HARDY’S INEQUALITY FOR FUNCTIONS VANISHING ON A PART OF THE BOUNDARY

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Abstract. We develop a geometric framework for Hardy’s inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary.

1. Introduction

Hardy’s inequality is one of the classical items in analysis [27, 42]. Two milestones among many others in the development of the theory seem to be the result of Necas [41] that Hardy’s inequality holds on strongly Lipschitz domains and the insight of Maz’ya [38], [39, Ch. 2.3] that its validity depends on measure theoretic conditions on the domain. Rather recently, the geometric framework in which Hardy’s inequality remains valid was enlarged up to the frontiers of what is possible – as long as the boundary condition is purely Dirichlet, see [25, 28], compare also [3, 31, 48]. Moreover, over the last years it became manifest that Hardy’s inequality plays an eminent role in modern PDE theory, see e.g. [7, 46, 43, 2, 13, 9, 16, 23, 32, 34].

What has not been treated systematically is the case where only a part \( D \) of the boundary of the underlying domain \( \Omega \) is involved, reflecting the Dirichlet condition of the equation on this part while on \( \partial \Omega \setminus D \) other boundary conditions may be imposed, compare [11, 20, 2, 24, 8] including references therein. The aim of this paper is to set up a geometric framework for the domain \( \Omega \) and the Dirichlet boundary part \( D \) that allow to deduce the corresponding Hardy inequality

\[
\int_\Omega \left| \frac{u}{\text{dist}_D} \right|^p \, dx \leq c \int_\Omega |\nabla u|^p \, dx.
\]

As in the well established case \( D = \partial \Omega \) we in essence only require that \( D \) is \( l \)-thick in the sense of [28]. In our context this condition can be understood as an extremely weak compatibility condition between \( D \) and \( \partial \Omega \setminus D \).

Our strategy of proof is first to reduce to the case \( D = \partial \Omega \) by purely topological means, provided two major tools are applicable: An extension operator \( E : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^d) \), the subscript \( D \) indicating the subspace of those Sobolev functions which vanish on \( D \) in an appropriate sense, and a Poincaré inequality on \( W^{1,p}_D(\Omega) \). This abstract result is established in Section 5. In a second step in Sections 6 and 7 these partly implicit conditions are substantiated by more geometric assumptions that can be checked – more or less – by appearance. In particular, we prove that under the mere assumption that \( D \) is closed, every linear continuous extension operator \( W^{1,p}_D(\Omega) \to W^{1,p}(\mathbb{R}^d) \) that is constructed by the usual procedure of gluing together local extension operators preserves the Dirichlet condition on \( D \). This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained in [4].

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It is of course natural to ask, whether Hardy’s inequality also characterizes the space $W^{1,p}_D(\Omega)$, i.e. whether the latter is precisely the space of those functions $u \in W^{1,p}_D(\Omega)$ for which $u/\text{dist}_D$ belongs to $L^p(\Omega)$. Under very mild geometric assumptions we answer this question to the affirmative in Section 8.

Finally, in Section 9 we attend to the naive intuition that the part of $\partial \Omega$ that is far away from $D$ should only be circumstantial for the validity of Hardy’s inequality and in fact we succeed to weaken the previously discussed geometric assumptions considerably.

2. Notation

Throughout we work in Euclidean space $\mathbb{R}^d$, $d \geq 1$. We use $x$, $y$, etc. for vectors in $\mathbb{R}^d$ and denote the open ball in $\mathbb{R}^d$ around $x$ with radius $r$ by $B(x,r)$. The letter $c$ is reserved for generic constants that may change their value from occurrence to occurrence. Given $F \subset \mathbb{R}^d$ we write $\text{dist}_F$ for the function that measures the distance to $F$ and $\text{diam}(F)$ for the diameter of $F$.

In our main results on Hardy’s inequality we denote the underlying domain and its Dirichlet part by $\Omega$ and $D$. The various side results that are interesting in themselves and drop off on the way are identified by the use of $\Lambda$ and $E$ instead.

Next, let us introduce the common first-order Sobolev spaces of functions ‘vanishing’ on a part of the closure of the underlying domain that are most essential for the formulation of Hardy’s inequality.

Definition 2.1. If $\Lambda$ is an open subset of $\mathbb{R}^d$ and $E$ is a closed subset of $\Lambda$, then for $p \in [1, \infty[$ the space $W^{1,p}_E(\Lambda)$ is defined as the completion of $C_\infty^0 E(\Lambda) := \{ v | \Lambda : v \in C_\infty^0(\mathbb{R}^d), \text{supp}(v) \cap E = \emptyset \}$ with respect to the norm $v \mapsto (\int_\Lambda |\nabla v|^p + |v|^p \, dx)^{1/p}$. More generally, for $k \in \mathbb{N}$ we define $W^{k,p}_E(\Lambda)$ as the closure of $C_\infty^0(\Lambda)$ with respect to the norm $v \mapsto (\int_\Lambda \sum_{\sum_j=0}^k |D^{j}v|^p \, dx)^{1/p}$.

The situation we have in mind is of course when $\Lambda = \Omega$ and $E = D$ is the Dirichlet part $D$ of the boundary $\partial \Omega$.

As usual, the Sobolev spaces $W^{k,p}(\Lambda)$ are defined as the space of those $L^p(\Lambda)$ functions whose distributional derivatives up to order $k$ are in $L^p(\Lambda)$, equipped with the natural norm. Note that by definition $W^{k,p}_0(\Lambda) = W^{k,p}_D(\Lambda)$ but in general $W^{k,p}_\phi(\Lambda) \subsetneq W^{k,p}(\Lambda)$, cf. [39] Sec. 1.1.6]

3. Main results

The following version of Hardy’s inequality for functions vanishing on a part of the boundary is our main result. Readers not familiar with the measure theoretic concepts used to describe the regularity of the Dirichlet part $D$ may refer to Section 4.1 beforehand.

Theorem 3.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $D \subset \partial \Omega$ be a closed part of the boundary and $p \in [1, \infty[$. Suppose that the following three conditions are satisfied.

(i) The set $D$ is $l$-thick for some $l \in [d-p, d]$.
(ii) The space $W^{1,p}_D(\Omega)$ can be equivalently normed by $\| \nabla \cdot \|_{L^p(\Omega)}$.
(iii) There is a linear continuous extension operator $E : W^{1,p}_D(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$.

Then there is a constant $c > 0$ such that Hardy’s inequality

$$\int_\Omega \left| \frac{u}{\text{dist}_D} \right|^p \, dx \leq c \int_\Omega |\nabla u|^p \, dx \tag{3.1}$$

holds for all $u \in W^{1,p}_D(\Omega)$. 

Theorem 3.4. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain and \( p \in [1, \infty[ \). Let \( D \subset \partial \Omega \) be \( l \)-thick for some \( l \in ]d-p, d[ \) and assume that for every \( x \in \partial \Omega \setminus D \) there is an open neighborhood \( U_x \) of \( x \) such that \( \Omega \cap U_x \) is a \( W^{1,p} \)-extension domain. Then there is a constant \( c > 0 \) such that
\[
\int_{\Omega} \left| \frac{u}{\text{dist}_D} \right|^p \, dx \leq c \int_{\Omega} |\nabla u|^p \, dx, \quad u \in W^{1,p}_D(\Omega).
\]

Remark 3.3. The assumptions in the above theorem are met for all \( p \in ]1, \infty[ \) if \( D \) is a \( (d-1) \)-set and for every \( x \in \partial \Omega \setminus D \) there is an open neighborhood \( U_x \) of \( x \) such that \( \Omega \cap U_x \) is a Lipschitz-set with respect to \( D \). Concerning the third condition note carefully that we require the extension operator to preserve the Dirichlet boundary condition on \( D \). Whereas extension of Sobolev functions is a well-established business, the preservation of traces is much more delicate and we devote Section 6.3 to this problem.

It is interesting to remark that under geometric assumptions very similar to those in Theorem 3.2 the space \( W^{1,p}_D(\Omega) \) is the largest subspace of \( W^{1,p}(\Omega) \) in which Hardy’s inequality can hold. This is made precise by our third main result.

Theorem 3.4. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain and \( p \in [1, \infty[ \). Let \( D \subset \partial \Omega \) be porous and \( l \)-thick for some \( l \in ]d-p, d[ \). Finally assume that for every \( x \in \partial \Omega \setminus D \) there is an open neighborhood \( U_x \) of \( x \) such that \( \Omega \cap U_x \) is a \( W^{1,p} \)-extension domain. If \( u \in W^{1,p}(\Omega) \) is such that \( u/\text{dist}_D \in L^p(\Omega) \), then already \( u \in W^{1,p}_D(\Omega) \).

Remark 3.5. (i) The assumption on \( D \) are met if \( D \) is an \( l \)-set for some \( l \in ]d-p, d[ \), see Remark 4.4 below.

(ii) In the case \( D = \partial \Omega \) the conclusion of Theorem 3.4 is classical [10, Thm. V.3.4] and remains true without any assumptions on \( \partial \Omega \).

In Section 5 we give the proof of the general Hardy inequality from Theorem 3.1. The proofs of Theorem 3.2 and 3.4 are postponed to the end of Sections 6 and 7 respectively.

4. Preliminaries

4.1. Regularity concepts for the Dirichlet part. For convenience we recall the notions from geometric measure theory that are used to describe the regularity of the Dirichlet part \( D \) in Hardy’s inequality. For \( l \in ]0, \infty[ \) the \( l \)-dimensional Hausdorff measure of \( F \subset \mathbb{R}^d \) is
\[
\mathcal{H}_l(F) := \liminf_{\delta \to 0} \left\{ \sum_{j=1}^{\infty} \text{diam}(F_j)^l : F_j \subset \mathbb{R}^d, \text{diam}(F_j) \leq \delta, F \subset \bigcup_{j=1}^{\infty} F_j \right\}
\]
and its centered Hausdorff content is defined by
\[
\mathcal{H}_t^\infty(F) := \inf \left\{ \sum_{j=1}^{\infty} r_j^t : x_j \in F, r_j > 0, F \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\}.
\]

**Definition 4.1.** Let \( l \in ]0, \infty[ \). A non-empty compact set \( F \subset \mathbb{R}^d \) is called \( l \)-thick if there exist \( R > 0 \) and \( \gamma > 0 \) such that
\[
\mathcal{H}_t^\infty(F \cap B(x, r)) \geq \gamma r^l
\]
holds for all \( x \in F \) and all \( r \in ]0, R[ \). It is called \( l \)-set if there are two constants \( c_0, c_1 > 0 \) such that
\[
c_0 r^l \leq \mathcal{H}_t(F \cap B(x, r)) \leq c_1 r^l
\]
holds for all \( x \in F \) and all \( r \in ]0, 1[ \).

**Remark 4.2.**
(i) If \([4.1]\) holds for constants \( R, \gamma \), then for all \( S \geq R \) it also holds with \( R \) and \( \gamma \) replaced by \( S \) and \( \gamma R^\frac{1}{S-1} \), respectively. For more information on this notion of \( l \)-thick sets the reader can refer to \([28]\).
(ii) The notion of \( l \)-sets is due to \([22] \) Sec. II.1. It can be extended literally to arbitrary Borel sets \( F \), see \([22] \) Sec. VII.1.1.

**Definition 4.3.** A set \( F \subset \mathbb{R}^d \) is porous if for some \( \kappa \leq 1 \) the following statement is true: For every ball \( B(x, r) \) with \( x \in \mathbb{R}^d \) and \( 0 < r \leq 1 \) there is \( y \in B(x, r) \) such that \( B(y, \kappa r) \cap F = \emptyset \).

**Remark 4.4.** It is known that a set \( F \subset \mathbb{R}^d \) is porous if and only if its so-called Assouad dimension is strictly less than the space dimension \( d \), see \([33] \) Thm. 5.2. Recently it was shown in \([30]\) that this notion of dimension coincides with the one introduced by Aikawa, that is the infimum of all \( t > 0 \) for which there exist \( c_t > 0 \) such that
\[
\int_{B(x, r)} \text{dist}(x, F)^{t-d} \, dx \leq c_t r^t, \quad x \in F, r > 0.
\]
In particular, each \( l \)-set, \( l \in ]0, d[ \), has Aikawa dimension equal to \( l \) and thus is porous \([29] \) Lem. 2.1.

For a later use we include a proof of the following two elementary facts. We remark that the first lemma is also implicit in \([0] \) Lem. 2.

**Lemma 4.5.** Let \( l \in ]0, \infty[ \). If \( F \subset \mathbb{R}^d \) is a compact \( l \)-set, then there are constants \( c_0, c_1 > 0 \) such that
\[
c_0 r^l \leq \mathcal{H}_t^\infty(F \cap B(x, r)) \leq c_1 r^l
\]
holds for all \( r \in ]0, 1[ \) and all \( x \in F \). In particular, \( F \) is \( l \)-thick.

**Proof.** We prove \( \mathcal{H}_t^\infty(A) \leq \mathcal{H}_t(A) \leq c \mathcal{H}_t^\infty(A) \) for all non-empty Borel subsets \( A \subset F \).

