QUANTUM CORRECTIONS TO MAXWELL ELECTRODYNAMICS IN A HOMOGENEOUS AND ISOTROPIC UNIVERSE WITH COSMOLOGICAL CONSTANT

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Some cosmological consequences of first order quantum corrections to Maxwell electrodynamics are investigated in the context of a spatially flat homogeneous and isotropic universe driven by a magnetic field plus a cosmological term $\Lambda$. The introduction of these quantum corrections may provide a more realistic model for the universe evolution. For a vanishing $\Lambda$, we derive the general solution corresponding to the particular one recently found by Novello et al. [gr-qc/9806076]. We also find a general solution for the case when $\Lambda$ is a non-vanishing constant. Both solutions describe a non-singular, bouncing universe that begins arbitrarily large, contracts to a minimum non-zero size $a_{\text{min}}$ and expands thereafter. However, we show that the first order correction to the electromagnetic Lagrangean density, in which the analysis is based, fails to describe the dynamics near $a_{\text{min}}$, since, at this point, the magnetic fields grows beyond the maximum strength allowed by the approximation used ($B \ll 8.6 \times 10^{-7}$ Tesla = 0.0086 Gauss). The time range where the first order approximation can be used is explicitly evaluated. These problems may be circumvented through the use of higher order terms in the effective Lagrangean, as numerical calculations performed by Novello et al. [gr-qc/9809080], for the vanishing $\Lambda$ case, have indicated. They could also be evaded in some models based on oscillatory behaviour of the fundamental constants. A third general solution corresponding to a constant magnetic field sustained by a time dependent $\Lambda$ is derived. The temporal behaviour of $\Lambda$ is univocally determined. This latter solution is capable of describing the whole cosmic history and describes a universe that, although with vanishing curvature ($K = 0$), has a scale factor that approaches zero asymptotically in the far past, reaches a maximum and then contracts back to an arbitrarily small size. The cosmological term decays during the initial expansion phase and increases during the late contraction phase, so as to keep $B$ constant throughout. An important feature of this model is that it presents an inflationary dynamics except in a very short period of time near its point of maximum size.

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I. INTRODUCTION

In a recent paper, Novello and collaborators [1] have analysed the cosmological consequences of quantum corrections to Maxwell electrodynamics. They have considered the quantum effects leading to the production of electron-positron pairs that were first derived by Heisenberg and Euler [2]. The analysis was a semiclassical one (quantum field in a classical general relativistic geometry), made in first order on the effective Lagrangean density (weak field limit) and applied to a spatially flat Friedmann-Robertson-Walker (FRW) model. The more interesting result found in Ref. [1] was the removal of the primordial singularity due to the appearance of a negative pressure in the early stages of the universe. The analytical solution derived shows that the energy density associated with the electromagnetic field vanishes at the point where the scale factor reaches its minimum. The non-singular behaviour of the model is unaffected by the presence of other types of ultrarelativistic matter obeying the equation of state $p_{\text{ur}} = \rho_{\text{ur}}/3$. In a subsequent paper [3], the analysis was extended beyond the first order approximation and the numerical solutions obtained show that the non-singular behaviour is preserved.

The conclusion reached by the authors of references [1,3] was that the cosmological singularity of FRW models is a distinguished feature of classical electrodynamics. This problem is overcome when quantum corrections are taken into account, leading to a more realistic description of the universe.

In the present paper we extend the analysis developed in [1]. We begin by showing that the analytical solution obtained in that paper is a particular one and write down the corresponding general solution. We then analyse how

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this solution is modified by the presence of a non-zero cosmological term \( \Lambda \) in two different situations: we consider both the case of a constant \( \Lambda \), which leads also to a non-singular universe, and the case of a time varying “cosmological constant” that supplies energy to a constant cosmological magnetic field. The equation that leads to a constant magnetic field as a possible solution to the field equations appeared already in Ref. [1] but was disregarded. The presence of a time varying \( \Lambda \) gives physical meaning to such a solution. The time dependence for \( \Lambda \) is univocally determined and represents a slightly modification of a form that has been suggested frequently in the literature \((\Lambda = 3H^2 - c_0 c_0 = \text{constant})\). The universe described by this latter solution begins arbitrarily small in the far past, expands towards a maximum size \( a_{\text{max}} \) at the point where the vacuum energy reaches its minimum and then contracts towards an arbitrarily small size in the distant future. A remarkable feature of this model is that it has an inflationary dynamics, except during a short period of time near \( a_{\text{max}} \). This scenario obviously violate the standard evolution law for the magnetic field \((B(t) \propto a^{-2})\), which is commonly used to rescale to present time the several constraints on primordial magnetic fields presented in the literature [5,6]

We also discuss the restrictions on the domain of validity of the solutions found, due to the use of the first order approximation for the effective electromagnetic Lagrangean.

