Linear system matrices of rational transfer functions

Froilán M. Dopico$^{a,1}$, María C. Quintana$^{a,1}$, Paul Van Dooren$^{b,2,*}$

$^a$Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain.

$^b$Department of Mathematical Engineering, Université catholique de Louvain, Avenue Georges Lemaître 4, B-1348 Louvain-la-Neuve, Belgium.

Abstract

In this paper we derive new sufficient conditions for a linear system matrix

$$S(\lambda) := \begin{bmatrix} T(\lambda) & -U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix},$$

where $T(\lambda)$ is assumed regular, to be strongly irreducible. In particular, we introduce the notion of strong minimality, and the corresponding conditions are shown to be sufficient for a polynomial system matrix to be strongly minimal. A strongly irreducible or minimal system matrix has the same structural elements as the rational matrix $R(\lambda) = W(\lambda) + V(\lambda)T(\lambda)^{-1}U(\lambda)$, which is also known as the transfer function connected to the system matrix $S(\lambda)$. The pole structure, zero structure and null space structure of $R(\lambda)$ can be then computed with the staircase algorithm and the $QZ$ algorithm applied to pencils derived from $S(\lambda)$. We also show how to derive a strongly minimal system matrix from an arbitrary linear system matrix by applying to it a reduction procedure, that only uses unitary equivalence transformations. This implies that numerical errors performed during the reduction procedure remain bounded. Finally, we show how to perform diagonal scalings to an arbitrary pencil such that its row and column norms are all of the order of 1. Combined with the fact that we use unitary transformation in both the reduction procedure and the computation of the eigenstructure, this guarantees that we computed the exact eigenstructure of a perturbed linear system matrix, but where the perturbation is of the order of the machine precision.

Keywords: linear system matrix, irreducibility, minimality, rational matrix, poles, zeros, null space structure.

*Corresponding author

Email addresses: dopico@math.uc3m.es (Froilán M. Dopico), maquinta@math.uc3m.es (María C. Quintana), paul.vandooren@uclouvain.be (Paul Van Dooren)

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1. Introduction

Already in the seventies, Rosenbrock [9] introduced the concept of a polynomial system matrix

\[ S(\lambda) := \begin{bmatrix} T(\lambda) & -U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix}, \]

where \( T(\lambda) \) is assumed to be regular. He showed that the finite pole and zero structure of its transfer function matrix \( R(\lambda) = W(\lambda) + V(\lambda)T(\lambda)^{-1}U(\lambda) \) can be retrieved from the polynomial matrices \( T(\lambda) \) and \( S(\lambda) \), respectively, provided it is irreducible or minimal, meaning that the matrices

\[ \begin{bmatrix} T(\lambda) & -U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix}, \]

have, respectively, full row and column rank for all finite \( \lambda \). This was already well known for state-space models of a proper transfer function \( R_p(\lambda) \), where the system matrix takes the special form

\[ S_p(\lambda) := \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix}, \]

where \((A, B)\) is controllable and \((A, C)\) is observable, meaning that \( S_p(\lambda) \) is minimal. That is, \( \begin{bmatrix} \lambda I - A & -B \end{bmatrix} \) and \( \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \) both satisfy the conditions in (2), respectively.

The poles of such a proper transfer function are all finite and are the eigenvalues of \( A \), while the finite zeros are the finite generalized eigenvalues of the pencil \( S_p(\lambda) \). The main advantage of using state-space models is that there are algorithms to compute the eigenstructure using unitary transformations only. There are also algorithms available to derive a minimal state-space models from a non-minimal one, and these algorithms are also based on unitary transformations only [11], [13].

When allowing generalized state space models, then all transfer functions can be realized by a system matrix of the type

\[ S_g(\lambda) := \begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix}, \]

since the matrix \( E \) is allowed to be singular. Moreover, when the pencils

\[ \begin{bmatrix} \lambda E - A & -B \\ C \end{bmatrix}, \begin{bmatrix} \lambda E - A \\ C \end{bmatrix}, \]

have, respectively, full row rank and column rank for all finite \( \lambda \), then we retrieve the irreducibility or minimality conditions of Rosenbrock in (2), which imply that the finite poles of \( R(\lambda) := D + C(\lambda E - A)^{-1}B \) are the finite eigenvalues of \( \lambda E - A \) and the finite zeros of \( R(\lambda) \) are the finite zeros of \( S_g(\lambda) \). It was shown in [15] that when imposing also the conditions that the pencil in (3) is strongly irreducible, meaning that the matrices in (4) have no zeros at finite points or at infinity, then also the infinite pole and zero structure of \( R(\lambda) \) can be retrieved from the infinite structure of \( \lambda E - A \) and \( S_g(\lambda) \), respectively, and that the left and right minimal indices of \( R(\lambda) \) and \( S_g(\lambda) \) are also the same. Moreover, a
A reduction procedure to derive a strongly irreducible generalized state-space model from a reducible one was also given in [11], [13], and it is also based on unitary transformations only.

In [16] these results were then extended to arbitrary polynomial models, but it required irreducibility tests that were more involved. In this paper we will show that these conditions can again be simplified (and also made more uniform) when the system matrix is linear, i.e.,

\[
S(\lambda) := \begin{bmatrix}
    A(\lambda) & -B(\lambda) \\
    C(\lambda) & D(\lambda)
\end{bmatrix} := \begin{bmatrix}
    \lambda A_1 - A_0 & B_0 - \lambda B_1 \\
    \lambda C_1 - C_0 & \lambda D_1 - D_0
\end{bmatrix}.
\]  

(5)

We will define the notion of strongly minimal polynomial system matrix, and we will prove that the strong minimality conditions imply the strong irreducibility conditions in [16]. We remark that, although the notions of irreducible or minimal polynomial system matrix refer to the same conditions in (2), the conditions for a polynomial system matrix to be strongly irreducible or strongly minimal are different in general. We will also show that when the strong minimality conditions are not satisfied, we can reduce the system matrix to one where they are satisfied, and this without modifying the transfer function. Such a procedure was already derived in [14], but only for linear system matrices that were already minimal at finite points. In this paper we thus extend this to arbitrary linear system matrices.

In the next Section we briefly recall the background material for this paper and introduce the basic notation. In Section 3 we also recall the definition of strongly irreducible polynomial system matrix in [16], and we introduce the notion of strong minimality. In addition, we establish the relation between them. We then give, in Section 4, an algorithm to construct a strongly minimal linear system matrix from an arbitrary one, and we discuss the computational aspects in Section 5. In Section 6 we develop diagonal scaling methods to arbitrary pencils such that their row and column norms remain “balanced”. Finally, we end with some numerical experiments in Section 7 and some concluding remarks in Section 8.

2. Background

We will restrict ourselves here to polynomial and rational matrices with coefficients in the field of complex numbers \( \mathbb{C} \). The set of \( m \times n \) polynomial matrices, denoted by \( \mathbb{C}[\lambda]^{m \times n} \) and the set of \( m \times n \) rational matrices, denoted by \( \mathbb{C}(\lambda)^{m \times n} \), can both be viewed as matrices over the field of rational functions with complex coefficients, denoted by \( \mathbb{C}(\lambda) \).

Every rational matrix can have poles and zeros and has a right and a left null space (these can be void). Via the local Smith-McMillan form, one can associate structural indices to the poles and zeros, and via the notion of minimal polynomial bases for rational vector spaces, one can associate so called right and left minimal indices to the right and left null spaces. We briefly recall here these different types of indices. Since we assumed (for simplicity) that the coefficients of the rational matrix are in \( \mathbb{C} \), the poles and zeros are in the same set.

**Definition 2.1.** A square rational matrix \( M(\lambda) \in \mathbb{C}(\lambda)^{m \times m} \) is said to be regular at a point \( \lambda_0 \in \mathbb{C} \) if the matrix \( M(\lambda_0) \) is bounded (i.e., \( M(\lambda_0) \in \mathbb{C}^{m \times m} \)) and is invertible.
This is equivalent to that both rational matrices $M(\lambda)$ and $M(\lambda)^{-1}$ have a convergent Taylor expansion around the point $\lambda = \lambda_0$. Namely,

\[
M(\lambda) := M_0 + (\lambda - \lambda_0) M_1 + (\lambda - \lambda_0)^2 M_2 + (\lambda - \lambda_0)^3 M_3 + \cdots,
\]

\[
M(\lambda)^{-1} := M_0^{-1} + (\lambda - \lambda_0) H_1 + (\lambda - \lambda_0)^2 H_2 + (\lambda - \lambda_0)^3 H_3 + \cdots.
\]

If $\lambda = \infty$, $M(\lambda)$ is said to be biproper or regular at infinity if the Taylor expansions above are in terms of $1/\lambda$ instead of the factor $(\lambda - \lambda_0)$.

