A localized boundary deformation which splits the spectrum of the Laplacian

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Abstract

For any Lipschitz domain we construct an arbitrarily small, localized perturbation which splits the spectrum of the Laplacian into simple eigenvalues. We use for this purpose a Hadamard’s formula and spectral stability results.

1 Introduction

In the seminal works [8] and [10], respectively Micheletti and Uhlenbeck showed that the eigenvalues of the Dirichlet Laplacian are generically simple in the space of smooth manifolds equipped with the $C^k$-topology (see also the survey papers [3, Section 4.3], [5, Section 1.3] and references therein for related works). In this paper we generalize this result to Lipschitz domains and show that a stronger, localized version holds as follows.

Theorem 1. For any Lipschitz domain $\Omega$, $\varepsilon > 0$, and $x$ on the boundary $\partial \Omega$, there exists a domain $\tilde{\Omega}$ whose symmetric difference with $\Omega$ is contained in the ball of radius $\varepsilon$ centered at $x$, and whose (Dirichlet, Neumann, or Robin) Laplacian eigenvalues are all simple. Moreover $\tilde{\Omega}$ can be constructed so that the Lipschitz constant of $\partial \tilde{\Omega}$ is arbitrarily near to the one of $\partial \Omega$.

More in detail the structure of the paper is the following. In Section 2 we review some preliminary material, in particular regarding spectral stability. In Section 3 we recall a Hadamard’s formula and study some independence properties of eigenfunctions and their gradients at the boundary. More in detail, Hadamard’s formula provides us with a first-order estimate on the shift of an eigenvalue $\lambda$ which depends on the value of

$$|\nabla u|^2 - cu^2$$

at the boundary of the domain considered, where $u$ is an eigenfunction associated to $\lambda$ and $c$ is a constant which depends only on the choice of boundary conditions. By showing that for two orthogonal eigenfunctions the corresponding values of (1) in any open subset of the boundary must differ at least at a point, we are able to construct a localized perturbation which splits any non-simple eigenvalue. However, even when small, this perturbation might cause the shift and the overlap of other eigenvalues. This possibility is ruled out in Section 4, where uniform bounds for the whole spectrum are adapted to our case from sharp stability estimates from [2]. In conclusion, these bounds allow the construction of a localized perturbation, which consists of a sequence of small “bumps” at the boundary of the domain considered, which proves Theorem 1.

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2 Notations and preliminary results

In this section we fix the main notation which will be used in the paper and recall some preliminary results on eigenvalues and eigenfunctions of the Laplacian. Regarding the notation:

- we say that $X$ is a domain if $X$ is an open, bounded, and connected subset of $\mathbb{R}^N$;
- we say that $\lambda$ is an eigenvalue of a domain $X$ with associated eigenfunction $u$ (assumed to be not constant zero) if
  \[ \Delta u + \lambda u = 0 \quad \text{in} \quad X, \]
  and either one of the following homogeneous boundary conditions is satisfied on $\partial X$:
  \[
  \begin{cases}
  u = 0 & \text{(Dirichlet),} \\
  \frac{\partial u}{\partial \nu} = 0 & \text{(Neumann),} \\
  \sigma u = \frac{\partial u}{\partial \nu} & \text{(Robin),}
  \end{cases}
  \]
  where $\sigma$ is a fixed non-zero constant and $\nu$ indicates the outward unit normal vector.
- we indicate as $\Omega$ a fixed domain with Lipschitz boundary.

We actually require (2) and (3) to be satisfied only in a weak sense, that is: $\lambda$ is an eigenvalue of $X$ with associated eigenfunction $u$, if $u$ is an element of a function space $V(X)$ and
  \[ Q(u, v) = \lambda \int_X uv, \quad \text{for every} \quad v \in V(X), \]
where, depending on the choice of boundary conditions, we have

| Boundary conditions | $Q(u, v)$ | $V(X)$ |
|---------------------|-----------|--------|
| Dirichlet           | $\int_X \nabla u \cdot \nabla v$ | \{u $\in H^1(X)$ : trace of u at $\partial X$ is 0\} |
| Neumann             | $\int_X \nabla u \cdot \nabla v$ | $H^1(X)$ |
| Robin               | $\int_X \nabla u \cdot \nabla v - \int_{\partial X} \sigma uv$ | $H^1(X)$ |

where $H^1$ is the space of square integrable functions with square integrable distributional gradient. However, from elliptic regularity theory, we know that Laplacian eigenfunctions are analytic inside any open domain. Thus (2) is satisfied also in the classical sense. Moreover if $\Sigma$ is a smooth (that is $C^\infty$) part of $\partial X$, $u$ is also smooth on $\Sigma$ (see for example [4, Section 6.3]).