The paper is organized as follows: In Section II we set down the basic equations. In Section III we generalize the solution derived in [1] which assumes a vanishing cosmological term and a time-dependent magnetic field. In Section IV we obtain a new solution that takes into account the presence of a constant \( \Lambda \). A new solution involving the presence of a \( \Lambda(t) \) plus a constant magnetic field is derived is Section V. Section VI contains a summary of our results and suggestions for future work.

In what follows Greek indices run from 0 to 4 and Latin indices run from 1 to 3. As in [1], we use Heaviside-Lorentz electromagnetic units, but, unlike that reference, and unless otherwise stated, we make \( \hbar = c = 1 \). This is the scheme used in [7] which seems to be more appropriate to our needs. In this system, the magnetic field \( \vec{B} = \vec{H} \) is measured in Tesla [7] and the fine structure constant is given by [7]

\[
\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036},
\]

so that the magnitude of the electron charge is

\[
e = \sqrt{4\pi \alpha} = 0.30282.
\]

(See the Appendix A of Ref. [7] and the Appendix on Units and Dimensions of Ref. [8] for further details.)

**II. FUNDAMENTAL EQUATIONS**

In the system of units used in this paper, the Lagrangean density for free fields in classical Maxwell electrodynamics is written as

\[
\mathcal{L}_{(\text{MAXWELL})} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{4} F^2,
\]

where \( F_{\mu\nu} \) is the electromagnetic field strength tensor. Canonical energy-momentum tensor is then given by

\[
T_{\mu\nu}^{(\text{MAXWELL})} = F_{\mu\alpha} F^{\alpha\nu} + \frac{1}{4} F_{\mu\nu}.
\]

The quantum corrections to classical electrodynamics that should be considered in order to describe the process of creation of electron-positron pairs by the electromagnetic field were evaluated by Heisenberg and Euler [9]. A clear derivation of the effective Lagrangean that describes these quantum effects can be found in [10], where one should be aware that the Gaussian system of electromagnetic units is used. The first order calculation yields for the effective Lagrangean density [10]

\[
\mathcal{L} = -\frac{1}{4} F^2 + \mu \left[ \frac{1}{4} F^2 + \frac{7}{16} (F^*)^2 \right],
\]

where

\[
F^* \equiv F^{*\mu\nu} F_{\mu\nu},
\]

\( F^{*\mu\nu} \) is the dual of \( F_{\mu\nu} \).
\[ \mu = \frac{8}{45} \frac{\alpha^2}{m_e^4} \approx 1.4 \times 10^8 \text{ GeV}^{-4}, \]  

and \( m_e = 5.110 \times 10^{-4} \text{ GeV} \) is the electron mass. [If one restore all units a factor \( \hbar^3/c^5 \) appears in (6).]

The domain of validity of the above expression for \( \mathcal{L} \) is that of small wave numbers and small electromagnetic field strength, i.e.,

\[ k \ll m_e, \]  

\[ E \ll E_{cr} \equiv \frac{m_e^2}{e} = \frac{m_e^2}{\sqrt{4\pi\alpha}}, \]  

\[ B \ll B_{cr} \equiv \frac{m_e^2}{e} = \frac{m_e^2}{\sqrt{4\pi\alpha}} = 8.6 \times 10^{-7} \text{ Tesla} = 0.0086 \text{ Gauss}. \]  

[Restoring all units, a factor of \( c/\hbar \) appears multiplying the right hand side of (8) and a factor of \( c^3/\hbar \) multiplies the right hand side of equations (9) and (10).]

The corresponding modified energy-momentum tensor becomes

\[ T_{\mu\nu} = -\frac{4}{3} \partial \mathcal{L} / \partial F F^\alpha_{\mu} F_{\nu}^{\alpha} + \left( \frac{\partial \mathcal{L}}{\partial (F^*_\mu F^*_\nu - \mathcal{L})} \right) g_{\mu\nu}. \]  

We will apply the above equations for a homogeneous and isotropic universe with line element

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]  

where \( K = 0, \pm 1 \).

