Chern-Ricci curvatures, holomorphic sectional curvature and Hermitian metrics

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Abstract We present some formulae related to the Chern-Ricci curvatures and scalar curvatures of special Hermitian metrics. We prove that a compact locally conformal Kähler manifold with the constant nonpositive holomorphic sectional curvature is Kähler. We also give examples of complete non-Kähler metrics with pointwise negative constant but not globally constant holomorphic sectional curvature, and complete non-Kähler metrics with zero holomorphic sectional curvature and nonvanishing curvature tensors.

Keywords Chern-Ricci curvatures, holomorphic sectional curvature, locally conformal Kähler metric, k-Gauduchon metric

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Introduction

This paper mainly concerns Hermitian manifolds with constant or pointwise constant holomorphic sectional curvature. For a general Hermitian manifold \((M,\omega)\), the (Chern) holomorphic sectional curvature \(H\) is defined by

\[ H(X) = \frac{R(X, X, X, X)}{|X|^4}, \]

where \(R\) is the curvature tensor of the Chern connection and \(X \in T^{1,0}_p(M)\) (see [19,42]). The holomorphic sectional curvature plays a fundamental role in complex geometry. Complete Kähler manifolds with constant holomorphic sectional curvature are called complex space forms [42]. They are natural analogues of complete Riemannian manifolds with constant sectional curvature. It is known (see [13,17]) that a simply connected complex space form is holomorphically isometric to the complex projective space \(\mathbb{CP}^n\), the complex hyperbolic space \(\mathbb{B}^n\) or \(\mathbb{C}^n\).

In [2], Balas and Gauduchon proved that a compact Hermitian surface with constant nonpositive holomorphic sectional curvature must be Kähler. In higher dimensions, it is known that there are examples of compact non-Kähler manifolds with \(H = 0\) (e.g., the Iwasawa manifold, see [1]). A natural question is: if \((M,\omega)\) is an \(n\)-dimensional compact Hermitian manifold with constant (or pointwise constant) negative holomorphic sectional curvature and \(n \geq 3\), is \((M,\omega)\) still Kähler?

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The method of Balas-Gauduchon depends heavily on $n = 2$. When $n \geq 3$, we restrict ourselves to locally conformal Kähler manifolds. There are rich examples of locally conformal Kähler manifolds, including the elliptic surfaces, diagonal Hopf manifolds and flat principal circle bundles over a compact Sasakian manifold [6]. We prove the following result.

**Theorem 1.1.** Let $(M, \omega)$ be a compact locally conformal Kähler manifold with constant nonpositive holomorphic sectional curvature. Then $(M, \omega)$ is Kähler. In particular, the universal cover of $(M, \omega)$ is the complex hyperbolic space $\mathbb{B}^n$ or $\mathbb{C}^n$.

The proof of Theorem 1.1 is based on a relation between the first and second Chern-Ricci curvatures for locally conformal Kähler metrics. In [23], Liu and Yang systematically studied a variety of Ricci curvatures on a Hermitian manifold. Among other results, they derived explicit relations between all kinds of Ricci curvatures on general Hermitian manifolds. In the locally conformal Kähler case, we can get a simpler formula (see Proposition 3.2). Then we are able to reduce the theorem to the conformally Kähler case. Actually we prove the following result under the more general pointwise constant condition.

**Theorem 1.2.** Let $(M, \omega)$ be a compact locally conformal Kähler manifold with pointwise nonpositive constant holomorphic sectional curvature. Then $(M, \omega)$ is globally conformal Kähler.

We remark that Vaisman [32] has proved that a locally conformal Kähler metric with pointwise constant (Chern) sectional curvature is either globally conformal Kähler or has vanishing first Chern class. The constancy of sectional curvature is of course stronger than the constancy of holomorphic sectional curvature. For example, the sectional curvatures of $\mathbb{P}^n$ and $\mathbb{B}^n (n \geq 2)$ are not pointwise constant.

An important class of locally conformal Kähler manifolds is called Vaisman manifolds, whose Lee forms are parallel with respect to the Levi-Civita connection. It is shown in [26] that a Vaisman metric on a compact manifold must be Gauduchon. Then we obtain the following corollary.

**Corollary 1.3.** A compact Vaisman manifold with pointwise nonpositive constant holomorphic sectional curvature is Kähler.

Considering the constancy of holomorphic sectional curvature, a natural question is: does the pointwise constancy of $H$ imply the global constancy?

When $\omega$ is Kähler and $n \geq 2$, it is always true by Schur’s lemma, as $\omega$ is Kähler-Einstein and $H$ is a constant multiple of the scalar curvature (see Proposition 3.4). If $\omega$ is non-Kähler, we construct counterexamples showing that the Schur type result does not hold in general (see Example 3.8).

**Proposition 1.4.** There exist non-Kähler, conformally flat metrics on $\mathbb{C}^n (n \geq 2)$ with pointwise negative constant (or pointwise positive constant) but not globally constant holomorphic sectional curvature. In the negative case, the metric is complete.

**Remark 1.5.** If the holomorphic sectional curvature is defined by using the Levi-Civita connection, it is Gray and Vanheche [11] who first discovered that the Schur type result does not hold in the non-Kähler setting. Since the Levi-Civita connection coincides with the Chern connection if and only if the metric is Kähler, our results are obviously different from theirs. We refer to [11, 27, 28] and the references therein for more results and development in that direction. Also see [21] for some recent results on almost Kähler 4-manifolds with constant nonnegative (Chern) holomorphic sectional curvature.

Using the conformal change technique, we also show the following (see Example 3.9).

**Proposition 1.6.** There exists a complete non-Kähler, conformal Kähler metric on $\mathbb{C}^n (n \geq 2)$ with zero holomorphic sectional curvature but a nonvanishing curvature tensor.

To the authors’ knowledge, all the previously known examples of non-Kähler manifolds with $H = 0$ have vanishing curvature (the quotient of complex Lie groups [3]). Our example implies that the holomorphic sectional curvature does not necessarily determine the curvature tensor of a Hermitian metric. Also, it shows that the compactness condition in Theorem 1.1 cannot be replaced by completeness in the $H = 0$ case. It would be an interesting question to study whether there exist similar examples to that in Proposition 1.4 on compact manifolds.

We also discuss another notion of special Hermitian metrics, which is called the $k$-Gauduchon metric.
It is introduced by Fu et al. [7] as a generalization of the Gauduchon metric. A $k$-Gauduchon metric is a Hermitian metric satisfying

$$\sqrt{-1} \partial \bar{\partial} \omega^k \wedge w^{n-k-1} = 0.$$ 

In particular, a pluriclosed metric (i.e., $\partial \bar{\partial} \omega = 0$) is 1-Gauduchon, while an Astheno-Kähler metric (i.e., $\partial \bar{\partial} \omega^{n-2} = 0$) is $(n-2)$-Gauduchon. We have the following characterization.

**Proposition 1.7.** Let $(M, \omega)$ be an $n$-dimensional compact Hermitian manifold ($n \geq 3$) and $k$ be an integer such that $1 \leq k \leq n-1$. Then the following are equivalent:

1. $\omega$ is $k$-Gauduchon;
2. $s - \hat{s} = \frac{k-1}{n-k} |\partial^* \omega|^2 + \frac{n-1-k}{n-2} |\partial \omega|^2$,

where $s$ is the Chern scalar curvature and $\hat{s}$ is a Riemannian type scalar curvature of $\omega$ with respect to the Chern connection.

