Double Unresolved Approximations to Multiparton Scattering Amplitudes

J. M. Campbell and E. W. N. Glover

Physics Department, University of Durham, Durham DH1 3LE, England

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Abstract

We present approximations to tree level multiparton scattering amplitudes which are appropriate when two partons are unresolved. These approximations are required for the analytic isolation of infrared singularities of \( n + 2 \) parton scattering processes contributing to the next-to-next-to-leading order corrections to \( n \) jet cross sections. In each case the colour ordered matrix elements factorise and yield a function containing the singular factors multiplying the \( n \) parton amplitudes. When the unresolved particles are colour unconnected, the approximations are simple products of the familiar eikonal and Altarelli-Parisi splitting functions used to describe single unresolved emission. However, when the unresolved particles are colour connected the factorisation is more complicated and we introduce new and general functions to describe the triple collinear and soft/collinear limits in addition to the known double soft gluon limits of Berends and Giele. As expected the triple collinear splitting functions obey an \( N = 1 \) SUSY identity. To illustrate the use of these double unresolved approximations, we have examined the singular limits of the tree level matrix elements for \( e^+e^- \to 5 \) partons when only three partons are resolved. When integrated over the unresolved regions of phase space, these expressions will be of use in evaluating the \( \mathcal{O}(\alpha_s^3) \) corrections to the three jet rate in electron positron annihilation.
1 Introduction

Electron-positron annihilation has proved a very clean and direct source of information about the nature of the strong interaction. In particular, three jet events observed at DESY gave the first clear indications of the existence of the gluon \([1]\), while more recently, the Casimirs that determine the gauge group of QCD have been measured \([2]\) using four jet data. On the theoretical side, a large amount of effort has been devoted to determining multi-jet rates within perturbative QCD. Leading order predictions for the production of up to five jets have been known for some time \([3, 4, 5, 6, 7]\). While the general features of events are generally well described by leading order estimates, significant improvement can be obtained by including higher order corrections at either fixed order or via resummation of the dominant logarithms or a mixture of both. The next-to-leading order corrections to three-jet like quantities were first computed in the early 1980’s \([5, 6]\) and systematically evaluated by Kunszt and Nason \([8]\) using a general purpose Monte Carlo program. In many cases the radiative corrections are significant and resummations of infrared logarithms \([9]\) must be employed to make sensible comparisons with experimental data. Such calculations have been used to extract a precise value of the strong coupling constant from three jet events and other hadronic event shapes \([10]\) with a global average of,

\[
\alpha_s(M_Z) = 0.121 \pm 0.005.
\]

A recent re-evaluation of lower energy data yields \([11]\),

\[
\alpha_s(M_Z) = 0.122^{+0.008}_{-0.006},
\]

while preliminary analyses using hadronic data above the Z-resonance from the OPAL Collaboration \([12]\) shows that the strong coupling evolves in the manner predicted by QCD.

Recently there has been much progress towards a more complete theoretical description of four jet events produced in electron-positron annihilation. The one-loop matrix elements relevant for four jet production have been evaluated by two groups \([13, 14, 15, 16]\) using contrasting methods. Bern et al calculate helicity amplitudes for the \(e^+e^- \rightarrow (\gamma^*, Z) \rightarrow 4 \) parton processes while in Refs. \([13, 15]\), the spin summed interferences between one-loop and tree-level \(\gamma^* \rightarrow 4 \) parton processes were evaluated. These results have been compared \([16]\) and found to agree for specific phase space points. Using these one-loop matrix elements and the earlier tree level five parton matrix elements of \([7]\), two general purpose Monte Carlo programs have been constructed \([17, 18]\) and used to compute the four-jet rate and other four-jet like hadronic observables at next-to-leading order (i.e. \(O(\alpha_s^3))\). The corrections are again large \([17, 18]\) indicating the need for resummations of infrared logarithms, but nevertheless, improved determinations of the Casimirs are possible \([19]\) together with more stringent bounds on the existence of a light supersymmetric partner of the gluon \([20]\).

One of the next steps is to evaluate the next-to-next-to-leading order corrections to the three-jet rate. Together with LEP/SLC data and hadronic events from the next linear
collider this would play a role in reducing the error on $\alpha_s(M_Z)$ from event shape analyses to the 2-3% level. However, to accomplish this, several ingredients are necessary. First, the two-loop $\gamma^* \rightarrow 3$ parton amplitudes must be determined including the evaluation of the two loop box graph with massless internal and external legs. Second, the real infrared singularities which occur when two particles are unresolved in the tree level process and when a single particle is unresolved in the one-loop process must be systematically isolated. In the case of two real unresolved particles, there are a variety of different configurations;

(a) two soft particles,

(b) two pairs of collinear particles,

(c) three collinear particles,

(d) one soft and two collinear,

while in the one-loop process one particle may be soft or two may be collinear as before. For next-to-leading order calculations, many methods have been developed to first isolate the infrared singularities associated with one unresolved particle analytically and then numerically combine the remaining finite real and virtual contributions [6, 21, 3, 22, 23]. All of these rely on suitable approximations to the real matrix elements in the soft and collinear limits. However, the infrared singular limits of the real and one-loop amplitudes relevant at next-to-next-to-leading order are rather less well known. In ref. [24], Bern et al., have developed appropriate splitting functions for one-loop processes where two external particles are collinear while Berends and Giele have examined the multiple soft gluon behaviour of QCD matrix elements in ref. [25].

In this paper, and as a first step towards the next-to-next-to-leading order 3 jet rate, we examine the double unresolved limits of multiparton scattering amplitudes, and find suitable approximations for the $e^+e^- \rightarrow 5$ parton matrix elements [7] when two particles are unresolved. We present results for all of the configurations (a)-(d) above. In section 2 we briefly review the structure of tree level scattering amplitudes when one particle is unresolved. In section 3 we write down general expressions for the structure of the $e^+e^- \rightarrow 5$ parton matrix elements. Sections 4 and 5 are organised according to whether the two unresolved particles are colour connected or not. The precise meaning of what colour connected means will be given in sect. 4. In the unconnected case, the singular limits are merely obtained by multiplying single unresolved factors. However, when the particles are colour connected, the structure is more involved (sect. 5) and we give explicit formulae detailing the double unresolved singular factors. In each case we write down expressions for the double unresolved limits of the five parton process. Our results are summarised in section 6.
2 Single unresolved limits

The soft gluon and collinear parton limits of multiparton scattering amplitudes are well known. For example, if we consider the matrix element for the tree level scattering of \( n \) gluons labelled 1...n, then, decomposing the amplitude according to the various colour structures [26], we find,

\[
M(1,2,\ldots,n) = 2ig^{n-2} \sum_{P(1,2,\ldots,n-1)} Tr(T^{a_1}T^{a_2}\ldots T^{a_n})A(1,2,\ldots,n). \tag{2.1}
\]

Here, \( a_i \) represents the colour of the \( i \)th gluon and the summation runs over the \((n-1)!\) non-cyclic permutations of the gluons. At leading order in the number of colours, \( N \), the squared matrix elements summed over all colours have the simple form,

\[
|M(1,2,\ldots,n)|^2 = \left( \frac{g^2N^2}{2} \right)^{n-2} (N^2 - 1) \sum_{P(1,2,\ldots,n-1)} |A(1,2,\ldots,n)|^2. \tag{2.2}
\]

Unlike QED [27], it is the colour ordered sub-amplitudes \( A \) rather than the squared matrix elements that have nice factorisation properties in the soft and collinear limits (see for example [28]). For example, in the limit where the \( n \)th gluon becomes soft, we have the QED-like factorisation into an eikonal factor multiplied by the colour ordered amplitude with gluon \( n \) removed, but the ordering of the hard gluons preserved,

\[
|A(1,2,\ldots,n-1,n)|^2 \to S(n-1)n_1(s_{1(n-1)}, s_{1n}, s_{n(n-1)}) |A(1,2,\ldots,n-1)|^2. \tag{2.3}
\]

with the eikonal factor given by,

\[
S_{abc}(s_{ac}, s_{ab}, s_{bc}) = \frac{4s_{ac}}{s_{ab}s_{bc}}. \tag{2.4}
\]

Similarly, in the limit where two gluons become collinear, the sub-amplitudes factorise. For example, if gluons \( a \) and \( b \) become collinear and form gluon \( c \), then adjacent gluons give a singular contribution,

\[
|A(1,\ldots,a,b,\ldots,n)|^2 \to P_{gg\to g}(z, s_{ab}) |A(1,\ldots,c,\ldots,n)|^2, \tag{2.5}
\]

while separated gluons do not,

\[
|A(1,\ldots,a,\ldots,b,\ldots,n)|^2 \to \text{finite}. \tag{2.6}
\]

In other words, there is no singular contribution as \( s_{ab} \to 0 \), and, when integrated over the small region of phase space where the collinear approximation is valid, this colour ordered amplitudes generates a negligible contribution to the cross section. In eq. (2.5), \( z \) is the fraction of the momentum carried by one of the gluons and, after integrating over the azimuthal angle of the plane containing the collinear particles with respect to the rest of the hard process, the collinear splitting function \( P_{gg\to g} \) is given by,

\[
P_{gg\to g}(z,s) = \frac{2}{s}P_{gg\to g}(z) \tag{2.7}
\]
where the usual Altarelli-Parisi splitting kernel \[29\] with the colour factor removed is given by,
\[
P_{gg\rightarrow g}(z) = \left( \frac{1 + z^4 + (1 - z)^4}{z(1 - z)} \right).
\]
(2.8)

In either soft or collinear limits, the process appears to involve only \( n - 1 \) hard gluons and the infrared singularities must precisely cancel \([30, 31]\) with those from the \((n - 1)\)-gluon one-loop amplitudes. Together, the real and virtual graphs form the next-to-leading order perturbative contribution for infrared safe quantities associated with the \((n - 1)\) gluon scattering process. Many methods have been developed to first isolate the infrared singularities analytically and then numerically combine the remaining finite real and virtual contributions \([1, 21, 5, 22, 23]\).

