Graduation formula: a new method to construct belief reliability distribution under epistemic uncertainty

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Abstract: In reliability engineering, the observations of the variables of interest are always limited due to cost or schedule constraints. Consequently, the epistemic uncertainty, which derives from lack of knowledge and information, plays a vital influence on the reliability evaluation. Belief reliability is a new reliability metric that takes the impact of epistemic uncertainty into consideration and belief reliability distribution is fundamental to belief reliability application. This paper develops a new method called graduation formula to construct belief reliability distribution with limited observations. The developed method constructs the belief reliability distribution by determining the corresponding belief degrees of the observations. An algorithm is designed for the graduation formula as it is a set of transcendental equations, which is difficult to determine the analytical solution. The developed method and the proposed algorithm are illustrated by two numerical examples to show their efficiency and future application.

Keywords: belief reliability, belief reliability distribution, epistemic uncertainty.

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1. Introduction

Typically, reliability is referred to as the capability that a component or system can perform a required function for a given period of time under stated operating conditions [1]. In reliability engineering, the quality of reliability evaluation is affected by uncertainties. Uncertainties could be classified into two types, aleatory uncertainty and epistemic uncertainty. Aleatory uncertainty describes the inherent variation associated with a physical system or environment while epistemic uncertainty derives from lack of knowledge and information [2–7]. The frequentist probabilistic methods are widely used in risk and reliability assessments to model aleatory uncertainty [8,9]. However, for systems that rarely fail (nuclear systems, chemical processes, railway systems, etc.) or components or systems that do not have enough time to generate sufficient amounts of reliability data, the frequentist approaches are not suitable for use [10–12]. For this reason, several methods were proposed to manage epistemic uncertainties such as interval theory-based reliability methods [13,14], possibility theory-based reliability methods [15,16], evidence theory-based reliability [17–19] and belief reliability [20].

Belief reliability is a new reliability metric proposed by Zhang et al. [20] to describe the reliability of components or systems affected by both aleatory uncertainty and epistemic uncertainty by the chance theory. The chance theory is founded by Liu [21] as a mixture of the uncertainty theory and the probability theory, to deal with problems affected by both aleatory uncertainty and epistemic uncertainty. More specifically, when the system is mainly affected by aleatory uncertainty, belief reliability degenerates to a probability theory-based reliability metric. On the other hand, when the system is mainly affected by epistemic uncertainty, belief reliability could be an uncertainty theory-based reliability metric.

The uncertainty theory is a new mathematical theory proposed in [22] to model epistemic uncertainty by introducing the uncertain measure, the uncertain variable, the uncertainty distribution and other concepts. Nowadays, the uncertainty theory has been successfully employed to model epistemic uncertainty in various areas, such as reliability [23], finance [24], game theory [25], project management [26] and social network [27].

In engineering practice, the observations of the variables of interest are always limited for many reasons, e.g., the components are very expensive, the tests are time-consuming, the systems fail rarely. As a consequence, the acquired observations are with great epistemic uncertainty. This makes it appropriate to employ belief reliability. As
probability density functions are used to represent aleatory uncertainty in the probabilistic reliability theory, belief reliability distribution is the one that is used to represent epistemic uncertainty in belief reliability. This paper focuses on the construction of belief reliability distribution with sparse observations. Since belief reliability would degenerate to the uncertainty theory-based reliability metric when the system is only affected by epistemic uncertainty, belief reliability distribution is essentially uncertainty distribution. Consequently, the aim of this presentation is to construct reasonable uncertainty distribution based on limited and sparse observations.

In the uncertainty theory, uncertainty distribution can be determined by uncertain statistic methods. Uncertainty statistics is a method for collecting and interpreting the expert’s empirical data, which was proposed in [22] and developed in [28–30]. In the uncertainty theory, empirical data are defined as a pair of observation and its corresponding belief degree. Consider that the variable of interest is denoted by $\xi$, an observation is $x$ and the corresponding belief degree is $\alpha$. Then $(x, \alpha)$ is a unit of empirical data. For example, let $\xi$ be the lifetime of a light bulb, $(10 \,000, 0.7)$ means that the belief degree of the lifetime of this light bulb smaller than 10 000 h is 0.7. In the extant literature, empirical data are collected by a questionnaire survey [22], which is developed for the reason that when no samples are available to estimate distribution functions, or some emergency arises, we have to invite some domain experts to evaluate the belief degree that each event will happen. Naturally, the corresponding uncertainty statistical methods are developed to interpret this kind of data. However, in practice, some observations, even not very much, could be obtained by some kind of methods. Besides, we cannot always invite domain experts to evaluate its belief degrees. Consequently, the existent uncertainty statistics methods are not effective to overcome these difficulties. Therefore, it is urgent to propose a new method to construct uncertainty distribution based on the uncertainty theory, so that the sparse data in engineering practice can be used for the reliability evaluation.

