Dinamic model of spherical perturbations in Friedmann Universe

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Abstract
A selfconsistent system of equations describing evolution of linear spherically symmetric perturbations in Fridmann world by an arbitrary equation of state is obtained. The perturbations singular part corresponding to massive particle-like source is extracted, an evolutionary equation for mass of this source is obtained and exactly solved. An exact solution of evolutionary equations of state for perturbations by an arbitrary equation of state is constructed.

1 Introduction
In a number of papers of one of the Authors collaboratively with A.A. Popov Ref.[1]-[3] a theory of spherical perturbations of the Friedmann world in connection with the necessity of developing the relativistic kinetic theory considering gravitational interactions was constructed. The procedure applied for the kinetic equations derivation of averaging local fluctuations of the gravitational field pointed out by one of the Authors an interesting fact: the average quadrature fluctuations of the gravitational field play a role of tensor of energy-momentum of an ideal fluid with extremely-hard equation of state [3]. In the paper of one of the Authors collaboratively with A.A. Popov exact solutions of the equations for the spherically symmetric perturbations of the ultrarelativistic Friedmann world with an arbitrary curve index [4] were obtained. However, the analysis of the obtained solutions and the calculation of the average quadrature corrections to the Friedmann metrics were not fulfilled. In connection with the problem of dark energy and dark matter in cosmology the task of possible alteration of macroscopic equation of state of the Friedmann universe by the local gravitational interactions becomes actual, as far as the potential energy on the gravitational interactions is negative, it can correspond to a negative macroscopic pressure. In the present work we start investigating this possibility by fulfilling subsequent maintaining a dynamic theory of the Friedmann universe subject to the local gravitational perturbations.
2 Spherically symmetric space-time

2.1 Space symmetry and energy-momentum tensor

Let us study the space-time with spherical symmetry which metrics in the isotropic frame of reference \( (r, \theta, \varphi, \eta) \), where \( \eta \) is the time coordinate and \( r \) is a derivative one, can be written in the form

\[
ds^2 = e^\nu d\eta^2 - e^\lambda [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)],
\]

(1)

where

\[
\lambda = \lambda(r, \eta); \quad \nu = \nu(r, \eta)
\]

are arbitrary scalar functions of their arguments. Isotropic coordinates are convenient because within these coordinates the three-dimension space metrics takes an explicitly conform plane form. As it is known the metrics (1) admits a group of \( G_3 \) rotations with the Killing’s vectors (see, for example Ref. [7])

\[
\xi^1 = (0, \sin \varphi, \theta \cos \varphi, 0), \quad \xi^2 = (0, -\cos \varphi, \theta \sin \varphi, 0),
\]

\[
\xi^3 = (0, 0, 1, 0),
\]

(2)

Therefore in consequence of the Einstein equations the space-time symmetry is taken after by the energy-impulse tensor

\[
\mathcal{L}_\xi T^{ik} = 0, \quad (\alpha = 1, 3),
\]

(3)

where \( \mathcal{L} \) is the Lie’s derivative in the direction of \( \xi \) (see, for example Ref. [7])

\[
\mathcal{L}_\xi A^i = A^i_k \xi^k + A^k \xi^i_k
\]

(4)

the other tensor derivatives are processed in the same way as the Lie’s derivative. Thus under the conditions of spherical symmetry of space-time the energy-impulse tensor, as it is known, takes the form of the energy-impulse tensor of an ideal isotropic fluid

\[
T^{ik} = (\varepsilon + p)u^i u^k - p g^{ik},
\]

(5)

Where the scalars \( \varepsilon(r, \eta) \) and \( p(r, \eta) \) are an energy density and a fluid pressure respectively and \( u^i \) is a singular time-like vector of this fluid dynamic velocity

\[
g_{ik} u^i u^k = 1,
\]

(6)

whereas

\[
u^i = (u^r(r, \eta), 0, 0, u^\eta(r, \eta)).
\]

(7)

