“NON-PERTURBATIVE METHODS” IN FIELD THEORY

KENNETH INTRILIGATOR

UCSD Physics Department, 9500 Gilman Drive, La Jolla CA 92037, USA

This talk is an overview of selected topics related to renormalization group flows and the phases of gauge theories.

1 Introduction

In an asymptotically free gauge theory, starting at small coupling \( g \), the coupling \( g(\mu/\Lambda) \to 0 \) in the ultra-violet and perturbation theory is valid. In the infra-red, on the other hand, \( g(\mu/\Lambda) \) becomes large and we might wonder if there are some “non-perturbative methods” (which was the assigned title of my talk) which can be applied to the problem. The analysis in the IR looks hard, but often that’s only because we’re making a mistake in trying to describe IR physics in terms of the UV variables. The physics of the IR is often better described using a weakly coupled, effective field theory for the light degrees of freedom, for example the chiral lagrangian for pions.

The question to ask, then, is “what are the correct variables and interactions in the IR?” There are two paths for answering this question. The first is to derive the answer directly from the ultraviolet lagrangian using some sort of non-perturbative methods. This path is extremely hard. While there have been important developments in such non-perturbative methods, there does not seem to be a systematic way to determine even when such methods are or are not applicable, even for answering only qualitative questions about the physics. (This possibly partly reflects my own ignorance concerning this path.)

The second path to answering the above question is to use symmetries, match to known results, use guesswork if needed, do some non-trivial cross-checks, then conjecture you’ve “solved” the theory! In other words, in this path you cheat, getting to the answer without doing all the hard work. This path has been very fruitful over the past few years (and longer, e.g. it’s the path from QCD to pions and the chiral lagrangian). In this talk, I will mostly discuss results obtained by this second path, especially using supersymmetry. Of course, in the long-term, we would also like to have non-perturbative methods which are powerful enough to be able to derive these results directly from the UV lagrangian. In any case, knowing the answers should prove useful in developing more direct non-perturbative methods.

2 Renormalization group flows to the IR

Theories can flow under the renormalization group to either a free or interacting, scale invariant, RG fixed point in the extreme IR. We can schematically picture the flows in theory space, with coordinates given by the various coupling constants. Intuitively, we can picture the RG flows as streams of water, flowing over mountains and through valleys, into lakes. The “lakes,” which are the fixed points, can be either free or interacting. Each fixed point is the end point of flows for all theories with coupling constants \( g_i \) in some particular basin of attraction. The basins of attraction do not overlap: a given theory will flow to a unique endpoint.

There are some flows which, when plotted in theory space, with the coupling constant coordinates, look funny. For example, while one flow ends up in one fixed point, another flow, which starts off parallel and near to the first flow, can end up veering away from the first flow and eventually flow into a different fixed point, possibly quite distant from that of the first flow. In other words, seemingly nearby points can be in very different basins of attraction. In the picture of streams of water, these funny looking flows are due to various mountain ridges not shown on the map. While the two flows described above initially seemed nearby, they were actually separated by a large mountain ridge and thus wound up flowing through different valleys and into different lakes.

The intuition behind the above picture of the renormalization group flow is that massive degrees of freedom decouple in RG flows to the infrared – that there is a thinning of degrees of freedom. Because of this, the RG flow is irreversible. The flow can not circle around to where it was before, and thus there are no limit cycles where the flow forever circles around in a closed loop. As with the streams of water, the intuition is that the RG always flows “downward.”

This was proven in 2d by Zamolodchikov for any unitary theory. He showed that there is a “c-function,” \( c(g_i) \), which monotonically decreases along all RG flows and is stationary at the RG fixed points. The beta functions \( \beta_i = d g_i / d(\log \mu) \), which give the “velocity vector,” \( v_i = -\beta_i \), of the flow to the IR, are the gradients of the c-function: \( \beta_i = \partial c / \partial g_i \). The c-function \( c(g_i) \), which counts the number of degrees of freedom of the theory,
thus corresponds to the notion of height in the picture of flowing water. It was also shown that there is a positive definite metric $G^{ij}(g)$ on theory space, which can be used to measure distances between theories. The metric allows the funny flows to be understood, as it gives the information about if there are mountains. Two points $g_i^{(1)}$ and $g_i^{(2)}$ which seem nearby on the map are actually separated by a mountain ridge if \( \int_{g_i^{(1)}}^{g_i^{(2)}} \sqrt{G^{kl}dg_kdg_l} \) is always large. The RG flows, as with water flows, are minimum distance, geodesics with respect to the metric $G^{ij}$.

We expect a similar situation in 4d, though there the proof has been elusive. A candidate c function analogous to the 2d case, has been conjectured by Cardy

$$c(g) = \frac{120}{\pi^2} \int_{S^4} \langle T_\mu^\nu \rangle \sqrt{g} d^4x.$$  

For free fields, this $c = 124N_1 + 11N_{1/2} + 2N_0$, where $N_j$ is the number of fields of spin $j$. The question, then, is if this function indeed monotonically decreases along all RG flows. A weaker statement to check is if the UV and IR endpoints of flows satisfy $c_{UV} < c_{IR}$. The fact that $c_{UV} > c_{IR}$ for Cardy’s c-function was checked in a number of “solved” supersymmetric gauge theory examples with various other candidate c-functions ruled out. A recent claim is that this longstanding problem has finally been solved and that Cardy’s $c(g_i)$ can indeed be proven to monotonically decrease along all RG flows for any unitary theory. It remains to be seen if the arguments of \[ \] which has been regarded with some scepticism by some\[ \] are really an airtight proof of the c-theorem. In any case, it seems likely that there is a c-theorem in 4d and that this is the correct c-function.

3 Anomaly matching

A useful constraint on where RG flows possibly end up are the ’t Hooft anomaly matching conditions\[\]. The ’t Hooft anomalies are the obstructions, $TrH$ and $TrH^3$, to gauging a global symmetry $H$. For unbroken $H$, these anomalies can be evaluated knowing only the massless fermion spectrum. ’t Hooft argued that these quantities are constant along RG flows. Thus the original UV theory and the IR fixed point must have the same ’t Hooft anomalies. Also, all theories which flow to the same RG fixed point must have the same ’t Hooft anomalies. This is a useful constraint for ruling out scenarios about where theories flow: if the ’t Hooft anomalies don’t match, it’s wrong!

