Non-uniform Dependence on Initial Data for the Generalized Camassa–Holm–Novikov Equation in Besov Space

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Communicated by A. Constantin

Abstract. Considered in this paper is the generalized Camassa–Holm–Novikov equation with high order nonlinearity, which unifies the Camassa–Holm and Novikov equations as special cases. We show that the solution map of generalized Camassa–Holm–Novikov equation is not uniformly continuous on the initial data in Besov spaces $B^{s}_{p,r}(\mathbb{R})$ with $s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$, $1 \leq p, r < \infty$ as well as in critical space $B^{\frac{3}{2}}_{2,2}(\mathbb{R})$. Our result covers and improves the previous work given by Li et al. (J Differ Equ 269:8686–8700, 2020; J Math Fluid Mech 22:4–50, 2020; J Math Fluid Mech 23:36, 2021).

Mathematics Subject Classification. 35B30, 35G25, 35Q53.

Keywords. Non-uniform dependence, Camassa–Holm equation, Novikov equation, High order nonlinearity, Besov spaces.

1. Introduction

In this paper, we consider the Cauchy problem for the generalized Camassa–Holm–Novikov (gCHN) equation with high order nonlinearity proposed by Anco et al. [1] as follows:

$$\begin{align*}
m_t + u^k m_x + (k + 1)u^{k-1}_x m &= 0, \\
m &= u - u_{xx}, \\
u(0, x) &= u_0(x).
\end{align*}$$

(1.1)

here $t > 0$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. The gCHN is an evolution equation with $(k + 1)$-order nonlinearities, and can be regarded as a subclass of the generalized Camassa–Holm (g-kbCH) equation considered in [15,22]

$$m_t + u^k m_x + bu^{k-1}_x m = 0, \quad k \in \mathbb{Z}^+, \quad b \in \mathbb{R}.$$

As shown in [1,15,22], the gCHN equation (1.1) admits a local conservation law, possesses single peakons of the form $u(x, t) = c e^{\frac{t}{2} e^{x-ct}} (c=\text{const.} \text{ is the wave speed})$ as well as multi-peakon solutions and exhibits wave breaking phenomena. The precise blow-up scenario and global strong solutions in the settings of Besov spaces have been investigated by Yan [35].

When $k = 1$, (1.1) reduces to the classical Camassa–Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

(1.2)

which was originally derived as a bi-Hamiltonian system by Fokas and Fuchssteiner [14] in the context of the KdV model and gained prominence after Camassa–Holm [3] independently re-derived it as an approximation to the Euler equations of hydrodynamics. Constantin and Lannes [10] later educed CH equation from the water waves equations. The CH equation is completely integrable in the sense of having a Lax pair, a bi-Hamiltonian structure as well as possessing an infinity of conservation laws, and it also admits exact peakon solutions of the form $ce^{-|x-ct|} [3,5,8,9,11,14]$. 
When $k = 2$, (1.1) becomes the famous Novikov equation \[32\]

\[
    u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}.
\]

(1.3)

It is shown in \[23\] that the Novikov equation with cubic nonlinearity shares similar properties with the CH equation, such as a Lax pair in matrix form, a bi-Hamiltonian structure, infinitely many conserved quantities and peakon solutions given by the formula $u(x, t) = \sqrt{2e^{\frac{-|x-ct|}{2}}}$. We should remark that the $H^1$ norm of solutions to the g-kbCH is conserved if and only if $b = k + 1$, which naturally excludes the Degasperis-Procesi equation for the case $k = 1$ and $b = 3$. Therefore, system (1.1) contains the CH and the Novikov equations but not the Degasperis-Procesi equation, and is referred as the gCHN equation \[1,22\].

Using the Green function $G(x) = \frac{1}{2}e^{-|x|}$ and the identity $(1 - \partial_x^2)^{-1}f = G \ast f$ for all $f \in L^2(\mathbb{R})$, we can express (1.1) in the following equivalent form

\[
\begin{aligned}
    u_t + u^k u_x &= -G \ast (\frac{2k-1}{2}u^{k-1}u_x^2 + u^{k+1}) - G \ast (\frac{k-1}{2}u^{k-2}u_x^3), \\
    u(0, x) &= u_0(x).
\end{aligned}
\]

(1.4)

Let $P(D) = -\partial_x(1 - \partial_x^2)^{-1}$ and $J(D) = -(1 - \partial_x^2)^{-1}$, we can continue to rewrite (1.4) as follows

\[
\begin{aligned}
    u_t + u^k u_x &= P(D)(\frac{2k-1}{2}u^{k-1}u_x^2 + u^{k+1}) + J(D)(\frac{k-1}{2}u^{k-2}u_x^3), \\
    u(0, x) &= u_0(x).
\end{aligned}
\]

(1.5)

In recent years, the question of well-posedness in different spaces for the CH type equations has become of great interest due to its abundant physical and mathematical properties and a series of achievements have been made in the study of the CH type equations. The Cauchy problem of both CH and Novikov equations in Sobolev spaces and Besov spaces have been investigated in \[6,7,12,13,16,25\] and \[21,31,33,34,36\] respectively. For general $k$ and $b$, using a Galerkin-type approximation scheme, Himonas and Holliman \[22\] established the local well-posedness of the g-kbCH equation in the Sobolev space $H^s$ on both the line and the circle. Zhao et al. \[37\] extended the above well-posedness result for gCHN equation to the Besov space $B^s_{p,r}(\mathbb{R})$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leq p, r \leq \infty$ (however, for $r = \infty$, the continuity of the data-to-solution map is established in a weaker topology). Subsequently, Chen et al. \[4\] solved the critical case for $(s, p, r) = (\frac{3}{2}, 2, 1)$.

