Turing Completeness of Finite, Epistemic Programs

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In this note, we present the proof of Lemma 1.1 of [8], namely that the class of epistemic programs [1] is Turing complete. Following preliminary definitions in Section 1, Section 2 states and proves the theorem.

1 Definitions

Let there be given a countable set Φ of atoms and a finite set I of agents. Where p ∈ Φ and i ∈ I, define the language L by

\[ \phi := \top \mid p \mid \neg \phi \mid \phi \land \phi \mid \square_i \phi. \]

We use relational semantics to evaluate formulas. A Kripke model for L is a tuple \( M = ([M], R, [\cdot]) \) where \([M]\) is a countable, non-empty set of states, \( R : I \to \mathcal{P}([M] \times [M]) \) assigns to each \( i \in I \) an accessibility relation \( R_i \), and \([\cdot] : \Phi \to \mathcal{P}([M])\) assigns to each atom a set of states. With \( s \in [M] \), call \( Ms = ([M], R, [\cdot], s) \) a pointed Kripke model. The used semantics are standard (see e.g. [4, 7]), including the modal clause:

\[ Ms \models \square_i \phi \text{ iff for all } t : sR_it \text{ implies } Mt \models \phi. \]

Pointed Kripke models may be updated using action models and product update [1–3, 5, 6]. We here invoke a set of mild, but non-standard, requirements to fit the framework of [8].

A multi-pointed action model is a tuple \( \Sigma_r = ([\Sigma], R, pre, \Gamma) \) where \([\Sigma]\) is a countable, non-empty set of actions. The map \( R : I \to \mathcal{P}([\Sigma] \times [\Sigma]) \) assigns an accessibility relation \( R(i) \) on \( \Sigma \) to each agent \( i \in I \). The map \( pre : [\Sigma] \to L \) assigns to each action a precondition. Finally, \( \emptyset \neq \Gamma \subseteq [\Sigma] \) is the set of designated actions.

Where \( X \) is a set of pointed Kripke models, call \( \Sigma_r \) deterministic if \( \models pre(\sigma) \land pre(\sigma') \to \bot \) for each \( \sigma \neq \sigma' \in \Gamma \).

Let \( \Sigma_r \) be deterministic over \( X \) and let \( Ms \in X \). Then the product update of \( Ms \) with \( \Sigma_r \), denoted \( Ms \otimes \Sigma_r \), is the pointed Kripke model \( ([M \Sigma], R', [\cdot]', s') \).

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with

\[ [M\Sigma] = \{(s, \sigma) \in [M] \times [\Sigma] : (M, s) \models \text{pre}(\sigma)\} \]

\[ R' = \{(s, \sigma), (t, \tau) : (s, t) \in R_i \text{ and } (\sigma, \tau) \in R_i\}, \text{ for all } i \in \mathbb{N} \]

\[ [p]'' = \{(s, \sigma) : s \in [p]\}, \text{ for all } p \in \Phi \]

\[ s' = (s, \sigma) : \sigma \in \Gamma \text{ and } M s \models \text{pre}(\sigma) \]

As \( \Sigma^r \) is assumed deterministic over \( X \) at most one suitable \( s' \) exists. If \( M s \models \neg \text{pre}(\sigma) \) for all \( \sigma \in \Gamma \), \( M s \otimes \Sigma^r \) is undefined.

2 Theorem and Proof

Call a finite, deterministic multi-pointed action an epistemic program.\(^1\) We then show:

**Theorem 1.** The set of epistemic programs is Turing complete.

**Remark 1.** The proof uses a strict sub-class of the mentioned action models, all with only equivalence relations as suited for multi-agent S5 logics, and requires only the use of finite, S5 pointed Kripke models.

**Preliminaries.** Define a Turing machine as a 7-tuple

\[ M = (Q, q_0, q_h, \Gamma, b, \Sigma, \delta) \]

where \( Q \) is a finite set of states with \( q_0 \in Q \) the start state and \( q_h \in Q \) the halt state, \( \Gamma \) a finite set of tape symbols with \( b \in \Gamma \) the blank symbol and \( \Sigma = \Gamma \setminus \{b\} \) the set of input symbols, and \( \delta \) a partial function

\[ \delta : Q \times \Gamma \to Q \times \Gamma \times \{l, h, r\} \]

with \( \delta(q_h, \gamma) \) undefined for all \( \gamma \in \Gamma \), called the transition function. If \( \delta(q, \gamma) \) is undefined, the machine will halt.