**Remark 1.8.** Another characterization in terms of scalar curvatures of the Bismut connection (also called the Strominger connection [41]) is obtained in [8]. A direct corollary is that if $(M, \omega)$ is $k$-Gauduchon, then $s \geq \hat{s}$.

We mention that recently there have been breakthroughs on Kähler manifolds with negative or positive holomorphic sectional curvature. A conjecture of Yau on negative holomorphic sectional curvature is Kobayashi hyperbolic [10]. Its canonical bundle is also conjectured to be ample (see [20, 39] for some recent progress).

The rest of this paper is as follows. In Section 2, we give some background of Hermitian geometry. In Section 3, we prove Theorems 1.1 and 1.2 and give the examples mentioned above (see Examples 3.8–3.10). Finally, we study some properties of the $k$-Gauduchon metric and prove Proposition 1.7 (see Proposition 4.5).

## 2 The torsion 1-form and Chern-Ricci curvatures

In this section, we give some background materials in Hermitian geometry. We will present some formulae related to the Chern connection. The readers are referred to [4, 22, 23, 33, 37, 42] for more details. We remark that we do not make use of any good coordinates in our discussion. We will use Einstein summation notation throughout the paper.

### 2.1 Operators on Hermitian manifolds

Let $(M, g)$ be a $2n$-dimensional Riemannian manifold. Write $g = g_{ij} dx^i dx^j$, where $(x^1, x^2, \ldots, x^{2n})$ is a local coordinate. Denote $(g^{ij})$ to be the inverse matrix of $(g_{ij}), 1 \leq i, j \leq 2n$. Then $g$ induces an inner product $\langle \cdot, \cdot \rangle$ on the cotangent bundle $T^* M$ by $\langle dx^i, dx^j \rangle = g^{ij}$. Let $\Lambda^k T^* M$, $1 \leq k \leq 2n$ be the bundle of real $k$-forms. The inner product induced by $g$ on $\Lambda^k T^* M$ is

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det((\alpha_i, \beta_j)), \quad \alpha_i, \beta_j \in T^* M. \quad (2.1)$$

Equivalently,

$$\langle \varphi, \psi \rangle = \frac{1}{k!} g^{i_1 j_1} \cdots g^{i_k j_k} \varphi_{i_1 \cdots i_k} \psi_{j_1 \cdots j_k}, \quad (2.2)$$

for

$$\varphi = \frac{1}{k!} \varphi_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \psi = \frac{1}{k!} \psi_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$
where \( \varphi_{i_1 \cdots i_k} \) is skew symmetric in \( i_1, \ldots, i_k \) and \( \psi_{j_1 \cdots j_k} \) is skew symmetric in \( j_1, \ldots, j_k \). For \( X \in TM \) and \( \varphi \in \Lambda^k T^* M \), define the contraction (or interior product) \( \iota_X \varphi \in \Lambda^{k-1} T^* M \) by

\[
\iota_X \varphi(X_1, \ldots, X_{k-1}) := \varphi(X, X_1, \ldots, X_{k-1}).
\]

We have

\[
\iota_X (\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha_i(X) \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_k.
\]

Denote \( \widetilde{X} = g(X, \cdot) \in T^* M \) to be the metric dual of \( X \). Then we have

\[
\langle \iota_X \varphi, \psi \rangle = \langle \varphi, \widetilde{X} \wedge \psi \rangle \tag{2.3}
\]

for \( \varphi \in \Lambda^{k+1} T^* M \) and \( \psi \in \Lambda^k T^* M \). Indeed, if \( \varphi = \alpha_1 \wedge \cdots \wedge \alpha_{k+1}, \quad \psi = \beta_1 \wedge \cdots \wedge \beta_k \),

where \( \alpha_i, \beta_j \in T^* M \), then from (2.1) we get

\[
\langle \varphi, \widetilde{X} \wedge \psi \rangle = \sum_{i=1}^{k+1} (-1)^{i-1} \langle \alpha_i, \widetilde{X} \rangle \langle \alpha_1 \wedge \cdots \wedge \alpha_{k+1}, \beta_1 \wedge \cdots \wedge \beta_k \rangle
\]

\[
= \left( \sum_{i=1}^{k+1} (-1)^{i-1} \alpha_i(X) \alpha_1 \wedge \cdots \wedge \alpha_{k+1}, \beta_1 \wedge \cdots \wedge \beta_k \right)
\]

\[
= \langle \iota_X \varphi, \psi \rangle.
\]

The general case follows by linear expansion.

Now assume that \( (M, J) \) is a complex manifold. If the Riemannian metric \( g \) satisfies \( g(X, Y) = g(JX, JY) \) for all \( X, Y \in TM \), then \( g \) is called a Hermitian metric. The fundamental \((1,1)\) form associated with \( g \) is given by \( \omega(X, Y) = g(JX, Y) \). Let \( TM^C = TM \otimes_R \mathbb{C} \) be the complexified tangent bundle. Denote \( h \) to be the \( \mathbb{C} \)-linear extension of \( g \) to \( TM^C \). Let \( (z^1, z^2, \ldots, z^n) \) be a local holomorphic coordinate and

\[
h_{i\bar{j}} = h \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right).
\]

Locally, we have

\[
\omega = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j.
\]

We also refer to \((M, \omega)\) as a Hermitian manifold.

Let \( \Omega^{p,q} M \) be the space of \((p, q)\) forms on \( M \), \( 1 \leq p, q \leq n \). Extend the inner product \( \langle \cdot, \cdot \rangle \) on \( \Lambda^k T^* M \) to \( \Omega^{p,q} M \) in the following way:

\[
\langle b \varphi_1 + c \varphi_2, \psi \rangle = b \langle \varphi_1, \psi \rangle + c \langle \varphi_2, \psi \rangle,
\]

\[
\langle \varphi, b \psi_1 + c \psi_2 \rangle = \overline{b} \langle \varphi, \psi_1 \rangle + \overline{c} \langle \varphi, \psi_2 \rangle,
\]

for \( b, c \in \mathbb{C} \) and \( \varphi, \psi \in \Lambda^k T^* M \). By this extension, for example,

\[
\langle dz^i, dz^j \rangle = h^{i\bar{j}},
\]

where \((h^{i\bar{j}})\) is the (transposed) inverse matrix of \((h_{i\bar{j}})\) (see [23] for more details about the relation between \( g_{i\bar{j}} \) and \( h_{i\bar{j}} \)). Also \( \langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle \). Write \((p, q)\) forms \( \varphi \) and \( \psi \) in local coordinates

\[
\varphi = \frac{1}{p!q!} \varphi_{i_1 \cdots i_p k_1 \cdots k_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q},
\]

\[
\psi = \frac{1}{p!q!} \psi_{j_1 \cdots j_p k_1 \cdots k_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q},
\]
where \( \varphi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \) is skew symmetric in \( i_1, \ldots, i_p \) and skew symmetric in \( l_1, \ldots, l_q \) (similarly for \( \psi_{j_1 \ldots j_p \bar{j}_1 \ldots \bar{j}_q} \)). Then

\[
\langle \varphi, \psi \rangle = \frac{1}{p!q!} h^{i_1 j_1} \ldots h^{i_p j_p} k^{k_1 \bar{k}_1} \ldots k^{k_q \bar{k}_q} \varphi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \psi_{j_1 \ldots j_p \bar{j}_1 \ldots \bar{j}_q}. \tag{2.4}
\]

This extends the formula (2.2).

