Truly Concurrent Process Algebra Is Reversible

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Abstract. Based on our previous process algebra for concurrency APTC, we prove that it is reversible with a little modifications. The reversible algebra has four parts: Basic Algebra for Reversible True Concurrency (BARTC), Algebra for Parallelism in Reversible True Concurrency (APRTC), recursion and abstraction.

Keywords: Reversible Computation; True Concurrency; Behavioral Equivalence; Bisimilarity

1. Introduction

Process algebra is a formal tool to capture computation, especially concurrency, such as CCS [1] [2] [3] and ACP [4]. Several years ago, we do some work on process algebra for true concurrency, such as APTC [5] and CTC [6], while traditional process algebra focuses on interleaving.

Reversible calculi [7] [8] [9] tries to describe reversible computation in the framework of process algebra. Based on CTC and APTC, we also did some work on reversible algebra called RCTC [10] and RAPTC [11]. But the axiomatization of RAPTC is imperfect, it is sound, but not complete. The main reason is that the existence of multi choice operator makes a sound and complete axiomatization can not be established.

In this paper, we try to use alternative operator to replace multi choice operator and we get a sound and complete axiomatization for reversible computation. The main reason of using alternative operator is that when an alternative branch is forward executing, the reverse branch is also determined and other branches have no necessaries to remain. But, when a process is reversed, the other branches disappear. We call the reversible algebra using alternative operator partially reversible algebra.

This paper is organized as follows. In section 2 we introduce some preliminaries on APTC, reversible semantics, and proof techniques. We introduce the whole sound and complete axiomatization in section 3. Finally, we conclude this paper in section 4.

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2. Backgrounds

2.1. APTC

In this subsection, we introduce the preliminaries on truly concurrent process algebra APTC [8], which is based on the truly concurrent bisimulation semantics. APTC has an almost perfect axiomatization to capture laws on truly concurrent bisimulation equivalence, including equational logic and truly concurrent bisimulation semantics, and also the soundness and completeness bridged between them.

APTC captures several computational properties in the form of algebraic laws, and proves the soundness and completeness modulo truly concurrent bisimulation/rooted branching truly concurrent bisimulation equivalence. These computational properties are organized in a modular way by use of the concept of conservation extension, which include the following modules, note that, every algebra are composed of constants and operators, the constants are the computational objects, while operators capture the computational properties.

1. BATC (Basic Algebras for True Concurrency). BATC has sequential composition \( \cdot \) and alternative composition \( + \) to capture causality computation and conflict. The constants are ranged over \( \mathbb{E} \), the set of atomic events. The algebraic laws on \( \cdot \) and \( + \) are sound and complete modulo truly concurrent bisimulation equivalences, such as pomset bisimulation \( \sim_p \), step bisimulation \( \sim_s \), history-preserving (hp-) bisimulation \( \sim_{hp} \) and hereditary history-preserving (hhp-) bisimulation \( \sim_{hhp} \).

2. APTC (Algebra for Parallelism for True Concurrency). APTC uses the whole parallel operator \( \parallel \), the parallel operator \( \| \) to model parallelism, and the communication merge \( | \) to model causality (communication) among different parallel branches. Since a communication may be blocked, a new constant called deadlock \( \delta \) is extended to \( \mathbb{E} \), and also a new unary encapsulation operator \( \partial_H \) is introduced to eliminate \( \delta \), which may exist in the processes. And also a conflict elimination operator \( \Theta \) to eliminate conflicts existing in different parallel branches. The algebraic laws on these operators are also sound and complete modulo truly concurrent bisimulation equivalences, such as pomset bisimulation \( \sim_p \), step bisimulation \( \sim_s \), history-preserving (hp-) bisimulation \( \sim_{hp} \). Note that, these operators in a process except the parallel operator \( \| \) can be eliminated by deductions on the process using axioms of APTC, and eventually be steadied by \( \sim \), \( + \) and \( \parallel \). This is also why bisimulations are called an truly concurrent semantics.

3. Recursion. To model infinite computation, recursion is introduced into APTC. In order to obtain a sound and complete theory, guarded recursion and linear recursion are needed. The corresponding axioms are RSP (Recursive Specification Principle) and RDP (Recursive Definition Principle). RDP says the solutions of a recursive specification can represent the behaviors of the specification, while RSP says that a guarded recursive specification has only one solution, they are sound with respect to APTC with guarded recursion modulo truly concurrent bisimulation equivalences, such as pomset bisimulation \( \sim_p \), step bisimulation \( \sim_s \), history-preserving (hp-) bisimulation \( \sim_{hp} \), and they are complete with respect to APTC with linear recursion modulo truly concurrent bisimulation equivalence, such as pomset bisimulation \( \sim_p \), step bisimulation \( \sim_s \), history-preserving (hp-) bisimulation \( \sim_{hp} \).

4. Abstraction. To abstract away internal implementations from the external behaviors, a new constant \( \tau \) called silent step is added to \( \mathbb{E} \), and also a new unary abstraction operator \( \tau_I \) is used to rename actions in \( I \) into \( \tau \) (the resulted APTC with silent step and abstraction operator is called \( APTC_\tau \)). The recursive specification is adapted to guarded linear recursion to prevent infinite \( \tau \)-loops specifically. The axioms for \( \tau \) and \( \tau_I \) are sound modulo rooted branching truly concurrent bisimulation equivalences (a kind of weak truly concurrent bisimulation equivalence), such as rooted branching pomset bisimulation \( \approx_s \), rooted branching history-preserving (hp-) bisimulation \( \approx_{hp} \). To eliminate infinite \( \tau \)-loops caused by \( \tau_I \) and obtain the completeness, CFAR (Cluster Fair Abstraction Rule) is used to prevent infinite \( \tau \)-loops in a constructible way.

APTC can be used to verify the correctness of system behaviors, by deduction on the description of the system using the axioms of APTC. Based on the modularity of APTC, it can be extended easily and elegantly. For more details, please refer to the manuscript of APTC [8].
2.2. Truly Concurrent Behavioral Semantics

The semantics of APTC is based on truly concurrent bisimulation/ rooted branching truly concurrent bisimulation equivalences, and the modularity of APTC relies on the concept of conservative extension, for the conveniences, we introduce some concepts and conclusions on them.

Definition 2.1 (Prime event structure with silent event). Let \( \Lambda \) be a fixed set of labels, ranged over \( a, b, c, \ldots \) and \( \tau \). A \((\Lambda\text{-labelled})\) prime event structure with silent event \( \tau \) is a tuple \( E = (X, \leq, \approx, \lambda) \), where \( X \) is a denumerable set of events, including the silent event \( \tau \). Let \( E = X \setminus \{\tau\} \), exactly excluding \( \tau \), it is obvious that \( \tau^* = \epsilon \), where \( \epsilon \) is the empty event. Let \( \lambda : X \rightarrow \Lambda \) be a labelling function and let \( \lambda(\tau) = \gamma \). And \( \leq, \approx \) are binary relations on \( E \), called causality and conflict respectively, such that:

1. \( \leq \) is a partial order and \( [e] = \{e' \in E | e' \leq e\} \) is finite for all \( e \in E \). It is easy to see that \( e \leq \tau^* \leq e' = e \leq \tau \leq \cdots \leq e', \) then \( e \leq e' \).
2. \( \approx \) is irreflexive, symmetric and hereditary with respect to \( \leq \), that is, for all \( e, e', e'' \in E \), if \( e \approx e' \leq e'', \) then \( e \approx e'' \).

Then, the concepts of consistency and concurrency can be drawn from the above definition:

1. \( e, e' \in E \) are consistent, denoted as \( e \sim e' \), if \( \neg(e \approx e') \). A subset \( X \subseteq E \) is called consistent, if \( e \sim e' \) for all \( e, e' \in X \).
2. \( e, e' \in E \) are concurrent, denoted as \( e \parallel e' \), if \( \neg(e \approx e') \), and \( \neg(e \approx e') \).

Definition 2.2 (Configuration). Let \( E \) be a PES. A (finite) configuration of \( E \) is a (finite) consistent subset of events \( C \subseteq E \), closed with respect to causality (i.e. \([C] = C\)). The set of finite configurations of \( E \) is denoted by \( C(E) \). We let \( \hat{C} = C \setminus \{\tau\} \).

A consistent subset of \( X \subseteq E \) of events can be seen as a pomset. Given \( X, Y \subseteq E \), \( X \sim Y \) if \( X \) and \( Y \) are isomorphic as pomsets. In the following of the paper, we say \( C_1 \sim C_2 \), we mean \( \hat{C}_1 \sim \hat{C}_2 \).

Definition 2.3 (Pomset transitions and step). Let \( E \) be a PES and let \( C \in C(E) \), and \( \emptyset \neq X \subseteq E \), if \( C \cap X = \emptyset \) and \( C' = C \cup X \in C(E) \), then \( C \overset{X}{\rightarrow} C' \) is called a pomset transition from \( C \) to \( C' \). When the events in \( X \) are pairwise concurrent, we say that \( C \overset{X}{\rightarrow} C' \) is a step.

Definition 2.4 (Weak pomset transitions and weak step). Let \( E \) be a PES and let \( C \in C(E) \), and \( \emptyset \neq X \subseteq E \), if \( C \cap X = \emptyset \) and \( C' = C \cup X \in C(E) \), then \( C \overset{X}{\Rightarrow} C' \) is called a weak pomset transition from \( C \) to \( C' \), where we define \( \overset{\tau}{\Rightarrow} X \overset{\tau'}{\Rightarrow} \tau'^* \). And \( \overset{\tau}{\Rightarrow} X \overset{\tau'}{\Rightarrow} \tau'^* \), for every \( e \in X \). When the events in \( X \) are pairwise concurrent, we say that \( C \overset{X}{\Rightarrow} C' \) is a weak step.

We will also suppose that all the PESs in this paper are image finite, that is, for any PES \( E \) and \( C \in C(E) \) and \( a \in \Lambda \), \( \{e \in E | C \overset{e}{\rightarrow} C' \wedge \lambda(e) = a\} \) and \( \{e \in E | C \overset{e}{\Rightarrow} C' \wedge \lambda(e) = a\} \) is finite.

Definition 2.5 (Pomset, step bisimulation). Let \( E_1, E_2 \) be PESs. A pomset bisimulation is a relation \( R \subseteq C(E_1) \times C(E_2) \), such that if \( (C_1, C_2) \in R \), and \( C_1 \overset{X}{\rightarrow} C_1' \) then \( C_2 \overset{X}{\rightarrow} C_2' \), with \( X \subseteq E_1 \), \( X \subseteq E_2 \), \( C_1 \sim C_2 \) and \( (C_1', C_2') \in R \), and vice-versa. We say that \( E_1, E_2 \) are pomset bisimilar, written \( E_1 \approx_R E_2 \), if there exists a pomset bisimulation \( R \), such that \( (\emptyset, \emptyset) \in R \). By replacing pomset transitions with steps, we can get the definition of step bisimulation. When PESs \( E_1 \) and \( E_2 \) are step bisimilar, we write \( E_1 \approx_s E_2 \).

Definition 2.6 (Weak pomset, step bisimulation). Let \( E_1, E_2 \) be PESs. A weak pomset bisimulation is a relation \( R \subseteq C(E_1) \times C(E_2) \), such that if \( (C_1, C_2) \in R \), and \( C_1 \overset{X}{\rightarrow} C_1' \) then \( C_2 \overset{X}{\rightarrow} C_2' \), with \( X \subseteq E_1 \), \( X \subseteq E_2 \), \( X \sim X \) and \( (C_1', C_2') \in R \), and vice-versa. We say that \( E_1, E_2 \) are weak pomset bisimilar, written \( E_1 \approx_p E_2 \), if there exists a weak pomset bisimulation \( R \), such that \( (\emptyset, \emptyset) \in R \). By replacing weak pomset transitions with weak steps, we can get the definition of weak step bisimulation. When PESs \( E_1 \) and \( E_2 \) are weak step bisimilar, we write \( E_1 \approx_s E_2 \).

Definition 2.7 (Posetal product). Given two PESs \( E_1, E_2 \), the posetal product of their configurations, denoted \( C(E_1) \times C(E_2) \), is defined as
Definition 2.11 (Branching pomset, step bisimulation) A subset \( R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2) \) is called a posetal relation. We say that \( R \) is downward closed when for any \( (C_1, f, C_2), (C'_1, f', C'_2) \in \mathcal{C}(E_1) \times \mathcal{C}(E_2) \), if \( (C_1, f, C_2) \subseteq (C'_1, f', C'_2) \) pointwise and \( (C_1, f, C_2) \in R \), then \( (C_1', f, C_2') \in R \).

For \( f : X_1 \rightarrow X_2 \), we define \( f[x_1 \rightarrow x_2] : X_1 \cup \{x_1\} \rightarrow X_2 \cup \{x_2\}, \) \( z \in X_1 \cup \{x_1\}, (1) \) \( f[x_1 \rightarrow x_2](z) = x_2 \), if \( z = x_1; (2) \) \( f[x_1 \rightarrow x_2](z) = f(z) \), otherwise. Where \( X_1 \subseteq E_1, X_2 \subseteq E_2, x_1 \in E_1, x_2 \in E_2 \).

Definition 2.8 (Weakly posetal product). Given two PESs \( E_1, E_2 \), the weakly posetal product of their configurations, denoted \( \mathcal{C}(E_1) \times \mathcal{C}(E_2) \), is defined as

\[
\{(C_1, f, C_2) | C_1 \in \mathcal{C}(E_1), C_2 \in \mathcal{C}(E_2), f : C_1 \rightarrow C_2 \ \text{isomorphism}\}.
\]

A subset \( R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2) \) is called a posetal relation. We say that \( R \) is downward closed when for any \( (C_1, f, C_2), (C'_1, f', C'_2) \in \mathcal{C}(E_1) \times \mathcal{C}(E_2) \), if \( (C_1, f, C_2) \subseteq (C'_1, f', C'_2) \) pointwise and \( (C_1, f, C_2) \in R \), then \( (C_1', f, C_2') \in R \).

For \( f : X_1 \rightarrow X_2 \), we define \( f[x_1 \rightarrow x_2] : X_1 \cup \{x_1\} \rightarrow X_2 \cup \{x_2\}, \) \( z \in X_1 \cup \{x_1\}, (1) f[x_1 \rightarrow x_2](z) = x_2 \), if \( z = x_1; (2) f[x_1 \rightarrow x_2](z) = f(z) \), otherwise. Where \( X_1 \subseteq E_1, X_2 \subseteq E_2, x_1 \in E_1, x_2 \in E_2 \).

Definition 2.9 ((Hereditary) history-preserving bisimulation). A history-preserving (hp-) bisimulation is a posetal relation \( R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2) \) such that if \( (C_1, f, C_2) \in R \), and \( C_1 \sim C'_1 \), then \( C_2 \sim C'_2 \), with \( (C'_1, f[e_1 \rightarrow e_2], C'_2) \in R \), and vice-versa. \( E_1, E_2 \) are history-preserving (hp-)bisimilar and are written \( E_1 \sim_{hp} E_2 \) if there exists a hp-bisimulation \( R \) such that \( (\emptyset, \emptyset, \emptyset) \in R \).

A hereditary history-preserving (hphp-)bisimulation is a downward closed hp-bisimulation. \( E_1, E_2 \) are hereditary-history-preserving (hphp-)bisimilar and are written \( E_1 \sim_{hphp} E_2 \).

Definition 2.10 (Weak (hereditary) history-preserving bisimulation). A weak history-preserving (hp-) bisimulation is a weak posetal relation \( R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2) \) such that if \( (C_1, f, C_2) \in R \), and \( C_1 \Rightarrow C'_1 \), then \( C_2 \Rightarrow C'_2 \), with \( (C'_1, f[e_1 \rightarrow e_2], C'_2) \in R \), and vice-versa. \( E_1, E_2 \) are weak history-preserving (hp-)bisimilar and are written \( E_1 \sim_{hp} E_2 \) if there exists a hp-bisimulation \( R \) such that \( (\emptyset, \emptyset, \emptyset) \in R \).

A weakly hereditary history-preserving (hphp-)bisimulation is a downward closed weak hp-bisimulation. \( E_1, E_2 \) are weakly hereditary-history-preserving (hphp-)bisimilar and are written \( E_1 \sim_{hphp} E_2 \).

Definition 2.11 (Branching pomset, step bisimulation). Assume a special termination predicate \( \downarrow \), and let \( \sqrt{\cdot} \) represent a state with \( \sqrt{\cdot} \downarrow \). Let \( E_1, E_2 \) be PESs. A branching pomset bisimulation is a relation \( R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2) \), such that:

1. if \( (C_1, C_2) \in R \), and \( C_1 \xrightarrow{\tau} C'_1 \) then
   - either \( X \equiv \tau^* \), and \( (C'_1, C_2) \in R \); or there is a sequence of \( \tau \)-transitions \( C_2 \xrightarrow{\tau^*} C'_2 \), such that \( (C_1, C'_2) \in R \) and \( C_2 \xrightarrow{X} C'_2 \) with \( (C'_1, C'_2) \in R \).

2. if \( (C_1, C_2) \in R \), and \( C_2 \xrightarrow{\tau} C'_2 \) then
   - either \( X \equiv \tau^* \), and \( (C_1, C'_2) \in R \); or there is a sequence of \( \tau \)-transitions \( C_1 \xrightarrow{\tau^*} C'_1 \), such that \( (C'_1, C_2) \in R \) and \( C'_1 \xrightarrow{X} C'_2 \) with \( (C'_1, C'_2) \in R \).

3. if \( (C_1, C_2) \in R \) and \( C_1 \downarrow \), then there is a sequence of \( \tau \)-transitions \( C_2 \xrightarrow{\tau^*} C'_2 \), such that \( (C'_1, C'_2) \in R \) and \( C'_2 \downarrow \).

4. if \( (C_1, C_2) \in R \) and \( C_2 \downarrow \), then there is a sequence of \( \tau \)-transitions \( C_1 \xrightarrow{\tau^*} C'_1 \), such that \( (C'_1, C'_2) \in R \) and \( C'_1 \downarrow \).
We say that $E_1, E_2$ are branching pomset bisimilar, written $E_1 \approx_{bp} E_2$, if there exists a branching pomset bisimulation $R$, such that $(\emptyset, \emptyset) \in R$.

By replacing pomset transitions with steps, we can get the definition of branching step bisimulation. When PESs $E_1$ and $E_2$ are branching step bisimilar, we write $E_1 \approx_{bs} E_2$.

**Definition 2.12 (Rooted branching pomset, step bisimulation).** Assume a special termination predicate $\downarrow$, and let $\sqrt{\cdot}$ represent a state with $\sqrt{\cdot}$. Let $E_1, E_2$ be PESs. A rooted branching pomset bisimulation is a relation $R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2)$, such that:

1. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{bp} C_2'$;
2. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{bp} C_2'$;
3. if $(C_1, C_2) \in R$ and $C_1 \downarrow$, then $C_2 \downarrow$;
4. if $(C_1, C_2) \in R$ and $C_2 \downarrow$, then $C_1 \downarrow$.

We say that $E_1, E_2$ are rooted branching pomset bisimilar, written $E_1 \approx_{rbp} E_2$, if there exists a rooted branching pomset bisimulation $R$, such that $(\emptyset, \emptyset) \in R$.

By replacing pomset transitions with steps, we can get the definition of rooted branching step bisimulation. When PESs $E_1$ and $E_2$ are rooted branching step bisimilar, we write $E_1 \approx_{rbs} E_2$.

**Definition 2.13 (Branching (hereditary) history-preserving bisimulation).** Assume a special termination predicate $\downarrow$, and let $\sqrt{\cdot}$ represent a state with $\sqrt{\cdot}$. A branching history-preserving (hp-) bisimulation is a weakly posetal relation $R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2)$ such that:

1. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{hp} C_2'$;
2. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{hp} C_2'$;
3. if $(C_1, C_2) \in R$ and $C_1 \downarrow$, then $C_2 \downarrow$;
4. if $(C_1, C_2) \in R$ and $C_2 \downarrow$, then $C_1 \downarrow$.

We say that $E_1, E_2$ are branching history-preserving (hp-)bisimilar and are written $E_1 \approx_{bhp} E_2$ if there exists a branching hp-bisimulation $R$ such that $(\emptyset, \emptyset) \in R$.

An arbitrary history-preserving (hp-) bisimulation is a downward closed branching hp-bisimulation. $E_1, E_2$ are branching hereditary history-preserving (bhp-)bisimilar and are written $E_1 \approx_{bhbp} E_2$.

**Definition 2.14 (Rooted branching (hereditary) history-preserving bisimulation).** Assume a special termination predicate $\downarrow$, and let $\sqrt{\cdot}$ represent a state with $\sqrt{\cdot}$. A rooted branching history-preserving (hp-) bisimulation is a weakly posetal relation $R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2)$ such that:

1. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{bhhp} C_2'$;
2. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{bhhp} C_2'$;
3. if $(C_1, C_2) \in R$ and $C_1 \downarrow$, then $C_2 \downarrow$;
4. if $(C_1, C_2) \in R$ and $C_2 \downarrow$, then $C_1 \downarrow$.

$E_1, E_2$ are rooted branching history-preserving (hp-)bisimilar and are written $E_1 \approx_{rhhp} E_2$ if there exists a rooted branching hp-bisimulation $R$ such that $(\emptyset, \emptyset, \emptyset) \in R$. 

Assume a special termination predicate $\downarrow$, and let $\sqrt{\cdot}$ represent a state with $\sqrt{\cdot}$. A branching step bisimulation is a relation $R \subseteq \mathcal{C}(E_1) \times \mathcal{C}(E_2)$, such that:

1. if $(C_1, C_2) \in R$, then $C_1 \xrightarrow{e_1} C_1'$ and $C_2 \xrightarrow{e_2} C_2'$ such that $(C_1, C_2) \in R$ and $C_1' \approx_{bp} C_2'$;
A rooted branching hereditary history-preserving (hp-)bisimulation is a downward closed rooted branching hp-bisimulation. \( E_1, E_2 \) are rooted branching hereditary history-preserving (hp-)bisimilar and are written \( E_1 \equiv_{\text{hp}} E_2 \).

**Definition 2.15** (Congruence). Let \( \Sigma \) be a signature. An equivalence relation \( R \) on \( T(\Sigma) \) is a congruence if for each \( f \in \Sigma \), if \( s_iR_t \) for \( i \in \{1, \ldots, \text{arity}(f)\} \), then \( f(s_1, \ldots, s_{\text{arity}(f)})Rf(t_1, \ldots, t_{\text{arity}(f)}) \).

**Definition 2.16** (Conservative extension). Let \( T_0 \) and \( T_1 \) be TSSs (transition system specifications) over signatures \( \Sigma_0 \) and \( \Sigma_1 \), respectively. The TSS \( T_0 \oplus T_1 \) is a conservative extension of \( T_0 \) if the LTSs (labeled transition systems) generated by \( T_0 \) and \( T_0 \oplus T_1 \) contain exactly the same transitions \( t \xrightarrow{a} t' \) and \( tP \) with \( t \in T(\Sigma_0) \).

