Finite temperature Casimir effect and dispersion in the presence of compactified extra dimensions

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Abstract
Finite temperature Casimir theory of the Dirichlet scalar field is developed assuming that there is a conventional Casimir setup in physical space with two infinitely large plates separated by a gap \( R \) and, in addition, an arbitrary number \( q \) of extra compactified dimensions. As a generalization of earlier theory, we assume in the first part of the paper that there is a scalar ‘refractive index’ \( N \) filling the whole of the physical space region. After presenting general expressions for free energy and Casimir forces, we focus on the low temperature case, because this is of major physical interest both for force measurements and for issues related to entropy and the Nernst theorem. Thereafter, in the second part we analyze dispersive properties, assuming for simplicity that \( q = 1 \), by taking into account the dispersion associated with the first Matsubara frequency only. The medium-induced contribution to the free energy, and pressure, is calculated at low temperatures.

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1. Introduction
Consider two infinite parallel plates separated by a gap \( R \), and assume that the field between the plates, as well as outside, is a scalar field obeying Dirichlet boundary conditions at \( z = 0 \) and \( z = R \) (\( z \) is the direction normal to the plates). In order to get a Casimir configuration, we have to take into account the field outside the plates also. Let one of the plates, say, the right one at \( z = R \), be termed a ‘piston’. Now generalize the situation such that \( p \) spatial dimensions are envisaged, together with \( q \) extra compactified dimensions. The spacetime dimension is thus \( D = p + q + 1 \). We are led to a Casimir piston model in which spacetime is flat. This model has attracted considerable attention in the recent literature [1–24]. One reason for the current interest is obviously the mathematical elegance of the formalism; the efficiency of regularization procedures such as zeta function regularization is quite striking. Typical for this kind of theory is that the field energy can be expressed in terms of Epstein-like zeta functions. The Casimir force between the two plates in physical space follows by taking the derivative of the energy (or free energy at finite temperature) with respect to \( R \).

Another motivation is of a more physical nature, namely to investigate constraints for non-Newtonian gravity from the Casimir effect (cf, for instance, [28, 29]). Present-day Casimir force experiments are so accurate that the possible influence from extra dimensions is taken seriously. The hypothetical extra force is usually taken to be of the Yukawa form.

In the present paper, we will however not consider possible Casimir-induced deviations from Newtonian mechanics, but focus instead on analysis of the Casimir free energy and force in the presence of extra compactified dimensions at a finite temperature \( T \). At the final stages of our work, we actually became aware of a series of papers of Teo, which go along similar lines as our considerations. There is necessarily some overlap between Teo’s papers and the present paper. However, since our results were for the most part obtained independently of each other, it may be of interest to make a closer comparison:

In [5], the finite temperature Casimir force in the presence of extra dimensions was calculated for the scalar field with Dirichlet boundary conditions. The free energy was not considered. The force was always found to be attractive. When the ratio \( R/L \) (in our notation) increased from 0.1 to 1, the
Casimir force was found to increase quite a lot, by a factor of about 3. In our approach, the Casimir force is calculated via the free energy. A satisfactory feature is that a detailed comparison (some calculation is needed) shows our results to be in agreement with each other.

A more extended version of this work was given in the next paper [6], considering also the free energy. In the third paper in the series [7], the free energy was studied for a massive scalar field with general curvature coupling subject to Dirichlet or Neumann boundary conditions—thus a more general setting than that considered in our present investigation.

Finally, in three recent papers Teo investigated the finite temperature effect for a scalar field in multidimensional space assuming Robin boundary conditions [22]; then Teo gave a comparison between the Kaluza–Klein and the Randall–Sundrum models with special emphasis on the non-reversal of sign in the presence of warped extra dimensions [23], and worked out the Casimir theory for a multi-layer dielectric slab model in piston geometry [24].

In the present paper, we focus on the following two issues:

1. We allow there to be a ‘refractive’ index, called $N$, experienced by the scalar field in the physical medium between the plates. This influences the Casimir free energy as well as the entropy, and is of interest in connection with the Nernst theorem as $T \to 0$. To our knowledge, such a generalization has not been considered before. The introduction of a refractive index is not so trivial as one might imagine if one compares it with electrodynamic theory: in that case, the Casimir free energy expression $F$ gets an extra factor $N$ in the denominator at $T = 0$, the finite $T$ corrections being more complicated. Here, the $T = 0$ behavior can be understood in a physical way by observing that the momentum of a photon in the medium can be written in the form \[ \sqrt{k_\perp^2 + \pi^2 m^2}/R^2, \] where $k_\perp$ is the photon momentum parallel to the plates whose separation is $R$, and $m$ is an integer (in turn, this is intimately linked to the assumption that the photon momentum is as given by the Minkowski energy–momentum tensor). Taking into account the volume element in phase space, one obtains the essential result for $F$. Cf [25] and also the remarks of Ravndal [26, 27].

2. Our second point is to take into account the frequency dispersion of the medium in a crude way, by including the refractive index for low frequencies only. Specifically, we cut off the influence from the medium, thus setting $N = 1$, for all frequencies exceeding the first Matsubara frequency $\zeta_1 = 2\pi T$. From a basic viewpoint, this procedure is permitted, since there is no conflict with the fact that the refractive index—like any other generalized susceptibility—has to be such that it decreases monotonically along the positive imaginary frequency axis. Moreover, in physical units, $\zeta_1$ is lying in the region $10^{14} \text{s}^{-1}$ and is thus quite high, making the model sensible physically. This generalization is made for the case $q = 1$ only.

We begin in the next section by considering the general case where the number $p$ of edges in physical space is allowed to take the values 1, 2 or 3. Thereafter we specialize to the case $p = 1$, corresponding to the conventional setup with two parallel plates separated by a gap $R$. Zeta function regularization is employed. Our main focus will be on low temperatures as mentioned, as this is of major physical interest. For $N$ constant, we evaluate the Casimir free energy, the corresponding pressure, and the entropy, and work out some simplifying limiting expressions. The dispersion generalization is considered in section 6.

Finally, we mention for completeness that all plates, in physical space as well as in the extra compactified space, are so large that edge effects are negligible. We use natural units, with $\hbar = c = k_B = 1$.

