Instantons, Hilbert Schemes and Integrability

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Abstract

We review the deformed instanton equations making connection with Hilbert schemes and integrable systems. A single \( U(1) \) instanton is shown to be anti-self-dual with respect to the Burns metric.

1 Introduction

The aim of the present review is to describe various settings surrounding the matrix equations

\[
[B_2, B_1] + IJ = \zeta_1 V, \quad (1)
\]
\[
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger = 2\zeta_2 V, \quad (2)
\]

where \( B_{1,2} \in M_v(\mathbb{C}), \ I \in M_w \times v(\mathbb{C}), \ J \in M_w \times v(\mathbb{C}) \). We will be interested in the space of solutions to these equations up to equivalence under an action of \( GL(\alpha) := GL(v, \mathbb{C}) \times GL(w, \mathbb{C}) \). These equations arise naturally in the context of integrable systems which will be recalled in the next section. The space of such matrices describes the phase space of the integrable system and we will refer to it as a “moduli space” as it describes the system at all energies and momenta. This space carries a natural hyper-Kähler structure and possesses several moment maps which are also reviewed.

Now the same moduli space also describes other interesting phenomena. Indeed, if there is to be a simple motto describing this talk it is: “Phase spaces of completely integrable systems give interesting moduli spaces for field theories”. This is well known in the context of Seiberg-Witten theory \([7, 40]\) but true more generally \([10, 31]\). In the present setting the same moduli space parametrises the (semistable) torsion free sheaves on \( \mathbb{C}P^2 \) whose restriction on the projective line \( \ell_\infty \) at infinity is trivial, as was shown first by Nakajima. The connection between the Calogero-Moser systems and instanton/sheaf moduli was noted by Nekrasov \([12]\) and Wilson \([54]\). When the right-hand-sides of these equations vanish we have that the matrices give the ADHM data for the construction of charge \( v \) \( SU(w) \) instantons on \( \mathbb{R}^4 \). For non-vanishing right-hand-side a modified ADHM construction yields instantons on a non-commutative space \([44]\). One can also apply the usual ADHM construction to the matrices above \([12]\). The gauge fields resulting will not of course be anti-self-dual with respect to the standard metric but

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one can ask whether they have any further nice properties. We shall perform this construction and in the process encounter several surprises. First we will see that $U(1)$ instantons exist and are well behaved. Such instantons do not exist on $\mathbb{R}^4$ but exist here because, as we shall discover, space-time is “blown-up”. On this blown-up space we will show there is a natural metric for which the charge one abelian instanton is in fact self-dual. The higher charge case will be dealt with elsewhere.

2 An Integrable System

The class of integrable systems we shall focus on here are of the Calogero-Moser family \cite{18, 20, 45}. These systems have a rather rich structure with connections to representation theory \cite{41}, functional equations and index theorems \cite{19, 14, 15, 4, 5, 16, 25, 31} to Seiberg-Witten and topological field theory \cite{28, 23, 38, 8, 9, 10, 11, 7, 40, 29}. The quantum mechanics of these systems has been well studied \cite{20, 46, 26, 50, 51, 47, 48, 53}. Many properties of these models can be found in the book \cite{22}. The Calogero-Moser systems are in many ways generic: given an integrable system with polynomial conserved quantities and suitable symmetry we arrive at these models. Thus for example we may characterise the $(a_n)$ Calogero-Moser system by

\begin{theorem}
Let $H$ and $P$ be the (natural) Hamiltonian and centre of mass momentum

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V, \quad P = \sum_{i=1}^{n} p_i.
$$

Denote by $Q$ an independent third order quantity

$$Q = \sum_{i=1}^{n} p_i^3 + \frac{1}{6} \sum_{i,j,k} d_{ijk} p_i p_j p_k + \sum_{i \neq j} d_{ij} p_i^2 p_j + \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j + \sum_{i} b_i p_i + c.
$$

If these are $S_n$ invariant and Poisson commute,

$$\{P, H\} = \{P, Q\} = \{Q, H\} = 0,$$

then $V = \frac{1}{6} \sum_{i \neq j} \varphi (x_i - x_j) + \text{const}$ and we have the Calogero-Moser system.

\end{theorem}

For our purposes here we will need only the simplest example. Following Olshanetsky and Perelomov we will derive these models as a coadjoint reduction of a simpler system \cite{13, 53}. Consider a Lie algebra $\mathfrak{g}$ and $A \in \mathfrak{g}$ moving freely on this vector space: $\ddot{A} = 0$. Thus $\dot{A} = a + bt$ for constant $a$, $b \in \mathfrak{g}$. We may conjugate $A$ to give a piece $X_{ss}$ lying in a given Cartan subalgebra and a constant (possibly vanishing) nilpotent piece $X_n$:

$$A = g X g^{-1},$$

with $X = X_{ss} + X_n$. For $\mathfrak{gl}(n)$ this is simply putting $A$ into Jordan form. Now

$$\dot{A} = g \left( \dot{X} + [M, X] \right) g^{-1} := g L g^{-1}, \quad M = g^{-1} \dot{g},$$

$$\ddot{A} = g \left( \ddot{L} + [M, L] \right) g^{-1} = 0.$$
Thus we obtain a Lax pair (without spectral parameter) $\dot{L} = [L, M]$ of the form $L = \dot{X} + [M, X]$ corresponding to geodesics $\dot{A} = 0$. Now consider the obviously conserved angular momentum

$$[A, \dot{A}] = g[X, L]g^{-1} := g\bar{C}g^{-1}$$

where clearly

$$0 = \dot{\bar{C}} + [M, \bar{C}].$$

When $X$ is semi-simple it is particularly easy to solve for $L$ in terms of $\bar{C}$ and $X$. Considering the case of $gl(v)$ and assuming $X$ semi-simple, or equivalently that $A(t)$ is diagonalisable, we obtain

$$L = \dot{X} + [M, X] = \dot{x} \cdot H + \sum_{\alpha \in \Phi} \alpha \cdot x E_{\alpha} = \sum_i \dot{x}_i H_i + \sum_{i \neq j} \frac{\bar{C}}{x_i - x_j} E_{ij}.$$

Here $\{H_i, E_{\alpha}\}$ form a Chevalley basis of $g$ with root system $\Phi$ and we have used a normalization $\text{Tr} E_{\alpha} E_{-\beta} = \delta_{\alpha, \beta}$. As we see from (4) the matrix $\bar{C}$ is in general time-dependent and this leads to the spin Calogero-Moser models. For particular angular momentum it is possible to have a simplification: $\bar{C}$ will be constant if and only if $[M, \bar{C}] = 0$. The usual $gl(w)$ (spinless) Calogero-Moser model corresponds to

$$\bar{C} = \zeta_C \sum \alpha E_{\alpha} = u^T u - \zeta_C 1_V$$

where $u = \sqrt{\zeta_C}(1, 1, \ldots, 1)$. This yields the Lagrangian

$$\frac{1}{2} \text{Tr} \dot{A}^2 = \frac{1}{2} \text{Tr} L^2 = \frac{1}{2} \sum_i \dot{x}_i^2 - \sum_{i < j} \frac{\zeta_C^2}{(x_i - x_j)^2}$$

and Hamiltonian

$$H = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} \frac{\zeta_C^2}{(x_i - x_j)^2}.$$  

This Hamiltonian is that of the completely integrable system we are interested in. By setting

$$B_1 = L, \quad B_2 = X, \quad I = u^T, \quad J = u$$

(or equivalently $B_1 = \dot{A}, B_2 = A, I = gu^T, J = ug^{-1}$) we have (1) and for the normal matrices we have assumed (for which $g$ is unitary) then (3) is identically satisfied. In this case $w = 1$ and clearly conjugation of $B_{1,2}$ by $GL(v)$ with an attendant action on $I$ and $J$ does not effect the reduced system. As $B_{1,2}$ are determined by the initial conditions the equivalence classes of solutions up to this action describe the phase space of the Calogero-Moser system. By considering the spin Calogero-Moser models we get $w > 1$.

