TWIST PERIODIC ORBITS FOR CONTINUOUS MAPS OF THE EIGHT SPACE

Associate professor Iftichar Mudhar Talb Al-shraa
Iraq, Babylon University,
College of Education for Pure Science

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Abstract. Let $g$ be a continuous map from 8 to itself has a fixed point at $(0,0)$, we prove that $g$ has a twist periodic orbit if there is a rational rotation number

1.1 Introduction:
In [1], Alseda,L. et.al introduced the definition of twist periodic orbit, they studied this type of periodic on continuous circle map of degree one which has a fixed point In.[4], Misurewicz,M studied the twist sets for circle maps. Let 8 be the one point union of two unit circles attached at $(0,0)$ and $g: 8 \rightarrow 8$ be a map has a fixed point at $(0,0)$, we say that $z \in 8$ is a periodic point of $g$ if there exists a positive integer $n$ such that $g^n(z) = z$. The period of $z$ is the smallest integer satisfying this relation. Let $p(g)$ be the set of periods of $g$ if $z \in 8$ is a periodic point of period $n$, then the orbit of $z$ is the set

Throughout this work, we will use $e$ as projection from $s^1$ onto $8$. There exists many projections from $s^1$ into $8$, but we define $e$ as:

This projection is one to one. We can find a lift map $f$ from $s^1$ to itself such that $eg = fe$. Let $G$ be a lift to $g$ and $e\pi(t) = e(cos2\pi t + isin2\pi t)$ a projection of $R \rightarrow 8$. It is clear $\pi$ is many to one, $G$ is not defined uniquely; that is if $G_1$ and $G$ are two lifting of $g$ then $G = G_1 + m$ with $m \in Z$. the degree of $g$ is the number $a$ which satisfy $G(t + 1) = G(t) + a, \forall t \in R$. In this work, we use $a=1$. If $z \in 8$ is a periodic point of $g$ of period $n$ and $e\pi(t) = z$ then $G^n(t) = t + k$; where $k \in Z$ this imply $k/n$ is the rotation number of $z$ we denoted it by $\rho(G,z)$. We denote by $L_G(g)$ the set of all rotation numbers of $g$ so $L_G(g) = \{\rho(G,z); z \in 8; G: R \rightarrow R\}$ is called the rotation set of $g$. Since $f$ is unique lift to $g$ then the properties of rotation number and rotation set on the circle satisfy on the eight space. So the rotation set does not depend on the choice of $t$, but it depends on the periodic orbit. Also, if $(a_i)$ is a convergent sequence of $a$ and $a_i \in L_G(g)$ then $a \in L_G(g)$.

The another property $\rho(G,z) = \rho(G_1,z) + k$ if $G = G_1 + k$. If $g$ has no periodic point the rotation number exist for all $t \in R$ and it is independent of $t$ and is irrational number there are more properties on circle maps satisfy on 8 (see [2] and [3]). Let $z_1,z_2 \in 8$ we denote $(z_1,z_2)$(resp. $[z_1,z_2]$) the open (resp. closed) arc of 8 from $z_1$ counter clockwise to $z_2$. Let $\frac{d}{m}$ be a rational number, we assume $m>0$ and let $T$ be a periodic orbit of period $m$ and rotation number $\frac{d}{m}$ with $(d,m) = 1$. Let $G$ be lift such that $G$ is order preserving on the set $(en)^{-1}T$ then we say that $T$ is a twist periodic orbit (for simplify we say TPO) of $g$ period $m$ and rotation number $\frac{d}{m}$. It is clear, every periodic orbit of period 1 is a TPO. If $\frac{d}{m} > 0$ then $(z_i,z_{i+1}) \cap T = \emptyset, \forall i = 1,2, ..., m-1$ and $(z_m,z_1) \cap T = \emptyset$, and if $\frac{d}{m} < 0$ then
Then notion of TPO of period m characterizes the simplest behavior of the graph a map which has the same rotation number.

### 1.2 Main Theorems

In this section, we prove two theorems which find the TPO, also we prove other properties of it.

**Proposition 1.2.1:** If \( \frac{k}{n} < c \); \( a, c \in L_0(g) \) and \( k, n \in \mathbb{Z} \) with \( n > 0 \) and \( \frac{k}{n} \in L_0(g) \) then \( n \in p(g) \).

