Bilinear Hilbert Transforms and (Sub)Bilinear Maximal Functions along Convex Curves

Junfeng Li and Haixia Yu

Abstract In this paper, we determine the $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedness of the bilinear Hilbert transform $H_\gamma(f, g)$ along a convex curve $\gamma$

$$H_\gamma(f, g)(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x-t)g(x-\gamma(t)) \frac{dt}{t},$$

where $p, q,$ and $r$ satisfy $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$, and $r > \frac{1}{2}, p > 1,$ and $q > 1$. Moreover, the same $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedness property holds for the corresponding (sub)bi linear maximal function $M_\gamma(f, g)$ along a convex curve $\gamma$

$$M_\gamma(f, g)(x) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x-t)g(x-\gamma(t))| \, dt.$$

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2010 Mathematics Subject Classification. Primary 42B20; Secondary 47B38.
Key words and phrases. Bilinear Hilbert transform, (sub)bi linear maximal function, convex curve, time frequency analysis.

Junfeng Li is supported by NSFC-DFG (# 11761131002), Haixia Yu is supported by "the Fundamental Research Funds for the Central University" (# 20lgpy144).

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1 Introduction

1.1 Main problem and main result

The bilinear Hilbert transform $H_\gamma(f, g)$ along a curve $\gamma$ is defined as

$$H_\gamma(f, g)(x) := p.v. \int_{-\infty}^{\infty} f(x - t) g(x - \gamma(t)) \frac{dt}{t}$$

for $f$ and $g$ in the Schwartz class $S(\mathbb{R})$. The corresponding (sub)linear maximal function $M_\gamma(f, g)$ is defined as

$$M_\gamma(f, g)(x) := \sup_{\epsilon > 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - t) g(x - \gamma(t))| dt.$$ 

The $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedness property for these two operators with some general curves $\gamma$ are of great interest to us. We start with a special case $\gamma := P$, a polynomial of degree $d$ with no linear term and constant term, where $d \in \mathbb{N}$ and $d > 1$. In [37], Li and Xiao set up the $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedness for $H_\gamma(f, g)$ and $M_\gamma(f, g)$ where $p, q, r$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $r > \frac{d+1}{d-1}$, $p > 1$, and $q > 1$. Moreover, they showed that $r > \frac{d+1}{d-1}$ is sharp up to the end point. By replacing $\gamma$ with a homogeneous curve $t^d$ with $d \in \mathbb{N}$, $d > 1$, the range of $r$ was extended by Li and Xiao to $r > \frac{1}{2}$. Furthermore, they believe that with some special conditions on $\gamma$, the full range boundedness of $H_\gamma(f, g)$ and $M_\gamma(f, g)$ can be obtained. Here and hereafter, we omit the relationship that $p, q$, and $r$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $p > 1$, and $q > 1$ and the fact that $d > 1$ and $d \in \mathbb{N}$.

We call the full range bounded if the range of $r$ is $(\frac{1}{2}, \infty)$. In this paper, we provide some sufficient conditions of $\gamma$ regarding this concern.

**Theorem 1.1.** Let $\gamma \in C^3(\mathbb{R})$ be either odd or even, with $\gamma(0) = \gamma'(0) = 0$, $\lim_{t \to 0^+} \gamma'(t)/\gamma''(t) = 0$, and convex on $(0, \infty)$. Furthermore, if

\begin{equation}
\text{(1.1) there exist positive constants } C_1 \text{ and } C_2 \text{ such that } C_1 \leq \left(\frac{\gamma'}{\gamma''}\right)'(t) \leq C_2 \text{ on } (0, \infty). \tag{1.1}
\end{equation}

Then, there exists a positive constant $C$ such that

$$\|H_\gamma(f, g)\|_{L^r(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.$$
for any \( f \in L^p(\mathbb{R}) \) and \( g \in L^q(\mathbb{R}) \), where \( p, q, r \) satisfy \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), and \( r > \frac{1}{q}, p > 1, q > 1 \).

**Theorem 1.2.** Under the same conditions of \( \gamma \), we have

\[
\|M_\gamma(f, g)\|_{L^r(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.
\]

**Remark 1.3.** It is easy to see that \( M_\gamma(f, g) \) does not map \( L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \). Moreover, it is trivial that \( M_\gamma(f, g) \) is bounded from \( L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \). Therefore, we can restrict the range of \( r \) as \( \frac{1}{q}, \infty \) in the rest of the paper.

**Remark 1.4.** The following are some curves satisfying the conditions of Theorem 1.1; we here write only the part for \( t \in [0, \infty) \) based on its odd or even property:

(i) for any \( t \in [0, \infty) \), \( \gamma_1(t) := t^\alpha \) under \( \alpha \in (1, \infty) \);

(ii) for any \( t \in [0, \infty) \), \( \gamma_2(t) := t^\alpha \log(1 + t) \) under \( \alpha \in (1, \infty) \);

(iii) for any \( t \in [0, \infty) \) and \( K \in \mathbb{N} \), \( \gamma_3(t) := \sum_{i=1}^{K} t^{\alpha_i} \) under \( \alpha_i \in (1, \infty) \) for all \( i = 1, 2, \ldots, K \).

**Remark 1.5.** For a more general curve \( \gamma \), Lie [39] introduced a set \( NF^C \) and obtained the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) boundedness of \( H_\gamma(f, g) \) for \( \gamma \in NF^C \). Later, it was extended to the \( L^p(\mathbb{R}) \times L^q(\mathbb{R}) \rightarrow L^r(\mathbb{R}) \) boundedness with \( r \geq 1 \) in [40]. Furthermore, Gaitan and Lie [20] obtained the same boundedness for \( M_\gamma(f, g) \). It is worth noting that these results are sharp in the sense that we cannot take \( \frac{1}{q} < r < 1 \), since the polynomial \( P \) stated in [37] belongs to the set \( NF^C \). More recently, Guo and Xiao [24] obtained the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) boundedness of \( H_\gamma(f, g) \) and \( M_\gamma(f, g) \), where \( \gamma \in F(-1, 1) \); the definition of the set \( F(-1, 1) \) can be found on P. 970 in [24].

The argument of this paper is based on the works of Guo and Xiao [24], Li [34], Li and Xiao [37] and Lie [39, 40], but we also make several contributions:

- Our conditions may be easier to check than \( NF^C \) in Lie [39, 40] and \( F(-1, 1) \) in Guo and Xiao [24]. Moreover, we require less regularity on \( \gamma \) than that in [39, 40, 24]. On the other hand, we only require that the curve \( \gamma \) belongs to \( C^3 \), but \( NF^C \subset C^4 \) and \( F(-1, 1) \subset C^5 \).

- We obtain the full range boundedness for \( H_\gamma(f, g) \) and \( M_\gamma(f, g) \). Therefore, our results extend the results of Li [34] and Li and Xiao [37, Theorem 3], which concern the homogeneous curve \( \gamma(t) := t^d \), to more general classes of curves.

- The main difference between this paper and the abovementioned works is a partition of unity. We split our multiplier by the following partition of unity; i.e.,

\[
\sum_{m,n,k \in \mathbb{Z}} \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^n} \right) \phi \left( \frac{\gamma (2^{-j})}{2^{n+j-k}} \right) = 1,
\]

see (2.25), instead of

\[
\sum_{m,n \in \mathbb{Z}} \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta (2^{-j})}{2^{n+j}} \right) = 1,
\]
where $\phi$ is a standard bump function supported on $\{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\}$ such that $0 \leq \phi(t) \leq 1$ and $\sum_{k \in \mathbb{Z}} \phi(2^{-k}t) = 1$ for all $t \neq 0$. The aim of this partition of unity (2.25) is to avoid using uniform paraproduct estimates at the low-frequency part. In [34], Li used a uniform paraproduct estimate, i.e., [34, Theorem 4.1], to bound the low-frequency part. In this paper, we present an easy way to dispose of the low-frequency part by using this partition of unity and the Littlewood-Paley theory together with the uniform estimates (3.4) and (3.5).

1.2 Background and motivation

There are rich backgrounds from which to study the boundedness property of $H_\gamma(f, g)$ and $M_\gamma(f, g)$.

♦ If we take $\gamma(t) := t$, the boundedness of these two operators is trivial. This follows from the boundedness of the classical Hilbert transform, the Hardy-Littlewood maximal function and the Hölder inequality.

♦ If $\gamma(t) := -t$, these operators turn out to be the standard bilinear Hilbert transform and the corresponding (sub)bilinear maximal function whose boundedness is not easy to obtain. Lacey and Thiele [30, 31] obtained the boundedness with $r > \frac{3}{2}$ for the standard bilinear Hilbert transform. For the same boundedness of the corresponding maximal function, we refer to Lacey [29]. In the same paper, a counterexample showed that if $\frac{1}{2} < r < \frac{2}{3}$ the boundedness fails for these operators.

♦ If we take $\gamma(t) := r^d$ or $\gamma := P$ or a more general curve $\gamma$, the boundedness of these two operators has been stated at the beginning of this paper and in Remark 1.5, where $P$ is a polynomial of degree $d$ with no linear term and constant term.

♦ There are some other types of bilinear Hilbert transforms. Let

$$H_{\alpha,\beta}(f, g)(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x - \alpha t)g(x - \beta t) \frac{dt}{t}.$$  

Grafakos and Li [23] set up the uniform boundedness for $r > 1$. Later, Li [35] extended the index to $r > \frac{2}{3}$. Recently, Dong [12] considered the bilinear Hilbert transform $H_{P,Q}(f, g)$ along two polynomials

$$H_{P,Q}(f, g)(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x - P(t)) g(x - Q(t)) \frac{dr}{t}$$

and the corresponding maximal operator $M_{P,Q}(f, g)$

$$M_{P,Q}(f, g)(x) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x - P(t)) g(x - Q(t))| dt.$$  

Here, $P$ and $Q$ are polynomials with no constant term. Dong proved that these operators are bounded for $r > \frac{d}{d+1}$, where $d$ is the correlation degree of these two polynomials $P$ and $Q$. For the definition of the correlation degree, we refer the reader to P. 2 in [12]. There are many other related works; see, for example, [11, 13, 14, 15, 18, 32].
The study of the boundedness of $H_\gamma(f, g)$ and $M_\gamma(f, g)$ originated from Calderón [4] in order to study the Cauchy transform along Lipschitz curves, but there have also been many other motivations:

- One of the motivations arises from ergodic theory. For instance, for $n \in \mathbb{N}$, the $L^r(\mathbb{R})$-norm convergence property of the non-conventional bilinear averages

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n) g(T^{n^2})$$

as $N$ tends to $\infty$. Here, $T$ is an invertible and measure-preserving transformation of a finite measure space. For more details, we refer to [10, 19, 27].

- Another motivation is offered by number theory. There are many various nonlinear extensions of Roth’s theorem for some sets with positive density; see, for example, [2, 3, 16].

- There have also been many developments during the last few years regarding the Hilbert transform $H_\gamma f$ along the curve $\gamma$ defined as

$$H_\gamma f(x_1, x_2) := \text{p. v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{dt}{t},$$

and the corresponding maximal function $M_\gamma f$ along the curve $\gamma$ defined as

$$M_\gamma f(x) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - \gamma(t))| \, dt.$$ 

These operators were initiated by Fabes and Rivière [17] and Jones [28] in order to understand the behavior of the constant-coefficient parabolic differential operators. Later, $H_\gamma f$ and $M_\gamma f$ were extended to cover more general classes of curves [5, 6, 7, 9, 41, 43]. $H_\gamma (f, g)$ is closely associated with $H_\gamma f$ since they have the same multiplier. Indeed, we can rewrite $H_\gamma (f, g)(x)$ as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} \left( \text{p. v.} \int_{-\infty}^{\infty} e^{-i\xi t} e^{-i\eta y(t)} \frac{dt}{t} \right) \, d\xi \, d\eta$$

and $H_\gamma f(x_1, x_2)$ can be rewritten as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi, \eta) e^{i\xi x_1} e^{i\eta y_2} \left( \text{p. v.} \int_{-\infty}^{\infty} e^{-i\xi t} e^{-i\eta y(t)} \frac{dt}{t} \right) \, d\xi \, d\eta.$$

Therefore, we can find many similarities between the approaches of $H_\gamma (f, g)$ and $H_\gamma f$.

### 1.3 Organization and notations

We now present the structure of the rest of this paper.
Acknowledgments

Y an for the many valuable comments and helpful discussions.

In Section 2 we give some preliminaries for our proof. Subsection 2.1 provides two inequalities about the curve γ, which will be used repeatedly in our proof. Subsection 2.2 is devoted to splitting \( H_\gamma(f, g) \) into the following three parts: the low-frequency part \( H^1_{\gamma}(f, g) \); the high-frequency part away from the diagonal part \( H^2_{\gamma}(f, g) \); and the high-frequency part near the diagonal part \( H^3_{\gamma}(f, g) \); see (2.32) and (2.33). At the same time, we split \( \frac{1}{\gamma} = \sum_{j \in \mathbb{Z}} 2j^2(2^j) \) and set up a uniform boundedness for each \( H_{\gamma,j}(f, g) \), where the corresponding kernel is \( 2^j(2^j) \). A similar decomposition of \( M_\gamma(f, g) \) can be found in Subsection 2.3. The second author would like to thank his postdoctoral advisor Prof. Lixin Yan for the many valuable comments and helpful discussions.

In Section 3, we establish the full range boundedness for \( H^2_{\gamma}(f, g) \). To this aim, we consider two cases according to the function that has the higher frequency.

In Section 4, we obtain the full range boundedness for \( H^1_{\gamma}(f, g) \) by using Taylor series expansion.

In Section 5, we prove the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) boundedness of \( H^3_{\gamma}(f, g) \). To this aim, we obtain a \( 2^{-\epsilon_{0m}} \) decay for \( H_m(f, g) \) (see (6.2)) defined on the frequency piece along the diagonal for some positive constants \( \epsilon_0 \). Here, we used the \( TT^* \) theorem [26, Theorem 1.1], the stationary phase method and \( \sigma \)-uniformity.

In Section 6, we obtain the full range weak-\( L^p(\mathbb{R}) \times L^q(\mathbb{R}) \rightarrow L'(\mathbb{R}) \) boundedness for \( H_m(f, g) \) with a bound \( m \). By interpolation with the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) estimate that has a decay bound of \( 2^{-\epsilon_{0m}} \), we can finish our proof. To obtain the weak boundedness, we need to split \( H_m(f, g) \) into the following three error terms: \( |H^{(1)}_m(f, g)| \), \( |H^{(2)}_m(f, g)| \), and \( |H^{(3)}_m(f, g)| \), see (6.8), and the major term \( |H^{(4)}_m(f, g)| \) in (6.9). Subsection 6.1 describes the estimation for these three error terms, and Subsection 6.2 is devoted to establishing the major term \( |H^{(4)}_m(f, g)| \) by using the method of time frequency analysis, which is the most difficult part.

In Section 7, we set up the full range boundedness of \( M_\gamma(f, g) \).

Throughout this paper, we denote by \( C \) a positive constant that is independent of the main parameters involved, whose exact value is allowed to change from line to line. The positive constants with subscripts, \( C_1 \) and \( C_2 \), are fixed constants. The symbol \( a \leq b \) or \( b \approx a \) means that there exists a positive constant \( C \) such that \( a \leq Cb \). \( a \approx b \) means \( a \leq b \) and \( b \leq a \). We use \( S(\mathbb{R}) \) to denote the Schwartz class on \( \mathbb{R} \). Let \( \mathbb{Z}_+ := \mathbb{Z} \setminus \mathbb{N} \) with \( \mathbb{N} := \{0, 1, 2, \ldots\} \). \( \hat{f} \) denotes the Fourier transform of \( f \), and \( \hat{\hat{f}} \) is the inverse Fourier transform of \( f \). For any \( 0 < p < \infty \), we denote \( p' \) as its conjugate index if \( \frac{1}{p} + \frac{1}{p'} = 1 \). It is obvious that \( p' < 0 \) if \( 0 < p < 1 \). For any set \( E \), we use \( \chi_E \) to denote its characteristic function. \( \#E \) denotes the cardinality of it. \( E^C \) indicates its complementary set.

Acknowledgments. The second author would like to thank his postdoctoral advisor Prof. Lixin Yan for the many valuable comments and helpful discussions.
2 Preliminaries

2.1 The curve \( \gamma \)

We first explain some simple properties of the curve \( \gamma \) in Theorem 1.1 which will be used in this paper. Since \( \gamma \in C^3(\mathbb{R}) \), \( \gamma(0) = \gamma'(0) = 0 \) and \( \gamma \) is convex on \((0, \infty)\), we have \( \gamma''(t) \geq 0 \) on \((0, \infty)\). By (2.1), we obtain \( \gamma''(t) \neq 0 \) on \((0, \infty)\) and thus \( \gamma''(t) > 0 \) on \((0, \infty)\). Therefore, \( \gamma' \) is strictly increasing and \( \gamma'(t) > 0 \) on \((0, \infty)\). On the other hand, let us set \( G_1(t) := 2C_2t - \gamma'(t)/\gamma''(t) \) and \( G_2(t) := \gamma'(t)/\gamma''(t) - C_1t/2 \), by (2.1), we then have \( G_1'(t) \geq C_2 \) and \( G_2'(t) \geq C_1/2 \) on \((0, \infty)\). This, combined with \( \lim_{t \to 0^+} \gamma'(t)/\gamma''(t) = 0 \), leads to \( G_1(t) \geq 0 \) and \( G_2(t) \geq 0 \) on \((0, \infty)\) and therefore

\[
2.1 \quad \frac{1}{2C_2} \leq \frac{t\gamma''(t)}{\gamma'(t)} \leq \frac{2}{C_1}, \quad \text{for any } t \in (0, \infty).
\]

Since \( \gamma(0) = \gamma'(0) = 0 \), by the Cauchy mean value theorem, for any \( t \in (0, \infty) \), there exists \( \tau_1 \in (0, t) \) such that \( \frac{t\gamma'(t)}{\gamma'(0)} = \frac{\gamma'(t) - \gamma'(0)}{\tau_1 - 0} = \frac{\gamma'(\tau_1)}{\gamma'(0)} \). Thus, by (2.1),

\[
2.2 \quad 1 + \frac{1}{2C_2} \leq \frac{t\gamma'(t)}{\gamma'(0)} \leq 1 + \frac{2}{C_1}, \quad \text{for any } t \in (0, \infty),
\]

which further implies that \( \gamma(t)/t \) is strictly increasing on \((0, \infty)\).

Let \( G_3(t) := \log_2 \gamma'(t) \) for any \( t \in (0, \infty) \), by (2.1), we then have \( 1/2C_2t \leq G_3'(t) \leq 2/C_1t \) for any \( t \in (0, \infty) \). By the Lagrange mean value theorem, there exists a constant \( \theta \in [1, 2] \) such that \( G_3(2t) - G_3(t) = G_3'(\theta t)t \in [1/4C_2, 2/C_1] \), which further leads to

\[
2.3 \quad 2^{\frac{1}{4C_2}} \leq \frac{\gamma'(2t)}{\gamma'(t)} \leq 2^{\frac{1}{2C_1}}, \quad \text{for any } t \in (0, \infty).
\]

By the Cauchy mean value theorem and \( \gamma(0) = 0 \), for any \( t \in (0, \infty) \), there exists \( \tau_2 \in (0, t) \) such that \( \frac{\gamma(2t)}{\gamma(0)} = \frac{\gamma(2\tau_2)}{\gamma(\tau_2)} \). This, combined with (2.3), implies

\[
2.4 \quad 2^{1 + \frac{1}{4C_2}} \leq \frac{\gamma(2t)}{\gamma(t)} \leq 2^{1 + \frac{1}{2C_1}}, \quad \text{for any } t \in (0, \infty).
\]

2.2 Decomposition of \( H_\gamma(f, g) \)

For \( H_\gamma(f, g) \), we rewrite it as

\[
H_\gamma(f, g)(x) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta)e^{i\xi x}e^{i\eta y}m(\xi, \eta) \, d\xi \, d\eta,
\]

where

\[
2.6 \quad m(\xi, \eta) := \text{p.v.} \int_{-\infty}^{\infty} e^{-i\xi t}e^{-i\eta y(t)} \frac{dt}{t}.
\]

Let \( \rho \) be an odd smooth function supported on \( \{ t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2 \} \) such that \( \frac{1}{t} = \sum_{j \in \mathbb{Z}} 2^j \rho(2^j t) \). Then,

\[
2.7 \quad m(\xi, \eta) = \sum_{j \in \mathbb{Z}} m_j(\xi, \eta),
\]
where

\begin{equation}
(2.8) \quad m_j(\xi, \eta) := \int_{-\infty}^{\infty} e^{-i2^j \xi t} e^{-i\eta(2^j t)} \rho(t) \, dt.
\end{equation}

Therefore, we split \( H_\gamma(f, g) \) as

\[ H_\gamma(f, g)(x) = \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta \gamma(2^j t)} m_j(\xi, \eta) \, d\xi \, d\eta =: \sum_{j \in \mathbb{Z}} H_{\gamma, j}(f, g)(x). \]

From Proposition 2.2 below, we need only to consider that \(|j| \) large enough. Before giving Proposition 2.2, we first state the following lemma which can be found in [25, Lemma 4.7].

**Lemma 2.1.** Let \( I \subset \mathbb{R} \) be an interval, \( k \in \mathbb{N} \), \( f \in C^k(I) \), and suppose that for some \( \sigma > 0 \), \(|j^{(k)}(x)| \geq \sigma \) for all \( x \in I \). Then there exists a positive constant \( C \) depending only on \( k \) such that

\[ \|x \in I : |f(x)| \leq \rho\| \leq C \left( \frac{p}{\sigma} \right)^\frac{1}{k} \]

for all \( \rho > 0 \).

**Proposition 2.2.** Let \( \gamma \) and \( p, q, r \) be the same as in Theorem 1.1. Then there exists a positive constant \( C \) independent of \( j \) such that

\[ \left\| H_{\gamma, j}(f, g) \right\|_{L^2(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})} \]

for all \( f \in L^p(\mathbb{R}) \) and \( g \in L^q(\mathbb{R}) \).

