Generalized KP hierarchy: Möbius Symmetry, Symmetry Constraints and Calogero-Moser System

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Abstract

Analytic-bilinear approach is used to study continuous and discrete non-isospectral symmetries of the generalized KP hierarchy. It is shown that Möbius symmetry transformation for the singular manifold equation leads to continuous or discrete non-isospectral symmetry of the basic (scalar or multicomponent KP) hierarchy connected with binary Bäcklund transformation. A more general class of multicomponent Möbius-type symmetries is studied. It is demonstrated that symmetry constraints of KP hierarchy defined using multicomponent Möbius-type symmetries give rise to Calogero-Moser system.

1 Introduction

It is a pleasure for us to dedicate this paper to Prof. V.E. Zakharov 60th birthday. The technique used in this work (analytic-bilinear approach, see [1], [2]) takes its origin in the ideas of the ∂-dressing method developed by V.E. Zakharov and S.V. Manakov [3]. It is interesting to note that the first paper formulating basic ideas of analytic-bilinear approach [4] was published in the volume dedicated to 55th birthday of Prof. Zakharov.

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Analytic-bilinear approach formalizes important features of the $\bar{\partial}$-dressing method connected with construction of integrable equations, leaving aside some details of the scheme of generating special classes of solutions. In this form the method becomes close to Grassmannian approach \cite{5}, \cite{6}, thus filling the gap between $\bar{\partial}$-dressing method and more abstract Grassmannian approach, preserving at the same time some useful structures characteristic of the original $\bar{\partial}$-dressing method.

In this work we use analytic-bilinear approach to study continuous and discrete non-isospectral symmetries of the generalized KP hierarchy. We demonstrate that M"obius symmetry on the level of KP singular manifold equations (KPSM) hierarchy, binary B"acklund transformations of KP hierarchy and solitonic transformations of the $\tau$-functions through the Date-Jimbo-Kashiwara-Miwa vertex operator are different manifestations of the same discrete symmetry. Considering continuous non-isospectral symmetries, we show that Calogero-Moser system can be obtained through the symmetry constraint of KP hierarchy.

2 Generalized KP Hierarchy

First we give a sketch of the picture of generalized KP hierarchy in frame of analytic-bilinear approach; for details we refer to \cite{1}, \cite{2}.

The formal starting point is Hirota bilinear identity for Cauchy-Baker-Akhiezer function,

$$\oint \chi(\nu, \mu; g_1)g_1^{-1}(\nu)\chi(\lambda, \nu; g_2)d\nu = 0 \quad \lambda, \mu \in D. \quad (1)$$

Here $\chi(\lambda, \mu; g)$ (the Cauchy kernel) is a function of two complex variables $\lambda, \mu \in D$, where $D$ is a unit disc, and a functional of the loop group element $g \in \Gamma^+$, i.e., of a complex-valued function analytic and having no zeros in $\mathbb{C} \setminus D$, equal to 1 at infinity; the integration goes over the unit circle. By definition, the function $\chi(\lambda, \mu)$ possesses the following analytical properties: as $\lambda \to \mu$, $\chi \to (\lambda - \mu)^{-1}$ and $\chi(\lambda, \mu)$ is an analytic function of two variables $\lambda, \mu \in D$ for $\lambda \neq \mu$. The function $\chi(\lambda, \mu; g)$ is a solution to (1) if it possesses specified analytic properties and satisfies (1) for all $\lambda, \mu \in D$ and some class of loops $g \in \Gamma^+$.

In another form, more similar to standard Hirota bilinear identity,
the identity (1) can be written as
\[ \oint \psi(\lambda, \mu; g_2) \psi(\nu, \mu; g_1) d\lambda = 0, \]  
(2)
where
\[ \psi(\lambda, \mu, g) = g(\lambda) \chi(\lambda, \mu, g) g^{-1}(\mu). \]

We call the function \( \psi(\lambda, \mu; g) \) a Cauchy-Baker-Akhiezer function.

Hirota bilinear identity (1) incorporates the standard Hirota bilinear identity for the Baker-Akhiezer (BA) and dual (adjoint) Baker-Akhiezer function of the KP hierarchy. Indeed, let us introduce these functions by the formulae
\[ \psi(\lambda; g) = g(\lambda) \chi(\lambda; 0), \]
\[ \tilde{\psi}(\mu; g) = g^{-1}(\mu) \chi(0; \mu). \]

Then for the Baker-Akhiezer function \( \psi(\lambda; g) \) and the dual Baker-Akhiezer function \( \tilde{\psi}(\mu; g) \), taking the identity (1) at \( \lambda = \mu = 0 \), we get the usual form of the Hirota bilinear identity
\[ \oint \tilde{\psi}(\nu; g_2) \psi(\nu; g_1) d\nu = 0. \]  
(3)

The only minor difference from the standard setting here is that we define the BA and dual BA function in the neighborhood of zero, not in the neighborhood of infinity.

