NORMAL FUNCTIONS OVER LOCALLY SYMMETRIC VARIETIES

RYAN KEAST AND MATT KERR

Abstract. We classify the irreducible Hermitian real variations of Hodge structure admitting an infinitesimal normal function, and draw conclusions for cycle-class maps on families of abelian varieties with a given Mumford-Tate group.

1. Introduction

Normal functions are holomorphic horizontal sections of the intermediate Jacobian bundle $J(V)$ associated to a variation of Hodge structure $V$ (of odd weight) over a complex analytic manifold. Given a family of smooth projective varieties $\mathcal{X} \rightarrow \mathcal{S}$ and a cycle $3 \in CH^p(\mathcal{X})$ whose restriction $Z_s := 3 \cdot X_s$ to each fiber is homologous to zero, the Abel-Jacobi images $AJ_X^p(Z_s)$ yield such a section $\nu_3$ of $J(V_{2p-1})$. These geometric normal functions and their singularities are at the heart of the reformulation of the Hodge Conjecture by Green and Griffiths [GG], which has brought about a renewal of interest (cf. [KP] and references therein).

The most famous example arises from the image $C^+$ of an algebraic curve $C$ in its Jacobian. Writing $\iota : J(C) \rightarrow J(C)$ for the map sending $u \mapsto -u$, and $C^-$ for $\iota(C^+)$, the Ceresa cycle $Z_C := C^+ - C^- \in CH_1(J(C))$ is homologous to zero, with $AJ_C^{g-1}(Z_C) \in J(H_{2g-3}(J(C)))$ of infinite order for very general $C$ of genus at least 3 [Ce]. By studying cohomology of the mapping class group, Hain has proved (up to a factor of 2) that the resulting section essentially generates all normal functions over $M_g$ [Ha]. By “essentially” we mean up to sections of Jacobians of level-1 variations, which are families of abelian varieties. In this paper we shall mostly ignore these, which in the geometric case corresponds to considering only the normal functions that factor through the Griffiths group of cycles modulo algebraic equivalence. We shall also work rationally, i.e. consider torsion normal functions to be zero.

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A natural question is what happens over $A_g$, or more generally over quotients of the Siegel domain $D = \Pi_g := Sp_{2g}(\mathbb{R})/U(g)$ by a congruence subgroup $\Gamma \leq Sp_{2g}(\mathbb{Q})$. However, Ceresa’s normal function does not even survive the (generically 2:1) map from $M_g$ to $A_g$, let alone extend to the latter. In fact, a result of Ragunathan [Ra] implies a much more general vanishing phenomenon: if $\mathcal{V}$ is a homogeneous VHS (cf. §2.3) over a locally symmetric variety $X := \Gamma\backslash D := \Gamma\backslash G(\mathbb{R})/K$, associated to a nontrivial $\mathbb{Q}$-irrep $V$ of $G$, and $G$ has $\mathbb{Q}$-rank $> 1$, then $J(\mathcal{V})$ admits no normal functions over any Zariski open subset of $X$. Moreover, when the $\mathbb{Q}$-rank is one, the boundary components in the smooth toroidal compactification of $X$ are smooth, and so any admissible normal function would have no singularities to study.

This situation improves somewhat when one considers étale neighborhoods of $X$ instead of Zariski ones, motivated by the example of $M_3$ minus its hyperelliptic locus. To simplify matters, assume that the group $G$ (of any $\mathbb{Q}$-rank), the Hermitian symmetric domain $D$, and the representation $V$ are as above, but also simple resp. irreducible over $\mathbb{R}$, with $V_{\mathbb{R}}$ of highest weight $\lambda$. (We also consider some cases where $G_{\mathbb{R}} = U(1)$ · simple.) One would like to classify the pairs $(D, \lambda)$ for which $\mathcal{V}$ has odd weight and the resulting $J(\mathcal{V}) \to X$ admits a normal function after base change to some étale neighborhood. For this to happen, a certain cohomology sheaf $\oplus j \geq 0 H^1(j)$ of infinitesimal normal functions on $D$ (cf. §2.1) must not vanish, and it is the pairs with this property that we shall classify in this paper using a result of Kostant [Ko] (cf. §§2.2, 2.4). We should note that this approach has already been carried out by Nori [No] in the Siegel domain case, so the results below involving $D = \Pi_g$ are not new.

