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Pseudo-Riemannian Lie groups admitting left-invariant conformal vector fields

Groupes de Lie pseudo-riemanniens admettant des champs vectoriels conformes invariants à gauche

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Abstract. Let $G$ be a Lorentzian Lie group or a pseudo-Riemannian Lie group of type $(n-2,2)$. If $G$ admits a non-Killing left-invariant conformal vector field, then $G$ is solvable.

Résumé. Soit $G$ un groupe de Lie lorentzien ou un groupe de Lie pseudo-riemannien de type $(n-2,2)$. Si $G$ admet un champ vectoriel invariant à gauche non-Killing, alors $G$ est résoluble.

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1. Introduction

Let $(M,g)$ be a pseudo-Riemannian manifold. The conformal transformation group of $(M,g)$, denoted by $\text{Conf}(M,g)$, is called essential if no metric in the conformal class of $g$ is preserved by $\text{Conf}(M,g)$. If $(M,g)$ is Riemannian of dimension $\geq 2$, then $\text{Conf}(M,g)$ is essential if and only if $(M,g)$ is conformally diffeomorphic to the standard sphere $\mathbb{S}^n$ or $\mathbb{R}^n$ with the canonical flat metric. This is the famous Lichnérowicz’s Conjecture, which was finally proved by J. Ferrand in [7]. However, in the pseudo-Riemannian case, the situation is quite different as shown in [1,8,11,14].
Thus it is meaningful to study the structure of a pseudo-Riemannian manifold \((M, g)\) with \(\text{Conf}(M, g)\) essential, for example a homogeneous pseudo-Riemannian manifold with a non-Killing conformal vector field. Here a vector field \(X\) on \((M, g)\) is said to be conformal, if

\[ L_X g = 2\rho g, \]  

(1)

where \(L_X\) is the Lie derivation and \(\rho\) is a smooth function on \(M\) called the conformal factor with respect to \(g\). If \(g\) is a Riemannian metric, the existence of the function \(\rho\) might give some information about the topological structure of the Riemannian manifold (see [5, 12]). As a subclass of conformal vector fields, a Yamabe soliton vector field, i.e. a vector field \(X\) satisfies

\[ L_X g = 2(\text{scal} - \lambda) g \]

where scal is the scalar curvature of the metric \(g\) and \(\lambda\) is a constant, plays an important role in exploring Yamabe flow (see [2–4, 6, 9]). If \(\rho = 0\), we call \(X\) a Killing vector field which provides a close link between the geometry of a manifold \(M\) and the algebra of \(I(M)\), where \(I(M)\) denotes the set of all isometries in \((M, g)\) (see [13]).

Here we restrict \((M, g)\) to be a pseudo-Riemannian Lie group which is a Lie group with a left-invariant pseudo-Riemannian metric. All Lie groups are assumed to be connected. Furthermore the Lie group is called type \((p, q)\) if the signature of the pseudo-Riemannian metric is of \((p, q)\).

As we know, there are some studies on Lorentzian Lie groups with non-Killing left-invariant conformal vector fields, i.e. pseudo-Riemannian Lie groups of type \((n - 1, 1)\). For example the results in [2, 4, 15, 17].

The paper is organized as follows. First, we recall some facts on non-Killing left-invariant conformal vector fields on pseudo-Riemannian Lie groups in Section 2, and then prove the following Theorem in Section 3.

**Theorem 1.** Let \(G\) be a Lorentzian Lie group or a pseudo-Riemannian Lie group of type \((n - 2, 2)\), where \(n \geq 4\). If \(G\) admits a non-Killing left-invariant conformal vector field, then \(G\) is solvable.

But it is unknown for general type \((p, q)\) for \(p, q \geq 3\), and we conjecture that Theorem 1 holds for any type \((p, q)\) for \(p, q \geq 3\).

In Section 4, we construct a class of pseudo-Riemannian solvable Lie groups of type \((p, q)\) which admit non-Killing left-invariant conformal vector fields, and then prove they are conformally flat. That is, they satisfy Lichnérowicz conjecture in the pseudo-Riemannian case.

