3D Simulation of acoustic waves propagating through porous domain using PUFEM tetrahedral elements

Mingming Yang¹,*, Cyrille Breard¹, Yipeng Cen¹ and Jean-Daniel Chazot²

¹ Shanghai Aircraft Design and Research Institute, COMAC, Shanghai, China
² Laboratoire Roberval, FRE UTC-CNRS 2012, Université de Technologie de Compiègne, 60205 Compiègne, BP 20529, France

*Corresponding author’s address: yangmingming@comac.cc

Abstract. Partition of Unity Finite Element Method is one of the new deterministic prediction techniques developed for tackling medium and high frequency wave problems, because it has the overwhelming advantage over traditional Finite Element Method due to its high efficiency and low computational cost properties. In this paper, PUFEM is further developed to simulate the 3D sound pressure field where porous material is included and modeled as an equivalent homogeneous fluid. Lagrange multipliers are applied at the air and porous material interface. The performance and efficiency of the established numerical model using 3D PUFEM elements are analyzed, it is shown that PUFEM technique can effectively reduce the computational burden while still maintaining the sufficient accuracy for the simulation results of 3D acoustic problems.

1. Introduction

PUFEM is one of the new deterministic prediction techniques that have been developed in the recent years to tackle the medium and high frequency wave problems. The main idea of this technique is to include a set of known wave functions that are the solutions to the partial differential equation into the finite element space. With this property, PUFEM technique has the overwhelming advantage over the traditional finite element method in terms of the computational resources, the adaptivity of the finite element space as well as the complexity of the resulting numerical model when we solve the medium and high frequency wave problems.

The technique of PUFEM is firstly proposed and presented in the work of Melenk and Babuška [1], then they continue to extend their work to prove the convergence of this method through some numerical tests [2]. Moreover, PUFEM enjoys its popularity due to the fact that it can be easily adapted into any conventional FEM mesh. This technique has already been employed to solve various acoustic wave problems including scattering problem [3], diffraction problems [4] and flow acoustic problem [5], we can also find the applications of PUFEM in two-dimensional interior sound field problems with absorbing material [6,7,8] and other wave propagation problems [9].

To the author’s knowledge, there is no related articles concerning with the three-dimensional wave transmission problems where the porous absorber is included. Therefore, we introduce and explain the fundamental idea and mathematical background of PUFEM for modelling 3D wave transmission problems with porous domain in this paper, then we continue to focus on the study of numerical performances in terms of degrees of freedoms (Dofs) for getting the solution of this problem. Note that the porous material hereby is modeled as an equivalent homogeneous fluid. Next, the obtained results have been verified and validated for a normal tube and then a tube with geometry singularities. Finally,
the advantage of numerical performance of PUFEM technique over the conventional finite element is investigated.

![Figure 1. General interior case of problem under consideration.](image)

**2. PUFEM with equivalent homogeneous fluid**

The general interior acoustic problem considering air and acoustic domain is shown in figure 1, where $\Omega_a$ represents the cavity filled with the air, $\Omega_p$ is the porous domain, and $\Omega=\Omega_a+\Omega_p$. For the sake of clarity, all quantities related to the air domain is denoted by the subscript $a$ and the domain associated with porous material is denoted by the subscript $p$. Because the porous media is modeled as an equivalent homogeneous fluid, so that the governing Helmholtz equation in each domain satisfies,

$$\Delta p_a + \kappa_a^2 p_a = 0 \quad \text{in} \quad \Omega_a$$

$$\Delta p_p + \kappa_p^2 p_p = 0 \quad \text{in} \quad \Omega_p$$

where $\omega$ is the harmonic angular frequency, $\kappa_a=\omega/c_a$ is the wavenumber in the air domain while $\kappa_p=\omega(\rho_p/K_p)^{1/2}$ represents the wavenumber in porous domain, and the mean density $\rho_p$ and dynamic bulk modulus $K_p$ can be calculated through Johnson-Champoux Allard’s model [6]. The coupling condition at the interface $\Gamma_{ca}$ and $\Gamma_{cp}$ from figure 1 need to satisfy the continuity condition of pressure $p_p=p_a$, and normal velocity condition,

$$\phi \frac{\partial p_p}{\rho_p \partial n_p} = -\frac{1}{\rho_a} \frac{\partial p_a}{\partial n_a} \quad \text{in} \quad \Gamma'_a$$