Such a geometry may be generated by electromagnetic fields only if these are considered in its average properties. Using the standard spatial average process we set

\[ \langle E_i \rangle = 0, \]  

\[ \langle B_i \rangle = 0, \]  

\[ \langle E_i E_j \rangle = -\frac{1}{3} E^2 g_{ij}, \]  

\[ \langle B_i B_j \rangle = -\frac{1}{3} B^2 g_{ij}, \]  

\[ \langle E_i B_j \rangle = 0. \]  

Equations (13) - (17) then imply

\[ \langle F_{\mu\alpha} F^\alpha_{\nu} \rangle = \frac{2}{3} (E^2 + B^2) U_{\mu} U_{\nu} + \frac{1}{3} (E^2 - 2B^2) g_{\mu\nu}, \]  

where

\[ U_{\mu} = \frac{dx_{\mu}}{ds}. \]  

Hence, for the classical Lagrangean (3) the average value of the energy-momentum tensor reduces to the form of a perfect fluid

\[ \langle T_{\mu\nu} \rangle = (\rho + p) U_{\mu} U_{\nu} - p g_{\mu\nu}. \]  

where the density \( \rho \) and pressure \( p \) have the well known form

\[ \rho = \frac{1}{2} (E^2 + B^2), \]  

\[ p = \frac{1}{3} \rho. \]
In order to analyse the modifications implied by the use of the modified Lagrangean \((5)\), we assume that the dominant material content is a primordial plasma in which only the average value of the squared magnetic field \(B^2\) survives, i.e., we use Eqs. \((13) - (17)\) with \(E^2 = 0\). Then Equation \((20)\) still holds, but the energy density and pressure are now given by

\[
\rho = \frac{1}{2} B^2 (1 - 2\mu B^2),
\]

\[
p = \frac{1}{6} B^2 (1 - 10\mu B^2) = \frac{1}{3} \rho - \frac{4}{3} \mu B^4.
\]

From Eqs. \((5)\), \((23)\) and \((24)\) we see that the weak energy condition \(\rho > 0\) is obeyed if

\[
B < \frac{1}{2\sqrt{\mu}} = 6.0 \times 10^{-5} \text{ Tesla} = 0.60 \text{ Gauss},
\]

whereas the pressure will reach negative values only if

\[
B > 2.7 \times 10^{-5} \text{ Tesla} = 0.27 \text{ Gauss}.
\]

Note, however, that expression \((5)\) will hold only if the condition \((10)\) is satisfied. Therefore, the conditions \((10)\) and \((26)\) are not compatible. We shall return to this point later.

The Einstein equations for the metric \((12)\) read

\[
\frac{\ddot{a}}{a} = \frac{\Lambda(t)}{3} - \frac{4\pi G}{3} (\rho + 3p),
\]

\[
\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda(t)}{3},
\]

from which the energy conservation equation can be written as

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = -\frac{\dot{\Lambda}}{8\pi G}.
\]

In the above equations we have allowed \(\Lambda\) to be time-dependent and the overdot means derivative with respect to the cosmic time \(t\).

Recent observations that have led to the so called age problem \([13 - 15]\) have driven attention to cosmological models with a non-vanishing \(\Lambda\). Estimates of the density parameter and some kinematical tests also point to the existence of an effective vacuum component \([16,17]\). Even more recently, some evidence for an accelerated cosmic expansion has arisen from measurements involving type Ia supernova at high redshifts \([18,19]\). On the other hand, quantum field theory and inflationary cosmologies have pointed out to the possibility of treating \(\Lambda\) as a dynamical quantity \([4,16,20 - 22]\). Several mechanisms have been identified as possible sources of fluctuating vacuum energy (see, for example, \([4]\) and references therein).

As it will be shown below, the field equations lead to a possible solution corresponding to a constant magnetic field plus a time-dependent cosmological term. In this case the time dependence of \(\Lambda\) is univocally determined.

Replacing \((23)\) and \((24)\) in the Einstein equations \((27) - (28)\) and in the energy conservation equation \((29)\), we get

\[
\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3} B^2 (1 - 6\mu B^2),
\]

\[
\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{4\pi G}{3} B^2 (1 - 2\mu B^2) + \frac{\Lambda}{3},
\]

\[
B \left(1 - 4\mu B^2\right) \left(\dot{B} + 2 \frac{\dot{a}}{a} B\right) = -\frac{\dot{\Lambda}}{8\pi G}.
\]

By solving any two of the above equations we find the complete cosmological solution for our model.
III. GENERAL SOLUTION FOR $\Lambda = 0$ AND A TIME-DEPENDENT MAGNETIC FIELD

This is the case studied in [1] where a particular solution for the scale factor was found. We will find the corresponding general solution and indicate how to recover this previous result.

