Comments on existence of $Z(2)$ kink soliton configuration in $N=2$ scalar field theory

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Abstract: In this article, we look at the existence of kink soliton configurations in $N=2$ scalar fields models analytically. Although solution of such classical configuration was given earlier in literature but such study was done rather numerically. Here, we first present the result which was known earlier then we will try to convince readers existence of such classical configuration can be proven using some simple analytical techniques without knowing detailed solution. Such study will be helpful to generalize the problem for higher number of field variables where one might not be able to find an analytic solution which is often the case. Then we present a new solution which is discontinuous in nature for one field variable and which has its own importance in terms of symmetry of the field theory, topology and contribution to energy of configuration.
1 Introduction

Solution of $\mathbb{Z}(2)$ kink soliton configuration is well known in literature. It was found in different physical situations like [1], [2], [3], [4]. And they are important because they are finite energy classical configurations which one can’t find perturbatively from trivial vacuum configuration. They have other properties which are important in their own rights.

From Derrick’s theorem it is well established that in presence of potential term in action soliton configurations can only be existed in $1+1$ dimensional field theories. But it does not say about or put any restriction on the number of scalar fields in the system which can lead to existence of soliton configurations even in $1+1$ dimension. For $\mathcal{N} = 1$ scalar field theories it is much trivial compared to $\mathcal{N} > 1$ scalar field theories because of integrability condition shown in [5], [6] and in other literatures.

Solution of $\mathbb{Z}(2)$ kink soliton configuration was given in [7] using something known as Euler coordinates for Montonen-Sarker-Trullinger-Bishop model also known as MSTB model. Solution that was given in the model is exact but the way they find the solution can’t be generalized to $\mathcal{N} > 2$ scalar field theories in $1+1$ dimension. But we want to
find a way of looking at the problem such that we can find the answer to existence of soliton configuration even we can’t find exact solution analytically because such approach can easily be generalized to \( \mathcal{N} > 2 \) scalar field theories.

In this article, we study \( \mathcal{N} = 2 \) scalar field theory in two models in which first is the generalized version of the potential used in \( \phi^4 \) kink configuration and second one is MSTB model. Through studying MSTB model we can show the existing solution matches all the properties that the exact soliton solutions have in \( 1 + 1 \) dimension.

\section{\( \mathcal{N} = 2 \) scalar field theory with \( SO(2) \) action invariant theory}

\subsection{Introduction to model}

Here we study a field theory which constituents are two scalar field variables and it is invariant under \( SO(2) \) group action in internal space. There are many such possibilities but we also want to generalize known \( \phi^4 \) kink configuration. So, we choose a system described by following action

\begin{equation}
S = \int d^4x \left[ \frac{1}{2} \sum_{a=1}^{2} \partial_{\mu} \phi_a \partial_{\mu} \phi_a - U(\{\phi_a\}) \right]
\end{equation}

where \( U(\{\phi_a\}) \) is chosen to be \( U(\phi_1, \phi_2) = \frac{\lambda}{4}(\phi_1^2 + \phi_2^2 - \eta^2)^2 \).

Euler-Lagrange equations of motion are following

\begin{align*}
\partial_{\mu} \partial^{\mu} \phi_1 &= -\lambda \phi_1(\phi_1^2 + \phi_2^2 - \eta^2) \\
\partial_{\mu} \partial^{\mu} \phi_2 &= -\lambda \phi_2(\phi_1^2 + \phi_2^2 - \eta^2)
\end{align*}

\subsection{Analysis of Static configuration}

Since we are interested in static configurations initially (because we can find dynamical solution by boosting it) we can write above equations as follows

\begin{align*}
\partial_x^2 \phi_1 &= \lambda \phi_1(\phi_1^2 + \phi_2^2 - \eta^2) \\
\partial_x^2 \phi_2 &= \lambda \phi_2(\phi_1^2 + \phi_2^2 - \eta^2)
\end{align*}

Note that this implies following

\begin{equation}
\phi_2 \partial_x^2 \phi_1 - \phi_1 \partial_x^2 \phi_2 = 0 \\
\implies \partial_x(\phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2) = 0 \\
\implies \phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2 = \text{const}
\end{equation}

Here, one need to remember that solitons are necessarily finite energy configurations, therefore in static situation following quantity should be finite