Recall from spectral theory that the eigenvalues of $\Omega$ have finite multiplicity and can be arranged in a non-decreasing sequence which tends to infinity, and which we will denote as
  \[ \lambda_1 \leq \lambda_2 \leq \ldots, \]
where each eigenvalue is repeated as many times as its multiplicity.

For future reference we record the following uniqueness result.

**Theorem 2.** Let $u$ be such that $\Delta u + \lambda u = 0$ in $\Omega$. If $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\Sigma$, an open and smooth subset of $\partial \Omega$, then $u$ is constant zero in the whole $\Omega$.

We briefly outline the classic argument to prove this fact from Holmgren’s uniqueness theorem. Let $B$ be an open ball such that $B \cap \partial \Omega \subseteq \Sigma$. Extending $u$ to 0 in $B \setminus \Omega$, it is easy to check that $-\Delta u = \lambda u$ in the distributional sense in $B$. By [7, Theorem 5.3.1], $u$ must be zero also in an open set inside $\Omega$. But then $u = 0$ on the whole $\Omega$ by analytic continuation.
2.1 Stability of eigenvalues of the Laplacian

We review some results that show that the spectrum of the Laplacian is continuous under domain perturbations, and give some useful quantitative estimates on the eigenvalues’ shifts.

First we recall a result of analyticity of eigenvalues and eigenfunctions with respect to a perturbation parameter, which is a consequence of the classic Rellich-Nagy Theorem [9, Theorem 1 at p. 33] (see also [3, Section 4.2] and references therein).

Theorem 3. Let \((\phi_t)_{t \in [0,t_0]}\) be a family of diffeomorphisms of \(\mathbb{R}^N\) such that \(\phi_t\) is analytic in \(t\), \(\phi_0\) is the identity, and \(\phi_t(\Omega) \supseteq \Omega\) for every \(t\). Let \(\lambda\) be an eigenvalue of \(\Omega\) of multiplicity \(m\). Then there exist \(\lambda^1_t \leq \cdots \leq \lambda^m_t\) and functions \(u^1_t, \ldots, u^m_t\) such that for \(j = 1, \ldots, m\),

- for any \(t\), \(\lambda^j_t\) is an eigenvalue of \(\Omega_t\) with associated eigenfunction \(u^j_t\);
- for any \(t\), \(\int_{\Omega_t} u^j_t u^i_t\) is 1 if \(j = i\) and is 0 otherwise;
- \(\lambda^j_t\) and \(u^j_t\) are analytic in \(t\);
- \(\lambda^j_0 = \lambda\) and \(u^j_0\) is an eigenfunction associated to \(\lambda\).

Moreover for any \(\delta > 0\) small enough, there is a \(T\) such that for any \(t < T\) the only eigenvalues of \(\phi_t(\Omega)\) in \((\lambda - \delta, \lambda + \delta)\) are \(\lambda^1_t, \ldots, \lambda^m_t\).

For our purposes we will also need a finer estimate on the variation of eigenvalues, as expressed in the following lemma.

Lemma 4. Let \(\phi\) be a diffeomorphism of \(\mathbb{R}^N\). Let \(\lambda_n\) be the \(n\)-th eigenvalue of \(\Omega\) and \(\tilde{\lambda}_n\) the \(n\)-th eigenvalue of \(\phi(\Omega)\). Then there exists a constant \(C\), which depends only on the Lipschitz constants of \(\partial \Omega\) and of \(\phi\), such that

\[
|\tilde{\lambda}_n - \lambda_n| \leq C \max\{\tilde{\lambda}_n, \lambda_n\}(\|\phi - id\|_{C^1(\overline{\Omega})}).
\]

The proof of this estimate can be obtained by following the same argument in the proof of [2, Lemma 6.1], substituting appropriately the bilinear form and the function space with the ones defined in (4), depending on the boundary conditions considered.

