Three lines proof of the lower bound for the matrix rigidity

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Abstract. The rigidity of a matrix describes the minimal number of entries one has to change to reduce matrix’s rank to \( r \). We give very simple combinatorial proof of the lower bound for the rigidity of Sylvester (special case of Hadamard) matrix that matches the best known result by de Wolf (2005) for Hadamard matrices proved by quantum information theoretical arguments.

1 Introduction

1.1 Problem
Changing some entries of a complex matrix can reduce its rank. The rigidity of a matrix \( M \) is the function \( R_M(r) \), which for a given rank \( r \), gives the minimum number of entries of \( M \) which one has to change in order to reduce \( M \)’s rank to \( r \) or less. More formally,

\[
R_M(r) = \min_{\text{rank}(\tilde{M}) \leq r} \{ \text{weight}(M - \tilde{M}) \},
\]

where \( \text{weight} \) denotes the number of non-zero entries. In other words, large rigidity shows that the matrix’s rank is stable under perturbations. It is easy to see that \( R_M(r) \geq n - r \) for any full rank matrix \( M \), because change of one entry reduces the rank by at most 1.

1.2 History
In this section I survey all known results for the matrix rigidity over infinite fields up to my best knowledge. There have been done a lot of work on rigidity on finite fields \[4\] and on restricted and generalized versions of the rigidity problem as well \[17\].

The rigidity was defined by Valiant \[23,24\]; a similar notion independently was proposed by Grigoriev \[6\]. The main motivation to study rigidity is that good lower bounds on rigidity would give important complexity results in other computational models, like linear algebraic circuits and communication complexity. For communication complexity \((0,1)\)-matrices are especially important. Valiant showed \( R_M(r) \geq (n - r)^2 \) for "almost all" matrices \( M \). Pudlak and Rodl \[10\]
showed a similar result for (0, 1)-matrices. However, to show a good lower bound for an explicit matrix still remains unsolved task.

The most interesting matrix probably is Hadamard matrix. Pudlak and Savicky showed that for any Hadamard matrix \( H \), \( R_H(r) = \Omega \left( \frac{n^2}{r \log r} \right) \), Razborov improved their result to \( R_H(r) = \Omega \left( \frac{n^2}{r \log r} \right) \). Grigoriev and Nisan independently observed an easy method to get lower bound for any totally non-singular matrix \( M \) (i.e. a matrix in which all submatrices are non-singular) \( R_M(r) = \Omega \left( \frac{n^2}{2r^3} \right) \). Similar strategy was used by Alon to improve rigidity of Hadamard matrix \( R_H(r) = \Omega \left( \frac{n^2}{r^2} \right) \), Lokam to give an alternate proof of the same result, Kashin and Razborov determined the exact value of \( T \), particularly, for a large \( n \) but small \( r \) it is like \( R_T(r) = \frac{n^2}{256r} \). Pudlak showed that \( R_M(r) = \Omega \left( \frac{n^2}{r^2} \right) \) if \( M \) belongs to a class of matrices called Densely Regular, that includes triangular matrix, Vandermonde matrices, shifters and parity shifters. Shokrollahi et al. showed that \( R_C(r) = \Omega \left( \frac{n^2}{r \log n} \right) \) for a Cauchy matrix \( C \). Codenotti et al. studied the rigidity of some matrices under combinatorial assumptions.

Lokam gives some quadratic lower bounds for ”less explicit” matrices. Landsberg et al. gave geometrical interpretation of matrix rigidity.

However, these results do not give a superlinear rigidity for an explicit matrix when \( r = O(n) \). Lokam observes that a method used in all those results (and this paper as well) by getting ”candidate” matrices that are close to full rank does not give a results like \( R(r) = \omega \left( \frac{n^2}{r \log \frac{n}{r}} \right) \).

Codenotti gives a survey paper on the matrix rigidity problem as well as some interesting problems.

In this paper we give a simple proof in ”three lines” of \( R_S(r) \geq \frac{n^2}{4r} \) for any Sylvester matrix \( S \) (special case of Hadamard matrix). The same proof works for other ”well behaved” matrices, like Discrete Fourier Transform. However, our main contribution is the simplicity of the proof.

1.3 Matrices

If \( A = (a_{ij}) \) and \( B = (b_{kl}) \) are matrices of size \( m \times n \) and \( p \times q \) respectively, the Kronecker product \( A \otimes B \) is the \( mp \times nq \) matrix made up of \( p \times q \) blocks, where the \((k, l)\) block is \( b_{kl}A \).

**Sylvester matrix** \( S(n) \) of order \( n := 2^k \) is \( n \times n \) matrix made by iterating Kronecker product of \( k \) copies of the following matrix
\[ S(2) = \begin{pmatrix} + & + & \cdot & \cdot \\ \cdot & \cdot & + & - \end{pmatrix} \]

where + and − denotes +1 and −1 respectively.

For example,
\[ S(4) = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & - \\ - & - & + & + \end{pmatrix} \]

Sylvester matrices are special case of Hadamard matrices. A real valued matrix \( H \) is called Hadamard matrix iff \( HH^T = nI \).

Discrete Fourier Transform is \( n \times n \) matrix \( \mathcal{F}N(n) = (f_{jk}) \) defined by
\[ f_{jk} = \omega^{(j-1)(k-1)}, \text{ where } \omega := e^{\frac{2\pi i}{n}} \text{ and } i := \sqrt{-1}. \]

2 Proof

**Theorem 1** If \( S(n) \) is a Sylvester matrix and \( r \leq n/2 \) is a power of 2 then

\[ R_{S(n)}(r) \geq \frac{n^2}{4r}. \]

In other words, for any \( n \times n \) matrix \( \tilde{S} \) such that \( \text{rank}(\tilde{S}) \leq r \) holds

\[ \text{weight}(S(n) - \tilde{S}) \geq \frac{n^2}{4r}. \]

**Proof.** Assume the opposite, \( \text{weight}(S(n) - \tilde{S}) < \frac{n^2}{4r} \). Let uniformly divide \( \tilde{S} \) in \( (\frac{n}{2r})^2 \) submatrices \( \tilde{S}_{ij} \) of size \( 2r \times 2r \). By a counting argument, there exists \( i,j \) s.t. \( \text{weight}(S(2r) - \tilde{S}_{ij}) < r \). Thus, \( \text{rank}(\tilde{S}) \geq \text{rank}(\tilde{S}_{ij}) > 2r - r = r \). Contradiction. \( \square \)

The same proof works for \( R_{\mathcal{F}T(n)}(r) \geq \frac{n^2}{4r} \), where \( \mathcal{F}T(n) \) denotes \( n \times n \) Discrete Fourier Transform matrix, because DFT matrix where columns with even indexes are written first is represented as a matrix
\[ \begin{pmatrix} \mathcal{F}T(n/2) & \omega^j \mathcal{F}T(n/2) \\ \mathcal{F}T(n/2) & -\omega^{-n/2} \mathcal{F}T(n/2) \end{pmatrix} \]

where \( j \) denote the index of a row. Since rows of a \( \mathcal{F}T(n) \) are orthogonal and multiplication by some constant does not change this property, each submatrix can be recursively divided again and again, by getting full rank submatrices. This is the only property of matrices we need in the proof.
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