Abstract. We provide new examples of rational maps in four dimensions with two rational invariants, which have unexpected geometric properties and fall outside classes studied by earlier authors. We can reconstruct the map from both invariants, but one of the invariants also defines a new map, which we call the shadow map. We show that both maps are integrable, the shadow map leading to nontrivial fibrations of an invariant 3-fold obtained from the original map.

1. Introduction

Rational maps in two dimensions with invariant curves form the starting point for many developments in algebraic geometry and integrable systems theory. Elliptic curves play a crucial role in the development of the field, see also [15, 18].

There is up to now no general framework to describe integrable maps in dimension higher than two, but there exists a number of examples, obtained by various methods.

The simplest is to construct periodic reductions of integrable lattice equations [17, 24], as well as symmetry reduction [16, 14], since such reductions are automatically integrable.

A different approach taken in [19, 20] was to start from integrable Hamiltonian systems, and use an appropriate discretisation to obtain birational maps, together with the adequate symplectic structure, ensuring integrability in the sense of Liouville.

Finally, a direct generalisation of the two dimensional case to four dimensions was given in [3], the idea being to start from a multiquadratic expression and construct generating involutions leaving these quantities invariant. Among the resulting maps are autonomous versions of members of hierarchies of $q$-discrete Painlevé equations [9].

In this paper, we provide new four-dimensional maps with two rational invariants. The invariants are not multiquadratic, different coordinates appear with different degrees, and therefore our maps do not fall into the class described in [3]. They arose from the autonomous limit of additive discrete Painlevé equations [4]. We show that they have vanishing algebraic entropy [2]. The vanishing of the entropy will be our test of integrability throughout the paper.

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1For example the autonomous version of equation (4.4) in [9] is the same as equation (4.29) in [3].
It is possible to reconstruct the maps from the invariants. While the original map can be recovered unambiguously from the lower degree invariant, the other invariant defines two maps, thus providing us with an alternate one which we called the shadow map. The shadow map is itself integrable and we show that it leads to a non-trivial fibration by curves of the 3-fold given by the higher degree invariant.

Based on these results, we propose a model of four-dimensional maps with two rational invariants, and give two more instances of such maps, one with a structure of invariants similar to the previous ones, and one presenting novel features.

The paper is organised as follows. In Section 2 we describe the autonomous limit of the second member of the hierarchy of the discrete first Painlevé equation (the equation with initial-value space $E_6^{(1)}$) [4]. The map is defined from a recurrence of order 4, and acts on $\mathbb{CP}^4$. We give its two invariants. We show that the shadow map is integrable by calculating its algebraic entropy, giving its three invariants and deducing a non-trivial elliptically fibered 3-fold from these results. In Section 3 we provide parallel results for the second member of the hierarchy of the discrete second Painlevé equation ($D_6^{(1)}$). Section 4 describes the geometry of the invariant surfaces, and gives a construction scheme for four dimensional maps with two algebraic invariants. Section 5 and 6 give two new recurrences, constructed along the lines of the scheme given in the previous section. Both are integrable, but with different characteristics, revealed by the analysis of the growth of the degrees of their iterates. This difference is reflected in the nature of the invariants of their shadow maps: the first shadow map possesses three independent rational invariants, while the second only has two rational invariants, but also one non-rational invariant, a situation already encountered in [5]. In Section 7 we introduce a notion of inflation, which allows us to produce from a recurrence of order $n$ a new recurrence of order $n+1$. We use this notion to analyse the model described in Section 6. We conclude with some hints for further studies.

2. Autonomous $d_4P^{(i)}$

In this section, we study the autonomous version of a fourth-order member of the hierarchy of the discrete first Painlevé equation [3], denoted by $d_4P^{(i)}$ (Equation (2.9) of [4]). We study the map in $\mathbb{CP}^4$, by providing invariants, constructing the shadow map and deducing further properties.