**Definition 2.2.** Let $R(\lambda)$ be an arbitrary $m \times n$ rational matrix of normal rank $r$. Then its *local Smith-McMillan form* at a point $\lambda_0 \in \mathbb{C}$ is the quasi diagonal matrix obtained under rational left and right transformations $M_\ell(\lambda)$ and $M_r(\lambda)$, that are regular at $\lambda_0$:

\[
M_\ell(\lambda) R(\lambda) M_r(\lambda) = \begin{bmatrix}
(\lambda - \lambda_0)^{d_1} & 0 & \cdots & 0 & O_{r,n-r} \\
0 & (\lambda - \lambda_0)^{d_2} & \ddots & \vdots & \\
\vdots & \ddots & \ddots & 0 & \\
0 & \cdots & 0 & (\lambda - \lambda_0)^{d_r} & \\
O_{m-r,r} & O_{m-r,n-r} & & & \\
\end{bmatrix}
\] (6)

where $d_1 \leq d_2 \leq \ldots \leq d_r$. If $\lambda_0 = \infty$, the basic factor $(\lambda - \lambda_0)$ is replaced by $1/\lambda$ and the transformation matrices are then biproper.

**Remark 2.3.** The normal rank of a rational matrix is the size of its largest nonidentically zero minor. The indices $d_i$ are unique and are called the *structural indices* of $R(\lambda)$ at $\lambda_0$. In particular, the strictly positive indices correspond to a zero at $\lambda_0$, and the strictly negative indices correspond to a pole at $\lambda_0$. The *zero degree* is defined as the sum of all structural indices of all zeros (infinity included), and the *polar degree* is the sum of all structural indices (in absolute value) of all poles (infinity included).

The above definitions of pole and zero structure of a rational matrix $R(\lambda)$ are those that are commonly used in linear systems theory (see [6]) and are due to McMillan. They describe the spectral properties of a rational matrix. But when applying them to matrix pencils $S(\lambda)$ we may wonder if they coincide with definitions of eigenvalues and generalized eigenvalues and their multiplicity, i.e., the Kronecker structure of $S(\lambda)$ (see [4]). For this comparison, we only need to look at zeros, since a pencil has only one pole (namely, infinity) and its multiplicity is the rank of the coefficient of $\lambda$. In other words, its polar structure is trivial. But what about the correspondence of the *zero structure* of $S(\lambda)$ (in the McMillan sense) and the eigenvalue structure of $S(\lambda)$ (in the sense of Kronecker)? It turns out that for finite eigenvalues of $S(\lambda)$ there is a complete isomorphism with the zero structure of $S(\lambda)$: every Jordan block of size $k$ at an eigenvalue $\lambda_0$ in the Kronecker canonical form of $S(\lambda)$ corresponds to an elementary divisor $(\lambda - \lambda_0)^k$ in the Smith-McMillan form of $S(\lambda)$. But for $\lambda = \infty$, there is a difference. It is well known (see [15]) that a Kronecker block of size $k$ at $\lambda = \infty$ corresponds to an elementary divisor $(\frac{1}{\lambda})^{(k-1)}$ in the Smith-McMillan form. For the point at infinity there is thus a shift of 1 in the structural indices. For this reason we want to make a clear distinction between both index sets. Whenever we talk about
zeros, we refer to the McMillan structure, and whenever we talk about eigenvalues, we refer to the Kronecker structure.

It is well known that every rational vector subspace \( \mathcal{V} \), i.e., every subspace \( \mathcal{V} \subseteq \mathbb{C}(\lambda)^n \) over the field \( \mathbb{C}(\lambda) \), has bases consisting entirely of polynomial vectors. Among them some are minimal in the following sense introduced by Forney [3]: a minimal basis of \( \mathcal{V} \) is a basis of polynomial vectors whose sum of degrees is minimal among all bases of \( \mathcal{V} \) consisting of polynomial vectors. The fundamental property [3, 6] of such bases is that the ordered list of degrees of the polynomial vectors in any minimal basis of \( \mathcal{V} \) is always the same. Therefore, these degrees are an intrinsic property of the subspace \( \mathcal{V} \). This leads to the definition of the minimal bases and indices of a rational matrix. An \( m \times n \) rational matrix \( R(\lambda) \) of normal rank \( r \) smaller than \( m \) and/or \( n \) has non-trivial left and/or right rational null-spaces, respectively, over the field \( \mathbb{C}(\lambda) \):

\[
\mathcal{N}_r(R) := \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T R(\lambda) \equiv 0^T \},
\]

\[
\mathcal{N}_r^*(R) := \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : R(\lambda)x(\lambda) \equiv 0 \}.
\]

Rational matrices with non-trivial left and/or right null-spaces are said to be singular. If the rational subspace \( \mathcal{N}_r^*(R) \) is non-trivial, it has minimal bases and minimal indices, which are called the left minimal bases and indices of \( R(\lambda) \). Analogously, the right minimal bases and indices of \( R(\lambda) \) are those of \( \mathcal{N}_r(R) \), whenever this subspace is non-trivial. Notice that an \( m \times n \) rational matrix of normal rank \( r \) has \( m - r \) left minimal indices \( \{ \eta_1, \ldots, \eta_{m-r} \} \), and \( n-r \) right minimal indices \( \{ \epsilon_1, \ldots, \epsilon_{n-r} \} \). It is well known (see [6]) that the McMillan degree \( \delta(R) \) of a rational matrix is the sum of all polar degrees. The following degree sum theorem was proven in [11], and also mentioned in [15], and relates the McMillan degree to the other structural elements of \( R(\lambda) \). In particular, it relates \( \delta(R) \) to the the zero degree \( \delta_z(R) \) of \( R(\lambda) \), that is the sum of all degrees of its zeros; to the left nullspace degree \( \delta_L(R) \) of \( R(\lambda) \), that is the sum of all left minimal indices and to the right nullspace degree \( \delta_r(R) \) of \( R(\lambda) \), that is the sum of all right minimal indices.

**Theorem 2.4.** Let \( R(\lambda) \in \mathbb{C}(\lambda)^{m \times n} \) be a rational matrix of normal rank \( r \). Then

\[
\delta(R) := \delta_\mu(R) = \delta_z(R) + \delta_L(R) + \delta_r(R).
\]

3. **Strong irreducibility and minimality**

In this section we recall the strong irreducibility conditions in [16] for polynomial system matrices, and we introduce the notion of strong minimality. Then, we study the relation between them for the case of linear system matrices.

**Definition 3.1.** A polynomial system matrix \( S(\lambda) \) as in (1) is said to be strongly controllable and strongly observable, respectively, if the polynomial matrices

\[
\begin{bmatrix}
T(\lambda) & -U(\lambda) & 0 \\
V(\lambda) & W(\lambda) & -I
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
T(\lambda) & -U(\lambda) \\
V(\lambda) & W(\lambda) \\
0 & I
\end{bmatrix},
\]

have no finite or infinite zeros. If both conditions are satisfied \( S(\lambda) \) is said to be strongly irreducible.
Let us now consider the transfer function matrix \( R(\lambda) = W(\lambda) + V(\lambda)T(\lambda)^{-1}U(\lambda) \) of the polynomial system matrix in (1). In such a case, we also say that the system quadruple \( \{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\} \) realizes \( R(\lambda) \). Moreover, we say that the system quadruple is strongly irreducible if the polynomial system matrix is strongly irreducible. It was shown in [16] that the pole/zero and null space structure of \( R(\lambda) \) can be retrieved from a strongly irreducible system quadruple \( \{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\} \) as follows.

**Theorem 3.2.** If the polynomial system matrix \( S(\lambda) \) in (1) is strongly irreducible, then

1. the zero structure of \( R(\lambda) \) at finite and infinite \( \lambda \) is the same as the zero structure of \( S(\lambda) \) at finite and infinite \( \lambda \),
2. the pole structure of \( R(\lambda) \) at finite \( \lambda \) is the same as the zero structure at \( \lambda \) of \( T(\lambda) \),
3. the pole structure of \( R(\lambda) \) at infinity is the same as the zero structure at infinity of
   \[
   \begin{bmatrix}
   T(\lambda) & -U(\lambda) & 0 \\
   V(\lambda) & W(\lambda) & -I \\
   0 & I & 0
   \end{bmatrix},
   \]
4. the left and right minimal indices of \( R(\lambda) \) and \( S(\lambda) \) are the same.

