CHARACTERISTIC POLYNOMIALS OF AUTOMORPHISMS OF HYPERELLIPTIC CURVES

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Abstract. Let $\alpha$ be an automorphism of a hyperelliptic curve $C$ of genus $g$ and let $\overline{\alpha}$ be the automorphism induced by $\alpha$ on the genus-0 quotient of $C$ by the hyperelliptic involution. Let $n$ be the order of $\alpha$ and let $\overline{n}$ be the order of $\overline{\alpha}$. We show that the characteristic polynomial $f$ of the automorphism $\alpha^*$ of the Jacobian of $C$ is determined by the values of $n$, $\overline{n}$, and $g$, unless $n = \overline{n}$, $n$ is even, and $(2g + 2)/n$ is even, in which case there are two possibilities for $f$. In every case we give explicit formulas for the possible characteristic polynomials.

1. Introduction

Let $\alpha$ be an automorphism of a genus-$g$ curve $C$ over a field $k$ and let $\alpha^*$ be the corresponding automorphism of the Jacobian of $C$. Let $n$ be the order of $\alpha$ and let $f$ be the characteristic polynomial of $\alpha^*$. The values of $n$ and $g$ provide some restrictions on the possible values of $f$, but in general they do not determine $f$; for example, a nontrivial involution of a genus-3 curve can have characteristic polynomial equal to $(x - 1)^i(x + 1)^{6-i}$ for $i \in \{0, 2, 4\}$, and all three possibilities occur.

If $C$ is hyperelliptic, with hyperelliptic involution $\iota$, then the automorphism $\alpha$ gives rise to an automorphism $\overline{\alpha}$ of the genus-0 quotient $C/\langle \iota \rangle$. Let $\overline{n}$ be the order of $\overline{\alpha}$, so that either $n = \overline{n}$ or $n = 2\overline{n}$. The triple $(g, n, \overline{n})$ still does not in general determine $f$: If $C$ has genus 3 and $\alpha$ and $\overline{\alpha}$ each have order 2, then $f$ can be either $(x - 1)^2(x + 1)^4$ or $(x - 1)^4(x + 1)^2$, and both possibilities occur.

The purpose of this note is to show that if $C$ is hyperelliptic, this ambiguity between two possible characteristic polynomials is the worst that can happen; furthermore, the triple $(g, n, \overline{n})$ determines $f$ completely unless $n = \overline{n}$, $n$ is even, and $(2g + 2)/n$ is an even integer.

Theorem 1. Let $C$ be a hyperelliptic curve of genus $g$ over a field $k$ and let $\alpha$, $\overline{\alpha}$, $n$, $\overline{n}$, and $f$ be as above.

(1) If $\overline{n}$ is odd and $n = \overline{n}$, then $2g \equiv 0, -1$, or $-2 \mod \overline{n}$, and

$$f = \begin{cases} 
\frac{(x^n - 1)^{(2g+1)/n}}{(x - 1)^2} & \text{if } 2g \equiv -2 \mod \overline{n}; \\
\frac{(x^n - 1)^{(2g+1)/n}}{(x - 1)} & \text{if } 2g \equiv -1 \mod \overline{n}; \\
(x^n - 1)^{2g/n} & \text{if } 2g \equiv 0 \mod \overline{n}.
\end{cases}$$

Date: 2 April 2008.

2000 Mathematics Subject Classification. Primary 14H37; Secondary 14H40.

Key words and phrases. Automorphism, hyperelliptic curve, characteristic polynomial.

The first author was partially supported by NSF grant DMS 0653873.
(2) If $\pi$ is odd and $n = 2\pi$, then $2g \equiv 0, -1, \text{ or } -2 \mod \pi$, and

$$f = \begin{cases} 
(x^n + 1)^{(2g+2)/\pi} & \text{if } 2g \equiv -2 \mod \pi; \\
(x+1)^2 & \text{if } 2g \equiv -1 \mod \pi; \\
(x^n + 1)^{(2g+1)/\pi} & \text{if } 2g \equiv 0 \mod \pi.
\end{cases}$$

(3) If $\pi$ is even and $n = \pi$, then $2g \equiv -2 \mod \pi$. Furthermore:
(a) if $(2g + 2)/\pi$ is odd, then
\[ f = \frac{(x^n - 1)^{(2g+2)/\pi}}{(x^2 - 1)}; \]
(b) if $(2g + 2)/\pi$ is even, then
\[ f = \frac{(x^n - 1)^{(2g+2)/\pi}}{(x - 1)^2} \text{ or } f = \frac{(x^n - 1)^{(2g+2)/\pi}}{(x + 1)^2}. \]

(4) If $\pi$ is even and $n = 2\pi$, then $2g \equiv 0 \mod \pi$ and
\[ f = (x^n + 1)^{2g/\pi}. \]

Remark. Note that in Statements (1) and (2) of the theorem, if $\pi = 1$ then the three expressions in the equality for $f$ are all the same.