Denoting the iterates by $x_n$, for each $n \in \mathbb{Z}$, we take homogeneous coordinates in four dimensions in $\mathbb{CP}^4$ to be $[x, y, z, u, t]$, which stands for $[x_n, x_{n-1}, x_{n-2}, x_{n-3}, 1]$ up to a common factor. The map then sends

$$[x_n, x_{n-1}, x_{n-2}, x_{n-3}, 1] \mapsto [x_{n+1}, x_n, x_{n-1}, x_{n-2}, 1],$$

up to common factors. We denote this map by

$$\varphi^{(i)} : [x, y, z, u, t] \mapsto [x', y', z', u', t'], \quad (2.1)$$

where

$$\begin{align*}
    x' &= -ay(x^2 + y^2 + z^2 + 2yz + 2xy + xz + zu) - bty(y + z + x) - cyt^2 + dt^3, \\
    y' &= ayx^2, \quad z' = ay^2, \quad u' = ayz, \quad t' = axy.
\end{align*}$$
It can be checked that the map \( \varphi^{(i)} \) has two invariants \( \Delta^{(i)}_4 \) and \( \Delta^{(i)}_5 \) which are:

\[
t^4 \Delta^{(i)}_4 = ay \left( -y^2 - 2yz - xy - z^2 - zu + xu \right) - bt yz (z + y),
- cyzt^2 + dt^3 (z + y)
\]

\[
t^5 \Delta^{(i)}_5 = ay \left( azu + xy + y^2 + 2yz + z^2 \right) (z + u + y + x) + c yz (z + u + y + x) t^2
- d (zu + xy + y^2 + 2yz + z^2) t^3 + byz (y + z + x) (u + y + z) t.
\]

Both invariants are unchanged by the involution

\[
i : [x, y, z, u, t] \mapsto [u, z, y, x, t].
\]

The sequence of degrees of the iterates of \( \varphi^{(i)} \),

\[
\{d_n\}^{(i)} = 1, 3, 6, 12, 21, 33, 47, 64, 83, 104, 128, 154, 183, 214, 248, 284, \ldots
\]

is fitted by the rational generating function

\[
g^{(i)}(r) = \frac{r^{10} - r^9 - r^6 + 2r^4 + 2r^3 + r + 1}{(r + 1)(1 - r)^3}.
\]

The distribution of the poles in (2.5) shows that the degrees \( d_n \) grow polynomially in \( n \) with quadratic growth, implying vanishing of the algebraic entropy.

Using the fact that the map is coming from a recurrence, we can recover the map \( \varphi^{(i)} \) from each invariant. In particular, \( \Delta^{(i)}_4([x', x, y, z, t]) - \Delta^{(i)}_4([x, y, z, u, t]) \) decomposes into two factors, one being trivial \((x - z)\) and the other giving back the map (2.1). The similar difference constructed with the invariant \( \Delta^{(i)}_5 \) also decomposes into two factors, both now giving nonlinear maps. One is the original map (2.1), while the other one turns out to be

\[
\varphi^{(i)}_s : [x, y, z, u, t] \mapsto [yz + z^2 + zu - x^2 - xy, x^2, xy, xz, tx],
\]

which we call the shadow map.

The integrability of the shadow map can be seen from the evaluation of its algebraic entropy. The sequence of degrees of the iterates of \( \varphi^{(i)}_s \) is

\[
\{d_n\}^{(i)} = 1, 2, 4, 7, 11, 17, 24, 32, 41, 52, 64, 77, 91, 107, 124, 142, \ldots,
\]

which has a generating function given by the rational fraction

\[
g^{(i)}(r) = \frac{1 + r^2 + r^3 + 2r^5}{(1 + r)(1 + r^2)(1 - r)^3},
\]

showing again quadratic growth and vanishing of the entropy.

This shadow map has some unexpected and intriguing properties. The image of a generic orbit hints at the existence of three invariants for \( \varphi^{(i)}_s \). See Figure 2.1 which is the projection of an orbit on a two-dimensional plane of coordinates.

