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ON HOFMANN’S BILINEAR ESTIMATE

PASCAL AUSCHER

Abstract. Using the framework of a previous article joint with Axelsson and McIntosh, we extend to systems two results of S. Hofmann for real symmetric equations and their perturbations going back to a work of B. Dahlberg for Laplace’s equation on Lipschitz domains. The first one is a certain bilinear estimate for a class of weak solutions and the second is a criterion which allows to identify the domain of the generator of the semi-group yielding such solutions.

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1. Introduction

S. Hofmann proved in [11] that weak solutions of

$$\text{div}_{t,x}A(x)\nabla_{t,x}U(t,x) = \sum_{i,j=0}^{n} \partial_i A_{i,j}(x) \partial_j U(t,x) = 0$$

on the upper half space $\mathbb{R}^{1+n}_+ := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n ; t > 0\}$, $n \geq 1$, where the matrix $A = (A_{i,j}(x))_{i,j=0}^{n} \in L_\infty(\mathbb{R}^n ; L(\mathcal{C}^{1+n}))$ is assumed to be $t$-independent and within some small $L_\infty$ neighborhood of a real symmetric strictly elliptic $t$-independent matrix, obey the following bilinear estimate

$$\left| \int \int_{\mathbb{R}^{1+n}_+} \nabla_{t,x}U \cdot \nabla dtdx \right| \leq C \|U_0\|_2 (\|t \nabla v\| + \|N_* v\|_2)$$

for all $\mathcal{C}^{1+n}$-valued field $v$ such that the right-hand side is finite. See below for the definition of the square-function $||| \|$ and the non-tangential maximal operator $N_*$. The trace of $U$ at $t = 0$ is assumed to be in the sense of non-tangential convergence a.e. and in $L_2(\mathbb{R}^n)$.

In addition, he proves that the solution operator $U_0 \rightarrow U(t, \cdot)$ defines a bounded $C_0$ semi-group on $L_2(\mathbb{R}^n)$ whose infinitesimal generator $\mathcal{A}$ has domain $W^{1,2}(\mathbb{R}^n)$ with $\|\mathcal{A}f\|_2 \sim \|\nabla f\|_2$.

Such results were first proved by B. Dahlberg [8] for harmonic functions on a Lipschitz domain. A version of the bilinear estimate for Clifford-valued monogenic functions was proved by Li-McIntosh-Semmes [16]. A short proof of Dahlberg’s estimate for harmonic functions and some applications appear in Mitrea’s work [17]. $L^p$ versions are recently discussed by Varopoulos [20].

Hofmann’s arguments for variable coefficients rely on the deep results of [1], and in particular Theorem 1.11 there where the boundedness and invertibility of the layer potentials are obtained from a $T(b)$ theorem, Rellich estimates in the case of real symmetric matrices and perturbation. This also generalizes somehow the case where $A_{0,i} = A_{i,0} = 0$ for $i = 1, \ldots, n$ corresponding to the Kato square root problem.
The recent works [3,4], pursuing ideas in [2], allow us to extend this further to systems, making clear in particular that specificities of real symmetric coefficients and their perturbations and of equations - in particular the De Giorgi-Nash-Moser estimates - are not needed: it only depends on whether the Dirichlet problem is solvable. We use the solution operator constructed in [3] and the proof using \(P_t - Q_t\) techniques of Coifman-Meyer from [7] makes transparent the para-product like character of this bilinear estimate. We also establish a necessary and sufficient condition telling when the domain of the infinitesimal generator \(A\) of the Dirichlet semi-group is \(W^{1,2}\).

We apologize to the reader for the necessary conciseness of this note and suggest he (or she) has (at least) the references [2, 3, 4] handy. In Section 2, we try to extract from them the relevant information. The proof or the bilinear estimate for variable coefficients systems is in Section 3. Section 4 contains the discussion on the domain of the Dirichlet semi-group.

2. Setting

We begin by giving a precise definition of well-posedness of the Dirichlet problem for systems. Throughout this note, we use the notation \(X \approx Y\) and \(X \lesssim Y\) for estimates to mean that there exists a constant \(C > 0\), independent of the variables in the estimate, such that \(X/C \leq Y \leq CX\) and \(X \leq CY\), respectively.

