The Distance Energy of Clique Trees

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Abstract

The distance energy of a simple connected graph \(G\) is defined as the sum of absolute values of its distance eigenvalues. In this paper, we mainly give a positive answer to a conjecture of distance energy of clique trees proposed by Lin, Liu and Lu [H. Q. Lin, R. F. Liu, X. W. Lu, The inertia and energy of the distance matrix of a connected graph, Linear Algebra Appl., 467 (2015), 29-39.].

Key words: Distance energy, Clique tree, Equitable partition, Spectral radius.

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1 Introduction

Graham and Pollak [12] in 1971 discovered an interesting and elegant result on the distance matrix of a tree that the determinant of distance matrix of any \(n\)–vertex tree is \((-1)^{n-1}(n-1)2^{n-2}\), which is independent with the structure of trees. Furthermore, Graham and Lovász in 1978 [11] derived the coefficients of the characteristic polynomial of the distance matrix of a graphs in term of as certain tied linear combinations of the numbers of various subgraphs of a graph. Recently, Cheng and Lin [8] presented a class of graph whose distance determinant are independent of their structures. These results motivated that properties of distance matrix of a graph have been investigated. There are some excellent surveys (see [1, 3, 21]) on this topic.

Let \(G\) be a connected graph of order \(n\) with vertex set \(V(G)\) and edge set \(E(G)\). Thus the distance matrix of \(G\) is defined as \(D(G) = (d_{uv})_{n \times n}\), where \(d_{uv}\) is the distance between vertices \(u\) and \(v\) in \(G\).
and \( v \) in graph \( G \). Clearly, \( D(G) \) is a symmetric matrix and its eigenvalues are real, denoted by \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \). Moreover, \( \lambda_1(G) \) is called the distance spectral radius of \( G \) and is denoted by \( \lambda(G) \). Further, Indulal, Gutman and Vijaykumar \[15\] in 2009 introduced distance energy which is defined as

\[
E_D(G) = \sum_{i=1}^{n} |\lambda_i(G)|.
\]

They obtained some sharp bounds for the distance spectral radius and D-energy of graphs with diameter 2. Andelić, Koledin and Zoran Stanić in \[2\] obtained the exact value of the distance energy of several types of graphs. Recently, Varghesea, So and Vijayakumar \[23\] studied how the distance energy of complete bipartite graphs changes when an edge is deleted. Vaidya and Popat \[22\] studied the distance energy of two particular graph compositions. Recently, Stevanović \[20\] gave more results on the graph composition.

A clique tree is a graph whose blocks are cliques (for example see \[4\]), a clique path \( P_{n_1, n_2, \cdots, n_k} \) is the graph obtained from the path \( P_{k+1} = v_1 \cdots v_{k+1} \) with order \( k + 1 \) replacing \( v_i v_{i+1} \) by \( K_{n_i} \) such that \( V(K_{n_i}) \cap V(K_{n_{i+1}}) = v_{i+1}, i = 1, 2, \cdots, k - 1 \) and \( V(K_{n_i}) \cap V(K_{n_j}) = \emptyset \) if \( i \neq j \).

Let \( n_+(D(G)), n_-(D(G)) \) denote the number of positive and negative eigenvalues of \( D(G) \), respectively. Lin, Liu and Lu \[17\] proved that

**Theorem 1.1 \[17\]** Let \( G \) be a clique tree with order \( n \). Then \( n_+(D(G)) = 1 \) and \( n_-(D(G)) = n - 1 \).

Furthermore, they proposed the following conjecture.

**Conjecture 1.2 \[17\]** Among all clique trees with cliques \( K_{n_1}, \cdots, K_{n_k} \), the graph attains the maximum distance energy is \( P_{\left\lfloor \frac{n-k+3}{2} \right\rfloor, 2, \cdots, 2, \left\lceil \frac{n-k+3}{2} \right\rceil} \).

From theorem 1.1 the distance energy of a clique tree \( G \) satisfies

\[
E_D(G) = \sum_{i=1}^{n} |\lambda_i(G)| = \lambda_1(G) - \sum_{i=2}^{n} \lambda_i(G) = 2\lambda_1(G),
\]

the last equation is from that the trace of \( D(G) \) is \( \sum_{i=1}^{n} \lambda_i(G) = 0 \). Thus it is sufficient to consider maximum spectral radius of clique trees instead of maximum distance energy of clique trees.

