Approximation of associated GBS operators by Szász-Mirakjan type operators

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\textbf{Abstract.} In this article, the approximation properties of the Szász-Mirakjan type operators are studied for the function of two variables and the rate of convergence of the bivariate operators is determined in terms of total and partial modulus of continuity. An associated GBS (Generalized Boolean Sum)-form of the bivariate Szász-Mirakjan type operators is considered for the function of two variables to find an approximation of $B$-continuous and $B$-differentiable function in the Bögel's space. Further, the degree of approximation of the GBS type operators is found in terms of mixed modulus of smoothness and functions belonging to the Lipschitz class as well as a pioneering result is obtained in terms of Peetre $K$-functional. Finally, the rate of convergence of the bivariate Szász-Mirakjan type operators and the associated GBS type operators are examined through graphical representation for the finite and infinite sum which shows that the rate of convergence of the associated GBS type operators is better than the bivariate Szász-Mirakjan type operators.

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1. Introduction

Approximation properties form an integral part in the study of approximation theory that includes convergence, rate of convergence, the order of approximation etc. Applications and convergence based discussion of the linear positive operators defined over different types of interval (finite or infinite) on $\mathbb{R}_+$, have been discussed by many researchers. In 1912, first of all, Bernstein proposed an operator, so-called Bernstein operator of one variable which approximates the functions defined over a finite interval $[0, 1]$.

In the study of [1,3,13,25–27], it is found that the Bernstein operators have been converted into bivariate Bernstein operators for function of two variables over $[0, 1] \times [0, 1]$ with their graphical representation in the study of the approximation properties for the function of two variables.

Many results related to approximations theory have also been discussed by many authors [24,33,37,38,46]. Despite of these, if we move towards the operators defined over an infinite interval, first of all, the Szász-Mirakjan operators were introduced and studied by Mirakjan and Szász [35,42] independently and so many work done in a bivariate direction of these operators to generalize and check the behavior of the operatots for the function of two variables. Later on Szász-Mirakjan operators have been discussed theoretically, numerically as well graphically by many authors [2,28,40,41,50] using bivariate extension for approximation of the functions of two variables.

Similarly, for the bivariate operators, one more property has been studied in Bögel space and that is the property of generalized boolean sum of the bivariate operators, so called GBS-type operators while the functions are considered to be $B$-continuous. In 1934 and in 1935, Bögel [6,7] introduced Bögel space, after that Dobrescu and Matei [29], estimated the rate of convergence of associated GBS-type operators of the bivariate Bernstein operators in the Bögel space. Similarly, Badea et al [8], proved the Korovkin type theorem for the function of two variables in the Bögel space. In 1988, Badea et al. [10] gave a quantitative
variant of Korovkin type theorem for $B$-continuous function and estimated the degree of approximation of $B$-continuous function as well as $B$-differentiable function for certain linear positive operators. After that, in 1991, quantitative and non-quantitative Korovkin type theorem was proved by Badea and Cottin \[9\] in the Bögel space. Similarly, the approximation properties of the bivariate Bernstein type operators and their associated GBS operators have been examined by many researchers (see \[18, 19, 30–32, 36, 43, 44\]).

In 2015, Bărboşu and Muraru \[20\] established some pioneer results through the associated GBS-type operators of Bernstein-Schurer-Stancu type operators using $q$-integers and some extension in terms of associated GBS-type operators of Kantorovich variant of a new kind of $q$-Bernstein-Schurer operators have been discussed by \[47\]. Similarly, Siddharth et al. \[48\] discussed the associated GBS-type operators. Bărboşu et al. \[21\] introduced GBS-Durrmeyer type operators based on $q$-integers, while \[23\] has discussed the properties of GBS-type operators. In 2016, Agrawal and Ispir \[4\] estimated the degree of approximation of the Chlodowsky-Szász-Charlier type operators for the function of two variables. In the same year, Agrawal et al. \[5\] studied the approximation properties and obtained the degree of approximation in terms of mixed modulus of smoothness.

Yadav et al. \[49\] proposed bivariate Szász-Mirakjan type operators for the function of two variables, where they studied the approximation properties as well as the rate of convergence of the proposed bivariate operators in a polynomial weighted space and obtained a Voronovskaya type theorem as well as discussed simultaneous approximation property for the bivariate GBS-type operators. In 2016, Agrawal and Ispir \[4\] discussed the associated GBS-type operators of the defined operators (1.1) with the GBS operators of the $q$-integers and some extension in terms of associated GBS operators have been examined by many researchers (see \[18, 19, 30–32, 36, 43, 44\]).

For our main results, we need some basic lemmas. Consider the function $e_{ij} = x^i y^j$ such that $i, j \in \{0, 1\}$ and $i + j \leq 2$. Then the following lemma hold:

**Lemma 1.1.** Let $x, y \geq 0$ and for each $m, n \in \mathbb{N}$. Then the following results hold:

\begin{align*}
\dot{Y}_{m,n,a}(e_{10}; x, y) &= \frac{x \log(a)}{m \left(a^{\frac{1}{m}} - 1\right)} \\
\dot{Y}_{m,n,a}(e_{01}; x, y) &= \frac{y \log(a)}{n \left(a^{\frac{1}{n}} - 1\right)} \\
\dot{Y}_{m,n,a}(e_{20}; x, y) &= \frac{x \log(a) \left(a^{\frac{1}{m}} + x \log(a) - 1\right)}{m^2 \left(a^{\frac{1}{m}} - 1\right)^2}
\end{align*}
Similarly, we can prove other results.

**Proof.** Here, we have \( x, y \geq 0 \) and \( m, n \in \mathbb{N} \) then

\[
\hat{Y}_{m,n,a}(e_{10}; x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} s_{m,n}(x, y) \frac{k_1}{m} = a \left( \frac{x}{-i+\pi} \right)^m a \left( \frac{y}{-i+\pi} \right) \frac{n2^m}{m} \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k-1} \right)^m.
\]

Similarly, we can prove other results. \( \square \)

**Lemma 1.2.** For every \( x, y \in X = [0, \infty) \times [0, \infty) \) and \( m, n \in \mathbb{N} \), it gives the following results:

1. \( \hat{Y}_{m,n,a}((t - x); x, y) = -\frac{x \left( ma^{\frac{1}{2}} - \log(a) - m \right)}{m (a^{\frac{1}{2}} - 1)} \)

2. \( \hat{Y}_{m,n,a}((s - y); x, y) = -\frac{y \left( na^{\frac{1}{2}} - \log(a) - n \right)}{n (a^{\frac{1}{2}} - 1)} \)

3. \( \hat{Y}_{m,n,a}((t - x)^2; x, y) = \frac{x \left( \frac{1}{2} \log(a) \right)^2 + x (\log(a))^2}{m^2 (a^{\frac{1}{2}} - 1)^2} \)

4. \( \hat{Y}_{m,n,a}((s - y)^2; x, y) = \frac{y \left( \frac{1}{2} \log(a) \right)^2 + y (\log(a))^2}{n^2 (a^{\frac{1}{2}} - 1)^2} \)
5. \( \hat{Y}_{m,n,a}((t-x)^4; x, y) = \frac{x}{m^4 \left( \frac{1}{a} - \frac{1}{m} \right)^4} \left\{ m^4 x^3 \left( \frac{1}{a} - \frac{1}{m} \right)^4 - \left( \frac{1}{a} - \frac{1}{m} \right)^3 (-1 + 4mx - 6m^2x^2 + 4m^3x^3) \log a \\
\quad + \left( \frac{1}{a} - \frac{1}{m} \right)^2 x(7 - 12mx + 6m^2x^2) (\log a)^2 \\
\quad - 2 \left( \frac{1}{a} - \frac{1}{m} \right)^2 x^2 (-3 + 2mx)(\log a)^3 + x^3(\log a)^4 \right\} \)

6. \( \hat{Y}_{m,n,a}((s-y)^4; x, y) = \frac{y}{n^4 \left( \frac{1}{a} - \frac{1}{n} \right)^4} \left\{ n^4 y^3 \left( \frac{1}{a} - \frac{1}{n} \right)^4 - \left( \frac{1}{a} - \frac{1}{n} \right)^3 (-1 + 4ny - 6n^2y^2 + 4n^3y^3) \log a \\
\quad + \left( \frac{1}{a} - \frac{1}{n} \right)^2 y(7 - 12ny + 6n^2y^2) (\log a)^2 \\
\quad - 2 \left( \frac{1}{a} - \frac{1}{n} \right)^2 y^2 (-3 + 2ny)(\log a)^3 + y^3(\log a)^4 \right\} \).

