Heisenberg uncertainty principle (UP) holds that the nearly unlimited precise observation of linear momentum is for any states, and in complementarity, states with nearly infinite position space and vice versa for a particle by an lower bound $\hbar/2$ on the product of two their underlying uncertainties$^1$. In a cylindrical system, however, only the states with integral charges $m\hbar$ allow the unlimitedly precise observation of orbital angular momentum (OAM), and in complementarity, angular position (AP) states with $2\pi$ range for round and doughnut light beams$^2$. Moreover, the correlation of OAM and AP of entangled photon pairs is one order of magnitude stronger that those with independent particles$^3$. Here we propose a UP of demonstrating arbitrary states of unlimitedly precise mean OAM refer to the azimuthal phase-gradient (PG), and in complementarity, states with infinite range AP, observed in a set of numerous singular light beams with simultaneously helical wavefront and phase-shift-front associated with three natures of wave, discrete OAM eigenmode and linear optics. These arbitrary OAM states of arbitrary phase-jump are extended by discrete precisions that consist of not only the OAM eigenstates but also the states with the max phase-jump. Furthermore, we theoretically demonstrate the double resolution limit of OAM eigenmodes and the super-$2\pi$ angle period for two overlap probabilities of $n$-sectional fractional OAM state with one and two PGs, respectively, which is used to illustrate the quantum nonlocality in the spontaneously parametric down-conversion (SPDC) process.
**UP for singular light with one phase-gradient (PG)**

In a cylindrical system with the angle coordinate $\phi$ range of $2\pi$, the orbital angular momentum (OAM) eigenstate $|m\in \mathbb{Z}\rangle$ is associated with the eigenvalue $m\hbar$ by $\hat{L}|m\rangle = m|m\rangle$ for a light beam of helical wavefront, where $\hat{L}$ is OAM operator and $m$ denotes the topological charge$^4$. The optical OAM intrinsically accompanies the azimuthal phase-gradient (PG) of the helical wavefront$^5$. Further, the known uncertainty principle (UP) demonstrates that the OAM is precisely observed as $(\hat{L} - m)|m\rangle = 0$ for optical vortexes those are in the system with the discrete OAM eigenmode of resolution $\hbar$ in Hilbert space $H_\times$, and round and doughnut beams of $2\pi$ angular position (AP) range (AP uncertainty is $\pi/\sqrt{3}$)$^2$. Indeed, the known-UP presents the unlimited lower bound of OAM uncertainty zero by the definition of OAM eigenstate precision. These OAM precisions only discretely exist for OAM eigenstates is attributed to its foundation restrain of the limited OAM resolution $\hbar$, as well as the limited AP range $2\pi$. For the non-precision observation of OAM, there are two conjugated uncertainties of OAM and AP in a sector light beam$^2$.

A singular light beam with the fractionally azimuthal PG, denoted by $M$, has phase-singularity at $n$ azimuthally symmetric APs, and this is termed as $n$-sectional fractional vortex (FV$n$), where $n$ is an integer. Its quantum state $|Mn(M \in \mathbb{Q})\rangle$ is noneigenstate and consists of OAM eigenstates of the basis state $|m\rangle$; its OAM mean is evaluated as follows

$$\overline{Mn}(M) = M - \left(\frac{n}{2\pi}\right) \sin \left(\frac{M \cdot 2\pi}{n}\right).$$  \hfill (1)

In Eq. (1), the $n$ times amplitude fluctuation compared between various FVns is owing to the eigenmode separation of discrete nature $n\hbar$ in $H_{\times/n}$. Based on its dominated probability weight in a range of OAM eigenmodes$^{6,7}$ except those of eigenstates $|Mn(M \mod m=0)\rangle = |m\rangle$, all light beams of $|Mn\rangle$ should have finite OAM uncertainty. However, OAM uncertainty calculated with root mean square (RMS) in light of the known-UP$^2$ is invalid in $|Mn\rangle$, because its OAM variance is divergent$^7$. 
Herein, this-UP presents advantages such as the unlimited precision for states with arbitrary OAM means, specified OAM uncertainty for FVn with $|Mn\rangle$. According to this-UP, there will be the unlimited precision observation in OAM with respect to azimuthal PG if $\hat{L}|Mn(M)\rangle = M|Mn(M)\rangle$ for a cylindrical system with phase-singularity, where we consider the intrinsic relation between OAM and azimuthal PG. Equation (1) indicates the precision observation can be realised when $M \mod n/2 = 0$, which presents the phase shift (PS) of equaling the integer multiple of half wavelength for the host light beam in a round cycle. One part of $M \mod n = 0$ is with the phase-singularity of minimum phase-jump zero, and the other part of $M \mod n = n/2$ is with the phase-singularity of maximum phase-jump $\pi$. While the former is the OAM eigenstate, it is noteworthy in the latter since its non-eigenstate. This non-eigenstate shows that the precision can be attributed to the symmetry of sine wave. In a sine wave round, $\pi$ PS departs equally between two PSs of zero and $2\pi$ and vice versa, which results in the symmetric OAM spectra for FVns with phase-singularities of zero and $\pi$ phase-jumps and their OAM observation precisions. Two functional relations between $\bar{Mn}(M) - M$ and $M$ are obtained for FV1 and FV3, as shown by blue curves in Fig. 1a, where green markings indicate the quantized amplitudes and red points those are intersected between the blue curve and the $M$ axis indicate precise OAM observations. In Fig. 1b, four symmetry spectra with OAM eigenmodes resolutions of $h$ and $3h$ are obtained for FV1 with $M = 0$ and $1/2$ and FV3 with $M = 0$ and $3/2$, using the equations obtained from ref. 6. This cylindrical system includes that without the phase-singularity as $|Mn(M \in \mathbb{Q})\rangle \supset |m \in \mathbb{Z}\rangle$, and the former doubles the observation precision number compared with the latter.

