ON MARKOVIAN SEMIGROUPS OF LÉVY DRIVEN SDES, SYMBOLS AND PSEUDO–DIFFERENTIAL OPERATORS

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Abstract. We analyse analytic properties of nonlocal transition semigroups associated with a class of stochastic differential equations (SDEs) in $\mathbb{R}^d$ driven by pure jump–type Lévy processes. First, we will show under which conditions the semigroup will be analytic on the Besov space $B^{m}_{p,q}(\mathbb{R}^d)$ with $1 \leq p, q < \infty$ and $m \in \mathbb{R}$. Secondly, we present some applications by proving the strong Feller property and give weak error estimates for approximating schemes of the SDEs over the Besov space $B^{m}_{\infty,\infty}(\mathbb{R}^d)$.

1. Introduction

The purpose of the article is to show smoothing properties for the Markovian semigroup generated by stochastic differential equations driven by pure jump–type Lévy processes. Let $L = \{L(t) : t \geq 0\}$ be a family of Lévy processes. In the second part of the article, we show some applications of our results. However, before we start to present our results let us introduce some notation. We consider the stochastic differential equations of the form

\begin{equation}
\begin{aligned}
dX^x(t) &= b(X^x(t-)) \, dt + \sigma(X^x(t-)) \, dL(t) \\
X^x(0) &= x, \quad x \in \mathbb{R}^d,
\end{aligned}
\end{equation}

(1.1)

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous. Under this assumption, the existence and uniqueness of a solution to equation (1.1) is well established, see for e.g. [2, p. 367, Theorem 6.2.3]. Let $(\mathcal{P}_t)_{t \geq 0}$ be the Markovian semigroup associated to $X$ defined by

\begin{equation}
(\mathcal{P}_t f)(x) := \mathbb{E}[f(X^x(t))], \quad t \geq 0, \quad x \in \mathbb{R}^d.
\end{equation}

(1.2)

Then, it is known that $(\mathcal{P}_t)_{t \geq 0}$ is a Feller semigroup (see [2] Theorem 6.7.2]) and its infinitesimal generator is given by

\begin{equation}
Au(x) = \int_{\mathbb{R}} e^{ix^T \xi} a(x, \xi) \hat{u}(\xi) \, d\xi \quad u \in \mathcal{S}(\mathbb{R}^d),
\end{equation}

where the symbol $a$ is defined by

\begin{equation}
a(x, \xi) := -\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left[e^{t(X^x(t)-x)^T \xi} - 1\right], \quad x \in \mathbb{R}^d.
\end{equation}

(1.3)

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In [16], the authors investigate the analytic properties of the Markovian semigroup generated by an SDE driven by a Lévy process (see Theorem 2.1 in [16]). These type of analytic results are used to solve several applications which arise in fields related to probability theory such as nonlinear filtering theory [17], or stochastic numerics (see section (6)). In this article we put a further step and investigate under which constraints the corresponding Markovian semigroup \((P_t)_{t \geq 0}\) driven by an SDE with pure jump noise forms an analytic semigroup in the Besov spaces \(B^{m}_{p,q}(\mathbb{R}^d)\) with \(1 \leq p,q < \infty\) and \(m \in \mathbb{R}\).

The analyticity property of the Markovian semigroup \((P_t)_{t \geq 0}\) in \(B^{m}_{p,q}(\mathbb{R}^d)\) is very useful to obtain the strong Feller property of \((P_t)_{t \geq 0}\) and studying the weak error of an approximation of an SDE driven by Lévy noise. It enables us to obtain an explicit estimate of the distance between the semigroups are associated with the original problem (1.1) and the one associated with the approximated scheme (6.3). The regularity of the Markovian semigroup associated with \(\mathbb{R}^d\)-valued SDEs plays an important role as far as the strong Feller property, long time behaviour or proving existence and uniqueness of invariant measure of SPDEs are concerned (e.g., see [37]). In this work, we obtain regularity of the Markovian semigroup \((P_t)_{t \geq 0}\) (e.g., see Corollary (5.1) and Corollary (5.4)) associated with equation (1.1) with quite weaker assumptions regarding on the coefficient functions \(b\) and \(\sigma\). In particular, if \(\sigma\) is bounded from below and above, \(b, \sigma \in C^{d+3}_b(\mathbb{R}^d)\), then the semigroup \((P_t)_{t \geq 0}\) is analytic on \(B^{m}_{p,q}(\mathbb{R}^d)\) for all \(1 \leq p, q < \infty\) and \(m \in \mathbb{R}\).

In [24, Theorem 2.2] and [38], they get some results on the density of the solution of an SDE driven by a Lévy process. These estimates are uniform in space and are related to our results, see Corollaries 5.2 and 5.5. In [4], they represent their main result as the propagation of the regularity of the Markovian semigroups induced by the solution process of an SDE driven by a Brownian motion and a Lévy process. In particular, they show that for all \(k \in \mathbb{N}\) there exists a constant \(C > 0\) such that

\[
\sup_{0 < t \leq T} \| P_t f \|_{W^k} \leq k \cdot (T, P) \| f \|_{W^k},
\]

for all \(f \in C^k_b(\mathbb{R}^d)\). Here \(\| f \|_{k, \infty}\) is the supremum norm of \(f\) and its first \(k\) derivatives. In the case of \(k = 0\), this means that the semigroup \((P_t)_{t \geq 0}\) is a Feller semigroup. Künn, [33], investigates the Feller property of the Markovian semigroup for unbounded diffusion coefficients. In [34], the analyticity of the Markovian semigroup \((O_t)_{t \geq 0}\) is proven for SDEs with only additive noise; here the noise has to have a very special form. Notice also that in [12], authors derived a Bismuth-Elworthy-Li type formula for the Lévy process in the Hilbert space setting. In addition to that, it was established the fractional gradient estimates of the semigroup associated with an SDE forcing by pure additive jump noise for the \(L^\infty\)-norm (see [12, Theorem 5.1]).

The paper is organised as follows. In section 2 we give a short review of the symbols associated with the SDEs driven by Lévy processes and introduce some notations. We present the invertibility of the pseudo-differential operator as a theorem with the detailed proof in section 3. In section 4 we study under which conditions on the symbol, the semigroup \((P_t)_{t \geq 0}\) is an analytic semigroup in general Besov space space \(B^{m}_{p,q}(\mathbb{R}^d)\), \(1 \leq p, q < \infty\), and fractional Sobolev space \(H^m_{W}(\mathbb{R}^d)\). From this results, as the first application, we can verify under which constrains the semigroup \((P_t)_{t \geq 0}\) is strong Feller in section 5. We show some weak error estimates in section 6. In particular, we investigate the Monte-Carlo error of the simulation
of SDEs driven by a pure Lévy process. In this approach, we apply pure analytic methods coming from harmonic analysis, respective from the theory of pseudo-differential operators. In the appendix, we give a short overview of pseudo-differential operators.

**Notation 1.1.** For a multiindex \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \) let \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \). For an element \( \xi \in \mathbb{R}^n \), let \( \xi^\alpha \) be defined by \( \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \). Moreover for a function \( f : \mathbb{R}^d \to \mathbb{C} \) we write \( \partial^\alpha f(x) \) for

\[
\frac{\partial^\alpha}{\partial x_1 \partial x_2 \cdots \partial x_d} f(x).
\]

\( < \cdot : \mathbb{R} \ni \xi \mapsto < \xi >^\rho := (1 + \xi^2)^{\rho/2} \in \mathbb{R} \). Following inequality, also called Peetre’s inequality is used on several places

\[
<x + y >^s \leq c_s < x >^s < y >^s, \quad x, y \in \mathbb{R}^d, \quad s \in \mathbb{R}.
\]

For a multiindex \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \) let \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \). For a multiindex \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \) and an element \( \xi \in \mathbb{R}^n \) let \( \xi^\alpha \) be defined by \( \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \).

Let \( X \) be a nonempty set and \( f, g : X \to [0, \infty) \). We set \( f(x) \preceq g(x), x \in X \), iff there exists a \( C > 0 \) such that \( f(x) \leq C g(x) \) for all \( x \in X \). Moreover, if \( f \) and \( g \) depend on a further variable \( z \in Z \), the statement for all \( z \in Z \), \( f(x, z) \preceq g(x, z) \), \( x \in X \) means that for every \( z \in Z \) there exists a real number \( C_z > 0 \) such that \( f(x, z) \leq C_z g(x, z) \) for every \( x \in X \). Also we set \( f(x) \asymp g(x), x \in X \), iff \( f(x) \preceq g(x) \) and \( g(x) \preceq f(x) \) for all \( x \in X \). Finally, we say \( f(x) \gtrsim g(x), x \in X \), iff \( g(x) \preceq f(x) \), \( x \in X \). Similarly as above, we handle the case if the functions depend on a further variable.

Let \( S(\mathbb{R}^d) \) be the Schwartz space of functions \( C_0^\infty(\mathbb{R}^d) \) where all derivatives decrease faster than any power of \( |x| \) as \( |x| \) to infinity. Let \( S^*(\mathbb{R}^d) \) be the dual of \( S(\mathbb{R}^d) \).

2. Symbols, their definitions and properties

In this section, we give a short review of symbols coming up Hoh’s and Lévy’s symbols in dealing with processes generated by Lévy processes. Besides, we introduce some notations. Throughout the remaining article, let \( L = \{L^x(t) : t \geq 0, x \in \mathbb{R}^d \} \) be a family of Lévy processes \( L^x \), where \( L^x \) is a Lévy process starting at \( x \in \mathbb{R}^d \). Then \( L \) generates a Markovian semigroup \( \{P_t \}_{t \geq 0} \) on \( C_0(\mathbb{R}^d) \) by

\[
P_t f(x) := \mathbb{E} f(L^x(t)), \quad f \in C_0(\mathbb{R}^d).
\]

Let \( A \) be the infinitesimal generator of \( \{P_t \}_{t \geq 0} \) acting on \( C_0^2(\mathbb{R}^d) \) defined by

\[
Af := \lim_{h \to 0} \frac{1}{h} (P_h - P_0) f, \quad f \in C_0^2(\mathbb{R}^d).
\]

Another way of defining \( A \) is done by Lévy symbols (see [23]). In particular let

\[
\psi(\xi) = \frac{1}{t} \ln(\mathbb{E} e^{i \xi \cdot L(t)}), \quad \xi \in \mathbb{R}^d.
\]

Observe that we have (see e.g. [21] P.42 and [40])

\[
\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i \xi \cdot z} - 1 - i(\xi, x) \mathbb{1}_{\{|z| \leq 1\}} \right) \nu(dz), \quad \xi \in \mathbb{R}^d.
\]
If $L$ is a pure jump process with symbol $\psi$, then the infinitesimal generator defined by (2.1) can also be written as
\begin{equation}
Af = -\int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \psi(\xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

The operator $A$, usually denoted in the literature by $\psi = \psi(D)$, is well defined in $C^2_b(\mathbb{R}^d)$, has values in $B_b(\mathbb{R}^d)$ (bounded Borel functions in $\mathbb{R}^d$) and satisfies the positive maximum principle (see e.g. [26, Theorem 4.5.13]). Therefore, $A$ generates a Feller semigroup on $C^\infty_b(\mathbb{R}^d)$ and a sub–Markovian semigroup on $L^2(\mathbb{R}^d)$ (see e.g. [27, Theorem 2.6.9 and Theorem 2.6.10]). To characterise the symbol, we introduce the generalised Blum–Getoor index.

**Definition 2.1.** Let $L$ be a Lévy process with symbol $\psi$ and $\psi \in C^{(k)}(\mathbb{R}^d \setminus \{0\})$ for some $k \in \mathbb{N}_0$. Then the Blumenthal–Getoor index of order $k$ is defined by
\[ s := \inf_{\lambda > 0} \left\{ \lambda : \lim_{|\xi| \to \infty} \frac{\partial^{\alpha} \psi(\xi)}{|\xi|^{\lambda-|\alpha|}} = 0 \right\}. \]

Here $\alpha$ denotes a multi-index. If $k = \infty$ then Blumenthal–Getoor index of infinity order is defined by
\[ s := \inf_{\lambda > 0} \left\{ \lambda : \lim_{|\xi| \to \infty} \frac{|\partial^{\alpha} \psi(\xi)|}{|\xi|^{\lambda-|\alpha|}} = 0 \right\}. \]

**Remark 2.1.** The Blumenthal–Getoor index of order infinity is defined for the sake of completeness. We are interested in weakening the assumption on the symbol, i.e., reducing the order $k$.

To analyse analytic properties of the Markovian semigroup $(P_t)_{t \geq 0}$ and to define the resolvent of the associated operator $\psi(D)$, the range of the symbol is of importance.

**Definition 2.2.** Let $\mathcal{R}g(\psi)$ be the essential range of $\psi$, i.e.
\[ \mathcal{R}g(\psi) := \{ y \in \mathbb{C} \mid \text{Leb}\{ |s| \in \mathbb{R}^d : |\psi(s) - y| < \epsilon\} > 0 \text{ for each } \epsilon > 0\}. \]

Finally, to characterize the spectrum of the associated operator, one can introduce the type of a symbol.

**Definition 2.3.** We call a symbol $\psi$ is of type $(\omega, \theta)$, $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, iff
\[ -\mathcal{R}g(\psi) \subset \mathbb{C} \setminus \omega + \Sigma_{\theta+\frac{\pi}{2}}. \]

**Remark 2.2.** If a symbol $\psi$ is of type $(0, \theta)$, then there exists a constant $c > 0$ such that
\[ |\Re(\psi(\xi))| \leq c \psi(\xi), \quad \xi \in \mathbb{R}^d. \]

This is called sector condition of the symbol $\psi$. \footnote{Here, Leb denotes the Lebesgue measure.}
Remark 2.3. For \( \lambda \in \mathbb{C} \setminus \text{Rg}(\psi) := \{ \zeta \in \mathbb{C} : \exists \xi \text{ with } \psi(\xi) = \zeta \} \) we have (see Theorem 1.4.2 of [20])
\[
\| R(\lambda, \psi(D)) \| \leq \frac{1}{\text{dist}(\text{Rg}(\psi), \lambda)}.
\]
Moreover, the set \( \text{Rg}(\psi) \) equals the spectrum of the generator \( A \).

For several examples of Lévy processes and their symbols, we refer to [16]. In case there is no dependence on the space variable, one can derive properties of the Markovian semigroup directly using the range of the symbol. Given a solution process of an SDE, usually, the associated infinitesimal generator or the Markovian semigroup depends on \( x \). In particular, let \( X = \{ X^x(t) : t \geq 0 \} \) be a \( \mathbb{R}^d \)-valued solution of the stochastic differential equation given in (1.1) and, as before let \( (P_t)_{t \geq 0} \) be the associated Markovian semigroup defined in (1.2). Let \( \psi \) be the Lévy symbol of the Lévy process \( L \). Then, one can show (see Theorem 3.1 [43]), that \( X^x \) has the symbol \( a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) given by
\[
(2.3) \quad a(x, \xi) = \psi(\sigma^T(x) \xi), \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.
\]

Note, due to the dependence on \( x \), given two symbols \( a_1(x, \xi) \) and \( a_2(x, \xi) \) the corresponding operators \( a_1(x, D) \) and \( a_2(x, D) \) do not necessarily commute. Hence, many tricks which are working for symbols without dependence on \( x \) does not work for operators coming from symbols being not independent of \( x \). Especially, tricks relying on the Boni’s paraproduct gets much more demanding.

