Strong Phase Correlations of Solitons of Nonlinear Schrödinger Equation

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Abstract

We discuss the possibility to suppress the collapse in the nonlinear 2+1 D Schrödinger equation by using the gauge theory of strong phase correlations. It is shown that invariance relative to $q$-deformed Hopf algebra with deformation parameter $q$ being the fourth root of unity makes the values of the Chern-Simons term coefficient, $k = 2$, and of the coupling constant, $g = 1/2$, fixed; no collapsing solutions are present at those values.

1. Introduction

The characteristic feature of the nonlinear Schrödinger equation

$$i\partial_t \nabla \Psi = -\frac{1}{2} \Delta \Psi - g|\Psi|^{2\alpha} \Psi$$

(1)

in the case of the positive coupling constant, $g$, and a sufficiently high level of nonlinearity, $\alpha \geq 2/d$, determined by spatial dimensionality $d$ of the problem, is existence of collapsing solution [1, 3]. At those self-focusing solutions [4] the Hamiltonian

$$H = \int_{\mathcal{M}} d^2 x \left( \frac{1}{2} |\Psi|^2 - \frac{g}{\alpha + 1} |\Psi|^{2(\alpha + 1)} \right)$$

(2)

is unbounded from below (for $d = 2$ and for a manifold $\mathcal{M}$ being the whole plane). This fact hinders using such a solution for calculating the partition function for the ensemble of nonlinear modes under consideration. In terms of the situation of general position there is no prohibition for existence of collapsing solution in model (1). At the same time, it is difficult to see the aforesaid phenomenon as an example
that leads us beyond the framework of general statistical principles in the case of multi-soliton field configurations. That means that within the theory of the field in plasma and in nonlinear optics, in which the nonlinear Schrödinger equation is actively used [2, 3, 4], there is a possibility to stabilize the collapse.

So far it has stayed absolutely unclear, what is the way to solve the problem associated with unboundedness of the Hamiltonian at collapsing solutions in the three-dimensional space. In $2 + 1D$ systems the situation is somewhat different. Peculiarity of $2 + 1D$ systems is connected primarily with realization of fractional statistics in them [3, 6]. The reason for realization of fractional statistics is purely topological. It is caused totally by the fact that the $2D$ configuration space, $M$, is multi-connected, due to which the fundamental group $\pi_1(M)$ coincides with the braid group $B$. From the $2 + 1D$ point of view, the complicated topological pattern manifests itself in the phenomenon of brading of world lines distributed over the plane of degrees of freedom. Generally speaking, correlations of that kind are so strong that properties of one realization of field distribution depend on all the rest configurations in the whole system.

Within the long-wave approximation the way, which is often used for account of such strong correlations, requires taking into account the Chern-Simons term in the Lagrangian and replacing usual derivatives with co-variant ones that contain gauge potential. Applying that approach to the nonlinear Schrödinger equation in the $2 + 1D$ case yielded unexpected effects. It turned out that the solution found by Jackiw and Pi [7] contains considerable contribution from the gauge field (see also Ref. [8]). It is thought generally [9] that the Chern-Simons interaction modifies statistics, while making no influence on dynamics properties of the system. Further we will give arguments showing that those two points of view [7, 9] do not contradict each other in reality.

One of the major results of the work of Jackiw and Pi [7] was the conclusion about existence of such a value for the coupling constant, $g = 1/k$, for which the dangerous contribution $-|\Psi|^4$ for $d = 2$ forming the collapsing solution is reduced. We would like to pay attention to this phenomenon for the fixed value of the coefficient before the Chern-Simons term, $k = 2$. It is corresponded by the maximum value of the coupling constant, $g = 1/2$, in this state of strong phase correlations. By this, we turn to a structure of deformed Hopf’s algebra, on which the Chern-Simons
interaction is based, that is, the so-called “quantum group” with the deformation parameter being a root of unity.

In terms of the most complete account of the symmetry realized in the ground state, the mentioned value of the coefficient, \( k = 2 \), makes the statistic and dynamic approaches self-consistent, as well as emphasized symmetry invariance of the model in the sense of deformed Hopf’s algebra.

Physical realization of the possibility of collapse suppression for classic wave fields is determined by a number of restrictions that will be discussed further. We do not separate now the classic and quantum cases and consider them from the common point of view. By that we mean that field distributions in the quantum region (after corresponding re-calculation of scales) are transferred into the classic region without variations in the qualitative picture. The reason for that being valid lies, in our opinion, in the topologic nature of strong phase correlations of 2 + 1D wave fields considered in the infrared limit.

The qualitative pattern of the phenomena under consideration in the 2 + 1D case will be valid, as we think, for any complex field. Specifically, interesting results for the \( SU(2) \times U(1) \) model for a similar approach, but from a somewhat different point of view have been recently obtained in Ref. [10]. Details of relativistic soliton configurations, as well as influence of boundary conditions were the object of recent studies in Refs. [11, 12], respectively.