When $D$ is of tube type, and ignoring variations of weight one, we obtain a complete classification: the pairs include all of the weight-3 Calabi-Yau variations

$$(I_{3,3}, \omega_3), (II_6, \omega_6), (III_3, \omega_3), (EVIII, \omega_7)$$

described by Gross [Gro] and Friedman-Laza [FL], as they obviously must (cf. §2.1). The remainder of the list is rather short: namely, we have

$$(I_{2,2}, \omega_1 + a\omega_2 \text{ or } \omega_3 + a\omega_2), (II_4, \omega_1 + a\omega_3 \text{ or } \omega_1 + a\omega_4),$$

$$(III_1, a\omega_1), (III_2, \omega_1 + a\omega_2), (IV_{2n-1}, a\omega_1 + \omega_n),$$

and $$(IV_{2n-2}, a\omega_1 + \omega_{n-1} \text{ or } a\omega_1 + \omega_n),$$

where $a$ is any positive integer. Our results in the non-tube cases are more subtle and partial; while we enlarge $G$ by a 1-torus and thus allow a variant of the half-twist construction [vG1] (cf. §4), we mainly restrict
to variations of Calabi-Yau type or appearing in the cohomology of abelian varieties of generalized Weil type.

Returning to \( \mathcal{A}_g \), we can ask once more about the loci “supporting” higher-weight normal functions. By this we mean that there is an irreducible homogeneous variation \( \mathcal{V} \) of odd level > 1 over \( \mathcal{A}_g \), a subvariety \( \mathcal{I} \to \mathcal{A}_g \) with étale neighborhood \( \mathcal{T} \to \mathcal{I} \), and a nontorsion horizontal section of \( J(j^*r^*\mathcal{V}) \) over \( \mathcal{T} \). Alternatively, one could consider only those \( \mathcal{V} \) appearing in the cohomology of the universal abelian variety. In either case, the loci \( \iota(\mathcal{I}) \) are proper subvarieties for \( g > 3 \) \([\text{No}]\), and are expected to be more like \( \mathcal{M}_g \) than locally symmetric subvarieties. Indeed, \( \mathcal{M}_g \) supports in this sense the Ceresa normal function for any \( g \); another example is the Fano normal function supported by the locus in \( \mathcal{A}_5 \) of intermediate Jacobians of cubic threefolds \([\text{CNP}]\).

The approach in this article is consequently analogous to the search for special subvarieties of \( \mathcal{M}_g \) \([\text{MO}]\), which are expected not to exist for large enough \( g \). So the following vanishing result, which follows from our classification, should not be surprising:

**Theorem.** The loci of Weil resp. quaternionic abelian varieties in \( \mathcal{A}_g \) do not support higher weight normal functions for \( g > 6 \) resp. 8. In particular, the image of the reduced Abel-Jacobi map

\[
\overline{\text{AJ}} : \text{Griff}^r(A)_\mathbb{Q} \to \overline{\mathcal{I}}(A)_\mathbb{Q}
\]

is zero for a very general Weil resp. quaternionic abelian variety \( A \) of dimension at least 8 resp. 10.

(See Thms. 3.3 and 3.5 in \$3.\) There are related results for cycles in certain codimensions for the generalized Weil abelian varieties over \( \text{I}_{p,q} \) \((p + q = n + 1, p < q)\) studied in \$4.1, which get weaker as \( |q - p| \) grows (cf. Theorem 4.3). However, both here and for the domains \( IV_{m} \) parametrizing “spin abelian varieties” (for instance, those arising from the Kuga-Satake construction), one has infinitesimal normal functions for arbitrarily large \( n \) and \( m \).

For each of the “infinitesimal normal functions” described in this paper, the next immediate problem is to determine whether it comes from an actual (admissible) normal function, and if so, to geometrically realize it and determine (outside the exceptional cases) its locus of support in \( \mathcal{A}_g \). We take a stab at this for \((\text{I}_{3,3}, \omega_3)\) in \$3.5, using the fact that for certain families of Weil abelian 6-folds, the general member is a Prym variety associated to an unramified 3:1 curve covering with base curve of genus \( g = 4 \). The normal function extends to the locus of these generalized Pryms in \( \mathcal{A}_6 \), and the construction works for \( g > 4 \) as well (but the closure of these loci don’t contain an \( \text{I}_{p,p} \)).
With an admissible normal function in hand, one can also try to determine its zero locus, and its singularities along intersections of toroidal boundary components. But even in the absence of this, one can probably still use Kostant’s result, replacing $\mathfrak{g}$ by subalgebras of the form $\mathfrak{sl}_2^{\mathbb{C}}$, to compute spaces of singularity classes for normal functions over homogeneous variations.

