INDECOMPOSABLE $K_1$ AND THE HODGE-D-CONJECTURE FOR $K3$ AND ABELIAN SURFACES

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Abstract. Let $X$ be a projective algebraic manifold, and $\text{CH}^k(X, 1)$ the higher Chow group, with corresponding real regulator $r_{k,1} \otimes \mathbb{R} : \text{CH}^k(X, 1) \otimes \mathbb{R} \to H^{2k-1}_D(X, \mathbb{R}(k))$. If $X$ is a general $K3$ surface or Abelian surface, and $k = 2$, we prove the Hodge-D-conjecture, i.e. the surjectivity of $r_{2,1} \otimes \mathbb{R}$. Since the Hodge-D-conjecture is not true for general surfaces in $\mathbb{P}^3$ of degree $\geq 5$, the results in this paper provide an effective bound for when this conjecture is true. We then apply these results to the space of indecomposables $\text{CH}_{\text{ind}}^k(X, 1; \mathbb{Q})$, specifically by proving that $\text{Level}(\text{CH}_{\text{ind}}^k(X, 1; \mathbb{Q})) \geq k - 2$ where $X$ is a general $k$-fold product of elliptic curves. This leads to a hard generalization of Mumford’s famous theorem on the kernel of the Albanese map on the Chow group of zero-cycles on a surface of positive genus.

1. Statement of results

Let $X$ be a projective algebraic manifold. This paper concerns the maps, called regulators, from $K_1$ of $X$ to real Deligne cohomology. More specifically, in terms of Bloch’s higher Chow groups $\text{CH}^k(X, m)$ [Blo1], we are interested in the case $m = 1$ and the map

$$r_{k,1} : \text{CH}^k(X, 1) \to H^{2k-1}_D(X, \mathbb{R}(k))$$

where $\mathbb{R}(k) = \mathbb{R}(2\pi \sqrt{-1})^k$ and

$$H^{2k-1}_D(X, \mathbb{R}(k)) \simeq H^{k-1,k-1}(X, \mathbb{R}(k - 1))$$

is Deligne cohomology. Beilinson’s Hodge-D-conjecture for real varieties would imply that

$$r_{k,1} \otimes \mathbb{R} : \text{CH}^k(X, 1) \otimes \mathbb{R} \to H^{2k-1}_D(X, \mathbb{R}(k))$$

is surjective (see [Ja]). That conjecture is now known to be false using the works of [No] and [G-S] (see [MS1]); although the corresponding conjecture

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for smooth projective varieties defined over number fields is still open. For example if $S \subset \mathbb{P}^3$ is a transcendentally generic surface of degree $d \geq 5$, then the image $\text{cl}_{2,1}(\text{CH}^2(S,1)) \subset H^3_{D}(S,\mathbb{R}(2))$ is contained in the image of $H^3_{D}(\mathbb{P}^3,\mathbb{R}(2)) \to H^3_{D}(S,\mathbb{R}(2))$. The Hodge-D-conjecture is trivially true for all smooth $S \subset \mathbb{P}^3$ of degree $d \leq 3$. This is because $H^2(S)$ is generated by algebraic cycles if $d \leq 3$. When $d = 4$, $S$ is a K3 surface, and one can ask about the status of the Hodge-D-conjecture here, and more generally for all K3 surfaces. In this paper we settle this question for general K3 surfaces $S$, where general means in the sense of the real analytic Zariski topology (see Sec. 2). In other words $H^{1,1}(S,\mathbb{R}(1)) \simeq H^{1,1}(S,\mathbb{R}(1))^\vee$ is generated by currents of the form

$$\omega \in H^{1,1}(S,\mathbb{R}(1)) \mapsto \frac{1}{2\pi \sqrt{-1}} \sum_j \int_{D_j} \omega \log |f_j|,$$

where $f_j \in \mathbb{C}(D_j)^\times$ and $\sum_j \text{div}(f_j) = 0$.

More specifically, we prove the following:

**Theorem 1.1.** (i) The Hodge-D-conjecture holds for general K3 surfaces in the real analytic Zariski topology.

(ii) The Hodge-D-conjecture holds for general Abelian surfaces in the real analytic Zariski topology, hence for general Kummer surfaces.

(iii) The Hodge-D-conjecture holds for general products $E_1 \times E_2$ of elliptic curves in the real analytic Zariski topology, hence for general “special Kummer surfaces” (see Sec. 2).

There is ample evidence why such a statement should be true. There are the works of A. Collino, S. Müller-Stach, C. Voisin, et al on nontrivial regulator calculations on K3 surfaces. See for example [Co2], [MS1] and the references cited there. Further, the second author proved a twisted version of the conjecture with twisted higher Chow groups and regulators [Lw1]. There are two key ingredients which make a proof for a general K3 surface $S$ possible. First of all, there are plenty of nontrivial higher Chow cycles on $S$ constructed out of rational curves. Take two rational curves $D_1$ and $D_2$ and two distinct points $p_1, p_2 \in D_1 \cap D_2$. Choose $f_i \in \mathbb{C}(D_i)^\times$ such that

$$(f_1) = p_1 - p_2 \quad \text{and} \quad (f_2) = p_2 - p_1$$

and then $(f_1, D_1) + (f_2, D_2)$ defines a class in $\text{CH}^2(S, 1)$. More generally, take $n$ rational curves $D_1, D_2, \ldots, D_n$ and pick $n$ distinct points $p_1, p_2, \ldots, p_n$ such that $p_i \in D_i \cap D_{i+1}$ (let $D_{n+1} = D_1$ and $p_{n+1} = p_1$). Choose $f_i \in \mathbb{C}(D_i)^\times$ such that

$$(f_i) = p_i - p_{i+1}$$

and then $\sum_{i=1}^n (f_i, D_i) \in \text{CH}^2(S, 1)$.

It is well known that there are rational curves on $S$. This statement was made more precise in [C1], that there are rational curves in every linear series on $S$. Indeed, we think the following is true:
Conjecture 1.2. The rational curves on $S$ are dense. That is, let $\Sigma = \bigcup D_\alpha$ be the union of all rational curves on $S$. Then the closure $\overline{\Sigma}$ of $\Sigma$ under the analytic topology is $S$.

The motivation for this conjecture is based on the analogy between rational curves on $K3$ surfaces with the density of torsion points on elliptic curves. These torsion points were instrumental in constructing nontrivial $K_2$ classes on general elliptic curves (see [Blo2]), and correspondingly, the rational curves on a general $K3$ surface play a role here in constructing nontrivial $K_1$ classes.

Actually, there are enough rational curves even only in the primitive class by the counting of [B-L] and [Y-Z] (a primitive class is a divisor which is not the multiple of another divisor; there is only one primitive class on a general algebraic $K3$, which has self-intersection $2g - 2$ with $g$ called the genus of the $K3$). For example, there are 3200 rational curves in the primitive class of a quartic $K3$, which seem enough to produce 20 generators of $H^{1,1}(S, \mathbb{R})$. It turns out in the end that we only need a fraction of these 3200 rational curves to realize the surjection of $\text{cl}_{2,1} \otimes \mathbb{R}$.

The second key fact is that Hodge-$D$-conjecture obviously holds on a $K3$ surface with maximum Picard number 20. This points us to a way to prove the conjecture for general $K3$ surfaces by degeneration. Actually, this is also the way in which the twisted version of the conjecture was proven [Lw1].

In Sec. 5 we turn our attention to indecomposability of $K_1$ for special classes of varieties. There we introduce the notion of Level, which measures the complexity of the Chow groups $\text{CH}^k(X, m)$, as well as the quotient group of indecomposables $\text{CH}^k_{\text{ind}}(X, m)$. By exploiting the results in Theorem 1.1, we arrive at the following:

**Theorem 1.3.** Let $X = E_1 \times \cdots \times E_k$ be a general product of $k$ elliptic curves, i.e. in the countable real analytic Zariski topology (see Sec. 2), and $\text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q})$ the space of indecomposables. Then:

(i) $\text{Level}(\text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q})) \geq k - 2$.

(ii) In particular for $k \geq 3$, there are an uncountable number of indecomposables in the kernel of the regulator

$$\text{cl}_{k,1} : \text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q}) \to H^2D_{k-1}(X, \mathbb{Q}(k)).$$

This paper has its origins in a discussion that the second author had with the first, where it was suggested by the second author that the Hodge-$D$-conjecture for $K3$ surfaces should be true, based on the abundance of rational curves, and that a proof should involve degeneration to a $K3$ surface of maximum Picard number. The second author is indeed very grateful to the first author for supplying the complete degeneration argument in Sec. 8, without which this paper would not have evolved in its present form.
One consequence of the results of this paper is a significant generalization of Theorems 1 and 2 on page 544 of [GL1] (and corresponding statements in [GL2]). Not only do we present correct proofs of these theorems, the results in this paper are much deeper. The second author is grateful to Morihiko Saito for pointing out the errors in the degeneration argument in [GL1], where the cycles constructed in Theorem 2.4 and Proposition 3.3 of [GL1] are regulator decomposable, contrary to the claims there. The problem in [GL1] has to do with the presence of singularities of a real 2-form after degeneration to a singular fiber. Thus this paper can also be seen as providing the correct proofs to the main results in [GL1] (and [GL2]).

2. Notation

Throughout this paper, $X$ is assumed to be a projective algebraic manifold of dimension $n$. If $A \subset \mathbb{R}$ is a subring, we put $A(k) = A(2\pi \sqrt{-1})^k$. For the higher Chow groups $\text{CH}^k(X, m)$ introduced in Sec. 3, and for $A$ above, we denote $\text{CH}^k(X, m) \otimes A$ by $\text{CH}^k(X, m; A)$.

The use of the terminology “general $X$” in this paper will have two possible meanings. Firstly, for a variety $Y$, a real analytic Zariski open set $U$ in $Y$ will be the complement of a real analytic subvariety of $Y$. If $Y$ parameterizes a family $\{X_t\}_{t \in Y}$ of projective algebraic manifolds, then a general such $X = X_t$ in the real analytic Zariski topology means that $t \in U$, for some real analytic Zariski open set $U$ in $Y$. Secondly, a countable real analytic open set $U_c$ of $Y$ is the complement of a countable union of real analytic subvarieties of $Y$. We say that $X = X_t$ is general in the countable real analytic Zariski topology, if $t \in U_c$.

Let $A$ be an Abelian surface, and $Y = A/\pm 1$ its corresponding Kummer counterpart [S-S, p. 550]. Following [S-S], we say that $Y$ is special if $A$ reducible, i.e. contains an elliptic curve.

3. Deligne cohomology and higher Chow groups

(a) Deligne cohomology. Let

$$\Omega^\bullet_X := \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \rightarrow 0,$$

be the holomorphic de Rham complex. The complex $\Omega^\bullet_X$ is filtered by the Hodge filtration

$$F^p\Omega^\bullet_X := 0 \rightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \Omega^{p+2}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \rightarrow 0.$$
Thus we arrive at the isomorphism
\[ F^pH^k_{\text{DR}}(X, \mathbb{C}) := H^k(F^p\Omega^*_X) \rightarrow H^k(\Omega^*_X) \]
is injective for all \( k \). Therefore if we put \( \Omega_X^{<p} := \Omega^*_X/F^p\Omega^*_X \), then
\[ H^k(\Omega^*_X^{<p}) = \frac{H^k_{\text{DR}}(X, \mathbb{C})}{F^pH^k_{\text{DR}}(X, \mathbb{C})} \]

For a subring \( A \subset \mathbb{R} \), we introduce the Deligne complex
\[ A_D(k) : A(k) \rightarrow \mathcal{O}_X \rightarrow \Omega^1_X \rightarrow \cdots \rightarrow \Omega^k_X. \]
call this \( \Omega^*_X^{<k} \)

**Definition 3.1.** Deligne cohomology is given by the hypercohomology:
\[ H^i_D(X, A(k)) = H^i(A_D(k)). \]

Applying \( H^*(-) \) to the short exact sequence:
\[ 0 \rightarrow \Omega^*_X^{<k}[-1] \rightarrow A_D(k) \rightarrow A(k) \rightarrow 0, \]
yields the short exact sequence:
\[ 0 \rightarrow \frac{H^{i-1}(X, \mathbb{C})}{H^{i-1}(X, A(k)) + F^kH^{i-1}(X, \mathbb{C})} \rightarrow H^i_D(X, A(k)) \rightarrow H^i(X, A(k)) \cap F^kH^i(X, \mathbb{C}) \rightarrow 0. \]
The cases of interest are \( A = \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \), for \( i = 2k - 1 \). For example, if \( A = \mathbb{Z} \) and \( i = 2k - 1 \), then we arrive at the short exact sequence
\[ 0 \rightarrow \frac{H^{2k-2}(X, \mathbb{C})}{F^kH^{2k-2}(X, \mathbb{C}) + H^{2k-2}(X, \mathbb{Z}(k))} \rightarrow H^2_{D}(X, \mathbb{Z}(k)) \rightarrow H^{2k-1}(X, \mathbb{Z}(k))_{\text{tor}} \rightarrow 0. \]
Next, if \( A = \mathbb{Q} \) and \( i = 2k - 1 \), then from Hodge theory,
\[ H^i(X, \mathbb{Q}(k)) \cap F^kH^i(X, \mathbb{C}) = 0. \]
Thus we arrive at the isomorphism
\[ H^{2k-1}_{D}(X, \mathbb{Q}(k)) \simeq \frac{H^{2k-2}(X, \mathbb{C})}{F^kH^{2k-2}(X, \mathbb{C}) + H^{2k-2}(X, \mathbb{Q}(k))}. \]
If \( A = \mathbb{R} \) and if we set
\[ \pi_{k-1} : \mathbb{C} = \mathbb{R}(k) \oplus \mathbb{R}(k-1) \rightarrow \mathbb{R}(k-1) \]
to be the projection, then we have the isomorphisms:
\[ H^{2k-m}_{D}(X, \mathbb{R}(k)) \simeq \frac{H^{2k-m-1}(X, \mathbb{C})}{F^kH^{2k-m-1}(X, \mathbb{C}) + H^{2k-m-1}(X, \mathbb{R}(k))} \]
\[ \pi_{k-1} \rightarrow \frac{H^{2k-m-1}(X, \mathbb{R}(k-1))}{\pi_{k-1}(F^kH^{2k-m-1}(X, \mathbb{C}))}. \]
For example if $A = \mathbb{R}$ and $i = 2k - 1$, we have
\[H_{pd}^{2k-1}(X, \mathbb{R}(k)) \simeq \frac{H^{2k-2}(X, \mathbb{C})}{F^k H^{2k-2}(X, \mathbb{C}) + H^{2k-2}(X, \mathbb{R}(k))} \xrightarrow{\pi_{k-1}} H^{k-1,k-1}(X, \mathbb{R}(k-1)) \]
\[=: H^{k-1,k-1}(X, \mathbb{R}(k-1)) \simeq \left\{ H^{n-k+1,n-k+1}(X, \mathbb{R}(n-k+1)) \right\}^\vee.