\begin{equation}
E[\phi_1, \phi_2] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \sum_{a=1}^{2} (\partial_x \phi_a)^2 + U(\phi_1, \phi_2) \right] \\
= \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \sum_{a=1}^{2} (\partial_x \phi_a)^2 + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2 - \eta^2)^2 \right] < \infty
\end{equation}
Above restriction necessary means that
\[ \partial_x \phi_0 |_{x = \pm \infty} = 0 \]  
\[ (\phi_1^2 + \phi_2^2) |_{x = \pm \infty} = \eta^2 \]  
(2.6)
Therefore,
\[ (\phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2) |_{x = \pm \infty} = 0 \]  
(2.7)
and since \( \phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2 = \text{const} \) which means
\[ \phi_2(x) \partial_x \phi_1(x) = \phi_1(x) \partial_x \phi_2(x) \]  
(2.8)
Therefore, two field variables are related in following way
\[ \phi_1(x) = C \phi_2(x) \]  
(2.9)
where \( C \) is constant. From eq.(2.3) we can write following
\[ \partial_x \phi_1 \partial_x^2 \phi_1 + \partial_x \phi_2 \partial_x^2 \phi_2 = \lambda (\phi_1 \partial_x \phi_1 + \phi_2 \partial_x \phi_2)(\phi_1^2 + \phi_2^2 - \eta^2) \]  
\[ \implies \frac{d}{dx} \left[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2 - \eta^2)^2 \right] = 0 \]  
(2.10)
Now using eq.(2.6) one can find that
\[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 = \frac{\lambda}{4} (\phi_1^2 + \phi_2^2 - \eta^2)^2 \]  
(2.11)
Note that
\[ \frac{\partial}{\partial x} \left[ \phi_1 \partial_x \phi_1 + \phi_2 \partial_x \phi_2 \right] \]  
\[ = (\partial_x \phi_1)^2 + (\partial_x \phi_2)^2 + \phi_1 \partial_x^2 \phi_1 + \phi_2 \partial_x^2 \phi_2 \]  
\[ = \frac{\lambda}{2} (\phi_1^2 + \phi_2^2 - \eta^2)^2 + \lambda (\phi_1^2 + \phi_2^2)(\phi_1^2 + \phi_2^2 - \eta^2) \]  
\[ \implies \partial_x^2 (\phi_1^2 + \phi_2^2) = \lambda (\phi_1^2 + \phi_2^2 - \eta^2)(3\phi_1^2 + 3\phi_2^2 - \eta^2) \]  
(2.12)
Now we define a field variable \( \chi(x) = \phi_1^2(x) + \phi_2^2(x) \). Then we can write above equation as follows
\[ \partial_x^2 \chi = \lambda (\chi - \eta^2)(3\chi - \eta^2) \]  
\[ = V'(\chi) \]  
\[ \implies V(\chi) = \lambda \int (\chi - \eta^2)(3\chi - \eta^2) d\chi \]  
\[ = \lambda (\chi^3 - 2\eta^2 \chi^2 + \eta^4 \chi) \]  
(2.13)
Therefore, we are able to write differential equation in single auxiliary field variable. And the solution of the above differential equation is
\[ (x - x_0) = \pm \frac{1}{\sqrt{2\lambda}} \int_0^\chi \frac{d\tilde{\chi}}{\sqrt{\tilde{\chi}^2 - 2\eta^2 \tilde{\chi}^2 + \eta^4 \tilde{\chi}}} \]  
\[ = \pm \frac{1}{\sqrt{2\lambda}} \int_0^\chi \frac{d\tilde{\chi}}{\sqrt{\tilde{\chi} (\tilde{\chi} - \eta^2)}} \]  
(2.14)
Now if we define $U^2 = \tilde{\chi}$, then we will find
\[
(x - x_0) = \pm \sqrt{\frac{2}{\lambda}} \int_0^{\sqrt{\tilde{\chi}}} \frac{dU}{U^2 - \eta^2} = \pm \sqrt{\frac{2}{\lambda \eta}} \tanh^{-1} \left( \frac{\sqrt{\tilde{\chi}}}{\eta} \right)
\]
(2.15)

\[
\Rightarrow \phi_1^2 + \phi_2^2 = (\eta \tanh \sigma(x - x_0))^2, \quad \sigma = \eta \sqrt{\frac{\lambda}{2}}
\]

Then according to eq.(2.9) we can write
\[
\phi_1(x) = \sin \theta \Phi(x) \\
\phi_2(x) = \cos \theta \Phi(x) \\
\Rightarrow C = \tan \theta \\
\Phi(x) = \eta \tanh \sigma(x - x_0)
\]
(2.16)

where $\theta$ is a constant.

Asymptotically vacuum manifold is a circle, $\theta$ defines the angle that the path makes with x-axis on $\phi_1 - \phi_2$ plane going from $x = -\infty \rightarrow x = \infty$. All of such paths are equivalent because they can be found out through rotation once we know one path. It is where invariance under $SO(2)$ group action plays an important role. These kink configuration only connects two opposite points located on the asymptotic vacuum manifold through a straight line passing through origin.

### 2.3 Analysis in different parametrization

In this section we choose different parametrization of field variables to analysis existence of soliton configuration.
In this case we define
\[ \phi_1(x) = \rho(x) \cos \theta(x), \quad \phi_2(x) = \rho(x) \sin \theta(x) \] (2.18)

In terms of these field variables we can rewrite the action in eq.(2.1) as follows
\[ S = \int dx \left[ \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda}{4} (\rho^2 - \eta^2)^2 \right] \] (2.19)

Note that equations of motion are
\[ \partial_\mu \partial^\mu \rho = \rho (\partial \theta)^2 - \frac{\partial}{\partial \rho} U(\rho) \]
\[ \partial_\mu (\rho^2 \partial^\mu \theta) = 0 \] (2.20)
\[ U(\rho) = \frac{\lambda}{4} (\rho^2 - \eta^2)^2 \]

And in static configuration these equations become
\[ \partial_x^2 \rho = \frac{\partial}{\partial \rho} U(\rho) - \rho (\partial_x \theta)^2 \]
\[ \rho^2 \partial_x^2 \theta = 0 \] (2.21)

Note that in terms of this parametrization we can write Energy of static configuration is
\[ E[\rho, \theta] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} (\partial_x \rho)^2 + \frac{1}{2} \rho^2 (\partial_x \theta)^2 + U(\rho) \right] \] (2.22)

Therefore, finite energy condition of soliton configuration implies
\[ \partial_x \rho \big|_{x=\pm \infty} = 0, \quad \partial_x \theta \big|_{x=\pm \infty} = 0 \]
\[ \rho^2(x) \big|_{x=\pm \infty} = \eta^2 \] (2.23)

Second equation of Euler-Lagrange equations with above condition implies \( \partial_x \theta(x) = 0 \) which lead to following Euler-Lagrange equation
\[ \partial_x^2 \rho = \frac{\partial}{\partial \rho} U(\rho) \] (2.24)

which exactly gives the \( \mathbb{Z}(2) \)-kink soliton solution but in \( \rho(x) \) configuration which is \( \rho(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)} \).

