POLES OF FINITE-DIMENSIONAL REPRESENTATIONS OF YANGIANS IN TYPE A

SACHIN GAUTAM AND CURTIS WENDLANDT

Abstract. Let $g$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, and let $Y_\hbar(g)$ be the Yangian of $g$. In this paper, we initiate the study of the set of poles of the rational currents defining the action of $Y_\hbar(g)$ on an arbitrary finite-dimensional vector space $V$. We prove that this set is completely determined by the eigenvalues of the commuting Cartan currents of $Y_\hbar(g)$, and therefore encodes the singularities of the components of the $q$-character of $V$. In type $A$, we explicitly determine the set of poles of every irreducible $V$ in terms of the roots of the underlying Drinfeld polynomials. In particular, our results yield a complete classification of the finite-dimensional irreducible representations of the Yangian double in type $A$.

Contents

1. Introduction 1
2. Yangians 4
3. Poles of finite–dimensional representations of Yangians 8
4. Representations of $DY_\hbar(g)$ 14
5. The Yangian and Yangian double of $\mathfrak{sl}_n$ 18
Appendix A. Verification of $Y_\hbar(\mathfrak{sl}_n)$ relations on $L_{\omega_m}(a)$ 24
References 26

1. Introduction

1.1. Let $g$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, and let $Y_\hbar(g)$ be the Yangian of $g$ [7] (here $\hbar \in \mathbb{C}^\times$ is fixed throughout). The category of finite–dimensional representations of $Y_\hbar(g)$, denoted by $\text{Rep}_{fd}(Y_\hbar(g))$, has a rich mathematical structure and has been extensively studied from various viewpoints (see e.g., [4, Ch. 12] and references therein; some of the important properties are recalled in §2 below). This paper is aimed at studying a discrete invariant of finite–dimensional representations of $Y_\hbar(g)$, called the poles of $V$, for every $V \in \text{Rep}_{fd}(Y_\hbar(g))$.

It is well-known that the action of $Y_\hbar(g)$ on a finite–dimensional vector space $V$ over $\mathbb{C}$ is determined by $\text{End}(V)$-valued rational functions $\{\xi_i(u), x_i^\pm(u)\}_{i \in \mathbf{I}}$ satisfying certain commutation relations (see [8, §3], or §2.6 below). Here, $\mathbf{I}$ is the set of vertices of the Dynkin diagram of $g$. Thus, to $V \in \text{Rep}_{fd}(Y_\hbar(g))$ we can associate

2020 Mathematics Subject Classification. Primary 17B37; Secondary 81R10.
a collection of finite subsets \( \{ \sigma_i(V) \}_{i \in I} \) of \( \mathbb{C} \), where, for each \( i \in I \), \( \sigma_i(V) \subset \mathbb{C} \) is the set of poles of \( \xi_i(u), x_i^\pm(u) \) (see §3.1 for details).

1.2. The set \( \sigma(V) = \bigcup_{i \in I} \sigma_i(V) \) was introduced in [8, §3.8] in order to define certain subcategories of \( \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \). This set gives constraints on some constructions in \( \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \), and appears in the description of singularities of \( R \)-matrices. More precisely, it has manifested itself in the following settings:

1. A representation \( V \in \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \) is said to be non-congruent if \( (\sigma(V) - \sigma(V)) \cap \mathbb{Z} = \{0\} \). The main construction of [8, §5.3] gives an action of the quantum loop algebra \( U_q(L\mathfrak{g}) \) on \( V \), under the assumption that \( V \) is non-congruent.

2. For \( V, W \in \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \), the Drinfeld tensor product \( V \otimes_D W \) is defined in [9, §4.5] under the assumption that \( \sigma(V) \cap \sigma(W) = \emptyset \). In fact, by [9, Thm. 4.6 (i) and (ii)], \( V(s) \otimes_D W \) is a rational family of \( Y_\hbar(\mathfrak{g}) \)-representations, with poles at \( s \in \sigma(W) - \sigma(V) \).

3. The diagonal part of the universal \( R \)-matrix of \( Y_\hbar(\mathfrak{g}) \), denoted by \( R^0(s) \), evaluated on a pair of finite-dimensional representations, \( V \) and \( W \), gives a divergent series, which is the asymptotic expansion of two different \( \text{End}(V \otimes W) \)-valued meromorphic functions \( R_{V,W}^{0,\pm}(s) \) related by a unitarity relation ([9, Thm 5.9] and [10, Thm 6.7]). Each \( R_{V,W}^{0,\pm}(s) \) can be viewed as a meromorphic commutativity constraint on \( \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \) endowed with the Drinfeld (rational) tensor product \( \otimes_D \). The set of poles of these meromorphic functions are given by:

\[
\left( \sigma(W) - \sigma(V) - \frac{\hbar}{2} \{ \ell + r \} \right) - \mathbb{Z}_{\geq 0} \ell \hbar, \\
\left( \sigma(W) - \sigma(V) - \frac{\hbar}{2} \{ \ell + r \} \right) + \mathbb{Z}_{\geq 0} \ell \hbar,
\]

where \( \ell \) is one half the eigenvalue of the Casimir element of \( \mathfrak{g} \) on the adjoint representation, and \( r \) ranges over a finite collection of integers, associated to \( \mathfrak{g} \) (see [9, Thm 5.9 (viii)]).

1.3. In this paper, we begin a systematic study of the set of poles of a finite-dimensional representation of \( Y_\hbar(\mathfrak{g}) \). To state our first main result, consider the Jordan decomposition \( \xi_i(u) = \xi_{i,S}(u) \xi_{i,U}(u) \), for each \( i \in I \) (see [8, Lemma 4.12] or §3.2 below). For a rational \( \text{End}(V) \)-valued function \( f(u) \) of a complex variable \( u \in \mathbb{C} \), let \( \sigma(f(u); V) \) be the set of poles of \( f \). Recall that \( \sigma_i(V) = \sigma(\xi_i(u); V) \cup \sigma(x_i^\pm(u); V) \). We prove the following (see Theorem 3.3):

**Theorem.** Let \( V \in \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \) and \( i \in I \). Then \( \sigma_i(V) = \sigma(\xi_{i,S}(u); V) \).

This theorem allows us to relate the sets \( \{ \sigma_i(V) \}_{i \in I} \) to the theory of \( q \)-characters [15] (see §3.7 below). It also immediately implies the following properties of \( \sigma_i(V) \), which are not directly obvious.

**Corollary.**

1. Let \( V, W \in \text{Rep}_fd(Y_\hbar(\mathfrak{g})) \). Then \( \sigma_i(V \otimes W) \subset \sigma_i(V) \cup \sigma_i(W) \), for each \( i \in I \).
(2) Let \( V \in \text{Rep}_{fd}(Y_h(\mathfrak{g})) \) and let \( \{V_1, \ldots, V_N\} \) be the simple factors of a composition series of \( V \). Then, for each \( i \in I \), we have
\[
\sigma_i(V) = \bigcup_{k=1}^{N} \sigma_i(V_k).
\]

1.4. Finite-dimensional irreducible representations of the Yangian \( Y_h(\mathfrak{g}) \) are classified by \( I \)-tuples of monic polynomials called Drinfeld polynomials \([7]\). Let \( L(P) \in \text{Rep}_{fd}(Y_h(\mathfrak{g})) \) denote the unique finite–dimensional irreducible representation with Drinfeld polynomials \( P = (P_i(u))_{i \in I} \). We propose the following problem:

**Problem.** Given an \( I \)-tuple of monic polynomials \( P \) and \( i \in I \), determine \( \sigma_i(L(P)) \) in terms of the sets of zeroes \( Z(P_j(u)) \) of the polynomials \( P_j(u) \) (for all \( j \in I \)).

Our second main result, obtained in Theorem 5.1, provides a complete solution to this problem in type \( A \). Explicitly, we prove the following theorem.

**Theorem.** Let \( n \geq 2 \) and assume that \( \mathfrak{g} = \mathfrak{sl}_n \) so that \( I = \{1, \ldots, n-1\} \). Let \( P = (P_i(u))_{i \in I} \) be a tuple of monic polynomials. Then, for each \( i \in I \), we have
\[
\sigma_i(L(P)) = \bigcup_{j \in I} Z(P_j(u)) + S_i^{(n)}(j),
\]
where \( S_i^{(n)}(j) \subset \frac{\mathbb{Z}}{2} \mathbb{Z} \) denotes the string of numbers
\[
S_i^{(n)}(j) = \hbar \left\{ \frac{i+j}{2} - k : k \in [i+j+1-n, i] \cap [1, j] \right\}.
\]

It is worth remarking that the sets \( S_i^{(n)}(j) \) have appeared in \([5, \text{Thm 6.2}] \) and \([12, \text{Thms. 5.17, 5.21}] \) in order to characterize cyclicity and irreducibility of tensor products of fundamental representations of \( Y_h(\mathfrak{sl}_n) \) (see also \([2]\) for the quantum loop case). This strongly suggests that such a characterization for arbitrary irreducible representations can be stated in terms of the poles of the individual tensor factors.

1.5. Now let \( DY_h(\mathfrak{g}) \) be the Yangian double of \( \mathfrak{g} \) (see \([14]\) and §4.1 below). As remarked in the introduction of \([14]\), finite-dimensional representations of \( DY_h(\mathfrak{g}) \) are closely related to those of \( Y_h(\mathfrak{g}) \). More precisely, we prove the following (see Proposition 4.4 below) result.

**Proposition.** \( \text{Rep}_{fd}(DY_h(\mathfrak{g})) \) is isomorphic to the subcategory \( \text{Rep}_{fd}^0(Y_h(\mathfrak{g})) \subset \text{Rep}_{fd}(Y_h(\mathfrak{g})) \) consisting of representations \( V \) such that \( 0 \notin \sigma(V) \).

Note that by Theorem 1.3, the subcategory \( \text{Rep}_{fd}^0(Y_h(\mathfrak{g})) \) is a tensor-closed, Serre subcategory (i.e., closed under taking sub/quotient objects and extensions). This allows us to define a tensor product on \( \text{Rep}_{fd}(DY_h(\mathfrak{g})) \) by transport of structure (see Corollary 4.5 below). Moreover, finite–dimensional irreducible representations of \( DY_h(\mathfrak{g}) \) are precisely those finite–dimensional irreducible representations \( V \) of \( Y_h(\mathfrak{g}) \) for which \( 0 \notin \sigma(V) \). This result is one of our motivations for proposing Problem 1.4 - i.e., obtaining \( \sigma(V) \) in terms of the zeroes of the Drinfeld polynomials of \( V \). Our solution to this problem when \( \mathfrak{g} = \mathfrak{sl}_n \) allows us to explicitly describe irreducible finite–dimensional representations of \( DY_h(\mathfrak{sl}_n) \). Namely, we obtain the following classification result in Corollary 5.1.
Corollary. The isomorphism classes of finite–dimensional irreducible representations of $\mathcal{D}Y_h(\mathfrak{sl}_n)$ are parametrized by tuples of monic polynomials $(P_i(u))_{1 \leq i \leq n-1}$ satisfying
\[ -\frac{h}{2}S_n \subset \mathbb{C} \setminus \mathbb{Z}(P_i(u)) \quad \forall i \in I, \]
where $S_n = \{0, 1, \ldots, n-2\}$.

For $n = 2$, this result is in agreement with a conjecture of Iohara [13, Conjecture 1] for the formal $h$-adic analogue of $\mathcal{D}Y_h(\mathfrak{sl}_n)$. It also clarifies how the situation diverges for all $n > 2$.

1.6. Outline. We review the basics of the representation theory of Yangians in §2. In §3, the poles of a finite–dimensional representation of $Y_h(\mathfrak{g})$ are introduced, and the main properties of these sets (Theorem 3.3) are established. In §4, we obtain the identification of the category of finite–dimensional representation of the Yangian double with the subcategory of $\text{Rep}_{fd}(Y_h(\mathfrak{g}))$ consisting of those representations for which 0 is not a pole (Proposition 4.4). The explicit determination of the set of poles in terms of the zeroes of the Drinfeld polynomials for $\mathfrak{g} = \mathfrak{sl}_n$ is given in §5 (Theorem 5.1).

1.7. Acknowledgments. The first author was supported through the Simons foundation collaboration grant 526947. The second author gratefully acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) provided via the postdoctoral fellowship (PDF) program.

2. Yangians

2.1. Let $\mathfrak{g}$ be a finite–dimensional, simple Lie algebra over $\mathbb{C}$, and let $(\cdot, \cdot)$ be the invariant bilinear form on $\mathfrak{g}$ normalized so as to have the squared length of every short root equal to 2. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\{\alpha_i\} \in \mathfrak{h}^* \subset \mathfrak{h}^*$ a basis of simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $a_{ij} = 2(\alpha_i, \alpha_j)/\langle \alpha_i, \alpha_j \rangle$ the entries of the corresponding Cartan matrix $A$. Set $d_i = (\alpha_i, \alpha_i)/\langle \alpha_i, \alpha_i \rangle \in \{1, 2, 3\}$, so that $d_i a_{ij} = d_j a_{ji}$, for every $i, j \in I$.