3 Colour-ordered matrix elements squared

There are two five parton processes,
\[
e^+e^- \rightarrow q\bar{q}ggg,
\]
(3.1)
and,
\[
e^+e^- \rightarrow q\bar{q}Q\bar{q}g,
\]
(3.2)
for which the lowest order matrix elements for process (3.1) can be written,
\[
\mathcal{M}(Q_1; 1, 2, 3; \overline{Q}_2) = \hat{S}_5^{\mu}(Q_1; 1, 2, 3; \overline{Q}_2)V^\mu,
\]
(3.3)
while for process (3.2) we have,
\[
\mathcal{M}(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = \hat{T}_5^{\mu}(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V^\mu.
\]
(3.4)

In these expressions, \( V^\mu \) represents the lepton current, while \( \hat{S}_5^{\mu} \) and \( \hat{T}_5^{\mu} \) are hadronic currents containing quarks and gluons. The gluon colour is denoted by the adjoint representation label \( a_1, \ldots, a_3 \) while that of the quark is \( c_1, \ldots, c_4 \).

The two quark-three gluon current \( \hat{S}_5^{\mu} \) may be decomposed according to the colour structure \([26]\),
\[
\hat{S}_5^{\mu}(Q_1; 1, 2, 3; \overline{Q}_2) = ieg^3 \sum_{P(1,2,3)} (T^{a_1}T^{a_2}T^{a_3})_{c_1c_2}S_\mu(Q_1; 1, 2, 3; \overline{Q}_2),
\]
(3.5)
where \( S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) \) represents the colourless subamplitude where the gluons are emitted in an ordered way from the quark line. The colour matrices are normalised such that,
\[
\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}.
\]
By summing over all permutations of gluon emission, all Feynman diagrams and colour structures are accounted for.

The squared matrix elements for eq. (3.1) are simply,

$$|\hat{S}_\mu^3 V^\mu|^2 = e^2 \left( \frac{g^2 N}{2} \right)^3 \left( \frac{N^2 - 1}{N} \right)$$

\[
\times \left[ \sum_{P(1,2,3)} \left( |S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 - \frac{1}{N^2} |S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 \right) + \left( \frac{N^2 + 1}{N^4} \right) |S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 \right].
\]

(3.6)

In the last two terms, the tilde indicates that that gluon should be inserted in all positions in the amplitude. In other words,

$$S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) = S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) + S_\mu(Q_1; 1, 3, 2; \overline{Q}_2) + S_\mu(Q_1; 3, 1, 2; \overline{Q}_2).$$

(3.7)

In this case, gluon 3 is effectively photon-like and the contribution from the triple and quartic gluon vertices drops out. For reference, the squared matrix elements for quark-antiquark production together with a single gluon can be written,

$$|\hat{S}_\mu^3 V^\mu|^2 = e^2 \left( \frac{g^2 N}{2} \right)^3 \left( \frac{N^2 - 1}{N} \right) |S_\mu(Q; G; \overline{Q}) V^\mu|^2.$$

(3.8)

The four quark-one gluon current may be decomposed according to its colour structure as follows,

$$\hat{T}_\mu^5(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = \frac{i g^3}{2}$$

\[
\times \left[ T_{c_1 c_4}^a \delta_{c_1 c_2} T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) + (Q_1 \leftrightarrow Q_3, \overline{Q}_2 \leftrightarrow \overline{Q}_4) - (Q_1 \leftrightarrow Q_3) - (\overline{Q}_2 \leftrightarrow \overline{Q}_4) \right],
\]

(3.9)

where the exchanges are understood to apply to the colour labels as well. The colour ordered sub-current can be written,

$$T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = \delta_{Q_1 Q_2} \delta_{Q_3 Q_4} T_\mu^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)$$

\[
+ \frac{1}{N} \delta_{Q_3 Q_4} T_\mu^B(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1),
\]

(3.10)

where,

$$T_\mu^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = A_{\mu}^{Q_1 Q_2}(Q_1; 1; \overline{Q}_2|Q_3; \overline{Q}_4) + A_{\mu}^{Q_3 Q_4}(Q_3; \overline{Q}_2|Q_1; 1; \overline{Q}_4)$$

$$T_\mu^B(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = B_{\mu}^{Q_1 Q_2}(Q_1; 1; \overline{Q}_2|Q_3; \overline{Q}_4) + B_{\mu}^{Q_3 Q_4}(Q_3; \overline{Q}_2|Q_1; 1; \overline{Q}_4).$$

(3.11)
Here, \( \delta_{Q_iQ_j} = 1 \) if quarks \( i \) and \( j \) have the same flavour. The functions \( A_{\mu}^{Q_1Q_2} \) and \( B_{\mu}^{Q_1Q_2} \) describe Feynman diagrams where the gauge boson couples to the \( Q_1Q_2 \) pair. However, in \( A_{\mu}^{Q_1Q_2} \), the colour flows along the gluon connecting the two quark pairs, so that \( Q_1 \) and \( \overline{Q}_4 \) are the endpoints of a colour antenna (and similarly \( Q_3 \) and \( \overline{Q}_2 \)) while in \( B_{\mu}^{Q_1Q_2} \), no colour is transmitted between the quark pairs and now \( Q_1 \) and \( \overline{Q}_2 \) form the endpoints of a colour antenna (and similarly \( Q_3 \) and \( \overline{Q}_4 \)). In each case, the gluon may be emitted from any position on the colour line.

Squaring the four quark-one gluon amplitude and summing over colours yields,

\[
|\tilde{T}_\mu V|^2 = e^2 \left( \frac{g^2 N_c}{2} \right)^3 \left( \frac{N_c^2 - 1}{N_c^2} \right) \times \left[ |T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu|^2 + |T_\mu(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) V^\mu|^2 \right.
\]

\[
- \frac{2}{N_c} \Re \left( T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu + T_\mu(Q_3, \overline{Q}_4; Q_1, \overline{Q}_2; 1) V^\mu \right) \left( T_\mu(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) V^\mu \right)^\dagger \right]
\]

\[
+ (Q_1 \leftrightarrow Q_3, \overline{Q}_2 \leftrightarrow \overline{Q}_4),
\]

or equivalently,

\[
|\tilde{T}_\mu V|^2 = e^2 \left( \frac{g^2 N_c}{2} \right)^3 \left( \frac{N_c^2 - 1}{N_c^2} \right) \times \left[ |T_\mu^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu|^2 \right.
\]

\[
+ \frac{1}{N_c^2} \left( |T_\mu^A(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) V^\mu|^2 - |T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu|^2 \right)
\]

\[
+ \frac{2 \delta_{Q_1Q_3}}{N_c} \Re \left( T_\mu^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu \right) \left( T_\mu^B(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu \right)^\dagger
\]

\[
- \frac{(N_c^2 + 1)}{2 N_c^3} \delta_{Q_2Q_4} \Re \left( T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu \right) \left( T_\mu(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) V^\mu \right)^\dagger
\]

\[
+ (Q_1 \leftrightarrow Q_3, \overline{Q}_2 \leftrightarrow \overline{Q}_4) + \delta_{Q_1Q_3}(Q_1 \leftrightarrow Q_3) + \delta_{Q_2Q_4}(Q_2 \leftrightarrow Q_4). \tag{3.13}
\]

Here we have introduced the shorthand notation,

\[
T_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) = T_\mu^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) + T_\mu^A(Q_3, \overline{Q}_4; Q_1, \overline{Q}_2; 1)
\]

\[
= T_\mu^B(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) + T_\mu^B(Q_3, \overline{Q}_2; Q_1, \overline{Q}_4; 1). \tag{3.14}
\]

Note that in the case of identical quarks, there is an extra symmetry factor of \( 1/4 \) multiplying the matrix elements.
4 Colour unconnected double unresolved

In the cases where the two unresolved particles are not colour connected, the factorisation of the amplitudes involves the well-known functions describing single soft and collinear emission. We first describe what is meant by colour connected and colour unconnected.