If one specializes this to the generalized state space model (3) one retrieves the results of [15], which are simpler and only involve the pencils \( (\lambda E - A) \), (3) and (4). We now show that the above conditions can be simplified when the system matrices are linear as in (5).

First, we present the definition of strongly minimal polynomial system matrix.

**Definition 3.3.** Let \( d \) be the degree of the polynomial system matrix \( S(\lambda) \) in (1). \( S(\lambda) \) is said to be strongly \( E \)-controllable and strongly \( E \)-observable, respectively, if the polynomial matrices

\[
\begin{bmatrix}
T(\lambda) & -U(\lambda) \\
V(\lambda) & W(\lambda) \\
0 & I & 0
\end{bmatrix},
\]

have no finite or infinite eigenvalues, considered as polynomial matrices of grade \( d \). If both conditions are satisfied \( S(\lambda) \) is said to be strongly minimal.

The letter \( E \) in the definition of strong \( E \)-controllability and \( E \)-observability refers to the condition of the matrices in (8) not having eigenvalues, finite or infinite. We prove in Proposition 3.7 that the strong irreducibility conditions hold if the strong minimality conditions are satisfied. For this, we need to recall Lemma 1 of [15], which we give here in its transposed form. Then, we prove Theorems 3.5 and 3.6, and Proposition 3.7 as a corollary of them.

**Lemma 3.4.** The zero structure at infinity of the pencil \( [\lambda K_1 - K_0 \mid -L_0] \) where \( K_1 \) has full column rank, is isomorphic to the zero structure at zero of the pencil \( [K_1 - \mu K_0 \mid -L_0] \). Moreover, if the pencil has full row normal rank, then it has no zeros at infinity, provided the constant matrix \( [K_1 \mid -L_0] \) has full row rank.

**Proof.** The first part is proven in [15]. The second part is a direct consequence of the first part, when filling in \( \mu = 0 \).

\[\square\]
Theorem 3.5. The pencil
\[
\begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1 & 0 \\
\lambda C_1 - C_0 & \lambda D_1 - D_0 & -I
\end{bmatrix},
\] (9)
where \(\lambda A_1 - A_0\) is regular, has no zeros at infinity if the pencil
\[
\begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1
\end{bmatrix}
\] (10)
has no eigenvalues at infinity.

Proof. Clearly the pencils in (9) and (10) have full (row or column) normal rank since \(\lambda A_1 - A_0\) is regular. We can thus apply the result of Lemma 3.4. If we use an invertible matrix \(V\) to “compress” the columns of the coefficient of \(\lambda\) in the following pencil
\[
\begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1 \\
\lambda C_1 - C_0 & \lambda D_1 - D_0
\end{bmatrix}
\] \[V\]
\[
\begin{bmatrix}
\lambda K_1 - K_0 & -L_0 & 0 \\
\lambda \hat{K}_1 - \hat{K}_0 & -L_0 & -I
\end{bmatrix},
\] such that the matrix \(K_1\) has full column rank, then this pencil has no zeros at infinity provided the constant matrix
\[
\begin{bmatrix}
K_1 & -L_0 & 0 \\
\hat{K}_1 & -\hat{L}_0 & -I
\end{bmatrix}
\] has full row rank. But if \(\begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1
\end{bmatrix}\) has no infinite eigenvalues, it follows that \(\begin{bmatrix}
A_1 & -B_1
\end{bmatrix}\) has full row rank. And since \(\begin{bmatrix}
A_1 & -B_1
\end{bmatrix}V = \begin{bmatrix}
K_1 & 0
\end{bmatrix}\), \(K_1\) must have full row rank as well (in fact, it is invertible). It then follows from Lemma 3.4 that the pencil in (9) has no zeros at infinity. \(\square\)

In the next theorem, we state without proof the transposed version of Theorem 3.5.

Theorem 3.6. The pencil
\[
\begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1 \\
\lambda C_1 - C_0 & \lambda D_1 - D_0 & 0 \\
0 & I
\end{bmatrix},
\] (12)
where \(\lambda A_1 - A_0\) is regular, has no zeros at infinity if the pencil
\[
\begin{bmatrix}
\lambda A_1 - A_0 \\
\lambda C_1 - C_0
\end{bmatrix}
\] (11)
has no eigenvalues at infinity.

Let us now consider a linear system matrix
\[
L(\lambda) := \lambda L_1 - L_0 := \begin{bmatrix}
\lambda A_1 - A_0 & B_0 - \lambda B_1 \\
\lambda C_1 - C_0 & \lambda D_1 - D_0
\end{bmatrix},
\] (12)
with \(\lambda A_1 - A_0\) regular. Notice that if \(L(\lambda)\) is minimal (i.e., satisfies (2)) and, in addition, satisfies the conditions in (10) and (11), then it is strongly minimal. By Theorems 3.5 and 3.6, we have that these conditions imply strong irreducibility on linear system matrices. We state such result in Proposition 3.7.
Proposition 3.7. A linear system matrix as in (12) is strongly irreducible if it is strongly minimal.

Remark 3.8. Notice that conditions (10) and (11) are only sufficient, not necessary. But they are easy to test, and also to obtain after a reduction procedure, as we show in Section 4.

Lemma 3.4, Theorems 3.5 and 3.6 and Proposition 3.7 can be extended to polynomial system matrices. However, we do not state these results here since, in this paper, we are focusing on linear system matrices. If we recapitulate the results of this section, we obtain the following theorem.

Theorem 3.9. A linear system pencil $L(\lambda)$ as in (12), realizing the transfer function $R(\lambda) := (\lambda D_1 - D_0) + (\lambda C_1 - C_0)(\lambda A_1 - A_0)^{-1}(\lambda B_1 - B_0)$, is strongly irreducible if it is strongly minimal. Moreover, if $L(\lambda)$ is strongly irreducible then

1. the zero structure of $R(\lambda)$ at finite and infinite $\lambda$ is the same as the zero structure of $S(\lambda)$ at finite and infinite $\lambda$,
2. the left and right minimal indices of $R(\lambda)$ and $S(\lambda)$ are the same,
3. the finite polar structure of $R(\lambda)$ is the same as the finite zero structure of $\lambda A_1 - A_0$, and
4. the infinite polar structure of $R(\lambda)$ is the same as the infinite zero structure of the pencil

$$
\begin{pmatrix}
\lambda A_1 - A_0 & -\lambda B_1 & 0 \\
\lambda C_1 & \lambda D_1 & -I \\
0 & I & 0
\end{pmatrix}.
$$

(13)

Remark 3.10. It follows from this theorem and the McMillan degree sum theorem for rational transfer functions (see [15]) that the rank of $L_1$ equals the McMillan degree of $R(\lambda)$, and that there can be no linear system matrix with a smaller rank of $L_1$ that satisfies Theorem 3.9.

It may look strange that there is such a difference in the treatment of finite and infinite poles of $R(\lambda)$ in Theorem 3.9, but it should be pointed out that the matrices $(B_1, C_1, D_1)$ contribute to the polar structure of $R(\lambda)$, and not to the finite polar structure. Notice that in (13) we have eliminated the matrices $B_0, C_0$ and $D_0$ with equivalence transformations using the identity matrices as pivots.

