Killing spinors as a characterisation of rotating black hole spacetimes

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Abstract

We investigate the implications of the existence of Killing spinors in a spacetime. In particular, we show that in vacuum and electrovacuum a Killing spinor, along with some assumptions on the associated Killing vector in an asymptotic region, guarantees that the spacetime is locally isometric to the Kerr or Kerr-Newman solutions. We show that the characterisation of these spacetimes in terms of Killing spinors is an alternative expression of characterisation results of Mars (Kerr) and Wong (Kerr-Newman) involving restrictions on the Weyl curvature and matter content.

Keywords: Spinorial methods, black holes, Killing spinors, Kerr-Newman solution, invariant characterisations

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1 Introduction

The Kerr spacetime, describing a rotating black hole in vacuum, is one of the most interesting exact solutions to the Einstein field equations. As well as having physical relevance, the existence of various incarnations of uniqueness theorems (see e.g. [8] and references within for a survey of this vast topic) has cemented its place as one of the most important vacuum solutions mathematically. There also exist generalisations to spacetimes containing restricted forms of matter—for example, the Kerr-Newman solution to the Einstein-Maxwell equations. Although less physically interesting than the vacuum case, these solutions still retain many interesting features of the Kerr solution, including uniqueness under further assumptions on the matter content. Thus, these generalisations still retain a mathematical importance.

The remarks in the previous paragraph justify the attention given to finding characterisations of the Kerr spacetime and its relations—see e.g. [15, 10]. Such characterisations can be used to study various open questions about these black hole spacetimes. For example, they can be used to reformulate uniqueness theorems and clarify relations between them; study the stability of the solutions, by indicating the behaviour of perturbations; and illustrate the special characteristics of these particular solutions, in particular through the use of symmetries—see e.g. [1] for a recent discussion on these and related ideas. The last of these is elegantly achieved through the use of Killing spinors. Closely related to Killing-Yano tensors, these spinorial objects represent “hidden symmetries” of the spacetime, which cannot be represented using Killing vectors. It has been shown previously (see [3, 4, 5]) that a vacuum spacetime admitting a Killing spinor, along with conditions on the Weyl curvature and an asymptotic condition, must be isometric to the Kerr spacetime. This result crucially depends on a result of Mars (see [16]) which uses the structure of

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the Weyl tensor, and its relation to the Killing vectors of the spacetime, to characterise the Kerr solution in a way that exploits to the maximum possible extent the asymptotic flatness of the spacetime —more precisely, it is required that the selfdual Killing form of the stationary Killing vector is an eigenform of the selfdual Weyl tensor.

The characterisation of the Kerr spacetime by Mars given in [16] relies on a previous characterisation of this solution to the vacuum Einstein field equations in terms of the vanishing of the so-called Mars-Simon tensor —see [15]. Interestingly, the latter characterisation has been generalised to the electrovacuum case by Wong [22] assuming some restrictions on the matter content. This characterisation is not optimal —in the sense that it assumes the existence of certain relations among the relevant geometric objects; by contrast, in [15], the existence of the vacuum counterpart of these relations is a consequence of the characterisation. Nevertheless, as a consequence of the analysis in [22], one may expect that the Kerr-Newman solution can be characterised by the use of Killing spinors in a similar way to the vacuum case. The characterisations in both [15] and [22] come in both a local version (in which certain constraints arising in the characterisation are fixed by evaluating them at finite points of the manifold) and a global version (in which asymptotic flatness is used to fix the value of the constants). Remarkably, the generalisation of the characterisation in [16] to the electrovacuum case has, so far, not been obtained.

The purpose of this article is to revisit the characterisation of the Kerr spacetime using Killing spinors and then generalise to the electrovacuum case using Wong’s result in [22]. Our analysis suggests that Wong’s result can be strengthened to obtain a characterisation of the Kerr-Newman spacetime more in the spirit of Mars’s original result in [15] and, in turn, use this result to obtain a generalisation of the analysis of [16] in which the Kerr-Newman spacetime is characterised in an optimal way by a combination of local and global assumptions.

Outline of the article. This paper is organised as follows. In section 2 we give an introduction to Killing spinors, their relation to Killing vectors and investigate the implications on the curvature of the spacetime. We will spend some time defining 1-forms and potentials which are useful in the characterisations later on. In section 3, we define the required asymptotic conditions needed for the characterisation theorems. Then, in section 4, we show that the conditions of the characterisation result of Mars [16] are satisfied when the spacetime admits an appropriate Killing spinor. Finally, section 5 shows the same for Wong’s characterisation of the Kerr-Newman spacetime —i.e. the existence of an appropriate Killing spinor on a electrovacuum spacetime guarantees that the solution is Kerr-Newman up to an isometry.

Conventions

In what follows, (\(\mathcal{M}, g\)) will denote an electrovacuum spacetime satisfying the Einstein equations with vanishing cosmological constant. The signature of the metric in this article will be \((+,-,-,-)\), to be consistent with most of the existing literature using spinors. We use the spinorial conventions of [18]. The lowercase Latin letters \(a, b, c, \ldots\) are used as abstract spacetime tensor indices while the uppercase letters \(A, B, C, \ldots\) will serve as abstract spinor indices. The Greek letters \(\mu, \nu, \lambda, \ldots\) will be used as spacetime coordinate indices while \(\alpha, \beta, \gamma, \ldots\) will serve as spatial coordinate indices.

Our conventions for the curvature are that
\[
\nabla_c \nabla_d u^b \nabla_c u^b = R_{dca}^\ b u^a .
\]
The curvature spinors \(\Psi_{ABCD}\) and \(\Phi_{A'B'C'}\) are defined by the relations
\[
\square_{AB} \xi^C = \Psi_{ABCD} \xi^D - 2 \lambda_{(A} \xi_{B)C}, \quad \square_{A'B'} \xi^C = \Phi_{C'DA'B'} ,
\]
where \(\square_{AB} \equiv \nabla_{A} (\nabla_B A')\). Given an antisymmetric rank 2 tensor \(F_{ab}\), its Hodge dual is defined by
\[
F^*_{ab} \equiv \frac{1}{2} \varepsilon_{ab}^{cd} F_{cd} .
\]
The self-dual version of \(F_{ab}\) is then defined by
\[
F_{ab} \equiv F_{ab} + i F^*_{ab} .
\]
2 Killing spinors

The purpose of this section is to provide a summary of the basic theory of Killing spinors in electrovacuum spacetimes —see [11, 12, 13]. Throughout we assume that \((\mathcal{M}, g)\) denotes an electrovacuum spacetime. In spinorial notation the Einstein-Maxwell equations read

\[
\Phi_{A'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \quad \Lambda = 0
\]

where \(\phi_{AB} = \phi_{(AB)}\) is the Maxwell spinor satisfying

\[
\nabla^A_A\phi_{AB} = 0.
\]

The Bianchi identity in electrovacuum spacetimes takes the form

\[
\nabla^A_A\Psi_{ABCD} = 2\bar{\phi}_{A'B'}\nabla_{B'}\phi_{CD}.
\]

2.1 Basic equations

A Killing spinor is a valence-2 symmetric spinor \(\kappa_{AB}\) satisfying the equation

\[
\nabla_A'(A\kappa_{BC}) = 0.
\]

By taking a further contracted derivative of this equation, it can be shown that a solution to equation (3) must also satisfy the integrability condition

\[
\kappa_{(A}F\Psi_{BCD)}F = 0
\]

where \(\Psi_{ABCD}\) is the Weyl spinor, a completely symmetric spinor which is the spinorial equivalent of the Weyl tensor. This condition restricts the form of the Weyl spinor as it requires that \(\Psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}\).