**Proof.** Using the Lemma 1.1, for every \( x, y \in X \) and for all \( m, n \in \mathbb{N} \), we have

1. \( \hat{Y}_{m,n,a}((t-x); x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} s_{m,n}(x, y) \left( \frac{k_1}{m} - x \right) \)

2. \( \hat{Y}_{m,n,a}((t-x)^2; x, y) = \hat{Y}_{m,n,a}((t^2 - 2tx + x^2); x, y) \)

3. \( \hat{Y}_{m,n,a}((t-x)^2; x, y) = \hat{Y}_{m,n,a}(e_{20}; x, y) - 2x \hat{Y}_{m,n,a}(e_{10}; x, y) + x^2 \)

Similarly, other equalities can be proved. \( \square \)

**Lemma 1.3.** For all \( x, y \geq 0 \), the following inequalities hold true:

\( \hat{Y}_{m,n,a}((t-x)^2; x, y) \leq \frac{x(x+1)}{m} = \delta_m^2(x) \)

\( \hat{Y}_{m,n,a}((s-y)^2; x, y) \leq \frac{y(y+1)}{m} = \delta_n^2(y) \)

**Proof.** For all \( m, n \in \mathbb{N} \) and \( x \geq 0 \), we have

\( \hat{Y}_{m,n,a}((t-x)^2; x, y) = \frac{x \left( m^2x \left( \frac{1}{a} - \frac{1}{m} \right)^2 - \left( \frac{1}{a} - \frac{1}{m} \right) \log(a)(2mx - 1) + x(\log a)^2 \right)}{m^2 \left( \frac{1}{m} - \frac{1}{a} \right)^2} \)
Similarly, it can be proved.

\[ \text{Remark 1.1. For all } (x, y) \in [0, c] \times [0, d], \text{ where } 0 \leq x \leq c \text{ and } 0 \leq y \leq d, \text{ we have} \]

\[ \hat{Y}_{m,n,a}((t-x)^2; x, y) \leq \frac{c(c+1)}{m} = \frac{\lambda_x}{m}, \]

(1.6)

\[ \hat{Y}_{m,n,a}((s-y)^2; x, y) \leq \frac{d(d+1)}{n} = \frac{\lambda_y}{n}, \]

(1.7)

where \( \lambda_x, \lambda_y \) are positive constants.

**Proof.** Using the Lemma 1.3, we can obtain the required results.

**Lemma 1.4.** For all \( x, y \in [0, c] \times [0, d] \) and \( m, n \in \mathbb{N} \), the following inequalities hold true

\[ \hat{Y}_{m,n,a}((t-x)^4; x, y) \leq \frac{M_x}{m^2}, \]

(1.8)

\[ \hat{Y}_{m,n,a}((s-y)^4; x, y) \leq \frac{M_y}{n^2}, \]

(1.9)

where \( M_x, M_y \) are positive constants.

**Proof.** By Lemma 1.2, we have

\[
\hat{Y}_{m,n,a}((t-x)^4; x, y) = x \left( \frac{\log a}{m^4 \left( \frac{a}{m} - 1 \right)} \right) + x^2 \left( \frac{\log a}{m^3 \left( \frac{a}{m} - 1 \right)} \right) \left( \frac{7 \log a}{m^3 \left( \frac{a}{m} - 1 \right)} - \frac{4}{m^2} \right)
\]

\[ + \frac{6 x^3 \log a}{m^2 \left( \frac{a}{m} - 1 \right)} \left( 1 - \frac{2 \log a}{m \left( \frac{a}{m} - 1 \right)} \right) + \frac{(\log a)^2}{m \left( \frac{a}{m} - 1 \right)} \]

\[ + x^4 \left( 1 - \frac{4 \log a}{m \left( \frac{a}{m} - 1 \right)} \right) + \frac{6 (\log a)^2}{m^2 \left( \frac{a}{m} - 1 \right)^2} - \frac{4 (\log a)^3}{m^3 \left( \frac{a}{m} - 1 \right)^3} + \frac{(\log a)^4}{m^4 \left( \frac{a}{m} - 1 \right)^4} \]

\[ \leq \frac{x}{m^3} + \frac{x^2}{n^2} \left( \frac{7 \log a}{m \left( \frac{a}{m} - 1 \right)} - 4 \right) + \frac{6 x^3}{n} \left( \frac{\log a}{m \left( \frac{a}{m} - 1 \right)} - 1 \right)^2 + x^4 \left( \frac{\log a}{m \left( \frac{a}{m} - 1 \right)} - 1 \right)^4 \]

\[ \leq \frac{x}{m^3} \left( x^4 + 7 x^2 + x \right) + \frac{x^3}{m^2} \left( 3 x^2 + 6 x^3 \right)
\]

\[ \leq \frac{1}{m^2} \left( x^4 + 10 x^2 + 6 x^3 \right)
\]

\[ \leq \frac{1}{m^2} \left( x^4 + 10 x^2 + 6 x^3 + c \right) = \frac{M_x}{m^2}.
\]

Similarly, it can be proved that

\[ \hat{Y}_{m,n,a}((s-y)^4; x, y) \leq \frac{M_y}{n^2}. \]
2. Basic properties of the bivariate operators

For finding the rate of convergence of the bivariate operators defined by (1.1) in terms of modulus of continuity, here we define the modulus of continuity. Let the function \( f(x, y) \in C_b(X = [0, \infty) \times [0, \infty)) \), be the space of all continuous and bounded function defined on \( X = [0, \infty) \times [0, \infty) \), then the total (complete) modulus of continuity for the function of two variables can be defined as:

\[
\omega(f, \delta) = \sup\{|f(t, s) - f(x, y)| : \sqrt{(t-x)^2 + (s-y)^2} \leq \delta, \ (t, s) \in X, \ \delta > 0\}
\]

and the partial modulus of continuity can be defined as [39]:

\[
\omega_1(f, \delta) = \sup\{|f(u_1, y) - f(u_2, y)| : |u_1 - u_2| \leq \delta, \ \delta > 0\},
\]

\[
\omega_2(f, \delta) = \sup\{|f(x, v_1) - f(x, v_2)| : |v_1 - v_2| \leq \delta, \ \delta > 0\}.
\]

The following theorem will show the rate of convergence of the bivariate operators (1.1) with the help of modulus of continuity.

