ERROR ESTIMATION OF THE
RELAXATION FINITE DIFFERENCE SCHEME
FOR THE NONLINEAR SCHRODINGER EQUATION

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Abstract. We consider an initial- and boundary-value problem for the nonlinear Schrödinger equation with homogeneous Dirichlet boundary conditions in the one space dimension case. We discretize the problem in space by a central finite difference method and in time by the Relaxation Scheme proposed by C. Besse [C. R. Acad. Sci. Paris Sér. I 326 (1998), 1427-1432]. We provide optimal order error estimates, in the discrete \( L^2_n(H^1_n) \) norm, for the approximation error for the time nodes and at the intermediate time nodes. In the context of the nonlinear Schrödinger equation, it is the first time that the derivation of an error estimate, for a fully discrete method based on the Relaxation Scheme, is completely addressed.

1. Introduction

1.1. Formulation of the problem. Let \( T > 0, I := [x_a, x_b] \) be a bounded closed interval in \( \mathbb{R} \), \( Q := [0, T] \times I \) and \( u : Q \rightarrow \mathbb{C} \) be the solution of the following initial and boundary value problem:

\[
\begin{align*}
&u_t = i u_{xx} + i g(|u|^2) u + f \quad \text{on} \ Q, \\
&u(t, x_a) = u(t, x_b) = 0 \quad \forall t \in [0, T], \\
&u(0, x) = u_0(x) \quad \forall x \in I,
\end{align*}
\]

where \( g \in C([0, +\infty), \mathbb{R}) \), \( f \in C(Q, \mathbb{C}) \) and \( u_0 \in C(I, \mathbb{C}) \) with

\[
u_0(x_a) = u_0(x_b) = 0.
\]

In addition, we assume that the problem above admits a unique solution \( u \in C^{1,2}_{t,x}(Q) \) and that the data \( g, f \) and \( u_0 \) are smooth enough and compatible in order to ensure that the solution and its higher derivatives are sufficiently, for our purposes, smooth on \( Q \).

1.2. The Relaxation Finite Difference method. Let \( N \) be the set of all positive integers and \( L := x_b - x_a \). For given \( N \in \mathbb{N} \), we define a uniform partition of the time interval \([0, T]\) with time-step \( \tau := \frac{T}{N} \), nodes \( t_n := n \tau \) for \( n = 0, \ldots, N \), and intermediate nodes \( t_{n+\frac{1}{2}} = t_n + \frac{\tau}{2} \) for \( n = 0, \ldots, N - 1 \). Also, for given \( J \in \mathbb{N} \), we consider a uniform partition of \( I \) with mesh-width \( h := \frac{L}{J+1} \) and nodes \( x_j := x_a + j h \) for \( j = 0, \ldots, J + 1 \). Then, we introduce the discrete spaces

\[
\mathbb{C}_h := \{(v_j)_{j=0}^{J+1} : v_j \in \mathbb{C}, j = 0, \ldots, J + 1 \}, \quad \mathbb{C}_h^0 := \{(v_j)_{j=0}^{J+1} \in \mathbb{C}_h : v_0 = v_{J+1} = 0 \},
\]

\[
\mathbb{R}_h := \{(v_j)_{j=0}^{J+1} : v_j \in \mathbb{R}, j = 0, \ldots, J + 1 \}, \quad \mathbb{R}_h^0 := \mathbb{C}_h^0 \cap \mathbb{R}_h,
\]

a discrete product operator \( \cdot \otimes \cdot : \mathbb{C}_h \times \mathbb{C}_h \rightarrow \mathbb{C}_h \) by

\[
(v \otimes w)_j = v_j w_j, \quad j = 0, \ldots, J + 1, \quad \forall v, w \in \mathbb{C}_h,
\]

a discrete Laplacian operator \( \Delta_h : \mathbb{C}_h^0 \rightarrow \mathbb{C}_h^0 \) by

\[
\Delta_h v_j := \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2}, \quad j = 1, \ldots, J, \quad \forall v \in \mathbb{C}_h^0,
\]

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In addition, we introduce operators $I_h : C(I, \mathbb{C}) \to \mathbb{C}_h$ and $\tilde{I}_h : C(I, \mathbb{C}) \to \mathbb{C}_h^*$, which, for given $z \in C(I, \mathbb{C})$, are defined, respectively, by $(I_h z)_j := z(x_j)$ for $j = 0, \ldots, J + 1$ and $(\tilde{I}_h z)_j := z(x_j)$ for $j = 1, \ldots, J$, and a Discrete Elliptic Projection operator $R_h : C^2(I; \mathbb{C}) \to \mathbb{C}_h^*$ (cf. [2]) by requiring

\begin{equation}
\Delta_h(R_h[v]) = \hat{v}''(v''') \quad \forall v \in C^2(I; \mathbb{C}).
\end{equation}

Finally, for $\ell \in \mathbb{N}$ and for any function $q : \mathbb{C}^{\ell} \to \mathbb{C}$ and any $w = (w_1, \ldots, w^\ell) \in (\mathbb{C}_h)^\ell$, we define $q(w) \in \mathbb{C}_h$ by $(q(w))_j := g\left(w_{j_1}, \ldots, w_{j_{\ell}}\right)$ for $j = 0, \ldots, J + 1$.

The Relaxation Finite Difference (RFD) method combines a standard finite difference method for space discretization, with the Besse Relaxation Scheme for time-stepping (cf. Section 5 in [6]). Its algorithm consists of the following steps:

**Step 1:** Define $W^0 \in \mathbb{C}_h^*$ by

\begin{equation}
W^0 := R_h[u_0]
\end{equation}

and find $W^{\frac{1}{2}} \in \mathbb{C}_h^*$ such that

\begin{equation}
W^{\frac{1}{2}} - W^0 = i \frac{\tau}{2} \Delta_h \left( \frac{W^{\frac{1}{2}} + W^0}{2} \right) + i \frac{\tau}{2} g(|u^0|^2) \otimes \left( \frac{W^{\frac{1}{2}} + W^0}{2} \right) + \frac{\tau}{2} I_h \left[ f(t_n) \right].
\end{equation}

**Step 2:** Define $\Phi^{\frac{1}{2}} \in \mathbb{R}_h$ by

\begin{equation}
\Phi^{\frac{1}{2}} := g(|W^{\frac{1}{2}}|^2)
\end{equation}

and find $W^1 \in \mathbb{C}_h^*$ such that

\begin{equation}
W^1 - W^0 = i \tau \Delta_h \left( \frac{W^{\frac{1}{2}} + W^0}{2} \right) + i \tau \Phi^{\frac{1}{2}} \otimes \left( \frac{W^{\frac{1}{2}} + W^0}{2} \right) + \tau I_h \left[ f(t_n) \right].
\end{equation}

**Step 3:** For $n = 1, \ldots, N - 1$, first define $\Phi^{n+\frac{1}{2}} \in \mathbb{R}_h$ by

\begin{equation}
\Phi^{n+\frac{1}{2}} := 2g(|W^n|^2) - \Phi^{n-\frac{1}{2}}
\end{equation}

and then find $W^{n+1} \in \mathbb{C}_h^*$ such that

\begin{equation}
W^{n+1} - W^n = i \tau \Delta_h \left( \frac{W^{n+\frac{1}{2}} + W^n}{2} \right) + i \tau \Phi^{n+\frac{1}{2}} \otimes \left( \frac{W^{n+\frac{1}{2}} + W^n}{2} \right) + \tau I_h \left[ f(t_{n+1}) \right].
\end{equation}

**Remark 1.1.** It is easily verified that the (RFD) method requires the numerical solution of a linear tridiagonal system of algebraic equation at every time step.

**Remark 1.2.** The (RFD) approximations are, unconditionally, well-posed (see Lemma 4.1).

**Remark 1.3.** In [18], we construct a second order approximation $\Phi^{\frac{1}{2}}$ of $g(|u(t^{\frac{1}{2}})|^2)$, instead of the the first order approximation $g(|u_0|^2)$ proposed in [6]. Later, we show that both choices yield a second order convergence of the numerical method at the time nodes (see Theorems 4.4 and 4.5).

1.3. **Related references and main results.** It is well known (see, e.g. [11], [12], [3], [12]) that the application of an implicit time-stepping method to a non linear Schrödinger equation, creates, at every time step, the need of approximating the solution to a system of complex, non linear algebraic equations via an iterative solver. One way to keep the computational complexity of the method in a predefine level, is to employ a linear implicit time-stepping method (see, e.g. [14], [5], [15], [13]) that handles the linear part of the equation implicitly and the non linear part of the equation explicitly or semi-implicitly and thus require, at every time step, the solution of a linear system of algebraic equations. Within this context, C. Besse [5] proposed the Relaxation Scheme (RS) which was a new, linear implicit, time-stepping method, conserving in a discrete way the charge and the energy. The Relaxation Scheme (RS) along with a finite element or a finite difference space discretization, is computationally efficient (see, e.g., [4], [12], [10]) and performs as a second order method (see, e.g., [6], [13]). Later, C. Besse [6], analysing the (RS) as a semidiscrete
in time method approximating the solution of the Cauchy problem for the nonlinear Schrödinger equation with power non-linearity, show its convergence for small final time $T$, without concluding a convergent rate with respect to the time step. C. Besse et al. [7] focusing on the cubic nonlinear Schrödinger equation, combine the stability results in [6] with a proper consistency argument based on the Taylor formula and bound the approximation error of the (RS) in the $H^s(\mathbb{R}^d)$--norm with a second order error with respect to the time-step, additive term. The error estimate obtained yields a first order convergence of the (RS) when $g(|u(T^*)|^2)$ is approximated by $g(|u_0|^2)$ (see, e.g., [5], [6]), and thus it is not able to explain its second order convergence that has been observed experimentally by several authors (see, e.g., [6], [13]). Also, the error analysis developed in [6] and [7] is based on the derivation of a priori bounds for the time-discrete approximations in the $H^{s+2}(\mathbb{R}^d)$--norm with $s > \frac{d}{2}$, (cf. Hypotheses 2 in [6]). However, this approach can not be adopted for the analysis of fully discrete methods, where the (RS) is coupled with a finite difference or finite element method for space discretization. The problem is coming from the fact that on one hand the finite element approximations are, usually, only $H^1$ functions, and on the other hand the finite difference approximations are not able to mimic, in a discrete way, all the compatibility conditions that the solution to the continuous problem satisfy. This indicates that the error estimation in the fully discrete case has to follow a different path.

Recently, considering the approximation of a semilinear heat equation in the one space dimension by the (RS) coupled with a central finite difference method, we provided a second order error estimate using energy-type techniques [17]. Unfortunately, the latter convergence analysis can not be extended in the case of the nonlinear Schrödinger equation, because it is based on the stability properties of the parabolic problems.

In the work at hands, our aim is to contribute to the understanding of the convergence nature of the (RS) by investigating the convergence of the (RFD) method formulated in (1.6)–(1.10) for the approximation of the solution to the problem (1.1). By building up a proper stability argument based on [6] and formulating a proper modified version of the numerical method based on the framework proposed in [16], we are able to prove a new, optimal, second order error estimate in a discrete $L^\infty_t(H^s_x)$--norm at the nodes and the intermediate time nodes, without restrictions on the final time $T$ and avoiding to impose coupling conditions on the mesh parameters. Also, considering the first order in time approximation $\Phi^j = g(|u_j[u_0]|^2)$, we show that convergence of the method is still second order at the nodes while remain first order in time at the intermediate nodes.

We close this section by giving a brief overview of the paper. In Section 2, we introduce notation and we prove a series of auxiliary results that we will often use later in the analysis of the numerical method. Section 3 is dedicated to the definition and estimations of the consistency errors along with the presentation of the approximation properties of the Discrete Elliptic Projection operator. Finally, Section 4 contains the convergence analysis of the (RFD) method via the construction and the analysis of a modified scheme.

2. Preliminaries

In this section, we introduce additional notation and present a series of basic auxiliary results that will be used often in the convergence analysis of the numerical method.

2.1. Additional notation. Let us define the discrete space $S_h^0 := \{(z_j)_{j=0}^J : z_j \in \mathbb{C}, j = 0, \ldots, J\}$ along with its real subset $S_h^0 := \{(z_j)_{j=0}^J : z_j \in \mathbb{R}, j = 0, \ldots, J\}$, and we define the discrete space derivative operator $\delta_h : C_h \to S_h^0$ by $\delta_h v_j := \frac{v_{j+1} - v_j}{h}$ for $j = 0, \ldots, J$ and $v \in C_h$. On $S_h^0$ we define the inner product $(\cdot, \cdot)_{0,h}$ by $(z, v)_{0,h} := h \sum_{j=0}^J z_j v_j$ for $z, v \in S_h^0$, and we will denote by $\| \cdot \|_{0,h}$ the corresponding norm, i.e., $\|z\|_{0,h} := \sqrt{(z, z)_{0,h}}$ for $z \in S_h^0$. Also, we define a discrete maximum norm on $S_h^0$ by $\|v\|_{\infty,h} := \max_{0 \leq j \leq J} |v_j|$ for $v \in S_h^0$. 

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We provide $C_h$ with the discrete inner product $(\cdot, \cdot)_{0,h}$ by $(v, z)_{0,h} := h \sum_{j=0}^{j-1} v_j \overline{z}_j$ for $v, z \in C_h$, and we shall denote by $\| \cdot \|_{0,h}$ its induced norm, i.e. $\|v\|_{0,h} := \sqrt{(v, v)_{0,h}}$ for $v \in C_h$. Also, we define on $C_h$ a discrete maximum norm $|\cdot|_{\infty,h}$ by $|w|_{\infty,h} := \max_{0 \leq j \leq J-1} |w_j|$ for $w \in C_h$, a discrete $H^1$-seminorm $|\cdot|_{1,h}$ by $|w|_{1,h} := |\delta h w|_{0,h}$ for $w \in C_h$ and a discrete $H^1$-norm $|\cdot|_{1,h}$ by $|w|_{1,h} := (|w|_{0,h}^2 + |w|_{1,h}^2)^{\frac{1}{2}}$ for $w \in C_h$.

2.2. Auxiliary results. It is easily seen that, for $v \in C_h$, the following inequalities hold

\begin{equation}
|v|_{\infty,h} \leq \sqrt{L} \|v\|_{1,h}
\end{equation}

\begin{equation}
\|v\|_{0,h} \leq L \|v\|_{1,h}
\end{equation}

\begin{equation}
|v|_{1,h} \leq 2 h^{-1} \|v\|_{0,h}.
\end{equation}

Under the light of (2.2), the seminorm $|\cdot|_{1,h}$ is a norm on $C_h$ which is equivalent to $\| \cdot \|_{1,h}$.

