On duality of the noncommutative supersymmetric Maxwell-Chern-Simons theory

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Abstract

We study the possibility of establishing the dual equivalence between the noncommutative supersymmetric Maxwell-Chern-Simons theory and the noncommutative supersymmetric self-dual theory. It turns to be that whereas in the commutative case the Maxwell-Chern-Simons theory can be mapped into the sum of the self-dual theory and the Chern-Simons theory, in the noncommutative case such a mapping is possible only for the theory with modified Maxwell term.
I. INTRODUCTION

The duality, allowing to construct mappings between different field theories is a very important aspect of three-dimensional field models. Initially the duality was observed for the example of the free Maxwell-Chern-Simons and self-dual theories [1]. Further, in a number of papers [2] different methods of implementing the duality were studied. The development of noncommutative field theories brought a question about possible generalization of duality in this situation. There, the ordering problem of product of fields turns out to be fundamental, at least in the application of the gauge embedding method [3].

One approach of implementing the duality for the noncommutative field theories is based on the use of the Seiberg-Witten map, as it was developed in [4]. We would like to point out that there is an alternative method of construction of dual models for the noncommutative theories which has been previously developed in [5] and successfully applied to the study of the duality. This method is based on an appropriate change of variables allowing to rewrite the action in a simpler form, with a decreased number of derivatives. As a result, the modified Maxwell-Chern-Simons action turns out to be mapped into two theories, one of them being the Chern-Simons theory whereas the other one is the self-dual model. Here, our aim is to study this method in the noncommutative case within the framework of the superfield formulation of supersymmetric field theories. By conveniently deforming the original Lagrangian in the Wess-Zumino gauge, we demonstrate that the mentioned duality holds in the physical sector.

II. THE MAXWELL-CHERN-SIMONS THEORY

The starting point of our study is the Maxwell-Chern-Simons theory whose action looks like (we follow the notations of [6, 7]):

\[
S = \frac{1}{2g^2} \int d^5 z \left[ -\frac{1}{2} W^\alpha * W_\alpha + m \left( A^\alpha * W_\alpha + \frac{i}{6} \{ A^\alpha, A^\beta \} * D_\beta A_\alpha + \frac{1}{12} \{ A^\alpha, A^\beta \} * \{ A_\alpha, A_\beta \} \right) \right],
\]

where

\[
W_\beta = \frac{1}{2} D^\alpha D_\beta A_\alpha - \frac{i}{2} [ A^\alpha, D_\alpha A_\beta ] + \frac{1}{6} [ A^\alpha, \{ A_\alpha, A_\beta \} ],
\]
is the superfield strength constructed on the base of the the spinor superpotential $A_\alpha$. In Eq. (1), the first and the second terms are the noncommutative Maxwell and Chern-Simons terms, respectively. Due to the noncommutativity, this action, although Abelian, includes the self-interactions for the gauge superfield. The parameter $m$ is the topologic; mass of the superfield. Hereafter it is implicitly assumed that all commutators and anticommutators are Moyal ones, that is, $[A, B] \equiv A \ast B \doteq B \ast A$, with
\[
A(x) \ast B(x) \equiv A(x) \exp \left(\frac{i}{2} \overrightarrow{\partial} \partial_{x}^{\mu} \Theta_{\mu\nu} \overrightarrow{\partial}_{x}^{\nu}\right) B(x),
\]
being the Moyal-Groenewald $\ast$-product.

### III. DUAL PROJECTING OF THE THEORY

Let us carry the dual projection of the above theory. To do it, we introduce the auxiliary spinor superfield $\pi^\alpha$ to lower the order of the Lagrangian, that is, the number of interacting fields in vertices. With this objective, it is natural to suggest that the equivalent form of this action is
\[
S = \frac{1}{2g^2} \int d^5 z \left(\frac{k_1}{2} (\pi^\alpha - A^\alpha) \ast (\pi_\alpha - A_\alpha) + k_2 \pi^\alpha D^\beta D_\alpha A_\beta + k_3 \pi^\alpha \ast [A^\beta, D_\beta A_\alpha] + l_1 \{A_\alpha, A_\beta\} \ast D_\beta A_\alpha\right).
\]
To simplify the situation, we employ the Wess-Zumino gauge in which the (Moyal) products of three and more spinor superfields which are not affected by the action of derivatives, as f.e. $A_\alpha \ast A_\beta \ast A_\gamma$, are equal to zero. Here $k_1, k_2, k_3, l_1$, are constants to be fixed. The equation of motion for the $\pi_\alpha$ is
\[
\pi_\alpha = A_\alpha - \frac{k_2}{k_1} D^\beta D_\alpha A_\beta - \frac{k_3}{k_1} [A^\beta, D_\beta A_\alpha].
\]
By substituting this $\pi_\alpha$ into the action (4), we arrive at
\[
S = \frac{1}{2g^2} \int d^5 z \left[ - \frac{k_2^2}{2k_1} D^\gamma D_\alpha A_\gamma D^\beta D_\alpha A_\beta - \frac{k_2 k_3}{k_1} D^\gamma D^\alpha A_\gamma \ast [A^\beta, D_\beta A_\alpha] - \frac{k_3^2}{2k_1} [A^\gamma, D_\gamma A^\alpha] \ast [A^\beta, D_\beta A_\alpha] + (k_2 A_\alpha D^\gamma D_\alpha A_\gamma + (k_3 + l_1) \{A^\alpha, A^\beta\} \ast D_\beta A_\alpha\right],
\]
whereas the expanded form of the Maxwell-Chern-Simons action in the Wess-Zumino gauge is (cf. [7])

