SPECTRAL FLOW AND THE UNBOUNDED KASPAROV PRODUCT

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ABSTRACT. We present a fairly general construction of unbounded representa-
tives for the interior Kasparov product. As a main tool we develop a theory of 
$C^1$-connections on operator $*$-modules; we do not require any smoothness as-
sumptions; our $\sigma$-unitality assumptions are minimal. Furthermore, we use work 
of Kucerovsky and our recent Local Global Principle for regular operators in 
Hilbert $C^*$-modules.

As an application we show that the Spectral Flow Theorem and more generally 
the index theory of Dirac-Schrödinger operators can be nicely explained in terms 
of the interior Kasparov product.


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1. Introduction

The Spectral Flow Theorem ([RoSa95], or quite recently [GLM^+11]) relates the spectral flow of a family, \( A(x) \), of unbounded selfadjoint Fredholm operators to the index of the Fredholm operator \( D = \frac{d}{dx} + A(x) \). D is an example of a so called Dirac-Schrödinger operator on the complete manifold \( \mathbb{R} \). Index theorems for such operators, at least in the special case where \( A(x) \) is a finite rank bundle morphism, were established in the 80s and 90s, e.g., Anghel [Ang93a], [Ang93b] or [Les97, Chap. IV] and the references therein.

The family \( \{ A(x) \}_{x \in \mathbb{R}} \) naturally defines a class \([F_1]\) in the first K-theory group of \( C_0(\mathbb{R}) \), while the Dirac-operator \(-i\frac{d}{dx}\) defines a class \([F_2]\) in the first K-homology group of the same C*-algebra. It follows from KK-theory that the spectral flow of \( A(x) \) can be recovered as the Kasparov product of the classes \([F_1]\) and \([F_2]\) via the natural identification \( KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z} \), cf. [BLA98, Sec. 18.10]. On the other hand classes in \( KK(\mathbb{C}, \mathbb{C}) \) are represented by Fredholm operators and therefore the Spectral Flow Theorem can be rephrased in the following way: The Dirac-Schrödinger operator \( D = \frac{d}{dx} + A(x) \), viewed as an unbounded C*-Kasparov module represents the interior Kasparov product of \([F_1] \in K_1(C_0(\mathbb{R}))\) and \([F_2] \in K^1(C_0(\mathbb{R}))\).

It is tempting to generalize this pattern by replacing the real line by a complete Riemannian manifold. The family then becomes parametrized by the manifold whereas a Dirac operator (or, slightly more generally, a first order elliptic operator with bounded propagation speed) on the complete manifold naturally replaces \(-i\frac{d}{dx}\). For the realization of this program it turns out that the existing theories of unbounded representatives for the KK-product, see e.g., [BAJu83], [Kuc97], [Mes09]), do not suffice. It is the purpose of this paper to establish an appropriate improvement of unbounded KK-theory which naturally covers Dirac-Schrödinger operators on complete manifolds.

In the paper [Mes09] Mesland develops a framework of smooth algebras and differentiable C*-modules equipped with smooth connections with the purpose of establishing a general formula for the unbounded KK-product. We pursue a less technical approach: Since unbounded Kasparov modules are abstractions of first order elliptic differential operators it is most natural to impose \( C^1 \)-conditions on both modules and connections. It turns out that the theory of operator modules and complete boundedness provides a good operator algebraic framework for treating such concepts.

More concretely, let us fix a pair of unbounded (odd) Kasparov modules \((X, D_1)\) and \((Y, D_2)\) over C*-algebras A-B and B-C, respectively. The C*-algebra B then
possesses a dense operator \(\ast\)-algebra (cf. Section 3) \(B_1\) which is the largest algebra for which the unbounded derivation defined by \(D_2\) yields bounded adjointable operators. This operator \(\ast\)-algebra defines a \(C^1\)-structure on the \(C^\ast\)-algebra \(B\). By Kasparov’s stabilization theorem [Kas80a], [Blag8, Sec. 13.6.2], [Lan95, Sec. 6.2] the Hilbert \(C^\ast\)-module \(X\) is a direct summand in the standard module \(B^\infty\) over \(B\). Let \(P \in \mathcal{L}(B^\infty)\) denote the projection with \(PB^\infty \cong X\). We say that \(X\) has a \(C^1\)-structure if the projection \(P\) descends to a completely bounded projection (see e.g., [ChSi87]) on the standard module \(B_1^\infty\) over the operator \(\ast\)-algebra \(B_1\). The image \(PB_1^\infty \subseteq PB^\infty\) is an operator \(\ast\)-module over \(B_1\). A \(C^1\)-structure on \(X\) gives rise to a Graßmann \(D_2\)-connection \(\nabla_{D_2}\) (Def. 4.6, cf. [Con80, p. 600]) which is an essential ingredient for the construction of the unbounded KK-product.

To formulate our main result we need to introduce one more technical device, namely that of a correspondence between \((X, D_1)\) and \((Y, D_2)\) (Def. 6.3). Roughly speaking a correspondence from \((X, D_1)\) to \((Y, D_2)\) is a pair \((X_1, \nabla^0)\) consisting of the operator \(\ast\)-module \(PB_1^\infty\) over \(B_1\) and a hermitian \(D_2\)-connection \(\nabla^0_{D_2} : PB_1^\infty \to X \hat{\otimes}_B \mathcal{L}(Y)\) such that

1. The commutator \([1 \otimes \nabla^0 D_2, a] : \mathcal{D}(1 \otimes \nabla^0 D_2) \to X \hat{\otimes}_B Y\) is well-defined and extends to a bounded operator on \(X \hat{\otimes}_B Y\) for all \(a \in A_1\).
2. The unbounded operator

\[
[D_1 \otimes 1, 1 \otimes \nabla^0 D_2]([D_1 \otimes 1 - i \cdot \mu]^{-1} : \mathcal{D}(1 \otimes \nabla^0 D_2) \to X \hat{\otimes}_B Y
\]

is well-defined and extends to a bounded operator on \(X \hat{\otimes}_B Y\), for all \(\mu \in \mathbb{R} \setminus \{0\}\).

Note that (2) is less restrictive than the commutator conditions imposed by Mesland in [Mes09, Definitions 4.9.1 and 4.9.5]; the latter in particular imply the boundedness of the commutator \([D_1 \otimes 1, 1 \otimes \nabla^0 D_2]\). Our weaker condition of relative boundedness of the commutator is already needed for the Spectral Flow Theorem over the real line in the context of [RoSa95].

The main result of this paper can then be stated as follows:

**Theorem 1.1.** Let \((X, D_1)\) and \((Y, D_2)\) be two odd unbounded Kasparov modules for \((A, B)\) and \((B, C)\) respectively. Suppose that there exists a correspondence \((X_1, \nabla^0)\) from \((X, D_1)\) to \((Y, D_2)\). Let \(\nabla_{D_2} : X_1 \to X \hat{\otimes}_B \mathcal{L}(Y)\) be any hermitian \(D_2\)-connection. Then the pair \((D_1 \times \nabla^0 D_2, (X \hat{\otimes}_B Y)^2)\) is an even unbounded Kasparov \(A\–C\) module which represents the interior Kasparov product of \((X, D_1)\) and \((Y, D_2)\).

Here \(D_1 \times \nabla^0 D_2\) is essentially the operator \(D_1 \otimes 1 + i \cdot \nabla^0 D_2\), see Eq. (5.1) below. Theorem 1.1 is a combination of Theorem 6.7 and Theorem 7.5.

Let us briefly outline the above mentioned application to Dirac-Schrödinger operators. Let \(M\) be a complete oriented manifold of dimension \(m\) and \((D_1(x))_{x \in M}\) a family of unbounded selfadjoint operators parametrized by the manifold. We assume that the domain \(W := \mathcal{D}(D_1(x))\) is independent of \(x\) and that the graph norms of the family are uniformly equivalent. Furthermore, the map
$D_1 : M \to \mathcal{L}(W,H)$ is assumed to be weakly differentiable with uniformly bounded derivative; see Subsection 8.3 for the precise formulation. On top of these conditions we will require that the inclusion $i : W \to H$ is compact and that the spectra of $D_1(x), x \in M$, are uniformly bounded away from zero outside a compact set $K \subseteq M$. These conditions are essentially those required by Robbin and Salomon in the one-dimensional scenario, [RoSa95, A1-A3]. On the other hand, we let $D_2 : \Gamma_\infty(M,F) \to L^2(M,F)$ be a first order formally selfadjoint elliptic differential operator with bounded propagation speed; Here $F$ is a hermitian vector bundle over $M$.

**Theorem 1.2.** Suppose that the conditions outlined before are satisfied. The Dirac–Schrödinger operator $\lambda D_1(x) + iD_2 : \mathcal{D}(D_1(x)) \cap \mathcal{D}(D_2) \to L^2(M,H \otimes F)$ is an unbounded Fredholm operator for $\lambda > 0$ large enough. Furthermore, its Fredholm index coincides with the integer given by the interior Kasparov product $[D_1(\cdot)] \hat{\otimes} C_0(M)[D_2] \in KK(C, C)$ under the canonical identification $KK(C, C) \cong \mathbb{Z}$.

Theorem 1.2 is proved in the final Section 8. The proof of Theorem 1.1 consists of two main steps. First of all one needs to prove that the pair $(D_1 \times_\nabla D_2, (X \hat{\otimes}_B Y)^2)$ is an unbounded Kasparov module. This includes the selfadjointness and regularity of the unbounded product operator $D_1 \times_\nabla D_2$. Once this is place we can apply the work of Kucerovsky [Kuc97] which establishes general criteria for recognizing an unbounded Kasparov module as a representative of the interior Kasparov product. The selfadjointness and regularity of the unbounded product operator is handled by the following result which the authors proved in a predecessor of this paper.

**Theorem 1.3** ([Kale11, Theorem 7.10]). Let $S$ and $T$ be two selfadjoint and regular operators on a Hilbert $C^*$–module $E$. Suppose that there exists a core $E$ for $T$ such that the following two conditions are satisfied:

1. We have the inclusions $(S - i \cdot \mu)^{-1}(\xi) \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and $T(S - i \cdot \mu)^{-1}(\xi) \in \mathcal{D}(S)$ for all $\mu \in \mathbb{R} \setminus \{0\}$ and all $\xi \in E$.
2. The unbounded operator $[S, T](S - i \cdot \mu)^{-1} : E \to E$ extends to a bounded operator on $E$ for all $\mu \in \mathbb{R} \setminus \{0\}$.

Then the unbounded anti–diagonal operator

$$D = \begin{pmatrix} 0 & S - i T \\ S + i T & 0 \end{pmatrix} : (\mathcal{D}(S) \cap \mathcal{D}(T))^2 \to E^2$$

is selfadjoint and regular. Here the power refers to the cartesian product (i.e., direct sum) of modules. In particular, the inclusions of domains $\mathcal{D}(D) \hookrightarrow \mathcal{D}(S), \mathcal{D}(D) \hookrightarrow \mathcal{D}(T)$ are continuous.

Finally, we briefly explain how the paper is organized:

In Section 2 we give a quick summary of the theory of operator spaces and introduce the notion of an operator $*$-algebra. As a geometric example we endow
the $\ast$-algebra of $C^1$-functions vanishing at infinity on a Riemannian manifold with an operator $\ast$-algebra structure.

Next, in Section 3 we discuss operator $\ast$-modules. As hinted at earlier an operator $\ast$-module is a direct summand in the standard module over an operator $\ast$-algebra. We think of operator $\ast$-modules as the analogues of Hilbert $C^*$-modules where the $C^*$-algebra is replaced by the more flexible notion of an operator $\ast$-algebra.

Section 4 is devoted to the theory of connections on operator $\ast$-modules. In analogy to the geometric theory of connections the space of connections is an affine space modelled on a certain space of completely bounded $\Lambda$-linear operators; there is a canonical connection, called the Graßmann connection, which arises from the operator $\ast$-module structure.

Section 5 is concerned with proving selfadjointness and regularity of the unbounded product operator $D_1 \times_\nabla D_2$. In the following Section 6 we prove that the pair consisting of the unbounded product operator and the interior tensor product of $X$ and $Y$ is an unbounded Kasparov module. In particular we show that the resolvent of the unbounded product operator is compact.

Kucerovsky’s criterion [Kuc97, Theorem 13], stated in detail as Theorem 7.2 below, is then applied in Section 7 to ultimately prove that our unbounded product construction yields an unbounded version of the interior Kasparov product.

The final Section 8 then treats the geometric example of Dirac-Schrödinger operators and proves Theorem 1.2.

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2. Operator $\ast$-algebras

The purpose of this section is to introduce the notion of an operator $\ast$-algebra. This concept will be used throughout the paper. We start by reviewing the relevant theory of operator spaces. This subject is well-treated in the literature. See for example Ruan [Rua88], Blecher [Ble96] and the references therein.

2.1. Preliminaries on operator spaces. Let $X$ be a Banach space over the complex numbers. The norm on $X$ will be denoted by $\| \cdot \|_X : X \to [0, \infty)$.

We will use the notation $M(\mathbb{C})$ for the $\ast$-algebra of infinite matrices over $\mathbb{C}$ with only finitely many entries different from 0. $M(\mathbb{C})$ can be identified with the direct
limit \( \lim_{n \to \infty} M_n(\mathbb{C}) \) where the limit is taken with respect to the inclusions

\[
i_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}), \quad i_n(v) = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}.
\]

(2.1)

We will refer to the \( \mathcal{C} \)-algebra \( M(\mathbb{C}) \) as the finite matrices over \( \mathbb{C} \).

For each \( n \in \mathbb{N} \) the \( \ast \)-algebra \( M_n(\mathbb{C}) \) is faithfully represented on the Hilbert space \( \mathbb{C}^n \). We can thus endow \( M_n(\mathbb{C}) \) with the operator norm coming from this faithful representation. The inclusions in Eq. (2.1) then become isometries and we obtain a well-defined norm on the direct limit \( M(\mathbb{C}) \). The norm on \( M(\mathbb{C}) \) will be denoted by \( \| \cdot \|_C : M(\mathbb{C}) \to [0, \infty) \). Alternatively, \( M(\mathbb{C}) \) is faithfully represented on the Hilbert space \( \ell^2(\mathbb{Z}_+) \) of square-summable sequences and \( \| \cdot \|_C \) is the norm induced by this representation.

For a Banach space \( X \) we denote by \( M(X) := M(\mathbb{C}) \otimes_\mathbb{C} X \) the algebraic tensor product of the finite matrices over \( \mathbb{C} \) and the Banach space \( X \). This vector space has the structure of a \( M(\mathbb{C}) \)-\( M(\mathbb{C}) \) bimodule in the obvious way. We will refer to the bimodule \( M(X) \) as the finite matrices over \( X \).

**Definition 2.1** (Operator Space). A Banach space \( (X, \| \cdot \|) \) is called an operator space if there exists a norm \( \| \cdot \|_X : M(X) \to [0, \infty) \) (a priori to be distinguished from the norm, \( \| \cdot \|, \) on \( X \)) on the finite matrices over \( X \) such that

(1) For any pair of finite matrices over \( \mathbb{C}, v, w \in M(\mathbb{C}) \), and any finite matrix over \( X, x \in M(X) \), we have the inequality

\[
\|v \cdot x \cdot w\|_X \leq \|v\|_C \cdot \|x\|_X \cdot \|w\|_C.
\]

(2) For any pair of projections, \( p, q \in M(\mathbb{C}) \), with \( pq = 0 \) and any finite matrices \( x, y \in M(X) \) we have the identity

\[
\|pxp + qyq\|_X = \max(\|pxp\|_X, \|qyq\|_X).
\]

(3) For any projection, \( p \in M(\mathbb{C}) \), of rank one and any element \( x \in X \) we have the identity \( \|pxx\|_X = \|x\| \).

The last condition ensures that the norm \( \| \cdot \|_X \) is compatible with the given norm on \( X \) and hence in the sequel we will always write \( \| \cdot \|_X \).

A closed subspace of a \( C^* \)-algebra is naturally an operator space. Conversely, every operator space is isometric to a subspace of the algebra \( \mathcal{L}(\mathcal{H}) \) of bounded operators on some Hilbert space, see [Rud88]. We will now review some standard constructions for operator spaces.

For \( m \in \mathbb{N} \) we can make the \( (m \times m) \)-matrices over the operator space \( X \) into an operator space as follows: we define the norm on the finite matrices over \( M_m(X) \) using an appropriate identification \( M_n(M_m(X)) \cong M_{nm}(X) \) of vector spaces for each \( n \in \mathbb{N} \). Furthermore, we let \( \overline{M}(X) \) denote the completion of \( M(X) \) in the operator norm. This normed space can be given the structure of an operator space by using the identification \( M_n(\overline{M}(X)) \cong \overline{M}(M_n(X)) \).
The direct sum \(X^m = \bigoplus_{i=1}^m X\) can be embedded into the \((m \times m)\)-matrices over \(X\) using the injective linear map

\[
\varphi : X^m \to M_m(X), \quad \varphi([x_i]) = \sum_{i=1}^m e_{i1} \otimes x_i = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ x_m & 0 & \ldots & 0 \end{pmatrix}.
\]

(2.2)

Here \(e_{i1} \in M_m(X)\) denotes the matrix with 1 in position \((i, 1)\) and zeros elsewhere. This embedding gives \(X^m\) the structure of an operator space. Finally, the infinite direct sum, \(X^\infty\), is defined as the completion of the finite sequences \(c_0(X) := \bigoplus_{i=1}^\infty X\) with respect to the norm of the matrix algebra \(M(X)\), i.e., the closure of \(c_0(X)\) inside \(\overline{M}(X)\). The operator space structure on \(X^\infty\) is given by the identification \(M_m(X^\infty) \cong M_m(X)\).