At this stage we have associated the rational (complexified) Calogero-Moser integrable system with the equations (1) and (2). Before looking at the this space of matrices more closely it is worth recording the connection with Seiberg-Witten theory. The moduli space of four-dimensional $N = 2$ SYM with adjoint matter is described by the elliptic Calogero-Moser model,

$$H = \frac{1}{2} \sum_i p_i^2 + \zeta_C^2 \sum_{i < j} \varphi(x_i - x_j).$$
Here the potential is described by the Weierstrass $\wp$-function which has degenerations $1/\sin^2 x$ and $1/x^2$. The free $N = 4$ theory corresponds to the limit $\zeta^2 \to 0$ and there is also a double scaling limit in which the resulting potential is that of the periodic Toda chain describes the pure $N = 2$ SYM gauge theory [33, 34].

The perturbative limit of the $N = 2$ theory with adjoint matter is described in terms of the potential $1/\sin^2 x$ while the perturbative limit of the pure $N = 2$ gauge theory is given by the non-periodic Toda chain. The rational degeneration we are considering also has a field theoretic interpretation: it arises in the perturbative limit of the $N = 2$ theory with massive adjoint matter, reduced down to three dimensions.

3 Moduli Spaces and Moment Maps

We shall now describe in more detail the space of solutions to (1) and (2). Let $V$ and $W$ be hermitian complex vector spaces of dimensions $v$ and $w$ respectively and call $\alpha = (v, w)$ the dimension vector. Let $B_1$ and $B_2$ be maps from $V$ to itself, let $I$ be a map from $W$ to $V$ and finally let $J$ be a map from $V$ to $W$. This data may be expressed by the quiver or directed graph below:

The space of matrices

$$V_{(v,w)} = \{(B_1, B_2, I, J) \mid B_{1,2} \in M_v(\mathbb{C}), I \in M_{v \times w}(\mathbb{C}), J \in M_{w \times v}(\mathbb{C})\}$$

appearing here is a (flat) hyper-Kähler manifold. We have a metric on $V_{(v,w)}$ coming from the hermitian inner product $\langle \alpha, \beta \rangle = \text{Tr} \alpha \beta^\dagger$ on $M_{r \times s}(\mathbb{C})$ matrices. With $x = (B_1, B_2, I, J)$ and $y = (\tilde{B}_1, \tilde{B}_2, \tilde{I}, \tilde{J})$ this is given by

$$g(x, y) = \frac{1}{2} \text{Tr} \left(B_1 \tilde{B}_1^\dagger + \tilde{B}_1 B_1^\dagger + B_2 \tilde{B}_2^\dagger + \tilde{B}_2 B_2^\dagger + I \tilde{I}^\dagger + \tilde{I} I^\dagger + J^\dagger \tilde{J} + \tilde{J} J^\dagger\right).$$

This metric is hermitian for the three complex structures

$$\hat{i} \cdot (B_1, B_2, I, J) = (iB_1, iB_2, iI, iJ),$$
$$\hat{j} \cdot (B_1, B_2, I, J) = (-B_2^\dagger, B_1^\dagger, -J^\dagger, I^\dagger),$$
$$\hat{k} = \hat{i} \hat{j} \hat{k},$$

which obey the usual relations of the quaternions. That is

$$g(x, y) = g(\hat{i}x, \hat{i}y) = g(\hat{j}x, \hat{j}y) = g(\hat{k}x, \hat{k}y),$$
and \( \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -1 \). We have associated to each of the complex structures the Kähler forms

\[
\begin{align*}
\omega_1(x, y) &= g(\hat{i}x, y) = \frac{i}{2} \text{Tr} \left( B_1 B_1^\dagger - B_1 B_1^\dagger + B_2 B_2^\dagger - B_2 B_2^\dagger + \hat{I}\hat{I}^\dagger - \hat{I}\hat{I}^\dagger - J^\dagger J + J^\dagger J \right), \\
\omega_2(x, y) &= g(\hat{j}x, y) = \frac{1}{2} \text{Tr} \left( \hat{B}_1 B_2 + B_2^\dagger \hat{B}_1^\dagger - \hat{B}_2 B_1 - B_1^\dagger \hat{B}_2^\dagger + \hat{I}\hat{J} + J^\dagger \hat{I}^\dagger - J\hat{I} + J^\dagger \hat{I}^\dagger \right), \\
\omega_3(x, y) &= g(\hat{k}x, y) = \frac{i}{2} \text{Tr} \left( \hat{B}_1 B_2 - B_2^\dagger \hat{B}_1^\dagger - \hat{B}_2 B_1 + B_1^\dagger \hat{B}_2^\dagger + \hat{I}\hat{J} - J^\dagger \hat{I}^\dagger - J\hat{I} + J^\dagger \hat{I}^\dagger \right).
\end{align*}
\]

We may express the first symplectic form as

\[
\omega_1 = \frac{i}{2} \text{Tr} \left( dB_1 \wedge dB_1^\dagger + dB_2 \wedge dB_2^\dagger + dI \wedge dI^\dagger - dJ^\dagger \wedge dJ \right) = \frac{i}{2} d\Theta_1,
\]

which shows it closed. Similarly for the other Kähler forms we have

\[
\begin{align*}
\omega_2 &= \frac{1}{2} d\text{Tr} \left( B_1 dB_2 + B_2 dB_1^\dagger + I dB^\dagger - J dB^\dagger \right) = d\Theta_2, \\
\omega_3 &= \frac{1}{2i} d\text{Tr} \left( B_1 dB_2 - B_2 dB_1^\dagger + I dB^\dagger + J dB^\dagger \right) = d\Theta_3.
\end{align*}
\]

It is also convenient to introduce

\[
\omega_C(x, y) = \omega_2(x, y) + i\omega_3(x, y) = \text{Tr} \left( B_2 B_2^\dagger - B_1 \hat{B}_2 + \hat{I}\hat{I} - J\hat{I} \right).
\]

We see from \( \omega_C(\hat{i}u, v) = g(\hat{j}u, v) + ig(\hat{k}u, v) = -\omega_2(u, v) + i\omega_3(u, v) = i\omega_C(u, v) \) that this is of type \((2, 0)\). We can write this complex symplectic form as

\[
\omega_C = \text{Tr} (dB_1 \wedge dB_2 + dI \wedge dJ) = d\text{Tr} (B_1 dB_2 + IdB) = d\Theta_C
\]

with \( \Theta_C = \Theta_2 + i\Theta_3 \).