**Definition 2.17** (Source-dependency). The source-dependent variables in a transition rule of \( \rho \) are defined inductively as follows: (1) all variables in the source of \( \rho \) are source-dependent; (2) if \( t \xrightarrow{a} t' \) is a premise of \( \rho \) and all variables in \( t \) are source-dependent, then all variables in \( t' \) are source-dependent. A transition rule is source-dependent if all its variables are. A TSS is source-dependent if all its rules are.

**Definition 2.18** (Freshness). Let \( T_0 \) and \( T_1 \) be TSSs over signatures \( \Sigma_0 \) and \( \Sigma_1 \), respectively. A term in \( T(T_0 \oplus T_1) \) is said to be fresh if it contains a function symbol from \( \Sigma_1 \setminus \Sigma_0 \). Similarly, a transition label or predicate symbol in \( T_1 \) is fresh if it does not occur in \( T_0 \).

**Theorem 2.19** (Conservative extension). Let \( T_0 \) and \( T_1 \) be TSSs over signatures \( \Sigma_0 \) and \( \Sigma_1 \), respectively, where \( T_0 \) and \( T_0 \oplus T_1 \) are positive after reduction. Under the following conditions, \( T_0 \oplus T_1 \) is a conservative extension of \( T_0 \). (1) \( T_0 \) is source-dependent. (2) For each \( \rho \in T_1 \), either the source of \( \rho \) is fresh, or \( \rho \) has a premise of the form \( t \xrightarrow{a} t' \) or \( tP \), where \( t \in T(\Sigma_0) \), all variables in \( t \) occur in the source of \( \rho \) and \( t' \), \( a \) or \( P \) is fresh.

### 2.3. Forward-reverse Truly Concurrent Bisimulations

Reversible computation is based on reverse semantics [5] [6] [7]. In this subsection, we introduce the reverse semantics for true concurrency, which are firstly introduced in our previous work on reversible process algebra [10] [11].

**Definition 2.20** (Forward-reverse (FR) pomset transitions and forward-reverse (FR) step). Let \( \mathcal{E} \) be a PES and let \( C \in \mathcal{E}(\mathcal{E}) \), \( \emptyset \neq X \subseteq \mathcal{E}, \mathcal{K} \subseteq \mathbb{N} \), and \( \mathcal{X}[\mathcal{K}] \) denotes that for each \( e \in X \), there is \( e[m] \in X[\mathcal{K}] \) where \( (m \in \mathcal{K}) \), which is called the past of \( e \), and we extend \( \mathcal{E} \) to \( \mathcal{E} \cup \tau \cup \mathcal{E}[\mathcal{K}] \). If \( C \cap \mathcal{X}[\mathcal{K}] = \emptyset \) and \( C' = C \cup \mathcal{X}[\mathcal{K}], X \in \mathcal{E} \), then \( C \xrightarrow{X[\mathcal{K}]} C' \) is called a forward pomset transition from \( C \) to \( C' \), and \( C' \xleftarrow{X[\mathcal{K}]} C \) is called a reverse pomset transition from \( C' \) to \( C \). When the events in \( X \) are pairwise concurrent, we say that \( C \xrightarrow{X[\mathcal{K}]} C' \) is a forward step and \( C' \xleftarrow{X[\mathcal{K}]} C \) is a reverse step.

**Definition 2.21** (Weak forward-reverse (FR) pomset transitions and weak forward-reverse (FR) step). Let \( \mathcal{E} \) be a PES and let \( C \in \mathcal{E}(\mathcal{E}) \), and \( \emptyset \neq X \subseteq \mathcal{E}, \mathcal{K} \subseteq \mathbb{N} \), and \( \mathcal{X}[\mathcal{K}] \) denotes that for each \( e \in X \), there is \( e[m] \in X[\mathcal{K}] \) where \( (m \in \mathcal{K}) \), which is called the past of \( e \). If \( C \cap \mathcal{X}[\mathcal{K}] = \emptyset \) and \( C' = C \cup \mathcal{X}[\mathcal{K}], X \in \mathcal{E} \), then \( C \xrightarrow{X[\mathcal{K}]} C' \) is called a weak forward pomset transition from \( C \) to \( C' \), where we define \( \xrightarrow{e \mapsto e' \mapsto e''} \) and \( \xrightarrow{X[\mathcal{K}]} \), for every \( e \in X \), and \( C' \xleftarrow{X[\mathcal{K}]} C \) is called a weak reverse pomset transition from \( C' \) to \( C \), where we define \( \xleftarrow{e \mapsto e' \mapsto e''} \) and \( \xleftarrow{X[\mathcal{K}]} \), for every \( e \in X \) and \( m \in \mathcal{K} \). When the events in \( X \) are pairwise concurrent, we say that \( C \xrightarrow{X[\mathcal{K}]} C' \) is a weak forward step and \( C' \xleftarrow{X[\mathcal{K}]} C \) is a weak reverse step.

We will also suppose that all the PESs in this paper are image finite, that is, for any PES \( \mathcal{E} \) and \( C \in \mathcal{E}(\mathcal{E}) \), and \( a \in \Lambda \), \( \{e \in \mathcal{E}[\mathcal{E}] C \xrightarrow{e} C' \wedge \lambda(e) = a\} \) and \( \{e \in \mathcal{E}[\mathcal{E}] C \xleftarrow{e} C' \wedge \lambda(e) = a\} \), and \( a \in \Lambda \), \( \{e \in \mathcal{E}[\mathcal{E}] \xrightarrow{e[m]} C \wedge \lambda(e) = a\} \) and \( \{e \in \mathcal{E}[\mathcal{E}] \xleftarrow{e[m]} C \wedge \lambda(e) = a\} \) are finite.

**Definition 2.22** (Forward-reverse (FR) pomset, step bisimulation). Let \( E_1, E_2 \) be PESs. An FR pomset
is a relation \( R \subseteq C(\mathcal{E}_1) \times C(\mathcal{E}_2) \), such that (1) if \((C_1, C_2) \in R\), and \( C_1 \overset{X_1}{\longrightarrow} C'_1 \) then \( C_2 \overset{X_2}{\longrightarrow} C'_2 \), with \( X_1 \subseteq \mathcal{E}_1 \), \( X_2 \subseteq \mathcal{E}_2 \), \( X_1 \sim X_2 \) and \((C'_1, C'_2) \in R\), and vice-versa; (2) if \((C'_1, C'_2) \in R\), and \( C'_2 \overset{X'_2[K_2]}{\longrightarrow} C_2 \), with \( X'_2 \subseteq \mathcal{E}_2 \), \( K_2 \subseteq \mathbb{N} \), \( X'_2 \sim X_2 \) and \((C_1, C_2) \in R\), and vice-versa. We say that \( \mathcal{E}_1, \mathcal{E}_2 \) are FR pomset bisimilar, written \( \mathcal{E}_1 \sim^p R \mathcal{E}_2 \), if there exists an FR pomset bisimulation \( R \), such that \((\varnothing, \varnothing) \in R\). By replacing FR pomset transitions with FR steps, we can get the definition of FR step bisimulation. When PESs \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are FR step bisimilar, we write \( \mathcal{E}_1 \sim^s_{fr} \mathcal{E}_2 \).

**Definition 2.23** (Weak forward-reverse (FR) pomset, step bisimulation). Let \( \mathcal{E}_1, \mathcal{E}_2 \) be PESs. A weak FR pomset bisimulation is a relation \( R \subseteq C(\mathcal{E}_1) \times C(\mathcal{E}_2) \), such that (1) if \((C_1, C_2) \in R\), and \( C_1 \overset{X_1}{\longrightarrow} C'_1 \) then \( C_2 \overset{X'_2}{{\xrightarrow{K_2}}} C'_2 \), with \( X_1 \subseteq \mathcal{E}_1 \), \( X'_2 \subseteq \mathcal{E}_2 \), \( X_1 \sim X'_2 \) and \((C'_1, C'_2) \in R\), and vice-versa; (2) if \((C'_1, C'_2) \in R\), and \( C'_1 \overset{X'_1[K_1]}{\longrightarrow} C_1 \), then \( C'_2 \overset{X'_2[K_2]}{\longrightarrow} C_2 \), with \( X'_1 \subseteq \mathcal{E}_1 \), \( X'_2 \subseteq \mathcal{E}_2 \), \( K_1, K_2 \subseteq \mathbb{N} \), \( X'_1 \sim X'_2 \) and \((C_1, C_2) \in R\), and vice-versa. We say that \( \mathcal{E}_1, \mathcal{E}_2 \) are weak FR pomset bisimilar, written \( \mathcal{E}_1 \sim^s_{fr} R \mathcal{E}_2 \), if there exists a weak FR pomset bisimulation \( R \), such that \((\varnothing, \varnothing) \in R\). By replacing weak FR pomset transitions with weak FR steps, we can get the definition of weak FR step bisimulation. When PESs \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are weak FR step bisimilar, we write \( \mathcal{E}_1 \sim^s_{fr} \mathcal{E}_2 \).

**Definition 2.24** (Forward-reverse (FR) (hereditary) history-preserving bisimulation). An FR history-preserving (hp-) bisimulation is a posetal relation \( R \subseteq C(\mathcal{E}_1) \times C(\mathcal{E}_2) \) such that (1) if \((C_1, f, C_2) \in R\), and \( C_1 \overset{e_1}{{\xrightarrow{m}}} C'_1 \), then \( C_2 \overset{e'_2}{{\xrightarrow{m}}} C'_2 \), with \( (C'_1, f[e_1 \mapsto e_2], C'_2) \in R\), and vice-versa; (2) if \((C'_1, f', C'_2) \in R\), and \( C'_1 \overset{e_1[m]}{\longrightarrow} C_1 \), then \( C'_2 \overset{e_2[n]}{\longrightarrow} C_2 \), with \( (C'_1, f'[e_1[m] \mapsto e_2[n]], C'_2) \in R\), and vice-versa. \( \mathcal{E}_1, \mathcal{E}_2 \) are FR history-preserving (hp-) bisimilar and are written \( \mathcal{E}_1 \sim^h_{hp} R \mathcal{E}_2 \) if there exists an FR hp-bisimulation \( R \) such that \((\varnothing, \varnothing, \varnothing) \in R\).

An FR hereditary history-preserving (hhp-)bisimulation is a downward closed FR hp-bisimulation. \( \mathcal{E}_1, \mathcal{E}_2 \) are FR hereditary history-preserving (hhp-)bisimilar and are written \( \mathcal{E}_1 \sim^h_{hhp} \mathcal{E}_2 \).

**Definition 2.25** (Weak forward-reverse (FR) (hereditary) history-preserving bisimulation). A weak FR history-preserving (hp-) bisimulation is a weakly posetal relation \( R \subseteq C(\mathcal{E}_1) \times C(\mathcal{E}_2) \) such that (1) if \((C_1, f, C_2) \in R\), and \( C_1 \overset{e_1}{{\xrightarrow{m}}} C'_1 \), then \( C_2 \overset{e'_2}{{\xrightarrow{m}}} C'_2 \), with \( (C'_1, f[e_1 \mapsto e_2], C'_2) \in R\), and vice-versa; (2) if \((C'_1, f', C'_2) \in R\), and \( C'_1 \overset{e_1[m]}{\longrightarrow} C_1 \), then \( C'_2 \overset{e_2[n]}{\longrightarrow} C_2 \), with \( (C'_1, f'[e_1[m] \mapsto e_2[n]], C'_2) \in R\), and vice-versa. \( \mathcal{E}_1, \mathcal{E}_2 \) are weak FR history-preserving (hp-) bisimilar and are written \( \mathcal{E}_1 \sim^w_{hhp} \mathcal{E}_2 \) if there exists a weak FR hp-bisimulation \( R \) such that \((\varnothing, \varnothing, \varnothing) \in R\).

A weak FR hereditary history-preserving (hhp-)bisimulation is a downward closed weak FR hp-bisimulation. \( \mathcal{E}_1, \mathcal{E}_2 \) are weak FR hereditary history-preserving (hhp-)bisimilar and are written \( \mathcal{E}_1 \sim^w_{hhp} \mathcal{E}_2 \).

**Definition 2.26** (Branching forward-reverse pomset, step bisimulation). Assume a special termination predicate \( \dagger \), and let \( \sqrt{ \downarrow } \) represent a state with \( \sqrt{ \downarrow } \). Let \( \mathcal{E}_1, \mathcal{E}_2 \) be PESs. A branching FR pomset bisimulation is a relation \( R \subseteq C(\mathcal{E}_1) \times C(\mathcal{E}_2) \), such that:

1. if \((C_1, C_2) \in R\), and \( C_1 \overset{X}{\longrightarrow} C'_1 \) then
   - either \( X = \tau^* \), and \((C'_1, C_2) \in R\);
   - or there is a sequence of (zero or more) \( \tau \)-transitions \( C_2 \overset{\tau^*}{\longrightarrow} C'_2 \), such that \((C_1, C_2) \in R\) and \( C_2 \overset{X}{\longrightarrow} C'_2 \) with \((C'_1, C'_2) \in R\);
2. if \((C_1, C_2) \in R\), and \( C_2 \overset{X}{\longrightarrow} C'_2 \) then
   - either \( X = \tau^* \), and \((C_1, C'_2) \in R\);
   - or there is a sequence of (zero or more) \( \tau \)-transitions \( C_1 \overset{\tau^*}{\longrightarrow} C'_1 \), such that \((C'_1, C_2) \in R\) and \( C_1 \overset{X}{\longrightarrow} C'_1 \) with \((C'_1, C'_2) \in R\);
3. if \((C_1, C_2) \in R\) and \(C_1 \downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C_2 \xrightarrow{\tau^*} C_0\) such that \((C_1, C_0) \in R\) and \(C_0 \downarrow\);
4. if \((C_1, C_2) \in R\) and \(C_2 \downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C_1 \xrightarrow{\tau^*} C_0\) such that \((C_1, C_0) \in R\) and \(C_0 \downarrow\);
5. if \((C'_1, C'_2) \in R\), and \(C'_1 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_1\) then
   - either \(X[\mathcal{K}] \equiv \tau^*\), and \((C_1, C'_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C'_2 \xrightarrow{\tau^*} C'_0\), such that \((C'_1, C'_0) \in R\) and \(C'_0 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_2\) with \((C_1, C_2) \in R\);
6. if \((C'_1, C'_2) \in R\), and \(C'_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_2\) then
   - either \(X[\mathcal{K}] \equiv \tau^*\), and \((C'_1, C_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C'_1 \xrightarrow{\tau^*} C'_0\), such that \((C'_1, C'_0) \in R\) and \(C'_0 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_1\) with \((C_1, C_2) \in R\);
7. if \((C'_1, C'_2) \in R\) and \(C'_1 \downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C'_2 \xrightarrow{\tau^*} C'_0\) such that \((C'_1, C'_0) \in R\) and \(C'_0 \downarrow\);
8. if \((C'_1, C'_2) \in R\) and \(C'_2 \downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C'_1 \xrightarrow{\tau^*} C'_0\) such that \((C'_1, C'_0) \in R\) and \(C'_0 \downarrow\).

We say that \(\mathcal{E}_1, \mathcal{E}_2\) are branching FR pomset bisimilar, written \(\mathcal{E}_1 \approx_{\text{bp}} \mathcal{E}_2\), if there exists a branching FR pomset bisimulation \(R\), such that \((\emptyset, \emptyset) \in R\).

By replacing FR pomset transitions with FR steps, we can get the definition of branching FR step bisimulation. When PESs \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are branching FR step bisimilar, we write \(\mathcal{E}_1 \approx_{\text{bs}} \mathcal{E}_2\).

**Definition 2.27** (Rooted branching forward-reverse (FR) pomset, step bisimulation). Assume a special termination predicate \(\downarrow\), and let \(\sqrt{\cdot}\) represent a state with \(\sqrt{\downarrow}\). Let \(\mathcal{E}_1\) and \(\mathcal{E}_2\) be PESs. A rooted branching FR pomset bisimulation is a relation \(R \subseteq \mathcal{C}(\mathcal{E}_1) \times \mathcal{C}(\mathcal{E}_2)\), such that:

1. if \((C_1, C_2) \in R\) and \(C_1 \xrightarrow{\mathcal{X}[\mathcal{K}]} C'_1\) then \(C_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C'_2\) with \(C'_1 \approx_{\text{bp}} C'_2\);
2. if \((C_1, C_2) \in R\) and \(C_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C'_2\) then \(C_1 \xrightarrow{\mathcal{X}[\mathcal{K}]} C'_1\) with \(C'_1 \approx_{\text{bp}} C'_2\);
3. if \((C'_1, C'_2) \in R\), and \(C'_1 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_1\) then \(C'_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_2\) with \(C_1 \approx_{\text{bs}} C'_2\);
4. if \((C'_1, C'_2) \in R\), and \(C'_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_2\) then \(C'_1 \xrightarrow{\mathcal{X}[\mathcal{K}]} C_1\) with \(C_1 \approx_{\text{bs}} C'_2\);
5. if \((C_1, C_2) \in R\) and \(C_1 \downarrow\), then \(C_2 \downarrow\);
6. if \((C_1, C_2) \in R\) and \(C_2 \downarrow\), then \(C_1 \downarrow\).

We say that \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are rooted branching FR pomset bisimilar, written \(\mathcal{E}_1 \approx_{\text{rbs}} \mathcal{E}_2\), if there exists a rooted branching FR pomset bisimulation \(R\), such that \((\emptyset, \emptyset) \in R\).

By replacing FR pomset transitions with FR steps, we can get the definition of rooted branching FR step bisimulation. When PESs \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are rooted branching FR step bisimilar, we write \(\mathcal{E}_1 \approx_{\text{rs}} \mathcal{E}_2\).

**Definition 2.28** (Branching forward-reverse (FR) (hereditary) history-preserving bisimulation). Assume a special termination predicate \(\downarrow\), and let \(\sqrt{\cdot}\) represent a state with \(\sqrt{\downarrow}\). A branching FR history-preserving (hp-) bisimulation is a weakly postsal relation \(R \subseteq \mathcal{C}(\mathcal{E}_1) \times \mathcal{C}(\mathcal{E}_2)\) such that:

1. if \((C_1, f, C_2) \in R\) and \(C_1 \xrightarrow{e_1} C'_1\) then
   - either \(e_1 \equiv \tau\), and \((C'_1, f[e_1 \mapsto \tau], C_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C_2 \xrightarrow{\tau^*} C'_2\), such that \((C_1, f, C'_2) \in R\) and \(C'_2 \xrightarrow{\mathcal{X}[\mathcal{K}]} C'_0\) with \((C'_1, f[e_1 \mapsto e_2], C_2) \in R\);
2. if \((C_1, f, C_2) \in R\), and \(C_2 \overset{e_2}{\rightarrow} C'_2\) then
   - either \(e_2 \equiv \tau\), and \((C_1, \{e_2 \mapsto \tau\}, C'_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C_1 \overset{\tau^*}{\rightarrow} C'_1\), such that \((C'_1, f, C_2) \in R\) and \(C'_0 \overset{e_1}{\rightarrow} C'_1\) with \((C'_1, \{e_2 \mapsto e_1\}, C'_2) \in R\);

3. if \((C_1, f, C_2) \in R\) and \(C_1 \Downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C_2 \overset{\tau^*}{\rightarrow} C'_2\) such that \((C_1, f, C'_2) \in R\) and \(C'_2 \Downarrow\);

4. if \((C_1, f, C_2) \in R\) and \(C_2 \Downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C_1 \overset{\tau^*}{\rightarrow} C'_1\) such that \((C'_1, f, C_2) \in R\) and \(C'_1 \Downarrow\);

5. if \((C'_1, f', C'_2) \in R\), and \(C_1 \overset{e_1[m]}{\rightarrow} C\) then
   - either \(e_1[m] \equiv \tau\), and \((C_1, f'[e_1[m] \mapsto \tau], C'_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C'_1 \overset{\tau^*}{\rightarrow} C'_0\), such that \((C'_0, f', C'_0) \in R\) and \(C'_0 \overset{e_1[n]}{\rightarrow} C'_1\) with \((C_1, f'[e_1[m] \mapsto e_2[n]], C'_2) \in R\);

6. if \((C'_1, f', C'_2) \in R\), and \(C'_2 \overset{e_2[n]}{\rightarrow} C\) then
   - either \(e_2[n] \equiv \tau\), and \((C'_1, f'[e_2[n] \mapsto \tau], C_2) \in R\);
   - or there is a sequence of (zero or more) \(\tau\)-transitions \(C'_1 \overset{\tau^*}{\rightarrow} C'_0\), such that \((C'_0, f', C'_2) \in R\) and \(C'_0 \overset{e_1[m]}{\rightarrow} C'_1\) with \((C_1, f[e_2[n] \mapsto e_1[m]], C'_2) \in R\);

7. if \((C'_1, f', C'_2) \in R\), and \(C'_1 \Downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C'_2 \overset{\tau^*}{\rightarrow} C'_0\) such that \((C'_0, f', C'_2) \in R\) and \(C'_0 \Downarrow\);

8. if \((C'_1, f', C'_2) \in R\), and \(C'_2 \Downarrow\), then there is a sequence of (zero or more) \(\tau\)-transitions \(C'_1 \overset{\tau^*}{\rightarrow} C'_0\) such that \((C'_0, f', C'_2) \in R\) and \(C'_0 \Downarrow\).

\(\mathcal{E}_1, \mathcal{E}_2\) are branching FR \(hp\)-bisimilar and are written \(\mathcal{E}_1 \equiv_{\text{fr}} \mathcal{E}_2\) if there exists a branching \(FR\) \(hp\)-bisimulation \(R\) such that \((\emptyset, \emptyset, \emptyset) \in R\).

A branching \(FR\) hereditary \(hp\)-bisimulation is a downward closed branching \(FR\) \(hp\)-bisimulation. \(\mathcal{E}_1, \mathcal{E}_2\) are branching \(FR\) hereditary \(hp\)-bisimilar and are written \(\mathcal{E}_1 \equiv_{\text{fr}h} \mathcal{E}_2\).

**Definition 2.29** (Rooted branching forward-reverse (FR) (hereditary) history-preserving bisimulation). Assume a special termination predicate \(\Downarrow\), and let \(\Diamond\) represent a state with \(\Downarrow\). A rooted branching \(FR\) history-preserving (hp-)bisimulation is a weakly posetal relation \(\mathcal{R} \subseteq \mathcal{C}(\mathcal{E}_1) \times \mathcal{C}(\mathcal{E}_2)\) such that:

1. if \((C_1, f, C_2) \in R\), and \(C_1 \overset{e_1}{\rightarrow} C'_1\), then \(C_2 \overset{e_2}{\rightarrow} C'_2\) with \(C'_1 \equiv_{\text{fr}h} C'_2\);
2. if \((C_1, f, C_2) \in R\), and \(C_2 \overset{e_2}{\rightarrow} C'_2\), then \(C_1 \overset{e_1}{\rightarrow} C'_1\) with \(C'_1 \equiv_{\text{fr}h} C'_2\);
3. if \((C'_1, f', C'_2) \in R\), and \(C'_1 \overset{e_1[m]}{\rightarrow} C_1\), then \(C'_2 \overset{e_2[n]}{\rightarrow} C_2\) with \(C'_1 \equiv_{\text{fr}h} C_2\);
4. if \((C'_1, f', C'_2) \in R\), and \(C'_2 \overset{e_2[n]}{\rightarrow} C_2\), then \(C'_1 \overset{e_1[m]}{\rightarrow} C_1\) with \(C'_1 \equiv_{\text{fr}h} C_2\);
5. if \((C_1, f, C_2) \in R\) and \(C_1 \Downarrow\), then \(C_2 \Downarrow\);
6. if \((C_1, f, C_2) \in R\) and \(C_2 \Downarrow\), then \(C_1 \Downarrow\).