3 Hadamard’s formula and boundary properties of eigenfunctions

In this section we study some independence properties of Laplacian eigenfunctions and of their gradients at the boundary. We first recall a Hadamard’s formula for the variation of eigenvalues under a deformation of the boundary. The dot superscript will indicate differentiation in \(t\).

Lemma 5. Let \((\phi_t)_{t \in [0,t_0]}\) be a family of diffeomorphisms such that \(\phi_t\) is analytic in \(t\) and \(\phi_0\) is the identity. Suppose that the support of \(\phi_t\) is contained in a fixed open set \(U\) for every \(t\), and that \(\partial \Omega \cap U\) is smooth. Let \(\lambda_t, u_t\) be an eigenvalue-eigenfunction couple of \(\phi_t(\Omega)\), and suppose both are differentiable in \(t\). Then

\[
\dot{\lambda}_0 = \int_{\partial \Omega} (|\nabla u_0|^2 - \lambda_0 u_0^2 + (\partial_{\nu_0} u_0)(H u_0 - 2 \partial_{\nu_0} u_0)) \nu_0 \cdot \dot{e}_0,
\]

where \(\nu_t\) indicates the outward unit normal vector, \(e_t\) the identity on \(\phi_t(\partial \Omega)\), and \(H\) is the mean curvature of \(\partial \Omega\).
Hereafter we briefly prove this fact in the case of homogeneous Dirichlet or Neumann boundary conditions. The case of Robin conditions requires a finer analysis of the dependence on $t$ of the surfaces $\phi_t(\partial \Omega)$, for which we refer to [1, Identities (69) and (57)].

**Proof.** Let \((\Omega_t)_{t \in [0,t_0]}\) be a family of domains such that $\Omega_t = \phi_t(\Omega)$ for every $t$. By the divergence theorem, the distributional gradient of the measure $\chi_{\Omega_t} dL^N$, where $\chi_{\Omega_t}$ is the characteristic function of $\Omega_t$ and $L^N$ is the $N$-dimensional Lebesgue measure, is given by $\nu_t \Sigma_t^{N-1}$, where $\Sigma_t^{N-1}$ is the surface measure on $\partial \Omega_t$. Therefore by the chain rule

$$
\frac{d}{dt}(\chi_{\Omega_t} L^N) = \nu_t \cdot \dot{\Sigma}_t^{N-1},
$$
so we have the following Leibniz’ formula:

$$
\frac{d}{dt} \left( \int_{\Omega_t} f_t \right) = \int_{\Omega_t} \dot{f}_t + \int_{\partial \Omega_t} f_t \nu_t \cdot \dot{\nu}_t.
$$

Consider now the identity

$$
\lambda_t = -\int_{\Omega_t} u_t \Delta u_t = \int_{\Omega_t} |\nabla u_t|^2.
$$

Differentiating in $t$ the first equality in (7) and using (6) we obtain

$$
2 \lambda_t \int_{\Omega_t} \dot{u}_t u_t = -\lambda_t \int_{\partial \Omega_t} u_t^2 \nu_t \cdot \dot{\nu}_t.
$$

In the case of Neumann boundary conditions, differentiating in $t$ the last term in (7), using (6), integrating by parts, and substituting (8), we have that

$$
\dot{\lambda}_t = \int_{\partial \Omega_t} (|\nabla u_t|^2 - \lambda_t u_t^2) \nu_t \cdot \dot{\nu}_t + 2 \int_{\partial \Omega_t} \dot{u}_t \frac{\partial u_t}{\partial \nu_t},
$$

which gives (5) since $\partial_{\nu_0} u_0 = 0$ on $\partial \Omega_0$. Proceeding in the same way for Dirichlet boundary conditions, only exchanging the roles of the functions in the integration by parts step, we obtain

$$
\dot{\lambda}_t = \int_{\partial \Omega_t} (|\nabla u_t|^2 - \lambda_t u_t^2) \nu_t \cdot \dot{\nu}_t + 2 \int_{\partial \Omega_t} u_t \frac{\partial \dot{u}_t}{\partial \nu_t} + 2 \dot{\lambda}_t,
$$

which gives (5) since $u_0 = 0$ on $\partial \Omega_0$.

