Noncommutative Geometry and Structure of Space-Time

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Abstract

I give a summary review of the research program using noncommutative geometry as a framework to determine the structure of space-time. Classification of finite noncommutative spaces under few assumptions reveals why nature chose the Standard Model and the reasons behind the particular form of gauge fields, Higgs fields and fermions as well as the origin of symmetry breaking. It also points that at high energies the Standard Model is a truncation of Pati-Salam unified model of leptons and quarks. The same conclusions are arrived at uniquely without making any assumptions except for an axiom which is a higher form of Heisenberg commutation relations quantizing the volume of space-time. We establish the existence of two kinds of quanta of geometry in the form of unit spheres of Planck length. We provide answers to many of the questions which are not answered by other approaches, however, more research is needed to answer the remaining challenging questions.

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1 Introduction

One of the main aims of theoretical physics is to explain and understand the fundamental laws of nature. Despite the apparent complexity, the laws of physics are expressed by concise mathematical formulas. Presently there are four known fundamental forces. The first force is gravity which is universal. It interacts with all particles with mass or energy. It is extremely weak and long range and mediated by the graviton, a tensor $g_{\mu\nu}$. Due to the absence of negative mass and the attractive nature of gravity, masses can become extremely large and the gravitational force sizable as in the examples of stars and black holes.

The electromagnetic force acts on charged particles, mediated by the photon $A_\mu$, where charges could be positive or negative and is much stronger ($\sim 10^{36}$) than the gravitational force. Despite the huge ratio, astronomical objects are mostly neutral, resulting in the dominance of gravity. The third force is the weak force covering the interactions of doublets and singlets in left-handed and right-handed sectors of leptons and quarks. The electromagnetic and weak interactions are unified in one force, the electroweak force at the energy scale of $\sim 100$ Gev mediated by the four vector bosons of the symmetry group $SU(2) \times U(1)$.

This symmetry is broken spontaneously when the scalar Higgs $SU(2)$ doublet acquires a vacuum expectation value (vev) with the electromagnetic symmetry $U(1)_{\text{em}}$ as the unbroken subgroup. The three vectors $W^{\pm}_\mu$ and $Z_\mu$ associated with the broken generators are heavy with masses of $\sim 80$ Gev and $\sim 90$ Gev, respectively, thus explaining the short range of weak interactions. The strong nuclear force acts on quarks made of three different colors and is thus governed by the symmetry group $SU(3)$. The force is mediated by the eight Gluon vectors, $V^{i}_\mu$, $i = 1, \cdots, 8$ and is $\sim 10^2$ stronger than the electromagnetic force. The fundamental fermions come in three identical families in representations of the symmetry group $SU(3) \times SU(2) \times U(1)$ where each member of the higher generation have a much larger mass ($\sim 10^3$) than the corresponding particle of the previous generation. There are 16 fermions per generation, each comprising of 8 left-handed and 8 (anti) right-handed two components chiral spinors. All masses are acquired through the couplings of the Higgs doublet to the left and right handed spinors. The 8 left-handed components of the first family comprise of a lepton $SU(2)$ doublet made from a neutrino and an electron as well as a quark doublet made from up and down quarks of three colors. The 8 right-handed components comprise of $SU(2)$ singlets made up of a neutrino, electron, an up quark and down quark of three colors. The neutrinos (which are electrically neutral) are mostly left-handed with extremely small masses which is explained by assuming that the right-handed components also acquire a Majorana type large mass, usually through coupling to a singlet scalar field with a large vev ($\sim 10^{11}$ Gev ). The particles listed above with the forces governing their interactions (without gravity) is known as the Standard Model of particle physics. The model has 19 parameters, fixed by experimental data, and is studied using the methods of quantum field theory. It agrees, so far, with all experimental tests and at present there are no hints of whether there is physics beyond the Standard Model. There are, in addition, parameters associated with the neu...
trinos, which are still not fully determined from experiment. The success of the Standard Model poses enormous challenge to understanding the reasons for its very specific structure. Out of infinite number of possibilities the question we must pose is to understand the reason why nature chose the configuration given by the Standard Model. One important question we will answer in this lecture is *Why the Standard Model?*

The monumental achievement of Einstein is that he showed that matter curves the four-dimensional space-time in such a way that the geometry becomes Riemannian. The main tool in Riemannian geometry is the four-dimensional metric tensor $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3, 4$ which is used in measuring distances