**Proof.** For \( r \geq 1 \), by the Minkowski inequality and Hölder inequality, we have

\[ \|H_{\gamma, j}(f, g)\|_{L^r(\mathbb{R})} \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|f(-t)\|_r^\frac{2}{r} \|g(-\gamma(t))\|_r^\frac{2}{r} 2^j |\rho(2^j t)| \, dt \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}. \]

For \( \frac{1}{2} < r < 1 \), we rewrite \( H_{\gamma, j}(f, g) \) as

\[ H_{\gamma, j}(f, g)(x) = \int_{-\infty}^{\infty} f(x - 2^{-j} t) g(x - \gamma(2^{-j} t)) \rho(t) \, dt. \]

Let

\[ \begin{aligned}
A_1 & := \{ 2^{-j} t : t \in \text{supp } \rho \}; \\
A_\gamma & := \{ \gamma(2^{-j} t) : t \in \text{supp } \rho \}; \\
E_\lambda & := \{ t \in \text{supp } \rho : 2^l < 2^{-j} |\gamma(2^{-j} t)| \leq 2 \cdot 2^l \},
\end{aligned} \]

where \( l \in \mathbb{Z} \). Furthermore, let \( A_1(\lambda) := \{ 2^{-j} t : t \in E_\lambda \} \) and \( A_\gamma(\lambda) := \{ \gamma(2^{-j} t) : t \in E_\lambda \} \); we have

\begin{equation}
(2.9) \quad H_{\gamma, j}(f, g)(x) = \sum_{\lambda \in \mathbb{Z}} \int_{E_\lambda} f(x - 2^{-j} t) g(x - \gamma(2^{-j} t)) \rho(t) \, dt =: \sum_{\lambda \in \mathbb{Z}} H_{\gamma, j}^\lambda(f, g)(x).
\end{equation}

To estimate \( H_{\gamma, j}(f, g) \), we consider the following three cases:

\[ 2^l \geq 2 \cdot 2^{-j}, \quad 2^l \leq \frac{1}{4} 2^{-j} \quad \text{and} \quad \frac{1}{2} 2^{-j} \leq 2^l \leq 2^{-j}. \]
Case I: $2^j \geq 2 \cdot 2^{-j}$

We observe that

$$|A_1(I)| = 2^{-j}|E_t| \leq 2^4|E_t| \quad \text{and} \quad |A_2(I)| \leq 2 \cdot 2^4|E_t|.$$ 

Without loss of generality, we may restrict $x \in I_1$ of length $2 \cdot 2^4|E_t|$. By the Cauchy-Schwarz inequality, we have

$$\left\| H^4_{\gamma, t}(f, g) \right\|_{L^2_+(\mathbb{R})} \leq |I_1|^{\frac{1}{2}} \left( \int_{I_1} \int_{E_t} |f(x - 2^{-j}t)g(x - \gamma(2^{-j}t))| \, dt \, dx \right)^{\frac{1}{2}}. \quad (2.10)$$

We change the variables $u := x - 2^{-j}t$ and $v := x - \gamma(2^{-j}t)$, and

$$\left| \frac{\partial(u, v)}{\partial(x, t)} \right| = |2^{-j} - 2^{-j}\gamma'(2^{-j}t)| \geq 2^4 - \frac{2^4}{2} = \frac{2^4}{2}$$

for all $t \in E_t$. Note that $|I_1| = 2 \cdot 2^4|E_t|$; we can control the last term in (2.10) by

$$\left( \frac{|I_1|}{2^{4}} \right)^{\frac{1}{2}} \left( \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \right)^{\frac{1}{2}} \approx |E_t|^{\frac{1}{2}} \left( \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \right)^{\frac{1}{2}}.$$

Henceforth, in this case,

$$\left\| H^4_{\gamma, t}(f, g) \right\|_{L^2_+(\mathbb{R})} \leq |E_t| \cdot \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}. \quad (2.11)$$

For $E_t$, by (2.1), we have

$$2^{-2j}y''(2^{-j}t) = 2^{-j} \left| \frac{2^{-j}ty''(2^{-j}t)}{\gamma'(2^{-j}t)} \cdot \frac{\gamma'(2^{-j}t)}{t} \right| \geq 2^{-j}y' \left( \frac{1}{2^{2j}} \right)$$

for $t \in \text{supp } \rho$. By Lemma 2.1 it implies

$$|E_t| \leq \frac{2^4}{2^{-j}y' \left( \frac{1}{2^{2j}} \right)}. \quad (2.12)$$

On the other hand, since $\gamma'$ is strictly increasing on $(0, \infty)$, we have

$$2^4 \leq 2^{-j}y'(2 \cdot 2^{-j}). \quad (2.13)$$

From (2.12), (2.13) and (2.3), we obtain

$$\sum_{j \in \mathbb{Z}: 2^j \geq 2 \cdot 2^{-j}} |E_t|^\frac{1}{2} \leq \left[ \frac{1}{2^{-j}y' \left( \frac{1}{2^{2j}} \right)} \right]^\frac{1}{2} \sum_{j \in \mathbb{Z}: 2^j \geq 2 \cdot 2^{-j}} 2^t \leq \left[ \frac{2^{-j}y'(2 \cdot 2^{-j})}{2^{-j}y' \left( \frac{1}{2^{2j}} \right)} \right]^\frac{1}{2} \lesssim 1. \quad (2.14)$$

Therefore, from (2.11), (2.14), we have

$$\left\| \sum_{j \in \mathbb{Z}: j \geq 2 \cdot 2^{-j}} H^4_{\gamma, t}(f, g) \right\|_{L^2_+(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}. \quad (2.15)$$
As in (2.14), we have \( \sum_{\lambda \in \mathbb{Z}} 2^{j+1/2} |E_{\lambda}| \leq 1 \), by the Hölder inequality, for all \( p > 1 \),

\[
(2.16) \quad \left\| \sum_{\lambda \in \mathbb{Z}} H_{\gamma,j}^\lambda(f, g) \right\|_{L^1(\mathbb{R})} \leq \sum_{\lambda \in \mathbb{Z}} |E_{\lambda}| \cdot \|f\|_{L^p(\mathbb{R})} \cdot \|g\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \cdot \|g\|_{L^{p'}(\mathbb{R})}.
\]

By interpolation between (2.15) and (2.16), for any \( \frac{1}{r} < 1 \), we obtain

\[
(2.17) \quad \left\| \sum_{\lambda \in \mathbb{Z}} H_{\gamma,j}^\lambda(f, g) \right\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \cdot \|g\|_{L^{p'}(\mathbb{R})}.
\]

**Case II:** \( 2^j \leq \frac{1}{4} 2^{-j} \)

Noting that

\[ |A_{\gamma}(\lambda)| \leq 2 \cdot 2^j |E_{\lambda}| \leq \frac{1}{2} 2^{-j} |E_{\lambda}| = \frac{1}{2} |A_1(\lambda)| \leq 2^{-j} |E_{\lambda}|, \]

we restrict \( x \) in an interval \( I_{\lambda} \) of length \( 2 \cdot 2^{-j} |E_{\lambda}| \). On the other hand, we have

\[
|\partial(u, v)| \left| \partial(x, t) \right| = |2^{-j} - 2^{-j} \gamma'(2^{-j} t)| \geq 2^{-j} - 2 \cdot 2^j \geq \frac{1}{2} 2^{-j}
\]

for all \( t \in E_{\lambda} \). As in Case I, we also have

\[
\left\| H_{\gamma,j}^\lambda(f, g) \right\|_{L^2(\mathbb{R})} \leq |E_{\lambda}| \cdot \|f\|_{L^1(\mathbb{R})} \cdot \|g\|_{L^1(\mathbb{R})}.
\]

Furthermore,

\[
(2.18) \quad \left\| \sum_{\lambda \in \mathbb{Z}} H_{\gamma,j}^\lambda(f, g) \right\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \cdot \|g\|_{L^{p'}(\mathbb{R})}
\]

for all \( \frac{1}{r} < 1 \).

**Case III:** \( \frac{1}{4} 2^{-j} \leq 2^j \leq 2^{-j} \)

We are free to assume that \( 2^{-j} \gamma'(2^{-j} t) > 0 \). Otherwise, it can be handled exactly in the same way as Case II, since

\[ |A_{\gamma}(\lambda)| \leq 2 \cdot 2^{-j} |E_{\lambda}|, \quad |A_1(\lambda)| = 2^{-j} |E_{\lambda}| \]

and

\[
|\partial(u, v)| \left| \partial(x, t) \right| = |2^{-j} - 2^{-j} \gamma'(2^{-j} t)| \geq 2^{-j}
\]

for \( t \in E_{\lambda} \). We now consider

\[
\mathbb{H}_{\gamma,j}(f, g)(x) := \int_{E_{\gamma}} f(x - 2^{-j} t) g(x - \gamma(2^{-j} t)) \rho(t) \, dt,
\]

where \( E_{\gamma} := \{ t \in \supp \rho : \frac{1}{4} < \gamma'(2^{-j} t) \leq 2 \} \). Let \( E_{\gamma}(h) := \{ t \in E_{\gamma} : 2^h < |\gamma'(2^{-j} t) - 1| \leq 2 \cdot 2^h \} \).

By simple calculation, we know that \( E_{\gamma} \subset \bigcup_{h \in \mathbb{Z}} E_{\gamma}(h) \). Let

\[
(2.19) \quad \mathbb{H}^h_{\gamma,j}(f, g)(x) := \int_{E_{\gamma}(h)} f(x - 2^{-j} t) g(x - \gamma(2^{-j} t)) \rho(t) \, dt.
\]
Noting that $|\{2^{-j}t : t \in E_x(h)\}| = 2^{-j}|E_x(h)|$ and $|\{\gamma(2^{-j}t) : t \in E_x(h)\}| \lesssim 2^{-j}|E_x(h)|$, without loss of generality, we restrict $x \in I_h$ of length $2^{-j}|E_x(h)|$. By the Hölder inequality, for all $p > 1$, we can bound $\|\mathbb{H}^h_{\gamma,j}(f, g)\|_{L^p(R)}$ by

\[
\int_{I_h} \left| \int_{E_x(h)} f(x - 2^{-j}t) g(x - \gamma(2^{-j}t)) \rho(t) \, dt \right| \, dx \leq |E_x(h)| \cdot \|f\|_{L^p(R)} \|g\|_{L^p(R)}.
\]

The Cauchy-Schwarz inequality allows us to obtain

\[
\left\| \mathbb{H}^h_{\gamma,j}(f, g) \right\|_{L^p_{\gamma,j}(R)} \leq |I_h|^{\frac{1}{2}} \left( \int_{I_h} \left( \int_{E_x(h)} |f(x - 2^{-j}t) g(x - \gamma(2^{-j}t))| \, dt \right)^2 \, dx \right)^{\frac{1}{2}}.
\]

Let $u := x - 2^{-j}t$ and $v := x - \gamma(2^{-j}t)$, noting that $|\frac{\partial(v)}{\partial(x)}| = |2^{-j} - 2^{-j}\gamma'(2^{-j}t)| \geq 2^h 2^{-j}$ for all $t \in E_x(h)$ and $|I_h| = 2^{-j}|E_x(h)|$, we control the last term in (2.21) by $(\frac{|E_x(h)|}{2^h})^\frac{1}{2}(\|f\|_{L^p(R)} \|g\|_{L^p(R)})^{\frac{1}{2}}$. Henceforth, in this case,

\[
\left\| \mathbb{H}^h_{\gamma,j}(f, g) \right\|_{L^p_{\gamma,j}(R)} \leq \frac{|E_x(h)|}{2^h} \|f\|_{L^p(R)} \|g\|_{L^p(R)}.
\]

Noting that $|E_x(h)| \leq 4$, from (2.20) and (2.22), by interpolation, we obtain the boundeness of $\mathbb{H}^h_{\gamma,j}(f, g)$ for all $\frac{1}{r} < r < 1$ for the case that $h = -1$. Noting that $E_x \subset \bigcup_{h \in \mathbb{Z}_-} E_x(h)$, in what follows, we will focus on the second case; i.e., $h \in \mathbb{Z}_-$ and $h < -1$.

From the fact that $\gamma'$ is strictly increasing on $(0, \infty)$ and (2.21), we have $2^{-j}|\gamma''(2^{-j}t)| \geq \gamma'(\frac{1}{2}2^{-j})$ for $t \in E_x(h)$. By Lemma 2.1 we have $|E_x(h)| \leq \frac{2^h}{\gamma'(2^{-j})}$. Furthermore, we have

\[
\gamma'(2 \cdot 2^{-j}) \geq |\gamma''(2^{-j}t)| = |\gamma'(2^{-j}t) - 1| \geq 1 - 2 \cdot 2^h \geq \frac{1}{2}
\]

for $t \in E_x(h)$. Therefore, (2.23) implies $|E_x(h)| \leq \frac{2^h}{\gamma'(2^{-j})} \frac{\gamma'(2^{-j})}{\gamma'(2^{-j})} \leq 2^h$. By interpolation (2.20) and (2.22), there exists a positive constant $e$ independent of $j$ such that

\[
\left\| \mathbb{H}^h_{\gamma,j}(f, g) \right\|_{L^p(R)} \leq 2^{eh} \|f\|_{L^p(R)} \|g\|_{L^p(R)}
\]

for $\frac{1}{r} < r < 1$, which leads to

\[
\left\| \sum_{h \in \mathbb{Z}_-} \mathbb{H}^h_{\gamma,j}(f, g) \right\|_{L^p(R)} \leq \|f\|_{L^p(R)} \|g\|_{L^p(R)}.
\]

Putting all the estimates together, we finish the proof of Proposition 2.2. \hfill $\Box$

Let $\phi$ be a standard bump function supported on $\{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\}$ such that $0 \leq \phi(t) \leq 1$ and $\sum_{k \in \mathbb{Z}} \phi(2^{-j}t) = 1$ for all $t \neq 0$. We decompose the unity as

\[
\sum_{m,n,k \in \mathbb{Z}} \phi \left( \frac{m}{2^{m+j}} \right) \phi \left( \frac{n}{2^k} \right) \phi \left( \frac{\gamma'(2^{-j})}{2^{n+j-k}} \right) = 1
\]
for all $\xi, \eta \neq 0$. Let

\begin{equation}
(2.26) \quad m_{j,m,n,k}(\xi, \eta) := m_j(\xi, \eta) \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^{n+j}} \right) \phi \left( \frac{\gamma(2^{-j})}{2^{n+j}} \right).
\end{equation}

Then

\begin{equation}
(2.27) \quad m_j(\xi, \eta) = \sum_{m,n,k \in \mathbb{Z}} m_{j,m,n,k}(\xi, \eta).
\end{equation}

We denote the diagonal as

\[ \triangle := \{(m, n) \in \mathbb{Z}^2 : m, n \geq 0, |m - n| \leq 2/C_1 + 1\}, \]

and split $m_j$ as the following three parts:

\begin{equation}
(2.28) \quad m_j(\xi, \eta) = m_j^1(\xi, \eta) + m_j^2(\xi, \eta) + m_j^3(\xi, \eta).
\end{equation}

Accordingly, the low-frequency part $m_j^1$ is

\begin{equation}
(2.29) \quad m_j^1(\xi, \eta) := \sum_{(m,n) \in (\mathbb{Z}_-)^2, k \in \mathbb{Z}} m_{j,m,n,k}(\xi, \eta).
\end{equation}

The high-frequency part away from the diagonal part $m_j^2$ is

\begin{equation}
(2.30) \quad m_j^2(\xi, \eta) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(\mathbb{Z}_-)^2 \cup \triangle\}, k \in \mathbb{Z}} m_{j,m,n,k}(\xi, \eta).
\end{equation}

The high-frequency part near the diagonal part $m_j^3$ is

\begin{equation}
(2.31) \quad m_j^3(\xi, \eta) := \sum_{(m,n) \in \triangle, k \in \mathbb{Z}} m_{j,m,n,k}(\xi, \eta).
\end{equation}

Accordingly, we can split $H_\gamma(f, g)$ into three parts:

\begin{equation}
(2.32) \quad H_\gamma(f, g)(x) = H_\gamma^1(f, g)(x) + H_\gamma^2(f, g)(x) + H_\gamma^3(f, g)(x),
\end{equation}

where

\begin{equation}
\begin{aligned}
H_\gamma^1(f, g)(x) &:= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} m_j^1(\xi, \eta) \, d\xi \, d\eta; \\
H_\gamma^2(f, g)(x) &:= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} m_j^2(\xi, \eta) \, d\xi \, d\eta; \\
H_\gamma^3(f, g)(x) &:= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} m_j^3(\xi, \eta) \, d\xi \, d\eta.
\end{aligned}
\end{equation}
2.3 Decomposition of $M_\gamma (f, g)$

$M_\gamma (f, g)$ is a positive operator, and we may assume that $f$ and $g$ are non-negative. By simple calculation, we deduce that

$$M_\gamma (f, g)(x) \leq \sup_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x - t)g(x - \gamma(t))2^{3/2}|\rho(2^j t)|\,dt =: \sup_{j \in \mathbb{Z}} M_{j, \gamma}(f, g)(x).$$  

As in Proposition 2.2 above, we will focus on the case where $|f|$ is large enough. We rewrite $M_{j, \gamma}(f, g)$ as

$$M_{j, \gamma}(f, g)(x) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta)e^{ix\xi}e^{i\eta\gamma}\hat{m}_j(\xi, \eta)\,d\xi\,d\eta,$$

where

$$\hat{m}_j(\xi, \eta) := \text{p.v.} \int_{-\infty}^{\infty} e^{-2^{-j}x\xi}e^{-i\eta(2^{-j}t)}|\rho(t)|\,dt.$$  

As in (2.26), we define $\hat{m}_{j, m, n, k}$ as

$$\hat{m}_{j, m, n, k}(\xi, \eta) := \hat{m}_j(\xi, \eta)\phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^n} \right) \phi \left( \gamma' \left( \frac{2^{-j}}{2^{n+j+k}} \right) \right),$$

and split $\hat{m}_j$ as

$$\hat{m}_j(\xi, \eta) = \hat{m}^1_j(\xi, \eta) + \hat{m}^2_j(\xi, \eta) + \hat{m}^3_j(\xi, \eta).$$

The low-frequency part $\hat{m}^1_j$ is

$$\hat{m}^1_j(\xi, \eta) := \sum_{(m, n) \in \mathbb{Z}^2, k \in \mathbb{Z}} \hat{m}_{j, m, n, k}(\xi, \eta).$$

The high-frequency part away from the diagonal part $\hat{m}^2_j$ is

$$\hat{m}^2_j(\xi, \eta) := \sum_{(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}, k \in \mathbb{Z}} \hat{m}_{j, m, n, k}(\xi, \eta).$$

The high-frequency part near the diagonal part $\hat{m}^3_j$ is

$$\hat{m}^3_j(\xi, \eta) := \sum_{(m, n) \in \mathbb{Z}, k \in \mathbb{Z}} \hat{m}_{j, m, n, k}(\xi, \eta).$$

Similarly,

$$M_\gamma (f, g)(x) \leq M^1_\gamma (f, g)(x) + M^2_\gamma (f, g)(x) + M^3_\gamma (f, g)(x).$$
\section{The boundedness of $H^1_\gamma(f, g)$}

For $m_j^1$ in \eqref{eq:2.29}, we employ the Taylor series expansion, i.e.,
\begin{equation}
\sum_{n}\frac{(-i)^{n+2}2n^2v}{u!v!} H^1_\gamma \sum_{m,n} \frac{(2n^2)\eta}{2n+1} \bar{\phi}_v(\frac{2n+1}{2n+1}) \bar{\phi}_v(\frac{2n+1}{2n+1}) \int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2j\gamma t)}{\gamma(2)} \right)^v \rho(t) dt.
\end{equation}

Therefore, $H^1_\gamma(f, g)(x)$ can be rewritten as
\begin{equation}
\sum_{j,k} \sum_{m,n} \frac{(-i)^{n+2}2n^2v}{u!v!} H^1_\gamma \sum_{m,n} \frac{(2n^2)\eta}{2n+1} \bar{\phi}_v(\frac{2n+1}{2n+1}) \bar{\phi}_v(\frac{2n+1}{2n+1}) \int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2j\gamma t)}{\gamma(2)} \right)^v \rho(t) dt.
\end{equation}

where $\hat{\phi}_v(x) := 2^j \hat{\phi}_v(2^j x)$ and $\hat{\phi}_v$ means the inverse Fourier transform of $\hat{\phi}_v$; by the Cauchy-Schwarz inequality, we dominate $H^1_\gamma(f, g)(x)$ by
\begin{equation}
\sum_{m,n} \sum_{u,v} \frac{2n^2v}{u!v!} \left| \int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2j\gamma t)}{\gamma(2)} \right)^v \rho(t) dt \right|
\end{equation}

For $r \geq 1$, by the triangle inequality and Hölder inequality, the $L'(\mathbb{R})$ norm of $H^1_\gamma(f, g)$ is at most
\begin{equation}
\sum_{m,n} \sum_{u,v} \frac{2n^2v}{u!v!} \left| \int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2j\gamma t)}{\gamma(2)} \right)^v \rho(t) dt \right|
\end{equation}

From \eqref{eq:2.3} and \eqref{eq:2.2}, it is easy to see that
\begin{equation}
\int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2j\gamma t)}{\gamma(2)} \right)^v \rho(t) dt \leq 1.
\end{equation}
We claim that

\[(3.4) \quad \sum_{j \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \right| \leq 1\]

holds uniformly for \(n \in \mathbb{Z}_-\) and \(k \in \mathbb{Z}\). Since \(0 \leq \phi \leq 1\), it is enough to show that the sum in (3.4) has at most a finite number of terms and the number independent of \(n \in \mathbb{Z}_-\) and \(k \in \mathbb{Z}\). Indeed, since \(\phi\) is supported on \(\{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\}\), then, for the \(j\)-th term in (3.4), it implies \(\frac{1}{2^{j/2}} \leq \frac{1}{\gamma(2^{j-k})} \leq \frac{2^{j/2}}{\gamma(2^{j-k})}\). For the \(j + K\)-th term in (3.4), it implies \(\frac{1}{2^{j/K}} \leq \frac{2^{j/K}}{\gamma(2^{j-k})} \leq \frac{1}{2^{j/2}} \gamma(2^{j-k})\), where \(K > 0\). Therefore, it suffices to show that there exists a positive constant \(K\) that depends only on \(\gamma\) such that \(2^{j/K} \leq \frac{1}{\gamma(2^{j-k})}\). From (3.2), this is a direct consequence of \(\gamma(2^{-j}) > 4 \frac{1+\frac{1}{j}}{1+\frac{1}{\sqrt{2}}}\).

From (3.4), we need only to take \(K\) that satisfies \(2^{(1+\frac{1}{\sqrt{2}})K} > 4 \frac{1+\frac{1}{j}}{1+\frac{1}{\sqrt{2}}}\).

As in (3.4), we also have that

\[(3.5) \quad \sum_{k \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \right| \leq 1\]

holds uniformly for \(n \in \mathbb{Z}_-\) and \(j \in \mathbb{Z}\). (3.5) and the Littlewood-Paley theory implies

\[(3.6) \quad \left\| \sum_{j,k \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \cdot \phi_{u,m+j} \cdot f \right|^2 \right\|_{L^p(\mathbb{R})} \leq \left\| \sum_{j \in \mathbb{Z}} \left| \phi_{u,m+j} \cdot f \right|^2 \right\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}.

From (3.4), as in (3.6),

\[(3.7) \quad \left\| \sum_{j,k \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \cdot \phi_{v,k} \cdot g \right|^2 \right\|_{L^q(\mathbb{R})} \leq \|g\|_{L^q(\mathbb{R})}.

From the fact that \(\sum_{m,n \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \frac{\gamma(2^{n-m})}{u!v!} \leq 1\) and the estimates (3.2), (3.3), (3.6) and (3.7), we have

\[(3.8) \quad \|H^1_\gamma(f,g)\|_{L^q(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.

For \(\frac{1}{2} < r < 1\), from (3.1), we bound \(\|H^1_\gamma(f,g)\|_{L^r(\mathbb{R})}\) by

\[(3.9) \quad \sum_{m,n \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \left( \frac{\gamma(2^{n-m})}{u!v!} \right)^r \left[ \left( \int_{-\infty}^{\infty} t^\mu \left( \frac{2\gamma(2^{-j})}{\gamma(2^{-j})} \right)^\nu \rho(t) dt \right)^r \right]^{\frac{1}{r}} \times \left( \sum_{j,k \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \cdot \phi_{u,m+j} \cdot f \right|^2 \right)^\frac{1}{2r} \left( \sum_{j,k \in \mathbb{Z}} \left| \phi \left( \frac{\gamma(2^{-j})}{2^{n+j-k}} \right) \cdot \phi_{v,k} \cdot g \right|^2 \right)^\frac{1}{2r}\right\|_{L^r(\mathbb{R})}.

The fact \(\sum_{m,n \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \left( \frac{\gamma(2^{n-m})}{u!v!} \right)^r \leq 1\) for all \(\frac{1}{2} < r < 1\) combining (3.3), (3.6), (3.7) and (3.9), leads to

\[(3.10) \quad \|H^1_\gamma(f,g)\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.

This is the desired estimate for the first item \(H^1_\gamma(f,g)\).
4 The boundedness of $H^2_\gamma(f, g)$

Noting that
\[ \int_{-\infty}^{\infty} \frac{d}{2^{-j}\xi + \eta 2^{-j}\gamma'(2^{-j}t)} e^{-i2^{-j}t \eta - i\eta y(2^{-j}t)} = e^{-i2^{-j}t \eta - i\eta y(2^{-j}t)}, \]
we split $m_{j,m,n,k}$ in (2.26) as the following two parts:

\[ m_{j,m,n,k}(\xi, \eta) = A_{j,m,n,k}(\xi, \eta) + B_{j,m,n,k}(\xi, \eta), \]

where $A_{j,m,n,k}$ is defined as

\[ (4.2) \quad \left( \int_{-\infty}^{\infty} e^{-i2^{-j}t \eta - i\eta y(2^{-j}t)} \frac{-i2^{-j}t}{2^{-j}\xi + \eta 2^{-j}\gamma'(2^{-j}t)} \right) \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^k} \right) \phi \left( \frac{\gamma'(2^{-j})}{2^{n+j-k}} \right) \]

and $B_{j,m,n,k}$ is defined as

\[ (4.3) \quad \left( \int_{-\infty}^{\infty} e^{-i2^{-j}t \eta - i\eta y(2^{-j}t)} \frac{2^{-j}t}{2^{-j}\xi + \eta 2^{-j}\gamma'(2^{-j}t)} \right) \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^k} \right) \phi \left( \frac{\gamma'(2^{-j})}{2^{n+j-k}} \right) \]

Based on this decomposition, we split $H^2_\gamma(f, g)$ as follows:

\[ (4.4) \quad \sum_{j \in \mathbb{Z}} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\{\xi\} \cup \Delta)} \sum_{k \in \mathbb{Z}} A_{j,m,n,k}(f, g)(x) + \sum_{j \in \mathbb{Z}} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\{\xi\} \cup \Delta)} \sum_{k \in \mathbb{Z}} B_{j,m,n,k}(f, g)(x) \]

\[ =: A^2_\gamma(f, g)(x) + B^2_\gamma(f, g)(x), \]

where

\[ (4.5) \quad \left\{ \begin{array}{l}
A_{j,m,n,k}(f, g)(x) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} A_{j,m,n,k}(\xi, \eta) \, d\xi \, d\eta;
B_{j,m,n,k}(f, g)(x) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} B_{j,m,n,k}(\xi, \eta) \, d\xi \, d\eta.
\end{array} \right. \]

Since $(m, n) \in \mathbb{Z}^2 \setminus (\{\xi\} \cup \Delta)$, there are two cases: $n > |m| + 2/C_1 + 1$ and $m > |n| + 2/C_1 + 1$.