**Proof:** Suppose \( \rho(G, z) = \frac{k}{n} \) and thus there exists \( z \in \mathbb{Z} \) such that there is \( t \in \mathbb{R} \) and \( e \pi(t) = z \) then by definition \( \rho(G, z) = \lim_{n \to \infty} \sup \frac{c_n(t) - t}{n} = \frac{k}{n} \) so there is \( \varepsilon > 0 \) such that \( -\varepsilon < \frac{c_n(t) - t}{n} < \frac{k}{n} < \varepsilon \) then \( \frac{k}{n} - \varepsilon < \frac{c_n(t) - t}{n} < \frac{k}{n} + \varepsilon \). Hence

\[
(k + t) - n\varepsilon < G^n(t) < (k + t) + n\varepsilon
\]

Since \( \varepsilon \) is an arbitrary and \( G^n(t) \in \mathbb{R} \) is any point then \( G^n(t) = t + k \) that is \( t \) is a periodic point of \( G \) of period \( n \) then by properties of the lift then \( G \) has periodic of period \( n \) then \( n \in p(g) \).

**Corollary 1.2.2:** \( L_0(g) \cap \mathbb{Z} \neq \emptyset \) if and only if \( z = 0 \) is the only fixed point of \( g \).

**Proof:** If \( z = 0 \) then there exists \( t \in \mathbb{Z} \) such that \( e \pi(t) = z = 0 \). Let \( L_0(g) \cap \mathbb{Z} \neq \emptyset \) so there is \( \frac{k}{n} \in L_0(g) \cap \mathbb{Z} \) hence there is \( z \in \mathbb{Z} \) such that \( \rho(G, z) = \frac{k}{n} \) so there is \( t \in \mathbb{Z} \), \( e \pi(t) = 0 \).

**Corollary 1.2.3:** \( L_0(g) \cap \mathbb{Z} \neq \emptyset \) if and only of \( g \) has a fixed point \( z = 0 \).

**Proof:** \( \Rightarrow \) Since \( L_0(g) \cap \mathbb{Z} \neq \emptyset \) then there is an integer number \( k \in L_0(g) \), so by proposition 1.2.1, \( 1 \in p(g) \). In other word \( g \) has a fixed point \( \Leftarrow \) since \( 1 \in p(g) \) then there \( z \in \mathbb{Z} \) such that \( g(z) = z \), so \( G(t) = t + k \); where \( k \in \mathbb{Z} \), thus \( \rho(G, z) = \frac{k}{1} \in L_0(g) \) then \( k \in L_0(g) \cap \mathbb{Z} \neq \emptyset \).

The proposition below gives a geometric interpretation of a TPO on the eight space:

**Proposition 1.2.4:** Let \( g: \mathbb{Z} \to \mathbb{Z} \) be a continuous map, \( T = \{z_1, z_2, ..., z_m\} \) be a TPO of period \( m \) and rotation number \( \frac{d}{m} \). Then \( \frac{d}{m} > 0 \) and \( \frac{d}{m} \) be the lifting of \( g \) for which \( \rho(G, z_i) = \frac{d}{m}(e \pi)^{-1} T = \{t_i; i \in \mathbb{Z}\} \) with \( \cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots \). Assume \( e \pi(t_1) = z_1 \). We claim \( t_{i+r} = t_i + r \), if \( t_i \in \mathbb{Z} \).

Since \( t_i \) is a periodic point of period \( m \) then \( e \pi(t_{i+r}) = e \pi(t_i) \). Thus \( \pi(t_{i+r}) = \pi(t_i) \); \( t_i \in \mathbb{Z} \), so \( t_{i+r} = t_i + r \). Also, we claim \( t_{i+r} = t_i + \frac{r}{2} \), if \( t_i \in \mathbb{Z} \). Then \( (t_{i+r}) = e - \pi(t_i) \) so \( \pi(t_{i+r}) = -\pi(t_i) \). Then \( t_{i+r} = t_i + \frac{r}{2} \). Case 1: \( e \pi(t_{i+r}) = e \pi(t_i) \); \( t_i \) for \( k \in \mathbb{Z} \) and \( i = 1, 2, ..., m \), since \( F \) on \( (e \pi)^{-1} T \) is one to one and by definition of TPO \( (e \pi)^{-1} T \) order preserving then \( G(t_i) = t_i+1 \), for some \( j \in \mathbb{Z} \) and all \( i \), so \( G^2(t_i) = G(G(t_i)) = G(t_{i+1}) = t_{i+2} \) thus by induction we get \( G^m(t_i) = t_{i+m} \). Since \( t_i \) is a periodic
point of period $m$ and $\rho(G,z) = \frac{d}{m}$ then $t_i + d = G^m(t_i)$

$= G^{m-1}(G(t_i)) = G^{m-1}(t_{i+j}) = \ldots = t_{i+mj} = t_i + j$

(by claim 1) then $d = j$ hence $g(z_i) = z_i + d (\mod m)$ \forall i = 1, 2, \ldots, m.

