Single and two-mode mechanical squeezing of an optically levitated nanodiamond via dressed-state coherence

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Abstract

Nonclassical states of macroscopic objects are promising for ultrasensitive metrology as well as testing quantum mechanics. In this work, we investigate dissipative mechanical quantum state engineering in an optically levitated nanodiamond. First, we study single-mode mechanical squeezed states by magnetically coupling the mechanical motion to a dressed three-level system provided by a nitrogen-vacancy center in the nanoparticle. Quantum coherence between the dressed levels is created via microwave fields to induce a two-phonon transition, which results in mechanical squeezing. Remarkably, we find that in ultrahigh vacuum quantum squeezing is achievable at room temperature with feedback cooling. For moderate vacuum, quantum squeezing is possible with cryogenic temperature. Second, we present a setup for two mechanical modes coupled to the dressed three levels, which results in two-mode squeezing analogous to the mechanism of the single-mode case. In contrast to previous works, our study provides a deterministic method for engineering macroscopic squeezed states without the requirement for a cavity.

1. Introduction

Optical levitation has been a powerful tool for trapping and manipulating small particles since its inception \cite{1}. Recent advances with optically levitated dielectric microscopic and nanoscopic particles have provided a promising platform for optomechanics \cite{2} with multiple degrees of freedom and ultrahigh mechanical quality factors \cite{3–6}. Motivated by testing quantum mechanics at the macroscopic scale and by potential applications in nanoscale sensing, many studies have been performed on the center of mass motion cooling \cite{7–13}, quantum state preparation \cite{14}, non-equilibrium dynamics \cite{15–18}, and ultra-sensitive metrology \cite{19–22} of optically trapped nanoparticles.

Recently, levitated nanoparticles with internal degrees of freedom, such as the nitrogen-vacancy (NV) center with a single spin, have been studied theoretically to test quantum wavefunction collapse models \cite{23, 24} and quantum gravity \cite{25} in vacuum. More recently, optical levitation of nanodiamonds in low vacuum has been demonstrated experimentally \cite{26, 27}, paving the way for preparing quantum states of mechanically oscillating levitated nanoparticles.

In this article, we propose a method for creating single- and two-mode squeezed states of mechanical oscillation of an optically levitated single NV center nanodiamond, motivated by the potential for the applications of such states to sensitive metrology \cite{28}. Generally, single-mode mechanical squeezing has been proposed theoretically \cite{29–38} and demonstrated experimentally \cite{39–42} in cavity-based optomechanical systems, for example, by driving an optomechanical cavity with two frequency tones \cite{38–42}. Also, two-mode mechanical squeezing has been studied via mechanisms such as dissipative reservoir engineering \cite{43–45}, quantum measurement backaction \cite{46}, and nondegenerate parametric modulation \cite{47–49}. More specifically, spin–mechanical systems \cite{50} have been studied extensively, using strained-induced coupling \cite{51–54}, or in the presence of a magnetic field gradient \cite{55–57}, for mechanical cooling \cite{57–60}, optomechanical spin control \cite{51, 54}, and mass spectrometry \cite{61}. Recently, single-mode mechanical squeezing was investigated via qubit
measurement [62] and feedback stabilization [63] in a spin-mechanical system. Even more recently, subthermal mechanical quadrature squeezing was reported in a levitated nanoparticle [64].

In the present work, the nanoparticle mechanical motion is coupled to the single NV center spin via a magnetic field gradient, without requiring a cavity [23, 24]. Distinct from the works cited above [62, 63], our method does not require a measurement-based technique, but instead relies on a microwave field-induced spin-state coherence for generating steady-state mechanical squeezing in both the single-mode and two-mode cases. By applying two microwave fields coupling the $|0\rangle$ and $|\pm 1\rangle$ states of the NV center ground-state triplet [57], a dressed three-level system is created to induce a two-phonon transition in the mechanical oscillator, an interesting effect which has not been studied before, to the best of our knowledge, in spin-optomechanical systems. Our work thus promotes the subarea of levitated nanomechanics with a new method of creating macroscopic nonclassical states [23–25].

To arrive at our results, we employ a master equation approach to describe the mechanical motion, by tracing out the spin degree of freedom in the Born–Markov approximation. This approach is enabled by applying optically induced dissipation [65] to the spin triplet states leading to relaxation rates much stronger than the spin–mechanical coupling. For the single-mode case, we find remarkably that quantum squeezing is achievable at room temperature with experimentally achievable ultrahigh vacuum and feedback cooling techniques [13]. For moderate vacuum, quantum squeezing is possible with precooled phonon occupation number. For the two-mode case, we propose a setup such that both modes are coupled to the dressed states in exactly the same way as for the single-mode case. The analytical results for both the single-mode and the two-mode squeezing are equivalent to each other. We also present numerical results in a wide range of parameters for single-mode squeezing, which are applicable to the two-mode case.

The analysis presented using an optically levitated nanodiamond is quite general, therefore the proposal can also be extended to related systems, such as, nanodiamonds using magneto-gravitational traps [66] or Paul traps [67, 68], which avoid optical scattering, or a single NV center coupled to a cantilever [57].

2. Single NV center coupled to one mechanical mode

2.1. The model
We consider a single NV center nanodiamond optically trapped in vacuum and executing harmonic center of mass motion along all three directions in space, as shown in figure 1. A magnetic field $B_x = B_0 x$ with the gradient $B_0$ is applied to couple the mechanical motion and the electron spin of the NV center. The magnetic field is assumed to be lying in the $z$-$y$ plane of the spin axes and making an angle $\varphi$ with the $z$ axis. The Hamiltonian of the system is

$$H = H_m + H_{NV} + H_{\text{int}},$$

$$H_m = \hbar \omega_m d^d \ddot{d},$$

\hspace{1cm} (1)

\hspace{1cm} (2)
\[ H_{\text{NV}} = -\hbar \Delta (|+1\rangle \langle +1| + |-1\rangle \langle -1|) + \frac{\hbar \Omega_0}{2} (|0\rangle \langle +1| + |0\rangle \langle -1| + \text{h.c.}) + \frac{\hbar \Omega_1}{2} (|-1\rangle \langle +1| + |+1\rangle \langle -1|), \]

\[ H_{\text{int}} = \hbar g \cos(\varphi) S_a (d^+ + d) + \hbar g \sin(\varphi) S_d (d^+ + d), \]  

where \( \omega_m \) is the mechanical oscillation frequency determined by the optical trap beam intensity and the nanoparticle mass \( m \) \([8]\), \( g = g_e \mu_B B_0 x_0 \hbar / \hbar \), \( g_2 \approx 2 \) is the Landé factor, \( \mu_B \) is the Bohr magneton, \( x = x_0(d^- + d) \), and \( x_0 = \sqrt{\hbar / 2m\omega_m} \). The creation (annihilation) operator of the mechanical motion along the \( \sigma^- \) axis is \( d^+ (d^-) \). The spin operator components are \( S_y = |+1\rangle \langle +1| - |-1\rangle \langle -1| \), and \( S_y = -i |+1\rangle \langle -1| + |i-1\rangle \langle +1| \). The NV center Hamiltonian has been obtained in the rotating-wave frame with two microwave driving frequencies which couple the \( |0\rangle \) and \( |\pm 1\rangle \) states of the spin-1 system with a detuning \( \Delta \) and Rabi frequency \( \Omega_0 \), and an effective driving field coupling between \( |+1\rangle \) and \( |-1\rangle \) states with a Rabi frequency \( \Omega_1 \), as shown in figure 2 (a). Note here the microwave field \( \Omega_0 \) is used in addition to the field \( \Omega_1 \). By going to the eigenbasis of \( H_{\text{NV}} \), we find that