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

A test particle in this space moves along a geodesic according to the parametric equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$$

where $\lambda$ is the affine parameter and $\Gamma^\mu_{\nu\rho}$ is the (symmetric) Christoffel connection determined from the metricity condition

$$0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma}$$

Dynamics of the gravitational field is dictated by the Einstein-Hilbert action

$$I_g = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R$$

where $R$ is the scalar curvature of the four-dimensional manifold and $G$ is Newton constant. The variational principle gives Einstein equations, which is a set of 10 non-linear second order differential equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor due to the contributions of matter. Newton’s law is recovered by setting $g_{00} = 1 + \phi$ and $g_{ij} = -\delta_{ij}$ where $\phi$ is the gravitational potential. This classical action has passed all experimental tests including the recent results on gravitational waves. This action is, however, not well behaved at the quantum level. At present, there is no satisfactory theory of quantum gravity. In this sense gravity cannot be unified with the other three fundamental interactions. A main difficulty is to quantize the metric $g_{\mu\nu}$ in a background independent way. On the other hand, a very important question to answer is whether there is a scale at which all the four interactions are unified? It is known, to a high degree of precision that if there is no new physics beyond the Standard Model, then the three coupling constants, which are energy dependent, do not all meet at one common energy scale. Instead of intersecting at a point, the intersection of the three lines (on a logarithmic scale) form a triangle in the energy range $10^{14} - 10^{17}$ Gev, which is
not far off from the Planck scale $10^{19}$ Gev where the gravitational force becomes strong. For a believer in the idea of unification, this suggests that there must be new physics beyond the Standard Model that makes unification a possibility and it is preferable to have the unification scale to be near the upper end of $10^{17}$ Gev. This would also point to the possibility of having all the fields, including the graviton, as part of a full unified system and not as separate independent parts. In addition, the possibility of unification with gravity at the Planck scale where the gravitational field becomes strong, suggests that Riemannian geometry would cease to be the appropriate framework to describe geometry of space-time. Some discretization must take place, and there must exist a minimal volume, or some form of quanta of geometry, out of which space-time is built.

The purpose of this summary is to show that it is possible to understand the deep reasons behind the choice of nature to the Standard Model and why the particles, their interactions and group representations are the way they are. The framework of Riemannian geometry must be replaced by that of noncommutative geometry combined with the condition of the existence of quanta of geometry. Noncommutative geometry is spectral in nature. It grew from formulation of quantum mechanics by von Neumann and later developed by Alain Connes [1]. The geometric data is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an associative algebra with unit 1 and involution $\ast$, $\mathcal{H}$ a complex Hilbert space carrying a faithful representation of the algebra $\mathcal{A}$ and $D$ is a self-adjoint operator on $\mathcal{H}$ with a compact resolvent of $(D - \lambda 1)^{-1}$, where $\lambda \notin \mathbb{R}$. In addition the real structure $J$ is an anti-unitary operator that sends the algebra $\mathcal{A}$ to its commutant $\mathcal{A}^o$ such that

$$[a, b^o] = 0, \quad a, b \in \mathcal{A}, \quad b^o = Jb^*J^{-1} \in \mathcal{A}^o. \quad (1)$$

The chirality operator $\gamma$ is a unitary operator in $\mathcal{H}$ defined in even dimensions such that $\gamma^2 = 1$ and commutes with $\mathcal{A}$

$$[\gamma, a] = 0 \quad \forall a \in \mathcal{A}. \quad (2)$$

There are commutativity or anti-commutativity relations between $D, J,$ and $\gamma$:

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' J, \quad D\gamma = -\gamma D, \quad (3)$$

where $\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}$. The operators $\gamma$ and $J$ are similar to the chirality and charge conjugation operators and to every fixed value of $\epsilon, \epsilon', \epsilon''$ is associated a KO dimension, which may be non-metric, and thus is defined only modulo 8. Riemann’s formula for the distance between two points

$$d(a, b) = \inf_{\gamma} \int_{\gamma} \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$$

where $\gamma$ is any path connecting $a$ and $b$, is replaced by

$$d(a, b) = \sup \{ ||f(a) - f(b)|| : f \in \mathcal{A}, \quad ||D, f|| \leq 1 \}$$
It is useful to get familiarized with the new geometry by noting that Riemannian geometry corresponds to the choice where $\mathcal{A} = C^\infty (M)$ the algebra of infinitely differentiable functions over $M$, $\mathcal{H} = L^2 (S, M)$, the Hilbert space of square integrable functions and