And note that \( \theta(x) = \text{const} \) throughout the soliton configuration which we found easily in this case compared to previous case.

3 MSTB model

3.1 Introduction to the model

MSTB model which is described in [8] can be described by adding a quadratic term in \( \phi_2 \) field in eq.(2.1) which breaks the \( SO(2) \) group action invariance. And remaining symmetry that the theory has is the discrete \( \mathbb{Z}(2) \) symmetry.
The resultant action looks something like this

\[
S[\phi_1, \phi_2] = \int d^4x \left[ \frac{1}{2} \sum_{a=1}^{2} \partial_{\mu} \phi_a \partial^{\mu} \phi_a - U(\{\phi_a\}) \right]
\]  

(3.1)

And the Euler-Lagrange equations of motion look following

\[
\partial_{\mu} \partial^{\mu} \phi_1 = -\lambda \phi_1 (\phi_1^2 + \phi_2^2 - \eta^2)
\]

\[
\partial_{\mu} \partial^{\mu} \phi_2 = -\lambda \phi_2 (\phi_1^2 + \phi_2^2 - \eta^2 + \frac{M^2}{\lambda})
\]

(3.2)

3.2 Analysis of static-configuration

Since we are interested in static-configuration we can write Euler-Lagrange equations as follows

\[
\partial^2 \phi_1 = \lambda \phi_1 (\phi_1^2 + \phi_2^2 - \eta^2)
\]

\[
\partial^2 \phi_2 = \lambda \phi_2 (\phi_1^2 + \phi_2^2 - \eta^2 + \frac{M^2}{\lambda})
\]

(3.3)

which leads to following equation

\[
\partial x \left[ \phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2 \right] = -M^2 \phi_1 \phi_2 \neq 0
\]

(3.4)

which is the non-homogeneous version of eq.(2.4).

Similarly in analogous way as we did in eq.(2.10) one can show that

\[
\partial x \left[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 \right] = \partial_x U(\phi_1, \phi_2)
\]

\[
\implies \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 = U(\phi_1, \phi_2) = C
\]

(3.5)

Let's first study the vacuum manifold which can be analyzed by writing down the following conditions

\[
\frac{\partial U}{\partial \phi_1} = \lambda \phi_1 (\phi_1^2 + \phi_2^2 - \eta^2) = 0
\]

\[
\implies \phi_1 = 0, \text{ or } \phi_1^2 + \phi_2^2 = \eta^2
\]

\[
\frac{\partial^2 U}{\partial \phi_1^2} = \lambda (3\phi_1^2 + \phi_2^2 - \eta^2)
\]

\[
\implies \left. \frac{\partial^2 U}{\partial \phi_1^2} \right|_{\phi_1=0} = \lambda (\phi_2^2 - \eta^2) \geq 0 \text{ or } \leq 0
\]

\[
\left. \frac{\partial^2 U}{\partial \phi_1^2} \right|_{\phi_1^2+\phi_2^2=\eta^2} = 2\lambda \phi_1^2 \geq 0
\]

(3.6)

Similarly

\[
\frac{\partial U}{\partial \phi_2} = \lambda \phi_2 (\phi_1^2 + \phi_2^2 - \eta^2 + \frac{M^2}{\lambda}) = 0
\]

\[
\implies \phi_2^2 + \phi_1^2 = \eta^2, \phi_2 = 0
\]

\[
\implies \phi_1 = \pm \eta, \phi_2 = 0
\]

(3.7)
is the vacuum manifold because

\[ \frac{\partial^2 U}{\partial \phi_1^2} = \lambda(\phi_1^2 + 3\phi_2^2 - \eta^2 + \frac{M^2}{\lambda}) \]

\[ \Rightarrow \frac{\partial^2 U}{\partial \phi_2^2} \bigg|_{\phi_1=\pm\eta, \phi_2=0} = 2\lambda(\phi_2^2 + \frac{M^2}{\lambda}) > 0 \] (3.8)

Therefore, we have proven that \((\phi_1 = \eta, \phi_2 = 0), (\phi_1 = -\eta, \phi_2 = 0)\) are the only two points on the circle of radius \(\eta\) in \(\phi_1 - \phi_2\) plane which is the possible vacuum manifold which is set of 2 discrete points.

Demanding finite energy configuration guarantees that

\[ \partial_x \phi_1 \big|_{x=\pm \infty} = 0 = \partial_x \phi_2 \big|_{x=\pm \infty} \]
\[ \phi_1(x = \pm \infty) = \pm \eta, \phi_2(x = \pm \infty) = 0 \] (3.9)

If we use the above condition and integrate eq.(3.5) from \(-\infty \rightarrow \infty\), we will find

\[ C = 0 \Rightarrow \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 = U(\phi_1, \phi_2) \] (3.10)

and eq.(3.4) leads to

\[ \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx = 0 \] (3.11)

