A REMARKABLE PROPERTY OF CONCIRCULAR VECTOR
FIELDS ON A RIEMANNIAN MANIFOLD

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ABSTRACT. In this paper, we show that given a nontrivial concircular vector
field $u$ on a Riemannian manifold $(M, g)$ with potential function $f$, there exists
a unique smooth function $\rho$ on $M$ that connects $u$ to the gradient of potential
function $\nabla f$, which we call the connecting function of the concircular vector
field $u$. Then this connecting function is shown to be a main ingredient in
obtaining characterizations of $n$-sphere $S^n(c)$ and the Euclidean space $E^n$. We
also show that the connecting function influences topology of the Riemannian
manifold.

1. Introduction

One of important topics in differential geometry of a Riemannian manifold $(M, g)$
is studying influence of special vector fields on its geometry as well as topology.
These special vector fields are geodesic vector fields, Killing vector fields, concircular
vector fields, Jacobi-type vector fields and conformal vector fields on a Riemannian
manifold. Moreover, it is well known that their existence have considerable impact
on the geometry of the Riemannian manifold and these vector fields are used in
finding characterizations of spheres as well as Euclidean spaces (cf. [3]-[12], [15]-
[17], [19]). In [11], Fialkow initiated the study of concircular vector fields on a
Riemannian manifold. A smooth vector field $u$ on a Riemannian manifold $(M, g)$
is said to be a concircular vector field if

$$\nabla_X u = fX, \quad X \in \mathfrak{X}(M),$$

where $\nabla$ is the Riemannian connection on the Riemannian connection on $(M, g)$
and $f : M \to R$ is a smooth function and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector
fields on $M$ (see also [13]). The smooth function appearing in the definition of the
concircular vector field $u$ is called the potential function of the concircular vector
field $u$. A concircular vector field $u$ is said to be a non-trivial concircular vector
field if the potential function $f \neq 0$.

Note that a concircular vector field is a closed conformal vector field, a natural
question arises, what is so special about a concircular vector field among closed
conformal vector fields? In this paper, we answer this question by showing that
to each non-trivial concircular vector $u$ with potential function $f$ on a connected
Riemannian manifold $(M, g)$, there exists a unique smooth function $\rho$ such that
$\nabla f = \rho u$, where $\nabla f$ is the gradient of the potential function $f$. Thus, this unique

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function $\rho$ connects the gradient $\nabla f$ of the potential function $f$ and the concircular vector vector field $u$ and therefore, we call $\rho$ the connecting function of the concircular vector field $u$. It is interesting to observe that connecting function $\rho$ is helpful in finding characterizations of the $n$-sphere $S^n(c)$ as well as the Euclidean space $E^n$ (cf. theorems, 3.2, 3.4, 4.1). Moreover, in the last section, we observe that the connecting function $\rho$ also influences topology of the Riemannian manifold (cf. theorems 5.1, 5.2).

2. Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian manifold $(M, g)$ and $u$ be a non-trivial concircular vector field on $(M, g)$ with potential function $f$. Then

$$\nabla_X u = fX, \quad X \in \mathfrak{X}(M),$$

and the curvature tensor field $R$ of the Riemannian manifold $(M, g)$ satisfies

$$R(X, Y) u = X(f)Y - Y(f)X, \quad X, Y \in \mathfrak{X}(M),$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

On an $n$-dimensional Riemannian manifold $(M, g)$, the Ricci tensor $Ric$ is given by

$$Ric(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on $M$. The Ricci operator $Q$ of the Riemannian manifold $(M, g)$ is a symmetric operator defined by

$$g(QX, Y) = Ric(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

The scalar curvature $S$ of the Riemannian manifold is defined by $S = TrQ$ the trace of the Ricci operator $Q$. The gradient $\nabla S$ of the scalar curvature satisfies (cf, [1, 2])

$$\frac{1}{2} \nabla S = \sum_{i=1}^{n} (\nabla Q)(e_i, e_i),$$

where the covariant derivative

$$(\nabla Q)(X, Y) = \nabla_X QY - Q\nabla_X Y.$$

Thus, using equations (2.2) and (2.3), we conclude

$$Ric(Y, u) = -(n - 1)Y(f).$$

Hence, we have

$$Q(u) = -(n - 1)\nabla f,$$

where $\nabla f$ is the gradient of the potential function $f$.