If $B$ is time-dependent and $\Lambda$ is a constant, Equation (32) can be easily integrated to give

$$B(t) = B_0 \left( \frac{a_0}{a} \right)^2,$$  

(33)

where $B_0$ is a constant of integration. In this paper, the subscript 0 does not indicate the present day value of a quantity. Rather, it indicates the value of that quantity in an arbitrary time $t_0$, which will appear in the general solution for $a(t)$ as a second constant of integration. It is this second constant $t_0$ that was arbitrarily chosen in [1].

Thus $B_0 = B(t_0)$ and $a_0 = a(t_0)$. (In Ref. [1] the authors normalized $a_0$, so that $a_0 = 1$.)

$$\beta_0 \equiv \sqrt{1 - 2 \mu B_0^2}.$$

(36)

In order to compare with the results of [1] we recast (34) in the form

$$a(t) = a_0 \left[ 4 \alpha_0^2 (t - t_0)^2 + 4 \alpha_0 \beta_0 (t - t_0) + 1 \right]^{1/4},$$

(37)

with

$$\gamma_0 \equiv \beta_0 - 2a_0t_0,$$

(38)

$$\delta_0 \equiv 4a_0\beta_0^2 \left( a_0t_0 - \beta_0 \right) + 1.$$

(39)

The linear term in $t$ inside the parenthesis of (37) does not appear in the solution given by the authors of Ref. [1]. This is due to the fact that they have arbitrarily chosen the constant of integration $t_0$ to be

$$t_0 = \frac{\beta_0}{2 \alpha_0} = \frac{1}{2 B_0} \sqrt{\frac{3 (1 - 2 \mu B_0^2)}{4 \pi G}}.$$

(40)

From the solution above we may determine the time behaviour of the magnetic field

$$B(t) = \frac{B_0}{\left( 4 \alpha_0^2 t^2 + 4 \alpha_0 \gamma_0 t + \delta_0 \right)^{1/2}}.$$  

(41)

The energy density and pressure are obtained through the use of Eqs. (23), (24), respectively.

Note also that the Hubble parameter is

$$H = \frac{\dot{a}}{a} = \frac{\alpha_0 \left[ 2a_0(t - t_0) + \beta_0 \right]}{[4\alpha_0^2(t - t_0)^2 + 4\alpha_0\beta_0(t - t_0) + 1]}.$$  

(42)

Hence,

$$H_0 \equiv H(t_0) = \alpha_0 \beta_0 = B_0 \sqrt{\frac{4 \pi G (1 - 2 \mu B_0^2)}{3}}.$$  

(43)

(In [1], the notation $H$ is used to represent the magnetic field.)
From (37) we see that, for large \( t \), we recover the usual solution for radiation dominated universes, \( a(t) \propto t^{1/2} \). However, the more interesting feature of (37) is that the quadratic function inside the parenthesis does not have real roots, being positive for any \( t \). Therefore, the model is non-singular with \( a(t) \) reaching the minimum value

\[
a_{\text{min}} = a_0 \left(2 \mu B_0^2\right)^{1/4}
\]

at

\[
t_{\text{min}} = -\frac{\gamma_0}{2a_0} = t_0 - \frac{\beta_0}{2a_0}.
\]

The universe thus obtained is a bouncing one: it begins arbitrarily large at \( t \ll t_{\text{min}} \), decreases until the minimum value (44) at \( t_{\text{min}} \) and then begins to expand.