We discovered that indeed the shadow map possesses three independent rational invariants:

\[
\Sigma^{(i)}_2 = \frac{zu + xy + y^2 + 2yz + z^2}{t^2},
\]

\[
\Sigma^{(i)}_3 = \frac{yz (z + u + y + x)}{t^3},
\]

\[
\Sigma^{(i)}_4 = \frac{yz (y + z + x) (u + y + z)}{t^4}.
\]
There is a simple algebraic relation between $\Delta_5^{(I)}$ and the invariants $\Sigma_i^{(I)}$:

$$\Delta_5^{(I)} = a \Sigma_2^{(I)} \Sigma_3^{(I)} + b \Sigma_4^{(I)} + c \Sigma_3^{(I)} - d \Sigma_2^{(I)}.$$  \hfill (2.9)

The three invariants $\Sigma_i^{(I)}$ define a non-trivial elliptic fibration of the 3-folds of constant $\Delta_5^{(I)}$. Indeed the curves in $\mathbb{C}P^4$ defined by $\Sigma_i^{(I)} = \text{constant}, i = 2, 3, 4$ possess an infinite group of automorphisms provided by $\varphi_s$ itself, and are consequently of genus $g \leq 1$. The discrete orbits shown by Figure 2.1 are restricted to these elliptic (or accidentally rational) curves.

Furthermore, the compositions $\tau = \iota \cdot \varphi^{(I)}$ and $\tau_s = \iota \cdot \varphi_s^{(I)}$ define further involutions, which moreover commute, i.e., $\tau \cdot \tau_s = \tau_s \cdot \tau$.

Remark: Note that any functional combination of $\Delta_5^{(I)}$ and $\Delta_4^{(I)}$ will also be an invariant of $\varphi^{(I)}$. Different choices will lead to different shadow maps. The choices we made above were based on two requirements: firstly, to produce invariants of minimal degree, and secondly to define the simplest possible shadow map.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{orbit.png}
\caption{An orbit of the shadow map for $d_4 P^{(I)}$.}
\end{figure}
3. Autonomous $d_4 P^{(II)}$

In this section, we study the autonomous version of a fourth-order member of the hierarchy of the discrete second Painlevé equation \( d_4 P \), denoted by $d_4 P^{(II)}$. (The latter is Equation (3.7) of \cite{1}.)

In $\mathbb{CP}^4$, this equation gives rise to the map

$$
\phi^{(II)} : \ [x, y, z, u, t] \mapsto [x', y', z', u', t'],
$$

with

$$
\begin{align*}
  x' &= d \ t^5 - a \ (t - y) \ (t + y) \ (ut^2 - yz^2 - uz^2 - 2yxz - x^2y) - c \ yt^4 \\
  -b \ t^2 \ (t - y) \ (t + y) \ (z + x), \\
  y' &= ax \ (t^2 - y^2) \ (t^2 - x^2), \\
  z' &= ay \ (t^2 - y^2) \ (t^2 - x^2), \\
  u' &= az \ (t^2 - y^2) \ (t^2 - x^2), \\
  t' &= at \ (t^2 - y^2) \ (t^2 - x^2).
\end{align*}
$$

The invariants $\Delta_6^{(II)}$ and $\Delta_8^{(II)}$ of $\phi^{(II)}$ are given by

$$
\begin{align*}
  t^6 \Delta_6^{(II)} &= a \ (t - z) \ (t + z) \ (t - y) \ (t + y) \ (ux - uz - xy - yz) \\
  &\quad -b \ t^2 \ (z^2t^2 + t^2y^2 - z^2y^2) - c \ t^4yz + d \ t^5 \ (z + y), \\
  t^8 \Delta_8^{(II)} &= a \ (t^5z^2 + z^2y^2 + x^2) \ t^6 - z^2y^2 \ (uz + xy + yz)^2 \\
  &\quad - (2yuz^2 + 2uzxy + x^2z^2 + 2xzy^2 + 2x^2y^2 + u^2y^2 + 2z^2y^2 + 2u^2z^2) \ t^4 \\
  &\quad + (2x^2y^2z^2 + 2uyz^2z^2 + 2xy^2z^3 + 2yuv^4 + z^2y^4 + y^2z^4 + 2u^2y^2z^2 + x^2y^4 \\
  &\quad + 2uxy^3z + 2uxyz^3 + 2xzy^4 + z^4u^2) \ t^2) \\
  &\quad + b \ t^2 \ (t - z) \ (t + z) \ (t - y) \ (t + y) \ (z + x) \ (u + y) \\
  &\quad + c \ t^4 \ (xzt^2 - z^2y^2 + yut^2 - yuz^2 - xzy^2) \\
  &\quad - d \ t^5 \ (xt^2 + zt^2 - yz^2 - xy^2 - uz^2 + ut^2 - yz^2 + yt^2).
\end{align*}
$$

Both invariants are again unchanged by the involution

$$
\nu : [x, y, z, u, t] \mapsto [u, z, y, x, t].
$$

In this case, the shadow map $\phi_s^{(II)}$, defined as above, is

$$
[x, y, z, u, t] \overset{\nu}{\mapsto} [x^2y - yz^2 - uz^2 + ut^2, x \ (t^2 - x^2), y \ (t^2 - x^2), z \ (t^2 - x^2), t \ (t^2 - x^2)],
$$

and it turns out to also have vanishing algebraic entropy.

