Exact solutions for the motion of spinning massive particles in conformally flat spacetimes

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Abstract. The motion of massive spinning test particles (tops) on gravitational backgrounds is studied based on a Lagrangian formulation developed some forty years ago. In particular, the (non-geodesical) equations of motions on some conformally flat spacetimes are solved exactly. Since tops follow non-geodesical paths, velocity and momentum vectors are, in general, non-parallel. Some peculiar features such as superluminal motion are exhibited and discussed in some detail.

1. Introduction
It is largely known that the dynamics of spinning massive particles (tops) is described by non-geodesic equations. Massive spinning particles have a completely different behavior as compared to spinless massive particles. One may intuitively understand this behavior noticing that the equivalence principle (interpreted as stating that test particles in a gravitational field follow geodesics) is valid only for spinless point test particles. As a point particle cannot spin, the top is an extended particle. Therefore, tops experience tidal forces and, therefore, follow non-geodesic paths.

The equations governing the motion of tops in gravitational fields were first derived by Mathisson [1, 2] and later by Papapetrou [3] as the limiting case of rotating fluids moving in curved spacetimes. However, a correct derivation of the spherical top motion in gravitational fields, (described by the metric field $g_{\mu\nu}$) has been obtained at Lagrangian level [4, 5].

The complete development of this theory can be found in Refs. [4, 5], or more recently in Refs. [6, 7]. Here we just limit ourselves to mention that in this classical theory, the velocity $u_\mu$ and the canonical momentum $P_\mu$ vectors are, in general, not parallel. This is similar to the behavior of velocity and momentum of particles described by quantum theories, as for example, in the Dirac equation. The Lagrangian theory [4, 5] gives the evolution equation for the top dynamics

$$\frac{DP_\mu}{D\lambda} = -\frac{1}{2} R_{\nu\alpha\beta\mu} u^\nu S^{\alpha\beta}, \quad \frac{DS^{\mu\nu}}{D\lambda} = P^\mu u^\nu - u^\mu P^\nu,$$

(1)

where the covariant derivatives are defined through an arbitrary parameter $\lambda$, and $R_{\nu\alpha\beta\mu}$ is the Riemann tensor. These results hold for an arbitrary Lagrangian (time-reparametrization...
invariant) function defined in terms of four scalars constructed in terms of the top’s velocity vector and angular velocity (antisymmetric) tensor (for detail, see Refs. [4, 5]). It is worthwhile mentioning that the system is constrained to restrict the spin tensor to generate rotations only. From the Lagrangian theory one can derive the constraint $S^{\mu\nu}P_\nu = 0$, known as the Tulczyjew constraint.

The main features of this Lagrangian theory is that it can be proved that the square of the mass $m^2 \equiv P_\mu P^\mu > 0$ is conserved. Thus the momentum vector is timelike along the motion. However, as we will show, the velocity field may become spacelike [6, 7]. On the other hand, the scalar spin $J^2 \equiv (1/2)S_{\mu\nu}S^{\mu\nu}$ is also conserved on any curved background, where $S_{\mu\nu}$ is the antisymmetric spin tensor. Besides, in general, if $\xi_\mu$ is a vector de Killing, then it can be shown from Eqs. (1) that the quantity

$$C_\xi \equiv P^\mu \xi_\mu - \frac{1}{2} S^{\mu\nu} \xi_{\mu;\nu} \tag{2}$$

is also conserved along the top motion Refs. [4, 5].

The top dynamics (1) has been studied for several different metrics in Refs. [6, 7]. Interestingly enough, it has been shown that several exact solutions for the motion can be found. In all of cases, the velocity is not restricted to be timelike. Depending on the relations between the constants of motion of the particle (mass, spin, angular momentum and energy), the top velocity can become spacelike, implying superluminal motion.

In this work, we study the top dynamics in the particular case of a static spherically symmetric conformally flat spacetime. We explore different conformal factors for universes composed by radiation and a static perfect fluid. There exist exact solutions for universes filled with uniform electromagnetic radiation only, matter-dominated universes and radiation-dominated universes. These systems were studied in Ref. [7], and here we extend the analysis of our results.

