Cosmology from the two-dimensional renormalization group acting as the Ricci flow

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The two-dimensional renormalization group acting as the Ricci flow $\Lambda \frac{\partial}{\partial \Lambda} g_{\mu \nu} = R_{\mu \nu}$ produces a specific 1+3 dimensional space-time metric which describes an expanding universe that starts with a big bang $a \sim t^{1/\sqrt{3}}$ then decelerates until $z = 0.2$ then accelerates until ending at $t_{\text{max}} = 1.6 t_H$ with a big blowup $a \sim (t_{\text{max}} - t)^{-1/\sqrt{3}}$. The only free parameters are the overall time scale and the value of the present time $t_0$. These are fixed by the Hubble constant $H_0 = t_{\text{H}}^{-1}$ and the present deceleration parameter $q(t_0)$. This crude calculation of cosmology omits all but the gravitational field. The only energy-momentum is purely gravitational dark matter and energy. This is a preliminary exploration towards a specific, comprehensive, testable calculation of cosmology from a fundamental theory in which physics is produced by a quantum version of the two-dimensional renormalization group.

The renormalization group of the two-dimensional general nonlinear model acts at leading order as the Ricci flow [1–3]

$$\Lambda \frac{\partial}{\partial \Lambda} g_{\mu \nu} = R_{\mu \nu}$$

The field of the nonlinear model takes its values in a Riemannian manifold $M$. The Riemannian metric $g_{\mu \nu}$ encodes the couplings of the model. The beta-function at leading order is the Ricci tensor $R_{\mu \nu}$. The renormalization group drives $g_{\mu \nu}$ to a fixed point (modulo at most a finite number of relevant parameters). The fixed points are the solutions of

$$R_{\mu \nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$$

where $v^\mu(x)\partial_\mu$ is a vector field on $M$. A vector field is an infinitesimal reparametrization $\dot{x}^\mu = v^\mu(x)$ of $M$. This is a field redefinition in the nonlinear model equivalent to the redundant perturbation $\dot{g}_{\mu \nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. Thus the fixed point equation (2) expresses physical 2-d scale invariance.

The 2-d quantum field theory is manifestly well defined when $g_{\mu \nu}$ has euclidean signature. Assume that $M$ is a four-dimensional Riemannian manifold of the form $I \times S^3$ where $I$ is a real interval and $S^3$ is the unit 3-sphere. Assume SO(4) invariance. Parametrize $I \times S^3$ as a spherical shell in $\mathbb{R}^4$. The flat euclidean metric is $\delta_{\mu \nu} dx^\mu dx^\nu = dr^2 + r^2 ds^2_{S^3}$ where $ds^2_{S^3}$ is the round metric on the unit 3-sphere. The general SO(4)-invariant metric has the form

$$ds_4^2 = \delta_{\mu \nu} dx^\mu dx^\nu + \delta_{\mu \nu} \frac{ds^2_{S^3}}{r^2}$$

The general nonlinear model (also called the nonlinear sigma model) was constructed in [1–3] in 2+\(\epsilon\) dimensions with fixed point equation $R_{\mu \nu} - \epsilon g_{\mu \nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. The solutions were called quasi-Einstein metrics. The Ricci flow was introduced in Mathematics in [4]. The solutions of $R_{\mu \nu} - \epsilon g_{\mu \nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ have been called Ricci solitons or Ricci flow solitons in the Mathematics literature.
The metric can be made conformally flat

\[ g_{\mu\nu} dx^\mu dx^\nu = F_1(r)^2 dr^2 + F_2(r)^2 ds^2_{S_3} \]

by reparametrizing \( r \to \tau(r) \) with \( dr/d\tau = F_2/F_1 \). The general SO(4)-invariant vector field is \( \nu^\tau(\tau)\partial_\tau \). Then [5]

\[
R_{\mu\nu} dx^\mu dx^\nu = (-3\partial_\tau f_\tau) d\tau^2 + (-\partial_\tau f_\tau + 2 - 2f_\tau^2) ds^2_{S_3}
\]

\[
(\nabla_\mu \nu_\nu + \nabla_\nu \nu_\mu) dx^\mu dx^\nu = (2\partial_\tau \nu_\nu - 2\nu_\tau f_\tau) d\tau^2 + (2\nu_\tau f_\tau) ds^2_{S_3}
\]  