A Turing machine acts on a bi-infinite tape with cells indexed by \( \mathbb{Z} \) and labeled with \( \Gamma \) such that only \( b \) occurs on the tape infinitely often. With the machine in state \( q \in Q \) and reading label \( \gamma \in \Gamma \), the transition function determines a possibly new state of the machine \( q' \in Q \), a symbol \( s' \) to replace \( s \) at the current position on the tape, and a movement of the metaphorical “read/write head”: Either one cell to the left (\( l \)), none (stay here, \( h \)), or one cell to the right (\( r \)).

A configuration of a machine is fully given by i) the current labeling of the tape, ii) the position of the read/write-head on the tape, and iii) the state of the machine. The space of possible configurations of a machine \( M \) is thus \( \mathcal{C} = \mathcal{T} \times \mathbb{Z} \times Q \), where \( \mathcal{T} \) is the set of bi-infinite strings \( t = (\ldots, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \ldots) \)

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\(^1\) The term stems from the seminal [1].
over \( \Gamma \) such that only \( b \) occurs infinitely often in \( t \). The transition function \( \delta \) of \( M \) may thus be recast as a partial function \( \delta : \mathcal{C} \to \mathcal{C} \).

We want to recast \( \delta \) in a slightly different manner. Each tape \( t \) has infinite head and tail consisting solely of \( bs \). Ignoring all but a finite segment of these yields a finite non-unique representation of the tape. Formally, for a string \( t = (\ldots, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \ldots) \) and \( k < k' \) let \( t_{|[k,k']} \) be the substring \((\gamma_k, \ldots, \gamma_{k'})\). The set of all such finite representations of \( \mathcal{T} \) is then given by

\[
T = \{ t = (\gamma_k, \ldots, \gamma_{k'}): \exists t \in \mathcal{T} \text{ s.t. } t = t_{|[k,k']} \text{ and } \forall j < k, \forall j' > k', t_j = t_{j'} = b \}.
\]

Each \( t \in T \) corresponds to a unique \( t \in \mathcal{T} \). Conversely, each configuration \( c = (t, i, q) \in \mathcal{C} \) may be represented by the equivalence class \( \{(t_{|[k,k']}, i, q): k < k'\} \) of its finite approximations. In each such equivalence class, there exists representatives for which the position \( i \) of the read-write head is “on the tape”, i.e., satisfies that \( \gamma_i \in t \). We impose this as a requirement and define a restricted equivalence class for each \( c = (t, i, q) \in \mathcal{C} \) by \( [c] = \{(t_{|[k,k']}, i, q): k \leq i \leq k'\} \).

With \( \mathcal{C} = \{[c] : c \in \mathcal{C}\} \), i.e., the set of equivalence classes of finite representations of configurations for which the read-write head is on the finite tape, the transition function may finally be recast as a partial function \( \delta : \mathcal{C} \to \mathcal{C} \).

**Remark 2.** The class of Turing machines with \( \Gamma = \{0, 1\}, b = 0 \), is Turing complete. Henceforth, we restrict attention to this sub-class. \( \square \)

**Proof**

To prove Theorem 1, it must be shown that any Turing machine can be simulated by an epistemic program. We show that as follows: First, we define an invertible operator \( K \) that for any finite representation of a configuration \( c \in [c] \in \mathcal{C} \) produces a pointed Kripke model \( K(c) \). Second, we define an epistemic program \( \Sigma r \) which satisfies that

\[
K^{-1}(K(c) \otimes \Sigma r) \in \delta([c]),
\]

for any \([c] \in \mathcal{C}\). Hence \( \Sigma r \) may be used to calculate the trajectory of \( \delta \).

**Machine, Language and Logic.** Fix a Turing machine \( M \) with states \( Q \), and fix from this a set of relation indices \( Q' = Q \cup \{a, b, 1\} \). Let the modal language \( \mathcal{L} \) be based on the single atom \( p \) and operators \( \square, i \in Q' \).

**Configuration Space.** Let \( \mathcal{C} = \{[c] : c \in \mathcal{C}\} \) be the set of equivalence classes of finite representations of configurations for which the read-write head is on the finite tape for \( M \) and let \( c = (t, i, q) \in \mathcal{C} \). We construct a pointed Kripke model \( K(c) \) representing \((t, i, q)\). We exemplify the construction to be in Fig. 1.