In this paper, a graduation formula and its approximation algorithm are proposed to address this problem. The graduation formula is put forward on the basis of the max entropy principle. In the uncertainty theory, Chen and Dai [31] proved that normal uncertainty distribution is the max entropy distribution when the uncertain variable has limited expected values and variances. The problem, then, is how to determine the expected value and variance of the variable of interest based on the limited observations. The basic idea of the graduation formula is to find a set of belief degrees that equal the derivation results of the normal uncertainty distribution with expected values and variances from the empirical moments determined by this set of belief degrees. As the graduation formula is a set of transcendental equations, an algorithm is developed to estimate the approximate solution of the graduation formula.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries on the uncertainty theory, the chance theory and belief reliability. The graduation formula and its approximation algorithm are introduced in Section 3 and Section 4, respectively. Subsequently, two numerical examples are employed to demonstrate the proposed model in Section 5. Finally, this paper is summarized in Section 6.

2. Preliminaries

2.1 Uncertainty theory

The uncertainty theory has been widely applied as a new tool for modeling epistemic uncertainty. In the uncertainty theory, belief degrees of events are quantified by defining uncertain measures.

**Definition 1** Uncertain measure [32]

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms.

**Axiom 1** Normality axiom

$\mathcal{M}\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Axiom 2** Duality axiom

$\mathcal{M}\{A\} + \mathcal{M}\{A^c\} = 1$ for any event $A \in \mathcal{L}$ where $A^c$ is the complementary set of event $A$.

**Axiom 3** Subadditivity axiom

For every countable sequence of events $A_1, A_2, \ldots$, we have

$$\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}. \quad (1)$$

**Axiom 4** Product axiom

Let $(\Gamma_i, \mathcal{L}_i, \mathcal{M}_i)$ be uncertainty spaces for $i = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{ \prod_{i=1}^{\infty} A_i \right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}\{A_i\} \quad (2)$$

where $\mathcal{L}_i$ is the $\sigma$-algebras over $\Gamma_i$ and $A_i$ is the arbitrarily chosen event from $\mathcal{L}_i$ for $i = 1, 2, \ldots$, respectively.

**Definition 2** Uncertain variable [32]

An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\xi \in \mathcal{B}$ is an event for any Borel set $\mathcal{B}$ of real numbers.

**Definition 3** Uncertainty distribution [32]

The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number $x$. 

Example 1 An uncertain variable $\xi$ is called a normal variable if it has a normal uncertainty distribution

$$\phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

(3)
denoted by $\mathcal{N}(e, \sigma)$, where $e$ and $\sigma$ are real numbers with $\sigma > 0$.

Theorem 1 Max entropy theorem [31]
Let $\xi$ be an uncertain variable with the finite expected value $e$ and variance $\sigma^2$. Then

$$H[\xi] \leq \frac{\pi\sigma}{\sqrt{3}}$$

(4)
and the equality holds if $\xi$ is a normal uncertain variable with the expected value $e$ and variance $\sigma^2$, i.e., $\mathcal{N}(e, \sigma)$.

2.2 Chance theory

The chance theory is founded by Liu [21] as a mixture of the uncertainty theory and the probability theory, to deal with problems affected by both aleatory uncertainty and epistemic uncertainty. The basic concept in the chance theory is the chance measure of an event in a chance space.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, A, Pr)$ be a probability space. Then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, A, Pr)$ is called a chance space.

Definition 4 Chance measure [21]
Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, A, Pr)$ be a chance space and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of $\Theta$ is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} dx.$$  

(5)

Definition 5 Uncertain random variable [21]
An uncertain random variable is a function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, A, Pr)$ to the set of real numbers such that $\xi \in B$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$ of real numbers.

