THE FIRST INTEGRAL METHOD FOR TWO FRACTIONAL NON-LINEAR BIOLOGICAL MODELS

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Abstract. Travelling wave solutions of the space and time fractional models for non-linear blood flow in large vessels and Deoxyribonucleic acid (DNA) molecule dynamics defined in the sense of Jumarie’s modified Riemann-Liouville derivative via the first integral method are presented in this study. A fractional complex transformation was applied to turn the fractional biological models into an equivalent integer order ordinary differential equation. The validity of the solutions to the fractional biological models obtained with first integral method was achieved by putting them back into the models. We observed that introducing fractional order to the biological models changes the nature of the solution.

1. Introduction. It is no new knowledge that most processes and phenomena that arise in plasma physics, mathematical physics, quantum mechanics, fluid mechanics, solid state physics, hydrodynamics, bio-genetics, chemical kinematics, etc., can be described by non-linear evolution equations. However, in recent times, it has been found that many of these physical, chemical and biological processes are governed by non-linear evolution equations of non-integer or fractional order [31, 28, 17, 36]. Various attempts by different authors at obtaining the exact solutions to these non-linear evolution equations in order to better understand the phenomena they describe have stimulated the discovery of an impressive array of methods. These methods include the inverse scattering transform [2], the Backlund transform [29, 34], the Darboux transform [27], the Hirota bilinear method [18], the tanh-function method [11], the sine-cosine method [26, 38], the exp-function method [16], the generalized Riccati equation [39], the Homogeneous balance method [12], the $\left(G'/G\right)$ expansion method [35, 41], the modified simple equation method [19, 42, 44].

The first integral method introduced by Feng [13] is a very effective tool for generating exact soliton and periodic solutions to non-linear evolution equations.

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The advantage of the first integral method is that it often produces new solutions due to its algorithm not depending on the use of an auxiliary equation, which is often the case in other methods. Raslan [33] applied the first integral method to solve the Fisher equation. Abbasbandy and Shirzadi [1] solved the modified Benjamin-Bona-Mahony equation with the first integral method. Gong, Tian and Wang [14] considered the non-linear Duffing van der Pol type oscillator system by means of the first integral method. Bekir and Unsal [5] showed the applicability of the method to the combined KdV-mKdV equation and the Pochhammer-Chree equation.

In this paper, we apply first integral method to two space and time fractional biological models (non-linear blood flow in large vessels and Deoxyribonucleic acid (DNA) molecule dynamics) defined in the sense of Jumaries modified Riemann-Liouville derivative via a fractional complex transformation. The Jumaries modified Riemann-Liouville derivative of order \( \sigma \) with respect to \( x \) is defined as [20]:

\[
D_x^\sigma f(x) = \begin{cases} \\
\frac{1}{\Gamma(1 - \sigma)} \int_0^x (x - \xi)^{-\sigma - 1} \left[ f(\xi) - f(0) \right] d\xi, & \sigma < 0 \\
\frac{1}{\Gamma(1 - \sigma)} \frac{d}{dx} \int_0^x (x - \xi)^{-\sigma} \left[ f(\xi) - f(0) \right] d\xi, & 0 < \sigma < 1 \\
\left[ f(\sigma - n)(x) \right]^{(n)}, & n \leq \sigma < n + 1, \ n \geq 1 
\end{cases}
\] (1)

Some useful properties of the modified Riemann-Liouville derivative are listed below [20, 21]:

\[
D_x^\sigma x^k = \frac{\Gamma(1 + k)}{\Gamma(1 + k - \sigma)} x^{k-\sigma} 
\] (2)

\[
D_x^\sigma (f(x)g(x)) = g(x)D_x^\sigma f(x) + f(x)D_x^\sigma g(x) 
\] (3)

This paper is organized as follows. In Section 2, the first integral method is introduced and explained. In Section 3, the first integral method is applied to obtain solutions to two space and time fractional biological models with non-linear blood flow in large vessels model and Deoxyribonucleic acid (DNA) molecule dynamics model presented in Section 3.1 and 3.2 respectively. Finally, a conclusion is given in Section 4.

2. Description of the first integral method. Here, we provide a brief explanation of the first integral method for finding travelling wave solutions of non-linear fractional evolution equations. Suppose the non-linear fractional evolution equation is in the following form:

\[
\wp \left( u, D_t^\gamma u, D_x^\sigma u, D_t^{2\gamma} u, D_x^{2\sigma} u, D_x^\sigma D_t^\gamma u, \ldots \right) = 0 \quad 0 < \sigma, \gamma \leq 1 
\] (4)

\( \wp \) is a polynomial of \( u(x, t) \) and its derivatives (integer and fractional) with respect to \( x \) and \( t \). \( \sigma \) and \( \gamma \) are parameters that describe the order of the space and time derivatives respectively.

Theorem 2.1. Fractional complex transformation. To transform Eq. 4 into a non-linear ordinary differential equation (ODE) of integer order by applying a fractional complex transformation proposed by Li and He [25]:

\[
u(x, t) = U(\xi), \quad \xi = \frac{x^\sigma}{\Gamma(1 + \sigma)} - \frac{V t^\gamma}{\Gamma(1 + \gamma)} 
\] (5)

where \( V \) is an arbitrary constant and Eq. 4 reduces to a non-linear integer order ODE of the form

\[
P \left( u, u', u'', \ldots \right) = 0 
\] (6)
Next, we introduce new independent variables $X(\xi)$ and $Y(\xi)$ such that
\[ X(\xi) = f(\xi), \quad Y(\xi) = \frac{\partial f}{\partial \xi} = f_\xi(\xi) \] (7)
This leads to a system of non-linear ordinary differential equations:
\[ X_\xi(\xi) = Y(\xi), \quad Y_\xi(\xi) = F(X(\xi), Y(\xi)) \] (8)
According to the qualitative theory of ordinary differential equations [9], if we can find the integrals to 8 under the same condition, then the general solutions to 8 can be solved directly. Generally, it is very difficult to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that informs us on how to obtain its first integrals, nor is there a logical way for telling us what these first integrals are. We apply the Division theorem to obtain one first integral to equation 8 which reduces equation 6 to a first-order integrable ordinary differential equation. This equation is then solved to obtain the exact solution to equation 4.