\(\mathcal{E}_1, \mathcal{E}_2\) are rooted branching \(FR\) history-preserving (hp-)bisimilar and are written \(\mathcal{E}_1 \equiv_{\text{fr}h} \mathcal{E}_2\) if there exists a rooted branching \(FR\) \(hp\)-bisimulation \(R\) such that \((\emptyset, \emptyset, \emptyset) \in R\).

A rooted branching \(FR\) hereditary history-preserving (hp-)bisimulation is a downward closed rooted branching \(FR\) \(hp\)-bisimulation. \(\mathcal{E}_1, \mathcal{E}_2\) are rooted branching \(FR\) hereditary history-preserving (hp-)bisimilar and are written \(\mathcal{E}_1 \equiv_{\text{fr}h} \mathcal{E}_2\).
No. Axiom
A1 \[ x + y = y + x \]
A2 \[ (x + y) + z = x + (y + z) \]
A3 \[ x + x = x \]
A41 \[ (x + y) \cdot z = x \cdot z + y \cdot z \]
A42 \[ x \cdot (y + z) = x \cdot y + x \cdot z \]
A5 \[ (x - y) \cdot z = x \cdot (y - z) \]

Table 1. Axioms of BARTC

2.4. Proof Techniques

In this subsection, we introduce the concepts and conclusions about elimination, which is very important in the proof of completeness theorem.

Definition 2.30 (Elimination property). Let a process algebra with a defined set of basic terms as a subset of the set of closed terms over the process algebra. Then the process algebra has the elimination to basic terms property if for every closed term \( s \) of the algebra, there exists a basic term \( t \) of the algebra such that the algebra \( \vdash s = t \).

Definition 2.31 (Strongly normalizing). A term \( s_0 \) is called strongly normalizing if does not an infinite series of reductions beginning in \( s_0 \).

Definition 2.32. We write \( s \triangleright_{\text{lpo}} t \) if \( s \rightarrow^+ t \) where \( \rightarrow^+ \) is the transitive closure of the reduction relation defined by the transition rules of a algebra.

Theorem 2.33 (Strong normalization). Let a term rewriting (TRS) system with finitely many rewriting rules and let \( > \) be a well-founded ordering on the signature of the corresponding algebra. If \( s \triangleright_{\text{lpo}} t \) for each rewriting rule \( s \rightarrow t \) in the TRS, then the term rewriting system is strongly normalizing.

3. Basic Algebra for Reversible True Concurrency

In this section, we will discuss the algebraic laws of the confliction \(+\) and causal relation \( \cdot \) based on reversible truly concurrent bisimulations. The resulted algebra is called Basic Algebra for Reversible True Concurrency, abbreviated BARTC.

3.1. Axiom System of BARTC

In the following, let \( e_1, e_2, e'_1, e'_2 \in E \), and let variables \( x, y, z \) range over the set of terms for true concurrency, \( p, q, s \) range over the set of closed terms. The predicate \( \text{Std}(x) \) denotes that \( x \) contains only standard events (no histories of events) and \( \text{NStd}(x) \) means that \( x \) only contains histories of events. The set of axioms of BARTC consists of the laws given in Table 1.

3.2. Properties of BARTC

Definition 3.1 (Basic terms of BARTC). The set of basic terms of BARTC, \( B(\text{BARTC}) \), is inductively defined as follows:
1. \( E \in B(\text{BARTC}) \);
2. if \( e \in E, t \in B(\text{BARTC}) \) then \( e \cdot t \in B(\text{BARTC}) \);
3. if \( [e]m \in E, t \in B(\text{BARTC}) \) then \( t \cdot [e]m \in B(\text{BARTC}) \);
4. if \( t, s \in B(\text{BARTC}) \) then \( t + s \in B(\text{BARTC}) \).

Theorem 3.2 (Elimination theorem of BARTC). Let \( p \) be a closed BARTC term. Then there is a basic BARTC term \( q \) such that \( \text{BARTC} \vdash p = q \).
Let $e$ after forward execution of the event $e$, the predicate $e[m]$ represents successful forward termination after forward execution of the event $e$, the predicate $\overleftrightarrow{e[m]}$ represents successful reverse termination after reverse execution of the event history $e[m]$.

### 3.3. Structured Operational Semantics of BARTC

In this subsection, we will define a term-deduction system which gives the operational semantics of BARTC. We give the forward operational transition rules of operators $\cdot$ and $+$ as Table 3 shows, and the reverse rules of operators $\cdot$ and $+$ as Table 4 shows. And the predicate $\overrightarrow{e[m]}$ represents successful forward termination after forward execution of the event $e$, the predicate $\overleftrightarrow{e[m]}$ represents successful reverse termination after reverse execution of the event history $e[m]$.

---

**Table 2. Term rewrite system of BARTC**

| No. | Rewriting Rule |
|-----|----------------|
| RA3 | $x + x \rightarrow x$ |
| RA41 | $(x + y) \cdot z \rightarrow x \cdot z + y \cdot z$ |
| RA42 | $x \cdot (y + z) \rightarrow x \cdot y + x \cdot z$ |
| RA5 | $(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z)$ |

**Table 3. Forward single event transition rules of BARTC**

| No. | Rewriting Rule |
|-----|----------------|
| 1   | $x \xrightarrow{e} e[m]$ |
| 2   | $x + y \xrightarrow{e} e[m]$ |
| 3   | $x + y \xrightarrow{e} x'$ |
| 4   | $x \cdot y \xrightarrow{e} e[m] \cdot y$ |
| 5   | $x \cdot y \xrightarrow{e} x' \cdot y$ |

**Proof.** (1) Firstly, suppose that the following ordering on the signature of BARTC is defined: $\cdot > +$ and the symbol $\cdot$ is given the lexicographical status for the first argument, then for each rewrite rule $p \rightarrow q$ in Table 2 relation $p \succ lpo q$ can easily be proved. We obtain that the term rewrite system shown in Table 2 is strongly normalizing, for it has finitely many rewriting rules, and $>$ is a well-founded ordering on the signature of BARTC, and if $s \succ lpo t$, for each rewriting rule $s \rightarrow t$ is in Table 2 (see Theorem 2.33).

(2) Then we prove that the normal forms of closed BARTC terms are basic BARTC terms. Suppose that $p$ is a normal form of some closed BARTC term and suppose that $p$ is not a basic term. Let $p'$ denote the smallest sub-term of $p$ which is not a basic term. It implies that each sub-term of $p'$ is a basic term. Then we prove that $p$ is not a term in normal form. It is sufficient to induct on the structure of $p'$:

- Case $p' \equiv e$. $e \in \mathbb{E}$, $p'$ is a basic term, which contradicts the assumption that $p'$ is not a basic term, so this case should not occur.
- Case $p' \equiv p_1 \cdot p_2$. By induction on the structure of the basic term $p_1$:
  - Subcase $p_1 \in \mathbb{E}$. $p'$ would be a basic term, which contradicts the assumption that $p'$ is not a basic term;
  - Subcase $p_1 \equiv e \cdot p'_1$. RA5 rewriting rule can be applied. So $p$ is not a normal form;
  - Subcase $p_1 \equiv p'_1 \cdot e[m]$. RA5 rewriting rule can be applied. So $p$ is not a normal form;
  - Subcase $p_1 \equiv p'_1 \cdot p'_2 \cdot e[m]$. RA41 and RA42 rewriting rule can be applied. So $p$ is not a normal form.

- Case $p' \equiv p_1 + p_2$. By induction on the structure of the basic terms both $p_1$ and $p_2$, all subcases will lead to that $p'$ would be a basic term, which contradicts the assumption that $p'$ is not a basic term.

$\square$
Proof. It is easy to see that FR pomset bisimulation is an equivalent relation on BARTC terms, we only need to prove that

\[ \sim \]

Theorem 3.3 (Congruence of BARTC with respect to FR pomset bisimulation equivalence). FR pomset bisimulation equivalence \( \sim_{fr} \) is a congruence with respect to BARTC.

Proof. It is easy to see that FR pomset bisimulation is an equivalent relation on BARTC terms, we only need to prove that \( \sim_{fr} \) is preserved by the operators \( \cdot \) and \(+\).

- Causality operator \( \cdot \). Let \( x_1, x_2 \) and \( y_1, y_2 \) be BARTC processes, and \( x_1 \sim_{fr} y_1, x_2 \sim_{fr} y_2 \), it is sufficient to prove that \( x_1 \cdot x_2 \sim_{fr} y_1 \cdot y_2 \).

By the definition of FR pomset bisimulation \( \sim_{fr} \) (Definition 2.22), \( x_1 \sim_{fr} y_1 \) means that

\[ x_1 \xrightarrow{X} x'_1 \quad y_1 \xrightarrow{Y} y'_1 \]

Table 4. Reverse single event transition rules of BARTC

\[
\begin{align*}
x \xrightarrow{e} x & \quad \xrightarrow{X} X[K] \quad (X \subseteq x) \\
x + y \xrightarrow{e} x & \quad x \xrightarrow{X} x' \quad (X \subseteq x) \\
x + y \xrightarrow{e} x' & \quad x \xrightarrow{Y} Y[K] \quad (Y \subseteq y) \\
x + y \xrightarrow{e} x' & \quad x \xrightarrow{Y} y' \quad (Y \subseteq y) \\
y \xrightarrow{e} y & \quad x \xrightarrow{X} x \cdot e \\
x \cdot y \xrightarrow{e} x \cdot y' & \quad x \cdot y \xrightarrow{e} x \cdot y' \\
Y \xrightarrow{e} Y[K] & \quad x \xrightarrow{X[K]} X \quad (X \subseteq x) \\
X \xrightarrow{Y[K]} Y & \quad x \xrightarrow{X[K]} x' \quad (X \subseteq x) \\
X \xrightarrow{Y[K]} Y & \quad y \xrightarrow{Y[K]} y' \quad (Y \subseteq y) \\
Y \xrightarrow{Y[K]} Y & \quad x \cdot y \xrightarrow{Y[K]} x \cdot y' \quad (Y \subseteq y) \\

\end{align*}
\]

Table 5. Forward pomset transition rules of BARTC

The forward pomset transition rules are shown in Table 4, and reverse pomset transition rules are shown in Table 6. Different to single event transition rules, the pomset transition rules are labeled by pomsets, which are defined by causality \( \cdot \) and conflict \( + \).

Table 5. Forward pomset transition rules of BARTC

The forward pomset transition rules are shown in Table 4, and reverse pomset transition rules are shown in Table 6. Different to single event transition rules, the pomset transition rules are labeled by pomsets, which are defined by causality \( \cdot \) and conflict \( + \).

Table 6. Reverse pomset transition rules of BARTC
with $X_1 \subseteq x_1$, $Y_1 \subseteq y_1$, $X_1 \sim Y_1$ and $x'_1 \sim^f_p y'_1$. The meaning of $x_2 \sim^f_p y_2$ is similar. By the pomset transition rules for causality operator $\cdot$ in Table 5 and Table 6 we can get
\[
x_1 \cdot x_2 \xrightarrow{X_1[K]} x_1' \quad y_1 \xrightarrow{Y_1[L]} y_1'
\]
with $X_1 \subseteq x_1$, $Y_1 \subseteq y_1$, $X_1 \sim Y_1$ and $x_2 \sim^f_p y_2$; $X_2 \subseteq x_2$, $Y_2 \subseteq y_2$, $X_2 \sim Y_2$ and $x_1 \sim^f_p y_1$ so, we get $x_1 \cdot x_2 \sim^f_p y_1 \cdot y_2$, as desired. Or, we can get
\[
x_1 \cdot x_2 \xrightarrow{X_2[K]} x_1 \cdot x_2' \quad y_1 \cdot y_2 \xrightarrow{Y_1[L]} y_1 \cdot y_2'
\]
with $X_1 \subseteq x_1$, $Y_1 \subseteq y_1$, $X_1 \sim Y_1$ and $x'_1 \sim^f_p y'_1$, $x_2 \sim^f_p y_2$; $X_2 \subseteq x_2$, $Y_2 \subseteq y_2$, $X_2 \sim Y_2$ and $x'_2 \sim^f_p y'_2$, $x_1 \sim^f_p y_1$, so, we get $x_1 \cdot x_2 \sim^f_p y_1 \cdot y_2$, as desired.

• Conflict operator $\oplus$. Let $x_1, x_2$ and $y_1, y_2$ be BARTC processes, and $x_1 \sim^f_p y_1$, $x_2 \sim^f_p y_2$, it is sufficient to prove that $x_1 + x_2 \sim^f_p y_1 + y_2$. The meanings of $x_1 \sim^f_p y_1$ and $x_2 \sim^f_p y_2$ are the same as the above case, according to the definition of FR pomset bisimulation $\sim^f_p$ in Definition 2.22. By the pomset transition rules for conflict operator $\oplus$ in Table 5 and Table 6 we can get four cases:
\[
x_1 + x_2 \xrightarrow{X_1[K]} x_1' \quad y_1 \xrightarrow{Y_1[L]} y_1'
\]
with $X_1 \subseteq x_1$, $Y_1 \subseteq y_1$, $X_1 \sim Y_1$, so, we get $x_1 + x_2 \sim^f_p y_1 + y_2$, as desired. Or, we can get
\[
x_1 + x_2 \xrightarrow{X_1[K]} x_1' \quad y_1 + y_2 \xrightarrow{Y_1[L]} y_1'
\]
with $X_1 \subseteq x_1$, $Y_1 \subseteq y_1$, $X_1 \sim Y_1$, and $x'_1 \sim^f_p y'_1$, so, we get $x_1 + x_2 \sim^f_p y_1 + y_2$, as desired. Or, we can get
\[
x_1 + x_2 \xrightarrow{X_2[K]} x_2' \quad y_1 \xrightarrow{Y_2[L]} y_2'
\]
with $X_2 \subseteq x_2$, $Y_2 \subseteq y_2$, $X_2 \sim Y_2$, so, we get $x_1 + x_2 \sim^f_p y_1 + y_2$, as desired. Or, we can get
\[
x_1 + x_2 \xrightarrow{X_2[K]} x_2 \quad y_1 + y_2 \xrightarrow{Y_2[L]} y_2'
\]
Theorem 3.4 (Soundness of BARTC modulo FR pomset bisimulation equivalence). Let \( x \) and \( y \) be BARTC terms. If \( \text{BARTC} \vdash x = y \), then \( x \sim^p_f y \).

Proof. Since FR pomset bisimulation \( \sim^p_f \) is both an equivalent and a congruent relation, we only need to check if each axiom in Table 1 is sound modulo FR pomset bisimulation equivalence.

- **Axiom A1.** Let \( p, q \) be BARTC processes, and \( p + q = q + p \), it is sufficient to prove that \( p + q \sim^p_f q + p \).

By the pomset transition rules for operator + in Table 5 and Table 6 we get

\[
\begin{align*}
\frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
p + q \xrightarrow{p} P \quad (P \subseteq p) && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
q \xrightarrow{Q} Q[L] \quad (Q \subseteq q) && \frac{q \xrightarrow{Q} Q[L]}{(Q \subseteq q)} \\
p + q \xrightarrow{Q} Q[L] \quad (Q \subseteq q) && \frac{q \xrightarrow{Q} Q[L]}{(Q \subseteq q)} \\
q \xrightarrow{Q} Q \quad (Q \subseteq q) && \frac{q \xrightarrow{Q} Q}{(Q \subseteq q)} \\
p + q \xrightarrow{Q} Q \quad (Q \subseteq q) && \frac{q \xrightarrow{Q}}{(Q \subseteq q)} \\
q \xrightarrow{Q} q' \quad (Q \subseteq q) && \frac{q \xrightarrow{Q}}{(Q \subseteq q)} \\
p + q \xrightarrow{Q} q' \quad (Q \subseteq q) && \frac{q \xrightarrow{Q}}{(Q \subseteq q)} \\
q \xrightarrow{Q} q' \quad (Q \subseteq q) && \frac{q \xrightarrow{Q}}{(Q \subseteq q)} \\
p + q \xrightarrow{Q} q' \quad (Q \subseteq q) && \frac{q \xrightarrow{Q}}{(Q \subseteq q)}
\end{align*}
\]

So, \( p + q \sim^p_f q + p \), as desired.

- **Axiom A2.** Let \( p, q, s \) be BARTC processes, and \( (p + q) + s = p + (q + s) \), it is sufficient to prove that \( (p + q) + s \sim^p_f p + (q + s) \).

By the pomset transition rules for operator + in Table 5 and Table 6 we get

\[
\begin{align*}
\frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
(p + q) + s \xrightarrow{p} P[K] \quad (P \subseteq p) && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
\frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
(p + q) + s \xrightarrow{p} P \quad (P \subseteq p) && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
\frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
(p + q) + s \xrightarrow{p} P \quad (P \subseteq p) && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
\frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)} \\
(p + q) + s \xrightarrow{p} P \quad (P \subseteq p) && \frac{p \xrightarrow{p} P[K]}{(P \subseteq p)}
\end{align*}
\]
\[
\frac{p \xrightarrow{P} p'}{(P \subseteq p)} \quad \frac{p \xrightarrow{p'} (P \subseteq p)}{p + (q + s) \xrightarrow{p'} (P \subseteq p)}
\]

\[
\frac{p \xrightarrow{P[K]} p'}{(P \subseteq p)} \quad \frac{p \xrightarrow{p'} (P \subseteq p)}{p + (q + s) \xrightarrow{p'} (P \subseteq p)}
\]

\[
\frac{q \xrightarrow{Q} Q[L]}{(Q \subseteq q)} \quad \frac{q \xrightarrow{Q} Q[L]}{(Q \subseteq q)}
\]

\[
\frac{q \xrightarrow{Q[L]} Q}{(p + q + s) \xrightarrow{Q[L]} Q} \quad \frac{q \xrightarrow{Q[L]} Q}{(p + q + s) \xrightarrow{Q[L]} Q}
\]

\[
\frac{s \xrightarrow{S[M]} S[M]}{(S \subseteq s)} \quad \frac{s \xrightarrow{S[M]} S[M]}{(S \subseteq s)}
\]

\[
\frac{s \xrightarrow{S[M]} S}{(S \subseteq s)} \quad \frac{s \xrightarrow{S[M]} S}{(S \subseteq s)}
\]

\[
\frac{s \xrightarrow{S[M]} S}{(S \subseteq s)} \quad \frac{s \xrightarrow{S[M]} S}{(S \subseteq s)}
\]

\[
\frac{s \xrightarrow{S[M]} S}{(S \subseteq s)} \quad \frac{s \xrightarrow{S[M]} S}{(S \subseteq s)}
\]

So, \((p + q) + s \xrightarrow{fr} p + (q + s)\), as desired.

**Axiom** A3. Let \(p\) be a BARTC process, and \(p + p = p\), it is sufficient to prove that \(p + p \xrightarrow{fr} p\). By the pomset transition rules for operator \(+\) in Table[5] and Table[6] we get

\[
\frac{p \xrightarrow{P[K]} P[K]}{(P \subseteq p)} \quad \frac{p \xrightarrow{P[K]} P[K]}{(P \subseteq p)}
\]

\[
\frac{p \xrightarrow{P[K]} P}{(P \subseteq p)} \quad \frac{p \xrightarrow{P[K]} P}{(P \subseteq p)}
\]

\[
\frac{p \xrightarrow{p'} (P \subseteq p)}{p + p \xrightarrow{p'} (P \subseteq p)} \quad \frac{p \xrightarrow{p'} (P \subseteq p)}{p + p \xrightarrow{p'} (P \subseteq p)}
\]
\[
\begin{align*}
\frac{p \xrightarrow{P[K]} p'}{(P \subseteq p)} & \quad \frac{p \xrightarrow{P[K]} p'}{(P \subseteq p)} \\
p + p \xrightarrow{P[K]} p' & \quad p + p \xrightarrow{P[K]} p'
\end{align*}
\]

So, \( p + p \sim_P f^r p \), as desired.

- **Axiom A41.** Let \( p, q, s \) be BARTC processes, \( \text{Std}(p), \text{Std}(q), \text{Std}(s) \), and \((p + q) \cdot s = p \cdot s + q \cdot s\), it is sufficient to prove that \((p + q) \cdot s \sim_P f^r p \cdot s + q \cdot s\). By the pomset transition rules for operators \(+\) and \(\cdot\) in Table 5 we get

\[
\begin{align*}
\frac{p \xrightarrow{\cdot} P[K]}{(p + q) \cdot s \xrightarrow{\cdot} P[K] \cdot s} & \quad \frac{p \xrightarrow{\cdot} P[K]}{p \cdot s + q \cdot s \xrightarrow{\cdot} P[K] \cdot s} \\
p \xrightarrow{p} p' & \quad p \xrightarrow{p'} (P \subseteq p) \\
q \xrightarrow{Q} Q[L] & \quad q \xrightarrow{Q} Q[L] \\
(p + q) \cdot s \xrightarrow{Q} Q[K] \cdot s & \quad (p + q) \cdot s \xrightarrow{Q} Q[K] \cdot s \\
q \xrightarrow{Q} q' & \quad q \xrightarrow{Q} q' \\
(p + q) \cdot s \xrightarrow{Q} q' & \quad (p + q) \cdot s \xrightarrow{Q} q' \\
\end{align*}
\]

So, \((p + q) \cdot s \sim_P f^r p \cdot s + q \cdot s\), as desired.

- **Axiom A42.** Let \( p, q, s \) be BARTC processes, \( N\text{Std}(p), N\text{Std}(q), N\text{Std}(s) \), and \( p \cdot (q + s) = p \cdot q + p \cdot s \), it is sufficient to prove that \( p \cdot (q + s) \sim_P f^r p \cdot q + p \cdot s \). By the pomset transition rules for operators \(+\) and \(\cdot\) in Table 6 we get

\[
\begin{align*}
\frac{q \xrightarrow{Q[L]} Q}{p \cdot (q + s) \xrightarrow{Q[L]} p \cdot Q} & \quad \frac{q \xrightarrow{Q[L]} Q}{p \cdot (q + s) \xrightarrow{Q[L]} p \cdot Q} \\
\frac{q \xrightarrow{Q[L]} q'}{p \cdot (q + s) \xrightarrow{Q[L]} p \cdot q'} & \quad \frac{q \xrightarrow{Q[L]} q'}{p \cdot (q + s) \xrightarrow{Q[L]} p \cdot q'} \\
\frac{s \xrightarrow{S[M]} S}{p \cdot (q + s) \xrightarrow{S[M]} p \cdot S} & \quad \frac{s \xrightarrow{S[M]} S}{p \cdot (q + s) \xrightarrow{S[M]} p \cdot S} \\
\frac{s \xrightarrow{S[M]} s'}{p \cdot (q + s) \xrightarrow{S[M]} p \cdot s'} & \quad \frac{s \xrightarrow{S[M]} s'}{p \cdot (q + s) \xrightarrow{S[M]} p \cdot s'}
\end{align*}
\]

So, \( p \cdot (q + s) \sim_P f^r p \cdot q + p \cdot s \), as desired.