4. Reducing to a strongly minimal linear system matrix

In this section we give an algorithm to reduce an arbitrary linear system matrix to a strongly minimal one. Given a linear system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$, where $A(\lambda) \in \mathbb{C}(\lambda)^{d \times d}$ is assumed to be regular, we describe first how to obtain a strongly E-controllable quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ of smaller size. For that, our reduction procedure deflates finite and infinite “uncontrollable eigenvalues” by proceeding in three different steps. Then the reduction to a strongly E-observable one is dual and can be obtained by mere transposition of the system matrix and application of the first method for obtaining a strongly E-controllable system.
We have obtained an equivalent system representation in which the (1, 1)-block, $X(\lambda)\tilde{W}_{11}$, can be deflated since it does not contribute to the transfer function. We then obtain a smaller linear system pencil:

$$\begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)E - \tilde{B}(\lambda) \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)E + D(\lambda)
\end{bmatrix}.$$
that has the same transfer function. One can also perform this elimination by another unitary transformation \( \tilde{W} \) constructed to eliminate \( W_{13} \):

\[
\begin{bmatrix}
\tilde{W}_{11} & 0 & W_{13}
\end{bmatrix}
\begin{bmatrix}
\tilde{W}_{11} & 0 & \tilde{W}_{13} \\
0 & I & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33}
\end{bmatrix}
= \begin{bmatrix} I & 0 & 0 \end{bmatrix},
\]

which then yields

\[
\begin{bmatrix}
X(\lambda)\tilde{W}_{11} & 0 & X(\lambda)W_{13} \\
\tilde{Y}(\lambda) & \tilde{A}(\lambda) & -\tilde{B}(\lambda) \\
\tilde{Z}(\lambda) & \tilde{C}(\lambda) & D(\lambda)
\end{bmatrix}
\begin{bmatrix}
\tilde{W}_{11} & 0 & \tilde{W}_{13} \\
0 & I & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33}
\end{bmatrix}
= \begin{bmatrix} X(\lambda) & 0 & 0 \\
\tilde{Y}(\lambda)\tilde{W}_{11} - \tilde{B}(\lambda)\tilde{W}_{31} & \tilde{A}(\lambda) & \tilde{Y}(\lambda)\tilde{W}_{13} - \tilde{B}(\lambda)\tilde{W}_{33} \\
\tilde{Z}(\lambda)\tilde{W}_{11} + \tilde{D}(\lambda)\tilde{W}_{31} & \tilde{C}(\lambda) & \tilde{Z}(\lambda)\tilde{W}_{13} + \tilde{D}(\lambda)\tilde{W}_{33}
\end{bmatrix}.
\]

Notice that the new transfer function has now changed, but only by a constant factor \( \tilde{W}_{33} \), which moreover is invertible. This follows from the identity

\[
\begin{bmatrix} E & I \end{bmatrix} \begin{bmatrix} \tilde{W}_{11} \ W_{13} \end{bmatrix} = \begin{bmatrix} \tilde{W}_{13} & \tilde{W}_{33} \end{bmatrix}
\]

expressing that both matrices span the null-space of the same matrix \( \begin{bmatrix} \tilde{W}_{11} & W_{13} \end{bmatrix} \).

This also implies that

\[
\begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)E - \tilde{B}(\lambda) \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)E + D(\lambda)
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & \tilde{W}_{33}
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)\tilde{W}_{13} - \tilde{B}(\lambda)\tilde{W}_{33} \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)\tilde{W}_{13} + \tilde{D}(\lambda)\tilde{W}_{33}
\end{bmatrix},
\]

which shows that their Schur complements are related by a constant factor \( \tilde{W}_{33} \).

**Step 3:** Finally, we show that the submatrix

\[
\begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)E - \tilde{B}(\lambda) \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)E + D(\lambda)
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & \tilde{W}_{33}
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)\tilde{W}_{13} - \tilde{B}(\lambda)\tilde{W}_{33} \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)\tilde{W}_{13} + \tilde{D}(\lambda)\tilde{W}_{33}
\end{bmatrix},
\]

has no finite or infinite eigenvalues anymore. For this, we first point out that the form of the following product

\[
\begin{bmatrix} I & 0 \\
0 & \tilde{V} \end{bmatrix} := \begin{bmatrix} I & 0 & 0 \\
0 & \tilde{V}_{22} & \tilde{V}_{23} \\
0 & \tilde{V}_{32} & \tilde{V}_{33} \end{bmatrix} := W \begin{bmatrix} V & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} \tilde{W}_{11} & 0 & \tilde{W}_{13} \\
0 & I & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33} \end{bmatrix}
\]

follows from the fact that it is unitary. This then implies the equality

\[
\begin{bmatrix}
X(\lambda) & 0 \\
\tilde{Y}(\lambda)\tilde{W}_{11} - \tilde{B}(\lambda)\tilde{W}_{31} & \tilde{A}(\lambda) \\
\tilde{Y}(\lambda)\tilde{W}_{13} - \tilde{B}(\lambda)\tilde{W}_{33} & 0
\end{bmatrix}
\]
where $\tilde{\mathbf{V}}$ such that the following identity holds
\[
\begin{bmatrix}
X(\lambda) & 0 & 0 \\
Y(\lambda) & \tilde{A}(\lambda) & -\tilde{B}(\lambda) \\
\mathbf{0} & 0 & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}
\mathbf{I} & 0 \\
0 & \tilde{V}
\end{bmatrix},
\]
which in turn implies that $\begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)\tilde{W}_{13} & -\tilde{B}(\lambda)\tilde{W}_{33}
\end{bmatrix}$ has no finite or infinite eigenvalues. We thus have shown that the system matrix
\[
S_c(\lambda) := \begin{bmatrix}
A_c(\lambda) & -B_c(\lambda) \\
C_c(\lambda) & D_c(\lambda)
\end{bmatrix}
:= \begin{bmatrix}
\tilde{A}(\lambda) & \tilde{Y}(\lambda)\tilde{W}_{13} - \tilde{B}(\lambda)\tilde{W}_{33} \\
\tilde{C}(\lambda) & \tilde{Z}(\lambda)\tilde{W}_{13} + D(\lambda)\tilde{W}_{33}
\end{bmatrix}
\]
is now strongly $E$-controllable and that its transfer function $R_c(\lambda)$ equals $R(\lambda)\tilde{W}_{33}$, where $R(\lambda)$ is the transfer function of the original quadruple and $\tilde{W}_{33}$ is invertible. We summarize the result obtained by the three-steps procedure above in the following theorem.

**Theorem 4.1.** Let $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ be a linear system quadruple, with $A(\lambda) \in \mathbb{C}[\lambda]^{d \times d}$ regular, realizing the rational matrix $R(\lambda) := C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$. Then there exist unitary transformations $U, V \in \mathbb{C}^{d \times d}$ and $\tilde{W} \in \mathbb{C}^{(d+n) \times (d+n)}$ such that the following identity holds
\[
\begin{bmatrix}
U & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
A(\lambda) & -B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}
\begin{bmatrix}
V & 0 \\
0 & I_n
\end{bmatrix}
\tilde{W} = \begin{bmatrix}
X_\tau(\lambda) & 0 & 0 \\
Y_\tau(\lambda) & A_c(\lambda) & -B_c(\lambda) \\
Z_\tau(\lambda) & C_c(\lambda) & D_c(\lambda)
\end{bmatrix},
\]
where $\tilde{W}$ is of the form $\tilde{W} := \begin{bmatrix}
\tilde{W}_{11} & 0 & \tilde{W}_{13} \\
0 & I_d & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33}
\end{bmatrix} \in \mathbb{C}^{(d_r+d_e+n) \times (d_r+d_e+n)}$, $d_\tau$ is the number of (finite and infinite) eigenvalues of $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$, and $X_\tau(\lambda) \in \mathbb{C}[\lambda]^{d_r \times d_\tau}$ is a regular pencil. Moreover,

a) the eigenvalues of $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$ are the eigenvalues of $X_\tau(\lambda)$,
b) $\begin{bmatrix} A_c(\lambda) & -B_c(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{d_e \times (d_r+d_e+n)}$ has no (finite or infinite) eigenvalues,
c) the quadruple $(A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda))$ is a realization of the transfer function $R_c(\lambda) := R(\lambda)\tilde{W}_{33}$, with $\tilde{W}_{33} \in \mathbb{C}^{n \times n}$ invertible, and

d) if $\begin{bmatrix} A(\lambda) \\
C(\lambda)
\end{bmatrix}$ has no finite or infinite eigenvalues, then $\begin{bmatrix} A_c(\lambda) \\
C_c(\lambda)
\end{bmatrix}$ also has no finite or infinite eigenvalues.

**Remark 4.2.** Notice that conditions b) and d) in Theorem 4.1 imply that the system quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ is strongly minimal.

**Proof.** The decomposition and the three properties a), b) and c) were shown in the discussion above. The only part that remains to be proven is property d). This follows from the identity (4), which yields
\[
\begin{bmatrix}
U & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A(\lambda) \\
C(\lambda)
\end{bmatrix}
V = \begin{bmatrix}
X(\lambda)\tilde{W}_{11} & 0 \\
\tilde{Y}(\lambda) & A_c(\lambda) \\
\tilde{Z}(\lambda) & C_c(\lambda)
\end{bmatrix}.
\]
This clearly implies that if \( \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \) has full rank for all \( \lambda \) (including infinity), then so does \( \begin{bmatrix} A_o(\lambda) \\ C_o(\lambda) \end{bmatrix} \).

