A FIELD GUIDE TO FORWARD-BACKWARD SPLITTING
WITH A FASTA IMPLEMENTATION

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Abstract. Non-differentiable and constrained optimization play a key role in machine learning, signal and image processing, communications, and beyond. For high-dimensional minimization problems involving large datasets or many unknowns, the forward-backward splitting method provides a simple, practical solver. Despite its apparently simplicity, the performance of the forward-backward splitting is highly sensitive to implementation details.

This article is an introductory review of forward-backward splitting with a special emphasis on practical implementation concerns. Issues like stepsize selection, acceleration, stopping conditions, and initialization are considered. Numerical experiments are used to compare the effectiveness of different approaches.

Many variations of forward-backward splitting are implemented in the solver FASTA (short for Fast Adaptive Shrinkage/Thresholding Algorithm). FASTA provides a simple interface for applying forward-backward splitting to a broad range of problems.

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1. Introduction

A large number of non-differentiable and constrained convex optimization problems have the form

\[
\text{minimize } h(x) = f(x) + g(x),
\]

where \( x \in \mathbb{R}^N \), \( f \) is differentiable, and \( g \) is an arbitrary (i.e., not necessarily smooth) convex function. Problems of the form (1) arise frequently in the many areas including machine learning, dictionary learning [1], online learning for streaming data [2], sparse coding [3, 4], spectral clustering [5], sparse covariance estimation [6], approximate nearest neighbor search and vector quantization [7], as well as compressive sensing [8].

In many situations, the function \( g \) is neither differentiable nor even finite-valued, in which case the problem (1) cannot be minimized using simple gradient descent methods. However, for a large class of functions \( g \) that arise in practice, one can evaluate the proximal operator

\[
\text{prox}_g(z, \tau) = \arg \min_x \tau g(x) + \frac{1}{2} \|x - z\|^2.
\]

The proximal operator finds a point close to the minimizer of \( g \) without straying too far from a starting point \( z \). This operator is often referred to as a backward (or implicit) gradient descent step with stepsize \( \tau \). If the proximal operator (2) can be evaluated easily, then one can solve (1) efficiently using the Forward-Backward Splitting (FBS) method (also called the proximal gradient method). FBS can handle non-differentiable objectives and convex constraints while maintaining the simplicity of gradient descent methods.

Due to the vast applications of FBS and its utility for sparse coding and regression, many variants of FBS have been developed to improve performance and ease of use. In its raw form, FBS requires the user to choose a number of convergence parameters that strongly effect both the performance and reliability of the algorithm. These include stepsizes, stopping condition parameters, acceleration schemes, stability conditions, and initialization. With the right modifications, FBS can be performed without substantial oversight from the user.

This article introduces and reviews FBS from a practical point of view. While this is not the first review article written on FBS, it differs from other reviews in that it focuses on implementation issues. For an excellent theoretical review of this subject, see [9].

In Section 2, we introduce the forward-backward splitting and discuss when it converges. Numerous example problems are discussed in Section 3, and for each we discuss the formulation and solution via FBS. In Section 4, we discuss practical issues related to the implementation of FBS. The performance of variants of FBS on different test problems is explored in 5.

The practical issues discussed in Section 4 are incorporated into a new reference implementation of FBS, called FASTA (short for Fast Adaptive Shrinkage/Thresholding Algorithm). FASTA provides a simple interface for applying forward-backward splitting to a broad range of problems.

1.1. A Word on Notation. The \( \ell_2 \) norm is denoted by \( \|x\| = \sqrt{\sum x_i^2} \). We use the “single-bar” notation to define the \( \ell_1 \) norm:

\[
|x| = \sum_i |x_i|.
\]

The transpose of a matrix/vector \( x \) is denoted \( x^T \). The inner-product of two vectors \( x \) and \( y \) is denoted \( \langle x, y \rangle = x^T y \). The inner product of two matrices \( X \) and \( Y \) is given by
\( (X, Y) = \text{trace}(X^TY) = \sum_{i,j} X_{i,j} Y_{i,j} \). We use the symbol \( \Re x \) to denote the real part of a (possibly) complex quantity. For example \( \Re x \) denotes the real part of \( x \).

The vector \( x^* \) that minimizes some objective function \( h \) is denoted \( x^* = \arg \min_x h(x) \). In contrast, the minimal value of \( f \) is denoted \( \min_x f(x) \).

### 2. Forward-Backward Splitting

Forward-Backward Splitting is a two-stage method that addresses each term in (1) separately. The FBS method is listed in Algorithm 1.

#### Algorithm 1 Forward-Backward Splitting

while not converged do
  \[
  \hat{x}_{k+1} = x_k - \tau_k \nabla f(x_k) \tag{3}
  \]
  \[
  x_{k+1} = \text{prox}_g(\hat{x}_{k+1}, \tau_k) = \arg \min_x \tau_k g(x) + \frac{1}{2} \|x - \hat{x}_{k+1}\|^2 \tag{4}
  \]
end while

Let’s examine each step of the algorithm in detail. Line (3) performs a simple forward gradient descent step on \( f \). This step begins at iterate \( x_k \), and then moves in the direction of the (negative) gradient of \( f \), which is the direction of steepest descent. The scalar \( \tau_k \) is the stepsize which controls how far the iterate moves along the gradient direction during iteration \( k \).