We write \((t, x)\) for the standard coordinates for \(\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n\), \(t\) standing for the vertical or normal coordinate. For vectors \(v = (v_\alpha)_{1 \leq \alpha \leq m} \in \mathbb{C}^{(1+n)m}\), we write \(v_0 \in \mathbb{C}^m\) and \(v_i \in \mathbb{C}^{nm}\) for the normal and tangential parts of \(v\), i.e., \(v_0 = (v_0^\alpha)_{1 \leq \alpha \leq m}\) whereas \(v_i = (v_i^\alpha)_{1 \leq \alpha \leq m}\).

For systems, gradient and divergence act as \((\nabla_{t,x} U)^\alpha_i = \partial_i U^\alpha\) and \((\text{div}_{t,x} F)^\alpha = \sum_{j=0}^n \partial_j F^\alpha_i\), with corresponding tangential versions \(\nabla_x U = (\nabla_{t,x} U)^\|\) and \((\text{div}_x F)^\alpha = \sum_{j=1}^n \partial_j F^\alpha_i\). With \(\text{curl}_x F_i = 0\), we understand \(\partial_j F_i^\alpha = \partial_i F_j^\alpha\), for all \(i, j = 1, \ldots, n, \alpha = 1, \ldots, m\).

We consider divergence form second order elliptic systems

\[
\sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i A_{i,j}^\alpha(x) \partial_j U^\beta(t,x) = 0, \quad \alpha = 1, \ldots, m,
\]

on the half space \(\mathbb{R}^{1+n}_+ := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \; ; \; t > 0\}\), \(n \geq 1\), where the matrix \(A = (A_{i,j}^\alpha(x))_{i,j=0,\ldots,n} \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))\) is assumed to be \(t\)-independent with complex coefficients and strictly accretive on \(N(\text{curl}_i)\), in the sense that there exists \(\kappa > 0\) such that

\[
\sum_{i,j=0}^n \sum_{\alpha,\beta=1}^m \int_{\mathbb{R}^n} \text{Re}(A^\alpha_{i,j}(x) f_j^\beta(x) \overline{f_i^\alpha(x)}) \, dx \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 \, dx,
\]

for all \(f \in N(\text{curl}_i) := \{g \in L_2(\mathbb{R}^n; \mathbb{C}^{(1+n)m}) \ ; \; \text{curl}_x(g_i) = 0\}\). This is nothing but ellipticity in the sense of Gårding. See the discussion in [3]. By changing \(m\) to \(2m\) we could assume that the coefficients are real-valued. But this does not simplify matters and we need the complex hermitean structure of our \(L_2\) space anyway.

Definition 2.1. The Dirichlet problem (Dir-\(A\)) is said to be well-posed if for each \(u \in L_2(\mathbb{R}^n; \mathbb{C}^m)\), there is a unique function

\[U_i(x) = U(t, x) \in C^1(\mathbb{R}_+; L_2(\mathbb{R}^n; \mathbb{C}^m))\]
where \( \tilde{\nabla} \) such that \( \nabla \) introduced in [13]. Then any function \( u \in L^p_\infty(\mathbb{R}^n; C_0^m) \), where \( U \) satisfies (2) for \( t > 0 \), \( \lim_{t \to 0} U_t = u \), \( \lim_{t \to \infty} U_t = 0 \), \( \lim_{t \to \infty} \nabla_{t,x} U_t = 0 \) in \( L_2 \) norm, and \( \int_0^{t_1} \nabla_{t,x} U_t \, ds \) converges in \( L_2 \) when \( t_0 \to 0 \) and \( t_1 \to \infty \). More precisely, by \( U \) satisfying (2), we mean that \( \int_0^\infty \langle (\nabla_{t,x} U_t), \nabla_{t,x} v \rangle ds = -\langle (\nabla_{t,x} U_t), v \rangle \) for all \( v \in C_0^\infty(\mathbb{R}^n; C_0^m) \).

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [11]. However, a sufficient condition is provided in [3] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem (see Section 4).

