Control Barrier Function Contracts for Vehicular Mission Planning under Signal Temporal Logic Specifications

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Abstract—We present a compositional control synthesis method based on assume-guarantee contracts with application to correct-by-construction design of vehicular mission plans. In our approach, a mission-level specification expressed in a fragment of signal temporal logic (STL) is decomposed into formulas whose predicates are defined on non-overlapping time intervals. The STL formulas are then mapped to aggregations of contracts associated with continuously differentiable time-varying control barrier functions. The barrier functions are used to constrain the lower-level control synthesis problem, which is solved via quadratic programming. Our approach can mitigate the conservatism of previous methods for task-driven control based on under-approximations. We illustrate its effectiveness on a case study motivated by vehicular mission planning under safety constraints as well as constraints imposed by traffic regulations under vehicle-to-vehicle and vehicle-to-infrastructure communication.

Index Terms—Signal temporal logic, control synthesis, contract-based design, control barrier functions.

I. INTRODUCTION

The necessity of ensuring mission safety of autonomous cyber-physical systems such as vehicles immersed in an urban setting\textsuperscript{1} has motivated the development of correct-by-construction, algorithmic control synthesis methods (see, e.g., \textsuperscript{2}, \textsuperscript{3}) to help ensure that a system fulfills its mission requirements while avoiding potentially hazardous configurations.

A major challenge to control synthesis stems from the heterogeneity of formalisms needed to design and analyze complex cyber-physical systems\textsuperscript{4}. Some of the efforts in the literature leverage symbolic approaches to effectively synthesize provably correct high-level task planners. However, by relying on discrete abstractions of the design space, these methods may be prone to scalability issues when applied to complex continuous systems. On the other hand, low-level feedback control synthesis methods have shown to be effective in enforcing invariance and simple reachability properties on continuous systems. They have, however, difficulty in capturing more complicated mission constraints, including logical constraints, often inducing discontinuities in the target safe sets. More recently, the representation of the mission specification in an expressive logic language, such as signal temporal logic (STL), together with mixed integer linear encodings of the STL formulas\textsuperscript{5} have been proposed to perform discrete-time trajectory planning in a model predictive control fashion for a wider class of objectives, including time-sensitive constraints. However, efficiently encompassing mission-level (logical) and control-level (dynamical) constraints within a unifying framework remains a challenge.

Compositional and hierarchical methods show the promise of harnessing the complexity due to the scale and heterogeneity of the control design problem, e.g., via a layered approach that can capture different kinds of constraints at different layers, without inducing excessive conservatism in the solutions. In this context, assume-guarantee (A/G) contracts have been employed\textsuperscript{6}, \textsuperscript{7} to support compositional synthesis under temporal logic specifications. A/G reasoning has also been explored to argue about the correct composition of lane keeping and cruise control for vehicular planning\textsuperscript{8}. However, an A/G contract framework that can effectively bridge high-level planning and continuous-time feedback control is an open research problem.

This paper addresses the above challenges by exploring a formalization of control barrier functions\textsuperscript{2} in terms of A/G contracts capable of bridging high-level task planning and low-level feedback control. Central to our approach is the characterization of time-varying safe sets via a composition of continuously differentiable time-varying control barrier function (C\textsuperscript{1} TV-CBF) contracts that can capture time-varying constraints including jump discontinuities. Our contributions can be summarized as follows:

- We formalize a notion of time-varying finite-time convergence control barrier function (TV-FCBF) as a contract providing an effective interface between task planning and feedback control synthesis.
- We determine necessary and sufficient conditions for the composition of TV-CBF contracts to generate a compatible contract, for which a controller is guaranteed to exist.
- By building on these abstractions, we introduce an algorithm that maps a mission-level specification expressed in a fragment of STL to an aggregation of CBF contracts from which a feedback controller can be designed via quadratic programming.

Our method is reminiscent of previous approaches to STL control synthesis using CBFs\textsuperscript{9}, in that we associate candidate CBFs with atomic STL predicates in the specification. However, our approach can mitigate the potential

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The conjunction of two contracts \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) is defined as the contract serving as the greatest lower bound which refines both. This can be used to represent a combination of requirements that must be satisfied simultaneously. The composition of contracts is, instead, used to derive a more complex contract that must be satisfied by a composition of components, each satisfying its local contract. A detailed exposition of all the terms summarized above may be found in the literature \([12]\).