We will say that a continuous linear map \(\alpha : X \to Y\) between the operator spaces \(X, Y\) is completely bounded if the supremum of operator norms, \(\sup_{n \in \mathbb{N}} \|M_n(\alpha)\|\), is finite. Here the notation \(M_n(\alpha)\) stands for the continuous linear map \(\text{id} \otimes \alpha : M_n(X) \cong M_n(\mathbb{C}) \otimes X \to M_n(\mathbb{C}) \otimes Y \cong M_n(Y)\) between the matrix spaces which is induced by \(\alpha\). The vector space of completely bounded linear maps from \(X\) to \(Y\) will be denoted by \(\text{CB}(X, Y)\). This vector space becomes a Banach space when equipped with the norm defined by \(\|\alpha\|_{\text{cb}} := \sup_{n \in \mathbb{N}} \|M_n(\alpha)\|\). In fact, it can be proved that the Banach space of completely bounded maps can be turned into an operator space. The norms on finite matrices come from the identification of vector spaces

\[M_n(\text{CB}(X, Y)) \cong \text{CB}(X, M_n(Y)), \quad \text{for } n \in \mathbb{N};\]

see [ErRu88, p. 140]. We remark that a map \(\alpha : X \to Y\) is completely bounded if and only if it induces a bounded map \(\alpha : \overline{M}(X) \to \overline{M}(Y)\); the latter is then automatically completely bounded.

Finally, we will say that the operator spaces \(X\) and \(Y\) are completely isomorphic if there exists a completely bounded vector space isomorphism \(\alpha : X \to Y\) with completely bounded inverse. Note that the complete boundedness of the inverse is not automatic here. It would follow if we knew that the induced map \(\alpha : \overline{M}(X) \to \overline{M}(Y)\) between Banach spaces were bijective. This, however, does of course not follow from the bijectivity and complete boundedness of \(\alpha : X \to Y\).

### 2.2. Operator \(*\)-algebras.

**Definition 2.2.** Let \(X\) be an operator space which is at the same time an algebra over the complex numbers. We will call \(X\) an operator algebra if the multiplication \(m : X \times X \to X\) is completely bounded. This means that there exists a constant \(K > 0\) such that \(\|x \cdot y\| \leq K \cdot \|x\| \cdot \|y\|\) for all \(x, y \in M(X)\).

We remark that a closed sub-algebra of a \(C^*\)-algebra is an operator algebra. Indeed, the norm on the finite matrices is induced by the unique \(C^*\)-norm on the matrices over the \(C^*\)-algebra. The converse is also true by a theorem of D. P.
Blecher [Ble95, Theorem 2.2], cf. also Christensen–Sinclair [ChSi87], and Paulsen–Smith [PaSm87].

We are now ready to introduce the concept of an operator *–algebra, cf. [Mes09, Def. 3.2.3] and [Iva11, Def. 3.3].

**Definition 2.3.** Suppose that \( X \) is an operator *–algebra. We will say that \( X \) is an operator *–algebra if \( X \) has a completely bounded involution \( \dagger : X \to X \). Here, the involution on matrices is defined as usual by transposing the matrix and replacing each entry \( x \) by \( x^\dagger \), i.e., \( \{x_{ij}\}^\dagger = \{x_{ji}\} \).

**Example 2.4.** An important example is provided by a closed subalgebra \( A \subseteq B \) of a C*–algebra \( B \) together with a *–automorphism \( \sigma : B \to B \) with square equal to the identity and with \( \sigma(x)^* \in A \) for all \( x \in A \). Defining a new involution on \( A \) by \( x^\dagger := \sigma(x)^* \) turns \( A \) together with \( \dagger \) into an operator *–algebra. This involution is actually completely isometric in the sense that \( \|x^\dagger\| = \|x\| \) for all \( x \in M(A) \). Note that \( A \) is not necessarily (neither with * nor with \( \dagger \)) a sub–*–algebra of \( B \).

For later reference we introduce the concept of \( \sigma \)–unitality for operator *–algebras.

**Definition 2.5.** An operator *–algebra \( X \) is called \( \sigma \)–unital if there exists a bounded sequence \( \{u_m\} \) in \( X \) such that

\[
\lim_{m \to \infty} \|u_m \cdot x - x\|_X = 0 = \lim_{m \to \infty} \|x \cdot u_m - x\|_X, \quad \text{for all } x \in X.
\]

We will refer to the sequence \( \{u_m\} \) as a (bounded) approximate unit for \( X \).

The boundedness of \( \{u_m\} \) and the complete boundedness of the map \( \dagger \) ensure that with \( \{u_m\} \) the sequences \( \{u_m^\dagger\} \) and \( \{u_m u_m^\dagger\} \) are bounded approximate units as well.

### 2.3. Geometric examples of operator *–algebras

We will start by discussing another structure which, by using Example 2.4, can be used to construct an operator *–algebra.

**Proposition 2.6.** Assume that we are given

1. A Hilbert C*–module \( E \) over the C*–algebra \( B \).
2. A *–algebra, \( A \), together with a *–homomorphism \( \pi : A \to \mathcal{L}(E) \) into the algebra of adjointable operators, \( \mathcal{L}(E) \), on \( E \).
3. A derivation \( \delta : A \to \mathcal{L}(E) \) which vanishes on \( \ker \pi \) and satisfies \( \delta(a^*) = U \delta(a)^* U \) for \( a \in A \); here, \( U \) is some unitary \( U \in \mathcal{L}(E) \) which commutes with the elements of \( A \). The derivation property means that \( \delta(a \cdot b) = \delta(a) \cdot \pi(b) + \pi(a) \cdot \delta(b) \) for \( a, b \in A \).

Let \( A_1 \subseteq \mathcal{L}(E) \) denote the completion of \( \pi(A) \) in the norm \( \| \cdot \|_1 : \pi(a) \mapsto \|\pi(a)\|_\infty + \|\delta(a)\|_\infty \); here \( \| \cdot \|_\infty \) denotes the operator norm on \( \mathcal{L}(E) \).
Then the sub-$*$-algebra $A_1 \subseteq \mathcal{L}(E)$ can be given the structure of an operator $*$-algebra by embedding it into $\mathcal{L}(E \oplus E)$ as follows:

$$\rho : a \mapsto \begin{pmatrix} a & 0 \\ \delta(a) & a \end{pmatrix} \in \mathcal{L}(E \oplus E).$$

**Remark 2.7.** 1. Since $\delta$ vanishes on $\ker \pi$ it descends to a derivation on $\pi(A)$ and, by continuity, to a derivation $A_1 \to \mathcal{L}(E)$ which is again denoted by $\delta$. By the very construction $\delta : A_1 \to \mathcal{L}(E)$ is completely bounded.

2. Let $A$ be the $C^*$-completion of $\pi(A)$, i.e., the completion of $\pi(A)$ with respect to the norm of $\mathcal{L}(E)$. Then the natural inclusion $A_1 \hookrightarrow A$ is completely bounded. Indeed, for $a \in M_n(A_1)$ and $\xi \in \mathbb{C}^n$ we have

$$\|a \cdot \xi\| \leq \left\| \begin{pmatrix} a & 0 \\ \delta(a) & a \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\| \leq \|a\|_1 \cdot \|\xi\|_1.$$

3. An important application of Proposition 2.6 is the following, cf. also the beginning of Sec. 4 below. Consider a triple $(A, E, D)$, where $A$ and $E$ are as in (1) and (2) of Proposition 2.6 and where $D$ is a selfadjoint densely defined unbounded operator in $E$ such that for each $a \in A$ the operator $\pi(a)$ maps the domain of $D$ into itself and the commutator $[D, \pi(a)]$ is in $\mathcal{L}(E)$.

Put $\delta : A \to \mathcal{L}(E), a \mapsto [D, \pi(a)] \in \mathcal{L}(E)$. Then $\delta(a^*) = -[D, a]^* = i[D, a]^*i$, thus (3) of Proposition 2.6 is satisfied with $U = i \cdot 1$. As a consequence we obtain a new triple $(A_1, E, D)$ satisfying (1) and (2) of Proposition 2.6 where now $A_1 \subseteq \mathcal{L}(E)$ is an operator $*$-algebra, the inclusion $A_1 \hookrightarrow A$ into its $C^*$-completion is completely bounded and $\delta = [D, \cdot] : A_1 \to \mathcal{L}(E)$ is completely bounded.

**Proof.** The derivation property of $\delta$ ensures that $\rho$ is an algebra homomorphism; it is injective since $A$ is embedded into $\mathcal{L}(E)$. Furthermore, the norm induced by $\mathcal{L}(E \oplus E)$ on $A_1$ is equivalent to $\| \cdot \|_1$.

$\rho$, however, does not preserve the involution. Instead, we have

$$\rho(a^*) = V\rho(a)^*V, \quad \text{with } V = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}. \quad (2.3)$$

$V$ is a unitary with $V^2 = 1$. Thus with the inner automorphism of $\mathcal{L}(E \oplus E)$ defined by $\sigma(\xi) = V\xi V$ we have $\sigma^2 = 1$ and $\rho(a^*) = \sigma(\rho(a))^* =: \rho(a)^\dagger$. Thus according to Example 2.4 the algebra $\rho(A_1)$ with involution $\rho(a)^\dagger := \sigma(\rho(a))^*$ is an operator $*$-algebra and $\rho$ is a $*$-isomorphism from $(A_1, *)$ onto $(\rho(A_1), \dagger)$. \hfill $\Box$

Remark that Proposition 2.6 is very much related to the example appearing in [MESO9, Sec. 3.1].

Next we apply Proposition 2.6 to the algebra of $C^1$–functions which vanish at infinity on an oriented Riemannian manifold $M^m$ of dimension $m$. The orientation assumption is made for convenience only to have the Hodge $*$ operator$^\dagger$ at

$^\dagger$At this point we have to deal with at least three mathematical objects whose standard notation is *: the involution, $\dagger$, on an operator $*$-algebra $A_1 \subseteq \mathcal{L}(E)$, which is to be distinguished from the
our disposal without having to deal with the orientation line bundle. With more notational effort the orientation assumption can be disposed. We will use the notation \( d \) for the exterior derivative of complex valued forms on \( M \). Furthermore, we let \( \dagger \) denote the involution on complex valued forms given by complex conjugation.

A section \( s \) in a hermitian vector bundle, \( \mathcal{E} \), over \( M \) is said to vanish at infinity if for each \( \varepsilon > 0 \) there exists a compact subset \( K \subset M \) such that \( \|s(x)\|_{\varepsilon} < \varepsilon \) for all \( x \in M \setminus K \). In short we write \( \lim_{x \to \infty} s(x) = 0 \). We denote by \( C^1_0(M) \) space of continuously differentiable complex valued functions on \( M \) for which

\[
\lim_{x \to \infty} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} df(x) = 0.
\]

Here, \( d \) is the exterior derivative. \( C^1_0(M) \) is a \( \ast \)-algebra with involution defined by \( f^\dagger(x) = \overline{f(x)} \). Denote by

\[
\Omega^p(M) := \Gamma^\infty(M, \Lambda^p T^*M \otimes \mathbb{C})
\]

the smooth complex valued \( p \)-forms, \( \text{i.e.,} \) the smooth sections of the hermitian vector bundle \( \Lambda^p T^*M \otimes \mathbb{C} \). Since this vector bundle is the complexification of a real vector bundle, complex conjugation is well-defined for forms and for \( \omega \in \Omega^p(M) \) we therefore put \( \omega^\dagger(x) := \overline{\omega(x)} \). The scalar product on the bundles \( T^*M \otimes \mathbb{C} \) is induced by the Riemannian metric. The metric and the Hodge star operator are tied together by the formula

\[
\langle \omega, \eta \rangle \, \text{vol}_x = \omega \wedge \ast \eta
\]

for \( \omega, \eta \in \Lambda^m T^*_x M \otimes \mathbb{C} \); here \( \text{vol}_x \in \Lambda^m T^*_x M \) denotes the Riemannian volume element.

The space of bounded continuous sections \( \Gamma_b(\Lambda^* T^*M \otimes \mathbb{C}) \) forms a graded commutative algebra which acts by left multiplication on the space \( L^2(\Lambda^* T^*M \otimes \mathbb{C}) \) of square-integrable sections of \( \Lambda^* T^*M \otimes \mathbb{C} \). This representation is faithful and hence we view \( \Gamma_b(\Lambda^* T^*M \otimes \mathbb{C}) \) as a sub-\( \ast \)-algebra of the \( \ast \)-algebra \( \mathcal{L}(L^2(\Lambda^* T^*M \otimes \mathbb{C})) \).

Note that \( C^1_0(M) \) is a sub-\( \ast \)-algebra of \( \Gamma_b(\Lambda^* T^*M \otimes \mathbb{C}) \subset \mathcal{L}(L^2(\Lambda^* T^*M \otimes \mathbb{C})) \).

The exterior derivative on functions now induces a natural derivation

\[
\delta : C^1_0(M) \rightarrow \mathcal{L}(L^2(\Lambda^* T^*M \otimes \mathbb{C})), \quad \delta f(\omega) := df \wedge \omega,
\]

for \( \omega \in L^2(\Lambda^* T^*M \otimes \mathbb{C}) \). \hspace{1cm} (2.7)

In order to apply Proposition 2.6 to this situation we need to find the unitary \( U \) which commutes with \( C^1_0(M) \), has square identity, and \( \delta f^\dagger = U(\delta f)^\dagger U \). \( U \) is provided by the Hodge \( \ast \) operator as follows:

\[\text{native involution, } \ast \text{, on the } \ast \text{-algebra } \mathcal{L}(\mathcal{E}) \text{ and finally the Hodge star operator, } \ast \text{. To distinguish the three objects notationally, we denote them by } \dagger, \ast, \text{ and } \ast, \text{ respectively.} \]
Let

\[ \lambda_0 := \sqrt{-1}^{m(m-1)/2}, \quad \text{and} \quad \lambda_p := (-1)^{p(p-1)/2} \lambda_0, \]

for \( p = 0, \ldots, m \). \( \text{(2.8)} \)

Then one checks the following identities:

\[
\begin{align*}
\lambda_p \cdot \lambda_{m-p} &= (-1)^{p(m-p)}, & \text{for } p = 0, \ldots, m, & \text{(2.9)} \\
\lambda_{p+1} &= (-1)^p \lambda_p, & \text{for } p = 0, \ldots, m-1, & \text{(2.10)} \\
\lambda_{p+q} \cdot \lambda_{m-q} &= (-1)^{p(p-1)/2 + q(m-p-q)}, & \text{for } p, q, p + q \in \{1, \ldots, m\}. & \text{(2.11)}
\end{align*}
\]

where Eq. (2.9) is a special case of Eq. (2.11). Define the modified \( * \) operator on \( p \)-forms by \( \tilde{\chi}_p := \tilde{\chi}_p \lambda_p \). Then \( \tilde{\chi} \) is a unitary which commutes with the action of \( C^*_0(M) \) and which satisfies \( \tilde{\chi}^2 = 1 \). Moreover, for any \( p \)-form, \( \omega \in \Omega^p(M) \), the adjoint of the operator \( \text{ext}(\omega) := \omega \wedge \cdot \) of exterior multiplication by \( \omega \) is given by

\[
\text{ext}(\omega)^* = (-1)^{p(p-1)/2} \tilde{\chi} \text{ext}(\omega) \tilde{\chi},
\]

in particular

\[
(\delta f)^* = \tilde{\chi} \delta (f^\dagger) \tilde{\chi}.
\]

In view of Proposition 2.6 we have proved.

Proposition 2.8. The map

\[
\pi : C^*_0(M) \to \mathcal{L}(L^2(\Lambda^*T^*M) \oplus L^2(\Lambda^*T^*M)), \quad \pi(f) = \begin{pmatrix} f & 0 \\ df & f \end{pmatrix}
\]

is a \( * \)-isomorphism from \( (C^*_0(M), \dagger) \) onto an operator \( * \)-subalgebra of \( \mathcal{L}(L^2(\Lambda^*T^*M) \oplus L^2(\Lambda^*T^*M)) \) with involution given by \( \pi(f)^\dagger = V \pi(f) V \), where \( V = \begin{pmatrix} 0 & \tilde{\chi} \\ \ast & 0 \end{pmatrix} \) and \( \ast \) is defined on \( \Lambda^p \) as \( \lambda_p \ast_p \) with \( \lambda_p \) from Eq. (2.8).

This gives \( C^*_0(M) \) naturally the structure of an operator \( * \)-algebra.

**Note 2.9** (Approximate unit on complete manifolds). For future reference we note that if the oriented Riemannian manifold \( M \) is complete then we can choose a sequence of smooth compactly supported functions \( \{\chi_k\} \) such that

1. The image of each \( \chi_k \) is contained in the interval \([0, 1]\), i.e., \( 0 \leq \chi_k \leq 1 \).
2. The exterior differentials, \( d\chi_k \), converge to \( 0 \) uniformly, more precisely it can be arranged that \( \|d\chi_k\|_{\infty} \leq 1/k \) for all \( k \in \mathbb{N} \).
3. For each compact subset \( K \subset M \) there is an index \( k_0 \) such that \( \chi_k(x) = 1 \) for all \( x \in K \) and all \( k \geq k_0 \).

