On Non-Autonomous Evolutionary Problems.

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MATH-AN-01-2013
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The paper extends well-posedness results of a previously explored class of
time-shift invariant evolutionary problems to the case of non-autonomous me-
dia. The Hilbert space setting developed for the time-shift invariant case can be
utilized to obtain an elementary approach to non-autonomous equations. The
results cover a large class of evolutionary equations, where well-known strate-
gies like evolution families may be difficult to use or fail to work. We exemplify
the approach with an application to a Kelvin-Voigt-type model for visco-elastic
solids.

Keywords and phrases: non-autonomous, evolutionary problems, extrapolation spaces, Sobolev lattices
Mathematics subject classification 2010: 35F05, 35F10, 35F15, 37L05, 74B99
0 Introduction

Contents

0 Introduction 4
1 Preliminaries 5
2 Space-time evolutionary equations 8
3 An application to a Kelvin-Voigt-type model in visco-elasticity 26

0 Introduction

In a number of studies it has been demonstrated that systems of the form

\[(\partial_0 \mathcal{M} + A) u = F,\] (0.1)

where \(\mathcal{M}\) is a continuous, linear mapping and the densely defined, closed linear operator \(A\) is such that \(A\) and \(A^*\) are maximal \(\omega\)-accretive for some suitable \(\omega \in \mathbb{R}\), cover numerous models from mathematical physics. Indeed, \(A\) skew-selfadjoint\(^1\) is a standard situation, which for simplicity we shall assume throughout. The well-posedness of (0.1) hinges on a positive definiteness assumption imposed on \(\partial_0 \mathcal{M}\) in a suitable space-time Hilbert space setting. Under this assumption the solution theory is comparatively elementary since \((\partial_0 \mathcal{M} + A)\) together with its adjoint are positive definite yielding that \((\partial_0 \mathcal{M} + A)\) has dense range and a continuous inverse.

In applications of this setting the operator \(A\) has a rather simple structure whereas the complexity of the physical system is encoded in the “material law” operator \(\mathcal{M}\). A simple but important case is given by

\[\mathcal{M} = M_0 + \partial_0^{-1} M_1,\]

where \(M_0, M_1\) are time-independent continuous linear operators. Here we have anticipated that in the Hilbert space setting to be constructed \(\partial_0^{-1}\) (time integration) has a

\(^1\)Two densely defined linear operators \(A, B\) are skew-adjoint (to each other) if \(A = -B^*\). If \(A = B\), we call \(A\) skew-selfadjoint (rather than self-skew-adjoint). The proper implications

\[\text{A selfadjoint} \implies \text{A symmetric} \implies \text{A Hermitean}\]

are paralleled by

\[\text{A skew-selfadjoint} \implies \text{A skew-symmetric} \implies \text{A skew-Hermitean}.\]

Frequently, “skew-adjoint” is used to mean “skew-selfadjoint”. We shall, however, not follow this custom for the obvious reason.
proper meaning. The positive definiteness assumption requires $M_0$ to be non-negative and selfadjoint and
\[ \varrho M_0 + \Re M_1 \geq c_0 \] (0.2)
for some $c_0 \in ]0, \infty[ \text{ and all sufficiently large } \varrho \in ]0, \infty[ $. Since we do not assume that $M_0$ is always strictly positive, (0.2) may imply constraints on $M_1$. If $M_0$ is positive definite it may seem natural, following the proven idea of first finding a fundamental solution (given by an associated semi-group), and then to obtain general solutions as convolutions with the data (Duhamel’s principle, variation of constants formula) and so proving well-posedness.

This is the classical method of choice in a Banach space setting, see e.g. [1, 4, 5, 8, 17] as general references. In comparison our approach is (currently) limited to a Hilbert space setting, however, apart from being conceptually more elementary, it allows to incorporate delay and convolution integral terms by a simple perturbation argument and, if $M_0$ has a non-trivial kernel, the system becomes a differential-algebraic systems, which to the above approach makes no difference, but cannot be conveniently analyzed within the framework of semi-group theory.

The purpose of this paper is to extend well-posedness results previously obtained for time-shift invariant material operators $M$ to cases, where $M$ is not time-shift invariant. This is the so-called time-dependent or non-autonomous case. The above-mentioned limitations of the semi-group approach carry over to the application of classical strategies based on evolution families introduced by Kato ([7]), for a survey see e.g. [16, 8], which are the corresponding abstract Green’s functions, in the non-autonomous case. The approach we shall develop here, by-passes the relative sophistication of the classical approach based on evolution families and extends, moreover, to differential-algebraic cases and allows to include memory effects, in a simple unified setting.

To keep the presentation self-contained we construct the Hilbert space setting in sufficient detail and formulate our results so that the autonomous case re-appears as a special case of the general non-autonomous situation.

In order to formulate the problem class rigorously and to avoid at the same time to incur unnecessary regularity constraints on data and domain for prospective applications, it is helpful to introduce suitable extrapolation Hilbert spaces (Sobolev lattices). This will be done in the next section. In Section 2 we shall describe a class of non-autonomous evolutionary equations and its solution theory. The paper concludes with the discussion of a particular application to a class of Kelvin-Voigt-type models for visco-elastic solids to exemplify the theoretical findings.

In the following let $H$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and induced norm $| \cdot |$.

1 Preliminaries

In this subsection we recall the construction of a short Sobolev chain associated with a normal, boundedly invertible operator $N$ on some Hilbert space $H$ as it was presented
1 Preliminaries

for instance in [11, Section 2.1]. We note that the construction can be generalized to the case of a closed, boundedly invertible operator on some Hilbert space. However, since the operators we are interested in are all normal, we may reduce ourselves to the slightly easier case of normal operators.

**Definition.** Let $H$ be a Hilbert space and $N: D(N) \subseteq H \to H$ be a normal operator with $0 \in \sigma(N)$. Then the sesqui-linear form

$\langle \cdot | \cdot \rangle_{H_1(N)}: D(N) \times D(N) \to \mathbb{C}$

$(f, g) \mapsto \langle Nf | Ng \rangle$,

defines an inner product on $D(N)$ and due to the closedness of $N$, $D(N)$ equipped with this inner product becomes a Hilbert space. We denote it by $H_1(N)$. Moreover, by

$\langle \cdot | \cdot \rangle_{H_{-1}(N)}: H \times H \to \mathbb{C}$

$(f, g) \mapsto \langle N^{-1}f | N^{-1}g \rangle$

we define an inner product on $H$ and we denote the completion of $H$ with respect to the induced norm by $H_{-1}(N)$. The triple $(H_1(N), H_0(N), H_{-1}(N))$ is called short Sobolev chain associated with $N$, where $H_0(N) := H$.

**Remark 1.1.** Note that the above construction can be done analogously for the $k$-th power of $N$ for $k \in \mathbb{N}$. The resulting sequence of Hilbert spaces $(H_k(N))_{k \in \mathbb{Z}}$, where we set $H_k(N) := H_1(N^k)$ and $H_{-k}(N) := H_{-1}(N^k)$ for $k \in \mathbb{N} \setminus \{0\}$ is called the Sobolev chain associated with $N$.

A simple estimate shows that $H_1(N)$ is densely and continuously embedded into $H_0(N)$, which itself is densely and continuously embedded into $H_{-1}(N)$. Thus we arrive at a chain of dense and continuous embeddings of the form

$H_1(N) \hookrightarrow H_0(N) \hookrightarrow H_{-1}(N)$.

We can establish the operator $N$ as a unitary operator acting on this chain.

**Proposition 1.2 ([11, Theorem 2.1.6]).** The operator $N: H_1(N) \to H_0(N)$ is unitary. Moreover, the operator $N: D(N) \subseteq H_0(N) \to H_{-1}(N)$ has a unitary extension to $H_0(N)$.

Since $N$ is boundedly invertible its adjoint $N^*$ has a bounded inverse, too. Hence we can do the same construction of a Sobolev chain associated with $N^*$. However, since $N$ is assumed to be normal we have that $D(N) = D(N^*)$, as well as $|Nx| = |N^*x|$ for each $x \in D(N)$. This yields that the Sobolev chains for $N$ and $N^*$ coincide. Hence, by Proposition 1.2 we obtain the following result.

---

We will not distinguish notationally between the operator $N$ and its unitary realizations on the Sobolev chain associated with $N$.
Corollary 1.3. The operator $N^* : D(N^*) \subseteq H_k(N) \to H_{k-1}(N)$ has a unitary extension to $H_k(N)$ for $k \in \{0, 1\}$.

Again this realization will be denoted by $N^*$ although this might cause confusion, since $N^*$ could be interpreted as the adjoint of the unitary realization of the operator $N$ as in Proposition 1.2. The adjoint would then be the respective inverse.

Note that for normal $N \in L(H, H) =: L^2(H)$, the space of continuous linear mappings from $H$ to $H$, we have that

$$H_k(N) = H$$

as topological linear spaces with merely different inner products (inducing equivalent norms) in the different Hilbert spaces $H_k(N)$, $k \in \{-1, 0, 1\}$. This indicates that considering continuous linear operators $N$ does not lead to interesting chains.

In the remaining part of this section we consider a particular example of a normal operator and its associated Sobolev chain, namely the time derivative in an exponentially weighted $L^2$-space (see [11, 12, 6] for more details).

For $\varrho \in \mathbb{R}$ we consider the Hilbert space

$$H_{\varrho,0}(\mathbb{R}) := \{ f \in L^2_{\text{s.o.}}(\mathbb{R}) \mid (x \mapsto \exp(-\varrho x)f(x)) \in L^2(\mathbb{R}) \}$$

equipped with the inner product

$$\langle f | g \rangle_{H_{\varrho,0}(\mathbb{R})} := \int_{\mathbb{R}} f(x)^* g(x) \exp(-2\varrho x) \, dx \quad (f, g \in H_{\varrho,0}(\mathbb{R})).$$

We define the operator $\partial_{0,\varrho}$ on $H_{\varrho,0}(\mathbb{R})$ as the closure of

$$\partial_{0,\varrho}|_{\hat{C}_c(\mathbb{R})} : \hat{C}_c(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) \to H_{\varrho,0}(\mathbb{R})$$

$$\phi \mapsto \phi',$$

where by $\hat{C}_c(\mathbb{R})$ we denote the space of arbitrarily often differentiable functions on $\mathbb{R}$ with compact support. In this way we obtain a normal operator with $\Re \partial_{0,\varrho} = \varrho$. Hence, for $\varrho \neq 0$ the operator $\partial_{0,\varrho}$ is boundedly invertible and one can show that $\|\partial_{0,\varrho}^{-1}\|_{L(H_{\varrho,0}(\mathbb{R}))} = 1/|\varrho|$. Thus for $\varrho \neq 0$ we can construct the Sobolev chain associated with $\partial_{0,\varrho}$ and we introduce the notation $H_{\varrho,k}(\mathbb{R}) := H_k(\partial_{0,\varrho})$ for $\varrho \neq 0$ and $k \in \{-1, 0, 1\}$. For $\Im \partial_{0,\varrho}$ we have as a spectral representation the Fourier-Laplace transform $\mathcal{L}_{\varrho} : H_{\varrho,0}(\mathbb{R}) \to L^2(\mathbb{R})$ given by the unitary extension of

$$\hat{C}_c(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) \to L^2(\mathbb{R})$$

$$\phi \mapsto \left( x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-ixy) \exp(-\varrho y) \phi(y) \, dy \right).$$

3The domain of $\partial_{0,\varrho}$ consists precisely of the functions $f \in H_{\varrho,0}(\mathbb{R})$ with distributional derivative lying in $H_{\varrho,0}(\mathbb{R})$. 

