Topologically inequivalent quantizations

G. Acquaviva, A. Iorio, and L. Smaldone

1 Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holeˇ soviˇ ck´ ach 2, 18000 Praha 8, Czech Republic.

2 Institute of Particle and Nuclear Physics, Faculty of Mathematics and Physics, Charles University, V Holeˇ soviˇ ck´ ach 2, 18000 Praha 8, Czech Republic.

We discuss the representations of the algebra of quantization, the canonical commutation relations, in a scalar quantum field theory with spontaneously broken \( U(1) \) internal symmetry, when a topological defect of the vortex type is formed via the condensation of Nambu–Goldstone particles. We find that the usual thermodynamic limit is not necessary in order to have the inequivalent representations needed for the existence of physically disjoint phases of the system. This is a new type of inequivalence, due to the nontrivial topological structure of the phase space, that appears at finite volume. We regard this as a first step towards a unifying view of topological and thermodynamic phases, and offer here comments on the possible application of this scenario to quantum gravity.

I. INTRODUCTION

Motivated by earlier research in quantum gravity [1, 2], we explore from an algebraic point of view the apparent dichotomy between phases due to spontaneous symmetry breaking (SSB) and topological phases, and we investigate whether it is possible to reconcile these two concepts.

Topological phases of matter are a relatively novel, exotic and intensely studied field of research, see, e.g., [3]. They are not seen as standard thermodynamic phases, governed by Landau’s theory of SSB, with an associated local order parameter, but rather as the effect of an order emerging from the nontrivial topology of fields, space or both [3]. On the other hand, in apparent contradiction with what just said, it has been known since long time that topologically ordered structures can be seen as the effect of SSB, see, e.g., [4].

Our investigation here focuses on the underlying mathematical structures that make SSB possible in the first place, hence the analysis applies equally well to field theory, condensed matter and quantum gravity. In fact, our main motivations lie in the latter, fundamental side. Indeed, the Bekenstein bound on the entropy of any physical system contained in a finite volume, could indicate that, at the most fundamental level, the Hilbert space is finite dimensional, as advocated for instance in [5] and [6]. This is also our view [11], which leads to a picture where both matter and space emerge from the underlying dynamics of these fermionic fundamental components, that with Feynman we call Xons [7].

This is an intriguing, and perhaps unavoidable picture, but the presence of finite degrees of freedom poses very puzzling questions: How can we have SSB in such a system? If the Xons indeed make everything, it is precisely their rearrangements into different phases the way we have to explain the world as it is now [11]. Without SSB, how can this be? Black hole evaporation, in the first place, could not be seen as a phase transition, and the whole thermodynamics of black holes would be standing on unsound bases. Is there a way a system with finite degrees of freedom can naturally have phases? Are they, perhaps, of topological kind?

To start answering these questions, at least partially, we need to recall that at the origin of the issue here there is an old and important theorem, due to Stone [8] and von Neumann [9] (SvN). It establishes that, for finite degrees of freedom, all representations of the canonical commutation relations (CCRs), the algebra of quantization, are equivalent up to a unitary transformation, see, e.g., [10]. In other words, SvN theorem establishes that a system with finite degrees of freedom only has one phase. It is then quantum infinite systems, with an infinite number of degrees of freedom, that can have more than one phase stemming from the same Hamiltonian/Lagrangian, see, e.g., [4, 11]. That is when SvN can happen [12].

In our case, not only the degrees of freedom are finite for Xons, but the associated Hilbert space is finite dimensional, and this makes the problem so hard to solve that we decided to attack it by incremental steps. The first step is to consider a simpler but related problem, that is to investigate the occurrence of phases for a system with finite degrees of freedom, but infinite-dimensional Hilbert space. Indeed SvN can also be evaded when the topology of the phase space is nontrivial, even though the degrees of freedom are finite, see, e.g., [13]. A mathematically idealized case where such “topological evasion” from SvN appears is the quantum particle on a circle, studied extensively in [14], see also [15] and [16].

This appears as a promising road to pursue: to explore whether SSB can emerge in the context of finite degrees of freedom, through the topological evasion of SvN. Nonetheless, even in this simpler setting the problem is too hard,
and we need to “simplify” the matter even more because we first need to clarify the issues raised earlier in this
Introduction, i.e., how SSB-driven phases and topological phases are related. The best setting we could envisage in
order to tackle such question is that of a quantum system: while we know that in such systems one can have
inequivalent quantizations (that is, the inequivalent representations of the CCRs) in the infinite volume limit, we
shall look also for the occurrence of topologically inequivalent quantizations. In this way we shall have in one place
both kinds of inequivalence, and this will make the comparison easier.

With this plan in mind, in what follows, we consider a quantum field theory in which vortices are introduced
through SSB implemented via the so-called boson transformation [16–20]. The presence of a vortex is formally
identified by a topological defect along an axis of the configuration space. We then show that the above is an improper
transformation, meaning that it gives rise to the wanted inequivalent representations of the CCRs. This is shown by
explicitly calculating the vacuum-to-vacuum amplitude and identifying the different contributions that make it
vanish. When topology is trivial, to take the limit of infinite volume, by removing the infrared regulators, would
be a sufficient condition for this to happen. However, the presence of the vortex introduces another, independent
condition which makes the amplitude vanish at finite volume, giving rise to the inequivalent representations of the
CCRs due to a non-trivial topology of space.

Note that the problem of SSB in a finite volume was also studied in [21]. There, by means of the Ward–Takahashi
relations, it is proved that SSB of $U(1)$ symmetry for a non-relativistic, weakly interacting Bose gas, can happen in
a box of volume $V$, and that the Goldstone theorem holds. Moreover, it is shown that violations of the SvN theorem
occurs even at finite volume, because ultra-violet divergences too could produce inequivalent representations of
CCRs [22], due to the infinite number of degrees of freedom.

In section II we first describe the procedure that leads to the formation of vortices in a scalar field theory. In
section III we derive an explicit expression for the vacuum-to-vacuum amplitude in the boson-transformed system in
the case of a global boson transformation, highlighting the fact that inequivalent representations of the CCRs are
obtained only in the infinite volume limit. In section IV we consider a specific local transformation that introduces
a linear vortex in the system: in this case the vacuum-to-vacuum amplitude presents an additional regularization
which, once removed, leads to topologically inequivalent representations of the CCRs at finite volume. In section V
we discuss how such topological nontriviality is the reason for the stability of vortices at finite volume, and we
propose a broader definition of SSB which encompasses both thermodynamic and topological phases. Finally, in
section VI we provide our conclusions and offer a discussion on the results in the light of possible applications in the
context of quantum gravitational theories.