Remark. The ambiguity in Statement (3b) is unavoidable. Suppose $\alpha$ is an automorphism of $C$ for which $n = \pi$, $\pi$ is even, and $(2g + 2)/\pi$ is even. Then $\alpha$ and $\iota \alpha$ give the same values of $n$ and $\pi$, but they have different characteristic polynomials.

One of our motivations for the work in this paper was Proposition 13.1 of [1], which is concerned with automorphisms $\alpha$ of supersingular genus-2 curves $C$ over finite fields of characteristic at least 5. The proposition says in part that if $\alpha$ is such an automorphism, and if $n$ and $\pi$ are as defined above, then the pair $(n, \pi)$ appears in the left-hand column of Table 1 and the characteristic polynomial of $\alpha^*$ is as given in the right-hand column of the table. Here we note that Theorem [1] shows that the same conclusion holds for automorphisms of arbitrary genus-2 curves over arbitrary fields, with the restrictions on the values of $n$ and $\pi$ coming from the congruence conditions in the theorem.

In Section 2 we prove two lemmas about quotients of hyperelliptic curves by cyclic groups. In Section 3 we use these lemmas to prove Theorem 1.

Conventions. In this paper, a curve will always mean a geometrically-irreducible one-dimensional nonsingular scheme over a field $k$; by the usual equivalence of categories, we could just as well phrase the entire paper in terms of one-dimensional function fields over $k$. When we speak of the projective line $\mathbf{P}^1$ over a field $k$, we will usually pick without comment a generator $x$ of its function field, so that we can identify the function field with $k(x)$.

2. Quotients of Hyperelliptic Curves

Our proof of Theorem [1] will depend on two lemmas concerning quotients of hyperelliptic curves, which we state and prove in this section. Throughout this section, $C$ will be a hyperelliptic curve over an algebraically-closed field $k$, $\iota$ will be
Table 1. Characteristic polynomials associated to possible values of \( n \) and \( \overline{n} \) for genus-2 curves [1, Table 4].

| \((n, \overline{n})\) | Polynomial          |
|------------------------|---------------------|
| (1, 1)                 | \((x - 1)^4\)       |
| (2, 1)                 | \((x + 1)^4\)       |
| (2, 2)                 | \((x - 1)^2(x + 1)^2\) |
| (3, 3)                 | \((x^2 + x + 1)^2\) |
| (4, 2)                 | \((x^2 + 1)^2\)     |
| (5, 5)                 | \(x^4 + x^3 + x^2 + x + 1\) |
| (6, 3)                 | \((x^2 - x + 1)^2\) |
| (6, 6)                 | \((x^2 - x + 1)(x^2 + x + 1)\) |
| (8, 4)                 | \(x^4 + 1\)         |
| (10, 5)                | \(x^4 - x^3 + x^2 - x + 1\) |

Before we begin the proof of the lemma we mention some facts about automorphisms of genus-0 curves that we will use repeatedly.

Suppose \( \gamma \) is a finite-order automorphism of a genus-0 curve \( X \) over an algebraically-closed field \( k \). By choosing an appropriate isomorphism \( X \cong \mathbb{P}^1 \), we may write the action of \( \gamma \) on the function field of \( \mathbb{P}^1 \) in one of two forms: either \( x \mapsto \xi x \) for a root of unity \( \xi \), or \( x \mapsto x + 1 \). In the first case the order of \( \gamma \) is not divisible by the characteristic \( p \) of \( k \). Furthermore, the quotient map from \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \) induced by \( \gamma \) gives a Kummer extension of function fields \( k(x) \to k(x) \) that can be written
as \( x \mapsto x^m \), where \( m \) is the order of \( \gamma \). This map has two ramification points, and each point ramifies totally. When \( \gamma \) can be written \( x \mapsto x + 1 \) the order of \( \gamma \) is equal to \( p \), and the associated quotient map \( \mathbb{P}^1_\mathbb{C} \to \mathbb{P}^1_\mathbb{C} \) gives an Artin-Schreier extension of function fields \( k(x) \to k(x) \) that can be written as \( x \mapsto x^p - x \). Only one point of \( \mathbb{P}^1_\mathbb{C} \) ramifies in this map, but again the ramification is total.

Proof of Lemma 3. Suppose \( m \) is odd, so that the Galois group \( G \) is cyclic. We know that if a \( Q \) ramifies in \( \psi \), then it ramifies completely. Thus, the image of \( H \) in \( \overline{G} \) is either trivial or all of \( \overline{G} \). The only subgroups of \( G \) that have these images in \( \overline{G} \) are the ones listed in first statement of the lemma.

