Relativistic Calogero-Moser Model as Gauged WZW Theory

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Abstract

We study quantum integrable systems of interacting particles from the point of view, proposed in [4]. We obtain Calogero-Moser and Sutherland systems as well as their relativistic-like generalization due to Ruijsenaars by a hamiltonian reduction of integrable systems on the cotangent bundles over semi-simple Lie algebras, their affine algebras and central extensions of loop groups respectively. The corresponding 2d field theories form a tower of deformations. The top of this tower is the gauged G/G WZW model on a cylinder with inserted Wilson line in appropriate representation. Its degeneration yields 2d Yang-Mills theory, whose small radius limit is Calogero model itself. We make some comments about the spectra and eigenstates of the models which one can get from their equivalence with the field theories.

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This paper should be considered as a natural continuation of [4]. The question of determination of spectra and eigenstates in some inegrable many-body systems will be addressed here. The main tool in our approach will be a hamiltonian reduction approach to the systems with symmetries. We shall consider two dimensional gauge theories, which can be treated as a topological theories and due to their topological nature only finite number of degrees of freedom form a "physical" Hilbert space of the theories. This space can be interpreted as a state space in some quantum mechanical problem. On the other hand, on can easily calculate path integrals in the topological field theories under consideration. It provides us with an information about the spectra and wave functions in the quantum mechanical problems.

We consider first the Sutherland model [21] at some distinguished values of coupling constants and show that it can be obtained even at quantum level as a result of hamiltonian reduction from the cotangent bundle to the affine Lie algebra \( \hat{g} \). Then we compare this reduction with the reduction which is present in the two dimensional Yang-Mills theory and realize that these two in fact coincide. This permits us to get the Harish-Chandra-like formulas for the wavefunctions and for the kernel in the Sutherland model (actually, we use here results of [3]). This is in the good relation with the idea of I.Cherednik to consider affine algebras to get exact answers for many-body integrable systems [11]. This result generalizes results of [6], while some methods of [6] could be applied in our situation too to give a satisfying treatment of the large \( N \) behaviour of our systems.

Recently relativistic-like Calogero-Moser-Sutherland models were found [12]. We show that these models (actually, their trigonometric version) appear in the Hamiltonian reduction procedure applied to the \( T^*\hat{G} \) (the continuation of approaches of [3], [8]), and that this reduction is equivalent to that in the gauged \( G/G \) Wess-Zumino-Witten theory (and also in the Chern-Simons theory on a threefold which is a suspension of the two dimensional surface by a circle). Thus, we are able to tell something about the spectra and their degeneracy in these case. To this end we need an expression for the path integral in the gauged \( G/G \) WZW model on a disc. This expression can, in principle, be extracted from [13].

1 Affine Lie algebra approach to Sutherland model

1.1 Hamiltonian reduction of \( T^*\hat{g} \)

Let us consider some semi-simple real Lie algebra \( g \) with the Killing form \( <,> \) denoted also as a \( \text{tr} \) and let us fix an isomorphism between \( g \) and \( g^* \) induced from the \( \text{tr} \). Affine Lie algebra \( \hat{g} \) is defined to be the central extension of the loop algebra \( \mathcal{L}g \) that is it is a space of the
pairs \((\phi, c)\) where \(\phi\) is a map from the circle \(S^1\) to the \(g\) and \(c\) is a real number. The Lie algebra structure on \(\hat{g}\) is introduced as follows:

\[
[(\phi_1(\varphi), c_1), (\phi_2(\varphi), c_2)] = ([\phi_1(\varphi), \phi_2(\varphi)], \int_{S^1} \langle \phi_1, \partial_{\varphi} \phi_2 \rangle),
\]

Now let us turn to the dual space to the \(\hat{g}\). This space \(\hat{g}^\ast\) consists of the pairs \((A, \kappa)\) where \(A\) is \(g\)-valued one-form (actually, a distribution) on the circle and \(\kappa\) is just the real number. The pairing between the \(\hat{g}\) and \(\hat{g}^\ast\) is

\[
< (A, \kappa); (\phi, c) > = \int_{S^1} \langle \phi, A \rangle + c\kappa.
\]

The direct sum \(\hat{g} \oplus \hat{g}^\ast\) we will denote as \(T^*\hat{g}\). On this cotangent bundle the natural symplectic structure is defined:

\[
\Omega = \int_{S^1} \text{tr}(\delta \phi \wedge \delta A) + \delta c \wedge \delta \kappa \quad (1.1)
\]

We can define adjoint and coadjoint actions of the loop group \(L G\) on the \(\hat{g}\) and \(\hat{g}^\ast\). The former is defined from the commutation relations and the latter from the pairing, that is the element \(g(\varphi)\) acts as follows:

\[
(\phi(\varphi), c) \to (g(\varphi)\phi(\varphi)g(\varphi)^{-1}, \int_{S^1} \text{tr}(-\phi g^{-1} \partial_{\varphi} g) + c) \quad (1.2)
\]

\[
(A, \kappa) \to (gAg^{-1} + \kappa g \partial_{\varphi} g^{-1}, \kappa) \quad (1.3)
\]

This action clearly preserves the symplectic structure \((1.1)\) and thus defines a moment map

\[
\mu : T^*\hat{g} \rightarrow \hat{g}^\ast
\]

which sends a 4-tuple \((\phi, c; A, \kappa)\) to the pair \((\kappa d\phi + [A, \phi], 0)\).

Now let us choose an appropriate level of moment map and make a hamiltonian reduction under this level. To guess, what element of \(\hat{g}^\ast\) we should choose it is convinient to return back to the finite-dimensional example of this procedure. Namely, if one considers a reduction of the \(T^*g\) with respect to the action of \(G\) by conjugation, then, to get a desired Calogero system, one takes an element \(J\) of \(g^\ast\), which has maximal stabilizer, different from the whole \(G\). It is easy to show, that the representative of the coadjoint orbit of this element has the following form:

\[
J_\nu = \nu \sum_{\alpha \in \Delta^+} (e_\alpha + e_{-\alpha}), \quad (1.4)
\]

where \(\nu\) is some real number, \(e_{\pm \alpha}\) are the elements of nilpotent subalgebras \(n_{\pm} \subset g\), which correspond to the roots \(\alpha\), and \(\Delta^+\) is the set of positive roots. In fact, if the corresponding Coxeter group has
more than one orbit in the set of roots (i.e. two) then there are more
coupling constants \(1\). Let us denote the coadjoint orbit of \(J_\nu\) by \(O_\nu\)
and by the \(R_\nu\) the representation of \(G\), arising upon the quantization
of \(O_\nu\). For generic \(J \in g^*\) let \(G_J\) will denote the stabilizer of \(J\) in
the coadjoint orbit \(O_J\) of \(J\), i.e. \(O_J = G/G_J\).

Now let us return back to the case of affine Lie algebra. We could
just take the formula (1.4) and replace everywhere usual roots by
affine ones, but we prefer to have more geometrical reasoning. Because
of the vanishing level \(\kappa_{\mu}\) of the moment map, the coadjoint orbit of
generic value of it is rather huge, while the stabilizer of the element
\((J(\varphi); 0) \in \hat{g}^*\) is a "continious product" \(\prod_{\varphi \in S^1} G_{J(\varphi)}\) which is very
small in comparison with the whole loop group \(LG\) unless for almost
all \(\varphi \in S^1\) the group \(G_{J(\varphi)}\) coincides with \(G\).