\[ H = \hbar \omega_{\text{osc}} d^+ d + \hbar \omega_{\text{osc}} |a\rangle \langle a| + \hbar \omega_\text{ph} |b\rangle \langle b| + \hbar \omega_{\text{osc}} |c\rangle \langle c| + \hbar g_2 \cos(\varphi) S_a (d^+ + d) + \hbar g_2 \sin(\varphi) S_d (d^+ + d), \]

where the coupling constants \( g_2 = -g_2 e^{i\varphi} \sin(\theta), g_4 = g_4 e^{i\varphi} \cos(\theta) \), and the dressed states are

|a\rangle = \sin(\theta)|0\rangle + \cos(\theta)|+\rangle,
|b\rangle = |-\rangle,
|c\rangle = \cos(\theta)|0\rangle - \sin(\theta)|+\rangle,

with \(|\pm\rangle = (|+1\rangle \pm |-1\rangle)/\sqrt{2} \), and \( \tan(2\theta) = -\sqrt{2} \Omega_0 / (\Delta - \Omega_1 / 2) \). The eigenvalues of the dressed states are \( \omega_{\text{osc}} = (-\Delta + \Omega_1 / 2 \pm \sqrt{(\Delta - \Omega_1 / 2)^2 + 2\Omega_0^2})^2 \) and \( \omega_{\text{ph}} = -\Delta - \Omega_1 / 2 \), respectively. The dressed states of equation (6) are shown in figure 2 (b), along with the oscillator phonons of energy \( \omega_{\text{osc}} \) which couple to the NV center via the terms in the second line of equation (5). The effective detunings \( \Delta_1 \) and \( \Delta_2 \) will be derived later in the text.

We note that in our model the coupling field \( \Omega_1 \) provides external control of the hybridization and eigenfrequencies of the single spin levels. In the eigenbasis of \( H_{\text{NV}} \), the mechanical motion couples to two transitions of the eigenstates, which is promising for creating mechanical squeezing because of the implied two-phonon transition. A related scheme has been considered for coherent three-level atoms coupled to a cavity field via a two-photon transition for quantum noise quenching and optical field squeezing \([69]\). Finally, we note that the orientation of the magnetic field gradient only adds a phase to the mechanical-spin coupling in the eigenbasis.

### 2.2. Driving-induced dissipation

The electron spin in the NV center is notable for its long coherence time (on the order of 1 ms) even at room temperature \([65, 70, 71]\). In order to induce fast dissipation in the spin system, which is necessary for generating steady state mechanical squeezing, we apply two optical fields with the same Rabi frequency \( \Omega_p \) driving the ground-state spin levels to the excited states \( |E_1\rangle \) and \( |E_2\rangle \) via spin-conserving transitions \([72]\), which de-excite to states \(|\pm 1\rangle\) with a decay rate \( \gamma_1 \), and to the state \(|0\rangle\) with an effective decay rate \( \gamma_0 \), as shown in figure 3. By

![Figure 2](image-url)
considering spin-mechanical couplings \((g_s, g_m)\) and microwave fields \((\Omega_0, \Omega_1)\) much weaker than the optical Rabi frequency \(\Omega_p\), we find the nonzero steady-state density matrix elements in the dressed eigenbasis of \(\{|a\}, \{|b\}, \{|c\}\}\) due to the dissipation mechanism as (see appendix A)

\[
\rho_{bb} = \frac{(\Gamma_1 - \Gamma_0)\Omega_0^2}{(2\Delta - \Omega_0)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0^2},
\]

\[
\rho_{bc} = \frac{-\left((\Delta - \Omega_0/2)/2\right)}{\sqrt{\left((\Delta - \Omega_0/2)^2 + 2\Omega_0^2\right)}} \frac{\Gamma_0(2\Delta - \Omega_0)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0^2}{\Gamma_0(2\Delta - \Omega_0)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0^2} + \frac{1 - \rho_{bb}}{2},
\]

\[
\rho_{bb} = \frac{\Gamma_0}{\sqrt{2}} \frac{\Gamma_1\Omega_0\gamma_0}{(\Delta - \Omega_0)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0^2},
\]

where \(\Gamma_0 = \frac{\Omega_0^2}{\gamma_0^2}(\gamma_1 + \gamma_0)^2\), and \(\Gamma_1 = \frac{\Omega_0^2}{\gamma_0^2}(\gamma_1 + \gamma_0)\). As can be seen from equations (7)–(10), dissipative driving can be used to control the the populations and coherences for the NV dressed states. The mechanical oscillator interacts with the NV spin due to the presence of the magnetic field. In the steady-state, therefore, the mechanical frequency is shifted by the NV spin, while mechanical motion can be engineered via the mechanical-spin interaction through the driving-induced dissipation. We substantiate these statements below.

2.3. The reduced master equation of the mechanical oscillator

In the interaction picture, we can write equation (5) as

\[
\hat{H}_f = -\hat{h}\Delta_0 |a\rangle\langle a| - \hat{h}\Delta_1 |b\rangle\langle b| + \hat{h}\langle g_c|c\rangle\langle b|d^\dagger + g_m|a\rangle\langle b| + \text{h.c.},
\]

where \(\Delta_0 = 2\omega_m - \omega_m, \Delta_1 = \omega_m - \omega_m\), and we have used the rotating-wave approximation. The approximation is valid when \(\Delta_0, \Delta_1, \Gamma_0, \Gamma_1 \ll \omega_m\). We now trace out the spin degree of freedom to obtain the reduced master equation for the mechanical oscillator density matrix \(\rho_m\), which is our system of interest i.e.