Suppose \( m \) is even and the characteristic \( p \) of \( k \) is not 2. Since the automorphism \( \bar{\beta} \) of \( \overline{C} \) has order \( m \), and \( m \neq p \), the facts we mentioned before the start of the proof show that that \( p \) does not divide \( m \). Thus \( p \) does not divide \( \#G = 2m \), so all ramification in \( \varphi \) is tame. In particular, the inertia group \( H \) is cyclic. The only cyclic subgroups of \( G \) whose images in \( \overline{G} \) are either trivial or all of \( \overline{G} \) are the four groups listed in the second statement. \( \square \)

Lemma 3. With notation and assumptions as above, let \( g \) be the genus of \( C \), let \( h \) be the genus of \( D \), and let \( e \) be the number of points of \( \overline{D} \) that ramify in both the right and the bottom maps of Diagram 1. Then \( e \in \{0, 1, 2\} \). If the characteristic of \( k \) is not 2, then the relationship between \( g \) and \( h \) depends on \( e \) and on the parity of \( m \) as follows:

| \( m \) odd | \( m \) even |
|---|---|
| \( e = 0 \) | \( 2h = (2g + 2)/m - 2 \) | \( 2h = (2g + 2)/m - 2 \) |
| \( e = 1 \) | \( 2h = (2g + 1)/m - 1 \) | \( 2h = (2g + 2)/m - 1 \) |
| \( e = 2 \) | \( 2h = 2g/m \) | \( 2h = (2g + 2)/m \) |

If \( k \) has characteristic 2, then \( m \) is equal to 2 if it is even, and the relationship between \( g \) and \( h \) depends on \( e \) and on the parity of \( m \) as follows:

| \( m \) odd | \( m = 2 \) |
|---|---|
| \( e = 0 \) | \( 2h = (2g + 2)/m - 2 \) | \( 2h = g - 1 \) |
| \( e = 1 \) | \( 2h = (2g + 1)/m - 1 \) | \( 2h = g \) if \( g \) is even, \( 2h = g + 1 \) if \( g \) is odd |
| \( e = 2 \) | \( 2h = 2g/m \) | \( \text{(not possible)} \) |

Proof. We know that at most two points ramify in the cover \( \psi : \overline{C} \to \overline{D} \), so it follows immediately that \( e \) is at most 2.

Let \( \mathcal{D}_C \) and \( \mathcal{D}_D \) denote the differentials of the double covers \( C \to \overline{C} \) and \( D \to \overline{D} \), respectively, and let \( \mathcal{D}_C \) and \( \mathcal{D}_D \) be the discriminants of these covers. Note that we have \( \deg \mathcal{D}_C = \deg \mathcal{D}_C \) and \( \deg \mathcal{D}_D = \deg \mathcal{D}_D \). More specifically, if \( P \) is a point of \( \overline{C} \) at which \( \mathcal{D}_C \) has positive order, then there is a unique point \( p \) of \( C \) over \( P \), and \( \text{ord}_p \mathcal{D}_C = \text{ord}_p \mathcal{D}_C \); the analogous statement holds for points of \( \overline{D} \). The Riemann-Hurwitz formula [2, Thm. 3.3.5], applied to the double covers \( C \to \overline{C} \) and \( D \to \overline{D} \), shows that

\[
\begin{align*}
g &= -1 + (1/2) \deg \mathcal{D}_C = -1 + (1/2) \deg \mathcal{D}_C \\
h &= -1 + (1/2) \deg \mathcal{D}_D = -1 + (1/2) \deg \mathcal{D}_D.
\end{align*}
\]
Therefore, to find the relationship between $g$ and $h$ we need only find the relationship between the degrees of $\mathcal{D}_C$ and $\mathcal{D}_D$.

Before we turn to the various cases summarized in the tables in the statement of the lemma, we will sketch out the general method we use to compare the degrees of these two discriminants. Throughout this introductory sketch, we will assume that we are not in the special case where $m$ is even and $k$ has characteristic 2.

Suppose $P$ is a point of $\overline{C}$ that ramifies in the double cover $C \to \overline{C}$. By looking at the lists in Lemma 2 of the possible ramification groups for the point $\psi(P)$ in the extension $\varphi : C \to \overline{D}$, we see that $\psi(P)$ must ramify in the double cover $D \to \overline{D}$. In other words, the support of $\mathcal{D}_C$ is contained in the inverse image under $\psi$ of the support of $\mathcal{D}_D$.

We divide the support of $\mathcal{D}_D$ into two sets: Let $E$ be the set of points of $\overline{D}$ that ramify both in $D \to \overline{D}$ and in $\psi$, and let $E'$ be the set of points of $\overline{D}$ that ramify in $D \to \overline{D}$ but not in $\psi$. Then we have $e = \#E$, and we set $e' := \#E'$. Let $\mathcal{D}_C'$ be the part of $\mathcal{D}_C$ supported on $\psi^{-1}(E')$, and let $\mathcal{D}_D'$ be the part of $\mathcal{D}_D$ supported on $E'$.