**Lemma 2.1.** For all $v, z \in C_h$ it holds that

\begin{equation}
(\Delta_h v, z)_{0,h} = -\left(\delta_h v, \delta_h z\right)_{0,h} = (v, \Delta_h z)_{0,h},
\end{equation}

\begin{equation}
(\Delta_h v, v)_{0,h} = -\|v\|_{1,h}^2.
\end{equation}

**Proof.** Let $v, z \in C_h$. First, we obtain (2.4) proceeding as follows

\[
(\Delta_h v, z)_{0,h} = \sum_{j=1}^{J} \left[ (\delta_h v)_j - (\delta_h v)_{j-1} \right] \overline{z}_j = \sum_{j=0}^{J} (\delta_h v)_j \overline{z}_j - \sum_{j=0}^{J-1} (\delta_h v)_j \overline{z}_{j+1} = -\left(\delta_h v, \delta_h z\right)_{0,h}.
\]

Then, we observe that (2.5) is a simple consequence of (2.4). \qed

**Lemma 2.2.** Let $\varepsilon > 0$, $g \in C^2(\mathbb{R}; \mathbb{R})$, $g'_v := \sup_{x \in [0, \varepsilon]} |g''(x)|$, $g''_v := \sup_{x \in [0, \varepsilon]} |g'''(x)|$ and $\mathcal{R}_h := \{v \in \mathbb{R}_h : |v|_{\infty,h} \leq \varepsilon\}$. Then, for $v, w \in \mathcal{R}_h$, it holds that

\begin{equation}
|g(v) - g(w)|_{0,h} \leq g'_v \|v - w\|_{0,h}
\end{equation}

and

\begin{equation}
|g(v) - g(w)|_{1,h} \leq g'_v \|v - w\|_{1,h} + g''_v \|\delta_h w\|_{\infty,h} \|v - w\|_{0,h}.
\end{equation}

**Proof.** Let $v, w \in \mathcal{R}_h$. Then, for $s \in [0, 1]$, we define $c^s \in \mathcal{R}_h$, by $c^s := s v + (1-s) w$ and $a^s, b^s \in S_h$ by $a_j^s := s v_{j+1} + (1-s) v_j$ and $b_j^s := s w_{j+1} + (1-s) w_j$ for $j = 0, \ldots, J$. Applying the mean value theorem, we have $g(v) - g(w) = (v-w) \otimes \left(\int_0^1 g''(c^s) \, ds\right)$, which, easily, yields (2.6). Applying, again, the mean value theorem, we conclude that

\[
|\delta_h (g(v) - g(w))_j| = \left| (\delta_h (v-w))_j \left( \int_0^1 g''(c^s) \, ds \right) \right| \\
\leq g''_v |(\delta_h v)_j| \int_0^1 |s(v_{j+1} - w_{j+1}) + (1-s)(v_j - w_j)| \, ds + g'_v |(\delta_h (v-w))_j| \\
\leq \frac{1}{2} g''_v |(\delta_h v)_j| (|v_{j+1} - w_{j+1}| + |v_j - w_j|) + g'_v |(\delta_h (v-w))_j|,
\]

which, easily, yields (2.7). \qed

**Lemma 2.3.** Let $\varepsilon > 0$, $\mathcal{R}_h := \{v \in \mathbb{R}_h : |v|_{\infty,h} \leq \varepsilon\}$, $g \in C^4(\mathbb{R}; \mathbb{R})$, $g'_v := \sup_{x \in [0, \varepsilon]} |g''(x)|$, $g''_v := \sup_{x \in [0, \varepsilon]} |g'''(x)|$. Then, for $v^a, v^b, z^a, z^b \in \mathcal{R}_h$, it holds that

\begin{equation}
\|g(v^a) - g(v^b) - g(z^a) + g(z^b)\|_{0,h} \leq g'_v |z^a - z^b|_{\infty,h} \|v^b - z^b\|_{0,h}
\end{equation}

\begin{equation}
+ \left( g'_v + g''_v \right) |z^a - z^b|_{\infty,h} \|v^a - v^b - z^a + z^b\|_{0,h}
\end{equation}
Also, for
\begin{equation}
\|g(v^a) - g(v^b) - g(z^a) + g(z^b)\|_{1,h} \leq (g'_e + g''_e |z^a - z^b|_{\infty,h})|v^a - v^b - z^a + z^b|_{1,h}
\end{equation}
\begin{equation}
+ \frac{1}{2} g''_e (|v^a|_{1,h} + |v^b|_{1,h})|v^a - v^b - z^a + z^b|_{\infty,h}
\end{equation}
\begin{equation}
+ g''_e |z^a - z^b|_{\infty,h}|v^b - z^b|_{1,h}
\end{equation}
\begin{equation}
+ \mathcal{F}_e(z^a, z^b) \left( \|v^a - v^b - z^a + z^b\|_{0,h} + \|v^b - z^b\|_{0,h} \right),
\end{equation}
where
\begin{equation}
\mathcal{F}_e(z^a, z^b) := g''_e \left| \delta_h(z^a - z^b) \right|_{\infty,h} + g''_e \left| z^a - z^b \right|_{\infty,h} \left[ \|\delta_h(z^a - z^b)\|_{\infty,h} + \|\delta_h z^b\|_{\infty,h} \right].
\end{equation}

Proof. Let $v^a, v^b, z^a, z^b \in \mathcal{R}$. We simplify the notation, first, by defining $a^s, b^s \in \mathcal{R}$ by $a^s := s v^a + (1 - s) v^b$ and $b^s := s z^a + (1 - s) z^b$ for $s \in [0, 1]$, and then, by introducing $f^a, f^b \in \mathcal{R}$ by
\begin{equation}
f^a := \int_0^1 g'(a^s) \, ds \quad \text{and} \quad f^b := \int_0^1 g'(a^s) - g'(b^s) \, ds.
\end{equation}
Also, we set $e^a := v^a - z^a$ and $e^b := v^b - z^b$.

**Part I.** First, we use the definition of $f^a$ and the mean value theorem, to get
\begin{equation}
\|f^a\|_{\infty,h} \leq g'_e
\end{equation}
and
\begin{equation}
\left| (\delta_h f^a)_j \right| \leq \frac{1}{h} \int_0^1 \left| g'(a^s) - g'(a^s) \right| \, ds
\end{equation}
\begin{equation}
\leq g''_e \int_0^1 \left| s \delta_h v^a + (1 - s) \delta_h v^b \right| \, ds
\end{equation}
\begin{equation}
\leq \frac{1}{2} g''_e \left( |\delta_h v^a| + |\delta_h v^b| \right),
\end{equation}
which, obviously, yields
\begin{equation}
\|f^a\|_{1,h} \leq \frac{1}{2} g''_e \left( |v^a|_{1,h} + |v^b|_{1,h} \right).
\end{equation}
Next, we use the definition of $f^a$ and the mean value theorem, to obtain
\begin{equation}
\|f^b\|_{0,h} \leq g''_e \int_0^1 \left| a^s - b^s \right| \, ds
\end{equation}
\begin{equation}
\leq g''_e \int_0^1 \left| s (v^a - v^b - z^a + z^b) + (v^b - z^b) \right| \, ds
\end{equation}
\begin{equation}
\leq g''_e \left( \left| (v^a - v^b - z^a + z^b) \right| + \left| (v^b - z^b) \right| \right),
\end{equation}
which, leads to
\begin{equation}
\|f^b\|_{0,h} \leq g''_e \left( e^a - e^b \right),
\end{equation}
which we use to obtain
\begin{equation}
\|f^b\|_{1,h} \leq \int_0^1 \left| g'(a^s) - g'(b^s) \right| \, ds
\end{equation}
\begin{equation}
\leq g''_e \left( |e^a - e^b|_{1,h} + |e^b|_{1,h} \right)
\end{equation}
\begin{equation}
+ g''_e \left( \|\delta_h(z^a - z^b)\|_{\infty,h} + \|\delta_h z^b\|_{\infty,h} \right) \left( |e^a - e^b|_{0,h} + |e^b|_{0,h} \right),
\end{equation}
**Part II.** Using the mean value theorem, we obtain
\begin{equation}
g(v^a) - g(v^b) - g(z^a) + g(z^b) = \mathcal{L}^A + \mathcal{L}^B,
\end{equation}
where \( \mathfrak{L}^A, \mathfrak{L}^B \in \mathbb{R}_h \) are defined by \( \mathfrak{L}^A := (v^a - v^b - z^a + z^b) \otimes f^A \) and \( \mathfrak{L}^B := (z^a - z^b) \otimes f^B \). Thus, using (2.10) and (2.12), we have

\[
\begin{align*}
\| \mathfrak{L}^A \|_{0,h} & \leq g'_c \| e^a - e^b \|_{0,h}, \\
\| \mathfrak{L}^B \|_{0,h} & \leq g''_c (z^a - z^b) \|_{\infty,h} \left( \| e^a - e^b \|_{0,h} + \| e^b \|_{0,h} \right).
\end{align*}
\]

Thus, (2.8) follows easily, from (2.14) and (2.15).

**Part III.** Observing that

\[
(\delta_h \mathfrak{L}^A)_j = f^A_{j+1} \delta_h (v^a - v^b - z^a + z^b)_j + (\delta_h f^A)_j (v^a - v^b - z^a + z^b)_j,
\]

\[
(\delta_h \mathfrak{L}^B)_j = f^B_{j+1} \delta_h (z^a - z^b)_j + (\delta_h f^B)_j (z^a - z^b)_j
\]

for \( j = 0, \ldots, J \), we obtain

\[
\begin{align*}
\| \mathfrak{L}^A |_{1,h} & \leq |f^A|_{\infty,h} \| e^a - e^b \|_{1,h} + |f^A|_{1,h} \| e^a - e^b \|_{\infty,h}, \\
\| \mathfrak{L}^B |_{1,h} & \leq |\delta_h (z^a - z^b)|_{\infty,h} \| f^B \|_{0,h} + |z^a - z^b|_{\infty,h} \| f^B \|_{1,h}.
\end{align*}
\]

Using (2.16), (2.10) and (2.11), we have

\[
\begin{align*}
\| \mathfrak{L}^A |_{1,h} & \leq g'_c \| e^a - e^b \|_{1,h} + \frac{1}{2} g''_c (|v^a|_{1,h} + |v^b|_{1,h}) \| e^a - e^b \|_{\infty,h}.
\end{align*}
\]

Combining (2.16), (2.12) and (2.13), we arrive at

\[
\begin{align*}
\| \mathfrak{L}^A |_{1,h} & \leq g'_c \| \delta_h (z^a - z^b) \|_{\infty,h} \left( \| e^a - e^b \|_{0,h} + \| e^b \|_{0,h} \right) \\
& \quad + |z^a - z^b|_{\infty,h} g''_c \left( \| \delta_h (z^a - z^b) \|_{\infty,h} + \| \delta_h z^b \|_{\infty,h} \right) \left( \| e^a - e^b \|_{0,h} + \| e^b \|_{0,h} \right) \\
& \quad + |z^a - z^b|_{\infty,h} g'_c \left( \| e^a - e^b \|_{1,h} + \| e^b \|_{1,h} \right).
\end{align*}
\]

Finally, (2.9) follows easily, in view of (2.14), (2.17) and (2.18). \( \square \)

**Lemma 2.4.** For \( v^a, v^b, z^a, z^b \in C_h \), it holds that

\[
\begin{align*}
\| |v^a|^2 - |z^a|^2 |_{0,h} & \leq \| |v^a|_{\infty,h} + |z^a|_{\infty,h} \| v^a - z^a \|_{0,h}, \\
\| |v^a|^2 - |z^a|^2 |_{1,h} & \leq \| |v^a|_{\infty,h} |v^a - z^a|_{1,h} + 2 \| \delta_h (z^a)|_{\infty,h} \| v^a - z^a \|_{0,h}, \\
\| |v^a|^2 - |v^b|^2 - |z^a|^2 + |z^b|^2 |_{0,h} & \leq 2 \| |v^a|_{\infty,h} + |v^b|_{\infty,h} + |z^a - z^b|_{\infty,h} \| v^a - v^b - z^a + z^b \|_{0,h}, \\
\| |v^a|^2 - |v^b|^2 - |z^a|^2 + |z^b|^2 |_{1,h} & \leq \left( G^A (v^a, v^b, z^a, z^b) \right) |v^a - v^b - z^a + z^b|_{1,h} \\
& \quad + \left( G^B (v^a, v^b, z^a, z^b) \right) |v^a - v^b - z^a + z^b|_{\infty,h} \\
& \quad + 2 \| \delta_h (z^a - z^b) |_{\infty,h} \| v^b - z^b |_{1,h} + 2 \| \delta_h (z^a - z^b) |_{\infty,h} \| v^b - z^b |_{0,h}
\end{align*}
\]

where

\[
\begin{align*}
G^A (v^a, v^b, z^a, z^b) & := |v^a|_{\infty,h} + |v^b|_{\infty,h} + |z^a - z^b|_{\infty,h}, \\
G^B (v^a, v^b, z^a, z^b) & := |v^a|_{1,h} + |v^b|_{1,h} + |z^a - z^b|_{1,h}.
\end{align*}
\]

**Proof.** Let \( v^a, v^b, z^a, z^b \in C_h \) and \( \zeta := v^a - v^b - z^a + z^b \). The inequalities (2.19), (2.20) and (2.21) follow easily by observing that

\[
\begin{align*}
(|v^a|^2 - |z^a|^2)_j &= \text{Re} \left[ \overline{(v^a + z^a)}_j (v^a - z^a)_j \right], \\
(|v^a|^2 - |v^b|^2 - |z^a|^2 + |z^b|^2)_j &= \text{Re} \left[ \overline{(v^a + v^b)}_j \zeta_j + (z^a - z^b)_j \zeta_j + 2 (z^a - z^b)_j (v^b - z^b)_j \right]
\end{align*}
\]
for \( j = 0, \ldots, J + 1 \), and
\[
\delta_h(|v^a|^2 - |z^a|^2)_j = \text{Re} \left[ \left( \frac{v^a_{j+1} + v^a_j}{2} \right) \delta_h(v^a - z^a)_j \right. \\
+ \left. \delta_h(z^a)_j \left( (v^a - z^a)_{j+1} + (v^a - z^a)_j \right) \right], \\
\delta_h(|v^b|^2 - |z^b|^2)_j = \text{Re} \left[ \left( \frac{v^b + v^b}{2} \right) \delta_h(v^b - z^b)_j \right. \\
+ \left. \delta_h(z^b)_j \left( (v^b - z^b)_{j+1} + (v^b - z^b)_j \right) \right] \\
+ 2 \text{Re} \left[ (v^b - z^b)_j \delta_h(v^b - z^b)_j + \delta_h(z^a - z^b)_j (v^b - z^b)_j \right]
\]
for \( j = 0, \ldots, J \).

2.3. A mollifier. Let \( \delta > 0 \), \( p_\delta : [\delta, 2\delta] \to \mathbb{R} \) be the unique polynomial of \( P^7[\delta, 2\delta] \) satisfying
\[
p_\delta(\delta) = \delta, \quad p_\delta'(\delta) = 1, \quad p''_\delta(\delta) = 0, \quad p_\delta(2\delta) = 2\delta, \quad p_\delta'(2\delta) = p''_\delta(2\delta) = p'''_\delta(2\delta) = 0,
\]
and \( n_\delta \in C^0(\mathbb{R}; \mathbb{R}) \) be an odd function (cf. [12], [16]) defined by
\[
n_\delta(x) := \begin{cases}
    x, & \text{if } x \in [0, \delta], \\
p_\delta(x), & \text{if } x \in (\delta, 2\delta], \\
    2\delta, & \text{if } x > 2\delta.
\end{cases}
\]

Then, we define a complex mollifier \( \gamma_\delta : \mathbb{C} \to \mathbb{C} \) (cf. [16]) by
\[
\gamma_\delta(z) := n_\delta(\text{Re}(z)) + i n_\delta(\text{Im}(z)) \quad \forall z \in \mathbb{C},
\]
for which it is, easily, verified that
\[
|\gamma_\delta(z)| \leq \sqrt{2} \sup_x |n_\delta| \quad \forall z \in \mathbb{C},
\]
(2.25)
\[
|\gamma_\delta(z_1) - \gamma_\delta(z_2)| \leq \sup_h |n'_\delta| |z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C}
\]
(2.26) and
\[
\gamma_\delta(z) = z \quad \forall z \in \{ z \in \mathbb{C} : |z| \leq \delta \}.
\]

Lemma 2.5. Let \( \delta > 0 \), \( n'_{\delta, \infty} := \sup_r |n'_\delta| \), \( n''_{\delta, \infty} := \sup_r |n''_\delta| \) and \( n'''_{\delta, \infty} := \sup_r |n'''_\delta| \). Then, for all \( v^a, v^b, z^a, z^b \in \mathbb{C}_h \), it holds that
\[
\|\gamma_\delta(v^a) - \gamma_\delta(v^b)\|_{0,h} \leq n'_{\delta, \infty} \|v^a - v^b\|_{0,h},
\]
(2.28)
\[
|\gamma_\delta(v^a) - \gamma_\delta(v^b)|_{1,h} \leq 2 n'_{\delta, \infty} \|v^a - v^b\|_{1,h} + 2 n''_{\delta, \infty} \|\delta_h v^b\|_{0,h} \|v^a - v^b\|_{0,h},
\]
(2.29)
\[
\|\zeta_\delta\|_{0,h} \leq 2 n''_{\delta, \infty} \|z^a - z^b\|_{\infty,h} \|v^b - b^b\|_{0,h} + 2 \left( n'_{\delta, \infty} + n''_{\delta, \infty} \|z^a - z^b\|_{\infty,h} \right) \|v^a - v^b - z^a + z^b\|_{0,h}
\]
(2.30) and
\[
|\zeta_\delta|_{1,h} \leq 2 \left( n'_{\delta, \infty} + n''_{\delta, \infty} \|z^a - z^b\|_{\infty,h} \right) \|v^a - v^b - z^a + z^b\|_{1,h} \\
+ n''_{\delta, \infty} \|v^a\|_{1,h} + \|v^b\|_{1,h} \|v^a - v^b - z^a + z^b\|_{1,h} \\
+ 2 n''_{\delta, \infty} \|z^a - z^b\|_{\infty,h} \|v^b - z^b\|_{1,h} \\
+ 2 F'_\delta(z^a, z^b) \left( \|v^a - v^b - z^a + z^b\|_{0,h} + \|v^b - z^b\|_{0,h} \right),
\]
(2.31)
where \( \zeta_\delta := \gamma_\delta(v^a) - \gamma_\delta(v^b) - \gamma_\delta(z^a) + \gamma_\delta(z^b) \in \mathbb{C}_h \) and
\[
F'_\delta(z^a, z^b) := n''_{\delta, \infty} \|\delta_h(z^a - z^b)\|_{\infty,h} + n'''_{\delta, \infty} \|z^a - z^b\|_{\infty,h} \left( \|\delta_h(z^a - z^b)\|_{\infty,h} + \|\delta_h z^b\|_{\infty,h} \right),
\]
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be linear operators defined by $A_h$ we obtain

and thus (2.33) is established. Since $\Delta_h\chi = 0$, which, along (2.7), (2.8) and (2.9) (with $\chi = 0$), yields that $A_h\chi = 0, or, equivalently $\|\chi\|_{0,h} = \|\chi\|_{0,h}$, which obviously yields (2.32).