See, e.g., [Wol73, Sec. 5], [LAM89, p. 117], or [LES97, Lemma 3.2.4]. In particular, the sequence \( \{\chi_k\} \) is a bounded approximate unit for the operator \( * \)-algebra \( C^*_0(M) \). Thus, in the case of an oriented complete manifold the operator \( * \)-algebra \( C^*_0(M) \) is \( \sigma \)-unital.
3. Operator \(*\text{-}\)modules

The purpose of this section is to introduce the notion of an operator \(*\text{-}\)module. Stated a little vaguely, an operator \(*\text{-}\)module is a direct summand in a standard module over an operator \(*\text{-}\)algebra. In particular, by Kasparov’s stabilization theorem each countably generated Hilbert \(C^\ast\)–module is an operator \(*\text{-}\)module \([\text{Kas80a}], [\text{Blag8}, \text{Sec. 13.6.2}], [\text{Lan95}, \text{Sec. 6.2}].\) However, the concept is more general. For example, we show that the Hilbert space-valued \(C^1\)–functions which vanish at infinity on an oriented Riemannian manifold form an operator \(*\text{-}\)module. We expect that, under reasonable assumptions, \(C^1\)–sections of Hilbert bundles which vanish at infinity are operator \(*\text{-}\)modules as well, cf. Remark 3.7 below.

We start by giving the main operator algebraic definitions.

**Definition 3.1.** Let \(A\) be an operator algebra in the sense of Definition 2.2. Furthermore, let \(X\) be a right-module over \(A\).

We will then say that \(X\) is a right operator module over \(A\) if \(X\) is equipped with the structure of an operator space such that the right action \(X \times A \to X\) is completely bounded. This means that there exists a constant \(K > 0\) such that

\[
\|\xi \cdot a\|_X \leq K \cdot \|\xi\|_X \cdot \|a\|_A, \quad \text{for all } \xi \in M(X) \text{ and all } a \in M(A).
\]

Operator modules are well-treated in the literature, see the survey \([\text{ChSi89}]\) and the references therein.

**Definition 3.2.** Suppose that \(A\) is an operator \(*\text{-}\)algebra in the sense of Definition 2.3 with involution \(\dagger\) and let \(X\) be a right operator module over \(A\) in the sense of Definition 3.1.

We will say that \(X\) is a hermitian operator module if there exists a completely bounded pairing \(\langle \cdot, \cdot \rangle_X : X \times X \to A\) satisfying the conditions

\[
\begin{align*}
\langle \xi, \eta \cdot \lambda + \rho \cdot \mu \rangle &= \langle \xi, \eta \cdot \lambda \rangle + \langle \xi, \rho \cdot \mu \rangle \\
\langle \xi, \eta \cdot x \rangle &= \langle \xi, \eta \rangle \cdot x \\
\langle \xi, \eta \rangle &= \langle \eta, \xi \rangle ^\dagger
\end{align*}
\]

(3.1)

for all \(\xi, \eta, \rho \in X\), \(x \in A\) and \(\lambda, \mu \in \mathbb{C}\).

The condition of complete boundedness means that the induced pairing of matrices

\[
\langle \cdot, \cdot \rangle_X : M(X) \times M(X) \longrightarrow M(A), \quad \langle \xi, \eta \rangle_{ij} = \sum_{k=1}^\infty \langle \xi_{ki}, \eta_{kj} \rangle
\]

(3.2)

is bounded in the sense that there exists a constant \(K > 0\) such that

\[
\|\langle \xi, \eta \rangle\|_A \leq K \cdot \|\xi\|_X \cdot \|\eta\|_X, \quad \text{for all } \xi, \eta \in M(X).
\]

(3.3)

Our first example of a hermitian operator module is the standard module over \(A\).
Definition 3.3. By the standard module over the operator \( \ast \)-algebra \( A \) we will understand the completion of the finite sequences \( c_0(A) \subseteq M(A) \) in the norm of \( M(A) \), thus the closure of \( c_0(A) \) in \( \overline{M}(A) \).

The standard module \( A^\infty \) is an operator module over \( A \). Furthermore, we can define the pairing

\[
\langle \cdot, \cdot \rangle : A^\infty \times A^\infty \to A \quad \langle [a_i], [b_i] \rangle = \sum_i a_i^\dagger \cdot b_i. \quad (3.4)
\]

To see that the sum is convergent we note that for \( [a_i], [b_i] \in A^\infty \) by definition the matrices, cf. Eq. (2.2),

\[
\xi := \begin{pmatrix} a_1 & 0 & \ldots \\ a_2 & 0 & \ldots \\ \vdots & \ddots & \ddots \end{pmatrix}, \quad \eta := \begin{pmatrix} b_1 & 0 & \ldots \\ b_2 & 0 & \ldots \\ \vdots & \ddots & \ddots \end{pmatrix}
\]

are in \( \overline{M}(A) \) and the pairing Eq. (3.2) yields

\[
\langle \xi, \eta \rangle_A = \xi^\dagger \cdot \eta = \begin{pmatrix} \sum_i a_i^\dagger \cdot b_i & 0 & \ldots \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}. \quad (3.6)
\]

The properties of the pairing Eq. (3.2), in particular Eq. (3.3), then imply not only the convergence of the rhs of Eq. (3.4) but also that the pairing \( \langle \cdot, \cdot \rangle \) in Eq. (3.4) is completely bounded. \( A^\infty \) is thus a hermitian operator module.

We are now ready for the main definition of this section.

Definition 3.4. Suppose that \( X \) is a hermitian operator module over the operator \( \ast \)-algebra \( A \). We will say that \( X \) is an operator \( \ast \)-module if it is completely isomorphic to a direct summand in the standard module \( A^\infty \). To be more precise, there exist a completely bounded selfadjoint idempotent \( P : A^\infty \to A^\infty \) and a completely bounded isomorphism of hermitian operator modules \( \alpha : X \to PA^\infty \). Here the selfadjointness of \( P \) means that \( \langle P[a_i], [b_i] \rangle = \langle [a_i], P[b_i] \rangle \) for all sequences \( \{a_i\}, \{b_i\} \in A^\infty \).

Suppose that \( X = PA^\infty \) and \( Y = QA^\infty \) are operator \( \ast \)-modules. A finite sequence \( \xi = \{\xi^k\}_{1 \leq k \leq N} \in c_0(X) \) may be thought of as a matrix \( \{\xi^k\}_{k=1}^\infty \) with entries in \( A \) where only finitely many rows contain nonzero entries, i.e.,

\[
\xi = \begin{pmatrix} \xi^1_1 & \xi^1_2 & \ldots \\ \vdots & \ddots & \ddots \\ \xi^N_1 & \xi^N_2 & \ldots \end{pmatrix}.
\]

(3.7)

Here, each row \( \{\xi^k_i\}_{i \in \mathbb{N}} \) lies in \( A^\infty \). The reader should be warned that the matrix \( \xi \) does not necessarily lie in \( \overline{M}(A) \). Rather, the transposed matrix, \( \xi^\dagger \), i.e., the infinite
matrix with columns given by $\xi_1^t, \xi_2^t, \ldots \in X \subseteq A^\infty$ is in $\overline{M}(A)$. The infinite matrix $(\xi^t)^\dagger \in \overline{M}(A)$ is obtained from $\xi^t \in \overline{M}(A)$ by applying the completely bounded involution $\dagger : \overline{M}(A) \to \overline{M}(A)$. Equivalently, $(\xi^t)^\dagger$ is obtained by replacing each entry $\xi_k^t$ of the matrix $\xi$ by $(\xi_k^t)^\dagger$.

For each pair of finite sequences $\xi, \eta \in c_0(X)$ and $\rho \in c_0(Y)$ we let $\theta_{\xi,\eta} : Y \to X$ denote the completely bounded module map defined by

$$\theta_{\xi,\eta}(\rho) := \xi^t \cdot \langle \eta^t, \rho \rangle = \sum_{k=1}^\infty \xi^t_k \cdot \langle \eta^t_k, \rho \rangle,$$

for $\rho \in Y$,

in fact

$$\|\theta_{\xi,\eta}\|_{cb} \leq C \cdot \|\xi\|_X \cdot \|\eta\|_Y$$

(3.8)

with some constant $C > 0$ independent of $\xi, \eta$. Note that $\theta_{\xi,\eta} : Y \to X$ is given by matrix multiplication by the infinite matrix $\xi^t \cdot (\eta^t)^\dagger \in \overline{M}(A)$.

**Proposition 3.5.** Suppose that $X$ is an operator $\sigma$–module over the $\sigma$–unital operator $\sigma$–algebra $A$. Then there exists a sequence $(w^m)_{m=1}^\infty$ of elements in $c_0(X)$ such that $\theta_{w^m,w^m}(\rho) \to \rho$ for all $\rho \in X$. Furthermore, the sequence can be chosen to be bounded in the sense that $\sup_{m \in \mathbb{N}} \| (w^m)^t \|_X < \infty$. This means that $\sup_{m \in \mathbb{N}} \| \theta_{w^m,w^m} \|_{cb} < \infty$.

**Proof.** It suffices to prove the claim for $X = A^\infty$; for if $(w^m)_{m=1}^\infty$ is such a sequence for $A^\infty$ then $(Pw^m)_{m=1}^\infty$ does the job for $X = PA^\infty$.

Let $(u^m)_{m=1}^\infty$ be an approximate unit for $A$ in the sense of Definition 2.5. For each $m \in \mathbb{N}$ we let $(v^m)^t = (e_1 u_m, \ldots, e_m u_m)$. Here $e_i u_m \in A^\infty$ denotes the sequence with $u_m$ in position $i$ and zeros elsewhere. Then

$$\| (v^m)^t \|_{A^\infty} = \| (e_1 u_m, \ldots, e_m u_m) \|_{A^\infty} = \| 1_m \otimes u_m \|_A = \| u_m \|.$$

Here $1_m \in M(\mathbb{C})$ denotes the $(m \times m)$ unit matrix viewed as an idempotent in $M(\mathbb{C})$. Furthermore, we have used item (3) of Definition 2.1. This proves that $\sup_{m \in \mathbb{N}} \| (v^m)^t \|_{A^\infty} < \infty$ since the approximate unit $(u_m)$ is bounded in $A$. Therefore, to prove that the sequence $\{ \theta_{v^m,v^m}(\alpha) \}$ converges strongly to the identity we only need to show that $\theta_{v^m,v^m}(\alpha) \to \alpha$ for each finite sequence $\alpha = \sum_{i=1}^k e_i a_i \in c_0(A)$.

For each $m \geq k$ we have

$$\| \theta_{v^m,v^m}(\alpha) - \alpha \|_{A^\infty} = \| v^m \langle v^m, \alpha \rangle - \alpha \|_{A^\infty} = \| \sum_{i=1}^m e_i u_m (e_i u_m, \alpha) - \alpha \|_{A^\infty}$$

$$= \| \sum_{i=1}^k e_i (u_m u_m^\dagger a_i - a_i) \|_{A^\infty} \leq \sum_{i=1}^k \| u_m u_m^\dagger \cdot a_i - a_i \|_A.$$

As remarked after Definition 2.5 the sequence $(u_m u_m^\dagger)$ is a bounded approximate unit for $A$ as well, hence the right hand side converges to $0$ and the claim about strong convergence follows. The last claim is a consequence of Eq. (3.8). $\square$
We expect that the theory of operator \( \mathcal{L} \)-modules fits nicely into Blecher’s theory of rigged modules [Ble96, Def. 3.1]. Obvious candidates for the structure maps are induced by the sequence \( \{w^m\} \) of elements in \( c_0(X) \) using the module structure as well as the completely bounded pairing.

Furthermore, each countably generated Hilbert \( C^* \)-module is an operator \( \mathcal{L} \)-module. This can be seen as a consequence of Kasparov’s stabilization theorem [Kas80a], [Bla98, Sec. 13.6.2], [Lan95, Sec. 6.2].

### 3.1. The standard module for \( C^0_0(M) \)

We end this section by computing the standard module for the algebra \( C^0_0(M) \), \( M \) an oriented Riemannian manifold, cf. Subsection 2.3. For a separable Hilbert space \( H \) we denote by \( C^0_0(M,H) \) the space of \( H \)-valued continuously differentiable maps \( f : M \to H \) satisfying Eq. (2.4). \( C^0_0(M,H) \) becomes a Banach space when equipped with the norm

\[
\|f\|_1 = \sup_{x \in M} |\langle f(x), f(x) \rangle|^{1/2} + \sup_{x \in M} |\langle df(x), df(x) \rangle|^{1/2}.
\]

Furthermore, pointwise multiplication gives \( C^0_0(M,H) \) the structure of a module over \( C^0_0(M) \).

**Proposition 3.6.** Let \( M \) be an oriented Riemannian manifold and let \( H \) be a separable Hilbert space. The standard module, \( (C^0_0(M))^\infty \), over the operator \( \mathcal{L} \)-algebra \( \Lambda = C^0_0(M) \) is then isomorphic to \( C^0_0(M,H) \).

**Proof.** Let \( \{e_i\} \) be an orthonormal basis for \( H \). Recall that the submodule \( \oplus_{i=1}^\infty C^0_0(M) \) of finite sequences is dense in the standard module \( (C^0_0(M))^\infty \). Likewise, the linear span, \( \text{span}_C \{e_i f | i \in \mathbb{N}, f \in C^0_0(M)\} \), is a dense submodule of \( C^0_0(M,H) \). These two submodules are isometric by the isomorphism \( \{f_i\} \mapsto \sum_{i=1}^\infty e_i f_i \). Indeed, the norm of multiplication by the matrices

\[
\begin{pmatrix}
  f_1 & 0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  f_k & 0 & \ldots & 0
\end{pmatrix},
\begin{pmatrix}
  df_1 & 0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  df_k & 0 & \ldots & 0
\end{pmatrix}
\]

on \( L^2(\Lambda^* T^* M \otimes \mathbb{C}) \) is easily seen to be \( \sup_{x \in M} (\sum_{i=1}^k |f_i(x)|^2)^{1/2} \), resp. \( \sup_{x \in M} (\sum_{i=1}^k |df_i(x), df_i(x)|)^{1/2} \). Consequently,

\[
\|f\|_1 = \|f_i\|_\infty + \|df_i\|_\infty,
\]

where the notations \( \|f_i\|_\infty \) and \( \|df_i\|_\infty \) are shorthand for the operator norms of the multiplication by the matrices in (3.10). Since the natural norm on \( (C^0_0(M))^\infty \) is given by the right hand side of Eq. (3.11) we reach the conclusion. \( \square \)

**Remark 3.7.** With the above result in mind it seems to be a worthwhile task to characterize the operator \( \mathcal{L} \)-modules over \( C^0_0(M) \), thus the “completely bounded” direct summands in the module \( C^0_0(M,H) \), e.g., in the case of a complete oriented manifold. We expect that many interesting Hilbert bundles will appear in this way. We hope to explore this in a subsequent publication.
4. Connections on operator $*$–modules

In order to ease reference to it we are going to introduce some standard notation in the form of a numbered proclaim.

Convention 4.1. Let $X_1 = PA_1^\infty$ be an operator $*$–module over an operator $*$–algebra $A_1$. We assume that $A_1 \subseteq A$ sits as a dense $*$–subalgebra inside a $C^*$–algebra $A$ and that the inclusion $i : A_1 \hookrightarrow A$ is completely bounded, cf. Proposition 2.6 and Remark 2.7. The operator $*$–algebra norm on $A_1$ will be denoted by $\| \cdot \|_1$ and the $C^*$–algebra norm on $A$ will be denoted by $\| \cdot \|_1$.

Given such an $X_1$ the completely bounded selfadjoint idempotent $P : A_1^\infty \to A_1^\infty$ extends to an orthogonal projection $P : A^\infty \to A^\infty$. Indeed,

$$\|P(a_i)\|^2 = \|(a_i, P(a_i))\| \leq \|\{a_i\}\| \cdot \|P\{a_i\}\|$$

for all sequences $\{a_i\} \subseteq A_1^\infty$. We let $X = PA^\infty$ denote the Hilbert $C^*$–module over $A$ defined by $P : A^\infty \to A^\infty$. The inclusion $X_1 \to X$ is then completely bounded and compatible with both the inner products and the module actions.

Convention 4.2. Let $A_1$ be an operator $*$–algebra as in Convention 4.1. Let $(A,Y,D)$ be a triple consisting of:

1. An $A$–$B$ Hilbert $C^*$–bimodule $Y$. That is, $Y$ is a Hilbert $C^*$–right module over the $C^*$–algebra $A$, together with a $*$–representation $\pi : A \to \mathcal{L}_B(Y)$.
2. A selfadjoint densely defined unbounded operator $D : D(D) \to Y$ in $Y$ such that
   a) Each $a \in A_1$ maps the domain of $D$ into itself,
   b) the commutator with $D$ yields a completely bounded map $[D, \cdot] : A_1 \to \mathcal{L}(Y)$ on the operator $*$–algebra $A_1 \subseteq A$.

cf. also [Kuc97, Def. 6]. If $B = \mathbb{C}$ these are, up to the requirement of complete continuity and a missing compactness assumption, the axioms for a spectral triple $(Y,A_1,D)$, cf. [Hig06, Def. 3.1 and Remark 3.2]. For unbounded operators one has to be careful with domains; the condition (2a), which is very important, is often slightly obscured in the literature. The assumption of complete boundedness in (2b) is not very restrictive, cf. Remark 2.7, 3.