4.1 Colour connection

In the previous section we have seen how tree level matrix elements can be decomposed into colour ordered subamplitudes which have nice factorisation properties in the infrared limits. Therefore it is useful to view the matrix elements in terms of the colour structure associated with the subamplitudes. For example, in eq. (3.3), the two quark-three gluon subamplitude, $S_\mu(Q_1; 1, 2, 3; \overline{Q}_2)$ is associated with the colour structure $(T^{a_1}T^{a_2}T^{a_3})_{c_1c_2}$. This is a colour antenna that ends on the quark/antiquark colour charges $c_1$ and $c_2$ with ordered emission of gluons with colour $a_1, \ldots, a_3$. Within this colour antenna, gluon 1 is colour connected to the quark $Q_1$ and gluon 2, but not to the antiquark $\overline{Q}_2$ or to gluon 3. In cases involving more than one quark-antiquark pair there can be many colour antennae. For example, the four quark amplitude $A_{\mu}^{Q_1Q_2}(Q_1; 1; \overline{Q}_4|Q_3; \overline{Q}_2)$ of section 3 describes a process with two separate colour antennae. In general, the particles in one antenna are not colour connected to the particles in one of the other antennae. However, there is one case where particles in adjoining antenna can usefully be thought of as colour connected. This is when there is an antiquark at the end of one antenna and a like flavour quark at the beginning of another,

$A(\ldots, \overline{Q}|Q, \ldots)$.

When this quark-antiquark pair are collinear, they combine to form a gluon $G$, which then connects, or pinches together, the two separate colour antennae, so that,

$|A(\ldots, \overline{Q}|Q, \ldots)|^2 \rightarrow P_{q\overline{q}\rightarrow G}(z, s_{Q\overline{Q}})|A(\ldots, a, G, b, \ldots)|^2$.

A useful definition of colour “connected” therefore includes these antennae pinching configurations along with the more straightforward colour connection within a single antenna. All other cases are colour “unconnected”.

4.2 Two collinear pairs

Two pairs of particles may become collinear separately, but with the particles in one or both of the pairs themselves not colour “connected”. In these cases, there are no singular contributions containing both of the vanishing invariants. For instance, if partons \{a, d\} and \{b, c\} are collinear then,

$|A(\ldots, a, \ldots, b, \ldots, c, \ldots, d, \ldots)|^2 \rightarrow$ less singular. (4.1)
By this we mean there is no contribution proportional to $1/s_{ad}s_{bc}$ and once again, when integrated over the small region of phase space relevant for this approximation yields a negligible contribution.

The situation where two pairs of colour “connected” particles are collinear is also rather trivial. If partons $a$ and $b$ form $P$, while $c$ and $d$ cluster to form $Q$, so that $P$ and $Q$ are themselves colour unconnected, then,

$$|A(\ldots, a, b, \ldots, c, d, \ldots)|^2 \rightarrow P_{ab \rightarrow P}(z_1, s_{ab}) \ P_{cd \rightarrow Q}(z_2, s_{cd}) |A(\ldots, P, \ldots, Q, \ldots)|^2.$$  \hfill (4.2)

Here, $z_1$ and $z_2$ are the momentum fractions carried by $a$ and $c$ respectively. A similar result holds if either of the pairs involves particles in separate antennae, but which are able to undergo antenna pinching. The collinear splitting functions are related to the (colourless) Altarelli-Parisi splitting kernels by eq. (2.7) which, in the conventional dimensional regularisation scheme [32] with all particles treated in $d = 4 - 2\epsilon$ dimensions, are given by [29],

$$P_{gg \rightarrow q}(z) = \left(1 + z^2 - \epsilon(1 - z)^2\right) , \hfill (4.3)$$

$$P_{gq \rightarrow g}(z) = \left(\frac{z^2 + (1 - z)^2 - \epsilon}{1 - \epsilon}\right), \hfill (4.4)$$

with $P_{gg \rightarrow g}$ given in eq. (2.8) and $P_{gq \rightarrow q}(z) = P_{gq}(1 - z)$. As before, azimuthal averaging of the collinear particle plane is understood.

### 4.2.1 Double collinear limit of $e^+e^- \rightarrow 5$ partons

In this limit where two pairs of partons are simultaneously collinear, the five parton matrix elements factorise into the three parton matrix elements multiplied by a combination of products of collinear splitting functions. Summing over all possible unconnected double collinear limits, for the two-quark currents we find,

$$|S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 \rightarrow \left(P_{Q_1 \rightarrow Q P_{23 \rightarrow G}} + P_{Q_1 \rightarrow Q P_{\overline{3}Q_3 \rightarrow \overline{Q}}} + P_{12 \rightarrow G P_{3\overline{Q}_3 \rightarrow \overline{Q}}} \right) |S_\mu^3 V^\mu|^2,$$

$$|S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 \rightarrow \left(P_{Q_1 \rightarrow Q P_{2\overline{Q}_2 \rightarrow \overline{Q}}} + P_{Q_1 \rightarrow Q P_{\overline{3}Q_3 \rightarrow \overline{Q}}} + P_{Q_1 \rightarrow Q P_{3\overline{Q}_3 \rightarrow \overline{Q}}} + P_{Q_{13} \rightarrow Q P_{12 \rightarrow G}} + P_{12 \rightarrow G P_{3\overline{Q}_3 \rightarrow \overline{Q}}} \right) |S_\mu^3 V^\mu|^2,$$

$$|S_\mu(Q_1; 1, 2, 3; \overline{Q}_2) V^\mu|^2 \rightarrow \sum_{P(1,2,3)} P_{Q_1 \rightarrow Q P_{\overline{2}\overline{Q}_2 \rightarrow \overline{Q}}} |S_\mu^3 V^\mu|^2,$$  \hfill (4.5)

whilst the only contributing pieces for the four-quark process are,

$$|T^A_\mu(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1) V^\mu|^2 \rightarrow \left(P_{Q_1 \rightarrow Q P_{\overline{4}Q_3 \rightarrow \overline{G}}} + P_{1\overline{Q}_4 \rightarrow \overline{q} P_{\overline{2}\overline{Q}_2 \rightarrow \overline{G}}} \right) |S_\mu^3 V^\mu|^2,$$

$$|T^B_\mu(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1) V^\mu|^2 \rightarrow \left(P_{Q_1 \rightarrow Q P_{\overline{4}Q_3 \rightarrow \overline{G}}} + P_{1\overline{Q}_2 \rightarrow \overline{q} P_{\overline{4}\overline{Q}_4 \rightarrow \overline{G}}} \right) |S_\mu^3 V^\mu|^2.$$
\[
\left| \mathcal{T}_\mu(Q_1, \bar{Q}_2; Q_3, \bar{Q}_4; 1)V_\mu^2 \right|^2 \rightarrow \left( P_{Q_1 \rightarrow q}P_{\bar{Q}_4 \rightarrow \bar{q}}P_{q_3 \rightarrow G} + P_{\bar{Q}_2 \rightarrow q}P_{q_4 \rightarrow \bar{q}}P_{\bar{q}_3 \rightarrow G} + P_{Q_3 \rightarrow q}P_{Q_1 \rightarrow \bar{q}}P_{\bar{q}_2 \rightarrow \bar{q}} \right) \left| S^3 V_\mu^2 \right|^2.
\]

(4.6)

For brevity we have dropped the arguments of the splitting functions. Explicitly, for \( n_F \) flavours of quark, we find,

\[
\frac{1}{3!} \left| \hat{S}_\mu^5 V_\mu^2 \right|^2 + \left| \hat{T}_\mu^5 V_\mu^2 \right|^2 = \left( \frac{g^2}{2} \right)^2 \left( \frac{N^2 - 1}{N^2} \right) \left| S^3 V_\mu^2 \right|^2 \\
\times \left( P_{Q_1 \rightarrow q}P_{Q_2 \rightarrow \bar{q}}P_{Q_3 \rightarrow \bar{q}} + P_{Q_1 \rightarrow q}P_{Q_2 \rightarrow \bar{q}}P_{Q_3 \rightarrow \bar{q}} + P_{Q_1 \rightarrow q}P_{Q_2 \rightarrow \bar{q}}P_{Q_3 \rightarrow \bar{q}} + P_{Q_1 \rightarrow q}P_{Q_2 \rightarrow \bar{q}}P_{Q_3 \rightarrow \bar{q}} \right).
\]

(4.7)

Note also that the identical gluon factor \( 1/3! \) is eliminated since each term in the sum over permutations produces an identical contribution. It is interesting to review the origin of the factor of \( n_F \). There are \( n_F(n_F - 1)/2 \) contributions from two unlike pairs of quarks, each of which generates two sets of singular limits - that indicated plus the symmetric term \( (Q_1 \leftrightarrow Q_3, \bar{Q}_2 \leftrightarrow \bar{Q}_4) \). In addition there are \( n_F \) like-quark pair contributions, which after the symmetries have been applied yield four singular limits. However, the identical quark contribution is multiplied by the identical particle factor \( 1/4 \) so that the net result is,

\[
\frac{n_F(n_F - 1)}{2} \times 2 + n_F \times 4 \times \frac{1}{4} = n_F^2.
\]

One factor of \( n_F \) is absorbed into the three parton matrix elements \( \left| \hat{S}_\mu^3 V_\mu^2 \right|^2 \), while the other appears as an explicit factor.

