Twisted Morava K-theory and connective covers of Lie groups

Hisham Sati * and Aliaksandra Yarosh†

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Abstract

Twisted Morava K-theory, along with computational techniques, including a universal coefficient theorem and an Atiyah-Hirzebruch spectral sequence, was introduced by Craig Westerland and the first author in [SW15]. We employ these techniques to compute twisted Morava K-theory of all connective covers of the stable orthogonal group and stable unitary group, and their classifying spaces, as well as spheres and Eilenberg-MacLane spaces. This extends to the twisted case some of the results of Ravenel and Wilson [RW80] and Kitchloo, Laures, and Wilson [KLW04a] for Morava K-theory. This also generalizes to all chromatic levels computations by Khorami [Kh11] (and in part those of Douglas [Do06]) at chromatic level one, i.e. for the case of twisted K-theory. We establish that for natural twists in all cases, there are only two possibilities: either that the twisted Morava homology vanishes, or that it is isomorphic to untwisted homology. We also provide a variant on the twist of Morava K-theory, with mod 2 cohomology in place of integral cohomology.

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∗hsati@nyu.edu
†alexandra.yarosh@gmail.com


1 Introduction

Twisted cohomology and covering spaces have a very intimate relationship. Twisted cohomology encodes additional data coming from a principal bundle on a space. The earliest incarnation is perhaps cohomology with local coefficients which allows, in particular, to define Poincaré duality for non-orientable manifolds, as long as we consider the cohomology with coefficients given by the local system of the orientation double cover. What we consider here is a generalization of the two concepts, namely that of a cover and that of a twisted cohomology, to generalized covers and to generalized cohomology theories. For the former we will consider groups and their classifying spaces arising from cokilling homotopy groups of the stable orthogonal and unitary groups, leading to their Whitehead towers. On the latter, we will consider twisted Morava K-theory $[SW15]$, of which twisted K-theory can essentially be viewed as a specific instance.

Morava K-theory $K(n)$ is in some sense an “extension” of K-theory: it is a complex-oriented cohomology theory that defined for every integer $n$ and prime $p$, where $n$ is the chromatic level, i.e., the height of the corresponding formal group law. See $[JW75]$ $[W82]$ $[W91]$ $[Ru98]$ $[RW80]$ $[Ra86]$ $[HRW98]$ $[HKR92]$. Note that $K(n)_*(X)$ is always a coalgebra. When $X$ is an H-space, it is in addition a Hopf algebra. This will be the case for all the examples for which we compute the Morava K-theory. Therefore, their corresponding Morava K-homology theories will be Hopf algebras.

The first author and Westerland $[SW15]$ show that Morava K-theory $K(n)$, Morava E-theory $E_n$, and some of their variants admit twists by $K(Z,n+1)$ bundles. The motivation for this theory came from string theory: it was conjectured by the first author $[Sa09]$ that a twisted form of Morava K-theory and E-theory should describe an extension of the untwisted setting in his work with Kriz on anomalies at chromatic level two $[KS04]$. A vast generalization of this conjecture is proved in $[SW15]$.

The coefficients of the theory $K(n)_* = \mathbb{Z}/p[v_n,v_n^{-1}]$ form a graded field, which implies that $K(n)_*(-)$ always has a very useful computational tool manifested in the Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

In our main examples, we do not quite have products, but rather fibrations. No similar general formula for calculations for fibrations exits. However, when the fiber is an Eilenberg-MacLane space, a very useful tool is available: The universal coefficient theorem (UCT) of Sati-Westerland $[SW15]$, which says that $K(Z,n+2)$-twisted Morava K-theory $K(n)$ of a space is isomorphic to the untwisted Morava K-theory of a certain $K(Z,n+1)$-bundle over that space. This is a generalization of a theorem by Khorami $[Kh11]$ in the case of twisted K-theory, i.e., for $n = 1$, where the fibers are $K(Z,2) \cong CP^\infty$.

The main examples that we consider arise from connective covers of classical Lie groups. The Whitehead tower is a way of approximating a space by a sequence of spaces of increasing connectivity that under mild assumptions converges to the original space. More precisely, it is a sequence
of fibrations $X^{m+1} \to X^m \to \cdots \to X$, such that each $X^m$ is $(m-1)$-connected (i.e., the first non-trivial homotopy group occurs in degree $m$), and the induced map $\pi_i(X^m) \to \pi_i(X)$ is an isomorphism for $i \geq m$ (i.e., we are successively killing homotopy groups of $X$, starting from the bottom). In particular, $X^2$ is the universal cover of $X$. In general $X^m$ is called the $m$-connective cover of $X$. The main example appearing in our work is the connective covers of the stable orthogonal group and its classifying space. Some of the connective covers have distinguished names arising from string theory and M-theory [SSS09][SSS12][Sa15].

\begin{equation}
\begin{array}{c}
K(\mathbb{Z},11) \longrightarrow BO(13) = BNinebrane \\
K(\mathbb{Z}/2,9) \longrightarrow BO(11) = B2\text{-}Spin \overset{\frac{1}{2\pi}p_3}{\longrightarrow} K(\mathbb{Z},12) \\
K(\mathbb{Z}/2,8) \longrightarrow BO(10) = B2\text{-}Orient \overset{\alpha_{10}}{\longrightarrow} K(\mathbb{Z}/2,10) \\
K(\mathbb{Z},7) \longrightarrow BO(9) = BFivebrane \overset{\alpha_9}{\longrightarrow} K(\mathbb{Z}/2,9) \\
K(\mathbb{Z},3) \longrightarrow BO(8) = BString \overset{\frac{1}{2}p_2}{\longrightarrow} K(\mathbb{Z},8) \\
K(\mathbb{Z}/2,1) \longrightarrow BO(4) = BSpin \overset{\frac{1}{2}p_1}{\longrightarrow} K(\mathbb{Z},4) \\
K(\mathbb{Z}/2,0) \longrightarrow BO(2) = BSO \overset{w_2}{\longrightarrow} K(\mathbb{Z}/2,2) \\
\text{BO} \overset{w_1}{\longrightarrow} K(\mathbb{Z},2,1)
\end{array}
\end{equation}

Notice that notation $BO(m)$ means $(BO)(m)$ rather than $B(O(m))$. In the above diagram, $w_1$ and $w_2$ are first and second Stiefel-Whitney classes, respectively, with the classes $\alpha_9$ and $\alpha_{10}$ their exotic generalizations (see [Sa15]) and $\frac{1}{2}p_i$ is the fractional $i$th Pontrjagin class. Unlike SO and Spin, which have classical descriptions as a Lie groups, String – as well as higher covers – is not a Lie group. There are various models for these groups in the literature, but we will not need such explicit characterizations in this paper, as we are mainly interested in the homotopy type.

We compute twisted Morava K-theory of all the connective covers of BO and BU in Section 3.1. We will consider various Eilenberg-MacLane (EM) fibrations arising in diagram (1.1) (and its complex counterpart),

\[ EM \longrightarrow X^m \longrightarrow X, \]

to which we apply the UCT of [SW15], leading to determination of the twisted Morava K-theory of $X$ from the untwisted Morava K-theory of $X^m$. The latter relies on recent extensive results of Kitchloo, Laures, and Wilson on Morava K-theory of spaces related to the classifying space BO [KLW04a][KLW04b], while the Morava K-theory of the fiber relies on the computations of Ravenel and Wilson of the Morava K-theory of Eilenberg-MacLane spaces [RW80].
The idea of calculating the twisted theory of a space via the untwisted theory of a related space also has other occurrences. Twisted K-theory of finite-dimensional projective spaces $\mathbb{R}P^n$ was studied in [BEJMS06], where they were shown to be computed, via T-duality, using the untwisted K-theory of the dual space (often a sphere). We will generalize the projective spaces to infinite dimensions and then to higher cohomological ranks, i.e., Eilenberg-MacLane spaces.

The instances where the fiber is a mod 2 Eilenberg-MacLane space require us to introduce twists by mod 2 Eilenberg-MacLane spaces (of different rank), and consider a mod 2 analogue of the universal coefficient theorem from [SW15], which we establish in Section 3.4. We observe that, in all cases considered, twisted Morava K-theory is either equal to untwisted Morava K-theory of these covers, or is zero altogether, with a transition occurring after height 2.

The computations using the UCT of [SW15] rely on twisted homology rather than on twisted cohomology versions of Morava K-theory. For $R$ a spectrum, it is generally easier to compute the twisted $R$-homology rather than the twisted $R$-cohomology of spaces. Consider bundles over a space $X$ with fiber the $R$-theory spectrum (satisfying certain conditions). Then twisted $R$-homology is naturally defined as the homotopy of the total space of the bundle, relative to the base [ABG10] [ABGHRT] [SW15].

Twisted K-theory (see [DK70] [Ro89] [BCMMS02] [AS04] [AS06]) twisted by $PU(\mathcal{H}) \cong K(\mathbb{Z}, 2)$ bundles also has a homological counterpart, namely twisted K-homology, discussed topologically in [Do06], geometrically in [Wa08], and analytically in [Me12]. Twisted geometric cycles for general CW-complexes is described in [BCW13]. Directly one prime at a time, a finite-dimensional (analytic) model for twisted mod $p$ K-theory by using twisted vectorial bundles with Clifford action is given in [Go12]. Morava K-theory at chromatic level one, $K(1)$, is one of $(p - 1)$-summands in the Adams decomposition of K-theory at a prime $p$. We will mainly be interested in the prime $p = 2$, so that the relation between $K(1)$ and complex K-theory is even more direct.

Twisted K-theory of compact Lie groups has been studied by physicists (see, e.g., [Br04] [MMS01] [GG04]) as well as by mathematicians, starting with Douglas [Do06]. The computation of the twisted K-groups was extended to Lie groups which are not necessarily compact simple and simply connected in [GG04] [MR17]. The results for twisted K-theory $K^*(G, h)$, for arbitrary choices of the twist $h$, are already rather complicated and hard to understand. In [MR17] Mathai and Rosenberg introduced a new method for computing twisted K-theory using the Segal spectral sequence, giving simpler computations of certain twisted K-theory groups of compact Lie groups relevant for physics. Twisted K-groups for Lie groups are shown to be trivial except in the simplest case of $SU(2)$ and $SO(3)$.