### 2. Preliminaries

Let \(G\) be a Lie group with the Lie algebra \(g\) consisting of left-invariant vector fields and let \(\langle \cdot, \cdot \rangle\) be a pseudo-Riemannian metric on \(G\). Assume that \(\nabla\) is the Levi-Civita connection associated with \(\langle \cdot, \cdot \rangle\). Then

\[ [X, Y] = \nabla_X Y - \nabla_Y X. \]  

(2)

If \(\langle \cdot, \cdot \rangle\) is left-invariant on \(G\), we have

\[ \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle = 0, \]  

(3)

for any \(X, Y, Z \in g\). By (2) and (3),

\[ \langle \nabla_X Y, Z \rangle = \frac{1}{2} (\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle), \]  

(4)

where \(X, Y, Z\) are all left-invariant vector fields. Assume that \(X \in g\) is a conformal vector field, by (1), we have

\[ 0 = L_X \langle X, X \rangle = 2\rho |X|^2. \]

Furthermore if \(\langle \cdot, \cdot \rangle\) is Riemannian, then \(\rho = 0\) or \(X = 0\). That is, \(X\) is Killing or trivial.

For left-invariant pseudo-Riemannian metrics \(\langle \cdot, \cdot \rangle\), we have
Lemma 2 ([2]). Let $G$ be a unimodular pseudo-Riemannian Lie group. Then any left-invariant conformal vector field on $G$ is a Killing vector field.

If $G$ is a non-unimodular pseudo-Riemannian Lie group, we have the following result.

Lemma 3 ([2]). Let $G$ be an $n$-dimensional non-unimodular pseudo-Riemannian Lie group of type $(p,q)$. If $\mathfrak{g}$ admits a non-Killing conformal vector field, then $\dim C(\mathfrak{g}) \leq \min(p,q)$ and $\dim[\mathfrak{g},\mathfrak{g}] \geq \dim \mathfrak{g} - \min(p,q)$.

Furthermore, for a Lorentzian Lie group, we have the following Lemma.

Lemma 4 ([15]). Let $G$ be a Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then $\dim[\mathfrak{g},\mathfrak{g}] = \dim \mathfrak{g} - 1$.

3. The proof of Theorem 1

In order to prove Theorem 1, we first recall two facts.

Lemma 5 ([10]). Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. If there is an invertible derivation on $\mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Lemma 6. For any matrix $H \in \text{so}(n,1)$, either $H$ has $n-1$ purely imaginary and two non-zero real eigenvalues $\pm r \in \mathbb{R}$, or $H$ has $n+1$ purely imaginary eigenvalues. Here, we consider $0$ as a purely imaginary number and $\text{so}(n,1)$ is defined by

$$\text{so}(n,1) = \left\{ \begin{pmatrix} A & C \\ C' & 0 \end{pmatrix} : A = -A' \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{n \times 1} \right\},$$

where $A'$ denotes the transpose of $A$.

Let $G$ be a pseudo-Riemannian Lie group whose Lie algebra is $\mathfrak{g}$ and let $X$ be a non-Killing left-invariant conformal vector field on $G$. Denote by $\langle \cdot, \cdot \rangle$ the pseudo-Riemannian metric of signature $(p,q)$ on $G$. Clearly $\langle X, X \rangle = 0$. By the definition of a conformal vector field (1), we have

$$\langle [X, U], V \rangle + \langle U, [X, V] \rangle = -2\rho \langle U, V \rangle,$$

where $U, V \in \mathfrak{g}$ and $0 \neq \rho$ is a constant.

Lemma 7 ([2]). Let $G$ be a Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then the restriction of $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate.

In fact, this lemma holds for any pseudo-Riemannian Lie group.

Lemma 8. Let $G$ be a pseudo-Riemannian Lie group admitting a non-Killing left-invariant conformal vector field $X$. Then the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate.

Proof. Assume that the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g},\mathfrak{g}]$ is non-degenerate. Then the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g},\mathfrak{g}]^\perp$ is non-degenerate. Thus there is a vector field $U \in [\mathfrak{g},\mathfrak{g}]^\perp$ such that $\langle U, U \rangle \neq 0$. By (5),

$$0 = \langle [X, U], U \rangle + \langle U, [X, U] \rangle = -2\rho \langle U, V \rangle.$$

Thus $\rho = 0$, it is a contradiction. So the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate.