4.1 Case 1: $n > |m| + 2/C_1 + 1$

Applying the Taylor series expansion, we have

\[ (4.6) \quad \frac{1}{2^{-j}\xi + \eta 2^{-j}\gamma'(2^{-j}t)} = \frac{1}{2^{-j}\xi + \eta 2^{-j}\gamma'(2^{-j}t)} \sum_{l \in \mathbb{N}} \frac{(-1)^l}{2^{l(n-m)}} \left( \frac{\xi}{2^{l(n+m)}} \right)^l \left( \frac{\eta}{2^{l(n+k)}} \right)^l \gamma'((2^{-j}t)^l). \]

Then, $A_{j,m,n,k}(f, g)(x)$ can be written as

\[ \frac{-i}{2^n} \sum_{l \in \mathbb{N}} \frac{(-1)^l}{2^{l(n-m)}} \hat{f}_{l-1} \hat{g}_{l} \hat{f}_{l+m+j} \hat{f}_{l+n+k} \hat{f}_{l+m+j} \hat{g}_{l} \int_{-\infty}^{\infty} e^{-i2^{-j}t \eta - i\eta y(2^{-j}t)} \rho(t) \gamma'((2^{-j}t)^l) \, dt. \]
Noting that \( \gamma \) is either odd or even, \( \gamma' \) is increasing on \((0, \infty)\), \( \rho \) is supported on \( \{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\} \), and by (2.3), we have

\[
(4.7) \quad \left| \int_{-\infty}^{\infty} e^{-i2^{-j}t-iy(2^{-j}t)} \rho'(t) \left( \frac{\gamma'(2^{-j}t)}{\gamma(2^{-j}t)} \right)^{j+1} \right| \leq 2^{\frac{12}{j+1}}.
\]

By the Cauchy-Schwarz inequality,

\[
(4.8) \quad A^2_\gamma(f, g)(x) \leq \sum_{(m, n) \in \mathbb{Z}^2 \setminus \{(\mathbb{Z}, \mathbb{Z}) \cup \Delta\}} \frac{1}{2^n} \sum_{\ell \in \mathbb{N}} \frac{2^{\frac{12}{j+1}}}{2^{(n-m)}} \left\| \sum_{j, k \in \mathbb{Z}} \phi_{-j-1} \left( \frac{\gamma'(2^{-j}t)}{2n+j-k} \right) \cdot \left| \phi_{l,m+j} * f(x) \right|^2 \right\|^\frac{1}{2} \times \left\| \sum_{j, k \in \mathbb{Z}} \phi_{-j-1} \left( \frac{\gamma'(2^{-j}t)}{2n+j-k} \right) \cdot \left| \phi_{l-1,k} * g(x) \right|^2 \right\|^\frac{1}{2}.
\]

For \( r \geq 1 \), by the triangle inequality and Hölder inequality, we bound \( \|A^2_\gamma(f, g)\|_{L^p(\mathbb{R})} \) by

\[
(4.9) \quad \sum_{(m, n) \in \mathbb{Z}^2 \setminus \{(\mathbb{Z}, \mathbb{Z}) \cup \Delta\}} \frac{1}{2^n} \sum_{\ell \in \mathbb{N}} \frac{2^{\frac{12}{j+1}}}{2^{(n-m)}} \left\| \sum_{j, k \in \mathbb{Z}} \phi_{-j-1} \left( \frac{\gamma'(2^{-j}t)}{2n+j-k} \right) \cdot \left| \phi_{l,m+j} * f(x) \right|^2 \right\|^{\frac{1}{2}} \times \left\| \sum_{j, k \in \mathbb{Z}} \phi_{-j-1} \left( \frac{\gamma'(2^{-j}t)}{2n+j-k} \right) \cdot \left| \phi_{l-1,k} * g(x) \right|^2 \right\|^{\frac{1}{2}}.
\]

On the other hand, from \( n > |m| + 2/C_1 + 1 \), it implies \( \sum_{\ell \in \mathbb{N}} \frac{2^{\frac{12}{j+1}}}{2^{(n-m)}} \leq 1 \). From \( n > |m| + 2/C_1 + 1 \), we have

\[
(4.10) \quad \sum_{(m, n) \in \mathbb{Z}^2 \setminus \{(\mathbb{Z}, \mathbb{Z}) \cup \Delta\}} \frac{1}{2^n} \sum_{\ell \in \mathbb{N}} \frac{1}{2^{\frac{12}{j+1}}} \leq 1.
\]

As in (3.8) and (3.6), we assert that

\[
(4.11) \quad \|A^2_\gamma(f, g)\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.
\]

As in (3.9), it is easy to see that (4.11) also holds for \( \frac{1}{2} < r < 1 \).

Applying the Taylor series expansion again, we have

\[
(4.12) \quad \frac{n2^{-2}r y''(2^{-j}t)}{(2^{-j}t)^2} = \frac{1}{2^n} \frac{1}{\xi^{2n}} \sum_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{2^{2(n-m)}} \left( \frac{\xi}{2^{2n}} \right)^{\ell+1} \left( \frac{y''(2^{-j}t)}{y'(2^{-j}t)} \right)^{\ell+1}.
\]

We can then write \( B_{y, l,m,n}(f, g)(x) \) as

\[
\frac{i}{2^n} \sum_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{2^{2(n-m)}} \phi_{-j-1} \left( \frac{y'(2^{-j}t)}{2n+j-k} \right) \phi_{l,m+j} * f(x) \cdot \phi_{l-1,k} * g(x)
\]
\[
\times \int_{-\infty}^{\infty} e^{-2^j t \xi - i \eta \gamma(2^j t)} 2^{-j} \gamma''(2^j t) (\frac{\gamma'(2^j t)}{\gamma(2^j t)})^{i+1} \rho(t) \, dt.
\]

Since \( \gamma \) is either odd or even, \( \gamma' \) is increasing on \((0, \infty)\), \( \rho \) is supported on \( \{ t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2 \} \), and by (2.1) and (2.3), it is easy to see that \( |\frac{2^{-j} \gamma''(2^j t)}{\gamma(2^j t)}| = |\frac{2^{-j} \gamma''(2^j t)}{\gamma'(2^j t)}| \leq 1 \). Thus

\[
(4.13) \quad \left| \int_{-\infty}^{\infty} e^{-2^j t \xi - i \eta \gamma(2^j t)} 2^{-j} \gamma''(2^j t) (\frac{\gamma'(2^j t)}{\gamma(2^j t)})^{i+1} \rho(t) \, dt \right| \leq 2^{\frac{2j}{r}}.
\]

As in (4.9) and (4.10), for \( r > \frac{1}{2} \), we assert that

\[
(4.14) \quad \|B_\gamma^2(f, g)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.
\]

### 4.2 Case 2: \( m > |n| + 2/C_1 + 1 \)

By the Taylor series expansion, we have

\[
(4.15) \quad \frac{1}{2^{-j} \xi + \eta 2^{-j} \gamma(2^j t)} = \frac{1}{2^m} \frac{1}{\xi^{m-j}} \sum_{l \in \mathbb{N}} (-1)^l \frac{\eta^{2^{-j} l}}{2^{l(m-j)}} \frac{\gamma''(2^{-j} l)}{\gamma(2^{-j} l)} \frac{\gamma'(2^{-j} l)}{\gamma(2^{-j} l)}.
\]

As in case 1, we have

\[
(4.16) \quad \|A_\gamma^2(f, g)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}
\]

for \( r > \frac{1}{2} \). Furthermore,

\[
(4.17) \quad \frac{\eta 2^{-j} \gamma''(2^{-j} t)}{(2^{-j} \xi + \eta 2^{-j} \gamma(2^{-j} t))^2} = \frac{1}{2^m} \frac{1}{\xi^{m+j}} 2^{-j} \gamma''(2^{-j} t) \sum_{l \in \mathbb{N}} (-1)^l \frac{\eta^{2^{-j} l}}{2^{l(m+j)}} \frac{\gamma''(2^{-j} l)}{\gamma(2^{-j} l)} \frac{\gamma'(2^{-j} l)}{\gamma(2^{-j} l)}.
\]

As in case 1, for \( r > \frac{1}{2} \), again we have

\[
(4.18) \quad \|B_\gamma^2(f, g)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.
\]

From (4.14), (4.11), (4.14), (4.16) and (4.18), we may obtain

\[
(4.19) \quad \|H_\gamma^2(f, g)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}
\]

for \( r > \frac{1}{2} \). This is the desired estimate for the second item \( H_\gamma^2(f, g) \).

### 5 The \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \) boundedness of \( H_\gamma^3(f, g) \)

Note that \( \triangle = \{(m, n) \in \mathbb{Z}^2 : m, n \geq 0, |m - n| \leq 2/C_1 + 1 \} \) and \( m_j^3 = \sum_{(m, n) \in \triangle, k \in \mathbb{Z}} m_{j, m, n, k} \); without loss of generality, we may write that

\[
(5.1) \quad m_j^3 = \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} m_{j, m, k}
\]
where \( m_{j,m,k} := m_{j,m,m,k} \). Therefore, we rewrite \( H_\gamma^3(f,g) \) as

\[
H_\gamma^3(f,g)(x) = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} H_{j,m,k}(f,g)(x),
\]

where

\[
H_{j,m,k}(f,g)(x) := \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{i\xi x} e^{i\eta y} m_j(\xi,\eta) \phi \left( \frac{\xi}{2^{m+j}} \right) \phi \left( \frac{\eta}{2^k} \right) d\xi d\eta
\]

with \( m_j \) as in (5.3).

We observe that in order to obtain

\[
\|H_\gamma^3(f,g)\|_{L^1(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})},
\]

it suffices to prove the following Proposition 5.1. Indeed, notice that \( \phi \) is a standard bump function supported on \( \{t \in \mathbb{R} : \frac{1}{8} \leq |t| \leq 2 \} \); let \( \Phi \) be a bump function supported on \( \{t \in \mathbb{R} : \frac{1}{8} \leq |t| \leq 8 \} \) such that \( \Phi(t) = 1 \) on \( \{t \in \mathbb{R} : \frac{1}{4} \leq |t| \leq 4 \} \); thus, it is safe to insert \( \Phi \) into \( H_{j,m,k}(f,g) \). In other words, recall that \( \psi_\lambda(\xi) = 2^\lambda \psi(2^\lambda \xi) \), we have

\[
H_{j,m,k}(f,g)(x) = \Phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) H_{j,m,k} \left( \Phi_{m+j} * f, \Phi_k * g \right)(x).
\]

By the triangle inequality, the Cauchy-Schwarz inequality and (5.6), it implies that \( \|H_\gamma^3(f,g)\|_{L^1(\mathbb{R})} \) can be bounded by

\[
\sum_{m \in \mathbb{N}} 2^{-\epsilon_0 m} \left[ \sum_{j,k \in \mathbb{Z}} \left| \Phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \right| \left\| \Phi_{m+j} * f \right\|_{L^2(\mathbb{R})}^2 \right]^{\frac{1}{2}} \cdot \left[ \sum_{j,k \in \mathbb{Z}} \left| \Phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \right| \left\| \Phi_k * g \right\|_{L^2(\mathbb{R})}^2 \right]^{\frac{1}{2}}.
\]

By this estimate with (3.4), (3.5) and the Littlewood-Paley theory, we may obtain (5.4).

**Proposition 5.1.** There exist positive constants \( C \) and \( \epsilon_0 \) such that

\[
\|H_{j,m,k}(f,g)\|_{L^1(\mathbb{R})} \leq C 2^{-\epsilon_0 m} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}
\]

holds uniformly for \( j, k \in \mathbb{Z} \).

As in (3.4), we define the bilinear operator \( B_{j,m,k}(f,g)(x) \) as

\[
2^\frac{j+m-k}{2} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \hat{f}(x-2^m t) \hat{g}(x-2^k \gamma(2^{-j} t)) \rho(t) dt \quad \text{if} \quad j \geq 0;
\]

and

\[
2^\frac{j-m-j}{2} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \hat{f}(x-2^m t) \hat{g}(x-2^k \gamma(2^{-j} t)) \rho(t) dt \quad \text{if} \quad j < 0.
\]

We observe that (5.6) is equivalent to

\[
\|B_{j,m,k}(f,g)\|_{L^1(\mathbb{R})} \leq 2^{-\epsilon_0 m} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.
\]
Indeed, let $\xi := 2^{m+j}\xi$ and $\eta := 2^k\eta$; we then write $H_{j,m,k}(f, g)(x)$ as

$$2^{m+j+k} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(2^{m+j}\xi)\hat{g}(2^k\eta)e^{i2^{m+j}\xi x} e^{i2^k\eta x} \left( \int_{-\infty}^{\infty} e^{-t2^m\xi t} e^{-t2^k\eta(2^{-j}t)} \rho(t) \, dt \right) \phi(\xi)\phi(\eta) \, d\xi \, d\eta.$$ 

For $j \geq 0$, let $x := 2^{-k}x$; we may therefore write

$$2^{-k}H_{j,m,k}(f, g)(2^{-k}x) = 2^{m+j+k} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(2^{m+j}\xi)\hat{g}(2^k\eta)e^{i2^{m+j-k}\xi x} e^{i2^k\eta x} \left( \int_{-\infty}^{\infty} e^{-t2^m\xi t} e^{-t2^k\eta(2^{-j}t)} \rho(t) \, dt \right) \phi(\xi)\phi(\eta) \, d\xi \, d\eta.$$ 

Note that $\|2^{m+j}\hat{f}(2^{m+j})\|_{L^2(\mathbb{R})} = \|\hat{\Psi}\|_{L^2(\mathbb{R})}$ and $\|2^k\hat{g}(2^k)\|_{L^2(\mathbb{R})} = \|\hat{\Theta}\|_{L^2(\mathbb{R})}$; we conclude that (5.6) is equivalent to (5.9), where $B_{j,m,k}(f, g)$ is defined in (5.7). For $j < 0$, the only difference is to make the variable change $x := \frac{x}{2^m\eta}$, and we omit the details.

We now turn to the proof of (5.9). In what follows, we will only focus on the first case, i.e., $j \geq 0$, since the second case, i.e., $j < 0$, is similar. From Proposition 2.2 and (5.9), we can assume that $j$ and $m$ are sufficiently large.

**Claim:** (5.9) is equivalent to Proposition 5.2 below.

**Proof of the Claim.** This claim is essentially [34, Lemma 5.1]. As in [34], let $\hat{\psi}$ be a nonnegative Schwartz function such that $\hat{\psi}$ is supported on $\{t \in \mathbb{R} : |t| \leq \frac{1}{100}\}$ and satisfies $\hat{\psi}(0) = 1$. Then, $B_{j,m,k}(f, g)(x)$ can be written as

$$2^{m+j+k} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{N \in \mathbb{Z}} \sum_{k_1, k_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \left( \chi_{[2^{-m}(N+k_1), 2^{m}(N+k_1+1)]} \ast \psi_m \ast \hat{\phi} \ast f \right) \left( 2^{m+j-k}x - 2^{m}t \right) \times \left( \chi_{[2^{k-j}(N+k_2), 2^{k-j}(N+k_2+1)]} \ast \hat{\phi} \ast g \right) \left( x - 2^{k}\gamma(2^{-j}t) \right) \rho(t) \, dt \cdot \chi_{[2^{k-j}(N), 2^{k-j}(N+1)]}(x).$$

We split

$$B_{j,m,k}(f, g)(x) := B_{j,m,k}^I(f, g)(x) + B_{j,m,k}^{II}(f, g)(x),$$

where $B_{j,m,k}^I(f, g)$ sums over $A := \{k_1, k_2 \in \mathbb{Z} : \max\{|k_1|, |k_2|\} \geq \Theta\}$ and $B_{j,m,k}^{II}(f, g)$ sums over $B := \{k_1, k_2 \in \mathbb{Z} : \max\{|k_1|, |k_2|\} < \Theta\}$, where $\Theta := 2^{\frac{1}{4}m}$ is sufficiently large.

For $B_{j,m,k}^I(f, g)$, we have $|\chi_{[2^{-m}(N+k_1), 2^{m}(N+k_1+1)]} \ast \psi_m(2^{m+j-k}x - 2^{m}t)| \leq \frac{1}{|k_1|^3}$. Note that $\frac{3\Theta}{2}$ is increasing on $(0, \infty)$; it implies $2^{j}\gamma(2^{-j}t)$ is sufficiently small if $j$ is sufficiently large, and we also have $|\chi_{[2^{k-j}(N+k_2), 2^{k-j}(N+k_2+1)]} \ast \hat{\phi}(x - 2^{k}\gamma(2^{-j}t))| \leq \frac{1}{|k_2|^3}$. Thus, $B_{j,m,k}^I(f, g)(x)$ is bounded by

$$2^{m+j+k} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{N \in \mathbb{Z}} \sum_{k_1, k_2 \in A} \frac{1}{|k_1|^3} \frac{1}{|k_2|^3} \int_{-\infty}^{\infty} \hat{\phi} \ast f \left( 2^{m+j-k}x - 2^{m}t \right) \times \hat{\phi} \ast g \left( x - 2^{k}\gamma(2^{-j}t) \right) \rho(t) \, dt \cdot \chi_{[2^{k-j}(N), 2^{k-j}(N+1)]}(x) \leq \frac{1}{\Theta} \frac{2^{m+j+k}}{2^{m+j-k}} \int_{-\infty}^{\infty} \hat{\phi} \ast f \left( 2^{m+j-k}x - 2^{m}t \right) \hat{\phi} \ast g \left( x - 2^{k}\gamma(2^{-j}t) \right) \rho(t) \, dt.$$
The last inequality is a result of the fact that \( \sum_{k \in \mathbb{Z}} \frac{1}{|k|^s} \lesssim \frac{1}{\Theta} \) and \( \sum_{k \in \mathbb{Z}} \frac{1}{|k|^s} \lesssim 1. \) By the Hölder and Young inequalities, it now follows that

\[
(5.12) \quad \left\| B^{I}_{j,m,k}(f, g) \right\|_{L^1(\mathbb{R})} \lesssim \frac{1}{\Theta} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}.
\]

For \( B^{II}_{j,m,k}(f, g) \), let \( j \) and \( m \) be sufficiently large; we have that

\[
\mathcal{F} \left( \chi_{[2^m(N+k_1),2^{m(N+k_1)+1}]} * \psi_{-m} * \tilde{\phi} * f \right) \quad \text{and} \quad \mathcal{F} \left( \chi_{[2^{k-1/(N+k_2),2^{k-1/(N+k_2)+1}]} * \psi_{j-k} * \tilde{\phi} * g \right)
\]

are supported on \( \{|t| : \frac{1}{4} \leq |t| \leq 4\} \), where \( \mathcal{F}(f) \) means the Fourier transform of \( f \). Then, \( B^{II}_{j,m,k}(f, g)(x) \) can be written as

\[
\sum_{N \in \mathbb{Z}} \sum_{k_1,k_2 \in B} B^{*}_{j,m,k} \left( \chi_{[2^m(N+k_1),2^{m(N+k_1)+1}]} * \psi_{-m} * \tilde{\phi} * f, \chi_{[2^{k-1/(N+k_2),2^{k-1/(N+k_2)+1}]} * \psi_{j-k} * \tilde{\phi} * g \right)(x) \times \chi_{[2^{k-1/(N+k_2),2^{k-1/(N+k_2)+1}]}(x).
\]

The definition of \( B^{*}_{j,m,k}(f, g) \) will be given in Proposition 5.2. From \( \sum_{N \in \mathbb{Z}} |\chi_{[2^m(N+k_1),2^{m(N+k_1)+1}]} * \psi_{-m}| \leq \| \psi \|_{L^1(\mathbb{R})} \) and \( \sum_{N \in \mathbb{Z}} |\chi_{[2^{k-1/(N+k_2),2^{k-1/(N+k_2)+1}]} * \psi_{j-k}| \leq \| \psi \|_{L^1(\mathbb{R})} \), by the Cauchy-Schwarz inequality and the Young inequality, \( (5.13) \), \( \| B^{II}_{j,m,k}(f, g) \|_{L^1(\mathbb{R})} \) can be bounded by

\[
(5.13) \quad \Theta^2 2^{-j \varepsilon_m} \left[ \sum_{N \in \mathbb{Z}} \left\| \chi_{[2^m(N+k_1),2^{m(N+k_1)+1}]} * \psi_{-m} * \tilde{\phi} * f \right\|_{L^2(\mathbb{R})}^2 \right]^{1 \over 2} \times \left[ \sum_{N \in \mathbb{Z}} \left\| \chi_{[2^{k-1/(N+k_2),2^{k-1/(N+k_2)+1}]} * \psi_{j-k} * \tilde{\phi} * g \right\|_{L^2(\mathbb{R})}^2 \right]^{1 \over 2} \lesssim \Theta^2 2^{-j \varepsilon_m} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}.
\]

From (5.12) and (5.13), note that \( \Theta = 2^{-\varepsilon_m} \); we obtain

\[
(5.14) \quad \| B_{j,m,k}(f, g) \|_{L^1(\mathbb{R})} \lesssim 2^{-\varepsilon_m} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}.
\]

This is (5.9) if we let \( \varepsilon_0 := \varepsilon_m^\rho \). \( \square \)

**Proposition 5.2.** There exist positive constants \( C \) and \( \varepsilon'_0 \) such that

\[
(5.15) \quad \left\| B^{*}_{j,m,k}(f, g) \cdot \chi_{[2^{k-1/(N+2^k-1)+1}]} \right\|_{L^1(\mathbb{R})} \leq C 2^{-\varepsilon'_0 m} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}
\]

holds uniformly for \( j \in \mathbb{Z} \) and \( N \in \mathbb{Z} \), where \( B^{*}_{j,m,k}(f, g) \) is defined as

\[
B^{*}_{j,m,k}(f, g)(x) := 2^m \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \tilde{\Phi} * f \left( 2^{m+j-k} x - 2^m t \right) \phi \left( x - 2^k \gamma(2^{-j}) t \right) \rho(t) \, dt
\]

and \( \Phi \) is the same bump function as in (5.9). We will prove Proposition 5.2 in three steps.
5.1 An estimate by using $TT^*$ argument

In this subsection, we show that

\begin{equation}
\left\| B_{j,m,k}^*(f,g) \cdot X_{2^k,2^{k-j}(N+1)} \right\|_{L^1(\mathbb{R})} \leq 2^{-\frac{2m+1-2}{6}} \left\| f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})}.
\end{equation}

By the Hölder inequality, it suffices to show that

\begin{equation}
\left\| B_{j,m,k}^*(f,g) \right\|_{L^2(\mathbb{R})} \leq 2^{-\frac{2m+1-2}{6}} \left\| f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})}.
\end{equation}

We rewrite $B_{j,m,k}^*(f,g)(x)$ as

\begin{equation}
2^{-\frac{m+1}{2}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+1-j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i \xi k} e^{i \pi x} I_{j,m,k}(\xi,\eta) \Phi(\xi) \Phi(\eta) \, d\xi \, d\eta,
\end{equation}

where

\begin{equation}
I_{j,m,k}(\xi,\eta) := \int_{-\infty}^{\infty} e^{-i 2^m \xi} e^{-i 2^j \eta} \rho(t) \, dt.
\end{equation}

Let

\begin{equation}
\varphi(t, \xi, \eta) := -\xi t - 2^{-m-j} \eta (2^{-j} t).
\end{equation}

We denote $\varphi'_1$ as the derivative of $\varphi$ with respect to the first variable $t$. Then

\begin{equation}
\varphi'_1(t, \xi, \eta) = -\xi - 2^{-m-j} \gamma(2^{-j} t) \eta \quad \text{and} \quad \varphi''_1(t, \xi, \eta) = -2^{-m-2-j} \gamma''(2^{-j} t) \eta.
\end{equation}

We define $t_0(\xi, \eta)$ as

\begin{equation}
\varphi'_1(t_0(\xi, \eta), \xi, \eta) = 0.
\end{equation}

For simplicity, we may further assume that $\frac{1}{2} \leq |t_0(\xi, \eta)| \leq 2$. By the method of the stationary phase, we assert that

\begin{equation}
I_{j,m,k}(\xi, \eta) = e^{2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} \left( \frac{2\pi}{-i 2^m \varphi''(t_0(\xi, \eta), \xi, \eta)} \right)^{\frac{1}{2}} \rho(t_0(\xi, \eta)) + O \left( 2^{-\frac{1}{2} m} \right).
\end{equation}

For $h \in L^2(\mathbb{R})$, we have $\int_{-\infty}^{\infty} B_{j,m,k}^*(f,g)(x)h(x) \, dx$ is equal to

\begin{equation}
2^{-\frac{m+1}{2}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+1-j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \Phi(\xi) \hat{g}(\eta) \Phi(\eta) I_{j,m,k}(\xi, \eta) \hat{h} \left( 2^{m+j-k} \xi + \eta \right) \, d\xi \, d\eta.
\end{equation}

Based on (5.23), we estimate $\int_{-\infty}^{\infty} B_{j,m,k}^*(f,g)(x)h(x) \, dx$ by considering the following two parts:

**Part A:** $O \left( 2^{-\frac{1}{2} m} \right)$

With some abuse of notation, we write $\int_{-\infty}^{\infty} B_{j,m,k}^*(f,g)(x)h(x) \, dx$ as

\begin{equation}
2^{-\frac{m+1}{2}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+1-j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \Phi(\xi) \hat{g}(\eta) \Phi(\eta) \hat{h} \left( 2^{m+j-k} \xi + \eta \right) \cdot O \left( 2^{-\frac{1}{2} m} \right) \, d\xi \, d\eta.
\end{equation}
Thus, by the Hölder inequality and Plancherel’s formula, we bound \( \int_{-\infty}^{\infty} B_{j,m,k}^*(f, g)(x) h(x) \, dx \) by
\[
2^{-\frac{1}{2}m}\, 2^{\frac{1}{2}m-j-k} \int_{-\infty}^{\infty} \left| \hat{f}(\xi) \Phi(\xi) \right| \int_{-\infty}^{\infty} \left| \hat{g}(\eta) \Phi(\eta) \right| h \left( 2^{m-j-k} \xi + \eta \right) \, d\eta \, d\xi \lesssim 2^{-\frac{1}{2}m}\, 2^{\frac{1}{2}m-j-k} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})} \| h \|_{L^2(\mathbb{R})}.
\]
Furthermore, let \( h := \text{sgn}(B_{j,m,k}^*(f, g)) \cdot \chi_{[2^{k-j} N, 2^{k-j}(N+1)]} \); we have
\[
\left\| B_{j,m,k}^*(f, g) \cdot \chi_{[2^{k-j} N, 2^{k-j}(N+1)]} \right\|_{L^1(\mathbb{R})} \lesssim 2^{-m} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}.
\]
This is (5.15) as desired.