(b) Higher Chow groups. The higher Chow groups $\text{CH}^k(X, m)$ were invented by Bloch [Blo1] (and independently by S. Landsberg). They are defined for all $k, m \geq 0$; moreover in [Blo1] there is proven a Riemann-Roch theorem, namely that the Chern character map
\[\text{ch} : \text{CH}^\bullet(X, m; \mathbb{Q}) \xrightarrow{\sim} K_m(X) \otimes \mathbb{Q},\]
is an isomorphism, generalizing the Grothendieck Riemann-Roch theorem in the case $m = 0$. Since in this paper, we are only interested in the case $m = 1$, we provide an abridged alternative version of the definition of Bloch’s higher Chow groups, for the cases $0 \leq m \leq 2$, that we acquire using a Gersten resolution. Let $K_{k,X}$ be the sheaf of $K$-groups on $X$, i.e. where $K_{k,X}$ is the sheaf associated to the presheaf in the Zariski topology,
\[U \subset X \mapsto K_k(\Gamma(U, \mathcal{O}_X)).\]

**Definition 3.2.** (See [MS1]) For $0 \leq m \leq 2$, define
\[\text{CH}^k(X, m) = H_{Zar}^{k-m}(X, K_{k,X}).\]

The best way to interpret the RHS in Definition 3.2 is via the Gersten resolution proven by Bloch for $k = 2$ and by Quillen for general $k$. This is the flasque resolution of $K_{k,X}$ given by
\[0 \to K_{k,X} \to K_k(\mathbb{C}(X)) \xrightarrow{i_{Z,*}} K_{k-1}(\mathbb{C}(Z)) \to \cdots \]
\[\xrightarrow{i_{Z,*}} K_2(\mathbb{C}(Z)) \to \oplus_{\text{cd}_X Z = k-2} i_{Z,*} K_1(\mathbb{C}(Z)) \]
\[\to \oplus_{\text{cd}_X Z = k} i_{Z,*} K_0(\mathbb{C}(Z)) \to 0\]

We recall that $K_0(\mathbb{C}(Z)) \simeq \mathbb{Z}$, $K_1(\mathbb{C}(Z)) = \mathbb{C}(Z)^\times$, and that $K_2(\mathbb{C}(Z))$ is generated by symbols. Taking global sections leads to a complex, whose last three terms are
\[\bigoplus_{\text{cd}_X Z = k-2} K_2(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{\text{cd}_X Z = k-1} \mathbb{C}(Z)^\times \xrightarrow{\text{div}} z^k(X),\]
where $T$ is the Tame symbol, and $\text{div}$ is the divisor map, and $z^k(X)$ is the free abelian group generated by subvarieties of codimension $k$ in $X$. Thus
for example,
\[
\text{CH}^k(X, 0) = \frac{z^k(X)}{\text{subgroup of principal divisors}} = \text{CH}^k(X),
\]
and
\[
\text{CH}^k(X, 1) = \frac{\left\{ \sum_j (f_j, Z_j) : \text{cd}_X Z_j = k - 1, f_j \in \mathbb{C}(Z_j)^\times, \sum_j \text{div}(f_j) = 0 \right\}}{\text{Image(Tame symbol)}}.
\]

4. A regulator

In this section, we recall the definition of the regulator
\[
c_{k, 1} : \text{CH}^k(X, 1) \rightarrow H^{2k-1}_D(X, \mathbb{Z}(k)),
\]
where we recall that \( H^{2k-1}_D(X, \mathbb{Z}(k)) \) fits in the short exact sequence:
\[
0 \rightarrow H^{2k-2}(X, \mathbb{C}) \rightarrow H^{2k-2}_D(X, \mathbb{C}) + H^{2k-2}(X, \mathbb{Z}(k)) \rightarrow H^{2k-1}_D(X, \mathbb{Z}(k)) \rightarrow H^{2k-1}(X, \mathbb{Z}(k))_{\text{tor}} \rightarrow 0.
\]

We define
\[
\text{CH}_{\text{hom}}^k(X, 1) = \ker \left( \text{CH}^k(X, 1) \rightarrow H^{2k-1}_D(X, \mathbb{Z}(k)) \right).
\]

Using the compatibility of Poincaré and Serre duality, there is an induced map
\[
\text{cl}_{k, 1} : \text{CH}_{\text{hom}}^k(X, 1) \rightarrow \left\{ F^{n-k+1}H^{2n-2k+2}(X, \mathbb{C}) \right\}^\vee / H^{2n-2k+2}(X, \mathbb{Z}(n-k)).
\]

The formula we use for \( \text{cl}_{k, 1} \) is this (see [Lev]): Consider
\[
\xi := \sum_j (f_j, D_j), \quad \sum_j \text{div}(f_j) = 0,
\]
where \( \text{codim}_X D_j = k - 1 \) and \( f_j \in \mathbb{C}(D_j)^\times \). Choose a branch of the log function on \( \mathbb{C}\setminus[0, \infty) \), and put \( \gamma_j = f_j^{-1}[0, \infty] \), and \( \gamma = \sum_j \gamma_j \). Then \( \partial \gamma = 0 \) on \( X \), and our assumption that \( \xi \in \text{CH}_{\text{hom}}^k(X, 1) \) means that \( \gamma = \partial \zeta \) is a boundary. For \( \omega \in F^{n-k+1}H^{2n-2k+2}(X, \mathbb{C}) \),
\[
\text{cl}_{k, 1}(\xi)(\omega) = \frac{1}{(2\pi \sqrt{-1})^{n-k}} \left( \sum_j \int_{D_j \setminus \gamma_j} \omega \log(f_j) + 2\pi \sqrt{-1} \int_{\xi} \omega \right).
\]

Now consider a smooth family of projective algebraic manifolds \( Y := \coprod_{t \in \Delta} X_t \) over a disk \( \Delta \subset \mathbb{C} \). We want to show that \( \text{cl}_{k, 1} \) varies holomorphically with respect to a family of \( K_1 \)-cycles in \( Y \) over \( \Delta \). Assume given an algebraic family of cycles \( \xi_t = \sum_j (f_{j,t}, D_{j,t}) \in \text{CH}_{\text{hom}}^k(X_t, 1), t \in \Delta \). As in [Gr], one can choose \( C^\infty \) differential forms \( \omega_1, \ldots, \omega_r \) on \( Y \) such that each \( \omega_i \) is of Hodge type \( (n, n-2k+2) + \cdots + (n-k+1, n-k+1) \), \( d\omega_i \wedge dt = 0 \), and
\{\omega_i \mid X_i : i = 1, \ldots, r\} gives a basis of \( F^{n-k+1}H^{2n-2k+2}(X_t, \mathbb{C}) \) for all \( t \in \Delta \).

Let \( \omega \) be any linear combination of \( \omega_1, \ldots, \omega_r \). This next result is probably well-known to experts, although a proof does not seem to be written down in the literature. For the convenience to the reader, we present a proof here.

**Proposition 4.1.** \( cl_{k,1}(\xi_t)(\omega) \) varies holomorphically in \( t \in \Delta \).

**Proof.** We base our proof on similar ideas in Appendix A of [Gr]. First, it is reasonably clear that \( cl_{k,1}(\xi_t)(\omega) \) varies continuously in \( t \in \Delta \). We use the criterion of holomorphicity via Morera’s theorem. Let \( \Gamma \subset \Delta \) be a simple-closed curve oriented counterclockwise. We need to show that \( \int_\Gamma cl_{k,1}(\xi_t)(\omega)dt = 0 \). This amounts to the calculation of

\[ \sum_j \int_\Gamma \left( \int_{D_{j,t} \setminus \gamma_{j,t}} \omega \log(f_{j,t}) \right) dt + 2\pi \sqrt{-1} \int_\Gamma \left( \int_{\xi_t} \omega \right) dt. \]

Note that \( \Gamma = \partial M \) for some region \( M \subset \Delta \).

(I) We first evaluate \( 2\pi \sqrt{-1} \int_\Gamma \left( \int_{\xi_t} \omega \right) dt \): Put \( \xi = \bigcup_{t \in \Gamma} \xi_t, \xi_M = \bigcup_{t \in M} \xi_t \) and \( \gamma_M = \bigcup_{t \in M} \gamma_t \). Then by Stokes’ theorem

\[ 2\pi \sqrt{-1} \int_\Gamma \left( \int_{\xi_t} \omega \right) dt = \int_{\xi} \omega \wedge dt - 2\pi \sqrt{-1} \int_{\gamma_M} \omega \wedge dt. \]

(II) Next we evaluate \( \sum_j \int_\Gamma \left( \int_{D_{j,t} \setminus \gamma_{j,t}} \omega \log(f_{j,t}) \right) dt \): Put

\[ D_{j,\Gamma} = \bigcup_{t \in \Gamma} D_{j,t} \text{ and } D_{j,M} = \bigcup_{t \in M} D_{j,t}. \]

Note that our assumptions on \( \omega \), (and holomorphicity of \( f_{j,t} \), away from the pole sets) imply that

\[ d\left( \log(f_{j,t}) \omega \wedge dt \right) = \frac{df_{j,t}}{f_{j,t}} \wedge \omega \wedge dt, \]

and that by Hodge type considerations alone,

\[ \left( \frac{df_{j,t}}{f_{j,t}} \wedge \omega \wedge dt \right) \bigg|_{D_{j,M}} = 0. \]

Thus by Stokes’ theorem, we have

\[ \sum_j \int_\Gamma \left( \int_{D_{j,t} \setminus \gamma_{j,t}} \omega \log(f_{j,t}) \right) dt = \sum_j \int_{D_{j,\Gamma}} \log(f_{j,t}) \omega \wedge dt \]

\[ = 2\pi \sqrt{-1} \int_{\gamma_M} \omega \wedge dt, \]

where we use the fact that we pick up a \( 2\pi \sqrt{-1} \) “period” from \( \log(f_{j,t}) \) as we cross \( \gamma_{j,t} \).

Finally, the resulting terms from (I) and (II) cancel, which establishes the proposition. \( \square \)
We note that the forms $\omega_1, \ldots, \omega_r$ define a holomorphic frame of the bundle

$$\prod_{t \in \Delta} F^{n-k+1} H^{2n-2k+2}(X_t, \mathbb{C})$$

over $\Delta$. One can define a frame of the $C^\infty \mathbb{R}(n-k+1)$-bundle

$$\prod_{t \in \Delta} H^{n-k+1, n-k+1}(X_t, \mathbb{R}(n-k+1))$$

in terms of a linear combinations of the forms $\omega_1, \ldots, \omega_r$. The coefficient functions of these linear combinations will be complex-valued combinations of real analytic functions. If we put $X = X_t$, then this follows from the fact that forms in $H^{n-k+1, n-k+1}(X, \mathbb{C})$ are by duality, precisely subspace of the forms in $F^{n-k+2} H^{2n-2k+2}(X, \mathbb{C})$ killed by the wedge products of forms in $F^k H^{2k-2}(X, \mathbb{C})$ (see Remark 4.3). Further, for any form $\omega \in H^{n-k+1, n-k+1}(X, \mathbb{R}(n-k+1))$,

$$\text{Re} \left( \frac{1}{(2\pi \sqrt{-1})^{n-k+1}} \left( \sum_j \int_{\gamma_j} \omega \log(f_j) + 2\pi \sqrt{-1} \int_{\xi} \omega \right) \right)$$

$$= \frac{1}{(2\pi \sqrt{-1})^{n-k+1}} \sum_j \int_{D_j} \omega \log |f_j|.$$  

We denote by $r_{k,1}$, the corresponding Beilinson real regulator

$$r_{k,1} : \text{CH}^k(X, 1) \to H^{2k-1}(X, \mathbb{R}(k))$$

$$\simeq H^{n-k+1, n-k+1}(X, \mathbb{R}(n-k+1))^\vee$$

given by

$$\xi = \sum_j (f_j, D_j) \mapsto r_{k,1}(\xi)(\omega) = \frac{1}{(2\pi \sqrt{-1})^{n-k+1}} \sum_j \int_{D_j} \omega \log |f_j|.$$  

**Corollary 4.2.** The corresponding real regulator defines a real analytic function in $t \in \Delta$. In particular, if we are given a smooth family of projective varieties $X := \coprod_{t \in \Delta} X_t \to S$, over a smooth quasiprojective base $S$, and family of of $K_1$ classes $\xi_t \in \text{CH}^k(X_t, 1)$ with nonvanishing real regulator value $r_{k,1}(\xi_t)$, then $r_{k,1}(\xi_t) \neq 0$ for $t$ in some nonempty real analytic Zariski open subset of $S$.