\[
\dot{\rho}_m = \text{Tr}_R\left(-\frac{i}{\hbar}[\hat{H}_f, \rho]\right) = -ig_c[d^\dagger, \rho_m] - ig_m[d^\dagger, \rho_m] + \text{h.c.},
\]

where \(\text{Tr}_R\) denotes the trace over the spin degree of freedom. We note that the steady-state density matrix elements \(\rho_{ba}\) and \(\rho_{bc}\), due to the fast dissipation and strong Rabi frequencies, are zero to the lowest order. The first-order perturbation of these quantities are given by the spin–mechanical interaction \(H_f\) in appendix B. By substituting for \(\rho_{bc}\) and \(\rho_{ba}\) in equation (12), we obtain
\[ \rho_m = A_m D[d] \rho_m + A_+ D[d^+] \rho_m - \frac{i\delta}{2} [d^d, \rho_m] + \frac{S_1}{2} (d^d \rho_m - d \rho_m d^d) + \frac{S_2}{2} (\rho_m d^2 - d \rho_m d) + \frac{S^*_2}{2} (d^2 \rho_m - d \rho_m d) + \gamma_m (n_{th} + 1) D[d^d] \rho_m + \gamma_m n_{th} D[d] \rho_m, \]  
\tag{13}

where \( D[d] \rho_m = (2d \rho_m d^d - d^d \rho_m - \rho_m d^2 d)/2 \) corresponds to the standard Lindblad operator, and \( \gamma_m \) is the effective decay rate of the mechanical oscillator and \( n_{th} = 1/(e^{\hbar \omega_0/\hbar \Gamma} - 1) \) is the effective mean phonon number due to both the surrounding gas and the trapping beam \cite{12}. The mechanical fluctuations due to the optical pump \( \Omega_p \) are negligible as shown in the following discussion. In equation (13), the terms proportional to \( A_- (A_+) \) describe the dissipation-induced cooling (heating) due to coupling of the mechanical motion to the transitions from \( |c \rangle \langle a| \) to \( |b\rangle \). The terms proportional to \( \delta \) are the mechanical frequency shifts due to the mechanical-spin interaction. The terms proportional to \( S_j \) or \( S_j^* \) denote the mechanical squeezing via a two-phonon transition using the single NV spin. The explicit expressions of \( A_- , A_+ , \delta \), and \( S_j \) are derived in the appendix B.

2.4. System dynamics—analytical results

To study the system dynamics of the mechanical oscillator, we derive from the reduced master equation (13) that

\[
\langle d^d \rangle = - (\gamma_m + A_- - A_+) \langle d^d \rangle + \frac{S_1}{2} - \frac{S^*_2}{2} (d^2) + (\gamma_m n_{th} + A_+), \tag{14}
\]

\[
\langle d^2 \rangle = - (\gamma_m + A_- + A_+ + i\delta) \langle d^2 \rangle + (S_1 - S_2) \langle d^d \rangle + S_1. \tag{15}
\]

The steady-state solutions to the above equations are given by

\[
\langle d^d \rangle_{ss} = \frac{(\gamma_m n_{th} + A_+) + \left| \frac{(S^*_2 - S^*_1) S_1}{\gamma_m + A_- - A_+ + i\delta} \right|}{(\gamma_m + A_- + A_+) + \left| \frac{|S_1 - S_2|^2}{\gamma_m + A_- - A_+ + i\delta} \right|}, \tag{16}
\]

\[
\langle d^2 \rangle_{ss} = \frac{(S_1 - S_2) \langle d^d \rangle_{ss} + S_1}{\gamma_m + A_- - A_+ + i\delta}. \tag{17}
\]

To obtain maximum mechanical squeezing, we define the quadrature variance rotated in the phase-space plane such that

\[
(\Delta x)^2 = \frac{1}{4} \left( \langle d^d \rangle_{ss} + \langle d^d \rangle_{ss} - |\langle d^2 \rangle_{ss}| - |\langle d^2 \rangle_{ss}| \right), \tag{18}
\]

and the criteria for quantum squeezing is given by

\[ (\Delta x)^2 < \frac{1}{4} \tag{19} \]

We consider \( \Delta_1 = 0 \) (\( \omega_m = \omega_{\parallel} \)), \( \Delta_2 = -\Delta - 3 \hbar \alpha /2 > 0 \) (\( \omega_{\perp} \approx \omega_{\parallel} \)), and \( |\Delta_2| \gg (\Gamma_0, g) \), such that the cooling transition \( (|c\rangle \rightarrow |b\rangle) \) is resonant and the heating transition \( (|a\rangle \rightarrow |c\rangle) \) is far off-resonant (see figure 2(b)). Therefore, steady-state squeezing is possible since the spin-mechanical cooling dominates over the spin-mechanical heating. We obtain to the first-order the quantities:

\[
A_- \approx \frac{4g_{\parallel}|\rho_{\parallel}|}{\Gamma_1 (1 + \sin^2 \theta)}, \tag{20}
\]

\[
S_1 \approx 2g_{g_{\parallel}} \sin \theta \cos \theta \rho_{\parallel} + \frac{4g_{g_{\parallel}} |\rho_{\parallel}|}{\Gamma_1 (1 + \sin^2 \theta)}, \tag{21}
\]

\[
S_2 \approx 2g_{g_{\parallel}} \sin \theta \cos \theta \rho_{\parallel} - \frac{4g_{g_{\parallel}} |\rho_{\parallel}|}{\Gamma_1 (1 + \sin^2 \theta)}, \tag{22}
\]

\[
\delta \approx -2g_{g_{\parallel}} \rho_{\parallel} \Delta_2. \tag{23}
\]

The other quantities \( \delta, A_+, S_2 \approx 0 \). The physical origins of these terms can be seen from the above expressions in this limiting case. The cooling term is due to the absorption of phonons from the mechanical oscillator and
The dressed-state spin coherence (proportional to $\rho_c$) gives rise to squeezing. The mechanical frequency shift is due to a non-resonant two-phonon process between the states $|a\rangle$ and $|b\rangle$. Here we also used the condition $\Gamma_0 \approx \Gamma_1$ which can be satisfied by applying a strong resonant microwave field coupling the excited spin triplet states [72, 73] to suppress the dissipation from states $|E_1\rangle$ and $|E_2\rangle$ to $|\pm 1\rangle$. Therefore, the steady-state mean phonon number due to the dissipative cooling is given by

$$\langle d^+ d \rangle = - \frac{\gamma_m n_{th}}{\gamma_m + A_-}.$$  \hspace{1cm} (24)

where the cooling rate is given by

$$A_- = \frac{g^2}{\Gamma_0} \frac{8 \cos^2 \theta \sin^2 \theta}{(1 + \sin^2 \theta)(1 + \cos^2 \theta)}$$ \hspace{1cm} (25)

which recovers the result in [57] when $\Gamma_0 \approx \Gamma_1 \ll \omega_m$. By using the above conditions, we obtain

$$\langle d^+ d \rangle = - \frac{S_1}{\gamma_m + A_- + i \delta} (\langle d^+ d \rangle + 1).$$  \hspace{1cm} (26)

The quadrature variance is then given by

$$(\Delta x)^2 \approx \frac{1}{4} \left(1 - \frac{2S_1}{\gamma_m + A_- + i \delta} \right) + \frac{1}{2} \left(1 - \frac{S_1}{\gamma_m + A_- + i \delta} \right) \langle d^+ d \rangle.$$  \hspace{1cm} (27)