### 4.3 Triple collinear factorisation

If three collinear particles are colour “unconnected” then there is no singularity. So if \( a, b \) and \( c \) all become collinear,

\[
|A(\ldots, a, \ldots, b, \ldots, c, \ldots)|^2 \rightarrow \text{finite},
\]

(4.8)

and there is no singular contribution involving the invariants \( s_{ab}, s_{bc} \) or \( s_{abc} \). As before, because the region of phase space where the triple collinear limit is valid is extremely small, this gives a negligible contribution to the cross section. When two of the three collinear particles are colour “connected” we find a singular result,

\[
|A(\ldots, a, \ldots, b, c, \ldots)|^2 \rightarrow 1/s_{bc}.
\]

(4.9)

However, when integrated over the triple collinear region of phase space that requires \( s_{ab}, s_{bc} \) or \( s_{abc} \) all to be small, we again obtain a negligible contribution that is proportional to the small parameter defining the extent of the triple collinear phase space. We therefore ignore contributions of this type.
4.4 Soft/collinear factorisation

Two particles may be unresolved if one of them is a soft gluon and another pair are collinear. When the soft gluon $g$ is not colour connected to either of the colour “connected” collinear particles $c$ and $d$, factorisation is straightforward,

$$|\mathcal{A}(\ldots, a, g, b, \ldots, c, d, \ldots)|^2 \rightarrow S_{agb}(s_{ab}, s_{ag}, s_{bg}) P_{cd \to P}(z, s_{cd}) |\mathcal{A}(\ldots, a, b, \ldots, P, \ldots)|^2. \quad (4.10)$$

4.4.1 Soft/collinear limit of $e^+e^- \rightarrow 5$ partons

In the soft/collinear limit, the five parton matrix elements again factorise into a singular factor multiplying the squared two-quark current relevant for three parton production,

$$\begin{align*}
|S_{\mu}(Q_1; 1, 2, 3; \bar{Q}_2)V^\mu|^2 &\rightarrow \left(S_{Q_{12}} P_{\bar{Q}_2 \to Q} + P_{Q_{1 \to Q} S_{23\bar{Q}_2}}\right) |S_3^\mu V^\mu|^2, \\
\left|S_{\mu}(Q_1; 1, 2, 3; \bar{Q}_2)V^\mu\right|^2 &\rightarrow \left(S_{Q_{12}} P_{\bar{Q}_2 \to Q} + P_{Q_{13 \to Q} S_{12\bar{Q}_2}} + P_{12 \to G S_{Q_{13\bar{Q}_2}}}\right) |S_3^\mu V^\mu|^2.
\end{align*} \quad (4.11)$$

Note that for $|S_{\mu}(Q_1; \hat{1}, \hat{2}, \hat{3}; \bar{Q}_2)V^\mu|^2$, the soft and collinear limits are considered to be overlapping and will be dealt with in section 5.2.

In the four-quark current case, the soft/collinear limit has only two colour-unconnected contributions. The first is given by,

$$|\mathcal{T}_B^\mu(Q_1; \bar{Q}_4; Q_3, \bar{Q}_2; 1)V^\mu|^2 \rightarrow S_{Q_{14}} P_{Q_3 \bar{Q}_4 \to G} |S_3^\mu V^\mu|^2, \quad (4.12)$$

whilst the limit of $|\mathcal{T}_\mu(Q_1, \bar{Q}_2; Q_3, \bar{Q}_4; 1)V^\mu|^2$, again involves both unconnected and connected factors and therefore discussion of this will also be deferred until section 5.2. The other subamplitudes vanish in the unconnected soft/collinear limit.

Applying these limits to the full five parton matrix elements is straightforward and, after removing identical particle factors where necessary, we find,

$$\begin{align*}
\frac{1}{3!} \left|\tilde{S}_3^\mu V^\mu\right|^2 + \left|\tilde{T}_\mu^5 V^\mu\right|^2 &= \left(g^2 N_c^2 / 2\right) \left|S_3^\mu V^\mu\right|^2 \\
\times &\left[\left(\frac{N_c^2 - 1}{N_c^2}\right) \left(S_{Q_{12}} P_{\bar{Q}_2 \to Q} + P_{Q_{1 \to Q} S_{23\bar{Q}_2}}\right) - \frac{1}{N_c^2} P_{12 \to G S_{Q_{13\bar{Q}_2}}} \\
+ \frac{n_F}{N_c^2} S_{Q_{14}} P_{Q_3 \bar{Q}_4 \to G}\right]. \quad (4.13)
\end{align*}$$
4.5 Two soft gluons

When two unconnected gluons are soft, the factorisation is again simple [25]. For gluons $g_1$ and $g_2$ soft we find,

$$|A(\ldots, a, g_1, b, \ldots, c, g_2, d, \ldots)|^2 \rightarrow S_{ab}(s_{ab}, s_{ag_1}, s_{bg_2}) S_{cd}(s_{cd}, s_{cg_1}, s_{dg_2})$$

$$\times |A(\ldots, a, b, \ldots, c, d, \ldots)|^2,$$

(4.14)

so that the singular factor is merely the product of two single soft gluon emission factors given by eq. (2.4). Note that $b = c$ is allowed.

4.5.1 Double soft limit of $e^+e^- \rightarrow 5$ partons

The sum over the unconnected double soft limits of the colour ordered subamplitudes can be easily read off,

$$|S_\mu(Q_1; 1, 2, 3; Q_2)V^\mu|^2 \rightarrow S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} |S_\mu^3 V^\mu|^2,$$

$$|S_\mu(Q_1; 1, 2, 3; Q_2)V^\mu|^2 \rightarrow (S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} + S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1}) |S_\mu^3 V^\mu|^2,$$

$$|S_\mu(Q_1; 1, 2, 3; Q_2)V^\mu|^2 \rightarrow \frac{1}{2} \sum_{P(1,2,3)} S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} |S_\mu^3 V^\mu|^2.$$

(4.15)

There is no contribution from the four-quark matrix elements. Inserting these limits into the full five parton matrix elements yields,

$$\frac{1}{3!} |\tilde{S}_\mu V^\mu|^2 + |\tilde{T}_\mu V^\mu|^2 = \left( \frac{g^2 N}{2} \right)^2 |\tilde{S}_\mu V^\mu|^2$$

$$\times \left[ S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} - \frac{1}{N^2} \left( S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} + S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} \right) + \left( \frac{N^2 + 1}{2N^4} \right) S_{Q_1\bar{Q}_2} S_{Q_2\bar{Q}_1} \right],$$

(4.16)

where once again the sum over permutations is eliminated by the identical particle factor.
5 Colour connected double unresolved

The factorisation that occurs when the two unresolved particles are colour “connected” is necessarily more involved than that in section 4. In particular, we will need to introduce new functions to describe this factorisation.

5.1 Triple collinear factorisation

When three colour “connected” particles cluster to form a single parent parton there are four basic clusterings,

\[ g g g \to G, \quad q g g \to Q, \]
\[ g \bar{q} q \to G, \quad q \bar{q} q \to Q, \]

and the colour ordered sub-amplitude squared for an \( n \)-parton process then factorises in the triple collinear limit,

\[ |A(\ldots, a, b, c, \ldots)|^2 \to P_{abc \to P}|A(\ldots, P, \ldots)|^2. \] (5.1)

As before, partons able to undergo antenna pinching are considered to be colour connected, so that there may be contributions from amplitudes such as \( A(\ldots, a, b|c, \ldots) \). The triple collinear splitting function for partons \( a, b \) and \( c \) clustering to form the parent parton \( P \) is generically,

\[ P_{abc \to P}(w, x, y, s_{ab}, s_{ac}, s_{bc}, s_{abc}), \] (5.2)

where \( w, x \) and \( y \) are the momentum fractions of the clustered partons,

\[ p_a = wp_P, \quad p_b = xp_P, \quad p_c = yp_P, \quad \text{with} \quad w + x + y = 1. \] (5.3)

In addition to depending on the momentum fractions carried by the clustering partons, the splitting function also depends on the invariant masses of parton-parton pairs and the invariant mass of the whole cluster. In this respect, they are different from the splitting functions derived in the jet-calculus approach [33], and implemented in the shower Monte Carlo NLLJET [34], which depend only on the momentum fractions.