This observation pertains to its contrast side. For OAM observation that is not as precise as $M \mod n/2 \neq 0$, an OAM uncertainty exists that is defined by the RMS of amplitude that is departed from the precision and constructed by the set of all phase-gradients:

$$\text{RMS}[\bar{Mn}(M) - M] = \frac{n}{2\pi} \times \text{RMS}\left[-\sin\left(M \frac{2\pi}{n}\right)\right] = \frac{n}{2\pi} \times \sqrt{\frac{1}{2}} = \frac{n}{2\sqrt{2\pi}} [h],$$

where the departure amplitude $nh/2\pi$ represents the OAM range. The OAM uncertainty is discrete as illustrated by Eq. (2), and its lower bound is $h/2\sqrt{2\pi}$ when $n = 1$. This definition is reasonable from the perspective of its conjugate observation,
AP. Further, FV\(_n\) has an AP range of isotropic intensity (except the phase-singularity) in the non-repeating helical wavefront, which is equal to the \(\phi\) range \(2\pi/n\). Two AP ranges of \(2\pi\) and \(2\pi/3\) are obtained for FV1 and FV3, respectively, as segregated by the phase-singularity shown in Fig. 1c. Because there is no truncated intensity raised by an obstacle in FV\(_n\), we assumed that its AP uncertainty is equal to that of the round beam by \(\pi/\sqrt{3n}\), where the variation \(n\) is applied by the linear optics suggested by this-UP. These two uncertainties of OAM and AP are inversely proportional with proportionality constant \(\hbar\pi/2\sqrt{6}\), as well as the proportionality constant \(\hbar\) of two observation ranges.\(^6\) This is a discretely quantified uncertainty relation, observed for respect \(n\) set of numerous FV\(_n\)s and associated with the three natures of waves, discretely infinite-dimensional OAM eigenmodes with resolution \(n\) and linear optics\(^6\). The upper and lower bounds of OAM and AP uncertainties from FV\(_n\) are infinity large and small, respectively, for the infinity large \(n\).

Fig. 1 | Proposed UP associated with three natures of wave, discrete OAM eigenmode and linear optics. a, Periodic functions of \(M\) produced by the difference between \(\overline{Mn}\) and \(M\hbar\) for FV1 and FV3 (blue curves), where green markings indicate the amplitudes of these periodic functions as \(\hbar/2\pi\) and \(3\hbar/2\pi\), and red points indicate precise OAM observations with \(M\) mod 1/2 and 3/2 = 0, respectively. b, Left: two OAM spectra of FV1 with \(M = 0\) and 1/2, where the interval between modes of nonzero weights is one; right: two OAM spectra FV3 with \(M = 0\) and 3/2, where the interval between modes of nonzero weights is three. c, Two intensity images of FV1 and FV3, whose AP ranges of non-repeating helical wavefront are equal to \(\phi\) ranges between APs of the phase-singularity \(2\pi\) and \(2\pi/3\), respectively. Both products of two OAM and AP ranges for FV1 and FV3 are \(\hbar\).
UP for singular light with two PGs

This UP expounds that the OAM uncertainty is not restrained by a discrete quantity of $\frac{n\hbar}{2\sqrt{2\pi}}$ by the employment of PS. A singular light beam with a superposition of two FVns with two different PGs $M_1$ and $M_2 = M_1 + \delta$ is expressed as

\[ |M_n(M_1, M_2)| = |M_n(M_1)| + |M_n(M_2)|, \]

where $\delta$ denotes the PS between these two gradients having units of radian. Its mean OAM is the average of the two OAM means of $|M_n(M_1)|$ and $|M_n(M_2)|$, given by,

\[
\overline{M_n}(M_1, M_2) = \sum_{m=-\infty}^{\infty} m^j \overline{P_m}[M_n(M_1, M_2)] = \frac{M_1 + M_2}{2} - \frac{n}{4\pi} \sin \left( M_1 \frac{2\pi}{n} \right) + \sin \left( M_2 \frac{2\pi}{n} \right).
\]

(3)

Let this singular beam have an equivalently azimuthal PG $M_{12} = (M_1 + M_2)/2$.