We will mostly suppress $\pi$ from the notation and write $a \cdot y$ for the action, $\pi(a)$, of $a \in A$ on $y \in Y$. The $C^*$–algebra of bounded adjointable operators on $Y$ is denoted by $\mathcal{L}_B(Y)$; if no confusion is possible we will also omit the subscript $B$.

In the sequel we will repeatedly use the interior tensor product “$\otimes_A$” of $C^*$–modules, see [Lang95, Prop. 4.5] and [Blg98, Sec. 13.5]. Recall that the Hilbert $B$–module $X \otimes_A Y$ is the completion of the algebraic tensor product $X \otimes_A Y$ with respect to the inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_B = \langle y_1, \langle x_1, x_2 \rangle_A y_2 \rangle_B, \quad \text{for } x_1, x_2 \in X, y_1, y_2 \in Y.$$
The C*-algebra $\mathcal{L}_B(Y)$ is, as every C*-algebra, a Hilbert module over itself. Thus $X \hat{\otimes}_A \mathcal{L}_B(Y)$ is a Hilbert $\mathcal{B}$-module. It is also an $\mathcal{A}$-right module via the representation $\pi : \mathcal{A} \to \mathcal{L}_B(Y)$. The action of $\mathcal{L}_B(Y)$ on $Y$ gives rise to the contraction map
\[
c : (X \hat{\otimes}_A \mathcal{L}(Y)) \otimes Y \to X \hat{\otimes}_A Y, \quad x \otimes T \otimes y \mapsto x \otimes Ty
\] (4.1)
and the inner product on $X$ induces the pairing
\[
X \times X \hat{\otimes}_A \mathcal{L}_B(Y) \to \mathcal{L}_B(Y), \quad (x, y \otimes T) = \langle x, y \rangle \cdot T
\] (4.2)

After these preparations we are ready to introduce the main concept of this section.

**Definition 4.3.** With the notation of Conventions 4.1 and 4.2 introduced before we call a completely bounded linear map $\nabla_D : X_1 \to X \hat{\otimes}_A \mathcal{L}(Y)$ a D–connection if
\[
\nabla_D (x \cdot a) = \nabla_D (x) \cdot a + x \otimes [D, a], \quad \text{for all } x \in X_1, a \in \mathcal{A}.
\]

A D–connection is called hermitian if additionally
\[
[D, \langle x_1, x_2 \rangle] = \langle x_1, \nabla_D (x_2) \rangle - \langle x_2, \nabla_D (x_1) \rangle^*, \quad \text{for all } x_1, x_2 \in X_1.
\]

Here $\langle \cdot, \cdot \rangle$ denotes the pairing introduced in Eq. (4.2).

Furthermore, we write $c(\nabla_D)$ for the composition of maps
\[
c(\nabla_D) : X_1 \otimes Y \xrightarrow{\nabla_D \otimes 1} (X \hat{\otimes}_A \mathcal{L}(Y)) \otimes Y \xrightarrow{c} X \hat{\otimes}_A Y.
\]

Before passing to the examples of hermitian D–connections we record.

**Lemma 4.4.** Assume Conventions 4.1, 4.2. Suppose that $\nabla_D : X_1 \to X \hat{\otimes}_A \mathcal{L}(Y)$ is a hermitian D–connection. Then we have the identity
\[
\langle c(\nabla_D)(x_1 \otimes y_1), x_2 \otimes y_2 \rangle = \langle x_1 \otimes y_1, c(\nabla_D)(x_2 \otimes y_2) \rangle - \langle y_1, [D, \langle x_1, x_2 \rangle](y_2) \rangle,
\]
between inner products on $X \hat{\otimes}_A Y$ and inner products on $Y$, for all $x_1, x_2 \in X_1$ and all $y_1, y_2 \in Y$.

**Proof.** The result follows from the computation
\[
\langle c(\nabla_D)(x_1 \otimes y_1), x_2 \otimes y_2 \rangle = \langle \nabla_D(x_1)(y_1), x_2 \otimes y_2 \rangle
\]
\[
= \langle (x_2, \nabla_D(x_1))(y_1), y_2 \rangle = \langle y_1, (x_2, \nabla_D(x_1))^*(y_2) \rangle
\]
\[
= \langle y_1, (x_1, \nabla_D(x_2))(y_2) \rangle - \langle y_1, [D, \langle x_1, x_2 \rangle](y_2) \rangle
\]
\[
= \langle x_1 \otimes y_1, c(\nabla_D)(x_2 \otimes y_2) \rangle - \langle y_1, [D, \langle x_1, x_2 \rangle](y_2) \rangle.
\]

Here we have in fact only used the hermitian property of $\nabla_D$. □
4.1. The Graßmann connection. We continue assuming Convention 4.1. We shall now see that our operator $*$–module $X_1 = \mathcal{A}_1^\infty$ carries a canonical hermitian $D$–connection provided that an extra mild condition is satisfied. This connection is basically the restriction of the commutator $[D, \cdot]$ to the operator $*$–module. We will thus term this connection the Graßmann $D$–connection.

In order to explain our extra condition we introduce the $C^*$–algebra of $D$–1–forms, cf. [CONG94, Chap. IV.1].

**Definition 4.5.** Let $(X_1, \Lambda_1)$ and $(\Lambda, Y, D)$ as in Conventions 4.1 and 4.2. Let $\Omega_D^1 \subseteq \mathcal{L}_B(Y)$ be the smallest $C^*$–subalgebra of $\mathcal{L}_B(Y)$ such that $[D, a_1]$ and $\pi(a) \in \Omega_D^1$ for all $a_1 \in \Lambda_1$ and all $a \in \Lambda$. We endow $\Omega_D^1$ with its natural structure of an $\Lambda$–$\Omega_D^1$ Hilbert–$C^*$–module. Let $\pi: \Lambda \to \mathcal{L}(\Omega_D^1)$ be the action of $\Lambda$ by left multiplication on $\Omega_D^1$.

The action $\pi: \Lambda \to \Omega_D^1$ is called essential if $\pi(\Lambda)\Omega_D^1$ is dense in $\Omega_D^1$; then also $\pi(\Lambda_1)\Omega_D^1$ is dense in $\Omega_D^1$.

From now on we assume that the action $\pi: \Lambda \to \Omega_D^1$ is essential. In particular, we have an isomorphism $\Lambda^\infty \widehat{\otimes}_\Lambda \Omega_D^1 \cong (\Omega_D^1)^\infty$ of Hilbert $C^*$–modules.

We are now ready to define the Graßmann $D$–connection.

**Definition 4.6 (Graßmann $D$–connection).** By the Graßmann $D$–connection on $X_1$ we will understand the completely bounded linear map $\nabla_D^{Gr}$ obtained as the composition

$$X_1 \rightarrow \Lambda_1^\infty \xrightarrow{[D, \cdot]} (\Omega_D^1)^\infty \cong \Lambda^\infty \widehat{\otimes}_\Lambda \Omega_D^1 \rightarrow \mathcal{L}(\Lambda) \xrightarrow{\pi} X \widehat{\otimes}_\Lambda \mathcal{L}(Y).$$

Here the second map, $[D, \cdot]$, is given by applying the completely bounded derivation $[D, \cdot]$ to each entry in a sequence.

**Proposition 4.7.** The Graßmann $D$–connection is a hermitian $D$–connection in the sense of Definition 4.3.

**Proof.** We start by proving that $\nabla_D^{Gr}$ is a $D$–connection. Thus, let $x = \{a_i\} \in X$ and let $a \in \Lambda_1$. Then

$$\nabla_D^{Gr}(x \cdot a) = (P \otimes 1)\{[D, a_i] \cdot a\}$$

$$= (P \otimes 1)\{[D, a_i]\} \cdot a + (P \otimes 1)(\{a_i\} \otimes [D, a])$$

$$= \nabla_D^{Gr}(x) \cdot a + x \otimes [D, a].$$

Here we have suppressed various maps in order to obtain a cleaner computation. To show that $\nabla_D^{Gr}$ is hermitian we compute for $\{a_i\}, \{b_i\} \in X$

$$[D, \{a_i\}, \{b_i\}] = \sum_{i=1}^\infty [D, a_i^* \cdot b_i] = \sum_{i=1}^\infty a_i^*[D, b_i] - (\sum_{i=1}^\infty b_i^*[D, a_i])^*$$

$$= \{a_i, ([D, b_i])\} - \{b_i, ([D, a_i])\}^*$$

$$= \{a_i, \nabla_D^{Gr}(b_i)\} - \{b_i, \nabla_D^{Gr}(a_i)\}^*,$$

which is the desired identity. \qed
4.2. Comparison of connections. We are now interested in comparing different hermitian $D$–connections. We shall apply a general result on automatic boundedness for operator $*$–modules.

**Proposition 4.8.** Let $(X_1,A_1)$ be as in Convention 4.1 with $A_1$ being $\sigma$–unital. Then any completely bounded $A_1$–linear map $\alpha : X_1 \to Z$ into an operator module $Z$ over $A$ extends to a completely bounded $A$–linear map, $\alpha : X \to Z$.

\[
X_1 = PA_1^\infty \xrightarrow{\alpha} Z \\
X = PA^\infty
\]

**Proof.** Let $\{\theta_{w^m_w}\}$ denote a sequence of completely bounded operators as in Proposition 3.5. Then we have for any finite matrix $x \in M_n(X_1)$ over $X_1$

\[
\alpha(x) = \lim_{m \to \infty} \alpha(\theta_{w^m_w}w(x)) = \lim_{m \to \infty} \alpha((1_n \otimes (w^m)^t) \cdot (1_n \otimes (w^m)^t 1_n, x)) = \lim_{m \to \infty} \alpha(1_n \otimes (w^m)^t) \cdot (1_n \otimes (w^m)^t, x).
\]

Here $(w^m)^t \otimes 1_n$ refers to the $(n \times n)$ diagonal matrix which has the row $(w^m)^t$ on the diagonal. It follows that

\[
||\alpha(x)||_Z \leq C_1 \cdot \limsup_{m \to \infty} \|\alpha\|_{cb} \cdot \|(w^m)^t\|_{X_1} \cdot \|(w^m)^t \otimes 1_n, x\|_A \\
\leq C_2 \cdot \limsup_{m \to \infty} \|\alpha\|_{cb} \cdot \|(w^m)^t\|^2_{X_1} \cdot ||x||_X,
\]

with constants $C_1, C_2 > 0$ independent of the size $n$ of the matrix. But this proves the claim since $\sup_{m \in \mathbb{N}} \|(w^m)^t\| < \infty$ is finite. \hfill $\Box$

**Proposition 4.9.** Let $(X_1,A_1)$ be as in Convention 4.1 with $A_1$ being $\sigma$–unital. Furthermore let $(A,Y,D)$ as in Convention 4.2. Consider two $D$–connections $\nabla_D, \tilde{\nabla}_D : X_1 \to X \otimes_A \mathcal{L}(Y)$. Then the difference $\nabla_D - \tilde{\nabla}_D$ extends to a completely bounded $A$–linear operator $\nabla_D - \tilde{\nabla}_D : X \to X \otimes_A \mathcal{L}(Y)$.

If additionally the two connections are hermitian then the completely bounded operator $c(\nabla_D - \tilde{\nabla}_D) : X \otimes A Y \to X \otimes A Y$ is selfadjoint. Here, $c$ is the contraction map defined in Eq. (4.1).

**Proof.** The first claim is a consequence of Proposition 4.8 since the connection property implies that the difference $\nabla_D - \tilde{\nabla}_D$ is $A_1$–linear.

Hence we get a completely bounded operator $c(\nabla_D - \tilde{\nabla}_D) : X \otimes A Y \to X \otimes A Y$. 

To prove the last claim let us fix some elements \(x_1, x_2 \in X_1\) and \(y_1, y_2 \in Y\). By Lemma 4.4 we can calculate as follows

\[
\langle c(\nabla_D - \bar{\nabla}_D)(x_1 \otimes y_1), x_2 \otimes y_2 \rangle \\
= \langle x_1 \otimes y_1, c(\nabla_D)(x_2 \otimes y_2) \rangle - \langle y_1, [D, (x_1, x_2)](y_2) \rangle \\
- \langle x_1 \otimes y_1, c(\bar{\nabla}_D)(x_2 \otimes y_2) \rangle + \langle y_1, [D, (x_1, x_2)](y_2) \rangle \\
= \langle x_1 \otimes y_1, c(\nabla_D - \bar{\nabla}_D)(x_2 \otimes y_2) \rangle.
\]

But this computation proves the proposition. \(\square\)

5. Selfadjointness and regularity of the unbounded product operator

Let \((X_1 = \mathcal{PA}_{12}^\infty, A_1)\) be as in Convention 4.1 with \(A_1\) being \(\sigma\)-unital; as usual \(X = \mathcal{PA}_{12}^\infty\) denotes the associated Hilbert \(C^*\)-module. Assume furthermore, that \(D_1 : \mathcal{D}(D_1) \rightarrow X\) is an unbounded selfadjoint and regular operator on \(X\).

Secondly, we assume that we are given a triple \((A, Y, D_2)\) as in Convention 4.2 and assume that the action \(\pi : A \rightarrow \mathcal{L}(Y)\) is essential. Furthermore, we will assume that the action \(\Lambda \rightarrow \mathcal{L}(\Omega_D^1)\) is essential as well, cf. Def. 4.5.

We assume that these assumptions are in effect for the remainder of this section. The aim of this section is to prove, under an additional commutator hypothesis, the selfadjointness and regularity of the unbounded product operator defined by

\[
D_1 \times_{\nabla} D_2 := \begin{pmatrix}
0 & D_1 \otimes 1 - \imath 1 \otimes \nabla D_2 \\
(D_1 \otimes 1 + \imath 1 \otimes \nabla D_2) & 0
\end{pmatrix}
\]

(5.1)

Here \(\nabla : X_1 \rightarrow X \otimes_A \mathcal{L}(Y)\) is any hermitian \(D_2\)-connection.

5.1. Selfadjointness and regularity of \(1 \otimes \nabla D_2\). We start with a discussion of the right leg \(1 \otimes \nabla D_2\) of the unbounded product operator for which no additional assumptions are needed.

The first thing to do is to investigate the situation where the hermitian \(D_2\)-connection is the Graßmann \(D_2\)-connection constructed in Subsection 4.1. The case of a general hermitian \(D_2\)-connection will then follow from the comparison result in Proposition 4.9.

We let \(\text{diag}(D_2) : \mathcal{D}(\text{diag}(D_2)) \rightarrow Y^\infty\) denote the selfadjoint and regular diagonal operator induced by \(D_2 : \mathcal{D}(D_2) \rightarrow Y\). Now, since the action \(\pi : A \rightarrow \mathcal{L}(Y)\) is assumed to be essential we have an isomorphism \(A^\infty \otimes_A Y \cong Y^\infty\) of Hilbert \(C^*\)-modules. In particular, we can make sense of the projection \(P \otimes 1 : Y^\infty \rightarrow Y^\infty\). We define the unbounded operator \(1 \otimes \nabla_{Gr} D_2\) as the composition

\[
1 \otimes \nabla_{Gr} D_2 : \mathcal{D}(1 \otimes \nabla_{Gr} D_2) \xrightarrow{\mathcal{D}(\text{diag}(D_2))} \mathcal{D}(\text{diag}(D_2)) \xrightarrow{\text{diag}(D_2)} Y^\infty \xrightarrow{[P \otimes 1]} X \hat{\otimes}_A Y.
\]

Here the domain is given by the intersection \(\mathcal{D}(1 \otimes \nabla_{Gr} D_2) := \mathcal{D}(\text{diag}(D_2)) \cap (X \hat{\otimes}_A Y)\).
Lemma 5.1. For each \( x \in X_1 \) and each \( y \in D(D_2) \) we have the explicit formula
\[
(1 \otimes \nabla c_2) D_2(x \otimes y) = x \otimes D_2(y) + c(\nabla c^\Gamma D_2)(x \otimes y).
\]
In particular, the unbounded operator \( 1 \otimes \nabla c_2 D_2 \) is densely defined.

Proof. Let us fix elements \( x = \{a_i\} \in X_1 = PA_1^\infty \) and \( y \in D(D_2) \). We then have that
\[
x \otimes D_2(y) + c(\nabla c^\Gamma D_2)(x \otimes y) = (P \otimes 1)[a_i \cdot D_2(y)] + (P \otimes 1)[D_2, a_i](y)
\]n
\[
= (P \otimes 1)[D_2(a_i \cdot y)] = (P \otimes 1)\text{diag}(D_2)[a_i \cdot y] = (1 \otimes \nabla c_2 D_2)(x \otimes y).
\]
But this is the desired identity. \( \Box \)

Before we continue we remark that each element \( T \in \overline{M}(A_1) \) determines both a completely bounded operator \( T : A_1^\infty \to A_1^\infty \) on the standard module and a bounded adjointable operator \( T : Y^\infty \to Y^\infty \). We will make use of this observation in the next lemmas.

