Classical and quantum superintegrability with applications

Willard Miller Jr\textsuperscript{1}, Sarah Post\textsuperscript{2} and Pavel Winternitz\textsuperscript{3}

\textsuperscript{1} School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
\textsuperscript{2} Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822, USA
\textsuperscript{3} Centre de Recherches Mathématiques et Département de Mathématiques et de Statistique, Université de Montréal, C.P. 6128-CV, Montréal, Québec H3C 3J7, Canada

E-mail: miller@ima.umn.edu, spost@hawaii.edu and wintern@crm.umontreal.ca

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Abstract

A superintegrable system is, roughly speaking, a system that allows more integrals of motion than degrees of freedom. This review is devoted to finite dimensional classical and quantum superintegrable systems with scalar potentials and integrals of motion that are polynomials in the momenta. We present a classification of second-order superintegrable systems in two-dimensional Riemannian and pseudo-Riemannian spaces. It is based on the study of the quadratic algebras of the integrals of motion and on the equivalence of different systems under coupling constant metamorphosis. The determining equations for the existence of integrals of motion of arbitrary order in real Euclidean space $E_2$ are presented and partially solved for the case of third-order integrals. A systematic exposition is given of systems in two and higher dimensional space that allow integrals of arbitrary order. The algebras of integrals of motions are not necessarily quadratic but close polynomially or rationally. The relation between superintegrability and the classification of orthogonal polynomials is analyzed.

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(Some figures may appear in colour only in the online journal)
1. Introduction

A standard way to gain insight into the behavior of a physical system is to construct a mathematical model of the system, analyze the model, use it to make physical predictions that follow from the model and compare the results with experiment. The models provided by classical and quantum mechanics have been and continue to be spectacularly successful in this regard. However, the systems of ordinary and partial differential equations (PDEs) provided by these models can be very complicated. Usually they cannot be solved analytically and solutions can only be approximated numerically. A relatively few systems, however, can be solved exactly with explicit analytic expressions that predict future behavior, and with adjustable parameters, such as mass or initial position, so that one can determine the effect on the system of changing these parameters. These are classical and quantum integrable Hamiltonian systems. A special subclass of these systems, called superintegrable, is extremely important for developing insight into physical principles, for they can be solved algebraically as well as analytically, and many of the simpler systems are featured prominently in textbooks. What distinguishes these systems is their symmetry, but often of a much subtler kind than just group symmetry. The symmetries in their totality form quadratic, cubic and other higher-order algebras, not necessarily Lie algebras, and are sometimes referred to as ‘hidden symmetries’.
Famous examples are the classical anharmonic oscillator (Lissajous patterns) and Kepler systems (planetary orbits), the quantum Coulomb system (energy levels of the hydrogen atom, leading to the periodic table of the elements) and the quantum isotropic oscillator. The Hohmann transfer, a fundamental procedure for the positioning of satellites and orbital maneuvering of interplanetary spacecraft is based on the superintegrability of the Kepler system.

Superintegrable systems admit the maximum possible symmetry and this forces analytic and algebraic solvability. The special functions of mathematical physics and their properties are closely related to their origin and use in providing explicit solutions for superintegrable systems, for instance via separation of variables in the PDEs of mathematical physics. These systems appear in a wide variety of modern physical and mathematical theories, from semiconductors [161] to black holes [53]. As soon as a system is discovered it tends to be implemented as a model, due to the fact that it can be solved explicitly. Perturbations of superintegrable systems are frequently used to study the behavior of more complex systems, e.g., the periodic table is based on perturbations of the superintegrable hydrogen atom system.

The principal research activity in this area involves the discovery, classification and solution of superintegrable systems, and elucidation of their structure, particularly the underlying symmetry algebra structure, as well as application of the results in a wide variety of fields. Superintegrability has deep historical roots, but the modern theory was inaugurated by Smorodinsky, Winternitz and collaborators in 1965 [51, 52, 121] who explored multiseparability in two-dimensional (2D) and three-dimensional (3D) Euclidean spaces. Wojciechowski seems to have coined the term ‘superintegrable’ and applied it about 1983 [197]. The earlier terminology was ‘systems with accidental degeneracy’, going back to Fock and Bargmann [13, 50]. Other terms used in this context were ‘higher symmetries’, or ‘dynamical symmetries’ [51, 52, 121]. Some explicit solutions of the $n$-body problem by Calogero [25–27, 172], dating from the late 1960s were crucial examples for the theory. The technique of coupling constant metamorphosis (CCM) to map between integrable and superintegrable systems was introduced in the mid-1980s [22, 72]. Interest increased greatly due to papers by Evans in about 1990 [45, 46, 48] which contained many examples that generalized fundamental solvable quantum mechanical systems in three dimensions. In about 1995, researchers such as Letourneau and Vinet [116] recognized the very close relationship between quasi-exact solvability (QES) [60, 189, 190] for quantum systems in one dimension and second-order superintegrable systems in two and higher dimensions. Beginning about 2000, the structure theory and classification of second-order superintegrable systems has been largely worked out, with explicit theorems that provide a concrete foundation for the observations made in explicit examples (Daskaloyannis, Kalnins, Kress, Miller, Pogosyan, etc). The quadratic algebras of symmetries of the second-order superintegrable systems, and their representation theory has been studied since about 1992 with results by Zhedanov, Daskaloyannis, Kalnins, Kress, Marquette, Miller, Pogosyan, Post, Vinet, Winternitz, etc. New applications of the theory to other branches of physics are appearing, e.g. work by Quesne on variable mass Hamiltonians [100, 161]. More recently, important examples of physically interesting third- and fourth-order quantum superintegrable systems were announced by Evans and Verrier, and by Rodriguez, Tempesta, and Winternitz [169, 170, 191].

Superintegrable systems of second-order, i.e., classical systems where the defining symmetries are second-order in the momenta and quantum systems where the symmetries are second-order partial differential operators, have been well studied and there is now a developing structure and classification theory. The classification theory for third-order systems that separate in orthogonal coordinate systems, i.e. that also admit a second-order integral, has begun and many new systems have recently been found, including quantum systems with no...
classical analogue and systems with potentials associated with Painlevé transcients. These are quantum systems that could not be obtained by quantizing classical ones. Their quantum limits are sometimes free motion. In other cases, the $\hbar$ going to 0 limit are is singular, in the sense that the quantum potential satisfies a partial differential equation in which the leading terms vanish for $\hbar$ going to 0.

However for general third- and higher-order superintegrable systems much less is known. In particular until very recently there were few examples and almost no structure theory and classification theory.

This situation has changed dramatically with the publication of the 2009 paper 'An infinite family of solvable and integrable quantum systems on a plane' by Tremblay, Turbiner and Winternitz [184, 185]. The authors’ paper had an immediate effect on the active field of classical and quantum superintegrable systems. Their examples and conjectures have led rapidly to new classes of higher-order superintegrable systems, thereby reinvigorating research activity and publications in the subject. The authors introduced a family of both classical and quantum mechanical potentials in the plane, parametrized by the constant $k$, conjectured and gave evidence that these systems were both classically and quantum superintegrable for all rational $k$, with orders that are arbitrarily large. It has now been verified that the conjectures were correct (Kalnins, Kress, Miller, Quesne, Pogosyan [59, 77, 81, 98, 163]). Higher-order superintegrable systems had been thought to be uncommon, but are now seen to be ubiquitous with a clear path to construct families of other candidates at will (e.g. [11, 19, 31, 32, 80, 81, 124, 126, 127, 151, 154, 167]). Tools are being developed for the verification of classical and quantum superintegrability of higher order that can be applied to a variety of Hamiltonian systems. A structure theory for these systems, classification results and applications are following.

This review is focused on the structure and classification of maximal superintegrable systems and their symmetry algebras, classical and quantum. Earlier reviews exist, including those on the group theory of the hydrogen atom and Coulomb problem [12, 43, 107], oscillators [117, 118, 138] and accidental degeneracy or symmetry in general [130, 136].

There are other interesting approaches to the theory that we do not address here. In particular there is a geometrical approach to the classical theory, based on foliations, e.g. [49, 134, 140, 180] or invariant theory of Killing tensors [1, 2]. Many authors approach classical and quantum superintegrable systems from an external point of view. They use elegant techniques such as $R$-matrix theory and coalgebra symmetries to produce superintegrable systems with generators that are embedded in a larger associative algebra with simple structure, such as a Lie enveloping algebra, e.g. [3, 6, 9, 10, 28, 108, 166, 168, 188]. Here we take an internal point of view. The fundamental object for us is the symmetry algebra generated by the system. A useful analogy is differential geometry where a Riemannian space can be considered either as embedded in Euclidean space or as defined intrinsically via a metric.

Let us just list some of the reasons why superintegrable systems are interesting both in classical and quantum physics.

1. In classical mechanics, superintegrability restricts trajectories to an $n - k$ dimensional subspace of phase space ($0 < k < n$). For $k = n - 1$ (maximal superintegrability), this implies that all finite trajectories are closed and motion is periodic [140].
2. At least in principle, the trajectories can be calculated without any calculus.
3. Bertrand’s theorem [17] states that the only spherically symmetric potentials $V(r)$ for which all bounded trajectories are closed are the Coulomb–Kepler system and the harmonic oscillator, hence no other superintegrable systems are spherically symmetric.
4. The algebra of integrals of motion is a non-Abelian and interesting one. Usually it is a finitely generated polynomial algebra, only exceptionally a finite dimensional Lie algebra or Kac–Moody algebra [36].
5. In the special case of quadratic superintegrability (all integrals of motion are at most quadratic polynomials in the moments), integrability is related to separation of variables in the Hamilton–Jacobi equation, or Schrödinger equation, respectively.

6. In quantum mechanics, superintegrability leads to an additional degeneracy of energy levels, sometimes called ‘accidental degeneracy’. The term was coined by Fock [50] and used by Moshinsky and collaborators [117, 135–139], though the point of their studies was to show that this degeneracy is certainly no accident.

7. A conjecture, born out by all known examples, is that all maximally superintegrable systems are exactly solvable [181]. If the conjecture is true, then the energy levels can be calculated algebraically. The wave functions are polynomials (in appropriately chosen variables) multiplied by some gauge factor.

8. The non-Abelian polynomial algebra of integrals of motion provides energy spectra and information on wave functions. Interesting relations exist between superintegrability and supersymmetry in quantum mechanics.

As a comment, let us mention that superintegrability has also been called non-Abelian integrability. From this point of view, infinite dimensional integrable systems (soliton systems) described e.g. by the Korteweg-de-Vries equation, the nonlinear Schrödinger equation, the Kadomtsev–Petviashvili equation, etc. are actually superintegrable. Indeed, the generalized symmetries of these equations form infinite dimensional non-Abelian algebras (the Orlov–Shulman symmetries) with infinite dimensional Abelian subalgebras of commuting flows [145–147].

Before we delve into the specifics of superintegrability theory, we give a simplified version of the requisite mathematics and physics governing Hamiltonian dynamical systems in section 2. Then in section 3 we study, as examples, the 2D Kepler system and the 2D hydrogen atom in detail, both in Euclidean space and on the 2-sphere. The examples illustrate basic features of superintegrability: complete solvability of the systems via the symmetry algebra, important applications to physics, and relation of superintegrable systems via contraction. These systems have been studied literally for centuries, but the pure superintegrability approach has novel features. In section 4 we sketch the structure and classification theory for second-order superintegrable systems, the most tractable class of such systems. Section 5 is devoted to the classification of higher-order systems in 2D Euclidean space, where the quantization problem first becomes serious. In sections 6 and 7 we present examples of higher-order classical and quantum systems and tools for studying their structure. Here great strides have been made but, as yet, there is no classification theory. Section 8 is devoted to the generalized Stäckel transform, an invertible structure preserving transformation of one superintegrable system to another that is basic to the classification theory. Since superintegrability is a concept that distinguishes completely solvable physical systems it should be no surprise that there are profound relations to the theory of special functions. In section 9.1 we make that especially clear by showing that the Askey scheme for orthogonal hypergeometric polynomials can be derived from contractions of 2D second-order superintegrable systems.

2. Background and definitions

2.1. Classical mechanics

The Hamiltonian formalism describes dynamics of a physical system in \( n \) dimensions by relating the time derivatives of the position coordinates and the momenta to a single function on the phase space, the Hamiltonian \( \dot{H} \). A physical system describing the position of a particle at time \( t \) involves \( n \) position coordinates \( q_j(t) \), and \( n \) momentum coordinates, \( p_j(t) \). The phase
space of a physical system is described by points \((p_j, q_j) \in F^{2n}\), where \(F\) is the base field, usually \(\mathbb{R}\) or \(\mathbb{C}\). In its simplest form, the Hamiltonian can be interpreted as the total energy of the system: \(\mathcal{H} = T + V\), where \(T\) and \(V\) are kinetic and potential energy, respectively. Explicitly,

\[
\mathcal{H} = \frac{1}{2m} \sum_{j,k} g^{jk}(q)p_j p_k + V(q)
\]

(2.1)

where \(g^{jk}\) is a contravariant metric tensor on some real or complex Riemannian manifold. That is \(g^{-1} = \det(g^{jk}) \neq 0\), \(g^{jk} = g^{kj}\) and the metric on the manifold is given by \(ds^2 = \sum_{j,k=1}^n g_{jk} dq^j dq^k\) where \((g_{jk})\) is the covariant metric tensor, the matrix inverse to \((g^{jk})\).

Under a local transformation \(q_j' = f_j(q)\) the contravariant tensor and momenta transform according to

\[
(g')^{jk} = \sum_{a,b} \frac{\partial q_j'}{\partial q_a} \frac{\partial q_b}{\partial q_j} g^{ab}, \quad (p')_k = \sum_{j=1}^n \frac{\partial q_j}{\partial q_k} p_j.
\]

(2.2)

Here \(m\) is a scaling parameter that can be interpreted as the mass of the particle. For the Hamiltonian (2.1) the relation between the momenta and the velocities is \(p_j = m \sum_{k=1}^n \delta_{jk} q_k\), so that \(T = \frac{1}{2m} \sum_{j,k} g^{jk}(q)p_j p_k = \frac{m}{2} \sum_{j,k=1}^n g^{jk}(q)q_j q_k\). Once the velocities are given, the momenta are scaled linearly in \(m\). In mechanics the exact value of \(m\) may be important but for mathematical structure calculations it can be scaled to any nonzero value. To make direct contact with mechanics we may set \(m = 1\); for structure calculations we will usually set \(m = 1/2\). The above formulas show how to rescale for differing values of \(m\).

The dynamics of the system are given by Hamilton’s equations, [5, 58]

\[
\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad j = 1, \ldots, n.
\]

(2.3)

Solutions of these equations give the trajectories of the system.

**Definition 1.** The Poisson bracket of two functions \(\mathcal{R}(p, q), \mathcal{S}(p, q)\) on the phase space is the function

\[
\{\mathcal{R}, \mathcal{S}\}(p, q) = \sum_{j=1}^n \left( \frac{\partial \mathcal{R}}{\partial p_j} \frac{\partial \mathcal{S}}{\partial q_j} - \frac{\partial \mathcal{R}}{\partial q_j} \frac{\partial \mathcal{S}}{\partial p_j} \right).
\]

(2.4)

The Poisson bracket obeys the following properties, for \(\mathcal{R}, \mathcal{S}, T\) functions on the phase space and \(a, b\) constants:

\[
\{\mathcal{R}, \mathcal{S}\} = -\{\mathcal{S}, \mathcal{R}\}, \quad \text{anti-symmetry}
\]

(2.5)

\[
\{\mathcal{R}, a\mathcal{S} + bT\} = a\{\mathcal{R}, \mathcal{S}\} + b\{\mathcal{R}, \mathcal{T}\}, \quad \text{bilinearity}
\]

(2.6)

\[
\{\mathcal{R}, \{\mathcal{S}, \mathcal{T}\}\} + \{\mathcal{S}, \{\mathcal{T}, \mathcal{R}\}\} + \{\mathcal{T}, \{\mathcal{R}, \mathcal{S}\}\} = 0, \quad \text{Jacobi identity}
\]

(2.7)

\[
\{\mathcal{R}, \mathcal{S}\mathcal{T}\} = \{\mathcal{R}, \mathcal{S}\}\mathcal{T} + \mathcal{S}\{\mathcal{R}, \mathcal{T}\}, \quad \text{Leibniz rule}
\]

(2.8)

\[
\{f(\mathcal{R}), \mathcal{S}\} = f'(\mathcal{R})\{\mathcal{R}, \mathcal{S}\}, \quad \text{chain rule.}
\]

(2.9)

With \(\delta_{jk}\) the Kronecker delta, coordinates \((q, p)\) satisfy canonical relations

\[
\{p_j, p_k\} = \{q_j, q_k\} = 0, \quad \{p_j, q_k\} = \delta_{jk}.
\]

(2.10)
A trajectory of an integrable system lies on the common intersection of the hypersurfaces

\[ \{P_j, P_k\} = \{Q_j, Q_k\} = 0, \quad \{P_j, Q_k\} = \delta_{jk}, \quad 1 \leq j, k \leq n. \]

Canonical coordinates are true coordinates on the phase space, i.e., they can be inverted locally to express \( q, p \) as functions of \( Q, P \). Furthermore, under this change of coordinates the Poisson bracket (2.4) maintains its form, i.e.,

\[ \{\mathcal{R}, S\}(P, Q) = \sum_{j=1}^{n} \left( \frac{\partial \mathcal{R}}{\partial P_j} \frac{\partial S}{\partial Q_j} - \frac{\partial \mathcal{R}}{\partial Q_j} \frac{\partial S}{\partial P_j} \right). \]

Any coordinate change of the form \( q'_j(q), p'_j(q, p) \), where \( q' \) depends only on \( q \) and \( p' \) is defined by (2.2), is always canonical.

In terms of the Poisson bracket, we can rewrite Hamilton’s equations as

\[ \frac{dq_j}{dt} = \{\mathcal{H}, q_j\}, \quad \frac{dp_k}{dt} = \{\mathcal{H}, p_k\}. \]

(2.11)

For any function \( \mathcal{R}(q, p) \), its dynamics along a trajectory \( q(t), p(t) \) is

\[ \frac{d\mathcal{R}}{dt} = \{\mathcal{H}, \mathcal{R}\}. \]

(2.12)

Thus \( \mathcal{R}(q, p) \) will be constant along a trajectory if and only \( \{\mathcal{H}, \mathcal{R}\} = 0 \).

If \( \{\mathcal{H}, S\} = 0 \), then \( S(q, p) \) is a constant of the motion.

A system with Hamiltonian \( \mathcal{H} \) is integrable if it admits \( n \) constants of the motion \( \mathcal{P}_1 = \mathcal{H}, \mathcal{P}_2, \ldots, \mathcal{P}_n \) that are in involution:

\[ \{\mathcal{P}_j, \mathcal{P}_k\} = 0, \quad 1 \leq j, k \leq n, \]

(2.13)

and are functionally independent.

Now suppose \( \mathcal{H} \) is integrable with associated constants of the motion \( \mathcal{P}_j \). Up to a canonical change of variables it is possible to assume that \( \det \left( \mathcal{P}_j, \mathcal{P}_k \right) \neq 0 \). Then by the inverse function theorem we can solve the \( n \) equations \( \mathcal{P}_j(q, p) = c_j \) for the momenta to obtain \( p_k = p_k(q, c) \), \( k = 1, \ldots, n \), where \( c = (c_1, \ldots, c_n) \) is a vector of constants. For an integrable system, if a particle with position \( q \) lies on the common intersection of the hypersurfaces \( \mathcal{P}_j = c_j \) for constants \( c_j \), then its momentum \( p \) is completely determined. Also, if a particle following a trajectory of an integrable system lies on the common intersection of the hypersurfaces \( \mathcal{P}_j = c_j \) at time \( t_0 \), where the \( \mathcal{P}_j \) are constants of the motion, then it lies on the same common intersection for all \( t \) near \( t_0 \). Considering \( p_j(q, c) \) and using the conditions (2.13) and the chain rule it is straightforward to verify \( \frac{\partial p_j}{\partial \mathcal{Q}_j} = \frac{\partial p_j}{\partial q_j} \). Therefore, there exists a function \( u(q, c) \) such that \( p_j = \frac{\partial u}{\partial \mathcal{Q}_j} \), \( j = 1, \ldots, n \). Note that \( \mathcal{P}_j(q, \frac{\partial u}{\partial q}) = c_j \), and, in particular, \( u \) satisfies the Hamilton–Jacobi equation

\[ \mathcal{H}\left( q, \frac{\partial u}{\partial q} \right) = E. \]

(2.14)
where $E = c_1$. By construction $\det \left( \frac{\partial u}{\partial q_i/\partial c_k} \right) \neq 0$, and such a solution of the Hamilton–Jacobi equation depending nontrivially on $n$ parameters $c$ is called a complete integral. This argument is reversible: a complete integral of (2.14) determines $n$ constants of the motion in involution, $P_1, \ldots, P_n$.

**Theorem 1.** A system is integrable $\iff$ (2.14) admits a complete integral.

A powerful method for demonstrating that a system is integrable is to exhibit a complete integral by using additive separation of variables.

It is a standard result in classical mechanics that for an integrable system one can integrate Hamilton’s equations and obtain the trajectories, [5, 58]. The Hamiltonian formalism is well suited to exploiting symmetries of the system and an important tool in laying the framework for quantum mechanics.

### 2.2. Quantum mechanics

We give a brief introduction to basic principles necessary to understand quantum superintegrable systems. We ignore such issues as the domains of unbounded operators and continuous spectra, and proceed formally. In any case, we will be mainly interested in bound states and their discrete spectra.

In quantum mechanics, physical states are represented as one-dimensional subspaces in a complex, projective Hilbert space: two states are equivalent if they differ by a constant multiplicative factor. A standard Euclidean space model for a quantum mechanical bound state system with $n$ degrees of freedom is the one where the state vectors are complex square integrable functions $\Phi_1(x, t)$ on $\mathbb{R}^n$ and the transition amplitude between two states $\Phi_1, \Psi_1$ is the inner product $\langle \Psi_1, \Phi_1 \rangle = \int_{\mathbb{R}^n} \Psi_1(x, t) \Phi_1(x, t) \, dx$. Usually, states are normalized: $||\Psi||^2 = \langle \Psi, \Psi \rangle = 1$. If $A$ is a self-adjoint operator (observable) then

$$\langle \Psi, A\Phi \rangle = \int_{\mathbb{R}^n} \Psi(x, t) A\Phi(x, t) \, dx = \int_{\mathbb{R}^n} (A\Psi(x))\Phi(x) \, dx = \langle A\Psi, \Phi \rangle.$$  

In this model $q_j \rightarrow X_j = x_j$, i.e., multiplication by Cartesian variable $x_j$, $p_j \rightarrow P_j = -i\hbar \partial_x$. In analogy to $H = \frac{1}{2m} \sum_{j=1}^{n} p_j^2 + V(q)$, we have the quantum Hamiltonian on $n$-dimensional Euclidean space,

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + V(x).$$ (2.15)

Observables correspond to quantities that can be measured. The probability of an arbitrary state, $\Phi$ being measured with a given eigenvalue $\lambda$ is determined by computing its transition amplitude, $|\langle \Psi_\lambda, \Phi \rangle|$, where $\Psi_\lambda$ is an eigenvector of an operator $A$ with eigenvalue $\lambda$. As in classical mechanics, we create most of our observables out of the quantities position and momentum. However, in quantum mechanics these quantities correspond to operators. That is, the self-adjoint operator $X_j$ gives the value of the coordinate $x_j$ while the self-adjoint operator $P_j$ gives the $j$th momentum. We can define a bilinear product on the operators by the commutator $[A, B] = AB - BA$. This product satisfies the same relations (2.5)–(2.8) as the Poisson bracket. The position and momentum operators satisfy the commutation relations,

$$[X_j, X_k] = [P_j, P_k] = 0, \quad [X_j, P_k] = i\hbar \delta_{jk},$$ (2.16)

where $\hbar$ is the reduced Planck constant and $i = \sqrt{-1}$. Equations (2.16) define a Heisenberg algebra $H_n$ in a $n$-dimensional space.
The time evolution of quantum mechanical states is determined by $H$: the dynamics is given by the time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt} \Phi(t) = H \Phi(t). \quad (2.17)$$

Using the Hamiltonian, we can define the time evolution operator $U(t) = \exp(iHt/\hbar)$ as a one parameter group of unitary operators determined by the self-adjoint operator $H$. A state at an arbitrary time $t$ will be given in terms of the state at time 0 by $\Phi(t) = U(t)\Phi(0)$. If a state at time 0 is an eigenvector for the Hamiltonian, say $H\Phi_E(0) = E\Phi_E(0)$, then the time evolution operator acts on an eigenvector by a (time dependent) scalar function, $\Phi(t) = U(t)\Phi_E(0) = e^{iEt/\hbar}\Phi_E(0)$. The eigenvalue problem

$$H\Phi = E\Phi \quad (2.18)$$

is called the time-independent Schrödinger equation. If we can find a basis of eigenvectors for $H$, we can expand an arbitrary vector in terms of this energy basis to get the time evolution of the state. Finding eigenvalues and eigenvectors for Hamiltonians is a fundamental task in quantum mechanics.

The expected value of an observable $A$ in a state $\Phi(t)$ obeys the relation

$$\frac{d}{dt}\langle \Phi, A\Phi \rangle = \frac{1}{\hbar} \langle \Phi, [H, A]\Phi \rangle. \quad (2.19)$$

This is in analogy with the classical relation (2.12). If $[H, A] = 0$, then we can choose a basis for the Hilbert space which is a set of simultaneous eigenfunctions of $A$ and $H$, [192]. It is important to determine a complete set of commuting observables to specify the system. This idea leads naturally to quantum integrability.

**Definition 6.** A quantum mechanical system in $n$ dimensions is integrable if there exist $n$ integrals of motion, $L_j, j = 1, \ldots, n$, that satisfy the following conditions.

- They are well-defined Hermitian operators in the enveloping algebra of the Heisenberg algebra $H_n$ or convergent series in the basis vectors $X_j, P_j, j = 1, \ldots, n$.
- They are algebraically independent in the sense that no Jordan polynomial formed entirely out of anti-commutators in $L_j$ vanishes identically.
- The integrals $L_j$ commute pair-wise.

The extension of these ideas from classical mechanics to quantum mechanics is straightforward, except for functional independence. There is no agreed-upon operator equivalence for this concept. In this review we use algebraic independence of a set of operators $S_1, \ldots, S_n$. We say the operators are algebraically independent if there is no nonzero Jordan polynomial that vanishes identically. That is, there is no symmetrized polynomial $P$ in $\hbar$ non-commuting variables such that $P(S_1, \ldots, S_n) \equiv 0$. By symmetrized we mean that if the monomial $\alpha S_j S_k$ appears in $P$ it occurs in the symmetric form via the anti-commutator $\alpha(S_j S_k + S_k S_j)$. Similarly third-order products are symmetrized sums of six monomials, etc.

If the classical Hamiltonian admits a constant of the motion $S(q, p)$ we also want to find a corresponding operator $S$, expressible in terms of operators $X_j, P_j$, that commutes with $H$. However, in quantum mechanics the position and momentum operators do not commute so the analogy with the function $S$ is not clear. We usually take symmetrized products of any terms with mixed position and momentum, though there are other conventions. The quantization problem of determining suitable operator counterparts to classical Hamiltonians and constants of the motion is very difficult, sometimes impossible to solve but, for many of the simpler superintegrable systems the solution of the quantization problem is unique and straightforward.
More generally we can consider quantum systems on a Riemannian manifold, in analogy with the classical systems (2.1):

\[ H = -\frac{\hbar^2}{2m\sqrt{g}} \sum_{j,k=1}^{n} \partial_{x_j} \left( \sqrt{g} g^{jk} \partial_{x_k} \right) + V(x) = -\frac{\hbar^2}{2m} \Delta_n + V \]  

(2.20)

where \( \Delta_n \) is the Laplace–Beltrami operator on the manifold [34, volume I]. Inner products are computed using the volume measure \( \sqrt{g} \, \text{d}x \). In the special case of Cartesian coordinates in Euclidean space, expression (2.20) reduces to (2.15).

We often consider the time-independent Schrödinger equation \( H \Psi = E \Psi \) for functions \( \Psi(x) \) on complex Riemannian manifolds with complex coordinates \( x_j \). In these cases we are going beyond quantum mechanics and Hilbert spaces and considering formal eigenvalue problems. However many concepts carry over, such as symmetry operators, integrability, superintegrability, relations to special functions, etc. While for direct physical application it is important to specify the mass \( m \) and to retain the Planck constant \( \hbar \), for many mathematical computations we can rescale these constants, without loss of generality. We rewrite the Schrödinger eigenvalue equation as

\[ \Delta_n \Psi(x) - \frac{2m}{\hbar^2} V(x) \Psi(x) = -\frac{2m}{\hbar^2} E \Psi(x) \]  

(2.21)

or \( (\Delta_n + \tilde{V}) \Psi = \tilde{E} \Psi \) where \( \tilde{V}, \tilde{E} \) are rescaled potential and energy eigenvalue, respectively. This is appropriate because many of the superintegrable systems we consider have potentials of the form \( V = \alpha V_0 \) where \( \alpha \) is an arbitrary parameter. Thus, we will make the most convenient choice of \( -\hbar^2/2m \), usually \(+1\) or \(-1/2\), with the understanding that the result can be scaled to reinsert \( \hbar \) and obtain any desired \( m \). In cases where \( V \) does not admit an arbitrary multiplicative factor or the quantization problem cannot be solved we will have to retain the original formulation with \( H \) given by (2.20). We shall see in section 5 that in quantum mechanics integrable and superintegrable systems exist that have no classical counterparts. The limit \( \hbar \to 0 \) corresponds to free motion, or this limit may be singular. This occurs when integrals of motion of order 3 or higher are involved. Thus, for the case of third- and higher-order integrals of motion, it is best to keep \( \hbar \) explicitly in all formulas in order to be able to pass to the classical limit.

2.3. Superintegrability

Let us first focus on the explicit solvability properties of classical systems \( \mathcal{H} = \sum g^{ij} p_i p_j + V \) and operator systems \( H = \Delta_n + V \). For this review, we will be focusing mainly on classical integrable and superintegrable systems that are polynomial in the momenta both for the purposes of quantization as well as simply for the fact that these are the most well studied. We thus define polynomial integrability and superintegrability for classical systems. The quantum analogy of these systems will then be integrals which are finite-order differential operators and hence we define quantum integrability and superintegrability of finite-order. For most of the review will omit the adjective ‘polynomial’ or ‘of finite-order’ and just speak of integrability and superintegrability.

Similarly, we define superintegrability as having more than \( n \) integrals of motion. There is thus a notion of minimal and maximal superintegrability. We will focus in this review on maximal superintegrability and often omit the adjective maximal. Maximal superintegrability requires that there exists \( 2n - 1 \) integrals of motion, one of which (the Hamiltonian) commutes with all of the others. In this sense, maximal superintegrability coincides with non-commutative integrability [134, 140] which requires \( 2n - r \) integrals, \( r \) of which commute with all of the integrals.
2.3.1. Classical superintegrability and the order of integrable systems.

**Definition 7.** A Hamiltonian system is (polynomially) integrable if it is integrable and the constants of the motion are each polynomials in the momenta globally defined (except possibly for singularities on lower dimensional manifolds).

Systems with symmetry beyond polynomial integrability are (polynomially) superintegrable; those with maximum possible symmetry are maximally (polynomially) superintegrable.

**Definition 8.** A classical Hamiltonian system in n dimensions is (polynomially) superintegrable if it admits \( n + k \) with \( k = 1, \ldots, n - 1 \) functionally independent constants of the motion that are polynomial in the momenta and are globally defined except possibly for singularities on a lower dimensional manifold. It is minimally (polynomially) superintegrable if \( k = 1 \) and maximally (polynomially) superintegrable if \( k = n - 1 \).

Every constant of the motion \( S \), polynomial or not, is a solution of the equation \( [H, S] = 0 \) where \( H \) is the Hamiltonian. This is a linear homogeneous first-order partial differential equation for \( S \) in \( 2n \) variables. It is a well-known result that every solution of such equations can be expressed as a function \( F(f_1, \ldots, f_{2n-1}) \) of \( 2n - 1 \) functionally independent solutions, [34]. Thus there always exist \( 2n - 1 \) independent functions, locally defined, in involution with the Hamiltonian, the largest possible number. However, it is rare to find \( 2n - 1 \) such functions that are globally defined and polynomial in the momenta. Thus superintegrable systems are very special.

Most authors require a superintegrable system to be integrable. We have not done so here because we know of no proof (or counter example) that every superintegrable system in our sense is necessarily integrable. It is a theorem that at most \( n \) functionally independent constants of the motion can be in mutual involution, [5]. However, several distinct \( n \)-subsets of the \( 2n - 1 \) polynomial constants of the motion for a superintegrable system could be in involution. In that case the system is *multi-integrable*.

Another feature of superintegrable systems is that the classical orbits traced out by the trajectories can be determined algebraically, without the need for integration. Along any trajectory each of the symmetries is constant: \( L_s = c_s, s = 1, \ldots, 2n - 1 \). Each equation \( L_s(q, p) = c_s \) determines a \((2n - 1)\)-dimensional hypersurface in the \( 2n \)-dimensional phase space, and the trajectory must lie in that hypersurface. Thus the trajectory lies in the common intersection of \( 2n - 1 \) independent hypersurfaces; hence it must be a curve. An important property of real superintegrable systems is loosely stated as ‘all bounded trajectories are periodic’ [182]. The formal proof is in [140].

The polynomial constants of motion for a system with Hamiltonian \( H \) are elements of the (polynomial) Poisson algebra of the system. Stated as a lemma we have:

**Lemma 1.** Let \( H \) be a Hamiltonian with constants of the motion \( L, K \). Then \( \alpha L + \beta K \), \( LK \) and \( \{L, K\} \) are also constants of the motion.

More generally, any set \( F_k \) of \( k \) polynomial constants of the motion \( L_1, \ldots, L_k \) will generate a symmetry algebra \( S_{F_k} \), a subalgebra of \( S_H \), simply by taking all possible finite combinations of scalar multiples, sums, products and Poisson brackets of the generators. (Since \( H \) is always a constant of the motion, we will always require that \( H \) must belong to the symmetry algebra generated by \( F_k \).) We will be particularly interested in finding sets of generators for which \( S_{F_k} = S_H \). The \( n \) defining constants of the motion of a polynomially integrable system do not generate a very interesting symmetry algebra, because all Poisson brackets of the generators vanish. However, for the \( 2n - 1 \) generators of a polynomial superintegrable system the brackets
cannot all vanish and the symmetry algebra has nontrivial structure. We always chose the generators to be irreducible, i.e. they are not products or powers of lower order polynomial integrals.

The \textit{order} \( O(\mathcal{L}) \) of a polynomial constant of the motion \( \mathcal{L} \) is its order as a polynomial in the momenta. (Note that the order is an intrinsic property of a symmetry: it does not change under a transformation from position coordinates \( \mathbf{q} \) to coordinates \( \mathbf{q}^{\prime} \).) Here \( \mathcal{H} \) has order 2. The \textit{order} \( O(F_k) \) of a set of generators \( F_k = \{ \mathcal{L}_1, \ldots, \mathcal{L}_k \} \), is the maximum order of the generators, excluding \( \mathcal{H} \).

Let \( S = S_k \) be a symmetry algebra of a Hamiltonian system generated by the set \( F_k \).

Clearly, many different sets \( F_k \) can generate the same symmetry algebra. Among all these there will be at least one set of generators \( F_{k0}^{(0)} \) for which \( \ell = O(F_{k0}^{(0)}) \) is a minimum. Here \( \ell \) is unique, although \( F_{k0}^{(0)} \) is not. We define the \textit{order} of \( S \) to be \( \ell \).

\textbf{2.3.2. Extension to quantum systems.} The extension of superintegrability to quantum systems is relatively straightforward. We state our definitions for systems determined by Schrödinger operators of the form \((2.20)\) in \( n \) dimensions.

\begin{definition}
A quantum system is \textit{integrable} (of finite-order) if it is integrable and the integrals of motion \( L_k \) are finite-order differential operators.
\end{definition}

\begin{definition}
A quantum system in \( n \) dimensions is superintegrable (of finite-order) if it admits \( n + k, k = 1, \ldots, n \) algebraically independent finite-order partial differential operators \( L_1 = H, \ldots, L_{n+k} \) in the variables \( \mathbf{x} \) globally defined (except for singularities on lower dimensional manifolds), such that \([H, L_j] = 0\). It is minimally superintegrable (of finite-order) if \( k = 1 \) and maximally superintegrable (of finite-order) if \( k = n - 1 \).
\end{definition}

Note that, unlike in the case of classical superintegrability, there is no proof that \( 2n - 1 \) is indeed the maximal number of possible algebraically independent symmetry operators. However, there are no counterexamples known to the authors, that the maximum possible number of algebraically independent symmetry operators for a Hamiltonian of form \((2.20)\) is \( 2n - 1 \).

In analogy with the classical case the symmetry operators for quantum Hamiltonian \( H \) form the symmetry algebra \( S_H \) of the quantum system, closed under scalar multiplication, multiplication and the commutator:

\begin{lemma}
Let \( H \) be a Hamiltonian with symmetries \( L, K \), and \( \alpha, \beta \) be scalars. Then \( \alpha L + \beta K, LK \) and \([L, K]\) are also symmetries.
\end{lemma}

We will use the term symmetries and integrals of motion interchangeably.

Any set \( F_k \) of \( k \) symmetry operators \( L_1, \ldots, L_k \) will generate a symmetry algebra \( S_{F_k} \), a subalgebra of \( S_H \), simply by taking all possible finite combinations of scalar multiples, sums, products and commutators of the generators. (Since \( H \) is always a symmetry, we will always require that \( H \) belongs to the symmetry algebra generated by \( F_k \).) We will be particularly interested in finding sets of generators for which \( S_{F_k} = S_H \). However, for the \( 2n - 1 \) generators of a superintegrable system the commutators cannot all vanish and the symmetry algebra has non-Abelian structure, [192].

The \textit{order} \( O(\mathcal{L}) \) of a symmetry \( \mathcal{L} \) is its order as a linear differential operator. The \textit{order} \( O(F_k) \) of a set of generators \( F_k = \{ L_1 = H, \ldots, L_k \} \), is the maximum order of the generators, excluding \( H \). Let \( S = S_k \) be a symmetry algebra of a Hamiltonian system generated by the set \( F_k \). Many different sets \( F_{k0}^{(0)} \) of symmetry operators can generate the same symmetry algebra. Among all these there will be a set \( F_{k0}^{(0)} \) for which \( \ell = O(F_{k0}^{(0)}) \) is a minimum. We define the \textit{order} of \( S \) to be \( \ell \).
3. Important examples

We present some simple but important examples of superintegrable systems and show, for classical systems, how the trajectories can be determined geometrically (without solving Newton’s or Hamilton’s equations) and, for quantum systems, how the energy spectrum can be determined algebraically (without solving the Schrödinger equation), simply by exploiting the structure of the symmetry algebra. We treat 2D versions of the Kepler and hydrogen atom systems in flat space and in positive constant curvature space and the hydrogen atom in 3D Euclidean space.

3.1. The classical Kepler system

The Kepler problem is a specific case of the two body problem for which one of the bodies is stationary relative to the other and the bodies interact according to an inverse square law. The motion of two isolated bodies satisfies this condition to good approximation if one is significantly more massive than the other. Kepler specifically investigated the motion of the planets around the sun and stated three laws of planetary motion: (1) Planetary orbits are planar ellipses with the Sun positioned at a focus. (2) A planetary orbit sweeps out equal areas in equal time. (3) The square of the period of an orbit is proportional to the cube of the length of the semi-major axis of the ellipse.

We will see that the precise mathematical statements of these laws are recovered simply via superintegrability analysis. Since planetary orbits lie in a plane we can write the Hamiltonian system in 2D Euclidean space with Cartesian coordinates:

\begin{equation}
H = \frac{L}{2} = \frac{1}{2} \left( \frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} \right), \quad \alpha > 0.
\end{equation}

(3.1)

Hamilton’s equations of motion are

\[ \dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = \frac{\alpha q_1}{(q_1^2 + q_2^2)^{3/2}}, \quad \dot{p}_2 = \frac{\alpha q_2}{(q_1^2 + q_2^2)^{3/2}}, \]

leading to Newton’s equations \( \ddot{q} = \frac{\alpha q}{q_1^2 + q_2^2} \). We shall not solve these equations to obtain time dependence of the trajectories, but rather show how superintegrability alone implies Kepler’s laws and determines the orbits. Here \( n = 2 \) and \( 2n - 1 = 3 \) so three constants of the motion are required for superintegrability. Kepler’s second law is a statement of the conservation of angular momentum. The conserved quantity is \( L_2 = q_1 p_2 - q_2 p_1 \) which can be verified by checking that \( \{H, L_2\} = 0 \). Angular momentum is conserved in any Hamiltonian system with potential that depends only on the radial distance \( r = \sqrt{q_1^2 + q_2^2} \). However, the gravitational potential (along with the isotropic oscillator potential) is special in that it admits a third constant of the motion, see the Bertrand theorem, [17, 58]. Consider the 2-vector

\[ \mathbf{e} = (L_3, L_4) = \left( \frac{\alpha q_1}{\sqrt{q_1^2 + q_2^2}}, -\frac{\alpha q_2}{\sqrt{q_1^2 + q_2^2}} \right) \]

(3.2)

in the \( q_1 - q_2 \) plane. One can check that \( \{H, L_3\} = \{H, L_4\} = 0 \), so the components of \( \mathbf{e} \) are constants of the motion and the classical Kepler system is superintegrable. The quantity \( \mathbf{e} \) is called the Laplace–Runge–Lenz vector. We will see that for an elliptical orbit or a hyperbolic trajectory the Laplace–Runge–Lenz vector is directed along the axis formed by the origin \((q_1, q_2) = (0, 0)\) and the perihelion (point of closest approach) of the trajectory to the origin.

The perihelion is time-invariant in the Kepler problem, so the direction in which \( \mathbf{e} \) points must
be a constant of the motion. The length squared of the Laplace vector is determined by the energy and angular momentum:

\[ \mathbf{e} \cdot \mathbf{e} \equiv \mathcal{L}_3^2 + \mathcal{L}_4^2 = 2\mathcal{L}_2^2\mathcal{H} + \alpha^2. \] (3.3)

We have found four constants of the motion and only \(2n - 1 = 3\) can be functionally independent. The functional dependence is given by (3.3). We can use the symmetries, \(\mathcal{L}_1, \ldots, \mathcal{L}_4\) to generate the symmetry algebra. The remaining nonzero Poisson brackets are

\[ \{\mathcal{L}_2, \mathcal{L}_3\} = -\mathcal{L}_4, \quad \{\mathcal{L}_2, \mathcal{L}_4\} = \mathcal{L}_3, \quad \{\mathcal{L}_3, \mathcal{L}_4\} = 2\mathcal{L}_2\mathcal{H}. \] (3.4)

(The first equations show that \(\mathbf{e}\) transforms as a 2-vector under rotations about the origin.) The structure equations do not define a Lie algebra, due to the quadratic term \(\mathcal{L}_2^2\mathcal{H}\). They, together with the Casimir (3.3), define a quadratic algebra, a Lie algebra only if \(\mathcal{H}\) is restricted to a constant energy. An alternative is to consider \(\mathcal{H}\) as a ‘loop parameter’ and then the operators \(\mathcal{L}_i\) generate a twisted Kac–Moody algebra [36, 37].