7
2 Space-time evolutionary equations

In other words, we have the unitary equivalence
\[ \text{Im} \partial_{0,e} = L^*_{e} m \mathcal{L}_{e}, \]
where \( m \) denotes the multiplication-by-argument operator in \( L^2(\mathbb{R}) \) with maximal domain, i.e. \((mf)(x) := xf(x)\) for a.e. \( x \in \mathbb{R}, f \in D(m) := \{ g \in L^2(\mathbb{R}) \mid (x \mapsto xg(x)) \in L^2(\mathbb{R}) \}\). The latter yields
\[ \partial_{0,e} = L^*_{e} (im + \varrho) \mathcal{L}_{e}. \]
For \( \varrho \neq 0 \) we can represent the resolvent \( \partial_{0,e} \) as an integral operator given by
\[ (\partial_{0,e}^{-1} u)(x) = \int_{-\infty}^{x} u(t) \, dt \quad (x \in \mathbb{R}, \text{a.e.}) \]
if \( \varrho > 0 \) and
\[ (\partial_{0,e}^{-1} u)(x) = -\int_{x}^{\infty} u(t) \, dt \quad (x \in \mathbb{R}, \text{a.e.}) \]
if \( \varrho < 0 \) for all \( u \in H_{\varrho,0}(\mathbb{R}) \). Since we are interested in the (forward) causal situation (see Definition 2.14 below), we assume \( \varrho > 0 \) throughout. Moreover, in the following we shall mostly write \( \partial_{0} \) for \( \partial_{0,e} \) if the choice of \( \varrho \) is clear from the context.

Let now \( N \) denote a normal operator in a Hilbert space \( H \) with 0 in its resolvent set. Then \( N \) has a canonical extension to the time-dependent case, i.e., to \( H_{\varrho,0}(\mathbb{R}; H) \cong H_{\varrho,0}(\mathbb{R}) \otimes H \), the space of \( H \)-valued functions on \( \mathbb{R} \), which are square-integrable with respect to the exponentially weighted Lebesgue-measure. Analogously we can extend \( \partial_{0} \) to an operator on \( H_{\varrho,0}(\mathbb{R}; H) \) in the canonical way. Then \( \partial_{0} \) and \( N \) become commuting normal operators and by combining the two chains we obtain a Sobolev lattice in the sense of [11, Sections 2.2 and 2.3] based on \( (\partial_{0}, N) \) yielding a family of Hilbert spaces
\[ (H_{\varrho,k}(\mathbb{R}; H_{s}(N)))_{k,s \in \{-1,0,1\}} \]
with norms \( | \cdot |_{\varrho,k,s} \) given by
\[ v \mapsto |\partial_{0}^{k} N^{s} v|_{H_{\varrho,0}(\mathbb{R}; H)} \]
for \( k, s \in \{-1,0,1\} \). The operators \( \partial_{0} \) and \( N \) can then be established as unitary mappings from \( H_{\varrho,k}(\mathbb{R}; H_{s}(N)) \) to \( H_{\varrho,k-1}(\mathbb{R}; H_{s}(N)) \) for \( k \in \{0,1\}, s \in \{-1,0,1\} \) and from \( H_{\varrho,k}(\mathbb{R}; H_{s}(N)) \) to \( H_{\varrho,k}(\mathbb{R}; H_{s-1}(N)) \) for \( k \in \{-1,0,1\}, s \in \{0,1\} \), respectively.

2 Space-time evolutionary equations

Well-posedness for a class of evolutionary problems

We are now ready to rigorously approach the well-posedness class we wish to present. We shall consider equations of the form
\[ (\partial_{0} M(m_{0}, \partial_{0}^{-1}) + A) u = F, \]
where for simplicity we assume that $A$ is skew-selfadjoint in a Hilbert space $H$ and $M(t, \cdot)$ is a material law function in the sense of [10] for almost every $t \in \mathbb{R}$. More specifically we assume that $M$ is of the form

$$M(m_0, \partial_0^{-1}) = M_0(m_0) + \partial_0^{-1}M_1(m_0),$$

where $M_0, M_1 \in L_s^\infty(\mathbb{R}; L(H))$, the space of strongly measurable uniformly bounded functions with values in $L(H)$. We understand $\partial_0 M_0(m_0) + M_1(m_0) + A$ as an (unbounded) operator in $H_{\psi,0}(\mathbb{R}; H)$ with maximal domain

$$D := \{u \in H_{\psi,0}(\mathbb{R}; H) \mid \partial_0 M_0(m_0)u + Au \in H_{\psi,0}(\mathbb{R}; H)\}.$$ 

Note that for $u \in H_{\psi,0}(\mathbb{R}; H)$ we have

$$\partial_0 M_0(m_0)u + M_1(m_0)u + Au \in H_{\psi,-1}(\mathbb{R}; H_{-1}(A+1))$$

and for this to be in $H_{\psi,0}(\mathbb{R}; H)$ is the constraint determining the maximal domain $D$.

**Hypotheses.** We say that $T \in L_s^\infty(\mathbb{R}; L(H))$ satisfies the property

(a) if $T(t)$ is selfadjoint ($t \in \mathbb{R}$),

(b) if $T(t)$ is non-negative ($t \in \mathbb{R}$),

(c) if the mapping $T$ is Lipschitz-continuous, where we denote the smallest Lipschitz-constant of $T$ by $|T|_{\text{Lip}}$, and

(d) if there exists a set $N \subseteq \mathbb{R}$ of measure zero such that for each $x \in H$ the function

$$\mathbb{R} \setminus N \ni t \mapsto T(t)x$$

is differentiable.\(^4\)

**Lemma 2.1.** Assume that $M_0$ satisfies properties (a) and (d). Then for each $t \in \mathbb{R}$ the mapping

$$\hat{M}_0(t) : H \to H$$

$$x \mapsto \begin{cases} (M_0(\cdot)x)'(t), & t \in \mathbb{R} \setminus N, \\ 0, & t \in N \end{cases}$$

is a bounded linear operator with $\|\hat{M}_0(t)\|_{L(H)} \leq |M_0|_{\text{Lip}}$. Thus, $\hat{M}_0 \in L_s^\infty(\mathbb{R}; L(H))$ gives rise to a multiplication operator $\hat{M}_0(m_0) \in L(H_{\psi,0}(\mathbb{R}; H))$ given by

$$(\hat{M}_0(m_0)u)(t) := (M_0(\cdot)u(t))'(t)$$

\(^4\)Note that $(\Phi(m_0)\varphi)(t) = \Phi(t)\varphi(t)$ for $t \in \mathbb{R}, \varphi \in \tilde{C}_\infty(\mathbb{R}; H)$ and $\Phi : \mathbb{R} \to L(H)$ strongly measurable and bounded. Hence, $\Phi(m_0) \in L(H_{\psi,0}(\mathbb{R}; H))$ and $\|\Phi(m_0)\|_{L(H_{\psi,0}(\mathbb{R}; H))} \leq \sup_{t \in \mathbb{R}} \|\Phi(t)\|_{L(H)}$ for each $m_0 \geq 0$.

\(^5\)If $H$ is separable, then the strong differentiability of $T$ on $\mathbb{R} \setminus N$ for some set $N$ of measure zero already follows from the Lipschitz-continuity of $T$. 
2 Space-time evolutionary equations

for \( u \in H_{\ell,0}(\mathbb{R}; H) \) and almost every \( t \in \mathbb{R} \). Moreover, for \( u \in D(\partial_0) \) the product rule

\[
\partial_0 M_0(m_0)u = \dot{M}_0(m_0)u + M_0(m_0)\partial_0 u \tag{2.1}
\]

holds. In particular, \( M_0(m_0) \in L(H_{\ell,-1}(\mathbb{R}; H)) \). If, in addition, \( M_0 \) satisfies property (m) then \( \dot{M}_0(m_0) \) is selfadjoint.

Proof. Let \( t \in \mathbb{R} \setminus N \). The linearity of \( \dot{M}_0(t) \) is obvious. For \( x \in H \) we estimate

\[
\frac{1}{\|h\|} |(M_0(t + h) - M_0(t)) x| \leq \|M_0\|_{\text{Lip}} |x|
\]

for each \( h \in \mathbb{R} \setminus \{0\} \). Thus, \( |\dot{M}_0(t)x| = |(M_0(\cdot):x)'(t)| \leq \|M_0\|_{\text{Lip}} |x| \), which shows that \( \dot{M}_0(t) \in L(H) \) with \( \|M_0(t)\|_{L(H)} \leq \|M_0\|_{\text{Lip}} \). Assuming property (a), we see that the self-adjointness of \( \dot{M}_0(t) \) follows from

\[
\langle \dot{M}_0(t)x|y \rangle = \lim_{h \to 0} \frac{1}{h} \langle (M_0(t + h) - M_0(t)) x|y \rangle
\]

for each \( x, y \in H \). It is left to show the product rule (2.1). To this end, let \( \phi \in \hat{C}_\infty(\mathbb{R}; H) \). Then we compute

\[
M_0(t + h)\phi(t + h) - M_0(t)\phi(t) = M_0(t + h) (\phi(t + h) - \phi(t)) + (M_0(t + h) - M_0(t)) \phi(t)
\]

for each \( t, h \in \mathbb{R} \). This yields

\[
\frac{1}{h} \left( M_0(t + h)\phi(t + h) - M_0(t)\phi(t) \right) = M_0(t + h) \left( \frac{1}{h} (\phi(t + h) - \phi(t)) \right) + \left( \frac{1}{h} (M_0(t + h) - M_0(t)) \right) \phi(t)
\]

for every \( t \in \mathbb{R}, h \in \mathbb{R} \setminus \{0\} \). The term on the right-hand side in the latter formula tends to

\[
M_0(t)\phi'(t) + (M_0(\cdot):\phi(t))'(t) = (M_0(m_0)\partial_0 \phi)(t) + (\dot{M}_0(m_0)\phi)(t)
\]

for \( t \in \mathbb{R} \setminus N \) as \( h \to 0 \). Thus, the left-hand side is differentiable almost everywhere and since \( \dot{M}_0(m_0)\phi + M_0(m_0)\partial_0 \phi \in H_{\ell,0}(\mathbb{R}; H) \) we derive that \( M_0(m_0)\phi \in D(\partial_0) \) and

\[
\partial_0 M_0(m_0)\phi = \dot{M}_0(m_0)\phi + M_0(m_0)\partial_0 \phi.
\]

The product rule (2.1) for functions in \( D(\partial_0) \) now follows by approximation. To show that the operator \( M_0(m_0) \) can be established as a bounded operator on \( H_{\ell,-1}(\mathbb{R}; H) \) we observe that

\[
\partial_0^{-1} M(m_0) - M(m_0)\partial_0^{-1} = \partial_0^{-1} (M(m_0)\partial_0 - \partial_0 M(m_0)) \partial_0^{-1} = -\partial_0^{-1} \dot{M}_0(m_0) \partial_0^{-1}
\]
and thus,
\[ |M(m_0)u|_{H_{e,-1}(\mathbb{R}; H)} = |\partial_0^{-1}M(m_0)u|_{H_{e,0}(\mathbb{R}; H)} \]
\[ \leq |\partial_0^{-1}M_0(m_0)\partial_0^{-1}u|_{H_{e,0}(\mathbb{R}; H)} + |M_0(m_0)\partial_0^{-1}u|_{H_{e,0}(\mathbb{R}; H)} \]
\[ \leq \left( \frac{1}{\varrho} |M_0|_{\text{Lip}} + \|M_0(m_0)\|_{L(H_{e,0}(\mathbb{R}; H))} \right) |u|_{H_{e,-1}(\mathbb{R}; H)} \]
for each \( u \in H_{e,0}(\mathbb{R}; H) \).

