LOCAL CONTRIBUTIONS TO DONALDSON-THOMAS INVARIANTS

ANDREA T. RICOLFI

ABSTRACT. Let $C$ be a smooth curve embedded in a smooth quasi-projective threefold $Y$, and let $Q^n_C = \text{Quot}_n(I_C)$ be the Quot scheme of length $n$ quotients of its ideal sheaf. We show the identity $\tilde{\chi}(Q^n_C) = (-1)^n \chi(Q^n_C)$, where $\tilde{\chi}$ is the Behrend weighted Euler characteristic. When $Y$ is a projective Calabi-Yau threefold, this shows that the DT contribution of a smooth rigid curve is the signed Euler characteristic of the moduli space. This can be rephrased as a DT/PT wall-crossing type formula, which can be formulated for arbitrary smooth curves. In general, such wall-crossing formula is shown to be equivalent to a certain Behrend function identity.

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1. INTRODUCTION

One of the conjectures in [MNOP06] stated that 0-dimensional Donaldson-Thomas (DT, for short) invariants of a smooth projective Calabi-Yau threefold equal the signed Euler characteristic of the moduli space. Now, the more general formula

\begin{equation}
\tilde{\chi}(\text{Hilb}^n Y) = (-1)^n \chi(\text{Hilb}^n Y)
\end{equation}

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is known to hold for any smooth threefold $Y$, proper or not [BF08, Thm. 4.11]. Here $\tilde{\chi} = \chi(-, \nu)$ is the Euler characteristic weighted by the Behrend function [Beh09]. The 0-dimensional MNOP conjecture is also solved with cobordism techniques in [Li06, LP09].

1.1. Main result. We propose a statement analogous to (1.1), again with no Calabi-Yau or properness assumption on the threefold $Y$, but where a curve is present. More precisely, we focus on the space of 1-dimensional subschemes $Z \subset Y$ whose fundamental class is the cycle of a fixed Cohen-Macaulay curve $C \subset Y$. A natural scheme structure on this space seems to be provided by the Quot scheme $Q^n_C = \text{Quot}_n(I_C)$ of 0-dimensional length $n$ quotients of $I_C$, the ideal sheaf of $C$. By identifying a surjection $I_C \twoheadrightarrow F$ with its kernel $I_Z$, we see that $Q^n_C$ parametrizes curves $Z \subset Y$ differing from $C$ by a finite subscheme of length $n$. Our main result, proved in §4, is the following weighted Euler characteristic computation.

**Theorem.** Let $Y$ be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. If $Q^n_C = \text{Quot}_n(I_C)$, then

\[
\tilde{\chi}(Q^n_C) = (-1)^n \chi(Q^n_C).
\]

The proof uses stratification techniques as in [BF08] and [BB07].

1.2. Applications. Let $Y$ be a smooth projective threefold. Let $I_m(Y, \beta)$ be the Hilbert scheme of curves $Z \subset Y$ in class $\beta \in H_2(Y, \mathbb{Z})$, with $\chi(\mathcal{O}_Z) = m$. Given a Cohen-Macaulay curve $C \subset Y$ of arithmetic genus $g$, embedded in class $\beta$, we show there is a closed immersion $i : Q^n_C \to I_{1-g+n}(Y, \beta)$. We define

\[
I_n(Y, C) \subset I_{1-g+n}(Y, \beta) = I
\]

to be its scheme-theoretic image. When $Y$ is Calabi-Yau, we define the contribution of $C$ to the full (degree $\beta$) DT invariant of $I$ to be the weighted Euler characteristic

\[
\text{DT}_{n,C} = \chi(I_n(Y, C), \nu_I).
\]

A first consequence of (1.2) is the identity

\[
\text{DT}_{n,C} = (-1)^n \chi(I_n(Y, C))
\]

when $C$ is a smooth rigid curve in $Y$, because in this case (1.3) is both open and closed.

1.2.1. Local DT/PT correspondence. Let $P_m(Y, \beta)$ be the moduli space of stable pairs introduced by Pandharipande and Thomas [PT09]. For a Calabi-Yau threefold $Y$ and a homology class $\beta \in H_2(Y, \mathbb{Z})$, the generating functions encoding the DT and PT invariants of $Y$ satisfy the “wall-crossing type” formula

\[
\text{DT}_\beta(Y, q) = M(-q)^{\chi(Y)} \cdot \text{PT}_\beta(Y, q).
\]

Here and throughout, $M(q)$ denotes the MacMahon function, the generating series of plane partitions, that is,

\[
M(q) = \sum_{\pi} q^{|\pi|} = \prod_{k \geq 1} (1 - q^k)^{-k}.
\]

The DT/PT correspondence stated above was first conjectured in [PT09] and later proved in [Bri11, Tod10]. In this paper we ask about a similar formula.
relating the local invariants, that is, the contributions of a single smooth curve $C \subset Y$ to the full DT and PT invariants of $Y$ in the class $\beta = [C]$.

If $C \subset Y$ is a fixed smooth curve of genus $g$, we consider the closed subscheme

$$P_n(Y, C) \subset P_{1-g+n}(Y, \beta) = P$$

of stable pairs with Cohen-Macaulay support equal to $C$. We use (1.2) and the isomorphism $P_n(Y, C) \cong \text{Sym}^n C$ to show the generating function identity

$$\sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n = M(-q)^{\chi(Y)} (1+q)^{2g-2},$$

which holds without any Calabi-Yau assumption.

For $Y$ a Calabi-Yau threefold, we consider the stable pair local contributions

$$\text{PT}_{n,C} = \chi(P_n(Y, C), \nu_P)$$

like we did in (1.4) for ideal sheaves. We assemble all the local invariants into generating functions

$$\text{DT}_C(q) = \sum_{n \geq 0} \text{DT}_{n,C} q^n$$

$$\text{PT}_C(q) = \sum_{n \geq 0} \text{PT}_{n,C} q^n.$$ 

The PT side has been computed [PT10, Lemma 3.4] and the result is

$$\text{PT}_C(q) = n_{g,C} \cdot (1+q)^{2g-2},$$

where $n_{g,C}$ is the BPS number of $C$. Therefore it is clear by looking at (1.5) that the DT/PT correspondence

$$\text{DT}_C(q) = M(-q)^{\chi(Y)} \cdot \text{PT}_C(q)$$

holds for $C$ if and only if, for every $n$, one has

$$\text{DT}_{n,C} = n_{g,C} \cdot \tilde{\chi}(I_n(Y, C)).$$

For instance, it holds when $C$ is rigid. In the last section, we discuss the plausibility to conjecture the identity (1.6) to hold for all smooth curves.

Conventions. All schemes are defined over $\mathbb{C}$, and all threefolds are assumed to be smooth. An ideal sheaf is a torsion-free sheaf with rank one and trivial determinant. For a smooth projective threefold $Y$, we denote by $I_n(Y, \beta)$ the moduli space of ideal sheaves with Chern character $(1, 0, -\beta, -m + \beta \cdot c_1(Y)/2)$. It is naturally isomorphic to the Hilbert scheme parametrizing closed subschemes $Z \subset Y$ of codimension at least 2, with homology class $\beta$ and $\chi(\mathcal{O}_Z) = m$. A Cohen-Macaulay curve is a scheme of pure dimension one without embedded points. The Calabi-Yau condition for us is simply the existence of a trivialization $\omega_Y \cong \mathcal{O}_Y$. We use the word rigid as a shorthand for the more correct infinitesimally rigid: for a smooth embedded curve $C \subset Y$, this means $H^0(C, N_{C/Y}) = 0$, where $N_{C/Y}$ is the normal bundle. Finally, we refer to [Beh09] for the definition and properties of the Behrend function and of the weighted Euler characteristic.
2. The Local Model

The global geometry of a fixed smooth curve in a threefold \( C \subset Y \) will be analysed through the local model \( A^1 \subset A^3 \) of a line in affine space. We get started by introducing the moduli space of ideal sheaves for this local model.

Let \( X \) be the resolved conifold, i.e. the total space of the rank two bundle \( O_{\mathbb{P}^1}(-1, -1) \rightarrow \mathbb{P}^1 \). It is a quasi-projective Calabi-Yau threefold. We let \( C_0 \subset X \) be the zero section, and \( A^3 \subset X \) a fixed chart of the bundle.

**Definition 2.1.** For any integer \( n \geq 0 \), we define

\[ M_n \subset I_{n+1}(X, [C_0]) \]

to be the open subscheme parametrizing ideal sheaves \( I_Z \subset O_X \) such that no associated point of \( Z \) is contained in \( X \setminus A^3 \).

