WHEN IS THE KOBAYASHI METRIC A KÄHLER METRIC?

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**Abstract.** We prove that if the Kobayashi metric on a strongly pseudoconvex domain with smooth boundary is a Kähler metric, then the universal cover of the domain is the unit ball.

1. INTRODUCTION

A bounded pseudoconvex domain in complex Euclidean space has a number of metrics: the Kobayashi metric, the Bergman metric, and the unique up to scaling Kähler-Einstein metric. All of these metrics coincide, up to a multiplicative constant, on the unit ball but there is no reason to think this would happen for a generic domain. This leads to the following natural question:

**Question.** What are the domains where the Kobayashi, Bergman, and Kähler-Einstein metrics are not pairwise distinct (up to scaling)?

This is a generalization of an old and well known problem: In 1979, Cheng [Che79] conjectured that on a strongly pseudoconvex domain the Bergman metric is Kähler-Einstein if and only if the domain is biholomorphic to the unit ball. More generally, Yau [Yau82, problem no. 44] asked (in a slightly different form) if it was possible to classify the pseudoconvex domains where the Bergman metric is Kähler-Einstein. Recently, Huang and Xiao established Cheng’s conjecture for strongly pseudoconvex domains with $C^\infty$ boundary.

**Theorem 1.1.** [HX21] If $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $C^\infty$ boundary, then the Bergman metric is Kähler-Einstein if and only if $\Omega$ is biholomorphic to the unit ball.

In this note we investigate the bounded domains in $\mathbb{C}^d$ where the Kobayashi metric coincides up to scaling with the Bergman or Kähler-Einstein metric, or more generally is a Kähler metric. In this direction, the best result appears to be from the 1980’s and is due to Stanton [Sta83]: if either the Kobayashi metric or Carathéodory metric is a smooth Hermitian metric and the two metrics coincide at a point, then the domain is biholomorphic to the unit ball. This in particular applies to convex domains.

We also note that Burns-Shnider [BS76] have constructed examples of bounded strongly pseudoconvex domains in $\mathbb{C}^d$ with non-trivial fundamental groups whose universal cover is the unit ball. For these examples, the Kobayashi metric and Kähler-Einstein metric coincide up to scaling. So as one moves beyond the case of convex domains, more examples appear.

For strongly pseudoconvex domains, we prove the following general result.

**Theorem 1.2.** Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $C^\infty$ boundary. Then the following are equivalent:

1. the Kobayashi metric on $\Omega$ is a Kähler metric,
2. the Kobayashi metric on $\Omega$ is a Kähler metric with constant holomorphic sectional curvature,
(3) the universal cover of $\Omega$ is biholomorphic to the unit ball.

As corollaries we obtain:

**Corollary 1.3.** Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $C^\infty$ boundary. Then the Bergman metric is a scalar multiple of the Kobayashi metric if and only if $\Omega$ is biholomorphic to the unit ball.

*Proof.* If the Bergman metric is a scalar multiple of the Kobayashi metric, then Theorem 1.2 implies that the Bergman metric has constant holomorphic sectional curvature. Then by a result of Lu [Lu66], $\Omega$ is biholomorphic to the unit ball.

Conversely, if $\Omega$ is biholomorphic to the ball, then it is well known that the Bergman metric is a scalar multiple of the Kobayashi metric. □

**Corollary 1.4.** Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $C^\infty$ boundary. Then the Kähler-Einstein metric is a scalar multiple of the Kobayashi metric if and only if the universal cover of $\Omega$ is biholomorphic to the unit ball.

*Proof.* If the Kähler-Einstein metric is a scalar multiple of the Kobayashi metric, then Theorem 1.2 implies that the universal cover of $\Omega$ is biholomorphic to the unit ball.