Let $\chi \in C_h^0$ and $v = B_h(\chi)$. Then, we have $A_h(v) = T_h(\chi)$ which is equivalent to $v = \chi = i \frac{T}{2} \Delta_h (v + \chi)$. In view of (2.31), we obtain $Re[(v - \chi, v + \chi)\|_{0,h}] = 0$, or, equivalently $\|v\|_{0,h}^2 = \|\chi\|_{0,h}^2$, and thus (2.33) is established. Since $\Delta_h(v - \chi) = i \frac{T}{2} \Delta_h (\Delta_h (v + \chi))$, in view of (2.1) and (2.3), we have

which, obviously, yields (2.34).

Finally, we obtain (2.36) proceeding as follows

$(I_h + B_h)^{-1} A_h^{-1} = [A_h^{-1}(A_h + T_h)]^{-1} A_h^{-1}
= (A_h + T_h)^{-1} A_h^{-1}
= (A_h + T_h)^{-1}
= (2 I_h)^{-1}
= \frac{1}{2} h.$

2.4. **Space discrete operators.** Let $I_h : C_h^0 \rightarrow C_h^0$ be the identity operator and $A_h, T_h : C_h^0 \rightarrow C_h^0$ be linear operators defined by $A_h := I_h - i \frac{T}{2} \Delta_h$, $T_h := I_h + i \frac{T}{2} \Delta_h$ and $B_h := A_h^{-1} T_h$.

**Lemma 2.6.** The operators $A_h$ and $T_h$ are invertible and the following relations hold

\begin{align}
\|A_h^{-1}(\chi)\|_{0,h} &\leq \|\chi\|_{0,h}, \\
\|B_h(\chi)\|_{0,h} &\leq \|\chi\|_{0,h}, \\
\|B_h(\chi)\|_{1,h} &\leq \|\chi\|_{1,h}
\end{align}

for $\chi \in C_h^0$, and

\begin{align}
(I_h + B_h)^{-1} A_h^{-1} &= \frac{1}{2} I_h.
\end{align}

**Proof.** First, we observe that (2.28) follows, easily, from (2.26). Now, let $\|\cdot\|_* = \|\cdot\|_{0,h}$ or $\|\cdot\|_{1,h}$. Then, using (2.24), we have

$$
\|\gamma_\delta(v^a) - \gamma_\delta(v^b)\|_* \leq \|n_\delta(Re(v^a)) - n_\delta(Re(v^b))\|_* + \|n_\delta(Im(v^a)) - n_\delta(Im(v^b))\|_*,
$$

and

$$
\|\zeta_\delta\|_* \leq \|n_\delta(Re(v^a)) - n_\delta(Re(v^b)) - n_\delta(Re(z^a)) + n_\delta(Re(z^b))\|_*
+ \|n_\delta(Im(v^a)) - n_\delta(Im(v^b)) - n_\delta(Im(z^a)) + n_\delta(Im(z^b))\|_*,
$$

which, along (2.7), (2.8) and (2.9) (with $\delta = n_\delta$), yield (2.29), (2.30) and (2.31). \qed
3. Discretization Errors

3.1. Consistency of the discretization in time. To simplify the notation, we set \( t^+ := \frac{t}{2} \), \( u^+ := l_h[u(t^+, \cdot)] \), \( u^n := l_h[u(t^n, \cdot)] \) for \( n = 0, \ldots, N \), and \( u^{n+\frac{1}{2}} := l_h[u(t^{n+\frac{1}{2}}, \cdot)] \) for \( n = 0, \ldots, N - 1 \). In view of the Dirichlet boundary conditions (1.2) and the compatibility conditions (1.4), it holds that \( u^+ \in C^0_h \), \( u^n \in C^0_h \) for \( n = 0, \ldots, N \) and \( u^{n+\frac{1}{2}} \in C^0_h \) for \( n = 0, \ldots, N - 1 \). Also, we simplify the notation by setting \( w^+ = g(|u|^2) \) and \( w^n = g(|u^n|^2) u \).

For \( n = 1, \ldots, N - 1 \), we define \( r^n \in C^0_h \) by

\[
(3.1) \quad \frac{1}{2} \left[ g(|u^{n+\frac{1}{2}}|^2) + g(|u^{n-\frac{1}{2}}|^2) \right] = g(|u^n|^2) + r^n.
\]

Then, applying the Taylor formula, in a standard way, we obtain

\[
(3.2) \quad \max_{1 \leq n \leq N-1} \|r^n\|_{0,h} \leq \bar{C}_{1,h} \tau^2 \max_{[0,T]\times I} |\partial_t^2 w^n|,
\]

\[
(3.3) \quad \max_{1 \leq n \leq N-1} |r^n|_{1,h} \leq \bar{C}_{1,h} \tau^2 \max_{[0,T]\times I} |\partial_x \partial_t^2 w^n|,
\]

\[
(3.4) \quad \max_{2 \leq n \leq N-1} \|r^n - r^{n-1}\|_{0,h} \leq \bar{C}_{2,h} \tau^3 \max_{[0,T]\times I} |\partial_x^3 w^n|,
\]

\[
(3.5) \quad \max_{2 \leq n \leq N-1} |r^n - r^{n-1}|_{1,h} \leq \bar{C}_{2,h} \tau^3 \max_{[0,T]\times I} |\partial_x \partial_t^2 w^n|.
\]

Let \( r^+ \in C^0_h \) be defined by

\[
(3.6) \quad u^+ - u^0 = \frac{i}{\tau} l_h \left[ \frac{u_x(t_1, \cdot) + u_x(t_0, \cdot)}{2} \right] + \frac{i}{\tau} g(|u^0|^2) \otimes \frac{u^+ + u^0}{2} + \frac{i}{\tau} l_h \left[ \frac{f(t^+, \cdot) + f(t^0, \cdot)}{2} \right] + \frac{i}{\tau} r^+ + \frac{i}{2} r^+.
\]

and \( r^{n+\frac{1}{2}} \in C^0_h \) be given by

\[
(3.7) \quad u^{n+1} - u^n = i \tau l_h \left[ \frac{u_x(t_{n+1}, \cdot) + u_x(t_n, \cdot)}{2} \right] + i \tau g(|u^{n+\frac{1}{2}}|^2) \otimes \frac{u^{n+1} + u^n}{2} + \tau l_h \left[ \frac{f(t_{n+1}, \cdot) + f(t_n, \cdot)}{2} \right] + \tau r^{n+\frac{1}{2}}
\]

for \( n = 0, \ldots, N - 1 \). Assuming that the solution \( u \) is smooth enough on \([0, T] \times I\), and using (1.4) and the Dirichlet boundary conditions (1.2), we conclude that \( u_{xx}(t, x) = -f(t, x) \) for \( t \in [0, T] \) and \( x \in \{x_1, x_3\} \). Thus, we have \( r^+ \in C^0_h \) and \( r^{n+\frac{1}{2}} \in C^0_h \) for \( n = 0, \ldots, N - 1 \). Combining (1.1) with a standard application of the Taylor formula we get the following formulas:

\[
(3.8) \quad r_{j}^{n+\frac{1}{2}} = \frac{\tau^2}{2} \int_{0}^{T} \left[ s^2 \partial_t^3 u(t_n + s \tau, x_j) + \left( \frac{1}{2} - s \right)^2 \partial_t^3 u(t^{n+\frac{1}{2}} + s \tau, x_j) \right] ds
\]

\[
- \frac{\tau^2}{2} \int_{0}^{T} \left[ s \partial_t^3 u(t_n + s \tau, x_j) + \left( \frac{1}{2} - s \right) \partial_t^3 u(t^{n+\frac{1}{2}} + s \tau, x_j) \right] ds
\]

\[
+ i \frac{\tau^2}{2} \int_{0}^{T} \left[ s \partial_t^3 w^{\theta}(t_n + s \tau, x_j) + \left( \frac{1}{2} - s \right) \partial_t^3 w^{\theta}(t^{n+\frac{1}{2}} + s \tau, x_j) \right] ds
\]

\[
- i \frac{g(|u^{(n+\frac{1}{2})}|^2)}{2} \tau^2 \int_{0}^{T} \left[ s u_{tt}(t_n + s \tau, x_j) + \left( \frac{1}{2} - s \right) u_{tt}(t^{n+\frac{1}{2}} + s \tau, x_j) \right] ds.
\]
for \( j = 0, \ldots, J + 1 \) and \( n = 0, \ldots, N - 1 \), and

\[
\begin{align*}
\tau_{j+}^n & = \frac{s^2}{2} \int_0^s \left[ s^2 \mathcal{D}^3 u(s, \tau, x_j) + \left( \frac{1}{4} - s \right)^2 \mathcal{D}^3 u(t_{j+}^n + s \tau, x_j) \right] \, ds \\
& - \frac{s^2}{2} \int_0^s \left[ s \mathcal{D}^3 u(s, \tau, x_j) + \left( \frac{1}{4} - s \right) \mathcal{D}^3 u(t_{j+}^n + s \tau, x_j) \right] \, ds \\
& - ig \left( |w_{j+}^n|^2 \right) \frac{s^2}{2} \int_0^s \left[ s \mathcal{D}^3 u(s, \tau, x_j) + \left( \frac{1}{4} - s \right) \mathcal{D}^3 u(t_{j+}^n + s \tau, x_j) \right] \, ds \\
& + i \frac{s^2}{2} \int_0^s \left[ s \mathcal{D}^3 u(s, \tau, x_j) + \left( \frac{1}{4} - s \right) \mathcal{D}^3 u(t_{j+}^n + s \tau, x_j) \right] \, ds \\
& + i \frac{s^2}{2} \int_0^s \mathcal{D}^3 u(s, \tau, x_j) \, ds
\end{align*}
\]

for \( j = 0, \ldots, J + 1 \). From (3.8) and (3.9), we obtain:

\[
\begin{align*}
\| \tau_{j+} \|_{0,h} & \leq \mathcal{C}_{3,\lambda} \tau \max_{[0,T] \times [1]} \left( |\mathcal{D}^2 u| + |\mathcal{D}^3 u| + |\mathcal{D}^3 w^n| \right), \\
\| \tau_{j+} \|_{1,h} & \leq \mathcal{C}_{3,\lambda} \tau \max_{[0,T] \times [1]} \left( |\mathcal{D}^2 u| + |\mathcal{D}^3 u| + |\mathcal{D}^3 w^n| \right), \\
\max_{0 \leq n \leq N-1} \| \tau_{j+} \|_{0,h} & \leq \mathcal{C}_{4,\lambda} \tau^2 \max_{[0,T] \times [1]} \left( |\mathcal{D}^2 u| + |\mathcal{D}^3 u| + |\mathcal{D}^3 w^n| \right), \\
\max_{0 \leq n \leq N-1} \| \tau_{j+} \|_{1,h} & \leq \mathcal{C}_{4,\lambda} \tau^2 \max_{[0,T] \times [1]} \left( |\mathcal{D}^2 u| + |\mathcal{D}^3 u| + |\mathcal{D}^3 w^n| \right), \\
\max_{0 \leq n \leq N-1} \| \tau_{j+} - \tau_{(n-1)+} \|_{0,h} & \leq \mathcal{C}_{5,\lambda} \tau^3 \max_{[0,T] \times [1]} \left( |\mathcal{D}^2 u| + |\mathcal{D}^3 u| + |\mathcal{D}^3 w^n| \right).
\end{align*}
\]

3.2. Approximation estimates for the Discrete Elliptic Projection. Let \( v \in C^4(I; \mathbb{C}) \). After applying the Taylor formula around \( x = x_j \), it follows that

\[
\Delta_h \left( \tilde{I}_h v \right) - \tilde{I}_h (v''') = \frac{h^2}{12} r^v(v),
\]

where \( r^v(v) \in \mathbb{C}_h^0 \) is defined by

\[
(\tilde{r}^v(v))_j := \int_0^1 \left[ (1-y)^3 v'''(x_j + h y) + y^3 v'''(x_{j-1} + h y) \right] \, dy, \quad j = 1, \ldots, J.
\]

Subtracting (1.5) from (3.16), we get the following error equation

\[
\Delta_h \left( \tilde{I}_h v - R_h [v] \right) = \frac{h^2}{12} r^v(v).
\]

Taking the inner product of both sides of (3.15) with \( (\tilde{I}_h(v) - R_h [v]) \) and using (2.5), the Cauchy-Schwarz inequality and (2.2), we obtain

\[
\| R_h [v] - \tilde{I}_h (v) \|_{1,h} \leq \frac{1}{h^{1/2}} h^2 \| r^v(v) \|_{0,h}
\]

which, along with (2.2) and (3.17), yields

\[
\| R_h [v] - \tilde{I}_h (v) \|_{1,h} \leq \frac{\sqrt{1+\frac{1}{24}} h^2}{24} h^2 \max_j |v'''|.
\]

We close the section with a useful lemma.

Lemma 3.1. Let \( w \in C^1_{*,x} (Q) \) and \( \partial_t w \in C^0_{*,x} (Q) \). Then, it holds that

\[
\| R_h \left[ \frac{w(t, \tau) - w(s, \tau)}{t-s} \right] - \tilde{I}_h \left[ \frac{w(t, \tau) - w(s, \tau)}{t-s} \right] \|_{1,h} \leq \frac{\sqrt{1+\frac{1}{24}} h^2}{24} h^2 \max_Q |\partial^2_x \partial_t w|
\]

for all \( t, s \in [0, T] \) with \( s < t \).
Proof. For \( t,s \in [0,T] \) with \( s < t \), and \( \psi_h := R_h \left[ \frac{w(t) - w(s)}{t-s} \right] - \hat{i}_h \left[ \frac{w(t) - w(s)}{t-s} \right] \). Using (2.2) and (3.19), we have
\[
\|\psi_h\|_{1,h} \leq \sqrt{1 + L^2} |\psi_h|_{1,h} \\
\leq \sqrt{1 + L^2} h^2 \left\| e^{\left[ \frac{w(t) - w(s)}{t-s} \right]} \right\|_{0,h} \\
\leq \sqrt{1 + L^2} h^2 (t-s)^{-1} \int_s^t \|e^{\left[ \partial_i w(s) \right]}\|_{0,h} ds \\
\leq \sqrt{1 + L^2} h^2 \max_q |\partial_s^q \partial_i w|.
\]
\( \square \)

4. Convergence

4.1. Existence and uniqueness of the (RFD) approximations. The following lemma establishes that the (RFD) approximations are well-defined.

Lemma 4.1. The (RFD) approximations \( W_n \) and \( (W^n)_n \) defined by (1.6), (1.7), (1.9) and (1.10) are well-defined.

Proof. Let \( \phi \in \mathbb{R}_{2h}, \varepsilon > 0 \) and \( \nu_h[\varepsilon, \phi] : C_0^2 \to C_0^1 \) be a discrete operator given by
\[
\nu_h[\varepsilon, \phi] \chi := \chi - i \varepsilon \tau \Delta_h(\chi) - i \varepsilon \tau \phi \otimes \chi \quad \forall \chi \in C_0^2.
\]
Then, using (2.3), we obtain that \( \text{Re}(\nu_h[\varepsilon, \phi] \chi)_{0,h} = \|\chi\|_{0,h}^2 \) for \( \chi \in C_0^2 \), which, easily, yields that \( \text{Ker}(\nu_h[\varepsilon, \phi]) = \{0\} \). Since, the space \( C_0^2 \) is finite dimensional, we conclude that \( \nu_h[\varepsilon, \phi] \) is invertible.