Algebraically general Killing spinors

In the case that the Killing spinor \(\kappa_{AB}\) is algebraically general, we can use the principal spinors \(\alpha_A\) and \(\beta_B\) to form a normalised spin dyad which we will denote by \(\{o^A, \iota^B\}\) and such that \(o_A\iota^A = 1\). The Killing spinor \(\kappa_{AB}\) is then expanded in terms of the basis as

\[
\kappa_{AB} = \kappa o_{(A}\iota_{B)}
\]

for some factor of proportionality \(\kappa\). Due to equation (4), the Weyl spinor can be expanded in a similar way as

\[
\Psi_{ABCD} = \psi o_{(A}\iota_{B}\iota_{C}\iota_{D)}
\]

for some factor of proportionality \(\psi\).

The substitution of expression (5) in the Killing spinor equation (3) implies restrictions on the Newman-Penrose (NP) spin connection coefficients. Namely, one has that

\[
k = \lambda = \nu = \sigma = 0,
\]

consistent with the fact that the spacetime is, at least, of Petrov type D.
2.2 The Killing vector associated to a Killing spinor

A Killing spinor $\kappa_{AB}$ can be used to define the spinorial counterpart $\xi_{AA'}$ of a (possibly complex) vector via the relation

$$\xi_{AA'} \equiv \nabla^C A^{A'} \kappa_{AC}. \quad (7)$$

It can be shown, using the Killing spinor equation (3) and commuting covariant derivatives, that $\xi_{AA'}$ satisfies the equation

$$\nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'} = -6 \kappa_{(A} \Phi_{B)C} A^{A'} B^{B'}. \quad (8)$$

Therefore, if

$$\kappa_{(A} \Phi_{B)C} A^{A'} B^{B'} = 0 \quad (9)$$

then $\xi_{AA'}$ is the spinorial counterpart of a (possibly complex) Killing vector in the spacetime. In what follows, we call condition (8) the matter alignment condition. In the particular case of an electrovacuum spacetime the matter alignment condition takes the form

$$\kappa_{(A} \phi_{B)C} = 0 \quad (10)$$

implying that the spinors $\kappa_{AB}$ and $\phi_{AB}$ are proportional to each other. Thus, in terms of the basis dyad $\{o, i\}$ used to express equation (5) one can write

$$\phi_{AB} = \varphi_{(A^i B)} \quad (10)$$

with $\varphi$ a proportionality constant.

As discussed in [19], the notion of a Lie derivative is, in general, not well defined for spinors. However, in the case of a Hermitian spinor $\xi^{AA'}$ associated to a real Killing vector, there exists a consistent expression which can be used to obtain the spinorial counterpart of the condition $\mathcal{L}_\xi F_{ab} = 0$, stating that the Killing vector $\xi^a$ is also a symmetry of the Faraday tensor —namely:

$$\mathcal{L}_\xi \phi_{AB} \equiv \xi^{CC'} \nabla_{CC'} \phi_{AB} + \phi_{C(A} \nabla_{B)C} \xi^{CC'}. \quad (11)$$

The Maxwell spinor will be said to inherit the symmetry generated by the Killing vector $\xi^a$ if $\mathcal{L}_\xi \phi_{AB} = 0$ —recall that the Maxwell spinor $\phi_{AB}$ is related to the Faraday tensor via the relation

$$\mathcal{F}_{AA'BB'} = 2 \phi_{AB} \epsilon_{A'B'}.$$

Remark. In Section 2.5.3 it will be shown that in an electrovacuum spacetime $\mathcal{M}, g, F$ endowed with a Killing spinor $\kappa_{AB}$ such that $\xi_{AA'}$ is Hermitian and $\phi_{AB}$ and $\kappa_{AB}$ satisfy the alignment condition (9) then $\phi_{AB}$ inherits the symmetry of the spacetime.

2.3 Relation to Killing-Yano tensors

If a spacetime $\mathcal{M}, g, F$ admits a Killing spinor $\kappa_{AB}$, and the vector $\xi^{AA'}$ defined by (7) satisfies $\xi^{AA'} = \bar{\xi}^{AA'}$ (i.e. is a real vector), then one can construct a real, valence-2 antisymmetric tensor $\bar{Y}_{ab}$ as the tensorial counterpart of the spinorial relation

$$Y_{AA'BB'} \equiv i(\kappa_{AB} \epsilon_{A'B'} - \bar{\kappa}_{A'B'} \epsilon_{AB})$$

which, as a consequence of (3), satisfies the Killing-Yano equation

$$\nabla_{(a} Y_{b)c} = 0.$$

Such a tensor is called a Killing-Yano tensor. Conversely, if a spacetime admits a Killing-Yano tensor $Y_{ab}$, one can construct a valence-2 symmetric spinor $\kappa_{AB}$ from the relation

$$\kappa_{AB} \equiv -\frac{i}{4} \epsilon^{AA'BB'} (Y_{AA'BB'} + iY^*_{AA'BB'}).$$

which satisfies the Killing spinor equation (3) —see e.g. [19], Section 6.7 page 107; also [17].

Remark. The existence of a Killing-Yano tensor for the Kerr-Newman spacetime is a key ingredient to show the integrability of the Hamilton-Jacobi equations for geodesic motion, and the separability of the Maxwell equations and the Dirac equation on the Kerr-Newman spacetime —see e.g. [14] or [21] for further details.
2.4 The Killing form

In the reminder of this section assume that the matter alignment condition (8) is satisfied, so that \( \xi_{AA'} \) is the spinorial counterpart of a Killing vector. Moreover, assume that \( \xi_{AA'} \) is a Hermitian spinor so that, in fact, it is the spinorial counterpart of a real vector. Then, define the spinorial counterpart of the Killing form of \( \xi^a \), namely

\[
H_{ab} \equiv \nabla_{[a} \xi_{b]} = \nabla_a \xi_b
\]

by

\[
H_{AA'BB'} \equiv \nabla_{AA'} \xi_{BB'}.
\]

As a consequence of the antisymmetry in the pairs \( AA' \) and \( BB' \) it can be decomposed into irreducible parts as

\[
H_{AA'BB'} = \eta_{AB} \epsilon_{A'B'} + \bar{\eta}_{A'B'} \epsilon_{AB}
\]

where \( \eta_{AB} \) is a symmetric spinor — the Killing form spinor. In the sequel, we will require the self-dual part of \( H_{AA'BB'} \), denoted by \( \mathcal{H}_{AA'BB'} \), and defined by

\[
\mathcal{H}_{AA'BB'} \equiv H_{AA'BB'} + i H^*_{AA'BB'}.
\]

A direct calculation then yields

\[
\mathcal{H}_{AA'BB'} = 2 \eta_{AB} \epsilon_{A'B'}.
\]

Using equation (12), the spinor \( \eta_{AB} \) can be expressed in terms of the Killing vector as

\[
\eta_{AB} = \frac{1}{2} \nabla_{AA'} \xi_{B'} A'.
\]

Then, by using (7), this can be expanded in terms of the Killing spinor so that

\[
\eta_{AB} = -\frac{3}{4} \Psi_{ABCD} \kappa^CD.
\]