Substituting this gradient in its state and in Eq. (3) results respectively in

\[ |M_n(M_1, M_2)| = |M_n(M_{12})| \]

and

\[
\overline{M_n}(M_{12}) = M_{12} - \cos \left( \frac{\delta \pi}{n} \right) \left[ \frac{n}{2\pi} \sin \left( M_{12} \frac{2\pi}{n} \right) \right].
\]

(4)

Equation (4) reveals an additional PS factor $\cos (\delta \pi/n)$ (compared to Eq. (1)), to the fluctuation amplitude of the sinusoidal function. This PS is such that this light has PS-singularity located at $n$ azimuthally symmetric orientations (see Methods). More generally, for a cylindrical system with PS-singularity located at $n \phi$s, there will be the unlimited precision observation in OAM $\overline{M_n}(M_{12})$ with respect to azimuthal PG $M_{12}$ if $\hat{L}[M_n(M_{12})] = M_{12} |M_n(M_{12})|$, where we consider the intrinsic relationship between OAM and azimuthal PG$^5$. According to equation (4), the precision observation can be realised when $M_{12} \mod n/2 = 0$, whose value is identical to that of one PG $n/2$. Similarly, for $M_{12} \mod n/2 \neq 0$, the OAM uncertainty is as follows:
By comparing between Eqs. (2) and (5), the equivalent OAM resolution for singular light with $M_{12}$ is decreased and unlimited by $h\left|\cos\left(\frac{\delta}{n}\pi\right)\right|$ while that is $h$ for FV with $M$. According to Eq. (5), the OAM uncertainty can be quantified continuously, and its lower bound is limited to zero when $\delta \mod n = n/2$. In this limit, all states with arbitrary $M_{12}$, i.e., those which are either asymmetrical or symmetrical OAM spectra, are precise in OAM observation. This limit leads to a singularity because, in contrast, only those states with symmetric OAM spectra are precise in observation for the case with nonzero OAM uncertainty. Figure 2 illustrates the superposition principle applied to a singular light beam with two PGs for $n = 1$ and three cases of $\delta = 1/3, 1/2$ and 1. In Fig. 2a, the two functions of $\bar{M}\left(M_1\right) - M_1$ versus $M_1$ and $\bar{M}\left(M_2\right) - M_2$ versus $M_2$ are averaged to a function $\bar{M}\left(M_{12}\right) - M_{12}$ versus $M_{12}$, where the red and green markings indicate these three $\delta$ and three amplitudes of the sinusoidal fluctuation by $h\left|\cos\left(\frac{\delta}{n}\pi\right)\right|/2\pi$, respectively.

The PS-gradient is a real phenomenon which presents the interference degree in its located system. In a cylindrical system, it is observed such that the intensity decreases in a sinusoidal square form in $\phi$ for a singular light beam with $M_{12}$. The intensity ratio between two $\phi$s those are differenced by $\Delta\phi \in [0, 2\pi]$ in this beam is evaluated as $\cos^2\left(\frac{\delta\Delta\phi}{2n}\right)$. For two $\phi$s at two edges of a non-repeating helical wavefront, its value is $\cos^2\left(\frac{\delta\pi}{n}\right)$. We raise the following question for examining the relationship between the variation intensity and the AP intrinsically: ‘How to define the AP range for a light beam whose intensity is decreasing in an azimuthal sinusoidal square form?’ Two APs, defined in near field, are between 0 and $2\pi$ for the isotropic intensity of a sector beam$^2$, and 0 and $2\pi/n$ for a non-repeating isotropic intensity of FV$n$. Both cases reveal that large AP range implies large AP uncertainty; in the latter, a proportional relationship exists based on the linear system suggested by this-UP. From intuition, the varying intensity of the azimuthally sinusoidal square decrease presents a larger AP uncertainty, and therefore larger AP range compared with the isotropic
intensity. Then, this introduces another question: ‘How the AP range expands for the singular light beam arises here?’ According to this-UP, AP uncertainty is inversely proportional to OAM uncertainty. Because OAM uncertainty ratio is $|\cos(\delta \pi/ n)|$ with a linear decrease (Eq. (5)), the AP uncertainty ratio and AP range ratio are both $1/|\cos(\delta \pi/ n)|$ with a linear increase, based on a comparison of two light beams with $M_{12}$ and $M$. By applying the linear AP space, the AP is mapped to an angle coordinate $\phi_a$ by $\phi_a/|\cos(\delta \pi/ n)|$ for this singular light with $M_{12}$.

Figure 2b shows three intensity images of singular light beams with $M_{12}$ made up of two FVs with respective PGs of $(1/3,2/3)$, $(1/3,5/6)$ and $(1/3,4/3)$. They are anisotropic, which is attributing to the azimuthally varying PSs. This PS variation results in an unintegrated phase-singularity, or PS-singularity (see Methods). As an example of $(1/3,5/6)$ in the middle column of Fig. 2b, the intensity in one side of PS-singularity is zero whereas that in the other side remains unchanged, because of the completely destructive and constructive interferences, respectively. The cases for the items of PS, intensity ratio between two edge sides of PS-singularity, and OAM and AP ranges and their products are displayed in Extended Data Table 1. The superposition made of three or more FVs with numerous PGs is reduced to that of two in the OAM mean and PS considered by the equivalently azimuthal PG based on superposition principle. This-UP reveals that the $n$-set of numerous singular light beams, each of which is with the superposition made of numerous FVs with various PGs, witnesses a continuously quantified uncertainty relation with the proportionality constant $h\pi/2\sqrt{6}$ on the product of two of their underlying uncertainties. Large OAM accuracy with respect to PG implies small AP accuracy (large AP range and large intensity variation azimuthally), for which the lower and upper bounds of the two uncertainties are zero and infinity, respectively.
**Fig. 2 | Unlimited precision OAM for arbitrary states and infinite AP.** a, Top and middle panels: three pairs of OAM mean deviations with PSs = 1/3, 1/2 and 1; bottom panel: three OAM mean deviations, each of which results from the average of the corresponding above two deviations; green markings indicate amplitudes of periodic functions of these deviations for $\theta/4\pi$, 0 and $\theta/2\pi$. b, Left, middle and right: three simulated intensity images with superpositions of the respective two FVs with $(1/3,2/3)$, $(1/3,5/6)$ and $(1/3,4/3)$, respectively. They vary smoothly with $\phi$ and result in one-half, completely vanishing and identical intensities compared at two edges of respective PS-singularities.