By (1.6) the initial approximation \( W^0 \) is clearly defined. According to (1.7), (1.9) and (1.10) we have
\[
W_n^r = \nu_h^{-1}[\frac{1}{\tau}, g(\left| u^0 \right|^2)] W^0 + i \frac{r}{\tau} \Delta_h(W^0) + i \frac{r}{\tau} g(\left| u^0 \right|^2) \otimes W^0 + \frac{\tau}{\tau} \hat{i}_h \left[ f(t_n^r, \cdot) + f(t_0^r, \cdot) \right]
\]
and
\[
W_n^{n+1} = \nu_h^{-1}[\frac{1}{\tau}, \Phi^{n+\frac{1}{2}}] W^n + i \frac{r}{\tau} \Delta_h(W^n) + i \frac{r}{\tau} \Phi^{n+\frac{1}{2}} \otimes W^n + \frac{\tau}{\tau} \hat{i}_h \left[ f(t_{n+1}^r, \cdot) + f(t_n^r, \cdot) \right]
\]
for \( n = 1, \ldots, N-1 \).
\( \square \)

4.2. The (MRFD) scheme. For given \( \delta > 0 \), the modified version of the (RFD) method derives \( \delta \)-dependent approximations of the solution \( u \) to (1.1)-(1.4) according to the steps below:

Step I: Define \( V_0^0 \in C_0^2 \) by
\[
V_0^0 := R_h[u_0]
\]
and then find \( V_{\delta}^0 \in C_0^2 \) such that
\[
V_{\delta}^0 = i \frac{\tau}{\tau} \Delta_h \left( \frac{V_{\delta}^0 + V_{\delta}^\perp}{2} \right) + i \frac{r}{\tau} g(\left| u^0 \right|^2) \otimes \frac{V_{\delta}^0 + V_{\delta}^\perp}{2} + \frac{\tau}{\tau} \hat{i}_h \left[ f(t_{\delta}^0, \cdot) + f(t_0^0, \cdot) \right].
\]

Step II: Define \( \Phi_{\delta}^0 \in \mathbb{R}_h \) by
\[
\Phi_{\delta}^0 := g(\left| \gamma_{\delta} (V_{\delta}^0) \right|^2)
\]
and then find \( V_1^0 \in C_0^2 \) such that
\[
V_1^0 - V_0^0 = i \tau \Delta_h \left( \frac{V_1^0 + V_{\delta}^\perp}{2} \right) + i \tau R_h(\Phi_{\delta}^0) \otimes \gamma_{\delta} \left( \frac{V_1^0 + V_{\delta}^\perp}{2} \right) + \tau \hat{i}_h \left[ f(t_{\delta}^0, \cdot) + f(t_0^0, \cdot) \right].
\]
Step III: For \( n = 1, \ldots, N - 1 \), define \( \Phi_{\delta}^{n+\frac{1}{2}} \in \mathbb{R}_h \) by
\[
(4.5) \quad \Phi_{\delta}^{n+\frac{1}{2}} := 2g(|\gamma \delta(V^u_n)|^2) - \Phi_{\delta}^{n-\frac{1}{2}}
\]
and then find \( V^{n+1}_\delta \in C_h^0 \) such that
\[
(4.6) \quad V^{n+1}_\delta - V^n_\delta = i \tau \Delta_h \left( \frac{V^{n+1}_\delta + V^n_\delta}{2} \right) + i \tau n_\delta (\Phi_{\delta}^{n+\frac{1}{2}}) \otimes \gamma \delta \left( \frac{V^{n+1}_\delta + V^n_\delta}{2} \right) + \tau^\mu_1 \left[ f(t_{n+1}) + f(t_n) \right].
\]

**Remark 4.1.** Under the light of Lemma 4.4 and in view of (4.3), (4.4), (4.7) and (4.2) it follows that \( V^0_\delta = W^0 \) and \( V^\frac{1}{2}_\delta = W^\frac{1}{2} \).

4.3. Existence of the (MRFD) approximations. First, we recall the following Brouwer-type fixed-point lemma, for a proof of which we refer the reader to [3].

**Lemma 4.2.** Let \((C, H, (\cdot, \cdot)_H)\) be a complex finite dimensional inner product space, \( \| \cdot \|_H \) be the associated norm, \( \mu : H \mapsto H \) be a continuous operator, \( S_\varepsilon := \{ z \in H : \| z \|_H = \varepsilon \} \) for \( \varepsilon > 0 \), and \( B_\varepsilon := \{ z \in H : \| z \|_H \leq \varepsilon \} \) for \( \varepsilon > 0 \). If there exists \( \alpha > 0 \) such that \( \text{Re}(\mu(z), z)_H \geq 0 \) for all \( z \in S_\alpha \), then there exists \( z_\ast \in B_\alpha \) such that \( \mu(z_\ast) = 0 \).

**Proposition 4.1.** For \( \delta > 0 \), there exist \( (V^n_\delta)^n_{n=1} \in C^0_h \) satisfying (4.3) and (4.4).

**Proof.** Let \( \varphi \in \mathbb{R}_h \), \( z \in C^0_h \) and \( \mu_h : C^0_h \mapsto C^0_h \) be a continuous non linear operator defined by
\[
\mu_h(\chi) := 2\chi - i \tau \Delta_h (\chi) - i \tau \left( \begin{array}{c}
\mu(z, \varphi) \\
\mu_\delta(\chi)
\end{array} \right) + z \quad \forall \chi \in C^0_h.
\]
Let \( \alpha > 0 \) and \( \chi \in C^0_h \) with \( \| \chi \|_{0,h} = \alpha \). Using (2.4), the Cauchy-Schwarz inequality and (2.25), we obtain
\[
\text{Re}(\mu_h(\chi), \chi)_{0,h} = 2 \| \chi \|_{0,h}^2 + \tau \text{Re}\left( \text{Im}(\chi) \right) \| \chi \|_{0,h}^2 + \text{Re}(\chi, \chi)_{0,h} \\
\geq \| \chi \|_{0,h}^2 \left( 2 - \tau n_\delta(\varphi) + \gamma \delta(\chi) \right) \| \chi \|_{0,h} - \| z \|_{0,h} \\
\geq \left( 2 - \tau \sqrt{2} \right) \| \chi \|_{0,h}.
\]
Choosing \( \alpha = \alpha_\ast := \frac{\sqrt{2}}{2} \sup \| \chi \|_{0,h} + \frac{1}{2} \| z \|_{0,h} + 1 \), (4.7) yields that \( \text{Re}(\mu_h(\chi), \chi)_{0,h} > 0 \), which, in view of Lemma 4.4 results that there exists \( \chi_\ast \in C^0_h \) such that \( \| \chi_\ast \|_{0,h} \leq \alpha_\ast \), and \( \mu_h(\chi_\ast) = 0 \).

We establish the existence of the modified approximations by induction. First, we observe that \( V^0_\delta \) is well defined. Next, we assume that there exists a modified approximation \( V^{n-1}_\delta \) for \( n \in \{0, \ldots, N-1\} \). Then, we choose \( \varphi = \Phi_{\delta}^{n-1} \) and \( z = 2V^n_\delta + \frac{\tau}{\tau_1} \left[ f(t_{n+1}) + f(t_n) \right] \), to obtain a root \( \chi^{n+1} \in C^0_h \) of the corresponding operator \( \mu_h \). Thus, \( V^{n+1}_\delta = 2\chi^{n+1} - V^n_\delta \) forms a solution to the nonlinear system (4.4) when \( n \geq 1 \), or, (4.4) when \( n = 0 \).

4.4. Convergence of the (MRFD) scheme. In this section we investigate the convergence properties of the modified (RFD) approximations.

**Theorem 4.2.** Let \( u_{\max} := \max_u |u|, \ g_{max} := \max_Q |g(|u|^2)| \) and \( \delta_\ast \geq \max\{u_{\max}, g_{\max}\} \). Then, there exist positive constants \( C_\delta^0 \), \( C_\delta^1 \) and \( C_\delta^2 \), independent of \( \tau \) and \( h \), such that: if \( \tau C_\delta^0 \leq \frac{1}{2} \), then
\[
(4.8) \quad \left\| u^{\frac{1}{2}}_\delta - V^\frac{1}{2}_\delta \right\|_{1,h} + \max_{0 \leq m \leq N} \left\| u^m - V^m_\delta \right\|_{1,h} \leq C^1_\delta \left( \tau^2 + h^2 \right)
\]
and
\[
(4.9) \quad \max_{0 \leq m \leq N-1} \left\| g(u^{m+\frac{1}{2}}) - \Phi_{\delta_\ast}^{m+\frac{1}{2}} \right\|_{1,h} \leq C^2_\delta \left( \tau^2 + h^2 \right).
\]

**Proof.** To simplify the notation, we set \( e_{\max} := g(|u^{m+\frac{1}{2}}|^2) - \Phi_{\delta_\ast}^{m+\frac{1}{2}} \in \mathbb{R}_h \) for \( m = 0, \ldots, N - 1 \), \( e^\mu := u^{\frac{1}{2}}_\delta - V^\frac{1}{2}_\delta \in \mathbb{R}_h \), \( e^\gamma := u^{\frac{1}{2}}_\delta - R_h[u(t^\ast, \cdot)] \in \mathbb{R}_h, \ e^\delta := \mu_h[u(t^\ast, \cdot)] ) - V^\delta_\delta \in \mathbb{R}_h, \ e^\mu := u^m - R_h[u(t_m, \cdot)] \in \mathbb{R}_h, \ e^\mu := R_h[u(t_m, \cdot)] - V^\mu_\delta \in \mathbb{R}_h \), \( e^\gamma := u^m - V^m_\delta \in \mathbb{R}_h \) for \( m = 0, \ldots, N \), and \( \partial \varphi^m := \frac{\partial \varphi^m - \varphi^{m-n}}{\tau^2} \in \mathbb{R}_h \) for \( m = 1, \ldots, N \). Also, we note that \( e^\mu = \varphi^\mu + \varphi^\gamma + \varphi^\delta \), \( e^\mu = \varphi^\mu + \varphi^\gamma + \varphi^\delta \) for \( m = 0, \ldots, N \), and that due to (4.1) we have \( \varphi^\delta = 0 \).
In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\tau$, $h$ and $\delta_*$, and may change value from one line to the other. Also, we will use the symbol $C_{\delta_*}$ (with or without additional symbols) to denote a generic constant that depends on $\delta_*$ but is independent of $\tau$, $h$, and may change value from one line to the other. We note that the constants $C$ and $C_{\delta_*}$ may depend on the solution $u$ and its derivatives.

**Part 1**: Combining (4.11), (4.12), (3.6) and (1.5), we get the following error equation:

$$
(4.10) \quad \dot{\vartheta}^\frac{1}{2} - \vartheta^0 - i \frac{\tau}{4} \Delta_h(\dot{\vartheta}^\frac{1}{2} + \vartheta^0) = \tau \sum_{\ell=1}^{3} A^\ell,
$$

where

$$
A^1 := \frac{i}{2} \left( R_h \left[ \frac{u(t^2_\ell) - u(t_\ell)}{(\tau/2)} \right] - \frac{u^\frac{1}{2}_\ell - u^0}{(\tau/2)} \right) \in \mathbb{C}_h,
$$

$$
A^2 := \frac{1}{2} \vartheta^\frac{1}{2} \in \mathbb{C}_h,
$$

$$
A^3 := \frac{1}{2}g \left( |u^0|^2 \right) \vartheta^\frac{1}{2} \in \mathbb{C}_h.
$$

**Part 2**: Since $\vartheta^0 = 0$, after taking the $(\cdot, \cdot)_{0,h}$-inner product of (4.10) with $\dot{\vartheta}^\frac{1}{2}$, and then using (2.5) and keeping the real parts of the relation obtained, we arrive at

$$
(4.11) \quad ||\dot{\vartheta}^\frac{1}{2}||^2_{0,h} = \tau \sum_{\ell=1}^{3} \text{Re}[(A^\ell, \dot{\vartheta}^\frac{1}{2})_{0,h}].
$$

Using (3.21) and (3.10), we obtain

$$
(4.12) \quad ||A^1||_{0,h} \leq \frac{1}{2} \left\| R_h \left[ \frac{u(t^2_\ell) - u(t_\ell)}{(\tau/2)} \right] - \frac{u^\frac{1}{2}_\ell - u^0}{(\tau/2)} \right\|_{0,h} \leq C h^2
$$

and

$$
(4.13) \quad ||A^2||_{0,h} \leq \frac{1}{2} ||\vartheta^\frac{1}{2}||_{0,h} \leq \tau C h.
$$

Combining the Cauchy-Schwarz inequality with (4.12) and (4.13), we get

$$
(4.14) \quad \text{Re}[(A^1, \dot{\vartheta}^\frac{1}{2})_{0,h}] + \text{Re}[(A^2, \dot{\vartheta}^\frac{1}{2})_{0,h}] \leq C (\tau + h^2) ||\dot{\vartheta}^\frac{1}{2}||_{0,h}.
$$

Observing that

$$
\text{Re}[(A^3, \dot{\vartheta}^\frac{1}{2})_{0,h}] = -\frac{1}{4} \text{Im}[g(\vartheta^0) \otimes (\dot{\vartheta}^\frac{1}{2} + \dot{\vartheta}^\frac{1}{2} + \vartheta^0 + \vartheta^0), \dot{\vartheta}^\frac{1}{2})_{0,h}] \\
= -\frac{1}{4} \text{Im}[g(\vartheta^0) \otimes (\dot{\vartheta}^\frac{1}{2} + \vartheta^0), \dot{\vartheta}^\frac{1}{2})_{0,h} - \frac{1}{4} \text{Im}[g(\vartheta^0) \otimes \dot{\vartheta}^\frac{1}{2}, \dot{\vartheta}^\frac{1}{2})_{0,h},
$$

we use the Cauchy-Schwarz inequality and (3.20), to get

$$
(4.15) \quad \text{Re}[(A^3, \dot{\vartheta}^\frac{1}{2})_{0,h}] \leq C \left( ||\dot{\vartheta}^\frac{1}{2}||_{0,h} + ||\vartheta^0||_{0,h} \right) ||\vartheta^\frac{1}{2}||_{0,h} \leq C h^2 ||\vartheta^\frac{1}{2}||_{0,h}.
$$

In view of (4.11), (4.14) and (4.15), we, easily, conclude that

$$
(4.16) \quad ||\dot{\vartheta}^\frac{1}{2}||_{0,h} \leq C (\tau^2 + \tau h^2).
$$

Taking the $(\cdot, \cdot)_{0,h}$–inner product of (4.11) with $\Delta_h(\dot{\vartheta}^\frac{1}{2})$, and then using (2.4) and keeping the real parts of the relation obtained, it follows that

$$
(4.17) \quad ||\dot{\vartheta}^\frac{1}{2}||^2_{1,h} = \tau \sum_{\ell=1}^{3} \text{Re}[\langle \delta_h A^\ell, \delta_h \dot{\vartheta}^\frac{1}{2} \rangle_{0,h}].
$$
Using the Cauchy-Schwarz inequality, (3.21), (3.11) and (3.20), we have

$$\text{Re}[\langle \delta_h A^1, \delta_h \varphi^\frac{1}{2} \rangle_{0,h}] + \text{Re}[\langle \delta_h A^2, \delta_h \varphi^\frac{1}{2} \rangle_{0,h}] \leq (|A^1|_{1,h} + |A^2|_{1,h}) |\varphi^\frac{1}{2}|_{1,h}$$

and

$$\text{Re}[\langle \delta_h (A^3), \delta_h \varphi^\frac{1}{2} \rangle_{0,h}] = -\frac{1}{2} \text{Im}[\langle \delta_h (g(u^0)^2) \otimes (\varphi^\frac{1}{2} + \varphi^0), \delta_h \varphi^\frac{1}{2} \rangle_{0,h}]$$

$$\leq |g(u^0)|^2 \otimes (\varphi^\frac{1}{2} + \varphi^0)_{1,h}$$

$$\leq C (|\varphi^\frac{1}{2}|_{1,h} + |\varphi^0|_{1,h}) |\varphi^\frac{1}{2}|_{1,h}$$