2. General formalism: the case when $p = 1, 2$ and 3

We assume finite temperatures from the outset. Let $p$ be the number of edges in physical space. Thus $p$ can take the values $p = 1$ (two plates), $p = 2$ (four plates) or $p = 3$ (box). The number $M$ of transverse dimensions in physical space is $M = 3 - p$. We let index $i$, refer to the transverse directions, so that $i \in [1, M]$. The number $q$ of extra compactified dimensions is at first assumed arbitrary. The total spacetime dimension $D$ is thus $D = 4 + q$. We assume there to be a scalar field $\Phi(x', y', t)$ in the bulk. As mentioned, Dirichlet boundary conditions are assumed on the plates in physical space. The plate separations are $R_i$, with $i \in [1, p]$. In the extra space with dimension $q$, we assume, in accordance with common usage, that a torus of circumference $2\pi L_j$, $j \in [1, q]$, is attached to each spacetime point. Periodic boundary conditions are assumed for the extra dimensions.

As mentioned, we will allow for a constant ‘refractive index’ $N$ in the physical bulk. This index describes the response from the homogeneous and isotropic medium to the scalar field, in the same way as the usual refractive index $\sqrt{\varepsilon \mu}$ does in a dielectric medium, $\varepsilon$ being the permitivity and $\mu$ the permeability. Our model is thus a sort of scalar electrodynamics, which has been considered occasionally in earlier studies also, for instance in [30], dealing with curvilinear space. In the ordinary case $D = 4$, our model implies that the scalar photons possess the momentum $\mathbf{k} = N \omega \mathbf{k}$, in accordance with the Minkowski momentum in dielectric media in ordinary electrodynamics as already mentioned, implying a spacelike total photon four-momentum.

The central differential operator in Euclidean space ($r = i\tau$) is $\Delta_E = N^2 \partial_x^2 + \sum_j \partial_j^2 + \sum_j q \partial_j^2$. The free energy $F$ can be found by using the zeta function for the operator $-\Delta_E$,

\[
\zeta_{-\Delta_E}(s) = \sum_j \lambda_j^{s_j},
\]

where in the present generalized case the eigenvalues are

\[
\lambda_j = k_\perp^2 + \sum_{i=1}^p \left( \frac{n_i \pi}{R_i} \right)^2 + \sum_j q \left( \frac{m_j}{L_j} \right)^2 + N^2 \zeta_i^2.
\]

Here $J$ is an index referring to all the indices $[n_i], [m_j], [k_j]$, as well as the Matsubara frequencies $\zeta_i = 2\pi i T$ with.
The Dirichlet conditions on the plates cause the first sum to run over positive \( n_1 \) only. Further, \( m_1 \) extends over all integers because of periodic boundary conditions. In (2), we have also introduced \( k_{\perp}^2 \) as the square of the transverse wave number \( k_{\perp} \), i.e. \( k_{\perp}^2 = \sum_{i=1}^{M} k_i^2 \). We emphasize that \( N \) refers to the physical space only. To introduce an analogous refractive index in the extra space would hardly have any physical meaning.

The general expression for the free energy reads

\[
F = -\frac{T}{2} \beta \langle \zeta_\perp \rangle (0).
\]

We first consider the interior free energy, called \( F_1 \), in the cavity. With \( V_\perp \) denoting the transverse volume, \( V_\perp = \prod_{l=1}^{M} R_l \), and with the quantity \( A \) defined as

\[
A = (2\pi NIT)^2 + \sum_{j=1}^{q} \left( \frac{m_{j}}{L_j} \right)^2 + \sum_{j=1}^{q} \left( \frac{m_{j}}{L_j} \right)^2,
\]

we have for the zeta function

\[
\zeta_{\perp\perp}(s) = \frac{V_\perp}{(2\pi)^M} \sum_{l=-\infty}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_1=-\infty}^{\infty} \int d^M k_{\perp}(k_{\perp}^2 + A)^{-s}.
\]

Here the notation \( \{n_j\} = 1 \) means that \( n_1 = 1, 2, \ldots, \infty, n_2 = 1, 2, \ldots, \infty \). The integral can be evaluated using the technique of generalized polar coordinate transformations [31]. This yields, after integrating over all angles,

\[
\zeta_{-\perp\perp}(s) = \frac{V_\perp}{(2\pi)^M} \frac{2\pi M^2}{\Gamma(M/2)} \sum_{l=-\infty}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_1=-\infty}^{\infty} \int d^M r_{\perp} r_{\perp}^{-1} (r^2 + A)^{-s}.
\]

After the variable change \( x = r^2/A \) the integral part can be recognized as the integral representation of the beta function \( B(v, u) = \Gamma(v) \Gamma(u)/\Gamma(v + u) \) and hence the zeta function is

\[
\zeta_{-\perp\perp}(s) = \frac{V_\perp}{(2\pi)^M} \frac{\pi M^2}{\Gamma(M/2)} \frac{\Gamma(s - M/2)}{\Gamma(s)} \sum_{l=-\infty}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_1=-\infty}^{\infty} A^{M/2-s}.
\]

The free energy can easily be found from the above expression if we recall that \( (g(z)/\Gamma(z))_{z=0} = g(0) \) for any function \( g(z) \).

The main point here is that cavity I is a squeezed region, in which the discretization of allowable wave numbers is essential. Cavity II, on the other hand, is non-squeezed, so that summations over wave numbers can be replaced by integrals. Similarly, the reference configuration (cavities III and IV) corresponding to \( \eta \sim 2 \) is also non-squeezed. The technique is explained in more detail in [32; chapter 4, for instance]. The conventional situation with parallel plates is retrieved by taking \( X \rightarrow \infty \).

Thus in \( F_II \) the sum over the discrete \( k_\perp = n_\perp \pi/R_\perp \) goes to an integral in the limit \( X \rightarrow \infty \). This integral can be evaluated together with the integrals over the transverse directions in equation (5). To find the free energy of cavity II we need only to replace \( M \) with \( M + 1 \), \( V_\perp \) with \( (X - R)V_\perp \), remove \( (n_\perp \pi/R_\perp)^2 \) in \( A \) and remove the sum over \( n_\perp \).