Continuity properties of the solution map is an important part of the well-posedness theory. In fact, the non-uniform continuity of data-to-solution map suggests that the local well-posedness cannot be established by the contraction mappings principle since this would imply Lipschitz continuity for the solution map. One method to deal with the lack of uniform continuity is based on the traveling wave solutions. This method is used to show that dependence of the periodic CH solutions on initial data in Sobolev spaces $H^s$ can not be better than continuous for $s \geq 2$ \[18\] and for $s = 1$ on both the circle and the line \[18\] by Himonas et al. Using precisely constructed peakon traveling wave solutions, Grayshan and Himonas \[15\] showed that the solution map of g-kbCH equation is not uniformly continuous in Sobolev spaces with exponent less than $\frac{3}{2}$. Another method is based on the approximate solutions, which differs from the former since it does not require the availability of two actual solutions sequences to confirm the conditions of non-uniform dependence. This method was used earlier by Koch and Tzvetkov \[24\] for the Benjamin–Ono equation, and was further developed by Himonas et al. in the Sobolev spaces $H^s (s > \frac{3}{2})$ for the CH equation on the line \[19\] and on the circle \[20\] and for the Novikov equation \[21\] as well as for the g-kbCH equation on both the line and the circle \[22\]. Recently, by developed a new approximation technique, Li et al. \[27,28,30\] showed that the solution maps of both the CH and Novikov equations are not uniformly continuous from $B^s_{p,r}(\mathbb{R})$ into $C([0, T]; B^s_{p,r}(\mathbb{R}))$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $(p, r) \in [1, \infty] \times [1, \infty)$ or $(s, p, r) = (\frac{3}{2}, 2, 1)$.

To the best of our knowledge, whether the data-to-solution map of the Cauchy problem (1.1) for general $k$ is not better than continuous in Besov spaces has not been studied yet. We should emphasize that, due to the fact that the structure of system (1.1) is much more complicated than that of the CH and Novikov equations, these results mentioned above for CH and Novikov equations do not extend cleanly to the gCHN equation. Inspired by \[30\], and establishing delicate estimates of the high order
nonlinearity in different Besov spaces, we are able to show that the solution map of the initial problem (1.1) is not uniformly continuous on the initial data in Besov spaces $B^s_{p,r}(\mathbb{R})$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}, (p,r) \in [1,\infty] \times [1,\infty)$ as well as in critical space $B^{3/2}_{2,1}(\mathbb{R})$.

Our main result reads as follows.

**Theorem 1.1.** Let

$$s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}, (p,r) \in [1,\infty] \times [1,\infty) \text{ or } (s,p,r) = \left(\frac{3}{2}, 2, 1\right).$$

The data-to-solution map $u_0 \mapsto S_t(u_0)$ of the Cauchy problem (1.5) is not uniformly continuous from any bounded subset of $B^s_{p,r}(\mathbb{R})$ into $C([0,T];B^s_{p,r}(\mathbb{R}))$. More precisely, there exist two initial sequences $g_n$ and $f_n$ such that

$$\|f_n\|_{B^s_{p,r}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|g_n\|_{B^s_{p,r}} = 0,$$

but

$$\liminf_{n \to \infty} \|S_t(f_n + g_n) - S_t(f_n)\|_{B^s_{p,r}} \gtrsim t, \quad t \in (0,T_0],$$

with small positive time $T_0$ for $T_0 \leq T$.

**Remark 1.1.** As mentioned above, the system (1.5) includes the Camassa–Holm equation and the Novikov equation, thus our result covers and improves the work given by Li et al. [27,28,30].

**Remark 1.2.** Compared the system considered in [29], there is one more term $(1 - \partial_x^2)^{-1}(k^{-1}u^{k-2}u_x^3)$ in system (1.5). However, it is not redundant. On the one hand, it connects the Novikov equation. On the other hand, it weakens the attenuation of the difference between the initial data and the corresponding solution, making its qualitative analysis more complicated. For more details, see Lemmas 3.2 and 3.3 and Propositions 3.1 and 3.2.

**Notations** Given a Banach space $X$, we denote the norm of a function on $X$ by $\| \cdot \|_X$, and

$$\|f\|_X = \sup_{0 \leq t \leq T} \|f(t)\|_X.$$

For $f = (f_1, f_2, \ldots, f_n) \in X$, we use the simplified notation

$$\|f\|_X = \|f_1\|_X + \|f_2\|_X + \cdots + \|f_n\|_X.$$

The symbol $A \lesssim B$ means that there exists a uniform positive constant $C$ independent of $A$ and $B$ such that $A \leq CB$.

## 2. Littlewood–Paley Analysis

In this section, we will review the definition of Littlewood–Paley decomposition and nonhomogeneous Besov space, and then list some useful properties which will be frequently used in the sequel. For more details, the readers can refer to [2].

There exists a couple of smooth functions $(\chi, \varphi)$ valued in $[0,1]$, such that $\chi$ is supported in the ball $B \triangleq \{ \xi \in \mathbb{R} : |\xi| \leq \frac{1}{2}\}$, $\varphi$ is supported in the ring $C \triangleq \{ \xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}$. Moreover,

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,$$

$$\chi(\xi) + \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1,$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}) \cap \text{Supp } \varphi(2^{-j'}) = \emptyset,$$

$$j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j}) = \emptyset.$$
3. Non-uniform Continuous Dependence

Then, we can define the nonuniform continuous dependence $\Delta_j$ as follows:

\[
\Delta_j u = 0, \quad \text{if } j \leq -2, \\
\Delta_{-1} u = \chi(D) u = F^{-1}(\chi F u), \\
\Delta_j u = \varphi(2^{-j}D) u = F^{-1}(\varphi(2^{-j}) F u), \quad \text{if } j \geq 0.
\]

**Definition 2.1** [2]. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^{s}_{p,r}(\mathbb{R})$ consists of all tempered distribution $u$ such that

\[
\|u\|_{B^{s}_{p,r}(\mathbb{R})} \triangleq \left\| \left(2^{2s}\|\Delta_j u\|_{L^p(\mathbb{R})} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.
\]

In the following, we list some basic lemmas and properties about Besov space which will be frequently used in proving our main result.

**Lemma 2.1** [2].

1. **Algebraic properties:** $\forall s > 0$, $B^{s}_{p,r}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ is a Banach algebra. $B^{s}_{p,r}(\mathbb{R})$ is a Banach algebra $\iff B^{s}_{p,r}(\mathbb{R}) \iff L^{\infty}(\mathbb{R}) \iff s > \frac{1}{p}$ or $s = \frac{1}{p}$, $r = 1$.

2. For any $s > 0$ and $1 \leq p, r \leq \infty$, there exists a positive constant $C = C(s, p, r)$ such that

$$
\|uv\|_{B^{s}_{p,r}(\mathbb{R})} \leq C Police \|v\|_{L^{\infty}(\mathbb{R})} \|u\|_{B^{s}_{p,r}(\mathbb{R})} + \|v\|_{L^{\infty}(\mathbb{R})} \|u\|_{B^{s}_{p,r}(\mathbb{R})}.
$$

3. For any $s \in \mathbb{R}$, $\mathcal{P}(D)$ is continuous from $B^{s}_{p,r}$ into $B^{s+1}_{p,r}$ and $\mathcal{J}(D)$ is continuous from $B^{s}_{p,r}$ into $B^{s+2}_{p,r}$.

