Double Periodicity and Frequency-Locking in the Langford Equation

Short running title:
Double Periodicity in the Langford Equation

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Abstract

The bifurcation structure of the Langford equation is studied numerically in detail. Periodic, doubly-periodic, and chaotic solutions and the routes to chaos via coexistence of double periodicity and period-doubling bifurcations are found by the Poincaré plot of successive maxima of the first mode $x_1$. Frequency-locked periodic solutions corresponding to the Farey sequence $F_n$ are examined up to $n = 14$. Period-doubling bifurcations appears on some of the periodic solutions and the similarity of bifurcation structures between the sine-circle map and the Langford equation is shown. A method to construct the Poincaré section for triple periodicity is proposed.

Keywords: bifurcation, chaos, double periodicity, Langford equation
1 Introduction

A quasi-periodicity route to turbulence due to Ruelle and Takens [7] is that transition could occur via three successive Hopf bifurcations, leading from a fixed point to a limit cycle, then to a torus and finally to a 3-torus. On the other hand, double periodicity is considered to be well modeled by the one dimensional sine-circle map, which indicates that frequency-locking and periodic solutions appear as the nonlinear parameter increases. Therefore, it is a natural question what actually happens on attractors of a simple set of ordinary differential equations (ODEs) having double periodicity, like the Langford equation [4–6]. Similar numerical studies have been done for the five dimensional ODEs modeling magnetoconvection [1, 2] and the six dimensional ODEs of the Gledzer shell model of turbulence [8]. In the latter case the parameter to be changed is the viscosity, or equivalently, the Reynolds number. Both studies show a bifurcation structure very similar to that of the sine-circle map, although triple periodicity is stated based on the numerically obtained Lyapunov exponents in [2]. The presented results correspond to a detailed study that extends Langford [4–6].

In Section 2, a method to construct the three dimensional Poincaré section is proposed. For triple periodicity expected by Ruelle and Takens [7], points lie on a surface. Frequency-locked double periodicity indicates points on a closed curve embedded on the surface. Section 3 gives the Langford equation with an explanation of the evolution of the energy and selected parameters. In Section 4, numerical results of bifurcation structures of the equation are shown. It is confirmed that, instead of triple periodicity, we have double periodicity, frequency-locking and period-doubling bifurcations. The structure is very complicated; frequency-locking corresponding to the Farey sequence with the index up to 14 is confirmed, and plural sequences of the period-doubling bifurcations are observed. Summary and further possibilities to explore complexity of dynamical systems modeling fluid dynamics and other high dimensional systems with the presented approach are described in Conclusions.

2 Triple periodicity

As a typical example of triple periodicity, we consider the following function which is a sum of three sinusoidal functions of \( t \);
\[ x(t; a, \omega) = \sin t + \sin \pi t + a \sin \omega t. \]  

A standard method to construct the Poincaré map used in [1,8] is to seek for successive local maxima \( x_n \) of \( x(t) \). Then we let \( z_n = x_n + ix_{n+1} \) and \( \theta_n = (2\pi)^{-1} \text{Arg}(z_n) \text{ (mod 1)} \). For double periodicity, \( \theta_n \) obeys the generalized sine-circle map [8], which gives a curve on the \((\theta_n, \theta_{n+1})\) plane. Similarly, we can anticipate a surface in the \((\theta_n, \theta_{n+1}, \theta_{n+2})\) space. To check this analogy, we show the Poincaré plot of Eq. (1) in the time interval \( 0 < t < 5000 \) for double periodicity \( a = 0 \) in Figure 1, triple periodicity \( a = 0.5, \omega = \sqrt{2} \) in Figure 2, and frequency-locked double periodicity \( a = 0.5, \omega = 1.4 \) in Figure 3. Points lie on a curve in Figure 1 (a) and on a torus in Figure 2 (a), whose three dimensional structure can be viewed by a new 3D graphics function in Mathematica 6. The latter indicates a possibility that \( \theta_{n+2} \) can be expressed by a function of \( \theta_n \) and \( \theta_{n+1} \). This Poincaré plot will give a method to identify triple periodicity among complicated time series of general numerical or observed data.