Suppose $Q$ is a point of $E'$, and let $P$ be one of the $m$ points in $\psi^{-1}(Q)$. Then locally at $P$ and at $Q$ the extensions $C \to \overline{C}$ and $D \to \overline{D}$ are isomorphic, so the order of $\mathcal{D}_C$ at $P$ is equal to the order of $\mathcal{D}_D$ at $Q$. This shows that $\deg \mathcal{D}_C' = m \deg \mathcal{D}_D'$.

All that remains is to find the relationship between the portion of $\mathcal{D}_C$ supported on $\psi^{-1}(E)$ and the portion of $\mathcal{D}_D$ supported on $E$.

Suppose $Q$ is a point of $E$. For each $i$ let $H_i$ be the $i$-th ramification group of $Q$ in the double cover $D \to \overline{D}$. By [2, Thm. 3.5.9], the order of $\mathcal{D}_D$ at $Q$ is equal to $\sum(\#H_i - 1)$, but since each $H_i$ has order 1 or 2, the value of this sum is simply the largest $i$ such that $H_i$ is nontrivial. Let $q$ be the point if $D$ lying over $Q$, and let $v$ be a uniformizer at $q$. According to [2 Lem. 3.5.6], the largest value of $i$ such that $H_i$ is nontrivial is the valuation of $v - \iota^*v$ at $q$. Thus, $\ord_Q \mathcal{D}_D = \val_q(v - \iota^*v)$.

Let $P$ be the unique point of $\overline{C}$ with $\psi(P) = Q$. If $P$ is unramified in the double cover $C \to \overline{C}$ then $\mathcal{D}_C$ has order 0 at $P$. If $P$ is ramified, let $p$ be the point of $C$ lying over it, and let $u$ be a uniformizer at $p$. Arguing as above, we find that $\ord_P \mathcal{D}_C = \val_p(u - \iota^*u)$.

With these formulas for $\ord_Q \mathcal{D}_D$ and $\ord_P \mathcal{D}_C$ in hand, we turn to the various cases listed in the lemma.

First suppose that the characteristic of $k$ is not 2 and that $m$ is odd. If $Q$ is a point in $E$, then the inertia group of $Q$ in $\varphi$ must be $G$. This shows that the unique point $P$ with $\psi(P) = Q$ is ramified in the double cover $C \to \overline{C}$. Since the characteristic of $k$ is not 2, the point $P$ is tamely ramified. Likewise, $Q$ is tamely ramified in $D \to \overline{D}$. Thus, $\mathcal{D}_C$ has order 1 at $P$ and $\mathcal{D}_D$ has order 1 at $Q$. It follows that

$$\deg \mathcal{D}_C - e = \deg \mathcal{D}_C' = m \deg \mathcal{D}_D' = m(\deg \mathcal{D}_D - e),$$

which gives $(2g + 2 - e) = m(2h + 2 - e)$, which is what is claimed in the left-hand column of the first table in Lemma 3.

Suppose that the characteristic of $k$ is not 2 and that $m$ is even. If $Q$ is a point of $E$, then the inertia group of $Q$ in the cover $\varphi$ must be $\langle \iota, \beta \rangle$. In this case we see that the unique point $P$ of $\overline{C}$ with $\psi(P) = Q$ does not ramify in the double cover $C \to \overline{C}$. This tells us that

$$\deg \mathcal{D}_C = \deg \mathcal{D}_C' = m \deg \mathcal{D}_D' = m(e + \deg \mathcal{D}_D),$$
which leads to the entries in the right-hand column of the first table in Lemma 3.

Now suppose that \( k \) has characteristic 2 and that \( m \) is odd, and suppose \( Q \) is a point of \( E \). Let \( P \) be the unique point of \( C \) with \( \psi(P) = Q \). The inertia group of \( Q \) in the cover \( \varphi : C \to \overline{C} \) must be \( G \), so \( P \) is ramified in the double cover \( C \to \overline{C} \). As above, let \( p \) be the point of \( C \) lying over \( P \) and let \( q \) be the point of \( D \) lying over \( Q \). We would like to compare the order of \( \mathcal{O}_C \) at \( P \) to the order of \( \mathcal{O}_D \) at \( Q \). Since these are locally-defined quantities, we may replace the curves in Diagram 1 with their completions at \( p \), \( q \), \( P \), and \( Q \), respectively. We can then choose a uniformizer \( u \) for \( p \) and a uniformizer \( v \) for \( q \) such that \( v = u^m \).

