LOCAL MOVES ON SPATIAL GRAPHS
AND FINITE TYPE INVARIANTS

KOUKI TANIYAMA
Department of Mathematics, School of Education, Waseda University
Nishi-Waseda 1-6-1, Shinjuku-ku, Tokyo 169-8050, Japan
e-mail: taniyama@mn.waseda.ac.jp

AKIRA YASUHARA
Department of Mathematics, Tokyo Gakugei University
Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan
Current address, October 1, 1999 to September 30, 2001:
Department of Mathematics, The George Washington University
Washington, DC 20052, USA
e-mail: yasuhara@u-gakugei.ac.jp

Abstract
We define $A_k$-moves for embeddings of a finite graph into the 3-sphere for each natural number $k$. Let $A_k$-equivalence denote an equivalence relation generated by $A_k$-moves and ambient isotopy. $A_k$-equivalence implies $A_{k-1}$-equivalence. Let $\mathcal{F}$ be an $A_{k-1}$-equivalence class of the embeddings of a finite graph into the 3-sphere. Let $\mathcal{G}$ be the quotient set of $\mathcal{F}$ under $A_k$-equivalence. We show that the set $\mathcal{G}$ forms an abelian group under a certain geometric operation. We define finite type invariants on $\mathcal{F}$ of order $(n;k)$. And we show that if any finite type invariant of order $(1;k)$ takes the same value on two elements of $\mathcal{F}$, then they are $A_k$-equivalent. $A_k$-move is a generalization of $C_k$-move defined by K. Habiro. Habiro showed that two oriented knots are the same up to $C_k$-move and ambient isotopy if and only if any Vassiliev invariant of order $\leq k - 1$ takes the same value on them. The ‘if’ part does not hold for two-component links. Our result gives a sufficient condition for spatial graphs to be $C_k$-equivalent.

2000 Mathematics Subject Classification: Primary 57M15; Secondary 57M25, 57M27
Short Running Title: Local Moves on Spatial Graphs and Finite Type Invariants
Introduction

K. Habiro defined a local move, $C_k$-move, for each natural number $k$ [2]. It is known that if two embeddings $f$ and $g$ of a graph into the three sphere are the same up to $C_k$-move and ambient isotopy, then $g$ can be deformed into a band sum of $f$ with certain $(k+1)$-component links and that changing position of a band and an arc, which is called a band trivialization of $C_k$-move, is realized by $C_{k+1}$-moves and ambient isotopy [17]. This is one of the most important properties of $C_k$-move. We consider local moves which have this property. We define $A_1$-move as the crossing change and $A_{k+1}$-move as a band trivialization of $A_k$-move; see Section 1 for the precise definition. So $A_k$-move is a generalization of $C_k$-move. In fact, the results for $A_k$-move in this paper hold for $C_k$-move.

Let $A_k$-equivalence denote an equivalence relation given by $A_k$-moves and ambient isotopy. Habiro showed that two oriented knots are $C_k$-equivalent if and only if they have the same Vassiliev invariants of order $\leq k-1$ [3],[4]. The ‘only if’ part of this result is true for $A_k$-move and for the embeddings of a graph, in particular for links (Theorem 5.1). However the ‘if’ part does not hold for two-component links. For example, the Whitehead link is not $C_3$-equivalent to a trivial link because they have different Arf invariants, see [16]. On the other hand, H. Murakami showed in [8] that the Vassiliev invariants of links of order $\leq 2$ are determined by the linking numbers and the second coefficient of the Conway polynomial of each component. Hence, the values of any Vassiliev invariant of order $\leq 2$ of these two links are the same. So we note that Vassiliev invariants of order $\leq k-1$ are not enough to characterize $C_k$-equivalent embeddings of a graph.

We will define in Section 1 a finite type invariant of order $(n;k)$ as a generalization of a Vassiliev invariant and see that if any finite type invariants of order $(1;k)$ takes the same value on two $A_{k-1}$-equivalent embeddings of a graph, then they are $A_k$-equivalent (Theorem 1.1). While a Vassiliev invariant is defined by the change in its value at every ‘wall’ corresponding to a crossing change, a finite type invariant of order $(n;k)$ is defined similarly by ‘walls’ corresponding to $A_k$-moves. A finite type invariant of order $(n;1)$ is a Vassiliev invariant of order $\leq n$.

It is shown that the set of $C_k$-equivalence classes of knots forms an abelian group under the connected sum [3],[4]. This is also true for $A_k$-equivalence classes. Since the connected sum is peculiar to knots, we cannot apply it to embeddings of a graph. In Section 2, we will define a certain geometric sum for the elements in an $A_{k-1}$-equivalence class of the embedding of a graph. Then we will see that the quotient set of the $A_{k-1}$-equivalence class
under $A_k$-equivalence forms an abelian group (Theorem 2.4).

It is not essential that $A_1$-move is the crossing change. This is a big difference between $A_k$-move and $C_k$-move. We will study a generalization of $A_k$-move in Section 4. For example, if we put $A_1$-move to be the $\#$-move defined by Murakami [3], then we get several results similar to that for original $A_k$-move.

1. $A_k$-Moves and Finite Type Invariants

Let $B^3$ be the oriented unit 3-ball. A tangle is a disjoint union of properly embedded arcs in $B^3$. A tangle is trivial if it is contained in a properly embedded 2-disk in $B^3$. A trivialization of a tangle $T = t_1 \cup t_2 \cup \cdots \cup t_k$ is a choice of mutually disjoint disks $D_1, D_2, \ldots, D_k$ in $B^3$ such that $D_i = (D_i \cap \partial B^3) \cup t_i$ for $i = 1, 2, \ldots, k$. It can be shown that in general a trivialization is not unique up to ambient isotopy of $B^3$ fixed on the tangle.

Let $T$ and $S$ be tangles, and let $t_1, t_2, \ldots, t_k$ and $s_1, s_2, \ldots, s_k$ be the components of $T$ and $S$ respectively. Suppose that for each $t_i$ there exists some $s_j$ such that $\partial t_i = \partial s_j$. Then we call the ordered pair $(T, S)$ a local move, which can be interpreted as substituting $S$ for $T$. Two local moves $(T, S)$ and $(T', S')$ are equivalent if there exists an orientation preserving homeomorphism $h : B^3 \to B^3$ such that $h(T) = T'$ and $h(S)$ is ambient isotopic to $S'$ relative to $\partial B^3$. We consider local moves up to this equivalence.