### B. Control Barrier Functions

We assume that a dynamical system, e.g., describing the ego vehicle, is governed by the dynamics

\[
\dot{x} = f(x) + g(x)u, \quad (1)
\]

where \( f, g \) are locally Lipschitz-continuous functions of the state variables \( x \in D \subseteq \mathbb{R}^n \), \( u \in U \subseteq \mathbb{R}^m \) is the input vector, and \( U \) is the set of allowable inputs. CBFs are used to provide safety guarantees for such systems.

**Definition 1** (Control Barrier Function \([2]\)). Let \( h(x) : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function and let \( C \subseteq \mathbb{R}^n \) be a compact superlevel set of \( h(x) \) such that \( C = \{x \in D : h(x) \geq 0\} \). We say that \( h \) is a Control Barrier Function (CBF) if there exists an extended \( K\infty \) class function \( \alpha \) such that, for all \( x \in D \), the following holds:

\[
\sup_{u \in U} |L_f h(x) + L_g h(x)u| \geq -\alpha(h(x)). \quad (2)
\]

\( L_f h(x) = \frac{dh(x)}{dt} f(x) \) and \( L_g h(x) = \frac{dh(x)}{dt} g(x) \) are the appropriate Lie derivatives. \( C \) is the safe set corresponding to the CBF \([2]\).

It can be proven \([2, 13]\) that any controller \( u \in U_{safe} = \{u \in U : L_f h(x) + L_g h(x)u \geq -\alpha(h(x))\} \) ensures that, if the system starts in \( C \), i.e., \( x(t_0) \in C \), then it will stay in \( C \). The existence of a CBF is then equivalent to ensuring the forward-invariance property of the safe set \( C \), hence rendering the system evolution safe, given safety conditions on its initial states. A notion of finite-time convergence CBF (FCBF) has also been proposed for time-invariant CBFs \([11]\) to guarantee finite-time convergence to a safe set. In this paper, we extend the concept of FCBF to time-varying CBF and formalize them as A/G contracts.

### C. Signal Temporal Logic

We represent the mission specification using STL \([14]\), which offers a rigorous formalism for the specification and analysis of temporal properties of real-valued dense-time signals. We assume that an STL atomic predicate \( \phi_h \) is evaluated over a real-valued predicate function \( h(x) : \mathbb{R}^n \to \mathbb{R} \). \( \phi_h \) evaluates to true (\( \top \)) if \( h(x) \geq 0 \) holds, and false (\( \bot \)) otherwise. We then consider a fragment of STL according to the following syntax:

\[
\psi := \top \mid \phi_h \mid \neg \phi_h \mid \mathcal{G}_T \phi_h \mid \mathcal{F}_T \phi_h \mid \neg \psi \mid \psi_1 \land \psi_2 \quad (3)
\]
where $\phi_i$ is an atomic predicate, $\psi, \psi_1, \psi_2$ are STL formulas. $G$ and $F$ are the globally and eventually temporal operators, respectively, and $\Gamma$ is a bounded time interval. Our fragment does not include the nesting of temporal operators. We say that $(x, t)$ satisfies $\psi$, written $(x, t) \models \psi$, if there exists a signal (trajectory) $x(t) \in \mathbb{R}^n$ such that $\psi$ holds at time $t$. We simply write $x \models \psi$ if $(x, 0) \models \psi$.

III. CONTROL BARRIER FUNCTION A/G CONTRACTS

We begin by extending classical results from finite-time convergence CBFs ([11]) to a new class of CBFs, which we call time-varying finite-time convergence CBFs (FCBFs). We show that FCBFs can be formalized as A/G contracts. Their composition leads, in general, to piecewise continuously differentiable ($C^1$) TV-CBF contracts for which a controller is guaranteed to exist.

Definition 2 (Time-Varying Finite-Time Convergence CBF (TV-FCBF)). Let $h(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Let $C(t) \subseteq D \subseteq \mathbb{R}^n$ be the compact upper level of $h(t, x)$. If for the system in [11] there exist $0 \leq \rho < 1$ and $\gamma > 0$ such that, $\forall t \geq 0, \forall x \in D$, we have

$$\sup_{u \in U} \left[ \frac{\partial}{\partial t} h(t, x) + L_f h(t, x) + L_g h(t, x) u + \gamma \sigma_{\text{sign}}(h(t, x)) |h(t, x)|^\rho \right] \geq 0,$$

then $h(t, x)$ is a Time-Varying Finite-Time Convergence CBF.

When $h(t, x) = h(x)$ the TV-FCBF reduces to an FCBF [11]. For a given $h(t, x)$, the set of safe inputs is

$$U_{\text{safety}}(t) = \{ u \in U : \frac{\partial}{\partial t} h(t, x) + L_f h(t, x) + L_g h(t, x) u + \gamma \sigma_{\text{sign}}(h(t, x)) |h(t, x)|^\rho \geq 0 \}.$$

The following theorem discusses the finite-time convergence property of a TV-FCBF.

