LAMINATED TIMOSHENKO BEAMS WITH INTERFACIAL SLIP AND INFINITE MEMORIES

A. Guesmia
Institut Elie Cartan de Lorraine, UMR 7502, Université de Lorraine, 3 Rue Augustin Fresnel
BP 45112, 57073 Metz Cedex 03, France
and
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
e-mail: aissa.guesmia@univ-lorraine.fr

J. E. Muñoz Rivera
Department of Mathematics, Federal University of Rio de Janeiro and National Laboratory
for Scientific Computation, Petrópolis, RJ, Brasil
e-mail: rivera@lncc.br

M. A. Sepúlveda Cortés
CI²MA and DIM, Universidad del Concepción, Concepción, Chile
e-mail: mauricio@ing-mat.udec.cl

O. Vera Villagrán
Departamento de Matemática, Universidad del Bio-Bío, Concepción, Chile
e-mail: overa@ubiobio.cl

Abstract. We study in this paper the well-posedness and stability of three structures with interfacial slip and two infinite memories effective on the transverse displacement and the rotation angle. We consider a large class of kernels and prove that the system has a unique solution satisfying some regularity properties. Moreover, without restrictions on the values of the parameters, we show that the solution goes to zero at infinity and give an information on its speed of convergence in terms of the growth of kernels at infinity. A numerical analysis of the obtained theoretical results will be also given.

keywords. Viscoelastic structure, interfacial slip, semigroups, stability, lyapunov functional, numerical analysis.

Mathematics Subject Classification. 35B40, 35L45, 74H40, 93D20, 93D15.

1. Introduction

We consider the well-posedness and stability of three structures with interfacial slip and two infinite memories effective on the transverse displacement and the rotation angle

\[
\begin{align*}
\rho_1 \varphi_{tt} + k(u - \varphi_x)_x + \int_0^\infty g_1(s) \varphi_{xx}(t - s) \, ds &= 0, \\
\rho_2 (v - u)_{tt} - b(v - u)_{xx} - k(u - \varphi_x) + \int_0^\infty g_2(s) (v(t - s) - u(t - s))_{xx} \, ds &= 0, \\
\rho_2 v_{tt} - b v_{xx} + 3k(u - \varphi_x) + 4\delta v + 4\gamma v_t &= 0
\end{align*}
\]

with boundary conditions

\[
\varphi(0, t) = \varphi(1, t) = u(0, t) = u_x(1, t) = v(0, t) = v_x(1, t) = 0
\]

and initial data

\[
\begin{align*}
(\varphi(x, -t), u(x, -t), v(x, -t)) &= (\varphi_0(x, t), u_0(x, t), v_0(x, t)), \\
(\varphi_t(x, 0), u_t(x, 0), v_t(x, 0)) &= (\varphi_1(x), u_1(x), v_1(x)),
\end{align*}
\]
where \((x, t) \in (0, 1) \times \mathbb{R}_+\), \(\varphi = \varphi(x, t)\) is the transversal displacement, \(u = u(x, t)\) represents the rotation angle, \(v = v(x, t)\) is proportional to the amount of slip along the interface, \(\rho_1, \rho_2, k, b, \delta, \gamma\) are positive constants and \(g_i : \mathbb{R}_+ \to \mathbb{R}_+\) is a given function, \(i = 1, 2\).

The structures known under the name laminated Timoshenko beams are composed of two layered identical beams of uniform thickness and attached together on top of each other subject to transversal and rotational vibrations, and taking account the longitudinal displacement. An adhesive layer of small thickness is bonding the two adjoining surfaces and creating a restoring force being proportional to the amount of slip and producing a damping. These structures are used in many practical fields; see, for example, [21] for more details. When the longitudinal displacement (slip) is ignored, the laminated Timoshenko beams are reduced to the well known Timoshenko beams [29]. During the last few years, these structures were the subject of several studies in the literature recovering well-posedness and stability by adding some kinds of (boundary or internal) controls.

When \(v\) is not taken in consideration: \(v = 0\), (1.1) is reduced to Timoshenko beams [29] and its stability question was widely treated in a huge number of works; see, for example, [7] and [17] and the references therein.

The authors of [30] proved the exponential stability through mixed homogeneous Dirichlet-Neumann boundary conditions and two boundary controls at \(x = 1\) provided that the speeds of wave propagations of the first two equations are different; that is

\[(1.4) \quad \sqrt{\frac{k}{\rho_1}} \neq \sqrt{\frac{b}{\rho_2}}.\]

It was also proved in [30] that the frictional damping \(4\gamma v_t\) is strong enough to stabilize asymptotically the structure but it is not able to stabilize the structure exponentially. The same exponential stability result of [30] was proved in [6] for the same model but through two boundary controls at \(x = 0\) and \(x = 1\). The exponential stability result of [30] was improved in [28] by assuming some weaker conditions on the parameters. The authors of [23] proved that the exponential stability holds if the boundary controls are replaced by a frictional damping acting on the first equation. The author of [26] proved that the exponential stability holds without any restriction on the parameters if the first two equations are also damped via frictional dampings. Recently, the authors of [27] proved that, without any restriction on the parameters, the polynomial stability holds under additional three dynamic boundary conditions. For the stability of laminated beams with Cattaneo’s or Fourier’s type heat conduction, we refer the readers to [1] and [19].

The stability in case of viscoelastic dampings represented by finite memory terms in the form of a convolution on \([0, t]\) (see, for example, [4], [21] and [23]) was treated in [21], [22] and [23]. Namely, under some restrictions on the parameters and with one or two kernels converging exponentially to zero at infinity, the exponential stability was proved in [21], [22] and [23].

For the stability of Bresse systems [5] with infinite memories, we refer the readers to [12], [13], [15], [16], and the references therein.

In [14], the first author of the present paper proved, under some restrictions on the parameters and with \(\gamma = 0\), some exponential and polynomial stability results using only one infinite memory with a kernel converging exponentially to zero at infinity.

From the cited results above, we see that the exponential and/or polynomial stability has been proved under some restrictions on the parameters and with kernels converging exponentially to zero at infinity. The first main objective of this paper is proving that two infinite memories guarantee the stability of the system without any restrictions on the parameters. Moreover, we show that the class of admissible kernels is much more larger than the one containing kernels converging exponentially to zero at infinity. Our second main objective is presenting a numerical analysis to illustrate our theoretical stability results.

The proof of the well-posedness and stability results are based on the semigroup theory and the energy method, respectively. However, the numerical results are proved using the finite difference approach (of second order in space and time).

The paper is organized as follows. In Section 2, we consider some assumptions on the relaxation functions and prove the well-posedness. In Sections 3, we prove our stability results. In Section 4, we
present our numerical analysis. We end our paper by giving some general comments and issues in section 5.

2. SETTING OF THE SEMIGROUP

We introduce the variable \( \psi = v - u \), and as in [8], we consider the variables \( \eta \) and \( z \) and their initial data given by

\[
\begin{align*}
\eta(x, t, s) &= \varphi(x, t) - \varphi(x, t - s), \quad x \in (0, 1), \ s, t \in \mathbb{R}_+ , \\
\eta_0(x, s) &= \varphi_0(x, 0) - \varphi_0(x, s), \quad x \in (0, 1), \ s, t \in \mathbb{R}_+ , \\
z(x, t, s) &= \psi(x, t) - \psi(x, t - s), \quad x \in (0, 1), \ s, t \in \mathbb{R}_+ , \\
z_0(x, s) &= v_0(x, 0) - u_0(x, 0) - (v_0(x, s) - u_0(x, s)), \quad x \in (0, 1), \ s, t \in \mathbb{R}_+ .
\end{align*}
\]

So the system (2.1) becomes

\[
\begin{align*}
\rho_1 \varphi_{tt} - k (\varphi_x + \psi - v)_x + g_1^0 \varphi_{xx} - \int_0^\infty g_1(s) \eta_{xx}(t - s) \ ds = 0, \\
\rho_2 \psi_{tt} - (b - g_2^0) \psi_{xx} + k (\varphi_x + \psi - v) - \int_0^\infty g_2(s) z_{xx} \ ds = 0, \\
\rho_2 v_{tt} - b v_{xx} - 3 k (\varphi_x + \psi - v) + 4 \delta v + 4 \gamma v_t = 0,
\end{align*}
\]

where

\[
g_i^0 = \int_0^\infty g_i(s) \ ds, \quad i = 1, 2,
\]

with boundary conditions

\[
\varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi_x(1, t) = v(0, t) = v_x(1, t) = 0, \quad t \in \mathbb{R}_+.
\]

The functionals \( \eta \) and \( z \) satisfy

\[
\begin{align*}
\eta_t(x, t, s) + \eta_x(x, t, s) - \varphi_t(x, t) &= 0, \quad x \in (0, 1), \ s, t > 0, \\
z_t(x, t, s) + z_x(x, t, s) - \psi_t(x, t) &= 0, \quad x \in (0, 1), \ s, t > 0, \\
\eta(x, 0, s) &= \eta_0(x, s), \quad z(x, 0, s) = z_0(x, s), \quad x \in (0, 1), \ s \in \mathbb{R}_+, \\
\eta(x, t, 0) &= z(x, t, 0) = z_x(1, t, s) = 0, \quad x \in (0, 1), \ t, s \in \mathbb{R}_+ .
\end{align*}
\]