Since \( C_0 \) is rigid, we can interpret \( M_n \) as the moduli space of “curves” in \( A^3 \), consisting of a fixed affine line \( L = C_0 \cap A^3 \) together with \( n \) roaming points.

The scheme \( M_n \) seems to be the perfect local playground for studying the enumerative geometry of a fixed curve (with \( n \) points) in a threefold. Exactly like \( \text{Hilb}^n A^3 \) was essential \([BF08]\) to unveil the Donaldson-Thomas theory of \( \text{Hilb}^n Y \), where \( Y \) is any Calabi-Yau threefold, the space \( M_n \) will help us to figure out the DT contribution of a fixed smooth rigid curve in a Calabi-Yau threefold (and, conjecturally, all smooth curves). Forgetting about the Calabi-Yau assumption, we will find out that understanding the local picture in \( A^3 \) gives information about arbitrary threefolds, in perfect analogy with the results of \([BF08]\).

In the rest of this section, we show that \( M_n \) is isomorphic to the Quot scheme of the ideal sheaf of a line, and we compute its DT invariant via equivariant localization.

Let \( L \) denote the line \( C_0 \cap A^3 \). Note that if \( Z \subset X \) corresponds to a point of \( M_n \), by definition its embedded points can only be supported on \( L \). Similarly, isolated points are confined to the chart \( A^3 \subset X \).

**Proposition 2.1.** There is an isomorphism of schemes \( M_n \cong \text{Quot}_n(\mathcal{I}_L) \).

**Proof.** Let \( T \) be a scheme and let \( i : A^3 \times T \rightarrow X \times T \) be the natural open immersion. If \( O_{X \times T} \rightarrow O_Z \) represents a \( T \)-valued point of \( M_n \), we can consider the sheaf \( \mathcal{F} = \mathcal{I}_{C_0 \times T}/\mathcal{I}_Z \), which by definition of \( M_n \) is supported on a subscheme of \( A^3 \times T \) which is finite of relative length \( n \) over \( T \). Restricting the short exact sequence

\[ 0 \rightarrow \mathcal{F} \rightarrow O_Z \rightarrow O_{C_0 \times T} \rightarrow 0 \]

to \( A^3 \times T \) gives a short exact sequence

\[ 0 \rightarrow i^* \mathcal{F} \rightarrow i^* O_Z \rightarrow O_{L \times T} \rightarrow 0 \]

with \( T \)-flat kernel, so we get a \( T \)-valued point \( \mathcal{I}_{L \times T} \rightarrow i^* \mathcal{F} \) of \( \text{Quot}_n(\mathcal{I}_L) \), since as we already noticed \( i^* \mathcal{F} \) has the same support as \( \mathcal{F} \).

Conversely, a \( T \)-flat quotient \( \mathcal{F} \) of the ideal sheaf \( \mathcal{I}_{L \times T} \) determines a flat family of subschemes

\[ Z \subset A^3 \times T \rightarrow T, \]
where \( L \times T \subset Z \). Taking closures inside \( X \times T \), we get closed immersions

\[
C_0 \times T \subset Z \subset X \times T.
\]

The support of \( \mathcal{F} \) is proper over \( T \), and since \( \mathbb{A}^3 \) and \( X \) are separated, we see that the inclusion maps of \( \text{Supp} \mathcal{F} \) in \( \mathbb{A}^3 \times T \) and \( X \times T \) are proper. This says that the pushforward \( _* \mathcal{F} \) is a coherent sheaf on \( X \times T \). It agrees with the relative ideal of the immersion \( C_0 \times T \subset Z \), and is supported exactly where \( \mathcal{F} \) is. Finally, the short exact sequence

\[
0 \to _* \mathcal{F} \to \mathcal{O}_Z \to \mathcal{O}_{C_0 \times T} \to 0
\]

says \( \mathcal{O}_Z \) is \( T \)-flat (being an extension of \( T \)-flat sheaves), therefore we get a \( T \)-valued point of \( M_n \). The two constructions are inverse to each other, whence the claim. \( \square \)

Keeping the above result in mind, we will sometimes silently identify \( M_n \) with \( \text{Quot}_n(\mathcal{I}_L) \), and we will switch from subschemes (or ideal sheaves) to quotient sheaves with no further mention.

**Remark 2.1.** The resolved conifold \( X \) plays little role here. In fact, the above proof shows the following. If there is an immersion \( \mathbb{A}^3 \to Y \) into some Calabi-Yau threefold \( Y \), such that the closure of a line \( L \subset \mathbb{A}^3 \) becomes a rigid rational curve \( C \subset Y \), then the Hilbert scheme \( I_{n+1}(Y, [C]) \) contains an open subscheme isomorphic to \( \text{Quot}_n(\mathcal{I}_L) \).

**2.1. The DT invariant.** The open subscheme \( M_n \subset I_{n+1}(X, [C_0]) \) inherits, by restriction, a torus-equivariant symmetric obstruction theory, and therefore an equivariant virtual fundamental class

\[
[M_n]^\text{vir} \in A^*_T(M_n) \otimes \mathbb{Q}(s_1, s_2, s_3).
\]

The torus \( T \subset (\mathbb{C}^\times)^3 \) we are referring to is the two-dimensional torus fixing the Calabi-Yau form on \( X \), and acting on \( X \) by rescaling coordinates. We refer the reader to [BB07, § 2.3] for more details on this action and for an accurate description of the fixed locus

\[
I_m(X, d[C_0])^T \subset I_m(X, d[C_0])
\]

for every \( d > 0 \). An ideal sheaf \( \mathcal{I}_Z \in M_n \) is \( T \)-fixed if it becomes a monomial ideal when restricted to the chosen chart \( \mathbb{A}^3 \subset X \). The fixed locus \( M_n^T \subset M_n \) is isolated and reduced, by [MNOP06, Lemma 6 and 8]. In the language of the topological vertex, a \( T \)-fixed ideal can be described as a way of stacking \( n \) boxes in the corner of the one-legged configuration \( (\square, \square, \square) \). We give an example in Figure 1.

**FIGURE 1.** A \( T \)-fixed ideal in \( M_n \). The “z-axis” has to be figured as infinitely long, corresponding to the line \( L = C_0 \cap \mathbb{A}^3 \).
The parity of the tangent space dimension at torus-fixed points of $I_m(X, d[C_0])$ was computed in [BB07, Prop. 2.7]. The result is $(-1)^{m-d}$ by an application of [MNOP06, Thm. 2]. In our case $m = n + 1$ and $d = 1$ so we get the sign $(-1)^n$ for $I_{n+1}(X, C_0)$. Since $M_n$ is open in this Hilbert scheme, the parity does not change and we deduce that

$$(-1)^{\dim T_{M_n}|_{\mathcal{I}}} = (-1)^n$$

for all fixed points $\mathcal{I} \in M_n^T$. After the Calabi-Yau specialization $s_1 + s_2 + s_3 = 0$ of the equivariant parameters, and by the symmetry of the obstruction theory, the virtual localization formula [GP99] reads

$$[M_n]^\text{vir} = (-1)^n [M_n^T] \in A_0(M_n),$$

where, as mentioned above, the sign

$$(-1)^n = \frac{e^T(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e^T(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} \in Q(s_1, s_2, s_3)$$

comes from [MNOP06, Thm. 2].

We define the Donaldson-Thomas invariant of $M_n$ by equivariant localization through formula (2.1). Hence we can compute it as

$$\text{DT}(M_n) = (-1)^n \chi(M_n),$$

where the Euler characteristic $\chi(M_n)$ counts the number of fixed points.

It is easy to see (see for instance the proof of [BB07, Lemma 2.9]) that

$$\sum_{n \geq 0} \chi(M_n)q^n = \frac{M(q)}{1-q}$$

where $M(q) = \prod_{m \geq 1}(1 - q^m)^{-m}$ is the MacMahon function, the generating series of plane partitions. In particular, the DT partition function for the moduli spaces $M_n$ takes the form

$$\sum_{n \geq 0} \text{DT}(M_n)q^{n+1} = q \frac{M(-q)}{1+q} = q(1 - 2q + 5q^2 - 11q^3 + \cdots).$$

In the sum, we have switched indices by one to follow the general convention of weighting the variable $q$ by the holomorphic Euler characteristic.

3. CURVES AND QUOT SCHEMES

3.1. Main characters. Let $C$ be a Cohen-Macaulay curve embedded in a quasi-projective variety $Y$ and let $\mathcal{I}_C \subset \mathcal{O}_Y$ denote its ideal sheaf. For an integer $n \geq 0$, let $Q = \text{Quot}_n(\mathcal{I}_C)$ be the Quot scheme parametrizing 0-dimensional quotients of $\mathcal{I}_C$, of length $n$. See [Nit05] for a proof of the representability of the Quot functor in the quasi-projective case. By looking at the full exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to F \to 0$$

for a given point $[\mathcal{I}_C \to F]$ of $Q$, we think of the Quot scheme as parametrizing curves $Z \subset Y$ obtained from $C$, roughly speaking, by adding a finite subscheme of length $n$.

