{\ell(n)}.$$

The rest of the induction hypothesis, follows exactly same as the last part of the proof of Theorem 5.2 in [7]. Thus, the proof of Theorem 3.1 is complete.

4. **Proof of the main theorem Theorem 1.5.** In this section, we determine the least upper bound of $r$ for which $\beta_n(I)$ forms an optimal set of $n$-means. It is known that $\beta_n(I)$ forms a CVT if $0 < r \leq \frac{5 - \sqrt{17}}{2}$ (see [13, Lemma 4.2]). By Proposition 2.8, $\gamma_n(I)$ forms a CVT if $0.3613249509 \leq r \leq 0.437625816 < \frac{5 - \sqrt{17}}{2}$. Let us now prove the following proposition.

**Proposition 4.1.** For $n \geq 2$, let $\beta_n(I)$ be the set defined by Definition 1.3, and $\gamma_n(I)$ be the set defined by Definition 2.7. Assume that $n$ is not of the form $2^{\ell(n)}$ for any positive integer $\ell(n)$. Then, $V(\beta_n(I)) > V(\gamma_n(I))$ if $0.4350411707 < r \leq 0.4376259168 < \frac{5 - \sqrt{17}}{2}$, and $V(\beta_n(I)) = V(\gamma_n(I))$ if $r = 0.4350411707$, where $V(\beta_n(I)) := V(P, \beta_n(I))$ and $V(\gamma_n(I)) := V(P, \gamma_n(I))$, respectively, denote the distortion errors for the CVTs $\beta_n(I)$ and $\gamma_n(I)$.

**Proof.** If $n$ is of the form $2^{\ell(n)}$ for some positive integer $\ell(n)$, then as $\beta_n(I) = \gamma_n(I)$, we have $V(\beta_n(I)) = V(\gamma_n(I))$ for all $0 < r < \frac{1}{2}$. Let us assume that $n$ is not of the
form $2^\ell(n)$ for any positive integer $\ell(n) \geq 2$. Then, the following three cases can aries:

**Case 1.** $n = 3$.

In this case we have,

\[
V(\beta_3(I)) = -\frac{(r-1)(r^4 + r^2)}{8(r+1)}, \quad \text{and}
\]

\[
V(\gamma_3(I)) = -\frac{r^7 + r^6 + 4r^5 - 2r^4 - 2r^3 - 8r^2 + 9r - 3}{48(r+1)}.
\]

Then, $V(\beta_3(I)) = V(\gamma_3(I))$ if $r = 0.4350411707$, and $V(\beta_3(I)) > V(\gamma_3(I))$ if $0.4350411707 < r < \frac{1}{2}$.

**Case 2.** $n \geq 4$ and $2^\ell(n) < n \leq 3 \cdot 2^\ell(n) - 1$.

Then, using Definition 2.7 and Proposition 2.9, we see that $V(\beta_n(I)) = V(\gamma_n(I))$ if

\[
\frac{1}{2^{l(n)}} r^{2^\ell(n)} V \left( 2^{\ell(n)+1} - n + r^2 (n - 2^{\ell(n)}) \right) = \frac{1}{2^{l(n)+1}} r^{2^\ell(n)-1} \left( V(\gamma_n(I)) (n - 2^{\ell(n)}) + V(\gamma_2(I)) (3 \cdot 2^{\ell(n)-1} - n) \right),
\]

which after simplification yields that $\frac{1}{2} r^2 (r^2 + 1) V = V(\gamma_n(I))$, i.e., $V(\beta_n(I)) = V(\gamma_n(I))$. Hence, by Case 1, we have $V(\beta_n(I)) = V(\gamma_n(I))$ if $r = 0.4350411707$, and $V(\beta_n(I)) > V(\gamma_n(I))$ if $0.4350411707 < r < \frac{1}{2}$.

