A UNIVERSAL PROBABILITY APPROXIMATION METHOD: MARKOV PROCESS APPROACH

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ABSTRACT. We view the classical Lindeberg principle in a Markov process setting to establish a universal probability approximation framework by Itô’s formula and Markov semigroup. As applications, we consider approximating a family of online stochastic gradient descents (SGDs) by a stochastic differential equation (SDE) driven by additive Brownian motion, and obtain an approximation error with explicit dependence on the dimension which makes it possible to analyse high dimensional models. We also apply our framework to study stable approximation and normal approximation and obtain their optimal convergence rates (up to a logarithmic correction for normal approximation).

Keywords: Probability approximation, Markov process, Itô’s formula, Markov semigroup, Online stochastic gradient descent, Stochastic differential equation, Stable approximation, Normal approximation.

1. INTRODUCTION

Lindeberg principle provides an elegant proof for the classical central limit theorem of the sum of independent random variables [52], it has been extensively applied to many research problems, see [24, 50, 70, 26, 18, 57, 27, 34, 49, 72, 8, 32] and the references therein. In this paper, we shall view the classical Lindeberg principle in a Markov process setting, and use the well developed tools in stochastic analysis, such as Itô’s formula, Markov semigroup and infinitesimal generator, to establish a universal probability approximation framework.

In order to interpret our method, we first briefly recall the classical Lindeberg principle by the following example. Let \((\xi_i)_{i\geq 1}\) be a sequence of independent and identically distributed (i.i.d.) \(\mathbf{R}\)-valued random variables with \(E\xi_i = 0\), \(E\xi_i^2 = 1\) and \(E|\xi_i|^3 < \infty\). Let \((\zeta_i)_{i\geq 1}\) be a sequence of independent standard normal distributed random variables and it is well known that \(\frac{\zeta_1 + \cdots + \zeta_n}{\sqrt{n}}\) is a standard normal distributed random variables for any \(n \in \mathbb{N}\). Write \(\xi_{n,i} = \frac{\xi_i}{\sqrt{n}}\), \(\zeta_{n,i} = \frac{\zeta_i}{\sqrt{n}}\), and denote

\[
X_n = \xi_{n,1} + \cdots + \xi_{n,n}, \quad Y_n = \zeta_{n,1} + \cdots + \zeta_{n,n}.
\]

Further denote \(Z_0 = X_n\) and \(Z_i = Z_{i-1} - \xi_{n,i} + \xi_{n,i} \) for \(i \geq 1\), we easily see that \(Z_i\) is obtained by swapping \(\zeta_{n,i}\) in \(Z_{i-1}\) with \(\xi_{n,i}\). For any bounded 3rd order differentiable function \(h\), we have

\[
|E[h(X_n)] - E[h(Y_n)]| \leq \sum_{i=1}^{n} |E[h(Z_{i-1})] - E[h(Z_i)]| \leq Cn^{-3/2}||h'||\sum_{i=1}^{n} [E|\xi_i|^3 + E|\zeta_i|^3] \leq Cn^{-1/2}||h'||,
\]

where \(||.||\) is the uniform norm of continuous function and the second inequality is obtained by a 3rd order Taylor expansion.

Let us now explain the Lindeberg’s proof from a perspective of Markov process and view the swap trick as a comparison of two Markov processes. Denote \(X_0 = 0, X_i = \)
\( \zeta_{n,1} + \ldots + \zeta_{n,i} \) and \( Y_0 = 0, Y_i = \zeta_{n,1} + \ldots + \zeta_{n,i} \) for \( i \geq 1 \), it is clear that \((X_i)_{0 \leq i \leq n}\) and \((Y_i)_{0 \leq i \leq n}\) are both Markov processes. Let \( j \geq i \geq 1 \), denote by \( X_j(i, x) \) the random variable \( X_j \) given \( X_i = x \in \mathbb{R} \), i.e., \( X_j(i, x) = x + \zeta_{n,i+1} + \ldots + \zeta_{n,j} \), it is obvious \( X_j(i, X_i) = X_j \) for \( j \geq i \). Similarly, we define \( Y_j(i, y) \) for \( j \geq i \). It is easy to see that \( Z_j = X_n(j, Y_j) = X_n(j, Y_j(j - 1, Y_{j-1})) \) and \( Z_{j-1} = X_n(j - 1, Y_{j-1}) = X_n(j, X_j(j - 1, Y_{j-1})) \) for each \( 1 \leq j \leq n \), thus

\[
|\mathbb{E}[h(X_n)] - \mathbb{E}[h(Y_n)]| \leq \sum_{j=1}^{n} |\mathbb{E}[h(Z_{j-1})] - \mathbb{E}[h(Z_j)]|
= \sum_{j=1}^{n} |\mathbb{E}[h(X_n(j, X_j(j - 1, Y_{j-1})))] - \mathbb{E}[h(X_n(j, Y_j(j - 1, Y_{j-1})))]|.
\]

Notice that \( \mathbb{E}[h(X_n(j, X_j(j - 1, Y_{j-1})))] \) and \( \mathbb{E}[h(X_n(j, Y_j(j - 1, Y_{j-1})))] \) are the functions of \( X_j(j - 1, Y_{j-1}) \) and \( Y_j(j - 1, Y_{j-1}) \), respectively, and compare these two new functions rather than directly compute \( |\mathbb{E}[h(Z_{j-1})] - \mathbb{E}[h(Z_j)]| \) in Lindeberg principle. Because \((X_i)_{0 \leq i \leq n}\) can be embedded into a Brownian motion \((B_t)_{0 \leq t \leq 1}\) which has a smoothening effect, we expect that Itô’s formula and the semigroup theory of Brownian motion will make \( \mathbb{E}[h(X_n(j, X_j(j - 1, Y_{j-1})))] \) and \( \mathbb{E}[h(X_n(j, Y_j(j - 1, Y_{j-1})))] \) have better regularity than \( h \), see more details in Subsection 4.3. Since the above procedure only depends on Markov property, this perspective of viewing Lindeberg principle can be extended to other Markov processes.

The novelty of this paper is the following two aspects. (1) We view the procedure of the classical Lindeberg principle as a special Markov process and extend this point of view to general Markov process setting, using Itô’s formula of Markov process and Markov semigroup (see, e.g., [41, 36, 51, 17, 61]), we establish a universal probability approximation framework. (2) We apply our framework to the three applications in the classical Wasserstein-1 distance: approximating a family of online stochastic gradient descent (SGD) in machine learning by a stochastic differential equation (SDE), \( \alpha \)-stable approximation, and normal approximation.

For the first application on approximating SGD by a stochastic differential equation (SDE), there have been many results, see for instance [69, 54, 45, 55, 3, 56, 39, 38, 20] and the references therein. To the best of our knowledge, the approximating SDEs in most of these works are driven by multiplicative Brownian motions, so one has to assume that the diffusion coefficient satisfies Lipschitz condition to guarantee their well-posedness. However, this Lipschitz condition assumption is often hard to verify for concrete models. On the other hand, most of the approximation results in previous works have test functions with high derivatives, from which one cannot obtain an approximation error bound in a probability metric. In contrast, our approximating SDE is driven by additive Brownian motion, which enables us to use our framework to get an explicit error bound in the classical Wasserstein-1 distance. Moreover, we can find an explicit dependence of approximation error on the dimension and thus make analyzing high dimensional models possible. The price of getting an approximating SDE with additive noise and an explicit error bound with respect to the dimension is to sacrifice the convergence rate. We give linear regression and penalized logistic regression as two examples of this theorem.

The second and third applications are the stable and normal approximations, which have recently been intensively studied by Stein’s method, see for instance, stable approximation [73, 29, 30, 21] and normal approximation [25, 65, 71, 28, 40, 68]. Using our framework and borrowing the regularity results of nonlocal PDEs in [73, 29, 30], we can obtain the convergence rates of stable approximation therein. For simplicity, in this paper we only
show the special case of the sum of Pareto distributed random variables to illustrate the idea, it is easy to see from its proof that our framework will work for the cases in [73, 29, 30]. For the third application, by a direct calculation under our framework, we get a \( \frac{1}{\sqrt{n}} \) convergence rate up to \( \log n \) correction for multivariate normal approximation, which was established in [40, 71]. For Stein’s method, we refer the reader to [67, 25, 65, 60, 71, 28, 40, 68] for normal approximation, [22, 14, 4, 15, 1] for Poisson approximation, [73, 29, 30, 31, 21] for \( \alpha \)-stable approximation and [5, 9, 43, 6, 7, 2, 53, 59, 42, 10, 48] for other approximations.

Besides the applications to online SGD, normal and stable approximations addressed in this paper, our new method can also be applied to many other probability approximations, e.g. diffusion approximation with constant step size and so on. However, this framework cannot be applied to study Poisson approximation and diffusion approximation with variable step size due to their inhomogeneity feature. We will study these research problems in the future paper.

In this paper, we focus on the approximation problems in Wasserstein-1 distance. However, it is clear to see from Theorem 2.1 that our approximation method also works for other metric. For instance, if we consider bound measurable function \( h \), then the approximation will be in total variation metric.

The organization of this paper is as follows. We shall introduce our probability approximation framework and main theorem in next section. In Section 3, we will give the results about the three applications to SGD, stable and normal approximations, their proofs are given in Subsections 4.1-4.3, respectively. Appendixes A and B are devoted to prove some auxiliary lemmas about the first application, while Appendixes C and D provide the proofs of auxiliary lemmas about the second and third applications, respectively.

**Notations.** We end this section by introducing some notations, which will be frequently used in sequel. The inner product of \( x, y \in \mathbb{R}^d \) is denoted by \( \langle x, y \rangle \) and the Euclidean metric is denoted by \( |x| \).

Let \( \mu \) and \( \nu \) be two probability distributions on \( \mathbb{R}^d \), their Wasserstein-1 distance is defined as
\[
d_W(\mu, \nu) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,
\]
where \( \text{Lip}(1) := \{ h : \mathbb{R}^d \to \mathbb{R} : |h(x) - h(y)| \leq |x - y|, \ X \text{ and } Y \text{ are two random variables with distributions } \mu \text{ and } \nu, \text{ respectively.} \)

For a random variable \( X \), we denote by \( \mathcal{L}(X) \) its probability law.

Let \( \mathcal{C}(\mathbb{R}^d, \mathbb{R}) \) denote the collection of all continuous functions \( f : \mathbb{R}^d \to \mathbb{R} \) and \( \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}) \), \( k \geq 1 \), denote the collection of all \( k \)-th order continuously differentiable functions. For \( f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}) \) and \( v, v_1, v_2, x \in \mathbb{R}^d \), the directional derivative \( \nabla_v f(x) \) and \( \nabla_{v_2} \nabla_{v_1} f(x) \) are respectively defined by
\[
\nabla_v f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon},
\]
\[
\nabla_{v_2} \nabla_{v_1} f(x) = \lim_{\epsilon \to 0} \frac{\nabla_{v_1} f(x + \epsilon v_2) - \nabla_{v_1} f(x)}{\epsilon}.
\]
\( \nabla f(x) \in \mathbb{R}^d, \ \nabla^2 f(x) \in \mathbb{R}^{d \times d} \) and \( \Delta f(x) \in \mathbb{R} \) are the gradient, the Hessian and the Laplacian of \( f \), respectively. It is known that \( \nabla_v f(x) = \langle \nabla f(x), v \rangle \) and \( \nabla_{v_2} \nabla_{v_1} f(x) = \langle \nabla^2 f(x), v_1 v_2 \rangle_{\text{HS}} \), where \( T \) is the transpose operator and \( \langle A, B \rangle_{\text{HS}} := \sum_{i,j=1}^d A_{ij} B_{ij} \) for \( A, B \in \mathbb{R}^{d \times d} \). Given a matrix \( A \in \mathbb{R}^{d \times d} \), its Hilbert-Schmidt norm is \( \| A \|_{\text{HS}} = \sqrt{\sum_{i,j=1}^d A_{ij}^2} = \).
\( \sqrt{\text{Tr}(A^T A)} \) and its operator norm is \( \|A\|_{op} = \sup_{|v| = 1} |Av| \). It is known that

\[
(1.2) \quad \|A\|_{op} = \sup_{|v_1|, |v_2| = 1} |\langle A, v_1 v_2^T \rangle|, \quad \|A\|_{op} \leq \|A\|_{HS} \leq \sqrt{d} \|A\|_{op}.
\]

We can also define \( \nabla_v f(x) \) and \( \nabla_v, \nabla_v f(x) \) for a second-order differentiable function \( f = (f_1, \ldots, f_d)^T : \mathbb{R}^d \to \mathbb{R}^d \) in the same way as above. Define \( \nabla f(x) = (\nabla f_1(x), \ldots, \nabla f_d(x))^T \) and \( \nabla^2 f(x) = (\nabla^2 f_i(x))_{i=1}^d \in \mathbb{R}^{d \times d} \). In this case, we have \( \nabla_v f(x) = \{ (\nabla f_1(x), v_1), \ldots, (\nabla f_d(x), v_d) \}^T \) and \( \nabla_v \nabla_v f(x) = \{ (\nabla^2 f_1(x), v_1 v_1^T), \ldots, (\nabla^2 f_d(x), v_1 v_2^T) \}_{HS} \}

Moreover, \( C_0(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \) denotes the set of all bounded continuous functions from \( \mathbb{R}^{d_1} \) to \( \mathbb{R}^{d_2} \) with the supremum norm defined by

\[ ||f|| = \sup_{x \in \mathbb{R}^{d_1}} |f(x)|. \]

Denote by \( C_{p_1, \ldots, p_k} \) some positive number depending on \( k \) parameters, \( p_1, \ldots, p_k \), whose exact values can vary from line to line.

### 2. The Framework and Main Theorem

Let \( (E, \rho) \) be a separable metric space (\( \rho \) is the metric), let \( (X_t)_{t \geq 0} \) be a continuous time homogeneous \( E \)-valued stochastic process, and let \( (Y_k)_{k \in \mathbb{Z}_+} \) be a discrete time homogeneous \( E \)-valued stochastic process (note \( \mathbb{Z}_+ = \{0\} \cup \mathbb{N} \)). If \( X_0 = x \in E \), we denote the process \( (X_t)_{t \geq 0} \) by \( (X^x_t)_{t \geq 0} \) to stress it starts from \( x \). Similarly for the notation \( (Y^y_k)_{k \in \mathbb{Z}_+} \) for \( y \in E \). In addition, for any \( d \)-dimensional random vectors \( \xi, \xi' \), we call \( \xi \equiv \xi' \) if for any \( A \in \mathcal{B}(E) \), the Borel set of \( E \), we have

\[ \mathbb{P}(\xi \in A) = \mathbb{P}(\xi' \in A). \]

For \( (X_t)_{t \geq 0} \) and \( (Y_k)_{k \in \mathbb{Z}_+} \), their infinitesimal generators are respectively defined as

\[ A^X f(x) := \lim_{t \to 0} \frac{\mathbb{E}[f(X^x_t)] - f(x)}{t} \quad \text{and} \quad A^Y f(y) := \mathbb{E}[f(Y^y_t) - f(y)], \]

where \( f \) belongs to a family of functions according to concrete approximation problems. Then, by [36, Proposition 1.7 in Chapter 4], we have the following Itô’s formula:

\[
(2.1) \quad f(X^x_t) - f(x) = \int_0^t A^X f(X^x_s) ds + \mathcal{M}_t
\]

where \( (\mathcal{M}_t)_{t \geq 0} \) is a martingale with \( \mathbb{E}[\mathcal{M}_t] = 0 \) for all \( t \geq 0 \). For a more thorough discussion on the infinitesimal generators and Itô’s formula, we refer the reader to [36, Chapter 1 and Chapter 4], [74, Chapter IX], [61, Chapter 4] and the references therein.

Our first main result is a framework of comparing the distributions of \( (X_t)_{t \geq 0} \) and \( (Y_k)_{k \in \mathbb{Z}_+} \), which can be fitted into many probability approximations arising in concrete applications. The key ingredients of the proof are Markov semigroup, Itô’s formula and infinitesimal generator in stochastic analysis.

**Theorem 2.1.** Let \( N \geq 2 \) be a natural number and let \( h : E \to \mathbb{R} \) be a measurable function such that: (1) \( \mathbb{E}[h(X^x_t)] < \infty \) and \( \mathbb{E}[h(Y^y_k)] < \infty \) for all \( x \in E, y \in E, t \leq N \) and \( k \leq N \); (2) the function \( u_k(x) := \mathbb{E}[h(X^x_t)] \) for \( k \geq 1 \) satisfies \( \mathbb{E}[A^X u_k(X^y_t)] < \infty \) and \( \mathbb{E}[A^X u_k(X^y_t)] < \infty \) for all \( 1 \leq j, k \leq N \) and \( 0 \leq t \leq 1 \). Then

\[
(2.2) \quad \mathbb{E}h(X_N) - \mathbb{E}h(Y_N) = I_h + II_h + III_h,
\]
where
\[ I_h = \sum_{j=1}^{N-1} \mathbb{E}[A^X u_{N-j}(Y_{j-1}) - A^Y u_{N-j}(Y_{j-1})]; \]
\[ II_h = \sum_{j=1}^{N-1} \mathbb{E} \int_0^1 [A^X u_{N-j}(X_s^{Y-1}) - A^X u_{N-j}(Y_{j-1})] ds, \]
\[ III_h = \mathbb{E}[h(X_1^{Y_N-1}) - h(Y_{N-1})] + \mathbb{E}[h(Y_N) - h(Y_{N-1})]. \]

In particular,
\[ d_W(\mathcal{L}(X_N), \mathcal{L}(Y_N)) \leq \sup_{h \in \text{Lip}(1)} (|I_h| + |II_h| + |III_h|). \]  

**Remark 2.2.** If the $h$ in (2.2) is chosen to be in other function family, the distance between $\mathcal{L}(X_N)$ and $\mathcal{L}(Y_N)$ will be different. For instance, if we consider bound measurable function $h$, then the approximation will be in total variation metric.