First, fix \( \varepsilon > 0 \) and let \( \{A_j\}_{j \in \mathbb{N}} \) be a covering of \( A \) by sets with diameter at most \( \varepsilon \). If \( A_j \cap A = \emptyset \), then \( A_j \) is contained in an open ball \( B_j \) centered in \( A \) and radius such that \( r_j^l = \text{diam}(A_j)^l + \varepsilon 2^{-j} \). The so-obtained countable covering \( \{B_j\} \) of \( A \) satisfies
\[
\sum_{j \in \mathbb{N}, A_j \cap A \neq \emptyset} \text{diam}(A_j)^l \geq \sum_{j \in \mathbb{N}, A_j \cap A \neq \emptyset} (r_j^l - \varepsilon 2^{-j}) \geq \mathcal{H}_t^\infty(A) - \varepsilon.
\]

Taking the infimum over all such coverings \( \{A_j\}_{j \in \mathbb{N}} \) and passing to the limit \( \varepsilon \to 0 \) afterwards, \( \mathcal{H}_t^\infty(A) \leq \mathcal{H}_t(A) \) follows. Conversely, let \( \{B_j\}_{j \in \mathbb{N}} \) be a covering of \( A \) by open balls with radii \( r_j \) centered in \( A \). If \( r_j \leq 1 \), then \( \mathcal{H}_t(F \cap B_j) \leq c r_j^l \) since by assumption \( F \) is an \( l \)-set, and if \( r_j > 1 \),
then certainly $\mathcal{H}_t(F \cap B_j) \leq \mathcal{H}_t(F) r_j^l$. Note carefully that $0 < \mathcal{H}_t(F) < \infty$ holds for $F$ can be covered by finitely many balls with radius $1$ centered in $F$. Altogether,

$$\sum_{j=1}^{\infty} r_j^l \geq c \sum_{j=1}^{\infty} \mathcal{H}_t(F \cap B_j) \geq c \mathcal{H}_t\left(F \cap \bigcup_{j=1}^{\infty} B_j\right) \geq c \mathcal{H}_t(A).$$

Passing to the infimum, $\mathcal{H}^\infty_t(A) \geq c \mathcal{H}_t(A)$ follows. \hfill \Box

**Lemma 4.6.** If $F \subset \mathbb{R}^d$ is $l$-thick, then it is $m$-thick for every $m \in ]0,l[$.  

**Proof.** Inspecting the definition of thick sets, the claim turns out to be a direct consequence of the inequality

$$\sum_{j=1}^{N} r_j^m \geq \left( \sum_{j=1}^{N} r_j^l \right)^{m/l}$$

for positive real numbers $r_1, \ldots, r_N$. \hfill \Box

### 4.2. Quasieverywhere defined functions. The results of Sections 4-8 rely on deep insights from potential theory and we shall recall the necessary notions beforehand. For further background we refer e.g. to [1].

**Definition 4.7.** Let $\alpha > 0$, $p \in ]1, \infty[\]$ and let $F \subset \mathbb{R}^d$. Denote by $G_\alpha := F^{-1}(1+|\xi|^2)^{-\alpha/2}$ the Bessel kernel of order $\alpha$. Then

$$C_{\alpha,p}(F) := \inf \left\{ \int_{\mathbb{R}^d} |f|^p : f \geq 0 \text{ on } \mathbb{R}^d \text{ and } G_{\alpha} * f \geq 1 \text{ on } F \right\}$$

is called $(\alpha,p)$-capacity of $F$. The corresponding Bessel potential space is

$$H^{\alpha,p}(\mathbb{R}^d) := \{ G_{\alpha} * f : f \in L^p(\mathbb{R}^d) \} \quad \text{with norm} \quad \| G_{\alpha} * f \|_{H^{\alpha,p}(\mathbb{R}^d)} = \| f \|_p.$$

It is well-known that for $k \in \mathbb{N}$ the spaces $H^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$ coincide up to equivalent norms [45 Sec. 2.3.3]. The capacities $C_{\alpha,p}$ are outer measures on $\mathbb{R}^d$ [1 Sec. 2.3]. A property that holds true for all $x$ in some set $E \subset \mathbb{R}^d$ but those belonging to an exceptional set $F \subset E$ with $C_{\alpha,p}(F) = 0$ is said to be true $(\alpha,p)$-quasieverywhere on $E$, abbreviated $(\alpha,p)$-q.e. A property that holds true $(\alpha,p)$-q.e. also holds true $(\beta,p)$-q.e. if $\beta < \alpha$. This is an easy consequence of [1 Prop. 2.3.13]. A more involved result in this direction is the following [1 Thm. 5.5.1].

**Lemma 4.8.** Let $\alpha, \beta > 0$ and $1 < p, q < \infty$ be such that $\beta q < \alpha p < d$. Then each $C_{\alpha,p}$-nullset also is a $C_{\beta,q}$-nullset.

There is also a close connection between capacities and Hausdorff measures, cf. [1 Ch. 5.] for an exhaustive discussion. Most important for us is the following comparison theorem. In the case $p \in ]1, d] \]$ this is proved in [1 Sec. 5] and if $p \in ]d, \infty[, \]$ then the result follows directly from [1 Prop. 2.6.1].

**Theorem 4.9** (Comparison Theorem). Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p < \infty$. Then every $C_{\alpha,p}$-nullset is also a $\mathcal{H}_l$- and thus a $\mathcal{H}^\infty_l$-nullset.

Bessel capacities naturally occur when studying convergence of average integrals for Sobolev functions. In fact, if $\alpha > 0$, $p \in ]1, \frac{d}{\alpha}]$ and $u \in H^{\alpha,p}(\mathbb{R}^d)$, then $(\alpha,p)$-quasievery $y \in \mathbb{R}^d$ is a Lebesgue point for $u$ in the $L^p$-sense, that is

$$\lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u(x) \, dx =: u(y)$$

and

$$\lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} |u(x) - u(y)|^p \, dx = 0$$
hold [1] Thm. 6.2.1]. The \((\alpha, p)\)-quasieverywhere defined function \(u\) reproduces \(u\) within its \(H^{\alpha, p}\)-class. It gives rise to a meaningful \((\alpha, p)\)-quasieverywhere defined restriction \(u|_E := u|_E\) of \(u\) to \(E\) whenever \(E\) has non-vanishing \((\alpha, p)\)-capacity. For convenience we agree upon that \(u|_E = 0\) is true for all \(u \in H^{\alpha, p}(\mathbb{R}^d)\) if \(E\) has zero \((\alpha, p)\)-capacity. Note also that these results remain true if \(p \in [\frac{d}{d-1}, \infty]\); since in this case \(u\) has a Hölder continuous representative \(u\) which then satisfies (4.2) and (4.3) for every \(y \in \mathbb{R}^d\).

We obtain an alternate definition for Sobolev spaces with partially vanishing traces.

**Definition 4.10.** Let \(k \in \mathbb{N}, p \in [1, \infty]\) and \(E \subseteq \mathbb{R}^d\) be closed. Define

\[
W^k_p(E) := \left\{ u \in W^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds for all multiindices } \beta, 0 \leq |\beta| \leq k-1 \right\}
\]

and equip it with the \(W^{k,p}(\mathbb{R}^d)\)-norm.

The following theorem of Hedberg and Wolff is also known as the \((k, p)\)-synthesis.

**Theorem 4.11 ([1] Thm. 9.1.3).** The spaces \(W^k_p(E)\) and \(W^k_p(\mathbb{R}^d)\) coincide whenever \(k \in \mathbb{N}, p \in [1, \infty]\) and \(E \subseteq \mathbb{R}^d\) is closed.

Hedberg and Wolff’s theorem manifests the use of capacities in the study of traces of Sobolev functions. However, if one invests more on the geometry of \(E\), e.g. if one assumes that it is an \(l\)-set, then by the subsequent recent result of Brewster, Mitrea, Mitrea and Mitrea capacities can be replaced by the \(l\)-dimensional Hausdorff measure at each occurrence.

**Theorem 4.12 ([1] Thm. 4.4, Cor. 4.5)).** Let \(k \in \mathbb{N}, p \in [1, \infty]\) and let \(E \subseteq \mathbb{R}^d\) be closed and additionally an \(l\)-set for some \(l \in [d-p, d]\). Then

\[
W^k_p(E) = W^k_p(\mathbb{R}^d) = \left\{ u \in W^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds } \mathcal{H}^{d-1}\text{-a.e. on } E \right\}
\]

for all multiindices \(\beta, 0 \leq |\beta| \leq k-1\),

where on the right-hand side \(D^\beta u|_E = 0\) means, as before, that for \(\mathcal{H}^{d-1}\text{-almost every } y \in E\) the average integrals \(\frac{1}{|B(y, r)|} \int_{B(y, r)} D^\beta u(x) \, dx\) vanish in the limit \(r \to 0\).

5. **Proof of Theorem 3.1**

We will deduce Theorem 3.1 from the following proposition that states the assertion in the case \(D = \partial \Omega\).

**Proposition 5.1 ([25], see also [25]).** Let \(\Omega \subseteq \mathbb{R}^d\) be a bounded domain and let \(p \in [1, \infty]\). If \(\partial \Omega\) is \(l\)-thick for some \(l \in [d-p, d]\), then Hardy’s inequality is satisfied for all \(u \in W^{1,p}_0(\Omega)\), i.e. (3.1) holds with \(\Omega\) replaced by \(\Omega\) and \(D\) by \(\partial \Omega\).

Below we will reduce to the case \(D = \partial \Omega\) by purely topological means, so that we can apply Proposition 5.1 afterwards. We will repeatedly use the following topological fact.

\((\blacksquare)\) Let \(\{M_\lambda\}_\lambda\) be a family of connected subsets of a topological space. If \(\bigcap_\lambda M_\lambda \neq \emptyset\), then \(\bigcup_\lambda M_\lambda\) is again connected.

As required in Theorem 3.1 let now \(\Omega \subseteq \mathbb{R}^d\) be a bounded domain and let \(D\) be a closed part of \(\partial \Omega\). Then choose an open ball \(B \supseteq \overline{\Omega}\) that, in what follows, will be considered as the relevant topological space. Consider

\[
C := \{ M \subset B \setminus D : M \text{ open, connected and } \Omega \subset M \}
\]

and for the rest of the proof put

\[
\Omega_\bullet := \bigcup_{M \in C} M.
\]
In the subsequent lemma we collect some properties of \( \Omega_* \). Our proof here is not the shortest possible, cf. [3] Lem. 6.4 but it has, however, the advantage to give a description of \( \Omega_* \) as the union of \( \Omega \) the boundary part \( \partial \Omega \setminus D \) and those connected components of \( B \setminus \overline{\Omega} \) whose boundary does not consist only of points from \( D \). This completely reflects the naive geometric intuition.

**Lemma 5.2.** It holds \( \Omega \subseteq \Omega_* \subseteq B \). Moreover, \( \Omega_* \) is open and connected and \( \partial \Omega_* = D \) in \( B \).

**Proof.** The first assertion is obvious. By construction \( \Omega_* \) is open. Since all elements from \( C \) contain \( \Omega \) the connectedness of \( \Omega_* \) follows by \( \Box \). It remains to show \( \partial \Omega_* = D \).

Let \( x \in D \). Then \( x \) is an accumulation point of \( \Omega \) and, since \( \Omega \subseteq \Omega_* \), of \( \Omega_* \). On the other hand, \( x \not\in \Omega_* \) by construction. This implies \( x \in \partial \Omega_* \) and so \( D \subseteq \partial \Omega_* \).

In order to show the inverse inclusion, we first show that points from \( \partial \Omega \setminus D \) cannot belong to \( \partial \Omega_* \). Indeed, since \( D \) is closed, for \( x \in \partial \Omega \setminus D \) there is a ball \( B_x \subseteq B \) around \( x \) that does not intersect \( D \). Since \( x \) is a boundary point of \( \Omega \), we have \( B_x \cap \Omega \neq \emptyset \). Both \( \Omega \) and \( B_x \) are connected, so \( \Box \) yields that \( \Omega \cup B_x \) is connected. Moreover, this set is open, contains \( \Omega \) and avoids \( D \), so it belongs to \( C \) and we obtain \( \Omega \cup B_x \subseteq \Omega_* \). This in particular yields \( x \in \Omega_* \), so \( x \notin \partial \Omega_* \) since \( \Omega_* \) is open.

Summing up, we already know that \( x \in \overline{\Omega} \) belongs to \( \partial \Omega_* \) if and only if \( x \in D \). So, it remains to make sure that no point from \( B \setminus \overline{\Omega} \) belongs to \( \partial \Omega_* \).

As \( B \setminus \overline{\Omega} \) is open, it splits up into its open connected components \( Z_0, Z_1, Z_2, \ldots \). There are possibly only finitely many such components but at least one. We will show in a first step that for all these components it holds \( \partial Z_j \subseteq \partial \Omega \). This allows to distinguish the two cases \( \partial Z_j \subseteq D \) and \( \partial Z_j \cap (\partial \Omega \setminus D) \neq \emptyset \). In Steps 2 and 3 we will then complete the proof by showing that in both cases \( Z_j \) does not intersect \( \partial \Omega_* \).

**Step 1:** \( \partial Z_j \subseteq \partial \Omega \) for all \( j \). First note that \( \partial Z_j \cap \Omega = \emptyset \) for all \( j \). Indeed, assuming this set to be none-empty and investing that \( \Omega \) is open, we find that the set \( \partial Z_j \cap \Omega \) cannot be empty either and this contradicts the definition of \( Z_j \).