- **Axiom A5.** Let \( p, q, s \) be BARTC processes, and \( (p \cdot q) \cdot s = p \cdot (q \cdot s) \), it is sufficient to prove that \((p \cdot q) \cdot s \sim_P f^r p \cdot (q \cdot s)\). By the pomset transition rules for operator \(\cdot\) in Table 5 and Table 6 we get

\[
\begin{align*}
\frac{p \xrightarrow{P[K]} P[K]}{(p \cdot q) \cdot s \xrightarrow{P[K]} (P[K] \cdot q) \cdot s} & \quad \frac{p \xrightarrow{P[K]} P[K]}{p \cdot (q \cdot s) \xrightarrow{P[K]} (P[K] \cdot (q \cdot s))}
\end{align*}
\]
Theorem 3.5 (Completeness of BARTC modulo FR pomset bisimulation equivalence). Let $p$ and $q$ be closed BARTC terms, if $p \sim_p^{fr} q$ then $p = q$.

Proof. Firstly, by the elimination theorem of BARTC, we know that for each closed BARTC term $p$, there exists a closed basic BARTC term $p'$, such that $\text{BARTC} \vdash p = p'$, so, we only need to consider closed basic BARTC terms.

The basic terms (see Definition 3.1) modulo associativity and commutativity (AC) of conflict + (defined by axioms A1 and A2 in Table 1), and this equivalence is denoted by $\equiv_{AC}$. Then, each equivalence class $s$ modulo AC of + has the following normal form

$$s_1 + \cdots + s_k$$

with each $s_i$ either an atomic event or of the form $t_1 \cdot t_2$, and each $s_i$ is called the summand of $s$.

Now, we prove that for normal forms $n$ and $n'$, if $n \sim_p^{fr} n'$ then $n \equiv_{AC} n'$. It is sufficient to induct on the sizes of $n$ and $n'$.

- Consider a summand $e$ of $n$. Then $n \equiv_{AC} e[|m|]$, so $n \sim_p^{fr} n'$ implies $n' \equiv_{AC} e[|m|]$, meaning that $n'$ also contains the summand $e$.
- Consider a summand $e[|m|]$ of $n$. Then $n \equiv_{AC} e$, so $n \sim_p^{fr} n'$ implies $n' \equiv_{AC} e$, meaning that $n'$ also contains the summand $e[|m|]$.
- Consider a summand $t_1 \cdot t_2$ of $n$. Then $n \equiv_{AC} t_1[|K|] \cdot t_2$, so $n \sim_p^{fr} n'$ implies $n' \equiv_{AC} t_1[|K|] \cdot t_2$ with $t_1[|K|] \cdot t_2 \sim_p^{fr} t_1[|K|] \cdot t_2'$, meaning that $n'$ contains a summand $t_1 \cdot t_2'$. Since $t_2$ and $t_2'$ are normal forms and have sizes no greater than $n$ and $n'$, by the induction hypotheses $t_2 \sim_p^{fr} t_2'$ implies $t_2 = AC t_2'$.
- Consider a summand $t_1 \cdot t_2[|L|]$ of $n$. Then $n \equiv_{AC} t_1[|L|] \cdot t_2$, so $n \sim_p^{fr} n'$ implies $n' \equiv_{AC} t_1[|L|] \cdot t_2$ with $t_1[|L|] \cdot t_2 \sim_p^{fr} t_1'[|L|] \cdot t_2$, meaning that $n'$ contains a summand $t_1'[|L|] \cdot t_2$. Since $t_2$ and $t_1'$ are normal forms and have sizes no greater than $n$ and $n'$, by the induction hypotheses $t_1 \sim_p^{fr} t_1'$ implies $t_1 = AC t_1'$.

So, we get $n \equiv_{AC} n'$.

Finally, let $s$ and $t$ be basic terms, and $s \sim_p^{fr} t$, there are normal forms $n$ and $n'$, such that $s = n$ and $t = n'$. The soundness theorem of BARTC modulo FR pomset bisimulation equivalence (see Theorem 3.3) yields $s \sim_p^{fr} n$ and $t \sim_p^{fr} n'$, so $n \sim_p^{fr} s \sim_p^{fr} t \sim_p^{fr} n'$. Since if $n \sim_p^{fr} n'$ then $n \equiv_{AC} n'$, $s = n \equiv_{AC} n' \equiv_{AC} t$, as desired.

The step transition rules are almost the same as the transition rules in Table 5 and Table 6, the difference is that events in the transition pomset are pairwise concurrent for the step transition rules, and we omit them.

Theorem 3.6 (Congruence of BARTC with respect to FR step bisimulation equivalence). Step bisimulation equivalence $\sim_p^{fr}$ is a congruence with respect to BARTC.
Proof. It is easy to see that FR step bisimulation is an equivalent relation on BARTC terms, we only need to prove that \( \sim_{fr} \) is preserved by the operators \( \cdot \) and +. The proof is almost the same as proof of congruence of BARTC with respect to FR pomset bisimulation equivalence, the difference is that events in the transition pomset are pairwise concurrent for FR step bisimulation equivalence, and we omit it.

**Theorem 3.7** (Soundness of BARTC modulo FR step bisimulation equivalence). Let \( x \) and \( y \) be BARTC terms. If \( \text{BARTC} \vdash x = y \), then \( x \sim_{fr} y \).

Proof. Since FR step bisimulation \( \sim_{fr} \) is both an equivalent and a congruent relation, we only need to check if each axiom in Table 1 is sound modulo FR step bisimulation equivalence. The soundness proof is almost the same as soundness proof of BARTC modulo FR pomset bisimulation equivalence, the difference is that events in the transition pomset are pairwise concurrent, and we omit it.

**Theorem 3.8** (Completeness of BARTC modulo FR step bisimulation equivalence). Let \( p \) and \( q \) be closed BARTC terms, if \( p \sim_{fr} q \) then \( p = q \).

Proof. The proof of completeness is almost the same as the proof of BARTC modulo FR pomset bisimulation equivalence, the only difference is that events in the transition pomset are pairwise concurrent, and we omit it.

The transition rules for (hereditary) FR hp-bisimulation of BARTC are the same as single event transition rules in Table 3, Table 4.

**Theorem 3.9** (Congruence of BARTC with respect to FR hp-bisimulation equivalence). Hp-bisimulation equivalence \( \sim_{fr}^{hp} \) is a congruence with respect to BARTC.

Proof. It is easy to see that history-preserving bisimulation is an equivalent relation on BARTC terms, we only need to prove that \( \sim_{fr}^{hp} \) is preserved by the operators \( \cdot \) and +.

The proof is similar to the proof of congruence of BARTC with respect to FR pomset bisimulation equivalence, we omit it.

**Theorem 3.10** (Soundness of BARTC modulo FR hp-bisimulation equivalence). Let \( x \) and \( y \) be BARTC terms. If \( \text{BARTC} \vdash x = y \), then \( x \sim_{fr}^{hp} y \).

Proof. Since FR hp-bisimulation \( \sim_{fr}^{hp} \) is both an equivalent and a congruent relation, we only need to check if each axiom in Table 1 is sound modulo FR hp-bisimulation equivalence.

The proof is similar to the proof of soundness of BARTC modulo FR pomset and step bisimulation equivalences, we omit it.

**Theorem 3.11** (Completeness of BARTC modulo FR hp-bisimulation equivalence). Let \( p \) and \( q \) be closed BARTC terms, if \( p \sim_{fr}^{hp} q \) then \( p = q \).

Proof. The proof is similar to the proof of completeness of BARTC modulo FR pomset and step bisimulation equivalences, we omit it.

**Theorem 3.12** (Congruence of BARTC with respect to FR hhp-bisimulation equivalence). Hhp-bisimulation equivalence \( \sim_{fr}^{hhp} \) is a congruence with respect to BARTC.

Proof. It is easy to see that FR hhp-bisimulation is an equivalent relation on BARTC terms, we only need to prove that \( \sim_{fr}^{hhp} \) is preserved by the operators \( \cdot \) and +.

The proof is similar to the proof of congruence of BARTC with respect to FR hp-bisimulation equivalence, we omit it.

**Theorem 3.13** (Soundness of BARTC modulo FR hhp-bisimulation equivalence). Let \( x \) and \( y \) be BARTC terms. If \( \text{BARTC} \vdash x = y \), then \( x \sim_{fr}^{hhp} y \).
Proof. Since FR hhp-bisimulation \( \sim_{hbp}^{fr} \) is both an equivalent and a congruent relation, we only need to check if each axiom in Table 1 is sound modulo FR hhp-bisimulation equivalence.

The proof is similar to the proof of soundness of BARTC modulo FR hhp-bisimulation equivalence, we omit it.

\[
\text{Table 7. Forward transition rules of parallel operator } \parallel
\]

\[
\begin{align*}
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} e_2[m] \\
x \parallel y & \xrightarrow{(e_1,e_2)} e_1[m] \parallel e_2[m] \\
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} y' \\
x \parallel y & \xrightarrow{(e_1,e_2)} e_1[m] \parallel y'
\end{align*}
\]

\[
\begin{align*}
x & \xrightarrow{e_1} x' \quad y \xrightarrow{e_2} e_2[m] \\
x & \xrightarrow{e_1} x' \parallel e_2[m] \\
x & \xrightarrow{e_1} x' \quad y \xrightarrow{e_2} y' \\
x & \xrightarrow{e_1} x' \parallel e_2[m] \parallel y'
\end{align*}
\]

\[
\text{Table 8. Reverse transition rules of parallel operator } \parallel
\]

\[
\begin{align*}
x & \xrightarrow{e_1[m]} e_1 \quad y \xrightarrow{e_2[m]} e_2 \\
x \parallel y & \xrightarrow{(e_1[m],e_2[m])} e_1 \parallel e_2 \\
x & \xrightarrow{e_1[m]} e_1 \quad y \xrightarrow{e_2[m]} y' \\
x \parallel y & \xrightarrow{(e_1[m],e_2[m])} e_1 \parallel y'
\end{align*}
\]

\[
\begin{align*}
x & \xrightarrow{e_1[m]} e_1 \quad y \xrightarrow{e_2[m]} e_2 \\
x \parallel y & \xrightarrow{(e_1[m],e_2[m])} e_1 \parallel e_2 \\
x & \xrightarrow{e_1[m]} e_1 \quad y \xrightarrow{e_2[m]} y' \\
x \parallel y & \xrightarrow{(e_1[m],e_2[m])} e_1 \parallel y'
\end{align*}
\]

4. Algebra for Parallelism in Reversible True Concurrency

In this section, we will discuss parallelism in reversible true concurrency. The resulted algebra is called Algebra for Parallelism in Reversible True Concurrency, abbreviated APRTC.

4.1. Parallelism

The forward transition rules for parallelism \( \parallel \) are shown in Table 7 and the reverse transition rules for \( \parallel \) are shown in Table 8.

The forward and reverse transition rules of communication \( | \) are shown in Table 9 and Table 10.

\[
\begin{align*}
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} e_2[m] \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \\
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} y' \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \cdot y'
\end{align*}
\]

\[
\begin{align*}
x & \xrightarrow{e_1} x' \quad y \xrightarrow{e_2} e_2[m] \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \cdot x' \\
x & \xrightarrow{e_1} x' \quad y \xrightarrow{e_2} y' \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \cdot x' \parallel y'
\end{align*}
\]

\[
\text{Table 9. Forward transition rules of communication } |
\]

\[
\begin{align*}
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} e_2[m] \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \\
x & \xrightarrow{e_1} e_1[m] \quad y \xrightarrow{e_2} y' \\
x \parallel y & \xrightarrow{(e_1,e_2)} \gamma(e_1,e_2)[m] \cdot y'
\end{align*}
\]
Table 10. Reverse transition rules of communication operator

| Rule | Description |
|------|-------------|
| $x \xrightarrow{e_1[m]} e_1$ | $(\gamma(e_1, e_2))$ |
| $\Theta(x) \xrightarrow{e_1[m]} e_1$ | $(\gamma(e_1, e_2))$ |
| $x \xrightarrow{e_1[m]} x'$ | $(\gamma(e_1, e_2))$ |
| $\Theta(x) \xrightarrow{e_1[m]} x'$ | $(\gamma(e_1, e_2))$ |
| $x \xrightarrow{e_1[m]} e_1$ | $\gamma(e_1, e_2)$ |
| $\Theta(x) \xrightarrow{e_1[m]} e_1$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} x'$ | $\gamma(e_1, e_2)$ |
| $\Theta(x) \xrightarrow{e_1[m]} x'$ | $\gamma(e_1, e_2)$ |

Table 11. Forward transition rules of conflict elimination

| Rule | Description |
|------|-------------|
| $y \xrightarrow{e_2[m]} e_2$ | $(\gamma(e_1, e_2))$ |
| $x \xrightarrow{e_1[m]} y$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y'$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_2[m]} y'$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} e_1$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_2[m]} e_2$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y \xrightarrow{e_2[m]} y'$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y' \xrightarrow{e_2[m]} y$ | $\gamma(e_1, e_2)$ |

The conflict elimination is also captured by two auxiliary operators, the unary conflict elimination operator $\Theta$ and the binary unless operator $\triangleleft$. The forward and reverse transition rules for $\Theta$ and $\triangleleft$ are expressed by ten transition rules in Table 11 and Table 12.

Theorem 4.1 (Congruence theorem of APRTC). FR truly concurrent bisimulation equivalences $\sim_{fr}^p$, $\sim_{fr}^s$, $\sim_{fr}^{hp}$ and $\sim_{fr}^{hkp}$ are all congruences with respect to APRTC.

Table 12. Reverse transition rules of conflict elimination

| Rule | Description |
|------|-------------|
| $y \xrightarrow{e_2[m]} e_2$ | $(\gamma(e_1, e_2))$ |
| $x \xrightarrow{e_1[m]} y$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y'$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_2[m]} y'$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} e_1$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_2[m]} e_2$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y' \xrightarrow{e_2[m]} y$ | $\gamma(e_1, e_2)$ |
| $x \xrightarrow{e_1[m]} y \xrightarrow{e_2[m]} y'$ | $\gamma(e_1, e_2)$ |
Proof. (1) Case FR pomset bisimulation equivalence $\sim_p^f$.

- Case parallel operator $\parallel$. Let $x_1, x_2$ and $y_1, y_2$ be ARPTC processes, and $x_1 \sim_p^f y_1, x_2 \sim_p^f y_2$, it is sufficient to prove that $x_1 \parallel x_2 \sim_p^f y_1 \parallel y_2$.

By the definition of FR pomset bisimulation $\sim_p^f$ (Definition 2.22), $x_1 \sim_p^f y_1$ means that

$$x_1 \stackrel{X_1}{\longrightarrow} x_1' \stackrel{Y_1}{\rightarrow} y_1$$

with $X_1 \subseteq x_1, Y_1 \subseteq y_1, X_1 \sim Y_1$ and $x_1' \sim_p^f y_1'$. The meaning of $x_2 \sim_p^f y_2$ is similar.

By the forward transition rules for parallel operator $\parallel$ in Table 7, we can get

$$x_1 \parallel x_2 \xrightarrow{(X_1,X_2)} X_1[K] \parallel X_2[K] \parallel y_2 \xrightarrow{(Y_1,Y_2)} Y_1[J] \parallel Y_2[J]$$

with $X_1 \subseteq x_1, Y_1 \subseteq y_1, X_2 \subseteq x_2, Y_2 \subseteq y_2, X_1 \sim Y_1$ and $X_2 \sim Y_2$, and the assumption $X_1[K] \parallel X_2[K] \sim_p^f Y_1[J] \parallel Y_2[J]$ and $X_1 \parallel X_2 \sim_p^f Y_1 \parallel Y_2$, so we get $x_1 \parallel x_2 \sim_p^f y_1 \parallel y_2$, as desired.

Or, we can get

$$x_1 \parallel x_2 \xrightarrow{(X_1,X_2)} x_1' \parallel X_2[K] \parallel y_2 \xrightarrow{(Y_1,Y_2)} Y_1[J] \parallel y_2'$$

with $X_1 \subseteq x_1, Y_1 \subseteq y_1, X_2 \subseteq x_2, Y_2 \subseteq y_2, X_1 \sim Y_1, X_2 \sim Y_2$, and the assumptions $x_1' \parallel X_2[K] \sim_p^f y_1' \parallel Y_2[J]$ and $x_1' \parallel X_2 \sim_p^f y_1' \parallel Y_2'$, so we get $x_1 \parallel x_2 \sim_p^f y_1 \parallel y_2$, as desired.

Or, we can get

$$x_1 \parallel x_2 \xrightarrow{(X_1,X_2)} X_1[K] \parallel x_2' \parallel y_2 \xrightarrow{(Y_1,Y_2)} Y_1[J] \parallel y_2'$$

with $X_1 \subseteq x_1, Y_1 \subseteq y_1, X_2 \subseteq x_2, Y_2 \subseteq y_2, X_1 \sim Y_1, X_2 \sim Y_2$, and the assumptions $X_1[K] \parallel x_2' \sim_p^f Y_1[J] \parallel y_2'$ and $x_1 \parallel x_2' \sim_p^f Y_1 \parallel y_2'$, so we get $x_1 \parallel x_2 \sim_p^f y_1 \parallel y_2'$, as desired.

- Case communication operator $|$. It can be proved similarly to the case of parallel operator $\parallel$, we omit it.

Note that, a communication is defined between two single communicating events.

- Case conflict elimination operator $\Theta$. It can be proved similarly to the above cases, we omit it. Note that the conflict elimination operator $\Theta$ is a unary operator.

- Case unless operator $\not\parallel$. It can be proved similarly to the case of parallel operator $\parallel$, we omit it. Note that, a conflict relation is defined between two single events.

(2) The cases of FR step bisimulation $\sim_s^f$, FR hp-bisimulation $\sim_h^f$ and FR hhp-bisimulation $\sim_h^f$ can be proven similarly, we omit them.
4.2. Axiom System of Parallelism

Definition 4.2 (Basic terms of APRTC). The set of basic terms of APRTC, \( B(\text{APRTC}) \), is inductively defined as follows:

1. \( E \in B(\text{APRTC}) \);
2. if \( e \in E \), \( t \in B(\text{APRTC}) \) then \( e \cdot t \in B(\text{APRTC}) \);
3. if \( e[m] \in E \), \( t \in B(\text{APRTC}) \) then \( t \cdot e[m] \in B(\text{APRTC}) \);
4. if \( t, s \in B(\text{APRTC}) \) then \( t + s \in B(\text{APRTC}) \);
5. if \( t, s \in B(\text{APRTC}) \) then \( t \parallel s \in B(\text{APRTC}) \).

We design the axioms of parallelism in Table 13, including algebraic laws for parallel operator \( \parallel \), communication operator \( | \), conflict elimination operator \( \Theta \) and unless operator \( \triangleleft \), and also the whole parallel operator \( \overline{\parallel} \). Since the communication between two communicating events in different parallel branches may cause deadlock (a state of inactivity), which is caused by mismatch of two communicating events or the imperfectness of the communication channel. We introduce a new constant \( \delta \) to denote the deadlock, and let the atomic event \( e \in E \cup \{ \delta \} \).

Based on the definition of basic terms for APRTC (see Definition 4.2) and axioms of parallelism (see Table 13), we can prove the elimination theorem of parallelism.

Theorem 4.3 (Elimination theorem of FR parallelism). Let \( p \) be a closed APRTC term. Then there is a basic APRTC term \( q \) such that \( \text{APRTC} \vdash p = q \).

Proof. (1) Firstly, suppose that the following ordering on the signature of APRTC is defined: \( \parallel > \cdot > + \) and the symbol \( \parallel \) is given the lexicographical status for the first argument, then for each rewrite rule \( p \rightarrow q \) in Table 14 relation \( p \succ_{tpo} q \) can easily be proved. We obtain that the term rewrite system shown in Table 14 is strongly normalizing, for it has finitely many rewriting rules, and \( > \) is a well-founded ordering on the signature of APRTC, and if \( s \succ_{tpo} t \), for each rewriting rule \( s \rightarrow t \) is in Table 14 (see Theorem 2.33).

(2) Then we prove that the normal forms of closed APRTC terms are basic APRTC terms.

