Ordering of the Unicyclic Signed Graphs With Perfect Matchings by Their Minimal Energies

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Abstract. Let \( U_n \) be the set of unicyclic signed graphs with perfect matchings having \( 2n \) vertices, where \( \sigma \) is a signing function from the edge set of the graphs considered to \( \{-1, 1\} \). The increasing order of the signed graphs among \( U_n \) according to their minimal energies is considered. A relationship between the energies of a unicyclic graph and of its signed graphs is derived. A new integral formula for comparing the energies of two signed graph is introduced. In \( U_n \) with \( n \geq 721 \), the first 18 signed graphs in the increasing order by their minimal energies are obtained.

1. Introduction

Let \( G = (V(G), E(G)) \) be a simple undirected graph of order \( n \), where \( V(G) \) and \( E(G) \) are the vertex set and the edge set of \( G \), respectively. Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \). A signed graph, denoted by \( S = (G, \sigma) \), is obtained from \( G \) by assigning a positive or negative label on the edge of \( G \), where \( \sigma : E(G) \to \{-1, 1\} \) is the signing function. Namely, for an edge \( v_\alpha v_\beta \) of \( G \), \( \sigma(v_\alpha v_\beta) = -1 \) or 1, where \( 1 \leq i, j \leq n \). We say that \( G \) is the underlying graph of \( S \) and \( S \) is the signed graph of \( G \). If each edge of \( E(G) \) has a positive label, then \( S \) becomes \( G \). The signed graph is of theoretical interest due to both their applications in modeling a variety of physical and socio-psychological processes and their interesting connections with many classical mathematical systems [6].

The sign of a signed cycle is defined to be the product of signs of its edges. A signed cycle is said to be positive/negative if its sign is positive/negative. Namely, a positive/negative signed cycle contains an even/odd number of negative edges. A signed graph is said to be balanced if each of its cycles is positive, and otherwise to be unbalanced.

The adjacency matrix of \( S \) is an \( n \times n \) matrix \( A(S) = (a_{ij}) \), where \( a_{ij} = \sigma(v_i v_j) \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \) otherwise. The characteristic polynomial of \( S \) is

\[
\phi_S(x) = \det[xI - A(S)] = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S),
\]

where \( I \) is the unit matrix of order \( n \) and \( a_1(S), \ldots, a_n(S) \) are the coefficients of \( \phi_S(x) \). The \( n \) roots of \( \phi_S(x) = 0 \) are denoted by \( \lambda_1, \ldots, \lambda_n \), which are called the eigenvalues of the corresponding signed graph \( S \). Since \( A(S) \) is a real symmetric matrix, all \( \lambda_i \) with \( 1 \leq i \leq n \) are real.
We denote the adjacency matrix of $G$ by $A(G) = (a_{ij})$, where $a_{ij} = a_{ji} = 1$ if $i$ and $j$ are adjacent, else $a_{ij} = a_{ji} = 0$ for $1 \leq i, j \leq n$. The energy of $G$, denoted by $E(G)$, as introduced by Gutman [8], is defined as the sum of the absolute values of all the eigenvalues of $A(G)$.

Let $M$ be an $m \times n$ complex matrix. The singular values of $M$ are the positive square roots of the eigenvalues of $MM^*$, where $M^*$ is the conjugate transpose of $M$. Nikiforov [16] first defined that the energy of a matrix $M$ is the sum of its singular values. For undirected graphs, there are various generalizations of the graph energy, for example, the matching energy by Gutman and Wagner [10], Laplacian energy by Gutman and Zhou [11], the incidence energy by Jooyandeh et al. [14], the distance energy by Indulal et al. [13] and Ramane et al. [19], and so on. For directed graphs, Peña and Rada [17] introduced the energy of a digraph, and Adiga et al. [2] proposed the skew energy of an oriented graph. The energy for the signed digraphs was defined by Bhat and Pirzada [3, 18].

The energy of the signed graph $S$ was first introduced by Germina et al. [6] and it can be reduced to

$$E(S) = \sum_{i=1}^{n} |\lambda_i|.$$  \hspace{1cm} (2)

Bhat and Pirzada [4] expressed $E(S)$ as the following Coulson integral formula

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \log \left( \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j}(S)x^{2j} \right)^2 + \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j+1}(S)x^{2j+1} \right)^2 \right) dx,$$  \hspace{1cm} (3)

where $b_i(S) = |a_i(S)|$ for $0 \leq i \leq n$. Note that $b_0(S) = 1$, $b_1(S) = 0$ and $b_2(S)$ equals the number of edges in $S$. It can be seen from (3) that $E(S)$ is a strictly monotonously increasing function of $b_1(S)$, where $0 \leq i \leq n$. Let $S_1$ and $S_2$ be two unicyclic signed graphs. We get

$$b_1(S_1) \geq b_1(S_2) \implies E(S_1) \geq E(S_2),$$  \hspace{1cm} (4)

where $E(S_1) = E(S_2)$ if and only if $b_1(S_1) = b_1(S_2)$ for all $0 \leq i \leq n$. We will refer to the relation (4) as the method of coefficient comparison. For the sake of conciseness, we introduce the symbols “$\rightarrow$” and “$\Leftarrow$”.

$$E(S_1) < E(S_2) \iff S_1 \nsim S_2,$$

where $E(S_1) = E(S_2) \iff S_1 \equiv S_2.$  \hspace{1cm} (5)

The Coulson integral formula for $E(G)$, as introduced in [9], can readily be written by replacing $S$ in (3) with $G$, where $b_2(G) = |a_2(G)|$ and $a_2(G)$ is the coefficients of the characteristic polynomial of $G$ for $0 \leq i \leq n$.

The relation (4) has successfully been employed in the study on the extremal values of energy for signed graphs. Among all the unicyclic signed graphs with $n$ vertices, Bhat and Pirzada [4] characterized the unicyclic signed graph with the minimal energy. Among all the bicyclic signed graphs with $n$ vertices, Bhat et al. [5] determined the bicyclic signed graphs with the first and the second minimal energies.

We denote by $U_{2n}$ the set of unicyclic graphs with perfect matchings having $2n$ vertices. Among $U_{2n}$, for the increasing order of graphs according to their minimal energies, Wang [20] obtained the first 7 graphs for $l = 2r + 1$ and $l = 4j + 2$ when $n \geq 45$, where $l$ is the girth of the graphs considered and $r, j$ are positive integers; and Zhu [25] derived the first 7 graphs when $n \geq 191$.

Let $U_{2n}^\sigma$ be the set of unicyclic signed graphs with perfect matchings having $2n$ vertices, where $\sigma$ is a signing function from the edge set of the graphs considered to $\{-1, 1\}$. In this paper, we will study the increasing order of the signed graphs among $U_{2n}^\sigma$ according to their minimal energies.

This paper is organized as follows. In Section 2, some graphs are introduced and necessary lemmas are presented. The results of Section 3 are divided into five parts. In Subsection 3.1, a relationship between the energies of a unicyclic graph and of its signed graphs is derived. In Subsection 3.2, a new integral formula for comparing the energies of two signed graph is introduced. The capped graphs, in which the number of 2-matchings is less than $2n - 3$, of the graphs among $U_{2n}$ are characterized in Subsection 3.3. In Subsection 3.4, by using the method of coefficient comparison in (4), the theorem of zero points and the new integral formula (presented in Theorem 3.3 in Subsection 3.2), we compare energies for the signed graphs whose
capped graphs have the number of 2-matchings being less than 2n − 3. In Subsection 3.5, the preceding 18 signed graphs in the increasing order by their minimal energies are obtained among $\mathcal{U}_n^{2n}$ with $n \geq 721$; and the preceding 12 graphs in the increasing order by their minimal energies are derived among $\mathcal{U}_{2n}$ with $n \geq 721$.

2. Preliminaries

A basic figure is a graph whose components are cycles or edges or both. In 1980, Acharya [1] studied the characteristic polynomial of a signed graph and got Lemma 2.1 as follows.

**Lemma 2.1.** [1] If $S$ is a signed graph with characteristic polynomial $\phi_S(x) = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S)$, then for all $j = 1, 2, \ldots, n$, we have

$$a_j(S) = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)}2^{c(L)} \prod_{Z \in c(L)} s(Z),$$

where $\mathcal{L}_j$ is the set of all basic figures $L$ of order $j$ in $S$, $p(L)$ the number of components of $L$, $c(L)$ the set of all cycles in $L$, and $s(Z)$ the sign of cycle $Z$.

For $u \in V(S)$, let $S - u$ be the graph obtained from $S$ by deleting $u$ and all the edges in $S$ which are incident with $u$. For $e \in E(S)$, let $S - e$ be the graph obtained from $S$ by deleting $e$.

**Lemma 2.2.** [7] Let $S$ be a unicyclic signed graph and $uv$ a pendent edge of $S$ with a pendent vertex $v$. Then

$$\phi_S(x) = x\phi_{(S-v)}(x) - \phi_{(S-v-u)}(x).$$

A $k$-matching of $G$ is a union of $k$ independent edges in $G$. Let $m(G, k)$ be the number of $k$-matchings in $G$, where $0 \leq k \leq n$. For $G$ and its signed graph $S$, it is obvious that $m(G, k) = m(S, k)$ for $0 \leq k \leq n$.

**Lemma 2.3.** [9] Let $e = uv$ be an edge of a graph $G$. Then we have

$$m(G, k) = m(G - e, k) + m(G - u - v, k - 1).$$

Let $T$ be a tree as shown in Fig. 1(a). Namely, $T$ is obtained from an edge $uv$ by attaching trees $T_1$ and $T_2$ at $u$ and $v$, respectively. Let $T'$ be the tree obtained from $T$ by first identifying $u$ and $v$ and then attaching a pendent edge at $u$. $T'$ is the tree as shown in Fig. 1(b). For the 2-matchings of $T$ and $T'$, by using Lemma 2.3, we can get Lemma 2.4 as follows. We omit the proof of Lemma 2.4 since it is straightforward.

**Lemma 2.4.** Let $T$ and $T'$ be two trees as shown in Fig. 1. We have $m(T, 2) > m(T', 2)$. 

![Figure 1: T and T' in Lemma 2.4.](image-url)
Lemma 2.5. [4] If $S$ is a signed graph on $n$ vertices, then
\[
\mathcal{E}(S) = \frac{1}{n} \int_{-\infty}^{\infty} \frac{1}{x^2} \log \left| x^2 \phi_S \left( \frac{i}{x} \right) \right| \, dx,
\]
where $i^2 = -1$.

For $G \in \mathcal{U}_{2n}$, the relationship between $b_i(G)$ and $m(G, i)$ is shown in Theorem 4 in [12] and Lemma 2 in [22].

Next, we will give a formula to calculate $m(G, i)$, where $G \in \mathcal{U}_{2n}$.

It is consistent to define $m(G, 0) = 1$. Obviously, $m(G, 1) = 2n$. Let $\overline{G} = G - M(G) - S_0$, where $M(G)$ is the perfect matching of $G$ and $S_0$ the set of isolated vertices in $G - M(G)$. It is clear that $|M(G)| = n$, where $|M(G)|$ is the number of edges in $M(G)$. We call $\overline{G}$ the capped graph of $G$ and $G$ the original graph of $\overline{G}$. Each $k$-matching $\Omega$ of $G$ can be partitioned into two parts: $\Omega = \Phi \cup \Psi$, where $\Phi$ is a matching in $\overline{G}$ and $\Psi \subset M(G)$. Thus, for $G \in \mathcal{U}_{2n}$, we have [24]
\[
m(G, k) = \sum_{i=0}^{k} m(\overline{G}, i) \binom{n-i}{k-i} = p + \sum_{i=2}^{k} m(\overline{G}, i) \binom{n-i}{k-i},
\]
where
\[
p = \binom{n}{k} + \binom{n-2}{k-1}
\]
and $j$ is the number of edges in $M(G)$ which are adjacent to the $i$-matching $\Phi$.

For $n \geq 2$, $P_n$ is a path with $n$ vertices, and the vertices of $P_n$ are labeled consecutively by $v_1, v_2, \ldots, v_n$.