Now, to prove the claim of Step 1, assume by contradiction that, for some \( j \), there is a point \( x \in \partial Z_j \) that does not belong to \( \partial \Omega \). By the observation above we then have \( x \notin \overline{\Omega} \) and consequently there is a ball \( B_x \) around \( x \) that does not intersect \( \overline{\Omega} \). Now, set \( B_x \cup Z_j \) is connected thanks to \( \Box \), avoids \( \overline{\Omega} \) and includes \( Z_j \) properly. However, this contradicts the property of \( Z_j \) to be a connected component of \( B \setminus \overline{\Omega} \).

**Step 2:** If \( \partial Z_j \subseteq D \), then \( \overline{\Omega_*} \cap Z_j = \emptyset \). We first note that it suffices to show \( \Omega_* \cap Z_j = \emptyset \). In fact, due to \( \overline{\Omega_*} = \partial \Omega_* \cup \overline{\Omega_*} \) we then get \( \Omega_* \cap Z_j = \emptyset \) since \( Z_j \) is open.

So, let us assume there is some \( x \in \Omega_* \cap Z_j \). Then \( \Omega_* \cap Z_j \) is connected due to \( \Box \). By assumption we have \( \partial Z_j \subseteq D \) and by construction the sets \( Z_j \) and \( \Omega_* \) are both disjoint to \( D \). So we can infer that \( \partial Z_j \cap (\Omega_* \cup Z_j) = \emptyset \) and this allows us to write

\[
\Omega_* \cup Z_j = (\Omega_* \cap Z_j) \cup (Z_j \cup (B \setminus \overline{Z_j})) = Z_j \cup (\Omega_* \cap (B \setminus \overline{Z_j})).
\]

This is a decomposition of \( \Omega_* \cup Z_j \) into two open and mutually disjoint sets, so if we can show that both are nonempty then this yields a contradiction to the connectedness of \( \Omega_* \cup Z_j \) and the claim of Step 2 follows. Indeed, even find

\[
\Omega_* \cap (B \setminus \overline{Z_j}) = \Omega_* \setminus Z_j = \Omega_* \setminus (\partial Z_j \cup Z_j) \subseteq \Omega \setminus (D \cup \overline{Z_j}) = \Omega \neq \emptyset,
\]

since both \( D \) and \( Z_j \) do not intersect \( \Omega \).

**Step 3:** If \( \partial Z_j \cap (\partial \Omega \setminus D) \neq \emptyset \), then \( Z_j \subseteq \Omega_* \). Let \( x \in \partial Z_j \cap (\partial \Omega \setminus D) \), and let \( B_x \) be a ball around \( x \) that does not intersect \( D \). The point \( x \) is a boundary point of \( Z_j \), so \( B_x \cap Z_j \neq \emptyset \) and we obtain that \( B_x \cup Z_j \) is connected by \( \Box \). By the same argument, also the set \( B_x \cup \Omega \) is connected and putting these two together a third reiteration of the argument yields that
(B_z ∪ Ω) ∪ (B_z ∪ Z_j) = Ω ∪ B_z ∪ Z_j is again connected. This last set is open and does not intersect D, so it belongs to C and we end up with Ω ∪ B_z ∪ Z_j ⊆ Ω. In particular we have Z_j ⊆ Ω. □

**Remark 5.3.** Conversely, it can be shown that the asserted properties characterize Ω* uniquely in the sense that if an open, connected subset Ξ ⊆ R of B additionally satisfies ∂Ξ = D, then necessarily Ξ = Ω. In fact, since Ξ ∩ D = ∅ one has Ξ ⊆ Ω*, due to the definition of Ω*. In order to obtain the inverse inclusion we write

\[(\Omega_*) \cap \Xi = (\Omega_*) \cap (\Xi \cap \partial \Xi) \cup (\Omega_* \cap (B \setminus \Xi)) = \Xi \cup (\Omega_* \cap (B \setminus \Xi)),\]

since Ω* ∩ ∂Ξ = Ω ∩ D = ∅. Both Ξ = Ξ ∩ Ω* and Ω* ∩ (B \ Ξ) are open in Ω*, and Ξ ∩ Ω is non-empty. Since Ω* is connected and Ξ = Ξ ∩ Ω* is clearly disjoint to Ω* ∩ (B \ Ξ), this latter set must be empty. Thus, (5.1) gives Ξ = Ω*.

**Corollary 5.4.** Consider Ω* as a subset of R^d. Then Ω* is open and connected. Moreover, either ∂Ω* = D or ∂Ω* = D ∪ ∂B.

**Proof.** It is clear that Ω* remains open. Assume that Ω* is not connected. Then there are disjoint open sets U, V ⊆ R^d such that Ω* = U ∪ V. However, the property Ω ⊆ B then gives Ω* = Ω ∩ B = (U ∩ B) ∪ (V ∩ B), where U ∩ B and V ∩ B are open in B and disjoint to each other. This contradicts Lemma 5.2.

For the last assertion consider an annulus A ⊆ B that is adjacent to ∂B and does not intersect ∂B. Let Z_j be the connected component of B \ Ω that contains A. We distinguish again the two cases of Step 2 and Step 3 in the proof of Lemma 5.2. If ∂Z_j ⊆ D, we have shown in Step 2 that Z_j is disjoint to Ω* and this implies ∂Ω* = ∂Ω* ∩ B = D. In the second case, we infer from Step 3 in the above proof that A ⊆ Z_j ⊆ Ω and this implies ∂Ω* = D ∪ ∂B.

Let us now conclude the proof of Theorem 3.1. We first observe that in both cases appearing in Corollary 5.4 the set ∂Ω* is m-thick for some m ∈ ]d - p, d - 1[. In fact, D is l-thick for some l ∈ ]d - p, d[ by assumption and using its local representation as the graph of a Lipschitz function, it can easily be checked that ∂B is a (d - 1)-set, hence (d - 1)-thick owing to Lemma 4.5. The claim follows from Lemma 4.6. Altogether, Proposition 5.1 applies to our special choice of Ω*.

Now, let E be the extension operator provided by Assumption (iii) of Theorem 3.1. In view of Corollary 5.4 we can define an extension operator E_{\Omega*} : W^{1,p}_D(Ω) → W^{1,p}_0(Ω*) as follows: If ∂Ω* = D, then we put E_{\Omega*} := E|_{\Omega*} and if ∂Ω* = D ∪ ∂B, then we choose η ∈ C^\infty(\overline{B}) with the property η ≡ 1 on \overline{Ω} and put E_{\Omega*} := (ηE)|_{\Omega*}. This allows us to apply Proposition 5.1 to the functions E_{\Omega*}u ∈ W^{1,p}_0(Ω*), where u is taken from W^{1,p}_D(Ω). With a final help of Assumption (ii) in Theorem 3.1, this gives

\[
\int_{\Omega} \left| \frac{u}{\partial D} \right|^p \, dx \leq \int_{\Omega} \left| \frac{u}{\partial \Omega} \right|^p \, dx \leq \int_{\Omega*} \left| E_{\Omega*}u \right|^p \, dx \leq c \int_{\Omega*} \left| \nabla (E_{\Omega*}u) \right|^p \, dx
\]

\[
\leq c \| E_{\Omega*}u \|_{W^{1,p}_0(\Omega*)}^p \leq c \| u \|_{W^{1,p}_D(\Omega)}^p \leq c \int_{\Omega} |\nabla u|^p \, dx
\]

for all u ∈ W^{1,p}_D(Ω) and the proof is complete.

**Remark 5.5.**

(i) At the first glance one might think that Ω* could always be taken as B \ D. The point is that this set need not be connected, as the following example shows. Take Ω = \{ x : 1 < |x| < 2 \} and D = \{ x : |x| = 1 \} ∪ \{ x : |x| = 2, x_1 ≥ 0 \}. Obviously, if a ball B contains Ω, then B \ D cannot be connected. In the spirit of Lemma 5.2 the set Ω* has here to be taken as B \ (D ∪ \{ x : |x| < 1 \}). Thus, the somewhat subtle, topological considerations above cannot be avoided in general.

(ii) One might suggest that the procedure of this work is not limited to the proof of Hardy’s inequality in the non-Dirichlet case. Possibly the combination of an application of the extension operator E/E_{\Omega*} and the construction of Ω* may serve for the reduction of other
problems on function spaces related to mixed boundary conditions to the pure Dirichlet case.

Finally, instead of its $l$-thickness we can also require that $D$ is an $l$-set – a condition that promises to be more common to applications. One access to such a result is to prove that the $l$-property of $\partial \Omega$ implies the $p$-fatness of $\mathbb{R}^d \setminus \Omega$ – a result which was first obtained by Maz’ya [40]. Knowing this, Hardy’s inequality may then be deduced from the results in [31] or [48]. Our approach is quite different and simply rests on Proposition 5.1 and Lemma 4.5. So we can record the following.

**Corollary 5.6.** The assertion of Theorem 3.1 remains valid if instead of its $l$-thickness we require that $D$ is an $l$-set.

6. **The extension operator**

In this section we discuss the second condition in our main result Theorem 3.1, that is the extendability for $W_{1,p}^D(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension.

- **Dirichlet cracks can be removed:** We open the possibility of passing from $\Omega$ to another domain $\Omega^\star$ with a reduced Dirichlet boundary part, while $\Gamma = \partial \Omega \setminus D$ remains part of $\partial \Omega^\star$. In most cases this improves the boundary geometry in the sense of Sobolev extendability, see the example in the following Figure.

![Figure 1](image)

**Figure 1.** The set $\Sigma$ does not belong to $\Omega$, and carries – together with the striped parts – the Dirichlet condition.

- **Sobolev extendability is a local property:** We show that only the local geometry of the domain around the boundary part $\Gamma$ plays a role for the existence of an extension operator.

- **Preservation of traces:** We prove under very general geometric assumptions that the extended functions do have the adequate trace behavior on $D$ for every extension operator.

We believe that these results are of independent interest and therefore decided to directly present them for higher-order Sobolev spaces $W_{k,p}^E$. In the end we review some feasible commonly used geometric conditions which together with our abstract principles really imply the corresponding extendability.

6.1. **Dirichlet cracks can be removed.** As in Figure 1 there may be boundary parts which carry a Dirichlet condition and belong to the inner of the closure of the domain under consideration. Then one can extend the functions on $\Lambda$ by 0 to such a boundary part, thereby enlarging the domain and simplifying the boundary geometry. In the following we make this precise.

**Lemma 6.1.** Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $E \subset \partial \Lambda$ be closed. Define $\Lambda^\star$ as the interior of the set $\Lambda \cup E$. Then the following hold true.
(i) The set $\Lambda\star$ is again a domain, $\Xi := \partial \Lambda \setminus E$ is a (relatively) open subset of $\partial \Lambda\star$ and $\partial \Lambda\star = \Xi \cup (E \cap \partial \Lambda\star)$.

(ii) Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Extending functions from $W^k_E(\Lambda)$ by 0 to $\Lambda\star$, one obtains an isometric extension operator $\text{Ext}(\Lambda, \Lambda\star)$ from $W^k_E(\Lambda)$ onto $W^k_E(\Lambda\star)$.

Proof. 

(i) Due to the connectedness of $\Lambda$ and the set inclusion $\Lambda \subset \Lambda\star \subset \overline{\Lambda}$, the set $\Lambda\star$ is also connected, and, hence a domain. Obviously, one has $\overline{\Lambda\star} = \overline{\Lambda}$. This, together with the inclusion $\Lambda \subset \Lambda\star$ leads to $\partial \Lambda\star \subset \partial \Lambda$. Since $\Xi \cap \Lambda\star = \emptyset$, one gets $\Xi \subset \partial \Lambda\star$. Furthermore, $\Xi$ was relatively open in $\partial \Lambda$, so it is relatively open also in $\partial \Lambda\star$.

The last asserted equality follows from $\partial \Lambda\star = (\Xi \cap \partial \Lambda\star) \cup (E \cap \partial \Lambda\star)$ and $\Xi \subset \partial \Lambda\star$.

(ii) Consider any $\psi \in C^\infty(\mathbb{R}^d)$ and its restriction $\psi|_\Lambda$ to $\Lambda$. Since the support of $\psi$ has a positive distance to $E$, one may extend $\psi|_\Lambda$ by 0 to the whole of $\Lambda\star$ without destroying the $C^\infty$-property. Thus, this extension operator provides a linear isometry from $C^\infty(\Lambda)$ onto $C^\infty(\Lambda\star)$ (if both are equipped with the $W^{k,p}$-norm). This extends to a linear extension operator $\text{Ext}(\Lambda, \Lambda\star)$ from $W^k_E(\Lambda)$ onto $W^k_E(\Lambda\star)$, see the two following commutative diagrams:

\[ \begin{array}{c}
\text{C}^{\infty}(\mathbb{R}^d) \quad \text{restrict}_{\text{g} \cdot \Delta \rightarrow \Lambda\star} \quad \text{C}^{\infty}(\Lambda) \\
\downarrow \text{restrict}_{\text{g} \cdot \Delta \rightarrow \Lambda} \quad \downarrow \text{extend}_{\text{g} \cdot \Delta \rightarrow \Lambda\star} \\
\text{C}^{\infty}(\Lambda\star) \\
\end{array} \]

\[ \begin{array}{c}
W^k_E(\mathbb{R}^d) \quad \text{restrict}_{\text{g} \cdot \Delta \rightarrow \Lambda\star} \quad W^k_E(\Lambda) \\
\downarrow \text{restrict}_{\text{g} \cdot \Delta \rightarrow \Lambda} \quad \downarrow \text{extend}_{\text{g} \cdot \Delta \rightarrow \Lambda\star} \\
W^k_E(\Lambda\star) \\
\end{array} \]

\[ \square \]

Remark 6.2. 