We state below a dual theorem that constructs, from an arbitrary linear system quadruple \( \{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\} \), a subsystem \( \{A_o(\lambda), B_o(\lambda), C_o(\lambda), D_o(\lambda)\} \) where \( \begin{bmatrix} A_o(\lambda) \\ C_o(\lambda) \end{bmatrix} \) has no finite or infinite eigenvalues. Its proof is obtained by applying the previous theorem on the transposed system \( \{A^T(\lambda), C^T(\lambda), B^T(\lambda), D^T(\lambda)\} \) and then transposing back the result.

**Theorem 4.3.** Let \( \{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\} \) be a linear system quadruple, with \( A(\lambda) \in \mathbb{C}[\lambda]^{d \times d} \) regular, realizing the rational matrix \( R(\lambda) := C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda) \in \mathbb{C}(\lambda)^{m \times n} \). Then there exist unitary transformations \( U, V \in \mathbb{C}^{d \times d} \) and \( \bar{W} \in \mathbb{C}^{(d+m) \times (d+m)} \) such that the following identity holds

\[
\bar{W} \begin{bmatrix} U & 0 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} V & 0 & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X_\pi(\lambda) & Y_\pi(\lambda) & Z_\pi(\lambda) \\ 0 & A_\pi(\lambda) & -B_\pi(\lambda) \\ 0 & 0 & C_\pi(\lambda) & D_\pi(\lambda) \end{bmatrix},
\]

where \( \bar{W} \) is of the form \( \bar{W} := \begin{bmatrix} \bar{W}_{11} & 0 & \bar{W}_{13} \\ 0 & I_{d_\pi} & 0 \\ \bar{W}_{31} & 0 & \bar{W}_{33} \end{bmatrix} \in \mathbb{C}^{(d_\pi+d_a+m) \times (d_\pi+d_a+m)} \), \( d_\pi \) is the number of (finite and infinite) eigenvalues of \( \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \), and \( X_\pi(\lambda) \in \mathbb{C}[\lambda]^{d_\pi \times d_\pi} \) is a regular pencil. Moreover,

a) the eigenvalues of \( \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \) are the eigenvalues of \( X_\pi(\lambda) \),

b) \( \begin{bmatrix} A_o(\lambda) \\ C_o(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(d_\pi+d_a) \times d_a} \) has no (finite or infinite) eigenvalues,

c) the quadruple \( \{A_o(\lambda), B_o(\lambda), C_o(\lambda), D_o(\lambda)\} \) is a realization of the transfer function \( R_o(\lambda) := \bar{W}_{33}R(\lambda) \), with \( \bar{W}_{33} \in \mathbb{C}^{m \times m} \) invertible,

d) if \( \begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} \) has no finite or infinite eigenvalues then \( \begin{bmatrix} A_o(\lambda) & -B_o(\lambda) \end{bmatrix} \) also has no finite or infinite eigenvalues.

In order to extract from the system quadruple \( \{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\} \) a subsystem \( \{A_o(\lambda), B_o(\lambda), C_o(\lambda), D_o(\lambda)\} \) that is both strongly E-controllable and E-observable (and hence also strongly minimal), we only need to apply the above two theorems one after the other. The resulting subsystem would then be a realization of the transfer function \( R_o = C_o(\lambda)A_o(\lambda)^{-1}B_o(\lambda) + D_o(\lambda) \in \mathbb{C}(\lambda)^{m \times n} \). Since the transfer function was changed only by left and right transformations that are constant and invertible, the left and right nullspace will be transformed by these invertible transformations, but their minimal indices will be unchanged.
5. Computational aspects

In this section we give a more “algorithmic” description of the procedure described in Section 4 to reduce a given system quadruple \( \{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\} \) to a strongly E-controllable quadruple \( \{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\} \) of smaller size. We describe the essence of the three steps that were discussed in that section.

**Step 1**: Compute the staircase reduction of the submatrix

\[
\begin{bmatrix}
A(\lambda) & -B(\lambda)
\end{bmatrix}
\]

\[
U \begin{bmatrix}
A(\lambda) & -B(\lambda)
\end{bmatrix} W^* = \begin{bmatrix}
x(\lambda) & 0 & 0 \\
y(\lambda) & A(\lambda) & -B(\lambda)
\end{bmatrix},
\]

**Step 2**: Compute the matrices \( V \) and \( \tilde{W} \) to compress the first block row of

\[
\begin{bmatrix}
W_{11} & W_{12} & W_{13}
\end{bmatrix}
\begin{bmatrix}
V & 0 & I \\
0 & I & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{W}_{11} & 0 & \tilde{W}_{13} \\
0 & I & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33}
\end{bmatrix} = \begin{bmatrix}
I & 0 & 0
\end{bmatrix}
\]

**Step 3**: Display the uncontrollable part \( X(\lambda) \) using the transformations \( U, V \) and \( \tilde{W} \)

\[
\begin{bmatrix}
U & 0 & I
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
A(\lambda) & -B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix} \begin{bmatrix}
V & 0 & I
\end{bmatrix} \tilde{W} = \begin{bmatrix}
X(\lambda) & 0 & 0 & 0 & 0 & 0 \\
\times & A_c(\lambda) & -B_c(\lambda)
\times & C_c(\lambda) & D_c(\lambda)
\end{bmatrix}
\]

where we have used the notations introduced in Section 4. The computational complexity of these three steps is cubic in the dimensions of the matrices that are involved, provided that the staircase algorithm is implemented in an efficient manner [1]. But it is also important to point out that the reduction procedure to extract a strongly minimal linear system matrix from an arbitrary one, can be done with unitary transformations only, and that only one staircase reduction is neeeded when one knows that the pencil \( \begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} \) has normal rank equal to its number of rows. Indeed, this pencil then does not have any left null space or left minimal indices and only the regular part has to be separated from the right null space structure. This can be obtained by performing one staircase reduction on the rotated pencil \( \begin{bmatrix} \tilde{A}(\mu) & -\tilde{B}(\mu) \end{bmatrix} \), where the coefficient matrices

\[
\begin{bmatrix}
\tilde{A}_0 \\
\tilde{A}_1
\end{bmatrix} = \begin{bmatrix}
cI & sI \\
-sI & cI
\end{bmatrix} \begin{bmatrix}
A_0 \\
A_1
\end{bmatrix},
\]

\[
\begin{bmatrix}
\tilde{B}_0 \\
\tilde{B}_1
\end{bmatrix} = \begin{bmatrix}
cI & sI \\
-sI & cI
\end{bmatrix} \begin{bmatrix}
B_0 \\
B_1
\end{bmatrix},
\]

\( c^2 + s^2 = 1 \)

correspond to a change of variable \( \lambda = (c\mu - s)/(s\mu + c) \). If one now chooses the rotation such that the rotated pencil has no eigenvalues at \( \mu = \infty \), then only the finite spectrum has to be separated from the right minimal indices, which can be done with one staircase reduction.

6. Scaling

Unitary transformations have the big advantage that error propagation remains bounded, but it is also important to make sure that all numerical quantities are well
balanced in terms of their norm. This is where balancing and scaling comes in. For a square regular pencil, this has been studied in [8, 17]. To the best of our knowledge this problem has not been considered yet for rectangular pencils, and developing some scaling techniques in the rectangular case is the goal of this section.

