Rotating and twisting locally rotationally symmetric spacetimes: a general solution

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In this paper we derive a general solution for the most general rotating and twisting locally rotationally symmetric spacetimes. This is achieved in three steps. First we decompose the manifold via 1+1+2 semi-tetrad formalism that yields a set of geometrical and thermodynamic scalars for the spacetime. We then recast the Einstein field equations in terms of evolution and propagation of these scalars. It is then shown that this class of spacetimes must possess self similarity and we use this property to solve for these scalars, thus obtaining a general solution. This solution has a number of very interesting cosmological or astrophysical consequences which we discuss in detail.

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I. INTRODUCTION

Locally Rotationally Symmetric (LRS) spacetimes are those that possess a continuous isotropy group at each point which generally implies the existence of a multiply-transitive isometry group acting on the spacetime manifold \([1, 2]\). It is well known that isotropies around a point in any spacetime with a fluid can be a 3-dimensional or 1-dimensional subgroup of the full group of isometries; they necessarily leave the normalised 4-velocity of the matter flow invariant. The 3-d case implies isotropy at every point, yielding the Friedmann-Lemaître-Robertson-Walker (FLRW) models, while the 1-d case corresponds to anisotropic and in general spatially inhomogeneous models. These have a preferred spacelike direction \(e^a\) orthogonal to the fluid flow 4-vector \(u^a\): all spatial directions orthogonal to \(e^a\) and \(u^a\) are geometrically identical.

In the case of a perfect fluid, these spacetimes are split into three classes as described in Section IV depending on whether the vorticity component \(\Omega\) along the direction \(e^a\) of the fluid and 2-dimensional twist \(\xi\) of the vector field \(e^a\) are zero or not (they cannot both be non-zero in this case). However for an imperfect fluid - for example if there is an entropy flux - both can be non-zero. In a previous paper [3] we obtained a set of field equations and integrability conditions for the imperfect fluid case. We also proved that the LRS spacetimes with nonzero rotation and spatial twist must be self similar. In this paper, we extend that work by obtaining a general solution to the field equations for this situation. This is achieved by using the property of

the self similarity. Also we show that we may specify an equation of state for the isotropic pressure at an initial Cauchy surface for particular applications.

In Section II, the semi-tetrad formalism used is introduced in a general form. In Section IV it is restricted to the case of LRS fluid spacetimes. In Section V a reduced set of field equations is obtained with self-similar variables. In section VI the general solution to the field equations is obtained for the case of LRS fluids with non-zero rotation and spatial twist. In Section VII their properties are discussed in both cosmological and astrophysical scenarios.

II. LRS SPACETIMES IN SEMI-TETRAD FORMALISM

Due to the symmetries of LRS spacetimes, a 1+1+2 semi tetrad covariant formalism (which is a natural extension of local 1+3 decomposition), is well suited for describing the geometry, as in this formalism the field equations become a set of coupled differential equations in covariantly defined scalar variables. In the next two subsections we briefly discuss the 1+3 and semitetrad formalisms, and in the third subsection we present the field equations in terms of covariantly defined geometrical scalar variables.

A. 1+3 decomposition of spacetime

The 1+3 decomposition provides a covariant description of the spacetime in terms of 3-vectors, scalars and projected symmetric trace-free (PSTF) 3-tensors [4]. This is helpful for understanding various physical and geometrical aspects of relativistic fluid flows. With respect to a timelike congruence, the spacetime can be locally decomposed into time and space parts. Such a timelike congruence can be defined by the matter flow lines, with
the four-velocity defined as
\[ u^a = \frac{dx^a}{d\tau} \text{, with } u^a u_a = -1, \tag{1} \]
where \( \tau \) is proper time along the flow lines. Given the four-velocity \( u^a \), we have unique parallel and orthogonal projection tensors
\[ U^a_b = -u^a u_b, \tag{2} \]
\[ h^a_b = g^a_b + u^a u_b, \tag{3} \]
where \( h^a_b \) is the projection tensor that projects any 4-d vector or tensor onto the local 3-space orthogonal to \( u^a \) which has volume element \( \epsilon_{abc} \). 