(i) Note that no assumptions on $E$ beside closedness are necessary.

(ii) Having extended the functions from $\Lambda$ to $\Lambda\star$, the 'Dirichlet crack' $\Sigma$ in Figure 1 has vanished, and one ends up with the whole cube. Here the problem of extending Sobolev functions is almost trivial. We suppose that this is the generic case – at least for problems arising in applications.

The above considerations suggest the following procedure: extend the functions from $W^k_E(\Lambda)$ first to $\Lambda\star$, and afterwards to the whole of $\mathbb{R}^d$. The next lemma shows that this approach is universal.

Lemma 6.3. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain, let $E \subset \partial \Lambda$ be closed and as before define $\Lambda\star$ as the interior of the set $\Lambda \cup E$. Every linear, continuous extension operator $\mathfrak{F} : W^k_E(\Lambda) \rightarrow W^k_E(\mathbb{R}^d)$ factorizes as $\mathfrak{F} = \mathfrak{F}\star \text{Ext}(\Lambda, \Lambda\star)$ through a linear, continuous extension operator $\mathfrak{F}\star : W^k_E(\Lambda\star) \rightarrow W^k_E(\mathbb{R}^d)$.

Proof. Let $\mathfrak{S}$ be the restriction operator from $W^k_E(\Lambda\star)$ to $W^k_E(\Lambda)$. Then we define, for every $f \in W^k_E(\Lambda\star)$, $\mathfrak{S}\star f := \mathfrak{S}\mathfrak{F}f$. We obtain $\mathfrak{F}\star \text{Ext}(\Lambda, \Lambda\star) = \mathfrak{S}\mathfrak{S}\star \text{Ext}(\Lambda, \Lambda\star) = \mathfrak{S}\mathfrak{F}$. This shows that the factorization holds algebraically. However, one also has

\[
\|\mathfrak{F}\star \text{Ext}(\Lambda, \Lambda\star)f\|_{W^k_E(\mathbb{R}^d)} = \|\mathfrak{F}\mathfrak{F}\star f\|_{W^k_E(\mathbb{R}^d)} \leq \|\mathfrak{F}\|_{L(W^k_E(\Lambda); W^k_E(\mathbb{R}^d))} \|f\|_{W^k_E(\Lambda)}
\]

\[ = \|\mathfrak{F}\|_{L(W^k_E(\Lambda\star); W^k_E(\mathbb{R}^d))} \|\text{Ext}(\Lambda, \Lambda\star)f\|_{W^k_E(\Lambda\star)}. \quad \square \]

Having extended the functions already to $\Lambda\star$, one may proceed as follows: Since $E$ is closed, so is $E\star := E \cap \partial \Lambda\star$. So, one can now consider the space $W^1_E(\Lambda\star)$ and has the task to establish an extension operator for this space – while afterwards one has to take into account that the original functions were 0 also on the set $E \cap \Lambda\star$ and have not been altered by the extension operator thereon. However, note carefully that $E\star := E \cap \partial \Lambda\star$ may have a worse geometry than $E$. For
example, take Figure 2 and suppose that this time only $\Sigma$ forms the whole Dirichlet part of the boundary. Then $E$ is a $(d-1)$-set whereas even $H_{d-1}(E)$ = 0 holds.

To sum up, if one aims at an extension operator $E : W^{k,p}_E(\Lambda) \to W^{k,p}_E(\mathbb{R}^d)$, one is free to modify the domain $\Lambda$ to $\Lambda_\star$. In most cases this improves the local geometry concerning Sobolev extensions and we do not have examples where the situation gets worse. Though we do not claim that this is, in a whatever precise sense, the generic case.

6.2. Sobolev extendability is a local property. Below, we make precise in which sense Sobolev extendability is a local property. We set up the following notation.

**Definition 6.4.** A domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$-extension domain for given $k \in \mathbb{N}$ and $p \in [1,\infty]$ if there exists a continuous extension operator $E_{k,p} : W^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$. If $\Lambda$ is a $W^{k,p}$-extension domain for all $k \in \mathbb{N}$ and all $p \in [1,\infty]$ in virtue of the same extension operator, then $\Lambda$ is a universal Sobolev extension domain.

**Proposition 6.5.** Let $k \in \mathbb{N}$ and $p \in [1,\infty]$. Let $\Lambda$ be a bounded domain and let $E$ be a closed part of its boundary. Assume that for every $x \in \partial \Lambda \setminus \overline{E}$ there is an open neighborhood $U_x$ of $x$ such that $\Lambda \cap U_x$ is a $W^{k,p}$-extension domain. Then there is a continuous extension operator $E_{k,p} : W^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$.

Moreover, if each local extension operator $E_{k,p}$ maps the space $W^{k,p}_E(\Lambda \cap U_x)$ into $W^{k,p}(\mathbb{R}^d)$, where $E_x := \overline{E \cap U_x} \subset \partial(\Lambda \cap U_x)$, then also $E_{k,p} : W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d)$.

**Proof.** For the construction of the extension operator let for every $x \in \partial \Lambda \setminus \overline{E}$ denote $U_x$ the open neighborhood of $x$ from the assumption. Let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcovering of $\partial \Lambda \setminus \overline{E}$. Since the compact set $\partial \Lambda \setminus \overline{E}$ is contained in the open set $\bigcup U_{x_j}$, there is an $\varepsilon > 0$, such that the sets $U_{x_1}, \ldots, U_{x_n}$, together with the open set $U := \{y \in \mathbb{R}^d : \text{dist}(y, \partial \Lambda \setminus \overline{E}) > \varepsilon\}$, form an open covering of $\Lambda$. Hence, on $\Lambda$ there is a $C_0^\infty$-partition of unity $\eta, \eta_1, \ldots, \eta_n$, with the properties $\text{supp}(\eta) \subset U$, $\text{supp}(\eta_j) \subset U_{x_j}$.

Assume $\psi \in C_0^\infty(\Lambda)$. Then $\eta \psi \in C_0^\infty(\Lambda)$. If one extends this function by 0 outside of $\Lambda$, then one obtains a function $\varphi \in C_0^\infty(\mathbb{R}^d) \subset C_0^\infty(\mathbb{R}^d) \subset W^{k,p}_E(\mathbb{R}^d)$ with the property $\|\varphi\|_{W^{k,p}(\mathbb{R}^d)} = \|\eta \psi\|_{W^{k,p}(\Lambda)}$.

Now, for every fixed $j \in \{1, \ldots, n\}$, consider the function $\psi_j := \eta_j \psi \in W^{k,p}(\Lambda \cap U_{x_j})$. Since $\Lambda \cap U_{x_j}$ is a $W^{k,p}$-extension domain by assumption, there is an extension of $\varphi_j$ to a $W^{k,p}(\mathbb{R}^d)$-function $\varphi_j$ together with an estimate $\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \leq c \|\psi_j\|_{W^{k,p}(\Lambda \cap U_{x_j})}$, where $c$ is independent from $\psi_j$. Clearly, one has a priori no control on the behavior of $\varphi_j$ on the set $\Lambda \setminus U_{x_j}$. In particular $\varphi_j$ may there be nonzero and, hence, cannot be expected to coincide with $\eta_j \psi$ on the whole of $\Lambda$. In order to correct this, let $\zeta_j \in C_0^\infty(\mathbb{R}^d)$-function which is identically 1 on $\text{supp}(\eta_j)$ and has its support in $U_{x_j}$. Then $\eta_j \varphi_j$ equals $\zeta_j \varphi_j$ on all of $\Lambda$. Consequently, $\zeta_j \varphi_j$ really is an extension of $\eta_j \psi$ to the whole of $\mathbb{R}^d$ which, additionally, satisfies the estimate

$$\|\zeta_j \varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \leq c \|\varphi\|_{W^{k,p}(\mathbb{R}^d)} \leq c \|\eta \psi\|_{W^{k,p}(\Lambda \cap U_{x_j})} \leq c \|\psi\|_{W^{k,p}(\Lambda)},$$

where $c$ is independent from $\psi$. Thus, defining $E_{k,p}(\psi) = \varphi + \sum_j \zeta_j \varphi_j$ one gets a linear, continuous extension operator from $C_0^\infty(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$. By density, $E_{k,p}$ uniquely extends to a linear, continuous operator $E_{k,p} : W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d)$.

Finally, assume that the local extension operators map $W^{k,p}_E(\Lambda \cap U_{x_j})$ into $W^{k,p}(\mathbb{R}^d)$. Using the notation above, this means that $\varphi_j$ can be approximated in $W^{k,p}(\mathbb{R}^d)$ by a sequence from...
$C^\infty_\alpha(\mathbb{R}^d)$. Since $\zeta_j$ is supported in $U_{x_j}$, multiplication by $\zeta_j \in C^\infty_\alpha(\mathbb{R}^d)$ maps $C^\infty_\alpha(U_{x_j}(\mathbb{R}^d))$ into $C^\infty_\alpha(\mathbb{R}^d)$ boundedly with respect to the $W^{k,p}(\mathbb{R}^d)$-topology. Hence, $\zeta_j \varphi_j \in W^{k,p}_E(\mathbb{R}^d)$. Since in any case $\varphi \in W^{k,p}_E(\mathbb{R}^d)$, the conclusion follows.

\[ \square \]

**Remark 6.6.** By construction one gets uniformity for $\mathcal{E}$ with respect to $p$ and $k$ if one invests the respective uniformity concerning the extension property for the local domains $\Lambda \cap U_x$. In particular, one obtains an extension operator that is bounded from $W^{k,p}_E(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in [1, \infty]$ if the local domains are universal Sobolev extension domains.

### 6.3. Preservation of traces.

Proposition 6.5 allows to construct Sobolev extension operators from $W^{k,p}_E(\Omega)$ into $W^{k,p}(\mathbb{R}^d)$ and gives a sufficient condition for preservation of the Dirichlet condition. In this section we prove that in fact every such extension operator has this feature. Recall that this is the crux of the matter in Assumption (iii) of Theorem 3.1. The key lemma is the following.

**Lemma 6.7.** Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $\Lambda \subset \mathbb{R}^d$ be a domain, let $E \subset \partial \Lambda$ be closed and let $\mathcal{E}_{k,p} : W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d)$ be a bounded extension operator. Any of the following conditions guarantees that $\mathcal{E}_{k,p}$ in fact maps into $W^{k,p}(\mathbb{R}^d)$:

(i) For $(k,p)$-quasievery $y \in E$ balls around $y$ in $\Lambda$ have asymptotically nonvanishing relative volume, i.e.

\[ \liminf_{r \to 0} \frac{|B(y,r) \cap \Lambda|}{r^d} > 0. \]

(ii) The set $E$ is an $l$-set for some $l \in |d-p,d|$ and (6.1) holds for $H_l$-almost every $y \in E$.

(iii) There exists $q > d$ such that $\mathcal{E}_{k,p}$ maps $C_\infty^\alpha(\Lambda)$ into $W^{k,q}(\mathbb{R}^d)$.

**Proof.** As $C_\infty^\alpha(\Omega)$ is dense in $W^{k,p}_E(\Lambda)$ and since $\mathcal{E}_{k,p}$ is bounded, it suffices to prove that given $v \in C_\infty^\alpha$ the function $u := \mathcal{E}_{k,p}v$ belongs to $W^{k,p}_E(\mathbb{R}^d)$. The proof of (i) is inspired by [50, pp. 190-192]. Easy modifications of the argument will yield (ii) and (iii).

(i) Fix an arbitrary multiindex $\beta$ with $|\beta| \leq k - 1$. Let $\mathcal{D}^\beta u$ be the representative of the distributional derivative $D^\beta u$ of $u$ defined $(k-|\beta|,p)$-q.e. on $\mathbb{R}^d$ via

\[ \mathcal{D}^\beta u(y) := \lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} D^\beta u(x) \, dx. \]

Recall from (4.3) that then

\[ \lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} |\mathcal{D}^\beta u(x) - \mathcal{D}^\beta u(y)| \, dx \]

\[ \leq \lim_{r \to 0} \left( \frac{1}{|B(y,r)|} \int_{B(y,r)} |D^\beta u(x) - D^\beta u(y)|^p \, dx \right)^{1/p} = 0. \]

holds for $(k-|\beta|,p)$-q.e. $y \in \mathbb{R}^d$. Since (6.1) holds for $(k,p)$-quasievery $y \in E$, it a fortiori holds for $(k-|\beta|,p)$-quasievery such $y$. Let now $N \subset \mathbb{R}^d$ be the exceptional set such that on $\mathbb{R}^d \setminus N$ the function $\mathcal{D}^\beta u$ is defined and satisfies (6.2) and such that (6.1) holds for every $y \in E \setminus N$. Owing to Theorem 4.11 the claim follows once we have shown $\mathcal{D}^\beta u(y) = 0$ for all $y \in E \setminus N$.