Two types of scalings can be applied to the pencil $S(\lambda) = \lambda A - B$. The first one is a change of variable $\tilde{\lambda} := d_\lambda \lambda$ to make sure that the scaled matrices $A$ and $d_\lambda B$ have approximately the same norm. This can be done without introducing rounding errors, by taking $d_\lambda$ equal to a power of 2. The staircase and the $QZ$ algorithm work independently on both matrices and this scaling can be restored afterwards, again without introducing any additional errors. One could therefore argue that this scaling is irrelevant, but we will see that it affects the second scaling procedure we will discuss. The second type of scaling is based on multiplication on the left and on the right by diagonal matrices $D$ and $S$, respectively, that are chosen to “balance” in some way the row and column norms of $A := D_\ell A D_r$ and $B := D_\ell B D_r$. In [8, page 259], for $n \times n$ regular pencils $\lambda A - B$, such matrices $D_\ell$ and $D_r$ are chosen to make

$$
\|\text{col}_j(\tilde{A})\|_2^2 + \|\text{col}_j(\tilde{B})\|_2^2 = \|\text{row}_i(\tilde{A})\|_2^2 + \|\text{row}_i(\tilde{B})\|_2^2 = \gamma, \text{ for } i, j = 1, \ldots, n.
$$

Then $\|\tilde{A} - \tilde{B}\|_F^2 = n \gamma$. However, this is no longer possible for $m \times n$ rectangular pencils since the number of rows and columns are different, and the best that can be achieved is

$$
\|\text{col}_j(\tilde{A})\|_2^2 + \|\text{col}_j(\tilde{B})\|_2^2 = \gamma_r, \text{ for } j = 1, \ldots, n, \text{ and }
\|\text{row}_i(\tilde{A})\|_2^2 + \|\text{row}_i(\tilde{B})\|_2^2 = \gamma_c, \text{ for } i = 1, \ldots, m,
$$

with $\|\tilde{A} - \tilde{B}\|_F^2 = n \gamma_r = m \gamma_c$. Therefore, we need to develop new methods which extend the procedure described in [8] to the non-square case. In addition, these new methods are related to the famous Sinkhorn-Knopp algorithm which scales the rows and columns of a square non-negative matrix $M$ to become doubly stochastic [7, 10]. This algorithm and its convergence conditions were extensively analyzed in [7]. It was shown in [10] that this problem is convex and that the scaling procedure converges to a global minimum, provided $M$ has total support, meaning that every nonzero element in $M$ must lie on a nonzero diagonal. If this is not the case, the scaling procedure may converge to unbounded or singular scaling matrices $D_\ell$ and $D_r$, even though the scaled matrix $D_\ell M D_r$ becomes doubly stochastic. Moreover, if $M$ is fully indecomposable then the scaling matrices $D_\ell$ and $D_r$ are unique up to a scalar multiple. That is, if $D_\ell M D_r = D_2 M D_r = S$, with $S$ doubly stochastic, then $D_\ell \ell = k D_\ell$ and $D_r = k D_r$ for some constant $k > 0$.

We consider two different approaches in order to solve the problem. The first approach is directly inspired from [8] and reduces to it in the case of square pencils. The second approach has the additional advantage that it constructs a unique and bounded scaling. However, this scaling does not exactly verify the equations in (15) but only approximations of these equations. In what follows, for any matrix $A = [a_{ij}]$, we denote by $|A|$ the matrix whose elements are of the form $(A)_{ij} = |a_{ij}|$. Moreover, $A^{\otimes 2}$ denotes the matrix $A \circ A$, where “$\circ$” is the Hadamard product.

**Approach 1:** Given two matrices $A$, $B$ of size $m \times n$, we consider the following constrained minimization problem over the set of non-negative and diagonal matrices
The corresponding unconstrained problem with Lagrange multipliers $\gamma$ is:

$$\inf_{\det D^2 = c_F, \det D^2 = c_r} (\|D_r A D_r\|_F^2 + \|D_r B D_r\|_F^2),$$

for some real numbers $c_F > 0$ and $c_r > 0$. If we define the matrix $M := |A|^{\alpha^2} + |B|^{\alpha^2}$, this optimization problem can be solved by a sequence of alternating scalings $D^2_F$ and $D^2_r$ that make the rows of $D^2_F(MD^2_r)$ have equal norms, and then the columns of $(D^2_F M)D^2_r$ have equal norms. This is very similar to the Sinkhorn-Knopp algorithm, except that it is for a nonsquare matrix. If one makes the change of variables for the elements of $D_F$ and $D_r$ as follows $d^2_{ij} = \exp(u_i)$, $d^2_{ij} = \exp(v_j)$, then the above minimization is equivalent to a convex minimization problem with linear constraints:

$$\inf \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \exp(u_i + v_j), \quad \text{subject to} \quad \sum_{i=1}^{m} u_i = \ln c_F, \sum_{j=1}^{n} v_j = \ln c_r. \quad (16)$$

The corresponding unconstrained problem with Lagrange multipliers $\gamma_F$ and $\gamma_r$, is:

$$\inf \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \exp(u_i + v_j) + \gamma_F (\ln c_F - \sum_{i=1}^{m} u_i) + \gamma_r (\ln c_r - \sum_{j=1}^{n} v_j).$$

The first order conditions of optimality are the equality constraints of (16) and the equations

$$\sum_{j=1}^{n} d^2_{ij} m_{ij} d^2_{ij} = \gamma_F, \quad \sum_{i=1}^{m} d^2_{ij} m_{ij} d^2_{ij} = \gamma_r, \quad (17)$$

which express exactly that the row norms of $D^2_F MD^2_r$ are equal to each other and that its column norms are equal to each other. The alternating row and column scaling, then amounts to coordinate descent applied to the minimization. This algorithm thus continues to decrease the cost function as long as the equalities (17) are not met.

With a suitable choice of the constants $c_F$ and $c_r$, the resulting scaled pencil will have row and columns norms that are all bounded by 1, and hence $S_0$ and $S_1$ will have squared Frobenius norms bounded by $\max(d + m, d + n)$. For such a pencil, one can bound the backward errors of the staircase algorithm and the subsequent $QZ$ algorithm, by the machine precision. This also guarantees that we will have eventually computed an irreducible quadruple for an original quadruple that is nearby in a relative sense. Unfortunately, this variant may have unbounded or singular scaling matrices as solution when $M$ is not dense. As a result, the algorithm may converge to a pencil for which the structural elements are changing, since the normal rank of the pencil may change. Such deficiencies are avoided in the next approach.

**Approach 2:** Given two matrices $A$, $B$ of size $m \times n$, we now consider the following constrained minimization problem:

$$\inf_{\det D^2_F = c_F, \det D^2_r = c_r} 2(\|D_r A D_r\|_F^2 + \|D_r B D_r\|_F^2) + \alpha^2 \left( \frac{1}{m^2} \|D^4_F\|_F^2 + \frac{1}{n^2} \|D^4_F\|_F^2 \right), \quad (18)$$

for some real number $c > 0$. If we denote by $1_k$ the $k$-vector of all 1’s, and also use the above definition of the matrix $M$, then we can rewrite this as follows:

$$\inf_{\det D^2_F = c_F, \det D^2_r = c_r} 1_m^T m+n \begin{bmatrix} \frac{\alpha^2}{m^2} & 1_m^T m_n D^2_F & 1_n^T m_n D^2_{F} & D^2_F M D^2_r & D^2_r M D^2_r \\ \frac{\alpha^2}{m} & 1_n^T m_n D^2_F & 1_n^T m_n D^2_{F} & \frac{\alpha^2}{n} & 1_n^T m_n D^2_r & 1_n^T m_n D^2_{F} \end{bmatrix} 1_{m+n}. \quad (19)$$
Observe, in addition, that the constrained minimization problems (18) and (19) minimize the Frobenius norm squared of the two-sided scaled matrix
\[
\begin{bmatrix}
D^\ell & 0 \\
0 & D^r
\end{bmatrix}
\begin{bmatrix}
D^\ell & 0 \\
0 & D^r
\end{bmatrix} M_\alpha,
\]
subject to \(\det D^\ell \det D^r = c\). Then we prove in Theorem 6.5 that this can be solved by the Sinkhorn–Knopp algorithm in a unique way (in contrast to approach 1 where uniqueness is not guaranteed). Before, we need to prove the auxiliary Lemma 6.4. In this lemma, we use the notions given in [2] of fully indecomposable matrix and bipartite graph associated to a matrix.

**Definition 6.1.** Let \(A\) be an \(n \times n\) matrix with \(n \geq 2\). \(A\) is said to be fully indecomposable if there are no permutation matrices \(P\) and \(Q\) such that
\[
PAQ = \begin{bmatrix}
A_1 & A_{12} \\
0 & A_2
\end{bmatrix},
\]
with \(A_1\) and \(A_2\) square matrices of size at least 1.

**Definition 6.2.** Let \(A = [a_{ij}]\) be an \(m \times n\) matrix. Let \(U = \{u_1, u_2, \ldots, u_m\}\) and \(V = \{v_1, v_2, \ldots, v_n\}\) be sets of cardinality \(m\) and \(n\), respectively, such that \(U \cap V = \emptyset\). The bipartite graph associated with \(A\), denoted by \(BG(A)\), is the graph with vertex set \(U \cup V\) and whose edges are the pairs \({u_i, v_j}\) for which \(a_{ij} \neq 0\).