There is also a natural action of \( GL(\alpha) := GL(v, \mathbb{C}) \times GL(w, \mathbb{C}) \) on \( V(v, w) \) via

\[
(g, h) : (B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gIh^{-1}, hJg^{-1})
\]

For \((g, h) \in U_v(\mathbb{C}) \times U_w(\mathbb{C})\) we have that \( \hat{j} \circ (g, h) = (g, h) \circ \hat{j} \) and consequently \( U_v(\mathbb{C}) \times U_w(\mathbb{C}) \) preserves the quaternionic structure of \( V(v, w) \). Observe that although the metric and \( \omega_1 \) are only \( U_v(\mathbb{C}) \times U_w(\mathbb{C}) \) invariant the complex symplectic form \( \omega_C \) is in fact \( GL(v, \mathbb{C}) \times GL(w, \mathbb{C}) \) invariant. Associated with the \( U_v(\mathbb{C}) \) action we have the tangent vector \( x_\xi = ([\xi, B_1], [\xi, B_2], \xi I, -J\xi) \) and we may associate a Hamiltonian and moment map to each of the symplectic structures via

\[
H_\xi^\Theta = \Theta_1(x_\xi) = \langle \xi^\dagger, \mu_i \rangle.
\]

Now we have

\[
\Theta_1(x_\xi) = \frac{i}{2} \text{Tr} \xi^\dagger \left( [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \right) = \frac{i}{2} \langle \xi^\dagger, [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \rangle,
\]

and

\[
\begin{align*}
\Theta_2(x_\xi) &= \frac{1}{2} \langle \xi^\dagger, [B_1, B_2] - [B_1, B_2]^\dagger + JJ^\dagger I^\dagger \rangle, \\
\Theta_3(x_\xi) &= -\frac{i}{2} \langle \xi^\dagger, [B_1, B_2] + [B_1, B_2]^\dagger + JJ^\dagger I^\dagger \rangle,
\end{align*}
\]
\[ \Theta_C(x_\xi) = \text{Tr} \xi^\dagger (A_1 + B_2 + IJ)^\dagger = \langle \xi^\dagger, [A_1, B_2] + IJ \rangle \]

(which does not require \( \xi^\dagger = -\xi \)) and so \( \mu_C = \mu_2 + i\mu_3 \). We thus have the moment map
\[ \mu : \mathbb{R}^3 \otimes u_v(\mathbb{C}) \rightarrow \mathbb{R}^3 \otimes u_v(\mathbb{C}) \]
\[ \mu_C(A_1, B_2, I, J) = [A_1, B_2] + IJ \]
\[ \mu_R(A_1, B_2, I, J) = [A_1, B_2] + IJ \]

(11)
\[ (12) \]

We recognize (1) and (2) as fixing these moment maps. Let us collect the numbers \((\zeta_R, \text{Re}\zeta_C, \text{Im}\zeta_C)\) into a three-vector \(\vec{\zeta} \in \mathbb{R}^3\). A standard result \([32]\) is that when \(G = U_v(\mathbb{C})\) acts freely on \(\mu^{-1}(\zeta)\) then \(\mu^{-1}(\zeta)/G\) is a smooth manifold with Riemannian metric and hyper-Kähler structure induced from those on \(V_{(v,w)}\).

Let us also record the moment map \(\tilde{\mu} : V_{(v,w)} \rightarrow \mathbb{R}^3 \otimes u_w(\mathbb{C})\) for the \(U_w(\mathbb{C})\) action given
\[ \tilde{\mu}(B_1, B_2, I, J) = \left( \frac{i}{2}(JJ^\dagger - I^\dagger I), \frac{1}{2}(I^\dagger J^\dagger - JJ), \frac{i}{2}(I^\dagger J^\dagger - JJ) \right). \]

Let us further consider the map \(\mu_C : V_{(v,w)} \rightarrow GL_v(\mathbb{C})\) given in (12). The differential \(d\mu_C\) may be determined from the the linear terms in an \(\epsilon\)-expansion about \((B_1 + \epsilon X, B_2 + \epsilon Y, I + \epsilon L, J + \epsilon M)\):
\[ d\mu_C|_{(B_1, B_2, I, J)}(X, Y, L, M) = [X, B_1] + [B_2, Y] + IM + LJ. \]

The image of \(d\mu_C\) is orthogonal to those matrices \(W\) such that
\[ < d\mu_C|_{(B_1, B_2, I, J)}(X, Y, L, M), W > = 0. \]

Using the nondegeneracy and cyclicity of \(\text{Tr}\) this is equivalent to
\[ \{W \in GL_v(\mathbb{C}) | [B_1, W^\dagger] = 0, [B_2, W^\dagger] = 0, W^\dagger I = 0, JW^\dagger = 0\}. \]

In particular the \(w\) columns of \(I\) are in \(\text{Ker} W\) and this space is stable under left multiplication by \(B_1\) and \(B_2\).

We remark that one may associate with any a quiver a path algebra \(\mathbb{M}\) with generators corresponding to the vertices and directed edges the associative algebra structure is induced by the concatenation of paths where possible and zero otherwise. Thus to the path algebra \(\mathbb{M}\) associated with
one has relations
\[
\begin{align*}
  e^2 &= e & f^2 &= f & e + f &= 1 \\
  e.x &= x & e.y &= y & e.u &= u & f.v &= v \\
  x.e &= x & y.e &= e & v.e &= v & u.f &= f
\end{align*}
\]
with all other products vanishing. By a representation \( \text{rep}_\alpha(M) \) of a quiver one means the assignment to each vertex a vector space and to each directed edge a linear map of the corresponding vector spaces. The dimension-vector \( \alpha \) of a representation is the integral vector containing the dimensions of the vertex spaces. Representations are defined up to equivalence of a change of basis in the vertex spaces. We have thus been describing the representations a particular quiver.

4 The Deformed Instanton Equations

We shall now describe perhaps the simplest deformation of the instanton equations. Form a sequence of linear maps
\[
V \xrightarrow{\sigma_z} V \otimes \mathbb{C}^2 \otimes W \xrightarrow{\tau_z} V
\]
where
\[
\sigma_z = \begin{pmatrix}
  -B_1 + z_1 \\
  B_2 - z_2 \\
  J
\end{pmatrix}, \quad \tau_z = \begin{pmatrix}
  B_1 - z_1 & B_1 - z_1 & I
\end{pmatrix}.
\]
Here \( z = (z_1, z_2) \in \mathbb{C}^2 \) are complex parameters and we let \( \sigma = \sigma_{(0,0)} \) etc.