Using equation (27), we can see that for $\left| \frac{S_1}{\gamma_m + A_- + i \delta} \right| < 1$, the quadrature $(\Delta x)^2$ can be smaller than 1/4 when $(\langle d^+ d \rangle \sim 0$, which demonstrates quantum squeezing of the mechanical motion near the ground state. To obtain quantum ground state cooling, we require the strong cooperativity condition, i.e., $g^2/(\Omega_0 \gamma_m n_{th}) \gtrsim 1$ from equations (24) and (25), and strong coupling condition [57], i.e., $g > \gamma_m$, $\Delta \omega_{bc}$. Here $\Delta \omega_{bc}$ is the frequency shift between the levels $|b\rangle$ and $|c\rangle$ due to the hyperfine interaction between the single spin and its surrounding nuclear bath. For a nuclear-spin interaction Hamiltonian $H_{bc} = \hbar \delta n S_z$ with a typical strength $\delta_n \sim 1$ MHz, the frequency shift is evaluated to the second-order as $\Delta \omega_{bc} = \delta_n^2/\omega_{bc} (1 + \sin^2 \theta - \tan^2 \theta)$. Similar to that in [57], the quadratic frequency shift can be completely suppressed by choosing a suitable dressed-state angle $\theta$ such that $1 + \sin^2 \theta - \tan^2 \theta = 0$. Distinct from [57], the favorable angle $\theta$ can be satisfied with many sets of parameters $\Delta$, $\Omega_0$ and $\Omega_1$ rather than one set of parameters. The reason is that we have three free parameters $\Delta$, $\Omega_0$ and $\Omega_1$ (instead of two in [57]) which are constrained by two conditions $\omega_{bc} = \omega_m$ and $1 + \sin^2 \theta - \tan^2 \theta = 0$. This is another advantage of having the extra control parameter $\Omega_1$. We consider numerical parameters explicitly in the next section for quantum ground state cooling and quantum squeezing.

2.5. System dynamics—numerical results

2.5.1. Case 1: $\Omega_1 = 0$

We first consider the case of no coupling ($\Omega_1 = 0$) between the $|\pm 1\rangle$ NV states (see figure 2) as this coupling is not essential to the physics, and only provides fine control as shown below. We plot the numerical results for $(\langle d^+ d \rangle, (\Delta x)^2, A_-, A_+)$ using the solutions equations (16) and (17). First, we observe that ground-state cooling [57] is possible with strong cooperativity, i.e., $g^2/(\Omega_0 \gamma_m n_{th}) \gtrsim 1$ as shown in figure 4(a). In this case the cooling processes dominate the heating. Second, we observe in figure 4(b) that the quadrature variance $(\Delta x)^2 < 1/4$, which implies quantum squeezing of the one quadrature of the mechanical oscillator. We find that the region for which the quantum squeezing occurs qualitatively agrees with the region $(\langle d^+ d \rangle \ll 1,$ as
discussed analytically in section 2.4. To understand the cooling and the squeezing, we plot $A_-$ and $A_+$ in figure 5. As $\Omega_0/\omega_m$ varies between 0 and $\sqrt{2}$, we see an optimal cooling limit is obtained by balancing the cooling and heating effects from the single spin.

2.5.2. Case 2: $\Omega_1 \neq 0$

For $\Omega_1 \neq 0$, we have an extra control over the single NV spin which couples to the mechanical oscillator. For an initial phonon number $n_0 = 10^3$ and mechanical quality factor $Q = 10^6$, we first plot the quantity $(\Delta x)^2 - 1/4$ versus the scaled Rabi frequencies $\Omega_0/\omega_m$ and $\Omega_1/\omega_m$ in figure 6(a). We observe that quantum squeezing, $(\Delta x)^2 - 1/4 < 0$, can be realized for a large range of parameters. With the extra control parameter $\Omega_1$, both the dressed-state angle $\theta$ and the populations of the dressed states can be controlled independently. Therefore, we see from equations (20)–(22) that the cooling rate and the squeezing rate can be optimally controlled. For example, an enhancement of the mechanical squeezing can be obtained for $\Omega_1 < 0$, which corresponds to $\Omega_1 = 0$ and a $\pi$ phase difference between the driving fields $\Omega_1$ and $\Omega_0$. Second, we plot the quantity $(\Delta x)^2 - 1/4$ versus $\Omega_0/\omega_m$ and $n_{th}$ in figure 6(b), where strong squeezing below 3 dB can be obtained, i.e. $(\Delta x)^2 - 1/4 < -1/8$. We find quantum squeezing can be achieved when $n_{th} \sim 3 \times 10^5$, which corresponds to an initial temperature $\sim 0.1$ K. This initial temperature of the mechanical oscillator may be achieved with cryogenic techniques or by using feedback cooling [8, 12, 13]. Furthermore, we plot the quantity $(\Delta x)^2 - 1/4$ versus $\Omega_0/\omega_m$ and $g/\omega_m$ keeping other parameters constant, in figure 6(c). We see from the figure that stronger $g$ is preferred for realizing quantum squeezing as long as the Born-Markov approximation is valid.

2.6. Experimental accessibility

Remarkably, we find, at initial room temperature environment for the mechanical oscillator, that quantum squeezing is feasible with our system for ultrahigh vacuum with feedback cooling. In ultrahigh vacuum ($< 10^{-8}$ mbar), as demonstrated recently for an optically levitated nanoparticle [13], the gas damping rate is on the order of $\gamma_m \sim 10^{-6}$ Hz, which corresponds to $\omega_m/\gamma_m \sim 10^{12}$. As an example, we consider an optically levitated nanodiamond with a radius 50 nm and a mechanical oscillation frequency of $\sim 10^{9}$ Hz. Hence the maximum magnetic field splitting on the levels $|\pm 1\rangle$ is on the order of $1$ MHz, much smaller than the zero-magnetic field splitting $2.88$ GHz between $|\pm 1\rangle$ and $|0\rangle$. Therefore, the strong magnetic field does not generate a strong magnetic field to change the groundstate frequencies appreciably in our model.

To obtain an optical-induced dissipation rate $\Gamma_0 = \omega_m/4 \approx 1.5$ MHz for the electron spin, we consider an optical pump Rabi frequency $\Omega_p \sim 8$ MHz and a typical excited state decay rate $\gamma_p \sim 40$ MHz [65]. For $\Omega_p \sim 8$ MHz, the corresponding optical pump power is smaller than 1 $\mu$W [75], which has a negligible effect on the mechanical motion fluctuation due to the optical scattering [3].

To reduce the mean phonon number of the mechanical oscillator due to both the surrounding gas and the optical trapping field, feedback cooling of the nanoparticle can be employed by introducing extra mechanical damping from feedback [8, 12, 13]. For completeness, we briefly discuss the implementation on feedback cooling of an optically levitated nanoparticle [8, 12, 13]. The optical trapping beam is scattered by the nanoparticle and the scattered light carries the information of the nanoparticle position. By doing an
interferometric detection of the scattered and unscattered light, the position of the nanoparticle can be measured [5]. The position information is then fed back to modulate the intensity of the trapping beam which controls the stiffness of the harmonic trap. Feedback cooling is achieved by increasing the trap stiffness when the nanoparticle moves away from the trap center and reducing it when the nanoparticle moves towards the center. The final phonon number of the nanoparticle is determined by its initial phonon number, the gas pressure, the detection efficiency, and the feedback strength [12]. We estimate that with a feedback-induced mechanical damping $\gamma_{bf} \sim 10^3$ Hz, quantum squeezing is achievable at room temperature when the initial phonon occupation number is reduced to $n_{th} \sim 2$. These numbers correspond to a strong cooperativity $g^2/\Omega_0 \gamma_{th} n_{th} \sim 10$. Our prediction is within the reach of a recent experiment, where a final phonon number of 63 has been demonstrated with feedback cooling [13].