where $n_p$ and $n_a$ denote the outward unit normal vectors of porous and air domain with the convention that $n_p=-n_a$, $\phi$ refers to the porosity of the absorber. A normal velocity $\nu_a$ can be applied on the rest part of the boundary of the air cavity $\Gamma'_a$ through the formula,

$$\frac{\partial p_a}{\partial n_a} = i \rho_a \omega \nu_a \quad \text{in} \quad \Gamma'_a$$

and the rigid wall boundary condition is applied on the porous absorber, which is,

$$\frac{\partial p_p}{\partial n_p} = 0 \quad \text{in} \quad \Gamma'_p$$

After applying the standard weight residual scheme to above two governing equations and doing the integration by parts, we can get the weak form of the problems in both domains ($\alpha=a, p$) as,
The core idea of PUFEM is to enrich the conventional finite element field by incorporating a priori knowledge of the homogeneous partial differential equation. Plane waves are usually chosen to be the expansion basis on the consideration of their convenience. The wave expansion process for three-dimensional air and porous domain can be written in an integrated format as,

\[
p_{\alpha}(x) = \sum_{j=1}^{4} \sum_{q=1}^{Q_j} A_{\alpha,jq} \exp(i \kappa_{\alpha} d_{\alpha,jq} \cdot x)
\]

where the sound pressure field \( p_{\alpha} \) is the function of \( x \), \( Q_j \) indicates the number of used plane wave directions on each node of the element, \( d_{\alpha,jq} \) is the directional vectors of the plane wave basis attached to node \( j \), the plane wave directions are evenly distributed on a unit sphere. The enrichment process of PUFEM on one node of tetrahedron element is illustrated in figure 2. \( N_j \) corresponds to the linear shape functions on tetrahedron element. Coefficients \( A_{\alpha,jq} \) stand for the amplitude of each wave function attached to the \( j^{th} \) node, which are also the unknowns of the current problem. Note that the weight functions \( \delta p_{\alpha} (\alpha=a,p) \) are selected from the same plane wave basis as shape functions.

Figure 2. Distribution of plane wave directions in 3D.

By employing the standard Lagrange multiplier technique, the continuity condition of pressure field across the air-porous interface can be weakly enforced as,

\[
\int_{\Gamma_c} (p_p - p_a) \delta \Lambda d\Gamma = 0
\]

where \( \delta \Lambda \) means an appropriate weight function, \( \Gamma_c \) denotes the air-porous common interface. The Lagrange multiplier takes the same expansion form of wave field in the porous domain. All above equations eventually lead to the following symmetric system,

\[
\begin{pmatrix}
K_a & 0 & -C_a \\
0 & K_p & C_p \\
-C_a^T & -C_p^T & 0
\end{pmatrix}
\begin{pmatrix}
A_a \\
A_p \\
\Lambda
\end{pmatrix}
=
\begin{pmatrix}
V_a \\
0 \\
0
\end{pmatrix}
\]

where \( K_{\alpha} (\alpha=a,p) \) are the plane wave finite element matrices for the Helmholtz problem, \( C_{\alpha} \) denote the coupling matrices, \( A_{\alpha} \) and \( \Lambda \) contain all the information of plane wave amplitudes.
3. Numerical example
In this section, two numerical examples are set up to investigate the numerical performance of the PUFEM tetrahedron element for solving the three-dimensional acoustic propagation and coupling problem. The absorbing materials are modeled as an equivalent homogeneous fluid, and the corresponding acoustic properties of the sound absorbing porous material are taken from literature [6]. The corresponding code is developed in Matlab with double precision and all numerical tests are carried out on Xeon X7542 2.67GHz win64 Pilcam server. This example also included an exact integration scheme introduced in reference [7] for the acceleration purpose.

3.1. Numerical model of a 3D standing wave tube
In figure 3, it is shown that the geometry of the tube is $0.03\times0.03\times0.15$ m and the whole length of the tube $L$ is evenly divided into three sections. The porous absorber is located in the middle region which is displayed in yellow. This numerical model of tube is partitioned into 72 PUFEM elements and contain 32 nodes. The characteristic length of this model refers to the longest edge of the current mesh configuration, which is $h_{\text{max}} = (0.05^2 + 0.03^2 + 0.03^2)^{1/2} = 0.066$ m. A prescribed velocity boundary condition $\partial p_a / \partial n_a = 1$ is applied on the front surface of the tube, and a rigid wall is established on the other end.

![Figure 3. Geometry and PUFEM mesh of the target problem.](image)

3.2. Performance of the method
Figure 4 shows the analytical solution and numerical solution for $\kappa h_{\text{max}} = 30$ and frequency $f = 21,000$ Hz, the simulation results of sound pressure level are also given in figure 5.