**Remark 2.2.** Note that the product rule
\[ \partial_0 M_0(m_0)\phi = \dot{M}_0(m_0)\phi + M_0(m_0)\partial_0 \phi \]
for \( \phi \in D(\partial_0) \) can be extended by continuity to \( \phi \in H_{e,0}(\mathbb{R}; H) \). Indeed, both the operators \( \partial_0 M_0(m_0) \) and \( \dot{M}_0(m_0) + M_0(m_0)\partial_0 \) are densely defined continuous mappings from \( H_{e,0}(\mathbb{R}; H) \) to \( H_{e,-1}(\mathbb{R}; H) \) coinciding on the dense subset \( H_{e,1}(\mathbb{R}; H) \subseteq H_{e,0}(\mathbb{R}; H) \).

**Corollary 2.3.** Assume that \( M_0 \) satisfies properties (c) and (d). Then
\[ D = \{ u \in H_{e,0}(\mathbb{R}; H) \mid \partial_0 M_0(m_0)u + Au \in H_{e,0}(\mathbb{R}; H) \} \]
\[ = \{ u \in H_{e,0}(\mathbb{R}; H) \mid M_0(m_0)\partial_0^*u - Au \in H_{e,0}(\mathbb{R}; H) \}. \]
Moreover, we have \((M_0(m_0)\partial_0^* - A)u = (-\partial_0 M_0(m_0) + 2\varrho M_0(m_0) + \dot{M}_0(m_0) - A)u\) for all \( u \in D \).

**Proof.** Let \( u \in H_{e,0}(\mathbb{R}; H) \). Then \( u \in D \) if and only if \((\partial_0 M_0(m_0) + A)u \in H_{e,0}(\mathbb{R}; H)\). Since \((-2\varrho M_0(m_0) - \dot{M}_0(m_0))u \in H_{e,0}(\mathbb{R}; H)\), we have that \((\partial_0 M_0(m_0) + A)u \in H_{e,0}(\mathbb{R}; H)\) if and only if
\[ H_{e,0}(\mathbb{R}; H) \ni (\partial_0 M_0(m_0) + A)u + (-2\varrho M_0(m_0) - \dot{M}_0(m_0))u \]
\[ = M_0(m_0)\partial_0 u + Au - 2\varrho M_0(m_0)u \]
\[ = -(M_0(m_0)\partial_0^* - A)u, \]
where we have used that \( \partial_0^* = -\partial_0 + 2\varrho \), which can be verified immediately.

**Corollary 2.4.** Let \( \varepsilon, \varrho > 0 \). Assume that \( M_0 \) satisfies the properties (c) and (d). Then for \( u \in H_{e,0}(\mathbb{R}; H) \) we have that
\[ \left( 1 + \varepsilon \partial_0 \right)^{-1} \partial_0 M_0(m_0)u = \partial_0 M_0(m_0)(1 + \varepsilon \partial_0)^{-1}u = \varepsilon \partial_0(1 + \varepsilon \partial_0)^{-1}\dot{M}_0(m_0)(1 + \varepsilon \partial_0)^{-1}u. \]

**Proof.** For \( u \in H_{e,0}(\mathbb{R}; H) \), we compute, invoking the product rule (2.1), that
\[ \left( (1 + \varepsilon \partial_0)^{-1} \partial_0 M_0(m_0) - \partial_0 M_0(m_0)(1 + \varepsilon \partial_0)^{-1} \right) u \]
\[ = (1 + \varepsilon \partial_0)^{-1} (\partial_0 M_0(m_0)(1 + \varepsilon \partial_0) - (1 + \varepsilon \partial_0)\partial_0 M_0(m_0)) (1 + \varepsilon \partial_0)^{-1}u \]
\[ = (1 + \varepsilon \partial_0)^{-1} (\partial_0 M_0(m_0)\varepsilon \partial_0 - \varepsilon \partial_0 (\partial_0 M_0(m_0))) (1 + \varepsilon \partial_0)^{-1}u \]
\[ = (1 + \varepsilon \partial_0)^{-1} \varepsilon \partial_0 (M_0(m_0)\partial_0 - \partial_0 M_0(m_0)) (1 + \varepsilon \partial_0)^{-1}u \]
\[ = \varepsilon \partial_0(1 + \varepsilon \partial_0)^{-1}(-\dot{M}_0(m_0))(1 + \varepsilon \partial_0)^{-1}u. \]
2 Space-time evolutionary equations

In the spirit of the solution theory in [11, Chapter 6], we will require the following positive
definiteness constraint on the operators $M_0, \dot{M}_0$ and $M_1$: there exists a set $N_1 \subseteq \mathbb{R}$ of
measure zero with $N \subseteq N_1$ such that

$$\exists c_0 > 0, \varrho_0 > 0 \forall t \in \mathbb{R} \setminus N_1, \varrho \geq \varrho_0 : \varrho M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) \geq c_0.$$  \hfill (2.2)

From this we derive the following estimate.

**Lemma 2.5.** Assume that $M_0$ satisfies properties (i)-(d). Assume that inequality (2.2) holds and let $\varrho \geq \varrho_0$. Then for $u \in D(\partial_0) \cap D(A)$ and $a \in \mathbb{R}$ we have that

$$\int_{-\infty}^{a} \Re \langle \partial_0 M_0(m_0) u + M_1(m_0) u + Au | u(t) \rangle e^{-2\varrho t} \, dt \geq c_0 \int_{-\infty}^{a} |u(t)|^2 e^{-2\varrho t} \, dt.$$  \hfill (2.3)

For the proof of the lemma, we need the following.

**Lemma 2.6.** Let $\varrho > 0$. Assume that $M_0$ satisfies the properties (i), (a) and (d). Then for $u \in D(\partial_0) \cap D(A)$ and $a \in \mathbb{R}$ the following equality holds:

$$\int_{-\infty}^{a} \Re \langle \partial_0 M_0(m_0) u + M_1(m_0) u + Au | u(t) \rangle e^{-2\varrho t} \, dt = \frac{1}{2} \langle u(a) | M_0(a) u(a) \rangle e^{-2\varrho a} + \int_{-\infty}^{a} \varrho \langle u(t) | M_0(t) u(t) \rangle e^{-2\varrho t} \, dt$$

$$+ \int_{-\infty}^{a} \left\langle \frac{1}{2} \dot{M}_0(t) u(t) + \Re M_1(t) u(t) \bigg| u(t) \right\rangle e^{-2\varrho t} \, dt.$$  \hfill (2.4)

**Proof.** Let $u \in D(\partial_0) \cap D(A)$ and $a \in \mathbb{R}$. Note that, since $A$ is skew-selfadjoint,

$$\Re \langle Au | u \rangle(t) = 0$$

for almost every $t \in \mathbb{R}$. Hence, the left-hand side in (2.4) equals

$$\int_{-\infty}^{a} \Re \langle \partial_0 M_0(m_0) u + M_1(m_0) u | u(t) \rangle e^{-2\varrho t} \, dt.$$
Using again the product rule (2.1) we obtain

\[
2 \int_{-\infty}^{a} \Re \langle \partial_1 M_0(m_0) u + M_1(m_0) u | u(t) \rangle e^{-2gt} dt
\]

\[=
2 \int_{-\infty}^{a} \Re \langle M_0(t) \partial_0 u(t) | u(t) \rangle e^{-2gt} dt + 2 \int_{-\infty}^{a} \Re \langle \dot{M}_0(t) u(t) + \Re M_1(t) u(t) | u(t) \rangle e^{-2gt} dt
\]

\[=
\int_{-\infty}^{a} \Re \langle \partial_0 u(t) | M_0(t) u(t) \rangle e^{-2gt} dt
\]

\[+
\int_{-\infty}^{a} \Re \langle M_0(t) \partial_0 u(t) | u(t) \rangle e^{-2gt} dt + 2 \int_{-\infty}^{a} \langle \dot{M}_0(t) u(t) + \Re M_1(t) u(t) | u(t) \rangle e^{-2gt} dt
\]

\[=\int_{-\infty}^{a} \langle u(\cdot) | (M_0(m_0) u)(\cdot) \rangle'(t) e^{-2gt} dt - \int_{-\infty}^{a} \Re \langle u(t) | (\partial_0 M_0(m_0) u)(t) \rangle e^{-2gt} dt
\]

\[+
\int_{-\infty}^{a} \Re \langle M_0(t) \partial_0 u(t) | u(t) \rangle e^{-2gt} dt + 2 \int_{-\infty}^{a} \langle \dot{M}_0(t) u(t) + \Re M_1(t) u(t) | u(t) \rangle e^{-2gt} dt.
\]

Using again the product rule (2.1) we obtain

\[
- \int_{-\infty}^{a} \Re \langle u(t) | (\partial_0 M_0(m_0) u)(t) \rangle e^{-2gt} dt + \int_{-\infty}^{a} \Re \langle M_0(t) \partial_0 u(t) | u(t) \rangle e^{-2gt} dt
\]

\[= - \int_{-\infty}^{a} \langle \dot{M}_0(t) u(t) | u(t) \rangle e^{-2gt} dt.
\]

Hence, we arrive at

\[
2 \int_{-\infty}^{a} \Re \langle \partial_0 M_0(m_0) u + M_1(m_0) u | u(t) \rangle e^{-2gt} dt
\]

\[=
\int_{-\infty}^{a} \langle u(\cdot) | (M_0(m_0) u)(\cdot) \rangle'(t) e^{-2gt} dt + 2 \int_{-\infty}^{a} \left( \frac{1}{2} \dot{M}_0(t) u(t) + \Re M_1(t) u(t) \right) u(t) e^{-2gt} dt
\]

\[= \langle u(a) | M_0(a) u(a) \rangle e^{-2ga} + \int_{-\infty}^{a} 2 \langle u(t) | M_0(t) u(t) \rangle e^{-2gt} dt
\]

\[+ 2 \int_{-\infty}^{a} \left( \frac{1}{2} \dot{M}_0(t) u(t) + \Re M_1(t) u(t) \right) u(t) e^{-2gt} dt,
\]
2 Space-time evolutionary equations

where we have used integration by parts.