Equation (4) is called the proximal step, or backward gradient descent step. To understand this terminology, we examine the proximal operator (2). Any \( x_p \) that minimizes (2) must satisfy the optimality condition

\[
0 = \tau G + (x_p - z) \tag{5}
\]

where \( G \in \partial g(x_p) \) is some sub-gradient of \( g \). Note that when \( g \) is differentiable we simply have \( G = \nabla g(x_p) \). Equation (5) rearranges to

\[
x_p = \text{prox}_g(z, \tau) = z - \tau G.
\]

This shows that \( x_p \) is obtained from \( z \) by marching down the sub-gradient of \( g \). For this reason, the proximal operator performs a gradient descent step. Because the sub-gradient \( G \) is evaluated at the final point \( x_p \) rather than the starting point \( z \), this is called backward gradient descent.

Equation (5) is equivalent to the set inclusion \( 0 \in \partial \tau g(x_p) + (x_p - z) \), which rearranges to

\[
z \in \tau \partial g(x_p) + x_p = (\tau \partial g + I)x_p.
\]

For this reason, the proximal operator (2) is sometimes written

\[
x_p = (\tau \partial g + I)^{-1} z = J_{\tau \partial g} z
\]

where \( J_{\tau \partial g} = (\tau \partial g + I)^{-1} \) is the resolvent operator of \( \tau \partial g \). The resolvent is simply another way to express the proximal operator. The proximal/resolvent operator, when applied to \( z \), performs a backward gradient descent step starting at \( z \).

Algorithm 1 alternates between forward gradient descent on \( f \), and backward gradient descent on \( g \). The use of a backward step for \( g \) is advantageous in several ways. First, it may be difficult to choose a sub(gradient) of \( g \) in cases where the sub-gradient \( \partial g \) has a complex form or is not unique. In contrast, it can be shown that problem (4) always has a
unique well-defined solution [10], and (as we will see later) it is often possible to solve this problem in simple closed form. Second, the backward step has an important effect on the convergence of FBS, which is discussed in the next section.

2.1. Convergence of FBS. The use of backward (as opposed to forward) descent for the second step of FBS is needed to guarantee convergence. To see this, consider a fixed point $x^\star$ of the FBS iteration. Such a point satisfies

$$x^\star = \text{prox}_g(x^\star - \tau_k \nabla f(x^\star), \tau_k) = x^\star - \tau_k \nabla f(x^\star) - \tau_k G(x^\star)$$

for some $G(x^\star) \in \partial g(x^\star)$. This simplifies to

$$0 = \nabla f(x^\star) + G(x^\star),$$

which is the optimality condition for (1). This simple argument shows that a vector is a fixed-point of the FBS iteration if and only if it is optimal for (1). This equation has a simple interpretation: when FBS is applied to an optimal point $x^\star$, the point is moved to a new location by the forward descent step, and the backward descent step puts it back where it started. Both the forward and backward step agree with one another because they both rely on the gradients evaluated at $x^\star$.

This simple fixed point property is not enough to guarantee convergence. FBS is only convergent when the stepsize sequence \{\tau_k\} satisfies certain stability bounds. In the case of a constant stepsize $\tau_k = \tau$, FBS is known to converge for

$$\tau < \frac{2}{L(\nabla f)}$$

where $L(\nabla f)$ is a Lipschitz constant of $\nabla f$ (i.e., $\|\nabla(x_2) - \nabla(x_1)\| < L\|x_2 - x_1\|$ for all $x_1, x_2$). An important property of FBS is that this stability condition does not depend on $g$. For non-constant stepsizes, it is known that convergence is guaranteed if the stepsizes satisfy $0 < l < \tau_k < u < 2/L(\nabla f)$ for some upper bound $u$ and lower bound $l$ (see theorem 3.4 in [10] for this result and its generalization).

In practice, one seldom has accurate knowledge of $L(\nabla f)$, and the best stepsize choice depends on both the problem being solved and the error at each iteration. For this reason, it is better in practice to choose the sequence \{\tau_k\} adaptively and enforce convergence using backtracking rules rather than the explicit stepsize restriction (6). These issues will be discussed in depth in Section 4.

3. Applications of Forward-Backward Splitting

In this section we study a variety of problems, and discuss how they are formulated and solved using forward-backward splitting. FBS finds use in a large number of fields, including machine learning, signal and image processing, statistics, and communication systems. Here, we briefly discuss a small subset of potential applications. In Section 5 we present numerical experiments using these test problems.

3.1. Convex Constraints and the Projected Gradient Method. The projected gradient (PG) method is a special case of FBS involving a convex constraint. Suppose we are interested in solving

$$\text{minimize } f(x) \text{ subject to } x \in C$$

where $L(\nabla f)$ is a Lipschitz constant of $\nabla f$ (i.e., $\|\nabla(x_2) - \nabla(x_1)\| < L\|x_2 - x_1\|$ for all $x_1, x_2$). An important property of FBS is that this stability condition does not depend on $g$. For non-constant stepsizes, it is known that convergence is guaranteed if the stepsizes satisfy $0 < l < \tau_k < u < 2/L(\nabla f)$ for some upper bound $u$ and lower bound $l$ (see theorem 3.4 in [10] for this result and its generalization).

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for some convex set $C$. We can rewrite this problem in the form (1) using the (non-smooth) characteristic function $\chi_C(x)$ of the set $C$, which is zero for $x \in C$ and infinity otherwise. As a consequence, the problem

$$\text{minimize } f(x) + \chi_C(x)$$

is then equivalent to (1) with $g(x) = \chi_C(x)$. To apply FBS, we must evaluate the proximal operator of $\chi_C$:

$$\text{prox}_{\chi_C}(z, \tau) = \arg \min_{x \in C} \frac{1}{2} \|x - z\|^2. \quad (8)$$

The solution to (8) is the element of $C$ closest to $z$; this is simply the orthogonal projection of $z$ onto the set $C$. The resulting PG method was originally studied by Goldstein, Levitin, and Polyak [11, 12, 13].