**UP for singular light with one PG and entanglement**

The overlap probability between two FV$n$ states with an azimuthal angle difference $\alpha$ in-between is

$$\left|\langle M^\alpha(n;0)|M\alpha^\alpha(n;0)\rangle\right|^2 = \left[\frac{\pi}{\pi^2} \frac{(2t-1)-n\alpha}{n^2}\right]^2 \sin^2\left(\frac{M\pi}{n}\right) + \cos^2\left(\frac{M\pi}{n}\right),$$

(6)

$$\frac{2\pi}{n} (t-1) \leq \alpha < \frac{2\pi}{n} t, \ t = 1, 2, ..., n.$$  

This probability presents orthogonal relations of coincident fringes used to prove quantum nonlocality experiments with high-dimensional two-particle entanglement$^8,9$, provided by two conditions of $\alpha = \pi \left(\frac{2t-1}{n}\right)$ and $M \mod n = n/2$. Substituting $\alpha = \pi \left(\frac{2t-1}{n}\right)$ in Eq. (6) gives

$$\left|\langle M^\alpha(n;0)|M\alpha^\alpha(n;0)\rangle\right|^2 = \cos^2\left(\frac{M\pi}{n}\right).$$

(7)
From Eq. (7), the maximum visibility of this overlap is obtained effectively by the PG range of \( n/2 \) \(( np/2 - n(p +1)/2, p = 0, 1, 2, ..., or \infty \). In comparison, for the independent (nonentangled) photons, the PG range is significantly considered as the period of \( \overline{Mn}(M) - M \) (see Eq. (1)), \( n \). The former range which originates in entangled photon pairs is one-half of that which in independent photons, and their limits are 1/2 and 1, respectively. The significance of these two ranges can be realised by OAM eigenmodes resolution. The quantum OAM eigenmodes resolution of the entangled state of photon-pairs by \( nh/2 \) is twice that of classic one of the pure state of single-photons by \( nh \). Nevertheless, no quantum’s OAM range is found through the former PG range, although the obtained classical OAM range \( nh/2 \pi \) is associated with the latter PG range. Substituting \( M \mod n = n/2 \) in equation (6) gives

\[
\left| \langle Mn(M \mod n = n/2; 0) | Mn(M \mod n = n/2; \alpha) \rangle \right|^2 = \left[ \frac{\pi (2n - 1) - n\alpha}{\pi^2} \right]^2. \tag{8}
\]

From equation (8), the angle periods of these probability functions are \( 2\pi/n \), which are equal to \( \phi \) ranges of \( | Mn \rangle \). They significantly present the experimentally coincidence fringes between SPDC photon pairs\(^{10,11}\), which is physically observable. Based on their discretely quantified properties and by applying the linear system property, the experimental angle period, as well as the theoretical one, is quantum’s AP observation ranges held by this-UP; however, we do not know their conjugated OAM ranges. Substituting \( n = 1 \) and 3 in Eqs. (7) and (8), respectively, produces four probability functions (represented by blue and red curves in Fig. 3a and 3b). In Fig. 3a, two maximum visibilities (= one) of two functions with \( \alpha = \pi \) and \( \pi/3 \) are obtained by the PG ranges of 1/2 and 3/2 (indicated by green markings), respectively. Their identical amplitudes do not gain anything in the OAM range for us. However, in Fig. 3b, two angle periods of the obtained functions with \( M \mod 1 = 1/2 \) and \( M \mod 3 = 3/2 \), as well as two quantum’s AP ranges, are \( \pi \) and \( \pi/3 \) (indicated by green markings), identical to those classical two in Fig. 1c.

The standard formula for experimental coincidence in the SPDC experiment is

\[
\left| \langle Mn(M; 0) | Mn(-M; \alpha) \rangle \right|^2 \tag{9,12}, \text{ where } |\Psi\rangle \text{ is the SPDC bi-photon state with high-dimensional two-photon entanglement } \sum_{\ell=-\infty}^{\infty} C_\ell |\ell\rangle - |\ell\rangle \text{ and } |\ell\rangle \text{ is its OAM basis mode with the probability amplitude } C_\ell, \text{ provided by the use of } \ell = 0 \text{ pump beam}^{13}.
\]
16. \(|\langle Mn(M;0)|Mn(M;\alpha)\rangle|^2\) and \(|\langle Mn(M;0)|Mn(-M;\alpha)\rangle|\Psi\rangle|^2\) will be identical only if all \(|\ell\rangle\) modes are equally weighted\(^9\). Nonetheless, they are identical in two ranges of PG and AP for maximum visibility and have little discrepancy in function profile for \(|Mn(M)\rangle\) with small \(M\). The reason for the latter is both the dominated OAM modes are small and few in \(|\Psi\rangle\) and \(|Mn(M \text{ is small})\rangle\)\(^9\).