**Proof.** (2.3) is an immediate corollary from (2.2) by the definition of Wasserstein-1 distance, so we only need to prove (2.2). For any $s \geq 0$ and any $x \in E$, let $t \geq s$, we denote by $X_t(s, x)$ the random variable $X_t$ given $X_s = x$. Similarly, we denote $Y_k(j, y)$ the random variable $Y_k$ given $Y_j = y$ for any $k \geq j \geq 0$ and $y \in E$. It is easily seen that
\[ X_t = X_t(s, X_s), \quad 0 \leq s \leq t < \infty; \]
\[ Y_k = Y_k(j, Y_j), \quad 0 \leq j \leq k < \infty. \]

By (2.4), we see that
\[ \mathbb{E}h(X_N) = \mathbb{E}h(X_N(1, X_1)) = \mathbb{E}h(X_N(1, X_1)) - \mathbb{E}h(X_N(1, Y_1)) + \mathbb{E}h(X_N(1, Y_1)). \]

When $N \geq 2$, by (2.4) again, $X_N(1, Y_1) = X_N(2, X_2(1, Y_1))$ and thus
\[ \mathbb{E}h(X_N(1, Y_1)) = \mathbb{E}h(X_N(2, X_2(1, Y_1))) = \mathbb{E}h(X_N(2, X_2(1, Y_1))) - \mathbb{E}h(X_N(2, Y_2)) + \mathbb{E}h(X_N(2, Y_2)). \]

Continuing this process and repeatedly using the relation $X_N(i+1, X_{i+1}(i, Y_i)) = X_N(i, Y_i)$, we finally obtain
\[ \mathbb{E}h(X_N) = \sum_{j=1}^{N-1} \left[ \mathbb{E}h(X_N(j, X_j(j-1, Y_{j-1}))) - \mathbb{E}h(X_N(j, Y_j)) \right] + \mathbb{E}h(X_N(N-1, Y_{N-1})), \]
and thus
\[ \mathbb{E}h(X_N) - \mathbb{E}h(Y_N) = \sum_{j=1}^{N-1} \left[ \mathbb{E}h(X_N(j, X_j(j-1, Y_{j-1}))) - \mathbb{E}h(X_N(j, Y_j)) \right] \]
\[ + \left[ \mathbb{E}h(X_N(N-1, Y_{N-1})) - \mathbb{E}h(Y_{N-1}) \right]. \]

Since $X_N(j, x) \eqdist X_N^{j-N}$, we have $u_{N-j}(x) = \mathbb{E}[h(X_N(j, x))]$. (Note that the relation $u_{N-j}(x)$ can be written as $P_{N-j}h(x)$ with $(P_t)_{t \geq 0}$ being the semigroup of $(X_t)_{t \geq 0}$). Thus we have $\mathbb{E}h(X_N(j, X_j(j-1, Y_{j-1}))) = \mathbb{E}u_{N-j}(X_j(j-1, Y_{j-1}))$ and
\[ \mathbb{E}h(X_N) - \mathbb{E}h(Y_N) = \sum_{j=1}^{N-1} \left[ \mathbb{E}u_{N-j}(X_j(j-1, Y_{j-1})) - \mathbb{E}u_{N-j}(Y_j) \right] \]
\[ + \left[ \mathbb{E}h(X_N(N-1, Y_{N-1})) - \mathbb{E}h(Y_{N-1}) \right]. \]
Let us now calculate each term in the sum on the right hand side. Rather than directly computing the differences in brackets, we interpolate a common term and use Itô’s formula.

More precisely, by the relation $X_i^{Y_{j-1}} \overset{\text{d}}{=} X_j(j-1, Y_{j-1})$ and $Y_i^{Y_{j-1}} \overset{\text{d}}{=} Y_j(j-1, Y_{j-1}) = Y_j$, we have for $1 \leq j \leq N - 1$,

$$
\mathbb{E}u_{N-j}(X_j(j-1, Y_{j-1})) - \mathbb{E}u_{N-j}(Y_j)
= \mathbb{E}[u_{N-j}(X_j(j-1, Y_{j-1})) - u_{N-j}(Y_{j-1})] - \mathbb{E}[u_{N-j}(Y_j) - u_{N-j}(Y_{j-1})]
= \mathbb{E}\left[u_{N-j}(X_i^{Y_{j-1}}) - u_{N-j}(Y_{j-1})\right] - \mathbb{E}\left[u_{N-j}(Y_i^{Y_{j-1}}) - u_{N-j}(Y_{j-1})\right].
$$

By (2.1), we have

$$
u_{N-j}(X_i^{Y_{j-1}}) - u_{N-j}(Y_{j-1}) = \int_0^1 A^X u_{N-j}(X_s^{Y_{j-1}}) ds + \mathcal{M}_1,
$$

where $(\mathcal{M}_t)_{0 \leq t \leq 1}$ is a martingale with mean 0, and thus

$$
\mathbb{E}\left[u_{N-j}(X_i^{Y_{j-1}}) - u_{N-j}(Y_{j-1})\right] = \mathbb{E}\left[\int_0^1 A^X u_{N-j}(X_s^{Y_{j-1}}) ds\right].
$$

On the other hand, by conditional probability, we obtain

$$
\mathbb{E}\left[u_{N-j}(Y_i^{Y_{j-1}}) - u_{N-j}(Y_{j-1})\right] = \mathbb{E}\left[A^Y u_{N-j}(Y_{j-1})\right].
$$

Hence, for $1 \leq j \leq N - 1$,

$$
\mathbb{E}u_{N-j}(X_j(j-1, Y_{j-1})) - \mathbb{E}u_{N-j}(Y_j)
= \mathbb{E}\int_0^1 [A^X u_{N-j}(X_s^{Y_{j-1}}) - A^Y u_{N-j}(Y_{j-1})] ds + \mathbb{E}\left[A^X u_{N-j}(Y_{j-1}) - A^Y u_{N-j}(Y_{j-1})\right].
$$

For the term (2.5), we have

$$
\mathbb{E}h(X_N(N-1, Y_{N-1})) - \mathbb{E}h(Y_N) = \mathbb{E}\left[h(X_1^{Y_{N-1}}) - h(Y_{N-1})\right] - \mathbb{E}\left[h(Y_N) - h(Y_{N-1})\right].
$$

Combining all the relations above, we immediately obtain the equality in the theorem, as desired. \qed

3. Three Applications

As we mentioned early, we focus on Wasserstein-1 distance in this paper, though Theorem 2.1 can be applied to approximation problems in other metrics, for instance, if we replace the $\text{Lip}(1)$ function family by bounded measurable function family, the approximation turns to be in total variation metric.

In order to apply Theorem 2.1 to study approximation problems, it suffices to bound the three terms $\mathcal{I}_h, \mathcal{II}_h, \mathcal{III}_h$ in (2.2). In this paper, we only consider three applications: SDE’s approximation to online SGD, stable approximation, and normal approximation. The other applications such as Poisson approximation will be studied in the forthcoming paper.

3.1. Application 1: Online SGD and SDEs ([33, 56]). For the first application, we concentrate on studying the approximation problem of a family of online SGDs with a special structure, which enables us to approximate these SGDs by an SDE driven by additive Brownian motion. Using our framework, we can find an explicit dependence of approximation error on the dimension that it makes it possible to analyze high dimensional models. We shall give two examples for Theorem 3.5 below, one is the least square regression and the other is penalized logistic regression. Moreover, these two statistical regressions can be
high dimensional (i.e., \(d\) is larger than \(N\) in Theorem 3.5 below). We refer to the reader to [62, 33, 69, 55, 56, 39] for more details of SGDs and online SGDs.

The problem approximating SGD by SDE is well studied, see for instance [69, 54, 45, 55, 3, 56, 39, 38, 20] and the references therein. To the best of our knowledge, the approximating SDEs in most of these works are driven by multiplicative Brownian motion, so one has to assume that the diffusion coefficient satisfies Lipschitz condition to guarantee their well-posedness. However, this Lipschitz condition assumption may be hard to verify for concrete models, even for linear regression model.

Now, we first introduce the Online SGD. Estimation of model parameters by minimizing an objective function is a fundamental idea in statistics. Let \(w^* \in \mathbb{R}^d\) be the true \(d\)-dimensional model parameters. In common models, \(w^*\) is the minimizer of a convex objective \(P(w) : \mathbb{R}^d \to \mathbb{R}\), i.e.,

\[
w^* = \arg \min \left( P(w) = \mathbb{E}_{\zeta \sim \Pi} \psi(w, \zeta) = \int \psi(w, \zeta) d\Pi(\zeta) \right),
\]

where \(\zeta\) denotes the random sample from a probability distribution \(\Pi\) and \(\psi(w, \zeta)\) is the loss function. The online SGD is a widely used optimization method for minimizing \(P(w)\).

The online SGD is an iterative algorithm, let \(w_0 = x\) and the \(k\)-th iterate \(w_k\) takes the following form,

\[
w_k = w_{k-1} - \eta \nabla \psi(w_{k-1}, \zeta_k), \quad k \geq 1,
\]

where \(\zeta_k\) is the \(k\)-th sample randomly drawn from the distribution \(\Pi\), and \(\nabla \psi(w_{k-1}, \zeta_k)\) denotes the gradient of \(\psi(w_{k-1}, \zeta_k)\) with respect to \(w\) at \(w = w_{k-1}\).

We first assume:

**Assumption A1**

i) We have \(\zeta \sim (a, b) \sim \Pi\), where \(a\) and \(b\) are an \(\mathbb{R}^{d_1}\)-valued and an \(\mathbb{R}^{d_2}\)-valued random vectors respectively, with \(1 \leq d_i \leq d\), \(i = 1, 2\).

ii) There exist two vector valued functions \(\nabla \phi(x, y) : \mathbb{R}^d \times \mathbb{R}^{d_1} \to \mathbb{R}^d\), \(\nabla \phi(y, z) : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^d\), such that

\[
\mathbb{E}[\nabla \phi(a, b)] = 0,
\]

\[
\nabla \psi(w, \zeta) = \nabla \phi(w, a) + \phi(a, b), \quad \forall w \in \mathbb{R}^d.
\]

It is easily seen that

\[
\nabla P(w) = \mathbb{E}[\nabla \psi(w, \zeta)] = \mathbb{E}[\nabla \phi(w, a)] + \mathbb{E}[\nabla \phi(a, b)] = \mathbb{E}[\nabla \phi(w, a)].
\]

Based on **Assumption A1**, the SGD (3.1) can be written as

\[
w_k = w_{k-1} - \eta \nabla \phi(w_{k-1}, a_k) - \eta \nabla \phi(a_k, b_k).
\]

In order to find an approximate SDE, rather than (3.3) itself, we consider an intermediate process as the following:

\[
\hat{w}_k = \tilde{w}_{k-1} - \eta \mathbb{E}[\nabla \phi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1}] - \eta \mathbb{E}[\nabla \phi(a_k, b_k) | b_k],
\]

by (3.2), the intermediate process (3.4) can also be written as

\[
\hat{w}_k = \tilde{w}_{k-1} - \eta \nabla P(\hat{w}_{k-1}) - \eta \mathbb{E}[\nabla \phi(a_k, b_k) | b_k]
\]

\[= \hat{w}_{k-1} - \eta \nabla P(\hat{w}_{k-1}) - \sqrt{\eta} V_\eta(b_k),\]

where \(V_\eta(b_k) = \sqrt{\eta} \mathbb{E}[\nabla \phi(a_k, b_k) | b_k]\). It is straightforward to check that

\(\mathbb{E}[V_\eta(b)] = 0\), \(\text{Cov}[V_\eta(b), V_\eta(b)] = \eta \Sigma\).
and 

\[ \Sigma = \mathbb{E}\left[ \mathbb{E}\left[ \nabla \phi(a,b) | b \right] \left( \mathbb{E}\left[ \nabla \phi(a,b) | b \right] \right)^T \right]. \]

Since (3.5) is also an SGD algorithm for finding \( \omega^* \), it is natural to consider the stochastic differential equation (SDE) as follows to approximate the original online SGD (3.1):

\[ d\hat{X}_t = -\nabla P(\hat{X}_t)dt + (\eta \Sigma)^{\frac{1}{2}}dW_t, \quad \hat{X}_0 = x, \]

where \( W_t \) is a \( d \)-dimensional Brownian motion.

The advantage of Eq. (3.6) is that it is driven by an additive Brownian motion (i.e., its diffusion coefficient is a constant matrix), which enables us to find an explicit dependence of approximation error on the dimension and thus make analyzing high-dimensional models possible.

For further use, we shall assume:

**Assumption A2** The mapping \( \nabla P(x) \) belongs to \( \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d) \) and there exist \( \theta_0 > 0 \) and \( \theta_1 \geq 0 \) such that

\[ \langle v, \nabla_v \nabla P(x) \rangle \geq \theta_0 |v|^2, \quad \forall v, x \in \mathbb{R}^d; \]

(3.7)

\[ |\nabla_{v_1} \nabla_{v_2} \nabla P(x)| \leq \theta_1 |v_1||v_2|, \quad \forall v_1, v_2, x \in \mathbb{R}^d. \]

(3.8)

**Remark 3.1.** By integration, (3.7) implies

\[ \langle x - y, \nabla P(x) - \nabla P(y) \rangle \geq \theta_0 |x - y|^2, \quad \forall x, y \in \mathbb{R}^d. \]

(3.9)

In addition, for any \( x, y \in \mathbb{R}^d \),

\[ \nabla P(x) - \nabla P(y) = \int_0^1 \nabla_{x-y} \nabla P(y + r(x-y)) dr, \]

and for any \( v, z \in \mathbb{R}^d \), we have

\[ \nabla_v \nabla P(z) - \nabla_v \nabla P(0) = \int_0^1 \nabla_z \nabla_v \nabla P(tz) dt, \]

these imply

\[ |\nabla P(x) - \nabla P(y)| \leq \int_0^1 \left| \nabla_{x-y} \nabla P(y + r(x-y)) \right| dr \]

\[ \leq \int_0^1 (|\nabla_{x-y} \nabla P(0)| + \int_0^1 |\nabla_{y+r(x-y)} \nabla_{x-y} \nabla P(ty + tr(x-y))| dt) dr. \]

Then, by Cauchy-Schwarz inequality and (3.8), we have

\[ |\nabla P(x) - \nabla P(y)| \leq (|\nabla^2 P(0)| + \theta_1 |x| + \theta_1 |y|)|x - y| \]

(3.10)

\[ \leq \theta_2 (1 + |x| + |y|)|x - y| \]

with \( \theta_2 = \max\{|\nabla^2 P(0)|, \theta_1\} \).

In order to ensure that the second moments of \( w_k \) and \( \tilde{w}_k \) are bounded, we shall assume:

**Assumption A3** There exists a constant \( \kappa > 0 \) such that

\[ \mathbb{E}|\nabla \phi(x, a_k)|^2 + \mathbb{E}|\nabla \phi(a_k, b_k)|^2 \leq \kappa^2 (1 + |x|^2). \]

(3.11)
Remark 3.2. Recall
\[ \nabla \psi(x, \zeta_k) = \nabla \varphi(x, a_k) + \nabla \phi(a_k, b_k), \]
by Young inequality and (3.11), for any \( x \in \mathbb{R}^d \), we have
\[ |\nabla \psi(x, \zeta_k)|^2 \leq 2(\mathbb{E}|\nabla \varphi(x, a_k)|^2 + \mathbb{E}|\nabla \phi(a_k, b_k)|^2) \leq 2\kappa^2(1 + |x|^2). \]
This along with (3.2) and Cauchy-Schwarz inequality yields that
\[ |\nabla P(x)| = |\mathbb{E}|\nabla \varphi(w, a_k)| | \leq \sqrt{\mathbb{E}|\nabla \varphi(x, a_k)|^2} \leq 2\kappa(1 + |x|). \]

To illustrate the online SGD recursion in (3.1) and the form of \( \varphi \) and \( \phi \), we consider the following two motivating examples. In appendix A, we will verify that the following two examples satisfy Assumption A2 and Assumption A3.