An equally intriguing prospect is to try to extend the computation of infinitesimal normal function spaces (or singularity classes) to the nonclassical case, which is more relevant to the Green-Griffiths program. Here the homogeneous families of Hodge structures only become variations upon restriction to the image of a period map. However, the computation only requires an infinitesimal variation as input, so it may already be interesting to start with tangent spaces to the Schubert VHS of $[R\ddot{o}]$.

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2. **Infinitesimal normal functions and Kostant’s theorem**

This section contains the abstract underpinnings of the calculations in §§3-4. Apart from §2.4, it is mostly expository and contains the material on infinitesimal invariants, Lie algebra cohomology, and Hermitian VHS that will be used subsequently.

2.1. **Normal functions and infinitesimal invariants.** Let $\mathcal{V} \to \mathcal{X}$ be a polarized $\mathbb{Q}$-VHS over a complex manifold, pure of weight $-1$. By abuse of notation, $\mathcal{V}$ will denote the underlying holomorphic vector bundle and its sheaf of sections, with Hodge filtration $\mathcal{F}^\bullet$; $\mathcal{V}$ is the underlying $\mathbb{Q}$-local system, $Q : \mathcal{V} \times \mathcal{V} \to \mathcal{Q}_{\mathcal{X}}$ the polarization, and $\nabla$ the Gauss-Manin connection.

Define a filtered complex of sheaves
\begin{equation}
C^\bullet := (\mathcal{V} \otimes \Omega^\bullet_{\mathcal{X}}, \nabla), \quad F^pC^\bullet := \left(\mathcal{F}^p \cdot \mathcal{V} \otimes \Omega^\bullet_{\mathcal{X}}, \nabla\right)
\end{equation}
on $\mathcal{X}$, so that the differential of each
\[Gr_F^pC^\bullet = \left(Gr_F^p \cdot \mathcal{V} \otimes \Omega^\bullet_{\mathcal{X}}, \nabla\right)\]
is $\mathcal{O}_{\mathcal{X}}$-linear. Consider the exact sequence
\begin{equation}
0 \to F^0C^\bullet \to C^\bullet / \mathcal{V} \to C^\bullet / (F^0C^\bullet + \mathcal{V}) \to 0;
\end{equation}
the sheaf of (quasi-)horizontal sections of the Jacobian bundle $J(\mathcal{V})$ is
\[ \mathcal{J}_{hor}^{\mathbb{Q}} = \mathcal{H}^0 \left( \mathbf{C}^* / (F^0 \mathbf{C}^* + \mathbb{V}) \right). \]
The space of normal functions associated to $\mathcal{V}$ is
\[ \text{NF}_\mathcal{S}(\mathcal{V}) := \Gamma \left( \mathcal{S}, \mathcal{J}_{hor}^{\mathbb{Q}} \right), \]
with infinitesimal invariant
\[ \delta : \text{NF}_\mathcal{S}(\mathcal{V}) = \mathbb{H}^0 \left( \mathcal{S}, \frac{\mathbf{C}^*}{F^0 \mathbf{C}^* + \mathbb{V}} \right) \to \mathbb{H}^1 \left( \mathcal{S}, F^0 \mathbf{C}^* \right) \to \Gamma \left( \mathcal{S}, \mathcal{H}_\mathcal{V}^1 (F^0 \mathbf{C}^*) \right) \]
arising from (2.2).

**Lemma 2.1.** If $H^0(\mathcal{S}, \mathcal{V}) = 0$, then $\delta$ is injective.

**Proof.** Since $H^0(\mathcal{S}, \mathcal{V}) = H^0(\mathcal{S}, \mathbb{V}) = H^0(\mathcal{S}, \mathbb{V}) \otimes \mathbb{Q}$, this follows from the long-exact sequence of (2.2). □

**Lemma 2.2.** If $\mathcal{H}_\mathcal{V}^0 (F^0 \mathbf{C}^*)$ and $\mathcal{H}_\mathcal{V}^0 (F^0 \mathbf{C}^*)$ vanish, then $\text{NF}_\mathcal{S}(j^* \mathcal{V}) = \{0\}$ for any étale neighborhood $j : \mathcal{T} \to \mathcal{S}$.