**Theorem 2.1.** If the bivariate operators \( \hat{Y}_{m,n,a}(f; x,y) \) defined by (1.1) are linear and positive, then the following relations hold:

\[
|\hat{Y}_{m,n,a}(f; x,y) - f(x,y)| \leq 2\omega(f; \delta_{m,n}),
\]

\[
|\hat{Y}_{m,n,a}(f; x,y) - f(x,y)| \leq 2(\omega_1(f, \delta_m) + \omega_2(f, \delta_n)),
\]

where \( \omega \) is the total modulus of continuity and \( \omega_1, \omega_2 \) are the partial modulus of continuity with respect to \( x, y \) respectively.

**Proof.** Using the definition of modulus of continuity, we can write

\[
|\hat{Y}_{m,n,a}(f; x,y) - f(x,y)| \leq \hat{Y}_{m,n,a}(|f(t,s) - f(x,y)|; x,y)
\]

\[
\leq \hat{Y}_{m,n,a}(\omega(\sqrt{(t-x)^2 + (s-y)^2}; x,y))
\]

\[
\leq \omega(f; \delta) \left(1 + \frac{1}{\delta} (\hat{Y}_{m,n,a}((t-x)^2 + (s-y)^2); x,y)\right)
\]

\[
\leq \omega(f; \delta) \left(1 + \frac{1}{\delta} (\hat{Y}_{m,n,a}((t-x)^2 + (s-y)^2); x,y)\right)^{\frac{1}{2}}
\]

\[
= \omega(f; \delta) \left\{1 + \frac{1}{\delta} \left(\frac{mx(a^\frac{1}{n} - 1)^2 - (a^\frac{1}{n} - 1)\log(a)(2mx - 1) + x(\log a)^2}{m^2 (a^\frac{1}{n} - 1)^2} + \frac{ny(a^\frac{1}{m} - 1)^2 - (a^\frac{1}{m} - 1)\log(a)(2ny - 1) + y(\log a)^2}{n^2 (a^\frac{1}{m} - 1)^2}\right)^{\frac{1}{2}}\right\}.
\]

upon considering

\[
\delta = \left\{1 + \frac{1}{\delta} \left(\frac{mx(a^\frac{1}{n} - 1)^2 - (a^\frac{1}{n} - 1)\log(a)(2mx - 1) + x(\log a)^2}{m^2 (a^\frac{1}{n} - 1)^2} + \frac{ny(a^\frac{1}{m} - 1)^2 - (a^\frac{1}{m} - 1)\log(a)(2ny - 1) + y(\log a)^2}{n^2 (a^\frac{1}{m} - 1)^2}\right)^{\frac{1}{2}}\right\} = \delta_{m,n},
\]

the next one step will give the required result.

Now to prove the second part of this theorem. Upon using the properties (2.2), (2.3) and with the help of Cauchy-Schwarz inequality, we get:

\[
|\hat{Y}_{m,n,a}(f; x,y) - f(x,y)| \leq \hat{Y}_{m,n,a}(|f(t,s) - f(x,y)|; x,y)
\]

\[
\leq \hat{Y}_{m,n,a}(|f(t,s) - f(t,y)|; x,y) + \hat{Y}_{m,n,a}(|f(t,y) - f(x,y)|; x,y))
\]
\[
\leq \omega_2(f, \delta_m) \left( 1 + \frac{1}{\delta_m} (Y_{m,n,a}(\sqrt{(s-y)^2};x,y) + \right) \\
+ \omega_1(f, \delta_n) \left( 1 + \frac{1}{\delta_n} (Y_{m,n,a}(\sqrt{(t-x)^2};x,y) \right),
\]
(2.6)

where

\[
\delta_m = \frac{x \left( m^2 x (a^\frac{1}{n} - 1)^2 - (a^\frac{1}{n} - 1) \log(a)(2mx - 1) + x(\log a)^2 \right)}{m^2 (a^\frac{1}{n} - 1)^2}
\]
(2.7)

\[
\delta_n = \frac{y \left( n^2 y (a^\frac{1}{n} - 1)^2 - (a^\frac{1}{n} - 1) \log(a)(2ny - 1) + y(\log a)^2 \right)}{n^2 (a^\frac{1}{n} - 1)^2}
\]
(2.8)

hence, by using inequality (2.6), the above result can be obtained. \[\square\]

2.1. Some basic definitions for associated GBS (Generalized Boolean Sum) operators. In recent years, the study of generalized Boolean sum (GBS) operators of certain linear positive operators is an interesting topic in approximation theory and function theory. In order to make analysis in multidimensional spaces, Karl Bögel introduced the concepts of B-continuous and B-differentiable function in [6, 7]. In [11], the authors discussed some significance role of the Bögel space. They proved that the space of all bounded Bögel functions is isometrically isometric with the completion of the blending function space with respect to suitable norm. Also the main importance of the Bögel space is that the functions which are not continuous in general but are B-continuous can also be approximated by the operators.

In this subsection, some basic definitions are defined for associated GBS-type operators in the Bögel space and their related properties are discussed.

**Definition 2.2.** B-Continuous: Consider two compact intervals \( \mathfrak{A}_1, \mathfrak{A}_2 \subset \mathbb{R} \), a function \( f : \mathfrak{A}_1 \times \mathfrak{A}_2 \to \mathbb{R} \) is said to be B-continuous function at a point \((u_0, v_0) \in \mathfrak{A}_1 \times \mathfrak{A}_2\), if

\[
\lim_{(u,v) \to (u_0,v_0)} \Delta f((u,v),(u_0,v_0)) = 0,
\]
(2.9)

where \( \Delta f((u,v),(u_0,v_0)) = f(u,v) - f(u_0,v) - f(u,v_0) + f(u_0,v_0) \) and the set of all B-continuous function is denoted by \( C_b(\mathfrak{A}_1 \times \mathfrak{A}_2) \).

**Definition 2.3.** B-Bounded: A real valued function \( f \) defined on \( \mathfrak{A}_1 \times \mathfrak{A}_2 \) is said to be B-Bounded, if there exist a positive constant \( M \) such that

\[
\Delta f((u,v),(u_0,v_0)) \leq M,
\]
(2.10)

i.e. denoted by \( B_b(\mathfrak{A}_1 \times \mathfrak{A}_2) \).

**Definition 2.4.** B-Differentiable: A function \( f \) is called B-Differentiable iff

\[
D_B f(u_0, v_0) = \lim_{(u,v) \to (u_0,v_0)} \frac{\Delta f((u,v),(u_0,v_0))}{(u-u_0)(v-v_0)},
\]
(2.11)

provided the limit exists and finite where the set of all B-differentiable functions is denoted by \( D_B(\mathfrak{A}_1 \times \mathfrak{A}_2) \).

For more details see [6, 7].

Motivated by cited papers in introduction part, here, we define the associated GBS-type operators of the above defined biavriate operators (1.1) to investigate their approximation properties in the Bögel space. The main motive of this part is to determine the convergence results of the GBS-type operators defined by (2.12) along with their properties by theoretical, numerical as well as graphical sense. The goodness of the GBS-type operators is that these operators have a better rate of convergence than the proposed bivariate operators (1.1) as well as the GBS operators of Mirakjan-Favard-Szász. So, before the discussion of their properties, first we construct here the GBS-type operators of the above bivariate operators (1.1).