Fig. 3 | Double OAM resolution by entanglement. a, Left and Right: overlap probability (blue curve) and normalized experiment coincidence (red points) versus \(M\) for \([1,\pi]\) and \([3,\pi/3]\), where green markings indicate the PG ranges of maximum visibility by 1/2 and 3/2, respectively. b, Left and Right: overlap probability (red curve) and normalized experiment coincidence (blue points) versus \(\alpha\) for \(\{1,3/2\}\) and \(\{3,3/2\}\), where green markings indicate the AP ranges by \(2\pi\) and \(2\pi/3\), respectively.

UP for singular light with two PGs and entanglement

The unnormalized quantum state for the \(n\)-sectional singular light beam with \(M_{12}\) is obtained by \(|Mn(M_1,M_2;\alpha)\rangle=|Mn(M_1;\alpha)\rangle+|Mn(M_2;\alpha)\rangle\). The unnormalized overlap probability between two \(n = 1\) light beams with \(M_{12}\) is given by \(|\langle M (M_1,M_2;0)|M (M_1,M_2;\alpha)\rangle|^2\). Its expanded form is the sum (see Eq. (M.1)) of
numerous compositions of four overlap amplitudes between two light fields with $M_1$ and $M_2$, which are formulated as Eqs. (M.3)–(M.7) (see Methods):

$$\left|\langle M_1(0)|M_1(\alpha)\rangle\right|^2 + \left|\langle M_2(0)|M_2(\alpha)\rangle\right|^2 + 2\left|\langle M_1(0)|M_2(\alpha)\rangle\right|^2$$

$$+ 2\text{Re}\left[\langle M_1(0)|M_1(\alpha)\rangle\langle M_2(0)|M_2(\alpha)\rangle^*\right] + 2\text{Re}\left[\langle M_1(0)|M_2(\alpha)\rangle\langle M_2(0)|M_1(\alpha)\rangle^*\right]$$

$$+ 2\text{Re}\left[\langle M_1(0)|M_1(\alpha)\rangle\langle M_2(0)|M_2(\alpha)\rangle^*\right] + 2\text{Re}\left[\langle M_2(0)|M_2(\alpha)\rangle\langle M_1(0)|M_2(\alpha)\rangle^*\right].$$

(9)

This probability can be normalized by referring to $\alpha = 0$ in Eq. (9):

$$\left|\langle M(M_1, M_2; 0)|M(M_1, M_2; \alpha)\rangle\right|^2$$

$$\left|\langle M(M_1, M_2; 0)|M(M_1, M_2; 0)\rangle\right|^2. \quad \text{(10)}$$

The probability normalization for $n = 1$ light beam with $M_{12}$ cannot be achieved by its normalized quantum state of $|M(M_1, M_2; \alpha)\rangle/\sqrt{\langle M(M_1, M_2; \alpha)|M(M_1, M_2; \alpha)\rangle}$, because the intersecting term in this denominator only produces to one PS (see Eq. (M.10)), which cannot be mapped to numerous intersecting terms in numerator between various pairs of two PGs of $M_1$ and $M_2$ in Eq. (9).

For comparison, we use the conditions of $\alpha = \pi$ and $M_{12} \mod 1 = 1/2$ for the overlap probability with $M_{12}$, which was used in the evaluation for that with $M$. The results obtained by substituting $\alpha = \pi$ in Eq. (10) are plotted as blue curves in Fig. 4a, for three cases of $\delta = 1/3, 1/2$ and 1. Some of the terms those are superpositions of numerous PGs with various phases in $\left|\langle M(M_1, M_2; 0)|M(M_1, M_2; \alpha)\rangle\right|^2$ are negative values, such that blue curves have the inverse part reflected from and the turning points intersected by x axis. If Eq. (9) is without the absolute value, not experimental fringe, the plotted curves will be plotted as the connection made up of the non-inverse solid and inverse dashed parts. As shown in Fig. 4a, three such curves all have PG ranges of 1/2 in $\delta = 1/3, 1/2$ and 1, similar to that with one PG ($\delta = 0$, left column of Fig. 3a). This identical 1/2 PG range between two probabilities with $M$ and $M_{12}$ in entangled
photon-pairs is similar to that of 1 PG range between two OAM mean deviations with $M$ and $M_{12}$ in single photons. Although their amplitudes differ (solid or dashed curves), on OAM uncertainty is recognised by us.