\[
S = \frac{1}{2g^2} \int d^5z \left[ -\frac{1}{8} D\gamma D^\alpha A_\gamma D^\beta D_\alpha A_\beta + \frac{i}{4} D^\gamma D^\alpha A_\gamma * [A^\beta, D_\beta A_\alpha] + \frac{1}{8} [A^\gamma, D_\gamma A^\alpha] * [A^\beta, D_\beta A_\alpha] + m \left( \frac{1}{2} A^\alpha D^\beta D_\alpha A_\beta - \frac{i}{3} \{A^\alpha, A^\beta\} * D_\beta A_\alpha \right) \right].
\] (6)

Comparing the above expressions we obtain \( k_1 = m^2, k_2 = \frac{m^2}{2} \) (which is easily found in the commutative case). Further, \( k_3 = -\frac{im}{2}, l_1 = \frac{im}{6} \). Thus, the action (4) is found to be:

\[
S = \frac{1}{2g^2} \int d^5z \left( \frac{m^2}{2} (\pi^\alpha - A^\alpha) * (\pi_\alpha - A_\alpha) + \frac{m}{2} \pi^\alpha D^\beta D_\alpha A_\beta - \frac{im}{2} \pi^\alpha * [A^\beta, D_\beta A_\alpha] + \frac{im}{6} A^\alpha * [A^\beta, D_\beta A_\alpha] \right).
\] (7)

Now, we introduce \( \pi_\alpha = f^+_\alpha + f^-_\alpha, A_\alpha = f^+_\alpha - f^-_\alpha \), so that the quadratic part of this action takes the form

\[
S_2 = \frac{1}{2g^2} \int d^5z (2m^2 f^+\alpha f^-\alpha - \frac{m}{2} f^+\alpha D^\beta D_\alpha f^-\alpha + \frac{m}{2} f^+\alpha D^\beta D_\alpha f^+_\beta),
\] (8)

which is a sum of the quadratic part of the self-dual action for the \( f^-_\alpha \) field and the quadratic part of the Chern-Simons action for the \( f^+_\alpha \) field. The interaction part, however, is much more complicated than in [5]. Note, however, that it involves terms only up to fourth order in the fields whereas the original Maxwell-Chern-Simons action involves terms up to the sixth order.

The vertex of third order in the fields in the action (7) looks like

\[
V_3 = -im \frac{1}{3} \int d^5 z (f^+\alpha + 2f^-\alpha) * [f^+\beta - f^-\beta, D_\beta f^+_\alpha - D_\beta f^-_\alpha],
\] (9)

from which we see that the \( f^+\alpha \) gives the Chern-Simons triple term with the correct coefficient (that is, \( -\frac{i}{3} \)), but \( f^-\alpha \) with a wrong one (that is, \( -\frac{2i}{3} \)). A similar situation was observed in [5]. We note also the presence of ”mixed” terms.

From this result, we conclude that the noncommutativity destroys duality in the ”pure” sense. To evade this situation, we introduce a deformed action in a way similar to [7],

\[
S_1 = \frac{1}{2g^2} \int d^5z \left( \frac{m^2}{2} (\pi^\alpha - A^\alpha) * (M^{-1})_{\alpha\beta} * (\pi^\beta - A^\beta) + \frac{m}{2} \pi^\alpha D^\beta D_\alpha A_\beta - \frac{im}{2} \pi^\alpha * [A^\beta, D_\beta A_\alpha] + \frac{im}{6} A^\alpha * [A^\beta, D_\beta A_\alpha] \right),
\] (10)
where \((M^{-1})_{\alpha\beta}\) is a matrix to be determined. The corresponding deformed Maxwell-Chern-Simons action is