3.2. Lasso Regression. One of the earliest sparse regression tools from high-dimensional statistics is the Lasso regression, which is was easily written in the form (7).

An important application of PG methods as in (7) is the Lasso regression problem [14], which is an important sparse regression tool. The Lasso regression is defined as follows:

$$\text{minimize } \frac{1}{2} \|Ax - b\|^2 \text{ subject to } \|x\|_1 \leq \lambda. \quad (9)$$

Here, the convex set $C = \{x : \|x\|_1 \leq \lambda\}$ is simply an $\ell_1$-norm ball. The corresponding proximal step, i.e., projection onto the $\ell_1$-norm ball, can be carried out efficiently using linear-time algorithms, such as the method proposed in [15]. We note that FBS has been used previously for the Lasso regression problem (9) in the popular SPGL1 solver [16, 17].

3.3. $\ell_1$-Norm Penalized (Logistic) Regression. One of the most common applications for FBS is the following $\ell_1$-norm penalized least squares problem:

$$\text{minimize } \mu \|x\|_1 + \frac{1}{2} \|Ax - b\|^2. \quad (10)$$

In statistics, problem (10) is used to find sparse solutions to under-determined least squares problems (CITE). Problem (10) is called basis pursuit denoising (BDPN) in the context of compressive sensing and sparse signal recovery [8, 3].

When the vector $b \in \{0, 1\}^M$ contains binary-valued entries representing the outcomes of random Bernoulli trials with success probability $P(b_i = 1 \mid x) = e^{Ax_i}/(1 + e^{Ax_i})$, one is interested in solving the so-called sparse logistic regression problem

$$\text{minimize } \mu \|x\|_1 + \text{logit}(Ax, b) \quad (11)$$

with the logit penalty function defined as

$$\text{logit}(z, b) = \sum_{i=1}^{M} \log(e^{z_i} + 1) - b_i z_i.$$

Both problems, (10) and (11), can be solved using FBS with $g(x) = \|x\|_1$. The proximal operator of $g$ is given by the well-known shrinkage operator $\text{shrink}(z, \mu \tau)$, whose $i$th element is obtained as

$$\text{shrink}(z_i, \mu \tau) = \text{sign}(z_i) \max \{|z_i| - \mu \tau, 0\}. \quad (12)$$
3.4. Low-Rank (1-bit) Matrix Completion. The goal of matrix completion is to recover a low-rank matrix $\hat{X}$ from a small number of linear measurements. One prominent formulation of matrix completion takes the form
\begin{equation}
\text{minimize} \quad \mu \|X\|_* + L(X, Y),
\end{equation}
where $\|X\|_*$ is the low-rank inducing nuclear norm of the matrix $X$ and $L(\cdot, Y)$ is a loss function that depends on the model of the observed data $Y$ [18].

Suppose that $Y$ contains noisy and punctured (or missing) data of the entries of $\hat{X}$, and let $\Omega$ denote the set of indices that have been observed. In this case, one can use
\begin{equation*}
L(X, Y) = \sum_{\alpha \in \Omega} (X_\alpha - Y_\alpha)^2
\end{equation*}
as an appropriate loss function. Another loss function arises in 1-bit matrix completion, which can be viewed as an instance of low-rank logistic regression [19]. Each observation $Y_\alpha$ is assumed to be a Bernoulli random variable with success probability $P(Y_\alpha = 1|\hat{X}_\alpha) = e^{\hat{X}_\alpha}/(1 + e^{\hat{X}_\alpha})$. For this case, the appropriate loss function is again the logit function:
\begin{equation*}
L(X, Y) = \logit(X, Y).
\end{equation*}

For both quadratic and logit link functions, (13) can be solved using FBS. Specifically, one needs to compute the proximal operator of the nuclear norm, defined as
\begin{equation}
\text{prox}_\tau(Z, \mu) = \arg \min_X \mu \|X\|_\infty + \frac{1}{2} \|X - Z\|^2,
\end{equation}
whose solution is given by the matrix $U \text{shrink}(S, \mu\tau)V^T$, where $Z = USV^T$ is the singular value decomposition (SVD) of $Z$ [19].

3.5. Phase Retrieval from Noisy Measurements. A variety of non-convex problems can be relaxed into convex problems involving symmetric positive semi-definite (SPSD) matrices. This idea was popularized by [20], in which it was shown that the NP-complete MaxCut problem has a convex relaxation with the form of a semi-definite program. Here, we consider the more recent PhaseLift algorithm, which can be used for phase retrieval applications [21].

Suppose we wish to recover a complex-valued signal $x \in \mathbb{C}^N$ from measurements of the form $|\langle a_i, x \rangle|^2 = b_i$ for some set of vectors $\{a_i\}_{i=1}^M$. In words, we take inner products of the vector $x$ with the measurement vectors $\{a_i\}$ and discard the phase information.