There has been a lot of activity in calculating the Morava K-theory of $BG$, for $G$ a finite group, due to the Hopkins-Kuhn-Ravenel conjecture on evenness of Morava K-theory of $BG$ [HKR92] and its counterexamples by Kriz [Kr97]. Note that all Eilenberg-MacLane spaces used in this paper have even Morava K-theory, by [RWY98]. Finite groups can support integral twists needed for Morava K-theory, as the integral cohomology of a $K(G, 1)$ space is in general nonzero, albeit all
torsion. However, the homotopy degree of such a space is not supported by the computational techniques that we use here. It would be interesting to investigate twisted Morava K-theory of such spaces via alternative methods.

Which chromatic level for Morava K-theory to use? The first effect we encounter here is acyclicity. From Ravenel-Wilson [RW80], the nth Morava K-theory ‘sees’ only the first n Eilenberg-MacLane spaces. As far as localization goes, mod p K-theory is the first in the Morava series K(n). From Anderson and Hodgkin [AH68], the space K(G, n) is KZ/p-acyclic for all n ≥ 3 and all G. This implies that, to the eyes of mod p K-theory, the higher Eilenberg-MacLane spaces might as well be points. For higher chromatic levels, Ravenel and Wilson [RW80] showed that for K(n) at odd primes p,

\[ \tilde{K}(n)_* K(G, n + 2) = 0, \quad \tilde{K}(n)_* K(A, n + 1) = 0, \]

for A any finite abelian group, in particular a cyclic group Z/p. Consequently, fibrations of connective covers stabilize in the sense that the K(n)-(co)homology would be the same for X(m) and X(m + 1) after some critical m. Note that this is a general feature in the sense that every homology theory E_\ast has a transitional dimension, at which all higher Eilenberg-MacLane spaces are E_\ast-acyclic (see [Bo82]).

For us this means that, in order to describe the connective covers involving EM spaces, we need a chromatic level that is at least as high as the homotopy degree of that space. That is, for String, BString, Fivebrane, and BFivebrane, we need at least K(2), K(3), K(6) and K(7), respectively. If we use lower chromatic levels then we will not be able to see the fiber in the fibration defining the connective covers. This then would mean that, to the eyes of those Morava K-theories with corresponding chromatic levels lower than the above threshold, the calculations essentially reduce to those of the total space, i.e., to Spin, BSpin, String and BString, respectively.

The second effect that we witness is the vanishing of twisted Morava K-theory of certain spaces. We consider Eilenberg-MacLane spaces in Section 3.2 as well as twists by fundamental classes of spheres in Section 3.3 and show that at height 2 and above the twisted Morava K-homology is zero. At first, it might seem surprising that the twisted homology would ever be zero at all – for untwisted homology, in the most trivial case of a point or contractible space one has that the homology is equal to the coefficient ring. However, reduced twisted homology of a point is zero, so vanishing twisted homology just means that the space behaves like a point in our setting. For the case of twisted K-theory, it was shown in [MR17] that the twisted K-groups of most Lie groups in fact vanish.

Directly tracing the essence of the proofs of the theorems in Section 3.1 this method can be captured in the following general vanishing theorem for twisted Morava K-homology.
**Theorem 1** (Vanishing theorem for twisted Morava K-theory). If a principal \( K(\mathbb{Z}, n + 1) \) bundle \( \xi : E \to B \) is such that the induced map on Morava homology is a map of Hopf algebras, then composition with the Bockstein map

\[
K(n)_* (K(\mathbb{Z}/2, n)) \overset{\delta p_1}{\to} K(n)_* (K(\mathbb{Z}, n + 1)) \to K(n)_* (E)
\]

is zero and hence twisted Morava homology of the base vanishes, \( K(n)_*(B, \xi) = 0 \).

Note that a map of Hopf algebra occurs, for example, when \( E \to B \) is a loop space map.

A third important phenomenon that we encounter is that of **untwisting**. We show in Section 3.1 that for all connective covers, with the twist given by the corresponding class defining the fibration, the twisted Morava K-homology is given by the underlying untwisted Morava K-homology. There is, however, one notable exception which, somewhat surprisingly, is the classifying space of a classical Lie group: we show in Proposition 20 that \( K(2) \) of \( B \text{Spin} \) with a twist given by the generator \( \frac{1}{2} p_1 \) in fact vanishes.

**Theorem 2** (Untwisting in the Whitehead tower). The twisted Morava K-homology of all groups in the Whitehead tower of the orthogonal and unitary groups, and their classifying spaces (except for \( B \text{Spin} \) in Proposition 20), with the canonical twist, is isomorphic to the underlying untwisted Morava K-homology.

This seems to be an instance of a more general phenomenon, and it does occur even in the case of K-homology. For \( G \) a compact, connected, simply connected, simple Lie group of rank \( n \), the twisted K-homology ring with non-zero twisting class \( k \in H^3(G; \mathbb{Z}) \cong \mathbb{Z} \) is an exterior algebra of rank \( n - 1 \) tensored with a cyclic group \([\text{Do06} \text{ Br04}] K_*(G) \cong \Lambda[x_1, \cdots, x_{n-1}] \otimes \mathbb{Z}/c(G, k)\), where \( c \) is a constant which is, interestingly, most involved for the case of \( \text{Spin}(m) \). This cyclic term can be viewed as the effect of the twist in the sense that one gets the untwisted groups when \( c \) is zero. This is certainly the case when the twist itself is zero, \( k = 0 \). However, investigating the expressions for \( c \) in [Do06] this can be zero for many nontrivial twists, hence effectively reducing the twisted theory to the untwisted one. For the “basic twist” \( k = 1 \) one has that \( c(\text{Spin}(4n - 1), 1), c(\text{Spin}(4n + 1), 1), c(\text{Spin}(4n + 2), 1), \text{and } c(\text{Spin}(4n), 1) \) are all equal to \( \gcd\{1, 0\} \) and hence are zero. This then implies that the cyclic factor is \( \mathbb{Z} \), so that the twisted K-homology reduces to the underlying exterior algebra. Therefore, for the basic twist, one has \( K_*(\text{Spin}(n)) \cong K_*(\text{Spin}(n)) \). The same holds for instance for other ‘special’ values, e.g. \( k = 2n - 1, 2n + 1 \) and \( 2n + 3 \). Douglas in [Do06] also finds that \( \text{Spin}(n) \) is very special in that, unlike other Lie groups, a detailed knowledge is needed of the twisted module structure on \( \mathbb{Z} \) required to identify the cyclic orders.

In addition to the universal coefficient theorem (UCT), another computational tool we will use is the twisted Atiyah-Hirzebruch spectral sequence (AHSS) from [SW15], which approximates a twisted generalized (co)homology theory by usual (co)homology of successive quotients arising from nested filtrations of the underlying space \( X \). This generalizes to higher chromatic levels the
AHSS for twisted K-theory \[\text{[Ro82][Ro89][AS04][AS06]}, \text{which in turn generalizes that of complex K-theory [AH61].}\]

Rationally, the integral version of Morava K-theory $K(n)$ with coefficients $K(n)_* = \mathbb{Z}[v_n, v_n^{-1}]$, is isomorphic to $v_n$-periodic rational cohomology $K(n)^*(X) \otimes \mathbb{Q} \cong H^*(X; K(n)_* \otimes \mathbb{Q})$, where $v_n$ is the generator of degree $2(p^n - 1)$ (see \[\text{[KS04][SW15][GS17]}\]). The rational computations of those connective covers that we consider have been studied in \[\text{[SW17]}\]. Here we consider one prime at a time, with the prime $p = 2$ playing a distinguished role, hence concentrating instead on torsion information.

This paper is organized as follows. We start in Section \[\text{2}\] by providing the setting and the main background that we need, recalling the results of Khorami on twisted K-homology in Section \[\text{2.1}\], which we aim to generalize in later sections. In Section \[\text{2.2}\] we recall the fundamental computations in untwisted Morava K-theory, mainly those of Ravenel and Wilson for Eilenberg-MacLane spaces and Kitchloo, Laures and Wilson for spaces related to BO. Then in Section \[\text{2.3}\] we recall the main constructions of twists of Morava K-theory, as well as the computational tools developed in \[\text{[SW15]}\], and which we will directly apply in later sections. We also provide a partial characterization and computational tools for twisted $K(1)$, or mod 2 K-theory, in Section \[\text{2.4}\]. Then in the main parts of the paper in Section \[\text{3}\] we present our computations and constructions. In Section \[\text{3.1}\] we compute the twisted Morava K-theory of those connective covers of the classifying spaces BO and BU which have fibers integral Eilenberg-MacLane spaces. Then in Section \[\text{3.2}\] and Section \[\text{3.3}\] we provide the computations for twisted K-theory of Eilenberg-MacLane spaces and of spheres, respectively. In order to complete the computation of connective covers from Section \[\text{3.1}\] we compute the twisted Morava K-theory of those covers in the tower \[\text{(1.1)}\] with fibers mod 2 Eilenberg-MacLane spaces in Section \[\text{3.4}\]. This requires a mild extension of some of the results of \[\text{[SW15]}\] to include mod 2 twists. Many of the computations in this paper have been part of the thesis of the second author \[\text{[Ya17]}\] under the guidance of the first author.

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\section{Computations in Morava K-theory and (variations on) its twists}

\subsection{Computations in twisted K-homology}

In this section we summarize the main results of Khorami \[\text{[Kh11]}\] which we will need. Among the results of this paper is a generalization of the main computations there, using the techniques from \[\text{[SW15]}\]. Consider a space $X$ with a twist $\tau \in H^3(X; \mathbb{Z})$ of degree three and consider a $\mathbb{C}P^\infty$ principal bundle $\mathbb{C}P^\infty \to P_\tau \to X$ induced by $\tau$. This setup leads to a spectral sequence (see
Even though $K_\ast$ is not a flat $K_\ast K(\mathbb{Z}, 2)$-module, the K-homology universal coefficient theorem says

**Theorem 3** (K-homology UCT [Kh11]).

$$K_\ast^+(X) \cong K_\ast(P_\tau) \otimes_{K_\ast(\mathbb{CP}^\infty)} \hat{K}_\ast,$$

where $\hat{K}_\ast$ is just the coefficient ring $K_\ast$ with the $K_\ast(\mathbb{CP}^\infty)$-module structure obtained from the action of $\mathbb{CP}^\infty$ on K-theory.

The module structure is important for the purpose of this paper, and the following is a quick summary from how that works in the case of K-homology [Kh11]. The module structure for the more general case of twisted Morava K-theory in Section 2.3 will be similar. The action of $\mathbb{CP}^\infty$ on $K$ is via tensor product with a complex line bundle $L$, which is the main reason why twisted K-theory is possible. The bundle $P_\tau$ admits a fiberwise action of $\mathbb{CP}^\infty$ given by $\mathbb{CP}^\infty \times P_\tau \to P_\tau$.