\(^1\)That is in the frame of reference in which the metrics of three-dimension space takes a conform plane form
Assuming
\[ u^r = vu^\eta e^{\frac{\nu - \lambda}{2}}, \quad v^2 < 1, \]
where \( v(r, \eta) \) is a radial three-dimension velocity of fluid, we get from (6) and
\[u^\eta = e^{-\frac{\nu}{2}} \frac{1}{\sqrt{1 - v^2}}; \quad (8)\]
\[T_4^1 = (\varepsilon + p)e^{(\nu - \lambda)/2} \frac{v}{1 - v^2}; \quad T_4^4 = \varepsilon + v^2 p; \]
\[T_1^1 = -\varepsilon v^2 + p \frac{1}{1 - v^2}; \quad T_2^2 = T_3^3 = -p. \quad (9)\]
Thus the following algebraic relations are valid
\[T_1^1 + T_4^4 = \varepsilon - p; \quad (10)\]
\[T_4^4 - T_1^1 = (\varepsilon + p) \frac{1 + v^2}{1 - v^2}; \]
\[T = T_1^1 + T_2^2 + T_3^3 + T_4^4 = \varepsilon - 3p. \]

2.2 Einstein Equations
Nontrivial Einstein equations relatively the metrics (11) have the form (see, for example Ref. [6])
\[\frac{1}{2} e^{-\lambda} \left( \frac{\lambda^2}{2} + \lambda' \nu' + \frac{2}{r} (\lambda' + \nu') \right) - e^{-\nu} \left( \frac{\nu}{2} + \frac{3}{4} \lambda^2 \right) = 8\pi \frac{\varepsilon v^2 + p}{1 - v^2} \quad (= -8\pi T_1^1); \quad (11)\]
\[\frac{1}{4} e^{-\lambda} \left[ 2(\nu'' + \nu') \nu' + \frac{2}{r} (\lambda' + \nu') \right] - e^{-\nu} \left( \frac{\nu}{2} + \frac{3}{4} \lambda^2 \right) = 8\pi p \quad (= -8\pi T_2^2), \quad (12)\]
\[-e^{-\lambda} \left( \frac{\lambda''}{4} + \frac{1}{4} \lambda^2 + \frac{2}{r} \lambda' \right) + \frac{3}{4} e^{-\nu} \lambda^2 = 8\pi \frac{\varepsilon + v^2 p}{1 - v^2} \quad (= 8\pi T_4^4); \quad (13)\]
\[\frac{1}{2} e^{-\lambda} (2\lambda' - \nu' \lambda) = 8\pi (\varepsilon + p)e^{(\nu - \lambda)/2} \frac{v}{1 - v^2} \quad (= 8\pi T_4^1), \quad (14)\]

\(^2\)In order to obtain these equations in the formulae of Ref. [6] it is necessary to assume \( \mu = \lambda + 2 \ln r \).
where $f'$ is a derivative from the function $f$ by the radial variable $r$ and $\dot{f}$ is a derivative by the time variable $\eta$ and the universal set of units is assumed everywhere in which $G = c = \hbar = 1$. Subtracting from the both parts of Eq. (11) the corresponding parts of Eq. (12) subject to (9) we get the consequence

$$
\frac{1}{2}e^{-\lambda} \left[ \frac{1}{2} \lambda'' + \lambda' \nu' - \frac{1}{2} \nu' + \frac{1}{r} (\lambda' + \nu') - (\lambda'' + \nu'') \right] = 8\pi(\varepsilon + p)\frac{\nu^2}{1 - \nu^2}.
$$

(15)

3 Background space-time

Within the isotropic spherical coordinates the nonperturbed gravitational field is described by the metrics of homogeneous isotropic universe

$$
ds^2 = a^2(\eta) \left( d\eta^2 - \frac{1}{\rho^2(r)} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \right),
$$

(16)

where

$$
\rho(r) = 1 + \frac{1}{4} kr^2,
$$

(17)

And the curvature index $k = 0$ is for the space-plane universe, $k = \pm 1$ is for the universe with positive and negative curvature of three-dimensional space respectively. Whereas $r$ and $\eta$ are dimensionless variables and the scale factor $a(\eta)$ has length dimension.