From (4.17), (4.18) and (4.19), it follows that

$$|\varphi^\frac{1}{2}|_{1,h} \leq C (\tau^2 + \tau h^2).$$

Since $\delta_\ast \geq u_{\text{max}}$, we use (2.3), (2.27), (2.25), (2.10) (with $g = g$ and $\varepsilon = 2 \sup_h |n_\delta|^2$), (4.19) and (2.28), to have

$$\|e^0_{\text{min}}\|_{0,h} = \|g(\gamma_{\delta_\ast}(u^\frac{1}{2}))^2 - g(\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2}))^2\|_{0,h}$$

$$\leq C_{\delta_\ast} \|\gamma_{\delta_\ast}(u^\frac{1}{2})^2 - |\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})^2\|_{0,h}$$

$$\leq C_{\delta_\ast} \left[ |\gamma_{\delta_\ast}(u^\frac{1}{2})|_{\infty,h} + |\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})|_{\infty,h} \right] \|\gamma_{\delta_\ast}(u^\frac{1}{2}) - \gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})\|_{0,h}$$

$$\leq C_{\delta_\ast} \|\gamma_{\delta_\ast}(u^\frac{1}{2}) - \gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})\|_{0,h}$$

$$\leq C_{\delta_\ast} \|e^\frac{1}{2}\|_{0,h}$$

$$\leq C_{\delta_\ast} (|\varphi^\frac{1}{2}|_{0,h} + |\varphi^0|_{0,h}),$$

which, along with (4.16) and (3.20), yields

$$\|e^0_{\text{min}}\|_{0,h} \leq C_{\delta_\ast} (\tau^2 + h^2).$$

Also, we use (2.26), (2.24) (with $g = g$ and $\varepsilon = 2 \sup_h |n_\delta|^2$), (2.27), (2.22), (2.29), (2.24) and (2.1), to get

$$|e^0_{\text{mid}}|_{1,h} = |g(\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2}))^2 - g(\gamma_{\delta_\ast}(u^\frac{1}{2}))^2\|_{1,h}$$

$$\leq C_{\delta_\ast} \left[ |\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})^2 - |\gamma_{\delta_\ast}(u^\frac{1}{2})^2\|_{1,h} + |\delta_h(|u^\frac{1}{2})^2\|_{\infty,h} \|\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})^2 - |\gamma_{\delta_\ast}(u^\frac{1}{2})^2\|_{0,h} \right]$$

$$\leq C_{\delta_\ast} \|\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})^2 - |\gamma_{\delta_\ast}(u^\frac{1}{2})^2|_{1,h}$$

$$\leq C_{\delta_\ast} \left[ 1 + |\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2})|_{\infty,h} + |\delta_h(u^\frac{1}{2})|_{\infty,h} \right] |\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2}) - \gamma_{\delta_\ast}(u^\frac{1}{2})|_{1,h}$$

$$\leq C_{\delta_\ast} \|\gamma_{\delta_\ast}(V_{\delta_\ast}^\frac{1}{2}) - \gamma_{\delta_\ast}(u^\frac{1}{2})\|_{1,h}$$

$$\leq C_{\delta_\ast} (1 + |\delta_h(u^\frac{1}{2})|_{\infty,h}) |\varphi^\frac{1}{2}|_{1,h}$$

$$\leq C_{\delta_\ast} (|\varphi^\frac{1}{2}|_{1,h} + |\varphi^0|_{1,h}),$$

which, along with (4.20) and (3.20), yields

$$|e^0_{\text{mid}}|_{1,h} \leq C_{\delta_\ast} (\tau^2 + h^2).$$
Since \( \delta \geq \max \{ u_{\text{max}}, g_{\text{max}} \} \), using (4.4), (4.6), (4.7) and (1.5), we get

\[
\vartheta^{n+1} - \vartheta^n - i \frac{\tau}{2} \Delta_h (\vartheta^{n+1} + \vartheta^n) = \tau \sum_{\ell=1}^4 B^{\ell,n}, \quad n = 0, \ldots, N - 1,
\]

where

\[
B^{1,n} := \left( R_h \left[ \frac{u(t, z_{n+1})}{\tau} - u(t, z_n) \right] - \frac{\vartheta^{n+1} - \vartheta^n}{\tau} \right) \in C_h^0,
\]

\[
B^{2,n} := i \vartheta^{n+1} \in C_h^0,
\]

\[
B^{3,n} := \text{in}_{\delta_n} \left( \Phi_n^{\delta_n} \right) \in C_h^0,
\]

\[
B^{4,n} := \text{in}_{\delta_n} \left( g(||u_n^{\delta_n}|^2) - n_{\delta_n} (\Phi_n^{\delta_n}) \right) \in C_h^0.
\]
Applying (2.25), (2.27), (2.8) with \(g = g\) and \(\varepsilon = 2 \sup_n |n\delta|^2\), (2.21), (2.19), (2.28) and (2.30), we obtain
\[
\|\sigma_n\|_{0,h} \leq C_{\delta}, \left[ \left\| \gamma_\delta, (V_{n_{\delta}}^0)^2 - |\gamma_\delta, (V_{n_{\delta}}^{n-1})|^2 - |\gamma_\delta, (u^n)|^2 + \gamma_\delta, (u^{n-1})^2 \right\|_{0,h} + \left\| u^n - u^{n-1} \right\|_{0,h} \right]
\leq C_{\delta}, \left[ \left\| \gamma_\delta, (V_{n_{\delta}}^0)^2 - |\gamma_\delta, (V_{n_{\delta}}^{n-1})|^2 - |\gamma_\delta, (u^n)|^2 \right\|_{0,h} \right]
\leq C_{\delta}, \left[ \left\| \gamma_\delta, (V_{n_{\delta}}^0)^2 - |\gamma_\delta, (V_{n_{\delta}}^{n-1})|^2 - |\gamma_\delta, (u^n)|^2 \right\|_{0,h} \right]
\leq C_{\delta}, \left[ \left\| \gamma_\delta, (V_{n_{\delta}}^0)^2 - |\gamma_\delta, (V_{n_{\delta}}^{n-1})|^2 - |\gamma_\delta, (u^n)|^2 \right\|_{0,h} \right]
\]
which, along with (3.20) and (3.21), yields
\[
(4.33) \quad \|\sigma_n\|_{0,h} \leq C_{\delta}, \tau (h^2 + \|\partial \varphi^n\|_{0,h} + \|\varphi^{n-1}\|_{0,h}), \quad n = 2, \ldots, N - 1.
\]
Taking the \((\cdot, \cdot)_{0,h}\) inner product of both sides of (4.31) with \((e_n^m + e_{n-1}^m, h)\), and then the Cauchy-Schwarz inequality, it follows that
\[
\left\| e_n^m \right\|_{0,h}^2 - \left\| e_{n-1}^m \right\|_{0,h}^2 \leq 2 \left( \left\| e_n^m \right\|_{0,h} + \left\| e_{n-1}^m \right\|_{0,h} \right) \left( \left\| e_n^m + e_{n-1}^m \right\|_{0,h} \right), \quad n = 2, \ldots, N - 1,
\]
which, along with (3.20) and (3.21), yields
\[
(4.34) \quad \left\| e_n^m \right\|_{0,h} - \left\| e_{n-1}^m \right\|_{0,h} \leq C_{\delta}, \tau \left( \left\| \partial \varphi^n \right\|_{0,h} + \left\| \varphi^{n-1} \right\|_{0,h} \tau^2 + h^2 \right), \quad n = 2, \ldots, N - 1.
\]
\textbf{Part 6}: Since \(\varphi^0 = 0\), after setting \(n = 0\) in (4.29), we conclude that there exists a constant \(C_{1,\delta} > 0\) such that
\[
\left\| \varphi^1 \right\|_{0,h} \leq C_{1,\delta}, \tau \left( \tau^2 + h^2 \right) + \left\| \varphi^0_0 \right\|_{0,h}.
\]
Assuming that \(\tau C_{1,\delta} \leq \frac{1}{2}\), the latter inequality yields
\[
(4.35) \quad \left\| \varphi^1 \right\|_{0,h} \leq C_{\delta}, \tau \left( \tau^2 + h^2 \right) + \left\| \varphi^0_0 \right\|_{0,h}.
\]
Setting \(n = 1\) in (4.30), and using (3.2), (2.6) with \(g = g\) and \(\varepsilon = 2 \sup_n |n\delta|^2\), (2.25), (2.19) and (2.28), we have
\[
\left\| e_1^1 \right\|_{0,h} \leq 2 \left\| g(\gamma_\delta (u^1)^2) - g(\gamma_\delta (V_{n_{\delta}}^1)^2) \right\|_{0,h} + \left\| e_0^0 \right\|_{0,h} + C \tau^2
\leq C_{\delta}, \left[ \left\| g(\gamma_\delta (u^1)^2) - g(\gamma_\delta (V_{n_{\delta}}^1)^2) \right\|_{0,h} + \tau^2 + \left\| e_0^0 \right\|_{0,h} \right]
\leq C_{\delta}, \left[ \left\| g(\gamma_\delta (u^1)^2) - g(\gamma_\delta (V_{n_{\delta}}^1)^2) \right\|_{0,h} + \tau^2 + \left\| e_0^0 \right\|_{0,h} \right]
\leq C_{\delta}, \left[ \left\| e_0^0 \right\|_{0,h} + \tau^2 + \left\| e_0^0 \right\|_{0,h} \right],
\]
which, along with (3.20) and (3.31), yields
\[
(4.36) \quad \left\| e_1^1 \right\|_{0,h} \leq C_{\delta}, \left( \left\| \varphi^1 \right\|_{0,h} + \left\| \varphi^1 \right\|_{0,h} + \tau^2 + h^2 \right) + \left\| e_0^0 \right\|_{0,h}.
\]
Now, set \(n = 1\) in (4.29) and then use (4.35) and (4.36), to conclude that there exists a constant \(C_{2,\delta} \geq C_{1,\delta}\) such that
\[
\left\| \varphi^2 \right\|_{0,h} \leq C_{2,\delta}, \tau \left( \left\| \varphi^2 \right\|_{0,h} + \tau^2 + h^2 \right) + \left\| e_0^0 \right\|_{0,h},
\]
from which, after assuming that \(\tau C_{2,\delta} \leq \frac{1}{2}\), we obtain
\[
(4.37) \quad \left\| \varphi^2 \right\|_{0,h} \leq C_{\delta}, \tau \left( \tau^2 + h^2 \right) + \left\| e_0^0 \right\|_{0,h}.
\]
Since $\vartheta^0 = 0$, we use (4.23) (with $n = 0$), (4.26), (4.27), (4.28) and (4.35), to get
\[
\|A_h(\partial \vartheta^1)\|_{0,h} = \tau^{-1} \|A_h(\vartheta^1)\|_{0,h}
\]
(4.38)
\[
\leq \frac{4}{\ell_1} \|B^{\ell,0}\|_{0,h}
\]
\[
\leq C_{\delta_1} (\tau^2 + h^2 + \|e^0_{\text{mid}}\|_{0,h}).
\]
Finally, in view of (4.23) (with $n = 1$), (4.38), (4.39), (4.40), (4.41), (4.42) and (4.43) we obtain
\[
\|A_h(\partial \vartheta^2)\|_{0,h} = \tau^{-1} \left[ \|A_h(\vartheta^2)\|_{0,h} + \|A_h(\vartheta^1)\|_{0,h} \right]
\]
(4.39)
\[
\leq C_{\delta_1} \tau^{-1} \left[ \|T_h(\vartheta^1)\|_{0,h} + \tau \sum_{\ell_1 = 1}^{4} \|B^{\ell,1}\|_{0,h} + \tau (\tau^2 + h^2 + \|e^0_{\text{mid}}\|_{0,h}) \right]
\]
\[
\leq C_{\delta_1} \tau^{-1} \left[ 2 \|\vartheta^1\|_{0,h} + \|A_h(\vartheta^1)\|_{0,h} + \tau (\tau^2 + h^2 + \|e^0_{\text{mid}}\|_{0,h}) \right]
\]
\[
\leq C_{\delta_1} (\tau^2 + h^2 + \|e^0_{\text{mid}}\|_{0,h}).
\]

**Part 7**: Since $\delta_1 > \max\{g_{\text{max}}, u_{\text{max}}\}$, from (4.23), we easily conclude that
\[
\partial \vartheta^{n+1} - \partial \vartheta^n = \Delta_h (\partial \vartheta^{n+1} + 2 \partial \vartheta^n + \partial \vartheta^{n-1}) + \sum_{\ell_1 = 1}^{6} \Gamma_{\ell,n}, \quad n = 2, \ldots, N - 1,
\]
where
\[
\Gamma_{1,n} := R_h \left[ \frac{u(t_{n-1}, \cdot) - u(t_n, \cdot) - u(t_{n-2}, \cdot)}{\tau} \right] - \frac{u^{n+1} - u^n - u^{n-1} - u^{n-2}}{\tau},
\]
\[
\Gamma_{2,n} := \frac{r - r^{(n-2)} + \frac{1}{2}}{r^{(n-2)} + \frac{1}{2}},
\]
\[
\Gamma_{3,n} := -i n_{\delta_1} (\Phi^{(n-2)}_{\delta_1}) \otimes \Gamma_{3,n},
\]
\[
\Gamma_{4,n} := \gamma_{\delta_1} \left( \frac{V_{\delta_1}^{n+1} + V_{\delta_1}^{n-2}}{2} \right) - \gamma_{\delta_1} \left( \frac{V_{\delta_1}^{n-1} + V_{\delta_1}^{n-2}}{2} \right) - \gamma_{\delta_1} \left( \frac{u^{n+1} + u^n}{2} \right) + \gamma_{\delta_1} \left( \frac{u^{n-1} + u^n}{2} \right),
\]
\[
\Gamma_{5,n} := -i \left[ g(|u|^{n+1}^{\frac{1}{2}}) - g(|u|^{(n-2)}^{\frac{1}{2}}) \right] \otimes \left[ \gamma_{\delta_1} \left( \frac{V_{\delta_1}^{n+1} + V_{\delta_1}^{n-2}}{2} \right) - \gamma_{\delta_1} \left( \frac{u^{n+1} + u^n}{2} \right) \right],
\]
\[
\Gamma_{6,n} := -i \gamma_{\delta_1} \left( \frac{V_{\delta_1}^{n+1} + V_{\delta_1}^{n-2}}{2} \right) \otimes \Gamma_{6,n},
\]
\[
\Gamma_{7,n} := n_{\delta_1} (\Phi^{(n-2)}_{\delta_1}) + n_{\delta_1} (\Phi^{(n-2)}_{\delta_1}) - n_{\delta_1} (g(|u|^{n+1}^{\frac{1}{2}})) + n_{\delta_1} (g(|u|^{(n-2)}^{\frac{1}{2}})).
\]

**Part 8**: Let $n \in \{2, \ldots, N - 1\}$. Observing that
\[
u(t_{n+1}, x) - u(t_n, x) - u(t_{n-1}, x) + u(t_{n-2}, x) = \int_0^\tau \left( \int_{t_{n-2} + s}^{t_{n+1} + s} u_{tt}(s', x) \, ds' \right) \, ds
\]
and using (3.20), (3.21), (2.28) and the mean value theorem, we obtain
\[
\|\Gamma_{1,n}\|_{0,h} \leq C \tau \int_0^\tau \left( \int_{t_{n-2} + s}^{t_{n+1} + s} \|R_h[u_{tt}(s', \cdot)] - I_h[u_{tt}(s', \cdot)]\|_{0,h} \, ds' \right) \, ds
\]
(4.42)
\[
\leq C \tau h^2,
\]
\[
\|\Gamma_{2,n}\|_{0,h} \leq \|r - r^{(n-1)} + \frac{1}{2}\|_{0,h} + \|r^{(n-1)} + \frac{1}{2} - r^{(n-2)} + \frac{1}{2}\|_{0,h}
\]
(4.43)
\[
\leq C \tau^3.
\]
which can be written in a vector operational form (cf. \[6\]) as follows

\[
\|\Gamma^{4,n}\|_{0,h} \leq C_{\delta}, \tau \|e^{n+1} + e^n\|_{0,h}
\]

(4.44)

\[
\leq C_{\delta}, \tau (\|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h} + \|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h})
\]

and

\[
\leq C_{\delta}, \tau (h^2 + \|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h})
\]

(4.45)

Also, we apply \((2.30)\), \((3.20)\) and \((3.21)\), to get

\[
\|\Gamma^{3,n}\|_{0,h} \leq C_{\delta}, \|\Gamma^{3,n}\|_{0,h}
\]

(4.46)

\[
\leq C_{\delta}, \tau \left( \|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h} \right)
\]

\[
+ C_{\delta}, \left( \|\vartheta^{n+1} - \vartheta^n\|_{0,h} + \|\vartheta^{n-1} - \vartheta^{n-2}\|_{0,h} \right)
\]

Combining \((4.42)\), \((4.43)\), \((4.44)\), \((4.45)\), \((4.46)\) and \((4.47)\), we arrive at

\[
\sum_{\ell=1}^{6} \|\Gamma^{\ell,n}\|_{0,h} \leq C_{\delta}, \tau \left( \tau^2 + h^2 + \|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h} + \|\vartheta^{n+1}\|_{0,h} \right)
\]