Recovery of $x$ from the phase-less measurements $b$ requires the solution of a system of non-linear equations. However, one can relax the equations into a convex problem by defining $A_j = a_ja_j^T$, and observing that $|\langle a_i, x \rangle|^2 = \langle A_j, xx^T \rangle = b_i$. Letting $A(xx^T)_i = \langle A_i, xx^T \rangle$, the set of phase-less measurements can be written compactly as $A(xx^T) = b$. Finally, note that any SPSD rank-1 matrix $X$ can be factorized as $X = xx^T$. This observation allows us to pose the recovery problem in the following form:
\begin{equation}
\text{minimize rank}(X) \quad \text{subject to } A(X) = b, \ X \succeq 0.
\end{equation}

Even though rank($\cdot$) is a non-convex function, one can form a convex relaxation of (15) by replacing rank($\cdot$) with the sparsity-inducing nuclear norm. The resulting problem, known as PhaseLift, is given by [21]:
\begin{equation}
\text{minimize} \ \|X\|_* \quad \text{subject to } A(X) = b, \ X \succeq 0.
\end{equation}
In the case where the measurement vector $b$ is contaminated by additive noise, we choose an $\ell_2$-norm penalty, which stems from a Gaussian noise model. Specifically, we can write the resulting phase-retrieval problem as
\[
\minimize \mu \|X\|_* + \|A(X) - b\|^2 \quad \text{subject to} \quad X \succeq 0,
\]
which can be solved using FBS. The relevant proximal operator corresponds to
\[
\minimize \tau \mu \|X\|_* + \|X - Z\|^2,
\]
whose solution is given by the matrix $U \text{shrink}(\Lambda, \mu \tau) U^T$, where $Z = U \Lambda U^T$ is the eigenvalue decomposition of $Z$.

3.6. Democratic Representations. The dynamic range of signals plays an important role in approximate nearest neighbor search and vector quantization [7], as well as communications and robotics/control [22]. Given a signal $b \in \mathbb{R}^M$, a low-dynamic range representation can be found by choosing a suitable matrix $A \in \mathbb{R}^{M \times N}$ with $M < N$, and by solving
\[
(17) \minimize \mu \|x\|_\infty + \frac{1}{2} \|Ax - b\|^2.
\]
The problem (17) often yields a so-called democratic representation $x^*$ that has small dynamic range and for which a large number of entries have equal (and small) magnitude [22].

The problem (17) can, once again, be solved using FBS. The only missing ingredient is the proximal operator for the $\ell_\infty$-norm, which can be computed in linear time using the method in [15].

4. Bells and Whistles

FBS in its raw form suffers from numerous problems including: (i) The convergence speed of FBS depends strongly on the choice of the stepsize parameters $\{\tau_k\}$. The best stepsize choices may not be intuitively obvious. (ii) Convergence is only guaranteed when the stepsizes satisfy the stability condition (6) which depends on the Lipschitz constant for $\nabla f$. In real applications one may have no knowledge of this Lipschitz constant and no practical way to estimate it. (iii) To automatically stop the FBS iteration, a good measure of convergence is needed.

In this section, we discuss practical methods for overcoming these problems. We begin with adaptive schemes for automatic stepsize selection. We then discuss backtracking methods that can guarantee stability even when the user has no explicit knowledge of $L(\nabla f)$. Finally, we discuss stopping conditions and measures of convergence.

4.1. Adaptive Stepsize Selection. The efficiency of FBS (and gradient methods in general) is very sensitive to the choice of the stepsize $\tau_k$. For this reason, much work has been devoted to studying adaptive stepsize strategies. Adaptive methods automatically tune stepsize parameters in real time (as the algorithm runs) to achieve fast convergence. In this section, we discuss spectral (also called Barzilai-Borwein) stepsize methods, and how they can be adapted to FBS.

Before considering the full-scale FBS method for (1), we begin by considering the case $g = 0$. In this case $h = f$ and FBS reduces to simple gradient descent of the form
\[
x_{k+1} = x_k - \tau_k \nabla f(x).
\]
Spectral schemes for gradient descent were proposed by Barzilai and Borwein [23], who model the function $f$ as the simple quadratic function

$$f(x) \approx \hat{f}(x) = \frac{a}{2} \|x\|^2 + \langle x, b \rangle. \quad (19)$$

It can be shown that the optimal stepsize choice for (19) is $\tau = 1/\alpha$. With this choice, the gradient descent method achieves a perfect minimizer of the simple quadratic (19) in one iteration. This motivates the following stepsize scheme for gradient descent: Before applying the descent step (18), approximate $f$ with a quadratic of the form (19). Then, take a step of length $\tau_k = 1/\alpha$.

Spectral stepsize rules have been generalized to handle FBS with specific choices of $g$. When $g$ is the characteristic function of a convex set, the marriage of FBS with an adaptive stepsize produces the spectral projected gradient method (SPG) [24]. SPG was further specialized to handle the case of $\ell_1$ constraints by van den Berg and Friedlander [16, 17]. The authors of [25] present a spectral stepsize rule that handles the case $g = |\cdot|$. The resulting method solves $\ell_1$ penalized least squares problems using a combination of spectral gradient descent and subspace optimization. This idea was revisited in [26], in which the authors present a modification of FBS for $\ell_1$ penalized least squares that implicitly performs conjugate gradient optimization inside of low-rank subspaces.

Spectral stepsizes for general FBS with arbitrary $f$ and $g$ were first studied in [27] for use with the solver SpaRSA. The method presented here is a hybrid of the spectral scheme [27] with the “adaptive” stepsize rule presented in [28].

It was observed in the introduction that the stepsize restriction for FBS depends only on $f$ and not on $g$. Spectral methods for FBS exploit this property. The idea is to build a quadratic approximation for $f$ of the form (19) at each iteration, and then choose the optimal gradient descent stepsize $\tau_k = 1/\alpha$. Let

$$\Delta x_k = x_k - x_{k-1}, \quad \Delta F_k = \nabla f(x_k) - \nabla f(x_{k-1}). \quad (20)$$

If we assume a quadratic model for $f$ of the form (19), then we have

$$\Delta F_k = \nabla f(x_k) - \nabla f(x_{k-1}) = a(x_k - x_{k-1}).$$

The function (19) is fit to $f$ using a least squares method which chooses $a$ to minimize either $\|\Delta F_k - a\Delta x_k\|^2$ or $\|a^{-1} \Delta F_k - \Delta x_k\|^2$. Just like in the case $g = 0$, we then select a stepsize of length $\tau_k = 1/\alpha$. The resulting stepsize choice is given by

$$\tau^s_k = \frac{\langle \Delta x_k, \Delta x_k \rangle}{\langle \Delta x_k, \Delta F_k \rangle}, \quad \text{and} \quad \tau^m_k = \frac{\langle \Delta x_k, \Delta F_k \rangle}{\langle \Delta F_k, \Delta F_k \rangle}, \quad (22)$$

respectively for each least squares problem. The value $\tau^s_k$ is known as the “steepest descent” stepsize, and $\tau^m_k$ is called the “minimum residual” stepsize [28].