Let $(T, S)$ be a local move such that $T$ and $S$ are trivial tangles. First choose a trivialization $D_1, D_2, \ldots, D_k$ of $T$. Each $D_i$ intersects $\partial B^3$ in an arc $\gamma_i$. Let $E_i$ be a small regular neighbourhood of $\gamma_i$ in $\partial B^3$. We devide the circle $\partial E_i$ into two arcs $\alpha_i$ and $\beta_i$ such that $\alpha_i \cap \beta_i = \partial \alpha_i = \partial \beta_i$. By slightly perturbing into $\alpha_i$ and into $\beta_i$ into the interior of $B^3$ on either side of $D_i$, we obtain properly embedded arcs $\bar{\alpha}_i$ and $\bar{\beta}_i$. We consider $k$ local moves $(S \cup \bar{\alpha}_i, S \cup \bar{\beta}_i)$ ($i = 1, 2, \ldots, k$) and call them the band trivializations of the local move $(T, S)$ with respect to the trivialization $D_1, D_2, \ldots, D_k$. Note that both $S \cup \bar{\alpha}_i$ and $S \cup \bar{\beta}_i$ are trivial tangles.

We now inductively define a sequence of local moves on trivial tangles in $B^3$ which depend on the choice of trivialization. An $A_1$-move is the crossing change shown in Figure 1.1. Suppose that $A_k$-moves are defined and there are $l$ $A_k$-moves $(T_1, S_1), (T_2, S_2), \ldots, (T_l, S_l)$ up to equivalence. For each $A_k$-move $(T_i, S_i)$ ($i = 1, 2, \ldots, l$), we choose a single trivialization $\tau_i = \{D_{i,1}, D_{i,2}, \ldots, D_{i,k+1}\}$ of $T_i$ and fix it. (The choice of $\tau_i$ is independent of the trivialization that is chosen to define $A_k$-move $(T_i, S_i)$.) Then the band trivializations of $(T_i, S_i)$ with respect to the trivialization $\tau_i$ are called $A_{k+1}(\tau_i)$-moves and these
$A_{k+1}(\tau_i)$-moves ($i = 1, 2, \ldots, l$) are called $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-moves. Note that the number of $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-moves is at most $l(k + 1)$ up to equivalence. Although the choice of trivializations is important for the definition of $A_k$-move, our proof is the same for every choice. Therefore the results of this paper hold for every choice of trivializations $\tau_1, \tau_2, \ldots, \tau_l$. So we denote $A_{k+1}(\tau_1, \tau_2, \ldots, \tau_l)$-move simply as $A_{k+1}$-move. It is known that $C_k$-move defined by Habiro is a special case of $A_k$-move for certain choices of trivializations; see [2], [10]. We will see that $A_k$-move, as well as $C_k$-move, has the property mentioned in Introduction (Proposition 2.1 and Lemma 2.2).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure1.png}
\caption{Figure 1.1}
\end{figure}

*Examples.* (1) The trivialization of a tangle in Figure 1.1 is unique up to ambient isotopy. Therefore we have any band trivialization of an $A_1$-move is equivalent to the local move in Figure 1.2-(i). Thus $A_2$-move is unique up to equivalence. It is not hard to see that an $A_2$-move is equivalent to the *delta move* in Figure 1.2-(ii) defined by H. Murakami and Y. Nakanishi [7], and then it is equivalent to the local move in Figure 1.2-(iii).

(2) If we choose a trivialization for the $A_2$-move as in Figure 1.3-(i), then, by the symmetry of the $A_2$-move, any $A_3$-move is equivalent to the local move in Figure 1.3-(ii).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure2.png}
\caption{Figure 1.2}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3.png}
\caption{Figure 1.3}
\end{figure}
A local move \((S,T)\) is called the inverse of a local move \((T,S)\). It is clear that the inverse of an \(A_1\)-move is again an \(A_1\)-move. By the definition of \(A_k\)-move, we see that the inverse of an \(A_k\)-move with \(k \geq 2\) is equivalent to itself.

Let \((T,S)\) be an \(A_k\)-move and \(D_1, D_2, \ldots, D_{k+1}\) the fixed trivialization of \(T = t_1 \cup t_2 \cup \cdots \cup t_{k+1}\). We set \(\alpha = \partial B^3 \cap (D_1 \cup D_2 \cup \cdots \cup D_{k+1})\) and \(\beta = S\). A link \(L\) in \(S^3\) is called type \(k\) if there is an orientation preserving embedding \(\varphi : B^3 \to S^3\) such that \(L = \varphi(\alpha \cup \beta)\). Then the pair \((\alpha, \beta)\) is called a link model of \(L\).

We now define an equivalence relation on spatial graphs by \(A_k\)-move. Let \(G\) be a finite graph. Let \(V(G)\) denote the set of the vertices of \(G\). Let \(f, g : G \to S^3\) be embeddings. We say that \(f\) and \(g\) are related by an \(A_k\)-move if there is an \(A_k\)-move \((T,S)\) and an orientation preserving embedding \(\varphi : B^3 \to S^3\) such that

\[
\begin{align*}
&\text{(i)} \quad \text{if } f(x) \neq g(x) \text{ then both } f(x) \text{ and } g(x) \text{ are contained in } \varphi(\text{int}B^3), \\
&\text{(ii)} \quad f(V(G)) = g(V(G)) \text{ is disjoint from } \varphi(\text{int}B^3), \quad \text{and} \\
&\text{(iii)} \quad f(G) \cap \varphi(\text{int}B^3) = \varphi(T) \text{ and } g(G) \cap \varphi(\text{int}B^3) = \varphi(S).
\end{align*}
\]

We also say that \(g\) is obtained from \(f\) by an application of \((T,S)\). We define \(A_k\)-equivalence as an equivalence relation on the set of all embeddings of \(G\) into \(S^3\) given by the relation above and ambient isotopy. For an embedding \(f : G \to S^3\), let \([f]_k\) denote the \(A_k\)-equivalence class of \(f\). By the definition of \(A_k\)-move we see that an application of an \(A_{k+1}\)-move is realized by two applications of \(A_k\)-move and ambient isotopy. Thus \(A_{k+1}\)-equivalence implies \(A_k\)-equivalence. In other words we have \([f]_1 \supset [f]_2 \supset \cdots \supset [f]_{k+1} \supset \cdots\).