Now suppose we have a specific solution of Hamilton’s equations with angular momentum \(\mathcal{L}_2 = \ell\) and energy \(\mathcal{H} = E\). Because of the radial symmetry we are free to choose the coordinate axes of the Cartesian coordinates centered at \((0, 0)\) in any orientation we wish. We choose peripatetic coordinates such that the Laplace vector corresponding to the solution is pointed in the direction of the positive \(q_1\)-axis. In these coordinates we have \(\mathcal{L}_4 = 0, \mathcal{L}_3 = e_1 > 0\) and \(e_1^2 = 2\ell^2E + \alpha^2\). Then the first two structure equations simplify to \(p_1 = -\frac{\alpha q_1}{\ell \sqrt{q_1^2 + q_2^2}},\) \(p_2 = \frac{\alpha}{\ell} + \frac{\alpha q_2}{\ell \sqrt{q_1^2 + q_2^2}}\). The original expression for \(\mathcal{L}_2\) allows us to write:

\[ \ell = \frac{q_1e_1}{\ell} + \frac{\alpha q_1^2}{\ell \sqrt{q_1^2 + q_2^2}} + \frac{\alpha q_2^2}{\ell \sqrt{q_1^2 + q_2^2}} = \frac{q_1e_1}{\ell} + \frac{\alpha \sqrt{q_1^2 + q_2^2}}{\ell}, \]

which after rearrangement and squaring becomes

\[ \left(1 - \frac{e_1^2}{\alpha^2}\right)q_1^2 + \frac{2\ell^2e_1}{\alpha^2}q_1 + q_2^2 = \frac{\ell^4}{\alpha^2}. \] (3.5)

As is well known from second year calculus, these are conic sections in the \(q_1 - q_2\) plane; our trajectories are ellipses, parabolas, and hyperbolas, depending on the discriminant of the equation, i.e., depending on the constants of the motion, \(e_1, \ell, E\). There are special cases of circles, stationary points, and straight lines. The three general cases are presented in figure 1.

Returning to Kepler’s laws, the first is now obvious. The only closed trajectories, or orbits, are elliptical. Kepler’s second law is a statement of conservation of angular momentum. Indeed, introduce polar coordinates such that \(q_1 = r \cos \phi, q_2 = r \sin \phi\) note that along the trajectory

\[ \ell = \mathcal{L}_2 = q_1(t)p_2(t) - q_2(t)p_1(t) = q_1 \frac{dq_2}{dt} - q_2 \frac{dq_1}{dt} = r^2 \frac{d\phi}{dt}. \]

The area traced out from time 0 to time \(t\) is \(A(t) = \frac{1}{2} \int_0^t r^2 d\phi = \frac{1}{2} \int_0^t r^2 d\phi\). Differentiating with respect to time: \(\frac{d}{dt}A(t) = \frac{1}{2}r^2 \frac{d\phi}{dt}\), so the rate is constant. Note that Kepler’s third law is only valid for closed trajectories: ellipses. We may write the period \(T\) of such an orbit in terms of the constants of the motion. Explicit evaluation for \(\phi(0) = 0, \phi(T) = 2\pi\), yields \(A(T) = \frac{T^2}{2}\) as the area of the ellipse. Kepler’s third law follows easily from equation (3.5) and the simple calculus expression for the area of an ellipse.
3.2. A Kepler analogue on the 2-sphere

There are analogues of the Kepler problem on spaces of nonzero constant curvature that are also superintegrable. We consider a 2-sphere analogue and show that superintegrability yields information about the trajectories, and that in a particular limit we recover the Euclidean space problem. It is convenient to consider the 2-sphere as a 2D surface embedded in Euclidean 3-space. Let \( s_1, s_2, s_3 \) be standard Cartesian coordinates. Then the equation
\[
s_1^2 + s_2^2 + s_3^2 = 1
\]
defines the unit sphere. The embedding phase space is now six-dimensional with conjugate momenta \( p_1, p_2, p_3 \). The phase space for motion on the 2-sphere will be a four-dimensional (4D) submanifold of this Euclidean phase space. One of the constraints is
\[
(s_1^2 + s_2^2 + s_3^2)^{3/2} = \frac{\alpha s_3}{s_1^2 + s_2^2}.
\]
with \( \alpha < 0 \) and \( J_1 = s_2 p_3 - s_3 p_2 \), where \( J_2, J_3 \) are obtained by cyclic permutations of 1, 2, 3.

If the universe has constant positive curvature, this would be a possible model for planetary motion about the Sun. \[175\]. Note that the \( J_k \) are angular momentum generators, although \( J_1, J_2 \) are not constants of the motion. Due to the embedding, we have
\[
\mathcal{H}' = p_1^2 + p_2^2 + p_3^2 + \frac{\alpha s_3}{(s_1^2 + s_2^2 + s_3^2)^{3/2}} = \mathcal{H} + \frac{(s_1 p_1 + s_2 p_2 + s_3 p_3)^2}{s_1^2 + s_2^2 + s_3^2},
\]
so we can use the usual Euclidean Poisson bracket \( \{\mathcal{F}, \mathcal{G}\} = \sum_{i=1}^{3} (-\partial_i \mathcal{F} \partial_{p_i} \mathcal{G} + \partial_{p_i} \mathcal{F} \partial_i \mathcal{G}) \) for our computations if at the end we restrict to the unit sphere. (Note that we have here normalized the parameters \( m/2 = 1 \).) The Hamilton equations for the trajectories \( s_j(t), p_j(t) \) in phase space are \( \frac{ds_j}{dt} = \{\mathcal{H}, s_j\}, \frac{dp_j}{dt} = \{\mathcal{H}, p_j\}, j = 1, 2, 3. \) The classical basis for the constants is
\[
L_1 = 2J_1 J_3 - \frac{\alpha s_1}{s_1^2 + s_2^2}, \quad L_2 = 2J_2 J_3 - \frac{\alpha s_2}{s_1^2 + s_2^2}, \quad \mathcal{X} = J_3.
\]
The structure and Casimir relations are

\[ \{ \mathcal{X}, L_1 \} = -L_2, \quad \{ \mathcal{X}, L_2 \} = L_1, \quad \{ L_1, L_2 \} = 4(H - 2\lambda^2)\mathcal{X}, \quad (3.9) \]

\[ L_1^2 + L_2^2 + 4\lambda^4 - 4H\lambda^2 - \alpha^2 = 0. \tag{3.10} \]

Here, \((L_1, L_2)\) transforms as a vector with respect to rotations about the \(s_3\)-axis. The Casimir relation expresses the square of the length of this vector in terms of the other constants of the motion: \(L_1^2 + L_2^2 = \kappa^2, \quad \kappa^2 = \alpha^2 + 42\lambda^2 - 4\lambda^4\), where \(\kappa \geq 0\). We choose the \(s_1, s_2, s_3\) coordinate system so that the vector points in the direction of the positive \(s_1\)-axis: \((L_1, L_2) = (\kappa, 0)\). Then

\[ J_1 J_3 = \frac{\alpha s_1}{2\sqrt{s_1^2 + s_2^2}} + \frac{\kappa}{2}, \quad J_2 J_3 = \frac{\alpha s_2}{2\sqrt{s_1^2 + s_2^2}}, \]

\[ J_3 = \mathcal{X}, \quad J_1^2 + J_2^2 + J_3^2 = H - \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}}. \tag{3.11} \]

Substituting the first three equations into the fourth, we obtain:

\[ \left( H\lambda^2 - \left( \frac{\alpha^2}{4} + \frac{\kappa^2}{4} \right) - \lambda^4 \right) (s_1^2 + s_2^2) - \alpha^2 \left( \frac{\kappa s_1}{2} + X^2 s_3 \right)^2 = 0. \tag{3.12} \]

For fixed values of the constants of the motion equation (3.12) describes a cone. Thus the orbit lies on the intersection of this cone with the unit sphere \(s_1^2 + s_2^2 + s_3^2 = 1\), a conic section. This is the spherical geometry analogue of Kepler’s first law. A convenient way to view the trajectories is to project them onto the \((s_1, s_2)\)-plane: \((s_1, s_2, s_3) \rightarrow (s_1, s_2)\). The projected points describe a curve in the unit disc \(s_1^2 + s_2^2 \leq 1\). This curve is defined by

\[ \left[ H\lambda^2 - \left( \frac{\alpha^2}{4} + \frac{\kappa^2}{4} \right) - \lambda^4 \right] (s_1^2 + s_2^2) - \alpha^2 \left( \frac{\kappa s_1}{2} \pm \lambda \sqrt{1 - s_1^2 - s_2^2} \right)^2 = 0. \tag{3.13} \]

The plus sign corresponds to the projection of the trajectory from the northern hemisphere, the minus sign to projection from the southern hemisphere.

Notice that the potential has an attractive singularity at the north pole: \(s_1 = s_2 = 0, s_3 = 1\) and a repulsive singularity at the south pole. In polar coordinates \(s_1 = r\cos \phi, s_2 = r\sin \phi\) the equation of the projected orbit on the \((s_1, s_2)\)-plane is given by

\[ r^2 = \frac{4\lambda^4}{4\lambda^2 + (-\alpha + \kappa \cos \phi)^2}, \quad 0 \leq \phi < 2\pi. \tag{3.14} \]

**Theorem 2.** For nonzero angular momentum, the projection sweeps out area \(A(t)\) in the plane at a constant rate \(\frac{dA}{dt} = \mathcal{X}\) with respect to the origin \((0,0)\).

**Theorem 3.** For nonzero angular momentum, the period of the orbit is

\[ T = \frac{4\lambda^3\pi}{\sqrt{2\sqrt{4\lambda^4 + \alpha^2 + \kappa^2} + 2\lambda}}, \quad (3.15) \]
3.2.1. Contraction to the Euclidean space Kepler problem. The sphere model can be considered as describing a 2D bounded ‘universe’ of radius 1. Suppose an observer is situated ‘near’ the attractive north pole. This observer uses a system of units with unit length ε where 0 < ε ≪ 1. The observer is unable to detect that she is on a 2-sphere; to her the universe appears flat. In her units, the coordinates are

\[ s_1 = \epsilon x, \quad s_2 = \epsilon y, \quad s_3 = 1 + O(\epsilon^2), \]

\[ p_1 = p_x / \epsilon, \quad p_2 = p_y / \epsilon, \quad p_3 = -(xp_x + yp_y) + O(\epsilon^2). \] (3.16)

Here, \( \epsilon^2 \) is so small that to the observer, it appears that the universe is the plane \( s_3 = 1 \) with local Cartesian coordinates \((x, y)\). We define new constants \( \beta, h \) by

\[ \alpha = \beta / \epsilon, \quad \mathcal{H} - \mathcal{X}^2 = h / \epsilon^2. \] (3.17)

Substituting into (3.7), we find \[ h / \epsilon^2 (p_x^2 + p_y^2) + \frac{\beta}{\epsilon^2 \sqrt{x^2 + y^2}} = h / \epsilon^2. \]

Multiplying both sides of this equation by \( \epsilon^2 \), we obtain the Hamiltonian for the Euclidean Kepler system:

\[ p_x^2 + p_y^2 + \frac{\beta}{\sqrt{x^2 + y^2}} = h. \] (3.18)

Using the same procedure we find that the constants of the motion become \( \mathcal{X} = xp_x - yp_y, \mathcal{L}_1 = \frac{2}{\epsilon} \), \( \mathcal{L}_2 = \frac{2}{\epsilon} \), where \((e_1, e_2)\) is the Laplace–Runge–Lenz vector for the Kepler system:

\[ e_1 = -2\mathcal{X} p_x - \frac{\beta x}{\sqrt{x^2 + y^2}}, \quad e_2 = 2\mathcal{X} p_y - \frac{\beta y}{\sqrt{x^2 + y^2}}. \]

The same procedure applied to the structure equations yields

\[ \{\mathcal{X}, e_1\} = -e_2, \quad \{\mathcal{X}, e_2\} = e_1, \quad \{e_1, e_2\} = 4h\mathcal{X}, \quad e_1^2 + e_2^2 - 4h \mathcal{X}^2 - \beta^2 = 0. \]

Thus the length of the Laplace vector is \( k = \kappa / \epsilon \) where \( k^2 = 4h \mathcal{X}^2 + \beta^2 \). To the observer, the trajectories lie in the plane \( s_3 = 1 \) and equation (3.13) for the paths of the trajectories is

\[ \left[ h \mathcal{X}^2 - \left( \frac{\beta^2}{4} + \frac{k^2}{4} \right) \right]^2 (x^2 + y^2) - \beta^2 \left( \frac{kx}{2} + \mathcal{X}^2 \right)^2 = 0. \] (3.19)

The solutions of the Kepler trajectory equations (3.19) are the usual conic sections: intersections of a plane and a cone.

3.2.2. Trajectory determination. To obtain a plot of a trajectory, defined by its constants of the motion and parametrized by time, we can integrate Hamilton’s equations numerically. To do this it is necessary to identify a point in six-dimensional phase space that lies on the trajectory, to serve as an initial point for integration. Thus for each set of constants of the motion we must find a distinguished point.

First we take the case \( \mathcal{X} \neq 0 \). We project the orbits onto the \( s_1 - s_2 \) unit disc. From (3.14), we see that \( r^2 \), is minimized when \( \phi = 0 \). This is the perihelion of the orbit, which implies that \( q_1 = r(0) = \sqrt{\frac{4 \mathcal{X}^2}{4 \mathcal{X}^2 + (-\kappa - \mathcal{X})^2}}, q_2 = 0, p_1 = 0, p_2 = \mathcal{X} / q_1, q_3 = \frac{(-\alpha + \kappa)}{\sqrt{4 \mathcal{X}^2 + (-\kappa - \mathcal{X})^2}} \) and \( p_3 = 0 \).

Note that aphelion occurs at \( q_1 = -r(\pi) = \sqrt{\frac{4 \mathcal{X}^2}{4 \mathcal{X}^2 + (-\kappa - \mathcal{X})^2}}, q_2 = 0, p_1 = 0, p_2 = -\mathcal{X} / q_1 \).

The Casimir relation implies \( \kappa^2 = \alpha^2 + 4(\mathcal{H} - \mathcal{X}^2) \mathcal{X}^2 > 0 \). If \( \mathcal{H} = \mathcal{X}^2 \), then \( \alpha^2 = \kappa^2 \) and \( -\alpha = \kappa \), so the aphelion is precisely unity, while perihelion is less than unity. If \( \mathcal{H} < \mathcal{X}^2 \), we have \( -\alpha < \kappa \), which implies aphelion occurs in the northern hemisphere. If \( \mathcal{H} > \mathcal{X}^2 \), we have \( -\alpha > \kappa \), which implies aphelion occurs in the southern hemisphere.

In the case of zero angular momentum, \( \mathcal{X} = 0 \), (3.11) implies \( s_2 = 0 \), and \( \alpha = \kappa \), so \( \mathcal{H} = \frac{2 \mathcal{X}^2}{\kappa} \). The projection onto the disc is a segment of the line \( s_2 = 0 \), so the motion occurs on a segment of a great circle on the 2-sphere. All trajectories crash into the north pole in finite time.
Figure 2. A hyperbolic type orbit determined by the parameters $q_1 = 2/\sqrt{580}$, $q_2 = 0$, $q_3 = 211/\sqrt{580}$, $p_1 = 0$, $p_2 = \sqrt{580}/2$, $p_3 = 0$, $\alpha = -8$, $\kappa = 16$, $X = 1$. To the left, the orbit on the sphere and the right, the projection of the orbit. Note that the orbit crosses into the southern hemisphere.

Figure 3. An elliptic type orbit determined by the parameters $q_1 = 2/\sqrt{580}$, $q_2 = 0$, $q_3 = 211/\sqrt{580}$, $p_1 = 0$, $p_2 = \sqrt{580}/2$, $p_3 = 0$, $\alpha = -8$, $\kappa = 16$, $X = 1$. To the left, the orbit on the sphere and the right, the projection of the orbit. Note that the orbit is restricted to the northern hemisphere.

**Classification of trajectories (see figures 2–4)**

**Case 1.** The trajectory is contained within the northern hemisphere. We have $r^2 = 4X^2/4X^2 + (-\alpha + \kappa \cos \phi)^2 < 1$ for all $\phi$, and $X^2 > H$.

**Case 2.** The projection has one point of tangency with the circle; the trajectory is contained in the closure of a hemisphere, $\xi = \pm 1$ so $H = X^2$.

**Case 3.** The projection has two points of tangency with the unit circle; the trajectory has points in both hemispheres. We have $H > X^2$.

We correlate these cases with the original Kepler orbits, using (3.19). Dividing both sides of the equation by $\beta^2 k^2$, we write the discriminant as $\Delta = (1 - \frac{\alpha^2}{\kappa^2})(\frac{\alpha}{\kappa}) = (1 - \frac{\alpha^2}{\kappa^2})(\frac{\alpha}{\kappa})$. When $\frac{\alpha^2}{\kappa^2} > 1$, $\Delta < 0$; when $\frac{\alpha^2}{\kappa^2} = 1$, $\Delta = 0$; and when $\frac{\alpha^2}{\kappa^2} < 1$, $\Delta > 0$. Thus cases 1, 2, and 3 are, respectively, analogies of ellipses, parabolas, and hyperbolas in the original Kepler problem.

**3.2.3. The Hohmann transfer on the 2-sphere.** The Hohmann transfer is a rocket maneuver used to place satellites in geocentric orbits about the earth and for interplanetary navigation,
[35]. It is based on the superintegrability of the classical Kepler system and uses impulse maneuvers to change trajectories. For an impulse maneuver the rocket engine is turned on for a very short time but with a powerful thrust. At the instant $t_0$ of firing, the rocket which was on a trajectory with present linear momentum $p^{(0)}$ and position $s^{(0)}$ immediately follows a new trajectory with initial position still $s^{(0)}$ but new linear momentum $p^{(1)}$ at time $t_0$. The change in momentum is $\Delta p = p^{(1)} - p^{(0)}$, and it completely determines the new trajectory of the rocket. This is an idealization, but for very brief rocket firing during a long mission it can be quite accurate and energy efficient. The mass of the rocket will decrease due to the engine firing; however, the mass of the rocket factors out of the orbit equations, so will not effect our calculations.

The Hohmann transfer, proposed by Hohmann in 1925 [35], uses impulse maneuvers to take a rocket from a near-Earth circular orbit to a higher circular orbit with an expenditure of minimum $\Delta(p)$. To explain the basic idea we apply it in the setting of the 2-sphere universe. Suppose we have a rocket with engines turned off that is in a counter-clockwise circular orbit around the north pole. Thus the Laplace vector is zero, i.e., $\kappa^{(0)} = 0$, the angular momentum is $\mathcal{X}^{(0)} > 0$ and the energy $\mathcal{H}^{(0)}$. The projection in the $(s_1, s_2)$-plane is the circle with radius $r^{(0)} = 2(\mathcal{X}^{(0)})^2/\sqrt{4(\mathcal{X}^{(0)})^4 + \alpha^2}$. We use impulse maneuvers to put the rocket in a new counter-clockwise circular orbit farther from the pole. The constants of the motion for the target orbit are given as $\kappa^{(1)} = 0$, $\mathcal{X}^{(1)} > 0$ and the energy $\mathcal{H}^{(1)}$. The projection of the target orbit in the $(s_1, s_2)$-plane is the circle with radius $r^{(1)} = 2(\mathcal{X}^{(1)})^2/\sqrt{4(\mathcal{X}^{(1)})^4 + \alpha^2}$ where $r^{(1)} > r^{(0)}$. We choose periaptic coordinates so that at time $t = 0$ the projection of the rocket orbit is crossing the positive $s_1$-axis at the point $s_1 = r^{(0)}$, $s_2 = 0$. The linear momentum is $p_1 = 0$, $p_2 = \mathcal{X}^{(0)}/r^{(0)}$. (On the 2-sphere we have $s_3 = \alpha/r^{(0)}$, $p_3 = 0$.) At $t = 0$ we fire the rocket motor briefly in a direction tangent to the orbit, to boost the momentum from $p_2$ to $p_2 + \Delta_1(p_2)$. (Here, the phase space coordinates $p_1$, $p_3$, $s_1$, $s_2$, $s_3$ remain unchanged. If $\Delta_1(p_2)$ is appropriately chosen the rocket will now follow an elliptical-type orbit in the $(s_1, s_2)$-plane with equation $r^2 = 4\mathcal{X}^4/(4\mathcal{X}^4 + (-\alpha + \kappa \cos \phi)^2)$ where $\mathcal{X}, \kappa$ are new constants of the motion. We design this orbit such that the perigee radius ($\phi = 0$) is $r_p = r^{(0)}$ and the apogee radius ($\phi = \pi$) is $r_a = r^{(1)}$, so that the ellipse will be tangent to the circular destination orbit at apogee. Thus we require

$$(r^{(0)})^2 = r_p^2 = \frac{4\mathcal{X}^4}{4\mathcal{X}^4 + (-\alpha + \kappa)^2}, \quad (r^{(1)})^2 = r_a^2 = \frac{4\mathcal{X}^4}{4\mathcal{X}^4 + (\alpha + \kappa)^2}.$$
Solving for $\mathcal{X}$ and $\kappa$ in these two equations we find

\[
\kappa = -\frac{\alpha}{r_a^2 - r_p^2} \left( r_a \sqrt{1 - r_p^2} - r_p \sqrt{1 - r_a^2} \right)^2, \quad \mathcal{X}^4 = -\frac{\kappa \alpha r_a^2 r_p^2}{r_a^2 - r_p^2}.
\]

Note that $0 < r_p < r_a \leq 1$. At perigee on the elliptical-type orbit we must have $p_2 = \mathcal{X}/r^{0(0)}$. Thus $\Delta(p_2) = (\mathcal{X} - \mathcal{X}^{(0)})/r^{0(0)}$. The time for the satellite to travel from perigee to apogee is half a period $T/2$ where $T$ is given by (3.15). At apogee we again fire the rocket engine, briefly, to put the satellite in the higher circular orbit. We orient the engine so it fires in direction of the tangent vector of the trajectory. At the instant of firing, $t = T/2$, the linear momentum at apogee is $p_1 = 0$, $p_2 = -\mathcal{X}/r^{(1)}$, $p_3 = 0$ and the momentum vector is tangent to the trajectory. At the instant just after firing the position is the same but the new momentum in the $s_2$-direction is $p_2 + \Delta_3(p_2)$. We require that $\Delta_3(p_2)$ is exactly the change in momentum required to put the satellite in the higher circular orbit, i.e., to make $\kappa^{(1)} = 0$ and the angular momentum equal to $\mathcal{X}^{(1)}$. Thus $p_2 + \Delta_3(p_2) = -\mathcal{X}^{(1)}/r^{(1)}$, so $\Delta_3(p_2) = (\mathcal{X} - \mathcal{X}^{(1)})/r^{(1)}$. The total delta-p is $\Delta_1(p_2) + \Delta_3(p_2)$ which can be shown to be the minimum required to move to the higher circular orbit. An example of the transfer is in figure 5. Similar Hohmann transfers can be designed to change to and from elliptic type and hyperbolic type trajectories.

3.3. Quantum Kepler–Coulomb on the 2-sphere

Our next example is an analogue of the hydrogen atom on the 2-sphere. This system is superintegrable and, just as in section 3.2.1, we show that the Euclidean quantum Kepler–Coulomb system can be obtained as a limit of the 2-sphere system. As in the classical case we

Figure 5. The Hohmann transfer trajectories projected onto the unit disc. Here $\alpha = -1$, $r_p = 0.3$, $r_a = 0.7$, $\kappa = 0.05142$ and $\mathcal{X} = 0.0567$. 

embed the 2-sphere in Euclidean 3-space. We define the Hamiltonian operator for the quantum
Kepler system as

\[ H = \sum_{i=1}^{3} J_i^2 + \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}}. \]

(3.20)

Where the operator \( J_i = s_2 \partial_2 - s_3 \partial_2 \) and \( J_2, J_3 \) are obtained by cyclic permutations of this expression. Our immediate goal is to determine the bound state eigenvalues of the Hamiltonian operator according to the equation \( H \Psi = E \Psi \) for functions \( \Psi(s) \) square integrable on the
2-sphere \( s_1^2 + s_2^2 + s_3^2 = 1 \). This system was first treated in [174] and [178], by different
methods. We compute in Euclidean 3-space with Cartesian coordinates \( s_1, s_2, s_3 \), but at the end we restrict our solutions to the 2-sphere. A basis for the symmetry operators is

\[ L_1 = J_1 J_3 + J_3 J_1 - \frac{\alpha s_1}{\sqrt{s_1^2 + s_2^2}}, \quad L_4 = J_1 J_2 + J_2 J_1 - \frac{\alpha s_1}{\sqrt{s_1^2 + s_2^2}}, \quad X = J_3, \]

along with \( H \). They satisfy the structure relations

\[ [X, L_1] = -L_2, \quad [X, L_3] = L_1, \quad [L_1, L_2] = 4HX - 8X^3 + X, \]
\[ L_1^2 + L_2^2 + 4X^4 - 4HX^2 + H - 5X^2 - \alpha^2 = 0. \]

(3.21)

We define a new basis of symmetry operators \( L_0 = iX, L^+ = L_1 - iL_2, L^- = L_4 + iL_2 \). Here
\( L^\pm \) are raising and lowering operators for the eigenvalues of \( L_0 \), just as in Lie algebra theory.
The structure and Casimir relations become

\[ [L_0, L^\pm] = \pm L^\pm, \quad [L^+, L^-] = 8HL_0 + 16L_0^3 + 2L_0, \]
\[ L^+ L^- - (4HL_0 + 8L_0^3 + L_0) = -4L_0^4 - 4HL_0^2 - H - 5L_0^2 + \alpha^2. \]

(3.22)

(3.23)

Equivalently (3.23) may be written as

\[ L^- L^+ + (4HL_0 + 8L_0^3 + L_0) = -4L_0^4 - 4HL_0^2 - H - 5L_0^2 + \alpha^2. \]

(3.24)

Now let \( V_E \) be the finite-dimensional eigenspace of vectors \( \Psi \) with eigenvalue \( E \). Thus
\( V_E \) is the vector space of all solutions \( \Psi \) of the Schrödinger equation \( H \Psi = E \Psi \). Since
all of the symmetry operators commute with \( H \), each maps the space \( V_E \) into itself. For
example, since \( [H, L_0] = 0 \), if \( \Psi \in V_E \) then \( H(L_0 \Psi) = L_0 H \Psi = EL_0 \Psi \), so \( L_0 \Psi \in V_E \). Suppose \( V_E \) is m-dimensional. Then we can find a basis of \( m \) vectors for \( V_E \) with respect to
which the action of the symmetry operators restricted to the eigenspace is given by \( m \times m \)
matrices. The matrix corresponding to \( H \) is just \( E \) I where \( I \) is the \( m \times m \) unit matrix. The
structure equations are now satisfied in the algebra of matrix products and linear combinations.
We have a representation of the symmetry algebra. Quantum superintegrable systems have
degenerate eigenvalues and the reason for this is closely related to the representations. We can
determine the degeneracies and compute the possible eigenvalues \( E \) through analysis of the
representations.

Here \( L_0 \) is formally self-adjoint and \( L^\pm \) are mutual adjoints. We will find an orthogonal
basis for \( V_E \) consisting of eigenvectors of \( L_0 \). For this basis, the matrix corresponding to \( L_0 \)
will be diagonal. The following property is critical. Let \( \Psi \in V_E \) be an eigenvector of \( L_0 \) with
eigenvalue \( \lambda \).

**Lemma 3.** Either \( L^+ \Psi = 0 \) (the zero vector) or \( L^- \Psi \) is an eigenvector of \( L_0 \) with eigenvalue
\( \lambda + 1 \). Furthermore, either \( L^- \Psi = 0 \) or \( L^+ \Psi \) is an eigenvector of \( L_0 \) with eigenvalue \( \lambda - 1 \).
3.3.1. Contraction to Euclidean space.

As for the classical case, we consider a system of units with unit length mechanical system, \[14, 15, 198\], a field whose study is now under intensive development.

\[ (3 \mu^2 + 22 J. Phys. A: Math. Theor. A) \]

A similar procedure applied to the symmetry operators yields \[E\] eigenspace with energy \(3(E+1)\). As in the classical case, the Hamiltonian \(H\) on the 2-sphere contracts to a Hamiltonian on Euclidean space

\[ h = (\partial_x^2 + \partial_y^2) + \frac{\beta}{\sqrt{x^2 + y^2}}. \]

A similar procedure applied to the symmetry operators yields

\[ \epsilon L_1 = \ell_1 = -\partial_y (x \partial_y - y \partial_x) - (x \partial_y - y \partial_x) \partial_y - \frac{\beta x}{\sqrt{x^2 + y^2}}, \]

\[ \epsilon L_2 = \ell_2 = -\partial_y (x \partial_y - y \partial_x) - (x \partial_y - y \partial_x) \partial_y - \frac{\beta y}{\sqrt{x^2 + y^2}}, \]

\[ \epsilon X = \chi = x \partial_x - y \partial_y. \]
The structure relations contract to
\[ [\hbar, \ell_1] = [\hbar, \ell_2] = [\hbar, \chi] = 0, \quad [\chi, \ell_1] = -\ell_2, \quad [\chi, \ell_2] = \ell_1, \]
\[ [\ell_1, \ell_2] = 4h\chi + \chi, \quad \ell_1^2 + \ell_2^2 + h - 5\chi^2 - \beta^2 = 0. \quad (3.31) \]

Setting \( \ell_0 = i\chi \), \( \ell^+ = \ell_1 - i\ell_2 \), and \( \ell^- = \ell_1 + i\ell_2 \), we obtain:
\[ [\hbar, \ell_0] = [\hbar, \ell^+] = [\hbar, \ell^-] = 0, \quad [\ell_0, \ell^\pm] = \pm \ell^\pm, \quad [\ell^+, \ell^-] = 2(4h + 1)\ell_0. \quad (3.32) \]

With the rescaling (3.17), the energy spectrum (3.28) becomes \( E_n = -\frac{\beta^2}{4}(n + 1)^2 + \frac{c^2}{4} - \frac{\beta^2}{4} \). Taking the limit as \( \epsilon \to 0 \), we recover the spectrum of the hydrogen atom confined to a plane,
\[ E_n = -\frac{\beta^2}{(n + 1)^2}. \quad (3.33) \]

3.4. Quantum Coulomb problem in Euclidean space \( E_3 \)

The Schrödinger equation for the hydrogen atom is solved in detail in every textbook on quantum mechanics. Here we shall briefly review its superintegrability [13, 50, 148]. The quantum Hamiltonian
\[ H = (\vec{p})^2 - \frac{\alpha}{r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \alpha > 0, \quad (3.34) \]
commutes with the angular momentum operators
\[ \vec{L} = \vec{r} \times \vec{p} \quad (3.35) \]
and the Laplace–Runge–Lenz vector
\[ \vec{A} = \vec{p} \times \vec{L} - \vec{L} \times \vec{p} - \frac{\alpha}{r} \vec{r}. \quad (3.36) \]
The commutation relations between these operators are
\[ [\vec{L}, H] = [\vec{A}, H] = 0, \quad (3.37) \]
\[ [L_j, L_k] = i\epsilon_{jkl} L_l, \quad [L_j, A_k] = i\epsilon_{jkl} A_l, \quad [A_j, A_k] = -i\epsilon_{jkl} L_l H, \quad (3.38) \]
where \( \epsilon_{jkl} \) is the completely antisymmetric tensor. Thus the angular momentum components \( L_j \) generate an \( o(3) \) algebra and the Laplace–Runge–Lenz vector transforms like a vector under rotations. However, (3.38) shows that the integrals of motion \([L_j, A_k]\) do not strictly speaking form a finite-dimensional Lie algebra. The standard procedure [13, 50, 148] is to restrict to the case of a fixed energy, i.e. put \( H = E \) where \( E \) is a fixed constant. For \( E < 0 \) (bound states) the relations (3.38) correspond to the Lie algebra \( o(4) \). For \( E > 0 \) (scattering states), we obtain the \( o(3, 1) \) Lie algebra. For \( E = 0 \), also in the continuous spectrum, we have the Euclidean algebra \( e(3) \). A different possibility, the one adopted in this review, is to view \([\vec{L}, \vec{A}, H]\) as the generators of a quadratic algebra. Finally, we mention the possibility of considering \( H \) as a ‘loop parameter’ as mentioned above. The relation (3.38) then leads to an infinite-dimensional Lie algebra, namely a twisted centerless Kac–Moody algebra [36, 37]. In any case, the representations of each of these algebras lead to the same formula for the spectrum of the hydrogen atom, namely the famous Balmer formula,
\[ E_n = -\frac{2me^4}{\hbar^2} \frac{1}{n^2}. \quad (3.39) \]
An interesting historical footnote is that Pauli in his famous 1926 paper [148] uses precisely the formulas (3.38) to derive the Balmer formula (3.39). He calls \( \vec{A} \) the Lenz vector and does not mention the group \( O(4) \) nor the algebra \( o(4) \) explicitly. That was done about 16 years later by Fock and Bargmann [13, 50]. Pauli obtained this result before the Schrödinger equation was known [173] using only the algebra, no calculus.
4. Second-order systems

The first papers to consider systematically what would now be called superintegrability began with a search for systems that admitted ‘dynamical symmetries’ of order 2, connected with separation of variables [51, 52, 121]. The structure and classification of second-order superintegrable systems is the best understood area of the theory. In this section, we discuss the classification of second-order superintegrable systems in 2D and 3D and give an indication in the final section how such an analysis could be extended into nD.

4.1. Second-order superintegrability in 2D

4.1.1. Determining equations, classical/quantum isomorphism. We start with the determining equations (Killing and Bertrand–Darboux) for second-order integrals of motion for Hamiltonians in two-dimensions. We show that there exists an isomorphism between classical and quantum superintegrable systems.

Consider a general Hamiltonian with a scalar potential defined on a manifold written in conformal coordinates so that \( ds^2 = \lambda(x, y) (dx^2 + dy^2) \). (Since all 2D manifolds are conformally flat there always exist such Cartesian-like coordinates \((x, y)\) and it is convenient to work with these.) The corresponding classical Hamiltonian takes the form

\[
H = \frac{1}{\lambda(x, y)} (p_x^2 + p_y^2) + V(x, y), \quad (x, y) = (x_1, x_2).
\]  

(4.1)

Since \( H \) is an even function of the momenta, any function on phase space which Poisson commutes with it can always be chosen to be an even or odd function of the momenta (hence no first-order terms). Thus, we can express any second-order integral of the motion as

\[
L = \sum_{i, j=1, 2} a^{ij}(x, y) p_j p_i + W(x, y).
\]  

(4.2)

The conditions that \( \{ L, H \} = 0 \) are equivalent to the *Killing equations*

\[
a^{ii} = -\frac{\lambda_1}{\lambda} a^{i1} - \frac{\lambda_2}{\lambda} a^{i2}, \quad i = 1, 2
\]  

(4.3)

\[
2a^{ij} + a^{ji} = -\frac{\lambda_1}{\lambda} a^{i1} - \frac{\lambda_2}{\lambda} a^{i2}, \quad i, j = 1, 2, \quad i \neq j.
\]

The subscripts denote partial derivatives. The determining equations for the functions \( W_j \) are

\[
W_j = \sum_{k=1}^2 \lambda(x, y) a^{jk} V_k,
\]  

(4.4)

with compatibility condition, \( \partial_1 W_2 = \partial_2 W_1 \), given by the *Bertrand–Darboux equation*

\[
(V_{22} - V_{11}) a^{12} + V_{12} (a^{11} - a^{22}) = \left[ \frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda} \right] V_1 + \left[ \frac{(\lambda a^{22})_1 - (\lambda a^{21})_2}{\lambda} \right] V_2.
\]  

(4.5)

Now, consider the determining equations for a second-order integral of a quantum Hamiltonian \( H \)

\[
H = \frac{1}{\lambda(x, y)} (p_x^2 + p_y^2) + V(x, y),
\]  

(4.6)

where in this case \( p_i \) are the conjugate momenta represented as \( p_i = -i\hbar \partial_i \). Analogously the most general form for a formally self-adjoint second-order differential operator which commutes with \( H \) is given by a sum of even terms, properly symmetrized

\[
L = \sum_{i, j=1, 2} \frac{1}{2} [a^{ij}(x, y), p_j p_i] + W(x, y).
\]  

(4.7)
It is a direct calculation to show that the commutation relation $[H, L] = 0$ holds whenever the Killing equations (4.3) and the Bertrand–Darboux equations (4.5) hold. Thus, we have proven the following theorem.

**Theorem 4.** If $L$ (4.2) is a second-order integral for a classical Hamiltonian $\mathcal{H}$ (4.1) then the second-order differential operator $L$ (4.7) commutes with the quantum Hamiltonian $H$ (4.6). In particular, if $\mathcal{H}$ (4.1) is superintegrable for a fixed potential $V$, then $H$ (4.6) will also be superintegrable for the same potential.

If the Hamiltonian is superintegrable there exist two second-order constants in involution with $\mathcal{H}$ and we get two B-D equations. These two equations can be used to obtain fundamental PDEs for the potential [86]:

$$
V_{22} - V_{11} = A^{22}(x)V_1 + B^{22}(x)V_2, \quad V_{12} = A^{12}(x)V_1 + B^{12}(x)V_2.
$$

(4.8)

If the integrability conditions for the PDEs (4.8) are satisfied identically, we say that the potential is nondegenerate. That means, at each regular point $x_0$ where the $A^{ij}, B^{ij}$ are defined and analytic, we can prescribe the values of $V, V_1, V_2$ and $V_{11}$ arbitrarily and there will exist a unique potential $V(x)$ with these values at $x_0$.

Nondegenerate potentials depend on these three parameters, not including the trivial additive parameter. Degenerate potentials depend on less than three parameters. It has been shown that all second-order superintegrable degenerate potentials in 2D are restrictions of nondegenerate ones, although they may admit an additional, first-order integral associated with group symmetry. In this case, the minimal generating set changes along with the structure of the algebra generated by the integrals [84].

### 4.1.2. Classification of nondegenerate systems, proof of the existence of quadratic algebras.

The structure theory for nondegenerate systems has been worked out by Kalnins et al [86, 87, 95, 96]. Assume that the basis symmetries take the form

$$
\mathcal{L}_i = \sum_{j,k=1,2} a^{ik}_{(j)}(x)p_j p_k + W^i(x).
$$

**Definition 11.** A set of symmetries $\{\mathcal{L}_i, i = 1, \ldots, \ell\}$ is said to be functionally linearly dependent if there exist $c_i(x), not identically 0, such that

$$
\sum_{i=1}^{\ell} c_i(x) \sum_{j,k=1}^{2} a^{ik}_{(j)}(x)p_j p_k = 0.
$$

The set is functionally linearly independent if it is not functionally linearly dependent.

This functional linear independence criterion splits superintegrable systems of all orders into two classes with different properties. In two-dimensions, there is only one functionally linearly dependent superintegrable system, namely $\mathcal{H} = p_x p_x + V(z)$, where $V(z)$ is an arbitrary function of $z$ alone. This system separates in only one set of coordinates $(z, \bar{z}) = (x+iy, x-iy)$. For functionally linearly independent 2D systems the theory is much more interesting. These are exactly the systems which separate in more than one set of orthogonal coordinates. From [86] we have the following results.

**Theorem 5.** Let $\mathcal{H}$ be the Hamiltonian of a 2D superintegrable functionally linearly independent system with nondegenerate potential.

- The space of second-order constants of the motion is three-dimensional.
- The space of third-order constants of the motion is one-dimensional.
- The space of fourth-order constants of the motion is six-dimensional.
The space of sixth-order constants is ten-dimensional (10D).

These results follow from a study of the integrability conditions for constants of the motion that gives these numbers as the upper bounds of the dimensions. To prove that the bounds are achieved we use the result:

**Theorem 6.** Let $K$ be a third-order constant of the motion for a superintegrable system with nondegenerate potential $V$:

$$K = \sum_{k,j=1}^{2} a^{kji}(x,y) p_k p_j p_i + \sum_{\ell=1}^{2} b^{\ell}(x,y)p_{\ell}.$$  

Then $b^{\ell}(x,y) = \sum_{j=1}^{2} f^{\ell j}(x,y) \frac{\partial V}{\partial x_j}(x,y)$ with $f^{\ell j} + f^{j\ell} = 0$, $1 \leq \ell, j \leq 2$. The $a^{ijk}$, $b^{\ell}$ are uniquely determined by the quantity $f^{1\ell}(x_0, y_0)$ at any regular point $(x_0, y_0)$ of $V$.

This follows from a direct computation of the Poisson bracket $\{\mathcal{H}, K\}$ using the B-D relations for the $a^{jk}$. Since $\mathcal{L}_1, \mathcal{L}_2$ is a nonzero third-order constant of the motion we can see that the number (1) is achieved and we can solve for the coefficients $a^{ijk}(x, y)$ and $b^{\ell}$. This result enables us to choose standard bases for second- and higher-order symmetries. Indeed, given any $2 \times 2$ symmetric matrix $A_{00}$, any regular point $(x_0, y_0)$ there exists one and only one second-order symmetry (or constant of the motion) $\mathcal{L} = a^{ijk}(x, y) p_k p_j + W(x, y)$ such that the matrix given by $a^{ijk}(x_0, y_0)$ is equal to $A_{00}$ and $W(x_0, y_0) = 0$. Further, if $\mathcal{L}_\ell = \sum a^{ijk}_\ell p_k p_j + W(\ell)$, $\ell = 1, 2$ are second-order constants of the motion and $A_{i0}(x, y) = \{a^{ijk}_i(x, y)\}$, $i = 1, 2$ are $2 \times 2$ symmetric matrix functions, then the Poisson bracket of these symmetries is given by

$$\{\mathcal{L}_1, \mathcal{L}_2\} = \sum_{k,j=1}^{2} a^{kji}(x,y) p_k p_j p_i + b^{\ell}(x,y)p_{\ell}.$$  

Using this relation we can solve for $f^{k\ell}$:

$$f^{k\ell} = 2\lambda \sum_j \left(\frac{a^{kj}_j a^{\ell j}^{(2)}}{a^{kj}_j a^{(1)}} - \frac{a^{kj}_j a^{\ell j}^{(1)}}{a^{kj}_j a^{(2)}}\right).$$  

Thus $\mathcal{L}_1, \mathcal{L}_2$ is uniquely determined by the skew-symmetric matrix

$$[A_{22}, A_{11}] = A_{22} - A_{11},$$  

hence by the constant matrix $[A_{22}(x_0, y_0), A_{11}(x_0, y_0)]$ evaluated at a regular point, by the Taylor series generated from the Killing equations (4.3) and the Bertrand–Darboux equations (4.5). This allows a standard structure by exploiting identification of the space of second-order constants of the motion with the space of $2 \times 2$ symmetric matrices and identification of third-order constants of the motion with the space of $2 \times 2$ skew-symmetric matrices.

Indeed given a basis for the 3D space of symmetric matrices, $\{A^{ij}\}$, we can define a standard set of basis symmetries $S_{ij\ell} = \sum a^{ij}(x) p_j p_i + W^{ij\ell}(x)$ corresponding to a regular point $x_0$. The third-order symmetry is defined by the commutator of the basis elements; these are necessarily skew-symmetric and hence the identification with $K$ with the skew-symmetric matrix. The operators $S_{ij\ell}$ form an alternative basis to basis $\mathcal{L}_4$ for second-order integrals. An intuitive choice of basis for the 3D space of symmetric matrices is

$$\{A^{ij}\} \in \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(4.9) corresponding to the basis of symmetries $\{S_{11}, S_{22}, S_{12}\}$. Notice, the identity element

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

corresponds to the Hamiltonian.

