Surface tension and viscosity of nuclei in liquid drop model

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Abstract. An analytical solution for the capillary oscillations of the charged drop in dielectric medium obtained with taking into account the damping due to viscosity. The model has been applied for the estimation of even-even spherical nuclei surface tension and nuclei viscosity. Attenuation factor to nuclear capillary oscillation frequency ratio has been found.

1. Introduction
In present the interest to the viscous hydrodynamics of strongly coupled matter increases [1, 2]. Liquid drop model (LDM) successfully being used for semi-empirical formulation of surface and coulomb terms in Bethe–Weizsacker mass formula [3, 4]. LDM is foundation for 5D harmonic oscillator collective nuclear model [5], which allows to interpret vibration excitations of near-spherical nuclei. In the original study of Rayleigh, devoted to drop capillary oscillations [6], the energy dissipation due to viscosity was not taken into account. In this paper an analytical solution for the capillary oscillations of the nucleus is presented with taking into account the damping due to viscosity and nuclear medium polarizability. The model has been applied for interpretation of even-even spherical nuclei vibration spectra. The ratio for attenuation decrement to the oscillation frequency was obtained which gives a limit for viscosity of the spherical nuclear quadrupole oscillations \( \eta \ll 1 \text{ MeV fm}^{-2} \text{ c}^{-1} \), meantime viscosity for octupole oscillations has significant values which grows from 2.8 up to 11.7 MeV fm\(^{-2}\) c\(^{-1}\) with increase of atomic number from 106 to 218.

2. Rayleigh’s coordinates for harmonic drop oscillations
Author [6] found that the normal coordinates for the drop capillary oscillations are coefficients \( a_n \) in expansion for drop surface radius over the Legendre polynomials

\[
R(\theta) = \sum_{n=2}^{\infty} a_n P_n(\cos \theta),
\]

where \( \theta \) is the polar angle shown in figure 1. The condition of drop volume invariability during deformation gives the relation between the coefficients \( a_n \) in the expansion (1).

\[
a_0 = a - \frac{1}{a} \sum_{n=2}^{\infty} \frac{a_n^2}{2n+1},
\]
Figure 1. Coordinates and radii used. $R$ and $a$ are radii of deformed and original drop respectively; $a_0$ is the radius corresponding to the first term in expansion (1); $\theta$ is the polar angle.

where $a$ is the radius of undeformed nucleus. The nuclear radius deviation from its unperturbed value $\Delta R$ can be represented in the forms

$$\Delta R = R - a = \delta R - \Delta a, \quad (3)$$

where $\Delta a = a - a_0$ and

$$\delta R(\theta) = \sum_{n=2}^{\infty} a_n P_n(\cos \theta), \quad (4)$$

The radius dependence from azimuthal variable $\phi$ is not essential for our task. LDM uses azimuth dependant coordinates $[4]$

$$R(\theta, \phi) = a \left(1 + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta, \phi)\right). \quad (5)$$

The model of 5D oscillator uses $l = 2$ only. To express all below energy relations through $\alpha_{lm}$ its enough to make following substitution

$$a_n^2 \rightarrow \frac{a^2}{4\pi} \sum_{m=-l}^{l} |\alpha_{lm}|^2. \quad (6)$$

3. Electrostatic energy of weakly deformed spherical nucleus in dielectric medium

Now we calculate the deviation of electrostatic energy of drop with charge $Ze$ in dielectric medium due to small deformation. The electric nuclear permittivity $\varepsilon_1$ and surrounding pion cloud permittivity $\varepsilon_2$ differ little from one $[7]$. Nevertheless, for general case we consider the case
of polarizable nuclear and outer volume. For spherical nucleus unperturbed electric potential equals
\[
\phi_0(r) = \begin{cases} 
\frac{1}{\epsilon_2} + \frac{1}{2\epsilon_1} \frac{Ze}{a} - \frac{1}{2\epsilon_1} \frac{Ze^2 r^2}{a^2}, & 0 < r < a; \\
\frac{Ze^2 r^2}{2\epsilon_2}, & a < r.
\end{cases} \tag{7}
\]

We have to solve Poisson equation that connects deviations of electrostatic potential and charge density
\[
\Delta (\epsilon \delta \phi(r)) = -4\pi \delta \rho_c(r). \tag{8}
\]
The charge density deviation due to deformation can be written as
\[
\delta \rho_c(r) = \rho_{c0} \left( \Theta(R(\theta) - a) - \Theta(a - r) \right), \tag{9}
\]
where \( \Theta(r) \) is Heaviside step function, \( \rho_{c0} \)—nuclear charge density.