The triple collinear splitting functions \( P_{abc \to P} \) are obtained by retaining terms in the full matrix element squared that possess two of the ‘small’ denominators \( s_{ab}, s_{ac}, s_{bc} \) and \( s_{abc} \). As before, we consider the explicit forms of the \( \gamma^* \to \text{four and five parton squared matrix elements} \) and work in conventional dimensional regularisation, with all external particles in \( d = 4 - 2\epsilon \) dimensions. Similar results could be derived using helicity methods or by examining the on-shell limits of the recursive gluonic and quark currents of ref. [28].

Although the splitting functions are universal, and apply to any process involving the same three colour connected particles, for processes involving spin-1 particles, there are additional (non-universal) azimuthal correlations due to rotations of the polarisation vectors.
These angular correlations do not contribute to the underlying infrared singularity structure and vanish after all azimuthal integrations have been carried out and we therefore systematically omit them.

A further check on our results is provided by the strong-ordered limit, where the particles become collinear sequentially rather than at the same time. In this limit these functions factorise into the product of two usual collinear splitting functions plus azimuthal terms, agreeing with the results of [35].

5.1.1 Three collinear gluons

Firstly, examining the sub-amplitudes for multiple gluon scattering, we find that the colour-ordered function $P_{ggg \rightarrow G}$ is given by,

$$P_{abc \rightarrow G}(w, x, y, s_{ab}, s_{bc}, s_{abc}) = 8 \times \left\{ \begin{array}{l}
+ \frac{(1 - \epsilon)}{s^2_{ab}s^2_{abc}} \frac{(x s_{abc} - (1 - y)s_{bc})^2}{(1 - y)^2} + \frac{2(1 - \epsilon)s_{bc}}{s^2_{ab}} + \frac{3(1 - \epsilon)}{2s^2_{abc}} \\
+ \frac{1}{s_{ab}s_{abc}} \left( \frac{(1 - y(1 - y))^2}{yw(1 - w)} - 2\frac{x^2 + xy + y^2}{1 - y} + \frac{xw - x^2y - 2}{y(1 - y)} + 2\epsilon \frac{x}{(1 - y)} \right) \\
+ \frac{1}{2s_{ab}s_{bc}} \left( 3x^2 - 2\frac{(2 - w + w^2)(x^2 + w(1 - w))}{y(1 - y)} + \frac{1}{yw} + \frac{1}{(1 - y)(1 - w)} \right) \\
+ (s_{ab} \leftrightarrow s_{bc}, w \leftrightarrow y) + \text{azimuthal terms} \end{array} \right\} \quad (5.4)$$

This splitting function is symmetric under the exchange of the outer gluons, and contains poles only in $s_{ab}$ and $s_{bc}$.

5.1.2 Two gluons with a collinear quark or antiquark

There are two distinct splitting functions representing the clustering of two gluons and a quark which depend on whether or not the gluons are symmetrised over. In the unsymmetrised case, there will be poles in $s_{g_1 g_2}$, due to contributions from the triple gluon vertex which are not present in the QED-like case. For the pure QCD splitting we find,

$$P_{qg_1 g_2 \rightarrow Q}(w, x, y, s_{qg_1}, s_{qg_2}, s_{g_1 g_2}, s_{qg_1 qg_2}) = 4 \times \left\{ \begin{array}{l}
+ \frac{1}{s_{qg_1} s_{g_1 g_2}} \left( (1 - \epsilon) \frac{1 + w^2}{y} + \frac{1 + (1 - y)^2}{(1 - w)} \right) + 2\epsilon \left( \frac{w}{y} + \frac{1 - y}{1 - w} \right) \\
+ \frac{1}{s_{qg_1} s_{qq_1 g_2}} \left( (1 - \epsilon) \frac{(1 - y)^3 + w(1 - x) - 2y}{y(1 - w)} \right) - \epsilon \left( \frac{2(1 - y)(y - w)}{y(1 - w)} - x \right) - \epsilon^2 x \end{array} \right\}$$

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Here, the singular contribution comes entirely from the first term where combining the two colour lines together. Explicit evaluation yields,

\[
\begin{align*}
&+ \frac{1}{s_{g_1g_2}s_{qg_1g_2}} \left( (1 - \epsilon) \left( \frac{(1 - y)^2(2 - y) + x^2 + 2xy - 2 - y}{y(1 - w)} \right) + 2\epsilon \left( \frac{fw - y - 2g}{y(1 - w)} \right) \right) \\
&+ (1 - \epsilon) \left( \frac{2(xs_{g_1g_2} - (1 - w)s_{qg_1})^2}{s_{g_1g_2}^2} \right) + \frac{1}{s_{qg_1g_2}^2} \left( \frac{4s_{g_1g_2}}{s_{g_1g_2}^2} + (1 - \epsilon) \frac{s_{g_1g_2}}{s_{qg_1}} + (3 - \epsilon) \right) \right),
\end{align*}
\]

while for the QED-like splitting where one or other or both gluons in the colour ordered amplitude are symmetrised over,

\[
\begin{align*}
P_{qg_1g_2 \to Q}(w, x, y, \bar{s}_{qg_1}, s_{g_1g_2}, \bar{s}_{qg_1g_2}) &= 4 \times \left\{ \right. \\
&+ \frac{1}{2s_{qg_1}w} \left( 1 + w^2 - \epsilon(x^2 + xy + y^2) - \epsilon^2 xy \right) \\
&+ \frac{1}{s_{qg_1}s_{g_1g_2}w} \left( w(1 + x \epsilon^2 xy) + (1 - y)^3 - \epsilon(1 - y)(x^2 + xy + y^2) + \epsilon^2 xy \right) \\
&- \frac{(1 - \epsilon)}{s_{g_1g_2}^2} \left( (1 - \epsilon) \frac{s_{g_1g_2}}{s_{qg_1}} - \epsilon \right) \right) + (s_{g_1g_2} \leftrightarrow s_{qg_1}, x \leftrightarrow y). \tag{5.6}
\end{align*}
\]

The function \(P_{qg_1g_2 \to Q}\) can be interpreted as the relevant triple collinear splitting function with one or both of the gluons replaced by photons. As such, this result echoes that found in \[51\] for \(P_{qg \gamma \to Q}\). Using charge conjugation, we see that the functions representing clustering of two gluons with an antiquark are simply,

\[
\begin{align*}
P_{g_1g_2 \to Q}(w, x, y, s_{g_1g_2}, s_{g_1q}, s_{g_1g_2q}) &= P_{g_1g_2 \to Q}(y, x, w, \bar{s}_{g_1g_2}, \bar{s}_{g_1q}, s_{g_1g_2q}), \\
P_{\bar{g}_1g_2 \to Q}(w, x, y, s_{g_1q}, s_{g_1g_2q}) &= P_{qg_1g_2 \to Q}(y, x, w, \bar{s}_{g_1g_2}, \bar{s}_{qg_1}, s_{g_1g_2q}). \tag{5.7}
\end{align*}
\]

### 5.1.3 A quark-antiquark pair with a collinear gluon

Similarly the clustering of a gluon with a quark-antiquark pair into a parent gluon again has two distinct functions. For example, there is a singular contribution from the four quark matrix elements when \(Q_4\), \(Q_3\) and the gluon cluster,

\[
\begin{align*}
|T^A_\mu(Q_1, \bar{Q}_2; Q_3, \bar{Q}_4; 1)V^\mu| &= |A_\mu^{Q_1Q_2}(Q_1; 1; \bar{Q}_1)Q_3)|Q_2) V^\mu + A_\mu^{Q_3Q_4}(Q_3; \bar{Q}_2; Q_4; 1)|Q_4) V^\mu|^2 \\
&\to P_{Q_4Q_3 \to Q} S^3 \mu V^\mu |^2.
\end{align*}
\]

Here, the singular contribution comes entirely from the first \(A\) term where combining \(\bar{Q}_4\), \(Q_3\) pinches the two colour lines together. Explicit evaluation yields,

\[
\begin{align*}
P_{gqq \to G}(w, x, y, s_{gq}, s_{qq}, s_{gqq}) &= 4 \times \left\{ \right. \\
&- \frac{1}{s^2_{gqq}} \left( 4s_{gq} + (1 - \epsilon) s_{qg} + (3 - \epsilon) \right) - \frac{2(xs_{gqq} - (1 - w)s_{gq})^2}{s^2_{qg} s^2_{gqq}(1 - w)^2} \\
&+ (1 - \epsilon) \left( \frac{2(xs_{g1g2} - (1 - w)s_{qg1})^2}{s_{g1g2}^2} \right) + \frac{1}{s_{qg1g2}^2} \left( \frac{4s_{g1g2}}{s_{g1g2}^2} + (1 - \epsilon) \frac{s_{g1g2}}{s_{qg1}} + (3 - \epsilon) \right) \right). \tag{5.5}
\end{align*}
\]
\[ + \frac{1}{s_{qq}s_{gq}} \left( \frac{(1-y)}{w(1-w)} - y - 2w - \epsilon - \frac{2x(1-y)(y-w)}{(1-\epsilon)w(1-w)} \right) \]
\[ + \frac{1}{s_{qg}s_{qq}} \left( \frac{x((1-w)^3-w^3)}{w(1-w)} - \frac{2x^2(1-\epsilon)}. \right) \]
\[ + \frac{1}{s_{qq}s_{gq}} \left( \frac{(1+w^3+4xw)}{w(1-w)} + \frac{2x(w(x-y)-y(1+w))}{(1-\epsilon)w(1-w)} \right) \}

+ azimuthal terms. (5.8)

Again applying charge conjugation yields the further relation,

\[ P_{\bar{q}gq\rightarrow G}(w, x, y, s_{\bar{q}g}, s_{gq}, s_{\bar{q}gq}) = P_{\bar{q}gq\rightarrow G}(y, x, w, s_{gq}, s_{\bar{q}g}, s_{\bar{q}gq}), \] (5.9)

describing instances where the gluon is colour connected to the quark rather than the antiquark.