For $l \geq 3$, $C_l$ is a cycle with $l$ vertices, and the vertices of $C_l$ are labeled consecutively by $u_1, u_2, \ldots, u_l$. For $n \geq 3$, $X_n$ is a star with $n$ vertices.

For $l = 3$, some graphs are introduced as follows.

For $n \geq 4$, $C_n^a$ is the graph obtained from $C_3$ by attaching $a$ and $n-a-3$ pendent edges to $u_1$ and $u_2$ respectively, where $0 \leq a \leq \left\lfloor \frac{n-3}{2} \right\rfloor$.

For $n \geq 5$, $Q_n$ is the graph obtained by identifying $u_1$ of $C_3$ with a pendent vertex of $X_{n-2}$.

For $n \geq 6$, let $H_n, S_n$ and $T_n$ be the graphs obtained by identifying $v_2, v_3$ and $v_4$ of $P_4$ of $l_{n-2}$ with $u_1$ of $C_3$ respectively, where $l_{n-2}$ with $n \geq 6$ is the graph obtained from $P_4 = v_1v_2v_3v_4$ by attaching $n-6$ pendent edges to $v_2$.

For $n \geq 6, R_n$ is the graph obtained from $C_3$ by attaching $n-5$ pendent edges to $u_1$ and a path of length 2 to $u_2$.

For $n \geq 6$, $W_n$ is the graph obtained from $C_3$ by attaching one pendent edge to $u_1$ and identifying $u_2$ of $C_3$ with a pendent vertex of $X_{n-3}$.

For $n \geq 6$, $Y_n$ is the graph obtained from $C_3$ by attaching one pendent edge to $u_1$ and $u_2$ and $n-5$ pendent edges to $u_3$.

Denote by $K = [C_a, Q_n, H_n, S_n, T_n, R_n, W_n, Y_n]$, where $0 \leq a \leq \left\lfloor \frac{n-3}{2} \right\rfloor$.

For $n \geq 7$, $O_n$ is the graph obtained from $P_6$ by attaching $n-6$ pendent edges to $v_2$. For $n \geq 6, Z_n$ is the graph obtained from $C_3$ by attaching $n-5$ pendent edges to $u_1$.

Let $\mathcal{H}_n$ be the set of unicyclic graphs with $n$ vertices. Let $\text{dia}(G)$ be the diameter of a graph $G$.

Lemma 2.6. [21, 23] Let $G \in \mathcal{H}_n$ with $n \geq 7$ and $C_l$ the cycle contained in $G$. If one of the three conditions holds: (i) $\text{dia}(G-e) \geq 5$ and $G-e \notin O_n$, where $e$ is an edge of $C_l$; or (ii) $l = 3$ and $G \notin K$; or (iii) $l \geq 5$ and $G \notin Z_n$, then we have $m(G, 2) \geq 3n-12$ and $m(G, 3) \geq 2n-12$, where the two equalities do not hold simultaneously.
3. Main results

3.1. Comparing the energies of a unicyclic graph and of its signed graphs

Bhat and Pirzada [4] obtained that for a unicyclic graph \( G \) with an odd girth, any two signed graphs on \( G \) have the same energy. In Theorem 3.2, we will extend this result to the relationship between the energies of a unicyclic graph and of its signed graphs. To obtain Theorem 3.2, Lemma 3.1 is needed. By Lemma 2.1, we can get Lemma 3.1 as follows.

Lemma 3.1. Let \( S \in \mathcal{U}_{2r}^c \) and \( l \) the length of the cycle contained in \( S \) and \( r \) and \( h \) integers with \( r, h \geq 1 \). Then

\[
b_{2i+1}(S) = \begin{cases} 
0, & l = 2r, l = 2r + 1 \& 2i + 1 < l; \\
2m(S - C_i, i - \frac{1}{2}), & l = 2r + 1 \& 2i + 1 \geq l;
\end{cases}
\]

\[
b_2(S) = \begin{cases} 
m(S, i), & l = 2r + 1, \\
m(S, i) + 2m(S - C_i, i - \frac{1}{2}), & l = 4h + 2 \& S \text{ is balanced and } l = 4h; \\
m(S, i) - 2m(S - C_i, i - \frac{1}{2}), & l = 4h \& S \text{ is unbalanced and } l = 4h + 2.
\end{cases}
\]

For a unicyclic graph \( G \), by Lemma 3.1 and (3), we can conclude that any two balanced graphs of \( G \) have the same energy and any two unbalanced graphs of \( G \) also have the same energy. If \( G \) is a unicyclic graph, we use, for simplicity, \( G^+ \) to denote its balanced graph and \( G^- \) its unbalanced graph throughout this paper. We have a relationship for \( E(G^+), E(G^-) \) and \( E(G) \), as shown in Theorem 3.2.

Theorem 3.2. Let \( G \) be a unicyclic graph with girth \( l \). For positive integers \( r, h \geq 1 \), we have

(i) \( E(G^+) = E(G^-) = E(G) \) for \( l = 2r + 1 \);
(ii) \( E(G^+) = E(G) < E(G^-) \) for \( l = 2r \) with \( r = 2h \);
(iii) \( E(G^+) = E(G) > E(G^-) \) for \( l = 2r \) with \( r = 2h + 1 \).

Proof. Let \( l = 2r + 1 \). By Lemma 2 in [22] and Lemma 3.1, we have \( b_{2i}(G^+) = b_{2i}(G^-) = b_{2i}(G) = m(G, i) \), \( b_{2i+1}(G^+) = b_{2i+1}(G^-) = b_{2i+1}(G) = 0 \) for \( 2i + 1 < l \), and \( b_{2i+1}(G^+) = b_{2i+1}(G^-) = b_{2i+1}(G) = 2m(G - C_i, i - r) \) for \( 2i + 1 \geq l \), where \( 0 \leq i \leq \lfloor n/2 \rfloor \). Furthermore, by comparing \( E(G), E(G^+) \) and \( E(G^-) \) in terms of their Coulson integral formula in (3), we get Theorem 3.2(i).

By the method similar to that for Theorem 3.2(i), we can get Theorem 3.2(ii) and (iii). \( \square \)

3.2. A new integral formula for comparing the energies of two signed graphs

From Lemma 2.5, we obtain a new integral formula for comparing the energies between two signed graphs by directly using their characteristic polynomials, which are shown in Theorem 3.3.

Theorem 3.3. Let \( \phi_{S_1}(x) \) and \( \phi_{S_2}(x) \) be the characteristic polynomials of signed graphs \( S_1 \) and \( S_2 \) respectively, where \( S_1 \) and \( S_2 \) have the same number of vertices. Then

\[
E(S_1) - E(S_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left| \frac{\phi_{S_1}(ix)}{\phi_{S_2}(ix)} \right| dx.
\]

Proof. From Lemma 2.5, we have

\[
E(S_1) - E(S_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log \left| \frac{x^2 \phi_{S_1}(\frac{1}{x})}{\phi_{S_2}(\frac{1}{x})} \right| dx.
\]

Obviously, we get

\[
E(S_1) - E(S_2) = \frac{1}{\pi} \left[ \int_{-\infty}^{0} \log \left| \frac{\phi_{S_1}(\frac{1}{x})}{\phi_{S_2}(\frac{1}{x})} \right| d\left( \frac{1}{x} \right) + \int_{0}^{\infty} \log \left| \frac{\phi_{S_1}(\frac{1}{x})}{\phi_{S_2}(\frac{1}{x})} \right| d\left( \frac{1}{x} \right) \right].
\]
Since \( 0 \) is a flaw, we obtain
\[
\int_{-\infty}^{0} \log \frac{\phi_S(\frac{t}{n})}{\phi_S(\frac{1}{n})} \frac{dt}{t} + \int_{0}^{\frac{1}{2}} \log \frac{\phi_S(\frac{t}{n})}{\phi_S(\frac{1}{n})} \frac{dt}{t} + \int_{\frac{1}{2}}^{1} \log \frac{\phi_S(\frac{t}{n})}{\phi_S(\frac{1}{n})} \frac{dt}{t} = \int_{0}^{1} \log \frac{\phi_S(t/n)}{\phi_S(t/n+1)} dt + \int_{1}^{\infty} \log \frac{\phi_S(t/n)}{\phi_S(t/n+1)} dt = \int_{0}^{\infty} \log \frac{\phi_S(t/n)}{\phi_S(t/n+1)} dt.
\]
where \( \frac{1}{2} = t \). Similarly, we have
\[
\int_{0}^{\infty} \log \frac{\phi_S(t/n)}{\phi_S(t/n+1)} \frac{dt}{t} = \int_{0}^{\infty} \log \frac{\phi_S(t/n)}{\phi_S(t/n+1)} dt.
\]
Substituting (17) and (18) into (16), we obtain (14). \( \square \)

3.3. Finding the graphs whose capped graphs have 2-matching less than \( 2n - 3 \)

In this section, among \( \mathcal{U}_{2n} \), the graphs whose capped graphs have 2-matching less than \( 2n - 3 \) are characterized in Lemma 3.8. To obtain Lemma 3.8, we need to introduce Lemmas 3.4–3.7 first.

Let \( \mathcal{T}_{n+1} \) be the set of trees with \( n + 1 \) vertices. Let \( E_{n+1}^{b,c} \) be the tree obtained from \( F_4 \) by attaching \( a \) and \( b \) pendant edges at \( v_3 \) and \( v_4 \) respectively, where \( 0 \leq a \leq \frac{n-3}{2} \) and \( b = n - a - 3 \). Let \( F_{n+1}^{b,c} \) be the tree obtained from \( P_5 \) by attaching \( a, b, c \) and \( c \) pendant edges at \( v_2, v_3 \) and \( v_4 \) respectively, where \( 0 \leq a, b, c \leq n - 4 \) and \( a + b + c = n - 4 \).

**Lemma 3.4.** Let \( G \in \mathcal{U}_{2n} \) and \( n \geq 9 \). If \( \widetilde{G} \in \mathcal{T}_{n+1} \) and \( \widetilde{G} \neq X_{n+1}, E_{n+1}^{0,n-3}, E_{n+1}^{1,n-4}, E_{n+1}^{0,0,n-4}, E_{n+1}^{0,n-4,0}, \) then \( m(\widetilde{G}, 2) \geq 2n - 3 \).

**Proof.** Let \( G \in \mathcal{U}_{2n} \) with \( n \geq 9 \) and \( \widetilde{G} \in \mathcal{T}_{n+1} \). We suppose \( \widetilde{G} \neq X_{n+1} \). Thus, \( \text{dia}(\widetilde{G}) \geq 3 \). Four cases are considered according to the value of \( \text{dia}(\widetilde{G}) \).

**Case (i).** \( \text{dia}(\widetilde{G}) = 3 \).
In this case, \( \widetilde{G} = E_{n+1}^{a,b} \). By (8), we have \( m(E_{n+1}^{b,c}, 2) = (a + 1)(n - 2 - a) = f_3(a) \). Therefore, \( m(E_{n+1}^{b,c}, 2) \) is an increasing function of \( a \) since \( f_3'(a) = -2a + n - 3 \geq 0 \), where \( 0 \leq a \leq \frac{n-3}{2} \). We have \( m(E_{n+1}^{0,n-3}, 2) = f_3(0) = n - 2 < 2n - 3, m(E_{n+1}^{1,n-4}, 2) = f_3(1) = 2(n - 3) < 2n - 3, \) and \( m(E_{n+1}^{0,0,n-4}, 2) = f_3(0) = 3(n - 4) \geq 2n - 3 \) for \( 2 \leq a \leq \frac{n-3}{2} \). Namely, in Case (i), if \( \widetilde{G} \neq E_{n+1}^{0,n-3}, E_{n+1}^{1,n-4}, E_{n+1}^{0,0,n-4}, \) then we have \( m(\widetilde{G}, 2) \geq 2n - 3 \).