Since $T$ is an orbit not a union of several orbits then $(d, m) = 1$

Case 2: If $t_i \in \frac{Z}{n}$, $e_{\pi}(t_i) = e\left(\frac{t_i + \frac{r}{2}}{2}\right) - e(\pi(t_i)) = -e\pi(t_i) = 0$. Since, $F$ on $(e\pi)^{-1}T$ is one to one and by definition of TPO $(e\pi)^{-1}T$ order preserving then $G(t_i) = t_{i+j}$; for some $j \in \frac{Z}{2}$ in the same way $t_i + d = G^m(t_i) = G^{m-1}(t_{i+j}) = \ldots = t_{i+mj} = t_i + j$ (by claim 2) thus $d = j$; where $j \in \frac{Z}{2}$ this imply $g(z_j) = f(0) = 0$, so $z_i$ is an eventually fixed point.

An A-graph of $g$ with respect to $A_m$ and $B_m$ is an oriented graph with vertices $I_1, I_2, \ldots, I_{m+n-1}$ such that if $I_a \beta$ covers $I_\beta$ $k$ time but not $k+1$ times then there are $k$ arrows from $I_a$ to $I_\beta$. a sequence $I_{i_0} \rightarrow I_{i_1} \rightarrow \ldots \rightarrow I_{i_r}$ is an A-graph of $g$ is called a path of length $r$ and the path is called a loop if $I_{i_0} = I_r$ a loop is called simple if $I_{i_j} \neq I_{i_k}$ for $0 \leq j < k < r$.

The lemma below generalize lemma in [2] on circle map:

**Lemma 1.2.5:** Let $g: S \rightarrow S$ be a continuous map of degree one. If $I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_{n-1} \rightarrow I_0$ is a loop in a A-graph of $g$ then there exists a fixed point $z$ of $g^n$ such that $g^n(z) \in I_i$ for $i = 1, 2, \ldots, n-1$.

**Lemma 1.2.6:** Let $g$ be an monotone map (not necessarily continuous) of a closed interval $I$ into itself. Then $g$ has a fixed point.

**Proof:** suppose $g$ is increasing map Let $t = \sup\{y: g(y) > y\} \neq 0$ since $g$ is increasing, if $t < g(t)$ then $\forall y \in (t, g(t))$ then $g(y) > g(t) > y$ but this contraction.

If $t > g(t)$ then $\forall y \in (t, g(t))$ then $g(y) < g(t) < y$ but this contraction. Thus $t = g(t)$ that is, $t$ is a fixed point. If $g$ is decreasing map then the proof is similar. □

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an invariant set that is $g(A) \subset A$, let $t_1 < t_2 < \cdots < t_n < t_{n+1}$ be such that $e\pi(t_i) = a_i$ for $i = 1, 2, \ldots, n$. Let $G$ be a lift of $g$, we denote by $\tilde{G}$ the map such that:

1) $\tilde{G}(t_i + k) = G(t_i + k)$ for $i = 1, 2, \ldots, n$ and $k \in Z$.

2) $\tilde{G}$ on $[t_i + k, t_{i+1} + k]$ is linear for $i = 1, 2, \ldots, n$ and $k \in Z$.

3) $\tilde{G}$ on $[t_n + k, t_{n+1} + k]$ is linear for $k \in Z$.

We call $\tilde{G}$ the A linearization of $G$. we denote by $\bar{g}$ the map of $8$ of degree one which has $\tilde{G}$ as lift, also we say $\tilde{G}$ the A linearization of $g$. If $\bar{g} = g$ that is $\tilde{G} = g$ then we say that $G$ and $g$ are linear.