First, in three steps, we construct the set of worlds: i) Construct slightly too many “tape cells”: Let \( \lceil u \rceil = \max\{|k|, |k'|\} \) if this is even, else let \( \lceil u \rceil = \max\{|k|, |k'|\} + 1 \) and take a set of worlds \( C = \{c_j : -((\lceil u \rceil + 5) \leq j \leq \lceil u \rceil + 5)\}. \)

ii) Represent the content of a cell: Add worlds \( S = \{s_j : \gamma_j = 1\} \) to indicate
“cells” with the unique non-blank “symbol” 1. Let iii) Add a “read/write head”: Let \( H = \{ h_j : j = r/w \} \). Finally, we define the set of worlds as \( W = C \cup S \cup H \).

Second, we add relations between the worlds, also in three steps. In the following let \( R^* \) denote the reflexive, symmetric, and transitive closure of the relation \( R \) on a given base set, here \( W \). In particular \( (w, w) \in R^* \) for all \( w \in W \). i) We structure the cells \( c_i \) into a tape using relations \( R_a \) and \( R_b \): \( R_a = \{ (c_j, c_{j+1}) : j \text{ is even} \}^* \), \( R_b = \{ (c_j, c_{j+1}) : j \text{ is odd} \}^* \). ii) We attach the non-blank symbols to the appropriate cells: Let \( R_1 = \{ (c_j, s_j) : s_j \in S \}^* \). iii) We mount the read/write head at the correct position and in the correct state, \( q \): Let \( R_q = \{ (c_j, h_j) : h_j \in H \}^* \). For the remaining states \( q' \in Q \setminus \{ q \} \), let \( R_{q'} = \{ \}^* \). Finally, let \( [p] = \{ c_j, s_j, h_j \in C \cup S \cup H : j \text{ is even} \} \) and the actual world be \( c_0 \).

We thus obtain a pointed Kripke model \( K(c) = (W, \{ R_j \}_{j \in Q}, [c], c_0) \) for the finite configuration representation \( c \) of Turing machine \( M \). Figure 1 illustrates this, depicting the model \( K(c) \) for configuration \( c = (t, 3, q_0) \). Given \( K(c) \), we may clearly invert the construction process and re-obtain an element of \( [c] \). Finally let \( C = \{ K(c) : c \in C \} \).

**Expressible Properties.** To construct an epistemic program that simulates \( \delta : C \rightarrow C \), i.e., satisfies Eq. (1), we take advantage of the fact that various properties of configurations are modally expressible. Hence, we can use these as preconditions. The relevant properties and formulas are summarized in Table 1.

**Epistemic Program.** We construct an epistemic program \( \Sigma r = (\Sigma, \{ R_j \}_{j \in Q'}, \text{pre}, \Gamma) \) that simulates \( \delta : C \rightarrow C \), cf. Eq. (1). An example of such an epistemic program is illustrated in Fig. 2. We argue for the adequacy of the epistemic program in parallel with its construction. In the following, the precondition of action \( \sigma_\varphi \) is the formula \( \varphi \).

**Actual actions, halting, and tape enlargement.** Let the set of actual actions be given by \( \Gamma = \{ \gamma_\varphi : \varphi \in \Phi \} \) with \( \Phi = \{ R, L, 2_{AM} \} \cup \{ h_{q_i}, l_{q_i}, r_{q_i} : q \in Q, i \in \{ 0, 1 \} \} \), cf. Table 1.

Then, for any \( K(c) \in C \), for every cell state \( c_j \in C \) of \( K(c) \), \( c_j \) will satisfy exactly one of the formulas in \( \Phi \). \( \Sigma r \) is thus deterministic over \( C \), and the actual world of \( K(c) \otimes \Sigma r \) is a cell. Finally, formulas from \( \Phi \) are only satisfied at cell