All this argumentation shows that an appropriate value of the mo-
ment map we are looking for is the following:

\[
\mu = (J[\mu], 0) : J[\mu](\varphi) = \delta(\varphi) J_\nu
\]

Here by \(\delta(\varphi)\) we mean a distribution (one-form) supported at some
marked point \(0 \in S^1\) and normalized in such a way, that \(\int_{S^1} \delta(\varphi) = 1\). The
coadjoint orbit of \(\mu\) is nothing, but the finite-dimensional orbit
\(O_\nu\).

Now let us complete the reduction. To this end we should resolve
the equation

\[
\kappa_\partial_\varphi \phi + [A, \phi] = J_\mu(\varphi) = \delta(\varphi) J_\nu
\]

modulo the action of stabilizer of \(\mu\), that is modulo subgroup of
\(LG\), consisting of \(g(\varphi) \in G\), such that \(g(0) \in G_\nu\). We can do it as
follows. First we use generic gauge transformation \(\tilde{g}(\varphi)\) to make \(A\)
to be the Cartan subalgebra \(t \subset g\)-valued constant one-form \(D\) (it
is always possible for non-vanishing \(\kappa\)). We are left with the freedom
to use the constant gauge transformations with values in the Cartan
subgroup \(T \subset G\), generated by \(D\) - these do not touch \(D\). Actually,
the choice of \(D\) is not unique. The only invariant of \(LG\) action is the
conjugacy class of monodromy \(\exp(\frac{2\pi}{\kappa} D) \in T \subset G\). Let us fix some
of these choices. Let \(ix_i\) will denote the entries of \(D = iX\). Let us
decompose the \(g\)-valued function \(\phi\) on the \(S^1\) on its Cartan-valued
part \(P(\varphi) \in t\) and let \(\phi_{\pm}(\varphi) \in n_{\pm}\) be its nilpotent-valued parts. Let
\(\phi_\alpha = <\phi, e_\alpha>\). Then the equation (1.6) will take the form:

\[
\kappa_\partial_\varphi P = \delta(\varphi)[J^g_\nu]_{\gamma}\tag{1.7}
\]

\[
\kappa_\partial_\varphi \phi_\alpha + <D, \phi_\alpha> = \delta(\varphi)[J^g_\nu]_{\alpha}\tag{1.8}
\]

where \(J^g_\nu\) is simply \(\text{Ad}_{\tilde{g}(0)}^*(J_\nu)\), \([J]_\gamma\) denotes the Cartan’s part of \(J\)
and
\([J]_\alpha = <J, e_\alpha>\).
From the equation (1.7) we deduce that $D = \text{constant}$ and $[J^\nu, \gamma] = 0$. This implies, that be Cartan-valued constant conjugation we can twist $J^\nu$ to the $J_\nu$ itself. Then, (1.8) implies that locally (at $\varphi \neq 0$) we can represent $\phi_\alpha(\varphi)$ as follows:

$$
\phi_\alpha(\varphi) = \exp(-\frac{\varphi}{\kappa} < D, \alpha >) \times M_\alpha
$$

(1.9)

where $M_\alpha$ is locally constant vector in $g$. Look at the right hand side of (1.8) leads us to the conclusion, that $M_\alpha$ jumps when $\varphi$ goes through 0. This jump is equal to

$$
\exp(-\frac{2\pi}{\kappa} < D, \alpha >) - 1 \times M_\alpha = [J^\nu]_\alpha
$$

(1.10)

The final answer for the reduction is the following: the physical degrees of freedom are in the $\exp(-\frac{2\pi i}{\kappa} X)$ and $P$ (note that $P$’s entries are purely imaginary) with the reduced symplectic structure:

$$
\omega = \frac{1}{2\pi i} \text{tr}(\delta P \wedge \delta X)
$$

(1.11)

and we have

$$
\phi_\alpha(\varphi) = \nu \frac{\exp(-\frac{i\varphi}{\kappa} < X, \alpha >)}{\exp(-\frac{2\pi}{\kappa} < X, \alpha >) - 1}
$$

Now if we would take some simple hamiltonian system on the $T^*\hat{g}$ whose Hamiltonian is invariant under (1.2),(1.3) actions, then we will get somewhat complicated system on the reduced symplectic manifold, i.e. on $T^*T$. For example, let us take any invariant polynomial $Q(\phi)$ on the Lie algebra $g$, that is $Q \in S^1(g^*)^G = S^1(t^*)^W$, where $W$ is the Weyl group of $G$. Then we can construct a Hamiltonian on $T^*(\hat{g})$ by the formula

$$
H_Q = \frac{1}{2\pi} \int_{S^1} d\varphi Q(\phi)
$$

with somehow fixed one-form $d\varphi$ (for example, it will be convinient for us to fix a normalization $\frac{1}{2\pi} \int_{S^1} d\varphi = 1$). For quadratic casimir we get

$$
H_2 = \frac{1}{4\pi} \int_{S^1} d\varphi < \phi, \phi >
$$

(1.12)

On the reduced manifold we are left with the Hamiltonian

$$
H_2 = -\frac{1}{2} \text{tr} P^2 + \sum_{\alpha \in \Delta_+} \frac{\nu^2}{\sin^2 < X, \alpha >}
$$

(1.13)

which coincides with the Hamiltonian of the Sutherland model. Here $\Delta_+$ denotes the set of positive roots of $g$. For example, for $G =$
SU(\(N\)) we have the Hamiltonian for pair-wise interaction between the particles with the potential:

\[
V_{ij}^A = \frac{g^2}{\sin^2(x_i - x_j)}
\]

for \(G = SO(2N)\) we have:

\[
V_{ij}^D = g_2^2\left[ \frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right] + g_1^2\left[ \frac{1}{\sin^2(x_i)} + \frac{1}{\sin^2(x_j)} \right]
\]

where \(g_1, g_2\) are coupling constants. In general case one has as many different coupling constants as many orbits in the root system has a Weyl group, i.e. two or one \([5]\).

### 1.2 Two-dimensional Yang-Mills theory

Let us consider some two dimensional surface \(\Sigma\) with fixed non-degenerate two-form \(\omega \in \Omega^2(\Sigma)\) but without fixed metrics. Then one can consider some principal \(G\)-bundle \(E\) over \(\Sigma\) and the space \(A\) of connections on \(E\). This space is a symplectic one and group \(G\) of gauge transformations acts on it preserving that symplectic structure. The moment map for this action in fact coincides with the curvature \(F = dA + A \wedge A\) of the connection \(A\) unless our surface \(\Sigma\) has holes. In that case one has to specify what are boundary conditions for the gauge transformations.

Let us take a surface \(\Sigma\) with some fixed contours \(C_{\beta}\) (in the case of holes one can consider non-closed contours with the ends on this holes). Then for each loop \(C\) a natural gauge invariant observable is defined:

\[
W_C = \chi_\alpha(P \exp \oint_C A)
\]

where \(\alpha\) is some irreducible representation of \(G\). For non-closed contours one should attach to each end of it a representation \(\alpha_{\Gamma}, \beta_{\Gamma}\), belonging to the ends of the same contour \(\Gamma\) are to be dual. Then, the observable we wish to construct takes its values in the tensor product \(\alpha_{\Gamma} \otimes \alpha_{\Gamma}^*\) (where \(\alpha_{\Gamma}\) belongs to that end from this this contour departs due to orientation - it is the group element \(P \exp \int_{\Gamma} A \in G\) in the representation \(\alpha_{\Gamma}\), i.e. \(W_{\Gamma} = T_{\alpha_{\Gamma}}(P \exp \int_{\Gamma} A)\). This is an example of the coloring of the (fat) graph in terminology of \([\mathbb{F}]\).