2. Exact solution for conformally flat spacetimes

We plan to find an exact solution of Eqs. (1) for conformally flat spacetimes

$$ds^2 = \Omega^2 \eta_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

with the spherical symmetric conformal factor $\Omega \equiv \Omega(r)$ and where $\eta_{\mu\nu}$ is the spherical flat metric.

The main procedures to obtain the solutions are outlined in [7]. For extended calculations we refer the readers to the mentioned work. Along these lines we will just display the main results. The conserved energy for the top motion can be calculated in theory using the conserved quantities (2) derived from the equations of motion (1). For the Killing vector $\xi_\mu = (c^2 \Omega^2, 0, 0, 0)$, we can find that the conserved energy is

$$E = P_t - c^2 \Omega \Omega' S^{tr}, \tag{4}$$

where $P_t$ is the time-component of the momentum and similarly for the spin tensor $S^{tr}$. Here the symbol ‘ refers to the derivative with respect to $r$. Also we have added explicitly here and after the speed of light $c$ in the expressions. On the other hand, using the Killing vector $\xi_\mu = (0, 0, 0, -r^2 \Omega^2)$, we can find another conserved quantity in the equatorial plane $\theta = \pi/2$. This conserved quantity turns to be the component of the angular momentum orthogonal to the equatorial plane. Using (2) we find

$$j = -\Omega (\Omega + r\Omega') rS^{\phi r} - P_{\phi}, \tag{5}$$

where $P_{\phi}$ is one of the angular component of the momentum.
Exact solutions for the momenta can be obtained using the equations of motion (1) and the conserved quantities (2) for the mass \( m \) and spin \( J \). Following the calculations outlined in [7], we can find the momenta in terms of constants of motion

\[
\begin{align*}
P_\phi(r) &= \frac{1}{1-\eta} \left(-j + \frac{EJ(\Omega + r\Omega')}{mc^2\Omega'}\right), \\
P_t(r) &= \frac{1}{1-\eta} \left(E \pm \frac{jJ\Omega'}{mr\Omega'}\right), \\
P_r(r) &= \pm \left[\frac{P_t^2}{c^2} - \frac{P_\phi^2}{r^2} - m^2c^2\Omega^2\right]^{1/2},
\end{align*}
\]

where \( \eta(r) = \Omega'(\Omega + r\Omega')J^2/(r\Omega^4m^2c^2) \). The two signs correspond to the two possible spin orientations. Using these expressions in Eqs. (1) we can find the evolution of the top velocity in the equatorial plane [7]

\[
\dot{\phi} = \frac{\zeta}{r^2} \left(\frac{P_\phi}{P_t}\right), \quad \dot{r} = \zeta c^2 \left(\frac{P_r}{P_t}\right),
\]

where we have defined the auxiliary functions

\[
\zeta = (\eta - 1) \left[\eta + 1 - \frac{J^2}{\Omega^4m^2c^2r} \left(2\Omega' + r\Omega''\right)\right]^{-1}, \quad \gamma = (1 - \eta)^{-1} \left[1 + \frac{J^2}{\Omega^4m^2c^2} \left((\Omega')^2 - \Omega\Omega''\right)\right].
\]

At the same time, the trajectories for the top motion can be calculated directly from Eqs. (7). The trajectory in a parametric representation can be obtained by solving the integral

\[
\phi(r) = \int dr \frac{\gamma P_\phi}{r^2 P_t}.
\]

These solutions allow us to evaluate the interval \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2\Omega^2dt^2 \left(1 - \frac{r^2/c^2 - r^2\dot{\phi}^2/c^2}{c^2}\right) \) along the trajectory of the top. Notice that the interval is a measure of the velocity of the particle. Therefore, if \( ds^2 > 0 \), the velocity of the top is expected to be timelike moving the top at subluminal speed. Ordinary (spinless) particle always follow timelike trajectories, however if a particle could achieve luminal or superluminal velocities, the interval will behave as \( ds^2 = 0 \) or \( ds^2 < 0 \) respectively. The evaluation of the top interval in conformally flat spacetime can be found using the solutions (7). This turns out to be

\[
\frac{ds^2}{c^2dt^2} = \Omega^2 \left(1 - \zeta^2\right) + \frac{m^2c^4\zeta^2}{P_t^2} \left(1 + \frac{P_\phi^2}{\Omega^2m^2r^2c^2}\right).
\]

It is worth remarking that the previous interval is not always timelike. Depending on the constants of motion (specially spin \( J \)) and initial conditions, the interval of the top could be lightlike or spacelike. In particular, if the spin is neglected, then \( ds^2/c^2dt^2 = \Omega^4(m^2c^4/E^2) > 0 \), and the velocity of the top is always timelike. The main implication of considering the spin of particles is that the top could reach superluminal velocities without violating any law of General Relativity.