We have 5:7 frequency-locking in the case of Figure 3. Correspondingly, points are again on a curve, which is more complicated than that in Figure 1. Actual data may contain higher harmonics in Eq. (1) but the qualitative behavior can be expected to be similar.

3 The Langford Equation

The Langford equation is the set of three ordinary differential equations for \( x_i(t), i = 1, 2, 3 \), given as follows:

\[
\begin{align*}
\frac{dx_1}{dt} &= \dot{x}_1 = (x_3 - b)x_1 - cx_2, \\
\frac{dx_2}{dt} &= \dot{x}_2 = cx_1 + (x_3 - b)x_2, \\
\frac{dx_3}{dt} &= \dot{x}_3 = d + ax_3 - \frac{x_3^3}{3} \\
&\quad - (x_1^2 + x_2^2)(1 + fx_3) + ex_3x_1^3.
\end{align*}
\]

The temporal evolution of the energy defined by \( E = (1/2) \sum_{i=1}^{3} x_i^2 \) is given by

\[
\frac{dE}{dt} = dx_3 + ax_3^2 - \frac{x_3^4}{3} - (b + fx_3^2)(x_1^2 + x_2^2) + ex_3^2x_1^3.
\]
The set of the parameters fixed in this paper is borrowed from [6] as

\[(a, b, c, d, f) = (1, 0.7, 3.5, 0.6, 0.25), \quad (6)\]

and \(e\) is the changing parameter. Because \(b > 0\) and \(f > 0\), the right hand side of (5) becomes negative if \(E\) is sufficiently large for \(e = 0\), leading to the proof that the solution is finite. If \(e \neq 0\), the finiteness of the solution is unknown but supported by the numerical solution for small \(e\).

The contraction rate of the volume element in the phase space is given by

\[
\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -x_3^2 + 2x_3 - 2b + a - f(x_1^2 + x_2^2) + ex_1^3. \quad (7)
\]

It depends on the position in the phase space. If \(e = 0\), it becomes negative for sufficiently large values of \(E\) since \(f\) is positive and the quadratic terms become dominant.

4 Bifurcation Structure

Before we show numerical results of the Langford equation, it is useful to review the bifurcation structure of the sine-circle map:

\[
\theta_{n+1} = f(\theta_n) = \theta_n + \Omega + \frac{K}{2\pi} \sin(2\pi \theta_n), \quad (\text{mod} \ 1). \quad (8)
\]

In order to compare the diagram of (8) with that of the Langford equation, the parameter \(\Omega\) and \(K\) are parametrized as

\[
\Omega = 0.5(1 + \alpha), \quad K = 4\alpha. \quad (9)
\]

The initial condition is \(\theta_1 = 0, 0 < \alpha < 1, \Delta \alpha = 0.001\), the total iteration is 400, and the last 200 steps are plotted in Figure 4. Although double periodicity and frequency-locking are illustrated by the sine-circle map in many literatures, the period-doubling bifurcation is also observed at \(\alpha \approx 0.6\) in Figure 4, which can make it easy to understand the similarity of the bifurcation structures between the sine-circle map and the Langford equation. The coexistence of two scenarios of routes to turbulence, quasi periodicity and period-doubling bifurcation, may be common in many of dynamical systems.
In order to study bifurcation structures of the Langford equation, sets of $x_1$ at its local maximum after transient states are plotted with various values of the parameter $e$. Eqs. 2-4 are solved by NDSolve command in Mathematica 5.2, which makes 100 to 500 different computations possible in a single program. The typical final time is $t_f = 2000 \sim 4000$ and the last period $t_f - t_d \leq t \leq t_f$ with $t_d = 200 \sim 600$ is picked up to identify attractors. In order to find the local maximum, the time period is divided into intervals with the width $\Delta t = 0.1$, and for the interval which can include the local maximum, the FindMaximum command is invoked. The typical initial condition for the numerical computation is $(x_1, x_2, x_3) = (0.01, 0, 0)$. We also choose the initial condition as the final state just before the parameter varied in order to examine the hysteresis. In some cases, multiple stable states are observed. About 40 different runs are performed with various regions of $e$ and suitable numerical parameters.