For sufficiently small oscillation amplitudes \( a_l \) with accuracy \( a_l^2 \) the nuclear charge density variation can be presented through delta function and its derivative
\[
\delta \rho_c(r) = \rho_{c0} \delta(a - r) \Delta R(\theta) - \frac{1}{2} \rho_{c0} (\Delta R)^2 \frac{d}{dr} \delta(a - r). \tag{10}
\]
The second term guarantees the fulfillment of the charge conservation condition with considered accuracy
\[
\int \delta \rho_c(r) dV = 0. \tag{11}
\]
Electrostatic potential variation for deformed nucleus inside \( \delta \phi \) and outside \( \tilde{\delta} \phi \) the radius \( a \) can be represented in the forms satisfying Laplace equation
\[
\delta \phi = \sum_{l=0}^{\infty} b_l r^l P_l(\cos \theta), \tag{11}
\]
\[
\tilde{\delta} \phi = \sum_{l=0}^{\infty} c_n r^{l+1} P_l(\cos \theta). \tag{12}
\]

Coefficients in potential expansions (11) and (12) have to satisfy the boundary condition of continuity on the drop surface
\[
\delta \phi|_{r=a+0} = \delta \phi|_{r=a-0}, \tag{13}
\]
which connects inside and outside potential coefficients
\[
c_l = a^{2l+1} b_l. \tag{14}
\]
The boundary condition for potential derivatives has the form
\[
\epsilon_2 \frac{\partial \tilde{\delta} \phi}{\partial r}
\bigg|_{r=a+0} - \epsilon_1 \frac{\partial \delta \phi}{\partial r}
\bigg|_{r=a-0} = -4\pi \rho_{c0} (\Delta R - \Delta a). \tag{15}
\]
Substitution of expansions (11) and (12) in (15) and using (14) gives coefficients \( b_l \). For \( l = 0 \),
\[
b_0 = \frac{4\pi}{\epsilon_2} \rho_{c0} a \Delta a \tag{16}
\]
and for \( l > 0 \)
\[
b_l = \frac{4\pi a^{1-l} \rho_{c0}}{\epsilon_1 l + \epsilon_2 (l+1)} a_l. \tag{17}
\]
Electrostatic energy deviation for deformed nucleus is
\[
\delta U_c = \frac{1}{2} \int \rho_{c0}(r) \delta \phi(r) dV + \frac{1}{2} \int \delta \rho_c(r) \delta \phi(r) dV. \tag{18}
\]
The integral \( \int \delta \rho_c(r) \delta \phi(r) dV = 0. \) Substitution of (3), (6), (11),(12) in (18) gives
\[
\delta U_c = \frac{3 Z^2 e^2}{2 \epsilon_2 a^3} \sum_l \frac{l \epsilon_1 + (l - 2) \epsilon_2}{(2l + 1)(l \epsilon_1 + (l + 1) \epsilon_2)} a_l^2. \tag{19}
\]
4. The equation of motion for nuclear capillary oscillation
For irrotational liquid motion the kinetic energy equals to [6]

\[ K = 2\pi \rho a^3 \sum_{l=2}^{\infty} \frac{a_l^2}{l(2l+1)}, \]  

(20)
where \( \rho \) is nuclear matter density. The change of potential energy associated with the deviation from the spherical shape of the drop is defined by the surface tension and can be written as [6]

\[ \delta U_s = 2\pi \sigma \sum_{l=2}^{\infty} \frac{(l-1)(l+2)}{2l+1} a_l^2, \]  

(21)
where \( \sigma \) is the surface tension of the liquid. For kinetic energy viscous dissipation can be expressed as [8]

\[ \dot{K} = -\eta \int \nabla(v^2) dS, \]  

(22)
where \( \eta \) is the dynamic viscosity of the fluid. Calculation of dissipative function \( F = -\frac{1}{2} \dot{K} \) [9,10]

\[ F = 4\pi \rho a \sum_{l=2}^{\infty} \frac{l-1}{l} a_l^2. \]  