Let \(f : G \to S^3\) be an embedding, \(L_i\) links of type \(k\) and \((\alpha_i, \beta_i)\) their link models \((i = 1, 2, \ldots, n)\). Let \(I = [0,1]\) be the unit closed interval. An embedding \(g : G \to S^3\) is called a band sum of \(f\) with \(L_1, L_2, \ldots, L_n\) if there are mutually disjoint embeddings \(b_{ij} : I \times I \to S^3\) \((i = 1, 2, \ldots, n, j = 1, 2, \ldots, k+1)\) and mutually disjoint orientation preserving embeddings \(\varphi_i : B^3 \to S^3 - f(G)\) with \(L_i = \varphi_i(\alpha_i \cup \beta_i)\) \((i = 1, 2, \ldots, n)\) such that the following (i) and (ii) hold:

\[
\begin{align*}
&\text{(i)} \quad b_{ij}(I \times I) \cap f(G) = b_{ij}(I \times I) \cap f(G - V(G)) = b_{ij}(I \times \{0\}) \quad \text{and} \\
&\quad b_{ij}(I \times I) \cap (\bigcup_i \varphi_i(B^3)) = b_{ij}(I \times \{1\}) \text{ is a component of } \varphi_i(\alpha_i) \text{ for any } i, j \quad (i = 1, 2, \ldots, n, j = 1, 2, \ldots, k+1). \\
&\text{(ii)} \quad f(x) = g(x) \text{ if } f(x) \text{ is not contained in } \bigcup_{i,j} b_{ij}(I \times \{0\}) \text{ and}
\end{align*}
\]
\[ g(G) = (f(G) \cup \bigcup_i L_i - \bigcup_{i,j} b_{ij}(I \times \partial I)) \cup \bigcup_{i,j} b_{ij}(\partial I \times I). \]

Then we denote \( g \) by \( F(f; \{L_1, L_2, \ldots, L_n\}, \{B_1, B_2, \ldots, B_n\}) \), where \( B_i = b_{i1}(I \times I) \cup b_{i2}(I \times I) \cup \cdots \cup b_{ik+1}(I \times I) \) (\( i = 1, 2, \ldots, n \)). We call each \( b_{ij}(I \times I) \) a band. We call each \( \varphi_i(B^3) \) an associated ball of \( L_i \). See Figure 1.4 for an example of a band sum of an embedding \( f \) with links \( L_1, L_2, L_3 \) of type 3.

\[ \text{Figure 1.4} \]

**Remark.** It follows from the definition that if \( g \) is a band sum of \( f \) with some links of type \( k \), then \( g \) is \( A_k \)-equivalent to \( f \). The converse is also true and will be shown in Proposition 2.1. In Lemma 2.2, we show that the position of a band is changeable up to \( A_{k+1} \)-equivalence. The origin of the name ‘band trivialization’ comes from this fact.

Let \( h : G \to S^3 \) be an embedding and \( H \) an abelian group. Let \( \varphi : [h]_{k-1} \to H \) be an invariant. We say that \( \varphi \) is a **finite type invariant of order** \( (n; k) \) if for any embedding \( f \in [h]_{k-1} \) and any band sum \( F(f; \{L_1, L_2, \ldots, L_{n+1}\}, \{B_1, B_2, \ldots, B_{n+1}\}) \) of \( f \) with links \( L_1, L_2, \ldots, L_{n+1} \) of type \( k - 1 \),

\[ \sum_{X \subset \{1, 2, \ldots, n+1\}} (-1)^{|X|} \varphi(F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})) = 0 \in H, \]

where the sum is taken over all subsets, including the empty set, and \( |X| \) is the number of the elements in \( X \).
In the next section we show the following theorem.

**Theorem 1.1.** Let $f, g : G \to S^3$ be $A_{k-1}$-equivalent embeddings. Then they are $A_k$-equivalent if and only if $\varphi(f) = \varphi(g)$ for any finite type $A_k$-equivalence invariant $\varphi$ of order $(1; k)$.

Note that finite type invariants of order $(n; 2)$ coincide with Vassiliev invariants of order $n$. It is shown in [5, Theorem 1.1, Theorem 1.3] that two embeddings of a finite graph $G$ into $S^3$ are $A_2$-equivalent if and only if they have the same Wu invariant [21]. It follows from [14, Section 2] that Wu invariant is a finite type invariant of order $(1; 2)$. Since two embeddings are always $A_1$-equivalent, we have the following corollary.

**Corollary 1.2.** Let $f, g : G \to S^3$ be embeddings. Then the following conditions are mutually equivalent.

(i) $f$ and $g$ are $A_2$-equivalent.

(ii) $f$ and $g$ have the same Wu invariant.

(iii) $\varphi(f) = \varphi(g)$ for any Vassiliev invariant $\varphi$ of order 1. □

In Section 5 we show the following proposition.

**Proposition 1.3.** Let $\varphi$ be a Vassiliev invariant of order $(n + 1)(k - 1) - 1$. Then $\varphi$ is a finite type invariant of order $(n; k)$.

2. $A_k$-Equivalence Group of Spatial Graphs

The following proposition is a natural generalization of [20, Lemma] and stems from the fact that a knot with the unknotting number $u$ can be unknotted by changing $u$ crossings of a regular diagram of it [12], [18].

**Proposition 2.1.** Let $f, g : G \to S^3$ be embeddings. If $f$ and $g$ are $A_k$-equivalent, then $g$ is ambient isotopic to a band sum of $f$ with some links of type $k$.

