A polytrope is a convex polytope that is expressed as the tropical convex hull of a finite number of points. Every bounded cell of a tropical linear space is a polytrope. Speyer conjectured that conversely every polytrope arises as a bounded cell of a tropical linear space. We develop general settings, and solve the conjecture for dimensions \( \leq 3 \). We also investigate vertices and edges of an arbitrary polytrope.

\section{Introduction}

Tropical geometry is geometry over min-plus or max-plus algebra, and in this paper our tropical semiring is assumed min-plus algebra. Many notions in classical geometry can be tropicalized, and when tropicalized they demonstrate interesting, but often intricate types of behavior. Convexity and linearity are two of such, and we study the relationship between their tropicalized notions. For standard tropical theory and terminology, we refer to [MS15]. Additionally, we refer to [DS04, JK10] for tropical convexity.

Let \( V = (v_1 \cdots v_k) \in \mathbb{R}^{k \times k} \) be a real square matrix of size \( k \), then \( V \) is tropically nonsingular if and only if the tropical convex hull \( P = \text{tconv} (v_1, \ldots, v_k) \subset \mathbb{R}^k / \mathbb{R} \mathbb{I} \) with \( \mathbb{I} = (1, \ldots, 1) \) has full-dimension, in which case \( P \) is called a tropical simplex. Every tropical simplex is decomposed into \textbf{polytropes}, that is, tropical polytopes that are convex polytopes at the same time, where a tropical polytope means the tropical convex hull of a finite number of points, cf. [DS04, Proposition 17].

Pick any \( k \) points \( v_1, \ldots, v_k \). As those points vary, their tropical convex hull \( \text{tconv} (v_1, \ldots, v_k) \) also varies along. If it has a full-dimensional polytrope \( P \), every vertex\(^1\) of \( P \) is the intersection of linear varieties \( V_i, i \in I \) for some nonempty proper subset \( I \subset [k] \) such that each \( V_i \) contains \( v_i \) and their codimensions \( c_i > 0 \) sum up to 2010 Mathematics Subject Classification. Primary 14T05; Secondary 05B35, 52B40, 52C35.

\(^1\)We mean by a vertex an ordinary vertex (pseudo-vertex).
to $k - 1$, cf. Section 4. The number of vertices of $P$ is at least $k$ and at most $\binom{2k - 2}{k - 1}$, [DS04, Proposition 19].

Let $M$ be a rank-$k$ loopless matroid on a set $\{n\} := \{1, \ldots, n\}$. The Dressian $\text{Dr}(M)$ is the moduli space of the $(k - 1)$-dimensional tropical linear spaces in the $(n - 1)$-dimensional tropical projective space, whose fiber is a balanced polyhedral complex dual to the loopless part of a coherent matroid subdivision of the matroid polytope $\text{BP}_M$, where a polyhedron is called loopless if it is not contained in any coordinate hyperplane. We will just say that the complex is dual to the subdivision for short, or vice versa. To each vertex of the tropical linear space, there corresponds a maximal matroid polytope of the subdivision.

Then, every bounded cell of a tropical linear space is a polytrope. David Speyer conjectured that the converse also holds. We reformulate the conjecture and call it Speyer’s conjecture:

**Conjecture 1.1.** Every polytrope up to tropical and affine isomorphisms arises as a bounded cell of a tropical linear space.

In other words, Speyer’s conjecture is equivalent to the following.

**Conjecture 1.2.** For any fixed dimension $d$ and for any $d$-dimensional polytrope $P$, there is another $d$-dimensional polytrope $P'$ that arises as a cell of a tropical linear space and that is isomorphic to $P$ under some tropical and affine isomorphism.

The conjecture is plainly true in dimension 1, and turns out true in dimension 2: Consider the $(k, n)$-hypersimplex $\Delta^k_n$ for positive integers $k$ and $n$ with $k \leq n$:

$$\Delta^k_n = \prod_{i=1}^n [0, 1] \cap \{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = k \}$$

where $[0, 1]$ is a closed interval in $\mathbb{R}$. For any full-dimensional matroid subdivision in the hypersimplex $\Delta^k_n$ with $k = 3$, the number of its maximal matroid polytopes that contain a fixed common ridge is at most 6, [Shi19, Theorem 3.21]. From this, it ultimately follows that all 2-dimensional polytopes, up to tropical and affine isomorphisms, arise as cells of tropical linear spaces, see Section 6.

We go a step further and show that the conjecture holds in dimension 3. Develin and Sturmfels showed polytopes come from coherent subdivisions of a product of two simplices, [DS04, Theorem 1], which is quite a common approach to polytopes. In this paper, however, we directly attack matroid subdivisions, and for any given polytrope we consider a coherent matroid subdivision such that the polytrope is a bounded cell of a polyhedral complex dual to the matroid subdivision, where the matroidal setting is indebted to [Shi19].

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2We mean by a matroid polytope a (matroid) base polytope, that is, the convex hull of indicator vectors of bases of a matroid, cf. Section 2.
2. Preliminaries

In this section, a few new lemmas are introduced at the end, and relevant notions and notations are offered beforehand. For more details or for a more comprehensive grasp, readers are suggested to refer to [Aig79, GS87, Oxl11, Sch03, Shi19].

Let $M$ be a (finite) matroid with rank function $r$. A pair $\{F, L\}$ of subsets of the ground set $E(M)$ is called a modular pair if:

$$r(F) + r(L) = r(F \cup L) + r(F \cap L).$$

A subset $A$ of $E(M)$ is called a separator of $M$ if $\{A, E(M) - A\}$ is a modular pair. Let $A_1, \ldots, A_{\kappa(M)}$ be all nonempty inclusionwise minimal separators of $M$ where $\kappa(M)$ is the number of those. Note that $\kappa$ is a $\mathbb{Z}_{\geq 0}$-valued function defined on the collection of matroids. Then, $M$ is written as:

$$M|_{A_1} \oplus \cdots \oplus M|_{A_{\kappa(M)}}$$

where all $M|_{A_i}$ with $i = 1, \ldots, \kappa(M)$ are called the connected components of $M$, and $\kappa(M)$ is the number of connected components of $M$. A matroid $M$ is called inseparable or connected if it has no proper separator, and separable or disconnected otherwise. A subset $A$ of the ground set $E(M)$ is called inseparable or separable if $M|_A$ is.\(^3\) For any $A \subseteq E(M)$, we denote:

$$M(A) := M|_A \oplus M/A.$$ 

For subsets $A_1, \ldots, A_m$ of $E(M)$, we write:

$$M(A_1)(A_2) \cdots (A_m) = (\cdots ((M(A_1))(A_2)) \cdots )(A_m).$$

A subset $A \subseteq E(M)$ is called non-degenerate if $\kappa(M(A)) = \kappa(M) + 1$.\(^4\)