Now let us write down the action of Yang-Mills-like gauge theory on \(\Sigma\). To this end we need an extra field \(\phi \in \text{Maps}(\Sigma, \mathfrak{g})\).

Let us take any invariant polynomial function on \(g\), i.e. \(Q \in S(g^*)^G\). Then the action is:

\[
S_{\Sigma, Q} = \int_\Sigma <\phi, F > + \omega Q(\phi)
\]

(1.14)
Let us locally decompose the tangent bundle to \( \Sigma \) to the time-like and space-like directions. This is a choice of real polarization - it has some difficulties when the surface has pants-like vertices. In the vicinity of generic point \( P \in \Sigma \) we can always find such a coordinates \( t, x \) that 2-form \( \omega \) will take its canonical form;

\[
\omega = dt \wedge dx
\]

We suppose the orbits of the vector field \( \frac{\partial}{\partial x} \) to be periodic almost everywhere. The symplectic structure on \( \mathcal{A} \) (more precisely, that part of it, which corresponds to the vicinity of \( P \)) is

\[
\Omega = \int_\Sigma \langle \delta A_t, \delta A_x \rangle dt \wedge dx
\]

(This symplectic structure, certainly, doesn’t depend on the \( \omega \) or whatever, it is simply \( \Omega = \int_\Sigma \text{tr}(\delta A \wedge \delta A) \).

Now let us suppose, that some contour \( \Gamma \) goes straight along the time-like direction \( \frac{\partial}{\partial t} \). Then we get the following path integral (we disregard now any kinds of \( \Sigma \) except spheres with two holes \( \gamma_1, \gamma_2 \), and on each hole points \( P_1 \) and \( P_2 \) respectively are marked. Contour \( \Gamma \) goes from \( P_1 \) to \( P_2 \):

\[
\int D A_t D A_x D \phi \exp[-S_{\mathcal{V} \times I, Q}] < v_1 | P \exp \int_\Gamma A | v_2 >
\]

here \( < v_1, | v_2 > \) are the vectors in the representations \( \alpha_\Gamma, \alpha_\Gamma^* \) respectively. We wish to take \( \alpha_\Gamma \) to be equal \( R_\nu \) from the previous section.

Let us remind some facts about \( < v_1 | P \exp \int_\Gamma A | v_2 > \). It follows ideologically the Kirillov constructions of the representations (and it was checked in [10]), that this matrix element can be represented as a path integral with boundary conditions. We have to take a coadjoint orbit \( \mathcal{O}_\nu \) which corresponds to the \( R_\nu \) choose one-form \( \theta = d^{-1} \omega_\mathcal{O} \), where \( \omega_\mathcal{O} \) is a natural Kirillov’s form on \( \mathcal{O}_\nu \) and then we can write a desired path integral:

\[
< v_1 | P \exp \int_\Gamma A | v_2 > \sim \int D \chi \exp[- \int (\theta - < A_t, \mu_\mathcal{O} >)]
\]

where integral in the r.h.s. needs some boundary conditions, which can be specified after choosing a polarization on the orbit \( \mathcal{O}_\nu \), \( \mu_\mathcal{O} : \mathcal{O}_\nu \rightarrow \mathfrak{g}^* \) is a moment map corresponding to the natural \( G \) action on the orbit. Finally, \( D \chi \) is a path integral measure, constructed from the \( \omega \). In the case we are interested in, (and for \( G = SU(N) \)) \( \mathcal{O}_\nu = \mathbb{C}P^{N-1} \) with the symplectic form \( \omega_\mathcal{O} = N \nu \times \text{Fubini-Shtudi} \) form. The corresponding moment map \( \mu_\mathcal{O} \) sends the point \( (z_0 : z_1 : \ldots : z_{N-1}) \in \mathbb{C}P^{N-1} \) to the matrix \( J_{ij} = \nu(\delta_{ij} - z_i \overline{z}_j) \in \mathfrak{su}(N)^* \). So, we can rewrite the matrix element of the Wilson line as the following path integral:
\[
\int \mathcal{D}z_i \mathcal{D}\bar{z}_i \exp\left[-\nu \int \sum_i z_i \partial_t z_i - \sum_{ij} A_{ij}(t) z_i \bar{z}_j \right]
\] (1.15)

Then, we put this expression into the Yang-Mills path integral and see, that the field \( A_t \) plays the role of Lagrangian multiplier, giving rise to the constraint \((1.6)\). Resolving it (modulo gauge transformations, preserving the point on the coadjoint orbit) we get the same finite-dimensional Hamiltonian for the eigenvalues of the monodromy around the space-like slice, as we had in the previous section. This proves the equivalence between these two theories.

We see, that the whole construction to work, we need the orbit \( O_\nu \) to be quantizable. This implies (in the case of \( G = SU(N) \)) that \( N\nu \) should be integer. Further we will get more restrictive condition.

### 1.3 Wavefunctions and amplitudes for the Sutherland model

What can we get from this equivalence? We can solve the Sutherland model due to possibility of simple solving of the 2d Yang-Mills theory. Namely, in 2d YM theory, we can calculate exactly the path integral on the discomputed with the help of the form \( \omega \) and \( a_D \) is the area of \( \mathcal{D} \), i.e. \( \int \mathcal{D} \omega \).

In our situation we have to attach a representation \( R_\nu \) to the outgoing end of the Wilson line and the dual to the incoming one. So, the boundary conditions we should fix are: vectors \( < v_1 | \in R_\nu^* \), \( | v_2 > \in R_\nu \) and the monodromies \( g_2 = P \exp \int_0^1 A \), \( g_1 = P \exp \int_0^1 A \) around the initial and final holes respectively.

Gauge transformations will rotate monodromies and vectors as follows: \( g_i \rightarrow hg_i h^{-1} \), \( | v_i > \rightarrow T_{R_\nu}(h)| v_i > \ i = 1, 2 \) and the answer will be invariant with respect to these transformations. The measure on \( g_i \) is the Haar measure. Using this gauge freedom, we can make \( g \) to be in the Cartan subgroup \( T \in G \). The measure on \( T \) induced from the modding out angle variables is the product of the Haar measure on the \( T - \prod_i d\theta_i \) and corresponding group-like VanderMonde determinant. To get the proper answer for the transition amplitude we should take a square root of this measure (to get a half-form). Finally, note that in the Sutherland model itself we have a quadratic \( Q(\phi) = \frac{1}{2} \text{tr}(\phi^2) \).

Now let us turn to the calculation. Let us cut our cylinder along the contour \( \Gamma \): we will get the disk, and on the edges, corresponding to \( \Gamma \) we have some group element \( h \in G \), which should be integrated out with the weight \( < v_1 | T_{R_\nu}(h)| v_2 > \). The whole monodromy around the disk is given by \( g_1 h g_2^{-1} h^{-1} \), so the answer for the transition amplitude (or evolution kernel) is the following:

\[
< g_2; v_2 | \exp[-tH]| g_1; v_1 > = \sum_h d_h e^{-tQ_2(h+\rho)} \int dh \chi_{\alpha_h}(g_1 h g_2^{-1} h^{-1}) < v_1 | T_{R_\nu}(h)| v_2 >
\] (1.16)
From this expression we can extract the spectrum of the model as well as the structure of the Hilbert space (this can be achieved also by looking at the $T^*G$): 

$$
\mathcal{H} = \bigoplus_h \text{Inv}(\alpha_h \otimes \alpha_h^* \otimes R_\nu) = \bigoplus \sum_{\Phi_h} C \cdot \Phi_h
$$

where $\Phi_h : \alpha_h \rightarrow \alpha_h \otimes R_\nu$ is an intertwiner.

$$
E_{\alpha_h} = \frac{1}{2} \langle \hat{h} + \rho, \hat{h} + \rho \rangle
$$

From the answer for the kernel we can realize that more restrictive conditions on coupling $\nu$ should be imposed to be possible to get the Sutherland system as quantum reduction: if we take $g_1, g_2$ to be $T$-valued, then the integral over $h$ will invariant under left and right independent multiplication of $h$ by the elements of $T$ - this lead to the condition $T_{R_\nu}(T)|v_i> = |v_i>$. It is the integral $\nu$, for which $R_\nu$ contains such a vector (vacuum vector of $R_\nu$). Moreover, there exist only one such a vector. Let us denote it as $|0>$. 