The twisted K-homology of $X$ is given as [Kh11]

$$K_\ast^+(X) \cong K_\ast(P_\tau) \otimes_{K_\ast(\mathbb{CP}^\infty)} \hat{K}_\ast.$$

The graded abelian group $\hat{K}_\ast$ is the same as $K_\ast$ with the $K_\ast(\mathbb{CP}^\infty)$-module structure being not the one obtained by collapsing $\mathbb{CP}^\infty$ to a point, $\mathbb{CP}^\infty \to \text{pt}$ (cf. Remark 5), but rather coming from the map $\mathbb{CP}^\infty \to \text{GL}_1(K)$ and then using the tautological action of the target on $K_\ast$. For any principal $\mathbb{CP}^\infty$ bundle $P_\tau \to X$, K-homology $K_\ast(P_\tau)$ is a $K_\ast(\mathbb{CP}^\infty)$-module, where the action of $\mathbb{CP}^\infty$ on the total space $P_\tau$ induces a map $K_\ast(\mathbb{CP}^\infty \times P_\tau) \to K_\ast(P_\tau)$. Since $K_\ast(\mathbb{CP}^\infty)$ is free over the coefficients $K_\ast$, one has an isomorphism $K_\ast(\mathbb{CP}^\infty \times P_\tau) \cong K_\ast(\mathbb{CP}^\infty) \otimes_{K_\ast} K_\ast(P_\tau)$ which gives the module structure $K_\ast(\mathbb{CP}^\infty) \otimes_{K_\ast} K_\ast(P_\tau) \to K_\ast(P_\tau)$.

Note that the K-homology of $\mathbb{CP}^\infty$ can be given explicitly as follows (see [Ad74]). From complex orientation, $K_\ast(\mathbb{CP}^\infty) = K_\ast(\text{pt})[[x]]$, where $x = L - 1$, where $L$ is the Hopf line bundle over $\mathbb{CP}^\infty$.

So there are unique elements $\beta_i \in K_{2i}(\mathbb{CP}^n)$, $1 \leq i \leq n$, such that $\langle x^k, \beta_i \rangle = \delta_i^k$, $1 \leq k \leq n$. The collection $\{\beta_0 = 1, \beta_1, \beta_2, \cdots\}$ forms a $K_\ast$-basis for $K_\ast(\mathbb{CP}^\infty)$

$$K_\ast(\mathbb{CP}^\infty) = K_\ast\{\beta_0, \beta_1, \cdots\} = \mathbb{Z}[t, t^{-1}]\{\beta_0, \beta_1, \cdots\}.$$

The main examples presented in [Kh11] are the following.

**Example 1** (Degree three integral Eilenberg-MacLane space $K(\mathbb{Z}, 3)$). For a twist $n : K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$, comparing the bundle $K(\mathbb{Z}, 2) \to P_n \to K(\mathbb{Z}, 3)$ with the path-loop fibration $K(\mathbb{Z}, 3) \to PK(\mathbb{Z}, 2) \cong * \to K(\mathbb{Z}, 2)$ identities $P_n$ with $K(\mathbb{Z}/n\mathbb{Z}, 2)$. Then, invoking a result of Anderson and Hodgkin [AH68] that $\hat{K}_\ast(K(\mathbb{Z}/n\mathbb{Z}, 2)) = 0$, gives the triviality of twisted K-homology $K_\ast^{(n)}(K(\mathbb{Z}, 3)) = 0$. 

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Example 2 (Three-Sphere \(S^3\)). Since \(H^3(S^3; \mathbb{Z}) \cong \mathbb{Z}\) any twist \(n : S^3 \to K(\mathbb{Z}, 3)\) corresponds to an integer. The differential in the Atiyah-Hirzebruch-Serre spectral sequence is identified as \(d_3(\sigma_3) = n\beta_1\), where \(\sigma_3\) is the natural generator of \(H_3(S^3, K(\mathbb{C}P^\infty, 3))\), corresponding to the natural generator of \(H_3(S^3; \mathbb{Z})\), and \(\beta_1\) is the degree one generator of \(K\)-homology of \(\mathbb{C}P^\infty\) above, interpreted cohomologically as a map \(S^2 = \mathbb{C}P^1 \to \mathbb{C}P^\infty\). The \(\mathbb{Z}/2\)-graded twisted \(K\)-homology is then

\[
K_*(^{(n)}(S^3) \cong (K_*(\mathbb{C}P^\infty)/n\beta_1) \otimes_{K_*(\mathbb{C}P^\infty)} \mathbb{K}_* / n = \mathbb{Z}/n\mathbb{Z},
\]

and vanishes for the basic twist \(n = 1\).

Example 2 can also be deduced from the calculations of twisted \(K\)-homology of Lie groups [Do06, Br04, MR17] since \(S^3 \cong \mathrm{SU}(2)\). We will generalize the above two examples to higher dimensions and higher chromatic levels in Section 3.2 and Section 3.3 respectively.

2.2 Computations in Morava \(K\)-theory

We briefly describe some computational tools used in this paper. First, just from the fact that \(K(n)\) is complex-oriented, the Morava \(K\)-theory cohomology of classifying spaces of the unitary group is given as [Ad74, Ra86]

\[
K(n)^*(\mathbb{C}P^\infty) \cong K(n)^*[x],
\]

\[
K(n)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong K(n)^*[x, y],
\]

\[
K(n)^*(\mathrm{BU}(n)) \cong K(n)^*[c_1, c_2 \ldots c_n],
\]

where \(|x| = |y| = 2\), and \(|c_k| = 2k\).

The fundamental computation here is the Morava \(K\)-theory of Eilenberg-MacLane spaces by Ravenel and Wilson. While the Hopf algebra \(K(n)^*K(\mathbb{Z}, n + 1)\) is a power series algebra on one generator, the dual Hopf algebra \(K(n)^*K(\mathbb{Z}, n + 1)\) turns out to be an algebra with generators \(x\) satisfying \(x^p = ux\) for a suitable unit \(u\).

Theorem 4 ([RWS80, Theorem 11.1, Theorem 12.1]). Let \(K(n)\) be Morava \(K\)-theory at prime \(p\).

\[\text{(i) } K(n)^*K(\mathbb{Z}/p^j, q) \cong K(n)^* \text{ for } q > n.\]

\[\text{(ii) } K(n)^*K(\mathbb{Z}/p^j, n) \cong \bigotimes_{i=0}^{j-1} R(a_i) \text{ and } K(n)^*(K(\mathbb{Z}/p^j, n)) \cong K(n)^*[x]/x^{p^j}, \text{ where the generator } x \text{ has dimension } |x| = 2^{p^j-1}, \text{ the element } a_k \text{ is dual to } (-1)^{k(n-1)}x^{p^j}, \text{ and } R(a_k) = \mathbb{Z}/p[a_k, v_n^{\pm 1}]/(a_k^{p^j} - (-1)^{n-1}v_n^{p^j}a_k).\]

\[\text{(iii) } K(n)^*K(\mathbb{Z}, q + 1) \cong K(n)^* \text{ for } q > n.\]

\[\text{(iv) Let } \delta : K(\mathbb{Z}/2^j, q) \to K(\mathbb{Z}, q + 1) \text{ be the Bockstein map, and let } b_i := \delta_*(a_i). \text{ Then } K(n)^*K(\mathbb{Z}, n + 1) \cong \bigotimes_{i=0}^{\infty} R(b_i) \quad \text{and} \quad K(n)^*(K(\mathbb{Z}/p^j, n)) \cong K(n)^*[x].\]

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Remark 1 (Notation). Note the difference in notation from [RW80]: Our $a_k$ and $b_k$ are originally $a_J$ and $b_J$, with $J = (nk, 1, 2, \ldots, n - 1)$, and our $x$ is $x_S$ with $S = (1, 2, \ldots, n - 1)$. Note also that in [RW80] this theorem is only proven for odd primes, but it was later extended to $p = 2$ in [JW83 Appendix]. See also [HL18, Section 2.2] for more details.

The following example should help in understanding the relation between generators arising from $K(1)$, viewed essentially as $K$-(co)homology, versus those arising from it being viewed as the first Morava K-theory.

Example 3 ($K(1)$ of $\mathbb{C}P^\infty$). We set $p = 2$. The formal group law arising from complex orientation gives that as an algebra $K(1)_* \mathbb{C}P^\infty$ is generated by elements $\beta(i) \in K(1)_{2i+1}(\mathbb{C}P^\infty)$, with relations $\beta(i)^2 = v_1^{2j} \beta(j)$, where $\beta(i) := \beta_{2i}$ (see [RW80]). We keep in mind that in the comparison the lowest degree element is $\beta(0) := \beta_1$.

| $K$-(co)homology | $(K(1))$-(co)homology |
|------------------|------------------------|
| $K^*(\mathbb{C}P^\infty) = K^*(pt)[[x]] = \mathbb{Z}[t^\pm][[x]], |x| = 2$ | $K(1)^*(\mathbb{C}P^\infty) = K(1)_*(pt)[[x]] = \mathbb{Z}/2[v_1^\pm][[x]], |x| = 2$ |
| $K_*(\mathbb{C}P^\infty) = \mathbb{Z}[t^\pm](\beta_0, \beta_1, \ldots), \beta_j$ dual to $x^j$ | $K(1)_*(\mathbb{C}P^\infty) = \bigoplus_{k \geq 0} R(b_k), |b_k| = 2^{k+1}, b_k$ dual to $x^2$ |

The indexing between the two columns in the table are related by $j \mapsto 2^k$. One consequence to keep in mind for bookkeeping the indices of the generators is that $j$ starts from zero while $2^k$ starts from 1. In going from the first column to the second we have mod 2 reduction, from coefficients $\mathbb{Z}[t^\pm]$ to $\mathbb{Z}/2[v_1^\pm]$, where we also map $t \mapsto v_1$ in the process.

When analyzing connective covers of BO in Section 3.1 we will need several results of Kitchloo, Laures and Wilson [KLW04a][KLW04b] about Morava K-theory of spaces related to BO.

Theorem 5 ([KLW04a Theorem 1.3]). Let $bo$, $BO$, $BSO$, $BSpin$ denote the connective $\Omega$-spectra with zeroth spaces $\mathbb{Z} \times BO$, BO, BSO, and BSpin, respectively. Let $E \to B$ be a connective cover with fiber $F$, and $B$ is one of the following: $bo$, for $i \geq 4$, $BO$, $BSO$, $BSpin$, for $i \geq 2$. Then the fibration $F \to E \to B$ induces the following short exact sequence of Hopf algebras

$$K(n)_* \to K(n)_*(F) \to K(n)_*(E) \to K(n)_*(B) \to K(n)_*, \quad (2.1)$$

where $K(n)$ is the Morava K-theory at prime $p = 2$.