The previous exact solution requires an explicit expression for the conformal factor. Here we study a Universe composed by uniform electromagnetic radiation (through the electrostatic potential \( F_{0r} \)) and a static perfect fluid with energy density \( \epsilon \) and pressure \( p \). In this case an exact form for the conformal factor can be obtained from Einstein equations [8]. The conformal factor is

\[
\Omega(r) = Q_1 \left(r^{-2/(1+3\alpha)} - Q_2\beta\right)^{(1+3\alpha)/2},
\]

where \( Q_1 \) and \( Q_2 \) are constants.
where $Q_1$ y $Q_2$ are constants, $\alpha = p/\epsilon$ and $\beta = (1 + 3\alpha)^{(3+3\alpha)/(1+3\alpha)}$. At the same time the energy density and the electromagnetic field could be obtained as

$$\epsilon = \frac{3c^2\beta Q_2}{4\pi G Q_1^2(1 + 3\alpha)} r^{-\frac{4+6\alpha}{1+3\alpha}} \left( r^{-2/(1+3\alpha)} - Q_2 \beta \right)^{-3-3\alpha},$$

$$(F_0r)^2 = \frac{c^4\Omega^2}{2GQ_1^2(1 + 3\alpha)} \left( r^{-\frac{2}{1+3\alpha}} - Q_2 \beta \right)^{-3-3\alpha} \left[ 2 + 6\alpha - 4(2 + 3\alpha)\beta Q_2 r^{-\frac{2}{1+3\alpha}} \right], \quad (12)$$

where $G$ is the gravitational constant and $Q_2 \geq 0$ is imposed in order to have positive semidefinite energies. Notice that the pressure is obtained as $p = \alpha \epsilon$, which corresponds to an equation of state that depends on the type of Universe studied.

3. Tops in a Universe with radiation only

When $Q_2 = 0$, both the energy density and the pressure vanish. Thus, there is only radiation in the Universe. The conformal factor acquires the form $\Omega = Q/r$, which is generally called the Bertotti-Robinson solution [9, 10, 11]. In this simple case, the momenta (6) becomes $P_\phi = -j$, $P_t = E \pm jJ/(mQr)$. Using this, and the velocities expressions (7) we can calculate the interval for the top in this kind of Universe

$$\frac{ds^2}{c^2dt^2} = \frac{1}{r^4P_t^2m^4c^2Q^2} \left[ 2m^2c^2Q^2j^2J^2 + m^6c^6Q^6 - j^2J^4 \right]. \quad (13)$$

Notice that the interval depends on constants of motion only. There will be exist certain values of $m$, $J$, $j$ and $Q$ for which it may happen that $ds^2 \leq 0$ (lightlike or spacelike trajectories respectively). This implies that depending on the characteristics of the top, the massive particle could travel at superluminal speed for all moments of its motion.

In Fig. (1) we plot the region (in blue shadow) where the interval (13) becomes negative or zero. The plot is made in terms of the dimensionless quantities $J/(mcQ)$ and $j/(mcQ)$, as the expression (13) depends only on the relative values of those constants. Notice that due to the existence of the spin of the top, the region when $ds^2 \leq 0$ is larger than the one when $ds^2 > 0$.

**Figure 1.** Behavior of the interval (13). The blue shadow region corresponds to the values of the constants of motion where $ds^2 \leq 0$. 
4. Tops in a matter-dominated Universe

If the energy density of the fluid dominates over the pressure then the universe is matter-dominated (with non-relativistic matter). We can model this choosing \( \alpha = 0 \) [12]. In this case we have that the conformal factor is \( \Omega = Q_1 (1/r^2 - Q_2)^{1/2} \), and the energy density and the strength of the electromagnetic field (12) are finite.