Theorem 1. Let $C(t)$ be the superlevel set of the continuously differentiable function $h(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ as in Definition 2 with corresponding $0 \leq \rho < 1$ and $\gamma > 0$, and let $x(t_0) = x_0 \in C(t_0)$ be the initial state, any controller $u \in U_{\text{safety}}(t)$ renders $C(t)$ forward-invariant $\forall t > t_0$. Moreover, if $x_0 \in D \setminus C(t_0)$, then $u \in U_{\text{safety}}(t)$ drives $x$ to $C(t)$ within a finite time $T = \frac{1}{\gamma(1-\rho)} \| h(t_0, x_0) \|^{1-\rho}$.

Proof. The proof follows the same line of reasoning as previous results [11]. Consider the candidate Lyapunov function $V(t, x) = \max(0, -h(t, x))$. When $x_0 \in C(t_0)$, then $h(t_0, x) > 0$ and $V(t, x) = 0$ hold. By the comparison lemma [15], we obtain $V(t, x) = 0$ for all $t > t_0$, hence $x(t) \in C(t) \forall t > t_0$. When $x_0 \in D \setminus C(t_0)$, then we have $V(t_0, x_0) > 0$ and $V(t, x) \leq -\gamma V(t, x) \forall t \geq t_0$. Again, by the comparison lemma, $x$ will converge to $C(t)$ within the finite time $T = \frac{1}{\gamma(1-\rho)} \| h(t_0, x_0) \|^{1-\rho}$, i.e., $x(t) \in C(t) \forall t \geq t_0 + T$.

We also say that $C_h(t) = \{ x \in D : h(t, x) \geq 0 \}$ is the safe set for the barrier function $h(t, x)$. TV-FCBFs can mitigate the conservatism of concatenating multiple CBFs via a min operator and taking a smooth approximation of the result, as discussed in the following example.

Example 1. For a simple vehicle model, $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, where $u \in \mathbb{R}$ is the input acceleration and $x = [X \ V]^T$, with $X \in \mathbb{R}$ and $V \in \mathbb{R}_{\geq 0}$ being the position and the velocity of the vehicle, let us consider the STL formula $\bigwedge_{i=1}^N G_{(t_i, t_i)} \phi_i$, where the predicates $\phi_i = V_{\text{max}, i} - V \geq 0$ are defined over $N$ non-overlapping adjacent intervals, that is, $\Gamma_i = [t_{i-1}, t_i)$ and $\Gamma_i \cap \Gamma_{i+1} = \emptyset$ for $1 \leq i \leq N$. The formula $\phi$ prescribes a maximum speed limit during $\Gamma_i$.

A method to synthesize a controller satisfying this formula could associate a CBF $h_i(x) := V_{\text{max}, i} - V$ with each predicate $\phi_i$ and combine them via a smooth under-approximation of the pointwise min operator [9]. However, this procedure would result into an overly conservative requirement, always yielding a speed limit that is stricter than $\min_i (V_{\text{max}, i} - V)$.

We show that a combination of TV-CBF and TV-FCBF formalized as contracts can mitigate this conservatism.

The notion of invariance can be formalized as a contract to be satisfied by the closed-loop system. The contract algebra can then be used to reason about the composition of invariance properties and derive conditions for control synthesis.

Definition 3 (TV-CBF Contract). A TV-CBF contract $C_h = (V, A_h, G_h)$ is defined as follows:

$$V = \{ t, x = (x_1, \ldots, x_n) \},$$
$$A_h := x(t_0) \in C_h(t_0),$$
$$G_h := \forall t > t_0 : x(t) \in C_h(t),$$

where $C_h(t)$ is the safe set of $h(t, x)$ and $t_0$ is the initial time.

