Minimal generating sets in wreath products

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May 3, 2014

Abstract

We find some sufficient conditions under which the permutational wreath product of two groups has a minimal generating set. In particular we prove that for a regular rooted tree the group of all automorphisms and the group of all finite-state automorphisms of such a tree satisfy these conditions.

1 Introduction

We consider the following problem

Problem 1. Let $T$ be a regular rooted tree. Do the group of all automorphisms and the group of all finite-state automorphisms of $T$ have any minimal generating set?

Originally such a problem was posted by B. Csákány and F. Gécseg in 1965. They asked whether or not the semigroup of all automaton transformations, the semigroup of all finite automaton transformations, the group of all bijective automaton transformations, and the group of all finite bijective automaton transformations over a fixed finite alphabet with at least two elements have a minimal generating set?

Answer for semigroups is negative. This result was obtained independently by S. Aleshin in 1970 and P. Dömösi in 1972. The question about groups (i.e. Problem 1) was also formulated in papers of P. Dömösi. Particularly it was in Problem 2.1 and Problem 2.31. Moreover this problem is mentioned in the papers Problem 2.1 and Problem 2.31.

Among works related to this problem we mention the result of Andriy Oliynyk from 2012. Namely, it was proven that finite-state wreath product of transformation semigroups is not finitely generated and in some cases doesn’t have a minimal generating set. We also mention papers devoted to the study of generating sets in projective limits of wreath products of groups.

We find some sufficient conditions under which the permutational wreath product of finite group and infinite group has a minimal generating set (Theorem 2.1). We also give a few examples of groups and classes of groups satisfying such conditions. In particular we prove that for a regular rooted tree the group of all automorphisms and the group of all finite-state automorphisms of such a tree satisfy these conditions (Theorems 3.6 and 3.8). Therefore these groups have minimal generating sets. Thus Problem 1 is solved positively.

Most results of this paper were announced without proofs in 2012. Application of Theorem 2.1 to wider class of wreath branch groups will be the subject of forthcoming paper.

The author gratefully acknowledges the many helpful suggestions of Ievgen Bondarenko, Andriy Oliynyk, and Wital Sushchansky during the preparation of the paper.

2 Main theorem

We first recall the notion of the permutational wreath product.

Let $(A, X)$ be a permutation group and let $H$ be a group. Then the permutational wreath product $(A, X) ≀ H$ is the semi-direct product $(A, X) ⋊ H^X$, where $(A, X)$ acts on the direct power $H^X$ by the respective permutations of the direct factors.

We will say that a permutation group $(A, X)$ satisfies the condition $PS$ if:
1. The group \((A, X)\) is finite and transitive.

2. There are subsets \(X_1, X_2\) of \(X\) and subgroups \(A_1, A_2\) of \(A\) with the following properties:
   - \((A_1, X)\) and \((A_2, X)\) act transitively on \(X_1\) and \(X_2\) respectively and act trivially on \(X \setminus X_1\) and \(X \setminus X_2\) respectively.
   - \(X_1\) and \(X_2\) do not intersect.
   - \(|X_1| \geq 2, |X_2| \geq 3\).
   - If \(|X_1| = 2\) then there is \(a \in A\) satisfying \(a(X_1) \cap X_2 \neq \emptyset\) and \(a(X_1) \not\subseteq X_2\).

We say that a group \(G\) satisfies the \(L\)-condition, if \(G\) is decomposed into permutational wreath product \(G = (A, X) \wr H\) and there are a normal subgroup \(H_0\) of \(H\) and an integer \(k > 1\) with the following properties:

1. The permutation group \((A, X)\) satisfies the condition \(PS\),
2. The quotient \(H/H_0\) has infinite minimal generating set,
3. \(|H/H_0| \geq |H_0|\),
4. Either \(H_0 < H'\) or \(H' \leq H_0 < H^k H'\), where \(H'\) is the commutator subgroup of \(H\) and \(H^k = \{h^k, h \in H\}\).
5. If \(H' \leq H_0 < H^k H'\) then there is a subset \(C\) of some minimal generating set of \(H/H_0\) such that
   - \(|C| = |H/H_0|\),
   - For every coset \(c \in C\) there is \(h \in c\) of order \(k\).