4. Let $1 \leq p, r \leq \infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ or $s = 1 + \frac{1}{p}$, $r = 1$. Then we have

$$
\|uv\|_{B^{s+2}_{p,r}(\mathbb{R})} \leq C \|u\|_{B^{s+1}_{p,r}(\mathbb{R})} \|v\|_{B^{s+1}_{p,r}(\mathbb{R})}.
$$

**Lemma 2.2** [2, 26]. Let $1 \leq p, r \leq \infty$. Assume that

$$
\sigma > - \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\} \quad \text{or} \quad \sigma > -1 - \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\} \quad \text{if} \quad \div v = 0.
$$

There exists a constant $C = C(p, r, \sigma)$ such that for any solution to the linear transport equation:

$$
\partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0,
$$

the following statement holds:

$$
\sup_{s \in [0,t]} \|f(s)\|_{B^{s}_{p,r}} \leq C e^{CV_p(v, t)} \left( \|f_0\|_{B^{s}_{p,r}} + \int_{0}^{t} \|g(\tau)\|_{B^{s}_{p,r}} d\tau \right),
$$

with

$$
V_p(v, t) = \begin{cases} 
\int_{0}^{t} \|\nabla v(s)\|_{B^{s+1}_{p,r}} ds, & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \{\sigma = 1 + \frac{1}{p} \text{ and } r = 1\}, \\
\int_{0}^{t} \|\nabla v(s)\|_{B^{s}_{p,r}} ds, & \text{if } \sigma = 1 + \frac{1}{p} \text{ and } r > 1, \\
\int_{0}^{t} \|\nabla v(s)\|_{B^{s}_{p,\infty \cap L^\infty}} ds, & \text{if } \sigma < 1 + \frac{1}{p}.
\end{cases}
$$

3. Non-uniform Continuous Dependence

In this part, we will show the proof of Theorem 1.1 in detail. To begin with, we shall establish several crucial estimates to show that for specially selected initial data $u_0$ in $B^{s}_{p,r}(\mathbb{R})$, the corresponding solution $S_t(u_0)$ can be approximated by $u_0 - t(u_0)^k \partial_x u_0 + t(\mathcal{P}(D)(u_0) + \mathcal{J}(D)(u_0))$ in a small time near $t = 0$.

Firstly, to estimate the high order nonlinearity term $u^k$, we establish the following
Lemma 3.1. Let $k, m \in \mathbb{Z}^+$, $m \leq k$, $1 \leq p, r \leq \infty$ and $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}, r = 1$. Then we have

$$\|u^k\|_{B_{p,r}^s} \leq C\|u\|^{k-1}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s} \quad \text{or} \quad \|u^k\|_{B_{p,r}^s} \leq C\|u\|^{k-1}_{L_\infty}\|u\|_{B_{p,r}^s},$$

and

$$\|u^{k-m}v^m\|_{B_{p,r}^s} \leq C\|u, v\|^{k-1}_{L_\infty}\|u, v\|_{B_{p,r}^s}.$$

Proof. For $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}, r = 1$, firstly using the product law (2) and then the Banach algebra property (1) of Lemma 2.1, by recurrence method one has

$$\|u^k\|_{B_{p,r}^s} \lesssim \|u\|_{L_\infty}\|u^{k-1}\|_{B_{p,r}^{s-1}} + \|u\|_{B_{p,r}^s}\|u^{k-1}\|_{L_\infty}$$

$$\lesssim \|u\|_{B_{p,r}^{s-1}}\|u^{k-1}\|_{B_{p,r}^{s-1}} + \|u\|_{B_{p,r}^s}\|u^{k-1}\|_{B_{p,r}^{s-1}}$$

$$\lesssim \|u\|_{B_{p,r}^{s-1}}\|u^{k-1}\|_{B_{p,r}^{s-1}} + \|u\|_{B_{p,r}^s}\|u^{k-1}\|_{B_{p,r}^{s-1}}$$

$$\vdots$$

$$\lesssim \|u\|_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s}.$$ 

The other two terms can be processed in a similar more relaxed way and they really hold for $s > 0$. Thus we finish the proof of Lemma 3.1.

According to Lemma 3.1, we have the estimates for $\mathcal{P}(u)$, $\mathcal{J}(u)$ and $u^k u_x$.

Lemma 3.2. Let $k \in \mathbb{Z}^+$, $1 \leq p, r \leq \infty$ and $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}, r = 1$. Then we have

$$\|\mathcal{P}(D)(u) - \mathcal{P}(D)(v)\|_{B_{p,r}^{s-1}} \lesssim \|u - v\|_{B_{p,r}^{s-1}}\|u, v\|^{k-1}_{B_{p,r}^{s-1}}\|u, v\|_{B_{p,r}^s},$$

$$\|\mathcal{J}(D)(u) - \mathcal{J}(D)(v)\|_{B_{p,r}^{s-1}} \lesssim \|u - v\|_{B_{p,r}^{s-1}}\|u, v\|^{k-2}_{B_{p,r}^{s-1}}\|u, v\|_{B_{p,r}^s},$$

$$\|\mathcal{P}(D)(u) - \mathcal{P}(D)(v)\|_{B_{p,r}^s} \lesssim \|u - v\|_{B_{p,r}^s}\|u, v\|^{k-1}_{B_{p,r}^{s-1}}\|u, v\|_{B_{p,r}^s}$$

$$+ \|u - v\|_{B_{p,r}^s}\|u, v\|^{k-2}_{B_{p,r}^{s-1}}\|u, v\|_{B_{p,r}^s}$$

$$\|\mathcal{J}(D)(u) - \mathcal{J}(D)(v)\|_{B_{p,r}^s} \lesssim \|u - v\|_{B_{p,r}^{s-1}}\|u, v\|^{k-2}_{B_{p,r}^{s-1}}\|u, v\|_{B_{p,r}^s}$$

and

$$\|\mathcal{P}(D)(u)\|_{B_{p,r}^{s+1}} \lesssim \|u\|^{k-1}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s}\|u\|_{B_{p,r}^{s+1}},$$