2. Soliton structure

To discuss the effect of collapse stabilization and symmetry properties of the main state we will briefly review basic expressions of Ref. [7]. The gauged 2+1D nonlinear Schrödinger equation is the equation for motion of the system with the following Lagrangean density:

\[
\mathcal{L} = -\frac{k}{8} \epsilon^{\mu\nu\lambda} a_\mu f_{\nu\lambda} + i \Psi^* (\partial_t + ia_0) \Psi - \frac{1}{2} |D\Psi|^2 + \frac{g}{2} |\Psi|^4.
\]

Here \( a_\mu \) is gauge potential that parametrizes the Chern-Simons term \( U(1) \), \( \hbar = e = m = c = 1 \), \( D\Psi = (\nabla - ia) \Psi \) is covariant gradient; the Lagrangian density of the inherent gauge field \( f^2_{\alpha\beta} \) is omitted for the sake of simplicity. Account of that term does not change the qualitative results [12].
The Hamiltonian for system (3)

$$H = \frac{1}{2} \int_{\mathcal{M}} d^2 x \left( |D\Psi|^2 - g|\Psi|^4 \right)$$

with the use of Bogomolny’s decomposition [13] have the following form:

$$H = \frac{1}{2} \int_{\mathcal{M}} d^2 x \left( |(D_1 - iD_2) \Psi|^2 - (g - k^{-1})|\Psi|^4 \right).$$

Term $\frac{1}{2} \nabla \times J$ with current density $J = (1/2i) (\Psi^* D \Psi - \Psi D \Psi^*)$, which was omitted when going from Eq. (4) to Eq. (5), yields zero contribution to Eq. (5) after integration.

We see that at $g = 1/k$ the Hamiltonian is determined positively. Moreover, it equals zero at the solutions of the self-duality equation,

$$D_1 \Psi = iD_2 \Psi.$$ (6)

Solutions of the self-duality equation, (6), have the form [7] of

$$\Psi (r, \vartheta) = \frac{2n\sqrt{k}}{r} \left( \left( \frac{r}{r_0} \right)^n + \left( \frac{r_0}{r} \right)^n \right)^{-1} e^{i(1-n)\vartheta},$$

$$a_i (r, \vartheta) = \frac{2n\epsilon_{ij}x_j}{r^2} \left( 1 + \left( \frac{r_0}{r} \right)^2 \right)^{-1}.\quad (7)$$

In Eqs. (7) and (8) $r_0$ is arbitrary scale, integer $n \geq 2$ determines flux $\Phi = \int d^2 x f_{12} = 4\pi n$ of the statistic gauge field and whole number of solutions.

The condition of $n \geq 2$ follows from the requirement that the conformal generator should be finite [7]. Using the conformal transformation, $t^{-1} \rightarrow t^{-1} + a$, $r \rightarrow r/(1 + at)$ and static solutions (4), one can obtain time-dependent distribution

$$\Psi (t, r) = (1 - at)^{-1} \exp[-iar^2/2(1 - at)] \Psi (r/(1 - at)).$$ (9)

Multi-soliton solution of Eqs. (6) contains [7] $4n$ parameters of the parameters that set the phase, scale, and two spatial coordinates for each of the $n$ solitons. For all those field configurations at $g = 1/k$ the energy equals zero. (Note that the Chern-Simons term in (3) contains first derivatives and yields, therefore, no contribution to the energy). That means that $4n$ parameters together with the
value of coefficient $k$ set dimensionality of the space of degeneration of the ground state, and, thus, the number of zeroth modes.

Calculation of the partition function,

$$Z = \sum_{\text{all states}} e^{-\beta H},$$

(10)

which is equal in this case to the statistic weight, is reduced to calculation of the contribution, which arises from the zeroth modes, to the determinant of the operator that describes small Gaussian deviations from solutions (7) and (8). The problem of calculating the statistic weight has not been solved yet and lies beyond the scope of the present paper, so we will concentrate on the problem of collapse stabilization.

In terms of calculating the partition function the dynamic details of field distributions (7) and (8) are not very important. Of essential significance are only the number of independent parameters and the fact that the Hamiltonian is bounded from below. Therefore, from the statistical point of view the problem of the influence of the gauge field on specific realization of field $\Psi$ becomes less acute. The main role of the gauge field is transferred to condition $g = k^{-1}$ that limits the Hamiltonian from below. That condition of compensation in the space of parameters is supplementary to the self-duality equations, which are usually sufficient for description of field distributions in the state with zero energy, with the topologic charge differing from zero, and with finite action. Let us clear up the reason for compensation of the nonlinear term, $-|\Psi|^4$ in Hamiltonian (4) in more detail.