Let \( dv \) be the volume form of \( g \). The total inner product is defined to be

\[
(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle dv.
\]

Denote \(|\varphi|^2 = \langle \varphi, \varphi \rangle\). The total norm is

\[
\|\varphi\|^2 = \int_M |\varphi|^2 dv.
\]

The Hodge \( * \) operator is the unique operator determined by \( g \) satisfying the following:

\[
* : \Lambda^{k}T^* M \to \Lambda^{2n-k}T^* M, \quad \varphi \wedge *\psi = \langle \varphi, \psi \rangle dv,
\]

where \( \varphi, \psi \in \Lambda^{k}T^* M \). It is extended \( \mathbb{C} \)-linearly to complex forms and satisfies

\[
* : \Omega^{p,q} M \to \Omega^{n-p,n-q} M, \quad \varphi \wedge *\psi = \langle \varphi, \psi \rangle dv
\]

for \( \varphi, \psi \in \Omega^{p,q} M \) and \( dv = \omega_n^2 \). Also, we have

\[
\overline{*\varphi} = *\overline{\varphi}, \quad * *\varphi = (-1)^{p+q}\varphi, \quad (*\varphi, *\psi) = \langle \varphi, \psi \rangle.
\]

The formal adjoint operators \( \partial^* \) and \( \bar{\partial}^* \) are given by

\[
(\partial\varphi, \psi) = (\varphi, \partial^* \psi) \quad \text{and} \quad (\bar{\partial}\varphi, \psi) = (\varphi, \bar{\partial}^* \psi).
\]

It is well known that

\[
\partial^* = - * \bar{\partial}^* \quad \text{and} \quad \bar{\partial}^* = -* \partial^*
\]

on compact Hermitian manifolds. Write \( \iota_j \varphi = \iota \frac{\partial}{\partial z_j} \varphi \) and \( \iota_j \psi = \iota \frac{\partial}{\partial \bar{z}_j} \psi \) for convenience. It follows from (2.3) that

\[
\langle \varphi, dz^i \wedge \psi \rangle = \langle h^{i j} \iota_j \varphi, \psi \rangle, \quad \langle \varphi, d\bar{z}^i \wedge \psi \rangle = \langle h^{i j} \iota_j \varphi, \psi \rangle. \tag{2.5}
\]

The Lefschetz operator \( L : \Omega^{p,q} M \to \Omega^{p+1,q+1} M \) and its adjoint \( \Lambda : \Omega^{p+1,q+1} M \to \Omega^{p,q} M \) are defined by

\[
L \varphi = \omega \wedge \varphi \quad \text{and} \quad \langle L \varphi, \psi \rangle = \langle \varphi, \Lambda \psi \rangle.
\]

From (2.5), in local coordinates, we get

\[
\Lambda = \sqrt{-1} h^{i j} \iota_i \iota_j. \tag{2.6}
\]

For \( \varphi \in \Omega^{p,q} M \) with \( p + q = k \), a direct computation gives (see also [33])

\[
\Lambda(\omega \wedge \varphi) = (n-k)\varphi + \omega \wedge (\Lambda \varphi),
\]

i.e.,

\[
[L, \Lambda] \varphi = (k-n)\varphi. \tag{2.7}
\]

Applying this equality repeatedly, we have

\[
[L^s, \Lambda] \varphi = [L^{s-1}, \Lambda] L^s \varphi + s(k-n+s-1)L^{s-1}\varphi.
\]

In particular, for \( s = r \),

\[
[L^r, \Lambda] \varphi = r(k-n+r-1)L^{r-1}\varphi. \tag{2.8}
\]
Definition 2.1. A \((p, q)\) form \(\varphi\) is called primitive if \(\Lambda \varphi = 0\).

For a primitive \(\varphi \in \Omega^{p,q} M\) with \(p + q = k\), we have

\[
\Lambda (\omega \wedge \varphi) = (n-k)\varphi, \quad |\varphi \wedge \omega|^2 = (n-k)|\varphi|^2, \quad (2.9)
\]

\[
\Lambda L^r \varphi = r(n-k-r+1)L^{r-1} \varphi. \quad (2.10)
\]

2.2 The torsion 1-form

Let \(\nabla\) be the Chern connection of a Hermitian manifold \((M, J, h)\). It is the unique connection such that \(\nabla J = 0\), \(\nabla h = 0\) and the torsion tensor \(T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\) has vanishing \((1,1)\) part. By using local coordinates, the Christoffel symbols \(\Gamma^k_{ij}\) and the torsion \(T\) of \(\nabla\) are given by (see, e.g., [30])

\[
\Gamma^k_{ij} = h^{kl} \partial_i h_{jl} \quad \text{and} \quad T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = h^{kl} T_{ijl}.
\]

As \(\omega = \sqrt{-1} h_{ij} dz^i \wedge d\bar{z}^j\), then

\[
\partial \omega = \sqrt{-1} T_{ijk} dz^i \wedge dz^j \wedge d\bar{z}^k. \quad (2.11)
\]

The torsion 1-form of the Chern connection is a \((1,0)\)-form defined by \(\tau = \tau_i dz^i\) with \(\tau_i = T^k_{ik} = h^{kl} T_{ikl}\). By using (2.6) and (2.11), a direct computation gives

\[
\tau = \Lambda \partial \omega = \sum_{k=1}^n T^k_{ik} dz^i. \quad (2.12)
\]

The following result is well known on Hermitian manifolds.

Lemma 2.2. Let \((M, \omega)\) be an \(n\)-dimensional Hermitian manifold. Then

\[
\partial \omega^{n-1} = \tau \wedge \omega^{n-1}. \quad (2.13)
\]

Proof. Define \(\alpha_0 = \partial \omega - \frac{1}{n-1} L (\Lambda \partial \omega)\). Then \(\Lambda \alpha_0 = 0\) by (2.9). This is equivalent to \(L^{n-2} \alpha_0 = 0\) (see [33]), i.e.,

\[
\left( \partial \omega - \frac{1}{n-1} L \Lambda \partial \omega \right) \wedge \omega^{n-2} = 0.
\]

Then

\[
\partial \omega^{n-1} = (n-1) \partial \omega \wedge \omega^{n-2} = L (\Lambda \partial \omega) \wedge \omega^{n-2} = \tau \wedge \omega^{n-1},
\]

where the last equality follows from (2.12).

Let \(\theta = \frac{1}{n-1} (\tau + \overline{\tau})\). It is called the Lee form of \(\omega\). Then (2.13) is also equivalent to

\[
d \omega^{n-1} = (n-1) \theta \wedge \omega^{n-1}.
\]

The following lemma will be used frequently. It may be obtained by the Bochner formula on Hermitian manifolds (see, e.g., [9,23]). We provide a naive proof here.

Lemma 2.3. Let \((M, \omega)\) be a compact Hermitian manifold and \(\tau = T^k_{ik} dz^i\) be the torsion 1-form of the Chern connection. Then

\[
\tau = \Lambda \partial \omega = -\sqrt{-1} \partial^* \omega. \quad (2.14)
\]
Proof. Let \( \varphi \) be any \((1,0)\) form. As \( *\omega = \frac{\omega^{n-1}}{(n-1)!} \), by (2.13),

\[
(\varphi, \bar{\partial}^* \omega) \frac{\omega^n}{n!} = \varphi \wedge \tau \wedge \frac{\omega^{n-1}}{(n-1)!}
\]

\[
= (\varphi \wedge \bar{\tau} \omega) \frac{\omega^n}{n!}
\]

\[
= (\varphi, \sqrt{-1} \tau) \frac{\omega^n}{n!}.
\]

Then we see \( \tau = -\sqrt{-1} \bar{\partial}^* \omega \). The first equality in (2.14) follows from (2.12).