**Proof.** Vanishing of $\mathcal{H}_\mathcal{V}^0 (F^0 \mathbf{C}^*) = \mathbb{V} \cap F^0$ implies $H^0(\mathcal{S}, j^* \mathcal{V}) = \{0\}$; now apply the previous Lemma. □

Now writing $\mathcal{H}_\mathcal{V}^k (j) := \mathcal{H}_\mathcal{V}^k (Gr^k_{\mathcal{F}} C^*)$, consider the spectral sequence
\[ \mathcal{E}_1^{q,p} := \left\{ \begin{array}{ll} \mathcal{H}^{p+q}(p), & p \geq 0 \\ 0, & p < 0 \end{array} \right. \]
converging to $\mathcal{H}_\mathcal{V}^k (F^0 \mathbf{C}^*)$. We recover at once the result of Green and Voisin [Gre], in the form stated by Nori [No]:

**Proposition 2.3.** If all the $\mathcal{H}_\mathcal{V}^0 (j)$ and $\mathcal{H}_\mathcal{V}^1 (j)$ vanish for $j \geq 0$, then $\text{NF}_\mathcal{S}(j^* \mathcal{V}) = \{0\}$ for any étale neighborhood $j : \mathcal{T} \to \mathcal{S}$.

Two obvious situations in which the vanishing conditions fail are those of VHS of level one, or level three and “Calabi-Yau type”. Recall that if
\[ \mathcal{V} = F_{p_{\min}} \mathcal{V} \supseteq F_{p_{\min} + 1} \mathcal{V} \supseteq \cdots \supseteq F_{p_{max}} \mathcal{V} \supseteq F_{p_{max} + 1} \mathcal{V} = \{0\} \]
then the level $\ell(\mathcal{V}) = p_{max} - p_{min}$; for weight $-1$ we have $\ell(\mathcal{V}) = 2p_{max} + 1 = -2p_{min} - 1$. Write $h^p := \dim(Gr^p_{\mathcal{F}} \mathcal{V})$ for the Hodge numbers, and $\underline{h} = (h_{p_{min}}, \ldots, h_{p_{max}})$; put $d := \dim_{\mathbb{C}} \mathcal{S}$. By naively computing ranks we have

**Example 2.4.** (level 1) If $\underline{h} = (n, n)$ and $d > 1$, then $\mathcal{H}^1(0) \neq \{0\}$. If $d = 1$, then $\mathcal{H}^1(1) \neq \{0\}$. 
Example 2.5. (level 3 CY) If \( h = (1, n, n, 1) \) and \( 1 < d < 2n \), then \( \mathcal{H}^1(0) \neq \{0\} \). If \( d = 1 \), then \( \mathcal{H}^1(2) \neq \{0\} \). (All CY VHS we consider have \( d \leq n \).)

This makes the broad vanishing results we obtain in this paper somewhat surprising. It also explains why we have to ignore the level-one cases below.

Finally, given a smooth family of varieties \( \pi : \mathcal{X} \to \mathcal{S} \), let \( Z \in Z^r(X_{s_0})_{Q, \text{hom}} \) be a homologically trivial cycle on a very general fiber. Suppose its class in the Griffiths group \( \text{Griff}^r(X_{s_0}) = \frac{Z^r(X_{s_0})_{Q, \text{hom}}}{Z^r(X_{s_0})_{Q, \text{alg}}} \) has nonzero image under \((2.4)\)

\[
\overline{AJ}_{X_{s_0}} : \text{Griff}^r(X_{s_0}) \to \mathcal{J}^r(X_{s_0}) := \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \overline{H}^{2r-1}(X_{s_0}, \mathbb{Q}(r)))
\]

where \( \overline{H}^{2r-1}(X_{s_0}) \) is the quotient of \( H^{2r-1}(X_{s_0}) \) by its maximal level-one sub-HS. Spreading out yields a cycle \( Z \in Z^r(X_T)_{Q} \) on an étale pullback \( \mathcal{X}_T \xrightarrow{\pi_T} T \) meeting fibers properly (with each \( Z \cdot X_t \equiv \text{hom} 0 \)), and a normal function

\[ \nu_{\mathcal{Z}} \in \text{NF}_{\mathcal{S}}(\mathcal{V}), \quad \mathcal{V} = \overline{R}^{2r-1}(\pi_{\mathcal{S}})_* Q_{\mathcal{X}_T}(r) \otimes O_{\mathcal{S}} \]

where \( \overline{R} \) denotes the quotient by the maximal level-one sub-VHS. So we have the