A number of variations of spectral descent are reviewed by Fletcher [29], several of which perform better in practice than the “classic” stepsize rules (22). We recommend the “adaptive” BB method [28], which uses the rule

$$\tau_k = \begin{cases} \tau^m_k, & \text{if } \tau^m_k / \tau^s_k > \frac{1}{2} \\ \tau^s_k - \frac{1}{2} \tau^m_k, & \text{otherwise.} \end{cases} \quad (23)$$

Note that (particularly for non-convex problems), the stepsize $\tau^m_k$ or $\tau^s_k$ may be negative. If this happens, the stepsize should be discarded and replaced with its value from the previous
iteration. When complex-valued problems are considered, it is important to use only the real part of the inner product in (23).

4.2. **Acceleration.** Adaptive stepsize selection helps to speed the convergence of the potentially slow FBS method. Another approach to dealing with slow convergence is to use predictor-corrector schemes that “accelerate” the convergence of FBS.

While several predictor-corrector variants of FBS have been proposed (see for example [30]), the algorithm FISTA has become quite popular because of its lack of tuning parameters and good worst-case performance. FISTA relies on a one-size-fits-all sequence of acceleration parameters that works for any objective. FISTA is listed in Algorithm 2. Step 2 of FISTA simply performs an FBS step. Step 4 performs prediction, in which the current iterate is advanced further in the direction it moved during the previous iteration. The aggressiveness of this prediction step is controlled by the scalar parameter $\alpha_k$. This acceleration parameter is updated in step 3, and increases on each iteration causing the algorithm to become progressively more aggressive. The FISTA method alternates between “predicting” an aggressive estimate of the solution, and “correcting” this estimate using FBS to attain greater accuracy.

One notable advantage of this approach is the worst-case convergence rate. It has been shown that FISTA decreases the optimality gap (the difference between the objective at iteration $k$ and the optimal objective value) with rate $O\left(\frac{1}{k^2}\right)$ [31]. In contrast, the worst-case performance of conventional FBS is known to be $O\left(\frac{1}{k}\right)$.

**Algorithm 2 FISTA**

```
Require: $y_1 = x_0 \in \mathbb{R}^N$, $\alpha_1 = 1$, $\tau < 1/L(\nabla G)$
1: for $k = 1, 2, 3, \ldots$ do
2: \hspace{1em} $x_k = \text{prox}_g(y_k - \tau \nabla f(y_k), \tau)$
3: \hspace{1em} $\alpha_{k+1} = \frac{(1 + \sqrt{1 + 4\alpha_k^2})}{2}$
4: \hspace{1em} $y_{k+1} = x_k + \frac{\alpha_k - 1}{\alpha_{k+1}}(x_k - x_{k-1})$
5: end for
```

In practice, FISTA is generally coupled with a backtracking line search such as those described in the next section.

4.3. **Backtracking Line Search.** Convergence of FBS can be guaranteed by enforcing the stability condition (6). In practice, though, the user generally has no knowledge of the global properties of $\nabla f$, and so the actual value of this stepsize restriction is unknown. In Section 4.1 we discuss adaptive methods that automatically choose stepizes for us. This does not free us from the need for stability conditions: spectral methods are, in general, not guaranteed to converge, even for convex problems [24].

Even without knowing the stepsize restriction (6), convergence can be guaranteed by incorporating a backtracking line search. Such methods proceed by checking a line search condition after each iteration of FBS. The search condition usually enforces that the objective has decreased sufficiently. If this condition fails to hold, then backtracking is performed – the stepsize is decreased and the FBS iteration repeated until the backtracking condition (i.e., sufficient decrease of the objective) is satisfied.

Line search methods were originally proposed for smooth problems [32], and later for projected gradient schemes [33, 24]. A backtracking scheme for general FBS was proposed by Beck and Teboulle for use with FISTA [31]. Many authors consider these line searches
Algorithm 3 Non-Monotone Line Search

while $x_k$ and $x_{k+1}$ violate condition (24) do
    $\tau_k \leftarrow \tau_k/2$
    $x_{k+1} \leftarrow \text{prox}_g(x_k - \tau \nabla f(x_k), \tau)$
end while

to be overly conservative, particularly for poorly conditioned problems [24], and prefer non-monotone line search conditions [34, 35]. Rather than insisting upon an objective decrease on every iteration, non-monotone line search conditions allow the objective to increase within limitations.

Non-monotone line search methods are advantageous for two reasons. First, for poorly conditioned problems (i.e., objective functions with long, narrow valleys) it may be the case that iterate $x_{k+1}$ lies much closer to the minimizer than $x_k$, despite $x_{k+1}$ having the larger objective value. A non-monotone line search prevents such iterates from being rejected.

Second, the objective function must be evaluated every time the line-search condition is tested. For complex problems where evaluating the objective is costly and multiple backtracking steps are necessary for each iteration, the line search procedure may dominate the runtime of the method. Non-monotone line search conditions are less likely to be violated and thus backtracking terminates faster, which alleviates this computational burden.