\[
S = \frac{1}{2g^2} \int d^5z \left[ -\frac{1}{2} W^\alpha \ast M_{\alpha\beta} \ast W^\beta + m \left( A^\alpha \ast W_\alpha + \frac{i}{6} \{ A^\alpha, A^\beta \}_\ast \ast D_\beta A_\alpha \right) \right],
\]

where the \(W_\alpha\) is restricted in the Wess-Zumino gauge by first two terms of (2). It is clear that the matrix \(M^{-1}\) should be of the form

\[
(M^{-1})_{\alpha\beta} = -\theta_{\alpha\beta} + \Lambda_{\alpha\beta}[f],
\]

with \(\Lambda_{\alpha\beta}[f]|_{f_0=0} = 0\).

Our aim is to fix the \(\Lambda_{\alpha\beta}[f]\) in a way allowing for the arisal of the Chern-Simons and self-dual actions, that is, we want to obtain \(S_1\) in the form

\[
S_1 = \frac{1}{2g^2} \int d^5z \left[ 2m^2 f^{-\alpha} + \frac{m}{2} f^{-\alpha} D^\beta D_\alpha f^- + \frac{im}{3} \{ f^{-\alpha}, f^{-\beta} \}_\ast \ast D_\beta f^- + \right.
\]

\[
+ \frac{1}{2g^2} \int d^5z \left[ \frac{m}{2} f^+ D^\beta D_\alpha f^+ - \frac{im}{3} \{ f^+, f^+ \}_\ast \ast D_\beta f^+ \right].
\]

We note that the quadratic part of this action already was obtained in Eq. \((10)\), so, it remains to fix the triple and quartic terms. This can be done via the undetermined coefficients method. So, the problem is to choose \(\Lambda_{\alpha\beta}\) to satisfy the relation

\[
2m^2 f^{-\alpha} \ast \Lambda_{\alpha\beta} \ast f^{-\beta} - \frac{im}{3} \{ f^+, f^+ \}_\ast \ast D_\beta f^- =
\]

\[
= \frac{im}{3} \{ f^{-\alpha}, f^{-\beta} \}_\ast \ast D_\beta f^- - \frac{im}{3} \{ f^+, f^+ \}_\ast \ast D_\beta f^+.
\]

It is easy to check that the terms involving only \(f^+\) fields in the left- and the right-hand sides of this equation exactly coincide. The terms with two or more \(f^-\) fields must be cancelled by the term \(f^{-\alpha} \ast \Lambda_{\alpha\beta} \ast f^{-\beta}\). The only remaining difficulty is related to the terms with only one \(f^-\) field (all other fields carry \(+\) signs). However, the sum of these “dangerous” terms vanishes. In fact, for triple terms we have

\[
\int d^5z \left[ -\frac{2im}{3} \{ f^{-\alpha}, f^+ \}_\ast \ast D_\beta f^+ + \frac{im}{3} \{ f^+, f^{-\beta} \}_\ast \ast D_\beta f^- + \right.
\]

\[
+ \frac{im}{3} \{ f^+, f^+ \}_\ast \ast D_\beta f^+ \right],
\]

or, in a more explicit form

\[
\frac{2m}{3} \int d^5z \int d^3k_1 d^3k_2 d^3k_3 (2\pi)^3 \delta(k_1 + k_2 + k_3) \sin(k_2 \wedge k_3) \times
\]

\[
\int \left( 2f^{-\alpha}(k_1) f^+\beta(k_2) D_\beta f^+(k_3) - f^+\alpha(k_1) f^{-\beta}(k_2) D_\beta f^-+(k_3) - f^{-\alpha}(k_1) f^+\beta(k_2) D_\beta f^-+(k_3) \right).
\]
After integration by parts this expression can be rewritten as

\[
\frac{2m}{3} \int d^2 \theta \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} (2\pi)^3 \delta(k_1 + k_2 + k_3) \sin(k_2 \land k_3) \times \\
\times \left( 2f^{-\alpha}(k_1) f^{+\beta}(k_2) D_\beta f^+_\alpha(k_3) - f^{+\alpha}(k_1) f^{-\beta}(k_2) D_\alpha f^+_\beta(k_3) - \\
- f^{+\alpha}(k_1) D_\beta f^{+\beta}(k_2) f^-_\alpha(k_3) + D_\beta f^{+\alpha}(k_1) f^{+\beta}(k_2) f^-_\beta(k_3) \right).
\]