**Proof.** We consider the embeddings up to ambient isotopy for simplicity. By the assumption there is a finite sequence of embeddings $f = f_0, f_1, ..., f_n = g$ and orientation preserving embeddings $\varphi_1, \varphi_2, ..., \varphi_n : B^3 \to S^3$ such that $(\varphi_i^{-1}(f_{i-1}(G)), \varphi_i^{-1}(f_i(G)))$ is an $A_k$-move for each $i$. We shall prove this proposition by induction on $n$. 7
First we consider the case $n = 1$. Let $D_1, D_2, ..., D_{k+1}$ be the fixed trivialization of the tangle $\varphi_1^{-1}(f_0(G))$ and $\gamma_j = D_j \cap \partial B^3$ ($j = 1, 2, ..., k + 1$). Then $L = \bigcup_j \varphi_1(\gamma_j) \cup (\varphi_1(B^3) \cap f_1(G))$ is a link of type $k$. By taking a small one-sided collar for each $\varphi_1(\gamma_j)$ in $S^3 - \varphi_1(\text{int } B^3)$, we have mutually disjoint embeddings $b_j : I \times I \rightarrow S^3$ ($j = 1, 2, ..., k + 1$) such that $b_j(I \times I) \cap \varphi_1(B^3) = b_j(I \times \{1\})$ and $b_j(I \times I) \cap f_0(G) = b_j(I \times I) \cap f_1(G) = b_j(\partial I \times I)$. Then we deform $f_0$ up to ambient isotopy keeping the image $f_1(G)$ so that neither the associated balls of $L$ nor the bands in $B$ intersect $\varphi_1(B^3)$. Note that this deformation is possible, since the tangle $\varphi_1^{-1}(f_1(G))$ is trivial. In fact, sweeping out the associated balls, band-slidings and sweeping out the bands are sufficient. See Figure 2.1. Then by the same arguments as that in the case $n = 1$, we find that $f_1$ is a band sum $F(f_0; \{L\}, \{B\})$. Then we have

$$F(F(f; \{L\}, \{B\}); L, B) = F(f; \{L\} \cup L, \{B\} \cup B).$$

This completes the proof. □

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1.png}
\caption{Figure 2.1}
\end{figure}

As we mentioned before, the origin of the name ‘band trivialization’ comes from the following lemma.

**Lemma 2.2.** The moves in Figures 2.2-(i), (ii), (iii) and (iv) are realized by $A_{k+1}$-moves.
Proof. The move in Figure 2.2-(i) is just a band trivialization of an $A_k$-move. Hence by the definition it is an $A_{k+1}$-move. It is easy to see that the moves in Figures 2.2-(ii) and (iii) are generated by the moves in Figure 2.2-(i). To see that the move in Figure 2.2-(iv) is realized by $A_{k+1}$-moves, we first slide the bands as illustrated in Figure 2.3, and then perform the moves in Figure 2.2-(i). \hfill \Box

Let $h : G \rightarrow S^3$ be an embedding and let $[f_1], [f_2] \in [h]_{k-1}/(A_k\text{-equivalence})$, where $[h]_{k-1}/(A_k\text{-equivalence})$ denotes the set of $A_k$-equivalence classes in $[h]_{k-1}$. Since both $f_1$ and $f_2$ are $A_{k-1}$-equivalent to $h$, by Proposition 2.1, there are band sums $F(h; \mathcal{L}_i, \mathcal{B}_i) \in [f_i]_k$ of $h$ with links $\mathcal{L}_i$ of type $k - 1 (i = 1, 2)$. Suppose that the bands in $\mathcal{B}_1$ and the associated balls of $\mathcal{L}_1$ are disjoint from the bands in $\mathcal{B}_2$ and the associated balls of $\mathcal{L}_2$. Note that up to slight ambient isotopy of $F(h; \mathcal{L}_2, \mathcal{B}_2)$ that preserves $h(G)$ we can always choose the bands and the associated balls so that they satisfy this condition. In the following we assume this condition without explicit mention. Then we have a new band sum $F(h; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$.
We define
\[ [f_1]_k +_h [f_2]_k = [F(h; L_1 \cup L_2, B_1 \cup B_2)]_k. \]

**Lemma 2.3.** The sum ‘+h’ above is well-defined.

**Proof.** It is sufficient to show for two embeddings \( F(h; L_1, B_1), F(h; L'_1, B'_1) \in [f]_k \) that \( F(h; L_1 \cup L_2, B_1 \cup B_2) \) and \( F(h; L'_1 \cup L_2, B'_1 \cup B_2) \) are \( A_k \)-equivalent. Consider a sequence of ambient isotopies and applications of \( A_k \)-moves that deforms \( F(h; L_1, B_1) \) into \( F(h; L'_1, B'_1) \).

We consider this sequence of deformations together with the links in \( L_2 \) and the bands in \( B_2 \). Whenever we apply an \( A_k \)-move we deform the associated balls of \( L_2 \) and the bands in \( B_2 \) up to ambient isotopy so that they are away from the 3-ball within which the \( A_k \)-move is applied. Thus \( F(h; L_1 \cup L_2, B_1 \cup B_2) = F(F(h; L'_1, B'_1); L_2, B_2) \) is \( A_k \)-equivalent to a band sum \( F(F(h; L'_1, B'_1); L'_2, B'_2) \) for some \( L'_2 \) and \( B'_2 \). Compare the band sums \( F(F(h; L'_1, B'_1); L'_2, B'_2) \) and \( F(h; L'_1 \cup L_2, B'_1 \cup B_2) = F(F(h; L'_1, B'_1); L_2, B_2) \). We have that the links in \( L'_2 \) are ambient isotopic to the links in \( L_2 \). It follows from Lemma 2.2 that the bands in \( B'_2 \) can be deformed into the position of the bands in \( B_2 \) by band slidings and \( A_k \)-moves. Thus these two are \( A_k \)-equivalent. \( \square \)

**Theorem 2.4.** The set \( [h]_k^{-1}/(A_k\text{-equivalence}) \) forms an abelian group under ‘+h’ with the unit element \( [h]_k \).

We denote this group by \( G_k(h; G) \) and call it the \( A_k \)-equivalence group of the spatial embeddings of \( G \) with the unit element \( [h]_k \).

**Remark.** Note that for any graph \( G \) and any embedding \( h : G \rightarrow S^3 \), \( [h]_1 \) is equal to the set of all embeddings of \( G \) into \( S^3 \). In [19], the second author called \( G_2(h; G) \) a graph homology group and gave a practical method of calculating this group.