Random variables and uncertain variables are two special cases of uncertain random variables. If an uncertainty random variable $\xi(\gamma, \omega)$ does not vary with $\gamma$, it degenerates to a random variable. If an uncertainty random variable $\xi(\gamma, \omega)$ does not vary with $\omega$, it degenerates to an uncertain variable.

Definition 6 Chance distribution [21]
Let $\xi$ be an uncertain random variable. Then its chance distribution is defined by

$$\phi(x) = Ch\{\xi \leq x\}$$

(6)
for any $x \in \mathbb{R}$.

2.3 Belief reliability

Belief reliability is a new reliability metric that takes into consideration both aleatory uncertainty and epistemic uncertainty and thus it is built on the chance theory.

Definition 7 Belief reliability [20]
Let a system state variable $\xi$ be an uncertain random variable, and $\Xi$ be the feasible domain of the products state. Then the belief reliability is defined as the chance that the system state is within the feasible domain, i.e.,

$$R_B = Ch\{\xi \in \Xi\}.$$  

(7)
If the state variable $\xi$ degenerates to an uncertain variable, then the belief reliability metric will be a belief degree. Let $R_B^{(U)}$ denote the belief reliability under the uncertainty theory. Then

$$R_B = R_B^{(U)} = M\{\xi \in \Xi\}.$$  

(8)
This means the system is mainly influenced by epistemic uncertainty, and the belief reliability degenerates to the uncertainty theory-based reliability metric.

Definition 8 Belief reliability distribution [20]
Assume that a system state variable is an uncertain random variable, then the chance distribution of $\xi$, i.e.,

$$\phi(x) = Ch\{\xi \leq x\}$$

(9)
is defined as the belief reliability distribution.
If the state variable $\xi$ is an uncertain variable, then the belief reliability distribution will be an uncertainty distribution. Then

$$\phi(x) = M\{\xi \leq x\}.$$  

(10)
The mean time to failure (MTTF) is an important belief reliability index that can be used to represent the expected life of the system or component.

Definition 9 MTTF [20]
Assume the system failure time $T$ is an uncertain random variable with a belief reliability function $R_B(t)$. The MTTF is defined as

$$MTTF = E[T] = \int_0^\infty Ch\{T > t\} dt = \int_0^\infty R_B(t) dt.$$  

(11)
3. Graduation formula

In engineering practice, the observations of the variables of interest are generally sparse due to cost or schedule constraints. Consequently, it is important to make full use of the obtained observations. Distribution with the max entropy is the one that makes the most of known information without extra assumptions. In the uncertainty theory, Chen and Dai [31] have proved that the normal uncertainty...
distribution is the max entropy distribution when the uncertain variable has limited expected values and variances. Therefore, we could determine the uncertainty distribution of an uncertain variable by estimating its expected values and variances based on its observations.

Let $\xi$ be the uncertain variable of interest and $x_1 < x_2 < \cdots < x_n$ be the observations of $\xi$.

Since we only acquire sparse observations rather than sparse observations together with their belief degrees, it is intuitive to calculate the expected value $e_0$ and variance $\sigma_0$ of the observations as follows:

\[
\begin{align*}
    e_0 &= \frac{1}{n} \sum_{i=1}^{n} x_i \\
    \sigma_0^2 &= \frac{1}{n} \sum_{i=1}^{n} (x_i - e_0)^2.
\end{align*}
\]

As a result, a normal uncertainty distribution $\mathcal{N}(e_0, \sigma_0)$ could be obtained.

Then the corresponding belief degree $\alpha_i$ of each observation $x_i$ could be estimated by

\[
\alpha_i = \left(1 + \exp\left(-\frac{n(e_0 - x_i)}{\sqrt{3} \sigma_0}\right)\right)^{-1}.
\]

Finally, $n$ pairs of empirical data, which are made up of the observations and their belief degrees, are shown as follows: $(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)$.

In the uncertainty theory, empirical moments could be calculated by empirical data and empirical uncertainty distribution.

**Definition 10** Empirical uncertainty distribution [32]

Assume that we have obtained a set of expert’s experimental data $(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)$ that meet the following consistence conditions (perhaps after a rearrangement):

\[
x_1 < x_2 < \cdots < x_n, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1.
\]

Then the empirical uncertainty distribution is

\[
\phi(x) = \begin{cases} 0, & x < x_1 \\ \alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \\ 1, & x > x_n \end{cases},
\]

**Definition 11** The $k$th empirical moment of the empirical uncertainty distribution is defined as

\[
\xi_k = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n)x_n^k.
\]

According to the definition of the empirical uncertainty distribution and empirical moments, the expected value $e_1$ and the second empirical moment $\xi_2$ could be calculated. Then the variance could be determined by $\sigma_1^2 \approx \xi_2 - e_1^2$. Thus, a new normal uncertainty distribution $\mathcal{N}(e_1, \sigma_1)$ could be obtained.