**Remark 4.3.** Here are some details on the existence of real analytic frames: One can assume that the first $\{\omega_1, \ldots, \omega_r\} \subset \{\omega_1, \ldots, \omega_r\}$ restrict to a basis for

$$F^{n-k+2} H^{2n-2k+2}(X_t, \mathbb{C}).$$

A corresponding holomorphic frame gives $C^\infty$ forms $\eta_1, \ldots, \eta_r$ restricting to a basis of $F^k H^{2k-2}(X_t, \mathbb{C})$. One can construct a $C^\infty$ frame restricting to a
basis of $H^{n-k+1, n-k+1}(X_t, \mathbb{R}(n-k+1))$ by finding the general solution of the linear system
\[ \langle a_1(t)\omega_1 + \cdots + a_r(t)\omega_r, \overline{\eta}_j \rangle = 0 \text{ for } j = 1, \ldots, \ell, \]

together with the nonsingularity of the matrix $(\langle \omega_i, \overline{\eta}_j \rangle)_{1 \leq i, j \leq \ell}$. After which, via the projection $\pi_{n-k+1} : \mathbb{C} \rightarrow \mathbb{R}(n-k+1)$, we arrive at the real analytic frame, twisted by $\mathbb{R}(n-k+1)$ as required.

### 5. Indecomposability

In [Lw2] we introduced the notion of the Level of a Chow group (see below). Similar notions appear elsewhere, such as in [Sa]. One should think of the Level as an integral invariant measuring the complexity of a given Chow group. Using this notion of Level, we will see that $\text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q})$ can be very large (and uncountable); moreover even in the kernel of the regulator map. Thus one arrives at an analogous result to Mumford’s famous theorem [Md].

The setting is this. Recall Bloch’s higher Chow group $\text{CH}^k(X, m)$ [Blo1]. As in the case $m \leq 1$, Bloch [Blo3] (as well as Beilinson) constructs a cycle class map
\[ \text{cl}_{k,m} : \text{CH}^k(X, m) \rightarrow H^{2k-m}_D(X, \mathbb{Z}(k)). \]
For $m \geq 1$, we have for example
\[ H^{2k-m}_D(X, \mathbb{Q}(k)) \simeq \frac{H^{2k-m-1}(X, \mathbb{C})}{F^k H^{2k-m-1}(X, \mathbb{C}) + H^{2k-m-1}(X, \mathbb{Q}(k))}. \]
One has products $\text{CH}^k(X, m) \cap \text{CH}^r(X, \ell) \subset \text{CH}^{k+r}(X, m + \ell)$ compatible with the product structure on Deligne cohomology.

We now assume $m \geq 1$.

**Definition 5.1.** (i) The subgroup of decomposables is given by
\[ \text{CH}^k_D(X, m) := \text{Image} \left( \left( \mathbb{C}^\times \right)^{\otimes m} \otimes \mathbb{Z} \text{CH}^k(X, 0) \xrightarrow{\cap} \text{CH}^k(X, m) \right), \]
where $\mathbb{C}^\times$ is identified with $\text{CH}^1(X, 1)$ via the identification
\[ \text{CH}^1(X, 1) = H^0_{\text{zar}}(X, \mathcal{K}_1, X) = H^0_{\text{zar}}(X, \mathcal{O}_X^\times). \]

(ii) The space of indecomposables is given by
\[ \text{CH}^k_{\text{ind}}(X, m; \mathbb{Q}) := \text{CH}^k(X, m; \mathbb{Q})/\text{CH}^k_D(X, m; \mathbb{Q}). \]
Remark 5.2. (i) There is an isomorphism
\[ \text{cl}_{1,1} : \text{CH}^1(X, 1) \xrightarrow{\sim} H^1_D(X, \mathbb{Z}(1)) \simeq H^0(X, \mathbb{C}/\mathbb{Z}(1)) \]

(ii) The product structure in Deligne cohomology implies that
\[ H^1_D(X, \mathbb{Z}(1)) \cup H^1_D(X, \mathbb{Z}(1)) = 0 \in H^2_D(X, \mathbb{Z}(2)). \]

Therefore \( \text{cl}_{k,m}(\text{CH}^k_D(X, m)) = 0 \) for \( m \geq 2 \).

(iii) In the case \( m = 1 \), we have
\[ \text{cl}_{k,1}(\text{CH}^k_D(X, 1; \mathbb{Q})) \subset \mathbb{C}_\mathbb{Q} \otimes H^{k-1,k-1}(X, \mathbb{Q}(k-1)), \]
with equality \( \Leftrightarrow \) the Hodge conjecture holds for \( X \).

(iv) A rigidity result of Beilinson [Bei1] implies that the image
\[ \text{cl}_{k,m}(\text{CH}^k(X, m; \mathbb{Q})) \]
is countable for \( m \geq 2 \). A variant of this rigidity argument [MS1] shows that the induced map
\[ \text{cl}_{k,1} : \frac{\text{CH}^k(X, 1; \mathbb{Q})}{\text{CH}^2_D(X, 1; \mathbb{Q})} \to \frac{H^{2k-1}_D(X, \mathbb{Q}(k))}{\mathbb{C}_\mathbb{Q} \otimes H^{k-1,k-1}(X, \mathbb{Q}(k-1))}. \]
has countable image.

Before stating our main results, we introduce some terminology.
\( N^*H^i(X, \mathbb{Q}) = \) filtration by coniveau, with graded piece
\[ \text{Gr}^N_i H^i(X, \mathbb{Q}) = \frac{N^i H^i(X, \mathbb{Q})}{\bigcap_{\ell=1}^{N^i} H^i(X, \mathbb{Q})}. \]

More explicitly,
\[ N^j H^i(X, \mathbb{Q}) = \ker : H^i(X, \mathbb{Q}) \to \lim_{Y \to X \text{ closed}} \text{codim}_X Y \geq j \]
\[ H^i(X - Y, \mathbb{Q}). \]

Definition 5.3. Let \( G \) be a subgroup of \( \text{CH}^k(X, m; \mathbb{Q}) \). Then

(i) Level(\( \text{CH}^k(X, m; \mathbb{Q})/G \)) is the smallest integer \( r \geq 0 \) such that there exists a closed subvariety \( i : Y \hookrightarrow X \) of \([\text{pure}] \) codimension \( k - r - m \) satisfying
\[ \text{CH}^k(X, m)_\mathbb{Q} = G + i_* \text{CH}^{r+m}(Y, m; \mathbb{Q}). \]

(ii) Level(\( G \)) is the smallest integer \( r \geq 0 \) such that there exists a closed subvariety \( i : Y \hookrightarrow X \) of \([\text{pure}] \) codimension \( k - r - m \) satisfying \( G \subset i_* \text{CH}^{r+m}(Y, m; \mathbb{Q}) \).
Let $S$ be a smooth projective variety of dimension $r$. We refer to the diamond below, where the upper diagonal arrows are given by Hodge-Künneth projections, and the lower arrows are defined by integration along $S$ (see [Lw2]).

\[\begin{align*}
\text{CH}^k(S \times X, m) & \quad \downarrow \\
H^{2k-m}_D(S \times X, \mathbb{R}(k)) & \quad \downarrow \\
H^{\ell-1,0}(S) \otimes H^{k-\ell,k-m}(X) & \quad \downarrow \\
\cap H^{r-\ell+1,r}(S) & \quad \downarrow \\
H^{k-\ell,k-m}(X) & \\
\end{align*}\]

(5.1)

**Definition 5.4.** (i) $H^{\{k,\ell,m\}}(X) = \mathbb{C}$-subspace of $H^{k-\ell,k-m}(X)$ generated by the image of $\text{CH}^k(S \times X, m)$ in $H^{k-\ell,k-m}(X)$ in the above diagram, and over all smooth projective algebraic $S$.

(ii) $H^{k-\ell,k-m}_N(X) = \mathbb{C}$-subspace of $H^{k-\ell,k-m}(X)$ generated by the Hodge projected image

\[N^{k-\ell}H^{2k-\ell-m}(X, \mathbb{Q}) \rightarrow H^{k-\ell,k-m}(X).\]

**Remark 5.5.** (i) As mentioned in [Lw2], it is always the case that

\[H^{\{k,\ell,0\}}(X) \subset H^{k-\ell,k}(X);\]

moreover, one can show that $H^{\{k,\ell,0\}}(X) = H^{k-\ell,k}_N(X)$ under the assumption of the hard Lefschetz conjecture. Further, under the assumption of the General Hodge Conjecture, one can show that

\[\text{Gr}^{k-\ell}_NH^{2k-\ell}(X, \mathbb{Q}) \neq 0 \iff H^{k-\ell,k}_N(X) \neq 0.\]

(ii) For $m \leq 2$, one can easily show that (see [Lw2])

\[H^{\{k-m,\ell-m,0\}}(X) \subset H^{\{k,\ell,m\}}(X).\]

**Theorem 5.6.** ([Lw2], abridged version) Let $X$ be a projective algebraic manifold and assume that $m \leq 2$ Then:

\[H^{\{k,\ell,m\}}(X)/H^{\{k-m,\ell-m,0\}}(X) \neq 0 \Rightarrow \text{Level} \left( \text{CH}^k_{\text{ind}}(X, m)_\mathbb{Q} \right) \geq \ell - m.\]
Remark 5.7. One can readily verify that in the above theorem (see [Lw2]),
\[ \ell - m \geq 1 \Rightarrow \text{CH}^k_{\text{ind}}(X, m; \mathbb{Q}) \] uncountable.

Thus if \( \text{CH}^k_{\text{ind}}(X, m; \mathbb{Q}) \) is uncountable, then by rigidity, it follows that there are an uncountable number of indecomposables in the kernel of
\[ \text{cl}_{k,m} : \text{CH}^k(X, m; \mathbb{Q}) \to H^{2k-m}_D(X, \mathbb{Q}(k)), \]
provided that the Hodge conjecture holds for \( X \) in the case \( m = 1 \), i.e. provided that \( H^{k-1,k-1}(X, \mathbb{Q}) \) is generated by algebraic cocycles.

6. Theorem 5.6

It is instructive to explain some of the ideas behind Theorem 5.6. Broadly speaking, the the relationship between [higher] Chow groups and Hodge theory is fortified by the following beautiful [generalized] conjectural formula of Beilinson:
\[ \text{Gr}_\ell^F \text{CH}^k(X, m; \mathbb{Q}) \simeq \text{Ext}^\ell_{\mathcal{MM}}(1, h^{2k-\ell-m}(X)(k)), \]
where \( \mathcal{MM} \) is some conjectural category of mixed motives, \( 1 \) is the trivial motive, and \( \text{Gr}_\ell^F \text{CH}^k(X, m; \mathbb{Q}) \) is the graded piece of a conjectured “Bloch-Beilinson” filtration on \( \text{CH}^k(X, m; \mathbb{Q}) \). One way to try to realize this is to construct a duality pairing between cohomology and higher Chow groups. In the case \( m = 0 \), this is Salberger’s duality, which was exploited in [Sa], and defined as follows: One first views \( X, S \) as defined over an algebraically closed field \( \overline{k} \) of finite transcendence degree over \( \mathbb{Q} \). Let \( \eta \) be the generic point of \( S \) and \( L = \overline{k}(\eta) \). Recall
\[ \text{CH}^k(X_L) = \lim_{U \subset S/\pi} \text{CH}^k(U \times X). \]

We define the top row arrow by imposing commutivity below:
\[
\begin{align*}
\text{CH}^k(U \times X) \times H^{2n-2k+\ell}(X) & \longrightarrow H^\ell(U) \\
\text{cl}_k & \downarrow \quad \downarrow \text{Pr}_2^* \\
H^{2k}(U \times X) \times H^{2n-2k+\ell}(U \times X) & \longrightarrow H^{2n+\ell}(U \times X),
\end{align*}
\]
where \( \int_X(-) \) is defined by integration along the fibers of \( \text{Pr}_1 : U \times X \to U \).