2.2. The Yangian $Y_h(\mathfrak{g})$. Let $h \in \mathbb{C}^\times$ be fixed throughout. The Yangian $Y_h(\mathfrak{g})$ is the unital, associative algebra over $\mathbb{C}$ generated by elements $\{\xi_{i,r}, x^\pm_{i,r}\}_{i,r} \in I, r \in \mathbb{Z}_{\geq 0}$, subject to the following relations.

(Y1) For every $i, j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we have $[\xi_{i,r}, \xi_{j,s}] = 0$.
(Y2) For every $i, j \in I$ and $s \in \mathbb{Z}_{\geq 0}$, we have $[\xi_{i,0}, x^\pm_{j,s}] = \pm d_i a_{ij} x^\pm_{j,s}$.
(Y3) For every $i, j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we have $[\xi_{i,r+1}, x^\pm_{j,s}] = [\xi_{i,r}, x^\pm_{j,s}] = \pm h \frac{d_i d_j a_{ij}}{2} (\xi_{i,r} x^\pm_{j,s} + x^\pm_{j,s} \xi_{i,r})$.
(Y4) For every $i, j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we have $[x^\pm_{i,r+1}, x^\pm_{j,s}] = [x^\pm_{i,r}, x^\pm_{j,s}] = \pm h \frac{d_i d_j a_{ij}}{2} (x^\pm_{i,r} x^\pm_{j,s} + x^\pm_{j,s} x^\pm_{i,r})$.
(Y5) For every $i, j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we have $[x^+_{i,r}, x^-_{j,s}] = \delta_{ij} \xi_{i,r+s}$. 

Let $i \neq j \in I$. Set $m = 1 - a_{ij}$. Then, for every $r_1, \ldots, r_m, s \in \mathbb{Z}_{\geq 0}$, we have:

$$\sum_{\pi \in \mathcal{S}_m} \left[ x_{i,r_1}^{\pm} \big| [x_{i,r_2}^{\pm}, \ldots, [x_{i,r_m}^{\pm}, x_{j,s}^{\pm}] \ldots] \right] = 0.$$ 

2.3. Remark. The relation (Y6) follows from (Y1)–(Y3) and the special case of (Y6) when $r_1 = \ldots = r_m = 0$ (see [16, Lemma 1.9]). The latter automatically holds on finite–dimensional representations of the algebra defined by relations (Y2) and (Y5) alone (see [8, Prop. 2.7]).

2.4. The inclusion $U(g) \subset Y_h(g)$. Let $\nu : h \to h^*$ be the isomorphism determined by $(\cdot, \cdot)$. For each fixed $i \in I$, set $h_i = \nu^{-1}(\alpha_i)/d_i$ and choose root vectors $x_i^{\pm} \in g_{\pm\alpha_i}$ such that $[x_i^+, x_i^-] = d_i h_i$. Then the defining relations (Y1)–(Y6) of $Y_h(g)$ imply that the assignment

$$x_i^{\pm} \mapsto x_{i,0}^{\pm}, \quad \text{and} \quad d_i h_i \mapsto \xi_{i,0} \quad \forall i \in I$$

extends to an algebra homomorphism $U(g) \to Y_h(g)$. It is a well-known consequence of the Poincaré–Birkhoff–Witt theorem for $Y_h(g)$ that this is an embedding (see [10, §2.7], for instance) and we shall freely use this fact and view $U(g) \subset Y_h(g)$.

2.5. Rational currents. For each $i \in I$, define $\xi_i(u), x_i^{\pm}(u) \in Y_h(g)[u^{-1}]$ by

$$\xi_i(u) = 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x_i^{\pm}(u) = \hbar \sum_{r \geq 0} x_{i,r}^{\pm} u^{-r-1}.$$ 

The following rationality property was obtained in [8, Prop. 3.6].

Proposition. Let $V$ be a $Y_h(g)$-module on which $h$ acts semisimply with finite–dimensional weight spaces. Then, for every weight $\mu \in h^*$ of $V$, the generating series

$$\xi_i(u) \in \text{End}(V_{\mu})[u^{-1}] \quad \text{and} \quad x_i^{\pm}(u) \in \text{Hom}(V_{\mu}, V_{\mu \pm \alpha_i})[u^{-1}]$$

are the expansions at $\infty$ of rational functions of $u$.

Explicitly, set $t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} \xi_{i,0}^2 \in Y_h(g)^b$. Then the relations

$$x_i^{\pm}(u) = \hbar \left( u \mp \frac{1}{2d_i} \text{ad}(t_{i,1}) \right)^{-1} \xi_{i,0}^{\pm} \quad \text{and} \quad \xi_i(u) = 1 + [x_i^+(u), x_i^-(u)]$$

determine $x_i^{\pm}(u)$ and $\xi_i(u)$ as the expansions of operator valued rational functions on each weight space of $V$.

2.6. Finite-dimensional representations. A finite–dimensional representation $V$ of $Y_h(g)$ is thus completely determined by rational, $\text{End}(V)$-valued functions $\{\xi_i(u), x_i^{\pm}(u)\}_{i \in I}$ satisfying $\xi_i(\infty) = \text{Id}_V$ and $x_i^{\pm}(\infty) = 0$. These rational functions are subject to the following relations (see [8, Prop. 2.3]).

(Y1) For every $i, j \in I$, $[\xi_i(u), \xi_j(v)] = 0$.

(Y2) For every $i, j \in I$, let $a = h d_i a_{ij}/2$. Then we have:

$$(u - v \mp a)\xi_j(u)x_i^{\pm}(v) = (u - v \pm a)x_j^{\pm}(v)\xi_i(u) \mp 2ax_j^{\pm}(u \mp a)\xi_i(u).$$
(3) For every $i, j \in I$, let $a = h d_i a_{ij}/2$. Then we have:

\[
(u - v \mp a)x_i^\pm(u)x_j^\pm(v) = (u - v \pm a)x_j^\pm(v)x_i^\pm(u) + \hbar \left( [x_i^\pm(u), x_j^\pm(v)] - [x_i, x_j^\pm] \right).
\]

(4) For every $i, j \in I$:

\[
[x_i^+(u), x_j^-(v)] = \delta_{ij} \frac{\hbar}{u - v} (\xi_i(v) - \xi_i(u)).
\]

2.7. **Shift automorphism.** The group of translations of the complex plane acts on $Y_h(g)$ by

\[
\tau_a(y_r) = \sum_{s=0}^{r} \left( \begin{array}{c} r \\ s \end{array} \right) a^{r-s} y_s,
\]

where $a \in \mathbb{C}$, $r \in \mathbb{Z}_{\geq 0}$ and $y$ is one of $\xi_i, x_i^\pm$. In terms of generating series, one has

\[
\tau_a(y(u)) = y(u - a).
\]

Given a representation $V$ of $Y_h(g)$ and $a \in \mathbb{C}$, we set $V(a) := \tau_a(V)$.

2.8. **Diagram automorphisms.** Let $\omega$ be an automorphism of the Dynkin diagram of $\mathfrak{g}$. That is, $\omega \in \mathfrak{S}_I$ such that $a_{ij} = a_{\omega(i), \omega(j)}$ for every $i, j \in I$. Then $\omega$ defines an automorphism of $Y_h(g)$:

\[
\xi_i(u) \mapsto \xi_{\omega(i)}(u) \quad \text{and} \quad x_i^+(u) \mapsto x_{\omega(i)}^+(u),
\]

for every $i \in I$.

For a representation $V$ of $Y_h(g)$, we denote the pull-back $\omega^*(V)$ by $V^\omega$.

2.9. **Cartan involution.** By [5, Prop. 2.4], the Cartan involution $x_i^\pm \mapsto x_i^\mp$ of $\mathfrak{g}$ extends to an involutive algebra automorphism $\varphi$ of $Y_h(g)$, determined by

\[
\varphi(\xi_i(u)) = \xi_i(-u) \quad \text{and} \quad \varphi(x_i^+(u)) = -x_i^-(u)
\]

for all $i \in I$. For a representation $V$ of $Y_h(g)$, we set $V^\varphi = \varphi^*(V)$.

Similarly, by [3, Prop. 2.9], the standard Cartan anti-involution of $\mathfrak{g}$ extends to an anti-involution $\theta$ of $Y_h(g)$, determined on $\xi_i(u)$ and $x_i^\pm(u)$, for each $i \in I$, by

\[
\theta(\xi_i(u)) = \xi_i(u) \quad \text{and} \quad \theta(x_i^+(u)) = x_i^-(u).
\]

2.10. **Cooprocess.** The Yangian $Y_h(g)$ is known to be a Hopf algebra with co-product $\Delta : Y_h(g) \to Y_h(g) \otimes Y_h(g)$ defined as follows. Recall from §2.5 that $t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} \xi_{i,0}^2$ for each $i \in I$. Then, the set of elements $\{\xi_{i,0}, x_{i,0}^+, t_{i,1}\}_{i \in I}$ generates $Y_h(g)$ as an algebra, and $\Delta$ is uniquely determined by the formulae

\[
\Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0, \quad \text{where} \quad y = \xi_i, x_i^\pm,
\]

\[
\Delta(t_{i,1}) = t_{i,1} \otimes 1 + 1 \otimes t_{i,1} - \hbar \sum_{\alpha \in R_+} (\alpha_i, \alpha) x_\alpha^- \otimes x_\alpha^+,
\]

for all $i \in I$, were $R_+$ is the set of positive roots of $\mathfrak{g}$ and $x_\alpha^\pm \in \mathfrak{g}_{\pm \alpha} \subset Y_h(g)$ are chosen so as to have $(x_\alpha^+, x_\alpha^-) = 1$. We refer the reader to [11, §4.2–4.5] for a proof that these formulæ indeed define a coassociative algebra homomorphism.

An explicit formula for $\Delta(\xi_i(u))$ is not known in general. However, for our purposes it would suffice to know that $\{\xi_i(u)\}_{i \in I}$ are all group-like modulo certain strictly triangular terms. To state this precisely, let $Y_{\leq 0}$ (resp. $Y_{\geq 0}$) be the
subalgebra of \( Y_h(\mathfrak{g}) \) generated by \( \{ \xi_{i,r}, x_i^+, x_i^- \} \in I, r \in \mathbb{Z}_{\geq 0} \) (resp. \( \{ \xi_{i,r}, x_i^+ \} \in I, r \in \mathbb{Z}_{\geq 0} \)). These algebras are graded by \( Q_+ \) (the positive cone in the root lattice of \( \mathfrak{g} \)). Then, we have the following (see, e.g., [3, Prop. 2.8]).

**Proposition.** For each \( i \in I \), \( \Delta(\xi_i(u)) \) satisfies

\[
\Delta(\xi_i(u)) = \xi_i(u) \otimes \xi_i(u) + \left( \bigoplus_{\beta > 0} Y_{\beta}^{\leq 0} \otimes Y_{\beta}^{\geq 0} \right) [u^{-1}].
\]

2.11. **Drinfeld polynomials.** A remarkable result established in the foundational work [7] of Drinfeld asserts that the finite-dimensional irreducible representations of \( Y_h(\mathfrak{g}) \) are classified by \( I \)-tuples of monic polynomials. More precisely, one has the following theorem.

**Theorem.** Let \( V \) be a finite-dimensional irreducible representation of \( Y_h(\mathfrak{g}) \). Then, there exists a unique (up to scalar) non-zero vector \( \Omega \in V \), and an \( I \)-tuple of monic polynomials \( (P_i(u))_{i \in I} \in \mathbb{C}[u]^I \) such that:

1. \( V \) is generated as a \( Y_h(\mathfrak{g}) \)-module by \( \Omega \).
2. For each \( i \in I \), one has

\[
x_i^+(u)\Omega = 0 \quad \text{and} \quad \xi_i(u)\Omega = \frac{P_i(u + d_i h)}{P_i(u)}\Omega.
\]

Conversely, given an \( I \)-tuple of monic polynomials \( (P_i(u))_{i \in I} \), there is a unique (up to isomorphism) finite-dimensional irreducible representation of \( Y_h(\mathfrak{g}) \) containing a non-zero vector \( \Omega \) which satisfies the properties listed above.

Given an \( I \)-tuple \( \underline{P} = (P_i(u))_{i \in I} \) of monic polynomials, the corresponding finite-dimensional irreducible representation of \( Y_h(\mathfrak{g}) \) will be denoted by \( L(\underline{P}) \). The polynomials \( (P_i(u)) \) are called the Drinfeld polynomials associated to \( L(\underline{P}) \).

2.12. **Restriction.** Let \( J \subset I \) be a non-empty set. Let \( Y_h(\mathfrak{g}_J) \) be the subalgebra of \( Y_h(\mathfrak{g}) \) generated by \( \{ \xi_{i,r}, x_i^+ \} \in J, r \in \mathbb{Z}_{\geq 0} \). The following lemma is from [3, Lemma 4.3].

**Lemma.** Let \( V \) be a finite-dimensional irreducible representation of \( Y_h(\mathfrak{g}) \) and \( \Omega \in V \) be the highest-weight vector. Then, the \( Y_h(\mathfrak{g}_J) \)-module generated by \( \Omega \) is an irreducible representation of \( Y_h(\mathfrak{g}_J) \).