$$D = e^\mu_a \gamma^a (\partial \mu + \omega_{\mu})$$

where $e^\mu_a$ is the square root of the inverse metric $g^\mu\nu$, $\gamma^a$ are anti-Hermitian Dirac matrices defining a Clifford algebra $\{\gamma^a, \gamma^b\} = -2 \delta^{ab}$, $a, b = 1, \ldots, d$ where $d$ is the dimension of $M$. The spin-connection $\omega_{\mu}$ is $SO (d)$ Lie algebra valued, $\omega_{\mu} = \frac{1}{2} \omega_{\mu}^b \gamma_{bc}$, where $\gamma_{bc} = \frac{1}{2} (\gamma^b \gamma^c - \gamma^c \gamma^b)$. The spectrum of the operator $D$ contains all the geometric invariants. In simpler terms, it is more useful to use spectrum of the Dirac operator, which we interpret as the inverse distance: $ds^{-1} = D$, to measure distances. The reality operator $J$ is the charge conjugation operator and $\gamma$, the chirality operator in even dimensions, is

$$\gamma = (i)^\frac{d}{2} \gamma_1 \cdots \gamma_d$$

Given the above identifications, results of Riemannian geometry over spin-manifolds are obtained with the aid of reconstruction theorems [4].

2 Classification and Why the Standard Model

Marc Kac [5] asked the question: Can we hear the shape of the drum? Geometrically this is equivalent to determining the geometry of a manifold from the spectrum of the Laplacian. Although the existence of isospectral manifolds showed that the answer is not unique [6], this deficiency is overcome in noncommutative geometry where the data defines uniquely the geometry. The question that Kac asked is relevant to determining the structure of space-time as defined by the spectrum of all the known elementary particles. Naturally our notion of space-time will get modified whenever more particles at higher energies, or equivalently small distances, are discovered. As alluded to in the introduction, there is no known connection between the elementary particles of the Standard Model and geometry. Although gauge theories can be thought of as vector bundles defined over the four-dimensional manifold, this description does not add any understanding to answer the question: Why the Standard Model? Fortunately the task of identifying the structure of space-time, in the down to up approach as function of energy, starting with the spectrum of known particles, could be answered in noncommutative geometry [7] [8]. This shows the need for a geometry that combines the four-dimensional Riemannian manifold which has continuous spectrum with the discrete spectrum of the Standard Model. We must therefore look for a noncommutative space admitting both discrete and continuous spectrum [10]. This situation is reminiscent of quantum mechanics, the inspiration behind noncommutative geometry, where physical operators such as the Hamiltonian, admit both discrete and continuous spectrum. As a first approximation we assume that the required noncommutative geometry is a tensor product of two geometries, the first corresponding to a continuous
four-dimensional Riemannian spin-manifold, and the second is discrete. The spectrum of the full geometry is then given by \[ A = C^\infty (M_4) \otimes A_F \]
\[ \mathcal{H} = L^2 (S, M_4) \otimes \mathcal{H}_F \]
\[ D = D_M \otimes 1 + \gamma_5 \otimes D_F \]
\[ J = J_M \gamma_5 \otimes J_F \]
\[ \gamma = \gamma_5 \otimes \gamma_F \]

where \((A_F, \mathcal{H}_F, D_F, J_F, \gamma_F)\) defines the geometry of the finite space, which to start with, is undetermined. Irreducible elements of the Hilbert space \(\Psi \in \mathcal{H}\) are chiral \(\gamma \Psi = \Psi\), however, acting on them by the reality operator \(J\) produce new independent elements. Physically speaking, this implies that all known fermions must have a mirror image with the same masses and this is not observed. Thus we must require the property that \(J \Psi\) will be identified with \(\Psi\). This turns out to be satisfied only when the KO dimension of the finite noncommutative space is 6 (mod 8). Classifying all finite noncommutative spaces with KO dimension 6 shows that the algebra \(A_F = M_{4n}(\mathbb{C}) \oplus M_{4n}(\mathbb{C})\), \(n \geq 1\) \[11\]. Making the further physical requirement that the first algebras \(M_{2n}(\mathbb{C})\) is subject to antilinear isometry symmetry, restricts it to the form \(M_{2n}(\mathbb{H})\) where \(\mathbb{H}\) is the field of quaternions. The simplest possibility is then to have the algebra \(A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})\) and when this is subjected to the condition of commutativity of elements of the algebra with the chirality operator \(\gamma_F\), the algebra reduces to \(A_F = (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})\). Thus an element \(\Psi \in \mathcal{H}\) is of the form \(\Psi = (\psi_A \psi_A')\)