Taking limits, we arrive at
\[ \text{CH}^k(X_L) \times H^{2n-2k+\ell}(X) \xrightarrow{\langle \cdot, \cdot \rangle} H^\ell(\mathbb{C}(S)) \]
But \( \text{CH}^k(S \times X) \to \text{CH}^k(X_L) \) is surjective. Thus the image of \( \langle \cdot, \cdot \rangle \) lies in
\[ \text{Image}(H^\ell(S, \mathbb{Q}) \to H^\ell(\mathbb{C}(S))) \simeq \frac{H^\ell(S, \mathbb{Q})}{N^1H^\ell(S, \mathbb{Q})}. \]
Thus:
\[
\langle \ , \rangle : \text{CH}^k(X_L) \otimes H^{2n-2k+\ell}(X, \mathbb{Q}) \to \frac{H^\ell(S, \mathbb{Q})}{\text{N}^1H^\ell(X, \mathbb{Q})}.
\]

The easiest way to explain the Theorem 5.6 is to relate it to a certain pairing between cohomology and higher Chow groups, generalizing the Salberger duality pairing above. A generalization of that pairing for \(\text{CH}^k(X, m; \mathbb{Q})\) appears [Lw3]. We present a simplified version, sufficient for our needs here. We refer to the notation in diamond diagram (5.1) above, with \(m \geq 1\). Then there is a pairing defined in the obvious way:

(6.1) \[
\langle \ , \rangle : \text{CH}^k(S \times X, m; \mathbb{Q}) \otimes H^{n-k+\ell,n-k+m}(X) \to H^{\ell-1,0}(S).
\]

The trick is to relate (6.1) to the level of \(\text{CH}^k(X, m; \mathbb{Q})\). In fact, if we view \(X\) and \(S\) as defined over an algebraically closed field \(k\) of finite transcendence degree over \(\mathbb{Q}\), and choose an embedding \(L := \overline{k}(S) \hookrightarrow \mathbb{C}\), and consider the [known injective] pullback \(\text{CH}^k(X_L, m; \mathbb{Q}) \hookrightarrow \text{CH}^k(X = X/\mathbb{C}, m; \mathbb{Q})\), one can argue that the pairing in (6.1) is zero if
\[
\text{Level}(\text{CH}^k(X, m; \mathbb{Q})) < \ell - m.
\]

Now let \(S\) and \(w \in \text{CH}^k(S \times X, m)\) be given such that the corresponding subspace in \(H^{\{k,\ell,m\}}(X)/H^{\{k-m,\ell-m,0\}}(X) \neq 0\). If
\[
\text{Level}(\text{CH}^k_{\text{incl}}(X, m; \mathbb{Q})) < \ell - m,
\]
then one can argue that
\[
\langle w, - \rangle : \{H^{\{k-m,\ell-m,0\}}(X)\}^\perp \to H^{\ell-1,0}(S),
\]
is zero, where
\[
\{H^{\{k-m,\ell-m,0\}}(X)\}^\perp \subset H^{n-k+\ell,n-k+m}(X)
\]
\[
= \{v \mid \langle v, H^{\{k-m,\ell-m,0\}}(X) \rangle = 0\}.
\]
But by Serre duality, this in turn violates the assumption that
\[
H^{\{k,\ell,m\}}(X)/H^{\{k-m,\ell-m,0\}}(X) \neq 0,
\]
a contradiction.

7. Basic strategy for Hodge-\(\mathcal{D}\) on K3

In the next three sections, we will prove the first part of Theorem 1.1, i.e., Hodge-\(\mathcal{D}\) conjecture for a general K3 surface. As mentioned at the beginning, our basic strategy is to degenerate a general K3 surface to a K3 surface with maximum Picard number and study the degeneration of the higher Chow cycles given by (1.1).

Let \(X/\Delta\) be a family of K3 surfaces over disk \(\Delta\), where the central fiber \(X_0\) has Picard number 20. Let \(F_1, F_2, \ldots, F_{20}\) be the generators of \(\text{Pic}(X_0) = \mathbb{Z}^{20}\).
On every fiber $X_t$ for $t \neq 0$, we construct a higher Chow cycle $\varepsilon_t$ in the way of (1.1). We will show that for each $\alpha$, there are some good choices of $\varepsilon_t$ such that the limit $\lim_{t \to 0} \text{cl}_{2,1}(\varepsilon_t) = c_1(F_\alpha)$.

Here we have to say something about taking limit of higher Chow cycles and regulators. Given a family of curves $D \subset X$, even if $D_t$ is reduced and irreducible, the limit $\lim_{t \to 0} D_t$ could very well be reducible and nonreduced. So instead of working with $D$, we prefer to work with its stable reduction. It naturally leads to the following definition.

Recall that a map $\phi : C \to S$ is called stable if

1. $C$ is of normal crossing, i.e., only has nodes as singularities;
2. every contractible component of $C$ under $\phi$ meets the rest of $C$ at no less than three distinct points.

We call $\phi$ prestable if we drop the second condition.

For a surface $S$, we define $\widetilde{\text{CH}}^2(S, 1)$, called the higher Chow group of prestable maps to $S$, which consists of cycles of the form

\[(7.1) \quad \sum_i (f_i, \phi_i : C_i \to S)\]

where $\phi_i : C_i \to S$ is a prestable map from a curve $C_i$ to $S$, $f_i$ is a rational function on $C_i$ such that $f_i \not\equiv 0$ when restricted to every irreducible component of $C_i$ and

\[(7.2) \quad \sum_i \sum_{M \subset C_i} \text{div} ((\phi_i)_* f_i|_M) = 0,\]

where $M$ runs over all irreducible components of $C_i$ with $(\phi_i)_* M \neq 0$ and $(\phi_i)_* f_i$ is defined as follows. Let $N = \phi_i(M)_{\text{red}}$ be the reduced image of $M$ under $\phi_i$ and where $(\phi_i)_* f_i|_M \in \mathbb{C}(N)^*$ is defined by

\[(7.3) \quad (\phi_i)_* f_i(p) = \prod_{\alpha} (f_i(q_\alpha))^{m_\alpha}\]

for $p \in N$, where $\phi_i^* p = \sum_{\alpha} m_\alpha q_\alpha$ with $m_\alpha \in \mathbb{Z}$ and $q_\alpha \in M$. Note that $(\phi_i)_* f_i$ is nothing but the norm of $f_i$ under the field extension $\mathbb{C}(M)/\mathbb{C}(N)$. Of course, if either $M$ or $N$ is singular, we pass the definition to its normalization.

We have the regulator map $\widetilde{\text{cl}}_{2,1} : \widetilde{\text{CH}}^2(S, 1) \to H^{1,1}(S, \mathbb{R})^{\vee}$ defined by

\[(7.4) \quad \widetilde{\text{cl}}_{2,1}(\varepsilon)(\omega) = \frac{1}{2\pi i} \sum_i \sum_{M \subset C_i} \int_M \phi_i^* \omega \log |f_i|\]

for $\varepsilon = \sum_i (f_i, \phi_i : C_i \to S)$ and $\omega \in H^{1,1}(S, \mathbb{R})$, where $M$ runs over all irreducible components of $C_i$.

There is a natural projection $\varphi : \widetilde{\text{CH}}^2(S, 1) \to \text{CH}^2(S, 1)$ given by

\[(7.5) \quad \varphi \left( \sum_i (f_i, \phi_i : C_i \to S) \right) = \sum_i \sum_{M \subset C_i} ((\phi_i)_* f_i, \phi_i(M)_{\text{red}})\]
where $M$ runs over all irreducible components of $C_i$ with $(\phi_i)_* M \neq 0$.

It is not hard to check that
\begin{equation}
\tilde{\text{cl}}_{2,1} = \text{cl}_{2,1} \circ \varphi
\end{equation}
and it is obvious that $\varphi$ is onto. So it suffices to prove that $\tilde{\text{cl}}_{2,1} \otimes \mathbb{R}$ is surjective for a general K3 surfaces.

Let $X/\Delta$ be a smooth family of projective surfaces. It is convenient to define the relative higher Chow group $\widetilde{\text{CH}}^2(X/\Delta, 1)$ of prestable maps, which consists of cycles in the form
\begin{equation}
\tilde{\varepsilon} = \sum_i (f_i, \phi_i : Y_i \to X)
\end{equation}
where $\phi_i : Y_i \to X$ is a flat family of prestable maps with the diagram
\begin{equation}
\begin{array}{ccc}
Y_i & \xrightarrow{\phi_i} & X \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\omega_t} & X
\end{array}
\end{equation}
$f_i \in C(Y_i)^*$ flat over $\Delta$ and $\tilde{\varepsilon}_t \in \widetilde{\text{CH}}^2(X_t, 1)$ for every $t \in \Delta$.

There is a good notion of taking limit of $\tilde{\text{cl}}_{2,1}$ over a family of surfaces $X/\Delta$. Namely, we have the following.

**Proposition 7.1.** Let $X/\Delta$ be a smooth family of projective surfaces and $\tilde{\varepsilon} \in \widetilde{\text{CH}}^2(X/\Delta, 1)$. Then
\begin{equation}
\lim_{t \to 0} \tilde{\text{cl}}_{2,1}(\tilde{\varepsilon}_t)(\omega_t) = \tilde{\text{cl}}_{2,1}(\tilde{\varepsilon}_0)(\omega_0)
\end{equation}
for any real $(1, 1)$ form $\omega_t$ of $X_t$ that varies continuously with respect to $t$.

### 8. Rational Curves on K3 Surfaces

#### 8.1. Degeneration of K3 surfaces.
Consider a K3 surface with Picard lattice
\begin{equation}
\begin{pmatrix}
-2 & 1 \\
1 & 0
\end{pmatrix}
\end{equation}
Such a surface $S$ can be realized as an elliptic fibration over $\mathbb{P}^1$ with fiber $F$ and a unique section $C$, where $C^2 = -2$, $C \cdot F = 1$ and $F^2 = 0$. There are exactly 24 nodal rational curves in the linear series $[F]$. Such surfaces were used by J. Bryan and N.C. Leung in the enumerative problems on K3 surfaces [B-L]. We will call $S$ a BL K3 surface as in [C2]. One reason why such surfaces are so useful in the study of curves on a general K3 surface is that it lies on every component of the moduli space of algebraic K3 surfaces. That is, a general K3 surface of genus $g$ (so the primitive class has self-intersection $2g - 2$) can be degenerated to a BL K3 surface with the primitive class degenerated to $C + gF$. In addition, a curve $D$ in the linear series $|C + gF|$ is very easy to describe; it is the union of $C$ and $g$ elliptic
“tails”, i.e., $D = C \cup F_1 \cup F_2 \cup ... \cup F_g$, where $F_i \in |F|$. Moreover, if $D$ is the limit of a family of rational curves, $F_i$ must be one of the 24 rational curves; $D$ could be nonreduced and it must be if $g > 24$.

As mentioned before, we need to degenerate a $K3$ surface to one with maximum Picard number. Let us consider a BL $K3$ surface with Picard number 20. The extra Picard number comes from $-2$ curves now appearing on the singular fibers of $S \to \mathbb{P}^1$. Suppose that $S$ have $s$ singular fibers $F_1, F_2, ..., F_s$ and each $F_i$ is a closed chain of $r_i$ rational curves (see Figure 1). The Picard number is $\sum r_i - s + 2 = 20$ and the total number of nodes of $F_i$ is $\sum r_i = 24$. Therefore $s = 6$. We will consider $S$ with $r_i = 4$ for $i = 1, 2, ..., 6$. That is, $S \to \mathbb{P}^1$ is an elliptic fibration with 6 singular fibers with each singular fiber a union of 4 rational curves (see Figure 2).

![Figure 1. A BL K3 surface with large Picard number](image)

![Figure 2. The BL K3 surface we will use](image)

Let $X/\Delta$ be a one-parameter family of $K3$ surfaces of genus $g$, where $\Delta$ is a disk parameterized by $t$. The central fiber $X_0 = S$ is a BL $K3$ surface described as above, given in fig. 2. We want to construct higher Chow cycles on $S$ using rational curves. However, a limiting rational curve (i.e. a curve on $X_0$ which is the limit of rational curves on the general fibers $X_t$) in $|C + gF|$ has the form $C + \sum_{i=1}^{6} m_i F_i$. So any two curves $D_1$ and $D_2$ in this
form meet improperly; indeed, \( D_1 \) and \( D_2 \) do not meet properly anywhere on \( S \). Thus we cannot construct a higher Chow cycle from \( D_1 \) and \( D_2 \) in the way of (1.1). To overcome this obstacle, we use a construction of \([C2]\): we blow up \( X \) along \( F_i \).

Let \( \widetilde{X} \) be the blowup of \( X \) along \( F_1, F_2, \ldots, F_6 \). Let us study the behavior of the blowup along \( E = F_i \).

8.2. The blowup of \( X \) along \( E \). Let \( N_{E/X} \) be the normal bundle of \( E \) in \( X \). Here normal bundles are defined as the duals of corresponding conormal bundles, as opposed to the cokernels of the maps on the tangent spaces. We have the following exact sequence

\[
0 \rightarrow N_{E/S} \rightarrow N_{E/X} \rightarrow N_{S/X}|_E \rightarrow 0
\]

Notice that \( \text{Ext}(\mathcal{O}_E, \mathcal{O}_E) = H^1(\mathcal{O}_E) = \mathbb{C} \) and hence the above sequence might not split. Actually, this is always the case as long as the family \( X \) is general enough. We will sketch the argument for this fact below. For details, please see \([C2]\).