II. LINEAR VORTICES FROM BOSON TRANSFORMATIONS

In order to set-up the stage and the notation, we recall here how vortices can be formed via SSB [11]. Consider a
scalar field Lagrangian density of the form

$$\mathcal{L}(x) = \partial_\mu \varphi^\dagger(x) \partial^\mu \varphi(x) - V(\varphi^\dagger(x) \varphi(x)),$$

invariant under the $U(1)$ rigid phase transformation

$$\varphi(x) \to e^{i\alpha} \varphi(x).$$

If this symmetry is spontaneously broken, we have (a) $\langle \varphi \rangle \equiv v \neq 0$, and (b) physically disjoint realizations of the
system, still governed by the same action, that are the “phases” induced by SSB [12]. Expanding \(\varphi\) linearly around \(v\) and using a polar decomposition \(\varphi(x) = (\bar{v} + \rho(x))e^{i\chi(x)}\), with \(\langle \rho(x) \rangle = \langle \chi(x) \rangle = 0, v = \bar{v}e^{i\chi(x)}\), the Lagrangian

$$\mathcal{L}(x) = \partial_\mu \rho(x) \partial^\mu \rho(x) + (\rho(x) + \bar{v})^2 \partial_\mu \chi(x) \partial^\mu \chi(x) - V(\rho(x)),$$

It can be seen from [3] that \(\rho\) decays into \(\chi\) particles, and external lines of Feynman diagrams will appear with
derivatives attached. Therefore, taking into account all-order contributions, the interacting \(\varphi\) can be generally
expanded in terms of asymptotic fields and

$$\varphi(x) = F[\partial \chi] e^{i\chi(x)}.$$

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1 Here the potential includes the mass term for \(\varphi\).

2 Actually this expression stems from \(\varphi(x) = F[\partial \chi^{\text{phys}}] e^{i\chi^{\text{phys}}(x)}\), where \(\chi^{\text{phys}}(x)\) are asymptotic in/out fields or
quasi-particle fields (in condensed matter physics). Here \(c\) is a constant \(c\)-number, depending on renormalization constants.
For simplicity, in the following, we drop the superscript \(\text{phys}\) and we put \(c = 1\). This does not affect our considerations in
any way.
Here $F$ is some functional of the asymptotic fields. Eq.\(^3\) is known as *dynamical map* or *Haag expansion* of $\mathcal{F}$ and it is a weak mapping. $\chi$ is the Nambu-Goldstone (NG) field, here called *phason*, and it satisfies massless Klein–Gordon equation

$$\Box \chi(x) = 0. \tag{5}$$

With these, transformation (2) amounts to the field translation $\chi(x) \to \chi(x) + c \alpha$, that, in order to control infrared singularities, in field theory should be actually regarded as

$$\chi(x) \to \chi(x) + \alpha f(x), \tag{6}$$

with $f$ being a square-integrable function, satisfying

$$\Box f = 0, \tag{7}$$

and the limit $f \to 1$ should be taken at the end of calculations. This transformation is known as *boson transformation*\(^{16,20}\). The case when $f$ is not square-integrable (e.g., when its Fourier transform does not exist), and when the limit $f \to 1$ is not taken, is particularly interesting for us, because in that case, as we shall show later, such canonical transformation leads to inequivalent representations of the CCRs, even when the infrared and ultraviolet regularizations are not removed – in particular, when the volume is kept finite.

Such an $f(x)$ is obtained for instance when, for some point $x$,

$$G_{\mu \nu}^I(x) \equiv [\partial_{\mu}, \partial_{\nu}] f(x) \neq 0, \tag{8}$$

$\partial_{\mu} f$ and $G_{\mu \nu}^I(x)$ being regular functions. It has been shown that boson transformation\(^6\) within the condition (8) is a useful tool to describe the formation of topological defects in systems with SSB (for a general review of the subject see Refs.\(^3,11\)).

Some points should be remarked:

- Eq.\(^6\) induces a local gauge transformation of $\mathcal{F}$: $\varphi(x) \to \varphi(x) e^{i \alpha f(x)}$. By performing the boson transformation\(^6\) into the Lagrangian\(^3\), one can see that $A_\mu \equiv \partial_\mu f$ behaves as a *classical pure gauge* field. Hence to impose Eq.\(^7\) is the same as to fix the *Lorentz gauge* $\partial_\mu A^\mu = 0$. On the other hand, the condition \(^8\) makes the gauge field nontrivial, i.e., it amounts to have a nontrivial field strength

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \neq 0, \tag{9}$$

at the core of the vortex, as is typical of this and other topological structures, such as instantons.\(^{23}\) (see also the discussion of section VI). This gives the physical intuition of what is going on here, and in a moment we shall make these considerations more rigorous.

- Field equations are invariant under transformation \(^6\) if $f$ satisfies the same equation as $\chi$ (i.e. the condition \(^7\)). In other words

$$\mathcal{F}_f(x) \equiv F[\partial \chi + \partial f] e^{i \alpha f(x)}, \tag{10}$$

is an *exact* solution of Heisenberg equation of motions. This is the statement of *boson transformation theorem*\(^25\). We remark that this result holds independently of the presence of gauge fields in the original Lagrangian (see e.g. Ref.\(^{26}\) for a proof of this theorem in the case of an Abelian Higgs-type Lagrangian), and it is strongly based on the “gauge” condition \(^7\). Moreover, this is a non-perturbative result.\(^4\)

- It was proved that Eq.\(^8\) can be satisfied only by boson condensation of massless fields, and it then implies Eq.\(^7\). In fact \(^11\)\(^11\)\(^20\), if $(\Box + M^2) f(x) = 0$, i.e. we consider boson condensation of a massive field, then

$$\partial^\mu G_{\mu \nu}^I(x) = (\Box + M^2) \partial_\nu f(x). \tag{12}$$

As anticipated, we assume that both $\partial_\mu f$ and $G_{\mu \nu}^I(x)$ are Fourier transformable. Then

$$\partial_\mu f(x) = \frac{1}{\Box + M^2} \partial^\lambda G_{\lambda \mu}^I(x). \tag{13}$$

By deriving (contracting both members by $\partial_\mu$), we get Eq.\(^7\) and then $M = 0$: topological defects via boson transformation can be only formed by the condensation of *massless* bosons (in the present case NG bosons).