There is a further contribution when the quark-antiquark and gluon combine to form a photon-like colour singlet. This occurs when,

\[ |T^B_{\mu}(Q_3, \overline{Q}_2; Q_1, \overline{Q}_4; 1)V^\mu|^2 = |B^Q_3,Q_4(Q_3; 1; \overline{Q}_2)|Q_1, \overline{Q}_4(V^\mu + B^Q_1,Q_2(Q_1; \overline{Q}_3; 1; \overline{Q}_4)V^\mu|^2 \rightarrow P_{Q_3\overline{Q}_4\rightarrow \hat{G}}|S^B_\mu V^\mu|^2. \]

In this case the singular contribution is produced by the second \( B \) term and is due to the entire \( Q_3; 1; \overline{Q}_4 \) antenna pinching to form a gluon which is then inserted in a symmetrised way (i.e. with a tilde) into the remaining colour antenna. This QED-like splitting function is given by,

\[ P_{qg\rightarrow \hat{G}}(w, x, y, s_{qg}, s_{gq}, s_{\bar{q}g}, s_{\bar{q}gq}) = 4 \times \left\{ \right. \]
\[ - \frac{1}{s_{qg}s_{\bar{q}gq}} \left( 1 - \epsilon \right) + \frac{1}{s_{qg}s_{gq}} \left( 1 + x^2 - \frac{x + 2wy}{1-\epsilon} \right) \]
\[ - \frac{1}{s_{qg}s_{\bar{q}gq}} \left( 1 + 2x + \epsilon - \frac{2(1-y)}{1-\epsilon} \right) \left\} + (s_{qg} \leftrightarrow s_{gq}, w \leftrightarrow y + \text{azimuthal terms}.(5.10) \right. \]

5.1.4 A quark-antiquark pair with a collinear quark or antiquark

Lastly, we consider the clustering of a quark-antiquark pair \((Q\overline{Q})\) and a quark \((q)\) to form a parent quark \(q'\) with the same flavour as \(q\). The splitting function depends upon whether or not the quarks are identical,

\[ P_{q\overline{Q}Q\rightarrow q'} = P_{q\overline{Q}Q\rightarrow q'}^{\text{non-ident.}} - \frac{\delta_{qQ}}{N} P_{q\overline{Q}Q\rightarrow q'}^{\text{ident.}}, \] (5.11)
where $\delta_{qQ} = 1$ for identical quarks. If quarks $Q_1, Q_3$ and $Q_4$ are clustered to form $Q$, then we find a non-identical quark contribution,

$$|T_\mu^A(Q_3; Q_4; Q_1, Q_2; 1)V^\mu|^2 = |A_{\mu Q}^{Q_4}(Q_3; 1; Q_2) V^\mu + A_{\mu Q}^{Q_3 Q_2}(Q_1; Q_4|Q_3; 1; Q_2) V^\mu|^2$$

$$\rightarrow P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{non-ident.} |S_\mu V^\mu|^2,$$

with,

$$P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{non-ident.}(w, x, y, s_{q\overline{q}}, s_{Q\overline{Q}}, s_{qQ}) = 4 \times \left\{ -\frac{1}{s_{q\overline{q}}^2} \left( (1 - \epsilon) + \frac{2w_{q\overline{q}}}{s_{q\overline{q}}^2} \right) - 2 \frac{(x s_{q\overline{q}} - (1 - w) s_{q\overline{q}})^2}{s_{Q\overline{Q}}^2 s_{q\overline{q}}^2 (1 - w)^2} + \frac{1}{s_{Q\overline{Q}} s_{q\overline{q}}^2 (1 - w)} \left( 1 + x + (x + w) \epsilon \right) - \epsilon (1 - w) \right\}. \quad (5.12)$$

The singular contribution is now generated by the square of the second $\mathcal{A}$ term; pinching $Q_3$ and $Q_4$ together connects the two colour antennae together and combining with $Q_1$ ensures that the vector boson couples to a flavour singlet $Q_1 Q_2$ pair. Precisely the same function describes the triple collinear limit of the $T^B$ functions. We find that in the same limit,

$$|T_\mu^B(Q_1; Q_4; Q_3, Q_2; 1) V^\mu|^2 = |B_{\mu Q}^{Q_3}(Q_1; 1; Q_2) V^\mu + B_{\mu Q}^{Q_3 Q_2}(Q_3; Q_4) V^\mu|^2$$

$$\rightarrow P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{non-ident.} |S_\mu V^\mu|^2,$$

where this time the first term alone contributes. Here $Q_3$ and $Q_4$ combine to form a photon which clusters with $Q_1$. As before, charge conjugation generates the associated function for an antiquark combining with a quark-antiquark pair,

$$P_{Q_1 Q_4 \rightarrow Q}^\text{non-ident.}(w, x, y, s_{q\overline{q}}, s_{Q\overline{Q}}, s_{qQ}) = P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{non-ident.}(w, x, y, s_{q\overline{q}}, s_{Q\overline{Q}}, s_{qQ}). \quad (5.13)$$

When the flavours of the clustering quarks are the same, there is an additional contribution coming from the interference terms of the four-quark matrix elements. For instance, when $Q_1, Q_4$ and $Q_3$ combine,

$$\Re \left( T_\mu^A(Q_3; Q_4; Q_1, Q_2; 1)V^\mu \right) \left( T_\mu^B(Q_3; Q_4; Q_1, Q_2; 1)V^\mu \right)^\dagger$$

$$\sim A_{\mu Q}^{Q_3 Q_3}(Q_3; 1; Q_2) V^\mu \left(B_{\mu Q}^{Q_3 Q_3}(Q_3; 1; Q_2) V^\mu \right)^\dagger$$

$$\rightarrow -\frac{1}{2} P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{idem.} |S_\mu V^\mu|^2,$$

where,

$$P_{Q_1 Q_2 Q_3 \rightarrow Q}^\text{idem.}(w, x, y, s_{q\overline{q}}, s_{Q\overline{Q}}, s_{qQ}) = 4 \times \left\{ \right\}$$
\[ - \frac{(1-\epsilon)}{s_{qQ}} \left( \frac{2s_{qq}}{s_{qQ}} + 2 + \epsilon \right) - \frac{1}{2s_{qQ}^2} \left( \frac{x(1+x^2)}{(1-y)(1-w)} - \epsilon x \left( \frac{2(1-y)}{(1-w)} + 1 + \epsilon \right) \right) \\
+ \frac{1}{s_{qQ}^2} \left( \frac{1+x^2}{(1-y)} + 2x \left( \frac{1}{1-w} \right) - \epsilon \left( \frac{(1-w)^2}{(1-y)} + (1+x) + \frac{2x}{(1-w)} + \epsilon(1-w) \right) \right) \}
+ \left( s_{qQ} \leftrightarrow s_{QQ}, y \leftrightarrow w \right). \tag{5.14} \]

Here, there are poles in the matrix elements when $Q_4$ clusters with both $Q_3$ and $Q_1$ and the triple collinear function is symmetric under $q \leftrightarrow Q$.

### 5.1.5 The $N = 1$ SUSY identity

These triple-collinear splitting functions, like the ordinary Altarelli-Parisi splitting kernels, can be related by means of an $N = 1$ supersymmetry identity. In unbroken supersymmetric theories, the masses of gluon and gluino are identical thereby ensuring that the self-energies of the two particles are equal. Considering all two particle cuts of the one-loop diagrams contributing to these self-energies, then in terms of the colour stripped Altarelli-Parisi kernels eq. 2.8 and eq. 4.4,

\[ P_{gg \to g}(z) + P_{q\bar{q} \to g}(z) = P_{g\bar{g} \to q}(z) + P_{g\bar{q} \to q}(z). \tag{5.15} \]

Here the quark plays the role of the gluino. Note that in conventional dimensional regularisation the number of degrees of freedom for the gluon and gluino are no longer equal and the supersymmetry is broken\(^1\). Therefore this result is not true away from four dimensions. Similarly, by considering the three particle cuts of the relevant two-loop contributions to the self energies, and omitting the invariants in the arguments of the functions, we find,

\[
\sum_{P(a,b,c)} \left( P_{ggg \to G}(a, b, c) + 2P_{g\bar{q} \to G}(a, b, c) + P_{q\bar{g} \to G}(a, b, c) \right) = \sum_{P(a,b,c)} \left( 2P_{ggg \to Q}(a, b, c) + P_{g\bar{g} \to Q}(a, b, c) + 2P_{q\bar{q} \to Q}^{\text{non-ident}}(a, b, c) + P_{q\bar{q} \to Q}^{\text{ident}}(a, b, c) \right), \tag{5.16} \]

provided we set $\epsilon = 0$. This non-trivial relation between the splitting functions is a further check on the results presented in this section.