See Definition 16 for short exact sequences of Hopf algebras. Several of the spaces used in the the above theorem correspond to spaces in the Whitehead tower of the orthogonal group (1.1): for $i = 0$, we have

$$bo_0 = BO(8) \to BSpin_0 = BSpin \to BSO_0 = BSO \to BO_0 = BO, \quad$$

$$K(\mathbb{Z}, 8) \downarrow K(\mathbb{Z}, 4) \downarrow K(\mathbb{Z}/2, 2) \downarrow K(\mathbb{Z}/2, 1)$$
and for \(i = 8\), we have

\[
\begin{align*}
B\text{Spin}_8 &= B\text{Ninebrane} & B\text{SO}_8 &= B2\text{-Spin} & B\text{O}_8 &= B2\text{-Orient}.
\end{align*}
\]

\(\xymatrix{
K(\mathbb{Z}, 12) \ar[d] & K(\mathbb{Z}/2, 10) \ar[d] & K(\mathbb{Z}/2, 9) \ar[d]
}

See also Remark 4 below.

**Remark 2.** It is worth noting that the maps in the short exact sequence (2.1) are precisely the maps induced by maps \(F \to E \to B\) defining the connective cover. While it is not explicitly mentioned in the statement of this theorem in [KLW04a], examination of the proof and the results upon which it relies ([KLW04a Theorem 4.2][RWY98 Proposition 2.0.1]) makes it clear. This fact is crucial to some of our computations in later sections.

To deal with base spaces outside of the range specified by Theorem 5, we will need another exact sequence.

**Theorem 6 ([KLW04a Theorem 1.5]).** Let \(K(n)\) be the Morava K-theory at \(p = 2\). There is a short exact sequence of Hopf algebras

\[
\xymatrix{
K(n)_* \ar[r] & K(n)_* K(\mathbb{Z}/2, 2) & K(n)_* K(\mathbb{Z}, 3) & K(n)_* B\text{String} \ar[l]_\delta
}
\]

where \(\delta_*\) is the map induced by Bockstein map \(K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}, 2)\), and \((\times 2)_*\) is the map induced by multiplication by 2 on \(K(\mathbb{Z}, 4)\).

A similar result holds for connective covers of the classifying space \(BU\) of the unitary group \(U\).

**Theorem 7 ([RWY98 Section 2.6],[KLW04a Theorem 1.2]).** Let \(BU\) denote the connective \(\Omega\)-spectrum with zeroth space \(BU\), and let \(E \to B\) is a connective cover with fiber \(F\), and \(B\) is \(BU_i\), for \(i \geq 2\). Then the fibration \(F \to B \to E\) induces the following short exact sequence of Hopf algebras:

\[
K(n)_* \to K(n)_*(F) \to K(n)_*(E) \to K(n)_*(B) \to K(n)_*,
\]

where \(K(n)\) is Morava K-theory at a prime \(p\).

**2.3 Twisted Morava K-theory**

We will be particularly interested in twists of Morava K-theory by Eilenberg-MacLane spaces. Given a map \(X \to K(\mathbb{Z}, n + 2)\) the induced bundle from the path-loop fibration over \(K(\mathbb{Z}, n + 2)\)
in the diagram

\[
\begin{array}{c}
K(Z, n + 1) \xrightarrow{P_H} PK(Z, n + 2) \cong \ast \xleftarrow{\cong} K(Z, n + 1)
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{\cong} K(Z, n + 2)
\end{array}
\]

is a principal \(K(Z, n+1)\)-bundle, which is used in [SW15] to twist \(K(n)\)-theory. The twisting of \(K(n)\) are constructed by studying the space of maps from \(K(Z, m)\) to \(BGL_1 K(n)\). Using obstruction theory and showing that the cobar spectral sequence collapses leads to the space of twistings \(tw_{K(n)}(K(Z, n+2))\) being \(\text{Hom}_{K(n)_* - \text{alg}}(K(n)_* K(Z, n+1), K(n)_*)\), hence is homotopically discrete. A careful identification of this space leads to the following.

**Theorem 8** ([SW15, Theorem 3.1, 3.3]). Consider twists of Morava K-theory \(K(n)\) by Eilenberg-MacLane spaces \(K(Z, m)\).

(i) There are no non-trivial twists for \(m > n + 2\).

(ii) For \(m = n + 2\), components of the space \(\text{Map}(K(Z, n + 2), BGL_1 K(n))\) are contractible.

(iii) If \(p \neq 2\) then the set of twists \(tw_{K(n)}(K(Z, n + 2)) \cong tw_{K(n)}(\ast)\).

(iv) If \(p = 2\), then the set of twists is a group isomorphic to dyadic integers, i.e., one has \(tw_{K(n)}(K(Z, n + 2)) \cong \mathbb{Z}_2\).

Because of this theorem, henceforth we restrict our attention to the case \(p = 2\) when dealing with Morava K-theory twisted by integral Eilenberg-MacLane spaces. We will also need the following definition from [SW15], of which we will provide a mild generalization in Section 3.4.

**Definition 9.** (i) A universal twist \(u\) is an element of \(tw_{K(n)}(K(Z, n + 2)) \cong \mathbb{Z}_2\) that is a topological generator.

(ii) Given a class \(H \in H^{n+2}(X; \mathbb{Z}) \cong [X, K(Z, n + 2)]\), the \(H\)-twisted Morava K-theory is defined as

\[
K(n)_*(X; H) := K(n)^u(H)(X),
\]

and similarly for cohomology.

Once and for all we make an arbitrary fixed choice of \(u\). Defined this way, twisted Morava K-theory has all the properties that we would like it to have [SW15, Theorem 4.1], including normalization, namely that if \(H = 0\) then \(K(n)^*(X; H) = K(n)^*(X)\), and the module property, i.e., \(K(n)^*(X; H)\) is a module over \(K(n)_0(X)\). \(\Box\)

Notice that any cohomology class in \(H^{n+2}(X; \mathbb{Z})\) can be interpreted as a \(K(Z, n + 1)\)-bundle. The main computational tool we employ is the relationship between twisted homology of the base and untwisted cohomology of the total space.

\(^1\)Note that here the notation with the superscript 0 means the untwisted Morava K-theory of \(X\), not its degree 0 part.
**Theorem 10** (Universal coefficient theorem [SW15, Theorem 4.3]). Let $H \in H^{n+2}(X)$, and $P_H \to X$ be the principal $K(\mathbb{Z}, n+1)$ bundle over $X$, classified by $H$. Then

$$K(n)^* _{X; H} \cong K(n)^* _{P_H} \otimes K(n)^* _{(K(\mathbb{Z}, n+1))} \rightarrow K(n)^* .$$

Here $K(n)^* _{P_H}$ is a $K(n)^* (K(\mathbb{Z}, n+1))$ module since $P_H$ is a $K(\mathbb{Z}, n+1)$ bundle, and $K(n)^* _{P_H}$ is made into $K(n)^* _{(K(\mathbb{Z}, n+1))}$ by sending $b_0$ to $v_n$ and $b_i$ to 0 for all $i > 0$, where we make use of Theorem 4 for the structure of $K(n)^* _{(K(\mathbb{Z}, n+1))}$.

The module structure here is also crucial, as in the case of chromatic level one, i.e. twisted K-homology, emphasized after Theorem 3. Since this is completely analogous with obvious changes, we will not repeat the discussion.

Since $K(n)$ of Eilenberg-MacLane spaces is known by the work of Ravenel-Wilson (Theorem 11 above), Theorem 10 theoretically reduces the problem of computing twisted homology to computing homology of the total space.

Another very useful computational tool at our disposal is the Atiyah-Hirzebruch spectral sequence (AHSS) for twisted Morava K-theory.

**Theorem 11** ([SW15, Theorem 5.1]). For $H \in H^{n+2}(X)$, there is a spectral sequence converging to twisted Morava K-theory

$$E_2^{p,q} = H^p(X, K(n)^0) \Rightarrow K(n)^* _{X; H}.$$

The first possible nontrivial differential is $d_{2n+3-1}$, given by

$$d_{2n+3-1}(x) = (Q_n(x) + (-1)^{|x|} x \cup (Q_{n-1} \cdots Q_1(H))) .$$

Here $Q_n$ is the cohomology operation known as $n$th Milnor primitive at the prime 2. It may be defined inductively as $Q_0 = Sq^1$, and $Q_{j+1} = Sq^2 Sq^1 - Q_j Sq^2$, where $Sq^j : H^n(X; \mathbb{Z}/2) \to H^{n+j}(X; \mathbb{Z}/2)$ is the $j$-th Steenrod square operation in mod 2 cohomology.

### 2.4 Twisted $K(1)$ and twisted mod 2 K-theory

The ‘default’ coefficients for a (co)homology theory is the integers. We are also interested in other coefficients, mainly the integers mod $p$. Morava K-theory $K(n)$ is defined one prime at a time, while K-theory $K$ does not depend on a prime. In order to be able to compare the latter to the former at chromatic level $n = 1$, we need to restrict K-theory to seeing one prime at a time.

Given a spectrum $E$ and any abelian group $G$, one can define a *spectrum with coefficients in $G$* (and, correspondingly, $E$-cohomology with coefficients in $G$) as the smash product $EG := E \wedge SG$, where $SG$ is a Moore spectrum of type $G$. This spectrum $EG$ satisfies the following short exact
or more generally, for any space $X$, the following holds at the level of $E$-homology

$$0 \longrightarrow \pi_n(E) \otimes G \longrightarrow \pi_n(EG) \longrightarrow \text{Tor}_1(\pi_{n-1}(E), G) \longrightarrow 0,$$

or more generally, for any space $X$, the following holds at the level of $E$-homology

$$0 \longrightarrow E_n(X) \otimes G \longrightarrow (EG)_n(X) \longrightarrow \text{Tor}_1(E_{n-1}, G) \longrightarrow 0,$$

where $(EG)_n$ denotes the homology theory corresponding to the spectrum $EG$.