Figure 5 shows the bifurcation diagram for the parameter range $0 \leq e \leq 0.2$. The step of the parameter $e$ is $\Delta e = 0.0001$. The region $e < 0.03$ includes doubly periodic solutions. Then 3:4 frequency-locking yields the periodic solution. The Feigenbaum period-doubling bifurcation occurs at $e \simeq 0.057$. There are many periodic windows as well as chaotic solutions after the period-doubling bifurcation. Its behavior is very similar to that of the one dimensional logistic map.

The range $0.02 < e < 0.03$ is enlarged in Figure 6, in order to see the most dominant frequency-locked periodic solutions. The solution is doubly periodic if $e < 0.02028$, even if $e$ is negative. As $e$ increases, a periodic solution with 17:23 resonance appears at $e \simeq 0.02028$. Periodic solutions shown by $m_2$ points are specified by the resonance condition $m_1 : m_2$ drawn at just below the upper frame of Figure 6. The region including double periodicity ends with the emergence of 3:4 resonance at $e \simeq 0.02935$. Doubly periodic solutions draw points whose numbers are controlled by the time range for plotting and typically much larger than the periodic solutions.

The Farey sequence $F_n$ for integers $n > 0$ is the set of irreducible rational numbers $a/b$ where $0 \leq a \leq b \leq n$ and the greatest common divisor of $a$ and $b$ is 1. The Farey sequence appears in frequency-locking of the sine-circle map. The resonance conditions between 17:23 and 3:4 are shown up to the Farey index $n = 14$ in Table 1.

For the 23:31 resonant periodic solution, an incomplete period-doubling bifurcation is ob-
Table 1: Farey sequence $F_n$ up to $n = 14$.

| $F_n$ | 0 : 1 |
|-------|--------|
| $F_2$ | 1 : 2 |
| $F_3$ | 1 : 3 |
| $F_4$ | 1 : 4 |
| $F_5$ | 1 : 5 |
| $F_6$ | 1 : 6 |
| $F_7$ | 1 : 7 |
| $F_8$ | 1 : 8 |
| $F_9$ | 1 : 9 |
| $F_{10}$ | 1 : 10 |
| $F_{11}$ | 1 : 11 |
| $F_{12}$ | 1 : 12 |
| $F_{13}$ | 1 : 13 |
| $F_{14}$ | 1 : 14 |
| $F_{15}$ | 1 : 15 |
| $F_{16}$ | 1 : 16 |
| $F_{17}$ | 1 : 17 |
| $F_{18}$ | 1 : 18 |
| $F_{19}$ | 1 : 19 |
| $F_{20}$ | 1 : 20 |
| $F_{21}$ | 1 : 21 |
| $F_{22}$ | 1 : 22 |
| $F_{23}$ | 1 : 23 |
| $F_{24}$ | 1 : 24 |
| $F_{25}$ | 1 : 25 |
| $F_{26}$ | 1 : 26 |
| $F_{27}$ | 1 : 27 |
| $F_{28}$ | 1 : 28 |
| $F_{29}$ | 1 : 29 |
| $F_{30}$ | 1 : 30 |
| $F_{31}$ | 1 : 31 |
| $F_{32}$ | 1 : 32 |
| $F_{33}$ | 1 : 33 |
| $F_{34}$ | 1 : 34 |
| $F_{35}$ | 1 : 35 |
| $F_{36}$ | 1 : 36 |
| $F_{37}$ | 1 : 37 |
| $F_{38}$ | 1 : 38 |
| $F_{39}$ | 1 : 39 |
| $F_{40}$ | 1 : 40 |
| $F_{41}$ | 1 : 41 |
| $F_{42}$ | 1 : 42 |
| $F_{43}$ | 1 : 43 |
| $F_{44}$ | 1 : 44 |
| $F_{45}$ | 1 : 45 |
| $F_{46}$ | 1 : 46 |
| $F_{47}$ | 1 : 47 |
| $F_{48}$ | 1 : 48 |
| $F_{49}$ | 1 : 49 |
| $F_{50}$ | 1 : 50 |
| $F_{51}$ | 1 : 51 |
| $F_{52}$ | 1 : 52 |
| $F_{53}$ | 1 : 53 |
| $F_{54}$ | 1 : 54 |
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| $F_{62}$ | 1 : 62 |
| $F_{63}$ | 1 : 63 |
| $F_{64}$ | 1 : 64 |
| $F_{65}$ | 1 : 65 |
| $F_{66}$ | 1 : 66 |
| $F_{67}$ | 1 : 67 |
| $F_{68}$ | 1 : 68 |
| $F_{69}$ | 1 : 69 |
| $F_{70}$ | 1 : 70 |
| $F_{71}$ | 1 : 71 |
| $F_{72}$ | 1 : 72 |
| $F_{73}$ | 1 : 73 |
| $F_{74}$ | 1 : 74 |
| $F_{75}$ | 1 : 75 |
| $F_{76}$ | 1 : 76 |
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| $F_{79}$ | 1 : 79 |
| $F_{80}$ | 1 : 80 |
| $F_{81}$ | 1 : 81 |
| $F_{82}$ | 1 : 82 |
| $F_{83}$ | 1 : 83 |
| $F_{84}$ | 1 : 84 |
| $F_{85}$ | 1 : 85 |
| $F_{86}$ | 1 : 86 |
| $F_{87}$ | 1 : 87 |
| $F_{88}$ | 1 : 88 |
| $F_{89}$ | 1 : 89 |
| $F_{90}$ | 1 : 90 |
| $F_{91}$ | 1 : 91 |
| $F_{92}$ | 1 : 92 |
| $F_{93}$ | 1 : 93 |
| $F_{94}$ | 1 : 94 |
| $F_{95}$ | 1 : 95 |
| $F_{96}$ | 1 : 96 |
| $F_{97}$ | 1 : 97 |
| $F_{98}$ | 1 : 98 |
| $F_{99}$ | 1 : 99 |
| $F_{100}$ | 1 : 100 |