The long exact sequence associated to (8.2) is

\[
0 \rightarrow H^0(N_{E/S}) \rightarrow H^0(N_{E/X}) \rightarrow H^0(N_{S/X}|_E) \rightarrow H^1(N_{E/S})
\]

Notice that \( \text{Ext}(\mathcal{O}_E, \mathcal{O}_E) = H^1(\mathcal{O}_E) = \mathbb{C} \) and hence the above sequence might not split. Actually, this is always the case as long as the family \( X \) is general enough. We will sketch the argument for this fact below. For details, please see \([C2]\).

The long exact sequence associated to (8.2) is

\[
0 \rightarrow H^0(N_{E/S}) \rightarrow H^0(N_{E/X}) \rightarrow H^0(N_{S/X}|_E) \rightarrow H^1(N_{E/S})
\]

Obviously, (8.2) splits if and only if the last map

\[
H^0(N_{S/X}|_E) \rightarrow H^1(N_{E/S})
\]

is zero. We have a natural identification \( H^0(N_{S/X}|_E) \) with \( T_{\Delta,0} \), the tangent space of \( \Delta \) at the origin. It is easy to see that (8.4) actually factors through the Kodaira-Spencer map associated to \( X \), i.e., we have

\[
H^0(N_{S/X}|_E) \cong T_{\Delta,0} \xrightarrow{\text{ks}} H^1(T_S) \rightarrow H^1(T_S|_E) \cong H^1(N_{E/S}) = \mathbb{C}
\]

where \( \text{ks} \) is the Kodaira-Spencer map. Here we claim that the last map \( H^1(T_S|_E) \rightarrow H^1(N_{E/S}) \) is an isomorphism by the following argument. First, we denote by \( \Omega_V \) the cotangent sheaf of a variety \( V \).

By the standard exact sequence

\[
0 \rightarrow N_{E/S}^\vee \rightarrow \Omega_S|_E \rightarrow \Omega_E \rightarrow 0,
\]

we have the exact sequence

\[
H^0(N_{E/S}) \rightarrow \text{Ext}(\Omega_E, \mathcal{O}_E) \rightarrow H^1(T_S|_E) \rightarrow H^1(N_{E/S}) \rightarrow 0.
\]

Notice that \( H^0(N_{E/S}) = \mathbb{C} \) classifies the embedded deformations of \( E \subset S \) and \( \text{Ext}(\Omega_E, \mathcal{O}_E) = \mathbb{C} \) classifies the versal deformations of \( E \). To show that \( H^0(N_{E/S}) \) maps nontrivially to \( \text{Ext}(\Omega_E, \mathcal{O}_E) \), it suffices to show that as \( E \) varies in the pencil \( |\mathcal{O}_S(E)| \), the corresponding Kodaira-Spencer map to the tangent space of the versal deformation space of \( E \) at the origin is nontrivial,
or equivalently, the map to the versal deformation space of $E$ is unramified over the origin. To see this has to be true, we only need to localize the problem at a node $p$ of $E$: if the map to the versal deformation space is ramified over the origin, then $S$ is locally given by $xy = t^a$ at $p$ for some $\alpha > 1$; however, this is impossible since $S$ is smooth at $p$. This justifies that $H^1(T_S|_E) \to H^1(N_{E/S})$ is an isomorphism.

Since the deformation of $X_0$ in $X$ preserves the primitive class $C + gF$, the Kodaira-Spencer class $\text{ks}(\partial/\partial t)$ lies in the subspace of $H^1(T_S)$ given by

$$V = \{ v \in H^1(T_S) : \langle v, c_1(C + gF) \rangle = 0 \},$$

the pairing $\langle \cdot, \cdot \rangle$ is given by Serre duality $H^1(T_S) \times H^1(\Omega_S) \to \mathbb{C}$. If $X$ is chosen general, $\text{ks}(\partial/\partial t)$ is general in $V$.

We claim that the kernel of the map $H^1(T_S) \to H^1(T_S|_E)$ is precisely the subspace

$$W = \{ w \in H^1(T_S) : \langle w, c_1(F) \rangle = 0 \}.$$

We have the exact sequence

$$H^1(T_S(-E)) \xrightarrow{f} H^1(T_S) \to H^1(T_S|_E) \cong H^1(N_{E/S}) = \mathbb{C}. $$

So the kernel of $H^1(T_S) \to H^1(T_S|_E)$ is the image $\text{Im } f$ of $f$. We claim that

$$\text{Im } f = W. $$

By Kodaira-Serre duality, we have the following commutative diagram:

$$
\begin{array}{ccc}
H^1(T_S(-E)) & \xrightarrow{\sim} & H^1(\Omega_S(E)) \\
\downarrow f & & \downarrow g \\
H^1(T_S) & \xrightarrow{\sim} & H^1(\Omega_S)^\vee \\
\end{array}
$$

So we may identify the map $f$ with $g : H^1(\Omega_S(-E)) \to H^1(\Omega_S)$, which is the same as

$$\frac{d}{d t} \frac{\partial}{\partial t} : H^1(\partial S(-E)) \to H^1(\partial S)$$

on the Dolbeaut cohomologies. For any $\psi \in H^1(\partial S(-E))$, we have

$$\int_S g(\psi) \wedge c_1(E) = \int_E g(\psi) = 0.$$

So $\text{Im } f \subset W$. Since $W$ has codimension one in $H^1(T_S)$ and (8.10) is exact, we necessarily have (8.11). This justifies our claim.

Obviously, $V \subset W$ and hence $\text{ks}(\partial/\partial t)$ maps nontrivially to $H^1(T_S|_E)$. Therefore, the map (8.4) is nonzero and (8.3) does not split for $X$ general. Actually, from the above argument we see that (8.4) splits if and only if $\text{ks}(\partial/\partial t) \in V \cap W$, which happens if the deformation of $X_0$ in $X$ preserves both $C$ and $F$, i.e., the general fibers $X_t$ are also BL $K3$ surfaces.

Let $R = \mathbb{P}N_{E/X} \subset \tilde{X}_0$ be the exceptional divisor over $E$, where $\tilde{X}_0$ is the central fiber of the blowup $\tilde{X}$, i.e., the total transform of $X_0$ under the
blowup. We continue to use $S$ to denote the proper transform of $S$ under $\tilde{X} \to X$. The two surfaces $S$ and $R$ intersect transversely along a curve, which maps isomorphically to $E$. Again, we continue to use $E$ for this curve. Obviously, $E = S \cap R$ corresponds to a nonzero section in $H^0(N^\vee_{E/X})$. Since (8.2) does not split, $H^0(N^\vee_{E/X}) = \mathbb{C}$ and $E$ is the only section in the linear series $\mathbb{P}H^0(O_R(E))$; as we will see, this is the key fact which makes the geometry of $R$ interesting.

If (8.2) were to split, then $R \cong \mathbb{P}N_{E/X}$ is simply $E \times \mathbb{P}^1$, which is the trivial ruled surface over $E$; in our case, it does not split so we call $R$ the twisted ruled surface over $E$.

Another important fact about $\tilde{X}$ is that it is singular and has exactly four rational double points over the four nodes of $E$. Let $E = E_0 \cup E_1 \cup E_2 \cup E_3$ and $q_i = E_{i-1} \cap E_i$ for $i \in \mathbb{Z}$, where we let $q_i = q_{i+4}$ and $E_i = E_{i+4}$ for convenience (see Figure 2). Then for each $i$, there is a rational double point $r_i$ of $\tilde{X}$ lying on the fiber of $R \to E$ over $q_i$.

### 8.3. Construction of the twisted ruled surface $R$

Let $p : R \to E$ be the projection and $R_i = p^{-1}E_i$. Obviously, $R_i \cong \mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$. Let $Q_i = p^{-1}(q_i)$ be the fiber over $q_i$. An alternative way to construct $R$ is to glue four copies of $\mathbb{F}_0$ along $Q_i$. The question is how to glue.

Let $\phi_{i,i+1} : Q_i \to Q_{i+1}$ be the map sending a point $x \in Q_i$ to $y = \phi_{i,i+1}(x) \in Q_{i+1}$ such that both $x$ and $y$ lie on a curve in the linear series $|E_i|$. If $y = \phi_{i,i+1}(x)$, we use the notation $\overline{xy}$ to denote the curve in $|E_i|$ passing through $x$ and $y$. Without causing any confusion, let us abbreviate $\phi_{i,i+1}$ to $\phi$ and let $\phi^{-1}$ be the inverse of $\phi$.

Consider $\phi^4 : Q_0 \to Q_0$. For a point $x_0 \in Q_0$, let $x_k = \phi^k(x_0)$. If $x_4 = x_0$, then $\sum_{i=1}^4 x_{i-1}x_i \in |E|$; however, $E$ is the only member in the linear series $|E|$. Therefore, $x_4 = x_0$ if and only if $x_0 = q_0$. Consequently, $\phi^4 \in \text{Aut}(Q_0) \cong PGL(2)$ has exactly one fixed point. If we represent $\phi^4$ by a $2 \times 2$ matrix, then this matrix has exactly one eigenvector and is hence equivalent to

\[
\begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}
\]

for some $\lambda \neq 0$. More explicitly, if we pick the coordinates on $Q_0 \cong \mathbb{P}^1$ such that $q_0 = \infty$ becomes the point at infinity, then $\phi^4$ is given by

\[
\phi^4(y) = y + \lambda.
\]

The figure below (Figure 3) shows how to glue four copies of $\mathbb{F}_0$ to obtain $R$.

Let $r_i^{(k)} = \phi^k(r_i)$ for $k \in \mathbb{Z}$. Consider the six points

\[
q_0, r_0, r_0^{(4)}, r_0^{(3)}, r_0^{(2)}, r_0^{(1)} \in Q_0 \cong \mathbb{P}^1.
\]

One can think of $(q_0, r_0, r_0^{(4)}, r_0^{(3)}, r_0^{(2)}, r_0^{(1)}) \in \mathcal{M}_{0,6}$, where $\mathcal{M}_{0,n}$ is the moduli space of $n$ ordered points on $\mathbb{P}^1$. We claim that the moduli of these
six points is determined by the Kodaira-Spencer class $ks(\partial/\partial t)$ of $X$ and if $ks(\partial/\partial t)$ is general in $V$ given by (8.8), then the corresponding point in $\mathcal{M}_{0,6}$ is general. Namely, we have the following lemma, which we will prove later.

**Lemma 8.1.** There is a well-defined rational map $\mathbb{P}V \to \mathcal{M}_{0,6}$ which sends

$$v \to (q_0, r_0^{(4)}, r_1^{(3)}, r_2^{(2)}, r_3^{(1)})$$

where $q_0, r_i, \phi$ are obtained from a family $X$ with Kodaira-Spencer class $v$. Furthermore $\mathbb{P}V$ dominates $\mathcal{M}_{0,6}$ under this map.

By the above lemma, we see that $r_i^{(k)} \neq r_j^{(l)}$ for all $k, l \in \mathbb{Z}$ provided $i \neq j$.

### 8.4. Construction of Limit Rational Curves on $\tilde{X}_0$.

Let $q = C \cap E$, where we recall that $C$ is the unique section $S \to \mathbb{P}^1$ with $C^2 = -2$. Without loss of generality, we assume that $q \in E_0$.

Let $\Gamma$ be a curve on $\tilde{R}$ satisfying the following conditions.

1. $\Gamma$ is reduced and $E_i, Q_i \not\subset \Gamma$ for all $E_i$ and $Q_i$.
2. Let $\Gamma_i = \Gamma \cap R_i$. Then $\Gamma_i \in |mE_i|$ for $R_i \neq R_0$; $\Gamma_0 = N \cup G$, where $N$ is an irreducible curve in $|Q_0 + \mu E_0|$, $q \in N$ and $G \in |(m - \mu)E_0|$, and where $m, \mu$ are non-negative integers.
3. $\Gamma$ consists of $N$ and chains of curves in the form

$$\sum_{j=k}^{l-1} r_i^{(j)} r_i^{(j+1)}$$

where $k \leq 0$, $l \geq 0$, $r_i^{(k)} \in Q_1$, $r_i^{(l)} \in Q_0$ (i.e. $i \equiv 1 - k \equiv -l \pmod{4}$) and $N$ passes through $r_i^{(k)}$ and $r_i^{(l)}$.
4. $N$ meets $Q_0$ and $Q_1$ transversely everywhere and if $N$ meets $Q_0$ or $Q_1$ at some point $u$, then $\Gamma$ contains a chain of curves in the form of (8.19) with $u = r_i^{(k)}$ or $r_i^{(l)}$. 

**Figure 3.** Glue $R_i$ to obtain $R$.
For examples of such $\Gamma$, please see the figures in the next section.

We claim that such $\Gamma$ is part of a limiting rational curve. More precisely, we have the following.