Let

\[
\begin{align*}
\varphi_t &= \ddot{\varphi}, \quad \psi_t = \ddot{\psi}, \quad v_t = \dddot{v}, \\
U &= (\varphi, \ddot{\varphi}, \psi, \ddot{\psi}, v, \dddot{v}, \eta, z), \\
U_0 &= (\varphi_0, \varphi_1, v_0 - u_0, v_1 - u_1, v_0, v_1, \eta_0, z_0).
\end{align*}
\]

Now, we can rewrite the system (1.3), (2.1) and (2.2) in the following initial value problem:

\[
\begin{align*}
U_t(t) &= \mathcal{A}U(t), \quad t > 0, \\
U(0) &= U_0,
\end{align*}
\]
where the operator $\mathcal{A}$ is defined by

(2.6) \[ \mathcal{A} U = \begin{pmatrix}
    \frac{1}{\rho_1} (\varphi_x + \psi - v)_x - \frac{1}{\rho_1} \varphi_{xx} + \frac{1}{\rho_1} \int_0^\infty g_1(s) \eta_{xx} \, ds \\
    \frac{1}{\rho_1} (b - g_2^0) \psi_{xx} - k (\varphi_x + \psi - v) + \frac{1}{\rho_2} \int_0^\infty g_2(s) \varphi_{xx} \, ds \\
    \frac{1}{\rho_2} [b v_{xx} + 3 k (\varphi_x + \psi - v) - 4 \delta v - 4 \gamma \tilde{v}] \\
    \tilde{v} \\
    \tilde{\psi} \\
    - \eta_x + \tilde{\varphi} \\
    - z_x + \tilde{\psi}
\end{pmatrix}.
\]

Let us consider the standard $L^2(0, 1)$ space with its classical scalar product $\langle \cdot, \cdot \rangle$ and generated norm $\| \cdot \|$. We consider also the phase Hilbert spaces

$$L_{g_1} = \left\{ v : \mathbb{R}_+ \to H_1, \int_0^\infty g_1(s) \| v_x(s) \|^2 \, ds < \infty \right\},$$

equipped with the inner product

$$\langle w, \tilde{w} \rangle_{L_{g_1}} = \int_0^\infty g_1(s) \langle w_x(s), \tilde{w}_x(s) \rangle \, ds,$$

$$\tilde{H}_1 = H_0^1(0, 1), \tilde{H}_2 = H_1^1(0, 1) \text{ and } H_1^1(0, 1) = \{ h \in H^1(0, 1) : h(0) = 0 \}.$$

The energy space is given by

$$\mathcal{H} = H_0^1(0, 1) \times L^2(0, 1) \times [H_1^1(0, 1) \times L^2(0, 1)]^2 \times L_{g_1} \times L_{g_2},$$

equipped with the inner product, for any

$$U_1 = (\varphi_1, \tilde{\varphi}_1, \psi_1, \tilde{\psi}_1, v_1, \tilde{v}_1, \eta_1, z_1), \quad U_2 = (\varphi_2, \tilde{\varphi}_2, \psi_2, \tilde{\psi}_2, v_2, \tilde{v}_2, \eta_2, z_2) \in \mathcal{H},$$

(2.7) \[ \langle U_1, U_2 \rangle_{\mathcal{H}} = 3 k \langle (\varphi_{1x} + \psi - v_1), (\varphi_{2x} + \psi - v_2) \rangle + 3 (b - g_2^0) \langle \psi_{1x}, \psi_{2x} \rangle - 3 g_1 \langle \varphi_{1x}, \varphi_{2x} \rangle + b \langle v_{1x}, v_{2x} \rangle + 4 \delta \langle v_1, v_2 \rangle + 3 \rho_1 \langle \tilde{v}_1, \tilde{v}_2 \rangle + 3 \rho_2 \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle + 3 \rho_2 \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle + 3 \langle \eta_1, \eta_2 \rangle_{L_{g_1}} + 3 \langle z_1, z_2 \rangle_{L_{g_2}}.\]

The domain of $D(\mathcal{A})$ is given by

(2.8) \[ D(\mathcal{A}) = \{ U \in \mathcal{H}, \mathcal{A} U \in \mathcal{H}, \psi_x(1) = v_x(1) = \eta(x, 0) = z(x, 0) = z_x(1, s) = 0 \}.\]

Now, to get our well-posedness results, we consider the following hypothesis:

(H1) Assume that the function $g_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, is differentiable, nonincreasing and integrable on $\mathbb{R}_+$ such that there exists a positive constant $k_0$ satisfying, for any

$$(\varphi, \psi, w) \in H_0^1(0, 1) \times H_1^1(0, 1) \times H_1^1(0, 1),$$

we have

(2.9) \[ k_0 (\| \varphi_x \|^2 + \| \psi_x \|^2 + \| w_x \|^2) \leq 3 k \| \varphi_x + \psi - v \|^2 + 3 (b - g_2^0) \| \psi_x \|^2 + b \| v_x \|^2 + 4 \delta \| v \|^2 - 3 g_1^0 \| \varphi_x \|^2.\]

Moreover, assume that there exists positive constants $\beta_1$ and $\beta_2$ such that

(2.10) \[ - \beta_i g_i(s) \leq g_i'(s), \quad s \in \mathbb{R}_+, \quad i = 1, 2.\]
Theorem 2.1. Assume that (H1) holds. Then, for any $U_0 \in \mathcal{D}(A)$, system (2.15) admits a unique solution $U$ satisfying

$$U \in C(\mathbb{R}_+; \mathcal{H}).$$

If $U_0 \in \mathcal{D}(A)$, then $U$ satisfies

$$U \in C^1(\mathbb{R}_+; \mathcal{H}) \cap C(\mathbb{R}_+; \mathcal{D}(A)).$$

Proof. First, under condition (2.9) and according to our homogeneous Dirichlet boundary conditions and Poincare’s inequality, the inner product of $\mathcal{H}$

$$\langle A U, U \rangle_{\mathcal{H}} = -4 \gamma \|v_1\|^2 + \frac{1}{2} \int_0^\infty g_1'(s) \|\eta_x\|^2 ds + \frac{1}{2} \int_0^\infty g_2'(s) \|z_x\|^2 ds \leq 0.$$ 

Hence, $A$ is a dissipative operator, since $g_1$ and $g_2$ are nonincreasing and (2.10) guarantees the boundedness of the integrals in (2.14).

Third, we prove that $I - A$ is surjective. Let $F = (f_1, \ldots, f_8)^T \in \mathcal{H}$. We prove that there exists $U \in \mathcal{D}(A)$ satisfying

$$U - AU = F.$$ 

First, the first, third and fifth equations in (2.15) are equivalent to

$$\dot{\varphi} = \varphi - f_1, \quad \dot{\psi} = \psi - f_3 \quad \text{and} \quad \dot{v} = v - f_5.$$ 

Second, from (2.16), we see that the last two equations in (2.15) are reduced to

$$\eta_s + \eta = \varphi + f_7 - f_1 \quad \text{and} \quad z_s + z = \psi + f_8 - f_3.$$ 

Integrating with respect to $s$ and noting that $\eta$ and $z$ should satisfy $\eta(0) = z(0) = 0$, we get

$$\eta(s) = (1 - e^{-s})(\varphi - f_1) + \int_0^s e^{-s} f_2(\tau) d\tau \quad \text{and} \quad z(s) = (1 - e^{-s})(\psi - f_3) + \int_0^s e^{-s} f_4(\tau) d\tau.$$ 

Third, using (2.16) and (2.18), we find that the second, fourth and sixth equations in (2.15) are reduced to

$$\begin{cases} 
\rho_1 \varphi - k (\varphi_x + \psi - v)_x + \tilde{g}_1 \varphi_{xx} = \rho_1 (f_1 + f_2) + \int_0^s g_1(s) \left((1 - e^{-s}) f_1 + \int_0^s e^{-s} f_2(\tau) d\tau\right) ds, \\
\rho_2 \psi - (b - \tilde{g}_2) \psi_{xx} + k (\varphi_x + \psi - v) = \rho_2 (f_3 + f_4) + \int_0^s g_2(s) \left((1 - e^{-s}) f_3 + \int_0^s e^{-s} f_4(\tau) d\tau\right) ds, \\
(\rho_2 + 4 \gamma + 4 \delta) v - b v_{xx} - 3 k (\varphi_x + \psi - v) = (\rho_2 + 4 \gamma) f_5 + \rho_2 f_6, 
\end{cases}$$

where

$$\tilde{g}_i = \int_0^\infty e^{-s} g_i(s) ds, \quad i = 1, 2.$$ 

We see that, if (2.19) admits a solution satisfying the required regularity in $\mathcal{D}(A)$, then (2.16) implies that $\dot{\varphi}, \dot{\psi}$ and $\dot{v}$ exist and satisfy the required regularity in $\mathcal{D}(A)$. On the other hand, (2.18) implies that $\eta$ and $z$ exist and satisfy the required regularity in $\mathcal{D}(A)$. Indeed, from (2.17), we remark that it is enough to prove that $\eta \in L_{g_1}$ and $z \in L_{g_2}$. We have