Conversely, if the universal cover $\tilde{\Omega}$ of $\Omega$ is biholomorphic to the unit ball, then on $\tilde{\Omega}$ the Kähler-Einstein metric is a scalar multiple of the Kobayashi metric. Since holomorphic covering maps between bounded domains are local isometries for both the Kähler-Einstein metric and the Kobayashi metric, we see that on $\Omega$ the Kähler-Einstein metric is a scalar multiple of the Kobayashi metric. □

We suspect that the boundary regularity conditions in Theorem 1.2 can be reduced to $C^2$ although this would require new ideas.

### 2. Preliminaries

We fix some basic notations:

- Throughout the paper, $d$ is an integer satisfying $d \geq 2$ and $(z_1, \ldots, z_d)$ denotes the standard coordinates in $\mathbb{C}^d$.
- We denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{C}^d$ and by $\mathbb{B}^d$ the unit ball $\mathbb{B}^d := \{ z \in \mathbb{C}^d : \|z\| < 1 \}$ of $\mathbb{C}^d$. When $d = 1$, we let $\mathbb{D} = \mathbb{B}^1$ denote the unit disk in $\mathbb{C}$.
- Given a domain $\Omega$ we let $k_\Omega$ denote the Kobayashi infinitesimal pseudo-metric and let $K_\Omega$ denote the Kobayashi pseudo-distance obtained by integrating $k_\Omega$ along piecewise smooth curves. When $\Omega$ is bounded, $k_\Omega$ is non-degenerate and $K_\Omega$ is a distance. We normalize the Kobayashi metric $k_{\mathbb{B}^d}$ on the unit ball to have holomorphic sectional curvature equal to -4.

#### 2.1. Distance to the boundary and tangential projections

If $\Omega \subset \mathbb{C}^d$ is a bounded domain and $z \in \Omega$, we denote

$$\delta_\Omega(z) = \min \{\|z - x\| : x \in \partial \Omega \}. $$

When $\Omega$ has $C^2$ boundary and $z \in \overline{\Omega}$ is sufficiently close to $\partial \Omega$, there is a unique point $\pi(z) \in \partial \Omega$ such that $\|z - \pi(z)\| = \delta_\Omega(z)$. In this case, we also let

$$P_z : \mathbb{C}^d \to T^\mathbb{C}_{\pi(z)} \partial \Omega$$

denote the (Euclidean) orthogonal projection and let $P_z^\perp = \text{Id} - P_z$. 
2.2. Complex geodesics in strongly pseudoconvex domains. In this section we recall some facts about complex geodesics in strongly pseudoconvex domains.

Given a domain $\Omega \subset \mathbb{C}^d$, a holomorphic map $\varphi : \mathbb{D} \to \Omega$ is a complex geodesic if
\[
K_\Omega(\varphi(z_1), \varphi(z_2)) = K_\mathbb{D}(z_1, z_2)
\]
for all $z_1, z_2 \in \mathbb{D}$.

Surprisingly, complex geodesics are abundant in strongly pseudoconvex domains. In particular, the following result of Huang states that complex geodesics exist at base points sufficiently close to the boundary and in directions sufficiently tangential.

**Theorem 2.1 ( [Hua94] Theorem 1 and [BFFW19] Theorem 1.1)).** Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathcal{C}^3$ boundary. For any $p \in \partial \Omega$ and any neighborhood $V$ of $p$ in $\mathbb{C}^d$, there exists $\epsilon > 0$ such that: if $z \in \Omega$, $\|z - p\| < \epsilon$, $v \in \mathbb{C}^d$ is non-zero, and
\[
\left\| P_z^\perp(v) \right\| < \epsilon \| P_z(v) \|,
\]
then there exists a complex geodesic $\varphi : \mathbb{D} \to \Omega$ with $\varphi(\mathbb{D}) \subset V$, $\varphi(0) = z$, and $\varphi'(0)\lambda = v$ for some $\lambda > 0$. Moreover, there exists a holomorphic map $\rho : \Omega \to \mathbb{D}$ with $\rho \circ \varphi = \text{id}_\mathbb{D}$.

**Remark 2.2.** The main assertion is Theorem 1 in [Hua94] and the “moreover” part appears in Theorem 1.1 in [BFFW19].

We also use the following existence result due to Bracci, Fornaess, and Fornaess-Wold [BFFW19].