Recall that a Hermitian metric is called balanced if \( d\omega^{n-1} = 0 \), namely, \( \tau = 0 \). We can easily get the following fact (also see [1]).

**Corollary 2.4.** Let \((M, \omega)\) be a compact Hermitian manifold. If the torsion 1-form \( \tau \) is holomorphic, then \((M, \omega)\) is balanced.

**Proof.** By (2.14), \( \bar{\partial} \tau = -\sqrt{-1} \bar{\partial} \bar{\partial}^* \omega = 0 \), then we have

\[
\| \bar{\partial}^* \omega \|^2 = (\bar{\partial} \bar{\partial}^* \omega, \omega) = 0.
\]

So \( \tau = -\sqrt{-1} \bar{\partial} \bar{\partial}^* \omega = 0 \) and \( \omega \) is balanced. \( \square \)

### 2.3 Curvatures on Hermitian manifolds

Let \( R_{i\bar{j}k\bar{l}} \) be the components of the curvature tensor of the Chern connection \( \nabla \). Then

\[
R_{i\bar{j}k\bar{l}} = -h_{p\bar{i}} \partial_j T^p_{k\bar{l}}.
\]

Also, the following commutative relations hold (see, e.g., [25, 29]):

\[
\begin{align*}
R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} &= -\nabla_j T_{i\bar{k}l}, \\
R_{i\bar{j}k\bar{l}} - R_{i\bar{k}\bar{l}j} &= -\nabla_i T_{j\bar{k}l}, \\
R_{i\bar{j}k\bar{l}} - R_{i\bar{k}\bar{l}j} &= -\nabla_l T_{j\bar{k}i} - \nabla_k T_{j\bar{i}l}.
\end{align*}
\]

The \( i \)-th Chern-Ricci form \( \rho^{(i)} = \sqrt{-1} \rho^{(i)}_j dz^i \wedge d\bar{z}^j \), \( 1 \leq i \leq 4 \) is defined by

\[
\begin{align*}
\rho^{(1)}_{ij} &= h^{kl} R_{i\bar{j}k\bar{l}}, \\
\rho^{(2)}_{ij} &= h^{kl} R_{i\bar{k}l\bar{j}}, \\
\rho^{(3)}_{ij} &= h^{kl} R_{i\bar{l}k\bar{j}}, \\
\rho^{(4)}_{ij} &= h^{kl} R_{i\bar{k}\bar{j}l}.
\end{align*}
\]

The two scalar curvatures are

\[
\begin{align*}
s &= \hat{h}^{i\bar{j}} h^{kl} R_{i\bar{j}k\bar{l}} \quad \text{and} \quad \hat{s} &= \hat{h}^{i\bar{j}} h^{kl} R_{i\bar{j}k\bar{l}},
\end{align*}
\]

where \( s \) is the usual Chern scalar curvature and \( \hat{s} \) is a Riemannian type scalar curvature. Both \( s \) and \( \hat{s} \) are real.

**Proposition 2.5** (See [9]). Let \((M, \omega)\) be a compact Hermitian manifold. Then

\[
s - \hat{s} = (\partial \partial^* \omega, \omega) = d^* \tau + |\tau|^2.
\]

**Proof.** By (2.14),

\[
\partial \partial^* \omega = -\sqrt{-1} \partial_i T^i_{j\bar{i}} dz^i \wedge d\bar{z}^j.
\]

Then from (2.16), we have

\[
s - \hat{s} = -h^{i\bar{j}} \partial_i T^i_{j\bar{i}} = (\partial \partial^* \omega, \omega).
\]

As \( \tau = -\sqrt{-1} \bar{\partial} \bar{\partial}^* \omega \), the last equality of (2.18) follows from the lemma below. \( \square \)
Lemma 2.6. Let $(M, \omega)$ be a compact Hermitian manifold and $\phi$ be a $(1,0)$ form on $M$. Then
\[
\partial^* \phi = \langle \sqrt{-1}\partial \phi, \omega \rangle + \langle \phi, \sqrt{-1}\partial^* \omega \rangle.
\] (2.21)

Proof. For any $(1,0)$ form $\varphi$, we have
\[
\varphi \wedge *\phi = \langle \sqrt{-1}\varphi \wedge \bar{\phi}, \omega \rangle \frac{\omega^n}{n!} = \sqrt{-1}\varphi \wedge \bar{\phi} \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]
Then we see
\[
*\phi = -\sqrt{-1}\phi \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]
By (2.13) and (2.14), we have
\[
\partial^* \phi = *\bar{\partial} \left( \sqrt{-1}\phi \wedge \frac{\omega^{n-1}}{(n-1)!} \right)
= * \left( \sqrt{-1}(\bar{\partial} \phi - \phi \wedge \varpi) \wedge \frac{\omega^{n-1}}{(n-1)!} \right)
= \langle \sqrt{-1}\bar{\partial} \phi, \omega \rangle + \langle \phi, \sqrt{-1}\partial^* \omega \rangle.
\]
This completes the proof. □

To compare the Ricci curvatures, we introduce some notions first. For each $1 \leq i \leq n$, define local $(2,0)$ forms $\xi^i = \frac{1}{2}T_{jk}^i dz^j \wedge dz^k$ as in [37] (see more details there). The column vector of the torsion 2-forms $(\xi^i)$ is denoted by $\xi$. Denote
\[
^t \xi \wedge h \check{\xi} = \frac{1}{4} h_{ijl}^k T_{jk}^l T_{ij}^l dz^j \wedge dz^k \wedge dz^r \wedge dz^s.
\]
Then $^t \xi \wedge h \check{\xi}$ is independent of the local coordinates. It is a global nonnegative $(2,2)$ form defined on $M$ and vanishes if and only if $T = 0$ (see [37]). By (2.6), we have
\[
\Lambda(\xi, \check{\xi}) = \sqrt{-1} h^l \xi^i T_{ijl}^k T_{jik}^m dz^k \wedge dz^l.
\] (2.22)

The following relation is established in [23, Theorem 4.1]. We give a direct proof here.

Proposition 2.7 (See [23]). Let $(M, h)$ be a compact $n$-dimensional Hermitian manifold and $^t \xi \wedge h \check{\xi}$ be the $(2,2)$ form defined as above. Then
\[
\rho^{(1)} - \rho^{(2)} = \Lambda(\sqrt{-1} \partial \check{\omega}, \omega + \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega - \Lambda(\xi, \check{\xi}).
\] (2.23)

Proof. From (2.17), we have
\[
\rho^{(1)}_{ij} - \rho^{(2)}_{ij} = h^{kl} (\nabla_k T_{ijkl} - \nabla_j T_{ikl}).
\] (2.24)
Recall that $\bar{\partial}^* \omega = \sum_{k=1}^n \sqrt{-1} T_{ik}^l dz^l$, so
\[
\bar{\partial} \bar{\partial}^* \omega = -\sqrt{-1} \sum_{k=1}^n \nabla_j T_{ik}^l dz^l \wedge dz^l.
\] (2.25)
Then (2.23) follows from (2.24) and the next lemma. □

