Dependence of Eigenvalues of Discontinuous Fourth-order Differential Operators with Eigenparameter Dependent Boundary Conditions

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Abstract
In this paper, we investigate a fourth-order differential operator with eigenparameter dependent boundary conditions and transmission conditions. To study the eigenvalues of the problem, we establish a new operator associated with the considered problem. Furthermore, we prove that the eigenvalues are differentiable depending on the parameters of the problem. Finally, the differential expressions of the eigenvalues with respect to all parameters are given.

Keywords Fourth-order differential operator · Eigenparameter-dependent boundary condition · Transmission condition · Dependence of eigenvalue · Differential expression

1 Introduction
As we all know, differential operator problems with interior discontinuity have important application prospects. Many actual physical and medical problems can be transformed into differential operator problems with interior discontinuity. For example, the diffraction problem of light, the vibration problem of string with node, the differential operator problem of potential functions are generalized functions, and the heat conduction and mass transfer problems [1–4]. In order to solve

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interior discontinuities, some conditions are imposed on the discontinuous points, and these conditions are usually called interface conditions, point interactions or transmission conditions. In addition, there are also some physical problems need to be transformed into interior discontinuity high-order differential operator problems. For example, the fourth-order differential equation describing the vibration of a grid of beams. It is necessary to consider the continuity at the nodes and the balance at both ends of the nodes, that is, the transmission conditions are added at these nodes to transform the problem into a differential operator with interior discontinuity. We have noticed that many mathematicians are also very interested in differential operator problems with transmission conditions in recent years, and some considerable results have been obtained [5–9]. Among them, the dependence of eigenvalues on the problem with transmission conditions plays an important role in the spectrum theory of differential operators. Recently, Üğurlu [10] and Zinsou [11] considered the dependence of eigenvalues of third-order and fourth-order differential equations with transmission conditions respectively, and obtained that the eigenvalues of the problems are differentiable functions of all the data.

In recent years, the differential operators with boundary conditions depending on eigenparameter are also widely used in acoustic scattering, quantum mechanics theory and so on. As far as we know, some partial differential equations can be transformed into differential operator problems with eigenparameter dependent boundary conditions through the method of separation of variables. Particularly, the fourth-order differential operators with eigenparameter dependent boundary conditions appear in elastic beam models, free bending vibrations of rod and so on [12, 13]. Up to now, the differential operators with boundary conditions depending on eigenparameter have become an important research topic and some excellent results have been obtained [14–17]. In particular, the dependence of eigenvalues on the problem with eigenparameter dependent boundary conditions is also concerned by researchers. Recently, Zhang and Li in [18] showed that the eigenvalues of Sturm-Liouville problems with eigenparameter dependent boundary conditions are differential functions of all the data.

The dependence of eigenvalues and eigenfunctions on problems has great significance in differential operator theory, which provides theoretical support for numerical calculation of eigenvalues and eigenfunctions [19, 20]. While some researches have been carried on the dependence of the eigenvalues for the differential equations [21–29], there is no research on a class of fourth-order differential operator with transmission conditions and containing eigenparameter in the boundary conditions at two endpoints. For such a problem, using the classical analysis techniques and spectral theory of linear operator, a new linear operator $\mathcal{F}$ associated with the problem in an appropriate Hilbert space $H$ is defined such that the eigenvalues of the problem coincide with those of $\mathcal{F}$. Furthermore, we not only prove that each of the eigenvalues of the problem can be embedded in a continuous eigenvalue branch but also obtain the differential expressions of the eigenvalues with respect to all data in the sense of Fréchet derivative.

The rest of this paper is organized as follows: In Sect. 2, we introduce a discontinuous fourth-order boundary value problem and define a new self-adjoint operator $\mathcal{F}$ such that the eigenvalues of such a problem coincide with those of $\mathcal{F}$. In Sect. 3,
we discuss the continuity of the eigenvalues and eigenfunctions. In Sect. 4, we give the differential expressions of the eigenvalues with respect to all the data.

## 2 Preliminaries and Lemmas

Consider the fourth-order differential equation

$$\left( p(x)f''(x) \right)'' - \left( q(x)f'(x) \right)' + q_0(x)f(x) = \mu w(x)f(x),$$

(2.1)

on $J = [a, c) \cup (c, b]$, with eigenparameter dependent boundary conditions at endpoints

$$\mu(\tau_1 f(a) + \tau_2 f'(a) + \tau_3 f^{[2]}(a) + \tau_4 f^{[3]}(a)) - (\tau_1 f(a) + \tau_2 f'(a) + \tau_3 f^{[2]}(a) + \tau_4 f^{[3]}(a)) = 0,$$

(2.2)

$$\mu(\sigma_1 f(a) + \sigma_2 f'(a) + \sigma_3 f^{[2]}(a) + \sigma_4 f^{[3]}(a)) - (\sigma_1 f(a) + \sigma_2 f'(a) + \sigma_3 f^{[2]}(a) + \sigma_4 f^{[3]}(a)) = 0,$$

(2.3)

$$\mu(\phi_1 f(b) + \phi_2 f'(b) + \phi_3 f^{[2]}(b) + \phi_4 f^{[3]}(b)) - (\phi_1 f(b) + \phi_2 f'(b) + \phi_3 f^{[2]}(b) + \phi_4 f^{[3]}(b)) = 0,$$

(2.4)

$$\mu(\psi_1 f(b) + \psi_2 f'(b) + \psi_3 f^{[2]}(b) + \psi_4 f^{[3]}(b)) - (\psi_1 f(b) + \psi_2 f'(b) + \psi_3 f^{[2]}(b) + \psi_4 f^{[3]}(b)) = 0,$$

(2.5)

and four transmission conditions at the point of discontinuity $x = c$

$$F(c+) = B \cdot F(c-),$$

(2.6)

where

$$-\infty < a < b < +\infty, \quad \frac{1}{p}, q, q_0, w \in L^1([a, c) \cup (c, b]), \quad p, w > 0 \text{ a.e. on J},$$

(2.7)

$$\tau_i, \sigma_i, \phi_i, \psi_i, \tau_i', \sigma_i', \phi_i', \psi_i' \in \mathbb{R}, \quad i = 1, 2, 3, 4,$$

$$F(x) = (f(x), f'(x), f^{[2]}(x), f^{[3]}(x))^T,$$

$$B = \begin{pmatrix}
\delta_1 & \alpha_1 & 0 & \alpha_2 \\
\delta_2 & \alpha_3 & \alpha_4 & 0 \\
\delta_3 & \beta_1 & \beta_2 & 0 \\
\delta_4 & \beta_3 & 0 & \beta_4
\end{pmatrix}$$

is a $4 \times 4$ real-valued matrix, and $\mu \in \mathbb{C}$ is the spectral parameter.
be $4 \times 4$ real-valued matrices, we assume that

$$\rho_0 = \begin{vmatrix} \delta_1 & \alpha_2 \\ \delta_4 & \beta_4 \end{vmatrix} = \begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 \end{vmatrix} > 0, \quad \rho_1 = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_3 & \beta_4 \end{vmatrix} = \begin{vmatrix} \delta_2 & \alpha_4 \\ \delta_3 & \beta_2 \end{vmatrix} = 0,$$