Suppose now that the matrices \( (B_1, B_2, I, J) \) obey the following equations:
\[
\begin{align*}
  \tau_z \sigma_z &= \zeta_c 1_V, \\
  \tau_z \sigma_z^t &= \Delta_z + \zeta_R 1_V, \\
  \sigma_z^t \sigma_z &= \Delta_z - \zeta_R 1_V.
\end{align*}
\]
These may be rewritten as
\[
\begin{align*}
  [B_1, B_2] + IJ &= \zeta_c 1_V, \\
  [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= 2\zeta_R 1_V,
\end{align*}
\]
which are precisely our (4) and (5), and here
\[
\Delta_z = II^\dagger + J^\dagger J + \frac{1}{2} \left[ (B_1 - z_1)(B_1^\dagger - \bar{z}_1) + (B_1^\dagger - \bar{z}_1)(B_1 - z_1) + (B_2 - z_2)(B_2^\dagger - \bar{z}_2) + (B_2^\dagger - \bar{z}_2)(B_2 - z_2) \right].
\]
We observe a consequence of these equations is that

\[ [B_2 - B_1^T, B_2^T + B_1] + (I + J)(I^T - J) = (2\zeta_R - \zeta_C + \bar{\zeta}_C)1_V. \]

(From this we can deduce (2) but not (1).) The previous section told us that the space of all matrices \((B_1, B_2, I, J)\) is a hyper-Kähler vector space and the equations (13-15) may be interpreted as \(U_v(\mathbb{C})\) hyper-Kähler moment maps \([22]\). We will denote by \(M_{v,w} = \mu^{-1}((\bar{\zeta})/U_v(\mathbb{C})\), the space of solutions to the equations (13-15) up to such a symmetry transformation.

When \(\bar{\zeta} = 0\) these equations, together with the injectivity and surjectivity of \(\sigma_z\) and \(\tau_z\) respectively, yield the standard ADHM construction. If one relaxes the injectivity condition then one gets the Donaldson compactification of the instanton moduli space \([24]\). In the nomenclature of Corrigan and Goddard \([21]\) describing charge \(v SU(w)\) instantons,

\[
\Delta = \begin{pmatrix} -B_1 & B_1^T \\ B_2 & B_2^T \\ J & I \end{pmatrix},
\]

and \(\Delta^T\Delta = \Delta \otimes I_2\) corresponds to the equations (13-15) when \(\bar{\zeta} = 0\). We are considering a deformation of the standard ADHM equations. The moduli space \(M_{v,w}\) is the space of freckled instantons on \(\mathbb{R}^4\) in the sense of \([57]\), a “freckle” simply being a point at which \(\sigma_z\) fails to be injective. One learns from \([39]\) that the deformed ADHM data parameterise the (semistable) torsion free sheaves on \(CP^2\) whose restriction on the projective line \(\ell_\infty\) at infinity is trivial. Each torsion free sheaf \(\mathcal{E}\) is included into the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}_Z \longrightarrow 0
\]

where \(\mathcal{F}\) is a holomorphic bundle \(\mathcal{E}^{**}\) and \(\mathcal{S}_Z\) is a skyscraper sheaf supported at points, the set \(Z\) of freckles \([57]\). From this exact sequence one learns that

\[
\text{ch}_i(\mathcal{E}) = \text{ch}_i(\mathcal{F}) - \#Z\delta_{i,2}.
\]

The same equations (13-15) also describe a further moduli space, those of instantons over a noncommutative \(\mathbb{R}^4\). In this work we have considered \(z_{1,2}\) as ordinary complex numbers and \(\bar{\zeta} \neq 0\). The same equations arise however by considering \(\bar{\zeta} = 0\) and the space-time coordinates having the following commutation relations:

\[
[z_1, z_2] = -\zeta_C, \quad [z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -2\zeta_R.
\]

An analogous construction to the ordinary (commutative) ADHM construction produces noncommutative instantons \([47]\).

Observe that by performing an \(SU(2)\) transformation

\[
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha B_1 - \beta B_2^T \\ \bar{\alpha} B_2 + \bar{\beta} B_1^T \end{pmatrix}, \quad \begin{pmatrix} I \\ J \end{pmatrix} \mapsto \begin{pmatrix} \alpha I - \beta J^T \\ \bar{\alpha} J + \bar{\beta} I \end{pmatrix},
\]

with \(|\alpha|^2 + |\beta|^2 = 1\), we can always rotate \(\bar{\zeta}\) into a vector \((\zeta_R, 0, 0)\). Such a transformation corresponds to singling out a particular complex structure on our data, for which \(z = (z_1, z_2)\) are the holomorphic coordinates on the Euclidean space-time. Further we may choose the complex structure such that \(\zeta_R > 0\). For \(\zeta_R > 0\) \([14]\) shows that \(\tau_z \tau_z^T\) is invertible and \(\tau_z\) is surjective. When \(\bar{\zeta} = (\zeta_R, 0, 0)\) and \(w = 1\) \([39]\) shows one can simplify the equations and set \(J = 0\). Then \(I^T I = 2v\zeta_R\) and \([B_1, B_2] = 0\). This situation yields a well-known moduli space, the Hilbert scheme of points on the plane which we now recall.
5 Hilbert Schemes of Points on the Plane

A Hilbert scheme $X^{[n]}$ or Hilb$_n(X)$ for a (suitable) manifold $X$ is roughly speaking the moduli space of $n$ points on $X$. It differs in general from the symmetric product $S^nX$ in that $X^{[n]}$ contains information on how points collide: “fat” points are points that have collided, and the Hilbert scheme retains the directions in which they coalesced. We have the (Hilbert-Chow) map $\pi : X^{[n]} \to S^nX$. When $\text{dim } X = 1$ we have $X^{[n]} = S^nX$ because there is only one direction for the points to collide. Thus for example the Hilbert scheme of $n$ points in the affine line $A$ (over an algebraically closed field $k$) is

$$A^{[n]} = \{ I \subset k[z] \mid I \text{ an ideal, } \dim_k k[z]/I = n \}$$

$$= \{ f(z) \in k[z] \mid f(z) = z^n + a_1z^{n-1} + \ldots + a_n, a_i \in k \}$$

$$= S^nA.$$

When $k = \mathbb{C}$ the Hilbert scheme often inherits nice properties possessed by the base space $X$. Thus if $X$ has a holomorphic symplectic form so does $X^{[n]}$ ($n \geq 2$). When $X$ is a K3 or an abelian surface then $X^{[n]}$ has a hyper-Kähler metric. These and many other results are described in [39].

For our purposes we will focus on the case when $\text{dim } X = 2$. Various connections have been made between integrable systems and Hilbert schemes in this dimension [27, 30, 52]. Consider the case when $X = \mathbb{A}^2$, the affine plane (over an algebraically closed field $k$) $\mathbb{A}$. Then

$$(\mathbb{A}^2)^{[n]} = \{ I \subset k[z_1, z_2] \mid I \text{ an ideal, } \dim_k k[z_1, z_2]/I = n \}$$

What is particularly convenient for us is the alternative description

**Theorem 2** There exists an isomorphism

$$(\mathbb{A}^2)^{[n]} \cong \frac{\{(B_1, B_2, I) \mid (i) \text{ } [B_1, B_2] = 0, \text{ } (ii) \text{ no subspace } S \subseteq k^n \text{ such that } B_1S \subset S \text{ and } \text{Image } I \subset S \}}{\text{GL}(n, k)}$$

where $B_{1,2} \in \text{End}(k^n)$ and $I \in \text{Hom}(k, k^n)$ with the $\text{GL}(n, k)$ action given by

$$g \cdot (B_1, B_2, I) = (gB_1g^{-1}, gB_2g^{-1}, gI).$$

The proof is constructive. Let $I$ be an ideal in $k[z_1, z_2]$ and define $V = k[z_1, z_2]/I$. Let $B_{1,2} \in \text{End}(V)$ be multiplication by $z_1, z_2$ mod $I$. Then $[B_1, B_2] = 0$. Define $I \in \text{Hom}(k, V)$ by $I(1) = 1$ mod $I$. Since 1 multiplied by products of $z_1$ and $z_2$ spans all of $k[z_1, z_2]$ then $k[B_1, B_2]I(1) = k^n$ and stability follows.