Taking $G = \mathbb{Z}/p$, we get “mod $p$” $E$-homology. For $E = K$, the complex $K$-theory spectrum, we can define $K$-theory with mod $p$ coefficients as a spectrum $K\mathbb{Z}/p = K \wedge S\mathbb{Z}/p$, where $S\mathbb{Z}/p$ is the Moore spectrum of type $\mathbb{Z}/p$. Moreover, there is the short exact sequence (2.3), in this case relating $K\mathbb{Z}/p$ homology and $K$-homology of a space $X$. On the other hand, by a classical result of Adams [Ad74], mod $p$ $K$-theory decomposes into a sum of $p - 1$ successive suspensions of $K(1)$. In our case of interest, $p = 2$, mod $2$ $K$-theory coincides with $K(1)$ since there is only one summand, and so $K(1)$ fits into the exact sequence

$$0 \longrightarrow K_n(X) \otimes \mathbb{Z}/2 \longrightarrow K(1)_n(X) \longrightarrow \text{Tor}_1(K_{n-1}(X), \mathbb{Z}/2) \longrightarrow 0.$$

We would like to establish a twisted version of this relationship.

**Theorem 12** (Exact sequence for twisted mod $p$ $K$-theory). Let $X$ be a topological space and $\tau_3 \in H^3(X; \mathbb{Z})$. Then we have the following exact sequence

$$0 \longrightarrow K^\tau_3_n(X) \otimes \mathbb{Z}/2 \longrightarrow K(1)_n(X; \tau_3) \longrightarrow \text{Tor}_1(K^\tau_{n-1}(X), \mathbb{Z}/2) \longrightarrow 0,$$

where $K^\tau_*(X)$ is $K$-theory twisted by $\tau$, as defined in Section 2.1.

Proof. Let $P$ be the total space of the principal $K(\mathbb{Z}, 2)$ bundle classified by a degree three class $\tau_3 \in H^3(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, 3)]$. The Thom spectrum corresponding to the twisted theory $K(1)(X; H)$ is

$$X^{u(H)} \cong \Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z}, 2)} K(1) \cong \Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z}, 2)} (K \wedge S\mathbb{Z}/2),$$

where $S\mathbb{Z}/2$ is the Moore spectrum of type $\mathbb{Z}/2$, as above. Now $K(\mathbb{Z}, 2)$ acts trivially on $S\mathbb{Z}/2$ and the associativity of a “mixed” smash product is established via [EKMM07, Proposition 3.4], so that $X^{u(\tau_3)} \cong (\Sigma^\infty P_+ \wedge_{K(\mathbb{Z}, 2)} K) \wedge S\mathbb{Z}/2$. On the other hand, from [ABG10, Section 7.2], the Thom spectrum for $K$-theory twisted by $\tau_3$ is exactly $\Sigma^\infty P_+ \wedge_{K(\mathbb{Z}, 2)} K \cong X^{T(\tau_3)}$, where $T : K(\mathbb{Z}, 3) \to K$ is the map defined in Section 2.1. Hence $X^{u(\tau_3)} \cong X^{T(\tau_3)} \wedge S\mathbb{Z}/2$. Then, using the exact sequence (2.3), we obtain

$$0 \longrightarrow \pi_n(X^{T(\tau_3)}) \otimes \mathbb{Z}/2 \longrightarrow \pi_n(X^{u(\tau_3)}) \longrightarrow \text{Tor}_1(\pi_{n-1}(X^{T(\tau_3)}), \mathbb{Z}/2) \longrightarrow 0.$$
But we defined twisted homology precisely as homotopy groups of the Thom spectrum. Therefore, we can rewrite this exact sequence as (2.4).

We will make use of this theorem in Section 3.3 when calculating the twisted Morava K-theory of spheres leading to Proposition 23. We will also consider a variation in Section 3.4, where the twists themselves will take values in cohomology with \( \mathbb{Z}/2 \) coefficients.

3 Computations of twisted Morava K-homology of spaces

3.1 Twisted Morava homology of connective covers of BO and BU

In this section we will apply our computations to the main examples, which are connective covers of the stable orthogonal and unitary groups (see diagram 1.1). Recall that we are working with Morava K-theory at \( p = 2 \) for the rest of the paper (except for the last section).

From [RW80] (Theorem 4 above) we have that \( K(n)_*(K(\mathbb{Z}, j)) \) is trivial for \( j > n + 1 \). With \( n = 1 \), it follows that \( K(1)_*(K(\mathbb{Z}, j > 2)) \) is trivial. Therefore, we immediately get that all connective covers of Spin, that is, String, Fivebrane etc. have the same twisted \( K(1)_* \)-homology, while all classifying spaces covering BSpin, that is, BString, BFivebrane, etc., have the same twisted \( K(1)_* \)-homology. More precisely, we have the following.

Lemma 13 (Twisted \( K(1)_* \)-homology of connective covers). For \( m > 3 \), let \( \text{Spin}(m) \) be a connective cover of Spin and \( B\text{Spin}(m) \) be the connective covers of the corresponding classifying space BSpin. Then, for any \( m' > m \), we have that

(i) The map \( \text{Spin}(m') \to \text{Spin}(m) \) is an isomorphism in twisted \( K(1)_* \).

(ii) The map \( B\text{Spin}(m') \to B\text{Spin}(m) \) is an isomorphism in twisted \( K(1)_* \).

In fact, in both cases the homology is necessarily untwisted.

Proof. From [HRW98, Lemma 3.3] one has that fibrations \( F \to E \to B \) with \( F \) \( K(n)_* \)-acyclic induce an isomorphism \( K(n)_*(E) \cong K(n)_*(B) \). By inspection, from the Whitehead tower 1.1, the fibers of Fivebrane and beyond begin with \( K(\mathbb{Z}, 6) \) through \( K(\mathbb{Z}/2, 7) \), \( K(\mathbb{Z}/2, 8) \), and \( K(\mathbb{Z}, 10) \) etc., all of which are \( K(1)_* \)-acyclic. Similarly for the fibers of the corresponding classifying spaces. This proves the lemma in the untwisted case. The twisted case follows trivially by noticing that, for \( m > 3 \), \( H^3(\text{Spin}(m); \mathbb{Z}) \) and \( H^3(B\text{Spin}(m); \mathbb{Z}) \) are both 0, yielding trivial twists.

The omitted case in Lemma 13(i), namely \( m = 3 \), corresponding to String, admits nontrivial twists. It is delicate and in the unstable case would use a careful analysis of twisted K-homology of the Lie group Spin(\( q \)) according to rank, as in [Do06]. With the mod 2 version of Khorami’s Theorem 3 (as in Section 3.4), we have

\[
K(1)_*(\text{Spin}(q)) \cong K(1)_*(\text{String}(q)) \otimes_{K(1)_*K(\mathbb{Z}/2)} K(1)_*.
\]
The homology of the classifying space $BO$ and let $\tau$ are given, respectively, as

(i) $K(n \geq 3)(BString)$, $K(n \geq 7)(Fivebrane)$, and $K(n \geq 8)(BFivebrane)$.

Note that this is somewhat related to [SW17]. There it was shown that, rationally, all higher
connective covers can be described using Spin rather than having to define a structure at level $r$
from that at level $r-1$. What we have is an analogue of rationalization, in the sense that Morava
$K(n)$-homology only sees Eilenberg-MacLane spaces of certain degrees, and the ones not seen can
be viewed as rationalized as far as $K(n)$-homology goes. Indeed, for a fibration $F \to E \to B$ with
fiber $F$ which is $K(n)_*$-acyclic, the map $j$ is a $K(n)_*$-equivalence (see [HRW98, Lemma 3.3]). From
[RW80], $K(Z,i)$ is $K(n)_*$-acyclic if and only of $i > n + 1$. Consequently, we immediately have the
following.

Lemma 14 ($K(n)$-equivalences for connective covers). We have the following
(i) $K(Z,3)$ is $K(1)_*$-acyclic so that $BSpin$ and $BString$ are $K(1)_*$-equivalent.
(ii) $K(Z,6)$ is $K(n)_*$-acyclic for $n < 5$ so that $String$ and $Fivebrane$ are $K(n < 5)_*$-equivalent.
(iii) $K(Z,7)$ is $K(n)_*$-acyclic for $n < 6$ so that $BString$ and $BFivebrane$ are $K(n < 6)_*$-equivalent.

Similarly one can go up the Whitehead tower [11] and deduce analogous statements for 2-Orient, 2Spin and Ninebrane, etc. This is compatible with Bousfield’s result that for any space $X$,
each $K(n)_*$-equivalence of spaces if a $K(m)_*$-equivalence for $1 \leq m \leq n$ [Bo99].

Our main result in this section is the following statement in the scope of Theorem 2

Theorem 15 (Twisted Morava K-homology of $BO\langle k \rangle$). Let $K(Z,n+1) \to BO\langle n+3 \rangle \to BO\langle n+2 \rangle$
be a fibration defining a connective cover of $BO$ (so $n = 2 \mod 8$ or $n = 6 \mod 8$) and $n \geq 6$,
and let $\tau_{n+2}$ denote the corresponding class in $H^{n+2}(BO\langle n+2 \rangle; \mathbb{Z})$. Then the twisted Morava K-homology of the classifying space $BO\langle n \rangle$ and the corresponding loop group $O\langle n-1 \rangle := \Omega BO\langle n \rangle$
are given, respectively, as

$$K(n)_*(BO\langle n+2 \rangle; \tau_{n+2}) \cong K(n)_*(BO\langle n+2 \rangle),$$
$$K(n-1)_*(O\langle n+1 \rangle; \tau_{n+1}) \cong K(n-1)_*(O\langle n+1 \rangle),$$

where $\tau_{n+2}$ is the twisting class and $\tau_{n+1}$ is its looping.

Our main tool in this section, which goes towards proving the above theorem, is the exact
sequence of Kitchloo-Laures-Wilson, Theorem 5 above, where $E \to B$ is a connective cover with
fiber $F$, and $B$ is one of the following: $BO_i$ for $i \geq 4$, $BO_1$, $BSO_i$, $BSpin_i$ for $i \geq 2$. It is worth noting
that the maps in this short exact sequence are precisely the maps induced by maps $F \to E \to B$
defining the connective cover. This fact is crucial to our computations.
Remark 4. Notice that if $\Omega BO$, $\Omega BO$, $\Omega BSO$, $\Omega BSpin$ denote the connective $\Omega$-spectra with zeroth spaces $\mathbb{Z} \times BO$, BO, BSO, and BSpin, then we have the following equivalences

$$BO(8k) \simeq \Omega BO_{8k}, \quad BO(8k + 1) \simeq \Omega BO_{8k}, \quad BO(8k + 2) \simeq \Omega BSO_{8k}, \quad BO(8k + 4) \simeq \Omega BSpin_{8k}.$$ 

In particular, $\Omega BString = BO(8) = \Omega BO_8$, $\Omega BFivebrane = BO(9) = \Omega BO_8$ and, since the spectra in question are $\Omega$-spectra, $String = \Omega BString \simeq \Omega BO_8 \simeq BO_7$ and $Fivebrane \simeq BO_7$.