It was recently argued that anomaly matching for discrete symmetries is also a useful constraint for ruling out various scenarios. In particular, it was argued there that the recently conjectured “chirally symmetric phase” of $\mathcal{N} = 1$ super Yang-Mills, which will be discussed further in sect. 6, be ruled out on the basis of matching anomalies for the discrete $Z_{2h} \subset U(1)_R$ left unbroken by instantons. This was criticized in \[\] on the basis that, if the discrete symmetry is the remnant of a spontaneously broken continuous symmetry, there will be Goldstone bosons present which ensure that the discrete anomalies are always matched. However, it is not clear why this criticism should be applicable for theories, such as the one under consideration, where the discrete symmetry does not come from spontaneously breaking a continuous symmetry and there is no Goldstone boson present.

Anomaly matching which does work for a given scenario, in a way which is non-trivial, can be regarded as non-trivial evidence that the scenario is correct. For example, confinement was thus argued to occur in $\mathcal{N} = 1$ supersymmetric $SU(2)$ gauge theory with a single matter field $Q$ in the 4 representation of $SU(2)$.

There are, however, some cautions to point out regarding anomaly matching. One is that global symmetries of one theory might not be manifest in another which flows to the same fixed point. This phenomenon, where a theory has a larger global symmetry in the infrared, is that of “accidental global symmetries.” This point was emphasized in \[\] and illustrated in a variety of supersymmetric examples. Another caution is that there are known examples of numerically miraculous, but physically misleading matching, suggesting confinement, in a class of models which are argued to definitely not confine but, rather, have interacting RG fixed points. For example, this is the case for $\mathcal{N} = 1$ supersymmetric $SO(N)$ with a single matter chiral superfield in the two-index symmetric tensor representation of $SO(N)$. There is a highly non-trivial anomaly matching, which holds for all $N$, suggesting confinement. Nevertheless, the theories actually do not confine.

4 Interacting RG fixed points

An interacting RG fixed point is a non-trivial confor- mal field theory. While there are many known conformal field theories in 2d, they were previously considered to be quite rare and exotic in 4d. A surprise which has been learned from studying supersymmetric theories is that they are actually very common! Indeed, they generi- cally occur if there is enough matter. This is not special to supersymmetry. A basic scenario for having a RG fixed point dates back to \[\]. Suppose the matter content is such that the one-loop beta function is negative (asymptotically free in UV) while the two-loop beta function is positive: $\beta(g) = -b_1g^2 + b_2g^3 + \ldots$. Then $\beta(g^*) \approx \sqrt{b_1/b_2} = 0$ and perturbation theory suggests
that the theory has a fixed point. As long as $g^* = \sqrt{b_1/b_2}$ is small, one is inclined to trust perturbation theory and believe that the fixed point actually exists.

For ordinary (non-supersymmetric) $SU(N_c)$ QCD with $N_f$ flavors of quarks in the $N_c + \bar{N_c}$, the above requirement on the signs of the one and two-loop contributions to the beta function are satisfied for $N_f$ just below $11N_c/2$. For example, for $N_f = \frac{11}{2}N_c - 1$ we have $b_1 \sim 1$ and $b_2 \sim N_c^2$ and thus expect a RG fixed point with $g^* \sim 1/N_c$. For large $N_c$, we can trust perturbation theory (the scaled coupling $(g^*)^2N_c \sim 1/N_c$ is also small) and are thus inclined to believe that the RG fixed point really exists. For $N_f$ any larger, $N_f \geq 11N_c/2$, the theory ceases to be asymptotically free and flows to a free theory in the infrared.

Decreasing $N_f$ below $N_f = \frac{11}{2}N_c - 1$, the value of $g^*$, where the perturbative beta function suggests a fixed point, tends to increase. For low enough $N_f$, we are then less inclined to believe perturbation theory and need non-perturbative methods to determine whether or not the theory continues to have a RG fixed point. The expectation is that for some entire range of flavors, $\frac{11}{2}N_c > N_f > N_f^*$, the theory has RG fixed points. Then, at some critical number of flavors, $N_f^*$, it goes over to a new phase. Eventually, for low enough $N_f$, the theory is expected to be in a confining phase.

As will be reviewed later, this flavor dependent phase structure has been well studied in the supersymmetric context using method two: symmetries and non-trivial cross checks.

For the present case of non-supersymmetric QCD, the phase structure has been analyzed using method one, with a variety of more direct approximations for the non-perturbative regime, both on the lattice and in the continuum. I will only mention here a result obtained in the continuum. In [15] the situation was considered where one starts at a non-trivial RG fixed point with some number $N_f$ massless flavors, and gives a mass to one flavor. In the infrared, this theory flows to QCD with $N_f - 1$ massless flavors. Analyzing such flows, it was argued that there is a direct transition from a RG fixed point, with no confinement or chiral symmetry breaking, to a phase with confinement with chiral symmetry breaking at $N_f^* \approx 4N_c$. This pattern differs from that seen in supersymmetric QCD where decreasing $N_f$, the phases and transitions are (RG fixed point) $\to$ (non-Abelian free magnetic phase) $\to$ (confinement without chiral symmetry breaking) $\to$ (confinement with chiral symmetry breaking).

In supersymmetric theories, the restrictions of conformal invariance become much more powerful. The conformal group and supersymmetry must combine into a single, super-conformal symmetry group. The symmetry group elements can be represented as a supermatrix

$$
\begin{pmatrix}
SO(d,2) & Q \\
Q^\dagger & J_R
\end{pmatrix},
$$

where $Q$ are the fermionic supersymmetry generators, including additional ones associated with the superconformal transformations, and $J_R$ are bosonic $R$-symmetries.
which rotate the supercharges. It is possible to show that such a supergroup, with \( Q \) in the spinor representation of \( SO(d, 2) \), can exist only for \( d \leq 6 \) \cite{18,19}, so superconformal theories are impossible above \( d = 6 \). The spectrum of operator dimensions are also constrained by the superconformal symmetry, e.g. it is possible to show that all operators satisfy

\[
\Delta \geq \frac{d - 1}{2} |q_R|,
\]

where \( \Delta \) is the dimension of the operator and \( q_R \) is its charge under a \( U(1) \) subgroup of the \( R \)-symmetry group.