From this, it follows that we have two well defined directional derivatives. The vector \( u^a \) is used to define the covariant time derivative along the flow lines (denoted by a dot) for any tensor \( S^{a..b..c..d..} \), given by
\[ \dot{S}^{a..b..c..d..} = u^c \nabla_c S^{a..b..c..d..}. \tag{4} \]
The tensor \( h^a_{bc..d..} \) is used to define the fully orthogonally projected covariant derivative \( D^a \) for any tensor \( S^{a..b..c..d..} \):
\[ D^a S^{a..b..c..d..} = h^a_f h^p_c..h^b_g h^q_d h^r_e \nabla_p S^{f..g..r..q..}. \tag{5} \]
with total projection on all free indices. In this way, the covariant derivative of \( u^a \) can be decomposed as
\[ \nabla_a u_b = -u_a A_b + \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \epsilon_{abc} \omega^c. \tag{6} \]
Here \( A_b = \dot{u}_b \) is the acceleration, \( \Theta = D_a u^a \) represents the expansion of \( u_a, \sigma_{ab} = (h^c_{\ (ab}) h^d_{\cdots c}) D_c u_d \) is the shear tensor that denotes the rate of distortion and \( \omega^c \) is the vorticity vector denoting the rotation.

The Weyl tensor is split relative to \( u^a \) into the electric and magnetic Weyl curvature parts as
\[ E_{ab} = C_{abcd} u^b u^d = E_{(ab)}, \tag{7} \]
\[ H_{ab} = \frac{1}{2} \epsilon_{abc} C^{de} b_{ce} u^d = H_{(ab)}, \tag{8} \]
where angle brackets represent the orthogonal symmetric trace-free part (so \( \sigma_{ab} = \sigma_{(ab)} \)).

The energy momentum tensor of matter can be decomposed similarly as
\[ T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}, \tag{9} \]
where \( p = (1/3) h_{ab} T_{ab} \) is the isotropic pressure, \( \mu = T_{ab} u^a u^b \) is the energy density, \( q_a = q_{(a)} = -h^c a T_{cd} u^d \) is the 3-vector that defines the heat flux, and \( \pi_{ab} = \pi_{(ab)} \) is the anisotropic stress.

B. 1+1+2 decomposition of spacetime

The 1+1+2 decomposition is a natural extension of the 1+3 decomposition in which the 3-space is further decomposed with respect to a given spatial direction, i.e., we now have another split along a preferred spatial direction such as the case which occur in LRS models \([8-10]\). We choose a spacelike vector field \( e^a \) such that
\[ e^a e_a = 0 \text{ and } e^a e_a = 1. \tag{10} \]
The new projection tensor is given by
\[ N^a_b \equiv h^a_b - e_a e^b = g^a_b + u_a u_b - e_a e^b. \tag{11} \]
This tensor projects vectors onto local 2-spaces orthogonal to both \( u^a \) and \( e^a \), defined as sheets. Thus
\[ e^a N_{ab} = 0 = u^a N_{ab}, \quad N^a a = 2. \tag{12} \]

This spacelike vector now naturally introduce two new derivatives, which for any tensor \( \psi_{a..b..c..d..} \),
\[ \dot{\psi}_{a..b..c..d..} \equiv e^f D_f \psi_{a..b..c..d..}, \tag{13} \]
\[ \delta_f \psi_{a..b..c..d..} \equiv N^a_p..N^b_q..N^c_i..N^d_j D_f \psi_{p..q..i..j..}. \tag{14} \]
The derivative \( \psi^a_{b..c..d..} \) along the \( e^a \) vector field in the surfaces orthogonal to \( u^a \) is called the hat-derivative, while the derivative \( \delta_f \psi_{a..b..c..d..} \) projected onto the sheet is called the \( \delta \)-derivative. This projection is orthogonal to \( u^a \) and \( e^a \) on every free index.