Example 2.1. Let \( L \) be a one dimensional \( \alpha \)-stable process and let \( \sigma \) and \( b \) be two Lipschitz continuous functions on \( \mathbb{R} \). Then the symbol
\[
a(x, \xi) := |\sigma(x)\xi|^\alpha + ib(x)\xi
\]
is of type \((0, \theta)\) if \( \alpha > 1 \). If \( \sigma \) is bounded away from zero, then the generalized Blumenthal–Getoor index is \( \alpha \). Let us assume that \( d = 1 \) and \( \sigma(x) = x \) for \( |x| \leq 1 \), \( \sigma \in \mathcal{C}^\infty_b(\mathbb{R}^d) \) and bounded from zero in \( \mathbb{R} \setminus (-1, 1) \). If \( b(x) \leq |x|^\alpha \), then the Blumenthal–Getoor index is again \( \alpha \).

3. INVERTIBILITY OF PSEUDO–DIFFERENTIAL OPERATORS

In this section, we study under which conditions the pseudo-differential operator is invertible. To investigate the inverse of a pseudo-differential operator one has to introduce the set of elliptic and hypoelliptic symbols. For reader’s convenience, we define the elliptic and hypoelliptic symbols in this section.

To treat pseudo-differential operators, different classes of symbols have been introduced. Here, we closely follow the definition of [1].

Definition 3.1. Let \( \rho, \delta \) two real numbers such that \( 0 \leq \rho \leq 1 \) and \( 0 \leq \delta \leq 1 \). Let \( \mathcal{S}^\rho_{\rho, \delta}(\mathbb{R}^d \times \mathbb{R}^d) \) be the set of all functions \( a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \), where
- \( a(x, \xi) \) is infinitely often differentiable, i.e. \( a \in \mathcal{C}^\infty_b(\mathbb{R}^d \times \mathbb{R}^d) \);
- for any two multi-indices \( \alpha \) and \( \beta \) there exist a constant \( C_{\alpha, \beta} > 0 \) such that
\[
\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle |\xi| \rangle^{m-\rho|\alpha|} \langle |x| \rangle^{|\delta|\beta}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.
\]
We call any function \( a(x, \xi) \) belonging to \( \bigcup_{m \in \mathbb{R}} S^m_{0,0}(\mathbb{R}^d \times \mathbb{R}^d) \) a symbol. For many estimates, one does not need that the function is infinitely often differentiable. It is often only necessary to know the estimates with respect to \( \xi \) and \( x \) up to a certain order. For this reason, one also introduces the following classes.

**Definition 3.2.** (compare [49, p. 28]) Let \( m \in \mathbb{R} \). Let \( A^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d) \) be the set of all functions \( a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \), where

- \( a(x, \xi) \) is \( k_1 \)-times differentiable in \( \xi \) and \( k_2 \) times differentiable in \( x \);
- for any two multi-indices \( \alpha \) and \( \beta \) with \( |\alpha| \leq k_1 \) and \( |\beta| \leq k_2 \), there exists a constant \( C_{\alpha, \beta} > 0 \) depending only on \( \alpha \) and \( \beta \) such that
  \[
  \left| \partial_\xi^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} (|\xi|)^{m-\rho|\alpha|} (|x|)^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.
  \]

Moreover, one can introduce a semi–norm in \( A^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d) \) by

\[
\|a\|_{A^m_{k_1,k_2;\rho,\delta}} = \sup_{|\alpha| \leq k_1, |\beta| \leq k_2} \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} \left| \partial_\xi^\alpha \partial_\xi^\beta a(x, \xi) \right| (|\xi|)^{\rho|\alpha|-m} (|x|)^{\delta|\beta|}, \quad a \in A^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d).
\]

We have seen in the introduction that, given a symbol, one can define an operator. In case the symbol \( \psi \) is a Lévy symbol, the operator defined by (2.2) is the infinitesimal generator of the semigroup of the Lévy process. In case one has an arbitrary symbol, the corresponding operator can be defined similarly.

**Definition 3.3.** (compare [49, p. 28, Def. 4.2]) Let \( a(x, \xi) \) be a symbol. Then, to \( a(x, \xi) \) corresponds an operator \( a(x, D) \) defined by

\[
(a(x, D) u)(x) := \int_{\mathbb{R}^d} e^{i(x, \xi) a(x, \xi)} \hat{u}(\xi) d\xi, \quad x \in \mathbb{R}^d, u \in \mathcal{S}(\mathbb{R}^d)
\]

and is called pseudo–differential operator.

In order to invert the to a symbol associated operator the symbol has to satisfy an elliptic property.

**Definition 3.4.** (compare [35, p. 35]) A symbol \( a \in S^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d) \) is called globally elliptic, if there exists a number \( r > 0 \),

\[
(|\xi|)^m \lesssim |a(x, \xi)|, \quad |\xi| \geq r, x \in \mathbb{R}^d.
\]

Later on, we will see that we need upper estimates not only for the symbol itself but also for its derivatives. A more sophisticated definition of the norm is given below.

**Definition 3.5.** (compare [35, p. 35]) Let \( m, \rho, \delta \) be real numbers with \( 0 \leq \delta < \rho \leq 1 \). The class \( Hyp^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d) \) consists of all functions \( a(x, \xi) \) such that

- \( a(x, \xi) \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d); \)
- there exists some \( r > 0 \) such that
  \[
  (|\xi|)^m \lesssim |a(x, \xi)|, \quad |\xi| \geq r.
  \]
and for an arbitrary multi-indices $\alpha$ and $\beta$ there exists a constant $C_{\alpha,\beta} > 0$ with
\[
|\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq C_{\alpha,\beta} (|\xi|)^{m-\rho|\alpha|} (|x|)^{\delta|\beta|}.
\]
for $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$.

In addition, for $k_1, k_2 \in N_0$, we define the following semi–norm given by
\[
\|a\|_{\mathcal{H}^m_{k_1, k_2, \rho, \delta}} = \sup_{|\alpha| \leq k_1, |\beta| \leq k_2} \sup_{x, \xi} \limsup_{|\xi| \to \infty} \left| \partial_x^\alpha \partial_{\xi}^\beta \left[ \frac{1}{a(x, \xi)} \right] |\xi|^{m+\rho|\alpha|} |x|^{\delta|\beta|} \right|.
\]

We are now interested, under which condition an operator $a(x, D)$ is invertible. To be more precise, we aim to answer the following questions. Given $f \in B^s_{p,r}(\mathbb{R}^d)$, does there exists an element $u \in S'(\mathbb{R}^d)$ such that
\[
a(x, D)u(x) = f(x), \quad x \in \mathbb{R}^d,
\]
and to which Besov space belongs $u$?

The invertibility is used for giving bounds of the resolvent of an operator $a(x, D)$. Here, one is interested not only in the invertibility of $a(x, D)$ but also in the invertibility of $\lambda + a(x, D)$, $\lambda \in \rho(a(x, D))$. In particular, we are interested in the norm of the operator $[\lambda + a(x, D)]^{-1}$ uniformly for all $\lambda$ belonging to the set of resolvents. However, executing a careful analysis, we can see that certain constants depend only on the first or second derivative on the symbol of $\lambda + a(x, D)$, which has the effect that this norm is independent of $\lambda$. Hence, it is necessary to introduce the following class.

**Definition 3.6.** Let $\rho, \delta$ be two real numbers such that $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. Let $m \in \mathbb{R}$ and $\kappa \in N_0$. Let $\mathcal{A}^{m, \kappa}_{k_1, k_2, \rho, \delta}(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where
- $a(x, \xi)$ is infinitely often differentiable, i.e. $a \in C^\infty_b(\mathbb{R}^d \times \mathbb{R}^d)$;
- for any two multi-indices $\alpha$ and $\beta$, with $|\alpha| \geq \kappa$, there exists $C_{\alpha,\beta}$ such that
  \[
  |\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq C_{\alpha,\beta} (|\xi|)^{m-\rho|\alpha|} (|x|)^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.
  \]

For $k_1, k_2 \in N_0$, we also introduce the following semi–norm by
\[
\|a\|_{\mathcal{A}^{m, \kappa}_{k_1, k_2, \rho, \delta}} = \sup_{\kappa \leq |\alpha| \leq k_1, |\beta| \leq k_2} \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} \left| \partial_x^\alpha \partial_{\xi}^\beta a(x, \xi) \right| (|\xi|)^{\rho|\alpha|-m} (|x|)^{\delta|\beta|}, \quad a \in \mathcal{A}^{m, \kappa}_{k_1, k_2, \rho, \delta}(\mathbb{R}^d \times \mathbb{R}^d).
\]

Now we are ready to state the main result of this section.

**Theorem 3.1.** Let $\kappa \geq 0$, $m \in \mathbb{R}$, $1 \leq p, r < \infty$. Let $a(x, \xi)$ be a symbol such that $a \in \mathcal{A}^{1-1}_{2d+4d+3,1}(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{H}^{m}_{2d+1, 0, 1, 0}(\mathbb{R}^d \times \mathbb{R}^d)$ for $k = \lceil \kappa \rceil$. Let $R \in \mathbb{N}$ such that
\[
(3.2) \quad R \geq 10 \times d \times \|a\|_{\mathcal{A}^{1-1}_{2d+4d+3,1}} \|a\|_{\mathcal{H}^{m}_{2d+1, 0, 1, 0}}
\]
and
\[
(3.3) \quad \langle |\xi| \rangle^\kappa \leq \frac{|a(x, \xi)|}{|a|_{\mathcal{A}^{0, 0, 0, 1}_{0, 1}}} \text{ for all } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d \text{ with } |\xi| \geq R.
\]
Then, there exists a bounded pseudo–differential operator $B : B^m_{p,r}(\mathbb{R}^d) \to B^m_{p,r}(\mathbb{R}^d)$ with symbol $b(x,D)$, such that

- $\{(\xi,x) \in \mathbb{R}^d \times \mathbb{R}^d : \sup_{\xi \in \mathbb{R}^d} b(x,\xi) > 0\} \subset \{|\xi| \leq 2R\}$,
- $B$ has norm $R^k$ on $B^m_{p,r}(\mathbb{R}^d)$ into itself,
- $a(x,D) = A + B$,

and, given $f \in B^m_{p,r}(\mathbb{R}^d)$, the problem

$$Au(x) = f(x), \quad x \in \mathbb{R}^d$$

has a unique solution $u$ belonging to $B^{m+k}_{p,r}(\mathbb{R}^d)$. In addition, there exists a constant $C_1 > 0$ such that for all $f \in B^m_{p,r}(\mathbb{R}^d)$ and $u$ solving (3.4) we have

$$|u|_{B^{m+k}_{p,r}} \leq C_1 \|a\|_{H^{p,r+1,0;0,1,0}} |f|_{B^m_{p,r}}, \quad f \in B^m_{p,r}(\mathbb{R}^d).$$

**Remark 3.1.** Since $a(x,\xi)$ is elliptic, we can find a number $R > 0$ satisfying (3.2) and (3.3).

**Remark 3.2.** In fact, analyzing the resolvent $[\lambda + a(x,D)]^{-1}$ of an operator $a(x,D)$, it will be important that we have here in the Theorem the norm of $A^{m,1}_{p,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ and not the norm of $\mathcal{A}^{m,1}_{p,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$. The reason is that calculating the norm in $A^{m,1}_{p,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ the first derivative has to be taken. Therefore, the norm in $A^{m,1}_{p,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ is independent of $\lambda$.

**Proof.** Note, that, for convenience for the reader, we summerized several definition and results necessary for the proof in the appendix. For simplicity, let $E = B^m_{p,r}(\mathbb{R}^d)$. Let $\chi \in C^\infty_b(\mathbb{R}^d)$ such that

$$\chi(\xi) = \begin{cases} 
0 & \text{if } |\xi| \leq 1, \\
1 & \text{if } |\xi| \geq 2, \\
\in (0,1) & \text{if } |\xi| \in (1,2).
\end{cases}$$

Let us put $\chi_R(\xi) := \chi(\xi/R)$, $\xi \in \mathbb{R}^d$. In addition, let us set

$$p_R(x,\xi) := a(x,\xi)\chi_R(\xi), \quad b(x,\xi) := a(x,\xi)(1 - \chi_R(\xi)), \quad \text{and} \quad q(x,\xi) := \frac{1}{a(x,\xi)}\chi_R(\xi).$$

Due to the condition on $R$, the function $q(x,\xi)$ is bounded and as a symbol, it is well defined.

Let us consider the following problem: Given $f \in B^m_{p,r}(\mathbb{R}^d)$, find an element $u \in S'(\mathbb{R}^d)$ such that we have

$$p_R(x,D)u(x) = f(x), \quad x \in \mathbb{R}^d.$$

Observe, that on one hand for a solution $u$ of (3.5) we have

$$[q(x,D)p_R(x,D)] u = q(x,D)f,$$

and, on the other hand, by Remark 3.3 the symbol for $q(x,D)p_R(x,D)$ is given by

$$(q \cdot p_R)(x,\xi) = q(x,\xi)p_R(x,\xi)$$

$$+ C(x,\xi) + \sum_{|\gamma|=\max(d-k+2,2)} \text{Os}^{-i(y,\eta)} \eta^\gamma r_{\gamma}(x,\xi,\eta) dy d\eta,$$
Secondly, by the Young inequality for a product, we know for $s > 0$
\[ \langle \xi + \theta \eta \rangle^{-2s} \leq \langle \xi \rangle^{-s} \langle \theta \eta \rangle^{-s}, \]
and by the Peetre inequality (see [1, Lemma 3.7, p. 44]), we know for $s > 0$
\[ \langle \xi + \theta \eta \rangle^s \leq \langle \xi \rangle^s \langle \theta \eta \rangle^s. \]
Next, straightforward calculations gives for \( s > d \)

\[
\int_{\mathbb{R}^d} (\eta)^{-s} \, d\eta \leq C.
\]

Using \( \partial_\eta^\rho e^{-i(y,\eta)} = (-y)^\rho e^{-i(y,\eta)} \), where \( \rho \) is a multiindex, and integration by parts, gives

\[
\partial_\xi^\rho m_R(x, \xi) = \sum \int \int (-y)^{-\rho} e^{-i(y,\eta)} \int_0^1 \partial_\xi^\rho \partial_\eta^\rho \left[ \partial_\xi^\rho q(x', \xi') \mid x'=x+y, \xi'=\xi+\theta \eta \right] \, d\theta \, dy \, d\eta
\]

\[
= \sum \int \int (-y)^{-\rho} e^{-i(y,\eta)} \int_0^1 \partial_\xi^\rho \theta^{d+1} \left[ \partial_\xi^\rho q(x', \xi') \mid x'=x+y, \xi'=\xi+\theta \eta \right] \, d\theta \, dy \, d\eta,
\]

Here the sum runs over all multiindex of the form \((d+1, 0, \ldots, 0), (0, d+1, \ldots, 0), \ldots, (0, \ldots, d+1)\). Analysing the proof of Theorem 3.9 [1] we see that we have to estimate

\[
|\partial_\xi^\rho m_R(x, \xi)| \leq \int \int \left[ \langle |x+\theta \eta|^{-\rho} \rangle^{-(d+1)} \left[ \langle |x+\theta \eta|^{-\alpha} \right] \right] \, d\theta \, dy \, d\eta
\]

\[
\leq \int \int \left[ \langle |x+\theta \eta|^{-\alpha} \right] \, d\theta \, dy \, d\eta
\]

\[
\leq \langle x \rangle^{-2\alpha - (d+1)} \int \int \theta^{d+1} \langle \theta \eta \rangle^{-\frac{1}{2}(\alpha + (d+1))} \, d\theta \, d\eta
\]

\[
\leq \langle x \rangle^{-2\alpha - (d+1)} \int \int \theta \langle \eta \rangle^{-\frac{1}{2}(\alpha + (d+1))} \, d\theta \, d\eta.
\]

The calculation above gives that for \( |\alpha| > d - k + 1 \), the integration with respect to \( \eta \) and \( \theta \) is finite. Taking into account Theorem \( \ref{thm:A.2} \) we can verify that

\[
\sup_{|\alpha| \leq d+1} \left| \partial_\xi^\rho m_R(x, \xi) \right| \langle |x| \rangle^{\alpha+1}
\]

\[
\lesssim \sup_{1 \leq |\alpha| \leq d+1} \sup_{1 \leq |\beta| \leq 2d+1} \left| \partial_\xi^\alpha \partial_\xi^\beta \left[ q_R(x, \xi) \right] \right| \sup_{|\delta| \leq d+1} \left| \partial_\xi^\delta p_R(x, \xi) \right|.
\]

Hence, by the generalized Leibniz rule (see [1] p. 200, (A.1)) we have

\[
\|m_R\|_{A_{d+1,0,1,0}}^{A_{d+1,0,1,0}} \leq \|q_R\|_{Hyp_{2d+1,0,1,0}}^{A_{2d+1,0,1,0}} \|p_R\|_{A_{2d+1,2d+1,1,0}}^{A_{2d+1,2d+1,1,0}},
\]

from what it follows that \( m_R(x, D) \) is a bounded operator with from \( B^{m}_{p,r}(\mathbb{R}^d) \) to \( B^{m+1}_{p,r}(\mathbb{R}^d) \).