**Case (ii).** \( \text{dia}(\widetilde{G}) = 4 \).
In this case, \( G \) may be viewed as a tree obtained from \( P_5 \) by attaching trees \( T_1, T_2 \) and \( T_3 \) at \( v_2, v_3 \) and \( v_4 \) respectively. Let \( |E(T_1)| = a, |E(T_2)| = b \) and \( |E(T_3)| = c \), where \( 0 \leq a, b, c \leq n - 4 \) and \( a + b + c = n - 4 \). By using Lemma 2.4 repeatedly, we get \( m(\widetilde{G}, 2) \geq m(F_{n+1}^{a,b,c}, 2) \). Three subcases are considered as follows.

**Subcase (ii.i).** \( a = 0, 0 \leq b, c \leq n - 4 \) and \( c = n - 4 - b \). By (8), we have
\[
m(F_{n+1}^{a,b,c}, 2) = (b + 2)(n - 3 - b) + b + 1 \geq f_2(2).
\]
We get \( f_2'(b) = -2b + n - 4 \). Thus, \( m(F_{n+1}^{a,b,c}, 2) \) is an increasing function of \( b \) with \( 0 \leq b \leq \frac{n-4}{2} \) and a decreasing function of \( b \) with \( \frac{n-4}{2} \leq b \leq n - 4 \). We have \( m(F_{n+1}^{0,n-4,2}, 2) = f_2(0) = 2n - 5 < 2n - 3, m(F_{n+1}^{0,0,n-4}, 2) = f_2(n - 4) = 2n - 5 < 2n - 3 \) and \( m(F_{n+1}^{0,n-4,0}, 2) = f_2(b) \) for \( b \geq \min(f_2(1), f_2(n - 5)) = 3n - 10 \geq 2n - 3 \), where \( 1 \leq b \leq n - 5 \).

Next, we use the same analysis as those for Case (i) or Subcase (ii.i) to derive the property of the 2-matching of the graphs considered.

**Subcase (ii.ii).** \( b = 0, 1 \leq a, c \leq \frac{n-4}{2} \) and \( c = n - 4 - a \).
By (8), we have \( m(F_{n+1}^{a,b,c}, 2) = (a + 2)(n - 3 - a) + a + 1 \geq f_3(1) = 3n - 10 \geq 2n - 3 \).
Subcase (ii.iii). \( c = 0, 0 \leq a, b \leq n - 4 \) and \( a = n - 4 - b \).
Subcase (ii.iii) is the same as that for Subcase (ii.i).

Subcase (ii.iv). \( 1 \leq a, c \leq \left\lceil \frac{n-4}{2} \right\rceil, 1 \leq b \leq n - 6 \) and \( c = n - 4 - a - b \).
If \( a = 1 \), then by (8), we obtain \( m(D_{n+1}^{1,b,c}) = (b + 1)(n - 4 - b) + 2(n - 3) \geq f_4(b) \geq \min\{f_3(1), f_4(n - 6)\} = 4n - 16 \geq 2n - 3 \). If \( 2 \leq a \leq \left\lceil \frac{n-5}{2} \right\rceil \), then by (8), we get

\[
m(D_{n+1}^{a,b,c}) = (a + 1)(n - a - 2) + b(n - 3 - a - b) + n - 3 - a - b
\geq (a + 1)(n - a - 2) \geq f_3(a) \geq f_3(2) = 3(n - 4) \geq 2n - 3.
\]

By the proofs of Subcases (ii.i)--(ii.iv), we have \( m(\widehat{G}, 2) \geq m(D_{n+1}^{a,b,c}) \geq 2n - 3 \) if \( \widehat{G} \neq D_{n+1}^{0,0,0,4}, F_{n+1}^{0,0,4,0} \).

Case (iii). \( \text{diam}(\widehat{G}) = 5 \).

In this case, \( \widehat{G} \) may be viewed as a tree obtained from \( P_6 \) by attaching trees \( T_1, T_2, T_3 \), and \( T_4 \) at \( v_2, v_3, v_4 \), and \( v_5 \) respectively. Let \( |E(T_1)| = a, |E(T_2)| = b, |E(T_3)| = c, \) and \( |E(T_4)| = d \), where \( 0 \leq a, b, c, d \leq n - 5 \) and \( a + b + c + d = n - 5 \). By using Lemma 2.4 repeatedly, we have

\[
m(\widehat{G}, 2) \geq m(C_{n+1}^{a,b,c,d}, 2),
\]

where \( C_{n+1}^{a,b,c,d} \) is the graph obtained from \( P_6 \) by attaching \( a, b, c, \) and \( d \) pendent edges to \( v_2, v_3, v_4, \) and \( v_5 \) respectively. By the methods similar to those for Case (iii), we can obtain \( m(\widehat{G}, 2) \geq m(C_{n+1}^{a,b,c,d}, 2) \geq 2n - 3 \).

Case (iv). \( \text{diam}(\widehat{G}) \geq 6 \).

In this case, \( \widehat{G} \) can be viewed as a connected tree obtained from \( P_{d+1} \) by attaching trees, say \( T_1, \ldots, T_{d-1} \), at \( v_2, \ldots, v_{d} \) respectively. By using Lemma 2.4 repeatedly and by the proof of Case (iii), we obtain

\[
m(\widehat{G}, 2) \geq m(C_{n+1}^{a,b,c,d}, 2) \geq 2n - 3.
\]

By combining the proofs of Cases (i)--(iv), we get Lemma 3.4. \( \square \)

For \( I = 4 \), some graphs are introduced as follows.

For \( n \geq 6, A_n \) is the graph obtained from \( C_4 \) by attaching one and \( n - 5 \) pendent edges to \( u_1 \) and \( u_2 \) respectively.

For \( n \geq 7, B_n \) is the graph obtained from \( C_4 \) by attaching a path \( P_3 \) to \( u_1 \) and \( n - 6 \) pendent edges to \( u_5 \).

For \( n \geq 6, C_n \) is the graph obtained from \( C_4 \) by identifying \( u_1 \) of \( C_4 \) with a pendent vertex of \( X_{n-3} \).

For \( n \geq 5, D_n^a \) is the graph obtained from \( C_4 \) by attaching \( a \) and \( n - a - 4 \) pendent edges to \( u_1 \) and \( u_3 \) respectively, where \( 0 \leq a \leq \left\lceil \frac{n-5}{2} \right\rceil \).

Lemma 3.5. Let \( G \in \mathcal{U}_{2n} \) and \( n \geq 10 \). If \( \widehat{G} \in \mathcal{H}_n \) and \( \widehat{G} \neq D_n^0, H_n, C_n, C_n^0, C_n^1 \), then \( m(\widehat{G}, 2) \geq 2n - 3 \).

Proof. Let \( \widehat{G} \in \mathcal{H}_n \) and \( n \geq 10 \), where \( G \in \mathcal{U}_{2n} \). Suppose that the cycle contained in \( \widehat{G} \) is \( C_4 \). Three cases are considered as follows.

Case (i). \( l \geq 5 \).
If \( \widehat{G} = Z_n \), then \( m(\widehat{G}, 2) = 3n - 10 > 2n - 3 \). If \( \widehat{G} \neq Z_n \), then by Lemma 2.6, we have \( m(\widehat{G}, 2) \geq 3n - 12 \geq 2n - 3 \).

Case (ii). \( l = 4 \).
If \( \widehat{G} \neq A_n, B_n, C_n, D_n^a \), then we can choose an edge \( e \) on \( C_4 \) of \( \widehat{G} \) such that \( \widehat{G} - e \neq O_n \) and \( \text{diam}(\widehat{G} - e) \geq 5 \).
Therefore, by Lemma 2.6, we have \( m(\widehat{G}, 2) \geq 3n - 12 \geq 2n - 3 \). By direct calculation, we get \( m(A_n, 2) = 3n - 11 \geq 2n - 3 \) and \( m(B_n, 2) = m(C_n, 2) = 4n - 16 \geq 2n - 3 \). By (8), we have

\[
m(D_n^a, 2) = (n - 3 - a)(a + 2) + a - f_6(a).
\]

Thus, \( f_6(a) \) is an increasing function of \( a \) since \( f'_6(a) = -2a + n - 4 \geq 0 \) for \( 0 \leq a \leq \left\lceil \frac{n-4}{2} \right\rceil \). If \( \widehat{G} = D_n^a \), then \( m(D_n^a, 2) = f_6(0) = 2n - 6 < 2n - 3 \). If \( \widehat{G} = D_n^a \) with \( 1 \leq a \leq \left\lceil \frac{n-4}{2} \right\rceil \), then \( m(D_n^a, 2) = f_6(a) \geq f_6(1) = 3n - 11 \geq 2n - 3 \).

Case (iii). \( l = 3 \).
If $\tilde{G} \notin \mathcal{K}$, then by Lemma 2.6, we have $m(\tilde{G}, 2) \geq 3n - 12 \geq 2n - 3$. When $n \geq 10$, we obtain $m(Q_n, 2) = m(K_{n, 2}) = 3n - 11 > 2n - 3$, $m(S_n, 2) = 4n - 18 > 2n - 3$, $m(T_n, 2) = 4n - 16 > 2n - 3$, $m(W_n, 2) = 4n - 17 > 2n - 3$, $m(Y_n, 2) = 3n - 12 > 2n - 3$, and $m(H_n, 2) = 2n - 6 < 2n - 3$. If $\tilde{G} = C_n^1$, then by (8), we have
\[
m(C_n^1, 2) = (n - 3 - a)(a + 1) + a \implies f_2(a).
\] (22)
Thus, $f_2(a)$ is an increasing function of $a$ since $f_2(a) = -2a + n - 3 \geq 0$ for $0 \leq a \leq \lfloor \frac{n-3}{2} \rfloor$. If $\tilde{G} = C_n^3$, then $m(C_n^3, 2) = f_2(0) = n - 3 < 2n - 3$. If $\tilde{G} = C_n^2$, then $m(C_n^2, 2) = f_2(1) = 2n - 7 < 2n - 3$. If $\tilde{G} = C_n^1$ with $2 \leq a \leq \lfloor \frac{n-3}{2} \rfloor$, then $f_2(a) \geq f_2(0) = 3n - 13 \geq 2n - 3$.

By the proofs of Cases (i)–(iii), we obtain that if $\tilde{G} \neq D_n^0, H_n, C_n^0, C_n^1$, then $m(\tilde{G}, 2) \geq 2n - 3$ for $n \geq 10$. □

Let the number of the components of $\tilde{G}$ be $o(\tilde{G})$.

**Lemma 3.6.** Let $G \in \mathcal{U}_{2n}$ and $n \geq 10$. If $o(\tilde{G}) \geq 3$ and each component of $\tilde{G}$ is a tree, then $m(\tilde{G}, 2) \geq 2n - 3$.

**Proof.** If $o(\tilde{G}) \geq 4$, then we concatenate $\tilde{G}$ together into a graph (denoted by $\tilde{G}$) in such a way that $o(\tilde{G}) = 3$. Obviously, $m(\tilde{G}, 2) > m(\tilde{G}, 2)$. We suppose that the numbers of vertices for the three components of $G$ are $a + 1, b + 1$ and $n - a - b + 1$. Using Lemma 2.4 repeatedly, we obtain $m(\tilde{G}, 2) \geq m(X_{a+1} \cup X_{b+1} \cup X_{n-a-b+1}, 2)$.

Suppose $1 \leq a \leq b$ and $a + b \leq n - 1$.

If $a = 1$, then we suppose $1 \leq b \leq \lfloor \frac{n-1}{2} \rfloor$. By (8), we have $m(X_2 \cup X_{b+1} \cup X_{n-a}, 2) = b(n - 1 - b) + n - 1 \implies f_2(b) \geq f_2(b) = 2n - 3$ since $f_2(b)$ is an increasing function of $b$ for $1 \leq b \leq \lfloor \frac{n-1}{2} \rfloor$. If $a \geq 2$, then by (8), we have
\[
m(X_{a+1} \cup X_{b+1} \cup X_{n-a-b+1}, 2) = m(X_{a+1} \cup X_{n-a+1}, 2) + b(n - a - b) \geq m(X_{a+1} \cup X_{n-a+1}, 2) + 2 \geq 2n - 3.
\] (23)
Therefore, we obtain Lemma 3.6. □

**Lemma 3.7.** Let $G \in \mathcal{U}_{2n}$ with $n \geq 10$. If $\tilde{G}$ is unconnected and $\tilde{G} \neq X_2 \cup C_n^1, X_2 \cup X_n, X_3 \cup X_{n-1}, X_2 \cup X_n^0$, then $m(\tilde{G}, 2) \geq 2n - 3$.