Definition 3.1. We denote by $W_n^0 \subset Q$ the closed subset parametrizing quotients $\mathcal{I}_C \to F$ such that $\text{Supp} F \subset C$, where $\text{Supp} F$ denotes the set-theoretic support of the sheaf $F$. We endow $W_n^0$ with the reduced scheme structure.
Given a point \([F] \in \mathcal{W}^n_C\), the support of \(F\) has the structure of a closed sub-scheme of \(Y\) but not of \(C\) in general; however, \(\text{Supp} F\) defines naturally an effective zero-cycle on \(C\). Sending \([F]\) to this cycle is a morphism, as we now show.

**Lemma 3.1.** There is a natural morphism \(u : \mathcal{W}^n_C \to \text{Sym}^n C\) sending a quotient to the corresponding zero-cycle.

**Proof.** Let \(T\) be a reduced scheme, which we take as the base of a valued point \(\mathcal{I}_{C \times T} \to \mathcal{I}\) of \(\mathcal{W}^n_C\). Let \(\pi : Y \times T \to T\) be the projection. Working locally on \(Y\) and \(T\) we see that by Nakayama’s lemma, \(\text{Supp} \mathcal{I} \cap \pi^{-1}(t) = \text{Supp} \mathcal{I}_t\) for every closed point \(t \in T\). Then the closed subscheme \(\text{Supp} \mathcal{I} \subset Y \times T\) is flat over \(T\) (because the Hilbert polynomial of the fibres \(\text{Supp} \mathcal{I}_t\) is the constant \(n\) and \(T\) is reduced), and hence defines a valued point \(T \to \text{Hilb}^n Y\). Composing with the Hilbert-Chow map \(\text{Hilb}^n Y \to \text{Sym}^n Y\) we get a morphism \(T \to \text{Sym}^n Y\) which factors through \(\text{Sym}^n C\), by definition of \(\mathcal{W}^n_C\).

For every partition \(\alpha\) of \(n\) there is a locally closed subscheme \(\text{Sym}^n C \subset \text{Sym}^n C\) parametrizing zero-cycles with multiplicities dictated by \(\alpha\). These subschemes form a stratification of \(\text{Sym}^n C\), which we can use together with the morphism \(u\) to stratify \(\mathcal{W}^n_C\) by locally closed subschemes

\[
(3.1) \quad \mathcal{W}^n_C = u^{-1}(\text{Sym}^n_C) \subset \mathcal{W}^n_C.
\]

In particular, since \(\text{Sym}^n_C(\alpha) \cong C\), there is a natural morphism

\[
(3.2) \quad \pi_C : \mathcal{W}^n_C(\alpha) \to C
\]

corresponding to the deepest stratum.

The main result of this section asserts that, when \(C\) is a smooth curve and \(Y\) is a smooth threefold, the map (3.2) is a Zariski locally trivial fibration. The proof is based on the Quot scheme adaptation of the results proven by Behrend and Fantechi for \(\text{Hilb}^n Y\) [BF08, §4].

Let us now introduce what will turn out to be the typical fibre of \(\pi_C\). Recall that \(X\) denotes the resolved conifold and \(C_0 \subset X\) is the zero section.

**Definition 3.2.** We denote by \(F_n \subset M_n\) the closed subset parametrizing subschemes \(Z \subset X\) such that the relative ideal \(\mathcal{I}_C / \mathcal{I}_Z\) is entirely supported at the origin \(0 \in L = C_0 \cap A^3\). We use the shorthand

\[
v_n = v_{M_n} |_{F_n}
\]

for the restriction of the Behrend function on \(M_n\) to \(F_n\).

We can think of \(F_n\) and all strata \(\mathcal{W}^n_C \subset \mathcal{W}^n C\) as endowed with the reduced scheme structure.

**Remark 3.1.** The morphism \(u : \mathcal{W}^n_C \to \text{Sym}^n C\) plays the role of the Hilbert-Chow map \(\text{Hilb}^n Y \to \text{Sym}^n Y\) in the 0-dimensional setting, and the subscheme \(F_n \subset M_n\) is the analogue of the punctual Hilbert scheme \(\text{Hilb}^n(A^3)_0 \subset \text{Hilb}^n A^3\) parametrizing finite subschemes supported at the origin.

**Proposition 3.1.** There is a natural isomorphism \(\mathcal{W}^n_L = L \times F_n\). Moreover, if \(p : \mathcal{W}^n_L \to F_n\) is the projection, we have the relation

\[
(3.3) \quad v_{M_n} |_{\mathcal{W}^n_L} = p^* v_n.
\]
PROOF. We view \( L \) as the additive group \( G_a \) and we let it act on itself by translation. This induces an action of \( L \) on \( M_n \). Restricting this action to \( F_n \) gives a map

\[
L \times F_n \to W_{L(n)}^{(n)}.
\]

This is an isomorphism, whose inverse is the morphism \( \pi_L \times \rho : W_{L(n)}^{(n)} \to L \times F_n \), where

\[
\rho : W_{L(n)}^{(n)} \to F_n
\]

takes a subscheme \( [Z] \in W_{L(n)}^{(n)} \) to its translation by \(-x \in G_a\), where \( x \in L = G_a \) is the unique embedded point on \( Z \). The identity (3.3) follows because the Behrend function is constant on orbits and for each \( P \in F_n \) the slice \( L \times \{ P \} \) is isomorphic to an orbit.

\[
\square
\]

3.2. Comparing Quot schemes. Let \( \varphi : Y \to Y' \) be a morphism of varieties, where \( Y \) is quasi-projective and \( Y' \) is complete. Let \( C' \subset Y' \) be a Cohen-Macaulay curve and let \( C = \varphi^{-1}(C') \subset Y \) denote its preimage. We assume \( C \) is a Cohen-Macaulay curve and \( C' \) is its scheme-theoretic image. In Lemma 3.2 we give sufficient conditions for this to hold.

Given an integer \( n \geq 0 \), we let \( Q = \text{Quot}_n(\mathcal{I}_C) \) and \( Q' = \text{Quot}_n(\mathcal{I}_{C'}) \).

We will show how to associate to these data a rational map

\[
\Phi : Q \dashrightarrow Q'.
\]

The rough idea is that we would like to “push down” the \( n \) points in the support of a sheaf \( [F] \in Q \) and still get \( n \) points, which would ideally form the support of the image sheaf \( \varphi_* F \). This only works, as one might expect, over the open subscheme \( V \subset Q \) parametrizing sheaves \( F \) such that \( \varphi|_{\text{Supp} F} \) is injective. Moreover, the resulting map \( \Phi : V \to Q' \) turns out to be étale whenever \( \varphi \) is. After extending this result to quasi-projective \( Y' \), we will be able to compare \( \text{Quot}_n(\mathcal{I}_C) \) with the local picture of \( M_n = \text{Quot}_n(\mathcal{I}_L) \), and pull back (étale-locally) the known results about \( \pi_L \) (Proposition 3.1) to deduce that the maps \( \pi_C \) defined in (3.2) are Zariski locally trivial, at least when \( C \) and \( Y \) are smooth.

LEMMA 3.2. Let \( \varphi : Y \to Y' \) be an étale morphism of varieties with image \( U \). If \( C' \subset Y' \) is a Cohen-Macaulay curve and \( U \cap C' \) is dense in \( C' \), then \( C = \varphi^{-1}(C') \) is Cohen-Macaulay and \( C' \) is its scheme-theoretic image.

Before proving the lemma, recall that a closed subscheme \( C' \) of a scheme \( Y' \) is said to have an embedded component if there is a dense open subset \( U \subset Y' \) such that \( U \cap C' \) is dense in \( C' \) but its scheme-theoretic closure does not equal \( C' \) scheme-theoretically. Recall that a curve is Cohen-Macaulay if it has no embedded points.

PROOF. Since the restriction \( C \to C' \) is étale and \( C' \) is Cohen-Macaulay, \( C \) is also Cohen-Macaulay. Moreover, \( U \) is open (because \( \varphi \) is étale) and dense (because \( Y' \) is irreducible), and since \( U \cap C' \subset C' \) is dense, the scheme-theoretic closure of \( U \cap C' \) agrees with \( C' \) topologically. But since \( C' \) has no embedded points, they in fact agree as schemes. On the other hand, the open subset \( U \cap C' \subset C' \) is the set-theoretic image of the étale map \( C \to C' \). Therefore its scheme-theoretic closure is the scheme-theoretic image of \( C \to C' \). So \( C' \) is the scheme-theoretic image of \( C \).

\[
\square
\]
NOTATION. For a scheme $S$, we will denote $\varphi_S = \varphi \times \text{id}_S : Y \times S \to Y' \times S$. The case $S = Q$ being quite special, we will let $\tilde{\varphi}$ denote $\varphi_Q = \varphi \times \text{id}_Q$.