Note that condition 2 of the definition of the \(L\)-condition imply that \(H\) is infinite groups.

The main result of the paper is the following theorem.

**Theorem 2.1.** A group with the \(L\)-condition has a minimal generating set.

**Proof of Theorem 2.1**

At first we fix some notations.

Let \(G = (A, X) \wr H\) and let \(H_0\) be a normal subgroup of \(H\). We assume that \(G, (A, X), H, H_0\) satisfy the conditions of Theorem 2.1 Let \(X = \{0, 1, \ldots, n\}\), \(X_1 = \{0, 1, \ldots, l_1\}\), and \(X_2 = \{l_2, l_2 + 1, \ldots, n\}\).

The group \(G\) is a semidirect product of its subgroups \(A, K\), where \(K\) is the direct product of \(n + 1\) copies of \(H\), i.e., \(K = H \times \cdots \times H\). We will also write whole subgroup \(K\) as \((H, \ldots, H)\). The conjugation of \(g = (g_0, \ldots, g_n)\) by an element of \((A, X)\) is a corresponding permutation of coordinates of the tuple.

Without loss of generality, we will make the following assumptions:

If there exists \(a \in A\) such that \(a(X_1) \cap X_2 \neq \emptyset\) and \(a(X_1) \not\subseteq X_2\) then let \(d_1 \in A\) be such that \(d_1(0) \notin X_2\) and \(d_1(1) = n\).

Otherwise, if \(a(X_1) \cap X_2 \neq \emptyset\) implies that \(a(X_1) \subseteq X_2\) for all \(a \in A\) then \(|X_1| \geq 3\) by the condition \(PS\). In this case let \(d_2 \in A\) be such that \(d_2(0) = n, d_2(1) = n - 1,\) and \(d_2(2) = n - 2\).

By the \(L\)-condition there is a minimal generating set \(\bar{F} = \{f_i \mid i \in I\}\) of \(H/H_0\), where \(I\) denotes a set of indices. Let \(\psi : H \to H/H_0\) be the canonical epimorphism. For every \(i \in I\) we fix some element \(f_i \in H\) such that \(\psi(f_i) = \bar{f_i}\). Denote

\[ F = \{f_i \mid i \in I\}. \]

Let \(I_1\) and \(I_2\) be subsets of \(I\) with the following properties:

- \(|I_2| = |I|\).
- \(I_1 = I \setminus I_2\).

Denote

\[ F^{I_j} = \{f_i \mid i \in I_j\} \text{ for } j = 1, 2. \]

In the case of \(H' \leq H_0 < H^k H'\) due to condition 5 of the \(L\)-condition we can assume that for every \(i \in I_2\) the following equality holds: \(f_i^n = e\). In the case of \(H_0 < H'\) we can assume that \(I_2 = I\).
Since $\bar{F}$ is an infinite set the set of the finite words over $\bar{F}$ has the same cardinality as $\bar{F}$. By the L-condition $|H/H_0| \geq |H_0|$. It follows that

$$|\mathcal{I}_2| = |\mathcal{I}| = |H/H_0| \geq |H_0|.$$ 

Therefore we can fix a surjection $\phi : \mathcal{I}_2 \to H_0$. We also define the set of elements of $G$:

$$S_k = \{ q_i = (f_i, e, \ldots, e, \phi(i)) \mid i \in \mathcal{I}_2 \} \cup \{ q_i = (f_i, e, \ldots, e) \mid i \in \mathcal{I}_1 \}.$$

Let us fix a minimal generating set of $A$: $S_A = \{ s_1, s_2, \ldots, s_r \}$. Let $S = S_K \cup S_A$ and $N = \langle S \rangle$. Note that $A = \langle S_A \rangle$ is contained in $N$.