In the \( 1+1+2 \) splitting, the 4-acceleration, vorticity and shear split in this way as
\[ \dot{u}^a = A e^a + A^a, \tag{15} \]
\[ \omega^a = \Omega e^a + \Omega^a, \tag{16} \]
\[ \sigma_{ab} = \Sigma (e_a e_b - \frac{1}{2} N_{ab}) + 2 \Sigma (e_a e^b) + \Sigma_{ab}. \tag{17} \]
For the electric and magnetic Weyl tensors we get
\[ E_{ab} = \mathcal{E} (e_a e_b - \frac{1}{2} N_{ab}) + 2 \mathcal{E} (e_a e^b) + \mathcal{E}_{ab}, \tag{18} \]
\[ H_{ab} = \mathcal{H} (e_a e_b - \frac{1}{2} N_{ab}) + 2 \mathcal{H} (e_a e^b) + \mathcal{H}_{ab}. \tag{19} \]
Similarly, the fluid variables \( q^a \) and \( \pi_{ab} \) are split as follows
\[ q^a = Q e^a + Q^a \tag{20} \]
\[ \pi_{ab} = \Pi (e_a e_b - \frac{1}{2} N_{ab}) + 2 \Pi (e_a e^b) + \Pi_{ab}. \tag{21} \]

By decomposing the covariant derivative of \( e^a \) in the directions orthogonal to \( u^a \) into its irreducible parts, we get
\[ D_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} + \zeta_{ab}, \tag{22} \]
where
\[ a_a \equiv e^d D_d e_a = \dot{e}_a, \tag{23} \]
\[ \phi \equiv \delta^a e_a \tag{24} \]
\[ \xi \equiv \frac{1}{2} \epsilon_{ab} \delta_a e_b \tag{25} \]
\[ \zeta_{ab} \equiv \delta_{(a} e_{b)} \tag{26} \]
III. LRS SPACETIMES AND FIELD EQUATIONS

The basic property of fluid filled LRS spacetimes is that there exists a unique, preferred spatial direction at every point, covariantly defined, which creates a local axis of symmetry. Hence the $1+1+2$ decomposition described in the previous section is ideally suited for the study of these spacetimes as we can immediately see that if we choose the spacelike unit vector $e^a$ along the preferred spatial direction, then by symmetry all the sheet vectors and tensors vanish identically:

$$A^a = \Omega^a = \Sigma_a = \mathcal{E}_a = H_a = Q^a = A_a = a_a = 0, \quad \Sigma_{ab} = \mathcal{E}_{ab} = H_{ab} = \Pi_{ab} = \zeta_{ab} = 0. \quad (27)$$

Thus the remaining variables are

$$D_1 := \{A, \Theta, \Omega, \Sigma, \mathcal{E}, H, \mu, p, Q, \Pi, \phi, \xi\} \quad (29)$$

$$D_{\text{matter}} := \{\mu, p, Q, \Pi\}, \quad (31)$$

are the matter variables that completely specify the energy momentum tensor of the matter. On the other hand

$$D_{\text{geometry}} := \{A, \Theta, \Omega, \Sigma, \mathcal{E}, H, \phi, \xi\}, \quad (32)$$

are the geometrical variables. By decomposing the Ricci identities for $u^a$ and $e^a$ and the doubly contracted Bianchi identities, we then get the following field equations for LRS spacetimes.