There are three different types of integrable discrete equations implied by identity (1), that, in the continuous limit, lead to the KP hierarchy in the usual form (in terms of potentials), to the modified KP hierarchy and to the hierarchy of the singular manifold equations. They arise for different types of functions connected with the Cauchy-Baker-Akhiezer function satisfying Hirota bilinear identity (see the derivation in [1], [2]).

On the first level, we have the equations for the diagonal of the regularized Cauchy kernel taken at zero (the potential)
\[ u(g) = \left( \chi(\lambda, \mu; g) - (\lambda - \mu)^{-1} \right)_{\lambda=0,\mu=0}, \]
on the second level, the equations for the Baker-Akhiezer and dual Baker-Akhiezer type wave functions (the modified equations)
\[ \Psi(g) = \int \psi(\lambda, g) \rho(\lambda) d\lambda, \]
\[ \Psi(g) = \int \tilde{\rho}(\mu)\tilde{\psi}(\mu, g) d\mu, \]

and on the third level – the equations for the Cauchy-Baker-Akhiezer type wave function
\[ \Phi(g) = \int \int (\psi(\lambda, \mu; g)) \rho(\lambda) \tilde{\rho}(\mu) d\lambda d\mu, \]

where \( \rho(\lambda), \tilde{\rho}(\mu) \) are some arbitrary weight functions. The equations of all three levels possess an infinite number of commuting symmetries and form in some sense a hierarchy of integrable discrete equations represented in the form of the general equation labeled by three continuous parameters (the lattice parameters).

To present discrete equations forming three levels of generalized KP hierarchy, we introduce difference and shift operators
\[ T_a f(g) = f(g \times g_a^{-1}), \]
\[ \Delta_a = T_a - 1, \]
\[ \tilde{T}_a f(g) = T_a^{-1} f(g) = f(g \times g), \]
\[ \tilde{\Delta}_a = 1 - \tilde{T}_a, \]

where elementary rational loop \( g_a(\nu) \) is defined as
\[ g_a(\nu) := \frac{\nu - a}{\nu} \]

We will use shift and difference operators with different values of lattice parameter \( a = a_i \) denoting
\[ T_i = T_{a_i}. \]

The first level of the generalized KP hierarchy is formed by equations for the potential \( u \)
\[ \sum_{(ijk)} \epsilon_{ijk} T_k \left( \frac{\Delta_i}{a_i} u - u T_i u \right) = 0, \tag{4} \]
where \( i \neq j \neq k \neq i; i, j, k \in \{1, 2, 3\} \), summation goes over different permutations of indices. Hirota bilinear identity (1) also implies linear equations
\[ \left( \frac{\Delta_i}{a_i} - \frac{\Delta_j}{a_j} \right) \tilde{\Psi}(g) = ((T_i - T_j)u(g)) \tilde{\Psi}(g) \tag{5} \]
and 
\[ \left( \frac{\Delta_i}{a_i} - \frac{\Delta_j}{a_j} \right) \Psi(g) = \left( (\bar{T}_i - \bar{T}_j)u(g) \right) \Psi(g), \quad (6) \]
that generate equation (4) as compatibility condition.

The second level of the generalized hierarchy is formed by equations for the wave functions of linear operators producing equations of the first level as compatibility conditions (the modified equations). It splits into two parts: equations for the dual wave function \( \tilde{\Psi} \)

\[ \sum_{(ijk)} \epsilon_{ijk} a_j a_k T_k \left( \tilde{\Psi}^{-1}(T_i \tilde{\Psi}) \right) = 0, \quad (7) \]
and equations for the wave functions \( \Psi \)

\[ \sum_{(ijk)} \epsilon_{ijk} a_j a_k \bar{T}_k \left( \Psi^{-1}(\bar{T}_i \Psi) \right) = 0. \quad (8) \]

The third level represents equations for the wave functions \( \Phi \) of linear operators of the second level (the singular manifold type equations)

\[ (T_j \Delta_i \Phi)(T_k \Delta_j \Phi)(T_i \Delta_k \Phi) = (T_j \Delta_k \Phi)(T_k \Delta_i \Phi)(T_i \Delta_j \Phi). \quad (9) \]
One could expect this chain to continue, but the wave functions of linear operators of the third level coincide with the wave functions of linear operators of the first level, and so the chain closes.