2.13. Consider the involution \( i \mapsto i^* \) of the Dynkin diagram of \( \mathfrak{g} \), induced by the longest element of the Weyl group: \( \alpha_i^* = -w_0(\alpha_i) \). Let \( \kappa \) be \( (1/4) \) times the eigenvalue of the Casimir element \( C \in U(\mathfrak{g}) \) on the adjoint representation. Explicitly, let \( \theta \) be the longest root of \( \mathfrak{g} \) and \( h^\vee \) be half the sum of positive roots of \( \mathfrak{g} \). Then:

\[
\kappa = \frac{1}{4} (\theta, \theta + 2\rho) = \frac{1}{2} (\theta, \theta) (1 + \rho(\theta^\vee)) = \frac{1}{2} m h^\vee,
\]

where \( m = 1, 2, 3 \) if \( \mathfrak{g} \) is of type \( \text{ADE} \), \( \text{BCF} \), \( \text{G} \) respectively, and \( h^\vee = 1 + \rho(\theta^\vee) \) is the dual Coxeter number of \( \mathfrak{g} \). The following table lists the value of \( \kappa \) in each type, where we follow Bourbaki’s convention for the labels of Dynkin diagrams [1].
The following result was obtained in [5, Prop. 3.5]. Recall that \( \varphi \) is the algebra automorphism of \( Y_h(\mathfrak{g}) \) defined in \( \S 2.9 \) above, and \( V^\varphi = \varphi^*(V) \) for any representation \( V \) of \( Y_h(\mathfrak{g}) \).

**Proposition.** Let \( L(P) \) be the finite-dimensional irreducible representation of \( Y_h(\mathfrak{g}) \), with Drinfeld polynomials \( P = (P_i(u))_{i \in I} \). Then, \( L(P)\varphi = L(P^\varphi) \) where

\[
P_i^\varphi(u) = (-1)^{\text{deg}(P_i)} P_i(-u + d_i h - \kappa h).
\]

### 3. Poles of finite–dimensional representations of Yangians

Let \( V \) be a finite–dimensional representation of \( Y_h(\mathfrak{g}) \). In this section we study the set of poles of the rational functions \( \{\xi_i(u), x_i^\pm(u)\}_{i \in I} \subset \text{End}(V)(u) \).

#### 3.1. Poles

For each \( i \in I \), let \( \sigma(\xi_i(u); V), \sigma(x_i^\pm(u); V) \subset \mathbb{C} \) be the finite set of poles of the rational functions \( \xi_i(u), x_i^\pm(u) \in \text{End}(V)(u) \). For \( z \in \mathbb{C} \), let \( \text{ord}_z(\xi_i(u); V) \) (resp. \( \text{ord}_z(x_i^\pm(u); V) \)) be the order of the pole of \( \xi_i(u) \) (resp. \( x_i^\pm(u) \)) at \( u = z \). By convention, we set \( \text{ord}_z(\xi_i(u); V) \) (resp. \( \text{ord}_z(x_i^\pm(u); V) \)) to be 0 if \( z \) is not a pole of the underlying function.

**Proposition.** For every \( i \in I \), we have

\[
\sigma(\xi_i(u); V) = \sigma(x_i^\pm(u); V) =: \sigma_i(V).
\]

Moreover, \( \text{ord}_z(\xi_i(u); V) = \text{ord}_z(x_i^\pm(u); V) \), for every \( z \in \mathbb{C} \).

**Proof.** This is clearly a statement about finite–dimensional representations of \( Y_h(\mathfrak{sl}_2) \). To simplify notation, we will drop the subscript \( i \). Using the relation \( \xi(u) = 1 + [x^+(u), x_0^-] = 1 + [x_0^+, x^-(u)] \), we conclude that

\[
\sigma(\xi(u); V) \subset \sigma(x^\pm(u); V) \quad \text{and} \quad \text{ord}_z(\xi(u); V) \leq \text{ord}_z(x^\pm(u); V)
\]

for every \( z \in \mathbb{C} \). To obtain the converse, let us assume that \( z \in \mathbb{C} \) is a pole of \( x^+(u) \). Let \( N = \text{ord}_z(x^+(u); V) \) and \( X := \lim_{u \to z} (u - z)^N x^+(u) \). Assume, for the sake of a contradiction, that \( \text{ord}_z(\xi(u); V) < N \).

Multiplying both sides of the relation \( \xi(u) = 1 + [x^+(u), x_0^-] \) by \( (u - z)^N \) and letting \( u \to z \), we get

\[
[X, x_0^-] = 0.
\]

We regard \( \text{End}(V) \) as a representation of \( \mathfrak{sl}_2 \) via:

\[
h \mapsto \text{ad}(\xi_0), \quad e \mapsto \text{ad}(x_0^+), \quad f \mapsto \text{ad}(x_0^-).
\]

Then \( X \in \text{End}(V) \) is a non-zero lowest-weight vector of weight +2 in the finite–dimensional \( \mathfrak{sl}_2 \)-representation \( \text{End}(V) \). This contradicts the finite-dimensionality of \( \text{End}(V) \).

\( \square \)
3.2. Jordan decomposition for commuting currents. Again, let \( V \) be a finite-dimensional representation of \( Y_\hbar(\mathfrak{g}) \). By [8, Lemma 4.12], we obtain the multiplicative Jordan decomposition of commuting elements \( \{\xi_i(u)\}_{i \in I} \) (see also [15]). We state and prove this result below, for the sake of completeness.

**Lemma.** For each \( i \in I \), the semisimple and unipotent parts of the multiplicative Jordan decomposition \( \xi_i(u) = \xi_{i,S}(u)\xi_{i,U}(u) \) are rational functions of \( u \). Moreover, \( V \) admits a basis with respect to which each \( \xi_i(u) \) is triangular.

**Proof.** Let \( a \subset \text{End}(V) \) be the abelian Lie subalgebra generated by \( \{\xi_{i,r} : i \in I, r \in \mathbb{Z}_{\geq 0}\} \). Since \( a \) is a finite-dimensional abelian Lie algebra, we have the direct sum decomposition \( V = \bigoplus_{\gamma \in a^*} V[\gamma] \) of \( V \) into generalized eigenspaces

\[
V[\gamma] = \{v \in V : (a - \gamma(a))^N v = 0, \ \forall a \in a, \ N \gg 0\}.
\]

An elementary result from linear algebra states that \( V[\gamma] \) admits an ordered basis in which every \( a \in a \) is triangular. This proves the second statement of the lemma.

Now let \( \mathbf{1}_\gamma : V \to V[\gamma] \) be the projection operator and, for each \( i \in I \), define

\[
\xi_{i,S}(u) = \sum_{\gamma \in a^*} \mathbf{1}_\gamma \circ \gamma(\xi_i(u)) \quad \text{and} \quad \xi_{i,U}(u) = \xi_{i,S}(u)^{-1}\xi_i(u).
\]

By uniqueness, this recovers the multiplicative Jordan decomposition of \( \xi_i(u) \), for each \( i \in I \). Moreover, it shows that the entries of the semisimple and unipotent parts, \( \xi_{i,S}(u) \) and \( \xi_{i,U}(u) \), are again rational functions of \( u \).

\[
\square
\]

3.3. Now we can state the main result of this section. If \( V \) is a finite-dimensional representation of \( Y_\hbar(\mathfrak{g}) \), let \( \sigma(\xi_{i,S}(u); V) \subset \mathbb{C} \) be the set of poles of the diagonal part of \( \xi_i(u) \). Recall the definition of \( \sigma_i(V) \) from Proposition 3.1.

**Theorem.** For each \( i \in I \), we have \( \sigma_i(V) = \sigma(\xi_{i,S}(u); V) \).

**Proof.** Note that it is sufficient to prove this statement for \( \mathfrak{g} = \mathfrak{sl}_2 \). Thus, in the remainder of this section we restrict our attention to \( Y_\hbar(\mathfrak{sl}_2) \).

We prove the theorem by induction on the length of a composition series of \( V \). The base case, i.e., when \( V \) is irreducible, is checked in §3.5. In §3.4 we carry out the induction step.

\[
\square
\]

3.4. Extensions. Let \( V_1, V_2 \) and \( V_3 \) be three finite-dimensional representations of \( Y_\hbar(\mathfrak{sl}_2) \). Assume that \( \sigma(V_\ell) = \sigma(\xi_{S}(u); V_\ell) \) for \( \ell = 1, 2 \), and that we have a short exact sequence of \( Y_\hbar(\mathfrak{sl}_2) \) representations:

\[
0 \to V_1 \to V_3 \to V_2 \to 0.
\]

**Proposition.** Under the hypotheses stated above, we have

\[
\sigma(V_3) = \sigma(\xi_{S}(u); V_3) = \sigma(\xi_{S}(u); V_1) \cup \sigma(\xi_{S}(u); V_2) = \sigma(V_1) \cup \sigma(V_2).
\]
Proof. Note that the equality $\sigma(\xi_S(u); V_3) = \sigma(\xi_S(u); V_1) \cup \sigma(\xi_S(u); V_2)$ is obvious, since $\xi_S(u)$ is semisimple and

$$\xi_S(u)|_{V_3} = \xi_S(u)|_{V_1} \oplus \xi_S(u)|_{V_2}.$$ 

The last equality $\sigma(\xi_S(u); V_1) \cup \sigma(\xi_S; V_2) = \sigma(V_1) \cup \sigma(V_2)$ holds by assumption. Moreover, since $V_3$ is an extension of $V_2$ by $V_1$, it is clear that $\sigma(V_1) \cup \sigma(V_2) \subset \sigma(V_3)$.

Thus, using Proposition 3.1, to prove $\sigma(V_3) = \sigma(\xi_S(u); V_3)$, it enough to show that $\sigma(x^+(u); V_3) \subset \sigma(V_1) \cup \sigma(V_2)$.

For the purposes of this proof, we write $V_3 = V_1 \oplus V_2$ as a $\mathfrak{sl}_2$–representation. Every element $y \in Y_h(\mathfrak{sl}_2)$ can be written in the block form

$$y = \begin{bmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{bmatrix},$$

where $y_{ij} : V_j \to V_i$, for each $i, j \in \{1, 2\}$.

Now suppose, towards a contradiction, that there is $z \in \sigma(x^+(u); V_3)$ such that $z \not\in \sigma(V_1) \cup \sigma(V_2)$. Let $N = \text{ord}_{\xi}(x^+(u); V_3) = \text{ord}_{\xi}(\xi(u); V_3)$ (see Proposition 3.1). Define the following elements of $\text{End}(V_3)$:

$$X^+ = \lim_{u \to z} (u - z)^N x^+(u) \quad \text{and} \quad H = \lim_{u \to z} (u - z)^N \xi(u).$$

Note that, since $z \not\in \sigma(V_1) \cup \sigma(V_2)$, the diagonal blocks of $X^+$ and $H$ are zero. We consider the $(1, 2)$ component of the relation (34):

$$x^+(u)_{11} x^-(v)_{12} + x^+(u)_{12} x^-(v)_{22} - x^-(v)_{11} x^+(u)_{12} - x^-(v)_{12} x^+(u)_{22} = \frac{\hbar}{u - v} (\xi(v)_{12} - \xi(u)_{12}).$$

Multiplying both sides by $(u - z)^N$ and letting $u \to z$, we obtain the identity

$$X^+ x^-(v)_{22} - x^-(v)_{11} X^+ = \frac{H}{v - z}.$$ 

Since the left–hand side of this equation is regular at $v = z$, so must be the right–hand side, which proves that $H = 0$. This is a contradiction to Proposition 3.1. □

3.5. Finite-dimensional irreducible representations of $Y_h(\mathfrak{sl}_2)$. Recall, from Theorem 2.11, that finite–dimensional irreducible representations of $Y_h(\mathfrak{sl}_2)$ are labeled by monic polynomials $P(u) \in \mathbb{C}[u]$. We denote by $L(P)$ the finite–dimensional irreducible representation with Drinfeld polynomial $P(u)$. Let $Z(P(u)) \subset \mathbb{C}$ denote the set of roots of $P(u)$.

Theorem. Let $P(u) \in \mathbb{C}[u]$ be a monic polynomial. Then the finite–dimensional irreducible representation $L(P)$ of $Y_h(\mathfrak{sl}_2)$ satisfies

$$\sigma(L(P)) = \sigma(\xi_S(u); L(P)) = Z(P(u)).$$
PROOF. We begin by considering the evaluation representations of $Y_h(\mathfrak{sl}_2)$. Let $a \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 0}$. Let $L_r(a)$ be the $r + 1$ dimensional representation of $Y_h(\mathfrak{sl}_2)$ with basis $\{v_0, \ldots, v_r\}$ and $Y_h(\mathfrak{sl}_2)$-action given by

\begin{align}
(3.1) \quad \xi(u)v_i &= \frac{(u - a - \hbar)(u - a + r\hbar)}{(u - a + (i - 1)\hbar)(u - a + i\hbar)}v_i, \\
(3.2) \quad x^+(u)v_i &= \frac{(r - i + 1)\hbar}{u - a + (i - 1)\hbar}v_{i-1}, \\
(3.3) \quad x^-(u)v_i &= \frac{(i + 1)\hbar}{u - a + i\hbar}v_{i+1},
\end{align}

where it is understood that $v_{-1} = v_{r+1} = 0$. It is clear by direct inspection that $L_r(a)$ is an irreducible representation, with Drinfeld polynomial

$$P_{r,a}(u) = \prod_{i=0}^{r-1} (u - a + i\hbar),$$

where $P_{0,a}(u) = 1$. For this representation, one has $\xi(u) = \xi_S(u)$ and

$$\sigma(L_r(a)) = \sigma(\xi_S(u); L_r(a)) = \mathbb{Z}(P_{r,a}(u)) = \{a - i\hbar : 0 \leq i \leq r - 1\} =: S_r(a).$$

In general, we will make use of the following result (see [6, Thm. 3.13] or [17, Prop. 3.3.2]). Given a monic polynomial $P(u) \in \mathbb{C}[u]$, there exists $a_1, \ldots, a_p \in \mathbb{C}$ and $r_1, \ldots, r_p \in \mathbb{Z}_{\geq 0}$ such that

$$L(P) = \bigotimes_{k=1}^{p} L_{r_k}(a_k) \quad \text{and} \quad P(u) = \prod_{k=1}^{p} P_{r_k,a_k}(u),$$

where the second condition is clearly a consequence of the first. Thus, to prove the theorem it suffices to verify the following two claims.