Evolution:

$$\dot{\phi} = \left(\frac{3}{2}\Theta - \Sigma\right)\left(A - \frac{1}{2}\phi\right) + 2\xi\Omega + Q, \quad (33)$$

$$\dot{\xi} = \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\xi + \left(A - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\mathcal{H}, \quad (34)$$

$$\dot{\Omega} = A\xi + \Omega\left(\Sigma - \frac{2}{3}\Theta\right), \quad (35)$$

$$\dot{\mathcal{H}} = -3\mathcal{E}\xi + \left(\frac{2}{3}\Sigma - \Theta\right)\mathcal{H} + \Omega Q + \frac{1}{2}\xi\Pi, \quad (36)$$

Propagation:

$$\dot{\phi} = -\frac{1}{3}\phi^2 + \left(\frac{3}{2}\Theta + \Sigma\right)(\frac{3}{2}\Theta - \Sigma) + 2\xi^2 - \frac{2}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi, \quad (37)$$

$$\dot{\xi} = -\phi\xi + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega, \quad (38)$$

$$\dot{\Sigma} - \frac{4}{3}\dot{\Theta} = -\frac{2}{3}\phi\Sigma - 2\xi\Omega - Q, \quad (39)$$

$$\dot{\mathcal{H}} = \left(\frac{1}{3}\Sigma - \frac{1}{3}\Theta\right)\Omega, \quad (40)$$

$$\dot{\mathcal{H}} - \frac{1}{2}\dot{\mu} + \frac{1}{2}\Pi = -\frac{1}{3}\phi\left(\mathcal{E} + \frac{1}{3}\Pi\right) + 3\mathcal{H} + \frac{1}{3}(\frac{1}{3}\Sigma - \frac{1}{3}\Theta)\Omega, \quad (41)$$

$$\dot{\Theta} = -3\xi\Sigma + \frac{1}{3}\left(\frac{1}{3}\phi + A\right)\Pi - \left(\frac{1}{3}\Theta + \Sigma\right)Q - \left(\mu + \Lambda\right)A, \quad (42)$$

Propagation/evolution:

$$\dot{A} - \dot{\Theta} = -\left(A + \phi\right)A + \frac{1}{3}\Theta^2 + \frac{1}{3}\Sigma^2 - 2\mu^2 + \frac{1}{3}(\mu + 3p - 2\Lambda), \quad (43)$$

$$\dot{\mu} + \dot{Q} = -\Theta(\mu + p) - (\phi + 2A)Q - \frac{1}{3}\Sigma\Pi, \quad (44)$$

$$\dot{Q} + \dot{\mu} + \dot{\Pi} = -\frac{1}{3}\left(\frac{1}{3}\phi + A\right)\Pi - \left(\frac{1}{3}\Theta + \Sigma\right)Q - \left(\mu + \Lambda\right)A, \quad (45)$$

$$\dot{\Sigma} - \frac{4}{3}\dot{\Theta} = \frac{1}{3}(2A - \phi)A - \left(\frac{1}{3}\Theta + \frac{1}{3}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 - \mathcal{E} + \frac{1}{3}\Pi, \quad (46)$$

$$\dot{\mathcal{H}} + \frac{1}{2}\dot{\Pi} + \frac{1}{2}\dot{Q} = + \left(\frac{1}{3}\Sigma - \Theta\right)\mathcal{E} - \frac{1}{2}(\mu + p)\Sigma - \frac{1}{3}\left(\frac{1}{3}\Theta + \frac{1}{3}\Sigma\right)\Pi + 3\xi\mathcal{H} + \frac{1}{3}(\frac{1}{3}\phi - 2A)Q, \quad (47)$$

Constraint:

$$\mathcal{H} = 3\Sigma - (2A - \phi)\Omega. \quad (48)$$

IV. MOST GENERAL CLASS OF LRS SPACETIMES

As described in [3], if we consider a perfect fluid form of matter with $Q = \Pi = 0$, then the propagation equations evolve consistently in time if and only if

$$\Omega\xi = 0. \quad (49)$$

The above relation then naturally divides perfect fluid LRS spacetimes in three distinct subclasses [2, 5]:

1. LRS class I: ($\Omega \neq 0, \xi = 0$) These are stationary inhomogeneous rotating solutions.

2. LRS class II: ($\xi = 0 = \Omega$) These are inhomogeneous orthogonal family of solutions that can be both static or dynamic. Spherically symmetric solutions are a subclass of this class.
3. LRS class III \((\xi \neq 0, \Omega = 0)\): These are homogeneous orthogonal models with a spatial twist.