where \(\psi_A\) is a 16 component \(L^2 (M, S)\) spinor in the fundamental representation of \(A_F\) of the form \(\psi_A = \psi_{\alpha I}\) where \(\alpha = 1, \cdots, 4\) with respect to \(M_2(\mathbb{H})\) and \(I = 1, \cdots, 4\) with respect to \(M_4(\mathbb{C})\) and where \(\psi_{A'} = C \psi_A^*\) is the charge conjugate spinor to \(\psi_A\) \[9\]. The chirality operator \(\gamma\) must commute with elements of \(A\) which implies that \(\gamma_F\) must commute with elements in \(A_F\). This predicts the number of fundamental fermions to be \(4^2\) chiral spinors in the representations \((2_R, 1_L, 4) + (1_R, 2_L, 4)\) of \(SU(2)_R \times SU(2)_L \times SU(4)\) symmetry. At this point it is important to note that the assumptions made are minimal and weak, except for one ad hoc assumption of the existence of antilinear isometry. Later we will show that working with a completely different starting point, in the form of an axiom, we arrive uniquely at the above algebra.

The operator \(J\) sends the algebra \(A\) to its commutant, and thus the full algebra acting on the Hilbert space \(\mathcal{H}\) is \(A \otimes A^0\). Under automorphisms of the algebra

\[ \Psi \rightarrow U \Psi, \quad (4) \]

where \(U = u \hat{u}\) with \(u \in A\), \(\hat{u} \in A^0\) with \([u, \hat{u}] = 0\), it is clear that the Dirac action is not invariant. This is similar to the situation in electrodynamics where the Dirac action is not invariant under local phase transformations but the invariance is easily restored by introducing the vector potential \(A_\mu\) through the
transformation $\gamma^\mu \partial_\mu \rightarrow \gamma^\mu (\partial_\mu + i e A_\mu)$. In our case, the Dirac operator $D$ is replaced with $D_A = D + A$, where the connection $A$ is given by

$$A = \sum a \tilde{a} [D, \tilde{b}].$$

It can be shown that under automorphisms $U$ of the algebra we have $D_A \rightarrow U D_A U^*$. The important special case occurs when the connection $A$ belongs to the algebra $\mathcal{A}$ but not to its commutant, and this occurs when the order one condition is satisfied

$$[a, [D, \tilde{b}]] = 0,$$

(6)

We then have

$$A = A = A^{(1)} + J A^{(1)} J^{-1}$$

where $A^{(1)} = \sum a [D, b]$ and the center of the algebra $Z(\mathcal{A})$ is non-trivial in such a way that the space is connected. This means that there is a mixing term between the fermions and their conjugates. The Dirac operator connects the spinors $\psi_A$ and their conjugates $\psi_A'$ so that

$$[D, Z(\mathcal{A})] \neq 0.$$  

(7)

In physical terms this would allow a Majorana mass term for the fermions. It was shown in $^{[11]}$ that the unique solution to this equation constrains the algebra $\mathcal{A}_F = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ to be restricted to a subalgebra

$$\mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C}),$$

(8)

so that an element of $\mathcal{A}$ takes the form

$$a = \begin{pmatrix} X \otimes 1_4 & \text{X} \otimes 1_4 \\ q \otimes 1_4 & 1_4 \otimes X \\ 1_4 \otimes m \end{pmatrix},$$

(9)

where $X \in \mathbb{C}$, $q \in \mathbb{H}$, $m \in M_3(\mathbb{C})$ and the operator $D_F$ have a singlet non-zero entry in the mixing term $(D_F)_A^A'$. The fermions are enumerated as

$$\psi_{11} = \nu_R, \quad \psi_{21} = e_R,$$

(10)

$$\psi_{a1} = l_a = \left( \begin{array}{c} \nu_L \\ e_L \end{array} \right),$$

(11)

$$\psi_{1i} = u_i R, \quad \psi_{2i} = d_{iR},$$

(12)

$$\psi_{ai} = q_{ia} = \left( \begin{array}{c} u_{iL} \\ d_{iL} \end{array} \right).$$

(13)
It is clear that the associated gauge group is \( U(1) \times SU(2) \times SU(3) \). The (finite) Dirac operator can be written in matrix form

\[
D_F = \begin{pmatrix}
D_A^R & D_A^B \\
D_A^B & D_A^B
\end{pmatrix},
\]

and must satisfy the properties

\[
\gamma_F D_F = -D_F \gamma_F \quad J_F D_F = D_F J_F,
\]

where \( J_F^2 = 1 \). We can show using elementary algebra, that the structure of the connection \( A \) is such that the block diagonal elements are of the form \( \gamma^\mu \) tensored with the gauge vectors of \( U(1) \times SU(2) \times SU(3) \) while the off-diagonal block elements are of the form \( \gamma_5 \) tensored with the Higgs doublet complex scalar field. There is, in addition, one singlet scalar field \( \sigma \) (real, or complex) \[13], \[14\], not present in the Standard Model whose vev gives mass to the right-handed neutrinos \[9\]. Dynamics of all the fermion fields is then governed by the Dirac action \[8\]

\[
(J \Psi, D_A \Psi).
\]

reproducing all the complicated structure present in the Standard Model. At present, there is no known explanation for the three generations that we must assume for the fermions and for the form of the Yukawa couplings responsible for the hierarchy of fermion masses.