We notice that considering

$$
c = \begin{cases} 
0 & \text{if } u|_{\partial \Omega} = 0, \\
\lambda_0 & \text{if } \partial_{\nu} u|_{\partial \Omega} = 0, \\
\lambda_0 + 2 \sigma^2 & \text{if } \sigma u|_{\partial \Omega} = \partial_{\nu} u|_{\partial \Omega},
\end{cases}
$$

if $\dot{e}_0$ is supported on a flat part of $\partial \Omega$, the integrand in (5) can be rewritten as $|\nabla u|^2 - cu^2$. In the following lemma we study such a quantity, in particular the behavior of its zeros.

**Lemma 6.** Let $c$ be a constant and let $u, \tilde{u}$ be two orthonormal eigenfunctions associated to the same eigenvalue. Let $\Sigma$ be an arbitrary smooth open subset of $\partial \Omega$. Then:

1. $|\nabla u|^2 - cu^2$ cannot be constant zero on $\Sigma$;
2. $|\nabla u|^2 - cu^2 - (|\nabla \tilde{u}|^2 - c\tilde{u}^2)$ cannot be constant zero on $\Sigma$. 

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Proof. The thesis for the case \( c = 0 \) is given by Theorem 2. Consider \( c \neq 0 \). Our approach is inspired to the treatment of [6, Chapter 6].

We first prove Point 1. Suppose by contradiction that \( |\nabla u|^2 = cu^2 \) on \( \Sigma \). We consider separately the different possible boundary conditions in (3).

i) If the Dirichlet condition holds then \( \partial u/\partial \nu = u = 0 \) on \( \Sigma \). By Theorem 2 then \( u = 0 \) on \( \Omega \), a contradiction.

ii) Suppose the Neumann condition holds. The eigenfunction \( u \) cannot be constant 0 on \( \Sigma \), otherwise we would be again in the situation of Case i, so there is \( x_0 \in \Sigma \) s.t. \( u(x_0) \neq 0 \). Let \( \gamma_t \) be a solution in \( \Sigma \) of the ODE

\[
\begin{cases}
\dot{\gamma}_0 = x_0, \\
\dot{\gamma}_t = C\nabla u(\gamma_t),
\end{cases}
\]

with \( C \) a constant to be determined. Then

\[
\frac{du(\gamma_t)}{dt} = C|\nabla u(\gamma_t)|^2 = Ccu(\gamma_t)^2,
\]

if \( \gamma_t \in \Sigma \). Therefore by choosing \( C \) large enough, there will be a time \( T \) at which \( \gamma_T \in \Sigma \) and \( |u(\gamma_t)| \to \infty \), which is a contradiction.

iii) If the Robin condition holds, then

\[
cu^2 = |\nabla u|^2 = \sigma^2 u^2 + |\nabla S u|^2 \quad \text{on} \quad \Sigma,
\]

where \( \nabla S u \) is the surface gradient of \( u \) on \( \partial \Omega \). If \( c \neq \sigma^2 \), we can build, as in Case ii, a curve \( \gamma \) on which the eigenfunction \( u \) blows up in short time, leading to a contradiction. If \( c = \sigma^2 \) then \( |\nabla S u| = 0 \) on \( \Sigma \), and this leads to the following chain of implications: \( u \) is constant on \( \Sigma \), \( \partial u/\partial \nu \) is constant on \( \Sigma \), \( u \) is constant in \( \Omega \) by Theorem 2, \( \partial u/\partial \nu \) is zero on \( \partial \Omega \), \( u \) is zero on \( \Omega \) by Theorem 2, a contradiction.