**Theorem 2.2. Division Theorem [13].** Suppose that $P(y, w)$, $Q(y, w)$ are polynomials in $C[y, w]$ and $P(y, w)$ is irreducible in $C[y, w]$. If $Q(y, w)$ vanishes at all zero points of $P(y, w)$, then there exists a polynomial $G(y, w)$ in $C[y, w]$ such that
\[ Q[y, w] = P[y, w]G[y, w] \] (9)
The Division theorem follows immediately from the Hilbert-Nullstellensatz Theorem of commutative algebra [6].

3. Application.

3.1. Model for blood flow in large vessels. Hashimuze [15] and Yomosa [40] working independently showed that blood flow is expected to behave like a non-linear wave or soliton due to its pulsatile nature and other physiological factors [7, 8, 23]. We consider the blood flow in the vessel to be Newtonian, viscous, homogeneous and incompressible. Therefore, the dynamics of blood flow in large vessels is governed by the Navier-Stokes and the continuity equations as [30, 10, 23]
\[ \frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(AW) = 0, \quad \frac{\partial W}{\partial t} + W \frac{\partial W}{\partial z} + \frac{\partial P}{\partial z} = 0 \] (10)
where $z$ is the axial coordinate, $t$ is time, $W$ is the axial component of the fluid velocity, $A$ is the cross-sectional area of the blood vessel and $P$ is the pressure inside the vessel. For the wall dynamics, applying Newton’s second law on a portion of the vessel wall gives the pressure as [30, 10, 23].
\[ P = \frac{2}{A+1} \frac{\partial^2 A}{\partial t^2} + 2(A-1) \frac{2+(A-1)}{(A+1)^2} \] (11)
The space and time fractional model for non-linear blood flow in large vessels is given by
\[ \frac{\partial^\gamma A}{\partial t} + \frac{\partial^\sigma (AW)}{\partial z} = 0, \] (12)
\[ \frac{\partial^\gamma W}{\partial t} + W \frac{\partial^\sigma W}{\partial z} + \frac{\partial^\sigma P}{\partial z} = 0 \] (13)
where $\sigma$ and $\gamma$ are parameters that describe the order of the space and time derivatives respectively. Here we apply the first integral method to this space and time fractional model. Substituting Eq. 11 into Eq. 13, we have

$$\frac{\partial^\sigma A}{\partial t^\sigma} + \frac{\partial^\sigma W}{\partial z} (AW) = 0, \quad (14)$$

$$\frac{\partial^\sigma W}{\partial t^\sigma} + W \frac{\partial^\sigma W}{\partial z} + \frac{\partial^\sigma}{\partial z} \left[ \frac{2}{A+1} \frac{\partial^{2\sigma} A}{\partial t^{2\sigma}} + 2 (A-1) \frac{2 + (A-1)}{(A+1)^2} \right] = 0. \quad (15)$$

We make the fractional complex transformation $W(z, t) = W(\xi), A(z, t) = A(\xi)$ by combining the independent variables $z$ and $t$ into one variable $\xi = z^\sigma/\Gamma(1+\sigma) - Vt^\gamma/\Gamma(1+\gamma)$. Then, Eqs. 14 and 15 becomes

$$-VA' + AW' = 0, \quad (16)$$

$$-VW' + WW' + 2 \left[ \frac{V^2 A''}{A+1} + (A-1) \frac{2 + (A-1)}{(A+1)^2} \right]' = 0. \quad (17)$$

Integrating Eqs. 16 and 17 once with respect to $\xi$ yields

$$-VA + AW + K_1 = 0, \quad (18)$$

$$-VW + \frac{W^2}{2} + 2 \frac{V^2 A''}{A+1} + 2 (A-1) \frac{2 + (A-1)}{(A+1)^2} + K_2 = 0. \quad (19)$$

where $K_1$ and $K_2$ are constants of integration. From Eq. 18, it is possible to express $W$ as a function of $A$. Substituting this into Eq. 19 and taking $K_2 = 0$ leads to

$$\frac{2V^2 A''}{A+1} + 2 (A-1) \frac{2 + (A-1)}{(A+1)^2} - \frac{V (VA - K_1)}{A} + \frac{(VA - K_1)^2}{2A^2} = 0$$

$$2V^2 A^2 (A+1) A'' + 2A^2 (A-1)(2 + (A-1)) - VA(A+1)^2 (VA - K_1)$$

$$+ \frac{1}{2} (A+1)^2 (VA - K_1)^2 = 0$$

$$A'' = \frac{2VA(A+1)^2 (VA - K_1) - 4A^2 (A^2 - 1) - (A+1)^2 (VA - K_1)^2}{4V^2 A^2 (A+1)} \quad (20)$$