Now observe following results

\[ \partial_x^2(\phi_1^2 + \phi_2^2) = 2((\partial_x \phi_1)^2 + (\partial_x \phi_2)^2) + 2[\phi_1 \partial_x^2 \phi_1 + \phi_2 \partial_x^2 \phi_2] \]
\[ = 2((\partial_x \phi_1)^2 + (\partial_x \phi_2)^2) + 4\lambda(\phi_1^2 + \phi_2^2)(\phi_1^2 + \phi_2^2 - \eta^2) + 2M^2 \phi_2^2 \]

\[ \Rightarrow \partial_x^2(\phi_1^2 + \phi_2^2) \big|_{x=\pm \infty} = 0, \Rightarrow \partial_x \phi_1(x) \big|_{x=\pm \infty} = 0 \]
\[ \partial_x^2(\phi_1^2 + \phi_2^2) = 2(3\partial_x \phi_1 \partial_x^2 \phi_1 + \phi_1 \partial_x^2 \phi_1 + 3\partial_x \phi_2 \partial_x^2 \phi_2 + \phi_2 \partial_x^2 \phi_2) \]
\[ \Rightarrow \partial_x^2(\phi_1^2 + \phi_2^2) \big|_{x=\pm \infty} = 2\phi_1(\pm \infty) \partial_x^2 \phi_1 \big|_{x=\pm \infty} \]
\[ = \lambda \partial_x \phi_1 \big|_{x=\pm \infty}(\phi_1^2 + \phi_2^2 - \eta^2) \big|_{x=\pm \infty} \]
\[ + 2\lambda \phi_1(x = \pm \infty)(\partial_x \phi_1 + \phi_2 \partial_x \phi_2) \big|_{x=\pm \infty} = 0 \]

\[ \Rightarrow \partial_x^3 \phi_1 \big|_{x=\pm \infty} = 0 \Rightarrow \partial_x^3(\phi_1^2 + \phi_2^2) \big|_{x=\pm \infty} = 0 \]
\[ \partial_x^3(\phi_1^2 + \phi_2^2) = 6(\partial_x^3 \phi_1)^2 + 8\partial_x \phi_1 \partial_x^2 \phi_1 + 2\phi_1 \partial_x^3 \phi_1 \]
\[ + 6(\partial_x^3 \phi_2)^2 + 8\partial_x \phi_2 \partial_x^2 \phi_2 + 2\phi_2 \partial_x^3 \phi_2 \]
\[ \Rightarrow \partial_x^3(\phi_1^2 + \phi_2^2) \big|_{x=\pm \infty} = 2\phi_1(x = \pm \infty) \partial_x^3 \phi_1 \big|_{x=\pm \infty} + 6(\partial_x^3 \phi_2)^2 \big|_{x=\pm \infty} \]
\[ \partial_x^2 \phi_2 \big|_{x=\pm \infty} = \lambda \phi_2(x = \pm \infty)(\phi_1^2 + \phi_2^2 - \eta^2) \big|_{x=\pm \infty} = 0 \]
\[ \partial_x^4 \phi_1 = \lambda \left[ \partial_x^2 \phi_1(\phi_1^2 + \phi_2^2 - \eta^2) + 2\partial_x \phi_1(\phi_1 \partial_x \phi_1 + \phi_2 \partial_x \phi_2) \right. \]
\[ + 2\partial_x \phi_1(\phi_1 \partial_x \phi_1 + \phi_2 \partial_x \phi_2) + 2\phi_1((\partial_x \phi_1)^2 + \phi_1 \partial_x^2 \phi_1) \]
\[ \left. + (\partial_x \phi_2)^2 + \phi_2 \partial_x \phi_2) \right] \]
\[ \Rightarrow \partial_x^4 \phi_1 \big|_{x=\pm \infty} = 0 \] (3.12)
This way one can show that there must be a finite order \( n \) upto which

\[
\partial_x^n (\phi_1^2 + \phi_2^2)|_{x=\pm\infty} = 0
\]  

(3.13)

which shows \( \phi_1^2 + \phi_2^2 \) near \( x \to \pm\infty \) becomes \( \eta^2 \) with flatness. And \( n \) can’t be infinite because then \( \phi_1^2 + \phi_2^2 = \eta^2 \) which will be a trivial case then.

### 3.3 Analysis using different field parametrization

If we now use the similar parametrization as we used earlier then we find following Lagrangian density

\[
\mathcal{L} = \frac{1}{2}(\partial x^2 + \frac{1}{2}\rho^2(\partial \theta)^2 - \frac{\lambda}{4}(\rho^2 - \eta^2)^2 - \frac{M^2}{2}\rho^2 \sin^2 \theta
\]  

(3.14)

Energy of static soliton configuration would be following

\[
E = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2}(\partial_x \rho)^2 + \frac{1}{2}(\partial_x \theta)^2 + \frac{\lambda}{4}(\rho^2 - \eta^2)^2 + \frac{M^2}{2}\rho^2 \sin^2 \theta \right]
\]  

(3.15)

And finite energy configuration means

\[
E < \infty \implies \partial_x \rho|_{x=\pm\infty} = 0, \partial_x \theta|_{x=\pm\infty} = 0
\]  

(3.16)

where \( m \in \mathbb{Z} \) is a topological number.