served in Figure 6. Figure 7 shows the diagram enlarged for $0.021 < e < 0.0235$. Figure 8 shows the diagram for $0.0243 < e < 0.0253$. The period-doubling bifurcations observed on both of the two parameter regions of 43:58 resonance.

The periodicity can be also judged by numerical computation of the square of the minimum distance

$$D = \min_{t>0} \sum_{i=1}^{3} (x_i(t) - x_i(0))^2,$$

indicating the accuracy of recurrence. In Figure 9 for $0.0272 < e < 0.0283$, the small value of $D$ indicates frequency-locked periodicity including resonances corresponding to $n = 15$ (35:47) and $n = 16$ (38:51 and 93:125) of $F_n$.

The width of resonance decreases as the index $n$ of $F_n$ increases, that implies scaling laws as noted by [1]. Summary of observed stable periodic windows is given in Table 2. Figure 10 shows the parameter $e$ versus the Farey index $n$ for the periodic windows. Many of resonances $m_1 : m_2$ corresponding to Farey sequence $F_n$ are confirmed, although some of them are not observed because of the possible lack of stability.
5 Conclusions

In conclusion, bifurcation structures similar to [1,2,8] is observed in the Langford equation; the coexistence of the double periodicity, frequency-locking and period-doubling bifurcations. The Langford equation would be one of the most illustrative ODE systems having double periodicity since it consists of only three variables. Each periodic attractor represents a corresponding knot in three dimensional space, as discussed in [3] for the Lorenz system and in [2] for the magnetoconvection system.

Recent progress of cost performance in computer hardware would make it possible to judge whether similar double periodicity and frequency-locking appear in the high dimensional Navier-Stokes turbulence and other chaotic systems. The method to judge quasi periodicity would be basically the same as that stated in the presented work.