We have mentioned before that the spectrum of the Dirac operator is given by geometric invariants. This led to the proposal that the dynamics of all bosonic fields, including graviton, gauge fields and Higgs fields is governed by the spectral action given by \[7\]

\[
\text{Tr} f \left( \frac{D_A}{\Lambda} \right)
\]

where \( f \) is a positive function and \( \Lambda \) is a cut-off scale. It turns out that the form of the function \( f \) is not important at scales below the Planck scale. At these scales, the action can be computed using the method of heat-kernel expansion. For the Dirac operator associated with the noncommutative geometry of the Standard Model, this computation yields \[9\]

\[
S_b = \frac{48}{\pi^2} f_1 \Lambda^4 \int d^4x \sqrt{g}
- \frac{4}{\pi^2} f_2 \Lambda^2 \int d^4x \sqrt{g} \left( R + \frac{1}{2} a \bar{H}H + \frac{1}{4} c \sigma^2 \right)
+ \frac{1}{2\pi^2} f_0 \int d^4x \sqrt{g} \left[ \frac{1}{30} \left( -18 C_{\mu\nu\rho\sigma}^2 + 11 R^\ast R^\ast \right) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 
+ \frac{1}{6} a \bar{H}H + b (\bar{H}H)^2 + a \left| \nabla_{\mu} H_d \right|^2 + 2 c \bar{H}H \sigma^2 + \frac{1}{2} d \sigma^4 + \frac{1}{12} e Rd^2 + \frac{1}{2} c (\partial_{\mu} \sigma)^2 \right]
+ \cdots
\]
where $a, c, d, e$ are defined in terms of the Yukawa couplings, $f_0 = f(0)$ and $f_k$ are the Mellin transforms of the function $f$

$$f_k = \int_0^\infty f(v) v^{k-1} dv, \quad k > 0.$$  

This action contains all bosonic interactions including gravity, gauge symmetries and those of the Higgs field and a scalar singlet. All couplings are related at unification scale. The only scale in the expansion is the cut-off scale $\Lambda$. The zeroth order term in the heat kernel expansion gives the cosmological constant, while the first order term gives the Einstein-Hilbert action and the scalar masses.

The second order term, which is conformal invariant, gives the Yang-Mills and scalar kinetic terms as well as the second order in curvature terms. A new feature in the spectral Standard Model is the existence of the singlet field $\sigma$ whose vev gives mass to the right-handed neutrino. In the Standard Model the Higgs coupling will become negative at some high energy scale of the order of $10^{11}$ Gev. Remarkably, this field $\sigma$ plays a very important role in stabilizing the Higgs coupling which will not become negative at very high energies as well as being consistent with a low Higgs mass of 125 Gev [13], [14]. The form of the gauge and Higgs kinetic terms and potential implies unification of the gauge couplings and the Higgs coupling. The universality of the Higgs couplings to all fermion masses implies a relation between the fermion masses and the gauge vectors masses. A study of the renormalization group equations shows that these relations are consistent with present experimental data and predict the top quark mass to be around 170 Gev. However, gauge couplings do approach each other, but do not meet at one scale, and thus form a triangle in the energy range $10^{14} - 10^{17}$ Gev. At present these couplings are measured to very high precision, and the non meeting of the couplings can be taken as a strong indication of the need for new physics beyond the Standard Model. We will address this point later.

It is then clear from the above results that the simple assumption that space-time is a noncommutative space which is a product of a continuous four-dimensional Riemannian spin-manifold tensored with a finite space of $KO$ dimension 6 gives an excellent geometric explanation of the intricate details of the Standard Model. We are able to explain and answer many open questions which are not answered simultaneously in other approaches:

1. Why there are 16 fermions per generation.
2. The gauge symmetry $U(1) \times SU(2) \times SU(3)$ and gauge fields as fluctuations of the Dirac operator along continuous directions.
3. Origin of the Higgs doublet field as fluctuations of the Dirac operator along discrete directions.
4. Why spontaneous symmetry breaking occurs.
5. Smallness of the neutrino masses due to see-saw mechanism.