Static field equations are following

\[
\partial_x^2 \rho = \rho(\partial_x \theta)^2 + \lambda \rho(\rho^2 - \eta^2) + M^2 \rho \sin^2 \theta
\]

\[
\partial_x^2 \theta + 2\rho \partial_x \rho \partial_x \theta = \frac{M^2}{2}\rho^2 \sin 2\theta
\]

\[
\implies \frac{1}{2}\partial_x(\partial_x \rho)^2 = \partial_x\left(\frac{\rho^2}{2}\right)(\partial_x \theta)^2 + \partial_x \left(\frac{\lambda}{4}(\rho^2 - \eta^2)\right) + M^2 \partial_x(\frac{\rho^2}{2}) \sin^2 \theta
\]  

(3.17)

\[
\frac{1}{2}\rho^2 \partial_x(\partial_x \theta)^2 + 2\rho \partial_x \rho(\partial_x \theta)^2 = \frac{M^2}{2}\rho^2 (\partial_x \sin^2 \theta)
\]

\[
\implies \partial_x \left[ \frac{1}{2}(\partial_x \rho)^2 + \frac{1}{2}\rho^2 (\partial_x \theta)^2 - \frac{\lambda}{4}(\rho^2 - \eta^2)^2 - \frac{M^2}{2}\rho^2 \sin^2 \theta \right] = 0
\]

If we put the asymptotic condition for vacuum manifold we will get

\[
\frac{1}{2}(\partial_x \rho)^2 + \frac{1}{2}\rho^2 (\partial_x \theta)^2 = \frac{\lambda}{4}(\rho^2 - \eta^2)^2 + \frac{M^2}{2}\rho^2 \sin^2 \theta
\]  

(3.18)

Note that if \( \rho(x) > \eta \) for all value of \( x \), then

\[
\partial_x^2 \rho = \rho(\partial_x \theta)^2 + \lambda \rho(\rho^2 - \eta^2) + M^2 \rho \sin^2 \theta > 0
\]  

(3.19)

But since \( \partial_x \rho|_{x=\pm\infty} = 0 \), \( \partial_x \rho \) must have to be increased then decreased or may follow reverse order.

But if \( \rho(x) < \eta \) then

\[
\partial_x^2 \rho = \rho(\partial_x \theta)^2 + M^2 \rho \sin^2 \theta - \lambda \rho(\rho^2 - \eta^2) | > 0 \text{ or } < 0
\]  

(3.20)

which means that depending on the value of \( x \), \( \partial_x \rho \) can either be increasing or decreasing, which shows \( \rho(x) \leq \eta \) must be a condition that the soliton configuration follows.
3.4 Exact soliton solutions

We start with first parametrization of field variables in which we first provide the exact solutions exist in literature we know of.

First kind solution is in a way trivial solution which is following
\[ \phi_1(x) = \pm \eta \tanh \sigma(x - x_0), \quad \phi_2(x) = 0, \quad \sigma = \eta \sqrt{\frac{\lambda}{2}} \]  (3.21)

Second non-trivial solution is following
\[ \phi_1(x) = \pm \eta \tanh M(x - x_0), \quad \phi_2(x) = \pm \lambda \sqrt{\eta^2 - \frac{M^2}{\lambda}} \text{sech}(M(x - x_0)) \]  (3.22)
given in [7]. Note that solution of \( \phi_2(x) \) is real provided \( |M| < \eta \sqrt{\lambda} \).

Energy of this configuration is also calculated in [7] which is \( 2M \lambda (1 - \frac{1}{3}M^2) \) which shows that this configuration can’t be perturbatively found because of \( \frac{1}{\lambda} \) dependence. In \( M \to 0 \) limit shows zero energy configuration which means \( M \to 0 \) does not analytically maps to \( \phi^4 \)-kink like solution.

In second parametrization we now propose a exact solution which is also a soliton configuration given by
\[ \theta(x) = \pi \theta(-x + x_0) \]
\[ \rho(x) = \eta \tanh \sigma(x - x_0) \]  (3.23)
which is a solution that does not depend on the parameter \( M \).

Note that
\[ \partial_x \theta = -\pi \delta(x - x_0) \]
\[ \implies \sin \theta(x) = 0 \]
\[ \implies \rho(x)(\partial_x \theta)^2 = 0 \]  (3.24)
Similarly \( \partial_x^2 \theta(x) = -\pi \delta'(x - x_0) \) which is zero for \( x \neq x_0 \) and does not exist at \( x = x_0 \) because right-hand and left-hand limit of the derivatives at \( x = x_0 \) are not equal. But since \( \partial_x^2 \theta(x) \) multiplied with \( \rho(x) \) which takes zero values at \( x = x_0 \) it solves the problem of existence of limit at \( x = x_0 \) from the field equations in eq.(3.17).

And energy density of such configuration is given by
\[ \mathcal{E}(\rho(x), \theta(x)) = (\partial_x \rho(x))^2 = (\eta \sigma)^2 \text{sech}^4 \sigma(x - x_0) \]  (3.25)
which is a localized function in \( x \) therefore energy of this configuration is finite.

Note that in general we can take solution to be following for multiple localized kinks and anti-kinks with 1 transition from \( -\eta \to \eta \)
\[ \theta(x) = m \pi \theta(x - x_0) + (m + 1) \pi \theta(-x + x_0) \implies \partial_x \theta(x) = -\pi \delta(x - x_0) \]  (3.26)
which does not change previous statements. Here \( m \) is the topological quantum number which defines what is the initial turn number around origin \( (\phi_1 = 0, \phi_2 = 0) \).
If we consider multi-kink configurations with even number of kinks and anti-kinks then if \( \rho(x) \) goes \( \eta \to \eta \) asymptotically then \( \theta(x) = 2m \pi \theta(-x + x_0) + 2n \pi \theta(x - x_0) \) and if \( \rho(x) \) goes \( -\eta \to -\eta \) asymptotically then \( \theta(x) = (2m + 1) \pi \theta(-x + x_0) + (2n + 1) \pi \theta(x - x_0) \) although these \( \rho(x) \)s are not exact solutions of the field equation with these \( \theta(x) \) functions. It essentially means how many loop has been created around \((\phi_1 = 0, \phi_2 = 0)\) to reach \( x \to \infty \) from \( x \to -\infty \).