The solutions for the momenta are now \( P_\phi = ( -j \pm E J Q_1^2 Q_2 / (mc^2 \Omega^2) ) / (1 - \eta) \), and \( P_t = ( E \mp j J Q_1^2 / (mr^2 \Omega^2) ) / (1 - \eta) \), where now \( \eta \) is reduced to \( \eta = J^2 Q_1^2 Q_2 / (r^4 m^2 c^2 \Omega^6) > 0 \). Similarly to the previous case, we can use the solutions to evaluate the interval, which for this case becomes

\[
\frac{ds^2}{c^2 dt^2} = \frac{\Omega^2}{(1 - \eta^2)^2 P_t^2} \left[ 3\eta(2 + \eta)(1 - \eta)^2 P_t^2 + m^2 c^4 \Omega^2 (1 - \eta)^4 - (1 - \eta)^2 P_\phi^2 \left( \frac{8\eta(1 + \eta)c^2}{r^2} + \left( \frac{J^2 Q_1^4}{\Omega^8 c^2 m^2 r^6} \right) \right) - \left( \frac{2 + 6\eta}{\Omega^4 m^2 r^6} \right) \right].
\]

(14)

Again, it is possible for some combination of the different top’s parameters to have \( ds^2 \leq 0 \) along the trajectories. The behavior of the interval depends on the relative strength of the following five dimensionless parameter \( Q_1/r, Q_1^2 Q_2, J\sqrt{Q_2}/(mc), j\sqrt{Q_2}/(mc) \), and \( E/(mc^2) \). On the contrary to the previous case (where the Universe is composed only by uniform radiation), now the behavior of \( ds^2 \) explicitly depends on the position \( r \) of the top.

In Fig. (2) we plot several examples of regions where \( ds^2 \) given in (14) could be zero or negative. As it is not possible to have a graphical representation of \( ds^2 \) using the five parameters, we have fixed two of them arbitrarily, \( Q_1^2 Q_2 \) and \( J\sqrt{Q_2}/(mc) \). Notice that these choices do not fix neither \( Q_1, Q_2 \) nor \( m \). In Figs. (2a), (2b) and (2c) we have fixed the values to be \( Q_1^2 Q_2 = 1 \) and \( J\sqrt{Q_2}/(mc) = 6 \), \( Q_1^2 Q_2 = 5 \) and \( J\sqrt{Q_2}/(mc) = 10 \), and \( Q_1^2 Q_2 = 7 \) and \( J\sqrt{Q_2}/(mc) = 16 \), respectively. We can notice that as both values increase, the size of the region where \( ds^2 \leq 0 \) diminishes. On the other hand, in Fig. (2d), we plot the region for the values \( Q_1^2 Q_2 = 15 \) and \( J\sqrt{Q_2}/(mc) = 3 \), showing that for large values of \( Q_1^2 Q_2 \), the region where \( ds^2 \leq 0 \) is bounded for large values of \( r \) (small values of \( Q_1/r \)).

5. Tops in a radiation-dominated Universe

Another type of Universe can be studied choosing the equation of state appropriately. If the composition of the Universe is such that radiation dominates over the fluid (for example in the very-early Universe), then \( \alpha = 1/3 \) [12]. In this case, the conformal factor reduces to \( \Omega = Q_1 (1/r - 4Q_2) \), while the momenta become \( P_\phi = ( -j \pm 4E J Q_1 Q_2 / (mc^2 \Omega^2) ) / (1 - \eta) \), and \( P_t = ( E \mp j J Q_1 / (mr^2 \Omega^2) ) / (1 - \eta) \), where \( \eta = 4J^2 Q_1^2 Q_2 / (r^3 m^2 c^2 \Omega^4) > 0 \).

With the solutions, the evaluation of the interval produces

\[
\frac{ds^2}{c^2 dt^2} = \frac{\Omega^2}{(1 - \eta^2)^2 P_t^2} \left[ 4\eta(1 - \eta)^2 P_t^2 + m^2 c^4 \Omega^2 (1 - \eta)^4 - (1 - \eta)^2 P_\phi^2 \left( \frac{3\eta(2 + \eta)c^2}{r^2} + \left( \frac{J^2 Q_1^4}{\Omega^8 m^4 c^2 r^6} \right) \right) - \frac{2 + 6\eta}{\Omega^4 m^2 r^6} \right]
\]

(15)

from where again we can see that \( ds^2 \) may become timelike, lightlike or spacelike depending on the top’s properties and its position.