Lemma 2.8. Let $(M, h)$ be a compact Hermitian manifold and $\omega = \sqrt{-1} h^i j dz^i \wedge dz^j$. Then
\[
\sqrt{-1} h^i j \nabla_i T_{jik}^l dz^k \wedge dz^l = \Lambda(\sqrt{-1} \partial \check{\omega}) + \partial \partial^* \omega - \Lambda(\xi, \check{\xi}).
\] (2.26)

Proof. A direct computation gives that
\[
\Lambda(\sqrt{-1} \partial \check{\omega}) = \sqrt{-1} h^i j l \sum_{p=1}^n \left( \frac{1}{4} (\partial_p T_{qil} - \partial_k T_{qil}) dz^p \wedge dz^k \wedge dz^q \wedge dz^l \right)
= \sqrt{-1} h^i j l (\partial_N T_{jik} - \partial_k T_{jil}) dz^k \wedge dz^l
= \sqrt{-1} h^i j l (\nabla T_{jik} + \nabla_k T_{jil} + h^{rs} T_{iks} T_{jir}) dz^k \wedge dz^l.
\]
Then (2.26) follows from (2.19) and (2.22). □
3 LCK metrics and holomorphic sectional curvature

In this section, we will discuss locally conformal Kähler (LCK for short) manifolds and constant (or pointwise constant) holomorphic sectional curvature. Then we prove Theorem 1.1 and give examples of non-Kähler manifolds with constant or pointwise constant holomorphic sectional curvature.

Let \((M, \omega)\) be an \(n\)-dimensional Hermitian manifold and \(n \geq 2\). A Hermitian metric \((M, \omega)\) is called locally conformal Kähler if

\[
d\omega = \theta \wedge \omega.
\]  

(3.1)

and \(\theta\) is closed. By (2.9) and (2.12), the real 1-form \(\theta\) is just the Lee form of \(\omega\) and (3.1) is equivalent to

\[
\partial \omega = \frac{1}{n-1} \tau \wedge \omega.
\]  

(3.2)

If in addition \(\theta\) is exact, then \((M, \omega)\) is globally conformal Kähler. Indeed, if \(\theta = df\) for some smooth function \(f\) on \(M\), then (3.1) implies that \(e^{-f} \omega\) is a Kähler metric.

Note that (3.2) is also valid on any compact Hermitian surfaces. We have the following simple observation.

**Proposition 3.1.** Let \((M, \omega)\) be a compact Hermitian surface. If the second Chern-Ricci form \(\rho^{(2)}\) is closed, then \((M, \omega)\) is a Kähler surface. In particular, if \(\rho^{(1)} = \rho^{(2)}\), then \(\omega\) is Kähler.

**Proof.** By [9, Lemme 1.12], we have

\[
\rho^{(1)} - \rho^{(2)} = \partial \partial^* \omega - \partial^* \partial \omega.
\]

As \(\rho^{(1)}\) is closed, if in addition \(\rho^{(2)}\) is closed, then

\[
\|\partial \partial^* \omega\|^2 = (\partial \partial^* \omega, \partial \omega) = 0.
\]

It follows that \((\partial^* \partial \omega, \omega) = \|\partial \omega\|^2 = 0\), i.e., \(\omega\) is Kähler. \(\square\)

For LCK manifolds or surfaces, we have the following relations between \(\rho^{(1)}\) and \(\rho^{(2)}\).

**Proposition 3.2.** Let \((M, \omega)\) be a compact locally conformal Kähler manifold. Then

\[
\rho^{(1)} - \rho^{(2)} = \frac{1}{n-1} ((\hat{s} - s) \omega + n \partial \partial^* \omega).
\]  

(3.3)

If \((M, \omega)\) is a compact Hermitian surface, then

\[
\rho^{(1)} - \rho^{(2)} = (\hat{s} - s) \omega + \partial \partial^* \omega + \partial \partial^* \omega.
\]  

(3.4)

**Proof.** If \(\omega\) is locally conformal Kähler or \(n = 2\), then

\[
\partial \omega = \frac{1}{n-1} \tau \wedge \omega.
\]

In local coordinates,

\[
\partial \omega = \frac{\sqrt{-1}}{2} T_{ijk} dz^i \wedge dz^j \wedge dz^k,
\]

\[
\tau \wedge \omega = \frac{\sqrt{-1}}{2} (h_{jk} T_{il} - h_{ik} T_{jl}) dz^i \wedge dz^j \wedge dz^k.
\]

Then we have

\[
(n-1) T_{ijk} = h_{jk} T_{il} - h_{ik} T_{jl}.
\]  

(3.5)

By (2.24),

\[
\rho^{(1)}_{ij} - \rho^{(2)}_{ij} = h^{lij} (\nabla_k T_{lj} - \nabla_j T_{kl}).
\]  

(3.6)
By (3.5), we get
\[
\sqrt{-1} h^{ik} \nabla_k T_{jkl} dz^i \wedge d\bar{z}^j = \frac{\sqrt{-1}}{n-1} h^{ik} \nabla_k (h_{ij} T_{is}^* - h_{il} T_{js}^*) dz^i \wedge d\bar{z}^j = \frac{1}{n-1} (\partial \partial^* \omega - (s - \bar{s}) \omega).
\]  
(3.7)

For a locally conformal Kähler metric, since \( d(\tau + \bar{\tau}) = 0 \), we have
\[
\partial \partial^* \omega - \bar{\partial} \bar{\partial}^* \omega = 0.
\]
Combining (3.6), (3.7) and (3.8), we get (3.2) and (3.4).

**Remark 3.3.** The relations between all kinds of Ricci curvatures on general Hermitian manifolds are obtained in [23]. We get the simpler formula (3.2) due to the locally conformal Kähler condition in our case.

Now we consider Hermitian metrics with constant holomorphic sectional curvature. For a Hermitian manifold \((M, \omega)\), the holomorphic sectional curvature is defined by
\[
H(X) = R(X, \bar{X}, X, \bar{X})/|X|^4,
\]
where \(X \in T_{p}^{1,0}(M)\) and \(R\) is the curvature tensor of the Chern connection. It is well known that when \(\omega\) is Kähler, \(H\) dominates the whole curvature tensor \(R\). In particular, \(H = c\) if and only if
\[
R_{ijkl} = \frac{1}{2} c(h_{ij} h_{kl} + h_{il} h_{kj}).
\]  
(3.8)

In this case, \((M, \omega)\) is isometric to a quotient of simply-connected complex space forms (see [42]). However, if \(\omega\) is not Kähler, (3.8) does not hold any more. Let \(K\) be the symmetric part of the curvature tensor \(R\) with components
\[
K_{ijkl} = \frac{1}{4} (R_{ijkl} + R_{kijl} + R_{ikjl} + R_{klij}).
\]
In [1], Balas proved the following proposition. For the reader’s convenience, we give a proof here.

**Proposition 3.4 (See [1]).** For a Hermitian manifold \((M, \omega)\), the holomorphic sectional curvature \(H = c\) at a point if and only if
\[
K_{ijkl} = \frac{1}{2} c(h_{ij} h_{kl} + h_{il} h_{kj}).
\]  
(3.9)

**Proof.** For \(p \in M\), let
\[
X = X^i \frac{\partial}{\partial z^i} \in T_{p}^{1,0} M.
\]
Then \(H = c\) at \(p\) if and only if
\[
R_{ijkl} X^i \bar{X}^j X^k \bar{X}^l = c|X|^4.
\]
We rewrite the above equality to get
\[
K_{ijkl} X^i \bar{X}^j X^k \bar{X}^l = c(h_{ij} X^i \bar{X}^j)^2 = \frac{1}{2} c(h_{ij} h_{kl} + h_{il} h_{kj}) X^i \bar{X}^j X^k \bar{X}^l.
\]
Note that both \(K_{ijkl}\) and \(h_{ij} h_{kl} + h_{il} h_{kj}\) are symmetric in \(i, k\) and also symmetric in \(j\) and \(l\). Thus the above equality holds for any \(X = X^i \frac{\partial}{\partial z^i}\) if and only if (3.9) is satisfied.