However, an outstanding significance arises from this probability by $M_{12}$ mod 1 = 1/2. Substituting $(1/3+2/3)/2$, $(1/4+3/4)/2$ and $(0+1)/2$ mod 1 = 1/2 in Eq. (10) produces results that are plotted as red curves in Fig 4b, for three cases of $\delta = 1/3$, 1/2 and 1, respectively. The profiles of the former two are asymmetric with respect to two reflection lines at $\pi$ radian, indicated by the green dashed lines, whereas that of the final one is symmetric. The difference in profile of these three overlap probabilities in an identical $\phi$ range provides a physically observable evidence for AP. The angle period, as well as the AP range, of this symmetric profile is $2\pi$, similar to that with one PG ($\delta = 0$, left column of Fig. 3b). Furthermore, this identical nature is similar to that of single photons. To understand what the asymmetrical profile infers, thinking over the physics meaning. A function of symmetric profile has more precise AP observation than that with asymmetric profile does. Namely, the latter has larger AP uncertainty, as well as larger AP range, compared to the former. In other words, an asymmetric profile with period $2\pi$ has AP range larger than $2\pi$, which is equal to AP range owned by a symmetric profile with period larger than $2\pi$. Because no such symmetry profile in a system with $\phi$ range $2\pi$, this asymmetry presents a super-$2\pi$ period. Similarly, there are the asymmetry profiles for all $n$ cases.

![Fig. 4](image_url)

**Fig. 4 | Super-$2\pi$ angle period of overlap probability in $\phi$ range $2\pi$. a, Overlap probability versus $M - \delta/2$ at $\alpha = \pi$ for $\delta = 1/3$, 1/2 and 1. b, Overlap probability versus $\alpha$ for $(1/3+2/3)/2$, $(1/4+3/4)/2$ and $(0+1)/2$ mod 1 = 1/2.**
Conclusions
This-UP explains the AP observation for the intrinsic variation intensity of a round beam and OAM observation for the $n$-set numerous singular light beams, associated with three natures of waves, infinitely discrete OAM eigenmodes and linear optics. It applies to all systems but is particularly important for cylindrical system with phase-singularity in near-field. A cylindrical system with phase-singularity and PS-singularity has an equivalent mapping to Heisenberg UP for arbitrary states with approximately nonlimited OAM precision and infinite AP. Its two fundamental elements of OAM eigenstates and discrete OAM eigenmodes are the cornerstones for the known-UP. Extended Data Table 2 compares numerous characteristics of these two UPs. This-UP theoretically demonstrates quantum correlations in OAM and AP in terms of double OAM eigenmode resolution and super-$2\pi$ period. A singular light beam with two PGs, whose unlimited OAM precision and infinite AP are held by this-UP classically and quantumly, possesses several exceptional characteristics, such as helical PS-front and precise observations of integer multiple of half wavelength. The $n$-sectional helical wavefront embedded with the two multiple PS functions, as well as the gained AP escaped from convention $2\pi$, promises the wider OAM degree of freedom in light beams, thereby extending the existing topics to make them applicable in various fields of science, such as optical spanner$^{17,18}$, communications$^{19,20}$, structured light$^{21}$, quantum information$^{22,23}$, microscopy$^{24}$, astronomy$^{25}$, interferometry$^{26}$, tomography$^{27}$, nonlinear optics$^{28,29}$ and singular optics$^{30}$. 
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Methods

Helical PS-front

PS is a wave property, which is used frequently for Gaussian light with planar wavefront, and holds an equivalent meaning of geometry phase (Berry phase) for optical vortex with helical wavefront. Namely, it is functional in intensity variation but none in profile for the both. However, PS is a function for connected the intensity and profile for FVn. Further, a light beam with a superposition made of two different fractional PGs $M_1$ and $M_2$ has the azimuthally continuous variation intensity which reveals the second function utilization of PS, in which its OAM and AP are held by this-UP. Similar to the phenomenon of the helical wavefront, the equal PS is helical round the beam propagation axis of this singular light beam as the helical PS-front. The gradient of this helical PS-front (named as PS-gradient), denoted $\delta M$, is equal to PS $\delta$ between these two PGs by $\delta M = M_2 - M_1 = \delta$. A PS-singularity exists, which is termed for the discontinuity of helical PS-front, at the n APs used exactly for the phase-singularity.

Extended Data

Fig. 1a depicts the relationship of beam cross-sections between the phases of two FVs with $M_1$ and $M_2$ and the PS of the superposed singular light with a PG-pair of $(M_1, M_2)$. Three PG-pairs of $(1/3, 2/3)$, $(1/3, 5/6)$ and $(1/3, 4/3)$ are used in three groups of respective three beam cross-sections (shown in left, middle and right columns of Extended Data Fig. 1a, respectively). Each group shows six and three texts, indicated respectively by the left two and right one beam cross-sections, present the phases for two FVs and the PSs for a singular light beam at two edge sides of phase-singularity and their symmetric orientation, respectively. The three differences for PS-pairs, or PS-jump, are obtained as $2\pi/3$, $\pi$ and $2\pi$. The left, middle and right plots shown in Extended Data Fig. 1b show six symmetric spectra obtained for six PG-pairs of $(-1/6, 1/6)$, $(1/3, 2/3)$, $(-1/4, 1/4)$, $(1/4, 3/4)$, $(-1/2, 1/2)$ and $(0, 1)^6$, where the three former and latter values correspond to OAM precisions with $M_1 = 0$ and $1/2$, respectively.