We find again that the shadow map has three independent rational invariants

$$
\begin{align*}
  \Sigma_3^{(II)} &= \frac{xzy^2 + zy^2 + uz^2 + yz^2 + (-z - y - x - u) \ t^2}{t^4}, \\
  \Sigma_4^{(II)} &= \frac{(zx + uy) \ t^2 - yz \ (uz + xy + yz)}{t^4}, \\
  \Sigma_6^{(II)} &= \frac{(t - z) \ (t + z) \ (t - y) \ (t + y) \ (z + x) \ (u + y)}{t^6}.
\end{align*}
$$

The invariant $\Delta_8^{(II)}$ has a simple algebraic relation to the $\Sigma_j^{(II)}$:

$$
\Delta_8^{(II)} = a \ (\Sigma_3^{(II)} - 2\Sigma_4^{(II)} - \Sigma_4^{(II)} - 2\Sigma_6^{(II)}) + b \ \Sigma_6^{(II)} + c \ \Sigma_4^{(II)} + d \ \Sigma_3^{(II)}.
$$

The situation is very similar to the previous section.
4. On the structure of the invariants

In this section, we study the structure of the rational invariants given in the previous two sections. Our starting point is the distribution of degrees shared by the pair of invariants arising from autonomous $d_4P^{(i)}$ and that arising from $d_4P^{(ii)}$. We show that their properties give rise to ruled 3-folds and elliptic 3-folds. We extend these properties to new rational invariants with similar structures.

The invariants in the previous sections are ratios of homogeneous polynomials. These polynomials have well defined degrees with respect to the homogeneous coordinates $[x, y, z, u, t]$. For both models we gave the simplest form of the invariants, where the denominators are just powers of $t$.

While the invariants of each model have different total degree (4 and 5 for $d_4P^{(i)}$, 6 and 8 for $d_4P^{(ii)}$), they share the same structure: the distribution of degrees with respect to each variable is similar for the numerators of $\Delta_4^{(i)}$ and $\Delta_6^{(ii)}$ (respectively for $\Delta_5^{(i)}$ and $\Delta_8^{(ii)}$). See Table 1.

|       | x | y | z | u | t | Total degree |
|-------|---|---|---|---|---|--------------|
| $\Delta_4^{(i)}$ | 1 | 3 | 3 | 1 | 3 | 4            |
| $\Delta_5^{(i)}$ | 1 | 3 | 3 | 1 | 5 | 6            |
| $\Delta_5^{(ii)}$| 2 | 4 | 4 | 2 | 3 | 5            |
| $\Delta_8^{(ii)}$| 2 | 4 | 4 | 2 | 7 | 8            |

Table 1. The distribution of degrees of invariants.

Notice that $\Delta_4^{(i)}$ and $\Delta_6^{(ii)}$ are linear in $x$ and $u$. Therefore, the varieties $\Delta_4^{(i)} = \text{constant}$ (resp. $\Delta_6^{(ii)} = \text{constant}$) have intersections with the $y$ and $z$-coordinate hyperplanes that are straight lines. Therefore, these varieties are ruled 3-folds.

On the other hand, notice that $\Delta_5^{(i)}$ and $\Delta_8^{(ii)}$ are quadratic in $x$ and $u$. Therefore, there is an elliptic fibration of the 3-dimensional varieties $\Delta_5^{(i)} = \text{constant}$ (resp. $\Delta_8^{(ii)} = \text{constant}$).

For both previous cases, the shadow maps provide us with a non trivial elliptic fibration of the varieties $\Delta_5^{(i)} = \text{constant}$ (resp. $\Delta_8^{(ii)} = \text{constant}$). Indeed their orbits are confined to algebraic curves defined by the invariants $\Sigma_j^{(i)}$ (resp. $\Sigma_k^{(ii)}$). We know these curves are elliptic (or accidentally rational) curves since they possess an infinite group of automorphisms: the iterates of the shadow map.