$$s \mapsto (1 - e^{-s})(\varphi - f_1) \in L_{g_1} \quad \text{and} \quad s \mapsto (1 - e^{-s})(\psi - f_3) \in L_{g_2}.$$
because $\varphi, f_1 \in H^1_0(0, 1)$ and $\psi, f_3 \in H^1_0(0, 1)$. On the other hand, using the Fubini theorem and Hölder inequalities, we get

$$\int_0^1 \int_0^\infty g_1(s) \left( \int_0^s e^{t-s} f_7(\tau) \, d\tau \right) \, ds \, dx$$

$$\leq \int_0^\infty e^{-2s} g_1(s) \left( \int_0^s e^t \, d\tau \right) \int_0^s e^t \|f_{72}(\tau)\|^2 \, d\tau \, ds$$

$$\leq \int_0^\infty e^{-s} \left( 1 - e^{-s} \right) g_1(s) \int_0^s e^t \|f_{72}(\tau)\|^2 \, d\tau \, ds$$

$$\leq \int_0^\infty e^{-s} g_1(s) \int_0^s e^t \|f_{72}(\tau)\|^2 \, d\tau \, ds$$

$$\leq \int_0^\infty e^t \|f_{72}(\tau)\|^2 \int_0^{+\infty} e^{-s} g_1(s) \, ds \, d\tau$$

$$\leq \int_0^\infty e^t g_1(\tau) \|f_{72}(\tau)\|^2 \int_0^{+\infty} e^{-s} \, ds \, d\tau$$

$$\leq \int_0^\infty g_1(\tau) \|f_{72}(\tau)\|^2 \, d\tau$$

$$\leq \|f_7\|^2_{L^2_0} < \infty.$$  

Then

$$s \mapsto \int_0^s e^{t-s} f_7(\tau) \, d\tau \in L_{g_1}.$$  

Similarly, we get

$$s \mapsto \int_0^s e^{t-s} f_5(\tau) \, d\tau \in L_{g_2}.$$  

Therefore $\eta \in L_{g_1}$ and $z \in L_{g_2}$. Finally, to prove that (2.19) admits a solution satisfying the required regularity in $D(\mathcal{A})$, we consider the variational formulation of (2.19) and using the Lax-Milgram theorem and classical elliptic regularity arguments. This proves that (2.19) has a unique solution $U \in D(\mathcal{A})$. By the resolvent identity, we have $\lambda I - \mathcal{A}$ is surjective, for any $\lambda > 0$ (see [20]). Consequently, the Lumer-Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $\mathcal{H}$.

3. Stability

In this section, we prove the stability of (2.5). Let $U_0 \in \mathcal{H}$, $U$ be the solution to (2.5) and $E$ be the energy of $U$ given by

$$E(t) = \frac{1}{2} \|U(t)\|^2_{\mathcal{H}}$$

$$= \frac{1}{2} \left( 3k \|\varphi_x + \psi - v\|^2 + 3(b - g_2^0)\|\psi_x\|^2 - 3g_0^0 \|\varphi_x\|^2 + b \|v_x\|^2 + 4\delta \|v\|^2 \right) + \frac{1}{2} \left( 3\rho_1 \|\varphi_t\|^2 + 3\rho_2 \|\psi_t\|^2 + \rho_1 \|v_t\|^2 + 3 \int_0^\infty (g_1(s)\|\eta_x\|^2 + g_2(s)\|z_x\|^2) \, ds \right).$$

According to (2.5), we have

$$E'(t) = \langle U_t(t), U(t) \rangle_{\mathcal{H}} = \langle \mathcal{A}U(t), U(t) \rangle_{\mathcal{H}},$$

where we use $'$ to denote the derivative with respect to $t$. So, using (2.14), we find

$$E'(t) = -4\gamma \|v_t\|^2 + \frac{1}{2} \int_0^{+\infty} (g_1' \|\eta_x\|^2 + g_2' \|z_x\|^2) \, ds \leq 0,$$

since $g_1$ and $g_2$ are nonincreasing.
To state our stability result, we consider the following additional hypothesis on the relaxation functions \( g_1 \) and \( g_2 \):

**H2** We assume that \( g_1^0 > 0 \), \( g_2^0 > 0 \) and there exist two positive constants \( \alpha_1 \) and \( \alpha_2 \) and an increasing strictly convex function \( G: \mathbb{R}_+ \to \mathbb{R}_+ \) of class \( C^1(\mathbb{R}_+) \cap C^2(0, \infty) \) satisfying

\[
G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} G'(t) = \infty
\]

such that, for any \( i = 1, 2 \),

\[
(3.4) \quad g_i'(s) \leq -\alpha_i g_i(s), \quad s \in \mathbb{R}_+
\]
or

\[
(3.5) \quad \int_0^\infty \frac{g_i(s)}{G^{-1}(-g_i'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g_i(s)}{G^{-1}(-g_i'(s))} < \infty.
\]

**Theorem 3.1.** Assume **H1** and **H2** hold true. Let \( U_0 \in \mathcal{H} \) such that, for any \( i = 1, 2 \),

\[
(3.6) \quad (3.4) \text{ holds or } \sup_{t \in \mathbb{R}_+} \int_0^\infty \frac{g_i(s)}{G^{-1}(-g_i'(s))} ||f_0x(\cdot, s-t)||^2 ds < \infty,
\]

where \( f_0 = \varphi_0 \) if \( i = 1 \), and \( f_0 = v_0 - u_0 \) if \( i = 2 \). Then there exist two positive constants \( c_1 \) and \( c_2 \) such that the solution \( U \) of (2.3) satisfies

\[
(3.7) \quad E(t) \leq c_2 G_1^{-1}(c_1 t), \quad t \in \mathbb{R}_+,
\]

where

\[
(3.8) \quad G_1(s) = \frac{1}{\int_0^1 \frac{1}{G_0(\tau)} d\tau} \quad \text{and} \quad G_0(s) = \begin{cases} s & \text{if } 3.4 \text{ holds for any } i = 1, 2, \\ sG'(s) & \text{otherwise.} \end{cases}
\]

**Remark 1.** The hypothesis (3.4) implies that \( g_i \) converges exponentially to zero at infinity. So, if both \( g_1 \) and \( g_2 \) converge exponentially to zero at infinity, then (3.7) leads to the exponential stability

\[
(3.9) \quad E(t) \leq c_2 e^{-c_1 t}, \quad t \in \mathbb{R}_+.
\]

However, the hypothesis (3.5), which was introduced by the second author in [9], allows \( s \to g_i(s) \) to have a decay rate at infinity arbitrarily close to \( \frac{1}{2} \). Indeed, for example, for \( g_i(s) = d_i (1 + s)^{-q} \) with \( d_i > 0 \) and \( q_i > 1 \), hypothesis (3.5) is satisfied with \( G(s) = s^r \), for all \( r > \max\left(\frac{2(1+q_i+q)}{q_i-1}, \frac{2q+1}{q-1}\right) \). And then (3.7) implies that

\[
E(t) \leq c_2 (t + 1)^{-\frac{r}{q-1}}, \quad t \in \mathbb{R}_+.
\]

For other examples, see [9] and [10]. In general, the decay rate of \( E \) depends on the decay rate of \( g_i \) which has the weaker decay rate.

**Proof of Theorem 3.1.** In order to prove Theorem 3.1, we will need to construct a Lyapunov functional equivalent to the energy \( E \). To simplify the computations, we denote by \( C \) and \( C_\lambda \) some positive constants which can be different from line to line and depend continuously on \( E(0) \) and on some positive constant \( \lambda \). In order to construct a Lyapunov functional \( R \) equivalent to the energy \( E \), we define first some functionals \( I_1, I_2, I_3, F \) and \( \mathcal{L} \) and prove some estimates on theirs derivatives. Let

\[
(3.10) \quad I_1(t) = -3 \rho_1 \int_0^1 \varphi_1 \int_0^\infty g_1(s) s dx, \quad I_2(t) = -3 \rho_2 \int_0^1 \psi_1 \int_0^\infty g_2(s) s dx,
\]

\[
I_3(t) = \int_0^1 (3 \rho_1 \varphi_1 + 3 \rho_2 \psi_1 + \rho_2 \psi_1) dx, \quad F(t) = I_1(t) + I_2(t) + I_3(t)
\]

\[
(3.11) \quad \text{and} \quad \mathcal{L}(t) = E(t) + \varepsilon F(t),
\]

where \( \varepsilon > 0 \) is a small parameter to be chosen later.
Lemma 3.2. For any $\delta_0 > 0$, there exists $C_{\delta_0} > 0$ such that the functionals $I_1$ and $I_2$ satisfy
\begin{equation}
(3.12) \quad I_1'(t) \leq -3 \rho_1 (g_1^0 - \delta_0) |||\varphi_t||| + \delta_0 (|||\varphi_x||| + |||\varphi_x + \psi - v|||) + C_{\delta_0} \int_0^\infty g_1(s) |||\eta_x|||^2 \, ds
\end{equation}
and
\begin{equation}
(3.13) \quad I_2'(t) \leq -3 \rho_2 (g_2^0 - \delta_0) |||\psi_t||| + \delta_0 (|||\psi_x||| + |||\varphi_x + \psi - v|||) + C_{\delta_0} \int_0^\infty g_2(s) |||z_x|||^2 \, ds.
\end{equation}