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\(^3\) A simple proof can be here sketched. Let us consider a scalar field $\Psi(x)$ satisfying the equation $\Lambda(\partial) \Psi(x) = F[\Psi(x)]$, with $\Lambda$ being a differential operator and $F$ being a generic functional of $\Psi$. By means of Yang–Feldman equation\(^{27}\), this can be expanded as

$$\Psi[\Psi_0; x] = \Psi_0(x) + \Lambda^{-1}(\partial) F[\Psi(x)], \tag{11}$$

$\Psi_0(x)$ being an asymptotic field, satisfying $\Lambda(\partial) \Psi_0(x) = 0$. Then it is easy to see that $\Lambda(\partial) \Psi^f(x) = F[\Psi^f(x)]$, with $\Psi^f(x) \equiv \Psi[\Psi_0 + f; x]$, and $\Lambda(\partial) f(x) = 0$. 

As an example, let us take \( f(x) \equiv \theta \) [11 13 17 20 25]:
\[
\chi(x) \rightarrow \chi(x) + \alpha \theta,
\]
where \( \theta \) is the azimuthal angle of a system of cylindrical coordinates, \((x_1 = r \cos \theta, x_2 = r \sin \theta, x_3 = z)\). This satisfies
\[
\nabla^2 \theta = 0, \quad \nabla \times \nabla \theta = 2\pi \delta(x_1)\delta(x_2).
\]
(then \( G_{12}^i(x) = -G_{21}^i(x) = 2\pi \delta(x_1)\delta(x_2) \)) Note that, since \( \theta \) is time independent, Laplace equation coincides with D’Alambert equation.

It can be proven [4] that the dynamical map of the Noether current associated to this \( U(1) \) symmetry reads
\[
j_{\mu}(x) = \partial_{\mu} \chi(x) + O(\partial^2 \chi),
\]
hence, under (14), the spatial part of the Noether current transforms into
\[
j(x) = \nabla \chi(x) + \alpha \nabla \theta.
\]
By taking the vacuum expectation value (VEV), we get
\[
J(x) \equiv \langle j(x) \rangle = \alpha \nabla \theta,
\]
which is a relativistic analogue of the superfluid vortex [11 29]. By taking the circuitation of \( J \) around the \( x_3 \)-axis we find
\[
Q_T = \int_{\Sigma} J \cdot dl = \int_{\Sigma} \partial \cdot \nabla \times J = 2\pi \alpha,
\]
where we used Stokes’ theorem and Eq. (15) (which gives \( \nabla \times J = 2\pi \alpha \partial \delta(x_1) \delta(x_2) \)).

\( Q_T \) is a topological charge [11], and as such its conservation does not rely on the dynamics, i.e., \( Q_T \) is conserved whether or not the fields are on-shell. Its general definition is given by the surface integral
\[
Q_T = \frac{1}{2} \int dS^{\alpha \mu} G^i_{\mu \alpha},
\]
as can be checked for the present case. Note that there is a line (string) of singularities along the \( x_3 \)-axis, therefore the \( E(2) \) symmetry of the plane is broken. Since under the phase transformation [2] also the VEV \( \varphi \) acquires a \( \theta \) dependent phase:
\[
v(\theta) = \tilde{v} e^{i \alpha \theta}
\]
then, when \( \theta \rightarrow \theta + 2\pi \), one has \( v(\theta) \rightarrow v(\theta) e^{i \alpha 2\pi} \). The condition of single-valuedness for \( v \) then constraints \( \alpha \in \mathbb{Z} \), and the topological charge is quantized accordingly (vortex quantization [11 29]).

Let us end this section with some remarks. In the usual, semiclassical treatment of Nielsen-Olesen-Landau-Ginzburg-Abrikosov (NOLGA) vortices [29 33 34] one finds that \( \varphi \rightarrow \tilde{v} e^{i \alpha \theta} \) when \( r \rightarrow \infty \), i.e., the field reaches the vacuum configuration at infinity. Therefore, the energy of a vortex diverges, e.g., in two-dimensions \( E \sim \tilde{v}^2 \int_{0}^{\infty} dr 1/r \sim \tilde{v}^2 \ln r \). The way to cure that is to make the theory local by adding a gauge field: \( L(\varphi, D_{\mu} \varphi, A_{\mu}) \). This is a Higgs-type model, which can be viewed as a relativistic analogue of Landau-Ginzburg model for superconductors (\( \varphi \) describes the condensate, \( A_{\mu} \) the electromagnetic potential), and the vortex in this case is the persistent current. The approach we are following here is fully quantum. The classical description, as we have seen above, emerges when taking the expectation values of quantum operators. Then a natural persistent (neutral) current \( J \) dynamically emerges from the boson transformation method, by taking the VEV of the Noether current \( j \) (see Eq. [15]). Moreover, its value coincides with the semiclassical result for the vacuum value of vector potential \( (A_{\mu} \rightarrow \alpha \partial_{\mu} \theta, \) when \( r \rightarrow \infty \)). However, the quantum approach does not require to perform the \( r \rightarrow \infty \) limit. This is particularly convenient because our result should work in the finite volume case. For the same reason here we do not care about energy divergences.

### III. REPRESENTATIONS OF THE CCRS IN PRESENCE OF A VORTEX: PRELIMINARIES.

The generator of the boson transformation [6] in the Schrödinger picture is
\[
Q = \int d^3x f(x) \bar{\chi}(x).
\]
The operator implementing such transformation is thus

\[ G \equiv \exp(i\alpha Q) = \exp \left( i\alpha \int d^3x f(x) \dot{\chi}(x) \right). \]  

(23)

so that

\[ G \chi(x) G^\dagger = \chi(x) + \alpha f(x), \]  

(24)

Note that here \( \dot{\chi}(x) \equiv \dot{\chi}(x)|_{t=0} \), where the reference time \( t = 0 \) was chosen arbitrarily because the charge \( Q \) is conserved.

To evaluate \( Q \) explicitly we have to expand the NG field in terms of creations and annihilation operators. Since we are interested in the formation of a linear vortex along an axis, we use cylindrical coordinates. Hence, Klein–Gordon equation reads

\[ \frac{\partial^2 \chi(x)}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \chi(x)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi(x)}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \chi(x)}{\partial z^2} + M^2 \chi(x) = 0, \]  

(25)

where we added a mass term for sake of generality.