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**Fig. 1.** An emulation of a Turing machine in state \( q_0 \) with the read/write head in position 3. Cells 1 and 2 are marked with 1 (or \( A \)), cells -1, 0 and 3 are not.
| Property                                              | Formula                                                                 |
|-------------------------------------------------------|-------------------------------------------------------------------------|
| Being a cell†                                        | \(c := (◊_u p ∨ ◊_u \neg p) \land ◊_u \neg p\)                         |
| Being a 1 symbol                                      | \(s := \neg c \land ◊_1 c\)                                           |
| Being a cell with symbol 1                           | \(1 := c \land ◊_1 \neg c\)                                          |
| Being a cell with symbol 0                           | \(0 := c \land \neg ◊_1 \neg c\)                                     |
| Being the cell of the \(r/w\)-head is while the machine is in state \(q\) | \(h_q := c \land ◊_q \neg c\)                                         |
| Being the cell immediately left of the \(r/w\)-head while the machine is in state \(q\) | \(l_q := c \land \neg h_q \land ((p \land ◊_u h_q ) \lor (\neg p \land ◊_u h_q ))\) |
| Being the cell immediately right of the \(r/w\)-head while the machine is in state \(q\) | \(r_q := c \land \neg h_q \land ((p \land ◊_u h_q ) \lor (\neg p \land ◊_u h_q ))\) |
| Being the cell of/im. left of/im. right of the \(r/w\)-head while the machine is in state \(j\) and the cell of the \(r/w\)-head contains a 1/0 | \(h_{q1}/l_{q1}/r_{q1}/h_{q0}/l_{q0}/r_{q0} : \)
Replace \(c\) in \(h_q/l_q/r_q\) with formula 1/0. |
| Being a cell at least two cells away from the \(r/w\)-head | \(h_{≥2} := c \land \bigwedge_{q \in Q} (\neg h_q \land \neg l_q \land \neg r_q)\) |
| Being the rightmost† cell                             | \(R := c \land \Box_a \neg p\)                                        |
| Being the leftmost† cell                              | \(L := c \land \Box_a \neg p\)                                        |
| Being the penultimate cell to the right               | \(PR := c \land \neg R \land ◊_a R\)                                 |
| Being the penultimate cell to the right               | \(PL := c \land \neg L \land ◊_b L\)                                 |
| Being at least two steps away from the \(r/w\)-head and not being the left- or rightmost cell | \(2_{AM} := h_{≥2} \land \neg R \land \neg L\)                       |

**Table 1.** Expressible properties used as preconditions. **Notes.** †: Recall that the extreme states of \(C\) are \(c_{−([u]+5)}\) and \(c_{[u]+5}\) with \([u]\) even.
Fig. 2. An illustration of the epistemic program \((\Sigma, \Gamma)\) for a Turing machine with transition function \(\delta(q, 0) = (q', 1, l)\) and \(\delta(q, 1) = (q, 0, r)\). That \(\Theta_{h_a0}R_q\gamma_{h_a0}\) ensures that on input \((q, 0)\) the \(r/w\)-head moves to the left and the machine is set to state \(q'\) and the relation \(\gamma_{h_a0}R_1\delta_{h_a0}\) ensures that the content of the current cell is set to 1. Similarly that \(\Theta_{h_1}q\gamma_{h_1}\) ensures that on input \((q, 1)\) the \(r/w\)-head moves to the right, the machine remains in state \(q\) and the absence of relation \(\gamma_{h_1}R_1\delta_{h_1}\) ensures that the content of the current cell is set to 0.

states of \(K(c)\). Jointly, this implies that \(\Gamma\) “copies” the set of tape cells from \(K(c)\) to \(K(c) \otimes \Sigma\).

The copied over tape may not be long enough for future operations, so we include a set of actions to preemptively enlarge it.\(^2\) To this end, let \(\Upsilon = \{\upsilon_L, \upsilon_{PL}, \upsilon_R, \upsilon_{PR}\}\). The precondition \(\phi\) of each \(\upsilon_\phi \in \Upsilon\) is satisfied by exactly one state \(c_j\) of \(K(c)\) which is a cell state. These cell state will thus have two successors in \(K(c) \otimes \Sigma\): \((c_j, \gamma_\phi)\) defined before and \((c_j, \upsilon_\phi)\). We thus gain four new cell states. Setting

\[
R_a = \{(\gamma_\phi, \gamma_\psi); \phi, \psi \in \Phi \setminus \{L\}\}^* \cup \{(\upsilon_{PR}, \upsilon_R), (\gamma_L, \upsilon_{PL})\}^*
\]

\[
R_b = \{(\gamma_\phi, \gamma_\psi); \phi, \psi \in \Phi \setminus \{R\}\}^* \cup \{(\gamma_R, \upsilon_{PR}), (\upsilon_{PL}, \upsilon_L)\}^*
\]

copies over the tape structure and suitably extends it to the new cell states, which are as the left most, penultimate left, penultimate right, and right most tape cells. Fig. 3 illustrates.