Equations of the second and third levels of the hierarchy and linear problems for them can be derived from the set of simple equations that directly follows from identity (1), namely

\[ \frac{\Delta_i}{a_i} \Phi = \tilde{\Psi} T_i \Psi \quad (10) \]
and, equivalently,

\[ \frac{\Delta_i}{a_i} \Phi = \Psi \bar{T}_i \tilde{\Psi}. \quad (11) \]

To reproduce second and third levels of the generalized KP hierarchy from the first, taking as a basic object equation (4) and without reference to bilinear identity, it is enough to notice that linear
equations (5) and (6) imply that there exists a function $\Phi(x)$ satisfying equations (10), (11). Indeed, using equations (5), (6) it is easy to check that cross-differences for the set of equations (10), (11) are equal, and the function $\Phi$ is well-defined on the lattice (and through some limit also as a function of continuous variables).

Connections between three different levels of the hierarchy of discrete equations may be described in terms of Miura maps and Cosmicure symmetry transformations, which are in some sense complementary.

3 From Discrete Equations to the Continuous Hierarchy

The loop $g \in \Gamma^+$ can be parametrized by the infinite set of complex variables $x_i, 1 \leq i \leq \infty$,

$$g(\lambda) = \exp(\sum_{i=1}^{\infty} x_i \lambda^{-i}),$$

and then the functionals of $g$ may be considered as functions of the infinite set of variables

$$\mathbf{x} = \{x_1, x_2, \ldots, x_n, \ldots\}.$$  

The transformation operators $T_i$ now look like $T_i : \mathbf{x} \to \mathbf{x} + [a_i]$, where $|a_i|_n = \frac{1}{n} a^n$. To compactify the notations, we will also use scaled difference operators

$$\Delta_i = a_i^{-1} \Delta_i, \quad \bar{\Delta}_i = a_i^{-1} \bar{\Delta}_i.$$  

The transformation operators $T_a$ can be represented in terms of differential operators in the form $T_a = \sum_{n=1}^{\infty} a^n p_n(\partial), \quad \bar{T}_a = \sum_{n=1}^{\infty} a^n p_n(-\partial)$, where $\tilde{\partial} = \left(\frac{\partial}{\partial x_1}, \frac{1}{2} \frac{\partial}{\partial x_2}, \ldots, \frac{1}{n} \frac{\partial}{\partial x_n}, \ldots\right)$, and $p_i$ are the Schur polynomials generated by the relation $\exp \left(\sum_{n=1}^{\infty} \lambda^n x_n\right) = \sum_{n=0}^{\infty} p_i(x) \lambda^n$. For the first three continuous variables we will use the notations $x = x_1$, $y = x_2$, $t = x_3$.

To demonstrate that the discrete form of KP hierarchy (1) written in terms of continuous variables as

$$\sum_{(ijk)} \epsilon_{ijk} T_k \left( a_i \Delta_i u(x) - u(x) T_i u(x) \right) = 0$$

(12)
generates equations of KP hierarchy in the standard form, we consider expansion of this equation into powers of parameters \(a_i, a_j, a_k\). The zeroth order of expansion of equation (12) into powers of the parameter \(a_i\) gives the equation containing two discrete transformations and one partial derivative

\[
T_k (\partial_x u - uu) - (\Delta_k u - u T_k u) + (\Delta_j u - u T_j u) - T_j (\partial_x u - uu) + T_j (\Delta_k u - u T_k u) - T_k (\Delta_j u - u T_j u) = 0. \quad (13)
\]

The first order of expansion of equation (13) into the powers of the parameter \(a_j\) represents an equation containing partial derivatives over two continuous variables and one discrete transformation,

\[
\left( \frac{1}{2} \left( \partial_y + \partial_y^2 \right) u - u \partial_x u \right) - \partial_x (\partial_x u - uu) + \partial_x (\Delta_k u - u T_k u) - T_k \left( \frac{1}{2} \left( \partial_y + \partial_y^2 \right) u - u \partial_x u \right) = 0. \quad (14)
\]

The final step is to take the first nontrivial order of the expansion into the powers of \(a_k\) (the second order) to get the potential form of the KP equation

\[
\partial_x \left( u_t - \frac{1}{4} u_{xxx} + \frac{3}{2} (u_x)^2 \right) = \frac{3}{4} u_{yy}, \quad (15)
\]

which reduces to standard KP equation for the function \(v = -2 \partial_x u\).

The higher orders of expansion of equation (14) will give us the higher equations of the KP hierarchy. This sequence of equations should be used recursively to get equations containing only partial derivatives over the highest order time, \(\partial_x\) and \(\partial_y\).

The interpretation of the chain of equations we have derived depends on the choice of the basic equation (i.e., in some sense on the point of reference).

A standard way is to take continuous equation (13) (or, rather, the KP hierarchy in the form of PDEs) as a basic system. Then the interpretation of the other objects is: 1) equation (14) defines a Bäcklund transformation for equation (12), 2) equation (13) is a superposition principle for two Bäcklund transformations, 3) discrete equation (12) provides an algebraic superposition principle for three Bäcklund transformations.