The following theorems show the existence of the quadratic algebra [86].
**Theorem 7.** The six distinct monomials
\[
(\mathcal{S}_{(11)})^2, (\mathcal{S}_{(22)})^2, (\mathcal{S}_{(12)})^2, \mathcal{S}_{(11)}\mathcal{S}_{(22)}, \mathcal{S}_{(12)}\mathcal{S}_{(21)}, \mathcal{S}_{(11)}\mathcal{S}_{(12)}, \mathcal{S}_{(12)}\mathcal{S}_{(22)},
\]
form a basis for the space of fourth-order symmetries.

We note that since \( \mathcal{R} = \{ \mathcal{L}_1, \mathcal{L}_2 \} \) is a basis for the third-order symmetries, \( \{ \mathcal{L}_i, \mathcal{R} \} \) are of fourth-order, so must be polynomials in the second-order symmetries.

**Theorem 8.** The ten distinct monomials
\[
(\mathcal{S}_{(ii)})^3, (\mathcal{S}_{(ij)})^3, (\mathcal{S}_{(ii)})^2\mathcal{S}_{(jj)}, (\mathcal{S}_{(ij)})^2\mathcal{S}_{(jj)}, (\mathcal{S}_{(ii)})^2, (\mathcal{S}_{(ij)})^2, \mathcal{S}_{(ii)}\mathcal{S}_{(12)}\mathcal{S}_{(22)},
\]
for \( i, j = 1, 2, i \neq j \) form a basis for the space of sixth-order symmetries.

Again, since \( \mathcal{R}^2 \) is a sixth-order constant we have immediately the existence of the Casimir relation which forces the algebra to close at order 6. This is a remarkable property of superintegrable systems and gives us finitely generated quadratic algebras. The analogous results for fifth-order symmetries follow directly from the Jacobi identity.

### 4.1.3. Stäckel transform: proof of constant curvature.

The Stäckel transform of a superintegrable system takes it to a new superintegrable system on a different space. The basic observation is, if we have a Hamiltonian \( \mathcal{H} \) and symmetry \( \mathcal{L} \), which separates as
\[
\mathcal{H} = \mathcal{H}_0 + \alpha V_0 + \tilde{\alpha} \quad \mathcal{L} = \mathcal{L}_0 + \alpha W_0
\]
with \( \{ \mathcal{H}_0, \mathcal{L}_0 \} = [\mathcal{H}, \mathcal{L}] = 0 \) then we have \( \{ \tilde{\mathcal{H}}, \tilde{\mathcal{L}} \} = 0 \) where
\[
\tilde{\mathcal{H}} = \frac{\mathcal{H}_0}{V_0} \quad \tilde{\mathcal{L}} = \mathcal{L}_0 - W_0 \tilde{\mathcal{H}}.
\]

After the transformation, the energy of the Hamiltonian \( \tilde{\mathcal{H}} \) is \( -\alpha \) and the original Hamiltonian is restricted to \( \tilde{\alpha} \). In this sense, the coupling constant \( \alpha \) has been exchanged with the energy, which can be represented by \( -\tilde{\alpha} \), and the Stäckel transform coincides with CCM \([72]\) for second-order superintegrable systems. See \([150]\) for more information on the two transformations.

We explain the process in greater detail in the case of a 2D classical system; for notational purposes, we denote \( \mathcal{H}_0 = \frac{1}{\lambda} (p^2 + p_y^2) \) and \( \mathcal{L}_0 = \sum a^i p_i p_j \). Suppose we have a nondegenerate superintegrable system
\[
\mathcal{H} = \frac{1}{\lambda(x, y)} (p_x^2 + p_y^2) + V(x, y) = \mathcal{H}_0 + V + \alpha U + \tilde{\alpha}
\]
where the potential \( V(x, y) \) as well as the separated potential piece \( U \) satisfy the differential equations \( (4.8) \). We transform to another superintegrable system \( \tilde{\mathcal{H}} \) via
\[
\tilde{\lambda} = \lambda U, \quad \tilde{V} = V/U.
\]

We then separate a complementary part of a symmetry \( \mathcal{L} \) as \( \mathcal{L}_U = \mathcal{L} - W_0 = \mathcal{L}_0 + W_0 \) where \( \{ \mathcal{L}_U, \mathcal{H}_0 + U \} = 0 \). Under this transformation, \( \tilde{\mathcal{L}} = \mathcal{L} - \frac{W_0}{\lambda} \mathcal{H} \) is in involution with \( \tilde{\mathcal{H}} \). Notice that our new superintegrable system is defined on a new manifold with metric \( \tilde{\lambda} = \lambda U \).

Although the new superintegrable system is on a (possibly) different manifold, the quadratic algebra remains unchanged except for the ordering of parameters. Indeed for a quadratic algebra as
\[
\mathcal{R}_2 = F(\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}, \alpha, \beta, \gamma), \quad \{ \mathcal{L}_i, \mathcal{R} \} = G_i(\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}, \alpha, \beta, \gamma)
\]
we have the transformed algebra
\[
\tilde{\mathcal{R}}_2 = F(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\alpha}, -\tilde{\mathcal{H}}, \beta, \gamma), \quad \{ \tilde{\mathcal{L}}_i, \tilde{\mathcal{R}} \} = G_i(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\alpha}, -\tilde{\mathcal{H}}, \beta, \gamma).
\]
Thus, it seems natural to view the quadratic algebra as the defining characteristic of a superintegrable system. Since the St"ackel transform is invertible, we can use it to classify equivalence classes whose defining characteristic will be the form of the quadratic algebra.

In 2D every superintegrable system is St"ackel equivalent to one on a constant curvature manifold. This was shown in detail in [87] whose results we summarize here. A basic fact is that given a Hamiltonian $\mathcal{H}$ with $\lambda(x, y)$ the metric, the differential equations for the leading order terms of the integral $\mathcal{L}$ (4.3) are compatible whenever

$$(\lambda_{22} - \lambda_{11})a_{12}^2 - \lambda_{12}(a_{22}^2 - a_{11}^2) = 3\lambda_1 a_{11}^2 - 3\lambda_2 a_{22}^2 + (a_{11}^2 - a_{22}^2)\lambda.$$  

(4.11)

Since the St"ackel transform of $\mathcal{L}$ does not change the function $a_{12}$ nor the difference $a_{22} - a_{11}$, any Hamiltonian which is St"ackel equivalent to a Hamiltonian $\mathcal{H}$ with metric $\lambda$, has a metric $\mu$ which also satisfies the (4.11). Thus, a study of the possible solutions for the metrics gives all the possible spaces which admit superintegrable systems. These are classified by the following theorem.

**Theorem 9.** If $ds^2 = \lambda(dx^2 + dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates $x$, $y$ such that $\lambda_{12} = 0$) then $\lambda = \mu$ is a solution of the system

$$\mu_{12} = 0, \quad \mu_{22} - \mu_{11} = 3\mu_1 (\ln a_{12}^2) - 3\mu_2 (\ln a_{22}^2) + \left(\frac{a_{11}^2 - a_{22}^2}{a_{12}^2}\right)\mu,$$

(4.12)

where either

I) $a_{12} = X(x)Y(y), \quad X'' = a^2 X, \quad Y'' = -a^2 Y,$

or

II) $a_{12} = \frac{2X'(x)Y'(y)}{C(X(x) + Y(y))^2},$

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y),$$

$$F(X) = \frac{a}{24} X^4 + \frac{\gamma_1}{6} X^3 + \frac{\gamma_2}{2} X^2 + \gamma_3 X + \gamma_4,$$

$$G(Y) = -\frac{a}{24} Y^4 + \frac{\gamma_1}{6} Y^3 - \frac{\gamma_2}{2} Y^2 + \gamma_3 Y - \gamma_4.$$  

Conversely, every solution $\lambda$ of one of these systems defines a nondegenerate superintegrable system. If $\lambda$ is a solution then the remaining solutions $\mu$ are exactly the nondegenerate superintegrable systems that are St"ackel equivalent to $\lambda$. Furthermore, if $\mathcal{V}$ is the potential for the superintegrable system with metric $\lambda$, then $\mathcal{V} = \lambda \mathcal{V}/\mu$ is the potential for the St"ackel equivalent system with metric $\mu$.

We note that these spaces are exactly the spaces classified by Koenigs in [111] where he identified the spaces which admit more that one second-order Killing tensor. As in his paper we write the requirements on $\mu$ and $a^\alpha$ as

$$a_{11}^2 + a_{22}^2 = 0, \quad \mu_{12} = 0, \quad a_{12}(\mu_{11} - \mu_{22}) + 3\mu_1 a_{11}^2 - 3\mu_2 a_{22}^2 + (a_{11}^2 - a_{22}^2)\mu = 0.$$

(4.13)

We can then see by direct computation the following theorem.

**Lemma 4.** Suppose $\mu = \lambda(x, y)$, $a_{12} = a(x, y)$ satisfy (4.13). Then $\mu = \tilde{a}(x, y)$, $a_{12} = \tilde{\lambda}(x, y)$ also satisfy (4.13) where

$$\tilde{a}(x, y) = a(x + iy, i(x - y)), \quad \tilde{\lambda}(x, y) = \lambda(x - iy, y - ix).$$

This transformation is invertible.
Using this lemma, we obtain the following theorem.

**Theorem 10.** System (4.13) characterizes a nondegenerate superintegrable system if and only if the metric $a^{12}(x, y)$ is of constant curvature. Equivalently, the system (4.13) characterizes a nondegenerate superintegrable system if and only if the symmetry $a^{12}$ is the image $a^{12} = \lambda$, where the metric $\lambda$ is of constant curvature (i.e. $\lambda_{12} = 0$).

**Proof.** System (4.13) characterizes a nondegenerate superintegrable system if and only if the symmetry $a^{12}$ satisfies the Liouville equation $(\ln a^{12})_2 = Ca^{12}$ for some constant $C$. (If $C = 0$ we have Case I, and if $C \neq 0$ we have Case II.) It is straightforward to check that this means that

$$\frac{a_{11}^{12} + a_{22}^{12}}{(\partial a^{12})^2} = \left(\frac{a_{11}^{12}}{a^{12}}\right)^2 + \left(\frac{a_{22}^{12}}{a^{12}}\right)^2 = 4iC,$$

so the scalar curvature of metric $\tilde{a}^{12}(dx^2 + dy^2)$ is constant. Similarly, if $\lambda$ is of constant curvature then $\lambda$ satisfies Liouville’s equation. \qed

**Theorem 11.** Every nondegenerate superintegrable 2D system is Stäckel equivalent to a superintegrable system on a constant curvature space.

This theorem greatly simplifies the task of finding superintegrable systems in 2D; we can restrict to the complex plane and the 2-sphere. Furthermore, we can use the symmetries for these spaces to find the symmetries for the superintegrable potential. We give examples in the following section.

### 4.1.4. The list

We fix some notation. Let $s_1^2 + s_2^2 + s_3^2 = 1$ be the embedding of the 2-sphere in 3D Euclidean space and $z = x + iy, \bar{z} = x - iy$. Define $J_1 = s_1 \partial_z - s_2 \partial_{\bar{z}}$ to be the generator of rotation about the $s_3$-axis, with $J_2, J_3$ obtained by cyclic permutation. On the Euclidean plane, we shall also use $J_3$ to denote the generator of rotations in the $x, y$ plane. Always $R = [L_1, L_2]$.

We use the notation for the constant curvature superintegrable systems found in [85]: $Ek$ is the $k$th complex Euclidean system on the list and $Sk$ is the $k$th system on the complex sphere.

All of these systems have the remarkable property that the symmetry algebras generated by $H, L_1, L_2$ for nondegenerate potentials close under commutation. Define the third-order commutation $R$ by $R = [L_1, L_2]$. Then the fourth-order operators $[R, L_1], [R, L_2]$ are contained in the associative algebra of symmetrized products of the generators:

$$[L_j, R] = \sum_{0 \leq \epsilon_1 + \epsilon_2 + \epsilon_3 \leq 2} M_{\epsilon_1, \epsilon_2, \epsilon_3}^{(j)} \{L_1^{\epsilon_1}, L_2^{\epsilon_2}\} H^{\epsilon_3}, \quad \epsilon_k \geq 0,$$

where $[L_1, L_2] = L_1 L_2 + L_2 L_1$ is the anti-commutator. The sixth-order operator $R^2$ is contained in the algebra of symmetrized products to third-order:

$$R^2 = \sum_{0 \leq \epsilon_1 + \epsilon_2 + \epsilon_3 \leq 3} N_{\epsilon_1, \epsilon_2, \epsilon_3} \{L_1^{\epsilon_1}, L_2^{\epsilon_2}\} H^{\epsilon_3} = 0. \quad (4.14)$$

In both equations the constants $M_{\epsilon_1, \epsilon_2, \epsilon_3}^{(j)}$ and $N_{\epsilon_1, \epsilon_2, \epsilon_3}$ are polynomials in the parameters $a_1, a_2, a_3$ of degree $2 - \epsilon_1 - \epsilon_2 - \epsilon_3$ and $3 - \epsilon_1 - \epsilon_2 - \epsilon_3$ respectively.

For systems with one-parameter potentials the situation is different [84]. There are four generators: one first-order $X$ and three second-order $H, L_1, L_2$. The commutators $[X, L_1], [X, L_2]$ are second-order and can be expressed as

$$[X, L_j] = \sum_{0 \leq \epsilon_1 + \epsilon_2 + \epsilon_3 \leq 1} P_{\epsilon_1, \epsilon_2, \epsilon_3}^{(j)} \{L_1^{\epsilon_1}, L_2^{\epsilon_2}, X^{2\epsilon_3}\} H^{\epsilon_3}, \quad j = 1, 2. \quad (4.15)$$
where \( [L_1^0, L_1^\alpha, X^{2\alpha+1}] \) is the symmetrizer of three operators and has six terms and \( X^0 = H^0 = I \). The commutator \([L_1, L_2]\) is third-order skew-adjoint and can be expressed as a polynomial in the generators via

\[
[L_1, L_2] = \sum_{0 \leq \alpha + \beta + \gamma \leq 2} Q_{\alpha, \beta, \gamma} [L_1^\alpha, L_2^\beta, X^{2\gamma+1}] H^\gamma.
\]

Finally, since there are at most three algebraically independent generators, there will be a polynomial identity satisfied by the four generators. It is of fourth-order:

\[
\sum_{0 \leq \alpha + \beta + \gamma + \delta \leq 2} S_{\alpha, \beta, \gamma, \delta} [L_1^\alpha, L_2^\beta, X^{2\gamma+1}] H^\delta = 0.
\]

The constants \( Q_{\alpha, \beta, \gamma}, S_{\alpha, \beta, \gamma, \delta} \) are polynomials in \( a_1 = 1 - e_1 - e_2 - e_3 - e_4, 1 - e_1 - e_2 - e_3 - e_4 \) and \( 1 - e_1 - e_2 - e_3 - e_4 \) respectively.

Below is a list of all quantum superintegrable systems in 2D constant curvature space. The corresponding classical systems and their structure equations are similar, but may differ with respect to the non-leading-order terms. The classification of Stäckel equivalent systems and notation comes from [113]. There are approximately 35 multiparameter families of superintegrable systems on Darboux and Koenig spaces, but all are Stäckel equivalent to these. We list nondegenerate systems; they depend on three parameters \( a_1, a_2, a_3 \) as well as a trivial additive parameter which we have dropped.

1. **Quantum S9.** This quantum superintegrable system is defined by

\[
H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1 a_2}{s_1^2} + \frac{a_2 a_3}{s_2^2} + \frac{a_3 a_1}{s_3^2},
\]

\[
L_1 = J_1^3 + \frac{a_3^2 s_1^2}{s_2^2} + \frac{a_2^2 s_2^2}{s_3^2}, \quad L_2 = J_2^3 + \frac{a_1^2 s_1^2}{s_3^2} + \frac{a_3^2 s_3^2}{s_1^2}. \]

The algebra is

\[
[L_1, R] = 4[L_1, L_2] - 4[L_1, L_3] - (8 + 16a_1)L_1 + (8 + 16a_2)L_2 + 8(a_j - a_k),
\]

\[
R^2 = \frac{1}{16} (L_1^2 L_2^2 - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \sum_{i=1}^{3} \sum_{j=i+1}^{3} (L_i L_j) + \sum_{i=1}^{3} (L_i L_i)) + \frac{1}{8} (16 + 16a_1)L_1 + \frac{1}{4} (16 + 16a_2)L_2 + \frac{1}{4} (16 + 16a_3)L_3 + \frac{1}{16} (a_1^2 + a_2^2 + a_3^2)
\]

To simplify the expressions, we have used \( L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3 \) and the indices \( \{i, j, k\} \) chosen as a cyclic permutation of \( \{1, 2, 3\} \).

This system is Stäckel equivalent to:

S7 : \( H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1 s_1}{\sqrt{s_2^2 + s_3^2}} + \frac{a_2 s_2}{\sqrt{s_1^2 + s_3^2}} + \frac{a_3 s_3}{\sqrt{s_1^2 + s_2^2}} \)

S8 : \( H = J_1^2 + J_2^2 + J_3^2 + a_1 \sqrt{s_1^2 + s_2^2 + s_3^2} + a_2 \frac{s_1 + i s_2 - s_3}{\sqrt{(s_1 + i s_2)(s_3 - i s_2)}} + a_3 \frac{s_1 + i s_2 + s_3}{\sqrt{(s_1 + i s_2)(s_3 + i s_2)}} \)

2. **Quantum E1.** The quantum system (Smorodinsky–Winternitz I [51]) is defined by

\[
H = \frac{a_1^2}{\lambda_x^2} + \frac{a_2^2}{\lambda_y^2} + a_1 (x^2 + y^2) + \frac{a_1 a_2}{\chi^2} + \frac{a_2 a_3}{\gamma^2},
\]

\[
L_1 = \frac{a_1^2}{\lambda_x^2} + a_1 y^2, \quad L_2 = (x \partial_y - y \partial_x)^2 + \left( \frac{a_2 y^2}{\lambda_x^2} + \frac{a_3 x^2}{\gamma^2} \right). \]
The algebra relations are
\[ [L_1, R] = 8L_1H - 8L_1^2 + 16a_1L_2 - 8a_1(1 + 2a_2 + 2a_3), \]
\[ [L_2, R] = 8[L_1, L_2] - 4L_2H - 16(1 + a_2 + a_3)L_1 + 8(1 + 2a_3)H, \]
\[ R^2 = \frac{8}{3} ([L_1, L_2, H] - [L_1, L_1, L_2]) - (16a_3 + 12)H^3 \]
\[ - \left( \frac{176}{3} + 16a_2 + 16a_3 \right) L_1^2 - 16a_1L_2^2 + \left( \frac{176}{3} + 32a_3 \right) L_1H \]
\[ + \frac{176a_1}{3} L_2 - \frac{16a_1}{3} (12a_2a_3 + 9a_2 + 9a_3 + 2). \] (4.22)

This system is St"{a}ckel equivalent to:
\[
E16 : \quad H = \partial_x^2 + \partial_y^2 + \frac{1}{\sqrt{x^2 + y^2}} \left( a_1 + a_2 \frac{x + \sqrt{x^2 + y^2}}{x - \sqrt{x^2 + y^2}} + a_3 \frac{1}{x - \sqrt{x^2 + y^2}} \right),
\]
\[
S2 : \quad H = J_1^2 + J_2^2 + J_3^2 + a_1 \frac{1}{(s_1 - i s_2)^2} + a_2 \frac{s_1 + i s_2}{(s_1 - i s_2)^2} + a_3 \frac{1}{s_3^2},
\]
\[
S4 : \quad H = J_1^2 + J_2^2 + J_3^2 + a_1 \frac{1}{(s_1 - i s_2)^2} + a_2 \frac{s_1 + i s_2}{(s_1 - i s_2)^2} + a_3 \frac{1}{s_3^2},
\]
\[
(4.23)
\]

The system E16 is also referred to as Smorodinsky–Winternitz III [51].

(3) Quantum E2. The system (Smorodinsky–Winternitz II [51]) is given by
\[ H = \partial_x^2 + \partial_y^2 + a_1(4x^2 + y^2) + a_2x + \frac{a_3}{y^2}, \]
\[ L_1 = \partial_x^2 + a_1y^2 + \frac{a_3}{y^2}, \] (4.24)
\[ L_2 = \frac{1}{2} \left\{ (x \partial_y - y \partial_x), \partial_x \right\} - y^2 \left( a_1x + \frac{a_2}{4} \right) + \frac{a_1x}{y^2}, \]
\[ [L_1, R] = -2a_2L_1 - 16a_1L_2, \]
\[ [L_2, R] = 6L_1^2 - 4L_1H + 2a_2L_2 - a_1(8a_3 + 6), \]
\[ R^2 = 4H \frac{L_1}{L_1} - 16a_1L_2^2 - 2a_2[L_1, L_2] \]
\[ - 4a_1(4a_3 + 3)H + 4a_1(4a_3 + 11)L_1 - \frac{a_1^2}{4} (4a_3 + 3). \] (4.25)

This system is St"{a}ckel equivalent to:
\[
S16 : \quad H = J_1^2 + J_2^2 + J_3^2 + a_1 \frac{1}{(s_1 - i s_2)^2} + a_2 \frac{s_2}{(s_1 - i s_2)^2} + a_3 \frac{1 - 4s_3^2}{s_3^2},
\]
\[
(4.26)
\]

(4) Quantum E3'. The quantum system is defined by
\[ H = \partial_x^2 + \partial_y^2 + a_1(x^2 + y^2) + a_2x + a_3y, \]
\[ L_1 = \partial_x^2 + a_1y^2 + a_3y, \]
\[ L_2 = \partial_x \partial_y + a_1xy + \frac{a_2y}{2} + \frac{a_3x}{2}, \] (4.27)
\[ [L_1, R] = -4a_1L_2 - a_2a_3, \]
\[ [L_2, R] = 4a_1L_1 - 2a_1H - \frac{(a_2 + a_3)(a_2 - a_3)}{2}, \]
\[ R^2 = 4a_1L_1(H - L_1) - 4a_1L_2^2 + (a_2^2 - a_3^2)L_1 - 2a_2a_3L_2 + a_3^2H - 4a_1. \] (4.28)

This system is St"{a}ckel equivalent to:
\[
E11 : \quad H = \partial_x^2 + \partial_y^2 + a_1z + a_2 \frac{z}{\sqrt{z}} + a_5 \frac{1}{\sqrt{z}}, \]
\[
E20 : \quad H = \partial_x^2 + \partial_y^2 + \frac{1}{\sqrt{x^2 + y^2}} \left( a_1 + a_2(x + \sqrt{x^2 + y^2}) + a_5(x - \sqrt{x^2 + y^2}) \right). \] (4.29)
The system \( E20 \) is (Smorodinsky–Winternitz IV [51]) while \( E3' \) is equivalent by a translation to the simple harmonic oscillator. However, it should be emphasized that without the two additional constants incorporated by the translation, the system would not be Stäckel equivalent to the other two.

(5) Quantum \( E10 \). The quantum system is defined by \( (\partial = \partial_z) \):

\[
H = 4\partial^2 \bar{\partial} + a_1 \left( z^2 - \frac{1}{2} \bar{z}^3 \right) + a_2 \left( z - \frac{3}{2} \bar{z}^2 \right) + a_3 \bar{z},
\]

\[
L_1 = -\partial^2 - \frac{a_1 z^2}{4} - \frac{a_3 \bar{z}}{2} + \frac{a_2}{12},
\]

\[
L_2 = \left[ (\partial - \bar{z} \bar{\partial}, \partial) - \bar{z}^2 - (2z + \bar{z}^2) \left( \frac{a_1(2z - 3\bar{z}^2)}{16} - \frac{a_2 z^2}{2} + \frac{a_1}{4} \right) \right].
\]

\([L_1, R] = 2a_1L_1 - \frac{a_1^2 a_3}{2} - \frac{a_1 a_3}{6},\]

\([L_2, R] = 24L_1^2 + 4a_3L_1 - 2a_1L_2 + a_3H,\]

\(R^2 = -16L_1^3 - \frac{a_1^2}{4}H^2 + 2a_1L_1L_2 - 2a_2L_1H - 4a_3L_1^2\)

\[- \left( \frac{a_1 a_2}{3} \right) L_2 - \frac{a_2 a_3}{3}H - a_1^3 - \frac{a_3^2}{27}.\]

This system is Stäckel equivalent to:

\(E9: H = \partial_z^2 + \bar{\partial}_z^2 + a_1 \frac{1}{z} + a_2(z + \bar{z}) + a_3 \frac{2\pi + z}{\sqrt{z}}.\)

(6) Quantum \( E8 \). The quantum system is defined by \( (\partial = \partial_z) \):

\[
H = 4\partial^2 + a_1 z \bar{z} + \frac{a_2 z}{\bar{z}^2} + \frac{a_3 \bar{z}}{z},
\]

\[
L_1 = -\partial^2 - \frac{a_1}{4} z^2 + \frac{a_3^2}{2z^2}, \quad L_2 = - \left( z \partial - \bar{z} \partial \right) \left( z^2 - \bar{z}^2 \right) + \frac{a_2 z^2}{z^2} + \frac{a_3 \bar{z}}{z^2}.
\]

\([L_1, R] = -8L_1^3 + 2a_1 a_2, \quad [L_2, R] = 8[L_1, L_2] - 16L_1 - 2a_3H,\]

\(R^2 = 8[L_1, L_2] - 176L_1^2 - a_2L_3H + a_2H^2 - 4a_1 a_2 L_2 - \frac{a_1(3a_1^2 - 4a_2)}{3}.\)

This system is Stäckel equivalent to:

\(E7: H = \partial_z^2 + \partial_{\bar{z}}^2 + a_1 \frac{1}{\sqrt{\bar{z}^2 - c^2}} + a_2 \frac{z}{\sqrt{\bar{z}^2 - c^2}} + a_3 \bar{z}, \quad c \in \mathbb{C},\)

\(E17: H = \partial_z^2 + \partial_{\bar{z}}^2 + a_1 \frac{1}{\sqrt{\bar{z}^2 - c^2}} + a_2 \frac{1}{z} + a_2 \frac{1}{z \sqrt{\bar{z}^2}},\)

\(E19: H = \partial_z^2 + \partial_{\bar{z}}^2 + a_1 \frac{1}{\sqrt{\bar{z}^2 - 4c^2}} + a_2 \frac{1}{\sqrt{\bar{z}(\bar{z} + 2)}} + a_3 \frac{1}{\sqrt{\bar{z}(\bar{z} - 2)}}.\)

The following are the degenerate systems; they depend on only one (other than the trivial additive) parameter and admit a Killing vector.

(7) Quantum \( S3 \) (Higgs oscillator). The system is the same as \( S9 \) with \( a_1 = a_2 = 0 \ a_3 = a \). The symmetry algebra is generated by

\[X = J_3, \quad L_1 = J_1^2 + \frac{a_1^2 s_2}{s_3^2}, \quad L_2 = \frac{1}{2}(J_1J_2 + J_2J_1) - \frac{a_1^2 s_2}{s_3^2}.\]
The structure relations for the algebra are given by
\[
[L_1, X] = 2L_2, \quad [L_2, X] = -X^2 - 2L_1 + H - a,
\]
\[
[L_1, L_2] = -(L_1X + XL_1) - \left( \frac{1}{2} + 2a \right) X,
\]
\[
0 = [L_1, X^2] + 2L_1^2 + 2L_2^2 - 2L_1H + \frac{5 + 4a}{2}X^2 - 2aL_1 - a.
\] (4.35)
This system is Stäckel equivalent to another constant curvature system.

This system is Stäckel equivalent to the following potential on the sphere
\[
S6 : \quad H = J_1^2 + J_2^2 + J_3^2 + a \frac{s_3}{\sqrt{s_1^2 + s_2^2}}.
\] (4.36)

(8) **Quantum E14.** The system is defined by
\[
H = \tilde{a}_x^2 + \tilde{a}_y^2 + \frac{a}{z^2}, \quad X = \partial_z,
\]
\[
L_1 = \tilde{z} \left( \tilde{z} \partial_z + \tilde{\bar{z}} \partial_{\bar{z}} \right) + \frac{a}{z}, \quad L_2 = (z \partial_z + \bar{z} \partial_{\bar{z}})^2 + \frac{a\bar{z}^2}{z},
\]
\[
[L_1, L_2] = -(X, L_2) - \frac{1}{2} X, \quad [X, L_1] = -X^2, \quad [X, L_2] = 2L_1.
\] (4.37)
\[
L_1^2 + XL_2X - bH - \frac{1}{4}X^2 = 0.
\]
This system is Stäckel equivalent to
\[
E12 : \quad H = \partial_x^2 + \partial_y^2 + \tilde{a} \frac{\bar{z}}{\sqrt{z^2 + c^2}}.
\] (4.38)

(9) **Quantum E6.** The system is defined by
\[
H = \tilde{a}_x^2 + \tilde{a}_y^2 + \frac{a}{x^2}, \quad X = \partial_y,
\]
\[
L_1 = \frac{1}{2} \left( x \partial_x - y \partial_y + \partial_x \right) - \frac{ay}{x^2}, \quad L_2 = (x \partial_x - y \partial_y)^2 + \frac{ay^2}{x^2},
\] (4.39)
with symmetry algebra
\[
[L_1, L_2] = (X, L_2) + \left( 2a + \frac{1}{2} \right) X, \quad [L_1, X] = -X^2, \quad [L_2, X] = 2L_1,
\]
\[
L_1^2 + \frac{1}{2} L_2 - \frac{1}{2} XL_2X - L_2H + \left( a + \frac{1}{4} \right) X^2 = 0.
\] (4.40)
This system is Stäckel equivalent to
\[
S5 : \quad H = J_1^2 + J_2^2 + J_3^2 + \frac{a}{(s_1 - is_2)^2}.
\] (4.41)

(10) **Quantum E5.** The system is defined by
\[
H = \tilde{a}_x^2 + \tilde{a}_y^2 + ax, \quad X = \partial_y,
\]
\[
L_1 = \partial_y \bar{z} + \frac{1}{2} ay, \quad L_2 = \frac{1}{2} (x \partial_x - y \partial_y + \partial_x) - \frac{1}{4} ay^2.
\] (4.42)
\[
[L_1, L_2] = 2X^3 - HX, \quad [L_1, X] = \frac{a}{2}, \quad [L_2, X] = L_1,
\]
\[
X^4 - HX^2 + L_1^2 + aL_2 = 0.
\] (4.43)
This system is not Stäckel equivalent to another constant curvature system.
(11) **Quantum E4.** The system is defined by

\[ H = \partial_x^2 + \partial_y^2 + a(x + iy), \quad X = \partial_x + i\partial_y, \quad (4.45) \]

\( L_1 = \partial_x^2 + ax, \quad L_2 = \frac{i}{2}(x\partial_y - y\partial_x, X) - \frac{a}{4}(x + iy)^2. \]

\[ [L_1, X] = a, \quad [L_2, X] = X^2, \quad [L_1, L_2] = X^3 + HX - [L_1, X], \quad (4.46) \]

\[ X^4 - 2[L_1, X^2] + 2HX^2 + H^2 + 4aL_2 = 0. \]

This system is Stäckel equivalent to \( E_{13} : H = \partial_x^2 + \partial_y^2 + \frac{a}{\sqrt{x^2 + y^2}}. \quad (4.47) \]

(12) **Quantum E3 (harmonic oscillator).** The system is determined by

\[ H = \partial_x^2 + \partial_y^2 + a(x^2 + y^2), \quad X = x\partial_x - y\partial_y, \quad (4.48) \]

\[ L_1 = \partial_y^2 - \omega^2 y^2, \quad L_2 = \partial_x + axy, \quad [L_2, X] = H - 2L_1, \quad [L_1, L_2] = -2aX, \]

\[ L_1^2 + L_2^2 - L_1H + aX^2 = a = 0. \]

This system is Stäckel equivalent to the Kepler–Coulomb system,

\[ E_{18} : H = \partial_x^2 + \partial_y^2 + \frac{a}{\sqrt{x^2 + y^2}}. \quad (4.49) \]

In this case, the Stäckel transform coincides with the Kustaanheimo–Stiefel transformation [114].

### 4.2. Second-order superintegrability in nD

#### 4.2.1. Classification in 3D conformally flat spaces.

In this section, we review the structure and classification of second-order superintegrable systems in 3D conformally flat space. For a more detailed proof and further analysis, see the original papers [88–90].

Consider a Hamiltonian in 3D conformally flat space, with metric \( ds^2 = \lambda(x_1, x_2, x_3)(dx_1^2 + dx_2^2 + dx_3^2) \), given by, in Cartesian-like coordinates,

\[ H = \frac{1}{\lambda(x_1, x_2, x_3)}(p_1^2 + p_2^2 + p_3^2) + V(x_1, x_2, x_3). \quad (4.50) \]

The second-order integrals of the motion are determined by the 3D version of the Killing equations,

\[ a_i^j = \sum_{j=1}^{3} -\frac{\lambda_j}{\lambda} a_i^j, \quad i, j = 1, 2 \]

\[ 2a_i^i + a_i^j = \sum_{k=1}^{3} -\frac{\lambda_k}{\lambda} a_i^k, \quad i, j = 1, 2, \quad i \neq j. \quad (4.51) \]

and the determining equations for the function \( W \),

\[ W_j = \sum_{k=1}^{3} \lambda(x, y) a_i^k V_k. \quad (4.52) \]

Thus, the requirements for the existence of a second-order integral of motion in 3D can be expressed in terms of the integrability conditions for these equations.
Theorem 12. A second-order function on phase space
\[ S = \sum_{1 \leq i,j \leq 3} a_{ij}(x_1, x_2, x_3) p_i p_j + W(x_1, x_2, x_3) \] (4.53)
is an integral of the motion for \( H \) (4.50) if and only if the following hold:

1. The leading order term \( S_0 = \sum_{1 \leq i,j \leq 3} a_{ij}(x_1, x_2, x_3) p_i p_j \) is a conformal symmetry of the free Hamiltonian on flat space \( H_0 = p_1^2 + p_2^2 + p_3^2 \); i.e. \( [H_0, S_0] = f H_0 \) for some function \( f = f(x_1, x_2, x_3) \).
2. The functions \( a_{ij} \) and \( \lambda \) satisfy the integrability conditions for the Killing equations (4.51),

\[
\begin{align*}
(\lambda_2 a_{12} + \lambda_3 a_{13})_2^2 &= (\lambda_1 a_{12} + [(a_{22} - a_{11})\lambda]_2 + \lambda_3 a_{23})_1, \\
(\lambda_2 a_{12} + \lambda_3 a_{13})_3^3 &= (\lambda_1 a_{13} + \lambda_2 a_{23} + [(a_{33} - a_{11})\lambda]_3)_1, \\
(\lambda_1 a_{12} + [(a_{22} - a_{11})\lambda]_2 + \lambda_3 a_{23})_2 &= (\lambda_1 a_{13} + \lambda_2 a_{23} + [(a_{33} - a_{11})\lambda]_3)_3.
\end{align*}
\] (4.54)

3. The potential satisfies the Bertrand–Darboux equations
\[ \sum_{k=1}^3 \left[ V_{jk} \lambda a^{jk} + V_{ik} \lambda a^{ik} + V_{j}(\lambda a^{jk} + (\lambda a^{kj})^\dagger) \right] = 0, \quad j = 1, 2, 3, \] (4.55)
the integrability conditions for the function \( W \) (4.52).

The classification of the Hamiltonians which satisfy these requirements breaks into two general classes, the degenerate and nondegenerate, but the full structure theory has been worked out only for the nondegenerate case and for systems with bases that are functionally independent.

Definition 12. The Hamiltonian (4.50) is nondegenerate if the Bertrand–Darboux equations (4.55) are satisfied identically as equations for the potential \( V \).

It can be shown that nondegenerate systems depend on exactly four parameters, not including the trivial additive parameter.

We have the following results for nondegenerate 3D superintegrable systems in conformally flat space; many of the proofs follow a similar structure as those for the 2D case.

Theorem 13. Let \( H \) be a classical, nondegenerate second-order superintegrable system in a 3D conformally flat space. Then

1. The classical Hamiltonian can be extended to a unique covariant quantum second-order superintegrable system. The potential of the quantum system differs from that of the classical system by an additive term depending linearly on the scalar curvature.
2. The corresponding Hamilton–Jacobi or Schrödinger equations allow separation of variables in multiple coordinate systems,
3. The system is Stäckel equivalent to either a system on flat space or the sphere. There are exactly ten systems in flat space and six on the sphere (though there is no published proof as yet in the latter case).

Theorem 14 (5 ⇒ 6). [88] Let \( H \) be a nondegenerate superintegrable system in a 3D conformally flat space, i.e. \( H \) admits five functionally independent second-order integrals satisfying the requirements of theorem 12. Then there exists an additional second-order integral such that the six are functionally linearly independent.
With the addition of this sixth integral, the algebra closes to a quadratic algebra.

**Theorem 15.** Let $\mathcal{H}$ be a nondegenerate superintegrable system in 3D conformally flat space which admits five (and hence six) second-order integrals of the motion. Then

1. The space of third-order integrals is of dimension 4 and spanned by the Poisson brackets of the six generators.
2. The space of fourth-order integrals is of dimension 21 and spanned by quadratic polynomials in the generators.
3. The space of sixth-order integrals is of dimension 56 and spanned by cubic polynomials in the generators.

Thus, the quadratic algebra closes at sixth-order. Additionally, there is an eighth-order functional relation between the six, second-order integrals. Furthermore, the theorem holds for quantum systems with Poisson brackets replaced by commutators and polynomials in the generators replaced by their appropriately symmetrized counterparts.

Let us consider an example of a quadratic superintegrable system which is nondegenerate, the singular isotropic oscillator:

$$
H = \hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2 + a^2\left(x_1^2 + x_2^2 + x_3^2\right) + \frac{b_1}{x_1^2} + \frac{b_2}{x_2^2} + \frac{b_3}{x_3^2}, \quad \hat{a}_i = \partial_{a_i}.
$$

A basis for the second-order constants of the motion is (with $H = M_1 + M_2 + M_3$)

$$
\begin{align*}
M_\ell &= \hat{a}_\ell^2 + a^2\frac{x_i^2}{x_\ell^2}, & \ell &= 1, 2, 3, \\
L_i &= (x_j\hat{a}_k - x_k\hat{a}_j)x_i^2 + \frac{b_j}{x_j^2}x_i^2 + \frac{b_k}{x_k^2}x_i^2,
\end{align*}
$$

where $i, j, k$ are pairwise distinct and run from 1 to 3. There are four linearly independent commutators of the second-order symmetries:

$$
\begin{align*}
S_1 &= [L_1, M_2] = [M_3, L_1], & S_2 &= -[M_3, L_2] = [M_1, L_2], \\
S_3 &= -[M_1, L_3] = [M_2, L_3], & R &= [L_1, L_2] = [L_2, L_3] = [L_3, L_1], \\
[M_i, M_j] &= [M_i, L_i] = 0, & 1 \leq i, j \leq 3.
\end{align*}
$$

The fourth-order structure equations are $[M_i, S_j] = 0$, $1 = 1, 2, 3$, and

$$
\begin{align*}
\epsilon_{ijk} [M_i, S_j] &= 8M_i M_k - 16a^2 L_j + 8a^2, & \epsilon_{ijk} [M_i, R] &= 8(M_j L_j - M_k L_k) + 4(M_k - M_j), \\
\epsilon_{ijk} [L_i, S_j] &= 8M_i L_j - 8M_i L_k + 4(M_k - M_j), & \epsilon_{ijk} [L_i, R] &= 4(L_i, L_k - L_j) - 16b_j M_k + 16b_k M_j + 8(M_k - M_j).
\end{align*}
$$

Here, $[F, G] = FG + GF$ and $\epsilon_{ijk}$ is the completely antisymmetric tensor. The fifth-order structure equations are obtainable directly from the fourth-order equations and the Jacobi identity. An example of the sixth-order equations is

$$
\begin{align*}
S_i^2 &= \frac{8}{3}[L_i, M_j, M_k] + 16a^2 L_i^2 + (16b_k + 12)M_i^2 + (16b_j + 12)M_i^2 - \frac{104}{3}M_j M_k \\
&\quad - \frac{176}{3}a^2 L_i - \frac{16}{3}a^2 (2 + 9b_j + 9b_k + 12b_j b_k) = 0.
\end{align*}
$$

The remainder of the structure equations can be found in [102], where representations for this quadratic algebra are obtained.
On the other hand, for true three parameter potentials (i.e. those which are not restrictions of a four parameter potential) the 5 ⇒ 6 theorem no longer holds, the space of third-order integrals is of dimension 3 and the there are sixth-order integrals which cannot be expressed as a polynomial in the generators. For example, consider the following extension of the Kepler–Coulomb system, here given as a classical system

\[ \mathcal{H} = p_1^2 + p_2^2 + p_3^2 + \frac{\alpha}{\sqrt{x_1^2 + x_2^2 + x_3}} + \frac{\beta}{x_1^2} + \gamma \frac{1}{x_2^2}, \]

\[ \mathcal{L}_1 = J_1^2 + \gamma \left( \frac{x_1^2 + x_3^2}{x_1^2} \right), \quad \mathcal{L}_2 = J_2^2 + \frac{\beta (x_1^2 + x_3^2)}{x_1^2}, \]

\[ \mathcal{L}_3 = J_3^2 + \left( \frac{\beta}{x_1^2} + \gamma \right) \left( \frac{\beta}{x_1^2} + \gamma \right), \]

\[ \mathcal{L}_4 = p_1 J_2 - p_2 J_1 + x_3 \left( \frac{\alpha}{2 \sqrt{x_1^2 + x_2^2 + x_3}} + \frac{\beta}{x_1^2} + \gamma \right). \] (4.61)

The third-order integrals are spanned by

\[ \mathcal{R}_1 = \{ \mathcal{L}_1, \mathcal{L}_2 \} = \{ \mathcal{L}_2, \mathcal{L}_3 \} = \{ \mathcal{L}_3, \mathcal{L}_1 \} = -4 J_1 J_2 J_3 + \cdots \]

\[ \mathcal{R}_2 = \{ \mathcal{L}_1, \mathcal{L}_4 \} = 2 p_1 J_1 J_3 - 2 p_2 J_1^2 + \cdots \]

\[ \mathcal{R}_3 = \{ \mathcal{L}_2, \mathcal{L}_5 \} = 2 p_2 J_2 J_3 - 2 p_3 J_2^2 + \cdots \]

\[ 0 \quad \{ \mathcal{L}_3, \mathcal{L}_5 \}, \]

unlike in the nondegenerate case where the dimension was 4. Also unlike in the nondegenerate case, there is a sixth-order integral which cannot be expressed as a polynomial in the generators \( \mathcal{H}, \mathcal{L}_1, \ldots, \mathcal{L}_5 \). Indeed, it is clear that the operator \( \mathcal{R}_2 \mathcal{R}_3 \) will have a leading order term of the form \( p_3 p_2 J_3^2 J_2^2 J_1^3 \) which cannot be obtained from any polynomial in the generators \( \mathcal{H}, \mathcal{L}_1, \ldots, \mathcal{L}_5 \).

This example demonstrates the difficulties which are encountered when trying to extend the analysis to ND, while at the same time restricting to second-order superintegrability. Indeed the structure of the previous system can be better understood by considering additional higher-order integrals \([82, 94, 179]\). This topic will be taken up in the following sections.