Now we prove the following result (see Theorem 1.2).
Theorem 3.5. Let \((M, \omega)\) be a compact Hermitian manifold. If \(\omega\) is locally conformal Kähler with pointwise nonpositive constant holomorphic sectional curvature, then \((M, \omega)\) is globally conformal Kähler.

Proof. Let \(H = c\), where \(c\) is a nonpositive smooth function on \(M\). Contract (3.9) with \(\omega\) to get (also see [1])

\[
\rho^{(1)} + \rho^{(2)} + 2\text{Re}\rho^{(3)} = 2(n + 1)c\omega, 
\]

\[
s + \dot{s} = n(n + 1)c. 
\]

As \(\omega\) is locally conformal Kähler, then \(\partial\bar{\partial}^*\omega = \bar{\partial}\partial^*\omega\). From (2.16) and (2.19), we have

\[
\rho^{(1)} = \rho^{(3)} + \partial\bar{\partial}^*\omega 
\]

and \(\rho^{(3)}\) is real. By Proposition 3.2, (3.10) and (3.11), we get

\[
\rho^{(3)} = \frac{1}{4(n - 1)}[(n + 1)(n - 2)c + 2\dot{s})\omega - (n - 2)\partial\bar{\partial}^*\omega].
\]

Also, from (3.12), \(\rho^{(3)}\) is a real closed \((1, 1)\) form. Take differential to (3.13) to get

\[
d(((n + 1)(n - 2)c + 2\dot{s})\omega) = 0.
\]

Let \(\phi = (n + 1)(n - 2)c + 2\dot{s}\). We will show that \(\phi\) is either everywhere nonzero or \(\phi \equiv 0\) (see also [32]).

1. If \(\phi\) is everywhere nonzero, then \(\omega\) is conformally Kähler and we are done.
2. Assume that \(\phi(p) = 0\) for some \(p \in M\). As \(\omega\) is locally conformal Kähler, we have

\[
d\phi + \phi\theta = 0
\]

and \(d\theta = 0\). Then \(\theta = du\) in a neighborhood of \(p\) for some real function \(u\). By (3.15), we have \(e^\theta d\phi + \phi d\theta = 0\). Then \(e^\theta \phi\) is constant near \(p\) and \(\phi \equiv 0\) in a neighborhood of \(p\). This implies that the set of points where \(\phi\) equals zero is open and closed. Thus \(\phi \equiv 0\) on \(M\).

We only need to discuss the case \(\phi \equiv 0\) to finish the proof.

(a) If \(c \equiv 0\), then \(s = \dot{s} = 0\) by (3.11). By (2.18),

\[
\int_M s dv = \int_M \dot{s} dv + \int_M |\tau|^2 dv.
\]

Then \(s = \dot{s}\) implies that \(\tau = 0\). As \(\omega\) is also locally conformal Kähler, we get \(d\omega = 0\) and \(\omega\) is Kähler.

(b) If \(c \leq 0\) and \(c < 0\) at least at one point on \(M\), as

\[
\dot{s} = -\frac{1}{2}(n + 1)(n - 2)c,
\]

by (3.11),

\[
s = \frac{1}{2}(n + 1)(3n - 2)c \leq 0 \leq \dot{s}.
\]

It follows that

\[
\int_M \dot{s} dv > 0 > \int_M s dv,
\]

which leads to a contradiction by (3.16).

Thus either \(\omega\) is Kähler or \(\phi\) is everywhere nonzero. Both imply that \(\omega\) is globally conformal Kähler.

By the uniqueness of the Gauduchon metric in each conformal class of a Hermitian metric, we also get the following corollary.

Corollary 3.6. Let \((M, \omega)\) be a compact Hermitian manifold. If \(\omega\) is locally conformal Kähler and Gauduchon with pointwise nonpositive constant holomorphic sectional curvature, then \((M, \omega)\) is Kähler.
In particular, any compact Vaisman manifold with pointwise nonpositive constant holomorphic sectional curvature is Kähler (see Corollary 1.3) as it is Gauduchon (see [26]).

Next, we consider holomorphic sectional curvature of globally conformal Kähler metrics. Assume \( \bar{\omega} = e^f \omega \), where \( f \) is a real smooth function. Denote \( \bar{R}, \bar{\rho}^{(i)}, \bar{s} \) and \( \tilde{s} \) to be the curvature tensor, Chern-Ricci curvatures and scalar curvatures of \( \bar{\omega} \). A direct computation gives that
\[
\bar{R}_{i j k l} = e^{f}(R_{i j k l} - \partial_i \partial_j f h_{k l}). \tag{3.17}
\]
Then we obtain
\[
\bar{\rho}^{(1)} = \rho^{(1)} - n\sqrt{-1}\partial \bar{\partial} f,
\bar{\rho}^{(2)} = \rho^{(2)} - \text{tr}_\omega(\sqrt{-1}\partial \bar{\partial} f)\omega,
\bar{\rho}^{(3)} = \rho^{(3)} - \sqrt{-1}\partial \bar{\partial} f. \tag{3.18}
\]
It follows that
\[
\bar{s} - \tilde{s} = e^{-f}(s - \hat{s}) - (n - 1)\text{tr}_\omega(\sqrt{-1}\partial \bar{\partial} f). \tag{3.19}
\]
When \( \omega \) is Kähler, we prove the following result.

**Proposition 3.7.** Let \((M, \bar{\omega})\) be a compact conformal Kähler manifold. If the holomorphic sectional curvature is nonpositive constant, then \((M, \bar{\omega})\) is Kähler.

**Proof.** Denote \( \bar{H} = c \) to be the holomorphic sectional curvature of \( \bar{\omega} \), where \( c \) is a nonpositive constant. The same argument as in Theorem 3.5 gives
\[
d((n + 1)(n - 2)c + 2\tilde{s})\bar{\omega}) = 0,
\bar{s} + \tilde{s} = n(n + 1)c.
\]
As \( \bar{\omega} = e^f \omega \) with \( d\omega = 0 \), we have
\[
((n + 1)(n - 2)c + 2\tilde{s})e^f = A
\]
for some constant \( A \). Then
\[
\tilde{s} - \hat{s} = n(n + 1)c - 2\tilde{s}
= 2(n - 1)(n + 1)c - Ae^{-f}.
\]
Putting it into (3.19), we get
\[
-(n - 1)\Delta f = 2(n - 1)(n + 1)ce^f - A, \tag{3.21}
\]
where \( \Delta f = \text{tr}_\omega \sqrt{-1}\partial \bar{\partial} f \). Let \( dv \) be the volume form of \( \omega \). Then
\[
\int_M \Delta f dv = \int_M \sqrt{-1}\partial \bar{\partial} f \wedge \frac{\omega^{n-1}}{(n - 1)!} = 0.
\]
If \( c = 0 \), then by (3.21), we see \( A = 0 \) and \( f \) is a constant. If \( c < 0 \), first we have
\[
\bar{\Delta} f = e^{-f} \Delta f = -\Delta e^{-f} + |\partial f|_\omega^2,
\tag{3.22}
\]
where \( \bar{\Delta} f = \text{tr}_\omega \sqrt{-1}\partial \bar{\partial} f \). From (3.21), we get
\[
2(n - 1)(n + 1)c - Ae^{-f} = (n - 1)(\Delta e^{-f} - |\partial f|_\omega^2). \tag{3.23}
\]
Integrating both sides of (3.21) with respect to the Kähler metric \( \omega \), we get
\[
A \text{vol}(M) = 2(n - 1)(n + 1)c \int_M e^f dv.
\]
As \( c < 0 \), we see \( A < 0 \). Then integrate both sides of (3.23) with respect to \( \omega \) to get
\[
A \int_M e^{-f} dv \geq 2(n-1)(n+1)\text{vol}(M).
\]
By the Cauchy-Schwarz inequality and the above equations, we have
\[
(\text{vol}(M))^2 \leq \int_M e^f dv \int_M e^{-f} dv \leq (\text{vol}(M))^2.
\]
The equality holds if and only if \( f \) is a constant. Therefore in either case, we derive that \( \omega \) is Kähler. \( \square \)