A crucial feature of the above results is that the magnetic field at \( t_{\text{min}} \) is

\[
B(t_{\text{min}}) = 1 \sqrt{2} \mu = 6.0 \times 10^{-5} \text{ Tesla} = 0.60 \text{ Gauss},
\]

a value much greater than the minimal critical value given by Eq. (10) which limits the domain of validity of the first order correction given by (5). We are forced to conclude that the above solution does not apply all the way back to the point of minimum size of the universe. This point has not been acknowledged explicitly by the authors of reference [1], although their numerical calculations [3,23] based on the exact effective Lagrangean density (beyond the first order approximation) seems to confirm the non-singular behaviour of the model. We should remark that, in [1], the requirement (5) is mentioned, but not the condition (10), which is established only in equation (6.84) of [10]. Note also that

\[
\rho(t_{\text{min}}) = 0.
\]

The times for which the constraint (10) is obeyed may be easily determined using (41). We find that the solution (37) fulfill the above requirement for

\[
t < t_{\text{min}} - t_{(1)} \quad \text{and} \quad t > t_{\text{min}} + t_{(1)},
\]

where

\[
t_{(1)} = m_{\text{pl}} \sqrt{\frac{3}{16 \pi}} \left(\frac{1}{B_0^2} - 2 \mu\right) \approx m_{\text{pl}} \frac{3}{4B_0} \sqrt{\frac{3}{\pi}} = 3.4 \times 10^{24} \text{ GeV}^{-1} = 2.2 \text{ s},
\]

where \( m_{\text{pl}} = G^{-1/2} = 1.2211 \times 10^{19} \text{ GeV} \) is the Planck mass.

This seems to be a disappointing conclusion about the capability of the first order quantum correction (5) to describe the very early stages of the universe. In the next two sections we try to evade this problem by allowing for the existence of a non-vanishing cosmological term, as it has been increasingly indicated by recent theoretical and observational results [17].

One could speculate that models with a time varying fine structure “constant” could perhaps evade the above mentioned problem. In fact, if \( \alpha \), and consequently \( \mu \), was greater at early times, the value of \( B(t_{\text{min}}) \) could reach values below the constraint (10). However, in spite of the fact that models with oscillatory variation of the fundamental constants have been proposed [24,27], recent calculations using the Keck Telescope data seems to indicate a negative variation of \( \alpha \) at high redshift (\( z > 1 \)): \( \Delta \alpha / \alpha = -1.1 \pm 0.4 \times 10^{-5} \) [20,27].

Before going to this next step, let us mention that, if (37) could describe the entire evolution of the universe in the distant past, it would imply the existence of an inflationary era (\( \ddot{a} > 0 \)) in the interval

\[
t_{\text{min}} - t_I < t < t_{\text{min}} + t_I,
\]

where

\[
t_I = \frac{m_{\text{pl}}}{2} \sqrt{\frac{3 \mu}{\pi}} \approx 7.1 \times 10^{22} \text{ GeV}^{-1} = 0.046 \text{ s}.
\]

Figure 1 shows the scale factor, the magnetic field, the energy density and the pressure as a function of time for a definite value of \( B_0 \). The time interval where the weak field approximation breaks down is also indicated.
IV. GENERAL SOLUTION FOR A CONSTANT NON-VANISHING \( \Lambda \) AND A TIME-DEPENDENT MAGNETIC FIELD

We will now extend the analysis made in [1] and investigate how the above conclusions are modified with the introduction of a constant cosmological term. For \( K = 0 \) and \( \Lambda = \text{constant} \neq 0 \), substitution of (33) into (31) leads to the equation

\[
\dot{Z}^2 = 16 \left[ \lambda Z^2 + \alpha_0^2 (Z - 2\mu B_0^2) \right],
\]

where we have defined

\[
Z \equiv \left( \frac{a}{a_0} \right)^4,
\]

\[
\lambda \equiv \frac{\Lambda}{3}.
\]

Equation (52) can be easily integrated to give

\[
a(t) = a_0 \left( \frac{1}{4\lambda} \right)^{1/4} \left[ C_0 e^{4\sqrt{\lambda} \left(t - t_0\right)} + \frac{D_0}{C_0} e^{-4\sqrt{\lambda} \left(t - t_0\right)} - 2 \alpha_0^2 \right]^{1/4},
\]

where

\[
C_0 \equiv \alpha_0^2 + 2\lambda + 2\sqrt{\lambda (\lambda + \alpha_0^2 - 2\alpha_0^2 \mu B_0^2)},
\]

\[
D_0 \equiv \alpha_0^3 (\alpha_0^2 + 8\lambda \mu B_0^2).
\]

The Hubble parameter is

\[
H(t) = 2B_0 \sqrt{\lambda} \left[ C_0 e^{4\sqrt{\lambda} \left(t - t_0\right)} + \frac{D_0}{C_0} e^{-4\sqrt{\lambda} \left(t - t_0\right)} - 2 \alpha_0^2 \right]^{-1/2}.
\]