We now prove Point 2. Suppose by contradiction that \( |\nabla u| - |\nabla \tilde{u}| = c(u^2 - \tilde{u}^2) \) on \( \Sigma \). Let \( x_0 \in \Sigma \) be a point where \( u(x_0) \) and \( \tilde{u}(x_0) \) are different (existence of such a point is guaranteed by the smoothness of eigenfunctions on \( \Sigma \) and Theorem 2). Let \( f_t = u(\gamma_t), \tilde{f}_t = -\tilde{u}(\tilde{\gamma}_t) \), where \( \gamma \) and \( \tilde{\gamma} \) solve

\[
\begin{cases}
\dot{\gamma}_t = C\nabla u(\gamma_t), \\
\dot{\tilde{\gamma}}_t = -C\nabla \tilde{u}(\tilde{\gamma}_t), \\
\gamma_0 = \tilde{\gamma}_0 = x_0,
\end{cases}
\]

and \( C \) is a constant to be determined. Then

\[
\dot{f}_t + \dot{\tilde{f}}_t = Cc(f_t^2 + \tilde{f}_t^2).
\]

Therefore \( \dot{f}_t \geq Cc f_t^2 \) or \( \dot{\tilde{f}}_t \geq Cc \tilde{f}_t^2 \) for \( t \) in a small neighborhood of 0. In conclusion, a choice of \( C \) large enough would lead to blow up in short time of \( u \) or \( \tilde{u} \), which is impossible. \( \square \)

4 Splitting of the spectrum

With the tools developed so far we can construct a localized boundary deformation which splits the eigenvalues perturbed from one eigenvalue as follows.
Proposition 7. Let $x \in \partial \Omega$, $B$ a ball centered at $x$, and $\Sigma = B \cap \partial \Omega$. Suppose $\Sigma$ is flat, that is $\Sigma$ is contained in a hyperplane. Then, under the same hypotheses and notation of Theorem 3, we can construct a family of diffeomorphisms $(\phi_t)_{t \in (0, t_0)}$ such that $\phi_t$ is the identity outside $B$, $|\phi_t - id|_{C^1}$ is arbitrarily small, and $\lambda^i_t \neq \lambda^j_t$ for any $i, j \in \{1, \ldots, m\}$ and for all $t \in (0, t_0)$.

Proof. Let $c$ be as in (9). By Point 2 of Lemma 6, there exists $y$ on $\Sigma$ such that

$$\left(||\nabla u^i_0||^2 - c(u^i_0)^2\right)(y) \neq \left(||\nabla u^j_0||^2 - c(u^j_0)^2\right)(y).$$

Then, by choosing a deformation of the boundary $\phi_t$ which is the identity outside an appropriately small neighborhood of $y$, we have

$$\int_{\partial \Omega} \left(||\nabla u^i_0||^2 - c(u^i_0)^2\right) \nu \cdot \dot{\phi}_0 \neq \int_{\partial \Omega} \left(||\nabla u^j_0||^2 - c(u^j_0)^2\right) \nu \cdot \dot{\phi}_0.$$

Such a perturbation can be constructed in many ways; for the sake of completeness, we give an explicit example hereafter.

By eventually reducing to a smaller $B$ and applying an invertible affine transformation, we can assume that $y = 0$ and $\Sigma = \{z \in B_1 : z_N = 0\}$, where $B_1$ is the unit ball. Let $\hat{z}$ indicate $(z_1, \ldots, z_{N-1})$ and let

$$\rho_c(\hat{z}) = \begin{cases} c^2 \exp \left( \frac{1}{|\hat{z}|^2 - 1} \right) & \text{if } |\hat{z}| < c, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that by construction $|\rho_c|_{C^1} \leq c$ for any $c \leq 1$. Let $\phi_t(z)$ be the extension of the map $z \mapsto (\hat{z}, t \rho_c(\hat{z}))$ from $\Sigma$ to a smooth function which is the identity outside $B$ and such that $|\phi_t - id|_{C^1} \leq |\rho_c|_{C^1}$. By construction, $\nu \cdot \dot{\phi}_0 = \rho_c(\hat{z})$ on $\Sigma$. Then by choosing $c$ small enough, by the smoothness of $u$ on $\Sigma$ and by (11), we have that (12) holds. Moreover we remark that it holds

$$|\phi_t - id|_{C^1} \leq c.$$ (13)