To construct new recurrences of the same type, we will proceed as follows.

1. Choose two polynomials $\Pi_l$ and $\Pi_h$ with respective degrees 1, 3, 3, 1 and 2, 4, 4, 2 in $x, y, z, u$ as in Table 1 and total degrees $d_l$ and $d_h$.
2. Impose the condition that both polynomials are invariant by the involution $\iota$ (Equation (2.4)), and define $\Delta_l = \Pi_l/t^{d_l}$ and $\Delta_h = \Pi_h/t^{d_h}$.
3. Assume that the conservation condition of $\Delta_l$ factors as
   
   \[ \Pi_l(x', x, y, z, t) - \Pi_l(x, y, z, u, t) = (x - z) Q(x', x, y, z, u, t). \]  
   (4.1)
   We thus get from Equation (4.1) a recurrence relation, and a birational map
   
   \[ \varphi : [x, y, z, u, t] \mapsto [x', y', z', u', t']. \]  
   (4.2)
4. Impose the condition that the higher degree polynomial $\Pi_h$ verifies:
   
   \[ \Pi_h(x', x, y, z, t) - \Pi_h(x, y, z, u, t) = Q(x', x, y, z, u, t) S(x', x, y, z, u, t). \]  
   (4.3)
The previous condition ensures the invariance of $\Delta_h$ under $\varphi$ and defines the shadow map $\varphi_s$ by solving $S(x', x, y, z, u, t) = 0$.

Remark: Denoting by $\delta(R, *)$ the discriminant of $R$ with respect to variable $*$, $\Pi_h$ verifies the necessary condition

$$\delta(\delta(\Pi_h, x), u) = 0.$$  \hspace{1cm} (4.4)

In what follows, we study examples of pairs of polynomials $(\Pi_1, \Pi_h)$ solving these conditions, both with the minimal total degrees $d_1 = 3$ and $d_h = 4$.

5. ANOTHER ELLIPTIC FIBRATION

In this section, we provide an example of a pair of polynomials $(\Pi_1, \Pi_h)$ satisfying the conditions given in Section 4. Define two polynomials

\begin{align*}
\Pi_1^{(1)} &= a t^2 (z + y) + b t (z^2 + y^2) + c (z + y) (xu - zu - xy - yz) \\
&\quad - t (xu - zu - xy - 3yz), \\
\Pi_h^{(1)} &= a c t^2 (uy + zu + xy + xz + 2yz) - a t^3 (z + u + y + x) \\
&\quad + b c t (z + y) (x + z) (u + y) - b t^2 (x + z) (u + y) \\
&\quad - c^2 \left(u^2 y^2 + 2u^2 yz + u^2 z^2 + 2uy^2 z + 2yz^2 u + x^2 y^2 + 2x^2 yz + x^2 z^2 + y^2 z + yz^2\right) \\
&\quad + 2c t \left(u^2 y + u^2 z + uy^2 + 2zuy + yx^2 + x^2 z + 2xyz + xz^2 + y^2 z + yz^2\right) \\
&\quad - t^2 \left(u^2 + 2uy + x^2 + 2xz + y^2 + z^2\right). \tag{5.1}
\end{align*}

The corresponding map $\varphi^{(1)}$ and shadow map $\varphi_s^{(1)}$ are given respectively by

$$\varphi^{(1)}(x, y, z, u, t) : \begin{cases} 
  x' = a t^2 + b t (x + z) - c (xy + yz + uy + zu) + t (2y + u), \\
  y' = c x (x + y) - x t, \\
  z' = c y (x + y) - y t, \\
  u' = c z (x + y) - z t, \\
  t' = c (x + y) t - t^2,
\end{cases} \tag{5.2}
$$

and

$$\varphi^{(1)}_s(x, y, z, u, t) : \begin{cases} 
  x' = -u t - c (xy - yz - uy - zu), \\
  y' = c x (x + y) - x t, \\
  z' = c y (x + y) - y t, \\
  u' = c z (x + y) - z t, \\
  t = c t (x + y) - t^2.
\end{cases} \tag{5.3}
$$