Definition 4 (TV-FCBF Contract). A TV-FCBF contract $C_h = (V, A_h, G_h)$ is defined as follows:

$$A_h := \exists t_{\text{conv}} > 0 : \forall t \in [t_0, t_0 + t_{\text{conv}}) : x(t) \notin C_h(t) \land x(t_0 + t_{\text{conv}}) \in C_h(t_0 + t_{\text{conv}}),$$
$$G_h := \forall t \geq t_{\text{conv}} + t_0 : x(t) \in C_h(t).$$

If a CBF contract holds, then there exists a controller which ensures the forward-invariance of $C_h(t)$ for the closed-loop system. Else, if a FCBF contract holds, there exists a controller which brings the system to the safe set $C_h(t)$ after time $t_{\text{conv}}$. We say that contract $C_h$ is imposed on the interval $\Gamma = [t_0, t_f]$, when $t_0$ is the initial time and $C_h(t)$ is the safe set for all $t \in \Gamma$. Let $C_h(t_i^+) = \{ x : \lim_{t \to t_i^-} h(t, x) \geq 0 \}$. In the following, we also assume that contracts are non-vacuous, that is, their assumptions are satisfiable. Using the above notions, we can combine contracts over neighboring intervals as stated by the following results.
Theorem 2. The composition of the TV-CBF contracts $\mathcal{E}_h \otimes \mathcal{E}_g$ imposed on $\Gamma_1 = [t_0, t_1)$ and $\Gamma_2 = [t_1, t_2)$, respectively, with $t_2 > t_1 > t_0$, is compatible if and only if $C_h(t_1^-) \cap C_g(t_1) \neq \emptyset$. Moreover, $\mathcal{E}_h \otimes \mathcal{E}_g$ is compatible for all $x(t_0) \in C_h(t_0)$ if and only if $C_h(t_1^-) \subseteq C_g(t_1)$ holds.

Intuitively, contracts $\mathcal{E}_h$ and $\mathcal{E}_g$ imposed on adjacent intervals can be composed, that is, at least one trajectory $x(t)$ satisfying the guarantees of $\mathcal{E}_h$ will also satisfy the assumptions of $\mathcal{E}_g$, if and only if the switching point of their intervals. On the other hand, for every trajectory $x(t)$ to remain safe throughout $\Gamma_1 \cup \Gamma_2$, the current safe set must be a subset of the upcoming safe set at the time of switching. This latter, stronger condition is equivalent to the condition for sequential composition of funnels [10] in Figure 1(a). We now present the corresponding result for FCBF contracts.

Theorem 3. The composition of TV-CBF contract $\mathcal{E}_h$ imposed on $\Gamma_1 = [t_0, t_1)$ and TV-FCBF contract $\mathcal{E}_g$ imposed on $\Gamma_2 = [\tau_g, t_2)$ is compatible if and only if (1) $C_h(t_1^-) \cap C_g(t_1) \neq \emptyset$ and (2) $\tau_g + t_{\text{conv},g} < t_1$ hold.

In other words, we can guarantee that every trajectory $x(t)$ remains in the safe sets of $\mathcal{E}_h$ and $\mathcal{E}_g$ throughout $\Gamma_1 \cup \Gamma_2$ by simply requiring that $C_h(t)$ overlaps with $C_g(t)$ at the time of switching, provided that $x(t)$ converges to $C_g(t)$ before $\Gamma_1$ has finished. Because $t_{\text{conv},g} \leq T_g$ by Theorem 1, if $\tau_g + T_g \leq t_1$, then condition (2) in Theorem 3 will also hold. We leverage this observation combined with Definition 2 and the results above to compute the set of safe inputs for a composition of contracts imposed on neighboring intervals.

Theorem 4. Let $\mathcal{E}_h = \bigotimes_{i=1}^N \mathcal{E}_{h,i}$ be imposed on non-overlapping intervals $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_N\}$ such that $i|: 1 \leq i \leq N$, $\Gamma_i = [t_{i-1}, t_i)$ and $t_{i-1} < t_i$. Let $\mathcal{E}_h$ be either a TV-CBF contract imposed over $\Gamma_i$ or $\mathcal{E}_{h,i} = \mathcal{E}_{h,i1} \otimes \mathcal{E}_{h,i2}$ where $\mathcal{E}_{h,i1}$ is a TV-CBF contract imposed over $\Gamma_i$ and $\mathcal{E}_{h,i2} = \mathcal{E}_{h,i+1}$ is a TV-FCBF contract imposed over $[\tau, t_i)$ for $t_{i-1} < \tau < t_i$. If $x(t_0) \in C_{h,i}(t_0)$, then any controller $u \in U_{\text{safe},h}(t)$ will make the closed-loop system satisfy $\mathcal{E}_h$, where

$$U_{\text{safe},h}(t) = \begin{cases} U_{\text{safe},h}(t) & (t_{i-1} < t \leq t_i) \lor \\
C_{h,i}(t^-) \subseteq C_{h,i+1}(t) & \end{cases}$$