Using equation 7, we get

$$X_\xi(\xi) = Y(\xi), \quad (21)$$

$$Y_\xi(\xi) = \frac{2VX(\xi)(X(\xi) + 1)^2 (VX(\xi) - K_1) - 4X(\xi)^2 (X(\xi)^2 - 1)}{4V^2X(\xi)^2 (X(\xi) + 1)}$$

$$- \frac{(X(\xi) + 1) (VX(\xi) - K_1)^2}{4V^2X(\xi)^2}. \quad (22)$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are non-trivial solutions of 21 and 22, and $q(X, Y) = \sum_{i=0}^m a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q [X(\xi), Y(\xi)] = \sum_{i=0}^m a_i(X)Y^i = 0 \quad (23)$$
\(a_i(X)\) \((i = 0, 1, 2, \ldots, m)\) are polynomials of \(X\) and \(a_m(X) \neq 0\). Eq. 23 is the first integral to 21 and 22. Due to the Division Theorem, there exists a polynomial \(g(X) + h(X)Y\) in the complex domain \(C[X, Y]\) such that

\[
\frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y) \sum_{i=0}^{m} a_i(X)Y^i
\]

(24)

\[
\sum_{i=0}^{m} ia_iY^{i-1} \left[ \frac{2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2}{4V^2X^2(X + 1)} \right] + \sum_{i=0}^{m} a_iY^{i+1} = [g(X) + h(X)Y] \sum_{i=0}^{m} a_iY^i
\]

(25)

Here, we take two different cases, corresponding to when \(m = 1\) and \(m = 2\) in Eq. 25.

**Case 1.** Suppose that \(m = 1\), by equating the coefficients of \(Y^i\) for \(i = 2, 1, 0\) on both sides of Eq. 25, we have

\[
\dot{a}_1(X) = h(X)a_1(X)
\]

(26)

\[
\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X)
\]

(27)

\[
a_1(X) \left[ 2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2 \right] = g(X)a_0(X) \left[ 4V^2X^2(X + 1) \right]
\]

(28)

Since \(a_i(X)\) \((i = 0, 1, 2, \ldots, m)\) are polynomials, then from Eq. 26 we deduce that \(a_1(X)\) is a constant and \(h(X) = 0\). For simplicity, take \(a_1(X) = 1\), then Eq. 27 becomes

\[
\dot{a}_0(X) = g(X)
\]

(29)

\[
[2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2] = g a_0 \left[ 4V^2X^2(X + 1) \right]
\]

(30)

Balancing the degrees of \(g(X)\) and \(a_0(X)\) in Eq. 30, we conclude that \(deg(g(X)) = 0\) only. Suppose that \(g(X) = B_0\), and \(B_0 \neq 0\), then we get \(a_0(X)\) as

\[
a_0(X) = B_0X + B_1
\]

(31)

Substituting \(a_0(X), a_1(X)\) and \(g(X)\) in Eq. 31 and setting all the coefficients of \(X^i\) \((i = 0, 1, 2, \ldots)\) to zero, we obtain a system of non-linear algebraic equations and by solving it, we obtain

\[
K_1 = 0, \quad B_1 = \frac{1 - 2B_0^2}{2B_0}, \quad V = \pm \frac{2}{\sqrt{1 - 4B_0^2}}
\]

(32)

Using Eq. 32 in Eq. 23, we obtain the first integral to Eqs. 21 and 22 as

\[
Y = -B_0X + B_0 - \frac{1}{2B_0}
\]

(33)

Solving Eq. 33, we get the exact solutions to the space and time fractional model for non-linear blood flow in large vessels as

\[
A(z, t) = \exp \left( -B_0z + C \right) + \frac{2B_0^2 - 1}{2B_0^2}
\]

(34)
where

\[ \chi = \frac{z^\gamma}{\Gamma(1+\sigma)} \pm \frac{2t^\gamma}{\Gamma(1+\gamma)\sqrt{1-4B_0^2}} \]

\[ P(z, t) = \frac{2}{3} \left[ \cosh(B_0\chi - C) - \sinh(B_0\chi - C) \right] \]

\[ + \frac{8B_0^2 \left[ \cosh(B_0\chi - C) - \sinh(B_0\chi - C) \right]}{(1-4B_0^2) \left[ 3 + \cosh(B_0\chi - C) - \sinh(B_0\chi - C) - \frac{1}{B_0^2} \right]} \]  \hfill (35)

where \( C \) is a constant of integration and the value of \( B_0 \) is neither 0 or 1/2.

**Case 2.** Suppose that \( m = 2 \), by equating the coefficients of \( Y^i(i = 3, 2, 1, 0) \) on both sides of Eq. 25, we have

\[ \dot{a}_2(X) = h(X)a_2(X) \]  \hfill (36)

\[ \dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X) \]  \hfill (37)

\[ \dot{a}_0(X) \left[ 4V^2X^2(X + 1) \right] \]

\[ + 2a_2(X) \left[ 2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2 \right] \]

\[ = (g(X)a_1(X) + h(X)a_0(X)) \left[ 4V^2X^2(X + 1) \right] \]  \hfill (38)

\[ a_1(X) \left[ 2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2 \right] \]

\[ = g(X)a_0(X) \left[ 4V^2X^2(X + 1) \right] \]  \hfill (39)

Since \( a_i(X) (i = 0, 1, 2) \) are polynomials, then from Eq. 36, we deduce that \( a_2(X) \) is a constant and \( h(X) = 0 \). For simplicity, we take \( a_2(X) = 1 \), then Eqs. 37 - 39 becomes

\[ \dot{a}_1(X) = g(X) \]  \hfill (40)

\[ \dot{a}_0(X) \left[ 4V^2X^2(X + 1) \right] \]

\[ + 2 \left[ 2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2 \right] \]

\[ = g(X)a_1(X) \left[ 4V^2X^2(X + 1) \right] \]  \hfill (41)

\[ a_1(X) \left[ 2VX(X + 1)^2(VX - K_1) - 4X^2(X^2 - 1) - (X + 1)^2(VX - K_1)^2 \right] \]

\[ = g(X)a_0(X) \left[ 4V^2X^2(X + 1) \right] \]  \hfill (42)