The sequence of degrees of the iterates are:

\begin{align*}
\{d_1\}^{(1)} &= 1, 2, 4, 8, 14, 22, 32, 44, 57, 72, 88, 106, 126, 148, 172, 198, \ldots \\
\{d_h\}^{(1)} &= 1, 2, 4, 8, 14, 21, 30, 40, 52, 66, 81, 98, 116, 136, 158, 181, \ldots
\end{align*}

fitted by the rational generating functions

\begin{align*}
g^{(1)}(r) &= \frac{r^{11} - r^{10} + r^9 - r^8 + r^3 + r^2 - r + 1}{(1 - r)^3} \quad \text{and} \\
g_s^{(1)}(r) &= \frac{2r^6 + 2r^4 + 2r^3 + r^2 + 1}{(r^4 + r^3 + r^2 + r + 1)(1 - r)^3},
\end{align*}

showing quadratic growth and, therefore, integrability.
Again, the shadow map has three independent rational invariants $\Sigma_i^{(1)}$

\[
t^2 \Sigma_2^{(1)} = c(uy + zu + xy + xz + 2yz) - t(z + u + y + x),
\]

\[
t^3 \Sigma_3^{(1)} = (x + z)(u + y)(c(z + y) - t),
\]

\[
t^4 \Sigma_4^{(1)} = c(t(u^2 + x^2 + y^2 z + yz^2) + c^2(xy + xz + yz)(uy + zu + yz)
- t^2(x + z)(u + y),
\]

and $\Delta_h^{(1)} = \Pi_{t}^{(1)} / t^h$ can be expressed in terms of the $\Sigma_i^{(1)}$:

\[
\Delta_h^{(1)} = a \Sigma_2^{(1)} - (\Sigma_2^{(1)})^2 + (b - 4) \Sigma_4^{(1)} + 2 \Sigma_4^{(1)}.
\]

The orbits of the shadow map are confined to elliptic curves which provide us with an elliptic fibration of the variety $\Delta_h^{(1)} = constant$. The situation is very similar to the one encountered in sections 2 and 3.

6. Beyond elliptic fibrations

In this section, we show that not all shadow maps arising from polynomials of the type defined in Section 4 result in elliptic fibrations. Consider the two polynomials:

\[
\Pi_t^{(2)} = a t(z + y)^2 + b(z + y)t^2 - c t(x - z)(u - y)
+ (z + y)(xu - zu - xy - 2y^2 - 3yz - 2z^2),
\]

\[
\Pi_h^{(2)} = a c t^2(xu + 2uy + zu + xy + 2xz + yz) - a t(z + y)(2y + x + z)(u + y + 2z)
+ b c(z + u + y + x)t^3 - b t^2(z + y)(x + 2z + u + 2y)
+ c^2 t^2(u^2 - 2uy + x^2 - 2xz + y^2 + z^2) + (z + y)^2(x + 2z + u + 2y)^2
- 2c t(z + y)(u^2 + xu + uy + 2zu + x^2 + 2xy + xz - yz).
\]

The corresponding map $\varphi^{(2)}$ and shadow map $\varphi^{(2)}_s$ are given by

\[
\varphi^{(2)}(x, y, z, u, t) : \begin{cases}
x' = -a t(2y + x + z) - b t^2 - c t(u - 2y) + 4y^2
+ 2(x^2 + xz + z^2) + 5y(x + z) + u(y + z), \\
y' = x(ct - x - y), \quad z' = y(ct - x - y), \\
u' = z(ct - x - y), \quad t' = t(ct - x - y),
\end{cases}
\]

and

\[
\varphi^{(2)}_s(x, y, z, u, t) : \begin{cases}
x' = c ut + 2(x^2 - z^2) + 3y(x - z) - u(y + z), \\
y' = x(ct - x - y), \quad z' = y(ct - x - y), \\
u' = z(ct - x - y), \quad t' = t(ct - x - y).
\end{cases}
\]