Thus in a nonperturbed state the introduced scalar metrics functions $\lambda$ and $\nu$ are equal to

$$
\nu_0 = \ln a^2(\eta); \quad \lambda_0 = \ln \left( \frac{a(\eta)}{\rho(r)} \right)^2,
$$

(18)

Whereas in consequence of time space homogeneity

$$
p = p_0(\eta); \quad \varepsilon = \varepsilon_0(\eta); \quad \nu_0 = 0.
$$

(19)

Substituting Eqs. (18) and (19) into the Einstein Eqs. (11)-(15) we obtain a set of equations describing the Friedmann universe dynamics

$$
\frac{1}{a^2} \left( \frac{\dot{a}^2}{a^2} + k \right) = \frac{8\pi}{3} \varepsilon;
$$

(20)

$$
\frac{1}{a^2} \left( 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + k \right) = -8\pi p.
$$

(21)

As it is known the second of these equations (21) can be substituted by the algebraic-differential consequence of 1 and 2 (see, for example Ref. [6])

$$
\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0.
$$

(22)
Here it is convenient to pass on from the time variable $\eta$ to the physical time $t$ by the formula

$$a(\eta)d\eta = dt, \quad \Rightarrow t = \int a(\eta)d\eta;$$

(23)

whereas

$$\frac{\partial}{\partial \eta} = a \frac{\partial}{\partial t} \rightarrow \dot{f} = a \dot{f}_t.$$  

(24)

Then the independent Einstein equations can be written in a more compact form (see, for example Ref. [6])

$$\frac{1}{a^2}(\dot{a}^2 + k) = \frac{8\pi}{3} \varepsilon;$$

(25)

$$\dot{\varepsilon} + 3 \frac{\dot{a}}{a}(\varepsilon + p) = 0,$$

(26)

If we know the equation of state, that is the relation of the form

$$p = p(\varepsilon),$$

(27)

then Eq. (22) is integrated in quadratures

$$-3 \ln a = \int \frac{d\varepsilon}{\varepsilon + p(\varepsilon)} + \text{Const.}$$

(28)

Substituting the solution (28) into Eq. (20) we obtain a closed differential equation of order 1 relatively $\varepsilon(\eta)$. In the case of barotropic equation of state

$$p = \kappa \varepsilon$$

(29)

Eq. (24) is easily integrated

$$\varepsilon = c_1 a^{-3(\kappa+1)},$$

(30)

and Eq. (20) is integrated in quadratures

$$\int \frac{da}{a \sqrt{\frac{8 \pi \kappa_1}{3} a^2 - 3(\kappa+1) - k}} = c_2 \eta,$$

(31)

where $c_1$ and $c_2$ are arbitrary constants. The pointed out equations are integrated in the elementary functions for the early Universe ($t \to 0$). As it is known in this case the behavior of the solutions does not depend on the curvature $k$ (see, for example Ref. [6]) and does not differ from the behavior of the solutions for a space plane Universe ($k = 0$)

$$a = a_1 \eta^{\frac{2}{3(\kappa+1)}}; \quad \varepsilon = c_1 a_1^{-3(1+\kappa)} \eta^{-\frac{6(1+\kappa)}{3(\kappa+1)}}, \quad \kappa + 1 \neq 0,$$

(32)

\[3\]We call the reader’s attention to the difference, $k$ is a curvature index and $\kappa$ (kappa) is a barotropic coefficient.
Where the constants $a_1$ and $c_1$ are connected by the relation

$$ a_1 = \left( 3\kappa + 1 \right) \sqrt{\frac{2\pi c_1}{3}} \frac{2}{3\kappa + 1}, \quad 1 + 3\kappa \neq 0. $$  \hspace{1cm} (33)

In doing so let us reduce a relation between the variables $t$ and $\eta$ from Eq. (23)

$$ t = a_1 \frac{3\kappa + 1}{3(1 + \kappa)} \eta^{\frac{3(1+\kappa)}{3\kappa + 1}}. $$  \hspace{1cm} (34)

Subject to (33) and (34) let us reduce from (32)

$$ \varepsilon = \frac{1}{2\pi(\kappa + 1)^2 t^2} $$  \hspace{1cm} (35)

Note the solution of the Einstein equations by $\kappa = -1/3$ is specific

$$ a = c_2 e^{\frac{2c_1}{\sqrt{8\pi} c_1} \eta}, \quad (\kappa = -\frac{1}{3}, \eta \in (-\infty, +\infty)). $$

However this property is a coordinate only. Really, by passing on from the time variable $\eta$ to the physical time $t$

$$ t = \frac{c_2}{\sqrt{8\pi} c_1} e^{\frac{2c_1}{\sqrt{8\pi} c_1} \eta} $$

we obtain $a \sim t$ and the energy dense formula (35) in which it is necessary to substitute $\kappa = -1/3$ only. By $\kappa = -1$ we get from (32) the so-called inflational solution

$$ a = a_1 e^{\Lambda t}; \quad \varepsilon = \frac{3\Lambda^2}{8\pi} = \text{const.} $$  \hspace{1cm} (36)

The solutions (33) corresponding to the values $\kappa < -1$ describe the so-called dark materia.