$$\|\mathcal{J}(D)(u)\|_{B_{p,r}^{s+1}} \lesssim \|u\|^{k-1}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s}\|u\|_{B_{p,r}^{s+1}},$$

$$\|u^k u_x\|_{B_{p,r}^{s-1}} \lesssim \|u\|^{k}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s}$$

$$\|u^k u_x\|_{B_{p,r}^{s+1}} \lesssim \|u\|^{k+1}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^s} + \|u\|^{k+2}_{B_{p,r}^{s-1}}\|u\|_{B_{p,r}^{s+1}},$$

$$\|u^k u_x - v^k v_x\|_{B_{p,r}^s} \lesssim \|u - v\|_{B_{p,r}^{s-1}}\|u, v\|^{k-1}_{B_{p,r}^{s-1}}\|v\|_{B_{p,r}^{s+1}}$$

$$+ \|u - v\|_{B_{p,r}^s}\|u, v\|^{k}_{B_{p,r}^{s+1}}.$$
Proof. For $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}, r = 1$, using (3), (4), (1) of Lemma 2.1 and the Young inequality, we have

$$\| \mathcal{P}(D)(u) - \mathcal{P}(D)(v) \|_{B^{s-1}_{p,r}} \lesssim \| u^{k-1} u_x^2 - v^{k-1} u_x^2 \|_{B^{s-1}_{p,r}} + \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

$$\lesssim \| u^{k-1} (u_x - v_x)(u_x + v_x) \|_{B^{s-1}_{p,r}} + \| (u^{k-1} - v^{k-1}) v_x^2 \|_{B^{s-1}_{p,r}}$$

$$+ \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

$$\lesssim \| u^{k-1} (u_x - v_x) \|_{B^{s-1}_{p,r}} + \| (u^{k-1} - v^{k-1}) v_x^2 \|_{B^{s-1}_{p,r}}$$

$$+ \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

$$\lesssim \| u^{k-1} \|_{B^{s-1}_{p,r}} \| u - v \|_{B^{s-1}_{p,r}} + \| (u^{k-1} - v^{k-1}) v_x^2 \|_{B^{s-1}_{p,r}}$$

Similarly,

$$\| \mathcal{P}(D)(u) - \mathcal{P}(D)(v) \|_{B^{s-1}_{p,r}} \lesssim \| u^{k-1} u_x^2 - v^{k-1} u_x^2 \|_{B^{s-1}_{p,r}} + \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

$$\lesssim \| u^{k-1} (u_x - v_x)(u_x + v_x) \|_{B^{s-1}_{p,r}} + \| (u^{k-1} - v^{k-1}) v_x^2 \|_{B^{s-1}_{p,r}}$$

$$+ \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

$$\lesssim \| u^{k-1} \|_{B^{s-1}_{p,r}} \| u - v \|_{B^{s-1}_{p,r}} + \| (u^{k-1} - v^{k-1}) v_x^2 \|_{B^{s-1}_{p,r}}$$

$$+ \| u^{k+1} - v^{k+1} \|_{B^{s-1}_{p,r}}$$

and by the interpolation inequality, we obtain

$$\| \mathcal{P}(D)(u) \|_{B^{s-1}_{p,r}} \lesssim \| u^{k-1} u_x^2 \|_{B^{s-1}_{p,r}} + \| u^{k+1} \|_{B^{s-1}_{p,r}}$$

The other terms can be processed in a similar way, here we omit the details. $

We can see the necessity of being bounded in $B^{s-1}_{p,r}$ from Lemma 3.2, however, for critical index $(s, p, r) = \left( \frac{3}{2}, 2, 1 \right)$, there is no estimates of solutions in $B^{s-1}_{p,r}$, we can use $C^{0,1}$ instead of $B^{s-1}_{p,r}$.

Lemma 3.3. Let $k \in \mathbb{Z}^+$ and $(s, p, r) = \left( \frac{3}{2}, 2, 1 \right)$. Then we have

$$\| \mathcal{P}(D)(u) - \mathcal{P}(D)(v) \|_{B^{s-1}_{p,r}} \lesssim \| u - v \|_{B^{s-1}_{p,r}}$$

$$\| \mathcal{J}(D)(u) - \mathcal{J}(D)(v) \|_{B^{s-1}_{p,r}} \lesssim \| u - v \|_{B^{s-1}_{p,r}}$$

$$\| \mathcal{P}(D)(u) \|_{B^{s+1}_{p,r}} \lesssim \| u \|_{C^{0,1}} \| u \|_{B^{s+1}_{p,r}}$$

$$\| \mathcal{J}(D)(u) \|_{B^{s+1}_{p,r}} \lesssim \| u \|_{C^{0,1}} \| u \|_{B^{s+1}_{p,r}}$$

$$\| u^k u_x \|_{B^{s+1}_{p,r}} \lesssim \| u \|_{B^{s+1}_{p,r}} \| u \|_{B^{s+1}_{p,r}} + \| u \|_{L^{\infty}}.$$
and
\[ \| u^k u_x - v^k v_x \|_{B_{p,r}^{s-1}} \lesssim \| u - v \|_{L^\infty} \| u, v \|_{C_0^1} + \| u - v \|_{B_{p,r}^{s+1}} \| u \|_{L^1}. \]

With the help of Lemmas 3.1–3.3, we can establish the estimates of the difference between the solution \( S_t(u_0) \) and initial data \( u_0 \) in different Besov norms.

**Proposition 3.1.** Let \( k \in \mathbb{Z}^+ \) and \( s > \max \{ 1 + \frac{1}{p}, \frac{3}{2} \} \), \((p, r) \in [1, \infty) \times [1, \infty) \). Assume that \( \| u_0 \|_{B_{p,r}^s} \lesssim 1 \), then we have
\[
\begin{align*}
\| S_t(u_0) - u_0 \|_{B_{p,r}^{s-1}} &\lesssim t \| u_0 \|_{B_{p,r}^{s-1}}^k \| u \|_{B_{p,r}^s}^2, \\
\| S_t(u_0) - u_0 \|_{B_{p,r}^{s+1}} &\lesssim t \| u_0 \|_{B_{p,r}^{s+1}}^k \| u \|_{B_{p,r}^s}^2, \\
\| S_t(u_0) - u_0 \|_{B_{p,r}^{s+1}} &\lesssim t \| u_0 \|_{B_{p,r}^{s+1}}^k \| u \|_{B_{p,r}^s}^2.
\end{align*}
\]