By our assumptions, $C' \times S$ is the scheme-theoretic image of $C \times S \subset Y \times S$ under $\varphi_S$, for any scheme $S$. Indeed, $\varphi$ is quasi-compact so the scheme-theoretic image commutes with flat base change.

**Remark 3.2.** Let $E$ be the universal sheaf on $Q$, with scheme-theoretic support $\Sigma \subset Y \times Q$. Since $\Sigma \to Q$ is proper (by the very definition of the Quot functor), and it factors through the (separated) projection $\pi : Y' \times Q \to Q$, necessarily the map $\Sigma \to Y' \times Q$ must be proper. Since $\tilde{\varphi}^*E$ is obtained as a pushforward from $\Sigma$, it is coherent. Therefore, pushing forward coherent sheaves supported on $\Sigma$ will still give us coherent sheaves, even if $\varphi$ is not proper.

**Remark 3.3.** Let $[F] \in Q$ be any point, and let $\mathcal{I}_Z \subset \mathcal{I}_C$ be the kernel of the surjection. Then we have closed immersions $C \subset Z \subset Y$ and $C' \subset Z' \subset Y'$, where $Z'$ denotes the scheme-theoretic image of $Z$. Using that $R^1\varphi_*F = 0$, we find a commutative diagram of coherent $\mathcal{O}_{Y'}$-modules

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{I}_C / \mathcal{I}_Z' \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{O}_{C'} \longrightarrow 0 \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow \\
0 & \longrightarrow & \varphi_*F \longrightarrow \varphi_*\mathcal{O}_Z \longrightarrow \varphi_*\mathcal{O}_C \longrightarrow 0
\end{array}
$$

having exact rows. The middle and right vertical arrows are monomorphisms by definition of scheme-theoretic image. For instance,

$$
\mathcal{I}_{C'} = \ker(\mathcal{O}_{Y'} \to \mathcal{O}_{C'}) = \ker(\mathcal{O}_{Y'} \to \mathcal{O}_{C'} \to \varphi_*\mathcal{O}_C)
$$

implies that $\mathcal{O}_{C'} \to \varphi_*\mathcal{O}_C$ is injective.

The previous remark can be made universal. Let $\mathcal{I}_{C \times Q} \to \mathcal{E}$ be the universal quotient, living over $Y \times Q$. Looking at its kernel $\mathcal{I}_Z$, we get a commutative diagram

$$
\begin{array}{ccc}
C \times Q & \longrightarrow & Z & \longrightarrow & Y \times Q \\
\downarrow & & \downarrow & & \downarrow \tilde{\varphi} \\
C' \times Q & \longrightarrow & Z' & \longrightarrow & Y' \times Q
\end{array}
$$

where the horizontal arrows are closed immersions, $\tilde{\varphi} = \varphi \times \text{id}_Q$ and $Z'$ denotes the scheme-theoretic image of $Z$. We also get a commutative diagram of coherent $\mathcal{O}_{Y' \times Q}$-modules

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{I}_{C \times Q} / \mathcal{I}_Z' \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{O}_{C \times Q} \longrightarrow 0 \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow \\
0 & \longrightarrow & \tilde{\varphi}_*\mathcal{E} \longrightarrow \tilde{\varphi}_*\mathcal{O}_Z \longrightarrow \tilde{\varphi}_*\mathcal{O}_{C \times Q} \longrightarrow 0
\end{array}
$$

having exact rows.

Let us consider the composition

$$(3.4) \quad \alpha : \mathcal{I}_{C \times Q} \to \mathcal{I}_{C \times Q} / \mathcal{I}_Z' \hookrightarrow \tilde{\varphi}_*\mathcal{E}$$

and let us write $\mathcal{K}$ for its cokernel. By Remark 3.2, $\tilde{\varphi}_*\mathcal{E}$ is coherent, hence $\mathcal{K} = \text{coker} \alpha$ is coherent, too. Thus $\text{Supp} \mathcal{K}$ is closed in $Y' \times Q$. Since $Y'$ is
complete, the projection \( \pi : Y' \times Q \to Q \) is closed. Therefore the complement
\[
(3.5) \quad Q \setminus \pi(\text{Supp } \mathcal{H}) \subset Q
\]
is an open subset of \( Q \).

**Proposition 3.2.** Let \([F] \in Q\) be a point such that \( \varphi \) is \'{e}tale in a neighborhood of \( \text{Supp } F \) and \( \varphi(x) \neq \varphi(y) \) for all distinct points \( x, y \in \text{Supp } F \). Then there is an open neighborhood \( U \subset Q \) of \([F]\) admitting an \'{e}tale map \( \Phi : U \to Q' \).

**Proof.** We first observe that we may reduce to prove the result after restricting \( Y \) to any open neighborhood of \( \text{Supp } F \) inside \( Y \). Indeed, if \( V \) is any such neighborhood, \( \text{Quot}_n(I_{\mathcal{C}|V}) \) is an open subscheme of \( Q \) that still contains \([F]\) as a point. We will take advantage of this freedom by choosing a suitable \( V \). We divide the proof in two steps.

**Step 1: Existence of the map.** Let \( \mathcal{Z} \subset Y \) be the closed subscheme determined by the kernel of \( I_{\mathcal{Z}} \). Let \( \mathcal{Z}' \subset Y' \) be its scheme-theoretic image. Since \( \varphi|_{\text{Supp } F} \) is injective and \( \varphi \) is \'{e}tale around \( \text{Supp } F \), the natural monomorphism \( I_{\mathcal{Z}'} \to \varphi^*F \) is an isomorphism and \( \varphi^*F \) is a sheaf of length \( n \), so that we get a well-defined point
\[
(3.6) \quad [\varphi_*F] \in Q'.
\]
Now let \( B \subset Y \) denote the support of \( F \) and let \( V \) be an open neighborhood of \( B \) such that \( \varphi \) is \'{e}tale when restricted to \( V \). We may assume \( V \) is affine, and in fact we may also assume \( Y = V \), by our initial remark.

In this situation, we have the Cartesian square
\[
\begin{array}{ccc}
Y \times [F] & \xleftarrow{i} & Y \times Q \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y' \times [F] & \xleftarrow{j} & Y' \times Q
\end{array}
\]
where the map \( \varphi \) is affine. Therefore, working affine-locally on \( Y' \times Q \), we see that the natural base change map \( j^*\varphi_*E \iso \varphi_*F \) is an isomorphism. This proves that the surjection \( \mathcal{J}_{C'} \to \varphi_*F \) defining the point \((3.6)\) is obtained precisely restricting \( \alpha : \mathcal{J}_{C' \times Q} \to \tilde{\varphi}_*E \), defined in \((3.4)\), to the slice
\[
j : Y' \times [F] \subset Y' \times Q.
\]
Letting \( U \subset Q \) denote the open subset defined in \((3.5)\), we see that \( \alpha \) restricts to a surjection
\[
\alpha|_{Y' \times U} : \mathcal{J}_{C' \times U} \twoheadrightarrow \varphi_*E|_U,
\]
where \( E|_U = E|_{Y' \times U} \). The target is a coherent sheaf, and it is flat over \( U \). Indeed, \( E \) is flat over \( Q \), thus \( \tilde{\varphi}_*E \) is also flat over \( Q \). But \( \varphi_*E|_U \) is naturally isomorphic to the pullback of \( \tilde{\varphi}_*E \) along the open immersion \( Y' \times U \subset Y' \times Q \), therefore it is flat over \( U \). Finally, the map \( \alpha|_{Y' \times U} \) restricts to length \( n \) quotients
\[
\mathcal{J}_{C'} \to \varphi_*E,
\]
for any closed point \([E] \in U \). Therefore we have just constructed a morphism
\[
\Phi : U \to Q', \quad [E] \mapsto [\varphi_*E].
\]
Step 2: Proving it is étale. We may shrink $Y$ further and replace it by any affine open neighborhood of $B = \text{Supp} F$ contained in $Y \setminus A$, where $A$ is the closed subset

$$A = \bigsqcup_{b \in B} \varphi^{-1} \varphi(b) \setminus \{b\} \subset Y.$$ 

After this choice, the preimage $Y_{\varphi(b)}$ is the single point $\{b\}$, for every $b \in B$. This condition implies that the natural morphism

$$\varphi^* \varphi_* F \cong F$$

is an isomorphism. Although this condition is not preserved in any open neighborhood of $[F]$, it is preserved infinitesimally, which is exactly what we need to establish étaleness.