We will also need to use some basic facts about Hopf algebras. Standard references include [Un11] [MM65] [Sw69]. From an algebraic point of view, exact sequences and extensions of Hopf algebras are described, for instance, in [AD95].

Definition 16. Let $A, B, C$ be commutative Hopf algebras over a field $k$. Suppose $i : A \to B$ is an injection of Hopf algebras, and $j : B \to C$ is a surjection of Hopf algebras. Then if $j$ induces the isomorphism $C \cong B/i(A^+)B$ of Hopf algebras, where $A^+$ denotes the augmentation ideal of $A$, we say that

$$k \to A \xrightarrow{i} B \xrightarrow{j} C \to k$$

is a short exact sequence of Hopf algebras.

Remark 5 (Hopf algebras and module structure). If we have an injective Hopf algebra morphism $i : A \to B$, we can view $B$ as an $A$-module, and then $B/i(A^+)B \cong B \otimes_A k$ (sometimes also denoted $B\!//A$). Therefore, in any short exact sequence of commutative $k$-Hopf algebras $k \to A \to B \to C \to k$, we always have $C \cong B \otimes_A k$. In particular, in the exact sequence (2.1), we have $K(n)_*(B) \cong K(n)_*(E) \otimes_{K(n)_*(F)} K(n)_*$, where the $K(n)_*(F)$-module structure on $K(n)_*$ is in this case via the mpa $F \to pt$ (cf. the discussion after Theorem 3).

Recall also the following classical result.

Theorem 17 ([MM65, Th. 4.4]). If $A$ is a connected Hopf algebra over a commutative ring with unity $K$, $B$ is a connected $A$-module coalgebra, $i : A \to B$, $\pi : B \to K \otimes_A B$ are the canonical morphisms, and the sequences $0 \to A \xrightarrow{i} B \xrightarrow{\pi} K \otimes_A B \to 0$ are split exact as sequences of graded $K$-modules, then there exists $h : B \to A \otimes_K (K \otimes_A B)$, which is an isomorphism of $A$-modules.

If we take the ring $K$ to be $K(n)_*$, then any module over $K(n)_*$ is free, and so any exact sequence is split automatically as $K(n)_*$-modules.

Proof of Theorem 17. Now consider the exact sequence (2.1) again. Applying Theorem 17 with $A = K(n)_*(F)$ and $B = K(n)_*(E)$, we obtain an isomorphism

$$K(n)_*(E) \cong K(n)_*(F) \otimes_{K(n)_*} K(n)_*(K(n)_*(E)).$$

Note the multiple uses of the notation $B$: as the base of the fibration as well as the middle Hopf algebra. We hope the distinction will be clear from the context.
But the latter term is precisely $K(n)_*(B)$, as mentioned above, so that we have the isomorphism $K(n)_*(E) \cong K(n)_*(F) \otimes_{K(n)_*} K(n)_*(B)$. Now, if $M$ is any $K(n)_*(F)$-module, we have

$$M \otimes_{K(n)_*(F)} K(n)_*(E) \cong M \otimes_{K(n)_*(F)} K(n)_*(F) \otimes_{K(n)_*} K(n)_*(B) \cong M \otimes_{K(n)_*} K(n)_*(B),$$

as $K(n)_*$-modules. We emphasize that here the actual $K(n)_*(F)$-module structure of $M$ is irrelevant.

Now take $F \to E \to B$ to be the connective cover $K(\mathbb{Z}, n + 1) \to \text{BO}(n + 3) \to \text{BO}(n + 2)$ for $n = 2 \text{ mod } 8$ or $n = 6 \text{ mod } 8$. If $n = 2 \text{ mod } 8$, then $n + 2 = 4 \text{ mod } 8$, and $\text{BO}(n + 2) \cong B\text{Spin}_{n-2}$, by Remark 4. If $n = 6 \text{ mod } 8$, then $n + 2 = 0 \text{ mod } 8$ and, by the same Remark, $\text{BO}(n + 2) \cong B\text{SO}_{n+2}$. So, as long as $n \geq 6$, $\text{BO}(n + 2)$ is one of the spaces that can serve as the base for the fibration in the sequence (3.1). So, in particular, for any $K(n)_*(K(\mathbb{Z}, n + 1))$-module $M$,

$$M \otimes_{K(n)_* K(\mathbb{Z}, n + 1)} K(n)_*(\text{BO}(n + 3)) \cong M \otimes_{K(n)_*} K(n)_*(\text{BO}(n + 2)).$$

(3.1)

The UCT (Theorem 10) for the bundle $K(\mathbb{Z}, n + 1) \to \text{BO}(n + 3) \to \text{BO}(n + 2)$ states that

$$K(n)_*(\text{BO}(n + 2); \tau_{n+2}) \cong K(n)_*(\text{BO}(n + 3)) \otimes_{K(n)_* K(\mathbb{Z}, n + 1)} K(n)_*,$$

with a special $K(n)_* K(\mathbb{Z}, n + 1)$-module structure on the latter factor from Theorem 10. But taking $M = K(n)_*$ in (3.1), we see that

$$K(n)_*(\text{BO}(n + 2); \tau_{n+2}) \cong K(n)_* \otimes_{K(n)_*} K(n)_*(\text{BO}(n + 2)) \cong K(n)_*(\text{BO}(n + 2)).$$

We highlight that Theorem 15 indicates that for the natural twists associated with connective covers of the orthogonal group and their classifying spaces, the twisted Morava K-homology coincides with the corresponding untwisted Morava K-homology. So, we see a drastic simplification for the following family of important spaces which arise often in the literature (see [SS90] [SSS09] [SSS12]).

**Corollary 18** (Twisted Morava K-homology of String and BString). For the String group and its classifying space, we have

$$K(5)_*(\text{String}; \tau_7) \cong K(5)_*(\text{String}),$$

$$K(6)_*(\text{BString}; \frac{1}{6}\tau_2) \cong K(6)_*(\text{BString}),$$

where $\frac{1}{6}\tau_2$ is the second fractional Pontrjagin class, classifying the fibration $K(\mathbb{Z}, 7) \to B\text{Fivebrane} \to \text{BString}$ and $\tau_7$ is its looping.

Notice that Theorem 7 provides us with a similar short exact sequence for connective covers of BU and so, similarly, we can make the same conclusion in that case.
Theorem 19 (Twisted Morava K-homology of $BU\langle k \rangle$). Let $n$ be an odd natural number, and let $K(\mathbb{Z}, n+1) \to BU\langle n+2 \rangle \to BU\langle n+1 \rangle$ be a fibration defining a connective cover of $BU$ and $\tau_{n+2}$ the corresponding class in $H^{n+2}(BU\langle n+1 \rangle; \mathbb{Z})$. Then

$$K(n)_*(BU\langle n+1 \rangle; \tau_{n+2}) \cong K(n)_*(BU\langle n+1 \rangle),$$

$$K(n-1)_*(U\langle n+1 \rangle; \tau_{n+1}) \cong K(n-1)_*(U\langle n+1 \rangle),$$

where again the class $\tau_{n+1}$ is the looping of $\tau_{n+2}$.

A more general class of spaces which will satisfy the same property is provided by [RWY98, Proposition 2.0.1].

Note that the restrictions on the base spaces in Theorem 5 prevent us from using the same argument for the fibration $K(\mathbb{Z}, 3) \to BString \to BSpin \to K(\mathbb{Z}, 4)$, as $BSpin = BSpin_0$, and Theorem 5 requires the base to be $BSpin$ with $i \geq 2$. However, this case can be handled using a different technique, leading to triviality of the twisted Morava K-homology of the classifying space of the Spin group.

Proposition 20 (Twisted Morava K-homology of $BSpin$). We have

$$K(2)_*(BSpin; \frac{1}{2}p_1) = 0,$$

where $\frac{1}{2}p_1 \in H^4(BSpin; \mathbb{Z})$ is the first fractional Pontrjagin class, classifying the fibration $K(\mathbb{Z}, 3) \to BString \to BSpin$.

Proof. By the UCT (Theorem 10), we need to compute $K(2)_*(BString) \otimes_{K(2)_*K(\mathbb{Z}, 3)} K(2)_*$. Recall also from there that the module structure on the second factor $K(2)_*$ is given by mapping $b_0 \in K(2)_*K(\mathbb{Z}, 3)$ to $v_2$ and $b_i \in K(2)_*K(\mathbb{Z}, 3)$ to $0$ for $i \geq 1$. Now consider the exact sequence in Theorem 6. From the Ravenel-Wilson computations (Theorem 4 above), the elements satisfy

$$b_i = \delta_\ast a_i,$$

where $K(n)_*K(\mathbb{Z}, 2^j/n) \cong \bigotimes_{i=0}^{j-1} R(a_i)$ and $R(a_k) = \mathbb{Z}/p[a_k, v_1^{\pm 1}]/(a_k^p - (-1)^{n-1}v_n^k a_k)$ for $k \geq 0$. So for $j = 1$, we have $K(2)_*K(\mathbb{Z}, 2) \cong R(a_0) \cong \mathbb{Z}/2[a_0, v_2\pm]/(a_0^2 + v_2 a_0)$, while $a_i = 0$ for $i \geq 1$. Since $b_i = \delta_\ast a_i$, we have $b_0 = \delta_\ast a_0$ while $b_i = 0$ for $i \geq 1$. Therefore, in $K(2)_*(BString) \otimes_{K(2)_*K(\mathbb{Z}, 3)} K(2)_*$, the relevant elements multiply as

$$1 \otimes v_2 = 1 \otimes b_0 = b_0 \otimes 1 = 0 \otimes 1 = 0.$$

Since the element $1 \otimes v_2$ is invertible, the whole ring must be zero. 

The sequence of infinite loop spaces $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 3) \xrightarrow{\alpha} BString \to BSpin$ induces, by [KLW04a], the exact sequence of Hopf algebras (2.2). It is emphasized in [KLW04a] that while, algebraically, there is a short exact sequence of Hopf algebras $K(2)_*K(\mathbb{Z}, 3) \xrightarrow{\gamma} K(2)_*BString \to K(2)_*BSpin$ which splits, the map $\gamma$ is not the map induced by $\alpha$ above. However, here we do not make explicit use of such splittings.
A similar argument as in the proof of Proposition 20 establishes Theorem 1 in the Introduction. In this case, the UCT (Theorem 10) gives \( K(n)_* (B, \xi) \cong K(n)_* (E) \otimes_K K(Z, n+1) K(n)_* \), with \( K(n)_* K(Z/2, n) \cong R(a_0) \cong \mathbb{Z}[a_0, v_0^2]/(a_0^2 + (-1)^n v_0 a_0) \) from Theorem 4. Again we have \( b_0 = \delta_* a_0 \) while \( b_i = 0 \) for \( i \geq 1 \), and the multiplicative structure is similar.