**Theorem 2.3.** ([BFFW19] Proposition 2.5) Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathcal{C}^3$ boundary. For any $p \in \partial \Omega$ and any neighborhood $V$ of $p$ in $\mathbb{C}^d$, there exists $\epsilon > 0$ such that: if $z, w \in \Omega$, $\|z - p\| < \epsilon$, $\|w - p\| < \epsilon$, and
\[
\left\| P_z^\perp(z - w) \right\| < \epsilon \| P_z(z - w) \|,
\]
then there exists a complex geodesic $\varphi : \mathbb{D} \to \Omega$ with $z, w \in \varphi(\mathbb{D}) \subset V$.

We also use the following well known results about the boundary behavior of complex geodesics.

**Theorem 2.4.** Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathcal{C}^2$ boundary.

1. If $\varphi : \mathbb{D} \to \Omega$ is a complex geodesic, then $\varphi$ extends continuous to a map $\hat{\varphi} : \overline{\mathbb{D}} \to \Omega$.
2. If $\varphi_n : \mathbb{D} \to \Omega$ is a sequence of complex geodesics converging locally uniformly to $\varphi : \mathbb{D} \to \Omega$, then $\varphi$ is a complex geodesic and the sequence $\hat{\varphi}_n : \overline{\mathbb{D}} \to \overline{\Omega}$ converges uniformly to $\hat{\varphi} : \overline{\mathbb{D}} \to \overline{\Omega}$.

**Proof.** One way to establish these results is to use a result of Balogh and Bonk [BB00] which states that $(\Omega, K_\Omega)$ is Gromov hyperbolic and the identity map $\Omega \to \overline{\Omega}$ extends to a homeomorphism of the Euclidean boundary and the Gromov boundary. The Gromov boundary is, by definition, equivalence classes of geodesic rays and hence

1. If $\gamma : [0, \infty) \to \Omega$ is a geodesic ray, then $\gamma(\infty) : = \lim_{t \to \infty} \gamma(t)$ exists in $\partial \Omega$.
2. If $\gamma_n : [0, \infty) \to \Omega$ is a sequence of geodesic rays converging locally uniformly to a geodesic ray $\gamma : [0, \infty) \to \Omega$, then $\lim_{n \to \infty} \gamma_n(\infty) = \gamma(\infty)$.

Now if $\varphi : \mathbb{D} \to \Omega$ is a complex geodesic, then $t \mapsto \varphi(t\tanh(t)e^{i\theta})$ is a geodesic for any $\theta \in \mathbb{R}$. So the two properties about geodesic rays above imply the theorem. \qed

We also use the following uniqueness result of Chang, Hu, and Lee.

**Theorem 2.5.** ([CHL88] Theorem 1) Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly convex domain with $\mathcal{C}^3$ boundary. If $z_1, z_2 \in \overline{\Omega}$ are distinct, then up to pre-composition by an element of $\text{Aut}(\mathbb{D})$ there exists at most one complex geodesic $\varphi$ with $\varphi(\mathbb{D}) \subset V$ and $z_1, z_2 \in \hat{\varphi}(\mathbb{D})$.

As a corollary we derive the following.
Corollary 2.6. Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathbb{C}^3$ boundary. For any $p \in \partial \Omega$ there exists a neighborhood $V$ of $p$ in $\mathbb{C}^d$ with the following properties:

1. If $z_1, z_2 \in V \cap \overline{\Omega}$ are distinct, then up to pre-composition by an element of $\text{Aut}(\mathbb{D})$ there exists at most one complex geodesic $\varphi$ with $\varphi(\mathbb{D}) \subset V$ and $z_1, z_2 \in \varphi(\mathbb{D})$.
2. If $z \in V \cap \Omega$ and $v \in T_z \Omega \simeq \mathbb{C}^d$ is non-zero, there exists at most one complex geodesic $\varphi$ with $\varphi(\mathbb{D}) \subset V$, $\varphi(0) = z$, and $\varphi'(0) \lambda = v$ for some $\lambda > 0$.