In addition, Zhang \[27\] studied the relation between the inertia and the distance energy for the line graph of unicyclic graphs. Moreover, Consolini and Todeschini \[9\] investigated the distance energy in terms of some invariants. Drury and Lin \[10\] characterized all connected graphs with distance energy in \([2n - 2, 2n]\). On the distance spectral radius of graphs, Liu \[19\] investigated the graphs with minimal distance spectral radius among the graphs with fixed vertex connectivity, matching number and chromatic number. Ilić \[14\] characterized the minimal spectral radius among trees given matching number. Bose, Nath and Paul \[5\] studied the connected graph with minimal distance spectral radius among all graphs fixed number of pendant vertices. Zhang \[25\] explored the graph with minimum distance spectral radius among all connected graphs with given diameter. Lin and Shu \[18\] gave some sharp lower and upper bounds for the distance spectral radius and characterized the graphs with the spectral radius attaining the bounds. Lin and Feng \[16\] characterized
the connected graphs with minimal or maximal distance spectral radius among all graphs with fixed independence number.

The main purpose of this paper is to give a positive answer of conjecture 1.2.

2 Proof of Conjecture 1.2

Let $A$ be a symmetric matrix, $Y = \{1, 2, \cdots, n\}$ be the index set of rows and columns of matrix $A$ and $\mathbf{1}_n$ be the all ones vector with $n$ elements, written by $\mathbf{1}$ if no ambiguity. Suppose $\{Y_1, Y_2, \cdots, Y_m\}$ is a partition of the set $Y$, the matrix $B = (b_{ij})$ is called quotient matrix of $A$ corresponding to the partition, where $A_{ij}$ is a submatrix of $A$ corresponding to the row index and column index $Y_i$ and $Y_j$ respectively and $b_{ij}$ is the average row sum of matrix $A_{ij}$. The $n \times m$ matrix $S = (s_{ij})$ is called characteristic matrix, where $s_{ij} = 1$ if $i \in Y_j$ and $s_{ij} = 0$ if $i \notin Y_j$. The partition is called equitable partition if $A_{ij}\mathbf{1} = b_{ij}\mathbf{1}$. For more details, the readers can refer to [6].

Lemma 2.1 [6] Let $B$ be the quotient matrix of $A$ corresponding to an equitable partition. If $v$ is an eigenvector of $B$ for an eigenvalue $\mu$, then $Sv$ is an eigenvector of $A$ for the same eigenvalues $\mu$, where $S$ is the characteristic matrix corresponding to the equitable partition.

Corollary 2.2

$$\lambda(P_{n+1,2,\cdots,2,n+1}) = \lambda(B(P_{n+1,2,\cdots,2,n+1})),$$

where

$$B(P_{n+1,2,\cdots,2,n+1}) = \begin{pmatrix} n_1 - 1 & u_{k-1}^T & kn_2 \\ n_1 u_{k-1} & D(P_{k-1}) & n_2 w_{k-1} \\ kn_1 & w_{k-1}^T & n_2 - 1 \end{pmatrix}, \quad \text{and}$$

$$u_k = (1, 2, \cdots, k)^T, \quad w_k = Qu_k,$$

$Q$ is a permutation matrix, the $(i, k+1-i)$-element of which is one, $1 \leq i \leq k$.

![Fig.1 $P_{n+1,2,\cdots,2,n+1}$](image)

Proof. Let $X_1, X_2, \cdots, X_k, X_{k+1}$ be a partition of $V(P_{n+1,2,\cdots,2,n+1})$ with $X_1 = V(K_{n_1+1}) \setminus \{v_2\}$, $X_i = \{v_i\}, i = 2, 3, \cdots, k$ and $X_{k+1} = V(K_{n_2+1}) \setminus \{v_k\}$. Then

$$D(P_{n+1,2,\cdots,2,n+1}) = \begin{pmatrix} (J - I)_{n_1 \times n_1} & \mathbf{1} u_{k-1}^T & kJ_{n_1 \times n_2} \\ u_{k-1} \mathbf{1}^T & D(P_{k-1}) & w_{k-1} \mathbf{1}^T \\ kJ_{n_2 \times n_1} & \mathbf{1} w_{k-1}^T & (J - I)_{n_2 \times n_2} \end{pmatrix}.$$
The quotient matrix of $D(P_{n_1+1,\cdots,2,n_2+1})$ corresponding to $X_1, X_2, \cdots, X_{k+1}$ is

$$B(P_{n_1+1,\cdots,2,n_2+1}) = \begin{pmatrix} n_1 - 1 & u_{k-1}^T & k n_2 \\ n_1 u_{k-1} & D(P_{k-1}) & n_2 w_{k-1} \\ k n_1 & w_{k-1}^T & n_2 - 1 \end{pmatrix}.$$}

Moreover, $\{X_1, X_2, \cdots, X_{k+1}\}$ is an equitable partition. By lemma 2.1, $\lambda(B(P_{n_1+1,\cdots,2,n_2+1})) = \lambda(B(P_{n_1+1,\cdots,2,n_2+1}))$. \hfill \Box

Now we consider the characteristic polynomial of the matrix $B(P_{n_1+1,\cdots,2,n_2+1})$.