Expansions for the algebraically general case

Assuming, now that \( \kappa_{AB} \) is algebraically general, using the basis expansions of both \( \kappa_{AB} \) and \( \Psi_{ABCD} \), we can find the basis expansion of \( \eta_{AB} \):

\[
\eta_{AB} = \frac{1}{4} \kappa \psi o(A'B) = \eta o(A'B)
\]

where

\[
\eta = \frac{1}{4} \kappa \psi.
\]

2.5 The Ernst forms and potentials

Throughout this section let \( \xi^a \) denote a real Killing vector on the electrovacuum spacetime \( (\mathcal{M}, g) \). A well-known consequence of the Killing equation

\[
\nabla_a \xi_b + \nabla_b \xi_a = 0
\]

and the definition of the Riemann tensor in terms of commutators of covariant derivatives is that

\[
\nabla_a \nabla_b \xi_c = R_{cba} d \xi_d.
\]

The Ernst form of the Killing vector \( \xi^a \) is defined as

\[
\chi_a = 2 \xi^b \mathcal{H}_{ba}
\]

Several properties of the Ernst form follow from the identity (17) recast as

\[
\nabla_a \mathcal{H}_{bc} = R_{cba} d \xi_d
\]
where \( \mathcal{R}_{abcd} \) denotes the self-dual Riemann tensor. From expression (19) it follows, using the identity
\[
^{*}R_{[abc]d} = \frac{1}{3} \varepsilon_{abce} R_{e d},
\]
that
\[
\nabla_{[a} \mathcal{H}_{bc]} = \frac{1}{3} \varepsilon_{abce} R_{e d} \xi^{d}, \quad \nabla^{a} \mathcal{H}_{ab} = -R_{ba} \xi^{a}.
\]
A further computation using the above identities and the definition of the Ernst form, equation (18), yields
\[
\nabla_{a} \chi_{b} - \nabla_{b} \chi_{a} = -2 \varepsilon_{cabc} R_{e d} \xi_{d}^{e}.
\]
(20)

2.5.1 The vacuum case

In vacuum \( \mathcal{R}_{abcd} = \mathcal{C}_{abcd} \), where \( \mathcal{C}_{abcd} \) denotes the self-dual Weyl tensor, and so from the symmetries of the Weyl tensor one concludes that
\[
\nabla_{a} \chi_{b} - \nabla_{b} \chi_{a} = 0.
\]
Consequently, in vacuum the Ernst form is closed and thus, locally exact so that there exists a scalar, the Ernst potential \( \chi \), such that
\[
\chi_{a} = \nabla_{a} \chi.
\]

Let now \( \xi_{AA'} \) denote the (Hermitian) spinorial counterpart of the real Killing vector \( \xi^{a} \). If \( \xi_{AA'} \) arises from a Killing spinor through relation (7), it follows from the spinor decomposition of \( \mathcal{H}_{AA'B'B'} \) that the spinorial counterpart \( \chi_{AA'} \) of the Ernst form \( \chi_{a} \) is given by
\[
\chi_{AA'} = 4 \eta_{AB} \xi_{A'}^{B},
\]
\[
= 3 \kappa^{CF} \Psi_{ABCF} \nabla_{DA'} \kappa_{DB}.
\]

2.5.2 The electrovacuum case

In the electrovacuum case the Ernst form is no longer exact — cf. equation (20). However, if the Faraday tensor inherits the symmetry of the spacetime — i.e. \( L_{\xi} F_{ab} = 0 \) — then it is possible to construct a further 1-form, the so-called electromagnetic Ernst form, which can be shown to be closed. In analogy to the definition in (18) one sets
\[
\varsigma_{a} \equiv 2 \xi^{b} F_{ba}.
\]
(21)

A computation then shows that
\[
\nabla_{a} \varsigma_{b} - \nabla_{b} \varsigma_{a} = 2 L_{\xi} F_{ab}.
\]
Thus, as claimed, if \( L_{\xi} F_{ab} = 0 \) then \( \varsigma_{a} \) is closed and, accordingly locally exact so that there exists a scalar, the electromagnetic Ernst potential \( \varsigma \) such that
\[
\varsigma_{a} = \nabla_{a} \varsigma.
\]

The spinorial version of equation (21) can be readily be found to be
\[
\sigma_{AA'} = 4 \phi_{AB} \nabla_{A'} T_{BQ}.
\]

2.5.3 Expansions in the algebraically general case

Consider now the case of an algebraically general spinor \( \kappa_{AB} \) such that \( \xi_{AA'} \) as given by equation (7) is Hermitian. In order to find the full basis expansions of \( \chi_{AA'} \) and \( \varsigma_{AA'} \) we need to calculate
the derivative of the proportionality factor $\kappa$. First, note the expressions for the derivatives of the spin basis vectors in terms of the spin coefficients of the Newman-Penrose formalism:

$$\nabla_{AA'}o_B = -\alpha o_A\partial_B\nu_A + \beta\partial_B\partial_A\nu_A + \gamma o_A\partial_B\partial_A - \epsilon\partial_B\partial_A\nu_A$$

$$\nabla_{AA'}\iota_B = \alpha o_A\partial_B\nu_A + \beta\partial_A\partial_B\nu_A - \gamma o_B\partial_A\partial_A - \epsilon\partial_A\partial_B\nu_A$$

Substituting the basis expansion for the Killing spinor into the Killing spinor equation, using expressions (22a)-(22b) and the relation $\epsilon_{AB} = o_A\iota_B - \iota_A o_B$, we find that

$$\nabla_{AA'}\kappa = \kappa (\mu o_A\partial_A - \pi o_A\nu_A - \tau\iota_A\partial_A + \rho\iota_A\nu_A).$$

The expressions obtained in the previous paragraphs allow one to obtain an expression of the Killing spinor in terms of the spin basis. A calculation starting from the definition (7) readily yields the expression

$$\xi_{AA'} = -\frac{3}{2} \kappa (\mu o_A\partial_A - \pi o_A\nu_A - \tau\iota_A\partial_A + \rho\iota_A\nu_A).$$

If $\xi_{AA'}$ is a Hermitian spinor, i.e. $\xi_{AA'} = \bar{\xi}_{AA'}$, then the previous expression implies

$$\bar{\mu}\kappa = \mu\kappa, \quad \bar{\tau}\kappa = \kappa\pi, \quad \bar{\rho}\kappa = \kappa\rho.$$  \hspace{1cm} (24)

The **vacuum case**. Using the previous expression along with the basis expansions for $\kappa_{AB}$ and $\Psi_{ABCD}$, in vacuum, the Ernst form can be expanded as

$$\chi_{AA'} = \frac{3}{4} \kappa^2 \psi (\mu o_A\partial_A - \pi o_A\nu_A + \tau\iota_A\partial_A - \rho\iota_A\nu_A).$$

(25)