Balancing the degrees of \( g(X) \) and \( a_0(X) \) in Eq. 42, we conclude that \( \text{deg}(g(X)) = 0 \) only. Suppose that \( g(X) = B_0 \), and \( B_0 \neq 0 \), then we obtain \( a_1(X) \) and \( a_0(X) \) as

\[ a_1(X) = B_0X + B_1 \]  \hfill (43)

\[ a_0(X) = \frac{K_1^2 \ln(X)}{2V^2} - \frac{K_1^2}{2V^2X} - 2X \left( \frac{4 + V^2}{4V^2} - \frac{B_0B_1}{2} \right) + X^2 \left( \frac{4 - V^2}{4V^2} + \frac{B_0^2}{2} \right) \]  \hfill (44)

Substituting \( a_0(X) \), \( a_1(X) \) and \( g(X) \) in Eq. 42 and setting all the coefficients of \( X^i(i = 0, 1, 2, \ldots) \) to zero, we obtain a system of non-linear algebraic equations and by solving it, we obtain

\[ K_1 = 0, \quad B_0 = \pm \sqrt{2}, \quad B_1 = \frac{1}{2} \left( 2B_0 - B_0^3 \right), \quad V = \pm 2\sqrt{B_0^2 - 3} \]  \hfill (45)
Using Eq. 45 in Eq. 23, we obtain the first integral to Eqs. 21 and 22 as

\[ Y^2 \pm \sqrt{2}XY + \frac{X^2}{2} = 0 \]  

(46)

Solving Eq. 46, we get the exact solutions to the space and time fractional model for non-linear blood flow in large vessels as

\[ P(z,t) = \frac{2}{\exp\left(\chi_{\sqrt{2}} + C\right) + 1} - \frac{4\gamma^2t^{2\gamma-2} \exp\left(\frac{\chi}{\sqrt{2}} + C\right)}{\left(\Gamma(1 + \gamma)\right)^2 \left(\exp\left(\frac{\chi}{\sqrt{2}} + C\right) + 1\right)} + \frac{2i\sqrt{2}t^{-2\gamma}(\gamma - 1) \exp\left(\frac{\chi}{\sqrt{2}} + C\right)}{\Gamma(1 + \gamma) \left(\exp\left(\frac{\chi}{\sqrt{2}} + C\right) + 1\right)} \]  

(47)

where

\[ \chi = \frac{z^\sigma}{\Gamma(1 + \sigma)} \pm i \frac{2t^\gamma}{\Gamma(1 + \gamma)}, \]

and \( C \) is a constant of integration. The solution obtained in Eq. 47 is complex in nature.

3.2. **Deoxyribonucleic (DNA) model.** The Deoxyribonucleic acid (DNA) molecule which is the carrier of information for life and reproduction of organisms consists of two long elastic poly-nucleotide chains or strands, connected to each other by an elastic membrane representing the hydrogen bonds between the pair of bases in the two chains. Recently, the investigation of DNA dynamics has successfully predicted inherent non-linear structures, which are responsible for forming localized waves that transport energy without dissipation [37, 32, 24, 3, 4, 43]. We consider a non-linear model of DNA molecule out-of-phase motion developed by [24] based on an earlier lattice model presented in [32]. The model is written in the form [24, 4]:

\[ \frac{\partial^2 \phi}{\partial t^2} - c_1^2 \frac{\partial^2 \phi}{\partial x^2} = A\phi^3 + B\phi^2 + C\phi \]  

(48)

where

\[ c_1^2 = \frac{Y}{\rho}, \quad A = \frac{4a^2\alpha - 2\alpha}{h^3}, \quad B = \frac{6\sqrt{2}a\alpha}{h^2}, \quad C = \frac{6\alpha}{h} - \frac{2\alpha}{l_0}, \quad \alpha = \frac{\mu l_0}{\rho \lambda} \]

Here \( \rho \), \( \lambda \) and \( Y \) are respectively the mass density, the area of transverse cross-section and the Young’s modulus of each strand; \( h \) is the distance between the two strands, \( \mu \) is the rigidity of the elastic membrane and \( l_0 \) is the height of the membrane in the equilibrium position. Recently, Knyazev and Knyazev [22] constructed solutions to Eq. 48 by using the Hirota bilinear method, Alka et. al [4] presented Riccati generalized solitary wave solutions to Eq. 48 via the elliptic equation method and Zayed and Arnous [43] applied the generalized Riccati equation mapping method to find many travelling wave solutions to the model.

In this section, we apply the first integral method to the space and time fractional DNA model given by

\[ \frac{\partial^{2\gamma} \phi}{\partial t^{2\gamma}} - c_1^2 \frac{\partial^{2\sigma} \phi}{\partial x^{2\sigma}} = A\phi^3 + B\phi^2 + C\phi \]  

(49)
We make the fractional complex transformation \( \phi(x, t) = \phi(\xi) \) by combining the independent variables \( x \) and \( t \) into one variable \( \xi = x^n/\Gamma(1 + \sigma) - t^n/\Gamma(1 + \gamma) \). Then Eq. 49 becomes

\[
V^2 \phi'' - c_1^2 \phi'' = A\phi^3 + B\phi^2 + C\phi
\]

Using equation 7, we get

\[
\phi'' = \left( \frac{A}{V^2 - c_1^2} \right) \phi^3 + \left( \frac{B}{V^2 - c_1^2} \right) \phi^2 + \left( \frac{C}{V^2 - c_1^2} \right) \phi
\]

(51)