There is another thing we want to point out that the fact such discontinuous solution of \( \theta(x) \) does not contribute anything in total energy of the configuration because of the fact that the action \( S[\rho, \theta] \) in eq.(3.14) apart from having \( \mathbb{Z}(2) \) symmetry w.r.t \( \rho(x) \) field also has discrete translational symmetry \( \theta(x) \to \theta(x) + n \pi \) in \( \theta(x) \) variable. And because of this reason the force calculation between two neighbouring kink, anti-kink configuration does not depend on the configuration of above kind of \( \theta(x) \) configurations. And the result will exactly match with result we have derived in [9].

So as we can see different parametrization of field variables captures different important informations which will eventually lead us to the properties of the solution even not a exact solution.

4 Force calculation in multi-kink configuration

4.1 Force calculation in first Configuration

First configuration is shown below where we chose kinks and anti-kinks in neighbouring position for both \( \phi_1, \phi_2 \) field variables.

Like in [5] we will calculate the time derivative of stress-energy tensor component \( T^{0i} \)

\[
T^{0i} = -T_{0i} = -(\partial_t \phi_1 \partial_x \phi_1 + \partial_t \phi_2 \partial_x \phi_2)
\]

\[
\Rightarrow \frac{dT^{0i}}{dt} = -\left( \partial_t^2 \phi_1 \partial_x \phi_1 + \partial_t \phi_1 \partial_x \partial_t \phi_1 + \partial_t^2 \phi_2 \partial_x \phi_2 + \partial_t \phi_2 \partial_x \partial_t \phi_2 \right)
\]

(4.2)
Figure 3: right figure shows $\phi_1$ configuration and left figure shows $\phi_2$ configuration

Now using the field equations we can write

$$\frac{dT^{\text{oi}}}{dt} = -\partial_x \left[ \frac{1}{2} (\partial_t \Phi_1)^2 + \frac{1}{2} (\partial_x \Phi_1)^2 + \frac{1}{2} (\partial_t \Phi_2)^2 + \frac{1}{2} (\partial_x \Phi_2)^2 - U(\Phi_1, \Phi_2) \right]$$

(4.3)

which means rate of change of momentum is

$$\frac{dP}{dt} = \frac{d}{dt} \int_{-a-R}^{-a+R} dT^{\text{oi}}(x)$$

where $R$ is the width of localized kink configuration in left where as $a$ is the position of the center of the kink.

This leads to force that anti-kink put on kink is given by following

$$F = -\frac{dP}{dt} = -\left[ \frac{1}{2} (\partial_t \Phi_1)^2 + \frac{1}{2} (\partial_x \Phi_1)^2 + \frac{1}{2} (\partial_t \Phi_2)^2 + \frac{1}{2} (\partial_x \Phi_2)^2 - U(\Phi_1, \Phi_2) \right]^{-a+R}_{-a-R}$$

(4.4)

Since we are interested in static configuration at a particular instant of time the time derivatives vanish and we will get

$$F = -\left[ \frac{1}{2} (\partial_x \Phi_1)^2 + \frac{1}{2} (\partial_x \Phi_2)^2 - U(\Phi_1, \Phi_2) \right]^{-a+R}_{-a-R}$$

(4.5)

Now we put additive ansatz which is

$$\Phi_1 \equiv \phi_1 + \phi_1 - \eta$$

$$\Phi_2 \equiv \phi_2 + \phi_2$$

(4.6)

Which means if we neglect the cross terms in spatial derivatives we will get

$$F = -\left[ \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_x \phi_2)^2 + \frac{1}{2} (\partial_x \phi_2)^2 - U(\phi_1 + \phi_1 - \eta, \phi_2 + \phi_2) \right]^{-a+R}_{-a-R}$$

$$= \left[ U(\phi_1 + \phi_1 - \eta, \phi_2 + \phi_2) - U(\phi_1, \phi_2) - U(\phi_1, \phi_2) - U(\phi_1, \phi_2) \right]^{-a+R}_{-a-R}$$

(4.7)
Now we define following quantities

\[
\begin{align*}
\phi_1^+ &= \phi_1(-a + R) = \eta \tanh \sigma R \\
\phi_1^- &= \phi_1(-a - R) = -\eta \tanh \sigma R \\
\phi_2^- &= \phi_2(-a - R) = \eta \tanh (2a - R) \\
\phi_2^- &= \phi_2(-a + R) = \eta \tanh (2a + R) \\
\phi_2^+ &= \phi_2(-a + R) = \sqrt{\eta^2 - \frac{M^2}{\lambda}} \sech \sigma R \\
\phi_2^- &= \phi_2(-a - R) = \sqrt{\eta^2 - \frac{M^2}{\lambda}} \sech \sigma R \\
\phi_2^- &= \phi_2(-a + R) = -\sqrt{\eta^2 - \frac{M^2}{\lambda}} \sech \sigma (2a - R) \\
\phi_2^- &= \phi_2(-a - R) = \sqrt{\eta^2 - \frac{M^2}{\lambda}} \sech \sigma (2a + R)
\end{align*}
\]

(4.8)