### 5.1.6 Triple collinear limit of $e^+e^- \to 5$ partons

Examining the two-quark currents in the triple collinear limit is now straightforward. Summing over all triple collinear limits, we find,

\[
|S_\mu(Q_1; 1, 2, 3; \bar{Q}_2)V^\mu|^2 \to \left( P_{Q_1 \to Q} + P_{123 \to G} + P_{23\bar{Q}_2 \to \bar{Q}} \right) |S_\mu^3V^\mu|^2, \]

\(^1\)We note that there are other variants of dimensional regularisation where the gluon and gluino degrees of freedom are equal.
The four-quark current contributions are more complicated, but in each case yield factors multiplying the two-quark current,

\[
|T_{\mu}^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu}|^2 \rightarrow \left( P_{1\overline{Q}_4Q_3\rightarrow G} + P_{\overline{Q}_2Q_1\rightarrow G} + P_{non-ident. \overline{Q}_2Q_3\rightarrow Q} \right) |S^{\mu}_{\mu}V_{\mu}|^2,
\]

\[
|T_{\mu}^B(Q_1, \overline{Q}_4; Q_3, \overline{Q}_2; 1)V_{\mu}|^2 \rightarrow \left( P_{Q_1\overline{Q}_2\rightarrow G} + P_{\overline{Q}_2Q_3\rightarrow Q} + P_{non-ident. \overline{Q}_2Q_3Q_4} \right) |S^{\mu}_{\mu}V_{\mu}|^2,
\]

\[
|T_{\mu}(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu}|^2 \rightarrow \left( P_{Q_1\overline{Q}_2\rightarrow G} + P_{\overline{Q}_2Q_3\rightarrow Q} + P_{non-ident. \overline{Q}_2Q_3Q_4} \right) |S^{\mu}_{\mu}V_{\mu}|^2,
\]

\[
\Re\left( T_{\mu}(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu} \right) \left( T_{\mu}(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu} \right)^\dagger \rightarrow -\frac{1}{2} \left( P_{ident. Q_1\overline{Q}_2Q_3\rightarrow Q} + P_{ident. \overline{Q}_2Q_3Q_4\rightarrow \overline{Q}} \right) |S^{\mu}_{\mu}V_{\mu}|^2,
\]

\[
\Re\left( T_{\mu}^A(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu} \right) \left( T_{\mu}^B(Q_1, \overline{Q}_2; Q_3, \overline{Q}_4; 1)V_{\mu} \right)^\dagger \rightarrow -\frac{1}{2} \left( P_{ident. Q_1\overline{Q}_2Q_3\rightarrow Q} + P_{ident. \overline{Q}_2Q_3Q_4\rightarrow \overline{Q}} \right) |S^{\mu}_{\mu}V_{\mu}|^2.
\]

Combining these limits and eliminating the identical particle factors where appropriate provides the triple collinear singular factor for the five parton squared matrix elements,

\[
\frac{1}{3!} \left| \tilde{S}^{\mu}_{\mu}V_{\mu} \right|^2 + \left| \tilde{T}_{\mu}^{\mu}V_{\mu} \right|^2 = \left( \frac{g^2N}{2} \right)^2 \left| S^{\mu}_{\mu}V_{\mu} \right|^2
\]

\[
\times \left[ P_{123\rightarrow G} + \left( \frac{N^2 - 1}{N^2} \right) P_{Q_1\overline{Q}_2\rightarrow Q} + \frac{N^2 - 1}{2N^4} P_{Q_1\overline{Q}_2\rightarrow Q} \right] \left( P_{Q_1\overline{Q}_2\rightarrow Q} + P_{23\overline{Q}_2\rightarrow Q} \right)
\]

\[
+ \frac{n_F}{N} \left( P_{Q_1Q_4\rightarrow G} + P_{\overline{Q}_4Q_1\rightarrow G} - \frac{1}{N} P_{Q_1\overline{Q}_2\rightarrow G} \right)
\]

\[
+ \frac{n_F}{N} \left( \frac{N^2 - 1}{N^2} \right) \left( P_{ident. Q_1Q_4\rightarrow Q} - \frac{1}{2N} P_{ident. Q_1Q_4\rightarrow Q} + P_{non-ident. Q_2Q_3Q_4\rightarrow Q} + \frac{1}{2N} P_{ident. Q_2Q_3Q_4\rightarrow Q} \right).
\]

5.2 Soft/collinear factorisation

We now examine the configurations where one gluon is soft and two particles are collinear. In this case, the sub-amplitudes factorise as,

\[
|A(\ldots, d, \ldots, a, b, c, \ldots)|^2 \rightarrow P_{dabc} |A(\ldots, d, \ldots, P, \ldots)|^2,
\]
where gluon $a$ is soft, partons $b$ and $c$ are collinear and either colour connected or able to undergo antenna pinching. Parton $d$ is the adjacent colour-connected hard parton in the antenna containing the soft gluon. If $a$ is symmetrised over ($\tilde{a}$, so QED-like) then $d$ is the quark in that colour-line; otherwise $d$ is simply the parton adjacent to $a$. In this limit the collinear partons form parton $P$ and carry momentum fractions,

$$p_b = xp_P, \quad p_c = (1-x)p_P,$$

and we write the soft/collinear factor as,

$$P_{d;abc}(x, s_{ab}, s_{bc}, s_{abc}, s_{ad}, s_{bd}, s_{cd}).$$

To determine the limiting behaviour, we again consider the explicit forms of the $\gamma^* \to$ four and five parton squared matrix elements. All terms that possess three of the ‘small’ denominators $s_{ab}, s_{ad}, s_{bc}$ and $s_{abc}$ contribute in the soft/collinear limit. Similar results could be derived using helicity methods or by examining the on-shell limits of the recursive gluonic and quark currents of ref. [28]. Alternatively, these limits can be obtained directly from the triple collinear limits of sect. 5.1, by keeping only terms proportional to $1/w$ and subsequently replacing $1/w$ by $(s_{bd} + s_{cd})/s_{ad}, 1/(1-w)$ by $1$ and $y$ by $1-x$.

In fact, in this limit we find a universal soft factor multiplied by a collinear splitting function,

$$P_{d;abc}(x, s_{ab}, s_{bc}, s_{abc}, s_{ad}, s_{bd}, s_{cd}) = S_{d;abc}(x, s_{ab}, s_{bc}, s_{abc}, s_{ad}, s_{bd}, s_{cd})P_{bc\to P}(x, s_{bc}),$$

where,

$$S_{d;abc}(x, s_{ab}, s_{bc}, s_{abc}, s_{ad}, s_{bd}, s_{cd}) = \frac{2(s_{bd} + s_{cd})}{s_{ab}s_{ad}} \left( z + \frac{(s_{ab} + zs_{bc})}{s_{abc}} \right).$$

A similar result holds for gluon $c$ becoming soft,

$$|A(\ldots, a, b, c, \ldots, e, \ldots)|^2 \to P_{abc;e}|A(\ldots, P, \ldots, e, \ldots)|^2,$$

where,

$$P_{abc;e} = P_{d;abc}(a \leftrightarrow c, d \leftrightarrow e).$$

In the case that $b$ is soft the matrix elements do not possess sufficient singularities since the collinear particles $a$ and $c$ are not directly colour-connected,

$$|A(\ldots, a, b, c, \ldots)|^2 \to \text{less singular}.$$
5.2.1 Soft/collinear limit of $e^+e^- \to 5$ partons

For the specific case of the two-quark currents, the sum over all soft/collinear limits is easily obtained,

\[
\left| S_\mu(Q_1; 1, 2, 3; Q_2) V^\mu \right|^2 \to \left( P_{Q_1;123} + P_{Q_1;123} + P_{123;Q_2} + P_{123;Q_2} \right) \left| S_\mu^3 V^\mu \right|^2,
\]

\[
\left| S_\mu(Q_1; 1, 2, 3; Q_2) V^\mu \right|^2 \to \left( P_{Q_1;123} + P_{1Q_1;3Q_2} + P_{3;Q_12} \right) \left| S_\mu^3 V^\mu \right|^2,
\]

\[
\left| S_\mu(Q_1; 1, 2, 3; Q_2) V^\mu \right|^2 \to \sum_{P(1,2,3)} \left( P_{Q_1;2Q_3} + P_{Q_1;3Q_2} \right) \left| S_\mu^3 V^\mu \right|^2.
\] (5.28)