Suppose that \( p \) is a normal form of some closed APRTC term and suppose that \( p \) is not a basic APRTC term. Let \( p' \) denote the smallest sub-term of \( p \) which is not a basic APRTC term. It implies that each sub-term of \( p' \) is a basic APRTC term. Then we prove that \( p \) is not a term in normal form. It is sufficient to induct on the structure of \( p' \):

- Case \( p' \equiv e \) or \( e[m] \), \( e \in E \). \( p' \) is a basic APRTC term, which contradicts the assumption that \( p' \) is not a basic APRTC term, so this case should not occur.
- Case \( p' \equiv p_1 \cdot p_2 \). By induction on the structure of the basic APRTC term \( p_1 \):
  - Subcase \( p_1 \in E \). \( p' \) would be a basic APRTC term, which contradicts the assumption that \( p' \) is not a basic APRTC term;
  - Subcase \( p_1 \equiv e \cdot p'_1 \). RR5 rewriting rule in Table 11 can be applied. So \( p \) is not a normal form;
  - Subcase \( p_1 \equiv p'_1 \cdot e[m] \). RA5 rewriting rule in Table 11 can be applied. So \( p \) is not a normal form;
  - Subcase \( p_1 \equiv p'_1 + p''_1 \). RA4 rewriting rule in Table 11 can be applied. So \( p \) is not a normal form;
  - Subcase \( p_1 \equiv p'_1 \parallel p''_1 \). \( p' \) would be a basic APRTC term, which contradicts the assumption that \( p' \) is not a basic APRTC term;
  - Subcase \( p_1 \equiv p'_1 | p''_1 \). RC11 and RRC11 rewrite rule in Table 12 can be applied. So \( p \) is not a normal form;
  - Subcase \( p_1 \equiv \Theta(p'_1) \). RCE19, RRC19 and RCE20 rewrite rules in Table 12 can be applied. So \( p \) is not a normal form.
- Case \( p' \equiv p_1 + p_2 \). By induction on the structure of the basic APRTC terms both \( p_1 \) and \( p_2 \), all subcases will lead to that \( p' \) would be a basic APRTC term, which contradicts the assumption that \( p' \) is not a basic APRTC term.
| No. | Axiom |
|-----|-------|
| A6  | $x + \delta = x$ |
| A7  | $\delta \cdot x = \delta(\text{Std}(x))$ |
| RA7 | $x \cdot \delta = \delta(\text{NStd}(x))$ |
| P1  | $x \parallel y = x \parallel (y + x) \parallel y$ |
| P2  | $x \parallel y = y \parallel x$ |
| P3  | $(x \parallel y) \parallel z = x \parallel (y \parallel z)$ |
| P4  | $e_1 \parallel (e_2 \cdot y) = (e_1 \parallel e_2) \cdot y$ |
| RP4 | $e_1[m] \parallel (y \cdot e_2[m]) = y \cdot (e_1[m] \parallel e_2[m])$ |
| P5  | $(e_1 \cdot x) \parallel e_2 = (e_1 \parallel e_2) \cdot x$ |
| RP5 | $(x \cdot e_1[m]) \parallel e_2[m] = x \cdot (e_1[m] \parallel e_2[m])$ |
| P6  | $(e_1 \cdot x) \parallel (e_2 \cdot y) = (e_1 \parallel e_2) \cdot (x \parallel y)$ |
| RP6 | $(x \cdot e_1[m]) \parallel (y \cdot e_2[m]) = (x \parallel y) \cdot (e_1[m] \parallel e_2[m])$ |
| P7  | $(x + y) \parallel z = (x \parallel z) + (y \parallel z)$ |
| P8  | $x \parallel (y + z) = (x \parallel y) + (x \parallel z)$ |
| P9  | $\delta \parallel x = \delta$ |
| P10 | $x \parallel \delta = \delta$ |
| C11 | $e_1 \cdot e_2 \cdot y = \gamma(e_1, e_2)$ |
| RC11 | $e_1[m] \parallel e_2[m] = \gamma(e_1, e_2)[m]$ |
| C12 | $e_1 \cdot (e_2 \cdot y) = \gamma(e_1, e_2) \cdot y$ |
| RC12 | $e_1[m] \parallel (e_2 \cdot y) = \gamma(e_1, e_2)[m]$ |
| C13 | $(e_1 \cdot x) \parallel e_2 = \gamma(e_1, e_2) \cdot x$ |
| RC13 | $(x \cdot e_1[m]) \parallel e_2[m] = x \cdot \gamma(e_1, e_2)[m]$ |
| C14 | $(e_1 \cdot x) \parallel (e_2 \cdot y) = \gamma(e_1, e_2) \cdot (x \parallel y)$ |
| RC14 | $(x \cdot e_1[m]) \parallel (y \cdot e_2[m]) = (x \parallel y) \cdot \gamma(e_1, e_2)[m]$ |
| C15 | $(x + y) \parallel z = (x \parallel z) + (y \parallel z)$ |
| C16 | $x \parallel (y + z) = (x \parallel y) + (x \parallel z)$ |
| C17 | $\delta \parallel x = \delta$ |
| C18 | $x \parallel \delta = \delta$ |
| CE19 | $\Theta(x) = e$ |
| RCE19 | $\Theta(e[m]) = e[m]$ |
| C20 | $\Theta(\delta) = \delta$ |
| CE21 | $\Theta(x + y) = \Theta(x) \parallel y + \Theta(y) \parallel x$ |
| CE22 | $\Theta(x \parallel y) = \Theta(x) \cdot \Theta(y)$ |
| CE23 | $\Theta(x \parallel y) = ((\Theta(x) \parallel y) \parallel y) + ((\Theta(y) \parallel x) \parallel x)$ |
| CE24 | $\Theta(x \parallel y) = (\Theta(x) \parallel y) + ((\Theta(y) \parallel x) \parallel x)$ |
| U25 | $(e_1, e_2) \parallel e_3 \parallel e_2 = \tau$ |
| RU25 | $(e_1[m], e_2[n]) \parallel e_3[m] \parallel e_2[n] = \tau$ |
| U26 | $(e_1, e_2, e_3 \leq e_1) \parallel e_3 = e_1$ |
| RU26 | $(e_1[m], e_2[n], e_3[n] \geq e_3[l]) \parallel e_1[m] \parallel e_3[l] = e_1[m]$ |
| U27 | $(e_1, e_2, e_3 \leq e_1) \parallel e_3 [l] = \tau$ |
| RU27 | $(e_1[m], e_2[n], e_3[n] \geq e_3[l]) \parallel e_3[l] \parallel e_1[m] = \tau$ |
| U28 | $e \parallel e = e$ |
| U29 | $\delta \parallel e = e \parallel \delta$ |
| U30 | $(x + y) \leq z = (x \leq z) + (y \leq z)$ |
| U31 | $(x \cdot y) \leq z = (x \leq z) \cdot (y \leq z)$ |
| U32 | $(x \parallel y) \leq z = (x \leq z) \parallel (y \leq z)$ |
| U33 | $(x \cdot y) \leq z = (x \leq z) \parallel (y \leq z)$ |
| U34 | $(x \parallel y) \leq z = (x \leq z) \parallel (y \leq z)$ |
| U35 | $(x \parallel (y \cdot z)) = (x \parallel y) \leq z$ |
| U36 | $(x \parallel (y \parallel z)) = (x \parallel y) \leq z$ |
| U37 | $(x \parallel (y \parallel z)) = (x \parallel y) \leq z$ |

Table 13. Axioms of parallelism

- Case $p' \equiv p_1 \parallel p_2$. By induction on the structure of the basic APRTC terms both $p_1$ and $p_2$, all subcases will lead to that $p'$ would be a basic APRTC term, which contradicts the assumption that $p'$ is not a basic APRTC term.
- Case $p' \equiv p_1 \mid p_2$. By induction on the structure of the basic APRTC terms both $p_1$ and $p_2$, all subcases will lead to that $p'$ would be a basic APRTC term, which contradicts the assumption that $p'$ is not a basic APRTC term.
- Case $p' \equiv \Theta(p_1)$. By induction on the structure of the basic APRTC term $p_1$, $RCE19 – RCE24$ rewrite rules in Table 13 can be applied. So $p$ is not a normal form.
- Case $p' \equiv p_1 \land p_2$. By induction on the structure of the basic APRTC terms both $p_1$ and $p_2$, all subcases
10. Rewriting Rule
RA6 \( x + \delta \rightarrow x \)
RA7 \( \delta - x \rightarrow \delta \)
RRRA7 \( x - \delta \rightarrow \delta \)
RP1 \( x \parallel y \rightarrow x \parallel y + x \parallel y \)
RP2 \( x \parallel y \rightarrow x \parallel y \)
RP3 \( (x \parallel y) \parallel z \rightarrow x \parallel (y \parallel z) \)
RP4 \( e_1 \parallel (e_2 \cdot y) \rightarrow (e_1 \parallel e_2) \cdot y \)
RRP4 \( e_1[m] \parallel (y \cdot e_2[m]) \rightarrow y \cdot (e_1[m] \parallel e_2[m]) \)
RP5 \( (e_1 \cdot x) \parallel e_2 \rightarrow (e_1 \parallel e_2) \cdot x \)
RRP5 \( (e_1 \cdot e_2[m]) \parallel e_2[m] \rightarrow x \cdot (e_1[m] \parallel e_2[m]) \)
RP6 \( (e_1 \cdot x) \parallel (e_2 \cdot y) \rightarrow (e_1 \parallel e_2) \cdot (x \parallel y) \)
RP7 \( (x \parallel e_1[m]) \parallel (y \cdot e_2[m]) \rightarrow (x \parallel y) \cdot (e_1[m] \parallel e_2[m]) \)
RP8 \( x \parallel (y + z) \rightarrow (x \parallel y) + (x \parallel z) \)
RP9 \( \delta \parallel x \rightarrow \delta \)
RP10 \( x \parallel \delta \rightarrow \delta \)
RC11 \( e_1 \cdot e_2 \rightarrow \gamma (e_1, e_2) \)
RC12 \( e_1 \cdot (e_2 \cdot y) \rightarrow \gamma (e_1, e_2) \cdot y \)
RC13 \( e_1 \cdot x \parallel e_2 \rightarrow \gamma (e_1, e_2) \cdot x \)
RC14 \( x \cdot e_1[m] \parallel e_2[m] \rightarrow x \cdot \gamma (e_1, e_2)[m] \)
RC15 \( (x + y) \parallel z \rightarrow (x \parallel z) + (y \parallel z) \)
RC16 \( x \parallel (y + z) \rightarrow (x \parallel y) + (x \parallel z) \)
RC17 \( \delta \parallel x \rightarrow \delta \)
RC18 \( x \parallel \delta \rightarrow \delta \)
RCE19 \( \Theta(e) \rightarrow e \)
RCE20 \( \Theta(e[m]) \rightarrow e[m] \)
RCE21 \( \Theta(x + y) \rightarrow \Theta(x) \cdot \delta \cdot \Theta(y) \cdot \delta \cdot x \)
RCE22 \( \Theta(x \cdot y) \rightarrow \Theta(x) \cdot \Theta(y) \)
RCE23 \( \Theta(x \parallel y) \rightarrow ((\Theta(x) \parallel y) \parallel y) \cdot ((\Theta(y) \parallel x) \parallel x) \)
RCE24 \( \Theta(x \parallel y) \rightarrow ((\Theta(x \parallel y) \parallel y) \cdot ((\Theta(y) \parallel x) \parallel x) \)
RR1725 \( (\Theta(e_1, e_2)) \cdot e_1 \parallel e_2 \rightarrow \tau \)
RU26 \( (\Theta(e_1, e_2)) \cdot e_1 \parallel e_2 \rightarrow \tau \)
RR26 \( (\Theta(e_1, e_2) \cdot e_2) \parallel e_1 \rightarrow \tau \)
RR27 \( (\Theta(e_1, e_2) \cdot e_2) \parallel e_1 \rightarrow \tau \)
RU28 \( e \parallel \delta \rightarrow e \)
RU29 \( \delta \parallel e \rightarrow \delta \)
RU30 \( (x + y) \parallel z \rightarrow ((x \parallel z) + (y \parallel z)) \)
RU31 \( (x \parallel y) \parallel z \rightarrow ((x \parallel z) \parallel (y \parallel z)) \)
RU32 \( (x \parallel y) \parallel z \rightarrow ((x \parallel z) \parallel (y \parallel z)) \)
RU33 \( (x \parallel y) \parallel z \rightarrow ((x \parallel z) \parallel (y \parallel z)) \)
RU34 \( x \parallel (y + z) \rightarrow ((x \parallel y) \parallel z) \)
RU35 \( x \parallel (y + z) \rightarrow ((x \parallel y) \parallel z) \)
RU36 \( x \parallel (y \parallel z) \rightarrow ((x \parallel y) \parallel z) \)
RU37 \( x \parallel (y \parallel z) \rightarrow ((x \parallel y) \parallel z) \)

Table 14. Term rewrite system of APRC

will lead to that \( p' \) would be a basic APRC term, which contradicts the assumption that \( p' \) is not a basic APRC term.

\[\square\]

4.3. Structured Operational Semantics of Parallelism

**Theorem 4.4** (Generalization of the algebra for parallelism with respect to BARTC). The algebra for parallelism is a generalization of BARTC.
Proof. It follows from the following three facts.

1. The transition rules of BARTC in section ?? are all source-dependent;
2. The sources of the transition rules for the algebra for parallelism contain an occurrence of $\downarrow$, or $\parallel$, or $\mid$, or $\Theta$, or $\triangleleft$;
3. The transition rules of APRTC are all source-dependent.

So, the algebra for parallelism is a generalization of BARTC, that is, BARTC is an embedding of the algebra for parallelism, as desired.

\textbf{Theorem 4.5} (Soundness of parallelism modulo FR step bisimulation equivalence). Let $x$ and $y$ be APRTC terms. If $\text{APRTC} \vdash x = y$, then $x \sim_{f_\downarrow}^r y$.

Proof. Since FR step bisimulation $\sim_{f_\downarrow}^r$ is both an equivalent and a congruent relation with respect to the operators $\downarrow$, $\parallel$, $\mid$, $\Theta$ and $\triangleleft$, we only need to check if each axiom in Table 13 is sound modulo FR step bisimulation equivalence.

The proof is similar to the proof of soundness of BARTC modulo FR step bisimulation equivalence, we omit it.

\textbf{Theorem 4.6} (Completeness of parallelism modulo FR step bisimulation equivalence). Let $p$ and $q$ be closed APRTC terms, if $p \sim_{f_\downarrow}^r q$ then $p = q$.

Proof. Firstly, by the elimination theorem of APRTC (see Theorem 4.3), we know that for each closed APRTC term $p$, there exists a closed basic APRTC term $p'$, such that $\text{APRTC} \vdash p = p'$, so, we only need to consider closed basic APRTC terms.

The basic terms (see Definition ??) modulo associativity and commutativity (AC) of conflict + (defined by axioms A1 and A2 in Table 1) and associativity and commutativity (AC) of parallel $\parallel$ (defined by axioms P2 and P3 in Table 13), and these equivalences is denoted by $=_{AC}$. Then, each equivalence class $s$ modulo AC of $+$ and $\parallel$ has the following normal form

$$s_1 + \cdots + s_k$$

with each $s_i$ either an atomic event or of the form

$$t_1 \cdots \cdot t_m$$

with each $t_j$ either an atomic event or of the form

$$u_1 \parallel \cdots \parallel u_n$$

with each $u_i$ an atomic event, and each $s_i$ is called the summand of $s$.

Now, we prove that for normal forms $n$ and $n'$, if $n \sim_{f_\downarrow}^r n'$ then $n =_{AC} n'$. It is sufficient to induct on the sizes of $n$ and $n'$.

- Consider a summand $e$ of $n$. Then $n \sim_e^r e[m]$, so $n \sim_{f_\downarrow}^r n'$ implies $n' \sim_e^r e[m]$, meaning that $n'$ also contains the summand $e$.
- Consider a summand $e[m]$ of $n$. Then $n \sim_e^r e[m]$, so $n \sim_{f_\downarrow}^r n'$ implies $n' \sim_e^r e[m]$, meaning that $n'$ also contains the summand $e[m]$.
- Consider a summand $t_1 \cdot t_2$ of $n$.
  - if $t_1 \equiv e'$, then $n \sim_{f_\downarrow}^r e'[m] \cdot t_2$, so $n \sim_{f_\downarrow}^r n'$ implies $n' \sim_{f_\downarrow}^r e'[m] \cdot t_2$ with $e'[m] \cdot t_2 \sim_{f_\downarrow}^r e'[m] \cdot t_2'$, meaning that $n'$ contains a summand $e' \cdot t_2'$. Since $t_2$ and $t_2'$ are normal forms and have sizes smaller than $n$ and $n'$, by the induction hypotheses if $t_2 \sim_{f_\downarrow}^r t_2'$ then $t_2 =_{AC} t_2'$;
  - if $t_2 \equiv e'[m]$, then $n \sim_{f_\downarrow}^r e'[m] \cdot t_1$, so $n \sim_{f_\downarrow}^r n'$ implies $n' \sim_{f_\downarrow}^r e'[m] \cdot t_1 \cdot e'$ with $t_1 \cdot e' \sim_{f_\downarrow}^r t_1' \cdot e'$, meaning that $n'$ contains a summand $t_1' \cdot e'$. Since $t_1$ and $t_1'$ are normal forms and have sizes smaller than $n$ and $n'$, by the induction hypotheses if $t_1 \sim_{f_\downarrow}^r t_1'$ then $t_1 =_{AC} t_1'$. 

\[\text{if } t_1 \equiv e_1 \parallel \cdots \parallel e_n, \text{ then } n \xrightleftharpoons{s}^{fr}(e_1[m] \parallel \cdots \parallel e_n[m]) \cdot t_2, \text{ so } n \xrightleftharpoons{s}^{fr} n' \text{ implies } n' \xrightleftharpoons{s}^{fr}(e_1[m] \parallel \cdots \parallel e_n[m]) \cdot t_2, \text{ with } t_2 \sim_{s}^{fr} t_2', \text{ meaning that } n' \text{ contains a summand } (e_1[m] \parallel \cdots \parallel e_n[m]) \cdot t_2'. \text{ Since } t_2 \text{ and } t_2' \text{ are normal forms and have sizes smaller than } n \text{ and } n', \text{ by the induction hypotheses if } t_2 \sim_{s}^{fr} t_2', \text{ then } t_2 =_{AC} t_2'.
\]

- if \(t_2 \equiv e_1[m] \parallel \cdots \parallel e_n[m] \), then \(n \xrightleftharpoons{s}^{fr}(e_1[m] \parallel \cdots \parallel e_n[m]) \cdot t_1 \sim_{s}^{fr} t_1' \), meaning that \(n' \) contains a summand \((e_1[m] \parallel \cdots \parallel e_n[m]) \cdot t_1'\). Since \(t_1 \) and \(t_1' \) are normal forms and have sizes smaller than \(n \) and \(n' \), by the induction hypotheses if \(t_1 \sim_{s}^{fr} t_1' \) then \(t_1 =_{AC} t_1' \).

So, we get \(n =_{AC} n' \).

Finally, let \(s \) and \(t \) be basic APRTC terms, and \(s \xrightleftharpoons{s}^{fr} t \), there are normal forms \(n \) and \(n' \), such that \(s = n \) and \(t = n' \). The soundness theorem of parallelism modulo FR step bisimulation equivalence (see Theorem 4.7) yields \(s \xrightleftharpoons{s}^{fr} n \) and \(t \xrightleftharpoons{s}^{fr} n' \), so \(n \xrightleftharpoons{s}^{fr} s \xrightleftharpoons{s}^{fr} t \xrightleftharpoons{s}^{fr} n' \). Since if \(n \xrightleftharpoons{s}^{fr} n' \) then \(n =_{AC} n' \), \(s = n =_{AC} n' = t \), as desired.

**Theorem 4.7** (Soundness of parallelism modulo FR pomset bisimulation equivalence). Let \(x \) and \(y \) be APRTC terms. If \(\text{APRTC} \vdash x = y \), then \(x \xrightleftharpoons{p}^{fr} y \).

**Proof.** Since FR pomset bisimulation \(\xrightleftharpoons{p}^{fr} \) is both an equivalent and a congruent relation with respect to the operators \(\parallel, |, \Theta, \triangleleft \), we only need to check if each axiom in Table 13 is sound modulo FR pomset bisimulation equivalence.

From the definition of FR pomset bisimulation (see Definition 2.22), we know that FR pomset bisimulation is defined by pomset transitions, which are labeled by pomsets. In a pomset transition, the events in the bisimulation equivalence.

Theorem 4.7, we can prove that each axiom in Table 13 is sound modulo FR pomset bisimulation equivalence, we omit them.

**Theorem 4.8** (Completeness of parallelism modulo FR pomset bisimulation equivalence). Let \(p \) and \(q \) be closed APRTC terms, if \(p \xrightleftharpoons{p}^{fr} q \) then \(p = q \).

**Proof.** The proof is similar to the proof of completeness of parallelism modulo FR step bisimulation equivalence, we omit it.

**Theorem 4.9** (Soundness of parallelism modulo FR hp-bisimulation equivalence). Let \(x \) and \(y \) be APRTC terms. If \(\text{APRTC} \vdash x = y \), then \(x \xrightleftharpoons{hp}^{fr} y \).

**Proof.** Since FR hp-bisimulation \(\xrightleftharpoons{hp}^{fr} \) is both an equivalent and a congruent relation with respect to the operators \(\parallel, |, \Theta, \triangleleft \), we only need to check if each axiom in Table 13 is sound modulo FR hp-bisimulation equivalence.

From the definition of FR hp-bisimulation (see Definition 2.24), we know that FR hp-bisimulation is defined on the posetal product \((C_1, f, C_2), f : C_1 \rightarrow C_2\) isomorphism. Two process terms \(s \) related to \(C_1 \) and \(t \) related to \(C_2 \), and \( \overline{f} : C_1 \rightarrow C_2 \) isomorphism. Initially, \((C_1, f, C_2) = (\emptyset, \emptyset, \emptyset)\), and \((\emptyset, \emptyset, \emptyset) \xrightleftharpoons{hp}^{fr} \).

When \(s \xrightarrow{e[m]} s' (C_1 \xrightarrow{e[m]} C_1') \), there will be \(t \xrightarrow{e[m]} t' (C_2 \xrightarrow{e[m]} C_2') \), and we define \(f' = f[e \rightarrow e] \). And when \(s \xrightarrow{e[m]} s' (C_1 \xrightarrow{e[m]} C_1') \), there will be \(t \xrightarrow{e[m]} t' (C_2 \xrightarrow{e[m]} C_2') \), and we define \(f' = f[e[m] \rightarrow e[m]] \). Then, if \((C_1, f, C_2) \xrightleftharpoons{hp}^{fr} \), then \((C_1', f', C_2') \xrightleftharpoons{hp}^{fr} \).

Similarly to the proof of soundness of parallelism modulo FR pomset bisimulation equivalence (see Theorem 4.7), we can prove that each axiom in Table 13 is sound modulo FR hp-bisimulation equivalence, we just need additionally to check the above conditions on FR hp-bisimulation, we omit them.
\[
\frac{x \xrightarrow{e} e[m]}{\partial_H(x) \xrightarrow{e} \partial_H(e[m])} \quad (e \notin H) \quad \frac{x \xrightarrow{e} x'}{\partial_H(x) \xrightarrow{e} \partial_H(x')} \quad (e \notin H)
\]
Table 15. Forward transition rules of encapsulation operator \(\partial_H\)

\[
\frac{x \xrightarrow{e} e}{\partial_H(x) \xrightarrow{e} \partial_H(x)} \quad (e \notin H) \quad \frac{x \xrightarrow{e} x'}{\partial_H(x) \xrightarrow{e} \partial_H(x')} \quad (e \notin H)
\]
Table 16. Reverse transition rules of encapsulation operator \(\partial_H\)

**Theorem 4.10** (Completeness of parallelism modulo FR hp-bisimulation equivalence). Let \(p\) and \(q\) be closed APTC terms, if \(p \sim_{hp} q\) then \(p = q\).

**Proof.** The proof is similar to the proof of completeness of parallelism modulo FR pomset bisimulation equivalence, we omit it.

### 4.4. Encapsulation

The mismatch of two communicating events in different parallel branches can cause deadlock, so the deadlocks in the concurrent processes should be eliminated. Like APTC \([8]\), we also introduce the unary encapsulation operator \(\partial_H\) for set \(H\) of atomic events, which renames all atomic events in \(H\) into \(\delta\). The whole algebra including parallelism for true concurrency in the above subsections, deadlock \(\delta\) and encapsulation operator \(\partial_H\), is called Reversible Algebra for Parallelism in True Concurrency, abbreviated APRTC.

The forward transition rules of encapsulation operator \(\partial_H\) are shown in Table 15, and the reverse transition rules of encapsulation operator \(\partial_H\) are shown in Table 16.

Based on the transition rules for encapsulation operator \(\partial_H\) in Table 15 and Table 16, we design the axioms as Table 17 shows.

**Theorem 4.11** (Conservativity of APRTC with respect to the algebra for parallelism). APRTC is a conservative extension of the algebra for parallelism.

**Proof.** It follows from the following two facts (see Theorem 2.19).

1. The transition rules of the algebra for parallelism in the above subsections are all source-dependent;
2. The sources of the transition rules for the encapsulation operator contain an occurrence of \(\partial_H\).

So, APRTC is a conservative extension of the algebra for parallelism, as desired.