The most powerful methods we know for construction of second-order systems in higher dimensional constant curvature spaces are based on orthogonal separation of variables for the zero-potential Helmholtz and associated Hamilton–Jacobi equations, particularly separation in ‘generic’ Jacobi ellipsoidal coordinates, \([76, 83, 97]\). We give an example for \( n = 3 \) and then generalize. Here, a ‘natural’ basis for first-order symmetries is given by \( p_1 \equiv p_x, p_2 \equiv p_y, p_3 \equiv p_z, J_1 \equiv y p_z - z p_y, J_2 \equiv z p_x - x p_z, J_3 \equiv x p_y - y p_x \). Among the separable systems for the Hamilton–Jacobi equation \( \mathcal{H} = p_x^2 + p_y^2 + p_z^2 = E \) there are seven ‘generic’ Euclidean systems, depending on a scaling parameter \( c \) and up to three parameters \( e_1, e_2, e_3 \). For each such set of coordinates there is exactly one nondegenerate superintegrable system that admits separation in these coordinates simultaneously for all values of the parameters \( c, e_j \). An example is the system \((23)\) in Böcher’s notation \([20]\),

\[ x - iy = \frac{1}{2} \left( \frac{u^2 + v^2 + w^2}{uvw} - \frac{1}{2} \frac{u^2 v^2 + u^2 w^2 + v^2 w^2}{u^3 v^3 w^3} \right), \]

\[ z = \frac{1}{2} c \left( \frac{u v}{w} + \frac{w v}{u} + \frac{w u}{v} \right), \quad x + iy = c u v w. \]

\[ \mathcal{L}_1 = J_1^2 + J_2^2 + J_3^2 + 2 e_2 (p_1 + ip_2) p_3, \quad \mathcal{L}_2 = -2 J_3 (J_1 + i J_2) + c^2 (p_1 + ip_2)^2. \]

The symmetries \( \mathcal{L}_1, \mathcal{L}_2 \) determine the separation in the variables \( u, v, w \). If a nondegenerate superintegrable system, with potential, separates in these coordinates for all values of the
parameter $c$, then the space of second-order symmetries must contain the five symmetries

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + V, \quad S_1 = J_1 \mathcal{H} + J_2 \mathcal{H} + J_3 \mathcal{H} + f_1, \quad S_2 = J_3(J_1 + iJ_2) + f_2;$$

$$S_3 = (p_x + ip_y)^2 + f_3, \quad S_4 = p_z(p_x + ip_y) + f_4.$$

Solving the Bertrand–Darboux equations for the potential we find the unique solution

$$V(x) := \alpha(x^2 + y^2 + z^2) + \frac{\beta}{(x + iy)^2} + \frac{\gamma z}{(x + iy)^3} + \frac{\delta(x^2 + y^2 - 3z^2)}{(x + iy)^4}.$$ 

Finally, we can use the symmetry conditions for this potential to obtain the full six-dimensional space of second-order symmetries. The other six generic cases yield corresponding results.

All second-order 3D Euclidean systems can be shown to be limits of these generic systems. Further these generic separable coordinates have analogues for all dimensions $n$ and lead to nondegenerate superintegrable systems. The number of distinct generic superintegrable systems for each integer $n \geq 2$ is $\sum_{j=0}^{n/2} p(j)$, where $p(j)$ is the number of integer partitions of $j$, given by the Euler generating function

$$\frac{1}{\log(1-t^2)} = \sum_{j=0}^{\infty} p(j)t^j.$$ 

By taking Stäckel transforms we can define superintegrable systems on many conformally flat spaces. Similarly, each of the five generic separable coordinates on the complex 3-sphere leads to a nondegenerate superintegrable system and this construction generalizes to $n$ dimensions. The number of distinct generic superintegrable systems for each integer $n \geq 2$ is $p(n+1)$ where $p(j)$ is the number of integer partitions of $j$.

There are many constructions and analyses of second-order systems in $n$ dimensions, e.g., [7–10, 83, 106, 171] but as yet no general theory. We finish this section with a discussion of methods that may give a clear way to obtain general results about second-order superintegrability in $n$ dimensions, those of algebraic geometry.

4.2.2. Classification: relation with algebraic geometry. In this section, we review the basic results of [91] without the details in order to give a general idea of the method and how it can be extended to higher dimensions. For a nondegenerate system in complex Euclidean space, the Bertrand–Darboux equations (4.55) depend on essentially ten different functions. From the assumption that the integrability conditions (4.54) are identically satisfied as well as the integrability conditions for the potential, the derivatives of these functions can be expressed as quadratic polynomials in the original functions as well as five quadratic identities for the ten functions.

If we consider the algebraic variety defined by these five polynomial functions in 10D, regular points of this variety will determine a unique superintegrable system as long as the system is closed under differentiation. This adds an additional quadratic function in the points, which, when adjoined to the original set of equations gives an algebraic variety, closed under differentiation each of whose regular points determine a nondegenerate superintegrable system in 3D Euclidean space. Two points in the variety correspond to the same system if and only if they lie on the same orbit under the action of the complex Euclidean group $E(3, \mathbb{C})$. In [92] it is shown that there are exactly ten orbit families, corresponding to the ten possible nondegenerate Euclidean systems. In [91] a similar analysis has been carried out for 2D Euclidean systems. The corresponding analysis for superintegrable systems on the complex sphere has not yet been carried out. For a start see [79].

The possibility of using methods of algebraic geometry to classify superintegrable systems is very promising and suggests a method to extend the analysis in arbitrary dimension as well as a way to understand the geometry underpinning superintegrable systems.
5. Higher-order determining equations on Euclidean space

In this section, we move beyond second-order and into higher-order superintegrability. We begin the section with some general theory about the structure of higher-order integrals for classical and quantum Hamiltonians in 2D real Euclidean space. A general form for the integrals is given as well as the determining equations.

In section 5.2, we compare these results to those given in the previous section by considering second-order integrals and the classification of the solutions to the determining equations in this case. In section 5.3, we give explicitly the determining equations for third-order integrals. These are nonlinear and very difficult to solve. One strategy is to assume separation of variables and this in turn implies the existence of an additional first or second-order integral of motion. All third-order superintegrable systems which admit separation of variables in Cartesian, polar and parabolic coordinates have been classified. The results of these investigations are reviewed in sections 5.3. and 5.4 discusses the fourth-order case.

5.1. Determining equations for higher-order integrals

In order to determine the equations for higher-order integrals, we focus our attention on 2D Hamiltonians with a real, scalar potential

\[ H = p_1^2 + p_2^2 + V(x, y). \]  

(5.1)

**Theorem 16.** A classical Nth order integral for the Hamiltonian (5.1) has the form

\[ X = \sum_{\ell=0}^{\lfloor \frac{N-2}{2} \rfloor} \sum_{j=0}^{N-2-2\ell} f_{j,2\ell}(p_1^{N-j-2\ell}, p_2^{N-j-2\ell}), \]  

(5.2)

where \( f_{j,k}(x, y) \) are real functions. The integral has the following properties:

(1) The functions \( f_{j,2\ell} \) and the potential \( V(x, y) \) satisfy the determining equations

\[ 0 = 2 \frac{\partial f_{j-1,2\ell}}{\partial x} + 2 \frac{\partial f_{j,2\ell}}{\partial y} - (j + 1) \frac{\partial V}{\partial x} f_{j+1,2\ell} - (N - 2\ell + 2 - j) \frac{\partial V}{\partial y} f_{j,2\ell}, \]  

(5.3)

(2) As indicated in (5.2), all terms in the polynomial \( X \) have the same parity.

(3) The leading terms in (5.2) (of order \( N \) obtained for \( \ell = 0 \)) are polynomials of order \( N \) in the enveloping algebra of the Euclidean (Poisson) Lie algebra \( E(2) \) with basis \( \{ p_1, p_2, L_3 \} \).

There are analogous results for quantum integrals although the proofs are more involved in the quantum case, see [156]. The results are as follows. Note that for higher-order integrals, there is a substantial difference between the classical and quantum case. In order to keep track of this difference, we will continue to normalize the mass to 2 but will leave the dependence of \( \hbar \) in the quantum system.

**Theorem 17.** A quantum Nth order integral for the Hamiltonian (5.1) has the form

\[ X = \frac{1}{2} \sum_{\ell} \sum_{j} (-i\hbar)^{N-2\ell} f_{j,2\ell, \ell} \partial_x^{N-2\ell} \partial_y^{N-2\ell-j}. \]  

(5.5)

\[ f_{j,k} = 0, \quad j < 0, k < 0, j + k > N, k = 2\ell + 1. \]  

(5.6)

where \( f_{j,k}(x, y) \) are real functions. The integral has the following properties:
where $Q_{j,2\ell}$ is a quantum correction term that is polynomial in $\hbar^2$ given below in (5.11).

(2) As indicated in (5.5), the symmetrized integral will have terms which are differential operators of the same parity.

(3) The leading terms in (5.5) (of order $N$ obtained for $\ell = 0$) are polynomials of order $N$ in the enveloping algebra of the Euclidean Lie algebra $E(2)$ with basis $\{p_1, p_2, L_3\}$.

Let us just mention that as in the classical case, the observation of the parity constraint reduces the possible functional coefficients by about half. Indeed, were one to express the integral in the standard form with all of the differential operators on the right, the integral would have $N + 1$ terms and be given by

$$X = \sum_{\ell = 0}^{\lfloor N/2 \rfloor} \sum_{j = 0}^{N-2\ell-1} \left( f_{j,2\ell} - \hbar^2 \phi_{j,2\ell} \right) \partial_x^{N-2\ell-j} (-i\hbar)^{N-2\ell}$$

$$- \frac{1}{i\hbar} \sum_{\ell = 0}^{\lfloor N/2 \rfloor} \sum_{j = 0}^{N-2\ell-1} \phi_{j,2\ell+1} \partial_x^{N-2\ell-j-1} (-i\hbar)^{N-2\ell-1},$$

where the $\phi_{j,\ell}$ are defined as

$$\phi_{j,2\ell} = \sum_{b = 0}^{\ell - 1} \sum_{a = 0}^{2b+2} \left( \frac{-\hbar^2}{2} \right)^{b} \binom{j + a}{a} \left( N - 2\ell + 2b + 2 - j - a \right) \frac{2b + 2 - a}{2b + 2 - a} f_{j+a,2\ell-2b-2}$$

$$\phi_{j,2\ell+1} = \sum_{b = 0}^{\ell} \sum_{a = 0}^{2b+1} \left( \frac{-\hbar^2}{2} \right)^{b} \binom{j + a}{a} \left( N - 2\ell + 2b - j - a \right) \frac{2b + 1 - a}{2b + 1 - a} f_{j+a,2\ell-2b}.$$  

Note that these are polynomial in $\hbar^2$. Without prior knowledge of the symmetrized structure, the operator $X$ would seem to depend on $N + 1$ functions instead of $\lfloor (N + 1)/2 \rfloor$.

Unlike in the classical case where it was clear that there were only $\lfloor (N + 2)/2 \rfloor$ sets of determining equations, since $[H, X]$ is a degree $N + 1$ polynomial with distinct parity, in the quantum case this is not immediately clear. As was shown in [156], every other term vanishes modulo the higher-order terms and so the only independent determining equations in the quantum case are the same as those in the classical case, up to a quantum correction $Q_{j,2\ell}$ given by

$$Q_{j,2\ell} = \left( 2\partial_x \phi_{j-1,2\ell} + 2\partial_y \phi_{j,2\ell} + \partial_x^2 \phi_{j,2\ell-1} + \partial_y^2 \phi_{j,2\ell-1} \right)$$

$$- \sum_{n = 0}^{\ell-2} \sum_{m = 0}^{2n+3} \left( -\hbar^2 \right)^{n} \binom{j + m}{m} \left( N - 2\ell + 2n + 4 - j - m \right) \frac{2n + 3 - m}{2n + 3 - m} f_{j+m,2\ell-2n+4}$$

$$- \sum_{n = 0}^{\ell-2} \sum_{m = 0}^{2n+2} \left( -\hbar^2 \right)^{n} \binom{j + m}{m} \left( N - 2\ell + 2n + 3 + j - m \right) \frac{2n + 2 - m}{2n + 2 - m} f_{j+m,2\ell-2n-3}$$

$$- \sum_{n = 0}^{\ell-1} \sum_{m = 0}^{2n+1} \left( -\hbar^2 \right)^{n} \binom{j + m}{m} \left( N - 2\ell + 2n + 2 - j - m \right) \frac{2n + 1 - m}{2n + 1 - m} f_{j+m,2\ell-2n-2}.$$
In this form, it is clear that the quantum correction terms are polynomial in \( \hbar \) and so the quantum determining equations go to the classical ones in the classical limit. In the integrable case, these quantum correct terms have been studied [67–72] and, as will be seen for third- and higher-order systems, there exist quantum integrable and superintegrable systems whose potentials either change or vanish in the classical limit.

5.2. Second-order determining equations

As an example, consider the case of an integral of second-order and compare these determining equations with those analyzed in section 4.1. There are two sets of determining equations. The equations which require the leading order terms to be in the enveloping algebra of \( E_2 \),

\[
0 = \frac{\partial f_{j-1.0}}{\partial x} + \frac{\partial f_{j.0}}{\partial y}, \quad j = 0, \ldots, 3, \tag{5.11}
\]

these are the Killing equations (4.3) from section 4 on Euclidean space. Note that there are no quantum correction terms for this equation since it corresponds to \( \ell = 0 \), and \( \phi_{j,0} \) and hence \( Q_{j,0} \) are identically 0. The next and final set of equations are given by the following equation for \( j = 0, 1 \)

\[
0 = 2 \frac{\partial f_{j-1.2}}{\partial x} + 2 \frac{\partial f_{j-2}}{\partial y} - (j + 1) \frac{\partial V}{\partial x} f_{j+1,0} - (2 - j) \frac{\partial V}{\partial y} f_{j,0} - \hbar^2 Q_{j,2}, \tag{5.12}
\]

with quantum correction terms

\[
Q_{j,2} = (2\partial_\ell\phi_{j-1,2\ell} + 2\partial_\ell\phi_{j,2\ell} + \partial_{\ell}^2\phi_{j,2\ell-1} + \partial_{\ell}^2\phi_{j,2\ell-1})
\]

\[
\phi_{j,1} = \sum_{a=0}^{1} \frac{1}{2} \left( j + a \right) \left( 2 - j - a \right) \partial_x^{j-a} \partial_y^{1-a} f_{j+a,0}. \tag{5.13}
\]

The equations (5.12) are the B–D equations (4.5) with \( f_{0,2} = W \) the zeroth-order term of the integral. However, recall from the previous section that the B–D equations (4.5) have no quantum correction term. For second-order superintegrable systems the quantum correction term at this level does not vanish identically but only when the higher-order terms satisfy the Killing equations (5.11).

Suppose now that the second-order terms satisfy these equations (5.11), so that the integral is of the form (without loss of generality we assume \( A_{0,0,2} = 0 \))

\[
X = A_{2,0,0}L_3^2 + A_{1,1,0}(L_3p_1 + p_1L_3) + A_{1,0,1}(L_3p_2 + p_2L_3) + A_{0,2,0}(p_1^2 - p_2^2) + 2A_{0,1,1}p_1p_2 + f_{0,2}(x_1, x_2). \tag{5.14}
\]

The remaining determining equations (5.12) for \( j = 0, 1 \) become

\[
\frac{\partial f_{0,2}}{\partial x} = -2 \left( A_{2,0,0}x^2 + 2A_{1,1,0}y + A_{0,2,0} \right) V_x + 2 \left( A_{2,0,0}xy + A_{1,1,0}x - A_{1,0,1} - A_{0,1,1} \right) V_y
\]

\[
\frac{\partial f_{0,2}}{\partial y} = -2 \left( A_{2,0,0}xy + A_{1,1,0}x - A_{1,0,1} - A_{0,1,1} \right) V_x - 2 \left( A_{2,0,0}x^2 + 2A_{1,0,1}x + 2A_{0,2,0} \right) V_y. \tag{5.15}
\]

The compatibility condition for the system (5.15) is

\[
\left( -A_{2,0,0}xy - A_{1,1,0}x + A_{1,0,1}Y + A_{0,1,1} \right)(V_{xx} - V_{yy}) - \left( A_{2,0,0}x^2 + y^2 \right) + 2A_{1,1,0}x + 2A_{1,0,1}y + 2A_{0,2,0}V_{xy}
\]

\[
- \left( A_{2,0,0}y + A_{1,0,1} \right) V_x + 3 \left( A_{2,0,0}x - A_{1,0,1} \right) V_y = 0. \tag{5.16}
\]
Equation (5.16) is exactly the same equation that we would have obtained if we had required that the potential should allow the separation of variables in the Schrödinger equation in one of the coordinate system in which the Helmholtz equation allows separation.

The Hamiltonian (5.1) is form invariant under Euclidean transformations, so we can classify the integrals $X$ into equivalence classes under rotations, translations and linear combinations with $H$. There are two invariants in the space of parameters $A_{j,k,l}$, namely

$$I_1 = A_{2,0,0}, \quad I_2 = (2A_{2,0,0}A_{0,2,0} - A_{1,1,0}^2 + A_{1,0,1}^2)^2 + 4(A_{2,0,0}A_{0,1,1} - A_{1,1,0}A_{1,0,1})^2.$$  (5.17)

Solving (5.16) for different values of $I_1$ and $I_2$ we obtain:

$${I_1 = I_2 = 0} \quad V_C = f_1(x) + f_2(y) \quad V_R = f(r) + \frac{1}{r^2}g(\phi) \quad x = r \cos \phi, \quad y = r \sin \phi$$

$${I_1 = 1, \quad I_2 = 0} \quad V_R = f(\xi) + g(\eta) \quad x = \frac{\xi^2 - \eta^2}{2}, \quad y = \xi \eta$$

$$I_1 = 0, \quad I_2 = 1 \quad V_P = \frac{f(\sigma) + g(\rho)}{\cos^2 \sigma - \cosh^2 \rho} \quad x = l \cosh \rho \cos \sigma, \quad y = l \sinh \rho \sin \sigma \quad 0 < l < \infty.$$  (5.18)

We see that $V_C, V_R, V_P$ and $V_E$ correspond to separation of variables in Cartesian, polar, parabolic and elliptic coordinates, respectively, and that second-order integrability (in $E_2$) implies separation of variables. For second-order superintegrability, two integrals of the form (5.14) exist and the Hamiltonian separates in at least two coordinate systems.

Four three-parameter families of superintegrable systems exist namely

$$V_I = \alpha (x^2 + y^2) + \beta \frac{\gamma}{\gamma^2} + \frac{\gamma}{\gamma^2}, \quad V_{II} = \alpha (x^2 + 4y^2) + \beta \frac{\gamma}{\gamma^2} + \gamma y$$

$$V_{III} = \frac{\alpha}{r} + \frac{1}{r^2} \left( \frac{\beta}{\cos^2 \phi} + \frac{\gamma}{\sin^3 \phi} \right), \quad V_{IV} = \frac{\alpha}{r} + \frac{1}{\sqrt{r}} \left( \beta \cos \frac{\phi}{2} + \gamma \sin \frac{\phi}{2} \right).$$  (5.19)

The classical trajectories, quantum energy levels and wave functions for all of these systems are known. The potentials $V_I$ and $V_{II}$ are isospectral deformations of the isotropic and an anisotropic harmonic oscillator, respectively, whereas $V_{III}$ and $V_{IV}$ are isospectral deformations of the Kepler–Coulomb potential. They correspond to cases E1, E2, E16, and E18 listed in section 4.1.4.

While the situation is completely understood for second-order integrals, it is less so for third-order ones. In the next section, we move onto the determining equations for the third-order case and the classification results that have been obtained.

5.3. Third-order integrals

There are three sets of determining equations corresponding to $\ell = 0, 1, 2$. The highest-order terms are the same as in all dimensions,

$$0 = \frac{\partial f_{j-1,0}}{\partial x} + \frac{\partial f_{j,0}}{\partial y} \quad j = 0, \ldots, 5$$  (5.20)

and lead to the requirement that the highest-order terms be from the enveloping algebra. They are solved by setting

$$f_{3,0} = -A_{300}y^3 + A_{210}x^2 - A_{120}xy + A_{030},$$

$$f_{2,0} = 3A_{300}xy^2 - 2A_{210}xy + A_{201}y^2 + A_{120}x - A_{111}y + A_{021}.$$
\[ f_{1,0} = -3A_{300}x^2y - 2A_{201}xy + A_{210}x^2 + A_{111}x - A_{102}y + A_{012}, \]
\[ f_{0,0} = A_{300}x^3 + A_{201}x^2 + A_{102}x + A_{030}. \]

In this case the quantum integral becomes
\[ X = \sum_{j+k+\ell=3} \frac{A_{jk\ell}}{2} \left[ \sum_{j+k+\ell=3} A_{jk\ell} \left( \frac{p_j^2}{2} \right) \right] + \frac{1}{2} \left\{ f_{1,2} - \hbar^2 A_{2,1,0}, p_1 \right\} + \frac{1}{2} \left\{ f_{0,2} - \hbar^2 A_{2,0,1}, p_2 \right\}. \] (5.21)

In analogy with other references [64, 65, 129, 186], we set \( f_{j,0} = F_{4-j} \) and
\[ G_1 \equiv f_{1,2} - \frac{3}{2} \hbar^2 A_{2,1,0} + 5\hbar^2 A_{3,0,0}, \quad G_2 \equiv f_{0,2} - \frac{3}{2} \hbar^2 A_{2,0,1} - 5\hbar^2 A_{3,0,0}. \]

The next set of equations, for \( \ell = 1 \) are
\[ 0 = 2 \frac{\partial f_{j-1,2}}{\partial x} + 2 \frac{\partial f_{j,2}}{\partial y} - (j + 1) \frac{\partial V}{\partial x} f_{j+1,0} - (3 - j) \frac{\partial V}{\partial y} f_{j,0} - \hbar^2 Q_{j,2}. \] (5.22)

Again, the quantum correction term vanishes on solutions of (5.20). The equations (5.22) become
\[ 2(G_1)_x = 3F_1V_x + F_2V_y, \quad j = 2, \] (5.23)
\[ 2(G_1)_y + 2(G_2)_x = 2(F_2V_x + F_3V_y), \quad j = 1, \] (5.24)
\[ 2(G_2)_y = F_3V_x + 3F_4V_y, \quad j = 0. \] (5.25)

Equations (5.23)–(5.25) satisfy the following linear compatibility conditions
\[ 0 = -F_3V_{xxx} + (2F_2 - 3F_4)V_{xxy} + (-3F_1 + 2F_3)V_{xy} - F_2V_{yy} \]
\[ + 2(F_2y - F_3y)V_{xx} + 2(-3F_1y + F_2x + F_3y - 3F_4x)V_{xy} + 2(-F_2y + F_3x)V_{yy} \]
\[ + (-3F_1y + 2F_2y - F_3x)V_x + (-F_2y + 2F_3y - 3F_4x)V_y. \] (5.26)

Finally, there is a single \( \ell = 2 \) equation and it is given by
\[ 0 = G_1V_x + G_2V_y - \frac{\hbar^2}{4} (F_1V_{xxx} + F_2V_{xxy} + F_3V_{xy} + F_4V_{yy} \]
\[ + (8A_{300}x + 2A_{210})V_x - (8A_{003}x - 2A_{201})V_y). \] (5.27)

Again, it is interesting to note that the quantum correction term \( Q_{0,4} \) has a nontrivial term depending on \( \hbar^4 \) but this term vanishes on solutions of (5.20).

In general, these equations are difficult to solve and, indeed, a full classification of their solutions is still an open question. Early research on third-order integrals of motion was first performed by Drach [41]. He considered the case of one third-order integral of motion (in addition to the Hamiltonian) in 2D complex space in classical mechanics. He found ten different complex potentials which allow a third-order integral. Later it was shown that seven of them are actually quadratically superintegrable and the third-order integral is reducible, i.e. is the Poisson commutator of two second-order integrals [164, 187].

One approach to solving these systems which has lead to the most complete classification is to assume separation of variables as well as the existence of a third-order integral; hence the systems are superintegrable with integrals of degree 2 and 3. This has been completed in Cartesian coordinate [64], polar coordinates [186] and parabolic coordinates [149].
5.3.1. Cartesian coordinates. This case was considered in [64]. Here we review these classification results. For uniformity we use the same approach as was used for the potentials separable in polar [186] and parabolic coordinates [149]. The results agree with those of [64]. Also note that the mass here is normalized to $m = 2$ while the results in [64] have $m = 1$.

Suppose that the Hamiltonian admits separation of variables in Cartesian coordinates as well as a third-order integral. In this case, the potential separates as $V = V_1(x) + V_2(y)$ and the linear compatibility condition (5.26) reduces to

$$-F_3 V_{1,xxx} + 2(F_{2,y} - F_{3,x}) V_{1,xx} - (3F_{1,yy} - 2F_{2,xy} + F_{3,xt}) V_{1,x}$$

$$= F_2 V_{2,yyy} + (F_{2,y} - F_{3,x}) V_{2,yy} + (F_{2,xy} - 2F_{3,xy} + 3F_{4,xt}) V_{2,y}.$$  

(5.28)

Differentiating (5.28) twice with respect to $x$ gives two linear ODEs for $V_1$:

$$(3A_{300} x^2 + 2A_{201} x + A_{102}) V_1^{(5)} + (36A_{300} + 12A_{201}) V_1^{(4)} + 84A_{300} V_1^{(3)} = 0,$$  

(5.29)

and

$$(-A_{111} x - A_{210} x^2 - A_{012}) V_1^{(5)} + (-12A_{210} x - 6A_{111}) V_1^{(4)} - 28A_{210} V_1^{(3)} = 0.$$  

(5.30)

Differentiating (5.28) twice with respect to $y$ gives two linear ODEs for $V_2$:

$$(3A_{300} y^2 - 2A_{210} y + A_{102}) V_2^{(5)} + (36A_{300} y^2 - 12A_{210}) V_2^{(4)} + 90A_{300} V_2^{(3)} = 0,$$  

(5.31)

and

$$(A_{201} y^2 - A_{111} y + A_{021}) V_2^{(5)} + (12A_{210} y^2 - 6A_{111}) V_2^{(4)} + 30A_{201} V_2^{(3)} = 0.$$  

(5.32)

$V_1$ will satisfy a linear ODE if either (5.29) or (5.30) are nontrivially satisfied. For these equations to be satisfied identically, we must require

$$A_{300} = A_{111} = A_{210} = A_{012} = A_{102} = 0.$$  

(5.33)

$V_2$ will also satisfy a linear equation unless equations (5.31) and (5.32) are satisfied identically when

$$A_{300} = A_{111} = A_{210} = A_{201} = A_{021} = A_{120} = 0.$$  

(5.34)

Due to the symmetry in the coordinates $x$ and $y$, there are essentially three possibilities for solutions of the determining equations:

Case 1: Both $V_1$ and $V_2$ satisfy nonlinear equations. This is the case when all four equations (5.29)–(5.32) are satisfied identically; this means that all the constants of (5.33) and (5.34) are zero. In this case, the only nonzero constants are $A_{003}$ and $A_{003}$, equations (5.23)–(5.25) can then be solved and the nonlinear equation (5.27) separates. Up to translation in $x$, $y$ and the potential, it remains to only solve the following nonlinear ODEs

$$12V_1^2 - h^2 V_1^{(3)} = A_{003} \sigma x,$$  

(5.35)

and

$$12V_2^2 - h^2 V_2^{(3)} = A_{003} \sigma y.$$  

(5.36)

If $\sigma = 0$, the solutions can be expressed as a sum of Weierstrass elliptic functions in $x$ and $y$, respectively. In this case the system is not superintegrable since the third-order integrals are algebraically dependent on the second-order ones. Otherwise, the solutions are given in terms of solutions of the first Painlevé equation ($\mathcal{P}_I$) [73]

$$V_0 = 2h^2 \omega_1^2 P_1(\omega_1, x) + 2h^2 \omega_2^2 P_1(\omega_2, x), \quad \omega_1 = \left(\frac{A_{003} \sigma}{h^3}\right)^{\frac{1}{2}}, \quad \omega_2 = \left(\frac{A_{003} \sigma}{h^3}\right)^{\frac{1}{2}}.$$  

(5.37)

In the classical case, $h = 0$, the potential becomes

$$V = \pm \sqrt{B_1} x \pm \sqrt{B_2} y.$$  

(5.38)
With these restrictions, the determining equations can be solved explicitly and the potential satisfies a nonlinear equation. This is the case when only the two equations (5.29) and (5.30) are satisfied identically; this means that all the constants of (5.33) are zero.

If $a = b = 0$, then the system is translation invariant in $x$; this case was addressed in [65].

If $a = b = 0$, then the system is translation invariant in $x$; this case was addressed in [65].

With these restrictions, the determining equations can be solved explicitly and the potential satisfies the equation,

$$h^2V_{2,yy} = 12V_2^3 + \sigma_1V_2 + \sigma_2,$$  \hspace{1cm} (5.39)

for $\sigma_1$ and $\sigma_2$ some constants of integration. In the classical case ($\hbar = 0$) the leading order term in (5.39) vanishes and the only solution is $V_2 = \text{const}$ and this corresponds to free motion. In the quantum case, we can integrate (5.39) to obtain a first-order equation of the form

$$h^2(V_2)^2 = 8(V - c_1)(V - c_2)(V - c_3),$$  \hspace{1cm} (5.40)

where $c_i$ are constant (roots of a cubic equation). If all of the $c_i$ are real and different we obtain, depending on the initial conditions we impose, two types of solutions in terms of Jacobi elliptic functions

$$V_b = 2(h\omega)^2k^2\text{sn}^2(\omega y, k), \quad \text{or} \quad V_c = \frac{2(h\omega)^2}{\text{sn}^2(\omega y, k)}.$$  \hspace{1cm} (5.41)

If the roots satisfy $c_{1,2} = p + iq$ with $p, q \in \mathbb{R}$, $q > 0$, $c_3 \in \mathbb{R}$, the solution is

$$V_d = \frac{2(h\omega)^2}{2 \text{cn}^2(\omega y, k) + 1},$$  \hspace{1cm} (5.42)

(the constants $\omega \in \mathbb{R}$ $0 < k < 1$ are expressed in terms of $c_1$, $c_2$, $c_3$). The potentials $V_c$ and $V_d$ are singular and all three are periodic.

If the two roots coincide (i.e. $k = 0$ or $k = 1$) we get elementary solutions. They yield superintegrable potentials of the form

$$V_e = \frac{2(h\omega)^2}{\cosh^2(\omega y, k)}, \quad V_f = \frac{(h\omega)^2}{\sin^2(\omega y, k)}, \quad V_g = \frac{2(h\omega)^2}{\sinh^2(\omega y, k)}.$$  \hspace{1cm} (5.43)

Here $V_e$ corresponds to the well-known soliton solution of the Korteweg–de Vries equation, $V_f$ is a 'singular soliton' solution and $V_f$ is periodic and singular. If all three roots coincide, we re-obtain a known superintegrable potential $2h^2/x^2$.

There are two cases remaining, namely $a \neq 0$ or $b \neq 0$ in the potential $V_1$. In the first case, (5.25)–(5.25) can then be solved and (5.27) reduces to two nonlinear equations. These equations yield trivial solutions (i.e. solutions which satisfy linear equations as in Case 3) unless $A_{030} = 0$ and $A_{120} \neq 0$. In this case, (5.27) reduced to a single nonlinear ODE

$$0 = -h^2V_1^{(4)} - 12a(xV_1)' + 12(V_1^2)'' - 2ax^2V_1''' + a^2x^2.$$  \hspace{1cm} (5.44)

As shown in [64], where the analysis is explained in more detail, the solutions can be expressed in terms of solutions of the fourth Painlevé equation $P_4$ [73]. The potential becomes

$$V_b = a(x^2 + \gamma^2) \pm h^2\sqrt{4aP_4} \left( x, \frac{-4\gamma}{h^2} \right) + 4aP_4 \left( x, \frac{-4\gamma}{h^2} \right) + 4aP_4 \left( x, \frac{-4\gamma}{h^2} \right).$$  \hspace{1cm} (5.45)

In the classical limit (as $h \rightarrow 0$) the potential satisfies

$$c^2x^2 - d^2 + 2d(V_1 - ax^2)(3V_1 + ax^2) = (9V_1 - ax^2)(V_1 - ax^2)^3,$$  \hspace{1cm} (5.46)

where $c$ and $d$ are arbitrary constants. It is interesting to note that it is possible to recover both the simple harmonic oscillator or the anisotropic oscillator with a ratio of 1 : 3 as special cases of both the classical and quantum system [64].
Finally, in the case that \( a = 0 \) and \( b \neq 0 \), again (5.23)–(5.25) can be solved and (5.27) reduces to two nonlinear equations. Depending on the choices of nonzero \( A_{ijk} \) (the leading terms in the third-order integral, the two nonlinear equations reduce to

\[
h^2 V''_i = 12V_i^2 + \lambda x + k, \quad A_{030} \neq 0
\]

or

\[
0 = \left(-3(V_i')^2 + \frac{h^2}{4}V_i^{(3)}\right)' + b \left((aV_i')' + 2V_i\right) \quad A_{030}A_{021} \neq 0.
\]

The solutions for this quantum system (5.47) are given in terms of solutions to the first Painlevé equation \( P_1 \), (\( V_i \) below) and the solutions for (5.48) are given in terms of solutions to the second Painlevé equation \( P_2 \), (\( V_j, V_k \) below)

\[
V_i = ay + 2h^2 a^2 P_1(awx),
\]

\[
V_j = bx + ay + 2(2hb)^2 P_1 \left(\frac{2b}{h^2}\right)x, 0
\]

\[
V_k = by + 2(2h^2a^2)\left(P_2(-4ah^2)x, \kappa) + P_2'(-(4ah^2)x, \kappa)\right).
\]

In the classical limit the solutions to (5.47) and (5.48) are given, respectively, by

\[
V_i = ay + b \sqrt{x}, \quad V_m = ay + V_i(x), \quad d = V_i(V_i - bx)^2.
\]

Throughout, \( a, b, c \) and \( d \) are arbitrary constants. We stress that the classical limit of the exotic potentials are always singular and we must take the limit of the determining equations rather than the limit of the solutions.

**Case 3: \( V_i \) and \( V_2 \) satisfy linear equations.** In the cases where at least one of the equations (5.29)–(5.32) is not satisfied identically, the potential functions satisfy linear equations and in particular are rational functions. In this case, the method of solving the system is to simply solve these linear ODEs for \( V_i \) and \( V_2 \), then solve (5.23)–(5.25) and replace the solutions into (5.27). The equation (5.27) then becomes a functional equation for the parameters of the system which must hold for all values of \( x \) and \( y \). This restricts the possible cases to the following:

\[
V = \begin{cases}
\frac{h^2}{4a}(x^2 + y^2) + \frac{2h^2}{(x-a)^2} + \frac{2h^2}{(x+a)^2}, & \text{also classical, 3:1 oscillator,} \\
\frac{h^2}{4a}(x^2 + y^2) + \frac{2h^2}{y^2} + \frac{2h^2}{(x+a)^2}, & \\
\frac{h^2}{4a}(9x^2 + y^2) + \frac{2h^2}{(y+a)^2} + \frac{2h^2}{(y-a)^2}.
\end{cases}
\]

Again, this list is taken from [64] although we have omitted the systems which are second-order superintegrable. Recall that if a system is second-order superintegrable then it will have an integral of third-order obtained from taking the commutator (or Poisson commutator) of the two integrals which are not the Hamiltonian. Also, several of these systems are classically superintegrable, where \( h \) is considered a constant not set to 0. For example, the harmonic
oscillator with rational frequencies is superintegrable even with the presence of singular terms [47, 169]. However the integrals will be higher-order polynomials in the momenta. The specifically quantum potentials here indicate the existence of a factorized form of the integrals for certain, \( h \)-dependent, choices of constants.

This completes the classification of third-order superintegrable systems which admit separation of variables in Cartesian coordinates.

5.3.2. Polar coordinates. Now, let us consider potentials allowing separation of variables in polar coordinates [186]

\[
V = R(r) + \frac{1}{r^2} S(\theta). \tag{5.54}
\]

The linear compatibility condition reduces to

\[
0 = r^4 F_1 r''' + \left(2 r^2 F_{3,r} - 2 r^2 F_{2,\theta} + 3 r \left(2 r^2 F_3 - F_1\right)\right) R''
+ \left(r^3 F_{3,rr} + 2 r^2 (3 F_{3,r} + 3 F_3 - F_{2,\theta}) - 4 r F_{2,\theta} + 3 F_{1,\theta}\right) R'
+ \frac{1}{r} F_2 S'' - \frac{1}{r^2} \left(2 r^2 F_{3,r} - 2 r F_{2,\theta} + 6 F_1\right) S'
+ \frac{1}{r^3} \left(3 r^3 F_{3,rr} + 6 r^2 F_{3,\theta} - 2 r^2 F_{2,\theta} + 3 r^2 F_{3,\theta} + r (F_{2,\theta} - 2 F_3) - 12 F_{1,\theta}\right) S
- \frac{1}{r^4} \left(2 r^2 F_{3,rr} - 12 r^2 F_3 + 4 r (3 F_{3,r} - F_{2,\theta}) + 4 r F_{2,\theta} + 6 F_{1,\theta} + 18 F_1\right) S, \tag{5.55}
\]

where the \( F_i \) are given by

\[
F_1 = A_1 \cos 3 \theta + A_2 \sin 3 \theta + A_3 \cos \theta + A_4 \sin \theta,
F_2 = -3 A_1 \sin 3 \theta + 3 A_2 \cos 3 \theta - A_3 \cos \theta + A_4 \sin \theta + \frac{B_1 \cos 2 \theta + B_2 \sin 2 \theta + B_0}{r},
F_3 = -3 A_1 \cos 3 \theta - A_2 \sin 3 \theta + A_3 \cos \theta + A_4 \sin \theta + \frac{-2 B_1 \sin 2 \theta + 2 B_2 \sin 2 \theta}{r},
F_4 = A_1 \sin 3 \theta - A_2 \cos 3 \theta - A_3 \sin \theta + A_4 \cos \theta - \frac{B_1 \cos 2 \theta + B_2 \sin 2 \theta + B_0}{r^2} - \frac{C_1 \sin \theta + C_2 \sin \theta}{r^3} + D_0,
\]

with

\[
A_1 = \frac{A_{030} - A_{012}}{4}, \quad A_2 = \frac{A_{021} - A_{003}}{4}, \quad A_3 = \frac{A_{030} + A_{012}}{4},
A_4 = \frac{3 A_{030} + A_{021}}{4}, \quad B_1 = \frac{A_{120} - A_{102}}{2}, \quad B_2 = \frac{A_{111}}{2}, \quad B_0 = \frac{A_{120} + A_{102}}{2},
C_1 = A_{210}, \quad C_2 = A_{201}, \quad D_0 = A_{300}.
\]

Here we summarize the results of [186]. Beginning with the linear compatibility condition, (5.55), it is possible to obtain by differentiating with respect to \( r \) several differential consequences which are ODEs for the function \( R(r) \). These are then solved and the possible choices are reduced to 4 cases: \( R(r) = a r^2, R(r) = a/r, R(r) = 0, \) or the equations for \( R(r) \) vanish by choice of the \( A_{jkl} \).

In the case that \( R(r) = a r^2 \), the only solution is the second-order superintegrable isotropic oscillator with singular terms (E1/Smorodinsky–Winternitz I). Similarly, the case where \( R(r) = a/r \) leads to a Kepler–Coulomb potential with singular term (E16/Smorodinsky–Winternitz III).
Finally, consider the cases that the equations for $R$ are satisfied identically. This can occur if $R(r)$ arbitrary and only $A_{300} \neq 0$ or $R(r) = 0$. In the first case, this leads to the potential [186]

$$V(r, \theta) = R(r) + \frac{2\hbar^2}{r^2} \mathcal{P}(\theta, t_2, t_3),$$

where $R(r)$ is arbitrary and $\mathcal{P}(\theta, t_2, t_3)$ is the Weierstrass elliptic function [24]. This potential allows a third-order integral, however it is algebraically related to the second-order one and the system is hence not superintegrable. However, if $R(r) = 0$ then the system admits two third-order integrals one of which is algebraically independent of the two second-order integrals.

The remaining cases when $R(r) = 0$ include the rational three-body Calogero system (or a special case of the TTW system [185])

$$V = \frac{a}{r^2 \cos^2(3\theta)},$$

which is known to be superintegrable [197]. There are also potentials which satisfy nonlinear equations. These occur when $R(r) = 0$ and the linear compatibility condition (5.55) for $S(\theta)$ are satisfied trivially, i.e. when all the constants except $A_{300}, A_{210}$ and $A_{201}$ are 0. This leads to a quantum superintegrable potential expressed in terms of the sixth Painlevé transcendent [73], $\mathcal{P}_6(\sin \theta / 2)$. The potential is given by

$$V(r, \theta) = 2 \frac{\hbar^2}{r^2} \left( \frac{\pm 8\hbar^2 \cos \theta W(x_\pm) + 4\beta_1 + \hbar^2}{4 \sin^2 \theta} \right), \quad x_\pm = \cos^2 \theta, \sin^2 \theta, \quad (5.56)$$

with

$$W(x) = \frac{x^2(x - 1)^2}{4\mathcal{P}_6(\mathcal{P}_6 - 1)(\mathcal{P}_6 - x)} \left[ \mathcal{P}_6' - \mathcal{P}_6 \left( \frac{\mathcal{P}_6(\mathcal{P}_6 - 1)}{x(x - 1)} \right)^2 + \frac{1}{8} \left( 1 - \sqrt{2\gamma_1} \right)^2 (1 - 2\mathcal{P}_6) - 1 \right]$$

$$- \frac{1}{4} \gamma_2 \left( 1 - \frac{2x}{\mathcal{P}_6} \right) - \frac{1}{4} \gamma_3 \left( 1 - \frac{2(\mathcal{P}_6 - 1)}{\mathcal{P}_6 - 1} \right) + \left( \frac{1}{8} - \frac{\gamma_2}{4} \right) \left( 1 - \frac{2(\mathcal{P}_6 - 1)}{\mathcal{P}_6 - x} \right). \quad (5.57)$$

The classical limit of this system $\hbar = 0$ is given by

$$V = \frac{T}{r^2}, \quad (5.58)$$

where $T$ satisfies a first-order linear ODE,

$$3z^2(1 + z^2)T'' + 2zT' - T^2 + 2(\beta_1 z^2 - \beta_2 z)T' + 2\beta_2 T + \frac{K_1}{2} = 0, \quad z = \tan \theta. \quad (5.59)$$

To summarize, for systems which admit separation of variables in polar coordinates as well as a third-order integral, there are the second-order superintegrable systems which separate in polar coordinates, the rational three-body Calogero system, as well as two new quantum systems depending either the Weierstrass elliptic function or the sixth Painlevé transcendent. Finally, there is also a new classically superintegrable system with the radial part of the potential satisfying a first-order nonlinear ODE (5.58), (5.59).