Then the question is whether the expected value and variance determined only by the observations, i.e., $(e_0, \sigma_0^2)$, and the expected value and variance determined only by the observations together with their belief degrees, i.e., $(e_1, \sigma_1^2)$, are the same.

This question could be answered by testing whether $\mathcal{N}(e_0, \sigma_0) \equiv \mathcal{N}(e_1, \sigma_1)$ for any set of observations. The answer is that two distributions are not always equal, and the details could be illustrated in Example 2.

**Example 2** Assume that the observations are $(0.20, 0.26, 0.27, 0.28)$.

**Step 1** Calculate $e_0$ and $\sigma_0$.

\[
\begin{align*}
    e_0 &= \frac{1}{4} \sum_{i=1}^{4} x_i = 0.2525 \\
    \sigma_0 &= \sqrt{\frac{1}{4} \sum_{i=1}^{4} (x_i - e_0)^2} = 0.0313
\end{align*}
\]

**Step 2** Calculate $e_1$ and determine empirical data $(x_i, \alpha_i)$.

As shown in (13), the empirical data could be obtained as follows: $(0.20, 0.0447)$, $(0.26, 0.6077)$, $(0.27, 0.7351)$, $(0.28, 0.8326)$.

**Step 3** Calculate $e_1$ and $\sigma_1$.

\[
\begin{align*}
    e_1 &= \alpha_1 x_1 + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=0}^{3} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{3-j} + (1 - \alpha_n)x_n \\
    \sigma_1 &= \sqrt{\frac{1}{3} \sum_{i=1}^{3} \sum_{j=0}^{3} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{3-j}}
\end{align*}
\]

Thus, we have $e_1 = 0.2459$ and $\sigma_1 = 0.2474$.

**Step 4** Compare $(e_0, \sigma_0)$ with $(e_1, \sigma_1)$.

$e_1 \neq e_0$ and $\sigma_1 \neq \sigma_0$. It is obvious that the two pairs of the expected value and variance are different. The possible reason for the difference is that $e_0$ and $\sigma_0$ are calculated based on the linear assumption as only pure observations are available while the uncertainty distribution to be determined is actually non-linear. Moreover, the new question is whether the belief degrees determined by $e_1$ and $\sigma_1$ could be used as the final choice to represent the indeterminate uncertainty distribution. Therefore, the graduation formula is proposed to address this problem.
**Definition 12** Graduation formula

Let $\xi$ be the uncertain variable of interest and $x_1 < x_2 < \cdots < x_n$ be the observations of $\xi$. Then the corresponding belief degree $\alpha_i$ of $x_i$ $(i = 1, 2, \ldots, n)$ could be determined by

$$\alpha_i = \left(1 + \exp \left(\frac{\pi(e - x_i)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

where

$$e = \frac{\alpha_1 + \alpha_2}{2}x_1 + \frac{\alpha_2 - \alpha_1}{2}x_2 + \sum_{i=2}^{n-1} \left(\frac{\alpha_{i+1} - \alpha_{i-1}}{2}\right)x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right)x_n$$

$$\sigma = \sqrt{\alpha_1(x_1 - e)^2 + \frac{1}{3}\sum_{i=1}^{n-2}(\alpha_{i+1} - \alpha_{i-1})(x_i - e)(x_{i+1} - e)^{2-j} + (1 - \alpha_n)(x_n - e)^2}.$$

The basic idea of this formula is to find a set of belief degrees $\alpha_i$ that equal the derivation results of the normal uncertainty distribution with parameters from the empirical moments determined by this set of belief degrees $\alpha_i$.

**4. Algorithm**

The graduation formula is a set of transcendental equations and thus it is difficult to determine its analytical solution. Therefore, an algorithm is developed to estimate the approximate solution of the graduation formula.