Notice that the bipartite graph associated with a matrix \(A\) is in general not a directed graph and is another way to represent the structure of nonzeros of the matrix \(A\), besides the adjacency graph or the incidence graph. Given a square matrix \(A\) with total support, Lemma 6.3 relates its bipartite graph to the condition of \(A\) being fully indecomposable. We use this result for proving Lemma 6.4.

**Lemma 6.3.** [2, Theorem 1.3.7] Let \(A\) be an \(n \times n\) matrix with \(n \geq 2\). Assume that \(A\) has total support. Then \(A\) is fully indecomposable if and only if its bipartite graph \(BG(A)\) is connected.

**Lemma 6.4.** \(M_\alpha^{\circ2}\) has total support. Moreover, if \(M \neq 0\) then \(M_\alpha^{\circ2}\) is fully indecomposable.

**Proof.** \(M_\alpha^{\circ2}\) has total support for all \(\alpha \neq 0\) since every nonzero element is an element of a positive diagonal (see [10]). To see that \(M_\alpha^{\circ2}\) is fully indecomposable, we apply Lemma 6.3. Then we consider \(BG(M_\alpha^{\circ2})\) the bipartite graph of \(M_\alpha^{\circ2}\). We assume without loss of generality that \(m_{1n}\) is a nonzero element of \(M := [m_{ij}]\). Then we consider the matrix
\[
N := \begin{bmatrix}
\frac{\alpha^2}{n^2} 1_m 1_m^T & 0 & m_{1n} \\
0 & m_{1n} & 0 \\
0 & 0 & \frac{\alpha^2}{n^2} 1_n 1_n^T
\end{bmatrix}.
\]
Notice that \(BG(N)\) is a sub-graph of \(BG(M_\alpha^{\circ2})\). Moreover, if \(\{u_1, u_2, \ldots, u_{m+n}\}\) and \(\{v_1, v_2, \ldots, v_{m+n}\}\) are the sets of vertices associated to the rows and columns of \(N\), respectively, then \(BG(N)\) is of the form
where the dashed edges correspond to the element $m_{1n}$. This proves that $BG(N)$ is connected. Therefore, $BG(M_{s}^{2})$ is connected and, by Lemma 6.3, $M_{s}^{2}$ is fully indecomposable. □

**Theorem 6.5.** Let $A$ and $B$ be $m \times n$ complex matrices and $\alpha, c > 0$ be real numbers. Let us consider the constrained minimization problem (18) over the set \{(D_{\ell}, D_{r}) : D_{\ell} := \text{diag}(\delta_{\ell,1}, \ldots, \delta_{\ell,m}), D_{r} := \text{diag}(\delta_{r,1}, \ldots, \delta_{r,n}), \delta_{\ell,i} > 0, \delta_{r,j} > 0\}. Then the following statements hold:

a) The optimization problem (18) is equivalent to the optimization problem (19).

b) The optimization problem (18) is equivalent to the optimization problem

\[
\inf_{\det D_{\ell}^{2} \det D_{r}^{2} = c} \left\| \begin{bmatrix} D_{\ell} & 0 \\ 0 & D_{r} \end{bmatrix} M_{0} \begin{bmatrix} D_{\ell} & 0 \\ 0 & D_{r} \end{bmatrix} \right\|_{F}^{2},
\]

where $M_{s}^{2} := \begin{bmatrix} \frac{\alpha^{2}}{m^{2}} \mathbf{1}_{m} \mathbf{1}_{m}^{T} & M \\ M^{T} & \frac{\alpha^{2}}{n^{2}} \mathbf{1}_{n} \mathbf{1}_{n}^{T} \end{bmatrix}$.

c) There exists a unique solution $\left(\tilde{D}_{\ell}, \tilde{D}_{r}\right)$ of (18). Moreover, $\left(\tilde{D}_{\ell}, \tilde{D}_{r}\right)$ is bounded and makes the matrix

\[
\begin{bmatrix} \tilde{D}_{\ell}^{2} & 0 \\ 0 & \tilde{D}_{r}^{2} \end{bmatrix} M_{s}^{2} \begin{bmatrix} \tilde{D}_{\ell}^{2} & 0 \\ 0 & \tilde{D}_{r}^{2} \end{bmatrix}
\]

a scalar multiple of a doubly stochastic matrix.

**Proof.** We have already seen statements a) and b) in this section. Then we only need to prove c). We make the change of variables $d_{\ell}^{2} = \exp(u_{i})$ and $d_{r}^{2} = \exp(v_{j})$ for the elements of $D_{\ell}$ and $D_{r}$, respectively. Then the optimization problem (18) is equivalent to the optimization problem:

\[
\inf 2 \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \exp(u_{i} + v_{j}) + \alpha^{2} \left( \frac{1}{m^{2}} \left( \sum_{i=1}^{m} \exp(u_{i}) \right)^{2} + \frac{1}{n^{2}} \left( \sum_{j=1}^{n} \exp(v_{j}) \right)^{2} \right),
\]

subject to $\sum_{i=1}^{m} u_{i} + \sum_{j=1}^{n} v_{j} = \ln(c)$.

(21)
The corresponding unconstrained problem with Lagrange multiplier $\gamma$ is:

$$\inf 2 \sum_{i=1}^{m} \sum_{j=1}^{n} m_{i,j} \exp(u_i + v_j) + \alpha^2 \left( \frac{1}{m^2} \left( \sum_{i=1}^{m} \exp(u_i) \right)^2 + \frac{1}{n^2} \left( \sum_{j=1}^{n} \exp(v_j) \right)^2 \right)$$

$$+ \gamma \left( \ln c - \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} v_j \right).$$

(22)

The first order conditions of optimality are the equality constraint of (21) and the equations

$$\frac{\alpha^2}{m^2} d_{t_i}^2 \sum_{i=1}^{m} d_{t_i}^2 + \sum_{j=1}^{n} d_{r_j}^2 m_{i,j} d_{r_j}^2 = \frac{\gamma}{2},$$

and

$$\frac{\alpha^2}{n^2} d_{r_j}^2 \sum_{j=1}^{n} d_{r_j}^2 + \sum_{i=1}^{m} d_{t_i}^2 m_{i,j} d_{t_i}^2 = \frac{\gamma}{2},$$

which express that the row sum and the column sum of

$$\begin{bmatrix} D_\ell^2 & 0 & 0 \\ 0 & D_r^2 \end{bmatrix} M^{\otimes 2} \begin{bmatrix} D_\ell^2 & 0 \\ 0 & D_r^2 \end{bmatrix}$$

are equal to $\frac{\gamma}{2}$. By Lemma 6.4, we know that $M^{\otimes 2}$ is fully indecomposable. Then, by the Sinkhorn–Knopp theorem, there exists a unique and bounded scaling $(E_\ell, E_r)$ that makes the matrix

$$\begin{bmatrix} E_\ell^2 & 0 & 0 \\ 0 & E_r^2 \end{bmatrix} M^{\otimes 2} \begin{bmatrix} E_\ell^2 & 0 \\ 0 & E_r^2 \end{bmatrix}$$

doubly stochastic. Assume that $\det E_\ell^2 \det E_r^2 = k$. We define $\tilde{D}_\ell := \left( \frac{\gamma}{k} \right)^{\frac{1}{m+n-1}} E_\ell$ and $\tilde{D}_r := \left( \frac{\gamma}{k} \right)^{\frac{1}{m+n-1}} E_r$. Then $\tilde{D}_\ell^2 \det \tilde{D}_r^2 = c$ and $(\tilde{D}_\ell, \tilde{D}_r)$ is the solution of (18). □

If one chooses $\alpha$ very large, the optimal scalings $D_\ell$ and $D_r$ will each tend to become proportional to the identity. If $\alpha$ is chosen to be small, more emphasis is put on making the column norms of $D_\ell^2 M D_r^2$ become equal and the row norms of $D_\ell^2 M D_r^2$ become equal. Finally, the Sinkhorn-Knopp algorithm computing this scaling has a complexity that is negligible with respect to the complexity of the reduction to a strongly irreducible system. The method computes $D_\ell$ and $D_r$ in an alternating fashion, until convergence is met. We refer to [7] for more information on the convergence of this algorithm.

**Remark 6.6.** The two above approaches and their convergence behaviour were described for real non-negative scalings. If one wants to perform scalings without introducing any rounding errors, one should restrict all scalings to powers of 2. The convergence behaviour and the convergence tests then need to be adapted, but here we do not go into such details here.