In addition, by the same analysis, we get

\[
\|m_R\|_{A_{d+1,0,1,0}}^{A_{d+1,0,1,0}} \leq \|q\|_{Hyp_{2d+1,0,1,0}}^{A_{2d+1,0,1,0}} \|p_R\|_{A_{2d+1,2d+1,1,0}}^{A_{2d+1,2d+1,1,0}},
\]

Observe, that

\[
\|p_R\|_{A_{2d+4,2d+1,2d+1,1,0}}^{A_{2d+4,2d+1,2d+1,1,0}} \leq \frac{1}{R} \|p_R\|_{A_{2d+4,2d+1,2d+1,1,0}}^{A_{2d+4,2d+1,2d+1,1,0}}.
\]
Therefore, analysing the symbols, \( m_R(x, D) \) is a bounded operator from \( B_{p,r}^m(\mathbb{R}^d) \) to \( B_{p,r}^m(\mathbb{R}^d) \) having norm

\[
\|m_R\|_{A_{d+1,0,1,0}^0} \leq \frac{1}{R} \|m_R\|_{A_{d+1,1,0,1,0}^{-1}}.
\]

Now, let us go back to a slightly modified problem to verify for a given \( f \in B_{p,r}^m(\mathbb{R}^d) \), the regularity of \( u \), where \( u \) solves

\[
p_R(x, D)u(x) = f(x), \quad x \in \mathbb{R}^d.
\]

From before, we know that

\[
(q \circ p_R)(x, \xi) = q(x, \xi)p_R(x, \xi) + C(x, \xi) + m_R(x, \xi) = I + C(x, \xi) + m_R(x, \xi).
\]

A careful analysis (see (3.8)) shows that \( C \in A_{d+1,0,1,0}^{-1} \), and

\[
\|C\|_{A_{d+1,0,1,0}^q} \leq \frac{1}{R} \|C\|_{A_{d+1,1,0,1,0}^{-1}}.
\]

Due to the assumption on \( R \) we know that \( R \) is such large that

\[
\left(\|C\|_{A_{d+1,0,1,0}^q} + \|m_R\|_{A_{d+1,1,0,1,0}^0}\right) \leq \frac{1}{6}.
\]

First, we will show that if \( f \in B_{p,r}^m(\mathbb{R}^d) \), then it follows that \( u \in B_{p,r}^m(\mathbb{R}^d) \). Suppose for any \( M \in \mathbb{N} \) we have \( |u|_{B_{p,r}^m} \geq M \). Since, from before we know that

\[
(I + C(x, D) + m_R(x, D))u = q(x, D)f,
\]

we get

\[
|I + C(x, D) + m_R(x, D)|_{B_{p,r}^m} \geq |u|_{B_{p,r}^m} - \frac{1}{R}|u|_{B_{p,r}^m} \geq \frac{5}{6}|u|_{B_{p,r}^m}.
\]

On the other side,

\[
|(I + C(x, D) + m_R(x, D))u|_{B_{p,r}^m} = |q(x, D)f|_{B_{p,r}^m} \leq \|q\|_{A_{d+1,0,1,0}^0} |f|_{B_{p,r}^m} < \infty,
\]

which leads to a contradiction, since we assumed that for any \( M \in \mathbb{N} \) we have \( |u|_{B_{p,r}^m} \geq M \). Hence, we know that \( u \in B_{p,r}^m(\mathbb{R}^d) \). In the next step, we will show that we have even \( u \in B_{p,r}^{m+\kappa}(\mathbb{R}^d) \) and calculate its norm in this space. Using similar arguments as above, we know

Theorem by [A.1] and Remark [A.2] that

\[
|q(x, D)f|_{B_{p,r}^{m+\kappa}} \leq \|q\|_{A_{d+1,0,1,0}^\kappa} |f|_{B_{p,r}^m}.
\]

Similar as in the proof of Theorem 3.24 in [11, p. 59] we define

\[
\tilde{q}(x, D) := \sum_{j=0}^k (-1)^j (C(x, D) + m_R(x, D))^j q(x, D),
\]

where

\[
(C(x, D) + m_R(x, D))^j = \underbrace{(C(x, D) + m_R(x, D)) \cdots (C(x, D) + m_R(x, D))}_{j \text{ times}}.
\]
Since the right hand side is an alternating sum, it follows by the identity
\[ q(x, D)p_R(x, D) = I + C(x, D) + m_R(x, D)u(x) \]
that
\begin{equation}
\tilde{q}(x, D)p_R(x, D) = I + (-1)^{k+1}(C(x, D) + m_R(x, D))^{k+1}.
\end{equation}
(3.10)
On the other side, since
\[ u(x) = q(x, D)f(x) - (C(x, D) + m_R(x, D))u(x), \]
we have
\[ q(x, D)f(x) = q(x, D)p_R(x, D)u(x) = (I + C(x, D) + m_R(x, D))u(x). \]
Since \( |m_R(x, \xi)|_{A_{d+1,0,1,0}} \leq \frac{1}{6} \), the sequence \( \{u_n : n \in \mathbb{N}\} \) defined by
\[ u_n(x) = \left( I + \sum_{k=1}^{N} (-1)^k(C(x, D) + m_R(x, D))^k \right) q(x, D)f(x), \]
is bounded and a Cauchy sequence. Therefore, there exists a \( u \) with \( u_n \to u \) strongly, and, therefore,
\[ \|u\|_{B^{m,p,r}_{\kappa,\kappa}} \lesssim \|q\|_{\text{Hyp}_{d+1,0,1,0}} \left( 1 + \sum_{k=1}^{\infty} \|C(x, D) + m_R\|_{A_{d+1,0,1,0}}^k \right) \|f\|_{B^{m,p,r}_{\kappa,\kappa}}. \]

This gives the assertion. \( \square \)

4. Analyticity of the Markovian semigroup in general Besov spaces

Given a function space \( X \) over \( \mathbb{R}^d \) we would be interested under which conditions on the coefficients \( \sigma, b \) and the symbol \( \psi \), the Markovian semigroup \( (P_t)_{t \geq 0} \) generates an analytic semigroup on \( X \). Here, one has first to verify that \( (P_t)_{t \geq 0} \) generates a strongly continuous semigroup. The necessary and sufficient conditions to fulfill a given semigroup is strongly continuous semigroup are given in Hille–Yosida Theorem. Let us assume that \( X \) be a Banach space. For an operator \( A \), let \( \rho(A) \) represent the resolvent set, i.e. \( \rho(A) = \{ \lambda \in \mathbb{C} : (\lambda I - A) \) is invertible \} and \( \sigma(A) = \mathbb{C} \setminus \rho(A) \). Now, if \( (A, D(A)) \) is closed, densely defined and for any \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \) one has \( \lambda \in \rho(A) \) (compare [13 Theorem 3.5, p. 73], or [36, Theorem 1.5.2]) and
\begin{equation}
\|R(\lambda, A)\|_{L(X,X)} \leq \frac{1}{\Re \lambda},
\end{equation}
(4.1)
then \( A \) generates a strongly continuous semigroup on \( \mathbb{X} \). Secondly, to show that this strongly continuous semigroup is an analytic one has to show either that (compare [36, Theorem 4.6, p. 101])

\[
M := \sup_{t>0} \|tAP_t\|_{L(\mathbb{X}, \mathbb{X})} < \infty,
\]

or

\[
\|R(\vartheta + i\tau : A)\|_{L(\mathbb{X}, \mathbb{X})} \leq \frac{C(A)}{|\tau|}, \quad \vartheta > 0, \vartheta, \tau \in \mathbb{R}.
\]

Let \( S(A) = \{\langle x^*, Ax \rangle : x \in D(A), x \in \mathbb{X}^*, \|x\| = 1, \|x^*\| = 1, \langle x^*, x \rangle = 1\} \) be the numerical range of an operator \( A \). If \( \mathbb{X} \) is a Hilbert space and \( \sigma \) constant, \( S(A) \) can be characterized by the \( \mathcal{R}_a(\psi) := \{\langle x, \xi \rangle \in \mathbb{C} : x, \xi \in \mathbb{R}^d\} \), where \( a = (x, \xi) := \psi(\sigma^T \xi) \). Since the range of \( \psi \) contains the numerical range \( S(A) \) of \( A \), we have (see Remark [2.3]),

\[
\|R(\lambda, A)\|_{L(\mathbb{X}, \mathbb{X})} \leq \frac{1}{\text{dist}(\lambda, S(A))}.
\]

Hence, for \( \mathbb{X} = H^m_2(\mathbb{R}^d) \) and \( \sigma(x) = \sigma_0 \), one can show by analyzing the numerical range, which is here given by

\[
S(a(x, D)) = \{\langle x, a(x, D)x \rangle : x \in \text{dom}(a(x, D)), |x|_{H^m_2(\mathbb{R}^d)} = 1, \langle x, x \rangle_{H^m_2(\mathbb{R}^d)} = 1\},
\]

and some purely geometric considerations the analyticity of the to \( A \) corresponding semigroup \( (P_t)_{t \geq 0} \) in \( \mathbb{X} \). Here \( \langle , \rangle \) represent the inner product in \( H^m_2(\mathbb{R}^d) \). In fact, choosing a complex number \( \lambda = \vartheta + i\tau \) with \( \vartheta > 0 \) and \( \tau \in \mathbb{R} \), and using that symbol \( \psi \) is of type \( (0, \theta) \), we obtain by as a application of (see Theorem 3.9 [36, Chapter I]),

\[
\|R(\lambda, a(x, D))\|_{L(H^m_2(\mathbb{R}^d), H^m_2(\mathbb{R}^d))} = \frac{1}{\text{dist}(\lambda, S(a(x, D)))} \leq \frac{1}{\text{dist}(\lambda, \rho(a(x, D)))} \leq \frac{1}{\text{dist}(\vartheta + i\tau, \rho(a(x, D)))} \leq \frac{1}{|\tau|} = \frac{C}{|\tau|},
\]

where \( C = \cos \theta \). These calculation implies that \( (P_t)_{t \geq 0} \) in \( \mathbb{X} \) is an analytic semigroup in \( \mathbb{X} \).

This result can be generalized to arbitrary Besov spaces. If one abandon the Hilbert space setting, the numerical range gets more complicated and it is better to use other methods. In the following Theorem we use the notations introduced in Appendix A.

**Remark 4.1.** Using the abstract theory on standard books such as [36, 13, 48], we would be able to prove the following theorems associated with the pseudo–differential operator induced by Lévy processes. In particular, we show that the all assumptions of the corresponding theorems (Theorem 5.2 [36, Chapter II], Theorem 3.9 [36, Chapter I], Theorem 2.3.3 [48, p.48], etc.) that we used to prove Theorem (2.1) are satisfied.

**Theorem 4.1.** Let us assume that the symbol \( a \in \mathcal{A}_2d+4,d+3,1,0(\mathbb{R}^d \times \mathbb{R}^d) \), where \( 1 < \delta < 2 \), is of type \( (0, \theta) \) and \( a \in \mathcal{H}^m_2d+4,d+3,1,0(\mathbb{R}^d \times \mathbb{R}^d) \). Then for all \( 1 \leq p, q < \infty \) and \( m \in \mathbb{R} \), the operator generates an analytic semigroup \( (P_t)_{t \geq 0} \) in \( B^m_p(\mathbb{R}^d) \).
Applying Theorem 4.1 to our special operator generated by a SDE give following Theorem.

**Theorem 4.2.** Let us assume that the symbol is of type $(0,\theta)$ and
\[ \psi \in A^{d+4,d+3,1,0}_2(\mathbb{R}^d \times \mathbb{R}^d) \cap H_{y}\psi_{2d+4,d+3,1,0}^\delta(\mathbb{R}^d \times \mathbb{R}^d), \]
where $1 < \delta < 2$ is the Blumenthal–Getoor index of order $2d + 4$ of $L$. In addition, let us assume that $\sigma \in C_b^{d+3}(\mathbb{R}^2)$ and $b \in C_b^{d+3}(\mathbb{R}^d)$, and that there exists a number $c > 0$ such that
\[ \inf_{x \in \mathbb{R}^d} |\sigma(x)| \geq c. \]
Then for all $1 \leq p, q < \infty$ and $m \in \mathbb{R}$, the Markovian semigroup is analytic in $B_{p,q}^m(\mathbb{R}^d)$.

**Remark 4.2.** The restriction that $p$ has to be strictly smaller than infinity comes from the fact, that the space of Schwarz functions $S(\mathbb{R}^d)$ is not dense in $B^m_{\infty,\infty}(\mathbb{R}^d)$.

**Proof of Theorem 4.2.** For simplicity, let us denote $B_{p,q}^m(\mathbb{R}^d)$ by $X$. Let us assume that the symbol $\psi$ and the coefficients $\sigma$ and $b$ are infinitely often differentiable. We first show that the operator $(\mathcal{P}_t)_{t \geq 0}$ generates a strongly continuous semigroup on $X$ by proving the required conditions in Hille–Yoshida Theorem. Theorem 2.3.3, p.48 in [18], gives us that Schwarz space $S(\mathbb{R}^d)$ is dense in $X$. In addition, it is straightforward that $S(\mathbb{R}^d) \subset \text{dom}(a(x,D))$. This immediately gives that $\text{dom}(a(x,D))$ is dense in $X$.

Before starting, let us split the operator $a(x,D)$ into two operator in the same way as it is done in Theorem 3.1. Let $R \in \mathbb{N}$ such that
\[ R \geq 6 \times \|a\|_{A^{-1,1}_{2d+4,d+3,1,0}} \]
and $\langle |\xi|^\delta\rangle \leq |a(x,\xi)|$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ with $|\xi| \geq R$. In addition, let $\chi \in C_b^\infty(\mathbb{R}^d_0)$ such that
\[ \chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq 1, \\ 1 & \text{if } |\xi| \geq 2. \end{cases} \]
and $b(x,\xi) := a(x,\xi)(1 - \chi(\xi/R))$ and $\tilde{a}(x,\xi) := a(x,\xi)\chi(\xi/R)$. We will show that $\tilde{A} = \tilde{a}(x,D)$ generates an analytic semigroup on $X$. Due to Theorem 2.1 [38 Chapter 3.2, p. 80], it follows that $A = \tilde{A} + B$ with $B = b(x,D)$ generates an analytic semigroup on $X$.