Proof of Lemma 2.5. Using Lemma 2.6 and the fact that $M_0(a)$ is non-negative we end up with

$$\int_{-\infty}^{a} \mathbb{R} \Re (\partial_0 M_0(m_0)u + M_1(m_0)u + Au|u(t)e^{-2\zeta t} dt \geq \int_{-\infty}^{a} \langle (\zeta M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) u) \big| u(t) \rangle e^{-2\zeta t} dt \geq c_0 \int_{-\infty}^{a} |u(t)|^2 e^{-2\zeta t} dt.$$ 

Our next goal is to show that (2.3) also holds for elements in $D$. For doing so, we need to approximate elements in $D$ by elements in $D(\partial_0) \cap D(A)$ in a suitable way.

Lemma 2.7. For each $u \in H_{\zeta,0}(\mathbb{R}; H)$ we have that $(1 + \varepsilon \partial_0)^{-1} u \rightarrow u$ as $\varepsilon \rightarrow 0^+$. 

Proof. Since the operator family $((1 + \varepsilon \partial_0)^{-1})_{\varepsilon>0}$ is uniformly bounded, it suffices to note that

$$(1 + \varepsilon \partial_0)^{-1} u - u = (1 + \varepsilon \partial_0)^{-1} \varepsilon \partial_0 u \rightarrow 0$$

as $\varepsilon \rightarrow 0^+$ for every $u \in H_{\zeta,1}(\mathbb{R}; H)$.

Remark 2.8. It should be noted that literally the same result holds true for $\partial_0$ replaced by $\partial_0^*$. The proof follows with obvious modifications.

Lemma 2.9. Let $\varepsilon > 0$ and let $u \in D$. Then $(1 + \varepsilon \partial_0)^{-1} u \in D(\partial_0) \cap D(A)$ and the following formula holds

$$u = (\partial_0 M_0(m_0) + M_1(m_0) + A) u \quad \text{(2.5)}$$

Moreover, we have

$$(\partial_0 M_0(m_0) + M_1(m_0) + A) (1 + \varepsilon \partial_0)^{-1} u \rightarrow (\partial_0 M_0(m_0) + M_1(m_0) + A) u \quad (\varepsilon \rightarrow 0^+)$$
in $H_{\zeta,0}(\mathbb{R}; H)$. 

14
Lemma 2.10. Assume that \(D\) in The most important step to generalize the statement of Lemma 2.5 to the case of elements Proof. With the help of Corollary 2.4 and the fact that \(A\) and \((1 + \varepsilon \partial_{\theta})^{-1}\) commute, the formula (2.5) follows. From (2.5) we read off that \((1 + \varepsilon \partial_{\theta})^{-1} u \in \mathcal{D} \). Moreover, since \((1 + \varepsilon \partial_{\theta})^{-1} u \in H_{\theta,1}(\mathbb{R}; H)\) we have \(\partial_{\theta} M_0(m_0)(1 + \varepsilon \partial_{\theta})^{-1} u \in H_{\theta,0}(\mathbb{R}; H)\). Defining
\[
(\partial_{\theta} M_0(m_0) + M_1(m_0) + A)(1 + \varepsilon \partial_{\theta})^{-1} u =: F \in H_{\theta,0}(\mathbb{R}; H),
\]
we get that
\[
F - (\partial_{\theta} M_0(m_0) + M_1(m_0))(1 + \varepsilon \partial_{\theta})^{-1} u = A(1 + \varepsilon \partial_{\theta})^{-1} u \in H_{\theta,0}(\mathbb{R}; H).
\]
Hence, \((1 + \varepsilon \partial_{\theta})^{-1} u \in D(A)\). According to Lemma 2.7 the left-hand side of equation (2.5) converges to \((\partial_{\theta} M_0(m_0) + M_1(m_0) + A) u\) and the last two terms on the right hand side cancel out as \(\varepsilon \to 0 +\). Moreover, since \(\left(\varepsilon \partial_{\theta}(1 + \varepsilon \partial_{\theta})^{-1} \hat{M}_0(m_0)(1 + \varepsilon \partial_{\theta})^{-1} u\right)_{\varepsilon > 0}\) is bounded in \(H_{\theta,0}(\mathbb{R}; H)\), there exists a weakly convergent subsequence. Using that \(\varepsilon \partial_{\theta}(1 + \varepsilon \partial_{\theta})^{-1} \hat{M}_0(m_0)(1 + \varepsilon \partial_{\theta})^{-1} u \to 0\) in \(H_{\theta,1}(\mathbb{R}; H)\), we deduce that
\[
\varepsilon \partial_{\theta}(1 + \varepsilon \partial_{\theta})^{-1} \hat{M}_0(m_0)(1 + \varepsilon \partial_{\theta})^{-1} u \to 0 \quad (\varepsilon \to 0+),
\]
and thus
\[
(\partial_{\theta} M_0(m_0) + M_1(m_0) + A)(1 + \varepsilon \partial_{\theta})^{-1} u \to (\partial_{\theta} M_0(m_0) + M_1(m_0) + A) u \quad (\varepsilon \to 0+).
\]

The most important step to generalize the statement of Lemma 2.5 to the case of elements in \(D\) is the following result.

Lemma 2.10. Assume that \(M_0\) satisfies the properties (\(\text{a}\)) and (\(\text{d}\)). Let \(\rho > 0\), \(a \in \mathbb{R} \cup \{\infty\}\) and let \(G : H_{\rho,0}(\mathbb{R}; H) \to \mathbb{R}\) be continuous. Moreover, assume that for \(u \in D(\partial_{\theta}) \cap D(A)\) we have that
\[
\int_{-\infty}^{\alpha} \Re((\partial_{\theta} M_0(m_0) + M_1(m_0) + A) u(t)e^{-2\rho t} dt \geq G(u).
\]
Then the latter inequality holds for all \(u \in \mathcal{D}\).

Proof. Let \(u \in \mathcal{D}\). According to Lemma 2.9 we have that
\[
\langle \chi_{[-\infty,a]}(m_0)u | (\partial_{\theta} M_0(m_0) + M_1(m_0) + A) u \rangle_{H_{\rho,0}(\mathbb{R}; H)} = \lim_{\varepsilon \to 0} \langle \chi_{[-\infty,a]}(m_0)(1 + \varepsilon \partial_{\theta})^{-1} u | (\partial_{\theta} M_0(m_0) + M_1(m_0) + A)(1 + \varepsilon \partial_{\theta})^{-1} u \rangle_{H_{\rho,0}(\mathbb{R}; H)},
\]
where we have used that the multiplication operator \(\chi_{[-\infty,a]}(m_0)\) with the cut-off function \(\chi_{[-\infty,a]}\) is a bounded operator on \(H_{\rho,0}(\mathbb{R}; H)\) and that \((1 + \varepsilon \partial_{\theta})^{-1}\) converges strongly to 1, by Lemma 2.7. With the assumed inequality and the fact that for \(\varepsilon > 0\) we have
2 Space-time evolutionary equations

\[(1 + \varepsilon \partial_0)^{-1} u \in D(\partial_0) \cap D(A) = H_{\varepsilon,1}(\mathbb{R}; H) \cap H_{\varepsilon,0}(\mathbb{R}; H_1(A + 1)),\]

by Lemma 2.9 we obtain that

\[
\Re \langle \chi_{[-\infty,0]}(m_0) u | \partial_0 M_0(m_0) + M_1(m_0) + A \rangle H_{\varepsilon,0}(\mathbb{R}; H) = \lim_{\varepsilon \to 0} \Re \langle \chi_{[-\infty,0]}(m_0)(1 + \varepsilon \partial_0)^{-1} u | \partial_0 M_0(m_0) + M_1(m_0) + A)(1 + \varepsilon \partial_0)^{-1} u \rangle H_{\varepsilon,0}(\mathbb{R}; H) \geq \lim_{\varepsilon \to 0} G((1 + \varepsilon \partial_0)^{-1} u) = G(u). \]

\[\tag{2.6}\]

\textbf{Corollary 2.11.} Assume that \(M_0\) satisfies properties (a)-(d). Assume that inequality (2.2) holds and let \(\varrho \geq \varrho_0\). Then for \(u \in D\) and \(a \in \mathbb{R} \cup \{\infty\}\) we have that

\[
\int_{-\infty}^{a} \Re \langle (\partial_0 M_0(m_0) + M_1(m_0) + A) u | u \rangle(t)e^{-2\varrho t} dt \geq c_0 \int_{-\infty}^{a} |u(t)|^2 e^{-2\varrho t} dt.
\]

\[\tag{2.6}\]

\textbf{Proof.} The statement is immediate from the Lemmas 2.5 and 2.10 with

\[G(u) = c_0 \int_{-\infty}^{a} |u(t)|^2 e^{-2\varrho t} dt = c_0 \langle \chi_{[-\infty,0]}(m_0) u | u \rangle H_{\varepsilon,0}(\mathbb{R}; H). \]

\[\tag{2.6}\]

\textbf{Lemma 2.12.} Assume that \(M_0\) satisfies the properties (a)-(d) and that inequality (2.2) holds. Let \(\varrho \geq \varrho_0\) and \(u \in \{v \in H_{\varepsilon,0}(\mathbb{R}; H) | (M_0(m_0)\partial_0^* - A) v \in H_{\varepsilon,0}(\mathbb{R}; H)\}\). Then the inequality

\[
\int_{\mathbb{R}} \Re \langle (M_0(m_0)\partial_0^* + M_1(m_0)^* - A) u | u \rangle(t)e^{-2\varrho t} dt \geq c_0 \int_{\mathbb{R}} |u(t)|^2 e^{-2\varrho t} dt
\]

holds.

\[\tag{2.6}\]

\textbf{Proof.} By Corollary 2.3 we deduce that \(u \in D\) and that for \(a \in \mathbb{R}\) we have

\[
\int_{-\infty}^{a} \Re \langle (M_0(m_0)\partial_0^* + M_1(m_0)^* - A) u | u \rangle(t)e^{-2\varrho t} dt
\]

\[
= \int_{-\infty}^{a} \Re \langle (-\partial_0 M_0(m_0) + 2\varrho M_0(m_0) + M_0(m_0) + M_1(m_0)^* - A) u | u \rangle(t)e^{-2\varrho t} dt. \tag{2.6}\]
With Lemma 2.6 we get for \( u \in D(\partial_0) \cap D(A) \) and \( a \in \mathbb{R} \) that

\[
\int_{-\infty}^{a} \Re \langle -\partial_0 M_0(m_0) + 2\varrho M_0(m_0) + \dot{M}_0(m_0) + M_1(m_0)^* - A \rangle u(t) e^{-2\varrho t} dt
= -\frac{1}{2} \langle u(a) | M_0(a) u(a) \rangle e^{-2\varrho a} - \int_{-\infty}^{a} \Re \langle u(t) | M_0(t) u(t) \rangle e^{-2\varrho t} dt
- \int_{-\infty}^{a} \left\langle \frac{1}{2} \dot{M}_0(t) u(t) - \Re M_1(t) u(t) \right| u(t) \rangle e^{-2\varrho t} dt
+ \int_{-\infty}^{a} \langle 2\varrho M_0(t) u(t) + \dot{M}_0(t) u(t) | u(t) \rangle e^{-2\varrho t} dt
\geq -\frac{1}{2} \langle u(a) | M_0(a) u(a) \rangle e^{-2\varrho a} + c_0 \int_{-\infty}^{a} \langle u(t) | u(t) \rangle e^{-2\varrho t} dt
\]

Letting \( a \to \infty \), we deduce that

\[
\int_{\mathbb{R}} \Re \langle -\partial_0 M_0(m_0) + 2\varrho M_0(m_0) + \dot{M}_0(m_0) + M_1(m_0)^* - A \rangle u(t) e^{-2\varrho t} dt
\geq c_0 \int_{\mathbb{R}} \langle u(t) | u(t) \rangle e^{-2\varrho t} dt.
\]

Now, Lemma 2.10 implies the latter inequality to hold for all \( u \in \mathcal{D} \). The assertion follows from equation (2.6).