Proposition 9. Assume that $G$ is a pseudo-Riemannian Lie group of type $(p,q)$. If $G$ admits a non-Killing left-invariant conformal vector field and $\dim[\mathfrak{g},\mathfrak{g}] = \dim \mathfrak{g} - \min(p,q)$, then $G$ is solvable.
**Proof.** Assume that \( X \in \mathfrak{g} \) is a non-Killing conformal vector field. Let \( V_0 = \{ y \in \mathfrak{g} \mid adX(y) = 0 \} \). By (5), for any \( u, v \in V_0 \) we have
\[
0 = \langle adX(u), v \rangle + \langle u, adX(v) \rangle = -2\rho \langle u, v \rangle.
\]
Since \( \rho \neq 0 \), it follows \( \langle u, v \rangle = 0 \) for any \( u, v \in V_0 \). So \( V_0 \) is an isotropy subspace of \((\mathfrak{g}, \langle \cdot, \cdot \rangle)\). Let \( k = \dim V_0 \). Since \( adX(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}] \), we have
\[
k \geq \dim \mathfrak{g} - \dim [\mathfrak{g}, \mathfrak{g}] = \min(p, q).
\]

Let \( m \) denote the multiplicity of the eigenvalue 0 of \( adX \). Clearly \( m \geq k \). We claim \( k = m \). Otherwise, \( m > k \). By the Jordan canonical form theory for a nilpotent matrix, there are non-zero vectors \( w \in \mathfrak{g}, v_0 \in V_0 \) satisfying \( adX(w) = v_0 \). Obviously, \( w \notin V_0 \). Since \( V_0 \) is an isotropy subspace, for any \( v \in V_0 \), we have
\[
0 = \langle adX(w), v \rangle + \langle w, adX(v) \rangle = -2\rho \langle w, v \rangle,
\]
which implies \( \langle w, v \rangle = 0 \) since \( \rho \neq 0 \). It follows that
\[
0 = \langle adX(w), w \rangle + \langle w, adX(w) \rangle = -2\rho \langle w, w \rangle,
\]
which implies \( \langle w, w \rangle = 0 \). Then we have an isotropy subspace of dimension \( \geq \min(p, q) + 1 \) spanned by \( V_0 \) and \( w \), which is impossible. That is, \( k = m \).

If \( adX \) isn't invertible on \([\mathfrak{g}, \mathfrak{g}]\), then we have
\[
k = m \geq \min(p, q) + 1.
\]
Namely \( \dim V_0 \geq \min(p, q) + 1 \). It is a contradiction since \( V_0 \) is an isotropy subspace of \((\mathfrak{g}, \langle \cdot, \cdot \rangle)\). Hence \( adX \) must be invertible on \([\mathfrak{g}, \mathfrak{g}]\). By Lemma 5, we know that \([\mathfrak{g}, \mathfrak{g}]\) is nilpotent which implies the solvability of \( \mathfrak{g} \). \( \square \)

**The proof of Theorem 1.** For the Lorentzian case, by Lemma 4, \( \dim [\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 1 \). Thus \( G \) is solvable by Proposition 9.

For the pseudo-Riemannian Lie group of type \((n - 2, 2)\), by Lemma 3, \( \dim [\mathfrak{g}, \mathfrak{g}] \geq \dim \mathfrak{g} - 2 \). By Lemma 2, we must have \( \dim [\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 2 \) or \( \dim [\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 1 \). If \( \dim [\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 2 \), then \( G \) is solvable by Proposition 9. If \( \dim [\mathfrak{g}, \mathfrak{g}] = n - 1 \), by Lemma 8, there is a basis \( \{ e_1, e_2, \ldots, e_{n-1}, e_n \} \) of \( \mathfrak{g} \) such that \( [\mathfrak{g}, \mathfrak{g}] = \langle e_1, e_2, \ldots, e_{n-1} \rangle \) and the metric matrix associated with this basis is defined by
\[
I_{n-3} \quad 0 \quad 0 \\
0 \quad -1 \quad 0 \\
0 \quad 0 \quad -1 \\
0 \quad 0 \quad 0
\]
(6)
Set \( adX(e_j) = \sum_{i=1}^{n} a_{ij} e_i, a_{ij} \in \mathbb{R}, 1 \leq i, j \leq n \). By (5), we know the matrix of \( adX \) associated with this basis is represented by
\[
\begin{pmatrix}
H - \rho I_{n-3} & 0 & \alpha \\
\beta & -2\rho & 0 \\
0' & 0 & 0
\end{pmatrix},
\]
where \( H \in so(n - 3, 1), \alpha \in \mathbb{R}^{(n-2) \times 1} \) and \( \beta \in \mathbb{R}^{1 \times (n-2)} \). By Lemma 6, we know the eigenvalues of \( (adX)_{[\mathfrak{g}, \mathfrak{g}]} \) are of the forms:
\[
-\rho, -2\rho, -\rho \pm \lambda, -\rho + ia (0 \neq a \in \mathbb{R}).
\]
If \( \lambda \neq \pm \rho \), then \( (adX)_{[\mathfrak{g}, \mathfrak{g}]} \) is invertible. By Lemma 5, \([\mathfrak{g}, \mathfrak{g}]\) is nilpotent which implies the solvability of \( \mathfrak{g} \). If \( \lambda = \rho \) or \( \lambda = -\rho \), then the eigenvalues of \( (adX)_{[\mathfrak{g}, \mathfrak{g}]} \) are of the forms:
\[
-\rho, -2\rho \ (\text{of multiplicity } 2), 0 \ (\text{of multiplicity } 1), -\rho + ia (0 \neq a \in \mathbb{R}).
\]
Assume that \( g = \mathfrak{s} \ltimes \mathfrak{r} \) is a Levi decomposition of \( g \) with \( \mathfrak{s} \neq 0 \). Let \( X = X_\mathfrak{s} + X_\mathfrak{r} \) be the corresponding decomposition of \( X \). Then we have