**Proof.** For simplicity, we set \( u(t) = S_t(u_0) \). Firstly, in view of the local well-posedness result [37], there exists a positive time \( T = T(\| u_0 \|_{B_{p,r}^s}, s, p, r) \) such that \( u(t) \) belongs to \( C([0, T]; B_{p,r}^s) \). Moreover, by Lemmas 2.1 and 2.2, for all \( t \in [0, T] \) and \( \gamma \geq s - 1 \), there holds
\[ \| u(t) \|_{B_{p,r}^\gamma} \leq C \| u_0 \|_{B_{p,r}^s}. \] (3.1)

For \( t \in [0, T] \), using the differential mean value theorem, the Minkowski inequality, Lemma 3.2 with \( v = 0 \) and the interpolation inequality, one has from (3.1) that
\[
\begin{align*}
\| u(t) - u_0 \|_{B_{p,r}^s} &\lesssim \int_0^t \| \partial_t u \|_{B_{p,r}^s} d\tau \\
&\lesssim \int_0^t \| \mathcal{P}(D)(u) \|_{B_{p,r}^s} d\tau + \int_0^t \| \mathcal{J}(D)(u) \|_{B_{p,r}^s} d\tau + \int_0^t \| u^k u_x \|_{B_{p,r}^s} d\tau \\
&\lesssim t \| u \|_{L^\infty_t(B_{p,r}^{s+1})} + \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \\
&\lesssim t \| u_0 \|_{B_{p,r}^{s-1}}^k + \| u_0 \|_{B_{p,r}^{s+1}}^k \| u_0 \|_{B_{p,r}^{s+1}}.
\end{align*}
\]

Similarly,
\[
\begin{align*}
\| u(t) - u_0 \|_{B_{p,r}^{s-1}} &\lesssim \int_0^t \| \partial_t u \|_{B_{p,r}^{s-1}} d\tau \\
&\lesssim \int_0^t \| \mathcal{P}(D)(u) \|_{B_{p,r}^{s-1}} d\tau + \int_0^t \| \mathcal{J}(D)(u) \|_{B_{p,r}^{s-1}} d\tau + \int_0^t \| u^k u_x \|_{B_{p,r}^{s-1}} d\tau \\
&\lesssim t \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \\
&\lesssim t \| u_0 \|_{B_{p,r}^{s-1}}^k + \| u_0 \|_{B_{p,r}^{s+1}}^k \| u_0 \|_{B_{p,r}^{s+1}},
\end{align*}
\]

and
\[
\begin{align*}
\| u(t) - u_0 \|_{B_{p,r}^{s-1}} &\lesssim \int_0^t \| \partial_t u \|_{B_{p,r}^{s-1}} d\tau \\
&\lesssim \int_0^t \| \mathcal{P}(D)(u) \|_{B_{p,r}^{s-1}} d\tau + \int_0^t \| \mathcal{J}(D)(u) \|_{B_{p,r}^{s-1}} d\tau + \int_0^t \| u^k u_x \|_{B_{p,r}^{s-1}} d\tau \\
&\lesssim t \| u \|_{L^\infty_t(B_{p,r}^{s+1})} + \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \| u \|_{L^\infty_t(B_{p,r}^{s+1})} \\
&\lesssim t \| u_0 \|_{B_{p,r}^{s-1}} + \| u_0 \|_{B_{p,r}^{s+1}} \| u_0 \|_{B_{p,r}^{s+1}}.
\end{align*}
\]

Thus, we finish the proof of Proposition 3.1. \( \square \)
Due to the lack of the estimate of solutions in $B_{2,1}^{1/2}(\mathbb{R})$, Proposition 3.1 is no longer valid for critical index $(s,p,r) = (\frac{3}{2}, 2, 1)$. We need to deal with the critical case by a different way.

**Proposition 3.2.** Assume that $(s,p,r) = (\frac{3}{2}, 2, 1)$ and $\|u_0\|_B^{s,p,r} \lesssim 1$. Then we have

\[
\|S_t(u_0) - u_0\|_{L^\infty} \lesssim t\|u_0\|_{C^{0,1}}, \\
\|S_t(u_0) - u_0\|_{B_{p,r}^{s,r}} \lesssim t(\|u_0\|_{B_{p,r}^{s,r}}^{k+1} + \|u_0\|_{C^{0,1}}^{k+1}), \\
\|S_t(u_0) - u_0\|_{B_{p,r}^{s,r+1}} \lesssim t(\|u_0\|_{B_{p,r}^{s,r+1}}^{k} + \|u_0\|_{C^{0,1}}^{k} \|u_0\|_{B_{p,r}^{s,r+2}}).
\]

**Proof.** For simplicity, we set $u(t) = S_t(u_0)$. By the local well-posedness result [4], there exists a positive time $T = T(\|u_0\|_{B_{p,r}^{s,r}}, s, p, r)$ such that $u(t)$ belongs to $C([0, T]; B_{p,r}^{s,r})$. Moreover, by Lemmas 2.1 and 2.2 and a bootstrap from [2] on the Camassa–Holm equation, there holds for all $t \in [0, T]$ and $\gamma \geq \frac{3}{2}$ that

\[
\|u(t)\|_{B_{2,1}^{1/2}} \leq C\|u_0\|_{B_{2,1}^{1/2}}, \quad \|u(t)\|_{C^{0,1}} \leq C\|u_0\|_{C^{0,1}}.
\]

Using the equivalent form (1.4) and convolution properties yields

\[
\|u(t) - u_0\|_{L^\infty} \lesssim \int_0^t \|\partial_t u\|_{L^\infty} d\tau \\
\lesssim \int_0^t \|u^{k} u_x\|_{L^\infty} d\tau + \int_0^t \|G_x * (u^{k-1} u_x^2 + u^{k+1})\|_{L^\infty} d\tau + (k - 1) \int_0^t \|G * (u^{k-2} u_x^2)\|_{L^\infty} d\tau \\
\lesssim \|u\|_{L^\infty_t(C^{0,1})}^{k+1} \|u\|_{L^\infty_t(B_{2,1}^{s,r})}^{k-1} \|u_0\|_{C^{0,1}} \|u_0\|_{B_{2,1}^{s,r+1}}.
\]