More generally, the reason why supersymmetric theories are often easier to “solve” via the second method is that all light fields, coupling constants, masses, even \( \Lambda_{QCD} \sim (e^{-\frac{\pi^2}{2} + i\theta})^{1/b} \) are complex (in \( d = 4 \)). Various quantities are holomorphic in the fields and coupling constants. This is the “power of holomorphy,” found by N. Seiberg \cite{20}. Since unbroken supersymmetry implies that \( E_{vac} = 0 \) always, there can be no first order phase transitions. The only possible phase transitions are second order, with some order parameter. In addition, supersymmetry implies that even the possible second order phase transitions occur at isolated points in the complex plane, and can thus always be avoided; there are no “walls” separating phases. (The closest thing to a “wall” in a supersymmetric theory is the curve of marginal stability \cite{21}, where otherwise stable (BPS) states can decay.)

For this reason, it is possible to obtain some exact results by matching to known results in various limits, e.g. weak coupling, and then analytically continuing in the various fields and coupling constants, to obtain the exact result everywhere. Only certain quantities can be obtained in this way – not all aspects of the theory are “solved.” But the solvable aspects concern the most interesting questions: the infrared physics. The exact results thus give useful insight into the strongly coupled dynamics of supersymmetric theories.

By using analytic continuation, with no phase transitions, in masses \( m \) or field expectation values \( \ell \), along with decoupling arguments, many results for different theories are interrelated. There is thus a growing web of interrelated results of different models, with many cross-checks. See, for example \cite{22} for a number of early examples and references.

The exact results for supersymmetric models have two types of applications for non-supersymmetric theories. One is for obtaining some qualitative insights into strong coupling phenomena. Another is as a testing grounds for general conjectures, e.g. the \( c \)-theorem, and non-perturbative techniques. For example instanton technology has been checked and extended by comparing with exact results obtained via supersymmetry \cite{22}. (Also subtleties concerning certain exact results, as well as new exact results, were obtained using the instanton technology discussed in \cite{23}; see references cited therein.)

One might wonder about obtaining more direct, quantitative, information for non-supersymmetric theories by starting with a supersymmetric theory, for which exact results can be obtained, and perturbing by adding supersymmetry breaking terms. This works for small, soft, supersymmetry breaking terms \( m_s \), but the \( m_s/|\Lambda| \) corrections are not under control. There can be phase transitions in \( m_s/|\Lambda| \), which can not be avoided, i.e. a \( m_s/\Lambda \) “wall.” There is, in fact, evidence for such phase transitions: the nearly supersymmetric, small \( m_s \), physics is qualitatively different from the non-supersymmetric, large \( m_s \) physics in various examples \cite{22,24}.

6 A quick tour of the 4d, \( \mathcal{N} = 1 \) susy gauge theory landscape

Returning to our analogy between the renormalization group and streams of water flowing over mountains and valleys, into lakes, we start our tour of the landscape at the bottom of a vast mountain range, with pure \( \mathcal{N} = 1 \) supersymmetric glue. This is at the bottom of the range because other theories, with vector-like matter, flow down to pure glue in the infrared upon adding masses for the matter fields.

Pure \( \mathcal{N} = 1 \) supersymmetric glue, with no additional matter fields, is the same as \( \mathcal{N} = 0 \) supersymmetric Yang-Mills, with gauge group \( G \), along with massless adjoint fermion matter fields \( \lambda \), which are the gluinos. The infrared physics of these theories is confinement, with a mass gap, and \( Z_{2h} \rightarrow Z_2 \) chiral symmetry breaking. Here \( h \equiv C_2(G) \) is the quadratic Casimir of the adjoint and \( Z_{2h} \) is the anomaly-free, discrete subgroup of the \( U(1) \) global \( \lambda \) fermion-number symmetry which is unbroken by instantons, which lead to \( \langle \prod_{i=1}^{h} (\alpha \lambda \alpha^*) (x_i) \rangle = (\text{const.}) \Lambda^{3h} \); this is independent of the positions \( x_i \) (as guaranteed by supersymmetry), and factorization for widely separated \( x_i \) suggests \( \langle \prod_{i=1}^{h} (\lambda \lambda) \rangle \rightarrow \langle \lambda \lambda \rangle^h \). The \( Z_2 \) is the subgroup left unbroken by gaugino condensation: \( \langle \alpha \lambda \alpha^* \rangle \sim e^{2\pi i k/h} \Lambda^3 \). Note that gaugino condensation has the quantum numbers of a “fractional instanton” and thus doesn’t correspond to any known, semi-classical, field configuration. Associated with the chiral symmetry breaking, there are \( h \) supersymmetric vacua, with mass gap, which are related by rotating the theta angle as \( \theta \rightarrow \theta + 2\pi \).

A “proof” of the above statements follows by adding vector-like matter, of a type so that this new theory is easier to “solve” than the original, pure-glue theory. Starting from the solved theory with additional matter,
we give the vector-like matter masses \( m \). Symmetries of the theory with added matter guarantee that the result of this procedure is always \( h \) supersymmetric vacua. Because there are no phase transitions in the complex mass parameter \( m \), the pure-glue theory obtained in the \( m \to \infty \) limit, where the added matter decouples, must also have \( h \) supersymmetric vacua. This method dates back to [25].

An old puzzle is that Witten’s original calculation of the index \( Tr(-1)^F \), which should be the number of supersymmetric vacua, gave \( r + 1 \) rather than \( h \), where \( r \) is the rank of the gauge group \( G \). For \( SU \) and \( Sp \) groups, the two results agree, as \( r + 1 = h \) for these cases, but for the other groups, \( SO, G_2, F_4, E_{6,7,8} \), the two answers disagree, as \( r + 1 \neq h \). This puzzle was recently resolved by Witten [26], who showed that the computation of \( Tr(-1)^F \) can miss contributions and verified that \( Tr(-1)^F = h \) for the \( SO \) groups, as well as the \( SU \) and \( Sp \) groups, which work as before. The \( G_2 \) case was similarly verified [26].