**Theorem 2.13** (Solution Theory). Let \( A : D(A) \subseteq H \to H \) be skew-selfadjoint and \( M_0, M_1 \in L^\infty_{\text{loc}}(\mathbb{R}; L(H)) \). Furthermore, assume that \( M_0 \) satisfies the hypotheses (A1)-(A3) and that (2.2) holds. Then the operator \( \partial_0 M_0(m_0) + M_1(m_0) + A \) is continuously invertible in \( H_{\varrho,0}(\mathbb{R}; H) \) for each \( \varrho \geq \varrho_0 \). A norm bound for the inverse is \( 1/c_0 \). Moreover, we get that

\[
(\partial_0 M_0(m_0) + M_1(m_0) + A)^* = (M_0(m_0) \partial_0^* + M_1(m_0)^* - A),
\]
where the latter operator is considered in \( H_{\varrho,0}(\mathbb{R}; H) \) with maximal domain.
2 Space-time evolutionary equations

Proof. Let \( q \geq q_0 \). By Corollary 2.1 we have that

\[
\Re \langle u | (\partial_0 M_0 (m_0) + M_1 (m_0) + A) u \rangle_{H_{p,0}(\mathbb{R}; H)} \geq c_0 \langle u | u \rangle_{H_{p,0}(\mathbb{R}; H)}
\]

for all \( u \in \mathcal{D} \). This implies that the operator \( \partial_0 M_0 (m_0) + M_1 (m_0) + A \) has a continuous inverse with operator norm less than or equal to the constant \( 1/c_0 \).

It remains to show that \( \partial_0 M_0 (m_0) + M_1 (m_0) + A \) maps onto \( H_{p,0}(\mathbb{R}; H) \). For this we compute the adjoint of

\[
\mathcal{B} := (\partial_0 M_0 (m_0) + M_1 (m_0) + A)
\]

considered as an operator in \( H_{p,0}(\mathbb{R}; H) \). Let \( f \in D(\mathcal{B}^*) \subseteq H_{p,0}(\mathbb{R}; H) \). Then for all \( u \in \mathcal{D} \) and \( \varepsilon > 0 \) we obtain with the help of equation (2.5) that

\[
\langle \mathcal{B} u | ((1 + \varepsilon \partial_0)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \nabla = \langle (1 + \varepsilon \partial_0)^{-1} \mathcal{B} u | f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
= \langle \mathcal{B} (1 + \varepsilon \partial_0)^{-1} u | f \rangle_{H_{p,0}(\mathbb{R}; H)} + \langle -\varepsilon \partial_0 (1 + \varepsilon \partial_0)^{-1} \dot{M}_0 (m_0) (1 + \varepsilon \partial_0)^{-1} u | f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
+ \langle (1 + \varepsilon \partial_0)^{-1} M_1 (m_0) u - M_1 (m_0) (1 + \varepsilon \partial_0)^{-1} u | f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
= \langle (1 + \varepsilon \partial_0)^{-1} u | \mathcal{B}^* f \rangle_{H_{p,0}(\mathbb{R}; H)} + \langle u | (1 + \varepsilon \partial_0)^{-1} \dot{M}_0 (m_0) (1 + \varepsilon \partial_0)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
+ \langle u | (1 + \varepsilon \partial_0)^{-1} M_1 (m_0) - M_1 (m_0) (1 + \varepsilon \partial_0)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)}.
\]

Hence, we deduce that \( (1 + \varepsilon \partial_0)^{-1} | D(\mathcal{B}^*) | \subseteq D(\mathcal{B}^*) \) for \( \varepsilon > 0 \). Moreover, we have \( (1 + \varepsilon \partial_0)^{-1} f \in D(\partial_0^*) \). Thus, for \( u \in H_{p,1}(\mathbb{R}; H_1(A+1)) \subseteq D(\mathcal{B}) \) and \( \varepsilon > 0 \) we get that

\[
\langle u | \mathcal{B}^* (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \nabla = \langle \mathcal{B} u | (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
= \langle \partial_0 M_0 (m_0) + M_1 (m_0) + A | u | (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \\
= \langle u | (M_0 (m_0) \partial_0^* + M_1 (m_0)^*) (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} + \langle Au | (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)}.
\]

Since \( H_{p,1}(\mathbb{R}; H_1(A+1)) \) is a core for \( A \), we deduce that \( (1 + \varepsilon \partial_0^*)^{-1} f \in D(A) \) for \( \varepsilon > 0 \). Moreover, we have

\[
\mathcal{B}^* (1 + \varepsilon \partial_0^*)^{-1} f = (M_0 (m_0) \partial_0^* + M_1 (m_0)^* - A) (1 + \varepsilon \partial_0^*)^{-1} f.
\]

Using (2.7) we can estimate

\[
| \mathcal{B}^* (1 + \varepsilon \partial_0^*)^{-1} f |_{H_{p,0}(\mathbb{R}; H)} = \sup \left\{ \left| \langle u | \mathcal{B}^* (1 + \varepsilon \partial_0^*)^{-1} f \rangle_{H_{p,0}(\mathbb{R}; H)} \right| \mid u \in \mathcal{D}, |u|_{H_{p,0}(\mathbb{R}; H)} \leq 1 \right\} \nabla \leq |\mathcal{B}^* f|_{H_{p,0}(\mathbb{R}; H)} + 2|M_0|_{Lip} |f|_{H_{p,0}(\mathbb{R}; H)} \\
+ 2|M_1 (m_0)||f|_{L(H_{p,0}(\mathbb{R}; H))}|f|_{H_{p,0}(\mathbb{R}; H)}.
\]
for every $\varepsilon > 0$ and thus, we find a weakly convergent subsequence in $H_{0,0}(\mathbb{R}; H) \subseteq H_{0,-1}(\mathbb{R}; H) \cap H_{0,0}(\mathbb{R}; H_{-1}(A+1))$. Moreover, note that $(M_0(m_0)\partial_0^* + M_1(m_0)^* - A)(1 + \varepsilon\partial_0^*)^{-1} f$ converges to $(M_0(m_0)\partial_0^* + M_1(m_0)^* - A) f$ in $H_{0,-1}(\mathbb{R}; H) \cap H_{0,0}(\mathbb{R}; H_{-1}(A+1))$. Thus, by the (weak) closedness of $\mathcal{B}^*$ we derive

$$\mathcal{B}^* f = (M_0(m_0)\partial_0^* + M_1(m_0)^* - A) f$$

and

$$D(\mathcal{B}^*) \subseteq \{ f \in H_{0,0}(\mathbb{R}; H) \mid (M_0(m_0)\partial_0^* + M_1(m_0)^* - A) f \in H_{0,0}(\mathbb{R}; H) \}.$$

We define

$$\mathcal{C} : D(\mathcal{C}) \subseteq H_{0,0}(\mathbb{R}; H) \to H_{0,0}(\mathbb{R}; H)$$

$$f \mapsto (M_0(m_0)\partial_0^* + M_1(m_0)^* - A) f,$$

where $D(\mathcal{C}) := \{ f \in H_{0,0}(\mathbb{R}; H) \mid (M_0(m_0)\partial_0^* - A) f \in H_{0,0}(\mathbb{R}; H) \}$. Lemma 2.12 ensures that $\mathcal{C}$ is one-to-one. Thus, so is $\mathcal{B}^*$. According to the projection theorem we have the orthogonal decomposition

$$H_{0,0}(\mathbb{R}; H) = N((\partial_0 M_0(m_0) + M_1(m_0) + A)^*) \oplus R(\partial_0 M_0(m_0) + M_1(m_0) + A)$$

$$= \{0\} \oplus R(\partial_0 M_0(m_0) + M_1(m_0) + A)$$

and this establishes the onto-property of $\partial_0 M_0(m_0) + M_1(m_0) + A$. Moreover, we get that $(\mathcal{B}^{-1})^* = (\mathcal{B}^*)^{-1} \subseteq \mathcal{C}^{-1}$. The first operator is left-total. Thus, $\mathcal{B}^* = \mathcal{C}$. 

\[\square\]

**Causality**

At first we give the definition of causality in our framework.

**Definition 2.14.** Let $H$ be a Hilbert space, $\varrho > 0$ and $G : D(G) \subseteq H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)$. Then $G$ is called (forward) causal, if for each $a \in \mathbb{R}$ and each $f, g \in D(G)$ the implication

$$\chi_{[-\infty,a]}(m_0)(f - g) = 0 \implies \chi_{[-\infty,a]}(m_0)(G(f) - G(g)) = 0$$

holds.

Now, we want to show that our solution operator $(\partial_0 M_0(m_0) + M_1(m_0) + A)^{-1}$ is causal in $H_{\varrho,0}(\mathbb{R}; H)$.

**Theorem 2.15** (causal solution operator). Under the assumptions of Theorem 2.13 the solution operator $(\partial_0 M_0(m_0) + M_1(m_0) + A)^{-1}$ is causal in $H_{\varrho,0}(\mathbb{R}; H)$ for each $\varrho \geq \varrho_0$. 

19
2 Space-time evolutionary equations

Proof. Let \( f \in H_{e,0}(\mathbb{R}; H) \) with \( \chi_{[-\infty,a]}(m_0)f = 0 \). We define \( u := (\partial_0 M_0(m_0) + M_1(m_0) + A)^{-1}f \in \mathcal{D} \) and estimate, using Corollary 2.11,

\[
0 = \int_{-\infty}^{\alpha} \Re \langle f | u \rangle(t) e^{-2e^t} \, dt \geq c_0 \int_{-\infty}^{\alpha} |u(t)|^2 e^{-2e^t} \, dt,
\]

which shows that \( \chi_{[-\infty,a]}(m_0)u = 0 \). Thus, due to linearity, the solution operator is causal. \( \square \)

An illustrative example

To exemplify what has been achieved so far, let us consider a somewhat contrived and simplistic example.