Assume hereafter that the compactified extra dimensions have all the same size \( (L_i = L) \). We introduce the Epstein-like zeta function

\[
E_\perp(a_1, a_2, \ldots, a_V; s) = \sum_{\{n_i\}} \left( \sum_{j=1}^{V} a_j n_j^2 \right)^{-s}.
\]

After some algebra we then obtain the following expression for the free energy, called \( F \), of the piston system:

\[
F = F_1 + F_{II} - F_{III} - F_{IV} = \frac{1}{2} T \pi^{M/2} V_\perp \frac{2\pi M^2}{(2\pi)^M} \sum_{k=0}^{q} \left( \frac{q}{k} \right) (2\pi)^{q-k} \times \left[ \Gamma \left( -\frac{M}{2} \right) \left( \frac{\pi^2}{R_\perp} \right)^2 \left( \frac{\pi^2}{R_\perp} \right)^2 \left( \frac{1}{L_1^2} \right)^2 \ldots \left( \frac{1}{L_2^2} \right)^2 - \frac{M}{2} \right]
\]

\[
+ 2E_{1+q} \left( (2\pi NIT)^2, \pi^2/R_\perp^2, \pi^2/R_\perp^2, \ldots, \frac{1}{L_1^2}, \ldots, \frac{1}{L_2^2} - \frac{M}{2} \right)
\]

\[
- \frac{R_\perp}{2\sqrt{\pi}} \frac{\Gamma \left( -\frac{M+1}{2} \right)}{2} \times \left[ E_{p-1+q} \left( \frac{\pi^2}{R_\perp^2}, \ldots, \frac{\pi^2}{R_{p-1}^2}, \frac{1}{L_1^2}, \ldots, \frac{1}{L_2^2} - \frac{M+1}{2} \right)
\]

\[
+ 2E_{p+q-k} \left( (2\pi NIT)^2, \pi^2/R_\perp^2, \pi^2/R_\perp^2, \ldots, \frac{1}{L_1^2}, \ldots, \frac{1}{L_2^2} - \frac{M+1}{2} \right)
\]

\[
- \frac{M+1}{2} \right] \right].
\]

This expression holds for arbitrary temperature, for arbitrary \( p \in [1, 3] \) and for arbitrary integers \( q \), when all the \( L_i \) are equal. As is usual, \( F \) refers to unit surface area. Recall that the dimensions of the plates in the transverse dimensions are assumed infinite.

### 3. The case when \( p = 1 \)

When \( p = 1 \), the system in physical space consists of two parallel plates separated by a gap \( R_p = R \). Thus \( M = 2 \). The value of \( q \) is at present kept arbitrary. With
The Casimir free energy per surface area is then
\[ E = \frac{T}{8\pi} \sum_{k=0}^{q} \left( \frac{q}{k} \right) (2)^{q-k} \times \left[ \Gamma(-1) \left\{ E_{1+q-k} \left( \frac{\pi^2}{R^2}, \frac{1}{L^2}, \ldots, \frac{1}{L^2}; -1 \right) \right\} 
+ 2E_{2+q-k} \left( (2\pi NT)^2, \frac{\pi^2}{R^2}, \frac{1}{L^2}, \ldots, \frac{1}{L^2}; -1 \right) \right] \]

with
\[ F_{q=0} = \frac{N^4 R^2}{90} - T \frac{\zeta(3)}{16\pi R^2} - T \frac{\Gamma(-1)}{4\pi} E_2 \left( (2\pi NT)^2, \frac{\pi^2}{R^2}; -1 \right). \]

The Epstein-like functions in equations (13) and (14) will be zeta function regularized [32, 33]. By repeatedly using
\[ E_v(a_1, a_2, \ldots, a_v; s) = -\frac{1}{2} E_{v-1}(a_2, a_3, \ldots, a_v; s) \]
\[ + \frac{1}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} E_{v-1} \left( a_2, a_3, \ldots, a_v; s - \frac{1}{2} \right) \]
\[ + \frac{2\pi^s}{\Gamma(s)^2} a_1^{-(s+1)/2} \sum_{n_1, n_2, \ldots, n_v = 1}^{\infty} n_1^{s-1/2} \left( \sum_{i=2}^{v} a_i n_i^2 \right)^{-s-1/2} \]
\[ \times K_{s-1/2} \left( \frac{2\pi}{\sqrt{a_1}} \sum_{i=2}^{v} a_i n_i^2 \right)^{1/2}, \]
we can express \( F \) in terms of the modified Bessel functions of the second kind, \( K_v \). We will now discuss limits of physical interest.

4. Approximations

As the above formalism is complicated, it is of interest to make mathematical simplifications applicable to situations of physical interest.

There is one simplification that is obvious, resulting from the fact that the separations in physical space are very much greater than those in the extra space. Casimir measurements are usually done with separations of the order of 1 \( \mu \)m or less. In the extra space, one can estimate the typical distances by looking for minima of the free energy. For instance, by making for a moment a digression into warped space, in [34] we found that at low temperatures there is a minimum of the free energy in a Randall–Sundrum model if the length \( r_c \) of the orbifold is of the order of \( 10^{-18} \) GeV\(^{-1} \sim 10^{-19} \) fm. It appears reasonable that we can set
\[ R/L \gg 1 \]
for all practical purposes.

4.1. Low temperatures

The most important limiting case is when \( T \) is low. This case gradually catches up the quantum mechanical zero-point
energy as $T \to 0$. We shall take the low temperature limit to mean
\begin{equation}
RT \ll 1.
\end{equation}

Obviously, it implies also that $LT \ll 1$. In dimensional terms, equation (17) implies $(k_B T)(R/hc) \ll 1$ or $RT \ll 0.002$ with $T$ given in kelvins and $R$ in meters. Thus to 1% accuracy, $R = 100 \text{ nm}$ corresponds to $T < 200 \text{ K}$, whereas $R = 1000 \text{ nm}$ corresponds to $T < 20 \text{ K}$.