**Theorem 4.12** (Congruence theorem of encapsulation operator \(\partial_H\)). Truly concurrent bisimulation equivalences \(\sim_p\), \(\sim_s\), \(\sim_{hp}\) and \(\sim_{hhp}\) are all congruences with respect to encapsulation operator \(\partial_H\).

| No. | Axiom |
|-----|-------|
| D1  | \(e \notin H \quad \partial_H(e) = 0\) |
| RD1 | \(e \notin H \quad \partial_H(e[m]) = e[m]\) |
| D2  | \(e \notin H \quad \partial_H(e) = \delta\) |
| RD2 | \(e \notin H \quad \partial_H(e[m]) = \delta\) |
| D3  | \(\partial_H(\delta) = \delta\) |
| D4  | \(\partial_H(x + y) = \partial_H(x) + \partial_H(y)\) |
| D5  | \(\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)\) |
| D6  | \(\partial_H(x \parallel y) = \partial_H(x) \parallel \partial_H(y)\) |

Table 17. Axioms of encapsulation operator
Table 18. Term rewrite system of encapsulation operator $\partial_H$

| No. | Rewriting Rule |
|-----|----------------|
| RD1 | $e \in H \quad \partial_H(e) \rightarrow e$ |
| RRD1 | $e \in H \quad \partial_H([e|m]) \rightarrow e[m]$ |
| RD2 | $e \in H \quad \partial_H(e) \rightarrow \delta$ |
| RRD2 | $e \in H \quad \partial_H([e|m]) \rightarrow \delta$ |
| RD3 | $\partial_H(\delta) \rightarrow \delta$ |
| RD4 | $\partial_H(x+y) \rightarrow \partial_H(x) + \partial_H(y)$ |
| RD5 | $\partial_H(x \cdot y) \rightarrow \partial_H(x) \cdot \partial_H(y)$ |
| RD6 | $\partial_H(x \parallel y) \rightarrow \partial_H(x) \parallel \partial_H(y)$ |

**Proof.** (1) Case FR pomset bisimulation equivalence $\sim^f_p$.

Let $x$ and $y$ be APRTCs, and $x \sim^f_p y$, it is sufficient to prove that $\partial_H(x) \sim^f_p \partial_H(y)$.

By the definition of FR pomset bisimulation $\sim^f_p$ (Definition 2.22), $x \sim^f_p y$ means that

$$x \xrightarrow{X} x' \quad y \xrightarrow{Y} y'$$

$$x \xrightarrow{[K]} x' \quad y \xrightarrow{[J]} y'$$

with $X \subseteq x$, $Y \subseteq y$, $X \sim Y$ and $x' \sim^f_p y'$.

By the FR pomset transition rules for encapsulation operator $\partial_H$ in Table 15 and Table 16, we can get

$$\partial_H(x) \xrightarrow{X} \partial_H(X[K])(X \not\subseteq H) \quad \partial_H(y) \xrightarrow{Y} \partial_H(Y[J])(Y \not\subseteq H)$$

$$\partial_H(x) \xrightarrow{[K]} \partial_H(X)(X \not\subseteq H) \quad \partial_H(y) \xrightarrow{[J]} \partial_H(Y)(Y \not\subseteq H)$$

with $X \subseteq x$, $Y \subseteq y$, and $X \sim Y$, and the assumptions $\partial_H(X[K]) \sim^f_p \partial_H(Y[J])$, $\partial_H(X) \sim^f_p \partial_H(Y)$, so, we get $\partial_H(x) \sim^f_p \partial_H(y)$, as desired.

Or, we can get

$$\partial_H(x) \xrightarrow{X} \partial_H(x')(X \not\subseteq H) \quad \partial_H(y) \xrightarrow{Y} \partial_H(y')(Y \not\subseteq H)$$

$$\partial_H(x) \xrightarrow{[K]} \partial_H(x)(X \not\subseteq H) \quad \partial_H(y) \xrightarrow{[J]} \partial_H(y')(Y \not\subseteq H)$$

with $X \subseteq x$, $Y \subseteq y$, $X \sim Y$, $x' \sim^f_p y'$ and the assumption $\partial_H(x') \sim^f_p \partial_H(y')$, so, we get $\partial_H(x) \sim^f_p \partial_H(y)$, as desired.

(2) The cases of FR step bisimulation $\sim^s_f$, FR hp-bisimulation $\sim^h_p$ and FR hhp-bisimulation $\sim^{hhp}_p$ can be proven similarly, we omit them.

**Theorem 4.13** (Elimination theorem of APRTCs). Let $p$ be a closed APRTC term including the encapsulation operator $\partial_H$. Then there is a basic APRTC term $q$ such that $APRTC \vdash p = q$.

**Proof.** (1) Firstly, suppose that the following ordering on the signature of APRTC is defined: $\gg \gg > +$ and the symbol $\|$ is given the lexicographical status for the first argument, then for each rewrite rule $p \rightarrow q$ in Table 18 relation $p \gg_{1po} q$ can easily be proved. We obtain that the term rewrite system shown in Table 18 is strongly normalizing, for it has finitely many rewriting rules, and $>_{1po}$ is a well-founded ordering on the signature of APRTC, and if $s >_{1po} t$, for each rewriting rule $s \rightarrow t$ is in Table 18 (see Theorem 2.33).

(2) Then we prove that the normal forms of closed APRTC terms including encapsulation operator $\partial_H$ are basic APRTC terms.

Suppose that $p$ is a normal form of some closed APRTC term and suppose that $p$ is not a basic APRTC term. Let $p'$ denote the smallest sub-term of $p$ which is not a basic APRTC term. It implies that each
sub-term of $p'$ is a basic APRTc term. Then we prove that $p$ is not a term in normal form. It is sufficient to induct on the structure of $p'$, following from Theorem 4.13, we only prove the new case $p' \equiv \partial_H(p_1)$:

- Case $p_1 \equiv e$. The transition rules RD1 or RD2 can be applied, so $p$ is not a normal form;
- Case $p_1 \equiv e[m]$. The transition rules RRD1 or RRD2 can be applied, so $p$ is not a normal form;
- Case $p_1 \equiv \delta$. The transition rules RD3 can be applied, so $p$ is not a normal form;
- Case $p_1 \equiv p'_1 + p''_1$. The transition rules RD4 can be applied, so $p$ is not a normal form;
- Case $p_1 \equiv p'_1 \cdot p''_1$. The transition rules RD5 can be applied, so $p$ is not a normal form;
- Case $p_1 \equiv p'_1 \parallel p''_1$. The transition rules RD6 can be applied, so $p$ is not a normal form.

\[\square\]

\textbf{Theorem 4.14} (Soundness of APRTc modulo FR step bisimulation equivalence). Let $x$ and $y$ be APRTc terms including encapsulation operator $\partial_H$. If $APRTc \vdash x = y$, then $x \sim^s_p y$.

\textbf{Proof.} Since FR step bisimulation $\sim^s_p$ is both an equivalent and a congruent relation with respect to the operator $\partial_H$, we only need to check if each axiom in Table 17 is sound modulo FR step bisimulation equivalence.

The proof is similar to the proof of soundness of the algebra of parallelism modulo FR step bisimulation equivalence, we omit it. \[\square\]

\textbf{Theorem 4.15} (Completeness of APRTc modulo FR step bisimulation equivalence). Let $p$ and $q$ be closed APRTc terms including encapsulation operator $\partial_H$, if $p \sim^s_p q$ then $p \equiv q$.

\textbf{Proof.} Firstly, by the elimination theorem of APRTc (see Theorem 4.13), we know that the normal form of APRTc does not contain $\partial_H$, and for each closed APRTc term $p$, there exists a closed basic APRTc term $p'$, such that $APRTc \vdash p \equiv p'$, so, we only need to consider closed basic APRTc terms.

Similarly to Theorem 4.10, we can prove that for normal forms $n$ and $n'$, if $n \sim^s_p n'$ then $n \equiv n'$.

Finally, let $s$ and $t$ be basic APRTc terms, and $s \sim^s_p t$, there are normal forms $n$ and $n'$, such that $s \equiv n$ and $t \equiv n'$. The soundness theorem of APRTc modulo FR step bisimulation equivalence (see Theorem 4.14) yields $s \sim^s_p n$ and $t \sim^s_p n'$, so $n \sim^s_p s \sim^s_p t \sim^s_p n'$. Since if $n \sim^s_p n'$ then $n \equiv n'$, $s \equiv n \equiv n' \equiv t$, as desired.

\[\square\]

\textbf{Theorem 4.16} (Soundness of APRTc modulo FR pomset bisimulation equivalence). Let $x$ and $y$ be APRTc terms including encapsulation operator $\partial_H$. If $APRTc \vdash x = y$, then $x \sim^p_p y$.

\textbf{Proof.} Since FR pomset bisimulation $\sim^p_p$ is both an equivalent and a congruent relation with respect to the operator $\partial_H$, we only need to check if each axiom in Table 17 is sound modulo FR pomset bisimulation equivalence.

From the definition of FR pomset bisimulation (see Definition 2.22), we know that FR pomset bisimulation is defined by pomset transitions, which are labeled by pomsets. In a pomset transition, the events in the pomset are either within causality relations (defined by $\cdot$) or in concurrency (implicitly defined by $\parallel$), of course, they are pairwise consistent (without conflicts). In Theorem 7?, we have already proven the case that all events are pairwise concurrent, so, we only need to prove the case of events in causality. Without loss of generality, we take a pomset of $P = \{e_1, e_2 : e_1 \cdot e_2\}$. Then the pomset transition labeled by the above $P$ is just composed of one single event transition labeled by $e_1$ succeeded by another single event transition labeled by $e_2$, that is, $P \equiv \{e_1, e_2\}$ or $P \equiv \{e_2[m] \cdot e_1[m]\}$.

Similarly to the proof of soundness of APRTc modulo FR step bisimulation equivalence (see Theorem 4.14), we can prove that each axiom in Table 17 is sound modulo FR pomset bisimulation equivalence, we omit them. \[\square\]

\textbf{Theorem 4.17} (Completeness of APRTc modulo FR pomset bisimulation equivalence). Let $p$ and $q$ be closed APRTc terms including encapsulation operator $\partial_H$, if $p \sim^p_p q$ then $p \equiv q$.

\textbf{Proof.} The proof can be proven similarly to the proof of completeness of APRTc modulo FR step bisimulation equivalence, we omit it. \[\square\]
Theorem 4.18 (Soundness of APRTC modulo FR hp-bisimulation equivalence). Let $x$ and $y$ be APRTC terms including encapsulation operator $\partial_H$. If $\texttt{APRTC} \vdash x = y$, then $x \sim_{hp}^f y$.

Proof. Since FR hp-bisimulation $\sim_{hp}^f$ is both an equivalent and a congruent relation with respect to the operator $\partial_H$, we only need to check if each axiom in Table 17 is sound modulo FR hp-bisimulation equivalence.

From the definition of FR hp-bisimulation (see Definition 2.24), we know that FR hp-bisimulation is defined on the posetal product $(C_1, f, C_2), f : C_1 \rightarrow C_2$ isomorphism. Two process terms $s$ related to $C_1$ and $t$ related to $C_2$, and $f : C_1 \rightarrow C_2$ isomorphism. Initially, $(C_1, f, C_2) = (\emptyset, \emptyset, \emptyset)$, and $(\emptyset, \emptyset, \emptyset) \sim_{hp}^f$.

When $s \overset{e}{\rightarrow} s' (C_1 \overset{e}{\rightarrow} C_1')$, there will be $t \overset{e}{\rightarrow} t' (C_2 \overset{e}{\rightarrow} C_2')$, and we define $f' = f[e \mapsto e]$. And when $s \overset{e}{\rightarrow} s' (C_1 \overset{e}{\rightarrow} C_1'), there will be $t \overset{e[m]}{\rightarrow} t' (C_2 \overset{e[m]}{\rightarrow} C_2')$, and we define $f' = f[e[m] \mapsto e[m]]$. Then, if $(C_1, f, C_2) \sim_{hp}^f$, then $(C_1', f', C_2') \sim_{hp}^f$.

Similarly to the proof of soundness of APRTC modulo FR pomset bisimulation equivalence (see Theorem 4.16), we can prove that each axiom in Table 17 is sound modulo FR hp-bisimulation equivalence, we just need additionally to check the above conditions on FR hp-bisimulation, we omit them.

Theorem 4.19 (Completeness of APRTC modulo FR hp-bisimulation equivalence). Let $p$ and $q$ be closed APRTC terms including encapsulation operator $\partial_H$, if $p \sim_{hp}^f q$ then $p = q$.

Proof. The proof is similar to the proof of completeness of APRTC modulo FR pomset bisimulation equivalence, we omit it.

5. Recursion

In this section, we introduce recursion to capture infinite processes based on APRTC. In the following, $E, F, G$ are recursion specifications, $X, Y, Z$ are recursive variables.

The behavior of the solution $(X_i|E)$ for the recursion variable $X_i$ in $E$, where $i \in \{1, \cdots, n\}$, is exactly the behavior of their right-hand sides $t_i(X_1, \cdots, X_n)$, which is captured by the two transition rules in Table 19.

Table 19. Transition rules of guarded recursion

\[
\begin{align*}
  & t_i((X_1|E), \cdots, (X_n|E)) \overset{e}{\rightarrow} \\
  & (X_i|E) \overset{e}{\rightarrow} \\
  & t_i((X_1|E), \cdots, (X_n|E)) \overset{y}{\rightarrow} \\
  & (X_i|E) \overset{y}{\rightarrow} 
\end{align*}
\]

Theorem 5.1 (Conservativity of APRTC with guarded recursion). APRTC with guarded recursion is a conservative extension of APRTC.

Proof. Since the transition rules of APRTC are source-dependent, and the transition rules for guarded recursion in Table 19 contain only a fresh constant in their source, so the transition rules of APRTC with guarded recursion are a conservative extension of those of APRTC.

Theorem 5.2 (Congruence theorem of APRTC with guarded recursion). Truly concurrent bisimulation equivalences $\sim_{hp}^f, \sim_{hp}^s$ and $\sim_{hp}^r$ are all congruences with respect to APRTC with guarded recursion.

Proof. It follows the following two facts:

1. in a guarded recursive specification, right-hand sides of its recursive equations can be adapted to the form by applications of the axioms in APRTC and replacing recursion variables by the right-hand sides of their recursive equations;
2. truly concurrent bisimulation equivalences $\sim_{hp}^f, \sim_{hp}^s$ and $\sim_{hp}^r$ are all congruences with respect to all operators of APRTC.
Table 20. Recursive definition and specification principle

| No. | Axiom |
|-----|-------|
| RDP | \((X_i|E) = t_i((X_1|E, \ldots, X_n|E))\) \((i \in \{1, \ldots, n\})\) |
| RSP | if \(y_i = t_i(y_1, \ldots, y_n)\) for \(i \in \{1, \ldots, n\}\), then \(y_i = (X_i|E)\) \((i \in \{1, \ldots, n\})\) |

5.1. Recursive Definition and Specification Principles

The RDP (Recursive Definition Principle) and the RSP (Recursive Specification Principle) are shown in Table 20.

**Theorem 5.3** (Elimination theorem of APRTC with linear recursion). Each process term in APRTC with linear recursion is equal to a process term \((X_1|E)\) with \(E\) a linear recursive specification.

**Proof.** By applying structural induction with respect to term size, each process term \(t_1\) in APRTC with linear recursion generates a process which can be expressed in the form of equations

\[ t_i = (a_{i11} \parallel \cdots \parallel a_{i1i})t_{i1} + \cdots + (a_{ik1} \parallel \cdots \parallel a_{ik_{ik}})t_{ik} + (b_{i11} \parallel \cdots \parallel b_{i1i1}) + \cdots + (b_{il1} \parallel \cdots \parallel b_{il_{il}}) \]

for \(i \in \{1, \ldots, n\}\). Or,

\[ t_i = t_{i1}(a_{i11}[m_{i1}] \parallel \cdots \parallel a_{i1i1}[m_{i1}]) + \cdots + t_{ik}(a_{ik1}[m_{ik}] \parallel \cdots \parallel a_{ik_{ik}}[m_{ik}]) + (b_{i11}[n_{i1}] \parallel \cdots \parallel b_{i1i1}[n_{i1}]) + \cdots + (b_{il1}[n_{il}] \parallel \cdots \parallel b_{il_{il}}[n_{il}]) \]

Let the linear recursive specification \(E\) consist of the recursive equations

\[ X_i = (a_{i11} \parallel \cdots \parallel a_{i1i1})X_{i1} + \cdots + (a_{ik1} \parallel \cdots \parallel a_{ik_{ik}})X_{ik} + (b_{i11} \parallel \cdots \parallel b_{i1i1}) + \cdots + (b_{il1} \parallel \cdots \parallel b_{il_{il}}) \]

or the equations,

\[ X_i = X_{i1}(a_{i11}[m_{i1}] \parallel \cdots \parallel a_{i1i1}[m_{i1}]) + \cdots + X_{ik}(a_{ik1}[m_{ik}] \parallel \cdots \parallel a_{ik_{ik}}[m_{ik}]) + (b_{i11}[n_{i1}] \parallel \cdots \parallel b_{i1i1}[n_{i1}]) + \cdots + (b_{il1}[n_{il}] \parallel \cdots \parallel b_{il_{il}}[n_{il}]) \]

for \(i \in \{1, \ldots, n\}\). Replacing \(X_i\) by \(t_i\) for \(i \in \{1, \ldots, n\}\) is a solution for \(E\). RSP yields \(t_1 = (X_1|E)\).

**Theorem 5.4** (Soundness of APRTC with guarded recursion). Let \(x\) and \(y\) be APRTC with guarded recursion terms. If APRTC with guarded recursion \(\vdash x = y\), then

1. \(x \sim_{fr}^s y\);
2. \(x \sim_{fr}^p y\);
3. \(x \sim_{hp} y\).

**Proof.** (1) Soundness of APRTC with guarded recursion with respect to FR step bisimulation \(\sim_{fr}^s\).

Since FR step bisimulation \(\sim_{fr}^s\) is both an equivalent and a congruent relation with respect to APRTC with guarded recursion, we only need to check if each axiom in Table 20 is sound modulo FR step bisimulation equivalence.

This can be proven similarly to the proof of soundness of APRTC modulo FR step bisimulation equivalence, we omit them.

(2) Soundness of APRTC with guarded recursion with respect to FR pomset bisimulation \(\sim_{fr}^p\).

Since FR pomset bisimulation \(\sim_{fr}^p\) is both an equivalent and a congruent relation with respect to the guarded recursion, we only need to check if each axiom in Table 20 is sound modulo FR pomset bisimulation equivalence.

From the definition of FR pomset bisimulation (see Definition 2.22), we know that FR pomset bisimulation is defined by pomset transitions, which are labeled by pomsets. In a pomset transition, the events in the
pomset are either within causality relations (defined by $\cdot$) or in concurrency (implicitly defined by $\cdot$ and $+$, and explicitly defined by $\mathcal{R}$), of course, they are pairwise consistent (without conflicts). In Theorem 7?, we have already proven the case that all events are pairwise concurrent, so, we only need to prove the case of events in causality. Without loss of generality, we take a pomset of $P = \{e_1, e_2 : e_1 \cdot e_2\}$. Then the pomset transition labeled by the above $P$ is just composed of one single event transition labeled by $e_1$ succeeded by another single event transition labeled by $e_2$, that is, $\xrightarrow{e_1 \cdots e_2}$ or $\xrightarrow{$e_2$[m]}$.

Similarly to the proof of soundness of $\text{APRTC}$ with guarded recursion modulo $\text{FR step bisimulation equivalence}$ (1), we can prove that each axiom in Table 20 is sound modulo $\text{FR pomset bisimulation equivalence}$, we omit them.

(3) Soundness of $\text{APRTC}$ with guarded recursion with respect to $\text{FR hp-bisimulation}$. $\sim^{fr}_{hp}$.

Since $\text{FR hp-bisimulation}$ $\sim^{fr}_{hp}$ is both an equivalent and a congruent relation with respect to guarded recursion, we only need to check if each axiom in Table 20 is sound modulo $\text{FR hp-bisimulation equivalence}$.

From the definition of $\text{FR hp-bisimulation}$ (see Definition 2.21), we know that $\text{FR hp-bisimulation}$ is defined on the posetal product $(C_1, f, C_2); f : C_1 \to C_2$ isomorphism. Two process terms $s$ related to $C_1$ and $t$ related to $C_2$, and $f : C_1 \to C_2$ isomorphism. Initially, $(C_1, f, C_2) = (\emptyset, \emptyset, \emptyset)$, and $(\emptyset, \emptyset, \emptyset) \sim^{fr}_{hp}$.

When $s \xrightarrow{e[m]} s'$ $(C_1 \xrightarrow{e[m]} C_1')$, there will be $t \xrightarrow{e[m]} t'$ $(C_2 \xrightarrow{e[m]} C_2')$, and we define $f' = f[e \mapsto e]$. And when $s \xrightarrow{e[m]} s'$ $(C_1 \xrightarrow{e[m]} C_1')$, there will be $t \xrightarrow{e[m]} t'$ $(C_2 \xrightarrow{e[m]} C_2')$, and we define $f' = f[e'[m] \mapsto e[m]]$. Then, $f \sim^{fr}_{hp}$, then $(C_1', f', C_2') \sim^{fr}_{hp}$.

Similarly to the proof of soundness of $\text{APRTC}$ with guarded recursion modulo $\text{FR pomset bisimulation equivalence}$ (2), we can prove that each axiom in Table 20 is sound modulo $\text{FR hp-bisimulation equivalence}$, we just need additionally to check the above conditions on $\text{FR hp-bisimulation}$, we omit them.

\textbf{Theorem 5.5 (Completeness of $\text{APRTC}$ with linear recursion).} Let $p$ and $q$ be closed $\text{APRTC}$ with linear recursion terms, then,

1. if $p \sim^{fr}_{s} q$ then $p = q$;
2. if $p \sim^{fr}_{p} q$ then $p = q$;
3. if $p \sim^{fr}_{hp} q$ then $p = q$.

\textbf{Proof.} Firstly, by the elimination theorem of $\text{APRTC}$ with guarded recursion (see Theorem 5.3), we know that each process term in $\text{APRTC}$ with linear recursion is equal to a process term $(X_1|E)$ with $E$ a linear recursive specification.

It remains to prove the following cases.

(1) If $(X_1|E_1) \sim^{fr}_{s} (Y_1|E_2)$ for linear recursive specification $E_1$ and $E_2$, then $(X_1|E_1) = (Y_1|E_2)$.

Let $E_1$ consist of recursive equations $X = t_X$ for $X \in X$ and $E_2$ consists of recursive equations $Y = t_Y$ for $Y \in Y$. Let the linear recursive specification $E$ consist of recursive equations $Z_{XY} = t_{XY}$, and $(X|E_1) \sim^{fr}_{s} (Y|E_2)$, and $t_{XY}$ consists of the following summands:

1. $t_{XY}$ contains a summand $(a_1 \mid \cdots \mid a_m)Z_{XY}$, if $t_X$ contains the summand $(a_1 \mid \cdots \mid a_m)Y'$ and $t_Y$ contains the summand $(a_1 \mid \cdots \mid a_m)X'$, such that $(X'|E_1) \sim^{fr}_{s} (Y'|E_2)$;
2. $t_{XY}$ contains a summand $(a_1[m] \mid \cdots \mid a_m[m])Z_{XY}$, if $t_X$ contains the summand $(a_1[m] \mid \cdots \mid a_m[m])Y'$ and $t_Y$ contains the summand $(a_1[m] \mid \cdots \mid a_m[m])X'$, such that $(X'|E_1) \sim^{fr}_{s} (Y'|E_2)$;
3. $t_{XY}$ contains a summand $b_1 \mid \cdots \mid b_n$ if $t_X$ contains the summand $b_1 \mid \cdots \mid b_n$, and $t_Y$ contains the summand $b_1 \mid \cdots \mid b_n$;
4. $t_{XY}$ contains a summand $b_1[n] \mid \cdots \mid b_n[n]$ if $t_X$ contains the summand $b_1[n] \mid \cdots \mid b_n[n]$, and $t_Y$ contains the summand $b_1[n] \mid \cdots \mid b_n[n]$.