**Proof.** First, we note that
\[
\partial_t \int_0^\infty g_1(s) \eta \, ds = \partial_t \int_0^t \int_0^\infty g_1(t-s) (\varphi(t) - \varphi(s)) \, ds \, ds
\]
dividing by $t$ and using the boundary conditions (2.2), we get
\begin{equation}
(3.14) \quad \partial_t \int_0^\infty g_1(s) \eta \, ds = \int_0^\infty g_1'(s) \eta \, ds + g_1^0 \varphi_t,
\end{equation}
that is,
\[
\partial_t \int_0^\infty g_1(s) \eta \, ds = \int_0^\infty g_1'(s) \eta \, ds + g_1^0 \varphi_t.
\]
Similarly, we have
\begin{equation}
(3.15) \quad \partial_t \int_0^\infty g_2(s) z \, ds = \int_0^\infty g_2'(s) z \, ds + g_2^0 \psi_t.
\end{equation}
Second, using Young’s and Hölder’s inequalities, we get the following inequality: for all $\lambda > 0$, there exists $C_{\lambda} > 0$ such that, for any $v \in L^2([0, 1])$ and $\eta \in \{\eta, \eta_x\}$ in case $i = 1$, and $\eta \in \{\zeta, \zeta_x\}$ in case $i = 2,$
\begin{equation}
(3.16) \quad \left| \int_0^1 v \int_0^\infty g_i(s) \eta \, ds \, dx \right| \leq \lambda \|v\|^2 + C_{\lambda} \int_0^\infty g_i(s) \|\eta\|^2 \, ds.
\end{equation}
Similarly,
\begin{equation}
(3.17) \quad \left| \int_0^1 v \int_0^\infty g_i'(s) \eta \, ds \, dx \right| \leq \lambda \|v\|^2 - C_{\lambda} \int_0^\infty g_i'(s) \|\eta\|^2 \, ds.
\end{equation}
Now, direct computations, using the first equation in (2.1), integrating by parts and using the boundary conditions (2.2) and (3.14), yield
\[
I_1'(t) = -3 \rho_1 g_1^0 |||\varphi_t|||^2 + 3 \left| \int_0^\infty g(s) \eta_x \, ds \right|^2 - 3 \rho_1 \int_0^1 \varphi_t \int_0^\infty g_1'(s) \eta \, ds \, dx
\]
\[
+ 3k \int_0^1 (\varphi_x + \psi - v) \int_0^\infty g(s) \eta_x \, ds \, dx - 3g_1^0 \int_0^1 \varphi_x \int_0^\infty g(s) \eta_x \, ds \, dx.
\]
Using (3.16) and (3.17) (with $i = 1$) for the last three terms of this equality, Poincaré’s inequality for $\eta$, Hölder’s inequality to estimate its second term, and (2.10) to estimate $g_1'$ by $\beta_1 g_1$, we get (3.12).

Similarly, using the second equation in (2.1), integrating by parts and using the boundary conditions (2.2) and (3.17) (with $i = 2$), we find (3.13).

Lemma 3.3. For any $\delta_0 > 0$, there exists $C_{\delta_0} > 0$ such that the functional $I_3$ satisfies
\begin{equation}
(3.18) \quad I_3'(t) \leq -3k |||\varphi_x + \psi - v|||^2 + 3 \left| g_1^0 |||\varphi_t||| - 3 \left( b - g_2^0 - \delta_0 \right) |||\psi_x|||^2 - (b - \delta_0) |||v_x|||^2 - 4 \delta |||v|||^2
\]
\[
+ 3 \rho_1 |||\varphi_t|||^2 + 3 \rho_2 |||\psi_t|||^2 + C_{\delta_0} |||v_t|||^2 + C_{\delta_0} \int_0^\infty \left( g_1(s) \|\eta_x\|^2 + g_2(s) \|z_x\|^2 \right) \, ds.
\]

**Proof.** Differentiating $I_3$, using the three equations in (2.1), integrating by parts and using the boundary conditions (2.2), we get
\[
I_3'(t) = -3k |||\varphi_x + \psi - v|||^2 + 3g_1^0 |||\varphi_t|||^2 - 3 \left( b - g_2^0 \right) |||\psi_x|||^2 - b |||v_x|||^2 - 4 \delta |||v|||^2 + \rho_2 |||v_t|||^2
\]
\[
+ 3 \rho_1 |||\varphi_t|||^2 + 3 \rho_2 |||\psi_t|||^2 - 4 \int_0^1 v v_t \, dx - 3 \int_0^1 \varphi_x \int_0^\infty g_1(s) \eta_x \, ds \, dx - 3 \int_0^1 \psi_x \int_0^\infty g_2(s) \, ds \, dx.
\]
Applying (3.10) and Young’s and Poincaré’s inequalities to estimate the last three integrals of this equality, we obtain (3.18).
Lemma 3.4. The functional $\mathcal{F}$ satisfies
\begin{equation}
\mathcal{F}'(t) \leq -E(t) + C \|v_t\|^2 + C \int_0^\infty g_1(s) \|\eta_x\|^2 ds + C \int_0^\infty g_2(s) \|z_x\|^2 ds.
\end{equation}
Moreover, there exists a positive constant $\mu_0$ such that
\begin{equation}
-\mu_0 E \leq F \leq \mu_0 E.
\end{equation}

**Proof.** By adding (3.12), (3.13) and (3.18), and using the definition of $E$, we have
\begin{align*}
\mathcal{F}'(t) & \leq -2E(t) + \delta_0 (2 \|\varphi_x + \psi - v\|^2 + 4 \|\varphi_x\|^2 + 4 \|\psi_x\|^2 + \|v_x\|^2 + 3 \rho_1 \|\varphi_t\|^2 + 3 \rho_2 \|\psi_t\|^2) \\
& + C\delta_0 \|v_t\|^2 + C\delta_0 \int_0^\infty (g_1(s) \|\eta_x\|^2 + g_2(s) \|z_x\|^2) ds.
\end{align*}
On the other hand, the Poincaré's inequality applied to $\psi$ and $v$, and the condition (4.10) imply that
\begin{equation}
\delta_0 (2 \|\varphi_x + \psi - v\|^2 + 4 \|\varphi_x\|^2 + 4 \|\psi_x\|^2 + \|v_x\|^2 + 3 \rho_1 \|\varphi_t\|^2 + 3 \rho_2 \|\psi_t\|^2) \leq \delta_0 C E(t).
\end{equation}
Then combining the last two inequalities and choosing $\delta_0 > 0$ small enough such that $\delta_0 C \leq 1$, we find (3.5).

From the Young’s and Poincaré’s inequalities, and the condition (2.9) lead to, for some $\mu_0 > 0$,
\begin{equation}
|F| \leq |I_1| + |I_2| + |I_3| \leq \mu_0 E,
\end{equation}

Lemma 3.5. There exits a positive constant $\varepsilon$ such that the functional $L$ satisfies
\begin{equation}
L'(t) \leq -\varepsilon E(t) + C \int_0^\infty g_1(s) \|\eta_x\|^2 ds + C \int_0^\infty g_2(s) \|z_x\|^2 ds,
\end{equation}
and there exist positive constants $\mu_1$ and $\mu_2$ such that
\begin{equation}
\mu_1 E \leq L \leq \mu_2 E.
\end{equation}

**Proof.** We differentiate $L$ from (4.4) with respect to time and use (115), together with the dissipation of energy (5.2) and the nonincreasingness of $g_1$ and $g_2$, we obtain
\begin{align*}
L'(t) & \leq -\varepsilon E(t) + (\varepsilon C - 4\gamma) \|v_t\|^2 + \varepsilon C \int_0^\infty g_1(s) \|\eta_x\|^2 ds + \varepsilon C \int_0^\infty g_2(s) \|z_x\|^2 ds.
\end{align*}
Then, for
\begin{equation}
0 < \varepsilon < \frac{4\gamma}{C},
\end{equation}
(3.21) holds. On the other hand, thanks to (4.10), we get
\begin{equation}
(1 - \varepsilon \mu_0) E \leq L \leq (1 + \varepsilon \mu_0) E.
\end{equation}
So, for
\begin{equation}
0 < \varepsilon < \frac{1}{\mu_0},
\end{equation}
we find (3.22) with $\mu_1 = 1 - \varepsilon \mu_0$ and $\mu_2 = 1 + \varepsilon \mu_0$. Finally, choosing
\begin{equation}
0 < \varepsilon < \min \left\{ \frac{4\gamma}{C}, \frac{1}{\mu_0} \right\},
\end{equation}
we get (3.21) and (3.22).