Corollary 2.6. If all the \( \mathcal{H}^0(j) \) and \( \mathcal{H}^1(j) \) vanish for \( j \geq 0 \), then \( \overline{AJ}_{X_{s_0}}(Z) = 0 \) in \((2.4)\).

While there is no converse result, nonvanishing of \( \mathcal{H}^1(0) \) in particular seems to be a good predictor of the existence of interesting cycles. Note that its nonvanishing for level-one VHS \( \mathcal{V} \) has a geometric “origin”. Namely, these VHS correspond to the \( H^1 \) (or \( H^{2D-1} \)) of families of abelian \( D \)-folds \( \mathcal{A} \xrightarrow{\pi} \mathcal{S} \). The existence of nontorsion points on the geometric generic fiber over \( \overline{C}^{(\mathcal{S})} \) yields nontrivial geometric normal functions in \( \text{NF}_{\mathcal{S}}(j^* \mathcal{V}) \) over some étale neighborhood \( \mathcal{S} \). This explains why we only consider the Griffiths group.

2.2. \( n \)-cohomology of finite-dimensional representations. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra of rank \( n \), \( \mathfrak{b} \supseteq \mathfrak{t} \) Borel and Cartan subalgebras, \( \Delta = \Delta(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^* \) the corresponding roots. Denote by \( \Delta^+ = \Delta(\mathfrak{b}) \) the positive roots, \( \Sigma = \{\sigma_1, \ldots, \sigma_n\} \subset \Delta^+ \) the simple roots, \( \Omega = \{\omega_1, \ldots, \omega_n\} \subset \mathfrak{t}^* \) the fundamental weights, and \( \Lambda \) the (weight) lattice they generate. The Killing form \( B(X, Y) = \text{Tr}(\text{ad}X \circ \text{ad}Y) \) on \( \mathfrak{g} \) induces a symmetric bilinear form \( \langle , \rangle \) on \( \Lambda \), a particular orthonormal
basis of which (as in [Kn App. C]) will be denoted by \( \{ e_i \} \). We have in particular \( \langle \omega_i, \sigma_j \rangle = \frac{1}{2} \langle \sigma_j, \sigma_j \rangle \delta_{ij} \).

By the Theorem of the highest weight, the irreducible representations \( \{ V^\lambda \} \) of \( g \) (of finite dimension) are parametrized by their highest weight \( \lambda \); there is a 1-to-1 correspondence between irreps and weights of the form \( \lambda = \sum_{i=1}^n m_i \omega_i \) with \( m_i \geq 0 \). Fix an element \( E \in \lambda \) with all \( \frac{1}{2} E(\sigma_i) \) non-positive and integral, and let \( g = \oplus_{j \in \mathbb{Z}} g^{j,-j} \) be the decomposition into \( \text{ad}(E) \)-eigenspaces with eigenvalue \( 2j \). For any representation \( (V, \rho) \), there is a corresponding decomposition into \( \rho(E) \)-eigenspaces, which can have odd or fractional eigenvalues. Write \( p = \oplus_{j \geq 0} g^{j,-j} \), \( n = \oplus_{j < 0} g^{j,-j} \) (so that \( \Delta(n) \subset \Delta^+ \)), and \( g^0 = g^{0,0} \). We also denote \( \Delta_0 = \Delta(g^0, t) \), \( \Delta_0^+ = \Delta_0 \cap \Delta^+ \), and \( \{ V_\xi^\lambda \} \) for the highest-weight irreps of \( g^0 \).