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Figure 1: Poincaré plot of (1) for the double periodicity $a = 0$ in (a) the $(x_n, x_{n+1}, x_{n+2})$ space, on (b) the $(\theta_n, \theta_{n+1})$ plane, and in (c) the $(\theta_n, \theta_{n+1}, \theta_{n+2})$ space. Points A, B and A', B' in Figures 1(a) and 1(b), and C, C' in Figure 1(c), which are separated due to modulus $2\pi$ in the argument, are the same points. In order to see which surface the points lie on, the rectangles including the points are added in Figure 1(c).
Figure 2: Poincaré plot of (1) for the triple periodicity $a = 0.5, \omega = \sqrt{2}$ in (a) the $(x_n, x_{n+1}, x_{n+2})$ space, on (b) the $(\theta_n, \theta_{n+1})$ plane, and in (c) the $(\theta_n, \theta_{n+1}, \theta_{n+2})$ space. The points are on the surface in Figure 2(a) and 2(c). As $a$ in Eq. (1) increases, the width of the surface becomes large.
Figure 3: Poincaré plot of (1) for the frequency-locked double periodicity $\alpha = 0.5, \omega = 1.4$ in (a) the $(x_n, x_{n+1}, x_{n+2})$ space and (b) the $(\theta_n, \theta_{n+1}, \theta_{n+2})$ space.
Figure 4: Bifurcation diagram of the sine-circle map (8). The parameters $\Omega$ and $K$ are related as (9). 40000 points are randomly selected from the data for a clear view.
Figure 5: Bifurcation diagram of the Langford equation for $0 < e < 0.2$. 40000 points are randomly selected from the data for a clear view. In the following figures, bifurcation diagrams are for the Langford equation.
Figure 6: Enlarged bifurcation diagram for $0.02 < e < 0.03$. 
Figure 7: Bifurcation diagram enlarged again for $0.021 < e < 0.0235$. 
Figure 8: Enlarged bifurcation diagram for $0.0243 < e < 0.0253$. Two parameter regions for $43:58$ and $83:112$ resonances are observed. The period-doubling bifurcation follows from $43:58$ periodic solutions.
Figure 9: The parameter $e$ versus the square of the minimum distance $D$ of numerical solutions showing its accuracy of recurrence for resonant periodicity.
Figure 10: The parameter $e$ versus the Farey index $n$ for periodic windows.
Table 2: Observed periodic windows. $m_1 : m_2$ denotes the resonance condition, $n$ the Farey index, and $e_l$ ($e_u$) the lower (upper) limit of $e$, respectively. NF denotes that the resonant periodic solutions are not found.

| $m_1 : m_2$ | $n$ | $e_l$     | $e_u$     |
|-------------|-----|----------|----------|
| 17:23       | 9   | 0.02028  | 0.02112  |
| 88:119      | 14  | 0.021625 | 0.021640 |
| 71:96       | 13  | 0.021794 | 0.021817 |
| 125:169     | 14  | 0.021924 | 0.021926 |
| 54:73       | 12  | 0.022056 | 0.022113 |
| 145:196     | 14  | 0.022217 | 0.022219 |
| 91:123      | 13  | 0.022287 | 0.022299 |
| 128:173     | 14  | 0.022373 | 0.022377 |
| 37:50       | 11  | 0.022513 | 0.02269  |
| 131:177     | 14  | 0.022813 | 0.022822 |
| 94:127      | 13  | 0.022873 | 0.022898 |
| 151:204     | 14  | 0.022937 | 0.022947 |
| 57:77       | 12  | 0.023013 | 0.023087 |
| 134:181     | 14  | 0.023147 | 0.023162 |
| 77:104      | 13  | 0.02320  | 0.023251 |
| 97:131      | 14  | 0.023301 | 0.023332 |
| 20:27       | 10  | 0.02350  | 0.02446  |
| 83:112      | 14  | 0.024650 | 0.024656 |
| 83:112      | 14  | 0.02489  | 0.024936 |
| 63:85       | 13  | 0.024594 | 0.024605 |
| 106:143     | 14  | 0.024698 | 0.0247   |
| 43:58       | 12  | 0.02476  | 0.02479  |
| 43:58       | 12  | 0.025015 | 0.025085 |
| 109:147     | 14  | NF       | NF       |
| 66:89       | 13  | NF       | NF       |
| 89:120      | 14  | 0.025098 | 0.025102 |
| 23:31       | 11  | 0.02521  | 0.0255   |
| 23:31       | 11  | 0.0259   | 0.02613  |
| 72:97       | 14  | NF       | NF       |
| 49:66       | 13  | NF       | NF       |
| 75:101      | 14  | 0.0261475| 0.026148 |
| 26:35       | 12  | 0.02626  | 0.02636  |
| 26:35       | 12  | 0.02705  | 0.02711  |
| 55:74       | 14  | NF       | NF       |
| 29:39       | 13  | 0.027675 | 0.02772  |
| 32:43       | 14  | 0.0274   | 0.02743  |
| 3:4         | 4   | 0.02935  | 0.057    |