Example 3.3. (Linear Regression). Under the classical linear regression setup, let the \( k \)-th sample be \( \tilde{z}_k = (\tilde{a}_k, \tilde{b}_k) \), where the input \( \tilde{a}_k \in \mathbb{R}^d \) is a sequence of random vectors independently drawn from a multivariate distribution with expectation \( \mu \) and finite second moment, and the response \( \tilde{b}_k \in \mathbb{R} \) follows a linear model, \( \tilde{b}_k = \tilde{a}_k^T \beta^* + \varepsilon_k \). Here \( \beta^* \in \mathbb{R}^d \) represents the true parameter of the linear model, and \( \varepsilon_k \) are i.i.d. centered random variables with finite variance, which are independent of \( \tilde{a}_k \). Given \( \tilde{z}_k = (\tilde{a}_k, \tilde{b}_k) \), the loss function at \( \beta \) is a quadratic one:
\[ \psi(\beta, \tilde{z}_k) = \frac{1}{2}(\tilde{a}_k^T \beta - \tilde{b}_k)^2, \]
the online SGD iterates in (3.1) become,
\[ \beta_k = \beta_{k-1} - \eta \tilde{a}_k (\tilde{a}_k^T \beta_{k-1} - \tilde{b}_k). \]
Since \( \tilde{b}_k = \tilde{a}_k^T \beta^* + \varepsilon_k \), we have
\[ \beta_k - \beta^* = \beta_{k-1} - \beta^* - \eta \tilde{a}_k (\beta_{k-1} - \beta^*) + \eta \tilde{a}_k \tilde{b}_k - \tilde{a}_k^T \beta^* \]
\[ = \beta_{k-1} - \beta^* - \eta \tilde{a}_k \tilde{a}_k^T (\beta_{k-1} - \beta^*) + \eta \tilde{a}_k \varepsilon_k. \]
Write \( w_k = \beta_k - \beta^* \), \( a_k = \tilde{a}_k \) and \( b_k = \varepsilon_k \), we have
\[ w_k = w_{k-1} - \eta \tilde{a}_k \tilde{a}_k^T w_{k-1} + \eta \tilde{a}_k \varepsilon_k, \]
which implies
\[ \nabla \varphi(w_{k-1}, a_k) = \tilde{a}_k \tilde{a}_k^T w_{k-1}, \quad \nabla \phi(a_k, b_k) = -\tilde{a}_k \varepsilon_k. \]
Since \( a_k \) is independent of \( \varepsilon_k \), we have
\[ \mathbb{E}[\nabla \phi(a_k, b_k) | b_k] = \mathbb{E}[-\tilde{a}_k \varepsilon_k | \varepsilon_k] = -\mathbb{E}[\tilde{a}_k] \varepsilon_k = -\mu \varepsilon_k, \]
and \( \mathbb{E}[\nabla \phi(a_k, b_k)] = -\mathbb{E}[\tilde{a}_k] \mathbb{E}[\varepsilon_k] = 0 \). What’s more, by (3.2), we have
\[ \nabla P(w) = \mathbb{E}[\nabla \varphi(w, a_k)] = \mathbb{E}[\tilde{a}_k \tilde{a}_k^T] w \]
and
\[ \Sigma = \mathbb{E} \left[ \mathbb{E} \left[ \nabla \phi(a, b) | b \right] \left( \mathbb{E} \left[ \nabla \phi(a, b) | b \right] \right)^T \right] = \mathbb{E} \left[ (\mu \varepsilon_k)(\mu \varepsilon_k)^T \right] = \mathbb{E}[\varepsilon_k^2] \mu \mu^T. \]

Example 3.4. (Penalized Logistic Regression). One of the most popular generalized linear model is the penalized logistic regression for binary classification problems. In particular, the logistic model assumes that the binary response \( b_k \in \{1, 0\} \) is generated by the following probabilistic model,
\[ P(b_k = 1 | a_k) = \frac{1}{1 + e^{-a_k^T w^*}}, \quad P(b_k = 0 | a_k) = \frac{1}{1 + e^{a_k^T w^*}}, \]
where $a_k$ is a sequence of i.i.d. random vectors. The loss function at $w$ is
\[
\psi(w, \zeta_k) = -\left[ b_k a_k^T w - \ln(1 + e^{a_k^T w}) \right] + \frac{\gamma}{2} w^T w,
\]
where $\gamma > 0$ is the tuning parameter (see, e.g., [46, (4.20) and (18.11)]). Given the loss function, the online SGD iterates in (3.1) become,
\[
\begin{align*}
    w_k &= w_{k-1} - \eta [a_k \frac{1}{1 + e^{-a_k^T w_{k-1}}} - b_k] + \gamma w_{k-1} \\
    &= w_{k-1} - \eta [a_k \frac{1}{1 + e^{-a_k^T w_{k-1}}} - \frac{1}{1 + e^{-a_k^T w^*}}] + \gamma w_{k-1} + \eta a_k (b_k - \frac{1}{1 + e^{-a_k^T w^*}}),
\end{align*}
\]
which implies
\[
\nabla \varphi(w_{k-1}, a_k) = a_k \left( \frac{1}{1 + e^{-a_k^T w_{k-1}}} - \frac{1}{1 + e^{-a_k^T w^*}} \right) + \gamma w_{k-1},
\]
\[
\nabla \phi(a_k, b_k) = -a_k (b_k - \frac{1}{1 + e^{-a_k^T w^*}}).
\]
Since $\mathbb{E}[b_k | a_k] = \frac{1}{1 + e^{-a_k^T w^*}}$, we have
\[
\mathbb{E}[\nabla \phi(a_k, b_k)] = \mathbb{E} \left[ \mathbb{E} \left[ \nabla \phi(a_k, b_k) | a_k \right] \right] = -\mathbb{E} \left[ a_k \left( \mathbb{E}[b_k | a_k] - \frac{1}{1 + e^{-a_k^T w^*}} \right) \right] = 0.
\]
What’s more, by (3.2), we have
\[
\nabla P(w) = \mathbb{E}[\nabla \varphi(w, a_k)] = \mathbb{E} \left[ a_k \left( \frac{1}{1 + e^{-a_k^T w}} - \frac{1}{1 + e^{-a_k^T w^*}} \right) \right] + \gamma w
\]
and
\[
\Sigma = \left( \frac{1}{\hat{p}} + \frac{1}{1 - \hat{p}} \right) \mathbb{E} \left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right] \mathbb{E} \left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right]^T,
\]
with $\hat{p} = \mathbb{E} \left[ \frac{1}{1 + e^{-a^T w^*}} \right] < 1$. Here, the second equality is proved in Lemma A.3 below.

Now, we are at the position to state our theorem of the first application.

**Theorem 3.5 (Online SGD v.s. SDE).** Keep the same notations as above. Let $N \geq 2$ be a natural number, let Assumption A1, Assumption A2 and Assumption A3 hold, and let $0 < \eta < \min \{ 1, \frac{\theta_0}{4\lambda} \}$. Then we have
\[
d_W(\mathcal{L}(\hat{X}_N), \mathcal{L}(w_N)) \leq C_{\theta, \alpha} \left[ 1 + |a|^2 + \| \nabla P(0) \|^2 + \| \Sigma^\frac{1}{2} \|^2_{HS} + d \mathbf{1}_{(0)}(\lambda_{min}) \right] \left[ (\lambda_{min} + \mathbf{1}_{(0)}(\lambda_{min}))^{-\frac{1}{2}} + 1 \right] \eta^\frac{1}{2},
\]
where $(\hat{X}_t)_{t \geq 0}$ is the diffusion process, which is defined by SDE (3.6), $(w_k)_{k \in \mathbb{Z}_+}$ is the SGD iteration process, which is defined by (3.1), and $\lambda_{min}$ is the smallest eigenvalue of $\Sigma$.

**3.2. Application 2: Multivariate $\alpha$-stable CLT for $\alpha \in (1, 2)$ ([44, 73, 30]).** The second application is the stable approximation, which has recently been intensively studied by Stein’s method, see for instance [73, 29, 30]. Using our framework and borrowing the regularity results of nonlocal PDEs in [73, 29, 30], we can obtain the convergence rates of stable approximation therein. For simplicity, here we only show the special case of the sum of Pareto distributed random variables to illustrate the idea, it is easy to see from its proof that our framework also works for general CLT established in [73, 29, 30].
Consider Pareto distribution with density

\[
\hat{p}(z) = \begin{cases} \frac{\Gamma((d-1)/2)}{\Gamma(d/2)} z^{\alpha - 1} \frac{1 - z}{|z|^\alpha}, & |z| \geq 1, \\ 0, & |z| < 1, \end{cases}
\]

where \( V(S^{d-1}) \) is the surface area of \( S^{d-1} \) and \( V(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \). Let random vectors \( \xi_1, \xi_2, \ldots \) be independent identically distributed with Pareto distribution defined as above. Set

\[
d_\alpha = \left( \int_0^\infty \frac{1 - \cos y}{y^\alpha + 1} dy \right)^{-1}, \quad \sigma = \left( \frac{\alpha}{V(S^{d-1})} \right)^{\frac{1}{\alpha}},
\]

where \( e \) is an unit vector and it is well known that \( d_\alpha = \frac{\alpha^2 \Gamma((d-1)/2)}{\Gamma(d/2) \Gamma((2-\alpha)/2)} \) (see, e.g., [13]), and let

\[
S_n = \frac{1}{\sigma} n^{-\frac{1}{\alpha}} (\xi_1 + \cdots + \xi_n).
\]

Let \((Z_t)_{t \geq 0}\) be the \( d \)-dimensional rotationally symmetric \( \alpha \)-stable process, i.e., \( \mathbb{E}[e^{iZ_t, \lambda}] = e^{-\|\lambda\|^\alpha} \), then we have \( Z_t \overset{d}{=} t^\frac{\alpha}{2} Z_1 \) (see, e.g., [66, Theorem 14.3]) and the corresponding generator is

\[
\Delta_{\alpha} f(x) = d_\alpha \int_{\mathbb{R}^d} \frac{f(x + y) - f(x)}{|y|^{\alpha + d}} dy, \quad f \in C_0^2(\mathbb{R}^d).
\]

We also state the definition of \( \alpha \)-stable law.

**Definition 3.6.** Let \( \alpha \in (1, 2) \) and \( \tau > 0 \). We say that \( Z \) is distributed according to the rotationally symmetric \( \alpha \)-stable law of parameter \( \tau \), and we write \( Z \sim S_\alpha S(\tau) \), to indicate that \( \mathbb{E}[e^{iZ, \lambda}] = e^{-\|\lambda\|^\alpha} \).

It is immediate to check that \( Z/\tau \sim S_\alpha S(1) \) iff \( Z \sim S_\alpha S(\tau) \). Therefore, starting from now and without loss of generality, we will only consider stable distributions for which \( \tau = 1 \). Moreover, from the definition above, if \( Z \sim S_\alpha S(1) \), then \( Z \overset{d}{=} Z_1 \).

Then, we have the following theorem:

**Theorem 3.7** (Multivariate Stable CLT with \( \alpha \in (1, 2) \)). Let \( Z \sim S_\alpha S(1) \) and \( S_n \) be defined by (3.16). Then, we have

\[
d_W(\mathcal{L}(Z), \mathcal{L}(S_n)) \leq C_{\alpha,d} n^{\frac{2 - \alpha}{\alpha}} + \left( \mathbb{E}|Z| + \frac{1}{\sigma} \mathbb{E}|\xi_n| \right) n^{-\frac{1}{\alpha}},
\]

where \( C_{\alpha,d} \) and \( \sigma \) are two explicit numbers defined by (4.23) and (3.15), respectively.

### 3.3. Application 3: Multivariate Normal CLT ([67, 28])

Finally, we apply Theorem 2.1 to the multivariate normal approximation, and recover the results in [40, 71]. Using the same trick in [40], one can remove the \( \log n \) term to obtain the optimal error bound.

In this application, we denote the \( d \)-dimensional Brownian motion by \((B_t)_{t \geq 0}\) and denote the \( d \)-dimensional standard normal distribution by \( N(0, I_d) \), that is, if \( B \sim N(0, I_d) \), then \( \mathbb{E}[e^{iB, \lambda}] = e^{-\|\lambda\|^2/2} \) for any \( \lambda \in \mathbb{R}^d \). Moreover, it is well known that \( B \overset{d}{=} B_1 \).

**Theorem 3.8** (Multivariate normal CLT). Let \( B \sim N(0, I_d) \) and \( S_n = \sum_{i=1}^n \xi_i \) with i.i.d. random vectors \( (\xi_i)_{i \in \mathbb{N}} \) satisfying \( \mathbb{E}\xi_i = 0, \mathbb{E}\xi_i^T I_d \) and \( \sup_i \mathbb{E}|\xi_i|^3 < \infty \). Then, we have

\[
d_W(\mathcal{L}(B), \mathcal{L}(S_n)) \leq \left( \frac{2}{3} d + 1 \right) \mathbb{E}|B| + \frac{1}{3} \mathbb{E}|\xi_i|^3 + \mathbb{E}|\xi_i| \right) n^{-\frac{d-1}{4}} (1 + \ln n).
\]
4. PROOFS OF THEOREMS 3.5, 3.7 AND 3.8

In this section, with the help of Theorem 2.1, we focus on proving the Theorem 3.5, Theorem 3.7 and Theorem 3.8.

4.1. Proof of Theorem 3.5. We first give the following upper bounds of the processes $w_k$ and $\tilde{w}_k$, which will be proved in Appendix A.

**Lemma 4.1.** Let $w_k$ and $\tilde{w}_k$ be defined by (3.1) and (3.4) with $w_0 = \tilde{w}_k = x \in \mathbb{R}^d$, respectively. Then, as $\eta < \min\{1, \frac{d}{d_w}\}$, we have

$$
\mathbb{E}|w_k|^2 \leq |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0k^2}{\theta_0^2},
$$

(4.1)

$$
\mathbb{E}|\tilde{w}_k|^2 \leq |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0k^2}{\theta_0^2},
$$

(4.2)

$$
\mathbb{E}|w_k - \tilde{w}_k|^2 \leq \frac{4\kappa^2(\theta_0^2|x|^2 + 2|\nabla P(0)|^2 + 2\theta_0k^2 + \theta_0^2)\eta}{\theta_0^3}.
$$

(4.3)

We cannot be sure that $\Sigma$ is positive definite, this will be crucial in the application of Bismut’s formula and Malliavin calculus. In order to solve this problem, we consider the following SDE:

$$
d\tilde{X}_t = -\nabla P(\tilde{X}_t)dt + (\eta\Sigma + \eta\delta I_d)\frac{d}{\sqrt{d}}dW_t, \quad \tilde{X}_0 = x,
$$

(4.4)

where $I_d$ is $d \times d$ identity matrix and $\delta \in (0, 1]$ is a constant which will be determined later. Then, we have the following estimate about $\tilde{X}_t$ and $\tilde{X}_t$, which will be proved in Appendix A.

**Lemma 4.2.** Let $\tilde{\tilde{X}}_t$ and $\tilde{X}_t$ be defined by SDEs (3.6) and (4.4), respectively. Denote the smallest eigenvalue of $\Sigma$ by $\lambda_{\min}$. Then, we have

$$
\mathbb{E}|\tilde{X}_t - \tilde{\tilde{X}}_t|^2 \leq \frac{\eta\delta^2d}{2\theta_0(\lambda_{\min} + \delta)}, \quad t \geq 0.
$$

(4.5)

With the help of Malliavin calculus and Bismut’s formula, we can obtain the following estimates, which will be proved in Appendix B.

**Lemma 4.3.** Let $\tilde{X}_t$ be the solution to the equation (4.4) and denote $P_th(x) = \mathbb{E}[h(\tilde{X}_t^x)]$ for $h \in \text{Lip}(1)$. Then, for any $x, v, v_1, v_2 \in \mathbb{R}^d$, we have

$$
|\nabla_v(P_th)(x)| \leq e^{-\theta_0t}|v|
$$

and

$$
|\nabla_{v_2}\nabla_{v_1}P_th(x)| \leq C_\theta e^{-\frac{2\theta_0t}{\sqrt{\theta_0\eta}} \left[ \frac{1}{(\lambda_{\min} + \delta) - \frac{\delta}{2}} + 1 \right]}|v_1||v_2|.
$$

(4.6)

(4.7)

In addition, we need to prove the following moment estimates, which will be proved in Appendix A.

**Lemma 4.4.** Let $\tilde{X}_t$ be the solution to the equation (4.4). Then, for any $t > 0$, we have

$$
\mathbb{E}|\tilde{X}_t|^2 < |x|^2 + \frac{|\nabla P(0)|^2 + \theta_0\eta(\Sigma\frac{\delta}{\|\|_H} + \delta d)}{\theta_0^2}
$$

and

$$
\mathbb{E}|\tilde{X}_t - x|^2 \leq 8\kappa^2t^2[|x|^2 + \frac{|\nabla P(0)|^2 + \theta_0\eta(\Sigma\frac{\delta}{\|\|_H} + \delta d)}{\theta_0^2}] + 2\eta t(\|\Sigma\|_{\|\|_H}^2 + \delta d).
$$

(4.8)

(4.9)
With the above results, we can give the proof of Theorem 3.5. 