It is straightforward to see that the term inside the square brackets of (53) is positive for all \( t \) and that the scale factor reaches its minimum value

\[
a_{\text{min}} = a_0 \left[ \frac{\alpha_0}{2\lambda} \left( \sqrt{\alpha_0^2 + 8\lambda \mu B_0^2} - \alpha_0 \right) \right]^{1/4}
\]

at

\[
t_{\text{min}} = t_0 + \frac{1}{8\sqrt{\lambda} \ln \left( \frac{D_0}{C_0^2} \right)}.
\]

As in the previous case, the universe bounces at \( t_{\text{min}} \) and, if the solution would effectively hold near \( t_{\text{min}} \), an inflationary phase would take place for all values of \( t \) such that

\[
C_0^2 x^4 - 8\alpha_0^2 C_0 x^3 + 14D_0 x^2 - 8\alpha_0^2 \frac{D_0}{C_0} x + \frac{D_0^2}{C_0^2} > 0,
\]

where

\[
x \equiv e^{4\sqrt{\lambda} \left(t - t_0\right)}.
\]

Nevertheless, the magnetic field at \( t_{\text{min}} \) is

\[
B(t_{\text{min}}) = \left[ \frac{\Lambda}{2\pi G \left( \sqrt{1 + \frac{2\Lambda \mu}{\pi G}} - 1 \right)} \right]^{1/2}.
\]
From the above expression we see that $B(t_{\text{min}}) \to \frac{1}{\sqrt{2} a}$ as $\Lambda \to 0$ and that $B(t_{\text{min}}) \to \infty$ as $\Lambda \to \infty$. Therefore this model suffers from the same problem as the previous one: the first order approximation (5) can not describe the dynamics all the way back to the point of minimum compression.

From the condition $B(t) < B_{\text{cr}}$, we find that the solution is valid for any time $t$ such that

$$t - t_{\text{min}} = \frac{1}{4} \sqrt{3} \ln \left[ \frac{A}{B_{\text{cr}}^2 \sqrt{4 \pi G (\pi G + 2 \mu \Lambda)}} \right],$$

(64)

where

$$A \equiv \Lambda + 2 \pi G B_{\text{cr}}^2 \pm \sqrt{\Lambda \left[ \Lambda + 4 \pi G B_{\text{cr}}^2 (1 - 2 \mu B_{\text{cr}}^2) \right]}.$$  

(65)

The domain of validity of the solution will then depend on the value of $\Lambda$.

Figure 2 and Figure 3 show the scale factor, the magnetic field, the energy density and the pressure as a function of time, for some values of $\Lambda$ and $B_0$.

V. GENERAL SOLUTION FOR A CONSTANT MAGNETIC FIELD AND A TIME-DEPENDENT $\Lambda$

We will now turn to the case when the magnetic field does not vary with time. This solution of Eq. (32) has not been analysed in Ref. [1]. If $\dot{\Lambda} = 0$, this leads to the well known models $p = \rho = 0$ or $p = -\rho = \text{constant}$. However, if we treat $\Lambda$ as a dynamical variable, then (32) with

$$B(t) = B_0 = \text{constant},$$

(66)

leads to

$$\dot{\Lambda} = 3K_0 \frac{\dot{a}}{a}$$

(67)

and hence

$$\Lambda(t) = \Lambda_0 + 3K_0 \ln \left( \frac{a}{a_0} \right),$$

(68)

where

$$\Lambda_0 \equiv \Lambda(t_0)$$

(69)

and

$$K_0 \equiv -\frac{16 \pi G}{3} B_0^2 (1 - 4 \mu B_0^2).$$

(70)

Substituting this result into (31), we get for the scale factor

$$a(t) = a_0 \exp \left[ \frac{K_0}{4} (t - t_0)^2 + H_0 (t - t_0) \right]$$

$$= a_0 \exp \left( \frac{K_0}{4} t^2 + \beta_1 t + \beta_0 \right),$$

(71)

where

$$H_0 \equiv \sqrt{\frac{\Lambda_0}{3} + \frac{4 \pi G}{3} B_0^2 (1 - 2 \mu B_0^2)}$$

$$= \sqrt{\frac{\Lambda_0}{3} + \frac{8 \pi G}{3} \rho},$$

(72)

$$\beta_1 \equiv -\left( \frac{K_0}{2} t_0 - H_0 \right),$$

(73)
and
\[ \beta_0 = t_0 \left( \frac{K_0}{4} t_0 - H_0 \right) . \]  

(74)