First, we will show that $(\tilde{a}(x,D), \text{dom}(\tilde{a}(x,D)))$ is closed in $X$. Let $\{v_n : n \in \mathbb{N}\} \subset \text{dom}(\tilde{a}(x,D))$ be a sequence such that $\lim_{n \to \infty} v_n = v$ in $\text{dom}(\tilde{a}(x,D))$ and $\lim_{n \to \infty} \tilde{a}(x,D)v_n = w$ in $X$. Then, to show that $(\tilde{a}(x,D), \text{dom}(\tilde{a}(x,D)))$ is closed in $X$, we have to show that $\tilde{a}(x,D)v = w$. Suppose that $|\tilde{a}(x,D)v - w|_X \neq 0$. In particular, there exists a constant $\hat{C} > 0$ such that $|\tilde{a}(x,D)v - w|_X \geq \hat{C}$. There exist a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have
\[ |v - v_n|_{\text{dom}(\tilde{a}(x,D))} < \frac{\hat{C}}{4\|\tilde{a}(x,D)\|_{L(\text{dom}(\tilde{a}(x,D)),X)}} \]
and
\[ |\tilde{a}(x,D)v_n - w|_X < \frac{\hat{C}}{4}. \]
Since $\tilde{a}(x, D)$ is linear and bounded operator on $\text{dom}(\tilde{a}(x, D))$, we have

$$|\tilde{a}(x, D)v - w|_X \leq |\tilde{a}(x, D)v - a(x, D)v_n|_X + |\tilde{a}(x, D)v_n - w|_X,$$

where in the Definition (3.5), it is considered for all $\lambda \in \Sigma \theta + \varpi$. Here, it is essential that $\tilde{a}(x, D)$ is linear and bounded on $\text{dom}(\tilde{a}(x, D))$. We will consider the case where $|\alpha| = |\beta| = 0$. Separating the real and imaginary part we set $\lambda = \lambda_1 + i\lambda_2$ and $\tilde{a}(x, \xi) = \psi_1(x, \xi) + i\psi_2(x, \xi)$. Now we have

$$\frac{1}{\lambda + \tilde{a}(x, \xi)} = \frac{\lambda_1 + \psi_1(x, \xi)}{(\lambda_1 + \psi_1(x, \xi))^2 + (\lambda_2 + \psi_2(x, \xi))^2} - i \frac{\lambda_2 + \psi_2(x, \xi)}{(\lambda_1 + \psi_1(x, \xi))^2 + (\lambda_2 + \psi_2(x, \xi))^2}.$$

In particular, simple calculations give

$$\left| \frac{1}{\lambda + \tilde{a}(x, \xi)} \right| \leq \frac{\sqrt{(\lambda_1 + \psi_1(x, \xi))^2 + (\lambda_2 + \psi_2(x, \xi))^2}}{(\lambda_1 + \psi_1(x, \xi))^2 + (\lambda_2 + \psi_2(x, \xi))^2} \leq \frac{1}{\lambda_2},$$

for $\lambda_1 \geq 1$. Next, we will consider the case where $|\alpha| = |\beta| = 1$, that is let $\alpha = k$ and $\beta = l$ with $k, l \in \{1, \ldots, d\}$. Then,

$$\partial_{x_i} \partial_{\xi_k} \left[ \frac{1}{\lambda + \tilde{a}(x, \xi)} \right] = - \frac{\partial^2 \psi_1(x, \xi) a(x, \xi)}{(\lambda + a(x, \xi))^3} + \frac{2\partial_x a(x, \xi) \partial_{\xi_k} \tilde{a}(x, \xi)}{(\lambda + a(x, \xi))^3}.$$

For simplicity, we will not separate real and imaginary part. Hence, we have

$$|\lambda| \left| \partial_{x_i} \partial_{\xi_k} \left[ \frac{1}{\lambda + \tilde{a}(x, \xi)} \right] \right| \leq |\lambda| \left( \frac{r^{-1}}{|\lambda + 1|^2} + \frac{r^{-1}}{|\lambda + 1|^3} \right) \leq C(r),$$

where in the Definition (3.5), it is considered for all $|\xi| \geq r$ for some $r > 0$. Similarly, we could get the bound for the general case where for multiindices $|\alpha| \leq 2d + 4$ and $|\beta| \leq d + 3$. Theorem 3.1 gives (4.5). That is

$$\|R(\lambda, \tilde{a}(x, D))\|_{L(X, X)} \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma \theta + \varpi.$$
for some \( C > 0 \). This gives us that \((P_t)_{t \geq 0}\) generates a strongly continuous semigroup over \( X \). Finally it remains to show that the Semigroup \((P_t)_{t \geq 0}\) is analytic over \( X \). Now pick \( \lambda = \vartheta + i\tau \in \Sigma_{\vartheta + \frac{i\tau}{\tau}} \) such that \( \vartheta > 0 \) and \( \tau \in \mathbb{R} \). From the estimate (4.6), we easily see that,

\[
\| R(\vartheta + i\tau, \tilde{a}(x, D)) \|_{L(X, X)} \leq \frac{C_{d, s}}{\tau}, \quad \lambda \in \Sigma_{\vartheta + \frac{i\tau}{\tau}},
\]

Then by applying the Theorem 5.2 [36, Chapter II] we could conclude that the Markovian semigroup is an analytic semigroup over \( X \).

In fact, analyzing the proof of [46, p. 58] one can see that the condition of the differentiability at the origin can be relaxed. Here, it is important to mention that the proof relies on the Theorem 2.5 [22, p. 120] (see also Theorem 4.23 [1]), from which one can clearly see the extension of the Theorem 9.7 of [49] to symbols, whose derivatives have a singularity at \( \{0\} \). Moreover, analyzing line by line of the proof of Theorem 9.7 in [49], one can give an estimate of the norm of the operator.

5. The first application: the strong Feller property

Let \( L = \{L(t) : t \geq 0\} \) be a family of Lévy processes. We consider the stochastic differential equations of the form

\[
\begin{cases}
  dX^x(t) = b(X^x(t-)) \, dt + \sigma(X^x(t-)) \, dL(t) \\
  X^x(0) = x, \quad x \in \mathbb{R}^d,
\end{cases}
\]

where \( \sigma : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d) \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \) are Lipschitz continuous. By \( C^0_b(\mathbb{R}^d) \) we denote the set of all real valued and uniformly continuous functions on \( \mathbb{R}^d \) equipped with the supremum–norm. The Markovian semigroup \((P_t)_{t \geq 0}\) of a process is called strong Feller, iff for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \), \( P_t f \in C^0_b(\mathbb{R}^d) \). In this section we will prove under certain assumptions the strong Feller property of the Markovian semigroup.

Before introducing the main results let us introduce some notation. If \( m \in \mathbb{N} \) we define

\[
C^m_b(\mathbb{R}^d) := \{ f \in C^0_b(\mathbb{R}^d) : D^\alpha f \in C^0_b(\mathbb{R}^d), \, |\alpha| \leq m \}
\]

dowered with the norm

\[
|f|_{C^m_b} := \sum_{|\alpha| \leq m} |D^\alpha f|_{C^0_b}.
\]

Let \( s \in \mathbb{R} \setminus \mathbb{N} \), then we put \( s = [s] + \{s\} \), where \( [s] \) is an integer and \( 0 \leq \{s\} < 1 \). Then

\[
C^s_b(\mathbb{R}^d) := \left\{ f \in C^s_b(\mathbb{R}^d) : \sum_{|\alpha| = [s]} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^{[s]}} < \infty \right\}
\]

dowered with the norm

\[
|f|_{C^s_b} := |f|_{C^s_b} + \sum_{|\alpha| = [s]} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^{[s]}}.
\]

Now, we can state our first result.
Let us assume that Corollary 5.2.

where $b = \gamma < n \delta$

the Besov spaces gives that

$6.4 \)), and, finally, we apply Theorem 5.1

Proof of Corollary 5.1:

Let $p_t(x, y) = (P_t \delta_x)(y)$, where $p_t$ denotes the density of the process $X$. Similarly, we get also estimates for the density.

Corollary 5.2. Let us assume that $L$ is a square integrable Lévy process with Blumenthal–Getoor index $1 < \delta < 2$ of order $2d + 4$, $\sigma \in C_b^{d+3}(\mathbb{R}^d)$, such that $\sigma$ is bounded away from zero, and $b \in C_b^{d+3}(\mathbb{R}^d)$. Then, the process defined by (6.1) is strong Feller. In particular, we have for all $\gamma < n \delta - d$, $n \in \mathbb{N}$ and $n \geq d$,

$$|P_t u|_{C^1_0(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} |u|_{L^\infty(\mathbb{R}^d)}.$$ 

Observe, for any $x, y \in \mathbb{R}^d$, we have

$$p_t(x, y) = (P_t \delta_x)(y),$$

where $p_t$ denotes the density of the process $X$. Similarly, we get also estimates for the density.

Corollary 5.2. Let us assume that $L$ is a square integrable Lévy process with Blumenthal–Getoor index $1 < \delta < 2$ of order $2d + 4$, $\sigma \in C_b^{d+3}(\mathbb{R}^d)$, such that $\sigma$ is bounded away from zero, and $b \in C_b^{d+3}(\mathbb{R}^d)$. Then, the process defined by (6.1) is strong Feller. In particular, we have for all $\gamma < n \delta - d$, $n \in \mathbb{N}$ and $n \geq d$,

$$|P_t u|_{C^1_0(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} |u|_{L^\infty(\mathbb{R}^d)}.$$ 

By means of [1] Excercise 6.25, Corollary 6.14,

$$B^\kappa_{p^*1}(\mathbb{R}^d) \hookrightarrow B^d_{p^*1}(\mathbb{R}^d) \hookrightarrow C^0_b(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d),$$

where $\kappa = n \delta - \frac{d}{p} - \gamma$ and $p^* = \frac{p}{p-1}$. Now by applying [1] Lemma 6.5 and duality property of the Besov spaces gives that

$$B^\kappa_{p^*1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \hookrightarrow B^{-\kappa}_{p,\infty}(\mathbb{R}^d) = B^{\gamma + \frac{d}{p} - n \delta}(\mathbb{R}^d) \hookrightarrow B^{\gamma + \frac{d}{p} - (n-1)\delta}_{p,1}(\mathbb{R}^d).$$
Finally by fixing \( q = 1 \) we would get,

\[
|\mathcal{P}_tu|_{C^\gamma_b(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} |u|_{B_{p,1}^{\gamma + \frac{d}{t} - (n-1)\delta}(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} |u|_{B_{p,1}^{\gamma + \frac{d}{t} - n\delta}(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} |u|_{L^\infty(\mathbb{R}^d)}.
\]

This gives the assertion.

\[ \square \]

**Proof of Corollary 5.2:** Fix \( p \in (1, \infty) \). We know \( \delta_x \in B_{p,\infty}^{-\frac{d}{p}}(\mathbb{R}^d) \), where \( p' \) is the conjugate of \( p \) (see [23, Formula B.2]). In addition, a function is \( \theta \) times continuous differentiable, if \( u \in C^{(\theta)}_b(\mathbb{R}^d) \). Since \( B_{p,1}^{\gamma_1}(\mathbb{R}^d) \hookrightarrow C^{(\theta)}_b(\mathbb{R}^d) \) for \( \gamma_1 = \theta + \frac{d}{p} \), we have to estimate \( |\mathcal{P}_t\delta_x|_{B_{p,q}^{\gamma_1}} \). Let \( m \) that large that \( m\delta > \theta + d \). Then \( \gamma_1 - m\delta < -(d - \frac{d}{p}) \). Now, we have

\[
|\mathcal{P}_t\delta_x|_{B_{p,q}^{\gamma_1}} \leq \left( \frac{mC}{t} \right)^m |\delta_x|_{B_{p,q}^{\gamma_1 - m\delta}} \leq \left( \frac{mC}{t} \right)^m |\delta_x|_{B_{p,1}^{\gamma_1 - 2\delta}}
\]

where \( \gamma_2 < -(d - \frac{d}{p}) \). Since \( \delta_x \in B_{p,\infty}^{-\frac{d}{p}}(\mathbb{R}^d) \), the right hand side is bounded.

\[ \square \]

**Proof of Theorem 5.1:** First, note that by W. Hoh (see [23]), the symbol \( \psi \) of the Lévy process is infinitely often differentiable. If the coefficient \( \sigma \) is independent from the space variable \( x \), then one can write the symbol of the semigroup \( (\mathcal{P}_t)_{t \geq 0} \) directly by \( (e^{t\phi(\xi)})_{t \geq 0} \) (see [3]). If \( \sigma \) depends on the space variable \( x \), one does not have such a nice representation of the symbol of the semigroup. We will use the representation of the semigroup \( (\mathcal{P}_t)_{t \geq 0} \) in terms of the contour integral as we have already successfully applied in [11], [21] or [13]. Let us remind, for any \( h \in \mathbb{R} \) such that \( \psi \) is of type \((h, \theta)\). Let \( \theta' \in (0, \theta), \rho \in (0, \infty), \) and

\[
\Gamma_{\theta'}(\rho, M) = \Gamma^{(1)}_{\theta', \rho} + \Gamma^{(2)}_{\theta', \rho} + \Gamma^{(3)}_{\theta', \rho},
\]

where \( \Gamma^{(1)}_{\theta', \rho} \) and \( \Gamma^{(2)}_{\theta', \rho} \) are the rays \( re^{-i(\frac{d}{2} + \theta')} \) and \( re^{i(\frac{d}{2} + \theta')} \), \( \rho \leq r \leq M < \infty \), and \( \Gamma^{(3)}_{\theta', \rho} = \rho^{-1} e^{i\alpha} \), \( \alpha \in [-\frac{d}{2} - \theta', \frac{d}{2} + \theta'] \). It follows from [39, Theorem 1.7.7] and Fubini’s Theorem that for \( t > 0 \) and \( v \in B_{p,q}^{\gamma}(\mathbb{R}^d) \) we have

\[
\mathcal{P}_tu = \lim_{M \to \infty} \frac{1}{2\pi i} \int_{\Gamma_{\theta'}(\rho, M)} e^{\lambda t} R(\lambda : a(x, D))v d\lambda,
\]

where \( R(\lambda : a(x, D)) \) denotes the inverse of \( a(x, \lambda, D) = \lambda I + a(x, D) \). Due to Theorem 4.2 we know that \( (\mathcal{P}_t)_{t \geq 0} \) is analytic in \( B_{p,q}^{\gamma}(\mathbb{R}^d) \). Therefore, for any element \( v \in B_{p,q}(\mathbb{R}^d) \), the limit exists and is well defined. Let \( u \in B_{p,\infty}^{\gamma - \delta}(\mathbb{R}^d) \) and \( \{v_n : n \in \mathbb{N}\} \) be a sequence such that
\(v_n \in B^\gamma_{p,q}(\mathbb{R}^d)\) and \(v_n \to u\) in \(B^{\gamma-\delta}_{p,q}(\mathbb{R}^d)\). By a change of variables, we obtain

\[
\lim_{M \to \infty} \left| \frac{1}{2\pi i} \int_{\Gamma(M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \right|_{B^\gamma_{p,q}} \\
\leq \lim_{M \to \infty} \left| \frac{1}{2\pi i} \int_{\Gamma(M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \right|_{B^\gamma_{p,q}} \\
+ \lim_{M \to \infty} \left| \frac{1}{2\pi i} \int_{\Gamma(M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \right|_{B^\gamma_{p,q}}
\]

The Minkowski inequality gives

\[
\ldots \leq \frac{1}{2\pi i} \int_{\Gamma(M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \left| d\lambda \right|_{B^\gamma_{p,q}} \\
+ \frac{1}{2\pi i} \int_{\Gamma(M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \left| d\lambda \right|_{B^\gamma_{p,q}}
\]

(5.4)

In particular, analysing the RHS of the estimate above means analysing the operator \(R(\lambda e^{i\beta}, a(x, D))\). Here, we apply Theorem [31]. Before doing that, we have to carefully calculate the semi-norm of \(\lambda + a(x, \xi)\) in the space of hypoelliptic operators. In doing this, first of all we require following estimate. As in p. 11 in [16], we can see that for \(\lambda \in \Sigma_{\theta + \frac{\pi}{2}}\),

\[
\langle |\lambda|^\frac{1}{\delta} + |\xi| \rangle^\delta \lesssim |\lambda + a(x, \xi)|.
\]

The above result is due to the fact that \(a \in \text{Hyp}_{d+1,0,1}^{\delta,\delta}(\mathbb{R}^d \times \mathbb{R}^d)\), (2.3) and since \(\sigma\) is bounded away from zero. Therefore

\[
|\lambda + a(x, \xi)|^{-\delta} \lesssim \langle |\lambda|^\frac{1}{\delta} + |\xi| \rangle^{-\delta} \lesssim \langle |\xi| \rangle^{-\delta},
\]

for some \(0 < R \leq |\xi|\).