The method proposed here is inspired by the monotone search proposed in [31] for general FBS, and generalizes the non-monotone strategy of [34] for SPG.

Let $M > 0$ be an integer line search parameter, and define $\hat{f}_k = \max\{f_k - 1, f_k - 2, \ldots, f_k - \min\{M, k\}\}$.

After each step of FBS, the following line search condition is checked.

\begin{equation}
    f(x_{k+1}) < \hat{f}_k + \Re\langle x_{k+1} - x_k, \nabla f(x_k) \rangle + \frac{1}{2\tau_k} \|x_{k+1} - x_k\|^2.
\end{equation}

If the backtracking condition (24) fails, the stepsize is decreased until (24) is satisfied. This process is formalized in Algorithm 3. Note that Algorithm 3 always terminates because condition (24) is guaranteed to hold whenever $\tau_k$ is less than the reciprocal of the Lipschitz constant of $\nabla f$.

A convergence proof for FBS with the line search described in Algorithm 3 is given in the appendix.

4.4. Stopping Conditions. While the accuracy of FBS becomes arbitrarily good as the number of iterations approaches infinity, we must of course stop after a finite number of iterations. A good stopping criteria should be strict enough to guarantee an acceptable degree of accuracy, but not be so conservative that it requires an excessive number of iterations.

Our stopping conditions will be based on the residual, which is simply the derivative of the objective function (or a sub-gradient in the case that $g$ is non-differentiable). Because $f$ is assumed to be smooth, we can differentiate this term in the objective directly. While we may not be able to differentiate $g$, we see from equation (5) that a sub-gradient is given by $(\hat{x}_{k+1} - x_{k+1})/\tau_k \in \partial g(x_{k+1})$. We now have the following formula for the residual $r_{k+1}$ at iterate $x_{k+1}$:

\begin{equation}
    r_{k+1} = \nabla f(x_{k+1}) + \frac{\hat{x}_{k+1} - x_{k+1}}{\tau_k}.
\end{equation}
A simple termination rule would stop the algorithm when \(|r_{k+1}| < tol\) for some small tolerance \(tol > 0\). However, this rule is problematic because it is not scale invariant. To understand what this means, consider the minimization of some objective function \(h(\cdot)\). This function can be re-scaled by a factor of 1000 to obtain \(h = 1000h\). The new rescaled objective has the same minimizer as the original, however the sub-gradient \(\partial h(x_k)\) is 1000 times larger than \(\partial h(x_k)\). A stopping parameter \(tol\) may be reasonable for minimizing \(h\) but overly strict for minimizing \(h\), even though the solutions to the problems are identical. Ideally, we would like scale invariant stopping rules that treat both of these problems equally.

One way to achieve scale invariance is by replacing the residual with the relative residual. To define the relative residual, we observe that (25) is small when

\[
\nabla f(x_{k+1}) \approx -\frac{\hat{x}_{k+1} - x_{k+1}}{\tau_k}.
\]

In plain words, the residual (25) measures the difference between the gradient of \(f\) and the negative sub-gradient of \(g\). The relative residual \(r_{k+1}^r\) measures the relative difference between these two quantities, which is given by

\[
r_{k+1}^r = \frac{\|\nabla f(x_{k+1}) + \frac{\hat{x}_{k+1} - x_{k+1}}{\tau_k}\|}{\max\{\|\nabla f(x_{k+1})\|, \|\frac{\hat{x}_{k+1} - x_{k+1}}{\tau_k}\|\} + \epsilon_r} = \frac{\|r_{k+1}\|}{\max\{\|\nabla f(x_{k+1})\|, \|\frac{\hat{x}_{k+1} - x_{k+1}}{\tau_k}\|\} + \epsilon_r}
\]

where \(\epsilon_r\) is some small positive constant to avoid dividing by zero.

Another more general scale invariant stopping condition uses the normalized residual, which is given by

\[
r_{k+1}^n = \frac{\|r_{k+1}\|}{\|r_1\| + \epsilon_n}
\]

where the small constant \(\epsilon_n\) prevents division by zero. Rather than being an absolute measure of accuracy, the normalized residual measures how much the approximate solution has improved relative to \(x_1\).

Both scale invariant conditions have advantages and disadvantages. The relative residual works well for a wide range of problems and is insensitive to the initial choice of \(x_0\). However, the relative residual loses scale invariance when \(\nabla f(x^*) = 0\) (in which case the denominator of (27) nearly vanishes). This happens, for example, when \(g\) is the characteristic function of a convex set and the constraint is inactive at the optimal point (i.e., the optimal point lies in the interior of the constraint set). In contrast, the normalized residual can be effective even if \(\nabla f(x^*) = 0\). However, this measure of convergence is potentially sensitive to the choice of the initial iterate, and so it requires the algorithm to be initialized in some consistent way. The strictness of this condition can also depend on the problem being solved.

For general applications, we suggest a combined stopping condition that terminates the algorithm when either \(r_{k+1}^r\) or \(r_{k+1}^n\) gets small. In this case, \(r_{k+1}^r\) terminates the iteration if a high degree of accuracy is attained, and \(r_{k+1}^n\) terminates the residual appropriately in cases where \(0 \in \partial g(x^*)\).

5. Numerical Results

We compare several variants of FBS and compare their performance using the test problems described in Section 3. The variants considered here are the original FBS with constant stepsize, and the accelerated variant FISTA described in Section 4.2. We also consider the
Figure 1. Sample converge curves for FBS, FISTA, and the adaptive method FASTA for four diverse test problems (see Sections 3 and 5 for details). The vertical axis shows the optimality gap.

adaptive method described in Section 4.1, which is implemented in the solver FASTA. This method uses the spectral stepsize rules proposed in [27] and [28].