**Lemma 2.3** If $x > \lambda(P_{k-1})$ with $k > 2$, then

1. $u_{k-1}^T [xI - D(P_{k-1})]^{-1} w_{k-1} = \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j u_{k-1}$

2. $u_{k-1}^T [xI - D(P_{k-1})]^{-1} u_{k-1} = \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j u_{k-1}$

3. $|u_{k-1} - w_{k-1}|^2 = \frac{(k-1)(k-2)}{3}$

4. $u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}) = u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1})$

**Proof.** (1) Since $x > \lambda(D(P_{k-1}))$ and $D(P_{k-1})$ is a nonnegative symmetric matrix, we have

$$[xI - D(P_{k-1})]^{-1} = \frac{1}{x} \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j,$$

which is also a nonnegative symmetric matrix. Then

$$u_{k-1}^T [xI - D(P_{k-1})]^{-1} w_{k-1} = \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j u_{k-1} > 0.$$

(2) By equation (2),

$$u_{k-1}^T [xI - D(P_{k-1})]^{-1} u_{k-1} > 0$$

and $w_{k-1}^T [xI - D(P_{k-1})]^{-1} w_{k-1} > 0$.

Note that $w_{k-1} = Q u_{k-1}$, $Q^T Q = I$ and $Q^T D(P_{k-1})Q = D(P_{k-1})$, we have

$$w_{k-1}^T [xI - D(P_{k-1})]^{-1} w_{k-1} = u_{k-1}^T Q^T [xI - D(P_{k-1})]^{-1} Q u_{k-1}$$

$$= u_{k-1}^T Q^T \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j Q u_{k-1}$$

$$= u_{k-1}^T \frac{1}{x} \sum_{j=0}^\infty \left( \frac{Q^T D(P_{k-1})Q}{x} \right)^j u_{k-1}$$

$$= u_{k-1}^T \frac{1}{x} \sum_{j=0}^\infty \left( \frac{D(P_{k-1})}{x} \right)^j u_{k-1}$$

$$= u_{k-1}^T [xI - D(P_{k-1})]^{-1} u_{k-1}.$$
**Case 1:** If $k$ is odd and $u_{k-1} - w_{k-1} = (2 - k, \ldots, -3, -1, 1, 3, \ldots, k - 2)^T$, then

$$
||u_{k-1} - w_{k-1}|| = \sqrt{2(1^2 + 3^2 + \cdots + (k - 2)^2)}
$$

$$
= \sqrt{2 \left(\frac{(k-1)k(2k-1)}{6} - 2^2 + 4^2 + \cdots + (k-1)^2\right)}
$$

$$
= \sqrt{2 \left(\frac{(k-1)k(2k-1)}{6} - \frac{4(1^2 + 2^2 + \cdots + (k-2)^2)}{2}\right)}
$$

$$
= \sqrt{2 \left(\frac{(k-1)k(2k-1)}{6} - \frac{(k-1)(k+1)}{6}\right)}
$$

$$
= \sqrt{\frac{(k-1)k(k-2)}{3}}.
$$

**Case 2:** If $k$ is even and $u_{k-1} - w_{k-1} = (2 - k, \ldots, -2, 0, 2, 4, \ldots, k - 2)^T$, then

$$
||u_{k-1} - w_{k-1}|| = \sqrt{2(2^2 + 4^2 + \cdots + (k - 2)^2)}
$$

$$
= \sqrt{2 \left(4(1^2 + 2^2 + \cdots + \left(\frac{k-2}{2}\right)^2)\right)}
$$

$$
= \sqrt{\frac{(k-1)k(k-2)}{3}}.
$$

(4). Since

$$
(u_{k-1} - w_{k-1})^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1})
$$

$$
= u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}) - u_{k-1}^T [xI - D(P_{k-1})]^{-1} (w_{k-1} - w_{k-1})
$$

$$
= u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}) - u_{k-1}^T [xI - D(P_{k-1})]^{-1} (w_{k-1} - w_{k-1})
$$

$$
+ u_{k-1}^T [xI - D(P_{k-1})]^{-1} w_{k-1},
$$

then

$$
(u_{k-1} - w_{k-1})^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}) = 2u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1})
$$

which is from (1) and (2). So

$$
u_{k-1}^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}) = \frac{1}{2} (u_{k-1} - w_{k-1})^T [xI - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1}).$$