(2.8)

Note that the quasi-derivatives associated with (2.1) are

$$f^{[0]} = f, \quad f^{[1]} = f', \quad f^{[2]} = pf'', \quad f^{[3]} = (pf'')' - qf'.$$

Let the direct sum space be defined as

$$H_1 = L^2_w((a, c), w) \oplus L^2_w((c, b), w)$$

with the inner product

$$\langle f, g \rangle_1 = \rho_0 \int_a^c f(x)g(x)w(x)dx + \int_c^b f(x)g(x)w(x)dx$$

for any $f, g \in H_1$. We define a new Hilbert space

$$H = H_1 \oplus \mathbb{C}^4$$

with the inner product

$$\langle F, G \rangle = \langle f, g \rangle_1 + \frac{\rho_0}{\theta_1} (f_1g_1 + f_2g_2) + \frac{1}{\theta_2} (f_3g_3 + f_4g_4)$$

for any $F = (f, f_1, f_2, f_3, f_4)^T, G = (g, g_1, g_2, g_3, g_4)^T \in H$, and

$$f_1 = \tau_1'f(a) + \tau_2'f'(a) + \tau_3'f^{[2]}(a) + \tau_4'f^{[3]}(a),$$
$$f_2 = \sigma_1'f(a) + \sigma_2'f'(a) + \sigma_3'f^{[2]}(a) + \sigma_4'f^{[3]}(a),$$
$$f_3 = \varphi_1'f(b) + \varphi_2'f'(b) + \varphi_3'f^{[2]}(b) + \varphi_4'f^{[3]}(b),$$
$$f_4 = \psi_1'f(b) + \psi_2'f'(b) + \psi_3'f^{[2]}(b) + \psi_4'f^{[3]}(b).$$

Define an operator $F$ as
with the domain
$$D(\mathcal{F}) = \{(f, f_1, f_2, f_3, f_4)^T \in H | f, f', f^{[2]}, f^{[3]} \in AC_{loc}([a, c) \cup (c, b]), \quad \mathcal{F}f = w^{-1}(pf''')' - (qf')' + q_0f = \mu f \in H_1, F(c+) = B \cdot F(c-)\}.$$  

**Lemma 2.1** The operator $\mathcal{F}$ is a self-adjoint operator in the Hilbert space $H$ if and only if

$$AP_2A^T = \theta_1 P_1, \quad CP_2C^T = \theta_2 P_1,$$

where

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$  

**Proof** The proof is similar to that of [7], the equation we considered is more complicated and the derivatives in boundary conditions are quasi-derivatives, here we omit the details. \hfill \Box

**Remark 2.1** The eigenvalues of the fourth-order boundary value problem (2.1)–(2.6) are the same as the eigenvalues of the operator $\mathcal{F}$, and the eigenfunctions are the first component of the corresponding vector eigenfunctions of the operator $\mathcal{F}$. Therefore, the study of the boundary value problem (2.1)–(2.6) can be transformed into that of the operator $\mathcal{F}$. Furthermore, the eigenvalues are all real-valued.

## 3 Continuity of Eigenvalues and Eigenfunctions

In this section, we give the continuity of eigenvalues and eigenfunctions on the parameters in the discontinuous fourth-order boundary value problem.

Let $\chi_i(x, \mu)$ and $\omega_i(x, \mu), i = 1, 2, 3, 4$ be respectively the linearly independent solutions of equation (2.1) respectively on $[a, c)$ and $(c, b]$ satisfying the following initial conditions
\[
\begin{pmatrix}
\chi_1(a, \mu) & \chi_2(a, \mu) & \chi_3(a, \mu) & \chi_4(a, \mu) \\
\chi_1'(a, \mu) & \chi_2'(a, \mu) & \chi_3'(a, \mu) & \chi_4'(a, \mu) \\
\chi_1''(a, \mu) & \chi_2''(a, \mu) & \chi_3''(a, \mu) & \chi_4''(a, \mu) \\
\chi_1'''(a, \mu) & \chi_2'''(a, \mu) & \chi_3'''(a, \mu) & \chi_4'''(a, \mu)
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix},
\]
and
\[
\begin{pmatrix}
\omega_1(b, \mu) & \omega_2(b, \mu) & \omega_3(b, \mu) & \omega_4(b, \mu) \\
\omega_1'(b, \mu) & \omega_2'(b, \mu) & \omega_3'(b, \mu) & \omega_4'(b, \mu) \\
\omega_1''(b, \mu) & \omega_2''(b, \mu) & \omega_3''(b, \mu) & \omega_4''(b, \mu) \\
\omega_1'''(b, \mu) & \omega_2'''(b, \mu) & \omega_3'''(b, \mu) & \omega_4'''(b, \mu)
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}.
\]

We define their Wronskian as
\[
\Psi(x, \mu) := \begin{pmatrix}
\chi_1(x, \mu) & \chi_2(x, \mu) & \chi_3(x, \mu) & \chi_4(x, \mu) \\
\chi_1'(x, \mu) & \chi_2'(x, \mu) & \chi_3'(x, \mu) & \chi_4'(x, \mu) \\
\chi_1''(x, \mu) & \chi_2''(x, \mu) & \chi_3''(x, \mu) & \chi_4''(x, \mu) \\
\chi_1'''(x, \mu) & \chi_2'''(x, \mu) & \chi_3'''(x, \mu) & \chi_4'''(x, \mu)
\end{pmatrix}, \quad x \in [a, c),
\]
and
\[
\Psi(x, \mu) := \begin{pmatrix}
\omega_1(x, \mu) & \omega_2(x, \mu) & \omega_3(x, \mu) & \omega_4(x, \mu) \\
\omega_1'(x, \mu) & \omega_2'(x, \mu) & \omega_3'(x, \mu) & \omega_4'(x, \mu) \\
\omega_1''(x, \mu) & \omega_2''(x, \mu) & \omega_3''(x, \mu) & \omega_4''(x, \mu) \\
\omega_1'''(x, \mu) & \omega_2'''(x, \mu) & \omega_3'''(x, \mu) & \omega_4'''(x, \mu)
\end{pmatrix}, \quad x \in (c, b].
\]