(23)
Euler–Lagrange equations for the normal coordinates \( a_l \) are the following

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{a}_l} - \frac{\partial L}{\partial a_l} = -\frac{\partial F}{\partial \dot{a}_l}, \]  

(24)
where Lagrangian \( L = K - \delta U_s - \delta U_c \). Substitution of (20),(21) and (23) into (24) gives the independent equations of motion for each normal coordinate \( a_l \)

\[ \ddot{a}_l + 2\gamma_l \dot{a}_l + \omega_{0l}^2 a_l = 0, \]  

(25)
where

\[ \omega_{0l}^2 = \frac{l}{a^3\rho} \left( (l-1)(l+2)\sigma - \frac{3Z^2e^2}{4\pi a^3 \varepsilon_2 (l\varepsilon_1 + (l-2)\varepsilon_2)} \right), \]  

(26)
and damping factor [9,10]

\[ \gamma_l = \frac{\eta}{\rho a^2} (l-1)(2l+1). \]  

(27)
The capillary oscillations frequency squared is

\[ \omega_l^2 = \omega_{0l}^2 - \gamma_l^2. \]  

(28)
5. Comparison with experiment
In case \( \varepsilon_1 = \varepsilon_2 = 1 \) expression (26) coincides with well known relation of LDM theory for hydrodynamic nuclear frequency oscillations [3].

As is seen from (28) the oscillation frequency decreases due to viscosity dissipation. The value of surface tension and viscosities \( \eta_2, \eta_3 \) can be obtained by applying equations (26)-(28) to quadrupole \( \omega_2 \) and octupole \( \omega_3 \) vibration frequencies

\[ \eta_3 = \frac{3}{4} \frac{M}{\pi a} \sqrt{\frac{35\omega_0^2 - 4\omega_2^2 + \frac{60E}{784}}{784 - 375\omega_2^2}}, \]  

(29)
### Table 1. Parameters for quadrupole and octupole oscillations of near-spherical nuclei.

| Nucleus | $a$, fm | $E_i$, MeV | $J_i^π$ | $\sigma$, MeV fm$^{-2}$ | $\eta_3$, MeV fm$^{-2}$ c$^{-1}$ | $\gamma_3/\omega_3$ |
|---------|---------|----------|--------|------------------|------------------------|------------------|
| $^{106}_{46}$Pd | 0 | 0.51 | 2$^+$ | 0.80 | 3.916 | 2.82 | 0.72 |
| $^{108}_{46}$Pd | 1.12 | 0.78 | 2$^+$ | 2.046 | 3$^-$ | 6.76 | 3.3 |
| $^{204}_{82}$Pb | 0.434 | 2$^+$ | 0.79 | 1.582 | 2$^+$ | 1.44 | 3.0 |
| $^{218}_{86}$Rn | 0.899 | 2$^+$ | 1.46 | 2.046 | 3$^-$ | 7.87 | 1.46 |
| $^{218}_{86}$Rn | 0.324 | 2$^+$ | 1.42 | 0.653 | 4$^+$ | 1.42 | 11.7 | 13.9 |

where $M$—nuclear mass, $E_c = \frac{3Z^2e^2}{5a}$, $\beta_{23} = \frac{\eta_2^2}{\eta_3^2}$.

Calculated values of surface tension and viscosity for even-even spherical nuclei are given in Table 1. Data for nuclear radii and energy spectrum were taken from [11] and [12]. The assumption was that $\eta_2 \ll \eta_3$. As is seen from the table, viscosities for octupole oscillations lay in the interval 2.8–11.7 MeV fm$^{-2}$ c$^{-1}$ for nuclei from $^{106}_{46}$Pd to $^{218}_{86}$Rn. Surface tension grows with mass number for the same nuclei from 0.8 to 1.46 MeV fm$^{-2}$. The Bethe–Weizsacker mass formula gives for nuclear surface tension the value

$$\sigma_0 = \frac{\alpha_s}{4\pi r_0^2},$$

where surface coefficient $\alpha_s = 17.8$ MeV [13] and $r_0 = 1.2$ fm. The range of low-lying collective excitations of even-even spherical nuclei are characterized by an energy gap, as in the case of superfluidity, which is manifested in our analysis in the absence of appreciable viscosity for quadrupole excitations.

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