**Lemma 1.2.7:** Let $g: S \rightarrow S$ be a continuous map and $A$ be an invariant subset under $g$. let $\tilde{G}$ be the A linearization of $g$ and suppose that $\tilde{G}$ has a TPO $T$ of period $s$ and $\rho(G, z) = \frac{r}{s}$ with $(r, s) = 1$; where $\tilde{G}$ is the lift of $G$ obtained by a linearization of $G$. If $T \not\subset A$ and $\tilde{G}$ is increasing at every point of $(e\pi)^{-1}T$ then $g$ has a TPO of period $s$ and $\rho(G, z) = \frac{r}{s}$.

**Proof:** Let $t_1 < t_2 < \cdots < t_n < t_{n+1}$ such that $e\pi\{t_1, t_2, \ldots, t_n\} = A$ and $I_i = [t_i, t_{i+1}], i = 1, 2, \ldots, n-1$ and $I_n = [t_n, t_{n+1} + 1]$ be subsets of $R$. Let $U$ be the partition of $R$ given by the points of $(e\pi)^{-1}A$, since $T \not\subset A$; $T$ is a periodic orbit and $A$ is invariant set for $\tilde{G}$ so $\tilde{G}(A) \subset A$ then $T$ and $A$ are disjoint that is $T \cap A = \emptyset$. We choose $z \in T$ this
imply there is unique \( i_1 \in \{1,2,\ldots,n\} \) such that there is 
\[ t \in I_{i_1} \text{ with } e\pi(t) = z \] hence there is unique \( i_1 \in \{1,2,\ldots,n\} \) and \( n_j \in \mathbb{Z} \) such that 
\[ \tilde{G}^{-1}(t) \in t_{i_j} + n_j = 1,2,\ldots,s + 1 \] since 
\[ (\tilde{G},z) = \frac{r}{s} : \tilde{G} \text{ has a TPO pf period } s. \text{ Then } \tilde{G}^s(t) = t + r \text{ thus } i_{s+1} = i_1 \text{ and } n_{s+1} = r \text{ In such a way we obtain the following path in the U-graph of } \tilde{G}^s : 
\[ \tilde{G} : I_{i_1} \to I_{i_2} + n_2 \to \cdots \to I_{i_s} + n_s \to I_{i_1} + r \]
We will define \( M_s : I_{i_1} + r \to I_{i_2} + n_2 \) and \( M_j : I_{i_j+1} + n_j \to I_{i_j} + n_j \) such that 
\[ M_j(t) = \inf \{ y \in I_{i_j} + n_j : G(y) = t \} \] Since \( \tilde{G} \) is increasing on every interval these maps \( M_j, M_s \) are increasing. Therefore \( M : I_{i_1} + r \to I_{i_1} + r \) such that 
\[ M_I = M_1 \circ M_2 \circ \cdots \circ M_s + r \] is clear \( M \) is increasing . By lemma 1.2.6, \( M \) has a fixed point \( p \in I_{i_1} + r \) let \( p_1 = p - r \in I_{i_1} \) thus 
\[ G^i(p_1) \in I_{i_{j+1}} + n_{j+1} \] for \( j = 0,1,\ldots,s - 1 \) and \( G^s(p_1) = p_1 + r \in I_{i_1} + r \) then the orbit 
\[ X = \{ e\pi(p_1), G(e\pi(p_1)), \ldots, G^{s-1}(e\pi(p_1)) \} \] is periodic of period \( s \) and 
\[ \rho(G, e\pi(p_1)) = \frac{r}{s} \] We claim \( X \) is a TPO, to show that : Since \( T = \{ z_1, z_2, \ldots, z_s \} \) periodic orbit on 
\[ (e\pi)^{-1}T \subset R \] that is 
\[ \{ G_j(p_1) : j = 0,1,\ldots,s - 1 \} \] is \( Z \subset R \) Now we will compare between 
\[ (e\pi)^{-1}T \] and \( X \). Let 
\[ g^i(e\pi(p_1)), g^j(e\pi(p_1)) \in X \] such that a set is to left and which to the right. The elements of the U-partition are :Case 1: If these intervals are different then 
\[ X = \{ g^i(e\pi(p_1)) : i = 0,1,\ldots,s - 1 \} \] are the end points of U-partition, therefore we get \( X \) is a TPO. Case 2: If they are the same , then one has to go along the orbits of these two points, since we only use increasing pieces of the maps \( G \) and \( \tilde{G} \) then 
\[ \{ G_j : j = 0,1,\ldots,s - 1 \} \] and \( (e\pi)^{-1}T \) in the same order, since \( T \) is a TPO so \( X \) is a TPO. ■ 