**Remark 2.2.** In the case of block matrices, i.e. \( A_{0,i}^\alpha(x) = 0 = A_{i,0}^\alpha(x), 1 \leq i \leq n, 1 \leq \alpha, \beta \leq m \), the second order system (2) can be solved using semi-group theory: \( V(t, \cdot) = e^{-tL^{1/2}} u_0 \) for \( L = -A_{00}^{-1} \text{div}_x A_{01} \xi_{x} \) acting as an unbounded operator on \( L_2(\mathbb{R}^n, C_0^m) \) (See below for the notation). This solution satisfies \( V_t = V(t, \cdot) \in C(\mathbb{R}_+; L_2(\mathbb{R}^n; C_0^m)) \cap C^1(\mathbb{R}_+; D(L^{1/2})), \lim_{t \to 0} V_t = u_0, \lim_{t \to \infty} V_t = 0 \) in \( L_2 \) norm, and (2) holds in the strong sense in \( \mathbb{R}^n \) for all \( t > 0 \) (and in the sense of distributions in \( \mathbb{R}_+^{1+n} \)). Hence, the two notions of solvability are not \( a \ priori \) equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for \( L; D(L^{1/2}) = W^{1,2}(\mathbb{R}^n, C_0^m) \) with \( \|L^{1/2}f\|_2 \sim \|\nabla x f\|_2 \). See [6] where this is explicitly proved when \( A_{00} \neq I \).

The following result is Corollary 3.4 of [3] (which, as we recall, furnishes a different proof of results obtained by combining [12] and [9] in the case of real symmetric matrices equations \( (m = 1) \)).

**Theorem 2.3.** Let \( A \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)n})) \) be a \( t \)-independent, complex matrix function which is strictly accretive on \( N(\text{curl}_x) \) and assume that \((\text{Dir}-A)\) is well-posed. Then any function \( U_t(x) = U(t, x) \in C^1(\mathbb{R}_+; L_2(\mathbb{R}^n; C_0^m)) \) solving (2), with properties as in Definition 2.1 has estimates

\[
\int_{\mathbb{R}^n} |u|^2 \, dx \approx \sup_{t > 0} \int_{\mathbb{R}^n} |U|^2 \, dx \approx \int_{\mathbb{R}^n} |\tilde{N}_x(U)|^2 \, dx \approx \|t \nabla_{t,x} U\|^2,
\]

where \( u = U|_{\mathbb{R}^n} \). If furthermore \( A \) is real (not necessarily symmetric) and \( m = 1 \), then Moser’s local boundedness estimate [18] gives the pointwise estimate \( \tilde{N}_x(U)(x) \lesssim N_x(U)(x) \), where the standard non-tangential maximal function is \( N_x(U)(x) := \sup_{|y-x| < c} |U(t, y)| \), for fixed \( 0 < c < \infty \).

We use the square-function norm

\[
\|F_t\|^2 := \int_0^{\infty} \|F_t\|^2 \, dt = \int_{\mathbb{R}_+^{1+n}} |F(t, x)|^2 \, \frac{dt \, dx}{t}
\]

and the following version \( \tilde{N}_x(F) \) of the modified non-tangential maximal function introduced in [13]

\[
\tilde{N}_x(F)(x) := \sup_{t > 0} t^{-(1+n)/2} \|F\|_{L_2(Q(t, x))},
\]

where \( Q(t, x) := [(1 - c_0) t, (1 + c_0) t] \times B(x; c_1 t) \), for some fixed constants \( c_0 \in (0, 1), c_1 > 0 \).

Next is Theorem 3.2 of [3], specialized to the Dirichlet problem.
Theorem 2.4. The set of matrices $A$ for which $(\text{Dir-} A)$ is well-posed is an open subset of $L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$. Furthermore, it contains

(i) all Hermitean matrices $A(x) = A(x)^*$ (and in particular all real symmetric matrices),

(ii) all block matrices where $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \leq i \leq n$, $1 \leq \alpha, \beta \leq m$, and

(iii) all constant matrices $A(x) = A$.

More importantly is the solution algorithm using an “infinitesimal generator” $T_A$. Write $v \in \mathbb{C}^{(1+n)m}$ as $v = [v_0, v_\parallel]$, where $v_0 \in \mathbb{C}^m$ and $v_\parallel \in \mathbb{C}^{nm}$, and introduce the auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{00} & A_0 \parallel \\ 0 & I \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 0 \\ A_{00} & A_0 \parallel \end{bmatrix}, \quad \text{if } A = \begin{bmatrix} A_{00} & A_0 \parallel \\ A_0 & A_{00} \parallel \end{bmatrix}$$

in the normal/tangential splitting of $\mathbb{C}^{(1+n)m}$. The strict accretivity of $A$ on $\mathcal{N}(\text{curl}_\parallel)$, as in (3), implies the pointwise strict accretivity of the diagonal block $A_{00}$. Hence $A_{00}$ is invertible, and consequently $\overline{A}$ is invertible [This is not necessarily true for $A$]. We define

$$T_A = \overline{A}^{-1} DA$$

as an unbounded operator on $L_2(\mathbb{R}^n, \mathbb{C}^{(1+n)m})$ with $D$ the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{bmatrix} 0 & \text{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