We note that using the driving field $\Omega_1$, it is possible to control the energy difference between dressed states. Our model requires $\omega_{bf} > 0$ for the rotating-wave approximation to be valid. We plot the value of $\omega_{bf}$ versus $\Omega_0$ and $\Omega_1$ in figure 6(d) and we find the condition is satisfied for the parameter regime where mechanical quantum squeezing can be engineered.

To summarize, single-mode quantum squeezed mechanical state is feasible using our model in ultrahigh vacuum, even at room temperature.

### 3. Single NV center coupled to two mechanical modes

#### 3.1. The model

The optically levitated nanodiamond has three harmonic oscillations independent of each other for small oscillation amplitudes, which is an excellent platform for multimode mechanical quantum state engineering. By applying magnetic field gradient in both $x$ and $y$ directions of the harmonic oscillations, as shown in figure 7, we can couple two mechanical modes to the single spin of the NV center nanodiamond. The magnetic field gradient...
are chosen such that \( \nabla \cdot B = 0 \), i.e., \( \frac{\partial B_y}{\partial x} = -\frac{\partial B_x}{\partial y} = B_0/2 \). We assume the magnetic fields \( B_x \) and \( B_y \) both have \( x \) and \( y \) position dependence. For simplicity, we consider \( \frac{\partial B_x}{\partial y} = -\frac{\partial B_y}{\partial x} = B_1/2 \). Therefore, the magnetic field vector is given by \( \vec{B} = (B_0x/2 + B_1y/2, -B_0y/2 - B_1x/2, 0) \). We assume \( x \) and \( y \) coordinate axes of the mechanical motions are in the plane of the spin operator components \( S_x \) and \( S_y \). The angles between \( B_x \) and \( S_x \), and between \( B_y \) and \( S_y \), are chosen such that this extra interaction term does not affect the superposed mode case. The second line in the above equation describes the interaction between the electron spin \( S \) and the mechanical motion \( \phi \). In the interaction-picture under the rotating-wave approximation as

\[
H_{\text{int}} = \hbar g_{SB} S_x (d_1^+ + d_1) + \hbar g_{SB} S_y (d_2^+ + d_2) + \hbar g_{SB} S_z (d_1^2 + d_2^2) + \hbar g_{SB} S_x (d_1^+ + d_1) S_x (d_2^+ + d_2),
\]

where \( d_1 \) (\( d_2 \)) is the annihilation operator in \( x \) (\( y \)) direction and \( g_{SB} \) is the spin-mechanical coupling strength for the two modes in the spin orientation defined in the appendix C. The electron spin dynamics is the same as in the single mechanical mode case, where the spin is driven by two microwave fields coupling between states \( |0\rangle \) and \( |\pm 1\rangle \), and an effective field coupling between states \( |+1\rangle \) and \( |-1\rangle \). In the eigenbasis of \( H_{NV} \), the total Hamiltonian is given by

\[
H = \hbar \omega_m d_1^+ d_1 + \hbar \omega_m d_2^+ d_2 + \hbar \omega_m |a\rangle \langle a| + \hbar \omega_m |b\rangle \langle b|
+ \hbar g_x |c\rangle \langle b| + g_x^2 |b\rangle \langle a| + \text{h.c.} (d_1^+ + d_2^+) \langle d_1^2 + d_2^2 + d_1 + d_2)
+ \hbar g_y (d_1^+ - d_1^0 + d_1 - d_2^0) \{ \cos^2(\theta) |c\rangle \langle c| + \sin^2(\theta) |a\rangle \langle a| - |b\rangle \langle b| \}
- \sin(\theta) \cos(\theta) |c\rangle \langle c| + |a\rangle \langle a| \}
\]

where we assume the two modes have the same frequency \( \omega_m \), and \( \phi = \pi/4 \) for simplicity such that \( g_x = -g_x \sin(\theta) \), \( g_y = g_y \cos(\theta) \), and \( g_z = \sqrt{2} \mu_0 / \hbar \). This configuration is possible for a nanoparticle trapped in an optical field, where the frequencies of two transverse modes can be made very close to each other [8]. The other quantities are the same as in the single mechanical mode case. The second line in the above equation describes the interaction between the electron spin \( S_z \) with the two mechanical modes in the dressed–state basis, similar to that of the single–mode case if we replace \( d \) with \( (d_1^0 + d_1^0)/\sqrt{2} \). The last term in equation (29) describes the coupling between the electron spin \( S_z \) with \( (d_1^0 - d_1^0) \) of the two mechanical modes. We readily see that this extra interaction term does not affect the superposed mode \( (d_1^0 + d_1^0)/\sqrt{2} \) directly because \( [d_1^0 + d_1^0, d_1^0 - d_1^0] = 0 \). However, the mechanical coupling to \( S_z \) perturbs the dressed states, \( |a\rangle \), \( |b\rangle \), \( |c\rangle \), which affect the superposed mode \( (d_1^0 + d_1^0)/\sqrt{2} \). Ideally, this extra interaction term can be eliminated by choosing the proper magnetic field gradients, i.e. \( B_0 = B_1 \), such that \( g_x = 0 \). Physically, this corresponds to a magnetic field with the proper position dependence along the direction of \( S_z \). In the following, we study how to engineer two-mode mechanical squeezed states under this condition. For slightly different magnetic field gradients \( B_0 \) and \( B_1 \), we will consider the first–order perturbation of the extra interaction term with both analytical and numerical results.

By considering the mechanical frequency \( \omega_m \) resonant coupled to the transition from \( |b\rangle \) to \( |c\rangle \) and far detuned from the other transition of \( |a\rangle \) to \( |b\rangle \), we can write \( H \), in the interaction–picture under the rotating-wave approximation as

---

**Figure 7.** The configuration considered in section 3. The green circle denotes an optically levitated nanodiamond oscillating in two separate harmonic potentials along the \( x \) coordinate (solid black curve) and \( y \) coordinate (dotted black curve), respectively. Magnetic field gradients are applied along both \( x \) and \( y \) directions. As shown in the figure, \( S_y \), \( S_x \) and \( B_x \), \( B_y \), all lie on the same plane, and the spin axes are not aligned with the coordinate axes. The spin component \( S_y \) (not shown) is perpendicular to \( B_x \), \( B_y \) and points into the plane.
where $\Delta_0 = 2\omega_m - \omega_{bc}$ and $\Delta_1 = \omega_m - \omega_{bc}$ are the same as the single mode case. The advantage of this configuration is such that the superposed mode $(d_1 + d_2)/\sqrt{2}$ can be cooled efficiently to its ground-state similar to the single mode case while squeezing process is engineered via the two-phonon transition of the superposed mode mediated by the dressed-state spin levels.