Let $\sigma$ map recursion variable $X$ in $E_1$ to $(X|E_1)$, and let $\psi$ map recursion variable $Z_{XY}$ in $E$ to $(X|E_1)$. So, $\sigma((a_1 \mid \cdots \mid a_m)X') \equiv (a_1 \mid \cdots \mid a_m)(X'|E_1) \equiv \psi((a_1 \mid \cdots \mid a_m)Z_{XY})$, or $\sigma((a_1[m] \mid \cdots \mid a_m[m])Y') \equiv (X'|E_1)(a_1[m] \mid \cdots \mid a_m[m]) \equiv \psi((a_1 \mid \cdots \mid a_m[m])Z_{XY})$, so by RDP, we get $(X|E_1) = \sigma(t_X) = \psi(t_{XY})$. Then by RSP, $(X|E_1) = (Z_{XY}|E)$, particularly, $(X_1|E_1) = (Z_{XY}|E)$. Similarly, we can obtain $(Y_1|E_2) = (Z_{XY}|E)$. Finally, $(X_1|E_1) = (Z_{XY}|E) = (Y_1|E_2)$, as desired.

(2) If $(X_1|E_1) \sim^{fr}_{p} (Y_1|E_2)$ for linear recursive specification $E_1$ and $E_2$, then $(X_1|E_1) = (Y_1|E_2)$.
Table 21. Transition rule of the silent step

\[
\begin{align*}
\tau & \xrightarrow{\delta} \sqrt{} \\
\tau & \xrightarrow{\tau} \sqrt{}
\end{align*}
\]

It can be proven similarly to (1), we omit it.

It can be proven similarly to (1), we omit it.

6. Abstraction

To abstract away from the internal implementations of a program, and verify that the program exhibits the desired external behaviors, the silent step $\tau$ and abstraction operator $\tau_I$ are introduced, where $I \subseteq E$ denotes the internal events. The transition rule of $\tau$ is shown in Table 21. In the following, let the atomic event $e$ range over $E \cup \{\delta\} \cup \{\tau\}$, and let the communication function $\gamma : E \cup \{\tau\} \times E \cup \{\tau\} \to E \cup \{\delta\}$, with each communication involved $\tau$ resulting in $\delta$.

**Theorem 6.1** (Conservativity of RAPTC with silent step). RAPTC with silent step is a conservative extension of RAPTC.

**Proof.** Since the transition rules of RAPTC are source-dependent, and the transition rules for silent step in Table 21 contain only a fresh constant $\tau$ in their source, so the transition rules of RAPTC with silent step is a conservative extension of those of RAPTC.

**Theorem 6.2** (Congruence theorem of RAPTC with silent step). Rooted branching FR truly concurrent bisimulation equivalences $\approx_{rbp}$, $\approx_{rbs}$ and $\approx_{rbhp}$ are all congruences with respect to RAPTC with silent step.

**Proof.** It follows the following two facts:

1. FR truly concurrent bisimulation equivalences $\approx_{fr}^p$, $\approx_{fr}^s$ and $\approx_{fr}^p$ are all congruences with respect to all operators of RAPTC, while FR truly concurrent bisimulation equivalences $\approx_{fr}^p$, $\approx_{fr}^s$ and $\approx_{fr}^p$ imply the corresponding rooted branching FR truly concurrent bisimulation $\approx_{rbp}$, $\approx_{rbs}$ and $\approx_{rbhp}$, so rooted branching FR truly concurrent bisimulation $\approx_{rbp}$, $\approx_{rbs}$ and $\approx_{rbhp}$ are all congruences with respect to all operators of RAPTC;

2. While $E$ is extended to $E \cup \{\tau\}$, it can be proved that rooted branching FR truly concurrent bisimulation $\approx_{rbp}$, $\approx_{rbs}$ and $\approx_{rbhp}$ are all congruences with respect to all operators of RAPTC, we omit it.

**6.1. Algebraic Laws for the Silent Step**

We design the axioms for the silent step $\tau$ in Table 22.
Proof. (1) Soundness of APRTC with silent step and guarded linear recursion with respect to rooted branching FR step bisimulation $\approx_{rbbs}$.

Since rooted branching FR step bisimulation $\approx_{rbbs}$ is both an equivalent and a congruent relation with respect to APRTC with silent step and guarded linear recursion, we only need to check if each axiom in Table 22 is sound modulo rooted branching FR step bisimulation equivalence.

Table 22. Axioms of silent step

| No. | Axiom $\vdash$  |
|-----|-----------------|
| RB1 | $e \cdot \tau = e$  |
| RB2 | $(x + y) \cdot \tau + x \cdot e[m] = (x + y) \cdot e[m]$  |
| B3  | $x \parallel \tau = x$  |

Table 22. Axioms of silent step

$t_i = (a_{i11} \parallel \cdots a_{ik_i})t_{i1} + \cdots + (a_{ik_i} \parallel a_{ik_i}1) t_{ik} + (b_{i11} \parallel \cdots b_{i11i}) + \cdots + (b_{il1} \parallel \cdots b_{il1})$

Or,

$t_i = t_{i1}(a_{i11}[m_{i1}] \parallel \cdots a_{i11}[m_{i1}]) + \cdots + t_{ik}(a_{ik_i1}[m_{ik}] \parallel \cdots a_{ik_i1}[m_{ik}]) + (b_{i11}[n_{i1}] \parallel \cdots b_{i11}[n_{i1}]) + \cdots (b_{il1}[n_{il}] \parallel \cdots b_{il1})$

for $i \in \{1, \ldots, n\}$. Let the linear recursive specification $E$ consist of the recursive equations

$X_i = (a_{i11} \parallel \cdots a_{i11i})X_{i1} + \cdots + (a_{ik_i1} \parallel \cdots a_{ik_i1})X_{ik} + (b_{i11} \parallel \cdots b_{i11i}) + \cdots + (b_{il1} \parallel \cdots b_{il1})$

Or,

$X_i = X_{i1}(a_{i11}[m_{i1}] \parallel \cdots a_{i11}[m_{i1}]) + \cdots + X_{ik}(a_{ik_i1}[m_{ik}] \parallel \cdots a_{ik_i1}[m_{ik}]) + (b_{i11}[n_{i1}] \parallel \cdots b_{i11}[n_{i1}]) + \cdots (b_{il1}[n_{il}] \parallel \cdots b_{il1})$

for $i \in \{1, \ldots, n\}$, Replacing $X_i$ by $t_i$ for $i \in \{1, \ldots, n\}$ is a solution for $E$, RSP yields $t_1 = \langle X_1 | E \rangle$. \(\Box\)

Theorem 6.4 (Soundness of APRTC with silent step and guarded linear recursion). Let $x$ and $y$ be APRTC with silent step and guarded linear recursion terms. If APRTC with silent step and guarded linear recursion $\vdash x = y$, then

1. $x \approx_{rbhs} y$;
2. $x \approx_{rbh} y$;
3. $x \approx_{rbs} y$.

Proof. (1) Soundness of APRTC with silent step and guarded linear recursion with respect to rooted branching FR step bisimulation $\approx_{rbbs}$.

Since rooted branching FR step bisimulation $\approx_{rbbs}$ is both an equivalent and a congruent relation with respect to APRTC with silent step and guarded linear recursion, we only need to check if each axiom in Table 22 is sound modulo rooted branching FR step bisimulation equivalence.

Though transition rules in Table 22 are defined in the flavor of single event, they can be modified into a step (a set of events within which each event is pairwise concurrent), we omit them. If we treat a single event as a step containing just one event, the proof of this soundness theorem does not exist any problem, so we use this way and still use the transition rules in Table 22.

- **Axiom B1.** Assume that $e \cdot \tau = e$, it is sufficient to prove that $e \cdot \tau \approx_{rbbs} e$. By the forward transition rules for operator $\cdot$ in Table 23 and $\tau$ in Table 24 we get

\[
\begin{align*}
e 
\quad \vdash e \cdot \tau \quad \vdash e 
\quad \vdash e \cdot \tau \quad \vdash e[m]
\end{align*}
\]

By the reverse transition rules for operator $\cdot$ in Table 24 and $\tau$ in Table 21 there are no transitions. So, $e \cdot \tau \approx_{rbbs} e$, as desired.
• **Axiom RB1.** Assume that $\tau \cdot e[m] = e[m]$, it is sufficient to prove that $\tau \cdot e[m] \approx_{rbs} e[m]$. By the forward transition rules for operator $\cdot$ in Table 3 and $\tau$ in Table 21 there are no transitions. By the reverse transition rules for operator $\cdot$ in Table 4 and $\tau$ in Table 21 we get

$$
\begin{align*}
\tau \cdot e[m] & \xrightarrow{\cdot} e[m] \\
\tau \cdot e[m] & \xrightarrow{\cdot} e \\
e[m] & \xrightarrow{\cdot} e \\
e[m] & \xrightarrow{\cdot} e
\end{align*}
$$

So, $\tau \cdot e[m] \approx_{rbs} e[m]$, as desired.

• **Axiom B2.** Let $p$ and $q$ be $RAPTC$ with silent step processes, and assume that $e \cdot (\tau \cdot (p+q)+p) = e \cdot (p+q)$, it is sufficient to prove that $e \cdot (\tau \cdot (p+q)+p) \approx_{rbs} e \cdot (p+q)$. There are several cases, we will not enumerate all. By the forward transition rules for operators $\cdot$ and $+$ in Table 4 and $\tau$ in Table 21 we get

$$
\begin{align*}
e & \xrightarrow{\cdot} e[m] \\
p & \xrightarrow{e_1} p' \\
q & \xrightarrow{e_1} q' \\
\tau \cdot (p+q)+p & \xrightarrow{e} e \cdot (p'+q') + p' \\
e & \xrightarrow{\cdot} e[m] \\
p & \xrightarrow{e_1} p'
\end{align*}
$$

By the reverse transition rules for operators $\cdot$ and $+$ in Table 4 and $\tau$ in Table 21 there are no transitions. So, $e \cdot (\tau \cdot (p+q)+p) \approx_{rbs} e \cdot (p+q)$, as desired.

• **Axiom RB2.** Let $p$ and $q$ be $RAPTC$ with silent step processes, and assume that $((x+y) \cdot \tau + x) \cdot e[m] = (x+y) \cdot e[m]$, it is sufficient to prove that $((x+y) \cdot \tau + x) \cdot e[m] \approx_{rbs} (x+y) \cdot e[m]$ There are several cases, we will not enumerate all. By the forward transition rules for operators $\cdot$ and $+$ in Table 21 there are no transitions. By the reverse transition rules for operators $\cdot$ and $+$ in Table 4 and $\tau$ in Table 21 we get

$$
\begin{align*}
((x+y) \cdot \tau + x) \cdot e[m] & \xrightarrow{\cdot} e[m] \\
(x+y) \cdot e[m] & \xrightarrow{\cdot} e \\
(x+y) \cdot e[m] & \xrightarrow{\cdot} e
\end{align*}
$$

So, $((x+y) \cdot \tau + x) \cdot e[m] \approx_{rbs} (x+y) \cdot e[m]$, as desired.

• **Axiom B3.** Let $p$ be an $RAPTC$ with silent step, and assume that $p \parallel \tau = p$, it is sufficient to prove that $p \parallel \tau \approx_{rbs} p$. By the forward transition rules for operator $\parallel$ in Table 7 and $\tau$ in Table 21 we get

$$
\begin{align*}
p & \xrightarrow{e} e[m] \\
p \parallel \tau & \xrightarrow{e} e[m] \\
p & \xrightarrow{e} p'
\end{align*}
$$

By the reverse transition rules for operator $\parallel$ in Table 8 and $\tau$ in Table 21 we get

$$
\begin{align*}
p & \xrightarrow{e[m]} \parallel \\
p \parallel e[m] & \xrightarrow{e[m]} e
\end{align*}
$$
Proof. Firstly, by the elimination theorem of APRTC with silent step and guarded linear recursion with respect to rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \) (see Table 22) is sound modulo rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \).

From the definition of rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \) (see Definition 2.27), we know that rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \) is defined by weak pomset transitions, which are labeled by pomsets with \( \tau \). In a weak pomset transition, the events in the pomset are either within causality relations (defined by \( \cdot \)) or in concurrency (implicitly defined by \( \cdot + \)) and explicitly defined by \( \downarrow \). Of course, they are pairwise consistent (without conflicts). In (1), we have already proven the case that all events are pairwise concurrent, so, we only need to prove the case of events in causality. Without loss of generality, we take a pomset of \( P = \{ e_1, e_2 : e_1 \cdot e_2 \} \). Then the weak pomset transition labeled by the above \( P \) is just composed of one single event transition labeled by \( e_1 \) succeeded by another single event transition labeled by \( e_2 \), that is, \( p \parallel \tau \xrightarrow{e_1} e_2 \).

Similarly to the proof of soundness of APRTC with silent step and guarded linear recursion modulo rooted branching FR step bisimulation \( \approx_{\text{rbs}} \) (1), we can prove that each axiom in Table 22 is sound modulo rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \), we omit them.

(3) Soundness of APRTC with silent step and guarded linear recursion with respect to rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \).

Since rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) is both an equivalent and a congruent relation with respect to APRTC with silent step and guarded linear recursion, we only need to check if each axiom in Table 22 is sound modulo rooted branching FR pomset bisimulation \( \approx_{\text{rbp}} \).

From the definition of rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) (see Definition 2.29), we know that rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) is defined on the weakly posetal product \( (C_1, f, C_2, f') : \hat{C}_1 \rightarrow C_2 \) isomorphism. Two process terms \( s \) related to \( C_1 \) and \( t \) related to \( C_2 \), and \( f : \hat{C}_1 \rightarrow C_2 \) isomorphism. Initially, \( (C_1, f, C_2) = (\emptyset, \emptyset, \emptyset) \), and \( (\emptyset, \emptyset, \emptyset, \emptyset) \) \( \approx_{\text{rbs}} \). When \( s \xrightarrow{e} s' \) \( (C_1 \xrightarrow{e} C_1') \), there will be \( t \xrightarrow{e} t' \) \( (C_2 \xrightarrow{e} C_2') \), and we define \( f' = f[e \mapsto e] \). And when \( s = s' \) \( (C_1 \xrightarrow{e} C_1') \), there will be \( t = t' \) \( (C_2 \xrightarrow{e} C_2') \), and we define \( f' = f[e[m] \mapsto e[m]] \). Then, if \( (C_1, f, C_2) \approx_{\text{rbs}} \), then \( (C_1', f', C_2') \approx_{\text{rbs}} \).

Similarly to the proof of soundness of APRTC with silent step and guarded linear recursion modulo rooted branching FR pomset bisimulation equivalence (2), we can prove that each axiom in Table 22 is sound modulo rooted branching FR hp-bisimulation equivalence, we just need additionally to check the above conditions on rooted branching FR hp-bisimulation, we omit them.

**Theorem 6.5** (Completeness of APRTC with silent step and guarded linear recursion). Let \( p \) and \( q \) be closed APRTC with silent step and guarded linear recursion terms, then,

1. if \( p \approx_{\text{rbs}} q \) then \( p = q \);
2. if \( p \approx_{\text{rbp}} q \) then \( p = q \);
3. if \( p \approx_{\text{rbhp}} q \) then \( p = q \).

Proof. Firstly, by the elimination theorem of APRTC with silent step and guarded linear recursion (see Theorem 6.3), we know that each process term in APRTC with silent step and guarded linear recursion is equal to a process term \( (X_1|E) \) with \( E \) a guarded linear recursive specification.
It remains to prove the following cases.

(1) If \( \langle X_1|E_1 \rangle s_{fr} \langle Y_1|E_2 \rangle \) for guarded linear recursive specification \( E_1 \) and \( E_2 \), then \( \langle X_1|E_1 \rangle = \{ Y_1|E_2 \} \).

Firstly, the recursive equation \( W = \tau + \cdots + \tau \) with \( W \neq X_1 \) in \( E_1 \) and \( E_2 \), can be removed, and the corresponding summands \( aW \) are replaced by \( a \), to get \( E'_1 \) and \( E'_2 \), by use of the axioms RDP, A3 and B1, RB1, and \( \langle X|E \rangle = \langle X'|E' \rangle, \langle Y|E \rangle = \langle Y'|E' \rangle \).

Let \( E_1 \) consists of recursive equations \( X = t_X \) for \( X \in X \) and \( E_2 \) consists of recursion equations \( Y = t_Y \) for \( Y \in Y \), and are not the form \( \tau + \cdots + \tau \). Let the guarded linear recursive specification \( E \) consists of recursion equations \( Z_{XY} = t_{XY} \), and \( \langle X|E \rangle s_{fr} \langle Y|E \rangle \), and \( t_{XY} \) consists of the following summands:

1. \( t_{XY} \) contains a summand \( (a_1 \parallel \cdots \parallel a_m)_{Z_{XY}}, \) if \( t_X \) contains the summand \( (a_1 \parallel \cdots \parallel a_m)_{X'} \) and \( t_Y \) contains the summand \( (a_1 \parallel \cdots \parallel a_m)_{Y'} \) such that \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
2. \( t_{XY} \) contains a summand \( Z_{XY} \) if \( XY \neq X_1Y_1, t_X \) contains the summand \( X' \) if \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
3. \( t_{XY} \) contains a summand \( \tau Z_{XY} \) if \( XY \neq X_1Y_1, t_X \) contains the summand \( X' \) and \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 

Since \( E_1 \) and \( E_2 \) are guarded, \( E \) is guarded. Constructing the process term \( u_{XY} \) consist of the following summands:

1. \( u_{XY} \) contains a summand \( (a_1 \parallel \cdots \parallel a_m)(X|E) \) if \( t_X \) contains the summand \( (a_1 \parallel \cdots \parallel a_m)_{X'} \) and \( t_Y \) contains the summand \( (a_1 \parallel \cdots \parallel a_m)_{Y'} \) such that \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
2. \( u_{XY} \) contains a summand \( Z_{XY} \) if \( XY \neq X_1Y_1, t_X \) contains the summand \( X' \) if \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
3. \( u_{XY} \) contains a summand \( \tau Z_{XY} \) if \( XY \neq X_1Y_1, t_X \) contains the summand \( X' \) and \( \langle X'|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 

Let the process term \( s_{XY} \) be defined as follows:

1. \( s_{XY} = \tau(X|E_1) + u_{XY} \) if \( XY \neq X_1Y_1, t_Y \) contains the summand \( \tau Y' \), and \( \langle X|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
2. \( s_{XY} = (X|E_1) \tau + u_{XY} \) if \( XY \neq X_1Y_1, t_Y \) contains the summand \( \tau Y' \), and \( \langle X|E_1 \rangle s_{fr} \langle Y'|E_2 \rangle \); 
3. \( s_{XY} = (X|E_1), \) otherwise.

So, \( \langle X|E_1 \rangle = \langle X|E_1 \rangle + u_{XY} \), and \( (a_1 \parallel \cdots \parallel a_m)(\tau(X|E_1) + u_{XY}) = \tau(X|E_1) + u_{XY} \), or \( (\tau(X|E_1) + u_{XY}) = \tau(X|E_1) + u_{XY} \), and \( \langle X|E_1 \rangle = \langle X|E_1 \rangle + u_{XY} \).

Let \( \sigma \) map recursion variable \( X \) in \( E_1 \) to \( \langle X|E \rangle \), and let \( \psi \) map recursion variable \( Z_{XY} \) in \( E \) to \( s_{XY} \). It is sufficient to prove \( s_{XY} = \psi(t_{XY}) \) for recursion variables \( Z_{XY} \) in \( E \). Either \( XY \equiv X_1Y_1 \) or \( XY \neq X_1Y_1 \), we all can get \( s_{XY} = \psi(t_{XY}) \). So, \( s_{XY} = (Z_{XY}|E) \) for recursive variables \( Z_{XY} \) in \( E \) is a solution for \( E \). Then by RSP, particularly, \( \langle X_1|E_1 \rangle = (Z_{X_1Y_1}|E) \). Similarly, we can obtain \( \langle Y_1|E_2 \rangle = (Z_{X_1Y_1}|E) \). Finally, \( \langle X_1|E_1 \rangle = (Z_{X_1Y_1}|E) = (Y_1|E_2) \), as desired.
Proof. Since the transition rules of APRT with silent step are source-dependent, and the transition rules for abstraction operator in Table 23 contain only a fresh operator \( \tau \), it can be proven similarly to (1), we omit it.

(3) If \((X_1|E_1) \approx_{ratb} (Y_1|E_2)\) for guarded linear recursive specification \(E_1\) and \(E_2\), then \((X_1|E_1) = (Y_1|E_2)\).

It can be proven similarly to (1), we omit it.

### 6.2. Abstraction

The unary abstraction operator \( \tau_I \) (\( I \in \mathbb{E} \)) renames all atomic events in \( I \) into \( \tau \). APRT with silent step and abstraction operator is called \( APRTC_\tau \). The transition rules of operator \( \tau_I \) are shown in Table 23.

#### Table 23. Transition rule of the abstraction operator

\[
\begin{array}{ll}
\begin{array}{c}
x \xrightarrow{e} \sqrt{\tau_I(x) \xrightarrow{e} \sqrt{}} \\
\end{array} & \begin{array}{l}
e \notin I \\
\end{array} \\
\begin{array}{c}
x \xrightarrow{e} \sqrt{\tau_I(x) \xrightarrow{e} \sqrt{}} \\
\end{array} & \begin{array}{l}
e \notin I \\
\end{array} \\
\begin{array}{c}
x \xrightarrow{e}\{m\} \sqrt{\tau_I(x) \xrightarrow{e}\{m\} \sqrt{}} \\
\end{array} & \begin{array}{l}
e \notin I \\
\end{array} \\
\begin{array}{c}
x \xrightarrow{e}\{m\} \sqrt{\tau_I(x) \xrightarrow{e}\{m\} \sqrt{}} \\
\end{array} & \begin{array}{l}
e \notin I \\
\end{array} \\
\end{array}
\]

(2) If \((X_1|E_1) \approx_{frb} (Y_1|E_2)\) for guarded linear recursive specification \(E_1\) and \(E_2\), then \((X_1|E_1) = (Y_1|E_2)\).

It can be proven similarly to (1), we omit it.

#### Theorem 6.6 (Conservativity of \( APRTC_\tau \)).

\( APRTC_\tau \) is a conservative extension of \( APRTC \) with silent step.

Proof. Since the transition rules of APRT with silent step are source-dependent, and the transition rules for abstraction operator in Table 23 contain only a fresh operator \( \tau_I \) in their source, so the transition rules of \( APRTC_\tau \) is a conservative extension of those of \( RAPTC \) with silent step.

#### Theorem 6.7 (Congruence theorem of \( APRTC_\tau \)).

Rooted branching FR truly concurrent bisimulation equivalences \( \approx_{ratb} \), \( \approx_{fr} \), and \( \approx_{ratbsp} \) are all congruences with respect to \( APRTC_\tau \).