To estimate the last two terms of (3.21), we adapt to our system a lemma introduced by the first author in [9] and improved in [11].
Lemma 3.6. There exist positive constants $d_1$ and $d_2$ such that, for any $\varepsilon_0 > 0$, the following two inequalities hold:

\begin{equation}
\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \int_0^\infty g_1(s) \|\eta_x\|^2 \, ds \leq -d_1 E'(t) + d_1 G_0(\varepsilon_0 E(t))
\end{equation}

and

\begin{equation}
\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \int_0^\infty g_2(s) \|z_x\|^2 \, ds \leq -d_2 E'(t) + d_2 G_0(\varepsilon_0 E(t)).
\end{equation}

**Proof.** If (3.19) holds, then we have from (3.2)

\begin{equation}
\int_0^\infty g_i(s) \|f_x\|^2 \, ds \leq -\frac{1}{\alpha_i} \int_0^\infty g_i'(t) \|f_x\|^2 \, ds \leq -\frac{2}{\alpha_i} E'(t),
\end{equation}

where $f = \eta$ in case $i = 1$, and $f = z$ in case $i = 2$. So (3.20) and (3.21) hold with $d_i = \frac{2}{\alpha_i}$ and $G_0(s) = s$.

When (3.5) is satisfied, we note first that, if $E(t_0) = 0$, for some $t_0 \in \mathbb{R}_+$, then $E(t) = 0$, for all $t \geq t_0$, since $E$ is nonnegative and nonincreasing, and consequently, (3.7) is satisfied, since $E$ is bounded. Thus, without loss of generality, we can assume that $E > 0$ on $\mathbb{R}_+$.

Because $E$ is nonincreasing, we have

\[ \|\eta_x\|^2 \leq 2 \left( \|\varphi_x(\cdot, t)\|^2 + \|\varphi_x(\cdot, t-s)\|^2 \right) \leq C E(0) + 2 \|\varphi_x(\cdot, t-s)\|^2 \]

\[ \leq \begin{cases} 
C E(0) & \text{if } 0 \leq s \leq t, \\
C E(0) + 2 \|\varphi_0\|^2 & \text{if } s > t \geq 0
\end{cases}
\]

so we conclude that

\begin{equation}
\|\eta_x\|^2 \leq M_1(t, s), \quad t, s \in \mathbb{R}_+.
\end{equation}

Similarly, we get

\begin{equation}
\|z_x\|^2 \leq \begin{cases} 
C E(0) & \text{if } 0 \leq s \leq t, \\
C E(0) + 2 \|v_0\|^2 - u_0(\cdot, s-t) - u_0(\cdot, s-t) & \text{if } s > t \geq 0
\end{cases}
\]

Let $\tau_1(t, s) = \tau_2(t, s) > 0$ (which will be fixed later on), $\varepsilon_0 > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$, for $s > 0$, and $K(0) = 0$, since (H2) implies that

\[ \lim_{s \to 0^+} \frac{s}{G^{-1}(s)} = \lim_{\tau \to 0^+} \frac{G(\tau)}{\tau} = G'(0) = 0. \]

The function $K$ is nondecreasing. Indeed, the fact that $G^{-1}$ is concave and $G^{-1}(0) = 0$ implies that, for any $0 \leq s_1 < s_2$,

\[ K(s_1) = G^{-1} \left( \frac{s_1 s_2}{s_2} + \left( 1 - \frac{s_1}{s_2} \right) 0 \right) \leq \frac{s_1}{s_2} \frac{s_1}{s_2} G^{-1}(s_2) + \left( 1 - \frac{s_1}{s_2} \right) G^{-1}(0) = \frac{s_2}{G^{-1}(s_2)} = K(s_2). \]

Then, using (3.28) and (3.29),

\begin{equation}
K(-\tau_2(t, s) g_1(s) \|\eta_x\|^2) \leq K(-M_1(t, s) \tau_2(t, s) g_1(s)), \quad t, s \in \mathbb{R}_+
\end{equation}

and

\begin{equation}
K(-\tau_2(t, s) g_2(s) \|z_x\|^2) \leq K(-M_2(t, s) \tau_2(t, s) g_2(s)), \quad t, s \in \mathbb{R}_+.
\end{equation}

Using (3.30), we arrive at

\[ \int_0^\infty g_1(s) \|\eta_x\|^2 \, ds = \frac{1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{1}{\tau_1(t, s)} G^{-1}(\tau_2(t, s) g_1(s) \|\eta_x\|^2) \]

\[ \times \tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s) K(-\tau_2(t, s) g_1(s) \|\eta_x\|^2) \, ds \]

\[ \leq \frac{1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{1}{\tau_1(t, s)} G^{-1}(\tau_2(t, s) g_1(s) \|\eta_x\|^2) \left( \frac{\tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s)}{-\tau_2(t, s) g_1(s)} \right) \]

\[ \times \left( \frac{\tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s)}{-\tau_2(t, s) g_1(s)} \right) K(-M_1(t, s) \tau_2(t, s) g_1(s)) \, ds \]

\[ \leq \frac{1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{1}{\tau_1(t, s)} G^{-1}(\tau_2(t, s) g_1(s) \|\eta_x\|^2) \frac{M_1(t, s) \tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s)}{-\tau_2(t, s) g_1(s)} ds. \]
Let \( G^*(s) = \sup_{\tau \in \mathbb{R}_+} \{ s \tau - G(\tau) \} \), for \( s \in \mathbb{R}_+ \), denote the dual function of \( G \). Thanks to (H2), we see that
\[
G^*(s) = s (G')^{-1}(s) - G((G')^{-1}(s)), \quad s \in \mathbb{R}_+. 
\]
Using Young's inequality: \( s_1 s_2 \leq G(s_1) + G^*(s_2) \), for
\[
s_1 = G^{-1}(- \tau_2(t, s) g_1'(s) \| \eta_x \|^2) \quad \text{and} \quad s_2 = \frac{M_1(t, s) \tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s)}{G^{-1}(- M_1(t, s) \tau_2(t, s) g_1'(s))},
\]
we get
\[
\int_0^\infty g_1(s) \| \eta_x \|^2 \, ds \leq \frac{1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{- \tau_2(t, s)}{\tau_1(t, s)} g_1'(s) \| \eta_x \|^2 \, ds \\
+ \frac{1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{1}{\tau_1(t, s)} G^* \left( \frac{M_1(t, s) \tau_1(t, s) G'(\varepsilon_0 E(t)) g_1(s)}{G^{-1}(- M_1(t, s) \tau_2(t, s) g_1'(s))} \right) \, ds.
\]
Using the fact that \( G^*(s) \leq s (G')^{-1}(s) \), we get
\[
\int_0^\infty g_1(s) \| \eta_x \|^2 \, ds \leq \frac{-1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{\tau_2(t, s)}{\tau_1(t, s)} g_1'(s) \| \eta_x \|^2 \, ds \\
+ \int_0^\infty \frac{M_1(t, s) g_1(s) G^{-1}(- M_1(t, s) \tau_2(t, s) g_1'(s))}{G^*} \left( G'(\varepsilon_0 E(t)) g_1(s) \right) \, ds.
\]
Then, using the fact that \( (G')^{-1} \) is nondecreasing and choosing \( \tau_2(t, s) = \frac{1}{M_1(t, s)} \), we get
\[
\int_0^\infty g_1(s) \| \eta_x \|^2 \, ds \leq \frac{-1}{G'(\varepsilon_0 E(t))} \int_0^\infty \frac{1}{M_1(t, s) \tau_1(t, s)} g_1'(s) \| \eta_x \|^2 \, ds \\
+ \int_0^\infty \frac{M_1(t, s) g_1(s) G^{-1}(- M_1(t, s) \tau_2(t, s) g_1'(s))}{G^*} \left( G'(\varepsilon_0 E(t)) g_1(s) \right) \, ds,
\]
where \( m_1 = \sup_{s \in \mathbb{R}_+} \frac{g_1'(s)}{G^{-1}(- g_1'(s))} < \infty \) (\( m_1 \) exists according to (K3)). Due to (K5) and the restriction on \( \varphi_0 \) in [3.4], we have
\[
\sup_{t \in \mathbb{R}_+} \int_0^\infty M_1(t, s) g_1(s) \, ds =: m_2 < \infty.
\]
Therefore, choosing \( \tau_1(t, s) = \frac{1}{m_1 M_1(t, s)} \) and using [3.2], we obtain
\[
\int_0^\infty g_1(s) \| \eta_x \|^2 \, ds \leq \frac{-m_1}{G'(\varepsilon_0 E(t))} \int_0^\infty g_1'(s) \| \eta_x \|^2 \, ds + \varepsilon_0 E(t) \int_0^\infty \frac{M_1(t, s) g_1(s)}{G^{-1}(- g_1'(s))} \, ds \\
\leq \frac{-2m_1}{G'(\varepsilon_0 E(t))} \varepsilon'(t) + \varepsilon_0 m_2 E(t),
\]
which gives [3.25] with \( d_1 = \max \{2 m_1, m_2 \} \) and \( G_0(s) = s G'(s) \). Repeating the same arguments, we get [3.20] with \( G_0(s) = s G'(s) \), \( d_2 = \max \{2 m_1, m_2 \} \).
\[
m_1 = \sup_{s \in \mathbb{R}_+} \frac{g_2(s)}{G^{-1}(- g_2'(s))}, \quad m_2 = \sup_{t \in \mathbb{R}_+} \int_0^\infty \frac{M_2(t, s) g_2(s)}{G^{-1}(- g_2'(s))} \, ds, \\
\tau_1(t, s) = \frac{1}{m_1 M_2(t, s)} \quad \text{and} \quad \tau_2(t, s) = \frac{1}{M_2(t, s)}.
\]