**Lemma 2.2.** The set $S$ is a minimal generating set of the group $N$.

**Proof.** Since $G$ is the semidirect product $A \ltimes K$ and $S_K \subset K$, any element $s$ of $S_A$ cannot be written as an product of elements of $S \setminus \{ s \}$ and their inverses.

Further, suppose, contrary to our claim, that the element $q_i$ for some $i \in \mathcal{I}$ is a product of elements of $S \setminus \{ q_i \}$ and their inverses. It is easy to check that this decomposition of $q_i$ can be transformed to the product of the form $q_i = (q_{i_1}^{e_1} \ldots (q_{i_m}^{e_m})^{e_m})$, where $i_1, \ldots, i_m \in \mathcal{I} \setminus \{ i \}$, $e_1, \ldots, e_m \in \{ -1, 1 \}$ and $a_1, \ldots, a_m \in A$. Consider $0$-th coordinate of $q_i$. We have that $f_i$ is a product of elements of $F \setminus \{ f_i \}$, their inverses and elements of $H_0$. Applying $\psi$ we conclude that $f_i$ is a product of elements of $F \setminus \{ f_i \}$ and their inverses. This contradicts to the fact that $\bar{F}$ is a minimal generating set of the quotient $H/H_0$.

We next show that the set $S$ is a generating set of $G$, i.e., we next show that $N = G$.

**Lemma 2.3.** For every $g \in \langle F \rangle$ the elements $u_{n-2} = (e, \ldots, e, g, g^{-1})$ and $u_{n-1} = (e, \ldots, e, g, g^{-1})$ are contained in $N$.

**Proof.** The element $g$ can be decomposed into the product of elements of $F$ and their inverses:

$$g = f_1^{e_1} \ldots f_m^{e_m},$$

where $e_1, \ldots, e_m \in \{ -1, 1 \}$. For every $j \in X_1 \setminus \{ 0 \}$ choose $b_j \in A_1$ such that $b_j(0) = j$. Then

$$t_j = b_j^{-1} q_{i_1}^{e_1} \ldots q_{i_m}^{e_m} b_j (q_{i_1}^{e_1} \ldots q_{i_m}^{e_m})^{-1} = (g^{-1}, e, \ldots, e, g, e, \ldots, e) \in N,$$

where $g$ is located on $j$-th coordinate of the tuple.

We consider all possible cases depending on the group $A$. We will use here the elements $d_1$ and $d_2$ which were defined at the beginning of the proof of the theorem.

1. There is $a \in A$ such that $a(X_1) \cap X_2 \neq \emptyset$ and $a(X_1) \not\subseteq X_2$. For every $m \in \{ n-2, n-1 \}$ choose $c_m \in A_2$ such that $c_m(n) = m$. Then $t_j^{d_1(t_j^{d_2} c_m)^{-1}} = u_m \in N$ for $m = n-2, n-1$.

2. For all $a \in A$, the inequality $a(X_1) \cap X_2 \neq \emptyset$ implies that $a(X_1) \subseteq X_2$. Then $|X_1| \geq 3$ and elements $u_{n-1} = t_j^{d_2}$ and $u_{n-2} = t_j^{d_2}$ are contained in $N$.