Consider two compact intervals \( \mathfrak{A}_1, \mathfrak{A}_2 \subset \mathbb{R} \) and for any point \((x,y) \in \mathfrak{A}_1 \times \mathfrak{A}_2\), the Boolean sum of the function \( f : \mathfrak{A}_1 \times \mathfrak{A}_2 \to \mathbb{R} \) can be defined as \( \Delta f((x,y),(t,s)) = f(x,y) - f(x,t) - f(s,y) + f(t,s) \) at a point...
(t, s) ∈ \mathbb{A}_1 \times \mathbb{A}_2. Then the associated GBS (Generalized Boolean Sum)-type operators of \( \hat{Y}_{m,n,a}(f; x, y) \) can be expressed as

\[
\hat{B} Y_{m,n}^a(f; x, y) = \hat{Y}_{m,n,a}(f(x, s) + f(t, y) - f(t, s))
\]

\[
(2.12) \quad = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \gamma_{m,n}(x, y) \left( f\left( x, \frac{k_2}{m} \right) + f\left( \frac{k_1}{m}, y \right) - f\left( \frac{k_1}{m}, \frac{k_2}{n} \right) \right),
\]

where \( f \in C_b(X_b = [0, c] \times [0, d]) \).

2.2. Degree of the approximation of the GBS-type operators. In this subsection, we discuss the rate of convergence of the GBS-type operators with the help of modulus of smoothness in a Bögel space, and get a relation using the mixed modulus of smoothness. Now to define the modulus of smoothness, we assume that the function \( f \in C_b(X_b = [0, c] \times [0, d]) \). The property of mixed modulus of smoothness is same as the modulus of continuity, which can be defined as

\[
(2.13) B(f; \delta_1, \delta_2) = \sup \{ |\Delta f(t, s; x, y)| : |t - x| < \delta_1, |s - y| < \delta_2, (x, y), (t, s) \in X_b = [0, c] \times [0, d] \},
\]

for any \((\delta_1, \delta_2) \in X = [0, \infty) \times [0, \infty) \) and having property

\[
(2.14) \quad \omega_B(f; \delta_m, \delta_n) \to 0, \text{ as } m, n \to \infty.
\]

**Remark 2.1.** The property of the modulus of smoothness can be defined as:

\[
(2.15) \quad \omega_B(f; \mu_1 \delta_1, \mu_2 \delta_2) = (1 + \mu_1)(1 + \mu_2) \omega_B(f; \delta_1, \delta_2), \quad \mu_1, \mu_2 > 0.
\]

**Theorem 2.5.** Let \( f \in C_b(X_b) \) and \( \hat{B} Y_{m,n}^a(f; x, y) \) be linear positive operators defined by (2.12), then the following inequality holds:

\[
(2.16) \quad |\hat{B} Y_{m,n}^a(f; x, y) - f(x, y)| \leq 4 \omega_B(f; \delta'_m, \delta'_n).
\]

**Proof.** Upon using the Remark 2.1, it can be written as:

\[
|\Delta f(t, s; x, y)| \leq \omega_B(f; \delta_1, \delta_2)
\]

\[
\leq \left( 1 - \frac{|t - x|}{\delta_1} \right) \left( 1 - \frac{|s - y|}{\delta_2} \right) \omega_B(f; \delta_1, \delta_2), \quad \delta_1, \delta_2 \geq 0.
\]

Using the property of the difference function \( \Delta f(t, s; x, y) \) and applying the operators (1.1), it gives

\[
(2.17) \quad \hat{B} Y_{m,n}^a(f; x, y) = f(x, y) \hat{Y}_{m,n,a}(1, x, y) - \hat{Y}_{m,n,a}(\Delta f(t, s; x, y), x, y),
\]

Upon using Cauchy-Schwartz inequality in equation (2.17), it obtains

\[
|\hat{B} Y_{m,n}^a(f; x, y) - f(x, y)| \leq \hat{Y}_{m,n,a}(\delta_1, |\Delta f(t, s; x, y)|; x, y)
\]

\[
\leq \left( \frac{1}{\delta_1} \hat{Y}_{m,n,a}(\delta_0, x, y) + \frac{1}{\delta_1} \hat{Y}_{m,n,a}(|t - x|; x, y) \right)
\]

\[
\times \left( \frac{1}{\delta_2} \hat{Y}_{m,n,a}(\delta_0, x, y) + \frac{1}{\delta_2} \hat{Y}_{m,n,a}(|s - y|; x, y) \right) \omega_B(f; \delta_1, \delta_2)
\]

\[
\leq \left( 1 + \frac{1}{\delta_1} \sqrt{\hat{Y}_{m,n,a}((t - x)^2; x, y)} + \frac{1}{\delta_2} \sqrt{\hat{Y}_{m,n,a}((s - y)^2; x, y)} \right) \omega_B(f; \delta_1, \delta_2).
\]

Now by using Lemma 1.3, and choosing \( \delta_1 = \delta'_m, \delta_2 = \delta'_n \), the desired results can be obtained. \( \square \)

Next we will find the degree of approximation of the GBS-type operators defined by (2.12), by means of B-continuous function belonging to the Lipschitz class and it can be defined as:

\[
(2.18) \quad \text{Lip}_M(\mu_1, \mu_2) = \{ f \in C_b(X_b) : |\Delta f((u, v), (u_0, v_0))| \leq M |u - u_0|^{\mu_1} |v - v_0|^{\mu_2}, \quad \mu_1, \mu_2 \in (0, 1) \},
\]

where \((u, v), (u_0, v_0) \in X_b \) and \( M > 0 \).
THEOREM 2.6. Let \( f \in \text{Lip}_M(\mu_1, \mu_2) \), then there exist a positive constant \( M \), such that
\[
(2.19) \quad |\hat{B}Y_{m,n}^{a}(f; x, y) - f(x, y)| \leq M\delta_m^a\delta_n^a.
\]

where \( \delta_m = \sqrt{\frac{x(x+1)}{m}} \), \( \delta_n = \sqrt{\frac{y(y+1)}{n}} \).

PROOF. By using the linearity property of GBS-type operators \( (2.12) \) and by definition of \( \hat{B}Y_{m,n}^{a}(f; x, y) \), we can write
\[
|\hat{B}Y_{m,n}^{a}(f; x, y) - f(x, y)| \leq \|\hat{B}Y_{m,n,a}\| \|\Delta f((t, s), (x, y))\| \leq M\hat{Y}_{m,n}(\mu_{1, 0}) + M\hat{Y}_{m,n}(\mu_{2, 0}) + M\hat{Y}_{m,n}(\mu_{2'}) + M\hat{Y}_{m,n}(\mu_{2'}). \]

Using Hölder’s inequality and by considering \( l_1 = \frac{2}{\mu_{1, 0}} \), \( r_1 = \frac{2}{\mu_{2, 0}} \), and \( l_2 = \frac{2}{\mu_{2'}} \), \( r_2 = \frac{2}{\mu_{2'}} \), in the next step, the required result can be obtained as,
\[
|\hat{B}Y_{m,n}^{a}(f; x, y) - f(x, y)| \leq M\hat{Y}_{m,n,a}(\mu_{1, 0}) + M\hat{Y}_{m,n,a}(\mu_{2, 0}) + M\hat{Y}_{m,n,a}(\mu_{2'}) + M\hat{Y}_{m,n,a}(\mu_{2'}). \]

Hence proved.

Next the rate of convergence of the above operators can be found, when the function is \( B \)-differentiable, and it is defined by \( (2.11) \).

LEMMA 2.1. For any \( x, y \geq 0 \) and for all \( m, n \in \mathbb{N} \), we have
\[
\hat{Y}_{m,n,a}((−x)^{2i}(*−y)^{2j}; x, y) = \hat{Y}_{m,n,a}((−x)^{2i}; x, y)\hat{Y}_{m,n,a}(*−y)^{2j}; x, y), \ \forall \ i, j \in \mathbb{N} \cup \{0\}.
\]

PROOF. Given that \( x, y \geq 0 \) and \( m, n \in \mathbb{N} \) then, we have
\[
\hat{Y}_{m,n,a}((−x)^{2i}(*−y)^{2j}; x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} s_{m,n}(x, y) \left( \frac{k_1}{m} - x \right)^{2i} \left( \frac{k_2}{n} - y \right)^{2j}
\]
\[
= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} s_{m,n}(x, y) \left( \frac{k_1}{m} - x \right)^{2i} \left( \frac{k_2}{n} - y \right)^{2j}
\]
\[
= \hat{Y}_{m,n,a}((−x)^{2i}; x, y)\hat{Y}_{m,n,a}(*−y)^{2j}; x, y).
\]

Hence proved.