3.2. Analytical results of the two-mode mechanical oscillator

At the steady-state of the spin states, we can trace out the spin degree of freedom to obtain the reduced master equation for the two-mode mechanical oscillator similar to equation (15)

\[
\dot{\rho}_m = \frac{A}{2} D[d_1 + d_1^\dagger] \rho_m + \frac{A}{2} D[d_2 + d_2^\dagger] \rho_m - \frac{i\delta}{4} [d_1^\dagger + d_2^\dagger] (d_1 + d_2), \rho_m \]

\[
+ \frac{S_1}{4} [(d_1^\dagger + d_2^\dagger)^2 \rho_m - (d_1^\dagger + d_2^\dagger) \rho_m (d_1^\dagger + d_2^\dagger)]
+ \frac{S_2}{4} [\rho_m (d_1^\dagger + d_2^\dagger)^2 - (d_1^\dagger + d_2^\dagger) \rho_m (d_1^\dagger + d_2^\dagger)]
+ \frac{S_3^*}{4} [\rho_m (d_1 + d_2)^2 - (d_1 + d_2) \rho_m (d_1 + d_2)]
+ \frac{S_4^*}{4} [(d_1 + d_2)^2 \rho_m - (d_1 + d_2) \rho_m (d_1 + d_2)]
+ \gamma_m (n_{th} + 1) D[d_1] \rho_m + \gamma_m n_{th} D[d_1^\dagger] \rho_m
+ \gamma_m (n_{th} + 1) D[d_2] \rho_m + \gamma_m n_{th} D[d_2^\dagger] \rho_m,
\]

where the coefficients are given in equations (B.5)–(B.9), and $\gamma_m = \gamma_m$ is the decay rate, assumed to be the same for both mechanical modes. The terms, such as the cooling, the heating, and the squeezing, in the reduced master equation for two mechanical modes are similar to those of the single-mode case. We are interested in the steady-state properties of the two-mode system and we find at the steady-state the relevant mean values are

\[
\langle (d_1^\dagger + d_2^\dagger)(d_1 + d_2) \rangle_{ss} = \frac{(2\gamma_m n_{th} + 2A_+)}{\gamma_m + A_+ - A_+ + i\delta},
\]

\[
\langle (d_1 + d_2)^2 \rangle_{ss} = \frac{(S_1 - 5\gamma_m)(\langle d_1^\dagger d_1^\dagger \rangle_{ss} + 2S_3^*)}{\gamma_m + A_+ - A_+ + i\delta}.
\]

To show two-mode mechanical squeezing, we consider the variance $\langle \Delta u^2 \rangle$ [76], where $u = (x_1^\dagger x_1 + x_2^\dagger x_2^\dagger)/2$, and $x_j^\dagger = (d_1 e^{-i\theta_j} + d_1^\dagger e^{i\theta_j})/\sqrt{2}$ $(j = 1, 2)$. To obtain the maximum degree of two-mode squeezing, we choose $\theta_1$ and $\theta_2$ such that the two-mode quadrature variance is given by

\[
\langle \Delta u^2 \rangle = \frac{1}{4} (\langle (d_1^\dagger + d_2^\dagger)(d_1 + d_2) \rangle_{ss} - |\langle (d_1 + d_2)^2 \rangle_{ss}| + 1).
\]

We find that in the two-mode case $\langle (d_1^\dagger + d_2^\dagger)(d_1 + d_2) \rangle_{ss} = 2 \langle d_1^\dagger d_2^\dagger \rangle_{ss}$ and $\langle (d_1 + d_2)^2 \rangle_{ss} = \langle d_2^2 \rangle_{ss}$ comparing with the single-mode results. Therefore, the two-mode quadrature variance under current configuration recovers that of the single-mode case, i.e.

\[
\langle \Delta u^2 \rangle = \langle \Delta x^2 \rangle.
\]

All the discussions about squeezing a single-mode mechanical oscillator apply to the two-mode case exactly under the assumption that the interaction between the spin component $S_3$ and the two mechanical modes are negligible. This assumption we made in the two-mode case is valid in the rotating-wave approximation. We also verified that $\langle d_1^\dagger d_1 \rangle_{ss} > 0$ for the parameter regime of interest for the requirement of steady-state of the two modes.

For nonzero $g_{sc} \ll \omega_{ph}, \omega_{bc}$, we see from equation (29) that the last term results in small frequency shifts of levels $|\alpha\rangle, |\beta\rangle, |c\rangle$. These frequency shifts do not change much the steady-state population $\rho_{ss}$ in the dressed-state basis since $g_{xc} \ll \Omega_b, \Omega_c$. However, the frequency shifts result in frequency uncertainty in $\omega_{bc}$, which may affect the resonant coupling between levels $|\beta\rangle, |c\rangle$ and $(b_1 + b_2)/\sqrt{2}$. To the first-order perturbation, the frequency uncertainty is evaluated to be
\[
\Delta \omega_{bc}(g_x) = 2g_x (1 + \sin^2 \theta) \sqrt{n_{bc}}.
\]  

For \( \Delta \omega_{bc}(g_x) < |g_x| \), the mechanical coupling to \( S_x \) has a small effect on the two-mode cooling and the squeezing. Below we will study numerically the effect of this term by adding the frequency uncertainty in the analytical expression. For \( \omega_{bc} \sim \omega_{qbc} \omega_{bq} \), we may also neglect the part proportional to \( (|c\rangle \langle a| + |a\rangle \langle c|) \) in the last term in equation (29) since \( \omega_{bc} \approx |\omega_{bc} - \omega_{bc}| \) under the rotating-wave approximation.

3.3. Numerical results of the two-mode mechanical oscillator

In this section, we present numerical results of the two mechanical modes coupled to the single NV center electron spin. In our model, the two modes couple to \( S_x \) for mechanical cooling and squeezing, and to \( S_y \) which leads to frequency shifts in the dressed states. Ideally, we find the proper magnetic field gradients which eliminates the coupling to \( S_y \). Numerically, we plot the results of the steady-state system variables for both the ideal case and non-ideal case for comparison.

In figure 8, we plot the final phonon number of the superposed mode \( b_1 + b_2 \) and the two-mode variance \((\Delta u)^2\) for \( g_x = 0 \) and \( g_x = g_x/50 \). First, when \( g_x = 0 \), we find that \((|b_1 + b_2|)^2(1 + 1/2) < 1\) for a range of parameters, which means the superposed mode \( b_1 + b_2 \) is indeed cooled close to its quantum ground state. We also find that the two-mode variance can be smaller than \( 1/4 \) meaning two-mode quantum squeezing, and the two-mode quadrature variance equals to the single-mode quadrature variance for the same parameters by comparing figure 4(b) with figure 8(b). Second, when \( g_x = g_x/50 \), we find both the final phonon number of the superposed mode and the two-mode variance are slightly reduced due to the frequency shifts.