On the other hand, the discrete equation (12) (in other words, the discrete form of the KP hierarchy) may be treated as a basic system as
well. Then formula (13) is a lowest order continuous symmetry for this system, equation (14) is a superposition principle for two continuous symmetries, and equation (15) is a superposition principle for three continuous symmetries of different orders.

Linear problems (3), (4) generating the discrete form of the KP hierarchy (2) as compatibility conditions in terms of continuous variables look like

\[ (\Delta_i - \Delta_j) \tilde{\Psi}(x) = ((T_i - T_j)u(x))\tilde{\Psi}(x), \]  
(16) \[ (\bar{\Delta}_i - \bar{\Delta}_j)\Psi(x) = ((\bar{T}_i - \bar{T}_j)u(x))\Psi(x). \]  
(17)

Both the set of equations (16) and the dual set (17) imply the same equation (2). Expansion of these linear equations into powers of parameters gives standard linear problems for KP hierarchy.

We will not use equations of the second level of generalized KP hierarchy in the present work. Equation for \( \Phi \) (discrete form of KP singular manifold equation hierarchy) in terms of continuous variables looks like

\[ (T_j \Delta_i \Phi(x))(T_k \Delta_j \Phi(x))(T_i \Delta_k \Phi(x)) \]
\[ = (T_j \Delta_k \Phi(x))(T_k \Delta_i \Phi(x))(T_i \Delta_j \Phi(x)), \]  
(18)

and, performing expansion into powers of parameters, we get the chain of equations connecting discrete and continuous case,

\[ (T_j \Phi_x)(T_k \Delta_j \Phi)\Delta_k \Phi = (T_k \Phi_x)(T_j \Delta_k \Phi)\Delta_j \Phi, \]  
(19)

\[ \frac{\partial}{\partial x} \ln \left( \frac{1}{\Phi_x} \frac{\Delta \Phi}{a} \right) = \frac{1}{2} \Delta \left( \frac{\Phi_y + \Phi_{xx}}{\Phi_x} \right), \]  
(20)

\[ \Phi_t = \frac{1}{4} \Phi_{xxx} + \frac{3}{8} \frac{\Phi_y \Phi_{xx} - \Phi_{xx}^2}{\Phi_x} + \frac{3}{4} \Phi_x W_y, \quad W_x = \frac{\Phi_y}{\Phi_x}. \]  
(21)

The last equation first arose in Painleve analysis of the KP equation as a singular manifold equation [9].

The interpretation of this chain of equations is similar to the interpretation given for equation (2).

Thus the integrable discrete equations written in terms of elementary rational loops encode the continuous hierarchy, the Bäcklund transformations and different types of superposition principles for them, and the discrete linear equations generate a hierarchy of linear problems, Darboux transformations and superposition principles for them.
4 Discrete and Continuous Non-Iso-

spectral Symmetries

The dynamics defined by Hirota bilinear identity (1) is connected with operator of multiplication by loop group element \( g \in \Gamma^+ \); this dynamics can be interpreted in terms of commuting flows corresponding to infinite number of ‘times’ \( x_n \). A general idea of introduction additional (in general, non-commutative) symmetries is to consider more general operators \( \hat{R} \) on the unit circle. Let us introduce symmetric bilinear form

\[
(f|g) = \oint f(\nu)g(\nu)d(\nu).
\]

In terms of this form identity (1) looks like

\[
(\chi(\ldots, \mu; g_1)g_1(\ldots)|g_2^{-1}(\ldots)\chi(\lambda, \ldots; g_2)) = 0 \quad \lambda, \mu \in D, \quad (22)
\]

or, for Cauchy-Baker-Akhiezer function \( \psi(\lambda, \mu; g) \),

\[
(\psi(\ldots, \mu; g_1)|\psi(\lambda, \ldots; g_2)) = 0 \quad \lambda, \mu \in D, \quad (23)
\]

where by dots we denote the argument which is involved into integration. Let some CBA function \( \psi(\lambda, \mu; g) \) satisfying Hirota bilinear identity be given. We define symmetry transformation connected with arbitrary invertible linear operator \( \hat{R} \) in the space of functions on the unit circle by the equations

\[
(\tilde{\psi}(\ldots, \mu; g_1)|\hat{R}\psi(\lambda, \ldots; g_2)) = 0,
\]

\[
(\psi(\ldots, \mu; g_1)|\hat{R}^{-1}\tilde{\psi}(\lambda, \ldots; g_2)) = 0.
\]