### 4 Linear spherically-symmetric perturbations of Friedmann space-time

#### 4.1 Equations for spherically-symmetric perturbations

Let us consider now the small spherically-symmetric perturbations of isotropic cosmological solution (18) assuming

$$ \lambda = \ln a^2(\eta) + \delta \lambda; \quad \nu = \ln a^2(\eta) + \delta \nu; $$

$$ p = p_0(\eta) + \left. \frac{dp}{d\varepsilon} \right|_{\varepsilon_0} \delta \varepsilon; \quad \varepsilon = \varepsilon_0(\eta) + \delta \varepsilon, $$ \hspace{1cm} (37)
Where the scalar functions $\delta \lambda(r, \eta)$, $\delta \nu(r, \eta)$, $\delta \varepsilon(r, \eta)$ and $\nu(r, \eta)$ will be assumed small of order 1 by smallness. Substituting (37) into Eq. (15) in the first approximation by smallness of the perturbations $\delta \lambda, \delta \nu, \delta \varepsilon$ and $\nu$ we obtain one closed equation relatively the function $\delta \lambda + \delta \nu$

\[
\frac{\partial}{\partial r} \left( \frac{\rho(r)}{r} (\lambda + \nu)' \right) = 0.
\]  

Integrating Eq. (38) we get (see also Ref. [4])

\[
\lambda + \nu = \begin{cases} 
C_1(\eta) + C_2(\eta) r^2, & k = 0; \\
C_1(\eta) + \frac{C_2(\eta)}{r^2}, & k = \pm 1,
\end{cases}
\]  

(39)

where $C_1(\eta)$ and $C_2(\eta)$ are arbitrary functions.

Further we shall search only the solutions of the Einstein equations of $C^1$ class, which out of some sphere coincide with homogeneous isotropic nonperturbed solution

\[
\lambda(r, \eta)|_{r=0} = \lambda_0(\eta), \quad \nu(r, \eta)|_{r=0} = \nu_0(\eta), \\
\lambda'(r, \eta)|_{r=0} = \lambda'_0(\eta), \quad \nu'(r, \eta)|_{r=0} = \nu'_0(\eta).
\]  

(40)

Such solutions correspond to the retarded solutions of hyperbolic type equations. The physical sense of the solutions we shall discuss later. Then according to (39) it should take place (see Ref. [4])

\[\delta \lambda + \delta \nu = 0 \Rightarrow \delta \lambda = -\delta \nu.\]  

(41)

In the paper [4] spherically symmetric perturbations are studied in the ultrarelativistic universe ($\kappa = 1/3$) only, however in doing that solutions of linearized Einstein equations for all types of Friedmann universe were obtained. In the present paper we will restrict ourselves with the case of the space-plane universe ($k = 0$), but the barotrop coefficient $\kappa$ will be considered to be arbitrary. Thus, subject to the background Einstein equations (20)-(21), which in the case of $k = 0$ have the consequence

\[
2 \frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} (1 - 3 \kappa),
\]  

(42)

we get a closed set of three differential Einstein equations linearized around the background solution (16), relatively the three unknown variables $\delta \nu(r, \eta)$, $\delta \varepsilon(r, \eta)$and $\nu(r, \eta)$

\[
\delta \ddot{\nu} + 3 \delta \dot{\nu} \frac{\dot{a}}{a} - 3 \kappa \delta \nu \frac{\dot{a}^2}{a^2} = 8 \pi a^2 \delta \rho; \quad (43)
\]

\[
3 \delta \ddot{\nu} \frac{\dot{a}}{a} + 3 \delta \dot{\nu} \frac{\ddot{a}}{a^2} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \delta \nu = -8 \pi a^2 \delta \varepsilon; \quad (44)
\]
\[
\frac{1}{a^3} \frac{\partial}{\partial \eta} a \delta \nu' = -8 \pi \varepsilon_0 (1 + \kappa) v. \tag{45}
\]

The latter of this set of equations (45) is a definition of the radial velocity \( v(r, \eta) \). One of the equations (43) and (44) determines the density energy perturbation \( \delta \varepsilon(r, \eta) \).