Claim 1. For any two finite–dimensional representations $V$ and $W$ of $Y_h(\mathfrak{sl}_2)$,

$$\sigma(V \otimes W) \subset \sigma(V) \cup \sigma(W).$$

Claim 2. For any $a_1, \ldots, a_p \in \mathbb{C}$ and $r_1, \ldots, r_p \in \mathbb{Z}_{\geq 0}$ we have

$$\bigcup_{k=1}^{p} S_{r_k}(a_k) \subset \sigma\left(\bigotimes_{k=1}^{p} L_{r_k}(a_k) \right).$$

Proof of Claim 1. We already know that $\sigma(V \otimes W) = \sigma(\xi(u); V \otimes W)$ by Proposition 3.1. Thus, we can prove our claim using the explicit formula

$$\Delta(\xi(u)) = \sum_{k \geq 0} (-1)^k(k + 1)x^-(u + \hbar)^k \xi(u) \otimes \xi(u)x^+(u + \hbar)^k$$

which may be found in [17, §3.5]. By Proposition 3.1, this formula will imply the claim provided the poles of the operators $x^-(u + \hbar)^k \xi(u)$ and $\xi(u)x^+(u + \hbar)^k$ are contained in $\sigma(\xi(u); V)$ and $\sigma(\xi(u); W)$, respectively. This is a consequence of the commutation relations

\begin{align}
(3.6) \quad \xi(u)x^+(u + \hbar)^k &= \frac{(-1)^k}{(k + 1)!} \text{ad}(x^+_0)^k \cdot \xi(u), \\
(3.7) \quad x^-(u + \hbar)^k \xi(u) &= \frac{1}{(k + 1)!} \text{ad}(x^-_0)^k \cdot \xi(u).
\end{align}
Note that the second relation can be recovered from the first by applying the Cartan anti-involution \( \theta \) from §2.9. To prove the first identity (3.6), note that the relation (\( \mathcal{Y}^2 \)) of §2.6 can be written as
\[
(u - v - \hbar)\xi(u)x^+(v) - (u - v + \hbar)x^+(v)\xi(u) \quad \text{is independent of } v.
\]
Setting \( v = u + \hbar \) gives us \(-2\hbar\xi(u)x^+(u + \hbar)\), while letting \( v \to \infty \) gives \( \hbar[x_0^+, \xi(u)] \).
Thus, we obtain
\[
(3.8) \quad \text{ad}(x_0^+) \cdot \xi(u) = -2\xi(u)x^+(u + \hbar)
\]
Now set \( u = v \) in the relation (\( \mathcal{Y}^3 \)) of §2.6 to obtain
\[
(3.9) \quad \text{ad}(x_0^+) \cdot x^+(u) = -x^+(u)^2.
\]
Together with the Leibniz rule, this yields the following equation, for every \( k \in \mathbb{Z}_{\geq 0} \):
\[
ad(x_0^+) \cdot x^+(u) = \hbar x^+(u)^{k+1}.
\]
Combining equations (3.8) and (3.9), we can prove (3.6) by induction on \( k \). For \( k = 0 \), there is nothing to prove. Assume (3.6) for \( k \in \mathbb{Z}_{\geq 0} \) and apply \( \text{ad}(x_0^+) \) on both of its sides to get:
\[
\text{ad}(x_0^+) \cdot x^+(u) = \hbar x^+(u)^{k+1}.
\]
This finishes the proof of Claim 1.

Proof of Claim 2. We will prove this claim by induction on \( p \), after first introducing some auxiliary notation. Recall that we are given \( a_1, \ldots, a_p \in \mathbb{C} \) and \( r_1, \ldots, r_p \in \mathbb{Z}_{\geq 0} \). We shall set
\[
W_{a_1, \ldots, a_p} := \bigotimes_{k=1}^p L_{r_k}(a_k).
\]
The basis of \( L_{r_k}(a_k) \) fixed in the beginning of the proof will now be denoted by \( \{v_{s}^{(k)}\} \). The module \( W_{a_1, \ldots, a_p} \) then has basis given by
\[
\{v_s : s = (s_1, \ldots, s_p) \in \mathbb{Z}_{\geq 0}^p \text{ such that } 0 \leq s_k \leq r_k, \forall 1 \leq k \leq p\},
\]
where \( v_s = v_{s_1}^{(1)} \otimes \cdots \otimes v_{s_p}^{(p)} \). Note that this is an ordered basis with respect to the standard lexicographic ordering. Moreover, equation (3.5) implies that the action of \( \xi(u) \) on \( W_{a_1, \ldots, a_p} \) in this basis is triangular, with diagonal entries given by (see (3.1) above):
\[
\xi_S(u)v_s = \lambda_{s_1, \ldots, s_p}(u)v_s
\]
where \( \lambda_{s_1, \ldots, s_p}(u) \) is the rational function
\[
\lambda_{s_1, \ldots, s_p}(u) = \prod_{k=1}^p \frac{(u - a_k - \hbar)(u - a_k + r_k\hbar)}{(u - a_k + (s_k - 1)\hbar)(u - a_k + s_k\hbar)}.
\]
We are now prepared to prove Claim 2 by induction on \( p \). The base case \( p = 1 \) is a consequence of equation (3.4) above. We assume that Claim 2 holds for some
fixed $p \geq 1$, and prove it for $p + 1$. This amounts to showing that every element of $z \in \bigcup_{k=1}^{p+1} S_{r_k}(a_k)$ is a pole of some $\lambda_{s_1,\ldots,s_{p+1}}(u)$. To this end, note that

$$\lambda_{s_1,\ldots,s_{p+1}}(u) = \frac{(u - a_{p+1} - h)(u - a_{p+1} + r_{p+1}h)}{(u - a_{p+1} + (s_{p+1} - 1)h)(u - a_{p+1} + s_{p+1}h)}.$$  

(3.10)

Let us first assume that $z$ belongs to the subset $\bigcup_{k=1}^p S_{r_k}(a_k)$. Then, by the induction hypothesis, $z$ is a pole of $\lambda_{s_1,\ldots,s_p}(u)$ for some $s = (s_1, \ldots, s_p)$. In light of the decomposition (3.10), there are three cases to consider:

- If $z \notin \{a_{p+1} + h, a_{p+1} - r_{p+1}h\}$, then $z$ is a pole of $\lambda_{s_1,\ldots,s_p,0}(u)$ for any $0 \leq s_{p+1} \leq r_{p+1}$.
- If $z = a_{p+1} + h$, then $z$ is a pole of $\lambda_{s_1,\ldots,s_p,0}(u)$.
- If $z = a_{p+1} - r_{p+1}h$, then $z$ is a pole of $\lambda_{s_1,\ldots,s_p,r_{p+1}}(u)$.

We are thus left to consider the case when $z \in S_{r_{p+1}}(a_{p+1}) \setminus \bigcup_{k=1}^p S_{r_k}(a_k)$. So, $z = a_{p+1} - th$ for some $0 \leq t \leq r_{p+1} - 1$. For each $1 \leq k \leq p$, set

$$s_k = \left\{ \begin{array}{ll} r_k & \text{if } z = a_k - r_kh \\ 0 & \text{otherwise} \end{array} \right.$$  

With this choice, it is clear that $z$ is not a zero of $\lambda_{s_1,\ldots,s_p}(u)$. Hence it is a pole of $\lambda_{s_1,\ldots,s_p,t}(u)$. This finishes the proof of Claim 2, and the theorem follows.  

3.6. Consequences of Theorem 3.3. The following properties of $\sigma_i(V)$ follow directly from Proposition 2.10, Theorem 3.3 and Proposition 3.4. 

Corollary.

1. For any two finite–dimensional representations $V, W$ of $Y_h(g)$, and $i \in \mathbf{I}$, we have

$$\sigma_i(V \otimes W) \subset \sigma_i(V) \cup \sigma_i(W).$$

2. Let $V$ be a finite–dimensional representation of $Y_h(g)$. Let $\{V_1, \ldots, V_N\}$ be the simple factors in a composition series of $V$. Then, for every $i \in \mathbf{I}$, we have

$$\sigma_i(V) = \bigcup_{k=1}^N \sigma_i(V_k).$$

3.7. Relation with characters. Let $\mathcal{L}$ denote the multiplicative subgroup $1 + u^{-1} \mathbb{C}[u^{-1}] \subset \mathbb{C}[u^{-1}]$. Given $V \in \text{Rep}_{fd}(Y_h(g))$ and $\mu = (\mu_i(u))_{i \in \mathbf{I}} \in \mathcal{L}^\mathbf{I}$, with

$$\mu_i(u) = 1 + h \sum_{r \geq 0} \mu_i,r u^{-r-1},$$

we define the generalized eigenspace $V[\mu]$ of $V$ by

$$V[\mu] = \{ v \in V : \forall i \in \mathbf{I}, k \geq 0, \exists N_k > 0 \text{ such that } (\xi_{i,k} - \mu_{i,k})^{N_k} v = 0 \}.$$  

Let $\mathbb{C}[\mathcal{L}^\mathbf{I}]$ be the group algebra of the direct product $\mathcal{L}^\mathbf{I}$ of $\mathbf{I}$-copies of $\mathcal{L}$. It is a $\mathbb{C}$-vector space with basis consisting of all formal exponentials $e(\mu)$, where $\mu \in \mathcal{L}^\mathbf{I}$, and multiplication defined on basis vectors by the rule

$$e(\mu) \cdot e(\nu) = e((\mu_i(u)\nu_i(u))_{i \in \mathbf{I}}).$$
**Definition.** [15] The $q$-character of $V$, denoted by $\chi_q(V)$, is defined as:

$$\chi_q(V) = \sum_{\mu \in \mathcal{L}^\times} \dim(V[\mu]) \ e(\mu) \in \mathbb{C}[\mathcal{L}].$$

By the rationality property (see Proposition 2.5 above) if $V[\mu] \neq 0$, then for every $i \in I$, $\mu_i(u)$ is the Taylor series expansion of a rational function of $u$, taking value 1 at $u = \infty$. Let $\sigma(\mu_i(u)) \subset \mathbb{C}$ be the finite set of poles of $\mu_i(u)$. Therefore, we can define the set of poles of $\chi_q(V)$ as:

$$\sigma(\chi_q(V)) = \bigcup_{\mu \in V[\mu] \neq 0} \left( \bigcup_{i \in I} \sigma(\mu_i(u)) \right) \subset \mathbb{C}.$$

Theorem 3.3 directly implies the following:

**Corollary.** For every $V \in \text{Rep}_{fd}(Y_h(\mathfrak{g}))$, we have

$$\sigma(\chi_q(V)) = \bigcup_{i \in I} \sigma(V) =: \sigma(V).$$

4. **Representations of $DY_h(\mathfrak{g})$**

4.1. **The Yangian double** $DY_h(\mathfrak{g})$. The Yangian double $DY_h(\mathfrak{g})$ is defined to be the unital associative algebra over $\mathbb{C}$, generated by $\{\xi_{i,r}, x^{\pm}_{i,r}\}_{i \in I, r \in \mathbb{Z}}$, subject to the relations (Y1)–(Y6) of $Y_h(\mathfrak{g})$ with the second index of each generator taking values in $\mathbb{Z}$.

This definition is such that the assignment $x^\pm_{i,r} \mapsto x^\pm_{i,r}$ and $\xi_{i,r} \mapsto \xi_{i,r}$, for each $i \in I$ and $r \in \mathbb{Z}_{\geq 0}$, extends to an algebra homomorphism

$$\iota : Y_h(\mathfrak{g}) \to DY_h(\mathfrak{g}).$$

As a consequence of Proposition 4.3 below, this homomorphism is injective, as suggested by our notation for generators of $DY_h(\mathfrak{g})$. The defining relations of $DY_h(\mathfrak{g})$ can be expressed equivalently in terms of generating series as follows. For each $i \in I$, introduce $\xi^\pm_i(u) \in DY_h(\mathfrak{g})[u^{\pm 1}]$ and $\chi^\pm_i(u) \in DY_h(\mathfrak{g})[u, u^{-1}]$ by

$$\xi^+_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1}, \quad \xi^-_i(u) = 1 - h \sum_{r < 0} \xi_{i,r} u^{-r-1}$$

and let $\delta(u) = \sum_{n \in \mathbb{Z}} u^n \in \mathbb{C}[u^{\pm 1}]$ denote the formal delta function. The following proposition is a straightforward consequence of the defining relations of $DY_h(\mathfrak{g})$; see [8, Prop. 2.3] and [18, Rem. 2.6], for instance.