Lemma 5.2. Let \( T \in \overline{M}(A_1) \). Then the associated bounded adjointable operator \( T : Y^\infty \to Y^\infty \) preserves the domain of \( \text{diag}(D_2) \) and the commutator \( [\text{diag}(D_2), T] = [D_2, T] : Y(\text{diag}(D_2)) \to Y^\infty \) extends to a bounded adjointable operator with the estimate \( \|\text{diag}(D_2, T)\| \leq \|T\|_{A_1} \cdot \|\text{diag}(D_2, \cdot)\|_{cb} \) on the operator norm.

Proof. The statement follows by approximating \( T \) with finite matrices and by using the complete boundedness of the commutator \( [D_2, \cdot] \). \( \Box \)

The result of Lemma 5.2 allows us to analyze the relation between the projection \( P \otimes 1 \) and the diagonal operator \( \text{diag}(D_2) \). We recall that our operator \( * \)-algebra \( A_1 \) is assumed to be \( \sigma \)-unital.

Lemma 5.3. The projection \( P \otimes 1 : Y^\infty \to Y^\infty \) preserves the domain of the diagonal operator \( \text{diag}(D_2) \) and the commutator
\[
[P \otimes 1, \text{diag}(D_2)] : D(\text{diag}(D_2)) \to Y^\infty
\]
extends to a bounded adjointable operator on the Hilbert \( C^* \)-module \( Y^\infty \).

Proof. By Proposition 3.5 there exists as sequence \( \{w^m\} \subseteq c_0(A_1^\infty) \) such that \( \sup \|\text{diag}(D_2)\|_{A_1^\infty} < \infty \) and \( \theta_{w^m}(\rho) \to \rho \) for all \( \rho \in A_1^\infty \). Proposition 3.5. Put \( \theta_m = (Pw^m)^1((w^m)^1)^* \in \overline{M}(A_1) \), \( cf. \) the paragraph before Proposition 3.5. Then, according to loc. cit., \( \theta_m(y) \to (P \otimes 1)(y) \) for all \( y \in Y^\infty \) and
\[
\sup_{m \in \mathbb{N}} \|\theta_m\|_{cb} \leq \sup_{m \in \mathbb{N}} \|\theta_m\|_{A_1} \leq C \cdot \|P\|_{cb} \cdot \sup_{m \in \mathbb{N}} \|\text{diag}(D_2)^1\|_{A_1^\infty}^2 < \infty.
\]
The statement of the lemma therefore follows from Lemma 5.2 if we can prove that the sequence \( \{[\text{diag}(D_2), \theta_m](y)\} \) converges for each \( y \) in a dense subspace of \( Y^\infty \), therefore it suffices to consider vectors of the form \( \alpha \otimes z \) where \( \alpha \in A_1^\infty \) and \( z \in Y \). However, for elements of this kind we have
\[
[D_2, \theta_m](\alpha \otimes z) = [D_2, \theta_m(\alpha)](z) - \theta_m[D_2, \alpha](z),
\]
where the right hand side converges to \( [D_2, P(\alpha)](z) - (P \otimes 1)[D_2, \alpha](z) \). \( \Box \)
We are now ready to prove the main result of this section: The selfadjointness and regularity of the unbounded operator $1 \otimes \nabla_{\text{Gr}} D_2$.

**Theorem 5.4.** Let $(X_1 = PA^1_1, \Lambda_1)$ be as in Convention 4.1 with $\Lambda_1$ being $\sigma$–unital and let the triple $(A,Y,D_2)$ satisfy the assumptions of Convention 4.2; furthermore, assume that the actions $\pi : A \to \mathcal{L}(Y)$ and $A \to \mathcal{L}(\Omega^1_{D_2})$ are essential.

Then for any hermitian $D_2$–connection $\nabla_{D_2} : X_1 \to X \hat{\otimes} A \mathcal{L}(Y)$ the unbounded operator

$$1 \otimes \nabla_{D_2} := 1 \otimes \nabla_{\text{Gr}} D_2 + c(\nabla_{D_2} - \nabla_{D_2}^{\text{Gr}}) : (X \hat{\otimes} A Y) \cap \mathcal{D}(\text{diag}[D_2]) \rightarrow X \hat{\otimes} A Y$$

is selfadjoint and regular. Furthermore we have the explicit formula

$$(1 \otimes \nabla_{D_2})(x \otimes y) = x \otimes D_2(y) + c(\nabla_{D_2})(x \otimes y)$$

whenever $x \in X_1$ and $y \in \mathcal{D}(D_2)$.

**Proof.** Let us first note that the class of selfadjoint regular operators is stable under bounded adjointable perturbations. This follows easily from a Neumann series argument and, e.g., [LAN95, Lemma 9.8], cf. also [WOR91]. To express it differently, in a slightly overblown fashion, it also follows from the Kato–Rellich Theorem for Hilbert $C^*$–modules [KALE11, Theorem 4.4].

We may assume that $\nabla$ is the Graßmann connection $\nabla_{\text{Gr}}$. The general case then follows from Proposition 4.9 and Lemma 5.1.

So let $Q = P \otimes 1$. The diagonal operator $\text{diag}[D_2]$ can be written

$$\text{diag}(D_2) = Q \text{diag}(D_2)Q + (1 - Q) \text{diag}(D_2)(1 - Q)$$

$$+ Q \text{diag}(D_2)(1 - Q) + (1 - Q) \text{diag}(D_2)Q$$

$$= Q \text{diag}(D_2)Q + (1 - Q) \text{diag}(D_2)(1 - Q)$$

$$+ [Q, \text{diag}(D_2)](1 - Q) - (1 - Q)[Q, \text{diag}(D_2)].$$

Now, from Lemma 5.3 we infer that

$$[Q, \text{diag}(D_2)](1 - Q) - (1 - Q)[Q, \text{diag}(D_2)] : Y^{\infty} \rightarrow Y^{\infty}$$

is selfadjoint and bounded. The operator

$$Q \text{diag}(D_2)Q + (1 - Q) \text{diag}(D_2)(1 - Q) : \text{diag}(D_2) \rightarrow Y^{\infty}$$

thus differs from the selfadjoint regular operator $\text{diag}(D_2)$ by a bounded selfadjoint operator. As noted at the beginning of this proof this implies the selfadjointness and regularity of the operator (5.2). But since $1 \otimes \nabla_{\text{Gr}} D_2$ is just the compression $Q \text{diag}(D_2)Q$ we reach the conclusion.

\[\square\]

### 5.2. Selfadjointness and regularity of the product operator

We are now in a position where we can prove selfadjointness and regularity results for unbounded product operators of the form $D_1 \times \nabla D_2$ where $\nabla_{D_2}$ is a hermitian $D_2$–connection. See the beginning of Section 5. Our main tool will be the Theorem 1.3 on selfadjointness and regularity for sums of operators:
In our case the roles of S and T are played by \( D_1 \otimes 1 \) and \( 1 \otimes_\nabla D_2 \). The Hilbert \( \text{C}^* \)-module \( E \) in loc. cit. is given by the interior tensor product \( X \hat{\otimes} A Y \). The selfadjointness and regularity of \( D_1 \) is easily seen to imply the selfadjointness and regularity of the unbounded operator \( D_1 \otimes 1 : \mathcal{D}(D_1) \hat{\otimes} A Y \to X \hat{\otimes} A Y \). The unbounded operator \( 1 \otimes_\nabla D_2 \) is selfadjoint and regular for any hermitian \( D_2 \)-connection \( \nabla_{D_2} \) by Theorem 5.4.

**Theorem 5.5.** In the situation of Theorem 5.4 let in addition \( D_1 \) be a selfadjoint regular operator on \( X \).

Suppose that there exists a hermitian \( D_2 \)-connection \( \nabla_{D_2}^0 : X_1 \to X \hat{\otimes} A \mathcal{L}(Y) \) such that the conditions in Theorem 1.3 are satisfied for \( S := D_1 \otimes 1 \) and \( T := 1 \otimes_\nabla D_2 \). Then the unbounded product operator

\[
D_1 \times_\nabla D_2 := \begin{pmatrix}
0 & D_1 \otimes 1 - i 1 \otimes_\nabla D_2 \\
(D_1 \otimes 1 + i 1 \otimes_\nabla D_2) & 0
\end{pmatrix}
: (\mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes_\nabla D_2))^2 \to (X \hat{\otimes} A Y)^2
\]

is selfadjoint and regular for any hermitian \( D_2 \)-connection \( \nabla_{D_2} : X_1 \to X \hat{\otimes} A \mathcal{L}(Y) \).

**Proof.** The selfadjointness and regularity of \( D_1 \times_\nabla D_2 \) is a consequence of Theorem 1.3. The statement for a general hermitian \( D_2 \)-connection follows since \( D_1 \times_\nabla D_2 - D_1 \times_\nabla D_2 \) extends to a selfadjoint bounded operator by Proposition 4.9. Here we use again the fact that the class of selfadjoint regular operators is stable under selfadjoint bounded perturbations as already mentioned at the beginning of the proof of Theorem 5.4. \( \square \)

6. **The interior product of unbounded Kasparov modules**

Let \( A, B \) and \( C \) be three \( \text{C}^* \)-algebras. Let us start by recalling some terminology from [BaJu83], cf. also [Kuc97].

**Definition 6.1.** By an unbounded Kasparov \( A \)-\( B \) module we will understand a pair \( (X, D) \) consisting of a countably generated \( A \)-\( B \) Hilbert \( \text{C}^* \)-bimodule \( X \) and an unbounded selfadjoint and regular operator \( D : \mathcal{D}(D) \to X \) on \( X \) such that:

1. There is a dense \(-\)-subalgebra \( A \subseteq A \) such that each \( a \in A \) maps \( \mathcal{D}(D) \to \mathcal{D}(D) \) and the commutator \( [D, a] : \mathcal{D}(D) \to X \) extends to a bounded operator.

2. The resolvent operator \( a \cdot (D - i)^{-1} \in \mathcal{K}(X) \) is \( B \)-compact for all \( a \in A \).

We will say that \( (X, D) \) is even when we have a \( \mathbb{Z}_2 \)-grading operator \( \gamma \in \mathcal{L}(X) \) such that \( \pi(a)\gamma - \pi(a)\gamma = 0 \) for all \( a \in A \) and \( D\gamma + \gamma D = 0 \).

An unbounded Kasparov \( A \)-\( B \) module without a grading operator is referred to as being odd.

We remind the reader of Proposition 2.6 and Remark 2.7, 3. although in this section we will not make use of the completion \( A_1 \) of \( A \).

Let us fix an odd unbounded Kasparov \( A \)-\( B \) module \( (X, D_1) \) and an odd unbounded Kasparov \( B \)-\( C \) module \( (Y, D_2) \). The aim of this section is to find sufficient
conditions for the existence of an “unbounded product” of \((X, D_1)\) and \((Y, D_2)\). When it exists, the unbounded product will be an even unbounded Kasparov \(\mathcal{A}\)–\(\mathcal{C}\) module which depends on the choice of a connection \(\nabla\) up to selfadjoint perturbations. The unbounded product operators will be of the “Dirac–Schrödinger–type” \(D_1 \times_{\nabla} D_2\) which we considered in the previous section. Let us give some relevant definitions.

**Definition 6.2.** \((Y, D_2)\) is called essential if the action of \(B\) on \(Y\) is essential and the derivation \([D_2, \cdot]\) is essential. That is, \(B \cdot Y\) is dense in \(Y\) and \(B \cdot \Omega^1_{D_2}\) is dense in the \(\Omega^1_{D_2}\). Recall from 4.5 that \(\Omega^1_{D_2} \subseteq \mathcal{L}(Y)\) denotes the smallest \(C^*\)-subalgebra such that \(b, [D_2, b] \in \Omega^1_{D_2}\) for all \(b\) in the dense \(*\)-subalgebra \(B \subseteq \mathcal{B}\) according to Def. 6.1.

The next definition is related to the notion of a correspondence which appears in the Ph.D. thesis of B. Mesland [Meso9, Sec. 6]. The conditions which we require are however substantially weaker than those advocated in loc. cit.

**Definition 6.3.** Suppose that \((Y, D_2)\) is essential. By a correspondence from \((X, D_1)\) to \((Y, D_2)\) we will understand a pair \((X_1, \nabla^0)\) consisting of an operator \(*\)-module \(X_1\) over a \(\sigma\)-unital operator \(*\)-algebra \(B_1\) and a completely bounded hermitian \(D_2\)-connection \(\nabla^0 : X_1 \rightarrow \hat{\otimes}_B \mathcal{L}(Y)\) such that

1. The operator \(*\)-module \(X_1 \subseteq X\) is a dense subspace of \(X\) and the operator \(*\)-algebra \(B_1 \subseteq \mathcal{B}\) is a dense \(*\)-subalgebra of \(\mathcal{B}\). The inclusions are completely bounded and compatible with module structures and inner products.
2. Each \(b \in B_1\) maps the domain of \(D_2\) into itself and the derivation \([D_2, \cdot] : B_1 \rightarrow \mathcal{L}(Y)\) is completely bounded on \(B_1\).
3. The commutator \([1 \otimes_{\nabla^0} D_2, a] : \mathcal{D}(1 \otimes_{\nabla^0} D_2) \rightarrow \hat{\otimes}_B \mathcal{L}(Y)\) is well-defined and extends to a bounded operator on \(\hat{\otimes}_B \mathcal{L}(Y)\) for all \(a \in \mathcal{A}\).
4. For any \(\mu \in \mathbb{R} \setminus \{0\}\) the unbounded operator
   \[ [D_1 \otimes 1, 1 \otimes_{\nabla^0} D_2] (D_1 \otimes 1 - i \cdot \mu)^{-1} : \mathcal{D}(1 \otimes_{\nabla^0} D_2) \rightarrow \hat{\otimes}_B \mathcal{L}(Y) \]
   is well-defined and extends to a bounded operator on \(\hat{\otimes}_B \mathcal{L}(Y)\).

Here, (4) is an abbreviation for the properties (1) and (2) in Theorem 1.3 for \(S = D_1 \otimes 1, T = 1 \otimes_{\nabla^0} D_2\).

Furthermore, we remark that the domain of \(1 \otimes_{\nabla^0} D_2\) can be replaced by a core for \(1 \otimes_{\nabla^0} D_2\) in requirement (3) and (4) of Definition 6.3.

**Definition 6.4.** Suppose that \((X_1, \nabla^0)\) is a correspondence from \((X, D_1)\) to \((Y, D_2)\). Let \(\nabla_{D_2} : X_1 \rightarrow \hat{\otimes}_B \mathcal{L}(Y)\) be a hermitian \(D_2\)-connection. By the unbounded interior product of \((X, D_1)\) and \((Y, D_2)\) with respect to \(\nabla_{D_2}\) we will understand the pair \(((\hat{\otimes}_B Y)^2, D_1 \times_{\nabla} D_2)\). Here \((\hat{\otimes}_B Y)^2\) is a \(\mathbb{Z}_2\)-graded \(\mathcal{A}\)-\(\mathcal{C}\) Hilbert \(C^*\)-bimodule. The grading is given by the grading operator \(\gamma := \text{diag}(1, -1)\).

We shall see that the unbounded interior product is an unbounded even Kasparov \(\mathcal{A}\)-\(\mathcal{C}\) bimodule. We remark that the selfadjointness and regularity condition was proved in Theorem 5.5. Furthermore, the boundedness of the commutator
\[|D_1 \times_{\nabla} D_2, a|\] for all \(a \in A\) follows from the third condition in Definition 6.3. The only real issue is therefore compactness of the resolvent. This problem will occupy the rest of the section. We remark that the unbounded interior product only depends on the choice of connection up to selfadjoint perturbations. This is a consequence of Proposition 4.9.

We start with a small compactness result.

**Lemma 6.5.** Suppose that \(B\) is \(\sigma\)-unital and that the action \(B \rightarrow Y\) is essential. Let \(K \in \mathcal{K}(B^\infty, X)\) be a \(B\)-compact operator. Then for \(z \in \mathbb{C} \setminus \mathbb{R}\) the bounded operator

\[
(K \otimes 1)(\text{diag}[D_2] - z)^{-1} \in \mathcal{K}(B^\infty \hat{\otimes}_B Y, X \hat{\otimes}_B Y)
\]

is \(C\)-compact. Here we have suppressed the isomorphism of Hilbert \(C^*\)-modules \(B^\infty \hat{\otimes}_B Y \cong Y^\infty\).

**Proof.** By the resolvent identity it suffices to prove the claim for \(z = 1\). Let \(\{u_m\}\) denote the countable approximate unit for \(B\). We then have a countable approximate unit \(\theta_m\) for the compact operators on \(B^\infty\) defined by \(\theta_m := \sum_{i=1}^m \theta_{e_i} u_m e_i u_m \in \mathcal{K}(B^\infty)\). Here \(e_i \cdot u_m \in B^\infty\) is the vector in the standard module with \(u_m \in B\) in position \(i \in \mathbb{N}\) and zeros elsewhere. We therefore only need to prove that the bounded operator \((\theta_m \otimes 1) \cdot (\text{diag}[D_2] - i)^{-1} \in \mathcal{L}(Y^\infty)\) is \(C\)-compact for all \(m \in \mathbb{N}\).