\[
ad X = \begin{pmatrix} (ad X_\mathfrak{s})_\mathfrak{s} & 0 \\ * & (ad X)_\mathfrak{r} \end{pmatrix} : \mathfrak{s} \ltimes \mathfrak{r} \to \mathfrak{s} \ltimes \mathfrak{r}.
\]

In particular, the eigenvalues of \( (ad X_\mathfrak{s})_\mathfrak{s} \) would be also eigenvalues of \( ad X \). It contradicts to \( \text{tr}(ad X_\mathfrak{s})_\mathfrak{s} = 0 \). So \( \mathfrak{s} = 0 \), i.e. \( g \) is solvable.

That is, we have Theorem 1. \( \square \)

In general, we have the following conjecture.

**Conjecture 10.** Let \( G \) be a pseudo-Riemannian Lie group of type \((p, q)\) where \( p, q \geq 3 \). If \( G \) admits a non-Killing left-invariant conformal vector field, then \( G \) is solvable.

### 4. Conformally flat pseudo-Riemannian Lie groups

The following example generalizes the Lorentzian case in [2] to type \((p, q)\).

**Example 11.** Consider the Lie algebra \( \mathfrak{g} \) defined by

\[
[e_n, e_i] = -\rho e_i, \ 1 \leq i \leq n-2, \ [e_n, e_{n-1}] = -2\rho e_{n-1},
\]

where \( \{e_1, e_2, \ldots, e_n\} \) is a basis of \( \mathfrak{g} \), and \( \rho \) is a non-zero constant. Clearly, \( \mathfrak{g} \) is a non-unimodular solvable Lie algebra with abelian derived algebra \([\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_1, e_2, \ldots, e_{n-1}\}\). Define an inner product \( \langle \cdot, \cdot \rangle \) of signature \((p, q)\) \((p, q \geq 1)\) on \( \mathfrak{g} \) associated with the basis \( \{e_1, e_2, \ldots, e_n\} \) by

\[
\begin{pmatrix}
I_{p-1} & -I_{q-1} \\
-I_{q-1} & 0 & 1 \\
1 & 0 
\end{pmatrix}.
\]

Let \( G \) denote the simply connected Lie group with the Lie algebra \( \mathfrak{g} \), and we also use the symbol \( \langle \cdot, \cdot \rangle \) to denote the induced left-invariant pseudo-Riemannian metric on \( G \). Then \( X \in \mathfrak{g} \) is a non-Killing conformal vector field on \((G, \langle \cdot, \cdot \rangle)\) if and only if

\[
\langle [X, e_i], e_j \rangle + \langle e_i, [X, e_j] \rangle = -2c \langle e_i, e_j \rangle,
\]

where \( 1 \leq i, j \leq n \), and \( c \) is a non-zero constant. By a straightforward computation, we know \( X = e_n \) is a left-invariant non-Killing conformal vector field on \((G, \langle \cdot, \cdot \rangle)\) satisfying

\[
L_X \langle \cdot, \cdot \rangle = 2\rho \langle \cdot, \cdot \rangle.
\]