**Proposition.** The defining relations of $DY_h(\mathfrak{g})$ are equivalent to the following formal series identities:

- **(D1)** For each $i, j \in I$, we have
  $$[\xi^+_i(u), \xi^\pm_j(v)] = 0 = [\xi^-_i(u), \xi^-_j(v)].$$

- **(D2)** For each $i, j \in I$, $\epsilon \in \{\pm\}$ and $a = h\delta_{i,j}/2$, we have
  $$(u - v \mp a)\xi^\pm_i(u)\chi^\pm_j(v) = (u - v \pm a)\chi^\pm_j(v)\xi^\pm_i(u).$$
(D3) For each \( i, j \in I \) and \( a = \hbar d_{aij}/2 \), we have
\[
(u - v \mp a)\varphi_i^\pm(u)\varphi_j^\pm(v) = (u - v \pm a)\varphi_j^\pm(v)\varphi_i^\pm(u).
\]

(D4) For each \( i, j \in I \), we have
\[
[\varphi_i^\pm(u), \varphi_j^\mp(v)] = \hbar \delta_{ij} u^{-1} \delta(u/v)(\xi_i^+(v) - \xi_i^-(v)).
\]

(D5) For each \( i \neq j \in I \) and \( m = 1 - a_{ij} \), we have
\[
\sum_{\pi \in S_m} \left[ \varphi_i^\pm(u_{\pi(1)}), \varphi_j^\mp(u_{\pi(2)}), \cdots \left[ \varphi_i^\pm(u_{\pi(m)}), \varphi_j^\mp(v) \right] \cdots \right] = 0.
\]

Here the relations (D1)–(D4) hold in the formal series space \( \text{DY}_\hbar(g)[u^{\pm 1}, v^{\pm 1}] \), while (D5) should be understood as a relation in \( \text{DY}_\hbar(g)[u_1^{\pm 1}, \ldots, u_m^{\pm 1}, v^{\pm 1}] \).

4.2. We shall further decompose each generating series \( \varphi_i^\pm(u) \) as a difference
\[
\varphi_i^\pm(u) = \varphi_i^\pm(u)_+ - \varphi_i^\pm(u)_-, \quad \text{where } \varphi_i^\pm(u) \in \text{DY}_\hbar(g)[u^{-1}] \text{ are defined by}
\]
\[
\varphi_i^\pm(u)_+ = \hbar \sum_{r \geq 0} x_i^{r+1} u^{-r-1} \quad \text{and} \quad \varphi_i^\pm(u)_- = -\hbar \sum_{r < 0} x_i^{r+1} u^{-r-1} \quad \forall i \in I.
\]

In particular, one has \( \iota(\xi_i(u)) = \xi_i^+(u) \) and \( \iota(x_i^\pm(u)) = \varphi_i^\pm(u) \) for all \( i \in I \).

Proposition 4.1 then admits the following corollary.

Corollary. For each \( i \in I \), define the operator
\[
T_i(u) := \text{Ad}(\xi_i^+(u))^{-1} - 1 : \text{DY}_\hbar(g) \to u^{-1} \text{DY}_\hbar(g)[u^{-1}].
\]

Then, for every \( i, j \in I \) and \( \epsilon \in \{\pm\} \), we have
\[
(2a \mp (u - v \mp a)T_i(u)) \varphi_j^\pm(v)_\epsilon = \mp T_i(u) \cdot h x_j^\epsilon
\]
in \( \text{DY}_\hbar(g)[v^{-\epsilon}, u^{-1}] \), where \( a = \hbar d_{aij}/2 \).

Proof. From the relation (D2) with \( \epsilon = + \), we obtain the identity
\[
(u - v \mp a)\varphi_j^\pm(v)_+ \varphi_i^\pm(u) - (u - v \mp a)\varphi_i^\pm(u)\varphi_j^\pm(v)_+ = (u - v \mp a)\varphi_j^\pm(v)_- \varphi_i^\pm(u) - (u - v \mp a)\varphi_i^\pm(u)\varphi_j^\pm(v)_-.
\]
The left-hand side belongs to \( \text{DY}_\hbar(g)[v^{-1}, u^{-1}] \) and the right-hand side belongs to \( \text{DY}_\hbar(g)[v, u^{-1}] \). Hence, the left-hand side (resp. right-hand side) is equal to the constant term of the right-hand side (resp. left-hand side) with respect to \( v \). Since these two constant terms themselves coincide and equal \( -\hbar [x_j^\epsilon, \xi_i^+(u)] \), we obtain
\[
(u - v \mp a)\varphi_j^\pm(v)_c \xi_i^+(u) - (u - v \mp a)\xi_i^+(u)\varphi_j^\pm(v)_c = -\hbar [x_j^\epsilon, \xi_i^+(u)].
\]
Left multiplying by \( \xi_i^+(u)^{-1} \) then yields the relation (4.1). \( \square \)

Note that, since the substitution \( u \mapsto v \mp a \) yields an algebra homomorphism
\( \text{DY}_\hbar(g)[v^{-1}, u^{-1}] \to \text{DY}_\hbar(g)[v^{-1}] \), the relation (4.1) with \( j = i \) implies that
\[
\varphi_i^\pm(v)_+ = \mp (2d_i)^{-1} T(v \mp \hbar d_i) \cdot x_i^{\epsilon}_0 \quad \forall i \in I.
\]
4.3. The formal shift operator. Let $\tau_z : Y_h(\g) \to Y_h(\g)[z]$ be the algebra embedding obtained by replacing $a \in \mathbb{C}$ by a formal variable $z$ in the definition of the shift automorphism $\tau_a$, given in Section 2.7. Let $Y_h(\g)[z; z^{-1}]$ denote the algebra of formal Laurent series in $z^{-1}$ with coefficients in $Y_h(\g)$. The following proposition is a consequence of Theorem 4.3 and Corollary 4.6 from [18].

**Proposition.** There is a unique algebra homomorphism

$$\Phi_z : DY_h(\g) \to Y_h(\g)[z; z^{-1}]$$

satisfying $\Phi_z \circ \iota = \tau_z$. It is determined by the formulae

$$\Phi_z(\xi_i^{-}(u)) = \exp(-u\partial_z)\xi_i(-z), \quad \Phi_z(\lambda_i^{\pm}(u)_-) = \exp(-u\partial_z)x_i^{\pm}(-z) \quad \forall \ i \in I.$$

As indicated in Section 4.1, this result implies that the natural homomorphism $\iota : Y_h(\g) \to DY_h(\g)$ is injective. Namely, $\tau_z$ is injective and one has $\tau_z = \Phi_z \circ \iota$.

4.4. Representations of $ DY_h(\g)$. Given $a \in \mathbb{C}$, let $\Rep_{fd}^a(Y_h(\g))$ denote the full subcategory of the category $\Rep_{fd}(Y_h(\g))$ of finite-dimensional $Y_h(\g)$-modules consisting of representations $V$ whose full set of poles

$$\sigma(V) = \bigcup_{i \in I} \sigma_i(V)$$

satisfies $\sigma(V) \subset \mathbb{C} \setminus \{a\}$. These categories are essentially independent of the choice of $a \in \mathbb{C}$ since, for any $b \in \mathbb{C}$, the pull-back functor $\tau_{b-a}^* : \Rep_{fd}(Y_h(\g)) \to \Rep_{fd}^a(Y_h(\g))$.

The following proposition provides the main result of this section.

**Proposition.** Let $V \in \Rep_{fd}^0(Y_h(\g))$. Then the $Y_h(\g)$-action on $V$ extends to a $DY_h(\g)$-action, uniquely determined by the property that, for each $i \in I$, the series

$$\xi_i^{-}(u) \in \End(V)[u] \quad \text{and} \quad \lambda_i^{\pm}(u)_- \in \End(V)[u]$$

are the expansions at 0 of the rational functions $\xi_i(u)$ and $x_i^{\pm}(u)$, respectively. Moreover, every finite-dimensional $DY_h(\g)$-module arises in this way.

**Proof.** Let $\pi_V : Y_h(\g) \to \End(V)$ be the underlying representation morphism. By Proposition 2.5, the assumption $\sigma(V) \subset \mathbb{C}\setminus a$, and the formulae of Proposition 4.3, the composite $\pi_V \circ \Phi_z : DY_h(\g) \to \End(V)[z; z^{-1}]$ satisfies

$$(\pi_V \circ \Phi_z)(DY_h(\g)) \subset \End(V) \otimes \mathbb{C}[z],$$

where $\mathbb{C}[z] \subset \mathbb{C}(z^{-1})$ is the localization of $\mathbb{C}[z]$ at the maximal ideal $z\mathbb{C}[z]$. Letting $\text{ev} : \mathbb{C}[z] \to \mathbb{C}$ denote the evaluation homomorphism $f(z) \mapsto f(0)$, we obtain a $DY_h(\g)$-module structure on $V$ given by the algebra homomorphism

$$\Gamma_V := (1 \otimes \text{ev}) \circ \pi_V \circ \Phi_z : DY_h(\g) \to \End(V).$$

Since $\Phi_z \circ \iota = \tau_z$ evaluates to the identity $\Gamma_{Y_h(\g)}$ at $z = 0$, we indeed have $\Gamma_V \circ \iota = \pi_V$. Moreover, the formulae of Proposition 4.3 show that, for each $i \in I$, $\xi_i^{-}(u)$ and $\lambda_i^{\pm}(u)$ operate as the Taylor expansions

$$\sum_{n \geq 0} \frac{\partial_u^n\xi_i(v)u^n}{n!} \bigg|_{v=0} \quad \text{and} \quad \sum_{n \geq 0} \frac{\partial_u^n\lambda_i^{\pm}(v)u^n}{n!} \bigg|_{v=0},$$

as expected.
respectively, of the rational functions $\xi_i(u)$ and $x_i^\pm(u)$ at 0. This completes the proof of the first part of the proposition.

Let us now turn to the second assertion. Let $V$ be an arbitrary finite-dimensional $\text{DY}_h(\mathfrak{g})$-module and fix $i \in I$. By virtue of Proposition 4.5, the series

$$\xi_i^+(u) \in \text{End}(V)[u^{-1}] \quad \text{and} \quad \mathcal{X}_i^+(u)^\pm \in \text{End}(V)[u^{-1}]$$

are the expansions at $\infty$ of rational functions of $u$, which we again denote by $\xi_i(u)$ and $x_i^+(u)$, respectively. To complete the proof, it is enough to show that $\xi_i^-(u)$ and $\mathcal{X}_i^+(u)^-$ are the expansions of these same rational functions at 0. Indeed, this will imply that the $\text{Y}_h(\mathfrak{g})$-module $\iota^*(V)$ belongs to $\text{Rep}^0_{fd}(\text{Y}_h(\mathfrak{g}))$ and that $V$ is equal to $\text{DY}_h(\mathfrak{g})$-module obtained by extending the $\text{Y}_h(\mathfrak{g})$-action on $\iota^*(V)$ as in the first part of the proposition.

Let $T_i(u) = \text{Ad}(\xi_i^+(u))^{-1} - 1$, as in Corollary 4.2. By the rationality of $\xi_i^+(u)$, there is a nonzero polynomial $P(u) \in \mathbb{C}[u]$ such that $T_i^P(u) := P(u)T_i(u)$ satisfies

$$T_i^P(u) \cdot X \in \text{End}(V)[u] \quad \forall \quad X \in \text{End}(V).$$

Left multiplying (4.1) by $P(u)$ and setting $j = i$ and $\epsilon = -$ , we obtain the identity

$$(2\hbar d_iP(u) \mp (u - v \pm \hbar d_i)T_i^P(u)) \mathcal{X}_i^+(v)^- = \mp T_i^P(u) \cdot x_i^+(v)^+ \in \text{End}(V)[u][v].$$

Applying the automorphism $u \mapsto u \mp \hbar d_i$ of $\text{End}(V)[u]$ followed by the homomorphism $\text{End}(V)[u][v] \rightarrow \text{End}(V)[v]$ determined by the evaluation $u \mapsto v$, we arrive at relation

$$P(v \mp \hbar d_i)\mathcal{X}_i^+(v)^- = \mp (2d_i)^{-1}T_i^P(v \mp \hbar d_i) \cdot x_i^+(v)^+ = P(v \mp \hbar d_i)x_i^+(v)^+ \in \text{End}(V)[v] \subset \text{End}(V)[v],$$

where the second equality is due to the relation (4.2). This implies that $\mathcal{X}_i^+(v)^-$ is the expansion of the rational function $x_i^+(v)^+$ at 0, as desired. Moreover, the relation (D4) of Proposition 4.4 implies that

$$\xi_i^+(u) = 1 + [\mathcal{X}_i^+(u)\pm, x_i^+(0)]^-, $$

from which it follows immediately that the series $\xi_i^-(u)$ is the expansion at 0 of the rational function $\xi_i(u)$. \hfill \Box

4.5. In categorical terms, Proposition 4.4 outputs an isomorphism of categories

$$\Gamma : \text{Rep}^0_{fd}(\text{Y}_h(\mathfrak{g})) \overset{\sim}{\rightarrow} \text{Rep}_{fd}(\text{DY}_h(\mathfrak{g}))$$

which commutes with the forgetful functor to vector spaces and has inverse given by the pull-back functor $\iota^*$. Three consequences of this interpretation relevant to our discussion are given by the following corollary.