Intuitively, one would expect it should be possible to express the Ernst form $\chi$ in terms of the scalars $\kappa$ and $\psi$. As it will be seen in Section 4, the characterisation of the Kerr spacetime given by Theorem 1 suggests that a combination of the form $\epsilon + \frac{3}{4} \kappa^2 \psi$ with $\epsilon$ a constant is a suitable candidate. In order to compute the derivative of this expression one needs an expression for $\nabla_{AA'}\psi$. The latter can be obtained from the vacuum Bianchi identity

$$\nabla^A A' \Psi_{ABCD} = 0.$$  

Substituting the basis expansion for the Weyl spinor into the above relation, using equations (22a) and (22b), collecting terms and finally making use of $\epsilon_{AB} = o_A\iota_B - \iota_A o_B$ one obtains

$$\nabla_{AA'}\psi = -3\psi (\mu o_A\partial_A - \pi o_A\nu_A + \tau\iota_A\partial_A - \rho\iota_A\nu_A).$$

(26)

Combining the latter with expression (23) for $\nabla_{AA'}\kappa$ one finds that

$$\nabla_{AA'} \left( \epsilon - \frac{3}{4} \kappa^2 \psi \right) = \chi_{AA'}$$

so that one can indeed set

$$\chi = \epsilon - \frac{3}{4} \kappa^2 \psi \quad \text{for some} \quad \epsilon \in \mathbb{C}.$$  

This expression can be simplified using the following observation: combining expressions for $\nabla_{AA'}\psi$ and $\nabla_{AA'}\kappa$ given by equations (26) and (23), respectively, one finds that

$$\nabla_{AA'} (\kappa^3 \psi) = 0;$$

accordingly, the combination $\kappa^3 \psi$ is a constant. Therefore, one has that

$$\kappa^3 \psi = 2\mathfrak{N}$$  \hspace{1cm} (27)
with $\mathfrak{M}$ a (possibly complex) constant and one has

$$\chi = \epsilon - \frac{3\mathfrak{M}}{4\kappa}. \quad (28)$$

**The electrovacuum case.** From the electrovacuum Bianchi identity in the form (2) a calculation yields

$$\nabla_{AA'}\psi = -3(\psi + 2\varphi \ddot{\varphi}) \mu A \bar{\ddot{A}}_{A'} + 3(\psi - 2\varphi \ddot{\varphi}) \pi o A \bar{\ddot{A}}_{A'}.$$

Similarly, from the Maxwell equations (1) and the derivatives of the basis vectors given by equations (22a) and (22b) one finds that

$$\nabla_{AA'}\varphi = -2\varphi (\mu A \ddot{A} - \pi o A \bar{\ddot{A}}_{A'} + \tau l A \ddot{A}_{A'} - \rho l A \bar{\ddot{A}}_{A'}). \quad (29)$$

Thus, a further calculation using the previous expressions yields the following explicit expression for the electromagnetic Ernst potential:

$$\zeta_{AA'} = 3\kappa \varphi (\mu o A \ddot{A}_{A'} - \pi o A \bar{\ddot{A}}_{A'} + \tau l A \ddot{A}_{A'} - \rho l A \bar{\ddot{A}}_{A'}).$$

In the electrovacuum case, assuming an algebraically general Killing spinor and that the Maxwell spinor and the Killing spinor satisfy the matter alignment condition (9), the characterisation of the Kerr-Newman spacetime given in Theorem 3 suggests an expression for $\zeta$ in terms of the scalars $\kappa$, $\psi$ and $\varphi$—namely $c' - \kappa \bar{\psi}/2 \bar{\varphi}$ with $c'$ a constant. Combining the above expressions one concludes that

$$\nabla_{AA'} \left( c' - \frac{\kappa \bar{\psi}}{2 \varphi} \right) = \zeta_{AA'}, \quad (30)$$

so that one can set

$$\zeta = c' - \frac{\kappa \bar{\psi}}{2 \varphi} \quad \text{for some} \quad c' \in \mathbb{C}. \quad (31)$$

Moreover, combining expression (23) for $\nabla_{AA'}\kappa$ with (29) one concludes that

$$\nabla_{AA'} (\kappa^2 \varphi) = 0$$

meaning the combination $\kappa^2 \varphi$ is constant. Thus, there exists a (possibly complex) constant $\Omega$ such that

$$\kappa^2 \varphi = \Omega. \quad (32)$$

In the electrovacuum case the relation between the scalars $\kappa$ and $\psi$ takes a more complicated form than in vacuum—cf. equation (27). Given a complex constant $\mathcal{C}'$, a calculation using expressions (23), (29) and relation (32) shows that

$$\nabla_{AA'} \left( \frac{\mathcal{C}}{\kappa} + \kappa^3 \psi \right) = -\left( \frac{6|\Omega|^2 \kappa \mu}{\kappa^2} + \frac{\mathcal{C}\mu}{\kappa} \right) o A \ddot{A}_{A'} - \left( \frac{6|\Omega|^2 \kappa \pi}{\kappa^2} + \frac{\mathcal{C}\pi}{\kappa} \right) o A \bar{\ddot{A}}_{A'}.$$

If the spinor $\xi_{AA'}$ is assumed to be Hermitian, then the previous expression reduces to

$$\nabla_{AA'} \left( \frac{\mathcal{C}}{\kappa} + \kappa^3 \psi \right) = -\frac{\kappa (\mathcal{C} + 6|\Omega|^2)}{\kappa^2} \left( \mu o A \ddot{A}_{A'} + \pi o A \bar{\ddot{A}}_{A'} - \tau l A \ddot{A}_{A'} - \rho l A \bar{\ddot{A}}_{A'} \right).$$

Thus, if one chooses $\mathcal{C} = -6|\Omega|^2$, then the combination $\mathcal{C}/\kappa + \kappa^3 \psi$ is a constant—that is, there exists $\mathfrak{M}' \in \mathbb{C}$ such that

$$\kappa^3 \psi - \frac{6|\Omega|^2}{\kappa} = \mathfrak{M}'.$$
Thus, the scalar $\psi$ can be expressed solely in terms of $\kappa$ as
\[
\psi = \frac{1}{\kappa^3} \left( \mathcal{M} + \frac{6|\Omega|^2}{\kappa} \right).
\]
(34)

Note that when the Maxwell field vanishes, then the constant $\Omega$ also vanishes and this equation reduces to the vacuum case given by (27).

Finally, it is observed that expanding expression (11) in terms of the spinor basis $\{o, \iota\}$ and using expressions (15) and (29) one concludes, after a calculation, that
\[
\mathcal{L}_\xi \phi_{AB} = 0
\]
—so that $\phi_{AB}$ inherits the symmetry generated by the Killing spinor $\kappa_{AB}$.

### 2.6 Spacetimes with an algebraically special Killing spinor

In this section we briefly consider electrovacuum spacetimes with an algebraically special Killing spinor. These spacetimes will not play a role in the remainder of this article. The reason for this is the following result:

**Lemma 1.** Let $(M, g)$ be a smooth electrovacuum spacetime with a matter content satisfying the matter alignment condition and admitting a valence-2 Killing spinor $\kappa_{AB}$ such that the associated field $\xi^{AA'}$ is a Hermitian spinor. If $\kappa_{AB}$ is algebraically special (i.e. $\kappa_{AB} = \alpha_A \alpha_B$ for some non-vanishing spinor $\alpha_A$) then $\xi^a = 0$.