Conversely, given $(B_1, B_2, I)$ as in the theorem define $\phi : k[z_1, z_2] \to k^n$ by $\phi(f(z_1, z_2)) = f(B_1, B_2)f(1)$. This is well defined by (i) and since Image $\phi$ is $B_{1,2}$ invariant and contains Image $I$ then by the stability (ii) it must be all of $k^n$. Thus $\phi$ is onto and Ker $\phi$ is then a codimension $n$ ideal $I \triangleleft k[z_1, z_2]$ yielding a point of $(\mathbb{A}^2)^{[n]}$. Thus

$$I = \{ f(z_1, z_2) \in k[z_1, z_2] \mid f(B_1, B_2)f(1) = 0 \} = \{ f(z_1, z_2) \in k[z_1, z_2] \mid f(B_1, B_2) = 0 \}$$

where the last equality follows from the stability condition.

**Example** Hilb$_1(\mathbb{C}^2)$: In this case $\text{dim } \mathbb{C} V = 1$ and so $B_1 = \lambda, B_2 = \mu$ are scalars. The stability condition means we require $I = I(1) \neq 0$ and using the $GL(1, \mathbb{C}) = \mathbb{C}^*$ invariance we may scale so that $I = 1$. Thus $\text{Hilb}_1(\mathbb{C}^2) \cong \{ (\lambda, \mu, 1) \in \mathbb{C}^3 \} \cong \mathbb{C}^2$. The ideal $I$ in this case is

$$I = \{ f(z_1, z_2) \in k[z_1, z_2] \mid f(\lambda, \mu) = 0 \} = z_1 - \lambda, z_2 - \mu >.$$
that is the maximal $m_p$ ideal corresponding to the point $p = (\lambda, \mu) \in \mathbb{C}^2$.

**Example** $\text{Hilb}_2(\mathbb{C}^2)$: Now $\dim_{\mathbb{C}} V = 2$ and $B_{1,2}$ are $2 \times 2$ matrices. We will consider two cases: $B_1$ diagonalisable with distinct eigenvalues and $B_1$ not diagonalisable.

First suppose $B_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_1 \neq \lambda_2$. The commutativity of $B_1$ and $B_2$ yields $B_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ where we do not demand the distinctness of $\mu_1, \mu_2$. Stability now yields that $I = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ where $\nu_1 \nu_2 \neq 0$, and using the group conjugation we may scale this so that $I = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus we have a representative of this orbit as

$$\left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$

The ideal corresponding to this is

$$\mathcal{I} = \{ f(z_1, z_2) \in k[z_1, z_2] \mid f(\lambda_1, \mu_1) = 0 = f(\lambda_2, \mu_2) \},$$

or $m_{p_1} \cap m_{p_2}$ which represents the two distinct points $p_1 = (\lambda_1, \mu_1)$ and $p_2 = (\lambda_2, \mu_2)$ in $\mathbb{C}^2$.

Next consider the situation when $B_1$ is not diagonalisable. Then $B_1$ can be taken to have Jordan form $B_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and $B_2$ is found to be $\begin{pmatrix} \mu & * \\ 0 & \mu \end{pmatrix}$. Similarly we find $I = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ where $\nu_2 \neq 0$. A representative for this orbit can then be taken to be

$$\left( \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad \text{where } [\alpha : \beta] \in \mathbb{C}P^1. \quad (17)$$

It remains to describe the ideal associated with this orbit type. Using

$$\left( \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \right)^k \left( \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix} \right)^t = \begin{pmatrix} \lambda^k \mu^t & k \alpha \lambda^{k-1} \mu^t + l \beta \lambda^k \mu^{t-1} \\ 0 & \lambda^k \mu^t \end{pmatrix},$$

we find that we can represent $\mathcal{I}$ as

$$\mathcal{I} = \{ f(z_1, z_2) \in k[z_1, z_2] \mid f(\lambda, \mu) = 0 = \alpha \partial_{z_1} f(z_1, z_2) \}_{(\lambda, \mu)} + \beta \partial_{z_2} f(z_1, z_2) \}_{(\lambda, \mu)} \}$$

$$\Rightarrow (z_1 - \lambda)^2, (z_1 - \lambda)(z_2 - \mu), (z_2 - \mu)^2, \beta(z_1 - \lambda) - \alpha(z_2 - \mu) > 0.$$ We can picture this as two points which have coalesced to the point $p = (\lambda, \mu)$ colliding with each other in the direction $\alpha \partial_{z_1} + \beta \partial_{z_2}$. For each point in $\mathbb{C}^2$ there is a family $[\alpha : \beta] \in \mathbb{C}P^1$ of such fat points.

Now the two cases just given in fact exhaust the possible orbit types of $\text{Hilb}_2(\mathbb{C}^2)$ up to the interchange of $B_1$ and $B_2$. If $B_1$ is in fact diagonal with equal eigenvalues then $B_2$ may be diagonal with distinct eigenvalues which is the first case above, or it may be non-diagonalisable and so in the second case. The only remaining possibility is that both $B_1$ and $B_2$ are scalar multiples of the identity, but this situation is ruled out by the stability requirement as here $\mathbb{C}[B_1, B_2]$ gives a one-dimensional subspace of $\mathbb{C}^2$.

The Hilbert-Chow map $\pi : \text{Hilb}_2(\mathbb{C}^2) \to S^2 \mathbb{C}^2$ in this example gives $\pi(B_1, B_2, I) = [p_1] + [p_2]$ for the first case and $2[p]$ for the second case. Away from the diagonal we have a one-to-one correspondence while on the diagonal the fibers are $\mathbb{C}P^1$. In fact $S^2 \mathbb{C}^2$ has singularities and $\text{Hilb}_2(\mathbb{C}^2)$ is smooth and gives a resolution of these singularities.
6 Constructing the gauge field

We shall now construct a gauge field corresponding to the deformed instanton equations following [12]. Our purpose is to further investigate the properties of these gauge fields. The fundamental object in the ADHM construction is the solution of

\[ \mathcal{D}_z^\dagger \Psi_z = 0, \quad \Psi_z : W \rightarrow V \otimes \mathbb{C}^2 \oplus W \]

where

\[ \mathcal{D}_z^\dagger = \left( \begin{array}{c} \tau_z \\ \sigma_z^\dagger \end{array} \right). \]

We shall need the components

\[ \Psi_z = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \Psi_{1,2} \in V, \quad \varphi \in V \otimes \mathbb{C}^2, \quad \chi \in W. \]  \hspace{1cm} (19)

The solution of (18) is not uniquely defined and one is free to perform a \( GL(w, \mathbb{C}) \) gauge transformation,

\[ \Psi_z \rightarrow \Psi_z g(z, \bar{z}), \quad g(z, \bar{z}) \in GL(w, \mathbb{C}). \]