THEOREM 2.7. Let \( f \in B_{\beta}(X_b) \) and \( \hat{D}_B f \in B_{\beta}(X_b) \), then there exist a positive constant \( M_1 \), such that
\[
(2.20) \quad |\hat{B}Y_{m,n}^{a}(f; x, y) - f(x, y)| \leq \frac{M_1}{\sqrt{mn}} \left\{ 3M_3\|\hat{D}_B f\| + \omega_B \left( \hat{D}_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right) \right\}
\]

PROOF. Using mean value theorem for \( B \)-differentiable functions, it can be written as
\[
(2.21) \quad D_B f(\beta, \gamma) = \frac{\Delta f((t, s), (x, y))}{(t-x)(s-y)}, \ \text{where} \ \beta \in (t, x); \ \gamma \in (s, y).
\]

By using the property \( \Delta f((t, s), (x, y)) \), it gives:
\[
(2.22) \quad D_B f(\beta, \gamma) = D_B f(\beta, (x, y)) + D_B f(x, \gamma) + D_B f(\beta, y) - D_B f(x, y),
\]

since, \( D_B f \in B_{\beta}(X_b) \), so by using equation \( (2.22) \) and the equality \( (2.21) \), it obtains
\[
|\hat{Y}_{m,n,a}(\Delta f((t, s), (x, y)); x, y)| \leq \hat{Y}_{m,n,a}((t-x)(s-y)D_B f(\beta, \gamma); x, y)
\]
\[
\leq \hat{Y}_{m,n,a}((t-x)(s-y)||\Delta f(\beta, \gamma), (x, y)||; x, y)
\]
\[
+ \hat{Y}_{m,n,a}(|t-x||s-y|)|D_B f(x, \gamma)| + |D_B f(\beta, y) - D_B f(x, y)|; x, y)
\]
\[
\leq \hat{Y}_{m,n,a}((t-x)|y-s|\omega_B(D_B f; |\beta-x|, |\gamma-y|); x, y)
\]
\[
+ 3\|D_B f\|\hat{Y}_{m,n,a}((t-x)(y-s); x, y),
\]

Hence proved.
as $\beta \in (x, t)$ and $\gamma \in (y, s)$ (already assumed) and with the property of modulus, for $h_m, h_n > 0$, we have

$$\omega_B(D_B f; |\beta - x|, |\gamma - y|) \leq \omega_B(D_B f; |t - x|, |s - y|)$$

$$\leq \left(1 + \frac{|t - x|}{h_m}\right)\left(1 + \frac{|s - y|}{h_n}\right)\omega_B(D_B f; h_m, h_n),$$

therefore,

$$|\tilde{Y}_{m,n}(\Delta f((t, s), (x, y)); x, y)| \leq \tilde{Y}_{m,n,a}\left(\left|t - x\right|\left|y - s\right|\left(1 + \frac{|t - x|}{h_m}\right)\left(1 + \frac{|s - y|}{h_n}\right)\omega_B(D_B f; h_m, h_n)\right): x, y$$

(2.23)

$$+3\|D_B f\|\tilde{Y}_{m,n,a}(\left|t - x\right|\left|y - s\right|): x, y,$$

since,

(2.24)

$$|\tilde{B}Y_{m,n}^a(f; x, y) - f(x, y)| \leq \tilde{Y}_{m,n}(\Delta f((t, s), (x, y)); x, y),$$

Upon using the inequalities (2.23), (2.24) and with the help of Cauchy-Schwarz inequality, we get

$$|\tilde{B}Y_{m,n}^a(f; x, y) - f(x, y)| \leq \left\{\left(\tilde{Y}_{m,n,a}(\left|t - x\right|^{2}(s - y)^{2}; x, y)\right)^{\frac{1}{2}} + h_m^{-1}\left(\tilde{Y}_{m,n,a}(\left|t - x\right|^{4}(s - y)^{2}; x, y)\right)^{\frac{1}{2}} \right\}\omega_B(D_B f; h_m, h_n)$$

$$+h_n^{-1}\tilde{Y}_{m,n,a}(\left|t - x\right|^{4}(s - y)^{2}; x, y)\omega_B(D_B f; h_m, h_n)$$

$$+3\|D_B f\|\tilde{Y}_{m,n,a}(\left|t - x\right|^{2}(s - y)^{2}; x, y).$$

Now using the inequalities (1.6), (1.7) and Lemma 1.4, we have

$$|\tilde{B}Y_{m,n}^a(f; x, y) - f(x, y)| \leq \left\{\sqrt{\frac{\lambda_x}{m}}\sqrt{\frac{\lambda_y}{n}} + h_m^{-1}\sqrt{\frac{M_x}{m^2}}\sqrt{\frac{M_y}{n}} + h_n^{-1}\sqrt{\frac{\lambda_x}{m}}\sqrt{\frac{M_y}{n^2}} \right\}\omega_B(D_B f; h_m, h_n)$$

$$+3\|D_B f\|\sqrt{\lambda_x}\sqrt{\lambda_y}.$$
\[
\frac{1}{\sqrt{mn}} \left\{ M_1M_2\omega_B \left( D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right) + 3M_3\|D_B f\| \right\},
\]

where \( M_1 = (\sqrt{\lambda_\epsilon} + \sqrt{\lambda_z}), \) \( M_2 = (\sqrt{\lambda_y} + \sqrt{\lambda_y}) \) and \( M_3 = \sqrt{\lambda_x\lambda_y} \) and \( M_4 = \max\{M_1M_2, M_3\}, \) Hence the Inequality gives

\[
|BY_{m,n}^a (f; x, y) - f(x, y)| \leq \frac{M_4}{\sqrt{mn}} \left\{ 3M_3\|D_B f\| + \omega_B \left( D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}} \right) \right\}.
\]

Hence, the proof is completed. \( \square \)

To improve the measure of smoothness, a mixed \( K \)-functional is introduced (see [12], [22]) and it is defined by

\[
K_B(f; x_1, x_2) = \{\|f - g_1 - g_2 - h\| + x_1\|D_B^{2,0} g_1\| + x_2\|D_B^{0,2} g_2\| + x_1x_2\|D_B^{2,2} h\|\},
\]

where \( g_1 \in D_B^{2,0}, g_2 \in D_B^{0,2}, h \in D_B^{2,2} \) and \( D^{i,j}_B \) represent the space of all functions \( f \in C_B(X_b) \) for \( 0 \leq i, j \leq 2 \) having mixed partial derivatives \( D^{i,j}_B f \) with \( 0 \leq \eta \leq i, 0 \leq \mu \leq j \) defined by

\[
D_x f(u, v) = D_B^{1,0} f(u, v) = \lim_{x \to u} \frac{\Delta_x f([u, x]; v)}{x - u},
\]

\[
D_y f(u, v) = D_B^{0,1} f(u, v) = \lim_{y \to v} \frac{\Delta_y f(u; [v, y])}{y - v},
\]

\[
D_y D_x f(u, v) = D_B^{0,1} D_B^{1,0} f(u, v) = \lim_{y \to v} \frac{\Delta_y (\Delta_x f(u; [v, y]))}{y - v},
\]

\[
D_x D_y f(u, v) = D_B^{1,0} D_B^{0,1} f(u, v) = \lim_{x \to u} \frac{\Delta_x (\Delta_y f([u, x]; v))}{x - u},
\]

where \( \Delta_x f([u, x]; v) = f(x, v) - f(u, v), \Delta_y f(u; [v, y]) = f(u, y) - f(u, v). \)