**Proof of Theorem 3.5.** Noticing that

\[ \mathbb{E}h(\tilde{X}_{\eta N}) - \mathbb{E}h(w_N) = \mathbb{E}[h(\tilde{X}_{\eta N}) - h(\tilde{X}_{\eta N})] + \mathbb{E}[h(\tilde{X}_{\eta N}) - h(w_N)] + \mathbb{E}[h(w_N) - h(w_N)], \]

by Cauchy-Schwarz inequality, Lemma 4.2 and (4.3), we have

\[ (4.10) \quad \mathbb{E}|h(\tilde{X}_{\eta N}) - h(\tilde{X}_{\eta N})| \leq \mathbb{E}|\tilde{X}_{\eta N} - \tilde{X}_{\eta N}| \leq \sqrt{\mathbb{E}|\tilde{X}_{\eta N} - \tilde{X}_{\eta N}|^2} \leq \left( \frac{\delta^2 d}{2\theta_0(\lambda_{\min} + \delta)} \right)^{\frac{1}{2}} \eta^\frac{1}{2} \]

and

\[ (4.11) \quad \mathbb{E}|h(\tilde{w}_N) - h(w_N)| \leq \mathbb{E}|\tilde{w}_N - w_N| \leq \sqrt{\mathbb{E}|\tilde{w}_N - w_N|^2} \leq C_{\theta, \eta}(|x| + |\nabla P(0)| + 1)\eta^\frac{1}{2}. \]

It remains to bound \( \mathbb{E}|h(\tilde{X}_{\eta N}) - h(\tilde{w}_N)| \). In order to apply Theorem 2.1, we need to identify the \( X_i \) and \( Y_k \) therein in our setting and compute the corresponding \( A^X \) and \( A^Y \). Let \( X_i = \tilde{X}_{\eta N}, Y_k = \tilde{w}_N, X_0 = Y_0 = x \in \mathbb{R}^d \) and \( N \geq 2 \). Then, for any \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \), it is easy to check that

\[ A^X f(x) = \lim_{t \to 0} \frac{\mathbb{E}(f(x + tX)) - f(x)}{t} \]

\[ = \eta \lim_{t \to 0} \frac{\mathbb{E}(f(x + \eta t)) - f(x)}{\eta t} = -\eta \langle \nabla f(x), \nabla P(x) \rangle + \frac{1}{2} \eta^2 \langle \nabla^2 f(x), \Sigma + \delta I_d \rangle_{HS}. \]

Recall

\[ \tilde{w}_k = \tilde{w}_{k-1} - \eta \nabla P(\tilde{w}_{k-1}) - \sqrt{\eta} V_\eta(b_k), \]

by Taylor expansion, we have

\[ A^Y f(x) = \mathbb{E}[f(Y^T_i) - f(x)] = \mathbb{E}\langle \nabla f(x), -\eta \nabla P(x) - \sqrt{\eta} V_\eta(b) \rangle + \mathcal{R}^f(x) \]

\[ = -\eta \langle \nabla f(x), \nabla P(x) \rangle + \mathcal{R}^f(x), \]

where

\[ \mathcal{R}^f(x) = \int_0^1 \int_0^r \mathbb{E}\langle \nabla^2 f(x - s\eta \nabla P(x) + \sqrt{\eta} V_\eta(b)), \eta \nabla P(x) + \sqrt{\eta} V_\eta(b) \rangle_{HS} ds dr. \]

Hence,

\[ (4.12) \quad |A^X f(x) - A^Y f(x)| \leq \frac{1}{2} \eta^2 |\langle \nabla^2 f(x), \Sigma + \delta I_d \rangle_{HS}| + |\mathcal{R}^f(x)|. \]

Now we apply Theorem 2.1 with \( u_k(x) = \mathbb{E}h(X^T_{\eta N}) = \mathbb{E}h(\tilde{X}_{\eta N}) \) for \( k \geq 1 \) to prove the theorem, it suffices to bound the three terms \( I_1, I_2, I_3, I_4, I_5, I_6 \) in (2.2).

For the term \( I_6 \), by (4.12) we have

\[ |I_6| \leq \frac{1}{2} \eta^2 \sum_{j=1}^N \mathbb{E} |\nabla^2 u_{N-j}(\tilde{w}_{j-1}), \Sigma + \delta I_d \rangle_{HS}| + \sum_{j=1}^N \mathbb{E} |\mathcal{R}^{u_{N-j}}(\tilde{w}_{j-1})| := I_{6,1} + I_{6,2}. \]

Noting that

\[ (4.13) \quad \eta \Sigma = \mathbb{E}[V_\eta(b) V_\eta(b)^T], \quad \langle \nabla^2 f(x), I_d \rangle_{HS} = \mathbb{E} \langle \nabla^2 f(x), \xi \xi^T \rangle_{HS}, \]

where \( \xi \) is a standard Gaussian variable.
where $\xi$ is a $d$-dimensional standard normal distribution, which is independent of $\tilde{X}_i$ and $\tilde{w}_k$. Hence, by (4.7) and Cauchy-Schwarz inequality, we have

\[ I_{h,1} \leq C \eta^2 \sum_{j=1}^{N} e^{-\frac{N}{\eta}(N-j)} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right) \mathbb{E} \left[ |V_{\eta}(b)|^2 + \eta \delta |\xi|^2 \right] \]

\[ \leq C \eta^2 \sum_{j=1}^{N} e^{-\frac{N}{\eta}(N-j)} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right) \left( \mathbb{E} |\nabla \phi(a, b)|^2 + \delta d \right) \]

\[ \leq C \eta^2 \sum_{j=1}^{N} e^{-\frac{N}{\eta}(N-j)} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right) (\kappa^2 + \delta d), \]

where the last inequality is by (3.11) with $x = 0$. Then, we have

\[ I_{h,1} \leq C_{\theta, \kappa} (1 + \delta d) \eta^2 \int_0^N e^{-\frac{N}{\eta}z} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\sqrt{\eta z}} + 1 \right) dz \]

\[ = C_{\theta, \kappa} (1 + \delta d) \eta^2 \int_0^{\eta N} e^{-\frac{N}{\eta}y} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\sqrt{\eta y}} + 1 \right) dy \]

\[ \leq C_{\theta, \kappa} (1 + \delta d) \left( (\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1 \right) \eta^\frac{1}{2}. \]

Moreover, by (4.7), (3.11), (3.13) and (4.2), we have

\[ I_{h,2} \leq C_{\theta} \sum_{j=1}^{N} e^{-\frac{N}{\eta}(N-j)} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right) \mathbb{E} [\eta^2 |\nabla P(\tilde{w}_{j-1})|^2 + \eta |V_{\eta}(b)|^2] \]

\[ \leq C_{\theta, \kappa} \eta^2 \sum_{j=1}^{N} e^{-\frac{N}{\eta}(N-j)} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right) \left[ 1 + \mathbb{E} |\tilde{w}_{j-1}|^2 \right] \]

\[ \leq C_{\theta, \kappa} (1 + |x|^2 + |\nabla P(0)|^2) \eta^\frac{1}{2} \int_0^{\eta N} e^{-\frac{N}{\eta}y} \left( \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\sqrt{\eta y}} + 1 \right) dy \]

(4.14)\[ \leq C_{\theta, \kappa} (1 + |x|^2 + |\nabla P(0)|^2) \left( (\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1 \right) \eta^\frac{1}{2}. \]

Therefore, we have

(4.15)\[ |I_h| \leq C_{\theta, \kappa} (1 + \delta d + |x|^2 + |\nabla P(0)|^2) \left[ (\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1 \right] \eta^\frac{1}{2}. \]

For the term $II_h$, we have

\[ |II_h| \leq \eta \sum_{j=1}^{N} \int_0^1 \mathbb{E} |\langle \nabla u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}, \nabla P(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) \rangle - \langle \nabla u_{N-j}(\tilde{w}_{j-1}), \nabla P(\tilde{w}_{j-1}) \rangle | ds \]

\[ + \frac{1}{2} \eta^2 \sum_{j=1}^{N} \int_0^1 \mathbb{E} |\nabla^2 u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) - \nabla^2 u_{N-j}(\tilde{w}_{j-1}), \Sigma + \delta I_d)|HS | ds \]

\[ := II_{h,1} + II_{h,2}. \]
By Cauchy-Schwarz inequality, we have
\[ II_{h,1} \leq \eta \sum_{j=1}^{N} \int_{0}^{1} \mathbb{E} \left[ \langle \nabla u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}), \nabla P(\tilde{w}_{j-1}) \rangle \right] ds \\
+ \eta \sum_{j=1}^{N} \int_{0}^{1} \mathbb{E} \left[ \langle \nabla u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}), \nabla P(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) \rangle - \nabla P(\tilde{w}_{j-1}) \rangle \right] ds \\
\leq \eta \sum_{j=1}^{N} \int_{0}^{1} \mathbb{E} \left[ \| \nabla u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) - \nabla u_{N-j}(\tilde{w}_{j-1}) \| \nabla P(\tilde{w}_{j-1}) \| ight] ds \\
+ \eta \sum_{j=1}^{N} \int_{0}^{1} \mathbb{E} \left[ \| \nabla u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) \| \nabla P(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}) - \nabla P(\tilde{w}_{j-1}) \| \right] ds \\
\]
then, by (4.7), (3.13), (4.6) and (3.10), we have
\[ II_{h,1} \leq C_{\theta,\kappa} \eta \sum_{j=1}^{N} \int_{0}^{1} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \mathbb{E} \left[ \| \tilde{X}_{\eta s}^{\tilde{w}_{j-1}} - \tilde{w}_{j-1} \| (1 + |\tilde{w}_{j-1}|) \right] \\
+ C_{\theta,\kappa} \eta \sum_{j=1}^{N} \int_{0}^{1} e^{-\theta_0 \eta(N-j)} \mathbb{E} \left[ (1 + |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}| + |\tilde{w}_{j-1}|) |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}} - \tilde{w}_{j-1}| \right] ds \\
\leq C_{\theta,\kappa} \eta \sum_{j=1}^{N} \int_{0}^{1} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \\
\times \mathbb{E} \left[ (1 + |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}| + |\tilde{w}_{j-1}|) |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}} - \tilde{w}_{j-1}| \right] ds, \\
\]
this along with Cauchy-Schwarz inequality, (4.2), (4.8), (4.9) and (4.14) yields that
\[ II_{h,1} \leq C_{\theta,\kappa} \eta \sum_{j=1}^{N} \int_{0}^{1} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \\
\sqrt{\mathbb{E} \left[ (1 + |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}| + |\tilde{w}_{j-1}|)^2 \right] \mathbb{E} \left[ |\tilde{X}_{\eta s}^{\tilde{w}_{j-1}} - \tilde{w}_{j-1}|^2 \right] ds \
\leq C_{\theta,\kappa} (1 + |x|^2 + |\nabla P(0)|^2 + ||\Sigma|^2||_{HS} + \delta d)^2 \eta^2 \sum_{j=1}^{N} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \\
\leq C_{\theta,\kappa} (1 + |x|^2 + |\nabla P(0)|^2 + ||\Sigma|^2||_{HS} + \delta d) \left[ (\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1 \right] \eta^2. \\
\]
Moreover, by (4.7), (4.13) and (4.14), we have
\[ II_{h,2} \leq \frac{1}{2} \eta^2 \sum_{j=1}^{N} \int_{0}^{1} \mathbb{E} \left[ \| \nabla^2 u_{N-j}(\tilde{X}_{\eta s}^{\tilde{w}_{j-1}}), \Sigma + \delta I_d \|_{HS} \right] + \| \nabla^2 u_{N-j}(\tilde{w}_{j-1}), \Sigma + \delta I_d \|_{HS} \right] ds \\
\leq C_\theta \eta^2 \sum_{j=1}^{N} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \mathbb{E} \left[ \| V_{\eta}(b) \|^2 + \eta \delta \| \xi \|^2 \right] \\
\leq C_\theta \eta^2 \sum_{j=1}^{N} e^{-\frac{\theta_0}{\eta} \eta(N-j)} \left[ \frac{(\lambda_{\min} + \delta)^{-\frac{1}{2}}}{\eta \sqrt{N-j}} + 1 \right] \left( \kappa^2 + \delta d \right) \\
\leq C_{\theta,\kappa} (1 + \delta d) \left[ (\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1 \right] \eta^2. \\
\]
Therefore, we have
\begin{equation}
|\mathcal{II}_h| \leq C_{\theta, \kappa}(1 + |x|) |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS} + \delta d)[(\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1] \eta \frac{1}{2}.
\end{equation}

It remains to estimate $\mathcal{III}_h$. By Cauchy-Schwarz inequality, (4.9) and (4.2), we have
\begin{equation}
\mathbb{E}|h(Y_{N-1}) - h(Y_{N-1})| = \mathbb{E}|\hat{X}^{Y_{N-1}}_\eta - \hat{w}_{N-1}|
\leq \sqrt{\mathbb{E}|\hat{X}^{Y_{N-1}}_\eta - \hat{w}_{N-1}|^2}
\leq C_{\theta, \kappa} \eta \sqrt{1 + |x|^2 + |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS} + \delta d}.
\end{equation}

Recall (3.4), by Cauchy-Schwarz inequality, (3.11) and (4.2), we have
\begin{equation}
\mathbb{E}|h(Y) - h(Y_{N-1})| \leq \mathbb{E}|\hat{w}_{N} - \hat{w}_{N-1}|
\leq \eta \sqrt{\mathbb{E}|\nabla \phi(a, b)|^2} + \mathbb{E}|\nabla \phi(a, b)| b_N|^2
\leq 2\kappa \eta \sqrt{1 + \mathbb{E}|\hat{w}_{N-1}|^2}
\leq 2\kappa \eta \sqrt{1 + |x|^2 + \frac{2|\nabla P(0)|^2 + \theta_0 \kappa^2}{\theta_0^2}}.
\end{equation}

These imply
\begin{equation}
|\mathcal{III}_h| \leq C_{\theta, \kappa} \eta \sqrt{1 + |x|^2 + |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS} + \delta d}.
\end{equation}

Therefore, by (4.10), (4.11), (4.15), (4.16) and (4.17), we immediately have
\begin{equation}
|\mathbb{E}h(\hat{X}_{\eta N}) - \mathbb{E}h(w_N)| \leq C_{\theta, \kappa}(1 + |x|^2 + |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS} + \delta d)[(\lambda_{\min} + \delta)^{-\frac{1}{2}} + 1] \eta \frac{1}{2}
+ \frac{\delta^2 d}{2\theta_0(\lambda_{\min} + \delta)} \eta \frac{1}{2}.
\end{equation}

Then, when $\lambda_{\min} > 0$, taking $\delta = 0$,
\begin{equation}
d_W(\mathcal{L}(\hat{X}_{\eta N}), \mathcal{L}(w_N)) \leq C_{\theta, \kappa}(1 + |x|^2 + |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS}) \left[\lambda_{\min}^{-\frac{1}{2}} + 1\right] \eta \frac{1}{2}.
\end{equation}

when $\lambda_{\min} = 0$, taking $\delta = 1$,
\begin{equation}
d_W(\mathcal{L}(\hat{X}_{\eta N}), \mathcal{L}(w_N)) \leq C_{\theta, \kappa}(1 + |x|^2 + |\nabla P(0)|^2 + \|\Sigma\|^2_{\HS} + d) \eta \frac{1}{2}.
\end{equation}

The proof is complete. \qed

4.2. **Proof of Theorem 3.7.** According to the heat kernel estimates of rotationally symmetric $\alpha$-stable process, we can get the following upper bounds for the semigroup of $\alpha$-stable process, which will be proved in Appendix C.

**Lemma 4.5.** Let $h \in \text{Lip}(1)$ and denote $P_t h(x) = \mathbb{E}h(Z_t^\alpha)$, then
\begin{equation}
|\nabla^2 (P_t h)|_{\HS} \leq 4\pi V_{\alpha, d+2} V(S^{d-1}) t^{-\frac{d}{2}},
\end{equation}
and
\begin{equation}
|\Delta^\alpha P_t h(x) - \Delta^\alpha P_t h(y)| \leq \frac{8\pi d \alpha V(S^{d-1})^2 V_{\alpha, d+2} t^{-\frac{d}{2}}}{\alpha(2 - \alpha)(\alpha - 1)} |x - y|^{2-\alpha},
\end{equation}
where
\begin{equation}
V_{\alpha, d} = \max \left\{ 2^{-d+1} \pi^{-\frac{d}{2}} \frac{\Gamma(d/\alpha)}{\Gamma(d/2)} \frac{\alpha 2^{\alpha-1} \sin \frac{\alpha \pi}{2} \Gamma((d + \alpha)/2) \Gamma(\alpha/2)}{\pi^{d/2 + 1}} \right\}.
\end{equation}
With the above results, we can give the proof of Theorem 3.7.

Proof of Theorem 3.7. In order to apply Theorem 2.1, we first need to identify the $X_{t}$ and $Y_{k}$ therein in our setting and compute the corresponding $\mathcal{A}^{X}$ and $\mathcal{A}^{Y}$. Let $X_{t} = Z_{t}^{x}$, $Y_{k} = S_{k} = \frac{\xi_{k}}{\sigma_{n}^{x}}$, $X_{0} = Y_{0} = 0$ and $N = n$. Then, for any $f \in C_{b}^{2}(\mathbb{R}^{d})$, it is straightforward to check that

$$
\mathcal{A}^{X} f(x) = \lim_{t \to 0} \frac{E f(X_{t}^{x}) - f(x)}{t} = \frac{1}{n} \lim_{t \to 0} \frac{E f\left(\frac{Z_{t}^{x}}{n}\right) - f(x)}{t} = \frac{1}{n} \Delta_{n}^{x} f(x),
$$

and

$$
\mathcal{A}^{Y} f(x) = \mathbb{E}[f(Y_{1}^{x}) - f(x)] = \frac{d_{x}}{n} \int_{|y| < (\sigma_{n}^{x})^{-1}} \frac{f(x + y) - f(x)}{|y|^{|\alpha + d|}} dy = \frac{1}{n} \Delta_{n}^{x} f(x) - R^{f}(x, n),
$$

where

$$
R^{f}(x, n) := \frac{d_{x}}{n} \int_{|y| < (\sigma_{n}^{x})^{-1}} \frac{f(x + y) - f(x)}{|y|^{|\alpha + d|}} dy.
$$

Now we apply Theorem 2.1 with $u_{k}(x) = \mathbb{E} h(X_{t}^{x}) = \mathbb{E} h(Z_{t}^{x})$ for $k \geq 1$ to prove the theorem, it suffices to bound the three terms $\mathcal{I}_{h}, \mathcal{I}_{II} h, \mathcal{I}_{III} h$ in (2.2).