Let us fix some \(m \in \mathbb{N}\). Using the identification \(Y^\infty \cong \hat{\bigotimes} Y\) where \(\hat{\bigotimes}\) denotes the exterior tensor product of Hilbert \(C^*\)-modules we get the identity

\[
(\theta_m \otimes 1) \cdot (\text{diag}[D_2] - i)^{-1} = p_m \otimes (u_m u_m^* \cdot (D_2 - i)^{-1}) \in \mathcal{L}(H \hat{\otimes} Y).
\]

(6.1)

Here \(p_m : H \rightarrow H\) denotes the finite rank orthogonal projection onto the subspace \(\text{span}_{C}(e_i)_{i=1}^m\) where \(\{e_i\}\) is an orthonormal basis for the separable Hilbert space \(H\). Since both of the factors in the tensor product on the rhs of (6.1) are \((C-\text{ resp. } C-)\) compact we get that \((\theta_m \otimes 1) \cdot (\text{diag}[D_2] - i)^{-1} \in \mathcal{K}(Y^\infty)\) is \(C\)-compact and the lemma is proved.

**Proposition 6.6.** Suppose that condition (1) and (2) in Definition 6.3 are satisfied. Let \(\nabla_{D_2} : X_1 \rightarrow X \hat{\otimes}_B L(Y)\) be a hermitian \(D_2\)-connection and let \(K \in \mathcal{K}(X)\) be a \(B\)-compact operator. Then the bounded adjointable operator \((K \otimes 1)(1 \otimes_{\nabla} D_2 - z)^{-1} \in \mathcal{K}(X \hat{\otimes}_B Y)\) is \(C\)-compact for all \(z \in \mathbb{C} \setminus \mathbb{R}\).

**Proof.** We start by recalling that by Proposition 4.9 the difference of unbounded operators \(1 \otimes_{\nabla} D_2 - 1 \otimes_{\nabla_{Gr}} D_2\) extends to a bounded selfadjoint operator. By the resolvent identity it is therefore sufficient to prove the claim for the Graßmann \(D_2\)-connection \(\nabla_{Gr} : X_1 \rightarrow X \hat{\otimes}_B L(Y)\) and \(z = i\).

By assumption, cf. Convention 4.1 resp. Proposition 4.8, we can assume that \(X_1 = PB_1^\infty\) and \(X = PB^\infty\) where \(P : B_1^\infty \rightarrow B_1^\infty\) is a completely bounded selfadjoint idempotent. Since \((Y, D_2)\) is assumed to be essential we have \(X \hat{\otimes}_B Y = PB^\infty \hat{\otimes}_B Y = \)
\[(P \otimes 1)(B^{\infty} \hat{\otimes} B)Y\) and \(Y^{\infty} \cong B^{\infty} \hat{\otimes} B Y.\) Put \(Q = P \otimes 1 : Y^{\infty} \cong B^{\infty} \hat{\otimes} B Y \rightarrow X \hat{\otimes} B Y.\) Then
\[(K \otimes 1)(1 \otimes \nabla D_2 - i)^{-1}Q = (K \otimes 1)(Q \text{diag}(D_2)Q - iQ)^{-1}Q = (KP \otimes 1)(Q \text{diag}(D_2)Q + (1 - Q) \text{diag}(D_2)(1 - Q) - i)^{-1}
\]

\[\text{Eq. (6.2)}\]

The difference \(Q \text{diag}(D_2)Q + (1 - Q) \text{diag}(D_2)(1 - Q) - \text{diag}(D_2)\) is bounded and selfadjoint by Lemma 5.3. Another application of the resolvent equation then shows that the \(C\text--\)compactness of the rhs of Eq. (6.2) is equivalent to that of

\[(KP \otimes 1)(\text{diag}(D_2) - i)^{-1} : B^{\infty} \hat{\otimes} B Y \rightarrow X \hat{\otimes} B Y.\]

Since \(KP : B^{\infty} \rightarrow X\) is \(B\text--\)compact the result follows from Lemma 6.5 \(\square\)

The next result will allow us to conclude compactness results for the resolvent \((D_1 \times \nabla D_2 - i)^{-1}\) by looking at operators of the form \((K \otimes 1)(1 \otimes \nabla D_2 - i)^{-1}\).

We are now ready to prove the main result of this section.

\textbf{Theorem 6.7.} Suppose that \((X_1, \nabla^0)\) is a correspondence between the unbounded odd Kasparov modules \((X_1, D_1)\) and \((Y, D_2)\). Let \(\nabla_{D_2} : X_1 \rightarrow X \hat{\otimes} B \mathcal{L}(Y)\) be any completely bounded hermitian \(D_2\text--\)connection. Then the pair \((D_1 \times \nabla D_2, X \hat{\otimes} B Y)\) is an even unbounded Kasparov A--C module which only depends on \(\nabla_{D_2}\) up to selfadjoint bounded perturbations.

\textit{Proof.} Let us fix some countable approximate unit \(\{\theta_m\}\) for the compact operators on \(X\).

As observed at the beginning of this section, Theorem 5.5 applies to \(D_1 \times \nabla D_2\). That means that Theorem 1.3 applies to \(S = D_1 \otimes 1\) and \(T = 1 \otimes \nabla D_2\). Since the difference \(1 \otimes \nabla D_2 - 1 \otimes \nabla D_2\) is bounded selfadjoint by Proposition 4.9 it follows in particular that we have the following commutative diagram of continuous inclusion maps of Hilbert \(C^*\text--\)modules:

\[
\begin{array}{c}
\mathcal{D}(1 \otimes \nabla D_2)^2 \\
\mathcal{D}(D_1 \times \nabla D_2)^2 \\
\mathcal{D}(D_1 \otimes 1)^2
\end{array}
\]

\begin{align*}
&\mathcal{D}(D_1 \times \nabla D_2)^2 \ar{r}{t_1} & \mathcal{D}(1 \otimes \nabla D_2)^2 \\
&\mathcal{D}(D_1 \times \nabla D_2)^2 \ar{r}{t_2} & (X \hat{\otimes} B Y)^2 \\
&\mathcal{D}(D_1 \otimes 1)^2 \ar{r}{t_3} & (X \hat{\otimes} B Y)^2 \\
&\mathcal{D}(D_1 \times \nabla D_2)^2 \ar{r}{t_4} & (X \hat{\otimes} B Y)^2
\end{align*}

We need to prove that for \(a \in A\) the bounded operator \(\pi(a) \circ t : \mathcal{D}(D_1 \times \nabla D_2) \rightarrow (X \hat{\otimes} B Y)^2\) is \(C\text--\)compact. Here, \(\pi(a) \in \mathcal{L}(X \hat{\otimes} B Y)^2\) is given by the action of \(A\) on the first component in the interior tensor product.

Let \(m \in \mathbb{N}\) and consider the composition \((\theta_m \otimes 1) \circ \pi(a) \circ t = (\theta_m \otimes 1) \circ \pi(a) \circ t_2 \circ t_1 : \mathcal{D}(D_1 \times \nabla D_2) \rightarrow (X \hat{\otimes} B Y)^2\), cf. Eq. (6.3).
The operator \( (\theta_m \otimes 1) \circ \pi(a) \circ t_2 : D(1 \otimes_D D_2)^2 \to (X \hat{\otimes}_B Y)^2 \) is C-compact by Proposition 6.6. Hence \( (\theta_m \otimes 1) \circ \pi(a) \circ t \in K(D(D_1 \times_D D_2), (X \hat{\otimes}_B Y)^2) \) is C-compact for all \( m \in \mathbb{N} \).

On the other hand we have the identity

\[
(\theta_m \otimes 1) \circ \pi(a) \circ t_2 = (\theta_m \otimes 1) \circ \pi(a) \circ t : D(D_1 \times_D D_2) \to (X \hat{\otimes}_B Y)^2.
\]

But the sequence of operators \( \{ (\theta_m \otimes 1) \circ \pi(a) \circ t_4 \} \) in \( L(D(D_1 \otimes 1)^2, (X \hat{\otimes}_B Y)^2) \) converges in operator norm to the bounded operator \( \pi(a) \circ t_4 \in L(D(D_1 \otimes 1)^2, (X \hat{\otimes}_B Y)^2) \). Indeed, this follows by noting that \( \pi(a) \circ (D_1 \otimes 1 - i)^{-1} \) is of the form \( K \otimes 1 \) where \( K \in K(X) \) is compact.

We have thus proved that \( \pi(a) \circ t \in L(D(D_1 \times_D D_2), (X \hat{\otimes}_B Y)^2) \) is the limit in operator norm of a sequence of compact operators. It is therefore compact and the theorem is proved. \( \square \)

7. Unbounded representatives for the interior Kasparov product

Let \( A, B \) and \( C \) be C*-algebras where \( A \) is separable and \( B \) is \( \sigma \)-unital. We then have the interior Kasparov product \( \hat{\otimes}_B : KK^1(A, B) \times KK^1(B, C) \to KK^0(A, C) \) which is a bilinear and associative pairing of abelian groups [Kas80b], [Blag8, Sec. 18]. The purpose of this section is to show that the unbounded interior product which we constructed in the last section is an unbounded version of the interior Kasparov product.

Let \( (X, D_1) \) and \( (Y, D_2) \) be odd unbounded Kasparov modules for \( (A, B) \) and \( (B, C) \) respectively. By the work of Baaj and Julg [BAJ83] the bounded transform \( F : (X, D) \mapsto (X, D(1 + D^2)^{-1/2}) \) provides classes \( F(X, D_1) \in KK^1(A, B) \) and \( F(Y, D_2) \in KK^1(B, C) \).

**Definition 7.1.** We say that an even unbounded Kasparov \( A-C \) bimodule \( (Z, D) \) represents the interior Kasparov product of \( (X, D_1) \) and \( (Y, D_2) \) if

\[
F(X, D_1) \hat{\otimes}_B F(Y, D_2) = F(Z, D)
\]

in the even KK–group \( KK^0(A, C) \).

We shall see that the existence of a correspondence \( (X_1, \nabla^0) \) from \( (X, D_1) \) to \( (Y, D_2) \) implies that the even unbounded Kasparov module \( (X \hat{\otimes}_B Y)^2, D_1 \times_D D_2 \) represents the interior Kasparov product for any hermitian \( D_2 \)-connection \( \nabla \). Our main tool will be a general result which is an adaption of a theorem of D. Kucerovsky to the case of the interior Kasparov product between two odd KK–theory groups. The result can thus be proved by an application of D. Kucerovsky’s theorem together with some understanding of formal Bott–periodicity in KK–theory, see for example [Blag8, Cor. 17.8.9].

For each \( x \in X \) we will use the notation \( T_x : Y^2 \to (X \hat{\otimes}_B Y)^2 \) for the multiplication operator \( T_x : (y_1, y_2) \mapsto ((x \otimes y_1), (x \otimes y_2)) \); \( T_x \) is bounded adjointable. Furthermore,
we let $\sigma_1, \sigma_2 \in M_2(\mathbb{C})$ denote the matrices
\[ \sigma_1 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

**Theorem 7.2 ([Kuc97, Theorem 13])**. Let $(\pi_1, X, D_1)$ and $(\pi_2, Y, D_2)$ be two odd unbounded Kasparov modules for $(A, B)$ and $(B, C)$, respectively. Let $(\pi_1 \otimes 1, (X \hat{\otimes}_B Y) \oplus (X \hat{\otimes}_B Y), D)$ be an even unbounded Kasparov $A$--$C$ module, where $(X \hat{\otimes}_B Y)^2$ is $\mathbb{Z}_2$--graded by the grading operator $\gamma = \text{diag}(1, -1)$. Suppose that the following conditions are satisfied:

1. The commutator
\[ \left[ \begin{pmatrix} D & 0 \\ 0 & D_2 \cdot \sigma_1 \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T^*_x \otimes 1 & 0 \end{pmatrix} \right] : \mathcal{D}(D) \oplus \mathcal{D}(D_2)^2 \to (X \hat{\otimes}_B Y)^2 \oplus Y^2 \]
is well--defined and extends to a bounded operator on $(X \hat{\otimes}_B Y)^2 \oplus Y^2$ for all $x$ in a dense subset of $A \cdot X$.

2. The domain of $D$ is contained in the domain of $(D_1 \otimes 1) \cdot \sigma_2$.

3. There exists a constant $C > 0$ such that
\[ \langle Dz, (D_1 \otimes 1) \cdot \sigma_2(z) \rangle + \langle (D_1 \otimes 1) \cdot \sigma_2(z), Dz \rangle \geq -C(z, z) \]
for all $z \in \mathcal{D}(D)$.

Then the even unbounded Kasparov module $(D, (X \hat{\otimes}_B Y)^2)$ represents the interior Kasparov product of $(D_1, X)$ and $(D_2, Y)$.

The second condition in the above theorem can be slightly weakened [Kuc97, Lemma 10]. However, for our purposes the stronger requirement on the domains is sufficient.

Let us fix two odd unbounded Kasparov modules $(X, D_1)$ and $(Y, D_2)$ for $(A, B)$ and $(B, C)$, respectively. Furthermore, we assume that we have a correspondence $(X^0, \nabla^0)$ from $(X, D_1)$ to $(Y, D_2)$. In particular, $(Y, D_2)$ is essential in the sense of Definition 6.2 and the operator $*$--algebra $B_1$ is $\sigma$--unital.

In the next lemmas we will show that $(X, D_1)$, $(Y, D_2)$ and $((X \hat{\otimes}_B Y)^2, D_1 \times \nabla^0 D_2)$ satisfy the conditions of Theorem 7.2. We remark that $((X \hat{\otimes}_B Y)^2, D)$ is an even unbounded Kasparov $A$--$C$ module by Theorem 6.7.

We let $\mathcal{F} := \pi_1(A) \cdot (D_1 - i)^{-1}(X_1)$ and remark that $\mathcal{F}$ is a dense subset of $A \cdot X$.

**Lemma 7.3.** For each $x \in \mathcal{F}$ we have that $T_x(\mathcal{D}(D_2)) \subseteq \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \nabla^0 D_2)$ and $T^*_x((D(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \nabla^0 D_2)) \subseteq \mathcal{D}(D_2)$.

**Proof.** Let $x := a \cdot (D_1 - i)^{-1}(\xi) \in \mathcal{F}$ with $\xi \in X$ and $a \in A$.

Let $y \in \mathcal{D}(D_2)$. We need to show that $x \otimes y \in \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \nabla^0 D_2)$. Since $\mathcal{F} \subseteq \mathcal{D}(D_1)$ we get that $T_x(y) = x \otimes y \in \mathcal{D}(D_1 \otimes 1)$. Next, by definition of $x \in \mathcal{F}$, we find $T_x(y) = a \cdot ((D_1 - i)^{-1} \otimes 1)(\xi \otimes y) = a \cdot ((D_1 - i)^{-1})(\xi \otimes y)$.

By Theorem 5.4 we have $\xi \otimes y \in \mathcal{D}(1 \otimes \nabla^0 D_2)$ because $\xi \in X$ and $y \in \mathcal{D}(D_2)$. Since $(X, \nabla^0)$ is a correspondence we infer that $((D_1 - i)^{-1} \otimes 1)(\mathcal{D}(1 \otimes \nabla^0 D_2)) \subseteq \mathcal{D}(D_2)$.


Finally, from Def. 6.3 (3) we conclude \( a \cdot ((D_1 \otimes 1 - i)^{-1})(\xi, \otimes y) \in \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \).

Now, let \( z \in \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \). We will show that then already \( T^*_\xi(z) \in \mathcal{D}(\mathcal{D}(D_2)) \). Since \( x = a \cdot (D_1 - i)^{-1}(\xi) \) we have that \( T^*_\xi(z) = T^*_\xi(D_1 \otimes 1 + i)^{-1}a^*(z) =: T^*_\xi z \). However, again, since \((X_1, \mathcal{V}_0)\) is a correspondence we get that \( \tilde{z} \in \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \).

The result of the lemma now follows from Lemma 5.2 since \( z \in \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \subseteq \mathcal{D}(\text{diag}(D_2)) \).

The next lemma implies that the first condition in Theorem 7.2 on the boundedness of the commutator is satisfied for our interior unbounded product \((D_1, (X \widehat{\otimes}_B Y)^2)\). Remark that the commutator is well–defined by Lemma 7.3.

**Lemma 7.4.** The commutator

\[
\left[ \begin{pmatrix} D_1 \otimes 1 \pm i \otimes \mathcal{V}_0 D_2 & 0 \\ 0 & \pm iD_2 \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T^*_x & 0 \end{pmatrix} \right] : (\mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \mathcal{V}_0 D_2)) \oplus \mathcal{D}(D_2) \to (X \widehat{\otimes}_B Y) \oplus Y
\]

extends to a bounded operator for all \( x \in \mathcal{F} \).

**Proof.** Let \( x = a \cdot (D_1 - i)^{-1}(\xi) \) with \( \xi \in X_1 \) and \( a \in A \). Then the identity

\[
(D_1 \otimes 1)T_x = (D_1 a (D_1 - i)^{-1} \otimes 1)T_\xi
\]

\[
= ([D_1, a] (D_1 - i)^{-1} \otimes 1)T_\xi + (a D_1 (D_1 - i)^{-1} \otimes 1)T_\xi
\]

proves that the operator \((D_1 \otimes 1)T_x : \mathcal{D}(D_2) \to X \widehat{\otimes}_B Y\) extends to a bounded operator. A similar calculation shows that \( T^*_\xi(D_1 \otimes 1) : \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \to Y \) extends to a bounded operator \( X \widehat{\otimes}_B Y \to Y \). We therefore only need to prove that the commutator

\[
\left[ \begin{pmatrix} 1 & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T^*_x & 0 \end{pmatrix} \right] : (\mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \mathcal{V}_0 D_2)) \oplus \mathcal{D}(D_2) \to (X \widehat{\otimes}_B Y) \oplus Y
\]

extends to a bounded operator.