**Case 3.** $n \geq 4$ and $3 \cdot 2^\ell(n-1) < n < 2^\ell(n)+1$.

Then, using Definition 2.7 and Proposition 2.9, we see that $V(\beta_n(I)) = V(\gamma_n(I))$ if

\[
\frac{1}{2^{l(n)}} r^{2^\ell(n)} V \left( 2^{\ell(n)+1} - n + r^2 (n - 2^{\ell(n)}) \right) = \frac{1}{2^{l(n)+1}} r^{2^\ell(n)-1} \left( V(\gamma_n(I)) (2^{\ell(n)+1} - n) + V(\gamma_4(I)) (n - 3 \cdot 2^{\ell(n)-1}) \right),
\]

which after simplification yields that $\frac{1}{2} r^2 (r^2 + 1) V = V(\gamma_n(I))$, i.e., $V(\beta_n(I)) = V(\gamma_n(I))$. Hence, by Case 1, we have $V(\beta_n(I)) = V(\gamma_n(I))$ if $r = 0.4350411707$, and $V(\beta_n(I)) > V(\gamma_n(I))$ if $0.4350411707 < r < \frac{1}{2}$.

Recall that both $\beta_n(I)$ and $\gamma_n(I)$ form CVTs if

\[
0.3613249509 \leq r \leq 0.4376259168 < \frac{5 - \sqrt{17}}{2}.
\]

Hence, by Case 1, Case 2, and Case 3, we see that if $n$ is not of the form $2^\ell(n)$ for any positive integer $\ell(n)$. Then, $V(\beta_n(I)) > V(\gamma_n(I))$ if $0.4350411707 < r \leq 0.4376259168 < \frac{5 - \sqrt{17}}{2}$, and $V(\beta_n(I)) = V(\gamma_n(I))$ if $r = 0.4350411707$. Thus, the proof of the proposition is complete. \qed

We now give the following definition.

**Definition 4.2.** For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n)$ be the unique natural number with $2^\ell(n) \leq n < 2^{\ell(n)+1}$. Let $\delta_n := \delta_n(I)$ be the set defined as follows: $\delta_2 := \{a(1), a(2)\}$, and

\[
\delta_3 := \{a(11, 121, 1221), a(1222, 21), a(22)\}, \quad \text{or}
\]

\[
\delta_3 := \{a(11), a(12, 2111), a(2112, 212, 22)\}.
\]

For $n \geq 4$, define $\delta_n := \delta_n(I)$ as follows:

\[
\delta_n(I) = \left\{ \begin{array}{ll}
\bigcup_{\omega \in I} S_\omega(\delta_3) \cup \bigcup_{\omega \in \{1, 2\}^{\ell(n)-1} \backslash I} S_\omega(\delta_2) & \text{if } 2^\ell(n) \leq n \leq 3 \cdot 2^{\ell(n)-1}, \\
\bigcup_{\omega \in \{1, 2\}^{\ell(n)-1} \backslash I} S_\omega(\delta_3) \cup \bigcup_{\omega \in I} S_\omega(\delta_4) & \text{if } 3 \cdot 2^{\ell(n)-1} < n < 2^{\ell(n)+1},
\end{array} \right.
\]
where $I \subset \{1, 2\}^{\ell(n)-1}$ with $\text{card}(I) = n - 2^{\ell(n)}$ if $2^{\ell(n)} \leq n \leq 3 \cdot 2^{\ell(n)-1}$; and $\text{card}(I) = n - 3 \cdot 2^{\ell(n)-1}$ if $3 \cdot 2^{\ell(n)-1} < n < 2^{\ell(n)+1}$.

The following proposition is due to [13].

**Proposition 4.3.** (see [13, Proposition 4.3]) Let $\delta_n(I)$ be the set defined by Definition 4.2, and $\beta_n(I)$ be the set defined by Definition 1.3. Suppose that $n$ is not of the form $2^{\ell(n)}$ for any positive integer $\ell(n)$. Then $V(P, \delta_n(I)) < V(P, \beta_n(I))$ if $0.4371985206 < r < \frac{5 - \sqrt{17}}{2}$, where $V(P, \delta_n(I))$ and $V(P, \beta_n(I))$, respectively, denote the distortion errors for the CVTs $\delta_n(I)$ and $\beta_n(I)$.