For the term $\mathcal{I}_{h}$, we have $|\mathcal{A}^{X} f(x) - \mathcal{A}^{Y} f(x)| \leq |R^{f}(x, n)|$ and by (4.20), we have

$$
|\mathbb{E} R^{u_{n-1}}(Y_{j-1}, n)| = \left| \frac{d_{x}}{n} \int_{|y| < (\sigma_{n}^{x})^{-1}} \frac{u_{n-1}(Y_{j-1} + y) - u_{n-1}(Y_{j-1}) - y \nabla u_{n-1}(Y_{j-1})}{|y|^{|\alpha + d|}} dy \right|
$$

$$
= \left| \frac{d_{x}}{n} \int_{S^{d-1}} \int_{0}^{(\sigma_{n}^{x})^{-1}} \frac{u_{n-1}(Y_{j-1} + r \theta) - u_{n-1}(Y_{j-1}) - r \theta \nabla u_{n-1}(Y_{j-1})}{r^{\alpha + 1}} dr d\theta \right|
$$

$$
= \left| \frac{d_{x}}{n} \int_{S^{d-1}} \int_{0}^{(\sigma_{n}^{x})^{-1}} \int_{0}^{1} \frac{r \theta \nabla u_{n-1}(Y_{j-1} + s r \theta) - \nabla u_{n-1}(Y_{j-1})}{r^{\alpha + 1}} ds dr d\theta \right|
$$

$$
\leq \frac{4 \pi V_{\alpha,d+2} V(S^{d-1}) d_{x}}{n} \left( n - j \right)^{-\frac{1}{2}} \int_{S^{d-1}} \int_{0}^{(\sigma_{n}^{x})^{-1}} \int_{0}^{1} r^{2} s ds dr d\theta,
$$

then, by integration, we have

$$
|\mathbb{E} R^{u_{n-1}}(Y_{j-1}, n)| \leq \frac{4 \pi V_{\alpha,d+2} V(S^{d-1}) d_{x}}{n} \left( n - j \right)^{-\frac{1}{2}} \frac{(\sigma n^{2})^{n-2}}{2(2 - \alpha)} V(S^{d-1})
$$

$$
= \frac{2 \alpha \pi V(S^{d-1}) V_{\alpha,d+2}}{(2 - \alpha) \sigma^{2}} \frac{1}{n} \left( n - j \right)^{-\frac{1}{2}}
$$

Hence, we have

$$
|\mathcal{I}_{h}| \leq \sum_{j=1}^{n-1} \frac{2 \alpha \pi V(S^{d-1}) V_{\alpha,d+2}}{(2 - \alpha) \sigma^{2}} \frac{1}{n} \left( n - j \right)^{-\frac{1}{2}}
$$

$$
= \frac{2 \alpha \pi V(S^{d-1}) V_{\alpha,d+2}}{(2 - \alpha) \sigma^{2}} \frac{1}{n} \sum_{j=1}^{n-1} \left( n - j \right)^{-\frac{1}{2}}
$$
where the last inequality is by the same argument as the proof of (18).

\[ I \leq \frac{2\alpha \pi V(S^{d-1})V_{\alpha,d+2}}{(2 - \alpha)\sigma^2} n^{-\frac{1}{2}} \int_0^n y^{-\frac{1}{2}} dy = \frac{2\alpha^2 \pi V(S^{d-1})V_{\alpha,d+2}}{(2 - \alpha)(\alpha - 1)\sigma^2} n^{-\frac{\alpha - 2}{2}}. \]

For the term \( II_h \), we have

\[ |II_h| \leq \frac{1}{n} \sum_{j=1}^{n-1} \int_0^1 E \left| \frac{\alpha}{2} u_{n-j}^2 (X^j_{t-})^i - \frac{\alpha}{2} u_{n-j}^2 (Y^j_{t-})^i \right| ds, \]

then, by (4.21) and the scaling property of \( Z_t \), i.e., \( Z_t \overset{d}{=} t^\frac{1}{\alpha} Z_1 \), we have

\[ II_h \leq \frac{1}{n} \sum_{j=1}^{n-1} \int_0^1 E \left| \frac{\alpha}{2} u_{n-j}^2 (Z^j_{t-})^i - \frac{\alpha}{2} u_{n-j}^2 (Y^j_{t-})^i \right| ds \]

\[ \leq \frac{1}{n} \sum_{j=1}^{n-1} \frac{8\pi d_{\alpha} V(S^{d-1})^2 V_{\alpha,d+2}}{\alpha(2 - \alpha)(\alpha - 1)} \left( \frac{n - j}{n} \right)^{\frac{1}{2}} \int_0^1 E \left| Z^j_{t-} \right|^{2-\alpha} ds \]

\[ = \frac{1}{n} \sum_{j=1}^{n-1} \frac{8\pi d_{\alpha} V(S^{d-1})^2 V_{\alpha,d+2}}{\alpha(2 - \alpha)(\alpha - 1)} \left( \frac{n - j}{n} \right)^{\frac{1}{2}} \frac{\alpha}{n^{\frac{\alpha - 2}{2}}} \int_0^1 s^{-\frac{\alpha}{2}} ds \]

\[ = \sum_{j=1}^{n-1} \frac{4\pi d_{\alpha} V(S^{d-1})^2 V_{\alpha,d+2}}{\alpha(2 - \alpha)(\alpha - 1)^2} E \left| Z^j_1 \right|^{2-\alpha} \frac{\alpha}{n^{\frac{\alpha - 2}{2}}}, \]

where the last inequality is by the same argument as the proof of (4.22).

It remains to estimate \( III_h \). By the scaling property of \( Z_t \), it is easily seen that

\[ E|h(X_i^{n-1}) - h(Y_{n-1})| \leq E|h(X_i^{n-1}) - Y_{n-1}| = E|Z^i_1| = n^{-\frac{1}{\alpha}} E|Z_1|, \]

\[ E|h(Y_n) - h(Y_{n-1})| \leq E|Y_n - Y_{n-1}| = E \frac{|\xi_{n}|}{\sigma n^{\frac{1}{\alpha}}}, \]

which imply

\[ III_h \leq \left( E|Z_1| + \frac{1}{\sigma} E|\xi_{n}| \right) n^{-\frac{1}{\alpha}}. \]

Collecting the estimates of \( I_h, II_h \) and \( III_h \), which hold true for all \( h \in \text{Lip}(1) \), we immediately obtain

\[ d_W(\mathcal{L}(Z_1), \mathcal{L}(S_n)) \leq C_{\alpha,d} n^{\frac{\alpha - 2}{\alpha}} + \left( E|Z_1| + \frac{1}{\sigma} E|\xi_{n}| \right) n^{-\frac{1}{\alpha}}, \]

where

\[ C_{\alpha,d} = \frac{2\alpha \pi V(S^{d-1})V_{\alpha,d+2}}{(2 - \alpha)(\alpha - 1)} \left[ \frac{2\pi V(S^{d-1})}{(\alpha - 1)\sigma^2} + \frac{\alpha}{\sigma^2} \right]. \]

Then, the desired result follows from the fact that \( Z \overset{d}{=} Z_1 \). \( \square \)
4.3. Proof of Theorem 3.8. In order to use Theorem 2.1, we need the following properties for the semigroup of Brownian motion, which will be proved in Appendix D.

**Lemma 4.6.** Let $h \in \text{Lip}(1)$ and denote $P_t h(x) = \mathbb{E} h(B_t^x)$, then for any $x, v, v_1, v_2 \in \mathbb{R}^d$, we have

$$|\langle \nabla^2 (P_t h)(x + v) - \nabla^2 (P_t h)(x), v v_T^T \rangle_{HS}| \leq \frac{2|v_1||v_2||v|}{t}$$

and

$$|\Delta (P_t h)(x + v) - \Delta (P_t h)(x)| \leq \frac{2d}{t}|v|.$$  

With the above results, we can give the proof of Theorem 3.8.

**Proof of Theorem 3.8.** In order to apply Theorem 2.1, we first need to identify the $X_t$ and $Y_t$ therein in our setting and compute the corresponding $A^X$ and $A^Y$. Let $X_t = B^X_t$, $Y_i = S_k = \sum_{i=1}^k \frac{\xi_i}{\sqrt{n}}$, where $(\xi_i)_{i \in \mathbb{Z}_+}$ is a sequence of i.i.d. random vectors satisfying $\mathbb{E} \xi_i = 0$, $\mathbb{E} \xi_i \xi_i^T = I_d$ and $\mathbb{E} |\xi|^3 < \infty$, $X_0 = Y_0 = 0$ and $N = n$. Then, for any $f \in C^2(\mathbb{R}^d)$, it is straightforward to check that

$$A^X f(x) = \lim_{t \to 0} \frac{E(f(X_t^x) - f(x))}{t} = \frac{1}{n} \lim_{t \to 0} \frac{E(f(B^X_{\frac{t}{n}}) - f(x))}{\frac{t}{n}} = \frac{1}{2n} \Delta f(x)$$

and

$$A^Y f(x) = E[f(Y_t^x) - f(x)] = E[f(Y_t^x) - f(x) - \langle \nabla f(x), \frac{\xi}{\sqrt{n}} \rangle]$$

$$= \frac{1}{n} E \left[ \int_0^1 \int_0^r \langle \nabla^2 f(x + s \frac{\xi}{\sqrt{n}}), \xi \xi^T \rangle_{HS} ds dr \right].$$

Now we apply Theorem 2.1 with $u_k(x) = \mathbb{E} h(X_k^x) = \mathbb{E} h(B^X_k)$ for $k \geq 1$ to prove the theorem, it suffices to bound the three terms $\mathcal{I}_h, \mathcal{II}_h, \mathcal{III}_h$ in (2.2).

For the term $\mathcal{I}_h$, noticing that $\mathbb{E} \xi \xi^T = I_d$ and $\langle \nabla^2 f(x), I_d \rangle_{HS} = \Delta f(x)$, we have

$$|A^X f(x) - A^Y f(x)| = \frac{1}{2n} \Delta f(x) - \frac{1}{n} E \left[ \int_0^1 \int_0^r \langle \nabla^2 f(x + s \frac{\xi}{\sqrt{n}}), \xi \xi^T \rangle_{HS} ds dr \right]$$

$$\leq \frac{1}{n} \int_0^1 \int_0^r E \langle \nabla^2 f(x + s \frac{\xi}{\sqrt{n}}) - \nabla^2 f(x), \xi \xi^T \rangle_{HS} ds dr.$$

Then, by (4.24), we have

$$|\mathcal{I}_h| \leq \frac{1}{n} \sum_{j=1}^{n-1} \int_0^1 \int_0^r E \langle \nabla^2 u_{n-j}(x + s \frac{\xi}{\sqrt{n}}) - \nabla^2 u_{n-j}(x), \xi \xi^T \rangle_{HS} ds dr$$

$$\leq \frac{2}{\sqrt{n}} E |\xi|^3 \int_0^1 \int_0^r s ds dr \sum_{j=1}^{n-1} \frac{1}{n-j}$$

$$= \frac{1}{3} \sqrt{n} E |\xi|^3 \sum_{j=1}^{n-1} \frac{1}{n-j} \leq \frac{1}{3} \sqrt{n} E |\xi|^3 (1 + \int_1^n \frac{1}{y} dy) = \frac{1}{3} \sqrt{n} E |\xi|^3 (1 + \ln n).$$

For the term $\mathcal{II}_h$, by (4.25) and the scaling property of $B_t$, i.e., $B_t \overset{d}{=} t^2 B_1$, we have

$$|\mathcal{II}_h| \leq \frac{1}{2n} \sum_{j=1}^{n-1} \int_0^1 E \left| \Delta u_{n-j}(X_{s}^{Y_j-1}) - \Delta u_{n-j}(Y_{j-1}) \right| ds.$$
\[
\leq d \int_0^1 \mathbb{E}|B_{\frac{t}{n}}| ds \sum_{j=1}^{n-1} \frac{1}{n-j} = \frac{d}{\sqrt{n}} \mathbb{E}|B_1| \int_0^1 s^\frac{1}{2} ds \sum_{j=1}^{n-1} \frac{1}{j} \leq \frac{2d}{3\sqrt{n}} \mathbb{E}|B_1|(1 + \ln n).
\]

It remains to estimate \( \mathcal{III}_h \). By the scaling property of \( B_t \), it is easily seen that
\[
\mathbb{E}|h(X_1^{Y_{n-1}}) - h(Y_{n-1})| \leq \mathbb{E}|X_1^{Y_{n-1}} - Y_{n-1}| = \mathbb{E}|B_{\frac{1}{n}}| = \frac{1}{\sqrt{n}} \mathbb{E}|B_1|,
\]
\[
\mathbb{E}|h(Y_n) - h(Y_{n-1})| \leq \mathbb{E}|Y_n - Y_{n-1}| = E \frac{\xi_n}{\sqrt{n}}
\]
which imply
\[
|\mathcal{III}_h| \leq (\mathbb{E}|B_1| + \mathbb{E}|\xi_n|) \frac{1}{\sqrt{n}}.
\]

Collecting the estimates of \( \mathcal{I}_h, \mathcal{II}_h \) and \( \mathcal{III}_h \), which hold true for all \( h \in \text{Lip}(1) \), we immediately obtain
\[
\begin{align*}
    d_W(\mathcal{L}(B_t), \mathcal{L}(S_n)) & \leq \left(\frac{2}{3}d + 1\right) \mathbb{E}|B_1| + \frac{1}{3} \mathbb{E}|\xi_1|^3 + E |\xi_1| \right) n^{-\frac{1}{2}} (1 + \ln n). \\

\end{align*}
\]

Then, the desired result follows from the fact that \( B \overset{d}{=} B_1 \).

\[ \square \]

**APPENDIX A. PROOFS OF LEMMAS IN SUBSECTIONS 3.1 AND 4.1**

**A.1. Verifying assumptions for two examples.** In this subsection, we verify Assumption A2 and Assumption A3 for Examples 3.3 and 3.4, respectively.

**Lemma A.1.** In Example 3.3, suppose \( A = \mathbb{E}[\tilde{a}_k \tilde{a}_k^T] \) is a positive definite matrix with smallest eigenvalue \( \tilde{\lambda}_{\min} \), \( \mathbb{E}|\tilde{a}_k|^4 < \infty \) and \( \mathbb{E}|\varepsilon_k|^2 < \infty \). Then, Assumption A2 and Assumption A3 hold for \( \theta_0 = \tilde{\lambda}_{\min}, \theta_1 = 0 \) and \( \kappa^2 = \max\{\mathbb{E}|\tilde{a}_k|^4, \mathbb{E}|\tilde{a}_k|^{\min}{\varepsilon_k}^2\} \).

**Proof.** Recall
\[
\nabla P(x) = \mathbb{E}[\tilde{a}_k \tilde{a}_k^T] x, \quad \nabla \varphi(x, a_k) = \tilde{a}_k \tilde{a}_k^T x, \quad \nabla \phi(a_k, b_k) = -\tilde{a}_k \varepsilon_k.
\]

Then, for any \( v, v_1, v_2, x \in \mathbb{R}^d \), it is easy to see that
\[
\langle v, \nabla_v \nabla P(x) \rangle = \langle v, Av \rangle \geq \tilde{\lambda}_{\min} |v|^2,
\]
\[
\nabla_v \nabla_v \nabla P(x) = 0,
\]
and noting that \( \tilde{a}_k \) and \( \varepsilon_k \) is independent, by Cauchy-Schwarz inequality, we have
\[
\mathbb{E}|\nabla \varphi(x, a_k)|^2 + \mathbb{E}|\nabla \phi(a_k, b_k)|^2 \leq \mathbb{E}|\tilde{a}_k|^4 |x|^2 + \mathbb{E}|\tilde{a}_k|^2 |\varepsilon_k|^2
\]
\[
\leq \max\{\mathbb{E}|\tilde{a}_k|^4, \mathbb{E}|\tilde{a}_k|^2 |\varepsilon_k|^2\} (1 + |x|^2).
\]

The proof is complete. \[ \square \]

**Lemma A.2.** In Example 3.4, suppose \( \mathbb{E}|a|^3 < \infty \). Then, Assumption A2 and Assumption A3 hold for \( \theta_0 = \gamma, \theta_1 = 3\mathbb{E}|a|^3 \) and \( \kappa^2 = \max\{3\mathbb{E}|a|^2, 2\gamma^2\} \).