**Theorem 2.8.** Let \( BY_{m,n}^a(f; x, y) \) be a GBS-type operator of \( \hat{Y}_{m,n,a}(f; x, y) \) for all \( x, y \in X_b = [0, c] \times [0, d] \) and for each function \( f \in C_B(X_b) \) with \( m, n \in \mathbb{N} \), we have

\[
|BY_{m,n}^a(f; x, y) - f(x, y)| \leq 2K_B \left( f, \frac{\lambda_x}{m}, \frac{\lambda_y}{n} \right).
\]

**Proof.** With the help of Taylor’s formula for the function \( g_1 \in C_B^{2,0}(X_b) \), we obtain

\[
g_1(t, s) - g_1(x, y) = (t - x)D_B^{1,0} g_1(x, y) + \int_x^t (t - \xi)D_B^{2,0} g_1(\xi, y) d\xi,
\]

Upon using the linearity and positivity properties of GBS-type operators and the definition of \( BY_{m,n}^a(f; x, y) \), it gives

\[
|BY_{m,n}^a(g_1; x, y) - g_1(x, y)| = \left| \hat{Y}_{m,n,a} \left( \int_x^t (t - \xi)D_B^{2,0} g_1(\xi, y) - D_B^{2,0} g_1(\xi, s) d\xi; x, y \right) \right|
\]

\[
\leq \hat{Y}_{m,n,a} \left( \int_x^t \|D_B^{2,0} g_1(\xi, y) - D_B^{2,0} g_1(\xi, s)\| d\xi; x, y \right)
\]

\[
\leq \|D_B^{2,0} g_1\| \|\hat{Y}_{m,n,a}((t - x)^2; x, y)\| < \|D_B^{2,0} g_1\| \frac{c}{m},
\]

similarly,

\[
|BY_{m,n}^a(g_2; x, y) - g_2(x, y)| < \|D_B^{0,2} g_2\| \frac{d}{n},
\]

let \( g_2 \in D_B^{0,2} \) then for \( h \in D_B^{2,2}, \) we have

\[
h(t, s) - h(x, y) = (t - x)D_B^{1,0} h(x, y) + (s - y)D_B^{0,1} h(x, y) + (t - x)(s - y)D_B^{1,1} h(x, y)
\]
\[ + \int_x^t (t - \xi) D_B^{2,0} h(\xi, y) d\xi + \int_x^t (s - \phi) D_B^{0,2} h(x, \phi) d\phi + \int_x^t (s - y)(t - \xi) D_B^{2,1} h(\xi, y) d\xi \\
+ \int_x^t (t - x)(s - \phi) D_B^{1,2} h(x, \phi) d\phi + \int_x^t (t - \xi)(s - \phi) D_B^{2,2} h(\xi, \phi) d\xi d\phi. \]

By using the definition of the GBS-type operators \( \hat{BY}_{m,n}^a(f; x, y) \) of the defined operators (1.1), we have

\[ \text{(2.33)} \]
\[ \hat{BY}_{m,n}^a((t - x); x, y) = 0, \hat{BY}_{m,n}^a((s - y); x, y) = 0, \]

in next step, we get

\[
|\hat{BY}_{m,n}^a(h; x, y) - h(x, y)| \leq \left| \hat{BY}_{m,n,a} \left( \int_x^s \int_y^t (s - \phi) D_B^{2,2} h(\xi, \phi) d\xi d\phi; x, y \right) \right|
\leq \hat{BY}_{m,n,a} \left( \int_x^s \int_y^t |(t - \xi)||s - \phi|| D_B^{2,2} h(\xi, \phi)| d\xi d\phi; x, y \right)
\leq \frac{1}{4}\|D_B^{2,2} h\| \|\hat{BY}_{m,n,a}((t - x)^2(s - y)^2; x, y)\|
\leq \|D_B^{2,2} h\| \frac{\lambda_x \lambda_y}{mn}.
\]

Now,

\[
|\hat{BY}_{m,n}^a(f; x, y) - f(x, y)| \leq |(f - g_1 - g_2 - h)(x, y)| + \left| (g_1 - \hat{BY}_{m,n,a} g_1)(x, y) \right| + \left| (g_2 - \hat{BY}_{m,n,a} g_2)(x, y) \right|
+ \left| (h - \hat{BY}_{m,n,a} h)(x, y) \right| + \left| \hat{BY}_{m,n,a}((f - g_1 - g_2 - h); x, y) \right|
\leq 2\|f - g_1 - g_2 - h\| + \|D_B^{2,0} g_1\| \frac{\lambda_x}{m} + \|D_B^{0,2} g_2\| \frac{\lambda_y}{n} + \|D_B^{2,2} h\| \frac{\lambda_x \lambda_y}{mn},
\]

by taking infimum over for all \( g_1 \in C_B^{2,0}, g_2 \in C_B^{0,2}, h \in C_B^{2,2} \), we get our desired result.

\[ \square \]

3. Graphical approach and Convergence based discussion

For validation of the results, the GBS-type operators are compared with the bivariate operators (1.1) and the rate of convergence is examined for finite sum over the interval \([0, 1]\) as well as for infinite sum over the interval \([0, \infty)\) through graphical representations along with their numerical approximation.

In this section, we discuss the behaviour of the operators with the function \( f(x, y) \) for particular values of \( k_1, k_2 \) and for an infinite series (i.e. for \( k_1 = 0, 1, \cdots, \infty \) and \( k_2 = 0, 1, \cdots, \infty \)). Also, check the behaviour of the operators (1.1) and (2.12) by comparison.
Example 3.1. Consider the function \( f(x,y) = x \sin \pi y \) (green). For the particular value of \( m = n = 10 \), \( k_1 = 9 = k_2 \), the corresponding operators are represented by \( \hat{Y}_{10,10}^a(f;x,y) \) (blue) and \( BY_{10,10}^a(f;x,y) \) (red) respectively. Upon considering the partitions as \( x_0 = 0, x_1 = \frac{1}{10}, \ldots, x_9 = \frac{9}{10} \) of \([0,1]\) and \( y_0 = 0, y_1 = \frac{1}{10}, \ldots, y_9 = \frac{9}{10} \) of \([0,1]\), the convergence approach of the operators \( \hat{Y}_{m,n}^a(f;x,y) \) and \( BY_{m,n}^a(f;x,y) \) to the function and their comparison are shown in Figure 6.

![Figure 1](image1.png)

**Figure 1.** The comparison of the convergence approach of the operators \( \hat{Y}_{m,n}^a(f;x,y) \) (blue) and \( BY_{m,n}^a(f;x,y) \) (red) to the function \( f(x,y) \) (green).

Now, we choose less numbers of partitions for the same function and for the same particular values of \( m = n = 10 \).

![Figure 2](image2.png)

**Figure 2.** The comparison of the convergence approach of the bivariate operators \( \hat{Y}_{m,n}^a(f;x,y) \) (blue) and \( BY_{m,n}^a(f;x,y) \) (red) to the function \( f(x,y) \) (green).