**Corollary.**

1. There is a unique tensor structure on $\text{Rep}_{fd}(\text{DY}_h(\mathfrak{g}))$ such that $\Gamma$ is a strict tensor functor. The tensor product $\otimes$ is given by

$$V \otimes W = \Gamma(\iota^*(V) \otimes \iota^*(W)).$$

2. A $\text{DY}_h(\mathfrak{g})$-module $V \in \text{Rep}_{fd}(\text{DY}_h(\mathfrak{g}))$ is irreducible if and only if there is an $\mathbf{I}$-tuple of Drinfeld polynomials $\mathbf{P} = (P_i(u))_{i \in \mathbf{I}}$ such that $V \cong \Gamma(L(\mathbf{P}))$ where $\sigma(L(\mathbf{P})) \subset \mathbb{C}^\times$. 

(3) Let $V \in \text{Rep}_{fd}(Y_h(\mathfrak{g}))$, with composition factors $\{V_1, \ldots, V_N\}$. Then the $Y_h(\mathfrak{g})$-action on $V$ extends to a $\text{DY}_h(\mathfrak{g})$-action if and only if
\[ \sigma(V_j) \subset \mathbb{C}^\times \quad \forall \quad 1 \leq j \leq N. \]

PROOF. The first assertion of Corollary 3.6 implies that $\text{Rep}_{fd}(Y_h(\mathfrak{g}))$ is a tensor subcategory of $\text{Rep}_{fd}(Y_h(\mathfrak{g}))$. This result, together with Proposition 4.4, implies Part (1) of the corollary.

Similarly, Parts (2) and (3) of the corollary follow from Proposition 4.4, Theorem 2.11 and, in the case of Part (3), the second assertion of Corollary 3.6. \qed

Note that Part (2) of this corollary implies that the isomorphism classes of finite-dimensional irreducible representations of $\text{DY}_h(\mathfrak{g})$ are parametrized by those $I$-tuples of monic polynomials $P = (P_i(u))_{i \in I}$ for which the condition
\[ \sigma(L(P)) \subset \mathbb{C}^\times \]
is satisfied. Both this observation and Part (3) of Corollary 4.5 motivate the problem of obtaining necessary and sufficient conditions on $P$ for which $\sigma(L(P)) \subset \mathbb{C}^\times$. When $\mathfrak{g} = sl_n$, Theorem 3.5 asserts that this condition is equivalent to $Z(P(u)) \subset \mathbb{C}^\times$, where $P = P(u)$. In this case, the above classification result is in agreement with the rank one instance of Conjecture 1 from [13] for the formal, $h$-adic, analogue of $\text{DY}_h(sl_n)$. The general situation diverges from this conjecture, and we demonstrate this in the next section by showing that an elegant solution exists in type $A_{n-1}$ for arbitrary $n$.

5. THE YANGIAN AND YANGIAN DOUBLE OF $sl_n$

In this section we restrict our attention to $\mathfrak{g} = sl_n$. Fix $n \geq 2$, and take $I = \{1, \ldots, n-1\}$. The entries of the Cartan matrix $A = (a_{ij})_{i,j \in I}$ are then given by
\[ a_{ij} = \begin{cases} 
2 & \text{if } i = j \\
-1 & \text{if } |i - j| = 1 \\
0 & \text{otherwise}
\end{cases} \]

In this case, the diagram automorphism $i \mapsto i^*$ induced by the longest element becomes $i^* = n - i$. Moreover, we have $\kappa = \frac{1}{2}$, where we recall from §2.13 that $\kappa$ is $(1/4)$ times the eigenvalue of the Casimir element on the adjoint representation.

5.1. For each $i, j \in I$, introduce the discrete interval $S^{(n)}_i(j) \subset \frac{n}{2}\mathbb{Z}$ by
\[ S^{(n)}_i(j) = \{ \frac{i+j}{2} - \ell : \ell \in [i + j + 1 - n, i] \cap [1, j] \} \]
and let $S_n \subset \mathbb{Z}$ denote the string of non-negative integers
\[ S_n = \{0, 1, \ldots, n - 2\} = I - \{1\}. \]

The following theorem provides the second main result of this article.

**Theorem.** Let $P = (P_i(u))_{i \in I}$ be an $I$-tuple of monic polynomials in $u$. Then the finite-dimensional irreducible $Y_h(sl_n)$-module $L(P)$ satisfies
\[ \sigma_i(L(P)) = \bigcup_{j \in I} (Z(P_j(u)) + S^{(n)}_i(j)) \quad \forall \quad i \in I. \]
Consequently, $L(P)$ has full set of poles $\sigma(L(P))$ given by

$$\sigma(L(P)) = \frac{h}{2}S_n + \bigcup_{i \in I} Z(P_i(u)).$$

As an immediate corollary to this theorem and Parts (2) and (3) of Corollary 4.5, we obtain the following classification results for the Yangian double $DY_h(\mathfrak{sl}_n)$.

Corollary.

1. The isomorphism classes of finite-dimensional irreducible representations of $DY_h(\mathfrak{sl}_n)$ are parametrized by $I$-tuples of monic polynomials $P = (P_i(u))_{i \in I}$ satisfying the condition

$$-\frac{\hbar}{2}S_n \subset \mathbb{C} \setminus Z(P_i(u)) \quad \forall i \in I.$$

2. Let $V$ be a finite-dimensional $Y_h(\mathfrak{sl}_n)$-module with composition factors given by $\{V_1, \ldots, V_N\}$. Then the $Y_h(\mathfrak{sl}_n)$-action on $V$ extends to a $DY_h(\mathfrak{sl}_n)$-action if and only if

$$-\frac{\hbar}{2}S_n \subset \mathbb{C} \setminus Z(P_i^j(u)) \quad \forall i \in I \text{ and } 1 \leq j \leq N,$

where, for each $1 \leq j \leq N$, $P = (P_i^j(u))_{i \in I}$ is the unique $I$-tuple of monic polynomials such that $V_j \cong L(P^j)$.

The rest of this section is devoted to a proof of Theorem 5.1, with most of our effort devoted to establishing the identity (5.2). In Section 5.2, we verify this relation on the fundamental representations of $Y_h(\mathfrak{sl}_n)$. This is then generalized to an arbitrary finite-dimensional irreducible representation $L(P)$ in Sections 5.3 and 5.4. Our proof of Theorem 5.1 is then completed in Section 5.5, where we deduce identity (5.3).

5.2. Fundamental representations. Let $1 \leq m \leq n-1$, and $a \in \mathbb{C}$. The fundamental representation $L_m(a)$ is the finite–dimensional irreducible representation of $Y_h(\mathfrak{sl}_n)$ with Drinfeld polynomials given by:

$$P_i(u) = \begin{cases} 1 & \text{if } i \neq m \\ u - a & \text{if } i = m \end{cases}$$

We give an explicit construction of $L_m(a)$ below. Let the standard basis of $\mathbb{C}^n$ be denoted by $\{|1\rangle, \ldots, |n\rangle\}$. Consider the subspace $V \subset (\mathbb{C}^n)^{\otimes m}$ defined by

$$V = \text{Span of } \{ |p_1\rangle \otimes \cdots \otimes |p_m\rangle : 1 \leq p_1 < \ldots < p_m \leq n \}.$$

For $p = (p_1, \ldots, p_m)$, we will use the notation $|p\rangle = |p_1\rangle \otimes \cdots \otimes |p_m\rangle$.

For each $i \in I$ and $1 \leq \ell \leq m$, define $a_{i,\ell} \in \mathbb{C}$ by

$$a_{i,\ell} = a + \frac{\hbar}{2}(m + i - 2\ell) \text{.}$$

We now define operators $\{\xi_i(u), x_i^\pm(u)\}_{i \in I}$ on $V$ by specifying their action on each tensor $|p\rangle$ as follows:

1. $\xi_i(u) |p\rangle = |p\rangle$, if either $\{p_1, \ldots, p_m\} \cap \{i, i+1\} = \emptyset$, or $\{i, i+1\} \subset \{p_1, \ldots, p_m\}$. 
• If \( i \in \{p_1, \ldots, p_m\} \) and \( i + 1 \not\in \{p_1, \ldots, p_m\} \), then:

\[
\xi_i(u) \, |\, p = \frac{u + h - a_{i,k}}{u - a_{i,k}} \, |\, p \rangle, \quad \text{where } p_k = i.
\]

• If \( i + 1 \in \{p_1, \ldots, p_m\} \) and \( i \not\in \{p_1, \ldots, p_m\} \), then:

\[
\xi_i(u) \, |\, p = \frac{u - h - a_{i,k}}{u - a_{i,k}} \, |\, p \rangle, \quad \text{where } p_k = i + 1.
\]

(2) \( x_i^+(u) \, |\, p = 0 \), if either \( i + 1 \not\in \{p_1, \ldots, p_m\} \), or \( \{i, i + 1\} \subset \{p_1, \ldots, p_m\} \). If \( |\, p \rangle \) is such that \( i + 1 \in \{p_1, \ldots, p_m\} \) (say, \( p_k = i + 1 \)) and \( i \not\in \{p_1, \ldots, p_m\} \), then:

\[
x_i^+(u) \, |\, p = \frac{h}{u - a_{i,k}} \, |\, p_1, \ldots, p_{k-1}, i, p_k, \ldots, p_m \rangle
\]

(3) \( x_i^- (u) \, |\, p = 0 \), if either \( i \not\in \{p_1, \ldots, p_m\} \), or \( \{i, i + 1\} \subset \{p_1, \ldots, p_m\} \). If \( |\, p \rangle \) is such that \( i \in \{p_1, \ldots, p_m\} \) (say, \( p_k = i \)) and \( i + 1 \not\in \{p_1, \ldots, p_m\} \), then:

\[
x_i^- (u) \, |\, p = \frac{h}{u - a_{i,k}} \, |\, p_1, \ldots, p_{k-1}, i+1, p_k, \ldots, p_m \rangle
\]

**Proposition.** The operators \( \{\xi_i(u), x_i^\pm(u)\}_{i \in I} \) on \( V \) given above satisfy the defining relations of \( Y_h(\mathfrak{sl}_n) \). The resulting representation is isomorphic to the fundamental representation \( L_{\varpi}(a) \). Moreover, for each \( i \in I \) we have

\[
\sigma_i(L_{\varpi}(a)) = \left\{ a + \frac{h}{2} (i + m - 2\ell) : \ell \in \{i + m + 1 - n, i\} \cap [1, m] \right\}
\]

\[
= Z(P_m(u)) + S_i^{(a)}(m).
\]

**Proof.** The relations of \( Y_h(\mathfrak{sl}_n) \) are verified in Appendix A below. One quick way to prove that \( V \) is irreducible is to notice that, when viewed as an \( \mathfrak{sl}_n \)-module via the inclusion \( \mathfrak{sl}_n \subset Y_h(\mathfrak{sl}_n) \), \( V \) is isomorphic to the exterior product \( \wedge^m \mathbb{C}^n \), with the canonical isomorphism given by

\[
|p_1, \ldots, p_m \rangle \mapsto |p_1 \wedge \cdots \wedge p_m \rangle.
\]

As \( \wedge^m \mathbb{C}^n \) is an irreducible representation of \( \mathfrak{sl}_n \), the same is true of \( V \). It follows automatically that \( V \) is irreducible as a \( Y_h(\mathfrak{sl}_n) \)-module.

It is also easy to see from the explicit expressions of \( \{\xi_i(u), x_i^\pm(u)\} \) given above, that \( \Omega = \{1, 2, \ldots, m\} \) is a highest-weight vector satisfying

\[
\xi_i(u) \Omega = \begin{cases} 
\Omega 
& \text{if } i \neq m \\
\frac{u + h - a_{m,m}}{u - a_{m,m}} \Omega 
& \text{if } i = m
\end{cases}
\]

Note that \( a_{m,m} = a \), therefore, the Drinfeld polynomials associated to \( V \) are \( P_i(u) = (u - a)^{a_{i,m}} \), as claimed.

It remains to compute \( \sigma_i(L_{\varpi}(a)) \) for \( 1 \leq i \leq n - 1 \). For this we can just locate the poles of \( x_i^+(u) \) (see Proposition 3.1 above). By definition, this operator has poles at \( a_{i,k} = a + \frac{h}{2} (m + i - 2k) \) for each \( 1 \leq k \leq m \) such that there exists \( p \) with \( p_k = i + 1 \) and \( p_{k-1} < i \). For such an increasing sequence to exist, it is necessary
and sufficient that \( k \leq i \) and \( m - k \leq n - i - 1 \), \( i.e. \), \( m + i + 1 - n \leq k \leq i \). This gives us

\[
\sigma_i(L_{\varpi_i}(a)) = \left\{ a + \frac{\hbar}{2}(m + i - 2k) : k \in [1, m] \cap [m + i + 1 - n, i] \right\},
\]
as claimed.