**Proof.** Let $G \in \mathcal{U}_{2n}$ and $n \geq 10$. Two cases are considered according to the types of the capped graph $\tilde{G}$.

**Case (i).** $\tilde{G}$ is an unconnected graph whose components are trees and a cycle.

If $\tilde{G}$ is composed of trees and a cycle, then we might concatenate them together into a connected unicyclic graph (denoted by $\tilde{G}_1$). If $\tilde{G} \notin \{D_n^0, H_n, C_n^0, C_n^1\}$, then by Lemma 3.5, $m(\tilde{G}, 2) > m(\tilde{G}_1, 2) \geq 2n - 3$. Otherwise, if $\tilde{G} \in \{D_n^0, H_n, C_n^0, C_n^1\}$, then four subcases are considered as follows.

**Subcase (i.1).** $\tilde{G}_1 = D_n^0$.

If $o(\tilde{G}) = 2$, then $\tilde{G} = X_{a+1} \cup D_n^0$ with $1 \leq a \leq n - 4$. By (8), we have
\[
m(X_{a+1} \cup D_n^0, 2) = (a + 2)(n - a) - 6 \implies f_2(a) \geq \min\{f_2(1), f_2(n-4)\} = 3n - 9 \geq 2n - 3.
\] (24)

If $o(\tilde{G}) \geq 3$, then we can concatenate $\tilde{G}$ into a graph with two components $X_{a+1} \cup D_{n-a}^0$, where $2 \leq a \leq n-4$. Thus, by (24), we get $m(\tilde{G}, 2) \geq m(X_{a+1} \cup D_{n-a}^0, 2) \geq 2n - 3$.

**Subcase (i.2).** $\tilde{G}_1 = C_n^0$.

If $o(\tilde{G}) = 2$, then $\tilde{G} = X_{a+1} \cup C_n^0$ with $1 \leq a \leq n - 3$. For $a = 1$, we have
\[
m(X_2 \cup C_n^0, 2) = 2n - 5 < 2n - 3.
\] (25)

For $2 \leq a \leq n - 3$, by (8), we get
\[
m(X_{a+1} \cup C_n^0, 2) = (a + 1)(n - a) - 3 \implies f_{10}(a) \geq \min\{f_{10}(2), f_{10}(n-3)\} = 3n - 9 \geq 2n - 3.
\] (26)
If \( \hat{G} \) is a graph in the plane with two components, where \( 2 \leq a \leq n-3 \). Therefore, by (26), we get \( m(\hat{G}, 2) \geq m(X_{a+1} \cup C_{n-a}^0) \geq 2n - 3 \).

**Subcase (i.iii).** \( \hat{G}_2 = H_n \).

If \( \hat{G} = 2 \), then \( \hat{G} \) may be one among the four types: \( X_2 \cup C_{n-1}^0, X_3 \cup C_{n-2}^0, X_{a+1} \cup H_{n-a} \) with \( 1 \leq a \leq n-6 \), and \( E_{n-a}^{0, a} \cup C_{n-a}^0 \) with \( 1 \leq a \leq n-5 \). By (25) and (26), we have \( m(X_2 \cup C_{n-1}^0) < 2n - 3 \) and \( m(X_3 \cup C_{n-2}^0) \geq 2n - 3 \) respectively. By (8), we get

\[
m(X_{a+1} \cup H_{n-a}, 2) = (a+2)(n-a) - 6 \leq f_{11}(a)
\geq \min\{f_{11}(1), f_{11}(n-5)\} = 3n - 9 \geq 2n - 3,
\]

\[
m(E_{a+1}^{0, a-1} \cup C_{n-a}^0, 2) = (a+2)(n-a-2) + n - 5 \leq f_{12}(a)
\geq \min\{f_{12}(1), f_{12}(n-5)\} = 4n - 14 \geq 2n - 3.
\]

If \( \hat{G} \geq 3 \), then we concatenate \( \hat{G} \) into a graph with two components such that the graph is not \( X_2 \cup C_{n-1}^0 \).

By the proof above, we get \( m(\hat{G}, 2) \geq 2n - 3 \).

**Subcase (i.iv).** \( \hat{G}_1 = C_n \).

If \( \hat{G} = 2 \), then \( \hat{G} \) may be \( X_2 \cup C_{n-1}^0 \) or \( X_{a+1} \cup C_{n-a}^0 \) with \( 1 \leq a \leq n-4 \). By (25), we have \( m(X_2 \cup C_{n-1}^0, 2) < 2n - 3 \). By (8), for \( 1 \leq a \leq n-4 \), we get

\[
m(X_{a+1} \cup C_{n-a}^0, 2) = (a+2)(n-a) - 7 \leq f_{13}(a)
\geq \min\{f_{13}(1), f_{13}(n-4)\} = 3n - 10 \geq 2n - 3.
\]

If \( \hat{G} \geq 3 \), then we concatenate \( \hat{G} \) into a graph with two components such that it is not \( X_2 \cup C_{n-1}^0 \).

Therefore, by (27), we get \( m(\hat{G}, 2) \geq 2n - 3 \).

By combining the proofs of Subcases (i.i)–(i.iv), we have \( m(\hat{G}, 2) \geq 2n - 3 \) if \( \hat{G} \neq X_2 \cup C_{n-1}^0 \).

**Case (ii).** \( \hat{G} \) is an unconnected graph whose components are trees only.

If \( \hat{G} \geq 3 \), then by Lemma 3.6, we have \( m(\hat{G}, 2) \geq 2n - 3 \). Next, we always assume \( \hat{G} = 2 \). We concatenate \( \hat{G} \) together into a tree, denoted by \( \hat{G}_2 \). Obviously, \( \hat{G}_2 \) in \( T_{n+1} \). If \( \hat{G}_2 \in \{X_{n+1}, E_{n-a}^{0, a-1} \cup C_{n-a}^0 \} \), then by Lemma 3.4, \( m(\hat{G}, 2) \geq m(\hat{G}_2, 2) \geq 2n - 3 \). Otherwise, if \( \hat{G}_2 \in \{X_{n+1}, E_{n-a}^{0, a-1} \cup C_{n-a}^0 \} \), then five subcases are considered as follows.

**Subcase (i.i).** \( \hat{G}_2 = X_{n+1} \).

As \( \hat{G} = 2 \), \( \hat{G} = X_{a+1} \cup X_{n-a+1} \) with \( 1 \leq a \leq \lfloor \frac{n}{2} \rfloor \). By (8), we have \( m(X_{a+1} \cup X_{n-a+1}, 2) = a(n-a) \pm f_{14}(a) \).

Thus, \( f_{14}(a) \) is an increasing function of \( a \) since \( f_{14}(a) \geq 0 \) for \( 1 \leq a \leq \lfloor \frac{n}{2} \rfloor \).

We have

\[
m(X_2 \cup X_n, 2) = f_{14}(1) = n - 1 < 2n - 3 \tag{28}
\]

\[
m(X_3 \cup X_{n-1}, 2) = f_{14}(2) = 2n - 4 < 2n - 3 \tag{29}
\]

\[
m(X_{a+1} \cup X_{n-a+1}, 2) = f_{14}(a) \geq f_{14}(3) = 3(n-3) \geq 2n - 3 \tag{30}
\]

for \( 3 \leq a \leq \lfloor \frac{n}{2} \rfloor \).

**Subcase (i.ii).** \( \hat{G}_2 = E_{n-a}^{0, a-1} \).

As \( \hat{G} = 2 \), \( \hat{G} \) may be one among the three types: \( X_2 \cup X_n, X_3 \cup X_{n-1} \) and \( X_{a+1} \cup E_{n-a}^{0, a-1} \) with \( 1 \leq a \leq n-3 \). By (28) and (29), we only need to consider the last type. By (8), we have \( m(X_{a+1} \cup E_{n-a}^{0, a-1}, 2) = a(n-a) + n - 2 - a = (a+1)(n-a) - 2 \pm f_{15}(a) \).

If \( a = 1 \), then we have

\[
m(X_2 \cup E_{1}^{0, n-4}, 2) = f_{15}(1) = 2n - 4 < 2n - 3. \tag{31}
\]

If \( 2 \leq a \leq n-3 \), then we get

\[
m(X_{a+1} \cup E_{n-a+1}^{0, n-4}, 2) = f_{15}(a) \geq \min\{f_{15}(2), f_{15}(n-3)\} = 3n - 8 \geq 2n - 3. \tag{32}
\]
Lemma 3.9. Let \( G = \mathcal{U}_{2n} \) with \( n \geq 10 \) and \( S \) its signed graph. Let \( l \) be the girth of \( G \) and \( r, j \) be integers with \( r, j \geq 1 \).

(i) For \( l = 2r + 1 \), if \( G \notin \mathcal{A}^* \), then \( \mathcal{E}(F_{2r+1}) < \mathcal{E}(G) = \mathcal{E}(G^*) = \mathcal{E}(G^-) \);

(ii) For \( l = 4j + 2 \), if \( S \) is balanced and \( S \notin \mathcal{A}^* \), then \( \mathcal{E}(F_{4j+2}) < \mathcal{E}(S) \);

(iii) For \( l = 4j \), if \( S \) is unbalanced and \( S \notin \mathcal{A}^* \), then \( \mathcal{E}(F_{4j}) < \mathcal{E}(S) \).
Proof. Obviously, \( \overline{F}_{2n,1} = D^0_n \). It is noted that \( m(D^0_n, 2) = 2n - 6 \). Since each 2-matching of \( D^0_n \) is adjacent to four edges of \( M(F^0_{2n, i}) \) and \( m(D^0_n, i) = 0 \) for \( 3 \leq i \leq n \), we have

\[
m(F^0_{2n, i}, k) = p + (2n - 6)\binom{n - 4}{k - 2}.
\]

From Lemma 3.1 and the fact that \( F^0_{2n,1} - C^+_4 \) is composed of \( n - 4 \) independent edges and four isolated vertices, we obtain

\[
b_2(F^0_{2n,1}) = m(F^0_{2n,1}, k) + 2m(F^0_{2n,1} - C^+_4, k - 2) \leq m(F^0_{2n,1}, k) + 2\binom{n - 4}{k - 2} = p + (2n - 4)\binom{n - 4}{k - 2}
\]

(i) \( l = 2r + 1 \).

By Lemma 3.1, we have

\[
b_2(S) = m(S, k) = p + m(S, 2)\binom{n - j}{k - 2} + \sum_{i=3}^{n} m(S, i)\binom{n - i}{k - i} \geq p + m(S, 2)\binom{n - 4}{k - 2}.
\]

If \( G \notin \mathcal{A} \), then by Lemma 3.8, we have \( m(S, 2) \geq 2n - 3 \) for \( n \geq 10 \), where \( S \) is the signed graph of \( G \). Therefore, it follows from (37) and (38) that \( b_2(F^0_{2n,1}) \leq b_2(S) \), where the equality does not hold for \( 0 \leq k \leq \lfloor n/2 \rfloor \). For example, \( b_4(F^0_{2n,1}) < b_4(S) \). Furthermore, from Lemma 3.1, we have \( b_{2k+1}(F^0_{2n,1}) = 0 \leq b_{2k+1}(S) \) for \( 0 \leq k \leq \lfloor n/2 \rfloor \). Thus, by (4) and Theorem 3.2, we get Lemma 3.9(i).

By the methods similar to those for \( l = 2r + 1 \), we have Lemma 3.9(ii) and (iii). \( \square \)

### 3.4. Comparing the energies of two signed graphs among \( \mathcal{A}^+ \cup \mathcal{A}^- \)

To obtain our final results, we need to compare their energies for the signed graphs in \( \mathcal{A}^+ \cup \mathcal{A}^- \). The method of coefficient comparison in (4) will be used as the first choice. For two unicyclic signed graphs in \( \mathcal{A}^+ \cup \mathcal{A}^- \), if their coefficients of the corresponding characteristic polynomials are incomparable, then their energies are directly performed on the basis of (2) by using the theorem of zero points, the integral formula in (14), and analytical techniques for the integral formula. In this section, the characteristic polynomial \( \phi_S(x) \) of \( S \) is rewritten by \( \phi(S, x) \).