**Part B:**
\[
e^{2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} \left( \frac{2\pi}{-2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} \right)^{\frac{1}{2}} \rho(t_0(\xi, \eta))
\]
From (2.11) and (2.3), we have \( \left| \frac{2^{1/2} \gamma(2^{j-1})}{\gamma(2^{j-1})} \right| = \left| \frac{2^{1/2} \gamma(2^{j-1})}{\gamma(2^{j-1})} \right| \approx 1 \). For simplicity, we write
\[
\int_{-\infty}^{\infty} B_{j,m,k}^*(f, g)(x) h(x) \, dx \text{ as}
\]
(5.25)
\[
2^{-\frac{1}{2}m} \phi \left( \frac{\gamma'(2^{-j})}{2^{m-j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \Phi(\xi) \hat{g}(\eta) \Phi(\eta) e^{2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} h \left( 2^{m-j-k} \xi + \eta \right) \, d\xi \, d\eta.
\]
Changing variables \( \xi := \frac{\xi + \eta}{2^{m-j-k}} \) and \( \eta := 2^{m-j-k} \eta \), we have
\[
2^{-\frac{1}{2}m} \phi \left( \frac{\gamma'(2^{-j})}{2^{m-j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \Phi(\xi) \hat{g}(\eta) \Phi(\eta) e^{2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} h \left( 2^{m-j-k} \xi + \eta \right) \, d\xi \, d\eta.
\]
From the Hölder inequality and Plancherel formula, it can be bounded by \( 2^{-\frac{1}{2}m} \| T_{j,m,k}(f, g) \|_{L^2(\mathbb{R})} \| h \|_{L^2(\mathbb{R})} \), where \( T_{j,m,k}(f, g)(\xi) \) is defined as
\[
\phi \left( \frac{\gamma'(2^{-j})}{2^{m-j-k}} \right) \int_{-\infty}^{\infty} \left( \hat{f}(\xi) \Phi(\xi) \right) \left( 2^{m-j-k} \xi + \eta \right) e^{2\pi \varphi(t_0(\xi, \eta), \xi, \eta)} \, d\xi \, d\eta.
\]
Therefore, (5.17) can be reduced to
(5.26)
\[
\| T_{j,m,k}(f, g) \|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{1}{2}m} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}.
\]
By the TT* argument, we obtain that \( \| T_{j,m,k}(f, g) \|_{L^2(\mathbb{R})} \) equals to
\[
\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^2 \left( \frac{\gamma'(2^{-j})}{2^{m-j-k}} \right) F(\xi, \eta_1, \eta_2) G(\xi, \eta_1, \eta_2) e^{2\pi \varphi(t_0(\xi, \eta_1, \eta_2), \xi, \eta_1, 2^{m-j-k} \xi + \eta_1)} \right.
\]
\[
\times e^{-2\pi \varphi(t_0(\xi, \eta_1, \eta_2), \xi, \eta_1, 2^{m-j-k} \xi + \eta_1)} \, d\eta_1 \, d\eta_2 \] \, d\xi,
\]
where
\[
\left\{ F(\xi, \eta_1, \eta_2) := (\hat{f}(\xi) \Phi(\xi)) (\hat{g}(\eta_1) \Phi(\eta_1)) (\hat{g}(\eta_2) \Phi(\eta_2)) \right\},
\]
\[
\left\{ G(\xi, \eta_1, \eta_2) := (\hat{f}(\xi) \Phi(\xi)) (2^{m-j-k} \eta_1) (\hat{g}(\eta_1) \Phi(\eta_1)(2^{m-j-k} \eta_1)) \right\}.
Let \( \eta_1 := \eta \) and \( \eta_2 := \eta + \tau \); then, \( \|T_{j,m,k}(f,g)\|_{L^2(\mathbb{R})}^2 \) is equal to
\[
\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^2 (2^{(j-m-j-k)}) \frac{\xi}{2^{m+j-k}} F_\tau (\frac{\xi}{2^{m+j-k}} - \eta) G_\tau (2^{m+j-k}) e^{i2^m \varphi (\frac{\xi}{2^{m+j-k}} - \eta, 2^{m+j-k} \eta)} \right. \\
\left. \times \left( e^{-i2^m \varphi (\frac{\xi}{2^{m+j-k}} - \eta, 2^{m+j-k} \eta)} \frac{\xi}{2^{m+j-k}} - \eta, 2^{m+j-k} \eta \right) \right] d\xi d\eta \right] d\tau,
\]
where \( F_\tau (\cdot) := (\hat{f}(\cdot) - \tau)(\hat{f}(\cdot) - \tau + 2^{m+j-k} \tau) \) and \( G_\tau (\cdot) := (\hat{g}(\cdot) + 2^{m+j-k} \tau) \). Furthermore, let \( u := \frac{\xi}{2^{m+j-k}} - \eta \) and \( v := 2^{m+j-k} \tau \); we have
\[
\|T_{j,m,k}(f,g)\|_{L^2(\mathbb{R})}^2 = \phi^2 \left( \frac{\gamma' (2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_\tau(u)G_\tau(v)e^{i2^m Q_\tau(u,v)} du dv \right) d\tau,
\]
where
\[
Q_\tau(u,v) = \varphi (t_0(u,v), u,v) - \varphi (t_0(u-\tau, v + 2^{m+j-k} \tau), u-\tau, v + 2^{m+j-k} \tau).
\]
To estimate the bilinear operator \( T_{j,m,k}(f,g) \) and obtain (5.26), we use the Hörmander theorem on the nondegenerate phase \[26\]. To this aim, we need to establish Proposition 5.3 below. Let us postpone the proof of Proposition 5.3 for the moment. We now turn to \( \|T_{j,m,k}(f,g)\|_{L^2(\mathbb{R})}^2 \) in (5.27).

Let \( \tau_0 := 2^{-\frac{m+j-k}{2}} \); noting that \( \Phi \) is supported on \( \{ t \in \mathbb{R} : \frac{1}{8} \leq |t| \leq 8 \} \), we split it as
\[
\|T_{j,m,k}(f,g)\|_{L^2(\mathbb{R})}^2 = \phi^2 \left( \frac{\gamma' (2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_\tau(u)G_\tau(v)e^{i2^m Q_\tau(u,v)} du dv \right) d\tau
\]
\[+ \phi^2 \left( \frac{\gamma' (2^{-j})}{2^{m+j-k}} \right) \int_{\tau_0 < |\tau| \leq 16} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_\tau(u)G_\tau(v)e^{i2^m Q_\tau(u,v)} du dv \right) d\tau.
\]

By the Hölder inequality and Plancherel formula, it is easy to see that \( \|F_\tau\|_{L^1(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} \) and \( \|G_\tau\|_{L^1(\mathbb{R})} \leq \|g\|_{L^2(\mathbb{R})} \). On the other hand, we have
\[
\left( \int_{\tau_0 < |\tau| \leq 16} \|F_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}} = \left( \int_{\tau_0 < |\tau| \leq 16} \left( \frac{\gamma' (2^{-j})}{2^{m+j-k}} \right)^2 dx d\tau \right)^{\frac{1}{2}} \leq \|f\|_{L^2(\mathbb{R})}^2,
\]
and \( \left( \int_{\tau_0 < |\tau| \leq 16} \|G_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}} \leq 2^{\frac{m+j-k}{2}} \|g\|_{L^2(\mathbb{R})}^2 \). With all the estimates, from (5.31), by the Hölder inequality and Hörmander \[26\] Theorem 1.1, we bound \( \|T_{j,m,k}(f,g)\|_{L^2(\mathbb{R})}^2 \) by
\[
\left( \int_{|\tau| \leq \tau_0} \|F_\tau\|_{L^1(\mathbb{R})} \|G_\tau\|_{L^1(\mathbb{R})} d\tau \right)^{\frac{1}{2}} \left( \int_{|\tau| \leq \tau_0} \|F_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}} + \left( \int_{|\tau| \leq \tau_0} \|F_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}} \left( \int_{|\tau| \leq \tau_0} \|G_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}}
\leq \tau_0 \|f\|_{L^2(\mathbb{R})}^2 \|g\|_{L^2(\mathbb{R})}^2 + (2^m |\tau_0|)^{\frac{1}{2}} \left( \int_{|\tau| \leq \tau_0} \|F_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}} \left( \int_{|\tau| \leq \tau_0} \|G_\tau\|_{L^2(\mathbb{R})}^2 d\tau \right)^{\frac{1}{2}}
\leq \left( \tau_0 + (2^m |\tau_0|)^{\frac{1}{2}} + 2^{\frac{m+j-k}{2}} \right) \|f\|_{L^2(\mathbb{R})}^2 \|g\|_{L^2(\mathbb{R})}^2 \leq 2 \frac{2^{m+j-k}}{\tau_0} \|f\|_{L^2(\mathbb{R})}^2 \|g\|_{L^2(\mathbb{R})}^2.
\]
Thus, we obtain (5.26), which leads to (5.16).
Proposition 5.3. Let $u, v, u - \tau, v + 2^{m+j-k}\tau \in \text{supp} \ \Phi$. Then, there exists a positive constant $C$ such that

\begin{equation}
\left| \frac{d^2 Q_r}{du dv}(u,v) \right| \geq C|\tau|,
\end{equation}

where $j$ is large enough.

Proof. Recall that

\begin{equation}
Q_r(u,v) = \varphi(t_0(u,v),u,v) - \varphi\left(t_0\left(u - \tau, v + 2^{m+j-k}\tau\right), u - \tau, v + 2^{m+j-k}\tau\right)
\end{equation}

and

\begin{equation}
\varphi(t,u,v) = -ut - v2^{k-m}\gamma(2^{-j}t).
\end{equation}

Let $\Psi(u,v) := \varphi(t_0(u,v),u,v)$. Then

\begin{equation}
\Psi(u,v) = -ut_0(u,v) - v2^{k-m}\gamma(2^{-j}t_0(u,v)),
\end{equation}

where $t_0(u,v)$ satisfies $\varphi'(t_0(u,v),u,v) = 0$. The definition of $t_0(u,v)$ leads to

\begin{equation}
-u - v2^{m-j+k}\gamma'(2^{-j}t_0(u,v)) = 0.
\end{equation}

Furthermore, we have

\begin{equation}
\frac{dt_0}{du}(u,v) = -\frac{1}{v} \frac{2^{m+2j-k}}{\gamma''(2^{-j}t_0(u,v))},
\end{equation}

and

\begin{equation}
\frac{dt_0}{dv}(u,v) = -\frac{1}{v} \frac{\gamma'(2^{-j}t_0(u,v))}{2^{-j}\gamma''(2^{-j}t_0(u,v))} = \frac{u}{v^2} \frac{2^{m+2j-k}}{\gamma''(2^{-j}t_0(u,v))}.
\end{equation}

Therefore,

\begin{equation}
\frac{d\Psi}{dv}(u,v) = -2^{k-m}\gamma(2^{-j}t_0(u,v))
\end{equation}

and from \ref{5.35},

\begin{equation}
\frac{d^2\Psi}{du dv}(u,v) = \frac{u}{v} \frac{dt_0}{du}(u,v).
\end{equation}

Then

\begin{equation}
\frac{d^3\Psi}{d^2 udv}(u,v) = \frac{1}{v^2} \frac{2^{m+2j-k}}{\gamma''(2^{-j}t_0(u,v))} \left( \frac{\gamma''' - (\gamma'')^2}{(\gamma'')^2} \right) (2^{-j}t_0(u,v)).
\end{equation}

From \ref{1.1}, we have

\begin{equation}
\left| \left( \frac{\gamma''' - (\gamma'')^2}{(\gamma'')^2} \right) (2^{-j}t_0(u,v)) \right| \approx 1.
\end{equation}
From (2.3) and (2.1), note that $\gamma$ is either odd or even, $\gamma'$ is increasing on $[0, \infty)$ and $\frac{\gamma'(2^{-j})}{2^{m+j-k}} \in \text{supp } \phi$; thus, we have

$$
\left| \frac{2^{m+2j-k}}{\gamma''(2^{-j}t_0(u,v))} \right| = \left| \frac{2^{m+j-k} \gamma'(2^{-j})t_0(u,v)}{\gamma'(2^{-j}) \gamma''(2^{-j}t_0(u,v))} \right| \approx 1.
$$

(5.42) Since $v \in \text{supp } \Psi$, from (5.41) and (5.42),

$$
\left| \frac{d^3\Psi}{d^2udv}(u,v) \right| \approx 1.
$$

(5.43) On the other hand, we write $\frac{d^3\Psi}{dvudv}(u,v)$ as

$$
\frac{1}{v^2} \frac{\gamma'(2^{-j}t_0(u,v))}{2^{-j} \gamma''(2^{-j}t_0(u,v))} \left( \frac{\gamma''' - (\gamma'')^2}{(\gamma'')^2} \right) (2^{-j}t_0(u,v)) - \frac{1}{v^2} \frac{\gamma'(2^{-j}t_0(u,v))}{2^{-j} \gamma''(2^{-j}t_0(u,v))}.
$$

(5.44) As in (5.43), we have

$$
\left| \frac{d^3\Psi}{dvudv}(u,v) \right| \leq 1.
$$

(5.45) By the mean value theorem, we rewrite $\frac{d^3Q_x}{dvudv}(u,v)$ as

$$
\frac{d^3\Psi}{d^2udv}(u - \theta_1 \tau, \tau) - \frac{d^3\Psi}{d^2udv}(u - \tau, v + \theta_2 2^{m+j-k} \tau) \cdot 2^{m+j-k} \tau,
$$

where $\theta_1, \theta_2 \in [0, 1]$. From (5.43), (5.45), and $2^{m+j-k} \approx \gamma'(2^{-j})$ when $j$ is large enough, we have

$$
\left| \frac{d^2Q_x}{dvudv}(u,v) \right| \geq (1 - 2^{m+j-k}) |\tau| \geq |\tau|.
$$

(5.46) This finishes the proof of Proposition 5.3. \hfill \Box

5.2 Another estimate by $\sigma$-uniformity and the $TT^*$ argument

In this subsection, we set up

$$
\left\| B_{j,m,k}(f,g) \right\|_{L^1(\mathbb{R})} \leq \left\{ \begin{array}{ll}
2^{-m} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
\Lambda_{j,m,k} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0,
\end{array} \right.
$$

(5.47) where $\Lambda_{j,m,k} := \max\{2^{\frac{m}{2} + \frac{1}{2} - \frac{k}{2}}, 2^{\frac{m}{2}} (\max\{2^{m+j-k}, 2^{-\frac{m}{2}}\})^\frac{1}{2}\}.$

We start by quoting a lemma stated in [16] Lemma 4.4] or [24] Lemma 3.3], which is a slight variant of (3.4) Theorem 7.1] and is called the $\sigma$-uniformity argument. Indeed, this argument can be traced back to Christ et al. [8] and Gowers [21]. Let $\sigma \in (0, 1]$, $\mathbb{I} \subset \mathbb{R}$ be a fixed bounded interval, and $U(\mathbb{I})$ be a nontrivial subset of $L^2(\mathbb{I})$ with $\|u\|_{L^2(\mathbb{I})} \leq C$ uniformly for every element of $u \in U(\mathbb{I})$. We say that a function $f \in L^2(\mathbb{I})$ is $\sigma$-uniform in $U(\mathbb{I})$ if

$$
\left| \int_{\mathbb{I}} f(x)u(x) \, dx \right| \leq \sigma \|f\|_{L^2(\mathbb{I})}
$$

for all $u \in U(\mathbb{I})$. 
Lemma 5.4. [32] Theorem 7.1 Let $\mathcal{L}$ be a bounded sublinear functional from $L^2(\mathbb{R})$ to $\mathbb{C}$, $S_\sigma$ be the set of all functions that are $\sigma$-uniform in $U(\mathbb{R})$,

$$\mathcal{A}_{\sigma} := \sup_{f \in S_\sigma} \frac{|\mathcal{L}(f)|}{\|f\|_{L^2(\mathbb{R})}} \quad \text{and} \quad \mathcal{M} := \sup_{u \in U(\mathbb{R})} |\mathcal{L}(u)|.$$

Then, for all $f \in L^2(\mathbb{R})$, we have

$$|\mathcal{L}(f)| \leq \max \{2\sigma^{-1} \mathcal{M}\},$$

We now turn to the proof of (5.47) by using Lemma 5.4. Let $\mathbb{I} := \supp \Phi$, and for any given $g \in L^2(\mathbb{R})$, let

$$\mathcal{L}(\chi \hat{f}) := \left\|B^*_{jm,k}(f, g) \cdot \chi_{[2^{-j}N, 2^{-j}(N+1)]}\right\|_{L^1(\mathbb{R})}.$$

Step 1: Estimates for $\mathcal{A}_{\sigma}$

We split the interval $[2^{-j}N, 2^{-j}(N+1)]$ as $\bigcup_{w=1}^{2m} I_w$, where

$$I_w := [a_w, a_{w+1}] := \left[2^{-j}N + \frac{w - 1}{2^{m+j-k}}, 2^{-j}N + \frac{w}{2^{m+j-k}}\right].$$

Furthermore, let us set

$$I'_w := \left[2^{-j}N + \frac{w - 1}{2^{m+j-k}} - 2^{-j}2^m, 2^{-j}N + \frac{w}{2^{m+j-k}} + 2^{-j}2^m\right].$$

It is easy to see that $x - 2^{-j}\gamma(2^{-j}) \in I'_w$ if $x \in I_w$, since $\gamma(2^{-j}) \leq 2^{-j}$ for all $t \in \sup \rho$. We write $B^*_{jm,k}(f, g)(x) \cdot \chi_{[2^{-j}N, 2^{-j}(N+1)]}(x)$ as

$$2^{m+j-k} \phi \left(\frac{\gamma(2^{-j})}{2^{m+j-k}}\right)^{2m} \sum_{w=1}^{2m} \mathcal{I}_{I_w}(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{\Phi}(\xi) \hat{g}_w(\eta) e^{i2^{m+j-k} \xi \cdot \mathcal{I}_{I_m,k}(\xi, \eta)} \xi d\eta d\xi,$$

where $I_{I_m,k}$ can be found in (5.19) and $\hat{g}_w := (\mathcal{I}_{I_m,k} \cdot \hat{\Phi} \ast g)$.

For $h \in L^2(\mathbb{R})$, based on $\xi \in \supp \Phi$ and $2^{m+j-k}|x - a_w| \leq 1$ for all $x \in I_w$, by Taylor’s theorem $e^{i2^{m+j-k} \xi(x-a_w)}$, it is safe to split $\int_{-\infty}^{\infty} B^*_{jm,k}(f, g)(x) \cdot \chi_{[2^{-j}N, 2^{-j}(N+1)]}(x) h(x) d\eta$ as the sum of $I$ and $II$, where

$$I := 2^{m+j-k} \phi \left(\frac{\gamma(2^{-j})}{2^{m+j-k}}\right)^{2m} \sum_{w=1}^{2m} \mathcal{I}_{I_w}(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{\Phi}(\xi) e^{i2^{m+j-k} \xi \cdot \mathcal{I}_{I_m,k}(\xi, \eta)} \xi d\eta d\xi,$$

$$II := 2^{m+j-k} \phi \left(\frac{\gamma(2^{-j})}{2^{m+j-k}}\right)^{2m} \sum_{w=1}^{2m} \mathcal{I}_{I_w}(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{\Phi}(\xi) e^{i2^{m+j-k} \xi \cdot \mathcal{I}_{I_m,k}(\xi, \eta)} \xi d\eta d\xi.$$

$G$ is a bump function supported on $\{t \in \mathbb{R} : 2^{-6} \leq |t| \leq 2^{6} \}$ such that $G(t) = 1$ on $\{t \in \mathbb{R} : 2^{-5} \leq |t| \leq 2^{5} \}$, and $F^{-1}(f)$ means the inverse Fourier transform of $f$. 

Bilinear Hilbert Transforms and (Sub)Bilinear Maximal Functions
For $I$. First, we obtain that $|I_{j,m,k}| \leq 2^{-m}$. Indeed, note that $|\eta| \leq 2^{-5-\frac{1}{r'}}$ or $|\eta| \geq 2^{5+\frac{1}{r'}}$, and $\xi \in \text{supp} \Phi$, $\frac{\chi'(\cdot)}{2^m}$ \in \text{supp} \phi; we have $|\varphi'_{\eta}(t,\xi,\eta)| \geq |\xi| - |2^{-m-j+k}y'(-2^{-j}t)| \geq \frac{1}{16}$ or $|\varphi'_{\eta}(t,\xi,\eta)| \geq |2^{-m-j+k}y'(-2^{-j}t)| - |\xi| \geq 8$. Note that $\varphi'_{\eta}(t,\xi,\eta) = -2^{-m-j+k}2^{-j}y'(-2^{-j}t), \eta$, by the Van der Corput lemma, for example, see [42] P. 332, Proposition 2), we have $|I_{j,m,k}| \leq 2^{-m}$. Therefore, by the Hölder inequality, Plancherel formula and Cauchy-Schwarz inequality, we bound $|I|$ by

$$2^{-m+j-k} \phi \left( \frac{(2^{-j})}{2^{m+j-k}} \right) \sum_{w=1}^{2^m} \sum_{l \in \mathbb{N}} \frac{8l}{l} 2^{-m} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| \hat{g}_w \right\|_{L^2(\mathbb{R})} \left\| (2^{m+j-k}(-a_w))' \chi_{I_n}(\cdot)h(\cdot) \right\|_{L^2(\mathbb{R})} \leq 2^{-m+j-k} \phi \left( \frac{(2^{-j})}{2^{m+j-k}} \right) \sum_{w=1}^{2^m} \sum_{l \in \mathbb{N}} \frac{8l}{l} 2^{-m} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| \hat{g}_w \right\|_{L^2(\mathbb{R})} \left\| (2^{m+j-k}(-a_w))' \chi_{I_n}(\cdot)h(\cdot) \right\|_{L^2(\mathbb{R})} \leq 2^{-m+j-k} \frac{1}{2} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})} ^{\frac{1}{2}} \left\| h \right\|_{L^2(\mathbb{R})},$$

where the last inequality is a result of the fact that $2^{m+j-k}|x-a_w| \leq 1$ for all $x \in I_w$. Based on the overlap property of $(I_w)_{w=1}^{2^m}$, we have

$$2^{-m} \left\{ \sum_{w=1}^{2^m} \left\| \hat{g}_w \right\|_{L^2(\mathbb{R})} ^2 \right\} ^{\frac{1}{2}} \leq \left\| g \right\|_{L^2(\mathbb{R})}, \quad \text{if } j-k+2m \leq 0,$$

$$2^{-m} \left\{ \sum_{w=1}^{2^m} \left\| \hat{g}_w \right\|_{L^2(\mathbb{R})} ^2 \right\} ^{\frac{1}{2}} \leq 2^{-m+j-k} \left\| g \right\|_{L^2(\mathbb{R})}, \quad \text{if } j-k+2m > 0.$$

Therefore,

$$|I| \lesssim \begin{cases} 2^{-\frac{m+j-k}{2}} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})} 2^{-\frac{j}{2}} \left\| h \right\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m \leq 0; \\ 2^{-\frac{m+j-k}{2}} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})} 2^{-\frac{j}{2}} \left\| h \right\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m > 0. \end{cases}$$

(5.48)

For $II$, we write $II$ as $II_1 + II_2$ by using the decomposition in (5.23). As in $I$, it is easy to see that

$$|II_1| \lesssim \begin{cases} 2^{-m} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})} 2^{-\frac{j}{2}} \left\| h \right\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m \leq 0; \\ 2^{-\frac{m+j-k}{2}} \left\| \chi f \right\|_{L^2(\mathbb{R})} \left\| g \right\|_{L^2(\mathbb{R})} 2^{-\frac{j}{2}} \left\| h \right\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m > 0. \end{cases}$$

(5.49)

$II_2$ can be written as

$$2^{-\frac{m+j-k}{2}} \phi \left( \frac{(2^{-j})}{2^{m+j-k}} \right) \sum_{w=1}^{2^m} \sum_{l \in \mathbb{N}} \frac{l}{l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) \Phi(\xi) e^{2^{m+j-k} \xi a_w} \widehat{g}_w(\eta) g(\eta) e^{2^m \varphi(t_0(\xi,\eta),\xi,\eta)} \rho(t_0(\xi,\eta)) \xi^{\ell} \mathcal{F}^{-1} \left( \left( 2^{m+j-k}(-a_w) \right)' \chi_{I_n}(\cdot)h(\cdot) \right)(\eta) \, d\xi \, d\eta \leq \left( \frac{2\pi}{\lambda} \right)^\frac{1}{t} 2^{-\frac{m+j-k}{2}} \phi \left( \frac{(2^{-j})}{2^{m+j-k}} \right) \sum_{w=1}^{2^m} \sum_{l \in \mathbb{N}} \frac{l}{l} \int_{-\infty}^{\infty} \Gamma(\eta) \widehat{g}_w(\eta) \mathcal{F}^{-1} \left( \left( 2^{m+j-k}(-a_w) \right)' \chi_{I_n}(\cdot)h(\cdot) \right)(\eta) \, d\eta,$$
where

\[
\begin{align*}
\mathcal{\Upsilon}(\eta) &= \int \nu(\xi, \eta) \hat{f}(\xi) e^{2\pi i \phi(0, \xi, \eta)} e^{2\pi i j - k} \xi^\eta d\xi; \\
\nu(\xi, \eta) &= \rho(t_0(\xi, \eta)) \left( \frac{1}{\gamma(t_0(\xi, \eta))} \right)^{\frac{1}{2}} G(\eta) \xi^\eta \Phi(\xi).
\end{align*}
\]

To apply Lemma 5.4 we first define

\[
U(\|) := \left\{ u_{\|, r}(\xi) \in L^2(\|) : r \in \mathbb{R}, 2^{-\frac{1}{4} \frac{2}{t^2}} \leq |\eta| \leq 2^{\frac{1}{4} \frac{2}{t^2}} \right\},
\]

where \( u_{\|, r}(\xi) := \nu(\xi, \eta) e^{-2\pi i \phi(0, \xi, \eta)} e^{-i\xi \eta}. \) From (2.1), (5.21) and \( \gamma'(2^{-j} \eta) \in \text{supp} \phi, \) it is easy to see that \( ||u_{\|, r}||_{L^2(\|)} \leq C \) uniformly for every element of \( U(\|). \) To estimate \( \mathcal{A}_{\sigma}, \) we first assume that \( \chi_1 f \in L^2(\|) \) is \( \sigma \)-uniform in \( U(\|), \) which further implies that

\[
(5.51) \quad |\mathcal{\Upsilon}(\eta)| \leq \sigma \|\chi_1 f\|_{L^2(\mathbb{R})}.
\]

Regarding \( I, \) let \( \sigma := 2^{-\frac{2}{k}}; \) it is easy to see that

\[
(5.52) \quad II \leq \begin{cases} 
2^{-\frac{2}{k}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} 2^{\frac{j-k}{2}} ||h||_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
2^{-\frac{2}{k}} 2^{\frac{m+j-k}{2}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} 2^{\frac{j-k}{2}} ||h||_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0.
\end{cases}
\]