Eq. (25) can be solved by separation of variables, as shown in Appendix A, and its solutions can be expanded as

\[ \chi(x) = \sum_{p,m,p_z} \mathcal{N} \left( a_{pmp_z} u_m(pr,p_z) + a_{pmp_z}^\dagger u_m^*(pr,p_z) \right), \]  

(26)

where we introduced the shorthand notation

\[ \sum_{p,m,p_z} \equiv \sum_{m=-\infty}^{\infty} \int_0^\infty dp \int_{-\infty}^{\infty} dp_z, \]  

(27)

and we defined

\[ u_m(pr,p_z) = \frac{\sqrt{r}}{2\pi} J_m(pr) e^{im\theta} e^{ip_z z}, \]  

(28)

with \( J_m(pr) \) the Bessel functions of the first kind. What we have in (26), then, is the scalar field expanded in cylindrical harmonics. In order to fix the normalization we require that the equal-time CCRs,

\[ [\chi(x), \dot{\chi}(x')] = i \delta^3(x - x'), \]  

(29)

are satisfied if

\[ [a_{pmp_z}, a_{kmk_z}^\dagger] = \delta(p-k) \delta_{mn} \delta(k_z - p_z). \]  

(30)

By using the Bessel functions closure relation \(^4\) 

\[ \int_0^\infty dp \int_0^\infty p dx \frac{J_n(px)}{J_n(py)} \frac{J_n(py)}{J_n(px)} = \frac{1}{x} \delta(x-y). \]  

(31)

and the delta function representations

\[ \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(x-y)} = \delta(x-y), \]  

(32)

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik(x-y)} = \delta(x-y), \]  

(33)

we can prove that the basis functions \( u_m(pr,p_z) \) satisfy the completeness relation

\[ \sum_{p,m,p_z} u_m(pr,p_z) u_m^*(pr',p_z') = \frac{1}{r} \delta(r-r') \delta(\theta-\theta') \delta(z-z') = \delta^3(x-x'), \]  

(34)

\(^4\) Actually, in [32, 33] this relation is proved for \( \Re n > 1/2 \). However, it holds true for every real \( m \), as reported in Wolfram functions site: https://functions.wolfram.com/Bessel-TypeFunctions/BesselJ/21/02/02/0006/.
and the orthonormality relation
\[ \int_0^{+\infty} dr \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dz \ u_m^*(p r, p z) u_n(k r, k z) = \delta(p - k) \delta_{m n} \delta(k_z - p_z). \] (35)

By choosing \( N = (2 E)^{-1/2} \), the equal-time CCRs (29) follow immediately.

Let us now come back to the charge (22), and use the expansion (26) there:
\[ Q = -i \int dr \int d\theta \int dz f(r, \theta, z) \ \sum_{p, m, p_z} \sqrt{E^2} \left( a_{p m p_z} u_m(p r, p z) - a_{p m p_z}^\dagger u_m^*(p r, p z) \right). \] (36)

With this the generator of the boson transformation can be written in the compact form
\[ G = \exp \left[ \sum_{p, m, p_z} \left( g_{p m p_z}^* a_{p m p_z} - g_{p m p_z} a_{p m p_z}^\dagger \right) \right], \] (37)

where
\[ g_{p m p_z} = \alpha \sqrt{E^2} \int dr \int d\theta \int dz f(r, \theta, z) u_m^*(p r, p z). \] (38)

The boson transformation (24), when written in terms of the ladder operators, reads
\[ a_{p m p_z} (g) = G a_{p m p_z} G^\dagger = a_{p m p_z} + g_{p m p_z}, \] (39)
\[ a_{p m p_z}^\dagger (g) = G a_{p m p_z}^\dagger G^\dagger = a_{p m p_z}^\dagger + g_{p m p_z}^*. \] (40)

These new ladder operators still satisfy the CCRs
\[ \left[ a_{p m p_z} (g), a_{k n k_z}^\dagger (g) \right] = \delta(p - k) \delta_{m n} \delta(p_z - k_z). \] (41)

Let us define the vacuum \( |0\rangle \) for the representation of the CCRs corresponding to \( g = 0 \) as usual
\[ a_{p m p_z} |0\rangle = 0. \] (42)

In the same way, the vacuum \( |0(g)\rangle \) for the representation corresponding to a nonzero \( g \) is defined through
\[ a_{p m p_z} (g) |0(g)\rangle = 0, \] (43)

and can be written as
\[ |0(g)\rangle = G |0\rangle. \] (44)

One can prove that \( G \) is unitarily implementable on \( \mathcal{H} \) if and only if \( |0(g)\rangle \in \mathcal{H} \). In other words, when the Hilbert spaces built on \( |0\rangle \) and \( |0(g)\rangle \) (denoted as \( \mathcal{H} \) and \( \mathcal{H}(g) \) respectively) are orthogonal, the canonical transformation is improper and the representations of CCRs for \( g = 0 \) and \( g \neq 0 \) are inequivalent.

It is then clear that the vacuum-to-vacuum amplitude
\[ \langle 0|0(g)\rangle = \langle 0|G|0\rangle, \] (45)

is the quantity that indicates whether the representations of the CCRs are equivalent, hence whether SSB indeed took place.

It can be proved either by means of Baker–Campbell–Hausdorff formula (4, 11, 35) or with functional integral techniques (36), that this amplitude explicitly reads
\[ \langle 0|0(g)\rangle = \exp \left( -\frac{1}{2} \sum_{p, m, p_z} |g_{p m p_z}|^2 \right). \] (46)

Let us first consider the case \( f(r, \theta, z) = 1 \). Then it is easy to compute that
\[ g_{p m p_z} = 2\pi \alpha \sqrt{E^2} \delta_{m 0} \delta(p) \delta(p_z). \] (47)
We now use the Dirac-delta representations Eqs. (31)-(33), obtaining:

\[ \sum_{p,m,p_z} g_{p,m,p_z}^2 = \sum_{k,n,k_z} \sum_{p,m,p_z} \delta(k - p) \delta_{m,n} \delta(p_z - k_z) g_{p,m,p_z} g_{k,n,k_z}. \]  

(48)