Proof. We can fix a neighborhood $U$ of $p$ and a holomorphic embedding $\Phi : U \hookrightarrow \mathbb{C}^d$ such that $U = \Phi(U \cap \Omega)$ is strongly convex with $\mathbb{C}^3$ boundary. Then fix a neighborhood $V$ of $p$ such that $V \subset U$.

If $\varphi : \mathbb{D} \to \Omega$ is a complex geodesic with $\varphi(\mathbb{D}) \subset V$, then $\Phi \circ \varphi : \mathbb{D} \to D$ is a complex geodesic. So part (1) follows immediately from Theorem 2.5.

For part (2) suppose $\varphi_0, \varphi_1$ are two complex geodesics with $\varphi_0(\mathbb{D}), \varphi_1(\mathbb{D}) \subset V$, $\varphi_0(0) = z = \varphi_1(0)$, and $\varphi_0'(0) \lambda_0 = v = \varphi_1'(0) \lambda_1$ for some $\lambda_0, \lambda_1 > 0$.

Then $\phi_0 = \Phi \circ \varphi_0$ and $\phi_1 = \Phi \circ \varphi_1$ are complex geodesics in $D$. Further, [Aba89] Corollary 2.6.20 implies that

$$\phi_0'(0) = \frac{\Phi'(z)v}{k_{\mathbb{D}}(\Phi(z), \Phi'(z)v)} = \phi_1'(0).$$

Next for $t \in [0, 1]$ consider $\phi_t = (1 - t)\phi_0 + t\phi_1$. Then $\phi_0'(0) = \phi_1'(0)$ and so by [Aba89] Corollary 2.6.20, each $\phi_t$ is a complex geodesic. So the line segment

$$\left[\phi_0(1), \phi_1(1)\right] = \left\{\hat{\phi}_t(1) : 0 \leq t \leq 1\right\}$$

is contained in $\partial D$. Since $\partial D$ contains no non-trivial line segments, we see that $\hat{\phi}_t(1) = \hat{\phi}_0(1)$ for all $t$ and hence by Theorem 2.5 the maps $\phi_t$ are all identical which implies that $\varphi_0 = \varphi_1$.  

\hfill \Box

3. Constant curvature near the boundary

Motivated by the work of Wong [Won77] (see [Sta83] for some corrections), we prove that if the Kobayashi metric is Kähler on a bounded strongly pseudoconvex domain, then the metric has constant holomorphic sectional curvature near the boundary.

Theorem 3.1. Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathbb{C}^3$ boundary. If $k_\Omega$ is a Kähler metric, then there exists a neighborhood $U$ of $\partial \Omega$ such that the holomorphic sectional curvature of $k_\Omega$ equals $-4$ on $\Omega \cap U$.

Remark 3.2. Theorem 3.1 is also true with $C^2$ boundary smoothness, but we assume $C^3$ smoothness to simplify the argument.

Proof of Theorem 3.1. For $v \in T_z \Omega \simeq \mathbb{C}^d$, let $H(z; v)$ denote the holomorphic sectional curvature at $v$.

Fix $p \in \Omega$ and then fix $\epsilon > 0$ satisfying Theorem 2.1 for $p$ and $V = \mathbb{C}^d$.

Lemma 3.3. For every $z \in \Omega$ and every non-zero $v \in T_z \Omega \simeq \mathbb{C}^d$ satisfying $\|z - p\| < \epsilon$ and $\|P_z^v(v)\| < \epsilon \|P_z^v(v)\|$, we have:

$$H(z; v) = -4.$$

Proof. Fix $z \in \Omega$ and $v \in T_z \Omega \simeq \mathbb{C}^d$ satisfying $\|z - p\| < \epsilon$ and $\|P_z^v(v)\| < \epsilon \|P_z^v(v)\|$. Let $\varphi : \mathbb{D} \to \Omega$ and $\rho : \Omega \to \mathbb{D}$ be as in Theorem 2.1. By the monotonicity property of the Kobayashi distance,

$$K_\Omega(\varphi(\zeta), \varphi(\eta)) \leq K_\mathbb{D}(\zeta, \eta) = K_\mathbb{D}(\rho \circ \varphi)(\zeta), (\rho \circ \varphi)(\eta)) \leq K_\Omega(\varphi(\zeta), \varphi(\eta))$$