Using Lemma 3.3 and $B_{2,1}^{3/2}(\mathbb{R}) \hookrightarrow C^{0,1}(\mathbb{R})$, following the same procedure as in the proof Proposition 3.1, we get

\[
\|u(t) - u_0\|_{B_{2,1}^{3/2}} \lesssim \int_0^t \|\partial_t u\|_{B_{2,1}^{3/2}} d\tau \\
\lesssim \int_0^t \|P(D)(u)\|_{B_{2,1}^{3/2}} d\tau + \int_0^t \|J(D)(u)\|_{B_{2,1}^{3/2}} d\tau + \int_0^t \|u^{k} u_x\|_{B_{2,1}^{3/2}} d\tau \\
\lesssim \int_0^t \|\partial_t u\|_{B_{2,1}^{3/2}} + \|u\|_{L^\infty_t(C^{0,1})} \|u\|_{L^\infty_t(B_{2,1}^{s,r})}^{1/2} \|u_0\|_{C^{0,1}} \|u_0\|_{B_{2,1}^{s,r+1}},
\]

\[
\|u(t) - u_0\|_{B_{2,1}^{5/2}} \lesssim \int_0^t \|\partial_t u\|_{B_{2,1}^{5/2}} d\tau \\
\lesssim \int_0^t \|P(D)(u)\|_{B_{2,1}^{5/2}} d\tau + \int_0^t \|J(D)(u)\|_{B_{2,1}^{5/2}} d\tau + \int_0^t \|u^{k} u_x\|_{B_{2,1}^{5/2}} d\tau \\
\lesssim \int_0^t \|\partial_t u\|_{B_{2,1}^{5/2}} + \|u\|_{L^\infty_t(C^{0,1})} \|u\|_{L^\infty_t(B_{2,1}^{s,r})}^{5/2} \|u_0\|_{C^{0,1}} \|u_0\|_{B_{2,1}^{s,r+1}}.
\]

Thus, we complete the proof of Proposition 3.2. \qed

Using the estimates of $u(t) - u_0$ with different norms, we can establish the following crucial estimates, which implies that for specially selected initial data $u_0$ in $B_{p,r}^s$, the corresponding solution $S_t(u_0)$ can be approximated by $u_0 - t(u_0)^k \partial_x u_0 + t(P(D)(u_0) + J(D)(u_0))$ in a small time near $t = 0$. 

\[
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\]
Proposition 3.3. Assume that $\|u_0\|_{B^s_{p,r}} \lesssim 1$. Under the assumptions of Theorem 1.1, then

- For $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and $(p, r) \in [1, \infty] \times [1, \infty)$, we have
  $$\|S_t(u_0) - u_0 - tv_0\|_{B^s_{p,r}} \lesssim t^2 \left(\|u_0\|_{B^s_{p,r}}^{k+1} + \|u_0\|_{B^s_{p,r}}^k \|u_0\|_{B^{s+1}_{p,r}} + \|u_0\|_{B^{s+1}_{p,r}}^2 \right);$$

- For $(s, p, r) = (\frac{3}{2}, 2, 1)$, we have
  $$\|S_t(u_0) - u_0 - tv_0\|_{B^s_{p,r}} \lesssim t^2 \left(\|u_0\|_{B^s_{p,r}}^{k+1} + \|u_0\|_{C^{0,1}} \|u_0\|_{B^{s+1}_{p,r}} + \|u_0\|_{C^{0,1}} \|u_0\|_{B^{s+1}_{p,r}}^2 \right),$$

here and in what follows we denote

$$v_0 = -\frac{1}{6} \frac{\partial_x u_0}{0} + \mathcal{P}(D)(u_0) + \mathcal{J}(D)(u_0).$$

Proof. Case 1 $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and $(p, r) \in [1, \infty] \times [1, \infty)$.

Using Lemma 3.2 and Proposition 3.1, we obtain from (3.1) that

$$\|u(t) - u_0 - tv_0\|_{B^s_{p,r}} \leq \int_0^t \|\partial_x u - \nu_0\|_{B^s_{p,r}, d\tau} \leq \int_0^t \|\mathcal{P}(D)(u) - \mathcal{P}(D)(u_0)\|_{B^s_{p,r}, d\tau} + \|\mathcal{J}(D)(u) - \mathcal{J}(D)(u_0)\|_{B^s_{p,r}, d\tau} + \int_0^t \|u^k \partial_x u - u^k \partial_x u_0\|_{B^s_{p,r}, d\tau} \leq \int_0^t \|u(\tau) - u_0\|_{B^s_{p,r}, d\tau} + \int_0^t \|u(\tau) - u_0\|_{B^{s+1}_{p,r}, d\tau} \leq \|u_0\|_{B^s_{p,r}}^{k+1} + \|u_0\|_{B^{s+1}_{p,r}} \|u_0\|_{B^{s+1}_{p,r}} + \|u_0\|_{B^{s+1}_{p,r}}^2 \|u_0\|_{B^{s+1}_{p,r}}^2.$$

Case 2 $(s, p, r) = (\frac{3}{2}, 2, 1)$.

Using Lemma 3.3 and Proposition 3.2, we obtain from (3.2) that

$$\|u(t) - u_0 - tv_0\|_{B^s_{p,r}} \leq \int_0^t \|\partial_x u - \nu_0\|_{B^s_{p,r}, d\tau} \leq \int_0^t \|\mathcal{P}(D)(u) - \mathcal{P}(D)(u_0)\|_{B^s_{p,r}, d\tau} + \|\mathcal{J}(D)(u) - \mathcal{J}(D)(u_0)\|_{B^s_{p,r}, d\tau} + \int_0^t \|u^k \partial_x u - u^k \partial_x u_0\|_{B^s_{p,r}, d\tau} \leq \int_0^t \|u(\tau) - u_0\|_{B^s_{p,r}, d\tau} + \int_0^t \|u(\tau) - u_0\|_{B^{s+1}_{p,r}, d\tau} \leq \|u_0\|_{B^s_{p,r}}^{k+1} + \|u_0\|_{B^{s+1}_{p,r}} \|u_0\|_{B^{s+1}_{p,r}} + \|u_0\|_{B^{s+1}_{p,r}}^2 \|u_0\|_{B^{s+1}_{p,r}}^2.$$

Thus, we complete the proof of Proposition 3.3.