To get the expression for the wavefunction, let us look once again on the integral over $h$. It can be taken by representing character in the orthonormal base of representation $\alpha_h$, so we will arrive to the expression:

$$
\Psi_{\alpha_h}(\{\theta_i\}) = N(\{\theta_i\}) \sum_{mn} C_{mn} \langle m | T_{\alpha_h}(\text{diag}[e^{i\theta_i}])|n \rangle
$$

(1.17)

here $N(\{\theta_i\})$ denotes the normalization factor, which comes from the taking into account the group-like Vandermonde determinant, $m, n$ run over the orthogonal base of the representation $\alpha$ and coefficients $C_{mn}$ are defined via:

$$
\int dh T_{\alpha}(h)_{mk} T_{\alpha}(h)_{nl}^{*} T_{R_\nu}(h)_{00} = C_{mn} C_{kl}^{*}
$$

so they are just Littlewood-Richardson (or Clebsh-Gordan) coefficients.

It is clear, that this consideration generalizes that of [8].

2 Deformation of Calogero-Moser models - trigonometric Ruijsenaars’s system.

2.1 Hamiltonian reduction of the cotangent bundle of a central extension of loop group

In this section we will consider the deformation of the previous constructions. Namely, let us consider the Hamiltonian reduction of the
cotangent bundle to the central extended loop group $\hat{G}$. This symplectic manifold is a space of 4-tuples

$$(g : S^1 \to G, c \in U(1); A \in \Omega^1(S^1) \otimes g^*, \kappa \in \mathbb{R}).$$

The action of an element $h \in L\hat{G}$ is the following:

$$g \to hgh^{-1}, \ A \to hAh^{-1} + \kappa h\partial h^{-1}$$

$$\kappa \to \kappa, \ c \to c \times S(g, h)$$

where $S$ is constructed from the $U(1)$-valued two-cocycle on the group $L\hat{G}$, $\Gamma(g, h)$ (it provides $L\hat{G}$ with a central extension) (the cocycle is supposed to be normalized : $\Gamma(g, g^{-1}) = 1$):

$$S(g, h) = \Gamma(h, g)\Gamma(hg, h^{-1})$$

As always, on this cotangent bundle exists a natural symplectic structure and the loop group action preserves it. Explicitly this form $\Omega$ can be written in some trivialization of the cotangent bundle $T^*\hat{G}$. Namely, we identify it with the direct product $\hat{g} \times \hat{G}$ with the help of left-invariant one-forms $\kappa\partial \varphi + A\varphi \in \Omega^1(\hat{G})$. The form $\Omega$ has the following structure:

$$\Omega = \int_{S^1} \text{tr}[A(g^{-1}\delta g)^2 + \delta A \wedge g^{-1}\delta g] +$$

$$\int_{S^1} \text{tr}[\kappa \partial \varphi g \cdot g^{-1}(\delta g \cdot g^{-1})^2 - \kappa \delta(\partial \varphi g) \cdot g^{-1}\delta g \cdot g^{-1}] +$$

$$+ c^{-1}\delta c \wedge \delta \kappa$$

The moment map of the action of loop group has the form:

$$\mu(g, c; A, \kappa) = (gAg^{-1} + \kappa g\partial g^{-1} - A, 0)$$

It is tempting to make the Hamiltonian reduction under some appropriate level of the moment map. It is clear that in this way we deform Sutherland system, presumably preserving its integrability. For simplicity we consider only $SU(N)$ case. The reasoning, similar to that in the YM case, suggests the level of moment map to be

$$\mu(g, c; A, \kappa) = i\nu(\frac{1}{N}Id - e \otimes e^+)\delta(\varphi)$$

As we did it in the previous sections, by general gauge transformation $H(\varphi)$ we can make $A$ to be constant diagonal matrix $D$, defined modulo affine Weyl group action, i.e. modulo permutations and addings of the integral diagonal matrices. Generally, such a transformation doesn’t respect the value of the moment map. Let $H$ denotes the value of $H(0)$ at the point 0. The matrix $H$ defines a point in the coadjoint orbit of $J$, i.e. $\mathcal{O}_J = \mathbb{C}P^{N-1}$. The semi-direct product of Cartan torus $U(1)^{N-1}$ and Weyl group $S_N$ acts on $\mathcal{O}_J$ by permutations of homogeneous coordinates and their multiplications by phases. The quotient $\mathcal{O}_J/U(1)^{N-1}$ coincides with $(N-1)$- simplex.
The equation (2.2) takes the form:

\[ gDg^{-1} + \kappa gdg^{-1} - D = i\nu\left(\frac{1}{N}Id - f \otimes f^+\right)\delta(\varphi) \quad (2.4) \]

Here \( f = He \) is some vector in \( \mathbb{C}^N \) with unit norm \( <f, f> = 1 \). We can assume that \( f \in \mathbb{R}^N \), due to the remaining possibility of the left multiplication \( H \to YH \) with \( Y \) being diagonal unitary matrix (it comes from the gauge freedom which survives after diagonalization of \( A \)). Equation (2.4) yields immediately:

\[ g = \exp\left(\frac{\varphi}{\kappa}D\right)G(\varphi)\exp\left(-\frac{\varphi}{\kappa}D\right); \quad \partial_\varphi G = -\frac{J}{\kappa}G\delta(\varphi) \]

where \( J = i\nu\left(\frac{1}{N}Id - f \otimes f^+\right) \). Let us also introduce a notation for the monodromy of connection \( D \):

\[ Z = \exp\left(-\frac{2\pi}{\kappa}D\right) = \text{diag}(z_1, \ldots, z_N), \quad \prod_i z_i = 1, \quad z_i = \exp\left(\frac{2\pi iq_i}{\kappa}\right) \]

In these notations the matrix \( \tilde{G} \) can be written as follows:

\[ \tilde{G}_{ij} = -\lambda^{-\frac{N-1}{2}}\frac{\lambda^{-N} - 1}{\lambda^{-1}z_i - z_j}e^{i\theta_i}\left(Q^+(z_i)Q^-(z_j)\right)^{1/2} \quad (2.6) \]

- the ratio of the finite difference and derivative of \( P \). When \( \lambda \to 1 \), the rational functions \( Q^\pm(z) \) tend to \( \frac{1}{N^i} \).