Our main computational tool will be a result of Kostant (cf. [Ko Thm. 5.14]). It computes the decomposition of the cohomologies of the natural complex

\[
0 \to V^\lambda \to n^\vee \otimes V^\lambda \to \wedge^2 n^\vee \otimes V^\lambda \to \cdots
\]

under the action of \( g^0 \). To state a version of it (Prop. 2.7 below), let \( W = W(g, t) \) and \( W_0 = W(g^0, t) \) be the Weyl groups, and consider the set

\[
W^0 := \{ w \in W \mid w(\Delta^+) \supseteq \Delta_0^+ \}
\]

of minimal-length representatives of cosets \( W_0 \backslash W \) (with length \( |w(\Delta^+) \cap \Delta^-| \)), which we partition into

\[
W^0(j) = \{ w \in W^0 \mid \text{length}(w) = j \}.
\]

Finally, write \( \rho := \frac{1}{2} \sum_{\delta \in \Delta^+} \delta = \sum_{i=1}^n \omega_i \), and

\[
w \cdot \lambda := w(\lambda + \rho) - \rho.
\]

**Proposition 2.7.** As a \( g^0 \)-module, \( H^k(n, V^\lambda) \cong \oplus_{w \in W^0(k)} V_0^{w \cdot \lambda} \).

Now suppose \( g \supset t \) is the complexification of \( g_\mathbb{R} \supset t_\mathbb{R} \), with \( t_\mathbb{R} \) compact (and \( g_\mathbb{R} \) not). Writing \( \Delta = \Delta_c \cup \Delta_n \) for the decomposition into compact and noncompact roots, we assume \( \frac{1}{2} E(\Delta_c) \subset 2\mathbb{Z} \) and \( \frac{1}{2} E(\Delta_n) \subset 2\mathbb{Z} + 1 \). Let \( \hat{V} \) be an irreducible representation (of finite dimension). We then have either \( \hat{V}_c \cong V^\lambda \) (real case) or \( \hat{V}_c \cong V^\lambda \oplus \overline{V^\lambda} \) as \( g \)-modules. Let \( w_0 \in W \) denote the unique element with \( w_0(\Delta^+) = \Delta^- \), and write \( -\tau \) for the induced involution on \( \Lambda \); for \( g \) simple of Hermitian type other than \( A_n, D_{2m+1} \), or \( E_6, \tau = \text{id}_\Lambda \). In the non-real case, we have \( \overline{V^\lambda} \cong V^{\tau(\lambda)} \), and we say \( V^\lambda \) is complex resp. quaternionic when \( \lambda \neq \tau(\lambda) \) resp. \( = \tau(\lambda) \). To distinguish the real and quaternionic cases, we use the following
Proposition 2.8. \([\text{GGK}]\) For \(\lambda = \sum M_i \sigma_i\) with \(\tau(\lambda) = \lambda\), \(V^\lambda\) is real iff \(\sum \sigma_i\) compact \(M_i \in \mathbb{Z}\).

Given \(\lambda = \sum m_i \omega_i\) \((m_i \geq 0)\), we shall write

\[
(2.7) \quad \tilde{V}^\lambda := \begin{cases} 
V^\lambda, & \text{real case} \\
V^\lambda \oplus V^{\tau(\lambda)}, & \text{cx./quat.}
\end{cases}
\]

It is to these representations that we shall apply Prop. 2.7. Note that since \(\tilde{V}^\lambda\) has an underlying \(\mathbb{R}\)-vector space \(\tilde{V}^\lambda_\mathbb{R}\) (which is irreducible as a representation of \(g_\mathbb{R}\)), it can be viewed (up to Tate twist) as an \(\mathbb{R}\)-Hodge structure via the action of \(E\), whose eigenvalues are regarded as “\(p - q\)” (on Hodge type \((p, q)\)).

Since differences of weights of \(V^\lambda\) lie in the root lattice (which \(E\) sends into \(2\mathbb{Z}\)) and \(-\tau(\lambda) = w_0(\lambda)\) is a weight of \(V^\lambda\), \(\tilde{V}^\lambda\) can be placed in weight \(-1\) iff \(E(\lambda)\) is odd. In this case

\[
(2.8) \quad (\tilde{V}^\lambda)^{p,-p-1} := \{(2p + 1)\text{-eigenspace of } E \text{ on } \tilde{V}^\lambda\},
\]

and (using \(\tilde{V}^\lambda \cong V^{\tau(\lambda)}\)) the level is

\[
(2.9) \quad \ell(\tilde{V}^\lambda) = \max \{-E(\lambda), -E(\tau(\lambda))\}.
\]

Moreover, when \(E(\lambda)\) is odd there exists (up to scale) a unique \(g\)-invariant alternating bilinear form \(Q\) on \(\tilde{V}^\lambda_\mathbb{R}\), which polarizes this Hodge structure. Note that in the non-real cases, \(Q\) pairs \(V^\lambda\) and \(V^{\tau(\lambda)}\).