The sequence of degrees of the iterates are now

\[
\{d_n\}^{(2)} = 1, 2, 4, 8, 13, 21, 31, 45, 61, 82, 106, 136, 169, 209, 253, 305, 361, 426, 496, 576, 661, \ldots
\]

and

\[
\{d_n\}^{(2)}_s = 1, 2, 4, 7, 12, 19, 28, 40, 55, 73, 95, 121, 151, 186, 226, 271, 322, 379, 442, 512, 589, \ldots
\]
These two sequence are fitted by the generating functions

\[ g^{(2)}(r) = \frac{1 + 2r^3 - r^4 + r^5 + r^6 - r^7}{(r^2 + 1)(r + 1)^2(r - 1)^4}, \]  
(6.3)

and

\[ g_s^{(2)}(r) = \frac{1 - r^2 + r^3 + 2r^4 - r^5}{(r^2 + r + 1)(r - 1)^4}. \]  
(6.4)

The fact that the poles of \( g(r) \) and \( g_s(r) \) at unity are of order 4 means that the two above sequences have cubic growth.
The shadow map has only two independent rational invariants $\Sigma^{(2)}_2$ and $\Sigma^{(2)}_3$:

\[ t^2 \Sigma^{(2)}_2 = c (z + u + y + x) t - (z + y) (x + 2z + u + 2y), \]
\[ t^3 \Sigma^{(2)}_3 = 2 c^2 t^2 (z + u + y + x) + c t (xy - zu - xy - 4y^2 - 7yz - 4z^2), \]
\[ - (z + y) (2y + x + z) (u + y + 2z) \]

and $\Delta^{(2)}_h = \Pi^{(2)}_h / t^h$ can be expressed in terms of the $\Sigma^{(2)}_j$:

\[ \Delta^{(2)}_h = (\Sigma^{(2)}_2)^2 + (2ac - 4c^2 - b) \Sigma^{(2)}_2 - (a - 2c) \Sigma^{(2)}_3. \] (6.5)

At this point drawing a typical orbit of the shadow map is extremely useful. See Figure 6.1 for a projection of an orbit on a 2-dimensional plane. The picture of a generic orbit shows that there exists an additional invariant of the shadow map. This third invariant cannot be rational. Indeed, if it was rational the orbits would be confined to elliptic curves and the growth of the degree of the iterates would be quadratic, not cubic \[1, 8\].

The orbits of the shadow map are not confined to elliptic curves, but to non algebraic curves drawn on the algebraic variety $\Delta^{(2)}_h = \text{constant}.$

7. An inflation process

The two maps of the previous section are related to algebraically integrable models, by a simple inflation process, defined as follows.

Given an integrable recurrence of order $N$ defined on a variable $x_n$, leading to a birational map $\varphi$ on $\mathbb{CP}_N$, one may “inflate” the recurrence to order $N + 1$ on a new variable $y_n$ related to $x_n$ by

\[ x_n = \frac{\alpha_1 y_n y_{n-1} + \alpha_2 y_n + \alpha_3 y_{n-1} + \alpha_4}{\alpha_5 y_n y_{n-1} + \alpha_6 y_n + \alpha_7 y_{n-1} + \alpha_8} \] (7.1)

with $\alpha_i, i = 1..8$ arbitrary constants.

Remark 1: With specific choices of the parameters $\alpha_i$, Equation (7.1) is known to appear in various transformations, among which are the definition of potential forms, the so-called discrete Cole-Hopf transformation \[13\], as well as the discrete Miura transformation (see for example Equation (1) in \[10\]). Such transformations act non-trivially, as they are not just coordinate transformations.

Remark 2: Going from the order $N$ recurrence on $x_n$ to the order $N+1$ recurrence on $y_n$ can be undone by a “deflation” transform, going from the recurrence on $y_n$ to the one on $x_n$. While inflation is always possible, deflation cannot be done on arbitrary recurrences.

The recurrence on $y_n$ defines a birational map $\varphi^+$ on $\mathbb{CP}_{N+1}$. Even if the entropy is preserved by inflation, the sequence of degrees of the iterates is not. In the integrable case, where the degrees of the iterates of $\varphi$ grow polynomially, the inflated map $\varphi^+$ will then still have vanishing entropy, but possibly with a different polynomial growth. This is what happens for the two maps of the previous section, with the simple redefinition

\[ x_n = y_n + y_{n-1}. \] (7.2)
The map (6.1) in $\mathbb{CP}_4$ is obtained by (7.2) from the following map in $\mathbb{CP}_4$:

$$[x, y, z, t] \mapsto [x', y', z', t'] \quad \text{with}$$

$$
\begin{align*}
x' &= -a t (x + y) - b c t^2 + t (x + y - z) + x^2 + 2 x y + y^2 + y z, \\
y' &= x (c t - x), \quad z' = y (c t - x), \quad t' = t (c t - x).
\end{align*}
$$