We also have a smooth function $h : M \to R$ on a Riemannian manifold $(M, g)$ associated to concircular vector field $u$, defined by

$$h = \frac{1}{2} \|u\|^2.$$

Then, using equation (2.1), we find the gradient $\nabla h$ of the smooth function $h$,

$$\nabla h = fu.$$
Note that the Hessian operator $A_\phi$ of a smooth function $\phi : M \to R$ on a Riemannian manifold $(M, g)$, and its Laplacian $\Delta \phi$ are defined by

$$A_\phi X = \nabla_X \nabla \phi, \quad \Delta \phi = \text{div} \nabla \phi = Tr A_\phi,$$

where

$$\text{div} X = \sum_{i=1}^n g(\nabla_{e_i} X, e_i).$$

The Hessian $\text{Hess}(\phi)$ is defined by

$$\text{Hess}(\phi)(X, Y) = g(A_\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Note that if $\phi$ is a nonconstant smooth function on a compact Riemannian manifold $(M, g)$ satisfying

$$\int_M \phi = 0,$$

then the minimal principle, gives

$$\int_M \| \nabla \phi \|^2 \geq \lambda_1 \int_M \phi^2,$$

where $\lambda_1$ is a first nonzero eigenvalue of the Laplace operator $\Delta$ acting on smooth functions of $M$.

Recall that the Laplace operator $\Delta$ acting on smooth vector fields on an $n$-dimensional Riemannian manifold $(M, g)$ is defined by

$$\Delta X = \sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X), \quad X \in \mathfrak{X}(M),$$

where $\{e_1, ..., e_n\}$ is an orthonormal frame on $M$. A smooth vector field $X$ is said to be harmonic if $\Delta X = 0$.

### 3. Connecting functions of concircular vector fields

In this section, first we show that for a non-trivial concircular vector field $u$ with potential function $f$ on a connected Riemannian manifold $(M, g)$, there exists a unique smooth function $\rho : M \to R$, which we call the connecting function of the concircular vector field $u$. Then, it is shown that the connecting function $\rho$ can be used to find characterizations of the $n$-sphere $S^n(c)$ as well as the Euclidean space $E^n$.

**Theorem 3.1:** Let $u$ be a non-trivial concircular vector field with potential function $f$ on a connected Riemannian manifold $(M, g)$. Then there exists a unique function $\rho : M \to R$ satisfying

$$\nabla f = \rho u.$$

**Proof.** Let $u$ be a non-trivial concircular vector field with potential function $f$ on a connected Riemannian manifold $(M, g)$. Then for the smooth function $h = \frac{1}{2} ||u||^2$, using equation (2.7), and (2.8), we find the following expression for the Hessian operator $A_h$

$$A_h(X) = X(f)u + f^2 X.$$
Thus, the Hessian $\text{Hess}(h)$ of the smooth function $h$ is given by

$$(3.1) \quad \text{Hess}(h)(X, Y) = X(f)g(u, Y) + f^2g(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Now, as the $\text{Hess}(h)$ is symmetric, equation (3.1) implies

$$X(f)g(u, Y) = Y(f)g(u, X),$$

and through which, we conclude

$$X(f)u = g(u, X)\nabla f.$$

Replacing $X$ by $u$ in above equation, we get

$$u(f)u = \|u\|^2 \nabla f,$$

which on taking the inner product with $\nabla f$, gives

$$\|u\|^2 \|\nabla f\|^2 = u(f)^2 = g(u, \nabla f)^2.$$

Above equation confirms that vector fields $u$ and $\nabla f$ are parallel. Hence, there exists a smooth function $\rho : M \to \mathbb{R}$ such that

$$(3.2) \quad \nabla f = \rho u.$$

If there is another function $\sigma : M \to \mathbb{R}$, satisfying $\nabla f = \sigma u$, then we have $(\rho - \sigma)u = 0$, which on connected $M$ implies that either $\rho = \sigma$ or $u = 0$. However, $u = 0$ in equation (2.1), gives $f = 0$, a contradiction to the fact that $u$ is a non-trivial concircular vector field. Hence, $\rho = \sigma$, that is $\rho$ is a unique function satisfying equation (3.3).

The unique function $\rho$ guaranteed by Theorem 3.1 that is associated to the non-trivial concircular vector field $u$ with potential function $f$ on a connected Riemannian manifold $(M, g)$ connects the vector field $\nabla f$ to the vector field $u$. Therefore, we call the function $\rho$ the connecting function of the non-trivial concircular vector field $u$. In the following results, we show that the connecting function $\rho$ can be used to characterize a $n$-sphere $S^n(c)$ of constant curvature $c$.