For the rest of the proof we fix $y \in E \setminus N$. For $r > 0$ we abbreviate $B(r) := B(y,r)$ and define

\[ W_j := \{ x \in \mathbb{R}^d \setminus N : |\mathcal{D}^\beta u(x) - D^\beta u(y)| > 1/j \} \]
Thanks to (6.2) for each $j \in \mathbb{N}$ we can choose some $r_j > 0$ such that $|B(r) \cap W_j| < 2^{-j}|B(r)|$ holds for all $r \in [0, r_j]$. Clearly, we can arrange that the sequence $\{r_j\}_j$ is decreasing. Now,

$$W := \bigcup_{j \in \mathbb{N}} \left\{ \left( B(r_j) \setminus B(r_{j+1}) \right) \cap W_j \right\}$$

has vanishing Lebesgue density at $y$, i.e. $r^{-d}|B(r) \cap W|$ vanishes as $r$ tends to 0: Indeed, if $r \in [r_{j+1}, r_j]$, then

$$|B(r) \cap W| \leq \left| \left( B(r) \cap W_j \right) \cup \bigcup_{j \geq l+1} \left( B(r_j) \cap W_j \right) \right|$$

$$\leq 2^{-l+1}|B(r)| + \sum_{j \geq l+1} 2^{-j}|B(r_j)| \leq 2^{-l+1}|B(r)|.$$

Now, (6.1) allows to conclude

$$\liminf_{r \to 0} \frac{|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W)|}{r^d} > 0.$$

Since $u$ is an extension of $v \in C^\infty_{\text{loc}}(\Lambda)$ and $y$ is an element of $E$ it holds $\mathcal{D}^\beta u = 0$ a.e. on $B(r) \cap \Lambda$ with respect to the $d$-dimensional Lebesgue measure if $r > 0$ is small enough. The previous inequality gives $|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W)| > 0$ if $r > 0$ is small enough. In particular, there exists a sequence $\{x_j\}_j$ in $\mathbb{R}^d \setminus W$ approximating $y$ such that $\mathcal{D}^\beta u(x_j) = 0$ for all $j \in \mathbb{N}$. Now, the upshot is that the restriction of $\mathcal{D}^\beta u$ to $\mathbb{R}^d \setminus W$ is continuous at $y$ since if $x \in \mathbb{R}^d \setminus W$ satisfies $|x - y| \leq r_j$ then by construction $|\mathcal{D}^\beta u(x) - \mathcal{D}^\beta u(y)| \leq 1/j$. Hence, $\mathcal{D}^\beta u(y) = 0$ and the proof is complete.

(ii) If $E$ is an $l$-set for some $l \in [d-p, d]$, then we can appeal to Theorem 4.12 rather than Theorem 4.11 and the same argument as in (i) applies.

(iii) By assumption $u \in W^{k,q}_E(\mathbb{R}^d)$, where $q > d$. By Sobolev embeddings each distributional derivative $D^\beta u$, $|\beta| \leq k-1$, has a continuous representative $\mathcal{D}^\alpha u$. As each $y \in E \cap \partial \Lambda$ is an accumulation point of $\Lambda \setminus E$ and since $D^\alpha u = D^\alpha v$ holds almost everywhere on $\Lambda$, the representative $\mathcal{D}^\alpha u$ must vanish everywhere on $E$ and Theorem 4.11 yields $u \in W^{k,p}_E(\mathbb{R}^d)$ as required.

\[ \square \]

Remark 6.8. If $\Lambda$ is a $d$-set and $E$ a $(d-1)$-set, then Lemma 6.7 is proved in [22, Sec. VIII.1].

We can now state and prove the remarkable result that every Sobolev extension operator that is constructed by localization techniques as in Proposition 6.5 preserves the Dirichlet condition.

Theorem 6.9. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $\Lambda$ be a bounded domain and let $E$ be a closed part of its boundary. Assume that for every $x \in \partial \Lambda \setminus E$ there is an open neighborhood $U_x$ of $x$ such that $\Lambda \cap U_x$ is a $W^{k,p}$-extension domain. Then there exists a continuous extension operator

$$\mathcal{E}_{k,p} : W^{k,p}_E(\Lambda) \rightarrow W^{k,p}_E(\mathbb{R}^d).$$

For the proof we recall the following result of Hailasz, Koskela and Tuominen.

Proposition 6.10 ([18, Thm. 2]). If a domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$-extension domain for some $k \in \mathbb{N}$ and $p \in [1, \infty]$, then it is a $d$-set.

\[ \text{Proof of Theorem 6.9.} \] According to Proposition 6.5 it suffices to check that each local extension operator $\mathcal{E}_x$ maps $W^{k,p}_E(\Lambda \cap U_x)$ into $W^{k,p}_E(\mathbb{R}^d)$, where $E_x := E \cap U_x \subset \partial(\Lambda \cap U_x)$. Owing to Proposition 6.10 the $W^{k,p}$-extension domain $\Lambda \cap U_x$ is a $d$-set and as such satisfies (6.1) around every of its boundary points. So, Lemma 6.7 (i) yields the claim. \[ \square \]
Remark 6.11. The extension operator in Theorem 6.9 is the same as in Proposition 6.5. Hence, the former result asserts that every Sobolev extension operator built by the common gluing-together of local extension operators automatically preserves the Dirichlet condition on $E$ under the mere assumption that this set is closed. Moreover, all uniformity properties as in Remark 6.6 remain valid.

6.4. Geometric conditions. In this subsection we finally review common geometric conditions on the boundary part $\partial \Lambda \setminus E$ such that the local sets $\Lambda \cap U_x$ really admit the Sobolev extension property required in Proposition 6.5.

A first condition, completely sufficient for the treatment of most real world problems, is the following Lipschitz condition.

Definition 6.12. A bounded domain $\Lambda \subset \mathbb{R}^d$ is called bounded Lipschitz domain if for each $x \in \partial \Lambda$ there is an open neighborhood $U_x$ of $x$ and a bi-Lipschitz mapping $\phi_x$ from $U_x$ onto a cube, such that $\phi_x(\Lambda \cap U_x)$ is the (lower) half cube and $\partial \Lambda \cap U_x$ is mapped onto the top surface of this half cube.

It can be proved by elementary means that bounded Lipschitz domains are $W^{1,p}$-extension domains for every $p \in [1, \infty]$, cf. e.g. [17] for the case $p = 2$. In fact, already the following $(\varepsilon, \delta)$-condition of Jones [21] assures the existence of a universal Sobolev extension operator.

Definition 6.13. Let $\Lambda \subset \mathbb{R}^d$ be a domain and $\varepsilon, \delta > 0$. Assume that any two points $x, y \in \Lambda$, with distance not larger than $\delta$, can be connected within $\Lambda$ by a rectifiable arc $\gamma$ with length $l(\gamma)$, such that the following two conditions are satisfied for all points $z$ from the curve $\gamma$:

$$l(\gamma) \leq \frac{1}{\varepsilon} \|x - y\|, \quad \text{and} \quad \frac{\|x - z\|}{\|x - y\|} \leq \frac{1}{\varepsilon} \text{dist}(z, \Lambda^c).$$

Then $\Lambda$ is called $(\varepsilon, \delta)$-domain.

Theorem 6.14 (Rogers). Each $(\varepsilon, \delta)$-domain is a universal Sobolev extension domain.

Remark 6.15. (i) Theorem 6.14 is due to Rogers [44] and generalizes the celebrated result of Jones [21]. Bounded $(\varepsilon, \delta)$-domains are known to be uniform domains, see [47, Ch. 4.2] and also [21, 36, 37, 35] for further information. In particular, every bounded Lipschitz domain is an $(\varepsilon, \delta)$-domain, see e.g. [12, Rem. 5.11] for a sketch of proof.

(ii) Although the uniformity property is not necessary for a domain to be a Sobolev extension domain [19] it seems presently to be the broadest known class of domains for which this extension property holds – at least if one aims at all $p \in [1, \infty]$.

Plugging in Rogers extension operator into Theorem 6.9 lets us re-discover [4, Thm. 1.3] in case of bounded domains and $p$ strictly between 1 and $\infty$. We even obtain a universal extension operator that simultaneously acts on all $W^{k,p}_E$-spaces and at the same time our argument reveals that the preservation of the trace is irrespective of the specific structure of Jones’ or Roger’s extension operators.

We believe that this sheds some more light also on [4, Thm. 1.3] though – of course – our argument cannot disclose the fundamental assertions on the support of the extended functions obtained in [4] by a careful analysis of Jones’ extension operator. We summarize our observations in the following theorem.

Theorem 6.16. Let $\Lambda$ be a bounded domain and let $E$ be a closed part of its boundary. Assume that for every $x \in \partial \Lambda \setminus E$ there is an open neighborhood $U_x$ of $x$ such that $\Lambda \cap U_x$ is a bounded Lipschitz or, more generally, an $(\varepsilon, \delta)$-domain for some values $\varepsilon, \delta > 0$. Then there exists a universal operator $E$ that restricts to a bounded extension operator $W^{k,p}_E(\Lambda) \to W^{k,p}_E(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in [1, \infty]$. 
In this section we will discuss sufficient conditions for Poincaré’s inequality, thereby unwinding Assumption \([10]\) of Theorem 3.1. Our aim is not greatest generality as e.g. in \([39]\) for functions defined on the whole of \(\mathbb{R}^d\), but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense geometric conditions, which may be checked more or less ‘by appearance’ – at least for problems arising from applications.

The next proposition gives a condition that assures that a closed subspace of \(W^{1,p}\) may be equivalently normed by the \(L^p\)-norm of the gradient of the corresponding functions only. We believe that this might also be of independent interest, compare also \([50, \text{Ch. 4}]\). Throughout \(\mathbb{I}\) denotes the function that is identically one.

**Proposition 7.1.** Let \(\Lambda \subset \mathbb{R}^d\) be a bounded domain and suppose \(p \in ]1, \infty[\). Assume that \(X\) is a closed subspace of \(W^{1,p}(\Lambda)\) that does not contain \(\mathbb{I}\) and for which the restriction of the canonical embedding \(W^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)\) to \(X\) is compact. Then \(X\) may be equivalently normed by \(v \mapsto (\int_{\Lambda} |\nabla v|^p \, dx)^{1/p}\).

**Proof.** First recall that both \(X\) and \(L^p(\Lambda)\) are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence \(\{v_k\}_k\) from \(X\) such that

\[
\frac{1}{k} \|v_k\|_{L^p(\Lambda)} \geq \|\nabla v_k\|_{L^p(\Lambda)}.
\]

After normalization we may assume \(\|v_k\|_{L^p(\Lambda)} = 1\) for every \(k \in \mathbb{N}\). Hence, \(\{\nabla v_k\}_k\) converges to 0 strongly in \(L^p(\Lambda)\). On the other hand, \(\{v_k\}_k\) is a bounded sequence in \(X\) and hence contains a subsequence \(\{v_{k_l}\}_l\) that converges weakly in \(X\) to an element \(v \in X\). Since the gradient operator \(\nabla : X \to L^p(\Lambda)\) is continuous, \(\{\nabla v_{k_l}\}_l\) converges to \(\nabla v\) weakly in \(L^p(\Lambda)\). As the same sequence converges to 0 strongly in \(L^p(\Lambda)\), the function \(\nabla v\) must be zero and hence \(v\) is constant. But by assumption \(X\) does not contain constant functions except for \(v = 0\). So, \(\{v_{k_l}\}_l\) tends to 0 weakly in \(X\). Owing to the compactness of the embedding \(X \hookrightarrow L^p(\Lambda)\), a subsequence of \(\{v_{k_l}\}_l\) tends to 0 strongly in \(L^p(\Lambda)\). This contradicts the normalization condition \(\|v_{k_l}\|_{L^p(\Lambda)} = 1\). \(\square\)

**Remark 7.2.** It is clear that in case \(X = W^{1,p}_D(\Omega)\) the embedding \(X \hookrightarrow L^p(\Lambda)\) is compact, if there exists a continuous extension operator \(\mathcal{E} : W^{1,p}_D(\Omega) \to W^{1,p}(\mathbb{R}^d)\). Hence, the compactness of this embedding is no additional requirement in view of Theorem 3.1.

In the case that \(E\) is \(l\)-thick, the following lemma presents two conditions that are particularly easy to check and entail \(1 \notin W^{1,p}_E(\Lambda)\). Loosely speaking, some knowledge on the common frontier of \(E\) and \(\partial \Lambda \setminus E\) is required: Either not every point of \(E\) should lie thereon or \(\partial \Lambda\) must not be too wild around this frontier.

**Lemma 7.3.** Let \(p \in ]1, \infty[\), let \(\Lambda\) be a bounded domain and let \(E \subset \partial \Lambda\) be \(l\)-thick for some \(l \in ]d - p, d]\). Both of the following conditions assure \(1 \notin W^{1,p}_E(\Lambda)\).

(i) The set \(E\) admits at least one relatively inner point \(x\). Here, ‘relatively inner’ is with respect to \(\partial \Lambda\) as ambient topological space.

(ii) For every \(x \in \partial \Lambda \setminus E\) there is an open neighborhood \(U_x\) of \(x\) such that \(\Lambda \cap U_x\) is a \(W^{1,p}\)-extension domain.