**Proof of Theorem 1.1.** The result follows from Theorem 3.5, Proposition 3.7 and the classical result for complex space forms \([13,17]\). \( \square \)

We use Proposition 3.4 and the conformal trick to construct complete non-Kähler metrics on \( \mathbb{C}^n \) with pointwise constant but not globally constant holomorphic sectional curvature. We also give an example of complete non-Kähler metrics on \( \mathbb{C}^n \) with zero holomorphic sectional curvature and nonvanishing curvature tensors. This gives the proofs of Propositions 1.4 and 1.6.

**Example 3.8.** Let \( \omega = \sum_{i=1}^n \sqrt{-1} dz^i \wedge d\bar{z}^i \) be the flat Euclidean metric on \( \mathbb{C}^n \) with curvature \( R = 0 \). Let
\[
f = c|z|^2 = c \sum_{i=1}^n |z_i|^2,
\]
where \( c \) is a nonzero real number and \( \tilde{\omega} = e^f \omega \). Then \( \partial_i \partial_j f = c \delta_{ij} \) and by (3.17), \( \tilde{R}_{ijkl} = -ce^f \delta_{ij} \delta_{kl} \).

The symmetric curvature tensor \( \tilde{K} \) is
\[
\tilde{K}_{ijkl} = \frac{1}{4} (\tilde{R}_{ijkl} + \tilde{R}_{kijl} + \tilde{R}_{iklj} + \tilde{R}_{ijkl})
\]
\[
= -\frac{ce^f}{2}(\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj})
\]
\[
= -\frac{ce^f}{2}(h_{ij} h_{kl} + h_{il}h_{kj}).
\]

By Proposition 3.4, the holomorphic sectional curvature \( \tilde{H} = -ce^{-f} \) is pointwise constant but not globally constant. When \( c > 0 \), we see that \( \tilde{\omega} = e^{|z|^2} \omega \) is complete.

**Example 3.9.** Let \( \omega = \sqrt{-1} \partial \bar{\partial} \log(1 + |z|^2) \) be the restriction of the Fubini-Study metric on \( \mathbb{C}^n \). Then
\[
h_{ij} = \frac{(1 + |z|^2) \delta_{ij} - \bar{z}_i z_j}{(1 + |z|^2)^2}.
\]
Also, the holomorphic sectional curvature of \( \omega \) is constant 2 (see \([42]\)), so we have
\[
R_{ijkl} = h_{ij} h_{kl} + h_{il} h_{kj}.
\]
Let \( f = 2 \log(1 + |z|^2) \) and \( \tilde{\omega} = e^f \omega \). As \( \partial_i \partial_j f = 2h_{ij} \), we have
\[
\tilde{R}_{ijkl} = e^f (h_{ij} h_{kl} + h_{il} h_{kj} - 2h_{ij} h_{kl}) = e^f (h_{ij} h_{kl} - h_{ij} h_{kl}).
\]
The symmetric curvature tensor is
\[
\tilde{K}_{ijkl} = \frac{1}{4} (\tilde{R}_{ijkl} + \tilde{R}_{kijl} + \tilde{R}_{iklj} + \tilde{R}_{ijkl}) = 0.
\]
So the holomorphic sectional curvature of \( \tilde{\omega} \) is zero, but the curvature is nowhere vanishing. Also,
\[
\tilde{\omega} = \sqrt{-1}((1 + |z|^2) \delta_{ij} - \bar{z}_i z_j)dz^i \wedge d\bar{z}^j
\]
is complete on \( \mathbb{C}^n \).
Example 3.10. Let \( \omega = -\sqrt{-1} \partial \bar{\partial} \log(1 - |z|^2) \) be the Bergman metric on the unit ball \( \mathbb{B}^n \). Then
\[
h_{ij} = \frac{(1 - |z|^2) \delta_{ij} + \bar{z}_i z_j}{(1 - |z|^2)^2}.
\]
Let \( f = 2 \log(1 - |z|^2) \) and \( \tilde{\omega} = e^f \omega \). The similar calculation to that in Example 3.9 gives that the holomorphic sectional curvature of \( \tilde{\omega} \) is zero and the curvature is nonzero everywhere. As
\[
\tilde{h}_{ij} = (1 - |z|^2) \delta_{ij} + \bar{z}_i z_j,
\]
\( \tilde{\omega} \) is not complete.

4 \( k \)-Gauduchon metrics

In this section, we study the \( k \)-Gauduchon metrics on complex manifolds.

Recall that a Gauduchon metric (or called standard metric) is a Hermitian metric satisfying \( \partial \bar{\partial} \omega \wedge \omega = 0 \). On a compact complex manifold, there exists a unique (up to a constant) Gauduchon metric in the conformal class of any Hermitian metric [9]. For \( 1 \leq k \leq n - 1 \), Fu et al. [7] considered the following equation:
\[
\partial \bar{\partial} \omega \wedge \omega - 1 = 0.
\]
(4.1)

A Hermitian metric is called \( k \)-Gauduchon if (4.1) is satisfied. We use the torsion 1-form and operators on Hermitian manifolds to study \( k \)-Gauduchon metrics. First, a direct computation gives
\[
\partial \partial \omega \wedge \omega - 1 - k(n - k - 1) \partial \omega \wedge \omega \wedge \omega^{n-3}.
\]
(4.2)

Lemma 4.1. Let \( (M, \omega) \) be a compact Hermitian manifold. Then
\[
*\left( \sqrt{-1} \partial \bar{\partial} \omega \wedge \omega \right) = (n - 1)! (\tilde{s} - s + |\partial^* \omega|^2).
\]
So \( \omega \) is Gauduchon if and only if \( s - \tilde{s} = |\partial^* \omega|^2 \).

Proof. As
\[
*\omega = \frac{\omega^{n-1}}{(n - 1)!},
\]
we have
\[
*\partial \bar{\partial} \omega^{n-1} = (n - 1)! \partial^* \tau = (n - 1)! \sqrt{-1} \partial^* \tau.
\]
Then the result follows from (2.18). \( \square \)

We will assume \( n \geq 3 \) in the following discussion.