**Theorem 3.2:** An $n$-dimensional compact and connected Riemannian manifold $(M, g)$ admits a non-trivial concircular vector field $u$ with potential function $f$ such that the connecting function $\rho$ is a constant along the integral curves of $u$, if and only if, $(M, g)$ is isometric to the $n$-sphere $S^n(c)$.

**Proof.** Suppose $(M, g)$ is an $n$-dimensional compact and connected Riemannian manifold admits a non-trivial concircular vector field $u$ with potential function $f$ such that the connecting function $\rho$ is a constant along the integral curves of $u$, that is, $u(\rho) = 0$. Then, using

$$\nabla f = \rho u,$$

the Hessian operator $A_f$ of the potential function $f$ is computed by taking covariate derivative in above equation, which is given by

$$(3.4) \quad A_fX = X(\rho)u + \rho fX, \quad X \in \mathfrak{X}(M).$$

Note that, using equation (3.1), we conclude $\text{div } u = nf$, and integrating this equation leads to

$$(3.5) \quad \int_M f = 0.$$
If $f$ is a constant, then above equation concludes that $f = 0$, which is contrary to the assumption that $u$ is a non-trivial circular vector field. Hence, the potential function $f$ is a nonconstant function. Now, using the symmetry of the Hessian operator $A_f$ in equation (3.4), we conclude that

$$X(\rho)g(u, Y) = Y(\rho)g(u, X),$$

and it implies that

$$X(\rho)u = g(u, X)\nabla \rho.$$

Replacing $X$ by $u$ in the above equation and using $u(\rho) = 0$, we conclude

$$\|u\|^2 \nabla \rho = 0.$$

However, $u \neq 0$ being a non-trivial concircular vector field, above equation on connected $M$, gives $\nabla \rho = 0$, that is, the connecting function $\rho$ is a constant. Moreover, the constant $\rho$ has to be a nonzero constant, for if $\rho = 0$, then Theorem 3.1 will imply $f$ is a constant, which is ruled out in the previous paragraph. Taking trace in equation (3.4), we get $\Delta f = \rho f$, that is, the nonconstant function $f$ is eigenfunction of the Laplace operator $\Delta$ acting on smooth function $s$ on $M$. Since, $M$ is compact, we conclude $\rho < 0$, that is, the nonzero constant $\rho < 0$. We put $\rho = -c$, $c > 0$ and we have

$$(3.6) \quad \nabla f = -cu.$$

Taking covariant derivative in equation (3.6) with respect to $X \in \mathfrak{X}(M)$ and using equation (2.1), we get

$$(3.7) \quad \nabla_X \nabla f = -cfX, \quad X \in \mathfrak{X}(M),$$

Hence, the nonconstant function $f$ satisfies the Obata’s differential equation (3.7) (cf. [15]) and thus, the Riemannian manifold $(M, g)$ is isometric to the sphere $S^n(c)$.

Conversely, we know that $S^n(c)$ is a hypersurface of the Euclidean space $E^{n+1}$ with unit normal $N$ and the Weingarten map $A = -\sqrt{c}I$. We take a nonzero constant vector field $Z$ on the Euclidean space $E^{n+1}$, whose restriction to $S^n(c)$, can be expressed as $Z = u + sN$, where $u$ is the tangential component of $Z$ and $s$ is a smooth function $s = \langle Z, N \rangle$ on the sphere $S^n(c)$, and $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $E^{n+1}$. Taking $X \in \mathfrak{X}(S^n(c))$, we get $X(s) = \langle Z, \sqrt{c}X \rangle = \sqrt{c}g(u, X)$, where $g$ is the induced metric on $S^n(c)$. Thus, we conclude

$$(3.8) \quad \nabla s = \sqrt{c}u.$$

Now, as $Z$ is a constant vector field, using the Euclidean connection $D$ on the Euclidean space $E^{n+1}$, we have $D_X Z = 0$. For $X, Y \in \mathfrak{X}(S^n(c))$, using the Gauss formula for hypersurface $D_X Y = \nabla_X Y - \sqrt{c}g(X, Y)N$, we compute

$$0 = \langle D_X Z, Y \rangle = X \langle Z, Y \rangle - \langle Z, \nabla_X Y - \sqrt{c}g(X, Y)N \rangle$$

$$= Xg(u, Y) - g(u, \nabla_X Y) + \sqrt{c}sg(X, Y)$$
$$= g(\nabla_X u, Y) + \sqrt{c}sg(X, Y),$$

and conclude

$$(3.9) \quad \nabla_X u = -\sqrt{c}u, \quad X \in \mathfrak{X}(S^n(c)).$$
Hence, \( u \) is concircular vector field on \( S^n(c) \), with potential function \( f = -\sqrt{cs} \), which gives \( \nabla f = -\sqrt{c} \nabla s \). Using equation (3.8), we conclude
\[
\nabla f = -cu. \tag{3.10}
\]