We are now ready to prove the main stability result [3.7]. Multiplying (3.21) by \( \frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \) and combining [3.25] and [3.20], we find, for \( c_0 = d_1 + d_2 \),
\[
\left( \frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \right) L'(t) + c_0 C E'(t) \leq - \left( \frac{\varepsilon}{\varepsilon_0} - c_0 C \right) G_0(\varepsilon_0 E(t)).
\]
We define our Lyapunov functional \( \mathcal{R} \) by
\[
\mathcal{R} = \tau_0 \left( \frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} L + c_0 C E \right),
\]
where $\tau_0$ is a positive constant that will be chosen later. Because $E$ is nonincreasing and $G$ is convex, then $\frac{G_0(\varepsilon_0 E)}{\varepsilon_0 E}$ is nonincreasing, and therefore, (3.32) and (3.33) lead to

$$R'(t) \leq - \tau_0 \left( \frac{\varepsilon}{\varepsilon_0} - \varepsilon_0 C \right) G_0(\varepsilon_0 E(t)).$$

Moreover, recalling that $\frac{G_0(\varepsilon_0 E)}{\varepsilon_0 E}$ is nonincreasing and using (3.32), we obtain

$$(3.35) \quad \tau_0 \varepsilon_0 C E \leq R \leq \tau_0 \left[ (1 + \varepsilon \mu_0) \frac{G_0(\varepsilon_0 E(0))}{\varepsilon_0 E(0)} + \varepsilon_0 C \right] E.$$

By choosing $0 < \varepsilon_0 < \frac{1}{\varepsilon_0 C}$, we deduce from (3.34) and (3.35) that $R$ is equivalent to $E$ and satisfies

$$(3.36) \quad R'(t) \leq - \tau_0 C G_0(\varepsilon_0 E(t)).$$

Thus, for $\tau_0 > 0$ such that

$$R \leq \varepsilon_0 E \quad \text{and} \quad R(0) \leq 1,$$

we get, for $c_1 = \tau_0 C$,

$$(3.37) \quad R'(t) \leq - c_1 G_0(R(t)).$$

Then (3.37) implies that $(G_1(R))' \geq c_1$, where $G_1$ is defined in (3.8). So, a direct integrating gives

$$(3.38) \quad G_1(R(t)) \geq c_1 t + G_1(R(0)).$$

Because $R(0) \leq 1$ and $G_1$ is decreasing, we obtain $G_1(R(t)) \geq c_1 t$ which implies that $R(t) \leq G_1^{-1}(c_1 t)$. Finally, from the equivalence of $R$ and $E$, the result (3.7) follows and the proof of Theorem 3.1 is complete.

4. Numerical analysis

In this section, we present some numerical results illustrating the asymptotic behavior of the energy for the exponential decay. For this, we use Finite Difference (of second order in space and time). Furthermore, the method of $\beta$–Newmark is a second order method preserving the discrete energy always when the discrete system of equations of motion is symmetric (i.e. matrices associated to the system should be symmetric).

4.1. Finite difference method. We consider $J$ an integer non-negative and $h = L/(J + 1)$ an spatial subdivision of the interval $(0, L)$ given by $0 = x_0 < x_1 < \ldots < x_J < x_{J+1} = L$, with $x_j = jh$ each node of the mesh. We use $\varphi_j(t), \psi_j(t), v_j(t)$, for all $j = 1, 2, \ldots, J$ and $t > 0$ to denote the approximate values of $\varphi(jh, t)$ and $\psi(jh, t)$, respectively. In addition, we denote the discrete operator $\Delta_h \varphi_j = \frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{h^2}$ and the $\theta$-scheme $Q_\theta \varphi_j = \theta \varphi_{j+1} + (1 - 2\theta) \varphi_j + \theta \varphi_{j-1}$, with $\theta = 1/4$. We assume the following finite difference scheme applied to system

$$(4.1) \begin{cases} 3 \left( \rho_1 \varphi''_j - k \Delta_h \varphi_j + \delta^+ \psi_j - \delta^- \psi_j \right) + \int_0^\infty g_1(s) \Delta_h \varphi_j(t - s) \, ds = 0, & j = 1, \ldots, J, \\ 3 \left( \rho_2 \psi''_j - (b - g_2^0) \Delta_h \psi_j + k \delta^+ \varphi_j + Q_1/4 \psi_j - Q_1/4 v_j \right) + \int_0^\infty g_2(s) \Delta_h \psi_j \, ds = 0, & j = 1, \ldots, J, \\ \rho_2 v''_j - b \Delta_h v_j - 3 k \delta^- \varphi_j + Q_1/4 \psi_j - Q_1/4 v_j + 4 \Theta_1/4 \delta v_j + 4 \gamma v'_j = 0, & j = 1, \ldots, J, \\ \varphi_0 = \varphi_J = \varphi_0 = v_0 = 0, & \psi_j = \psi_{J+1}, & v_j = v_{J+1}. \end{cases}$$

4.2. Equation of time and time discretization. The system (4.1) can be rewritten as

$$(4.2) \quad \mathbf{M} \begin{bmatrix} \dot{\varphi}_h \\ \dot{\psi}_h \end{bmatrix} + \mathbf{C} \begin{bmatrix} \dot{\varphi}_h \\ \dot{\psi}_h \end{bmatrix} + \mathbf{K} \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} + \mathbf{G} \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} = \mathbf{0},$$

where $\phi_h = (\phi_1, \ldots, \phi_J), \psi_h = (\psi_1, \ldots, \psi_J), v_h = (v_1, \ldots, v_J) \in \mathbb{R}^J$. The mass, damping, and stiffness matrices $(\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathcal{M}_{J,J}(\mathbb{R}))$, are given by

$$\mathbf{M} = \begin{bmatrix} 3 \rho_1 \mathbf{I} & 0 & 0 \\ 0 & 3 \rho_2 \mathbf{I} & 0 \\ 0 & 0 & \rho_2 \mathbf{I} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \gamma \mathbf{I} \end{bmatrix},$$

where $\mathbf{I}$ is the identity and $\mathbf{0}$ is the null matrix of $\mathcal{M}_{J,J}(\mathbb{R})$. \(\square\)
\[
\mathbf{D}^- = \frac{1}{h} \begin{bmatrix}
1 & 0 & \ldots & \ldots \\
-1 & 1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & -1 & 1
\end{bmatrix}, \quad \text{and } \mathbf{D}^+ = (\mathbf{D}^-)^t.
\]

We note that

\[
\mathbf{D}_0^2 = \mathbf{D}^+ \mathbf{D}^- - \frac{1}{h^2} \mathbf{J}^{jj} = \mathbf{D}^+ \mathbf{D}^- - \frac{1}{h^2} \mathbf{J}^{jj},
\]

where \(\mathbf{J}^{jj}\) is the single-entry matrix of one, for \(i\)-row, \(j\)-column. Additionally,

\[
\mathbf{Q} = \frac{1}{h} \begin{bmatrix}
2 & 1 & \ldots & \ldots \\
1 & 2 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 1 & 1
\end{bmatrix} = \mathbf{P}^+ \mathbf{P}^-, \quad \text{where } \mathbf{P}^- = \frac{1}{2} \begin{bmatrix}
1 & 0 & \ldots & \ldots \\
1 & 1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 1 & 1
\end{bmatrix} \quad \text{and } \mathbf{P}^+ = (\mathbf{P}^-)^t.
\]

The memory terms are given by, \(\mathbf{G}_1(\varphi_h) = \int_0^\infty g_1(s) \mathbf{D}_0^2 \varphi_j(t-s) \, ds\) and \(\mathbf{G}_2(\psi_h) = \int_0^\infty g_2(s) \mathbf{D}^+ \mathbf{D}^- \psi_j(t-s) \, ds\).