**Lemma 3.1** The number \( \mu \) is an eigenvalue of (2.1)–(2.6) if and only if
\[
\Delta(\mu) = \det \left( Q_1 \begin{pmatrix} Q_1 & Q_2 \\ B \Psi(c-, \mu) & -\Psi(c+, \mu) \end{pmatrix} \right) = 0,
\]
where
\[
Q_1 = \begin{pmatrix}
\mu \tau_1 - \tau_1 & \mu \tau_2 - \tau_2 & \mu \tau_3 - \tau_3 & \mu \tau_4 - \tau_4 \\
\mu \sigma_1 - \sigma_1 & \mu \sigma_2 - \sigma_2 & \mu \sigma_3 - \sigma_3 & \mu \sigma_4 - \sigma_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
and
\[
Q_2 = \begin{pmatrix}
\mu \phi_1 + \phi_1 & \mu \phi_2 + \phi_2 & \mu \phi_3 + \phi_3 & \mu \phi_4 + \phi_4 \\
\mu \psi_1 + \psi_1 & \mu \psi_2 + \psi_2 & \mu \psi_3 + \psi_3 & \mu \psi_4 + \psi_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**Proof** Let \( \mu \) be an eigenvalue of (2.1)–(2.6), then there exists a non-trivial solution
\[ f(x, \mu) = \begin{cases} 
\quad l_1 \chi_1(x, \mu) + l_2 \chi_2(x, \mu) + l_3 \chi_3(x, \mu) + l_4 \chi_4(x, \mu), & x \in [a, c), \\
\quad l_5 \omega_1(x, \mu) + l_6 \omega_2(x, \mu) + l_7 \omega_3(x, \mu) + l_8 \omega_4(x, \mu), & x \in (c, b] 
\end{cases} \]  
(3.1)

of (2.1), where \( l_i, i = 1, 2, \ldots, 8 \) are not all zero. Since \( f(x, \mu) \) satisfies the boundary conditions (2.2)–(2.5) and transmission conditions (2.6), we obtain

\[
Q_1 \Psi(a, \mu) \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} + Q_2 \Psi(b, \mu) \begin{pmatrix} l_5 \\ l_6 \\ l_7 \\ l_8 \end{pmatrix} = 0, \quad (3.2)
\]

and

\[
B \Psi(c-, \mu) \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} - \Psi(c+, \mu) \begin{pmatrix} l_5 \\ l_6 \\ l_7 \\ l_8 \end{pmatrix} = 0. \quad (3.3)
\]

Since \( l_i, i = 1, 2, \ldots, 8 \) are not all zero, then \( \Delta(\mu) = 0 \).

On the other hand, if \( \Delta(\mu) = 0 \), then Eqs. (3.2) and (3.3) have non-trivial solution \((l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8)\). We can choose a solution and define \( f(x, \mu) \) as in (3.1). Then \( f(x, \mu) \) satisfies (2.1)–(2.6) and is an eigenfunction. Hence, \( \mu \) is an eigenvalue of (2.1)–(2.6). \( \square \)

Consider a Banach space

\[ X = M_{4 \times 4}(\mathbb{R}) \times M_{4 \times 4}(\mathbb{R}) \times M_{4 \times 4}(\mathbb{R}) \times L(J) \times L(J) \times L(J) \times L(J) \]

with its norm

\[
\| \xi \| = \| A \| + \| B \| + \| C \| + \int_a^c \left( \frac{1}{| p |} + | q | + | q_0 | + | w | \right) \\
\quad + \int_c^b \left( \frac{1}{| p |} + | q | + | q_0 | + | w | \right)
\]

for any \( \xi = (A, B, C, \frac{1}{p}, q, q_0, w) \in X \). Let

\[ \Omega = \{ \xi \in X : (2.7)(2.8) \text{ hold} \}. \]

**Theorem 3.1** Let \( \tilde{\xi} = (\tilde{A}, \tilde{B}, \tilde{C}, \frac{1}{\tilde{p}}, \tilde{q}, \tilde{q}_0, \tilde{w}) \in \Omega \) and \( \mu(\tilde{\xi}) \) be an isolated eigenvalue of (2.1)–(2.6) with \( \tilde{\xi} \). Then \( \mu \) is continuous at \( \tilde{\xi} \). That is, given any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
| \mu(\xi) - \mu(\tilde{\xi}) | < \varepsilon
\]

if \( \xi = (A, B, C, \frac{1}{p}, q, q_0, w) \) satisfies

\( \text{ Springer} \)
\[
|\xi - \tilde{\xi}| = \|A - \tilde{A}\| + \|B - \tilde{B}\| + \|C - \tilde{C}\| + \int_a^c \left( \frac{1}{p} - \frac{1}{\tilde{p}} \right) + |q - \bar{q}| + |q_0 - \bar{q}_0| + |w - \bar{w}| + \int_c^b \left( \frac{1}{p} - \frac{1}{\tilde{p}} \right) + |q - \bar{q}| + |q_0 - \bar{q}_0| + |w - \bar{w}| < \delta.
\]

**Proof** By Lemma 2.1, $\mu(\xi)$ is an eigenvalue of (2.1)–(2.6) if and only if $\Delta(\xi, \mu(\xi)) = 0$. For any $\xi \in \Omega$, $\Delta(\xi, \mu)$ is an entire function of $\mu$ and is continuous in $\xi$ (see [30, Theorem 2.7, 2.8]). It is easy seen that $\Delta(\tilde{\xi}, \mu)$ is not a constant in $\mu$ because $\mu(\tilde{\xi})$ is an isolated eigenvalue. Hence there exists $\zeta > 0$ such that $\Delta(\xi, \mu) \neq 0$ for $\mu \in S_\zeta := \{ \mu \in \mathbb{C} : |\mu - \mu(\tilde{\xi})| = \zeta \}$. By the continuity of the roots of an equation as a function of parameters (see [31, (9.17.4)]), the statement follows.

By a normalized eigenvector $(m, m_1, m_2, m_3, m_4)^T \in H$, we mean $m$ satisfies the problem (2.1)–(2.6), and

\[
\|(m, m_1, m_2, m_3, m_4)^T\|^2 = \langle (m, m_1, m_2, m_3, m_4)^T, (m, m_1, m_2, m_3, m_4)^T \rangle
= \rho_0 \int_a^c m\bar{m} \, dx + \int_c^b m\bar{m} \, dx + \frac{\rho_0}{\theta_1}(m_1\bar{m}_1 + m_2\bar{m}_2)
+ \frac{1}{\theta_2}(m_3\bar{m}_3 + m_4\bar{m}_4) = 1.
\]

Now we give a result for normalized eigenfunctions.