$$U_{\text{safe},h}(t) \cap (t_i < t < t_{i+1})$$

$$U_{\text{safe},h,i+1}(t) = \begin{cases} C_{h,i}(t^-) \not\subseteq C_{h,i+1}(t) \land \\
C_{h,i}(t^-) \cap C_{h,i+1}(t) \neq \emptyset & \end{cases}$$

$$U_{\text{safe},h,i}(t) = \{u \in U : \frac{\partial}{\partial t} h_i(t, x) + L_f h_i(t, x) + L_g h_i(t, x) u + \alpha h_i(t, x) \geq 0\}$$

$$U_{\text{safe},e,i+1}(t) = \{u \in U : \frac{\partial}{\partial t} h_i+1(t, x) + L_f h_i+1(t, x) + L_g h_i+1(t, x) u + \gamma_i+1 \text{sign}(h_i+1(t, x)) \cdot |h_i+1(t, x)|^\beta \geq 0\}$$

provided that

$$T_{i+1} = \left| h_i+1(t_i, x(t_i)) \right|^{1-\rho_{i+1}} + \frac{\rho_i}{1-\rho_{i+1}} < t_i - t_i - t_{i-1}.$$
Algorithm 1: Control Synthesis Algorithm

Data:
\[ \Psi = \bigwedge_{i=1}^{n_c} \mathcal{G}_i(h_i(x) \geq 0) \land \bigwedge_{j=1}^{n_a} \mathcal{F}_j(f_j(x) \geq 0) \]
times of satisfaction \( t = t_{s,1}, \ldots, t_{s,i} \)
Result: \( u_{safe} \) or failure

\[ \Psi' = \bigwedge_{i=1}^{n_c} \mathcal{G}_i(h_i(x) \geq 0) \land \bigwedge_{j=1}^{n_a} \mathcal{F}_j(f_j(x) \geq 0) \]
//Divide STL into formulas \( \mathcal{G}_i \) whose predicates are defined on non-overlapping intervals \( \Gamma_1, \ldots, \Gamma_{N'_e}, \Gamma_i = [t_{i-1}, t_i] \)
\( \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k = \text{group}(\Psi') \)
for \( \mathcal{G}_j \in \{ \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k \} \) do

for \( h_i \in \mathcal{G}_j \) do

\[ \text{comp} = \text{chk - compatibility}(C_{h,i} \ominus C_{h,i+1}) \]
// Check (1) \( C_{h,i}(t_i) \subseteq C_{h,i+1}(t_i) \) or (2) \( C_{h,i}(t_i) \cap C_{h,i+1}(t_i) \neq \emptyset \) and \( t_{i+1} < t_i - t_{i-1} \)
if \( \text{comp} = \perp \) then

return failure

for \( t \leq T_{max} \) do

\[ u_{safe,f}(t) = \bigwedge_{j=1}^{k} u_{safe,f,j}(t) \]
//\( u_{safe,f,j}(t) \) is calculated as in Theorem 4
if \( u_{safe,f}(t) \neq \perp \) then

\[ u_{safe} = \arg\min_{u \in \mathcal{U}_{safe,f}(t)} \|u - u_{nom}\|_2 \]
else

return failure

\end{algorithm}

Let \( F_r = c_0 + c_1 V_f + c_2 V_f^2 \) be the sum of all frictional and aerodynamic forces on the ego vehicle, where \( c_0, c_1, \) and \( c_2 \) are parameters. Let \( u \in U \) be the input wheel force, \( x = (X_f, V_f, X_i)^T \), \( f(x) = (V_f - \frac{1}{m} F_r, V_f)^T \), and \( g(x) = (0, \frac{1}{m}, 0)^T \). The longitudinal dynamics of the system are given by \( \dot{x} = f(x) + g(x)u \) while \( t_{N_e} \) is the time horizon of the mission.

A. STL Specifications

The mission includes safety and regulatory requirements as follows:

1) The ego vehicle must maintain a safe distance from the lead vehicle. Let \( h_1(x) \) be the spacing error between the ego vehicle and the lead vehicle, defined by \( h_1 := X_r - h V_f - S_0 - \frac{V_f - V_f^2}{2a_{max}} \) where \( h \) is a constant time-headway (time required by the ego vehicle to cover the distance \( X_r \)), \( S_0 > 0 \) is the relative distance between ego and lead vehicle when they are in the rest position, and \( a_{max} \) is the absolute value of the maximum braking capability. We then require \( G_{[0,t_{N_e}],h_1}(h_1(x) \geq 0) \).