**Proof.** It follows directly from the existence of a non-vanishing algebraically special Killing spinor that the spacetime $(M, g)$ must be of Petrov type N —see equation (6). That is, one has that
\[
\Psi_{ABCD} = \psi \alpha_A \alpha_B \alpha_C \alpha_D
\]
(35)

for some function $\psi$. As the matter alignment condition holds by assumption, the Hermitian spinor $\xi^{AA'}$ is the spinorial counterpart of a real Killing vector $\xi^a$. The content of the Killing form of $\xi^a$ is encoded in the symmetric spinor $\eta_{AB}$. Substituting the expansions (35) and $\kappa_{AB} = \alpha_A \alpha_B$ into equation (14), it follows directly that $\eta_{AB} = 0$. Thus, the Killing form $H_{ab}$ of $\xi^a$ vanishes. Accordingly, $\xi^a$ is a covariantly constant vector on $(M, g)$ —i.e. one has
\[
\nabla_a \xi^b = 0.
\]
(36)

In order to further investigate the consequences of this observation we introduce a normalised spin dyad $\{o^A, i^A\}$ with $o_A = \alpha_A$ and $o_A i^A = 1$. One can then write
\[
\kappa_{AB} = o_A o_B, \quad \phi_{AB} = \varphi o_A o_B.
\]

Substituting the first of the above expressions into the Killing spinor equation $\nabla_{A'}(\kappa_{BC}) = 0$ one finds that
\[
\gamma = \alpha = \sigma = \kappa = 0, \quad \rho + \epsilon = 0, \quad \tau + \beta = 0.
\]
(37)

Moreover, one finds that the Hermitian spinor $\xi^{AA'}$ can be expressed as
\[
\xi^{AA'} = -3\beta o_A \bar{o}_{A'} + 3\epsilon o_A i_{A'}.
\]

The spinorial version of equation (36) implies $D\xi^{AA'} = 0$, $\Delta\xi^{AA'} = 0$, $\delta\xi^{AA'} = 0$ and $\tilde{\delta}\xi^{AA'} = 0$. In particular, from $\Delta\xi^{AA'} = 0$ and $\delta\xi^{AA'} = 0$, expanding in terms of the basis one finds that $\beta \tau = 0$ and $\epsilon \rho = 0$. Combining this expression with the third and fourth conditions in (37) one concludes that
\[
\tau = \beta = \epsilon = \rho = 0.
\]

It follows then that
\[
\xi^{AA'} = 0.
\]
As we want to use the asymptotics of the Killing vector $\xi_{A'A}$ in the characterisation of the Kerr and Kerr-Newman spacetime, we will rule out the algebraically special case and assume that the Killing spinor is algebraically general —i.e. $\kappa_{AB}R^{AB} \neq 0$.

**Remark.** Note that as we have $\Psi_{ABCD} \propto \kappa_{(AB}R^{CD)}$, then the conditions $\Psi_{ABCD} \Psi^{ABCD} \neq 0$, $\Psi_{ABCD} \neq 0$ imply that the Killing spinor is algebraically general and non-zero, i.e. $\kappa_{AB}R^{AB} \neq 0$, $\kappa_{AB} \neq 0$. These two conditions on the curvature are precisely the ones assumed in Theorem 6 of [3], and so the characterisation of Kerr in terms of Killing spinors presented in that article is essentially the same as the one presented here. Despite this, we reproduce the result in this paper for completeness and ease of comparison with the electrovacuum case. We do this using the local result of Mars given in [15], whereas the proof in [3] uses the global result from [16]. In the absence of a generalisation to the electrovacuum case of the characterisation of [16], our analysis of the Kerr-Newman spacetime must make use of the local result by Wong [22].

### 3 Boundary conditions

This section provides a brief discussion of the boundary conditions which will be used in conjunction with the properties of Killing spinors to characterise the Kerr and Kerr-Newman spacetimes.

#### 3.1 Stationary asymptotically flat ends

In the remainder of this article we will be particularly interested in spacetimes with a stationary asymptotically flat 4-end —see e.g. [22].

**Definition 1.** A stationary asymptotically flat 4-end in an electrovacuum spacetime $(\mathcal{M}, g, F)$ is an open submanifold $\mathcal{M}_\infty \subset \mathcal{M}$ diffeomorphic to $I \times (\mathbb{R}^3 \setminus B_R)$ where $I \subset \mathbb{R}$ is an open interval and $B_R$ is a closed ball of radius $R$. In the local coordinates $(t, x^a)$ defined by the diffeomorphism the components $g_{\mu\nu}$ and $F_{\mu\nu}$ of the metric $g$ and the Faraday tensor $F$ satisfy

\[
\begin{align*}
|g_{\mu\nu} - \eta_{\mu\nu}| + |r\partial_\nu g_{\mu\nu}| &\leq Cr^{-1}, \\
|F_{\mu\nu}| + |r\partial_\nu F_{\mu\nu}| &\leq C'r^{-2}, \\
\partial_t g_{\mu\nu} &= 0, \\
\partial_t F_{\mu\nu} &= 0,
\end{align*}
\]

where $C$ and $C'$ are positive constants, $r \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, and $\eta_{\mu\nu}$ denote the components of the Minkowski metric in Cartesian coordinates.

**Remark 1.** It follows from condition (38c) in Definition 1 that the stationary asymptotically flat end $\mathcal{M}_\infty$ is endowed with a Killing vector $\xi^a$ which takes the form $\partial_t$ —a so-called time translation. From condition (38d) one has that the electromagnetic field inherits the symmetry of the spacetime —that is $\mathcal{L}_\xi F = 0$, with $\mathcal{L}_\xi$ the Lie derivative along $\xi^a$.

Of particular interest will be those stationary asymptotically flat ends generated by a Killing spinor:

**Definition 2.** A stationary asymptotically flat end $\mathcal{M}_\infty \subset \mathcal{M}$ in an electrovacuum spacetime $(\mathcal{M}, g, F)$ endowed with a Killing spinor $\kappa_{AB}$ is said to be generated by a Killing spinor if the spinor $\xi_{A'A} \equiv \nabla^B A' \kappa_{AB}$ is the spinorial counterpart of the Killing vector $\xi^a$.

**Remark 2.** Stationary spacetimes have a natural definition of mass in terms of the Killing vector $\xi^a$ that generates the isometry —the so-called Komar mass $m$ defined as

\[
m \equiv -\frac{1}{8\pi} \lim_{r \to 0} \int_{S_r} \epsilon_{abcd} \nabla^c \xi^d dS^{ab},
\]

where $S_r$ is the sphere of radius $r$ centred at $r = 0$ and $dS^{ab}$ is the binormal vector to $S_r$. Similarly, one can define the total electromagnetic charge of the spacetime via the integral

\[
q = -\frac{1}{4\pi} \lim_{r \to \infty} \int_{S_r} F_{ab} dS^{ab}.
\]
Remark 3. In the asymptotic region the components of the metric can be written in the form

\[ g_{00} = 1 - \frac{2m}{r} + O(r^{-2}), \]
\[ g_{0\alpha} = \frac{4\epsilon_{\alpha\beta\gamma}S^\beta x^\gamma}{r^3} + O(r^{-3}), \]
\[ g_{\alpha\beta} = -\delta_{\alpha\beta} + O(r^{-1}), \]

where \( m \) is the Komar mass of \( \xi^a \) in the end \( M_\infty \), \( \epsilon_{\alpha\beta\gamma} \) is the flat rank 3 totally antisymmetric tensor and \( S^\beta \) denotes a 3-dimensional tensor with constant entries. For the components of the Faraday tensor one has that

\[ F_{0\alpha} = \frac{q}{r^2} + O(r^{-3}), \]
\[ F_{\alpha\beta} = O(r^{-3}) \]

—see e.g. [20]. Thus, to leading order any stationary electrovacuum spacetime is asymptotically a Kerr-Newman spacetime.