In a recent paper \cite{3} we established the existence of and found the necessary and sufficient conditions for the general class of solutions of Locally Rotationally Symmetric spacetimes that have non vanishing rotation and spatial twist simultaneously: that is for this class of spacetimes we have by definition

\[
\Omega \xi \neq 0. \tag{50}
\]

By the above, these solutions must be imperfect fluid models. We also provided a brief algorithm indicating how to solve the system of field equations with the given Cauchy data on an initial spacelike Cauchy surface. The important features of this class of spacetimes are as follows:

1. The necessary condition for a LRS spacetime to have non-zero rotation and spatial twist simultaneously is the presence of non-zero heat flux \(Q\) which is bounded from both sides.

2. In these spacetimes all scalars \(\Psi\) obey the following consistency relation:

\[
\forall \Psi, \quad \hat{\Psi} \Omega = \hat{\Psi} \xi, \tag{51}
\]

This equation can be easily derived by noting that for any scalar \(\Psi\) in a general LRS spacetime we have \(\nabla_a \Psi = -\hat{\Psi} u_a + \hat{\Psi} e_a \) and \(\hat{\Psi} \nabla_a \nabla_b \Psi = 0\). Also the above equation \((51)\), which is required by \((50)\), implies self-similarity, for it applies to all scalars, and is unchanged under the transformation \(\tau \to a \tau, \rho \to a \rho\), where \(\tau\) and \(\rho\) are the curve parameters of the integral curves of \(u\) and \(e\) respectively.

3. The above symmetries generate further constraints and hence the total set of constraint equations are now

\[
\mathcal{C} = \{C_1, C_2, C_3, C_4\}, \tag{52}
\]

where

\[
C_1 := \mathcal{H} = 3 \xi \Sigma - \left(2 \Omega + \frac{\Omega}{\xi} \left(\Sigma - \frac{2}{3} \Theta\right)\right) \Omega, \tag{53}
\]

\[
C_2 := \phi = -\frac{\Omega}{\xi} \left(\Sigma - \frac{2}{3} \Theta\right), \tag{54}
\]

\[
C_3 := Q = -\frac{\Omega}{1 + \left(\frac{\xi}{\sqrt{\Omega}}\right)^2} (\mu + p + \Pi), \tag{55}
\]

\[
C_4 := \mathcal{E} = \frac{\Omega}{\xi} \left(\Sigma - \frac{2}{3} \Theta\right) - \Sigma^2 + \frac{1}{3} \Theta \Sigma + \frac{2}{9} \Theta^2
+ 2 \left(\xi^2 - \Omega^2\right) + \frac{\left(\frac{\xi}{\sqrt{\Omega}}\right)^2}{\left(1 + \left(\frac{\xi}{\sqrt{\Omega}}\right)^2\right)} (\mu + p + \Pi)
- \frac{1}{2} \Pi - \frac{2}{3} \mu. \tag{56}
\]

It is important to verify that all these new constraints evolve consistently in time. This is indeed the case, as these constraints are derived by taking all the scalars \(\Psi \in \mathcal{D}_1\) and using the equation \((51)\) (which is true for all epochs) together with the field equations. Therefore the time derivatives of these new constraints will identically vanish using \((50)\) and the field equations as we feed the solutions back to the same system. Therefore solving for the set of variables

\[
\mathcal{D}_2 := \{\mathcal{A}, \Theta, \xi, \Sigma, \Omega, \mu\}, \tag{57}
\]

will automatically specify the rest

\[
\mathcal{D}_3 := \{Q, \phi, \mathcal{E}, \mathcal{H}, p\}, \tag{58}
\]

where we assume an equation of state for \(p\) of the form

\[
p = p(\mu, \Pi, Q). \tag{59}
\]

We note that the anisotropic pressure \(\Pi\) is not restricted by the constraints: there is no algebraic equation linking it to other thermodynamic variables. Hence this quantity should be specified at any initial Cauchy surface separately (subject to the energy conditions) and it would then evolve in time, via the field equations.