Then we have

\[
\left| \frac{1}{\lambda + a(x, \xi)} \right| \leq \left| \frac{1}{\lambda + \psi(\sigma(x)^T \xi)} \right| \leq \left| \frac{1}{\lambda + (\sigma(x)^T \xi)^{\delta}} \right| \leq C(\sigma, \delta) \langle |\xi| \rangle^{-\delta}.
\]

let \(k \in \{1, \ldots, d\}\). Then

\[
\left| \partial_{x_k} \left[ \frac{1}{\lambda + a(x, \xi)} \right] \right| \leq \left| \frac{\partial_x a(x, \xi)}{(\lambda + a(x, \xi))^2} \right| \leq \left| \frac{\langle |\xi| \rangle^{\delta-1}}{(\lambda + (|\xi|)^{\delta})^2} \right| \leq C(\sigma, \delta) \langle |\xi| \rangle^{-\delta-1}.
\]
Next, let $k, l \in \{1, \ldots, d\}$. Then,
\[
\partial_{\xi_k} \partial_{\xi_l} \left[ \frac{1}{\lambda + a(x, \xi)} \right] = -\frac{\partial_{\xi_k}^2 \partial_{\xi_l} a(x, \xi)}{(\lambda + a(x, \xi))^2} + \frac{2\partial_{\xi_k} a(x, \xi) \partial_{\xi_l} a(x, \xi)}{(\lambda + a(x, \xi))^3}.
\]
Hence, we have
\[
|\partial_{\xi_k} \partial_{\xi_l} \left[ \frac{1}{\lambda + a(x, \xi)} \right]| \leq C(\sigma, \delta) \left\{ \frac{\langle |\xi| \rangle^{\delta - 2}}{(\lambda + \langle |\xi| \rangle)^2} + \frac{\langle |\xi| \rangle^{\delta - 2}}{(\lambda + \langle |\xi| \rangle)^2} \right\} \leq C(\sigma, \delta) \langle |\xi| \rangle^{-\delta - 2}.
\]
Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a multiindex. By observing the pattern of the above derivative we could identify the general derivative $\partial_\xi^\alpha \left[ \frac{1}{\lambda + a(x, \xi)} \right]$ and get the following estimate. There exist $C_1, C_2, \ldots, C_{|\alpha|} > 0$ depending on $\sigma$ and $\delta$ such that
\[
\left| \partial_\xi^\alpha \left[ \frac{1}{\lambda + a(x, \xi)} \right] \right| \leq C_1 |\lambda + a|^{-|\alpha| - 1} \langle \xi \rangle^{\delta|\alpha| - |\alpha|}
+ C_2 |\lambda + a|^{-|\alpha|} \langle \xi \rangle^{\delta(|\alpha| - 1) - |\alpha|} + C_3 |\lambda + a|^{-|\alpha| + 1} \langle \xi \rangle^{\delta(|\alpha| - 2) - |\alpha|} + \cdots + C_{|\alpha|} |\lambda + a|^{-2} \langle \xi \rangle^{\delta - |\alpha|}.
\]
Therefore
\[
\left| \partial_\xi^\alpha \left[ \frac{1}{\lambda + a(x, \xi)} \right] \right| \langle \xi \rangle^{-|\alpha|} \leq C_1 |\lambda + a|^{-|\alpha| - 1} \langle \xi \rangle^{\delta |\alpha| - \delta}
+ C_2 |\lambda + a|^{-|\alpha|} \langle \xi \rangle^{\delta |\alpha| - 2\delta} + C_3 |\lambda + a|^{-|\alpha| + 1} \langle \xi \rangle^{\delta |\alpha| - 3\delta} + \cdots + C_{|\alpha|} |\lambda + a|^{-2}.
\]
Hence by using the fact that
\[
|\lambda + a(x, \xi)|^{-1} \lesssim \langle |\xi| \rangle + |\xi|^{-\delta} \lesssim \langle \xi \rangle^{-\delta},
\]
for some $0 < R \leq |\xi|$, we would get,
\[
\left| \partial_\xi^\alpha \left[ \frac{1}{\lambda + a(x, \xi)} \right] \right| \langle \xi \rangle^{-|\alpha|} \leq (C_1 + C_2 + \ldots C_{|\alpha|}) \langle |\xi| \rangle^{-2\delta} \lesssim \langle |\xi| \rangle^{-2\delta} \leq C(\sigma, \delta) R^{-2\delta}.
\]
This conclude that $\lambda + a(x, \xi) \in \text{Hyp}_{d+1,0,1,0}(\mathbb{R}^d \times \mathbb{R}^d)$.

It remains to estimate the norm of $\lambda + a(x, \xi)$ in $\tilde{\mathcal{A}}_{k_1,k_2,\rho,\delta}^m(\mathbb{R}^d \times \mathbb{R}^d)$ with $k_1 = 2d + 4, k_2 = d + 3$. Due to the fact that one have to take at least once the derivative with respect to $\xi$, the constant $\lambda$ has no influence on the norm in $\tilde{\mathcal{A}}$. Hence
\[
\|\lambda + a\|_{\tilde{\mathcal{A}}_{k_1,k_2,\rho,\delta}^m}
= \sup_{1 \leq |\alpha| \leq k_1, |\beta| \leq k_2} \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} \left| \partial_\xi^\alpha \partial_\xi^\beta (\lambda + a(x, \xi)) \right| \langle |\xi| \rangle^{\rho|\beta| - m} \langle |x| \rangle^{\delta|\alpha|}, \quad a \in \mathcal{A}_{k_1,k_2,\rho,\delta}^m(\mathbb{R}^d \times \mathbb{R}^d),
\]
with $k_1 = 2d + 4, k_2 = d + 3$ and $\rho = 1, \delta = 0$, it is easy to see that $\lambda + a(x, \xi) \in \mathcal{A}_{2d+4,d+3,1,0}^{-1}$. 

Going back to (5.4), we can write
\[
\lim_{M \to \infty} \left| \frac{1}{2\pi} \int_{\Gamma_{\rho}^*(\rho, M)} e^{\lambda t} R(\lambda : a(x, D)) v_n d\lambda \right|_{B_{p,q}^\gamma} 
\leq C(\sigma, \delta) \int_0^\infty e^{-r \sin \theta'} |v_n|_{B_{p,q}^\gamma-\delta} dr 
+ \frac{C(\sigma, \delta) \rho^{-1}}{2t\pi} \int_{\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho \cos \beta} |v_n|_{B_{p,q}^\gamma-\delta} d\beta 
\leq \frac{C(\sigma, \delta)}{2t\pi} |v_n|_{B_{p,q}^\gamma-\delta}.
\]

Taking the limit \( n \to \infty \) we get
\[
\lim_{M \to \infty} \left| \frac{1}{2\pi} \int_{\Gamma_{\rho}^*(\rho, M)} e^{\lambda t} R(\lambda : a(x, D)) u d\lambda \right|_{B_{p,q}^\gamma} \leq \frac{C(\sigma, \delta)}{2t\pi} |u|_{B_{p,q}^\gamma},
\]
which is the assertion.

The following corollary is a consequence of the above Theorem \( 5.1 \).

**Corollary 5.3.** Let \( L \) be a square integrable Lévy process with Blumenthal–Getoor index \( 1 < \delta < 2 \) of order \( 2d + 4 \). Let \( \sigma \in C_b^{d+3}(\mathbb{R}^d) \) be bounded away from zero and \( b \in C_b^{d+3}(\mathbb{R}^d) \). In addition, let \( m(D) \) be a pseudo–differential operator such that \( m(\xi) \in S_{1,0}^\kappa(\mathbb{R}^d \times \mathbb{R}^d) \) with \( \kappa \leq 1 \). Then we have for \( \gamma < \frac{\delta - \kappa}{4} \)
\[
|P_t m(D)u|_{C_b^\gamma(\mathbb{R}^d)} \leq \frac{C}{t} |u|_{L^\infty(\mathbb{R}^d)},
\]
with \( \frac{4d}{\delta - \kappa} < p < \infty \) and \( 1 < q < \infty \).

**Proof.** The proof is similarly to the proof of Theorem \( 5.1 \) therefore we include only essential points in this proof. We already shown that \( \lambda + a(x, \xi) \in \text{Hyp}^{\delta,\delta}_{d+1,0:1,0}(\mathbb{R}^d \times \mathbb{R}^d) \cap A^{-1}_{2d+4,d+3:1,0} \). On the other hand it is easy to see that \( \frac{1}{m(\xi)} \in \text{Hyp}^{\delta,\delta}_{d+1,0:1,0}(\mathbb{R}^d \times \mathbb{R}^d) \cap A^{-1}_{2d+4,d+3:1,0} \). As in above theorem (Theorem \( 5.1 \)), now we need to analyze
\[
|P_t m(D)u|_{B_{p,q}^{\gamma+d}} = \lim_{M \to \infty} \left| \frac{1}{2\pi} \int_{\Gamma_{\rho}^*(\rho, M)} e^{\lambda t} R(\lambda : a(x, D)) m(D) u d\lambda \right|_{B_{p,q}^{\gamma+d}}.
\]
Consider,

\[
\lim_{M \to \infty} \left| \frac{1}{2\pi} \int_{\Gamma_0'(\rho, M)} e^{\lambda} R(\lambda : a(x, D)) u d\lambda \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} \leq \lim_{M \to \infty} \left| \frac{1}{2\pi it} \int_{\rho}^{M} e^{-i(t^{\frac{2}{3}} + \theta')} R(t^{-1} e^{-i(t^{\frac{2}{3}} + \theta')}, a(x, D)) m(D) u e^{i(t^{\frac{2}{3}} + \theta')} dr \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} \\
+ \lim_{M \to \infty} \left| \frac{1}{2\pi it} \int_{\rho}^{M} e^{i(t^{\frac{2}{3}} + \theta')} R(s^{-1} e^{i(t^{\frac{2}{3}} + \theta')}, a(x, D)) m(D) u e^{-i(t^{\frac{2}{3}} + \theta')} dr \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} \\
+ \left| \frac{1}{2\pi it} \int_{-\frac{t^2}{2} - \theta'}^{t^{\frac{2}{3}} + \theta'} e^{\rho \sigma} R(\frac{s}{t} e^{i\beta}, a(x, D)) m(D) u \rho^{-1} e^{i\beta} d\beta \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} \leq \frac{1}{2\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} \left| R(t^{-1} e^{-i(t^{\frac{2}{3}} + \theta')}, a(x, D)) m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} dr \\
+ \frac{1}{2\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} \left| R(t^{-1} e^{-i(t^{\frac{2}{3}} + \theta')}, a(x, D)) m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} dr \\
+ \frac{\rho^{-1}}{2\pi} \int_{-\frac{t^2}{2} - \theta'}^{t^{\frac{2}{3}} + \theta'} e^{\rho \cos \beta} \left| R(\frac{s}{t} e^{i\beta}, a(x, D)) m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} d\beta,
\]

Now we will apply one of our main theorems (Theorem 3.1) to above estimate to get the following result,

\[\cdots \leq \frac{1}{2\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} \left| R(t^{-1} e^{-i(t^{\frac{2}{3}} + \theta')}, a(x, D)) m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} dr \]

since semi-norms \(\|\lambda + a\|_{\text{Hyp}_{d+1,0,1,0}^{\pm, \pm}(\mathbb{R}^{d} \times \mathbb{R}^{d})}\), \(\|\lambda + a\|_{A^{-1}_{d+4,d+3,1,0}}\) do not depend on \(\lambda\), we could reduce the above estimate to

\[\cdots \leq \frac{1}{2\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} \left| m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} dr \]

\[+ \frac{1}{2\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} \left| m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} dr + \frac{\rho^{-1}}{2\pi} \int_{-\frac{t^2}{2} - \theta'}^{t^{\frac{2}{3}} + \theta'} e^{\rho \cos \beta} \left| u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} d\beta,\]

\[\gamma \geq \left[ \frac{1}{\pi} \int_{\rho}^{\infty} e^{-r \sin \theta'} dr + \frac{1}{2\pi} \int_{-\frac{t^2}{2} - \theta'}^{t^{\frac{2}{3}} + \theta'} e^{\rho \cos \beta} d\beta \right] \left| m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}} \leq \frac{C}{t} \left| m(D) u \right|_{B^{\gamma + \frac{d}{p}}_{p, q}}.\]
Again we apply Theorem (3.1) for the last term in the above estimate to get,
\[ \frac{C}{t} |m(D)u|_{B_{p,q}^{\gamma+\frac{d}{p} - \delta}} \leq \frac{C}{t} \|m\|_{\text{Hyp}_{d+1,0,1,0}} |u|_{B_{p,q}^{\gamma+\frac{d}{p} - \delta + \kappa}} \]
\[ \leq \frac{C}{t} |u|_{B_{p,q}^{\gamma+\frac{d}{p} - \delta + \kappa}}. \]

The last inequality holds due to the fact that \( \gamma + \frac{d}{p} - \delta + \kappa < 0 \).