From (5.48), (5.49), and (5.52), \( |\int_0^\infty B^*_{j,m,k}(f, g) \cdot \chi_{[2^{j-1}N, 2^{j-1}(N+1)]}(x) h(x) \, dx| \) can be bounded by

\[
\begin{cases} 
2^{-\frac{2}{k}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} 2^{\frac{j-k}{2}} ||h||_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
2^{\frac{m+j-k}{2}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} 2^{\frac{j-k}{2}} ||h||_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0.
\end{cases}
\]

Let \( h := \text{sgn}(B^*_{j,m,k}(f, g)) \cdot \chi_{[2^{j-1}N, 2^{j-1}(N+1)]} \cdot ||B^*_{j,m,k}(f, g) \cdot \chi_{[2^{j-1}N, 2^{j-1}(N+1)]}||_{L^1(\mathbb{R})} \) can be bounded by

\[
\begin{cases} 
2^{-\frac{2}{k}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
2^{\frac{m+j-k}{2}} \|\chi_1 f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0.
\end{cases}
\]

It is easy to see that by the definition of \( \mathcal{L}(\chi_1 f) \) that

\[
(5.53) \quad \mathcal{A}_{\sigma} \leq \begin{cases} 
2^{-\frac{2}{k}} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
2^{\frac{m+j-k}{2}} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0.
\end{cases}
\]

**Step 2: Estimates for \( \mathcal{M} \)**

For \( \sigma \in L^\infty(\mathbb{R}), \) let \( x := 2^{k-j} x + 2^k \gamma(2^{-j} t); \) \( \int_0^\infty B^*_{j,m,k}(f, g)(x) \sigma(x) \, dx \) can be written as

\[
(5.54) \quad 2^{\frac{m+j-k}{2}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \int_0^\infty \int_0^\infty \Phi \ast f \left( 2^m x + 2^{m+j} \gamma(2^{-j} t) - 2^m t \right) \times \Phi \ast g \left( 2^{k-j} x + 2^k \gamma(2^{-j} t) \right) \rho(t) \, dt \, dx.
\]
Therefore,

\[
(5.55) \quad \left| \int_{-\infty}^{\infty} B^*_{j,m,k}(f,g)(x) \sigma(x) \, dx \right| \leq \|g\|_{L^2(\mathbb{R})} \|\mathcal{T}(\sigma)\|_{L^2(\mathbb{R})},
\]

where \(\mathcal{T}(\sigma)(x)\) is defined as

\[
2^m \Phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \Phi \ast f \left( 2^m x + 2^m + j \gamma(2^{-j} \tau) - 2^m \tau \right) \sigma \left( 2^k x + 2^k \gamma(2^{-j} \tau) \right) \rho(t) \, dt.
\]

Let \(\chi_2(\xi') \hat{f}(\xi) := \nu(\xi, \eta) e^{-i2m \varphi(t_0(\xi, \eta), \xi, \eta)} e^{-ir\xi} \) and \(x := x + 2^{-m} r, \|\mathcal{T}(\sigma)\|_{L^2(\mathbb{R})}^2\) equals to

\[
(5.56) \quad 2^{m\phi^2} \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \kappa(x,t) \sigma \left( 2^k x + 2^{-j} \gamma(2^{-j} \tau) \right) \rho(t) \, dt \right|^2 \, dx,
\]

where

\[
\kappa(x,t) := \int_{-\infty}^{\infty} \Phi(\xi) \nu(\xi, \eta) e^{i2m (2^{-j} \gamma(2^{2m-1}) \xi - t \xi - \varphi(t_0(\xi, \eta), \xi, \eta))} \, d\xi.
\]

Furthermore, let us set

\[
(5.57) \quad K(\xi, x, t) := x \xi + 2^2 \gamma(2^{-j} \tau) \xi - t \xi - \varphi(t_0(\xi, \eta), \xi, \eta),
\]

we have \(K'_1(\xi, x, t) := x + 2^j \gamma(2^{-j} \tau) - t + t_0(\xi, \eta)\). There exists a positive constant \(\delta\) such that \(2^j \gamma(2^{-j} \tau) - t + t_0(\xi, \eta) \leq \delta\) for \(j\) large enough. Therefore, we can split (5.56) as

\[
(5.58) \quad \|\mathcal{T}(\sigma)\|_{L^2(\mathbb{R})}^2 =: \|\mathcal{T}(a)\|_{L^2(\mathbb{R})}^2 + \|\mathcal{T}(b)\|_{L^2(\mathbb{R})}^2
\]

by \(1 = (1 - \Delta(x)) + \Delta(x)\) on the right-hand side of (5.56), where \(\Delta\) is a bump function supported on \(\{x \in \mathbb{R} : |x| \leq \delta\}\) such that \(\Delta(x) = 1\) on \(\{x \in \mathbb{R} : |x| \leq \frac{\delta}{2}\}\). For \(\|\mathcal{T}(a)\|_{L^2(\mathbb{R})}^2\), we have

\[
|K'_1(\xi, x, t)| \geq |x| - |2^j \gamma(2^{-j} \tau) - t + t_0(\xi, \eta)| \geq |x| - \frac{\delta}{2} \geq \frac{\delta}{4}
\]

by the Van der Corput lemma, for example, see [42, P. 332, Proposition 2]. We have that \(|K(x, t)| \leq 2^{-m} |x|^{-1}\). Then

\[
(5.59) \quad \|\mathcal{T}(a)\|_{L^2(\mathbb{R})}^2 \leq 2^m \left[ \int_{|x| \geq \frac{\delta}{4}} \left( 2^{-m} |x|^{-1} \right)^2 \, dx \right] \|\sigma\|_{L^\infty(\mathbb{R})}^2 \leq 2^{-m} \|\sigma\|_{L^\infty(\mathbb{R})}^2.
\]

We now turn to \(\|\mathcal{T}(b)\|_{L^2(\mathbb{R})}^2\); it has been defined as

\[
(5.60) \quad 2^m \phi^2 \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \kappa(x,t) \sigma \left( 2^k x + 2^{-j} \gamma(2^{-j} \tau) \right) \rho(t) \, dt \right|^2 \Delta(x) \, dx.
\]

For \(\kappa(x,t)\), whose phase function is \(2^m K(\xi, x, t)\). Let \(\xi(x, t)\) satisfy \(K'_1(\xi(x, t), x, t) = 0\); in other words,

\[
(5.61) \quad x + 2^j \gamma(2^{-j} \tau) - t + t_0(\xi(x, t), \eta) = 0.
\]
By the stationary phase method, we assert that

\begin{equation}
(5.62) \quad \kappa(x, t) = e^{2m K(\xi(x, t), x)} \left(\frac{2\pi}{-t2m K''(\xi(x, t), x)}\right)^{\frac{1}{2}} \Phi(\xi(x, t)) \cdot \nu(\xi(x, t), \eta) + O(2^{-\frac{7}{2}m}).
\end{equation}

From (5.62), we split \(\|T(\sigma), b\|_{L^2(\mathbb{R})}^2\) as

\begin{equation}
(5.63) \quad \|T(\sigma), b\|_{L^2(\mathbb{R})}^2 =: \|T(\sigma), b, I\|_{L^2(\mathbb{R})}^2 + \|T(\sigma), b, II\|_{L^2(\mathbb{R})}^2.
\end{equation}

Furthermore, it is easy to see that

\begin{equation}
(5.64) \quad \|T(\sigma), b, II\|_{L^2(\mathbb{R})}^2 \lesssim 2^{-2m}\|\sigma\|_{L^m(\mathbb{R})}^2.
\end{equation}

For \(\|T(\sigma), b, I\|_{L^2(\mathbb{R})}^2\), which can be written as

\begin{equation}
(5.65) \quad \phi \left(\frac{\gamma'(-j)}{2m+j-k}\right) \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{2m K(\xi(x, t), x)} \left(\frac{1}{K''(\xi(x, t), x)}\right)^{\frac{1}{2}} \Phi(\xi(x, t)) \right| \nu(\xi(x, t), \eta) \left(2^{k-j}x + 2^{k-j-m}r + 2^k \gamma(2^{-j}t)\right) \rho(t) \, dt \right|^2 \Delta(x) \, dx =: \|T(\sigma)\|_{L^2(\mathbb{R})}^2,
\end{equation}

where \(T(\sigma)(x)\) is defined as

\begin{equation}
(5.66) \quad \phi \left(\frac{\gamma'(-j)}{2m+j-k}\right) \int_{-\infty}^{\infty} \sqrt{\Delta(x)} e^{2m K(\xi(x, t), x)} \left(\frac{1}{K''(\xi(x, t), x)}\right)^{\frac{1}{2}} \Phi(\xi(x, t)) \cdot \nu(\xi(x, t), \eta) \left(2^{k-j}x + 2^{k-j-m}r + 2^k \gamma(2^{-j}t)\right) \rho(t) \, dt.
\end{equation}

Let

\begin{equation}
(5.67) \quad y(x, t) := x + 2^j \gamma(2^{-j}t) - t.
\end{equation}

Note that from (5.20), (5.61), and (5.57), we have \(K(\xi(x, t), x) = 2^{k-m} \eta \gamma(-2^{-j}y(x, t)).\) Let \(Q_j(t) := \frac{2^j \gamma(2^{-j}t)}{\gamma(2^{-j})}\) and \(u := Q_j(t).\) We rewrite \(T(\sigma)(x)\) as

\begin{equation}
\phi \left(\frac{\gamma'(-j)}{2m+j-k}\right) \int_{-\infty}^{\infty} \sqrt{\Delta(x)} e^{2m 2^{k-m-j} y'(2^{-j}) \eta Q_j(-y(x, Q_j^{-1}(u)))} \left(\frac{1}{K''(\xi(x, Q_j^{-1}(u)), x, Q_j^{-1}(u))}\right)^{\frac{1}{2}} \Phi(\xi(x, Q_j^{-1}(u))) \cdot \nu(\xi(x, Q_j^{-1}(u)), \eta) \left(2^{k-j}x + 2^{k-m-j}r + 2^m 2^{k-m-j} \gamma'(2^{-j}u)\right) \rho(Q_j^{-1}(u)) \, du.
\end{equation}

Furthermore, denote \(\mathcal{K}(x, u)\) as

\begin{equation}
(5.68) \quad \phi \left(\frac{\gamma'(-j)}{2m+j-k}\right) \sqrt{\Delta(x)} \left[\frac{1}{K''(\xi(x, Q_j^{-1}(u)), x, Q_j^{-1}(u))}\right]^{\frac{1}{2}} \Phi(\xi(x, Q_j^{-1}(u))) \cdot \nu(\xi(x, Q_j^{-1}(u)), \eta) \frac{\rho(Q_j^{-1}(u))}{(Q_j)'(Q_j^{-1}(u))}.
\end{equation}
We now consider two cases, i.e.,

\[(5.72) \quad \exists \rho \text{ such that }\]

\[(5.71) \quad \exists \text{ a positive constant } \rho \text{ sup }\]

From the definition of \( \rho \), we have

\[(5.70) \quad \exists \gamma \text{ such that }\]

By the \( T \) argument, \( \| T(\sigma) \|_{L^2(\mathbb{R})}^2 \) is equal to

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(x) \left( \int_{-\infty}^{\infty} K_{\tau,x}(v) e^{i 2^n k - m_j \gamma(2^{-j}) \partial y} P_{\tau,x}(v) \, dv \right) \, dx \, dr,
\]

where

\[
K_{\tau,x}(v) := K(x - y(2^{-j}) v, x + v) \cdot \overline{K(x - y(2^{-j}) v, v)};
\]

\[
W_x(x) := \overline{\sigma \left( 2^{k} x + 2^{k - m_j} r + 2^{m} 2^{k - m_j} \gamma(2^{-j}) u \right) \cdot \overline{\sigma \left( 2^{k} x + 2^{k - m_j} r + 2^{m} 2^{k - m_j} \gamma(2^{-j}) u \right)}};
\]

\[
P_{\tau,x}(v) := Q_j \left( -y \left( x - y(2^{-j}) v, v \right) - Q_j \left( -y \left( x - y(2^{-j}) v, v \right) \right) \right).
\]

From the definition of \( y(x, t) \) in (5.67), we have

\[
\begin{align*}
-y(x - y(2^{-j}) v, Q_j^{-1}(v)) & = -x + Q_j^{-1}(v) \\
-y(x - y(2^{-j}) v, Q_j^{-1}(v + \tau)) & = -x - y(2^{-j}) \tau + Q_j^{-1}(v + \tau).
\end{align*}
\]

Furthermore, let \( P(x, v) := Q_j \left( -x + Q_j^{-1}(v) \right) \). Then

\[
P_{\tau,x}(v) = P(x + y(2^{-j}) \tau, v + \tau) - P(x, v).
\]

We now turn to \( \| T(\sigma) \|_{L^2(\mathbb{R})}^2 \). Indeed, from the definition of \( \mathcal{K} \) in (5.68), we have \( Q_j^{-1}(v) \in \text{supp } \rho \), and \( |v| \leq 1 \) for \( j \) large enough. At the same time, we have \( |v + \tau| \leq 1 \). Therefore, there exists a positive constant \( \epsilon_0 > 1 \) such that

\[
|\tau| \leq \epsilon_0.
\]

Furthermore, we also have \( |K_{\tau,x}(v)| \leq 1 \), which leads to

\[
\left| \int_{-\infty}^{\infty} K_{\tau,x}(v) e^{i 2^n k - m_j \gamma(2^{-j}) \partial y} P_{\tau,x}(v) \, dv \right| \leq 1.
\]

We now consider two cases, i.e.,

\[
\frac{\gamma'(2^{-j})}{|x|} \geq \frac{1}{240} \quad \text{and} \quad \frac{\gamma'(2^{-j})}{|x|} < \frac{1}{240}.
\]

For the first case, note that \( \gamma'(2^{-j}) \) in supp \( \phi \); it is easy to see that

\[
\| T(\sigma) \|_{L^2(\mathbb{R})}^2 \leq 2^{m+j-k} \| \sigma \|_{L^2(\mathbb{R})}^2.
\]

For the second case, i.e., \( \frac{\gamma'(2^{-j})}{|x|} < \frac{1}{240} \). We need the following Lemma 5.5 and Proposition 5.6.
Lemma 5.5. [42] Lemma 2.1] Suppose \( \phi \) is real-valued and smooth in \((a, b)\) and that both \(|\phi'(x)| \geq \sigma_1\) and \(|\phi''(x)| \leq \sigma_2\) for any \( x \in (a, b) \). Then,
\[
\left| \int_a^b e^{i\phi(t)} dt \right| \leq \frac{2}{\sigma_1} + (b - a) \frac{\sigma_2}{\sigma_1^2}.
\]

Proposition 5.6. Let all of the variables \( x, \tau, v, \) and \( \eta \) be the same as in (5.70). If \( \frac{\gamma'(2^{-j})}{2^m} \leq \frac{1}{2e_0} \), then there exists a positive constant \( C \) such that
\[
(5.74) \quad \left| \frac{dP_{\tau,x}(v)}{dv} \right| \geq C|x||\tau| \quad \text{and} \quad \left| \frac{d^2P_{\tau,x}(v)}{dv^2} \right| \leq C|x|
\]
where \( j \) is large enough and \( e_0 \) can be found in (5.71).

Let us postpone the proof of Proposition 5.6 for the moment. By Lemma 5.5 similar to the Corollary on P. 334 in Stein’s book [42], with (5.74) in Proposition 5.6 and the fact that \( \frac{\gamma'(2^{-j})}{2^m} \in \text{supp} \phi \) and \( 2^{-6 - \frac{m}{2^j}} \leq |\eta| \leq 2^{6 + \frac{m}{2^j}} \), we have
\[
(5.75) \quad \left| \int_{-\infty}^{\infty} K_{\tau,x}(v)e^{2^m2^{-m-j}\gamma'(2^{-j})\eta}dP_{\tau,x}(v) \right| \leq \frac{1}{2^m|x||\tau|} + \frac{1}{2^m|x||\tau|^2}.
\]
From (5.72) and (5.75), we can estimate
\[
(5.76) \quad \left| \int_{-\infty}^{\infty} K_{\tau,x}(v)e^{2^m2^{-m-j}\gamma'(2^{-j})\eta}dP_{\tau,x}(v) \right| \leq \left( \frac{1}{2^m|x||\tau|} \right)^{\frac{1}{2}} + \left( \frac{1}{2^m|x||\tau|^2} \right)^{\frac{1}{2}}.
\]
Note that \( |v| \leq 1 \), from the definition of \( \mathcal{K} \), we have \( |x| \leq 1 \) for \( j \) large enough. From (5.71) and (5.76), we have
\[
(5.77) \quad \|T(\sigma)\|^2_{L^2(\mathbb{R})} \leq \int_{|\tau| \leq e_1} \int_{|x| \leq 1} \left( \frac{1}{2^m|x||\tau|} \right)^{\frac{1}{2}} + \left( \frac{1}{2^m|x||\tau|^2} \right)^{\frac{1}{2}} \ dx \ d\tau \cdot \|\sigma\|^2_{L^{\infty}(\mathbb{R})} \leq 2^{-\frac{m}{2}} \|\sigma\|^2_{L^{\infty}(\mathbb{R})}.
\]
By combining (5.73), (5.68), (5.63), (5.64), (5.58) and (5.59), we have
\[
(5.78) \quad \|T(\sigma)\|^2_{L^2(\mathbb{R})} \leq \left( 2^{m+j-k} + 2^{-\frac{m}{2}} + 2^{-2m} + 2^{-m} \right) \|\sigma\|^2_{L^{\infty}(\mathbb{R})}.
\]
By (5.55), we have
\[
(5.79) \quad \left| \int_{-\infty}^{\infty} B_{jm,n}(f,g)(x)dx \right| \leq \left( 2^{m+j-k} + 2^{-\frac{m}{2}} + 2^{-2m} + 2^{-m} \right) \|g\|_{L^2(\mathbb{R})} \|\sigma\|_{L^{\infty}(\mathbb{R})}.
\]
Let \( \sigma := \text{sgn} \left( B_{jm,n}(f,g) \cdot \chi_{[2^{k-j}/N,2^{k-j}/(N+1)]} \right) \). We see that
\[
(5.80) \quad \left\| B_{jm,n}^+(f,g) \cdot \chi_{[2^{k-j}/N,2^{k-j}/(N+1)]} \right\|_{L^1(\mathbb{R})} \leq \left( 2^{m+j-k} + 2^{-\frac{m}{2}} + 2^{-2m} + 2^{-m} \right) \|g\|_{L^2(\mathbb{R})}.
\]
Furthermore, it is clear by the definition of \( \mathcal{L}((\chi f)) \) that
\[
(5.81) \quad \mathcal{M} \leq \begin{cases} 
2^{-\frac{m}{2}} \|g\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m \leq 0, \\
\left( \max \left( 2^{m+j-k}, 2^{-\frac{m}{2}} \right) \right)^2 \|g\|_{L^2(\mathbb{R})}, & \text{if } j-k+2m > 0.
\end{cases}
\]
We now state the proof of Proposition 5.6.
Proof. By simple calculation, we have
\[
\frac{dP}{dv}(x,v) = (Q_j)'(Q_j^{-1}(v) - x) \cdot (Q_j)^{-1}(v) = \frac{(Q_j)'(Q_j^{-1}(v) - x)}{(Q_j)'(Q_j^{-1}(v))}.
\]
Here, we used the fact that \((Q_j^{-1})'(v) \cdot (Q_j)'(Q_j^{-1}(v)) = 1\). Furthermore, we have
\[
\frac{d^2P}{dx dv}(x,v) = -\frac{(Q_j)''(Q_j^{-1}(v) - x)}{(Q_j)'(Q_j^{-1}(v))} = \frac{2^{-j}y''(2^{-j}(Q_j^{-1}(v) - x))}{\gamma'(2^{-j}Q_j^{-1}(v))}.
\]
From (5.61), we have \(Q_j^{-1}(v) - x = t_0(\xi(x - \gamma'(-2^j) v, Q_j^{-1}(v), \eta)\), which implies that \(\frac{1}{2} \leq |Q_j^{-1}(v) - x| \leq 2\). It is also easy to see that \(\frac{1}{2} \leq |Q_j^{-1}(v)| \leq 2\). From (2.1) and (2.3), \(\frac{d^2P}{dx dv}(x,v)\) can be bounded by
\[
\left|\frac{2^{-j}y''(2^{-j}(Q_j^{-1}(v) - x))\gamma'(2^{-j}(Q_j^{-1}(v) - x))}{\gamma'(2^{-j}Q_j^{-1}(v))}\right| \leq 1.
\]
On the other hand, by the mean value theorem,
\[
\left|\frac{\gamma''(2^{-j}(Q_j^{-1}(v) - x))\gamma'(2^{-j}Q_j^{-1}(v) - \vartheta x)}{(\gamma'(2^{-j}Q_j^{-1}(v)))^2}\right| \leq 1,
\]
where \(\vartheta \in [0, 1]\). Noting that \(\frac{\gamma''(t)}{\gamma'(t)^2} = -\frac{\gamma''(t)}{\gamma'(t)}\frac{\gamma'(t)}{\gamma'(t)}\frac{2}{\gamma'(t)}\), which, together with (1.1), (2.1) and (2.3), shows that
\[
\left|\frac{d^2P}{dv^2}(x,v)\right| \approx |x|.
\]
By the mean value theorem, we write
\[
\frac{d^2P_{x\tau}}{dv}(v) = \frac{d^2P}{dv^2}(x + \vartheta_1 \gamma'(-2^j) \tau, v + \tau) \cdot \gamma'(-2^j) \tau + \frac{d^2P}{dv^2}(x, v + \vartheta_2 \tau) \cdot \tau,
\]
where \(\vartheta_1, \vartheta_2 \in [0, 1]\). From (5.84) and (5.86), note that \(\frac{\gamma''(2^{-j})}{|\tau|} \leq \frac{1}{M_0} < \frac{1}{2}\); it follows that
\[
\left|\frac{d^2P_{x\tau}}{dv}(v)\right| \geq \left|\frac{d^2P}{dv^2}(x, v + \vartheta_2 \tau)\right| \cdot |\tau| - \left|\frac{d^2P}{dv^2}(x + \vartheta_1 \gamma'(-2^j) \tau, v + \tau)\right| \cdot \gamma'(-2^j)|\tau| \geq \frac{|x||\tau|}{2}.
\]
This is the desired result regarding \(\left|\frac{d^2P_{x\tau}}{dv}(v)\right|\). For \(\left|\frac{d^2P_{x\tau}}{dv^2}(v)\right|\), from (5.86) and \(\frac{\gamma''(2^{-j})}{|\tau|} \leq \frac{1}{2}, |\tau| \leq 1\), it implies
\[
\left|\frac{d^2P_{x\tau}}{dv^2}(v)\right| \leq \left|\frac{d^2P}{dv^2}(x + \gamma'(-2^j) \tau, v + \tau)\right| + \left|\frac{d^2P}{dv^2}(x, v)\right| \leq |x|.
\]
This finishes the proof of Proposition 5.6. □

Step 3: Estimates for (5.47)

From (5.53) and (5.81), note that \(\sigma = 2^{-\frac{n}{2}}\) and \(\|\chi_t f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}\); by Lemma 5.4 and the definition of \(\mathcal{L}(\chi_t \tilde{f})\), for any \(\chi_t \tilde{f} \in L^2(\mathbb{R})\), it is easy to obtain (5.47).
5.3 Proof of proposition 5.2

In Subsection 5.1, we obtained (see (5.16))

\[ \left\| B^*_{j,m,k}(f,g) \cdot \mathcal{X}[2i^{-j}N,2i^{-j}(N+1)] \right\|_{L^1(\mathbb{R})} \lesssim 2^{\frac{3m+j-\epsilon}{6}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \]  

In Subsection 5.2, we bounded \( \|B^*_{j,m,k}(f,g) \cdot \mathcal{X}[2i^{-j}N,2i^{-j}(N+1)] \|_{L^1(\mathbb{R})} \) (see (5.47)) by

\[
\begin{cases}
2^{-\frac{m}{6}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m \leq 0, \\
\Lambda_{j,m,k} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}, & \text{if } j - k + 2m > 0,
\end{cases}
\]

where \( \Lambda_{j,m,k} := \max\{2^{\frac{3m+j-\epsilon}{6}} \}, 2^{\frac{m}{6}} \max\{2^{m+j-k}, 2^{-\frac{\epsilon}{4}}\}\). Our goal is to prove that (see Proposition 5.2 (5.15))

\[ \left\| B^*_{j,m,k}(f,g) \cdot \mathcal{X}[2i^{-j}N,2i^{-j}(N+1)] \right\|_{L^1(\mathbb{R})} \lesssim 2^{-\epsilon_0 m} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \]

Indeed, for the case that \( j - k + 2m \leq 0 \), (5.90) is true with \( \epsilon_0' := \frac{1}{16} \). For the case that \( j - k + 2m > 0 \), we have

\[ \left\| B^*_{j,m,k}(f,g) \cdot \mathcal{X}[2i^{-j}N,2i^{-j}(N+1)] \right\|_{L^1(\mathbb{R})} \lesssim 2^{\frac{15m}{16} + \frac{j}{4} - \frac{\epsilon}{4}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \]

Thus, combined with (5.89), we have \( \|B^*_{j,m}(f,g) \cdot \mathcal{X}[\frac{m}{\gamma'(2^{-j})}, \frac{m}{\gamma'(2^{-j})}(N+1)] \|_{L^1(\mathbb{R})} \) is bounded by

\[ \left(2^{\frac{15m}{16} + \frac{j}{4} - \frac{\epsilon}{4}}\right)^{\frac{1}{2}} \left(2^{-\frac{3m+j-\epsilon}{6}}\right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{\epsilon}{16}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \]

This is (5.90) with \( \epsilon_0' := \frac{1}{16} \). These are the proof of Proposition 5.2.

We obtain the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R}) \) estimate for \( H^3_{\gamma}(f,g) \).