Remark 2.9. The level of the complex summands (as complex Hodge structures) is

\[
(2.10) \quad \ell(V^\lambda) \left(= \ell(V^{\tau(\lambda)})\right) = -\frac{1}{2}(E(\lambda) + E(\tau(\lambda))).
\]

This is the minimal level that can be achieved using half-twists, which are discussed in §4.

2.3. Homogeneous VHS over locally symmetric varieties. Let \(G\) be a semisimple \(\mathbb{Q}\)-algebraic group of Hermitian type, such that \(G_\mathbb{R}\) has a compact maximal torus \(T_\mathbb{R}\); and write \(g_\mathbb{R} \supset t_\mathbb{R}\) for the Lie algebras (with complexifications \(g \supset t\)). Choose a cocharacter \(\chi_0 : \mathbb{G}_m \to T_\mathbb{C}\) such that \(E := \chi'_0(1)\) satisfies

\[
(2.11) \quad E(\Delta_c) = 0, \quad E(\Delta_n) = \{\pm 2\}.
\]

Denoting the composition \(\mathbb{G}_m \xrightarrow{\chi_0} T_\mathbb{C} \hookrightarrow G_\mathbb{C}\) by \(\varphi_0\), the corresponding Hermitian symmetric domain is the orbit under conjugation by the

\[\text{footnote}\]

\(^1\)One should also assume \(E\) doesn’t project to zero in any simple factor of \(g\), but in the calculations below \(g\) will be simple.
group of real points $G(\mathbb{R})$:

$$D := G(\mathbb{R}).\varphi_0 \cong G(\mathbb{R})/G^0(\mathbb{R}),$$

with $G^0(\mathbb{R})$ a maximal compact subgroup. Taking $\Gamma \leq G(\mathbb{Q})$ a torsion-free congruence subgroup, the locally symmetric variety $X := \Gamma \backslash D$ is in fact a projective variety by the Baily-Borel theorem.

The $G(\mathbb{R})$-orbit of $\text{Ad} \circ \varphi_0 : \mathbb{G}_m \to \text{Aut}(\mathfrak{g}_\mathbb{C}, -B)$ gives a $(-B)$-polarized $\mathbb{Q}$-VHS of weight zero and level two over $D$, with Hodge decompositions $\mathfrak{g} = \bigoplus_{j=-1,0,1} \mathfrak{g}_{\text{Ad}(g)}^{j,-j}$ at $g\varphi_0 g^{-1} \in D$; this descends to $X$. More generally (with $\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$), given an $\mathbb{A}$-representation $\rho : G \to \text{Aut}(\mathcal{V}_\mathbb{A}, Q)$ ($Q$ a $(-1)^k$-symmetric bilinear form) such that $\rho \circ \varphi$ is an $\mathbb{A}$-PHS, we get a homogeneous $\mathbb{A}$-PVHS $\mathcal{V}_\mathbb{A}$ (with weight of parity $k$) over $X$, called a Hermitian VHS. In every case the Hodge decomposition is induced by $d\rho(\text{Ad}(g))$; that is, a weight subspace $V_\xi \subset V^{p,q} \iff E(\xi) = p - q$.

For groups other than $E_6$ and $E_7$, most Hermitian $\mathbb{Q}$-PVHS arise from the relative cohomology of various canonical families of abelian varieties; in some cases (cf. [FL, ?]) one has also families of Calabi-Yau varieties. While these structures may matter for questions of geometric origin (i.e. algebraic cycles), the behavior of the $\{H^k(j)\}$ depends only on $\mathcal{V}_\mathbb{R}$. Moreover, any Hermitian PVHS of weight $-1$ on $X$ decomposes over $\mathbb{R}$ into a direct sum of the irreducible homogeneous real variations $\mathcal{V}_\mathbb{R}^\lambda$ arising (as above) from the PHS on $(\mathcal{V}_\mathbb{R}^\lambda, \rho^\lambda)$ described at the end of §2.2. Indeed, since $\Gamma \leq G(\mathbb{R})$ is Zariski-dense, we have an equivalence between real Hermitian VHS $\mathcal{V}_\mathbb{R}$ over $X$ and representations of $G(\mathbb{R})$.