It is possible to show that if both these equations for the transformed CBA function \( \tilde{\psi}(\lambda, \mu; g) \) are solvable, then the solution for them is the same (and unique), and it satisfies identity (23). In this case the symmetry transformation connected with operator \( \hat{R} \) is correctly defined. It is also possible to define one-parametric groups of transformations by the equation

\[
(\psi(\ldots, \mu; g_1, \Theta_1)|\exp((\Theta_1 - \Theta_2)\hat{r})|\psi(\lambda, \ldots; g_2, \Theta_2)) = 0. \quad (24)
\]

Taking the generators \( \hat{r}_{mn} = \lambda^n \partial_\lambda^m \), we get noncommutative symmetries in the form proposed by Orlov and Shulman [7]. In our work we will consider non-isospectral symmetries connected with operators
with degenerate kernel, and, in particular, generators with the kernel of the form
\[
r_{\alpha \beta}(\nu, \nu') = \frac{1}{2\pi i} \delta(\alpha - \nu) \delta(\beta - \nu'),
\]
(25)
where \(\alpha, \beta\) belong to the unit circle, or, more generally,
\[
r_{\rho \tilde{\rho}}(\nu, \nu') = \frac{1}{2\pi i} \tilde{\rho}(\nu') \rho(\nu),
\]
(26)
where for simplicity we put \((\tilde{\rho} | \rho) = 0\).

**Remark.** To make a transformation from the generators \(\hat{r}_{\alpha \beta}\) to the generators \(\hat{r}_{mn}\) used by Orlov and Shulman, it is enough to note that operator \(\hat{r}_{\alpha \beta}\) can be represented as a composition of the shift operator \(T_{\beta - \alpha} : \nu \to \nu + \beta - \alpha\) and operator of multiplication by the function \(\delta(\nu - \alpha)\). Then, expanding shift operator and \(\delta\)-function into powers of parameters, it is possible to make a transformation from one set of generators to the other.

Using simple identity
\[
\exp(\Theta_{\alpha \beta} \hat{r}_{\alpha \beta}) = I + \Theta_{\alpha \beta} \hat{r}_{\alpha \beta},
\]
which is satisfied due to nilpotence of the generators, and performing integration in the equation (24) taken for \(g_1 = g_2\), which in this case reads
\[
\oint \oint d\nu d\nu' \psi(\nu, \mu; g, \Theta_1)(\delta(\nu - \nu') + \frac{\Theta_1 - \Theta_2}{2\pi i} \delta(\beta - \nu') \delta(\alpha - \nu))
\times \psi(\lambda, \nu'; g, \Theta_2)) = 0,
\]
(27)
we get equation for the CBA function
\[
\psi(\lambda, \mu; x, \Theta_{\alpha \beta} + \Delta \Theta_{\alpha \beta}) = \psi(\lambda, \mu; x, \Theta_{\alpha \beta})
+ \Delta \Theta_{\alpha \beta} \psi(\lambda, \beta; x, \Theta_{\alpha \beta}) \psi(\alpha, \mu; x, \Theta_{\alpha \beta} + \Delta \Theta_{\alpha \beta}).
\]
(28)
It is possible to resolve this equation and express \(\psi(\lambda, \mu; x, \Theta_{\alpha \beta} + \Delta \Theta_{\alpha \beta})\) through \(\psi(\lambda, \mu; x, \Theta_{\alpha \beta})\). First we take equation (28) at \(\lambda = \alpha\) and get the expression for \(\psi(\alpha, \mu; x, \Theta_{\alpha \beta} + \Delta \Theta_{\alpha \beta})\),
\[
\psi(\alpha, \mu; x, \Theta_{\alpha \beta} + \Delta \Theta_{\alpha \beta}) = \frac{\psi(\alpha, \mu; x, \Theta_{\alpha \beta})}{1 - \Delta \Theta_{\alpha \beta} \psi(\alpha, \beta; x, \Theta_{\alpha \beta})}.
\]
(29)
Substituting (29) into (28), we finally get
\[
\psi(\lambda, \mu; x, \Theta_{\alpha\beta} + \Delta \Theta_{\alpha\beta}) = \psi(\lambda, \mu; x, \Theta_{\alpha\beta}) + \Delta \Theta_{\alpha\beta} \frac{\psi(\lambda, \beta; x, \Theta_{\alpha\beta}) \psi(\alpha, \mu; x, \Theta_{\alpha\beta})}{1 - \Delta \Theta_{\alpha\beta} \psi(\alpha, \beta; x, \Theta_{\alpha\beta})}.
\] (30)

In particular, this formula expresses the function \(\psi(\lambda, \mu; x, \Theta_{\alpha\beta})\) through the initial data \(\psi_0(\lambda, \mu; x) = \psi(\lambda, \mu; x, \Theta_{\alpha\beta} = 0)\), thus giving explicit formula for the action of non-isospectral symmetry connected with the generator (25) on the CBA function.