**Proof.** We treat both cases separately.

(i) Assume the assertion was false and \(1 \in W^{1,p}_E(\Lambda)\). Let \(x\) be the inner point of \(E\) from the hypotheses and let \(B := B(x, r)\) be a ball that does not intersect \(\partial \Lambda \setminus E\). Put \(\frac{1}{2}B := B(x, \frac{r}{2})\) and let \(\eta \in C^\infty_0(B)\) be such that \(\eta \equiv 1\) on \(\frac{1}{2}B\). We distinguish whether or not \(x\) is an interior point of \(\overline{X}\).
First, assume it is not. For every \( \psi \in C^\infty_0(\Lambda) \) the function \( \eta \psi \) belongs to \( W^{1,p}_0(\Lambda \cap B) \) and as such admits a \( W^{1,p} \)-extension \( \eta \widetilde{\psi} \) by zero to the whole of \( \mathbb{R}^d \). In particular,
\[
\eta \widetilde{\psi}(y) = \begin{cases} 
\psi(y), & \text{if } y \in \frac{1}{2}B \cap \Lambda \\
0, & \text{if } y \in \frac{1}{2}B \setminus \Lambda
\end{cases}
\]
and consequently,
\[
\| \nabla \eta \tilde{\psi} \|_{L^p(\frac{1}{2}B)} = \| \nabla \psi \|_{L^p(\frac{1}{2} B \cap \Lambda)}.
\]
Since by assumption \( 1 \) is in the \( W^{1,p}(\Lambda) \)-closure of \( C^\infty_0(\Lambda) \) and since the mappings \( W^{1,p}_0(\Lambda) \ni \psi \mapsto \nabla \eta \tilde{\psi} \in L^p(\frac{1}{2}B) \) and \( W^{1,p}_0(\Lambda) \ni \psi \mapsto \nabla \psi \in L^p(\Lambda \cap \frac{1}{2}B) \) are continuous, the previous equation extends to \( \psi = 1 \):
\[
\| \nabla \eta 1 \|_{L^p(\frac{1}{2}B)} = \| \nabla 1 \|_{L^p(\frac{1}{2} B \cap \Lambda)} = 0.
\]
On the other hand \( x \) is not an inner point of \( \overline{\Lambda} \) so that in particular \( \frac{1}{2}B \setminus \overline{\Lambda} \) is non-empty. Since this set is open, \( |\frac{1}{2}B \setminus \overline{\Lambda}| > 0 \). Recall that by construction \( \eta \hat{1} \in W^{1,p}(B) \) vanishes a.e. on \( \frac{1}{2}B \setminus \overline{\Lambda} \). Hence, for some \( c > 0 \) the Poincaré inequality
\[
\| \eta \hat{1} \|_{L^p(\frac{1}{2}B)} \leq c \| \nabla \eta \hat{1} \|_{L^p(\frac{1}{2}B)},
\]
holds, cf. [50, Thm. 4.4.2]. However, we already know that the right hand side is zero, whereas the left hand side equals \( |\frac{1}{2}B \cap \Lambda|^{1/p} \), which is nonzero since \( \frac{1}{2}B \cap \Lambda \) is nonempty and open – a contradiction.

Now, assume \( x \) is contained in the interior of \( \overline{\Lambda} \). Upon diminishing \( B \) we may assume \( B \subset \overline{\Lambda} \). For every \( \psi \in C^\infty_0(\mathbb{R}^d) \) we have \( \eta \psi \in C^\infty_0(\mathbb{R}^d) \) with an estimate
\[
\| \eta \psi \|_{W^{1,p}(\mathbb{R}^d)} \leq c \| \psi \|_{W^{1,p}(B)} = c \left( \int_B |\psi|^p + |\nabla \psi|^p \, dx \right)^{1/p}
\]
is some constant \( c > 0 \) depending only on \( \eta \) and \( p \). By our choice of \( B \) split
\[
B = B \cap \overline{\Lambda} = (B \cap \Lambda) \cup (B \cap \partial \Lambda) = (B \cap \Lambda) \cup (B \cap E).
\]
Since \( \psi \) vanishes in a neighborhood of \( E \),
\begin{equation}
\| \eta \psi \|_{W^{1,p}(\mathbb{R}^d)} \leq c \left( \int_{B \cap \Lambda} |\psi|^p + |\nabla \psi|^p \, dx \right)^{1/p} \leq c \| \psi \|_{W^{1,p}(\Lambda)}.
\end{equation}
Taking into account \( \eta \equiv 1 \) on \( \frac{1}{2}B \), the same reasoning gives
\begin{equation}
\int_{\frac{1}{2}B} \| \nabla (\eta \psi) \|^p \, dx = \int_{\frac{1}{2}B} |\nabla \psi|^p \, dx \leq \int_{\Lambda} |\nabla \psi|^p \, dx.
\end{equation}
By assumption there is a sequence \( \{ \psi_j \}_j \subset C^\infty_0(\Lambda) \) tending to \( 1 \) in the \( W^{1,p}(\Lambda) \)-topology. Due to (7.1) and the choice of \( \eta \), the sequence \( \{ \eta \psi_j \}_j \subset C^\infty_0(\mathbb{R}^d) \) then tends to some \( u \in W^{1,p}_0(\mathbb{R}^d) \) satisfying \( u = 1 \) a.e. on \( \frac{1}{2}B \cap \Lambda \). Due to (7.2), \( \nabla u = 0 \) a.e. on \( \frac{1}{2}B \), meaning that \( u \) is constant on this set. Since \( \frac{1}{2}B \cap \Lambda \) as a non-empty open set has positive Lebesgue measure, all this can only happen if \( u = 1 \) a.e. on \( \frac{1}{2}B \). Hence,
\[
\lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u \, dx = 1
\]
for every \( y \in \frac{1}{2}B \cap E \), which by Theorem 4.11 is only possible if \( C_{1,p}(\frac{1}{2}B \cap E) = 0 \). By Theorem 4.11 this in turn implies \( \mathcal{H}^\infty_0(\frac{1}{2}B \cap E) = 0 \) in contradiction to the \( l \)-thickness of \( E \).
(ii) Again assume the assertion was false. Then by (i) there exists some $x \in E$ that is not an inner point of $E$ with respect to $\partial \Lambda$. Hence $x$ is an accumulation point of $\partial \Lambda \setminus E$ and by assumption there is a neighborhood $U = U_x$ of $x$ such that $\Lambda \cap U$ is a $W^{1,p}$-extension domain. We denote the corresponding extension operator by $\mathcal{E}$. We shall localize the assumption $1 \in W_{E}^{1,p}(\Lambda)$ within $U$ to arrive at a contradiction.

To this end, let $r_0 > 0$ be such that $B(x,r_0) \subset U$ and let $\eta \in C^{\infty}_c(U)$ be such that $\eta \equiv 1$ on $B(x,r_0)$. Then also $\eta \in W_{E}^{1,p}(\Lambda)$ and in particular $\eta|_{\Lambda \cap U}$ belongs to $W_{E}^{1,p}(\Lambda \cap U)$, where $F := B(x,r_0/2) \subset E \subset \partial(\Lambda \cap U)$. Recall from Proposition 6.10 that the $W^{1,p}$-extension domain $\Lambda \cap U$ satisfies in particular

$$\liminf_{r \to 0} \frac{|B(y,r) \cap \Lambda \cap U|}{r^d} > 0.$$ 

around every $y \in \partial(\Lambda \cap U)$. Thus, Lemma 6.7(i) yields $u := \mathcal{E}(\eta|_{\Lambda \cap U}) \in W_{E}^{1,p}(\mathbb{R}^d)$.

On the other hand, similar to the proof of Lemma 6.6, let $u$ be the representative of $u$ that is defined by limits of integral means on the complement of some exceptional set $N$ with $C_{1,p}(N) = 0$ and fix $y \in F \setminus N$. Take $W$ as in (6.3) and (6.4). Repeating the arguments in the proof of Lemma 6.7 reveals that the restriction of $u$ to $\mathbb{R}^d \setminus W$ is continuous at $y$ and that $|B(y,r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W)| > 0$ if $r > 0$ is small enough. By construction $u = 1$ a.e. on $B(y,r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W)$ if $r < r_0$. Hence, there is a sequence $\{x_j\}_j$ approximating $y$ such that $u(x_j) = 1$ for every $j \in \mathbb{N}$. By continuity $u(y) = 1$ follows. This proves that $u = 1$ holds (1,$p$)-quasieverywhere on $F$.

By Theorem 4.11 this can only happen if $C_{1,p}(F) = 0$, which as in (i) contradicts the $l$-thickness of $E$. \qed

Remark 7.4.

(i) The proof of (i) shows that $1 \notin W_{E}^{1,p}(\Lambda)$ if $E$ is merely closed and contains a relatively inner point that is not an inner point of $\Lambda$.

(ii) Of course the Poincaré inequality holds in the case $E = \partial \Lambda$ irrespective of any geometric considerations as long as $\Lambda$ is bounded. This can be rediscovered by the results of this section. Indeed, $E$ then only consists of relatively inner points and as $\emptyset \neq \partial \Lambda \subset \partial \Lambda = E$ holds, it cannot be contained in the interior of $\Lambda$. Hence $1 \notin W_{E}^{1,p}(\Lambda)$. The compactness of the embedding $W_{0}^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and Theorem 7.1 gives the claim.

Under the second assumption of Lemma 7.3 there exists a linear continuous Sobolev extension operator $\mathcal{E} : W_E^{1,p}(\Lambda) \to W_E^{1,p}(\mathbb{R}^d)$, see Theorem 6.9. Then the compactness of the embedding $W_E^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and owing to Theorem 7.1 we can record the following special Poincaré inequality.

Proposition 7.5. Let $p \in ]1,\infty[ \setminus \mathbb{N}$ and let $\Lambda$ be a bounded domain. Suppose that $E \subset \partial \Lambda$ is $l$-thick for some $l \in [d-p,d]$ and that for each $x \in \partial \Lambda \setminus E$ there is an open neighborhood $U_x$ of $x$ such that $\Lambda \cap U_x$ is a $W^{1,p}$-extension domain. Then $W_E^{1,p}(\Lambda)$ may equivalently be normed by $v \mapsto (\int_{\Lambda} |(\nabla v)^p dx|^{1/p})^\frac{1}{p}$.

Now, also Theorem 3.2 follows. In fact, this result is just the synthesis of the above proposition with Theorems 3.1 and 6.9.

8. Proof of Theorem 3.3

The strategy of proof is to write $u$ as the sum of $v \in W^{1,p}(\Omega)$ with $v/\text{dist}_{\partial \Omega} \in L^p(\Omega)$ and $w \in W^{1,p}$ with support within a neighborhood of $\partial \Omega \setminus \overline{D}$. Then $v$ can be handled by the following classical result.

Proposition 8.1 ([33] Thm. V.3.4). Let $\emptyset \subset \Lambda \subset \mathbb{R}^d$ be open and let $p \in ]1,\infty[$. Then if $u \in W^{1,p}(\Lambda)$ and $u/\text{dist}_{\partial \Lambda} \in L^p(\Lambda)$, it follows $u \in W_{0}^{1,p}(\Lambda)$. 

For \( w \) we can – since local extension operators are available – rely on the techniques developed in Section 6. A key observation is an intrinsic relation between the property \( \frac{u}{\text{dist}_D} \in L^p(\Omega) \) and Sobolev regularity of the function \( \log(\text{dist}_D) \). In fact, a formal computation gives

\[
\nabla (u \log(\text{dist}_D)) = \log(\text{dist}_D) \nabla u + \frac{u}{\text{dist}_D} \nabla \text{dist}_D.
\]

Details are carried out in the following five consecutive steps.

**Step 1: Splitting \( u \) and handling the easy term.** As in the proof of Proposition 6.5 for every \( x \in \partial \Omega \setminus D \), let \( U_x \) be the open neighborhood of \( x \) from the assumption, let \( U_{x_1}, \ldots, U_{x_n} \) be a finite subcovering of \( \partial \Omega \setminus D \) and let \( \varepsilon > 0 \) be such that the sets \( U_{x_1}, \ldots, U_{x_n} \), together with \( U := \{ y \in \mathbb{R}^d : \text{dist}(y, \partial \Omega \setminus D) > \varepsilon \} \), form an open covering of \( \Omega \). Finally, let \( \eta, \eta_1, \ldots, \eta_n \) be a subordinated \( C_0^\infty \)-partition of unity. The described splitting is \( u = v + w \), where \( v := \eta u \) and \( w := \sum_{j=1}^n \eta_j u = (1 - \eta) u \). Since

\[
\text{dist}_{\partial \Omega}(x) \geq \min\{ \varepsilon, \text{dist}_{\partial \Omega}(x) \} \geq \min\{ \varepsilon \text{diam}(\Omega)^{-1}, 1 \} \cdot \text{dist}_D(x)
\]

holds for every \( x \in \text{supp}(\eta) \cap \Omega \), the function \( v \in W^{1,p}(\Omega) \) satisfies

\[
\int_{\Omega} \frac{v^p}{\text{dist}_{\partial \Omega}} \, dx \leq c \int_{\Omega} \frac{v^p}{\text{dist}_D} \, dx \leq c \int_{\Omega} \frac{u^p}{\text{dist}_D} \, dx < \infty
\]

by assumption on \( u \). Now, Proposition 8.1 yields \( v \in W^{1,p}_0(\Omega) \subset W^{1,p}_D(\Omega) \).