Suppose, \( f = 0 \), which will imply \( s = 0 \) and in view of equation (3.8), \( u = 0 \), that is, \( Z = 0 \) on \( S^n(c) \). As \( Z \) is a constant vector field, we get \( Z = 0 \) on \( E^{n+1} \), which gives a contradiction to the fact that \( Z \) is a nonzero constant vector field. Hence, \( f \neq 0 \), that is, \( u \) is a non-trivial concircular vector field with potential function \( f \). Then equation (3.10), implies that the connecting function \( \rho = -c \), which is a constant.

**Theorem 3.3:** An \( n \)-dimensional complete and simply connected Riemannian manifold \((M, g)\) admits a non-trivial concircular vector field \( u \) with potential function \( f \) such that \( \Delta u = -\lambda u \) for a constant \( \lambda > 0 \), if and only if, \((M, g)\) is isometric to the \( n \)-sphere \( S^n(\lambda) \).

**Proof.** Suppose \( u \) is a non-trivial concircular vector field on \((M, g)\) with potential function \( f \) and connecting function \( \rho \) such that \( \Delta u = -\lambda u \), \( \lambda > 0 \). Using equation (2.1) and a local orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( M \), by a straightforward computation, we get \( \Delta u = \nabla f \). Thus, theorem 3.1, gives \( -\lambda u = \rho u \), that is, \( (\rho + \lambda) u = 0 \). Since, a simply connected \( M \) is also connected and \( u \) being a non-trivial concircular vector field \( u \neq 0 \), we must have \( \rho = -\lambda \) and consequently, theorem 3.1, implies \( \nabla f = -\lambda u \), which on using equation (2.1), gives
\[
\nabla_X \nabla f = -\lambda f X, \quad X \in \mathfrak{X}(M). \tag{3.11}
\]
If potential function \( f \) is a constant, then above equation will imply, \( f = 0 \) (as \( \lambda > 0 \)), which is contrary to the assumption that \( f \) is potential function of the non-trivial concircular vector field \( u \). Hence, equation (3.11) is Obata’s differential equation for nonconstant function \( f \) and positive constant \( \lambda \), which proves that \((M, g)\) is isometric to \( S^n(\lambda) \).

Conversely, on \( S^n(c) \), as in the proof of theorem 3.2, there is a non-trivial concircular vector field \( u \) with potential function \( f \) and connecting function \( \rho \) that satisfy equations (3.8)-(3.10), which imply \( \Delta u = -cu \). \( \square \)

**Theorem 3.4:** An \( n \)-dimensional complete and simply connected Riemannian manifold \((M, g)\) admits a non-trivial concircular vector field \( u \) with potential function \( f \) and connecting function \( \rho \) satisfying (i) \( g(\nabla f, \nabla \rho) = 0 \) and (ii) \( \text{Ric}(\nabla f, \nabla f) > 0 \), if and only if, \((M, g)\) is isometric to the \( n \)-sphere \( S^n(c) \).

**Proof.** Suppose \( u \) is a non-trivial concircular vector field with potential function \( f \) and connecting function \( \rho \) on an \( n \)-dimensional Riemannian manifold \((M, g)\) satisfying
\[
g(\nabla f, \nabla \rho) = 0 \quad \text{and} \quad \text{Ric}(\nabla f, \nabla f) > 0. \tag{3.12}
\]
Then using theorem 3.1, in the above equations, we conclude
\[
\rho u(\rho) = 0 \quad \text{and} \quad \rho^2 \text{Ric}(u, u) > 0, \tag{3.13}
\]
that is, $u(\rho) = 0$. Using the symmetry of Hessian operator in equation (3.4), we have

$$X(\rho)g(u, Y) = Y(\rho)g(u, X), \quad X, Y \in \mathfrak{X}(M)$$

and taking $X = u$, in above equation, yields $Y(\rho)\|u\|^2 = 0, \ Y \in \mathfrak{X}(M)$. As $u$ is a non-trivial concircular vector field, we must have $Y(\rho) = 0, \ Y \in \mathfrak{X}(M)$, that is, $\rho$ is a constant and in view of second equation in equation (3.13), constant $\rho \neq 0$.