Remark 4. In the case of the exact Kerr-Newman spacetime with mass \( m \), angular momentum \( a \) and charge \( q \) a NP frame \( \{l^a, n^a, m^a, \bar{m}^a\} \) with associated spin dyad \( \{o^A, \iota^A\} \) such that the scalars \( \kappa, \varphi \) and \( \psi \) introduced in equations (5), (10) and (6), respectively, take the form

\[ \kappa = \frac{2}{3}(r - ia \cos \theta), \]
\[ \varphi = \frac{q}{(r - ia \cos \theta)^2}, \]
\[ \psi = \frac{6}{(r - ia \cos \theta)^3} \left( \frac{q^2}{r + ia \cos \theta} - m \right), \]

where \( r \) denotes the standard Boyer-Lindquist radial coordinate —see [1] for more details.

3.2 Killing spinor and Killing vector asymptotics

In general, the spinor \( \xi_{AA'} \) obtained from a Killing spinor \( \kappa_{AB} \) using formula (7) is not Hermitian. It is, however, well know that for the Kerr-Newman spacetime \( \xi_{AA'} \) is indeed the spinorial counterpart of a real Killing vector \( \xi^a \)—see e.g. [1]. More generally, this observation applies to any electrovacuum spacetime with a stationary asymptotically flat end. To see this, we first notice the following:

Lemma 2. Let \((\mathcal{M}, g, F)\) be a smooth electrovacuum spacetime with a stationary asymptotically flat end \( \mathcal{M}_\infty \), admitting a complex Killing vector field \( \xi^a \). If \( \xi^a \) tends to a time translation at infinity in \( \mathcal{M}_\infty \), then \( \xi^a \) is in fact a real vector in \( \mathcal{M}_\infty \).

Proof. The complex Killing vector can be written \( \xi^a = \xi_1^a + i\xi_2^a \) for two real vectors \( \xi_1^a, \xi_2^a \), which are also Killing vectors by linearity of the Killing vector equation. As a time translation \( (\partial_t)^a \) is a real vector, we have \( \xi_1^a \rightarrow (\partial_t)^a \) and \( \xi_2^a \rightarrow 0 \) as \( r \rightarrow \infty \) in the asymptotically flat end \( \mathcal{M}_\infty \). However, it is well known that there are no non-trivial real Killing vectors which vanish at infinity —see e.g. [6, 7]. Therefore, \( \xi_2^a = 0 \) on \( \mathcal{M}_\infty \), and \( \xi^a = \xi_1^a \) is a real Killing vector. \( \square \)

Therefore, by assuming that the Killing vector \( \xi^a \) is asymptotically a time translation, then we can drop the assumption requiring its spinorial equivalent \( \xi_{AA'} \) to be a Hermitian spinor. In fact, it is possible to replace this condition on the Killing vector with an asymptotic condition on the Killing spinor, as described in the following proposition:

Proposition 1. Let \((\mathcal{M}, g, F)\) denote an electrovacuum spacetime with a stationary asymptotically flat end \( \mathcal{M}_\infty \) generated by a Killing spinor \( \kappa_{AB} \). Then the functions \( \kappa, \varphi \) and \( \psi \) as defined
by equations (5), (6) and (10) satisfy

\begin{align*}
\kappa &= \frac{2}{3} r + O(1), \\
\varphi &= \frac{q}{r^2} + O(r^{-3}), \\
\psi &= -\frac{6m}{r^3} + O(r^{-4}).
\end{align*}

Moreover, one has that

\[ \xi^2 \equiv \xi_{AA'}\xi^{AA'} = 1 + O(r^{-1}). \]

**Proof.** The analysis in [20] shows that to leading order the electrovacuum spacetime \((M, g, F)\) coincides on \(M_\infty\) with the Kerr-Newman spacetime. Thus, the expansions for the fields \(\kappa, \varphi\) and \(\varphi\) must coincide, to leading order with their expressions for the Kerr-Newman spacetime —see [1].

\[ \square \]

4 **Characterisations of the Kerr spacetime**

The motivation behind our analysis is the following theorem by M. Mars —see [16):

**Theorem 1.** Let \((M, g)\) be a smooth, vacuum spacetime admitting a Killing vector \(\xi^a\) with selfdual Killing form \(\mathcal{H}_{ab}\). Let \(M\) satisfy:

(i) there exists a non-empty region \(M_* \subset M\) where

\[ \mathcal{H}^2 = \mathcal{H}_{ab}\mathcal{H}^{ab} \neq 0; \]

(ii) The selfdual Killing form and the Weyl tensor are related by

\[ C_{abcd} = H \left( \mathcal{H}_{ab}\mathcal{H}_{cd} - \frac{1}{3} \mathcal{H}^2 \mathcal{I}_{abcd} \right) \]  \hspace{1cm} (39)

where

\[ \mathcal{I}_{abcd} = \frac{1}{4} (g_{ac}g_{bd} - g_{ad}g_{bc} + i\epsilon_{abcd}) \]

and \(H\) is a complex scalar function.

Then there exist two complex constants \(l\) and \(c\) such that

\[ H = \frac{6}{c - \chi}, \quad \mathcal{H}^2 = -l(c - \chi)^4. \]

If, in addition, \(c = 1\) and \(l\) is real positive, then \((M, g)\) is locally isometric to the Kerr spacetime.

**Remark 1.** It is important to emphasise that in the above Theorem the existence of the constants \(c\) and \(l\) and the functional dependence of \(H\) and \(\mathcal{H}^2\) with respect to \(\chi\) are a consequence of the hypotheses of the theorem —this should be contrasted with the electrovacuum case in which the existence of the analogue constants needs to be assumed.

**Remark 2.** A particular case of Theorem 1 occurs when \((M, g)\) is a priori assumed to have an stationary asymptotically flat end \(M_\infty\) with the Killing vector \(\xi^a\) tending asymptotically to a time translation at infinity and such that the Komar mass associated to \(\xi^a\) is non-zero. These assumptions ensure that \(\mathcal{H}^2 \neq 0\) in a region of the spacetime —namely, in \(M_\infty\). Thus, one only needs to verify condition (39) to conclude that

\[ H = \frac{6}{1 - \chi} \]
and that the spacetime is locally isometric to the Kerr spacetime —see Theorem 2 in [15].

**Remark 3.** In the subsequent discussion we will make use of the spinorial transcription of the conditions in the previous Theorem. In particular, we notice that the content of the combination

\[ H_{ab}H_{cd} - \frac{1}{3}H^2I_{abcd} \]

can be encoded in terms of the spinor \( \eta_{AB} \) as defined in equation (13) as

\[
\left( 4\eta_{AB}\eta_{CD} - \frac{2}{3}\eta_{EF}\eta_{EF}(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD}) \right) \epsilon_{A'B'C'D'} = 4\eta_{(AB)}\eta_{(CD)}\epsilon_{A'B'C'D'} \]

where the last expression follows from a decomposition in irreducible terms. Thus, condition (39) can be concisely expressed in terms of spinors as

\[
\Psi_{ABCD} = 2H(\eta_{AB}\eta_{CD}) \]

Finally, it is noticed that the condition \( H^2 \neq 0 \) can be expressed as

\[
8\eta_{AB}\eta^{AB} \neq 0.
\]

### 4.1 Killing spinors and the Mars characterisation

In what follows we analyse the extent to which existence of a Killing spinor on a vacuum spacetime implies the hypotheses of the characterisation of Kerr given in Theorem 1. For bookkeeping purposes we explicitly state the assumptions to be made in the remainder of this section:

**Assumption 1.** Let \((\mathcal{M},g)\) be a smooth vacuum spacetime and let \( \mathcal{K} \subset \mathcal{M} \) such that:

(i) on \( \mathcal{K} \) there exists an algebraically general Killing spinor \( \kappa_{AB} \);

(ii) the spinor \( \xi_{AA'} \equiv \nabla_{BA'}\kappa_{AB} \) is on \( \mathcal{K} \) the spinorial counterpart of a real Killing spinor \( \xi^a \) —i.e. \( \xi_{AA'} \) is Hermitian.