In figure 9, we plot the range of \((\Delta u)^2 < 1/4\) for different values of \( \Omega_0/\omega_{bc} \) and \( \Omega_1/\omega_{bc} \) at \( g_x = 0 \) and \( g_x = g_x/50 \). By comparing the two figures, we find the range of quantum squeezing is slightly reduced due to the mechanical coupling to \( S_x \). Therefore, the two-mode mechanical squeezed states can be engineered with our model even if in the presence of imperfections, such as coupling to \( S_y \).

3.4. Discussion

In summary, we presented a method for engineering two-mode mechanical squeezed states under similar conditions required for the single-mode case by using the proper magnetic field gradients. Therefore, the discussions on the experimental accessibility in the single-mode case are applicable to the two-mode case here, such as room temperature squeezing with feedback cooling. The mechanical coupling to \( S_x \) due to the imperfect magnetic field gradients results in small dressed-state frequency uncertainty and weak mechanical squeezing reduction. As an application, the two-mode mechanical squeezed states are useful for sensitive phase measurement beyond the standard quantum limit in an interferometric setup [49, 77].

4. Conclusion

In this paper, we have investigated quantum state engineering of an optically levitated nanodiamond coupled to a single NV center ground-state electron spin. We considered quantum state engineering of both single-mode and two-mode mechanical motions. Both analytical and numerical results have been obtained to show that single-mode squeezed states of the mechanical oscillator is feasible with the state-of-art experiments even at
room temperature. We have shown that our scheme for single-mode squeezing can be readily extended to the case of two-mode squeezing, which is of interest for precision measurements.

In conclusion, we presented an experimentally realizable method for engineering both single-mode and two-mode mechanical squeezed states in an optically levitated nanodiamond via dressed-state coherence. Our work advances macroscopic quantum state engineering in cavity-free systems, and paves the way for sensitive metrology with squeezed mechanical states.

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Appendix A. Driving induced dissipation

The coupling Hamiltonian for optical pumping is given by $H_D = \hbar \Omega_p / 2(\ket{E_1} + \ket{E_2} + \h.c.)$ (see figure 3). We assume effective dissipation paths from the excited states $|E_1\rangle$ to the $|\pm1\rangle$ and $|0\rangle$ with dissipation rates $\gamma_1$ and $\gamma_0$, respectively. We consider the situation that the driving fields and the decay rates are much faster than the spin–mechanical coupling such that we can treat the dynamics of the spin separately from the mechanical motion. The master equation of the spin system is given by

$$\dot{\rho}_{E_iE_i} = -(\gamma_1 + \gamma_0) \rho_{E_iE_i} + \frac{\Omega_p^2}{2} (\rho_{E_i+1} - \rho_{-1E_i}), \quad (A.1)$$

$$\dot{\rho}_{E_i+1} = -\frac{\gamma_1 + \gamma_0}{2} \rho_{-1E_i+1} - \frac{i\Omega_p}{2} (\rho_{E_i+1} - \rho_{E_iE_i}), \quad (A.2)$$

$$\dot{\rho}_{E_iE_2} = -(\gamma_1 + \gamma_0) \rho_{E_iE_2} + \frac{i\Omega_p}{2} (\rho_{E_i-1} - \rho_{E_iE_i}), \quad (A.3)$$

$$\dot{\rho}_{E_i-1} = -\frac{\gamma_1 + \gamma_0}{2} \rho_{E_i-1} + \frac{i\Omega_p}{2} (\rho_{E_i+1} - \rho_{E_iE_i}), \quad (A.4)$$

At the steady-state, we find from equations (A.1)–(A.4) that

$$\rho_{E_iE_i} = \Omega_p^2 \rho_{E_i+1} + [\gamma_1 + \gamma_0] \rho_{E_iE_i},$$

$$\rho_{E_i+1} = -\Omega_p^2 \rho_{-1E_i+1} + [\gamma_1 + \gamma_0] \rho_{E_i+1},$$

$$\rho_{E_iE_2} = -\Omega_p^2 \rho_{E_iE_2} + [\gamma_1 + \gamma_0] \rho_{E_iE_2},$$

$$\rho_{E_i-1} = -\Omega_p^2 \rho_{E_i-1} + [\gamma_1 + \gamma_0] \rho_{E_i-1}.$$}

We then find the equation of motion of the density matrix elements for the ground-state spin levels due to the microwave fields as

$$\dot{\rho}_{E_i+1} = -\Gamma_0 \rho_{E_i+1} - \frac{i\Omega_p}{2} (\rho_{1E_i+1} - \rho_{1E_i}), \quad (A.5)$$

$$\dot{\rho}_{E_iE_2} = -\Omega_p^2 \rho_{E_iE_2} + [\gamma_1 + \gamma_0] \rho_{E_iE_2},$$

$$\dot{\rho}_{E_i-1} = -\Omega_p^2 \rho_{E_i-1} + [\gamma_1 + \gamma_0] \rho_{E_i-1}.$$
\[ \dot{\rho}_{-1-1} = -\Gamma_0 (\rho_{-1-1} - \rho_{-1+1}) - \frac{i \Omega_0}{2} (\rho_{0-1} - \rho_{-1-1}) + \frac{i \Omega_1}{2} (\rho_{-1+1} - \rho_{+1-1}), \]  
\[ \dot{\rho}_{00} = \Gamma_0 (\rho_{+1+1} + \rho_{-1-1}) + \frac{i \Omega_0}{2} (\rho_{0-1} - \rho_{+1+1}) + \frac{i \Omega_0}{2} (\rho_{0+1} - \rho_{-1+1}), \]  
\[ \dot{\rho}_{+1+1} = -\Gamma_0 (\rho_{+1+1} + \rho_{-1-1}) + \frac{i \Omega_0}{2} (\rho_{0+1} - \rho_{+1+1}) + \frac{i \Omega_1}{2} (\rho_{+1-1} - \rho_{1-1}), \]  
where \( \Gamma_0 = \Omega_0^2 \gamma_0 / (\gamma_1 + \gamma_0)^2 + \Omega_1^2 \gamma_0 / (\gamma_1 + \gamma_0)^2 \), and \( \Gamma_1 = \Omega_1^2 / (\gamma_1 + \gamma_0) \). We find the steady-state solutions to equations (4.5)–(4.10) as

\[ \rho_{00} = \frac{\Gamma_0 (2\Delta - \Omega_1)^2 + (\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0 \Gamma_1^2}{\Gamma_0 (2\Delta - \Omega_1)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0 \Gamma_1^2}, \]  
\[ \rho_{-1+1} = \frac{\Gamma_1 \Omega_0^2}{\Gamma_0 (2\Delta - \Omega_1)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0 \Gamma_1^2}, \]  
\[ \rho_{-1-1} = \frac{\Gamma_1 \Omega_0^2}{\Gamma_0 (2\Delta - \Omega_1)^2 + (3\Gamma_1 + \Gamma_0)\Omega_0^2 + \Gamma_0 \Gamma_1^2}, \]  
\[ \rho_{-1+1} = \frac{\Gamma_0 \Omega_0 (2\Delta - \Omega_1) + i\Gamma_1}{\Gamma_0 (2\Delta - \Omega_1) + i\Gamma_1}, \]  
\[ \rho_{1-1} = \frac{\Gamma_0 \Omega_0 (2\Delta - \Omega_1)}{\Gamma_0 (2\Delta - \Omega_1) + i\Gamma_1}, \]  
\[ \rho_{1+1} = \frac{\Gamma_0 \Omega_0 (2\Delta - \Omega_1) - i\Gamma_1}{\Gamma_0 (2\Delta - \Omega_1) + i\Gamma_1}. \]  