The Newmark algorithm \([22]\) is based on a set of two relations expressing the forward displacement \([\varphi_h^{n+1}, \psi_h^{n+1}, \psi_h^{n+1}]^T\) and velocity \([\dot{\varphi}_h, \dot{\psi}_h, \dot{\psi}_h]^T\). The method consists in updating the displacement, velocity and acceleration vectors from current time \(t^n = n \delta t\) to the time \(t^{n+1} = (n+1) \delta t\),

\[
\begin{align*}
\dot{\varphi}_h^{n+1} &= \dot{\varphi}_h^n + (1 - \zeta) \delta t \varphi_h^n + \zeta \delta t \varphi_h^{n+1} \\
\varphi_h^{n+1} &= \varphi_h^n + \delta t \dot{\varphi}_h^n + \left(\frac{1}{2} - \beta\right) \delta t^2 \varphi_h^n + \beta \delta t^2 \varphi_h^{n+1} \\
\dot{\psi}_h^{n+1} &= \dot{\psi}_h^n + (1 - \zeta) \delta t \psi_h^n + \zeta \delta t \psi_h^{n+1} \\
\psi_h^{n+1} &= \psi_h^n + \delta t \dot{\psi}_h^n + \left(\frac{1}{2} - \beta\right) \delta t^2 \psi_h^n + \beta \delta t^2 \psi_h^{n+1} \\
\dot{v}_h^{n+1} &= \dot{v}_h^n + (1 - \zeta) \delta t \psi_h^n + \zeta \delta t \psi_h^{n+1} \\
v_h^{n+1} &= v_h^n + \delta t \dot{v}_h^n + \left(\frac{1}{2} - \beta\right) \delta t^2 v_h^n + \beta \delta t^2 v_h^{n+1},
\end{align*}
\]

where \(\beta\) and \(\zeta\) are parameters of the methods that will be fixed later. Approximating \(\mathbf{G}_1(\varphi_h)\) and \(\mathbf{G}_2(\psi_h)\) by \(\tilde{\mathbf{G}}_1(\varphi_h^n) = \sum_{j=0}^N \delta t g_1^j (\mathbf{D}_0^2)^{n-j}\) and \(\tilde{\mathbf{G}}_2(\psi_h^n) = \sum_{j=0}^N \delta t g_2^j (\mathbf{D}^+ \mathbf{D}^-)^{n-j}\) (for \(N\) large enough), and replacing \([4.4]-[4.9]\) in the equation of motion \([4.2]\), we obtain

\[
(\mathbf{M} + \zeta \delta t \mathbf{C} + \beta \delta t^2 (\mathbf{K} + \mathbf{G}_0)) \begin{bmatrix}
\varphi_h^{n+1} \\
\psi_h^{n+1} \\
v_h^{n+1}
\end{bmatrix} = -\mathbf{C} \begin{bmatrix}
\varphi_h^n \\
\psi_h^n \\
v_h^n
\end{bmatrix} + (1 - \zeta) \delta t \begin{bmatrix}
\varphi_h^{n+1} \\
\psi_h^{n+1} \\
v_h^{n+1}
\end{bmatrix} - \sum_{j=1}^N \mathbf{G}_j \begin{bmatrix}
\varphi_h^{n+1-j} \\
\psi_h^{n+1-j} \\
v_h^{n+1-j}
\end{bmatrix},
\]

where

\[
\mathbf{K} = \begin{bmatrix}
-3kD_h^2 & -3kD_h^- & 3kD_h^- \\
-3kD_h^- & -6D_h^+D_h^- + 3kQ & -3kD_h^- \\
-3kD_h^- & -3kQ & -bD^+D^- + (3k + 4\delta)Q
\end{bmatrix}, \quad \text{where, } D_h^0 = \frac{1}{h^2} \begin{bmatrix}
-2 & 1 & \ldots & \ldots \\
1 & -2 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 1 & -2
\end{bmatrix},
\]
where \( G_j = \begin{bmatrix} \delta t g^1 D_0^j & 0 & 0 \\ 0 & \delta t g^2 D^+ D^- & 0 \\ 0 & 0 & 0 \end{bmatrix} \), \( j = 0, \ldots, N \). We introduce the variables

\[
\eta^{n,j} := \varphi^n - \varphi^{n-j} \\
z^{n,j} := \psi^n - \psi^{n-j}
\]

which verify

\[
\eta^{n,j} - \eta^{n,j-1} = \varphi^{n+1-j} - \varphi^n, \\
z^{n,j} - z^{n,j-1} = \psi^{n+1-j} - \psi^n.
\]

Thus, from (4.4) and (4.7), we obtain

\[
\delta t \dot{\varphi}^{n+\frac{1}{2}} = \eta^{n+1,j} - \eta^{n,j-1} + \delta t^2 \left( \beta - \frac{1}{2} \right) (\varphi^{n+1} - \varphi^n)
\]

\[
\delta t \dot{\psi}^{n+\frac{1}{2}} = z^{n+1,j} - z^{n,j-1} + \delta t^2 \left( \beta - \frac{1}{2} \right) (\psi^{n+1} - \psi^n)
\]

where \( \vartheta^{n+\frac{1}{2}} := \vartheta^n + \vartheta^{n+1} \), for all \( \vartheta^n \), with \( n \in \mathbb{Z} \). We will need the following Lemma:

**Lemma 4.1.** Let \( \beta = \frac{1}{2} \varsigma \). Then

\[
\sum_{j=1}^{N} \chi^j D_0^j \varphi^{n+1-j} \cdot \varphi^{n+\frac{1}{2}} = \frac{1}{2\delta t} \left[ - \left( \sum_{j=1}^{N} \chi^j \right) \left( \|D^+ \varphi\|^2 + \left( \varphi^1 \right)^2 \right) \right. \\
+ \sum_{j=1}^{N} \chi^j \left( \|D^+ \eta^{n,j}\|^2 + \left( \eta^{n,j} \right)^2 \right) \left. \right]^{n+1}_{n} \\
- \frac{1}{2\delta t} \sum_{j=1}^{N} \left( \chi^{j+1} - \chi^j \right) \left( \|D^+ \eta^{n,j}\|^2 + \left( \eta^{n,j} \right)^2 \right) \\
+ \frac{1}{2\delta t} \chi^{N+1} \left( \|D^+ \eta^{n,N}\|^2 + \left( \eta^{n,N} \right)^2 \right),
\]

\[
\sum_{j=1}^{N} \chi^j D^+ D^- \psi^{n+1-j} \cdot \psi^{n+\frac{1}{2}} = \frac{1}{2\delta t} \left[ - \left( \sum_{j=1}^{N} \chi^j \right) \|D^- \psi\|^2 + \sum_{j=1}^{N} \chi^j \|D^- z^{n,j}\|^2 \right]^{n+1}_{n} \\
- \frac{1}{2\delta t} \sum_{j=1}^{N} \left( \chi^{j+1} - \chi^j \right) \|D^- z^{n,j}\|^2 + \frac{1}{2\delta t} \chi^{N+1} \|D^- z^{n,N}\|^2,
\]

for all \( (\chi^j)_{j \in \mathbb{N}} \) with \( \chi^j \in \mathbb{R} \).
Proof. Let \( I(\psi^n) := \sum_{j=1}^{N} \chi^j D^+ D^- \psi_{n-j} \cdot \dot{\psi}^n \), and \( I(\dot{\psi}^n) := \sum_{j=1}^{N} \chi^j D^+ D^- \dot{\psi}^n \cdot \dot{\psi}^n \). Then, using (4.11), (4.12) and (4.13), with \( \beta = \frac{1}{2} \zeta \), we obtain

\[
I(\dot{\psi}^n) = - \sum_{j=1}^{N} \chi^j D^+ \left( \dot{\psi}^n \frac{\dot{\psi}^{n+1}}{\delta t} - z^{j,n+\frac{1}{2}} \right) \cdot D^- \frac{\dot{\psi}^{n+1} - \dot{\psi}^n}{\delta t}
\]

\[
= \frac{1}{2\delta t} \left( \sum_{j=1}^{N} \chi^j \right) \left( \|D^- \dot{\psi}^{n+1}\|^2 - \|D^- \dot{\psi}^n\|^2 \right) + \frac{1}{\delta t} \sum_{j=1}^{N} \chi^j D^- z^{n+\frac{1}{2}} \cdot D^- \left( z^{n+1,j} - z^n,j \right)
\]

(4.16)

\[
+ \frac{1}{\delta t} \sum_{j=1}^{N} \chi^j D^- z^{n+\frac{1}{2}} \cdot D^- \left( z^{n,j} - z^{n,j-1} \right).
\]

The term \(-A + B\) corresponds to the right hand side of the term in bracket \( [ \cdot ]_{n}^{n+1} \) in (4.15). On the other hand, the term \( C \) can be written as

\[
C = \frac{1}{2\delta t} \sum_{j=1}^{N} \chi^j D^- \left( z^{n,j} + z^{n,j-1} \right) \cdot D^- \left( z^{n,j} - z^{n,j-1} \right)
\]

\[
+ \frac{1}{2\delta t} \sum_{j=1}^{N} \chi^j D^- \left( z^{n+1,j} - z^{n,j-1} \right) \cdot D^- \left( z^{n,j} - z^{n,j-1} \right)
\]

\[
= - \frac{1}{2\delta t} \sum_{j=1}^{N} \left( \chi^j - \chi^j \right) \|D^- z^{n,j}\|^2 + \frac{1}{2\delta t} \chi^{n+1} \|D^- z^{n,N}\|^2 + I(\psi^n) - I(\dot{\psi}^n)
\]

Thus, observing that the left hand side of (4.16) is given by \( I(\dot{\psi}^{n+1}) = 2I(\dot{\psi}^{n+\frac{1}{2}}) - I(\dot{\psi}^n) \), and replacing this last expression in (4.16), we obtain (4.15). Repeating the same calculations for the couple \((\varphi, \eta)\), and considering (4.9), it follows (4.10).