(4)

\[ f_\tau = \partial_\tau f \quad \nu_\tau = g_{\tau\tau} \nu^\tau = a^2 \nu^\tau \]

The fixed point equation (2) becomes the ordinary differential equation

\[ \partial_\tau f_\tau = -2f_\tau \nu_\tau - 2f_\tau^2 + 2 \quad \partial_\tau \nu_\tau = 4f_\tau \nu_\tau + 3f_\tau^2 - 3 \]

(5)

We shall see below that there is an essentially unique solution of (5) which analytically continues in \( \tau \) to real time \( T = i^{-1} \tau \)

\[ ds^2 = a^2 (-dT^2 + ds^2_{S_3}) \]

(6)

The real-time ode is

\[ \partial_T f_T = -2f_T \nu_T - 2f_T^2 - 2 \quad \partial_T \nu_T = 4f_T \nu_T + 3f_T^2 + 3 \]

\[ \partial_T = i\partial_\tau \quad f_T = if_\tau = \partial_\tau a/a \quad \nu_T = i\nu_\tau \]

(7)

The solution is

\[ f_T = \frac{\cos 2T + \sqrt{3}}{\sin 2T} \quad \nu_T = \frac{-\sqrt{3}}{\sin 2T} \quad a = t'_0 \sin^{1 + \nu} T \cos^{-\nu} T \quad T \in (0, \pi/2) \]

(8)

with \( \nu = \sqrt{3}/2 - 1/2 = 0.3660 \ldots \). The only free parameter is the overall time scale \( t'_0 \). In co-moving time \( t \)

\[ ds^2 = -dt^2 + a^2 ds^2_{S_3} \quad dt = adT \]

\[ \frac{t}{t'_0} = \int_0^T \sin^{1 + \nu} T' \cos^{-\nu} T' dT' = \frac{1}{2} B_{\sin^2 T} \left( \frac{2 + \nu}{2}, \frac{1 - \nu}{2} \right) \]

(9)

\[ t \in (0, t_{\text{max}}) \quad \frac{t_{\text{max}}}{t'_0} = \frac{1}{2} B \left( \frac{2 + \nu}{2}, \frac{1 - \nu}{2} \right) = 1.470 \ldots \]

\( B(p, q) \) is the Euler beta function, \( B_2(p, q) \) the incomplete beta function. At the limits

\[ T \to 0 \quad \frac{t}{t'_0} \to \frac{T^{2 + \nu}}{2 + \nu} \quad a \to t'_0 (2 + \nu)^{1/\sqrt{3}} \left( \frac{t}{t'_0} \right)^{1/\sqrt{3}} \]

(10)

\[ T \to \frac{\pi}{2} \quad \frac{t_{\text{max}} - t}{t'_0} \to \frac{(\frac{\pi}{2} - T)^{1 - \nu}}{1 - \nu} \quad a \to t'_0 (1 - \nu)^{-1/\sqrt{3}} \left( \frac{t_{\text{max}} - t}{t'_0} \right)^{-1/\sqrt{3}} \]
This is an expanding universe which begins with a big bang \( a \sim t_{\text{max}}^{-0.577}\ldots \) and ends with a big blowup \( a \sim (t_{\text{max}} - t)^{-0.577}\ldots \). The Hubble parameter \( H = \frac{\partial t a / a = f_T / a}{} \) is

\[
H = \frac{1}{t_0} (\cos^2 T + \nu) \sin^{-2-\nu} T \cos^{-1+\nu} T
\]

The deceleration parameter \( q = -a \partial^2 a / (\partial_t a)^2 = -\partial_T f_T / f_T^2 \) is

\[
q = \frac{2 \left( \sqrt{3} \cos 2T + 1 \right)}{(\cos 2T + \sqrt{3})^2} \quad T_{q=0} = \frac{1}{2} \arccos(-1/\sqrt{3}) = (0.6959\ldots)\frac{\pi}{2}
\]

The expansion decelerates \((q > 0)\) until \( T = T_{q=0} \) then accelerates \((q < 0)\) until the end.