**Proof.** We consider embeddings up to ambient isotopy for simplicity. It is sufficient to show that for any \( [f]_k \in [h]_k^{-1}/(A_k\text{-equivalence}) \), there is an inverse of \( [f]_k \). Since \( f \) and \( h \) are \( A_{k-1} \)-equivalent, by Proposition 2.1, \( f \) and \( h \) are band sums \( F(h; \mathcal{L}, \mathcal{B}) \) and \( F(f; \mathcal{L}', \mathcal{B}') \) respectively, where \( \mathcal{L} \) and \( \mathcal{L}' \) are sets of links of type \( k-1 \). Thus we have \( h = F(F(h; \mathcal{L}, \mathcal{B}); \mathcal{L}', \mathcal{B}') \). Then, by using Lemma 2.2, we deform the associated balls of \( \mathcal{L}' \) and the bands in \( \mathcal{B}' \) up to \( A_k \)-equivalence so that they are disjoint form the associated balls of \( \mathcal{L} \) and the bands in \( \mathcal{B} \). Thus we see that \( h = F(F(h; \mathcal{L}, \mathcal{B}); \mathcal{L}', \mathcal{B}') \) is \( A_k \)-equivalent to a band sum \( F(h; \mathcal{L} \cup \mathcal{L}'', \mathcal{B} \cup \mathcal{B}'') \) for some \( \mathcal{L}'' \) and \( \mathcal{B}'' \) (for example see Figure 2.4). Thus we have
\[ [f]_k +_h [F(h; \mathcal{L}'', \mathcal{B}'')]_k = [F(h; \mathcal{L}, \mathcal{B})]_k + [F(h; \mathcal{L}'', \mathcal{B}'')]_k = [F(h; \mathcal{L} \cup \mathcal{L}'', \mathcal{B} \cup \mathcal{B}'')]_k = [h]_k. \]
This implies that \([F(h; L', B')]_k\) is an inverse of \([f]_k\). □

\[\text{Figure 2.4}\]

**Theorem 2.5.** Let \(h_1, h_2 : G \rightarrow S^3\) be \(A_{k-1}\)-equivalent embeddings. Then the groups \(G_k(h_1; G)\) and \(G_k(h_2; G)\) are isomorphic.

**Proof.** We define a map \(\phi : G_k(h_1; G) \rightarrow G_k(h_2; G)\) by \(\phi([f]_k) = [f]_k - h_2 [h_1]_k\), where \([x]_k - h_2 [y]_k\) denotes \([x]_k + h_2 (-[y]_k)\). Clearly this map is a bijection. We shall prove that \(\phi\) is a homomorphism. Let \([f_i]_k \in G_k(h_1; G)\) \((i = 1, 2)\). Then \(f_i = F(h_i; \mathcal{L}_i, \mathcal{B}_i)\) where \(\mathcal{L}_i\) is a set of links of type \(k - 1\) \((i = 1, 2)\). Since \(h_1\) and \(h_2\) are \(A_{k-1}\)-equivalent we see that \(h_1 = F(h_2; \mathcal{L}, \mathcal{B})\) where \(\mathcal{L}\) is a set of links of type \(k - 1\). Thus we have \(f_i = F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)\) \((i = 1, 2)\). By using Lemma 2.2, we deform \(f_i\) up to \(A_k\)-equivalence so that the associated balls of \(\mathcal{L}_i\) and the bands in \(\mathcal{B}_i\) are disjoint from the associated balls of \(\mathcal{L}\) and the bands in \(\mathcal{B}\) for \(i = 1, 2\). We may further assume that the
associated balls of $\mathcal{L}_1$ and the bands in $\mathcal{B}_1$ are disjoint from the associated balls of $\mathcal{L}_2$ and the bands in $\mathcal{B}_2$. Then we have

$$\phi([f_1]_k + h_1 [f_2]_k) = \phi([F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k)$$

$$= [F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k - h_2 [f_1]_k$$

$$= [F(h_2; \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2)]_k - h_2 [F(h_2; \mathcal{L}, \mathcal{B})]_k$$

$$= [F(h_2; \mathcal{L} \cup \mathcal{L}_1, \mathcal{B} \cup \mathcal{B}_1)]_k - h_2 [F(h_2; \mathcal{L}, \mathcal{B})]_k$$

and for each $i (i = 1, 2)$,

$$\phi([f_i]_k) = \phi([F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)]_k)$$

$$= [F(F(h_2; \mathcal{L}, \mathcal{B}); \mathcal{L}_i, \mathcal{B}_i)]_k - h_2 [f_1]_k$$

$$= [F(h_2; \mathcal{L} \cup \mathcal{L}_i, \mathcal{B} \cup \mathcal{B}_i)]_k - h_2 [F(h_2; \mathcal{L}, \mathcal{B})]_k$$

$$= [F(h_2; \mathcal{L}, \mathcal{B}_i)]_k.$$  

Thus we have $\phi([f_1]_k + h_1 [f_2]_k) = \phi([f_1]_k) + h_2 \phi([f_2]_k).$  

\[\Box\]

**Proposition 2.6.** The projection $p : [h]_{k-1} \longrightarrow [h]_{k-1}/(A_k\text{-equivalence}) = \mathcal{G}_k(h; G)$ is a finite type $A_k\text{-equivalence invariant of order } (1; k).$

**Proof.** It is clear that $p$ is an $A_k$-equivalence invariant. We shall prove that $p$ is finite type of order $(1; k)$. Let $f \in [h]_{k-1}$ be an embedding and $F(f; \{L_1, L_2\}, \{B_1, B_2\})$ a band sum of $f$ with links $L_1, L_2$ of type $k - 1$. Then it is sufficient to show that

$$\sum_{X \subset \{1, 2\}} (-1)^{|X|} p(F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})) = [h]_k.$$  