**Lemma 2.4** If $x > \lambda(P_{k-1})$, then

$$
det(xI - B(P_{n1+1,2, \ldots, n2+1}))
$$

$$
= [(x + 1)^2 - (n1 + n2)(1 + \alpha_k-1)(x + 1) + n1n2(1 + \alpha_k-k + \beta_k-k)(1 - k + \gamma_{k-1})]
$$

$$
\cdot det(xI - D(P_{k-1})),
$$

where

$$
\alpha_k-k = u_{k-1}^T (xI - D(P_{k-1}))^{-1} u_{k-1}, \quad \beta_k-k = u_{k-1}^T (xI - D(P_{k-1}))^{-1} w_{k-1},
$$

$$
\gamma_k-k = \frac{1}{2} (u_{k-1} - w_{k-1})^T (xI - D(P_{k-1}))^{-1} (u_{k-1} - w_{k-1}).
$$
Moreover, \( \alpha_{k-1} - \beta_{k-1} = u_{k-1}^T (xI - D(P_{k-1}))^{-1} (u_{k-1} - w_{k-1}) \)
\[
= \frac{1}{2} u_{k-1} - w_{k-1} \cdot (xI - D(P_{k-1}))^{-1} (u_{k-1} - w_{k-1}) = \gamma_{k-1}.
\]

Then
\[
det(xI - B(\mathbb{F}_{n_1+1,\ldots,2,n_2+1})) = [(x + 1)^2 - (n_1 + n_2)(1 + \alpha_{k-1})(x + 1) + n_1 n_2(1 + \alpha_{k-1})^2 - n_1 n_2(k + \beta_{k-1})^2] 
\cdot det(xI - D(P_{k-1})).
\]

Moreover,
\[
\lambda_{17} \geq \lambda_{18}.
\]

**Lemma 2.5** Let \( \mathbb{F}_{n_1+1,\ldots,2,n_2+1} \) be a clique path with \( k \) cliques, where \( n_1 \) and \( n_2 \) are nonnegative integers. Then
\[
\lambda(\mathbb{F}_{n_1+1,\ldots,2,n_2+1}) \geq \lambda(P_k) = \frac{(k - 1)(k + 1)}{3}.
\]

**Proof.** Since \( D(P_k) \) is a principal submatrix of \( D(\mathbb{F}_{n_1+1,\ldots,2,n_2+1}) \), we have \( \lambda(\mathbb{F}_{n_1+1,\ldots,2,n_2+1}) \geq \lambda(P_k) > \frac{(k - 1)(k + 1)}{3} \).

**Lemma 2.6** Among all clique trees with cliques \( K_{n_1}, \ldots, K_{n_k} \), the graph attains the maximum distance spectral radius is \( P_{n_1'+1,\ldots,2,n_k'+1} \).

Now we are ready to prove conjecture 1.2.
Furthermore, by Courant-Fischer theorem (refer to theorem 4.2.6 in [13]),

Thus $X$ where $QX$ implies that $QX$ is also an unit positive eigenvector of $D(P_{n-1})$ corresponding to $\lambda(P_{n-1})$. By the Perron-Frobenius theorem, there is an unique unit positive eigenvector $D$ corresponding to $\lambda(P_{n-1})$. By lemma 2.6, the graph which attains the maximum distance spectral radius is the clique path $P_{n_1, 1, 2, \ldots, 2, n_2+1}$. If $n_1, n_2, n_1', n_2'$ are positive integers such that

$$\max\{n_1, n_2\} < \max\{n_1', n_2'\}$$

and $n_1 + n_2 = n_1' + n_2' = n - 1 - k,$

which implies that $n_1 n_2 > n_1' n_2'$, we will show that

$$\lambda(P_{n_1, 1, 2, \ldots, 2, n_2+1}) > \lambda(P_{n_1' + 1, 2, \ldots, 2, n_2'+1}).$$