Let \( i > 1 \) be the valuation at \( p \) of \( u - \iota^i u \). Then we can write
\[
i^i u = u + cu^i + (\text{higher-order terms}),
\]
and raising both sides to the \( m \)-th power we find that
\[
i^i v = v + mcu^{m-1+i} + (\text{higher-order terms}).
\]
Since the valuation of \( v - i^i v \) at \( p \) is \( m - 1 + i \), the valuation \( j \) of \( v - i^i v \) at \( q \) must be \((m - 1 + i)/m\). In other words, \( i = mj - m + 1 \). If we let \( I \) denote the degree of the portion of \( \mathcal{O}_C \) supported on \( \psi^{-1}(E) \), and \( J \) the degree of the portion of \( \mathcal{O}_D \) supported on \( E \), then \( I = mJ - (m - 1)e \).

We find that
\[
\deg \mathcal{O}_C = \deg \mathcal{O}'_C + I = m \deg \mathcal{O}'_D + mJ - (m - 1)e = m \deg \mathcal{O}_D - (m - 1)e,
\]
so that \( \deg \mathcal{O}_C - e = m(\deg \mathcal{O}_D - e) \), which again leads to \( 2g + 2 - e = m(2h + 2 - e) \).

Finally, we consider the case where \( k \) has characteristic 2 and \( m \) is even. As we noted before the proof of Lemma 2, an even-order automorphism of a genus-0 curve in characteristic 2 must have order 2, so \( m = 2 \). Once again, we define \( E \) to be the set of points of \( \overline{D} \) that ramify in \( \psi : C \to \overline{C} \) and in the double cover \( D \to \overline{D} \) (so that \( e = \#E \)), and we define \( E' \) to be the set of points that ramify in \( D \to \overline{D} \) but not in \( \psi \). As before, we define \( \mathcal{O}'_C \) to be the part of \( \mathcal{O}_C \) supported on \( \psi^{-1}(E') \) and \( \mathcal{O}'_D \) to be the part of \( \mathcal{O}_D \) supported on \( E' \), and as before, we have
\[
\deg \mathcal{O}'_C = 2 \deg \mathcal{O}'_D.
\]

Since \( m = 2 \), the map \( \psi \) is ramified at a single point, and \( e \leq 1 \). If \( e = 0 \) then
\[
\deg \mathcal{O}_C = \deg \mathcal{O}'_C = 2 \deg \mathcal{O}'_D = 2 \deg \mathcal{O}_D,
\]
and it follows from the Riemann-Hurwitz formula that \( 2h = g - 1 \), as claimed in the second table in Lemma 3.

On the other hand, if \( e = 1 \) we may choose isomorphisms \( C \cong \mathbb{P}^1 \) and \( \overline{D} \cong \mathbb{P}^1 \) so that \( \psi \) corresponds to the function field map \( x \mapsto x^2 + x \); then \( \infty \) ramifies in the double cover \( D \to \overline{D} \cong \mathbb{P}^1 \). The function field \( k(D) \) of \( D \) is an Artin-Schreier extension of \( k(x) \), so it contains an element \( y \) not in \( k(x) \) such that \( y^2 + y \) lies in \( k(x) \). The completion of \( k(x) \) at \( \infty \) is the ring of Laurent series in \( 1/x \), and in this completion we can write
\[
y^2 + y = a_0 x^n + \sum_{i=-\infty}^{n-1} a_i x^i
\]
for some integer \( n \) and elements \( a_i \) of \( k \) with \( a_0 \neq 0 \). If \( n = 0 \) we can replace \( y \) with \( y + b \) for a constant \( b \in k \) with \( b^2 + b = a_0 \), which has the effect of replacing \( n \) with nonzero integer. If \( n \) is even and nonzero we may replace \( y \) with \( y + \sqrt{a_0} x^{n/2} \),...
which has the effect of replacing \( n \) by \( n/2 \); repeating this reduction, we find that we may assume that \( n \) is odd, say \( n = 2m - 1 \). If \( n \) were negative the point \( \infty \) would split in \( D \), contrary to assumption, so \( n \) must be positive.

Let \( q \) be the unique point of \( D \) lying over \( \infty \). It is easy to check that \( v = y/x^n \) is a uniformizer for \( q \). As before, the order of \( \deg D \) at \( \infty \) is equal to the valuation of \( v - t^*v \) at \( q \). Since \( t^*v = (y + 1)/x^m \), we find that
\[
\text{val}_q(v - t^*v) = \text{val}_q(1/x^m) = 2 \text{val}_\infty(1/x^m) = 2m.
\]