V. THE REDUCED SET OF FIELD EQUATIONS FOR SELF SIMILAR VARIABLES

We will now use the property of self similarity for the most general class of LRS spacetimes to further reduce the set of independent field equations. Let us consider the set of variables

\[
\mathcal{D}_4 := \{\mathcal{A}, \Theta, \xi, \Sigma, \Omega\} \subset \mathcal{D}_2, \tag{60}
\]

Then from the kinematical equations for LRS spacetimes

\[
\nabla_a u_b = -u_a e_b \mathcal{A} + e_a e_b \left(\frac{1}{3} \Theta + \Sigma\right) + \Omega e_{ab}, \tag{61}
\]

\[
D_a e_b = \frac{1}{2} \phi N_{ab} + \xi e_{ab}, \tag{62}
\]

it is clear that for any element \(f \in \mathcal{D}_4\), we must have

\[
f(\tau, \rho) = af(a \tau, a \rho), \tag{63}
\]

as \(u^a, e^a, N^{ab}\) and \(e^{ab}\) are dimensionless. Hence without any loss of generality, all these quantities can be written as

\[
f = \frac{f_0(z)}{\rho}, \tag{64}
\]

where

\[
z = \frac{\tau}{\rho}, \tag{65}
\]

and \(f_0\) is dimensionless. Also, from the Einstein field equations \(G_{ab} = T_{ab}\), we can easily see, as before, that all elements \(g \in \mathcal{D}_5\), where

\[
\mathcal{D}_5 := \{\mu, \Pi\} = \mathcal{D}_2 - \mathcal{D}_4, \tag{66}
\]

\[
\begin{align*}
\nabla_a u_b &= -u_a e_b A + e_a e_b \left(\frac{1}{3} \Theta + \Sigma\right) + \Omega e_{ab}, \\
D_a e_b &= \frac{1}{2} \phi N_{ab} + \xi e_{ab},
\end{align*}
\]

\[
f(\tau, \rho) = af(a \tau, a \rho),
\]

\[
f = \frac{f_0(z)}{\rho},
\]

\[
z = \frac{\tau}{\rho},
\]

and \(f_0\) is dimensionless. Also, from the Einstein field equations \(G_{ab} = T_{ab}\), we can easily see, as before, that all elements \(g \in \mathcal{D}_5\), where

\[
\mathcal{D}_5 := \{\mu, \Pi\} = \mathcal{D}_2 - \mathcal{D}_4, \tag{65}
\]
must satisfy
\[ g(\tau, \rho) = a^2 g(a\tau, a\rho). \] (66)

Therefore these quantities can be generally written as
\[ g \equiv \frac{g_0(z)}{\rho^2}. \] (67)

Now the dot and hat derivatives of all these elements can be written in terms of the dimensionless variable \( z \), in the following way: for \( f \in \mathcal{D}_4, \)
\[ \dot{f} = \frac{f_0, z}{\rho^2}, \] (68)
\[ \hat{f} = -\left(\frac{f_0 + z f_0, z}{\rho^2}\right), \] (69)

and for \( g \in \mathcal{D}_5 \)
\[ \dot{g} = \frac{g_0, z}{\rho^3}, \] (70)
\[ \hat{g} = -\left(\frac{2g_0 + z g_0, z}{\rho^3}\right). \] (71)