The acceleration \( [\dot{\varphi}^{n+1}, \dot{\psi}^{n+1}, \dot{v}^{n+1}]^\top \) is computed from (4.10), and the velocities \( [\dot{\varphi}^{n+1}, \dot{\psi}^{n+1}, \dot{v}^{n+1}]^\top \) are obtained from (4.3) and (4.10), respectively. Finally, the displacement \( [\varphi_h^{n+1}, \psi_h^{n+1}, v_h^{n+1}]^\top \) follows from (4.5) and (4.7), by simple matrix operations. Thus, the fully discrete energy of the system (4.4)-(4.10) is given by

(4.17) \( \mathcal{E}_h^n := \frac{1}{2} \left( \begin{array}{c} \dot{\varphi}_h^n \ 
\dot{\psi}_h^n \ 
\dot{v}_h^n \end{array} \right) \mathbf{M} \left[ \begin{array}{c} \varphi_h^n \
\psi_h^n \n\psi_h^n \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} \dot{\varphi}_h^n \ 
\dot{\psi}_h^n \n\dot{v}_h^n \end{array} \right] \mathbf{K} \left[ \begin{array}{c} \varphi_h^n \
\psi_h^n \n\psi_h^n \end{array} \right] + \mathbf{G} \left[ \begin{array}{c} \varphi_h^n \
\psi_h^n \n\psi_h^n \end{array} \right] \]
where $G(\cdot, \cdot)$ is the bilinear form derived from the memory term and described in the following line, and $g_{j,2}^{-} = \frac{1}{2}(g_{j,2}^{1} - g_{j,2}^{2})$. This is an approximation of energy for the continuous case. The increment of this energy can be expressed in terms of mean values and increments of the displacement and velocity. Then, we choose $\varsigma = \frac{1}{2}$ and $\beta = \varsigma^{2}$, reducing the above expression to

$$
E_{n+1} - E_{n} = -4 \delta t \gamma \| v_{h}^{n+1,2} \|^{2} \left\{ \frac{1}{4} \sum_{j=1}^{N} \left( g_{j,2}^{1+1} - g_{j,2}^{1} \right) \left( 2 \| D^{+} \eta^{n,j} \|^{2} + \left( \frac{\eta_{j,1}^{1}}{h^{2}} \right)^{2} \right) + \frac{1}{4} \sum_{j=1}^{N} \left( g_{j,2}^{2+1} - g_{j,2}^{2} \right) \left( 2 \| D^{-} z^{n,j} \|^{2} + \left( \frac{\eta_{j,1}^{1}}{h^{2}} \right)^{2} \right) - \frac{1}{2} g_{2}^{N+2} \| D^{-} z^{n,N} \|^{2} - \frac{1}{2} g_{2}^{N} \| D^{-} z^{n,1} \|^{2} \} \leq 0.
$$

With this, the fully discrete energy obtained by the $\beta$–Newmark method is decreasing and we expect that its asymptotic behavior be a reflection of the continuous case (see [18] and also [3, 2]).

5. Comments and issues

**Comment 1.** The speeds of wave propagations of the both last two equations in (1.1) are equal to $\sqrt{\frac{b}{\rho}}$. Our results hold true when the last two equations in (1.1) have different speeds of wave propagations (that is when $b$ in the last equation in (1.1) is replaced by $\tilde{b} > 0$).

**Comment 2.** Our results hold true when $\delta = 0$.

**Comment 3.** The last equation in (1.1) can be controlled via an infinite memory

$$
\int_{0}^{\infty} g_{3}(s) v_{xx}(t - s) \, ds
$$

instead of the linear frictional damping $4 \gamma v_{t}$, where $g_{3} : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is a given relaxation function satisfying the same hypotheses as $g_{1}$ and $g_{2}$. To prove the well-posedness results, we introduce a third variable $w$ similar to $\eta$ and $z$ given by

$$
w(x, t, s) = v(x, t) - v(x, t - s),$$

FIGURE 1. $\varphi(x, t)$ and $\varphi_{t}(x, t)$
we define its space $L_{g_3}$ as $L_{g_2}$ and we do some logical modifications. For the stability result, we add to $E$ in (3.1) the term

$$\frac{1}{2} \int_0^\infty g_3(s) \|w_x\|^2 ds$$

and we add to the definition of $F$ in (4.4) the integral

$$-\rho_2 \int_0^1 v_1 \int_0^\infty g_3(s) w ds dx.$$

Acknowledgment. This work was initiated during the visit in July-August 2017 of the first author to LNCC and RJ university, Brazil, and Bio-Bio and Concepción universities, Chile, and finished during the visit in June 2018 of the third author to Lorraine-Metz university, France, and the visit in August 2018 of the first author to Bio-Bio and Concepción universities, Chile. The first and third authors thank LNCC, Lorraine-Metz, RJ, Bio-Bio and Concepción universities for their kind support and hospitality. This work was supported FONDECYT grant no. 1180868, and by ANID-Chile through the project CENTRO DE MODELAMIENTO MATEMÁTICO (AFB170001) of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal.
REFERENCES

[1] M. S. Alves, P. Gamboa, G. C. Gorain, A. Ramaud and O. Vera, Asymptotic behavior of a flexible structure with Cattaneo type of thermal effect, Indagationes Mathematicae, 27 (2016), 821-834.
[2] M. Alves, J. Muñoz-Rivera, M. Sepúlveda and O. Vera, Exponential and the lack of exponential stability in transmission problems with localized Kelvin-Voigt dissipation, SIAM J. Appl. Math., 74 (2014), 354-365.
[3] M. Alves, J. Muñoz-Rivera, M. Sepúlveda, O. Vera and M. Zegarra, The asymptotic behaviour of the linear transmission problem in viscoelasticity, Math. Nachr., 287 (2014), 483-497.
[4] C. F. Beards and I. M. A. Imam, The damping of plate vibration by interfacial slip between layers, Int. J. Mach. Tool. Des. Res., 18 (1978), 131-137.
[5] J. A. C. Bresse, Cours de Mécanique Appliquée, Mallet Bachelier, Paris, 1859.
[6] X. G. Cao, D. Y. Liu and G. Q. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, J. Dynamical Control Syst., 13 (2007), 313-336.
[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, F. A. Falcao Nascimento, I. Lasiecka and J. H. Rodrigues, Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping, Z. Angew. Math. Phys., 65 (2014), 1189-1206.
[8] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Rational Mech. Anal., 37 (1970), 297-308.
[9] A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl., 382 (2011), 748-760.
[10] A. Guesmia, On the stabilization for Timoshenko system with past history and frictional damping controls, Palestine J. Math., 2 (2013), 187-214.
[11] A. Guesmia, Asymptotic behavior for coupled abstract evolution equations with one infinite memory, Applicable Analysis, 94 (2015), 184-217.
[12] A. Guesmia, Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements, Mediterr. J. Math., 14 (2017), 19 pages.
[13] A. Guesmia, Non-exponential and polynomial stability results of a Bresse system with one infinite memory in the vertical displacement, Nonauton. Dyn. Syst., 4 (2017), 78-97.
[14] A. Guesmia, Well-posedness and stability results for laminated Timoshenko beams with interfacial slip and infinite memory,IMA J. Math. Control and Information, DOI: 10.1093/imamci/dnz002.
[15] A. Guesmia and M. Kafini, Bresse system with infinite memories, Math. Meth. Appl. Sci., 38 (2015), 2389-2402.
[16] A. Guesmia and M. Kirane, Uniform and weak stability of Bresse system with two infinite memories, Z. Angew. Math. Phys., 67 (2016), 1-39.
[17] A. Guesmia, S. Messaoudi and A. Soufyane, On the stabilization for a linear Timoshenko system with infinite history and applications to the coupled Timoshenko-heat systems, Elect. J. Diff. Equa., 2012 (2012), 1-45.
[18] S. Krenk, Energy conservation in Newmark based time integration algorithms, Comput. Methods Appl. Mech. Engrg., 195 (2006), 6110-6124.
[19] W. Liu and W. Zhao, Exponential and polynomial decay for a laminated beam with Fourier’s type heat conduction, Preprints 2017, 2017020058, doi: 10.20944/preprints201702.0058.v1.
[20] Z. Liu and S. Zheng, Semigroups associated with dissipative systems, CRC Research notes in Mathematics, Chapmans & Hall/CRC, New York, 1999.
[21] A. Lo and N-E Tatar, Stabilization of laminated beams with interfacial slip, Elec. J. Diff. Equa., 2015 (2015), 1-14.
[22] A. Lo and N. E. Tatar, Uniform stability of a laminated beam with structural memory, Qual. Theory Dyn. Syst., 15 (2016), 517-540.
[23] A. Lo and N. E. Tatar, Exponential stabilization of a structure with interfacial slip, Discrete Contin. Dyn. Syst., 36 (2016), 6285-6306.
[24] N. M. Newmark, A method of computation for structural dynamics, J. Engrg. Mech. Div. ASCE, 85 (1959), 67-94.
[25] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer - Verlag, New York, 1983.
[26] C. A. Raposo, Exponential stability for a structure with interfacial slip and frictional damping, Appl. Math. Lett., 53 (2016), 85-91.
[27] C. A. Raposo, O. V. Villagrán, J. E. Muñoz Rivera and M. S. Alves, Hybrid laminated Timoshenko beam, J. Math. Phys., 58 (2017), 11 pages.
[28] N. E. Tatar, Stabilization of a laminated beam with interfacial slip by bounday controls, Boundary Values Problems, 2015 (2015), 11 pages.
[29] S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismaticbars, Philosophical magazine, 41 (1921), 744-746.
[30] J. M. Wang, G. Q. Xu and S. P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, SIAM J. Control Optim., 44 (2005), 1575-1597.