6. Stabilizing of the Higgs coupling at large scales.

7. Prediction of the top quark mass.

8. Unification of all fundamental forces including gravity.

These results are quite impressive, especially since there are no alternative explanations capable of producing them simultaneously. This shows that adopting the framework of noncommutative geometry for unification of all fundamental interactions is the right approach. On the other hand, it is also clear that this is not the final answer because there are still many unanswered questions, such as why there are only three generations, and the reason for the hierarchy of the fermions masses. We have also seen that the gauge coupling constants do not quite meet at a unified scale. This indicates that there must be new physics beyond the Standard Model. In our classification of noncommutative spaces, at one point, we made a simplifying assumption that the connection arising from inner fluctuations of the Dirac operator belongs to the algebra $\mathcal{A}$ but not to its commutant $\mathcal{A}^\circ$, which in turn required the Dirac operator to satisfy the order one condition. This condition need not be satisfied at high energies, and in seeking to find out whether the Standard Model is only an approximation of a more unified theory, such possibility must be explored. This brings us back to the finite algebra taken to be $\mathcal{A}_F = (\mathbb{H} \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})$. Inner fluctuations of the Dirac operator would then yield the connection

$$A = A_{(1)} + JA_{(1)}J^{-1} + A_{(2)},$$

where

$$A_{(1)} = \sum a [D, b]$$
$$A_{(2)} = \sum \hat{a} \left[ A_{(1)}, \hat{b} \right],$$

which satisfies $JA_{(2)}J^{-1} = A_{(2)}$. The connection $A_{(2)}$ is not linear, and corresponds to scalar fields along the off-diagonal components and are in mixed representations with respect to the gauge symmetry $SU(2)_R \times SU(2)_L \times SU(4)$. The gauge vector fields correspond to inner fluctuations along continuous directions and are those of the Pati-Salam model where the symmetry group $SU(4)$ assigns the lepton number as the fourth color [15]. This symmetry then achieves the unification of leptons and quarks. Depending on the form of the initial Dirac operator before fluctuations, the most general representation of the Higgs scalar fields is given by [15]

$$\Sigma_{aI} = (\mathcal{2}_R, 2_L, 1) + (\mathcal{2}_R, 2_L, 15)$$
$$H_{a1bJ} = (1_R, 1_L, 6) + (1_R, 3_L, 10)$$
$$H_{a1bJ} = (1_R, 1_L, 6) + (3_R, 1_L, 10).$$
The other possibilities are special cases of the above configuration, obtained by truncations. The Standard Model is also a special case where $\Sigma_{abI}^j$ is expressed in terms of the Higgs doublet $H$ and the field $H_{abI}$ is expressed in terms of $\sigma$ only. Analysis of the renormalization group equations for the Pati-Salam models show that it is possible to get gauge couplings unification and this is achieved at energy scales of the order $10^{16}$ GeV \cite{16}. It is then very gratifying to know that general classification of finite noncommutative spaces of $KO$ dimension 6 with only one extra assumption made on one of the algebras to have antilinear isometry symmetry, picks uniquely a Pati-Salam model or one of its truncations, including the Standard Model. This result is a major step in geometrizing the unification program and realizing Einstein’s dream of finding the right framework for obtaining a geometric theory where all interactions are dictated by metric fluctuations, which are now defined through the Dirac operator.

3 Volume Quantization and Quanta of Geometry

In the classification given above, the algebra $A_F = (H_R \oplus H_L) \oplus M_4(C)$ was the first possibility out of many of the form $A_F = (M_n(H_R) \oplus M_n(H_L)) \oplus M_4n(C)$, in addition to making the ad hoc assumption about the antilinear isometry that reduced the algebra $M_4n(C)$ to $(M_n(H_R) \oplus M_n(H_L))$ \cite{11}. It is then natural to try to derive the above results in a unique way without the need of making few assumptions. We will now do this adopting a different strategy. We start with the observation that at very small distances, of the order of Planck length, we expect the discrete nature of space-time to manifest itself in the form of having a minimal distance. At present the volume of the universe is $\sim 10^{60}$ in Planck units \cite{17}. In the noncommutative approach, and because of the need to have compact spectrum for the Dirac operator, space-time is taken to be of Euclidean signature. We therefore expect the volume of the four-dimensional space-time to be quantized as integer multiple of the volume of a unit Planck sphere. Such quantum number is present in the study of maps from the four-manifold $M_4$ to the four-sphere $S^4$ in the form of winding number. We, therefore, must identify the 4–volume form with the 4–form constructed from the 5 constrained coordinates on the sphere $Y^A$ such that $Y^AY^A = 1$, $A = 1, \ldots, 5$

$$\omega = \sqrt{g}dx^1 \wedge \cdots \wedge dx^4 = \frac{1}{4!}f_{ABCD}EY^AdY^B \wedge \cdots \wedge dY^E$$