Let $f(n_1, n_2, x) = \det(x I - B(P_{n_1, 1, 2, \ldots, 2, n_2+1}))$ be the characteristic polynomial of matrix $B(P_{n_1, 1, 2, \ldots, 2, n_2+1})$, where $B(P_{n_1, 1, 2, \ldots, 2, n_2+1})$ is a $(k + 1) \times (k + 1)$ matrix, similarly define $f(n_1', n_2', x)$. By the definition of $f(n_1, n_2, x), f(n_1', n_2', x)$ and lemma 2.4

$$f(n_1, n_2, x) - f(n_1', n_2', x) = (n_1 n_2 - n_1' n_2')(1 + \alpha_{k-1} + k + \beta_{k-1})(1 - k + \gamma_{k-1})]$$

$$\cdot \det(x I - D(P_{k-1})).$$

(4)

Suppose $x \geq \lambda(P_k) > \lambda(P_{k-1})$, by lemma 2.5

$$\alpha_{k-1}, \beta_{k-1} > 0 \text{ and } \det(x I - D(P_{k-1})) > 0.$$ (5)

By the Perron-Frobenius theorem, there is an unique unit positive eigenvector $X = (x_1, x_2, \ldots, x_{k-1})^T$ of $D(P_{k-1})$ corresponding to $\lambda(P_{k-1})$. Then

$$\lambda(P_{k-1}) = X^T Q^T D(P_{k-1}) Q X = X^T D(P_{k-1}) X,$$

which implies that $Q X$ is also an unit positive eigenvector of $D(P_{k-1})$ corresponding to $\lambda(P_{k-1})$, where $Q$ is a $(k - 1) \times (k - 1)$ permutation matrix whose $(i, k - i)$-element is one for $1 \leq i \leq k - 1$. Thus $X = Q X$, that is, $x_i = x_{k-i}, 1 \leq i \leq k - 1$. Hence

$$X^T (u_{k-1} - w_{k-1}) = 0.$$

Further, by Courant-Fischer theorem (refer to theorem 4.2.6 in [13]),

$$\gamma_{k-1} = \frac{1}{2} (u_{k-1} - w_{k-1})^T [x I - D(P_{k-1})]^{-1} (u_{k-1} - w_{k-1})$$

$$\leq \frac{1}{2} (u_{k-1} - w_{k-1})^T \frac{1}{x - \lambda_2(D(P_{k-1}))}.$$

(6)

By $x \geq \lambda(P_k) > \lambda(P_{k-1})$ and lemma 2.5 we have

$$x \geq \lambda(P_k) > \frac{(k - 1)(k + 1)}{3}.$$ (7)

By theorem 1.1 and lemma 2.3

$$\lambda_2(D(P_{k-1})) < 0 \text{ and } ||u_{k-1} - w_{k-1}||^2 = \frac{(k - 1)(k + 2)}{3}.$$ (8)

Inequalities (6), (7) and (8) implies that

$$\gamma_{k-1} \leq \frac{(k - 1)(k - 2)}{6} \frac{1}{x} \leq \frac{(k - 1)(k - 2)}{6} \frac{1}{\frac{(k - 1)(k + 1)}{3}} < k - 1.$$ (9)
Take inequalities (5), (9) and \( n_1n_2 > n'_1n'_2 \) into (4),
\[
f(n_1, n_2, x) - f(n'_1, n'_2, x) > 0.
\]
Which implies that \( f(n_1, n_2, x) - f(n'_1, n'_2, x) > 0 \) for \( x \geq \lambda(P_k) \). Since \( \lambda(P_{n_1+1,2,\ldots,2,n_2+1}) \geq \lambda(P_k) \)
and \( \lambda(P_{n'_1+1,2,\ldots,2,n'_2+1}) \geq \lambda(P_k) \), then
\[
\lambda(P_{n_1+1,2,\ldots,2,n_2+1}) > \lambda(P_{n'_1+1,2,\ldots,2,n'_2+1}).
\]
Since \( \lceil \frac{n-k+1}{2} \rceil, \lceil \frac{n-k+1}{2} \rceil \) are the only numbers satisfying that, for every positive integers \( n'_1, n'_2 \) and \( n'_1 + n'_2 = n + 1 - k \) such that
\[
\max\{\lceil \frac{n-k+1}{2} \rceil, \lceil \frac{n-k+1}{2} \rceil \} \leq \max\{n'_1, n'_2\} \text{ and } n'_1 + n'_2 = n + 1 - k.
\]
Thus \( P_{\lceil \frac{n-k+1}{2} \rceil+1,2,\ldots,2,\lceil \frac{n-k+1}{2} \rceil+1} = P_{\lceil \frac{n-k+1}{2} \rceil+1,2,\ldots,2,\lceil \frac{n-k+1}{2} \rceil+1} \) has the maximum spectral radius in the clique tree.

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