Then Eq. 49 becomes

\[
V^2 \phi'' - c_1^2 \phi'' = A\phi^3 + B\phi^2 + C\phi
\]

(50)

Using equation 7, we get

\[
X_\xi(\xi) = Y(\xi),
\]

(52)

\[
Y_\xi(\xi) = \left( \frac{A}{V^2 - c_1^2} \right) X(\xi)^3 + \left( \frac{B}{V^2 - c_1^2} \right) X(\xi)^2 + \left( \frac{C}{V^2 - c_1^2} \right) X(\xi).
\]

(53)

According to the first integral method, we suppose that \( X(\xi) \) and \( Y(\xi) \) are nontrivial solutions of 52 and 53, and \( q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i \) is an irreducible polynomial in the complex domain \( C[X, Y] \) such that

\[
q [X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0
\]

(54)

where \( a_i(X) \) (\( i = 0, 1, 2, \ldots, m \)) are polynomials of \( X \) and \( a_m(X) \neq 0 \). Eq. 54 is the first integral to 52 and 53. Due to the Division Theorem, there exists a polynomial \( g(X) + h(X)Y \) in the complex domain \( C[X, Y] \) such that

\[
\frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y) \sum_{i=0}^{m} a_i(X)Y^i
\]

(55)

\[
\sum_{i=0}^{m} a_i Y^{i+1} + \sum_{i=0}^{m} ia_i Y^{i-1} \left[ \left( \frac{A}{V^2 - c_1^2} \right) X^3 + \left( \frac{B}{V^2 - c_1^2} \right) X^2 + \left( \frac{C}{V^2 - c_1^2} \right) X \right] = \left[ g(X) + h(X)Y \right] \sum_{i=0}^{m} a_i Y^i
\]

(56)

Here, we take two different cases, corresponding to when \( m = 1 \) and \( m = 2 \) in Eq. 56.

Case 1. Suppose that \( m = 1 \), by equating the coefficients of \( Y^i (i = 2, 1, 0) \) on both sides of Eq. 56, we have

\[
\dot{a}_1(X) = h(X)a_1(X)
\]

(57)

\[
\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X)
\]

(58)

\[
a_1(X) \left[ \left( \frac{A}{V^2 - c_1^2} \right) X^3 + \left( \frac{B}{V^2 - c_1^2} \right) X^2 + \left( \frac{C}{V^2 - c_1^2} \right) X \right] = g(X)a_0(X)
\]

(59)

Since \( a_i(X) \) (\( i = 0, 1, 2, \ldots, m \)) are polynomials, then from Eq. 57 we deduce that \( a_1(X) \) is a constant and \( h(X) = 0 \). For simplicity, take \( a_1(X) = 1 \), then Eq. 58 becomes

\[
\dot{a}_0(X) = g(X)
\]

(60)

\[
\left( \frac{A}{V^2 - c_1^2} \right) X^3 + \left( \frac{B}{V^2 - c_1^2} \right) X^2 + \left( \frac{C}{V^2 - c_1^2} \right) X = g(X)a_0(X)
\]

(61)
Balancing the degrees of \( g(X) \) and \( a_0(X) \) in Eq. 61, we conclude that \( \text{deg}(g(X)) = 1 \) only. Suppose that \( g(X) = B_0X + B_1 \), and \( B_0 \neq 0 \), then we get \( a_0(X) \) as

\[
a_0(X) = \frac{1}{2} B_0^2 X^2 + B_1 X + B_2 \tag{62}
\]

Substituting \( a_0(X) \), \( a_1(X) \) and \( g(X) \) in Eq. 61 and setting all the coefficients of \( X^i (i = 0, 1, 2, \ldots) \) to zero, we obtain a system of non-linear algebraic equations and by solving it, we obtain

\[
a = \pm \frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}}, \quad B_1 = \frac{\sqrt{2} a B_0 h}{2a^2 - 1}, \quad B_2 = 0, \quad V = \pm \sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0)}{h^3 \lambda pB_0^2 (h - l_0)}} \tag{63}
\]

Using Eq. 63 in 54, we obtain the first integral to Eq. 52 and 53 as

\[
Y = \frac{1}{2} B_0 X \left( -X \pm \frac{h \sqrt{h - 3l_0} \sqrt{h - l_0}}{l_0} \right) \tag{64}
\]

Solving Eq. 64, we get the exact solutions to the space and time fractional DNA model as

\[
\phi(x, t) = \frac{\Delta \exp \left[ \frac{\Delta B_0}{2l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right]}{l_0 \exp \left[ \frac{\Delta B_0}{2l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right] - \exp(\Delta k)} \tag{65}
\]

where \( a = \frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}} \), \( V = \sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0)}{h^3 \lambda pB_0^2 (h - l_0)}} \), \( \Delta = h \sqrt{h - 3l_0} \sqrt{h - l_0} \) and \( k \) is a constant of integration.

\[
\phi(x, t) = \frac{-\Delta \exp(\Delta k)}{l_0 \exp(\Delta k) - \exp \left[ \frac{\Delta B_0}{2l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right]} \tag{66}
\]

where \( a = -\frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}} \), \( V = -\sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0)}{h^3 \lambda pB_0^2 (h - l_0)}} \), \( \Delta = h \sqrt{h - 3l_0} \sqrt{h - l_0} \) and \( k \) is a constant of integration.