**Proposition 8.2.** Let $D$ be a curve on $\tilde{X}_0$ with $\nu_* D \in |C + gF|$, where $\nu$ is the blowup map $\tilde{X} \to X$. Assume that for each $E = F_s$ ($s = 1, 2, ..., 6$), the corresponding $\Gamma = D \cap R$ satisfies all the conditions (1)-(4) listed above. Then $D$ can be deformed to rational curves on the general fibers $\tilde{X}_t$. More precisely, there exists a family of stable rational maps $\pi : Y \to \tilde{X}$ with the diagram

\[ Y \xrightarrow{\tilde{\pi}} \tilde{X} \]

\[ \pi \downarrow \]

\[ \downarrow \]

\[ X \downarrow \]

\[ \Delta \]

such that $D = \tilde{\pi}_* Y_0$.

The proof of the above proposition is not hard since $D$ is reduced and has only nodes as singularities. We just have to figure out when $D$ deforms, which nodes of $D$ remain as nodes and which are smoothed out. Using the deformational argument in [C1] and [C2], we see that $N \cap G$ and $r_i \in \Gamma$ remain as nodes and the rest are smoothed out. As a consequence, we can describe $Y_0$ as follows.

**Proposition 8.3.** Let $Y, D, \Gamma$ be given as in Proposition 8.2 and let $\tilde{\Gamma} \subset Y_0$ be the pre-image of $\Gamma$. Then $\tilde{\Gamma}$ meets the rest of $Y_0$ at a point over $q$. And

1. $\tilde{\Gamma}$ consists of $\tilde{N}$, which dominates $N$, and chains of rational curves attached to $\tilde{N}$;
2. each chain of curves (8.19) on $\Gamma$ breaks into (at most) two chains of curves on $\tilde{\Gamma}$: one dominates

\[ \sum_{j=k}^{1} r_i^{(j)} r_i^{(j+1)} \]

the other dominates

\[ \sum_{j=0}^{l-1} r_i^{(j)} r_i^{(j+1)} \]

and they meet $\tilde{N}$ at points over $r_i^{(k)}$ or $r_i^{(l)}$.

We certainly did not give all possible limiting rational curves in Proposition 8.2. The curve $D$ described there only represents a small fraction of all possible degenerations of rational curves on the general fiber. Using the
argument in [C2], one can classify all limiting rational curves. But there is no need for that here. We only need those \( D \)'s described above.

9. Proof of Hodge-D-conjecture for General K3 Surfaces

9.1. Construction of higher Chow cycles. Consider a pair of curves \( \Gamma, \Sigma \subset R \), satisfying (1)-(4) in 8.4:

\[
\Gamma = N_\Gamma + \sum_{j=-3}^{-1} r_0^{(j)} r_0^{(j+1)}
\]

where \( N_\Gamma \subset R_0 \) is the unique curve in \( |E_0 + Q_0| \) passing through \( r_0^{(-3)}, r_0, q \);

\[
\Sigma = N_\Sigma + \sum_{j=0}^{2} r_1^{(j)} r_1^{(j+1)} + \sum_{j=-1}^{1} r_2^{(j)} r_2^{(j+1)}
\]

where \( N_\Sigma \subset R_0 \) is the unique curve in \( |E_0 + 2Q_0| \) passing through the five points \( r_1, r_1^{(3)}, r_2^{(-1)}, r_2^{(2)}, q \) (see Figure 4 and 5).

Obviously, \( \Gamma \) and \( \Sigma \) meet at three points \( \{u, v, q\} = N_\Gamma \cap N_\Sigma \).

By Proposition 8.2, there exists families of stable rational maps \( \bar{\pi}_Y : Y \to \tilde{X} \) and \( \bar{\pi}_Z : Z \to \tilde{X} \) with the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\bar{\pi}_Y} & \tilde{X} \\
\downarrow{\pi_Y} & & \downarrow{\pi_Z} \\
X & & Z \\
\downarrow{\pi_2} & & \downarrow{\pi_Z} \\
\Delta & & \\
\end{array}
\]

such that \( (\pi_Y)_*Y_0, (\pi_Z)_*Z_0 \in |C + gF| \), \( \Gamma = R \cap (\bar{\pi}_Y)_*Y_0 \) and \( \Sigma = R \cap (\bar{\pi}_Z)_*Z_0 \).

Figure 4 shows \( \Gamma \) and its pre-image on \( Y_0 \) and Figure 5 shows \( \Sigma \) and its pre-image on \( Z_0 \).

\[ \text{Figure 4. Curve } \Gamma \]
Also \( \tilde{\pi}_Y(Y_0) \) and \( \tilde{\pi}_Z(Z_0) \) meet transversely at \( u \) and \( v \). Therefore, \( \tilde{\pi}_Y(Y_t) \)
and \( \tilde{\pi}_Z(Z_t) \) meet at two points in the neighborhood of \( u \) and \( v \), respectively.
That is, there exist sections \( U_Y \subset Y \) of \( Y/\Delta \) and \( U_Z \subset Z \) of \( Z/\Delta \) such that
\( u \in \tilde{\pi}_Y(U_Y) = \tilde{\pi}_Z(U_Z) \); there exist sections \( V_Y \subset Y \) of \( Y/\Delta \) and \( V_Z \subset Z \) of
\( Z/\Delta \) such that \( v \in \tilde{\pi}_Y(V_Y) = \tilde{\pi}_Z(V_Z) \).

We construct a cycle
\[
(9.4) \quad \bar{\epsilon} = (f_Y, \pi_Y : Y \to X) + (f_Z, \pi_Z : Z \to X)
\]
of \( \text{CH}^2(X/\Delta, 1) \) with
\[
(9.5) \quad (f_Y) = (U_Y) - (V_Y) \quad \text{and} \quad (f_Z) = (V_Z) - (U_Z).
\]

9.2. Computation of \( \bar{\epsilon}_0 \). Thanks to our knowledge about \( R \) obtained in
8.3, we can make explicit calculation of \( \bar{\epsilon}_0 \).

Let \( R_0 = E_0 \times Q_0 \) be parametrized by affine coordinates \((x, y)\) where \( x \)
and \( y \) are the affine coordinates of \( E_0 \) and \( Q_0 \), respectively. We choose \((x, y)\)
such that \( E_0 = \{y = \infty\}, Q_0 = \{x = 0\}, Q_1 = \{x = \infty\} \) and \( q = (1, \infty) \).

Assume \( r_1 = (\infty, y_1), r_2^{(-1)} = (\infty, y_2) \) and \( r_0^{(-3)} = (\infty, y_0) \). And assume
that \( \phi^4 \) is given by (8.16). We may choose \((x, y)\) such that \( \lambda = 1 \). Then
\( r_1^{(3)} = (0, y_1 + 1), r_2^{(2)} = (0, y_2 + 1) \) and \( r_0 = (0, y_0 + 1) \).

Then \( N_\Gamma \) is given by
\[
(9.6) \quad x(y - y_0) - (y - y_0 - 1) = 0
\]
and \( N_\Sigma \) is given by
\[
(9.7) \quad x(y - y_1)(y - y_2) - (y - y_1 - 1)(y - y_2 - 1) = 0.
\]

Then the \( y \)-coordinates of \( u \) and \( v \) are the two roots of the equation
\[
(9.8) \quad \frac{y - y_0 - 1}{y - y_0} = \frac{(y - y_1 - 1)(y - y_2 - 1)}{(y - y_1)(y - y_2)}.
\]

Let \( \alpha_0 \) and \( \beta_0 \) be the roots of the above equation. Then the restrictions of
\( f_Y \) and \( f_Z \) to \( N_\Gamma \) and \( N_\Sigma \) are
\[
(9.9) \quad g_0(y) = \frac{y - \alpha_0}{y - \beta_0} \quad \text{and} \quad \frac{1}{g_0(y)} = \frac{y - \beta_0}{y - \alpha_0}
\]
and the restrictions of \( f_Y \) and \( f_Z \) to the rest of \( Y_0 \) and \( Z_0 \) are constants (see Figure 4 and 5).

So \( \tilde{\varepsilon}_0 \) looks roughly like this:

\[
\tilde{\varepsilon}_0 = (g_0, N_\Gamma) + \sum \text{(constant, other component of } Y_0 \text{)}
\]

\[
\quad + \left( \frac{1}{g_0}, N_\Sigma \right) + \sum \text{(constant, other component of } Z_0 \text{)},
\]

(9.10)

where the “constants” are the values of \( g_0 \) and \( 1/g_0 \) at the intersections between \( N_\Gamma, N_\Sigma \) and the other components of \( Y_0, Z_0 \). Rigorously, we should write e.g. \( (g_0, N_\Gamma) \) as \( (g_0, \pi_Y : N_\Gamma \to X_0) \) but we want to keep our expression simple.

¿From Figure 4, \( N_\Gamma \) meets \( E_1 \cup E_2 \cup E_3 \) at a point over \( r_0^{(-3)} = (\infty, y_0) \) and the value of \( g_0 \) at \( r_0^{(-3)} \) is \( g(y_0) \). So the restriction of \( f_Y \) to \( E_1 \cup E_2 \cup E_3 \) is the constant \( g(y_0) \). Hence \( \tilde{\varepsilon}_0 \) contains the cycle

\[
(g(y_0), E_1 + E_2 + E_3).
\]

(9.11)

Rigorously, \( E_i \) \( (i = 1, 2, 3) \) is a component of \( Y_0 \) dominating \( E_i \subset X_0 \). But we would not use different notations as it would just make our answer a lot messier than necessary. Also notice that \( N_\Gamma \) meets \( C \) at \( q = (1, \infty) \), where \( g_0 \) has value 1. So the restriction of \( f_Y \) to the rest of \( Y_0 \) is 1. That is, these cycles are killed under the regulator map.

¿From Figure 5, \( N_\Sigma \) meets \( E_3 \cup E_2 \cup E_1 \) at a point over \( r_1^{(3)} = (0, y_1 + 1) \). Hence \( \tilde{\varepsilon}_0 \) contains the cycle

\[
-(g(y_1 + 1), E_3 + E_2 + E_1).
\]

(9.12)

Here, for convenience, we extend the group law of \( \text{CH}^2(S, 1) \) by letting \( (f, D_1) + (f, D_2) = (f, D_1 + D_2) \) and \( n(f, D) = (f^n, D) \) for \( n \in \mathbb{Z} \). Obviously, this is compatible with the regulator map.

Similarly, \( N_\Sigma \) meets \( E_3 \cup E_2 \) at a point over \( r_2^{(2)} = (0, y_2 + 1) \) and \( N_\Sigma \) meets \( E_1 \) at a point over \( r_2^{(-1)} = (\infty, y_2) \). Hence \( \tilde{\varepsilon}_0 \) contains the cycles

\[
-(g(y_2 + 1), E_3 + E_2) - (g(y_2), E_1).
\]

(9.13)

Combining (9.11)-(9.13), we have

\[
\tilde{\varepsilon}_0 = (g_0, N_\Gamma) + (g(y_0), E_1 + E_2 + E_3)
\]

\[
\quad - (g_0, N_\Sigma) - (g(y_1 + 1), E_3 + E_2 + E_1)
\]

\[
\quad - (g(y_2 + 1), E_3 + E_2) - (g(y_2), E_1) + (1, *)
\]

(9.14)

where by \((1, *)\) we mean that the rest terms are of the form \((1, D)\).

Recall that \( \varphi \) is the projection \( \text{CH}^2(S, 1) \to \text{CH}^2(S, 1) \) defined in Sec. 7, where \( S = X_0 \). Now we push \( \tilde{\varepsilon}_0 \) forward to \( \varepsilon_0 = \varphi(\tilde{\varepsilon}_0) \). Note that

\[
\varphi(g_0, N_\Gamma) = ((\pi_Y)_* g_0, E_0) \quad \text{and} \quad \varphi(g_0, N_\Sigma) = ((\pi_Z)_* g_0, E_0).
\]

(9.15)

Obviously, \( \text{div}(\pi_Y)_* g_0 = \text{div}(\pi_Z)_* g_0 \). Therefore, \((\pi_Y)_* g_0)/((\pi_Z)_* g_0)\) is a constant. To find this constant, it is enough to evaluate \((\pi_Y)_* g_0\) and \((\pi_Z)_* g_0\)
at some point on $E_0$, say $q$. Since $\pi_Y : N_\Gamma \to E_0$ is one-to-one, the preimage of $q$ is itself and hence $(\pi_Y) \ast g_0(q) = g_0(q) = 1$. And $\pi_Z : N_\Sigma \to E_0$ is two-to-one, the preimage of $q$ consists of $q$ and a point with $y$-coordinate $(y_1 + y_2 + 1)/2$ (solve (9.7) after setting $x = 1$). Therefore,

\[(9.16) \frac{(\pi_Y) \ast g_0}{(\pi_Z) \ast g_0} = \frac{1}{g_0((y_1 + y_2 + 1)/2)}.\]

In conclusion, we have

\[(9.17) \quad \varepsilon_0 = -\left(\frac{g_0(y_1 + y_2 + 1)}{2}, E_0\right)\]

\[+ \left(\frac{g_0(y_0)}{g_0(y_1 + 1)g_0(y_2 + 1)}, E_1 + E_2 + E_3\right)\]

\[+ \left(\frac{g_0(y_2 + 1)}{g_0(y_2)}, E_1\right) + (1, \ast).\]

Therefore, $\text{cl}_{2,1}(\varepsilon_0)$ is

\[(9.18) \quad -\log \left|\frac{g_0(y_1 + y_2 + 1)}{2}\right| c_1(E_0)\]

\[+ \log \left|\frac{g_0(y_0)}{g_0(y_1 + 1)g_0(y_2 + 1)}\right| c_1(E_1 + E_2 + E_3)\]

\[+ \log \left|\frac{g_0(y_2 + 1)}{g_0(y_2)}\right| c_1(E_1).\]

Next, we will change $\Gamma$ and $\Sigma$ to produce more classes in $H^{1,1}(S, \mathbb{R})^\vee$. Let $\Gamma$ and $\Sigma$ be given as follows:

\[(9.19) \quad \Gamma = N_\Gamma + \sum_{j=0}^{2} r_1^{(j)} r_1^{(j+1)}\]

where $N_\Gamma \subset R_0$ is the unique curve in $|E_0 + Q_0|$ passing through $r_1, r_1^{(3)}, q$ and

\[(9.20) \quad \Sigma = N_\Sigma + \sum_{j=-3}^{-1} r_0^{(j)} r_0^{(j+1)} + \sum_{j=-1}^{1} r_2^{(j)} r_2^{(j+1)}\]

where $N_\Sigma \subset R_0$ is the unique curve in $|E_0 + 2Q_0|$ passing through the four points $r_0^{(-3)}, r_0, r_2^{(-1)}, r_2^{(2)}, q$ (compare (9.19) and (9.20) with (9.1) and (9.2)).