We now use the infinitesimal criterion to show $\Phi$ is étale at the point $[F]$. Let $\iota : T \to \overline{T}$ be a small extension of fat points. Assume we have a commutative square

$$
\begin{array}{ccc}
T & \xrightarrow{i} & \overline{T} \\
\downarrow{g} & & \downarrow{b} \\
U & \xrightarrow{\Phi} & Q' \\
\end{array}
$$

where $g$ sends the closed point $0 \in T$ to $[F]$. Then we want to find a unique arrow $v$ making the two induced triangles commutative. Rephrasing this in terms of families of sheaves, let $\mathcal{I}_{C \times T} \to \mathcal{G}$ and $\mathcal{I}_{C' \times \overline{T}} \to \mathcal{H}$ be the families corresponding to $g$ and $h$, living over $Y \times T$ and $Y' \times \overline{T}$ respectively. We are after a unique $U$-valued family $\mathcal{I}_{C \times T} \to \mathcal{V}$ over $Y \times T$ with the following properties.

$(\ast)$ The condition $\Phi \circ v = h$ means we can find a commutative diagram

$$
\begin{array}{ccc}
\mathcal{I}_{C \times T} & \xrightarrow{\varphi_{T, *} \mathcal{Y}} & \mathcal{H} \\
\downarrow{\text{id}} & \approx & \downarrow{\text{id}} \\
\mathcal{I}_{C' \times \overline{T}} & \xrightarrow{\varphi_{\overline{T}, *} \mathcal{V}} & \mathcal{H} \\
\end{array}
$$

of sheaves on $Y' \times \overline{T}$.

Let us explain the condition in detail. We use, in the following, the notation $\tilde{p} = 1_Y \times p$ and $\overline{p} = 1_{Y'} \times p$, for a given map $p$. Looking at the diagram

$$
\begin{array}{ccc}
Y \times \overline{T} & \xrightarrow{\varphi_{\overline{T}}^*} & Y' \times \overline{T} \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
Y \times U & \xrightarrow{\varphi_{U, *}} & Y' \times U \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
Y \times Q & & Y' \times Q' \\
\end{array}
$$

we should require

$$\mathcal{H} \cong \overline{p}^* \overline{\Phi} \mathcal{E}',$$

where $\mathcal{E}'$ is the universal quotient sheaf on $Y' \times Q'$. However,

$$\overline{p}^* \overline{\Phi} \mathcal{E}' \cong \overline{\varphi_{U, *}} \mathcal{E} \cong \varphi_{T, *} \mathcal{V},$$

where we have used “affine base change” again.
(*** Looking at \( Y \times T \xrightarrow{\varphi_T} Y' \times T \)

\[
\begin{array}{ccc}
Y \times T & \xrightarrow{\varphi_T} & Y' \times T \\
\uparrow & & \uparrow \\
Y \times T & \xrightarrow{\varphi_T} & Y' \times T,
\end{array}
\]

the condition \( v \circ i = g \) means we can find a commutative diagram

\[
\begin{array}{ccc}
\Gamma^* \mathcal{I}_{C \times T} & \xrightarrow{\iota^*} & \Gamma^* \mathcal{V} \\
\downarrow & \cong & \downarrow \\
\mathcal{I}_{C \times T} & \xrightarrow{\iota^*} & \mathcal{G}
\end{array}
\]

of sheaves on \( Y \times T \).

We observe that

(i) the isomorphism \( \varphi_T^* \mathcal{V} \xrightarrow{\sim} \mathcal{H} \) defining (*), and

(ii) the isomorphism \( \varphi_T^* \varphi_T^* \mathcal{V} \xrightarrow{\sim} \mathcal{V} \), the “infinitesimal thickening” of (3.7),

together determine \( v \) uniquely: it is the unique arrow corresponding to the isomorphism class of the surjection

\[
\mathcal{I}_{C \times T} = \varphi_T^* \mathcal{I}_{C' \times T} \twoheadrightarrow \varphi_T^* \mathcal{H} = \mathcal{V}.
\]

To check that condition (*** is fulfilled by this family, we use that \( \Phi \circ g = h \circ i \).

In other words, there is a commutative diagram

\[
\begin{array}{ccc}
\Gamma^* \mathcal{I}_{C' \times T} & \xrightarrow{\iota^*} & \Gamma^* \mathcal{H} \\
\downarrow & \cong & \downarrow \\
\mathcal{I}_{C' \times T} & \xrightarrow{\iota^*} & \varphi_T^* \mathcal{G}
\end{array}
\]

As before, we have noted that the family corresponding to \( \Phi \circ g \) is

\[
\begin{array}{c}
\overline{\mathcal{G}} \varphi_{U*} \delta_U \cong \varphi_T^* \mathcal{G},
\end{array}
\]

where \( \overline{\mathcal{G}} \) is the map \( \text{id}_{Y' \times T} : Y' \times T \to Y' \times U \). Now we can compute

\[
\Gamma^* \mathcal{V} = \Gamma^* \varphi_T^* \mathcal{H} \cong \varphi_T^* \mathcal{H} \cong \varphi_T^* \varphi_T^* \mathcal{G} \cong \mathcal{G}.
\]

This finishes the proof.

\[ \square \]

**Corollary 3.1.** Let \( \varphi : Y \to Y' \) be an étale map of quasi-projective varieties, \( C' \subset Y' \) a Cohen-Macaulay curve with preimage \( C \). Let \( V \subset Q \) be the open subset parametrizing quotients \( \mathcal{I}_C \to F \) such that \( \varphi(x) \neq \varphi(y) \) for all \( x \neq y \in \text{Supp} \ F \). Then there is an étale map \( \Phi : V \to Q' \).

**Proof.** To apply Proposition 3.2, we need the target to be complete. Therefore, after completing \( Y' \) to a proper variety \( \overline{Y'} \), let us denote by \( \overline{C'} \) the scheme-theoretic closure of \( C' \). Then, Proposition 3.2 gives us an étale map \( \Phi : V \to \overline{Q'} \), where the target is the scheme of length \( n \) quotients of \( \mathcal{I}_{\overline{C'}} \). The map sends \( [F] \mapsto [i_* \varphi_* F] \), where \( i : Y' \to \overline{Y'} \) is the open immersion. However, the support of \( i_* \varphi_* F \) can be identified with \( \text{Supp}(\varphi_* F) \subset Y' \) for all \( [F] \), so that \( \Phi \) actually factors through \( Q' \). \[ \square \]
3.3. Applications to threefolds. In this section we assume $Y$ and $Y'$ are quasi-projective threefolds. All the other assumptions and notations from the previous sections remain unchanged here.

If $\varphi : Y \to Y'$ is an étale map, we see that the induced morphism

$$\Phi : V \to Q'$$

of Corollary 3.1, when restricted to the closed stratum $W_C^{(n)} \subset V$, appears in a Cartesian diagram

$$W_C^{(n)} \xrightarrow{\pi_C} C \xleftarrow{\Phi} V \xrightarrow{\text{open}} Q$$

(3.8)

where the horizontal maps were defined in (3.2). Let $V' \subset Q'$ be the image of the étale map $\Phi : V \to Q'$. Then the commutative diagram

$$W_C^{(n)} \xrightarrow{\text{open}} V \xrightarrow{\text{et}} Q$$

$$W_C^{(n)} \xrightarrow{\pi_C'} C'$$

yields the relation

$$\nu_Q|_{W_C^{(n)}} = \Phi^* \left( \nu_{Q'}|_{W_C^{(n)}} \right),$$

which will be useful in the next proof.

PROPOSITION 3.3. Let $\varphi : Y \to \mathbb{A}^3$ be an étale map of quasi-projective threefolds, and let $L \subset \mathbb{A}^3$ be a line.

(i) If $C = \varphi^{-1}(L) \subset Y$, we have a natural isomorphism $W_C^{(n)} = C \times F_n$.

(ii) The restricted Behrend function $\nu_Q|_{W_C^{(n)}}$ agrees with the pullback of $\nu_n$ under the natural projection to $F_n$.