We now use the Dirac-delta representations Eqs. (31)-(33), obtaining:

\[ \sum_{p,m,p_z} g_{p,m,p_z}^2 = \frac{1}{(2\pi)^2} \int dz \int d\theta \int dr \sum_{k,n,k_z} \sum_{p,m,p_z} \sqrt{k} \sqrt{\rho} J_0(kr) J_0(pr) e^{i(m-n)\theta} e^{i(k_z-p_z)z} g_{p,m,p_z} g_{k,n,k_z}. \]  

(50)

By means of the explicit expression

\[ \sum_{p,m,p_z} g_{p,m,p_z}^2 = \frac{\alpha^2 MV}{2}, \]  

(51)

which goes to zero when \( V \to \infty \). In that limit \( \langle 0|0(g) \rangle \) does not belong to the Fock space built on \( |0\rangle \) and we have unitarily inequivalent representations of CCR \[37\]. In other words, we reproduce here the standard result that for SSB to occur we need an infinitely extended system \[12\], i.e., if the system has a finite volume there is no SSB, no NG modes are produced to condense as a result, and the system can only describe one phase.

Note that we introduced a mass \( M \), which has to be sent to zero after the infinite volume limit is performed. An analogous result could be achieved by using the prescription \[6\] with \( f(x) \) being a square integrable function \[38\]. This is simply a way to have a well-defined mathematical procedure, that seems to convey the idea that, in usual SSB, \( V \) goes to zero when \( V \to \infty \), and that inequivalent representations of CCR can occur also in that case \[21\]. In this explicit example, we see a crucial aspect of why SSB-driven phases and topological phases appear to be different phenomena. The former are only possible at infinite volume, and have as distinctive feature the appearance of massless NG bosons. The latter may exist in compact, finite volumes, provided the boundary conditions are favorable, and do not require NG modes. Clearly, to search for a common ground, we need to understand the role of the infinite volume in SSB.

On this many authors have extensively commented, from different perspectives, see, e.g., \[4, 11, 12, 22, 39\]. In the words of \[12\]: to have true SSB one needs physically disjoint realizations of the system. A physical operation on the system is necessarily a local operation and hence, to have two physically disjoint realizations (phases) means that no local operation must be able to change one realization into another – otherwise they are one and the same phase. Therefore, since field configurations realized at infinity cannot be changed by a local operation, a sufficient condition for SSB is the infinite volume \[22\], with associated condensation of the massless NG modes.

In what follows, we shall investigate whether SSB can occur in a finite volume when a topological defect is present.

IV. TOPOLOGICAL INEQUIVALENT REPRESENTATIONS IN QFT

In this section we set \( M = 0 \), from the very beginning, and show that different phases occurs anyway, i.e. there is no need of the small effective mass.

In the case \( f(r, \theta, z) = \theta \) one has

\[ G \equiv \exp(i\alpha Q) = \exp\left(i\alpha \int dr \, d\theta \, dz \, \theta \bar{\chi}(r, \theta, z) \right), \]  

(53)

and

\[ g_{p,m,p_z} \equiv \frac{\alpha}{\sqrt{2(2\pi)^2}} \int dr \, d\theta \, dz \, \theta \, u_m^*(pr, p_z z). \]  

(54)

5 Acturally we use Eq. (31) in the form \[32\]: 

\[ \sqrt{k} \int_0^\infty dr \, r \, J_0(kr) \, J_0(pr) = \delta(k-p). \]  

(49)

6 Note that here we should also deform \( f \) so that \((\Box + M^2)f = 0 \). However, the error we commit is very small and we get the correct result \[52\] (see, e.g., Ref. \[37\]). This discrepancy is not present for \( f(r, \theta, z) = \theta \), because mass of NG field will be set to zero from the very beginning.
Here the integrals over $\theta$ and $z$ can be easily performed:

$$\theta_m = \frac{1}{2\pi} \int_0^{2\pi} \sin m \theta e^{-im \theta} = \delta_{0m} \pi + (1 - \delta_{0m}) \frac{i}{m}, \quad (55)$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sin p_z z = \delta(p_z). \quad (56)$$

The radial integral

$$f_m(p) = \int_0^\infty dr \, r J_m(pr), \quad (57)$$

is the Hankel (or Fourier–Bessel) transform of 1, explicitly solved in Appendix B

$$f_m(p) = \left( \frac{m}{p^2} + \frac{\delta(p)}{p} \right) H(m) + (1 - H(m)) \left( -1 \right)^m \left( \frac{\delta(p)}{p} - \frac{m}{p^2} \right), \quad (58)$$

where $H(m)$ is the Heaviside step function

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}. \quad (59)$$

We thus find

$$g_{pm_{p_z}} = \alpha \sqrt{2} \pi p \theta_m f_m(p) \delta(p_z). \quad (60)$$

where we used that $M = 0$ and that $E = \delta(p_z) = E|_{p_z=0}$.

Now it is convenient to notice that

$$\sum_{p,m,p_z} |g_{pm_{p_z}}|^2 = \sum_{m,n=-\infty}^{-1} \sum_{p,p_z,k,k_z} g_{pm_{p_z}} g_{knk_z} \delta_{mn} \delta(p-k) \delta(p_z-k_z)$$

$$+ \sum_{m,n=1}^{+\infty} \sum_{p,p_z,k,k_z} g_{pm_{p_z}} g_{knk_z} \delta_{mn} \delta(p-k) \delta(p_z-k_z) + \sum_{p,p_z,k,k_z} g_{\rho_{p_z}} g_{\rho_{k_z}} \delta(p-k) \delta(p_z-k_z)$$

$$= (i) + (ii) + (iii). \quad (61)$$

First, the computation of the term (iii) ($n = 0, m = 0$) is analogue to that for $f(r, \theta, z) = 1$, hence it goes to zero when $M$ is set to zero from the beginning. Then we focus on (ii), as the computation for (i) is very similar.