Here, we take the partitions within six terms like as \( x_0 = 0, x_1 = \frac{1}{10}, \ldots, x_5 = \frac{5}{10} \) of \([0,1]\) and \( y_0 = 0, y_1 = \frac{1}{10}, \ldots, y_5 = \frac{5}{10} \) of \([0,1]\) as shown in Figure 7. It can be seen from Figure 7 that the error gap between the function and operators are maximum in Figure 7 rather than in Figure 6.

Finally, it can be observed from Figures (6) and (7) that the accuracy approach of the GBS-type operators (2.12) to the function \( f(x,y) \) is better than the bivariate operators (1.1) but it depends on the number of partitions of \([0,1]\). By observing Figure 7 and Figure 6, it can be seen that for large number of partitions i.e as the length of the partition be small, the approximation is better as compared to less number of partitions of the interval i.e for larger length of partitions. It can also be concluded that the approach of the operators to the function will be good upon using large number of partitions as compared to less numbers of partitions for the same interval. On other the hand, the approach of the GBS-type operators (2.12) is better than the bivariate operators (1.1). So, finally we can say that the convergence rate of the GBS-type operators is better than the convergence rate of the bivariate operators in any case.
Remark: In general, if we consider \([x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i]\) and \([y_0, y_1], [y_1, y_2], \ldots, [y_{j-1}, y_j]\), are the sub-intervals of \([0, 1]\), provided each \(x_i, y_j\) are the some form of \(\frac{i}{m}, \frac{j}{n}\) respectively, where \(i = 1, 2, \ldots, k_1\), \(j = 1, 2, \ldots, k_2\) while \(k_1 \leq m, k_2 \leq n\), then the following concluding remarks can be obtained.

Concluding Remark:

- If the number of sub-intervals are maximum i.e., the sub-length \(x_i - x_{i-1}, y_j - y_{j-1}\) are small, then the approximation is good.
- If the number of sub-interval are minimum i.e., the sub-length \(x_i - x_{i-1}, y_j - y_{j-1}\) are large, then the approximation is not good.

Note: In above both conditions, the approach of the GBS-type operators (2.12) to the function is better than the bivariate operators as defined by (1.1).

Example 3.2. Consider a function defined by \(f(x, y) = x \sin \pi y\) (green). For the particular value of \(m = n = 10\), the corresponding operators \(\hat{Y}_{10,10,a}(f; x, y)\) and \(\hat{B}Y_{10,10}^a(f; x, y)\) are shaded by blue and red colors respectively as given in Figure 8. Here, it can be seen the approximation of the function defined by the operators (1.1), (2.12) and the error determined by the GBS-type operators to the function is minimum than the bivariate operators (2.12).

![Figure 3](image-url)  

Figure 3. Comparison of the convergence for both bivariate operators \(\hat{Y}_{m,n,a}(f; x, y)\) (blue) and \(\hat{B}Y_{m,n}^a(f; x, y)\) (red) to the function \(f(x, y)\) (green).

Concluding result: From Figure 8, it can be concluded that the convergence behavior of the GBS-type operators defined by (2.12) is better than the bivariate operators defined by (1.1).

Example 3.3. Consider a function \(f(x, y) = \sin(x + y)\) (green). On choosing the value of \(m = n = 10, 15\) for the GBS-type operators, the corresponding GBS operators can be represented as \(\hat{B}Y_{10,10}^2(f; x, y)\) (blue), \(\hat{B}Y_{15,15}^2(f; x, y)\) (yellow). It can be observed from Figure 4 that the error becomes smaller as the value of \(m\) and \(n\) be increases.

![Figure 4](image-url)  

Figure 4. The convergence of the GBS-type operators \(\hat{B}Y_{m,m}^a(f; x, y)\) to the function \(f(x, y)\).
From Figure 4, it can be seen the convergence behaviour of the GBS-type operators with the small value of the parameters (as $m = n = 10, 15$) where as Figure 5 represents the convergence behaviour of the GBS-type operators $\hat{B}Y^2_{15,15}(f; x, y)$ for $m = n = 15$ (yellow color) to the function in more closer form as compare to Figure 4.

**Figure 5.** The convergence of the GBS-type operator $\hat{B}Y^2_{15,15}(f; x, y)(\text{yellow})$ to the function $f(x, y)(\text{green})$.

**Concluding result:** It can be concluded from the graphical representations of the operators $\hat{B}Y^a_{m,n}(f; x, y)$ and $\hat{Y}_{m,n,a}(f; x, y)$ that the rate of convergence of the GBS-type operators (2.12) is better than the bivariate operators (1.1).

### 3.1. Numerical approach.

Next, we discuss the absolute error of the GBS-type operators (2.12) as well as the bivariate operators (1.1) to the function $f(x, y)$ and compare these operators with their numerical errors at different points and for different values of $m, n$.

Let $G^a_{m,n}(f; x, y) = |\hat{B}Y^a_{m,n}(f; x, y) - f(x, y)|$ and $S^a_{m,n,a}(f; x, y) = |\hat{Y}_{m,n,a}(f; x, y) - f(x, y)|$, then the given Table 1 represents the numerical approximations of the GBS-type operators (2.12) and bivariate operators (1.1).

| $m=n$ | $SY_{m,n,a}(f; x, y)$ | $G^a_{m,n}(f; x, y)$ |
|-------|------------------------|----------------------|
| 10    | 0.00887548             | 0.0000358021         |
| 15    | 0.00589734             | 0.0000160308         |
| 25    | 0.0035283              | $5.80313 \times 10^{-6}$ |
| 50    | 0.00176016             | $1.45647 \times 10^{-6}$ |
| 100   | 0.000879051            | $3.64803 \times 10^{-7}$ |

**Table 1.** A Comparison of the GBS-type operators and bivariate operators to the function $f(x, y)$

**Concluding remark:** From Table 1, it can be observed that the approximation by the GBS-type operators (2.12) to the function is better than the bivariate operators (1.1).

### 3.2. A comparison of the bivariate operators (1.1) with bivariate Kantorovich operators.

In this subsection, we show the graphical representation for the comparison of convergence of the bivariate operators (1.1) with the bivariate Kantorovich operators of Szász-Mirakjan. In 2006, Muraru [34] gave a quantitative approximation of Kantorovich-Szász bivariate operators, defined as

$$\hat{K}_{m,n}f : L_1([0, \infty) \times [0, \infty)) \to B([0, \infty) \times [0, \infty)), \ (m, n) \in \mathbb{N} \times \mathbb{N};$$

$$\hat{K}_{m,n}(f; x, y) = mne^{-m x -ny} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(mx)^{k_1}}{k_1!} \frac{(nx)^{k_2}}{k_2!} \int_0^{k_1+1} \int_0^{k_2+1} f(u, v) \, dudv.$$  

(3.1)

There are following computational examples, which represent the comparison.
Example 3.4. Let the function \( f(x) = x^2y(x-1)\sin(2\pi y) \) (green), for all \( 0 \leq x, y \leq 2 \) and choose the value of \( m, n = 10 \), for which the bivariate operators (yellow) defined by (1.1) show the better rate of convergence than the Kantorovich-Szász bivariate operators \( \hat{K}_{m,n}(f; x, y) \) (red) defined by (3.1), graphical representation can be seen by the Figure 6.

\[ \text{Figure 6. The comparison of the convergence of the operators } \hat{Y}_{m,n,a}(f; x, y) \text{ (yellow) and } \hat{K}_{m,n}(f; x, y) \text{ (red) to the function } f(x) \text{ (green)} \]