This is equivalent to the boundedness of the two operators

\[
(1 \otimes \mathcal{V}_0 D_2)T_x - T_x D_2 : \mathcal{D}(D_2) \to X \widehat{\otimes}_B Y,
\]

\[
T^*_x(1 \otimes \mathcal{V}_0 D_2) - D_2 T^*_x : \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes \mathcal{V}_0 D_2) \to Y.
\]

To prove the boundedness of (7.1) we calculate

\[
(1 \otimes \mathcal{V}_0 D_2)T_x = (1 \otimes \mathcal{V}_0 D_2)a((D_1 - i)^{-1} \otimes 1)T_\xi
\]

\[
= [1 \otimes \mathcal{V}_0 D_2, a]((D_1 - i)^{-1} \otimes 1)T_\xi + a[1 \otimes \mathcal{V}_0 D_2, (D_1 - i)^{-1} \otimes 1]T_\xi
\]

\[
+ a((D_1 - i)^{-1} \otimes 1)(1 \otimes \mathcal{V}_0 D_2)T_\xi.
\]
Since \((X_1, \nabla^0)\) is a correspondence the first two summands on the rhs are bounded and we can thus restrict our attention to the unbounded operator \((I \otimes_{\nabla^0} D_2^\xi - T_\xi D_2) : \mathcal{D}(D_2) \to X \hat{\otimes}_B Y\). However, by Theorem \ref{thm:adjointability} the latter equals \(c \circ T_{\nabla^0}(\xi)\) where \(c : X \hat{\otimes}_B \mathcal{L}(Y) \hat{\otimes}_B Y \to X \hat{\otimes}_B Y\) is the evaluation map. This proves that the commutator in \((\ref{eq:commutator})\) extends to a bounded operator.

The boundedness of \((\ref{eq:boundary})\) would follow from that of \((\ref{eq:commutator})\) if we knew that \((\ref{eq:commutator})\) is adjointable. Since this is not established yet we need to prove the boundedness of \((\ref{eq:boundary})\) separately. By a computation similar to the one carried out in \((\ref{eq:boundary})\) we get that it suffices to prove that the unbounded operator \(T_\xi^* (1 \otimes_{\nabla^0} D_2) - D_2 T_\xi^* : \mathcal{D}(D_1 \otimes 1, \nabla^0) \cap \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(D_2) \to Y\) extends to a bounded operator. Furthermore, by Proposition \ref{prop:representation} we may replace the connection \(\nabla^0\) by the Grassmann \(D_2\)-connection \(\nabla_{D_2}^{Gr}\). We then have the identity

\[
T_\xi^* (1 \otimes_{\nabla^0} D_2) - D_2 T_\xi^* = T_\xi^* \text{diag}(D_2) - D_2 T_\xi^*.
\]

Notice that we think of \(\xi \in X_1 \subseteq B_1^\infty\) as an element of \(B_1^\infty\) in the last identity. Furthermore, we are suppressing the inclusion \(\mathcal{D}(1 \otimes_{\nabla^0} D_2) \subseteq \mathcal{D}(\text{diag}(D_2))\). But the right hand side of \((\ref{eq:boundary})\) is bounded by Lemma \ref{lemma:boundedness} and the proof is complete. \(\Box\)

We are now ready to state the main theorem of this paper.

**Theorem 7.5.** Let \((X, D_1)\) and \((Y, D_2)\) be two odd unbounded Kasparov modules for \((A, B)\) and \((B, C)\) respectively. Suppose that there exists a correspondence \((X_1, \nabla^0)\) from \((X, D_1)\) to \((Y, D_2)\). Let \(\nabla : X_1 \to X \hat{\otimes}_B \mathcal{L}(Y)\) be any completely bounded hermitian connection. Then the even unbounded Kasparov \(A-C\) module \(((X \hat{\otimes}_B Y)^2, D_1 \times_{\nabla} D_2)\) represents the Kasparov product of \((X, D_1)\) and \((Y, D_2)\).

**Proof.** \(((X \hat{\otimes}_B Y)^2, D_1 \times_{\nabla} D_2)\) is an even unbounded Kasparov module by Theorem \ref{thm:representation}. For any even unbounded Kasparov module \((D, Z)\) and any odd selfadjoint operator \(R \in \mathcal{L}(X)\) the perturbed unbounded Kasparov module \((D + R, Z)\) gives rise to the same class in KK-theory under the bounded transform. Therefore, by Proposition \ref{prop:representation} it suffices to prove the Theorem for \(\nabla = \nabla^0\).

We apply Theorem \ref{thm:adjointability}. Its first condition is satisfied by Lemma \ref{lemma:boundedness}. The second condition in Theorem \ref{thm:adjointability} is satisfied since \(\mathcal{D}(D_1 \times_{\nabla^0} D_2) = (\mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes_{\nabla^0} D_2))^2\). Finally, the third condition in Theorem \ref{thm:adjointability} follows from \([\text{KALE11, Lemma}\ 7.6]\), which implies that there exists a constant \(C > 0\) such that

\[
\langle (D_1 \otimes 1 \pm i 1 \otimes_{\nabla^0} D_2)(z), (D_1 \otimes 1 \pm i 1 \otimes_{\nabla^0} D_2)(z) \rangle \geq \frac{1}{2} \left( \langle (D_1 \otimes 1)(z), (D_1 \otimes 1)(z) \rangle + \langle (1 \otimes_{\nabla^0} D_2)(z), (1 \otimes_{\nabla^0} D_2)(z) \rangle - C \langle z,z \rangle \right)
\]

for all \(z \in \mathcal{D}(D_1 \otimes 1) \cap \mathcal{D}(1 \otimes_{\nabla^0} D_2)\). \(\Box\)
8. Application: Dirac–Schrödinger operators on complete manifolds

8.1. Standing assumptions. Let \( M^m \) be a complete oriented Riemannian manifold \( M \) of dimension \( m \) and let \( H \) be a separable Hilbert space. We thus have the operator \(*\)-module \( C^1_0(M, H) \) over the operator \(*\)-algebra \( C^1_0(M) \). This is a consequence of Proposition 2.8 and Proposition 3.6. The operator \(*\)-algebra \( C^1_0(M) \) sits as a dense \(*\)-subalgebra inside the \( C^*\)-algebra \( C_0(M) \) of continuous functions vanishing at infinity and the inclusion is completely bounded. Thus with the pair \((X = C^1_0(M, H), C^1_0(M))\) we are in the situation of Convention 4.1. Remark that the completeness of the manifold entails that the operator \(*\)-algebra \( C^1_0(M) \) is \( \sigma \)-unital, see Note 2.9.

Furthermore, let \( D_{2,0} : \Gamma_{c}^{\infty}(M, F) \to L^2(M, F) \) be a first order elliptic differential operator acting on the sections of the smooth hermitian vector bundle \( F \to M \) over \( M \). We assume that \( D_{2,0} \) is symmetric with respect to the scalar product of \( L^2(M, F) \) and that \( D_{2,0} \) has bounded propagation speed, that is the symbol, \( \sigma_{D_{2}} : T^*M \to \text{End}(F) \) satisfies

\[
\sup_{\xi \in T^*M, \|\xi\| \leq 1} \|\sigma_{D_{2}}(\xi)\| =: C_{ps} < \infty. \tag{8.1}
\]

By the classical Theorem of Chernoff [CHE73] the completeness of \( M \) together with the bounded propagation speed assumption imply the essential selfadjointness of \( D_{2,0} \). By \( D_2 \) we then denote its selfadjoint closure.

The set-up outlined in this Subsection 8.1 will be in effect during the remainder of this Section 8.

8.2. Hermitian \( D_2 \)-connections. We shall now see that the composition of the exterior differential and the symbol of the first order differential operator \( D_2 \) is an example of a hermitian \( D_2 \)-connection. In fact we will interpret this composition as a Graßmann \( D_2 \)-connection.

The following small lemma, which should be well-known will be useful for proving complete boundedness of the commutator [\( D_2, \cdot \)].

**Lemma 8.1.** Let \( V \) be a \( N \)-dimensional Hilbert space and let \( Z \) be an operator space. Then any linear map \( \alpha : V \to Z \) is completely bounded with \( \|\alpha\|_{cb} \leq N \cdot \|\alpha\| \).

**Proof.** We remark that \( V \) has the structure of an operator space using the matrix norm \( \|\xi\| = \|\langle \xi, \xi \rangle\|_C, \xi \in M(V) \). Let \( \{e_i\}_{i=1}^N \) be an orthonormal basis for \( V \) and let \( \xi \in M_n(V) \) be an \((n \times n)\)-matrix. We can then write the matrix \( \xi \) as the sum \( \xi = \sum_{i=1}^N \xi_i \cdot e_i \) for some unique matrices \( \xi_i \in M_n(\mathbb{C}) \). In particular, we get the inequalities

\[
\|\alpha(\xi)\|_Z = \left\| \sum_{i=1}^N \xi_i \cdot \alpha(e_i) \right\|_Z \leq \sum_{i=1}^N \|\xi_i\|_C \|\alpha(e_i)\|_Z \leq N \cdot \|\xi\|_V \cdot \|\alpha\|,
\]

which in turn prove the lemma. \( \square \)
Proposition 8.2. The symbol of $D_2$ determines a completely bounded operator

$$\sigma_{D_2} : \Gamma_0(T^*M) \to \mathcal{L}(L^2(M, F)).$$

Here $\Gamma_0(T^*M)$ denotes the Hilbert $C_0(M)$–module of continuous one–forms which vanish at infinity.

Proof. We remark that $\Gamma_0(T^*M)$ is a Hilbert space when equipped with the matrix norm $\|\omega\| := \sup_{x \in M}\|\langle \omega(x), \omega(x)\rangle\|^{1/2}$, $\omega \in \mathcal{M}(\Gamma_0(T^*M))$. The symbol of $D_2$ defines a linear map $\sigma_{D_2} : T^*_xM \to \mathcal{L}(F_x)$ for each $x \in M$. The bounded propagation speed assumption Eq. (8.1) together with Lemma 8.1 then implies for each $\omega \in \mathcal{M}(\Gamma_0(T^*M))$

$$\|\sigma_{D_2}(\omega)\| \leq \sup_{x \in M}\|\sigma_{D_2}(\omega(x))\|_{\mathcal{L}(F_x)} \leq m \cdot C_{ps} \cdot \sup_{x \in M}\|\omega(x)\|_{T^*_xM} = m \cdot C_{ps} \cdot \|\omega\|,$$

hence the claim. \qed

Corollary 8.3. The commutator with $D_2$ determines a completely bounded map $[D_2, -] : C^1_b(M) \to \mathcal{L}(L^2(M, F)).$

Proof. Since $D_2$ is a first order operator we have $[D_2, f] = \sigma_{D_2}(df)$ for each $f \in C^1_b(M)$. The result then follows from Proposition 8.2 since $d : C^1_b(M) \to \Gamma_0(T^*M)$ is completely bounded by construction, \textit{cf.} Remark 2.7, 1.

We emphasize that the operator $*-$algebra structure on $C^1_b(M)$ comes from the exterior derivative $d$ and not from $D_2$. Thus the above Corollary is not immediate from Remark 2.7. Rather the bounded propagation speed assumption enters crucially in Prop. 8.2.

We remark that the left action $C_0(M) \to \mathcal{L}(\Omega^1_{D_2})$, \textit{cf.} Def. 4.5 and the paragraph thereafter, is essential in this case. It thus follows from the above results that we have the Graßmann $D_2$–connection

$$\nabla_{D_2}^G : C^1_b(M, H) \to C_0(M, H) \hat{\otimes}_{C_0(M)} \mathcal{L}(L^2(M, F)).$$

(8.2)

It is not hard to see that $\nabla_{D_2}^G$ coincides with the composition of the exterior differential and the symbol of $D_2$. In particular this composition is a hermitian $D_2$–connection by Proposition 4.7.

8.3. Dirac–Schrödinger operators. In addition to the Standing Assumptions 8.1 we assume that $W$ is another Hilbert space which is continuously and densely embedded in $H$ such that the inclusion map $W \hookrightarrow H$ is compact. Fix a family of selfadjoint operators $\{D_1(x)\}_{x \in M}$ parametrized by the manifold $M$ such that the following conditions are satisfied:

(A 1) The map $D_1 : M \to \mathcal{L}(W, H)$ is weakly differentiable. This means that the map $x \mapsto \langle D_1(x)\xi, \eta \rangle$ is differentiable for all $\xi, \in W$ and $\eta \in H$. Furthermore, we suppose that the weak derivative $d(D_1)(x) : W \to H \otimes T^*_x(M)$ is bounded for each $x \in M$ and that the supremum $\sup_{x \in M}\|d(D_1)(x)\| =: K < \infty$ is finite.
SPECTRAL FLOW AND THE UNBOUNDED KASPAROV PRODUCT

A2) The domain, $\mathcal{D}(\Delta) = W$, is independent of $x \in M$ and equals $W$. Moreover, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|\xi\|_W \leq \|\xi\|_{\mathcal{D}(\Delta)} \leq C_2 \|\xi\|_W$$

(8.3)

for all $\xi \in W$ and all $x \in M$. Thus the graph norms are uniformly equivalent to the norm $\|\cdot\|_W$ of $W$.

Remark 8.4. 1. These assumptions correspond to the assumptions (A-1), (A-2) of [RoSa95] in the one-dimensional case. When comparing, note that in our (A-2) it suffices to assume the first inequality in Eq. (8.3). The second then follows from the Closed Graph Theorem and the assumption $\mathcal{D}(\Delta) = W$.

2. (A 1) implies that $M \ni x \mapsto \Delta(x) \in \mathcal{L}(W, H)$ is a continuous map from $M$ into the bounded linear operators $W \to H$. To see this we note that for $\xi, \eta \in H$ and $x, y$ in a geodesic coordinate system, with $\gamma(x, y)$ denoting the unique shortest smooth path from $x$ to $y$,

$$\left|\langle (\Delta(x) - \Delta(y))\xi, \eta \rangle \right| = \left|\int_{\gamma(x, y)} \langle d(\Delta(s))\xi, \eta \rangle \right| \leq \sup_{s \in \Gamma} \|d(\Delta(s))\| \cdot \|\xi\| \cdot \|\eta\|_H,$$

(8.4)

hence at least locally $\|\Delta(x) - \Delta(y)\| \leq K \cdot \text{dist}(x, y)$.

The assumption (A 2) clearly implies that the supremum $\sup_{x \in M} \|\Delta(x)\|$ is finite.

As a consequence of these observations we get that the assignment $\Delta(f)(x) := \Delta(x)(f(x))$ defines a bounded operator $\Delta : \mathcal{C}_0(M, W) \to \mathcal{C}_0(M, H)$, which may also be viewed as an unbounded operator in $\mathcal{C}_0(M, H)$ with domain $\mathcal{C}_0(M, W)$. It is not hard to verify that our conditions on the family $\{\Delta(x)\}_{x \in M}$ imply that $\Delta$ is a well-defined selfadjoint and regular operator, cf. [KaLe11, Theorem 4.2. 2. and Theorem 5.8].

We remind the reader of the Standing Assumptions 8.1 and the Graßmann $D_2$-connection $\nabla_{D_2}^{Gr}$, Eq. (8.2). We have identifications

$$\mathcal{C}_0(M, H) \otimes \mathcal{C}_0(M) L^2(M, F) \cong (L^2(M, F))^\infty \cong L^2(M, H \otimes F) \cong H \otimes L^2(M, F)$$

(8.5)

of Hilbert spaces and hence, by slight abuse of notation $1 \otimes_{\nabla_{D_2}^{Gr}} D_2 = \text{diag}(D_2) = D_2$. We notice that $\Gamma_c^\infty(M, H \otimes F)$ is a core for $D_2 = 1 \otimes_{\nabla_{D_2}^{Gr}} D_2$.

We will use the notation $\Delta(\cdot) := \Delta(\cdot)1$ for the selfadjoint operator on $L^2(M, H \otimes F)$ associated to $\Delta$. This notation is very suggestive since for a function $f \in L^2(M, W \otimes F)$ we have the pointwise identity $(\Delta(\cdot)f)(x) = \Delta(x)f(x)$ for a.e. $x \in M$.

We are now going to prove the selfadjointness of the product operator $\Delta_1 \times_{\nabla_{D_2}^{Gr}} D_2$. To this end we need to verify the conditions in Theorem 5.5. The core is given by the smooth compactly supported sections, $\mathcal{E} := \Gamma_c^\infty(M, H \otimes F)$. We start with the first condition.
We will use the notation $d(D_1(\cdot)) : \mathcal{L}^2(W \otimes F) \to \mathcal{L}^2(H \otimes T^*M \otimes F)$ for the exterior derivative which is defined fiberwise by $d(D_1)(x) \otimes I : W \otimes F_x \to H \otimes T^*_xM \otimes F_x$. Furthermore, by slight abuse of notation we let $\sigma_{D_2} : \mathcal{L}^2(M, H \otimes F) \to \mathcal{L}^2(M, H \otimes F)$ denote the map which is defined fiberwise by $\xi \otimes \omega \otimes \eta \mapsto \xi \otimes \sigma_{D_2}(\omega)(\eta)$. Both $d(D_1(\cdot))$ and $\sigma_{D_2}$ are bounded operators by (A.1) and Proposition 8.2.