5.3.3. **Parabolic coordinates.** In the final case considered here, let us assume that the potential separates in parabolic coordinates, $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi \eta$,

$$H = -\frac{\hbar^2}{\xi^2 + \eta^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + V(\xi, \eta), \quad (5.60)$$

$$V(\xi, \eta) = \frac{W_1(\xi) + W_2(\eta)}{\xi^2 + \eta^2}. \quad (5.61)$$
The compatibility condition (5.26) becomes

\[
0 = F_3 V_{\xi\xi\xi} + (3 F_4 - 2 F_2) V_{\xi\xi\eta} + (3 F_1 - 2 F_3) V_{\xi\eta\eta} + F_2 V_{\eta\eta\eta}
\]

\[
+ \left( 2(F_{3\xi} - F_{2\eta}) - \frac{3\xi F_1 - 6\eta F_2 + 7\xi F_3}{\xi^2 + \eta^2} \right) V_{\xi\xi}
\]

\[
+ \left( 2(F_{2\eta} - F_{3\xi}) - \frac{3\xi F_4 - 6\xi F_5 + 7\eta F_2}{\xi^2 + \eta^2} \right) V_{\eta\eta}
\]

\[
+ \left( 2(3F_1 - F_{2\xi} - F_{3\eta} + 3F_{4\xi}) - \frac{21\eta F_1 - 5\eta F_3 - 5\xi F_2 + 21\xi F_4}{\xi^2 + \eta^2} \right) V_{\xi\eta}
\]

\[
+ AV_{\eta} + BV_{\xi},
\]

where

\[
A = F_{2\eta\eta} - 2F_{3\eta\xi} + 3F_{4\xi\xi} - \frac{7\eta F_{2\eta} - \xi F_{2\xi} + 6\xi F_{3\eta} + 6\eta F_{3\xi} - 3\eta F_{4\eta} - 21\xi F_{4\xi}}{\xi^2 + \eta^2}
\]

\[
+ 2 \frac{21\xi^2 F_4 + F_2 \xi^2 + 7\eta^2 F_2 - 12\xi \eta F_5 + 3F_4 \eta^2}{(\xi^2 + \eta^2)^2},
\]

\[
B = 3F_{1\eta\eta} - 2F_{3\eta\xi} + F_{3\xi\xi} - \frac{21\eta F_{1\eta} + 3\xi F_{1\xi} - 6\xi F_{2\eta} - 6\eta F_{2\xi} + \eta F_{3\eta} - 7\xi F_{3\xi}}{\xi^2 + \eta^2}
\]

\[
+ 2 \frac{3F_1 \eta^2 + 3F_1 \xi^2 + 21\xi F_2 - 12\xi \eta F_2 + 7\xi^2 F_3}{(\xi^2 + \eta^2)^2}.
\]

As explained in [149], these equations can be solved in a similar manner to the previous cases of Cartesian and polar coordinates. However, unlike the previous cases, all such superintegrable systems are second-order superintegrable. It is not difficult to see that the linear compatibility condition (5.62) vanishes identically only when all of the \(A_{\mu\nu}\) are zero. Thus, the potential terms \(W_1(\xi), W_2(\eta)\) always satisfy some nontrivial linear ODE. All that remains is to show by brute force that the only possible solutions of the determining equations are known second-order superintegrable systems.

Thus, we have shown that for systems which admit a third-order integral and separation of variables, there are new, truly third-order systems in the case of Cartesian and polar coordinates although not for parabolic. It is unknown what happens in the case of general elliptic coordinates, though we conjecture that, in this case, there will be no nontrivial solutions for the linear compatibility conditions, in analogy with the parabolic cases. Hence, it would seem that the existence of potentials which satisfy nonlinear equations exist only when there is separation in subgroup type coordinates. This is still an open question.

5.3.4. Summary of the case of third-order integrals. First of all, let us sum up the cases of third-order superintegrability that have been obtained. In classical mechanics they are given in table 1. We omit those for which the third-order integral is a Poisson commutator of two second-order integrals, since such systems are already second-order superintegrable. The potentials (C.4) and (4.21) were already known (see [75, 197]). The others are new. The finite trajectories for all of these potentials have been shown to be periodic (as they must be) [128, 129].

The results in the quantum case are considerably richer and are reproduced in table 2. The systematic study of superintegrable systems with at least one third-order integral of motion confirms in a striking manner the differences between classical and quantum integrability. In the case of second-order integrability and superintegrability, it suffices to determine all classical superintegrable systems. The quantum systems can be obtained from
the classical ones by the usual quantization procedure. The potentials are the same in both cases. The integrals of motion are also the same up to symmetrization of the operators. For third and higher-order integrals the situation is quite different. As we have shown in section 5.1 and especially in section 5.3 the determining equations in the quantum case contain terms proportional to $\hbar^2$ (or powers of $\hbar$). Thus the potentials in the quantum case can be quite different than in the classical one, as can the integrals of motion. An extreme case of this occurs when the potential is proportional to $\hbar^2$ and in the classical limit corresponds to free motion.

The Painlevé transcendent potentials have interesting polynomial algebras [124, 128]. Their representations can be used to calculate energy levels. There exist interesting connections with supersymmetric quantum mechanics that make it possible to calculate the wave functions in such potentials [122, 123, 129]. This has so far been carried out for the potential expressed in terms of the Painlevé transcendent $P_4$ [123]. Interestingly it turns out that this potential is an isospectral deformation of the harmonic oscillator: the spectrum depends linearly on one quantum number with integer values and hence coincides with that of the isotropic 2D harmonic oscillator. The degeneracy of the energy levels is however different and resembles that of an anisotropic harmonic oscillator. An analysis of these questions goes beyond the scope

| $V$ | Original reference |
|-----|---------------------|
| $a(9x^2 + y^2)$, $\pm \sqrt{b_1 x} \pm \sqrt{b_2 y}$, | From [64], C.4 |
| $a y^2 + V_1$ where $V_1$ satisfies equation (5.46), | C.6 |
| $a y + b \sqrt{x}$, | C.7 |
| $a y^2 + V_1$ where $(V_1 - bx)^2 V_1 = d$, | C.8 |
| $\frac{\mu}{\Sigma}$ where $T$ satisfies equation (5.59), | From [186], (4.21) |

| $V$ | Original reference |
|-----|---------------------|
| $\frac{\hbar^2}{2}(x^2 + y^2) + \frac{2\hbar^2}{(x^2 + y^2)} + \frac{2\hbar^2}{(x^2 + y^2)}$, | From [64], (Q.5) |
| $\frac{\hbar^2}{2}(x^2 + y^2) + \frac{2\hbar^2}{(x^2 + y^2)} + \frac{2\hbar^2}{(x^2 + y^2)}$, | (Q.6) |
| $2\hbar^2 \omega_1^2 \rho_1(\omega_1 \rho_1) + 2\hbar^2 \omega_2^2 \rho_2(\omega_2 \rho_2)$, | (Q.7) |
| $a(x^2 + y^2) \pm \hbar^2 \sqrt{4a P_4''(x, \frac{\mu}{\Sigma}) + 4a P_4''(x, \frac{\mu}{\Sigma}) + 4a x P_4(x, \frac{\mu}{\Sigma})}$, | (Q.8) |
| $b x + a y + 2(2b \hbar^2) \frac{\hbar}{2} P_2(\frac{\mu}{\Sigma})$, | (Q.9) |
| $a y + 2(2b \hbar^2) \frac{\hbar}{2} P_2(\frac{\mu}{\Sigma})$, | (Q.10) |
| $2(h \omega)^2 k^2 \sin^2(\omega_1 \rho_1)$, | From [65], (4.11) |
| $\frac{\hbar^2}{\Sigma} P(\theta, t_2, t_3)$, | From [186], (4.21) |

| $V$ | Original reference |
|-----|---------------------|
| $\frac{\hbar^2}{2}(x^2 + y^2) + \frac{2\hbar^2}{(x^2 + y^2)} + \frac{2\hbar^2}{(x^2 + y^2)}$ | From [64], (Q.5) |
| $\frac{\hbar^2}{2}(x^2 + y^2) + \frac{2\hbar^2}{(x^2 + y^2)} + \frac{2\hbar^2}{(x^2 + y^2)}$, | (Q.6) |
| $2\hbar^2 \omega_1^2 \rho_1(\omega_1 \rho_1) + 2\hbar^2 \omega_2^2 \rho_2(\omega_2 \rho_2)$, | (Q.7) |
| $a(x^2 + y^2) \pm \hbar^2 \sqrt{4a P_4''(x, \frac{\mu}{\Sigma}) + 4a P_4''(x, \frac{\mu}{\Sigma}) + 4a x P_4(x, \frac{\mu}{\Sigma})}$, | (Q.8) |
| $b x + a y + 2(2b \hbar^2) \frac{\hbar}{2} P_2(\frac{\mu}{\Sigma})$, | (Q.9) |
| $a y + 2(2b \hbar^2) \frac{\hbar}{2} P_2(\frac{\mu}{\Sigma})$, | (Q.10) |
| $2(h \omega)^2 k^2 \sin^2(\omega_1 \rho_1)$, | From [65], (4.11) |
| $\frac{\hbar^2}{\Sigma} P(\theta, t_2, t_3)$, | From [186], (4.21) |
of the present review, though for $\mathcal{P}_2$ they are treated in [123]. The fact that the Schrödinger equation with a Painlevé function potential can be solved algebraically lends further credence to the conjecture that all (Euclidean) maximally superintegrable potentials are exactly solvable [181].

5.4. Fourth-order integrals

We consider the fourth-order classical operator defined by

$$X = \sum_{j=0}^{4} f_{j,0} p_{2}^{4-j} + \sum_{j=0}^{2} f_{j,2} p_{2}^{2-j} + f_{0,4}. \quad (5.63)$$

Again, the highest-order determining equations require that the term $f_{j,0} p_{2}^{4-j}$ be in the enveloping algebra of $E(2, \mathbb{R})$ and so we can rewrite it as

$$\sum_{j=0}^{4} f_{j,0} p_{2}^{4-j} = \sum_{i+j+k=4} A_{ijk} p_{i}^{4} L_{j}^{k}, \quad L_{3} = xp_{2} - yp_{1}. \quad (5.64)$$

In analogy with the third-order case and to draw comparisons with the standard form of the third-order equations in the literature, we make the following definitions:

$$f_{22} = g_{1}, \quad f_{12} = g_{2}, \quad f_{0,2} = g_{3}, \quad (5.65)$$

and

$$F_{1} \equiv 4(-A_{004} y^{4} + A_{103} y^{3} - A_{202} y^{2} + A_{301} y - A_{400}), \quad (5.66)$$

$$F_{2} \equiv 4A_{004} y^{3} + A_{013} y^{2} - 3A_{103} y^{2} - A_{112} y^{2} + 2A_{202} y - A_{211} y - A_{301} x - A_{310}, \quad (5.67)$$

$$F_{3} \equiv -6A_{004} y^{2} - 3A_{013} y^{2} + 3A_{103} y^{2} - A_{002} y^{2} - A_{102} y^{2} - A_{211} y - A_{202} y - A_{212} y, \quad (5.68)$$

$$F_{4} \equiv 4A_{004} y - A_{013} x^{2} + 3A_{013} x^{2} - A_{112} x^{2} + 2A_{202} x - A_{212} x - A_{211} x, \quad (5.69)$$

$$F_{5} \equiv -4(A_{004} x^{4} + A_{013} x^{3} + A_{022} x^{2} + A_{031} x + A_{040}). \quad (5.70)$$

In this form, the determining equations for the classical systems are given by

$$0 = 2g_{1,x} + F_{1} V_{x} + F_{2} V_{y}, \quad (5.66)$$

$$0 = 2g_{2,x} + 2g_{1,y} + 3F_{2} V_{x} + 2F_{3} V_{y}, \quad (5.67)$$

$$0 = 2g_{3,x} + 2g_{2,y} + 2F_{3} V_{x} + 3F_{4} V_{y}, \quad (5.68)$$

$$0 = 2g_{3,y} + F_{2} V_{x} + F_{3} V_{y}, \quad (5.69)$$

and

$$f_{04,x} = g_{1} V_{x} + \frac{1}{2} g_{2} V_{y}, \quad (5.70)$$

$$f_{04,y} = \frac{1}{2} g_{2} V_{x} - g_{3} V_{y}. \quad (5.71)$$

The compatibility equations for (5.66)–(5.69) are linear and given by

$$0 = \partial_{xy} (F_{1} V_{x} + F_{2} V_{y}) - \partial_{xy} (3F_{2} V_{x} + 2F_{3} V_{y}) + \partial_{xy} (2F_{3} V_{x} + 3F_{4} V_{y}) - \partial_{xxx} (F_{2} V_{x} + F_{3} V_{y}). \quad (5.72)$$

Equations (5.70) and (5.71) have nonlinear compatibility equations. For the quantum integral, we take the following choice of symmetrization

$$X = \sum_{j+k+l=4} A_{jkl} l_{j}^{k} p_{l}^{k} + \frac{(-i\hbar)^{2}}{2} (\{g_{1}, \delta_{1}^{2}\} + \{g_{2}, \delta_{2}^{2}\} + \{g_{3}, \delta_{3}^{2}\}) + f_{04}. \quad (5.73)$$
Note again that, as shown in [156], a different choice would lead to an $\hbar^2$ correction term in $g_1$, $g_2$ or $g_3$. The determining equations are given by the first four equations for the classical system given (5.66)–(5.69) and the last two equations with a correction term of order $\hbar^2$

$$f_{04,x} = 2g_1 V_x + g_2 V_y + \frac{\hbar^2}{4}((F_2 + F_4)V_{xx} - 4(F_1 - F_3)V_{xy} - (F_2 - F_4)V_{yy} + (F_2,y - F_{3,x})V_{x} - (13F_{1,y} + F_{4,x})V_{y} - 4(F_2,y - F_{3,x})V_{yy} + 2(6A_{400}x^2 + 6A_{400}y^2 + 3A_{301}x - 29A_{310}y + 9A_{220} + 3A_{202})V_x + 2(56A_{400}xy + 13A_{310}y - 13A_{301}y + 3A_{211})V_y),$$

$$(5.73)$$

$$f_{04,y} = g_2 V_x + 2g_3 V_y + \frac{\hbar^2}{4}(-(F_2 + F_4)V_{xx} + 4(F_1 - F_3)V_{xy} + (F_2 - F_4)V_{yy} + 4(F_1,y - F_{3,x})V_{x} - (F_2,y + 13F_{3,x})V_{y} - (F_1,y - 3F_{3,x})V_{yy} + 2(56A_{400}xy - 13A_{310}y + 13A_{301}y + 3A_{211})V_x + 2(62A_{400}x^2 + 6A_{400}y^2 + 29A_{310}x - 3A_{301}y + 9A_{220} + 3A_{220})V_y).$$

$$(5.74)$$

Note that the terms on the second line of each equation are the quantum corrections terms $Q_{0,4}$ and $Q_{1,4}$ and, similarly to case $N = 2, 3$, the terms proportional to $\hbar^4$ in (5.11) vanish identically.

The only known attempt to solve these equations directly, was given in [155] where a nonseparable superintegrable system was constructed with a third and fourth-order integral. What resulted was the following system, which is a quantum version of a special case of the Drach systems:

**Theorem 18.** The operator triplet $(H, X, Y)$ with

$$H = \frac{1}{2}(p_1^2 + p_2^3) + \frac{\alpha y}{x^2} = \frac{5\hbar^2}{72\alpha^2}$$

$$X = 3p_1^2p_2 + 2p_2^3 + \left\{ \frac{9\alpha}{2x^2}, p_1 \right\} + \left\{ \frac{3\alpha}{x^2}, \frac{5\hbar^2}{24\alpha^2}, p_2 \right\}$$

$$Y = p_1^4 + \left\{ \frac{2\alpha y}{x^3}, -\frac{5\hbar^2}{36\alpha^2}, p_1 \right\} - \left\{ 6x^4 \alpha, p_1 p_2 \right\} = \frac{2\alpha^2(9x^2 - 2y^2)}{x^2} - \frac{5\alpha \hbar^2y}{9\alpha^2} + \frac{25\hbar^4}{1296\alpha^4}$$

constitutes a quantum superintegrable system that does not allow multiplicative separation of variables in the Schrödinger equation in any system of coordinates.

The work on third-order superintegrable systems which admit separation of variables in one coordinate system represents the only successful effort to classify solutions for higher-order superintegrability directly from the structure equations. The study of third-order superintegrability has lead to qualitatively new types of superintegrable quantum potentials, the Painlevé transcendent ones. These are solutions of second-order nonlinear ODEs and they do not satisfy any linear ODE. The system of determining equations obtained in section 5.1 is overdetermined. Hence there will always exist compatibility conditions for the potential, some of them linear. In the case of potentials allowing separation of variables in subgroup type coordinates (Cartesian and polar) the compatibility conditions could be satisfied trivially. This was done by setting some of the leading coefficients in the third-order integral equal to 0 without out annihilating completely the highest-order term. For potentials separating in parabolic coordinates this was not possible. A study of higher-order superintegrability, in particular the fourth-order case, could lead to interesting results, such as new transcendent functions with the Painlevé property.

As demonstrated by the example of fourth-order integrals, the number of determining equations and unknowns grow quickly and it becomes computationally inefficient to
address these determining equations directly. For a long time research into higher-order superintegrability was stymied by the intractability of the determining equations for such higher-order integrals. However, as we will see in the next sections, new methods of construction and verification of higher-order integrals, beyond brute force analysis of the determining equations, has led to a flourishing of new results.

6. Higher-order classical superintegrable systems

6.1. The problems and the breakthroughs

Until very recently, few examples beyond the anisotropic oscillator were known of superintegrable systems with orders \( N > 3 \) and there was almost no structure theory in either the classical or quantum case. There were two basic issues: (1) How to construct numerous examples of systems of higher order and in many dimensions, to develop enough insight to produce a classification theory. (2) How to compute efficiently the commutators of symmetry operators of arbitrarily high order, to verify superintegrability and determine the structure equations. A breakthrough for the first issue came with the publication of two papers by Tremblay, Turbiner and Winternitz, [184, 185], the first in 2009. They started with the second-order superintegrable system

\[
H = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta}^2 - \omega^2 (x^2 + y^2) + \frac{\alpha}{x^2} + \frac{\beta}{y^2},
\]

with a classical analogue also superintegrable. (Actually, they used the physical version of the Hamiltonian rather than the one used here, so that the sign of the energy flips.) They wrote the Schrödinger equation in polar coordinates \( r, \theta \), separable for this system, and replaced \( \theta \) by \( k \theta \) where \( k = p/q \) and \( p, q \) are relatively prime integers. After renormalization, the TTW system and its classical analogue become

\[
H = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta}^2 - \omega^2 r^2 + \frac{1}{r^2} \left( \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)} \right),
\]

\[
H = \frac{p^2}{r^2} + \frac{1}{r^2} \partial_{\theta}^2 + \omega^2 r^2 + \frac{1}{r^2} \left( \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)} \right).
\]

The authors noted that the Hamilton–Jacobi equation \( H = E \) is still separable in \( r, \theta \) coordinates so it admits \( H \) and the second-order symmetry \( L_2 \) responsible for the separation as symmetries. The TTW system is superintegrable if there exists a third constant of the motion, which may be of arbitrarily high order, dependent on \( k \). They noted that for small values of \( p, q \) these systems were already known to be superintegrable (but usually represented in Cartesian coordinates). For \( k = 1 \) this is the caged isotropic oscillator, for \( k = 1/2 \) it is [E16] in the constant curvature system list in [85], for \( k = 2 \) it is a Calogero system on the line, [25, 30], etc. They conjectured and gave strong evidence that the TTW systems were classically and quantum superintegrable for all rational \( k \). This conjecture had broad influence and led to a flurry of activity to prove the conjectures. It showed how an infinite number of higher-order superintegrable systems could be generated from a single second-order system. It was soon applied to generate numerous other families of higher-order systems and in \( n > 2 \) variables, [77, 78, 80].

The basic problem for proof of the conjectures was exhibiting symmetries of arbitrarily high order. For simple choices such as \( k = 6 \) the expression for the symmetry operator required several printed pages. The problem was solved for odd \( k \) in [163] and first solved in general in the papers [80, 98]. There have been other proofs of the conjecture using different methods:
using the methods of differential Galois theory, [59, 66] by direct computation of the action angle variables and [165] using factorization of the integrals. In [93] a general method was introduced that enabled explicit structure equations to be calculated. We here discuss the present state of the classical theory from this point of view.

6.1.1. The construction tool for classical systems. There are far more verified superintegrable Hamiltonian systems in classical mechanics than was the case in 2009. The principal method for constructing and verifying these new systems requires the assumption that the root system, such as (6.3), admits orthogonal separation of variables. For a Hamiltonian system in 2n-dimensional phase space the variable separation gives us n second-order constants of the motion in involution. We first review a general procedure, essentially the construction of action angle variables, which yields an additional n − 1 constants, such that the set of 2n − 1 is functionally independent. This is particularly of interest when it is possible to extract n new constants of the motion that are polynomial in the momenta. We show how this can be done in many cases, starting with the construction of action angle variables for nD Hamiltonian systems.

Consider a classical system in n variables on a Riemannian manifold that admits separation in orthogonal separable coordinates x. Then there is an n × n nonsingular St¨ackel matrix S = (Sij(xj)), [177] and the Hamiltonian is

\[ H = L_1 = \sum_{i=1}^{n} T_{ii} (p_i^2 + v_i(x_i)) = \sum_{i=1}^{n} T_{ii} p_i^2 + V(x_1, \ldots, x_n), \]

where \( V = \sum_{i=1}^{n} T_{ii} v_i(x_i) \), and T is the matrix inverse to S:

\[ T_{ij} S_{jk} = \sum_{j=1}^{n} S_{ij} T_{jk} = \delta_{ik}, \quad 1 \leq i, k \leq n. \] (6.4)

Here, we must require \( \Pi_{i=1}^{n} T_{ii} \neq 0 \). We define the quadratic constants of the motion \( L_k, \quad k = 1, \ldots, n \) by

\[ L_k = \sum_{i=1}^{n} T_{ii} (p_i^2 + v_i), \quad k = 1, \ldots, n, \quad \text{or} \quad p_i^2 + v_i = \sum_{j=1}^{n} S_{ij} L_j, \quad 1 \leq i \leq n. \] (6.5)

As is well known, \{L_j, L_k\} = 0, \quad 1 \leq j, k \leq n. Furthermore, by differentiating identity (6.4) with respect to \( x_h \) we obtain

\[ \partial_h T_{ik} = -\sum_{j=1}^{n} T_{ih} S_{kj} T_{jk}, \quad 1 \leq h, i, \ell \leq n, \quad S_{\ell h} \equiv \partial_h S_{\ell j}. \]

Now we define nonzero functions \( M_{kj}(x_j, L_1, \ldots, L_n) \) by the requirement \( \{M_{kj}, L_i\} = T_{ij} S_{jk}, \quad 1 \leq k, j, \ell \leq n \). It is straightforward to check that these conditions are equivalent to the differential equations \( 2p_i \partial_j M_{kj} = S_{jk} \). Set \( \tilde{L}_q = \sum_{j=1}^{n} M_{lj}, \quad 1 \leq q \leq n \). Then we have \( \{\tilde{L}_q, L_i\} = \sum_{j=1}^{n} T_{ij} S_{jq} = \delta_{iq} \). This shows that the 2n − 1 functions \( \tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_n \), are constants of the motion and they are functionally independent.

In the special case of \( n = 2 \) dimensions we can always assume that the St¨ackel matrix and its inverse are of the form

\[ S = \begin{pmatrix} f_1 & 1 \\ f_2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 \\ f_1 + f_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ f_2 & -f_1 \end{pmatrix}, \] (6.6)
where \( f_j \) is a function of the variable \( x_j \) alone, \([42]\). The constants of the motion \( \mathcal{L}_1 = \mathcal{H} \) and \( \mathcal{L}_2 \) are given to us via variable separation. We want to compute a new constant of the motion \( \tilde{\mathcal{L}}_2 \) functionally independent of \( \mathcal{L}_1, \mathcal{L}_2 \). Setting \( M_{21} = M, M_{22} = -N \), we see that

\[
2p_1 \frac{d}{dx_1}M = 1, \quad 2p_2 \frac{d}{dx_2}N = 1. \tag{6.7}
\]

from which we determine \( M, N \). Then \( \tilde{\mathcal{L}}_2 = M - N \) is the constant we seek.

6.1.2. Application for \( n = 2 \): the TTW system. As we have seen, for \( n = 2 \) and separable coordinates \( x_1 = x, x_2 = y \) we have

\[
\mathcal{H} = \mathcal{L}_1 = \frac{1}{f_1(x) + f_2(y)} (p_x^2 + p_y^2 + v_1(x) + v_2(y)),
\]

\[
\mathcal{L}_2 = \frac{f_2(y)}{f_1(x) + f_2(y)} (p_x^2 + v_1(x)) - \frac{f_1(x)}{f_1(x) + f_2(y)} (p_y^2 + v_2(y)). \tag{6.8}
\]

The constant of the motion \( \tilde{\mathcal{L}}_2 = M - N \) constructed by solving equations (6.7) is usually not a polynomial in the momenta. We describe a procedure for obtaining a polynomial from \( M - N \), based on the observation that integrals

\[
M = \frac{1}{2} \int \frac{dx_1}{\sqrt{f_1 \mathcal{H} + \mathcal{L}_2 - v_1}}, \quad N = \frac{1}{2} \int \frac{dx_2}{\sqrt{f_2 \mathcal{H} - \mathcal{L}_2 - v_2}},
\]

can often be expressed in terms of multiples of the inverse hyperbolic sine or cosine, and the hyperbolic functions satisfy addition formulas. Thus we will search for functions \( f_j, v_j \) such that \( M \) and \( N \) possess this property. There is a class of prototypes for this construction, namely the second-order superintegrable systems. We start our construction with one of these second-order systems and add parameters to get a family of higher-order superintegrable systems. We illustrate this by considering the TTW system in detail.

The TTW Hamiltonian (6.3) admits a second-order constant of the motion corresponding to separation of variables in polar coordinates, viz

\[
\mathcal{L}_2 = p_\theta^2 + \frac{\alpha}{\cos^2 k \theta} + \frac{\beta}{\sin^2 k \theta}. \tag{6.9}
\]

To demonstrate superintegrability we need to exhibit a third polynomial constant of the motion that is functionally independent of \( \mathcal{H}, \mathcal{L}_2 \). We apply the action angle variable construction above. In terms of the new variable \( r = e^{k \theta} \) the Hamiltonian assumes the canonical form

\[
\mathcal{H} = e^{-2R} \left( p_R^2 + p_\theta^2 + \omega^2 e^{4R} + \frac{\alpha}{\cos^2(k \theta)} + \frac{\beta}{\sin^2(k \theta)} \right).
\]

To find extra invariants we first construct a function \( M(R) \) such that \( [M, \mathcal{H}] = e^{-2R} \), or \( 2p_R \partial_R M = 1 \). This equation has a solution \( M = \frac{1}{\sqrt{L_2}} B \) where

\[
\sinh B = i \frac{(2 \mathcal{L}_2 e^{-2R} - \mathcal{H})}{\sqrt{\mathcal{H}^2 - 4 \omega^2 L_2}}, \quad \cosh B = \frac{2 \sqrt{\mathcal{L}_2} e^{-2R} p_R}{\sqrt{\mathcal{H}^2 - 4 \omega^2 L_2}},
\]

and we also have the relation \( p_R^2 + \mathcal{L}_2 + \omega^2 e^{4R} - e^{2R} \mathcal{H} = 0 \). Similarly \( N(\theta) \) satisfies \( [N, \mathcal{H}] = e^{-2R} \), or \( 2p_\theta \partial_\theta N = 1 \), with solution \( N = -\frac{1}{\sqrt{L_2}} A \) where

\[
\sinh A = i \frac{-\beta + \alpha - \mathcal{L}_2 \cos(2k \theta)}{\sqrt{(L_2 - \alpha - \beta)^2 - 4 \alpha \beta}}, \quad \cosh A = \frac{\sqrt{L_2} \sin(2k \theta) p_\theta}{\sqrt{(L_2 - \alpha - \beta)^2 - 4 \alpha \beta}}.
\]

Then \( M - N \) is a constant of the motion and, since it satisfies \( [M - N, \mathcal{L}_2] \neq 0 \), it is functionally independent of \( \mathcal{L}_2 \).
From these expressions for $M$ and $N$ we see that if $k = p/q$ (where $p, q$ are relatively prime integers) then $-4ip\sqrt{L_2}[N - M] = qA + pB$ and any function of $qA + pB$ is a constant of the motion (not polynomial in the momenta). Rather than dealing with sinh$x$, cosh$x$, we make use of an observation in [59] and work with the exponential function. We have
\begin{equation}
 e^A = \cosh A + \sinh A = \mathcal{X}/U, \quad e^{-A} = \cosh A - \sinh A = \mathcal{X}/U,
\end{equation}
\begin{equation}
 e^B = \cosh B + \sinh B = Y/S, \quad e^{-B} = \cosh B - \sinh B = Y/S,
\end{equation}
\begin{equation}
 X = \sqrt{L_2} \sin(2k\theta)p_0 + i(-\beta + \alpha - L_2 \cos(2k\theta)), \quad Y = 2\sqrt{L_2} e^{-2R} p_R + i(2L_2 e^{-2R} - \mathcal{H}),
\end{equation}
\begin{equation}
 \mathcal{X} = \sqrt{L_2} \sin(2k\theta)p_0 - i(-\beta + \alpha - L_2 \cos(2k\theta)), \quad \mathcal{Y} = 2\sqrt{L_2} e^{-2R} p_R - i(2L_2 e^{-2R} - \mathcal{H}),
\end{equation}
\begin{equation}
 U = \sqrt{(\mathcal{L}_2 - 2 \alpha - \beta)^2 - 4x}, \quad S = \sqrt{\mathcal{R}^2 - 4\alpha^2 \mathcal{L}_2}.
\end{equation}

Now note that $e^{xA+B}$ and $e^{-(xA+B)}$ are constants of the motion, where
\begin{equation}
 e^{xA+B} = (e^A)^q(e^B)^p = \frac{X^q Y^p}{U^qS^p}, \quad e^{-(xA+B)} = (e^{-A})^q(e^{-B})^p = \frac{X^q Y^p}{U^qS^p}.
\end{equation}
Moreover, the identity $e^{xA+B}e^{-(xA+B)} = 1$ can be expressed as
\begin{equation}
 X^q (\mathcal{X})^q Y^p (\mathcal{Y})^p = U^qS^p = P(\mathcal{L}_2, \mathcal{H})
\end{equation}
where $P$ is a polynomial in $\mathcal{L}_2$ and $\mathcal{H}$.

We restrict to the case $p + q$ even; the case $p + q$ odd is very similar, [93]. Let $a, b, c, d$ be nonzero complex numbers and consider the expansion
\begin{equation}
 \frac{1}{2}[((\sqrt{L}_2 a + ib)^q(\sqrt{L}_2 c + id)^p + (\sqrt{L}_2 a - ib)^q(\sqrt{L}_2 c - id)^p]
\end{equation}
\begin{equation}
 = \sum_{0 \leq \ell \leq q, 0 \leq s \leq p} \binom{q}{\ell} \binom{p}{s} b^\ell d^s e^{\ell s} \cdot \frac{L^{(q+p-\ell-s)/2}}{2} [T^{\ell s} + (-i)^{\ell s}],
\end{equation}
If $p + q$ is even then the sum (6.14) takes the form $T_{\text{even}}(\mathcal{L}_2)$, a polynomial in $\mathcal{L}_2$. Similarly, expansion
\begin{equation}
 \frac{1}{2}[((\sqrt{L}_2 a + ib)^q(\sqrt{L}_2 c + id)^p - (\sqrt{L}_2 a - ib)^q(\sqrt{L}_2 c - id)^p]
\end{equation}
\begin{equation}
 = \sum_{0 \leq \ell \leq q, 0 \leq s \leq p} \binom{q}{\ell} \binom{p}{s} b^\ell d^s e^{\ell s} \cdot \frac{L^{(q+p-\ell-s)/2}}{2} [T^{\ell s} - (-i)^{\ell s}],
\end{equation}
takes the form $\sqrt{L_2} V_{\text{even}}(\mathcal{L}_2)$ where $V_{\text{even}}$ is a polynomial in $\mathcal{L}_2$. Let
\begin{equation}
 \mathcal{L}_4 = \frac{1}{i\sqrt{L_2}}(\mathcal{L}^+ - \mathcal{L}^-), \quad \mathcal{L}_3 = \mathcal{L}^+ + \mathcal{L}^-,
\end{equation}
where $\mathcal{L}^\pm$ are defined as $\mathcal{L}^+ = X^q Y^p$, $\mathcal{L}^- = (\mathcal{X})^q (\mathcal{Y})^p$. Then we see from (6.14), (6.15) that $\mathcal{L}_3, \mathcal{L}_4$ are constants of the motion, polynomial in the momenta. Moreover, the identity $\mathcal{L}^+ \mathcal{L}^- = P(\mathcal{L}_2, \mathcal{H})$ is applicable, as are $[\mathcal{L}_2, \mathcal{L}^\pm] = \mp 4ip\sqrt{L_2} \mathcal{L}^\pm$. Now we obtain $[\mathcal{L}_2, \mathcal{L}_3] = -4p\mathcal{L}_2 \mathcal{L}_4$, $[\mathcal{L}_3, \mathcal{L}_4] = 4p\mathcal{L}_3$. Similarly, since
\begin{equation}
 \mathcal{L}_2^2 = \frac{1}{L_2} [(\mathcal{L}^+)^2 - 2 \mathcal{L}^+ \mathcal{L}^- + (\mathcal{L}^-)^2], \quad \mathcal{L}_2^3 = [(\mathcal{L}^+)^2 + 2 \mathcal{L}^+ \mathcal{L}^- + (\mathcal{L}^-)^2],
\end{equation}
can derive $\mathcal{L}_3^2 = -\mathcal{L}_2 \mathcal{L}_4^2 + 4P(\mathcal{L}_2, \mathcal{H})$. Using $[\mathcal{L}^+, \mathcal{L}^-] = 4ip\sqrt{L_2} \frac{dP}{\sqrt{L_2}}$, we derive $[\mathcal{L}_2, \mathcal{L}_4] = -4p\mathcal{L}_2^2 + 8p^2 \frac{\partial P}{\partial \mathcal{L}_2}$. Hence the structure equations are
\begin{equation}
 [\mathcal{L}_2, \mathcal{L}_4] = \mathcal{R}, \quad [\mathcal{L}_2, \mathcal{R}] = -16p^2 \mathcal{L}_2 \mathcal{L}_4, \quad [\mathcal{L}_4, \mathcal{R}] = 16p^2 \mathcal{L}_2^2 - 32p^2 \frac{\partial P}{\partial \mathcal{L}_2}, \quad \mathcal{R}^2 = -16p^2 \mathcal{L}_2 \mathcal{L}_4^2 + 64p^2 P(\mathcal{L}_2, \mathcal{H})
\end{equation}
6.1.3. The classical constant of the motion $L_5$. As for the classical TTW system with $k = 1$, our method does not give the generator of minimal order. In fact, for all rational $k$, we can extend our algebra by adding a generator of order one less than $L_4$. To see this, we consider the case $p + q$ even, leaving out the similar odd case. Note that $L_3$ is a polynomial in $L_2$ with constant term $2(-1)^{(q-p)/2}(\alpha - \beta)\mathcal{H}^0$. Thus
\[
L_5 = \frac{L_3 - 2(-1)^{(q-p)/2}(\alpha - \beta)\mathcal{H}^0}{L_2}
\]
is a constant, polynomial in the momenta. For example, if $p = q = 1$ then
\[
L_5 = \frac{L_3 - 2(\alpha - \beta)\mathcal{H}}{L_2},
\]
is a second-order constant of the motion. Just as before $L_2$, $L_5$, $\mathcal{H}$ generate a closed symmetry algebra that properly contains the original algebra.

6.2. The extended Kepler–Coulomb system

Now we apply our construction to an important 3D potential, the four-parameter extended Kepler–Coulomb system. We write it in the form
\[
\mathcal{H} = p_r^2 + \frac{\alpha}{r} + \frac{\beta}{r^2} + \frac{\gamma}{r} + \frac{\delta}{r^3}.
\]
Almost simultaneously Verrier and Evans [191] and Rodríguez, Tempesta and Winternitz [170] showed this system is fourth-order superintegrable and, later, Tanoudis and Daskaloyannis [179] showed in the quantum case that, if a second fourth-order symmetry is added to the generators, the symmetry algebra closes polynomially in the sense that all second commutators of the generators can be expressed as symmetrized polynomials in the generators. We introduce an analogue of the TTW construction and consider an infinite class of extended Kepler–Coulomb systems indexed by a pair of rational numbers $(k_1, k_2)$. We construct explicitly a set of generators, show these systems to be superintegrable and determine the structure of the generated symmetry algebras. The symmetry algebras close rationally; only for systems admitting extra discrete symmetries is polynomial closure achieved.

The extended Hamiltonian is
\[
\mathcal{H} = p_r^2 + \frac{\alpha}{r} + \frac{\beta}{r^2} + \frac{\gamma}{r} + \frac{\delta}{r^3}
\]
where
\[
L_2 = p_{\theta_1}^2 + \frac{L_3}{\sin^2(k_1\theta_1)}, \quad L_3 = p_{\phi_2}^2 + \frac{\beta}{\cos^2(k_2\phi_2)} + \frac{\gamma}{\sin^2(k_2\phi_2)}.
\]
Here $L_2$, $L_3$ are constants, in involution. They determine additive separation in the variables $r, \theta, \phi$. Let, $k_1 = \frac{p_1}{q_1}$, $k_2 = \frac{p_2}{q_2}$ where $p_1, q_1$, resp. $p_2, q_2$, are relatively prime. Applying the construction to get two new constants of the motion we note that $x_1 = r, x_2 = \theta_1, x_3 = \phi_2$ and $f_1 = \frac{1}{q_1}, f_2 = \sqrt{\frac{1}{\sin^2(k_1\theta_1)}}, f_3 = 0, V_1 = \frac{\alpha}{r}, V_2 = \frac{\beta}{r^2}$. We construct explicitly a set of generators, show these systems to be superintegrable and determine the structure of the generated symmetry algebras. The symmetry algebras close rationally; only for systems admitting extra discrete symmetries is polynomial closure achieved.

The extended Hamiltonian is
\[
\mathcal{H} = p_r^2 + \frac{\alpha}{r} + \frac{\beta}{r^2} + \frac{\gamma}{r} + \frac{\delta}{r^3}
\]
and $p_1q_2\alpha_2 - p_2q_1\alpha_2$, $q_1\alpha_1 - 2p_1\alpha_2$, are two constants such that the full set of five constants of the motion is functionally independent. We find
\[
e^{-\phi} = \cosh A_j - \sinh A_j = X_j/U_j, \quad e^{\phi} = \cosh A_j + \sinh A_j = X_j/U_j,
\]
and $p_1q_2\alpha_2 - p_2q_1\alpha_2$, $q_1\alpha_1 - 2p_1\alpha_2$, are two constants such that the full set of five constants of the motion is functionally independent. We find
\[
e^{\phi} = \cosh A_j - \sinh A_j = X_j/U_j, \quad e^{-\phi} = \cosh A_j + \sinh A_j = X_j/U_j,
\]
and $p_1q_2\alpha_2 - p_2q_1\alpha_2$, $q_1\alpha_1 - 2p_1\alpha_2$, are two constants such that the full set of five constants of the motion is functionally independent. We find
\[
e^{\phi} = \cosh A_j - \sinh A_j = X_j/U_j, \quad e^{-\phi} = \cosh A_j + \sinh A_j = X_j/U_j,
\]
and $p_1q_2\alpha_2 - p_2q_1\alpha_2$, $q_1\alpha_1 - 2p_1\alpha_2$, are two constants such that the full set of five constants of the motion is functionally independent. We find
\[
e^{\phi} = \cosh A_j - \sinh A_j = X_j/U_j, \quad e^{-\phi} = \cosh A_j + \sinh A_j = X_j/U_j,
\]
and $p_1q_2\alpha_2 - p_2q_1\alpha_2$, $q_1\alpha_1 - 2p_1\alpha_2$, are two constants such that the full set of five constants of the motion is functionally independent. We find
\[
e^{\phi} = \cosh A_j - \sinh A_j = X_j/U_j, \quad e^{-\phi} = \cosh A_j + \sinh A_j = X_j/U_j,
\]
\[ Y_1 = 2\sqrt{L_3}p_r - i\left(\alpha + \frac{L_2}{r}\right), \]
\[ Y_2 = -2L_3\cos^2(k_1\theta_1) + (L_2 - L_3 - \delta) - 2i\sqrt{L_3}\cot(k_1\theta_1)p_{\theta_1}, \]
\[ U_1 = \sqrt{L_2^2 - 2L_3(L_2 + \delta) + (L_2 - \delta)^2}, \quad U_2 = \sqrt{(\beta - \gamma - L_3)^2 - 4\gamma L_3}, \]
\[ S_1 = \sqrt{\alpha^2 + 4\gamma L_3}, \quad S_2 = \sqrt{L_2^2 - 2L_3(L_2 + \delta) + (L_2 - \delta)^2}, \]

where \( X_j, Y_j \) are obtained from \( X_i, Y_i \) by replacing \( \text{i} \) by \(-\text{i}\).