**Symbol transfer.** We copy all symbols from the old tape to the new, safe for the symbol at the current position of the \(r/w\)-head. To this end, add an action \(\pi_\phi\) with \(\phi = s \land \neg \diamond_1(\bigvee_{q \in Q} h_q)\). The formula \(\phi\) is then satisfied in \(K(c)\) exactly at the symbols states \(s_j \in S\) on which the \(r/w\)-head is not. Let \(\Gamma' = \{\gamma_\phi; \phi \in \Phi\}\) with \(\Phi = \{R, L, 2_{AM}\} \cup \{l_q, r_q; q \in Q, i \in \{0, 1\}\}\). Requiring that \((\Gamma' \times \{\pi_\phi\})^* \subseteq R_3\)

\(^2\)To save tape, this could be done in a more economical manner, only creating extra cells where actually needed.
ensures that the symbol states copied over to $K(c) \otimes \Sigma r$ are connected to the correct cell world. We give the precise definition of $R_1$ below.

**Symbol writing.** We implement the symbol writing part of the transition function $\delta$. Define a new set of actions by

$$\Delta = \{\delta_{hqi} : q \in Q, i \in \{0, 1\} \text{ and } \delta(i, q) \text{ is defined}\}.$$  

At most one action from $\Delta$ will have its precondition satisfied at any $K(c)$ and just in case $\delta(c)$ is defined. The world satisfying this precondition is a cell world, $c_j$, which will have two successors in $K(c) \otimes P$: a cell world successor $(c_j,_{\gamma_{hqi}})$ defined above and a symbol world successor $(c_j,_{\delta_{hqi}})$ defined here. We ensure that the emulation writes the correct symbol by connecting $(c_j,_{\delta_{hqi}})$ to $(c_j,_{\gamma_{hqi}})$ by $R_1$ or not: Let

$$R_{tmp} = \{(\delta_{hqi},_{\gamma_{hqi}}) : \gamma \in \Gamma \mid \delta(i, q) = (\cdot, 1, \cdot)\}$$

and let $R_1 = ((\Gamma' \times \{\pi_{\varphi}\}) \cup R_{tmp})^\ast$. This and the above ensures that the emulation produces a correctly labeled tape.

**State change and head repositioning.** We finally implement the state change and head repositioning encoded by $\delta$. To this end, define a set of events

$$\Theta = \{\theta_{hqi} : q \in Q, i \in \{0, 1\} \text{ and } \delta(i, q) \text{ is defined}\}.$$  

Again, at most one action from $\Theta$ will have its precondition satisfied at any $K(c)$ and just in case $\delta(c)$ is defined. The world satisfying this precondition is a cell world, $c_j$, which will hence have two successors in $K(c) \otimes \Sigma r$: a cell world successor $(c_j,_{\gamma_{\varphi}})$ defined above and a $r/w$-head world successor $(c_j,_{\theta_{hqi}})$ defined here. We “mount” the $r/w$-head world at the correct position and in the correct state using the relations $\{R_{q'}\}_{q' \in Q}$: For all $q' \in Q$, let

$$R_{q'} = \{(\gamma_{xq'},_{\theta_{hqi}}) : \delta(q, i) = (q', \cdot, x), i \in \{0, 1\}, q \in Q\}^\ast.$$  

The definition of $\{R_{q'}\}_{q \in Q}$ ensures that the $r/w$-head is moved and changes state appropriately, whenever $\delta(i, q)$ is defined. When $\delta(i, q)$ is not defined, the

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3 Possibly three, see below.
4 Possibly three, cf. the above.
\( r/w \)-head world \((c_j, \theta_{h,i})\) will be disconnected from the tape cell worlds. In that case, \(K(c) \otimes P\) will not be in \(\mathcal{C}\), and the emulation is said to halt. This concludes the construction and proof.

QED

Remark 3. The proof generalizes to \(k\)-tape Turing machines or bigger input symbol sets by replacing modality \(\Box_1\) with \(\Box_1, \Box_k\) and the corresponding formula 1 with \(1, \ldots, k\).

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