Now consider the line segment $L$ in $\mathbb{C}^d$ parametrized by $t \in [0, 1]$ as $L(t) = t z + (1 - t) p$. For each $t \in [0, 1]$, we have $L(t) \in \Omega$ and $L(t) \not\in \partial \Omega$. Since $k_\Omega$ is Kähler, we have $H(L(t); v) = -4$. By the monotonicity property of the Kobayashi distance, we get

$$K_\Omega(\varphi(L(t)), \varphi(L(t))) \leq K_\mathbb{D}(L(t), L(t)) = K_\mathbb{D}(z, z) \leq K_\Omega(\varphi(z), \varphi(z))$$

For all $t \in [0, 1]$. Therefore, for all $t \in [0, 1]$,

$$H(z; v) = -4.$$
for every $\zeta, \eta \in \mathbb{D}$. Thus $\varphi : (\mathbb{D}, K_{\mathbb{D}}) \to (\Omega, K_{\Omega})$ is an isometric embedding and hence
\[ H|_{\varphi(\mathbb{D})} \equiv -4. \]

Now if $z \in \Omega$ and $\|z - p\| < \varepsilon$, then $H(z; v)$ equals $-4$ on the open set
\[ \{ v \in T_z\Omega : v \neq 0 \text{ and } \|P^\perp_z(v)\| < \varepsilon \|P_z(v)\| \}. \]

Since the map $v \in T_z\Omega \mapsto H(z; v) \in \mathbb{R}$ is rational ($z$ is fixed), we then see that $H(z; v) = -4$ for all non-zero $v \in T_z\Omega$. So there exists a neighborhood $U_p$ of $p$, namely the open ball of radius $\varepsilon$, such that the holomorphic sectional curvature of $k_{\Omega}$ equals $-4$ on $U_p \cap \Omega$.

Since $p \in \partial \Omega$ was arbitrary, there exists a neighborhood $U$ of $\partial \Omega$ such that the holomorphic sectional curvature of $k_{\Omega}$ equals $-4$ on $\Omega \cap U$.

\[ \square \]

4. Proof of Theorem 1.2

Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $C^\infty$-smooth boundary.

It is well known that the Kobayashi metric on the unit ball is Kähler. Further, holomorphic covering maps are local isometries for the Kobayashi metric. Hence we see that (3) $\Rightarrow$ (2). Also, (2) $\Rightarrow$ (1) by definition.

The proof that (1) $\Rightarrow$ (3) is more involved. Suppose that the Kobayashi metric of $\Omega$ is Kähler. We will show that $\partial \Omega$ is spherical by using the Cartan-Ambrose-Hicks theorem from Riemannian geometry to construct for each $p \in \partial \Omega$ a CR-diffeomorphism from an open set of $S^{2d-1}$ to an open set in $\partial \Omega$ containing $p$. This general strategy is inspired by an argument due to Seshadri and Verma in [SV06].

Fix $p \in \partial \Omega$. Fix a neighborhood $V$ of $p$ satisfying Corollary 2.6. Using Theorem 3.1 and possibly shrinking $V$, we may also assume that $k_{\Omega}$ has constant holomorphic sectional curvature $-4$ on $V \cap \Omega$. Then fix $\varepsilon > 0$ which satisfies both Theorem 2.1 and Theorem 2.3 for $p$ and the neighborhood $V$.