Let $\phi : \mathcal{G}_k(f; G) \longrightarrow \mathcal{G}_k(h; G)$ be the isomorphism defined by $\phi([g]_k) = [g]_k - h [f]_k$. Then we have

$$\phi([F(f; \emptyset, \emptyset)]_k - f [F(f; \{L_1\}, \{B_1\})]_k - f [F(f; \{L_2\}, \{B_2\})]_k + f [F(f; \{L_1, L_2\}, \{B_1, B_2\})]_k)$$

$$= ([F(f; \emptyset, \emptyset)]_k - h [f]_k) - h ([F(f; \{L_1\}, \{B_1\})]_k - h [f]_k)$$

$$- h ([F(f; \{L_2\}, \{B_2\})]_k - h [f]_k) + h ([F(f; \{L_1, L_2\}, \{B_1, B_2\})]_k - h [f]_k)$$

$$= [F(f; \emptyset, \emptyset)]_k - h [F(f; \{L_1\}, \{B_1\})]_k - h [F(f; \{L_2\}, \{B_2\})]_k + h [F(f; \{L_1, L_2\}, \{B_1, B_2\})]_k$$

$$= \sum_{X \subset \{1, 2\}} (-1)^{|X|} p(F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})).$$  

Since

$$[F(f; \{L_1, L_2\}, \{B_1, B_2\})]_k = [F(f; \{L_1\}, \{B_1\})]_k + f [F(f; \{L_2\}, \{B_2\}]_k,$$

we have

$$\phi([F(f; \emptyset, \emptyset)]_k - f [F(f; \{L_1\}, \{B_1\})]_k - f [F(f; \{L_2\}, \{B_2\})]_k + f [F(f; \{L_1, L_2\}, \{B_1, B_2\})]_k)$$

$$= \phi([f]_k) = [h]_k.$$  

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This completes the proof. □

Proof of Theorem 1.1. The ‘only if’ part is clear. We show the ‘if’ part. Let \( f \) and \( g \) be embeddings in \([h]_{k-1}\). Suppose that any finite type invariant of order \((1; k)\) takes the same value on \( f \) and \( g \). Then by Proposition 2.6 we have \( p(f) = p(g) \), where \( p : [h]_{k-1} \rightarrow [h]_{k-1}/(A_k\text{-equivalence}) = G_k(h; G) \) is the projection. Hence we have \([f]_k = [g]_k\). This completes the proof. □

3. A_k-Equivalence Group of Knots

In this section we only consider the case that the graph \( G \) is homeomorphic to a disjoint union of circles. Let \( G = S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu \). Then there is a natural correspondence between the ambient isotopy classes of the embeddings of \( G \) into \( S^3 \) and the ambient isotopy classes of the ordered oriented \( \mu \)-component links in \( S^3 \). Therefore instead of specifying an embedding \( h : S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu \rightarrow S^3 \), we denote by \( L \) the image \( h(S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu) \) and consider it together with the orientation of each component and the ordering of the components. Thus \( G_k(L) \) denotes the \( A_k \)-equivalence group \( G_k(h; S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu) \) with the unit element \([h]_k\).

Theorem 3.1. Let \( O \) be a trivial knot. Then for any oriented knot \( K \), \( G_k(O) \) and \( G_k(K) \) are isomorphic.

Remark. For a graph \( G(\neq S^1) \) and embeddings \( h, h' : G \rightarrow S^3 \), \( G_k(h; G) \) and \( G_k(h'; G) \) are not always isomorphic. In fact there are two-component links \( L_1 \) and \( L_2 \) such that \( G_3(L_1) \cong \mathbb{Z} \oplus \mathbb{Z} \) and \( G_3(L_2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) [10].

Proof. We define a map \( \phi : G_k(O) \rightarrow G_k(K) \) by \( \phi([F(O; \mathcal{L}, \mathcal{B})]_k) = [K\#F(O; \mathcal{L}, \mathcal{B})]_k \) for each \([F(O; \mathcal{L}, \mathcal{B})]_k \in G_k(O)\), where \( \mathcal{L} \) is a set of links of type \( k-1 \) and \( \# \) means the connected sum of oriented knots. Clearly this is well-defined. By Lemma 2.2, any band sum \( F(K; \mathcal{L}, \mathcal{B}) \) of \( K \) with links \( \mathcal{L} \) of type \( k-1 \) is \( A_k \)-equivalent to \( K\#F(O; \mathcal{L}', \mathcal{B}') \) for some links \( \mathcal{L}' \) of type \( k-1 \) and \( \mathcal{B}' \). Hence \( \phi \) is surjective. For \([F(O; \mathcal{L}_i, \mathcal{B}_i)]_k \in G_k(O) \ (i = 1, 2)\), we have

\[
\phi([F(O; \mathcal{L}_1, \mathcal{B}_1)]_k +_O [F(O; \mathcal{L}_2, \mathcal{B}_2)]_k) = \phi([F(O; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k) = [K\#F(O; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k = [F(K; \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{B}_1 \cup \mathcal{B}_2)]_k = [F(K; \mathcal{L}_1, \mathcal{B}_1)]_k +_K [F(K; \mathcal{L}_2, \mathcal{B}_2)]_k = [K\#F(O; \mathcal{L}_1, \mathcal{B}_1)]_k +_K [K\#F(O; \mathcal{L}_2, \mathcal{B}_2)]_k = \phi([F(O; \mathcal{L}_1, \mathcal{B}_1)]_k) +_K \phi([F(O; \mathcal{L}_2, \mathcal{B}_2)]_k).
\]
This implies that \( \phi \) is a homomorphism. In order to complete the proof, we show that \( \phi \) is injective. Suppose that \( [K \# F(O; \mathcal{L}, \mathcal{B})]_k = [K]_k \). By Lemma 3.2, there is a knot \( K' \) such that \( [K' \# K]_k = [O]_k \). Then we have

\[
[F(O; \mathcal{L}, \mathcal{B})]_k = [(K' \# K) \# F(O; \mathcal{L}, \mathcal{B})]_k = [K' \# (K \# F(O; \mathcal{L}, \mathcal{B}))]_k = [K' \# K]_k = [O]_k.
\]

This implies that \( \ker \phi = \{[O]_k\} \). \( \square \)

Habiro originated 'clasper theory' and showed Lemma 3.2 for \( C_k \)-moves \([3, 4]\). The following proof is a translation of his proof in terms of band sum description of knots.