Formula (30) can be rewritten as
\[
\psi(\lambda, \mu; x, \Theta_{\alpha\beta}) = \psi_0(\lambda, \mu; x) \frac{1 - \Theta_{\alpha\beta} \psi_0(\alpha, \beta; x)}{1 - \Theta_{\alpha\beta} \psi(\alpha, \beta; x)},
\] (31)

where
\[
\det_{\alpha\beta} \psi_0(\lambda, \mu; x) = \det \left( \begin{array}{cc}
\psi_0(\lambda, \mu; x) & \psi_0(\lambda, \beta; x) \\
\psi_0(\alpha, \mu; x) & \psi_0(\alpha, \beta; x)
\end{array} \right).
\] (32)

Recalling determinant formula for the transformation of CBA function under the action of a rational loop (see [4, 1]),
\[
\psi_0(\alpha, \beta; x + [\mu] - [\lambda]) = \frac{\det_{\lambda\mu} \psi_0(\alpha, \beta; x)}{\psi_0(\lambda, \mu; x)},
\] (33)

we get another representation of the transformation (30),
\[
\psi(\lambda, \mu; x, \Theta_{\alpha\beta}) = \psi_0(\lambda, \mu; x) \frac{1 - \Theta_{\alpha\beta} \psi_0(\alpha, \beta; x + [\mu] - [\lambda])}{1 - \Theta_{\alpha\beta} \psi_0(\alpha, \beta; x)}.
\] (34)

Comparing this formula with the formula connecting the CBA function and the \(\tau\)-function (which in fact defines the \(\tau\)-function through the CBA function)
\[
\psi(\lambda, \mu, x) = g(\lambda) g(\mu)^{-1} \frac{1}{\lambda - \mu} \frac{\tau(x + [\mu] - [\lambda])}{\tau(x)},
\] (35)

we come to the conclusion that the \(\tau\)-function corresponding to the transformed CBA function \(\psi(\lambda, \mu; x, \Theta_{\alpha\beta})\) is given by the expression
\[
\tau(x, \Theta_{\alpha\beta}) = \tau_0(x) \frac{1 - \Theta_{\alpha\beta} \psi_0(\alpha, \beta; x)}{1 - \Theta_{\alpha\beta} \psi_0(\alpha, \beta; x)}.
\] (36)
Thus we have explicitly defined action of non-isospectral symmetry with the generator \((25)\) on KP \(\tau\)-function. Transformation \((36)\) coincides with the solitonic transformation of the \(\tau\)-function defined through Date-Jimbo-Kashiwara-Miwa vertex operator \(\mathcal{S}\). Below we will demonstrate that in terms of potential this is just a binary Bäcklund transformation, and for some choice of KPSM solution \(\Phi(x)\) this is a Möbius transformation.

For the function \(\Phi_{\alpha\beta} = \psi(\alpha, \beta; x)\) satisfying singular manifold equation \((18)\) from the formula \((30)\) we get especially simple transformation,

\[
\Phi_{\alpha\beta}(x, \Theta_{\alpha\beta}) = \frac{\Phi_{\alpha\beta}^0(x)}{1 - \Theta_{\alpha\beta}\Phi_{\alpha\beta}^0(x)},
\]

and this is nothing more than one-parametric subgroup of the Möbius group. Taking this formula at \(\Theta_{\alpha\beta} \to \infty\), we get (up to a constant) transformation of inversion \(\Phi_{\alpha\beta} \to \Phi_{-\alpha\beta}^{-1}\).

It is easy to check that the same derivation holds for the generators \((26)\) \(\hat{r}_\rho \tilde{r}_\rho\), and in this case we get Möbius transformation

\[
\Phi(x, \Theta) = \frac{\Phi^0(x)}{1 - \Theta \Phi^0(x)},
\]

for the solution of KPSM equation \((18)\) corresponding to the weight functions \(\rho(\nu), \tilde{\rho}(\nu)\)

\[
\Phi(x) = \oint \oint (\psi(\lambda, \mu; x))\rho(\lambda)\tilde{\rho}(\mu)d\lambda d\mu.
\]

5 Möbius Symmetry

In this section we will concentrate on Möbius symmetry of KPSM equation \((18)\), using only equations of generalized hierarchy and connections between them, without explicit use of bilinear technique underlying the construction. We will demonstrate that Möbius symmetry on the level of KPSM hierarchy generates binary Bäcklund transformations on the level of the basic KP hierarchy.