Proposition 2.5. Let $A \in L_\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{(1+n)m}))$ be a $t$-independent, complex matrix function which is strictly accretive on $\mathcal{N}(\text{curl}_{\parallel})$.

(1) The operator $T_A$ has quadratic estimates and a bounded holomorphic functional calculus on $L_2(\mathbb{R}^n, \mathbb{C}^{(1+n)m})$. In particular, for any holomorphic function $\psi$ on the left and right open half planes, with $z\psi(z)$ and $z^{-1}\psi(z)$ qualitatively bounded, one has

$$\|\psi(tT_A)f\|_2 \lesssim \|f\|_2.$$

(2) The Dirichlet problem $(\text{Dir-} A)$ is well-posed if and only if the operator

$$S : \mathcal{R}(\chi_+(T_A)) \to L_2(\mathbb{R}^n, \mathbb{C}^m), f \mapsto f_0$$

is invertible. Here, $\chi_+ = 1$ on the right open half plane and 0 on the left open half plane.

Item (1) is [3, Corollary 3.6] (and see [3] for an explicit direct proof) and item (2) can be found in [3, Section 4, proof of Theorem 2.2].

Lemma 2.6. Assume that $(\text{Dir-} A)$ is well-posed. Let $u_0 \in L_2(\mathbb{R}^n, \mathbb{C}^m)$. Then the solution $U$ of $(\text{Dir-} A)$ in the sense of Definition [2,7] is given by

$$U(t, \cdot) = (e^{-tT_A}f)_0, \quad f = S^{-1}u_0 \in \mathcal{R}(\chi_+(T_A))$$

and furthermore

$$\nabla_{t,x} U(t, \cdot) = \partial_t e^{-tT_A}f.$$

Proof. [3, Lemma 4.2] (See also [2, Lemma 2.55] with a slightly different formulation of the Dirichlet problem).
3. The bilinear estimate

We are now in position to state and prove the generalisation of Hofmann’s result.

**Theorem 3.1.** Assume that (Dir-A) is well-posed. Let $u_0 \in L^2(\mathbb{R}^n, C^m)$ and $U$ be the solution to (Dir-A) in the sense of Definition 2.1. Then for all $v: \mathbb{R}_+^{1+n} \to C(1+n)^m$ such that the right-hand side is finite,

$$\left| \int \int_{\mathbb{R}_+^{1+n}} \nabla_t x U \cdot v \, dt \, dx \right| \leq C \| u_0 \|_2 (\| t \nabla_{t,x} v \| + \| N_* v \|_2).$$

The pointwise values of $v(t, x)$ in the non-tangential control $N_* v$ can be slightly improved to $L^1$ averages on balls having radii $\sim t$ for each fixed $t$. See the end of proof.

**Proof.** It follows from the previous result that there exists $f \in \mathcal{R}(\chi^+(T_A))$ such that $U(t, \cdot) = (e^{-tTA}f)_0$ and

$$\nabla_{t,x} U(t, \cdot) = \partial_t F = -T_A e^{-tTA} f, \quad F = e^{-tTA} f.$$ Integrating by parts with respect to $t$, we find

$$\int \int_{\mathbb{R}_+^{1+n}} \nabla U \cdot v \, dt \, dx = - \int \int_{\mathbb{R}_+^{1+n}} t \partial_t F \cdot \nabla v \, dt \, dx - \int \int_{\mathbb{R}_+^{1+n}} t \partial^2 F \cdot \nabla v \, dt \, dx.$$ The boundary term vanishes because $t \partial_t F$ goes to 0 in $L^2$ when $t \to 0, \infty$ (this uses $f \in \mathcal{R}(\chi^+(T_A))$) and $\sup_{t>0} \| v(t, \cdot) \|_2 < \infty$ from $\| N_* v \|_2 < \infty$.