**Lemma 2.4.** The subgroup $(E, \ldots, E, H')$ of $K$ is contained in $N$.

**Proof.** Let $h_1, h_2 \in H$. Then there exist $g_j \in \langle F \rangle, i_j \in \mathcal{I}_2$ for $j = 1, 2$ such that $h_j = g_j \phi(i_j)$. By the construction, the set $S$ contains elements $q_j = (f_{i_j}, e, \ldots, e, \phi(i_j))$ for $j = 1, 2$. Let $a \in A_1$ be such that $a(0) = 1$. Then $q_j^{a} = (e, f_{i_j}, e, \ldots, e, \phi(i_j)) \in N$. By Lemma 2.2 elements $t_1 = (e, \ldots, e, g_1, e, g_1)$ and $t_2 = (e, \ldots, e, g_2, g_2)$ are contained in $N$. Therefore $h_1^{t_1} = t_1 q_{i_1}^{a} = (f_{i_1}, e, \ldots, e, g_1, g_1)$ and $h_2^{t_2} = t_2 q_{i_2}^{a} = (e, f_{i_2}, e, \ldots, e, g_2, g_2)$ are contained in $N$. Hence $h_1^{t_1} h_2^{t_2} h_1^{-1} h_2^{-1} = (e, \ldots, e, h_1 h_2 h_1^{-1} h_2^{-1}) \in N$. Thus $(E, \ldots, E, H') \not\subseteq N$.

**Lemma 2.5.** If $H' \subseteq H_0 < H^k H'$ then $(E, \ldots, E, H^k) \not\subseteq N$.

**Proof.** If $H' \subseteq H_0 < H^k H'$ then for every $i \in \mathcal{I}_2$ the following equality holds: $f_i^{k} = e$.

Let $h \in H$. Then $h = g_{0} h_{0}$ for some $g \in \langle F \rangle$ and $h_{0} \in H_{0}$. Since $H_{0} > H'$ and $F_{\mathcal{I}_2} \subseteq F_{\mathcal{I}_2}$, $g \in \langle F_{\mathcal{I}_2} \rangle$, and $h_{1} \in H'$ such that $g = g_{1} h_{1}$. Since $H_{0} > H'$ there exists $i \in \mathcal{I}_2$ satisfying $\phi(i) = h_{1} h_{0}$. Thus we have $h = g_{1} g_{2} \phi(i)$. By the construction, the set $S$ contains the element $q_{i} = (f_{i}, e, \ldots, e, \phi(i))$. By Lemma 2.2 element $t_2 = (e, \ldots, e, g_2, g_2)$ is contained in $N$. Let $a \in A$ be such that $a(0) = n$. Note that element $t_1 = a^{-1}(g_1, e, \ldots, e) = (e, \ldots, e, g_1)$ is contained in $N$. Therefore $h' = t_1 t_2 q_{i} = (f_{i}, e, \ldots, e, g_2, g_1 g_2 \phi(i))$ is contained
in \( N \). By the condition of the lemma there is \( h_2 \in H' \) such that \( a_2 h_2 = b_2 \). Let \( a_3 \in A_2 \) be such that \( a_3 (n-1) = a_n \). Then \( (e, \ldots, e, h_2, e) = a_3^{-1} (e, \ldots, e, h_2) a_3 \in N \) by Lemma 2.4. Therefore \( h^k (e, \ldots, e, h_2, e) = (e, \ldots, e, e, h^k) \in N \). Thus \( (E, \ldots, E, H^k) < N \). 

\[ \square \]

Lemma 2.6. \( N = G \).

Proof. If \( H_0 < H' \) then \( (E, \ldots, E, H_0) < N \) by Lemma 2.3. If \( H_0 > H' \) then the conditions of Lemma 2.5 hold by condition 5 of the L-condition. Thus in this case \( (E, \ldots, E, H_0) < N \) too. By construction \( (F, E, \ldots, E) \) is contained in \( \langle S_k, \ldots, E, H_0 \rangle \). Therefore the set \( (F, E, \ldots, E) \) is contained in \( N \). Let \( a \in A \) be such that \( a(0) = n \). Then \( a^{-1} (F, E, \ldots, E) a = (E, \ldots, E, F) \subseteq N \). Since \( \langle F \rangle H_0 = H \) we have \( (E, \ldots, E, H) < N \). Also by transitivity of \( (A, X) \) we get \( (H, \ldots, H) < N \). It follows that \( N = G \). 

Now the assertion of Theorem 2.1 follows immediately from Lemma 2.2 and Lemma 2.6.