(4.48)

\[
+ \|\vartheta^{n+1}\|_{0,h} + \|\vartheta^n\|_{0,h} + \|\vartheta^{n-1}\|_{0,h} \right).
\]

**Part 9**: Let \(I_h: \mathbb{C}_h \rightarrow \mathbb{C}_h\) be the identity operator, \(A_h := I_h - i \frac{\tau}{2} \Delta_h\), \(T_h := I_h + i \frac{\tau}{2} \Delta_h\) and \(B_h := A_h^{-1} T_h\). In view of Lemma \((2.30), (4.48)\) is equivalent to

\[
\vartheta^{n+1} = (B_h - I_h)(\vartheta^n) + B_h(\vartheta^{n-1}) + \sum_{\ell=1}^{6} A_h^{-1}(\Gamma^{\ell,n}), \quad n = 2, \ldots, N - 1,
\]

which can be written in a vector operational form (cf. \([6\]) as follows

\[
\left[ \frac{\partial \vartheta^{n+1}}{\partial \vartheta^n} \right] = G \left[ \frac{\partial \vartheta^n}{\partial \vartheta^{n-1}} \right] + \left[ F^n \right], \quad n = 2, \ldots, N - 1,
\]

(4.49)

where

\[
G := \begin{bmatrix} B_h - I_h & B_h \\ I_h & 0 \end{bmatrix} \quad \text{and} \quad F^n := \sum_{\ell=1}^{6} A_h^{-1}(\Gamma^{\ell,n}).
\]

Then, a simple induction argument yields that

\[
\left[ \frac{\partial \vartheta^{m+1}}{\partial \vartheta^m} \right] = G^{m-1} \left[ \frac{\partial \vartheta^2}{\partial \vartheta^1} \right] + \sum_{\ell=2}^{m} G^{m-\ell} \left[ F^\ell \right], \quad m = 2, \ldots, N - 1,
\]

(4.50)
Combining (4.50) and (4.52), we arrive at

\[ G = \begin{bmatrix} -I_h & I_h \\ I_h & B_h^{-1} \end{bmatrix} \begin{bmatrix} -I_h & 0 \\ 0 & B_h \end{bmatrix} \begin{bmatrix} -I_h & I_h \\ I_h & B_h^{-1} \end{bmatrix}^{-1} \]

we conclude that

\[ G^\kappa = \begin{bmatrix} -I_h & I_h \\ I_h & B_h^{-1} \end{bmatrix} \begin{bmatrix} (-1)^{\kappa} I_h & 0 \\ 0 & B_h \end{bmatrix} \begin{bmatrix} -I_h & I_h \\ I_h & B_h^{-1} \end{bmatrix}^{-1} \]

(4.51)

Observing that

\[ G \]

which, along with (4.38), (4.39) and (4.48), yields

\[ \forall \kappa \in \mathbb{N}. \]

(4.55)

It is easily seen that

\[ \left[ \frac{-I_h}{I_h} \right]^{-1} = \left[ \frac{-I_h + B_h^{-1}}{B_h} \right]^{-1} \left[ \frac{I_h + B_h^{-1}}{B_h} \right]^{-1} \]

\[ = \left[ \frac{-I_h + B_h^{-1}}{B_h} \right]^{-1} \left[ \frac{B_h(I_h + B_h)^{-1}}{B_h} \right], \]

which, along with (4.51) and (2.35), yields

(4.52)

\[ G^\kappa = \frac{1}{2} \left[ \frac{(-1)^{\kappa} I_h + B_h^{m+1}}{(-1)^{\kappa} I_h + B_h^m} \right] A_h \left[ \frac{(-1)^{\kappa} B_h + B_h^{m+1}}{(-1)^{\kappa} B_h + B_h^m} \right] A_h \quad \forall \kappa \in \mathbb{N}. \]

Combining (4.50) and (4.52), we arrive at

\[ \partial \vartheta^{m+1} = \left[ (-1)^{m-1} I_h + B_h^m \right] A_h(\partial \vartheta^2) + \left[ (-1)^{m-1} B_h + B_h^m \right] A_h(\partial \vartheta^1) \]

(4.53)

and

\[ \partial \vartheta^m = \left[ (-1)^{m-1} I_h + B_h^{m-1} \right] A_h(\partial \vartheta^2) + \left[ (-1)^{m-1} B_h + B_h^{m-1} \right] A_h(\partial \vartheta^1) \]

(4.54)

for \( m = 2, \ldots, N - 1. \)

Part 10: Applying the discrete norm \( \| \cdot \|_{0,h} \) on both sides of (4.53) and (4.54), and then using (2.38), it follows that

\[ \| \partial \vartheta^{m+1} \|_{0,h} + \| \partial \vartheta^m \|_{0,h} \leq 4 \left[ \| A_h(\partial \vartheta^2) \|_{0,h} + \| A_h(\partial \vartheta^1) \|_{0,h} \right] + 2 \sum_{\ell=2}^m \sum_{\ell'=1}^6 \| \Gamma^{\ell',\ell} \|_{0,h}, \quad m = 2, \ldots, N - 1, \]

which, along with (4.38), (4.39) and (4.48), yields

\[ \| \partial \vartheta^{m+1} \|_{0,h} + \| \partial \vartheta^m \|_{0,h} \leq C_d, \left( \tau^2 + h^2 + \epsilon_{\text{mid}}^0 \right) \| \vartheta^0 \|_{0,h} + \tau \| \vartheta^{m+1} \|_{0,h} + \tau \| \vartheta^m \|_{0,h} \]

(4.55)

\[ + C_d, \tau \sum_{n=2}^m \left( \| \epsilon_{\text{mid}}^{n-2} \|_{0,h} + \| \vartheta^n \|_{0,h} + \| \vartheta^{n-1} \|_{0,h} \right) \]

for \( m = 2, \ldots, N - 1. \)

Observing that \( \| \vartheta^{n-1} \|_{0,h} \leq \tau \| \vartheta^n \|_{0,h} \) and combining (4.29) and (1.31), we obtain

\[ \| \vartheta^{n+1} \|_{0,h} + \| \vartheta^n \|_{0,h} + \| \epsilon_{\text{mid}}^n \|_{0,h} \leq \| \vartheta^n \|_{0,h} + \| \epsilon_{\text{mid}}^{n-1} \|_{0,h} + \| \epsilon_{\text{mid}}^{n-2} \|_{0,h} + C_d, \tau \left( \tau^2 + h^2 \right) \]

\[ + C_d, \tau \left( \| \vartheta^{n+1} \|_{0,h} + \| \vartheta^n \|_{0,h} + \| \vartheta^n \|_{0,h} + \| \epsilon_{\text{mid}}^n \|_{0,h} \right) \]

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for \( n = 2, \ldots, N - 1 \). Now, sum both sides of the latter inequality with respect to \( n \) (from 2 up to \( m \)) and use (4.37) and (4.36), to get
\[
\| \vartheta^{m+1} \|_{0,h} + \| \vartheta^{m}_n \|_{0,h} + \| \vartheta^{m-1} \|_{0,h} \leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h} + \tau \| \vartheta^{m+1} \|_{0,h} + \tau \| \vartheta^{m}_n \|_{0,h}),
\]
(4.56)
\[
+ C_{\delta, \tau} \sum_{n=2}^{m} (\| \vartheta^{m-1} \|_{0,h} + \| \vartheta^0 \|_{0,h} + \| \vartheta^{m}_n \|_{0,h})
\]
for \( m = 2, \ldots, N - 1 \). Introducing the error quantities below
\[
\gamma^n_n := \| \vartheta^{m-1} \|_{0,h} + \| \vartheta^0 \|_{0,h} + \| \vartheta^0 \|_{0,h} + \| \vartheta^{m}_n \|_{0,h}, \quad n = 2, \ldots, N,
\]
from (4.56) and (4.57), we conclude that there exists a constant \( C_{\delta, \tau} \geq C_{2, \delta} \), such that
\[
(1 - C_{\delta, \tau}) \gamma^n_n \leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h}) + C_{\delta, \tau} \sum_{n=2}^{m} \gamma^n_n, \quad m = 2, \ldots, N - 1.
\]
Assuming that \( \tau C_{\delta, \tau} \leq 1/2 \) and applying a standard discrete Gronwall argument, (4.58) yields that
\[
\max_{2 \leq m \leq N} \gamma^n_m \leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h} + \gamma^2_h).
\]
Using (4.57), (4.36), (4.37) and (4.33), we obtain
\[
\gamma^n_n = \| \vartheta^{m-1} \|_{0,h} + \| \vartheta^0 \|_{0,h} + \| \vartheta^0 \|_{0,h} + \| \vartheta^{m}_n \|_{0,h}
\]
\[
\leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h} + \| \vartheta^0 \|_{0,h}) + \tau^{-1} (\| \vartheta^0 \|_{0,h} + 2 \| \vartheta^1 \|_{0,h})
\]
\[
\leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h}).
\]
Thus, (4.59), (4.60) and (4.33), yield
\[
\max_{0 \leq m \leq N} \| \vartheta^m \|_{0,h} + \max_{0 \leq m \leq N} \| \vartheta^m \|_{0,h} + \max_{1 \leq m \leq N} \| \partial \vartheta^m \|_{0,h} \leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h}).
\]
Taking the \((\cdot, \cdot)_{0,h}\)-inner product of (4.23) with \( (\vartheta^{m+1} - \vartheta^m) \), and then using (2.4) and keeping the imaginary parts of the relation obtained, we have
\[
| \vartheta^{n+1} |^2_{1,h} - | \vartheta^n |^2_{1,h} = 2 \tau \sum_{\ell=1}^{4} \text{Im}[(B^{\ell,n}, \partial \vartheta^{n+1})_{0,h}], \quad n = 0, \ldots, N - 1,
\]
which, along with the Cauchy-Schwarz inequality, (4.26), (4.27), (4.28) and (4.61), yields
\[
| \vartheta^{n+1} |^2_{1,h} - | \vartheta^n |^2_{1,h} \leq 2 \tau \| \partial \vartheta^{n+1} \|_{0,h} \sum_{\ell=1}^{4} | B^{\ell,n} \|_{0,h}
\]
\[
\leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h})^2, \quad n = 0, \ldots, N - 1.
\]
Since \( \vartheta^0 = 0 \), after applying a standard discrete Gronwall argument on (4.62), we arrive at
\[
\max_{0 \leq m \leq N} | \vartheta^m |_{1,h} \leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h}).
\]
Combining (4.63), (4.61), (4.20) and (4.21), we obtain
\[
\max_{0 \leq n \leq N} \| \vartheta^n \|_{1,h} \leq \max_{0 \leq n \leq N} (| \vartheta^n |_{1,h} + | \vartheta^n |_{1,h})
\]
\[
\leq C_{\delta, \tau} (\vartheta^2 + h^2 + \| \vartheta^0 \|_{0,h})
\]
\[
\leq C_{\delta, \tau} (\vartheta^2 + h^2),
\]
which, along with (4.20), (4.16) and (4.20), establishes (4.3).

Part 11: For \( n = 2, \ldots, N - 1 \), let \( \zeta^n_\delta \in \mathbb{R}^*_h \) and \( \psi^n_\delta \in C^*_h \) be defined by
\[
\zeta^n_\delta := | \gamma_\delta, (V^{\ell}_\delta) |^2 - | \gamma_\delta, (V^{n-1}_\delta) |^2 - | \gamma_\delta, (u^n) |^2 + | \gamma_\delta, (u^{n-1}) |^2,
\]
\[
\psi^n_\delta := \gamma_\delta, (V^n) - \gamma_\delta, (V^{n-1}) - \gamma_\delta, (u^n) + \gamma_\delta, (u^{n-1}).
\]
Under the light of (1.32), (2.9) with \((g = g_1\) and \(\varepsilon = 2\sup_{\delta} n_{\delta}^2)\), (2.1), (2.2), (2.20), (2.25) and (2.29), and (4.61), (3.20), (2.22), (2.31) and (3.21), we have

\[
|\sigma_{\delta}^n|_{1,h} \leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_1) + |\gamma_\delta, (V_{\delta}^{n-1})|_1 \right) |\zeta_{\delta}^n|_{1,h} \\
+ \tau |\gamma_\delta, (V_{\delta}^{-1})|_2 - |\gamma_\delta, (u^{-1})|_1 \right) \leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_\infty, h + |\gamma_\delta, (V_{\delta}^{n-1})|_\infty, h) |\gamma_\delta, (V_{\delta}^{n-1})|_1 \right) |\zeta_{\delta}^n|_{1,h} \\
(4.65)
\]

\[
|\zeta_{\delta}^n|_{1,h} \leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_1, h + |\gamma_\delta, (V_{\delta}^{n-1})|_1, h) |\psi_{\delta}^n|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^{n-1}) - \gamma_\delta, (u^{-1})|_1, h \right) \\
\leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_1, h + |\gamma_\delta, (V_{\delta}^{n-1})|_1, h) |\psi_{\delta}^n|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^{n-1})|_1, h \right) \\
\leq C_{\delta}, \left[ |\zeta_{\delta}^n|_{1,h} + \tau (\tau^2 + h^2 + \|e_{\text{mid}}^0\|_0, h) \right] \\
(4.66)
\]

\[
|\psi_{\delta}^n|_{1,h} \leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_1, h + |\gamma_\delta, (V_{\delta}^{n-1})|_1, h) |\psi_{\delta}^n|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^{n-1})|_1, h \right) \\
\leq C_{\delta}, \left( (1 + |\gamma_\delta, (V_{\delta}^n)|_1, h + |\gamma_\delta, (V_{\delta}^{n-1})|_1, h) \right) |\psi_{\delta}^n|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^{n-1})|_1, h \right) \\
\leq C_{\delta}, \left[ |\zeta_{\delta}^n|_{1,h} + \tau (\tau^2 + h^2 + \|e_{\text{mid}}^0\|_0, h) \right] \\
(4.67)
\]

for \(n = 2, \ldots, N - 1\). Thus, (4.65), (4.66) and (4.67), yield that

\[
|\sigma_{\delta}^n|_{1,h} \leq C_{\delta}, \tau \left( \tau^2 + h^2 + \|e_{\text{mid}}^0\|_0, h + |\vartheta|_{1, h} \right), \quad n = 2, \ldots, N - 1. \\
(4.68)
\]

Taking the \((\cdot, \cdot)_0, h\) inner product of both sides of (1.34) with \(\Delta_h (e_{\text{mid}}^n + e_{\text{mid}}^{-2})\), and then applying (2.4), keeping the real part of the obtained relation and using the Cauchy-Schwarz inequality, it follows that

\[
|e_{\text{mid}}^n|_{1,h} - |e_{\text{mid}}^{-2}|_{1,h} \leq 2 \left( |\sigma_{\delta}^n|_{1,h} + |r^n - r_{n-1}^n|_{1, h} \right) \left( |e_{\text{mid}}^n|_{1,h} + |e_{\text{mid}}^{-2}|_{1,h} \right), \quad n = 2, \ldots, N - 1. \\
\]

After using (5.5) and (1.68), the latter inequality yields

\[
|e_{\text{mid}}^n|_{1,h} - |e_{\text{mid}}^{-2}|_{1,h} \leq C_{\delta}, \tau \left( \tau^2 + h^2 + \|e_{\text{mid}}^0\|_0, h + |\vartheta|_{1, h} \right), \quad n = 2, \ldots, N - 1. \\
(4.69)
\]

Also, using (4.30) (with \(n = 1\), (3.3), (2.7) (with \(g = g_1\) and \(\varepsilon = 2\sup_{\delta} n_{\delta}^2\)), (2.2), (2.20), (2.25) and (2.29), we get

\[
|e_{\text{mid}}^1|_{1,h} \leq C_{\delta}, \left[ |e_{\text{mid}}^0|_{1,h} + |r^1|_{1, h} + |g| \left( |\gamma_\delta, (V_{\delta}^1)|^2 - g |\gamma_\delta, (u^1)|^2 \right) \right] \leq C_{\delta}, \left[ \tau^2 + |e_{\text{mid}}^0|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^1)|^2 - |\gamma_\delta, (u^1)|^2 \right] \leq C_{\delta}, \left[ \tau^2 + |e_{\text{mid}}^0|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^1)|^2 - |\gamma_\delta, (u^1)|^2 \right] \leq C_{\delta}, \left[ \tau^2 + |e_{\text{mid}}^0|_{1,h} + \tau |\gamma_\delta, (V_{\delta}^1)|^2 - |\gamma_\delta, (u^1)|^2 \right], \\
\]

which, along with (4.64), (4.21) and (4.22), yields

\[
|e_{\text{mid}}^1|_{1,h} \leq C_{\delta}, \left( \tau^2 + h^2 + \|e_{\text{mid}}^0\|_1, h \right) \leq C_{\delta}, \left( \tau^2 + h^2 \right). \\
(4.70)
\]
Summing both sides of (1.69) with respect to $n$ (from 2 up to $m$) and using (4.70), we arrive at