This equation forces the volume of the four manifold to be an integer multiple of the unit sphere $S^4$ \cite{18}. It turns out, however, that this condition restricts the topology of $M_4$ because the volume form $\omega$ would then not vanish anywhere, and thus the pullback of the maps $Y$ are non-singular \cite{20}. This cannot be always satisfied because the sphere is simply connected. We will see shortly how in noncommutative geometry this unacceptable restriction is avoided. We first have to write the volume quantization condition in an index free notation, using
the noncommutative data. In analogy with the Dirac operator which is constructed by tensoring the connection with the Clifford algebra, the coordinates \( Y^A \) are tensored by Dirac gamma matrices so that \( Y = Y^A \Gamma_A \) where

\[
\Gamma_A \in C_\kappa, \quad \{ \Gamma_A, \Gamma_B \} = 2\kappa \delta_{AB}, \quad (\Gamma_A)^* = \kappa \Gamma_A. \tag{22}
\]

Here \( \kappa = \pm 1 \) and \( C_+ = M_2(\mathbb{H}) \oplus M_2(\mathbb{H}) \) while \( C_- = M_4(\mathbb{C}) \) \cite{19}. Since we will be dealing with irreducible representations we take \( C_+ = M_2(\mathbb{H}) \). We then have

\[
Y^2 = \kappa, \quad Y^* = \kappa Y. \tag{23}
\]

and the volume quantization condition takes the simple form \cite{22}

\[
\frac{1}{2^2(4!)} \left< Y [D, Y]^4 \right> = \sqrt{\kappa}\gamma, \tag{24}
\]

where \( \left< \right> \) means taking the trace over the Clifford algebra. Notice that the two conditions in equation \(23\) can be combined into one relation \( Y^4 = 1 \). Our first observation is that condition \(24\) involves the commutator of the Dirac operator \( D \) and the coordinates \( Y \). In momentum space \( D \) is the Feynman-slashed \( \not{p} = \gamma^\mu p_\mu \) momentum and \( Y \) are the Feynman-slashed coordinates. This suggests that the quantization condition is a higher form of Heisenberg commutation relation quantizing the phase space formed by coordinates and momenta. We first notice that although the quantization condition is given in terms of the noncommutative data, the operator \( J \) is the only one missing. We therefore modify the condition to take \( J \) into account. We first define the projection operator \( e = \frac{1}{2} (1 + Y) \) satisfying \( e^2 = e \) \cite{3} but now there are two possibilities, \( Y \) corresponding to the case \( \kappa = 1 \) and \( Y' \) to the case \( \kappa = -1 \). Thus let \( Y = Y^A \Gamma_A \) and \( Y' = iJYJ^{-1} \) and \( \Gamma'_A = iJ\Gamma_A J^{-1} \) so that we can write

\[
Y = Y^A \Gamma_A, \quad Y' = Y'^A \Gamma'_A, \tag{25}
\]

satisfying \( Y^2 = 1 \) and \( Y'^2 = 1 \). The projection operators \( e = \frac{1}{2} (1 + Y) \) and \( e' = \frac{1}{2} (1 + Y') \) satisfy \( e^2 = e, \quad e'^2 = e' \) with \( e \) and \( e' \) commuting. This allows to define the projection operator \( E = ee' \) and the associated field

\[
Z = 2E - 1, \tag{26}
\]

satisfying \( Z^2 = 1 \). The conjectured quantization condition takes the elegant form of a two-sided relation \cite{21}, \cite{22}

\[
\left< Z [D, Z]^4 \right> = \gamma. \tag{27}
\]

One of the remarkable properties of four dimensions is that the quantization condition in terms of the \( Z \) coordinates splits into the sum of two pieces, one in terms of \( Y \) and the other in terms of \( Y' \)

\[
\left< Z [D, Z]^4 \right> = \left< Y [D, Y]^4 \right> + \left< Y' [D, Y'^4 \right>. \tag{28}
\]
This property is only shared with dimension $2$ but not with higher dimensions where interference terms do arise. This implies that the volume form of the $4$-dimensional Riemannian manifold is the sum of two $4$-forms and thus

$$\omega = \frac{1}{4!} \epsilon_{ABCDE} \left( Y^A dY^B \wedge \cdots \wedge dY^E + Y'^A dY'^B \wedge \cdots \wedge dY'^E \right)$$