Under the above assumptions, it follows from combining the basis expansion for \( \Psi_{ABCD} \) and \( \eta_{AB} \), equations (6) and (15), respectively, that

\[
\Psi_{ABCD} = \frac{16}{\kappa^2\psi}\eta_{(AB)}\eta_{(CD)} .
\]

Thus, hypothesis (ii) of Theorem 1 is satisfied with

\[
H = \frac{8}{\kappa^2\psi} ,
\]

—cf. equation (40). Using the expression for the Ernst potential predicted by the theory of Killing spinors, equation (28), one obtains that

\[
H = \frac{6}{\epsilon - \chi} ,
\]

which is precisely the form for \( H \) predicted by Theorem 1. From this expression one further concludes that

\[
\mathcal{H}^2 = -\frac{nR}{3} \left( \frac{4}{3nR} \right)^3 (\epsilon - \chi)^4 .
\]

This, again, is the form predicted by Theorem 1.

The above observations allow us to formulate the following *Killing spinor version* of Theorem 1:

**Proposition 2.** Let \((\mathcal{M},g)\) denote a smooth vacuum spacetime endowed with a Killing spinor \( \kappa_{AB} \) with \( \kappa_{AB}k^{AB} \neq 0 \) such that the spinor \( \xi_{AA'} \equiv \nabla_{BA'}\kappa_{AB} \) is Hermitian. Then there exist two complex constants \( l \) and \( c \) such that

\[
\mathcal{H}^2 = -l(\epsilon - \chi)^4 .
\]

If, in addition, \( \epsilon = 1 \) and \( l \) is real positive, then \((\mathcal{M},g)\) is locally isometric to the Kerr spacetime.
4.1.1 A characterisation using asymptotic flatness

In this subsection we simplify the previous discussion by assuming that the set \( K \subset M \) contains a stationary asymptotically flat end with the Killing spinor \( \kappa_{AB} \) generating the time translation Killing vector.

From Proposition 1 it readily follows that
\[
(c - \chi)^4 = \frac{16m^4}{r^4} + O(r^{-5}).
\]

Similarly, one has, using equation (15), that
\[
\mathcal{H}^2 = -4\eta^2 = -\frac{4m^2}{r^4} + O(r^{-5}).
\]

Thus, by consistency with the required asymptotic behaviour of the Ernst potential, one has to set \( c = 1 \) and the constant \( l \) in Proposition 2 is given by \( l = 1/4m^2 \).

We can summarise the discussion of the previous section in the following:

**Theorem 2.** Let \((M, g)\) be a smooth vacuum spacetime containing a stationary asymptotically flat end \( M_{\infty} \) generated by a Killing spinor \( \kappa_{AB} \). Let the Komar mass associated to the Killing vector \( \xi_{AA'} = \nabla_{A'}^B \kappa_{AB} \) in \( M_{\infty} \) be non-zero. Then, \((M, g)\) is locally isometric to the Kerr spacetime.

**Remark.** As observed in [1] the requirement on the non-vanishing of the Komar mass can be replaced by an assumption on the existence of a horizon.

5 Characterisations of the Kerr-Newman spacetime

In this section we discuss characterisations of the Kerr-Newman spacetime through Killing spinors. Our starting point is the following result —see [22]:

**Theorem 3.** Let \((M, g, F)\) be a smooth, electrovacuum spacetime admitting a real Killing vector \( \xi^a \). Assume further that \( \xi^a \) is timelike somewhere in \( M \) and that \( F_{ab} \) is non-null on \( M \) (i.e. \( F^2 \equiv F_{ab}F^{ab} \neq 0 \)) and that it inherits the symmetry of the spacetime —i.e.

\[
\mathcal{L}_\xi F_{ab} = 0. \tag{41}
\]

Assume, furthermore, that there exists a complex scalar \( P \), a normalisation for \( \varsigma \) and a complex constant \( c_1 \) such that:

\[
P^{-4} = -\varsigma_1^2 F^2, \tag{42a}
\]

\[
\mathcal{H}_{ab} = -\frac{1}{2} \varsigma F_{ab}, \tag{42b}
\]

\[
\mathcal{C}_{abcd} = 3P \left( \frac{1}{2} F_{ab} H_{cd} + \frac{1}{2} F_{ab} H_{cd} - \frac{1}{3} L_{abcd} F_{ef} H^{ef} \right). \tag{42c}
\]

Then there exist complex constants \( c_2 \) and \( c_3 \) such that:

\[
P = \frac{2}{c_2 - \varsigma}, \tag{43a}
\]

\[
4\varsigma^2 - |\varsigma|^2 = c_3. \tag{43b}
\]

If, further, \( c_2 \) is such that \( \varsigma_1 c_2 \) is real and \( c_3 \) is such that \(|c_2|^2 + c_3 = 4\), then \((M, g, F)\) is locally isometric to a Kerr-Newman spacetime.
Remark 1. As in Section 4, we will make use of a reformulation of the conditions in Theorem 3 in spinorial formalism. A direct computation shows that (42a) can be rewritten as

\[ P^{-4} = -8c_1^2 \phi_{AB} \phi^{AB}. \]

Similarly, condition (42b) can be readily expressed in terms of the spinors \( \eta_{AB} \) and \( \varphi_{AB} \) as

\[ \eta_{AB} = -\frac{1}{2} \bar{\zeta} \phi_{AB}, \]

while, finally, equation (42c) is equivalent to

\[ \Psi_{ABCD} = 6P \eta_{(AB} \phi_{CD)}. \]

5.1 Killing spinors and Wong’s characterisation

In this section we investigate some further consequences of the existence of Killing spinors on electrovacuum spacetimes. For easy reference we state the assumptions to be made in what follows:

Assumption 2. Let \((M, g)\) be a smooth electrovacuum spacetime and let \(K \subset M\) such that:

(i) on \(K\) there exists an algebraically general Killing spinor \(\kappa_{AB}\);

(ii) the spinor \(\xi_{AA'} \equiv \nabla^{B_A} \kappa_{AB}\) is on \(K\) the spinorial counterpart of a real Killing spinor \(\xi^a\) —i.e. \(\xi_{AA'}\) is Hermitian;

(iii) the Killing spinor \(\kappa_{AB}\) and the Maxwell spinor \(\varphi_{AB}\) satisfy the alignment condition \(\kappa_{(AQ} \varphi_{B)Q} = 0\) —that is, they are proportional.