PROOF. With the help of (3.8), we find a diagram

$$W_C^{(n)} \xrightarrow{\pi_C} C \xleftarrow{\Phi} Y$$

$$F_n \xrightarrow{p} W_L^{(n)} \xrightarrow{\pi_L} L \xleftarrow{\text{et}} \mathbb{A}^3$$

so that the first claim follows by the isomorphism $W_L^{(n)} = L \times F_n$ of Proposition 3.1. As for Behrend functions, we have, using (3.9) and (3.3),

$$\nu_Q|_{W_C^{(n)}} = \Phi^* \left( \nu_{M_n}|_{W_L^{(n)}} \right) = \Phi^* \left( p^* \nu_n \right).$$

The claim follows. □

The following can be viewed as the analogue of [BF08, Cor. 4.9].
COROLLARY 3.2. Let $Y$ be a smooth quasi-projective threefold. If $C \subset Y$ is a smooth curve, the map

$$\pi_C : W_C^{(n)} \rightarrow C$$

is a Zariski locally trivial fibration with fibre $F_n$. More precisely, there exists a Zariski open covering $C_i \subset C$ such that for all $i$ one has an isomorphism

$$(\pi_C^{-1}(C_i), v_Q) \cong (C_i, 1) \times (F_n, v_n)$$

of schemes with constructible functions on them.

PROOF. Cover $Y$ with open affine subschemes $U_i$ such that, for each $i$, the closed immersion $C_i = C \cap U_i \subset U_i$ is given, when $C_i$ is nonempty, by the vanishing of two equations. We can do this because $C$ is a local complete intersection. Possibly after shrinking each $U_i$, we can find étale maps $U_i \rightarrow \mathbb{A}^3$ and (using the smoothness of $C$) Cartesian diagrams

$$
\begin{array}{ccc}
C_i & \longrightarrow & U_i \\
\downarrow & & \downarrow \text{ét}
\end{array}
\quad
\begin{array}{ccc}
L & \longrightarrow & \mathbb{A}^3
\end{array}
$$

where $L$ is a fixed line in $\mathbb{A}^3$. Combining (3.8) with (both statements of) Proposition 3.3 yields Cartesian diagrams

$$
\begin{array}{ccc}
C_i \times F_n & \longrightarrow & C_i \\
\downarrow & & \downarrow \\
W_C^{(n)} & \longrightarrow & C
\end{array}
$$

and the claimed decomposition (3.10). \hfill \square

4. THE WEIGHTED EULER CHARACTERISTIC OF $Q^n_C$

The goal of this section is to prove the following result, anticipated in the Introduction.

THEOREM 4.1. Let $Y$ be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. If $Q^n_C = \text{Quot}_n(I_C)$, then

$$\chi(Q^n_C) = (-1)^n \chi(Q^n_C).$$

4.1. Ingredients in the proof. We briefly discuss the main tools used in the proof of the above formula.

4.1.1. Stratification. We start by observing that we have a stratification

$$Q^n_C = \coprod_{0 \leq j \leq n \atop \alpha \vdash j} \text{Hilb}^{n-j}(Y \setminus C) \times W_C^{(n)}$$

by locally closed subschemes, “separating” the points away from the curve from those embedded on the curve. We think of a partition $\alpha \vdash j$ as a tuple of positive integers

$$\alpha_1 \geq \cdots \geq \alpha_r \geq 1$$
such that $\sum \alpha_i = j$. Here $r_\alpha$ is the number of distinct parts of $\alpha$. Recall that

$$W_C^\alpha \subset Q_C^j,$$

defined for the first time in (3.1), parametrizes configurations of $r_\alpha$ distinct embedded points on $C$, having respective multiplicities $\alpha_1, \ldots, \alpha_{r_\alpha}$. According to (4.1), it is natural to expect the number

$$\tilde{\chi}(Q^n_C) = \chi(Q^n_C, \nu Q^n_C)$$

to be computed combining the following data.

First of all, contributions from $\text{Hilb}^{n-j}(Y \setminus C)$ are taken care of by $[BF08, \text{Thm. 4.11}]$, which implies the formula

$$\tilde{\chi}(\text{Hilb}^k(Y \setminus C)) = (-1)^k \chi(\text{Hilb}^k(Y \setminus C)).$$

Secondly, contributions from $W_C^\alpha \subset W_C^j$ will be fully expressed (thanks to the content of the previous section) in terms of the deepest stratum. The only relevant character here is the "punctual" locus $F_n$. It will be enough to know that

$$\chi(F_j, \nu_j) = (-1)^j \chi(F_j),$$

which follows from $[BF08, \text{Cor. 3.5}]$. Note that here $\chi(F_j) = \chi(M_j)$ counts the number of fixed points of the torus action we have recalled in § 2.1.

4.1.2. The Behrend function. According to $[Beh09]$, any complex scheme $Z$ carries a canonical constructible function $\nu_Z : Z \to Z$ and one can consider the weighted Euler characteristic

$$\tilde{\chi}(Z) = \chi(Z, \nu_Z) = \sum_{k \in Z} k \chi(\nu_Z^{-1}(k)).$$

Given a morphism $f : Z \to X$, one also has the relative weighted Euler characteristic

$$\tilde{\chi}(Z, X) = \chi(Z, f^* \nu_X).$$

We now list its main properties following $[Beh09, \text{Prop. 1.8}]$. First of all, it is clear that $\tilde{\chi}(Z) = \tilde{\chi}(Z, Z)$ through the identity map on $Z$.

(B1) If $Z = Z_1 \sqcup Z_2$ for $Z_i \subset Z$ locally closed, then

$$\tilde{\chi}(Z, X) = \tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X).$$

(B2) Given two morphisms $Z_i \to X_i$, $i = 1, 2$, we have

$$\tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2) = \tilde{\chi}(Z_1, X_1) \cdot \tilde{\chi}(Z_2, X_2).$$

(B3) Given a commutative diagram

$$\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
W & \longrightarrow & Y
\end{array}$$

with $X \to Y$ smooth and $Z \to W$ finite étale of degree $d$, we have

$$\tilde{\chi}(Z, X) = d(-1)^{\dim X/Y} \tilde{\chi}(W, Y).$$

(B4) This is a special case of (B3): if $X \to Y$ is étale (e.g. an open immersion), then $\tilde{\chi}(Z, X) = \tilde{\chi}(Z, Y)$. 
4.2. The computation. We can start the proof of Theorem 4.1. Let us shorten $Y_0 = Y \setminus C$ for convenience. After fixing a partition $\alpha \vdash j$, let

$$V_\alpha \subset \prod_i Q^\alpha_C$$

denote the open subscheme consisting of tuples $(F_1, \ldots, F_r)$ of sheaves with pairwise disjoint support. According to Corollary 3.1, we can use the étale cover $\varprojlim_i Y \to Y$ to produce an étale morphism

$$f_\alpha : V_\alpha \to Q^j_C.$$ 

It is given on points by taking the “union” of the 0-dimensional supports of the sheaves $F_i$. Letting $U_\alpha$ be the image of $f_\alpha$, we can form the diagram

$$
\begin{array}{ccc}
Z_\alpha & \xleftarrow{\text{open}} & V_\alpha \\
\downarrow{\text{Galois}} & & \downarrow{f_\alpha} \\
W^\alpha_C & \xleftarrow{\text{open}} & U_\alpha \\
\end{array}
$$

where the Cartesian square defines the scheme $Z_\alpha$. The morphism on the left is Galois with Galois group $G_\alpha$, the automorphism group of the partition $\alpha$. It is easy to see that in fact

$$Z_\alpha = \prod_i W^\alpha_C \setminus \Delta$$

also fits in the Cartesian square

$$
\begin{array}{ccc}
Z_\alpha & \xrightarrow{\text{open}} & \prod_i W^\alpha_C \\
\downarrow{\text{open}} & & \downarrow{\pi_\alpha} \\
C^\alpha \setminus \Delta & \xrightarrow{\text{open}} & C^\alpha \\
\end{array}
$$

(4.4)

where $W^\alpha_C \subset Q^\alpha_C$ is the deep stratum, $\Delta$ denotes the “big diagonal” (where at least two entries are equal), and the vertical map $\pi_\alpha$ is the product of the fibrations $\pi_C : W^\alpha_C \to C$, for $i = 1, \ldots, r_\alpha$.

We need two identities before we can finish the computation.

First identity. We have

$$\chi(W^\alpha_C) = |G_\alpha|^{-1} \chi(C^\alpha \setminus \Delta) \prod_i \chi(F_{a_i}).$$

(4.5)

Indeed, for each $\alpha$, the map

$$\pi_\alpha : Z_\alpha \to C^\alpha \setminus \Delta$$

appearing in (4.4) is Zariski locally trivial with fiber $\prod_i F_{a_i}$ by Corollary 3.2. Formula (4.5) follows since $W^\alpha_C$ is the free quotient $Z_\alpha / G_\alpha$.