We will define these maps as: let \( a, b \in R \) and \( G \) be a lift of \( g \) 
\[ G_r(a) = \sup \{ G(b) : b \leq a \} \] and 
\[ G_l(a) = \inf \{ G(b) : b \geq a \} \] We call \( g_r : 8 \to 8 \) such that \( G_r \) is a lift of \( g \) and \( g_l : 8 \to 8 \) such that 
\( G_l \) is a lift of \( g_l \) this mean \( g_r, g_l \) are continuous maps of degree one. 
**Lemma 1.2.8:** Let \( g : 8 \to 8 \) be a continuous map of degree one and \( W \) be a periodic orbit of \( g \) of period \( m \) and \( \rho(G, z) = \frac{d}{m} \) such that \( 0 < \frac{d}{m} < 1 \) and \( (d, m) = 1 \) . suppose that 
\( W \) is not a TPO . then 
\begin{enumerate}
\item \( i \)- \( g \) has a TPO \( R \) of period \( s \) and rotation number \( \frac{r}{s} \) with \( (r, s) = 1 \) and \( \frac{d}{m} < \frac{r}{s} \).
\item \( ii \)- \( g \) has a TPO \( L \) of period \( n \) and rotation number \( \frac{j}{n} \) with \( (j, n) = 1 \) and \( \frac{j}{n} < \frac{d}{m} \).
\end{enumerate}
**Proof:** Without loss of generality,If we assume \( g \) is linear on \( W \) and \( F \) is increasing on \( (e\pi)^{-1}R \) then this don’t for generality. Then by lemma 1.2.7, let \( w \) be a periodic orbit of \( g \) of period \( m \) and 
\[ (G, z) = \frac{d}{m} \] let \( U \) be the partition of \( 8 \) by elements of \( W \),since \( g \) is onto then for all 
\[ I \in U \exists j \in U \] Such that 
\[ g_r - covers I \] since the number of intervals of \( U \) is finite so we have a loop of length \( s \) in the U-graph of \( g_r \) with 
\[ 1 \leq s \leq m \] Assume that the loop is the shortest one of the U-graph of \( g_r \) since \( W \) is not a TPO at least on interval 
\[ I \in U \] satisfy \( (g_r, I) = \text{constant} \) then \( s < m \) By lemma 1.2.5, this loop gives us a periodic orbit \( R \) of 
\( g_r \) of period \( s \) and rotation number \( \frac{r}{s} \) since \( s < m \) then \( R \neq W \) all intervals on which \( G_r \) is non – decreasing (constant or increasing) then \( R \) is a TPO for \( g_r \) this imply \( R \) is a TPO for \( f \). By proposition 1.2.4, we get \( (r,s)=1 \). Now ,we need prove that
Theorem 1.2.9: Let \( g : \mathbb{S} \rightarrow \mathbb{S} \) be a continuous map of degree one. If \( \frac{d}{m} \in L(g) \) such that \( \frac{d}{m} \) is an end point of \( L(g) \) then all periodic orbits of \( g \) of period \( m \) and rotation number \( \frac{d}{m} \) are TPO.

Proof: Either \( \frac{d}{m} \in \mathbb{Z} \) such that \( (d,m)=1 \) then \( m=1 \) then by proposition 1.2.4, \( g \) has a TPO. or \( \frac{d}{m} \in \mathbb{Z} \) let \( F \) be a lifting of \( g \) such that \( \frac{d}{m} \) is the right end point of \( L_F(f) = [a, \frac{d}{m}] \). Let \( G' = G - E\left(\frac{d}{m}\right) = \frac{d}{m} - E\left(\frac{d}{m}\right) \in L_G(g) \) then \( (d,m)=1 \) and \( \frac{d}{m} \in (0,1) \) then by lemma 1.2.8, we get all the periodic orbits of period \( m \) and \( \rho(F',z) = \frac{d'}{m} \) are TPO.

\[ \rho(G,z) = \rho(G',z) + E\left(\frac{d}{m}\right) = \frac{d}{m} + E\left(\frac{d}{m}\right) = \frac{d}{m} \]

so \( T \) is TPO of period \( m \) and \( \rho(G,z) = \frac{d}{m} \).

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