With this information in hand we can write

\[
F = \left[ U(\phi_1^+ + \phi_1^- - \eta, \phi_2^+ + \phi_2^-) - U(\phi_1^+ + \phi_1^- - \eta, \phi_2^- + \phi_2^-) - U(\phi_1^+ + \phi_2^+) + U(\phi_1^+, \phi_2^-) \\
- U(\phi_1^+, \phi_2^+) + U(\phi_1^-, \phi_2^-) \right]
\]

\[
= \left[ U(\eta + \Delta \phi_1^+ + \Delta \phi_1^- + \Delta \phi_2^+ + \Delta \phi_2^-) - U(-\eta + \Delta \phi_1^- + \Delta \phi_1^- + \Delta \phi_2^-) - U(\eta + \Delta \phi_1^+, \Delta \phi_2^-) \\
+ U(\eta + \Delta \phi_1^+, \Delta \phi_2^-) - U(\eta + \Delta \phi_1^- + \Delta \phi_2^-) + U(\eta + \Delta \phi_1^-, \Delta \phi_2^-) \right]
\]

\[
= \left[ \frac{U^{(2,0)}}{2} \left[ (\Delta \phi_1^+ + \Delta \phi_1^-)^2 - (\Delta \phi_1^- + \Delta \phi_1^-)^2 \right] + \frac{U^{(0,2)}}{2} \left[ (\Delta \phi_2^+ + \Delta \phi_2^+)^2 - (\Delta \phi_2^- + \Delta \phi_2^-)^2 \right] \right]
\]

\[
+ \frac{U^{(1,1)}}{2} \left[ (\Delta \phi_1^+ + \Delta \phi_1^-)(\Delta \phi_2^+ + \Delta \phi_2^-) - (\Delta \phi_1^- + \Delta \phi_1^-)(\Delta \phi_2^- + \Delta \phi_2^-) \right]
\]

\[
+ \frac{U^{(2,0)}}{2} \left[ - (\Delta \phi_1^+)^2 + (\Delta \phi_1^-)^2 - (\Delta \phi_1^-)^2 + (\Delta \phi_1^-)^2 \right]
\]

\[
+ \frac{U^{(0,2)}}{2} \left[ - (\Delta \phi_2^+)^2 + (\Delta \phi_2^-)^2 - (\Delta \phi_2^-)^2 + (\Delta \phi_2^-)^2 \right]
\]

\[
+ \frac{U^{(1,1)}}{2} \left[ \Delta \phi_1^+ \Delta \phi_2^- + \Delta \phi_1^- \Delta \phi_2^- + \Delta \phi_1^+ \Delta \phi_2^- \Delta \phi_1^- \Delta \phi_2^- \right]
\]

(4.9)

After algebraic cancellation one would following expression for force

\[
F = U^{(2,0)}(\Delta \phi_1^+ \Delta \phi_2^- - \Delta \phi_1^- \Delta \phi_2^-) + U^{(0,2)}(\Delta \phi_2^+ \Delta \phi_2^- - \Delta \phi_2^- \Delta \phi_2^-) + U^{(1,1)}(\Delta \phi_1^- \Delta \phi_2^- \Delta \phi_1^- \Delta \phi_2^-)
\]

(4.10)

where

\[
U^{(2,0)}(-\eta, 0) = 2 \lambda \eta^2, \quad U^{(0,2)}(-\eta, 0) = M^2, \quad U^{(1,1)}(-\eta, 0) = 0
\]

(4.11)
Therefore,

\[ F = 2\lambda^2 (\Delta \phi_1^+ \Delta \phi_1^- + \Delta \phi_2^+ \Delta \phi_2^-) + M^2 (\Delta \phi_1^+ \Delta \phi_1^- - \Delta \phi_2^+ \Delta \phi_2^-) \]

\[ = m_1^2 (\Delta \phi_1^+ \Delta \phi_1^- - \Delta \phi_2^+ \Delta \phi_2^-) + m_2^2 (\Delta \phi_1^+ \Delta \phi_2^+ - \Delta \phi_2^+ \Delta \phi_2^-) \]  

(4.12)

where

\[ \Delta \phi_1^\pm = \phi_1^\pm - \phi_1(\pm \infty) \]

\[ \Delta \phi_2^\pm = \phi_2^\pm \]

\[ \Delta \phi_2^\pm = \phi_2^\pm \]

\[ \Rightarrow \Delta \phi_1^+ = \eta (\tanh \sigma R - 1) \]

\[ \Rightarrow \Delta \phi_2^+ = \eta (\tanh (2a - R) - 1) \]

\[ \Delta \phi_1^- = \eta (1 - \tanh \sigma R) \]

\[ \Rightarrow \Delta \phi_2^- = \eta (\tanh (2a + R) - 1) \]

\[ \Delta \phi_2^+ = \sqrt{\eta^2 - \frac{M^2}{2}} \tanh \sigma R \]

\[ \Delta \phi_2^- = \sqrt{\eta^2 - \frac{M^2}{2}} \tanh \sigma R \]

\[ \Delta \phi_2^+ = -\sqrt{\eta^2 - \frac{M^2}{2}} \tanh (2a - R) \]

\[ \Delta \phi_2^- = -\sqrt{\eta^2 - \frac{M^2}{2}} \tanh (2a + R) \]  

(4.13)