Second identity. We have

$$\bar{\chi}(Z_\alpha, \prod_i Q^\alpha_C) = \chi(C^\alpha \setminus \Delta) \prod_i \chi(F_{a_i}, v_{a_i}).$$

(4.6)
Indeed, by Corollary 3.2, we can find a Zariski open cover \( \{ B_s \} \) of \( C^r \setminus \Delta \) such that

\[
(\pi^{-1}_r B_s, v) \cong (B_s, 1_{B_s}) \times \left( \prod_i F_{a_i} \prod_i v_{a_i} \right).
\]

In the left hand side, \( v \) denotes the Behrend function restricted from \( \prod_i Q^a_i \). We can refine this to a locally closed stratification \( \bigcup \ell U_\ell = C^r \setminus \Delta \) such that each \( U_\ell \) is contained in some \( B_s \). Therefore,

\[
\tilde{\chi}(Z_0, \prod_i Q^a_i) = \sum_\ell \tilde{\chi}(\pi^{-1}_r U_\ell, \prod_i Q^a_i) \quad \text{by (B1)}
\]

\[
= \sum_\ell \chi(U_\ell \times \prod_i F_{a_i} \times \prod_i v_{a_i}) \quad \text{by (B2)}
\]

\[
= \chi(C^r \setminus \Delta) \prod_i \chi(F_{a_i} v_{a_i}),
\]

and (4.6) is proved.

Note that combining (4.1) and (4.5) we get

\[
\chi(Q^n_C) = \sum_{j, \alpha} \chi(\text{Hilb}^{n-j} Y_0) \cdot |G_\alpha|^{-1} \chi(C^r \setminus \Delta) \prod_i \chi(F_{a_i}).
\]

We now have all the tools to finish the computation. Let us fix \( j \) and a partition \( \alpha \vdash j \). We define

\[
D_\alpha \subset \text{Hilb}^{n-j} Y \times \prod_i Q^a_i
\]

to be the set of tuples \( (Z_0, F_1, \ldots, F_n) \) such that \( (F_1, \ldots, F_n) \in V_\alpha \) and the support of \( Z_0 \) does not meet the support of any \( F_i \). Then \( D_\alpha \) is an open subscheme. The Galois cover \( 1 \times f_\alpha : \text{Hilb}^{n-j} Y_0 \times Z_0 \rightarrow \text{Hilb}^{n-j} Y_0 \times W^a_C \) extends to an étale map \( D_\alpha \rightarrow Q^n_C \), so that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hilb}^{n-j} Y_0 \times Z_\alpha & \longrightarrow & D_\alpha \\
1 \times f_\alpha \downarrow & & \downarrow \text{ét} \\
\text{Hilb}^{n-j} Y_0 \times W^a_C & \longrightarrow & Q^n_C.
\end{array}
\]
Therefore we can start computing \( \tilde{\chi}(Q^n_C) = \chi(Q^n_C, \nu_{Q^n_C}) \) as follows:

\[
\tilde{\chi}(Q^n_C) = \sum_{j,a} \tilde{\chi}({\text{Hilb}}^{n-j} Y_0 \times W^n_{C_j}, Q^n_C) \quad \text{by (B1) applied to (4.1)}
\]

\[
= \sum_{j,a} |G_{\alpha}|^{-1} \tilde{\chi}({\text{Hilb}}^{n-j} Y_0 \times Z_{a}, D_{a}) \quad \text{by (B3) applied to (4.8)}
\]

\[
= \sum_{j,a} |G_{\alpha}|^{-1} \tilde{\chi}({\text{Hilb}}^{n-j} Y_0 \times Z_{a}, {\text{Hilb}}^{n-j} Y \times \prod_i Q^n_C) \quad \text{by (B4)}
\]

\[
= \sum_{j,a} |G_{\alpha}|^{-1} \tilde{\chi}({\text{Hilb}}^{n-j} Y_0, {\text{Hilb}}^{n-j} Y) \cdot \tilde{\chi}(Z_{a}, \prod_i Q^n_C) \quad \text{by (B2)}
\]

\[
= \sum_{j,a} |G_{\alpha}|^{-1} \tilde{\chi}({\text{Hilb}}^{n-j} Y_0) \cdot \chi(C^n_\alpha \setminus \Delta) \prod_i \chi(F_{a_i}) \quad \text{by (B4) and (4.6)}
\]

\[
= (-1)^n \sum_{j,a} \chi({\text{Hilb}}^{n-j} Y_0) \cdot |G_{\alpha}|^{-1} \chi(C^n_\alpha \setminus \Delta) \prod_i \chi(F_{a_i}) \quad \text{by (4.2) and (4.3)}
\]

\[
= (-1)^n \chi(Q^n_C) \quad \text{by (4.7)}.
\]

This completes the proof of Theorem 4.1.

**QUESTION 4.1.** It would be nice to determine whether the Behrend function on \( M_n = \text{Quot}_n(\mathcal{I}_L) \) is the constant sign \((-1)^n\). As far as we know, this is still open even when the curve is absent, i.e., for \( \text{Hilb}^n \mathbb{A}^3 \).

5. IDEALS, PAIRS AND QUOTIENTS

In this section we give some applications of the formula

\[
\tilde{\chi}(Q^n_C) = (-1)^n \chi(Q^n_C).
\]

We show that the DT/PT correspondence holds for the contribution of a smooth rigid curve in a projective Calabi-Yau threefold. We discuss, at a conjectural level, the case of an arbitrary smooth curve.

5.1. Local contributions. We fix a smooth projective threefold \( Y \) and a Cohen-Macaulay curve \( C \subset Y \) of arithmetic genus \( g = 1 - \chi(\mathcal{O}_C) \), embedded in class \( \beta \in H_2(Y, \mathbb{Z}) \). We will use the Quot scheme to endow the closed subset

\[
\{ Z \subset Y \mid C \subset Z, \chi(\mathcal{I}_C / \mathcal{I}_Z) = n \} \subset I_{1-g+n}(Y, \beta)
\]

with a natural scheme structure.

**LEMMA 5.1.** There is a closed immersion \( i : Q^n_C \to I_{1-g+n}(Y, \beta) \).

**PROOF.** Let \( \mathcal{I}_C \subset T \) be a flat family of quotients parametrized by a scheme \( T \). Letting \( Z \subset Y \times T \) be the subscheme defined by the kernel of the surjection, we get an exact sequence

\[
0 \to \mathcal{F} \to \mathcal{O}_Z \to \mathcal{O}_{C \times T} \to 0.
\]

The middle term is flat over \( T \), therefore it determines a point in the Hilbert scheme of \( Y \). The discrete invariants \( \beta \) and \( \chi = 1 - g + n \) are the right ones, as can be seen by restricting the above short exact sequence to closed points of \( T \). Therefore we get a morphism

\[
i : Q^n_C \to I_{1-g+n}(Y, \beta).
\]
The correspondence at the level of functor of points is injective, and the morphism is proper (since the Quot scheme is proper, as $Y$ is projective). Therefore $i$ is a closed immersion.

**Definition 5.1.** We define

$$I_n(Y, C) \subset I_{1-g+n}(Y, \beta)$$

(5.1) to be the scheme-theoretic image of $i : Q^n_C \to I_{1-g+n}(Y, \beta)$.

**Remark 5.1.** The closed subset $|I_n(Y, C)| \subset I_{1-g+n}(Y, \beta)$ also has a scheme structure induced by GIT wall-crossing $[\iota]$. Another scheme structure is defined in the recent paper [BK16]. See in particular Definition 4, where the notation used is $\text{Hilb}^q(Y, C)$. We believe both these scheme structures agree with the one of our Definition 5.1, in which case they describe schemes isomorphic to $Q^n_C$.

Assume $Y$ is a projective Calabi-Yau threefold. By the main result of [Beh09], the degree $\beta$ curve counting invariants

$$\text{DT}_{m, \beta} = \int_{[I_n(Y, \beta)]^{\text{vir}}} 1, \quad \text{PT}_{m, \beta} = \int_{[P_n(Y, \beta)]^{\text{vir}}} 1$$

(5.2) can be computed as weighted Euler characteristics of the corresponding moduli spaces, since the obstruction theories defining the virtual cycles are symmetric. One can define the contribution of $C$ to the above invariants as

$$\text{DT}_{n, C} = \chi(I_n(Y, C), v_I), \quad \text{PT}_{n, C} = \chi(P_n(Y, C), v_P).$$

Here we have set $I = I_{1-g+n}(Y, \beta)$ and $P = P_{1-g+n}(Y, \beta)$. The subscheme $P_n(Y, C) \subset P$ consists of stable pairs with Cohen-Macaulay support equal to $C$. Note that these integers remember how $C$ sits inside $Y$, since the weight is the Behrend function coming from the full moduli space.