The indicator vector of a subset $A \subseteq [n] := \{1, 2, \ldots, n\}$ is defined as a vector $1^A \in \mathbb{R}^n$ whose $i$-th entry is 1 if $i \in A$, and 0 otherwise. The convex hull of the indicator vectors $1^B$ of bases $B$ of a matroid $M$ is called a matroid polytope or a base polytope of $M$ and denoted by $BP_M$ while $M$ is called the matroid of $BP_M$. The dimension of $BP_M$ is:

$$\dim BP_M = |E(M)| - \kappa(M)$$

where $|E(M)|$ denotes the cardinality of $E(M)$, and again, $\kappa(M)$ is the number of connected components of $M$. Note that $BP_M$ is full-dimensional if and only if $M$ is inseparable.

Every face of a matroid polytope $BP_M$ is again a matroid polytope. The matroid of a face of $BP_M$ is called a face matroid of $M$. For any vector $w \in \mathbb{R}^S$, consider the face of the matroid polytope $BP_M$ at which $w$ is maximized. The matroid of the face is called the initial matroid of $M$ with respect to $w$ and denoted by $M_w$, see [MS15, Chapter 4.2].

\(^3\)“Inseparable” was used in [Sch03] to indicate a subset $A$ of $E(M)$ for a matroid $M$ such that the restriction matroid $M|_A$ is connected. In this paper, along the convention of [Shi19] we use inseparable (preferred) or connected for both inseparable subsets and connected matroids.

\(^4\)The definition of non-degenerate subsets was originally given in [GS87] only for inseparable matroids, and generalized to the current form in [Shi19].
For the nonempty ground set $S$, we denote by $\mathbb{R}^S$ the product of $|S|$ copies of $\mathbb{R}$ labeled by the elements of $S$, one for each. A partition $\bigcup_{i \in [k]} A_i$ of $S$ is said to be a $k$-partition. For any nonempty subset $I \subseteq [k]$, we denote:

$$(2.3) \quad A_I = \bigcup_{i \in I} A_i.$$ 

For any subset $A$ of $S$, we denote the formula:

$$x(A) = \sum_{i \in A} x_i$$

where $x_i$ are understood as coordinate functions in $\mathbb{R}^S$. Also, for any vector $v \in \mathbb{R}^S$ whose $i$-th entry is $v_i$ we write:

$$v(A) = \sum_{i \in A} v_i.$$ 

Let $W$ be a linear subspace of $\mathbb{R}^S$, and consider a quotient map $q : \mathbb{R}^S \rightarrow \mathbb{R}^S/W$. For any subset $U \subseteq \mathbb{R}^S$, we say that $q(U)$ equals $U$ modulo $W$ or vice versa. We also say that $U$ equals $U'$ modulo $W$ or vice versa if $q(U) = q(U')$.

Let $Q$ be a polyhedron with a set $Q$ of describing equations and inequalities. If the ambient space is understood, we simply write $Q$ for $Q$. For instance, the $(k, S)$-hypersimplex $\Delta^k_S \subseteq \mathbb{R}^S$ is defined as:

$$\Delta^k_S := [0, 1]^S \cap \{x(S) = k\}$$

where $[0, 1] \subseteq \mathbb{R}$ is a closed interval. For a nonempty polytope $Q$, denote by Vert($Q$) the set of all vertices of $Q$, by Aff($Q$) the affine span of $Q$, and by Aff$^0(Q)$ the linear span of $Q - \{p\}$ for some point $p \in Q$.

Let $Q, \hat{Q} \subseteq \mathbb{R}^S$ be two polytopes such that $Q$ is a nonempty proper face of $\hat{Q}$. Let $q : \mathbb{R}^S \rightarrow \mathbb{R}^S/\text{Aff}_0(Q)$ be a quotient map and $t : \mathbb{R}^S \rightarrow \mathbb{R}^S$ a transition map defined by $x \mapsto x - p$ for some $p \in Q$. Then, the image of $\hat{Q}$ under the map $q \circ t$ is called the quotient polytope of $Q$ modulo $Q$ and denoted by $\hat{Q}/Q$ or simply $[\hat{Q}]$ using square bracket when the context is clear, cf. [Max84]. We say that two polytopes are face-fitting if their intersection is a common face of both.

A $(k, S)$-tiling $\Sigma$ is a finite collection of polytopes in the $(k, S)$-hypersimplex $\Delta^k_S$ that are pairwise face-fitting. If all members of the tiling are matroid polytopes, it is called a matroid tiling. The support of $\Sigma$ is the union of its members. The dimension of $\Sigma$ is the dimension of the support of $\Sigma$. Throughout the paper, a matroid tiling is assumed equidimensional, that is, all of its members have the same dimension. A matroid subdivision of a matroid polytope is a matroid tiling whose support is the matroid polytope. When mentioning cells of $\Sigma$, we identify $\Sigma$ with the polytopal complex that its matroid polytopes generate with intersections.

Let $Q$ be a nonempty common cell of the polytopes of a tiling $\Sigma$. The collection of quotient polytopes of the members of $\Sigma$ modulo $Q$ is said to be the quotient tiling of $\Sigma$ modulo $Q$, and denoted by $\Sigma/Q$ or simply $[\Sigma]$.