Lemma 3.10 obtained by Li and Li [15] and Lemmas 3.11 and 3.12 derived by Wang [20] are simply quoted here.

**Lemma 3.10.** [15] If \( n \geq 5 \), then \( M_{2n,2}^+ \rightarrow A_{2n,1}^- \).

**Lemma 3.11.** [20] \( A_{2n,1} \rightarrow B_{2n,1}^- \) for \( n \geq 7 \) while \( B_{2n,1}^+ \rightarrow A_{2n,1}^- \) for \( 3 \leq n \leq 6 \).

**Lemma 3.12.** [20] \( B_{2n,1}^+ \rightarrow G_{2n,1} \rightarrow B_{2n,2}^- \) for \( n \geq 4 \).

**Lemma 3.13.** \( G_{2n,1} \rightarrow B_{2n,4}^- \) for \( n \geq 4 \).

**Proof.** By (7), we get

\[
\phi(G_{2n,1}, x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (n + 1)x^4 + 2x^3 - (n + 4)x^2 + 1],
\]

(39)

\[
\phi(B_{2n,4}, x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 4)x^4 + 6x^3 - (n + 4)x^2 - 2x + 1].
\]

(40)

By (39), (40) and (4), we have \( G_{2n,1} \rightarrow B_{2n,4}^- \) for \( n \geq 4 \). \( \square \)
Lemma 3.14. $B_{2n,4} \to B_{2n,2}$ for $n \geq 4$.

Proof. By (7), we obtain

$$\phi(B_{2n,4},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 4)x^4 + 6x^3 - (n + 4)x^2 - 2x + 1],$$

$$\phi(B_{2n,2},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 4)x^4 + 4x^3 - (n + 5)x^2 - 2x + 1].$$

In (14), let $S_1 = B_{2n,4}$ and $S_2 = B_{2n,2}$. Therefore, by (14), we obtain

$$E(B_{2n,4}) - E(B_{2n,2}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{p_1^2(x) + q_1^2(x)}{p_2^2(x) + q_2^2(x)} \, dx,$$

(41)

where $p_1(x) = x^8 + (n + 4)x^6 + (3n + 4)x^4 + (n + 4)x^2 + 1$, $q_1(x) = 2x^5 + 6x^3 + 2x$, $p_2(x) = x^8 + (n + 4)x^6 + (3n + 4)x^4 + (n + 5)x^2 + 1$, and $q_2(x) = 2x^5 + 4x^3 + 2x$. For $n \geq 4$, we get $[p_1^2(x) + q_1^2(x)] - [p_2^2(x) + q_2^2(x)] = -x^2[2x^8 + 2nx^6 + (6n - 12)x^4 + (2n + 1)x^2 + 2] \leq 0$. Therefore, it follows from (41) that $B_{2n,4} \to B_{2n,2}$ for $n \geq 4$. □

Lemma 3.15. $B_{2n,2} \to I_{2n,3}$ for $n \geq 4$.

Proof. By (7), we deduce

$$\phi(B_{2n,2},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 4)x^4 + 4x^3 - (n + 5)x^2 - 2x + 1],$$

(42)

$$\phi(I_{2n,3},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 5)x^4 + 6x^3 - (n + 5)x^2 - 2x + 1].$$

(43)

By (42), (43) and (4), we get $B_{2n,2} \to I_{2n,3}$ for $n \geq 4$. □

Lemma 3.16. $I_{2n,3} \to B_{2n,1}^{+}$ for $n \geq 4$.

Proof. It follows from (7) that

$$\phi(I_{2n,3},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (3n + 5)x^4 + 6x^3 - (n + 5)x^2 - 2x + 1],$$

(44)

$$\phi(B_{2n,1}^{+},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 + (3n + 6)x^4 - (n + 7)x^2 + 1].$$

(45)

In (14), let $S_1 = I_{2n,3}$ and $S_2 = B_{2n,1}^{+}$. Therefore, by (14), we obtain

$$E(I_{2n,3}) - E(B_{2n,1}^{+}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{p_1^2(x) + q_1^2(x)}{p_2^2(x)} \, dx,$$

(46)

where $p_1(x) = x^8 + (n + 4)x^6 + (3n + 5)x^4 + (n + 5)x^2 + 1$, $q_1(x) = 2x^5 + 6x^3 + 2x$ and $p_2(x) = x^8 + (n + 4)x^6 + (3n + 6)x^4 + (n + 7)x^2 + 1$. Since $[p_1^2(x) + q_1^2(x)] - p_2^2(x) = -x^2[2x^8 + 2(n + 8)x^6 + (10n + 3)x^4 + (14n - 10)x^2 + 4n + 2] \leq 0$ for $n \geq 4$, it follows from (46) that $I_{2n,3} \to B_{2n,1}^{+}$ for $n \geq 4$. □

Lemmas 3.17 and 3.18 are simply quoted, which are derived by Wang [20] and Zhu [25] respectively.

Lemma 3.17. [20] $B_{2n,3} \to F_{2n,1}^{+}$ for $n \geq 45$ while $F_{2n,1}^{+} \to B_{2n,3}$ for $4 \leq n \leq 44$.

Lemma 3.18. [25] If $n \geq 4$, then $B_{2n,2} \to I_{2n,1}$.

Lemma 3.19. $B_{2n,1}^{+} \to I_{2n,2}^{+}$ for $n \geq 43$ while $I_{2n,2}^{+} \to B_{2n,1}$ for $4 \leq n \leq 42$.

Proof. By (7), we have

$$\phi(B_{2n,1}^{-},x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 + (3n + 6)x^4 - (n + 7)x^2 + 1]$$

$$\pm (x^2 - 1)^{n-4}g_1(x),$$

(47)

$$\phi(I_{2n,2}^{-},x) = x^2(x^2 - 1)^{n-3}[x^4 - (n + 3)x^2 + 2n]$$

$$\pm x^2(x^2 - 1)^{n-3}g_2(x).$$

(48)
Obviously, we get

\[ g_1(1 - \frac{1}{2n}) = \frac{1 - 8n + 22n^2 - 44n^3 + 8n^4}{16n^4} > 0 \quad (n \geq 5), \]

\[ g_1(1 - \frac{1}{n}) = \frac{1 - 4n + 5n^2 - 4n^3}{n^4} < 0 \quad (n \geq 4), \]

\[ g_1(\sqrt{0.38}) = -0.992237 - 0.001672n < 0 \quad (n \geq 4), \]

\[ g_1(\sqrt{0.385}) = -1.01195 + 0.00260837n > 0 \quad (n \geq 388), \]

\[ g_1(\sqrt{2.61}) = -1.11102 + 0.046719n > 0 \quad (n \geq 24), \]

\[ g_1(\sqrt{2.62}) = -0.972525 - 0.011528n < 0 \quad (n \geq 4), \]

\[ g_1(\sqrt{n}) = 1 - 7n + 5n^2 - n^3 < 0 \quad (n \geq 4), \]

\[ g_1(\sqrt{n + 1}) = -3 - 2n + 2n^2 > 0 \quad (n \geq 4). \]

According to the theorem of zero points, we obtain

\[ E(B_{2n,1}^{-}) = 2(n - 4) + \sum_{i=1}^{n} |x_i| < 2(n - 4) + 2(\sqrt{\frac{1}{n}} + \sqrt{0.385} + \sqrt{2.62} + \sqrt{n + 1}), \quad (n \geq 388). \]  \hspace{1cm} (49)

Since \( g_2(\sqrt{1.99}) = -2.0099 + 0.01n > 0 \) for \( n \geq 201 \), \( g_2(\sqrt{2}) = -2 < 0 \), \( g_2(\sqrt{n + 1}) = -2 < 0 \), and \( g_2(\sqrt{n + 2}) = -2 + n > 0 \) for \( n \geq 3 \), we have

\[ 2(n - 3) + 2(\sqrt{1.99} + \sqrt{n + 1}) < E(J_{2n,2}^{+}), \quad (n \geq 201). \]  \hspace{1cm} (50)

It follows from \( 2(n - 4) + 2(\sqrt{\frac{1}{n}} + \sqrt{0.385} + \sqrt{2.62} + \sqrt{n + 1}) < 2(n - 3) + 2(\sqrt{1.99} + \sqrt{n + 1}) \) that the right-handed side of (49) is less than the left-handed side of (50) for \( n \geq 388 \). Therefore, \( B_{2n,1}^{-} \rightarrow J_{2n,2}^{+} \) for \( n \geq 388 \). The calculation yields \( B_{2n,1}^{-} \rightarrow J_{2n,2}^{+} \) for \( 43 \leq n \leq 387 \) while \( J_{2n,1}^{+} \rightarrow B_{2n,1}^{-} \) for \( 4 \leq n \leq 42 \). \( \Box \)

**Lemma 3.20.** \( J_{2n,2}^{+} \rightarrow B_{2n,3} \) for \( n \geq 3 \).

**Proof.** By (7), we have

\[ \phi(J_{2n,2}^{+}, x) = x^2(x^2 - 1)^{n-3}[x^4 - (n + 3)x^2 + 2n], \]  \hspace{1cm} (51)

\[ \phi(B_{2n,3}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 - 2x^3 + (2n + 1)x^2 + 2x - 1]. \]  \hspace{1cm} (52)

It follows from (51), (52) and (4) that \( J_{2n,2}^{+} \rightarrow B_{2n,3} \) for \( n \geq 3 \). \( \Box \)

**Lemma 3.21.** \( B_{2n,3} \rightarrow J_{2n,4} \) for \( n \geq 3 \).

**Proof.** By (7), we get

\[ \phi(B_{2n,3}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 - 2x^3 + (2n + 1)x^2 + 2x - 1], \]  \hspace{1cm} (53)

\[ \phi(J_{2n,4}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 - 2x^3 + (2n + 2)x^2 + 4x - 1]. \]  \hspace{1cm} (54)

It follows from (53), (54) and (4) that \( B_{2n,3} \rightarrow J_{2n,4} \) for \( n \geq 3 \). \( \Box \)

**Lemma 3.22.** \( J_{2n,4} \rightarrow J_{2n,2}^{-} \) for \( n \geq 3 \).
Proof. By (7), we obtain

\[
\phi(J_{2n,4}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 - 2x^3 + (2n + 2)x^2 + 4x - 1],
\]

\[\phi(J_{2n,2}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 + (2n + 4)x^2 - 4].\]  

In (14), let \(S_1 = J_{2n,4}\) and \(S_2 = J_{2n,2}\). Therefore, by (14), we obtain

\[\mathcal{E}(J_{2n,4}) - \mathcal{E}(J_{2n,2}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{p_1^2(x) + q_1^2(x)}{p_2^2(x)} dx,\]  

where \(p_1(x) = x^6 + (n + 3)x^4 + (2n + 2)x^2 + 1, q_1(x) = 2x^3 + 4x\) and \(p_2(x) = x^6 + (n + 3)x^4 + (2n + 4)x^2 + 4\). For \(n \geq 3\), we have \([p_1^2(x) + q_1^2(x)] - p_2^2(x) = -4x^6 - (4n + 14)x^6 - (14n + 14)x^4 - (12n + 12)x^2 - 15 < 0\). Therefore, by (57), we get \(J_{2n,4} \rightarrow J_{2n,2}\) for \(n \geq 3\). \(\Box\)

**Lemma 3.23.** \(J_{2n,2} \rightarrow C_{2n,1}\) for \(n \geq 721\) while \(C_{2n,1} \rightarrow J_{2n,2}\) for \(4 \leq n \leq 720\).