**Input:** Original observations of the variable of interest $(x_1, x_2, \ldots, x_n)$.

**Output:** Corresponding belief degrees of observations $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Part one: pretreatment

**Step 1** Calculate the arithmetic average $e_0$ of the original observations,

$$e_0 = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

**Step 2** Calculate the standard deviation $\sigma_0$ of $x_i$ as

$$\sigma_0 = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - e_0)^2}.$$

**Step 4** Calculate the corresponding belief degrees $\alpha_i^0$.

$$\alpha_i^0 = \left(1 + \exp \left(\frac{\pi(0 - x_i^0)}{\sqrt{3}\sigma_0}\right)\right)^{-1}$$

Part two: iteration process

For the $(t + 1)$th round of iteration, $t = 0, 1, 2, \ldots$.

**Step 1** Calculate the empirical expected value $e_{t+1}$,

$$e_{t+1} = \frac{\alpha_1^t + \alpha_2^t}{2}x_1^t + \sum_{i=2}^{n-1} \left(\frac{\alpha_{i+1}^t - \alpha_{i-1}^t}{2}\right)x_i^t + \left(1 - \frac{\alpha_{n-1}^t + \alpha_n^t}{2}\right)x_n^t.$$

**Step 2** Perform a coordinate system translation with a translation distance $e_{t+1}$ and calculate the new coordinate values $x_i^{t+1} (i = 1, 2, \ldots, n)$,

$$x_i^{t+1} = x_i - e_{t+1}.$$

**Step 3** Calculate the empirical standard deviation $\sigma_{t+1}$ of $x_i^{t+1} (i = 1, 2, \ldots, n)$ as

$$\sigma_{t+1} = \sqrt{\frac{\alpha_1^t(x_1^{t+1})^2 + \sum_{i=1}^{n-1} \sum_{j=0}^{2} (\alpha_{i+1}^t - \alpha_{i-1}^t)(x_i^{t+1})^2(x_{i+1}^{t+1})^{2-j} + (1 - \alpha_n^t)(x_n^{t+1})^2}}.$$
demonstrate the developed formula and the proposed algorithm. The first example is to assess the MTTF with limited observations and the other is to determine the belief reliability distribution of the degraded increment, which is important in the principle exploration of accelerated degradation testing. It could be learned from the two examples that the developed graduation formula and the proposed algorithm are accessible and efficient.

5.1 Assessment of MTTF

In reliability engineering, the MTTF is used as essential information in logistics for planning of the replacement or non-repairable items, warranties, planning for part obsolescence, and so on. Further, it is a very frequent practice that the MTTF is understood to be the universal attribute of a non-repairable item, and in the gross simplification it is often assumed to be the indicator of the expected life of that item. However, in practice, the information that could be used to assess the MTTF is always limited with the improvement of system reliability. In such cases, the results assessed by the probability theory are becoming less convincing as the theoretical basis of the probability theory is the law of large numbers. Belief reliability is a new reliability metric aiming for addressing the problem of the reliability evaluation with limited information. The graduation formula is a new method that is designed for determination of the belief reliability distribution when the observations are sparse.

Consider that the observations of the failure time of a complex system are 444 h, 538 h, 790 h and 1 006 h, respectively. The aim is to assess the MTTF of this kind of system based on these four observations.

First, we calculate the corresponding belief degree of the four observations by the graduation formula.

\[
\begin{align*}
\alpha_1 &= \left(1 + \exp \left(\frac{\pi(e - 444)}{\sqrt{3}\sigma} \right) \right)^{-1} \\
\alpha_2 &= \left(1 + \exp \left(\frac{\pi(e - 538)}{\sqrt{3}\sigma} \right) \right)^{-1} \\
\alpha_3 &= \left(1 + \exp \left(\frac{\pi(e - 790)}{\sqrt{3}\sigma} \right) \right)^{-1} \\
\alpha_4 &= \left(1 + \exp \left(\frac{\pi(e - 1 006)}{\sqrt{3}\sigma} \right) \right)^{-1}
\end{align*}
\]

where

\[
\begin{align*}
e &= \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{3} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_3 + \alpha_4}{2}\right) x_4 \\
\sigma &= \sqrt{\alpha_1 (x_1 - e)^2 + \frac{1}{3} \sum_{i=1}^{3} \sum_{j=0}^{2} (\alpha_{i+1} - \alpha_i) (x_i - e)^j (x_{i+1} - e)^2 - j + (1 - \alpha_4) (x_4 - e)^2}
\end{align*}
\]

The equation set (17) could be solved by the proposed algorithm and the results are

\[
\begin{align*}
\alpha_1 &= 0.0429 \\
\alpha_2 &= 0.1082 \\
\alpha_3 &= 0.7309 \\
\alpha_4 &= 0.9724
\end{align*}
\]

Therefore, the belief reliability distribution of the failure time is

\[
\Phi(x) = \left(1 + \exp \left(\frac{\pi(705.7671 - x)}{\sqrt{3} \times 152.9962} \right) \right)^{-1}.
\]

Finally, the MTTF of this system is 705.767 1 h.