The only non-vanishing contributions in the soft/collinear limit from the four-quark currents are,

\[
\left| T^A_\mu(Q_1, Q_2; Q_3, Q_4; 1) V^\mu \right|^2 \to \left( P_{Q_1;Q_3Q_4} + P_{Q_2;Q_1Q_4} \right) \left| S_\mu^3 V^\mu \right|^2,
\]

\[
\left| T_\mu(Q_1, Q_2; Q_3, Q_4; 1) V^\mu \right|^2 \to \left( S_{Q_1;Q_2} P_{Q_3Q_4 \to G} + S_{Q_1;Q_3} P_{Q_1Q_2 \to G} \right)
+ P_{Q_1;Q_3Q_4} - P_{Q_1;Q_4Q_2} - P_{Q_1;Q_3Q_4} + P_{Q_2;Q_1Q_4} + P_{Q_3;Q_2Q_1} - P_{Q_4;Q_2Q_1} + P_{Q_4;Q_1Q_2} \left| S_\mu^3 V^\mu \right|^2,
\] (5.29)

where the second term also includes both the unconnected soft/collinear contribution and interferences amongst the various subamplitudes. This is akin to the case of single soft gluon emission where we have,

\[
\mathcal{A}(\ldots, a, g, b, \ldots) \to \left( \frac{p_a \cdot \epsilon}{p_a \cdot p_g} - \frac{p_b \cdot \epsilon}{p_b \cdot p_g} \right) \mathcal{A}(\ldots, a, b, \ldots),
\]

and therefore,

\[
\Re(\mathcal{A}(\ldots, a, g, b, \ldots, c, d, \ldots)) (\mathcal{A}(\ldots, a, b, \ldots, c, g, d, \ldots))^\dagger
\to \frac{1}{2} \left( S_{agc} - S_{agd} - S_{bgc} + S_{bgd} \right) |\mathcal{A}(\ldots, a, b, \ldots, c, d, \ldots)|^2.
\]

Here, the soft factors are generated by the interference of the two eikonal factors.

\[
\sum_{\text{pols}} \left( \frac{p_a \cdot \epsilon}{p_a \cdot p_g} - \frac{p_b \cdot \epsilon}{p_b \cdot p_g} \right) \left( \frac{p_c \cdot \epsilon^*}{p_c \cdot p_g} - \frac{p_d \cdot \epsilon^*}{p_d \cdot p_g} \right).
\]

Adding up these limits for the five parton squared matrix elements gives,

\[
\frac{1}{3!} \left| \tilde{S}_\mu^5 V^\mu \right|^2 + \left| \tilde{T}_\mu^5 V^\mu \right|^2 = \left( \frac{g^2 N}{2} \right)^2 \left| S_\mu^3 V^\mu \right|^2
\]

20
\[ \times \left[ P_{Q_1;12} + P_{Q_1;123} + P_{123;Q_2} + P_{1;23;Q_2} \right. \\
- \frac{1}{N^2} \left( P_{Q_1;12;Q_2} + P_{Q_1;12;Q_2} + P_{3Q_1;12} + P_{3Q_2;1} \right) + \frac{1}{N^4} \left( P_{1Q_1;2;Q_2} + P_{Q_1;2;Q_2;3} \right) \\
+ \frac{2n_F}{N} P_{Q_1;1;Q_2} \\
- \frac{2n_F}{N^3} \left( S_{Q_1;1;Q_2} P_{3;Q_4} \rightarrow G + P_{Q_1;1;Q_2;Q_4} - P_{Q_1;1;Q_2;Q_4} - P_{Q_2;1;Q_3;Q_4} + P_{Q_2;1;Q_3;Q_4} \right) \right] \] 

(5.30)

5.3 Two soft gluons

Finally, we consider the contributions where two colour connected gluons are simultaneously soft. This was first studied by Berends and Giele [25] and we include this contribution here for the sake of completeness. Similar results have been discussed by Catani [37]. For gluons \( b \) and \( c \) soft the colour ordered subamplitudes factorise,

\[ |A(\ldots, a, b, c, d, \ldots)|^2 \rightarrow S_{abcd}(s_{ad}, s_{ab}, s_{cd}, s_{bc}, s_{abc}, s_{bcd}) |A(\ldots, a, d, \ldots)|^2. \] 

where the connected double soft gluon function is given by,

\[ S_{abcd}(s_{ad}, s_{ab}, s_{cd}, s_{bc}, s_{abc}, s_{bcd}) = \frac{8s_{ad}^2}{s_{bc}} \left( \frac{1}{s_{ab}s_{cd}} + \frac{1}{s_{cd}s_{abc}} - \frac{4}{s_{bc}s_{bcd}} \right) + \frac{8(1 - \epsilon)}{s_{bc}^2} \left( \frac{s_{ab} + s_{cd}}{s_{abc} + s_{bcd} - 1} \right)^2. \] 

(5.32)

Here \( a \) and \( d \) are the hard partons surrounding the soft pair and may either be gluons or quark/antiquarks. In four dimensions, the double soft factor can be extracted from [25] by squaring and summing the helicity amplitudes for two adjacent soft gluons. Alternatively, it can be obtained by explicitly taking the double soft limit of squared matrix elements for processes involving more than two gluons.

5.3.1 Double soft limit of \( e^+e^- \rightarrow 5 \) partons

As before, for the specific case of the two-quark currents, the connected double soft limit is easily obtained and summing over all contributions we find,

\[ |S_{\mu}(Q_1;1,2,3;Q_2)V^\mu|^2 \rightarrow \left( S_{Q_1;123} + S_{12;Q_2} \right) |S_{\mu}^3V^\mu|^2, \]
\[ |S_{\mu}(Q_1;1,2,3;Q_2)V^\mu|^2 \rightarrow S_{Q_1;12Q_2} |S_{\mu}^3V^\mu|^2, \]
\[ |S_{\mu}(Q_1;\bar{1},\bar{2},\bar{3};Q_2)V^\mu|^2 \rightarrow \text{less singular}. \] 

(5.33)
Combining these limits yields the double soft singular factor for the full squared matrix elements,

\[ \frac{1}{3!} |S^S_{\mu} V^\mu|^2 + |\tilde{S}^S_{\mu} V^\mu|^2 = \left( \frac{g^2 N}{2} \right)^2 |S^S_{\mu} V^\mu|^2 \left[ S_{Q123} + S_{123Q2} - \frac{1}{N^2} S_{Q12Q2} \right]. \]  

(5.34)

6 Summary

In this paper we have examined the factorisation properties of squared tree level matrix elements when two particles are unresolved within the framework of QCD. At next-to-next-to-leading order, this knowledge is required for the analytic isolation of infrared singularities of \( n + 2 \) parton scattering processes and subsequent numerical combination with the one-loop \( n + 1 \) parton (single unresolved particle) and two-loop \( n \) parton contributions.

The unresolved particles may be either soft gluons or groups of collinear particles or combinations of both. There are four double unresolved cases; two soft gluons, three simultaneously collinear particles, two independent pairs of collinear particles and one soft gluon together with a collinear pair. In section 4 we reviewed the (trivial) factorisation that occurs when the unresolved particles are colour “unconnected”. Such factorisation is well known and involves only the familiar eikonal and Altarelli-Parisi splitting kernels used to describe single unresolved emission (see sect. 2).

When the unresolved particles are all colour “connected”, we find a similar factorisation. In section 5 we introduced new functions to describe the triple collinear and soft/collinear limits in addition to recalling the known double soft gluon limits of Berends and Giele [25]. These functions are universal and apply to general multiparton scattering amplitudes. As a check on our results, we find that the triple collinear splitting functions obey an expected \( N = 1 \) SUSY identity. In addition, in the strong ordered limit, where one particle is much more unresolved than the other, these factors become simple products of single unresolved factors, one associated with each unresolved particle.

To illustrate the use of these double unresolved approximations, we have examined the singular limits of the tree level matrix elements for \( e^+e^- \rightarrow 5 \) partons [7]. In each case, we find that in the singular limit, the matrix elements can be approximated by a singular factor multiplying the tree level \( e^+e^- \rightarrow 3 \) parton matrix elements. These approximations will be of use in evaluating the \( O(\alpha_s^3) \) corrections to the three jet rate in electron positron annihilation. To achieve this however, much work still remains to be carried out. One important ingredient is to analytically integrate the approximations over the unresolved regions of phase space. A first step in this direction has been carried out in ref. [10] where the hybrid subtraction method of [23] has been used to evaluate the double unresolved singular contributions associated with a photon-gluon-quark cluster. There is in principle no reason why this approach should not be extended to the more general cases discussed here. A further ingredient is the evaluation of the two loop \( e^+e^- \rightarrow 3 \) parton matrix elements.
This is a formidable task in its own right and requires analytic expressions for the two loop box graph with massless internal and external legs for arbitrary dimension. So far, no such expression has been found.

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