**Theorem 3.2** Assume that $\mu(\xi)$ is an eigenvalue of (2.1)–(2.6) with $\xi \in \Omega$ and $(m, m_1, m_2, m_3, m_4)^T \in H$ is the corresponding normalized eigenvector for $\mu(\tilde{\xi})$. Then there exists a normalized eigenvector $(n, n_1, n_2, n_3, n_4)^T \in H$ for $\mu(\tilde{\xi})$ with $\tilde{\xi} \in \Omega$, which is specified in Theorem 3.1, such that

\[
n(x) \to m(x), n^{11}(x) \to m^{11}(x), n^{12}(x) \to m^{12}(x), n^{13}(x) \to m^{13}(x),
n_1(x) \to m_1(x), n_2(x) \to m_2(x), n_3(x) \to m_3(x), n_4(x) \to m_4(x),
\]

as $\tilde{\xi} \to \xi$ both uniformly on $J$.

**Proof** (i) We know that $\mu(\xi)$ is an isolated eigenvalue of multiplicity $j(j = 1, 2, 3, 4)$ for all $\xi$ in some neighborhood $\mathcal{N}$ of $\tilde{\xi}$ in $\Omega$. Suppose $\mu(\tilde{\xi})$ is simple, then there exists a neighborhood $\mathcal{N}$ of $\tilde{\xi}$ such that $\mu(\xi)$ is simple for each $\xi \in \mathcal{N}$. Let $(f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi))^T$ be an eigenvector for $\mu(\xi)$ with

\[
\|f(x, \xi)\| = \int_a^b f(x, \xi) \bar{f}(x, \xi) \, dx = 1.
\]

By Theorem 3.1, there exists $\mu(\tilde{\xi})$ such that
\[ \mu(\tilde{\xi}) \to \mu(\xi) \quad \text{as} \quad \tilde{\xi} \to \xi. \]

Then we can obtain the boundary condition matrix

\[ (Q_1, Q_2)(\tilde{\xi}) \to (Q_1, Q_2)(\xi) \quad \text{as} \quad \tilde{\xi} \to \xi. \]

By Theorem 3.2 in [21], we can obtain an eigenfunction \( f(x, \tilde{\xi}) \) for \( \mu(\tilde{\xi}) \) such that \( \|f(x, \tilde{\xi})\| = 1 \) and

\[
\begin{align*}
 f(x, \tilde{\xi}) &\to f(x, \xi), \\
 f^{[1]}(x, \tilde{\xi}) &\to f^{[1]}(x, \xi), \\
 f^{[2]}(x, \tilde{\xi}) &\to f^{[2]}(x, \xi), \\
 f^{[3]}(x, \tilde{\xi}) &\to f^{[3]}(x, \xi),
\end{align*}
\]

as \( \tilde{\xi} \to \xi \) both uniformly on \( J \). Then we obtain

\[
\begin{align*}
 f_1(\tilde{\xi}) &\to f_1(\xi), \\
 f_2(\tilde{\xi}) &\to f_2(\xi), \\
 f_3(\tilde{\xi}) &\to f_3(\xi), \\
 f_4(\tilde{\xi}) &\to f_4(\xi) \quad \text{as} \quad \tilde{\xi} \to \xi.
\end{align*}
\]

Let

\[
\begin{align*}
(m, m_1, m_2, m_3, m_4)^T &= \frac{(f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi))}{\|f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi)\|^T}, \\
(n, n_1, n_2, n_3, n_4)^T &= \frac{(f(x, \tilde{\xi}), f_1(\tilde{\xi}), f_2(\tilde{\xi}), f_3(\tilde{\xi}), f_4(\tilde{\xi}))}{\|f(x, \tilde{\xi}), f_1(\tilde{\xi}), f_2(\tilde{\xi}), f_3(\tilde{\xi}), f_4(\tilde{\xi})\|^T}, \\
m^{[1]} &= \frac{f^{[1]}(x, \xi)}{\|f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi)\|^T}, \\
n^{[1]} &= \frac{f^{[1]}(x, \tilde{\xi})}{\|f(x, \tilde{\xi}), f_1(\tilde{\xi}), f_2(\tilde{\xi}), f_3(\tilde{\xi}), f_4(\tilde{\xi})\|^T}, \\
m^{[2]} &= \frac{f^{[2]}(x, \xi)}{\|f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi)\|^T}, \\
n^{[2]} &= \frac{f^{[2]}(x, \tilde{\xi})}{\|f(x, \tilde{\xi}), f_1(\tilde{\xi}), f_2(\tilde{\xi}), f_3(\tilde{\xi}), f_4(\tilde{\xi})\|^T}, \\
m^{[3]} &= \frac{f^{[3]}(x, \xi)}{\|f(x, \xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi)\|^T}, \\
n^{[3]} &= \frac{f^{[3]}(x, \tilde{\xi})}{\|f(x, \tilde{\xi}), f_1(\tilde{\xi}), f_2(\tilde{\xi}), f_3(\tilde{\xi}), f_4(\tilde{\xi})\|^T}.
\end{align*}
\]

Then (3.4) holds by (3.5) (3.6).

(ii) Assume that \( \mu(\tilde{\xi}) \) is an eigenvalue of multiplicity \( j (j = 2, 3, 4) \). Then we can choose eigenfunctions of \( \mu(\tilde{\xi}) \) such that all of them satisfy the same initial conditions
at $c_0$ for some $c_0 \in J$ since a linear combination of $j$ linearly independent eigenfunctions can be chosen to satisfy arbitrary initial conditions. Similarly, we obtain (3.4) as (i). This completes the proof.

\[\square\]

4 Differential Expression of Eigenvalues

In this section, we show that the eigenvalues of the operator $F$ are differentiable functions with respect to all data and give expressions for their derivatives.

**Definition 4.1** [21] A map operator $F$ from a Banach space $X$ into a Banach space $Y$ is differentiable at a point $x \in X$, if there exists a bounded linear operator $dF_x : X \to Y$ such that for $h \in X$

\[|F(x + k) - F(x) - dF_x(k)| = o(k), \text{ as } k \to 0.\]

**Theorem 4.1** Assume that $\mu(\xi)$ is an eigenvalue of (2.1)–(2.6) with $\xi \in \Omega$ and $M = (m, m_1, m_2, m_3, m_4)^T \in H$ is the corresponding normalized eigenvector for $\mu(\xi)$. Suppose $\mu(\xi)$ is a simple eigenvalue or $\mu(\sigma)$ is an eigenvalue of multiplicity $j$ ($j = 2, 3, 4$) for each $\sigma$ in some neighborhood $N \subset \Omega$ of $\xi$. Then $\mu$ is differentiable with respect to all the parameters in $\xi$.