Using the above results, the non-trivial field equations become the following ordinary differential equations:
\[ \phi_{0, z} = \left[\frac{2}{3} \Theta_0 - \Sigma_0\right] A_0 - \frac{1}{3} \phi_0 + 2\xi_0 \Omega_0 + Q_0, \] (72)
\[ \xi_{0, z} = \left[\frac{1}{2} \Sigma_0 - \frac{1}{3} \Theta_0\right] \xi_0 + \left[A_0 - \frac{1}{3} \phi_0\right] \Omega_0 + \frac{1}{2} \hat{\xi}_0, \] (73)
\[ \Omega_{0, z} = A_0 \xi_0 + \Omega_0 \left[\Sigma_0 - \frac{2}{3} \Theta_0\right], \] (74)
\[ \mathcal{H}_{0, z} = -3\xi_0 \xi_0 + \left[\frac{3}{2} \Sigma_0 - \Theta_0\right] \mathcal{H}_0 + \Omega_0 Q_0 + \frac{1}{2} \xi_0 \Pi_0, \] (75)
\[ \Sigma_{0, z} - \frac{2}{3} \Theta_0, z = -\phi_0 A_0 + \frac{2}{3} \Theta_0^2 + \xi_0 \xi_0 - 2\Omega_0^2 + \frac{1}{2} \mu_0 + \phi_0 + \frac{1}{3} \hat{\Theta}_0 \Sigma_0 - \xi_0 \] \[ + \frac{1}{2} \Pi_0, \] (76)
\[ \xi_{0, z} + \frac{1}{3} \mu_0, z + \frac{1}{2} \Pi_{0, z} = \left[\frac{3}{2} \Sigma_0 - \Theta_0\right] \xi_0 + 3\xi_0 \mathcal{H}_0 - \frac{1}{2} (\mu_0 + \phi_0) + \frac{1}{3} Q_0 \phi_0 \] \[ - (\frac{1}{4} \Theta_0 - \frac{1}{4} \xi_0) \Pi_0 - \frac{1}{2} (\mu_0 + \phi_0) \Sigma_0. \] (77)

It can be shown that the rest of the field equations become redundant when the following set of dimensionless constraints \( \mathcal{C} \equiv \{ C_1, C_2, C_3, C_4 \} \) hold, which are easily derived by using equations \( \{63\} \) and \( \{67\} \) on the set of origi-
equations
\[ \frac{f_0,z}{f_0} = -\frac{A}{Az + B}, \tag{88} \]
\[ \frac{g_0,z}{g_0} = -\frac{2A}{Az + B}. \tag{89} \]

The general solutions for the equations (88) and (89) are given by
\[ f_0 = \frac{C_f}{Az + B}, \tag{90} \]
\[ g_0 = \frac{C_g}{(Az + B)^2}. \tag{91} \]

Here \( C_f \) and \( C_g \) are integration constants related to each of the kinematic and dynamic variables \( f_0 \) and \( g_0 \). Thus the set ‘C’ of arbitrary integration constants that we must specify to obtain the general solution for the most general LRS spacetime is given by:
\[ C \equiv (A, B, C_A, C_\Theta, C_\Sigma, C_\mu, C_\Pi), \tag{92} \]
where we must have \( A \neq 0 \) and \( B \neq 0 \) for the equation (50) to be true. The rest of the variables can then be easily obtained by using the constraint equations.

For example, using the constraint \( C_1 \) (equation (78)) we get the magnetic part of the Weyl scalar as follows:
\[ \mathcal{H} = \frac{C_\mathcal{H}}{(Az + B)^2}, \tag{93} \]
where we have
\[ C_\mathcal{H} = -3C_\Sigma + \left( 2C_A + \frac{B}{A}(C_\Sigma - \frac{2}{3}C_\Theta) \right) \frac{B}{A}. \tag{94} \]

Again, using the constraint \( C_2 \) (equation (79)) we get
\[ \phi_0 = \frac{C_\phi}{Az + B}; \quad C_{\phi} = -\frac{B}{A}(C_\Sigma - \frac{2}{3}C_\Theta). \tag{95} \]

The variables \( Q_0 \) and \( \xi_0 \) can similarly be obtained using equations (50) and (51) subject to the dimensionless algebraic equation of state \( p_0 = p_0(\mu_0, Q_0, \Pi_0) \), which must be provided separately along with the field equations. Once an equation of state in form of (58) is given, it is in principle possible to obtain such a dimensionless equation of state, as all the elements of \( D_{\text{matter}} \) have the same symmetries as (60) and hence the dimensionless part can be extracted from all of them.