An unresolved puzzle is the normalization of \( \langle (\lambda \lambda)^h \rangle \). There are two methods to compute the normalization. The first is a direct instanton calculation in the strongly coupled, pure-glue, theory. The second is to extract it from a different instanton calculation, in a weakly coupled theory with additional, massive, vector-like matter. The two methods disagree ...

A recent claim [27] is that there are additional vacua, with unbroken chiral symmetry: \( \langle \lambda \lambda \rangle = 0 \). If this is true, it would have dramatic consequences for the entire web of interrelated theories. All supersymmetric gauge theory results (SQCD [28], Seiberg-Witten [24], etc.) would need modification. It seems quite difficult (probably impossible) to consistently modify everything to allow for this possibility, and for this reason, I personally find such a chirally symmetric vacuum to be quite unlikely (and I also find the motivation to be not so compelling). In any case, the subject perhaps deserves further investigation.

It has recently been appreciated [29] that there are domain walls between the various supersymmetric vacua with different \( \langle \lambda \lambda \rangle \). For example, for \( x_3 \to +\infty \), the vacuum can be in the \( \langle \lambda \lambda \rangle = 3 \lambda^3 \) vacuum while, and for \( x_3 \to -\infty \) it could be in the \( \langle \lambda \lambda \rangle = e^{2\pi i/h} A^3 \) vacuum, with the two phases separated by a stable domain wall. These domain walls can saturate a BPS bound. Using a connection with string theory, it was argued [29] that a flux-tube string, which usually connects a quark charge \( q \) to an anti-quark charge \( \bar{q} \), can end on the domain walls.

We now consider some general aspects of \( N = 1 \) supersymmetric theories with matter. The landscape depends on \( h = C_2(G) \) versus \( \mu = \sum_f C_2(R_f) \), where the sum is over all matter fields \( f \) and \( R_f \) is the representation of the gauge group which that matter field is in. Taking into account the one-loop beta function, the theories are asymptotically free for \( \mu < 3h \).

For \( \mu \geq h \), there is an exactly degenerate “moduli space” of physically inequivalent vacua. Although this degeneracy is not protected by any standard symmetry, it is ensured by holomorphy constraint coming from supersymmetry, along with the boundary condition that the theory behave properly at weak coupling. Associated with the continuously degenerate vacua, there are exactly massless moduli fields, which generally are not Goldstone bosons (though some could be).

For \( \mu < h \), i.e. less matter than in the situation described above, there is a classical vacuum degeneracy, similar to that described above. But for \( \mu < h \), at the quantum level, this degeneracy is generically lifted by non-perturbative effects, which dynamically generate a superpotential \( W_{\text{dyn}} \neq 0 \). \( W_{\text{dyn}} \) can be exactly calculated using the holomorphy constraints, along with symmetries, connecting to known limits, and a universal, weakly coupled, \( SU(2) \) instanton calculation [31-33]. Asymptotic freedom implies that the potential associated with \( W_{\text{dyn}} \) is large at small field expectation values and slopes to zero at large expectation values. Thus the theory with \( W_{\text{dyn}} \neq 0 \) actually has no stable vacuum for \( W_{\text{tree}} = 0 \). By adding a \( W_{\text{tree}} \neq 0 \), it is possible to obtain a stable vacuum. In some models, this stable vacuum dynamically breaks supersymmetry; see [33] for a recent review.

Interestingly, some special models with \( \mu < h \) have inequivalent “branches” i.e. phases of the theory. The branches are labeled by a discrete quantum parameter. This only occurs where there are no matter fields in faithful representations of the center of the gauge group \( G \). This is the same condition as for having distinct Higgs, confining, or oblique confining phases: there must be some external test charges, charged under the gauge group \( G \), which can not be screened by the dynamical matter fields. Wilson or ‘t Hooft loops involving these test charges can then have either area or perimeter law dependence, serving as order parameters for inequivalent phases. The different branches correspond to the different phases.

The landscape for all theories with simple gauge group \( G \) and matter \( \mu \leq h \) has now been completely charted out. In several recent works [34-37], these theories have been comprehensively discussed, with all remaining, previously unsolved, cases analyzed.

Theories with \( \mu = h + 1 \) always have a quantum moduli space of vacua, which coincides with the classical moduli space of vacua. Often, as in the classic case of \( SU(N) \) gauge theory with \( N_f = N_c + 1 \), the low energy theory is free field theory, with no gauge fields, everywhere on the moduli space. The correct variables for the free fields are the confined meson or baryon moduli. At the origin, there are some additional massless confined meson or baryon moduli fields and the sigma model
metric is flat in terms of these confined moduli. Away from the origin, the additional massless fields get a mass via a superpotential, which is “dangerously irrelevant” at the origin. The landscape of such “s-confining” models based on simple gauge groups has been systematically surveyed, with previously unsolved models analyzed.

Not all models with $\mu = h + 1$ are so simple. Generally, models with no matter fields in a faithful representation of the gauge group can exhibit other, more interesting, types of phenomena. For example, $SO(N_c)$ with $N_f = N_c - 1$ matter fields in the $N_c$ representation of $SO(N_c)$ has $\mu = h + 1$, but the matter is not in a faithful representation of the gauge group, as test charges in the spinor rep of $SO(N_c)$ test charges can not be screened by the vector rep matter. Rather than a theory of free chiral superfields, the theory at the origin has a non-trivial RG fixed point for $N_c = 3$; this is an interesting theory with electric-magnetic-dyonic triality. For $N_c \geq 4$, the theory at the origin has free-magnetic, composite gauge invariance, with gauge group $SO(3)$ and $N_f = N_c - 3$ flavors. The phases of these theories were recently further analyzed by starting with a theory with additional, vector-like, matter in a faithful representation of the center of the gauge group, whose addition means that distinct phases do not occur, and then decoupling this field by giving it a large mass.

For $h + 1 < \mu < 3h$ and no tree-level superpotential, $W_{\text{free}} = 0$, all theories have RG fixed points or possibly free-magnetic phases (reviewed below) at the origin of their moduli space. The dynamics of the theory at the origin is generally poorly understood, and the landscape remains largely uncharted.