The starting point of our presentation is the \((1+1)\)-dimensional wave equation

\[
\partial_0^2 u - \partial_1^2 u = f \text{ on } \mathbb{R} \times \mathbb{R}.
\]

As usual we rewrite this equation as a first order system of the form

\[
\begin{pmatrix}
\partial_0
& 1 & 0 \\
0 & -\partial_1
\end{pmatrix}
\begin{pmatrix}
u
v
\end{pmatrix}
= \begin{pmatrix}
\partial_0^{-1} f
0
\end{pmatrix}. \quad (2.8)
\]

In this case we can compute the solution by Duhamel’s principle in terms of the unitary group generated by the skew-selfadjoint operator

\[
\begin{pmatrix}
0 & -\partial_1 \\
-\partial_1 & 0
\end{pmatrix}.
\]

This would be the simplest autonomous case. Let us now, based on this, consider a slightly more complicated situation, which is, however, still autonomous:

\[
\begin{pmatrix}
\partial_0
& & \\
& \partial_0^{-1} f
\end{pmatrix}, \quad (2.9)
\]

where \( \chi_I(m_1) \) denotes the spatial multiplication operator with the cut-off function \( \chi_I \), i.e.

\[
(\chi_I(m_1)f)(t,x) = \chi_I(x)f(t,x) \text{ for almost every } (t,x) \in \mathbb{R} \times \mathbb{R}, \text{ every } f \in H_{e,0}(\mathbb{R}; L^2(\mathbb{R})) \text{ and } I \subseteq \mathbb{R}. \]

In the notation of the previous section we have

\[
M_0(m_0) := \begin{pmatrix}
\chi_{[0,-\epsilon]}(m_1)
0 \\
0 & \chi_{[0,\epsilon]}(m_1)
\end{pmatrix}
\]
and

\[ M_1 (m_0) := \begin{pmatrix} \chi_{|\xi|<\epsilon} (m_1) & 0 \\ 0 & \chi_{|\xi|>\epsilon} (m_1) \end{pmatrix} \]

and both are obviously not time-dependent. Note that our solution condition \([2.2]\) is satisfied and hence, according to our findings, problem \([2.9]\) is well-posed in the sense of Theorem \(2.13\). By the dependence of the operators \(M_0(m_0)\) and \(M_1(m_0)\) on the spatial parameter, we see that \([2.9]\) changes its type from hyperbolic to elliptic to parabolic and back to hyperbolic and so standard semigroup techniques are not at hand to solve the equation. Indeed, in the subregion \([-\epsilon,0]\) the problem reads as

\[
\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_0^{-1} f \\ 0 \end{pmatrix},
\]

which may be rewritten as an elliptic equation for \(u\) of the form

\[ u - \partial_1^2 u = \partial_0^{-1} f. \]

For the region \([0,\epsilon]\) we get

\[
\begin{pmatrix} \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_0^{-1} f \\ 0 \end{pmatrix},
\]

which yields a parabolic equation for \(u\) of the form

\[ \partial_0 u - \partial_1^2 u = \partial_0^{-1} f. \]

In the remaining subdomain \(\mathbb{R} \setminus [-\epsilon,\epsilon]\) the problem is of the original form \([2.8]\), which corresponds to a hyperbolic problem for \(u\).

To turn this into a genuinely time-dependent problem we now make a modification to problem \([2.9]\). We define the function

\[ \varphi(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 < t \leq 1, \\ 1 & \text{if } 1 < t \end{cases} \quad (t \in \mathbb{R}) \]

and consider the material-law operator

\[ M_0 (m_0) = \varphi(m_0) \begin{pmatrix} \chi_{\mathbb{R} \setminus [-\epsilon,0]} (m_1) & 0 \\ 0 & \chi_{\mathbb{R} \setminus [-\epsilon,\epsilon]} (m_1) \end{pmatrix}, \]

which now also degenerates in time. Moreover we modify \(M_1(m_0)\) by adding a time-dependence of the form

\[ M_1 (m_0) = \begin{pmatrix} \chi_{[-\infty,0]} (m_0) + \chi_{[0,\infty]} (m_0) \chi_{[-\epsilon,0]} (m_1) & 0 \\ 0 & \chi_{[-\infty,0]} (m_0) + \chi_{[0,\infty]} (m_0) \chi_{[-\epsilon,\epsilon]} (m_1) \end{pmatrix}. \]
2 Space-time evolutionary equations

We show that this time-dependent material law still satisfies our solvability condition. To this end let \( \varrho > 0 \). Note that

\[
\varphi'(t) = \begin{cases} 1 & \text{if } t \in ]0, 1[, \\ 0 & \text{otherwise} \end{cases}
\]

and thus, for \( t \leq 0 \) we have

\[
\varrho M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq 1.
\]

For \( 0 < t \leq 1 \) we estimate

\[
\varrho M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) \\
= \left( \frac{1}{2} + \varrho t \right) \begin{pmatrix} \chi_{\mathbb{R}\setminus[-\varepsilon,0]}(m_1) & 0 \\ 0 & \chi_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(m_1) \end{pmatrix} + \begin{pmatrix} \chi_{[-\varepsilon,0]}(m_1) & 0 \\ 0 & \chi_{[-\varepsilon,\varepsilon]}(m_1) \end{pmatrix} \geq \frac{1}{2}
\]

and, finally, for \( t > 1 \) we obtain that

\[
\varrho M_0(t) + \frac{1}{2} \dot{M}_0(t) + \Re M_1(t) \\
= \varrho \left( \begin{pmatrix} \chi_{\mathbb{R}\setminus[-\varepsilon,0]}(m_1) & 0 \\ 0 & \chi_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(m_1) \end{pmatrix} \right) + \begin{pmatrix} \chi_{[-\varepsilon,0]}(m_1) & 0 \\ 0 & \chi_{[-\varepsilon,\varepsilon]}(m_1) \end{pmatrix} \geq \min\{\varrho, 1\}.
\]

Remark 2.16. Note that the spatial operator \( \begin{pmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{pmatrix} \) in the previous example can be substituted by every skew-selfadjoint operator. In applications, it turns out that this operator typically is a block operator matrix of the form \( \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \), where \( C \) is a densely defined closed linear operator between two Hilbert spaces. Indeed, even the one-dimensional transport equation shares this form, if one decomposes the functions in their even and odd parts (see [13, p. 17 f.]). Moreover, it should be noted that the block structures of the operator \( \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \) and of the operators \( M_0(m_0) \) and \( M_1(m_0) \) need not to be comparable (it turns out that this naturally arises in the study of boundary control systems, cf. [14], [15]). In those cases the semi-group approach for showing well-posedness is not applicable, without further requirements on the block structures of the involved operators.

Some perturbation results

In applications, it is useful to have a perturbation result at hand. To this end, we assume we are given a linear mapping

\[
M_\infty : D(M_\infty) \subseteq \bigcap_{\varrho \geq \varrho_0} H_{\varrho,0}(\mathbb{R}; H) \rightarrow \bigcap_{\varrho \geq \varrho_0} H_{\varrho,0}(\mathbb{R}; H)
\]
for some $\varrho_0 > 0$ in the way that for all $\varrho \geq \varrho_0$ we have that $D(M_\infty) \subseteq H_{\varrho,0}(\mathbb{R}; H)$ is dense and $M_\infty$ considered as a mapping from $H_{\varrho,0}(\mathbb{R}; H)$ to $H_{\varrho,0}(\mathbb{R}; H)$ is continuous. The assumptions give rise to a continuous extension, denoted with the same symbol. A straightforward consequence of our previous findings is the following.

**Theorem 2.17.** Let $A \colon D(A) \subseteq H \to H$ be skew-selfadjoint and $M_0, M_1 \in L_\infty^0(\mathbb{R}; L(H))$. Furthermore, assume that $M_0$ satisfies the properties (a)-(d) and that (2.2) holds. Assume that

$$\limsup_{\varrho \to \infty} \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} < c_0.$$  

Then there exists $\varrho_1 > 0$ such that the operator $\partial_0 M_0 (m_0) + M_1 (m_0) + M_\infty + A$ is continuously invertible in $H_{\varrho,0}(\mathbb{R}; H)$ for each $\varrho \geq \varrho_1$. If, in addition, $M_\infty$ is causal, then so is $(\partial_0 M_0 (m_0) + M_1 (m_0) + M_\infty + A)^{-1}$.

**Proof.** Let $\varrho_1 > 0$ be such that $\| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} < c_0$ for all $\varrho \geq \varrho_1$. Let $f \in H_{\varrho,0}(\mathbb{R}; H)$. Then, the mapping

$$\Phi : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)$$

$$u \mapsto (\partial_0 M_0 (m_0) + M_1 (m_0) + A)^{-1} (f - M_\infty u)$$

is a strict contraction, by Theorem 2.13. Observing that $u \in H_{\varrho,0}(\mathbb{R}; H)$ satisfies

$$(\partial_0 M_0 (m_0) + M_1 (m_0) + M_\infty + A) u = f$$

if and only if it is a fixed point of $\Phi$, we get existence and uniqueness of a solution with the help of the contraction mapping principle. If $M_\infty$ is causal, then so is $\Phi$ as a composition and a sum of causal mappings. Hence, $(\partial_0 M_0 (m_0) + M_1 (m_0) + M_\infty + A)^{-1}$ is causal.

**Remark 2.18.** Note that this perturbation result applies similarly to the case of non-linear perturbations if the best Lipschitz constant $|M_\infty|_{\varrho, \text{Lip}}$ of the perturbation $M_\infty$ considered as an operator in $H_{\varrho,0} (\mathbb{R}; H)$, $\varrho \in ]0, \infty[,$ satisfies

$$\limsup_{\varrho \to \infty} |M_\infty|_{\varrho, \text{Lip}} < c_0.$$  

It is possible to derive the following more sophisticated perturbation result, which needs little more effort. We introduce the following notation: For a closed subspace $V \subseteq H$ we denote by $\iota_V : V \to H$ the canonical embedding of $V$ into $H$. It turns out that then the adjoint $\iota_V^* : H \to V$ is the orthogonal projection onto $V$. Consequently $P_V := \iota_V \iota_V^* : H \to H$ becomes the orthogonal projector on $V$ and $1 - P_V = P_{V^\perp} = \iota_{V^\perp} \iota_{V^\perp}^*$.\footnote{Note that as an example $C_\infty(\mathbb{R}; H)$ is dense in $H_{\varrho,0}(\mathbb{R}; H)$ for all $\varrho > 0.$}
2 Space-time evolutionary equations

**Theorem 2.19.** Let $A : D(A) \subseteq H \rightarrow H$ be skew-selfadjoint and $M_0, M_1 \in L^\infty(\mathbb{R}; L(H))$. Furthermore, assume that $M_0$ satisfies the properties (iii)-(iv). Moreover, assume $t \mapsto N(M_0(t))$ to be time-independent, i.e., for all $t \in \mathbb{R}$ we have

$$N(M_0(t)) = N(M_0(0)) =: V.$$  

We further assume that for some set of measure zero $N_1 \subseteq \mathbb{R}$ the following estimates hold:

$$\exists c_0 > 0, \varrho_0 > 0 \forall t \in \mathbb{R} \setminus N_1 : \imath \nu \Re M_1(t) \nu \geq c_0, \quad (2.10)$$  

and

$$\exists c_1 > 0 \forall t \in \mathbb{R} \setminus N_1 : \imath \nu \perp M_0(t) \nu \geq c_1. \quad (2.11)$$

Furthermore, assume that $\limsup_{\varrho \to \infty} \|M_{\infty}\|_{L(H_{\varrho,0}(\mathbb{R}; H))} < \infty$ and there exist $\varrho, \varepsilon > 0$ such that for all $\varrho \geq \varrho$ and $u \in D(M_{\infty})$

$$\langle \Re M_{\infty} P_V u | P_V u \rangle_{H_{\varrho,0}(\mathbb{R}; H)} > (\varepsilon - c_0) \|P_V u\|^2_{H_{\varrho,0}(\mathbb{R}; H)}.$$  

Then there exists $\varrho_1 > 0$ such that the operator $\partial_0 M_0 (m_0) + M_1 (m_0) + M_{\infty} + A$ is continuously invertible in $H_{\varrho,0}(\mathbb{R}; H)$ for every $\varrho \geq \varrho_1$. If, in addition, $M_{\infty}$ is causal, then so is $(\partial_0 M_0 (m_0) + M_1 (m_0) + M_{\infty} + A)^{-1}$.