For a first estimate of the time scale \( t_0' \) and the present time \( t_0 \) use the deceleration parameter at the present time \( q_0 = q(T_0) \approx -0.6 \) in equation (12) to obtain \( T_0 = 0.77 \pi / 2 \). Then equate the Hubble constant \( H_0 \) to \( H(T_0) \) in (11) to obtain \( t_0' = 1.1 t_H \) where \( t_H = 1 / H_0 \approx 1.4 \times 10^{10} y \). Then (9) gives \( t_{\text{max}} = 1.6 t_H, t_0 = 0.73 t_H \) and (8) gives \( a_0 = a(t_0) = 1.5 t_H \).

Einstein’s equation

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

is satisfied with energy-momentum tensor

\[
T_{\mu\nu} = \frac{1}{8\pi G} (\nabla_\mu v_\nu + \nabla_\nu v_\mu - \nabla_\sigma v^\sigma g_{\mu\nu})
\]

which is that of a perfect fluid of density \( \rho(t) \) and pressure \( p(t) \)

\[
T_{\mu\nu} dx^\mu dx^\nu = \rho(t) dt^2 + p(t) a^2 ds_3^2
\]

\[
8\pi G \rho(t) = a^{-2} (3 f_T^2 + 3) \quad 8\pi G p(t) = a^{-2} (4 f_T v_T + 3 f_T^2 + 3)
\]

The energy-momentum is purely gravitational, tautologically dark. The equation-of-state parameter \( w(t) = p / \rho \) and the density parameter \( \Omega(t) = 8\pi G \rho / 3 H^2 = 1 + 1 / f_T^2 \) are

\[
w = \frac{\cos 2T}{3 \cos 2T + 2 \sqrt{3}} \quad \Omega = 1 + \left( \frac{\sin 2T}{\cos 2T + \sqrt{3}} \right)^2
\]

The estimate \( T_0 = 0.77 \pi / 2 \) gives present values \( w_0 = -0.6, \Omega_0 = 1.5 \). The cosmological parameters are graphed as functions of \( t \) in Figure 1. Past values for a selection of redshifts \( z = a_0 / a - 1 \) are listed in Table I.

We analyze the real-time ode (7) by adapting some of the methods that were used in [7] to analyze the ode analogous to (5) for the euclidean-signature SO(3)-invariant fixed point equation in three dimensions. The real-time ode (7) has time-reflection symmetry \((T, f_T, v_T) \rightarrow (-T, -f_T, -v_T)\) and constant of motion

\[
C = \frac{1}{a^2} \left( \frac{2}{3} v_T^2 + 2 f_T v_T + f_T^2 + 1 \right) \quad \partial_T C = 0
\]

Changing variables to \( h_+ = f_T + (1 \pm 1 / \sqrt{3}) v_T \) the constant of motion and ode become

\[
h_+ h_- + 1 = C a^2 \quad \partial_T h_\pm = -(1 + h_\pm^2) + \lambda_\pm (h_+ h_- + 1) \quad \lambda_\pm = 2 \pm \sqrt{3}
\]
FIG. 1. The cosmological parameters as functions of the co-moving time $t$. The present is $t_0$.

| $z$ | $H/H_0$ | $q$ | $w$ | $\Omega$ |
|-----|---------|-----|-----|---------|
| $z \gg 1$ | $1000$ | $1.2 \times 10^5$ | 0.73 | 0.16 | 1.0 |
| $z \gg 1$ | $100$ | $2.2 \times 10^3$ | 0.73 | 0.15 | 1.0 |
| $z \gg 1$ | $10$ | 48 | 0.74 | 0.15 | 1.0 |
| $T = \frac{1}{2} T_0$ | $1.7$ | 4.1 | 0.74 | 0.08 | 1.2 |
| $T = \frac{1}{2} T_0$ | $1.0$ | 2.4 | 0.69 | 0.02 | 1.3 |
| $q = 0$ | $0.2$ | 1.1 | 0 | $-0.3$ | 1.5 |
| $t = t_0$ | $0$ | 1 | $-0.6$ | $-0.6$ | 1.5 |

The cosmological solution (8) is $C = 0$, $h_- = \cot T$, $h_+ = -\tan T$.