For the first term, we use Cauchy-Schwarz inequality and that $\| t \partial_t F \| \lesssim \| u_0 \|_2$ from Theorem 2.3.

For the second term, we use the following identity: $T_A = \overline{A}^{-1} D B \overline{A}$ with $B = \overline{A}^{-1}$, which, by [3, Proposition 3.2], is strictly accretive on $N(\text{curl}_0)$, and observe that

$$t^2 \partial^2 F = \overline{A}^{-1} (tDB)^2 e^{-tDB}(\overline{A} f)$$

$$= \overline{A}^{-1} (tDB)(I + (tDB)^2)^{-1} \psi(tDB)(\overline{A} f)$$

$$= \overline{A}^{-1} (tDB)(I + (tDB)^2)^{-1} \overline{A} \psi(tA)(f)$$

with

$$\psi(z) = z(1 + z^2) e^{-(\text{sgn Re}z)z}.$$ Thus,

$$\int \int_{\mathbb{R}_+^{1+n}} t \partial^2 F \cdot v \, dt \, dx = \int \int_{\mathbb{R}_+^{1+n}} \overline{A} \psi(tA)(f) \cdot Q_t v_t \, dt \, dx.$$ with $Q_t = \Theta_t \overline{A}^{-1}$ and $\Theta_t = (tB^*D)(I + (tB^*D)^2)^{-1}$ acting on $v_t \equiv v(t, \cdot)$ for each fixed $t$. [The notation $\overline{A}$ has nothing to do with complex conjugate and we apologize for any conflict this may cause.] It follows from the quadratic estimates of Proposition 2.5 that

$$\| \psi(tA)(f) \| \lesssim \| f \|_2.$$ It remains to estimate $\| Q_t v_t \|$. To do that we follow the principal part approximation of [4] - which is an elaboration of the so-called Coifman-Meyer trick [7] - applied
to $Q_t$ instead of $\Theta_t$ there. That is, we write
\begin{equation}
Q_t \nu_t = Q_t \left( \frac{I - P_1}{t(-\Delta)^{1/2}} \right) t(-\Delta)^{-1/2} \nu_t + (Q_t P_t - \gamma_t S_t P_t) \nu_t + \gamma_t S_t P_t \nu_t
\end{equation}
where $\Delta$ is the Laplacian on $\mathbb{R}^n$, $P_t$ is a nice scalar approximation to the identity acting componentwise on $L_2(\mathbb{R}^n, C^{(1+n)m})$ and $\gamma_t$ is the element of $L^2_{\text{loc}}(\mathbb{R}^n; \mathcal{L}(C^{(1+n)m}))$ given by
$$
\gamma_t(x) = (Q_t \nu_t)(x)
$$
for every $\nu_t \in C^{(1+n)m}$. We view $\nu_t$ on the right-hand side of the above equation as the constant function valued in $C^{(1+n)m}$ defined on $\mathbb{R}^n$ by $\nu_t(x) := \nu_t$. We identify $\gamma_t(x)$ with the (possibly unbounded) multiplication operator $\gamma_t : f(x) \mapsto \gamma_t(x)f(x)$. Finally, the dyadic averaging operator $S_t : L_2(\mathbb{R}^n, C^{(1+n)m}) \to L_2(\mathbb{R}^n, C^{(1+n)m})$ is given by
$$
S_t \nu(x) := \frac{1}{|Q|} \int_Q \nu(y) \, dy
$$
for every $x \in \mathbb{R}^n$ and $t > 0$, where $Q$ is the unique dyadic cube in $\mathbb{R}^n$ that contains $x$ and has side length $\ell$ with $\ell/2 < t \leq \ell$.

With this in hand, we apply the triple bar norm to (4),

Using the uniform $L_2$ boundedness of $Q_t$ and that of $\frac{1}{t(-\Delta)^{1/2}}$, the first term in the RHS is bounded by $\|t(-\Delta)^{1/2} \nu_t\| \leq \|\nabla_x \nu_t\|$.

Following exactly the computation of Lemma 3.5 in [4], the second term in the RHS is bounded by $C \|t\nabla_x P_t \nu_t\| \leq C \|\nabla_x \nu_t\|$ using the uniform $L_2$ boundedness of $P_t$. This computation makes use of the off-diagonal estimates of $\Theta_t$, hence of $Q_t$, proved in [4, Proposition 3.11].