Characteristic feature of singular manifold equation \((18)\) is its invariance under Möbius transformation

\[
\Phi \to \frac{a\Phi + b}{c\Phi + d},
\]
which can be easily checked. Now we are going to find the symmetry of the basic KP hierarchy (12), which corresponds to the Möbius transformation on the level of the singular manifold equation. To do that, we define the transformations of the wave functions \( \Psi, \tilde{\Psi} \) using the equations (10), and then we substitute the wave functions into linear equations (16) and (17) to find the transformation of the potential \( u \). Generic Möbius transformation can be represented as composition of translation, scaling and inversion. Translation and scaling of \( \Phi \) do not change the potential \( u \) (translation doesn’t change wave functions, and scaling of wave functions doesn’t change the potential), and so in principle our problem is to find the transformation of potential \( u \) corresponding to inversion \( \Phi \to \Phi^{-1} \). Transformations of the wave functions, according to equations (10), look like \( \tilde{\Psi} \to -\Phi^{-1}\tilde{\Psi}, \Psi \to \Phi^{-1}\Psi \), and, substituting them into linear equations (16) and (17), we get the formula for the transformation of potential \( u \),

\[
u(\Phi^{-1}) = u(\Phi) - \Psi\Phi^{-1}\tilde{\Psi}.
\]

(39)

Taking into account equation

\[
\partial_x \Phi = \tilde{\Psi}\Psi,
\]

(40)
arising in the zeroth order of expansion of equation (10), it is possible to rewrite formula (39) as

\[
u(\Phi^{-1}) = u(\Phi) - \partial_x \ln \Phi,
\]

(41)

which is a well-known binary Bäcklund transformation. Thus we have shown that inversion on the level of KPSM equation hierarchy leads to binary Bäcklund transformation on the level of the basic KP hierarchy. The connection between Möbius transformation and binary Bäcklund transformation was discovered in the framework of Painleve analysis [9].

Let us consider a one-parametric subgroup of the Möbius group

\[
\Phi(\Theta) = \frac{\Phi_0}{1 - \Theta\Phi_0},
\]

(42)

characterized by the equation

\[
\partial_\Theta \Phi = \Phi^2.
\]

(43)
Using equation (41), it is easy to find continuous symmetry of KP hierarchy corresponding to this subgroup. First, directly from (41) we get a formula

\[ u(\Theta) := u \left( \frac{\Phi_0}{1 - \Theta \Phi_0} \right) = u \left( \frac{1 - \Theta \Phi_0}{\Phi_0} \right) - \partial_x \ln \left( \frac{1 - \Theta \Phi_0}{\Phi_0} \right). \]  

(44)

Taking into account that

\[ u \left( \frac{1 - \Theta \Phi_0}{\Phi_0} \right) = u(\Phi_0^{-1}), \]

and transforming \( u(\Phi_0^{-1}) \) using formula (41), we finally find symmetry transformation of potential \( u \) depending on continuous parameter \( \Theta \),

\[ u(\Theta) = u_0 - \partial_x \ln(1 - \Theta \Phi_0). \]  

(45)

Potential \( u \) satisfies differential relation

\[ \partial_\Theta u = \partial_x \Phi, \]  

(46)

or, taking into account formula (40),

\[ \partial_\Theta u = \Psi \bar{\Psi}. \]  

(47)

Expression \( \Psi \bar{\Psi} \) represents infinitesimal (in general non-isospectral) symmetry of KP hierarchy [13], and the formula (45) defines a one-parametric group of transformations connected with this symmetry (specified by extra relation (43)). Exactly this form of symmetry generator is used to define constrained KP hierarchy [11], [12], which will be one of the objects of our study.

We will also consider more general symmetry transformations of KPSM hierarchy (18), which we call multicomponent Möbius-type transformations. We have used an arbitrary pair of wave functions \( \Psi, \bar{\Psi} \) to define the function \( \Phi \) through the set of equations (10). Let us fix a set of wave functions and dual wave functions \( \Psi_k, \bar{\Psi}_k, 1 \leq k \leq \infty \) (in terms of Hirota bilinear identity we should fix a set of weight functions \( \rho_k(\nu), \bar{\rho}_k(\nu) \)). Then equations (10) define a matrix of solutions of equation (18) \( |\Phi| \) connected with the same solution \( u \) of KP hierarchy (12). Matrix entries \( \Phi_{kp} \) satisfy the equations

\[ \Delta_i \Phi_{kp} = \bar{\Psi}_k T_i \Psi_p, \]  