The stepsize parameter for all methods was initialized by first estimating the Lipschitz constant $L(\nabla f)$. Two random vectors $x_1, x_2$ were generated, and the estimate $\tilde{L} = \|\nabla f(x_2) - \nabla f(x_1)\|/\|x_2 - x_1\| \leq L(\nabla f)$ was formed. The initial stepsize was then $\tau_0 = 10/\tilde{L}$. This stepsize is guaranteed to be at least an order of magnitude larger than the true stepsize restriction for FISTA ($1/L$). The backtracking scheme in Section 4.3 was then used to guarantee convergence. We found that this implementation enables faster convergence than implementations that explicitly require a Lipschitz constant for $\nabla f$.

At each iteration of the algorithms considered, the relative residual (27) was computed and used as a measure of convergence. Time trials for all methods were terminated on the first iteration that satisfied $r_{k+1}^r < 10^{-4}$.

5.1. Test Problem Details.

Lasso: We generated problems of the form (9) using random Gaussian matrices $A \in \mathbb{R}^{M \times N}$ with $N = 1000$. Experiments were done with both $M = 500$ (moderately under-sampled) and $M = 100$ (highly under-sampled). The true signal $x_0$ had 20 entries of unit magnitude; the remaining entries were zero. The indices of the support set were chosen at random. The measurement vector was generated using the formula $b = Ax_0$ and then contaminated by additive Gaussian noise to achieve an signal-to-noise ratio (SNR) of 13 dB. Recovery was performed with $\lambda = 15$. 
Table 1. Complexity comparison of FBS, FISTA, and FASTA. We show the average number of iterations (and time per problem in seconds) to reach convergence. FASTA clearly outperforms FBS and FISTA for the considered test problems.

| Problem          | FBS       | FISTA    | FASTA    |
|------------------|-----------|----------|----------|
| BDPN 100         | 500.0 (0.193) | 407.8 (0.160) | 80.9 (0.035) |
| BDPN 500         | 59.4 (0.040)  | 50.4 (0.035)  | 16.6 (0.010)  |
| Lasso 100        | 500.0 (0.238) | 364.8 (0.177) | 69.4 (0.037) |
| Lasso 500        | 38.0 (0.029)  | 35.3 (0.028)  | 10.5 (0.007)  |
| Logistic         | 143.4 (0.099) | 75.5 (0.055)  | 19.2 (0.012)  |
| Matrix comp.     | 94.0 (9.71)   | 60.0 (6.34)   | 16.0 (1.60)   |
| PhaseLift        | 256.0 (12.5)  | 125.1 (6.19)  | 26.0 (1.124)  |
| Democracy        | 71.1 (0.057)  | 46.3 (0.0395) | 17.4 (0.013)  |

\(\ell_1\)-Norm Penalized Least Squares: We generated problems of the form (10) using a similar procedure as for Lasso. This time, however, with the noise scaled to achieve SNR = 20dB. Recovery was performed with \(\mu = 0.1\).

Logistic Regression: We generated random matrices \(A \in \mathbb{R}^{M \times N}\) of dimension \(N = 1000\) and \(M = 500\) with zero-mean Gaussian entries of variance of 4. Once again, the support of the true signal comprised 20 randomly chosen elements. Given \(z = Ax\), \(b\) was constructed as a set of random Bernoulli variables with probability of success given by the logistic function \(P(b_i = 1) = e^{z_i}/(1 + e^{z_i})\). The problem (11) was solved with \(\mu = 20\).

Matrix Completion: We generated a \(200 \times 1000\) matrix \(X\) with i.i.d. random Gaussian entries of standard deviation 10. The SVD of \(X\) was then computed, and all but the largest 5 singular values were set to zero to obtain a rank-5 matrix. The sigmoidal logistic function was then applied element-wise to \(X\) to obtain a matrix of probabilities; a random Bernoulli matrix \(Y\) was drawn from the resulting distribution. Problem (13) was then solved using the logistic penalty function with \(\mu = 25\).

Phase Retrieval: We generated random vectors of length \(N = 200\) with random Gaussian real and imaginary parts. The set \(\{a_i\}\) was created by randomly drawing complex Gaussian vectors. The measurement vector \(b\) of length \(M = 600\) was creating by setting \(b_i = |\langle a_i, x \rangle|^2\), and then, contaminating the resulting measurements with real-valued Gaussian noise to have SNR = 13dB. Problem (17) was then solved with parameter \(\mu = 15\), which was chosen to be large enough that the solution matrix had rank 1.

Democratic Representations: We generated frames of dimension \(500 \times 1000\) by randomly selecting subset of rows from a unitary discrete Fourier transform matrix. The signal \(b\) was a complex-valued random Gaussian vector of length 500. Equation (17) was solved with \(\mu = 300\).

5.2. Results and Discussion. We applied FBS, FISTA, and FASTA to all of the test problems described in Section 5.1 for 100 Monte–Carlo trials. The resulting average number of iterations (and time in seconds) is reported in Table 1. Convergence curves are shown in Figure 1. We observe that FASTA is approximately 3×-to-8× faster than both FBS and FISTA for all of the considered test problems.

We note that the complexity advantage of FASTA is more pronounced for problems where \(f\) is non-quadratic, i.e., for matrix completion and logistic regression. In this case,
the Hessian of $f$ varies over the problem domain, and the stability condition and optimal stepsize for FBS varies with it. For such problems, an adaptive scheme like FASTA is able to effectively match the stepsize to the local structure of the objective, resulting in outstanding convergence.