**Lemma 3.2.** For any knot \( K \) and any integer \( k \geq 1 \), there is a knot \( K' \) such that \( K' \# K \) is \( A_k \)-equivalent to a trivial knot.

**Proof.** We shall prove this by induction on \( k \). The case \( k = 1 \) is clear. Suppose that there is a knot \( K' \) such that \( K' \# K \) is \( A_{k-1} \)-equivalent to a trivial knot \( O \) \((k > 1)\). By Proposition 2.1, we may assume that \( O = F(K' \# K; \mathcal{L}, \mathcal{B}) \), where \( \mathcal{L} \) is a set of links of type \( k - 1 \). Then, by Lemma 2.2, we see that \( F(K' \# K; \mathcal{L}, \mathcal{B}) \) is \( A_k \)-equivalent to some \( K \# F(K'; \mathcal{L}, \mathcal{B'}) \). This completes the proof. \( \square \)

Let \( \mathcal{K}_k \) be the set of \( A_k \)-equivalence classes of all oriented knots. For \( [K]_k, [K']_k \in \mathcal{K}_k \), we define \( [K]_k + [K']_k = [K \# K']_k \). Then the following, shown by Habiro \([3, 4]\) in the case that \( A_k \)-moves coincide with \( C_k \)-moves, is an immediate consequence of Lemma 3.2.

**Theorem 3.3.** The set \( \mathcal{K}_k \) forms an abelian group under ‘+’ with the unit element \([O]_k\), where \( O \) is a trivial knot. \( \square \)

### 4. Generalized \( A_k \)-Move

In this section, we define a generalized \( A_k \)-move. For this move, several results similar to that in Sections 1, 2 and 3 hold.

Let \( T \) and \( S \) be trivial tangles such that \((T, S)\) and \((S, T)\) are equivalent. Let \( t_1, t_2, ..., t_n \) and \( s_1, s_2, ..., s_n \) be the components of \( T \) and \( S \) respectively. An \( A_1(T, S) \)-move is this local move \((T, S)\). Suppose that \( A_k(T, S) \)-moves are defined. For each \( A_k(T, S) \)-move \((T_k, S_k)\) we choose a trivialization of \( T_k \) and fix it. Then the band trivializations of \((T_k, S_k)\) with respect to the trivialization are called \( A_{k+1}(T, S) \)-moves. Let \((T_k, S_k)\) be an \( A_k(T, S) \)-move and \( D_1, D_2, \cdots, D_{n+k-1} \) the fixed trivialization of \( T_k = t_1 \cup t_2 \cup \cdots \cup t_{n+k-1} \). We set \( \alpha = \partial B^3 \cap (D_1 \cup D_2 \cup \cdots \cup D_{n+k-1}) \) and \( \beta = S_k \). A link \( L \) in \( S^3 \) is called type \((k; (T, S))\) if there is an orientation preserving embedding \( \varphi : B^3 \to S^3 \) such that \( L = \varphi(\alpha \cup \beta) \).
Then the pair \((\alpha, \beta)\) is called a link model of \(L\). As in Section 1, \(A_k(T, S)\)-move gives an equivalence relation, \(A_k(T, S)\)-equivalence, on the set of all embeddings of \(G\) into \(S^3\). For an embedding \(f : G \rightarrow S^3\), let \([f]_k\) denote the \(A_k(S, T)\)-equivalence class of \(f\). Let \(h : G \rightarrow S^3\) be an embedding and \(H\) an abelian group. Let \(\varphi : [h]_{k-1} \rightarrow H\) be an invariant. We can define that \(\varphi\) is a finite type invariant of order \((n; k; (T, S))\) as in Section 1.

By the arguments similar to that in Sections 1, 2 and 3, we have the following five theorems.

**Theorem 4.1.** Let \(f, g : G \rightarrow S^3\) be \(A_{k-1}(T, S)\)-equivalent embeddings. Then they are \(A_k(T, S)\)-equivalent if and only if \(\varphi(f) = \varphi(g)\) for any finite type \(A_k(T, S)\)-equivalence invariant \(\varphi\) of order \((1; k; (T, S))\). \(\square\)

Let \(h : G \rightarrow S^3\) be an embedding. For \([f_1]_k, [f_2]_k \in [h]_{k-1}/(A_k(T, S)\)-equivalence), we can define \([f_1]_k + h [f_2]_k\) as in Section 2, and we have

**Theorem 4.2.** The set \([h]_{k-1}/(A_k(T, S)\)-equivalence\) forms an abelian group under \(\cdot + h\) with the unit element \([h]_k\). \(\square\)

We denote this group by \(\mathcal{G}_{k(T, S)}(h; G)\) and call it the \(A_k(T, S)\)-equivalence group of the spatial embeddings of \(G\) with the unit element \([h]_k\).

**Theorem 4.3.** Let \(h_1, h_2 : G \rightarrow S^3\) be \(A_{k-1}(T, S)\)-equivalent embeddings. Then the groups \(\mathcal{G}_{k(T, S)}(h_1; G)\) and \(\mathcal{G}_{k(T, S)}(h_2; G)\) are isomorphic. \(\square\)

For an embedding \(h : S^1 \rightarrow S^3\), let \(K = h(S^1)\) and let \(\mathcal{G}_{k(T, S)}(K)\) denote the \(A_k(T, S)\)-equivalence group \(\mathcal{G}_{k(T, S)}(h; S^1)\) with the unit element \([h]_k\).

**Theorem 4.4.** Let \(O\) be a trivial knot. If any two knots are \(A_1(T, S)\)-equivalent, then for any oriented knot \(K\), \(\mathcal{G}_{k(T, S)}(O)\) and \(\mathcal{G}_{k(T, S)}(K)\) are isomorphic. \(\square\)

Let \(\mathcal{K}_{k(T, S)}\) be the set of \(A_k(T, S)\)-equivalence classes of all oriented knots. For \([K]_k, [K']_k \in \mathcal{K}_{k(T, S)}\), we define \([K]_k + [K']_k = [K \# K']_k\).