**Proof.** Recall
\[
\nabla P(x) = \mathbb{E}[a \left( \frac{1}{1 + e^{a^T x}} - \frac{1}{1 - e^{a^T x}} \right) ] + \gamma x,
\]
then, for any \( v, v_1, v_2, x \in \mathbb{R}^d \), it is easy to see that
\[
\langle v, \nabla_v \nabla P(x) \rangle = \langle v, \mathbb{E}\left[ e^{-a^T x} a a^T \right] v + \gamma v \rangle \geq \gamma |v|^2,
\]
\[ |\nabla v_1 \nabla v_2 \nabla P(x)| \leq \mathbb{E}\left[ \frac{e^{-a^T x}}{(1 + e^{-a^T x})^2} |a|^2 |v_1| |v_2| \right] + 2\mathbb{E}\left[ \frac{e^{-2a^T x}}{(1 + e^{-a^T x})^3} |a|^3 |v_1| |v_2| \right] \leq 3\mathbb{E}|a|^3 |v_1| |v_2|. \]

and recall
\[ \nabla \varphi(x, a) = a\left( \frac{1}{1 + e^{-a^T x}} - \frac{1}{1 + e^{-a^T w^*}} \right) + \gamma x, \]
\[ \nabla \phi(a, b) = -a(b - \frac{1}{1 + e^{-a^T w^*}}), \]

which implies
\[ \mathbb{E}|\nabla \varphi(x, a)|^2 + \mathbb{E}|\nabla \phi(a, b)|^2 \leq 2\mathbb{E}|a|^2 + 2\gamma^2 |x|^2 + \mathbb{E}|a|^2 \leq \max\{3\mathbb{E}|a|^2, 2\gamma^2\}(1 + |x|^2). \]

The proof is complete. \[ \square \]

Lemma A.3. Let the binary response \( b \in \{1, 0\} \) is generated by the following probabilistic model,
\[ \mathbb{P}(b = 1|a) = \frac{1}{1 + e^{-a^T w^*}}, \quad \mathbb{P}(b = 0|a) = \frac{1}{1 + e^{-a^T w^*}}, \]
where \( a \) is a random vector and \( w^* \) is a constant. Set \( \nabla \phi(a, b) = -a(b - \frac{1}{1 + e^{-a^T w^*}}) \), then
\[ \Sigma := \mathbb{E}\left[ \mathbb{E}\left[ \nabla \phi(a, b) |b\right] \left( \mathbb{E}[\nabla \phi(a, b) |b]\right)^T \right] \]
\[ = \left( \frac{1}{\hat{p}} + \frac{1}{1 - \hat{p}} \right) \mathbb{E}\left[ \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right] \left( \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right]\right)^T \right], \]
where \( \hat{p} = \mathbb{E}\left[\frac{1}{1 + e^{-a^T w^*}}\right] < 1. \)

Proof. By conditional probability formula, we have
\[ \mathbb{P}(b = 1) = \mathbb{E}[\mathbb{P}(b = 1|a)] = \mathbb{E}\left[ \frac{1}{1 + e^{-a^T w^*}} \right] = \hat{p} < 1, \]
and \( \mathbb{P}(b = 0) = 1 - \hat{p}. \) Then, by Bayes’ formula, we have
\[ \mathbb{E}\left[ -a(b - \frac{1}{1 + e^{-a^T w^*}}) |b = 1 \right] = \mathbb{E}\left[ \frac{-a}{1 + e^{-a^T w^*}} \frac{\mathbb{P}(b = 1|a)}{\mathbb{P}(b = 1)} \right] = \frac{-1}{\hat{p}} \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right], \]
and
\[ \mathbb{E}\left[ -a(b - \frac{1}{1 + e^{-a^T w^*}}) |b = 0 \right] = \mathbb{E}\left[ \frac{a}{1 + e^{-a^T w^*}} \frac{\mathbb{P}(b = 0|a)}{\mathbb{P}(b = 0)} \right] = \frac{1}{1 - \hat{p}} \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right]. \]

Therefore, we have
\[ \mathbb{E}\left[ \mathbb{E}\left[ \nabla \phi(a, b) |b\right] \left( \mathbb{E}[\nabla \phi(a, b) |b]\right)^T \right] \]
\[ = -\mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right] \left( \frac{-1}{\hat{p}} \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right]\right)^T \]
\[ + \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right] \left( \frac{1}{1 - \hat{p}} \mathbb{E}\left[ \frac{a}{(1 + e^{a^T w^*})(1 + e^{-a^T w^*})} \right]\right)^T \]
These imply we have

\[ \begin{align*}
\frac{1}{\rho} + \frac{1}{1 - \rho} & \mathbb{E} \left[ \frac{a}{(1 + e^{a^2 w^*})(1 + e^{-a^2 w^*})} \right] \\
& \mathbb{E} \left[ \frac{a}{(1 + e^{a^2 w^*})(1 + e^{-a^2 w^*})} \right]^T,
\end{align*} \]

the desired result follows. \( \square \)

A.2. Proof of Lemma 4.1. Recall (3.1), it is easily seen that

\[ \mathbb{E} |w_k|^2 = \mathbb{E} |w_{k-1}|^2 - 2\eta \mathbb{E} \left[ |\nabla \psi(w_{k-1}, \zeta_k)| w_{k-1} \right] + \eta^2 \mathbb{E} |\nabla \psi(w_{k-1}, \zeta_k)|^2. \]

Since \( \zeta_k \) is independent of \( w_{k-1} \), we have

\[ \begin{align*}
\mathbb{E} \left[ (\nabla \psi(w_{k-1}, \zeta_k), w_{k-1}) \right] &= \mathbb{E} \left[ (\mathbb{E} \left[ \nabla \psi(w_{k-1}, \zeta_k) | w_{k-1}, w_{k-1} \right], w_{k-1}) \right] \\
& = \mathbb{E} \left[ (\nabla P(w_{k-1}), w_{k-1}) \right] \\
& = \mathbb{E} \left[ (\nabla P(w_{k-1}) - \nabla P(0), w_{k-1}) \right] + \mathbb{E} \left[ (\nabla P(0), w_{k-1}) \right] \\
& \geq \theta_0 \mathbb{E} |w_{k-1}|^2 + \mathbb{E} \left[ (\nabla P(0), w_{k-1}) \right],
\end{align*} \]

where we used (3.9). In addition, by Cauchy-Schwarz inequality and (3.12), we have

\[ \mathbb{E} \left[ (\nabla P(0), w_{k-1}) \right] \leq \frac{\theta_0}{4} \mathbb{E} |w_{k-1}|^2 + \frac{|\nabla P(0)|^2}{\theta_0}, \]

\[ \mathbb{E} |\nabla \psi(w_{k-1}, \zeta_k)|^2 = \mathbb{E} \left[ |\nabla \psi(w_{k-1}, \zeta_k)|^2 | w_{k-1} \right] \leq 2\kappa^2 \mathbb{E} |w_{k-1}|^2 + 2\kappa^2. \]

These imply

\[ \begin{align*}
\mathbb{E} |w_k|^2 & \leq \left( 1 - \frac{3}{2} \theta_0 \eta + 2\kappa^2 \eta^2 \right) \mathbb{E} |w_{k-1}|^2 + \frac{2\eta |\nabla P(0)|^2}{\theta_0} + 2\kappa^2 \eta^2 \\
& \leq \left( 1 - \theta_0 \eta \right) \mathbb{E} |w_{k-1}|^2 + 2 \left( \frac{|\nabla P(0)|^2}{\theta_0} + \kappa^2 \right) \eta,
\end{align*} \]

(A.1)

where the second inequality is by the fact \( \eta \leq \min\{1, \frac{\theta_0}{4\kappa^2} \} \). Therefore,

\[ \begin{align*}
\mathbb{E} |w_k|^2 & \leq \left[ 1 - \theta_0 \eta \right] \mathbb{E} |w_{k-1}|^2 + 2 \left( \frac{|\nabla P(0)|^2}{\theta_0} + \kappa^2 \right) \eta \sum_{j=0}^{k-1} \left[ 1 - \theta_0 \eta \right]^j \\
& \leq |w_0|^2 + \frac{2 |\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2},
\end{align*} \]

(A.2)

(4.1) is proved.

Recall (3.4)

\[ \hat{w}_k = \tilde{w}_{k-1} - \eta \mathbb{E} \left[ \nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] - \eta \mathbb{E} \left[ \nabla \phi(a_k, b_k) | b_k \right], \]

we have

\[ \begin{align*}
\mathbb{E} |\hat{w}_k|^2 & = \mathbb{E} |\tilde{w}_{k-1}|^2 - 2\eta \mathbb{E} \left[ (\tilde{w}_{k-1}, \mathbb{E} \left[ \nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ \nabla \phi(a_k, b_k) | b_k \right]) \right] \\
& + \eta^2 \mathbb{E} \left[ |\nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ |\nabla \phi(a_k, b_k) | b_k \right].
\end{align*} \]

Since \( b_k \) is independent of \( \tilde{w}_{k-1} \) and \( \mathbb{E} \left[ \nabla \phi(a_k, b_k) \right] = 0 \), by (3.2), we have

\[ \begin{align*}
\mathbb{E} \left[ (\tilde{w}_{k-1}, \mathbb{E} \left[ \nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ \nabla \phi(a_k, b_k) | b_k \right]) \right] \\
= \mathbb{E} \left[ (\tilde{w}_{k-1}, \mathbb{E} \left[ \nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ \nabla \phi(a_k, b_k) | b_k \right]) \right] = \mathbb{E} \left[ (\nabla P(w_{k-1}, w_{k-1}) \right],
\end{align*} \]

and by Cauchy-Schwarz inequality, we have

\[ \begin{align*}
\mathbb{E} \left[ |\nabla \varphi(\tilde{w}_{k-1}, a_k) | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ |\nabla \phi(a_k, b_k) | b_k \right] \leq 2 \left( \mathbb{E} \left[ |\nabla \varphi(\tilde{w}_{k-1}, a_k) |^2 | \tilde{w}_{k-1} \right] + \mathbb{E} \left[ |\nabla \phi(a_k, b_k) |^2 \right] \right) \leq 2\kappa^2 \mathbb{E} |\tilde{w}_{k-1}|^2 + 2\kappa^2.
\end{align*} \]
Therefore, by the same argument as the proof of the inequality (A.1), we have
\[ E|\hat{w}_k|^2 \leq (1 - \theta_0 \eta) E|\hat{w}_{k-1}|^2 + 2 \left( \frac{\nabla P(0)}{\theta_0} + \kappa^2 \right) \eta, \]
then, by (A.2), we immediately have
\[ E|\hat{w}_k|^2 \leq |\hat{w}_0|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2}, \]
(4.2) is proved.
Recall (3.1) and (3.4), the difference vector
\[ \hat{w}_k = w_k - \hat{w}_k = \hat{w}_{k-1} - \eta \left( \nabla \psi(w_{k-1}, \zeta_k) - E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] - E \left[ \nabla \phi(a_k, b_k) | b_k \right] \right), \]
with \( \hat{w}_0 = 0 \). It is easily seen that
\[ E|\hat{w}_k|^2 = E\|\hat{w}_{k-1}\|^2 - 2\eta E\langle \hat{w}_{k-1}, \nabla \psi(w_{k-1}, \zeta_k) \rangle - E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] - E \left[ \nabla \phi(a_k, b_k) | b_k \right] \]
\[ + \eta^2 E\|\nabla \psi(w_{k-1}, \zeta_k) \|^2 - E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] - E \left[ \nabla \phi(a_k, b_k) | b_k \right] \|^2. \]
Since \( \zeta_k = (a_k, b_k) \) is independent of \( w_{k-1}, \hat{w}_{k-1} \), we have
\[ E\langle \hat{w}_{k-1}, \nabla \psi(w_{k-1}, \zeta_k) \rangle - E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] - E \left[ \nabla \phi(a_k, b_k) | b_k \right] \]
\[ = E\langle \hat{w}_{k-1}, \nabla P(w_{k-1}) - \nabla P(\hat{w}_{k-1}) \rangle \geq \theta_0 E\|\hat{w}_{k-1}\|^2, \]
where we used (3.2) and (3.9). In addition, by (3.11), (3.12), (4.1) and (4.2), we have
\[ E\|\nabla \psi(w_{k-1}, \zeta_k) \|^2 \leq 2 \left[ E\|\nabla \psi(w_{k-1}, \zeta_k) \|^2 + E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] + E \left[ \nabla \phi(a_k, b_k) | b_k \right] \|^2 \right] \]
\[ = 2 \left[ E\|\nabla \psi(w_{k-1}, \zeta_k) \|^2 + E \left[ \nabla \varphi(\hat{w}_{k-1}, a_k) | \hat{w}_{k-1} \right] + E \left[ \nabla \phi(a_k, b_k) | b_k \right] \|^2 \right] \]
\[ \leq 4\kappa^2 \left[ |w_{k-1}|^2 + |\hat{w}_{k-1}|^2 \right] + 8\kappa^2 \leq 8\kappa^2 \left( |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2} + 1 \right). \]
These imply
\[ E|\hat{w}_k|^2 \leq (1 - 2\theta_0 \eta)|\hat{w}_{k-1}|^2 + 8\kappa^2 \left( |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2} + 1 \right) \eta^2 \]
\[ \leq (1 - 2\theta_0 \eta)^k E|\hat{w}_0|^2 + 8\kappa^2 \left( |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2} + 1 \right) \eta^2 \sum_{j=0}^{k-1} (1 - 2\theta_0 \eta)^j \]
\[ \leq 4\kappa^2 \left( |x|^2 + \frac{2|\nabla P(0)|^2 + 2\theta_0 \kappa^2}{\theta_0^2} + 1 \right) \eta \leq \frac{4\kappa^2 \left( |x|^2 + 2|\nabla P(0)|^2 + 2\theta_0 \kappa^2 + \theta_0^2 \right) \eta}{\theta_0^2}, \]
the desired result follows.

\[ \square \]

A.3. Proof of Lemma 4.2. Recall SDEs (3.6) and (4.4), the difference vector
\[ \hat{Y}_t := \hat{X}_t - \hat{X}_t \]
solves the SDE
\[ d\hat{Y}_t = \nabla P(\hat{X}_t) - \nabla P(\hat{X}_t) dt + \left[ (\eta \Sigma)^{\frac{1}{2}} - (\eta \Sigma + \eta \delta I_d)^{\frac{1}{2}} \right] dW_t, \quad \hat{Y}_0 = 0. \]
Then, by Itô’s formula (see, e.g., [61, Theorem 4.2.1]) and (3.9), we have
\[ \frac{d}{ds} E |\hat{Y}_s|^2 = 2E \langle \nabla P(\hat{X}_s) - \nabla P(\hat{X}_s), \hat{Y}_s \rangle + \| (\eta \Sigma)^{\frac{1}{2}} - (\eta \Sigma + \eta \delta I_d)^{\frac{1}{2}} \|^2_{HS} \]
which implies
\[
\frac{d}{ds} \mathbb{E} |\tilde{Y}_s|^2 \leq -2\theta_0 \mathbb{E} |\tilde{Y}_s|^2 + \frac{\eta \delta^2 d}{\lambda_{\min} + \delta}.
\]
This inequality, together with \(\tilde{Y}_0 = 0\), implies
\[
\mathbb{E} |\tilde{Y}_t|^2 \leq \frac{\eta \delta^2 d}{\lambda_{\min} + \delta} \left( e^{-2\theta_0 (t-s)} - 1 \right) \leq \frac{\eta \delta^2 d}{2\theta_0 (\lambda_{\min} + \delta)}.
\]
The proof is complete. \(\square\)

A.4. **Proof of Lemma 4.4.** Recall (4.4), by Itô’s formula, (3.9) and Cauchy-Schwarz inequality, we have
\[
\frac{d}{ds} \mathbb{E} |\tilde{X}_s|^2 = 2\mathbb{E} \left( \tilde{X}_s^s - \nabla P(\tilde{X}_s^s) \right) + \eta \| (\Sigma + \delta I_d)^\frac{1}{2} \|^2_{HS} \\
= -2\mathbb{E} \left( \tilde{X}_s^s, \nabla P(\tilde{X}_s^s) - \nabla P(0) \right) - 2\mathbb{E} \left( \tilde{X}_s^s, \nabla P(0) \right) + \eta \| (\Sigma + \delta I_d)^\frac{1}{2} \|^2_{HS} \\
\leq -2\theta_0 \mathbb{E} |\tilde{X}_s|^2 + 2\theta_0 \mathbb{E} |\tilde{X}_s|^2 + \frac{\| \nabla P(0) \|^2}{\theta_0} + \eta \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d) \\
\leq -\theta_0 \mathbb{E} |\tilde{X}_s|^2 + \frac{\| \nabla P(0) \|^2}{\theta_0} + \eta \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d).
\]
This inequality, together with \(\tilde{X}_0 = x\), implies
\[
\mathbb{E} |\tilde{X}_t|^2 \leq e^{-\theta_0 t} |x|^2 + \frac{\| \nabla P(0) \|^2}{\theta_0} + \theta_0 \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d)] \int_0^t e^{-\theta_0 (t-s)} ds \\
\leq |x|^2 + \frac{\| \nabla P(0) \|^2 + \theta_0 \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d) \theta_0}{\theta_0}.
\]
(4.8) is proved.

By Cauchy-Schwarz inequality, Itô isometry and (3.13), we have
\[
\mathbb{E} |\tilde{X}_t - x|^2 \leq 2\mathbb{E} \left( \int_0^t -\nabla P(\tilde{X}_s^s) dr \right)^2 + 2\mathbb{E} \left( \int_0^t (\eta \Sigma + \eta \delta I_d)^\frac{1}{2} dW_t \right)^2 \\
\leq 2t \left( \int_0^t \mathbb{E} |\nabla P(\tilde{X}_s^s)|^2 dr \right)^2 + 2\eta \left( \int_0^t \| (\Sigma + \delta I_d)^\frac{1}{2} \|^2_{HS} dr \right)^2 \\
\leq 4\kappa^2 t \left( \mathbb{E} (1 + |\tilde{X}_s^s|)^2 dr \right)^2 + 2\eta t \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d) \\
\leq 8\kappa^2 t \left( \mathbb{E} (1 + |\tilde{X}_s^s|)^2 dr \right)^2 + 2\eta t \| (\Sigma^\frac{1}{2})^2 \|^2_{HS} + \delta d),
\]
which, together with (4.8), implies (4.9). The proof is complete. \(\square\)

**Appendix B. Proof of Lemma 4.3.**

Under Assumption A2 and some derivations in Subsection 3.1, we recall some preliminary of Malliavin calculus and derive standard estimates related to Malliavin calculus and SDE, which will be applied to prove the Lemma 4.3 in subsection 4.1.
B.1. **Malliavin calculus of SDE (4.4) ([40])**. For simplicity, denote \( B(x) := -\nabla P(x) \), \( \sigma_\delta := (\Sigma + \delta I_d)^{\frac{1}{2}} \). Then SDE (4.4) can be written as the following form:

(B.1) \[
d\tilde{X}_t = B(\tilde{X}_t)dt + \sqrt{\tau}\sigma_\delta dW_t, \quad \tilde{X}_0 = x.\]

What’s more, **Assumption A2** in Subsection 3.1 can be rewritten as the following form:

The mapping \( B(x) \) belongs to \( C^2(\mathbb{R}^d, \mathbb{R}^d) \) and there exist \( \theta_0 > 0 \) and \( \theta_1 \geq 0 \) such that

(B.2) \[
\langle v, \nabla_v B(x) \rangle \leq -\theta_0 |v|^2, \quad \forall v, x \in \mathbb{R}^d;
\]

(B.3) \[
|\nabla_{v_1} \nabla_{v_2} B(x)| \leq \theta_1 |v_1||v_2|, \quad \forall v_1, v_2, x \in \mathbb{R}^d.
\]

**Remark B.1.** Recall \( \sigma_\delta := (\Sigma + \delta I_d)^{\frac{1}{2}} \), it is easy to check that

(B.4) \[
\sigma_\delta \geq (\lambda_{\text{min}} + \delta)^{\frac{1}{2}} I_d,
\]

i.e., for any \( x \in \mathbb{R}^d \), we have

\[
x^T \sigma_\delta x \geq (\lambda_{\text{min}} + \delta)^{\frac{1}{2}} |x|^2.
\]

Under the **Assumption A2**, there exists a unique solution to the SDE (B.1) and the SDE (B.1) has a unique non-degenerate invariant measure (see, e.g., [19, 23, 35, 47, 63]).