5.2 Accelerated degradation testing

Accelerated degradation testing aids the reliability and lifetime evaluation for highly reliable products. In engineering applications, the number of test items are generally small due to the constraints of finance or testing resources constraints. This makes it appropriate to explore accelerated degradation principles in belief reliability. An important step in the exploration of accelerated degradation principles is to determine the belief reliability distribution of the degraded increment based on limited test units. Therefore, in this section, a numerical example on accelerated degradation testing is given to demonstrate the developed graduation formula and the proposed algorithm.

Here are two sets of observations of the degraded increment from a constant stress accelerated degradation test on five LED chips shown in Table 1.

| Temperature/ Sample number | K | 1# | 2# | 3# | 4# | 5# |
|---------------------------|---|----|----|----|----|----|
| K=303                    | 233| 4   | 3   | 1   | 2   | 5   |
| K=383                    | 4   | 3   | 1   | 2   | 4   | 5   |

First, we should rearrange the original observations and then get ordered observations (1,2,3,4,5) and (4,5,7,9,10).

Then we calculate the corresponding belief degree \(\alpha_i\) of the ordered observations by the graduation formula and results are shown in Table 2.
Table 2  Belief degrees

| Data | 1   | 2   | 3   | 4   | 5   |
|------|-----|-----|-----|-----|-----|
| Belief degree | 0.0235 | 0.1344 | 0.5000 | 0.8656 | 0.9765 |

| Data | 4   | 5   | 7   | 9   | 10  |
|------|-----|-----|-----|-----|-----|
| Belief degree | 0.0274 | 0.0847 | 0.5000 | 0.9153 | 0.9726 |

Finally, the belief reliability distributions of two sets are

\[
\begin{align*}
\Phi_1(x) &= \left(1 + \exp\left(-\frac{\pi(3 - x)}{\sqrt{3} \times 0.9742}\right)\right)^{-1} \\
\Phi_2(x) &= \left(1 + \exp\left(-\frac{\pi(7 - x)}{\sqrt{3} \times 1.5253}\right)\right)^{-1}
\end{align*}
\]  (19)

and shown in Fig. 1.

As shown in (19), the variance of the first data set is 0.9742 and the variance of the other data set is 1.5253. It could be easily learned from the variance that the uncertainty of the first data set is smaller than that of the second one.

Moreover, we employ the data set \{1,2,3,4,5\} to illustrate the efficiency of the proposed algorithm. It could be learned from Fig. 2 that the proposed algorithm has good convergence and high computational efficiency.

6. Conclusions

In this paper, a new method called the graduation formula is proposed to determine the belief reliability distribution with sparse data. In engineering practice, the observations of the variables of interest are generally sparse due to cost or schedule constraints. Consequently, it is important to make full use of the obtained observations. Distributions with the max entropy are the ones that make the most of known information without extra assumptions. The normal uncertainty distribution is the max entropy distribution in the uncertainty theory when the expected value and variance of the variable of interest are known. Therefore, the difficulty is to determine the expected value and variance of the variable of interest when only limited observations of the interest are available. The graduation formula is designed to assess the expected value and variance by the empirical uncertainty distribution and empirical moments. The basic idea of this approach is to find a set of belief degrees that equal the derivation results of the normal uncertainty distribution with parameters from the empirical moments determined by this set of belief degrees. As the graduation formula is a set of transcendental equations, an algorithm is developed to estimate the approximate solution of the graduation formula. Finally, two numerical examples are employed to demonstrate the applicability and validity of the developed formula and the proposed algorithm.

The belief reliability distribution is vital to the application of belief reliability. However, there are few methods to determine the belief reliability distribution. The proposed graduation formula is an efficient method that addresses the problem of determination of the belief reliability distribution with sparse data and thus could promote the application and development of belief reliability in engineering.

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Biographies

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