This gauge freedom can be partially fixed by normalising the vector \( \Psi_z \) as follows:

\[ \Psi_z^\dagger \Psi_z = 1_W. \]  \hspace{1cm} (20)

With this normalisation the \( U(w) \) gauge field is given by

\[ A = \Psi_z^\dagger d\Psi_z, \]

and its curvature is given by

\[ F = \Psi_z^\dagger d\mathcal{D}_z - \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} d\mathcal{D}_z^\dagger \Psi_z. \]  \hspace{1cm} (22)

More explicitly,

\[ \mathcal{D}_z^\dagger \mathcal{D}_z = \Delta_z \otimes 1 - 1_V \otimes \zeta^a \sigma_a, \]

hence

\[ \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} = \frac{1}{\Delta_z^2 - \zeta^2} (\Delta_z \otimes 1 + 1_V \otimes \zeta^a \sigma_a). \]

Formula (22) makes sense for \( z \in X^o \equiv \mathbb{R}^4 \setminus Z \), where \( X^o \) is the complement in \( \mathbb{R}^4 \) to the set \( Z \) of points (freckles) at which

\[ \text{Det} \left( \Delta_z^2 - \zeta^2 \right) = 0. \]  \hspace{1cm} (23)

More explicitly (22) is

\[ F = \varphi^\dagger \frac{\Delta_z \otimes \sigma_a}{\Delta_z^2 - \zeta^2} \varphi (d\bar{z}_1 \wedge dz_1 - d\bar{z}_2 \wedge dz_2) + 2 \varphi^\dagger \frac{\Delta_z \otimes \sigma_a}{\Delta_z^2 - \zeta^2} \varphi dz_1 \wedge dz_2 \]

\[ + 2 \varphi^\dagger \frac{\Delta_z \otimes \sigma_a}{\Delta_z^2 - \zeta^2} \varphi d\bar{z}_2 \wedge dz_1 + 2i \varphi^\dagger \frac{1}{\Delta_z^2 - \zeta^2} \varphi \hat{\zeta}. \]  \hspace{1cm} (24)
where $\hat{\zeta} = \zeta_R \overline{\omega}_R + \zeta_C \overline{\omega}_C + \zeta_C \omega_C$, and

$$\overline{\omega}_R = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \quad \overline{\omega}_C = dz_1 \wedge dz_2. \quad (25)$$

With respect to the orientation given by $\star 1 = -dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = -dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ where $z_1 = (x^1 + i x^2)/\sqrt{2}$ and $z_2 = (x^3 + i x^4)/\sqrt{2}$ we have a basis of anti-self-dual 2-forms given by

- $\lambda_1^1 = dx_0 \wedge dx^1 + dx_2 \wedge dx^3 = i(d\bar{z}_2 \wedge dz_1 - dz_2 \wedge d\bar{z}_1)$,
- $\lambda_2^1 = dx_0 \wedge dx_2 + dx_3 \wedge dx^1 = d\bar{z}_2 \wedge dz_1 + dz_2 \wedge d\bar{z}_1$,
- $\lambda_3^1 = dx_0 \wedge dx^3 + dx_1 \wedge dx^2 = i(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2)$,

and a basis for the self-dual 2-forms given by

- $\lambda_1^2 = dx_0 \wedge dx^1 - dx_2 \wedge dx^3 = i(dz_1 \wedge dz_2 - d\bar{z}_1 \wedge d\bar{z}_2)$,
- $\lambda_2^2 = dx_0 \wedge dx_2 - dx_3 \wedge dx^1 = dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2$,
- $\lambda_3^2 = dx_0 \wedge dx^3 - dx_1 \wedge dx^2 = i(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_2)$.

With this orientation we see $F$ is anti-self-dual when $\hat{\zeta} = 0$. One then has on $X^o$ that

$$F^+ := \frac{1}{2} (F + \ast F) = 4i \varphi^1 \frac{1}{\Delta_2^1 - \zeta_2^2} \varphi \hat{\zeta}. \quad (26)$$

Thus with respect to the standard complex coordinates the gauge field we have constructed is neither self-dual nor anti-self-dual. We can ask whether it has other nice properties. For example, do other coordinates exist for which it is either self-dual or anti-self-dual? If $\zeta^C = 0$ then (26) implies that $F^{0,2} = 0$, i.e. the $A_{z_1}, A_{z_2}$ define a holomorphic structure on the bundle $\mathcal{E}_z = \ker D_{z}^1$ over $X^o$. As we have a unitary connection, $F^{2,0} = F^{0,2} = 0$.

From (18) the holomorphic bundle $\mathcal{E}$ extends to a holomorphic bundle $\mathcal{F}$ on the whole of $\mathbb{R}^4$. We will now construct a compactification $X$ of $X^o$ with a holomorphic bundle $\mathcal{E}$ over $X$ such that $\mathcal{E}|_{X^o} = \mathcal{E}$, and whose connection $\mathcal{A}$ is a smooth continuation of the connection $\mathcal{A}$ over $X^o$. This compactification $X$ projects down to $\mathbb{C}^2$ via a map $p : X \to \mathbb{C}^2$. The pull-back $p^* \mathcal{F}$ is a holomorphic bundle over $X$ which differs from $\mathcal{E}$. This difference is localised at the exceptional variety, which is the preimage $p^{-1}(Z)$ of the set of freckles.

### 7 The Abelian case in detail

Let us rotate $\hat{\zeta}$ so that $\zeta^C = 0, \zeta_R = \zeta > 0$ and consider the case $w = 1$ or $U(1)$ or abelian instantons. As we have already remarked, Nakajima [8] shows that $J = 0$. Hence, $I^I I = 2v\zeta$ and $[B_1,B_2] = 0$. When the matrices $B_{1,2}$ and $I$ satisfy the stability criterion given earlier, the moduli space we are describing is the Hilbert scheme $\text{Hilb}_n(\mathbb{C})$. At the outset note that abelian instantons do not exist for $\mathbb{R}^4$.

We can now solve the equations (18) rather explicitly:

$$
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix}
= - \begin{pmatrix}
(B_{1}^1 - \bar{z}_1) & (B_{2}^1 - \bar{z}_2) \\
(B_{1}^2 - z_1) & (B_{2}^2 - z_2)
\end{pmatrix}
G \chi,
$$

where

$$G^{-1} = (B_1 - z_1)(B_1^1 - \bar{z}_1) + (B_2 - z_2)(B_2^1 - \bar{z}_2). \quad (28)$$
and

\[ \chi = \frac{1}{\sqrt{1 + I^\dagger G I}}. \tag{29} \]

Let \( P(z) = \text{Det} G^{-1} \). It is a polynomial in \( z, \bar{z} \) of degree \( v \). Clearly (29) implies that:

\[ \chi^2 = \frac{P(z)}{Q(z)} \]

where \( Q(z) = P(z) + I^\dagger \tilde{G}^{-1} I \) is another degree \( v \) polynomial in \( z, \bar{z} \), \( \tilde{G}^{-1} \) being the matrix of minors of \( G^{-1} \).