(48)
or, in matrix form,
\[
\frac{\Delta_i}{a_i}|\Phi| = |\bar{\Psi}|\mathcal{T}_i\langle\Psi|. \tag{49}
\]
It is easy to check that matrix inversion $|\Phi| \to |\Phi|^{-1}$ leads to the same equation \[49\] with transformed $|\bar{\Psi}|$, $\langle\Psi|$, $|\bar{\Psi}| \to |\Phi|^{-1}|\bar{\Psi}|$, $\langle\Psi| \to \langle\Psi||\Phi|^{-1}$. Substituting transformed vectors of wave functions into linear equations \[13\], \[17\), we come to the conclusion that all components of the wave functions give the same transformed potential
\[
u \to u - \langle\Psi||\Phi|^{-1}|\bar{\Psi}|. \tag{50}
\]
Taking into account that equation $\partial_x|\Phi| = |\bar{\Psi}\rangle\langle\Psi|$ imply the identity
\[
\langle\Psi||\Phi|^{-1}|\bar{\Psi}| = \partial_x \ln \det |\Phi|,
\]
we get another form of transformation of potential corresponding to multicomponent Möbius-type transformation,
\[
u(|\Phi|^{-1}) = u(|\Phi|) - \partial_x \ln \det |\Phi|, \tag{51}\]
which represents a composition formula for several binary Bäcklund transformations.

Multicomponent continuous Möbius-type symmetry
\[
|\Phi(\Theta)| = |\Phi_0|(I - \Theta|\Phi_0|)^{-1}, \quad \partial_\Theta|\Phi| = |\Phi|\langle\Phi|,
\]
leads to continuous symmetry for the potential
\[
u(\Theta) = u_0 - \partial_x \ln \det(I - \Theta|\Phi_0|), \tag{52}
\]
\[
\partial_\Theta u = \partial_x \text{tr} |\Phi| = \sum_{i=1}^N \Psi_i\bar{\Psi}_i. \tag{53}
\]

6 Symmetry Constraints and Calogero-Moser System

The concept of generalized hierarchy is rather effective tool in the study of symmetry constraints. A standard symmetry constraint for KP hierarchy is \[11\], \[12\]
\[
u_x = \Psi\bar{\Psi}, \tag{54}\]
and it was shown in \[11\] that it leads to AKNS hierarchy for the wave functions. It is possible also to derive two-dimensional equation for one function (either $u$ or $\Phi$). Indeed, it was shown above that $\Phi_x = \Psi \bar{\Psi}$, so for the constrained hierarchy

$$u_x = \Phi_x.$$  

Thus $u$ and $\Phi$ represent almost the same object, and the meaning of the constraint is that it glues the first and the third level of generalized hierarchy. We know that $u$ satisfies KP equation (15), and $\Phi$ satisfies KPSM equation, but, using the constraint, we can write two equations for both of these functions. Combining these equations, it is easy to eliminate the terms containing partial derivative over $t$ and get two-dimensional differential relation, for $\Phi$ it looks like (see also \[11\])

$$\partial_x \left( \partial_x\left( \frac{3}{2} \Phi_x^2 \right) - \frac{3}{8} \frac{\Phi_y^2}{\Phi_x} \right) + \frac{3}{4} \left( \frac{\Phi_y \Phi_{xy}}{\Phi_x} - \frac{3}{4} \Phi_{xx} W_y \right) = 0, \quad W_x = \frac{\Phi_y}{\Phi_x}.$$  

Let us consider one-parametric group of symmetry transformations of potential $u$ defined by the formula (52); we have shown that $u$ satisfies differential relations

$$\partial_\Theta u = \partial_x \text{tr} |\Phi| = \sum_{i=1}^N \bar{\Psi}_i \Psi_i.$$  

According to these relations, standard constrains of the type (54) can be interpreted as an equation

$$u_\Theta = u_x, \quad \Theta = 0.$$  

There is stronger symmetry constraint, for which relation (55) is required to be satisfied for all $\Theta$, not only at the origin. The dependence of $u$ on extra time $\Theta$ is rational, so the constraints of this type impose rational dependence of $u$ on $x$. In this way we come to rational Calogero-Moser system. Indeed, let us make a simple transformation of the formula (52) using relation (51),

$$u(\Theta) = u(|\Phi_0|^{-1}) - \partial_x \ln \det(|\Phi_0|^{-1} - \Theta),$$  

and substitute the result to the equation (55). Comparing the singularities, we come to the conclusion that

$$v = -2u_x = \sum_{i=1}^N \left( \phi_i(y, \cdots) - x - \Theta \right)^2,$$  

16
where $\phi_i$ are eigenvalues of the matrix $|\Phi(x = 0, y, t, \cdots)|^{-1}$. Due to the constraint the eigenvalues of this matrix should depend linearly on $x$, $\phi_i(x) = \phi_i(0) - x$, and also $\partial_x u(\Phi_0|^{-1}) = 0$. The substitution for the potential [57] characterizes Calogero-Moser integrable system of particles on the line [14]. Thus we have demonstrated that this system can be obtained through the symmetry constraint of KP hierarchy.

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