The intersection of base collections of two matroids $M_1$ and $M_2$ is called the base intersection of $M_1$ and $M_2$, and denoted by $M_1 \cap M_2$. When $M_1 \cap M_2$ is the base collection of a matroid, we denote the matroid by $M_1 \cap M_2$ abusing notation. For instance, if $M_1$ and $M_2$ are face matroids of the same matroid, then $M_1 \cap M_2$ is a matroid. For a collection $A$ of subsets of $S$, denote by $P_A$ the convex hull of the indicator vectors $1^A \in \mathbb{R}^S$ of all $A \in A$. Then, [Sch03, Corollary 41.12d] says:

$$\text{BP}_{M_1} \cap \text{BP}_{M_2} = P_{M_1 \cap M_2}.$$
We borrow some lemmas from [Shi19] and adjust them to our context.

**Lemma 2.1** ([Shi19]). Let $M = (r; S)$ be a matroid of rank $\geq 3$.

1. Let $F$ and $L$ be two subsets of $S$. Then, $M(F) \cap M(L) \neq \emptyset$ if and only if $\{F, L\}$ is a modular pair.

2. Suppose that $F_1, \ldots, F_m$ are subsets of $S$ such that $\bigcap_{i \in [m]} M(F_i)$ is a nonempty loopless matroid. Then, for any permutation $\sigma$ on $[m]$ one has:

$$\bigcap_{i \in [m]} M(F_i) = M(F_{\sigma(1)}) \cdots (F_{\sigma(m)}).$$

Further, every member of the Boolean algebra generated by $F_1, \ldots, F_m$ with unions and intersections is a flat of $M$.

3. Suppose in addition that $M$ is an inseparable matroid. Let $F$ and $L$ be two distinct non-degenerate flats with $r(F) \geq r(L)$ such that $BP_M(F) \cap BP_M(L) = BP_M(F \cap L)$ is a codimension-2 face of $BP_M$. Then, precisely one of the following three cases happens.

| $F \cap L = \emptyset$ | $M(F) \cap M(L) = M(F \cup L)$ with $M(F \cup L) = M|_F \oplus M|_L$ |
|----------------------|------------------------------------------------------------------|
| $F \cup L = S$ | $M(F) \cap M(L) = M(F \cap L)$ with $M(F \cap L) = M|_F \oplus M|_L$ |
| $F \supseteq L$ | $M(F) \cap M(L) = M/F \oplus M|_F / L \oplus M|_L$ |

### 3. Matroid Subdivisions of the Hypersimplex $\Delta^4_S$

Matroids can be identified with some 0/1-polytopes, that is, convex polytopes whose vertices are indicator vectors contained in the hypersimplex $\Delta^k_S$ for some positive integer $k$ and some finite set $S$, whose edge lengths\(^5\) are all 1, cf. [GGMS87, Theorem 4.1], [GGS87, Theorem 1], and [Sch03, Theorem 40.6]. Note that a matroid polytope can be obtained from a product of hypersimplices (which is also a matroid polytope) by cutting off corners.

In general, it is a difficult problem to describe how to cut a matroid polytope for producing another matroid polytope. In this section, we may restrict our interests to matroid subdivisions of the hypersimplex $\Delta^4_S$ whose matroid polytopes have a nonempty common face of codimension 3 that is contained in the interior\(^6\) of $\Delta^4_S$.

For full-dimensional matroid polytopes in $\Delta^4_S$, there is a characteristic property as follows, which will be used in the latter half of this section.

**Lemma 3.1.** Let $M = (r; S)$ be a rank-4 inseparable matroid with a rank-2 non-degenerate flat $F$. If $L$ is a non-degenerate flat of $M$ such that $BP_M(F) \cap BP_M(L)$ is a codimension-2 face of $BP_M$, then $r(L) \neq 2$.

**Proof.** Suppose $r(L) = 2$, then $L \neq F$ by assumption. Since $BP_M(F) \cap BP_M(L)$ is nonempty, $\{F, L\}$ is a modular pair by Lemma 2.1 (1), that is,

$$r(F \cup L) + r(F \cap L) = r(F) + r(L) = 4.$$

\(^5\)For a line segment $1^A1^B \subset \Delta^k_S$ with $A, B \subseteq S$, the $L^1$-norm of the vector $1^A - 1^B$ or $1^B - 1^A$ is $|A \cup B - A \cap B|$, and we mean by the length of $1^A1^B$ the number $\frac{1}{2}|A \cup B - A \cap B|$.

\(^6\)The interior of a polyhedron $Q$ is $Q - \partial Q$ which is denoted by $\text{int}(Q)$. 

Moreover, by Lemma 2.1 (3), one has either $F \cap L = \emptyset$ or $F \cup L = S$. But, the above formula tells that both of them happen at the same time, a contradiction to Lemma 2.1 (3). Therefore we conclude $r(L) \neq 2$. \hfill \Box

Now, we study matroid subdivisions of $\Delta^4_S$ of our interest. Let $S$ be a (finite) ground set of cardinality $\geq 8$. Fix as large a field $k$ as possible, for instance an infinite field such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$, and consider planes in $\mathbb{P}^3$ over $k$ as follows.

Let $F$ be a nonempty subset of $S$ with $|S-F| \geq 4$. Consider $|S-F| + 1$ planes in general position and label $|S-F|$ of them by elements of $S-F$, one for each, and label the remaining plane by (all the elements of) $F$; this defines a plane arrangement. Let $M$ be the corresponding matroid, then $F$ is its unique non-degenerate flat of size $> 1$ and its simplification is isomorphic to $U^4_{|S-F|+1}$. Since there are 5 planes in general position, $M$ is inseparable, cf. [Shi19, Lemma 4.14]. The matroid polytope $\text{BP}_M$ is given by:

$$\text{BP}_M = \Delta^4_S \cap \{x(F) \leq 1\}.$$ 

Moreover, $\Delta^4_S \cap \{x(S-F) \leq 3\}$ is also a full-dimensional matroid polytope, and let $M'$ be its inseparable matroid:

$$\text{BP}_{M'} = \Delta^4_S \cap \{x(S-F) \leq 3\}.$$ 

Which plane arrangement has this matroid structure $M'$? Consider $|S-F|$ distinct planes in $\mathbb{P}^3$ meeting at a point such that no 3 of them meet in a line. Generically embed them in another copy of $\mathbb{P}^3$ with $|F|$ planes in general position. The resulting plane arrangement has matroid structure $M'$ with $S-F$ a unique non-degenerate flat of size $> 1$.