### 4.2 Derivation of particlelike solutions

Further assuming the investigation of particle-like solutions of perturbation equations also, let us study canonic equations of motion of gravitating classical point particle in the gravitational field to which the \( \delta \)-like energy density corresponds. As a result of the two competitive processes – the accretion of material environment and the reverse process – evaporation of substance the mass of the classic point particle in the material environment cannot be constant. Therefore let us write the Hamilton invariant function of massive particle in the form

\[
H(x, P) = \sqrt{g^{ik} P_i P_k} - m, \quad (= 0), \tag{46}
\]

where \( m = m(s) \) is a scalar function. From (45) we get the normalization ratio

\[
(P, P) = m^2(s). \tag{46}
\]

The relativistic canonical equations of particle motion take the form

\[
\frac{dx^i}{ds} = \frac{\partial H}{\partial P_i}; \quad \frac{dP_i}{ds} = -\frac{\partial H}{\partial x^i}. \tag{47}
\]

From the first couple of the canonical equations subject to the normalization ratio we obtain

\[
\frac{dx^i}{ds} = \frac{P^i}{m} \Rightarrow g^{ik} \frac{du^i}{ds} \frac{du^k}{ds} = 1. \tag{48}
\]

The second couple of the canonical equations of motion gives the Lagrangian equations of classical massive particle of variable mass

\[
\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = (\ln m)_k \left( g^{ik} - \frac{du^i}{ds} \frac{du^k}{ds} \right). \tag{49}
\]

In spherically symmetric metrics the motion equations solution (48) which does not break the spherical symmetry, is a time line which corresponds to the particle rest state at the origin of the coordinates

\[
r = 0, \quad x^4 = \eta, \tag{50}
\]

\(^4\)See the details in Ref. [8].
in doing so the mass at rest can be an arbitrary function of the coordinate time
\[ m = m(\eta). \]  
(51)

Let us write down the density energy corresponding to its singular part in invariant form
\[ \delta \varepsilon_m = m(\eta)\delta(r), \]  
(52)

where \( \delta(r) \) is Dirac invariant \( \delta \)-function in spherical coordinates perceived in the sense of integral ratio
\[ \int d^3V \delta^3(x) = a^3 \int d\Omega \int_0^\infty r^2 dr \delta(r) = 4\pi a^3 \int_0^\infty \delta(r) r^2 dr = 1, \]  
(53)

so that
\[ \int d^3V \delta \varepsilon_m = m(\eta). \]  
(54)

Temporally abandoning the time derivatives by \( \eta \) in the left part of Eq. (43) we get then the following equation for the singular part corresponding to the singular part of density
\[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \delta \nu = 8\pi a^2 m(\eta) \delta(r). \]  
(55)

Multiplying the both parts (55) by \( ar^2 dr \) and integrating, then integrating by parts in the equation left part and at the same time assuming
\[ \lim_{r \to 0} r^2 \frac{\partial \delta \nu}{\partial r} = 0 \]  
(56)

we obtain
\[ ar^2 \frac{\partial \delta \nu}{\partial r} = 2m(\eta). \]  
(57)

Integrating this equation one more time we get
\[ \delta \nu = -\frac{2m(\eta)}{ar}. \]

Thus the ratio similar to the known one subject to redetermination of the invariant \( \delta \)-function (53) takes place
\[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \left( -\frac{m}{r} \right) = 4\pi a^3 m \delta(r). \]  
(58)

Therefore in order to extract the particle-like singular part of the solution further on it is convenient to introduce a new field function \( \psi(r, \eta) \) such as [9]
\[ \delta \nu = -\delta \lambda = 2 \frac{\psi(r, \eta) - m(\eta)}{ar} \equiv 2 \frac{\Phi(r, \eta)}{ar}, \]  
(59)
At that according to (56) the relation must be fulfilled

\[ \lim_{r \to 0} \left| \frac{\psi}{r} \right| < \infty. \]  

(60)

Extracting the singular part of energy density in the right part of Eq. (43), substituting the function \( \delta \nu \) in the form (59) into Eqs. (42)-(44) and excluding the singular part subject to the relations (58) and (60) we get the linear equations set relatively the function \( \Phi \) and perturbations of energy density and velocity

\[ \ddot{\Phi} + \frac{\dot{a}}{a} \dot{\Phi} - \frac{3}{2} \left(1 + \kappa\right) \frac{a^2}{a^2} \dot{\Phi} = 4 \pi r a^3 \kappa \delta \varepsilon, \]  