Lemma 4.1. There exists $z_0 \in \Omega$ such that
\[ \|z_0 - p\| < \varepsilon \quad \text{and} \quad \|P^\perp_{z_0}(z_0 - p)\| < \varepsilon \|P_{z_0}(z_0 - p)\|. \]

Proof. Fix a sequence $\{p_n\}$ in $\partial \Omega$ converging to $p$ such that
\[ \lim_{n \to \infty} \frac{P_n - p}{\|P_n - p\|} = v \in T_z^C \partial \Omega. \]

Then
\[ \lim_{n \to \infty} \frac{\|P^\perp_p(p_n - p)\|}{\|P_p(p_n - p)\|} = 0 \]
and
\[ \lim_{n \to \infty} \frac{\|P_n - p\|}{\|P_p(p_n - p)\|} = 1. \]

Since $\partial \Omega$ is smooth, there exists $C > 0$ such that $\|P_p - P_{p_n}\| \leq C \|p - p_n\|$. Fix $z_n \in \Omega$ with $\pi(z_n) = p_n$ and $\|z_n - p_n\| \leq \min\{\|p_n - p\|^2, \varepsilon/2\}$. Then
\[
\|P^\perp_{z_n}(z_n - p)\| = \|P^\perp_{z_n}(z_n - p)\|
\leq \|P^\perp_{z_n}(z_n - p_n)\| + \|P^\perp_p(p_n - p)\| + \|(P^\perp_{z_n} - P^\perp_p)(p_n - p)\|
\leq (1 + C) \|p - p_n\|^2 + \|P^\perp_p(p_n - p)\|
\]
and
\[ \|P_{z_n}(z_n - p)\| = \|P_{p_n}(z_n - p)\| \]
\[ \geq \|P_p(p_n - p)\| - \|P_{p_n}(z_n - p_n)\| - \|(P_{n} - P_p)(p_n - p)\| \]
\[ \geq \|P_p(p_n - p)\| - (1+C)\|p - p_n\|^2. \]
So
\[ \lim_{n \to \infty} \frac{\|P_{z_n}(z_n - p)\|}{\|P_{z_n}(z_n - p)\|} = 0. \]
Also
\[ \|z_n - p\| \leq \|z_n - p_n\| + \|p_n - p\| \leq \varepsilon/2 + \|p_n - p\|. \]
So for \( n \) large, \( z_n \) satisfies the lemma.

Next fix an open neighborhood \( O_p \) of \( p \) such that: if \( z \in O_p \cap \Omega \), then
\[ \max\{\|z - p\|, \|z - z_0\|\} < \varepsilon \]
and
\[ \|P_{z_0}(z_0 - z_0)\| < \varepsilon\|P_{z_0}(z - z_0)\|. \]
Then by Theorem 2.3 for each \( z \in O_p \cap \Omega \) there exists a unique complex geodesic \( \varphi_z \) with \( \varphi_z(0) = z_0 \), \( \varphi_z(\lambda) = z \) for some \( \lambda > 0 \), and \( \varphi_z(\mathbb{D}) \subset V \). By Theorem 2.4 each \( \varphi_z \) extends to a continuous map \( \hat{\varphi}_z : \overline{\mathbb{D}} \to \overline{\Omega} \).

**Lemma 4.2.** For any \( q \in O_p \cap \partial \Omega \) there exists a unique complex geodesic \( \varphi_q \) with \( \varphi_q(\mathbb{D}) \subset V \), \( \varphi_q(0) = z_0 \), and \( \hat{\varphi}_q(1) = q \).

**Proof.** By Corollary 2.6 we just have to verify existence. Pick a sequence \( \{z_n\} \) in \( O_p \cap \Omega \) converging to \( q \). Then \( \lambda_{z_n} \to 1 \). Passing to a subsequence we can suppose that \( \varphi_{z_n} \) converges locally uniformly to a complex geodesic \( \varphi : \mathbb{D} \to \Omega \). By Theorem 2.4, \( \hat{\varphi}(1) = \lim_{n \to \infty} \varphi_{z_n}(\lambda_{z_n}) = q \).