Next, we briefly recall Bismut’s approach to Malliavin calculus, which is crucial to prove **Lemma 4.3**.

We first consider the derivative of \( \tilde{X}_t^x \) with respect to initial value \( x \). Let \( v \in \mathbb{R}^d \) and \( \nabla_v \tilde{X}_t^x \) is defined by

\[
\nabla_v \tilde{X}_t^x = \lim_{\epsilon \to 0} \frac{\tilde{X}_t^{x+\epsilon v} - \tilde{X}_t^x}{\epsilon}, \quad t \geq 0.
\]

The above limit exists and satisfies

(B.5) \[
d\nabla_v \tilde{X}_t^x = \nabla B(\tilde{X}_t^x) \nabla_v \tilde{X}_t^x dt, \quad \nabla_v \tilde{X}_0^x = v,
\]

which is solved by

\[
\nabla_v \tilde{X}_t^x = \exp \left\{ \int_0^t \nabla B(\tilde{X}_r^x) dr \right\} v.
\]

For further use, we denote

\[
J^x_{s,t} = \exp \left\{ \int_s^t \nabla B(\tilde{X}_r^x) dr \right\}, \quad 0 \leq s \leq t < \infty.
\]

It is easy to see that \( J^x_{0,s} J^x_{s,t} = J^x_{0,t} \) for all \( 0 \leq s \leq t < \infty \) and

\[
\nabla_v \tilde{X}_t^x = J^x_{0,t} v.
\]

For \( v_1, v_2 \in \mathbb{R}^d \), we can define \( \nabla_{v_2} \nabla_{v_1} \tilde{X}_t^x \), which satisfies

(B.6) \[
d\nabla_{v_2} \nabla_{v_1} \tilde{X}_t^x = \nabla B(\tilde{X}_t^x) \nabla_{v_2} \nabla_{v_1} \tilde{X}_t^x dt + \nabla^2 B(\tilde{X}_t^x) \nabla_{v_2} \tilde{X}_t^x \nabla_{v_1} \tilde{X}_t^x dt,
\]

with \( \nabla_{v_2} \nabla_{v_1} \tilde{X}_0^x = 0 \).

Then, we have the following estimates:

**Lemma B.2.** For all \( v, v_1, v_2, x \in \mathbb{R}^d \), we have the following (deterministic) estimates:

(B.7) \[
|\nabla_{v_2} \tilde{X}_t^x| \leq e^{-\theta_0 t} |v|,
\]

(B.8) \[
|\nabla_{v_2} \nabla_{v_1} \tilde{X}_t^x| \leq C_\rho e^{-\frac{\theta_0 t}{2}} |v_1||v_2|.
\]
Proof. Recall $\theta_0 > 0$, by (B.5) and (B.2), we have
\[
\frac{d}{dt} |\nabla v \tilde{X}_t|^2 = 2 \langle \nabla v \tilde{X}_t, \nabla B(\tilde{X}_t) \nabla v \tilde{X}_t \rangle \leq -2\theta_0 |\nabla v \tilde{X}_t|^2,
\]
which implies
\[
|\nabla v \tilde{X}_t|^2 \leq e^{-2\theta_0 t} |v|^2.
\]
Writing $\varsigma(t) = \nabla v_2 \nabla v_1 \tilde{X}_t$, by (B.6), (B.2), (B.3), (B.7) and Young inequality, we have
\[
\frac{d}{dt} |\varsigma(t)|^2 = 2 \langle \varsigma(t), \nabla B(\tilde{X}_t) \varsigma(t) \rangle + 2 |\varsigma(t)| \nabla^2 B(\tilde{X}_t) \nabla v_2 \tilde{X}_t \nabla v_1 \tilde{X}_t
\leq -2\theta_0 |\varsigma(t)|^2 + 2\theta_1 |\varsigma(t)| e^{-2\theta_0 t} |v_1| |v_2|
\leq -2\theta_0 |\varsigma(t)|^2 + \frac{\theta_1^2}{\theta_0^2} e^{-4\theta_0 t} |v_1|^2 |v_2|^2.
\]
Recall that $\varsigma(0) = 0$, we further have
\[
|\varsigma(t)|^2 \leq C\theta \int_0^t e^{-\theta_0 (t-s)} e^{-4\theta_0 s} ds |v_1|^2 |v_2|^2 = C\theta e^{-\theta_0 t} \int_0^t e^{-3 \theta_0 s} ds |v_1|^2 |v_2|^2
\leq C\theta e^{-\theta_0 t} |v_1|^2 |v_2|^2,
\]
the desired result follows. \qed

Next, we use Bismut’s approach to Malliavin calculus for SDE (B.1)(58). Let $u \in L^2_{loc}([0, \infty) \times (\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$, i.e., $\mathbb{E} \int_0^t |u(s)|^2 ds < \infty$ for all $t > 0$. Further assume that $u$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t := \sigma(W_s : 0 \leq s \leq t)$; i.e., $u(t)$ is $\mathcal{F}_t$ measurable for $t \geq 0$. Define
\begin{equation}
U_t = \int_0^t u(s) ds, \quad t \geq 0.
\end{equation}
For a $t > 0$, let $F_t : C([0, t], \mathbb{R}^d) \to \mathbb{R}$ be a $\mathcal{F}_t$ measurable mapping. If the following limit exists
\[
D_U F_t(W) = \lim_{\epsilon \to 0} \frac{F_t(W + \epsilon U) - F_t(W)}{\epsilon}
\]
in $L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R})$, then $F_t(W)$ is said to be Malliavin differentiable and $D_U F_t(W)$ is called the Malliavin derivative of $F_t(W)$ in the direction $u$.

Let $F_t(W)$ and $G_t(W)$ both be Malliavin differentiable, then the following product rule holds:
\begin{equation}
D_U(F_t(W)G_t(W)) = F_t(W)D_U G_t(W) + G_t(W)D_U F_t(W).
\end{equation}
When
\[
F_t(W) = \int_0^t \langle a(s), dW(s) \rangle,
\]
where $a(s) = (a_1(s), \ldots, a_d(s))$ is a $d$-dimensional stochastic process adapted to the filtration $\mathcal{F}_s$ such that $\mathbb{E} \int_0^t |a(s)|^2 ds < \infty$ for all $t > 0$, it is easy to check that
\begin{equation}
D_U F_t(W) = \int_0^t \langle a(s), U(s) \rangle ds + \int_0^t \langle D_U a(s), dW_s \rangle.
\end{equation}
The following integration by parts formula is called Bismut’s formula. For Malliavin differentiable $F_t(W)$ such that $F_t(W), D_U F_t(W) \in L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R})$, we have

$$
\mathbb{E}[D_U F_t(W)] = \mathbb{E}[F_t(W) \int_0^t \langle u(s), dW_s \rangle].
$$

(B.12)

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz and let $F_t(W) = (F_t^1(W), \ldots, F_t^d(W))$ be a $d$-dimensional Malliavin differentiable functional. The following chain rule holds:

$$
D_U \phi(F_t(W)) = \langle \nabla \phi(F_t(W)), D_U F_t(W) \rangle = \sum_{i=1}^d \partial_i \phi(F_t(W)) D_U F_t^i(W).
$$

Now, we come back to the SDE (B.1). Fixing $t \geq 0$ and $x \in \mathbb{R}^d$, the solution $\tilde{X}_t^x$ can be considered to be a $d$-dimensional functional of Brownian motion $(W_s)_{0 \leq s \leq t}$.

The following Malliavin derivative of $\tilde{X}_t^x$ along the direction $U$ exists in $L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$ and is defined by

$$
D_U \tilde{X}_t^x(W) = \lim_{\epsilon \to 0} \frac{\tilde{X}_t^x(W + \epsilon U) - \tilde{X}_t^x(W)}{\epsilon}.
$$

We drop the $W$ in $D_U \tilde{X}_t^x(W)$ and write $D_U \tilde{X}_t^x = D_U \tilde{X}_t^x$ for simplicity. By (B.11), it satisfies the equation

$$
dD_U \tilde{X}_t^x = \nabla B(\tilde{X}_t^x) D_U \tilde{X}_t^x dt + \sqrt{\eta} \sigma_\delta u(t) dt, \quad D_U \tilde{X}_0^x = 0,
$$

and the equation has a unique solution:

$$
D_U \tilde{X}_t^x = \int_0^t J_{r,t}^x \sqrt{\eta} \sigma_\delta u(r) dr.
$$

Noticing that $\nabla \tilde{X}_t^x = J_{0,t}^x v$, if we take

$$
u(s) = \frac{1}{\sqrt{\eta t}} \sigma_\delta^{-1} \nabla \tilde{X}_s^x, \quad 0 \leq s \leq t,
$$

then (B.7) implies $u \in L^2_{\text{loc}}([0, \infty) \times (\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}^d)$. Since $\nabla \tilde{X}_r^x = J_{0,r}^x v$ and $J_{0,r}^x \cdot J_{r,t}^x = J_{0,t}^x$, for all $0 \leq r \leq t$, we have

$$
D_U \tilde{X}_t^x = \nabla \tilde{X}_t^x
$$
and

$$
D_U \tilde{X}_s^x = \frac{s}{t} \nabla \tilde{X}_s^x, \quad 0 \leq s \leq t.
$$

Let $v_1, v_2 \in \mathbb{R}^d$, and define $u_i$ and $U_i$ as (B.13) and (B.9), respectively, for $i = 1, 2$. We can similarly define $D_{U_2} \nabla_{v_1} \tilde{X}_s^x$, which satisfies the following equation: for $s \in [0, t]$,

$$
dD_{U_2} \nabla_{v_1} \tilde{X}_s^x = [\nabla B(\tilde{X}_s^x) D_{U_2} \nabla_{v_1} \tilde{X}_s^x + \nabla^2 B(\tilde{X}_s^x) D_{U_2} \tilde{X}_s^x \nabla_{v_1} \tilde{X}_s^x] ds
$$

(B.16)

with $D_{U_2} \nabla_{v_1} \tilde{X}_0^x = 0$, where the second equality is by (B.13) and (B.15).

For further use, we define

$$
\mathcal{I}_{v_1}(t) := \frac{1}{\sqrt{\eta t}} \int_0^t \langle \sigma_\delta^{-1} \nabla \tilde{X}_s^x, dB_s \rangle
$$
and

$$
\mathcal{R}_{v_1,v_2}^x(t) := \nabla_{v_2} \nabla_{v_1} \tilde{X}_t^x - D_{U_2} \nabla_{v_1} \tilde{X}_t^x.
$$
Then, by the same argument as the proof of (B.8), we have the following upper bounds on Malliavin derivatives.

**Lemma B.3.** Let \( v_i \in \mathbb{R}^d \) for \( i = 1, 2 \), and let

\[
U_{i,s} = \int_0^s u_i(r)dr, \quad 0 \leq s \leq t,
\]

where \( u_i(r) = \frac{1}{\sqrt{n}} \sigma_i^{-1} \nabla v_i \dot{X}^r_t \) for \( 0 \leq r \leq t \). Then, we have

\[
|D_{U_{2}} \nabla v_i \dot{X}^r_t| \leq C_0 e^{-\frac{\alpha_0}{2}t} |v_1||v_2|.
\]

(B.17)

Based on the above results, we have the following lemma:

**Lemma B.4.** Let \( v_1, v_2 \in \mathbb{R}^d \) and \( x \in \mathbb{R}^d \). For all \( p \geq 1 \), we have

\[
\mathbb{E}|\mathcal{I}^{x}_{v_1}(t)|^{2p} \leq \frac{C_p}{t^{2p}p(\lambda_{\min} + \delta)^p} |v_1|^{2p}
\]

(B.18)

and

\[
|\mathcal{R}^{x}_{v_1,v_2}(t)| \leq C_0 e^{-\frac{\alpha_0}{4}t} |v_1||v_2|.
\]

(B.19)

**Proof.** By the Burkholder-Davis-Gundy inequality [64, p.160], (B.4), (B.7) and (1.2), we have

\[
\mathbb{E}|\mathcal{I}^{x}_{v_1}(t)|^{2p} = \mathbb{E}\left[ \frac{1}{\sqrt{n}} \int_0^t \left( \sigma^{-1}_i \nabla v_i \dot{X}^r_s, dB_s \right)^2 \right]^{2p} \leq \frac{C_p}{t^{2p}p(\lambda_{\min} + \delta)^p} \left( \int_0^t |\nabla v_i \dot{X}^r_s|^2 ds \right)^p.
\]

Recall \( \mathcal{R}^{x}_{v_1,v_2}(t) = \nabla v_2 \nabla v_1 \dot{X}^r_t - D_{U_{2}} \nabla v_1 \dot{X}^r_t \), by (B.8) and (B.17), we have

\[
|\mathcal{R}^{x}_{v_1,v_2}(t)| \leq |\nabla v_2 \nabla v_1 \dot{X}^r_t| + |D_{U_{2}} \nabla v_1 \dot{X}^r_t| \leq C_0 e^{-\frac{\alpha_0}{4}t} |v_1||v_2|.
\]

The proof is complete.

**B.2. Proof of Lemma 4.3.** Recall \( P_t h(x) = \mathbb{E}[h(\dot{X}^r_t)] \) for \( h \in \text{Lip}(1) \), by Lebesgue’s dominated convergence theorem and (B.7), we have

\[
|\nabla v_i \mathbb{E}[h(\dot{X}^r_t)]| = |\mathbb{E}[\nabla h(\dot{X}^r_t) \nabla v_i \dot{X}^r_t]| \leq ||\nabla h|| ||\nabla \dot{X}^r_t|| = e^{-\alpha_0 t} |v|,
\]

(4.6) is proved.

Denote

\[
h_{\epsilon}(x) = \int_{\mathbb{R}^d} f_{\epsilon}(y) h(x-y) dy,
\]

with \( \epsilon > 0 \) and \( f_{\epsilon} \) is the density of the normal distribution \( N(0, \epsilon^2 I_d) \). It is easy to see that \( h_{\epsilon} \) is smooth, \( \lim_{\epsilon \to 0} h_{\epsilon}(x) = h(x) \), \( \lim_{\epsilon \to 0} \nabla h_{\epsilon}(x) = \nabla h(x) \) and \( |h_{\epsilon}(x)| \leq C(1 + |x|) \) for all \( x \in \mathbb{R}^d \) and some \( C > 0 \). Moreover, \( ||\nabla h_{\epsilon}|| \leq ||\nabla h|| \leq 1 \). Then, by Lebesgue’s dominated convergence theorem, we have

\[
\nabla v_2 \nabla v_1 \mathbb{E}[h_{\epsilon}(\dot{X}^r_t)] = \mathbb{E}[\nabla^2 h_{\epsilon}(\dot{X}^r_t) \nabla v_2 \dot{X}^r_t \nabla v_1 \dot{X}^r_t] + \mathbb{E}[\nabla h_{\epsilon}(\dot{X}^r_t) \nabla v_2 \nabla v_1 \dot{X}^r_t],
\]

(B.20)
by (B.13) and (B.14), we further have
\begin{align*}
E [\nabla^2 h_t (\tilde{X}_t^x) \nabla v_2 \tilde{X}_t^x \nabla v_1 \tilde{X}_t^x] &= E [\nabla^2 h_t (\tilde{X}_t^x) D_{U_2} \tilde{X}_t^x \nabla v_1 \tilde{X}_t^x] \\
&= E [D_{U_2} (\nabla h_t (\tilde{X}_t^x)) \nabla v_1 \tilde{X}_t^x] \\
&= E [D_{U_2} (\nabla h_t (\tilde{X}_t^x) \nabla v_1 \tilde{X}_t^x)] - E [\nabla h_t (\tilde{X}_t^x) D_{U_2} \nabla v_1 \tilde{X}_t^x] \\
&= E [\nabla h_t (\tilde{X}_t^x) \nabla v_1 \tilde{X}_t^x \mathcal{F}_{U_2} (t)] - E [\nabla h_t (\tilde{X}_t^x) D_{U_2} \nabla v_1 \tilde{X}_t^x],
\end{align*}
where the last equality is by Bismut’s formula (B.12). These imply
\begin{equation}
(\text{B.21}) \quad \nabla v_2 \nabla v_1 E [h_t (\tilde{X}_t^x)] = E [\nabla h_t (\tilde{X}_t^x) \nabla v_1 \tilde{X}_t^x \mathcal{F}_{U_2} (t)] + E [\nabla h_t (\tilde{X}_t^x) \mathcal{R}_{v_1,v_2}^x (t)].
\end{equation}

Therefore, by Lebesgue’s dominated convergence theorem, (B.7), (B.19), Cauchy-Schwarz inequality and (B.18), we have
\begin{align*}
|\nabla v_2 \nabla v_1 E [h_t (\tilde{X}_t^x)]| &= |\lim_{\epsilon \to 0} \nabla v_2 \nabla v_1 E [h_t (\tilde{X}_t^x)]| \\
&\leq \lim_{\epsilon \to 0} \left| E [\nabla h_t (\tilde{X}_t^x) \nabla v_1 \tilde{X}_t^x \mathcal{F}_{U_2} (t)] + E [\nabla h_t (\tilde{X}_t^x) \mathcal{R}_{v_1,v_2}^x (t)] \right| \\
&\leq e^{-\theta_0 t} |v_1| E [\mathcal{F}_{U_2} (t)] + C \theta e^{-\theta_0 t} |v_1| |v_2| \\
&\leq C \theta e^{-\theta_0 t} \left( |v_1| \sqrt{E [\mathcal{F}_{U_2} (t)]^2} + |v_1| |v_2| \right) \\
&\leq C \theta e^{-\theta_0 t} \left( \frac{\lambda_{\min} + \delta}{\sqrt{t}} \right) + 1 |v_1| |v_2|,
\end{align*}
the desired result follows. \( \square \)

APPENDIX C. PROOFS OF LEMMA 4.5

In order to prove Lemma 4.5, we first give the following estimates.