**Case 2.** Suppose that \( m = 2 \), by equating the coefficients of \( Y^i (i = 3, 2, 1, 0) \) on both sides of Eq. 56, we have

\[
\dot{a}_2(X) = h(X) a_2(X) \tag{67}
\]

\[
\dot{a}_1(X) = g(X) a_2(X) + h(X) a_1(X) \tag{68}
\]

\[
\dot{a}_0(X) + 2a_2(X) \left[ \left( \frac{A}{V^2 - c^2_1} \right) X(\xi)^3 + \left( \frac{B}{V^2 - c^2_1} \right) X(\xi)^2 + \left( \frac{C}{V^2 - c^2_1} \right) X(\xi) \right] = g(X) a_1(X) + h(X) a_0(X) \tag{69}
\]

\[
a_1(X) \left[ \left( \frac{A}{V^2 - c^2_1} \right) X(\xi)^3 + \left( \frac{B}{V^2 - c^2_1} \right) X(\xi)^2 + \left( \frac{C}{V^2 - c^2_1} \right) X(\xi) \right] = g(X) a_0(X) \tag{70}
\]

Since \( a_i(X) (i = 0, 1, 2) \) are polynomials, then from Eq. 67, we deduce that \( a_2(X) \) is a constant and \( h(X) = 0 \). For simplicity, we take \( a_2(X) = 1 \), then Eqs. 67 - 70 becomes

\[
\dot{a}_1(X) = g(X) \tag{71}
\]

\[
\dot{a}_0(X) + 2 \left[ \left( \frac{A}{V^2 - c^2_1} \right) X(\xi)^3 + \left( \frac{B}{V^2 - c^2_1} \right) X(\xi)^2 + \left( \frac{C}{V^2 - c^2_1} \right) X(\xi) \right] = g(X) a_1(X) \tag{72}
\]
\[ a_1(X) \left[ \left( \frac{A}{V^2 - c_1^2} \right) X(\xi) + \left( \frac{B}{V^2 - c_1^2} \right) X(\xi)^2 + \left( \frac{C}{V^2 - c_1^2} \right) X(\xi) \right] = g(X)a_0(X) \]  

(73)

Balancing the degrees of \( g(X) \) and \( a_0(X) \) in Eq. 73, we conclude that \( \text{deg}(g(X)) = 1 \) only. Suppose that \( g(X) = B_0X + B_1 \), and \( B_0 \neq 0 \), then we obtain \( a_0(X) \) and \( a_1(X) \) as

\[ a_1(X) = \frac{1}{2} B_0^2 X^2 + B_1 X + B_2 \]  

(74)

\[ a_0(X) = \left[ \frac{B_0^2}{8} - \frac{A}{2(V^2 - c_1^2)} \right] X^4 + \left[ \frac{B_0 B_1}{2} - \frac{2B}{3(V^2 - c_1^2)} \right] X^3 \]

\[ + \left[ \frac{1}{3} (B_1^2 + B_0 B_2) - C \right] X^2 + B_1 B_2 X \]  

(75)

Substituting \( a_0(X) \), \( a_1(X) \) and \( g(X) \) in Eq. 73 and setting all the coefficients of \( X^i (i = 0, 1, 2, \ldots) \) to zero, we get a system of non-linear algebraic equations and by solving it, we obtain

\[ a = \pm \frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}}, \quad B_1 = \frac{\sqrt{2aB_0h}}{2a^2 - 1}, \quad B_2 = 0, \quad V = \pm \sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0) - 32 \mu l_0^2}{h^3 \lambda \rho B_0^2 (h - l_0)}} \]  

(76)

Using Eq. 76 in 54, we obtain the first integral to Eq. 52 and 53 as

\[ \frac{B_0^2 X^2 \left( h^4 - 2l_0 \left( 2h^3 \pm X h \sqrt{h - 3l_0 \sqrt{h - l_0}} \right) + l_0^2 \left( 3h^2 + X^2 \right) \right)}{16l_0^3} \]

\[ - \frac{1}{2} Y^2 B_0 \left( -X \pm \frac{h \sqrt{h - 3l_0 \sqrt{h - l_0}}}{l_0} \right) + Y^2 = 0 \]  

(77)

Solving Eq. 77, we get the exact solutions to the space and time fractional DNA model as

\[ \phi(x, t) = \frac{\Delta \exp \left[ \frac{\Delta B_0}{4l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right]}{l_0 \exp \left[ \frac{\Delta B_0}{4l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right] - \exp(\Delta k)} \]  

(78)

where \( a = \frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}}, V = \sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0) - 32 \mu l_0^2}{h^3 \lambda \rho B_0^2 (h - l_0)}}, \Delta = h \sqrt{h - 3l_0 \sqrt{h - l_0}} \) and \( k \) is a constant of integration.

\[ \phi(x, t) = \frac{-\Delta \exp(\Delta k)}{l_0 \exp(\Delta k) - \exp \left[ \frac{\Delta B_0}{2l_0} \left( \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{V t^\gamma}{\Gamma(1+\gamma)} \right) \right]} \]  

(79)

where \( a = -\frac{\sqrt{h - 3l_0}}{\sqrt{2h - 2l_0}}, V = -\sqrt{\frac{h^3 \lambda Y B_0^2 (h - l_0) - 32 \mu l_0^2}{h^3 \lambda \rho B_0^2 (h - l_0)}}, \Delta = h \sqrt{h - 3l_0 \sqrt{h - l_0}} \) and \( k \) is a constant of integration.

**Remark 1.** The solution to the time and space fractional model for non-linear blood flow obtained in Eq. 47 is complex in nature and therefore not valid for the current application.