(7.3)

The latter has quadratic growth and two rational invariants $\Gamma_1$ and $\Gamma_2$

$$
\begin{align*}
t^3 \Gamma_1 &= -a t y^2 - b t^2 y - c (y - z) (x - y) t + y (x y - x z + y^2 + y z), \\
t^4 \Gamma_2 &= -a^2 t^2 y^2 - a b t^3 y + 2 a c t^2 x z - 2 a t x y z + b c t^3 (x + z) \\
&+ c^2 t^2 (x^2 - 2 x y + 2 y^2 - 2 y z + z^2) - 2 c y t (x^2 + x y - x z + z^2) \\
&- b t^2 y (x + y + z) + y^2 (x + y + z)^2.
\end{align*}
$$

One may further reduce the order, eliminating $z$ by specifying the value of the invariant $\Gamma_1 = k_1$. One obtains a birational map of $\mathbb{CP}_2 [x, y, t] \mapsto [x', y', t']$:

$$
\begin{align*}
x' &= -a t x^2 - b t x^2 + c x t (x - y) - k_1 t^3 + x^3 + x^2 y, \\
y' &= x (x - y) (c t - x), \quad t' = t (x - y) (c t - x).
\end{align*}
$$

(7.4)

This map possesses an invariant $I = J/t^4(x - y)^2$ with

$$
\begin{align*}
J &= a^2 t^2 x^2 y^2 + a b (x + y) x y t^2 + 2 a t (k_1 t^3 - k_1^2 t^2 y + k_1 t y^2 - x^3 y^2 - x^2 y^2) \\
&+ b^2 t^4 x y + b c (x + y) (x - y)^2 t^2 + b t^2 (x + y) (k_1 t - x) - x^2 y^2 + c^2 (x - y)^4 t^2 \\
&- 2 c t x y (x - y) (x - y)^2 + (k_1 t - x - y)^2 + 2 a c x y (x - y)^2 t^2.
\end{align*}
$$

It is interesting to notice that defining a recurrence in $\mathbb{CP}_2$ from the invariant $I$ by imposing $I(x', x, t) = I(x, y, t)$ gives (7.4) as well as non rational maps, due to its degree distribution. This indicates that it may be of a non QRT type [6].

The shadow map (6.2) may be understood in a similar way. The reduction to three dimensions obtained by eliminating $u$ through the invariance condition $\Sigma_2^{(2)} = k_2$ yields the map in $\mathbb{CP}_3 [x, y, z, t] \mapsto [x', y', z', t']$

$$
\begin{align*}
x' &= -c (x + y + z) t - k_2 t^2 + 2 (x + y)^2 + (x + y) z, \\
y' &= x (c t - x - y), \quad z' = y (c t - x - y), \quad t' = t (c t - x - y).
\end{align*}
$$

(7.5)

for which the sequence of degrees has cubic growth. One gets (7.5) as the result of the inflation process (7.2) applied to the known recurrence

$$
x_{n+1} + x_{n-1} = \frac{k_2 - x_n^2}{c - x_n}.
$$

(7.6)

In fact, taking $x_n = c - w_n$, we find

$$
w_{n+1} + w_n + w_{n-1} = \frac{\alpha}{w_n} + 2 w_n
$$

with $\alpha = c^2 - k_2$, which is an autonomous version of the discrete first Painlevé equation $dP_1$ [7]. Remark: (7.4), (7.5) and (7.6) can be made non-autonomous by varying $k$ and $k_2$ respectively. This turns (7.6) into the first discrete Painlevé equation $dP_1$. 
8. Conclusion

By examining the autonomous limits of the first members of hierarchies of discrete Painlevé equations we have exhibited maps in three and four dimensions which generalise the known two-dimensional discrete integrable maps.

The simple scheme we have described provides a plethora of interesting cases which we plan to examine further, in particular to characterise higher dimensional invariant varieties completely, provide Lax pairs, relate non-autonomous generalisations to the results of [12], and study the inflation process in more detail.

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