In conclusion, by Lemma 5, (12) implies that $\lambda^i_0 \neq \lambda^j_0$. Since $\lambda^i_t$ and $\lambda^j_t$ are both analytic in $t$, there exists a small $t_0$ such that $\lambda^i_t \neq \lambda^j_t$ for $t \in (0, t_0)$. \qed

Remark 8. The flatness assumption of $\Sigma$, although making the argument simpler, is not really necessary in the proof of Proposition 7, as one might build a boundary deformation such that (12) holds even if $\Sigma$ is not flat; the idea would be the same, only some care would be required to manage the mean curvature term which is present in (5). On the other hand, if our aim is to find a local perturbation as in Theorem 1, the flatness assumption is not restrictive. In fact, if $\Sigma$ is not contained in a hyperplane, by eventually considering a smaller $B$ and changing basis, we can assume that $\Sigma$ is the graph of a Lipschitz function $\phi$ such that $\phi(0) = x = 0$. Let $B_r, B_R$ be two balls centered in 0 such that $B_r \subset B_R \subset B$, and let $\eta$ be a smooth function which is 0 in $B_r$ and 1 outside $B_R$. Then the graph of $\phi \eta$ will be flat in $B_r$. Notice also that as $r \to 0$, $\eta$ can be chosen so that the Lipschitz constant of $\phi \eta$ converges to the Lipschitz constant of $\phi$. Thus for any $\delta > 0$, we can build a Lipschitz domain which differs from $\Omega$ only in $B_r$, is flat in $B_r$ (for a certain $r$ which depends on $\delta$), and whose Lipschitz constant differs from the Lipschitz constant of $\Omega$ by less than $\delta$.

We further remark that although Proposition 7 shows how to split one eigenvalue, the perturbation chosen might cause a couple of two other eigenvalues to overlap, creating a new repeated eigenvalue. To avoid this problem we need a finer control on the behavior of the whole spectrum; this is what is achieved in the following lemma.
Lemma 9. Consider $\varepsilon > 0$, $x$ a point on the boundary $\partial \Omega$, and $\lambda_r$ the first eigenvalue of $\Omega$ of multiplicity $m \geq 2$. Then for any $M > 0$ there exists a Lipschitz domain $\hat{\Omega}$, whose eigenvalues we indicate as $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \ldots$, such that:

1. the symmetric difference $\hat{\Omega} \triangle \Omega$ is contained in the ball of radius $\varepsilon$ centered at $x$;

2. for all $i \leq r + m + 1$, it holds $|\hat{\lambda}_i - \lambda_i| \leq M d_r$, where $d_r$ is the minimum positive number of the set $\{\lambda_{j+1} - \lambda_j : j = 1, \ldots, r + m\}$;

3. the multiplicity of $\hat{\lambda}_r$ is strictly smaller than the multiplicity of $\lambda_r$;

4. for all $i > r + m$, it holds $\hat{\lambda}_i > \lambda_r$.

Proof. Let $B_x$ be the ball of radius $\varepsilon$ centered at $x$ and let $\Sigma = B_x \cap \partial \Omega$. With the same construction of Remark 8 and of the proof of Proposition 7, we can build $(\Omega_t)_{t \in (0, t_0)}$ a family of perturbations of $\Omega$ obtained by a deformation of the boundary of $\Omega$ localized in $B_x$. Let $\lambda^t_1, \lambda^t_2, \ldots$ indicate the sequence of eigenvalues of $\Omega_t$, with associated eigenfunctions $u^t_1, u^t_2, \ldots$. By Theorem 3 we can assume that $\lambda^t_i, u^t_i$ are analytic in $t$, that $\lambda^0_i = \lambda_i$, and that $u^0_r, \ldots, u^0_{r+m}$ is an orthonormal basis for the eigenspace of $\lambda_r$. By Proposition 7, there are two distinct indices $i$ and $j$ among $\{r, \ldots, r+m\}$, such that for $t_0$ small enough

$$\lambda^t_i \neq \lambda^t_j, \quad \text{for } t \in (0, t_0). \tag{14}$$

By the eigenvalue stability estimate of Lemma 4, there is a $t_0$ small enough such that

$$|\lambda^t_i - \lambda_i| \leq M d_r, \quad \forall t < t_0, \forall i \in \{1, \ldots, r + m + 1\}. \tag{15}$$