Now we are following the same argument of the proof of corollary (5.1) to complete this argument. Fix \( \gamma < (n-1)\delta - \kappa - d \) and let \( p \geq 1 \) such that \( \frac{d}{p} < (n-1)\delta - \kappa - \gamma \). Fix \( 1 \leq q < \infty \) arbitrary. Then, we know, firstly (see [39, p. 14])
\[ C^\gamma(\mathbb{R}^d) = B_{\infty,\infty}^\gamma(\mathbb{R}^d). \]

Secondly, we apply the embedding \( B_{p,q}^{\gamma+\frac{d}{p}}(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^\gamma(\mathbb{R}^d) \) (see [39] Chapter 2.2.3, [1] Section 6.4), and, finally, we apply Theorem (5.1) \( n \) times to get,
\[ \left| \mathcal{P}_t m(D)u \right|_{C_b^\gamma(\mathbb{R}^d)} \leq \left| \mathcal{P}_t m(D)u \right|_{B_{\infty,\infty}^\gamma(\mathbb{R}^d)} \leq \left| \mathcal{P}_t m(D)u \right|_{B_{p,q}^{\gamma+\frac{d}{p}}(\mathbb{R}^d)} \]
\[ = \left| \left( \mathcal{P}_n \right)^n m(D)u \right|_{B_{p,q}^{\gamma+\frac{d}{p}}(\mathbb{R}^d)} \leq \left( \frac{nC}{t^n} \right)^n |u|_{B_{p,q}^{\gamma+\frac{d}{p} - n\delta + \kappa}(\mathbb{R}^d)}. \]

By means of [11, Exercise 6.25, Corollary 6.14],
\[ B_{p,1}^{\theta}(\mathbb{R}^d) \hookrightarrow B_{p,1}^{\theta}(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d), \]
where \( \theta = n\delta - \frac{d}{p} - \gamma + \kappa \) and \( p^* = \frac{p}{p-1} \). Now by applying [11, Lemma 6.5] and duality property of the Besov spaces gives that
\[ B_{p,1}^{\theta}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{-\theta}(\mathbb{R}^d) = B_{p,\infty}^{\gamma+\frac{d}{p} - n\delta + \kappa}(\mathbb{R}^d) \hookrightarrow B_{p,1}^{\gamma+\frac{d}{p} - (n-1)\delta + \kappa}(\mathbb{R}^d). \]

Finally by fixing \( q = 1 \) we would get,
\[ \left| \mathcal{P}_t m(D)u \right|_{C_b^\gamma(\mathbb{R}^d)} \leq \left( \frac{nC}{t^n} \right)^n |u|_{B_{p,1}^{\gamma+\frac{d}{p} - (n-1)\delta + \kappa}(\mathbb{R}^d)} \leq \left( \frac{nC}{t^n} \right)^n |u|_{L^\infty(\mathbb{R}^d)}. \]

This completes the proof. \( \square \)

If \( L \) is an \( \alpha \) stable process, the problem appears that only the moments up to \( p < \alpha \) are bounded. Therefore, the symbol is in higher dimensions not uniformly differentiable up to order \( d+1 \) in any neighborhood of \( \xi = 0 \). However, if \( \alpha > 0 \), then this problem can be solved.

**Corollary 5.4.** Let \( L \) be a Lévy process \( L \) with Blumenthal–Getoor index \( 1 < \delta < 2 \) of order \( 2d + 4 \). Let \( \sigma \in C_b^{d+3}(\mathbb{R}^d) \) be bounded away from zero and \( b \in C_b^{d+3}(\mathbb{R}^d) \). Then the process defined by (6.1) is strong Feller.

Similarly, we get also estimates for the density.
Corollary 5.5. Let us assume that $L$ is a Lévy process with Blumenthal–Getoor index $1 < \delta < 2$ of order $2d + 4$, $\sigma \in C_0^{d+3}(\mathbb{R}^d)$ is bounded away from zero, and $b \in C_0^{d+3}(\mathbb{R}^d)$. Then, the density of the process is $n$ times differentiable. In particular, for any $\theta = [n\delta - d]$, $n \in \mathbb{N}$ and $n \geq d$, we have

$$\frac{\partial^\alpha}{\partial y^\alpha} p_t(x, y) \leq \frac{C(n, d)}{t^n},$$

where $\alpha$ is a multiindex of length $\theta$.

Proof of Corollary 5.5. In order to deal with the large jumps we decompose the Lévy process into one Lévy process recollecting the jumps smaller than one and one Lévy process, recollecting the jumps larger than one. Therefore, let $\nu_0$ be the Lévy measure defined by

$$\nu_0 : \mathcal{B}(\mathbb{R}^d) \ni U \mapsto \nu(U \cap Z_0),$$

and $\nu_1$ be the Lévy measure defined by

$$\nu_1 : \mathcal{B}(\mathbb{R}^d) \ni U \mapsto \nu(U \cap Z_1),$$

where $Z_0 = \{ z \in \mathbb{R}^d : |z| \leq 1 \}$ and $Z_1 = \{ z \in \mathbb{R}^d : |z| > 1 \}$. Since the proof of this theorem mainly rely on the analysis of the decomposition of the small and large jumps it is important to decompose the probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$. Let $\tilde{n}_0$ be a compensated Poisson random measure on $(Z_0 \times \mathbb{R}_+, \mathcal{B}(Z_0) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^0 = (\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}_{t \in [0, T]}, \mathbb{P}^0)$ with intensity measure $\nu_0$ where

$$\mathcal{F}^0 = \sigma \{ \eta(B, [0, s]) : B \in \mathcal{B}(Z_0), s \in [0, T] \}$$

and for $0 \leq t \leq T$

$$\mathcal{F}^0_t = \sigma \{ \eta(B, [0, s]) : B \in \mathcal{B}(Z_0), s \in [0, t] \}.$$

Furthermore, let $\tilde{n}_1$ be a compensated Poisson random measure on $(Z_1 \times \mathbb{R}_+, \mathcal{B}(Z_1) \otimes \mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^1 = (\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t \in [0, T]}, \mathbb{P}^1)$ with finite intensity measure $\nu_1$ where

$$\mathcal{F}^1 = \sigma \{ \eta(B, [0, s]) : B \in \mathcal{B}(Z_1), s \in [0, T] \}$$

and for $0 \leq t \leq T$

$$\mathcal{F}^1_t = \sigma \{ \eta(B, [0, s]) : B \in \mathcal{B}(Z_1), s \in [0, t] \}.$$

Let $\Omega := (\Omega^0 \times \Omega^1)$, $\mathcal{F} := \mathcal{F}^0 \otimes \mathcal{F}^1$, $\mathcal{F}_t := \mathcal{F}^0_t \otimes \mathcal{F}^1_t$, $P = \mathbb{P}^0 \otimes \mathbb{P}^1$ and $\mathbb{E} = \mathbb{E}^0 \otimes \mathbb{E}^1$. We denote the to $\nu_0$ and $\nu_1$ associated Lévy processes by $L_0$ and $L_1$. It is clear by the independent scattered property of a Poisson random measure, that $L_0$ and $L_1$ are independent. Since $\nu_1$ is a finite measure, $L_1$ can be represented as a sum of its jumps. In particular, let $\rho = \nu_1(\mathbb{R}^d)$, $\{\tau_n : n \in \mathbb{N}\}$ be a family of independent exponential distributed random variables with parameter $\rho$,

$$T_n = \sum_{j=1}^n \tau_j, \quad n \in \mathbb{N},$$

(5.6) and $\{N(t) : t \geq 0\}$ be the counting process defined by

$$N(t) := \sum_{j=1}^\infty 1_{[\tau_j, \infty)}(t), \quad t \geq 0.$$
Observe, for any $t > 0$, $N(t)$ is a Poisson distributed random variable with parameter $\rho t$. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of independent, $\nu_0/\rho$ distributed random variables. Then the Lévy process $L_1$ given by (see [10] Chapter 3)

$$L_1(t) = \int_0^t \int_{Z_1} z \, \tilde{n}_1(dz, ds), \quad t \geq 0,$$

can be represented as

$$L_1(t) = \begin{cases} -z_0 t & \text{for } N(t) = 0, \\ \sum_{j=1}^{N(t)} Y_j - z_0 t & \text{for } N(t) > 0, \end{cases}$$

where $z_0 = \int_{\mathbb{R}^d} z \nu_1(dz)$. Let $(\mathcal{P}_t^0)$ the Markovian semigroup of the solution process $X^x_0$ given by

$$\left\{ \begin{array}{ll} dX_t^0(t) &= b(X_0(t-)) \, dt + \sigma(X_0(t-)) \, dL_0(t) \\ X_0^0(0) &= x, \quad x \in \mathbb{R}^d. \end{array} \right.$$

Now we have for $u(t) = \mathbb{E}(\phi(X(t)))$, where $X$ is the solution to the original equation (6.1) and the following identity

$$u(t) = \mathbb{E}(\mathcal{P}_t^0 \phi)(x) + \sum_{i=1}^{N(t)} \mathcal{P}_{t-T_i}^0 B_{Y_i} \mathbb{E}(\mathcal{P}_{T_i-}^0 \phi)(x) - \int_0^t \mathcal{P}_{t-s}^0 Du(s)[z_0] \, ds,$$

where $(B_y \phi)(x) = \phi(x + y) - \phi(x)$. To verify formula (5.8), observe that in the time interval $[0, T_1)$ the solution of $u$ is given by

$$u(t) = (\mathcal{P}_t^0 \phi)(x) + \int_0^t \mathcal{P}_{t-s}^0 Du(s)[z_0] \, ds, \quad t \in [0, T_1].$$

In particular, $u$ solves on the time interval $[0, T_1)$

$$\left\{ \begin{array}{ll} \dot{u}(t) &= a(x, D)u(t) + Du(t)z_0, \quad t \in [0, T_1), \\ u(0) &= \phi. \end{array} \right.$$

Let us denote the solution of (5.9) on the first time interval $[0, T_1]$ by $u_1$. At time $T_1$ the first large jump occurred. Hence, on the time interval $[T_1, T_2)$, $u$ solves

$$\left\{ \begin{array}{ll} \dot{u}(t) &= a(x, D)u(t) + Du(t)z_0, \quad t \in (T_1, T_2), \\ u(T_1) &= \mathbb{E}u_1(T_1, \cdot + Y_1). \end{array} \right.$$

Let us denote the solution of (5.10) by $u_2$. The variation of constant formula gives for $t \in (T_1, T_2)$

$$u_2(t) = \mathcal{P}_{t-T_1}^0 u_2(T_1) + \int_{T_1}^t \mathcal{P}_{t-s}^0 Du_2(s) \, z_0 \, ds.$$

Let us put

$$u(t) := \begin{cases} u_1(t), & \text{if } t \in [0, T_1), \\ u_2(t), & \text{if } t \in [T_1, T_2). \end{cases}$$

Since

$$u_1(T_1) = \mathcal{P}_{T_1}^0 \phi + \int_0^{T_1} \mathcal{P}_{T_1-s}^0 Du(s) \, z_0 \, ds,$$
and $u_2(T_1, x) = u_1(T_1, x + Y_1)$, $x \in \mathbb{R}^d$, it follows
\[
\begin{align*}
    u(t) &= \mathcal{P}_{t-T_1}^0 \mathcal{P}_{T_1}^0 \phi + \mathcal{P}_{t-T_1}^0 \int_0^{T_1} \mathcal{P}_{T_1-s}^0 D u(s) z_0 \, ds \\
    &+ \int_{T_1}^t \mathcal{P}_{t-s}^0 D u(s) z_0 \, ds + \mathbb{E}[\mathcal{P}_{t-T_1}^0 u(T_1^-) + Y_1] - \mathbb{E}[\mathcal{P}_{t-T_1}^0 u(T_1^-)].
\end{align*}
\]

Thus, we get for
\[
\begin{align*}
    \mathbb{E} \sum_{i=1}^{N(t)} \mathcal{P}_{t-s}^0 B_{Y_i} u(T_i) &= \sum_{k=1}^{\infty} \mathbb{P}(N(t) = k) \mathbb{E} \left[ \sum_{i=1}^{k} \mathcal{P}_{t-T_i}^0 B_{Y_i} u(T_i) \mid N(t) = k \right] \\
    &= \sum_{k=1}^{\infty} \mathbb{P}(N(t) = k) \mathbb{E} \left[ \sum_{i=1}^{k} \mathcal{P}_{T_i-T_i}^0 B_{Y_i} u(T_i) \right] \\
    &= \sum_{k=1}^{\infty} \mathbb{P}(N(t) = k) \mathbb{E} \left[ \int_0^t \mathcal{P}_{t-s}^0 B_{Y} u(s) \right], \end{align*}
\]

where $Y$ is distributed as $\nu_0/\rho$. Thus, we get for $\gamma \leq \delta - 1$
\[
|u(t)|_{B^\gamma_{\infty, \infty}} \leq \frac{1}{t} |\phi|_{B^0_{\infty, \infty}} + \int_0^t |\mathcal{P}_{t-s}^0 B_y u(s)|_{B^{\gamma}_{\infty, \infty}} \, ds + |z_0| \int_0^t |\mathcal{P}_{t-s}^0 D u(s)|_{B^{\gamma}_{\infty, \infty}} \, ds
\]

This leads to
\[
|u(t)|_{B^{\gamma}_{\infty, \infty}} \leq \frac{1}{t^{p+1}} |\phi|_{B^{p+1}_{\infty, \infty}} + \frac{C_1}{t^{p+1}} \left[ \int_0^t |u(s)|_{B^{p+1}_{\infty, \infty}} \, ds \right]^{\frac{\gamma-1}{p}}.
\]
A simple application of the Grönwall’s Lemma gives

$$|u(t)|_{B^0_{\infty,\infty}} \leq C(t, p, n)|\phi|_{B^0_{\infty,\infty}}.$$ 

By the definition of $B^0_{\infty,\infty}(\mathbb{R}^d)$, it follows that the process is strong Feller.

6. The second application: weak error of estimates

Given the intensity measure, in most of the cases one does not know the distribution of $L_t$ for a fixed time point $t \geq 0$. However, simulating stochastic differential equations driven by Lévy noise by the explicit or implicit Euler-Maruyama scheme, one has to simulate the increments $\Delta^n L := L(n\tau) - L((n-1)\tau)$ for $\tau > 0$ small. Here, one can apply several strategies to simulate the random variables $\Delta^n L$, $n \in \mathbb{N}$, to generate a so called Lévy walk given by

$$L : [0, \infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}^d} \zeta \eta(d\zeta, ds).$$

One way is to cut off the jumps being smaller than a given $\varepsilon$ and to simulate the corresponding compound Poisson process directly. One may replace the small jumps by a Gaussian random variable. Here, one gets a new Lévy process, denoted by $\hat{L}_\varepsilon$.

Cutting off the small jumps can be done in different ways. E.g. if the measure is decomposable, one would choose to cut off all jumps being in the unit ball with radius $\varepsilon$, denoted in the following by $B_\varepsilon$. Then, $\Delta^n L$ is the sum over $N$ random variables $\{Y_1, \ldots, Y_n\}$, where $N$ is Poisson distributed with parameter $\nu(\mathbb{R}^d \setminus B_\varepsilon)$ and the random variables $\{Y_1, \ldots, Y_n\}$ are identical and independent distributed with

$$Y_i = \frac{\nu(\cdot \cap B_\varepsilon)}{\nu(\mathbb{R}^d \setminus B_\varepsilon)}.$$ 

Now, one can replace the neglected small jumps by increments of a Wiener process. Here, the rate of convergence will not improved. However, since the Markovian semigroup of the approximate process preserves the analyticity, the error estimate is valid for a wider range of functions. In particular, if the small jumps are approximated by a Wiener process, we can estimate the error of the entity $\mathbb{P}(X \geq a) = \mathbb{E}1_{[a, \infty)}(X)$, where $X$ solves the stochastic differential equation

(6.1) \[
\begin{align*}
    dX^x(t) &= b(X^x(t-))dt + \sigma(X^x(t-))dL(t) \\
    X^x(0) &= x, \quad x \in \mathbb{R}^d.
\end{align*}
\]

Since $1_{[a, \infty)}$ is not differentiable function, the error estimate with respect to $\varepsilon$ will not be better if we replace the small jumps by a Wiener noise. However, due to the fact that the approximated Semigroup is smoothing, one can treat functions which are less regular. Here $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous.