For the third term in the RHS, we observe that $\gamma_t(x) = \Theta_t(\overline{A^{-1}} \nu_t)(x)$. Hence, the square-function estimate on $\Theta_t$ proved in [4, Theorem 1.1], the off-diagonal estimates of $\Theta_t$ and the fact that $\overline{A^{-1}}$ is bounded imply that $|\gamma_t(x)|^2 \text{d}t \text{d}x$ is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes $\|N_s(S_t P_t \nu_t)\|_2$, which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening.

\section{4. The domain of the Dirichlet semi-group}

Assume (Dir-A) in the sense of Definition 2.1 is well-posed. If we set
$$
P_t u_0 = (e^{-tTA} f)_0, \quad f = S^{-1} u_0 \in \overline{R(\chi_+(T_A))}
$$
for all $t > 0$, then Lemma 2.6 implies that $(P_t)_{t>0}$ is a bounded $C_0$-semigroup on $L_2(\mathbb{R}^n, C^m)$ [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].

Furthermore, our definition of well-posedness of the Dirichlet problem, the domain of the infinitesimal generator $A$ of this semi-group is contained in the Sobolev space $W^{1,2}(\mathbb{R}^n, C^m)$ and $\|\nabla x u_0\|_2 \leq \|A u_0\|_2$. Indeed, from Lemma 2.6 we have for all $t > 0$, $\partial_t e^{-tA} f = \nabla_{t,x} U(t, \cdot)$. Also $\partial_t e^{-tA} f \in \overline{R(\chi_+(T_A))}$ and the invertibility of $S$ tells that $\nabla_{t,x} U(t, \cdot) = S^{-1}(\partial_t U(t, \cdot))$. Therefore
$$
\|\nabla_{t,x} U(t, \cdot)\|_2 \leq \|\partial_t U(t, \cdot)\|_2.
$$
By definition of $A$, $\partial_t U(t, \cdot) = AU(t, \cdot)$, thus we have for all $t > 0$
$$
\|\nabla_{t,x} U(t, \cdot)\|_2 \leq \|A U(t, \cdot)\|_2.
$$
The conclusion for the domain follows easily.

The question of whether this domain coincides with $W^{1,2}(\mathbb{R}^n, C^m)$ is answered by the following theorem

**Theorem 4.1.** Assume that $(\text{Dir}-A)$ and $(\text{Dir}-A^*)$ are well-posed. Then the domain of the infinitesimal generator $A$ of $(\mathcal{P}_t)_{t>0}$ coincides with the Sobolev space $W^{1,2}(\mathbb{R}^n, C^m)$ and $\|\nabla_x u_0\|_2 \sim \|Au_0\|_2$.

This theorem applies to the three situations listed in Theorem 2.4.

**Proof.** Combining [4, Lemma 4.2] (which says that $(\text{Dir}-A^*)$ is equivalent to an auxiliary Neumann problem for $A^*$), [2, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for $A$: this is non trivial) with the proof of Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for $A$), we have that $(\text{Dir}-A^*)$ is well-posed if and only if

$$\mathcal{R} : \overline{\mathcal{R}(\chi_+(T_A))} \to L^2(\mathbb{R}^n, C^{nm}), f \mapsto f_||$$

is invertible. This implies that for $f \in \overline{\mathcal{R}(\chi_+(T_A))}$, we have that

$$\|f\|_2 \sim \|f_||\|_2.$$

Therefore, the conjunction of well-posedness for $(\text{Dir}-A)$ and $(\text{Dir}-A^*)$ gives

$$\|f_0\|_2 \sim \|f_||\|_2, \quad f \in \overline{\mathcal{R}(\chi_+(T_A))}.$$

From this, it is easy to identify the domain of $A$ by an argument as before. \hfill \Box

We have seen that invertibility of $S$ reduces to that of $\mathcal{R}$ (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that $\|f_||\|_2 \sim \|(Af)_0\|_2$ for all $f \in \overline{\mathcal{R}(\chi_+(T_A))}$. This method was first used in the context of Laplace equation on Lipschitz domains by Vechota [21]. This depends strongly of $A$. Various relations between Dirichlet, regularity and Neumann problems for $L^p$ data in the sense of non tangential approach for second order real symmetric equations are studied in [13, 14] and more recently in [15, 19].

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