Let $L$ be a nonempty subset of $S$ such that $|L| \geq 3$ and $|S-L| \geq 3$. Consider $|L|$ distinct planes in $\mathbb{P}^3$ meeting in a line and generically embed them in another copy of $\mathbb{P}^3$ with $|S-L|$ planes in general position. Let $M''$ be the corresponding matroid, then $L$ is a unique non-degenerate flat of size $> 1$. Its matroid polytope $\text{BP}_{M''}$ is given by:

$$\text{BP}_{M''} = \Delta^4_S \cap \{x(L) \leq 2\}.$$ 

Let $\sqcup_{i\in[4]} A_i$ be a 4-partition of $S$ with $|A_i| \geq 2$ for all $i \in [4]$, and recall the notation (2.3). Consider a polyhedral subdivision $\hat{\Sigma}$ of $\Delta^4_S$ obtained by cutting $\Delta^4_S$ with 4 planes $\{x(A_i) = 1\}$, $i \in [4]$. For any $i \in [4]$, denote:

$$\text{BP}_{M_i} := \Delta^4_S \cap \{x(A_{[4]-\{\ell\}}) \leq 3 : \ell \in [4] - \{i\}\} ,$$ 

$$\text{BP}_{M_{(i)}} := \Delta^4_S \cap \{x(A_{\ell}) \leq 1 : \ell \in [4] - \{i\}\} .$$

Also, for any $i, j \in [4]$ with $i \neq j$, denote:

$$\text{BP}_{M_{ij}} := \Delta^4_S \cap \left( \bigcup_{i \in \{i,j\}} \{x(A_i) \leq 1\} \right) \cap \left( \bigcap_{\ell \in [4]-\{i,j\}} \{x(A_{[4]-\{\ell\}}) \leq 3\} \right) .$$

Then, by definition,

$$\text{BP}_{M_{ij}} = \text{BP}_{M_{ij}} .$$

Then, the subdivision $\hat{\Sigma}$ consists of four $\text{BP}_{M_i}$’s, four $\text{BP}_{M_{(i)}}$’s and six $\text{BP}_{M_{ij}}$’s, and hence 14 polytopes in total:

$$\hat{\Sigma} = \{\text{BP}_{M_i} : i \in [4]\} \cup \{\text{BP}_{M_{(i)}} : i \in [4]\} \cup \{\text{BP}_{M_{ij}} : 1 \leq i < j \leq 4\} .$$

Those 14 polytopes are matroid polytopes by the above argument, and $\hat{\Sigma}$ is a matroid subdivision of $\Delta^4_S$. 

Let $Q = \cap \tilde{\Sigma} \subset \Delta^4_4$, then $Q$ is also a matroid polytope whose matroid is a direct sum of rank-1 uniform matroids:

$$U^1_{A_1} \oplus U^1_{A_2} \oplus U^1_{A_3} \oplus U^1_{A_4}.$$ Consider the quotient polytope $[\Delta^S_4] = \Delta^4_4/Q$ which is a 3-simplex and also the quotient tiling $[\tilde{\Sigma}] = \tilde{\Sigma}/Q$, see Figures 3.1 and 3.2 for the visualizations, where the black dots stand for the quotient polytope $[Q]$.

![Figure 3.1. The quotient tiling $[\tilde{\Sigma}]$.](image)

![Figure 3.2. The three kinds of maximal cells of $[\tilde{\Sigma}]$.](image)

Observe that $\text{BP}_{M_i}$ and $\text{BP}_{M_{i\ell}}$ cannot be further split into matroid polytopes by cutting with those planes containing $Q$. But, $\text{BP}_{M_{ij}}$ can be split so with one of two planes $\{x(A_{i,j}) = 2\}$ and $\{x(A_{i,j}) = 2\}$ for some $\ell \in [4]-\{i,j\}$ where actually these planes are unique two such planes by Lemma 3.1 while there are only three planes of the form $\{x(A_I) = 2\}$ with $I \subset [4]$ of cardinality 2:

$$\{x(A_{1,2}) = 2\} = \{x(A_{3,4}) = 2\},$$

$$\{x(A_{1,3}) = 2\} = \{x(A_{2,4}) = 2\},$$

$$\{x(A_{1,4}) = 2\} = \{x(A_{2,3}) = 2\}.$$ Denote:

$$\text{BP}_{M_{i\ell}(\ell)} := \Delta^4_n \cap \{x(A_i) \leq 1\} \cap \{x(A_{i,j}) \leq 2\} \cap \{x(A_{[4]-\ell}) \leq 3\}.$$
Then,
\[ (3.2) \quad \text{BP}_{M_{ij}(\ell)} = \text{BP}_{M_{ij}} \cap \{ x(A_{ij,\ell}) \leq 2 \}. \]
Note that
\[ \text{BP}_{M_{ij}(\ell)} \neq \text{BP}_{M_{ji}(\ell)}. \]
Moreover, \( \text{BP}_{M_{ij}(\ell)} \) and \( \text{BP}_{M_{ji}(\ell')} \) with \( \{ \ell' \} = [4] - \{ i, j, \ell \} \) are face-fitting through their common facet which is contained in \( \{ x(A_{ij,\ell}) = 2 \} \), and their union is:
\[ (3.3) \quad \text{BP}_{M_{ij}(\ell)} \cup \text{BP}_{M_{ji}(\ell')} = \text{BP}_{M_{ij}}. \]
See Figure 3.3 for the visualization of the quotient polytopes. Note that \( A_{\{ j,\ell \}} \) and \( A_{\{ i,\ell' \}} \) are non-degenerate flats of the matroids \( M_{ij}(\ell) \) and \( M_{ji}(\ell') \), respectively.

![Figure 3.3. The two splits of \([\text{BP}_{M_{ij}}]\).](image)

Splitting all \( \text{BP}_{M_{ij}} \) with \( 1 \leq i < j \leq 4 \) as above produces a matroid subdivision of \( \Delta_4^4 \), say \( \Sigma \), which is a refinement of \( \tilde{\Sigma} \) with \( Q = \cap \Sigma \). Then, \( \Sigma \) has 20 maximal matroid polytopes, where the quotient tiling \([\Sigma]\) has 4 parallelepipeds, 4 tetrahedra and 12 triangular prisms. Note that there are \( 2^6 \) different choices for \( \Sigma \).