The gauge field (21) is calculated to be

\[ A = (\partial - \bar{\partial}) \log \chi, \tag{30} \]

and its curvature is

\[ F = \partial \bar{\partial} \log \chi^2. \tag{31} \]

The formula (30) provides a well-defined one-form on the complement \( X^\circ \) in \( \mathbb{R}^4 \) to the set \( Z \) of zeroes of \( P(z) \). This is just where \( B_1 - z_1 \) and \( B_2 - z_2 \) fail to be invertible (and so \( \sigma_z \) fails to be injective), that is a “freckle”. Here we will only study the case of one point in detail and record that the higher charge case can be dealt with similarly, with integrable systems calculations helping greatly [12]. The case of just one freckle already yields a surprise: abelian instantons exist. Let us examine what is going on.

### 7.1 Charge one instantons.

To see what happens at such a point consider the case \( v = 1 \). Then (after shifting \( \bar{z}_1 \) by \( B_1^\dagger \), etc.)

\[ \Psi_z = \frac{1}{r \sqrt{r^2 + 2\zeta}} \left( \bar{z}_1 \sqrt{2\zeta} \bar{z}_2 \sqrt{2\zeta} \right), \quad \chi = \frac{r}{\sqrt{r^2 + 2\zeta}}, \tag{32} \]

where \( r^2 = |z_1|^2 + |z_2|^2 \). Thus in this case

\[ P(z) = z_1 \bar{z}_1 + z_2 \bar{z}_2, \quad Q(z) = z_1 \bar{z}_1 + z_2 \bar{z}_2 + 2\zeta. \]

The gauge field is given by (setting \( 2\zeta = 1 \)):

\[ A = \frac{1}{2r^2(1 + r^2)} (z_1 dz_1 - \bar{z}_1 d\bar{z}_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2), \tag{33} \]

and

\[ F = \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{r^2(1 + r^2)} - \frac{1 + 2r^2}{r^4(1 + r^2)^2} \sum_{i,j} z_i \bar{z}_j dz_j \wedge d\bar{z}_i. \tag{34} \]
7.2 The first blowup

To examine (33) further let us rewrite $A$ as follows:

$$A = A_0 - A_\infty,$$

$$A_0 = \frac{1}{2r^2} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2),$$

$$A_\infty = \frac{1}{2(1 + r^2)} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2).$$

The form $A_\infty$ is regular everywhere in $\mathbb{R}^4$. The form $A_0$ has a singularity at $r = 0$. Nevertheless, as we now show, this becomes a well-defined gauge field on $\mathbb{R}^4$ blown up at one point $z = 0$.

Let us describe the blowup in some detail. We start with $\mathbb{C}^2$ with coordinates $(z_1, z_2)$. The space blown up at the point $0 = (0, 0)$ is simply the space $X$ of pairs $(z, \ell)$, where $z \in \mathbb{C}^2$, and $\ell$ is a complex line which passes through $z$ and the point 0. $X$ projects to $\mathbb{C}^2$ via the map $p(z, \ell) = z$. The fiber over each point $z \neq 0$ consists of a single point while the fiber over the point 0 is the space $\mathbb{C}P^1$ of complex lines passing through the point 0.

In our applications we shall need a coordinatization of the blowup. The total space of the blowup is a union $X = U \cup U_0 \cup U_\infty$ of three coordinate patches. The local coordinates in the patch $U_0$ are $(t, \lambda)$ such that

$$z_1 = t, \quad z_2 = \lambda t. \quad (35)$$

In this patch $\lambda$ parameterises the complex lines passing through the point 0, which are not parallel to the $z_1 = 0$ line. In the patch $U_\infty$ the coordinates are $(s, \mu)$, such that

$$z_1 = \mu s, \quad z_2 = s. \quad (36)$$

There is also a third patch $U$, where $(z_1, z_2) \neq 0$. This projects down to $\mathbb{C}^2$ such that over each point $(z_1, z_2) \neq 0$ the fiber consists of just one point. The fiber over the point $(z_1, z_2) = 0$ is the projective line $\mathbb{C}P^1 = \{\lambda\} \cup \infty$. We now show that on this blown up space our gauge field is well defined.

On $U \cap U_0$ we may write

$$A_0 = \frac{td\bar{\ell} - \bar{t}dt}{2|t|^2} + \frac{\lambda d\bar{\lambda} - \bar{\lambda} d\lambda}{2(1 + |\lambda|^2)}. \quad (37)$$

Define $A_{U_0, \infty}$ as

$$A_{U_0} = \frac{\lambda d\bar{\lambda} - \bar{\lambda} d\lambda}{2(1 + |\lambda|^2)},$$

$$A_{U_\infty} = \frac{\mu d\bar{\mu} - \bar{\mu} d\mu}{2(1 + |\mu|^2)}. \quad (38)$$

Now $A_0$ is a well-defined one-form on $U$. On the intersections $U \cap U_0$ the one-forms $A_0$ and $A_{U_0}$ are related via a gauge transformation

$$i d \arg t.$$

On the intersection $U_0 \cap U_\infty$ the one-forms $A_{U_0}$ and $A_{U_\infty}$ are related via

$$i d \arg \lambda = -i d \arg \mu.$$
gauge transformations. Finally on \( U \cap U_\infty \) the one-forms \( A_0 \) and \( A_{U_\infty} \) are related via the gauge transformation

\[
i d \arg s.
\]

We have shown therefore that \( A_0 \) is a well-defined gauge field on \( X \). Observe also that at infinity \( A \to 0 \) as \( o(r^{-3}) \), which yields a finite action. In fact the gauge field \( (33) \) has a non-trivial Chern class \( ch_2 \):

\[
F \wedge F = - \frac{2}{r^2(1 + r^2)^3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2
\]

so that

\[
\frac{1}{4\pi^2} \int F \wedge F = 1.
\]

Finally, the restriction of \( A \) on the exceptional divisor \( E \), defined by the equation \( t = 0 \) in \( U_0 \) and \( s = 0 \) in \( U_\infty \), has non-trivial first Chern class:

\[
\frac{1}{2\pi i} \int_E F = -1.
\]

The reason that an abelian instanton exists is that space-time is blown up, and now there are noncontractible 2-spheres. The space under discussion is not \( \mathbb{C}^2 \). We should also remark that since the curvature of our gauge field has type the \((1, 1)\) and is non-degenerate on the blowup it can be used as a \( \text{Kähler} \) form. Then, tautologically, the gauge field has the same self-duality property as the \( \text{Kähler} \) form (it is either self-dual or anti-self-dual depending on the choice of orientation). The complex coordinates for this are in general complicated expressions of the coordinates \( z_{1,2} \) we have employed.

### 7.3 Comparison with the Born-Infeld instanton

Notice the similarity of the solution \( (33) \) to the formulae \( (4.56), (4.61) \) of the paper \[55\]. It has the same asymptotics both in the \( r^2 \to 0 \) and \( r^2 \to \infty \) limits. Of course the formulae in \[55\] were meant to hold only for slowly varying fields and that is why we don’t get precise agreement. Nevertheless, we conjecture that all our gauge fields are the transforms of the non-commutative instantons from \[44\] under the field redefinition described in \[55\]. From our analysis it follows that one has to modify the topology of space in order to make non-singular the corresponding gauge fields of the ordinary gauge theory.