The result follows by adapting the method of proof of Theorem 2.13. The crucial estimate to conclude the proof of Theorem 2.19 is given in the following lemma.

**Lemma 2.20.** Let $\varrho \geq \varrho_0$. Assume that $M_0$ satisfies the properties (iii)-(iv), that $t \mapsto N(M_0(t))$ is time-independent and that the inequalities (2.10)-(2.11) hold. Then for all $\varepsilon \in ]0, c_0[$ there exists $c > 0$ such that for all sufficiently large $\varrho$ and for $u \in D$ and $a \in \mathbb{R}$ we have that

$$\int_{-\infty}^{a} \Re \langle (\partial_0 M_0 (m_0) + M_1 (m_0) + A) u | u \rangle (t) e^{-2\varepsilon t} \, dt$$

$$\geq c \varepsilon \int_{-\infty}^{a} |P_V u(t)|^2 e^{-2\varepsilon t} \, dt + (c_0 - \varepsilon) \int_{-\infty}^{a} |P_V u(t)|^2 e^{-2\varepsilon t} \, dt. \quad (2.12)$$

**Proof.** In order to prove (2.12) observe that by Lemma 2.10 it suffices to verify the inequality for $u \in D(\partial_0) \cap D(A)$. Moreover, by Lemma 2.6 we only need to estimate

$$\frac{1}{2} \langle (u(t) M_0(a) u(t)) e^{-2\varepsilon t} + \int_{-\infty}^{a} \Re M_0(t) u(t) + \frac{1}{2} \dot{M}_0(t) u(t) + \Re M_1(t) u(t) | u(t) \rangle e^{-2\varepsilon t} \, dt.$$  

Since $M_0(a)$ is non-negative, we are reduced to showing an estimate for

$$\langle \dot{M}_0(t) \phi + \frac{1}{2} \dot{M}_0(t) \phi + \Re M_1(t) \phi | \phi \rangle.$$
for all \( t \in \mathbb{R} \) and \( \phi \in H \). Using that \( M_0(t) \) vanishes on \( V = N(M_0(0)) \), we get that

\[
\left\langle \varrho M_0(t) \phi + \frac{1}{2} M_0(t) \phi + \Re M_1(t) \phi \right\rangle \phi \\
= \varrho \langle M_0(t) P_{V^\perp} \phi | P_{V^\perp} \phi \rangle + \frac{1}{2} \langle M_0(t) P_{V^\perp} \phi | P_{V^\perp} \phi + P_V \phi \rangle \\
+ 2 \langle \Re M_1(t) P_{V^\perp} \phi | P_V \phi \rangle + \langle \Re M_1(t) P_{V^\perp} \phi | P_{V^\perp} \phi \rangle \\
\geq \left( \varrho c_1 - \frac{1}{2} |M_0|_{\text{Lip}} \| M_1(m_0) \|_{L(H_{\varrho,0}(\mathbb{R}; H))} \right) |P_{V^\perp} \phi|^2 \\
- \left( |M_0|_{\text{Lip}} + 2 \| M_1(m_0) \|_{L(H_{\varrho,0}(\mathbb{R}; H))} \right) |P_{V^\perp} \phi| |P_V \phi| + c_0 |P_V \phi|^2.
\]

The assertion follows now by applying the trivial inequality \( 2ab \leq \frac{1}{2}a^2 + \delta b^2 \) for \( a, b, \delta > 0 \). \( \square \)

**Proof of Theorem 2.13.** We denote \( B := \partial_0 M_0(m_0) + M_1(m_0) + A + M_\infty \) considered as an operator in \( H_{\varrho,0}(\mathbb{R}; H) \). Note that the maximal domain in \( H_{\varrho,0}(\mathbb{R}; H) \) coincides with \( D \). Moreover, since \( M_\infty \) is continuous, we have, by Theorem 2.13, that

\[
B^* = M_0(m_0) \partial_0^* + M_1(m_0)^* - A + M_\infty^*
\]

with domain being equal to \( D \), by Lemma 2.13. Let \( \varepsilon > 0 \) and choose \( \bar{\varrho} \) such that \( \sup_{\varrho \geq \bar{\varrho}} \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} < \infty \) and

\[
\langle \Re M_\infty P_V u | P_V u \rangle_{H_{\varrho,0}(\mathbb{R}; H)} \geq (\varepsilon - c_0) |P_V u|_{H_{\varrho,0}(\mathbb{R}; H)}^2
\]

for each \( \varrho \geq \bar{\varrho} \) and \( u \in D(M_\infty) \). Note that due to continuity the latter inequality also holds for every \( u \in H_{\varrho,0}(\mathbb{R}; H) \). For \( u \in D \) there exists, according to Lemma 2.20, a constant \( c > 0 \) such that for all sufficiently large \( \varrho \geq \bar{\varrho} \) we have

\[
\Re \langle Bu | u \rangle_{H_{\varrho,0}(\mathbb{R}; H)} = \Re \langle (\partial_0 M_0(m_0) + M_1(m_0) + A) u | u \rangle_{H_{\varrho,0}(\mathbb{R}; H)} + \Re \langle M_\infty u | u \rangle_{H_{\varrho,0}(\mathbb{R}; H)} \\
\geq \varrho c |P_{V^\perp} u|_{H_{\varrho,0}(\mathbb{R}; H)}^2 + \left( c_0 - \frac{\varepsilon}{2} \right) |P_V u|_{H_{\varrho,0}(\mathbb{R}; H)}^2 + (\varepsilon - c_0) |P_V u|_{H_{\varrho,0}(\mathbb{R}; H)}^2 \\
- 2 \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} |P_V u|_{H_{\varrho,0}(\mathbb{R}; H)} |P_{V^\perp} u|_{H_{\varrho,0}(\mathbb{R}; H)} \\
- \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} |P_{V^\perp} u|_{H_{\varrho,0}(\mathbb{R}; H)}^2 \\
\geq \left( \varrho c - \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))} - \frac{1}{\delta} \| M_\infty \|_{L(H_{\varrho,0}(\mathbb{R}; H))}^2 \right) |P_{V^\perp} u|_{H_{\varrho,0}(\mathbb{R}; H)}^2 \\
+ \left( \frac{\varepsilon}{2} - \delta \right) |P_V u|_{H_{\varrho,0}(\mathbb{R}; H)}^2
\]
3 An application to a Kelvin-Voigt-type model in visco-elasticity

for each $0 < \delta < \frac{\varepsilon}{2}$. By possibly increasing $\varrho$ such that

$$\varrho c - \| M_{\infty} \|_{L(H_{\varrho,0}(\mathbb{R};H))} - \frac{1}{\delta} \| M_{\infty} \|_{L(H_{\varrho,0}(\mathbb{R};H))^2} \geq \tilde{c} > 0$$

we deduce that for all $u \in \mathcal{D}$ the estimate

$$\Re \langle B u | u \rangle_{H_{\varrho,0}(\mathbb{R};H)} \geq \tilde{c} | u |^2_{H_{\varrho,0}(\mathbb{R};H)}$$

(2.13)

holds for all sufficiently large $\varrho$. By $\Re \langle B u | u \rangle_{H_{\varrho,0}(\mathbb{R};H)} = \Re \langle B^* u | u \rangle_{H_{\varrho,0}(\mathbb{R};H)}$ and $D(B) = \mathcal{D} = D(B^*)$, we deduce that $B^*$ is one-to-one. Hence, $B$ is continuously invertible and onto. For showing the causality of $B^{-1}$ in case of a causal operator $M_{\infty}$ it suffices to prove, that an inequality of the form

$$\Re \int_{-\infty}^{a} \langle B u(t) e^{-2\varrho t} \rangle dt \geq \tilde{c} \int_{-\infty}^{a} | u(t) |^2 e^{-2\varrho t} dt$$

holds for every $a \in \mathbb{R}$ and $u \in \mathcal{D}$. The latter can be shown as above, observing that due to the causality of $M_{\infty}$ we have

$$\Re \langle M_{\infty} u | \chi_{[-\infty,a]}(m_0) u \rangle_{H_{\varrho,0}(\mathbb{R};H)} = \Re \langle \chi_{[-\infty,a]}(m_0) M_{\infty} u | \chi_{[-\infty,a]}(m_0) u \rangle_{H_{\varrho,0}(\mathbb{R};H)}$$

$$= \Re \langle M_{\infty} \chi_{[-\infty,a]}(m_0) u | \chi_{[-\infty,a]}(m_0) u \rangle_{H_{\varrho,0}(\mathbb{R};H)}.$$

3 An application to a Kelvin-Voigt-type model in visco-elasticity

Although, applications are obviously abundant by simply extending well-known autonomous problems to the time-dependent coefficient case, we intend to give a more explicit application here to illustrate some of the issues that may appear in the non-autonomous case. A more straight-forward application would be for example solving Maxwell’s equations in the presence of a moving body, which reduces via suitable transformations to solving Maxwell’s equations with the body at rest but coefficients depending on time, [3]. Applying the above theory to this case avoids the intricacies of Kato’s method of evolution systems employed in [3]. As a by-product, the assumptions needed are considerably less restrictive.

As a more intricate application we would like to elaborate on here, we consider a time-dependent Kelvin-Voigt material in visco-elasticity. In [2] such a material is considered in connection with modeling a solidifying visco-elastic composite material and discussing homogenization issues. We shall use this as a motivation to analyze well-posedness in the presence of such a material under less restrictive assumptions.

In this model we have the equation

$$\partial_0 \eta(m_0) \partial_0 u - \text{Div} T = f$$
linking stress tensor field $T$ with the displacement vector field $u$, accompanied by a material relation of the form

$$T = (C(m_0) + D(m_0) \partial_0) E$$  \hspace{1cm} (3.1)

where $\text{Div}$ is the restriction of the tensorial divergence operator $\text{div}$ to symmetric tensors of order 2 and $E := \text{Grad} u$ with $\text{Grad}$ denoting the symmetric part of the Jacobian matrix $d \otimes u$ of the displacement vector field $u$. The operators $C, D$ and $\eta$ are thought of as material dependent parameters. Here the case $D(m_0) = 0$ would correspond to purely elastic behavior. Introducing $v := \partial_0 u$ as a new unknown we arrive, by differentiating (3.1), at

$$\partial_0 \eta(m_0) v - \text{Div} T = f$$

$$\partial_0 (C(m_0) + D(m_0) \partial_0)^{-1} T = \text{Grad} v,$$

where we can choose $\varrho$ large enough, such that

$$(C(m_0) + D(m_0) \partial_0)^{-1}$$

gets boundedly invertible. Assuming for sake of definiteness vanishing of the displacement $u$ on the boundary as a boundary condition we obtain an evolutionary equation of the form

$$\begin{pmatrix} \partial_0 (\eta(m_0) & 0 \\ 0 & (C(m_0) + D(m_0) \partial_0)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix} \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$  \hspace{1cm} (3.2)

where the choice of boundary condition amounts to replacing $\text{Grad}$ by the closure $\hat{\text{Grad}}$ of the restriction of $\text{Grad}$ to vector fields with smooth components vanishing outside of a compact subset of $\Omega$. The underlying Hilbert space is the subspace $L^2(\Omega)^3 \oplus L^2_{3\times3,\text{sym}}(\Omega)$ of $L^2(\Omega)^3 \oplus L^2(\Omega)^{3\times3}$ with its natural norm, where the second block-component space $L^2_{3\times3,\text{sym}}(\Omega)$ denotes the restriction of $L^2(\Omega)^{3\times3}$ to symmetric matrices with entries in $L^2(\Omega)$. Note that then $\begin{pmatrix} 0 & -\text{Div} \\ -\hat{\text{Grad}} & 0 \end{pmatrix}$ is skew-selfadjoint (see e.g. [11, Section 5.5.1]).