Some interesting phenomena have been observed in a hodge-podge of examples, with no general understanding. One is that two different looking theories, with different gauge groups and matter fields, can flow to the same renormalization group fixed point. At the fixed point, both give exactly the same physics. The original example of this for 4d asymptotically free (as opposed to finite) theories is Seiberg’s SQCD duality, between $SU(N_c)$ with $N_f$ fundamental flavors and $SU(N_f - N_c)$ with $N_f$ flavors, some singlet fields to be identified as the mesons of the original theory, and a superpotential. For $3N_c > N_f > \frac{2}{7}N_c$, both theories flow to the same, interacting, renormalization group fixed point. The $SU(N_c)$ theory is scale invariant at some coupling constant $\tilde{g}^{N_c,N_f}$, and the dual $SU(N_f - N_c)$ is the same scale invariant theory at some coupling $\tilde{g}^{N_f - N_c,N_f}$. Which description of the fixed point is more useful depends on which $g^*$ is small. Generally, smaller $g^*$ must correspond to larger $\tilde{g}^*$, though the precise map between the two is not known. It is worth emphasizing that duality is inherently quantum mechanical: the two dual theories are completely inequivalent at the classical level.

An entirely new phenomenon, which was also discovered in Seiberg’s seminal paper on duality, is the existence of a non-Abelian, free magnetic phase in four dimensional $\mathcal{N} = 1$ supersymmetric theories. There are low-energy fields, which are essentially solitons of the UV theory, which behave as quarks and gluons of a non-Abelian gauge theory which is IR free. The magnetic quarks and gluons are the solution for the low energy spectrum. The composite gluons show that gauge invariance does not have to be fundamental – there can be composite gauge invariance.

There are many examples of duality and free-magnetic phases, all found via some guess-work and many non-trivial cross-checks, fitting into a growing web of interrelated examples. There is no known general criteria which can be generally applied to determine whether a given theory has an interacting RG fixed point, a free-magnetic phase, or something else. At present, one has to work on a case-by-case, basis. It is also not generally known, if a theory does have a RG fixed point, whether it should have a dual description, and what that dual description should be. There are still many confusing examples which remain unsolved, and all examples should be understood at a deeper level.

For $SU$, $SO$, and $Sp$ groups $G$ with (only) fundamental matter, the duality has been “derived” by renormalization group flows from their $\mathcal{N} = 2$ supersymmetric analogs. Using exact results in the $\mathcal{N} = 2$ supersymmetric theories, the dual gauge group $\tilde{G}$ is infrared free, and its gauge group and matter content is directly seen. Breaking to $\mathcal{N} = 1$ supersymmetry at scale $m_s$, we have two different flows. The first has $m_s \gg \Lambda$ and flows first very close to the $\mathcal{N} = 1$ theory with gauge group $G$, and eventually to the fixed point of that theory. The second type of flow occurs for $m_s \ll \Lambda$. Then the dynamics is off controlled by the $\mathcal{N} = 2$ theory, and the theory first flows very close to the $\mathcal{N} = 2$ theory with dual gauge group $\tilde{G}$ and matter content. Eventually, the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ breaking due to $m_s$ kicks in, and the theory flows close to the $\mathcal{N} = 1$ supersymmetric theory with dual gauge group $\tilde{G}$ and matter content. Eventually, that theory flows to some fixed point. Now, assuming that the two different flows really are close to the two dual $\mathcal{N} = 1$ theories, and since $\mathcal{N} = 1$ supersymmetry should prohibit phase transitions in $m_s/\Lambda$, the two dual theories must, in fact, flow to the same fixed point.

This “proof” has a direct analog in the recent brane constructions of 4d gauge theories. However, this requires several more assumptions about the dynamics of branes and string theory, so perhaps this is better referred to as a “relation” than a “proof.”
7 Results in other dimensions and connections with string theory

There have also been a variety of results for a variety of gauge theories in other dimensions. Note that for $d \neq 4$ the gauge coupling is dimensionful. The effective, dimensionless, gauge coupling at an energy scale $E$ is $g_{eff} = gE^{(d-4)/2}$. Because of this classical scale dependence, all gauge theories are asymptotically free for $d < 4$ and infra-red free (i.e. “nonrenormalizable”) for $d > 4$.

For $d = 3$, there are many interesting RG fixed points with various supersymmetries and dualities. As an example with $N = 2$ supersymmetry in 3d (this has the same number of supercharges as $N = 1$ in 4d), a Wess-Zumino theory with a single chiral superfield $X$ and superpotential $W = X^3$ flows to an interacting RG fixed point. As another set of examples, SQED, with $(U(1))$ gauge group and $N_f > 0$ flavor of fields with charges $\pm1$ flow to RG fixed points. For the case $N_f = 1$, this SQED fixed point has a dual description in terms of a Wess-Zumino theory with chiral superfields $X, Y$, and $Z$, with superpotential $W = XYZ$. There are also “mirror symmetry” dual descriptions of RG fixed points with $N = 4$ supersymmetry in 3d: this duality exchanges: Higgs $\leftrightarrow$ Coulomb branches, classical $\leftrightarrow$ quantum, masses $\leftrightarrow$ Fayet-Iliopoulos terms, and manifest global symmetries $\leftrightarrow$ hidden quantum symmetries.

More surprisingly, it has been found (via string theory) that non-trivial RG fixed points exist for various (supersymmetric) field theories in $d = 5$ and $d = 6$. For $d = 5$ there is a complete classification of all supersymmetric gauge theories which exist via flows from 5d RG fixed points. The fixed points themselves are not well understood. Perturbing them by a relevant operator, which corresponds to the 5d gauge kinetic term, they flow to particular, IR free, 5d gauge theories which, in this sense, “exist.” For example, for $SU(N_c)$ gauge group, the theory exists for $N_f \leq 2N_c$ flavors; this is the analog of the constraint for asymptotic freedom in 4d. In 5d, RG fixed points only exist with minimal $N = 1$ supersymmetry (which has the same number of supercharges as $N = 2$ in 4d).