1. Let all the data of $\xi$ be fixed except the boundary condition parameter matrix $A$ and $\mu(A) := \mu(\xi)$. Then

\[d\mu_A(K) = \rho_0 M^T(a) [E - A(A + K)^{-1}] P_1 \bar{M}(a)\]

(4.1)

for all $K$ satisfying $\det(A + K) = \det A = \theta_1^2$ and $(A + K) P_2(A + K)^T = \theta_1 P_1$.

2. Let all the data of $\xi$ be fixed except the boundary condition parameter matrix $C$ and $\mu(C) := \mu(\xi)$. Then

\[d\mu_C(K) = -M^T(b) [E - C(C + K)^{-1}] P_1 \bar{M}(b)\]

(4.2)

for all $K$ satisfying $\det(C + K) = \det C = \theta_2^2$ and $(C + K) P_2(C + K)^T = \theta_2 P_1$.

3. Let all the data of $\xi$ be fixed except $B$ and $\mu(B) := \mu(B + K)$. Then

\[d\mu_B(K) = -M^*(c - ) K^* P_1 BM(c - )\]

(4.3)

where $M^*$ stands for the conjugate transpose of the matrix $M$.

4. Let all the data of $\xi$ be fixed except $p$ and $\mu(\frac{1}{p}) := \mu(\xi)$. Then

\[d\mu_{\frac{1}{p}}(k) = -\rho_0 \int_0^c |p m''|^2 k dx - \int_c^b |p m''|^2 k dx, \quad k \in L(J, \mathbb{R}).\]

(4.4)
5. Let all the data of \( \xi \) be fixed except \( q \) and \( \mu(q) \) := \( \mu(\xi) \). Then
\[
d\mu_q(k) = \rho_0 \int_a^c |m'|^2 k dx + \int_c^b |m'|^2 k dx, \quad k \in L(J, \mathbb{R}). \quad (4.5)
\]

6. Let all the data of \( \xi \) be fixed except \( q_0 \) and \( \mu(q_0) \) := \( \mu(\xi) \). Then
\[
d\mu_{q_0}(k) = \rho_0 \int_a^c |m|^2 k dx + \int_c^b |m|^2 k dx, \quad k \in L(J, \mathbb{R}). \quad (4.6)
\]

7. Let all the data of \( \xi \) be fixed except \( w \) and \( \mu(w) \) := \( \mu(\xi) \). Then
\[
d\mu_w(k) = -\mu(w) \cdot (\rho_0 \int_a^c |m|^2 k dx + \int_c^b |m|^2 k dx), \quad k \in L(J, \mathbb{R}). \quad (4.7)
\]

Proof Let all the data of \( \xi \) be fixed except one and \( \mu(\tilde{\xi}) \) be the eigenvalue satisfying Theorem 3.1 when \( \| \tilde{\xi} - \xi \| < \epsilon \) for sufficiently small \( \epsilon > 0 \). For the above seven cases, we replace \( \mu(\tilde{\xi}) \) by \( \mu(A + K), \mu(C + K), \mu(B + K), \mu(\frac{1}{p} + k), \mu(q + k), \mu(q_0 + k), \mu(w + k) \), respectively. Let \( N = (n, n_1, n_2, n_3, n_4)^T \) be the corresponding normalized eigenvector.

(1) By (2.1) we have
\[
(pm'')' - (qm')' + q_0 m = \mu(A)w_m, \quad (4.8)
\]
\[
(p\tilde{n}'')' - (q\tilde{n}')' + q_0 \tilde{n} = \mu(A + K)w_{\tilde{n}}. \quad (4.9)
\]

It follows from (4.8) and (4.9) that
\[
[\mu(A + K) - \mu(A)]m\tilde{n}w = (p\tilde{n}'')'m - (q\tilde{n}')'m - (pm'')'\tilde{n} + (qm')'\tilde{n}.
\]
Then
\[
[\mu(A + K) - \mu(A)](m, n)_1 = [\mu(A + K) - \mu(A)][\rho_0 \int_a^c m\tilde{n}wdx + \int_c^b m\tilde{n}wdx]
= \rho_0[m\tilde{n}^{[3]} - m^{[3]}\tilde{n} - m'\tilde{n}^{[2]} + m^{[2]}\tilde{n}'_{\tilde{c}+}]
+ [m\tilde{n}^{[3]} - m^{[3]}\tilde{n} - m'\tilde{n}^{[2]} + m^{[2]}\tilde{n}'_{\tilde{c}+}]. \quad (4.10)
\]

Let \( A + K = \begin{pmatrix}
\tilde{\tau}_1 & \tilde{\tau}_2' & \tilde{\sigma}_1 & \tilde{\sigma}_1' \\
\tilde{\tau}_2 & \tilde{\tau}_2' & \tilde{\sigma}_2 & \tilde{\sigma}_2' \\
\tilde{\tau}_3 & \tilde{\tau}_3' & \tilde{\sigma}_3 & \tilde{\sigma}_3' \\
\tilde{\tau}_4 & \tilde{\tau}_4' & \tilde{\sigma}_4 & \tilde{\sigma}_4'
\end{pmatrix} \). Then by the boundary condition (2.2), we have
Similarly, the boundary conditions (2.4) and (2.5) imply that

\[ \mu(A + K)(\tau_1 \tilde{n}(a) + \tau_2 \tilde{n}'(a) + \tau_3 \tilde{n}^{[2]}(a) + \tau_4 \tilde{n}^{[3]}(a)) \]

Thus

\[ [\mu(A + K) - \mu(A)] \frac{\rho_0}{\theta_1} m_1 \tilde{n}_1 \]

\[ = \frac{\rho_0}{\theta_1} [\tau_1 \tilde{n}(a) + \tau_2 \tilde{n}'(a) + \tau_3 \tilde{n}^{[2]}(a) + \tau_4 \tilde{n}^{[3]}(a)] \]

\[ [\tau_1' m(a) + \tau_2' m'(a) + \tau_3' m^{[2]}(a) + \tau_4' m^{[3]}(a)] \]

Analogously, the boundary condition (2.3) implies that

\[ \mu(A + K)(\tilde{\tau}_1 \tilde{n}(a) + \tilde{\tau}_2 \tilde{n}'(a) + \tilde{\tau}_3 \tilde{n}^{[2]}(a) + \tilde{\tau}_4 \tilde{n}^{[3]}(a)) \]

\[ = \tilde{\tau}_1 \tilde{n}(a) + \tilde{\tau}_2 \tilde{n}'(a) + \tilde{\tau}_3 \tilde{n}^{[2]}(a) + \tilde{\tau}_4 \tilde{n}^{[3]}(a). \]