2) The ego vehicle must follow variable speed limits. Let the mission interval \( \Gamma_v = [t_0, t_{N_v}] \) be divided into \( N_v \) consecutive intervals such that \( \Gamma_v = \bigcup_{1 \leq i \leq N_v} [t_{i-1}, t_i] \) and \( t_{i-1} < t_i \) for all \( i \). Let the speed limit \( V_{max}(t) \) be a discrete-valued function of time, described in a piecewise manner as \( V_{max}(t) = V_{max,i} \) \( \forall t \in \Gamma_{vi} = [t_{i-1}, t_i], 1 \leq i \leq N_v \). We define \( h_{vi}(x) := V_{max,i}(t) - V_f \) and \( \phi_{h_v} := h_{vi}(x) \geq 0 \). We then require \( G_{[0,t_{N_v}],\phi_{h_v}}(t) \) \( \forall t \leq t_{i-1} \).

3) The ego vehicle must follow the traffic signals. Let \( N \) be the number of signals, and \( s_i \in \{ \text{Red, Yellow, Green} \} \) be the state of the \( i^{th} \) traffic signal. Let \( g_{ij} \) be the instants at which the \( i^{th} \) signal turns green, yellow, and red for the \( f^{th} \) time, respectively, with \( g_{ij} < g_{ij} < g_{ij} < g_{i,j+1} \). We obtain that \( s_i = \text{Green} \) \( \forall t \in \Gamma_{ij} \), \( s_i = \text{Yellow} \) \( \forall t \in \Gamma_{ij} \), \( s_i = \text{Red} \) \( \forall t \in \Gamma_{ij} \).

Let \( h_{ir_1}(x) = P_i X_f - \beta V_f - S_0 \) and \( h_{ir_2}(x) = P_i X_f - \beta V_f - S_0 \), where \( P_i \) is the position of the \( i^{th} \) traffic signal, \( \beta > 0 \), and \( P_i - 1 < X_f < P_i \). If \( s_i = \text{Red} \), then \( \phi_{h_{r_1}r_1} := (h_{r_1}(x) \geq 0) \) must hold; if \( s_i = \text{Red} \), then \( \phi_{h_{r_2}(x) \geq 0} \) must hold. Let \( I_{B}(x) : R^n \rightarrow \{0, 1\} \) be an indicator function such that \( I_{B}(x) = 0 \) if and only if \( x \in B \subset R^n \). Let \( T_i = \{ x : P_i - 1 < X_f < P_i \} \), for \( 1 \leq i \leq N \). We can encode the traffic rules by \( \bigwedge_{i=1}^{N} \left[ \bigwedge_{j \in \mathcal{N}} \left[ \mathcal{G}_{ir_1j}(x) \ominus \mathcal{G}_{ir_2j}(x) \right] \right] \). Therefore, \( 3k-1 \leq k \leq N \), we define \( I_{B}(x) = 0 \) for \( 1 < i < N - 1 \).

4) The lead vehicle communicates its velocity \( V_l \) and acceleration \( a_l \) to the ego vehicle.

We organize the specifications above into STL formulas satisfying the conditions in Algorithm 1 and map them to compositions of contracts.

B. Mapping STL Formulas to Contracts

The overall STL specification before pre-processing is given by \( \phi_m = \bigwedge_{i=1}^{3} G_{i} \), where \( G_1 = \mathcal{G}_{[0,t_{N_e}],h_1} \), \( G_2 = \bigwedge_{i=1}^{n_c} \mathcal{G}_{[t_{i-1}, t_i],h_2} \), \( G_3 = \bigwedge_{i=1}^{n_c} \mathcal{G}_{[t_{i-1}, t_i],h_3} \), \( G_4 = \bigwedge_{i=1}^{n_c} \mathcal{G}_{[t_{i-1}, t_i],h_4} \), and \( G_5 = \bigwedge_{i=1}^{n_c} \mathcal{G}_{[t_{i-1}, t_i],h_5} \). The predicates in the specification are allocated to three formulas such that the predicates in each formula are defined on non-overlapping intervals. We first consider \( G_1 \). Since \( h_1(x) \) is \( C^1 \) and there is only one STL predicate over the whole horizon, we generate the CBF contract \( C_1 \) with the corresponding safe set \( C_1 \).

\( G_2 \) can be associated to a CBF contract \( C_2 = \bigotimes_{i=1}^{n_c} C_{h_2,i} \), where \( C_{h_2,i} \) can either be a CBF contract over interval \( \Gamma_i \) or \( C_{h_2,i} = \bigotimes_{i=1}^{n_c} C_{h_2,i} \), where \( C_{h_2,i} \) is a CBF contract for the safe set \( C_{h_2,i} \), and \( C_{h_2,i} \) is a CBF contract for \( C_{h_2,i+1} \), imposed over the intervals \( \Gamma_i \) and \( [t_{i-1}, T_{conv,v}, t_i] \).
respectively. Given \( h_{v,i}(x) = V_{\text{max},i} - V_f \), we set \( h_v(t, x) = h_{v,i}(t, x) \) for \( t \in \Gamma_i \forall i \in \mathcal{N}_v \).