AP range viewed by phase-singularity integration

The uncertainty ratio between two singular light beams with $M_{12}$ and $M$ also can be proven from the viewpoint of phase-singularity. A phase-singularity exists conventionally in FVn for the identical intensity in its two edge sides. With regarding to the conventional phase-singularity, the phase-singularity is not integrated for the inequality intensity in its two edge sides, each of which contributes a factor $r$. Meaningfully, this ratio is equivalently to the uncertainty ratio. The product with two edge-intensities equals the product of two factors $1 \times \cos^2 (\delta \pi/n) = r \times r$ and $r = \cos (\delta \pi/n)$. Large unintegrated phase-singularity implies large AP range, and phase-singularity is destroyed completely as $\delta = n/2$.

Expand for $\left| M(M_1, M_2; 0) M(M_1, M_2; \alpha) \right|^2$

The unnormalized quantum state for $n = 1$ singular light beam with two phase-gradients is given by $|M(M_1, M_2; \alpha)\rangle = |M(M_1; \alpha)\rangle + |M(M_2; \alpha)\rangle$. Its overlap amplitude is given by
\[
\langle M(M_0, M_1, M_2; \alpha) | M(M_1, M_2; \alpha) \rangle = \left( \langle M(M_1, M_2; 0) | M(M_1, M_2; 0) \rangle \right) \left( \langle M(M_1, \alpha) | M(M_1, \alpha) \rangle \right) \left( \langle M(M_2, \alpha) | M(M_2, \alpha) \rangle \right)
\]
\[
= \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle.
\]

Its overlap probability is the square of this overlap amplitude:
\[
\left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 = \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2
\]
\[
= \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2
\]
\[
= \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2
\]
\[
= \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2 + \left[ \langle M(M_1, 0) | M(M_1, \alpha) \rangle \langle M(M_2, 0) | M(M_2, \alpha) \rangle \right]^2
\]
\[
\tag{M.1}
\]

Formulas of overlap amplitudes between different fractional OAM states

Two fractional charges are \( M_i = m_i + \mu_i, \ i = 1 \) and 2. They differ by a PS charge \( \delta \) as \( M_2 = M_1 + \delta \).

The azimuthal part of a fractional OAM light field (or FV state) can be defined as
\[
\langle \phi | M(M_1, \alpha) \rangle = e^{i\phi} e^{i\delta f_\alpha(\phi - \alpha)},
\tag{M.2}
\]
where \( f_\alpha(\phi) = \begin{cases} 
1, & 0 \leq \phi < \alpha \\
0, & \alpha \leq \phi < 2\pi
\end{cases} \). Based on the completeness relation and Eq. (M.2), four overlap amplitudes of different angle coordinates 0 and \( \alpha \) between two FV states with identical PGs \( M_i \), different PGs \( M_1 \) and \( M_2 \), and \( M_2 \) and \( M_1 \) are as follows:

\[
\langle M(M_1, 0) | M(M_2, 0) \rangle = \frac{1}{2\pi} \left[ e^{i\phi} e^{i\delta f_\alpha(\phi - \alpha)} \right],
\tag{M.3}
\]

\[
\langle M(M_1, 0) | M(M_2, \alpha) \rangle = \frac{e^{i\phi}}{2\pi i\delta} \left[ e^{i\phi} \left( e^{i\alpha} - 1 \right) + \left( e^{i2\alpha} - e^{i\alpha} \right) \right]
\]
and

\[
\langle M(M_2, 0) | M(M_1, \alpha) \rangle = \frac{e^{i\phi}}{2\pi i\delta} \left[ e^{i\phi} \left( e^{-i\alpha} - 1 \right) + \left( e^{-i2\alpha} - e^{-i\alpha} \right) \right].
\tag{M.4}
\]
Eq. (M.4) or (M.5) with $\delta = 0$ reduces to Eq. (M.3), where L'Hôpital's rule is used. Four overlap probabilities, and the square modulus of Eqs. (M.3), (M.4) and (M.5), are respectively given by,

$$\left| \langle M; \delta \rangle | M; \alpha \rangle \right|^2 = \left( 1 - \frac{\alpha}{\pi} \right)^2 \sin^2 (M, \pi) + \cos^2 (M, \pi)$$

(M.6)

$$\left| \langle M; 0 \rangle | M; \alpha \rangle \right|^2 = \left| \langle M; 0 \rangle | M; \alpha \rangle \right|^2 = \frac{1}{2\pi^2 \delta^2} \left[ 2 - \cos(\alpha \delta) - \cos(2 \pi - \alpha \delta) + \cos(2 \pi M_1 + \alpha \delta) + \cos(2 \pi M_1 - \alpha \delta) - \cos(2 \pi M_1) - \cos(2 \pi M_2) \right].$$

(M.7)

Similarly, those of the identical angle coordinate $\alpha$ are,

$$\langle M; \alpha \rangle | M; \alpha \rangle = 1,$$

$$\langle M; \alpha \rangle | M; \alpha \rangle = \frac{1}{2\pi \delta} \left( e^{i2\pi \delta} - 1 \right)$$

and

$$\langle M; \alpha \rangle | M; \alpha \rangle = \frac{i}{2\pi \delta} \left( e^{-i2\pi \delta} - 1 \right).$$

(M.8)