6 Weak-\( L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R}) \) boundedness of \( H_m(f,g) \)

We begin with a lemma which can be found in [1] Lemma 5.4:

**Lemma 6.1.** [1] Lemma 5.4 Let \( 0 < p < \infty \) and \( A > 0 \). Then, the following statements are equivalent:

(i) \( \|f\|_{L^p(\mathbb{R})} \leq A \).

(ii) For every Lebesgue measurable set \( E \) with \( 0 < |E| < \infty \), there exists a subset \( E' \subset E \) with \( |E'| \geq \frac{|E|}{2} \) such that \( |\{f, \chi_{E'}\}| \leq A|E|^{\frac{1}{p}} \).

Recall that \( \psi_{\Lambda}(\xi) = 2^{\Lambda} \psi(2^{\Lambda} \xi) \); we rewrite \( H^3_{\gamma}(f,g) \) as

\[ H^3_{\gamma}(f,g)(x) = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma' (2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \phi_{m+j} * f(x - 2^{-j}t) \phi_k * g(x - \gamma(2^{-j}t)) \rho(t) \, dt. \]
Let

\[
H_m(f, g)(x) := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \hat{\delta}_{m+j} \ast f \left( x - 2^{-j} \right) \hat{\varphi}_k \ast g \left( x - \gamma(2^{-j}) \right) \rho(t) \, dt.
\]

In this section, we show that

\[
\|H_m(f, g)\|_{L^{p}(\mathbb{R})} \leq m\|f\|_{L^{p}(\mathbb{R})}\|g\|_{L^{p}(\mathbb{R})}
\]

for \( r > \frac{1}{2} \). We put the absolute value inside the integral and define

\[
|H_m(f, g)(x)| := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \hat{\delta}_{m+j} \ast f \left( x - 2^{-j} \right) \hat{\varphi}_k \ast g \left( x - \gamma(2^{-j}) \right) \rho(t) \, dt.
\]

Indeed, our goal is to obtain

\[
\|H_m(f, g)\|_{L^{p}(\mathbb{R})} \leq m\|f\|_{L^{p}(\mathbb{R})}\|g\|_{L^{p}(\mathbb{R})}.
\]

The main tool is Lemma 6.1 above. Therefore, we may assume that \( f := \chi_{F_1} \) and \( g := \chi_{F_2} \) throughout this section, where \( F_1 \) and \( F_2 \) are a Lebesgue measurable set satisfying \( 0 < |F_1|, |F_2| < \infty \). Furthermore, let us set \( F_3 \) as a Lebesgue measurable set with \( 0 < |F_3| < \infty \) and define

\[
\Omega := \left\{ x \in \mathbb{R} : M\chi_{F_1}(x) > C \frac{|F_1|}{|F_3|} \right\} \cup \left\{ x \in \mathbb{R} : M\chi_{F_2}(x) > C \frac{|F_2|}{|F_3|} \right\},
\]

where \( M \) is the Hardy-Littlewood maximal function. From the weak-(1, 1) boundedness of the uncentered Hardy-Littlewood maximal function \( M \), we may assume that \( |\Omega| < \frac{|F_2|}{2} \) with \( C \) large enough. Let \( F'_3 := F_3 \setminus \Omega \); we have \( |F'_3| > \frac{|F_2|}{2} \). By Lemma 6.1 for \( r > \frac{1}{2} \), it suffices to prove

\[
|\langle H_m(f, g), \chi_{F'} \rangle| \leq m|F_1|^{\frac{1}{r}}|F_2|^{\frac{1}{r}}|F_3|^{\frac{1}{r}}.
\]

Recall that \( \psi \) is a nonnegative Schwartz function such that \( \hat{\psi} \) is supported on \( \{ t \in \mathbb{R} : |t| \leq \frac{1}{100} \} \) and satisfies \( \hat{\psi}(0) = 1 \), and \( \psi_{\lambda}(x) = 2^{2\lambda} \hat{\psi}(2^{2\lambda} x) \). Furthermore, let \( \Omega_{\lambda} := \{ x \in \Omega : \text{dist}(x, \Omega^c) \geq 2^{-\lambda} \} \) and \( \hat{\psi}_{\lambda} := \chi_{\Omega^c_{\lambda}} \ast \psi_{\lambda} \). Therefore, we may split \( \hat{\delta}_{m+j} \ast f \) as:

\[
F_{m,j}(x, t) := \left( \hat{\delta}_{m+j} \ast \hat{\varphi}_{m+j} \ast f \right) \left( x - 2^{-j} \right) \quad \text{and} \quad F_{m,j}^C(x, t) := \left( (1 - \hat{\delta}_{m+j}) \ast \hat{\varphi}_{m+j} \ast f \right) \left( x - 2^{-j} \right).
\]

Similarly, we split \( \hat{\phi}_k \ast g \) as:

\[
G_{k,j}(x, t) := \left( \hat{\varphi}_k \ast \hat{\delta}_k \ast g \right) \left( x - \gamma(2^{-j}) \right) \quad \text{and} \quad G_{k,j}^C(x, t) := \left( (1 - \hat{\varphi}_k) \ast \hat{\delta}_k \ast g \right) \left( x - \gamma(2^{-j}) \right).
\]

Then, \( |H_m(f, g)| \) can be split into three error terms:

\[
\left\{ \begin{array}{l}
|H_m^1(f, g)(x)| := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} F_{m,j}^C(x, t) G_{k,j}(x, t) \rho(t) \, dt; \\
|H_m^2(f, g)(x)| := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} F_{m,j}^C(x, t) G_{k,j}^C(x, t) \rho(t) \, dt; \\
|H_m^3(f, g)(x)| := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} |F_{m,j}(x, t) G_{k,j}(x, t) - F_{m,j}^C(x, t) G_{k,j}(x, t)| \rho(t) \, dt;
\end{array} \right.
\]

and a major term:

\[
|H_m^4(f, g)(x)| := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'^{(2^-j)}}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} |F_{m,j}(x, t) G_{k,j}(x, t)| \rho(t) \, dt.
\]
6.1 Error terms \(|H_m^i|(f, g), |H_m^2|(f, g), |H_m^3|(f, g)|

In this subsection, we want to prove that

\[(6.10)\quad |\langle |H_m^i|(f, g), \chi_{E'}\rangle| \leq m|F|_p^\frac{1}{p}|F_2|_q^\frac{1}{q}|F_3|^{\frac{2}{r}}\]

for all \(i \in \{1, 2, 3\}\) and \(r > \frac{1}{2}\).

We will set up \((6.10)\) for \(|H_m^1|(f, g)|. The proofs for \(|H_m^2|(f, g)| and \(|H_m^3|(f, g)| are similar. Let \(K \in \mathbb{N}\) be large enough; for any \(x, y \in \mathbb{R}\) and any Lebesgue measurable set \(E \subset \mathbb{R}\), define

\[
\gamma_{x, y} := \frac{1}{(1 + 2^{j+m}(|x| - |y|))^{\frac{1}{K}}}
\]

and

\[
\delta_{j, k}(x, E) := \frac{1}{(1 + 2^{j+m}\text{dist}(x, E))^{\frac{1}{K}}}.
\]

Noting that \(\bar{\psi}_{m+j}(x - 2^{-j}t) = \int_{\Omega_{j+m}} 2^{j+m}\psi(2^{j+m}(x - 2^{-j}t - y))\,dy\) and \(\int_{\Omega_{j+m}} 2^{j+m}\psi(2^{j+m}(x - 2^{-j}t - y))\,dy = 1\) since \(\psi(0) = 1\), we have \((1 - \bar{\psi}_{m+j})(x - 2^{-j}t) = \int_{\Omega_{j+m}} 2^{j+m}\psi(2^{j+m}(x - 2^{-j}t - y))\,dy \leq \int_{\Omega_{j+m}} 2^{j+m}\delta_{j, k}(x - 2^{-j}t, y)\,dy\). Therefore, \(|\langle |H_m^1|(f, g), \chi_{E'}\rangle| can be bounded by

\[
(6.11)\quad \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi_{m+j-k} \int_{\Omega} \int_{-\infty}^{\infty} \int_{\Omega_{j+m}} 2^{j+m} \delta_{j, k}(x - 2^{-j}t, y) \times \left| \bar{\phi}_{m+j} \ast f(x - 2^{-j}t) \cdot \phi_{k} \ast g(x - \gamma(2^{-j}t)) \rho(t) \right| \,dy \,dt \,dx.
\]

There are two cases: \(x - 2^{-j}t \in \Omega^C\) and \(x - 2^{-j}t \in \Omega\).

**Case 1:** \(x - 2^{-j}t \in \Omega^C\)

It is easy to see that \(Mf(x) = M\chi_{F_1}(x) \leq 1\) for all \(x \in \mathbb{R}\); from the definition of \(\Omega\), it implies

\[
|\gamma_{x, y}| \leq \left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{2}}.
\]

Furthermore, let us set \(u := x - 2^{-j}t\) for any given \(t\), then \(x - \gamma(2^{-j}t) = u + tr_j(t)\). We also have

\[
\tilde{\phi}_k \ast g \leq M\chi_{F_2} \quad \text{and} \quad \delta_{j, k}(x - 2^{-j}t, y) \leq \delta_{j, k}(y, \Omega^C) \cdot \delta_{j, k}(u, y).
\]

Therefore, \((6.11)\) is dominated by

\[
\left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{2}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi_{m+j-k} \int_{\Omega_{j+m}} \delta_{j, k}(y, \Omega^C) \int_{\Omega^C} 2^{j+m} \delta_{j, k}(u, y) \left(\int_{-\infty}^{\infty} M\chi_{F_2} (u + tr_j(t)) |\rho(t)| \,dt \right) \,du \,dy.
\]

For any given \(u \in \Omega^C\), let \(r := u + tr_j(t)\). Then, for \(j > 0\) large enough, we have

\[
dr \leq \frac{dr}{2^{-j} - 2^{-j} \gamma(2^{-j}t)} \leq 2^j \,dr.
\]

On the other hand, note that \(\gamma(t)\) and \(\gamma(t)\) are increasing on \((0, \infty)\) and \((2, 4); we have

\[
|2^{-j}t - \gamma(2^{-j}t)| \leq 2 \cdot 2^{-j} + 2^1 + 2^{1/2} \gamma(2^{-j}) \leq 2^{-j}.
\]
Therefore, by the fact that \((M_XF_2)^{\frac{1}{p}}\) is an \(A_1\) weight, we conclude that

\[
(6.12) \quad \int_{-\infty}^{\infty} M_XF_2 \left(u + \text{tr}_j(t)\right) |\rho(t)| \, dt \leq 2^j \int_{u-2^{-j}}^{u+2^{-j}} (M_XF_2(\tau))^{\frac{1}{p}} \, d\tau \leq (M_XF_2(u))^{\frac{1}{p}}.
\]

Note that \(u \in \Omega^C\), which further implies that

\[
\int_{-\infty}^{\infty} M_XF_2(u + \text{tr}_j(t)) |\rho(t)| \, dt \leq \left(\frac{|F_2|}{|F_3|}\right)^{\frac{1}{q}}.
\]

Then, (6.11) can be bounded by

\[
(6.13) \quad \left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{p}} \left(\frac{|F_2|}{|F_3|}\right)^{\frac{1}{q}} \sqrt{\sum_{j \in \mathbb{Z}_exp} \sum_{k \in \mathbb{Z}} \phi \left(\frac{\gamma'(2^{-j})}{2^{m-j-k}}\right) \int_{\Omega_{j+m}} \delta_j \phi(y, \Omega^C) \int_{\Omega^C} 2^{j+m} \delta_j \phi(u, y) \, du \, dy.
\]

When \(j < 0\) and \(|j|\) are large enough, note that \(\frac{\gamma'(t)}{2^j}\) is strictly increasing on \((0, \infty)\) and \(\gamma'(0) = 0\); we have \(\frac{\gamma'(j_0)}{2^j} > \frac{\gamma'(j_0)}{2^j}\) with \(\frac{\gamma'(j_0)}{2^j} > 2^{j+m} (1 + \frac{1}{2}C^2)\) for \(|j|\) large enough. From (2.4) and (2.2), we have

\[
d\tau \leq \frac{2^{-j} \gamma'(2^{-j})}{\gamma'(2^{-j})} \frac{2^{-j} \gamma(2^{-j})}{\gamma(2^{-j})} \leq \frac{dr}{\gamma(2^{-j})}.
\]

It is also easy to see that \(|2^{-l} - \gamma(2^{-j})| \leq \gamma(2^{-j})\). Therefore, we can control (6.11) by (6.13), as in the case \(j > 0\).

As in (3.5) and noting that \(\int_{\Omega^C} 2^{j+m} \frac{du}{(1 + 2^{j+m}|u-y|)^{\frac{1}{p}}} \leq 1\), (6.13) can be bounded by

\[
\left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{p}} \left(\frac{|F_2|}{|F_3|}\right)^{\frac{1}{q}} \sqrt{\sum_{j \in \mathbb{Z}_exp} \int_{\Omega_{j+m}} \frac{1}{(1 + 2^{j+m}\text{dist}(y, \Omega^C))^\frac{1}{p}} \int_{\Omega^C} 2^{j+m} \frac{du}{(1 + 2^{j+m}|u-y|)^{\frac{1}{p}}} \, dy \leq \left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{p}} \left(\frac{|F_2|}{|F_3|}\right)^{\frac{1}{q}} \int_{\Omega} dy.
\]

This is the desired estimate, since \(|\Omega| < \frac{|F_3|}{|F_3|}\). The last inequality above follows from the fact

\[
\sum_{j \in \mathbb{Z}} \int_{\Omega_{j+m}} \frac{1}{(1 + 2^{j+m}\text{dist}(y, \Omega^C))^\frac{1}{p}} \, dy \leq \int_{\Omega} \sum_{j \in \mathbb{Z}_exp} \sum_{k \in \mathbb{Z}} \frac{1}{(1 + 2^{j+m}\text{dist}(y, \Omega^C))^\frac{1}{p}} \, dy \leq |\Omega| \sum_{l \in \mathbb{N}} \frac{1}{(1 + 2^l)^{\frac{1}{p}}} \leq |\Omega|.
\]

Here, we used the fact that \(\# \{ j \in \mathbb{Z} : 2^{-j-m+l} \leq \text{dist}(y, \Omega^C) \leq 2^{-j-m+l+1} \} \leq 1\) for any given \(y \in \Omega\) and \(l, m \in \mathbb{N}\).

**Case II:** \(x - 2^{-j} \in \Omega\)

In this case, we use the Whitney decomposition theorem to the open set \(\Omega\) (see, for example, [22, P. 609]). Let \(F\) be a collection of pairwise disjoint dyadic interval \(J\)'s such that \(\Omega = \bigcup_{j \in F} J\).
Then, for each $J \in \mathcal{F}$, we have $|J| \leq \text{dist}(J, \Omega^c) \leq 4|J|$; thus, $10J$ meets $\Omega^c$. Furthermore, for each $J \in \mathcal{F}$ and $i \in \{1, 2\}$, we have

$$\frac{1}{|10J|} \int_{10J} \chi_{F_i}(x) \, dx \leq \frac{|F_i|}{|F_3|}.$$  

We now introduce the following two lemmas which can be found in [37]:

**Lemma 6.2.** [37] Lemma 8.1] Let $I_1$, $I_2$ be two intervals in $\mathcal{F}$. Suppose that $a \in I_1$ and $b \in I_2$. If $\text{dist}(I_1, I_2) \geq 100 \min(|I_1|, |I_2|)$, then

$$\delta_{j,K}(a, b) \leq \delta_{j, \Psi}(a, \Omega^c) \cdot \delta_{j, \Psi}(b, \Omega^c).$$  

**Lemma 6.3.** [37] Lemma 8.2] Let $I_1$, $I_2$ be two intervals in $\mathcal{F}$. Suppose that $\text{dist}(I_1, I_2) \leq 100 \min(|I_1|, |I_2|)$, then

$$\frac{|I_2|}{2000} \leq |I_1| \leq 2000|I_2|.$$  

We now turn to (6.11); we rewrite it as

(6.14)  

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{I_1, I_2 \in \mathcal{F}} \int_{\Omega^c} \int_{-\infty}^{\infty} \int_{\Omega_{j,m}} 2^{j+m} \chi_{I_1}(y) \chi_{I_2}(x - 2^{-j}t, y)$$  

$$\times \left[ \hat{\phi}_{m+j} * f(x - 2^{-j}t) \hat{\phi}_{k} * g\left(x - \gamma(2^{-j}t)\rho(t)\right) \right] \, dy \, dt \, dx.$$  

**Case IIa:** $\text{dist}(I_1, I_2) \geq 100 \min(|I_1|, |I_2|)$

In this case, by Lemma 6.2, we have $\delta_{j,K}(x - 2^{-j}t, y) \leq \delta_{j, \Psi}(x - 2^{-j}t, \Omega^c) \cdot \delta_{j, \Psi}(y, \Omega^c)$. Then, (6.14) can be bounded by

(6.15)  

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{I_1, I_2 \in \mathcal{F}} \int_{\Omega^c} \int_{-\infty}^{\infty} \int_{\Omega_{j,m}} 2^{j+m} \chi_{I_1}(y) \delta_{j, \Psi}(y, \Omega^c) \, dy$$  

$$\times \chi_{I_2}(x - 2^{-j}t) \cdot \delta_{j, \Psi}(x - 2^{-j}t, \Omega^c) \left[ \hat{\phi}_{m+j} * f(x - 2^{-j}t) \hat{\phi}_{k} * g\left(x - \gamma(2^{-j}t)\rho(t)\right) \right] \, dx \, dt.$$  

For each $x - 2^{-j}t \in I_2$, we choose $z \in \Omega^c$ such that $\text{dist}(x - 2^{-j}t, \Omega^c) = |x - 2^{-j}t - z|$. The definition of $\Omega$ implies that $\delta_{j, \Psi}(x - 2^{-j}t, \Omega^c) \cdot |\hat{\phi}_{m+j} * f(x - 2^{-j}t)|$ can be bounded by

(6.16)  

$$\int_{-\infty}^{\infty} f(w) \frac{2^{j+m}}{(1 + 2^{j+m}|x - 2^{-j}t - w|)^{\frac{\sigma}{2}}} \, dw \leq \frac{1}{(1 + 2^{j+m}|x - 2^{-j}t - z|)^{\frac{\sigma}{2}}} \leq M f(z) \leq \left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{\theta}}.$$  

Furthermore, let $u := x - 2^{-j}t$; we have $x - \gamma(2^{-j}t) = u + tr_j(t)$. Note that $\hat{\phi}_{k} * g \lesssim M \chi_{F_2}$; we can bound (6.15) by

(6.17)  

$$\left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{\theta}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{I_1, I_2 \in \mathcal{F}} \int_{-\infty}^{\infty} \int_{\Omega_{j,m}} 2^{j+m} \chi_{I_1}(y) \delta_{j, \Psi}(y, \Omega^c) \, dy$$  

$$\leq \left(\frac{|F_1|}{|F_3|}\right)^{\frac{1}{\theta}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \sum_{I_1, I_2 \in \mathcal{F}} \int_{-\infty}^{\infty} \int_{\Omega_{j,m}} 2^{j+m} \chi_{I_1}(y) \delta_{j, \Psi}(y, \Omega^c) \, dy.$$
\[ x \chi_{I_2}(u) \cdot M_{X_F}^2 \left( u + \text{tr}_j(t) \right) |\rho(t)| \, dt \, du. \]

Note that (6.12) holds for all \( j \in \mathbb{Z} \); we have

\[ \sum_{I_2 \in F} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{I_2}(u) \cdot M_{X_F}^2 \left( u + \text{tr}_j(t) \right) |\rho(t)| \, dt \, du \]

\[ \leq \sum_{I_2 \in F} |10I_2| \inf_{w \in 10I_2} M \left( (M_{X_F})^\frac{1}{2} \right)(w) \leq \sum_{I_2 \in F} |10I_2| \inf_{w \in 10I_2} (M_{X_F})^\frac{1}{2} (w) \leq |\Omega| \left( \frac{|F_2|}{|F_3|} \right)^\frac{1}{2}. \]

On the other hand, \( |I_1| \leq \text{dist}(I_1, \Omega^C) \leq 4|I_1| \) implies that \( |I_1| \leq \text{dist}(y, \Omega^C) \leq 5|I_1| \) for all \( y \in I_1 \). Combining it with the fact that \( \# \left\{ j \in \mathbb{Z} : 2^{-j-m+1} \leq \text{dist}(y, \Omega^C) \leq 2^{-j-m+1} \right\} \leq 1 \) and \( I_1 \) is a dyadic interval, we obtain

\[ \sum_{j \in \mathbb{Z}} \sum_{I_2 \in F} \int_{\Omega_{j-m}} 2^{j+m} \chi_{I_2}(y) \cdot \delta_{j, \Omega}(y, \Omega^C) \, dy \]

\[ \leq \sum_{I_1 \in F} \int_{\Omega_{j-m}} \left( 1 \sum_{j \in \mathbb{Z}} 2^{-j-m+1} \leq \text{dist}(y, \Omega^C) \leq 2^{-j-m+1} \right) \frac{1}{(1 + 2^{j+m} \cdot 2^{-j-m+1})^\frac{1}{2}} \sum_{I_2 \in F} \left( 1 + 2^{j+m}|I_1| \right)^\frac{1}{2} \, dy \]

\[ \leq \sum_{m \in \mathbb{N}} \frac{1}{(1 + 2^m)^\frac{1}{2}} \sum_{j \in \mathbb{Z}} \left( \sum_{I_1 \in F} \left( 1 + 2^{j+m}|I_1| \right)^\frac{1}{2} \right. \]

\[ \sum_{I_2 \in F} \left. \right| \chi_{I_2}(u) \cdot M_{X_F}^2 \left( u + \text{tr}_j(t) \right) |\rho(t)| \, dt \, du \]

Therefore, for this case, we also have

\[ |\langle |H_m^2(f, g) \chi_{E^c} \rangle | \leq \left( \frac{|F_1|}{|F_3|} \right)^\frac{1}{2} \left( \frac{|F_2|}{|F_3|} \right)^\frac{1}{2} |\Omega|. \]

**Case IIb:** \( \text{dist}(I_1, I_2) \leq 100 \min(|I_1|, |I_2|) \)

By Lemma [6.3] we have \( \frac{|I_1|}{2000} \leq |I_1| \leq 2000|I_2| \) in this case. We may assume that \( \max(|I_1|, |I_2|) \leq 2^{50}2^{-j} \). Otherwise, we have \( |I_2| \geq 2^82^{-j} \). Since \( x - 2^{-j}t \in I_2 \) and \( x \in \Omega^C \), we have

\[ \text{dist}(x, I_2) \leq |x - (x - 2^{-j}t)| \leq 2 \cdot 2^{-j} \leq \frac{|I_2|}{2^j} \]

which further implies that \( \text{dist}(\Omega^C, I_2) \leq \frac{|I_1|}{2^j} \). This is a contradiction from \( I_2 \in F \).

On the other hand, we may also assume that \( |I_1| \geq 2^{-10}2^{-j-m} \). If not, \( y \in I_1 \cap \Omega \) implies that \( 2^{-j-m} \leq \text{dist}(y, \Omega^C) \leq 4|I_1| \leq 2^{-8}2^{-j-m} \). It is a contradiction. Therefore, for any \( I_1 \in F \), we have

\[ 2^{m+10||I_1||} \leq 2^{j} \leq \frac{2^{50}}{|I_1|}. \]

Then, for any given \( I_1 \in F \), there are at most 10\( m \) many \( j \)'s when \( m \) is large enough.

Furthermore, for any given \( I_1 \in F \), from \( \frac{|I_1|}{2000} \leq |I_1| \leq 2000|I_2| \), the number of interval \( I_2 \)'s with \( \text{dist}(I_1, I_2) \leq 100 \min(|I_1|, |I_2|) \) is finite and independent of \( m \). Let \( u := x - 2^{-j}t \), then \( x - \gamma(2^{-j}t) = u + \text{tr}_j(t) \). Without loss of generality, (6.14) can be controlled by

\[ \sum_{I_1 \in F} \sum_{j \in \mathbb{Z}} \sum_{2^{-j} \leq |I_2| \leq 2^{-j+1}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\Omega_{j+1}} 2^{j+m} \chi_{I_2}(y) \chi_{I_1}(u) \chi_{\Omega^C}(u + 2^{-j}t) \]
\[ \times \delta_{j,K}(u,y) \left| \tilde{\phi}_{m+j} * f(u) \cdot \tilde{\phi}_k \ast g \left( u + \text{tr}_j(t) \right) \rho(t) \right| \, dy \, du. \]

As in (6.16), we have \( \delta_{j,k}(u,y) \cdot |\tilde{\phi}_{m+j} * f(u)| \leq (M_{X_F})^{\frac{1}{2}}(y) \). Noting that \( \tilde{\phi}_k \ast g \leq (M_{X_F})^{\frac{1}{2}} \) and (3.5), we bound (6.21) by

\[ \sum_{I_t \in F} \sum_{j \in \mathbb{Z}} \sum_{\frac{1}{2} \leq 2^j \leq 2^{j_0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\Omega_{j,m}} 2^{j+m} \chi_{I_t}(y) \chi_{I_t}(u) \chi_{\Omega^c}(u + 2^{-j} t) \times \delta_j \phi_k(u,y) \cdot (M_{X_F})^{\frac{1}{2}}(y) \cdot (M_{X_F})^{\frac{1}{2}}(u + \text{tr}_j(t)) \cdot |\rho(t)| \, dy \, du. \]

As in (6.12), by the fact that \( 2^{-j} t \leq 2^{-j} \) for \( j > 0 \) and \( 2^{-j} t \leq \gamma(2^{-j}) \) for \( j \leq 0 \), it is easy to see that

\[ \int_{-\infty}^{\infty} \chi_{\Omega^c}(u + 2^{-j} t) \cdot (M_{X_F})^{\frac{1}{2}}(u + \text{tr}_j(t)) \cdot |\rho(t)| \, dt \leq (M_{X_F})^{\frac{1}{2}}(w) \]

for some \( w \in \Omega^c \). From the definition of \( \Omega \), it implies that

\[ \int_{-\infty}^{\infty} \chi_{\Omega^c}(u + 2^{-j} t) \cdot (M_{X_F})^{\frac{1}{2}}(u + \text{tr}_j(t)) \cdot |\rho(t)| \, dt \leq \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}}. \]

We also have

\[ \int_{-\infty}^{\infty} 2^{j+m} \chi_{I_t}(u) \cdot \delta_j \phi_k(u,y) \, du \leq M_{X_{I_t}}(y) \leq 1. \]

Therefore, as in (6.18), we bound (6.22) by

\[ \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}} \sum_{I_t \in F} \sum_{j \in \mathbb{Z}} \int_{\frac{1}{2} \leq 2^j \leq 2^{j_0}} \int_{\Omega_{j,m}} \chi_{I_t}(y) (M_{X_F})^{\frac{1}{2}}(y) \, dy \]

\[ \leq m \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}} \sum_{I_t \in F} \int_{10 I_t} \int_{10 I_t} (M_{X_F})^{\frac{1}{2}}(y) \, dy \leq \frac{m |\Omega|}{10} \left( \frac{|F_1|}{|F_3|} \right)^{\frac{1}{2}} \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}}. \]

This is the desired estimate, since \( |\Omega| \leq \frac{|F_3|}{10} \). Therefore, we obtain (6.10).