Similarly to \( \mathcal{C}_2 \), we can form \( \mathcal{C}_3 \) for \( \mathcal{S}_2 \), which specifies the traffic signals’ constraints for the system when the ego vehicle is approaching the \( i^{th} \) traffic signal. When \( P_{t-1} < X_f \leq P_t \), then we have \( C_{\text{pos},i}(t) = \{ x \in \mathbb{R}^n : h_{\text{pos},i}(t, x) \geq 0 \} \).

At \( t = r_{ij} \), we only require that \( C_{\text{pos},i}(t = r_{ij}) \cap C_{\text{pos},i}(t = r_{ij}) \neq \emptyset \) rather than \( C_{\text{pos},i}(t = r_{ij}) \subseteq C_{\text{pos},i}(t = r_{ij}) \).

C. Control Synthesis

We derive the set of constraints that will be used to find the control law guaranteeing the satisfaction of contracts \( \mathcal{C}_1, \mathcal{C}_2, \) and \( \mathcal{C}_3 \). By Definition 1 and the related treatment, the set of inputs ensuring the satisfaction of \( \mathcal{C}_1 \) is given by \( \mathcal{U}_{h_{i}} = \{ u \in U : u \leq \frac{m_{\text{max}}}{m_{\text{max}} + V_f} (h_1 + V_f + \frac{v_i}{\rho_i} a_i) + F_r \} \), where \( a_i \) is the acceleration of the lead vehicle received by V2V communication.

Let \( t_{i-1,0} \) be the time at which the ego vehicle crosses the \( (i-1)^{th} \) traffic signal, serving as the initial condition as the vehicle approaches the \( i^{th} \) traffic signal. Let \( \mathcal{U}_{g_{ij},i}(t) \) denote the set of safe inputs when \( s_i = \text{Green}, \) i.e., when \( t \in \Gamma_{ij,0} = \{ \max(g_{ij}, t_{i-1,0}), y_{ij} \} \), and \( \mathcal{U}_{r_{ij},i}(t) \) the set of safe inputs when \( s_i = \text{Red}, \) i.e., over \( \Gamma_{ij,r} \). For all \( j \in \mathbb{N} \), for all \( t \in \Gamma_{ij,g} \), if \( \max(y_{ij}, t_{i-1,0}) \leq t < r_{ij} \), according to Theorem 4, the set of safe inputs is

\[
U_{h_{\text{pos},i}}(t) = \begin{cases} U_{h_{r,i}}(t) & \text{if } g_{ij} \leq t \leq y_{ij} \\ U_{h_{r,i}}(t) \cap U_{h_{r,i}}(t) & \text{if } y_{ij} < t < r_{ij} \\ U_{h_{r,i}}(t) & \text{if } r_{ij} < t < g_{ij} + 1 \end{cases}
\]

\[
U_{h_{r,i}}(t) = \{ u \in U : u \leq \frac{m}{\beta} \left( (h_{\text{r},i}(x) - V_f) + F_r \right) \}
\]

\[
U_{h_{r,i}}(t) = \{ u \in U : u \leq \frac{m}{\beta} \left( \gamma_{r,j+1} \text{sign}(h_{\text{r},i}(x)) \right) + F_r \}
\]

We have \( \gamma_{r,j+1} = \frac{|h_{\text{r},i}(x_{t_{ij}}, x_{t_{ij}+1})|^{1-\rho_j+1}}{1-\rho_j} \) for \( t_{ij} = \max(y_{ij}, t_{i-1,0}) \). The set inputs \( U_{h_v}(t) \) ensuring the satisfaction of \( \mathcal{C}_{h_v} \) can be finally calculated using Theorem 4.

\[
U_{h_v}(t) = \begin{cases} U_{h_{\text{pos},i}}(t) & \text{if } t_{j-1} < t \leq t'_{j} \\ U_{h_{\text{pos},i}}(t) \cap U_{h_{j-1}}(t) & \text{if } t'_{j} < t < t_j \end{cases}
\]

\[
U_{h_{j-1}}(t) = \{ u \in U : u \leq \frac{m}{\beta} (h_{v,j} + F_r) \}
\]

\[
U_{h_{j+1}}(t) = \{ u \in U : u \leq \frac{m}{\beta} (\gamma_{v,j+1} \text{sign}(h_{v,j+1}) + F_r) \}
\]

We have \( \gamma_{v,j+1} = \frac{|h_{v,j}(x_{t_{ij}}, x_{t_{ij}+1})|^{1-\rho_{j+1}}}{1-\rho_{j+1}} \) for \( t_{ij} = T_{\text{conv},i} \), which ensures that the system converges from \( C_{v,j} \) to \( C_{v,j+1} \) within \( T_{\text{conv},i} \).