Now consider the curve \( C \), which is the fiber product of the double cover \( D \to \overline{D} \) with \( \psi : \overline{C} \to \overline{D} \). Locally at \( \infty \), we obtain \( C \) by taking the equality
\[
y^2 + y = a_n x^n + \sum_{i=-\infty}^{n-1} a_i x^i
\]
and replacing \( x \) with \( x^2 + x \). This gives us
\[
y^2 + y = a_n x^{2n} + a_n x^{2n-1} + (\text{terms in } x^i \text{ with } i < 2n - 1).
\]
If \( n > 1 \), then replacing \( y \) with \( y + \sqrt{a_n} x^n \) gives us
\[
y^2 + y = a_n x^{2n-1} + (\text{terms in } x^i \text{ with } i < 2n - 1),
\]
and we find that \( \text{ord}_\infty \mathcal{O}_C = 2n = 4m - 2 \). But if \( n = 1 \), the same substitution gives
\[
y^2 + y = (a_n + \sqrt{a_n}) x + (\text{terms in } x^i \text{ with } i < 1).
\]
In this case, \( \text{ord}_\infty \mathcal{O}_C = 2 = 4m - 2 \) if \( a_n \neq \sqrt{a_n} \), and \( \text{ord}_\infty \mathcal{O}_C = 0 = 4m - 4 \) otherwise.

Applying the Riemann-Hurwitz formula to the double covers \( D \to \overline{D} \) and \( C \to \overline{C} \) and using the relation \( \deg \mathcal{O}_C = 2 \deg \mathcal{O}_D \), we find that
\[
2h = \deg \mathcal{O}_D' + 2m - 2
\]
\[
g = \begin{cases} 
\deg \mathcal{O}_D' + 2m - 2 & \text{if } \infty \text{ ramifies in } C \to \overline{C} \cong \mathbb{P}^1; \\
\deg \mathcal{O}_D' + 2m - 3 & \text{if } \infty \text{ is unramified in } C \to \overline{C} \cong \mathbb{P}^1.
\end{cases}
\]
It follows that
\[
2h = \begin{cases} 
g & \text{if } \infty \text{ ramifies in } C \to \overline{C} \cong \mathbb{P}^1; \\
g + 1 & \text{if } \infty \text{ is unramified in } C \to \overline{C} \cong \mathbb{P}^1.
\end{cases}
\]
Clearly \( g \) is even in the first case and odd in the second, so we get the result given in the second column of the second table of Lemma \( \text{[3]} \).

Remark. One could also prove Lemma \( \text{[3]} \) by using explicit equations and the standard formulas for the genus of a hyperelliptic curve in terms of its defining equation \( [2] \text{ Cor. 3.6.3, Cor. 3.6.9} \). For example, suppose that the characteristic \( p \) of the base field is not 2, that \( m \) is odd and not divisible by \( p \), and that \( e = 2 \). By choosing appropriate isomorphisms \( \overline{C} \cong \mathbb{P}^1 \cong \overline{D} \) and an appropriate model for \( D \), we can assume that \( \psi \) is the map \( x \mapsto x^m \) and that \( D \) is given by \( y^2 = xf(x) \) for a separable even-degree polynomial \( f(x) \) with \( f(0) \neq 0 \). Then \( C \) has a singular model of the form \( y^2 = x^m f(x^m) \) and a nonsingular model of the form \( z^2 = xf(x^m) \). In this case one checks that \( h = (\deg f)/2 \) and \( g = (m \deg f)/2 \), so \( 2h = 2g/m \), as claimed by the lemma.
3. Proof of Theorem 1

In this section we prove Theorem 1. Let us begin by explaining the basic idea of the proof.

Since the conclusions of Theorem 1 are completely geometric, we may assume that $k$ is algebraically closed. The characteristic polynomial $f$ of $\alpha^n$ has degree $2g$; let its complex roots be $\zeta_1, \ldots, \zeta_{2g}$, so that the $\zeta$ are all $n$-th roots of unity. For each divisor $d$ of $n$ let $N_d$ denote the number of the $\zeta$ that are primitive $d$-th roots of unity and let $M_d$ denote the number of the $\zeta$ that satisfy $\zeta^d = 1$. Then we have

\begin{equation}
M_d = \sum_{e|d} N_e, \quad N_d = \sum_{e|d} \mu(d/e) M_e, \quad \text{and} \quad f = \prod_{d|n} \Phi_d^{N_d/\phi(d)},
\end{equation}

where $\mu$ is the M"obius function, $\phi$ is the Euler $\phi$-function, and $\Phi_d$ is the $d$-th cyclotomic polynomial. So to determine $f$, it is enough to determine the $M_d$.

For every divisor $d$ of $n$, let $f_d$ be the characteristic polynomial of the automorphism $\alpha^d$ of $C$. Then the complex roots of $f_d$ are the $d$-th powers of the complex roots of $f$, so $M_d$ is equal to the multiplicity of 1 as a root of $f_d$. This multiplicity is equal to twice the dimension of the part of the Jacobian on which $\alpha$ acts trivially, and this dimension is equal to the genus of the quotient of $C$ by $\langle \alpha^d \rangle$. We see that computing $M_d$ is equivalent to computing the genus of this quotient curve. If the hyperelliptic involution $\iota$ lies in $\langle \alpha^d \rangle$ then the genus of the quotient is 0; if not, then the genus is determined by Lemma 3. To prove the theorem, all we must do is verify that the values of $M_d$ predicted by the putative characteristic polynomials given in the theorem agree with the values we compute by applying Lemma 3.