Now, equation (2.5) and theorem 3.1, imply

$$Ric(u, u) = -(n - 1)\rho \|u\|^2.$$  

Combining equations (3.13) and (3.14), we conclude that the non-zero constant $\rho < 0$. Taking $\rho = -c, \ c > 0$, theorem 3.1, gives $\nabla f = -cu$, where $f$ has to be nonconstant, for otherwise we shall have $u = 0$, which is ruled out. Hence, using equation (2.1), we get the Obata’s differential equation

$$\nabla_X \nabla f = -cfX, \quad X \in \mathfrak{X}(M),$$

proving that $(M, g)$ is isometric to $S^n(c)$.

Converse trivially follows through the proof of theorem 3.2.

\[\square\]

4. CHARACTERIZATIONS OF EUCLIDEAN SPACES

In this section, we are interested in finding characterizations of a Euclidean space using non-trivial concircular vector fields.

**Theorem 4.1:** An $n$-dimensional complete and connected Riemannian manifold $(M, g)$ admits a non-trivial concircular vector field $u$ with potential function $f$ satisfying $Ric(\nabla f, \nabla f) = 0$, if and only if, $(M, g)$ is isometric to the Euclidean space $E^n$.

**Proof.** Suppose $(M, g)$ is an $n$-dimensional complete and connected Riemannian manifold $(M, g)$ that admits a non-trivial concircular vector field $u$ with potential function $f$, connecting function $\rho$ and the Ricci curvature satisfies

$$Ric(\nabla f, \nabla f) = 0.$$  

Using equation (2.5) and theorem 3.1, we have

$$Q(\nabla f) = -(n - 1)\rho \nabla f,$$

which in view of equation (4.1), gives

$$-(n - 1)\rho \|\nabla f\|^2 = 0.$$  

Note that, if $\rho = 0$, then theorem 3.1, gives $\nabla f = 0$, that is, $f$ is a constant. Thus, as $M$ is connected, equation (4.2), in its both outcomes, implies that $f$ is a constant. Now, observe that the constant $f \neq 0$, owing to the fact that $u$ is non-trivial. Using equation (2.7), for the function $h = \frac{1}{2}\|u\|^2$, we find the following expression for its Hessian operator

$$A_h X = f^2 X,$$

and consequently, we have

$$Hess(h) = cg,$$

where $c = f^2$ is a nonzero constant. Notice through equation (2.7), that the function $h$ is not a constant, for if $h$ were to be a constant, as $f \neq 0$, it would imply $u = 0$ a contradiction. Hence, the nonconstant function $h$ satisfies equation (4.3) for a
nonzero constant $c$, proves that the complete and connected Riemannian manifold $(M, g)$ is isometric to the Euclidean space $\mathbb{E}^n$ (cf. Theorem-1, [17]).

Conversely, consider the position vector field $u = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ on the Euclidean space $\mathbb{E}^n$, where $x^1, \ldots, x^n$ are Euclidean coordinates. which satisfies $\nabla_X u = X, X \in \mathfrak{X}(\mathbb{E}^n)$, where $\nabla$ is the Euclidean connection on $\mathbb{E}^n$. Thus, $u$ is a non-trivial concircular vector field on $\mathbb{E}^n$ with potential function $f = 1$ and connecting function $\rho = 0$, which satisfies the condition in the statement of the theorem. □

Our next result shows that harmonic concircular vector fields characterize Euclidean spaces.

**Theorem 4.2:** An $n$-dimensional complete and connected Riemannian manifold $(M, g)$ admits a non-trivial concircular vector field $u$ that satisfies $\Delta u = 0$, if and only if, $(M, g)$ is isometric to the Euclidean space $\mathbb{E}^n$.

**Proof.** Suppose $u$ is a non-trivial concircular vector field with potential function $f$ on an $n$-dimensional complete and connected Riemannian manifold $(M, g)$, which satisfies $\Delta u = 0$. Using equation (2.1), we compute

$$\Delta u = \nabla f.$$ 

Hence, the potential function $f$ is a constant and this constant $f \neq 0$ as $u$ is a non-trivial concircular vector field. Now, equation (2.7), with $f$ a constant gives

$$\text{Hess}(h) = cg,$$

where $c = f^2$ is a nonzero constant. Hence, $(M, g)$ is isometric to the Euclidean space $\mathbb{E}^n$.