From Eq. (46), we have

$$(i) = \sum_{k,n,k_z,p,m,p_z} \frac{\alpha^2}{2} \int dz \int d\theta \int dr \, r J_0(kr) J_0(pr) e^{i(m-n)\theta} e^{i(k_z - p_z)z} \frac{\delta(p_z) \delta(k_z)}{\sqrt{p_k}} \quad (62)$$

$$= \sum_{k,n,p,m} \frac{\alpha^2 h}{2} \int d\theta \int dr \, r J_0(kr) J_0(pr) e^{i(m-n)\theta} \frac{1}{\sqrt{p_k}}, \quad (63)$$

where we introduced the symbol $\sum_{k,n,p,m}$ for sums running over positive integers, and the second expression is the result of integrating over $k_z$ and $p_z$, and on a finite range over $z$, such that $\int_{-h/2}^{h/2} dz = h$. To proceed we first use the relation

$$\sum_{n=1}^\infty e^{\pm i n \theta} = \frac{1}{e^{\mp i \theta} - 1}, \quad (64)$$

7 To check that such an expression gives back Eq. (14), let us insert Eqs. (39) and (40) in the field expansion (26), to obtain

$$\chi(x) \rightarrow \chi(x) + \alpha + \alpha \sum_{m\neq0} \int_{-h/2}^{h/2} dp J_m(pr) f_m(p) \frac{\sin (m \theta)}{m} \sin (m \theta).$$

The second term on r.h.s. comes from the $\delta_{m0}$ contribution in $\theta_m$. We can now perform the integral over momenta, by means of the integral (59), and summing over $m$, using

$$\sum_{m=1}^\infty \frac{\sin (m \theta)}{m} = \frac{\theta + 2k \pi}{2} - \frac{\pi}{2}, \quad k \in \mathbb{Z}.$$
which thus leads to

\[(ii) = \frac{\alpha^2 h}{2} \int d\theta \int dp \frac{1}{2p^2 (1 - \cos \theta)}. \tag{65}\]

The integral over \( p \) can be easily regularized by introducing an infrared cut-off

\[\int_{\lambda}^{+\infty} dp \frac{1}{p^2} = \frac{1}{\lambda}. \tag{66}\]

The regularized integral over the azimuthal angle is the most interesting:

\[\frac{1}{2} \int_{2\pi-\varepsilon}^{2\pi-\varepsilon} d\theta \frac{1}{1 - \cos \theta} = \cot \left(\frac{\varepsilon}{2}\right). \tag{67}\]

Here we introduced a small parameter \( \varepsilon \), which geometrically corresponds to cut a thin slice around \( 0 = \theta = 2\pi \) direction (see Fig.1). Removing the cut one has \( \cot(\varepsilon/2) \to +\infty \).

In the same way, by means of

\[\sum_{n=1}^{\infty} e^{\pm in\theta} (-1)^n = -\frac{1}{e^{\pm i\theta} + 1}, \tag{68}\]

one can find the contributions (i) \((n < 0, m < 0)\)

\[(i) = \frac{\alpha^2 h}{2} \int d\theta \int dp \frac{1}{2p^2 (1 + \cos \theta)}. \tag{69}\]

and by a change of variables, \( \theta \to \theta + \pi \), this gives the same result \(\tag{67}\)

\[\frac{1}{2} \left( \int_{\pi}^{2\pi-\varepsilon} d\theta \frac{1}{1 - \cos \theta} + \int_{2\pi+\varepsilon}^{3\pi} d\theta \frac{1}{1 - \cos \theta} \right) = \cot \left(\frac{\varepsilon}{2}\right), \tag{70}\]

regularized, once more, by means of a cut along \( 0 = \theta = 2\pi \).

The final result is then

\[\langle 0|0(g)(\varepsilon, h, \lambda) = \exp \left\{-\frac{\alpha^2 h}{2} \frac{1}{\lambda} \cot \left(\frac{\varepsilon}{2}\right)\right\}. \tag{71}\]

where \( h \) and \( R = 1/\lambda \) are the height and radius of the cylinder, respectively (see Fig.1). This is what we were looking for, and our main new result: to have in one expression both kinds of inequivalence, the standard one, driven by the thermodynamic (infinite volume) limit

\[\lim_{h\to\infty} \langle 0|0(g)(\varepsilon, h, \lambda) = 0 = \lim_{\lambda\to0^+} \langle 0|0(g)(\varepsilon, h, \lambda), \tag{72}\]

and the inequivalence induced by the nontrivial topology

\[\lim_{\varepsilon\to0^+} \langle 0|0(g)(\varepsilon, h, \lambda) = 0. \tag{73}\]

The latter inequivalence, although of topological nature and occurring also at finite volume, it is generated here through the same mechanism as the standard inequivalences emerging in the thermodynamic (infinite volume) limit. The difference with the case \( f = 1 \), discussed in the previous section, is in the second term of the r.h.s. of Eq.\((\ref{eq:55})\): in the case \( f = 1 \), the sums over \( m, n \) are killed by \( \delta_{m0}, \delta_{n0} \), while in the present case we have to re-sum the infinite series, to take into account that we can wind around the singularity at \( r = 0 \) an infinite number of times.

Let us finally notice that the introduction of an upper bound on momenta, i.e. a cut-off \( \Lambda \), so that \( \lambda \leq p \leq \Lambda \) does not change our previous considerations. In other words, in the present example, ultra-violet divergences do not play any role (for the inequivalence). This would suggest that topological inequivalence here appears even when the system has a finite number of degrees of freedom.

\[8 \text{ We change the summation index as } n \to -n, m \to -m \text{ when they run through negative values.}\]
V. VORTICES AT FINITE VOLUME: TOPOLOGY AND SSB

A topological defect of the vortex type is stable only when the field \( \varphi(x) \) on the boundary (where it takes its vacuum configurations) is a map with nontrivial fundamental group. In our case

\[ \varphi : \text{boundary} \rightarrow U(1) \sim S^1. \tag{74} \]

With reference to Fig. 1, let us then focus on the configuration space, where the vortex is a line of singularities, surrounded by a (punctured) solid cylinder. When we perform the cut along \( \theta = 0 \), in order to regularize the integrals \( (65) \) and \( (69) \), the resulting space is topologically equivalent to a solid sphere or ball \( B_3 \), as indicated in Fig. 1(a). Therefore, the map \( (74) \) is clearly trivial, \( \pi_1(B_3) = e \), the vortex cannot stabilize and no SSB occurs.

On the other hand, see Fig. 1(b), when the \( \varepsilon \)-regularization is removed, the vortex is fully surrounded by the punctured solid cylinder, and the latter is homeomorphic to the solid torus, i.e. to the composition \( S^1 \times D^2 \) of a circle and a disk. The disk is fully contractible, hence the first homotopy group of the solid torus is isomorphic to the one of the circle

\[ \pi_1(S^1 \times D^2) \cong \pi_1(S^1) = \mathbb{Z}. \tag{75} \]

Therefore, keeping \( h \) and \( R = 1/\lambda \) finite but performing the limit \( \varepsilon \to 0 \), we have \( \langle 0|0(g) \rangle_{\varepsilon,h,\lambda} \to 0 \), hence the representations become inequivalent, and the vortex at finite volume stabilizes.