Proof of Theorem 1.1. Let $\hat{\phi} \in C_c^\infty(\mathbb{R})$ be a real-valued radial function on $\mathbb{R}$ and satisfy

$$\hat{\phi}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{4}, \\
0, & \text{if } |\xi| \geq \frac{1}{2}.
\end{cases}$$
Define the high frequency and low frequency functions $f_n$ and $g_n$ as
\[ f_n = 2^{-ns} \phi(x) \sin \left( \frac{17}{12} 2^n x \right), \quad g_n = \frac{12}{17} 2^{-\frac{n}{2}} \phi(x), \quad n \geq 1. \]
and it shows in [27] that $\|f_n\|_{B^\sigma_{p,r}} \lesssim 2^{n(\sigma-s)}$.

Setting $u^n_0 = f_n + g_n$, we consider system (1.5) with initial data $u^n_0$ and $f_n$, respectively. Obviously, we have
\[ \|u^n_0 - f_n\|_{B^0_{p,r}} = \|g_n\|_{B^0_{p,r}} \leq C 2^{-\frac{n}{2}}, \]
which means that
\[ \lim_{n \to -\infty} \|u^n_0 - f_n\|_{B^0_{p,r}} = 0. \]
It is easy to show that
\[ \|u^n_0, f_n\|_{B^{s-1}_{p,r}} \lesssim 2^{-\frac{n}{2}}, \]
\[ \|u^n_0, f_n\|_{C^{s-1}_0} \lesssim 2^{-\frac{n}{2}} \]
and
\[ \|u^n_0, f_n\|_{B^s_{p,r}} \leq C 2^{sn} \quad \text{for } s \geq 0, \]
which yield
\[ \|u^n_0\|^{k+1}_{B^{k}_{p,r}} + \|u^n_0\|^{k}_{B^{k}_{p,r}} + \|u^n_0\|^{2k}_{B^{k}_{p,r}} \lesssim 1, \]
\[ \|f_n\|^{k+1}_{B^{k}_{p,r}} + \|f_n\|^{k}_{B^{k}_{p,r}} + \|f_n\|^{2k}_{B^{k}_{p,r}} \lesssim 1, \]
and
\[ \|u^n_0\|^{k+1}_{C^{k+1}_0} + \|u^n_0\|^{k}_{C^{k+1}_0} + \|u^n_0\|^{2k}_{C^{k+1}_0} \lesssim 1, \]
\[ \|f_n\|^{k+1}_{C^{k+1}_0} + \|f_n\|^{k}_{C^{k+1}_0} + \|f_n\|^{2k}_{C^{k+1}_0} \lesssim 1. \]

Notice that
\[ S_t(u^n_0) = S_t(u^n_0) - t \nu_0(u^n_0) + u^n_0 + t \left( - (u^n_0)^k \partial_x u^n_0 + \mathcal{P}(D)(u^n_0) + \mathcal{J}(D)(u^n_0) \right), \]
\[ S_t(f_n) = S_t(f_n) - t \nu_0(f_n) + f_n + t \left( - (f_n)^k \partial_x f_n + \mathcal{P}(D)(f_n) + \mathcal{J}(D)(f_n) \right), \]
then according to Proposition 3.3 and Lemmas 3.2 and 3.3, we can deduce that
\begin{equation}
S_t(u^n_0) - S_t(f_n) \geq t \| (u^n_0)^k \partial_x u^n_0 - (f_n)^k \partial_x f_n \|_{B^0_{p,r}} - t \| \mathcal{P}(D)(u^n_0) - \mathcal{P}(D)(f_n) \|_{B^0_{p,r}} - t \| \mathcal{J}(D)(u^n_0) - \mathcal{J}(D)(f_n) \|_{B^0_{p,r}} - C t^2 \approx t \| (u^n_0)^k \partial_x u^n_0 - (f_n)^k \partial_x f_n \|_{B^0_{p,r}} - C 2^{-\frac{n}{2}} - C t^2. \tag{3.3}
\end{equation}
Moreover,
\[ (u^n_0)^k \partial_x u^n_0 - (f_n)^k \partial_x f_n = (g_n)^k \partial_x f_n + (u^n_0)^k \partial_x g_n + ((u^n_0)^k - (f_n)^k) \partial_x f_n. \]

With the aid of Lemma 2.1, we find that
\[ \|(u^n_0)^k - f_n^k - g_n^k\|_{\partial_x f_n} \lesssim \left( \| (u^n_0)^k - (f_n)^k \|_{L^\infty} \| f_n \|_{B^{s+1}_{p,r}} \right) + \left( \| (u^n_0)^k - (g_n)^k \|_{B^0_{p,r}} \| \partial_x f_n \|_{L^\infty} \right) \lesssim 2^{-n(s-1)}, \]
\[ \| (u^n_0)^k \partial_x g_n \|_{B^0_{p,r}} \lesssim \| u^n_0 \|_{B^0_{p,r}} \| g_n \|_{B^{s+1}_{p,r}} \lesssim 2^{-\frac{n}{2}}. \]
However, using the fact that
\[ \Delta_j((g_n)^k \partial_x f_n) = 0, \quad j \neq n \]
and
\[ \Delta_j((g_n)^k \partial_x f_n) = (g_n)^k \partial_x f_n, \quad n \geq 5, \]
direct calculation shows that for \( n \gg 1 \),
\[ \| (g_n)^k \partial_x f_n \|_{B^s_{p,r}} = 2^n \| (g_n)^k \partial_x f_n \|_{L^p} \]
\[ = \left\| \left( \frac{12}{17} \right)^{k-1} \phi^k \partial_x \phi \sin \left( \frac{17}{12} 2^n x \right) + \frac{12}{17} \phi \cos \left( \frac{17}{12} 2^n x \right) \right\|_{L^p} \]
\[ \gtrsim \left\| \left( \frac{12}{17} \right)^{k-1} \phi^k \cos \left( \frac{17}{12} 2^n x \right) \right\|_{L^p} - 2^{-n} \rightarrow \left( \frac{12}{17} \right)^{k-1} \left( \frac{2^n}{2\pi} \right) \frac{1}{2} \| \phi^k \|_{L^p}, \]
as \( n \to \infty \) by the Riemann Theorem.