(4.11)

Similarly, the boundary conditions (2.4) and (2.5) imply that

\[ [\mu(A + K) - \mu(A)] \frac{\rho_0}{\theta_1} m_2 \tilde{n}_2 \]

\[ = \frac{\rho_0}{\theta_1} M^T(a) \left( \begin{array}{cccc}
\tau_1' & \tau_2' & \tau_3' & \tau_4' \\
\tau_1 & \tau_2 & \tau_3 & \tau_4 \\
\tau_3 & \tau_3 & \tau_3 & \tau_4 \\
\tau_4 & \tau_4 & \tau_4 & \tau_4 \\
\end{array} \right) \left( \begin{array}{cccc}
-\tilde{\tau}_4 & -\tilde{\tau}_3 & -\tilde{\tau}_2 & -\tilde{\tau}_1 \\
\tilde{\tau}_4 & \tilde{\tau}_3 & \tilde{\tau}_2 & \tilde{\tau}_1 \\
\tilde{\tau}_4 & \tilde{\tau}_3 & \tilde{\tau}_2 & \tilde{\tau}_1 \\
\tilde{\tau}_4 & \tilde{\tau}_3 & \tilde{\tau}_2 & \tilde{\tau}_1 \\
\end{array} \right) P_1 \tilde{N}(a). \]

(4.12)
\[ \frac{1}{\theta_2}(m_3 \tilde{n}_1 + m_4 \tilde{n}_4) \]
\[ - \frac{1}{\theta_2} \left[ \varphi_1 \tilde{n}(b) + \varphi_2 \tilde{n}'(b) + \varphi_3 \tilde{n}^{[2]}(b) + \varphi_4 \tilde{n}^{[3]}(b) \right] \]
\[ [\varphi_1' m(b) + \varphi_2' m'(b) + \varphi_3' m^{[2]}(b) + \varphi_4' m^{[3]}(b)] \]
\[ + \frac{1}{\theta_2} \left[ \varphi_1 m(b) + \varphi_2 m'(b) + \varphi_3 m^{[2]}(b) + \varphi_4 m^{[3]}(b) \right] \]
\[ [(\varphi_1' \tilde{n}(b) + \varphi_2' \tilde{n}'(b) + \varphi_3' \tilde{n}^{[2]}(b) + \varphi_4' \tilde{n}^{[3]}(b))] \]
\[ - \frac{1}{\theta_2} \left[ \psi_1 \tilde{n}(b) + \psi_2 \tilde{n}'(b) + \psi_3 \tilde{n}^{[2]}(b) + \psi_4 \tilde{n}^{[3]}(b) \right] \]
\[ [\psi_1' m(b) + \psi_2' m'(b) + \psi_3' m^{[2]}(b) + \psi_4' m^{[3]}(b)] \]
\[ + \frac{1}{\theta_2} \left[ \psi_1 m(b) + \psi_2 m'(b) + \psi_3 m^{[2]}(b) + \psi_4 m^{[3]}(b) \right] \]
\[ [(\psi_1' \tilde{n}(b) + \psi_2' \tilde{n}'(b) + \psi_3' \tilde{n}^{[2]}(b) + \psi_4' \tilde{n}^{[3]}(b))] \]
\[ = - \frac{1}{\theta_2} M^T(b) \left( \begin{array}{c} \varphi_j' \\ \varphi_j \\ \varphi_4' \\ \varphi_4 \end{array} \right) (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \tilde{N}(b) + \frac{1}{\theta_2} M^T(b) \left( \begin{array}{c} \varphi_1' \\ \varphi_2' \\ \varphi_3' \\ \varphi_4' \end{array} \right) (\varphi_1, \varphi_2', \varphi_3', \varphi_4') \tilde{N}(b) \]
\[ - \frac{1}{\theta_2} M^T(b) \left( \begin{array}{c} \psi_j' \\ \psi_j \\ \psi_4' \\ \psi_4 \end{array} \right) (\psi_1, \psi_2, \psi_3, \psi_4) \tilde{N}(b) + \frac{1}{\theta_2} M^T(b) \left( \begin{array}{c} \psi_1' \\ \psi_2' \\ \psi_3' \\ \psi_4' \end{array} \right) (\psi_1', \psi_2', \psi_3', \psi_4') \tilde{N}(b) \]
\[ = \frac{1}{\theta_2} M^T(b) \left( \begin{array}{cccc} \varphi_1' & \varphi_1 & \psi_1' & 1 \\ \varphi_2' & \varphi_2 & \psi_2' & \psi_2 \\ \varphi_3' & \varphi_3 & \psi_3' & \psi_3 \\ \varphi_4' & \varphi_4 & \psi_4' & \psi_4 \end{array} \right) \left( \begin{array}{cccc} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \varphi_1' & \varphi_2' & \varphi_3' & \varphi_4' \\ -\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4 \\ -\psi_1 - \psi_2 - \psi_3 - \psi_4 \end{array} \right) \tilde{N}(b) \]
\[ = \frac{1}{\theta_2} M^T(b) CP_2 C^T \tilde{N}(b) \]
\[ = \frac{1}{\theta_2} M^T(b) \theta_2 P_1 \tilde{N}(b) \]
\[ = M^T(b) P_1 \tilde{N}(b) \]
\[ = m^{[3]}(b) \tilde{n}(b) - m^{[2]}(b) \tilde{n}'(b) + m'(b) \tilde{n}^{[2]}(b) - m(b) \tilde{n}^{[3]}(b). \]  

(4.13)

Then by (2.6), we have
\[ [m \tilde{n}^{[3]} - m^{[3]} \tilde{n} - m^{[2]} \tilde{n}'](c+) = \rho_0 [m \tilde{n}^{[3]} - m^{[3]} m^{[2]} + m^{[2]} \tilde{n}'](c-) \]
(4.14)

From (4.10)–(4.14), we get
Dividing both sides of (4.15) by \(K\) and taking the limit as \(K \to 0\), by Theorem 3.2 we get

\[
\begin{align*}
\frac{d\mu_A(K)}{dK} &= \rho_0 M^T(a)(E - A(A + K)^{-1}) P_1 \tilde{N}(a).
\end{align*}
\]

Then (4.1) follows. In a similar discussion, we can obtain (4.2).