Nonlinear term $-|\Psi|^4$ in the Hamiltonian exists as a quasi-classic limit of the interaction Hamiltonian for the second-quantized description of Bose particles with the pairwise attracting $\delta$-function interaction. It is well known from energy estimations from below [14, 15] that there is no prohibition for a collapse in the Bose system. Meanwhile, for fermions Pauli’s principle of exception limits the energy from below due to fermion repulsion [13] and prevents a collapse.

In $2+1D$ systems the classification of field distributions is known as fractional or intermediate statistics and is built in correspondence to irreducible representations of the braid group (see, e.g., Ref. [14]). Its Abelian and, in the general case, non-Abelian irreducible representations are parametrized by the phase $\exp(2\pi i \Phi)$, determined by flux $\Phi$ of the gauge field. In the non-Abelian case irreducible one-dimensional representations that belong to a discrete center of the group are parametrized by flux
\( \Phi = \frac{P}{Q} \) with mutually prime numbers \( P \) and \( Q \). The flux of the gauge field describes coherent adiabatic rotation dynamics of Bose particles with hard cores. The condition of the hard core reflects the effect of *distinguishability of loop links* constructed by the linked world lines projected on a plane. The chiral character of motion, which arises due to spontaneous breaking of the angular momentum projection being orthogonal to the plane, leads to partial Bose repulsion depending on the linking parameter. That effect of strong phase correlations with the repulsive character inherent for them prevents a collapse under the condition of compensation, \( g = k^{-1} \).

Are there any outstanding values of \( k \) in equation \( g = k^{-1} \)? In other words, what is the value of \( g \) under the conditions of collapse suppression? To obtain an answer, let us turn to analysis of symmetry properties of system (3).

Besides a wide choice of spatio-temporal symmetries [7], the Chern-Simons sector of model (3) contains an algebraic structure of Hopf’s deformed algebra. In terms of physics the latter describes the laws of addition of dynamic integrals of motion for fusion of strongly correlated fluctuations. As the deformation parameter, \( q \), of Hopf’s algebra in our case serves the adiabatic phase, \( \exp(2\pi i \Phi) \), which is equal, at \( \Phi = \frac{P}{Q} \), to the \( Q \)th root of unity. Further we will discuss the connection between numbers \( Q \) and \( k \), and now we will emphasize specific character of representations of Hopf’s deformed algebra with the deformation parameter being a root of unity.

For \( q \) being a root of unity, Hopf’s \( q \)-deformed algebra contains a rich center [18] that “maintains” its irreducible representations. By that, in the general case the central elements are not independent but satisfy some polynomial relations [18]. There is also come consequence of \( q \)-deformed algebras obtained from the initial one by division into invariant relations of central elements.

Generally speaking, there is no reversible universal \( R \)-matrix of Hopf’s deformed algebra at roots of unity. However, for the above quotients there are always transpositions of tensor products of representations, i.e., there are always \( R \)-matrices on representations. Some of those quotients have finite dimensionality and a universal \( R \)-matrix.

In a seminal paper [19] Arnaudon has obtained the answer to the following question: on which condition are the different quotients of initial Hopf’s algebra Hopf-equivalent? In the case of them being equivalent, the universal \( \mathcal{R} \)-matrix of
some of them can be transformed into a universal $\mathcal{R}$-matrix of the others. In Ref. [13] it was proved that it happens only when $Q = 4$. Moreover, Ref. [19] gave accurate expressions for the automorphisms and for the transformed universal $\mathcal{R}$-matrices for that case.

When studying the connection between the conformal field theory and Chern-Simons’s topological theory it was stated [20] that for $SU(N)$ of the gauge field $Q = k + N$. For $Q = 4$ and minimal limit, $N = 2$, of the non-Abelian symmetry we have $k = 2$. In the absence of the effect of shift to the value of $N = 2$ coefficient $k$ would have equaled 4. Note also that at $k = 2$ both the angular momentum, $J$, measured in units of $2\pi$, and the number of particles, $N = \int |\Psi|^2 \, d^2x = 4\pi |k| n$, coincides with the number $4n$ of zeroth modes.

An integral value of coefficient $k$ for the non-Abelian Chern-Simons term does not cause any objections, since it is dictated by invariance of the additional term in it as related to global gauge transformations. We deal with an Abelian gauge field, for which there is no complete and final proof of coefficient $k$ being an integer. Realization of various possibilities in that respect depends primarily on the boundary conditions related to global gauge transformations determined over all the $2 + 1D$ manifold, i.e., on its topological structure. In that situation of a certain arbitrary rule it is natural to see an Abelian gauge field as a field belonging to the center of its non-Abelian $SU(2)$ predecessor, and use, having applied that reduction, $N = 2$.