The operator families $(C(t))_{t \in \mathbb{R}}$ and $(D(t))_{t \in \mathbb{R}}$ are assumed to be uniformly bounded in $L^2_{3\times3,\text{sym}}(\Omega)$. Further constraint will of course be required to satisfy the assumptions of our solution theory above. We are led to a material law operator of the form

$${\mathcal{M}}(\partial_0^{-1}) = \begin{pmatrix} \eta(m_0) & 0 \\ 0 & (C(m_0) + D(m_0) \partial_0)^{-1} \end{pmatrix}.$$  

To deal with the term $(C(m_0) + D(m_0) \partial_0)^{-1}$ we need a projection technique. For this we recall that for a closed subspace $V$ of the underlying Hilbert space $L^2_{3\times3,\text{sym}}(\Omega)$, $\iota_V$ denotes the canonical injection of $V$ into $L^2_{3\times3,\text{sym}}(\Omega)$. The solution theory for the Kelvin-Voigt-type model is then summarized in the following theorem.
3 An application to a Kelvin-Voigt-type model in visco-elasticity

Theorem 3.1. Let \( \Omega \subseteq \mathbb{R}^3 \) be open and \( V \) be a closed subspace of \( L^2_{3 \times 3,\text{sym}}(\Omega) \). Let \( C \in L^\infty_s(\mathbb{R} ; L^2_{3 \times 3,\text{sym}}(\Omega)) \), \( \eta \in L^\infty_s(\mathbb{R} ; L^2(\Omega)^3) \) and \( B \in L^\infty_s(\mathbb{R} ; L(V)) \). Assume that \( C, \eta \) satisfy the properties \((\text{a})-(\text{d})\). We set
\[
D(t) := \begin{pmatrix} B(t) & 0 \\ 0 & 0 \end{pmatrix} \in L(V \oplus V^\perp)
\]
for all \( t \in \mathbb{R} \) and we assume the existence of \( c > 0 \) such that for all \( t \in \mathbb{R} \) we have
\[
\Re B(t) \geq c, \quad \iota_{V \perp}^* C(t) \iota_V \geq c, \quad \eta(t) \geq c.
\]
Then for all sufficiently large \( \varrho \) we have that for all \( F \in H_{\varrho,0}(\mathbb{R} ; L^2(\Omega)^3 \oplus L^2_{3 \times 3,\text{sym}}(\Omega)) \) there exists a unique solution \((v, T) \in H_{\varrho,0}(\mathbb{R} ; L^2(\Omega)^3 \oplus L^2_{3 \times 3,\text{sym}}(\Omega)) \) of the equation
\[
\begin{pmatrix}
\partial_t \begin{pmatrix} \eta(m_0) \\ 0 \end{pmatrix} \\
\begin{pmatrix} C(m_0) + D(m_0) \partial_0 \end{pmatrix}^{-1} \end{pmatrix} + \begin{pmatrix}
0 \\
\begin{pmatrix} -\text{Grad} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} v \\ T \end{pmatrix} = F.
\end{pmatrix}
\]
The solution depends continuously on the data. The solution operator, mapping any right-hand side \( F \) to the corresponding solution of the latter equation, is causal.

Proof. The proof rests on the perturbation result Theorem 2.17. Since the top left corner in the system under consideration clearly satisfies the solvability condition \[(2.22)\], we only have to discuss the lower right corner. For this we have to find a more explicit expression for \((C(m_0) + D(m_0) \partial_0)^{-1}\). An easy computation shows that
\[
(C(m_0) + D(m_0) \partial_0)^{-1} = \begin{pmatrix}
C(m_0) + D(m_0) \partial_0 \\
C(m_0) + D(m_0) \partial_0 \end{pmatrix}^{-1} - C(m_0) \partial_0 - D(m_0) \partial_0
\]
with
\[
W(m_0) = \begin{pmatrix}
C(m_0) + D(m_0) \partial_0 & C(m_0) \partial_0 \\
C(m_0) + D(m_0) \partial_0 & C(m_0) \partial_0
\end{pmatrix}^{-1}.
\]
Denoting \( S(m_0) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), we see that
\[
(C(m_0) + D(m_0) \partial_0)^{-1} = S(m_0)W(m_0)S(m_0)^*.
\]
by the selfadjointness of \( C(m_0) \). Now, the top left corner of \( W(m_0) \) may be expressed with the help of a Neumann expansion in the following way
\[
(B(m_0) \partial_0 + \iota_{V \perp}^* C(m_0) \iota_V - \iota_{V \perp}^* C(m_0) \iota_V \iota_{V \perp}^* (\iota_{V \perp}^* C(m_0) \iota_V)^{-1} \iota_{V \perp}^* C(m_0) \iota_V)^{-1}
\]
\[
= \partial_0^{-1} B(m_0)^{-1} \left( 1 + \partial_0^{-1} B(m_0)^{-1} (\iota_{V \perp}^* C(m_0) \iota_V - \iota_{V \perp}^* C(m_0) \iota_V \iota_{V \perp}^* (\iota_{V \perp}^* C(m_0) \iota_V)^{-1} \iota_{V \perp}^* C(m_0) \iota_V) \right)^{-1}
\]
\[
= \partial_0^{-1} B(m_0)^{-1} + \partial_0^{-1} \tilde{M}_\infty
\]

28
for some suitable $\widetilde{M}_\infty$, satisfying $\|\widetilde{M}_\infty\|_{L(H_{e,0}(\mathbb{R};V \oplus V^\perp))} \to 0$ as $\varrho \to \infty$. Thus, we arrive at

\[
(C(m_0) + D(m_0) \partial_0)^{-1}
= S(m_0) \begin{pmatrix} 0 & 0 \\ 0 & (\iota_{V^\perp}^* C(m_0)\iota_{V^\perp})^{-1} \end{pmatrix} S(m_0)^* + S(m_0) \begin{pmatrix} \partial_0^{-1} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^* + S(m_0) \begin{pmatrix} \partial_0^{-1} \widetilde{M}_\infty & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^*.
\]

The first term on the right-hand side can be computed as follows:

\[
S(m_0) \begin{pmatrix} 0 & 0 \\ 0 & (\iota_{V^\perp}^* C(m_0)\iota_{V^\perp})^{-1} \end{pmatrix} S(m_0)^* = \begin{pmatrix} 0 & 0 \\ 0 & (\iota_{V^\perp}^* C(m_0)\iota_{V^\perp})^{-1} \end{pmatrix}.
\]

For the second and third term on the right-hand side we observe that $S \in L^\infty_s(\mathbb{R}; L(V \oplus V^\perp))$. Moreover, $S$ is Lipschitz-continuous. Indeed, observing that

\[
\|((\iota_{V^\perp}^* C(t)\iota_{V^\perp})^{-1} - (\iota_{V^\perp}^* C(s)\iota_{V^\perp})^{-1}\|_{L(V^\perp)}
\leq \|((\iota_{V^\perp}^* C(t)\iota_{V^\perp})^{-1}\|_{L(V^\perp)}\|\iota_{V^\perp}^* C(s)\iota_{V^\perp}\|_{L(V^\perp)}
\leq \|((\iota_{V^\perp}^* C(t)\iota_{V^\perp})^{-1} - (\iota_{V^\perp}^* C(s)\iota_{V^\perp})^{-1}\|_{L(V^\perp)}\|\iota_{V^\perp}^* C(s)\iota_{V^\perp}\|_{L(V^\perp)}
\leq (c^{-1}|C|_{\text{Lip}} + c^{-2}|C|_{\text{Lip}}|C|_{\infty}) |s - t|,
\]

for $s, t \in \mathbb{R}$, we derive the Lipschitz-continuity of $S$. Thus, $S$ satisfies the hypothesis (4), since $L^2(\mathbb{R}; \mathcal{M})$ is separable. Hence, we can compute the second term on the right-hand side of (3.3) as follows by using the product rule (2.1)

\[
S(m_0) \begin{pmatrix} \partial_0^{-1} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^* = \partial_0^{-1} \partial_0 S(m_0) \begin{pmatrix} \partial_0^{-1} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^* = \partial_0^{-1} S(m_0) \begin{pmatrix} \partial_0^{-1} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^* + \partial_0^{-1} S(m_0) \begin{pmatrix} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^*.
\]

Defining $\tilde{M}_0(m_0) := \begin{pmatrix} 0 & 0 \\ 0 & (\iota_{V^\perp}^* C(m_0)\iota_{V^\perp})^{-1} \end{pmatrix}$, $\tilde{M}_1(m_0) := S(m_0) \begin{pmatrix} B(m_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^*$ and $\tilde{M}_\infty := S(m_0) \begin{pmatrix} \partial_0^{-1} \tilde{M}_\infty & 0 \\ 0 & 0 \end{pmatrix} S(m_0)^*$, we get that

\[
\partial_0 (C(m_0) + D(m_0) \partial_0)^{-1} = \partial_0 \tilde{M}_0(m_0) + \tilde{M}_1(m_0) + \tilde{M}_\infty.
\]
Clearly, $\widetilde{M}_\infty$ can be considered as a perturbation as in Theorem 2.17, since
\[
\|\widetilde{M}_\infty\|_{L(H_\varphi^0(\mathbb{R};V \oplus V^\perp))} \to 0 \quad (\varphi \to \infty).
\]
Moreover, one easily obtains that $\Re \langle \widetilde{M}_1(t) \iota_V \phi | \iota_V \phi \rangle_{V \oplus V^\perp} \geq (c/|B|_{L^\infty}) |\phi|^2_V$ for all $t \in \mathbb{R}$ and $\phi \in V$. Since $\widetilde{M}_0(t)$ is strictly positive on $V^\perp$ and $\widetilde{M}_0(t)$ is uniformly bounded in $t$, we get estimate (2.2) for sufficiently large $\varphi_0$. Theorem 2.17 concludes the proof. \qed

**Remark 3.2.** The assumption on the subspace $V$ in the above theorem expresses the fact that the null space of $D(m_0)$ is non-trivial due to degeneracies in various regions in $\Omega$, but it is implied that these regions do not vary in time. In other words there may be stationary areas in which the material exhibits purely elastic behavior and others showing different forms of visco-elastic behavior.

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