3 Applications and Examples

We first give natural constructions of groups with property PS.

Proposition 3.1. The following groups satisfy PS:

1. The symmetric group of degree \( m \geq 5 \).
2. The permutational wreath product \( B_1, Y_1 \wr (B_2, Y_2) \), where \( B_1, Y_1 \) and \( B_2, Y_2 \) are finite transitive permutation groups and \( |Y_1| \geq 2, |Y_2| \geq 3 \).
3. The permutational wreath product \( B_1, Y_1 \wr (B_2, Y_2) \wr (B_3, Y_3) \), where \( B_i, Y_i \) is finite transitive permutation group and \( |Y_i| \geq 2 \) for \( i = 1, 2, 3 \).

Now we formulate two corollaries from Theorem 2.1 which are more applicable.

Proposition 3.2. Let \( G = (A, X) \wr H \) and there is an integer \( k > 1 \) with the following properties:

1. \( (A, X) \) satisfies PS.
2. \( H \) is an infinite group.
3. \( H' > H^k \).
4. \(|H'| \leq |H/H'| \).

Then the group \( G \) satisfies the L-condition.

Proof. Due to Theorem 2.1 we only need to show that \( H/H' \) has infinite minimal generating set. By the conditions of the theorem \( H/H' \) has exponent \( p \). We conclude from results of \cite{IS} Theorem 3.1 and Lemma 4.3] that an abelian group of bounded exponent has minimal generating set, and the proposition follows.

Proposition 3.3. Let \( G = (A, X) \wr H \) and there is an infinite subgroup \( M \) of \( H \) with the following properties:

1. \( (A, X) \) satisfies PS.
2. \( M \) has exponent \( 2 \).
3. \(|M| = |H| \).
4. \( M \cap H^2 = E \).

Then the group \( G \) satisfies the L-condition.

Proof. Put \( H_0 = H^2 \). Then \( H/H_0 \) and \( M \) have minimal generating sets (Hamel basis) as vector spaces over the field with two elements.

Let \( I \) be an index set, and let \( B = \{ b_i \mid i \in I \} \) be a minimal generating set of \( M \). Let also \( \psi : H \to H/H^2 \) be the canonical epimorphism. Since \( M \cap H^2 = E \) the restriction \( \psi | M \) is a bijection and the set \( \psi(B) \) is a minimal generating set of \( \langle \psi(B) \rangle \). Since \( B \) is infinite we have \(|\psi(B)| = |B| = |M| = |H| \). Therefore we have \(|H| = |H/H_0| \) and \( \psi(B) \) can be complemented to Hamel basis \( F \) of the space \( H/H_0 \). It is also evident that \( b_i^2 = e \) for every \( b_i \in B \). Note that inclusion \( H' < H^2 \) is always true. Thus the group \( G \) satisfies the L-condition.
Remark that we use existence of Hamel basis of a vector space over a field in the proof of Proposition 3.3. Hence this proof uses the axiom of choice in some cases.

In the next section we will apply Proposition 3.3 to some groups of automorphisms of rooted trees, and particularly give positive answer to Problem 1.

**Automorphism groups of rooted trees**

We first recall necessary definitions related to rooted trees and groups acting on rooted trees. All notions which will be defined in this section are well-known, see for instance [10, 19, 20] for more details.

Let us fix our notations. Let $X = (X_1, X_2, \ldots )$ be a sequence of finite sets $X_i = \{0, 1, \ldots , n_i\}$ (we assume $n_i \geq 1$ for all $i$). Let $X^n$ denote the set of all words of the form $x_1 x_2 \ldots x_n$, where $x_i \in X_i$ for $i = 1, \ldots , n$. Let $X^*$ denote the set which consist of the empty word $\emptyset$ and all words of the form $x_1 x_2 \ldots x_n$, where $n \in \mathbb{N}$ and $x_i \in X_i$ for $i = 1, \ldots , n$. Let $X^n$ denote the set of all infinite words of the form $x_1 x_2 \ldots$, where $x_i \in X_i$. We denote by $X(k)$ the infinite sequence $(X_k, X_{k+1}, \ldots )$.