Thus the volume of the four-manifold (in multiples of unit Planck spheres) is the sum of two integers, the winding numbers associated with the two maps $Y$ and $Y'$. The restriction on the topology of $M_4$ to be that of a sphere is now removed because the sum of the pullbacks of $Y$ and $Y'$ does not vanish anywhere and thus each one of them could vanish separately, but not simultaneously. We have shown that for a compact connected smooth oriented manifold with $n \leq 4$ one can find two maps $\phi_+^\# (\alpha)$ and $\phi_-^\# (\alpha)$ whose sum does not vanish anywhere, satisfying equation (29) such that $\int_M \omega \in \mathbb{Z}$. The proof for $n = 4$ is more difficult and there is an obstruction unless the second Stiefel-Whitney class $w_2$ vanishes, which is satisfied if $M$ is required to be a spin-manifold and the volume to be larger than or equal to five units [22], [21].

We therefore, take as our starting point the higher form of Heisenberg commutation relations with the field $Z$ defining two separate maps from $M_4$ to the sphere $S^4$

$$Z = \frac{1}{2} (Y + Y') (Y' + 1) - 1, \quad Y \in M_2 (\mathbb{H}), \quad Y' \in M_4 (\mathbb{C})$$

and belonging to the finite algebra

$$\mathcal{A}_F = M_2 (\mathbb{H}) \oplus M_4 (\mathbb{C}),$$

However, the maps $Y$ and $Y'$ are functions of the coordinates of the manifold $M_4$ and therefore the algebra associated with this space must be

$$\mathcal{A} = C^\infty (M, \mathcal{A}_F) = C^\infty (M) \otimes \mathcal{A}_F.$$ (30)

The associated Hilbert space is

$$\mathcal{H} = L^2 (M, S) \otimes \mathcal{H}_F.$$ (31)

The Dirac operator mixes the finite space and the continuous manifold non-trivially

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$ (32)

where $D_F$ is a self adjoint operator in the finite space. The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$, and the anti-unitary operator $J$ is given by $J = J_M \gamma_5 \otimes J_F$. At this point we realize that we have obtained exactly the same noncommutative space obtained before and all subsequent analysis becomes identical.
4 Conclusions

We have thus identified the noncommutative space that corresponds to the structure of space-time. We started first by classifying all possible noncommutative spaces which are tensor products of continuous four-dimensional space, times finite spaces with $KO$ dominions 6, as dictated by the absence of mirror fermions. We showed that the algebra of the finite space is the sum of two matrix algebras. We then assumed that one of the algebras satisfy a certain antilinear isometry. This yielded a class of spaces, the first of which is the one that produces the Standard Model with all its intricacies. The result comes out in a unique way by assuming a differential condition on the Dirac operator where the connection obtained by inner fluctuations is restricted to the algebra but not its commutant and is linear. The Standard Model will hold as a unified model up to very high energies, however, it is not adequate at energies higher than $10^{14}$ Gev as evidenced from the failure of the gauge coupling constants to unify. This suggests that at very high energies the order one condition on the Dirac operator be removed. In an alternative approach, we started with an axiom representing a higher Heisenberg commutation relation quantizing the phase space of Dirac operators and coordinates provided by the two maps from the manifold to four spheres. The advantage of the second approach, is that there is no need to make various assumptions, physical, or ad hoc. This axiom of quantization is enough to determine fully the noncommutative space defining the structure of space-time. We have the physically satisfying result that the volume of space-time is quantized in terms of two kinds of quanta associated with the two maps $Y$ and $Y'$. Inner fluctuations of the Dirac operator over the algebra $\mathcal{A}$ then results in gauge and Higgs fields of the Pati-Salam model with the symmetry of $SU(2)_R \times SU(2)_L \times SU(4)$. The 16 fermions of each family are in the fundamental representation of the Hilbert space. The Standard Model is a special truncation of the Pati-Salam model. We have thus presented very strong evidence that noncommutative geometry is the correct framework to define the structure of space-time. We have determined the precise noncommutative space that reproduces all known particle interactions including gravity. Confidence in these results is strengthened by the fact that two different strategies, although completely not correlated, give the same answer. The advantage of the second approach is that in this case the noncommutative space is obtained uniquely without the need to make more assumptions or restrict ourselves to the simplest possibility. This shows that the idea of volume quantization is a fundamental one, and not only determine the noncommutative space defining space-time, but also defines the two different quanta needed to construct the four dimensional manifold.

We have, therefore succeeded in removing many of the mysteries associated with the Standard Model and pointed the way to the physics beyond. We have also answered many of the questions that defied explanations for so long. Naturally, many more questions remain, and it is a big challenge to extend the ideas presented here to uncover the remaining mysteries.
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