Lemma 4.2. Let \( (M, \omega) \) be a compact Hermitian manifold. Then
\[
* \left( \sqrt{-1} \partial \omega \wedge \partial \omega \wedge \omega^{n-3} (n - 3)! \right) = |\partial^* \omega|^2 - |\partial \omega|^2,
\]
(4.3)

or equivalently,
\[
\Lambda^3(\sqrt{-1} \partial \omega \wedge \partial \omega) = 6(|\partial^* \omega|^2 - |\partial \omega|^2).
\]

Proof. First, as \( \Lambda \partial \omega = -\sqrt{-1} \partial^* \omega \) and \( \Lambda \partial \partial^* \omega = (n - 1) \partial \partial^* \omega \), we see the (2,1) form
\[
\partial \omega + \frac{\sqrt{-1}}{n - 1} \Lambda \partial \partial^* \omega
\]
is primitive. By [33, Proposition 6.29], for a primitive $\alpha \in \Omega^{p,q}$ with $p + q = k$,

$$\ast \alpha = (-1)^p (\sqrt{-1})^k \frac{L^{n-k} \alpha}{(n-k)!}. \quad (4.4)$$

So

$$\ast \partial \omega = \sqrt{-1} \partial \omega \wedge \frac{\omega^{n-3}}{(n-3)!} - \frac{n-2}{(n-1)!} \bar{\partial} \ast \omega \wedge \omega^{n-2} - \frac{\sqrt{-1}}{n-1} \ast L \bar{\partial} \ast \omega. \quad (4.5)$$

By $\ast L = \Lambda \ast$ and (4.4),

$$\frac{\sqrt{-1}}{n-1} \ast L \bar{\partial} \ast \omega = \frac{1}{n-1} \Lambda \left( \bar{\partial} \ast \omega \wedge \frac{\omega^{n-1}}{(n-1)!} \right)$$

$$= \bar{\partial} \ast \omega \wedge \frac{\omega^{n-2}}{(n-1)!},$$

where we use (2.10) in the last equality.

Therefore,

$$\ast \partial \omega = \sqrt{-1} \partial \omega \wedge \frac{\omega^{n-3}}{(n-3)!} - \bar{\partial} \ast \omega \wedge \frac{\omega^{n-2}}{(n-2)!}. \quad (4.6)$$

Thus with

$$\ast \bar{\partial} \ast \omega = \frac{\partial \omega^{n-1}}{(n-1)!},$$

we get

$$|\partial \omega|^2 dv = -\sqrt{-1} \partial \omega \wedge \bar{\partial} \omega \wedge \frac{\omega^{n-3}}{(n-3)!} - \partial \omega \wedge \partial \ast \omega \wedge \frac{\omega^{n-2}}{(n-2)!}$$

$$= -\sqrt{-1} \partial \omega \wedge \bar{\partial} \omega \wedge \frac{\omega^{n-3}}{(n-3)!} + |\partial \omega|^2 dv.$$

Consequently,

$$\ast \left( \sqrt{-1} \partial \omega \wedge \bar{\partial} \omega \wedge \frac{\omega^{n-3}}{(n-3)!} \right) = |\partial \ast \omega|^2 - |\partial \omega|^2. \quad (4.7)$$

This completes the proof.

Then we have the following lemma.

**Lemma 4.3.** Let $(M, \omega)$ be a compact Hermitian manifold and $k$ be an integer such that $1 \leq k \leq n-1$. Then

$$\ast \left( \sqrt{-1} \partial \omega^k \wedge \omega^{n-k-1} \right) = k(n-2)! \left( \frac{k-1}{n-2} |\partial \ast \omega|^2 + \frac{n-k-1}{n-2} |\partial \omega|^2 - s + \hat{s} \right). \quad (4.8)$$

**Proof.** It is a combination of (4.2), (2.18) and Lemma 4.2.

**Remark 4.4.** An integral version of the above two lemmas is given by [23, Proposition 5.1]. We are curious about the pointwise equality and do the above calculation. Similar formulae are also obtained in [8,18].

Consequently, we have the following characterization.

**Proposition 4.5.** Let $(M, \omega)$ be a compact Hermitian manifold and $k$ be an integer such that $1 \leq k \leq n-1$. Then the following are equivalent:

1. $\omega$ is $k$-Gauduchon;
2. $\Lambda^{k+1}(\sqrt{-1} \partial \omega^k) = 0$;
3. $s - \hat{s} = \frac{k-1}{n-3} |\partial \ast \omega|^2 + \frac{n-k-1}{n-2} |\partial \omega|^2$.
Proof. As $\ast \omega^k = \frac{k!}{(n-k)!} \omega^{n-k}$ (see, e.g., [42]), we have

$$
\frac{1}{(k+1)!} \Lambda^{k+1}(\sqrt{-1} \partial \bar{\partial} \omega^k) = \sqrt{-1} \partial \bar{\partial} \omega^k \wedge \frac{\omega^{n-k-1}}{(n-k-1)!}.
$$

(4.9)

Then the result follows from Lemma 4.3.

Corollary 4.6. If $(M, \omega)$ is $k$-Gauduchon, then $s \geq \hat{s}$. In particular, if $(M, \omega)$ is pluriclosed or Astheno-Kähler, then $s \geq \hat{s}$.

Corollary 4.7. Let $(M, \omega)$ be a compact Hermitian manifold and $k$ be an integer such that $1 \leq k \leq n-2$. Then the following are equivalent:

1. $\omega$ is $k$-Gauduchon for all $k$, $1 \leq k \leq n-1$;
2. $|\partial \omega|^2 = |\partial^* \omega|^2 = \langle \partial \partial^* \omega, \omega \rangle$.

This follows from Lemma 4.1 and Proposition 4.5.

Remark 4.8. Note that

$$
\sqrt{-1} \partial \bar{\partial} \omega^k \wedge \omega^{n-k-1} = k (\partial \bar{\partial} \omega \wedge \omega^{n-2} + (k-1) \partial \omega \wedge \bar{\partial} \omega \wedge \omega^{n-3}).
$$

So for $1 \leq p, q \leq n-1$, if $\omega$ is $p$-Gauduchon and $q$-Gauduchon for $p \neq q$, then $\omega$ is $k$-Gauduchon for all $k$.

Using Proposition 4.5, we are able to obtain the following result which gives a slight generalization of [18, Proposition 3.8].

Proposition 4.9. Let $(M, \omega)$ be a compact Hermitian manifold and $k$ be an integer such that $1 \leq k \leq n-2$. If $\omega$ is locally conformal Kähler satisfying

$$
\int_M \Lambda^{k+1}(\sqrt{-1} \partial \bar{\partial} \omega^k) dv = 0,
$$

(4.10)

then $(M, \omega)$ is Kähler.

Proof. By (4.2) and (4.9) (see also [23, (5.1)]), for $1 \leq k \leq n-2$,

$$
\frac{1}{(k+1)!} \int_M \Lambda^{k+1}(\sqrt{-1} \partial \bar{\partial} \omega^k) dv
$$

$$
= - \frac{k}{(n-k-2)!} \int_M \sqrt{-1} \partial \omega \wedge \bar{\partial} \omega \wedge \omega^{n-3}
$$

$$
= \frac{(n-3)!k}{(n-k-2)!} (||\partial \omega||^2 - ||\partial^* \omega||^2).
$$

If (4.10) is satisfied, then

$$
||\partial \omega||^2 = ||\partial^* \omega||^2 = ||\tau||^2.
$$

As $\omega$ is locally conformal Kähler, then by (2.9) and (3.2),

$$
||\partial \omega||^2 = \frac{1}{n-1} ||\tau||^2.
$$

So $\tau = 0$ for $n \geq 3$ and $\omega$ is Kähler.

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