**Proof.** By (7), we deduce

\[
\phi(J_{2n,2}, x) = (x^2 - 1)^{n-3}[x^6 - (n + 3)x^4 + (2n + 4)x^2 - 4],
\]

\[\phi(C_{2n,1}, x) = (x^2 - 1)^{n-3}[x^8 - (n + 4)x^6 + (4n - 2)x^4 - (n + 3)x^2 + 1].\]  

By (14), we obtain

\[\mathcal{E}(J_{2n,2}) - \mathcal{E}(C_{2n,1}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \log f_1(n, x) dx = - \frac{2}{\pi} \int_{0}^{\infty} \log f_1(n, x) dx,\]  

where

\[f_1(n, x) = \frac{(x^2 + 1)[x^6 + (n + 3)x^4 + (2n + 4)x^2 + 4]}{x^8 + (n + 4)x^6 + (4n - 2)x^4 + (n + 3)x^2 + 1}.\]  

Then

\[
\frac{\partial f_1(n, x)}{\partial n} = -\frac{x^2(x^2 + 2)(x^2 + 1)(x^6 + 9x^4 + 6x^2 + 1)}{(x^8 + (n + 4)x^6 + (4n - 2)x^4 + (n + 3)x^2 + 1)^2} \leq 0.
\]

Thus \(f_1(n, x)\) is a decreasing function of \(n\). When \(n \geq 721\), it follows from (60) and (62) that

\[\mathcal{E}(J_{2n,2}) - \mathcal{E}(C_{2n,1}) = \frac{2}{\pi} \int_{0}^{\infty} \log f_1(n, x) dx \leq \frac{2}{\pi} \int_{0}^{\infty} \log f_1(721, x) dx \leq \frac{2}{\pi} \times (-0.0000107) < 0.
\]

It follows from (63) that \(J_{2n,2} \rightarrow C_{2n,1}\) for \(n \geq 721\) while the calculation yields \(C_{2n,1} \rightarrow J_{2n,2}\) for \(4 \leq n \leq 720\). \(\Box\)

**Lemma 3.24.** \(C_{2n,1} \rightarrow F_{2n,1} \rightarrow D_{2n,2} \rightarrow I_{2n,1}\) for \(n \geq 4\).

**Proof.** By (7), we get

\[
\phi(C_{2n,1}', x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 + (4n - 2)x^4 - (n + 3)x^2 + 1],
\]

\[\phi(F_{2n,1}', x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 + (4n - 2)x^4 + (n + 4)x^2 + 1],\]

\[\phi(D_{2n,2}', x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 + (4n - 1)x^4 - (n + 4)x^2 + 1],\]

\[\phi(I_{2n,1}', x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (4n - 1)x^4 + 2x^3 - (n + 4)x^2 + 1].\]  

By (64)–(67) and (4), we get Lemma 3.24. \(\Box\)
Lemma 3.25. $I_{2n,1} \rightarrow F_{2n,1}^{-}$ for $n \geq 4$.

Proof. By (7), we have

$$\phi(I_{2n,1}, x) = (x^2 - 1)^{n-1}[x^8 - (n + 4)x^6 - 2x^5 + (4n - 1)x^4 + 2x^3 - (n + 4)x^2 + 1],$$

$$\phi(F_{2n,1}^{-}, x) = (x^2 - 1)^{n-1}[x^8 - (n + 4)x^6 + 4n + 2)x^4 - (n + 4)x^2 + 1].$$

In (14), let $S_1 = I_{2n,1}$ and $S_2 = F_{2n,1}^{-}$. By (14), we get

$$\mathcal{E}(I_{2n,1}) - \mathcal{E}(F_{2n,1}^{-}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{p_1^2(x) + q_1^2(x)}{p_2^2(x)} dx,$$

where $p_1(x) = x^8 + (n + 4)x^6 + (4n - 1)x^4 + (n + 4)x^2 + 1$, $q_1(x) = 2x^5 + 2x^3$, and $p_2(x) = x^8 + (n + 4)x^6 + (4n + 2)x^4 + (n + 4)x^2 + 1$. For $n \geq 4$, since $[p_1^2(x) + q_1^2(x)] - p_2^2(x) = -x^4(6x^8 + (6n + 20)x^6 + (24n - 5)x^4 + (6n + 20)x^2 + 6) \leq 0$, it follows from (70) that $I_{2n,1} \rightarrow F_{2n,1}^{-}$ for $n \geq 4$. □

Lemma 3.26. $F_{2n,1}^{-} \Rightarrow L_{2n,1}^{-}$ for $n \geq 4$.

Proof. By (7), we obtain $\phi(F_{2n,1}^{-}, x) = (x^2 - 1)^{n-1}[x^8 - (n + 4)x^6 + (4n + 2)x^4 - (n + 4)x^2 + 1] = \phi(L_{2n,1}^{-}, x)$. Obviously, we get Lemma 3.26. □

Lemma 3.27. $F_{2n,1}^{-} \rightarrow E_{2n,3}^{-}$ for $n \geq 35$ while $E_{2n,3}^{+} \rightarrow F_{2n,1}^{-}$ for $6 \leq n \leq 34$.

Proof. By (7), we get

$$\phi(F_{2n,1}^{-}, x) = (x^2 - 1)^{n-1}[x^8 - (n + 4)x^6 + (4n + 2)x^4 - (n + 4)x^2 + 1],$$

$$\phi(E_{2n,3}^{+}, x) = (x^2 - 1)^{n-1}[x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n - 1)x^6 + (6n + 5)x^4 - (n + 5)x^2 + 1].$$

By (14), we obtain

$$\mathcal{E}(F_{2n,1}^{-}) - \mathcal{E}(E_{2n,3}^{+}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log f_2(n, x) dx = \frac{2}{\pi} \int_{0}^{\infty} \log f_2(n, x) dx,$$

where

$$f_2(n, x) = \frac{(x^2 + 1)^2[x^8 + (n + 4)x^6 + (4n + 2)x^4 + (n + 4)x^2 + 1]}{\phi_2}$$

with $\phi_2 = x^{12} + (n + 6)x^{10} + (6n + 8)x^8 + (11n - 1)x^6 + (6n + 5)x^4 + (n + 5)x^2 + 1$. Then

$$\frac{\partial f_2(n, x)}{\partial n} = -x^4(4x^2 + 1)(x^2 + 1)(x^4 + 4x^2 + 1)(x^2 + 1)^2 \leq 0.$$

Thus $f_2(n, x)$ is a decreasing function of $n$. When $n \geq 35$, we get

$$\mathcal{E}(F_{2n,1}^{-}) - \mathcal{E}(E_{2n,3}^{+}) = \frac{2}{\pi} \int_{0}^{\infty} \log f_2(n, x) dx \leq \frac{2}{\pi} \int_{0}^{\infty} \log f_2(35, x) dx \leq \frac{2}{\pi} \times (-0.0011112) < 0.$$

Therefore, by (75), we obtain $F_{2n,1}^{-} \rightarrow E_{2n,3}^{+}$ for $n \geq 35$ while the calculation yields $E_{2n,3}^{+} \rightarrow F_{2n,1}^{-}$ for $6 \leq n \leq 34$. □
Lemma 3.28. $E_{2n,3}^+ \rightarrow E_{2n,2}^+ \rightarrow D_{2n,3}^+$ for $n \geq 6$.

Proof. By (7), we have

\[
\phi(E_{2n,3}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n - 1)x^6 + (6n + 5)x^4 - (n + 5)x^2 + 1],
\]

(76)

\[
\phi(E_{2n,2}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n - 2)x^6 + (7n + 4)x^4 - (n + 6)x^2 + 1],
\]

(77)

\[
\phi(D_{2n,3}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n + 2)x^6 + 8nx^4 - (2n + 2)x^2 + 1].
\]

(78)

It follows from (76)–(78) and (4) that Lemma 3.28 holds. □

Lemma 3.29. $E_{2n,3}^+ \rightarrow H_{2n,1}$ for $n \geq 6$.

Proof. By (7), we get

\[
\phi(E_{2n,3}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n - 1)x^6 + (6n + 5)x^4 - (n + 5)x^2 + 1],
\]

(79)

\[
\phi(H_{2n,1}, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} - 2x^9 + (6n + 9)x^8 + 8x^7 - (11n + 3)x^6 - 8x^5 + (6n + 9)x^4 + 2x^3 - (n + 6)x^2 + 1].
\]

(80)

It follows from (79), (80) and (4) that $E_{2n,3}^+ \rightarrow H_{2n,1}$ for $n \geq 6$. □

Lemma 3.30. $E_{2n,3}^+ \rightarrow M_{2n,7}^+$ for $n \geq 6$.

Proof. By (7), we obtain

\[
\phi(E_{2n,3}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 8)x^8 - (11n - 1)x^6 + (6n + 5)x^4 - (n + 5)x^2 + 1],
\]

(81)

\[
\phi(M_{2n,7}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 9)x^8 - (11n + 3)x^6 + (6n + 9)x^4 - (n + 6)x^2 + 1].
\]

(82)

Thus, $E_{2n,3}^+ \rightarrow M_{2n,7}^+$ for $n \geq 6$ follows from (81), (82) and (4). □

Lemma 3.31. $M_{2n,7}^+ \rightarrow M_{2n,3}^+ \rightarrow M_{2n,4}^+$ for $n \geq 6$.

Proof. By (7), we get

\[
\phi(M_{2n,7}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 9)x^8 - (11n + 3)x^6 + (6n + 9)x^4 - (n + 6)x^2 + 1],
\]

(83)

\[
\phi(M_{2n,3}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 9)x^8 - (11n + 4)x^6 + (7n + 5)x^4 - (n + 6)x^2 + 1],
\]

(84)

\[
\phi(M_{2n,4}^+, x) = (x^2 - 1)^n x^{12} - (n + 6)x^{10} + (6n + 9)x^8 - 12nx^6 + (9n - 3)x^4 - (2n + 2)x^2 + 1].
\]

(85)

Thus, Lemma 3.31 directly follows from (83)–(85) and (4). □

Lemma 3.32. $F_{2n,1}^- \rightarrow M_{2n,5}^+$ for $n \geq 31$ while $M_{2n,5}^+ \rightarrow F_{2n,1}^-$ for $5 \leq n \leq 30$. 

Proof. By (7), we have
\[
\phi(F_{2n,1}^{-}, x) = (x^2 - 1)^{n+4}[x^8 - (n + 4)x^6 + (4n + 2)x^4 - (n + 4)x^2 + 1]
\ualph (x^2 - 1)^{n+4}g_5(x),
\]
\[
\phi(M_{2n,5}^{-}, x) = x^2(x^2 - 1)^{n+5}[x^8 - (n + 5)x^6 + (5n + 4)x^4 - (7n - 6)x^2 + 2n - 2]
\ualph x^2(x^2 - 1)^{n+5}g_5(x). \tag{86}
\]

By calculation, we have \(g_5(\sqrt{\frac{1}{n}}) > 0\) for \(n \geq 4\), \(g_5(\sqrt{\frac{1}{n}}) < 0\) for \(n = 8\), \(g_5(\sqrt{0.27}) < 0\) for \(n = 5\), \(g_5(\sqrt{0.27}) > 0\) for \(n = 4\), \(g_5(\sqrt{3.73}) > 0\) for \(n = 4\), \(g_5(\sqrt{3.74}) < 0\) for \(n = 5\), \(g_5(\sqrt{n - 1}) < 0\) for \(n = 5\), and \(g_5(\sqrt{n}) > 0\) for \(n \geq 4\). According to the theorem of zero points, we have
\[
\mathcal{E}(F_{2n,1}^{-}) = 2(n - 4) + \sum_{i=1}^{n} |x_i| < 2(n - 4) + 2(\sqrt{\frac{2}{n}} + \sqrt{0.27} + \sqrt{3.74} + \sqrt{n}), \quad (n \geq 8). \tag{88}
\]