**Remark.** We will verify that the operators defined above satisfy the relations of \( Y_h(\mathfrak{sl}_n) \) in Appendix A. Here we just want to remark as to how this action was computed. Consider the \( Y_h(\mathfrak{sl}_n) \)-action on \( \mathbb{C}^n \), depending on \( b \in \mathbb{C} \):

\[
x^+_i(u) | j \rangle = \delta_{i+1,j} \frac{\hbar}{u - b_i} | i \rangle,
\]

\[
x^-_i(u) | j \rangle = \delta_{i,j} \frac{\hbar}{u - b_i} | i + 1 \rangle,
\]

\[
\xi_i(u) | j \rangle = (1 - \delta_{i,j} - \delta_{i+1,j}) | j \rangle + \delta_{i,j} \frac{u + \hbar - b_i}{u - b_i} | i \rangle
\]

\[
+ \delta_{i+1,j} \frac{u - \hbar - b_i}{u - b_i} | i + 1 \rangle,
\]

where \( b_i = b + \frac{\hbar}{2}(i - 1) \). This representation, denoted by \( \mathbb{C}^n(b) \), is the fundamental representation \( L_{\varpi_1}(b) \), with

\[
\sigma_i(L_{\varpi_1}(b)) = \left\{ b + \frac{\hbar}{2}(i - 1) \right\} \quad \forall \ i \in I.
\]

The representation \( V \) given above is the cyclic subrepresentation generated by \( \Omega = | 1 \rangle \otimes \cdots \otimes | m \rangle \), in the (Drinfeld) tensor product (as defined in [9, §4.5]):

\[
\mathbb{C}^n \left( a + \frac{\hbar}{2}(m - 1) \right) \otimes_D \mathbb{C}^n \left( a + \frac{\hbar}{2}(m - 3) \right) \otimes_D \cdots \otimes_D \mathbb{C}^n \left( a - \frac{\hbar}{2}(m - 1) \right).
\]

It is worth pointing out that the Drinfeld tensor product \( V \otimes_D W \) of two finite-dimensional representations of \( Y_h(\mathfrak{g}) \) is defined in [9, §4.5] under the assumption that \( \sigma(V) \cap \sigma(W) = \emptyset \). However, for this definition and the proof of [9, Thm. 4.6], it is enough to assume the weaker condition that \( \sigma_i(V) \cap \sigma_i(W) = \emptyset, \) for every \( i \in I \). Otherwise, the tensor product written above would not be defined.

**5.3. Proof of (5.2):** \( (\subseteq) \). Recall that \( L(\underline{P}) \) is the finite-dimensional irreducible representation of \( Y_h(\mathfrak{sl}_n) \) associated to Drinfeld polynomials \( \underline{P} = (P_j(u))_{j \in I} \). For each \( j \in I \), let \( \{a^{(j)}_1, \ldots, a^{(j)}_{r_j}\} \) be the set of roots of \( P_j(u) \), listed with multiplicity. Consider the following tensor product of fundamental representations

\[
\mathcal{V} = \bigotimes_{j \in I} \left( \bigotimes_{s=1}^{r_j} L_{\varpi_j} \left( a^{(j)}_s \right) \right),
\]

where the ordering of tensor factors is immaterial for our argument. Let \( \Omega \in \mathcal{V} \) be the tensor product of the highest-weight vectors of each fundamental representation appearing above. Then \( L(\underline{P}) \) is a quotient of the subrepresentation of \( \mathcal{V} \) generated
by $\Omega$. Hence, for every $i \in I$, we have

$$\sigma_i(L(P)) \subset \sigma_i(V) \subset \bigcup_{1 \leq s \leq r_j} \sigma_i\left(\sum_{j \in I} a_s^{(j)}\right)$$

$$= \bigcup_{j \in I} \left\{a_1^{(j)}, \ldots, a_r^{(j)}\right\} + S_i^{(n)}(j)$$

$$= \bigcup_{j \in I} Z(P_j(u)) + S_i^{(n)}(j).$$

In the first line, we have used Corollary 3.6 (1), and in the second Proposition 5.2.

5.4. **Proof of (5.2):** (⊇). We will now show the reverse inclusion. That is, we will prove that

$$Z(P_j(u)) + S_i^{(n)}(j) \subset \sigma_i(L(P))$$

for all $i, j \in I$. Note that for $n = 2$ the identity (5.2), and thus (5.4), has already been established in §3.5. Indeed, in this case one has $S_1^{(2)}(1) = \{0\}$ and (5.2) collapses to statement of Theorem 3.5. We will now assume that $n \geq 3$ and that the relation (5.4) holds for $Y_h(\mathfrak{sl}_{n-1})$. Below, we will consider $Y_h(\mathfrak{sl}_{n-1}) \subset Y_h(\mathfrak{sl}_n)$ as the subalgebra generated by $\{\xi_k, r_k, r_k^+\}_{1 \leq k \leq n-2, r \in \mathbb{Z}_{\geq 0}}$. We now proceed with the proof of (5.4) for $Y_h(\mathfrak{sl}_n)$.

**Case 1:** $i = 1$. When $i = 1$, $S_1^{(n)}(j)$ is the singleton

$$S_1^{(n)}(j) = \left\{\frac{h}{2}(j-1)\right\} \quad \forall \ 1 \leq j \leq n-1.$$

Assume first that $1 \leq j \leq n-2$. Let $V \subset L(P)$ be $Y_h(\mathfrak{sl}_{n-1})$-module generated by the highest-weight vector $\Omega \in L(P)$. Then, by Lemma 2.12, $V$ is an irreducible representation of $Y_h(\mathfrak{sl}_{n-1})$ with Drinfeld polynomials $(P_1(u), \ldots, P_{n-2}(u))$. Thus, by the inductive hypothesis, we have

$$Z(P_j(u)) + \frac{h}{2}(j-1) \subset \sigma_1(V) \subset \sigma_1(L(P)) \quad \forall \ 1 \leq j \leq n-2.$$

To complete the proof of (5.4) for $i = 1$, it only remains to check that $ZP_{n-1}(u) + \frac{h}{2}(n-2) \subset \sigma_1(L(P))$. We are going to need Proposition 2.13 for this. Recall that $\varphi$ is the Cartan involution of $Y_h(\mathfrak{sl}_n)$ introduced for general $\mathfrak{g}$ in §2.9.

The already established case of the inclusion (5.4) for $i = j = 1$, takes the following form on the representation $L(P)^\varphi$:

$$Z(P_1^\varphi(u)) \subset \sigma_1(L(P))^\varphi = -\sigma_1(L(P)).$$

By Proposition 2.13, we have $Z(P_1^\varphi(u)) = -Z(P_{n-1}(u)) - \frac{h}{2}(n-2)$, since $P_1^\varphi(u) = (\pm 1)P_{n-1}(-u + \frac{h}{2} - \frac{h}{2})$. Hence,

$$-Z(P_{n-1}(u)) - \frac{h}{2}(n-2) \subset -\sigma_1(L(P)),$$

which is exactly (5.4) for $i = 1$ and $j = n-1$.

**Case 2:** $2 \leq i \leq n-2$. Let us remark that the $j = 1$ and $j = n-1$ cases of (5.4) can be shown easily by the induction argument as in Case 1 (in either of these
cases $S_i^{(n)}(j)$ is a singleton, independent of $n$). Therefore, it is sufficient to consider $2 \leq j \leq n - 2$. By the induction hypothesis, we have
\[
\mathcal{Z}(P_j(u)) + S_i^{(n-1)}(j) \subset \sigma_i(L(P)).
\]
On the other hand, using this inclusion for $L(P)^{\ell}$ with $j$ replaced by $n - j$, we obtain
\[
\mathcal{Z}(P_{n-j}(u)) + S_i^{(n-1)}(n-j) \subset \sigma_i(L(P)^{\ell}) = -\sigma_i(L(P)).
\]
Combining this with Proposition 2.13 then yields
\[
\mathcal{Z}(P_j(u)) + \left( \frac{\hbar}{2}(n-2) - S_i^{(n-1)}(n-j) \right) \subset \sigma_i(L(P)).
\]
It remains to observe that, by (5.1), we have the equalities
\[
S_i^{(n-1)}(j) = \hbar \left\{ \frac{i+j}{2} - \ell : \ell \in [i+j+2-n,i] \cap [1,j] \right\},
\]
\[
\frac{\hbar}{2}(n-2) - S_i^{(n-1)}(n-j) = \hbar \left\{ \frac{i+j}{2} - \ell : \ell \in [1,j-1] \cap [i+j+1-n,i] \right\},
\]
from which we can conclude that
\[
\left( \mathcal{Z}(P_j(u)) + S_i^{(n-1)}(j) \right) \cup \left( \mathcal{Z}(P_j(u)) + \left( \frac{\hbar}{2}(n-2) - S_i^{(n-1)}(n-j) \right) \right)
\]
\[
= \mathcal{Z}(P_j(u)) + S_i^{(n)}(j) \subset \sigma_i(L(P))
\]
as claimed.

Finally, we note that the $i = n - 1$ case can be deduced from the $i = 1$ case using the diagram automorphism $\omega$ given by $\omega(j) = n - j$ for every $j \in I$ (see §2.8).

5.5. **Proof of (5.3).** We now complete our proof of Theorem 5.1 by establishing the identity (5.3). By (5.2) and the definition of $\sigma(L(P))$, we have
\[
\sigma(L(P)) = \bigcup_{i \in I} \sigma_i(L(P)) = \bigcup_{i,j \in I} (\mathcal{Z}(P_j(u)) + S_i^{(n)}(j)) = \bigcup_{j \in I} (\mathcal{Z}(P_j(u)) + S_n(j)),
\]
where $S_n(j) := \bigcup_{i \in I} S_i^{(n)}(j)$. Therefore, to prove the first part of the theorem, it suffices to show that
\[
S_n(j) = \frac{\hbar}{2} S_n \quad \forall j \in I.
\]
To this end, observe that the the sets $S_i^{(n)}(j)$ satisfy the relation
\[
S_i^{(n)}(j) = \begin{cases} 
S_{i-1}^{(n-1)}(j-1) & \text{if } i > n-j, \\
S_{i-1}^{(n-1)}(j-1) \cup \frac{\hbar}{2}(i+j-2) & \text{if } i \leq n-j,
\end{cases}
\]
where $S_0^{(n-1)}(j-1) = \emptyset$. Consequently, $S_n(j)$ satisfies the recursive identity
\[
S_n(j) = S_{n-1}(j-1) \cup \frac{\hbar}{2} \bigcup_{i=1}^{n-j} \{i+j-2\} = S_{n-1}(j-1) \cup \frac{\hbar}{2} \{n-2\}
\]
It follows by induction on $j$ that (5.5) will hold for all pairs $(j, n) \in \mathbb{N} \times \mathbb{Z}_{>j}$ provided it is satisfied when $j = 1$. In this case, one has $[i+j+1-n,i] \cap [1,j] = \{1\}$, and so $S_i^{(n)}(1) = \frac{\hbar}{2} \{i-1\}$ for all $i \in I$, from which (5.5) follows immediately.
APPENDIX A. Verification of \( Y_h(\mathfrak{sl}_n) \) relations on \( L_{\mathbf{w}_m}(a) \)

A.1. Recall that \( 1 \leq m \leq n - 1 \), and \( V \) is the following \( \binom{n}{m} \)-dimensional subspace of \( (\mathbb{C}^n)^{\otimes m} \):

\[
V = \text{Span of } \{ |p\rangle : p = (p_1, \ldots, p_m) \text{ where } 1 \leq p_1 < \ldots < p_m \leq n \}.
\]

For each \( 1 \leq i \leq n - 1 \), and \( 1 \leq k \leq m \), let \( a_{i,k} = a + \frac{\hbar}{2}(m + i - 2k) \). We defined the operators \( \xi_{i,r}, x^\pm_{i,r} \) on \( V \), for every \( r \in \mathbb{Z}_{\geq 0} \), by the formulae

\[
(\text{A.1}) \quad \xi_{i,r} |p\rangle = \begin{cases} 
0 & \text{if, } \{p_1, \ldots, p_m\} \cap \{i, i + 1\} = \emptyset \text{ or } \{i, i + 1\} \\
a_{i,k}^+ |p\rangle & \text{if, } p_k = i \text{ and } p_{k+1} > i + 1, \\
-a_{i,k}^- |p\rangle & \text{if, } p_k = i + 1 \text{ and } p_{k-1} < i.
\end{cases}
\]

\[
(\text{A.2}) \quad x^+_{i,r} |p\rangle = a_{i,k}^+ |p_1, \ldots, p_{k-1}, i, p_{k+1}, \ldots, p_m\rangle \text{ if } p_k = i + 1 \text{ and } p_{k-1} < i,
\]

and \( x^-_{i,r} |p\rangle = 0 \) if there is no such \( k \), \( i.e. \), either \( i + 1 \notin \{p_1, \ldots, p_m\} \), or \( \{i, i + 1\} \subset \{p_1, \ldots, p_m\} \).

\[
(\text{A.3}) \quad x^-_{i,r} |p\rangle = a_{i,k}^- |p_1, \ldots, p_{k-1}, i + 1, p_{k+1}, \ldots, p_m\rangle \text{ if } p_k = i \text{ and } p_{k+1} > i + 1,
\]

and \( x^+_{i,r} |p\rangle = 0 \) if there is no such \( k \), \( i.e. \), either \( i \notin \{p_1, \ldots, p_m\} \), or \( \{i, i + 1\} \subset \{p_1, \ldots, p_m\} \).