The next step is to consider some simple Hamiltonian system on \( T^*\hat{G} \), which is invariant with respect to the loop group adjoint action and make a reduction. It is obvious that any \( Ad\)-invariant function \( \chi : G \to \mathbb{R} \) defines a Hamiltonian

\[ H_\chi = \int_{S^1} d\varphi \chi(\varphi) \]
For example, $\chi_{\pm}(g) = \text{Tr}(g \pm g^{-1})$ (here $\text{Tr}$ is a trace in the $N$-dimensional fundamental representation of $SU(N)$) give the Hamiltonians (up to a constant factor, independent of $q_i$):

$$H_{\pm} = \sum_i (e^{i\theta_i} \pm e^{-i\theta_i}) \prod_{j \neq i} f(q_{ij})$$

(2.7)

where the function $f(q)$ is given by:

$$f^2(q) = \left[1 - \frac{\sin^2(\pi \nu/\kappa N)}{\sin^2(\pi \nu/\kappa N)}\right].$$

For detailed elaboration of different limits of the Hamiltonian see, for example [12]. In particular, taking an appropriate scaling limit $\kappa \to \infty$ will return us back to Sutherland model. Note also, that in order to have a real-valued Hamiltonian we need $\nu < q_{ij}$ for any $i \neq j$, thus $\nu/\kappa$ should be restricted from above.

2.2 On Gauged $G/G$ WZW Model

Starting from the very symplectic form (2.1) and the moment map (2.2) we can write a geometric bulk action for the field theory. It will be a sum of the term like $\int pdq$ and a moment value fixing term $\int A_t \mu dt$, where $A_t$ is going to be a time-like component of a gauge field, serving as a Lagrange multiplier. Thus, the action is (we omit the term $c^{-1} \partial_t c \kappa$, while fix the non-vanishing level $\kappa$):

$$S(A, g) = \int d\varphi dt \text{tr} [-A_\varphi g^{-1} \partial_t g - \kappa \partial_t g \cdot g^{-1} \cdot \partial_\varphi g \cdot g^{-1} +$$

$$+ \kappa d^{-1} (\partial_\varphi g \cdot g^{-1} (dg \cdot g^{-1})^2) +$$

$$+ A_t (\kappa g^{-1} \partial_\varphi g + g^{-1} A_\varphi g - A_\varphi)]$$

This is the action of $G/G$ gauged WZW model [22], a topological gauge theory recently shown to give a Verlinde formula for the dimension of the space of conformal blocks in WZW theory on the surface [13].

Our theory will contain also a Wilson line in the representation $R_\nu$. To compute the path integral on a cylinder, as in the case of Yang-Mills theory, one cuts a cylinder along the Wilson line and the answer for the evolution kernel will have the form (1.16). In fact the representations will run only through integrable representations of $SU(N)_k$. We conjecture that the answer can be obtained also from $U_q(SL_N)$ representation theory [13], thus establishing the relation between this quantum group and gauged WZW theory once again.
2.3 Chern-Simons interpretation

Now we present the interpretation of the previous section construction in terms of Chern-Simons field theory, deformed by some Hamiltonian. Let us consider a Chern-Simons theory with gauge group $SU(N)$ on the space-time manifold being a product of the interval and a two-torus $X = I \times T^2$. Its action is given by:

$$S_{CS} = \frac{i\kappa}{4\pi} \int_X \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$  \hspace{1cm} (2.8)

We know that the phase space of Chern-Simons theory on a $\Sigma \times I$ with $I$ being an interval is a moduli space of flat connections. In the case of inserted into the path integral Wilson lines, let us say just one Wilson line in the representation $R$, when the integral in hands looks like:

$$\mathcal{D}A < v_1 | TR(P \exp \int A) | v_2 > \exp(-S_{CS}(A))$$

the answer will be slightly changed, namely we will have to consider a moduli space of connections on the punctured surface with prescribed conjugacy class of monodromy of connection around the puncture. This class $U$ is related to the highest weight $\hat{h}$ of the representation $R_\nu$ like $U = \exp(\frac{2\pi i}{\kappa+N}(\text{diag}(\hat{h}))$ If our surface is a two-torus $\Sigma = T^2$, then we can write immediately the condition on the monodromies $g_A, g_B$ of connection around $A$ and $B$ cycles and the monordomy $g_C$ around the puncture:

$$g_A g_B g_A^{-1} g_B^{-1} = g_C$$

Keeping in mind the condition on the conjugacy class of $g_C$, taking into account that for the representation $R_\nu$ we have a signature like $\nu \times \text{diag}(N,0,\ldots,0)$ we arrive to the equation (2.5). The (non-rigorous) explanation of this relation between the connections on the two-torus and $T^*G$ is the following. One can consider a hamiltonian reduction of the symplectic space of connections on the two-torus by the subgroup of the group of gauge transformations, which are equal to identity at some (non-contractible) contour $C$ on the torus. If one chooses a singular co-adjoint orbit of the moment map $F = B\delta_C$, where $B$ is a one-form on the $C$ and $\delta_C$ is a delta-function in the direction, normal to the contour $C$, i.e. the connection $A$ is allowed to have a jump at the $C$, then after a reduction one is left with the limiting values of connection $A_+$ and $A_-$ and with a $G$-valued function on $C$ - monodromy $g(x)$ ($x$ is a coordinate on $C$) of connection around the cycle which begins and ends at the point $x \in C$ and which is transversal to $C$ in homology $H^1(T^2, \mathbb{Z})$. The boundary values $A_+$ and $A_-$ are related via the gauge transformation $g(x)$, i.e. $A_2 = A_1 g$. 

More rigorous, but less transparent explanation involves a consideration of the space of connections on the annulus, where the gauge

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1We are grateful to A.Alexeiev for a discussion on this point
group acts with a non-trivial cocycle in a Poisson brackets of its Hamiltonians. This cocycle vanishes on a "diagonal" subgroup, which is a group of gauge transformations on the annulus, whose boundary values are related by some diffeomorphism \( \varphi \) of one component of the boundary onto another, i.e. \( g(x) = g(\varphi(x)) \), with \( x \) being a point on the one component and \( \varphi(x) \) is its image in the other. Then it is easy to see, that the Hamiltonian reduction of the space of connections by the action of this group can be done in two steps; first, we reduce by gauge transformations, equal to identity at the boundary of the annulus, then we take a symplectic quotient with respect to this "diagonal" subgroup, isomorphic to the loop group \( \mathcal{L}G \).

Also we could point out the equivalence of the sector of the space of observables in the Chern-Simons theory on \( \Sigma \times S^1 \), namely, observables, which depend on \( A_t \) with \( \partial_t \) being a direction, tangent to \( S^1 \) and the gauged \( G/G \) WZW theory. This can be proved either considering a gauge fixing \( A_t = \text{const} \) and diagonal, or by pointing out that the wavefunctions in the Chern-Simons theory are those in the theory with an action \( \int dt \int_{S^1} \text{tr}(A \partial_t \bar{A}) \) projected onto the space of functionals, invariant under the gauge group action. This involves taking into account the non-invariance of the polarization \( \delta \bar{A} \) and this is why the WZW part of the action appears (see [13] for detailed discussion).