We consider the four statements of the theorem in turn.

Proof of Statement (i): In this case our assumption is that $n = \pi$ and $\pi$ is odd. For each divisor $d$ of $n$ let $D_d$ be the quotient of $C$ by $\langle \alpha^d \rangle$. Then we have a diagram

\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{n/d} & D_d & \xrightarrow{d} & D_1 \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{n/d} & D_d & \xrightarrow{d} & D_1.
\end{array}
\end{equation}

As in Section 2 we see that the map $\varphi$ from $C$ to $\overline{D_1}$ is a Galois cover with group $G = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Let $\varphi_d$ be the map from $C$ to $\overline{D_d}$, let $E_d$ be the set of points of $\overline{D_d}$ that ramify going up to $D_d$ and going up $\overline{C}$, and let $e_d = \# E_d$. We will show that $e_d$ is determined by $e_1$.

Since $\pi$ is odd, Lemma 2 tells us that the inertia group of a point of $\overline{D_1}$ in the extension $\varphi : C \to \overline{D_1}$ is either trivial, or $\langle 1 \rangle$, or $\langle \alpha \rangle$, or all of $G$. A point in $E_1$ must have ramification group $G$, and so must lie under a unique point in $E_d$. Likewise, any point in $E_d$ must lie over a point of $\overline{D_1}$ that has ramification group $G$, and that therefore lies in $E_1$. Thus, for every $d$ we have $e_d = e_1$.

If $e_1 = 0$ then Lemma 3 shows that $M_d = (2g + 2)d/\pi - 2$ for all $d$. In particular we see that $\pi$ divides $2g + 2$. Also, we check that the polynomial $f = (x^n - 1)(2x^{n+2}/\pi)/(x - 1)^2$ produces the correct values of $M_d$. If $e_1 = 1$ then we have $M_d = (2g + 1)d/\pi - 1$, so $\pi$ divides $2g + 1$, and the polynomial $f = (x^n - 1)(2x^{n+1}/\pi)/(x - 1)$ gives the correct values of $M_d$. Finally, if $e_1 = 2$ then $M_d = 2gd/\pi$, so that $\pi$ divides $2g$, and the polynomial $f = (x^n - 1)^{2g}/\pi$ produces the required values of $M_d$. 
Proof of Statement (2): In this case $n = 2\pi$ and $\pi$ is odd, and we see that $\nu = \alpha^\pi$. Let $\alpha_0 = \alpha\nu$, so that $\alpha_0$ has order $\pi$ and induces an automorphism of order $\pi$ on $C$. Then Statement (1) tells us the characteristic polynomial $f_0$ of $\alpha_0^n$; furthermore, since $\alpha^\ast = -\alpha_0^\ast$, we have $f(x) = f_0(-x)$. This agrees with what is claimed in Statement (2).

Proof of Statement (3): In this case $n = \pi$ and $\pi$ is even, and the analysis is very much like that for Statement (1). For every divisor $d$ of $n$ we let $D_d$ be the quotient of $C$ by $\langle \alpha^d \rangle$, and Diagram (3) is again a diagram of Galois extensions, with the total Galois group $G$ being $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. However, since $n$ is even, $G$ is no longer a cyclic group. As before, we let $\phi$ be the map from $C$ to $\overline{D}_d$, we let $\phi_d$ be the map from $C$ to $\overline{D}_d$, we let $E_d$ be the set of points of $\overline{D}_d$ that ramify going up to $D_d$ and going up to $\overline{C}$, and we let $e_d = \#E_d$.

Let us first consider the case in which the characteristic of the base field is not equal to 2. Then according to Lemma 2 the ramification group of a point $Q$ of $\overline{D}_1$ in the cover $\phi$ is either trivial, the group $\langle \iota \rangle$, the group $\langle \alpha \rangle$, or the group $\langle \alpha \iota \rangle$. If a point has one of the first two inertia groups it will not lie in $E_d$, because it is not ramified in the bottom row of Diagram (3). If a point has inertia group $\langle \alpha \rangle$, then it will not lie in $E_d$ because it is not ramified in the extension $D_d \to \overline{D}_d$. But if a point has inertia group $\langle \alpha \iota \rangle$, it will lie in $E_d$ if $d$ is odd, and will not lie in $E_d$ if $d$ is even.