An immediate consequence of Theorem 4.1 is a formula for the DT contribution of a smooth rigid curve.

**Theorem 5.1.** Let $Y$ be a projective Calabi-Yau threefold, $C \subset Y$ a smooth rigid curve. Then

$$\text{DT}_{n, C} = (-1)^n \chi(I_n(Y, C)).$$

**Proof.** The inclusion (5.1) is both open and closed thanks to the infinitesimal isolation of $C$. Then $v_I|_{I_n(Y, C)} = v_{I_n(Y, C)}$, thus

$$\text{DT}_{n, C} = \tilde{\chi}(I_n(Y, C)) = (-1)^n \chi(I_n(Y, C)),$$

as claimed. □

**Remark 5.2.** In the rigid case, $\text{DT}_{n, C}$ is a DT invariant in the classical sense, namely it is the degree of the virtual class $[I_n(Y, C)]^{\text{vir}}$ obtained by restricting the one on $I_{1-g+n}(Y, \beta)$.

Theorem 5.1 can be seen as an instance of the following more general result, which is also a direct consequence of Theorem 4.1.

**Proposition 5.1.** Let $Y$ be a smooth projective threefold. If $C \subset Y$ is a smooth curve of genus $g$, then

$$\sum_{n \geq 0} \tilde{\chi}(I_n(Y, C))q^n = M(-q)^{\chi(Y)}(1 + q)^{2g-2}.$$  

(5.3)
PROOF. For any smooth threefold $X$ we have Cheah’s formula [Che96]

$$\sum_{n \geq 0} \chi(\text{Hilb}^n X) q^n = M(q)^{\chi(X)}.$$

We use this with $X = Y_0 = Y \setminus C$, together with formula (4.7), to compute

$$\sum_{n \geq 0} \chi(I_n(Y, C)) q^n = M(q)^{\chi(Y \setminus C)} \cdot \left( \sum_{n \geq 0} \chi(F_n) q^n \right)^{\chi(C)}$$

$$= M(q)^{\chi(Y \setminus C)} \cdot \left( \sum_{n \geq 0} \chi(M_n) q^n \right)^{\chi(C)}$$

$$= M(q)^{\chi(Y \setminus C)} \cdot \left( \frac{M(q)}{1 - q} \right)^{\chi(C)} \quad \text{by (2.2)}$$

$$= M(q)^{\chi(Y)} (1 - q)^{2g - 2}.$$

The claimed formula follows by Theorem 4.1. \qed

Remark 5.3. Formula (5.3) can be rewritten as

(5.4) $\sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n = M(-q)^{\chi(Y)} \sum_{n \geq 0} \tilde{\chi}(P_n(Y, C)) q^n.$

Indeed $P_n(Y, C) = \text{Sym}^n C$ is smooth of dimension $n$, thus $\tilde{\chi} = (-1)^n \chi$. The latter identity can be seen as the $v$-weighted version of the “local” wall-crossing formula between ideals and stable pairs, which was already established for a single Cohen-Macaulay curve at the level of Euler characteristics [ST11, Thm. 1.5]. In other words, (5.4) is precisely what happens to the Stoppa-Thomas identity

$$\sum_{n \geq 0} \chi(I_n(Y, C)) q^n = M(q)^{\chi(Y)} \sum_{n \geq 0} \chi(P_n(Y, C)) q^n$$

when we replace $q$ by $-q$.

5.2. DT/PT wall-crossing at a single curve. Let $C$ be a smooth curve of genus $g$, embedded in class $\beta$ in a smooth projective Calabi-Yau threefold $Y$. Let us define the generating series

$$\text{DT}_C(q) = \sum_{n \geq 0} \text{DT}_{n,C} q^n$$

$$\text{PT}_C(q) = \sum_{n \geq 0} \text{PT}_{n,C} q^n$$

encoding the local contributions defined in (5.2). The stable pair side has already been computed [PT10, Lemma 3.4]. The result is

(5.5) $\text{PT}_C(q) = n_{g,C} \cdot (1 + q)^{2g - 2},$

where $n_{g,C}$ is the $g$-th BPS number of $C$. For instance, if $C$ is rigid, then $n_{g,C} = 1$ and thanks to Theorem 5.1 we see that (5.3) can be rewritten as

$$\text{DT}_C(q) = M(-q)^{\chi(Y)} \cdot \text{PT}_C(q).$$

This formula can be seen as a “local DT/PT correspondence”, or local wall-crossing formula at $C$. We next prove that such formula, for arbitrary $C$, is equivalent to the following conjecture.

Conjecture 1. Let $C$ be a smooth curve in a projective Calabi-Yau threefold $Y$. Let $I = I_{1-g}(Y, \beta)$ be the Hilbert scheme where the ideal sheaf of $C$ lives as a point. Then, for all $n$, one has

$$\text{DT}_{n,C} = v_I(\mathcal{I}_C) \cdot \tilde{\chi}(I_n(Y, C)).$$
Remark 5.4. An equivalent formula has been conjectured by Bryan and Kool in their recent paper [BK16]. See Conjecture 18 in loc. cit. for the precise (more general) setting.

**Theorem 5.2.** Let $Y$ be a projective Calabi-Yau threefold, $C \subset Y$ a smooth curve. Then Conjecture 1 is equivalent to the wall-crossing identity

$$\DT_C(q) = M(-q)^{\chi(Y)} \cdot \PT_C(q).$$

**Proof.** Combining (5.5) with (5.3), we see that the right hand side of the formula equals

$$n_{g,C} \cdot \sum_{n \geq 0} \bar{\chi}(I_n(Y, C)) q^n.$$

Therefore the DT/PT correspondence holds at $C$ if and only if

$$\DT_n, C = n_{g,C} \cdot \bar{\chi}(I_n(Y, C)).$$

We are then left with proving that $v_T(\mathcal{I}_C) = n_{g,C}$. Recall that the moduli space of ideal sheaves is isomorphic to the moduli space of stable pairs along the open subschemes parametrizing pure curves. Moreover, the map $\phi : P_{-g}(Y, \beta) \to \mathcal{M}$ to the moduli space of stable pure sheaves considered in [PT10], defined by forgetting the section of a stable pair, satisfies the relation

$$v_{P_{-g}(Y, \beta)} = (-1)^{g} v_M$$

by [PT10, Thm. 4]. Hence

$$v_T(\mathcal{I}_C) = v_{\text{pure}}(\mathcal{I}_C)$$

$$= v_{P_{-g}(Y, \beta)}([\mathcal{O}_Y \to \mathcal{O}_C])$$

$$= (-1)^{g} v_M(\mathcal{O}_C)$$

$$= n_{g,C}$$

where the last equality is [PT10, Prop. 3.6]. □

**Remark 5.5.** Thanks to the identity $v_T(\mathcal{I}_C) = n_{g,C}$, proved in the course of Theorem 5.2, Conjecture 1 can be rephrased as

$$\DT_{n,C} = v_P|_{P_{e}(Y, C)} \cdot \bar{\chi}(I_n(Y, C)),$$

where $v_P|_{P_{e}(Y, C)}$ is the constant $(-1)^n \cdot n_{g,C} = (-1)^{n-g} v_M(\mathcal{O}_C)$. In particular the conjecture says that the DT and PT contributions of $C$ differ from the Euler characteristic of the corresponding moduli space by the same constant.

We end the paper with some speculations, indicating plausibility reasons why Conjecture 1 should hold true.

Suppose we were able to show that, given a point $\mathcal{I}_Z \in I_n(Y, C) \subset I$, a formal neighborhood of $\mathcal{I}_Z$ in $I$ is isomorphic to a product $U \times V$, where $U$ is a formal neighborhood of $\mathcal{I}_C$ in $I$ and $V$ is a formal neighborhood of $\mathcal{I}_Z$ in $I_n(Y, C)$. Then, since the Behrend function value $v(P)$ only depends on a formal neighborhood of $P$ [Jia], this would immediately lead to the Behrend function identity

$$v_T|_{I_n(Y, C)} = v_T(\mathcal{I}_C) \cdot v_{P_{e}(Y, C)},$$

from which Conjecture 1 follows after integration. One reason to believe in a product decomposition as above is the following. At least when the maximal
purely 1-dimensional part $C \subset Z$ is smooth, one may expect to be able to “separate” infinitesimal deformations of $C$ (the factor $U$) from those deformations of $Z$ that keep $C$ fixed (the factor $V$ in the Quot scheme). This decomposition is manifestly false when $C$ acquires a singularity, and we do not know of any counterexample in the smooth case.

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KJELL ARHOLMS 41, 4021 STAVANGER (NORWAY)

E-mail address: andrea.ricolfi@uis.no