### 7.4 Comparison with the non-commutative instanton

It is instructive to compare the solutions above with those defined on the noncommutative space. Traditionally one describes the moduli space with non-vanishing parameters \( \zeta \) as corresponding to the instantons on noncommutative \( \mathbb{R}^4 \) where all four coordinates are non-commuting. In order to make these solutions as close to the commutative ones as possible we shall consider the four dimensional space which is a product of the ordinary commutative plane, with the coordinates \( z, \bar{z} \) and the noncommutative plane, i.e. the algebra \([a, a^\dagger] = 1\). This would correspond to the parameters \( \zeta_C = 0 \) and \( \zeta_R = 1 \). For simplicity, set \( B_2 = 0 \), that is, consider the instantons, elongated at \( a^\dagger a = 0 \), along the \( z \) direction. Then the application of the ADHM construction yields the following formulae for the elongated instantons on this
Here, $\tilde{\partial} = d\bar{z} \frac{\partial}{\partial z} + da \{ , a^\dagger \}$, and $a^\dagger a = n$. The solution (40) is non-singular, without any topology change. However, there is a noncommutative indication of the blowup. It is in the phase of the function $\xi$. For $n > 0$ it vanishes, while for $n = 0$ it is winding around the zeroes of $\text{Det}(B_1 - z)$. In the commutative description this would have been described as the local gauge transformation, patching the regions with $a^\dagger a \sim 0$ and those with $a^\dagger a \gg 0$. The winding of this gauge transformation

$$\exp i \text{argDet}(B_1 - z)$$

is related with the number of the points, blown up in the commutative description.

8 The Burns metric

We shall now show that there is a particularly nice metric on the blow-up of $\mathbb{C}^2$ for which our charge one instanton is anti-self-dual. This is the Burns metric which is scalar flat with anti-self-dual Weyl curvature $W^+ = 0$. We remark that $F^+ = 0$ is the correct equation to go with $W^+ = 0$ if there is to be twistor correspondence.

Consider the Kähler form on $\mathbb{C}^2 - \{0\}$,

$$\Omega = -\frac{i}{2} \partial \bar{\partial} \left( |z|^2 + m \log |z|^2 \right).$$

We have a volume element

$$\frac{1}{2} \Omega \wedge \Omega = -\frac{1}{4} \left( 1 + \frac{m}{r^2} \right) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

Let $r^2 = |z|^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2$. Then with $g(Ix, y) = \Omega(x, y)$ and $Idz_1 = idz_1$ etc we get

$$g_{1\bar{1}} = g_{\bar{1}1} = \frac{1}{2} \left( 1 + \frac{m}{r^2} - \frac{m|z_1|^2}{r^4} \right)$$

$$g_{2\bar{2}} = g_{\bar{2}2} = \frac{1}{2} \left( 1 + \frac{m}{r^2} - \frac{m|z_2|^2}{r^4} \right)$$

$$g_{1\bar{2}} = g_{\bar{2}1} = -\frac{m}{2r^4} z_2 \bar{z}_1$$

$$g_{2\bar{1}} = g_{\bar{1}2} = -\frac{m}{2r^4} z_1 \bar{z}_2$$

For $m = 0$ this gives us the usual flat conventions. With $^*\Omega = \Omega$ we have $\{ \Omega, dz_1 \wedge dz_2, d\bar{z}_1 \wedge d\bar{z}_2 \} \in \Lambda_2^* T^* M$.

Now

$$\Lambda^2 T^* M = \Lambda^+_2 T^* M \oplus \Lambda^-_2 T^* M.$$
and $\Lambda^2 T^* M$ consists of the $(1,1)$ forms orthogonal to $\Omega$. With

$$\Omega \wedge *\alpha = -\frac{1}{2}(\Omega, \alpha) \Omega \wedge \Omega$$

we see that if $F \wedge \Omega = 0$ then $F \in \Lambda^2 T^* M$. We have explicitly calculated the abelian instanton. In terms of $\chi$ determined from the ADHM data we have

$$A = (\partial - \bar{\partial}) \log \chi,$$

and

$$F = \partial \bar{\partial} \log \chi^2.$$ 

With

$$\chi = \frac{r}{\sqrt{r^2 + m}},$$

(here $m = 2\zeta$ in our notation) we find $F \wedge \Omega = 0$. Thus $F = *F$ with this metric. Observe that $\text{Ricc} \wedge \Omega = 0$ and so our metric is self dual as stated. For both these calculations it is convenient to note

$$\partial \bar{\partial} f(r^2) \wedge \partial \bar{\partial} h(r^2) = 2r^2 h' f' \left( \frac{f''}{f'} + \frac{h''}{h'} + \frac{2}{r^2} \right) * 1$$

where $r^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2$ and $f' = df(x)/dx$ and so forth.

9 Discussion

Thus far we have identified the phase space of the (complexified $a_\infty$) Calogero-Moser systems with the moduli spaces of deformed instantons and that of instantons on a non-commutative space-time. We have shown that there is a very nice metric, the Burns metric, on the one-instanton space-time for which these deformed instanton equations are in fact anti-self-dual, i.e. solve the ordinary instanton equation. Thus far our discussion has focused on the real structure of these spaces, and as real spaces they are diffeomorphic. There is more to the story however. These spaces have complex structures, and as we described earlier, these are different. A choice of complex structure (or a $B$-field) effects this description. We will conclude by briefly recording some of these differences.

We have already described the Hilbert scheme of points in terms of codimension $v$ ideals in $A_0 = \mathbb{C}[x, y]$. The Calogero-Moser phase space has a description in terms of ideals in the 1-st Weyl algebra $A_1$. The crucial difference here is that $A_1$ has no finite dimensional representations (for $\text{Tr}([L, X] - 1_V) = 0$ yields $v = 0$). However by letting $C_v$ be the space for which (up to conjugation) $\{\text{rank}([L, X] - 1_V)\}$ has at most 1, Berest and Wilson show $C = \bigcup_{v \geq 0} C_v$ is equivalent to the isomorphism classes of right ideals in $A_1$. Further Ginzburg has shown there is an infinite algebraic group $G_1$ acting on $C$ such that it acts transitively on $C_v$. Thus $C_v$ is the coadjoint orbit of $G_1$. This parallels the results that hold for the Hilbert
scheme. These similarities and difference are summarized in the following:

\[
\begin{align*}
\mu_C &= [L, X] + u^T u = \zeta_C 1_V \\
\mu_R &= 0 \\
A_1 &= \mathbb{C} < x, y > \frac{[x, y]}{[x, y] - 1} \\
C_v &= \{(\text{rank}(L, X) - 1_V) \leq 1\} \\
C &= \bigsqcup_{v \geq 0} C_v \\
\Phi_p(x) &= x - p'(y) \\
\Phi_q(x) &= x + q'(x) \\
p, q &\text{ polynomials}
\end{align*}
\]

We conclude by stressing that at the moment the physical, D-brane interpretation of the Burns metric is far from being clear. The “commutative” description of the noncommutative instantons, following from the analysis of the Dirac-Born-Infeld action for the gauge theory on the D-brane in the presence of background magnetic field is usually performed in the static gauge. The singularity of the naive expressions for the gauge field may signal the invalidity of this gauge, suggesting that the topology of the worldvolume of the brane (not of the ambient flat ten dimensional space-time) is non-trivial.

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