We now use equation (15) to find the finite expression for $F_{q=0}$:
\begin{equation}
F_{q=0} = N^3 T^4 \frac{R \pi^2}{90} - T \frac{\zeta(3)}{16 R^2} + \frac{T}{8 \pi} \Gamma(-1) E_1 \left( \frac{\pi^2}{2 R^2}, -1 \right) - \frac{1}{12 \pi N} E_1 \left( \frac{\pi^2}{R^2}, -\frac{3}{2} \right) - \frac{1}{\sqrt{2}} \left( \frac{NT}{R} \right)^{3/2} \times \sum_{n,l=1}^{\infty} \left( \frac{n}{T} \right)^{3/2} K_{3/2} \left( \frac{\pi ln(NT)}{R} \right) .
\end{equation}

Note that $\frac{T}{8 \pi} \Gamma(-1) E_1 \left( \frac{\pi^2}{2 R^2}, -1 \right)$ cancels $-T \frac{\zeta(3)}{16 R^2}$, coming from the Matsubara integer equal to zero $(l = 0)$, in view of equation (12). This equation so far does not imply the low temperature limit. But since the argument in $K_{3/2}$ is large, we need only the first term of the asymptotic expansion $K_{3/2}(z) = (\pi/2z)^{1/2} e^{-z}$. Due to the product $nl$ in the exponent of the expansion, we only need the dominant term $n = l = 1$ in the double sum. Thus,
\begin{equation}
F_{q=0} = -\frac{\pi^2}{1440 N R^3} + N^3 T^4 \frac{R \pi^2}{90} - N T^2 \frac{e^{-\pi/NRT}}{2R} ,
\end{equation}
in the low temperature limit. Here, $T^4 \frac{R \pi^2}{90}$ is the Stefan–Boltzmann term corresponding to the vacuum energy in empty $3+1$ spacetime and $-\frac{N T^2 e^{-\pi/NRT}}{2R}$ is the leading correction term. One may note that there is no term proportional to $T^3 \zeta(3)$, this term being characteristic of electromagnetic fields.

As regards the dependence upon $N$, the following points may be noted. At $T = 0$ the (free) energy is one-half the sum over eigenfrequencies $\omega$. In view of the Dirichlet boundary conditions on the plates, this sum becomes transferable to a sum over integers, which may conveniently be done in view of zeta function regularization. As $\omega = k/N$, the proportionality factor $1/N$ in the first term in equation (19) necessarily follows. The second term in the same equation involves the thermal occupation number $1/(e^{\omega/T} - 1)$. This does not contain $N$ in itself, but the refractive index occurs again in the phase space element $d^3k/(2\pi)^3$, which brings in a factor $N^3$. This is seen to agree with equation (19). The other terms in equation (18) are more complicated, but it is useful to observe that $F$ is generally for finite $T$ equal to $T$ multiplied by the logarithm of the partition function. As $N$ occurs only in the combination $(NT)$ in the quantity $A$ defined in equation (4), which in turn is closely related to the eigenvalues of the basic $\lambda_j$ operator in equation (2), the ratio $F/T$ has to be a function of $(NT)$. We see that this is the case for all terms in equations (18) and (19). Cf also the remarks of Ravndal [26, 27].

To find the low temperature limit of the part of the free energy depending on $L$ and $q$, it is convenient to evaluate the remaining part of equation (13) in pairs. In this way, we obtain after a lengthy algebra (details omitted) the following expression for the free energy in the low temperature limit:
\begin{equation}
F = -\frac{\pi^2}{1440 N R^3} + \sum_{k=0}^{q-1} \left( \frac{q}{k} \right) (2)^{q-k} \times \left[ \frac{R}{32 \pi^2 N} \Gamma(-2) E_{q-k} \left( \frac{1}{L^2}, \ldots, \frac{1}{L^2}; -2 \right) - \frac{1}{12 \pi N} E_{1+q-k} \left( \frac{\pi^2}{R^2}, \ldots, \frac{\pi^2}{L^2}; -\frac{3}{2} \right) + N^3 T^4 \frac{R \pi^2}{90} - N T^2 \frac{e^{-\pi/NRT}}{2R} + 2q R T \left( \frac{NT}{2 \pi L} \right)^{3/2} e^{-1/NTL} - q^2 N^2 T^2 \exp \left( -\frac{1}{\pi \sqrt{R}} \left( \frac{\pi^2}{2 R} + 1 \right) \right) \right] .
\end{equation}

This is our main result. At zero temperature, only the first three terms remain. The result agrees with the paper [11] when $N = 1$, except for the terms in that reference leading to repulsive forces. We are, as mentioned, concerned with the piston model, where the repulsive terms are removed. This is also the situation considered in [10].

Again using equation (15), we can rewrite the zero-temperature energy $E$ per unit area as
\begin{equation}
E = -\frac{\pi}{1440 N R^3} + \sum_{k=0}^{q-1} \left( \frac{q}{k} \right) (2)^{q-k} \times \left[ \frac{1}{24 \pi N} E_{q-k} \left( \frac{1}{L^2}, \ldots, \frac{1}{L^2}; -2 \right) - \frac{1}{8 L^2 R^2 N} \right] \times \sum_{m_1, m_2, \ldots, m_n} \left( \begin{array}{c} \pi \kappa_{m_j} \kappa_{m_j} \kappa_{m_j} \kappa_{m_j} \kappa_{m_j} \kappa_{m_j} \\ 2n L \end{array} \right) K_2 \left( 2n L \left( \sum_{j=1}^{q-k} \sum_{m_j} m_j^2 \right)^{1/2} \right)
\end{equation}

(at $T = 0$ the thermodynamic energy $E$ is the same as the free energy $F$). The Epstein-like terms are independent of $E$ and $T$ and will not influence the Casimir force. The arguments of the modified Bessel functions are proportional to $R/L \gg 1$ and will only give exponentially small corrections to the first term.

The two last terms in equation (20) are low-temperature corrections from the compactified extra dimensions. They are proportional to the number $q$ and are exponentially decreasing in $1/T$.

The Casimir force $P$ per unit area can be found as $P = -\partial F/\partial R$. Since $K_j'(z) = -\frac{1}{2}(K_{j-1}(z) + K_{j+1}(z))$, we find,
We shall consider only briefly the case of high temperatures.