Now, Lemma 3.1 tells that no more such splitting is possible, and \( \Sigma \) has the largest number of maximal cells.

4. Vertices and Edges of Polytropes

4.1. Vertices of polytropes. Fix an integer \( k \geq 3 \), and consider the tropical projective space \( \mathbb{R}^{[k]} / \mathbb{R} \mathbb{I} \) with coordinates \( (x_1, \ldots, x_k) \). For any \( i \in [k] \) let \( E_i \) be the convex cone spanned over \( \mathbb{R}_{\geq 0} \) by standard basis vectors \( e_j, j \in [k] - \{ i \} \), where \( \mathbb{R}_{\geq 0} \) denotes the set of all nonnegative real numbers:
\[ E_i := \mathbb{R}_{\geq 0} (e_j : j \in [k] - \{ i \}). \]
Let \( P \) be a polytrope in \( \mathbb{R}^{[k]} / \mathbb{R} \mathbb{I} \). We may assume that \( P \) is full-dimensional, cf. [DS04, Proposition 17], so there are points \( \mathbf{v}_0 \in \text{int}(P) \) and \( \mathbf{v}_i \in \text{int}(E_i + \mathbf{v}_0) \) for all \( i \in [k] \), such that \( P \) is written as follows, cf. [JK10, Proposition 15] and [MS15, Proposition 5.2.10]:
\[ P = \text{tconv} (\mathbf{v}_1, \ldots, \mathbf{v}_k). \]
We further assume that \( P \) has the maximal number of vertices, which is \( \binom{2k-2}{k-1} \), see [DS04, Proposition 19].

We begin with an observation that by the classical convexity of \( P \) any fixed vertex \( \mathbf{v} \) of \( P \) is the intersection of hyperplanes in \( \mathbb{R}^{[k]} / \mathbb{R} \mathbb{I} \) such that the number of those hyperplanes is larger than or equal to \( k - 1 \) and each of them passes through exactly one of \( m \) distinct points \( \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m} \) for some \( m \in [k - 1] \) due to the maximality (of the number of vertices) of \( P \). For convenience, we may let \( \{ i_1, \ldots, i_m \} = [m] \) without loss of generality.
But, then, since $P$ is expressed as a tropical convex hull of the $k$ points $v_1, \ldots, v_k$, the point $v$ is the intersection of \textbf{max-plus} tropical hyperplanes with vertices $v_i$, $i \in [m]$, cf. [MS15, Section 5.2], and hence the vertex figure of $P$ at any vertex $v$ is a $(k - 1)$-simplex.

For each $i \in [m]$ let $V_i = V_i(v)$ be the intersection of those hyperplanes passing through $v_i$, then $V_i$ is a linear variety in $\mathbb{R}^k/\mathbb{R}$ and in an “\textbf{tropical affine piece}” $\mathbb{R}^k \setminus \{i\}$ it is described by linear equations $e_j \cdot (x - v_i) = 0$ for all $j$ contained in some nonempty subset $C_i$ of $[k] \setminus \{i\}$:

$$C_i = C_i(v) \subseteq [k] \setminus \{i\}.$$  

The codimension of $V_i$ is $|C_i|$ and all those codimensions sum up to $k - 1$, that is,

$$\sum_{i \in [m]} |C_i| = k - 1.$$

We are tempted to say that $C_i$, $i \in [m]$, are disjoint, which is not a valid reasoning. However, by the convexity and the maximality of $P$, we can say that each $V_i$ with $i \in [m]$ contains no $v_j$ with $j \in [k] \setminus \{i\}$. Now, let:

$$D_i = D_i(v) := [k] - C_i \cup \{i\}.$$  

Then,

$$V_i = v_i + \mathbb{R} \langle e_j : j \in D_i \rangle.$$  

Define $V_i^- \subset V_i$ as:

$$V_i^- = V_i^-(v) := v_i + \mathbb{R}_{\geq 0} \langle -e_j : j \in D_i \rangle = v_i + \bigcap_{j \in C_i \cup \{i\}} (-E_j)$$

where $(-E_j) = \mathbb{R}_{\geq 0} \langle -e_\ell : \ell \in [k] \setminus \{j\} \rangle$. By convention we write $V_i^- = v_i$ when $D_i = \emptyset$. Then, the vertex $v$ is written as the intersection of $V_i^-$’s for all $i \in [m]$:

$$v = \bigcap_{i \in [m]} V_i^-.$$  

By classical Bézout’s theorem, the above expression of $v$ in terms of $v_i$ and $C_i$ is uniquely determined. We define $C_i = \emptyset$ for $i \in [k] - [m]$, and introduce a notation.

\textbf{Notation 4.1.} We denote:

$$v = v_{i_1}^{C_{i_1}(v)} \cdots v_{i_m}^{C_{i_m}(v)}$$

where $i_j \in [k]$ and $C_{i_j}(v) \neq \emptyset$ for all $j \in [m]$. Then, $v_i = v_i^{[k] \setminus \{i\}}$ for all $i \in [k]$.

Note that $\bigcap_{i \in [m]} D_i = \emptyset$ and $\bigcup_{i \in [m]} D_i = k - 1$. Further, $D_i = \emptyset$ for some $i \in [m]$ if and only if $m = 1$.