(61)

\[ 3 \frac{\dot{a}}{a} \dot{\Phi} - \psi'' = -4 \pi r a^3 \delta \varepsilon, \]  

(62)

\[ \frac{\partial}{\partial r} \dot{\Phi} r = -4 \pi r a^3 \left(1 + \kappa \right) \varepsilon_0 v. \]  

(63)

Multiplying Eq. (62) by \( \kappa \) and adding its both parts to the corresponding parts of Eq. (61) we obtain a closed equation

\[ \ddot{\Phi} + \frac{\dot{a}}{a} (1 + 3 \kappa) \dot{\Phi} - \frac{3}{2} \left(1 + \kappa \right) \frac{a^2}{a^2} \dot{\Phi} - \kappa \psi'' = 0. \]  

(64)

Then assuming according to (59)

\[ \Phi(r, \eta) = \psi(r, \eta) - m(\eta). \]  

(65)

And dividing the variables in Eq. (64), we get two equations for the functions \( m(\eta) \) and \( \psi(r, \eta) \)

\[ \ddot{m} + \frac{\dot{a}}{a} (1 + 3 \kappa) \dot{m} - \frac{3}{2} \left(1 + \kappa \right) \frac{a^2}{a^2} m = \Theta(\eta); \]  

(66)

\[ \ddot{\psi} + \frac{\dot{a}}{a} (1 + 3 \kappa) \dot{\psi} - \frac{3}{2} \left(1 + \kappa \right) \frac{a^2}{a^2} \dot{\psi} - \kappa \psi'' = -\Theta(\eta), \]  

(67)

where \( \Theta(\eta) \) is an arbitrary function of its argument.

4.3 Basic theorem

Further let \( m = M(\Theta, \eta) \) be the private Eq. (66) solution corresponding to the given function \( \Theta(\eta) \). Then in consequence of Eq. (67) linearity the function \( \psi_1 = -M(\Theta, \eta) \) is a private solution of this equation. Then Eq. (67) general solution can be written in the form

\[ \psi(r, \eta) = \psi_0(r, \eta) - M(\Theta, \eta), \]  

(68)
where $\Psi(r, \eta)$ is general solution of the corresponding homogeneous equation

$$\ddot{\Psi} + \frac{\dot{a}}{a}(1 + 3\kappa)\dot{\Psi} - \frac{3}{2}(1 + \kappa)\frac{a^2}{a^2}\dot{\Psi} - \kappa \Psi'' = 0.$$  \hspace{0.5cm} (69)

Further on in consequence of the Eq. (66) linearity its general solution is sum of the corresponding homogeneous equation $\mu(\eta)$ and private solution of inhomogeneous one

$$m(\eta) = \mu(\eta) + M(\Theta, \eta).$$  \hspace{0.5cm} (70)

But then

$$\Phi(r, \eta) = \psi(r, \eta) - m(\eta) = \Psi(r, \eta) - \mu(\eta).$$  \hspace{0.5cm} (71)

where $\mu(\eta)$ is the homogeneous equation general solution

$$\ddot{\mu} + \frac{\dot{a}}{a}(1 + 3\kappa)\dot{\mu} - \frac{3}{2}(1 + \kappa)\frac{a^2}{a^2}\dot{\mu} = 0$$  \hspace{0.5cm} (72)

The rest equations of the set (61)-(63) describe the evolution of nonsingular part of energy density and perturbations velocity

$$\delta \varepsilon = -\frac{1}{4\pi r a^3} \left(3\frac{\dot{a}}{a}(\dot{\Psi} - \dot{\mu}) - \Psi''\right),$$  \hspace{0.5cm} (73)

$$\frac{\partial}{\partial r} \frac{\Psi - \mu}{r} = -4\pi r a^3(1 + \kappa)\varepsilon_0\nu.$$  \hspace{0.5cm} (74)

Thus we proved the theorem

**Theorem.** Linear spherically symmetric perturbations of Friedmann metrics are described by a set of two independent linear homogeneous equations (69) and (72) relatively two functions $\mu(\eta)$ and $\psi(r, \eta)$ which are nonsingular at the origin of the coordinates. Spherically symmetric perturbations of energy density and velocity are determined through metrics perturbations by the relations (73)-(74).