### 2.4 The spectrum and wavefunctions of the model

From this equivalence we can derive a spectrum of the Ruijsenaars model. Namely we will show that it is contained in the spectrum of the free relativistic fermionic system on the circle. By free relativistic model of fermions on the circle we mean the system of particles on the circle with coordinates \( 0 < q_i \leq 1 \) and conjugate momenta (rapidities) \( \theta_i \) living also on the circle, with the Hamiltonian

\[
H_+ = \sum_i \cos(\theta_i)
\]

and symplectic structure

\[
\omega = \frac{1}{2\pi} (\kappa + N) \sum \delta \theta_i \wedge \delta q_i
\]

(in fact, the shift \( \kappa \to \kappa + N \) is a quantum effect and can be deduced from [13], so we don’t consider its derivation here). The fermionic nature of these particles amounts to the phase space \( (T^{N-1} \times T^{N-1})/\mathcal{S}_N \) - just the moduli space of flat connections on the torus. Here \( T^{N-1} \) is a Cartan torus for \( SU(N) \) (center of mass is fixed) and \( \mathcal{S}_N \) is a symmetric group. We call them fermions, because their wavefunction vanishes on the diagonals \( z_i = z_j \) in the \( N-1 \)-torus (it contains a factor like \( \Delta(e^{i\theta_i}) \) - a group-like Vandermonde). It is obvious, that

\footnote{We are grateful to V.Fock for valuable discussions on this point}
the corresponding Schrödinger equation will be a difference equation, because $\cos\left(\frac{2\pi i}{\kappa + N}\partial_q\right)$ is a finite difference operator. Its eigenfunctions on the circle are exponents $e^{2\pi inq}$ ($n$ is necessarily integral) with eigenvalue

$$E_n = \cos\left(\frac{2\pi n}{\kappa + N}\right)$$

and the total spectrum will be given by the sum

$$\sum_i E_{n_i}$$

where we have to impose the following conditions, coming from the invariance $E_n = E_{n + \kappa + N}$ and the symmetric group action:

$$(\kappa + N) > n_N > \ldots > n_i > \ldots > n_1 \geq 0 \quad (2.9)$$

$$\sum_i n_i < (\kappa + N) \quad (2.10)$$

In our situation fermions on the circle are not completely ”free” and the spectrum is restricted. For our purposes will be more convenient to form a vector $\hat{h} = (n_i - i), i = 1, 2, \ldots, N$. It will correspond to the dominant weights of the $SU(N)$ irreducible representations. Now we can write an answer for the spectrum:

$$E_{\hat{h}} = \sum_{i} \cos\left(\frac{2\pi \hat{h}_i}{\kappa + N}\right)$$

and $\hat{h}$ satisfy the selection rule: there exists non-trivial intertwiner

$$\Phi_{\hat{h}} : \alpha_{\hat{h}} \rightarrow \alpha_{\hat{h}} \otimes R_{\nu}$$

(essentially it implies, that $\hat{h} = \lambda + \nu \rho$, where $\lambda_i \geq 0, i = 1, \ldots, N$.

An explanation of this spectrum can be derived from the properties of gauged $G/G$ WZW model [13]. Namely, to get a partition function of our interest we have to compute a path integral on the two-torus:

$$\int DA Dg \exp(S_{GW ZW}(g, A) + \int_{T^2} \text{Tr}(g + g^{-1})) \chi_{R_{\nu}}(P \exp \int_C A)$$

(2.11)

where $C$ is a non-contractible contour on the torus, $\chi_{R_{\nu}}$ is a character in the representation $R_{\nu}$. By the standard arguments we can calculate it by cutting the torus along $C$ and computing first the path integral in the theory $S_{GW ZW}(g, A) + \int \text{Tr}(g + g^{-1})$ on the annulus with coinciding monodromies $\hat{h}$ of the restrictions of the connection $A$ on the boundaries of the annulus and then integrate the answer over $H$ with the weight $\chi_{R_{\nu}}(\hat{h})$. To make a further progress one needs some information about the path integral on the annulus (or cylinder, what is the same). In principle, this can be deduced from [13]. We will not present here the whole derivation, just note that the path integral on the cylinder can be localized (either by some equivariant...
cohomology-like technique for $SU(N)$ gauge group itself or by using a gauge, where $g$ is diagonal and then use the usual localization for abelian gauge theory [13] on the $g$ with special values of eigenvalues, namely those which correspond to the integrable representations of $SU(N)_{\kappa}$, i.e. with the signature, given by numbers $n_i$, satisfying the same restrictions (2.9), (2.10). In arbitrary gauge theory in two dimensions the path integral on the disk and on the cylinder can be written in terms of sum over all irreducible representations of the gauge group via the characters in these representations. We conjecture that taking into account Wilson line in the representation $R_\nu$ will lead us to Macdonald polynomials [18]. We hope to return to more detailed investigation in the future.

3 Conclusions and discussion

In this paper we have considered some particular integrable deformations of Calogero-Moser systems, which preserve their trigonometric form, but make the quantum Schrödinger operator a finite difference operator. We have observed an equivalence of this theory with a gauged $SU(N)/SU(N)$ WZW model with Hamiltonian on the cylinder with inserted Wilson line. The level of WZW model and the representation in which Wilson line is taken are encoded in the coupling constant of the model and in the mass scale.

There are some important and intriguing questions, which remained beyond the scope of this paper. The first one is the elliptic analogue of the topological theories like gauged WZW or its degeneration - Yang-Mills theory. We expect this elliptic generalization to exist, because in the corresponding quantum integrable systems of particles there is such an analogue [16]. When elliptic curve degenerates one gets trigonometric version of integrable model. One of the possible ways is to consider double-loop algebras [19], [17] and/or Sklyanin algebra and its multivariable generalizations as the algebras of symmetries of corresponding theories. Some indications of the appearence of Sklyanin algebra are in the Askey-Wilson polynomials, which are particular examples of the wave-functions of systems under consideration [12], [14].

Another interesting question concerns investigation of different limits of our theories when coupling constants go to some extremal values. In fact, we have two important parameters in hands:

$$q = e^{\frac{2\pi i}{N}}$$

and

$$t = q^{\nu + 1}$$

We have conjectured that in the gauged $G/G$ WZW theory with the Wilson line in the representation $R_\nu$ the answers for path integrals

\footnote{See [20] for the progress in this direction}
on the surfaces with boundaries can be expressed in terms of Macdonald polynomials $P_h(x;q,t)$. These polynomials have very interesting limits (for example, they are responsible for the zonal spherical functions on the $p$-adic groups, namely Mautner-Cartier polynomials when $q = 0, t = 1/p$) if $q, t$ tend to some exceptional values. Unfortunately, in our approach $q$ should be a root of unity and $t$ should be an integral power of $q$, so there is no obvious way how to recognize all this beauty of Macdonald polynomials in gauged $G/G$ WZW theory for $SU(N)$ group. Presumably $SL(N,C)$ can overcome this problem.

It is known that Ruijsenaars system is closely related with the solitonic sector of Toda like theories, in particular asymptotics of the wave functions provides S-matrix for scattering in the solitonic sector [23]. The qualitative explanation of this relation looks as follows. Lagrangian approach for Toda theories is based on GWZW actions with some additional dependence on the spectral parameter. In the consideration above we added Hamiltonians and choosed some peculiar level of the momentum map. Thus the final action is nothing but the generating functional for GWZW while the level of the momentum map fixes the particular solitonic solutions. Hence we considered in essence the generating functional and solitonic S-matrix naturally appears in this context. Certainly this relation needs for further clarification, especially in context of $(q)KZ$ equations describing the soliton scattering amplitudes. We plan to consider this problem elsewhere.

Recently, in the works of Dunkl, Opdam and Heckmann [1] a powerful method of solving Sutherland and Calogero - type models was found. It is based on the introducing of operators acting on the group algebra of the symmetric group $C[S_N]$. This method was applied also in a seria of papers to a more general systems, including spin long-range interactions (see, e.g. [2]). To obtain a meaning of these operators in a gauge theory remains to be quite important question. They could be related somehow to the problem of Gribov copies in a gauge fixing in a gauge theory and, therefore can be of some imprtance in a four-dimensional gauge theory.