When $m \geq 2$, there are $i, j \in [m]$ with $i \neq j$. Then, at least one of two statements $j \in D_i$ and $i \in D_j$ is true since otherwise $V_i^-$ and $V_j^-$ do not intersect each other. Actually, both of them hold true by the convexity and the maximality of $P$ since otherwise $v_i$ and $v_j$ are contained in a linear subvariety of positive codimension, a contradiction. Then,

$$[m] \subset D_i \cup \{i\} \quad \text{and} \quad C_i \cap [m] = \emptyset$$

which are also true for $m = 1$ and hence true for all $m \in [k - 1]$. This begets:

$$\bigcup_{i \in [m]} C_i = [k] - [m] \quad \text{and} \quad \bigcap_{i \in [m]} C_i = [k] - \bigcup_{i \in [m]} D_i.$$
Likewise, for the rest of this subsection, all the computations are elementary set-theoretic computations. Since \(|\cup_{i \in [m]} D_i| = k - 1\), we have \(|\cap_{i \in [m]} C_i| = 1\). Then, by pigeonhole principal, we have a disjoint union:
\[
\cup_{i \in [m]} (C_i - \cap_{\ell \in [m]} C_\ell) = [k] - [m] - \cap_{\ell \in [m]} C_\ell
\]
which is not necessarily an \(m\)-partition, that is, it is possible that \(C_i = \cap_{\ell \in [m]} C_\ell\) for some \(i \in [m]\). Now, for all \(i \in [m]\), let:
\[
D_i^* = D_i^*(v) := C_i \cup \{i\} - \cap_{\ell \in [m]} C_\ell.
\]
Then,
\[
[k] = D_i \cup D_i^* \cup \left(\cap_{\ell \in [m]} C_\ell\right).
\]
We have another disjoint union:
\[
\cup_{i \in [m]} D_i^* = [k] - \cap_{\ell \in [m]} C_\ell = \cup_{i \in [m]} D_i.
\]
Note that for all \(i \in [m]\),
\[
1 \leq |D_i^*| < k - 1.
\]
The Boolean algebra generated by \(D_i, i \in [m]\), with intersections and unions is the same as that generated by \(D_i^*\)’s, \(i \in [m]\). Every nonempty member of the Boolean algebra is expressed as a union of \(D_i^*\)’s. Further, for any \(\emptyset \neq I \subset [m]\),
\[
\cup_{i \in I} D_i^* = \cap_{j \in [m] \setminus I} D_j.
\]
In particular, for all \(i \in [m]\),
\[
D_i^* = \cap_{j \in [m] \setminus \{i\}} D_j.
\]
Therefore, let:
\[
V_i^* = V_i^*(v) := \cap_{j \in [m] \setminus \{i\}} V_j \not\ni v_i.
\]
Then, there is a face of the vertex figure of \(P\) at \(v\) whose affine span is \(V_i^*\). Then, any intersection of two distinct those faces is the point \(\{v\}\), and both \(P\) and the convex hull of those faces have the same vertex figure at \(v\).

Remark 4.2. Note that \(D_i^*(v)\) and \(V_i^*(v)\) are defined only when \(m \geq 2\).

4.2. Directed edges of polytopes. Fix a vertex, say \(v = v_{i_1}^{C_{i_1}(v)} \ldots v_{i_m}^{C_{i_m}(v)}\) with a subset \(I = \{i_1, \ldots, i_m\} \subset [k]\) of size \(m \leq k - 1\) where \(C_i(v) \neq \emptyset\) for all \(i \in I\), see Notation 4.1. Let \(w\) be a vertex of \(P\) such that the line segment \(\overrightarrow{vw}\) connecting \(v\) and \(w\) is an edge of \(P\). Consider a directed edge originating from \(v\) with a direction vector \(\overrightarrow{vw}\) and denote it by a pair \((\overrightarrow{vw}, \overrightarrow{vw})\) of the edge and the direction vector. Now that \(\overrightarrow{vw} = \overrightarrow{vw}\) and \(\overrightarrow{vw} = -\overrightarrow{vw}\), it is natural to define:
\[
(\overrightarrow{vw}, \overrightarrow{vw}) = (\overrightarrow{vw}, \overrightarrow{vw}) := -(\overrightarrow{vw}, \overrightarrow{vw}).
\]
From now on, if both start and end points of a direction vector are displayed in an expression of the vector, we use this expression to denote the directed edge unless it causes confusion. That is, we simply write:
\[
\overrightarrow{vw} \text{ for } (\overrightarrow{vw}, \overrightarrow{vw}) \text{ and } \overrightarrow{vw} = -\overrightarrow{vw} \text{ for } (\overrightarrow{vw}, \overrightarrow{vw}).
\]
Since the vertex figure of \(P\) is a \((k - 1)\)-simplex, there are exactly \(k - 1\) edges and also exactly \(k - 1\) directed edges originating from \(v\).

Let \(v\) be a vertex of \(P\) that is different from \(v_1, \ldots, v_k\), then \(m = |I| \geq 2\). Consider the vertex figure of \(P\) at \(v\). Any edge \(\overrightarrow{vw}\) of \(P\) arises as an affine-span-generator of a line that is the intersection of the linear variety \(V_i^*(v)\) for some \(i \in I\).
and certain hyperplanes $H_1, \ldots, H_t$ passing through the point $v_i$ whose number is $t = |C_i(v)| - 1 = |D_i^v(v)| - 1$:

$$\mathbb{R} \langle \mathbf{w} \rangle = V_i^v \cap H_1 \cap \cdots \cap H_t.$$  

Every $a \in D_i^v(v)$ determines a directed edge of $P$ originating from $v$ as follows.

- For $a \neq i$, those hyperplanes are described by linear equations $e_j \cdot (x - v_i) = 0$ in $\mathbb{R}^{[k]-\{i\}}$ for all $j \in C_i(v) - \{a\}$, respectively, and:

  $$\mathbf{w} = \mathbb{R}_{\geq 0}(-e_a) = \mathbb{R}_{\geq 0} \left( \sum_{j \in [k]-\{a\}} e_j \right) = \mathbb{R}_{\geq 0} 1^{[k]-\{a\}}.$$  

- For $a = i$, those hyperplanes are described by linear equations $e_j \cdot (x - v_i) = 0$ in $\mathbb{R}^{[k]-\{i\}}$ for all $j \in C_i(v) - \cap_{\ell \in I} C_\ell(v) = D_i^v(v) - \{i\}$, respectively, and:

  $$\mathbf{w} = \mathbb{R}_{\geq 0} \left( \sum_{j \in D_i^v(v)} e_j \right) = \mathbb{R}_{\geq 0} 1^{D_i^v(v)}.$$  

Since $\sum_{i \in [m]} |D_i^v(v)| = k - 1$, this classifies all $k - 1$ directed edges $\mathbf{w}$ originating from the vertex $v = v_{i_1}^{C_{i_1}(v)} \cdots v_{i_m}^{C_{i_m}(v)}$ for $m \geq 2$: every direction vector $\mathbf{w}$ is a positive constant multiple of either:

- $1^{[k]-\{i\}}$ for some $i \in [k] - I - \cap_{\ell \in I} C_\ell(v)$, or

- $1^{D_i^v(v)}$ for some $i \in I$.