Let $C, C'$ indicate two constants which depend only on the dimension $N$, the Lipschitz constant of $\partial \Omega$ and the area of $\Omega$. By Weyl’s asymptotic law, $\lambda_n = C n^{2/N} + o(n^{2/N})$ for any $n$. Then, from the uniform estimate of Lemma 4, for $i > r + m$ it holds

$$\lambda^t_i - \lambda_r \geq (\lambda^t_i - \lambda_i) + \lambda_i - \lambda_r \geq C'(-C ci^{2/N} + i^{2/N} - r^{2/N}),$$

where $c > 0$ is a bound on the deformation magnitude (which we can choose arbitrarily small) as in (13). Therefore for $t_0$ and $c$ small enough,

$$\lambda^t_i - \lambda_r > 0, \quad \forall t < t_0, \forall i > r + m. \tag{16}$$

In conclusion, taking $\hat{\Omega} := \Omega_t$ for a certain $t$ small enough, Point 1 of the thesis holds by construction while Points 2-3-4 are consequences of (15)-(14)-(16).

The construction in the previous proof gives us a method to split the first non-simple eigenvalue without altering the simplicity of smaller eigenvalues. In fact by taking $M < 1/2$, from Points 2 and 4 of Lemma 9 we have that the eigenvalues $\hat{\lambda}_i$ perturbed from $\lambda_i$:

- lie in disjoint neighborhoods of $\lambda_i$, for $i < r$;
- are not further than $d_r/2$ from $\lambda_i$, for $r \leq i \leq r + m$;
- are larger than $\lambda_r$, for $i > r + m$.

Therefore $\hat{\lambda}_1, \ldots, \hat{\lambda}_{r-1}$ must still be simple. We can iterate this procedure to split the whole spectrum as in the following proof.
Proof of Theorem 1. Denoting as $B_\varepsilon$ the ball of radius $\varepsilon$ centered at $x$, let $\Sigma = B_\varepsilon \cap \partial \Omega$. As in Remark 8, for any $\delta > 0$, we can modify $\Sigma$ into $\Sigma'$ so that an open subset of $\Sigma'$ is contained in a hyperplane and the Lipschitz constant of $\Sigma'$ differs from the Lipschitz constant of $\Sigma$ by less than $\delta$. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of disjoint balls of radius $c2^{-n}$ with centers on $\Sigma'$ and contained in $B_\varepsilon$, with $c$ small enough so that $\Sigma' \cap \bigcup_n B_n$ is flat. In each $B_n$ we deform $\Sigma'$ with a diffeomorphism $\phi_n$ built as in the proof of Proposition 7. We obtain this way a sequence of domains $(\Omega_n)_{n \in \mathbb{N}}$ such that the thesis of Lemma 9 holds with $\Omega, \tilde{\Omega}, B, M$ replaced respectively by $\Omega_n, \Omega_{n+1}, B_n, M_n$ for each $n$, where for $M_n$ we take a constant smaller than $1/2^{n+1}$. Additionally, we can take $\phi_n$ such that $|\phi_n - id|_{C^1} \leq \delta/n$. And thus as $n \to \infty$, $\Omega_n$ converges to a domain $\tilde{\Omega}$ with Lipschitz constant not farther than $\delta$ from the Lipschitz constant of $\Omega$.

Let $r_n$ be the index of the first non-simple eigenvalue of $\Omega_n$. By Points 2 and 4 of Lemma 9 we have that all eigenvalues with index smaller than $r_n$ are simple for any $n$. Moreover $r_n$ is a non-decreasing sequence of integers which cannot be definitely constant; in fact by Point 3 of Lemma 9, $r_{n+j}$ can be equal to $r_n$ for at most $j \in \{1, \ldots, r_n\}$. Therefore $r_n \to \infty$ as $n \to \infty$, and thus $\tilde{\Omega}$ can have only simple eigenvalues. □

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