### 3.2 Twisted \( K(n) \)-homology of Eilenberg-MacLane spaces

We now look at bundles of Eilenberg-MacLane spaces with the base space also given by Eilenberg-MacLane spaces, thereby generalizing calculations of Khorami [Kh11]. We will generalize Example 1 from \( n = 1 \) to any natural number \( n \). Note that \( n \) essentially plays the role of the homotopic degree of \( K(Z, n+1) \) as well as the chromatic level of the Morava K-theory \( K(n) \) being used. The proof will follow similar strategies to the ones taken in [Kh11] for the case of twisted \( K \)-homology.

**Theorem 21** (Twisted Morava K-theory of \( K(Z, n) \)). Let \( k : K(Z, n+2) \to K(Z, n+2) \) be the map induced by multiplication by \( k \) on \( Z \), for \( k \geq 1 \). Then

\[
K(n)_* (K(Z, n+2); k) = 0 .
\]

**Proof**. Consider the identity map, \( id : K(Z, n+2) \to K(Z, n+2) \), and let \( P_{id} \) be the total space of the corresponding \( K(Z, n+1) \) bundle. Notice that \( P_{id} \) is contractible by construction (it is the total space of the universal principal \( K(Z, n+1) \) bundle), so that \( K(n)_* (P_{id}) \cong K(n)_* \).

Now consider the “multiplication by \( k \)” map \( Z \xrightarrow{k} Z \). Let \( k : K(Z, n+2) \to K(Z, n+2) \) be the induced map on the Eilenberg-MacLane spaces and \( P_k \) be the total space of the corresponding \( K(Z, n+1) \) bundle. Then the long exact sequence on homotopy groups for the principal fibration \( K(Z, n+1) \to P_k \to K(Z, n+2) \) reduces to

\[
0 \to \pi_{n+2}(P_k) \to \pi_{n+2}(K(Z, n+2)) \cong Z \to \pi_{n+1}(K(Z, n+1)) \cong Z \to \pi_{n+1}(P_k) \to 0 ,
\]

from which we observe that \( P_k \) has at most two non-trivial homotopy groups. To see how the multiplication by \( k \) fits into this picture, consider the map between \( P_k \) and the universal \( K(Z, n+1) \) bundle

\[
\begin{array}{ccc}
K(Z, n+1) & \xrightarrow{id} & K(Z, n+1) \\
\downarrow & & \downarrow \\
P_k & \xrightarrow{} & * \\
\downarrow & & \downarrow \\
K(Z, n+2) & \xrightarrow{k} & K(Z, n+2)
\end{array}
\]

Here the map on base spaces is “multiplication by \( k \)” by definition, and the map on the fibers is the identity map. This induces a map of exact sequences

\[
\begin{array}{cccccccc}
0 & \to & \pi_{n+2}(P_k) & \to & Z & \to & Z & \to & \pi_{n+1}(P_k) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & Z & \to & Z & \to & \pi_{n+1}(P_k) & \to & 0
\end{array}
\]

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from which we see that the top map $Z \to Z$ has to be multiplication by $k$. Consequently, $\pi_{n+1}(P_k) = 0$ and $\pi_{n+1}(P_k) = Z/k$, hence we have $P_k \simeq K(Z/k, n+1)$.

Since $K(n)$ at prime 2 is a 2-local theory, $K(n)_* (K(Z/k, n+1))$ is trivial for $k$ odd. Together with Künneth isomorphism this implies that it is sufficient to look at $K(n)_*(K(Z/2^j, n+1))$. From Theorem 4, we see that $K(n)_*(K(Z/2^j, n+1)) \cong K(n)_*$ and, therefore, for all $n \geq 1$ we have

$$K(n)_*(K(Z, n+2); k) \cong K(n)_* \otimes_{K(n)_*(K(Z, n+1))} K(n)_*.$$  

Now recall that, in the module structure on the second factor, $b_0$ from $K(n)_* K(Z, n+1)$ is mapped to 1. On the other hand, $b_0$ maps to 0 in $K(n)_*(K(Z/2^j, n+1)) \cong K(n)_*$, so that we finally arrive at $K(n)_*(K(Z, n+2); k) = 0$, for any $k > 0$.  

\[\square\]

### 3.3 Twisted $K(n)$-(co)homology of spheres

We now generalize Example 2 from $n = 1$ to any natural number $n$, which plays the role of the dimension of the sphere (minus 2) as well as the chromatic level. This time the results will differ and the proof will depart drastically from those of [Kh11]; instead will use the twisted Atiyah-Hirzebruch spectral sequence (AHSS) of [SW15], i.e., Theorem 11. We find that the twisted Morava K-theory is essentially trivial, with the meaning of triviality depending on chromatic level. For levels greater than 1, it is given by the underlying integral cohomology tensored with the coefficients of the theory, while for level 1, it is trivial in the sense of being actually zero.

**Theorem 22** (Twisted Morava K-theory of spheres). Fix an integer $n > 1$ and let $\sigma_{n+2}$ be the generator of the group $H^{n+2}(S^{n+2}; Z) = [S^{n+2}, K(Z, n+1)] \cong Z$. Then the twisted $n$th Morava K-cohomology of the $(n+2)$-sphere with twist $\tau_{n+2}$ given by any multiple of $\sigma_{n+2}$ is given by cohomology tensored with the coefficient ring:

$$K(n)^* (S^{n+2}; \tau_{n+2}) = H^*(S^{n+2}; Z) \otimes K(n)^* .$$

Proof. Let $P$ again denote the total space of the corresponding $K(Z, n+1)$ bundle. Recall from Theorem 11 that the first non-trivial differential in the twisted AHSS for Morava K-theory is

$$d_{2^n-1}(x) = Q_n(x) + Q_{n-1} \cdots Q_1(H) \cup x .$$

Because the differential is a module homomorphism, it suffices to consider cases $x = 1$ and $x$ dual to the fundamental class of $S^{n+2}$. In both cases $Q_n(x) = 0$ by dimension, since $H^*(S^{n+2}, K(n)^*)$ concentrated in degree $n + 2$, so the target of $Q_n$ is trivial.

When $n > 1$, we do have the part $Q_{n-1} \cdots Q_1(H)$ in the second term. Then, by the same dimension argument, $Q_{n-1} \cdots Q_1(H) = 0$ since the target of $Q_{n-1} \cdots Q_1$ is of higher degree cohomology. Therefore, the first possibly non-trivial differential is actually trivial. Also note that all the subsequent differentials must vanish too (they are even longer so will land in even higher cohomology groups). Since this holds for every $x$, the spectral sequence collapses.  

\[\square\]
The case $n = 1$ can be handled separately, by either using the non-twisted AHSS to compute $K(n)_* P$, or using the computations of twisted $K$-theory [Kh11] together with Theorem [12]: by the computations in Section [2.1] twisted $K$-homology for $S^3$ twisted by the generator $\sigma_3 \in H^3(S^3; \mathbb{Z})$ is $K_*^{\sigma_3}(S^3) = 0$. But, from Theorem [12] twisted $K(1)$ fits into the short exact sequence

$$0 \longrightarrow K_n^{\sigma_3}(S^3) \otimes \mathbb{Z}/2 \longrightarrow K(1)_n(S^3; \sigma_3) \longrightarrow \mathrm{Tor}_1(K_{n-1}^*(X), \mathbb{Z}/2).$$

Since $K_*^{\sigma_3}(S^3) = 0$, both the first and the third term of this exact sequence are zero, so that the middle term is zero as well. On the other hand, starting with a twist which is a multiple of the generator we can use $K_*(S^3; n\sigma_3) = \mathbb{Z}/n\mathbb{Z}$. Therefore, using the relation between Morava $K(1)$-homology at the prime 2 and $K$-homology, we arrive at the following.

**Proposition 23** (Twisted first Morava $K$-homology of the 3-sphere). The Morava $K(1)$-homology at $p = 2$ (i.e. mod 2 $K$-homology) of the 3-sphere with a twist a multiple of the generator $\sigma_3$ is

$$K(1)_*(S^3; n\sigma_3) = \mathbb{Z}/(2, n)[v_1^{\pm 1}],$$

while it vanishes for $n = 1$, $K(1)_*(S^3; \sigma_3) = 0$.

This is the mod 2 version of Khorami’s theorem [Kh11] – see Example [2]. Notice that this is consistent with the pattern we observed before: for ‘nice enough’ spaces, twisted Morava $K$-theory groups are either zero or equal to untwisted groups.

### 3.4 Twists by mod 2 Eilenberg-MacLane spaces and torsion connective covers

We would like to complete our investigations of connective covers of $BO$. So far we have been focusing solely on those covers which can be viewed as bundles of integral Eilenberg-MacLane spaces, i.e., those levels of the Whitehead tower of $BO$ in diagram (1.1) which have maps to $K(\mathbb{Z}, m)$. We would like to perform a similar analysis for the remaining “non-integral” covers, for instance, the analogue of orientation $K(\mathbb{Z}/2, 8) \rightarrow BO\langle 10 \rangle = B2\text{-Orient} \rightarrow BO\langle 9 \rangle = B\text{Fivebrane}$ and the analogue of Spin structure $K(\mathbb{Z}/2, 9) \rightarrow BO\langle 11 \rangle = B2\text{-Spin} \rightarrow BO\langle 10 \rangle = B2\text{-Orient}$, which are $K(\mathbb{Z}/2, m)$-bundles. However, so far we have been lacking the definition of Morava $K$-theory twisted by non-integral Eilenberg-MacLane spaces. The purpose of this section is to fill that gap.

Instead of focusing solely on $p = 2$, we will discuss twists of $K(n)$ by $K(\mathbb{Z}/p^j, m)$ for all primes $p$ and $j \geq 1$. From the description of Morava $K$-theory in Theorem [3] we see $K(n)_* K(\mathbb{Z}/p^j, n)$ is one of the factors of $K(n)_* K(\mathbb{Z}, n + 1)$, and $K(n)_* K(\mathbb{Z}/p^j, m) = K(n)_* K(\mathbb{Z}, m) = K(n)_*$ for $m > n$. Therefore, we should expect a similar theory as for twists by integral Eilenberg-MacLane spaces. In fact, the proofs in [SW15] transport to the mod $p$ case with little to no modification, and so we only outline them.