We can consider the set of words $X^*$ as rooted tree $T_X$ which can be defined as follows: a vertex $x_1 x_2 \ldots x_n$ is adjacent to $x_1 x_2 \ldots x_{n-1}$, $\emptyset$ is the root. For rooted tree $T_X$ we also define vertex subtree $T_v (v \in X^*)$ which vertices are the words of the form $v X^*$. We call the set of vertices $X^n$ the $n$-th level of $T_X$.

Let $\text{Aut}_T X$ be the group of all automorphisms of the tree $T_X$. Let $G < \text{Aut}_T X$. We recall definition of some standard subgroups of $G$:

- The subgroup of all elements of $G$ fixing every vertex of $n$-th level, denoted by $\text{Stab}_G(n)$, is called the stabilizer of the $n$-th level.
- For every $v \in X^*$ the subgroup of all elements of $G$ fixing every vertex outside the subtree $T_v$, denoted by $\text{rist}_G(v)$, is called the rigid stabilizer of the vertex $v$.
- The group generated by the set $\bigcup_{v \in X^*} \text{rist}_G(v)$, denoted by $\text{Rist}_G(n)$, is called the rigid stabilizer of the $n$-th level.

Let $\text{Aut}_T X$ be the subgroup of $\text{Aut}_T X$ such that an automorphism $g$ of $T_X$ is in $\text{Aut}_T X$ if and only if the equality $g(uv) = g(u)v$ is valid for any $u \in X^k$ and any $v \in X^*$. Note that $\text{Aut}_T X$ acts by permutations faithfully on the set $X^k$. Note also that $\text{Stab}_{\text{Aut}_T X}(k) = \text{Rist}_{\text{Aut}_T X}(k)$. Therefore the group $\text{Aut}_T X$ can be decomposed into semidirect product of its subgroups $\text{Aut}_k T_X \rtimes \text{Stab}_{\text{Aut}_T X}(k)$. It follows that $\text{Aut}_T X$ is isomorphic to the permutational wreath product $(\text{Aut}_k T_X, X^k) \rtimes \text{Stab}_{\text{Aut}_T X}(k)$, where $v \in X^k$ and $\text{Rist}_{\text{Aut}_T X}(k)$ is the base subgroup of this wreath product.

We define subgroup $M_0 < \text{Aut}_T X$ as infinite direct product: $M_0 = \prod_{i \geq 0} C^{(i)}_2$, where each $C^{(i)}_2$ is isomorphic to the group of order 2. The action of elements of $M_0$ on the tree $T_X$ can be defined in the following way. A nontrivial element of $C^{(i)}_2$ acts as follows $00\ldots 011v, 00\ldots 011v \rightarrow 00\ldots 011v$, $00\ldots 011v \rightarrow 00\ldots 010v$ for every $v \in X^{(i+1)}$, and $w \rightarrow w$ for the other words of $X^*$.

**Lemma 3.4.** Intersection $M_0 \cap (\text{Aut}_T X)^2$ is trivial.

**Proof.** For every $g \in \text{Aut}_T X$ and $n \geq 0$ we can write $g = g_n(g_{n-1}, \ldots , g_0)$, where $g_n \in \text{Aut}_n T_X$, $(g_{n-1}, \ldots , g_0) \in \text{Rist}_{\text{Aut}_T X}(n)$, and $\{v_1, \ldots , v_k\} = X^n$. Write $\Pi_n g = \prod_{v \in X^n} g_v$ for every $n \geq 0$.