After relabelling indices, the last term in the parentheses of this expression takes the form:

\[-f^{-\alpha}(k_1) f^{+\beta}(k_2) D_\beta f^+_\alpha(k_3), \text{ and the whole Eq. (17) is rewritten as}
\[
\frac{2m}{3} \int d^2 \theta \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} (2\pi)^3 \delta(k_1 + k_2 + k_3) \sin(k_2 \land k_3) \times \\
\times \left[ f^{-\alpha}(k_1) f^{+\beta}(k_2) \left( D_\beta f^+\alpha(k_3) - D_\alpha f^+\beta(k_3) \right) - f^{+\alpha}(k_1) f^-\beta(k_2) D^\alpha f^+_\beta(k_3) \right].
\]

Taking into account that \( D_\beta f^+\alpha - D_\alpha f^+\beta = C_{\alpha\beta} D^\gamma f^+_\gamma \), we find that this term identically vanishes.

After carrying out simplifications in the remaining terms, suggesting that \( \Lambda_{\alpha\beta} \) of first order in the \( f^{\pm\alpha} \) fields, we find

\[
\int d^5 z \left[ 2mf^{-\alpha} \Lambda_{\alpha\beta} f^{-\beta} + \frac{i}{3} \left\{ f^{-\alpha}, f^{+\beta} \right\} - \left\{ f^{+\alpha}, f^{-\beta} \right\} \right] D_\beta f^-_\alpha + \\
+ \frac{2i}{3} \left\{ f^{-\alpha}, f^{-\beta} \right\} D_\beta f^+_\alpha = i \int d^5 z \left\{ f^{-\alpha}, f^{-\beta} \right\} D_\beta f^-_\alpha.
\]

From this equation one can find \( \Lambda_{\alpha\beta} \) (which depends on phase factors).

First, one can write down a more explicit form of (19):

\[
\int d^2 \theta \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} (2\pi)^3 \delta(k_1 + k_2 + k_3) \left[ 2mf^{-\alpha}(k_1) \Lambda_{\alpha\beta}(k_2) f^{-\beta}(k_3) - \\
- \frac{1}{3} \left[ 4 \sin(k_1 \land k_2) f^{-\alpha}(k_1) f^{+\beta}(k_2) - 2 \sin(k_1 \land k_2) f^{+\alpha}(k_1) f^{-\beta}(k_2) \right] D_\beta f^-_\alpha(k_3) - \\
- \frac{4}{3} \sin(k_1 \land k_2) f^{-\alpha}(k_1) f^{-\beta}(k_2) D_\alpha f^+_\beta(k_3) \right] = \\
- 2 \int d^2 \theta \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} (2\pi)^3 \delta(k_1 + k_2 + k_3) \sin(k_1 \land k_2) f^{-\alpha}(k_1) f^{-\beta}(k_2) D_\beta f^-_\alpha(k_3).
\]

By comparing of the left- and right-hand sides of Eq. (20) we find:

\[
\int d^5 z f^{-\alpha} \Lambda_{\alpha\beta} f^{-\beta} = - (2\pi)^3 \sin(k_1 \land k_2) \delta(k_1 + k_2 + k_3) f^{-\alpha}(k_1) \times \\
\times \left[ - \frac{1}{2m} \left( D_\alpha f^-\beta(k_2) + D_\beta f^-\alpha(k_2) \right) + \\
+ \frac{1}{3m} \left( - f^+_\beta(k_2) D_\alpha - 2D_\beta f^+_\alpha(k_2) + 2C_{\alpha\beta} f^{+\gamma}(k_2) D_\gamma \right) \right] f^-\beta(k_3).
\]
which within the expression \([19]\) looks like:

\[
\Lambda_{\alpha\beta}[f] = \frac{i}{2m}(D_{\alpha}f_{\beta} + D_{\beta}f_{\alpha}) - \frac{i}{3m}(-f_{\beta}D_{\alpha} - 2D_{\beta}f_{\alpha}^{+} + 2C_{\alpha\beta}f^{+\gamma}D_{\gamma}).
\]  

(22)

So, the manifest form of \(\Lambda_{\alpha\beta}\) was found. Thus the dual projection of the modified noncommutative Maxwell-Chern-Simons theory was constructed.

IV. SUMMARY

We have succeeded in mapping the noncommutative supersymmetric Maxwell-Chern-Simons theory with the modified Maxwell term into the sum of the noncommutative Chern-Simons theory and noncommutative self-dual theory in the Wess-Zumino gauge. The essential result is that to achieve this mapping we must modify the Maxwell term introducing the nontrivial matrix \(M_{\alpha\beta}\). The appearance of this matrix is a natural implication of noncommutativity (and, thus, of a nontrivial self-interaction). In principle, this modification can be treated as some nonlinear extension of the initial Maxwell-Chern-Simons theory.

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