6. FASTA: A Handy Forward-Backward Solver

The solver FASTA (Fast Adaptive Shrinkage/Thresholding) implements forward-backward splitting for arbitrary test problems. Many improvement to FBS are implemented in FASTA including adaptively, acceleration, and a variety of stopping conditions. This solver has a simple interface that only requires the user to supply the gradient of $f$ and the proximal mapping (i.e., backward gradient descent operator) for $g$. 
Appendix A. Convergence Proof for Non-Monotone Line Search

We now consider the convergence of the backtracking line search discussed in section 4.3.

**Theorem 1.** Suppose that FBS is applied to (1) with convex $g$ and differentiable $f$. Suppose further that $h = f + g$ is proper, lower semi-continuous, and has bounded level sets. If $\{\tau_k\}$ is bounded below by a positive constant and

\begin{equation}
 f(x_{k+1}) - \hat{f}_k < \langle x_{k+1} - x_k, \nabla f(x_k) \rangle + \frac{1}{2\tau_k} \|x_{k+1} - x_k\|^2
\end{equation}

then $\lim_{k \to \infty} h(x_k) = h^*$, where $h^*$ denotes the minimum value of $h$.

**Proof.** From the optimality condition (4), we have $0 \in \tau_k \partial g(x_{k+1}) + x_{k+1} - \bar{x}_{k+1}$ and so

\[
 0 = \tau_k G_{k+1} + x_{k+1} - \bar{x}_{k+1} = \tau_k G_{k+1} + x_{k+1} - (x_k - \tau F_k)
\]

for some $G_{k+1} \in \partial g(x_{k+1})$ and $F_k = \nabla f(x_k)$. From this we arrive at

\begin{equation}
 x_k - x_{k+1} = \tau_k (G_{k+1} + F_k)
\end{equation}

and also

\begin{equation}
 \langle x_{k+1} - x_k, F_k + G_{k+1} \rangle = -\frac{1}{\tau_k} \|x_{k+1} - x_k\|^2.
\end{equation}

Now, because $g$ is convex

\begin{equation}
 g(x_k) \geq g(x_{k+1}) + \langle x_k - x_{k+1}, G_{k+1} \rangle.
\end{equation}

Subtracting (32) from (29) and applying (31) yields

\[
 h(x_{k+1}) = f(x_{k+1}) + g(x_{k+1}) \\
 \leq \hat{f}_k + g(x_k) + \langle x_{k+1} - x_k, F_k + G_{k+1} \rangle \\
 + \frac{1}{2\tau_k} \|x_{k+1} - x_k\|^2 \\
 = \hat{f}_k + g(x_k) - \frac{1}{2\tau_k} \|x_{k+1} - x_k\|^2
\]

\begin{equation}
 = \tilde{h}_k - \frac{1}{2\tau_k} \|x_{k+1} - x_k\|^2.
\end{equation}

where $\tilde{h}_k = \max\{h_{k-1}, h_{k-2}, \ldots, h_{k-\text{min}\{M,k\}}\}$. Note that $\{\tilde{h}_k\}$ is a monotonically decreasing bounded sequence, and thus has a limit $\tilde{h}^*$.

We aim to show that $\tilde{h}^*$ is the minimal value of $h$. Observe that $\tilde{h}_k = \tilde{h}_{k'}$ for some $k'$ with $k - M \leq k' \leq k$. It is clear from (33) that there must exist a sub-sequence $\{x_{k(i)}\}$ with $h(x_{k(i)}) = \tilde{h}_k$ such that

\begin{equation}
 \lim_{i \to \infty} \frac{1}{2\tau_k} \|x_{k(i)+1} - x_{k(i)}\|^2 = 0
\end{equation}

(otherwise, equation (33) would imply $\tilde{h}_k \to -\infty$ for large $k$). Note that the level sets of $h$ are assumed to be bounded, and by equation (33) $\{h(x_k)\}$ is bounded as well. By compactness, we may assume without loss of generality that $\{x_{k(i)}\}$ is a convergent sub-sequence of iterates with limit point $x^*$.

Now, from (30), we have

\begin{equation}
 \frac{1}{2\tau_k} \|x_{k(i)+1} - x_{k(i)}\|^2 = \frac{\tau_k}{2} \|G_{k(i)+1} + F_{k(i)}\|^2.
\end{equation}
Equation (35), together with (34) and the fact that $\tau_k$ is bounded away from zero, implies that
\[
\lim_{i \to \infty} \|G_{k(i)+1} + F_{k(i)}\|^2 = 0.
\]
Because $\nabla f$ is Lipschitz continuous and $\|x_{k(i)+1} - x_{k(i)}\| \to 0$, we also conclude that $x_{k(i)+1} \to x^*$ and
\[
\lim_{i \to \infty} \|G_{k(i)+1} + F_{k(i+1)}\|^2 = 0.
\]
Note that $G_{k(i)+1} + F_{k(i+1)} \in \partial h(x_{k(i)+1})$. Because the sub-differential of a convex function is continuous\(^1\), (36) implies that $0 \in \partial h(x^*)$, and so $x^*$ is a minimizer of $h$.

We have shown that $\lim_{k \to \infty} \hat{h}_k = h(x^*) = h^*$. Because $h^* \leq h(x_k) \leq \hat{h}_k$, we arrive at the conclusion
\[
\lim_{k \to \infty} h(x_k) = h^*.
\]

\[\square\]

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\(^1\)More formally, the convex sub-differential is a multi-valued function which is upper semicontinuous in a topological sense. See [36, 37].
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