In the case of \( N \) vortices (\( N \) line singularities in space) the bounded 3-manifold \( \mathcal{M} \) is a connected sum of \( N \) solid tori

\[ \mathcal{M} = (S^1 \times D^2) \# \ldots \# (S^1 \times D^2), \tag{76} \]

or, equivalently, a solid \( N \)-torus, a ball \( B^3 \) with \( N \) solid handles, or a so-called three-dimensional handlebody of genus \( g = N \). The fundamental group of \( \mathcal{M} \) is hence the free group of \( N \) generators\[^{40}\].

The question we face here is whether what we have found is true SSB. In the vast literature on the topic SSB is often identified by an invariant dynamics (Lagrangian, Hamiltonian, equations of motion) together with a noninvariant vacuum configuration. To our knowledge, though, the true SSB is identified by the concurrence of the following five instances:
\begin{itemize}
    \item invariance of the dynamics,
    \item non-invariance of the vacuum,
    \item disjoint realizations of the dynamics (non-equivalent representations),
    \item NG (massless) bosons,
    \item thermodynamic (infinite volume) limit,
\end{itemize}

as we have seen in the previous sections.

A way to summarize the main result of this paper is to say that in the case of the formation of vortices, \( f(r, \theta, z) = \theta \), which is a prototypical topological defect, we can in fact have SSB \textit{even at finite volume}, because: 1. the disjoint realizations are obtained already in the limit \( \varepsilon \to 0 \), keeping \( h, R \) finite, and 2. the massless NG bosons appear. That is, \textit{we have topological phases}, rather than \textit{thermodynamic phases}, but nonetheless all the other concurring instances that identify SSB are valid, as we have shown. This leads us to propose a more relaxed definition of SSB, where the last condition is replaced by a different one, that we now proceed to formulate. In this way topological and Landau’s phases both will stem from SSB.

In the above recalled picture of \([12]\) (see also, e.g., \([3, 11, 22, 39]\)), to have true SSB one needs be in the conditions that no local operation must be able to change one realization of the dynamics into another, otherwise there is always one and the same phase, even though they may appear different (i.e., associated to different vacua). Of course, since field configurations realized at infinity cannot be changed by a local operation, the infinite volume, along with the other conditions, is a sufficient condition. In our case here, we understand the inequivalence as the effect of being unable to change one topological phase into another through a local physical operation, because the operation leading from the solid torus to the solid cylinder of Fig. 1 by definition, is not a local operation. Therefore, it does not matter whether the boundary is at infinity or not, what matters is that to change the configuration of the fields at the boundary one needs a singular operation, not a local one.

Thus, the last point of the list of five concurring instances should be changed to

\begin{itemize}
    \item vacuum boundary (\textit{global}) conditions that cannot be changed by a \textit{local} operation.
\end{itemize}

Clearly, this definition includes both the thermodynamic/infinite-volume limit, regardless of the topology (corresponding, in our particular case, to \( h \to \infty \) or \( R \to \infty \), regardless of the fate of \( \varepsilon \)), and the finite volume with nontrivial topology (corresponding, in our particular case, to \( \varepsilon \to 0 \), and \( h \) and \( R \) finite). This unifies the two types of phases, thermodynamic and topological, at least in this case. Notice that, of course, it may happen that both inequivalences might be at work together, making the SSB stronger.

\section*{VI. CONCLUSIONS AND OUTLOOK}

Let us conclude by commenting on the possible applications of the above described findings to the quantum gravity scenarios evoked in \([1, 2]\), which was our main motivation in the first place. As recalled earlier, the physics of black-holes makes it natural to assume that the fundamental degrees of freedom of nature (that is those making matter, as well as space itself, Xons), are finite in number \([1]\). Indeed, this, together with a Pauli principle, explains the Bekenstein bound with fundamental arguments that do not assume pre-existing spatiotemporal concepts \([2]\). Although fascinating, and perhaps unavoidable, this picture has many open problems, the most important being the difficulty of explaining the rich world around us made of many phases, all stemming from a single underlying dynamics, through SSB. In fact, the disjoint realizations of the Xons dynamics could only come from topological inequivalence, that is the other way to avoid the chains of the SvN, and topological phases are on different footings than SSB phases \([3]\).

Therefore, what we have done here is a first step (of a many-step road), towards the necessary reconciliation of these two types of phases. There are still many aspects to be understood, though. The first, is that the volume \( V \), that is central to the discussion above, is the portion of \( \mathbb{R}^3 \) where the field \( \varphi \) is immersed, while, at the fundamental level, the Xons make space as well as matter, hence the above picture needs a drastic rethinking. The direction to go comes from condensed matter, and is intimately related to topological defects of the dislocation and disclination kind. We are referring to the emerging (Cartan) gravity description as a theory of defects in elastic media, put go comes from condensed matter, and is intimately related to topological defects of the dislocation and disclination kind. We are referring to the emerging (Cartan) gravity description as a theory of defects in elastic media, put forward by Kleinert \([41]\) (see also \([42]\), and the exhaustive \([43, 44]\)), and to the recent analog model of gravity on Dirac materials \([45, 48]\) (see also the reviews \([49, 50]\)).

In that approach, singular translations, \( u^a(x) \), and singular rotations, \( \omega^{ab}(x) \), in an elastic medium, play the role of our \( f(x) \) in \([3]\), giving rise to the field strengths \( T_{\mu\nu} \sim \partial_{\mu} u^{a} \) and \( R^{\mu\nu}_{\mu\nu} \sim \partial_{\mu} \omega^{ab} \), respectively (cf. Eq. \([9]\)). Here \( T_{\mu\nu} \) and \( R^{\mu\nu}_{\mu\nu} \) are the torsion and Riemann curvature tensors, respectively, written in the mixed notation: tangent space indices \( a, b, \ldots \) (that refer to the symmetry group to be gauged) and manifold indices \( \mu, \nu, \ldots \). With this in mind, to apply to that arena the results of this paper, which focuses on \textit{internal} \( U(1) \) symmetry, we
need to move to the gauge-gravity theory program, based on spatiotemporal groups [5, 6]. $A_\mu = e^{a}_\mu \mathbb{P}_a + \omega^{ab}_\mu \mathbb{J}_{ab}$: that is $\text{ISO}(n-1,1)$ for Poincaré, $\text{SO}(n,1)$ for de Sitter, $\text{SO}(n-1,2)$ for Anti de Sitter, or others, including the conformal group $\text{SO}(n,2)$. $A_\mu = e^{a}_\mu \mathbb{P}_a + \omega^{ab}_\mu \mathbb{J}_{ab} + e^{a}_\mu \mathbb{K}_a + \lambda_\mu \mathbb{D}$, see, e.g., [55, 56]. Before doing so, we should consider the fate of the SvN for the discrete structures of the quantum gravity based on Xons, and an interesting link between SvN and deformed CCRs with a natural discrete structure is in [57].