So when the characteristic of the base field is not 2, we see once again that the value of $e_d$ is determined by the value of $e_1$: We have $e_d = e_1$ if $d$ is odd, and $e_d = 0$ if $d$ is even. If $d$ is odd then $n/d$ is even, so Lemma 3 tells us that $M_d = (2g + 2)d/n - 2 + e_1$. (Note that since $M_1$ is twice the genus of $D_1$, we find that $n$ divides $2g + 2$ and that the parity of $e_1$ is equal to the parity of $(2g + 2)/n$.) If $d$ is even then Lemma 8 shows that $M_d = (2g + 2)d/n - 2$. These values for $M_d$ are consistent with

$$ f = \begin{cases} \frac{(x^n - 1)(2g+2)/n}{(x - 1)^2} & \text{if } e_1 = 0; \\ \frac{(x^n - 1)(2g+2)/n}{(x^2 - 1)} & \text{if } e_1 = 1; \\ \frac{(x^n - 1)(2g+2)/n}{(x + 1)^2} & \text{if } e_1 = 2, \end{cases} $$

so these must be the correct values of $f$.

As we noted above, the parity of $(2g + 2)/n$ is equal to that of $e_1$. Thus, if $(2g + 2)/n$ is odd then $e_1 = 1$, and we find the value of $f$ given in Statement (3a). If $(2g + 2)/n$ is even then $e_1$ is either 0 or 2, and we find that $f$ must have one of the two values given in Statement (3b).

Finally, we turn to the case in which the base field has characteristic 2. In this case we must have $n = 2$, so we only have to determine the value of $M_1$ (since we already know that $M_2 = 2g$). But Lemma 3 tells us the possibilities for this value: If $g$ is even then $M_1 = g$, while if $g$ is odd then $M_1$ is either $g - 1$ or $g + 1$. We check that the values of $f$ given in Statements (3a) and (3b) agree with these values of $M_1$ and $M_2$. 


Proof of Statement (4): In this case $n = 2\pi$ and $\pi$ is even, and we have $\iota = \alpha^{\pi}$.

Taking the quotient of $C$ by $\langle \alpha \rangle$ gives us a Galois extension

$$C \xrightarrow{2} C \xrightarrow{\pi} D$$

with group $G = \mathbb{Z}/n\mathbb{Z}$, where $C$ and $D$ are curves of genus 0.

Consider a point $Q$ of $D$ that ramifies going up to $C$. Then $Q$ must be totally ramified in this extension, so the inertia group of $Q$ in the total extension $C \to D$ is a subgroup of $G$ that surjects onto the Galois group of $C \to \Omega$. The only such subgroup is $G$ itself, so any point of $D$ that ramifies going up to $C$ must ramify totally in the extension $C \to \Omega$.

We see that if $d$ is a divisor of $n$ such that $n/d$ is even, then $\iota$ lies in the subgroup $\langle \alpha^d \rangle$, the genus of the quotient of $C$ by this subgroup is 0, and $M_d = 0$. If $d$ is a divisor of $n$ such that $n/d$ is odd, let $e_d$ be the number $e$ associated to $\alpha^d$ as in Lemma 3, then $e_d$ is equal to the number of points of $D$ that ramify in the degree-$\pi$ extension $C \to \Omega$, and this value is either 1 or 2, depending on whether or not $\pi$ is divisible by the characteristic of the base field.

Suppose the characteristic of the base field is not equal to 2. Then, since $\pi$ is even, the degree-$\pi$ map $C \to \Omega$ of genus-0 curves does not give an Artin-Schreier extension of function fields; rather, it gives a Kummer extension, and it follows that there are two points of $D$ that totally ramify going up to $C$. Any other points of $D$ that ramify going up to $C$ must have ramification groups of order 2. If there are $r$ of these points, then the Riemann-Hurwitz formula for Galois extensions tells us that

$$2g - 2 = n(-2 + 2(1 - 1/n) + r(1 - 1/2)) = -2 + r\pi,$$

and it follows that $\pi$ divides 2g.

Also, since we have two points of $D$ that ramify totally in $C \to \Omega$, we see that $e_d = 2$ whenever $n/d$ is odd, and it follows from Lemma 3 that $M_d = 2gd/n$ when $n/d$ is odd. Combining this with the observation that $M_d = 0$ when $n/d$ is even, we see that the polynomial $f = (x^\pi + 1)^{2g/\pi}$ gives the correct values of $M_d$.

Now suppose that the characteristic of the base field is equal to 2. Then $\pi$ must be equal to 2, and $\alpha$ has order 4. The diagram (4) shows that the quotient of $C$ by $\langle \alpha \rangle$ has genus 0, the quotient of $C$ by $\langle \alpha^2 \rangle$ has genus 0, and the quotient of $C$ by $\langle \alpha^4 \rangle$ has genus $g$. Thus $M_1 = M_2 = 0$ and $M_4 = 2g$. The polynomial that gives rise to these values of $M_d$ is $(x^2 + 1)^g$, which is the polynomial given in Statement (4).

□

References

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