It is evident that $\Pi_n g^2$ is an even permutation of $X_{n+1}$ for every $g \in \text{Aut}_T X$ and $n \geq 0$. Therefore $\Pi_n h$ is an even permutation of $X_{n+1}$ for every $h \in (\text{Aut}_T X)^2$ and $n \geq 0$. But for every nontrivial element $g \in M_0$ there is $m \geq 0$ such that $\Pi_m g$ is an odd permutation of $X_{m+1}$. Thus the intersection $M_0 \cap (\text{Aut}_T X)^2$ is trivial. \[ \square \]

**Proposition 3.5.** Let $G$ be an infinite automorphism group of $T_X$ and there is a positive integer $p$ with the following properties:

1. The group $G$ can be decomposed into a semidirect product of its subgroups $(G \cap \text{Aut}_k T_X, X^k) \rtimes \text{Rist}_G(k)$ provided the group $(G \cap \text{Aut}_k T_X, X^k)$ satisfies PS.
2. $|M_0 \cap G| = |G|$.
Then the group $G$ satisfies the L-condition.

Proof. Let $G$ be a group and $k \in \mathbb{N}$ be such that all conditions of the statement are satisfied. Let $v = 0 \ldots 0 \in X^k$. Then $G$ is isomorphic to the permutational wreath product $(G \cap \text{Aut}_T X_k, X^k) \wr \text{Rist}_G(v)$.

Consider the subgroup $M = M_0 \cap \text{Rist}_G(v)$ of the group $G$. It is obvious that $M$ has exponent 2. Since $M_0 \cap G = \langle M_0 \cap G \cap \text{Aut}_k T \rangle \times M$ we have $|M| = |M_0 \cap G|$. Combining it with the second condition of the statement we obtain $|M| = |G|$. It follows that $(M_0 \cap \text{Rist}_G(v))^2 < M_0 \cap (\text{Aut}_T X)^2 = E$. Thus the group $G$ satisfies the L-condition by Proposition 3.3 and the statement follows.

Examples of uncountable groups of automorphisms with the L-condition

We remind definitions for some classes of automorphisms of $T_X$.

- An automorphism $g$ is called finitary if there is a positive integer $n$ such that the equality $g(uv) = g(u)v$ is valid for any $u \in X^n$ and any $v \in X^n$.
- An automorphism $g$ is called weakly finitary if for any $w \in X^w$ there are $n \in \mathbb{N}$, $u \in X^n$, and $v \in X^w$ such that $w = uv$ and $g(uv) = g(u)v$.
- Two words $w_1, w_2 \in X^w$ are called cofinal if there are $n \in \mathbb{N}$, $u_1, u_2 \in X^n$, $v \in X^w$ satisfying $w_1 = u_1v$ and $w_2 = u_2v$.
- An automorphism $g$ is called cofinal if it maps cofinal words to cofinal words.
- An automorphism $g$ is called bicofinal if both $g$ and $g^{-1}$ are cofinal.

Denote by $\text{Aut}_f T_X$, $\text{Aut}_{w_f} T_X$, $\text{Aut}_b T_X$ the sets of all finitary, weakly finitary and bicofinal automorphisms of $T_X$ respectively. All of these sets are groups.

Note that, by definitions, we have the following inclusions:

$$\text{Aut}_f T_X < \text{Aut}_{w_f} T_X < \text{Aut}_b T_X.$$ 

For more details on these groups we refer the reader to [14].

**Theorem 3.6.** The groups $\text{Aut} T_X$, $\text{Aut}_{w_f} T_X$ and $\text{Aut}_b T_X$ satisfy the L-condition and so have minimal generating sets.

Proof. Let $G$ be one of the above groups. Then $G$ can be decomposed into semidirect product of its subgroups $(\text{Aut}_b T_X, X^k) \ltimes \text{Rist}_G(k)$ for a positive integer $k$. The permutation group $(\text{Aut}_b T_X, X^k)$ satisfies the condition $\text{PS}$ for $k \geq 3$ by Proposition 3.3. It is clear that $M_0 < \text{Aut}_{w_f} T_X < \text{Aut}_b T_X < \text{Aut} T_X$. It follows that $G$ satisfies all conditions of Proposition 3.3 and the statement follows.