In $d = 6$ there can be chiral supercharges and non-trivial RG fixed points exist for the minimal $\mathcal{N} = (1, 0)$ supersymmetry (which has the same number of supersymmetries as $N = 2$ in 4d) and for $\mathcal{N} = (2, 0)$ supersymmetry. The $\mathcal{N} = (1, 1)$ theories are necessarily free and theories with higher $\mathcal{N}$ have fields with spins up to two, i.e. necessarily include gravity. The $\mathcal{N} = (2, 0)$ theories do not include standard gauge fields, which are not allowed by the $(2, 0)$ supersymmetry, but rather chiral 2-form gauge fields, with self-dual field strengths: $A_{\mu \nu}$, with $dA = \ast dA$. The $\mathcal{N} = (1, 0)$ theories can include gauge fields, but are either free or anomalous unless self-dual, two-form gauge fields $A_{\mu \nu}$ are also present.

In all cases with 6d non-trivial fixed points, there are BPS strings, which appear to become tensionless at the origin of the “Coulomb branch,” where scalar moduli partners of $A_{\mu \nu}$ vanish. Nevertheless, it seems possible to interpret the theory at the origin as in interacting, 6d RG fixed point field theory rather than requiring some new and unknown kind of “tensionless string theory.”

The $\mathcal{N} = (2, 0)$ theory can be constructed from type IIB string theory compactified down to 6d on a space which is allowed to become singular or in IIA string theory or M-theory via branes. The world-volume of $N_c$ type IIA or M theory 5-branes has a 2-form version of $SU(N_c)$, with $N_c$ abelian 2-forms $A_{\mu \nu}^a$, with $dA = \ast dA$, corresponding to the Cartan of the $SU(N_c)$ and 6d strings corresponding to the $W$-bosons. At the origin of the moduli space, where the strings appear to become tensionless, there is the interacting $\mathcal{N} = (2, 0)$ RG fixed point conformal field theory.

The interacting 6d $\mathcal{N} = (2, 0)$ theory gives an ordinary gauge theory when reduced to $d < 6$ on a circle. Going to 4d by making two directions circles of radii $R_1$ and $R_2$ yields a 4d, $\mathcal{N} = 4$ SU($N_c$) gauge theory with $g^2_{YM} = R_1/R_2$. Because we can obviously exchange the names of the two circles, this construction makes the $g_{YM} \leftrightarrow 1/g_{YM}$ Montonen-Olive duality of this theory manifest. It is also possible to obtain a theory similar to 4d, non-supersymmetric, QCD from the 6d theory by changing the boundary conditions on the circles. The 4d, non-supersymmetric theory thus obtained is referred to as “MQCD” and it is hoped to be in the same universality class as real-world QCD, without phase transitions.

There has been a fruitful interplay between gauge theories and string theories over the past few years. It is possible to get composite gauge invariance in string theory in a variety of (related) ways: singular compactification geometry, zero size instantons, and branes. There is a correspondence between results in the field theory thus obtained and results concerning string theory. Generally the correspondence between field theory and string theory results involve opposite limits: where one side is known well (perhaps weakly coupled), the other side is often poorly understood. An example of a nice interplay between field theory and string theory is.

In this way, known results on one side translate into new results on the other side or, in the case where both sides are understood, cross-checks of the correspondence are obtained.

An example by which composite gauge invariance is obtained in string theory is via D-branes, which can be thought of as being similar to solitons of string theory. These theories have supersymmetric gauge theories living in their world-volume; see for an extensive review with
references. A simple example is $N_c$ parallel D3 branes of type IIB string theory, which has 4d, $\mathcal{N} = 4$, $SU(N_c)$ gauge theory living in its world-volume. (I ignore the subtle issue about if this theory is $U(N_c)$ or $SU(N_c)$.) Various other 4d (and other $d$) theories, with fewer supersymmetries, can be obtained via more complicated arrangements of branes. It is possible to use known field theory results to obtain the rules governing $D$-branes. Knowing these rules, along with various string dualities, $D$-branes prove to be a powerful tool to generalize to new examples and obtain new information about field theory. See \cite{Denef:2000nb} for an extensive review, with references.

8 Renormalization group fixed points and AdS.

The Maldacena conjecture \cite{Maldacena:1997re,Gubser:1998bc,Witten:1998qj}, in the generalized sense of \cite{Gubser:1998bc}, is that a gravity theory in $d + 1$ dimensional anti de Sitter space, $AdS_{d+1}$, is dual to a $d$ dimensional conformal field theory on the boundary of $AdS_{d+1}$. $d + 1$ dimensional anti de Sitter space is a solution of Einstein’s equations with negative cosmological constant, $\Lambda < 0$, and has a time-like boundary, which is ordinary $d$-dimensional Minkowski spacetime. The extra space dimension associated with the bulk of the $d + 1$ dimensional $AdS_{d+1}$ is to be thought of as roughly the renormalization group parameter of the boundary field theory.

A motivation for such a duality is that the $d$ dimensional conformal group is the same as the symmetry group of $AdS_{d+1}$. This symmetry group is inherited by the conformal field theory on the boundary of $AdS_{d+1}$. A concrete relation between the two dual theories is \cite{Malda:1998}:

$$Z_{\text{gravity}}[\Phi_4|\partial(AdS)] = Z_{\text{CFT}}[J_i(x)] = e^{\sum_i \int d^dx J_i(x)\mathcal{O}_i(x)}_{\text{CFT}},$$

where $\Phi_4$ are $AdS_{d+1}$ gravity fields, $\mathcal{O}_i$ are associated CFT operators, and $J_i(x)$ are arbitrary source functions. There is, in this way, a map between all operators $\mathcal{O}_i(x)$ of the $d$ dimensional boundary conformal field theory and the fields $\Phi_4$ of the gravity theory.

This duality realizes \cite{Malda:1998} the “holography” of gravity, due to ’t Hooft, Thorn, and Susskind \cite{Malda:1998}. The physics of the $d + 1$ dimensional, bulk, gravity theory is encoded in that of a $d$ dimensional boundary field theory. In a theory of gravity, one dimension can be regarded as a holographic illusion.