(2) From (2.1) and integration by parts we obtain

\[
\begin{align*}
[m(A + K) - \mu(A)](m, n)_1 &= \mu_0 \rho_0 \left[ (m_1 \tilde{n}_1 + m_2 \tilde{n}_2) + \frac{\rho_0}{\theta_1}(m_3 \tilde{n}_3 + m_4 \tilde{n}_4) \right] \\
&= \rho_0 [m^{[3]}(a) \tilde{n}(a) - m^{[2]}(a) \tilde{n}'(a) + m'(a) \tilde{n}^{[2]}(a) - m(a) \tilde{n}^{[3]}(a)] \\
&\quad + \frac{\rho_0}{\theta_1} M^T(a) \left( \begin{array}{c}
\tau_1' - \tau_1 \\
\tau_2' - \tau_2 \\
\tau_3' - \tau_3 \\
\tau_4' - \tau_4
\end{array} \right) \left( \begin{array}{c}
\tilde{\sigma}_1 - \sigma_1 \\
\tilde{\sigma}_2 - \sigma_2 \\
\tilde{\sigma}_3 - \sigma_3 \\
\tilde{\sigma}_4 - \sigma_4
\end{array} \right) P_1 \tilde{N}(a) \\
&\quad + \frac{\rho_0}{\theta_1} M^T(a) \left( \begin{array}{c}
\tau_1' - \tau_1 \\
\tau_2' - \tau_2 \\
\tau_3' - \tau_3 \\
\tau_4' - \tau_4
\end{array} \right) \left( \begin{array}{c}
\tilde{\sigma}_1 - \sigma_1 \\
\tilde{\sigma}_2 - \sigma_2 \\
\tilde{\sigma}_3 - \sigma_3 \\
\tilde{\sigma}_4 - \sigma_4
\end{array} \right) P_1 \tilde{N}(a)
\end{align*}
\]

(4.15)

\[= \rho_0 M^T(a) EP_1 \tilde{N}(a) \]

(4.16)
\[ [\mu(B + K) - \mu(B)] \left( (m, n) + \frac{\rho_0}{\theta_1} (m, n) + \frac{1}{\theta_1} (m, n) + \frac{1}{\theta_2} (m, n) \right) \]

\[ = \rho_0 [m(c-)\bar{n}^{(3)}(c-) - m^{(3)}(c+)\bar{n}(c_) - m'(c-)\bar{n}^{(2)}(c-) + m^{(2)}(c-)\bar{n}'(c-)] \]

\[ \quad - [m(c+)\bar{n}^{(3)}(c+) - m^{(3)}(c+)\bar{n}(c+) - m'(c+)\bar{n}^{(2)}(c+) + m^{(2)}(c+)\bar{n}'(c+)]. \]

By (2.6), we have

\[ M(c+) = B \cdot M(c-) \]

\[ \bar{N}(c+) = (B + K) \cdot \bar{N}(c-). \]

Thus

\[ [\mu(B + K) - \mu(B)] \left( (m, n) + \frac{\rho_0}{\theta_1} (m, n) + \frac{1}{\theta_1} (m, n) + \frac{1}{\theta_2} (m, n) \right) \]

\[ = \rho_0 N^*(c-)P_1M(c-) - N^*(c+)P_1M(c+) \]

\[ = \rho_0 N^*(c-)P_1M(c-) - N^*(c-)(B + K)^*P_1BM(c-) \]

\[ = \rho_0 N^*(c-)P_1M(c-) - N^*(c-)B^*P_1BM(c-) - N^*(c-)K^*P_1BM(c-) \]

\[ = -N^*(c-)K^*P_1BM(c-). \]

(4.16)

Dividing both sides of (4.16) by \( h \) and taking the limit as \( h \to 0 \), by Theorem 3.2 we get

\[ d\mu_\theta(K) = -M^*(c-)K^*P_1BM(c-). \]

(3) For \( k \in L(J, \mathbb{R}) \), let \( \frac{1}{p} + k = \frac{1}{\rho} \). From (2.1), we have

\[ \left[ \mu \left( \frac{1}{p} + k \right) - \mu \left( \frac{1}{p} \right) \right] (m, n) \]

\[ = \left[ \mu \left( \frac{1}{p} + k \right) - \mu \left( \frac{1}{p} \right) \right] \left[ \rho_0 \int_a^c m\bar{n}dx + \int_a^b m\bar{n}dx \right] \]

\[ = \rho_0 [m\bar{n}^{(3)} - m^{(3)}\bar{n} - m'(\bar{n})^{(2)} + m^{(2)}\bar{n}']_a^{(c-)} \]

\[ + [m\bar{n}^{(3)} - m^{(3)}\bar{n} - m'(\bar{n})^{(2)} + m^{(2)}\bar{n}']_b^{(c+)} \]

\[ + \rho_0 \int_a^c (\bar{p} - p)m''\bar{n}''dx + \int_a^b (\bar{p} - p)m''\bar{n}''dx, \]

where \( \bar{n}^{(2)} = \bar{p}\bar{n}'', \bar{n}^{(3)} = (\bar{p}\bar{n}'')' - q\bar{n}' \).

Then by the boundary conditions (2.2)-(2.5) and transmission condition (2.6), we have
Dividing both sides of (4.17) by $k$ and taking the limit as $k \to 0$, by Theorem 3.2 we get

$$\frac{d \mu_1}{p}(k) = -\rho_0 \int_a^c |p|m''^2kdx - \int_c^b |p|m''^2kdx.$$  

Then (4.4) follows. Similar to the proof of (4.4), we can obtain (4.5).

(4) For $k \in L(J, \mathbb{R})$. From (2.1), we have

$$[\mu(q_0 + k) - \mu(q_0)] \langle m, n \rangle_1 = [\mu(q_0 + k) - \mu(q_0)] \left[ \rho_0 \int_a^c m\tilde{n}dx + \int_c^b m\tilde{n}dx \right]$$

$$= m(b)\tilde{n}^{(3)}(b) - m(a)\tilde{n}^{(3)}(a) - m^{(3)}(b)\tilde{n}(b) + m^{(3)}(a)\tilde{n}(a)$$

$$- m'(b)\tilde{n}^{(2)}(b) + m'(a)\tilde{n}^{(2)}(a) + m^{(2)}(b)\tilde{n}'(b) - m^{(2)}(a)\tilde{n}'(a)$$

$$+ \rho_0 \int_a^c km\tilde{n}dx + \int_c^b km\tilde{n}dx.$$

Using the boundary conditions and transmission condition (2.2)–(2.6), we have

$$[\mu(q_0 + k) - \mu(q_0)] \left[ \langle m, n \rangle_1 + \frac{\rho_0}{\theta_1}(m_1\tilde{n}_1 + m_2\tilde{n}_2) \right.$$

$$+ \frac{1}{\theta_2}(m_3\tilde{n}_3 + m_4\tilde{n}_4) \bigg] = \rho_0 \int_a^c km\tilde{n}dx + \int_c^b km\tilde{n}dx.$$  

Then (4.6) follows. The proof of (4.7) is similar as that of (4.6), hence we omit the details. \hfill \square

5 Conclusions

This paper gives a detailed characterization of dependence of eigenvalue on spectral data for fourth-order differential operator with both endpoints dependent on spectral parameter and a transmission conditions at inner discontinuity. A operator formulae coinciding the considered problem is given, which is obtained by defining a suitable Hilbert Space and exact operator relation. The continuity dependence of eigenvalues
and eigenfunctions, and differentiating dependence to the spectral data are obtained by using Fréchet derivatives. Since both endpoints are containing spectral parameter, these kinds of spectral problems are more complicated and have profound theoretical importance.