A.2. Relation (Y1). Since \( V \) admits a basis of joint eigenvectors of \( \{\xi_{i,r}\}_{i \in I,r \in \mathbb{Z}_{\geq 0}} \), we get that \( [\xi_{i,r}, \xi_{j,s}] = 0 \), for every \( i, j \in I \) and \( r, s \in \mathbb{Z}_{\geq 0} \).

A.3. Relation (Y2). By (A.1) with \( r = 0 \), we get that the weight of \( |p\rangle \) is given by:

\[
\text{wt}(|p\rangle) = \varepsilon_{p_1} + \cdots + \varepsilon_{p_m},
\]

where \( \varepsilon_i : \mathbb{C}^n \to \mathbb{C} \) is the \( \ell^{th} \)-coordinate function, viewed as a linear form on the Cartan subalgebra \( \mathfrak{h} \subset \mathbb{C}^n \) \( (b = \text{Ker}(\varepsilon_1 + \ldots + \varepsilon_n)) \). Thus it is clear that from the definition of \( x_{j,s}^\pm \) that, either \( x_{j,s}^\pm |p\rangle = 0 \), or \( \text{wt}(x_{j,s}^\pm |p\rangle) = \pm (\varepsilon_j - \varepsilon_{j+1}) = \pm a_j \).

A.4. Relation (Y3). Let us verify this relation for the + case. For \( i, j \in I \) and \( r, s \in \mathbb{Z}_{\geq 0} \), we have to show that

\[
(\text{A.4}) \quad [\xi_{i,r+1}, x^+_{j,s}] - [\xi_{i,r}, x^+_{j,s+1}] = \frac{\hbar}{2} a_{js} (\xi_{i,r} x^+_{j,s} + x^+_{j,s} \xi_{i,r}).
\]

Note that, when \( |i - j| \geq 2 \), \( \xi_{i,r} \) and \( x^+_{j,s} \) acting on \( V \) commute, and the relation holds trivially.

Case 1. \( i = j \). Again, from (A.1) and (A.2), we notice that both sides of (A.4) evaluated on \( |p\rangle \) are zero, unless there is \( 1 \leq k \leq m \) such that \( p_k = i + 1 \) and \( p_{k-1} < i \). Assuming this is the case, we have:

\[
\begin{align*}
\xi_{i,r} x^+_{j,s} |p\rangle &= a_{i,k}^+ |p_1, \ldots, p_{k-1}, i, p_{k+1}, \ldots, p_m\rangle, \\
x^+_{i,r} \xi_{j,s} |p\rangle &= -a_{i,k}^- |p_1, \ldots, p_{k-1}, i, p_{k+1}, \ldots, p_m\rangle.
\end{align*}
\]

Let us write \( |p'| = |p_1, \ldots, p_{k-1}, i, p_{k+1}, \ldots, p_m\rangle \), so that we get:

\[
[\xi_{i,r+1}, x^+_{j,s}] |p\rangle = [\xi_{i,r}, x^+_{j,s+1}] |p\rangle = 2a_{i,k}^+ |p'|.
\]
Hence, both sides of (A.4) evaluated on \(|p\rangle\) are zero, and the relation holds.

**Case 2.**  \(j = i + 1\). Again the only vectors on which \(x_{i+1,r}^+\) acts as a non-zero operator are \(|p\rangle\) such that for some \(1 \leq k \leq m\), we have \(p_k = i + 2\), and \(p_{k-1} < i\). We will assume this is the case, and write \(|p\rangle = |p_1, \ldots, p_{k-1}, i+1, p_{k+1}, \ldots, p_m\rangle\). We have to consider two situations: \(p_{k-1} = i\) or \(p_{k-1} < i\).

- If \(p_{k-1} = i\), we get \(\xi_{i,r}x_{i+1,s}^+|p\rangle = 0\), and \(x_{i+1,s}^+\xi_{i,r}|p\rangle = a_{i,k-1}^+a_{i+1,k}^+|p\rangle\).

The relation to verify (A.4) becomes:

\[
-a_{i,k-1}^+a_{i+1,k}^+ + a_{i,k-1}^+a_{i+1,k}^+ = -\frac{\hbar}{2}a_{i,k-1}^+a_{i+1,k}^+.
\]

This is clearly equivalent to \(a_{i+1,k}^--a_{i,k-1} = -\frac{\hbar}{2}\) which follows from the definition \(a_{i,k} = a + \frac{\hbar}{2}(m + i - 2\ell)\).

- If \(p_{k-1} < i\), we get that \(x_{i+1,s}^+\xi_{i,r}|p\rangle = 0\), and \(\xi_{i,r}x_{i+1,s}^+|p\rangle = -a_{i,k}^+a_{i+1,k}^+\).

The relation (A.4) becomes:

\[
-a_{i,k}^+a_{i+1,k}^+ + a_{i,k}^+a_{i+1,k}^+ = \frac{\hbar}{2}a_{i,k}^+a_{i+1,k}^+.
\]

which is again readily verified.

**Case 3.**  \(j = i - 1\). This is entirely analogous to the previous one and hence its verification is omitted here.

**A.5. Relation (Y4).** Again we verify this relation for the + case only. Let \(i, j \in I\) and \(r, s \in \mathbb{Z}_{\geq 0}\). Then, we want to prove that

\[
(A.5) \quad [x_{i,r}^+, x_{j,s}^+] - [x_{i,r}^+, x_{j,s+1}^+] = \frac{\hbar}{2}a_{ij}(x_{i,r}^+x_{j,s}^+ + x_{j,s}^+x_{i,r}^+).
\]

In \(\text{End}(V)\), we have \(x_{i,r}^+x_{i,s}^+ = 0\), from which (A.5) follows when \(i = j\). This relation is also clear when \([i - j] \geq 2\). Thus it suffices to consider the case \(j = i + 1\).

1. From (A.2) it is clear that \(x_{i,r}^+, x_{i+1,s}^+|p\rangle \neq 0\) only for those basis vectors for which there exists \(k\) with \(p_k = i + 2\) and \(p_{k-1} < i\). In this case, \(x_{i+1,s}^+x_{i,r}^+|p\rangle = 0\), and:

\[
x_{i,r}^+x_{i+1,s}^+|p\rangle = a_{i,k}^+a_{i+1,k}^+|p_1, \ldots, p_{k-1}, i, p_{k+1}, \ldots, p_m\rangle.
\]

The relation to check (A.5), evaluated on such a basis vector \(|p\rangle\), becomes:

\[
a_{i,k}^+a_{i+1,k}^+ - a_{i,k}^+a_{i+1,k}^+ = -\frac{\hbar}{2}a_{i,k}^+a_{i+1,k}^+;
\]

i.e., \(a_{i,k}^+ - a_{i+1,k} = -\frac{\hbar}{2}\) which is true.

2. Similarly, \(x_{i+1,s}^+x_{i,r}^+|p\rangle \neq 0\) only for those basis vectors for which there exists \(k\) with \(p_k = i + 1\), \(p_{k-1} < i\) and \(p_{k+1} = i + 2\). In this case, \(x_{i,r}^+x_{i+1,s}^+|p\rangle = 0\), and:

\[
x_{i+1,s}^+x_{i,r}^+|p\rangle = a_{i,k}^+a_{i+1,k+1}^+|p_1, \ldots, p_{k-1}, i, i + 1, p_{k+2}, \ldots, p_m\rangle.
\]
The relation (A.5) evaluated on such a $|p\rangle$ becomes:

$$-a_{i,k}^{r+1}a_{i+1,k+1}^{r} + a_{i,k}^{r}a_{i+1,k+1}^{r+1} = -\frac{\hbar}{2}a_{i,k}^{r}a_{i+1,k+1}^{r+1},$$

i.e., $a_{i+1,k+1} - a_{i,k} = -\frac{\hbar}{2}$, which holds.

A.6. Relation (Y5). Let $i, j \in \mathbb{I}$ and $r, s \in \mathbb{Z}_{\geq 0}$. We have to show:

(A.6) \[ [x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{ij} \xi_{i,r+s}. \]

Again, for $i, j$ such that $|i-j| \geq 2$, the operators $x^{+}_{i,r}$ and $x^{-}_{j,s}$ commute and this relation holds.

Case 1. $i = j$. Note that $x^{+}_{i,r} x^{-}_{i,s} |p\rangle \neq 0$ only when there exists $k$ such that $p_{k} = i$, and $p_{k+1} > i+1$. In this case $x^{-}_{i,s} x^{+}_{i,r} |p\rangle = 0$ and we get:

$$[x^{+}_{i,r}, x^{-}_{i,s}] |p\rangle = a^{r+s}_{i,k} |p\rangle = \xi_{i,r+s} |p\rangle.$$  

Similarly, $x^{-}_{i,s} x^{+}_{i,r} |p\rangle \neq 0$ only on those basis vectors $|p\rangle$ for which there is a $k$ such that $p_{k} = i+1$ and $p_{k-1} < i$. For such vectors $x^{+}_{i,r} x^{-}_{i,s} |p\rangle = 0$ and we get:

$$[x^{+}_{i,r}, x^{-}_{i,s}] |p\rangle = -a^{r+s}_{i,k} |p\rangle = \xi_{i,r+s} |p\rangle.$$  

Case 2. $j = i+1$. We claim that both $x^{+}_{i,r} x^{-}_{i+1,s}$ and $x^{-}_{i+1,s} x^{+}_{i,r}$ are zero on $V$. Let us prove this claim for the former (latter being similar is omitted). In order for $x^{-}_{i+1,s} |p\rangle$ to be non-zero, it is necessary and sufficient that there exists $k$ with $p_{k} = i+1$ and $p_{k+1} > i+2$. However, since $x^{-}_{i+1,s}$ replaces $p_{k}$ with $i+2$, the resulting vector

$$x^{-}_{i+1,s} |p\rangle = a^{s}_{i+1,k} |p_{1}, \ldots, p_{k-1}, i+2, p_{k+1}, \ldots, p_{m}\rangle$$

does not contain $i+1$. Hence, $x^{+}_{i,r} x^{-}_{i+1,s} |p\rangle = 0$.

The case when $j = i-1$ is similar to the preceding one, hence omitted.

References

1. N. Bourbaki, Lie groups and Lie algebras Chapters 4,5,6, Springer-Verlag, 2002.
2. V. Chari, Braid group actions and tensor products, Int. Math. Res. Not. (2002), no. 7, 357–382. MR 1883181
3. V. Chari and A. Pressley, Fundamental representations of Yangians and singularities of $R$-matrices, J. Reine Angew. Math. 417 (1991), 87–128. MR 1103907
4. , A guide to quantum groups, Cambridge University Press, 1994.
5. , Yangians, integrable quantum systems and Dorey’s rule, Comm. Math. Phys. 181 (1996), no. 2, 265–302. MR 1414834
6. , Yangians: their representations and characters, Acta Appl. Math. 44 (1996), no. 1-2, 39–58. Representations of Lie groups, Lie algebras and their quantum analogues. MR 1407039
7. V. G. Drinfeld, A new realization of Yangians and quantum affine algebras, Soviet Math. Dokl. 36 (1988), no. 2, 212–216.
8. S. Gautam and V. Toledano Laredo, Yangians, quantum loop algebras, and abelian difference equations, J. Amer. Math. Soc. 29 (2016), no. 3, 775–824. MR 3486172
9. , Meromorphic tensor equivalence for Yangians and quantum loop algebras, Publ. Math. Inst. Hautes Etudes Sci. 125 (2017), 267–337. MR 3668651
10. S. Gautam, V. Toledano Laredo, and C. Wendlandt, The meromorphic $R$-matrix of the Yangian, arXiv:1907.03525 (2019), 48 pages (To appear in Progress in Mathematics, volume dedicated to N. Reshetikhin’s 60th birthday).
11. N. Guay, H. Nakajima, and C. Wendlandt, *Coproduct for Yangians of affine Kac-Moody algebras*, Adv. Math. **338** (2018), 865–911. MR 3861718

12. N. Guay and Y. Tan, *Local Weyl modules and cyclicity of tensor products for Yangians*, J. Algebra **432** (2015), 228–251. MR 3334147

13. K. Iohara, *Bosonic representations of Yangian double $DY_h(g)$ with $g = gl_N, sl_N$*, J. Phys. A **29** (1996), no. 15, 4593–4621.

14. S. M. Khoroshkin and V. N. Tolstoy, *Yangian double*, Lett. Math. Phys. **36** (1996), 373–402.

15. H. Knight, *Spectra of tensor products of finite dimensional representations of Yangians*, J. Algebra **174** (1995), 187–196.

16. S. Z. Levendorskii, *On generators and defining relations of Yangians*, Journal of Geometry and Physics **12** (1992), 1–11.

17. A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, vol. 143, A.M.S., 2007.

18. C. Wendlandt, *The formal shift operator on the Yangian double*, [arXiv:2008.10590](https://arxiv.org/abs/2008.10590), 2020.

Department of Mathematics, The Ohio State University, Columbus, OH 43210 (USA)

E-mail address: gautam.42@osu.edu

Department of Mathematics, The Ohio State University, Columbus, OH 43210 (USA)

E-mail address: wendlandt.4@osu.edu