(4.71)  
\[ |e^m_{mid}|_{1,h} + |e^{m-1}_{mid}|_{1,h} \leq C_\delta, \left( \tau^2 + h^2 + \|e^0_{mid}\|_{1,h} \right) + C_\delta, \tau \sum_{\ell=2}^m |\partial \partial^\ell|_{1,h}, \quad m = 2, \ldots, N - 1. \]

**Part 12:** First, we observe that (1.61), (1.21) and the inverse inequality (2.23) yield that

(4.72)  
\[ \max_{0 \leq m \leq N-1} |e^m_{mid}|_{1,h} \leq C_\delta, \left[ h + \left( \frac{\tau}{h} \right) \tau \right]. \]

From (1.30), (3.3), (2.7), (2.2), (2.20) and (2.29), we obtain

\[
|e^n_{mid}|_{1,h} \leq 2 \left| g((\gamma \delta, (V^n_{\delta^2}))^2) - g((\gamma \delta, (u^n)^2)) \right|_{1,h} + 2 |e^0_{mid}|_{1,h} + |e^{n-1}_{mid}|_{1,h} \\
\leq C_\delta, \left[ \tau^2 + |\gamma \delta, (V^n_{\delta^2})^2| - |\gamma \delta, (u^n)^2| \right]_{1,h} + |e^{n-1}_{mid}|_{1,h} \\
\leq C_\delta, \left[ \tau^2 + |\gamma \delta, (V^n_{\delta^2}) - \gamma \delta, (u^n)^2| \right]_{1,h} + |e^{n-1}_{mid}|_{1,h} \\
\leq C_\delta, \left( \tau^2 + |e^0_{mid}|_{1,h} \right) + |e^{n-1}_{mid}|_{1,h}, \quad n = 1, \ldots, N - 1,
\]

which, along with (4.64), yields

\[ |e^n_{mid}|_{1,h} \leq |e^{n-1}_{mid}|_{1,h} + C_\delta, \left( \tau^2 + h^2 \right), \quad n = 1, \ldots, N - 1. \]

Now, summing with respect to $n$ (from 1 up to $m$) and using (1.22), we get

(4.73)  
\[ \max_{0 \leq m \leq N-1} |e^m_{mid}|_{1,h} \leq |e^0_{mid}|_{1,h} + C_\delta, \tau^{-1} \left( \tau^2 + h^2 \right) \]

\[ \leq C_\delta, \left[ h + \left( \frac{\tau}{h} \right) + \tau + \tau^2 + h^2 \right]. \]

Observing that $\min\left\{ \frac{h}{\tau}, \frac{1}{\tau} \right\} \leq 1$ (cf. (19)) and combining (1.72) and (4.73), we conclude, easily, that there exists a positive constant $C^{b,n}_{\delta^t}$ which is independent of $h$ and $\tau$, and such that

(4.74)  
\[ \max_{0 \leq m \leq N-1} |e^m_{mid}|_{1,h} \leq C^{b,n}_{\delta^t}. \]

**Part 13:** Let $n \in \{2, \ldots, N - 1\}$. Using (4.11), (3.20) and (3.15), we get

(4.75)  
\[ |\Gamma^{1,n}|_{1,h} + |\Gamma^{2,n}|_{1,h} \leq C \tau \left( \tau^2 + h^2 \right). \]

Also, using the mean value theorem, (2.21), (2.22), (2.23), (1.64) and (2.7) (with $g = n_{\delta^t}$), we obtain

(4.76)  
\[ |\Gamma^{4,n}|_{1,h} \leq \left| g((u^{n+\frac{1}{2}})^2) - g((u^{n-1}+\frac{1}{2})^2) \right|_{\infty,h} \left| \gamma \delta, \left( \frac{V^{n+1}_\delta+V^n_\delta}{2} \right) \right|_{1,h} \\
+ \left| g((u^{n+\frac{1}{2}})^2) - g((u^{n-1}+\frac{1}{2})^2) \right|_{\infty,h} \left| \gamma \delta, \left( \frac{V^{n+1}_\delta+V^n_\delta}{2} \right) \right|_{1,h} \\
\leq C \tau \left| \gamma \delta, \left( \frac{V^{n+1}_\delta+V^n_\delta}{2} \right) \right|_{1,h} \\
\leq C_\delta, \tau \left( 1 + \|\delta_h (u^{n+1} + u^n)\|_{\infty,h} \right) |e^{n+1}|_{1,h} \\
\leq C_\delta, \tau \left( \tau^2 + h^2 + \|e^0_{mid}\|_{0,h} \right) \\
\text{and} \\
|\Gamma^{5,n}|_{1,h} \leq \left| (u^{n+1} + u^n) - (u^n + u^{n-1}) \right|_{\infty,h} \left| n_{\delta^t}, (\Phi_{\delta^t}(n^{n-2}+\frac{1}{2})) - n_{\delta^t}, (g((u^{n-1}+\frac{1}{2})^2)) \right|_{1,h} \\
+ \left| (u^{n+1} + u^n) - (u^n + u^{n-1}) \right|_{1,h} \left| n_{\delta^t}, (\Phi_{\delta^t}(n^{n-2}+\frac{1}{2})) - n_{\delta^t}, (g((u^{n-1}+\frac{1}{2})^2)) \right|_{\infty,h} \\
\leq C \tau \left| n_{\delta^t}, (\Phi_{\delta^t}(n^{n-2}+\frac{1}{2})) - n_{\delta^t}, (g((u^{n-1}+\frac{1}{2})^2)) \right|_{1,h} \\
\leq C_\delta, \tau \left| \|e^{n-2}_{mid}\|_{1,h} + \|\delta_h (g((u^{n-1}+\frac{1}{2})^2))\|_{\infty,h} \cdot \|e^{n-2}_{mid}\|_{0,h} \right| \\
\leq C_\delta, \tau |e^{n-2}_{mid}|_{1,h}. \]
Applying (2.9) (with $g = n_\delta$), (2.7), (2.22), (1.74), (1.31), (1.68), (1.5), (2.31), (1.64) and (3.21) it follows that

\[
|\Gamma^{6,n}_{1,h}| \leq C_\delta \left[ 1 + |\Phi^{(n-2)+\tau}_{\delta}\mid_{1,h} + |\Phi^{(n-2)+\tau}_{\delta}\mid_{1,h} \right] |\varepsilon^{n-2}_{\text{mid}}|_{1,h} + C_\delta \tau |e^{n-2}_{\text{mid}}|_{1,h}
\]

(4.78)

and

\[
|\Gamma^{3,n}_{1,h}| \leq C_\delta \left[ 1 + |V^{n+1}_{\delta}\mid_{1,h} + |V^{n-1}_{\delta}\mid_{1,h} \right] |\varepsilon^{n+1}_{\text{mid}}|_{1,h} + \tau |\varepsilon^{n+1}_{\text{mid}}|_{1,h}
\]

(4.79)

Then, we use (2.23), (2.31), (2.22), (1.64), (1.73), (2.23), (1.74) and (4.79), to get

\[
|\Gamma^{6,n}_{1,h}| \leq C_\delta \left[ 1 + |V^{n+1}_{\delta}\mid_{1,h} + |V^{n-1}_{\delta}\mid_{1,h} \right] |\varepsilon^{n+1}_{\text{mid}}|_{1,h} + |\varepsilon^{n-1}_{\text{mid}}|_{1,h}
\]

(4.80)

and

\[
|\Gamma^{3,n}_{1,h}| \leq C_\delta \left[ 1 + |V^{n+1}_{\delta}\mid_{1,h} + |V^{n-1}_{\delta}\mid_{1,h} \right] |\varepsilon^{n+1}_{\text{mid}}|_{1,h} + C_\delta \tau |\varepsilon^{n-1}_{\text{mid}}|_{1,h}
\]

(4.81)

Thus, from (4.73), (4.76), (4.77), (4.80) and (4.81), we arrive at

\[
|\Gamma^{6,n}_{1,h}| \leq C_\delta \tau |\varepsilon^{n+1}_{\text{mid}}|_{1,h} + |\varepsilon^{n+1}_{\text{mid}}|_{1,h} + |\varepsilon^{n-1}_{\text{mid}}|_{1,h}
\]

(4.82)

Part 14: Let $\epsilon \in (0,1)$. Taking the $\langle \cdot, \cdot \rangle_{0,h}$–inner product of (4.23) with $\Delta_h(\vartheta^{n+1} + \vartheta^n)$ and then using (2.24) and keeping the real parts of the relation obtained, it follows that

\[
|\vartheta^{n+1}_{1,h} - |\vartheta^n_{1,h}| = \tau \sum_{\ell=1}^4 \text{Re}[\langle \delta_h B^{\ell,n} \mid_{0,h} \rangle_{0,h}]
\]

which, along with the use of the Cauchy-Schwarz inequality, yields

\[
|\vartheta^{n+1}_{1,h} - |\vartheta^n_{1,h}| \leq \tau \sum_{\ell=1}^4 |B^{\ell,n}_{1,h}|
\]

(4.83)

First, we observe that (3.21) and (3.13), easily yield the following bound

\[
|B^{1,n}_{1,h} + |B^{2,n}_{1,h}| \leq C (\tau^2 + h^2)
\]

(4.84)
Combining (2.23), (2.1), (4.74), (2.29), (2.2), (4.64), (2.7) (with $g = m_d$) and (4.70), we have

\[
|B^{3,n}|_{1,h} \leq |n_d, \left( F^{3,n}_{\delta_s} \right)\rangle_{\infty,h} \mid \gamma_{\delta_s} \left( u^{n+\frac{1}{2}} \right) - \gamma_{\delta_s} \left( \frac{V_{\delta_s}^{n+1} + V_{\delta_s}^{n}}{2} \right) \mid_{1,h} \\
+ |n_d, \left( F^{3,n}_{\delta_s} \right)\rangle_{1,h} \mid \gamma_{\delta_s} \left( u^{n+\frac{1}{2}} \right) - \gamma_{\delta_s} \left( \frac{V_{\delta_s}^{n+1} + V_{\delta_s}^{n}}{2} \right) \mid_{\infty,h} \\
\leq C_{\delta_s} \tau \left[ 1 + \sup_{|x| \leq R} |n_d, \left( F^{3,n}_{\delta_s} \right)\rangle_{1,h} \mid \gamma_{\delta_s} \left( u^{n+\frac{1}{2}} \right) - \gamma_{\delta_s} \left( \frac{V_{\delta_s}^{n+1} + V_{\delta_s}^{n}}{2} \right) \mid_{1,h} \\
\leq C_{\delta_s} \left| e^{n+1} + e^n \mid 1,h \leq C_{\delta_s} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{0,h}) \right.
\]

and

\[
|B^{4,n}|_{1,h} \leq |u^{n+1} + u^n\rangle_{\infty,h} \mid n_d, \left( g(|u^{n+\frac{1}{2}}|^2) \right) - n_d, \left( F^{4,n}_{\delta_s} \right)\rangle_{1,h} \\
+ |n_d, \left( F^{4,n}_{\delta_s} \right)\rangle_{1,h} \mid n_d, \left( g(|u^{n+\frac{1}{2}}|^2) \right) - n_d, \left( F^{4,n}_{\delta_s} \right)\rangle_{0,h} \\
\leq C_{\delta_s} \left[ \left| n_d, \left( F^{4,n}_{\delta_s} \right) - n_d, \left( g(|u^{n+\frac{1}{2}}|^2) \right) \mid_{1,h} + \|e_{\text{mod}}\|_{0,h} \right. \\
\leq C_{\delta_s} \left( \|e_{\text{mod}}\|_{1,h} + \|e_{\text{mod}}\|_{0,h} \right) \\
\leq C_{\delta_s} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}).
\]

Since $\vartheta^0$, using (4.83), (4.84), (4.85) and (4.86), we conclude that

\[
|\vartheta^1|_{1,h} + |\vartheta^0|_{1,h} \leq C_{\delta} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}).
\]

Combining (4.23) (with $n = 0$), (4.83), (4.85), (4.86), we get

\[
|A_h(\partial \vartheta^1)|_{1,h} = \tau^{-1} |A_h(\vartheta^1)|_{1,h} \\
\leq \frac{1}{2} |B^0|_{1,h} \\
\leq C_{\delta_s} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}).
\]

Finally, in view of (4.23) (with $n = 1$), (4.88), (4.84), (4.85), (4.86) and (4.63), we obtain

\[
|A_h(\partial \vartheta^2)|_{1,h} = \tau^{-1} \left[ |A_h(\vartheta^2)|_{1,h} + |A_h(\vartheta^1)|_{1,h} \right] \\
\leq C_{\delta_s} \tau^{-1} \left[ |T_h(\vartheta^1)|_{1,h} + \tau \sum_{\ell = 1}^{5} |B^{\ell,1}|_{1,h} + \tau (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}) \right] \\
\leq C_{\delta_s} \tau^{-1} \left[ |2 \vartheta^1 - A_h(\vartheta^1)|_{1,h} + \tau (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}) \right] \\
\leq C_{\delta_s} \tau^{-1} \left[ |2 \vartheta^1|_{1,h} + |A_h(\vartheta^1)|_{1,h} = \tau (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}) \right] \\
\leq C_{\delta_s} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h}).
\]

Part 15: Apply the discrete norm $|\cdot|_{1,h}$ on both sides of (4.53) and (4.54), and then use (2.34), (4.88), (4.89) and (4.82), to have

\[
|\partial \vartheta^m|_{1,h} + |\partial \vartheta^m|_{1,h} \leq 4 \left[ |A_h(\partial \vartheta^2)|_{1,h} + |A_h(\partial \vartheta^1)|_{1,h} \right] + 2 \sum_{\ell = 2}^{m} \sum_{\ell = 1}^{6} |\Gamma^{\ell,1}|_{1,h} \\
\leq C_{\delta_s} (\tau^2 + h^2 + \|e_{\text{mod}}\|_{1,h} + \tau |\partial \vartheta^m|_{1,h}) \\
+ \tau \sum_{\ell = 2}^{m} \left( |\partial \vartheta^m|_{1,h} + |\partial \vartheta^{m-1}|_{1,h} + \|e_{\text{mod}}^{m-2}\|_{1,h} \right), \quad m = 2, \ldots N - 1.
\]
Introducing the following discrete \( H^1 \)-error quantities:
\[
\hat{s}_m^n = |e_{\text{mid}}^{m-1}|_{1,h} + |e_{\text{mid}}^{m-2}|_{1,h} + |\partial \theta^n|_{1,h} + |\partial \theta^{n-1}|_{1,h}, \quad n = 2, \ldots, N,
\]
from (4.90) and (4.74) we conclude that there exists a constant \( C_{4,\delta} \geq C_{3,\delta} \) such that
\[
(4.92) \quad (1 - C_{4,\delta}) \hat{s}_m^{m+1} \leq C_{4,\delta} (\tau^2 + h^2 + \|u^0_\text{mid}\|_{1,h}) + C_{\delta} \tau \sum_{\ell=2}^{m} \hat{s}_\ell, \quad m = 2, \ldots, N - 1.
\]
Assuming that \( \tau C_{4,\delta} \leq \frac{1}{2} \) and applying a standard discrete Gronwall argument, (4.92) yields that
\[
(4.93) \quad \max_{0 \leq m \leq N-1} \hat{s}_m^n \leq C_{\delta} (\tau^2 + h^2 + \|u^0_\text{mid}\|_{1,h} + \hat{s}_0^n).
\]
Using (4.91), (4.70) and (4.87), we obtain
\[
\hat{s}_m^n = \max_{0 \leq m \leq N-1} |e_{\text{mid}}^{m-1}|_{1,h} + |e_{\text{mid}}^{m-2}|_{1,h} + |\partial \theta^n|_{1,h} + |\partial \theta^{n-1}|_{1,h}
\leq C_{\delta} (\tau^2 + h^2 + \|u^0_\text{mid}\|_{1,h}) + \tau^{-1} (\|g^2|_{1,h} + 2 |\partial \theta|_{1,h})
\leq C_{\delta} (\tau^2 + h^2 + \|u^0_\text{mid}\|_{1,h}).
\]
From, (4.93) and (4.94), follows that
\[
(4.95) \quad \max_{0 \leq m \leq N-1} \|e_{\text{mid}}^{m+1}|_{1,h} + \max_{0 \leq m \leq N-1} |\partial \theta^{1,m}|_{1,h} \leq C_{\delta} \tau^2 + h^2 + \|u^0_\text{mid}\|_{1,h}.
\]
Thus, (4.9) follows easily from (4.95), (4.21) and (4.22).