Thus we obtain the solution for all the scalar variables of the set \( D_1 \) which completes the general solution. One can in principle obtain the metric elements from the definition of these covariant scalars. However it is important to note that all the physical properties of the LRS spacetime can be obtained directly from these covariant scalars as all of them have well defined geometrical and physical meaning. In the next section we will discuss some of the physical properties of these solutions for both astrophysical and cosmological scenarios.

VII. COSMOLOGICAL AND ASTROPHYSICAL PROPERTIES OF THIS GENERAL SOLUTION

This class of solutions have some very interesting properties, for both cosmological and stellar collapse scenarios which we list below. We can immediately see that there is a spacetime singularity along the curve \( B\rho + A\tau = 0 \), which is similar to the cosmological singularity of the FLRW or Lemaitre-Tolman-Bondi universes (or corresponding black hole singularities if we take the collapsing branch of the solutions). Apart from this, there are no other singular points on the manifold.

1. The most interesting feature of the singularity in this class of spacetime is it can be made timelike, spacelike or null by choice of the ratio of the constants \( A \) and \( B \). In other words, the ratio of rotation (\( \Omega \)) and spatial twist (\( \xi \)) at any initial Cauchy surface completely determines the nature of the initial (or final) singularity and this gives a range of different possibilities.

2. For the cosmological scenario, let us consider both \( A \) and \( B \) to be greater than zero. In that case the initial singularity is along the line \( B\rho + A\tau = 0 \). This ‘Big Bang’ is no longer instantaneous, and can be spacelike, timelike or null. Thus the section of the manifold that depicts the universe is given by
\[ \rho > 0, \quad \tau > -(B/A)\rho. \tag{96} \]

For an expanding universe with positive energy density, we must have \( \Theta > 0 \) and \( \mu > 0 \), and hence we must choose the constants
\[ C_\Theta > 0; \quad C_\mu > 0. \tag{97} \]

For the cosmological case we can choose dustlike matter with
\[ p_0 = 0, \tag{98} \]
\[ C_\Pi = 0 \Rightarrow \Pi_0 = 0. \tag{99} \]

Now we can immediately see that in this case \( \Theta < 0, \dot{\mu} < 0 \). There is no bounce in this cosmology as the expansion goes to zero asymptotically. Furthermore it is interesting to note that at spacelike infinity ‘i-‘ (where \( \rho \to \infty \)), timelike infinity ‘i+’ (where \( \tau \to \infty \)) and future null infinity \( T_+ \), all the kinematical and dynamical quantities vanish, making the spacetime asymptotically Minkowski. Hence, we get a cosmology that is Future asymptotically simple.

3. Another interesting case happens when the curves \( B\rho + A\tau = \text{const.} \) are null. In this case the initial singularity is incoming null. Then for any observer on the worldline \( \rho = 0, (\tau > 0) \), observation along the past null cone will depict a universe with
homogeneous density, in contrast to the fact that on a given time slice the density is inhomogeneous.

4. A similar picture can be obtained for collapsing stellar configurations with $A < 0$ and $B > 0$. In that case the section of the manifold $\rho > 0$ and $\tau < (B/|A|)\rho$ depicts the regular collapsing region which is Past asymptotically simple. To get a collapsing branch of the solution with positive matter density we must have $\Theta < 0$ and $\mu > 0$. Hence we choose

$$C_\Theta < 0 \quad ; \quad C_\mu > 0 . \quad (100)$$

Also here we should specify the equation of state linking the isotropic pressure to other thermodynamic variables and separately specify the constant $C_\Pi$ at the initial Cauchy surface subject to the energy conditions. We can easily check that in this case $\dot{\Theta} < 0, \dot{\mu} > 0$. Hence the collapse continues till $\Theta \to -\infty$ and $\mu \to \infty$. This is a final singularity at $\tau = (B/|A|)\rho$ and we can easily see that this singularity can be timelike, spacelike, or null, which will have important consequences in terms of the cosmic censorship conjecture.

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