The corresponding $\text{cl}_{2,1}(\varepsilon_0)$ is

\[(9.21) \quad \varepsilon_0 = -\left(\frac{g_1(y_2 + y_0 + 1)}{2}, E_0\right)\]

\[+ \log \left|\frac{g_1(y_1 + 1)}{g_1(y_0)g_1(y_2 + 1)}\right| c_1(E_1 + E_2 + E_3)\]

\[+ \log \left|\frac{g_1(y_2 + 1)}{g_1(y_2)}\right| c_1(E_1).\]
where
\begin{equation}
(9.22) \quad g_1(y) = \frac{y - \alpha_1}{y - \beta_1}
\end{equation}

and \(\alpha_1\) and \(\beta_1\) are the two roots of
\begin{equation}
(9.23) \quad \frac{y - y_1 - 1}{y - y_1} = \frac{(y - y_2 - 1)(y - y_0 - 1)}{(y - y_2)(y - y_0)}.
\end{equation}

We may produce one more class by choosing \(\Gamma\) and \(\Sigma\) to be
\begin{equation}
(9.24) \quad \Gamma = N_\Gamma + \sum_{j=-1}^{1} r_2^{(j)} r_2^{(j+1)}
\end{equation}

where \(N_\Gamma \subset R_0\) is the unique curve in \(|E_0 + Q_0|\) passing through \(r_2^{(-1)}, r_1^{(2)}, q\) and
\begin{equation}
(9.25) \quad \Sigma = N_\Sigma + \sum_{j=-3}^{-1} r_0^{(j)} r_0^{(j+1)} + \sum_{j=0}^{2} r_1^{(j)} r_1^{(j+1)}
\end{equation}

where \(N_\Sigma \subset R_0\) is the unique curve in \(|E_0 + 2Q_0|\) passing through the four points \(r_0^{(-3)}, r_0, r_1^{(3)}, q\).

The corresponding \(\text{cl}_{2,1}(\varepsilon_0)\) is
\begin{equation}
(9.26) \quad - \log \left| \frac{g_2\left(\frac{y_0 + y_1 + 1}{2}\right)}{c_1(E_0)} \right| + \log \left| \frac{g_2(y_2 + 1)}{g_2(y_2) g_2(y_1 + 1)} \right| c_1(E_1 + E_2 + E_3)
\end{equation}

where
\begin{equation}
(9.27) \quad g_2(y) = \frac{y - \alpha_2}{y - \beta_2}
\end{equation}

and \(\alpha_2\) and \(\beta_2\) are the two roots of
\begin{equation}
(9.28) \quad \frac{y - y_2 - 1}{y - y_2} = \frac{(y - y_0 - 1)(y - y_1 - 1)}{(y - y_0)(y - y_1)}.
\end{equation}

Finally, we need to show that the \(3 \times 3\) matrix formed by the coefficients of \((9.18), (9.21)\) and \((9.26)\) is invertible. This is easy thanks to Lemma 8.1, which tells us that \(y_0, y_1, y_2\) can be chosen arbitrarily. With the help of a computer, one can easily find some numerical values of \(y_0, y_1, y_2\) for which the matrix is invertible. For example, we find \(y_0 = 0, y_1 = 1/8, y_2 = 1/2\) a good choice. The Maple program we used is available upon request.

Therefore, \(c_1(E_0), c_1(E_1 + E_2 + E_3)\) and \(c_1(E_1)\) are in the image of \(\text{cl}_{2,1} \otimes \mathbb{R}\). Change \(r_2\) to \(r_4\) and by the same argument we will obtain \(c_1(E_3)\) in the image of \(\text{cl}_{2,1} \otimes \mathbb{R}\). Since \(E\) can be chosen to be any of the six singular fibers...
$F_1, F_2, ..., F_6$, we obtain all the classes in Pic($S$) and we are done. It only remains to prove Lemma 8.1.

9.3. Proof of Lemma 8.1. We have the surjective map

$$N_{E/X} \rightarrow \text{Ext}^1(\Omega_E, \mathcal{O}_E) \rightarrow 0$$

(9.29)

where $\text{Ext}^1(\Omega_E, \mathcal{O}_E) = T^1(\mathcal{E})$ is usually called the $T^1$ sheaf of $E$, which is a sheaf supported on the singular locus of $E$, i.e., $q_i$. It is easy to see that $r_i \in Q_i$ is induced by the map

$$N_{E/X} \mid_{q_i} \rightarrow T^1(E) \mid_{q_i}.$$

(9.30)

Therefore, we see that $(q_0, r_0, \phi^4(r_0), \phi^3(r_1), \phi^2(r_2), \phi(r_3)) \in \mathcal{M}_{0,6}$ only depends on the Kodaira-Spencer class of $X$ and the map (8.18) is well defined.

Instead of $\mathcal{M}_{0,6}$, we consider the map $f : \mathbb{P}V \rightarrow (\mathbb{P}^1)^3$ sending $v \in V$ to $(\phi^4(r_1), \phi^3(r_2), \phi(r_3))$ with $(q_0, r_0, \phi^4(r_0))$ fixed at $(0, 1, \infty)$. Obviously, $f$ is dominant if and only if the original map $\mathbb{P}V \rightarrow \mathcal{M}_{0,6}$ is dominant.

Let $V_3 \subset V_2 \subset V_1 \subset V_0 = V$ be a filtration, where

$$V_i = \{v \in V : \langle v, c_1(E_1) \rangle = \langle v, c_1(E_2) \rangle = ... = \langle v, c_1(E_i) \rangle = 0 \}.$$

(9.31)

We claim the following

Claim 9.1. $r_i = \phi(r_{i+1})$ if and only if $\langle v, c_1(E_i) \rangle = 0$.

If the above claim is true, then we have

$$f(\mathbb{P}V_3) \subset f(\mathbb{P}V_2) \subset f(\mathbb{P}V_1) \subset f(\mathbb{P}V_0)$$

and it follows that dim $f(\mathbb{P}V_0) = 3$, i.e., $f$ is dominant. So it remains to justify our claim.

Without the loss of generality, take $E_i = E_1$. Let $Z$ be the blowup of $X$ along $E_1$ with the exceptional divisor $M \cong \mathbb{P}N_{E_1/X}$. By the same argument as in 8.2, we can show that the exact sequence

$$0 \rightarrow N_{E_1/S} \rightarrow N_{E_1/X} \rightarrow N_{S/X} \mid_{E_1} \rightarrow 0$$

(9.33)

splits if and only if $\langle v, c_1(E_1) \rangle = 0$. Therefore, $M \cong \mathbb{F}_2$ if $\langle v, c_1(E_1) \rangle = 0$ and $M \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ otherwise.

Let $\tilde{Z}$ be the blowup of $Z$ along the proper transform of $E_0 \cup E_2 \cup E_3$ under $Z \rightarrow X$. Then $\tilde{Z}$ is actually the resolution of $X$ at the rational double points $r_1$ and $r_2$. We have the diagram

$$\begin{array}{ccc}
\tilde{Z} & \longrightarrow & X \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}$$

(9.34)
Let \( \tilde{M} \subset \tilde{Z} \) be the pullback of \( M \) under \( \tilde{Z} \to Z \). Then we have the diagram
\[
\begin{array}{ccc}
\tilde{M} & \rightarrow & R_1 \\
\downarrow & & \downarrow \\
M & & \\
\end{array}
\]
Both the maps \( \tilde{M} \to M \) and \( \tilde{M} \to R_1 \) are the blowdowns of two \((-1)\)-curves. But the two sets of \((-1)\)-curves are distinct. They can be describe explicitly as follows.

Starting with \( M \), we blow up \( M \) at two points \( p_1 \) and \( p_2 \) to obtain \( \tilde{M} \). Obviously, \( p_1 \) and \( p_2 \) are over \( q_1 \) and \( q_2 \), respectively. Let \( G_1 \) and \( G_2 \) be the two fibers of \( M \to E_1 \) that contain \( p_1 \) and \( p_2 \), respectively, and let \( \tilde{G}_1 \) and \( \tilde{G}_2 \) be the proper transforms of \( G_1 \) and \( G_2 \) under the blow up. It is easy to see that the map \( \tilde{M} \to R_1 \) blows down \( \tilde{G}_1 \) and \( \tilde{G}_2 \) to \( r_1 \) and \( r_2 \). Under such description, one can easily see that \( \phi(r_1) = r_2 \) if \( M \cong \mathbb{F}_2 \) and \( \phi(r_1) \neq r_2 \) if \( M \cong \mathbb{F}_0 \). [Here \( \mathbb{F}_n = \mathbb{F}(O \oplus O(n)) \).]

10. Proof of Theorem 1.1

In [S-S] there is constructed a 19-dimensional “universal” family \( S \) of marked \( K^3 \) surfaces, where roughly speaking, a marked \( K^3 \) surface is a \( K^3 \) surface together with the even unimodular lattice \( L := H_2(X, \mathbb{Z}) \), and with choice of polarization \( \xi \in L \). The special Kummer surfaces form a dense subset of that family. However we learned from the previous sections that the Hodge-\( \mathcal{D} \)-conjecture holds over a real Zariski open subset of that family. Thus it is clear that the Hodge-\( \mathcal{D} \)-conjecture holds for general special Kummer surfaces, as well as for general Kummer surfaces as well. By “lifting” these results to the corresponding Abelian surfaces, we can now prove the following. [General will mean as in Sec. 2, in the real analytic Zariski topology.]

**Theorem 10.1.** (i) The Hodge-\( \mathcal{D} \)-conjecture holds for surfaces of the form \( E_1 \times E_2 \), where \( E_1 \), \( E_2 \) are general elliptic curves.

(ii) The Hodge-\( \mathcal{D} \)-conjecture holds for general Abelian surfaces.

**Proof.** Let \( A \) be an Abelian surface, \( Y = A/\pm 1 \), be the quotient space after applying the involution \( x \mapsto -x \), and \( X \) the corresponding Kummer surface resulting from the blow-up of the 16 double points in \( Y \). It is clear that the cohomology \( H^2(X) \), which is of rank 22, comes from the cohomology of \( H^2(A) \) (of rank 6) together with the 16 rational curves resulting from the aforementioned blow-up process. Let \( w \subset A \times X \) be the correspondence induced by the above process. Then
\[
[w]^* : H^2(X, \mathbb{R}(1)) \to H^2(A, \mathbb{R}(1)),
\]
is a surjective morphism of Hodge structures. However by functoriality, there is a commutative diagram:

\[
\begin{array}{ccc}
\text{CH}^2(X, 1; \mathbb{R}) & \xrightarrow{w^*} & \text{CH}^2(A, 1; \mathbb{R}) \\
\downarrow r_{2,1}^X & & \downarrow r_{2,1}^A \\
H^{1,1}(X, \mathbb{R}(1)) & \xrightarrow{[w]^*} & H^{1,1}(A, \mathbb{R}(1))
\end{array}
\]

But \(r_{2,1}^X\) and \([w]^*\) are surjective, whence \(r_{2,1}^A\) is surjective. \(\square\)

11. Proof of Theorem 1.3

The construction in Sec. 7-9 dealt with \(K_1\) classes on a general \(K_3\) surface \(X\) that degenerate to \(K_0\) classes on a special \(K_3\) surface \(X_0\) with maximum Picard number. Now suppose that we are given \(K_3\) surfaces \(X_1, \ldots, X_M\), and \(K_1\) classes \(\xi_j = \sum_{\alpha_j} (f_{\alpha_j}, D_{\alpha_j})\) with \(\sum_{\alpha_j} \text{div}(f_{\alpha_j}) = 0\) on \(X_j\), \(j = 1, \ldots, M\). Then we can form a \(K_1\) cycle on the product \(X_1 \times \cdots \times X_M\) by the prescription

\[
\xi = \sum_{\alpha_1, \ldots, \alpha_M} (\text{Pr}_1^* f_{\alpha_1} \cdots \text{Pr}_M^* f_{\alpha_M}, D_{\alpha_1} \times \cdots \times D_{\alpha_M}),
\]

where \(\text{Pr}_j : D_{\alpha_1} \times \cdots \times D_{\alpha_M} \to D_{\alpha_j}\) is the canonical projection. Note that \(\xi\) determines a class in \(\text{CH}^{M+1}(X_1 \times \cdots \times X_M, 1)\), and further note that as the \(X_j\)'s degenerate to the special \(X_0\), the class \(\xi\) degenerates to an algebraic cycle lying in \(H^{1,1}(X_0, \mathbb{R}(1))^{\otimes M}\). Indeed one can find \(20^M\) such \(\xi\)'s, which degenerate to a basis of \(H^{1,1}(X_0, \mathbb{R}(1))^{\otimes M}\). Thus for general \(X_1, \ldots, X_M\), \(H^{1,1}(X_1, \mathbb{R}(1)) \otimes \cdots \otimes H^{1,1}(X_M, \mathbb{R}(1))\) lies in the image of the real regulator \(r_{M+1,1} : \text{CH}^{M+1}(X_1 \times \cdots \times X_M, 1; \mathbb{R}) \to H^{2M+1}_D(X_1 \times \cdots \times X_M, \mathbb{R}(M+1))\).