Since \(g_4(\sqrt{0.38}) > 0\) for \(n = 24\), \(g_4(\sqrt{0.39}) < 0\) for \(n = 24\), \(g_4(\sqrt{1.99}) < 0\) for \(n = 204\), \(g_4(\sqrt{2}) > 0\) for \(n \geq 5\), \(g_4(\sqrt{2.6}) > 0\) for \(n = 65\), \(g_4(\sqrt{2.62}) < 0\) for \(n = 5\), \(g_4(\sqrt{n}) < 0\) for \(n = 5\), and \(g_4(\sqrt{n + 1}) > 0\) for \(n = 5\), we have
\[
2(n - 5) + 2(\sqrt{0.38} + \sqrt{1.99} + \sqrt{2.6} + \sqrt{n}) < \mathcal{E}(M_{2n,5}^{+}), \quad (n \geq 204). \tag{89}
\]

It follows from \(2(n - 4) + 2(\sqrt{\frac{2}{n}} + \sqrt{0.27} + \sqrt{3.74} + \sqrt{n}) < 2(n - 5) + 2(\sqrt{0.38} + \sqrt{1.99} + \sqrt{2.6} + \sqrt{n})\) that the right-handed side of (88) is less than the left-handed side of (89) for \(n \geq 204\). Therefore, \(F_{2n,1}^{-} \to M_{2n,5}^{+}\) for \(n \geq 204\). The calculation yields \(F_{2n,1}^{-} \to M_{2n,5}^{+}\) for \(31 \leq n \leq 204\) while \(M_{2n,5}^{+} \to F_{2n,1}^{-}\) for \(5 \leq n \leq 30\). \(\qed\)

**Lemma 3.33.** \(M_{2n,5}^{+} \to M_{2n,6}^{+} \equiv L_{2n,2}^{+}\) for \(n \geq 5\).

**Proof.** By (7), we obtain
\[
\phi(M_{2n,5}^{+}, x) = x^2(x^2 - 1)^{n+5}[x^8 - (n + 5)x^6 + (5n + 4)x^4 - (7n - 3)x^2 + 3n - 3]
\ualph \phi(L_{2n,2}^{+}, x). \tag{90}
\]

Therefore, Lemma 3.33 directly follows from (90), (91) and (4). \(\qed\)

**Lemma 3.34.** \(F_{2n,1}^{-} \to C_{2n,5}^{+}\) for \(n \geq 4\).

**Proof.** It follows from (7) that
\[
\phi(C_{2n,5}^{+}, x) = (x^2 - 1)^{n+4}[x^8 - (n + 4)x^6 - 2x^5 + 4nx^4 + (2n - 2)x^3 - (n + 4)x^2 - 2x + 1]
\ualph (x^2 - 1)^{n+4}g_5(x). \tag{92}
\]

By calculation, we have \(g_5(-\sqrt{n + 1}) > 0\) for \(n \geq 4\), \(g_5(-\sqrt{n}) < 0\) for \(n \geq 4\), \(g_5(-1.49) < 0\) for \(n \geq 4\), \(g_5(-1.48) > 0\) for \(n \geq 382\), \(g_5(-1) = 0\), \(g_5(\sqrt{\frac{1}{n}}) < 0\) for \(n = 5\), \(g_5(\sqrt{\frac{1}{n}}) > 0\) for \(n \geq 4\), \(g_5(\sqrt{\frac{1}{n}}) > 0\) for \(n \geq 7\), \(g_5(\sqrt{n}) < 0\) for \(n \geq 4\), \(g_5(0.31) < 0\) for \(n \geq 4\), \(g_5(0.32) > 0\) for \(n \geq 32\), \(g_5(2.17) > 0\) for \(n \geq 7399\), \(g_5(2.18) < 0\) for \(n \geq 4\), \(g_5(\sqrt{n} + 1) > 0\) for \(n \geq 7\), and \(g_5(\sqrt{n} - 1) > 0\) for \(n \geq 7\), we have
\[
2(n - 4) + \sqrt{n} + 1.48 + 1 + \sqrt{\frac{1}{n}} + \sqrt{\frac{1}{2n}} + 0.31 + 2.17 + \sqrt{n} < \mathcal{E}(C_{2n,5}^{+}), \quad (n \geq 7399). \tag{93}
\]
Lemma 3.35. \( C_{2n,5} \rightarrow D_{2n,6} \) for \( n \geq 4 \).

**Proof.** By (7), we get

\[
\phi(C_{2n,5}, x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + 4nx^4 + (2n - 2)x^3]
\]

\[-(n + 4)x^2 - 2x + 1], \quad (94)
\]

\[
\phi(D_{2n,6}, x) = (x^2 - 1)^{n-4}[x^8 - (n + 4)x^6 - 2x^5 + (4n + 1)x^4 + (2n - 2)x^3]
\]

\[-(n + 5)x^2 - 2x + 1]. \quad (95)
\]

Thus, \( C_{2n,5} \rightarrow D_{2n,6} \) for \( n \geq 4 \) directly follows from (94), (95) and (4). \( \square \)

3.5. The preceding 18 signed graphs in the increasing order by their minimal energies in \( \mathcal{U}_{2n}^L \) with \( n \geq 721 \)

By Lemmas 3.10–3.25, we get the first 18 signed graphs in the increasing order according to their minimal energies among \( \mathcal{U}_{2n}^L \) with \( n \geq 721 \), which are showed in Theorem 3.36. In the proof of Theorem 3.36, we need to compare \( E(F_{2n,1}) \) with the energies of some signed graphs among \( \mathcal{A}^+ \cup \mathcal{A}^- \) by using the method of coefficient comparison. Due to the tedious calculations for the coefficients of their characteristic polynomials, we directly list their coefficients in (A1)–(A37) in Appendix B, which are derived by using Lemma 3.1.

**Theorem 3.36.** Let \( S \in \mathcal{U}_{2n}^L \) and \( G \neq I_{2n,1}, C_{2n,2}, D_{2n,1}, D_{2n,4}, D_{2n,7}, C_{2n,3} \), where \( G \) is the underlying graph of \( S \). Let \( l \) be the girth of \( S \) and \( r, j \) be integers with \( r, j \geq 1 \). If one of the following three conditions holds: (i) \( l = 2r + 1 \); (ii) \( l = 4j + 2 \) and \( S \) is balanced; (iii) \( l = 4j \) and \( S \) is unbalanced, then for \( n \geq 721 \), we have

\[
M^+_{2n,2} \rightarrow A_{2n,1} \rightarrow B^+_{2n,1} \rightarrow C_{2n,1} \rightarrow B_{2n,4} \rightarrow B_{2n,2} \rightarrow I_{2n,3} \rightarrow B_{2n,1} \rightarrow I^+_{2n,2} \rightarrow B_{2n,3}
\]

\[
I_{2n,4} \rightarrow I_{2n,3} \rightarrow C^+_{2n,1} \rightarrow F^+_{2n,1} \rightarrow D^+_{2n,2} \rightarrow I_{2n,1} \rightarrow F^+_{2n,1} \equiv L^+_{2n,1} \rightarrow S. \quad (96)
\]

**Proof.** Let \( n \geq 721 \). By Lemmas 3.10–3.16 and 3.19–3.25, we have the first to the sixteenth inequalities in (96). By Lemma 3.26, we have \( F_{2n,1} \leftarrow L^-_{2n,1} \) in (96). Next, we prove the last inequality \( F_{2n,1} \leftarrow S \) in (96), where \( S \) does not contain the preceding terms in (96).

Let \( \mathcal{B} \) be the set of \( I_{2n,1}, C_{2n,2}, D_{2n,3}, D_{2n,4}, D_{2n,7}, \) and all the 18 graphs listed in (96) before \( S \). Let \( \mathcal{C} = (\mathcal{A}^+ \cup \mathcal{A}^-) \setminus \mathcal{B} \). We divide \( \mathcal{C} \) into three subsets according to the girth of the graph. Namely \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \), where

\[
\mathcal{C}_1 = \{ C_{2n,4}, C_{2n,5}, D_{2n,5}, D_{2n,6}, E_{2n,1}, F_{2n,1} (4 \leq i \leq 7) \} \cup \{ H_{2n,1} \}
\]

\[
\cup \{ K_{2n,1} (1 \leq i \leq 3), L_{2n,1} (3 \leq i \leq 10), M_{2n,1} (8 \leq i \leq 19) \},
\]

\[
\mathcal{C}_2 = \{ C_{2n,1}^+, D_{2n,1}^+, D_{2n,3}^+, E_{2n,1}^+, E_{2n,2}^+, E_{2n,3}^+ \}
\]

\[
\cup \{ L_{2n,1}^+, L_{2n,2}^+, M_{2n,1}^+ (3 \leq i \leq 7), M_{2n,1}^+ (2 \leq i \leq 7) \},
\]

\[
\mathcal{C}_3 = \{ L_{2n,1}^-, M_{2n,1}^-, M_{2n,1}^- \}.
\]

It is noted that all the graphs in \( \mathcal{C}_1 \) have odd girth 3 or 5. All the graphs in \( \mathcal{C}_2 \) and in \( \mathcal{C}_3 \) have even girths 4 and 6, respectively. Three cases are considered according to the girth \( l \) of the signed graph.

**Case (i).** \( l = 2r + 1 \).

Since \( l = 2r + 1 \), by Theorem 3.2, for a fixed graph \( G \in \mathcal{A} \), we have \( \mathcal{E}(G^-) = \mathcal{E}(G^+) = \mathcal{E}(G) \). Therefore, we only need to prove \( F_{2n,1} \rightarrow G \) in (96), where \( G \) has girth \( 2r + 1 \) and \( G \) does not contain the preceding terms in (96).

If \( G \notin \mathcal{A} \), then by Lemma 3.9(i), we have \( F^-_{2n,1} \rightarrow G \). If \( G \in \mathcal{A} \setminus \mathcal{B} \), then \( G = C_1 \) since \( G \) has girth \( l = 2r + 1 \). Next, we prove \( F^-_{2n,1} \rightarrow G \) for \( G \in C_1 \).
By Lemmas 3.34 and 3.35, we have $F_{2n,1}^− \rightarrow C_{2n,5} \rightarrow D_{2n,6}$. By Lemmas 3.27 and 3.29, we have $F_{2n,1}^− \rightarrow E_{2n,1}^+ \rightarrow H_{2n,1}$. Furthermore, by using the method of coefficient comparison and comparing (A2) with (A9), (A10) and (A11), we obtain $F_{2n,1}^− \rightarrow E_{2n,1}^+ \rightarrow G$, where $G \in \{E_{2n,1}, E_{2n,4}, E_{2n,5}\}$. By comparing (A3) with (A4), (A5), (A12), (A13), (A6), (A7), and (A8), we get $F_{2n,1}^− \rightarrow H_{2n,1} \rightarrow G$, where $G \in \{C_{2n,4}, D_{2n,5}, E_{2n,6}, E_{2n,7}, K_{2n,1} \}$ ($1 \leq i \leq 3$). By comparing (A1) with (A14)–(A33), we obtain $F_{2n,1}^− \rightarrow G$, where $G \in \{L_{2n,1}, (3 \leq i \leq 10), M_{2n,1} \}$ ($8 \leq i \leq 19$).

Case (ii). $l = 4j$ and $S$ is unbalanced.

If $S \in \mathcal{A}^−$, then by Lemma 3.9(iii), we have $E(F_{2n,1}^−) < E(S)$. If $S \in \mathcal{A}^+ \setminus \mathcal{B}$, since $S$ has girth $l = 4j$ and $S$ is unbalanced, then we only need to prove $F_{2n,1}^− \rightarrow S$, where $S \in \{C_{2n,1}, D_{2n,2}, D_{2n,3}, F_{2n,2}, F_{2n,3}^− \} \cup \{L_{2n,2}, M_{2n,3}^− \}$ ($2 \leq i \leq 7$).