Another important direction is the finite-dimensional verification of equivalences, obtained between finite-dimensional system and (at first sight) infinite-dimensional. In the case of Sutherland model one can check, that there is an intermediate step in the procedure of Hamiltonian reduction, leaving us with the cotangent bundle to the group $SU(N)$. In the case of affine group instead of affine Lie algebra the situation seems to be slightly complicated and one hopes to get as a finite-dimensional analogue of $T^*SU(N)$ one of the $SU(N)$ doubles, most probably the Heisenberg double, which has an open dense symplectic leaf. But, instead of Hamiltonian action of $SU(N)$ on its cotangent bundle in this situation we have a Poissonian action of Poisson-Lie group, which is slightly more involved situation [3], [10]. Still it seems to be interesting question to answer. We think that in this way one can understand the relation of the quantum group
representation theory to our problems [18].

Let us finally note that finite-dimensional Ruijsenaars system can be consistently quantized. To this end one should find a proper normal ordered form of the Hamiltonian. For the $N = 2$ case it has the following form

$$H = \Phi_+ T_- \Phi_- + \Phi_- T_+ \Phi_+$$

$$T_\pm \Psi(x) = \Psi(x \pm \frac{1}{k+N})$$

where $\Phi_+ = \Phi_1, \Phi_- = \Phi_2$. In our approach we have the very normal ordering which preserves the integrability.

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References

[1] G.J. Heckmann, Invent. Math. 98 (1991) 341; C.F. Dunkl, Trans. Amer. Math. Soc. 311 (1989); E.M. Opdam, Invent. Math. 98 (1989) 1;
[2] D. Bernard, M. Gaudin, D. Haldane, V. Pasquier, SPhT-93-006; G. Felder and A.P. Veselov, ETH (1993); L. Brink, T.H. Hansson, M. Vassiliev, Phys. Lett. B 286 (1992);
[3] A. Migdal, ZHETP, 69(1975), 810; V. Rusakov, Mod. Phys. Lett. Bf A5 (1990) 693; E. Witten, preprint IASSNS-HEP-92/15; D. Gross, W. Taylor, Nucl. Phys. B400(1993)181; D. Gross, W. Taylor, preprint CERN-TH-6843/93, PUPT-1382,LBL-33767, UCB-PTH-93/09;
[4] A. Gorsky and N. Nekrasov, Hamiltonian systems of Calogero type and two dimensional Yang-Mills theory, UUITP-6/93, ITEP-20/93, hep-th/9304047, to appear in Nucl. Phys. B;
[5] M. Olshanetsky, A. Perelomov, Phys. Peps. 71, 313 (1981);
[6] D. Kazhdan, B. Kostant and S. Sternberg, Comm. on Pure and Appl. Math., Vol. XXXI, 481-507 (1978)
[7] A. Polychronakos, Phys. Rev. Lett. 69 (1992) 703;
[8] J.A.Minahan and A.P.Polychronakos, Phys.Lett.B302(1993)265;
M.Douglas, Conformal Field Theory Techniques in Large $N$
Yang-Mills theory, hepth/9311130
[9] V.Fock, A. Rosly, Moduli of flat connections and classical $r$-
matrix,ITEP-27/93;
[10] A.Alekseev, S.Shatashvili, CMP,128(1990)197
[11] I.Cherednik, RIMS 91 Integration of many body problems by
affine KZ equations
[12] S. N. M. Ruijsenaars, CMP 110 (1987) 191-213;
S.N.M. Ruijsenaars and H.Schneider, Ann. Phys. (NY) 170(1986)
370;
J.F. van Diejen, Commuting difference operators with polynomial
eigenfunctions, funct-an/9306002.
[13] M.Blau, G. Thompson, Nucl.Phys. B 408 (1993) 345-390;
A.Gerasimov, Localization in GWZW and Verlinde formula
UUITP-16/93, hepth/9305090;
[14] A.Gorsky, A.Zabrodin,Jorn.Phys.A26(1993),L635
[15] S.Shatashvili, Correlation functions in the Itzykson-Zuber model,
IASSNS-HEP-92/61;
[16] A.Alekseev, A.Malkin, Symplectic structure on the moduli
space of flat connections on the Riemann surface, UUITP-32/93,
hepth/9312004;
A. Alekseev, Integrability in Hamiltonian Chern-Simons theory,
hepth/9311074
[17] P.Etingof, I.Frenkel, Central extensions of current groups in two
dimensions,Yale priprint ,april 1992
[18] P.Etingof, A.Kirillov,Jr. , Macdonald polynomials and represen-
tations of quantum groups, hepth/9312103
[19] A.Gerasimov, D.Lebedev, A.Morozov, Int.Journ.of Mod.Phys.
A6(1991)977;
[20] A.Gorsky and N.Nekrasov, Elliptic Calogero-Moser System from
Two Dimensional Current Algebra, in preparation;
[21] F.Calogero,Jorn.Math.Phys.12(1971),419;
B.Sutherland,Phys.Rev.A5(1972),1372
[22] E.Witten,CMP,144(1992);
J.Sonnershtein,S.Yankelowicz,Nucl.Phys.B393(1993)301
[23] S.N.Ruijsenaars,in "Integrable and superintegrable sys-
tems",ed.B.Kupershmidt(World Scientific,Singapore,1990)165;
P.Freund.A.Zabrodin,CMP,147(1992)277;
O.Babelon,D.Bernard,Phys.Lett.B317(1993),363
Appendix A

In this section we prove the relation (2.6).

The necessary condition for the equation (2.5) to hold is that the eigenvalues of matrices $Z$ and

$$\exp(J)Z = \lambda (Id + (\lambda^{-N} - 1)f \otimes f^+)Z$$

coincide.

It leads to the system of equations

$$1 + (\lambda^{-N} - 1)\sum_{i=1}^{N} \frac{\Phi_i}{1 - \frac{\lambda}{z_k}} = 0,$$

$$k = 1, \ldots, N; \Phi_i = |f_i|^2$$

Also we have to satisfy the normalization condition $\sum_i \Phi_i = 1$. Let us note that our equations (0.1) resemble those arising in the definition of the Gelfand-Tzetlin coordinates [15]. The reason is obvious: Gelfand-Tzetlin coordinates are related to the imbeddings of $SU(N-1) \times U(1)$ into $SU(N)$, the quotient $SU(N)/SU(N-1) \times U(1))$ is precisely $\mathbb{CP}^{N-1}$ - our coadjoint orbit. Also let us note that the solution of these equation (written on a next line) looks like expression in the right hand side of the Bethe ansatz equations - the reason for this is to be understood. Our claim is: $\Phi_i = Q^+(z_i)$. We prove this formula by twofold evaluation of the following contour integrals in the complex plane around infinity:

$$I_0 = \frac{1}{2\pi i} \oint_{C_{\infty}} dz \frac{P(\lambda z)}{zP(z)}$$

and

$$I_k = \frac{1}{2\pi i} \oint_{C_{\infty}} dz \frac{P(\lambda z)}{(z - \lambda^{-1}z_k)P(z)}$$