In Fig. (3) we plot different regions where (15) could be non-positive. In this case, the values of the interval (15) will depend on the five dimensionless parameter \( Q_1/r, Q_1 Q_2, J/(mc Q_1), j/(mc \Omega_1) \), and \( E/(mc^2) \). For the graphical representation (3) we have arbitrarily fixed the parameters \( Q_1 Q_2 \) and \( J/(mc Q_1) \). Notice that for the radiation-dominated Universe, the region
Figure 2. Behavior of the interval (14). The region shows when $ds^2 \leq 0$ along the top motion. The region depends on the position $r$ of the top through the parameter $Q_1/r$. (a) Interval (14) with chosen values of $Q_2^2 = 1$ and $J\sqrt{Q_2}/(mc) = 6$. (b) Interval (14) for $Q_2^2 = 5$ and $J\sqrt{Q_2}/(mc) = 10$. (c) Interval (14) for $Q_2^2 = 7$ and $J\sqrt{Q_2}/(mc) = 16$. (d) Interval (14) for $Q_2^2 = 15$ and $J\sqrt{Q_2}/(mc) = 3$.

for superluminal motion is restricted to some values of $r$. In Figs. (3a), (3b) and (3c) we have fixed the parameter values to be $Q_1Q_2 = 1$ and $J/(mcQ_1) = 3$, $Q_1Q_2 = 5$ and $J/(mcQ_1) = 10$, and $Q_1Q_2 = 9$ and $J/(mcQ_1) = 19$, respectively. One sees that as both parameter values increase, the region where $ds^2 \leq 0$ increases for smaller values of $r$ (larger values of $Q_1/r$). In Fig. (3d), we have plotted the region $ds^2 \leq 0$ for the values $Q_1Q_2 = 18$ and $J/(mcQ_1) = 4$, which shows that superluminal motion can be achieved even for smaller values of $r$. 
Figure 3. Behavior of the interval (15). The region shows when $ds^2 \leq 0$ in the top motion. Again it depends on the position $r$. (a) Interval (15) with chosen values of $Q_2^2 = 1$ and $J\sqrt{Q_2}/(mc) = 3$. (b) Interval (15) for $Q_2^2 = 5$ and $J\sqrt{Q_2}/(mc) = 10$. (c) Interval (15) for $Q_2^2 = 9$ and $J\sqrt{Q_2}/(mc) = 19$. (d) Interval (15) for $Q_2^2 = 18$ and $J\sqrt{Q_2}/(mc) = 4$.

6. Conclusions
Along this work we have studied the exact solutions for the motion of the top in spherical conformally flat spacetimes. We have shown that the spin of the massive particles plays an extremely important role changing the dynamics as compared to a spinless particle case.

The inclusion of the spin implies that the massive particle follows non-geodesic paths. This has as a consequence that the top trajectories could be lightlike or spacelike as well as timelike. This behavior depends on the values of the constants of motion of the top and on the spacetime metric.

We review three kind of metrics for a Universe filled with uniform radiation only, a matter-
dominated Universe, and a radiation-dominated Universe. In all three Universes, superluminal motion of the top can be achieved. This seems to be a robust characteristic of Eqs. (1) which has already been studied for different spacetimes [6, 7].

For the case of a Universe filled with uniform radiation only, the lightlike or spacelike trajectories will depend only on the features of the massive spinning particle, and not on its position. This means that the nature of the top (whether it is a bradyon or a tachyon) is completely determined by the values of the constant of motion of the particle. In the matter-dominated and radiation-dominated Universes, the type of trajectories depends on the position of the top. In the plots (2) and (3), we depict that there are a huge number of possibilities for superluminal behavior of the top relying on the different values of its constants of motion on its position. In particular we have shown that in a radiation-dominated Universe it is possible to have superluminal motion for small values of \(r\), which may be relevant for the study of the early Universe.

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