**Theorem 4.5.** If any two knots are \(A_1(T, S)\)-equivalent, then the set \(\mathcal{K}_{k(T, S)}\) forms an abelian group under \(\cdot + \cdot\) with the unit element \([O]_k\), where \(O\) is a trivial knot. \(\square\)

**Remark.** If \((T, S)\) is the \#-move defined by Murakami \(\mathbb{[5]}\), then \(\mathcal{K}_{k(T, S)}\) is an abelian group.

5. \(A_k\)-Moves and Vassiliev Invariants

Let \(G\) be a finite graph. We give and fix orientations of the edges of \(G\). Let \(\mathcal{E}\) be the set of the ambient isotopy classes of the embeddings of \(G\) into \(S^3\). Let \(\mathbb{Z}\mathcal{E}\) be the free abelian
group generated by the elements of $E$. A crossing vertex is a double point of a map from $G$ to $S^3$ as in Figure 5.1. An $i$-singular embedding is a map from $G$ to $S^3$ whose multiple points are exactly $i$ crossing vertices. By the formula in Figure 5.2 we identify an $i$-singular embedding with an element in $E$. Let $R_i$ be the subgroup of $E$ generated by all $i$-singular embeddings. Note that $R_i$ is independent of the choices of the edge orientations. Let $H$ be an abelian group. Let $\varphi : E \rightarrow H$ be a map. Let $\tilde{\varphi} : E \rightarrow H$ be the natural extension of $\varphi$. We say that $\varphi$ is a Vassiliev invariant of order $n$ if $\tilde{\varphi}(R_{n+1}) = \{0\}$. Let $\iota : E \rightarrow E$ be the natural inclusion map and $\pi_i : E \rightarrow E/R_i$ the quotient homomorphism. Let $u_{i-1} = \pi_i \circ \iota : E \rightarrow E/R_i$ be the composition map. Then $\varphi$ is a Vassiliev invariant of order $n$ if and only if there is a homomorphism $\hat{\varphi} : E/R_{n+1} \rightarrow H$ such that $\varphi = \hat{\varphi} \circ u_n$.

In the following we sometimes do not distinguish between an embedding and its ambient isotopy class so long as no confusion occurs.

**Theorem 5.1.** Let $f, g : G \rightarrow S^3$ be $A_{k+1}$-equivalent embeddings. Then $u_k(f) = u_k(g)$.

By using induction on $k$, we see that an $A_k$-move $(T, S)$ is a $(k+1)$-component Brunnian local move, i.e., $T - t$ and $S - s$ are ambient isotopic in $B^3$ relative $\partial B^3$ for any $t \in T$ and $s \in S$ with $\partial t = \partial s$. It is not hard to see that if two embeddings $f$ and $g$ are related by a $(k+1)$-component Brunnian local move, then $f$ and $g$ are $k$-similar, where $k$-similar is an equivalence relation defined by the first author. Therefore, we note that Theorem 5.1 follows from [1] or [9]. However we give a self-contained proof here.

Let $T$ be a tangle. Let $\mathcal{H}(T)$ be the set of all (possibly nontrivial) tangles that are homotopic to $T$ relative to $\partial B^3$. Let $\mathcal{E}(T)$ be the quotient of $\mathcal{H}(T)$ by the ambient isotopy relative to $\partial B^3$. Then $Z\mathcal{E}(T)$, $i$-singular tangles and $\mathcal{R}_i(T) \subset Z\mathcal{E}(T)$ are defined as above.

**Proof.** It is sufficient to show for each $A_k$-move $(T, S)$ that $T - S$ is an element in $\mathcal{R}_k(T)$. We show this by induction on $k$. The case $k = 1$ is clear. Recall that an $A_k$-move $(T, S)$ is a band trivialization of an $A_{k-1}$-move, say $(T', S')$. Then we have that $T - S = X_1 - X_2$ where $X_1$ and $X_2$ are 1-singular tangles in Figure 5.3. Let $Y_1$ and $Y_2$ be 1-singular tangles in Figure 5.4. It is clear that $Y_1 - Y_2 = 0$. Thus we have $T - S = (X_1 - Y_1) - (X_2 - Y_2)$. By the induction hypothesis we see that $S' - T'$ is an element of $\mathcal{R}_{k-1}(S')$. Therefore both $X_1 - Y_1$ and $X_2 - Y_2$ are elements of $\mathcal{R}_k(T)$. 

![Figure 5.1](image1.png) ![Figure 5.2](image2.png)
Proof of Proposition 1.3. It is sufficient to show that

\[
\sum_{X \subset \{1, 2, \ldots, n+1\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})
\]

is an element of \(R_{(n+1)(k-1)}\). We show this together with some additional claims by induction on \(n\). First consider the case \(n = 0\). Then we have by Theorem 5.1 and its proof that \(F(f; \emptyset, \emptyset) - F(f; \{L_1\}, \{B_1\})\) is a sum of \((k-1)\)-singular embeddings each of which has all crossing vertices in the associated ball. Note that these \((k-1)\)-singular embeddings are natural extensions of the \((k-1)\)-singular tangles that express the difference of the \(A_{k-1}\)-move, and these \((k-1)\)-singular tangles depends only on the link \(L_1\). Next we consider the general case. Note that

\[
\sum_{X \subset \{1, 2, \ldots, n\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})
\]

\[
= \sum_{X \subset \{1, 2, \ldots, n\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})
\]

\[
- \sum_{X \subset \{1, 2, \ldots, n\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\} \cup \{L_{n+1}\}, \bigcup_{i \in X} \{B_i\} \cup \{B_{n+1}\}).
\]

By the hypothesis we have both

\[
\sum_{X \subset \{1, 2, \ldots, n\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})
\]

and

\[
\sum_{X \subset \{1, 2, \ldots, n\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\} \cup \{L_{n+1}\}, \bigcup_{i \in X} \{B_i\} \cup \{B_{n+1}\})
\]

are sums of \(n(k-1)\)-singular embeddings and they differ only by the band sum of \(L_{n+1}\). Therefore we have

\[
\sum_{X \subset \{1, 2, \ldots, n+1\}} (-1)^{|X|} F(f; \bigcup_{i \in X} \{L_i\}, \bigcup_{i \in X} \{B_i\})
\]
is a sum of \((n(k - 1) + (k - 1))\)-singular embeddings. □

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