Using above information we can write expression for force as following

\[ F = m_1^2 \eta^2 \left[ (1 - \tanh MR) \left( 1 - \tanh (2a - R) \right) + (1 - \tanh MR) \left( 1 - \tanh (2a + R) \right) \right] \]

\[ + m_2^2 \left( \eta^2 - \frac{M^2}{2} \right) \left[ - \tanh MR \tanh (2a - R) + \tanh MR \tanh (2a + R) \right] \]

\[ = 4m_1^2 \eta^2 \frac{e^{-2Ma}}{e^{2MR} + e^{-2MR}} (e^{2MR} + e^{-2MR}) - 4m_2^2 \left( \eta^2 - \frac{M^2}{2} \right) \frac{e^{-4Ma}}{e^{2MR} + e^{-2MR}} (e^{2MR} - e^{-2MR}) \]

\[ \approx 4m_1^2 \eta^2 e^{-4Ma} e^{2MR} - 4m_2^2 \left( \eta^2 - \frac{M^2}{2} \right) e^{-2Ma} \]

\[ = 4(m_1^2 e^{2MR} - m_2^2 e^{2Ma}) \eta^2 e^{-4Ma} + 4m_2^2 \frac{M^2}{2} e^{-2Ma} \]

(4.14)

So, the above expression is positive definite if \( m_1^2 > e^{M(2a-R)} \) \( m_2^2 \) \( \Rightarrow \lambda > \frac{M^2}{2} e^{M(2a-R)} \) and we also have a constraint which parameters must need to satisfy \( \left( \eta^2 - \frac{M^2}{2} \right) > 0 \) \( \Rightarrow \lambda > \frac{M^2}{2} \). And if \( e^{M(2a-R)} > 2 \) then \( \lambda > \frac{M^2}{2} \Rightarrow \lambda > \frac{M^2}{2} e^{M(2a-R)} \).

On the other hand expression is negative iff \( m_1^2 \frac{M^2}{2} e^{-2Ma} < (m_1^2 e^{2Ma} - m_1^2 e^{MR}) \eta^2 e^{4Ma} \Rightarrow x^2 e^{-2Ma} - xe^{2Ma} + 2 e^{MR} < 0 \) and we can safely take \( e^{4Ma} \gg e^{2MR} e^{2Ma} \) therefore inequality
becomes $x^2 e^{-2M a} - x e^{-2M a} < 0$ where $x = \frac{M^2}{\lambda \eta^2}$ which leads to condition $\frac{M^2}{2 \eta^4} e^{M(2a-R)} > \lambda > \frac{M^2}{\eta^2}$.

Therefore, if the first condition is satisfied by the parameter $\lambda$ then $F > 0$ implies force between kink and anti-kink is attractive in nature and if second condition is satisfied by $\lambda$ then $F < 0$ implies force between kink and anti-kink is repulsive in nature.

4.2 Force calculation in Second Configuration

Now we look at the force when in $\phi_2$ we have two neighbouring localized kinks but $> \phi_1$ remains as earlier which is shown in figure 4.

In this expression of the force is changed due to the term whose coefficient is $m^2_{\phi_2}$ in eq.(4.14). That term is changed into following expression

$$m^2_{\phi_2} \left( \eta^2 - \frac{M^2}{\lambda} \right) \tanh M R$$

which is positive definite because we need $\left( \eta^2 - \frac{M^2}{\lambda} \right) > 0$ and therefore in this case also net force is attractive in nature.
4.3 Nature of force

In this subsection we will discuss on the reason behind the expression of force we got in earlier subsections. Note that Potential term in the action eq.(3.1) can be rewritten in following way about asymptotic vacuum manifold $\phi_1 = \eta, \phi_2 = 0$

$$U(\phi_1, \phi_2) = \frac{\lambda}{4} (\phi_1^2 + \phi_2^2 - \eta^2)^2 + \frac{1}{2} M^2 \phi_2^2$$

$$U(\varphi_1, \varphi_2) = \frac{\lambda}{4} (\varphi_1^2 + 2\eta \varphi_1 + \varphi_2^2)^2 + \frac{1}{2} M^2 \varphi_2^2$$

$$= \frac{\lambda}{4} (\varphi_1^4 + \varphi_2^4 + 2\varphi_1^2 \varphi_2^2 + 4\eta \varphi_1^3 + 4\eta \varphi_1 \varphi_2^2) + \frac{\lambda \eta^2 \varphi_1^2}{2} + \frac{1}{2} M^2 \varphi_2^2,$$

which clearly shows kinks and anti-kinks interacted through exchanging $\varphi_1$ and $\varphi_2$ excitations or particles of square of masses $2\lambda \eta^2$ and $M^2$ respectively through Yukawa like interaction which we have found during calculation.

5 Conclusion

Studying dynamics of one dimensional soliton [10], [11], [12] is one of the important branch in modern physics. But very few articles in literature actually study coupled solitons or solitons in two interacting scalar fields like in [13], [14], [15] and they are mostly restricted to numerical approaches to study scattering phenomena in different background potential. People also have studied stability of such solitons [16], moduli space of kinks in 2 scalar fields model [17].

In this article we are not only able to provide a way to look at the existence problem of solitons in 2 scalar field theory which can be appropriately generalized to higher number of scalar field variables but we also provide a new solution of MSTB which is not there in literature. We also explained why this new soliton is important in terms of discrete translational symmetry existed in the model and in terms of zero energy change during discontinuous transition.

In the last section we have calculated amount of force that kinks or anti-kinks exerted on each other. Unlike $\phi^4$-kinks here we found a region in parameter space where kinks and anti-kinks can repel each other. We have also explain the origin of nature of the force. These forces are calculated in long range approximation which basically is that the distance between two neighbouring kinks, anti-kinks are much larger than their individual width.

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