Here, \( e^{\theta_1A_1-2p_rB_1} \) and \( e^{-q_1A_1+2p_rB_1} \) are constants of the motion, where
\[
e^{\theta_1A_1-2p_rB_1} = (e^{A_1})^{Y_1}(e^{-B_1})^{2p_r} = \frac{X_1^{Y_1}Y_1^{2p_r}}{U_1^{q_1}S_1^{2p_r}},
\[
e^{-q_1A_1+2p_rB_1} = X_1^{q_1}Y_1^{2p_r}.
\]

The identity \( e^{\theta_1A_1-2p_rB_1} e^{-q_1A_1+2p_rB_1} = 1 \) can be expressed as
\[
X_1^{Y_1}Y_1^{2p_r} = U_1^{2q_1}S_1^{2p_r} = P_1 = \left[ L_3^2 - 2L_3(L_2 + \delta) + (L_2 - \delta)^2 \right]^{q_1} [\alpha^2 + 4\gamma L_3]^{2p_r}
\]
where \( P_1 \) is a polynomial in \( \mathcal{H}, L_2 \) and \( L_3 \). Similarly, \( e^{\theta_1A_1-2p_qB_1} e^{-q_1A_1+2p_qB_1} \) are constants of the motion, where
\[
e^{\theta_1A_1-2p_qB_1} = (e^{A_1})^{p_q}(e^{-B_1})^{p_q} = \frac{X_1^{p_2}Y_1^{2p_2}}{U_2^{p_q}S_2^{2p_2}},
\[
e^{-q_1A_1+2p_qB_1} = X_1^{p_q}Y_1^{2p_q}.
\]

The identity \( e^{\theta_1A_1-2p_qB_1} e^{-q_1A_1+2p_qB_1} = 1 \) becomes
\[
X_2^{p_q}Y_2^{2p_q} = U_2^{p_q}S_2^{2p_q} = P_2(\mathcal{H}, L_2, L_3) = \left[ (\beta - \gamma - L_3)^2 - 4\gamma L_3 \right]^{p_q} [L_3^2 - 2L_3(L_2 + \delta) + (L_2 - \delta)^2]^{p_q}.
\]

We define basic raising and lowering symmetries
\[
\mathcal{J}^+ = X_1^{Y_1}Y_1^{2p_r}, \quad \mathcal{J}^- = X_1^{q_1}Y_1^{2p_q}, \quad \mathcal{K}^+ = X_2^{p_2}Y_2^{p_q}, \quad \mathcal{K}^- = X_2^{p_q}Y_2^{p_2},
\]
and restrict to the case where each of \( p_1, q_1, p_2, q_2 \) is an odd integer. Let
\[
\mathcal{J}_1 = \frac{1}{\sqrt{2}}(\mathcal{J}^+ + \mathcal{J}^-), \quad \mathcal{J}_2 = \mathcal{J}^+ - \mathcal{J}^-, \quad \mathcal{K}_1 = \frac{1}{\sqrt{\gamma}}(\mathcal{K}^+ + \mathcal{K}^-), \quad \mathcal{K}_2 = \mathcal{K}^- - \mathcal{K}^+.
\]

We see from the explicit expressions for the symmetries that \( \mathcal{J}_1, \mathcal{J}_2, \mathcal{K}_1, \mathcal{K}_2 \) are constants of the motion, polynomial in the momenta. Moreover, the identities \( \mathcal{J}^+ \mathcal{J}^- = P_1, \mathcal{K}^\pm \mathcal{K}^\mp = P_2 \) hold. The following relations are straightforward to derive from the definition of the Poisson bracket:
\[
\{ L_3, X_1 \} = \{ L_3, Y_1 \} = \{ L_2, Y_1 \} = \{ L_3, Y_2 \} = 0,
\]
\[
\{ L_2, X_1 \} = -4ik_1\sqrt{L_3}X_1, \quad \{ L_3, X_2 \} = -4ik_2\sqrt{L_3}X_2,
\]
\[
\{ L_2, X_2 \} = -\frac{4ik_2}{\sin^2(k_1\theta_1)}\sqrt{L_3}X_2, \quad \{ L_2, Y_2 \} = -\frac{4ik_1\sqrt{L_3}}{\sin^2(k_1\theta_1)}Y_2,
\]
\[
\{ \mathcal{H}, X_1 \} = -\frac{4ik_1\sqrt{L_2}}{r^2}\sin^2(k_1\theta_1)X_1, \quad \{ \mathcal{H}, X_2 \} = -\frac{4ik_2}{r^2}\sin^2(k_1\theta_1)\sqrt{L_3}X_2,
\]
\[
\{ \mathcal{H}, Y_1 \} = -\frac{2i\sqrt{L_2}}{r^2}Y_1, \quad \{ \mathcal{H}, Y_2 \} = -\frac{4ik_1\sqrt{L_3}}{r^2}\sin^2(k_1\theta_1)Y_2,
\]
with the corresponding results for \( \overline{X}_j, \overline{Y}_j \) obtained by replacing \( \text{i} \) by \(-\text{i}\).
Commutators relating the $\mathcal{J}$ and $\mathcal{K}$ follow from
\[
[X_1, X_2] = -\frac{4k_1}{\sqrt{L_2}} \left( i \cot(k_1 \theta_1) p_{\theta_1} + \sqrt{L_2} \cot^2(k_1 \theta_1) \right) X_2.
\]
\[
[Y_1, Y_2] = \frac{4ik_1 \sqrt{L_3}}{\sin^2(k_1 \theta_1)} \left( \frac{p_r}{\sqrt{L_2}} + \frac{2i}{r} \right) Y_2,
\]
\[
[X_1, Y_2] = 4ik_1 \sqrt{L_2} X_1 + \frac{4k_1 \sqrt{L_3}}{\sin^2(k_1 \theta_1)} \left( \frac{\sin(2k_1 \theta_1) p_{\theta_1}}{2 \sqrt{L_2}} - i \cos(2k_1 \theta_1) \right) Y_2
\quad + 8k_1 \cot^2(k_1 \theta_1) \left( \frac{i \sqrt{L_2 L_3} \sin(2k_1 \theta_1)}{2} p_{\theta_1} - \sqrt{L_2 L_3} \sin^2(k_1 \theta_1) - \sqrt{L_2 L_3} \right),
\]
\[
[X_2, Y_1] = \frac{4ik_2 \sqrt{L_3}}{\sin^2(k_1 \theta_1)} \left( \frac{p_r}{\sqrt{L_2}} + \frac{2i}{r} \right) X_2.
\]
From these results we find
\[
\left\{ \mathcal{J}^+, \mathcal{K}^+ \right\} = \frac{4iq_1 p_1 p_2 (\sqrt{L_2} - \sqrt{L_3})(L_2 + 2 \sqrt{L_2 L_3} + L_3 - \delta)}{(L_3 - L_2 - \delta)^2 - 4\delta L_2}.
\]
\[
\left\{ \mathcal{J}^+, \mathcal{K}^- \right\} = \frac{4iq_1 p_1 p_2 (\sqrt{L_2} + \sqrt{L_3})(L_2 - 2 \sqrt{L_2 L_3} + L_3 - \delta)}{(L_3 - L_2 - \delta)^2 - 4\delta L_2}.
\]
These relations prove closure of the symmetry algebra in the space of functions polynomial in $\mathcal{J}^\pm, \mathcal{K}^\pm$, rational in $\mathcal{L}_2, \mathcal{L}_3, \mathcal{H}$ and linear in $\sqrt{L_2}, \sqrt{L_3}$.

6.2.1. Structure relations for polynomial symmetries of the four-parameter potential. From the preceding section we have the structure relations (for $1 \leq j, k \leq 2$)
\[
\left\{ \mathcal{J}_j, \mathcal{K}_k \right\} = \frac{4q_1 p_1 p_2}{(L_3 - L_2 - \delta)^2 - 4\delta L_2}
\quad \times \left[ \mathcal{J}_{j-i} \mathcal{K}_k (-L_3)^{i-1} (L_2 - L_3 - \delta) + \mathcal{J}_j \mathcal{K}_{k+1} (-L_3)^{k+1} (L_2 + L_3 + \delta) \right],
\]
\[
\left\{ \mathcal{L}_2, \mathcal{J}_2 \right\} = 4p_1 \mathcal{L}_2 \mathcal{J}_1, \quad \left\{ \mathcal{L}_2, \mathcal{J}_1 \right\} = -4p_1 \mathcal{J}_2, \quad \left\{ \mathcal{L}_3, \mathcal{J}_1 \right\} = \left\{ \mathcal{L}_3, \mathcal{J}_2 \right\} = 0,
\]
\[
\left\{ \mathcal{L}_3, \mathcal{K}_2 \right\} = 4p_1 p_2 \mathcal{L}_3 \mathcal{K}_1, \quad \left\{ \mathcal{L}_3, \mathcal{K}_1 \right\} = -4p_1 p_2 \mathcal{K}_2, \quad \left\{ \mathcal{L}_2, \mathcal{K}_2 \right\} = \left\{ \mathcal{L}_2, \mathcal{K}_1 \right\} = 0,
\]
\[
\mathcal{J}_2^2 = -\mathcal{L}_2 \mathcal{J}_1^2 + 4p_1 (\mathcal{H} \mathcal{L}_2 \mathcal{J}_3), \quad \mathcal{K}_2^2 = \mathcal{L}_3 \mathcal{K}_1^2 + 4p_2 (\mathcal{H} \mathcal{L}_2 \mathcal{K}_3),
\]
\[
\left\{ \mathcal{J}_1, \mathcal{J}_2 \right\} = -2p_1 \mathcal{J}_1^2 + 8p_1 \frac{\partial \mathcal{P}_1}{\partial \mathcal{L}_2}, \quad \left\{ \mathcal{K}_1, \mathcal{K}_2 \right\} = -2p_1 p_2 \mathcal{K}_1^2 + 8p_1 p_2 \frac{\partial \mathcal{P}_2}{\partial \mathcal{L}_3}.
\]

The generators for the polynomial symmetry algebra produced so far are not of minimal order. Here $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are of order 2 and the orders of $\mathcal{J}_1, \mathcal{K}_1$ are one less than the orders of $\mathcal{J}_2, \mathcal{K}_2$ respectively. We will construct symmetries $\mathcal{J}_0, \mathcal{K}_0$ of order 1 less than $\mathcal{J}_1, \mathcal{K}_1$, respectively. (In the standard case $k_1 = k_2 = 1$ it is easy to see that $\mathcal{J}_0$ is of order 5 and $\mathcal{J}_2$ is of order 6, whereas $\mathcal{K}_1$ is of order 3 and $\mathcal{K}_2$ is of order 4. Then $\mathcal{J}_0, \mathcal{K}_0$ will be of orders 4 and 2, respectively, which we know corresponds to the minimal generators of the symmetry algebra in this case, [191].) The symmetry $\mathcal{J}_2$ is a polynomial in $\mathcal{L}_2$ with constant $D_1 = 2(-1)^{(\nu-1)/2}(\delta - L_3)\nu^{\nu/2}$, itself a constant of the motion. Thus $\mathcal{J}_0 = \frac{D_1}{D_2}$ is a polynomial symmetry of order 2 less than $\mathcal{J}_2$. We have the identity $\mathcal{J}_2 = \mathcal{L}_2 \mathcal{J}_0 + D_1$. From this, $\left\{ \mathcal{L}_2, \mathcal{J}_2 \right\} = \mathcal{L}_2 \mathcal{J}_0$. We already know that $\mathcal{L}_2 \mathcal{J}_0 = 4p_1 \mathcal{L}_2 \mathcal{J}_1$ so $\mathcal{L}_2 \mathcal{J}_0 = 4p_1 \mathcal{J}_1$, $\mathcal{L}_3 \mathcal{J}_0 = 0$.

The same construction works for $\mathcal{K}_2$. It is a polynomial in $\mathcal{L}_3$, with constant $D_2 = 2(-1)^{(\nu-1)/2} \nu^{\nu/2}$, $D_2$ is a polynomial symmetry of order 2 less than $\mathcal{K}_2$ and we have the identity $\mathcal{K}_2 = \mathcal{L}_3 \mathcal{K}_0 + \mathcal{D}_2$. Further, $\left\{ \mathcal{L}_3, \mathcal{K}_0 \right\} = 4p_1 p_2 \mathcal{K}_1$, $\left\{ \mathcal{L}_2, \mathcal{K}_0 \right\} = 0$.
Now we choose \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_0, \mathcal{K}_0 \) as the generators of our algebra. We define the basic nonzero commutators as
\[
\mathcal{R}_1 = \{ \mathcal{L}_2, \mathcal{J}_0 \} = 4p_1 \mathcal{J}_1, \quad \mathcal{R}_2 = \{ \mathcal{L}_3, \mathcal{K}_0 \} = 4p_1 p_2 \mathcal{K}_1, \quad \mathcal{R}_3 = \{ \mathcal{J}_0, \mathcal{K}_0 \}.
\]

Then we have
\[
\frac{\mathcal{R}_1^2}{16 p_1^2} = \mathcal{J}_1^2 = -\mathcal{L}_2 \mathcal{J}_0^2 - 2 \mathcal{D}_1 \mathcal{J}_0 + \frac{4p_1 - D_1}{L_2},
\]
where the right-hand term is polynomial in generators \( \mathcal{H}, \mathcal{L}_2, \mathcal{L}_3 \). Further,
\[
\frac{\mathcal{R}_2^2}{16 p_1^2 p_2^2} = \mathcal{K}_1^2 = -\mathcal{L}_3 \mathcal{K}_0^2 - 2 \mathcal{D}_2 \mathcal{K}_0 + \frac{4p_2 - D_2}{L_3},
\]
which again can be verified to be a polynomial in the generators. Note that this symmetry algebra cannot close polynomially in the usual sense. If it did close then the product \( \mathcal{R}_1 \mathcal{R}_2 \) would be expressible as a polynomial in the generators. The preceding two equations show that \( \mathcal{R}_1^2 \mathcal{R}_2^2 \) is so expressible, but that the resulting polynomial is not a perfect square. Thus the only possibility to obtain closure is to add new generators to the algebra, necessarily functionally dependent on the original set. Continuing in this way, see [94] for details, one finds that \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{K}_0, \mathcal{J}_0 \) generate a symmetry algebra that closes rationally. In particular, each of the commutators \( \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \) satisfies an explicit polynomial equation in the generators.

6.2.2. The special case \( k_1 = k_2 = 1 \). In the case \( k_1 = k_2 = 1 \) we are in Euclidean space and our system has additional symmetry. In terms of Cartesian coordinates \( x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta \), the Hamiltonian is
\[
\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2} + \frac{\delta}{x^2 + y^2 + z^2}.
\]
Note that any permutation of the ordered pairs \((x, \beta), (y, \gamma), (z, \delta)\) leaves the Hamiltonian unchanged. This leads to additional structure in the symmetry algebra. The basic symmetries are
\[
\mathcal{L}_3 = \mathcal{I}_{xy} = (xp_y - yp_x)^2 + \frac{\beta(x^2 + y^2)}{x^2} + \frac{y(x^2 + y^2)}{y^2},
\]
\[
\mathcal{L}_2 = \mathcal{I}_{xy} + \mathcal{I}_{xz} + \mathcal{I}_{yz} - (\beta + \gamma + \delta).
\]
Permutation symmetry of the Hamiltonian shows \( \mathcal{I}_{xy}, \mathcal{I}_{xz}, \mathcal{I}_{yz} \) are constants, and
\[
\mathcal{K}_0 = 4\mathcal{I}_{xy} + 2\mathcal{L}_3 - 2(\mathcal{L}_2 + \beta + \gamma + \delta) = 2(\mathcal{I}_{yz} - \mathcal{I}_{xz}).
\]
Here \( \mathcal{J}_0 \) is fourth-order:
\[
\mathcal{J}_0 = -16 \left( \mathcal{M}_1 + \frac{\delta (xp_y + yp_x + zp_z)^2}{z^2} \right) + 8\mathcal{H} (\mathcal{I}_{xz} + \mathcal{I}_{yz} - \beta - \gamma - \delta) + 2\alpha^2,
\]
\[
\mathcal{M}_3 = (yp_z - zp_y)p_x - (zp_x - xp_z)p_y - z \left( \frac{\alpha}{2x} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \right).
\]
If \( \delta = 0 \) then \( \mathcal{M}_3 \) is itself a constant of the motion.

The symmetries \( \mathcal{H}, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_0, \mathcal{K}_0 \) form a generating (rational) basis for the constants of the motion. Under the transposition \((x, \beta) \leftrightarrow (z, \delta)\) this basis is mapped to an alternate basis \( \mathcal{H}', \mathcal{L}_1', \mathcal{L}_3', \mathcal{J}_0', \mathcal{K}_0' \) where
\[
\mathcal{L}_2' = \mathcal{L}_2, \quad \mathcal{L}_3' = \frac{1}{4} \mathcal{K}_0 + \frac{1}{2} \mathcal{L}_3 - \frac{1}{2} \mathcal{L}_3 + \frac{\beta + \gamma + \delta}{2},
\]
\[
\mathcal{K}_0' = \frac{1}{2} \mathcal{K}_0 - 3 \mathcal{L}_3 - (\beta + \gamma + \delta), \quad \mathcal{R}_1' = \{ \mathcal{L}_2, \mathcal{J}_0' \}, \quad \mathcal{R}_2' = \{ \mathcal{L}_3', \mathcal{K}_0' \} = \frac{-5}{4} \mathcal{R}_2,
\]
\[
\mathcal{R}_3' = \{ \mathcal{J}_0', \mathcal{K}_0' \} = 2\mathcal{R}_1' - 2(\mathcal{L}_3, \mathcal{J}_0').
\]
All the identities in section 6.2.1 hold for the primed symmetries. The $\mathcal{K}'$ symmetries are simple polynomials in the $\mathcal{L}$, $\mathcal{K}$ symmetries already constructed, e.g., $\mathcal{K}'_1 = \frac{1}{2}(\mathcal{L}_3, \mathcal{K}_0') = -\frac{3}{2}K_1$. However, the $\mathcal{J}'$ symmetries are new.

$$\mathcal{J}'_0 = -16 \left( \mathcal{M}_1^2 + \frac{\beta(x^2 + y^2 + z^2)}{x^2} \right) + 8\mathcal{H}(I_{xy} + I_{xz} - \beta - \gamma - \delta) + 2\alpha^2, \quad (6.27)$$

$$\mathcal{M}_1 = (yp_x - xp_y)p_y - (xp_z - zp_x)p_z - x \left( \frac{\alpha}{2r} + \frac{\beta}{y^2} + \frac{\gamma}{z^2} + \frac{\delta}{r^2} \right).$$

Note that the transposition $(y, \gamma) \leftrightarrow (z, \delta)$ does not lead to anything new. Indeed, under the symmetry we would obtain a constant of the motion $\mathcal{J}'''_0$ but it is straightforward to check that $\mathcal{J}_0 + \mathcal{J}'_0 + \mathcal{J}'''_0 = 2\alpha^2$, so that the new constant depends linearly on the previous constants.

In the paper [179], Tanoudis and Daskaloyannis show that the quantum symmetry algebra generated by the six functionally dependent symmetries $\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3, J_0, K_0$ and $J'_0$ closes polynomially, in the sense that all double commutators of the generators are again expressible as polynomials in the generators, strong evidence that the classical analogue also closes polynomially. The complicated details in proving this can be found in [94] where it is shown that the six generators obey a functional identity of order 12.

6.2.3. Further results. Using the methods described above, many more classical systems have been shown to be higher-order superintegrable, and in some cases the structure of the symmetry algebras has been worked out. In [94] a extended version of the three-parameter Kepler–Coulomb system was studied for all rational $k_1, k_2$ and the structure of the symmetry algebra exposed. This is the restriction of the four-parameter potential to $\delta = 0$. It is of interest because the symmetry algebra of the restriction is larger than the original symmetry algebra; the system is second-order superintegrable in the $k_1 = k_2 = 1$ case. Always the symmetry algebra closes rationally, but not polynomially.

In [78] the scope of the construction given here is explored for $n = 2$ to find all superintegrable systems with rational $k_i$ where the $k = 1$ case is second-order superintegrable on a constant curvature space, separating in spherical coordinates. In [80] the construction of extended superintegrable systems for rational $k_i$ is considered in $n$ dimensions for systems that admit separation in subgroup coordinates, i.e., polyspherical coordinates. For $n = 4$ the first known examples of superintegrable systems on non-conformally flat spaces are presented.

The method of construction of higher-order superintegrable systems presented here always yields separable systems. However, recently examples of nonseparable superintegrable systems have been produced: [120], using differential Galois theory, and [155], which also has a quantum version. A general method of construction of such nonseparable systems and determination of their structure algebras has not yet been achieved.

7. Higher-order quantum superintegrable systems

The problem of finding and verifying higher-order quantum superintegrability is analogous to that of classical superintegrability, only harder. In addition to the problems of finding numerous examples of these systems and working out ways to compute the commutators of symmetry operators of arbitrarily high order, there is the quantization problem. The relationship between classical and quantum superintegrability is not 1 − 1, the difference shows up with greater frequency in superintegrable systems in dimensions > 2, or order > 2.

As for the classical case, a breakthrough came with the publication of the Tremblay, Turbiner and Winternitz papers, [184, 185]. They provided strong evidence for the conjecture
that the system, which we call the TTW system, (6.2), was quantum superintegrable for all rational $k$. There was a flurry of activity to prove the conjectures; the first results were obtained in [163] where Dunkl operators were used to prove quantum superintegrability for odd integer $k$. The problem for all rational $k$ was solved first in [77] by a very general approach that verified superintegrability but did not give information about the structure equations, see also [124]. Then in [81] a method was introduced that enabled superintegrability to be proved and explicit structure equations to be calculated. This is the recurrence approach. It applies to large families of quantum systems that permit separation in coordinates yielding hypergeometric eigenfunctions and works for all $n$. It has been conjectured that all maximally superintegrable Euclidean systems are exactly solvable, thus it should always apply for these systems [181].

Here is the procedure for $n = 2$ where one new symmetry is needed: (1) Require that the system admit a second-order symmetry that determines a separation of variables. (2) The formal eigenspaces of the Hamiltonian are invariant under action of any symmetry, so the operator must induce recurrence relations for the basis of separated eigenfunctions. (3) Suppose the separated eigenfunctions are of hypergeometric type. Use the known recurrence relations for hypergeometric functions to reverse this process and determine a symmetry operator from the recurrences. (4) Compute the symmetry operators and structure equations by restricting to a formal ‘basis’ of separated eigenfunctions. (5) Appeal to a general theory of canonical forms for symmetry operators to show that results obtained on a formal eigenbasis hold as true identities for purely differential operators defined independent of basis.

We sketch how this method works for the TTW system, (6.2), $H \Psi = E \Psi$, $H = L_1$. A formal eigenbasis takes the form
\[
\Psi = e^{-\tilde{z}'^2}k^{(2n+a+b+1)}L_n^{(2n+a+b+1)}(\text{cos}(k\theta))(\sin(k\theta))^{n+\frac{1}{2}}(\cos(k\theta))^{b+\frac{1}{2}}P_n^a(\cos(2\theta)),
\]
where $\alpha = k^2\left(\frac{1}{4} - a^2\right)$ and $\beta = k^2\left(\frac{1}{4} - b^2\right)$. The $L_n^a(z)$ are associated Laguerre functions and the $P_n^a(w)$ are Jacobi functions, [4]. We express the eigenfunctions in the form
\[
\Psi = (\sin(k\theta))^{n+\frac{1}{2}}(\cos(k\theta))^{b+\frac{1}{2}} \Pi \text{ with }
\]
\[
\Pi = e^{-\tilde{z}'^2} k^{(2n+a+b+1)}L_m^{(2n+a+b+1)}(\text{cos}(\omega))^m \Psi = Y_m^k(r)X_n^a(\theta),
\]
where $x = \text{cos}(2k\theta)$, $\mu = 2n + a + b + 1$. The energy eigenvalue is
\[
E = -2\omega [2(m+n+k) + 1 + (a + b + 1)k]
\]
and the symmetry operator responsible for the separation of variables is
\[
L_2\Psi = \left(\frac{\partial^2}{\partial \theta^2} + \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)}\right) \Psi = -k^2\mu^2 \Psi
\]
where $\mu = 2n + a + b + 1$, $k = p/q$. Recall that the separation constant is just the eigenvalue of the symmetry operator $L_2$. The energy values for the Hamiltonian are negative because here we have the non-physical convention of setting $-\hbar^2/2m = 1$. The transformation to the un-scaled Hamiltonian is given by (2.21).

The strategy is to look for recurrences that change $m$, $n$ but fix $u = m + nk$. This will map eigenfunctions of $H$ with eigenvalue $E$ to eigenfunctions with the same $E$. Two transformations achieving this are

1) $n \rightarrow n + q$, $m \rightarrow m - p$,  
2) $n \rightarrow n - q$, $m \rightarrow m + p$.

First, consider $X_n^{a,b}(x)$. Raise or lower $n$ with
1) $J_n^+ X_n^{a,b} = (2n + a + b + 2)(1 - x^2) \partial_x X_n^{a,b} + (n + a + b + 1)$
$\times (-2n + a + b + 2)x - (a - b))X_n^{a,b} = 2(n + 1)(n + a + b + 1)X_{n+1}^{a,b},$
2) $J_n^- X_n^{a,b} = -(2n + a + b)(1 - x^2) \partial_x X_n^{a,b}$
$-n((2n + a + b)x - (a - b))X_n^{a,b} = 2(n + a)(n + b)X_{n-1}^{a,b}$. 
For \( J_{m}^{\mu} (R) = \alpha^{k \mu / 2} Y_{m}^{k \mu} (r) \) where \( R = r^2 \), change \( m \) by

\[
K_{m,m}^{\pm} J_{m}^{\mu} = \begin{cases} 
(k \mu + 1) - \frac{E}{4} - \frac{1}{2R} k \mu (k \mu + 1) & \text{if } \mu > 0 \\
(k \mu + 1) - \frac{E}{4} + \frac{1}{2R} k \mu (1 - k \mu) & \text{if } \mu < 0 
\end{cases}
\]

\( K_{m,m}^{\pm} Y_{m}^{\mu} = -\omega Y_{m+1}^{\mu+2} \).

From these recurrences we construct the two operators

\[
\Xi^{\pm} = K_{m+2(p-1),m+2(p-1)}^{-} \cdots K_{m,m+2}^{-} F_{m}^{\pm} \cdots F_{n}^{\pm}.
\]

For fixed \( u = m + kn \), we have

\[
\Xi^{+} \Psi_{n} = 2^{q} (1)^{q} \omega^{q} (n + 1 + k n + 1) \Psi_{n} = \Xi^{+} \Psi_{n-q}.
\]

\[
\Xi^{-} \Psi_{n} = 2^{q} \omega^{q} (n - a) \Psi_{n} (n + kn + 1) \Psi_{n} = \Xi^{-} \Psi_{n+q}.
\]

These are basis dependent operators. However, under the transformation \( n \rightarrow -n - a - b - 1 \), i.e., \( \mu \rightarrow -\mu \), we have \( \Xi^{+} \rightarrow \Xi^{-} \) and \( \Xi^{-} \rightarrow \Xi^{+}. \) Thus \( \Xi^{+} = \Xi^{-} + \Xi^{+} \) is a polynomial in \( \mu^{2} \). Therefore, we can replace in \( (2n + a + b + 1)^{2} = \mu^{2} \) by \( L_{2}/k^{2} \) and \( E \) by \( H \) everywhere they occur, and express \( \Xi \) as a pure differential symmetry operator. Note also that under the transformation \( n \rightarrow -n - a - b - 1 \), i.e., \( \mu \rightarrow -\mu \) the operator \( \Xi^{+} \rightarrow \Xi^{-} \) changes sign, hence \( \Xi = (\Xi^{+} - \Xi^{-}) / (1/\mu) \equiv (\Xi^{+} - \Xi^{-}) (1/\mu) \) is unchanged. This defines \( \Xi \) as a symmetry operator. We set

\[
L_{3} = \Xi, \quad L_{4} = \Xi^{-}.
\]

We have shown that \( L_{3}, L_{4} \) commute with \( H \) on any formal eigenbasis. In fact, we have constructed pure differential operators which commute with \( H \), independent of basis. This takes proof, which is provided in [77] via a canonical form for higher-order differential operators. Thus operator relations verified on formal eigenbases must hold identically.

### 7.1. The structure of the TTW algebra

Taking a product of raising and lowering operators we obtain

\[
\Xi^{+} \Xi^{-} \Psi_{n} = (-1)^{q} 2^{q} \omega^{q} (n + 1 + k n + 1) \Psi_{n} (n + a + b + 1)
\]

\[
	imes (n + a + b + 1) \Psi_{n} (n + k n + 1) \Psi_{n} = \xi_{n} \Psi_{n},
\]

\[
\Xi^{+} \Xi^{-} \Psi_{n} = (-1)^{q} 2^{q} \omega^{q} (-n - a) \Psi_{n} (n + a + b + 1)
\]

\[
	imes (-n - a - b) \Psi_{n} (n - k n + 1) \Psi_{n} = \eta_{n} \Psi_{n}.
\]

Thus \( \Xi^{(+)} = \Xi^{+} \Xi^{-} + \Xi^{-} \Xi^{+} \) multiplies any basis function by \( \xi_{n} + \eta_{n} \). However, the transformation \( \mu \rightarrow -\mu \) maps \( \Xi_{+} \Xi_{-} \leftrightarrow \Xi_{-} \Xi_{+} \) and \( \xi_{n} \leftrightarrow \eta_{n} \). Thus \( \Xi^{(+)} \) is an even polynomial operator in \( \mu \), polynomial in \( u \), and \( \xi_{n} + \eta_{n} \) is an even polynomial function in \( \mu \), polynomial in \( u \). Furthermore, each of \( \Xi \), \( \Xi_{+} \), and \( \Xi_{-} \) is unchanged under \( u \rightarrow -u - (a + b + 1) \) hence a polynomial of order \( p \) in \( [2u + (a + b + 1)]^{2} \). We conclude

\[
\Xi^{(+) = P^{(+)}(H^{2}, L_{2}, \omega^{2}, a, b)}.
\]

Similarly

\[
\Xi^{(-)} = (\Xi_{-} \Xi_{+} - \Xi_{+} \Xi_{-}) / \mu = P^{(-)}(H^{2}, L_{2}, \omega^{2}, a, b).
\]
Here, \( \beta \) where \( J. \text{Phys. A: Math. Theor.} 46 \) find the lowest order generator we look for a symmetry operator \( L \) relative parities of \( p \). To find \( \beta \) we take the limit as \( n \to -q \). There are three similar cases, depending on the relative parities of \( p \) and \( q \). For \( p, q \) both odd we have

\[
\beta_n \equiv Q(H) = -\frac{H(a^2 - b^2)}{4} \Pi_{\ell=1}^{(p-1)/2} \left[ (-\omega^2) \left( \frac{H}{4\omega} - \ell \right) \left( \frac{H}{4\omega} + \ell \right) \right] 
\times \Pi_{\ell=1}^{(q-1)/2} \left[ \frac{1}{4}(-a - b + 2s)(a + b + 2s)(a - b + 2s)(-a + b + 2s) \right].
\]

Note that the raising and lowering operators \( \Xi_{\pm} \) are key to the success of this method. See also [124]. They themselves are not symmetries, not even pure differential operators. However, from them the true symmetries and structure equations can be constructed explicitly. In [29] the authors give a unified ladder operator, basis dependent, construction of the dynamical symmetries of the TTW system that includes both classical and quantum systems.

### 7.2. The 3D Kepler–Coulomb system

Now we extend the recurrence method to an important class of systems in three dimensions, the quantum analogue of the classical extended Kepler problem. This is a Stäckel transform of the extended oscillator system that is the natural 3D analogue of the TTW system. An interesting feature of this system is that the quantum potential now differs from the classical potential on nonflat spaces. The extended Kepler–Coulomb system is

\[
H = \frac{\partial_r^2}{r^2} + \frac{2}{r} \partial_r + \frac{\alpha}{r} + \frac{1 - \frac{k^2_1}{4r^2}}{r^2} + L_2, \\
L_3 = \frac{\partial_{\theta_1}^2}{\cos^2(k_2 \theta_2)} + \frac{\gamma}{\sin^2(k_2 \theta_2)}, \\
L_2 = \frac{\partial_{\theta_1}^2}{\sin^2(k_1 \theta_1)} + \frac{\delta}{\cos^2(k_1 \theta_1)}.
\]

Here, \( L_2, L_3 \) are symmetry operators that determine multiplicative separation of the Schrödinger eigenvalue equation \( H \Psi = E \Psi \), and \( k_j = p_j/q_j \) where \( p_j, q_j \) are nonzero relatively prime
positive integers for \( j = 1, 2 \), respectively. Note that \( L_2 \) and \( L_3 \) commute: \([L_2, L_3] = 0\). The scalar potential is

\[
\hat{V} = \frac{\alpha}{r} + \frac{1 - k_1^2}{4r^2} + \frac{1}{r^2} \left( \frac{\beta}{\sin^2(k_1 \theta_1) \cos^2(k_2 \theta_2)} + \frac{\gamma}{\sin^2(k_1 \theta_1) \sin^2(k_2 \theta_2)} + \frac{\delta}{\cos^2(k_1 \theta_2)} \right).
\]

It differs from the classical potential \( V \) by the term \( \frac{1 - k_1^2}{4r^2} \), proportional to the scalar curvature. It is relatively easy to prove superintegrability of the system, but the computations to determine the structure algebra are quite involved. The main message that we want to communicate is that it is indeed possible to determine this structure.

The separation equations for the equations, \( H \Psi = E \Psi \), \( L_3 \Psi = -\mu^2 \Psi \), \( L_2 \Psi = \lambda \Psi \) with \( \Psi = R(r) \Theta(\theta_1) \Phi(\theta_2) \), are:

\[
\left( \frac{\delta_{n}^2}{\sin^2(k_1 \theta_1)} + \frac{k_1^2 (\frac{1}{2} - b^2)}{\cos^2(k_2 \theta_2)} + \frac{k_1^2 (\frac{1}{2} - c^2)}{\sin^2(k_2 \theta_2)} \right) \Phi(\theta_2) = -\mu^2 \Phi(\theta_2), \tag{7.4}
\]

where we have taken \( \beta = k_1^2 (\frac{1}{2} - b^2) \), \( \gamma = k_1^2 (\frac{1}{2} - c^2) \), and written the separation constant \( \mu = k_2 (2m + b + c + 1) \). If we look for solutions \( \Theta(\theta_1) = \frac{\Psi(\theta_1)}{\sqrt{\sin(k_1 \theta_1)}} \), \( R(\Theta) = S(\Theta)/R \) with \( R = 2\sqrt{E}r \) the equations satisfied by \( \Psi \) and \( S \) are

\[
\left( \frac{\delta_{n}^2}{\sin^2(k_1 \theta_1)} + \frac{k_1^2 \mu^2}{\cos^2(k_1 \theta_1)} + \frac{\delta}{\cos^2(k_1 \theta_1)} + \frac{k_1^2}{4} - \lambda \right) \Psi(\theta_1) = 0, \tag{7.5}
\]

\[
\left( \frac{\delta_{k}^2}{4R^2} + \frac{1 - k_1^2 \mu^2}{2\sqrt{E} - 1} \right) S(R) = 0. \tag{7.6}
\]

In (7.6) we have set \( \lambda = \frac{k_1^2}{4} (1 - \rho^2), \delta = k_1^2 \left( \frac{1}{4} - d^2 \right) \). The separated solutions are

\[
\Xi_{p,m,n} = R_{p}^{(k_1)}(r) \Phi_{m}^{(c,b)}(\cos(2k_2 \theta_2)) \frac{\Psi_{n}^{(\mu/k_1,d)}(\cos(k_1 \theta_1))}{\sqrt{\sin(k_1 \theta_1)}},
\]

\[
\Phi_{m}^{(c,b)}(\cos(2k_2 \theta_2)) = \sin^{n+1/2}(k_2 \theta_2) \cos^{b+1/2}(k_2 \theta_2) P_{m}^{(c,b)}(\cos(2k_2 \theta_2)),
\]

where the \( P_{m}^{(c,b)}(\cos(2k_2 \theta_2)) \) are Jacobi functions, \([4]\);

\[
\Psi_{n}^{(\mu/k_1,d)}(\cos(k_1 \theta_1)) = \sin^{n+1/2}(k_1 \theta_1) \cos^{d+1/2}(k_1 \theta_1) P_{n}^{(\mu/k_1,d)}(\cos(2k_1 \theta_1)),
\]

where the \( P_{n}^{(\mu/k_1,d)}(\cos(2k_1 \theta_1)) \) are Jacobi functions;

\[
R_{p}^{(k_1)}(r) = \frac{S(r)}{2\sqrt{E}}, \quad S(r) = e^{-\sqrt{E}r/k_1^{1/2} + 1/2} L_{p}^{k_1}(2\sqrt{E}r),
\]

and \( \rho = 2(2n + 1/2 + d + 1) \), where the \( L_{p}^{k_1}(2\sqrt{E}r) \) are associated Laguerre functions, \([4]\), and the relation between \( E, \rho \) is the quantization condition

\[
E = \frac{\alpha^2}{(2p + k_1 \rho + 1)^2} = \frac{\alpha^2}{(2[2p + 2k_1 n + 2k_2 m] + 2k_1 [d + 1] + 2k_2 [b + c + 1] + 1)^2}. \tag{7.7}
\]

Note: as in the computations with the TTW potentials we are only interested in the space of generalized eigenfunctions, not the normalization of any individual eigenfunction. Thus the relations to follow are valid on generalized eigenspaces and do not necessarily agree with the normalization of common polynomial eigenfunctions.

There are transformations that preserve \( E \) and imply quantum superintegrability. Indeed for \( k_1 = p_1/q_1, k_2 = p_2/q_2 \) the transformations

1. \( p \to p + p_1, m \to m, n \to n - q_1 \),
2. \( p \to p - p_1, m \to m, n \to n + q_1 \),
3. \( p \to p, m \to m - p_1 q_2, n \to n + q_1 p_2 \),
4. \( p \to p m \to m + p_1 q_2, n \to n - q_1 p_2 \),

will accomplish this.
To effect the $r$-dependent transformations (1) and (2) we use $Y(1)_\pm$:

$$Y(1)_\pm^p R_p^\pm(r) = \left[2(\pm k_1 \rho + 1) \partial_r + \left(2\alpha + \frac{1-k_1^2\rho^2}{r}\right)\right] R_p^\pm(r)$$

(7.8)

$$Y(1)_\pm^p R_p^\pm(r) = -\frac{2\alpha}{2p+k_1\rho + 1} [(1 \pm 1) + (1 \mp 1)(p+1)(p+k_1\rho)] R_p^{\pm\pm 2}(r).$$

To incorporate the $\theta_1$-dependent parts of (1) and (2) we use the following recurrence formulas for the functions $\Psi_{n}^{\mu/k_1,d}(z)$ where $z = \cos(2k_1\theta_1)$:

$$Z(1)_\pm^n \frac{\Psi_{n}^{\mu/k_1,d}(z)}{\sin(k_1\theta_1)} = -2 \left(\frac{\mu}{k_1} + \frac{d}{n} + n\right) \frac{\Psi_{n-1}^{\mu/k_1,d}(z)}{\sin(k_1\theta_1)}$$

$$\equiv \left((1 - z^2) \left(\frac{\rho}{2} - 1\right) \partial_z + \frac{1}{2} \left(\frac{\rho}{2} - 1\right) \left(\frac{\rho}{2} - 1\right) z + \left(\frac{\mu^2}{k_1} - d^2\right)\right)$$

$$\times \frac{\Psi_{n}^{\mu/k_1,d}(z)}{\sin(k_1\theta_1)}.$$ (7.9)

With the identification $\rho = 2(2n + \frac{d}{2} + d + 1)$ we see that the operators $Z(1)_\pm$ depend on $\mu^2$ (which can be interpreted as a differential operator) and are polynomial in $\rho$. We now form the two operators

$$J^+ = (Y(1))^{\frac{\rho+k_1\rho}{4}} Y(1)^{\frac{\rho+k_1\rho}{2}} \cdots Y(1)^{\frac{\rho+k_1\rho}{2}} Y(1)^{\frac{\rho+k_1\rho}{2}} Z(1)^{\frac{\rho+k_1\rho}{2}} \cdots Z(1)^{n}.$$ (7.10)

Since $J^+$ and $J^-$ switch places under the reflection $\rho \to -\rho$ we see that

$$J_2 = J^+ + J^-, \quad J_1 = (J^+ - J^+)/\rho$$

are even functions in both $\rho$ and $\mu$, hence, pure differential operators.

To implement the $\theta_1$-dependent parts of (3) and (4) we set $w = \sin^2(k_1\theta_1)$ and consider $\Psi_{n}^{\mu/k_1,d}$ as a function of $w$. The relevant recurrences are

$$W_n^w(1) \frac{\Psi_{n}^{\mu/k_1,d}}{w^{1/4}} = \left[1 + \frac{\mu}{k_1}\right] \left((w - 1) \frac{d}{dw} - \frac{\mu}{4k_1} \left(1 - \frac{2}{w}\right) + \frac{d^2 - \frac{\mu^2}{4}}{4}\right] \frac{\Psi_{n}^{\mu/k_1,d}}{w^{1/4}}$$

$$= \frac{\left(\frac{\mu}{k_1} + \frac{d}{2} + d + 1\right) \frac{\mu}{k_1} + \frac{d}{2} + d + 1}{4\left(\frac{\mu}{k_1} + \frac{d}{2} + 2\right)} \frac{\Psi_{n-1}^{\mu/k_1,d+2,d}}{w^{1/4}}.\quad(7.11)$$

$$W_n^w(1) \frac{\Psi_{n}^{\mu/k_1,d}}{w^{1/4}} = \left[1 - \frac{\mu}{k_1}\right] \left((w - 1) \frac{d}{dw} + \frac{\mu}{4k_1} \left(1 - \frac{2}{w}\right) + \frac{d^2 - \frac{\mu^2}{4}}{4}\right] \frac{\Psi_{n}^{\mu/k_1,d}}{w^{1/4}}$$

$$= \frac{1}{4} \left(\frac{\mu}{k_1} + \frac{d}{2} - d + 1\right) \left(\frac{\mu}{k_1} + \frac{d}{2} - d + 1\right) \left(\frac{\mu}{k_1} - 1\right) \frac{\Psi_{n+1}^{\mu/k_1,d-2,d}}{w^{1/4}}.$$
To implement the $h_2$-dependent parts of (3) and (4) we use the recurrences $X(2)^m_c$.

$$X(2)^m_c \Phi_m^{(c,b)}(z) = 2(m + c + b + 1) \Phi_m^{(c,b)}(z)$$

$$\equiv \left[ (M + 1)(1 - z^2) \frac{d}{dz} - \frac{1}{2}(c^2 - b^2) \right] \Phi_m^{(c,b)}(z),$$

$$X(2)^m_c \Phi_m^{(c,b)}(z) = 2(m + c)(m + b) \Phi_m^{(c,b)}(z)$$

$$\equiv \left[ -(M - 1)(1 - z^2) \frac{d}{dz} - \frac{1}{2}(c^2 - b^2) \right] \Phi_m^{(c,b)}(z). \quad (7.12)$$

Here $z = \cos(2k_2\theta_2)$, $M = 2m + b + c + 1$. We define

$$K^\pm = (W(1)^{\pm \alpha}(p; q, 1^{-1}) \ldots W(1)^{\pm \alpha}(q; p, 1^{-1}) \ldots X(2)^m_c).$$

From the form of these operators we see that they are even functions of $\rho^2$ and they switch places under the reflection $\mu \rightarrow -\mu$. Thus

$$K_2 = K^+ + K^-, \quad K_1 = (K^+ - K^-)/\mu$$

are even polynomial functions in both $\rho$ and $\mu$, hence, pure differential operators.

We have now constructed partial differential operators $J_1, J_2, K_1, K_2$, each of which commutes with the Hamiltonian $H$ on its eight-dimensional formal eigenspaces. However, to prove that they are true symmetry operators we must show that they commute with $H$ when acting on any analytic functions, not just separated eigenfunctions. This is established in [82] via a canonical form, so that the $J_i, K_i$ are true symmetry operators. We will work out the structure of the algebra generated by $H, L_2, L_3, J_1, K_1$ and from this it will be clear the system is superintegrable.