Note that in particular, in light of the previous section, this implies that for a general product \(E_1 \times \cdots \times E_{2M}\) of elliptic curves, \(H^{1,1}(E_1 \times E_2, \mathbb{R}(1)) \otimes \cdots \otimes H^{1,1}(E_{2M-1} \times E_{2M}, \mathbb{R}(1))\) lies in the image of the real regulator. Let general mean as in Sec. 2, with respect to the countable real analytic Zariski topology. We are now in a position to prove the following.

**Theorem 11.1.** Let \(X = E_1 \times \cdots \times E_k\) be a general product of \(k\) elliptic curves, i.e. in the countable real analytic Zariski topology, and \(\text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q})\) the space of indecomposables. Then:

(i) \(\text{Level}(\text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q})) \geq k - 2\).

(ii) In particular for \(k \geq 3\), there are an uncountable number of indecomposables in the kernel of the regulator

\[
\text{cl}_{k,1} : \text{CH}^k_{\text{ind}}(X, 1; \mathbb{Q}) \to H^{2k-1}_D(X, \mathbb{Q}(k)).
\]
Proof. Using the notation just preceeding (11.1), as well as the terminology in (5.5) and (5.7), we put \( k = M + 1, \ X = E_1 \times \cdots \times E_k, \) and let \( S = E_{k+1} \times \cdots \times E_{2k−2}. \) Note that \( \dim S = \ell − 1, \) where \( \ell = k − 1. \) Then the regulator map under consideration is \( \text{cl}_{k,1} : \text{CH}^k(X,1) \to H^{2k−1}_D(X,\mathbb{Q}(k)). \)

Note that for a general product of elliptic curves, \[ N^1H^k(X,\mathbb{Q}) = \sum_{j=1}^{k} \left[ \frac{dx_jd\bar{x}_j}{|y_j|^2} \right] \bigcup H^{k-2}(X,\mathbb{Q}), \]

where \( E_j \) is defined in affine coordinates by the equations \( y_j^2 = h_j(x_j), \) for general cubic polynomials \( h_j(x_j), \) and \( j = 1, \ldots, k. \) It is obvious that \( H^{1,k-1}(X) \neq H^1_N(X). \) Thus by Theorem 5.6,

\[ \text{Level}(	ext{CH}^{k}_{\text{ind}}(X,1;\mathbb{Q})) \geq \ell − 1 = k − 2. \]

Since the Hodge conjecture is known for products of elliptic curves, it follows by Remark 5.7 that \( \ker \text{cl}_{k,1} \) contains an uncountable number of indecomposable elements, provided that \( k > 2. \)

Appendix A

The analogue of a \( K3 \) surface in dimension 1 is an elliptic curve, and the group of interest is \( K_2. \) We prove a version of the Hodge-D-conjecture for general elliptic curves. Let \( X \) be a compact Riemann surface of genus \( g, \) and let \( f, g \in \mathbb{C}(X)^\times. \) For a real form \( \omega \in H^1(X,\mathbb{R}), \) the integral

\[ \{f,g\} \mapsto \int_X \log |f|d\log |g| \wedge \omega \]

induces a map on \( K_2(X); \) more explicitly a map

\[ r_{2,2} : \text{CH}^2(X,2;\mathbb{R}) := H^0_{\text{Zar}}(X,\mathcal{K}_{2,X}) \otimes \mathbb{R} \to H^1(X,\mathbb{R})^\vee \]

\[ \simeq H^1(X,\mathbb{R}(1)) \simeq H^2_F(X,\mathbb{R}(2)). \]

Up to a multiplicative constant and real isomorphism, \( r_{2,2} \) is the real Beilinson regulator \([Lw1]. \) It is well-known that \( r_{2,2} \) is zero for general curves of genus \( g > 1 \) \([Co1], \) is nontrivial for the case of general elliptic curves \((g = 1) \) \([Blo2], [Co1], \) and is trivially surjective for \( g = 0. \) We sketch a proof of:

**Theorem** (Hodge-D for Elliptic Curves). If \( X \) is a general elliptic curve in the real analytic Zariski topology, then \( r_{2,2} \) is surjective.

**Proof.** Let \( X \) be an elliptic curve given in affine coordinates by the equation \( y^2 = h(x), \) where \( h(x) \) is a cubic polynomial with distinct roots. A basis for \( H^1(X,\mathbb{R}) \) is given by

\[ \omega_1 := \frac{dx}{y} + \frac{d\bar{x}}{\bar{y}} \ ; \ \ \omega_2 := \sqrt{-1}\left( \frac{dx}{y} - \frac{d\bar{x}}{\bar{y}} \right). \]
Next, we consider
\[ f_1 := y + x\sqrt{-1} \quad f_2 = y + x \quad g_1 = g_2 = x. \]

We claim that for general \( X \),
\[
\begin{align*}
\text{det} & \left[ \int_X \log |f_1|d\log |g_1| \wedge \omega_1 \quad \int_X \log |f_1|d\log |g_1| \wedge \omega_2 \right] \\
& \neq 0.
\end{align*}
\]

Now let us first assume that \( X \) is given for which (1) holds, and note that the rational functions \( f_1, f_2, g_1, g_2 \) can each be expressed in the form \( L_1/L_2 \), where \( L_j \) are homogeneous linear polynomials in the homogeneous coordinates of \( \mathbb{P}^2 \) (and where \( X \subset \mathbb{P}^2 \)). Since \( X \) has a dense subset of torsion points \( X_{tor} \), and by Abel’s theorem, one can find \( \tilde{L}_j \) “close” to \( L_j, j = 1, 2 \), such that \( \tilde{L}_j \cap X \subset X_{tor} \). Thus \( \tilde{L}_1/\tilde{L}_2 \) is “close” to \( L_1/L_2 \). Thus one can find \( \tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2 \) for which
\[
\left\{ |\text{div}(\tilde{f}_1)| \bigcup |\text{div}(\tilde{f}_2)| \bigcup |\text{div}(\tilde{g}_1)| \bigcup |\text{div}(\tilde{g}_2)| \right\} \subset X_{tor},
\]
and that by continuity considerations
\[
\begin{align*}
\text{det} & \left[ \int_X \log |\tilde{f}_1|d\log |\tilde{g}_1| \wedge \omega_1 \quad \int_X \log |\tilde{f}_1|d\log |\tilde{g}_1| \wedge \omega_2 \right] \\
& \neq 0.
\end{align*}
\]

From the general mechanism in [Blo2], one can complete \( \{\tilde{f}_1, \tilde{g}_1\}, \{\tilde{f}_2, \tilde{g}_2\} \) to classes \( \xi_1, \xi_2 \in \text{CH}^2(X, 2) \), for which
\[
\begin{align*}
\text{det} & \left[ r_{2,2}(\xi_1)(\omega_1) \quad r_{2,2}(\xi_1)(\omega_2) \right] \\
& \neq 0.
\end{align*}
\]

Thus modulo the claim in (1), we are done. We sketch a proof of the claim.

With regard to a volume element \( dV \):
\[
\begin{align*}
d\log |x| \wedge \omega_1 &= \frac{1}{2} \left( \frac{1}{x\bar{y}} - \frac{1}{\bar{x}y} \right) dx \wedge d\bar{x} = \frac{\text{Im}(\bar{xy})}{|x|^2|y|^2} dV \\
d\log |x| \wedge \omega_2 &= -\frac{\sqrt{-1}}{2} \left( \frac{1}{x\bar{y}} + \frac{1}{\bar{x}y} \right) dx \wedge d\bar{x} = -\frac{\text{Re}(\bar{xy})}{|x|^2|y|^2} dV
\end{align*}
\]

Now let us degenerate \( X \) to the rational elliptic curve \( X_0 \) given by \( y^2 = x^3 \).

Note that \( X_0 \) is given parametrically by \( (x, y) = (z^2, z^3), z \in \mathbb{C} \). Thus \( \bar{xy} = |z|^4z, \) and up to a real positive multiplicative constant times the standard volume element on \( \mathbb{C} \), which we will denote by \( dV_0 \), (4) and (5) become:
\[
\begin{align*}
d\log |x| \wedge \omega_1 &= \frac{\text{Im}(z)}{|z|^4} dV_0 \\
d\log |x| \wedge \omega_2 &= -\frac{\text{Re}(z)}{|z|^4} dV_0.
\end{align*}
\]
Let $H = \{ z \in \mathbb{C} \mid \text{Im}(z) \geq 0 \}$ be the upper half plane. Now one has the following formal calculations after degenerating to $X_0$, and using symmetry arguments:

\begin{align*}
(7) \quad & \int_{X_0} \log |f_1| d\log |g_1| \wedge \omega_1 \\
& = \int_{\mathbb{C}} \log |z| d\log |g_1| \wedge \omega_1 = \int_{\mathbb{C}} \log |z + \sqrt{-1}| \frac{\text{Im}(z)}{|z|^4} dV_0 \\
& = \int_{H} \log \left| \frac{z + \sqrt{-1}}{\bar{z} + \sqrt{-1}} \right| \frac{\text{Im}(z)}{|z|^4} dV_0 \to +\infty,
\end{align*}

using the fact

\[ \left| \frac{z + \sqrt{-1}}{\bar{z} + \sqrt{-1}} \right| > 1 \iff \text{Im}(z) > 0. \]

\begin{align*}
(8) \quad & \int_{X_0} \log |f_2| d\log |g_2| \wedge \omega_1 = -\int_{\mathbb{C}} \log |z + \sqrt{-1}| \frac{\text{Re}(z)}{|z|^4} dV_0 = 0. \\
(9) \quad & \int_{X_0} \log |f_2| d\log |g_2| \wedge \omega_1 = \int_{\mathbb{C}} \log |z + 1| \frac{\text{Im}(z)}{|z|^4} dV_0 = 0.
\end{align*}

For the final calculation, put $w = z\sqrt{-1}$, and note that $\text{Re}(z) = \text{Im}(w)$, and that $|z + 1| = |w + \sqrt{-1}|$. Then

\begin{align*}
(10) \quad & \int_{X_0} \log |f_2| d\log |g_2| \wedge \omega_2 = -\int_{\mathbb{C}} \log |z + 1| \frac{\text{Re}(z)}{|z|^4} dV_0 \\
& = -\int_{\mathbb{C}} \log |w + \sqrt{-1}| \frac{\text{Re}(w)}{|w|^4} dV_0 \\
& = -\int_{H} \log \left| \frac{z + \sqrt{-1}}{\bar{z} + \sqrt{-1}} \right| \frac{\text{Im}(z)}{|z|^4} dV_0 \to -\infty.
\end{align*}

Notice that the singularities in the integrals in (7) and (10) occur over the singular point $z = 0$ of the singular curve $X_0$, as expected. By using the Lebesgue theory of integration, we can make the calculations in (7)–(10) above more precise. First, by using the projection $(x, y) \mapsto x$, we have a double covering $X \to \mathbb{P}^1$. Thus for $f, g \in \mathcal{C}(X)$, and $\omega = \omega_1$ or $\omega = \omega_2$, we can express $\int_X \log |f| d\log |g| \wedge \omega$ as the integral of some Lebesgue integrable function $H(x)$ over $\mathbb{P}^1$. Next, by converting to polar coordinates, viz. $x = e^{i\sqrt{-1}}$, we can Fubini integrate in $t \in [0, 2\pi]$ and $r \in [0, \infty]$. Let $h(r)$ be the result of integrating $H(x)$ with respect to $t$ over $[0, 2\pi]$. As $X$ degenerates to $X_0$, we can construct a sequence $\{ h_n(r) \}$ which limits to $h_\infty(r)$ over $X_0$. In the cases of (7)–(10), we have that $h_\infty(r)$ is either zero, nonnegative, or nonpositive. By using the the standard Lebesgue integral limit theorems, we arrive at the claim in (1), and hence the theorem above. \qed
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