Taking the above estimates into (3.3) yields
\[ \liminf_{n \to \infty} \| S_t(u_0^n) - S_t(f_n) \|_{B^s_{p,r}} \gtrsim t \quad \text{for } t \text{ small enough.} \]
This completes the proof of Theorem 1.1. \( \square \)

**Acknowledgements.** The authors would like to thank the anonymous referees for their comments and suggestions which greatly improved the paper. Y. Yu is supported by the National Natural Science Foundation of China under Grant 12101011 and Natural Science Foundation of Anhui Province under Grant 1908085QA05. Y. Xiao is supported by the National Natural Science Foundation of China under Grant 11901167.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest

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**References**

[1] Anco, S.C., da Silva, P.L., Freire, I.L.: A family of wave-breaking equations generalizing the Camassa–Holm and Novikov equations. J. Math. Phys. 56, 091506 (2015)

[2] Bahouri, H., Chemin, J., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Springer, Berlin (2011)

[3] Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 71, 1661–1664 (1993)

[4] Chen, D., Li, Y., Yan, W.: On the Cauchy problem for a generalized Camassa–Holm equation. Discr. Contin. Dyn. Syst. 35(3), 871–889 (2015)

[5] Constantin, A.: The Hamiltonian structure of the Camassa–Holm equation. Expo. Math. 15(1), 53–85 (1997)

[6] Constantin, A., Escher, J.: Global existence and blow-up for a shallow water equation. Ann. Scuola Norm. Sup. Pisa Classe Sci. 26, 303–328 (1998)

[7] Constantin, A., Escher, J.: Well-posedness, global existence, and blow up phenomena for a periodic quasi-linear hyperbolic equation. Commun. Pure Appl. Math. 51, 475–504 (1998)

[8] Constantin, A.: On the scattering problem for the Camassa–Holm equation. Proc. R. Soc. Lond. Ser. A 457, 953–970 (2001)

[9] Constantin, A., Escher, J.: Particle trajectories in solitary water waves. Bull. Am. Math. Soc. 44, 423–431 (2007)

[10] Constantin, A., Lannes, D.: The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. Arch. Ration. Mech. Anal. 192, 165–186 (2009)

[11] Constantin, A., Escher, J.: Analyticity of periodic traveling free surface water waves with vorticity. Ann. Math. 173, 559–568 (2011)

[12] Danchin, R.: A few remarks on the Camassa–Holm equation. Differ. Integral Equ. 14, 953–988 (2001)
[13] Danchin, R.: A note on well-posedness for Camassa–Holm equation. J. Differ. Equ. 192, 429–444 (2003)
[14] Fokas, A., Fuchssteiner, B.: Symplectic structures, their Bäcklund transformation and hereditary symmetries. Phys. D 4, 47–60 (1981/1982)
[15] Grayshan, K., Himonas, A.: Equations with peakon traveling wave solutions. Adv. Dyn. Syst. Appl. 8, 217–232 (2013)
[16] Guo, Z., Liu, X., Molinet, L., Yin, Z.: Ill-posedness of the Camassa–Holm and related equations in the critical space. J. Differ. Equ. 266, 1698–1707 (2019)
[17] Himonas, A., Misiołek, G.: High-frequency smooth solutions and well-posedness of the Camassa–Holm equation. Int. Math. Res. Notices 51, 3135–3151 (2005)
[18] Himonas, A., Misiołek, G., Ponce, G.: Non-uniform continuity in $H^1$ of the solution map of the CH equation. Asian J. Math. 11, 141–150 (2007)
[19] Himonas, A., Kenig, C.: Non-uniform dependence on initial data for the CH equation on the line. Differ. Integral Equ. 22, 201–224 (2009)
[20] Himonas, A., Kenig, C., Misiołek, G.: Non-uniform dependence for the periodic CH equation. Commun. Partial Differ. Equ. 35, 1145–1162 (2010)
[21] Himonas, A., Holliman, C.: The Cauchy problem for the Novikov equation. Nonlinearity 25, 449–479 (2012)
[22] Himonas, A., Holliman, C.: The Cauchy problem for a generalized Camassa–Holm equation. Adv. Differ. Equ. 19, 161–200 (2014)
[23] Home, A., Wang, J.: Integrable peakon equations with cubic nonlinearity. J. Phys. A 41(372002), 1–11 (2008)
[24] Koch, H., Tzvetkov, N.: Nonlinear wave interactions for the Benjamin–Ono equation. Int. Math. Res. Not. 30, 1833–1847 (2005)
[25] Li, J., Yin, Z.: Remarks on the well-posedness of Camassa–Holm type equations in Besov spaces. J. Differ. Equ. 261, 6125–6143 (2016)
[26] Li, J., Yin, Z.: Well-posedness and analytic solutions of the two-component Euler–Poincaré system. Monatsh. Math. 183, 509–537 (2017)
[27] Li, J., Yu, Y., Zhu, W.: Non-uniform dependence on initial data for the Camassa–Holm equation in Besov spaces. J. Differ. Equ. 269, 8686–8700 (2020)
[28] Li, J., Li, M., Zhu, W.: Non-uniform dependence for Novikov equation in Besov spaces. J. Math. Fluid Mech. 22, 4–50 (2020)
[29] Li, J., Wu, X., Zhu, W., Guo, J.: Non-uniform continuity of the generalized Camassa–Holm equation in Besov spaces. arXiv: 2008.00647v1
[30] Li, J., Wu, X., Yu, Y., Zhu, W.: Non-uniform dependence on initial data for the Camassa–Holm equation in the critical Besov space. J. Math. Fluid Mech. 23, 36 (2021)
[31] Ni, L., Zhou, Y.: Well-posedness and persistence properties for the Novikov equation. J. Differ. Equ. 250, 3002–3021 (2011)
[32] Novikov, V.: Generalization of the Camassa–Holm equation. J. Phys. A 42, 342002 (2009)
[33] Wu, X., Yin, Z.: Well-posedness and global existence for the Novikov equation. Ann. Sc. Norm. Super. Pisa Classe Sci. Ser. V 11, 707–727 (2012)
[34] Wu, X., Yin, Z.: A note on the Cauchy problem of the Novikov equation. Appl. Anal. 92, 1116–1137 (2013)
[35] Yan, K.: Wave breaking and global existence for a family of peakon equations with high order nonlinearity. Nonlinear Anal. Real World Appl. 45, 721–735 (2019)
[36] Yan, W., Li, Y., Zhang, Y.: The Cauchy problem for the integrable Novikov equation. J. Differ. Equ. 253, 298–318 (2012)
[37] Zhao, Y., Li, Y., Yan, W.: local well-posedness and persistence property for the generalized Novikov equation. Discr. Contin. Dyn. Syst. 34(2), 803–820 (2014)

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(accepted: September 30, 2021; published online: October 12, 2021)