Singling out from the non-Abelian $SU(N)$ gauge theory its Abelian discrete center $Z_n$ is given in Ref. [17]. Some other arguments for integral (even) values of $k$ in the Abelian case are contained in Refs. [21].

Even in the case of $k = 2$ it is still unclear, what is the value of $g$, since seemingly, applying only to Eq. (1), we could always change the scales so that coefficient $g$ would equal any number. So, let us supplement the motion equation limited with solutions of the self-duality equation [6],

$$\nabla^2 \ln |\Psi|^2 = -\frac{2}{k} (2kg - 1)|\Psi|^2, \quad (11)$$

with the constraint that reflects the Gauss law, $\varepsilon_{\alpha\beta} f_{\alpha\beta} = k^{-1} |\Psi|^2$, i.e.,

$$\nabla^2 \ln |\Psi|^2 = -\frac{2}{k} |\Psi|^2. \quad (12)$$

We see that at $k = 2$ Eqs. (11) and (12) are compatible only in the case of $g = 1/2$. The same value of parameters satisfy the condition of $kg = 1$ of collapse prevention.
3. Conclusion

When \( d = 2, \ g = 1/2, \) and

\[
2i\partial_t \Psi + \nabla^2 \Psi + |\Psi|^2 \Psi = 0,
\]

Eq. (13) can be seen as a local limit of description of more realistic situations. One of such examples,

\[
2i\partial_t \Psi + \nabla^2 \Psi + \rho \Psi = 0,
\]

\[
\rho_{tt} - \nabla^2 \rho + \nabla^2 |\Psi|^2 = 0,
\]
describes sound motion of density \( \rho \) in the presence of an external ponderomotive force \( \rho \). Within a weak limit (\( |\Psi|^2 \to 0 \)) distribution of density relaxes towards the quasi-equilibrium value, \( \rho = |\Psi|^2 \), and we come back to Eq. (13).

Another situation arises when scale \( l \) of variation of density \( \rho \) does not coincide with the scale of the field, \( \Psi \). It is corresponded by the system of equations that includes Eq. (14) and equation

\[
\rho - l^2 \nabla^2 \rho - |\Psi|^2 = 0.
\]

At \( l \gg 1 \) Eqs. (14) and (16) describe such wide-scale distributions of concentration, \( \rho \), that scale \( l \) houses several solitary distributions of field \( \Psi \).

This situation is corresponded by the integral connection of field \( \rho \) with field \( \Psi \) in Eq. (14), and hence, by effective weakening of the nonlinearity degree, \( \alpha \). Therefore, at \( l \neq 0 \) and \( d = 2 \) the nonlinearity degree \( \alpha < 2 \), and the problem of collapse does not arise.

However, in connection with the possibility of sound generation (see Eqs. (14) and (15)) another difficulty appears. The matter is that in some Chern-Simons system there is no soft mode. Such a situation occurs, e.g., in the systems, in which the Fractional Quantum Hall Effect is realized. The sound mode is replaced by a quasi-particle with a finite gap in the long-wave limit, a so-called magnetoroton. Its existence reflects rigidness of the system of random and, at the same time, coherent vortex distributions of the field.

In order to retain phase coherence, which may be broken by sound reduction of the described topologic rigidness, in our case, one has to neglect the first term in Eq. (15) to reconstruct local connection \( \rho = |\Psi|^2 \). It follows from comparison of the
terms in Eqs. (14) and (15) that it is possible when $r_0 \ll 1$ and $|\Psi|^2 r_0^2 \sim 1$. Those conditions may be satisfied only in the case of a great number, $n \gg 1$, of solitons, and, therefore, at a sufficiently large number, $N \sim n$, of particles. For the statistical description the latter condition is assumed satisfied.

Finally, we would like to make the following comment: as it has been said already, the universal integer, $n$, numbers both the solitons and their angular momenta. For field distributions, in Eqs. (7) and (8) the angular momentum $n \geq 2$. The monopole value, $n = 0$, is excluded, since this value corresponds to the absence of vortices and consideration of the plane as a non-punctured manifold. On the other hand, the traditional collapse [1, 2] is singularity of the solutions of the nonlinear Schrödinger equation in the $n = 0$ channel. It is possible that the case of $n = 0$ corresponds to the phase state with vortex singularities compensated due to disorientation, and the collapse is prevented, e.g., due to saturation of nonlinearity. However, in the framework of the considered model without nonlinearity saturation we have to assume that $n \geq 2$.

In conclusion, we used hidden symmetry of the gauged 2+1D nonlinear Schrödinger equation relative to action of $q$-deformed Hopf algebra with deformation parameter $q = \exp \left[2\pi i/(2 + k)\right]$, and gave arguments for the fact that at the minimum possible value, $k = 2$, the value of coupling constant, $g$, equals $1/2$, for which the condition $gk = 1$ of collapse suppression is fulfilled.

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