Next, we present how the convergence result of Theorem 4.2 changes when \( \Phi_{\frac{2}{3}} \) is a first order approximation of \( g(|u(t^+,\cdot)|^2) \).

**Theorem 4.3.** Let \( u_{\text{max}} := \max_Q |u|, g_{\text{max}} := \max_Q |g(|u|^2)| \) and \( \delta_{\text{s}} \geq \max\{u_{\text{max}}, g_{\text{max}}\} \). If
\[
(4.96) \quad \Phi_{\frac{2}{3}} = g(|u|^{\frac{1}{2}}),
\]
then, there exist positive constants \( C_{\delta_{s}} \) and \( \tilde{C}_{\delta_{s}} \), independent of \( \tau \) and \( h \), such that: if \( \tau \tilde{C}_{\delta_{s}} \leq \frac{1}{2} \), then
\[
(4.97) \quad \max_{0 \leq m \leq N-1} \|g(u^{m+\frac{1}{2}}) - \Phi_{\frac{2}{3}}^{m+\frac{1}{2}}|_{1,h} + \max_{0 \leq m \leq N} \|u^m - W^m|_{1,h} \leq \tilde{C}_{\delta_{s}} (\tau + h^2).
\]
**Proof.** Here, for simplicity, we keep the notation and the notation convection of the proof of Theorem 4.2

Under the choice (4.96), we obtain \( \|e_{\text{mid}}^0\|_{1,h} = O(\tau) \) and hence (4.97) follows easily, moving along the lines of the proof of Theorem 4.2 (see (4.64) and (4.95)).

**4.5. Convergence of the (RFD) method.** In this section we show how we can use the convergence results for the (MRFD) scheme to conclude convergence of the (RFD) method.

**Theorem 4.4.** Let \( u_{\text{max}} := \max_Q |u|, g_{\text{max}} := \max_Q |g(u)|, \delta_{\text{s}} \geq 2 \max\{u_{\text{max}}, g_{\text{max}}\} \), \( C_{\delta_{s}} \), \( \tilde{C}_{\delta_{s}} \), and \( \hat{C}_{\delta_{s}} \) be the constants specified in Theorem 4.2. If \( \tau C_{\delta_{s}} \leq \frac{1}{2} \) and
\[
(4.98) \quad \max_{0 \leq m \leq N-1} \|g(u^{m+\frac{1}{2}}) - \Phi_{\frac{2}{3}}^{m+\frac{1}{2}}|_{1,h} + \|u^m - W^m|_{1,h} \leq C_{\delta_{s}} (\tau^2 + h^2).
\]
**Proof.** The convergence estimates (4.8) and (4.9), the mesh size condition (4.98) and (4.21) imply that
\[
(4.99) \quad \max_{0 \leq m \leq N-1} \|g(u^{m+\frac{1}{2}}) - \Phi_{\frac{2}{3}}^{m+\frac{1}{2}}|_{1,h} + g_{\text{max}} \leq \sqrt{C_{\delta_{s}}} (\tau^2 + h^2) + \frac{\delta_{s}}{2} \leq \delta_{s}.
\]
and

$$\max \{ |V_{\delta_n}^{\frac{1}{2}}|_{1,h}, \max_{0 \leq m \leq N} |V_{\delta_n}^m|_{1,h} \} \leq \max \{ |u^{\frac{1}{2}} - V_{\delta_n}^{\frac{1}{2}}|_{1,h}, \max_{0 \leq m \leq N} |u^m - V_{\delta_n}^m|_{1,h} \} + u_{\text{max}}$$

$$\leq \sqrt{N} \max \{ \|u^{\frac{1}{2}} - V_{\delta_n}^{\frac{1}{2}}\|_{1,h}, \max_{0 \leq m \leq N-1} \|u^m - V_{\delta_n}^m\|_{1,h} \} + \frac{\delta}{2}$$

$$\leq \sqrt{N} C_{\delta_n}^{\frac{3}{2}} (r^2 + h^2) + \frac{\delta}{2}$$

which, along with (4.100) and (4.101), yields \( \gamma_{\delta_n}(V_{\delta_n}^{\frac{1}{2}}) = V_{\delta_n}^{\frac{1}{2}}, n_\delta(\Phi_{\delta_n}^{m+\frac{1}{2}}) = \Phi_{\delta_n}^{m+\frac{1}{2}} \) for \( m = 0, \ldots, N-1 \), and \( \gamma_{\delta_n} \left( \frac{V_{\delta_n}^{n+1} + V_{\delta_n}^n}{2} \right) = \frac{V_{\delta_n}^{n+1} + V_{\delta_n}^n}{2} \) for \( n = 0, \ldots, N-1 \). Thus, we conclude that for \( \delta = \delta_n \), the (MFD) approximations are (RFD) approximations, i.e. \( W^\frac{1}{2} = V_{\delta_n}^{\frac{1}{2}}, W^n = V_{\delta_n}^n \) for \( n = 0, \ldots, N \), and \( \Phi_{\delta_n}^{n+\frac{1}{2}} = \Phi_{\delta_n}^{n+\frac{1}{2}} \) for \( n = 0, \ldots, N-1 \). Thus, we obtain (4.39) as a simple outcome of (4.8) and (4.9).

**Theorem 4.5.** Let \( u_{\text{max}} := \max_{\Omega} |u|, g_{\text{max}} := \max_{\Omega} |g(u)|, \delta_n \geq 2 \max \{ u_{\text{max}}, g_{\text{max}} \}, \mathcal{C}_\delta \) and \( \mathcal{C}_\delta \) be the constants specified in Theorem 4.3. \( \Phi^\frac{1}{2} = g(|u|^2) \), \( \tau \mathcal{C}_\delta \leq \frac{1}{2} \) and \( \mathcal{C}_\delta \sqrt{r^2 + h^2} \leq \frac{\delta}{2} \). Then, it holds that

\[
\max_{0 \leq m \leq N-1} \|g(|u^m|^\frac{1}{2}) - \Phi^m + \frac{1}{2}\|_{1,h} \leq \mathcal{C}_\delta (r^2 + h^2).
\]

Also, there exists a positive constants \( \mathcal{C}_\delta \geq \mathcal{C}_\delta \) and \( \tilde{C}_\delta \), independent of \( \tau \) and \( h \), such that: if \( \tau \mathcal{C}_\delta \leq \frac{1}{2} \), then

\[
\max_{0 \leq m \leq N} \|u^m - W^m\|_{1,h} \leq \tilde{C}_\delta (r^2 + h^2).
\]

**Proof.** Here, for simplicity, we keep the notation and the notation convection of the proof of Theorem 4.2.

Using our assumptions and moving along the lines of the proof of Theorem 4.4, we conclude that \( \gamma_{\delta_n}(V_{\delta_n}^{\frac{1}{2}}) = V_{\delta_n}^{\frac{1}{2}}, n_\delta(\Phi_{\delta_n}^{m+\frac{1}{2}}) = \Phi_{\delta_n}^{m+\frac{1}{2}} \) for \( m = 0, \ldots, N-1 \), and \( \gamma_{\delta_n} \left( \frac{V_{\delta_n}^{n+1} + V_{\delta_n}^n}{2} \right) = \frac{V_{\delta_n}^{n+1} + V_{\delta_n}^n}{2} \) for \( n = 0, \ldots, N-1 \). Thus, for \( \delta = \delta_n \), the (MFD) approximations are (RFD) approximations, and (4.97) inherits (4.100) and (4.102)

\[
\max_{0 \leq m \leq N} \|u^m - W^m\|_{1,h} = \max_{0 \leq m \leq N} \|u^m - V_{\delta_n}^m\|_{1,h} \leq C_\delta (r + h^2).
\]

Taking the \( (\cdot, \cdot)_{0,h} \)-inner product of (4.23) with \( \Delta_h(\vartheta^{n+1} + \vartheta^n) \) and then using (2.4) and keeping the real parts of the relation obtained, it follows that

\[
|\vartheta^{n+1}|_{1,h}^2 - |\vartheta^n|_{1,h}^2 = \sum_{\ell=1}^4 \mathcal{Z}^{\ell,n}, \quad n = 0, \ldots, N-1,
\]

where

\[
\mathcal{Z}^{\ell,n} := \tau \text{Re} \left[ (\delta_h B^{\ell,n}, \delta_h(\vartheta^{n+1} + \vartheta^n))_{0,h} \right], \quad \ell = 1, 2, 3, 4.
\]

Then, we sum with respect to \( n \) (from \( n = 0 \) up to \( n = m \)) to get

\[
|\vartheta^{m+1}|_{1,h}^2 = \sum_{n=0}^m \sum_{\ell=1}^4 \mathcal{Z}^{\ell,n}, \quad m = 0, \ldots, N-1.
\]
In view of (4.24), (3.21) and (3.13), after the application of the Cauchy-Schwarz inequality, we get

\[
\sum_{n=0}^{m} (Z^{1,n} + Z^{2,n}) \leq C \tau \sum_{n=0}^{m} (\tau^2 + \tau h^2) (|\vartheta^{m+1}|_{1,h} + |\vartheta^n|_{1,h})
\]

(4.104)

\[
\leq C \tau \sum_{n=0}^{m} ((\tau^2 + \tau h^2)^2 + |\vartheta^{m+1}|_{1,h}^2 + |\vartheta^n|_{1,h}^2)
\]

\[
\leq C \left[ (\tau^2 + \tau h^2)^2 + \tau |\vartheta^{m+1}|_{1,h}^2 + \tau \sum_{n=0}^{m} |\vartheta^n|_{1,h}^2 \right], \quad m = 0, \ldots, N - 1.
\]

Combining (4.24), (2.23), (2.1), (4.100) and (3.20), we have

\[
\sum_{n=0}^{m} Z^{3,n} \leq \frac{\tau}{2} \sum_{n=0}^{m} \left[ |n_2, (\Phi_{\delta,2}^n)|_{\infty,h} |e^{n+1} + e^n|_{1,h} \right.
\]

\[
+ |n_2, (\Phi_{\delta,2}^n)|_{1,h} |e^{n+1} + e^n|_{\infty,h} \left. |\vartheta^{n+1} + \vartheta^n|_{1,h} \right]
\]

\[
\leq C_{\delta, \tau} \tau \sum_{n=0}^{m} \left[ 1 + \sup_{h} |n_2, |\Phi_{\delta,2}^n|_{1,h} \left( |e^{n+1}|_{1,h} + |e^n|_{1,h} \right) (|\vartheta^{n+1}|_{1,h} + |\vartheta^n|_{1,h}) \right]
\]

(4.105)

\[
\leq C_{\delta, \tau} \tau \sum_{n=0}^{m} (|e^{n+1}|_{1,h} + |\vartheta^n|_{1,h} + |\vartheta^{n+1}|_{1,h} + |\vartheta^n|_{1,h}) (|\vartheta^{n+1}|_{1,h} + |\vartheta^n|_{1,h})
\]

\[
\leq C_{\delta, \tau} \tau \sum_{n=0}^{m} (h^2 + |\vartheta^{n+1}|_{1,h} + |\vartheta^n|_{1,h}) (|\vartheta^{n+1}|_{1,h} + |\vartheta^n|_{1,h})
\]

\[
\leq C_{\delta, \tau} h^4 + \tau |\vartheta^{n+1}|_{1,h}^2 + \tau \sum_{n=0}^{m} |\vartheta^n|_{1,h}^2, \quad m = 0, \ldots, N - 1.
\]

Now, from (4.24), it follows that

(4.106)

\[
\sum_{n=0}^{m} Z^{4,n} = Z^m_A + Z^m_B + Z^m_C, \quad m = 0, \ldots, N - 1,
\]

where

\[
Z^m_A := - \frac{\tau}{2} \text{Im} \left[ \{ \delta_h (e_{mid}^m \otimes (u^{m+1} + u^m)), \delta_h \vartheta^{m+1} \} \right],
\]

\[
Z^m_B := - \frac{\tau}{2} \text{Im} \left[ \sum_{n=1}^{m} \{ \delta_h (e_{mid}^{n-1} + e_{mid}^n) \otimes (u^n + u^{n-1})), \delta_h \vartheta^n \} \right],
\]

\[
Z^m_C := - \frac{\tau}{2} \text{Im} \left[ \sum_{n=1}^{m} \{ \delta_h (e_{mid}^{n-1} \otimes (u^n - u^{n-1})), \delta_h \vartheta^n \} \right].
\]

Using the Cauchy-Schwarz inequality, (4.104), (4.30), (2.6), (2.7), (4.102), (2.1), (3.2), (3.3) and (3.20), we have

(4.107)

\[
Z^m \leq \frac{\tau}{2} \left[ \| \delta_h (u^{m+1} + u^m) \|_{\infty,h} \| e_{mid}^m \|_{0,h} + |u^{m+1} + u^m|_{\infty,h} \| e_{mid}^m \|_{1,h} \right] |\vartheta^{m+1}|_{1,h}
\]

\[
\leq C \tau \left[ e_{mid}^m \|_{1,h} \left| \vartheta^{m+1} \right|_{1,h}
\right]
\]

\[
\leq C_{\delta, \tau} \left( \tau^2 + \tau h^2 \right) |\vartheta^{m+1}|_{1,h}
\]

\[
\leq C_{\delta, \tau} \left( \tau^2 + \tau h^2 \right)^2 + \frac{1}{27} |\vartheta^{m+1}|_{1,h}^2, \quad m = 0, \ldots, N - 1,
\]
\[ Z^m_c \leq \frac{\delta}{2} \sum_{n=1}^{m} \left[ \| \delta_h(u^{n+1} - u^{n-1}) \|_{\infty, h} \| e^{n}_{mid} \|_{0, h} + |u^{n+1} - u^{n-1}|_{\infty, h} \| e^{n}_{mid} \|_{1, h} \right] |\vartheta^n|_{1, h} \]

\[ \leq C \tau \sum_{n=1}^{m} \| e^{n}_{mid} \|_{1, h} |\vartheta^n|_{1, h} \]

\[ \leq C_{\delta, \tau} \sum_{n=0}^{m} \left( (\vartheta^2 + \tau h^2) \right) |\vartheta^n|_{1, h} \]

\[ \leq C_{\delta, \tau} \left[ (\vartheta^2 + \tau h^2) + \tau \sum_{n=0}^{m} |\vartheta^n|_{1, h} \right], \quad m = 1, \ldots, N - 1, \]

and

\[ Z^n_h \leq C \tau \sum_{n=0}^{m} \left( |e^{n}_{mid} + e^{-n}_{mid}|_{1, h} + \| e^{n}_{mid} + e^{-n}_{mid} \|_{0, h} \right) |\vartheta^n|_{1, h} \]

\[ \leq C \tau \sum_{n=0}^{m} \left( \| \vartheta^n \|_{1, h} + \| g(|u^n|^2) - g(|V_{\delta, h}^n|^2) \|_{1, h} \right) |\vartheta^n|_{1, h} \]

\[ \leq C_{\delta, \tau} \sum_{n=0}^{m} \left( (\vartheta^2 + \tau h^2) \right) |\vartheta^n|_{1, h} \]

\[ \leq C_{\delta, \tau} \left[ (\vartheta^2 + \tau h^2) + \tau \sum_{n=0}^{m} |\vartheta^n|_{1, h} \right], \quad m = 1, \ldots, N - 1. \]

Now, from \(4.108\), \(4.109\), \(4.110\), \(4.111\), \(4.112\), \(4.113\) and \(4.114\), we conclude that there exists a positive constant \(C_{\delta, \tau} \geq \frac{1}{2} C_{\delta, \tau}^0\) such that

\[ (\frac{1}{2} - \tau \tilde{C}_{\delta, \tau}) |\vartheta^{m+1}|_{1, h}^2 \leq C_{\delta, \tau} \left[ (\vartheta^2 + \tau h^2) + \tau \sum_{n=0}^{m} |\vartheta^n|_{1, h} \right], \quad m = 0, \ldots, N - 1. \]

Assuming that \(2 \tau \tilde{C}_{\delta, \tau} \leq \frac{1}{2} \) and applying a discrete Gronwall argument we arrive at

\[ \max_{0 \leq m \leq N} |\vartheta^n|_{1, h} \leq C_{\delta, \tau} (\vartheta^2 + h^2), \]

which, along with \(3.20\) and \(2.2\), yields \(4.101\). □

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