### 7. Numerical results

We illustrate the results of this paper with two numerical experiments: a polynomial one and a rational one.
Example 7.1. Let us consider the $2 \times 2$ polynomial matrix

$$P(\lambda) = \begin{bmatrix} e_5(\lambda) & 0 \\ 0 & e_1(\lambda) \end{bmatrix}$$

where $e_5(\lambda)$ is a polynomial of degree 5 and $e_1(\lambda)$ is a polynomial of degree 1. Expanding this fifth order polynomial matrix as

$$P(\lambda) = P_0 + P_1 \lambda + \cdots + P_5 \lambda^5,$$

a linear system matrix $S_P(\lambda)$ of $P(\lambda)$ is given by the following $10 \times 10$ pencil:

$$S_P(\lambda) = \begin{bmatrix} I_2 & -\lambda I_2 & -\lambda I_2 & P_1 \\ I_2 & -\lambda I_2 & I_2 & P_2 \\ -\lambda I_2 & I_2 & I_2 & P_3 \\ P_4 + \lambda P_5 & \cdots & \cdots & P_0 \end{bmatrix}.$$

The six finite Smith zeros of $P(\lambda)$ are clearly those of the scalar polynomials $e_1(\lambda)$ and $e_5(\lambda)$. These are also the finite zeros of $S_P(\lambda)$, since $S_P(\lambda)$ is minimal. However, $S_P(\lambda)$ is not strongly minimal if $P_5$ is singular and, in fact, it has an additional 4 eigenvalues at infinity (which are called infinite zeros in [5]). But in the McMillan sense, $P(\lambda)$ has no infinite zeros. The deflation procedure that we derived in this paper precisely gets rid of the extraneous infinite eigenvalues (infinite zeros in the sense of the Gohberg-Lancaster-Rodman approach). The numerical tests show that the sensitivity of the true McMillan zeros also can benefit from this.

In this example we compare the roots computed by four different methods:

1. computing the roots of the scalar polynomial and appending four $\infty$ roots
2. computing the generalized eigenvalues of the scaled pencil $S_P(\lambda)$
3. computing the roots of $QS_P(\lambda)Z$ for random orthogonal matrices $Q$ and $Z$
4. computing the roots of the minimal pencil obtained by our method

In this example we have a very large eigenvalue of the order of $10^5$ that is sensitive because of the presence of a Jordan block at infinity in $S_P(\lambda)$. The $QZ$ algorithm applied to $S_P(\lambda)$ only gets 11 digits of relative accuracy for that eigenvalue, but when applying first an orthogonal equivalence transformation $QS_P(\lambda)Z$, this eigenvalue gets completely perturbed because of the presence of that Jordan block: this results in 5 eigenvalues of the order of $10^5$, but none of them are close to the true eigenvalue. On the other hand, when deflating first the four uncontrollable eigenvalues at infinity, the reduced problem recovers a relative accuracy of 14 digits!
\[ \lambda_i \quad \delta^{(2)}_i \quad \delta^{(3)}_i \quad \delta^{(4)}_i \]

| \lambda_i          | \delta^{(2)}_i | \delta^{(3)}_i | \delta^{(4)}_i |
|-------------------|----------------|----------------|----------------|
| 8.3473e-05        | 7.9323e-17     | 1.2215e-16     | 1.4542e-17     |
| -3.0592e-02 + 2.7017e+00i | 3.4266e-15 | 6.7878e-15     | 1.9389e-15     |
| -3.0592e-02 - 2.7017e+00i | 3.4266e-15 | 6.7878e-15     | 1.9389e-15     |
| -2.9070e+00       | 2.2204e-15     | 1.3323e-15     | 3.1086e-15     |
| 2.9681e+00        | 4.4409e-15     | 1.7319e-14     | 4.8850e-15     |
| -2.7706e+05       | 6.0968e-06     | 1.4431e+05     | 2.9104e-09     |
| Inf               | Inf            | Inf            | (Inf)          |
| Inf               | Inf            | Inf            | (Inf)          |
| Inf               | Inf            | Inf            | (Inf)          |
| Inf               | Inf            | Inf            | (NaN)          |

Table 1: The correct generalized \( \lambda_i \) and the corresponding accuracies \( \delta^k_i \) for the three different calculations

**Example 7.2.** The second example is the rational matrix

\[
R(\lambda) = \begin{bmatrix}
\frac{e_5(\lambda)}{1/\lambda} & 0 \\
\frac{e_1(\lambda)}{1/\lambda} & 1
\end{bmatrix} = P_0 + P_1 \lambda + \cdots + P_5 \lambda^5 + \begin{bmatrix}
0 & 0 \\
1/\lambda & 0
\end{bmatrix},
\]

by using the notation of the example above. We consider the \( 12 \times 12 \) linear system matrix

\[
S_R(\lambda) = \begin{bmatrix}
\lambda I_2 - A & I_2 & -\lambda I_2 & -\lambda I_2 & -\lambda I_2 \\
I_2 & -\lambda I_2 & I_2 & -\lambda I_2 & I_2 \\
C & -\lambda I_2 & I_2 & -\lambda I_2 & I_2 \\
-\lambda I_2 & I_2 & -\lambda I_2 & I_2 & -\lambda I_2 \\
-\lambda I_2 & I_2 & -\lambda I_2 & I_2 & -\lambda I_2
\end{bmatrix}
\begin{bmatrix}
-B \\
P_0 \\
P_1 \\
P_2 \\
P_3 \\
-\lambda P_5
\end{bmatrix},
\]

where

\[
A = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

is a non-minimal realization of the strictly proper rational function \( \frac{1}{\lambda} \). In fact, the realization triple \((A, B, C)\) has two eigenvalues at \( \lambda = 0 \), of which one is uncontrollable since \( \frac{1}{\lambda} \) only has a pole at 0 of order 1. This is an artificial example since we could have realized the strictly proper part by using a minimal triple \((A, B, C)\) by removing the uncontrollable eigenvalue, but this is precisely what our reduction procedure does simultaneously for finite and infinite uncontrollable eigenvalues.

In this example the \( QZ \) algorithm applied to \( S_R(\lambda) \) recovers well all generalized eigenvalues. When applying the \( QZ \) algorithm to an orthogonally equivalent pencil \( QS_R(\lambda)Z \), the Jordan block at 0 gets perturbed to two roots of the order of the square root of the machine precision, which can be expected. But when deflating the uncontrollable eigenvalue at 0, this Jordan block is reduced to a single eigenvalue and part of the accuracy gets restored.

These two examples show that deflating uncontrollable eigenvalues may improve the sensitivity of the remaining eigenvalues which may improve the accuracy of their computation.
Table 2: The correct generalized $\lambda_i$ and the corresponding accuracies $\delta_i^k$ for the three different calculations

| $\lambda_i$          | $\delta_i^{(2)}$  | $\delta_i^{(3)}$  | $\delta_i^{(4)}$  |
|----------------------|-------------------|-------------------|-------------------|
| 0                    | 0                 | 7.9486e-09        | (1.4687e-16)      |
| 0                    | 2.4321e-18        | 7.9529e-09        | 4.6998e-13        |
| 1.6458e-03           | 3.0939e-15        | 4.1851e-12        | 4.6256e-13        |
| -7.9640e-02          | 3.0115e-15        | 8.1962e-14        | 7.4524e-15        |
| -4.3049e-01 + 7.1808e-01i | 1.0053e-15      | 9.4857e-16        | 4.4409e-16        |
| -4.3049e-01 - 7.1808e-01i | 1.0053e-15      | 9.4857e-16        | 4.4409e-16        |
| 4.3069e-01 + 7.1887e-01i | 1.0934e-15      | 1.4444e-15        | 5.6610e-16        |
| 4.3069e-01 - 7.1887e-01i | 1.0934e-15      | 1.4444e-15        | 5.6610e-16        |
| Inf                  | NaN               | NaN               | (Inf)             |
| Inf                  | NaN               | NaN               | (NaN)             |
| Inf                  | NaN               | NaN               | (NaN)             |

8. Conclusion

In this paper we looked at quadruple realizations $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ for a given rational transfer function $R(\lambda) = C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda)$, where the matrices $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are pencils, and where $A(\lambda)$ is assumed to be regular. We showed that under certain minimality assumptions on this quadruple, the poles, zeros and left and right null space structure of the rational matrix $R(\lambda)$ can be recovered from the generalized eigenstructure of two block pencils constructed from the quadruple. We also showed how to obtain such a minimal quadruple from a non-minimal one, by applying a reduction procedure that is based on the staircase algorithm. These results extend those previously obtained for generalized state space systems and polynomial matrices.

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