To determine the structure relations it is sufficient to establish them on the generalized eigenbases. Then a canonical form argument shows that the relations hold for general analytic functions. We start by using the definitions (7.10) and computing on a generalized eigenbasis:

$$J^\pm \Xi_{p,m,n} = \frac{(2)^{4p_1+q_1}(-1)^{q_1} \alpha^{2p_1}(n + 1)q_1, (\mu/k_1 + d + n + 1)q_1}{(2p + k_1, p + 1)_{2p_1}} \Xi_{p-2p_1,m,n+q_1}, \quad (7.13)$$

$$(-1)^{q_1} \alpha^{2p_1}(2p + k_1, p + 1)_{2p_1} J^\pm \Xi_{p,m,n} = (2)^{4p_1+q_1}(-\mu/k_1 - n)_{q_1} \times (-d - n)_{q_1} (p + 1)_{2p_1} (-p - k_1, p)_2, \Xi_{p+2p_1,m,n+q_1}, \quad (7.14)$$

$$(-1)^{q_1} \alpha^{2p_1} K^\pm \Xi_{p,m,n} = (n + 1)_{q_1, p_2} \left( -\frac{\mu}{k_1} - n \right)_{q_1, p_2} \times \left( -\frac{\mu}{k_1} \right)_{2q_1, p_2} (-m - c)_{p_1, q_2}, \Xi_{p,m-p_1, q_2, n+q_1, p_2}, \quad (7.15)$$

$$(-1)^{q_1} \alpha^{2p_1} K^\pm \Xi_{p,m,n} = \left( \frac{\mu}{k_1} + d + n + 1 \right)_{q_1, p_2} \Xi_{p,m,n+q_1, p_2}. \quad (7.16)$$

From these definitions we obtain:

$$[J_1, L_2] = 2k_1, J_2 + 4p_1^2J_1, \quad [L_2, J_2] = -4q_1 [J_1, L_2] = 4q_1^2k_1^2L_2 + 2k_1^2J_1(1 - 8q_1^2)J_1,$$

$$[J_1, L_3] = [J_2, L_2] = 0, \quad [K_1, L_2] = [K_2, L_2] = 0,$$

$$[K_1, L_3] = 4p_1J_2K_2 + 4p_1^2K_1, \quad [K_2, L_3] = -2p_1K_2L_3, [K_1] = -4p_1^2K_2 - 8p_1^3K_1.$$
Further we find
\[
4^{-q_1} E^{-p_1} J^+ J^- \Xi_{p,m,n} = \left( \frac{\rho/2 - \frac{\mu}{E}}{2} \right)_{q_1} \left( \frac{\rho/2 + \frac{\mu}{E} + d + 1}{2} \right)_{q_1} \\
\times \left( \frac{k_1 \rho - \frac{\mu}{\sqrt{E}}}{2} \right)_{q_1} \Xi_{p,m,n,2}. 
\]

\[
4^{-q_1} E^{-p_1} J^+ J^- \Xi_{p,m,n} = \left( \frac{-\rho/2 - \frac{\mu}{E} - d + 1}{2} \right)_{q_1} \left( \frac{-\rho/2 + \frac{\mu}{E} + d + 1}{2} \right)_{q_1} \\
\times \left( \frac{-k_1 \rho - \frac{\mu}{\sqrt{E}}}{2} \right)_{q_1} \Xi_{p,m,n,2}. 
\]

\[
2^{-p_1 q_1} K^+ K^- \Xi_{p,m,n} = I(p, m, n)(m + 1)_{p,q_1} (m + c + b + 1)_{p,q_1} \\
\times (m + c + 1)_{p,q_1} (m + b + 1)_{p,q_1} \Xi_{p,m,n}, 
\]

\[
2^{-p_1 q_1} K^+ K^- \Xi_{p,m,n} = I(p, m, n)(-m)_{p,q_1} (-m - c - b)_{p,q_1} \\
\times (-m - c)_{p,q_1} (-m - b)_{p,q_1} \Xi_{p,m,n}, 
\]

\[
I(p, m, n) = \prod_{q = 1}^{\rho} \left( \frac{\rho}{2} + \frac{\mu}{E} + d + 1 \right)_{q_1} \left( \frac{\rho}{2} + \frac{\mu}{E} - d + 1 \right)_{q_1} \Xi_{p,m,n,2}. 
\]

From these expressions it is easy to see that each of \( J^+ J^- J^+ J^- \) is a polynomial in \( \mu \) and \( E \) and that these operators switch places under the reflection \( \rho \to -\rho \). Thus \( P_1 (H, L_2, L_3) = J^+ J^- J^+ J^- \) and \( P_2 (H, L_2, L_3) = (J^+ J^- J^+ J^-) / \rho \) are each polynomials in \( H, L_2, L_3 \). Similarly, each of \( K^+ K^- K^+ K^- \) is a polynomial in \( \rho^2 \) and in \( \mu \) and these operators switch places under the reflection \( \mu \to -\mu \). Thus \( P_3 (L_2, L_3) = K^+ K^- K^+ K^- \) and \( P_2 (L_2, L_3) = (K^+ K^- K^+ K^-) / \mu \) are each polynomials in \( L_2, L_3 \).

Straightforward consequences of these formulas are the structure relations

\[
[J_1, J_2] = -2q_1 J_1^2 - 2P_2, \quad J_1^2 \left( \frac{1}{4} - k_1^2 L_2 \right) + 2q_1 J_1 J_2 = J_2^2 - 2p_1, \\
K_1^2 + K_2^2 = 2P_3 + 2p_1 p_2 K_1 K_2, \quad [K_1, K_2] = -2p_1 p_2 K_2^2 - 2P_4. 
\]

### 7.2.1. Lowering the orders of the generators.

Just as for the classical analogues, we can find generators that are of order one less than \( J_1 \) and \( K_1 \). First we look for a symmetry operator \( J_0 \) such that \( [L_2, J_0] = J_1 \) and \( [L_3, J_0] = 0 \). A straightforward computation yields the solution

\[
J_0 = -\frac{1}{2k_1^2 q_1} \left( J^- \frac{\rho}{\rho + 2q_1} + J^+ \frac{\rho}{\rho - 2q_1} \right) + \frac{S_1 (H, L_3)}{\rho^2 - 4q_1^2}. 
\]

From this it is easy to show that \( 2k_1^2 q_1 (\rho^2 - 4q_1^2) J_0 = -J_2^2 + 2q_1 J_1 + 2k_1^2 q_1 S_1 (H, L_3) \). To determine \( S_1 \) we evaluate both sides of this equation for \( \rho = -2q_1 \): \( S_1 (H, L_3) = \frac{1}{2k_1^2 q_1} J_{\rho=-2q_1} \).
The detailed computation can be found in [82]:

\[ 2k_1^2q_1(-4)^{(1-q)/2}4^{-p}S_1(H, L_s) = \left( \frac{L_3}{k_1^2} + d^2 \right) \Pi_{j=0}^{p-1} (\alpha^2 - (1 + 2s)^2H) \Pi_{j=0}^{(q-1)/2} \]

\[ \times \left( \left( s - \frac{d}{2} \right)^2 + \frac{L_3}{4k_1^2} \right) \left( s + \frac{d}{2} \right)^2 + \frac{L_3}{4k_1^2} \right). \]

\[ [L_2, J_0] = J_1, \quad [L_3, J_0] = 0, \quad 2k_1^2q_1J_0 \left( 1 - \frac{4L_2}{k_1^2} - 4q_1^2 \right) = -J_2 + 2q_1J_1 + 2k_1^2q_1S_1. \]

Next we look for a symmetry operator $K_0$ such that $[L_3, K_0] = K_1$ and $[L_2, K_0] = 0$. A straightforward computation yields the solution

\[ K_0 = -\frac{1}{4p_1p_2} \left( \frac{K^-}{\mu(\mu + p_1p_2)} + \frac{K^+}{\mu(-\mu - p_1p_2)} \right) + \frac{S_2(L_2)}{\mu^2 - p_1^2p_2^2}, \]

where the symmetry operator $S_2$ is to be determined. From this it is easy to show that $4p_1p_2K_0(\mu^2 - p_1^2p_2^2) = -K_2 + p_1p_2K_1 + 4p_1p_2S_2(L_2)$. To determine $S_2$ we evaluate both sides of this equation for $\mu = -p_1p_2$: $S_2(L_2) = \frac{1}{4p_1p_2}K_2K_2 - p_1p_2S_2$. The result is [82]:

\[ S_2(L_2) = \frac{\left( d^2 - \frac{\rho^2}{4} \right) 2p_1q_2 \left( b^2 - \frac{c^2}{4} \right) p_1p_2}{\Pi_{j=0}^{(q-1)/2} \left( \frac{d + \rho}{2} - s \right)^2 - \left( \frac{d - \rho}{2} - s \right)^2} \]

\[ \times \Pi_{j=0}^{(q-1)/2} \left( s + \frac{c + b}{2} \right)^2 \left( s - \frac{c + b}{2} \right)^2 \left( s - 1 - \frac{b - c}{2} \right)^2 \left( s - 1 + \frac{b - c}{2} \right)^2, \]

\[ [L_3, K_0] = K_1, \quad [L_2, K_0] = 0, \quad 4p_1p_2K_0(L_3 + p_1^2p_2^2) = K_2 - p_1p_2K_1 - 4p_1p_2S_2. \]

7.2.2. The structure equations. Now we determine the $J$, $K$-operator commutators. We write

\[ J^+ \Xi_{p,m,n} = J^+(p, m, n) \Xi_{p,m,n} = K^+(m, n) \Xi_{p,m,n}, \]

\[ K^+ \Xi_{p,m,n} = K^+(m, n) \Xi_{p,m,n}, \]

where $J^\pm, K^\pm$ are defined by the right-hand sides of (7.14), (7.15). We find

\[ [J^+, K^+] \Xi_{p,m,n} = \frac{1}{1 + A(\mp \rho, \mu)} [J^+, K^+] \Xi_{p,m,n} \equiv C(\mp \rho, \mu) [J^+, K^+] \Xi_{p,m,n}, \]

\[ [J^+, K^-] \Xi_{p,m,n} = C(\mp \rho, -\mu) [J^+, K^-] \Xi_{p,m,n}, \]

\[ A(\rho, \mu) = \frac{1}{(\frac{q}{2} + \frac{\mu}{2q} + \frac{1}{2} - \frac{s}{2})_q_1 \frac{(q}{2} + \frac{\rho}{2q} + \frac{1}{2} + \frac{s}{2})_q_2}. \]

From this we can compute the relations

\[ [J_1, K_2] = Q_{12}^1 [J_1, K_1] + Q_{12}^2 [J_2, K_2] + Q_{12}^3 [J_1, K_2] + Q_{12}^4 [J_2, K_1], \]

where $1 \leq j, k \leq 2$ and the $Q_{12}^h$ are rational in $\rho^2, \mu^2$:

\[ 4\rho^{-j} \mu^{-k} \frac{Q_{12}^h}{h} = (-1)^{j+k+i+\ell} C(-\rho, -\mu) + (-1)^{k+i+\ell} C(\rho, \mu) + (-1)^{i+j+\ell} C(-\rho, \mu) + C(\rho, -\mu). \]

The relations can be cast into the pure operator form

\[ [K_1, J_0] = Q_{11}^1 [J_1, K_1] + Q_{11}^2 [J_2, K_2] + Q_{11}^3 [J_1, K_2] + Q_{11}^4 [J_2, K_1], \]

where $h, \ell \leq 1, 2$, and $Q_{11}^h$ are polynomials in $L_2, L_3$. In particular,

\[ Q = B(\rho, \mu) B(-\rho, -\mu), \]

\[ B(\rho, \mu) = \Sigma_{h=0,1} \left( \frac{\rho}{4} + \frac{\mu}{2k_1} + \frac{1}{2} - \epsilon q_1 p_2 - \frac{d}{2} \right)_q_1 \left( \frac{\rho}{4} + \frac{\mu}{2k_2} + \frac{1}{2} - \epsilon q_1 p_2 + \frac{d}{2} \right)_q_1. \]
on a generalized eigenbasis. Thus for general $k_1, k_2$ the symmetry algebra closes algebraically but not polynomially. The basis generators are $H, L_2, L_3, J_0, K_0$ and the commutators $J_1, K_1$ are appended to the algebra.

7.2.3. The special case $k_1 = k_2 = 1$. In the case $k_1 = k_2 = 1$ we are in Euclidean space and just as for the classical system we have additional permutation symmetry. The extra symmetry gives rise to a new fourth-order symmetry operator $J_0'$ and the six functionally dependent symmetries $H, L_2, L_3, K_0$ (second order) and $J_0', J_0''$ (fourth order) generate a symmetry algebra that closes polynomially. The complicated details can be found in [82, 179]. The algebraic relation obeyed by the six generators is of order 12, [82].

8. Generalized St"ackel transform

In section 4 we introduced the St"ackel transform on second-order systems. Here we review the fundamentals of CCM and St"ackel transform, and apply them to map superintegrable systems to other such systems on different manifolds. In general, CCM does not preserve structure but we study specializations which do preserve polynomials and symmetry structures in both the classical and quantum cases. Details of the proofs can be found in [101].

8.1. Coupling constant metamorphosis for classical systems

The basic tool that we employ follows from ‘CCM’, a general fact about Hamiltonian systems, [72]. Let $\mathcal{H}(x, p) + \alpha U(x)$ define a Hamiltonian system in $2n$-dimensional phase space. Thus the Hamilton–Jacobi equation is $\mathcal{H}(x, p) + \alpha U(x) = E$.

**Theorem 19.** (CCM) Assume that the system admits a constant of the motion $K(\alpha)$, locally analytic in parameter $\alpha$. The Hamiltonian $\mathcal{H}' = (\mathcal{H} - E)/U$ admits the constant of the motion $K' = K(-\mathcal{H}')$, where now $E$ is a parameter.

**Proof.** If $F, G$ are functions on phase space of the form $G(x, p)$, $F = F(\alpha) = F(\alpha, x, p)$ where $\alpha = \alpha(x, p)$ then

$$[F, G] = [F(\alpha), G]|_{\alpha=\alpha(x, p)} + \partial_\alpha F(\alpha)|_{\alpha=\alpha(x, p)} \{\alpha, G\}.$$  

By assumption, $\{K(\alpha), \mathcal{H}\} = -\alpha \{K(\alpha), U\}$ for any value of the parameter $\alpha$. Thus

$$\{K(\alpha), \mathcal{H}'\} = \frac{\partial K(\alpha)}{U}(\mathcal{H}' + \alpha),$$  

$$\{K(-\mathcal{H}'), \mathcal{H}'\} = \left[ \partial_\alpha K(\alpha) \{\mathcal{H}', \mathcal{H}'\} + \frac{U}{U} \frac{K(\alpha)}{U} \{\mathcal{H}' + \alpha\} \right]_{\alpha=-\mathcal{H}'} = 0.$$

**Corollary 1.** Let $K_1(\alpha), K_2(\alpha)$ be constants of the motion for the system $\mathcal{H}(x, p) + \alpha U(x)$. Then $\{K_1(-\mathcal{H}'), K_2(-\mathcal{H}')\}$ is also a constant of the motion and

$$\{K_1(-\mathcal{H}'), K_2(-\mathcal{H}')\} = \{K_1, K_2\}(-\mathcal{H}').$$

Clearly CCM takes superintegrable systems to superintegrable systems. We are concerned with the case where $K$ is polynomial in the momenta and

$$\mathcal{H} = \sum_{i,j=1}^n g^{ij} p_i p_j + V(x) + \alpha U(x) \equiv \mathcal{H}_0 + V + \alpha U.$$  

(8.1)

For second-order constants of the motion there is special structure: the second-order constants are typically linear in $\alpha$, so they transform to second-order symmetries again. Then CCM agrees with the St"ackel transform. However, in general the order of constants of the motion is not preserved by CCM.

70
Example 1. The system $\mathcal{H} = p_1^2 + p_2^2 + b_1\sqrt{x_1} + b_2\sqrt{x_2}$ admits the second-order constant $\mathcal{K}^{(2)} = p_3^2 + b_2\sqrt{x_2}$ and the third-order constant $\mathcal{K}^{(3)} = p_3^3 + 3b_1\sqrt{x_1}p_1 - \frac{3b_2^2}{4}p_2$. ([64, 128, 129] and references contained therein). If we choose $\alpha U = \alpha x_2$ then the transform of $\mathcal{K}^{(3)}$ will be fifth order. If we choose $\alpha U = \alpha x_2$ then the transform of $\mathcal{K}^{(3)}$ will be nonpolynomial. To obtain useful structure results from CCM we need to restrict the generality of the transform action.

8.2. The Jacobi transform

Here we study a specialization of CCM to the case $V = 0$ in (8.1). This special version takes $N$th order constants of the motion for Hamiltonian systems to $N$th order constants. An $N$th order constant $\mathcal{K}(x, p)$ for the system

$$\mathcal{H} = \sum_{i,j=1}^{n} g_{ij} p_i p_j + U(x) = \mathcal{H}_0 + U$$

(8.2)

is a function on the phase space such that $\{\mathcal{K}, \mathcal{H}\} = 0$ where

$\mathcal{K} = \mathcal{K}_N + \mathcal{K}_{N-2} + \mathcal{K}_{N-4} + \cdots + \mathcal{K}_0, N$ even, \quad $\mathcal{K} = \mathcal{K}_N + \mathcal{K}_{N-2} + \cdots + \mathcal{K}_1, N$ odd.

Here, $\mathcal{K}_N \neq 0$ and $\mathcal{K}_j$ is homogeneous in $p$ of order $j$. This implies

$\{\mathcal{K}_N, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_N-2k, U\} + \{\mathcal{K}_N-2k, \mathcal{H}_0\} = 0, \quad k = 0, 1, \ldots, [N/2] - 1,$

and, for odd $\mathcal{K}_j$, $\{\mathcal{K}_j, U\} = 0$.

The case $N = 1$ is very special. Then $\mathcal{K} = \mathcal{K}_1$ and the conditions are $\{\mathcal{K}, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}, U\} = 0$, so $\mathcal{K}$ is a Killing vector and $U$ is invariant under the local group action generated by the Killing vector.

For $N = 2, \mathcal{K} = \mathcal{K}_2 + \mathcal{K}_0$ and the conditions are

$$\{\mathcal{K}_2, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_2, U\} + \{\mathcal{K}_0, \mathcal{H}_0\} = 0,$$

(8.3)

so $\mathcal{K}_2$ is a second-order Killing tensor and $U$ satisfies Bertrand–Darboux conditions.

For $N = 3, \mathcal{K} = \mathcal{K}_3 + \mathcal{K}_1$ and the conditions are

$$\{\mathcal{K}_3, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_3, U\} + \{\mathcal{K}_1, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_1, U\} = 0.$$

Integrability conditions for the last two equations lead to nonlinear PDEs for $U$.

Theorem 20. Suppose the system (8.2) admits an $N$th order constant of the motion $\mathcal{K}$ where $N \geq 1$. Then

$$\mathcal{\hat{K}} = \sum_{j=0}^{[N/2]} \left( \frac{\mathcal{H}_0 - E}{U} \right)^j \mathcal{K}_{N-2j}$$

is an $N$th order constant of the motion for the system $(\mathcal{H}_0 - E)/U$.

Corollary 2. Suppose the system $\mathcal{H}_0 + U$ is $N$th order superintegrable. Then the free system $\mathcal{H}_0/U$ is also $N$th order superintegrable.

We call $\mathcal{\hat{K}}$ a Jacobi transform of $\mathcal{K}$, since it is related to the Jacobi metric, [115, p 172]. Note that it is invertible.

Corollary 3. The Jacobi transform satisfies the properties $\{\mathcal{\hat{K}}, \mathcal{L}\} = \{\hat{K}, \hat{\mathcal{L}}\}, \mathcal{\hat{K}} \mathcal{\hat{L}} = \hat{K}\hat{L}$, and, if $\mathcal{K}$, $\mathcal{L}$ are of the same order, $a\hat{K} + b\mathcal{L} = a\hat{K} + b\hat{L}$, Thus it defines a homomorphism from the graded symmetry algebra of the system $\mathcal{H}_0 + U$ to the graded symmetry algebra of the system $(\mathcal{H}_0 - E)/U$. 

71
Example 2. Consider the system of Example 1: $H = p_1^2 + p_2^2 + b_1 \sqrt{x_1} + b_2 x_2 + b_3,$ and let $U = b_1 \sqrt{x_1} + b_2 x_2 + b_3$ for some fixed $b_1, b_2, b_3$ with $b_1 b_2 \neq 0.$ The Jacobi transforms of $H,$ $\hat{K}^{(2)}, \hat{K}^{(3)}$ are

$$\hat{H} = \frac{p_1^2 + p_2^2 - E}{b_1 \sqrt{x_1} + b_2 x_2 + b_3}, \quad \hat{K}^{(2)} = p_2^2 - b_2 x_2 \left( \frac{p_1^2 + p_2^2 - E}{b_1 \sqrt{x_1} + b_2 x_2 + b_3} \right),$$

$$\hat{K}^{(3)} = p_1^2 - \left( \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3 b_1^2}{4 b_2^2} p_1^2 - \frac{3 b_1^2 + p_2^2 - E}{b_1 \sqrt{x_1} + b_2 x_2 + b_3} \right).$$

8.3. The Stöckel transform

We use the same notation as in the previous section, and a particular nonzero potential $U = V(x, b_0).$ The Stöckel transform for a second-order system was treated in section 4.1.3. It is not a special case of CCM, although the two transforms are closely related. However in the situation where the potential functions $V(x, b)$ form a finite-dimensional vector space, usual in the study of second-order superintegrability, the transforms coincide. In this case, by redefining parameters if necessary, we can assume $V$ is linear in $b.$

Now we investigate extensions of the Stöckel transform to higher-order constants, assuming $V(x, b)$ is linear in $b = (b_0, b_1, \ldots, b_M),$ $U$ is of the form $U(x) = V(x, b_0^0)$ and the potentials $V(x, b)$ span a space of dimension $M + 1$:

$$V(x, b) = b_0 + \sum_{i=1}^{M} U^{(i)}(x) b_i$$

where the set of functions $[1, U^{(1)}(x), \ldots, U^{(M)}(x)]$ is linearly independent. In the study of second-order superintegrability, typically the second-order constants are linear in the $b$ and the algebra generated by these symmetries via products and commutators has the property that a constant of order $N$ depends polynomially on the parameters with order $\leq [N/2].$ Thus we consider only those higher-order constants of the motion of order $N$ of the form

$$\mathcal{K} = \sum_{j=0}^{[N/2]} \mathcal{K}_{N-2j}(p, b)$$

where $\mathcal{K}_{N-2j}(ap, b) = a^{N-2j} \mathcal{K}_{N-2j}(p, b)$ and $\mathcal{K}_{N-2j}(p, ab) = a^j \mathcal{K}_{N-2j}(p, b)$ for any parameter $a.$ Let $\mathcal{K}(b)$ be such an $N$th order constant. Then

$$\mathcal{K}(ab) \equiv \mathcal{K}(p, b + ab^0)$$

is an $N$th order constant for $H_0 + V(x, b) + aU(x).$ Applying theorem 19:

**Theorem 21.** Let $\mathcal{K}$ be an $N$th order constant of the motion for the system $H_0 + V(x, b)$ where $V$ is of the form (8.4) and $\mathcal{K}$ is of the form (8.5). Let $\mathcal{K}(a)$ be defined by (8.6). Then

$$\tilde{\mathcal{K}} = \mathcal{K} \left( \frac{H_0 + V(x, b)}{U(x)} \right) = \sum_{j=0}^{[N/2]} \tilde{\mathcal{K}}_{N-2j}(p, b)$$

is an $N$th order constant for the system $(H_0 + V(x, b))/U(x),$ whenever

$$\tilde{\mathcal{K}}_{N-2j}(ap, b) = a^{N-2j} \tilde{\mathcal{K}}_{N-2j}(p, b), \quad \tilde{\mathcal{K}}_{N-2j}(p, ab) = a^j \tilde{\mathcal{K}}_{N-2j}(p, b).$$

(8.7)
Example 3. Let, \[ \{185\}, \]
\[ \mathcal{H} = p_1^2 + p_2^2 + a(x_1^2 + x_2^2) + b \frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)} + c \frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2}. \]
There are two basic constants of the motion,
\[ \mathcal{K}_2 = (x_1 p_2 - x_2 p_1)^2 + 4b \frac{x_1^2 x_2^2}{(x_1^2 - x_2^2)^2} + c \frac{x_1^2 + x_2^2}{x_1^2 x_2^2}, \]
\[ \mathcal{K}_4 = (p_1^2 - p_2^2)^2 + \left[ 2a x_1^2 + 2b \frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} - 2c \frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2} \right] p_1^2 \]
\[ + \left[ -4a x_1 x_2 + 8b \frac{x_1 x_2}{(x_1^2 - x_2^2)^2} \right] p_1 p_2 + \left[ 2a x_2^2 + 2b \frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + 2c \frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2} \right] p_2^2 \]
\[ + a^2 (x_1^2 - x_2^2)^2 + b^2 \frac{x_1^2}{(x_1^2 - x_2^2)^2} + c^2 \frac{x_2^2}{x_1^2 x_2^2} + 8ab \frac{x_1 x_2}{(x_1^2 - x_2^2)^2} + 2bc \frac{x_1 x_2}{x_1^2 x_2^2}. \]
Then the transformed system also has second- and fourth-order constants:
\[ \tilde{\mathcal{H}} = p_1^2 + p_2^2 + a(x_1^2 + x_2^2) + b \frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)} + d \frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2} + \frac{E}{(x_1^2 - x_2^2)^2} + \frac{F}{x_1^2 x_2^2} + D. \]

Example 4. In \[ \{154\} \] it is shown that the TTW system is Stäckel equivalent to the caged isotropic oscillator. We sketch the corresponding construction in the 3D case. Consider the caged isotropic oscillator
\[ \mathcal{H}' = p_\phi^2 + \alpha' R^2 + \frac{L'_2}{R^2}. \]
\[ L'_2 = p_\phi^2 + \frac{L'_3}{\sin^2(j_1 \phi_1)} + \frac{\delta}{\cos^2(j_1 \phi_1)}, \]
\[ L'_3 = p_\phi^2 + \frac{\beta'}{\cos^2(j_2 \phi_2)} + \frac{\gamma'}{\sin^2(j_2 \phi_2)}. \]
Here \( L'_2, L'_3 \) are constants of the motion that determine additive separation in the spherical coordinates \( R, \phi_1, \phi_2 \). Also, \( j_1, j_2 \) are nonzero rational numbers. If \( j_1 = j_2 = 1 \), then in Cartesian coordinates we have \( \mathcal{H}' = p_1^2 + p_2^2 + p_3^2 + \alpha' R^2 + \beta'/x^2 + \gamma'/y^2 + \delta'/z^2 \), and \( \{8.8\} \) can be considered as a 3-variable analogue of the TTW system, flat space only if \( j_1 = 1 \). Consider the Hamilton–Jacobi equation \( \mathcal{H}' = E \) and take the Stäckel transform corresponding to division by \( R^2 \). Then, for \( r = R^2, 2\phi_1 = \theta_1, 2\phi_2 = \theta_2 \), we obtain the new Hamilton–Jacobi equation \( \mathcal{H} = E \) where \( \mathcal{H} = p_\phi^2 + \frac{\alpha}{r} + \frac{\delta}{r^2} \) with
\[ L_2 = p_\phi^2 + \frac{L_3}{\sin^2(k_1 \theta_1)} + \delta \frac{\cos^2(k_1 \theta_1)}{\sin^2(k_1 \theta_1)}, \]
\[ L_3 = p_\phi^2 + \frac{\beta}{\cos^2(k_2 \theta_2)} + \gamma \frac{\sin^2(k_2 \theta_2)}{\sin^2(k_2 \theta_2)}, \]
\[ E = -\alpha'/4, \alpha = -E'/4, \beta = \beta'/4, \gamma = \gamma'/4, \delta = \delta'/4, k_1 = j_1/2, k_2 = j_2/2. \]
This is the extended Kepler–Coulomb system. Stäckel transforms preserve structure so all structure results apply to the caged oscillator. Note that \( k_1 = k_2 = 1 \) for Kepler–Coulomb corresponds to \( j_1 = j_2 = 2 \) for the oscillator.

8.4. Coupling constant metamorphosis for quantum symmetries

Unlike the case of classical Hamiltonian, CCM for quantum systems is not guaranteed to preserve integrals of the motion. In general, it is not clear how to replace the coupling constant by the corresponding operator as you would for the function in classical mechanics.
However, we can isolate the cases where it is possible to preserve some of the symmetries. In particular, for second-order superintegrable systems in 2D, it is known that all potential and the integrals of motion depend linearly on at most four constants. Thus, CCM is well defined on these systems, see [90].

For higher-order integrals of the motion, we require that the Hamiltonian admit a part of the potential parameterized by a coupling constant and that the integrals of the motion be polynomial in this coupling constant. In this case, the CCM preserves integrals of the motion. The results are given in the following theorem from [101].

**Theorem 22.** Let

\[ H(\alpha) = H(0) + \alpha U \]

be a Hamiltonian operator with integral of motion of the form

\[ K(\alpha) = \sum_{j=0}^{[N/2]} K_{N-2j}\alpha^j, \quad [H(\alpha), K(\alpha)] = 0 \]

where \( H(0) \) and \( K_{N-j} \) are independent of the coupling constant \( \alpha \). Then the operator \( \tilde{K} = \sum_{b=0}^{[N/2]} (-1)^b K_{N-2b}(U^{-1}(H + b))^b \) is a well-defined finite-order differential operator which commutes with the Hamiltonian \( \tilde{H} = U^{-1}(H + b) \).

Furthermore, as with many known cases, if each \( K_{N-2j} \) is a differential operator of order \( N - 2j \) then the transformed operator \( \tilde{K} \) will remain \( N \)th order. This is the case in the following example.

**Example 5** (The 3-1 anisotropic oscillator). Let \( H(\alpha) = \partial_{11} + \partial_{22} + \alpha(9x_1^2 + x_2^2) \). This is a superintegrable system with generating second- and third-order symmetries

\[ L = \partial_{22} + \alpha x_2^2, \quad K = \{ x_1 \partial_2 - x_2 \partial_1, \partial_{22} \} + \frac{\alpha}{3}(\{ x_2^3, \partial_1 \} - 9\{ x_1 x_2^2, \partial_2 \}), \]

where \( \{ S_1, S_2 \} \equiv S_1 S_2 + S_2 S_1 \). Let \( U = (9x_1^2 + x_2^2) + c \). It follows that system \( \tilde{H} = \frac{1}{(9x_1^2 + x_2^2) + c} (\partial_{11} + \partial_{22} + b) \) has a second- and a third-order symmetry.

We also mention that as in the classical case, CCM preserves the structure of the symmetry algebras as shown in the following corollary.

**Corollary 4.** Let \( K(\alpha) \) \( L(\alpha) \) be \( N \)th and \( M \)th order operator symmetries, respectively, of \( H(\alpha) \), each satisfying the conditions of theorem 22. Then \( [L, \tilde{K}] = [L, \tilde{K}] \), \( \tilde{L} \tilde{K} = \tilde{K} \tilde{L} \).

Note that theorem 22 does not require that the quantum system go to a classical system, only that a scalable potential term can be split off. Thus it applies to ‘hybrid’ quantum systems that have a classical part which depends linearly on an arbitrary (scalable) parameter and a quantum part depending on \( \hbar^2 \).

**Example 6** (The hybrid 3-1 anisotropic oscillator). Let \( H = -\hbar^2(\partial_{11} + \partial_{22}) + a(9x_1^2 + x_2^2) + 2\hbar^2/x_2^2 \). This is a superintegrable system with generating second- and third-order symmetries,

\[ L = \partial_{22} + ax_2^2, \quad K = \{ x_1 \partial_2 - x_2 \partial_1, \partial_{22} \} + \left\{ \frac{a}{3} x_1^3 + \frac{\hbar^2}{\lambda_2^2}, \partial_1 \right\} - \left\{ 3x_1 \left( ax_2^2 + \frac{\hbar^2}{\lambda_2^2} \right), \partial_2 \right\}. \]

Let \( U = (9x_1^2 + x_2^2) + c \). Then the system also has one second- and one third-order symmetry:

\[ \tilde{H} = \frac{-\hbar^2}{(9x_1^2 + x_2^2) + c} \left( \partial_{11} + \partial_{22} + a(9x_1^2 + x_2^2) + \frac{2\hbar^2}{x_2^2} + b \right). \]
Example 7 (A translated hybrid 3-1 anisotropic oscillator). This is a slight modification of Example 6. Let $H = -\hbar^2 (\partial_{11} + \partial_{22}) + a(9x_1^2 + x_2^2) + cx_1 + \frac{2\hbar^2}{x_1^2}$. This is a superintegrable system with generating second- and third-order symmetries. Let $U = x_1$. It follows that the system
\[
\tilde{H} = \frac{-\hbar^2}{x_1} \left( \partial_{11} + \partial_{22} + a\left(9x_1^2 + x_2^2\right) + cx_1 + \frac{2\hbar^2}{x_1^2} + b\right)
\]
is superintegrable with one second- and one third-order symmetry.

Note that the general anisotropic oscillator with singular terms
\[
H = -\hbar^2 (\partial_{11} + \partial_{22}) + a(k^2x_1^2 + x_2^2) + b/x_1^2 + c/x_2^2,
\tag{8.9}
\]
is superintegrable for all integers $k$ [169], however it is only with the specific choices of constants as in examples 6 and 7 that the additional higher-order integral of motion is of third-order. In this sense, the systems in examples 6 and 7 are truly quantum since in the classical limit the potential reduces to the 3-1 anisotropic oscillator and the classical Hamiltonian with the potentials as in the quantum case ($\hbar$ is then considered an arbitrary constant) has different symmetry generators from the quantum case. Thus, we see that CCM can be extended to quantum systems which differ from their classical analogue as long as the potential has a term proportional to an arbitrary constant.

For simplicity, we have restricted our quantum constructions to 2D manifolds though some partial results hold in $n$ dimensions. There appears to be no insurmountable barrier to extending these results to 3D and higher conformally flat manifolds, but the details have not yet been worked out. Clearly gauge transformations are required and the gauge will depend on the curvature of the manifold. In the classical case a multiple parameter extension of the Stäckel transform has been studied [176].

9. Polynomial algebras and their irreducible representations

Just as the representation theory for Lie algebras has seen wide applications in mathematical physics and in special functions, it is natural that the representation theory for the algebras generated by superintegrable systems would have interesting analysis and applications. Indeed, the study of polynomial algebras and their representations continues to be a rich field of current inquiry, see e.g. [21, 38, 61–63, 116, 124, 125, 160, 161]. In particular, the irreducible representations give important information about the spectrum of the Hamiltonian and other symmetry operators, as well as the multiplicities of the bound state energy levels. In this section, we survey results concerning only quadratic algebras with three and five functionally independent operators. In the former case, discussed in section 9.1, the representations of second-order superintegrable systems in 2D is directly connected with the Askey scheme of orthogonal polynomials and the limits are obtained from limits of the physical systems which in turn give contractions of the algebras. The case of quadratic algebras with five generators is less well understood: we discuss only one model in section 9.2.1 as a generalization of the 2D results and an indication of possible areas of future research.

9.1. Quadratic algebras: contractions and Askey scheme

Special functions arise as solutions of exactly solvable problems, so it should not be surprising that the Askey scheme for organizing hypergeometric orthogonal polynomials is a consequence of contractions of superintegrable systems. We describe how all second-order superintegrable systems in two dimensions are limiting cases of a single system: the generic three-parameter
potential on the 2-sphere, $S^2$ in our listing. This implies that the quadratic symmetry algebras of these systems are contractions of that of $S^2$. The irreducible representations of $S^2$ have a realization in terms of difference operators in one variable, the structure algebra for the Wilson and Racah polynomials: the Askey–Wilson algebra for $q = 1$. Recently, Genest, Vinet and Zhedanov [55] have given an elegant proof of the equivalence between the symmetry algebra for $S^2$ and the Racah problem of $su(1, 1)$. By contracting the representations of $S^2$ we can obtain representations of the quadratic symmetry algebras of other systems we obtain the full Askey scheme, [109, 112]. This ties the scheme directly to physical phenomena. It is more general: it applies to all special functions that arise from these systems via separation of variables, not just those of hypergeometric type, and it extends to higher dimensions.

The special functions of mathematical physics are associated with realizations of the irreducible representations of the quadratic symmetry algebras of second-order superintegrable systems, [21, 38, 61–63, 160]. Since the structures of these algebras are essentially preserved irreducible representations of the quadratic symmetry algebras of second-order superintegrable potentials but one is an isolated Euclidean singleton unrelated to the Askey scheme. Since every second-order 2D superintegrable system is Stäckel equivalent to a constant curvature potential on the 2-sphere, and Zhedanov [55] have given an elegant proof of the equivalence between the symmetry algebras of second-order superintegrable systems, [21, 38, 61–63, 160]. Since the structures of these algebras are essentially preserved under the Stäckel transform it is sufficient to study only one system in each equivalence class.

Each of the 12 superintegrable systems related to the Askey scheme restricts to a free first-order superintegrable system when the potential is set to 0:  

1. The complex 2-sphere. Here $s_1^2 + s_2^2 + s_3^2 = 1$ is the embedding of the 2-sphere in complex Euclidean space, and the Hamiltonian is $H = J_1^2 + J_2^2 + J_3^2$, where $J_3 = s_1 \partial_{s_3} - s_2 \partial_{s_2}$ and $J_2, J_3$ are obtained by cyclic permutations of 1, 2, 3. The basis symmetries are $J_1, J_2, J_3$. They generate the Lie algebra so(3) with relations $[J_1, J_2] = -J_3, [J_2, J_1] = -J_1, [J_3, J_1] = -J_2$ and Casimir $H$.

2. The complex Euclidean plane. Here $H = \partial_1^2 + \partial_2^2$ with basis symmetries $P_1 = \partial_1, P_2 = \partial_2$ and $M = \partial_1 \partial_2 - \partial_2 \partial_1$. The symmetry Lie algebra is $e(2)$ with relations $[P_1, P_2] = 0, [P_1, M] = P_2, [P_2, M] = -P_1$ and Casimir $H$. As will be shown in a forthcoming article [105], all of the contractions of the quadratic algebras are in fact generated by the contractions of these Lie algebras, which have been classified [193].

9.1.1. Contractions of superintegrable systems. A detailed treatment of contractions will appear elsewhere, [105]. Here we just describe ‘natural’ contractions. Suppose we have a nondegenerate superintegrable system with generators $H, L_1, L_2$ and structure equations (4.14), defining a quadratic algebra $Q$. If we make a change of basis to new generators $\tilde{H}, \tilde{L}_1, \tilde{L}_2$ and parameters $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ such that

$$
\begin{pmatrix}
\tilde{L}_1 \\
\tilde{L}_2 \\
\tilde{H}
\end{pmatrix} =
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} & A_{2,3} \\
0 & 0 & A_{3,3}
\end{pmatrix}
\begin{pmatrix}
L_1 \\
L_2 \\
H
\end{pmatrix} +
\begin{pmatrix}
B_{1,1} & B_{1,2} & B_{1,3} \\
B_{2,1} & B_{2,2} & B_{2,3} \\
B_{3,1} & B_{3,2} & B_{3,3}
\end{pmatrix}
\begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{pmatrix},
$$

$$
\begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{pmatrix} =
\begin{pmatrix}
C_{1,1} & C_{1,2} & C_{1,3} \\
C_{2,1} & C_{2,2} & C_{2,3} \\
C_{3,1} & C_{3,2} & C_{3,3}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix},
$$

for some $3 \times 3$ constant matrices $A = (A_{ij}), B, C$ such that $\det A \cdot \det C \neq 0$, we will have the same system with new structure equations of the form (4.14) for $\tilde{R} = [\tilde{L}_1, \tilde{L}_2], [\tilde{L}_1, \tilde{H}], \tilde{R}^2$, but...
with transformed structure constants. We choose a continuous one-parameter family of basis transformation matrices \( A(\epsilon), B(\epsilon), C(\epsilon) \), \( 0 < \epsilon \leq 1 \) such that \( A(1) = C(1) \) is the identity matrix, \( B(1) = 0 \) and \( \det A(\epsilon) \neq 0, \det C(\epsilon) \neq 0 \). Now suppose as \( \epsilon \to 0 \) the basis change becomes singular, (i.e., the limits of \( A, B, C \) either do not exist or, if they exist do not satisfy \( \det A(0) \det C(0) \neq 0 \)) but the structure equations involving \( A(\epsilon), B(\epsilon), C(\epsilon) \), go to a limit, defining a new quadratic algebra \( Q' \). We call \( Q' \) a contraction of \( Q \) in analogy with Lie algebra contractions [74].

For a degenerate superintegrable system with generators \( H, X, L_1, L_2 \) and structure equations (4.15), (4.17), defining a quadratic algebra \( Q \), a change of basis to new generators \( \tilde{H}, X, \tilde{L}_1, \tilde{L}_2 \), parameter \( a \) such that \( \tilde{a} = Ca \), and

\[
\begin{pmatrix}
\tilde{L}_1 \\
\tilde{L}_2 \\
\tilde{H} \\
\tilde{X}
\end{pmatrix} =
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & 0 \\
A_{2,1} & A_{2,2} & A_{2,3} & 0 \\
0 & 0 & A_{3,3} & 0 \\
0 & 0 & 0 & A_{4,4}
\end{pmatrix}
\begin{pmatrix}
L_1 \\
L_2 \\
H \\
X
\end{pmatrix} +
\begin{pmatrix}
B_1 \\
B_2 \\
B_3 \\
0
\end{pmatrix} a
\]

for some \( 4 \times 4 \) matrix \( A = (A_{i,j}) \) with \( \det A \neq 0 \), complex 4-vector \( B \) and constant \( C \neq 0 \) yields the same superintegrable system with new structure equations of the form (4.15), (4.17) for \( [\tilde{X}, \tilde{L}_1], [\tilde{L}_1, \tilde{L}_2], \) and \( \tilde{G} = 0 \), but with transformed structure constants. Suppose we choose a continuous one-parameter family of basis transformation matrices \( A(\epsilon), B(\epsilon), C(\epsilon) \), \( 0 < \epsilon \leq 1 \) such that \( A(1) \) is the identity matrix, \( B(1) = 0, C(1) = 1 \), and \( \det A(\epsilon) \neq 0, C(\epsilon) \neq 0 \). Now suppose as \( \epsilon \to 0 \) the basis change becomes singular, (i.e., the limits of \( A, B, C \) either do not exist or, exist and do not satisfy \( C(0) \det A(0) \neq 0 \), but that the structure equations involving \( A(\epsilon), B(\epsilon), C(\epsilon) \), go to a finite limit, thus defining a new quadratic algebra \( Q' \). We call \( Q' \) a contraction of \( Q \).

It has been established that all second-order 2D superintegrable systems can be obtained from system S9 by limiting processes in the coordinates and/or a Stäckel transformation, e.g. [83, 86, 87, 106]. All systems listed in section 4.1.4 are limits of S9. It follows that the quadratic algebras generated by each system are contractions of the algebra of S9. (However, an abstract quadratic algebra may not be associated with a superintegrable system, and a contraction of a quadratic algebra associated with one superintegrable system to a quadratic algebra associated with another superintegrable system does not necessarily imply that this is associated with a coordinate limit process.)

9.1.2. Models of superintegrable systems. A representation of a quadratic algebra is a homomorphism of the algebra into the associative algebra of linear operators on some vector space. In this review a model is a faithful representation in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space. We will study classes of irreducible representations realized by these models. Suppose a superintegrable system with quadratic algebra \( Q \) contracts to a superintegrable system with quadratic algebra \( Q' \) via a continuous family of transformations indexed by the parameter \( \epsilon \). If we have a model of a representation of \( Q \) we can try to ‘save’ this representation, as did Wigner for Lie algebra representations [74], by passing through a continuous family of representations of \( Q(\epsilon) \) in the model to obtain a representation of \( Q' \) in the limit. We will show that as a byproduct of contractions from S9 for which we save representations in the limit, we obtain the Askey scheme for hypergeometric orthogonal polynomials. (The full details can be found in [104].) In all the models to follow the polynomials we classify are eigenfunctions of formally self-adjoint or formally skew-adjoint operators. The weight functions for the orthogonality can be found in [110]. They can be derived by requiring
that the second-order operators $H$, $L_1$, $L_2$ are formally self-adjoint and the first-order operator $X$ is formally skew-adjoint. See [103] for examples.

9.1.3. The S9 model. There is no differential model for S9 but a difference operator model yielding structure equations for the Racah and Wilson polynomials [99], defined as

$$w_n(t^2) \equiv w_n(t^2, \alpha, \beta, \gamma, \delta) = (\alpha + \beta)_n(\alpha + \gamma)_n(\alpha + \delta)_n \times {}_4F_3\left(\begin{array}{r} -n, \alpha + b + c + d + n - 1, \alpha - t, \alpha + t \end{array} \alpha + b, b, c, c \alpha + c, c + d \right)$$

$$= (a + b)_n(a + c)_n(a + d)_n \Phi_n^{(a,b,c,d)}(t^2),$$

(9.1)

where $(\alpha)_n$ is the Pochhammer symbol and ${}_4F_3(1)$ is a hypergeometric function of unit argument [4]. The polynomial $w_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$. For the finite-dimensional representations the spectrum of $t^2$ is $(\alpha + k)^2$, $k = 0, 1, \ldots, m$ and the orthogonal basis eigenfunctions are Racah polynomials. In the infinite-dimensional case they are Wilson polynomials. They are eigenfunctions for the difference operator $\tau^* \tau$ defined via $E^{(1)}_t F(t) = \tilde{F}(t + A)$ and

$$\tau = \frac{1}{2t} (E_t^{1/2} - E_t^{-1/2}),$$

(9.2)

$$\tau^* = \frac{1}{2}[(a + t)(b + t)(c + t)(d + t)E_t^{1/2} - (a - t)(b - t)(c - t)(d - t)E_t^{-1/2}].$$

(9.3)

A finite or infinite-dimensional bounded below representation is defined by $H = E$, $a_1 = \frac{1}{2} - a_1^2$ and

$L_1 = -4t^* \tau - 2(a_2 + 1)(a_3 + 1) + \frac{1}{2}$, $L_2 = -4t^* \tau - 2(a_2 + 1)(a_3 + 1) + \frac{1}{2}$, $E = -(m + 1)(m + 1 + \alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \frac{1}{4}$.