To be more precise, fix a truncation parameter $0 < \varepsilon < 1$ and let $B_\varepsilon = \{z \in \mathbb{R}^d : |z| \leq \varepsilon\}$, where $| \cdot |$ defined a metric on $\mathbb{R}^d$. Let us define the approximate Lévy measure

$$\nu^\varepsilon : \mathcal{B}(\mathbb{R}) \ni B \mapsto \nu\left(B \cap \left(\mathbb{R}^d \setminus B_\varepsilon\right)\right).$$
Let \( \hat{L} \) be the Lévy process induced by truncating small jumps, i.e., the Lévy process having intensity \( \nu^\xi \). So far, small jumps are truncated, but may be approximated by a Wiener process. For this purpose, at each time-step \( k \in \mathbb{N} \), we generate a Gaussian random variable \( \Delta_k^\varepsilon W_\varepsilon \), where

\[
(6.2) \quad \Delta_k^\varepsilon W_\varepsilon \sim \mathcal{N}(0, \Sigma^2(\varepsilon)\tau), \quad \text{and} \quad \Sigma(\varepsilon) = \int_{[-\varepsilon,\varepsilon]^d} \langle y, y \rangle \nu(dy) .
\]

Then, the increments of the Lévy process are approximated by

\[
\left( \hat{\Delta}_0^\varepsilon L_{\varepsilon,1} + \Delta_0^\varepsilon W_\varepsilon, \hat{\Delta}_1^\varepsilon L_{\varepsilon,1} + \Delta_1^\varepsilon W_\varepsilon, \hat{\Delta}_2^\varepsilon L_{\varepsilon,1} + \Delta_2^\varepsilon W_\varepsilon, \ldots, \hat{\Delta}_k^\varepsilon L_{\varepsilon,1} + \Delta_k^\varepsilon W_\varepsilon, \ldots \right).
\]

In the following we give an error estimate of the two processes \( X^\varepsilon \) and \( \hat{X}^\varepsilon \), where \( X^\varepsilon \) solves (6.1) and \( \hat{X}^\varepsilon \) solves

\[
(6.3) \quad \left\{ \begin{array}{ll} dX^\varepsilon(t) &= b(X^\varepsilon(t-))dt + \sigma(X^\varepsilon(t-))d\hat{L}^\varepsilon(t) + \sigma(X^\varepsilon(t-))dW_\varepsilon(t) \\ X^\varepsilon(0) &= x \end{array} \right., \quad x \in \mathbb{R}^d .
\]

Again let us define the corresponding Markovian semigroups. Let \( (\mathcal{P}_t)_{t \geq 0} \) be the Markovian semigroup of the process (6.1), i.e.

\[ \mathcal{P}_t \phi(x) := \mathbb{E} \phi(X^\varepsilon(t)), \quad t \geq 0. \]

and let \( (\hat{\mathcal{P}}^\varepsilon_t)_{t \geq 0} \) be the Markovian semigroup of the process (6.3), i.e.

\[ \hat{\mathcal{P}}^\varepsilon_t \phi(x) := \mathbb{E} \phi(\hat{X}^\varepsilon(t)), \quad t \geq 0. \]

The first proposition shows that the semigroup \( (\hat{\mathcal{P}}^\varepsilon_t)_{t \geq 0} \) of the approximation \( \hat{X}^\varepsilon \) is indeed analytic.

**Proposition 6.1.** Let us assume that \( \sigma \in \mathcal{C}^{d+3}(\mathbb{R}^d) \), \( b \in \mathcal{C}^{d+3}_b(\mathbb{R}^d) \), and \( \sigma \) is bounded away from zero. Then for all \( 1 \leq p,q < \infty \), the Markovian semigroup \( (\hat{\mathcal{P}}^\varepsilon_t)_{t \geq 0} \) is an analytic semigroup in \( B^{m}_{p,q}(\mathbb{R}^d) \) for all \( m \in \mathbb{R} \). Let \( m(D) \) be a pseudo–differential operator with symbol \( m \), where \( m(\xi) \in S^\kappa_{1,0}(\mathbb{R}^d \times \mathbb{R}^d) \) with \( \kappa < 2 \). Then we have for for \( u \in B^{m}_{p,q}(\mathbb{R}^d) \)

\[
(6.4) \quad \left\| \hat{\mathcal{P}}^\varepsilon_t m(D)u \right\|_{B^{m}_{p,q}(\mathbb{R}^d)} \leq \frac{C}{t} \| u \|_{B^{m}_{p,q}(\mathbb{R}^d)} .
\]

**Theorem 6.1.** Let us assume that \( \sigma \in \mathcal{C}^{d+3}_b(\mathbb{R}^d) \), \( b \in \mathcal{C}^{d+3}_b(\mathbb{R}^d) \), and \( \sigma \) is bounded away from zero, then for \( \delta \in (1,2) \) (Blumenthal–Getoor index of \( L \)), \( r_1,r_2 \in (0,1) \) such that \( r_1 + r_2 > 1 \) and \( 2r_1 > r_2 \) with \( \delta_1 = \frac{\delta r_1}{2} \) and \( \delta_2 = \delta(r_1 - \frac{r_2}{2}) \),

\[
\left\| \mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t \right\|_{L(B^{-\delta_2}_{\infty,\infty},B^{\delta_1}_{\infty,\infty})} \leq C t^{\delta(r_1 + r_2) - \frac{1}{2}} \varepsilon^{(2-\delta)} .
\]

Note, that \( (\mathcal{P}_t)_{t \geq 0} \) has generator given by the symbol \( a(x,\xi) = \psi(\sigma(x)^T\xi) \) and \( (\hat{\mathcal{P}}^\varepsilon_t)_{t \geq 0} \) has generator given by the symbol \( \hat{a}_\varepsilon(x,\xi) = \psi_\varepsilon(\sigma(x)^T\xi) - \Sigma(\varepsilon)\langle \xi, \sigma^T(x)\xi \rangle \), where

\[
\psi_\varepsilon(\xi) = \int_{\mathbb{R}^d \setminus B_\varepsilon} \left( e^{i\langle y,\xi \rangle} - 1 \right) \nu(dy), \quad \xi \in \mathbb{R}^d ,
\]
where \((y,y)\) is a varying point of the above estimate is due to the first estimate in the proof of the Lemma 15.1.7 in [29].

Since \(\nu(\cdot)\) is a \(\delta\)-stable \(\mathbb{L}^\gamma\) measure, we could apply result 14.7 in p.79 [40] to get the last inequality from the top of the above estimate is due to the first estimate in the proof of the Lemma 15.1.7 in [29]. Since \(\nu(\cdot)\) is a \(\delta\)-stable \(\mathbb{L}^\gamma\) measure, we could apply result 14.7 in p.79 [40] to get the last estimate.

Now let us consider \(\gamma = 1\). For \(1 \leq j \leq d\),

\[
\left| \frac{d}{d\xi_j} [a(x, \xi) - a_x(x, \xi)] \right| = \left| \frac{d}{d\xi_j} \left[ \psi(\sigma(x)^T \xi) - \psi_x(\sigma(x)^T \xi) + \psi(\sigma(x)^T \xi) \right] \right|
\]

Proof. Firstly, we have to show that \(|a(x, \xi) - a_x(x, \xi)| \leq C \varepsilon^{2-\delta}\). In particular, we have to show for any multiindex \(\gamma\) with \(|\gamma| \leq d + 1\), we have \(|\frac{\partial^\gamma}{\partial \xi^\gamma} [a(x, \xi) - a_x(x, \xi)]| \leq C \varepsilon^{|\gamma| - \delta}\). That means \(a(x, \xi) - a_x(x, \xi) \in S^2_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)\). Throughout this proof, \(C\) is denoted as a varying constant. Let us start with \(\gamma = 0\).

\[
|a(x, \xi) - a_x(x, \xi)| = |\psi(\sigma(x)^T \xi) - \psi_x(\sigma(x)^T \xi) + \psi(\sigma(x)^T \xi) - \psi_x(\sigma(x)^T \xi)|
\]

\[
= \left| \int_{\mathbb{R}^d \setminus \{0\}} \left[ e^{i(y, \sigma(x)^T \xi)} - 1 - i(y, \sigma(x)^T \xi) \right] \nu(dy) \right| - \left| \int_{\mathbb{R}^d \setminus B_\varepsilon} \left[ e^{i(y, \sigma(x)^T \xi)} - 1 \right] \nu(dy) \right|
\]

\[
+ \left| \int_{B_\varepsilon} \left( \xi, \sigma(x)^T \xi \right)(y, y) \nu(dy) \right|
\]

\[
= \left| \int_{B_\varepsilon} \left[ e^{i(y, \sigma(x)^T \xi)} - 1 - i(y, \sigma(x)^T \xi) \right] \nu(dy) \right|
\]

\[
\leq \int_{B_\varepsilon} \left[ e^{i(y, \sigma(x)^T \xi)} - 1 - i(y, \sigma(x)^T \xi) \right] \nu(dy) + C_1(d) \int_{B_\varepsilon} \left( \xi, \sigma(x)^T \xi \right) \int_{B_\varepsilon} |y|^2 \nu(dy)
\]

\[
\leq \left[ |\xi|^2 |\sigma(x)^T|^2 + C_1(d) |\xi|^2 |\sigma(x)^T| \right] \int_{B_\varepsilon} |y|^2 \nu(dy)
\]

\[
= C |\xi|^2 \int_{B_\varepsilon} |y|^2 \nu(dy) \leq C |\xi|^2 \int_{B_\varepsilon} |y|^2 \lambda(d\eta) \int_{B_\varepsilon} \varepsilon^2 \frac{d\eta}{\eta^{1+\delta}} \leq C |\xi|^2 \varepsilon^{2-\delta},
\]

where \((y,y) = [y_1 y_m]_{1 \leq i \leq d, 1 \leq m \leq d}\) and \(|y| = \sqrt{y_1^2 + \cdots + y_d^2}\). The second inequality from the top of the above estimate is due to the first estimate in the proof of the Lemma 15.1.7 in [29]. Since \(\nu(\cdot)\) is a \(\delta\)-stable \(\mathbb{L}^\gamma\) measure, we could apply result 14.7 in p.79 [40] to get the last estimate.
\[
\begin{align*}
&= \left| \int_{B_{\varepsilon}} \sum_{k=1}^{d} \left[ i y_k \sigma_{j_k}(x) \left[ e^{i(y,\sigma(x)^{T}\xi)} - 1 \right] + 2 \left[ \xi_k \sigma_{j_k}(x) \right] (y, y) \right] \nu(dy) \right| \\
\leq & \int_{B_{\varepsilon}} \sum_{k=1}^{d} \left[ |y_k \sigma_{j_k}(x)| \left| e^{i(y,\sigma(x)^{T}\xi)} - 1 - i(y, \sigma(x)^{T}\xi) \right| \\
& \quad + |y_k \sigma_{j_k}(x)| |(y, \sigma(x)^{T}\xi)| + 2 \left[ \xi_k \sigma_{j_k}(x) \right] (y, y) \right] \nu(dy) \\
\leq & \int_{B_{\varepsilon}} \sum_{k=1}^{d} \left[ |y_k \sigma_{j_k}(x)| |(y, \sigma(x)^{T}\xi)|^2 + |y_k \sigma_{j_k}(x)| |(y, \sigma(x)^{T}\xi)| + 2 \left[ \xi_k \sigma_{j_k}(x) \right] (y, y) \right] \nu(dy) \\
\leq & \int_{B_{\varepsilon}} \sum_{k=1}^{d} |y_k \sigma_{j_k}(x)| \left[ |(y, \sigma(x)^{T}\xi)| + |(y, \sigma(x)^{T}\xi)|^2 \right] \nu(dy) + |\xi| \left| \sigma(x)^{T}\right| \int_{B_{\varepsilon}} |y|^2 \nu(dy) \\
\leq & C \sum_{n=2}^{3} \int_{B_{\varepsilon}} |\xi|^n |y|^n \nu(dy) \leq C \sum_{n=2}^{3} \int_{\mathbb{R}^d} |\eta|^n \lambda(d\eta) \int_{0}^{\varepsilon} r^n \frac{dr}{r^{1+\delta}} \leq C \sum_{n=2}^{3} |\xi|^n \varepsilon^{n-\delta},
\end{align*}
\]

where \((y, y) = [y_1 y_m]_{1 \leq i \leq d, 1 \leq m \leq d}\) and \(|y| = \sqrt{y_1^2 + \cdots + y_d^2}\). Above estimate also completed according to the similar steps as in previous estimate. Now we estimate \(|\frac{\partial\gamma}{\partial\alpha_1 \xi_1 \cdots \partial\alpha_d \xi_d} [a(x, \xi) - a_{\varepsilon}(x, \xi)]|\) with \(2 \leq |\gamma| \leq d + 1\).

\[
\begin{align*}
&\left| \frac{\partial\gamma}{\partial\alpha_1 \xi_1 \cdots \partial\alpha_d \xi_d} [a(x, \xi) - a_{\varepsilon}(x, \xi)] \right| \\
&= \left| \frac{\partial\gamma}{\partial\alpha_1 \xi_1 \cdots \partial\alpha_d \xi_d} \left[ \psi(\sigma(x)^{T}\xi) - \psi(\sigma(x)^{T}\xi) + \Sigma(\varepsilon)(\xi, \sigma(x)^{T}\xi) \right] \right| \\
&= \left| \int_{B_{\varepsilon}} \frac{\partial\gamma}{\partial\alpha_1 \xi_1 \cdots \partial\alpha_d \xi_d} \left[ e^{i(y,\sigma(x)^{T}\xi)} - 1 - i(y, \sigma(x)^{T}\xi) + (\xi, \sigma(x)^{T}\xi)(y, y) \right] \nu(dy) \right| \\
&\leq \left| \int_{B_{\varepsilon}} \sum_{j=1}^{d} \Pi_{j=1} \left[ y_k \sigma_{j_k}(x) \right] e^{i(y,\sigma(x)^{T}\xi)} \nu(dy) \right| + |C(x)| \int_{B_{\varepsilon}} |(y, y)| \nu(dy) \\
&\leq C \int_{B_{\varepsilon}} \sum_{j=1}^{d} |y_j|^{\gamma_j} \nu(dy) + |C(x)| \int_{B_{\varepsilon}} |(y, y)| \nu(dy) \\
&\quad + C \int_{B_{\varepsilon}} |y|^{\gamma_j} \nu(dy) + |C(x)| \int_{B_{\varepsilon}} |y|^2 \nu(dy) \\
&\leq C \int_{B_{\varepsilon}} |y|^{\gamma_j} \nu(dy) \leq C \int_{\mathbb{R}^d} |\eta|^{\gamma_j} \lambda(d\eta) \int_{0}^{\varepsilon} r^{\gamma_j} \frac{dr}{r^{1+\delta}} \leq C \varepsilon^{\gamma_j - \delta},
\end{align*}
\]

where \((y, y) = [y_1 y_m]_{1 \leq i \leq d, 1 \leq m \leq d}, |y| = \sqrt{y_1^2 + \cdots + y_d^2}\) and \(C(x) = f(\sigma_j(x))\) if \(|\gamma| = 2\) and \(C(x) = 0\) if \(|\gamma| \geq 3\).
Observe [36, Proposition 3.1.2, in p.77] then we can write
\[ \left[ \mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t \right] u = \int_0^t \hat{\mathcal{P}}^\varepsilon_{t-s} \left[ a(x,D) - a_\varepsilon(x,D) \right] \mathcal{P}_s u \, ds. \]