Examples of countable groups of automorphisms with lifting condition

**Proposition 3.7.** Let $G$ be a countable automorphism group of rooted tree $T_X$ with the following properties:

- $\text{Aut}_f T_X < G$.
- $\text{Rist}_G(k) = \text{Stab}_G(k)$ for some integer $k \geq 3$.

Then the group $G$ satisfies the L-condition.

Proof. By the condition of the proposition $M_0 \cap G > M_0 \cap \text{Aut}_f T_X$. Since intersection $M_0 \cap \text{Aut}_f T_X$ is countable, $|M_0 \cap G| = |G|$. Since $\text{Aut}_f T_X < G$, $\text{Aut}_b T_X < G$. Therefore $G$ is decomposed into semidirect product $(\text{Aut}_b T_X, X^k) \ltimes \text{Rist}_G(k)$ of its subgroups. The permutation group $(\text{Aut}_b T_X, X^k)$ satisfies condition $\text{PS}$ for $k \geq 3$ by Proposition 3.3. It follows that $G$ satisfies all conditions of Proposition 3.3 and the statement follows.

From now we assume that $X_1 = X_2 = \ldots$. In this case $T_X$ is called regular rooted tree. A vertex subtree $T_v$ of $T_X$ for every $v \in X^n$ can be naturally identified with the whole tree $T_X$:

$$\pi_v : \{x_{n+1} \ldots x_m \mapsto x_{n+1} x_{n+2} \ldots x_m\}.$$ 

6
Thus for any $g \in \text{Aut}_T X$ and $v \in X^*$ we can define automorphism $g|_v \in \text{Aut}_T X$ in the following way: $g|_v(w) = w$ if and only if $g(vu) = g(v)w$ for every $u, w \in X^*$.

An automorphism $g \in \text{Aut}_T X$ is 	extit{finite-state} automorphism if the set of its states is finite. All finite-state automorphisms of the tree $T_X$ form the group $\text{FAut}_T X$ of 	extit{finite-state automorphisms} of the tree $T_X$.

Let us define the number $\Theta_n(g) = \# \{ v \in X^n \mid g|_v \neq e \}$ for every $g \in \text{Aut}_T X$.

The set of all finite-state automorphisms $g \in \text{FAut}_T X$ such that the sequence $\Theta_n(g)$ is bounded by a polynomial of degree $m$ form the group $\text{Pol}(m)$ of polynomial automorphisms of degree $m$ of the tree $T_X$. The group $\text{Pol}(0)$ is also called the group of bounded automorphisms.

The group of polynomial automorphisms, denoted by $\text{Pol}(\infty)$, is defined to be the union of increasing chain of groups: $\text{Pol}(\infty) = \bigcup_{m=0}^{\infty} \text{Pol}(m)$.

A subgroup $G$ of $\text{Aut}_T X$ is self-similar provided $g|_e \in G$ for all $g \in G$ and $v \in X^*$. The group $\text{RAut}_T X$ of functionally recursive automorphisms of $T_X$ can be defined as the union of all finitely generated self-similar subgroups of $\text{Aut}_T X$.

We refer the reader to [20, 5, 19] for details concerning groups defined above.

**Theorem 3.8.** The groups $\text{FAut}_T X$, $\text{Pol}(m)$ ($m \geq 0$), $\text{Pol}(\infty)$, and $\text{RAut}_T X$ satisfy the $L$-condition and so have minimal generating sets.

**Proof.** Let $G$ be one of the above groups. It is well known that $G$ is countable group. By definition, the group $G$ contains the group $\text{Aut}_T X$. Furthermore, we have $\text{Stab}(k) = \text{Rist}(k)$ for all positive integer $k$. It follows that $G$ satisfies the $L$-condition by Proposition 3.7 and the theorem follows. \hfill \Box

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