Recall from [ABG10] [ABGHR14] that a twist of theory $R$ by a space $Y$ is an element of $[Y, BGL_1 R]$. The following fact provides us with an obstruction-theoretic way to classify these maps.

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Proposition 24 ([SW15] Proposition 1.6). Let \( R \) be an \( A_{\infty} \) ring spectrum, \( Z = \Omega X \) and \( R_*(Z) \) is flat over \( R_* \). If the obstruction groups \( \operatorname{Ext}^k_{R_*(Z)}(R_*, \Omega^s R_*) \) vanish for \( s = k - 1, k - 2 \) and any \( k \geq 1 \), then there is a bijection \( \text{tw}_R(X) \leftrightarrow \operatorname{Hom}_{R_*-\text{alg}}(R_*(Z), R_*) \). Moreover, the obstruction groups lie in the \( E_2 \)-term of the cobar spectral sequence

\[
\operatorname{Ext}^k_{R_*(Z)}(R_*, \Omega^s R_*) \implies E^{k-s}(BZ).
\] (3.2)

Notice that when \( R = K(n) \), the flatness requirement is automatically satisfied for any \( Z \), since any \( K(n)_*(Z) \) is free over \( K(n)_* \). Now we can establish the following mod 2 analogue of Theorem 8.

Theorem 25 (Morava K-theory twisted by mod \( p \) Eilenberg-MacLane spaces). We have:

(i) There are no non-trivial twists of \( K(n) \) by \( K(\mathbb{Z}/p^j, m) \) for \( m > n + 1 \);

(ii) There are no non-trivial twists of \( K(n) \) by \( K(\mathbb{Z}/p^j, n + 1) \) at \( p \neq 2 \);

(iii) \( \text{tw}_{K(n)}(K(\mathbb{Z}/2^j, n + 1)) \cong \mathbb{Z}/2^j \).

Remark 6. Notice the shift in degree compared to integral Eilenberg-MacLane spaces. It is the same shift in degree that occurs in Theorem 8.

Proof. (Outline) We will use Proposition 24 with \( X = K(\mathbb{Z}/p^j, m), Z = K(\mathbb{Z}/p^j, m - 1) \), and \( R = K(n) \). From Theorem 8 if \( m > n + 1 \) then \( K(n)_*(K(\mathbb{Z}/p^j, m - 1)) = K(n)_* \). Consequently, the obstruction group is \( \operatorname{Ext}^k_{R_*}(R_*, \Omega^s R_*) = 0 \), so that the twists are given as

\[
\text{tw}_{K(n)}(K(\mathbb{Z}/2, m)) = \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*, K(n)_*) = \{*\}.
\]

Just as in the integral case, the spectral sequence (3.2) collapses by the work of Ravenel and Wilson [RWS0], and the obstruction groups vanish, leading to

\[
\text{tw}_{K(n)}(K(\mathbb{Z}/p^j, n + 1)) = \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*K(\mathbb{Z}/p^j, n + 1), K(n)_*).
\]

Since twistings of \( K(\mathbb{Z}, n + 2) \) restrict (along the Bockstein) to twistings of \( K(\mathbb{Z}/p^j, n + 1) \), this is a quotient of \( \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*K(\mathbb{Z}, n + 2), K(n)_*) = \text{tw}_{K(n)}(K(\mathbb{Z}, n + 2)) \), the twisting space for \( K(\mathbb{Z}, 2) \) from Theorem 8. But we know that, for \( p > 2 \), the latter is trivial.

Now fix \( p = 2 \), and recall from Theorem 8 that \( K(n)_*K(\mathbb{Z}/p^j, n) \cong \bigotimes_{i=0}^{j-1} R(a_i) \). As in the proof of [SW15] Theorem 3.3, \( \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*K(\mathbb{Z}/2^j, n + 1), K(n)_*) \) is determined by the images of the elements \( a_i \), for \( 0 \leq i \leq j - 1 \). By degree reasons, there is only one possible target for each \( a_i \) in the coefficient ring \( K(n)_* \). So an element of \( \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*K(\mathbb{Z}/2^j, n + 1), K(n)_*) \) is determined by the \( j \) elements among the \( a_i \) which are mapped to zero, and there are \( 2^j \) elements. By identifying \( \operatorname{Hom}_{K(n)_*^{-\text{alg}}}(K(n)_*K(\mathbb{Z}/2^j, n + 1), K(n)_*) \) with a subset of \( K(n)^*[x]/x^{2^j} \) (closed under multiplication, being the group-like elements of the Hopf algebra) it is possible to obtain a group structure on it. \( \square \)
This allows us to seek direct analogues of the constructions in [SW15], as recalled in Section 2.3. In particular, since \( \text{tw}_{\mathbb{K}(n)}(K(\mathbb{Z}/2, n+1)) \cong \mathbb{Z}/2 \), we can present an analogue of Definition 9.

**Definition 26.** (i) The universal twist of \( \mathbb{K}(n) \) by the mod 2 Eilenberg-MacLane space \( \mathbb{K}(\mathbb{Z}/2, n+1) \) is the non-zero element of \( \text{tw}_{\mathbb{K}(n)}(K(\mathbb{Z}/2, n+1)) \).

(ii) Let \( h \in H^n(X; \mathbb{Z}/2) \) be a mod 2 cohomology class. Then Morava K-theory of \( X \) twisted by \( h \) is defined to be \( K(n)_*(X; h) := K(n)^{\mu(h)}_*(X) \).

The universal coefficient theorem analogue of Theorem 10 is also true in this case, and the proof follows the proof of that theorem with obvious changes (hence we omit it to avoid repetition).

**Theorem 27** (UCT for Morava K-theory twisted by mod 2 EM spaces). If \( h \in H^n(X; \mathbb{Z}/2) \), and \( P \) denotes the total space of the bundle classified by \( h \), then

\[
K(n)_*(X; h) \cong K(n)_*(P) \otimes_{K(n)_*K(\mathbb{Z}/2,n)} K(n)_* .
\]

Equipped with this result, we can conclude our investigation of the Whitehead tower of \( \text{BO} \). In diagram (1.1), the notation introduced in [Sa15] for the \( \mathbb{Z}/2 \)'s in the Whitehead tower refers to the Bott periodicity cycles of length 8. That is, \( B(m\text{-Orient}) \) and \( B(m\text{-Spin}) \) are those mod 8 analogues of orientation and Spin structure that occur in Bott periodicity cycle \( m \). So we have a sequence \( B(1\text{-Orient}) = BSO \) and \( B(1\text{-Spin}) = BSpin \) in the first cycle, \( B(2\text{-Orient}) := BO\langle 10 \rangle \) and \( B(2\text{-Spin}) := BO\langle 11 \rangle \) in the second cycle and so on (see the Whitehead tower, Diagram (1.1)).

Applying Theorem 5 for \( G = BSO_i \) and \( G = BO_i \) (with \( i \geq 2 \)), we get the following mod 2 version of Theorem 15 with a basically identical proof.

**Theorem 28** (Twisted Morava K-theory for \( B(m\text{-Orient}) \) and \( B(m\text{-Spin}) \) structures). Let \( BO\langle n \rangle \) be a connective cover of \( BO \) with \( n = 1 \mod 8 \) or \( n = 2 \mod 8 \), and let \( h_{n+1} \) be the class in \( H^{n+1}(BO\langle n \rangle; \mathbb{Z}/2) \) classifying the connective cover fibration. Then

\[
K(n)_*(BO\langle n \rangle; h_{n+1}) \cong K(n)_*(BO\langle n \rangle) \quad \text{for any } n \geq 8 ,
\]

\[
K(n)_*(O\langle n \rangle; h_n) \cong K(n)_*(O\langle n \rangle) \quad \text{for any } n \geq 7 ,
\]

where \( h_n \in H^n(O\langle n \rangle; \mathbb{Z}/2) \) is the looping of \( h_{n+1} \).

The cohomology groups \( H^{n+1}(BO\langle n \rangle; \mathbb{Z}/2) \), in which our twists live, have been determined by Stong [St63]. They are isomorphic to \( H^{n+1}(K(\pi_n(BO), n); \mathbb{Z}/2)/I \otimes \mathbb{Z}/2[\theta_i|L(i) > \phi(n)] \), where \( I \) is some ideal depending on \( n \), \( \theta_i \) is a class in \( H^i(BO; \mathbb{Z}/2) \), and \( L(i) \) and \( \phi(n) \) are integers constructed from \( i \) and \( n \), respectively. Specific values can be read off from the above paper of Stong.

Note that the only connective covers of \( O \) and \( BO \) that we have not investigated so far are \( \text{Spin} \), \( \text{SO} \), and \( BSO \). The first two are not directly interesting for our purposes: \( \text{Spin} \) is defined via a map to \( K(\mathbb{Z}/2, 1) \) and \( \text{SO} \) is defined via a map to \( K(\mathbb{Z}/2, 0) \). This would mean that the corresponding twisted Morava K-theory has to be at height 0, i.e., rational cohomology, for which the result is
classical. This is the statement that $H^*(SO; t) \cong H^*(\text{Spin}; \mathbb{Z})$, where $t \in H^1(SO; \mathbb{Z}/2)$ classifies the double cover Spin → SO. For BSO, the Whitehead tower of the orthogonal group, Diagram \(\text{[1.1]}\), gives us the fibration

\[ K(\mathbb{Z}/2, 1) \to \text{BSpin} \to \text{BSO} \xrightarrow{w_2} K(\mathbb{Z}/2, 2), \]

where $w_2$ is the second Stiefel-Whitney class. However, as shown in \[\text{[KLTW04a, Section 5.3]}\], the induced map $K(n)_*(K(\mathbb{Z}/2, 1)) \to K(n)_*(\text{BSpin})$ has to be trivial, so it sends $b_0$ to 0 in $K(n)_*(\text{BSpin})$. Therefore, by the same argument as in Theorem 6, we have the following at chromatic level one.

**Corollary 29** (Mod 2 twisted Morava $K(1)$ of BSO at the prime 2). *With the twist given by the second Stiefel-Whitney class, we have*

\[ K(1)_*(\text{BSO}; w_2) \cong 0. \]

Throughout, we have only considered specific twists. It would be interesting to consider other types of twists or (further) arbitrary multiples of the twisting class. It would also be interesting to consider groups and their connective covers unstably. These will undoubtedly be subtle, as witnessed by the computations in \[\text{[Ra90]}\] \[\text{[Ra97]}\] \[\text{[Ni01]}\] of Morava K-theory of SO$(m)$ and Spin$(m)$ and by \[\text{[SS15]}\] in studying connective covers of these unstable groups.

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