The original example of this duality was obtained \cite{Malda:1998} by considering $D$-branes vs. the throat geometry of the associated black holes which carry the same quantum numbers. In this way, it was argued that 4d $\mathcal{N} = 4$, $SU(N_c)$ super-Yang-Mills theory is dual to type IIB string theory on $AdS_5 \times S^5$, with $N_c$ units of $F_5$ flux on the $S^5$. The string description is weakly coupled for the limit of small $g_{YM}$, with $\lambda = g_{YM}^2 N_c$ large. The Yang-Mills theory, on the other hand, is weakly coupled in the limit of $\lambda$ small. Thus, as is always the case, where one description of the physics is weakly coupled, the dual description is strongly coupled. Here the gravity or string theory can be regarded as the large $N_c$ master-field for $\lambda \gg 1$.

The duality has been generalized to $\mathcal{N} = 4$ theories with $SO(N)$ and $Sp(N)$ gauge groups \cite{Malda:1998} and to theories with $\mathcal{N} = 2,1,0$ supersymmetry \cite{Malda:1998}. Because the $AdS$ space remains untouched in these constructions, the resulting theories with $\mathcal{N} = 2,1,0$ susy also have a line of RG fixed points, i.e. are finite theories with $\beta(g) = 0$, at least in the limit of large $N_c$. It is also possible to break supersymmetry via finite temperature $T$, going from the 4d supersymmetric theory to a 3d non-supersymmetric theory by putting the theory on a Euclidean circle. It is also possible to obtain in this way the 4d non-supersymmetric MQCD via the 6d theory with $\mathcal{N} = (2,0)$ supersymmetry, which is dual \cite{Malda:1998} to $\mathcal{N}$ theory on $AdS_7 \times S^4$.

Wilson loop correlation functions are computed in the $AdS$ dual via \cite{Malda:1998}:

$$\langle \prod_i W(C_i) \rangle \sim e^{-\text{Area}[S(C_i)]},$$

where $S(C_i)$ is the minimal area, 2d world-sheet living in the $d + 1$ dimensional bulk whose boundary is $\partial C_i$, the Wilson loops $C_i$ living in the $d$ dimensional boundary. In the 4d $\mathcal{N} = 4$ theory conformal phase, in the limit $g_{YM} \to 0$ with $g_{YM}^2 N$ large, where the gravity dual is weakly coupled, this gives \cite{Malda:1998} for the potential between two test charge sources separated by distance $R$:

$$V(R) = -\frac{4\pi^2}{\Gamma(\frac{d+1}{2})} \sqrt{g_{YM}^2 N} \frac{\text{Area}}{R}.$$

Note that the $g_{YM}$ dependence differs from the $V(R) \sim g^2/R$ expected for small $g_{YM}^2 N$: the above result is thus interpreted as a non-trivial, new prediction for the theory in the limit of large $g_{YM}^2 N$. Presumably, there is some function of $g_{YM}$ and $N$ which interpolates between the two results.

Applying to 4d non-susy MQCD, obtained from higher dimensions via finite $T$, the potential is found to be \cite{Malda:1998}:

$$V(R) = \sigma R + O(e^{-RT}), \quad \sigma \sim (g_{YM}^2 N) T^2.$$

Also glueball masses have been analyzed in this limit \cite{Malda:1998}. A peculiar result, as emphasized by \cite{Malda:1998}, is that $M_{\text{glueball}}$ is $g_{YM}$ independent, so the above expression for the string tension implies that $M_{\text{glueball}}/\sqrt{\sigma} \to 0$ in the large $g_{YM}^2 N$ limit.

8

V(R) = σR + O(e^{-RT}), \quad σ \sim (g_{YM}^2 N) T^2.
9 Summary

To summarize, there are lots of predictions for strong coupling. It would be nice to have a deeper understanding, and also non-perturbative methods to directly check them! This is a challenge for the future.

Acknowledgments

I would like to thank the organizers for inviting me to speak at this interesting conference. I am especially grateful for their extensive assistance and patience, and for the many friendly reminders to complete this contribution. I am supported by UCSD DOE grant DOE-FG03-97ER40546 and an Alfred Sloan Foundation Fellowship. This contribution was written while I was a visitor at the IAS, and supported in part by the W.M. Keck Foundation.