When $v = v_j$ for some $j \in [k]$, that is, when $m = 1$, the $k - 1$ direction vectors are positive constant multiples of $(-e_i) = 1^{[k]-\{i\}}$ for all $i \in [k] - \{j\}$, respectively.

Now, at every vertex $v$ of $P$ and for all directed edges $\mathbf{w}$ originating from $v$, we define $\Lambda^v(\mathbf{w})$ by the following:

$$\Lambda^v(\mathbf{w}) = \begin{cases} [k] - \{i\} & \text{if } \mathbf{w} = 1^{[k]-\{i\}} \text{ for some } i \in [k], \\ D_i^v(v) & \text{if } \mathbf{w} = 1^{D_i^v(v)} \text{ for some } i \in [k]. \end{cases}$$

This notion is well-defined by the above argument. Note that:

$$\Lambda^v(\mathbf{w}) \cup \Lambda^w(\mathbf{w}) = [k].$$

5. **General Settings**

Fix an integer $k \geq 3$. Let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^{[k]} / \mathbb{R}1$ be a full-dimensional polytope. Assume that $P$ has the maximal number of vertices as in Section 4 and suppose that $P$ is a cell of a tropical linear space dual to a matroid subdivision of dimension $\geq k - 1$ whose support is a rank-$d$ loopless matroid polytope contained in the hypersimplex $\Delta^d_{S}$ for an integer $d \geq k$ and a finite set $S$. Note that $P$ is not assumed a maximal cell of the tropical linear space, and also that the dual matroid subdivision is not necessarily full-dimensional in $\Delta^d_{S}$, that is, its dimension can be less than $\dim \Delta^d_{S} = |S| - 1$.

Then, to every vertex $v$ of $P$, there corresponds a matroid polytope, say $BP_{M^v}$. Let $\Sigma$ be the set of those matroid polytopes, then $\Sigma$ is an equidimensional matroid tiling, and $\cap \Sigma$ is a nonempty loopless common face of those matroid polytopes:

$$\Sigma = \{BP_{M^v} : v \in \text{Vert}(P)\}.$$  

\[\text{This is a combinatorial analog of logarithm.}\]
where \( \text{Vert}(P) \) denotes the set of vertices of \( P \). Moreover, \( \cap \Sigma \) has codimension \( k - 1 \) in the support of \( \Sigma \). The matroid of \( \cap \Sigma \), say \( M_\cap \), is a direct sum of \( \kappa(M^\vee) + k - 1 \) connected components, that is,

\[
\kappa(M_\cap) = \kappa(M^\vee) + k - 1
\]

where \( \kappa \) denotes the number of connected components. Further, by Lemma 2.1 (2), the matroid \( M_0 \) can be written as

\[
M_0 = M_0|_{A_1} \oplus \cdots \oplus M_0|_{A_k}
\]

for some \( k \)-partition \( \sqcup_{i \in [k]} A_i \) of \( S \) such that for any vertex \( v \) of \( P \), the matroid \( M_0 \) is obtained as

\[
M_0 = M^\vee(F_1^\vee) \cdots (F_k^\vee)
\]

for some non-degenerate flats \( F_1^\vee, \ldots, F_k^\vee \) of \( M^\vee \) that are contained in the Boolean algebra generated by \( A_1, \ldots, A_k \), cf. (2.1) and (2.2). Recall the notation (2.3).

Fix an arbitrary vertex \( v \) of \( P \). For any directed edge \( \overrightarrow{vw} \) of \( P \) originating from the vertex \( v \) there is a non-degenerate flat \( F \) of \( M^\vee \) such that \( \mathbb{R}_{\geq 0} \overrightarrow{vw} \) equals \( \mathbb{R}_{\geq 0}^{1_{S-F}} \) modulo \( \text{Aff}_0(\cap \Sigma) \), cf. [DS04, Proposition 17], and hence

\[
A_{\Lambda^\vee(\overrightarrow{vw})} = S - F
\]

where the facet matroid \( M^\vee(F) \) of \( M^\vee \) equals the initial matroid \( (M^\vee)_{1_{S-F}} \) of \( M^\vee \) with respect to the indicator vector \( 1_{S-F} \in \mathbb{R}^S \) of \( S - F \):

\[
M^\vee(F) = (M^\vee)_{1_{S-F}}.
\]

By symmetry, \( S - F \) is a non-degenerate flat of \( M^w \) and:

\[
M^w(S - F) = (M^w)_{1_F}.
\]

The union of the rays \( \mathbb{R}_{\geq 0} \overrightarrow{vw} \) for all directed edges \( \overrightarrow{vw} \) of \( P \) originating from \( v \) is the support of a subcomplex of the 1-skeleton of the Bergman fan on trop \((M^\vee)\) modulo \( \text{Aff}_0(\cap \Sigma) \), cf. [MS15, Chapter 4.2].

Consider an involution \( f = \mathbb{1} - \text{id} \) defined on \( \mathbb{R}^S \) by \( f(x) = \mathbb{1} - x \). The face-fitting matroid polytopes of \( \Sigma \) in \( \Delta^{|S|}_d \) are transferred via \( f \) into face-fitting matroid polytopes in \( \Delta^{|S|}_d \) and vice versa, and therefore \( f \) transfers the matroid tiling \( \Sigma \) in \( \Delta^{|S|}_d \) into a matroid tiling in \( \Delta^{|S|}_d \), say \( \Sigma^* \), where \( \text{BP}_{M^\vee} \) and \( f(\text{BP}_{M^\vee}) = \text{BP}_{(M^\vee)^*} \) are congruent for all \( v \in \text{Vert}(P) \):

\[
\Sigma^* = \left\{ \text{BP}_{(M^\vee)^*} \subset \Delta^{|S|}_d : v \in \text{Vert}(P) \right\}.
\]

In the previous paragraph, \( A_{\Lambda^\vee(\overrightarrow{vw})} = S - F \) and \( A_{\Lambda^w(\overrightarrow{vw})} = F \) are non-degenerate flats of \((M^\vee)^*\) and \((M^w)^*\), respectively.