**Remark 2.** The plots to the solution of the time and space fractional model for non-linear blood flow obtained in Eq. 35 are presented in Figures 1 and 2 for selected values of the model parameters at different \( \sigma \) and \( \gamma \) values. From Figures 1 and 2, we observe that the nature of the sign on \( V = \pm \frac{2}{\sqrt{1-4B_0^2}} \) creates different shapes of
the solution in Eq. 35. For $V = \frac{2}{\sqrt{1-4B_0^2}}$ at different $\sigma$ and $\gamma$ values, the plot of Eq. 35 in Figure 1 is generally flat characterized with $P(z,t) = 2.5$ and a sharp drop in $P(z,t)$ near $z,t = 0$. The effect of the different $\sigma$ and $\gamma$ values is a change in the nature of the drop in $P(z,t)$ near $z,t = 0$ with $\sigma = 1$ and $\gamma = 1$ corresponding to the integer order case. In Figure 2, for $V = -\frac{2}{\sqrt{1-4B_0^2}}$, the plots are generally characterized by spikes with the different $\sigma$ and $\gamma$ values having a considerable effect on the nature and position of the spikes. Hence, introducing fractional order to the model for non-linear blood flow changes the nature of the solution.

**Remark 3.** Figures 3-6 represent the plots of Eqs. 65, 66, 78 and 79 respectively for selected values of the model parameters at different $\sigma$ and $\gamma$ values. We observe also that the different $\sigma$ and $\gamma$ values have a considerable effect on the nature of the plot.

**Remark 4.** All the travelling wave solutions to the time and space fractional models for non-linear blood flow in large vessels and Deoxyribonucleic acid (DNA) molecule dynamics obtained using the first integral method were checked by putting them back into the equations with the aid of Mathematica.

**4. Conclusion.** In this study, we have showed the efficacy of the first integral method for generating exact soliton and periodic solutions to fractional non-linear evolution equations. The time and space fractional models for non-linear blood flow in large vessels and Deoxyribonucleic acid (DNA) molecule dynamics were defined in the sense of Jumarie’s modified Riemann-Liouville derivative via a fractional complex transformation. The validity of the solutions to the fractional biological models obtained with first integral method was achieved by putting them back into the models with the aid of Mathematica. Thus, we conclude that the first integral method can be extended to solve other non-linear biological problems.

**REFERENCES**

[1] S. Abbasbandy and A. Shirzadi, The first integral method for modified benjamin-bona-mahony equation, *Communications in Nonlinear Science and Numerical Simulation*, 15 (2010), 1759–1764.

[2] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, Pa, USA, 1981.

[3] M. Aguero, M. Najera and M. Carrillo, Non classic solitonic structures in dna vibrational dynamics, *Int. J. Modern Phys. B.*, 22 (2008), 2571–2582.

[4] W. Alka, A. Goyal and C. N. Kumar, Nonlinear dynamics of DNA-Riccati generalized solitary wave solutions, *Physics Letters A*, 375 (2011), 480–483.

[5] A. Bekir and O. Unsal, Analytic treatment of nonlinear evolution equations using first integral method, *Pramana*, 79 (2012), 3–17.

[6] N. Bourbaki, *Commutative Algebra*, Addison-Wesley, Paris, 1972.

[7] H. Demiray, Weakly nonlinear waves in a viscous fluid contained in a viscoelastic tube with variable cross-section, *Eur J. Mech. A, Solid*, 24 (2005), 337–347.

[8] H. Demiray, Variable coefficient modified kdv equation in fluid-filled elastic tubes with stenosis: Solitary waves, *Chaos, Solitons and Fractals*, 42 (2009), 358–364.

[9] T. R. Ding and C. Z. Li, *Ordinary Differential Equations*, Peking University Press, Peking, 1996.

[10] B. Eliasson and P. K. Shukla, Formation and dynamics of finite amplitude localized pulses in elastic tubes, *Phys. Rev. E.*, 71 (2005), 067302.

[11] E. Fan, Extended tanh-function method and its applications to nonlinear equations, *Physics Letters A*, 277 (2000), 212–218.

[12] E. G. Fan, Two new applications of the homogeneous balance method, *Physics Letters A*, 265 (2000), 353–357.
[13] Z. S. Feng, Exact solution to an approximate sine-gordon equation in (n+1)-dimensional space, *Physics Letters A*, **302** (2002), 64–76.

[14] X. Gong, J. Tian and J. Wang, First integral method for an oscillator system, *Electronic Journal of Differential Equations*, **96** (2013), 1–12.

[15] Y. Hashimuze, Nonlinear pressure waves in a fluid-filled elastic tube, *J. Phys. Soc. Jpn.*, **54** (1985), 3305–3312.

[16] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons and Fractals*, **30** (2006), 700–708.

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, New Jersey, NJ, USA, 2000.

[18] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, *Journal of Mathematical Physics*, **14** (1973), 805–809.

[19] A. J. M. Jawad, M. D. Petković and A. Biswas, Modified simple equation method for nonlinear evolution equations, *Applied Mathematics and Computation*, **217** (2010), 869–877.

[20] G. Jumarie, Modified riemann-liouville derivative and fractional taylor series of nondifferentiable functions further results, *Computers and Mathematics with Applications*, **51** (2006), 1367–1376.

[21] G. Jumarie, Fractional partial differential equations and modified riemann-liouville derivative new methods for solution, *Journal of Applied Mathematics and Computing*, **24** (2007), 31–48.

[22] M. A. Knyazev and D. M. Knyazev, New kink-like solutions for nonlinear equation describing the dynamics of dna, *Journal of Physical Studies*, **16** (2012), 1001–1004.

[23] G. R. Kol and C. B. Tabi, Application of the $g'/g$ expansion method to nonlinear blood flow in large vessels, *IC*, **26** (2010), 1–9.

[24] D. X. Kong, S. Y. Lou and J. Zeng, Nonlinear dynamics in a new double chain-model of DNA, *Commun. Theor. Phys.*, **36** (2001), 737–742.