By combining all the constraints above, we get \( U_{\text{safe},i}(t) = U_{\text{safe},i}(t) \cap U_{\text{h}_{pos,i}}(t) \cap U_{h_v}(t) \). A PID controller is chosen as a nominal controller \( u_{\text{nom}} = m \left( k_1 V_r + k_2 \phi_i + k_3 \int_0^t \phi_i \, dx \right) + F_r \) to achieve the goal that \( V_f \to V_i \) and \( \phi_i \to 0 \), where \( k_1, k_2, \) and \( k_3 \) are the PID gains.

VI. Simulations and Results

We simulate a road scenario with the ego and lead vehicles. The traffic is regulated by ten traffic signals. The distance between any two consecutive traffic signals is not equal. The timing cycles of different traffic signals are also different. The speed limits changes every 50 s, which is possibly unrealistic but helps validate our methodology in extreme conditions. The variable speed limit \( V_{\text{max}}(t) \) has three values: 30 m/s, 25 m/s, and 10 m/s. We consider the worst-case scenario in which the lead vehicle may violate the rules. We use \( g = 9.8 \frac{m}{s^2}, \) \( m = 1650 \, Kg, \) \( c_0 = 0.1 \, N, \) \( c_1 = 5 \frac{N}{m/s^2}, \) \( c_2 = 0.25 \frac{N}{m^2/s}, \) \( a_{\text{max}} = 0.4g, \) and \( \rho_{r,j+1} = 0.9 \) and \( \rho_{v,j+1} = 0.91 \) for all \( j \in \mathbb{N} \). \( Y_{ij} \) is the convergence time for \( \mathcal{C}_3 \) and \( T_{\text{conv},i} = 5 \, s \) is the convergence time within each interval \( T_{ij} \) in \( \mathcal{C}_3 \). Fig. 2a shows that \( g_{\left(0, t_{ij}\right)} h_1(x) \geq 0 \) holds, i.e., the ego vehicle maintains a safe distance from the lead vehicle during the entire trip. The relative speed profile of the ego vehicle, the lead vehicle, and the variable speed limit \( V_{\text{max}}(t) \) are summarized in Fig. 2b, showing that \( V_f \leq V_{\text{max}}(t) \) always holds. Finally, as shown in Fig. 3, the ego vehicle complies with the traffic signals. The height of the horizontal lines represents the relative position of the traffic signals with respect to the origin. The colors represent the states of the traffic signals with respect to time. The signals are not synchronized. The ego vehicle never crosses a traffic signal when the state is Red. Moreover, since \( h_{\text{pos},i}(t, x) \) is always non-negative in Fig. 2c, the traffic signal contract is satisfied. Fig. 2d provides the control input \( u \) as a function of time.

VII. Conclusions

We presented a compositional control synthesis method mapping a mission-level STL specification to an aggregation of contracts defined via continuously differentiable time-varying control barrier functions. The barrier functions are used to constrain the lower-level control synthesis problem, which is solved via quadratic programming. We illustrated the effectiveness of the proposed algorithm on a case study motivated by vehicular mission planning under safety constraints as well as constraints imposed by traffic regulations. Future work includes investigating extensions of the approach to more expressive fragments of STL as well as multi-agent planning.

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(a) $G_{[0,t,N_v]}h_1(x) \geq 0$. The ego vehicle always maintains a safe distance from the lead vehicle.

(b) $G_{[0,N_v]}h_1(t,x) \geq 0$. Speed profile of the lead vehicle ($V_l$) and ego vehicle ($V_f$), and speed limit $V_{\text{max}}(t)$.

(c) $G_{[0,N_v]}h_{\text{pos}}(t,x) \geq 0$. The ego vehicle obeys the traffic signals.

(d) Normalized input $u$.

Fig. 2: Simulation results: all safety and regulatory constraints are satisfied by the controlled vehicle.

Fig. 3: Position of the ego vehicle $X_f$ vs. time. The horizontal lines show the position of the traffic signals while the colors show the states at a given time.

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