**Step 2: Extending \( w \).** By assumption the sets \( \Omega \cap U_{x_j}, 1 \leq j \leq n \), are \( W^{1,p} \)-extension domains. Since \( w = (1 - \eta) u \), where \( (1 - \eta) \) has compact support in the union of these domains, an extension \( \hat{w} \in W^{1,p}(\mathbb{R}^d) \) of \( w \in W^{1,p}(\Omega) \) with compact support within \( \bigcup_{j=1}^n U_{x_j} \) can be constructed just as in the proof of Proposition 6.5. Now, if we can show \( w \in W^{1,p}_D(\Omega) \), then by Step 1 also \( u = v + w \) belongs to this space.

**Step 3: Estimating the trace of \( \hat{w} \).** To prove \( \hat{w} \in W^{1,p}_D(\mathbb{R}^d) \) we rely once more on the techniques used in the proof of Lemma 6.7. So, let \( \hat{w} \) be the representative of \( \hat{w} \) defined on \( \mathbb{R}^d \setminus N \) via

\[
\hat{w}(y) := \lim_{r \to 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} \hat{w} \, dx,
\]

where the exceptional set \( N \) is of vanishing \((1, p)\)-capacity. Put

\[
U_\star := \bigcup_{j=1}^n U_{x_j}, \quad \Omega_\star := \Omega \cap U_\star, \quad \text{and} \quad D_\star = \overline{D \cap U_\star} \subset \partial \Omega_\star.
\]

Since \( \hat{w} \) has support in \( U_\star \), it holds \( \hat{w}(y) = 0 \) for every \( y \in D \setminus D_\star \). For the rest of the step let \( y \in D_\star \setminus N \).

By Proposition 6.10 each set \( \Omega \cap U_{x_j} \) is a \( d \)-set and it can readily be checked that this property inherits to their union \( \Omega_\star \). Hence, \( \Omega_\star \) satisfies the asymptotically nonvanishing relative volume condition 6.11 around \( y \) with a lower bound \( c > 0 \) on the lines inferior that is independent of \( y \) and – just as in the proof of Lemma 6.7 – a set \( W \subset \mathbb{R}^d \) can be constructed such that the restriction of \( \hat{w} \) to \( \mathbb{R}^d \setminus W \) is continuous at \( y \) and such that \( |B(y, r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)| \geq cr^d/2 \) if
Suppose $u$. Lemma 8.2.

Proof of Lemma 8.2. Using Remark 8.4 it suffices to construct an extension in proofs of these results can be found e.g. in [45, Sec. 2.3.2/2.5.1].

The Besov spaces are nested with the Bessel potential spaces in the sense that

$$\log(\text{dist}_D)^{-1},$$

which is bounded above in absolute value by $|\log r|^{-1}$ on $B(y, r)$ if $r < 1$. It follows

$$|\tilde{w}(y)| \leq \limsup_{r \to 0} \left( \frac{1}{r^d} \int_{B(y, r) \cap \Omega^s} |w| \log(\text{dist}_D) \, dx \right).$$

So, since $|\log r|^{-1} \to 0$ as $r \to 0$ the function $\tilde{w}$ vanishes at every $y \in D^s \setminus N$ for which the mean value integrals on the right-hand side remain bounded as $r$ tends to zero.

**Step 4: Intermezzo on $w \log(\text{dist}_D)$:** In this step we prove the following result.

**Lemma 8.2.** Let $p \in [1, \infty]$, let $\Lambda \subset \mathbb{R}^d$ be a bounded d-set, and let $E \subset \partial \Lambda$ be closed and porous. Suppose $u \in W^{1,p}(\Lambda)$ has an extension $v \in W^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{\text{dist}_D} \in L^p(\Lambda)$. If $r \in [1, p]$ and $s \in ]0, 1[$, then the function $|u \log(\text{dist}_E)|$ defined on $\Lambda$ has an extension in the Bessel potential space $H^{s,p}(\mathbb{R}^d)$ that is positive almost everywhere.

For the proof we need the following extension result of Jonsson and Wallin.

**Proposition 8.3 ([22 Thm. V.1.1]).** Let $s \in ]0, 1[$, $p \in [1, \infty]$ and let $\Lambda \subset \mathbb{R}^d$ be a d-set. Then there exists a linear operator $E$ that extends every measurable function $f$ on $\Lambda$ that satisfies

$$\|f\|_{L^p(\Lambda)} + \left( \int_{x,y \in \Lambda} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dx \, dy \right)^{1/p} < \infty$$

to a function $Ef$ in the Besov space $B^{s,p}(\mathbb{R}^d)$ of all measurable functions $g$ on $\mathbb{R}^d$ such that

$$\|g\|_{L^p(\mathbb{R}^d)} + \left( \int_{x,y \in \mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} \, dx \, dy \right)^{1/p} < \infty.$$

**Remark 8.4.** The Besov spaces are nested with the Bessel potential spaces in the sense that $B^{s,p}(\mathbb{R}^d) \subset H^{s-\varepsilon,p}(\mathbb{R}^d)$ for each $s > 0$ and every $\varepsilon \in ]0, s]$. Moreover, $W^{1,p}(\mathbb{R}^d) \subset B^{s,p}(\mathbb{R}^d)$. Proofs of these results can be found e.g. in [45] Sec. 2.3.2/2.5.1.

**Proof of Lemma 8.2.** Using Remark 8.4 it suffices to construct an extension in $B^{s,p}$ with the respective properties. Moreover, by the reverse triangle inequality it is enough to construct any extension $f \in B^{s,p}(\mathbb{R}^d)$ of $u \log \text{dist}_E$ and then $|f|$ can be used as the required extension of $|u \log \text{dist}_E|$. These considerations and Proposition 8.3 show that the claim follows provided

$$\|u \log(\text{dist}_D)\|_{L^r(\Lambda)} + \left( \int_{x,y \in \Lambda} \frac{|u(x) \log(\text{dist}_E(x)) - u(y) \log(\text{dist}_E(y))|^r}{|x - y|^{d+sr}} \, dx \, dy \right)^{1/r}$$

is finite.

To bound the $L^r$ norm on the left-hand side of (8.2) choose $q \in ]1, \infty]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder’s inequality

$$\|u \log(\text{dist}_E)\|_{L^r(\Lambda)} \leq \|u\|_{L^p(\Lambda)} \|\log(\text{dist}_D)\|_{L^q(\Lambda)}.$$
For the second term on the right-hand we use that the Aikawa dimension of the porous set \( E \) is strictly smaller than \( d \), see Remark 4.4. This entails for some \( \alpha < d \) and some \( x \in E \) the estimate
\[
\int_{\Lambda} \text{dist}_E(x)^{\alpha - d} \, dx \leq \int_{B(x, 2 \text{diam } \Lambda)} \text{dist}_E(x)^{\alpha - d} \, dx \leq c_\alpha (2 \text{ diam } \Lambda)^\alpha < \infty.
\]

Hence, some negative power of \( \text{dist}_E \) is integrable on \( \Lambda \) and by subordination of logarithmic growth \( \log(\text{dist}_E) \in L^q(\Lambda) \) follows. Altogether, \( u \log(\text{dist}_E) \in L^r(\Lambda) \) taking care of the first term in (8.2).

By symmetry the domain of integration for the second term on the left-hand side of (8.2) can be restricted to \( \text{dist}_E(x) \geq \text{dist}_E(y) \). By adding and subtracting the term \( u(y) \log(\text{dist}_E(x)) \) it in fact suffices to prove that
\[
(8.3) \qquad \left( \int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^r}{|x - y|^{d + sr}} \left| \log(\text{dist}_E(x)) \right|^r \, dx \, dy \right)^{1/r}
\]
and
\[
(8.4) \qquad \left( \int_{\Lambda} |u(y)|^r \int_{\text{dist}_E(x) \geq \text{dist}_E(y)} \frac{|\log(\text{dist}_E(x)) - \log(\text{dist}_E(y))|^r}{|x - y|^{d + sr}} \, dx \, dy \right)^{1/r}
\]
are finite. Fix \( s < t < 1 \), write (8.3) in the form
\[
\left( \int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^r}{|x - y|^{d + tp}} \left| \log(\text{dist}_E(x)) \right|^r \, dx \, dy \right)^{1/r}
\]
and apply Hölder’s inequality with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) to bound it by
\[
\leq \left( \int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^p}{|x - y|^{d + tp}} \, dx \, dy \right)^{1/p} \left( \int_{\Lambda} \int_{\Lambda} \frac{\left| \log(\text{dist}_E(x)) \right|^q}{|x - y|^{d + ts - tr}} \, dx \, dy \right)^{1/q}
\]
\[
\leq \| \log(\text{dist}_E) \|_{L^q(\Lambda)} \left( \int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^p}{|x - y|^{d + tp}} \, dx \, dy \right)^{1/p} \left( \int_{|y| \leq \text{diam}(\Lambda)} \frac{1}{|x|^{d + ts - tr}} \, dx \right)^{1/q}
\]

Now, \( \log(\text{dist}_E) \in L^q(\Lambda) \) has been proved above and the third integral is absolutely convergent since \( d + (s - t)q < d \). Finally note that by assumption \( u \) has an extension \( v \in W^{1,p}(\mathbb{R}^d) \). Since \( W^{1,p}(\mathbb{R}^d) \subset B^{p,v}(\mathbb{R}^d) \) the middle term above is finite as well, see Remark 8.4.

It remains to show that the most interesting term (8.4) is finite. Here, the additional assumptions on \( u, s \) and \( r \) enter the game. By the mean value theorem for the logarithm and since \( \text{dist}_E \) is a contraction, the \( r \)-th power of this term is bounded above by
\[
\int_{\Lambda} |u(y)|^r \int_{\text{dist}_E(x) \geq \text{dist}_E(y)} \frac{|\text{dist}_E(x) - \text{dist}_E(y)|^r}{\text{dist}_E(y)^r |x - y|^{d + sr}} \, dx \, dy
\]
\[
\leq \int_{\Lambda} \frac{|u(y)|}{\text{dist}_E(y)} \int_{\Lambda} \frac{|x - y|^r}{|x - y|^{d + sr}} \, dx \, dy
\]
\[
\leq \int_{\Lambda} \frac{|u(y)|}{\text{dist}_E(y)} \int_{|y| \leq \text{diam}(\Lambda)} \frac{1}{|x|^{d + r(s - 1)}} \, dx.
\]
Now, the integral with respect to \( x \) is finite since \( r(s - 1) < 0 \). The integral with respect to \( y \) is finite since by assumption \( \frac{u}{\text{dist}_E} \) is \( p \)-integrable on the bounded domain \( \Lambda \) and thus \( r \)-integrable for every \( r < p \). \( \square \)

On noting that by Definition 4.3 a subset of a porous set is again porous, Lemma 8.2 applies to the bounded \( d \)-set \( \Omega \) and the porous set \( D \subset \Omega \). Moreover, \( w = (1 - \eta)u \in W^{1,p}(\Omega) \) has
the extension \( \hat{w} \in W^{1,p}(\mathbb{R}^d) \) and satisfies
\[
\int_{\Omega_\star} \left| \frac{w(x)}{\text{dist}_D(x)} \right|^p \, dx \leq \|1 - \eta\|_\infty \int_{\Omega} \left| \frac{u(x)}{\text{dist}_D(x)} \right|^p < \infty.
\]
Hence we can record:

**Corollary 8.5.** For every \( r \in [1, p] \) and every \( s \in [0, 1] \) the function \(|w \log(\text{dist}_D)|\) defined on \( \Omega_\star \) has an extension \( f_{s,r} \in H^{s,r}(\mathbb{R}^d) \) that is positive almost everywhere.

**Step 5:** Re-inspecting the right-hand side of (8.1). We return to (8.1). Given \( r \in [1, p] \) and \( s \in [0, 1] \) let \( f_{s,r} \in H^{s,r}(\mathbb{R}^d) \) be as in Corollary 8.5. By (4.3) we can infer
\[
\limsup_{r \to 0} \frac{1}{r^d} \int_{B(y,r) \cap \Omega_\star} |w \log(\text{dist}_D)| \, dx \leq \limsup_{r \to 0} \frac{1}{r^d} \int_{B(y,r)} f_{s,r} \, dx < \infty
\]
for \((s, r)\)-quasievery \( y \in D_\star \setminus N \). By the conclusion of Step 3 this implies \( \hat{w}(y) = 0 \) for \((s, r)\)-quasievery \( y \in D_\star \setminus N \). To proceed further, we distinguish two cases:

(i) It holds \( p \leq d \). Since the product \( sr < p \leq d \) can get arbitrarily close to \( p \), Lemma 4.8 yields for every \( r \in [1, p] \) that \( \hat{w} = 0 \) holds \((1, r)\)-quasieverywhere on \( D_\star \setminus N \). Moreover, since \( C_{1,p}(N) = 0 \) by definition, \( \hat{w} = 0 \) holds even \((1, r)\)-quasieverywhere on \( D_\star \).