We are now ready to give the proof of the main theorem Theorem 1.5.

**Proof of Theorem 1.5.** Recall that $\beta_n(I)$ forms a CVT if $0 < r < \frac{5 - \sqrt{17}}{2}$. In [7], it is shown that $\beta_n(I)$ forms an optimal set of $n$-means if $r = \frac{1}{3} < 0.4350411707$. Theorem 3.1 implies that $\beta_n(I)$ also forms an optimal set of $n$-means if $r = 0.4350411707$. Proposition 4.1 implies that $V(P, \beta_n(I)) = V(P, \gamma_n(I))$ if $r = 0.4350411707$, and $V(P, \beta_n(I)) > V(P, \gamma_n(I))$ if $0.4350411707 < r \leq 0.4376259168 < \frac{5 - \sqrt{17}}{2}$. By Proposition 4.3, it follows that if $0.4371985206 < r < \frac{5 - \sqrt{17}}{2}$, then $V(P, \beta_n(I)) > V(P, \delta_n(I))$. Hence, if $0.4350411707 < r \leq \frac{5 - \sqrt{17}}{2}$, then the set $\beta_n(I)$ forms a CVT but does not form an optimal set of $n$-means. Thus, the least upper bound of $r$ for which $\beta_n(I)$ forms an optimal set of $n$-means is given by $r = 0.4350411707$, and this completes the proof of the theorem.

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**REFERENCES**

[1] E. F. Abaya and G. L. Wise, Some remarks on the existence of optimal quantizers, Statistics & Probability Letters, 2 (1984), 349–351.

[2] C. P. Dettmann and M. K. Roychowdhury, Quantization for uniform distributions on equilateral triangles, Real Analysis Exchange, 42 (2017), 149–166.

[3] Q. Du, V. Faber and M. Gunzburger, Centroidal voronoi tessellations: Applications and algorithms, SIAM Review, 41 (1999), 637–676.

[4] A. Gersho and R. M. Gray, Vector Quantization and Signal Compression, Kluwer Academic publishers: Boston, 1992.

[5] R. M. Gray, J. C. Kieffer and Y. Linde, Locally optimal block quantizer design, Information and Control, 45 (1980), 178–198.

[6] S. Graf and H. Luschgy, Foundations of Quantization for Probability Distributions, Lecture Notes in Mathematics, 1730, Springer, Berlin, 2000.

[7] S. Graf and H. Luschgy, The quantization of the cantor distribution, Math. Nachr., 183 (1997), 113–133.

[8] R. M. Gray and D. L. Neuhoff, Quantization, IEEE Transactions on Information Theory, 44 (1998), 2325–2383.

[9] A. György and T. Linder, On the structure of optimal entropy-constrained scalar quantizers, IEEE Transactions on Information Theory, 48 (2002), 416–427.

[10] J. Hutchinson, Fractals and self-similarity, Indiana Univ. J., 30 (1981), 713–747.

[11] D. Pollard, Quantization and the Method of $k$-Means, IEEE Transactions on Information Theory, 28 (1982), 199–205.

[12] M. K. Roychowdhury, Optimal quantizers for some absolutely continuous probability measures, Real Analysis Exchange, 43 (2017), 105–136.

[13] M. K. Roychowdhury, Quantization and centroid Voronoi tessellations for probability measures on dyadic Cantor sets, Journal of Fractal Geometry, 4 (2017), 127–146.

[14] P. L. Zador, Asymptotic quantization error of continuous signals and the quantization dimension, IEEE Transactions on Information Theory, 28 (1982), 139–149.
[15] R. Zam, *Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation, and Multiuser Information Theory*, Cambridge University Press, 2014.

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