By comparing (A1) with (A36), (A37) and (A35), we get $F_{2n,1}^− \rightarrow C_{2n,1}^−, D_{2n,2}^−, M_{2n,2}^−$ for $n \geq 6$. We have the following inequalities: $F_{2n} \rightarrow D_{2n,3}^− \rightarrow D_{2n,3}$ (by Lemmas 3.27, 3.28 and Theorem 3.2(ii)), $F_{2n} \rightarrow E_{2n,2}$ (by Lemmas 3.27, 3.28 and Theorem 3.2(ii)), $F_{2n} \rightarrow E_{2n,3}$ (by Lemma 3.27 and Theorem 3.2(ii)), $F_{2n} \rightarrow L_{2n,2}$ (by Lemmas 3.32, 3.33 and Theorem 3.2(ii)), $F_{2n,1} \rightarrow M_{2n,5}^−$ and $F_{2n,1} \rightarrow M_{2n,4}^−$ (by Lemmas 3.27, 3.30, 3.31, and Theorem 3.2(ii)), $F_{2n,1} \rightarrow M_{2n,5}^−$ (by Lemmas 3.32, 3.30 and Theorem 3.2(ii)), and $F_{2n,1} \rightarrow M_{2n,7}^−$ (by Lemmas 3.27, 3.30 and Theorem 3.2(ii)). Case (iii). $l = 4j + 2$ and $S$ is balanced.

If $S \in \mathcal{A}^+$, then by Lemma 3.9(iii), we have $F_{2n,1}^− \rightarrow S$. If $S \in \mathcal{A}^+ \setminus \mathcal{B}$, since $S$ has girth $l = 4j + 2$ and $S$ is balanced, then $S \in \{L_{2n,1}, M_{2n,1}^−\}$. By Lemma 3.26 and Theorem 3.2(ii), we get $F_{2n,1} \Rightarrow L_{2n,1} \rightarrow L_{2n,1}$. By (A1) and (A34) and by Theorem 3.2(iii), we obtain $F_{2n,1}^− \rightarrow M_{2n,1}^− \rightarrow M_{2n,1}^+$.

By the proofs of Cases (ii)–(iii), we get Theorem 3.36.

It should be pointed out that when $6 \leq n \leq 721$, for the graphs among $\mathcal{U}_{2n}^\cup$, the increasing order according to their minimal energies are complicated and irregular. Bearing $E(F_{2n,1}^+) = E(F_{2n,1})$ in mind, we can, by Theorems 3.2 and 3.36, directly obtain Theorem 3.37 as follows. Namely, among $\mathcal{U}_{2n}^\cup$, the first $12$ graphs in the increasing order according to their minimal energies are obtained. In 2011, Wang [20] obtained the first $7$ graphs, which are shown in Theorem 1 in [20]. Therefore, Theorem 3.37 extends the results of Theorem 1 in [20] and updates the proof of Theorem 1 in [20].

**Theorem 3.37.** Let $G \in \mathcal{U}_{2n}^\cup$ with $n \geq 721$ and $G \notin J_{2n,1}, C_{2n,2}, D_{2n,1}, D_{2n,4}, D_{2n,7}, C_{2n,3}$. Let $l$ be the girth of $G$ and $r, j$ be integers with $r, j \geq 1$. If $l = 2r + 1$ or $l = 4j + 2$, then we have

$$M_{2n,2} \rightarrow A_{2n,1} \rightarrow B_{2n,1} \rightarrow C_{2n,1} \rightarrow B_{2n,4} \rightarrow B_{2n,2} \rightarrow J_{2n,3} \rightarrow J_{2n,2} \rightarrow B_{2n,3} \rightarrow J_{2n,4} \rightarrow C_{2n,1} \rightarrow F_{2n,1} \rightarrow G.$$  \hspace{1cm} (97)

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Appendix A : Figures for signed graphs among $\mathcal{A}^+ \cup \mathcal{A}^-$

![Diagrams of signed graphs](image-url)
Appendix B: Expressions for coefficients of characteristic polynomials of some signed graphs among $\mathcal{A}^+ \cup \mathcal{A}^-$

\[ \begin{align*}
\{ & b_2(F_{2n,1}) = p + (2n - 4)\binom{n-4}{i-2}, \\
& b_{2i+1}(F_{2n,1}) = 0. \tag{A1} \}
\]

\[ \begin{align*}
\{ & b_2(E_{2n,3}) = p + (2n - 7)\binom{n-5}{i-2} + (n - 5)\binom{n-6}{i-3} - \binom{n-5}{i-3}, \\
& b_{2i+1}(E_{2n,3}) = 0. \tag{A2} \}
\]
\begin{align*}
\begin{cases}
    b_2(H_{2n,1}) = p + (2n - 6)(\binom{n-3}{i-2}) + (n - 5)(\binom{n-6}{i-3}), \\
    b_{2i+1}(H_{2n,1}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-3}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A3}

\begin{cases}
    b_2(C_{2n,4}) = p + (2n - 6)(\binom{n-4}{i-2}) + (n - 3)(\binom{n-3}{i-3}), \\
    b_{2i+1}(C_{2n,4}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-3}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A4}

\begin{cases}
    b_2(D_{2n,5}) = p + (2n - 5)(\binom{n-3}{i-2}) + (2n - 5)(\binom{n-4}{i-3}), \\
    b_{2i+1}(D_{2n,5}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-3}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A5}

\begin{cases}
    b_2(K_{2n,1}) = p + (2n - 5)(\binom{n-5}{i-2}) + 2(\binom{n-4}{i-3}) + (n - 4)(\binom{n-3}{i-3}), \\
    b_{2i+1}(K_{2n,1}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-4}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A6}

\begin{cases}
    b_2(K_{2n,2}) = p + (2n - 5)(\binom{n-4}{i-2}) + (n - 2)(\binom{n-5}{i-3}) + (n - 4)(\binom{n-3}{i-3}), \\
    b_{2i+1}(K_{2n,2}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-4}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A7}

\begin{cases}
    b_2(K_{2n,3}) = p + (2n - 5)(\binom{n-5}{i-2}) + (\binom{n-4}{i-3}) + (n - 5)(\binom{n-6}{i-3}) + (n - 4)(\binom{n-3}{i-3}), \\
    b_{2i+1}(K_{2n,3}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-5}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A8}

\begin{cases}
    b_2(E_{2n,1}) = p + (2n - 5)(\binom{n-5}{i-2}) + (\binom{n-4}{i-3}) + (n - 4)(\binom{n-6}{i-3}), \\
    b_{2i+1}(E_{2n,1}) = \begin{cases}
        0, & 2i + 1 < 5 \\
        2(\binom{n-4}{i-2}), & 2i + 1 \geq 5.
    \end{cases}
\end{cases} 
\tag{A9}

\begin{cases}
    b_2(E_{2n,4}) = p + (2n - 5)(\binom{n-4}{i-2}) + 3(\binom{n-4}{i-3}) + (n - 4)(\binom{n-5}{i-3}), \\
    b_{2i+1}(E_{2n,4}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A10}

\begin{cases}
    b_2(E_{2n,5}) = p + (2n - 5)(\binom{n-3}{i-2}) + (n - 3)(\binom{n-4}{i-3}) + (n - 4)(\binom{n-5}{i-3}), \\
    b_{2i+1}(E_{2n,5}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-4}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A11}

\begin{cases}
    b_2(E_{2n,6}) = p + (2n - 5)(\binom{n-4}{i-2}) + (\binom{n-3}{i-3}) + (n - 5)(\binom{n-6}{i-3}) + (n - 4)(\binom{n-3}{i-3}), \\
    b_{2i+1}(E_{2n,6}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-3}{i-1}) + 2(\binom{n-4}{i-2}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A12}

\begin{cases}
    b_2(E_{2n,7}) = p + (2n - 5)(\binom{n-5}{i-2}) + (n - 4)(\binom{n-5}{i-3}), \\
    b_{2i+1}(E_{2n,7}) = \begin{cases}
        0, & 2i + 1 < 3 \\
        2(\binom{n-4}{i-2}) + 4(\binom{n-4}{i-3}) + 2(\binom{n-6}{i-3}), & 2i + 1 \geq 3.
    \end{cases}
\end{cases} 
\tag{A13}

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\[ b_{25}(L_{2n,3}) = p + (2n - 4)\binom{n-4}{i-2} + 3\binom{n-4}{i-3}. \] (A14)

\[ b_{25}(L_{2n,4}) = p + (2n - 4)\binom{n-4}{i-2} + (n - 1)\binom{n-4}{i-3}. \] (A15)

\[ b_{25}(L_{2n,5}) = p + (2n - 4)\binom{n-4}{i-2} + (n - 2)\binom{n-4}{i-3}. \] (A16)

\[ b_{25}(L_{2n,6}) = p + (2n - 4)\binom{n-4}{i-2} + \binom{n-4}{i-3}. \] (A17)

\[ b_{25}(L_{2n,7}) = p + (2n - 4)\binom{n-4}{i-2} + (2n - 4)\binom{n-4}{i-3}. \] (A18)

\[ b_{25}(L_{2n,8}) = p + (2n - 4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3}. \] (A19)

\[ b_{25}(L_{2n,9}) = p + (2n - 4)\binom{n-4}{i-2} + (2n - 4)\binom{n-4}{i-3}. \] (A20)

\[ b_{25}(L_{2n,10}) = p + (2n - 4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3}. \] (A21)

\[ b_{25}(M_{2n,8}) = p + (2n - 4)\binom{n-4}{i-2} + 3\binom{n-4}{i-3} + (n - 4)\binom{n-5}{i-3}. \] (A22)

\[ b_{25}(M_{2n,9}) = p + (2n - 4)\binom{n-4}{i-2} + \binom{n-4}{i-3} + (n - 4)\binom{n-5}{i-3}. \] (A23)

\[ b_{25}(M_{2n,10}) = p + (2n - 4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + 2\binom{n-5}{i-3} + (n - 5)\binom{n-6}{i-3}. \] (A24)

\[ b_{25}(M_{2n,11}) = p + (2n - 4)\binom{n-4}{i-2} + (n - 2)\binom{n-4}{i-3} + (n - 3)\binom{n-5}{i-3}. \] (A25)

\[ b_{25}(M_{2n,12}) = p + (2n - 4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + (n - 3)\binom{n-5}{i-3}. \] (A26)

\[ b_{25}(M_{2n,13}) = p + (2n - 4)\binom{n-4}{i-2} + \binom{n-4}{i-3} + (n - 3)\binom{n-5}{i-3}. \] (A27)
\[ b_2(M_{2n,14}) = p + (2n - 4) \binom{n - 4}{i - 2} + \binom{n - 4}{i - 3} + \binom{n - 5}{i - 3} + (n - 4) \binom{n - 6}{i - 3}. \]  
(A28)

\[ b_2(M_{2n,15}) = p + (2n - 4) \binom{n - 4}{i - 2} + 4 \binom{n - 4}{i - 3} + (n - 4) \binom{n - 5}{i - 3}. \]  
(A29)

\[ b_2(M_{2n,16}) = p + (2n - 4) \binom{n - 4}{i - 2} + 2 \binom{n - 4}{i - 3} + 2 \binom{n - 5}{i - 3} + (n - 5) \binom{n - 6}{i - 3}. \]  
(A30)

\[ b_2(M_{2n,17}) = p + (2n - 4) \binom{n - 4}{i - 2} + (2n - 4) \binom{n - 4}{i - 3}. \]  
(A31)

\[ b_2(M_{2n,18}) = p + (2n - 4) \binom{n - 4}{i - 2} + (2n - 4) \binom{n - 4}{i - 3}. \]  
(A32)

\[ b_2(M_{2n,19}) = p + (2n - 4) \binom{n - 4}{i - 2} + (n - 1) \binom{n - 4}{i - 3} + (n - 4) \binom{n - 5}{i - 3}. \]  
(A33)

\[ b_2(M_{2n,1}^+) = p + (2n - 4) \binom{n - 4}{i - 2} + \binom{n - 4}{i - 3} + (n - 4) \binom{n - 5}{i - 3}. \]  
(A34)

\[ b_2(M_{2n,2}^+) = p + (2n - 2) \binom{n - 4}{i - 2} + \binom{n - 3}{i - 3} + (n + 3) \binom{n - 4}{i - 3}. \]  
(A35)

\[ b_2(C_{2n,1}) = p + (2n - 4) \binom{n - 4}{i - 2} + 3 \binom{n - 4}{i - 3}. \]  
(A36)

\[ b_2(D_{2n,2}) = p + (2n - 3) \binom{n - 4}{i - 2} + 4 \binom{n - 4}{i - 3}. \]  
(A37)