(ii) It holds \( p > d \). Then \( \hat{w} \) is the continuous representative of \( \hat{w} \in W^{1,p}(\mathbb{R}^d) \) and \( N \) is empty; see the beginning of Step 3. Moreover, we can choose \( s \) and \( r \) such that \( d - l \) \( \leq \) \( sr \) and conclude from the comparison theorem, Theorem 4.9, that \( \hat{w} \) vanishes \( H^\infty \)-a.e. on \( D_\star \). Since \( D \) is \( l \)-thick and \( U_\star \) is open, for each \( y \in D \cap U_\star \) the set \( B(y, r) \cap D \cap U_\star \) coincides with \( B(y, r) \cap D \) provided \( r > 0 \) is small enough and thus has strictly positive \( H^\infty \)-measure. So, the continuous function \( \hat{w} \) has to vanish everywhere on \( D \cap U_\star \) as well as on the closure of the latter set – which by definition is \( D_\star \).

Summing up, \( \hat{w} = 0 \) has been shown to hold \((1, r)\)-quasieverywhere on \( D_\star \) for every \( r \in [1, p] \).

From the beginning of Step 3 we also know that \( \hat{w} \) vanishes everywhere on \( D \setminus D_\star \) and as \( \hat{w} \in W^{1,p}(\mathbb{R}^d) \) has compact support, Hölder’s inequality yields \( \hat{w} \in W^{1,r}(\mathbb{R}^d) \). Combining these two observations with Theorem 4.11 we are eventually led to
\[
\hat{w} \in W^{1,p}(\mathbb{R}^d) \cap \bigcap_{1 < r < p} W^{1,r}(\mathbb{R}^d).
\]

We continue by quoting the following result of Hedberg and Kilpeläinen.

**Proposition 8.6 ([20, Cor. 3.5]).** Let \( p \in [1, \infty] \) and let \( \Lambda \subset \mathbb{R}^d \) be a bounded domain whose boundary is \( l \)-thick for some \( l \in [d - p, d] \). Then
\[
W^{1,p}(\Lambda) \cap \bigcap_{1 < r < p} W^{1,r}(\Lambda) \subset W^{1,p}_0(\Lambda).
\]

**Remark 8.7.** In [20] the requirement on \( \Lambda \) is that its complement is uniformly \( p \)-fat – a property that by the ingenious characterization in [28, Thm. 1] holds for every bounded set with \( l \)-thick boundary provided \( l \in [d - p, d] \).

In order to apply this result to the case of mixed boundary conditions, we proceed similarly to the proof of Theorem 3.1. With \( B \subset \mathbb{R}^d \) an open ball that contains the compact support of \( \hat{w} \) define again
\[
\mathcal{C} := \{ M \subset B \setminus D : M \text{ open, connected and } \Omega \subset M \}
\]
and
\[
\Omega_\star := \bigcup_{M \in \mathcal{C}} M.
\]
Then $\partial \Omega_\star \in \{D, D \cup \partial B\}$ by Corollary 5.4 subsequent to which it is also shown that $\partial \Omega_\star$ is $m$-thick for some $m \in [d - p, d]$. Finally, let $\eta \in C_0^\infty(B)$ be identically one on the support of $\tilde{w}$. As $\varphi \mapsto (\eta \varphi)|_{\Omega_\star}$ induces a bounded operator $W_1^{1,p}(\mathbb{R}^d) \to W_0^{1,p}(\Omega_\star)$, it follows from (8.5) that

$$\tilde{w}|_{\Omega_\star} = (\eta \tilde{w})|_{\Omega_\star} \in W_1^{1,p}(\Omega_\star) \cap \bigcap_{1 \leq r < p} W_0^{1,r}(\Omega_\star)$$

and thus $\tilde{w}|_{\Omega_\star} \in W_0^{1,p}(\Omega_\star)$ thanks to Proposition 8.6. Since by construction $\Omega \subset \Omega_\star$ and $D \subset \partial \Omega_\star$, we eventually conclude

$$w = \tilde{w}|_\Omega \in W_1^{1,p}(\Omega)$$

and the proof is complete. \(\square\)

9. A Generalization

If one asks: ‘What is the most restricting condition in Theorem 3.1?’, the answer doubtlessly is the assumption that a global extension operator shall exist. Certainly, this excludes all geometries that include cracks not belonging completely to the Dirichlet boundary part as in the subsequent Figure.

Since the distance function $\text{dist}_D$ measures only the distance to the Dirichlet boundary part $D$, points in $\partial \Omega$ that are far from $D$ should not be of great relevance in view of the Hardy inequality (3.1). In the following considerations we intend to make this precise. Let $U, V \subset \mathbb{R}^d$ be two open, bounded sets with the properties

$$D \subset U, \quad \overline{V} \cap D = \emptyset, \quad \overline{\Omega} \subset U \cup V.$$  

The philosophy behind this is to take $U$ as a small neighborhood of $D$ which – desirably – excludes the ‘nasty parts’ of $\partial \Omega \setminus D$. More properties of $U, V$ will be specified below. Let $\eta_U \in C_0^\infty(U), \eta_V \in C_0^\infty(V)$ be two functions with $\eta_U + \eta_V = 1$ on $\overline{\Omega}$. Then one can estimate

$$\left( \int_{\Omega} |u|^p \text{dist}_D^{-p} \, dx \right)^{1/p} \leq \left( \int_{U \cap \Omega} |\eta_U u|^p \text{dist}_D^{-p} \, dx \right)^{1/p} + \left( \int_{V \cap \Omega} |\eta_V u|^p \text{dist}_D^{-p} \, dx \right)^{1/p}.$$  

Since $\text{dist}_D$ is larger than some $\varepsilon > 0$ on $\text{supp}(\eta_V) \subset V$, the second term can be estimated by $\frac{1}{\varepsilon} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p}$. If one assumes, as above, Poincaré’s inequality, then this term may be estimated as required. In order to provide an adequate estimate also for the first term, we introduce the following assumption.

**Assumption 9.1.** The set $U$ from above can be chosen in such a way that $\Lambda := \Omega \cap U$ is again a domain and if one puts $\Gamma := (\partial \Omega \setminus D) \cap U$ and $E := \partial \Lambda \setminus \Gamma$, then there is a linear, continuous extension operator $\mathfrak{G} : W_1^{1,p}(\Lambda) \to W_1^{1,p}(\mathbb{R}^d)$.  

---

**Figure 2.** The domain $\Omega$ is the cube minus the triangle $\Sigma$. The Dirichlet boundary part $D$ consists exactly of the six outer sides of the cube minus the droplet $\Upsilon$ on the cover plate.
Clearly, this assumption is weaker than Condition [iii] in Theorem 3.1 in other words: Condition [iii] in Theorem 3.1 requires Assumption 9.1 to hold for an open set $U \supset \Omega$.

Let us discuss the sense of Assumption 9.1 in extenso. Philosophically spoken, it allows to focus on the extension not of the functions $\eta \v v$, which live on a set whose boundary does (possibly) not include the ‘nasty’ parts of $\partial \Omega \setminus D$ that are an obstruction against a global extension operator. In detail: one first observes that, for $\eta \v v \in C^0_c(U)$ and $v \v v \in W^{1,p}_D(\Omega)$, the function $\eta v|\Lambda$ even belongs to $W^{1,p}_E(\Lambda)$, see [19, Thm. 5.8]. Secondly, we have by the definition of $E$

$$\partial U \cap \Omega = (\partial U \cap \Omega) \setminus \Gamma \subset \partial \Lambda \setminus \Gamma = E.$$  

This shows that the ‘new’ boundary part $\partial U \cap \Omega$ of $\Lambda$ belongs to $E$ and is, therefore, uncritical in view of extension. Thirdly, one has $D = D \cap U \subseteq \partial \Omega \cap U \subset \partial \Lambda$, and it is clear that the ‘new Dirichlet boundary part’ $E$ includes the ‘old’ one $D$. Hence, the extension operator $\eta v$ may be viewed also as a continuous one between $W^{1,p}_D(\Omega)$ and $W^{1,p}_E(\Lambda)$. Thus, concerning $v = \eta v = \eta' v$ one is – mutatis mutandis – again in the situation of Theorem 3.1. Namely, $\eta v \in W^{1,p}_E(\Lambda)$ admits an extension $\eta v : W^{1,p}_D(\Omega) \subset W^{1,p}_E(\Lambda)$ satisfies the estimate $\|\eta v\|_{W^{1,p}_E(\Lambda)} \leq c\|v\|_{W^{1,p}_D(\Omega)}$, the constant $c$ being independent from $v$. This leads, as above, to a corresponding (continuous) extension operator $\eta v : W^{1,p}_D(\Omega) \rightarrow W^{1,p}_E(\Lambda)$. Here, of course, $\Lambda_\star$ has again to be defined as the connected component of $B \setminus D$ that contains $\Lambda$. Thus one may proceed again as in the proof of Theorem 3.1 and gets, for $v \in W^{1,p}_D(\Omega)$,

$$\int_{\Omega} \left( \frac{\eta v}{\text{dist}_D} \right)^p \, dx = \int_{\Lambda} \left( \frac{\eta v}{\text{dist}_D} \right)^p \, dx \leq \int_{\Lambda_\star} \left( \frac{\eta v}{\text{dist}_{\partial \Lambda_\star}} \right)^p \, dx \leq c\|v\|_{W^{1,p}_D(\Lambda_\star)} \leq c\|v\|_{W^{1,p}_D(\Omega)} \leq c\|v\|_{L^p(\Omega)}.$$  

since $\eta v$ belongs to $W^{1,p}_E(\Lambda) \subset W^{1,p}_D(\Lambda)$. Exploiting a last time Poincaré’s inequality, whose validity will be discussed in a moment, one gets the desired estimate.

When aiming at Poincaré’s inequality, it seems convenient to follow again the argument in the proof of Proposition 7.1 as pointed out above, the property $1 \not\in W^{1,p}_D(\Omega)$ to do only with the local behavior of $\Omega$ around the points of $D$, cf. Lemma 7.3. Hence, this will not be discussed further here.

Concerning the compactness of the embedding $W^{1,p}_D(\Omega) \hookrightarrow L^p(\Omega)$, one does not need the existence of a global extension operator $\mathcal{E} : W^{1,p}_D(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$. In fact, writing for every $v \in W^{1,p}_D(\Omega)$ again $v = \eta v + \eta' v$ and supposing Assumption 9.1 one gets the following:

If $\{v_k\}_k$ is a bounded sequence in $W^{1,p}_E(\Omega)$, then the sequence $\{\eta v_k\}_k$ is bounded in $W^{1,p}_E(\Lambda)$. Due to the extendability property, this sequence contains a subsequence $\{\eta v_{k_l}\}_l$ that converges in $L^p(\Lambda)$ to an element $v_\Lambda$. Thus, $\{\eta v_{k_l}\}_l$ converges to the function on $\Omega$ that equals $v_\Lambda$ on $\Lambda$ and 0 on $\Omega \setminus \Lambda$. The elements $\eta v_{k_l}$ in fact live on the set $\Pi := \Omega \cap V$ and are zero on $\Omega \setminus V$. In particular they are zero in a neighborhood of $D$. Moreover, they form a bounded subset of $W^{1,p}(\Pi)$. Therefore it makes sense to require that $\Pi$ is again a domain, and, secondly that $\Pi$ meets one of the well-known compactness criteria $W^{1,p}(\Pi) \hookrightarrow \text{Lip}(\Pi)$, cf. [33, Ch. 1.4.6]. Keep in mind that such requirements are much weaker than the global $W^{1,p}$-extendability, and in particular include the example in Figure 2 as long as the triangle $\Sigma$ has a positive distance to the six outer sides of the cube. Resting on these criteria, one obtains again the convergence of a subsequence $\{\eta v_{k_l}\}_l$ that converges in $L^p(\Pi)$ towards a function $v_\Pi$. The sequence $\{\eta v_{k_l}\}_l$ then converges in $L^p(\Omega)$ to a function that equals $v_\Pi$ on $\Pi$ and zero on $\Omega \setminus V$.

Altogether, we have extracted a subsequence of $\{v_k\}_k$ that converges in $L^p(\Omega)$.

Remark 9.2. In fact one does not really need that $\Pi$ is connected. By similar arguments as above it suffices to demand that it splits up in at most finitely many components $\Pi_1, \ldots, \Pi_n$, such that each of these admits the compactness of the embedding $W^{1,p}(\Pi_j) \hookrightarrow L^p(\Omega)$. 

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We summarize these considerations in the following theorem.

**Theorem 9.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $D \subset \partial \Omega$ be a closed part of the boundary.

Suppose that the following three conditions are satisfied:

(i) The set $D$ is $l$-thick for some $l \in [d - p, d]$.
(ii) The space $W^{1,p}_D(\Omega)$ can be equivalently normed by $\| \nabla \cdot \|_{L^p(\Omega)}$.
(iii) There are two open sets $U, V \subset \mathbb{R}^d$ that satisfy (9.1) and $U$ satisfies Assumption 9.1.

Then there is a constant $c > 0$ such that Hardy's inequality

$$
\int_{\Omega} \frac{|u|}{\text{dist}_D}^p \, dx \leq c \int_{\Omega} |\nabla u|^p \, dx
$$

holds for all $u \in W^{1,p}_D(\Omega)$.

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