![Figure 4. Pressure field of the current problem for $\kappa h_{\text{max}} = 30$: analytical solution (left), PUFEM solution with $Q = 305$ (right).](image)
Figure 5. Analytical (blue) and PUFEM solution (red) of SPL in the standing wave tube for the case of $\kappa_a h_{\text{max}} = 30$ and $Q = 305$.

It is shown in figure 5 that the PUFEM solution is extremely close to the analytical solution due to the high convergence rate of this method, and a stationary wave can be observed in the quiet region. To be more precise, the numerical error is around 1% when the average plane wave directions per node $Q = 305$ in this example, and this corresponds to a total number of degrees of freedom $N_{\text{dof}} = 12784$, including the Dofs of Lagrange multipliers. It is worth noting that more than two millions degrees of freedom, namely, finite element nodes will be required to attain the same accuracy level in this case. Around 2 degrees of freedom per wavelength is enough to provide accurate solutions, indicating that the computational cost has been drastically reduced to a low level by using PUFEM technique.

3.3. Standing wave tube example with geometric singularity

This example is carried out to evaluate the numerical performance of the PUFEM element for handling the geometric singularities. The geometry and mesh partitioning of the standing wave tube with singularity is shown in figure 6, in which 101 PUFEM elements and where 41 nodes are involved in this numerical model.

Figure 6. Geometry and PUFEM mesh of the model with geometric singularity.

In figure 7, the sound pressure level in the tube for $\kappa_a h_{\text{max}} = 30$ and no material is displayed. It can be observed that a transverse mode in the tube is generated by the scattering of waves due to the edge singularity. However, this mode disappears when we place a porous absorber in the middle region of the tube, this effect is clearly shown in figure 8. The average number of plane wave functions per node $Q$ is set as 308 for both two cases with and without porous material, and the total degree of freedom $N_{\text{dof}}$ is equal to 15676. To achieve the accuracy level for engineering purpose, similar number of finite element nodes as the previous example are used in this example, therefore, it has been shown that the PUFEM has the ability to cope with geometrical singularities.
4. Conclusion
This paper investigated the numerical performances of the PUFEM technique for solving 3D wave transmission problems when the absorbing materials is considered. By comparing with analytical solutions, the obtained results have been verified and validated for the normal tube and the tube with geometry singularities. Besides, the numerical performance shows that the PUFEM technique has the advantage of fast convergence rate compared to the conventional finite element. Finally, the PUFEM solutions were found to be around 1% error when only 2 Dofs per wavelength is adopted in the modelling process, this is sufficient for engineering usages.

Acknowledgments
The Authors wishing to acknowledge the financial support from Shanghai Pujiang Talent Project (18PJ1433600).

References
[1] Melenk J M and Babuška I 1996 The Partition of unity finite element method: Basic theory and applications. Comput Meth Appl Mech Eng vol 139 pp 289-314
[2] Melenk J M and Babuška I 1997 The partition of unity method. Int J Numer Meth Engng vol 40 pp 727-758
[3] Laghrouche O, Bettess P, Perrey-Debain E and Trevelyan J 2003 Plane wave basis for wave scattering in three dimensions. Commun Numer Meth Eng vol 19 pp 715–723
[4] Laghrouche O, Bettess P and Astley R J 2002 Modelling of short wave diffraction problems using approximating systems of plane waves. Int J Numer Meth Engng, vol 54(10) pp 1501–1533
[5] Astley R J and Gamallo P 2005 Special short wave elements for flow acoustics. Comput Meth Appl Mech Eng vol 194 pp 341-353
[6] Chazot J D, Nennig B and Perrey-Debain E 2013 Performances of the Partition of Unity Finite Element Method for the two-dimensional analysis of interior sound field with absorbing material J Sound Vib vol 332 pp 1930-1946
[7] Yang M, Perrey-Debain E, Nennig B and Chazot J D 2018 Development of 3D PUFEM with linear tetrahedral elements for the simulation of acoustic waves in enclosed cavities. Comput Meth Appl Mech Eng vol 335 pp 403-418
[8] Chazot J D, Perrey-Debain E and Nennig B 2014 The Partition of Unity Finite Element Method for the numerical solution of waves in air and poroelastic media. J Acoust Soc Am vol 135 pp 724-733
[9] Huttunen T and Monk P 2007 The use of plane waves to approximate wave propagation in anisotropic media. J Comput Math vol 25 pp 1072–1092