C.1. Heat kernel estimates of \( \alpha \)-stable process. Let \( p(t,x) \) be the transition probability density of rotationally symmetric \( \alpha \)-stable process \( (Z)_{t>0} \), which has characteristic function \( e^{-\|x\|^{\alpha}} \). It is well known that
\[ p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x), \quad t > 0, \quad x \in \mathbb{R}^d. \]

We have the following heat kernel estimates.

**Lemma C.1.** Let \( p(1,x) \) be the transition probability density of \( Z_1 \), we have
\[ p(1,x) \leq 2^{-d+1} \pi^{-d/2} \Gamma(d/\alpha) / \alpha \Gamma(d/2), \quad p(1,x) \leq \frac{\alpha^{d-1} \sin \frac{\alpha \pi}{2} \Gamma((d+\alpha)/2) / \Gamma(\alpha/2)}{\pi^{d/2+1}|x|^{d+\alpha}}, \]
and
\[ |\nabla p(1,x)| \leq 2\pi|x| p(d+2)(1,\tilde{x}), \]
and
\[ ||\nabla^2 p(1,x)||_{HS} \leq 2\pi dp_{(d+2)}(\tilde{x}) + 4\pi^2 |x|^2 p_{(d+4)}(\tilde{x}), \]
where \( \tilde{x} \in \mathbb{R}^{d+2} \) is such that \( |\tilde{x}| = |x| \) and \( p_{d+2}(1, \tilde{x}) \) is the transition density of the rotationally symmetric \( \alpha \)-stable process \( Z_1 \) in dimension \( d+2 \) and \( \tilde{x} \in \mathbb{R}^{d+4} \) is such that \( |\tilde{x}| = |x| \), and \( p_{d+4}(1, \tilde{x}) \) is the transition density of the rotationally symmetric \( \alpha \)-stable process \( Z_1 \) in dimension \( d+4 \).
Proof. For notational simplicity, we write $p(x) = p(1, x)$. By the [66, Proposition 2.3 (XII)], we have

$$p(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i \langle x, t \rangle} e^{-|t|^\alpha} dt = (2\pi)^{-d} \int_{\mathbb{R}^d} \cos(\langle x, t \rangle) e^{-|t|^\alpha} dt,$$

since $\cos(\langle x, t \rangle) \leq 1$, we have

$$p(x) \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-|t|^\alpha} dt = (2\pi)^{-d} \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{d-1} e^{-r^\alpha} dr d\theta = (2\pi)^{-d} V(\mathbb{S}^{d-1}) \int_0^\infty r^{d-1} e^{-r^\alpha} dr = (2\pi)^{-d} V(\mathbb{S}^{d-1}) \frac{1}{\alpha} \int_0^\infty y^{\frac{d}{\alpha}-1} e^{-y} dy = (2\pi)^{-d} V(\mathbb{S}^{d-1}) \frac{\Gamma(d/\alpha)}{\alpha},$$

where the last second equality is by taking $y = r^\alpha$. Recall $V(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, we have

$$p(x) \leq 2^{-d+1} \pi^{-\frac{d}{2}} \frac{\Gamma(d/\alpha)}{\alpha \Gamma(d/2)}.$$

Using the Fourier inversion theorem for radial functions [12, (2.1)], we have

$$p(x) = (2\pi)^{-\frac{d}{2}} |x|^{-\frac{d}{2}+1} \int_0^\infty e^{-r^\alpha} \frac{r^{d/2}}{2} J_{(d-2)/2}(|x|t) dt,$$

where $J_m$ denotes the Bessel function of first kind of order $m$. Then, let $r = |x|t$ in the above integral term, we have

$$p(x) = (2\pi)^{-\frac{d}{2}} |x|^{-d} \int_0^\infty e^{-r^\alpha} \frac{r^{d/2}}{2} J_{(d-2)/2}(r) dr.$$

From [11, section 7.2.8 (50)], we have $\frac{\partial}{\partial t}(t^m J_m(t)) = t^m J_{m-1}(t)$. Hence, using integration by parts, we have

$$p(x) = \alpha (2\pi)^{-\frac{d}{2}} |x|^{-d-\alpha} \int_0^\infty e^{-r^\alpha} \frac{r^{d/2+\alpha-1}}{2} J_{d/2}(r) dr \leq \alpha (2\pi)^{-\frac{d}{2}} |x|^{-d-\alpha} \int_0^\infty r^{\frac{d}{2}+\alpha-1} J_{d/2}(r) dr \leq \frac{\alpha^{2\alpha-1} \sin \frac{\alpha \pi}{2} \Gamma((d + \alpha)/2) \Gamma(\alpha/2)}{\pi^{d/2+1}|x|^{d+\alpha}}.$$

where the last inequality comes from the proof of [12, Theorem 2.1].

Furthermore, from [16, (11)], we have $\nabla p(x) = -2\pi x p_{(d+2)}(\hat{x})$, so

$$|\nabla p(x)| \leq 2\pi |x| p_{(d+2)}(\hat{x}).$$

In addition, we have $\nabla^2 p(x) = -2\pi p_{(d+2)}(\hat{x}) I_d - 2\pi x (\nabla p_{(d+2)}(\hat{x}))^T$, which implies

$$||\nabla^2 p(x)||_{HS} \leq 2\pi dp_{(d+2)}(\hat{x}) + 4\pi^2 |x|^2 p_{(d+4)}(\hat{x}).$$

The proof is complete. \qed
Lemma C.3. Fix $x, y$. Then, for any 
and

Recall (C.2)

Remark C.2. For the sake of convenience, we denote

$$V_{α,d} = \max \left \{ 2^{-d+1}π^{\frac{d}{2}} \frac{Γ(d/α)}{αΓ(d/2)}, \frac{α^{2α-1}sin(απΓ((d + α)/2)Γ(α/2))}{π^{d/2}} \right \},$$

then from Lemma C.1, we have

$$p(1, x) \leq V_{α,d} \left (1 \wedge \frac{1}{|x|^{α+d}} \right ),$$

and

(C.1) \hspace{1cm} |∇x p(1, x)| \leq 2πV_{α,d+2}|x|t^{\frac{d}{α}-\frac{d+2}{α}}(1 \wedge \frac{t^{\frac{α+d}{α}}}{|x|^{α+d+2}}) \leq 2πV_{α,d+2}t^{-\frac{1}{α}}(t^{\frac{d}{α}} \wedge \frac{t}{|x|^{α+d}}).

C.2. Inequality about fractional Laplacian operator $Δ^\frac{α}{2}$. The fractional Laplacian operator $Δ^\frac{α}{2}$ can transform $C^2_b(\mathbb{R}^d)$-functions into $(2 - α)$-Hölder continuous functions.

Lemma C.3. Fix $α \in (1, 2)$ and let $f \in C^2_b(\mathbb{R}^d)$. Then, for any $x, y \in \mathbb{R}^d$, we have

(C.2) \hspace{1cm} |(Δ^\frac{α}{2} f)(x) - (Δ^\frac{α}{2} f)(y)| \leq \frac{2d_α V(\mathbb{S}^{d-1})∥∇^2 f∥_{HS}}{α(2 - α)(α - 1)}|x - y|^{2-α}.

Proof. Recall (3.17), we have by the symmetric property that

$$Δ^\frac{α}{2} f(x) = d_α ∫_{\mathbb{S}^{d-1}} ∫_0^∞ f(x + rθ) - f(x) - rθ ∇f(x)) drdθ \hspace{1cm} = d_α ∫_{\mathbb{S}^{d-1}} ∫_0^∞ ∫_0^r ⟨θ, ∇f(x + ωs) - ∇f(x)⟩ r^{α+1} dsdrdθ \hspace{1cm} = d_α ∫_{\mathbb{S}^{d-1}} ∫_0^∞ ∫_0^r ⟨θ, ∇f(x + ωs) - ∇f(x)⟩ r^{α+1} dsdrdθ \hspace{1cm} = d_α ∫_{\mathbb{S}^{d-1}} ∫_0^∞ ⟨θ, ∇f(x + ωs) - ∇f(x)⟩ s^{α} dsdθ.$$

Then, for any $x, y \in \mathbb{R}^d$, we have

$$|Δ^\frac{α}{2} f(x) - Δ^\frac{α}{2} f(y)| \hspace{1cm} ≤ \frac{d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^∞ |∇f(x + ωs) - ∇f(x) - ∇f(y + ωs) + ∇f(y)| s^{α} dsdθ \hspace{1cm} ≤ \frac{d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^∞ |∇f(x + ωs) - ∇f(x) - ∇f(y + ωs) + ∇f(y)| s^{α} dsdθ \hspace{1cm} + \frac{d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^infinity |∇f(x + ωs) - ∇f(x) - ∇f(y + ωs) + ∇f(y)| s^{α} dsdθ.$$

Hence, one can write

$$\frac{d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^∞ |∇f(x + ωs) - ∇f(x) - ∇f(y + ωs) + ∇f(y)| s^{α} dsdθ \hspace{1cm} ≤ \frac{d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^∞ |∇f(x + ωs) - ∇f(x + ωs) + ∇f(y)| s^{α} dsdθ \hspace{1cm} ≤ \frac{2d_α}{α} ∫_{\mathbb{S}^{d-1}} ∫_0^∞ |∇^2 f|_{HS} |x - y|dsdθ = \frac{2d_α V(\mathbb{S}^{d-1})∥∇^2 f∥_{HS}}{α(α - 1)} |x - y|^{2-α},$$
whereas
\[ d_\alpha \int_{S^{d-1}} \int_0^{|x-y|} |\nabla f(x + \theta s) - \nabla f(x) - \nabla f(y + \theta s) + \nabla f(y)| dsd\theta \]
\[ \leq d_\alpha \int_{S^{d-1}} \int_0^{|x-y|} |\nabla f(x + \theta s) - \nabla f(x)| + |\nabla f(y + \theta s) - \nabla f(y)| dsd\theta \]
\[ \leq 2d_\alpha \|\nabla^2 f\|_{HS} \int_{S^{d-1}} \int_0^{|x-y|} \frac{1}{s^{\alpha-1}} dsd\theta = \frac{2d_\alpha V(S^{d-1}) \|\nabla^2 f\|_{HS}}{\alpha(2 - \alpha)} |x - y|^{2-\alpha}. \]
The desired result follows. \[\square\]

C.3. Proof of Lemma 4.5. Let \(p(t, x)\) be the transition probability density of rotationally symmetric \(\alpha\)-stable process \(Z_t\), then noticing that
\[ p(t, x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i(y-x,\lambda)} e^{-\|\lambda\|^\alpha} d\lambda, \]
we have by integration by parts that
\[ \nabla (P_t h)(x) = \int_{\mathbb{R}^d} \nabla_x p(t, x, y) h(y) dy = - \int_{\mathbb{R}^d} \nabla_y p(t, x, y) h(y) dy \]
\[ = \int_{\mathbb{R}^d} p(t, x, y) \nabla h(y) dy, \]
which implies
\[ \|\nabla^2 (P_t h)(x)\|_{HS} \leq \|\nabla h\| \int_{\mathbb{R}^d} |\nabla_x p(t, x, y)| dy \]
\[ \leq 2\pi V_{\alpha, d+2} t^{-\frac{1}{2}} \int_{\mathbb{R}^d} (t - \frac{d}{\alpha} + \frac{t}{|y-x|^{\alpha+d}}) dy \leq 4\pi V_{\alpha, d+2} V(S^{d-1}) t^{-\frac{1}{2}}, \]
where the last two inequalities is by (C.1). Furthermore, we have by (C.2) that
\[ |\Delta \Phi P_t h(x) - \Delta \Phi P_t h(y)| \leq \frac{2d_\alpha V(S^{d-1}) \|\nabla^2 (P_t h)\|_{HS}}{\alpha(2 - \alpha)(\alpha - 1)} |x - y|^{2-\alpha} \]
\[ \leq \frac{8\pi d_\alpha V(S^{d-1})^2 V_{\alpha, d+2} t^{-\frac{1}{2}}}{\alpha(2 - \alpha)(\alpha - 1)} |x - y|^{2-\alpha}. \]
The proof is complete. \[\square\]

Appendix D. Proof of Lemma 4.6

In this section, we use the semigroup of \(B_t^\alpha\) and the formula of integration by parts to prove Lemma 4.6.

Proof of Lemma 4.6. Recall
\[ P_t h(x) = \mathbb{E} h(B_t^\alpha) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} h(y) dy. \]
For any \(v, x_1, x_2 \in \mathbb{R}^d\) and \(f \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})\), denote the directional derivative of \(f(x_1, x_2)\) with respect to \(x_i\) by \(\nabla_{v_i} f(x_1, x_2)\), \(i = 1, 2\), respectively. Then, we have
\[ \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla_{v_1}^x e^{-\frac{|y-x|^2}{2t}} h(y) dy \]
\[ = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{2t}} \left( \frac{y - x}{t}, v_1 \right) h(y) dy, \]
which implies
\[
\nabla_{v_2} \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}^d} \nabla_{v_2}^x (e^{-\frac{|x-y|^2}{2t}} \langle y - x, v_1 \rangle h(y) dy \\
= \frac{1}{(2\pi t)^2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} \langle y - x, v_2 \rangle \langle y - x, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle h(y) dy.
\]

Hence, by integration by parts, we have
\[
\nabla_{v_2} \nabla_{v_1} P_t h(x) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}^d} \nabla_{v_2}^x \left[ e^{-\frac{|x-y|^2}{2t}} \left( \langle y - x, v_2 \rangle \langle y - x, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \right) \right] h(y) dy \\
= -\frac{1}{(2\pi t)^2} \int_{\mathbb{R}^d} \nabla_{v_2}^x \left[ e^{-\frac{|x-y|^2}{2t}} \left( \langle y - x, v_2 \rangle \langle y - x, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \right) \right] h(y) dy \\
= \frac{1}{(2\pi t)^2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} \langle y - x, v_2 \rangle \langle y - x, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \nabla_{v_2} h(y) dy \\
= \nabla \left[ \langle \langle B_t \rangle, v_2 \rangle \langle \langle B_t \rangle, v_1 \rangle - \frac{1}{t} \langle v_2, v_1 \rangle \right] \nabla h(B_t^2, v_3)
\]
then, by Cauchy-Schwarz inequality, we have
\[
\nabla \nabla h \nabla P_t h \leq \nabla \nabla h \nabla \left[ \frac{1}{t^2} \mathbb{E} \langle B_t, v_2 \rangle \langle B_t, v_1 \rangle + \frac{1}{t} \langle v_2, v_1 \rangle \right] \\
\leq \langle v_3 \rangle \left[ \frac{1}{t^2} \sqrt{\mathbb{E} \langle B_t, v_2 \rangle ^2 \mathbb{E} \langle B_t, v_1 \rangle ^2 + \frac{1}{t} \langle v_2, v_1 \rangle } \right] \\
= \langle v_3 \rangle \left[ \frac{1}{t} ||v_1|| ||v_2|| + \frac{1}{t} \langle v_2, v_1 \rangle \right] = \frac{2}{t} ||v_1|| ||v_2|| ||v_3||,
\]
then, (4.24) is proved.

Next, noticing that
\[
\Delta P_t h = \langle \nabla^2 P_t h, I_d \rangle
\]
and \(I_d = \mathbb{E}[WW^T]\) with \(W \sim N(0, I_d)\), it follows from (4.24) that
\[
\left| \Delta(P_t h(x+v) - \Delta(P_t h(x)) \right| = \langle \nabla^2(P_t h(x+v) - \nabla^2(P_t h(x)), I_d \rangle_{HS} \rangle \\
\leq \frac{2}{t} \mathbb{E}[|W|^2] |v| = \frac{2d}{t} |v|.
\]
The proof is complete. \(\square\)

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