The following is to prove that \((G, \langle \cdot, \cdot \rangle)\) is conformally flat. We first recall some definitions and a theorem of Weyl. For a pseudo-Riemannian manifold \((M, g)\), denote by \( \nabla, R, \text{Ric} \) and \( \text{scal} \) the Levi-Civita connection, the Riemann curvature tensor, the Ricci tensor and the scalar curvature respectively. For symmetric \((0, 2)\)-type tensor fields \( h, k \) on \((M, g)\), define the *Kulkarni–Nomizu product* as the \((0, 4)\)-type tensor field by

\[
h \circ k(v_1, v_2, v_3, v_4) = \frac{1}{2} (h(v_1, v_4) k(v_2, v_3) + h(v_2, v_3) k(v_1, v_4)) - \frac{1}{2} (h(v_1, v_3) k(v_2, v_4) + h(v_2, v_4) k(v_1, v_3)).
\]

The *Schouten tensor* for \( n > 2 \) is given by

\[
P = \frac{2}{n-2} \text{Ric} - \frac{\text{scal}}{(n-1)(n-2)} \cdot g,
\]

where \( \text{Ric} \) denotes the Ricci curvature tensor and \( \text{scal} \) the scalar curvature.
and the Weyl conformal curvature tensor $W$ is defined by

$$R = P \circ g + W,$$

(12)

where $R$ is the (0,4)-type Riemann curvature tensor and $\circ$ is the Kulkarni–Nomizu product. The following result of Weyl is well-known.

**Theorem 12 (16).** A pseudo-Riemannian manifold $M$ of dimension $\geq 4$ is conformally flat if and only if the Weyl conformal curvature tensor $W$ vanishes identically.

Let the notations as Example 11 for $n \geq 4$. Denote by $B$ the left-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ and denote by $\nabla$ the Levi-Civita connection as usual. Let

$$B_{ij} = B(e_i, e_j), [e_i, e_j] = C_{ij}^k e_k, \nabla e_i, e_j = \Gamma_{ji}^k e_k,$$

$$R(e_i, e_j)e_k = R_{ijk}^l e_l, R_{ijkl} = R_{ijkl}^s B_{sl}, H_{ijkl} = (P \circ B)(e_i, e_j, e_k, e_l).$$

Using the Koszul’s formula, we have

$$\Gamma_{ij}^k = \frac{1}{2}(-C_{ij}^s B_{sl} - C_{jt}^s B_{si} + C_{ti}^s B_{sj}) B^{lk}.$$

where $(B^{ij})$ denotes the inverse matrix of $(B_{ij})$. Furthermore, we have

$$R_{ijkl} = \Gamma_{kj}^s \Gamma_{sl}^l - \Gamma_{kl}^s \Gamma_{sj}^l - C_{ij}^s \Gamma_{ks}^l.$$

By the definition of $g$ in Example 11, the non-zero $C_{ij}^k$ are

$$C_{in}^i = \rho = -C_{ni}^i, i \leq n-2; C_{n-1,n}^{n-1} = 2\rho = -C_{n,n-1}^n.$$ 

Set $\epsilon_i = B(e_i, e_j) \in \{\pm 1\}$ for $i \leq n-2$. By a straightforward computation, the non-zero $\Gamma_{ij}^k$ are

$$\Gamma_{ii}^{n-1} = -\rho \epsilon_i, \Gamma_{ni}^i = \rho, i \leq n-2; \Gamma_{n-1,n}^{n-1} = -2\rho, \Gamma_{nn}^n = 2\rho,$$

and by the symmetry of $R$ corresponding to subscript, the fundamental non-zero $R_{ijkl}^l$ are

$$R_{inn}^{n-1} = -\rho^2 \epsilon_i, R_{inn}^i = \rho^2, i \leq n-2.$$

and consequently the fundamental non-zero $R_{ijkl}$ are

$$R_{inni} = \rho^2 \epsilon_i, i \leq n-2.$$

In particular, the Ricci tensor is

$$\text{Ric}(e_i, e_j) = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & (n-2)\rho^2
\end{pmatrix},$$

and the scalar curvature vanishes. Then by (11), we know that the Schouten tensor $P = \frac{2}{n-2} \text{Ric}$, and the fundamental nonzero $H_{ijkl} = (P \circ B)(e_i, e_j, e_k, e_l)$ are

$$H_{inni} = \rho^2 \epsilon_i, i \leq n-2.$$

Since the Weyl conformal curvature $W_{ijkl} = R_{ijkl} - H_{ijkl}$, by Theorem 12, we know that $(G, \langle \cdot, \cdot \rangle)$ is conformally flat.

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