4.2. High temperatures

For instance, in order to satisfy this condition with 1% accuracy when the separation is \( R = 1000 \text{ nm} \), the temperature must be quite high, \( T > 200 \text{ 000 K} \). At such temperatures, ordinary solid bodies do not exist. This limit is thus of less physical interest than that of low temperatures. Let us consider, however, the surface force density. Our starting point is again equation (13), from which we extract terms pairwise. Omitting all details we present only the final result for the high temperature limit of the Casimir pressure:

\[
P = -\frac{T \varepsilon_0(3)}{8\pi R^3} - \frac{2\pi T^3 N^2}{R} e^{-4\pi RNT} \left( 1 + 2\pi T \frac{N T L}{L^2} \right) \exp \left( -2\frac{R}{L} \sqrt{1 + (2\pi T N L)^2} \right). \tag{25}
\]

When \( N = 1 \) our results are in accordance with [5] although here we have included exponentially small corrections as well, not just the terms linear in temperature. In [5], the size of each extra compactified dimension is assumed arbitrary.

5. Entropy and the Nernst theorem

Of interest is here the low temperature limit. To evaluate the entropy \( S = -\frac{\partial F}{\partial T} \) we extract once again terms from expression (13) pairwise. As above we abstain from giving the details of the calculation and present only the result:

\[
S = -\frac{\partial}{\partial T} \left( \sum_{k=0}^{q-1} \left( \frac{q}{k} \right) (2)^{q-k} \right) - \frac{T \sqrt{2\pi NT}}{2\pi^2} \times \sum_{m_1, m_2, \ldots, m_q} \frac{1}{L^n} \left( \frac{\pi^2 n^2}{R^2} + \sum_{j=0}^{m_j} \frac{m_j^2}{L^2} \right)^{3/4} \times K_{3/2} \left( \frac{L}{T} \left( \sum_{j=0}^{m_j} \frac{m_j^2}{L^2} \right)^{1/2} \right) + N^3 T^4 R^2 \frac{\pi^2}{90} \times \sum_{n_{l=1}}^{\infty} \frac{1}{n_{l=1}} \frac{1}{L^2} \times K_{3/2} \left( \frac{\pi ln NTR}{NT} \right) \right) \tag{26}
\]

(terms that do not contribute in \( F \) are omitted). As the derivatives of \( K_n \) will produce terms containing \( K_n \) with the same argument, we see that, when \( T \to 0 \), the \( K_n \) will decay exponentially, implying that \( S \to 0 \). The Nernst theorem is satisfied. This is as we should expect, in the present case with idealized boundary conditions. The same property is known to hold for a metal without extra dimensions, when ideal boundary conditions are assumed for all frequencies. (The current discussion about thermal Casimir corrections relates to the case where dissipation is included; for a discussion of these issues see, for instance, [36].)
6. On dispersive properties, when $p = q = 1$

As the second novel element in our analysis, we will give a simplified analysis of the change in free energy due to frequency dispersion in the medium filling the physical space. As before, we assume $p = 1$, so that there are two infinite plates in physical space separated by a gap $R$. To simplify the formalism, we assume moreover that $q = 1$, so that there is only one extra dimension (containing a vacuum, not a medium). This formal simplification will still allow us to show the characteristics of the dispersive behavior.

Now write the free energy $F$ as a sum of its $q = 0$ part (i.e. ordinary space, with no extra dimension) and a $q = 1$ part,

$$F = F_{q=0} + F_{q=1}. \quad (27)$$

Here the first term is given by equation (11), for arbitrary temperature. The Matsubara frequencies are $\Omega = 2\pi l T$ with $l = 0, \pm 1, \pm 2, \ldots$. The first finite value, corresponding to $l = 1$, is in dimensional terms $\Omega_1 = 2\pi k_B T / \hbar = 2.4 \times 10^{14} \text{s}^{-1}$. This value is quite high, and it is therefore physically reasonable, as mentioned in the introduction, to ignore the dispersive properties for higher values of $l$. We accordingly adopt in the following a dispersive model in which $N(\Omega) = N_0$ = constant for $l = 1$ in the Epstein-like function $E_\Omega$, and then falls off abruptly such that we may put $N(\Omega) = 1$ for $l \geq 2$. This point simplifies the formalism, but still allows us to see the characteristics of the dispersive behavior.

Starting from the general expression for $E_\Omega$, we can thus obtain the following effective substitution:

$$E_2 \left( (2\pi NT)^2, \frac{\pi^2}{R^2}; -1 \right) = \sum_{n_1=1}^\infty \left[ (2\pi NT)^2 n_1^2 + \pi^2 n_1^2 \right]$$

$$\rightarrow E_2 \left( (2\pi T)^2, \frac{\pi^2}{R^2}; -1 \right) + E_2^{2\pi T N_0} \left( \frac{\pi^2}{R^2}; -1 \right) - E_2^{2\pi T} \left( \frac{\pi^2}{R^2}; -1 \right), \quad (28)$$

where $E_\Omega(c_1, a_2, \ldots, a_V; s) = \sum_{|n| = 1}^{\infty} \left( c^2 + \sum_{j=1}^V a_j n_j^2 \right)^{-s}$

is a generalized Epstein-like zeta function. The first term in equation (11) ($F_{q=0}$) does not refer to the dielectric properties of the physical space. As for the first term and all other terms not containing $N(\Omega)$, we get no change from the previous sections.

Then, for the third term with $E_1$ in equation (11), we make use of the same effective substitution as above, extracting the $l = 1$ term,

$$E_1 \left( (2\pi NT)^2; -\frac{1}{2} \right) \rightarrow E_1 \left( (2\pi T)^2; -\frac{1}{2} \right) + (2\pi T)^3 (N_0^3 - 1). \quad (30)$$

No regularization of the $N_0$ term is here necessary. Combining all three terms in $F_{q=0}$, we obtain

$$F_{q=0} = F_{N=0} + \frac{T}{4\pi} \left( E_2^{2\pi T N_0} \left( \frac{\pi^2}{R^2}; -1 \right) - E_2^{2\pi T} \left( \frac{\pi^2}{R^2}; -1 \right) \right) + \frac{TR}{6\pi} (2\pi T)^3 (N_0^3 - 1). \quad (31)$$