ACKNOWLEDGEMENTS

We would like to thank G.F. Aldi and P. Pais for useful discussions. A.I. and L.S. acknowledge support from Charles University Research Center (UNCE/SCI/013).

Appendix A: Klein–Gordon equation in cylindrical coordinates

In this appendix we briefly show how to solve the Klein–Gordon equation (25) by means of separation of variables. Firstly, we look at solutions of the form

$$\chi(x) = \phi(x) e^{\pm iEt}.$$  

(A1)

This leads to

$$-\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi(x)}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \phi(x)}{\partial \theta^2} - \frac{\partial^2 \phi(x)}{\partial z^2} = \left( E^2 - M^2 \right) \phi(x),$$  

(A2)

We now factorize $\phi(x)$ as

$$\phi(x) = R(r) \Theta(\theta) Z(z).$$  

(A3)

Therefore, we get:

$$-\frac{1}{R(r)} \frac{d}{dr} \left( r \frac{dR(r)}{dr} \right) - \frac{1}{\Theta(\theta) r^2} \frac{d^2 \Theta(\theta)}{d\theta^2} - \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = E^2 - M^2.$$  

(A4)

We now equate the last term in the l.h.s. to a negative separation constant (we are looking at wave solutions):

$$\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = -p_z^2,$$  

(A5)

which is solved by $Z(z) = e^{\pm ip_z z}$. Eq. (A4) now becomes:

$$-\frac{r}{R(r)} \frac{d}{dr} \left( r \frac{dR(r)}{dr} \right) - \frac{1}{\Theta(\theta)} \frac{d^2 \Theta(\theta)}{d\theta^2} = p^2 r^2,$$  

(A6)

where we defined $p^2 = E^2 - p_z^2 - M^2$. We further equate the second term on the l.h.s. to a negative integer (in order to enforce periodicity in $\theta$):

$$\frac{1}{\Theta(\theta)} \frac{d^2 \Theta(\theta)}{d\theta^2} = -m^2,$$  

(A7)

which has solution $\Theta(\theta) = e^{\pm im\theta}$. We are finally left with the equation:

$$\left( \frac{1}{r} \frac{d}{dr} + \frac{d^2}{dr^2} \right) R(r) = \left( -p^2 + \frac{m^2}{r^2} \right) R(r),$$  

(A8)

which is the Bessel differential equation. The solutions are the Bessel functions (of integer order) $J_m(pr), \ N_m(r)$ of the first and second type, respectively. We choose $J_m(pr)$, also called cylindrical harmonics, which are regular both at $r = 0$ and $r = \infty$. In fact, we require singularities at $r = 0$ to be imputed only to a singular boson transformation, as in Eq. (14).

9 As well known, this programme works particularly well in $n = 3$, where Chern-Simons forms of various kinds govern the gravity theories for the different settings [51, 53], see also [54, 55].
Appendix B: Hankel transform of $1$

The Hankel transform of a function $a(r)$ is defined as \[ A_n(p) = \int_0^\infty dr \, r J_n(pr) a(r), \] (B1)

and it satisfies the relation \[ a(r) = \int_0^\infty dp \tilde{A}_n(p) J_n(pr), \quad \tilde{A}_n(p) \equiv p A_n(p). \] (B2)

We now want to evaluate the integral (B1) in the case in which $a(r) = 1$. The idea is to rewrite the required expression as:

\[ \int_0^\infty dr \, J_n(pr) r = \frac{d}{d\alpha} \left( \int_0^\infty dr \, J_n(pr) e^{\alpha r} \right)_{\alpha=0}. \] (B3)

The r.h.s can be now integrated:

\[ \int_0^\infty dr \, J_n(pr) e^{\alpha r} = \frac{p^n \left( \sqrt{\alpha^2 + p^2} - \alpha \right)^{-n}}{\sqrt{\alpha^2 + p^2}}, \quad n \geq 0, \, p > 0, \, \alpha \leq 0. \] (B4)

Deriving the r.h.s. with respect to $\alpha$ and posing $\alpha = 0$, one gets:

\[ \int_0^\infty dr \, J_n(pr) r = \frac{n}{p^2}, \quad n \geq 0, \, p > 0. \] (B5)

Moreover, we know that for $n = 0$ \[ \int_0^\infty dr \, J_0(pr) = \frac{\delta(p)}{p}. \] (B6)

Therefore, it is reasonable to assume that

\[ \int_0^\infty dr \, J_n(pr) r = \frac{n}{p^2} + \frac{\delta(p)}{p}, \quad n \geq 0. \] (B7)

This result can be verified by substituting it in Eq.(B2) and using the fact that \[ \int_0^\infty dp \, \frac{e^{-ap}}{p} J_n(pr) = \frac{(\sqrt{a^2 + p^2} - a)^n}{n r^n}, \] (B8)

which, for $a = 0$, reduces to

\[ \int_0^\infty dp \, \frac{J_n(pr)}{p} = \frac{1}{n}. \] (B9)

Finally, we notice that, as a consequence of the well-known identity $J_n(x) = (-1)^n J_{-n}(x)$, we can solve the integral also for negative values of $n$. In fact

\[ \int_0^\infty dr \, J_n(pr) r = (-1)^n \int_0^\infty dr \, J_{-n}(pr) r. \] (B10)

If $n < 0$, the r.h.s. is solved by Eq.(B7). This concludes our proof of Eq.(58).

As final remark we note that when $n \rightarrow 0$ the result (B7) reduces to the known (B6). This is clear for $p \neq 0$; then one must keep in mind that the limit $n \rightarrow 0$ has to be performed before the limit $p \rightarrow 0$, giving in this way the correct result.
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