As already discussed in Section 2.5.3, under the above assumptions it follows that \(\mathcal{L}_\xi \varphi_{AB} = 0\) which, in turn, implies that \(\mathcal{L}_\xi \mathcal{F}_{ab} = 0\). Thus, the electromagnetic field inherits the symmetry generated by the Killing spinor \(\kappa_{AB}\).

From the discussion in Sections 2.1 and 2.4 it follows that

\[ \Psi_{ABCD} = \psi_{(AB} \phi_{CD)} , \quad \eta_{(AB} \phi_{CD)} = \eta \varphi_{0(A} \phi_{B)C} \psi_{D)}. \]

Thus, the spinorial version of condition (42c) in Theorem 3 is satisfied with a proportionality function \(P\) given by

\[ P = -\frac{2}{3c'} \varsigma. \]

Now, making use of expressions (31), (32) and (34) to rewrite \(P\) in terms of the electromagnetic Ernst potential one finds that

\[ P = \frac{2}{c_2 - \varsigma}, \quad c_2 = c' - \frac{2R}{2Q}. \]

Thus, under the Assumptions 2, hypothesis (42c) and conclusion (43a) in Theorem 3 are satisfied.

Moreover, from the discussion in Section 2.5.3 it follows that the spinors \(\eta_{AB}\) and \(\varphi_{AB}\) are proportional to each other with a proportionality function \(\zeta\) given by

\[ \zeta = -\frac{\varsigma \psi}{2\varphi}. \]

The calculations of Section 2.5, cf. equation (30) in particular, show that \(\varsigma\) satisfies the properties to be expected from the electromagnetic Ernst potential. Therefore, by setting the constant \(c'\) in the definition of \(\varsigma\) given by (31) to zero (and thereby fixing the normalisation of the potential), condition (42b) is satisfied. A similar remark holds for condition (42a) with the constant \(c_1\) given by

\[ c_1^2 = \frac{81}{64} Q^2. \]
In the presence of a Killing spinor, the norm $\xi^2 \equiv \xi_a \xi^a$ of the associated Killing vector is related to the electromagnetic form $\varsigma$. To see this consider
\[
\nabla_{AA'}\xi^2 = 2\xi^B\nabla_{AA'}\xi_{BB'} = -2\eta_{AB}\xi^B_{A'} - 2\tilde{\eta}_{A'B'}\xi^B_{A'},
\]
where in the second line it has been used that
\[
\nabla_{AA'}\xi_{BB'} = \eta_{AB}\epsilon_{A'B'} + \tilde{\eta}_{A'B'}\epsilon_{AB}.
\]
As the spinors $\eta_{AB}$ and $\phi_{AB}$ are proportional to each other, cf. the previous paragraph, one can write
\[
\nabla_{AA'}\xi^2 = \tilde{\varsigma}\xi_{BB'}\phi_{AB} + \varsigma\xi^B_{A'}\phi_{A'B'} = \frac{1}{4}(\varsigma\nabla_{BB'}\varsigma + \varsigma\nabla_{BB'}\tilde{\varsigma}) = \frac{1}{4}\nabla_{BB'}|\varsigma|^2.
\]
Thus, locally there exists a constant $c_3$ such that
\[
4\xi^2 - |\varsigma|^2 = c_3.
\]
Thus, conclusion (43b) in Theorem 3 is also a consequence of the existence of a Killing spinor.

We can summarise the discussion of this section with the following Killing spinor version of Theorem (3):

Proposition 3. Let $(\mathcal{M}, g, F)$ denote a smooth electrovacuum spacetime satisfying the matter alignment condition, endowed with a Killing spinor $\kappa_{AB}$ with $\kappa_{AB}\kappa^{AB} \neq 0$ such that the spinor $\xi_{AA'} \equiv \nabla^B_{A'}\kappa_{AB}$ is Hermitian. Then there exist two constants $c_2$ and $c_3$ such that
\[
(c_2 - \varsigma)^4 = -\left(\frac{9}{8}\Omega\right)^2 \mathcal{F}^2, \quad 4\xi^2 - |\varsigma|^2 = c_3.
\]
If, further, $c_2$ is such that $\bar{c}_2\Omega$ is real and $c_3$ is such that $|c_2|^2 + c_3 = 4$, then $(\mathcal{M}, g, F)$ is locally isometric to a Kerr-Newman spacetime.

5.1.1 A characterisation using asymptotic flatness

In this section we assume that the domain $\mathcal{K} \subset \mathcal{M}$ considered in the Assumptions 3 contains an stationary asymptotic flat end with the Killing spinor $\kappa_{AB}$ generating the time translation Killing vector. We use this further assumption to determine the values of the constants in Proposition 3.

Combining the asymptotic expansions of Proposition 1 with the relation (32) one readily concludes that
\[
\Omega = \frac{4}{9} q \in \mathbb{R}.
\]
Similarly, using equation (33) one concludes that
\[
\mathfrak{m}' = -\frac{16}{9} m.
\]
A further computation using equation (31) and (32), respectively, show that
\[
(c_2 - \varsigma)^4 = \left(c_2 - \frac{2m}{q} + O(r^{-1})\right)^4 = \left(\frac{9}{8}\Omega\right)^2 \mathcal{F}^2 = -\frac{q^4}{r^4} + O(r^{-5}).
\]
Thus, for consistency, one has to set
\[
c_2 = \frac{2m}{q}.
\]
—thus, one has that \( \tilde{c}_2 \Omega \in \mathbb{R} \). From the previous discussion it follows that \( \zeta = 2m/q + O(r^{-1}) \) so that, together with \( \zeta^2 = 1 + O(r^{-1}) \) one concludes that \( c_3 \) as defined by equation (43b) is given by

\[
     c_3 = 4 \left( 1 - \frac{m^2}{q^2} \right).
\]

Accordingly one has that \( |c_2|^2 + c_3 = 4 \) as required.

One can summarise the discussion of the previous paragraphs in the following:

**Theorem 4.** Let \((\mathcal{M}, g, F)\) be a smooth, electrovacuum spacetime satisfying the matter alignment condition, with a stationary asymptotically flat end \( M_\infty \) generated by a Killing spinor \( \kappa_{AB} \). Let both the Komar mass associated to the Killing vector \( \xi_{AA'} = \nabla_{A'} \kappa_{AB} \) and the total electromagnetic charge in \( M_\infty \) be non-zero. Then \((\mathcal{M}, g, F)\) is locally isometric to the Kerr-Newman spacetime.

### 6 Applications

The advantage of the Killing spinor characterisation of the Kerr and Kerr-Newman solutions is that the existence of such a spinor is a geometric condition, with only reasonable asymptotic conditions needing to be further assumed for the results presented above. This geometric condition can be converted into initial data for a spacelike Cauchy surface, in a way compatible with the constraint equations. This can then be exploited to construct a geometric invariant for the initial data set, which parametrises the deviation of the resulting global development of the initial data set from the exact Kerr or Kerr-Newman solution. Various versions of this construction analysis have been considered in [2, 3, 4, 5] for the vacuum case. A generalisation of these constructions to the electrovacuum case is given in [9].

Finally we point out that the results of this article suggest that the characterisations of the Kerr-Newman spacetime given by Wong in [22] can be improved to an optimal theorem in which condition (42a) in Theorem 3 is a consequence of the other hypothesis. An optimal result of this type is desirable if one is to attempt to use this type of characterisations to construct an alternative approach to the uniqueness of black holes.

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