[25] Z. B. Li and J. H. He, Fractional complex transform for fractional differential equations, *Mathematical and Computational Applications*, **15** (2010), 970–973.

[26] W. Malfliet, The tanh method: A tool for solving certain classes of non-linear pdes, *Mathematical Methods in the Applied Sciences*, **28** (2005), 2031–2035.

[27] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, Germany, 1991.

[28] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[29] M. R. Miura, *Backlund Transformation*, Springer-Verlag, Berlin-New York, 1976.

[30] S. Noubissie and P. Woofo, Dynamics of solitary blood waves in arteries with prostheses, *Phys. Rev. E.*, **67** (2003), 041911.

[31] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.

[32] M. Peyrard and A. Bishop, Statistical mechanics of a nonlinear model of dna denaturation, *Phys. Rev. Lett.*, **62** (1989), 2755–2758.

[33] K. R. Raslan, The first integral method for solving some important nonlinear partial differential equations, *Nonlinear Dynamics*, **53** (2008), 281–286.

[34] C. Rogers and W. F. Shadwick, *Backlund Transformation*, Academic Press, New York, NY, USA, 1982.

[35] M. L. Wang, X. Li and J. Zhang, The $(frac(g'/g))$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Physics Letters A*, **372** (2008), 417–423.

[36] B. J. West, M. Bologna and P. Grigolini, *Physics of Fractal Operators*, Springer, New York, NY, USA, 2003.

[37] L. V. Yakushevich, Nonlinear dna dynamics: A new model, *Physics Letters A*, **136** (1989), 413–417.

[38] C. T. Yan, A simple transformation for nonlinear waves, *Physics Letters A*, **224** (1996), 77–84.

[39] Z. Y. Yan and H. Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for whitham-broer-kaup equation in shallow water, *Physics Letters A*, **285** (2001), 355–362.

[40] S. Yomosa, Solitary waves in large blood vessels, *J. Phys. Soc. Jpn.*, **56** (1987), 506–520.

[41] E. M. E. Zayed, Traveling wave solutions for higher dimensional nonlinear evolution equations using the $(g'/g)$- expansion method, *J. Phys. A*, **42** (2009), 195202, 13 pp.

[42] E. M. E. Zayed, A note on the modified simple equation method applied to sharma-tasso-olver equation, *Applied Mathematics and Computation*, **218** (2011), 3962–3964.
[43] E. M. E. Zayed and A. H. Arnous, Many exact solutions for nonlinear dynamics of dna model using the generalized riccati equation mapping method, Scientific Research and Essays, 8 (2013), 340–346.

[44] E. M. E. Zayed and S. A. H. Ibrahim, Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method, Chinese Physics Letters, 29 2012.

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Figure 1. Figure showing $P(z, t)$ with $\chi = \frac{z^\sigma}{\Gamma(1+\sigma)} + 2t^{\gamma}/\Gamma(1+\gamma)\sqrt{1 - 4B_0^2}$ and $0 \leq z, t \leq 20$ for (a) $\sigma = 1, \gamma = 1$, (b) $\sigma = 1, \gamma = 0.5$, (c) $\sigma = 0.5, \gamma = 1$, (d) $\sigma = 0.5, \gamma = 0.5$.

Figure 2. Figure showing $P(z, t)$ with $\chi = \frac{z^\sigma}{\Gamma(1+\sigma)} - 2t^{\gamma}/\Gamma(1+\gamma)\sqrt{1 - 4B_0^2}$ and $0 \leq z, t \leq 20$ for (a) $\sigma = 1, \gamma = 1$, (b) $\sigma = 1, \gamma = 0.5$, (c) $\sigma = 0.5, \gamma = 1$, (d) $\sigma = 0.5, \gamma = 0.5$. 
Figure 3. Figure showing $\phi(x,t)$ (Eq. 65) with $B_0, l, k, Y, \lambda, \mu = 1, h = 3.33, \rho = 0.85, \text{ and } 0 \leq x, t \leq 5$ for (a) $\sigma = 1, \gamma = 1$, (b) $\sigma = 1, \gamma = 0.5$, (c) $\sigma = 0.5, \gamma = 1$, (d) $\sigma = 0.5, \gamma = 0.5$

Figure 4. Figure showing $\phi(x,t)$ (Eq. 66) with $B_0, l, k, Y, \lambda, \mu = 1, h = 3.33, \rho = 0.85, \text{ and } 0 \leq x, t \leq 5$ for (a) $\sigma = 1, \gamma = 1$, (b) $\sigma = 1, \gamma = 0.5$, (c) $\sigma = 0.5, \gamma = 1$, (d) $\sigma = 0.5, \gamma = 0.5$
Figure 5. Figure showing $\phi(x,t)$ (Eq. 78) with $B_0, l, k, Y, \lambda, \mu = 1$, $h = 3.33$, $\rho = 0.85$, and $0 \leq x, t \leq 5$ for (a) $\sigma = 1$, $\gamma = 1$, (b) $\sigma = 1$, $\gamma = 0.5$, (c) $\sigma = 0.5$, $\gamma = 1$, (d) $\sigma = 0.5$, $\gamma = 0.5$

Figure 6. Figure showing $\phi(x,t)$ (Eq. 79) with $B_0, l, k, Y, \lambda, \mu = 1$, $h = 3.33$, $\rho = 0.85$, and $0 \leq x, t \leq 5$ for (a) $\sigma = 1$, $\gamma = 1$, (b) $\sigma = 1$, $\gamma = 0.5$, (c) $\sigma = 0.5$, $\gamma = 1$, (d) $\sigma = 0.5$, $\gamma = 0.5$. 