Example 3.5. Consider the function \( f(x) = x^2y\cos(\pi y) \) (green), for all \( 0 \leq x, y \leq 4 \) and choose \( m, n = 10 \), for which the bivariate operators (yellow) defined by (1.1) present the better rate of convergence than the bivariate Kantorovich operators \( \hat{K}_{m,n}(f; x, y) \) (red) defined by (3.1), the graphical representation is illustrated by Figure 7.

\[ \text{Figure 7. The comparison of the convergence of the operators } \hat{Y}_{m,n,a}(f; x, y) \text{ (yellow) and } \hat{K}_{m,n}(f; x, y) \text{ (red) to the function } f(x) \text{ (green)} \]

Example 3.6. Let the function \( f(x) = y^2\cos(2\pi x) \) (green), for all \( 0 \leq x, y \leq 4 \) and consider \( m, n = 20 \), for which the graphical representation of the bivariate operators \( \hat{Y}_{m,n}(f; x, y) \) (yellow) defined by (1.1) and the bivariate Kantorovich operators \( \hat{K}_{m,n}(f; x, y) \) (red) defined by (3.1) is illustrated in Figure 8.
and for 

\[\text{by:}\]

\[\text{stated as:}\]

operators \(\hat{B}\)

functions considered to be 

so called GBS operators of Mirakjan-Favard-Szász and gave an approximation of the 
examples.

\[(3.4)\]

\(L\)

GBS-modification operators (3.2) are the GBS-form of the operators 

In 2008, Pop \[45\] introduced an associated GBS-type operators of the linear positive operators defined 

by an infinite sum, which can be defined as follows:

Then for 

f

\[(3.3)\]

\(L\)

\[(3.2)\]

\(f\)

Concluding Remark: By the above figures (6, 7, 8), we can say the rate of convergence of the bivariate 
operators \(Y_{m,n.a}(f; x, y)\) (1.1) is better than biavaraite Kantorovich operators \(K_{m,n}(f; x, y)\) defined by (3.1).

3.3. Comparison of associated GBS operators with the GBS-type operators of an infinite sum. In 2008, Pop \[45\] introduced an associated GBS-type operators of the linear positive operators defined 

by an infinite sum, so called GBS operators of Mirakjan-Favard-Szász and gave an approximation of the 
functions considered to be \(B\)-continuous and \(B\)-differentiable. The defined associated GBS operators can be 

stated as:

Let \(m, n \in \mathbb{N}\), the operators \(UL_{m,n}^*: E(I \times I) \to F(J \times J)\) are defined for any function \(f \in E(I \times I)\) 

and for \((x, y) \in J \times J\) such that

\[(3.2) \quad UL_{m,n}^*(f; x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \psi_{m,k_1}(x)\psi_{n,k_2}(y)[f(x_{m,k_1}, y) + f(x, x_{n,k_2}) - f(x_{m,k_1}, x_{n,k_2})],\]

where \((x_{m,k_1})_{k_1 \in \mathbb{N}}\) \(m \geq 1\), \((x_{n,k_2})_{k_2 \in \mathbb{N}}\) \(n \geq 1\) are the sequences of nodes and the functions \(\psi_{m,k_1} : I \to \mathbb{R}, \psi_{n,k_2} : J \to \mathbb{R}\) with the properties, \(\psi_{m,k_1} \geq 0, \psi_{n,k_2} \geq 0\), where \(I, J \subset \mathbb{R}, I \cap J \neq \emptyset\). The above 

GBS-modification operators (3.2) are the GBS-form of the operators \(L^*-\)type operators \[45\] and are given by:

\[(3.3) \quad L_{m,n}^*(f; x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \psi_{m,k_1}(x)\psi_{n,k_2}(y)f(x_{m,k_1}, x_{n,k_2}), \quad (x, y) \in J \times J,\]

where \(m, n \in \mathbb{N}\), \(f \in E(I \times I)\) and \(L_{m,n}^*: E(I \times I) \to F(J \times J)\).

For the particular case, Pop \[45\] determined the convergence properties for the GBS operators of 

Mirakjan-Favard-Szász. Here, if \(\psi_{m,k_1}(x) = \frac{k_1}{m}, \psi_{n,k_2}(x) = \frac{k_2}{n}\) and \(\psi_{m,k_1} = e^{-mx\frac{(mx)^{k_1}}{k_1!}}, \psi_{n,k_2} = e^{-ny\frac{(ny)^{k_2}}{k_2!}}\) 

then for \(f \in C([0, \infty) \times [0, \infty))\), the above operators (3.2) can be reduced to GBS operators of Mirakjan-Favard-Szász, which can be defined as follows:

\[(3.4) \quad US_{m,n}^*(f; x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} e^{-mx-ny\frac{(mx)^{k_1}}{k_1!} \frac{(ny)^{k_2}}{k_2!}} [f\left(\frac{k_1}{m}, y\right) + f\left(x, \frac{k_2}{n}\right) - f\left(\frac{k_1}{m}, \frac{k_2}{n}\right)] .\]

This subsection is very crucial from the comparison point of view of the GBS type operators as defined 

by (2.12) with the GBS operators of the Mirakjan-Favard-Szász type (3.4), which is shown by the following 
examples.
Example 3.7. Let the function be defined by \( f(x, y) = e^{x+y} \) (green). A comparison for the convergence of the GBS-type operators \( \hat{B}_{m,n}^{a}(f; x, y) \) (red) with the GBS operators of Mirakjan-Favard-Szász \( U_{m,n}^{*}(f; x, y) \) (black) to the function \( f(x, y) \) is illustrated in Figure 9 for \( m = n = 2 \). It can be observed that the GBS-type operators defined by (2.12) have a better rate of convergence than the GBS operators of Mirakjan-Favard-Szász as defined by (3.4).

![Figure 9](image)

**Figure 9.** A comparison of the rate of convergence of the GBS-type operators \( \hat{B}_{m,m}^{a}(f; x, y) \) and GBS operators of the Mirakjan-Favard-Szász to the function \( f(x, y) \).

Example 3.8. For the same function \( f(x, y) = e^{x+y} \) and at a certain point \((x, y)\), the error estimation of the GBS-type operators \( \hat{B}_{m,n}^{a}(f; x, y) \) and GBS operators of Mirakjan-Favard-Szász \( U_{m,n}^{*}(f; x, y) \) has been computed in Table 2.

| Error in the approximation for \( \hat{B}_{m,n}^{a}(f; x, y) \) and \( U_{m,n}^{*}(f; x, y) \) to the function \( f(x, y) \) |
|---|---|---|
| \( m=n \) | \( |U_{m,n}^{*}(f; x, y) - f(x, y)| \) | \( |\hat{B}_{m,n}^{a}(f; x, y) - f(x, y)| \) |
| 10 | \( 3.28277 \times 10^{-5} \) | \( 3.00798 \times 10^{-6} \) |
| 20 | \( 7.91371 \times 10^{-6} \) | \( 7.35189 \times 10^{-7} \) |
| 50 | \( 1.23907 \times 10^{-6} \) | \( 1.16048 \times 10^{-7} \) |
| 100 | \( 3.07549 \times 10^{-7} \) | \( 2.88814 \times 10^{-8} \) |

**Table 2.** A Comparison of the GBS-type operators \( \hat{B}_{m,n}^{a}(f; x, y) \) and GBS operators of Mirakjan-Favard-Szász \( U_{m,n}^{*}(f; x, y) \) to the function \( f(x, y) \).

**Concluding Remark:** It can be concluded from Table 2 that the error arising in the approximation at a certain point by GBS-type operators defined by (2.12) to the function is much smaller than the GBS operators of Mirakjan-Favard-Szász as defined by (3.4). Hence our GBS-type operators have a better rate of convergence than the GBS operators of Mirakjan-Favard-Szász type.

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