The steady-state solutions can be rearranged to give the results in equations (7)–(10) in the eigenbasis. Under the condition \( \Gamma_0 \approx \Gamma_1 \ll \omega_m \), we obtain from equations (8)–(10) that

\[ \rho_{cc} \approx \frac{(1 + \cos 2\theta)^2}{2(1 + \cos^2 \theta)}, \]  
\[ \rho_{ac} \approx \frac{(1 - \cos 2\theta)^2}{2(1 + \cos^2 \theta)}, \]  
\[ \rho_{ac} \approx \frac{-i \sin 2\theta}{\sqrt{1 + \cos^2 \theta}} \frac{\Gamma_0/2}{\sqrt{2(\Delta - \Omega_1/2)^2 + 2\Omega_0^2}}. \]  

**Appendix B. Reduced master equation**

The first-order perturbation of \( \rho_{bc} \) and \( \rho_{ba} \) are given by the spin-mechanical interaction \( H_I \) as

\[ \dot{\rho}_{bc} \otimes \rho_m \approx \left(i \Delta_2 - \frac{\Gamma_1}{2} \frac{1}{(1 + \sin^2 \theta)} \right) \rho_{bc} \otimes \rho_m + \frac{\Gamma_1}{2} \sin(\theta) \cos(\theta) \rho_{bc} \otimes \rho_m \]  
\[ - i g_x^* \left(d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m \right) \]  
\[ - ig_x^* \left(d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m \right)^d \]  
\[ \dot{\rho}_{ba} \otimes \rho_m \approx \left( -i \Delta_2 - \frac{\Gamma_1}{2} \frac{1}{(1 + \cos^2 \theta)} \right) \rho_{ba} \otimes \rho_m + \frac{\Gamma_1}{2} \sin(\theta) \cos(\theta) \rho_{bc} \otimes \rho_m \]  
\[ - i g_x^* \left(d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m \right) \]  
\[ - ig_x^* \left(d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m \right)^d \]  
where \( \Delta_2 = \Delta_0 - \Delta_1 = \omega_m - \omega_{ab} \), and the approximation is made on the decay rates of \( \rho_{bc} \) and \( \rho_{ba} \) by assuming \( \Gamma_0 \approx \Gamma_1 \). For \( |\Delta_1|, \Gamma_0 \gg g_x, g_y, g_z \) at the steady-state we find

\[ \rho_{bc} \otimes \rho_m = -ig_x^* \left[k_b^2 (d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m)^d + k_b^2 d \rho_{bc} \otimes \rho_m \right] \]  
\[ - ig_x^* \left[k_a^2 (d \rho_{bc} \otimes \rho_m - \rho_{bb} \otimes \rho_m)^d + k_a^2 d \rho_{bc} \otimes \rho_m \right]. \]
\[
\rho_{ba} \otimes \rho_m = -ig_c^* \left[ \frac{k_s}{M} (d^\dagger \rho_{cc} \otimes \rho_m - \rho_{bb} \otimes \rho_{m} d^\dagger) + \frac{k_s}{M} d \rho_{ac} \otimes \rho_m \right] \\
- ig_c^* \left[ \frac{k_s}{M} (d\rho_{cc} \otimes \rho_m - \rho_{bb} \otimes \rho_{m} d) + \frac{k_s}{M} d \rho_{ac} \otimes \rho_m \right],
\]  

(B.4)

where \( M = k_b k_c - k_j^2 \), \( k_i = 1/2 \Delta_c + \frac{1}{2} (1 + \sin^2 \theta) \), \( k_s = 1/2 \Delta_a + \frac{1}{2} (1 + \cos^2 \theta) \), and \( k_d = \frac{1}{2} \sin \theta \cos \theta \). By substituting \( \rho_{bc} \) and \( \rho_{ba} \) in equation (12), we obtain the result of the reduced master equation in equation (13) with the coefficients given by

\[
\delta = 2 \text{Im} \left[ \frac{|g_c|^2}{M} (k_b \rho_{bc} - k_d \rho_{bb} + k_s \rho_{ac}) + \frac{|g_c|^2}{M} (k_i \rho_{aa} - k_d \rho_{bb} + k_s \rho_{ac}) \right],
\]

(B.5)

\[
A_- = 2 \text{Re} \left[ \frac{|g_c|^2}{M} (k_b \rho_{bc} + k_d \rho_{bb}) + \frac{|g_c|^2}{M} k_i \rho_{aa} \right],
\]

(B.6)

\[
A_+ = 2 \text{Re} \left[ \frac{|g_c|^2}{M} (k_b \rho_{bc} + k_d \rho_{bb}) + \frac{|g_c|^2}{M} k_i \rho_{aa} \right],
\]

(B.7)

\[
S_1 = 2|g_c|g_{1c} \left( \frac{k_s}{M} \rho_{aa} + \frac{k_i}{M} \rho_{bb} + \frac{k_s}{M} \rho_{ac} \right),
\]

(B.8)

\[
S_2 = 2|g_c|g_{1c} \left( \frac{k_s}{M} \rho_{ac} + \frac{k_i}{M} \rho_{bb} + \frac{k_s}{M} \rho_{ac} \right).
\]

(B.9)

### Appendix C. Two-mode interaction Hamiltonian

Assuming the magnetic field \( \vec{B} = (B_0x/2 + B_0y/2, -B_0y/2 - B_1x/2, 0) \) and the spin orientation in figure 7, the two-mode interaction Hamiltonian is given by

\[
\hat{H}_{int} = g_s \mu_B \left( \frac{B_0x + B_1x \cos \varphi}{2} + \frac{B_0y + B_1x \sin \varphi}{2} \right) S_z \\
+ g_s \mu_B \left( \frac{B_0x + B_1y \sin \varphi}{2} - \frac{B_0y + B_1x \cos \varphi}{2} \right) S_x
\]

(C.1)

where \( g_{s1} = g_s \mu_B (B_0 \cos \varphi + B_1 \sin \varphi) x_0 / 2h \), \( g_{s2} = g_s \mu_B (B_0 \sin \varphi - B_1 \cos \varphi) x_0 / 2h \), \( \varphi_{s1} = \mu s \mu (B_0 \sin \varphi - B_1 \cos \varphi) y_0 / 2h \), \( \varphi_{s2} = \mu s \mu (B_0 \sin \varphi + B_1 \cos \varphi) y_0 / 2h \), and \( x_0, y_0 \) is the zero-point fluctuation in \( x(y) \) direction, and \( d_1(d_2) \) is the annihilation operator in \( x(y) \) direction.

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