Here we have collected all terms that give the free energy without dispersion, i.e. $F_{N=0}$, is equal to equation (11) with $N = 1$ for all $l$. Hence, we can use all expressions obtained above and only concentrate here on the additional terms introduced by the dispersion model. Since $E_2^{2\pi T N_0} (\pi^2/R^2; -1) = \pi^2/R^2 \zeta_{EH}(2\pi T N_0^2); -1)$, where $\zeta_{EH}$ is the Epstein–Hurwitz zeta function, we might recall the conventional expansion

$$\zeta_{EH}(p; s) = \sum_{n=1}^{\infty} (n^2 + p)^{-s}$$

$$= -\frac{1}{2} p^{-s} + \frac{\sqrt{\pi} \Gamma(1/2 - s)}{2\Gamma(s)} p^{1/2 + s/2} - \frac{2\pi^2}{\Gamma(s)} \sum_{n=1}^\infty n^{s-1/2} \zeta_{EH}(s - 1/2; 2\pi n \sqrt{p}); \quad (32)$$

cf [33, p 81]. Although this series is for many purposes a convenient expression, it is in our case with small arguments better to write $\zeta_{EH}$ in another way: as $\pi s$ is an integer we make use of equation (3.24) on p 37 in [32]1 to write

$$E_1(1; s) = \zeta_{EH}(c^2; s)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + s)}{k! \Gamma(s)} \zeta_{EH}(2k + 2s) e^{2k}$$

$$- \frac{\pi^{1/2}}{2\Gamma(s)} \Gamma \left( s - \frac{1}{2} \right) c^{1-2s}. \quad (33)$$

With this we find the finite expression

$$-\frac{T \pi}{4R^2} \Gamma(-1) \zeta_{EH}(2\pi T N_0^3; -1)$$

$$= -\frac{T \pi}{4R^2} \left( \Gamma(-1) \zeta_{EH}(2) - \pi^{-1/2} \Gamma \left( \frac{3}{2} \right) (2\pi T N_0^3) \right)$$

$$= -\frac{T \zeta_{EH}(3)}{8\pi R^2} + \frac{4\pi^2}{3} RT^4 N_0^3. \quad (34)$$

We can now insert these expressions into equation (31) to evaluate $F_{q=0}$ at arbitrary temperature. Moreover, to simplify the formalism somewhat, we restrict ourselves to the low-temperature approximation, $RT \ll 1$. From equations (19), (31) and (34), we find

$$F_{q=0} = \frac{8\pi^2}{3} RT^4 (N_0^3 - 1) - \frac{\pi^2}{1440 R^3} + \frac{\pi^2}{90} RT^4$$

$$- \frac{T \zeta_{EH}(3)}{2R}, \quad RT \ll 1. \quad (35)$$

where the influence from the medium is shown explicitly in the first term.

1 Equation (3.24) on p 37 in [32] gives three different expressions for the Epstein–Hurwitz formula. One is for $s \notin \mathbb{N}$ and $1/2 - s \notin \mathbb{N}$, the second for $s \in \mathbb{N}$ and the third for $1/2 - s \notin \mathbb{N}$. According to our recalculation, two of these expressions are correct, but in the case ($s \in \mathbb{N}$) the expression should be different and as given above. See the appendix. It should also be noted that an assumption for these expressions is that $c^2$ is small. This assumption is fulfilled in the low temperature limit since $c^2 = (2\pi T N_0^2)^2 \ll 1$. 


Consider next the term coming from the extra dimension, \( q = 1 \). From equation (13) we first obtain

\[
F_{q=1} = \frac{RT}{6\pi} E_1 \left( \frac{1}{L^2}; -\frac{3}{2} \right) - \frac{T}{4\pi} \Gamma(-1) E_2 \left( \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right)
\]

\[
+ \frac{RT}{3\pi} E_2 \left( (2\pi NT)^2, \frac{1}{L^2}; -\frac{3}{2} \right) - \frac{T}{2\pi} \Gamma(-1) E_3 \left( (2\pi NT)^2, \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right),
\]

where the terms are written in order of increasing complexity. The last two terms depend on \( N \) and can be rewritten in the same manner as equation (28). Again, we write the free energy as the free energy of a medium with \( N = 1 \) (for all \( l \)) plus correction terms from \( l = 1 \),

\[
F_{q=1} = F_{q=1}^N + \frac{RT}{\pi} \left( E_1^{2\pi N T} \left( \frac{1}{L^2}; -\frac{3}{2} \right) - E_2^{2\pi N T} \left( \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right) \right) - \frac{T}{2\pi} \Gamma(-1) E_3 \left( (2\pi NT)^2, \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right) \]

(36)

From p 48 in [33], we adopt a formula similar to equation (15) (with some re-writing)

\[
E_{\nu}(a_1, a_2, \ldots, a_{\nu}; s) = -\frac{1}{2} E_{\nu-1}(a_2, a_3, \ldots, a_{\nu}; s) + \frac{\pi^s}{\Gamma(s)} \int_a \frac{1}{(s-1/2)} E_{\nu-1} \left( a_2, a_3, \ldots, a_{\nu}; s - \frac{1}{2} \right)
\]

\[
+ \frac{2\pi^s}{\Gamma(s)} \left( \sum_{n_1, n_2, \ldots, n_{\nu-1}} n_1^{-1/2} \left( c^2 + \sum_{i=2}^{\nu} a_i n_i^2 \right)^{(s-1)/2} \right) \times K_{s-1/2} \left( \frac{2\pi}{\sqrt{a_1}} n_1 \left( c^2 + \sum_{i=2}^{\nu} a_i n_i^2 \right)^{1/2} \right).
\]

(38)

Then, we can write

\[
\Gamma(-1) E_2^{2\pi N T} \left( \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right) = -\frac{1}{2} \Gamma(-1) E_1^{2\pi N T} \left( \frac{1}{L^2}; -1 \right)
\]

\[
+ \frac{R}{\sqrt{T}} \Gamma \left( \frac{3}{2} \right) E_1^{2\pi N T} \left( \frac{1}{L^2}; -\frac{3}{2} \right) + \frac{2}{\sqrt{T}} \Gamma \sum_{m,n=1}^{\infty} n^{-3/2}
\]

\[
\times \left( \frac{m^2}{L^2} + (2\pi N T)^2 \right)^{3/4} K_{3/2} \left( 2Rn \left( \frac{m^2}{L^2} + (2\pi N T)^2 \right) \right). \]