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Conflict of interest All authors declare no conflicts of interest in this paper.

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References

1. Likov, A.V., Yu A.M.: The Theory of Heat and Mass Transfer Qosenergaizdat (1963)
2. Tikhonov, A.N., Samarskii, A.A.: Equations of Mathematical Physics. Pergramon, Oxford and New York (1963)
3. Buschmann, D., Stolz, G., Weidmann, J.: One-dimensional schrodinger operators with localpoint interactions. J. Reine Angew. Math. 467, 169–186 (1995)
4. Voitovich, N.N., Katsenelbaum, B.Z., Sivov, A.N.: Generalized Method of Eigenvibration in the Theory of Diffraction. Nauka, Moskov (1997)
5. Mukhtarov, OSh., Kadakal, M.: On a Sturm-Liouville type problem with discontinuous in two-points. Far East J. Appl. Math. 19(3), 337–352 (2005)
6. Zhang, M., Wang, Y.: Dependence of eigenvalues of Sturm-Liouville problems with interface conditions. Appl. Math. Comput. 265, 31–39 (2015)
7. Zhang, X., Sun, J.: Green function of fourth-order differential operator with eigenparameter-depend-ent boundary and transmission conditions. Acta Math. Appl. Sin. Engl. Ser. 33(2), 311–326 (2017)
8. Cai, J., Zheng, Z.: Matrix representations of Sturm-Liouville problems with coupled eigenparam-eter-dependent boundary conditions and transmission conditions. Math. Methods Appl. Sci. 41(9), 3495–3508 (2018)
9. Mukhtarov, OSh., Aydemir, K.: Two-linked periodic Sturm-Liouville problems with transmission conditions. Math. Methods Appl. Sci. 44(18), 14664–14676 (2021)
10. Uğurlu, E.: Third-order value transmission problems. Turk. J. Math. 43, 1518–1532 (2019)
11. Zinsou, B.: Dependence of eigenvalues of fourth-order boundary value problems with transmission conditions. Rocky Mt. J. Math. 50, 369–381 (2020)
12. Currie, S., Love, A.D.: Hierarchies of difference boundary value problems II-Applications. Quaest. Math. 37(3), 371–392 (2014)
13. Aliyev, Z.S., Guliyeva, S.B.: Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load. J. Differ. Equ. 263, 5830–5845 (2017)
14. Binding, P.A., Browne, P.J., Watson, B.A.: Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. J. Comput. Appl. Math. 148(1), 147–168 (2002)
15. Gao, C., Li, X., Ma, R.: Eigenvalues of a Linear fourth-order differential operator with squared spectral parameter in a boundary condition. Mediterr. J. Math. 15(3), 1–14 (2018)
16. Zhang, L., Ao, J.: On a class of inverse Sturm-Liouville problems with eigenparameter-dependent boundary conditions. Appl. Math. Comput. 362, 124553 (2019)
17. Li, K., Bai, Y., Wang, W., Meng, F.: Self-adjoint realization of a class of third-order differential operators with eigenparameter dependent boundary conditions. J. Appl. Anal. Comput. 10(6), 2631–2643 (2020)
18. Zhang, M., Li, K.: Dependence of eigenvalues of Sturm-Liouville problems with eigenparameter dependent boundary conditions. Appl. Math. Comput. 378, 1–10 (2020)
19. Greenberg, L., Marletta, M.: The code SLEUTH for solving fourth order Sturm-Liouville problems. ACM Trans. Math. Softw. 23(4), 453–493 (1997)
20. Bailey, P.B., Everitt, W.N., Zettl, A.: The SLEIGN2 Sturm–Liouville code. ACM Trans. Math. Softw. 27(2), 143–192 (2001)
21. Kong, Q., Zettl, A.: Eigenvalues of regular Sturm–Liouville problems. J. Differ. Equ. 131, 1–19 (1996)
22. Uğurlu, E.: Regular third-order boundary value problems. Appl. Math. Comput. 343, 247–257 (2019)
23. Li, K., Sun, J., Hao, X.: Eigenvalues of regular fourth-order Sturm-Liouville problems with transmission conditions. Math. Methods Appl. Sci. 40, 3538–3551 (2017)
24. Zheng, Z., Ma, Y.: Dependence of eigenvalues of 2nth-order spectral problems. Bound. Value Probl. 2017, 1–35 (2017)
25. Yang, Q.X., Wang, W.Y., Gao, X.C.: Dependence of eigenvalues of a class of higher-order Sturm–Liouville problems on the boundary. Math. Probl. Eng. 2015, 686102 (2015)
26. Bao, Q., Sun, J., Hao, X., Zettl, A.: New canonical forms of self-adjoint boundary conditions for regular differential operators of order four. J. Appl. Anal. Comput. 9, 2190–2211 (2019)
27. Uğurlu, E., Bairamov, E.: Fourth order differential operators with distributional potentials. Turk. J. Math. 44, 825–856 (2020)
28. Zhang, H., Ao, J., Li, M.: Dependence of eigenvalues of Sturm-Liouville problems with eigenparameter-dependent boundary conditions and interface conditions. Mediterr. J. Math. (2022). https://doi.org/10.1007/s00009-021-01943-x
29. Zhang, H., Ao, J., Mu, D.: Eigenvalues of discontinuous third-order boundary value problems with eigenparameter-dependent boundary conditions. J. Math. Anal. Appl. 506(2), 125680 (2022)
30. Kong, Q., Zettl, A.: Linear ordinary differential equations. In: Agarwal, R.P. (ed.) Inequalities and Applications, vol. 3, pp. 381–397. Singapore, WSSIAA (1994)
31. Dieudonné, J.: Foundations of Modern Analysis. Academic Press, New York (1969)