Now let \( r_1, r_2 \in (0, 1) \) such that \( r_1 + r_2 > 1 \) and \( 2r_1 > r_2 \). Then for \( u \in B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}(\mathbb{R}^d) \) we get
\[
\left| \left[ \mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t \right] u \right|_{B_{\infty,\infty}^{\delta r_2}} \leq \int_0^t \left| \hat{\mathcal{P}}^\varepsilon_{t-s} \left[ a(x,D) - a_\varepsilon(x,D) \right] \mathcal{P}_s u \right|_{B_{\infty,\infty}^{\delta r_2}} \, ds \\
\leq \int_0^t \left\| \hat{\mathcal{P}}^\varepsilon_{t-s} \left[ a(x,D) - a_\varepsilon(x,D) \right] \right\|_{L\left(B_{\infty,\infty}^{\delta r_2},B_{\infty,\infty}^{\delta r_2}\right)} \left\| a(x,D) - a_\varepsilon(x,D) \right\|_{L\left(B_{\infty,\infty}^{\delta r_2},B_{\infty,\infty}^{\delta r_2}\right)} \times \left| \mathcal{P}_s \right|_{B_{\infty,\infty}^{-\delta r_1}} \int_0^t (t-s)^{-\delta r_2} s^{-\delta r_1} \, ds \\
\leq \left\| a(x,D) - a_\varepsilon(x,D) \right\|_{L\left(B_{\infty,\infty}^{\delta(r_1-r_2)-1},B_{\infty,\infty}^{\delta r_2}\right)} \left| \mathcal{P}_s \right|_{B_{\infty,\infty}^{-\delta(r_1-r_2)-1}} \int_0^t (t-s)^{-\delta r_2} s^{-\delta r_1} \, ds \\
\leq t^{\delta(r_1+r_2)-1} \left\| a(x,D) - a_\varepsilon(x,D) \right\|_{L\left(B_{\infty,\infty}^{\delta r_2},B_{\infty,\infty}^{\delta r_2}\right)} \left| \mathcal{P}_s \right|_{B_{\infty,\infty}^{-\delta(r_1-r_2)-1}} B(1 - \delta r_1, 1 - \delta r_2) \\
\leq Ct^{\delta(r_1+r_2)-1} \left| \mathcal{P}_s \right|_{B_{\infty,\infty}^{-\delta(r_1-r_2)-1}}.
\]

Since the first result that we obtained in this proof \( \left| a(x, \xi) - a_\varepsilon(x, \xi) \right| \leq |\xi|^2 \varepsilon^{(2-\delta)} \), we could confirm that from the Theorem 6.19 in [1] that for any \( m \in \mathbb{R} \)
\( (a(x,D) - a_\varepsilon(x,D)) : B_{2+2m,\infty}(\mathbb{R}^d) \to B_{\infty,\infty}^m(\mathbb{R}^d) \)
is a bounded operator. Therefore we have
\[ \left\| a(x,D) - a_\varepsilon(x,D) \right\|_{L\left(B_{\infty,\infty}^{\delta r_2},B_{\infty,\infty}^{\delta r_2}\right)} \leq C \varepsilon^{(2-\delta)}. \]

Then we get the last inequality of the above calculation. Hence for some positive small parameters \( \delta_1 = \frac{\delta r_2}{2} \) and \( \delta_2 = \delta(r_1-\frac{r_2}{2}) \)
\[
\left| \mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t \right|_{C^0_b} \leq \left| \mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t \right|_{B_{\infty,\infty}^{\delta r_2}} \leq t^{\delta(r_1+r_2)-1} \varepsilon^{(2-\delta)} |\mathcal{P}_s|_{C^0_b}.
\]
\( \square \)
APPENDIX A. Symbol classes and pseudo–differential operators

In this section we shortly introduce pseudo–differential operators and their symbols. In addition we introduce the definitions and Theorems which are necessary to our purpose. For a detailed introduction on pseudo–differential operators and their symbols in the context of partial differential equations we recommend the books [1, 35, 45, 49], in the context of Markov processes we recommend the books [26, 27, 28] or the survey [4].

In order to treat pseudo–differential operators different classes of symbols have been introduced. Here, we closely follow the definition of [1].

Definition A.1. Let $\rho, \delta$ two real numbers such that $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. Let $S^m_{\rho, \delta}(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x, \xi)$ is infinitely often differentiable, i.e. $a \in C^\infty_b(\mathbb{R}^d \times \mathbb{R}^d)$;
- for any two multi-indices $\alpha$ and $\beta$ there exists $C_{\alpha, \beta} > 0$ such that
  $$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} |\langle |\xi| \rangle|^{m-\rho|\alpha|} |\langle |x| \rangle|^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$  

We can any function $a(x, \xi)$ belonging to $\cup_{m \in \mathbb{R}} S^m_{0, 0}(\mathbb{R}^d, \mathbb{R}^d)$ a symbol. For many estimates, one does not need that the function is infinitely often differentiable. In fact, it is often only necessary to know the estimates with respect to $\xi$ and $x$ up to a certain order. For this reason, one introduces also the following classes.

Definition A.2. (compare [49, p. 28]) Let $m \in \mathbb{R}$. Let $A^m_{k_1, k_2; \rho, \delta}(\mathbb{R}^d, \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x, \xi)$ is $k_1$–times differentiable in $\xi$ and $k_2$ times differentiable in $x$;
- for any two multi-indices $\alpha$ and $\beta$ with $|\alpha| \leq k_1$ and $|\beta| \leq k_2$, there exists a constant $C_{\alpha, \beta} > 0$ depending only on $\alpha$ and $\beta$ such that
  $$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} |\langle |\xi| \rangle|^{m-\rho|\alpha|} |\langle |x| \rangle|^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$  

Moreover, one can introduce a semi–norm in $A^m_{k_1, k_2; \rho, \delta}(\mathbb{R}^d, \mathbb{R}^d)$ by

$$\|a\|_{A^m_{k_1, k_2; \rho, \delta}} := \sup_{|\alpha| \leq k_1, |\beta| \leq k_2} \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| |\langle |\xi| \rangle|^{\rho|\alpha| - m} |\langle |x| \rangle|^{\delta|\beta|}, \quad a \in A^m_{k_1, k_2; \rho, \delta}(\mathbb{R}^d \times \mathbb{R}^d).$$

Remark A.1. For $m_1 \leq m_2$ it follows that $S^{m_1}_{\rho, \delta}(\mathbb{R}^d, \mathbb{R}^d) \supseteq S^{m_2}_{\rho, \delta}(\mathbb{R}^d, \mathbb{R}^d)$ and $A^{m_1}_{k_1, k_2; \rho, \delta}(\mathbb{R}^d, \mathbb{R}^d) \supseteq A^{m_2}_{k_1, k_2; \rho, \delta}(\mathbb{R}^d, \mathbb{R}^d)$, $k_1, k_2 \in \mathbb{N}$.

Definition A.3. (compare [49, p.28, Def. 4.2]) Let $a(x, \xi)$ be a symbol. Then, to $a(x, \xi)$ corresponds an operator $a(x, D)$ defined by

$$a(x, D) u(x) := \int_{\mathbb{R}^d} e^{i(x, \xi)} a(x, \xi) \hat{u}(\xi) d\xi, \quad u \in S(\mathbb{R}^d)$$

and called pseudo–differential operator.

Clearly, $a(x, D)$ is bounded from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$.
Corollary A.1. Let \( u \in H^m_2(\mathbb{R}^d) \) for all \( m \in \mathbb{R} \). Then
\[
a(x, D) u(x) := \int_{\mathbb{R}^d} e^{i(x, \xi)} a(x, \xi) \hat{u}(\xi) \, d\xi,
\]
is well defined with \( a(x, D) \) being a pseudo-differential operator.

Proof. Let \( v, \phi \in \mathcal{S}(\mathbb{R}^d) \). Then consider,
\[
(a(x, D)v, \phi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x, \xi)} a(x, \xi) \hat{v}(\xi) \, d\xi \, \overline{\phi(x)} \, dx
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix^T \xi} a(x, \xi) \phi(x) \, dx \, \hat{v}(\xi) \, d\xi.
\]
\[
= \int_{\mathbb{R}^d} \hat{v}(\xi) \int_{\mathbb{R}^d} e^{i(x, \xi)} a(x, \xi) \phi(x) \, dx \, d\xi,
\]
where we used Fubini theorem and the fact that \( \phi, \hat{v} \in \mathcal{S}(\mathbb{R}^d) \). In Lemma 3.31 in [1] showed that
\[
w(\xi) = \int_{\mathbb{R}^d} e^{i(x, \xi)} a(x, \xi) \phi(x) \, dx \in \mathcal{S}(\mathbb{R}^d),
\]
where \( a(x, \xi) \in S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d) \) with \( m \in \mathbb{R} \). Therefore we have,
\[
(a(x, D)v, \phi)_{L^2(\mathbb{R}^d)} = (v, a^*(x, D)\phi)_{L^2(\mathbb{R}^d)},
\]
such that \( a^*(x, D)\phi \in \mathcal{S}(\mathbb{R}^d) \). Now let \( u \in H^m_2(\mathbb{R}^d) \). There exist \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d) \) such that (see Corollary 3.42 in [1]),
\[
\lim_{n \to \infty} \langle u_n - u, \phi \rangle = 0,
\]
for any \( \phi \in \mathcal{S}(\mathbb{R}^d) \). Therefore due to the above facts we have
\[
\lim_{n \to \infty} (a(x, D)u_n, \phi) = \lim_{n \to \infty} \langle u_n, a^*(x, D)\phi \rangle = \langle u, a^*(x, D)\phi \rangle = \langle a(x, D)u, \phi \rangle < \infty.
\]
Then we could conclude that the Fourier integral representation of \( a(x, D)u \) is well defined in \( H^m_2(\mathbb{R}^d) \) with \( m \in \mathbb{R} \). See Theorem 3.41 in [1] as well. \( \square \)

One can easily see under which conditions \( a(x, D) \) is also bounded from \( L^p(\mathbb{R}^d) \) into \( L^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \). To see it, first observe that the operator can also be represented by a kernel of the form
\[
a(x, D)f(x) = \int_{\mathbb{R}^d} k(x, x - y) f(y) \, dy, \quad x \in \mathbb{R}^d,
\]
where the kernel is given by the inverse Fourier transform
\[
k(x, z) = \mathcal{F}_{\xi \rightarrow z} [a(x, \xi)] (z) \]
Differentiation gives the following estimate
\[
|k(x, z)| \leq C \left| \partial_{\xi}^\alpha P(x, \xi) \right| |z|^{-\alpha}.
\]
\[\text{For } \mathcal{F}_{\xi \rightarrow z}[f(x, \xi)](z) = \int_{\mathbb{R}^d} e^{-2\pi i (z, \xi)} a(x, \xi) \, d\xi.\]
By this estimate and the Young inequality for convolutions one can calculate bounds of the operator between Lebesgue spaces, like
\[
|a(x,D)f|_{L^q} \leq \|a\|_{A^0_{γ,0,1,0}} |f|_{L^q},
\]
for \(γ ≥ d + 1\). In case, we have additional regularity of the functions, or the function is a distribution, it is not that obvious. The next Theorem gives characterize the action of a pseudo-differential operator on Besov spaces.

**Theorem A.1.** (compare [1] Theorem 6.19, p. 164) Let \(κ, m ∈ \mathbb{R}, a(x,ξ) ∈ S^κ_{1,0}(\mathbb{R}^d × \mathbb{R}^d)\) and \(1 ≤ p, r ≤ ∞\). Then, \(a(x,D) : B^{κ+m}_{p,r}(\mathbb{R}^d) → B^m_{p,r}(\mathbb{R}^d)\) is a linear and bounded operator.

**Remark A.2.** Tracing step by step of the proof of Theorem 6.19 in [1] p. 164, one can see that for all \(κ, m ∈ \mathbb{R}, a(x,ξ) ∈ S^κ_{1,0}(\mathbb{R}^d × \mathbb{R}^d)\) and \(1 ≤ p, r ≤ ∞\) and any \(κ ≥ d + 1\) the following inequality holds
\[
|a(x,D)f|_{B^m_{p,r}} ≤ \|a\|_{A^κ_{κ,0,0}} |f|_{B^{κ+m}_{p,r}}.
\]

To analyze the composition of two operators of given symbols one has to evaluate a so called oscillatory integral. In particular, for any \(χ ∈ S(\mathbb{R}^d × \mathbb{R}^d)\) with \(χ(0,0) = 1\) and \(a ∈ S(\mathbb{R}^d × \mathbb{R}^d)\), we define the oscillatory integral by
\[
\text{Os} - \int \int e^{-i\eta \cdot y} a(y, \eta) \, dy \, d\eta := \lim_{\varepsilon \to 0} \int \int_{\mathbb{R}^d × \mathbb{R}^d} \chi(\varepsilon y, \varepsilon \eta) e^{-i(y,\eta)} a(y, \eta) \, dy \, d\eta.
\]
To calculate the oscillatory integral, the following Theorem is essential.

**Theorem A.2.** (compare [1] Theorem 3.9, p. 46) Let \(m ∈ \mathbb{R}, a ∈ A^m_{\gamma,0,d+1/0,0}(\mathbb{R}^d × \mathbb{R}^d)\), and let \(χ ∈ S(\mathbb{R}^d × \mathbb{R}^d)\) with \(χ(0,0) = 1\). Then the oscillatory integral
\[
\text{Os} - \int \int e^{-i(y,\eta)} a(y, \eta) \, dy \, d\eta
\]
exists and
\[
\left| \text{Os} - \int \int e^{-i(y,\eta)} a(y, \eta) \, dy \, d\eta \right| ≤ C_{m,d} \|a\|_{A^m_{\gamma,0,d+1/0,0}}.
\]

**Corollary A.2.** (compare [1] Corollary 3.10, p. 48) Let \(a_j ∈ S^m_{1,0}(\mathbb{R}^d × \mathbb{R}^d)\) be a bounded sequence in \(A^m_{\gamma,0,d+1/0,0}(\mathbb{R}^d × \mathbb{R}^d)\) such that there exists some \(a ∈ A^m_{\gamma,0,d+1/0,0}(\mathbb{R}^d × \mathbb{R}^d)\)
\[
\lim_{j → ∞} \partial^\alpha_\eta \partial^\beta_y a_j(y, \eta) = \partial^\alpha_\eta \partial^\beta_y a(y, \eta),
\]
for any \(|α| ≤ d + m + 1, |β| ≤ d + 1, y ∈ \mathbb{R}^d and \eta ∈ \mathbb{R}^d\). Then
\[
\lim_{j → ∞} \text{Os} - \int \int e^{-i(y,\eta)} a_j(y, \eta) \, dy \, d\eta = \text{Os} - \int \int e^{-i(y,\eta)} a(y, \eta) \, dy \, d\eta.
\]

With the help of the oscillatory integral, one can show that the composition of two pseudo-differential operators is again a pseudo-differential operator. Using formal calculations, an application of the Taylor formula leads to the following characterization.
Moreover, it can be expanded asymptotically as follows

\[ a(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \left( \partial_{\xi}^\alpha a_1(x, \xi) \right) \left( \partial_\xi^\alpha a_2(x, \xi) \right). \]

To be more precise, equation (A.1) means that

\[ \int [a_1 \cdot a_2](x, \xi) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \left( \partial_{\xi}^\alpha a_1(x, \xi) \right) \left( \partial_\xi^\alpha a_2(x, \xi) \right) \]

belongs to \( S^{(1,0)}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d) \) for every positive integer \( N \).

**Remark A.3.** Following the proof of Theorem 3.16 [1, p. 53], one observes that

\[ [a_1 \cdot a_2](x, \xi) = (N + 1) \sum_{|\alpha| = N+1} \frac{1}{\alpha!} \int e^{-i(y, \eta)} r_\alpha(x, \xi, y, \eta) \, dy \, d\eta, \]

with

\[ r_\alpha(x, \xi, y, \eta) = \int_0^1 \left[ \partial_{\xi}^\alpha p_1(x', \xi') \mid_{x' = x + y \theta} \partial_\xi^\alpha p_2(x', \xi') \mid_{x' = x + y} (1 - \theta)^N \right] \, d\theta. \]

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