The square of the denominator of the normalized quantum state for the $n = 1$ singular light beam with two PGs is,

$$\langle M_1, M_2; \alpha \rangle | M_1, M_2; \alpha \rangle = \langle M_1; \alpha \rangle | M_1; \alpha \rangle + \langle M_2; \alpha \rangle | M_2; \alpha \rangle + 2 \Re \langle M_1; \alpha \rangle | M_2; \alpha \rangle.$$

(M.9)

Substituting Eq. (M.8) in Eq. (M.9) results in,

$$\langle M_1, M_2; \alpha \rangle | M_1, M_2; \alpha \rangle = 2 \left[ 1 + \sin(2 \pi \delta) \right].$$

(M.10)

**Extended Data Fig. 1 | Schematic showing helical PS-front and spectra of symmetry**

**a**, Left and middle: relative phases for two beam cross-sections with PGs of (1/3,2/3), (1/3,5/6) and (1/3,4/3). Right: relative PSs for the beam cross-section superposed with two phases of the left and middle cross-sections. The differences between two PSs at two edge sides of phase-singularity of these three cross-sections, or three PS jumps, for these three phase-pair are $2\pi/3$, $\pi$ and $2\pi$, respectively. **b**, Left, middle and right: two OAM spectra for $\delta = 1/3$, $1/2$ and $1$, respectively, with singular light beams having $M_{iz} = 0$ and $1/2$. All of them are symmetric.
Extended Data Table 1 | Items for $n$-sectional singular light with helical PS-front.

| PS $\delta$ | Intensity ratio | OAM range | AP range | OAM range $\times$ AP range |
|-------------|----------------|-----------|----------|-----------------------------|
| $n \times 0$ | 1              | $n$       | $\frac{1}{n}$ | $h$                         |
| $\frac{n}{4}$ | $\frac{1}{2}$ | $\frac{n}{\sqrt{2}}$ | $\frac{\sqrt{2}}{n}$ | $h$                         |
| $\frac{n}{3}$ | $\frac{1}{4}$ | $\frac{n}{2}$ | $\frac{2}{n}$ | $h$                         |
| $\frac{n}{2}$, singularity | 0 | 0 | $\frac{\infty}{n}$ | $0 \times \infty$ |
| $\frac{2n}{3}$ | $\frac{1}{4}$ | $\frac{n}{2}$ | $\frac{2}{n}$ | $h$                         |
| $\frac{3n}{4}$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{n}$ | $h$                         |
| $n$          | 1              | $n$       | $\frac{1}{n}$ | $h$                         |
# Extended Data Table 2 | Comparison between this UP and known UP

| UP for OAM and AP | Proposed UP | Known UP |
|------------------|------------|----------|
| $\delta \in \mathbb{Q}$ | $\delta = 0$, $M \in \mathbb{Q}$ | $\delta \neq 0$, $M_{12} = (M_1 + M_2)/2 \in \mathbb{Q}$ | $m \in \mathbb{Z}$ |
| Precise OAM State and its OAM spectrum | $Mn \mod n = \frac{n}{2}$; symmetry | $Mn \mod n = \frac{n}{2}$, $\delta \neq \frac{n}{2}$; symmetry, $\delta \neq \frac{n}{2}$; asymmetry except zero and $\pi$ phase jumps, $\delta = \frac{n}{2}$ | OAM eigenstate $|m\rangle$; symmetry |
| OAM precision value | integer multiple of half wavelength, OAM eigenvalue and OAM mean with max phase jump | integer multiple of half wavelength, OAM eigenvalue and OAM mean with max phase jump for $\delta \neq n/2$. Any OAM mean for $\delta = n/2$; | integer multiple of full wavelength, OAM eigenvalue |
| Represented wavefront | fractional helical | fractional helical with helical PS-front | helical |
| OAM uncertainty for OAM eigenstates | including in fractional OAM states to have $\frac{\hbar}{2\sqrt{2\pi}}$ | including in fractional OAM states to have $\frac{\hbar}{2\sqrt{2\pi}} \cos \left(\frac{\delta \pi}{n}\right) = \frac{\hbar}{2\sqrt{2\pi}} \sim \frac{\hbar}{2\sqrt{2\pi}} \times 0$ | 0 |
| AP uncertainty for OAM eigenstates [radian] | including in fractional OAM states to have $\frac{\pi}{\sqrt{3n}}$ | including in fractional OAM states to have $\frac{\pi}{\sqrt{3n}} \cos \left(\frac{\delta \pi}{n}\right) = \frac{\pi}{\sqrt{3n}} \sim \frac{\pi}{\sqrt{3n}} \times \infty$ | $\frac{\pi}{\sqrt{3}}$ |
| Quantities of OAM and AP uncertainty | Discrete | Continuous | Continuous |
| Intensity image in near-field | Round beam with phase-singularity located at $n$ azimuthally | Round beam with phase-singularity and PS-singularity located at $n$ azimuthally sinusoidal square | Sector beam |
| Symmetric AP (FVn) | Decrease intensity |
|-------------------|-------------------|
| Profile for theoretical overlap amplitude or experimentally coincident fringe | Symmetry for angle period \( \frac{2\pi}{n} \) | \( \begin{cases} 
\text{asymmetry for} \\
\text{super} \frac{2\pi}{n} \text{ period, } \delta \neq n \\
\text{symmetry for} \frac{2\pi}{n} \text{ period, } \delta = n 
\end{cases} \) | Constant |