The phase portrait of the ode is shown in Figure 2. Every $C \neq 0$ trajectory asymptotes to a $C = 0$ trajectory. The separatrix $S$ and its time-reflection $S'$ each does so at one end; all the other $C \neq 0$ trajectories do so at both ends. The asymptotic behavior is one of

$$T \to 0^\pm 
\begin{align*}
h_- &\to \frac{1}{T} \\
h_+ &\to -T - \frac{1}{3} T^3 + c T (T^2)^{1+\nu} \\
&\quad \frac{c}{C} > 0 \\
h_+ &\to \frac{1}{T} \\
h_- &\to c' T (T^2)^{-\nu} \\
&\quad \frac{c'}{C} > 0
\end{align*}
(19)

The presence of an irrational power of $T$, namely $(T^2)^{1+\nu}$ or $(T^2)^{-\nu}$, implies that such a solution cannot be the analytic continuation of a solution in imaginary time $\tau = iT$. 

FIG. 2. Phase portrait of the real-time ode (7,18) in terms of \((f_T, v_T)\) on the left, \((h_-, h_+)\) on the right. The trajectories are the solutions. The arrows point in the direction of increasing \(T\). The interval between points in the trajectories is \(\Delta T = 0.01\). The universe is expanding when \(f_T = \partial_T a\) is positive. The expansion is accelerating when \(f_T\) is increasing. The two red trajectories are the \(C = 0\) solutions. The one in the lower right quadrant labeled \(C=0\) is the cosmological solution (8). Its time reflection is labeled \(C=0'\). The separatrix is labeled \(S\). Its time-reflection is labeled \(S'\).

The trajectories that start from \(h_- = \infty, h_+ = 0\) at \(T = 0\) are characterized by the numerical invariant

\[
c = (h_+ h_- + 1)(h_2)^{1+\nu} e^{\int_0^T \frac{2 a_T}{h_2(1/h_2)}} = \lim_{T \to 0} (h_+ h_- + 1)(h_2)^{1+\nu}
\]  

(20)
The separatrix \(S\) has \(c_S = 0.284 \ldots\). The \(c < c_S\) trajectories — the trajectories below \(S\) — are all roughly realistic as cosmologies. They all start with a big bang \(a \sim t^{1/\sqrt{3}}\) then decelerate to a big blow-up \(a \sim t^{-1/\sqrt{3}}\), all driven by purely gravitational dark energy-momentum. The gap between the \(C=0\) cosmological solution and the separatrix \(S\) means that the cosmological solution (8) is stable against real-time perturbations. The zone of stability on the side \(C > 0\) shrinks to zero as \(T \to 0\).

The two-dimensional renormalization group was developed into a comprehensive fundamental theory of physics in which the 2-d renormalization group acts not on a space of classical space-time fields but rather on measures on that space, producing not a solution of the classical field equations but rather a measure — a functional integral — expressing a solution of the quantum field equations [8–10]. The present exploratory calculations of cosmology from the 2-d renormalization group will need to be connected to this fundamental theory. Assumptions such as \(SO(4)\) space-time symmetry will need to be justified by properties of the 2-d renormalization group flow — by stability properties of the fixed point and/or as the dynamical result of a distinguished trajectory of the rg flow.

The analyticity assumption needs precise characterization and justification. The assumption here that \(f_T(T)\) and \(v_T(T)\) should be analytic in \(T\) singles out the \(C = 0\) cosmological solution. Analyticity serves as a selection principle providing specificity. It should express a physical principle.
The next exploratory steps will add standard model fields to the calculation, then spatial fluctuations and quantum effects, hoping to find analytic solutions of 2-d renormalization group fixed point equations that capture more and more qualitative features of the cosmology of the real world, aiming to find a solution that can be tested in quantitative detail. This would provide support for the fundamental theory [8] in which a quantum version of the two-dimensional renormalization group acts mechanically to produce physics and would provide guidance for deriving real world cosmology from that fundamental theory.

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[5] Calculations are shown in the accompanying supplemental material which consists of a note Calculations for "Cosmology ... " and three SageMath [6] notebooks of numerical calculations. The supplemental material can be found at http://www.physics.rutgers.edu/pages/friedan/papers/2019/Cosmology_I_supplementary_material/ or at https://share.cocalc.com/share/6a7035ba-9879-4b05-b3d0-688b1309a21c/Cosmology_I_supplementary_material/?viewer=share/.
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