Contour $C_{\infty}$ is oriented counterclockwise and goes around infinity. Both of them are obviously equal to $\lambda^{N}$ (residue at infinity). On the other hand, $I_0, I_k$ are given by the sum of residues at poles of the integrands inside of $C_{\infty}$. It gives

$$I_0 = 1 + \sum_i (\lambda^{N} - 1)Q^+(z_i)$$

$$I_k = \sum_i Q^+(z_i) \frac{\lambda^{N} - 1}{1 - \frac{\lambda}{z_k}}$$

and we obtain $j$ This is equivalent to $\lambda(Zv_i + (\lambda^{-N} - 1)f, Zv_i > f) = z_i v_i$, which leads to

$$v_i = g_i(\lambda^{-N} - 1)(z_i \lambda^{-1} - Z)^{-1} f$$
where \( g_i \) are some constants to be found from the normalization
\[
<v_i, v_j> = \delta_{ij}
\]
To this end we have to calculate an expression
\[
\Psi_i^{-1} = |g_i|^{-2} = (2 - \lambda^N - \lambda^{-N}) \sum_j \frac{\Phi_j}{(z_i \lambda^{-1} - z_j)(z_i^{-1} \lambda - z_j^{-1})} \tag{0.2}
\]
This expression can be evaluated with help of residues of the following contour integral
\[
J_i = \frac{1}{2\pi i} \oint_{C_\infty} \frac{P(\lambda z)}{(z - z_i \lambda^{-1})^2 P(z)}
\]
It is obvious that \( J_k = 0 \), because at infinity the integrand behaves as \( z^{-2} \). On the other hand, it implies:
\[
\frac{\lambda P'(z_k)}{P(\lambda^{-1} z_k)} = -\sum_i \text{Res}_{z = z_i} \frac{P(\lambda z)}{(z - z_i \lambda^{-1})^2 P(z)}, \tag{0.3}
\]
thus,
\[
\Psi_i = Q^{-}(z_i)
\]
Now we are in a position to write down an expression for matrix \( \tilde{G} \):
\[
\tilde{G}_{ij} = e^{i \theta_i \lambda^{N-1} \lambda^{-N} - 1} \frac{\lambda^{-N} - 1}{z_i - \lambda^{-1} z_j} (\Psi_i \Phi_j)^{1/2}
\]
Finally, after substituting here (0.2), (0.3) we just get (2.6). The phases \( \theta_i \) are restricted according to the condition \( \det(\tilde{G}) = 1 \), so this is a condition, imposed on the \( \sum_i \theta_i \), which doesn’t take us here. Still, for completeness, let us calculate this determinant:
\[
\det(\tilde{G}) = \sum_{\sigma \in S_N} (-)^\sigma \prod_i \tilde{G}_{i, \sigma(i)} =
\]
\[
e^{i \sum_j \theta_j \lambda^{N-1} \prod_j (Q^+(z_j)Q^-(z_j))^{1/2} \sum_{\sigma \in S_N} (-)^\sigma \prod_i \frac{\lambda^{-N} - 1}{\lambda^{-1} z_i - z_{\sigma(i)}} =
\]
\[
e^{i \sum_j \theta_j \lambda^{N-1} (\lambda^{-N} - 1)^N \left[ \frac{(\lambda - 1)(\lambda^{-1} - 1)}{(\lambda^N - 1)(\lambda^{-N} - 1)} \right]^{1/2} \times \prod_j (Q^+(z_j)Q^-(z_j))^{1/2} || \frac{1}{z_i - \lambda^{-1} z_j} || =
\]
\[
(Cauchy identity)
\]
\[
e^{i \sum_j \theta_j \lambda^{N-1} \prod_j \frac{\lambda^{-N} - 1}{z_j (1 - \lambda^{-1})} \prod_{j<k} \frac{(z_j - z_k)}{z_j - z_k (1-\lambda^{-1})(z_j \lambda^{-1} - z_k)}
\]
\[
\frac{(\lambda z_j - z_k)(\lambda^{-1}z_j - z_k)}{(z_j - z_k)^2} = e^{i}\sum_{j} \theta_j = 1
\]

The determinant \(\det|\frac{1}{z_i - \lambda z_j}|\) can be evaluated by representing it as a correlator of free fermions on a complex plane inserted into points \(z_i\) and \(\lambda z_i\) - it is a "physical" proof of Cauchy identities used also in \([12]\). So, the answer for this determinant cancels precisely all \(z_i\) dependent factors in the product (this could be also deduced from the absence of poles of this meromorphic function, therefore, it should be constant).

Appendix B

Here we derive an expression for the wavefunctions of Calogero model via the large \(\kappa\) limit in those for Sutherland model. We start from the expression for the transition kernel

\[
K(g_2, v_2|g_1, v_1) = \sum_{\alpha} \exp\left(-\frac{T}{\kappa^2}Q(\alpha)\right)K_\alpha(g_2, v_2|g_1, v_1)
\]

\[
K_\alpha(g_2, v_2|g_1, v_1) = \int_{SU(N)} dh \chi_\alpha(hg_1h^{-1}g_2^{-1}) < v_2|TR_{\alpha}(h)|v_1 > \quad (0.1)
\]

Let us write \(g_{1,2} = \exp(\frac{i}{\kappa}Q_{1,2})\), then \(hg_1h^{-1}g_2^{-1} = \exp(\frac{i}{\kappa}\Phi)\) with \(\Phi = hQ_1h^{-1} - Q_2 + O(\frac{1}{\kappa})\).

Now we can write down an expression for the eigenvalues of the matrix \(hg_1h^{-1}g_2^{-1}\) as \(e^{i\Phi_j}\) where \(\Phi_j\) are eigenvalues of \(\Phi\). By the Weyl character formula we have:

\[
dim(\alpha) = \frac{\Delta(\bar{p}_i)}{\Delta(i)}
\]

\[
\chi_\alpha(U) = \frac{\det|\beta_j^{\bar{p}_i}|}{\Delta(\beta_j)} \quad (0.2)
\]

where \(U\) is conjugate to \(diag(\beta_1, \ldots, \beta_N)\). Here collection of \(\bar{p}_i\)’s is the highest weight \(\bar{h}\) of the irreps \(\alpha_{\bar{h}}\).

The formula we will derive uses the representation of the large \(\kappa\) limit of the normalized character through the Harish-Chandra(-Itsykson-Zuber) integral, namely, if \(\bar{h} = \bar{P}\) is fixed, then

\[
\lim_{\kappa \to \infty} \frac{\chi_\alpha(U)}{\dim(\alpha)} = \int_{SU(N)} dh \exp(iTr(\bar{P}h\bar{Q}h^{-1})
\]

\[\text{We thank A.Marshakov for pointing out this useful application of Wick theorem.}\]
where \( \exp(i\bar{Q}) = \text{diag}(U) \), and \( \text{diag}(U) \) denotes any diagonalized form of the matrix \( U \). Substituting this expression into the integral in (0.1), we get an integral of the type:

\[
K_{\bar{P}}(g_2, v_2 | g_1, v_1) = \\
\Delta(\bar{P}) \kappa^{N(N-1)/2} \times \\
\int \int_{SU(N)} dh dh' \exp(i Tr(h'Q_1 h^{-1} - Q_2)h'^{-1} \bar{P}) \\
<v_2|T_{R_\nu}(h)|v_1>
\]

Finally, we can extract a wavefunction from the condition:

\[
\Psi^\dagger_{P}(\bar{Q}_2)\Psi_{P}(\bar{Q}_1) = \int \int_{SU(N)} dh dh' \kappa_{P}(g^h_2, v^h_2 | g'^h_1, v'^h_1)
\]

where \( g^h = hgh^{-1} \), \( v^h = T_{R_\nu}(h)v \).

It yields essentially:

\[
\Delta(\bar{P}) \Delta(\bar{Q}) \int_{SU(N)} dh \exp(i Tr(PhQh^{-1}) < 0|T_{R_\nu}(h)|0>
\]

In the case of \( N = 2 \) this integral reduces to the integral representation for the Bessel function. It is interesting to note, that the initial expression of the wavefunction for the Sutherland model could be regarded as the same formula applied to the central extended loop group – in that case we would get some path integral, which we can evaluate using, for example, localization technique, or, by choosing appropriate coordinates on the co-adjoint orbit of affine algebra ([11]).