### 6.2 Major term \( |H_m^4|(f,g) \)

In this subsection, we want to prove that

\[ \langle |H_m^4(f,g), X_E \rangle \rangle \leq m |F_1|^{\frac{1}{2}} |F_2|^{\frac{1}{2}} |F_3|^{\frac{1}{2}} \]

for \( r > \frac{1}{2} \). It is easy to see that \( \langle |H_m^4(f,g), X_E \rangle \rangle \) can be bounded by

\[ \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{m+j+k} \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \left( \psi_{m+j} \cdot \tilde{\phi}_{m+j} * f \right) \left( x - 2^{-j} t \right) \left( \tilde{\psi}_k \cdot \tilde{\phi}_k * g \right) \left( x - \gamma(2^{-j} t) \right) \rho(t) \right| \, dy \, dx. \]

It will be bounded by \( m |F_1|^{\frac{1}{2}} |F_2|^{\frac{1}{2}} |F_3|^{\frac{1}{2}} \). In what follows, we give the proof for the case \( j > 0 \). The case \( j \leq 0 \) can be handled similarly. According to the value range of \( r \), we consider the following two cases: \( r \geq 1 \) and \( \frac{1}{2} < r < 1 \). The case \( r \geq 1 \) follows from the following proposition.
Proposition 6.4. There exists a positive constant $C$ such that

$$|\langle |H_m^4(f, g), \chi_{E'}| \rangle | \leq C |F_1|^\theta |F_2|^\theta |F_3|^\theta$$

holds for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $r \geq 1$, $p > 1$, $q > 1$.

Proof. Let $u := x - \gamma(2^{-i})$; then $x - 2^{-j}t = u - \text{tr}_j(t)$. We can bound $|\langle |H_m^4(f, g), \chi_{E'}| \rangle |$ by

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| (\tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (u - \text{tr}_j(t)) (\tilde{\psi}_k \cdot \tilde{\phi}_k \ast g) (u) \rho(t) \right| dt du.$$

Noting that $|\text{tr}_j(t)| \leq |t|$, we have

$$\int_{-\infty}^{\infty} \left| (\tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (u - \text{tr}_j(t)) \rho(t) \right| dt \leq M(\tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f)(u).$$

By the Cauchy-Schwarz inequality, $|\langle |H_m^4(f, g), \chi_{E'}| \rangle |$ can be bounded by

$$\int_{-\infty}^{\infty} \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) |M(\tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f)(u)|^2 \right\|^{\frac{1}{2}} \cdot \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) \left| \tilde{\psi}_k \cdot \tilde{\phi}_k \ast g(u) \right|^2 \right\|^{\frac{1}{2}} du.$$

Furthermore, we also have $|\tilde{\psi}_{m+j}(u)| \leq \sup_{k \in \mathbb{Z}} |\tilde{\psi}_k(u)| \leq M(\chi_{\Omega})(u) \leq 1$. By the Hölder inequality, then $|\langle |H_m^4(f, g), \chi_{E'}| \rangle |$ is controlled by

$$\left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) |M(\tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f)|^2 \right\|_{L^p(\mathbb{R})} \cdot \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) \left| \tilde{\phi}_k \ast g \right|^2 \right\|_{L^q(\mathbb{R})},$$

which can be further bounded by $\|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})} \|M(\chi_{\Omega})\|_{L^r(\mathbb{R})}$ by the Fefferman-Stein inequality, the Littlewood-Paley Theory, (3.4) and (3.5). Noting that $f = \chi_{F_1}$, $g = \chi_{F_2}(x)$ and $|\Omega| < |F_2|$, from the $L^r(\mathbb{R})$ boundedness of $M$ for all $1 < r' \leq \infty$, $|\langle |H_m^4(f, g), \chi_{E'}| \rangle |$ can be bounded by $|F_1|^\theta |F_2|^\theta |F_3|^\theta$. Therefore, we complete the proof of Proposition 6.4. \hfill \Box

The rest of this subsection is devoted to the case $\frac{1}{r} < r' < 1$. Let $\theta$ be a nonnegative Schwartz function such that $\tilde{\theta}$ is supported on $\{ t \in \mathbb{R} : |t| \leq 2^{-10} \}$ and $\tilde{\theta}(0) = 1$, $\theta_A(x) := 2^A \theta(2^A x)$, $A \in \mathbb{Z}$. Let $I_{t,j} := \{ \frac{2}{2^j}, \frac{2^j + 1}{2^j} \}$ and $\chi_{t,j}^* := \chi_{I_{t,j}} \ast \theta_{j+m}$. We can make a partition of unity $1 = \sum_{n \in \mathbb{Z}} \chi_{t,j}^*(x)$. Denote

$$F_{n,m,j}(x,t) := (\chi_{t,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f)(x - 2^{-j}t)$$

and

$$G_{n,m,jk}(x,t) := (\chi_{t,j}^* \cdot \tilde{\psi}_k \cdot \tilde{\phi}_k \ast g)(x - \gamma(2^{-j}t)).$$

and define

$$T_m(f,g)(x) := \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \left| \sum_{n \in \mathbb{Z}} F_{n,m,j}(x,t) \cdot \sum_{k \in \mathbb{Z}} G_{n,m,jk}(x,t) \right| \rho(t) dt.$$

Next, we introduce the definition of a tree.
**Definition 6.5.** Let \( S \subset S_0 := \{(j,n); j \in \mathbb{N}, n \in \mathbb{Z}\} \). A subset \( T \subset S \) is called a tree of \( S \) with top \((j_0, n_0) \in S\) if \( I_{n,j} \subset I_{n_0,j_0} \) for all \((j,n) \in T\). \( T \) is called a maximal tree with top \((j_0, n_0) \in S\) if there is no tree in \( T' \subset S \) with the same top but strictly containing \( T \).

We still need several notations. For any fixed set \( S \subset S_0 \), we abuse the notation \( j \in S \) if and only if \((j,n) \in S\). For any \( j \in S \), we denote \( S_j := \{n \in \mathbb{Z} : (j,n) \in S\} \). An operator \( \Lambda_S[f,g] \) based on the set \( S \) is defined as

\[
\sum_{j \in S} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n \in S_j} F_{n,m,j}(x,t) \right| \cdot \left| \sum_{n \in S_j} G_{n,m,j,k}(x,t) \right| \cdot \rho(t) \, dt \, dx.
\]

We can use this philosophy to define other operators based on any set \( U \subset S_0 \). Then, our aim is to show that

\[
\Lambda_{S_0}[f,g] \lesssim m|F_1|^\frac{1}{2} |F_2|^\frac{1}{2} |F_3|^\frac{1}{2}.
\]

Let \( T \) be a tree; we rewrite \( \Lambda_T[f,g] \) as

\[
\sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n \in T_j} f_{n,m,j}(u - t \mathbf{r}_j(t)) \right| \cdot \left| \sum_{n \in T_j} g_{n,m,j,k}(u) \right| \cdot \rho(t) \, dt \, du.
\]

where \( f_{n,m,j} := \chi_{I_{n,j}} \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_k \ast f \) and \( g_{n,m,j,k} := \chi_{I_{n,j}} \cdot \tilde{\psi}_k \cdot \tilde{\phi}_k \ast g \). Noting that \( |t \mathbf{r}_j(t)| \lesssim |t| \), we have

\[
\int_{-\infty}^{\infty} \left| \sum_{n \in T_j} f_{n,m,j}(u - t \mathbf{r}_j(t)) \right| \rho(t) \, dt \lesssim M \left( \sum_{n \in T_j} f_{n,m,j} \right)(u).
\]

By the Cauchy-Schwarz inequality and Hölder inequality, it is bounded by

\[
\left\| \sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) M \left[ \sum_{n \in T_j} f_{n,m,j} \right]^{\frac{1}{2}} \right\|_{L^{1/2}(\mathbb{R})} \left\| \sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \sum_{n \in T_j} g_{n,m,k,j} \right\|_{L^{1/2}(\mathbb{R})}^{\frac{1}{2}}.
\]

By the Fefferman-Stein inequality and \( \mathbb{S} \), it is controlled by

\[
(6.27) \quad \left\| \sum_{j \in T} \sum_{n \in T_j} f_{n,m,j} \right\|_{L^{1/2}(\mathbb{R})} \left\| \sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \sum_{n \in T_j} g_{n,m,k,j} \right\|_{L^{1/2}(\mathbb{R})}^{\frac{1}{2}} =: ||S_{1,T}(f)||_{L^{1/2}(\mathbb{R})} ||S_{2,T}(g)||_{L^{1/2}(\mathbb{R})},
\]

where we have defined \( S_{1,T}(f) \) and \( S_{2,T}(g) \) by, respectively,

\[
(6.28) \quad \left\{ \begin{array}{l}
S_{1,T}(f)(x) := \left[ \sum_{j \in T} \left| \sum_{n \in T_j} f_{n,m,j}(x) \right|^{1/2} \right]^{1}; \\
S_{2,T}(g)(x) := \left[ \sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma'(2^{-j})}{2^{m+j-k}} \right) \left| \sum_{n \in T_j} g_{n,m,k,j}(x) \right|^{1/2} \right]^{1/2}.
\end{array} \right.
\]
6.2.1 Sizes and BMO estimates

For any positive integer $K$ large enough, we have

$$\chi_{I_{n,j}}^*(x) \lesssim \int_{I_{n,j}} \frac{1}{(1 + 2^{j+m}|x - y|)^K} \, dy =: \chi_{I_{n,j}}^{**}(x) \quad \text{and} \quad \chi_{I_{n,j}}^*(x) \lesssim \frac{1}{(1 + 2^{j+m}\text{dist}(x, I_{n,j}))^K}.$$

**Definition 6.6.** Let $T \subset S$ be a tree, $I_T$ be the time interval of the top of tree $T$, and $Df$ be the derivative of $f$. We define the following **1-size**$(T)$ and **2-size**$(T)$ as

$$1\text{-size}(T) := |I_T|^{-\frac{1}{2}} \left\{ \left\| \sum_{j \in I_T} \left( \sum_{m \in I_j} \phi_{m+j} \cdot \hat{\varphi}_{m+j} \cdot f \right) \right\|_{L^p(\mathbb{R})}^{2 \frac{1}{q}} \right. + \left. \left\| \sum_{j \in I_T} \left( \sum_{m \in I_j} \phi_{m+j} \cdot \hat{\varphi}_{m+j} \cdot (D\varphi)_{m+j} \cdot f \right) \right\|_{L^p(\mathbb{R})}^{2 \frac{1}{q}} \right. + \left. \left\| \sum_{j \in I_T} \left( \sum_{m \in I_j} \phi_{m+j} \cdot \hat{\varphi}_{m+j} \cdot (D\varphi)_{m+j} \cdot \hat{\varphi}_{m+j} \cdot f \right) \right\|_{L^p(\mathbb{R})}^{2 \frac{1}{q}} \right\},$$

$$2\text{-size}(T) := |I_T|^{-\frac{1}{2}} \left\{ \left\| \sum_{j \in I_T} \sum_{k \in \mathbb{Z}} \phi_{j-k} \left( \frac{1}{2^{m+j-k}} \right) \sum_{m \in I_j} \left( \chi_{I_{n,j}}^{**} \cdot \hat{\psi}_k \cdot \hat{\varphi}_k \cdot f \right) \right\|_{L^p(\mathbb{R})}^{2 \frac{1}{q}} \right\}. $$

**Definition 6.7.** For any subset $U \subset S_0$ and $i \in \{1, 2\}$, the **size**$_i(U)$ is defined as

$$\text{size}_i(U) := \sup_{T \subset U} |i\text{-size}(T)|,$$

where $T \subset U$ is a tree.

**Lemma 6.8.** Let $\rho \in (1, \infty)$, $M_{\rho,f} := (M(|f|^\rho))^{\frac{1}{\rho}}$. Then, for any tree $T \subset S$, there exists a positive constant $C$ such that

$$1\text{-size}(T) \leq C \inf_{x \in I_T} M_{\rho,f}(Mf)(x) \quad \text{and} \quad 2\text{-size}(T) \leq C \inf_{x \in I_T} M_{\rho,g}(Mg)(x).$$

**Proof.** By the Littlewood-Paley theory,

$$\left\| \sum_{j \in I_T} \sum_{m \in I_j} \left( \chi_{I_{n,j}}^{**} \cdot \hat{\varphi}_{m+j} \cdot \hat{\varphi}_{m+j} \cdot f \right) \right\|_{L^p(\mathbb{R})}^{\frac{1}{q}} \leq \left\| \sum_{j \in I_T} \left( \phi_{m+j} \cdot f \right) \right\|_{L^p(\mathbb{R})}^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R})}.$$
For the first part of (6.32), we split \( f \) into \( f\chi_{2I_T} \) and \( f\chi_{(2I_T)^c} \). It is easy to see that

\[
(6.34) \quad \left\| \sum_{j \in T} \sum_{n \in T_j} \left( \chi_{I_{n,j}}^* \cdot \chi_{\Omega_{m+j}^c} \ast \psi_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{2I_T}) \right) \right\|_{L^2(\mathbb{R})} \leq \|f\chi_{2I_T}\|_{L^p(\mathbb{R})} \leq |I_T|^{\frac{1}{p}} \inf_{x \in I_T} M_p f(x),
\]

and

\[
(6.35) \quad \left\| \sum_{j \in T} \sum_{n \in T_j} \left( \chi_{I_{n,j}}^* \cdot \chi_{\Omega_{m+j}^c} \ast \psi_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{2I_T}) \right) \right\|_{L^2(\mathbb{R})} \leq \sum \sum \left\| \chi_{I_{n,j}}^* \cdot \chi_{\Omega_{m+j}^c} \ast \psi_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{2I_T}) \right\|_{L^2(\mathbb{R})}.
\]

On the other hand, \( (\chi_{I_{n,j}}^* \cdot \chi_{\Omega_{m+j}^c} \ast \psi_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{2I_T})) \) can be bounded by

\[
1 \quad (1 + 2^{j+m} \text{dist}(x, I_{n,j})) \frac{1}{2} \frac{1}{(1 + 2^{j+m} \text{dist}(I_{n,j}, (2I_T)^c))^{\frac{1}{2}}} M f(x).
\]

(6.35) is bounded by

\[
(6.36) \quad \sum \sum \int_{-\infty}^{\infty} \frac{1}{1 + 2^{j+m} \text{dist}(x, I_{n,j})^{\frac{1}{2}}} \frac{1}{1 + 2^{j+m} \text{dist}(I_{n,j}, (2I_T)^c))^{\frac{1}{2}}} (M f)^p(x) \, dx \left( \frac{1}{2} \right).
\]

For the integral in the square bracket above, which is further bounded by the sum of

\[
\int_{I_T} \frac{1}{1 + 2^{j+m} \text{dist}(x, I_{n,j})^{\frac{1}{2}}} \frac{1}{1 + 2^{j+m} \text{dist}(I_{n,j}, (2I_T)^c))^{\frac{1}{2}}} (M f)^p(x) \, dx =: \Theta
\]

and

\[
\sum \int_{2^{j+1}I_T \setminus 2I_T} \frac{1}{1 + 2^{j+m} \text{dist}(x, I_{n,j})^{\frac{1}{2}}} \frac{1}{1 + 2^{j+m} \text{dist}(I_{n,j}, (2I_T)^c))^{\frac{1}{2}}} (M f)^p(x) \, dx =: \sum \Theta_I.
\]

Therefore, (6.36) is bounded by

\[
(6.37) \quad \sum \sum \Theta_{\frac{1}{2}} + \sum \sum \left( \sum \Theta_I \right)^{\frac{1}{2}}.
\]

For the first part in (6.37), we have

\[
\frac{|I_T|}{|I_{n,j}|} \left( \frac{1}{1 + 2^{j+m} \text{dist}(I_{n,j}, (2I_T)^c))^{\frac{1}{2}}} \right) \leq \frac{|I_T|}{|I_{n,j}|} \left( \frac{1}{1 + \frac{|I_T|}{|I_{n,j}|}} \right)^{\frac{1}{2}} \leq 1 \quad \text{and} \quad \frac{1}{|I_T|} \int_{I_T} (M f)^p(x) \, dx \leq \inf_{x \in I_T} (M_p(M f))^p(x).
\]
Thus
\[
\Theta \leq \frac{|I_{n,j}|}{(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}} |I_T|} \int_{I_T} (Mf)^p(x) \, dx
\]
\[
\leq \frac{|I_{n,j}|}{(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}}} \inf_{x \in I_T} (M_P(Mf))^p(x).
\]
Furthermore, we have
\[
\sum_{j \in T} \sum_{n \in T_j} \Theta^\frac{1}{p} \leq \sum_{j \in T} \sum_{n \in T_j} \frac{|I_{n,j}|^{\frac{1}{p}}} {(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}}} \inf_{x \in I_T} M_P(Mf)(x).
\]
It suffices to show that
\[
\sum_{j \in T} \sum_{n \in T_j} \frac{|I_{n,j}|^{\frac{1}{p}}} {(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}}} \leq |I_T|^\frac{1}{p}.
\]
Indeed, note that \(\text{dist}(I_{n,j}, (2I_T)^C) \geq |I_T|\); the left-hand side of (6.38) is bounded by
\[
\sum_{j \in T} \sum_{n \in T_j} \frac{|I_{n,j}|^{\frac{1}{p}}} {(1 + |I_T|/|I_{n,j}|)^{\frac{mp}{p'}}} \leq \sum_{j \in T} \sum_{n \in T_j} \frac{(2^{i+1}|I_T|)^{\frac{1}{p}}} {(1 + |I_T|/2^{i+1}|I_T|)^{\frac{mp}{p'}}} \leq |I_T|^\frac{1}{p} \sum_{j \in T} 2^{-i} \frac{(2^{i+1})^{\frac{1}{p}}} {(1 + 1/2^{i+1})^{\frac{mp}{p'}}} \leq |I_T|^\frac{1}{p}.
\]
This is the desired estimate.

For the second part in (6.37), note that
\[
\left\{ \begin{array}{ll}
\sum_{k \in \mathbb{N}} \left( 1 + \frac{2^i|I_T|}{|I_{n,j}|} \right)^{\frac{mp}{p'}} \frac{2^{i+1}|I_T|}{|I_{n,j}|} \leq 1; \\
\frac{1}{2^{i+1}|I_T|} \int_{2^{i+1}|I_T| \setminus 2|I_T|} (M_P(f))^p(x) \, dx \leq \inf_{x \in 2^{i+1}|I_T| \setminus 2|I_T|} (M_P(Mf))^p(x) \leq \inf_{x \in I_T} (M_P(Mf))^p(x).
\end{array} \right.
\]
Then, \( \sum_{j \in T} \Theta_j \) can be bounded by
\[
\frac{|I_{n,j}|}{(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}} \sum_{k \in \mathbb{N}} \left( 1 + 2^i|I_T| \right)^{\frac{mp}{p'}} \frac{2^{i+1}|I_T|}{|I_{n,j}|} \frac{1}{2^{i+1}|I_T|} \int_{2^{i+1}|I_T| \setminus 2|I_T|} (Mf)^p(x) \, dx}
\]
\[
\leq \frac{|I_{n,j}|}{(1 + 2^{i+m}\text{dist}(I_{n,j}, (2I_T)^C))^{\frac{mp}{p'}}} \inf_{x \in I_T} (M_P(Mf))^p(x).
\]
As in the first part in (6.37), we can obtain (6.32).

We now turn to the second part of (6.32). From (3.4) and the Littlewood-Paley theory
\[
(6.39)
\left\| \sum_{j \in T} \sum_{k \in \mathbb{Z}} \phi \left( \frac{\gamma(j - 2^{m-j-k})}{2^{m-j-k}} \right) \left( X_{I_{n,j}} \cdot \hat{\phi}_k \cdot \hat{\phi}_k \cdot g \right) \right\|_{L^p(\mathbb{R})}^{\frac{1}{2}} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \hat{\phi}_k \cdot g \right)^2 \right\|_{L^p(\mathbb{R})}^{\frac{1}{2}} \leq \|g\|_{L^q(\mathbb{R})}.
\]
We also split $g$ into $g\chi_{2I_j}$ and $g\chi_{(2I_j)^c}$. As in (6.34),
\[ \left\| \sum_{\ell \in \mathcal{A}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{(2^j-1/2m+j-k)}{2} \right) \right\| \left\| \sum_{n \in \mathcal{B}_{I_j}} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_k \cdot \tilde{\phi}_k \ast (g\chi_{2I_j}) \right) \right\|_{L^p}^{1/2} \lesssim |I_j|^{1/2 \cdot \inf_{x \in I_j} M_q(Mg)(x)).
\]

For the $g\chi_{(2I_j)^c}$ part, from the fact that $\frac{\gamma'(2^j)}{2m+j} \in \text{supp } \phi$, $\gamma$ is strictly increasing on $(0, \infty)$, $j > 0$, it implies that $\frac{1}{2^j} \leq \frac{\gamma'(2^j)}{2m+j} \leq \frac{\gamma'(1)}{2m+j}$, which further implies that $2m+j \leq 2^j$. Therefore,
\[ \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_k \cdot \tilde{\phi}_k \ast (g\chi_{(2I_j)^c}) \right)(x) \lesssim \frac{1}{(1 + 2^{j+m+\text{dist}(x, I_{n,j})})^{\frac{1}{2}}} \frac{1}{(1 + 2^{j+m+\text{dist}(I_{n,j}, (2I_j)^c)})^{\frac{1}{2}}} Mg(x).
\]

As in (3.3), we also have $\sum_{k \in \mathbb{Z}} |\phi \frac{(2^j-1/2m+j-k)}{2}| \leq 1$. As in (6.36), we have
\[ \left\| \sum_{\ell \in \mathcal{A}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{(2^j-1/2m+j-k)}{2} \right) \right\| \left\| \sum_{n \in \mathcal{B}_{I_j}} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_k \cdot \tilde{\phi}_k \ast (g\chi_{(2I_j)^c}) \right) \right\|_{L^p}^{1/2} \lesssim |I_j|^{1/2 \cdot \inf_{x \in I_j} M_q(Mg)(x)).
\]

Therefore, we obtain (6.32). Hence, we complete the proof of Lemma 6.8. }

**Lemma 6.9.** For the general subset $U$ of $S_0$ and $p \in (1, \infty)$, there exists a positive constant $C$ such that
\[ (6.40) \quad \|S 1_{U}(f)\|_{BMO} \leq C \min \left\{ 1, \frac{|F|}{|F|^p} \right\}.
\]

**Proof.** Let $J$ be a dyadic interval of length $2^{-j}$. It suffices to bound the following formula:
\[ (6.41) \quad \inf_{c \in \mathbb{R}} \int_{\mathbb{R}} \left| \sum_{\ell \in \mathcal{A}} \sum_{n \in U_j} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right)(x) \right|^{2 \cdot \frac{1}{p}} - c \ |dx|,
\]
which further is bounded by a sum of the following two parts:
\[ (6.42) \quad J_1 := \int_{\mathbb{R}} \sum_{\ell \in \mathcal{A}} \left| \sum_{n \in U_j} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right)(x) \right|^{2 \cdot \frac{1}{p}} \ dx;
\]
\[ J_2 := \inf_{c \in \mathbb{R}} \int_{\mathbb{R}} \left| \sum_{\ell \in \mathcal{A}} \sum_{n \in U_j} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right)(x) \right|^{2 \cdot \frac{1}{p}} - c \ |dx|.
\]

For $J_1$ in (6.42), we bound it by $J_{1,1} + J_{1,2}$, where
\[ J_{1,1} := \int_{\mathbb{R}} \sum_{\ell \in \mathcal{A}} \sum_{n \in U_j} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{2I_j}) \right)(x) \right|^{2 \cdot \frac{1}{p}} \ dx;
\]
\[ J_{1,2} := \int_{\mathbb{R}} \sum_{\ell \in \mathcal{A}} \sum_{n \in U_j} \left( \chi_{\infty,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f\chi_{(2I_j)^c}) \right)(x) \right|^{2 \cdot \frac{1}{p}} \ dx.
\]
For $J_{1,1}$, by the Hölder inequality, which is bounded by

$$|J|^\frac{1}{p} \left\| \sum_{j \in U: J \leq j+m} \left( \sum_{m \in U_j} (\chi_{I_{m,j}} \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast (f \chi_{2J}) \right)^2 \right\|_{L^p(J)}^{\frac{1}{2}}.$$ 

As in (6.33), the above expression is bounded by

$$|J|^{\frac{1}{p}} \|f \chi_{2J}\|_{L^p(\mathbb{R})} \sup_{j \in U: J \leq j+m} \|\tilde{\psi}_{m+j}\|_{L^\infty(J)}.$$ 