**Lemma 8.5.** Let $s \in \Gamma^\infty(M, H \otimes F)$ be a smooth compactly supported section. The vector $(D_1(\cdot) - i \cdot \mu)^{-1}(s)$ then lies in the domain of $D_2$. Furthermore we have for $\mu \in \mathbb{R} \setminus \{0\}$ the explicit formula

$$D_2(D_1(\cdot) - i \cdot \mu)^{-1}(s) = (D_1(\cdot) - i \cdot \mu)^{-1}D_2(s)$$

$$- (D_1(\cdot) - i \cdot \mu)^{-1}\sigma_{D_2}(d(D_1(\cdot)))(D_1(\cdot) - i \cdot \mu)^{-1}(s)$$

In particular we also have that $D_2(D_1(\cdot) - i \cdot \mu)^{-1}(s) \in \mathcal{D}(D_1(\cdot))$.

**Proof.** Let us consider a smooth compactly supported section $t \in \Gamma^\infty(M, H \otimes F)$ with support contained in a single coordinate patch $U \subseteq M$ with coordinates $(x_1, \ldots, x_m)$.

We start by noting that the function $x \mapsto \langle (D_1(x) - i \cdot \mu)^{-1}s(x), t(x) \rangle$ is differentiable with partial derivatives given by

$$\frac{\partial}{\partial x_j}\langle (D_1(x) - i \cdot \mu)^{-1}s(x), t(x) \rangle$$

$$= \langle (D_1(x) - i \cdot \mu)^{-1}\frac{\partial s(x)}{\partial x_j}, t(x) \rangle + \langle (D_1(x) - i \cdot \mu)^{-1}s(x), \frac{\partial t(x)}{\partial x_j} \rangle$$

$$- \langle (D_1(x) - i \cdot \mu)^{-1}\frac{\partial D_1(x)}{\partial x_j}(D_1(x) - i \cdot \mu)^{-1}t(x), s(x) \rangle.$$

Now, suppose that $D_2$ is given by the local formula $\sum_{j=1}^m A_j \frac{\partial}{\partial x_j} + B$ over $U$. Using the above computation, we then get that

$$\int_M \langle (D_1(x) - i \cdot \mu)^{-1}s(x), D_2t(x) \rangle \, d\text{vol}$$

$$= \sum \int_M \langle (D_1(x) - i \cdot \mu)^{-1}A_j(x)^*s(x), \frac{\partial t(x)}{\partial x_j} \rangle \, d\text{vol}$$

$$+ \int_M \langle (D_1(x) - i \cdot \mu)^{-1}B(x)^*s(x), t(x) \rangle \, d\text{vol}.$$
we have that

\[ \langle (D_1(x) - i \cdot \mu)^{-1}D_2s(x), t(x) \rangle \text{ dvol} \]

\[ + \sum_j \int_M \langle (D_1(x) - i \cdot \mu)^{-1} \frac{\partial D_1(x)}{\partial x_j}(D_1(x) - i \cdot \mu)^{-1}A_j(x)^*s(x), t(x) \rangle \text{ dvol} \]

\[ = \int_M \langle (D_1(x) - i \cdot \mu)^{-1}D_2s(x), t(x) \rangle \text{ dvol} - \int_M \langle (D_1(x) - i \cdot \mu)^{-1}\sigma_{D_1}(dD_1(x))(D_1(x) - i \cdot \mu)^{-1}s(x), t(x) \rangle \text{ dvol} \]

The claim of the lemma now follows by a partition of unity argument. \( \square \)

**Theorem 8.6.** Under the standing assumptions and (A 1), (A 2) the Dirac–Schrödinger operator

\[
\begin{pmatrix}
0 & D_1(\cdot) - iD_2 \\
D_1(\cdot) + iD_2 & 0
\end{pmatrix} : \left( \mathcal{D}(D_1(\cdot)) \cap \mathcal{D}(D_2) \right)^2 \to L^2(M, H \otimes F)^2
\]

associated with the family of unbounded operators \( \{D_1(x)\}_{x \in M} \) and the differential operator \( D_1 \) agrees with the unbounded product operator \( D_1 \times_{\mathcal{V}} D_2 \) and it is selfadjoint.

Note that for operators in Hilbert spaces (i.e., Hilbert C*-modules over \( \mathbb{C} \)) regularity is not an issue. Therefore, \( D_1 \times_{\mathcal{V}} D_2 \) is automatically regular.

**Proof.** By Theorem 5.5 and Lemma 8.5 we only need to prove that the operator \( [D_1(\cdot), D_2][D_1(\cdot) - i \cdot \mu]^{-1} : \Gamma^\infty(H \otimes F) \to L^2(H \otimes F) \) extends to a bounded operator. Now, by an application of Lemma 8.5 we have that

\[
[D_1(\cdot), D_2][D_1(\cdot) - i \cdot \mu]^{-1} = (D_1(\cdot) - i \cdot \mu)D_2[D_1(\cdot) - i \cdot \mu]^{-1} - D_2
\]

\[
= -\sigma_{D_2}d(D_1(\cdot))(D_1(\cdot) - i \cdot \mu)^{-1}
\]

and the desired boundedness result follows since \( d(D_1(\cdot)) : L^2(W \otimes F) \to L^2(H \otimes T^*M \otimes F) \) and \( \sigma_{D_2} : L^2(H \otimes T^*M \otimes F) \to L^2(H \otimes F) \) are bounded operators. \( \square \)

### 8.4. The index of Dirac–Schrödinger operators on complete manifolds.

We continue in the setting of the Standing Assumptions 8.1 and (A 1), (A 2). On top of these conditions we require

(A 3) that there exist a compact set \( K \subseteq M \) and a constant \( c > 0 \) such that the spectrum \( \text{spec}(D_1(x)) \subseteq (\infty, -c] \cup [c, \infty) \) is uniformly bounded away from zero for all \( x \in M \setminus K \).

This condition corresponds to (A-3) in [RoSa95], however we do not assume that \( D_1(x) \) has limits as \( x \) approaches infinity.

The ellipticity of \( D_2 \) implies that the composition

\[
\mathcal{D}(D_2) \longrightarrow L^2(M, F) \xrightarrow{f} L^2(M, F)
\]

of the inclusion and multiplication with any \( f \in C_0(M) \) is compact. This is immediate for compactly supported smooth \( f \) and then follows since \( C_0^\infty(M) \) is dense in \( C_0(M) \).
It is then not hard to see that the conditions on our differential operator $D_2$ imply that the pair $(D_2, L^2(M, F))$ is an odd unbounded Kasparov $C_0(M)\text{-}C$-module. We let $[D_2] := F(D_2, L^2(M, F)) \in KK^1(C_0(M), C) \cong K^1(C_0(M))$ denote the odd K-homology class obtained from the differential operator $D_2$ under the bounded transform, cf. the beginning of Sec. 7.

We shall now see that the family $\{D_1(x)\}$ gives rise to an odd unbounded Kasparov $C\text{-}C_0(M)$ module after a small modification.

**Proposition 8.7.** Let $\psi \in C_0^1(M)$ be a $C^1$-function which vanishes at infinity such that $\psi(x) > 0$ for all $x \in M$ and $\psi(x) = 1$ for all $x \in K$. Then the family $(\psi^{-1}(x) \cdot D_1(x))_{x \in M}$ defines an odd unbounded Kasparov $C\text{-}C_0(M)$ module $(\psi^{-1} \cdot D_1, C_0(M, H))$.

**Proof.** We define the unbounded operator

$$\psi^{-1} \cdot D_1 : D(\psi^{-1} \cdot D_1) \rightarrow C_0(M, H), \quad (\psi^{-1} \cdot D_1)(f)(x) = \psi^{-1}(x) \cdot D_1(x)(f(x)).$$

The domain is given by $D(\psi^{-1} \cdot D_1) = \{ f \in C_0(M, W) \mid D_1(f) \in \psi \cdot C_0(M, H) \}$.

We start by proving that $\psi^{-1} \cdot D_1$ is selfadjoint and regular. $\psi^{-1} \cdot D_1$ is certainly symmetric. To see that it is closed we let $\{f_n\}$ be a sequence in the domain such that $f_n \rightarrow f$ and $(\psi^{-1} \cdot D_1)(f_n) \rightarrow g$ is convergent in $C_0(M, H)$. It follows that $\{D_1(f_n)\}$ is convergent. But $D_1$ is closed so $f \in D(D_1)$ with $D_1(f) = \psi \cdot g$. This proves that $\psi^{-1}D_1$ is closed. The selfadjointness and regularity now follows by [Kale11, Theorem 4.2, 2. and Theorem 5.8]. Indeed, the localized unbounded operator at $x \in M$ is simply given by $\psi^{-1}(x) \cdot D_1(x) : W \rightarrow H$ which is selfadjoint by assumption.

Finally we show that the resolvent $(\psi^{-1} \cdot D_1 - i)^{-1} = \psi \cdot (D_1 - i\psi)^{-1}$ is compact. To this end we recall that the compact operators on the Hilbert $C^*$-module $C_0(M, H)$ are given by $K(C_0(M, H)) = C_0(M, K(H))$. Now, the operator $(D_1(x) - i\psi(x))^{-1} \in K(H)$ is compact for all $x \in M$ since the inclusion $W \rightarrow H$ is compact and it depends continuously on the parameter $x \in M$ by Remark 8.4, 2. and the resolvent identity. Furthermore, in view of (A 3) and the spectral theorem for unbounded selfadjoint operators we get that $\sup_{x \in M} \| (D_1(x) - i\psi(x))^{-1} \| < \infty$. Altogether this implies that $(\psi^{-1} \cdot D_1 - i)^{-1} = \psi \cdot (D_1 - i\psi)^{-1}$ lies in $C_0(M, K(H))$, proving the claim. □

We will use the notation $[D_1] := F(C_0(M, H), \psi^{-1} \cdot D_1) \in KK^1(C, C_0(M)) \cong K_1(C_0(M))$ for the odd K-theory class obtained from the parametrized family $\{\psi^{-1}(x)D_1(x)\}_{x \in M}$ under the bounded transform. As the notation suggests, the class $[D_1]$ is independent of the choice of function $\psi \in C_0^1(M)$ as long as $\psi(x) > 0$ for all $x \in M$ and $\psi|_K = 1$.

**Lemma 8.8.** Let $\psi$ and $\phi \in C_0^1(M)$ be two strictly positive $C^1$-functions which vanish at infinity and with $\psi|_K = 1 = \phi|_K$. The odd unbounded Kasparov modules $(\psi^{-1}D_1, C_0(M, H))$ and $(\phi^{-1}D_1, C_0(M, H))$ then represent the same class in $KK^1(C, C_0(M))$ under the bounded transform.
Proof. This follows by noting that the difference \( D_1(\psi^2 + D_1^2)^{-1/2} - D_1(\phi^2 + D_1^2)^{-1/2} \in C_0(M, K(H)) \) of bounded transforms is a compact operator. Indeed, we have that the difference
\[
D_1(x)(\psi^2(x) + D_1(x)^2)^{-1/2} - D_1(x)(\phi^2(x) + D_1^2(x))^{-1/2} \in K(H)
\] (8.6)
is compact for all \( x \in M \) since the function \( t \mapsto t(\psi^2(x) + t^2)^{-1/2} - t(\phi^2(x) + t^2)^{-1/2} \) lies in \( C_0(\mathbb{R}) \). Furthermore by [Les05, Prop. 2.2] and Remark 8.4, 2, we get that the vanishing at infinity follows from (A 3) and the spectral theorem for unbounded selfadjoint operators. \( \square \)

We can now make a sensible definition of spectral flow.

**Definition 8.9.** By the spectral flow of the family \( \{D_1(x)\}_{x \in M} \) with respect to the differential operator \( D_2 : \Gamma_c^\infty(M, F) \to L^2(M, F) \) we will understand the integer given by the interior KK–product \( [D_1 \otimes C_0(M)]D_2 \in KK(C, C) \) under the identification \( KK(C, C) \cong \mathbb{Z} \). We will apply the notation \( SF(D_1, D_2) \in \mathbb{Z} \) for the spectral flow.

We remark that the interior Kasparov product between odd K–theory and odd K–homology can be identified with the index pairing [Blaq8, Sec. 18.10].

In order to describe the above spectral flow as the index of an unbounded Fredholm operator we need to construct a correspondence between the odd unbounded Kasparov modules \( (\psi^{-1} \cdot D_1, C_0(M, H)) \) and \( (D_2, L^2(M, F)) \). To be able to deduce the commutator condition Def. 6.3, (4) for the pair \( \psi^{-1}D_1, D_2 \) from that for the pair \( D_1, D_2 \) we have to assume additionally that \( d\psi^{-1} \) is globally bounded. Let us first show that there are sufficiently many such functions.

**Lemma 8.10.** Let \( M \) be a complete Riemannian manifold, \( K \subset M \) a compact subset. Then there exists \( \psi \in C^\infty(M) \) strictly positive and vanishing at infinity such that

1. \( \psi(x) = 1 \) for all \( x \in K \),
2. \( \sup_{x \in M} |d\psi^{-1}(x)| < \infty \).

**Proof.** Fix \( x_0 \in M \). Let \( \phi \in C^\infty(M) \) be a smooth approximation of the distance function \( \text{dist}(\cdot, x_0) \) [Gaf59, Sec. 3] in the sense that \( |\phi(x) - \text{dist}(x, x_0)| \leq 1 \), \( |d\phi(x)| \leq 2 \) for all \( x \in M \). Furthermore, let \( \varphi \in C^\infty_c(M) \) be a compactly supported cut-off function with \( \varphi(x) = 1 \) for \( x \) in a neighborhood of \( K \). Then \( \psi^{-1}(x) := \rho(x) + (1 - \rho(x))(2 + \phi(x)) \) does the job. \( \square \)

**Proposition 8.11.** Let \( \psi \in C_0^1(M) \) be a strictly positive \( C^1 \)–function vanishing at infinity such that \( \psi^{-1} \) satisfies conditions (1) and (2) of Lemma 8.10. Then the pair \( (C_0^1(M, H), \nabla_2^{Gr}) \) given by the operator \( * \)–module \( C_0^1(M, H) \) over the operator \( * \)–algebra \( C_0^1(M) \) and the Graßmann \( D_2 \)–connection is a correspondence from \( (\psi^{-1}D_1, C_0(M, H)) \) to \( (D_2, L^2(M, F)) \).

**Proof.** Lemma 8.10 ensures that \( \psi \) with the desired properties exists. Then the result is mainly a consequence of Theorem 5.4, Lemma 8.5 and the proof of Theorem 8.6.
Note that
\[ [\psi^{-1} D_1, D_2] = -\sigma_{D_2}(d\psi^{-1}) D_1 + \psi^{-1}[D_1, D_2]. \] (8.7)

Bounded propagation speed Eq. (8.4), the global boundedness of \( d\psi^{-1} \) and the proof of Theorem 8.6 now show that \( [\psi^{-1} D_1(\cdot), D_2](D_1(\cdot) - i\cdot \mu)^{-1} \) extends to a bounded operator.

The main result of this section now follows from Theorem 7.5. See also Theorem 8.6.

**Corollary 8.12.** Under the assumptions of the previous Proposition the Dirac–Schrödinger operator \( \psi^{-1} D_1(\cdot) + iD_2 : D(\psi^{-1} D_1(\cdot)) \cap D(D_2) \rightarrow L^2(M, H \otimes F) \) is an unbounded Fredholm operator and the index

\[ \text{ind}(\psi^{-1} D_1(\cdot) + iD_2) = \text{SF}(D_1, D_2) \]

coincides with the spectral flow of the family \( \{D_1(x)\} \) with respect to the differential operator \( D_2 \).

**8.4.1. Proof of Theorem 1.2.** Finally, we make the link to the index of Dirac–Schrödinger operators as in [Ang93a, Prop. 1.4]. In loc. cit. operators of the form \( D + i\lambda A \), with \( D \) being a Dirac-type operator and \( A \) being a selfadjoint bundle homomorphism, are considered. \( D \) corresponds to our \( D_2 \) and \( A(x) \) corresponds to our \( D_1(x) \) in the special case of a finite-dimensional Hilbert space \( H \). Then under assumptions which are similar but slightly stronger than our (A 1), (A 2), (A 3) it is proved that \( D + i\lambda A \) is Fredholm for \( \lambda \) large enough.

In fact, it is not hard to see that under our assumptions (A 1), (A 2), (A 3) it follows from [KAL11, Lemma 7.6] that given \( C > 0 \) there exists a \( \lambda_0 = \lambda_0(C) > 0 \) large enough such that the operator \( \phi D_1(\cdot) + iD_2 \) is Fredholm for any \( C^1 \)-function \( \phi \in C^1(M) \) which satisfies \( \sup_{x \in M} |d\phi(x)| \leq C \) and \( \phi(x) \geq \lambda_0 \) for \( x \in M \setminus K \).

It then follows from the stability of the Fredholm index under deformations in the graph topology, cf., e.g., [Cola63], that for any such \( \phi \) the index of \( \phi D_1(\cdot) + iD_2 \) coincides with the spectral flow \( \text{SF}(D_1, D_2) \). This argument in particular applies to the function \( \phi \equiv \lambda \) for \( \lambda \geq \lambda_0(1) \).

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