6. Solution to Speyer’s Conjecture for Dimensions \( \leq 3 \)

Assume the setting of Section 5. Speyer’s conjecture is plainly true in dimension 1, that is, when \( k = 2 \). When \( P \) has dimension 2 with \( k = 3 \), we have a theorem that classifies all those full-dimensional matroid subdivisions \( \Sigma \) in the hypersimplex \( \Delta^3_n \) such that \( \cap \Sigma \) is a codimension 2 common face of the matroid polytopes of \( \Sigma \) that is contained in \( \text{int}(\Delta^3_n) \), see [Shi19, Theorem 3.21]. Then, in the same way as in Theorem 6.1 below, one can show that every 2-dimensional polytrope up to tropical and affine isomorphisms arises as a cell of a tropical linear space.
Note that similarly one can compute the 7 types of generic tropical planes in tropical projective space $\mathbb{T}P^3$ only with pen and paper, cf. [HJJS09, Figure 1] and [Shi19, Example 5.9].

Now, let $P$ be any 3-dimensional polytrope with $k = 4$ such that the number of its vertices is maximal, say $|2^4-1^2| = 8$. In the next theorem, we construct a matroid subdivision $\Sigma$ of the hypersimplex $\Delta^4_n$ for any positive integer $n \geq 8$ whose matroid polytopes have a nonempty common face of codimension 3 that is contained in the interior of $\Delta^3_n$, such that $P$ is a unique 3-dimensional cell of a tropical linear space dual to $\Sigma^*$ of (5.2).

Moreover, degeneration of $P$ in $\mathbb{R}^4/\mathbb{R}I$ is governed by appropriately merging matroid polytopes of $\Sigma$ into another matroid polytope, and there is a criterion for legitimate such merging, see [Shi19, Lemma 3.15]. Therefore, the theorem proves 3-dimensional Speyer’s conjecture.

**Theorem 6.1.** Speyer’s conjecture holds in dimension 3.

**Proof.** Let $P = \text{conv}(v_1, \ldots, v_4) \subset \mathbb{R}^4/\mathbb{R}I$ be a full-dimensional polytrope with the maximal number of vertices, and choose an integer $n \geq 8$. Let $\cap_{i \in [4]} A_i$ be a partition of $[n]$ with $|A_i| \geq 2$ for all $i \in [4]$. Let $Q$ be the matroid polytope of the direct sum of uniform matroids $U^1_{A_1} \oplus U^1_{A_2} \oplus U^1_{A_3} \oplus U^1_{A_4}$:

$$Q = \text{BP}_{U^1_{A_1} \oplus U^1_{A_2} \oplus U^1_{A_3} \oplus U^1_{A_4}}.$$

Observe that every vertex of $P$ is connected by an edge to a vertex of the form:

$$v_{i_1}^{C_1} v_{i_2}^{C_2} \quad \text{with} \quad i_1 \neq i_2.$$

So, we may let:

$$v = v_1^{(3,4)} v_2^{(4)}$$

without loss of generality and consider edges $\overrightarrow{vw}$. Then, the vertex $w$ is one of the following 3 vertices:

$$v_1^{(2,3,4)}, \; v_1^{(4)} v_2^{(4)} v_3^{(4)}, \; \text{and} \; v_1^{(3)} v_2^{(3,4)}.$$  

• If $w = v_1^{(4)} v_2^{(4)} v_3^{(4)}$, then $\overrightarrow{vw} \in \mathbb{R}_{\geq 0}1^{[4]}(4).$

• If $w = v_1^{(2,3,4)}$, then $\overrightarrow{vw} \in \mathbb{R}_{\geq 0}1^{[2]}(3).$

• Else if $w = v_1^{(3)} v_2^{(3,4)}$, then $\overrightarrow{vw} \in \mathbb{R}_{\geq 0}1^{[1,3]}.$

Consider the matroid subdivision $\tilde{\Sigma}$ of $\Delta^4_n$ studied in Section 3. Assign matroid polytopes $\text{BP}_{M_1}$ and $\text{BP}_{M_{(4)}}$ to vertices $v_1^{(2,3,4)}$ and $v_1^{(4)} v_2^{(4)} v_3^{(4)}$, respectively. Split $\text{BP}_{M_{12}}$ with the hyperplane $\{x(A_{1,3}) = 2\} = \{x(A_{2,4}) = 2\}$, and assign matroid polytopes $\text{BP}_{M_{12}(3)}$ and $\text{BP}_{M_{12}(4)}$ to $v_1^{(3,4)} v_2^{(4)}$ and $v_1^{(3)} v_2^{(3,4)}$, respectively, cf. formulas (3.1)–(3.3). Likewise, assign matroid polytopes to all vertices of $P$.

Then, all those assigned matroid polytopes form a matroid subdivision of $\Delta^4_n$, say $\Sigma$, satisfying that:

$$\cap \Sigma = Q \subset \text{int}(\Delta^4_n).$$

Up to both affine and tropical isomorphisms, the polytrope $P$ is dual to the matroid subdivision $\Sigma^*$ of (5.2), which completes the proof. $\square$
Remark 6.2. (1) The construction of the matroid subdivision $\Sigma^*$ of Theorem 6.1 is universal in the sense that it is a coarsest matroid subdivision to which a 3-dimensional polytrope is dual, cf. Lemma 2.1 (2).

(2) Choices of the splits into triangular prisms for the 6 polytopes $[BP_{M_{i,j}}]$, $1 \leq i < j \leq 4$, determines a whole matroid subdivision of $\Delta^4_{n}$, see Figures 3.1, 3.2 and 3.3. Every polytrope with 20 vertices is obtained from a coherent one, and there are up to symmetry 5 such, see [JK10, Figure 5].

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