(39)

There occurs a cancellation since \(-T/(2\pi R) + (2\pi R)\times \Gamma(-3/2) E_1^{2\pi N T} \left( 1/L^2; -3/2 \right) = -RT/(3\pi) E_1^{2\pi N T} \left( 1/L^2; -3/2 \right)\) with the aid of equation (33) and the reflection formula for \( \zeta \), we see that

\[
-\frac{1}{2} \Gamma(-1) E_1^{2\pi N T} \left( \frac{1}{L^2}, -1 \right) = -\frac{\zeta(3)}{4\pi^2} + \frac{8}{3} \pi^4 T^3 LN^3 \]

(40)

When all is put together, we obtain

\[
F_{q=1} = F_{q=1}^{N=1} - \frac{4}{3} \pi^2 T^4 L(N_0^3 - 1) - \frac{1}{\sqrt{\pi^3 R}}
\]

\[
\times \sum_{m,n=1}^{\infty} n^{-3/2} \left( \frac{m^2}{L^2} + (2\pi N T)^2 \right)^{3/4}
\]

\[
\times K_{3/2} \left( 2Rn \left( \frac{m^2}{L^2} + (2\pi N T)^2 \right) \right).
\]

(41)

Since \( R/L \gg 1 \), we only need to take into account the \( n = m = 1 \) term of the double-sum. Some calculation yields, for the medium-independent \( F_{q=1}^{N=1} \) terms,

\[
\Gamma(-1) E_3 \left( (2\pi T)^2, \frac{\pi^2}{R^2}, \frac{1}{L^2}; -1 \right) = -\frac{1}{2} \Gamma(-1) E_2
\]

\[
\times \left( \frac{\pi^2}{R^2}; -1 \right)
\]

\[
\times \frac{3R\xi_R(5)}{32\pi^5 T L^3} + \frac{1}{2\pi RT L^2}
\]

\[
\times \sum_{m,n=1}^{\infty} \frac{m^2}{n^2} K_2 \left( 2\pi n R/L \right) \left( \frac{2\pi^2 n^2}{R^2} + \frac{m^2}{L^2} \right) \times K_{3/2} \left( \frac{1}{T} \left( \frac{\pi^2 n^2}{R^2} + \frac{m^2}{L^2} \right) \right).
\]

(42)

and

\[
E_2 \left( (2\pi T)^2, \frac{\pi^2}{R^2}, \frac{1}{L^2}; -\frac{3}{2} \right) = -\frac{1}{240 L^3} + \frac{9}{64\pi^5 T L^4} \xi_R(5)
\]

\[
+ \frac{3T}{\pi L^2} \sum_{m=1}^{\infty} \frac{m^2}{L^2} K_2 \left( \frac{1m L}{T} \right).
\]

(43)

At last, we have the final low-temperature expression for \( L/R \ll 1 \):

\[
F_{q=1} = -\frac{4}{3} \pi^3 L^4 (N_0^3 - 1) - \frac{1}{\sqrt{1440\pi L^3}} - \frac{1}{8(\pi RL)^{3/2}} \times e^{-2R/L}
\]

\[
- \frac{T^2}{\sqrt{\pi^3 R^2 + \frac{L^2}{T^2}}} e^{-\frac{1}{T\sqrt{\pi^3 R^2 + \frac{L^2}{T^2}}}}
\]

\[
- \frac{T}{2\pi R} \sqrt{\frac{1}{T^2} + (2\pi T N_0)^2} e^{-2R \sqrt{\frac{1}{\pi^3 R^2} + (2\pi T N_0)^2}}
\]

\[
+ \frac{T}{2\pi R} \sqrt{\frac{1}{T^2} + (2\pi T)^2} e^{-2R \sqrt{\frac{1}{\pi^3 R^2} + (2\pi T)^2}}.
\]

\[
RT \ll 1, \quad L/R \ll 1.
\]

(44)
Adding equation (35) and (44), we obtain the total free energy per unit surface in physical space to the lowest order in $RT$ and $R/L$

$$F = \frac{1}{440\pi L^3} \left[ 1 - \left( \frac{\pi L}{R} \right)^3 \right] + \frac{\pi^2}{90} RT^4$$

$$+ \left( \frac{2R}{L} - L \right) \frac{4}{3} \pi^3 T^4 (N_0^3 - 1)$$

$$- T^2 e^{-\pi/RT} \frac{2R}{2R} - \frac{1}{8(\pi RL)^{3/2}} e^{-2R/L}$$

$$- \frac{T}{\pi} \sqrt{\frac{\pi^2}{R^2} + \frac{1}{L^2}} e^{-\frac{1}{2} \sqrt{\frac{\pi^2}{R^2} + \frac{1}{L^2}}}$$

$$- \frac{T}{2\pi R} \sqrt{\frac{1}{L^2} + (2\pi T N_0)^2} e^{-2R/\sqrt{\pi^2 + (2\pi T N_0)^2}}$$

$$+ \frac{T}{2\pi R} \sqrt{\frac{1}{L^2} + (2\pi T)^2} e^{-2R/\sqrt{\pi^2 + (2\pi T)^2}},$$

$$RT \ll 1, \quad L/R \ll 1.$$  \hspace{1cm} (45)

For the pressure $P = -\partial F/\partial R$, we obtain, by omitting the small correction terms,

$$P = -\frac{\pi^2}{480 R^4} - \frac{\pi^2 T^4}{90} \left[ 240 (N_0^3 - 1) + 1 \right]$$

$$+ T e^{-\pi/RT} \frac{2R}{2R} - \frac{1}{4L(\pi RL)^{3/2}} e^{-2R/L} + \frac{\pi T}{R^3} e^{-\frac{1}{2} \sqrt{\frac{\pi^2}{R^2} + \frac{1}{L^2}}}$$

$$- \frac{T}{\pi R} \left( \frac{1}{L^2} + (2\pi T N_0)^2 \right) e^{-2R/\sqrt{\pi^2 + (2\pi T N_0)^2}}$$

$$+ \frac{T}{\pi R} \left( \frac{1}{L^2} + (2\pi T)^2 \right) e^{-2R/\sqrt{\pi^2 + (2\pi T)^2}}.$$  \hspace{1cm} (46)

Here the first term gives the conventional attractive force in physical space. The second term is attractive and can be traced back to the $q = 0$ contribution. In the force expression, there are Bessel functions with the same arguments as in the free energy. There are several sums over Bessel functions with different arguments; we have kept the largest term from each sum to show the relative size of terms. Note that since $N_0 > 1$ the contribution from the extra dimension is attractive. It is seen that cross terms, containing contributions from both the $R$ and the $L$ spaces, are all exponentially small.