References

1. A.B. Zamolodchikov, JETP Lett. 43 (1986) 730.
2. J.L. Cardy, Phys. Lett. B 215, 749 (1998).
3. D. Anselmi, J. Erlich, D.Z. Freedman, A. Johansen, hep-th/9711032, Phys. Rev. D 57, 7570 (1998); D. Anselmi, D.Z. Freedman, M.T. Grisaru, A.A. Johansen, hep-th/9708043, Nucl. Phys. B 526, 543 (1998).
4. S. Forte, J.I. Latorre, hep-th/9805013, Nucl. Phys. B to appear.
5. Anonymous expert, private communication.
6. G. ’t Hooft, in “Recent Developments in Gauge Theories,” eds. G. ’t Hooft et. al. (Plenum Press, New York, 1980), 135.
7. C. Csaki, H. Murayama, hep-th/9710103, Nucl. Phys. B 515, 114 (1998).
8. A. Kovner, M. Shifman, hep-th/9702174, Phys. Rev. D 56, 2396 (1997).
9. I. Kogan, A. Kovner, M. Shifman, hep-th/9712046, Phys. Rev. D 57, 5195 (1998).
10. K. Intriligator, N. Seiberg, S. Shenker, hep-th/9410203, Phys. Lett. B 342, 152 (1995).
11. R. Leigh, M.J. Strassler, hep-th/9611020, Nucl. Phys. B 496, 132 (1997).
12. J.H. Brodie, P. Cho, K. Intriligator, hep-th/9802092, Phys. Lett. B 429, 319 (1998).
13. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).
14. T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).
15. T. Appelquist, J. Terning, L.C.R. Wijewardhana, hep-ph/9602385, Phys. Rev. Lett. 77, 1214 (1996); T. Appelquist, A. Ratnaweera, J. Terning, L.C.R. Wijewardhana, hep-ph/9806472, Phys. Rev. D 58, 105017 (1998).
16. P. Pouliot, M.J. Strassler, hep-th/9510228, Phys. Lett. B 370, 76 (1996).
17. G. Mack Commun. Math. Phys. 55, 1 (1977).
18. W. Nahm, Nucl. Phys. B 135, 149 (1978).
19. S. Minwalla, hep-th/9712074.
20. N. Seiberg, hep-ph/9309333, Phys. Lett. B 318, 469 (1993).
21. K. Intriligator and N. Seiberg, hep-th/9509066, Nucl. Phys. B. Proc. Suppl. 45B (1996) 1.
22. V.V. Khoze, M.P. Mattis, M.J. Slater, hep-th/9804009.
23. H.C. Cheng, Y. Shadmi, hep-th/9801148, Nucl. Phys. B to appear.
24. N. Arkani-Hamed, R. Rattazzi, hep-th/9804088.
25. M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 296, 445 (1988).
26. E. Witten Nucl. Phys. B 202, 253 (1982).
27. E. Witten, hep-th/9712028, J. High Energy Phys. 02, 006 (1998).
28. A.V. Smilga, hep-th/9801078, Phys. Rev. D 58, 105014 (1998).
29. V. Novikov, M. Shifman, A. Vainshtein, V. Zakharov, Nucl. Phys. B 260, 157 (1985).
30. N. Seiberg, hep-th/9402044, Phys. Rev. D 49, 6857 (1994).
31. N. Seiberg and E. Witten, hep-th/9407087, Nucl. Phys. B 426, 19 (1994).
32. G. Dvali, M. Shifman, hep-th/9612128, Phys. Lett. B 396, 64 (1997); A. Kovner, M. Shifman, A. Smilga, hep-th/9706089, Phys. Rev. D 56, 7978 (1997).
33. E. Witten, hep-th/9706100, Nucl. Phys. B 507, 658 (1997).
34. I. Affleck, M. Dine, N. Seiberg, Nucl. Phys. B 241, 493 (1984); Nucl. Phys. B 256, 557 (1985).
35. E. Poppitz, S. Trivedi, hep-th/9803104.
36. B. Grinstein, D.R. Noile, hep-th/9710001, Phys. Rev. D 57, 6471 (1998); hep-th/9803139, Phys. Rev. D 58, 045012 (1998).
37. G. Dotti, A.V. Manohar, hep-th/9712010, Phys. Rev. Lett. 80, 2758 (1998); G. Dotti, A.V. Manohar, W. Skiba, hep-th/9803087.
38. C. Csaki, W. Skiba, hep-th/9801173, Phys. Rev. D 58, 045008 (1998).
39. C. Csaki, M. Schmaltz, and W. Skiba, hep-th/9612205, Phys. Rev. D 55, 7870 (1997).
40. K. Intriligator and N. Seiberg, hep-th/9503173, Nucl. Phys. B 444, 125 (1995); hep-th/9506083.
41. M.J. Strassler, hep-th/9709081.
42. P.C. Argyres, M.R. Plesser, N. Seiberg, hep-th/9603042, Nucl. Phys. B 471, 159 (1995); P.C. Argyres, M.R. Plesser, A.D. Shapere, hep-th/9608129, Nucl. Phys. B 483, 172 (1997).
43. S. Elitzur, A. Giveon, D. Kutasov, hep-th/9702014.
44. O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M.J. Strassler, hep-th/9703110, Nucl. Phys. B 499, 67 (1997).
45. K. Intriligator, N. Seiberg, hep-th/9607207, Phys. Lett. B 387, 513 (1996).
46. N. Seiberg, hep-th/9608111, Phys. Lett. B 388, 753 (1996); K. Intriligator, D.R. Morrison, N. Seiberg, hep-th/9702198, Nucl. Phys. B 497, 56 (1997).
47. N. Seiberg, E. Witten, hep-th/9603003, Nucl. Phys. B 471, 121 (1996).
48. N. Seiberg, hep-th/9609161, Phys. Lett. B 390, 169 (1997).
49. K. Intriligator, hep-th/9702038, Nucl. Phys. B 496, 177 (1997).
50. L. Susskind, unpublished.
51. E. Witten, hep-th/9507121.
52. A. Strominger, hep-th/9512059, Phys. Lett. B 383, 1996 (44).
53. A. Sen, hep-th/9605150, Nucl. Phys. B 475, 562 (1996); T. Banks, M.R. Douglas, N. Seiberg, hep-th/9605199, Phys. Lett. B 387, 278 (1996).
54. J. Polchinski, hep-th/9611050.
55. A.Giveon, D. Kutasov, hep-th/9802076.
56. J.M. Maldacena, hep-th/9711200, Adv. Theor. Math. Phys. 2, 231 (1998).
57. S.S. Gubser, I. Klebanov, and A.M. Polyakov, hep-th/9802109, Phys. Lett. B 428, 10 (1998).
58. E. Witten, hep-th/9802150, Adv. Theor. Math. Phys. 2, 253 (1998).
59. G. ’t Hooft gr-qc/9310026.
60. C. Thorn, Sakharov Conf. Proc. (1991) p. 447.
61. L. Susskind, hep-th/9409098, J. Math. Phys. 36, 6377, (1995).
62. E. Witten, hep-th/9805122, J. High Energy Phys. 9807, 006 (1998).
63. S. Kachru and E. Silverstein, hep-th/9802183, Phys. Rev. Lett. 80, 4855 (1998); A. Lawrence, N. Nekrasov, and C. Vafa, hep-th/9803013.
64. E. Witten, hep-th/9803131, Adv. Theor. Math. Phys. 2, 505 (1998).
65. J.M. Maldacena, hep-th/9803002, Phys. Rev. Lett. 80, 4859 (1998); S.J. Rey, J. Yee, hep-th/9803001.
66. S.J. Rey, S. Theisen, J.T. Yee, hep-th/9803135, Nucl. Phys. B 527, 171 (1998); A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Yankielowicz, hep-th/9803263, J. High Energy Phys. 06, 001, (1998).
67. C. Csaki, H. Ooguri, Y. Oz, J. Terning, hep-th/9806021, R. Koch, A. Jevsek, M. Mihalecsu, J.P. Nunes, hep-th/9806127, Phys. Rev. D 58, 105009 (1998); D. Gross, H. Ooguri, hep-th/9805129, PRD 58, 106002 (1998).