Proof. (1) Case rooted branching FR pomset bisimulation equivalence \( \approx_{ratb} \).

Let \( x \) and \( y \) be \( APRTC_\tau \) processes, and \( x \approx_{frb} y \), it is sufficient to prove that \( \tau_I(x) \approx_{frb} \tau_I(y) \).

By the transition rules for operator \( \tau_I \) in Table 23, we can get

\[
\begin{align*}
\tau_I(x) & \xrightarrow{X} X[K](X \notin I) & \tau_I(y) & \xrightarrow{Y} Y[J](Y \notin I) \\
\end{align*}
\]

\[
\begin{align*}
\tau_I(x) & \xrightarrow{X[K]} X(X \notin I) & \tau_I(y) & \xrightarrow{Y[J]} Y(Y \notin I) \\
\end{align*}
\]

with \( X \subseteq x, Y \subseteq y \), and \( X \sim Y \).

Or, we can get

\[
\begin{align*}
\tau_I(x) & \xrightarrow{\tau_I(x')(X \notin I)} & \tau_I(y) & \xrightarrow{\tau_I(y')(Y \notin I)} \\
\end{align*}
\]

\[
\begin{align*}
\tau_I(x) & \xrightarrow{X[K]} \tau_I(x')(X \notin I) & \tau_I(y) & \xrightarrow{Y[J]} \tau_I(y')(Y \notin I) \\
\end{align*}
\]

with \( X \subseteq x, Y \subseteq y \), and \( X \sim Y \) and the hypothesis \( \tau_I(x') \approx_{frb} \tau_I(y') \).

Or, we can get
Table 24. Axioms of abstraction operator

| No. | Axiom                                      |
|-----|--------------------------------------------|
| T1  | $e \notin I \quad \tau_I(e) = e$           |
| RT1 | $e[m] \notin I \quad \tau_I(e[m]) = e[m]$ |
| T2  | $e \notin I \quad \tau_I(e) = \tau$       |
| RT2 | $e[m] \notin I \quad \tau_I(e[m]) = \tau$ |
| T3  | $\tau_I(\delta) = \delta$                |
| T4  | $\tau_I(x + y) = \tau_I(x) + \tau_I(y)$  |
| T5  | $\tau_I(x \cdot y) = \tau_I(x) \cdot \tau_I(y)$ |
| T6  | $\tau_I(x \parallel y) = \tau_I(x) \parallel \tau_I(y)$ |

Theorem 6.8 (Soundness of $APRTC_\tau$ with guarded linear recursion). Let $x$ and $y$ be $APRTC_\tau$ with guarded linear recursion terms. If $APRTC_\tau$ with guarded linear recursion $\vdash x = y$, then

1. $x \equiv^{fr}_{rbs} y$;
2. $x \equiv^{fr}_{rbp} y$;
3. $x \equiv^{fr}_{rhp} y$.

Proof. (1) Soundness of $APRTC_\tau$ with guarded linear recursion with respect to rooted branching FR step bisimulation $\equiv^{fr}_{rbs}$.

Since rooted branching FR step bisimulation $\equiv^{fr}_{rbs}$ is both an equivalent and a congruent relation with respect to $APRTC_\tau$ with guarded linear recursion, we only need to check if each axiom in Table 24 is sound modulo rooted branching FR step bisimulation equivalence.

The proof is similar to the proof of soundness of APRTC with silent step and guarded linear recursion, we omit them.

(2) Soundness of $APRTC_\tau$ with guarded linear recursion with respect to rooted branching FR pomset bisimulation $\equiv^{fr}_{rbp}$.

Since rooted branching FR pomset bisimulation $\equiv^{fr}_{rbp}$ is both an equivalent and a congruent relation with respect to $APRTC_\tau$ with guarded linear recursion, we only need to check if each axiom in Table 24 is sound modulo rooted branching FR pomset bisimulation $\equiv^{fr}_{rbp}$.

From the definition of rooted branching FR pomset bisimulation $\equiv^{fr}_{rbp}$ (see Definition 2.27), we know that
rooted branching FR pomset bisimulation \( \approx_{\text{rbhp}} \) is defined by weak pomset transitions, which are labeled by pomsets with \( \tau \). In a weak pomset transition, the events in the pomset are either within causality relations (defined by \( \cdot \)) or in concurrency (implicitly defined by \( \cdot + \), and explicitly defined by \( \cdot \)), of course, they are pair-wise consistent (without conflicts). In (1), we have already proven the case that all events are pair-wise concurrent, so, we only need to prove the case of events in causality. Without loss of generality, we take a pomset of \( P = \{ e_1, e_2 : e_1 \cdot e_2 \} \). Then the weak pomset transition labeled by the above \( P \) is just composed of one single event transition labeled by \( e_1 \) succeeded by another single event transition labeled by \( e_2 \), that is, \( \Rightarrow e_1 \Rightarrow e_2 \) or \( \Leftrightarrow e_1 \Leftrightarrow e_2 \).

Similarly to the proof of soundness of \( \text{APRTC}_\tau \) with guarded linear recursion modulo rooted branching FR step bisimulation \( \approx_{\text{rbhp}}^\tau \) (1), we can prove that each axiom in Table [24] is sound modulo rooted branching FR pomset bisimulation \( \approx_{\text{rbhp}} \). Since rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) is both an equivalent and a congruent relation with respect to \( \text{APRTC}_\tau \) with guarded linear recursion, we only need to check if each axiom in Table [24] is sound modulo rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \).

From the definition of rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) (see Definition [2.29]), we know that rooted branching FR hp-bisimulation \( \approx_{\text{rbhp}} \) is defined on the weakly posetal product \((C_1, f, C_2), f : C_1 \rightarrow C_2 \) isomorphism. Two process terms \( s \) related to \( C_1 \) and \( t \) related to \( C_2 \), and \( f : C_1 \rightarrow C_2 \) isomorphism. Initially, \((C_1, f, C_2) = (\emptyset, \emptyset, \emptyset)\), and \((\emptyset, \emptyset, \emptyset) \approx_{\text{rbhp}} \). When \( s \overset{\text{rbhp}}{\leftrightarrow} s' \), there will be \( t \overset{\text{rbhp}}{\leftrightarrow} t' \), and we define \( f' = f t e \rightarrow e e \). And when \( s \overset{\text{rbhp}}{\leftrightarrow} s' \), there will be \( t \overset{\text{rbhp}}{\leftrightarrow} t' \), and we define \( f' = f t e \rightarrow e e \). Then, if \((C_1, f, C_2) \approx_{\text{rbhp}} \), then \((C_1', f', C_2') \approx_{\text{rbhp}} \).

Similarly to the proof of soundness of \( \text{APRTC}_\tau \) with guarded linear recursion modulo rooted branching FR pomset bisimulation equivalence (2), we can prove that each axiom in Table [24] is sound modulo rooted branching FR hp-bisimulation equivalence, we just need additionally to check the above conditions on rooted branching FR hp-bisimulation, we omit them.

Though \( \tau \)-loops are prohibited in guarded linear recursive specifications (see Definition ??) in specifiable way, they can be constructed using the abstraction operator, for example, there exist \( \tau \)-loops in the process term \( \tau_1 (X = \alpha X) \). To avoid \( \tau \)-loops caused by \( \tau_1 \) and ensure fairness, the concept of cluster and CFAR (Cluster Fair Abstraction Rule) are still valid in true concurrency, we introduce them below.

**Definition 6.9 (Cluster).** Let \( E \) be a guarded linear recursive specification, and \( I \subseteq E \). Two recursion variable \( X \) and \( Y \) in \( E \) are in the same cluster for \( I \) iff there exist sequences of transitions \((X|E) \overset{\{b_{11} \ldots b_{i1}\}}{\rightarrow} \ldots \overset{\{b_{n1} \ldots b_{ni}\}}{\rightarrow} (X|E) \overset{\{c_{11} \ldots c_{i1}\}}{\rightarrow} \ldots \overset{\{c_{n1} \ldots c_{ni}\}}{\rightarrow} (X|E) \), or \((X|E) \overset{\{b_{1j} \ldots b_{1m}\}}{\rightarrow} \ldots \overset{\{b_{mj} \ldots b_{mi}\}}{\rightarrow} (Y|E) \overset{\{c_{11} \ldots c_{j1}\}}{\rightarrow} \ldots \overset{\{c_{n1} \ldots c_{j1}\}}{\rightarrow} (X|E) \), where \( b_{1j}, \ldots , b_{mj} \), \( c_{1j}, \ldots , c_{nj} \), \( b_{11}, \ldots , b_{1m} \), \( c_{11}, \ldots , c_{1j} \) \( I \cup \{ \tau \} \).\( a_1 \parallel \cdots \parallel a_k \), or \((a_1 \parallel \cdots \parallel a_k) X \), or \( a_1 m \parallel \cdots \parallel a_k m \), or \( X(a_1 m) \parallel \cdots \parallel a_k m \) \( m \) \( a_1 m \) \( C \) is an exit for the cluster \( C \) iff: (1) \( a_1 \parallel \cdots \parallel a_k \), or \((a_1 \parallel \cdots \parallel a_k) X \), or \( a_1 m \parallel \cdots \parallel a_k m \), or \( X(a_1 m) \parallel \cdots \parallel a_k m \) \( m \) \( a_1 m \) \( C \) is an exit for the cluster \( C \) iff: (1) \( a_1 \parallel \cdots \parallel a_k \), or \((a_1 \parallel \cdots \parallel a_k) X \), or \( a_1 m \parallel \cdots \parallel a_k m \), or \( X(a_1 m) \parallel \cdots \parallel a_k m \) \( m \) \( a_1 m \) \( \tau \) is a summand at the right-hand side of the recursive equation for a recursion variable in \( C \), and (2) in the case of \( a_1 \parallel \cdots \parallel a_k \) \( X \), and \( X(a_1 m) \parallel \cdots \parallel a_k m \) \( m \) \( a_1 m \) \( I \cup \{ \tau \} \) either \( a_1 m \) \( \notin \tau \) \( I \cup \{ \tau \} \) \( \{ 1, 2, \ldots , k \} \) \( X \neq C \).

**Theorem 6.10 (Soundness of CFAR).** CFAR is sound modulo rooted branching FR truly concurrent bisimulation equivalences \( \approx_{\text{rbhp}} \), \( \approx_{\text{rbhp}} \) and \( \approx_{\text{rbhp}} \).

**Proof.** (1) Soundness of CFAR with respect to rooted branching FR step bisimulation \( \approx_{\text{rbhp}} \). Let \( X \) be in a cluster for \( I \) with exits \( \{a_{11} \parallel \cdots \parallel a_{1l} \}, \ldots , \{a_{m1} \parallel \cdots \parallel a_{ml} \}, \{b_{1j} \parallel \cdots \parallel b_{1l} \}, \ldots \), \( \{b_{nj} \parallel \cdots \parallel b_{nl} \} \) and \( \{b_{11} \parallel \cdots \parallel b_{1l} \}, \ldots , \{b_{m1} \parallel \cdots \parallel b_{ml} \} \). Then \( (X|E) \) can execute a string of atomic events from \( I \cup \{ \tau \} \) inside the cluster of \( X \), followed
Table 25. Cluster fair abstraction rule

| No. | Axiom |
|------|-------|
| 1    | If \( X \) is in a cluster for \( I \) with exits \( \{(a_{11} \parallel \cdots \parallel a_{1i})Y_{1} \parallel \cdots \parallel (a_{m1} \parallel \cdots \parallel a_{mi})Y_{m}, b_{11} \parallel \cdots \parallel b_{1j}, \cdots, b_{nj} \parallel \cdots \parallel b_{nj}\} \), then \( \tau \cdot \tau_{I}((X|E)) = \tau \cdot \tau_{I}((a_{11} \parallel \cdots \parallel a_{1i})Y_{1} \parallel \cdots \parallel (a_{m1} \parallel \cdots \parallel a_{mi})Y_{m} \parallel b_{11} \parallel \cdots \parallel b_{1j} \parallel \cdots \parallel b_{nj} \parallel \cdots \parallel b_{nj}) \) |
| 2    | Or exists, \( \{Y_{1}(a_{11}[m] \parallel \cdots \parallel a_{1i}[1]), \cdots, Y_{m}(a_{m1}[m] \parallel \cdots \parallel a_{mi}[m]), b_{11}[1] \parallel b_{1j}[1] \parallel \cdots \parallel b_{nj}[1] \parallel \cdots \parallel b_{nj}[m] \parallel \cdots \parallel b_{nj}[m]\} \), then \( \tau_{I}((Y|E)) \cdot \tau = \tau_{I}((a_{11}[m] \parallel \cdots \parallel a_{1i}[1]) + \cdots + (Y_{m}(a_{m1}[m] \parallel \cdots \parallel a_{mi}[m]) + b_{11}[1] \parallel \cdots \parallel b_{1j}[1] \parallel \cdots \parallel b_{nj}[m] \parallel \cdots \parallel b_{nj}[m]) \cdot \tau \) |

Table 25. Cluster fair abstraction rule

by an exit \( (a_{i1} \parallel \cdots \parallel a_{im})Y_{i} \) for \( i' \in \{1, \cdots, m\} \) or \( b_{i1} \parallel \cdots \parallel b_{im} \) for \( j' \in \{1, \cdots, n\} \), or \( Y_{i}(a_{i1}[m] \parallel \cdots \parallel a_{im}[m]) \) for \( i' \in \{1, \cdots, m\} \) or \( b_{i1} \parallel b_{im} \) for \( j' \in \{1, \cdots, n\} \). These \( \tau' \) are non-initial in \( \tau_{I}((X|E)) \) and \( \tau_{I}((X|E)) \cdot \tau \), so they are truly silent by the axiom B1 and RB1, we obtain \( \tau_{I}((X|E)) \cdot \tau \overset{rbhp}{\Rightarrow} \tau_{I}((a_{i1}[m] \parallel \cdots \parallel a_{im}[m]) + b_{i1} \parallel \cdots \parallel b_{im} \parallel \cdots \parallel b_{nj}[m] \parallel \cdots \parallel b_{nj}[m]) \cdot \tau \) as desired.

(2) Soundness of CFAR with respect to rooted branching FR pomset bisimulation \( \sim_{rbhp}^{fr} \).

From the definition of rooted branching FR pomset bisimulation \( \sim_{rbhp}^{fr} \) (see Definition 2.27), we know that rooted branching FR pomset bisimulation \( \sim_{rbhp}^{fr} \) is defined by weak pomset transitions, which are labeled by pomsets with \( \tau \). In a weak pomset transition, the events in the pomset are either within causality relations (defined by \( \cdot \)) or in concurrency (implicitly defined by \( + \), and explicitly defined by \( \parallel \)) of course, they are pairwise consistent (without conflicts). In (1), we have already proven the case that all events are pairwise concurrent, so, we only need to prove the case of events in causality. Without loss of generality, we take a pomset of \( P = \{e_{1}, e_{2} : e_{1}, e_{2}\} \). Then the weak pomset transition labeled by the above \( P \) is just composed of one single event transition labeled by \( e_{1} \) succeeded by another single event transition labeled by \( e_{2} \), that is, \( P \overset{e_{1}}{\Rightarrow} e_{2} \) or \( P \overset{e_{2}}{\Rightarrow} e_{1} \).

Similarly to the proof of soundness of CFAR modulo rooted branching FR step bisimulation \( \sim_{rbhp}^{fr} \) (1), we can prove that CFAR in Table 25 is sound modulo rooted branching FR pomset bisimulation \( \sim_{rbhp}^{fr} \), we omit them.

(3) Soundness of CFAR with respect to rooted branching FR hp-bisimulation \( \sim_{rbhp}^{fr} \).

From the definition of rooted branching FR hp-bisimulation \( \sim_{rbhp}^{fr} \) (see Definition 2.29), we know that rooted branching FR hp-bisimulation \( \sim_{rbhp}^{fr} \) is defined on the weakly posetal product \( (C_{1}, f, C_{2}) \) : \( C_{1} \rightarrow C_{2} \) isomorphism. Two process terms \( s \) related to \( C_{1} \) and \( t \) related to \( C_{2} \), and \( f : C_{1} \rightarrow C_{2} \) isomorphism. Initially, \( (C_{1}, f, C_{2}) = (s, \emptyset, \emptyset, s, \emptyset, \emptyset) \). When \( s \overset{e}{\Rightarrow} s' \) \( (C_{1} \overset{e}{\Rightarrow} C_{1}') \), there will be \( t \overset{e}{\Rightarrow} t' \) \( (C_{2} \overset{e}{\Rightarrow} C_{2}') \), and we define \( f' = f[e \mapsto e] \). And when \( s \overset{e}{\Rightarrow} e \) \( (C_{1} \overset{e}{=} C_{1}') \), there will be \( t \overset{e}{=} t' \) \( (C_{2} \overset{e}{=} C_{2}') \), and we define \( f' = f[e \mapsto e] \). Then, if \( (C_{1}, f, C_{2}) \overset{e}{\Rightarrow} (C_{1}', f', C_{2}') \), then \( (C_{1}', f', C_{2}') \overset{e}{\Rightarrow} (C_{1}, f, C_{2}) \).

Similarly to the proof of soundness of CFAR modulo rooted branching FR pomset bisimulation equivalence (2), we can prove that CFAR in Table 25 is sound modulo rooted branching FR hp-bisimulation equivalence, we just need additionally to check the above conditions on rooted branching FR hp-bisimulation, we omit them.

**Theorem 6.11** (Completeness of \( \text{APRTC}_{r} \) with guarded linear recursion and CFAR). Let \( p \) and \( q \) be closed \( \text{APRTC}_{r} \) with guarded linear recursion and CFAR terms, then,

1. if \( p \sim_{rbhp}^{fr} q \) then \( p = q \);
2. if \( p \sim_{rbhp}^{fr} q \) then \( p = q \);
3. if \( p \sim_{rbhp}^{fr} q \) then \( p = q \).
Proof. (1) For the case of rooted branching FR step bisimulation, the proof is following.

Firstly, in the proof the Theorem 6.5, we know that each process term \( p \) in APRTC with silent step and guarded linear recursion is equal to a process term \( \langle X \mid E \rangle \) with \( E \) a guarded linear recursive specification. And we prove if \( \langle X_1 \mid E_1 \rangle \approx_{fr} \langle Y_1 \mid E_2 \rangle \), then \( \langle X_1 \mid E_1 \rangle = \langle Y_1 \mid E_2 \rangle \).

The only new case is \( p \equiv \tau(q) \). Let \( q = \langle X \mid E \rangle \) with \( E \) a guarded linear recursive specification, so \( p = \tau((X\mid E)) \). Then the collection of recursive variables in \( E \) can be divided into its clusters \( C_1, \ldots, C_N \) for \( I \). Let

\[
(a_{i,\tau} \mid \ldots \mid a_{k,\tau})Y_1 + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})Y_{im} + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}} \mid \ldots \mid b_{l_{1,\tau}}
\]

or,

\[
Y_1(a_{i,\tau_m}[m1] \mid \ldots \mid a_{k,\tau_m}[m1]) + \ldots + Y_{im}(a_{1,\tau_m}[mm] \mid \ldots \mid a_{k,\tau_m}[mm]) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}
\]

be the conflict composition of exits for the cluster \( C_i \), with \( i \in \{1, \ldots, N\} \).

For \( Z, Z' \in C_i \) with \( i \in \{1, \ldots, N\} \), we define

\[
s_Z = (a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_1\mid E)) + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_{im}\mid E)) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}^\prime
\]

and

\[
s_Z' = (a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_1\mid E)) + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_{im}\mid E)) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}^\prime
\]

For \( Z, Z' \in C_i \) and \( a_1, \ldots, a_j \in E \cup \{\tau\} \) with \( j \in \mathbb{N} \), we have

\[
(a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_1\mid E)) + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})\tau_I((Y_{im}\mid E)) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}^\prime
\]

Let the linear recursive specification \( F \) contain the same recursive variables as \( E \), for \( Z, Z' \in C_i \), \( F \) contains the following recursive equation

\[
Z = (a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})Y_1 + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})Y_{im} + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}
\]

Let the linear recursive specification \( F' \) contain the same recursive variables as \( E \), for \( Z, Z' \in C_i \), \( F \) contains the following recursive equation

\[
Z' = Y_1(a_{i,\tau_m}[m1] \mid \ldots \mid a_{k,\tau_m}[m1]) + \ldots + Y_{im}(a_{1,\tau_m}[mm] \mid \ldots \mid a_{k,\tau_m}[mm]) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}
\]

It is easy to see that there is no sequence of one or more \( \tau \)-transitions from \( \langle Z\mid F \rangle \) and \( \langle Z'\mid F' \rangle \) to itself, so \( F \) and \( F' \) is guarded.

For

\[
s_Z = (a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})Y_1 + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})Y_{im} + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}
\]

is a solution for \( F \). So, \( (a_{i} \mid \ldots \mid a_{j})\tau_I((Z\mid E)) = (a_{i} \mid \ldots \mid a_{j})s_Z = (a_{i} \mid \ldots \mid a_{j})(Z\mid F) \).

So,

\[
\langle Z\mid F \rangle = (a_{i,\tau_m} \mid \ldots \mid a_{k,\tau_m})(Y_1\mid F) + \ldots + (a_{1,\tau_m} \mid \ldots \mid a_{k,\tau_m})(Y_{im}\mid F) + b_{11} \mid \ldots \mid b_{i_{1,\tau}} + \ldots + b_{1im} \mid \ldots \mid b_{l_{1,\tau}}
\]

For
\[ s'_{Z} = Y_{i1}(a_{i1}[m1] \parallel \cdots \parallel a_{k_{i1}}[m1]) \cdots + Y_{im}(a_{im}[mm] \parallel \cdots \parallel a_{k_{im}}[mm]) + b_{i1}[n1] \parallel \cdots \parallel b_{im}[nn] \parallel \cdots + b_{lm}[nn], \]

is a solution for \( F' \). So, \( \tau_{I}(\langle Z'/\text{divid} \rangle_{S.alt0}E) = s'_{Z}(a_{1}[m] \parallel \cdots \parallel a_{j}[m]) = (Z'/F')(a_{1}[m] \parallel \cdots \parallel a_{j}[m]). \)

So,

\[ \langle Z'/F' \rangle = \langle Y_{i1}|F|(a_{i1}[m1] \parallel \cdots \parallel a_{k_{i1}}[m1]) \cdots + Y_{im}|F|(a_{im}[mm] \parallel \cdots \parallel a_{k_{im}}[mm]) + b_{i1}[n1] \parallel \cdots \parallel b_{im}[nn] \parallel \cdots + b_{lm}[nn] \rangle. \]

Hence, \( \tau_{I}(\langle X|E \rangle = (Z|F) \rangle) \), as desired.

(2) For the case of rooted branching FR pomset bisimulation, it can be proven similarly to (1), we omit it.

(3) For the case of rooted branching FR hp-bisimulation, it can be proven similarly to (1), we omit it. \( \square \)

7. Conclusions

Based on our previous process algebra for concurrency APTC, we prove that it is reversible with a little modifications. The reversible algebra has four parts: Basic Algebra for Reversible True Concurrency (BARTC), Algebra for Parallelism in Reversible True Concurrency (APRTC), recursion and abstraction.

This work can be used to verify the behavior of computational systems in a reversible flavor.

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