7. Summary

We began by developing the finite temperature Casimir theory of the scalar field, assuming Dirichlet boundary conditions for two large parallel plates with a gap $R$ in physical space endowed with a medium with constant ‘refractive index’ $N$, and in addition an arbitrary number $q$ of extra compactified dimensions. Zeta function regularization of the Epstein-like functions was used throughout. In the main part of the formalism the radii $L$ of the compactified dimensions were taken to be equal. For general values of $T$, the free energy $F$ was given by equation (13). For low $T$ values (i.e. $RT \ll 1$), which was the case of major physical interest, the approximate expression for $F$ was given by equation (20), and the corresponding Casimir pressure was given by equation (23). When $N = 1$, agreement was found with Teo’s paper [5]. The Nernst theorem was found to be satisfied; cf the low-temperature entropy expression (26). The case of high temperatures was briefly discussed. In all cases of physical interest, the inequality $R/L \gg 1$ could be assumed.

In the dispersive part of the theory considered in section 6 (assuming $q = 1$), we introduced a rough model implying neglect of the dispersive property at all frequencies higher than the first Matsubara frequency, $\zeta_1 = 2\pi T$. It is worth noting that for the Epstein–Hurwitz zeta function, we found it convenient to use expansion (33), instead of the more conventional expansion (32). We found (33) to be the most suitable form for low temperatures. For $RT \ll 1$, we calculated the free energy in equation (45), showing contributions from physical space as well as from the extra dimension. The corresponding pressure in equation (46) contains both attractive and repulsive terms. The common property of the cross terms, i.e. terms containing both $R$ and $L$ contributions, is that these terms are exponentially small.

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Appendix. Regularization of the Epstein–Hurwitz zeta function

In this appendix, we will argue that the correct regularized expression for the Epstein–Hurwitz zeta function should be

$$E^\gamma_1(1; s) = \zeta_{EH}(c^2; s)$$

$$= \sum_{k=0}^{-s} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta_R(2k+2s)c^{2k}$$

$$- \frac{\pi^{1/2}}{2\Gamma(s)} \Gamma\left(s - \frac{1}{2}\right)c^{1-2s}$$  \hspace{1cm} (A.1)

for $-s \in \mathbb{N}$. As noted above, this expression is a bit different from the one given in equation (3.24) in [32]:

$$E^\gamma_1(1; s) = \sum_{k=0}^{-s} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta_R(2k+2s)c^{2k}.$$  \hspace{1cm} (A.2)

It is seen that the two expressions differ with respect to two terms: the first is $k = -s$ from the sum over $k$ and the second is a term proportional to $c^{1-2s}$. Referring to the basic calculation in [32] we will argue why the $k = -s$ term should be omitted and the $c^{1-2s}$ term should be present.

We start with equation (3.7) of [32]:

$$E^\gamma_1(1; s) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta_R(2k+2s)c^{2k} + \text{Res}_c,$$  \hspace{1cm} (A.3)

where

$$\text{Res}_c \equiv \text{Res} \left[ \phi(a) \Gamma(a+1) \frac{\zeta_R(2s+2a)c^{2a}}{\Gamma(s)} \times \csc \pi a ; \ a = z \right]$$  \hspace{1cm} (A.4)
for $z = 1/2 - s$ and $z = -s$. (The function $\phi(a)$ comes from the asymptotic behavior of the integrand when the sum is converted into an integral; cf [32].) The sum over $k$ runs from 0 to $\infty$, but skips $k = 1/2 - s$ (if $1/2 - s \in \mathbb{N}$) to avoid the pole $\zeta_R(1)$ and $k = -s$ (if $-s \in \mathbb{N}$) to avoid the pole $\Gamma(0)$. The next step is to calculate the residue at $a = 1/2 - s$ and $a = -s$. We quote the results from [32]:

$$\text{Res}_{1/2-s} = \begin{cases} 
-\frac{\pi^{1/2}}{2\Gamma(s)} \Gamma \left( s - \frac{1}{2} \right) e^{1-2z}, & \text{if } 1/2 - s \notin \mathbb{N}, \\
(1-z/2) \frac{\pi^{1/2}}{\Gamma(s) \Gamma(1-s)} e^{1-2z} \left[ \frac{1}{2} \psi \left( \frac{1}{2} \right) \right], & \text{if } 1/2 - s \in \mathbb{N}, \\
-\left( \frac{3}{2} - s \right) + \log c^2 + \gamma, & \text{if } 1/2 - s \in \mathbb{N}, 
\end{cases}$$

(A.5)

where $\psi(a) = \frac{d}{da} \log \Gamma(a)$. For $a = -s$ we have

$$\text{Res}_{-s} = \begin{cases} 
\frac{1}{2} e^{2z}, & \text{if } -s \notin \mathbb{N}, \\
0, & \text{if } -s \in \mathbb{N}. 
\end{cases}$$

(A.6)

The results can be divided into three: $1/2 - s \notin \mathbb{N}$ and $-s \notin \mathbb{N}$; $1/2 - s \in \mathbb{N}$; and $-s \in \mathbb{N}$. We are interested in $-s \in \mathbb{N}$. Combining equations (A.3), (A.5) and (A.6), we find this to be equal to equation (A.1). The sum over $k$ only runs to $-s - 1$ since all terms with $k > -s$ are equal to zero since $\zeta_R(-2n) = 0$ for $n = 1, 2, 3, \ldots$ and $k = -s$ is excluded for $-s \in \mathbb{N}$. The $e^{1-z}$ term must be included since $-s \in \mathbb{N}$ implies $1/2 - s \notin \mathbb{N}$.

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