The Hubble parameter is
\[ H(t) = \frac{K_0}{2} (t - t_0) + H_0 \]
\[ = \frac{K_0}{2} t + \beta_1 , \]  

(75)

and we have \( H = 0 \) for
\[ t_c = -\frac{2\beta_1}{K_0} = t_0 - \frac{2H_0}{K_0} . \]  

(76)

At this point, the scale factor reaches the value
\[ a(t_c) = a_0 e^{-H_0^2/K_0} , \]  

(77)

whereas the cosmological term is
\[ \Lambda(t_c) = \Lambda_0 - 3H_0^2 = -4 \pi G B_0^2 (1 - 2 \mu B_0^2) = -8 \pi G \rho . \]  

(78)

The behaviour of the solution will depend on the sign of the constant \( K_0 \) (for \( K_0 = 0 \) we get the de Sitter solution). For \( K_0 > 0 \), that is, for \( B_0 > 1/(2\sqrt{\mu}) \), the universe begins arbitrarily large as \( t \to -\infty \), contracts until reaches its minimum size at \( t_c \) and begins to expands without limit from this point on. The solution is always accelerated (\( \ddot{a} > 0 \)). Note that this case violates condition (10).

A much more interesting solution is the one corresponding to \( K_0 < 0 \) (\( B_0 < 1/(2\sqrt{\mu}) \approx 4.2 \times 10^{-5} \) Tesla = 0.42 Gauss), since we can make \( B_0 \) satisfy (10) in this case. For this range of \( B_0 \), \( a(t) \) approaches zero asymptotically as \( t \to -\infty \), reaches a maximum at \( t_c \) and begins to contract, approaching zero as \( t \to +\infty \). Moreover, \( \ddot{a} > 0 \) for
\[ t < t_c - \sqrt{-\frac{2}{K_0}} \quad \text{and} \quad t > t_c + \sqrt{-\frac{2}{K_0}} . \]  

(79)

It is worthy remarking that the time interval \( \Delta t_{(NI)} \), prior to \( t_c \), for which the solution is not inflationary depends on the value of \( B_0 \) as
\[ \Delta t_{(NI)} = \sqrt{-\frac{2}{K_0}} = \frac{1}{B_0} \sqrt{\frac{3}{8 \pi G (1 - 4 \mu B_0^2)}} . \]  

(80)

This time interval reaches a minimum at \( B_0^2 = 1/(8\mu) \) and goes to infinity as \( B_0 \to 0 \) or \( B_0^2 \to 1/(4\mu) \). Remembering that, for consistency, condition (10) must be obeyed, we see that this time interval may be as small as 3.2 s. Hence we have a cosmological model that could describe a universe that inflates through much of its history.

Note that, as far as \( \Lambda(t) \) is concerned, \( t_c \) is a point of minimum for both \( K_0 > 0 \) and \( K_0 < 0 \). For the case of interest, \( (K_0 < 0) \), the evolution proceeds in the following way (note that the weak energy condition \( (\rho > 0) \) is automatically guaranteed if \( K_0 < 0 \)). In the far past, the universe is arbitrarily small, with an arbitrarily large vacuum energy. It expands in such a way that during this expansion phase, the vacuum energy decays so as to keep \( \rho \) constant. The vacuum energy eventually becomes negative at the time
\[ t_1 = t_c + \frac{2}{K_0} \sqrt{H_0^2 - \frac{\Lambda_0}{3}} , \]  

(81)

reaching its minimum value
\[ \rho_{\text{vac}}^{(\text{min})} = \frac{\Lambda_0 - 3H_0^2}{8 \pi G} \]  

(82)

at \( t_c \), where the universe reaches its maximum size. Thereafter, the contraction phase takes place. The vacuum energy then grows, becoming positive at
\[ t_2 = t_c - \frac{2}{K_0} \left( \frac{H_0^2 - \Lambda_0}{3} \right), \]  

whereas \( \rho \) remains constant. The expected tendency of \( \rho \) growing adiabatically with the decreasing \( a \) is compensated by the increase in \( \rho_{vac} \). The universe would end up arbitrarily small as \( t \to \infty \) with an arbitrarily large vacuum energy.

Figure 4 shows the scale factor and the cosmological term as a function of time for \( K_0 > 0 \) and some values of \( B_0 \) and \( \lambda_0 \), where we have further defined

\[ \lambda_0 = \frac{\Lambda_0}{3}. \]  

Figure 5 shows the same quantities for \( K_0 < 0 \).