At $\kappa > 0$ the homogeneous Eq. (69) is hyperbolic, at $\kappa < 0$ it is elliptic, at $\kappa = 0$ this equation coincides with Eq. (72).

### 5 Evolitional equations for perturbations at constant barotrop coefficient

#### 5.1 Mass evolution of particle-like source

Let us study the cosmological mass evolution of particle-like source. Passing on from the variable $\eta$ to the variable $a(\eta)$ subject to the relation (42) in the mass evolution equation (72) let us reduce it to the form

$$\frac{d^2\mu}{da^2} + \frac{3(1 + \kappa)\mu}{2a} - \frac{3}{2}(1 + \kappa)\frac{\mu}{a^2} = 0.$$  \hspace{0.5cm} (75)
The general solution of this equation can be easily obtained
\[ \mu = C_+ a + C_- a^{-\frac{3}{2}(1+\kappa)}, \]  
(76)

where \( C_+ \) and \( C_- \) are arbitrary constants. The \( C_+ \) coefficient term in this solution corresponds to accretion processes, the \( C_- \) coefficient one corresponds to evaporation processes. Substituting in Eq. (76) the scale factor relation to the time variable \( \eta \) and using the relation (34) between the time variable \( \eta \) and the physical time \( t \) we obtain in an explicit form the law of mass evolution of particle-like source
\[ \mu = \tilde{C}_+ \frac{2}{\eta^{3(1+\kappa)}} + \tilde{C}_- t^{-1}, \quad (1 + \kappa) \neq 0, \]  
(77)

Where \( \tilde{C}_+ \) and \( \tilde{C}_- \) are new arbitrary constants. From the solutions (76) and (77) it is seen \( C_- = \tilde{C}_- 0 \) corresponds to the final particle mass at the moment of time \( t = 0 \). The solutions (76) - (77) summarize the solutions obtained in the previous papers [3]-[4] for the two private values of adiabatic curve coefficient \( \kappa = 0 \) and \( \kappa = 1/3 \) which correspond to nonrelativistic and ultrarelativistic equation of state correspondingly. In the pointed out cases we get from (77)
\[ \mu = \tilde{C}_+ \frac{2}{\eta^{3(1+\kappa)}} + \tilde{C}_- t^{-1}, \quad \kappa = 0, \]  
(78)
\[ \mu = \tilde{C}_+ t^{\frac{1}{2}} + \tilde{C}_- t^{-1}, \quad \kappa = \frac{1}{3}, \]  
(79)

Let us study a numerical example. Let the particle-like source mass equal to Planck mass at the moment of time \( t = t_{Pl} \) then at the contemporary moment of time \( t \sim 10^{60} t_{Pl} \) according to (77) the contemporary mass of “particle” varies within the limits \( 10^{-18} M_\odot \) at \( \kappa = 1 \) till \( 10^2 M_\odot \) at \( \kappa = 0 \). At negative values of the adiabate curve index the perturbation mass rapidly increases and at \( \kappa \approx -0.5 \) in order of value it is compared with the visible universe mass (see Fig.1).

In the case of the inflation solution (36) the evolutional equation (72) takes the form
\[ \ddot{\mu} = -\frac{2}{\eta} \dot{\mu} = 0. \]  
(80)

Solving (80) we obtain
\[ m = C_+ + \frac{C_-}{\eta}. \]  
(81)

Now using the relation of the time constant \( \eta \) to the physical time \( t \) for the case of inflation (36) for this case finally we get
\[ \mu = C_+ + C_- e^{-\Lambda t}. \]  
(82)
In this case $C_- = 0$ corresponds to the final mass at $t \to -\infty$ again.

**Fig.1.** Relationship of contemporary mass value of particle-like source in Friedmann world which had Planck mass at Planck moment of time to barotropic coefficient $\kappa$. On the ordinate axis the mass logarithm values are marked off in Solar mass units $M_\odot \approx 2 \cdot 10^{33}$ g.