(9.4)

and the constants of the Wilson polynomials are chosen as

$a = -\frac{1}{2}(\alpha_1 + \alpha_3 + 1) - m, \quad d = \alpha_2 + m + 1 + \frac{1}{2}(\alpha_1 + \alpha_3 + 1),$

$b = \frac{1}{2}(\alpha_1 + \alpha_3 + 1), \quad c = \frac{1}{2}(-\alpha_1 + \alpha_3 + 1).$

Here $n = 0, 1, \ldots, m$ if $m$ is a nonnegative integer and $n = 0, 1, \ldots$ otherwise.

For the basis $f_{n,m} \equiv \Phi_n^{(a,b,c,d)}(t^2)$, the model action is $H f_{n,m} = E f_{n,m},$

$L_{1,f_{n,m}} = -\left(4n^2 + 4n[\alpha_2 + \alpha_3 + 1] + 2[\alpha_2 + 1][\alpha_3 + 1] - \frac{1}{2}\right) f_{n,m},$

$L_{2,f_{n,m}} = K_{n+1,n} f_{n+1,m} + K_{n-1,n} f_{n-1,m} + \left(K_{n,n} + \alpha_1^2 + \alpha_3^2 - \frac{1}{2}\right) f_{n,m},$

$K_{n+1,n} = \frac{(\alpha_1 + 1 + \alpha_2 + n)(m - n)(m - n + \alpha_1)(1 + m + n)}{(\alpha_3 + 1 + \alpha_2 + 2n)(\alpha_3 + 2 + \alpha_2 + 2n)}.$

$K_{n-1,n} = \frac{n(\alpha_3 + n)(\alpha_3 + \alpha_3 + 1 + \alpha_2 + m + n)(1 + \alpha_1 + \alpha_2 + m + n)}{(\alpha_3 + 1 + \alpha_2 + 2n)(\alpha_3 + \alpha_2 + 2n)}.$

$K_{n,n} = \left(\frac{1}{2}(\alpha_1 + \alpha_3 + 1) - m\right)^2 - K_{n+1,n} - K_{n-1,n}.$

Note that these models give the possible energy eigenvalues of the quantum Hamiltonians and their multiplicities.
9.1.4. Some $S9 \to E1$ contractions. There are at least two ways to take this contraction; it is possible to contract the sphere about the point $(0, 1, 0)$ which gives continuous dual Hahn polynomials as limits of Wilson polynomials. Contracting about the point $(1, 0, 0)$ leads to continuous Hahn polynomials or Jacobi polynomials. The dual Hahn and continuous dual Hahn polynomials correspond to the same superintegrable system but they are eigenfunctions of different generators. For the finite-dimensional restrictions ($m$ a positive integer) we have the restrictions of Racah polynomials to dual Hahn and Hahn respectively.

(1) Wilson $\to$ continuous dual Hahn. For the first limit, in the quantum system, we contract about $(0, 1, 0)$ so that the points of our 2D space lie in the plane $(x, 1, y)$. We set $s_1 = \sqrt{1 - x^2 - y^2} \approx 1 - \frac{x^2}{2} (x^2 + y^2), s_3 = \sqrt{y},$ for small $\epsilon$. The coupling constants transform as

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

and we get $E1$ as $\epsilon \to 0$. This gives the quadratic algebra contraction

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{H} \end{pmatrix} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & -\epsilon & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ H \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$ (9.5)

As in S9, it is advantageous in the model to express the three coupling constants as quadratic functions of other parameters, so that with $\alpha_2 \to \infty$,

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} - \beta_1^2 \\ \frac{1}{4} - \beta_2^2 \\ \frac{1}{4} - \beta_3^2 \end{pmatrix} = \begin{pmatrix} \frac{\epsilon^2}{4} - \epsilon^2 \alpha_1^2 \\ \frac{1}{4} - \alpha_1^2 \\ \frac{1}{4} - \alpha_2^2 \end{pmatrix}.$$ (9.6)

In the contraction limit the operators tend to $H' = \lim_{\epsilon \to 0} \tilde{H} = E'$ and

$$L_1' = \lim_{\epsilon \to 0} \tilde{L}_1 = -4\tau^* \tau' - 2\beta_1 (\beta_3 + 1), \quad L_2' = \lim_{\epsilon \to 0} \tilde{L}_2 = -4\tau^2 + \beta_2^2 + \beta_3^2 - \frac{1}{2}, \quad E' = -2\beta_1 (2m + 2 + \beta_2 + \beta_3).$$

The eigenfunctions of $L_1$, the Wilson polynomials, transform in the contraction limit to the eigenfunctions of $L_1'$, the dual Hahn polynomials $S_n$,

$$S_n(-t^2, a', b', c') = (a' + b')_{n} (a' + c')_{n} F_2 \begin{pmatrix} -n, & a' + t, & a' - t; 1 \end{pmatrix},$$

$$a' = -\frac{1}{2} (\beta_2 + \beta_3 + 1) - m, \quad b' = \frac{1}{2} (\beta_2 + \beta_3 + 1), \quad c' = \frac{1}{2} (\beta_2 + \beta_3 + 1).$$

Again, $n = 0, 1, \ldots, m$ if $m$ is a nonnegative integer and $n = 0, 1, \ldots$ otherwise. The operators $\tau^*$ and $\tau'$ are given by

$$\tau' = \tau = \frac{1}{2 \tau} (E_1^{1/2} - E_1^{-1/2}),$$

$$\tau^* = \frac{\beta_1}{2 \tau} [(a' + t)(b' + t)(c' + t)E_1^{1/2} - (a' - t)(b' - t)(c' - t)E_1^{-1/2}].$$

79
The action on the basis \( f'_{n,m} \equiv S_n(-t^2, a', b', c')/(a' + b' + c') \) is
\[
L'_1 f'_{n,m} = -2\beta_1 (2n + \beta_3 + 1) f'_{n,m}, \quad H' f'_{n,m} = E f'_{n,m}.
\]
\[
L'_2 f'_{n,m} = K'_{n+1,n} f'_{n+1,m} + K'_{n-1,n} f'_{n-1,m} + \left( K'_{n,n} + \beta_2^2 + \beta_3^2 - \frac{1}{2} \right) f'_{n,m},
\]
\[
K'_{n+1,n} = (m - n) (m - n + \beta_2), \quad K'_{n-1,n} = n(n + \beta_3)
\]
\[
K'_{n,n} = \left( \frac{1}{2} (\beta_2 + \beta_3 + 1) - m \right)^2 - K'_{n+1,n} - K'_{n-1,n}.
\]

(2) Wilson \( \rightarrow \) continuous Hahn. Next, we contract about \((1, 0, 0)\) so that the points of our 2D space lie in the plane \((1, x, y)\). We set \(s_1 = \sqrt{1 - x^2 - y^2} \approx 1 - \frac{x^2}{2} (x^2 + y^2), s_2 = \sqrt{x}, s_3 = \sqrt{y}, \) for small \(\epsilon\).

\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2 \\
\hat{a}_3
\end{pmatrix} =
\begin{pmatrix}
\epsilon^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}, \quad (9.8)
\]

\[
\begin{pmatrix}
\hat{L}_1 \\
\hat{L}_2 \\
\hat{H}
\end{pmatrix} =
\begin{pmatrix}
0 & \epsilon & 0 \\
1 & 0 & 0 \\
0 & 0 & \epsilon
\end{pmatrix}
\begin{pmatrix}
L_1 \\
L_2 \\
H
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-\epsilon & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}. \quad (9.9)
\]

In terms of the constants of (9.7) the transformation gives, with \(\alpha_1 \to \infty\),

\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2 \\
\hat{a}_3
\end{pmatrix} =
\begin{pmatrix}
-\beta_1^2 \\
\frac{1}{4} - \beta_2^2 \\
\frac{1}{4} - \beta_3^2
\end{pmatrix}
\begin{pmatrix}
\epsilon^2 & -\epsilon^2 \alpha_1^2 \\
\frac{1}{4} & \alpha_1^2 \\
\frac{1}{4} & \alpha_1^2
\end{pmatrix}.
\]

Solving a representation: We set \(t = -x + \frac{\theta_0}{\sqrt{2}} + m + \frac{1}{2} (\beta_3 + 1)\). In the contraction limit, the operators go as \(L'_{1,2} = \lim_{\epsilon \to 0} \hat{L}_{1,2}\),

\[
L'_1 = 2\beta_1 (2x - 2m - \beta_3 - 1), \quad H' = -2\beta_1 (2m + 2 + \beta_2 + \beta_3),
\]

\[
L'_2 = -4 (B(x) E_x + C(x) E^{-1} - B(x) - C(x)) - 2(\beta_2 + 1) (\beta_3 + 1) + \frac{1}{2}
\]

with \(B(x) = (x - m)(x + \beta_2 + 1), C(x) = x(x - m - 1 - \beta_3)\). The operators \(L'_1, L'_2\) and \(H'\) satisfy the algebra relations in (4.22). The eigenfunction of \(L'_1\), the Wilson polynomials, transform in the contraction limit to the eigenfunctions of \(L'_2\), which are the Hahn polynomials, \(f'_{n,m,n} = Q_n\).

\[
Q_n(x; \beta_2, \beta_3, m) = 3F_2 \left( \begin{array}{c}
-n, \beta_2 + \beta_3 + n + 1, -x \\
-m, \beta_2 + \beta_3 + 1
\end{array} ; 1 \right).
\]

The action of the operators on this basis is given by
\[
L'_{1,2} f'_{n,m,n} = K'_{n+1,n} f'_{n+1,m,n} + K'_{n,n,m} f'_{n,n,m} + K'_{n-1,n} f'_{n-1,m,n},
\]
\[
L'_{1,2} f'_{n,m,n} = -4 \beta_1 (2n + \beta_2 + \beta_3 + 1) (2n + \beta_2 + \beta_3 + 1) f'_{n,m,n},
\]
\[
K'_{n+1,n} = -4 \beta_1 (2n + \beta_2 + \beta_3 + 1) (n + \beta_2 + 1) (n + \beta_2 + 1)
\]
\[
K'_{n-1,n} = -4 \beta_1 (2n + \beta_2 + \beta_3 + 1) (n + \beta_2 + 1) (n + \beta_2 + 1)
\]
\[
K'_{n,n} = -4 \beta_1 (2n + \beta_2 + \beta_3 + 1) (n + \beta_2 + 1) - K'_{n+1,n} - K'_{n-1,n}.
\]
The eigenfunctions for \( L \) of the measure in the finite case. Also, the commutator of \( X \) directly as a contraction from symmetry leads to a new second-order symmetry. In [105] we will show how this follows \( \) \( \). However, after restriction one second-order generator \( L \) becomes a perfect square \( L = X^2 \). The spectrum of \( L \) is nonnegative but that of \( X \) can take both positive/negative values. This results in a virtual doubling of the support of the measure in the finite case. Also, the commutator of \( X \) and the remaining second-order symmetry leads to a new second-order symmetry. In [105] we will show how this follows directly as a contraction from \( R^2 \).

\[ \text{For } m = \sqrt{-E^2} - 1 + \frac{\beta_2 + \beta_3}{2}, \quad t = \sqrt{-E^2} \sqrt{\frac{1 + x}{2}} \] for \( E' \) a constant and letting \( m \to \infty \). Then (9.9) gives a contraction of the model for \( S_9 \) to a differential operator model for \( E_1 \) with \( \beta_1 = 0 \):

\[ L'_1 = \frac{E'}{2}(x + 1), \quad H' = E', \]

\[ L'_2 = 4(1 - x^2)\frac{d^2}{dx^2} + 4[\beta_3 - \beta_2 - (\beta_2 + \beta_3 + 2)x]d_x - 2(\beta_2 + 1)(\beta_3 + 1) + \frac{1}{2}. \]

The eigenfunctions for \( L_1 \), the Wilson polynomials, tend in the limit to eigenfunction of \( L'_2 \), the Jacobi polynomials:

\[ p_n^{\beta_2, \beta_3}(x) = \frac{(\beta_2 + 1)n}{n!} \binom{-n, \beta_2 + 3 + n + \frac{1}{2}; \frac{x - 1}{2}}{\beta_2 + 1}. \]

In terms of the basis \( f_n = \frac{n!}{\beta_2 + 1} p_n^{\beta_2, \beta_3}(x) \), the action of the operators is

\[ L'_1 f_n = K'_{n+1, n} f'_{n+1} + K'_{n-1, n} f_{n-1} + K'_{n, n} f_n, \]

\[ L'_2 f_n = -4n(n + \beta_2 + \beta_3 + 1) - 2(\beta_2 + 1)(\beta_3 + 1) + \frac{1}{2}, \]

\[ K'_{n+1, n} = \frac{E' (\beta_2 + \beta_3 + n + 1)(\beta_2 + n + 1)}{(\beta_2 + \beta_3 + 2n + 1)(\beta_2 + \beta_3 + 2n + 2)}, \]

\[ K'_{n-1, n} = \frac{E'n(n + \beta_3)}{(\beta_2 + \beta_3 + 2n)(\beta_2 + \beta_3 + 2n + 1)}, \quad K'_{n, n} = E' - K'_{n+1, n} - K'_{n-1, n}. \]

**9.1.5. A nondegenerate \( \) \( \) degenerate limit.** This appears initially a mere restriction of the three-parameter potential to one-parameter. However, after restriction one second-order generator \( L_1 \) becomes a perfect square \( L = X^2 \). The spectrum of \( L_1 \) is nonnegative but that of \( X \) can take both positive/negative values. This results in a virtual doubling of the support of the measure in the finite case. Also, the commutator of \( X \) and the remaining second-order symmetry leads to a new second-order symmetry. In [105] we will show how this follows directly as a contraction from \( R^2 \).

(1) **Wilson \( \) \( \) special dual Hahn (first model).** The quantum system \( E3 \) (4.1.4) is given in the singular limit from system \( S9 \), (9.1) by

\[ a_2 = a_3 = \epsilon \to 0, \quad a_1 = a_1, \quad X^2 = \lim_{\epsilon \to 0} L_1, \quad L'_2 = \lim_{\epsilon \to 0} L_2, \quad L'_1 = [X', L'_2]. \]

The operators in this contraction differ from those given in subsection 4.1.4 by a cyclic permutation of the coordinates \( s_i \to s_{i+1} \).

Now we investigate how the difference operator realization of \( S9 \) contracts to irreducible representations of the \( S3 \) algebra. This is more complicated since the original restricted algebra is now a proper subalgebra of the contracted algebra. The contraction is realized in the model by setting \( a_2 = a_3 = -1/2 \) and \( a_1 = \alpha \) (the subscript is dropped since now is a sole \( \alpha \)). The restricted operators become \( H' = E' \) with
The eigenfunctions for $X^2$, the Wilson polynomials, become

$$
\Phi_{2m}(t^2) = 4F_3 \begin{pmatrix} -n, -n, & -4m + 2\alpha + 1 & -t, \frac{1}{2} \\ -m, -m - \alpha, & 4 \end{pmatrix} \left( \begin{array}{c} 4m + 2\alpha + 1 + t \\ 4 \end{array} \right) ; 1
$$

(9.12)

Here $n = 0, 1, \ldots, m$ if $m$ is a nonnegative integer and $n = 0, 1, \ldots$ otherwise. For finite-dimensional representations, the spectrum of $t$ is the set \( \left\{ \frac{n}{2} + \frac{1}{2} + m-k \mid k = 0, 1, \ldots, m \right\} \).

The restricted polynomial functions (9.12) are no longer the correct basis functions for the contracted superintegrable system:

$$
L_2 f_{n,m} = K_{n+1,n} f_{n+1,m} + K_{n-1,n} f_{n-1,m} + \left( K_{n,n} + \alpha^2 \right) f_{n,m}
$$

$$
K_{n+1,n} = \frac{1}{4}(m - n + \alpha)(m - n), \quad K_{n-1,n} = \frac{1}{4}(m + n + \alpha)(m + n),
$$

$$
K_{n,n} = \left( \frac{1}{4}(\alpha + 1) - m \right)^2 - K_{n+1,n} + K_{n-1,n}.
$$

Now $K_{n-1,n}$ no longer vanishes for $n = 0$, so $f_{0,m}$ is no longer the lowest weight eigenfunction; note $f_{-1,m} = f_{1,m}$ is still a polynomial in $t^2$. To understand the contraction we set $n = N - M/2$ where $N$ is a nonnegative integer and $M = 2m$. Then the equations for the $K$ become

$$
K(N + 1, N) = \frac{1}{4}(M - N + \alpha)(M - N), \quad K(N - 1, N) = \frac{1}{4}N(N + \alpha),
$$

$$
K(N, N) = \frac{1}{4}(\alpha + 1 - M)^2 - K(N + 1, N) - K(N - 1, N).
$$

The three term recurrence relation gives a new set of basis orthogonal polynomials for representations of $S3$. The lowest eigenfunction occurs for $N = 0$; if $M$ is a nonnegative integer the representation is $(2m-1)$-dimensional with highest eigenfunction for $N = M$.

The new basis functions are

$$
f_{N,M}(t^2) = \frac{(\alpha + 1)_N}{(-\alpha - M)_N} F_2 \left( \begin{array}{cc} -N, -s & s + 2\alpha + 1 \\ -M, 1 + \alpha \end{array} \right) \left( \begin{array}{c} 1 \end{array} \right).
$$

(9.13)

Here $f_N$ is a polynomial of order $2N$ in $s$ and of order $n$ in $\lambda(s) = (s + 2\alpha + 1)$, a special case of dual Hahn polynomials. These dual Hahn polynomials admit

$$
X = i(B(s)E_x + C(s)E_x^{-1}), \quad B(s) + C(s) = M,
$$

$$
B(s) = \frac{(s + 2\alpha + 1)(M - s)}{2s + 2\alpha + 1}, \quad C(s) = \frac{s(s + M + 2\alpha + 1)}{2s + 2\alpha + 1}.
$$

(9.14)

The operators which form a model for the algebra (4.1.4) are $X$ (9.14),

$$
L_1 = -\left( s + \alpha + \frac{1}{2} \right)^2 + \alpha^2 - \frac{1}{4}, \quad L_2 = [L_1, X].
$$

For finite-dimensional representations the spectrum of $s$ is $\{0, 1, \ldots, M\}$. From the operator $L_1$ we can determine the relation $s = 2t - \alpha - 1/2$.

What is the relation between the functions (9.12) and the proper basis functions (9.13)? Note this model $X, L_1, L_2$ can be obtained from the contracted model $X', L'_1, L'_2$ by conjugating by the ‘ground state’ of the contracted model $\Phi_{2\frac{M}{2}}(t^2)$. We find explicitly the gauge function $\Phi_{2\frac{M}{2}}(t^2) = \frac{(\frac{1}{2} + \frac{1}{2})^2}{(\frac{1}{2} + \frac{1}{2})^2 \cdot (\frac{1}{2} + \frac{1}{2})^2}$, when $t$ is evaluated at the weights $t = \frac{M}{2} + \frac{1}{2} + \frac{1}{2} = k, k = 0, 1, \ldots, \frac{M}{2}$, so the operator $X'$ is related to $X$ via conjugation by $\Phi_{2\frac{M}{2}}(t^2)$. 
Note that the functions $\Phi_a(t^2)$ are only defined for discrete values of $t$. However, on this restricted set the functions $\Phi_{-\frac{a}{2}+N}$ and $f_{N,M}$ satisfy exactly the same three term recurrence formula under multiplication by $-4t^2-a$, with the bottom of the weight ladder at $N=0$. From this we find

$$\Phi_{-\frac{a}{2}+N}(t^2)f_{N,M}(t^2) = \Phi_{-\frac{a}{2}+N}(t^2), \quad t = \frac{a}{2} + \frac{1}{4} + \frac{M}{2} - k.$$  \hspace{1cm} (9.15)

Since $\Phi_{-\frac{a}{2}}(t^2) = \Phi_{\frac{a}{2}}(t^2)$, $f_{M,M}(t^2) = 1$ restricted to the spectrum of $t$.

(2) Wilson $\rightarrow$ special Hahn (second model). The quantum system $S3$ (4.35) can also be obtained from system $S9$, (9.1) by

$$a_1 = a_3 = \epsilon \rightarrow 0, \quad a_2 = \frac{1}{4} - \alpha^2, \quad L'_1 = \lim_{\epsilon \rightarrow 0} L_1, \quad X' = \lim_{\epsilon \rightarrow 0} L_2, \quad L'_2 = [X', L'_1].$$

Again, the physical model obtained by this contraction is related to the that given in subsection 4.1.4 by a cyclic permutation of the coordinates $s_i \rightarrow s_{i-1}$.

In this limit, the operator $X^2$ can be immediately factorized to obtain the skew-adjoint operator $X = 2i\epsilon$. Taking $x = t + m$, we find

$$L'_1 = -\left[B(x)E_x + C(x)E_x^{-1} - B(x) - C(x)\right] - \alpha - \frac{1}{4}, \quad X' = 2i(x - m),$$

$$B(x) = (x - 2m)(x + \alpha + 1), \quad C(x) = x(x - 2m - \alpha - 1), \quad L'_2 = [X', L'_1],$$

which is diagonalized by Hahn polynomials

$$\tilde{f}_{k,m} = \frac{3F2}{2}\left(-k, k + 2\alpha + 1, -x; \alpha + 1, -2m; 1\right) = Q_k(x; B_2, B_2, 2m),$$

$$L'_1 \tilde{f}_{k,m} = \frac{3F2}{2}\left(-k + \alpha + \frac{1}{2}, \alpha^2 - \frac{1}{4}\right) \tilde{f}_{k,m}, \quad k = 0, 1, \ldots, 2m.$$  \hspace{1cm} (9.16)

These polynomials satisfy special relations not obeyed by general Hahn polynomials. The dimension of the space has jumped from $m + 1$ to $2m + 1$. Comparing these eigenfunctions with the limit of the Wilson polynomials,

$$\lim_{\epsilon \rightarrow 0} L'_1 f_{n,m} = \left(-\left(2n + \alpha + \frac{1}{2}\right)^2 + \alpha - \frac{1}{4}\right) f_{n,m}, \quad n = 0, 1, \ldots, m,$$

$$f_{n,m}(t) = 4F3\left(-n, n + \alpha + \frac{1}{2}, -m - t, -m + t; \alpha + 1; \frac{1}{2} - m\right),$$

we see that in the limit only about half of the spectrum is uncovered. Here, the functions $f_{n,m}$ are even functions of $t$ whereas $\tilde{f}_{k,m}(t) = (-1)^k \tilde{f}_{k,m}(t)$.

The recurrences for multiplication by $2i\epsilon$ and $-4t^2$ are compatible, so we obtain the following identity obeyed by special Wilson polynomials:

$$4F3\left(-n, n + \alpha + \frac{1}{2}, -m - t, -m + t; \alpha + 1; \frac{1}{2} - m\right)\right), \quad n = 0, 1, \ldots, m.$$  \hspace{1cm} (9.17)

9.1.6. The scheme and final comments. The full set of contractions leading to the Askey scheme can be found in [104]. The top half of figure 6 shows the standard Askey scheme indicating which orthogonal polynomials can be obtained by pointwise limits from other polynomials and, ultimately, from the Wilson or Racah polynomials. The bottom half of figure 6 shows how each of the superintegrable systems can be obtained by a series of
contractions from the generic system $S9$. Not all possible contractions are listed, partly due to complexity and partly to keep the graph from being too cluttered. (For example, all nondegenerate and degenerate superintegrable systems contract to the Euclidean system $H = \partial_x x + \partial_y y$.) The singular systems are superintegrable in the sense that they have three algebraically independent generators, but the coefficient matrix of the second-order terms in the Hamiltonian is singular. They follow naturally as contractions of nonsingular systems. Figure 7 shows which orthogonal polynomials are associated with models of which quantum superintegrable system and how contractions enable us to reach all of these functions.
Figure 7. The Askey contraction scheme.

from S9. Again not all contractions have been exhibited, but enough to demonstrate that the Askey scheme is a consequence of the contraction structure linking second-order quantum superintegrable systems in 2D. Forthcoming papers will simplify considerably the complexity of this approach, [105]. Indeed the structure equations for nondegenerate superintegrable systems can be derived directly from the expression for $R^2$ alone, and the structure equations for degenerate superintegrable systems can be derived, up to a multiplicative factor, from the
Casimir alone. It will also be demonstrated that all of the contractions of quadratic algebras in the Askey scheme can be induced by natural contractions of the Lie algebras $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$.

There is a close association between the models and the symmetry operators in the original physical quantum systems that describe bases of eigenfunctions via separation of variables. All of the quantum systems are multiseparable. Some separable systems are exactly solvable in the sense that the physical solutions are products of hypergeometric functions. The special functions arising in the models we consider can be described as the coefficients in the expansion of a separable eigenbasis for the original quantum system in terms of another separable eigenbasis. The functions in the Askey scheme are all hypergeometric polynomials that arise as the expansion coefficients relating two separable eigenbases that are both of hypergeometric type. Special polynomials in the Bannai–Ito classification have been associated with superintegrability, see [151, 152]. The method obviously extends to second-order systems in more variables; a start can be found in [103]. Examples of models for higher-order superintegrable systems can be found in [82]. The eigenfunctions are now rational in general, rather than polynomial.

To extend the method to Askey–Wilson polynomials we would need to find appropriate $q$-quantum mechanical systems with $q$-symmetry algebras and, so far, this has not been done.

### 9.2. Quadratic algebras of 3D systems

As described in the previous section 9.1, the representation theory for quadratic operators with two functionally independent generators (i.e. those associated with second-order superintegrable systems in 2D) is now well understood. Although the analysis is not complete for 3D, there has been some work in this area, namely models of the singular isotropic oscillator [102], and models for the generic system on the 3-sphere [103]. It is this latter model that we present here.

#### 9.2.1. The system on the 3-sphere

The Hamiltonian operator is defined via the embedding of the unit 3-sphere in 4D flat space, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

$$H = \sum_{1 \leq i < j \leq 4} (x_i \partial_j - x_j \partial_i) + \sum_{k=1}^{4} a_k \frac{\partial_i \equiv \partial_k}{x_k^2}.$$  \hspace{1cm} (9.16)

A basis for the second-order integrals of the motion is given by

$$L_{ij} \equiv L_{ji} = (x_i \partial_j - x_j \partial_i) + \frac{a_i x_j^2}{x_i^2} + \frac{a_j x_i^2}{x_j^2},$$  \hspace{1cm} (9.17)

for $1 \leq i < j \leq 4$. While there are 6 such operators which commute with the Hamiltonian (9.16), there is a functional relations which appears at eighth order as well as the linear relation,

$$H = \sum_{1 \leq i < j \leq 4} L_{ij} + \sum_{k=1}^{4} a_k.$$  

Thus, there are indeed six linearly independent integrals and five functionally independent, in agreement with the 5 implies 6 theorem.

Let us now consider the algebra generated by these operators. In the following $i, j, k, \ell$ are pairwise distinct integers such that $1 \leq i, j, k, \ell \leq 4$, and $\epsilon_{ijk}$ is the completely skew-symmetric tensor such that $\epsilon_{ijk} = 1$ if $i < j < k$. There are four linearly independent commutators of the second-order symmetries (no sum on repeated indices):

$$R_{\ell} = \epsilon_{ijk} [L_{ij}, L_{jk}].$$  \hspace{1cm} (9.18)
This implies, for example, that
\[ R_1 = [L_{23}, L_{34}] = -[L_{24}, L_{34}] = -[L_{23}, L_{24}] \]

Also,
\[ [L_{ij}, L_{k\ell}] = 0. \]

Here we define the commutator of linear operators \( F, G \) by \([F, G] = FG - GF\).

The fourth-order structure equations are
\[
[L_{ij}, R_j] = 4\epsilon_{ikl}(L_{ik, L_{jl}} - [L_{il, L_{jk}}] + L_{il} - L_{ik} + L_{jk} - L_{jl})
\]
(9.19)

Here, \([L_{ij}, R_k] = 4\epsilon_{ijk}(L_{ij, L_{il}} - (2 + 4a_i)L_{il} - (2 + 4a_i)L_{il} + 2a_i - 2a_j). \)
(9.20)

Here, \([F, G] = FG + GF\). The fifth-order structure equations are obtainable directly from the fourth-order equations and the Jacobi identity.

The sixth-order structure equations are
\[
R_i^2 = \frac{8}{3}[L_{ij}, L_{ik}, L_{jk}] - (12 + 16a_k)L_{ij}^2 - (12 + 16a_i)L_{jk}^2 - (12 + 16a_j)L_{ik}^2
\]
\[
+ \frac{52}{3}(L_{ij, L_{ik} + L_{jk}}) + [L_{ik, L_{jk}}, L_{ij}] + \left(\frac{16}{3} + \frac{176}{3}a_i\right)L_{ij}
\]
\[
+ \left(\frac{16}{3} + \frac{176}{3}a_j\right)L_{jk} + \left(\frac{16}{3} + \frac{176}{3}a_j\right)L_{ij} + 64aa_ia_ja_k
\]
\[
+ 48(a_i a_j + a_j a_k + a_k a_i) + \frac{32}{3}(a_i + a_j + a_k).
\]
(9.21)

\[
\frac{\epsilon_{ikl}\epsilon_{jkl}}{2}\{R_i, R_j\} = \frac{4}{3}([L_{il, L_{jl}, L_{ik}}, L_{jl}] + [L_{ik, L_{jk}, L_{il}} - [L_{ij, L_{kl}, L_{kl}}])
\]
\[
+ \frac{26}{3}[L_{ik, L_{jk}}] + \frac{26}{3}[L_{il, L_{jk}}] + \frac{44}{3}[L_{ij, L_{kl}}, L_{kl}] + 4L_{ij}^2
\]
\[- \frac{32}{3}L_{ik} - \left(\frac{8}{3} - 8a_i\right)(L_{ik} + L_{jl}) - \left(\frac{8}{3} - 8a_j\right)(L_{jl} + L_{ik})
\]
\[
+ \left(\frac{16}{3} + 24a_i + 24a_j + 32a_ka_l\right)L_{ij} - 16(a_i a_j + a_i + a_j).
\]
(9.22)

Here, \([A, B, C] = ABC + A(BC) + B(AC) + C(AB) + C(AB).

The eighth-order functional relation is
\[
\sum \left[ \frac{1}{8}L_{ij}^2L_{kl}^2 - \frac{1}{92}[L_{ik, L_{jl}}, L_{jl}, L_{jk}] - \frac{1}{36}[L_{ij, L_{ik}, L_{kl}}]
\]
\[- \frac{7}{62}[L_{ij, L_{jl}, L_{kl}}] + \frac{1}{6}(\frac{1}{2} + \frac{2}{3}a_i)[L_{ij}L_{ik}]
\]
\[
+ \frac{2}{3}L_{ij}L_{kl} - \left(\frac{3}{4} - \frac{3}{4}a_i - \frac{3}{4}a_i\right) + \frac{2}{3}a_ia_i + 3a_ia_i + 7a_ia_i)
\]
\[
+ \left(\frac{1}{3} + \frac{1}{6}a_i\right)[L_{ik}, L_{jl}] + \left(\frac{4}{3}a_i + \frac{4}{3}a_i + \frac{7}{3}a_ia_i\right)L_{ij}
\]
\[
+ \frac{2}{3}a_ia_ia_i + 2a_ia_ia_i + \frac{4}{3}a_ia_i\right] = 0
\]
(9.23)

Here, \([A, B, C, D] \) is the 24 term symmetrizer of 4 operators and the sum is taken over all pairwise distinct \(i, j, k, \ell\).
We note here that the algebra described above contains several copies of the algebra generated by the corresponding potential on the 2-sphere. Namely, if \( \mathcal{A} \) is defined to be the algebra generated by all the operators \( \{L_{ij}, \mathcal{T}\} \) for all \( i, j = 1, \ldots, 4 \) where \( \mathcal{T} \) is the identity. Then, the subalgebras \( \mathcal{A}_k \) generated by \( \{L_{ij}, \mathcal{T}\} \) for \( i, j \neq k \) are exactly those associated with the 2D analogue of this system. For example, the algebra \( \mathcal{A}_4 \) generated by \( \{L_{12}, L_{13}, L_{23}\} \) admits the following operator \( \mathcal{H} = L_{12} + L_{13} + L_{23} + (3/4 - b_1^2 - b_2^2 - b_3^2) \mathcal{T} \), which is the Hamiltonian for the associated system on the 2-sphere and which is in the center of \( \mathcal{A}_4 \). Thus, the representation of \( \mathcal{A}_4 \) will be used as a basis for the representation of \( \mathcal{A} \).

9.2.2. The model. As described above, a representation of \( \mathcal{A} \) can be obtained by extending the representations for the subalgebras \( \mathcal{A}_k \), namely the representation of the system \( S \) in terms of Wilson polynomials. In order to extend this representation, we note that the operator \( \tilde{\mathcal{H}} = L_{12} + L_{13} + L_{23} + (3/4 - b_1^2 - b_2^2 - b_3^2) \mathcal{T} \) is in the center of \( \mathcal{A}_4 \) but not \( \mathcal{A} \) and so it is no longer the energy of the system but instead a variable. Thus, if we begin with a representation of \( \mathcal{A}_4 \) in terms of a variable \( \tau \), we adjoin a new variable \( s \) associated with the operator \( \tilde{\mathcal{H}} \). Thus, the model for the algebra \( \mathcal{A} \) will consist of difference operators in two variables.

The model is given by

\[
\begin{align*}
H &= -\left( 2M + \sum_{j=1}^{4} b_j + 3 \right)^2 + 1, \quad \mathcal{T} = \frac{1}{4} - 4s^2, \\
L_{13} &= -4\tau^* \sigma^2, \quad L_{12} = -4\tau^* \tau_1 - 2(b_1 + 1)(b_2 + 1) + 1/2, \\
L_{24} &= -4\tau^* \tau - 2(b_2 + 1)(b_4 + 1) + \frac{1}{2}, \\
L_{34} &= A(s)S(\sigma_\alpha \sigma_\beta), + B(s)S^{-1}(\sigma_\alpha \sigma_\beta), + C(s)4\tau^* \tau, + D(s),
\end{align*}
\]

where capital letters \( S^k \) and \( T^k \) are the shift operators in \( S \) and \( T \) respectively. The operators \( \tau \) and \( \tau^* \) are correspond to the factorization of the eigenvalue equation for Wilson polynomials (9.2) and (9.3) in the variable indicated by the subscript, with parameters for \( \tau^* \) given by

\[
\begin{align*}
\alpha &= \frac{b_2 + 1}{2} + s, \quad \beta = \frac{b_1 + b_3 + 1}{2}, \quad \gamma = \frac{b_1 - b_1 + 1}{2}, \quad \delta = \frac{b_2 - b_1 + 1}{2} - s \quad (9.25)
\end{align*}
\]

and parameters for \( \tau \) given by

\[
\begin{align*}
\tilde{\alpha} &= \frac{b_2 + 1}{2}, \quad \tilde{\beta} = -M - \frac{b_1 + b_2 - b_3}{2} - 1, \\
\tilde{\gamma} &= M + b_4 + \frac{b_1 + b_2 + b_3}{2} + 2, \quad \tilde{\delta} = -m + \frac{b_3 + 1}{2}. \quad (9.26)
\end{align*}
\]

The coefficient functions in \( L_{34} \) are

\[
\begin{align*}
A(s) &= -\left( 2M + b_1 + b_2 + b_3 - 2s + 2 \right) \left( 2M + b_1 + b_2 + b_3 + 2b_4 + 2s + 4 \right), \\
B(s) &= -\left( 2M + b_1 + b_2 + b_3 + 2s + 2 \right) \left( 2M + b_1 + b_2 + b_3 + 2b_4 - 2s + 4 \right), \\
C(s) &= -2 + \frac{2(2M + b_1 + b_2 + b_3 + 3)(2M + b_1 + b_2 + b_3 + 2b_4 + 3)}{2s(2s + 1)}, \\
D(s) &= 2s^2 - 2 \left( \frac{2M + b_1 + b_2 + b_3 + 2s + 4}{2} \right)^2 - \frac{(b_1 + b_2)^2}{2} + \frac{b_3^2 + b_4^2}{2} + \frac{(b_1 + b_2 + 1)^2 - b_4^2}{2} + \frac{(b_1 + b_2 + 1)(2M + b_1 + b_2 + b_3 + 3)(2M + b_1 + b_2 + b_3 + 2b_4 + 3)}{2(4s^2 - 1)}.
\end{align*}
\]
and the operators
\[ \sigma_{\mu,\nu} = -\frac{1}{2t} \left[ \left( \frac{\mu - 1}{2} + t \right) \left( \nu - \frac{1}{2} + t \right) T^\frac{1}{2} - \left( \frac{\mu - 1}{2} - t \right) \left( \nu - \frac{1}{2} - t \right) T^{-\frac{1}{2}} \right], \quad (9.27) \]
shift the parameters of Wilson polynomials, see [103] for the action of these operators on the basis.

There are several bases forming finite-dimensional irreducible representations for this model, namely
\[ d_{\ell,m}(s,t) = \delta(t - t_\ell)\delta(s - s_m), \quad 0 \leq \ell \leq m \leq M, \quad (9.28) \]
\[ f_{n,m}(s,t) = w_n(t^2, \alpha, \beta, \gamma, \delta) \delta(s - s_m), \quad 0 \leq n \leq m \leq M, \quad (9.29) \]
\[ g_{\ell,k}(s,t) = w_k(s^2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) \delta(t - t_\ell), \quad 0 \leq \ell \leq k + \ell \leq M, \quad (9.30) \]
the latter two being given in terms of Wilson polynomials in one-variable. Another basis, for which the operators \( L_{12} \) and \( L_{14} + L_{24} \) are diagonal is given in terms of two-variable Wilson polynomials constructed by Tratnik [57, 183].

The relation between this system and this two-variable generalization of the Wilson polynomials leads directly to at least two open question. First, whether there is an analogous ‘Askey-tableau’ for two variable orthogonal polynomials. Second, though related, whether the \( n \)-variable version of these polynomials gives representations for the associated system on the \( n \)-sphere and to consider their contractions.

10. Conclusions

In this review we have concentrated on certain aspects of superintegrability. First of all, we have restricted ourselves to finite-dimensional nonrelativistic classical and quantum Hamiltonian systems. The Hamiltonian always has the form \( H = T + V \) where \( T \) is the kinetic energy and \( V \) a scalar potential. Physically this corresponds to a scalar particle in a scalar potential field, or the interaction between two scalar particles.

We first consider the case of quadratic integrability where all integrals of motion are at most second-order polynomials in the momenta (sections 3 and 4). In 2D, the classification of such systems is complete and the results are presented. This is combined with a review of the structure of the quadratic algebras of their integrals of motion. This classification is performed for 2D Riemannian, pseudo-Riemannian or complex Riemannian spaces of constant or non-constant curvature that allow at least two Killing tensors (Darboux spaces) [111]. An important role in the study of superintegrability is played by the Stäckel transform which relates physically different systems, often even in different spaces, to each other.

Sections 5–7 are devoted to higher-order superintegrability, mainly in 2D and 3D real Euclidean spaces. In section 5, we present the determining equations for the existence of integrals of finite order \( N \geq 2 \) in the Euclidean space \( E_2 \). We review the case of \( N = 2 \) and verify the results of the previous section that the classical and quantum integrable (and superintegrable) potentials coincide and the integrals are the same, up to symmetrization. Starting from \( N = 3 \), the determining equations acquire non-vanishing quantum corrections and hence the classical and quantum potentials no longer necessarily coincide. The general solution to the determining equations is not known for \( N > 2 \). However, we present all third-order superintegrable systems that also allow a first-order integral or a second-order one leading to separation of variables in Cartesian, polar or parabolic coordinates. (The case of separation in elliptic coordinates has not yet been studied.) We also present the determining equations for fourth-order integrals and a new fourth-order superintegrable potential which does not admit separation of variables.
For higher-order integrals of motion, the determining equations quickly become intractable. However, a breakthrough in this area was in the discovery of a family (since named the TTW system) of exactly-solvable systems which was conjectured to be superintegrable with integrals of arbitrarily high order. The proof of this conjecture has led to novel approaches to construction and analysis of superintegrable systems. In section 6, the system is introduced and a method for constructing additional integrals of motion for classical Hamiltonians admitting separation of variables is introduced. This method is then applied to prove the superintegrability of the TTW system as well as to construct an infinite family of superintegrable systems containing the extended Kepler–Coulomb systems in 3D. Section 7 presents a method for proving the superintegrability of systems with higher-order integrals via recurrence relations for the separable solutions. This method makes use of the conjectured exact solvability of superintegrable systems to construct differential operators which fix energy eigenstates of the Hamiltonian. This method is used to prove the superintegrability of the TTW system and an extension of the 3D Kepler–Coulomb system and to construct their symmetry algebras.

Section 8 reviews the Stäckel transform introduced in section 4 and its relation to coupling constant metamorphosis (CCM). While CCM is well defined for Hamiltonians in classical mechanics, for quantum systems there are additional requirements on the form of the integral in order for it to be preserved as a symmetry under the transformation, namely it be a polynomial in the coupling constant. Section 9 is devoted to a more mathematical aspect of superintegrability, namely the relation between the representation theory of quadratic algebras and special function theory. Just as Lie algebra relations were exploited to determine energy levels of the hydrogen atom, so can the representation of the polynomial algebras give energy levels and expansion coefficients from one separable basis to another. These representations give a direct connection between second-order superintegrability systems in 2D and the Askey scheme of orthogonal polynomials. Contractions of the algebras correspond to the limits between the families of classical orthogonal polynomials.

Some aspects of superintegrability theory are not included in this review because of (self-imposed) restrictions on space and time. They include the study of systems not of the form $H = \Delta + V(\vec{x})$, for example velocity dependent potentials

$$H = (\vec{\rho})^2 + V(\vec{r}) + (\vec{A}(\vec{r}), \vec{p}) + (\vec{p}, \vec{A}(\vec{r})),$$

(10.1)

where $\vec{A}(\vec{r})$ is a vector potential. The Hamiltonian (10.1) describes the motion of a spin zero particle in a magnetic field, for instance. Integrable and superintegrable systems of this type in $E_2$ have been studied systematically with a restriction to first- and second-order integrals of the motion [16, 18, 33, 40, 44, 131, 159].

Another class of Hamiltonians that have been investigated from the point of view of integrability and superintegrability is of the form

$$H = (\vec{\rho})^2 + V_0(\vec{r}) + V_1(\vec{r})(\vec{\sigma} \vec{L})$$

(10.2)

and describes the interaction of a spin 1/2 particle with a spin 0 one (e.g. pion–nucleon interaction) [39, 194–196]. For further relevant articles on superintegrability for particles with spin see e.g. [141–144, 157, 158]. Finally, we mention a few other superintegrable systems where the Hamiltonian is of a different form than those studied here including those related to Dunkl oscillators and quantum spin lattices [54, 56, 132, 133, 152] as well as those associate with super-symmetric systems [23, 153, 162].

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