If one expects any such model to properly describe the evolution of the real universe, it would be advisable to take into account other matter fields, such as ultrarelativistic matter, scalar fields or dust. In [1], it was demonstrated, for the case \( \Lambda = 0 \), \( B = B(t) \), that the presence of ultrarelativistic matter with an equation of state \( p_{(ur)} = \rho_{(ur)}/3 \) would just amount for a reparametrization of the constants \( B_0 \) and \( \mu \). The effects of other matter fields in the models analysed in this section are currently under investigation.

VI. SUMMARY AND CONCLUSIONS

We have examined some consequences of considering first order quantum corrections to Maxwell electrodynamics in zero curvature FRW universes. We have derived general analytical solutions under three different assumptions. When the cosmological term \( \Lambda \) is identically zero and the dynamics is driven by a time dependent magnetic field, we have obtained the general form of the solution found previously by Novello et al. [1]. We have found a new solution when \( \Lambda \) is a non-vanishing constant. In both cases, the universe is non-singular, bouncing at a critical time when it reaches its minimum size. However, we have shown that, near this critical time, the magnetic field increases beyond the value allowed by the use of the first order approximation to the effective Lagrangean density. The time range where this weak field approximation does apply was evaluated. On the other hand, numerical work, that takes into account higher order terms, indicates that the non-singular behaviour is preserved [3,23].

We have derived a third solution that describes a universe driven by a time dependent \( \Lambda \) that sustains a constant magnetic field. In this case, the time behaviour of the cosmological term is univocally determined and depends on the logarithm of the scale factor. For small enough values of the magnetic field strength (so that the first order corrections can be used safely), the universe begins arbitrarily small as \( t \to \infty \), expands to a maximum size \( a_{max} \) at \( t_c \) and then contracts back to zero size as \( t \to \infty \). The energy density associated with \( \Lambda \) is arbitrarily large at \( t \to -\infty \), reaches a minimum at \( t_c \) and grows without limit as \( t \to \infty \). It therefore decays during the expansion phase and increases during the contraction era, so as to keep the magnetic energy constant. The dynamics is inflationary during most of the cosmic history, except near \( t_c \).

We are presently attempting to generalize our results for universes with non-zero curvature as well as analysing how our models would be modified by the presence of other matter fields. Another possibility is to study how the time evolution for the magnetic field, to be derived from (32), would be modified by imposing a definite decaying law for \( \Lambda \). This could be chosen among the several forms presented in the literature (see, for example, [4]). More accurate results could possibly be found by considering higher order terms in the effective Lagrangean.

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FIG. 1. The upper panel shows the scale factor (solid line) and the magnetic field (dashed line) while the lower panel shows the energy density (solid line) and the pressure (dashed line) for the model with $\Lambda = 0$. The vertical bars indicate the time interval $t_{\text{min}} - t(1) < t < t_{\text{min}} + t(1)$, during which the constraint (10) is not obeyed. $B_0$ has been chosen such that $\sqrt{2\mu}B_0 = 0.2$ and $B_0/B_c = 0.5$.

FIG. 2. The upper panel shows the scale factor (solid line) and the magnetic field (dashed line) while the lower panel shows the energy density (solid line) and the pressure (dashed line) for the model with a constant non-vanishing $\Lambda$. The values for $\Lambda$ and $B_0$ are such that $\sqrt{\lambda/\alpha_0} = 2 \times 10^{-4}$ and $\sqrt{2\mu}B_0 = 0.2$. 
FIG. 3. As in Figure 2 but for $\sqrt{\lambda}/\alpha_0 = 5 \times 10^{-5}$ and $\sqrt{2\mu B_0} = 0.05$.

FIG. 4. The scale factor (solid line) and the cosmological term (dashed line) for the model with constant magnetic field, time-dependent $\Lambda$ and $K_0 > 0$ ($\sqrt{2\mu B_0} = 1$). In the upper panel $\sqrt{\lambda_0}/\alpha_0 = 1$ and the lower panel is for $\sqrt{\lambda_0}/\alpha_0 = 0.5$. 
FIG. 5. As in Figure 4 but for $K_0 < 0$ ($\sqrt{2\mu B_0} = 0.1$). In the upper panel $\sqrt{\lambda_0}/\alpha_0 = 1$ and the lower panel is for $\sqrt{\lambda_0}/\alpha_0 = 0.5$. 