### 5.2 Evolutional Equation for nonsingular mode of perturbations

From the general nonperturbed solution (32) we obtain useful ratios

\[ \frac{\dot{a}}{a} = \frac{2}{3\kappa + 1} \frac{1}{\eta}; \quad \frac{\ddot{a}}{a} = \frac{2(1 - 3\kappa)}{(3\kappa + 1)^2} \frac{1}{\eta^2}, \quad (1 + \kappa \neq 0), \]

(83)

using which we reduce Eq. (69) for nonsingular mode of perturbations to the form

\[ \ddot{\Psi} + \frac{2}{\eta} \dot{\Psi} - \frac{6(1 + \kappa)}{(1 + 3\kappa)^2} \frac{\Psi}{\eta^2} - \kappa \Psi'' = 0. \]

(84)

In the case of $(1 + \kappa) = 0$ Eq. (74) is no longer an equation for defining a radial velocity of perturbations, but becomes a differential equation relatively the function $\psi$

\[ \frac{\partial}{\partial r} \left( \frac{\psi - \mu}{r} \right) = 0. \]

(85)
It is necessary to solve this equation simultaneously with Eq. (84) which in this case takes the form

\[ \ddot{\Psi} + \frac{2}{\eta} \dot{\Psi} + \Psi'' = 0. \] (86)

5.3 General solution of evolution equation for nonsingular mode of perturbations

Assuming

\[ \Psi(r, \eta) = R(r) \Theta(\eta) \]

and dividing variables in Eq. (84) we obtain ordinary differential equations

\[ \kappa R'' + e \alpha^2 R = 0, \] (87)

\[ \eta^2 \dddot{\Theta} + 2\eta \ddot{\Theta} + \left[ e \alpha^2 \eta^2 - 6 \frac{1 + \kappa}{(1 + 3\kappa)^2} \right] \Theta = 0. \] (88)

To guarantee similarity of the solutions the sign of the division constant should be opposite to the barotrop coefficient

\[ e = -\text{sgn}(\kappa). \] (89)

Solving Eq. (87) we get

\[ R = C_1 \sin \frac{\alpha}{\sqrt{\kappa}} r + C_2 \cos \frac{\alpha}{\sqrt{\kappa}} r. \] (90)

In order the function do not contain peculiarities at the origin of the coordinates \( \Psi(r, \eta) \) (condition (?) it is necessary and sufficient in the solution \( C_2 = 0 \), thus

\[ R(r) = C(\alpha) \sin \frac{\alpha}{\sqrt{\kappa}} r, \] (91)

where \( C(\alpha) \) is an arbitrary constant.

At \( \kappa > 0 \) Eq. (88) has its solution

\[ \Theta(\eta) = \frac{\tilde{C}_1}{\sqrt{\eta}} J_s(\alpha \eta) + \frac{\tilde{C}_2}{\sqrt{\eta}} Y_s(\alpha \eta), \] (92)

where \( J_s(z) \) and \( Y_s(z) \) are the Bessel functions of the first and second genera respectively and

\[ s = \frac{1}{2} \left[ \frac{5 + 3\kappa}{1 + 3\kappa} \right], \quad \kappa \in [-1, +\infty). \] (93)
At $\kappa < 0$ the Eq. (88) solution is expressed by the way of the Bessel functions of imaginary argument $I_s(z)$ and $K_s(z)$ (see, e.g. [10]):

$$\Theta(\eta) = \frac{C_1}{\sqrt{\eta}} I_s(\alpha \eta) + \frac{C_2}{\sqrt{\eta}} K_s(\alpha \eta).$$

(94)

As far as the functions $J_s(z)/\sqrt{z}$ and $K_s(z)/\sqrt{z}$ tend to infinity at $z \to 0$ in order to get restricted ones at $\eta \to 0$ it is necessary to set in the formulae (92) and (94) $C_2 = 0$.

$$\tilde{C}_2 = 0.$$  

(95)

Thus nonsingular solution of evolutionary equation for perturbations can be written in the form

$$\Phi(r, \eta) = \frac{1}{\sqrt{\eta}} \int_0^\infty C(\alpha) \sin \frac{\alpha}{\sqrt{|\kappa|}} r J_s(\alpha \eta) d\alpha, \quad \kappa > 0.$$  

(96)

At $\kappa < 0$ we obtain simultaneously

$$\Phi(r, \eta) = \frac{1}{\sqrt{\eta}} \int_0^\infty C(\alpha) \sin \frac{\alpha}{\sqrt{|\kappa|}} r I_s(\alpha \eta) d\alpha, \quad \kappa < 0.$$  

(97)

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