Let $s \in \mathbb{N}$ be the least integer such that $2^s J \cap \Omega^c \neq \emptyset$, where $2^s J$ denotes the interval of length $2^s |J|$ whose center is the same as that of $J$, then

$$|J|^{-\frac{1}{p}} \|f \chi_{2J}\|_{L^p(\mathbb{R})} \leq 2^\frac{s}{p} \inf_{x \in 2^s J} M_p f(x) \leq 2^\frac{s}{p} \min \left\{ 1, \left| \frac{F_1}{|F_3|} \right| \right\}.$$ 

On the other hand, $|J| = 2^{-J} \geq 2^{-j-m}$; it implies

$$\sup_{j \in U: J \leq j+m} \|\tilde{\psi}_{m+j}\|_{L^\infty(J)} \leq \sup_{j \in U: J \leq j+m} \frac{1}{\left( 1 + 2^{j+m} \text{dist}(x, \Omega^c_{j+m}) \right)^{\frac{K}{2}}} \leq \frac{1}{(1 + 2^{j+m} 2^s |J|)^{\frac{K}{2}}} \leq 2^{-Ks}.$$ 

From (6.43), (6.44) and (6.45), we have

$$J_{1,1} \leq |J| \min \left\{ 1, \left| \frac{F_1}{|F_3|} \right| \right\}. $$

For $J_{1,2}$, for each $x \in J$, we choose $z \in \Omega^c$ such that $\text{dist}(x, \Omega^c) \approx |x-z|$. It implies that $\tilde{\phi}_{m+j} \ast (f \chi_{(2J)^c})(x)$ can be bounded by

$$\int_{(2J)^c} |f(y)| 2^{j+m} \delta_{j,K}(x, y) dy \leq \int_{(2J)^c} |f(y)| \delta_{j,\xi}(z, x) \cdot \delta_{j,\xi}(x, y) 2^{j+m} \delta_{j,K}(x, y) dy.$$ 

On the other hand, for each $x \in J$ and $y \in (2J)^c$, it implies that $|x-y| \geq 2^{J-1}$. Furthermore,

$$\tilde{\phi}_{m+j} \ast (f \chi_{(2J)^c})(x) \leq M f(z) \left( \frac{1 + 2^{j+m} \text{dist}(x, \Omega^c)}{1 + 2^{j+m} |x-y|} \right)^{\frac{K}{2}} \leq M f(z) \left( \frac{1 + 2^{j+m} \text{dist}(x, \Omega^c)}{1 + 2^{j+m-J-1}} \right)^{\frac{K}{2}}.$$ 

Note that $M f(z) \leq \min(1, \left| \frac{F_1}{|F_3|} \right|)$ and $\tilde{\phi}_{m+j}(x) (1 + 2^{j+m} \text{dist}(x, \Omega^c))^\xi \leq 1$; we have

$$J_{1,2} \leq \min \left( 1, \left| \frac{F_1}{|F_3|} \right| \right)^{\frac{1}{p}} \cdot \int_J \left| \sum_{j \in U: J \leq j+m} \frac{1}{(1 + 2^{j+m-J-1})^{\frac{K}{2}} \xi} \right| dx \leq |J| \min \left( 1, \left| \frac{F_1}{|F_3|} \right| \right)^{\frac{1}{p}}.$$
By the Hölder inequality and Poincaré inequality, $J_2$ in (6.42) is bounded by

\begin{align}
(6.48) \quad |J|^2 \inf_{c \in \mathbb{R}} \left\{ \int \left| \sum_{j \in U : J > j+m} \left( \sum_{n \in U_j} (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) \right)^2 \right|^{\frac{1}{2}} \ dx \right\}^{\frac{1}{2}} \\
\leq |J| \left\{ \int D \left( \sum_{j \in U : J > j+m} \left( \sum_{n \in U_j} (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) \right)^2 \right) \ dx \right\}^{\frac{1}{2}}.
\end{align}

Note that $|D \left( \sum_{n \in U_j} (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) \right)^2 (x)|$ can be written as $2 | \sum_{n \in U_j} (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) \cdot | \sum_{n \in U_j} D(\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) |$. Regarding $J_{1,2}$, we have $| \tilde{\phi}_{m+j} \ast f(x) \leq Mf(z) \left( 1 + 2^{j+m} \text{dist}(x, \Omega_{\mathbb{C}}) \right)^\frac{\bar{p}}{2}$, where $z \in \Omega_{\mathbb{C}}$, which further implies that $| \sum_{n \in U_j} (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) | \leq Mf(z) \leq \min \{1, \frac{|E_1|}{|F_3|} \}$. At the same time, we also have that $| \sum_{n \in U_j} D(\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) | \leq 2^{j+m} \min \{1, \frac{|E_1|}{|F_3|} \}$. Therefore, from (6.48),

\begin{align}
(6.49) \quad J_2 \leq |J| \left\{ \int \sum_{j \in U : J > j+m} 2^{j+m} \left( \min \left\{ 1, \frac{|E_1|}{|F_3|} \right\} \right) \ dx \right\}^{\frac{1}{2}} \leq |J| \min \left\{ 1, \frac{|E_1|}{|F_3|} \right\}^{\frac{1}{2}}.
\end{align}

From (6.46), (6.47) and (6.49), we obtain (6.40).

**Lemma 6.10.** There exists a positive constant $C$ such that

\begin{align}
(6.50) \quad \left\| \chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right\|_{BMO} \leq C 2^m \text{size}_1(I_{n,j}).
\end{align}

**Proof.** Let $J$ be a dyadic interval. It suffices to bound the following formula:

\begin{align}
(6.51) \quad \inf_{c \in \mathbb{R}} \int \left| (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) - c \right| \ dx.
\end{align}

If $|I_{n,j}| \leq |J|$, by the Hölder inequality, we have

\begin{align}
\inf_{c \in \mathbb{R}} \int \left| (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) - c \right| \ dx \leq \left\| \chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right\|_{L^p(\mathbb{R})} |J|^{\frac{1}{p}} \leq |J| \text{size}_1(I_{n,j}).
\end{align}

If $|I_{n,j}| > |J|$, by the Poincaré inequality, we have

\begin{align}
\inf_{c \in \mathbb{R}} \int \left| (\chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f) (x) - c \right| \ dx \leq |J| \int D \left( \chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (x) \ dx.
\end{align}

By the Hölder inequality, without loss of generality, the right-hand side of (6.52) can be bounded by

\begin{align}
|J|^{2^{j+m}|I_{n,j}|} \left\| \chi_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot f \right\|_{L^p(\mathbb{R})} |J|^{-\frac{1}{p}} |I_{n,j}|^{\frac{1}{p}} \leq |J|^{2^m \text{size}_1(I_{n,j})}.
\end{align}

Therefore, we obtain (6.50).
Lemma 6.11. Let \( T \subset S_0 \) be a tree; then there exists a positive constant \( C \) such that

\[
|S_{1,T}(f)|_{\text{BMO}} \leq C2^n \text{size}_1(T).
\]

Proof. Let \( J \) be a dyadic interval and \( T_J := \{(j,n) \in T : I_{n,j} \subset 3J\} \). It suffices to bound the following formula:

\[
\inf_{c \in \mathbb{R}} \int_J \left[ \sum_{j \in T} \left| \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (x) \right|^2 \right]^{\frac{1}{2}} - c \, dx.
\]

Furthermore, let

\[
T_J^1 := \{(j,n) \in T \setminus T_J : |I_{n,j}| \leq |J|\} \quad \text{and} \quad T_J^2 := \{(j,n) \in T \setminus T_J : |I_{n,j}| > |J|\}.
\]

Then, (6.54) can be bounded by a sum of the following three parts:

\[
\begin{align*}
J^1 &:= \int_J \left[ \sum_{j \in T} \left( \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (x) \right)^2 \right]^{\frac{1}{2}} \, dx; \\
J^2 &:= \int_J \left[ \sum_{j \in T} \left( \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (x) \right)^2 \right]^{\frac{1}{2}} \, dx; \\
J^3 &:= \inf_{c \in \mathbb{R}} \int_J \left[ \sum_{j \in T} \left( \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (x) \right)^2 \right]^{\frac{1}{2}} - c \, dx.
\end{align*}
\]

By the Hölder inequality, \( J^1 \) can be bounded by

\[
\left\| \left( \sum_{j \in T} \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) \right)^2 \right\|_{L^p(J)}^{\frac{1}{2}} \leq \text{size}_1(T) |J|^\frac{1}{p} |J|^\frac{1}{q} = \text{size}_1(T)|J|.
\]

By the Hölder inequality, \( J^2 \) can be bounded by

\[
\left\| \sum_{j \in T} \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right)^2 \right\|_{L^p(J)}^{\frac{1}{2}} \leq \text{size}_1(T) \sum_{j \in T} \sum_{n \in T_{j,k}} \frac{|I_{n,j}|^{\frac{1}{2}}}{(1 + 2^{j+m} \text{dist}(J, I_{n,j}))^\frac{1}{r}} |J|^\frac{1}{q} \leq \text{size}_1(T)|J|.
\]

By the Hölder inequality and Poincaré inequality, as in (6.48), we bound \( J^3 \) by

\[
|J| \left\{ \int_J \left[ \sum_{j \in T} \left( \sum_{n \in T_{j,k}} \left( X_{n,j}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) \right)^2 \right] (x) \right\} \frac{1}{2}.
\]
Without loss of generality, by the Cauchy-Schwarz inequality, \( D(\sum_{n \in T_j} \langle \chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \rangle)^2 \) can be bounded by the sum of

\[
2^\frac{m}{J} \left\{ \int_j \left( \sum_{n \in T_j} \left| \sum_{n \in T_j} \frac{1}{|I_{n,j}|^\frac{1}{2}} |\chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f(x)| \right|^2 \right)^\frac{1}{2} \right\} ^\frac{2}{J}.
\]

and

\[
2^\frac{m}{J} \left\{ \int_j \left( \sum_{n \in T_j} \left| \sum_{n \in T_j} \frac{1}{|I_{n,j}|^\frac{1}{2}} |\chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot (D\tilde{\phi})_{m+j} \ast f(x)| \right|^2 \right)^\frac{1}{2} \right\} ^\frac{2}{J}.
\]

It is suffices to bound (6.56). Note that

\[
\left\| \chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right\|_{L^p(\mathbb{R})} \leq |I_{n,j}|^{\frac{1}{p}} \text{size}_1(I_{n,j})
\]

and (6.50), by interpolating, we obtain

\[
\left\| \chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right\|_{L^2(\mathbb{R})} \leq 2^\frac{m}{J} |I_{n,j}|^{\frac{1}{p}} \text{size}_1(I_{n,j}).
\]

By the Hölder inequality, (6.56) can be bounded by

\[
2^\frac{m}{J} \left\| \chi_{I_{n,j}}^* \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right\|_{L^2(\mathbb{R})} \leq 2^\frac{m}{J} \text{size}_1(I_{n,j}) \leq 2^m |J| \text{size}_1(T).
\]

The last inequality follows from

\[
\sum_{j \in T_j} \sum_{n \in T_j} \frac{|I_{n,j}|^{\frac{1}{p}}}{(1 + 2^{j+m} \text{dist}(J, I_{n,j}))^\frac{k}{p}} \leq 1,
\]

which can be found in Li (36). Hence, we complete the proof of Lemma 6.11.

From (6.28) to (6.31), we have

\[
\|S_{1,T}(f)\|_{L^p(\mathbb{R})} \leq \text{size}_1(T) \cdot |I_T|^{\frac{1}{p}} \quad \text{and} \quad \|S_{2,T}(f)\|_{L^p(\mathbb{R})} \leq \text{size}_2(T) \cdot |I_T|^{\frac{1}{p}}.
\]

From (6.40) in Lemma 6.9 and (6.53) Lemma 6.11, by interpolation with (6.59), respectively, we obtain

\[
\begin{cases}
\|S_{1,T}(f)\|_{L^p(\mathbb{R})} \leq |I_T|^{\frac{1}{p}} \text{size}_1(T) \min \left(1, \frac{|I_T|}{|I_{n,j}|} \right)^{\frac{1}{2}} \cdot |I_T|^{\frac{1}{p}}; \\
\|S_{1,T}(f)\|_{L^p(\mathbb{R})} \leq |I_T|^{\frac{1}{p}} \text{size}_1(T) \min \left(1, \frac{|I_T|}{|I_{n,j}|} \right)^{\frac{1}{2}} \cdot |I_T|^{\frac{1}{p}}.
\end{cases}
\]
Furthermore, we have

\[
\|S_{1,T}(f)\|_{L^p'([0,1])} \lesssim |T|^\frac{\eta}{p} \text{size}_1^*(T),
\]

where \(\text{size}_1^*(T) := \min\{\text{size}_1(T)^\eta \min\{1, |E_i|^{\frac{1}{2m} \frac{1}{2m}}\}, \text{size}_1(T)^{2m(1-\frac{1}{2m})}\}\).

### 6.2.2 The estimates for \(|H^4_n|(f, g)\)

For \(S \subset S_0\), we rewrite \(\Lambda_S[f, g]\) as

\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y'(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n \in S_j} f_{n,m,j}(x - tr_j(t)) \right| \left| \sum_{n \in S_j} g_{n,m,j,k}(x) \right| \cdot \rho(t) \, dx \, dt.
\]

**Lemma 6.12.** Let \(T \subset S_0\) be a tree and \(P \subset S_0\) be a subset; if \(T \cap P = \emptyset\) and \(T\) is a maximal tree in \(P \cup T\), then there exists a positive constant \(C\) such that

\[
|\Lambda_{P \cup T}[f, g] - \Lambda_P[f, g] - \Lambda_T[f, g]| \leq C \text{size}_1^*(P \cup T) \text{size}_2(P \cup T)|T|,
\]

where \(\text{size}_1^*(P \cup T)\) is defined naturally as \(\text{size}_1^*(T)\).

**Proof.** We begin our proof by a definition and a lemma. Let

\[
d_j(S_1, S_2) := \left(1 + 2^{j+m} \text{dist}(S_1, S_2)\right)^{-K}
\]

for any \(S_1, S_2 \subset S_0\),

where \(S_1\) and \(S_2\) are defined as the union of all intervals \(I_{n,j}\) with \((j, n) \in S_1\) and \((j, n) \in S_2\), respectively. As in [37, Lemma 10.1], if \(T \cap P = \emptyset\), we have

\[
\sum_{j \in \mathbb{N}} \sum_{I_j \in \mathbb{Z}} |I_j| |d_j(S_1, T_j) d_j(I, P_j)| \leq |T|;
\]

\[
\sum_{j \in \mathbb{N}} \sum_{I_j \in \mathbb{Z}} |I_j| |d_j(S_1, T_j) d_j(I, T_j)| \leq |T|.
\]

Since \(T\) is a maximal tree in \(P \cup T\), then the left hand side of (6.62) can be bounded by

\[
A_1 + B := \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y'(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n \in T_j} f_{n,m,j}(x - tr_j(t)) \right| \left| \sum_{n \in T_j} g_{n,m,j,k}(x) \right| \cdot \rho(t) \, dx \, dt
\]

\[
+ \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y'(2^{-j})}{2^{m+j-k}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n \in T_j} f_{n,m,j}(x - tr_j(t)) \right| \left| \sum_{n \in T_j} g_{n,m,j,k}(x) \right| \cdot \rho(t) \, dx \, dt.
\]

We here give the estimate of \(A_1; B\) can be handled similarly. From the Hölder inequality, \(A_1\) is bounded by

\[
\sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \phi \left( \frac{y'(2^{-j})}{2^{m+j-k}} \right) \sum_{I_j \in \mathbb{Z}} \left| \int_{-\infty}^{\infty} \left| \sum_{n \in T_j} f_{n,m,j}(x - tr_j(t)) \right| \cdot \rho(t) \, dt \right| \left| \phi \left( \frac{y'(2^{-j})}{2^{m+j-k}} \right) \sum_{n \in T_j} g_{n,m,j,k} \right|_{L^p(I_j)}.
\]
For \( x \in I \), note that \( \text{tr}_j(t) \lesssim 2^{-j} \); without loss of generality, we may write \( \chi_{I_{n,j}}^+ (x - \text{tr}_j(t)) \lesssim d_j(SI, I_{n,j}) \cdot \chi_{I_{n,j}}^+ (x - \text{tr}_j(t)). \) From (6.60), \( \| \sum_{n \in P_j} f_{n,m,j} (\cdot - \text{tr}_j(t)) \cdot \rho(t) \|_{L^\rho(I)} \) is controlled by

\[
\sum_{m \in P_j} d_j(SI, I_{n,j}) \left\| \int_{-\infty}^{\infty} \left( \chi_{I_{n,j}}^+ \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) (\cdot - \text{tr}_j(t)) \cdot \rho(t) \, dt \right\|_{L^\rho(I)}
\leq \sum_{n \in P_j} d_j(SI, I_{n,j}) \left\| M \left( \chi_{I_{n,j}}^+ \cdot \tilde{\psi}_{m+j} \cdot \tilde{\phi}_{m+j} \ast f \right) \right\|_{L^\rho(\mathbb{R})} \leq d_j(SI, P_j) |I\|^\frac{1}{2} \text{size}_1(P \cup T).
\]

It is easy to see that

\[
\left\| \phi^\frac{1}{2} \left( \frac{\gamma(2^{-j})}{2m+1-k} \right) \cdot \sum_{n \in I_j} g_{n,m,j,k} \right\|_{L^\rho(I)} \leq d_j(I, T_j)|I|^{\frac{1}{2}} \text{size}_2(P \cup T).
\]

As in (3.5), we have \( \sum_{k \in \mathbb{Z}} \phi^\frac{1}{2} \left( \frac{\gamma(2^{-j})}{2m+1-k} \right) \leq 1. \) This, combined with (6.64), (6.65) and (6.63), implies that Part A can be bounded by

\[
\sum_{j \in \mathbb{N}} \sum_{I: |I| = 2^{-j}} |I|d_j(SI, P_j) d_j(I, T_j) \text{size}_1^*(P \cup T) \text{size}_2(P \cup T) \leq \text{size}_1^*(P \cup T) \text{size}_2(P \cup T) |I_T|.
\]

Hence, we complete the proof of Lemma 6.12. \( \square \)

For any \( S \subset S_0, k \in \{1, 2\} \), as in [37, Lemma 6.12], by (6.32) in Lemma 6.8 we can always split \( S \) into \( S_1 \) and \( S_2 \):

(i) \( S_1 := \bigcup_{T \in F} T \) with \( \bigcup_{T \in F} |I_T| \leq \frac{|F_1|}{\text{size}_1(S \rho)} \) and \( \bigcup_{T \in F} |I_T| \leq \frac{|F_1|}{\text{size}_2(S \rho)} \), where \( T \) is maximal tree;

(ii) \( S_2 := S \setminus S_1 \) with \( \text{size}_1(S_2) \leq (\frac{1}{2})^\frac{1}{2} \text{size}_1(S) \) and \( \text{size}_2(S_2) \leq (\frac{1}{2})^\frac{1}{2} \text{size}_2(S) \),

which further implies that we can write \( S_0 \) as

\[
S_0 = \bigcup_{\sigma \leq 1} S_\sigma,
\]

where \( \sigma \) ranges over positive dyadic numbers, and \( S_\sigma \) is a union of maximal trees such that for each \( T \in S_\sigma \), we have

\[
\text{size}_1(T) \leq \sigma^{\frac{1}{2}} \left( \frac{|F_1|}{|F_3|} \right)^{\frac{1}{2}} \quad \text{and} \quad \text{size}_2(T) \leq \sigma^{\frac{1}{2}} \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}},
\]

and

\[
\text{size}_1(T) \geq \left( \frac{\sigma}{2} \right)^{\frac{1}{2}} \left( \frac{|F_1|}{|F_3|} \right)^{\frac{1}{2}} \quad \text{or} \quad \text{size}_2(T) \geq \left( \frac{\sigma}{2} \right)^{\frac{1}{2}} \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{2}}.
\]
We now turn to the proof of (6.26). Indeed, it is easy to see that
\[(6.69) \quad \sum_{T \in S_x} |l_T| \leq \left( \left( \frac{\|f_1\|}{\|T\|} \right)^{\frac{1}{\sigma}} + \left( \frac{\|f_2\|}{\|T\|} \right)^{\frac{1}{\sigma}} \right)^{\frac{1}{\sigma}} \approx \frac{|F_3|}{\sigma}.
\]
Furthermore, the fact that \(\Lambda_T[f, g]\) is bounded by (6.27), combined with the second part in (6.59) and (6.60), gives
\[(6.70) \quad \Lambda_T[f, g] \leq |l_T| \text{size}_1^*(T) \text{size}_2(T).
\]
From Lemma 6.12, (6.66) and (6.70), we conclude
\[(6.71) \quad \Lambda S_T[f, g] \leq \sum_{\sigma \leq 1} \sum_{T \in S_x} |l_T| \text{size}_1^*(T) \text{size}_2(T).
\]
From the definition of \(\text{size}_1^*(T)\), (6.67) and (6.69), the above expression can be bounded by
\[
|F_1|^\frac{\sigma}{2} |F_2|^\frac{\sigma}{2} |F_3|^\frac{\sigma}{2} \sum_{\sigma \leq 1} \min \left\{ 2^{m(\frac{1}{\sigma} - \frac{1}{\varphi})}, \sigma^{\frac{1}{\varphi}}, \sigma^{-\frac{1}{\varphi}} \right\} \leq m |F_1|^\frac{\sigma}{2} |F_2|^\frac{\sigma}{2} |F_3|^\frac{\sigma}{2}.
\]
This is the desired estimate.

7 The boundedness of \(M_\gamma(f, g)\)

For \(M_\gamma^1(f, g)\), as \(H_\gamma^1(f, g)\), we write \(\tilde{m}_T^1(\xi, \eta)\) as
\[
\sum_{m,n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \frac{(-i)^{u+v+2mu+2nv}}{u!v!} \tilde{\phi}_u \left( \frac{\xi}{2^m} \right) \tilde{\phi}_v \left( \frac{\eta}{2^k} \right) \int_{-\infty}^{\infty} t^\mu \left( \frac{2^j \gamma(2^j t^\gamma)}{2^{2\nu j - k}} \right)^\nu |\varrho(t)| \, dt.
\]
Furthermore, \(M_\gamma^1(f, g)\) can be written as
\[
M_\gamma^1(f, g)(x) = \sup_{j \in \mathbb{Z}} \sum_{m,n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \frac{(-i)^{u+v+2mu+2nv}}{u!v!} \tilde{\phi}_u \left( \frac{\gamma(2^j)}{2^{2\nu j - k}} \right) \int_{-\infty}^{\infty} t^\mu \left( \frac{2^{j+\gamma(2^j t^\gamma)}}{2^{2\nu j - k}} \right)^\nu |\varrho(t)| \, dt \tilde{\phi}_{u,m+j} * f(x) \cdot \tilde{\phi}_{v,k} * g(x).
\]
From [42 P. 24 Proposition], there exists a positive constant \(C\) such that \(\sup_{j \in \mathbb{Z}} |\tilde{\phi}_{u,m+j} * f| \leq C^m Mf\) and \(|\tilde{\phi}_{v,k} * g| \leq C^m Mg\). As in (3.3) and (3.5), combining with the fact that \(\sum_{m,n \in \mathbb{Z}} \sum_{u,v \in \mathbb{N}} \frac{c_{2mu+2nv}}{u!v!} \leq 1\), we conclude that
\[
M_\gamma^1(f, g)(x) \leq Mf(x) Mg(x).
\]
For \(r > \frac{1}{2}\), by the Hölder inequality, it leads to
\[(7.1) \quad \|M_\gamma^1(f, g)\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.
\]
For $M_3^2(f, g)$, as in $M_3^1(f, g)$, we also have $M_3^2(f, g)(x) \leq Mf(x)Mg(x)$. Therefore, for all $r > \frac{1}{2}$,
\begin{equation}
(7.2) \quad \|M_3^2(f, g)\|_{L^r(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.
\end{equation}

For $M_3^2(f, g)$, without loss of generality, we may assume that $m = n$. Therefore, we rewrite $M_3^2(f, g)$ as
\begin{equation}
(7.3) \quad M_3^2(f, g)(x) = \sup_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta) e^{i\xi x} e^{in\eta} \tilde{m}_j^3(\xi, \eta) \, d\xi d\eta,
\end{equation}
where $\tilde{m}_j^3(\xi, \eta) := \sum_{m \in \mathbb{N}, k \in \mathbb{Z}} \tilde{m}_{j,m,k}(\xi, \eta)$. Then (7.3) can be bounded by
\begin{equation}
(7.4) \quad \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta) e^{i\xi x} e^{in\eta} \tilde{m}_{j,m,k}(\xi, \eta) \, d\xi d\eta.
\end{equation}

As in (5.5) and (5.6), we have that there exists a positive constant $c_0$ such that
\begin{equation}
(7.5) \quad \left\| \sum_{j,k \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta) e^{i\xi x} e^{in\eta} \tilde{m}_{j,m,k}(\xi, \eta) \, d\xi d\eta \right\|_{L^1(\mathbb{R})} \lesssim 2^{-c_0m}\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}
\end{equation}
holds uniformly for $j, k \in \mathbb{Z}$. As in (5.3), we have
\begin{equation}
(7.6) \quad \left\| \sum_{j,k \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\eta) e^{i\xi x} e^{in\eta} \tilde{m}_{j,m,k}(\xi, \eta) \, d\xi d\eta \right\|_{L^p(\mathbb{R})} \lesssim m\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}
\end{equation}
holds uniformly for $j, k \in \mathbb{Z}$. By interpolating between (7.5) and (7.6), we conclude that
\begin{equation}
(7.7) \quad \|M_3^2(f, g)\|_{L^r(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}
\end{equation}
for $r > \frac{1}{2}$. From (7.1), (7.2) and (7.7), we obtain the $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ boundedness for the (sub)bilinear maximal function $M_3(f, g)$ for $r > \frac{1}{2}$.

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Bilinear Hilbert Transforms and (Sub)Bilinear Maximal Functions

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