Converse is trivial, as the position vector field $u$ on the Euclidean space $\mathbb{E}^n$ is harmonic. □

5. **Influence of concircular vector fields on topology**

In this section, we observe that because of the connecting function, we can exhibit the influence of non-trivial concircular vector fields on topology of the Reimannian manifolds. Our observations depend on already known results and therefore results in this sections are simply trivial applications of known results in differential topology. Recall that by Reeb’s theorem, if a compact smooth manifold $M$ admits a smooth function $F : M \to \mathbb{R}$ with exactly two critical points which are non-degenerate, then $M$ is homeomorphic to an $n$-sphere $S^n$. Moreover, it is later observed by Milnor (cf. theorem 1 p. 166 , [14] ) that this result holds even if the two critical points are degenerate. Using this modified Reeb’s theorem, we have the following trivial consequence:

**Theorem 5.1:** If an $n$-dimensional compact and connected Riemannian manifold $(M, g)$ admits a non-trivial concircular vector field $u$ with potential function $f$ and connecting function $\rho$ such that $\rho(p) \neq 0$ for each $p \in M$ and vector field $u$ has only two zeros, then $M$ is homeomorphic to an $n$-sphere.
Proof. Using theorem 3.2, we have $\nabla f = \rho u$, and the vector field $u$ has two zeros say at $p, q \in M$. Then as connecting function $\rho(x) \neq 0$ on $M$, points $p, q$ are critical points of the potential function $f$. Thus, the smooth function $f$ has exactly two critical points, which proves that $M$ is homeomorphic to $n$-sphere. \[\square\]

Consider a non-trivial concircular vector field $u$ that is nowhere zero on an $n$-dimensional connected Riemannian manifold $(M, g)$ with potential function $f$ and connecting function $\rho(p) \neq 0$, $p \in M$. Then by theorem 3.1, the potential function $f$ has no critical points. If we define a smooth vector field $\xi$ on $M$ by $\xi = \frac{\nabla f}{||\nabla f||^2}$, then, as $\xi(f) = 1$, the local flow $\{\phi_t\}$ of $\xi$ satisfies

(5.1) $f(\phi_t(p)) = f(p) + t,$

which on using escape lemma (cf. [13]), proves that $\xi$ is a complete vector field and $\{\phi_t\}$ is the global flow. Moreover, observe that $f : M \to R$ is a submersion, consequently, the level set $M_p = f^{-1}\{f(p)\}$ is a compact hypersurface of $M$. Now, we have the following:

**Theorem 5.2:** If an $n$-dimensional connected Riemannian manifold $(M, g)$ admits a non-trivial concircular vector field $u$, $u(p) \neq 0, p \in M$, with potential function $f$ and connecting function $\rho$ such that $\rho(p) \neq 0$ for each $p \in M$, then $M$ is diffeomorphic to $N \times R$ for some compact smooth manifold $N$.

Proof. For $p \in M$, we denote by $M_p$ the level set $f^{-1}\{f(p)\}$ of $f$, which is a compact hypersurface of $M$. We define $F : M_p \times R \to M$ by

$$F(q, t) = \phi_t(q),$$

which is a smooth map. First, we shall show that $F$ is a surjective: Take $x \in M$, then we can find $s \in R$, such that $\phi_s(x) = m \in M_p$, with $x = \phi_{-s}(m)$. Consequently,

$$F(m, -s) = x.$$ 

Next, we show that $F$ is an injective: Take $(q_1, t_1), (q_2, t_2) \in M_p \times R$ such that $F(q_1, t_1) = F(q_2, t_2)$. Then we have $\phi_{t_1}(q_1) = \phi_{t_2}(q_2)$ and using equation (5.1), we get

$$f(q_1) + t_1 = f(q_2) + t_2.$$

However, as $q_1, q_2 \in M_p$, we have $f(q_1) = f(q_2)$. Thus, we get $t_1 = t_2$. and $\phi_{t_1}(q_1) = \phi_{t_1}(q_2)$ implies $q_1 = q_2$. Hence $F$ is an injective. Finally, we have

$$F^{-1}(x) = (m, -s) = (\phi_s(x), -s)$$

is also smooth. Hence, $F$ is a diffeomorphism. \[\square\]

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