For each \( q \in O_p \cap \partial \Omega \), let \( v_q := \varphi'_q(0) \in T_{z_0} \Omega \cong \mathbb{C}^d \). By uniqueness, see Corollary 2.6 the map \( q \mapsto v_q \) is a homeomorphism onto its image. Hence, by the invariance of domain theorem,
\[ S = \{v_q : q \in O_p \cap \partial \Omega\} \]
is open in the unit sphere \( \{v \in T_{z_0} \Omega : k_\Omega(z_0, v) = 1\} \). Let
\[ C = \cup_{v \in S} \mathbb{R}^+ \cdot v. \]

For \( z \in \Omega \) let
\[ \exp_z : T_z \Omega \to \Omega \]
denote the exponential map associated to the Kähler metric \( k_\Omega \) and for \( z \in \mathbb{B}^d \) let
\[ \tilde{\exp}_z : T_z \mathbb{B}^d \to \mathbb{B}^d \]
denote the exponential map associated to the Kähler metric \( k_{\mathbb{B}^d} \). Fix a complex linear isometry \( \iota : (T_0 \mathbb{B}^d, k_{\mathbb{B}^d}(0, \cdot)) \to (T_{z_0} \Omega, k_\Omega(z_0, \cdot)) \). Then let
\[ \tilde{C} = \cup_{v \in S} \mathbb{R}^+ \cdot \iota^{-1}(v) \]
and \( D = \tilde{\exp}_0(\tilde{C}) \subset \mathbb{B}^d \). Then define
\[ F := \exp_{z_0} \circ \iota \circ \tilde{\exp}_0^{-1} : D \to \Omega. \]

For \( v \in S \), let \( \phi_v \) denote the complex geodesics with \( \phi_v(\mathbb{D}) \subset V \), \( \phi_v(0) = z_0 \), and \( \phi_v'(0) = v \) (i.e. \( \phi_v = \varphi_q \) where \( q \in O_p \cap \partial \Omega \) is the unique point with \( v = v_q \)). Then
\[ \exp_{z_0}(tv) = \phi_v\left(t \tanh\left(t\right)\right) \in V. \]
and so $F(D) \subset V$. Since $k_{\Omega}$ has constant holomorphic sectional curvature $-4$ on $V$ and $k_{\mathbb{R}^d}$ has constant holomorphic sectional curvature $-4$, the proof of the Cartan-Ambrose-Hicks theorem, see Theorem 1.42 in [CE08], implies that $F$ a holomorphic local isometry. (We note that this argument requires that parallel transport commutes with complex multiplication and hence, for this step, it is essential that $k_{\Omega}$ is a Kähler metric).

Since $\exp_0(v) = \tanh(\|v\|) \frac{v}{\|v\|}$ when $v \in T_0\Omega \simeq \mathbb{C}^d$ is non-zero,

$$F(z) = \phi_i\left(\frac{z}{\|z\|}\right)\left(\|z\|\right)$$

when $z \neq 0$. Thus Theorem 2.4 implies that $F$ extends continuously to

$$D \cup (\overline{D} \cap \mathbb{S}^{2d-1})$$

and, by construction, $F(\overline{D} \cap \mathbb{S}^{2d-1}) = O_p \cap \partial \Omega$.

Then by Corollary 1.4 in [Suk94] the map $F$ extends to a $C^\infty$ map

$$\overline{D} \cap \mathbb{S}^{2d-1} \to O_p \cap \partial \Omega.$$

Thus $\partial \Omega$ is spherical at $p$.

Since $p \in \partial \Omega$ was arbitrary, we see that the entire boundary is spherical. Then by Corollary 1.4 in [HX21] there exists a complete Kähler metric on $\Omega$ with constant negative holomorphic sectional curvature. Then a classical result of Hawley [Haw53] and Igusa [Igu54] says that the universal cover of $\Omega$ is biholomorphic to the unit ball (also see Chapter IX, Section 7 in [KN96]).

(Alternatively, if $\Omega$ has real analytic boundary one can use Theorem A.2 in [NS05] to deduce that the universal cover of $\Omega$ is biholomorphic to the unit ball.)

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