Before turning to our main calculation, we remind the reader what forms $D$ and $E$ can take when $G_G$ is simple. First, (2.11) forces $\Delta_n \cap \Sigma$ to be a singleton $\{\sigma_I\}$, which must additionally be a special simple root. That is, if we write $\mathfrak{g} = V^{\text{ad}}, \lambda_{\text{ad}} = \sum_{i=1}^n M_i^{\text{ad}} \sigma_i$, then $\sigma_I$ must be one of the simple roots $\sigma_i$ for which $M_i^{\text{ad}} = 1$. Further, the choice of $\sigma_I$ determines: the decomposition $\Delta = \Delta_c \amalg \Delta_n$, and thus the real form $\mathfrak{g}_\mathbb{R}$; and the Hodge structure at the base point $\varphi_0$, by

$$E(\sigma_I) = -2, \quad E(\sigma_j) = 0 \quad (\forall j \neq I).$$

\[\text{\textsuperscript{2}}\text{see }\S3.4 \text{ for discussion of one possible exception}\]
This leads to the classification (up to isomorphism) of the irreducible Hermitian symmetric domains of noncompact type:

| $D$ | $(R, \sigma_1)$ | $G(\mathbb{R})$ | $d$ | range |
|-----|-----------------|-----------------|-----|-------|
| $I_{p,n-p+1}$ | $(A_n, \sigma_p)$ | $SU(p, n - p + 1)$ | $p(n - p + 1)$ | $n \geq 2$ |
| $\Pi_n$ | $(D_n, \sigma_n)$ | $Spin^*(2n)$ | $\frac{1}{2}n(n-1)$ | $n \geq 4$ |
| $\Pi_n$ | $(C_n, \sigma_n)$ | $Sp(2n, \mathbb{R})$ | $\frac{1}{2}n(n+1)$ | $n \geq 1$ |
| $IV_{2n-1}$ | $(B_n, \sigma_1)$ | $Spin(2, 2n-1)$ | $2n-1$ | $n \geq 3$ |
| $IV_{2n-2}$ | $(D_n, \sigma_1)$ | $Spin(2, 2n-2)$ | $2n-2$ | $n \geq 4$ |
| $EIII$ | $(E_6, \sigma_1)$ | $E_{6,3}$ | 16 | — |
| $EVIII$ | $(E_7, \sigma_7)$ | $E_{7,3}$ | 27 | — |

Here $R$ is the root system, $G(\mathbb{R})$ is the simply connected group (which has all $\check{V}^\lambda$ as representations), and $d = \dim_{\mathbb{C}} D = \dim_{\mathbb{C}} X$. (See [LZ] for more details.)

**Remark 2.10.** We have omitted some cases to avoid redundancy due to exceptional isomorphisms and conjugate-isomorphisms. The latter are induced by the action of $\tau$ on $\sigma_1$. For $A_n$, $\tau$ exchanges $\sigma_i$ and $\sigma_{n+1-i}$ ($\forall i$); for $D_n$, $n$ odd, $\tau$ exchanges $\sigma_{n-1}$ and $\sigma_n$; for $E_6$, $\tau$ exchanges $\sigma_6 \leftrightarrow \sigma_1$ and $\sigma_7 \leftrightarrow \sigma_3$; and in all other cases the action is trivial.

The exceptional isomorphisms are $III_1 \cong I_{1,1} \cong IV_1 \cong II_2$, $III_2 \cong IV_3$, $I_{2,2} \cong IV_4$, $II_3 \cong I_{1,3}$, $II_4 \cong IV_6$ (triality for $D_4$), and $IV_2 \cong III_1 \times III_1$. For this reason we consider only $A_{n\geq 2}$, $B_{n\geq 3}$, $C_{n\geq 1}$, and $D_{n\geq 4}$.

In each case, the real variations arising from $H^1$ of the canonical families of abelian varieties over $X$ are (half-twists of the $\check{V}_\lambda$ with $-\frac{1}{2}(E(\lambda) + E(\tau(\lambda))) = 1$ (cf. [2.10]). All such $\lambda$ take the form $\omega_i$, with the possibilities corresponding to so-called “symplectic nodes” (cf. [LZ]). This will be recalled case by case where relevant in §§3-4. The variations of Calabi-Yau type are even simpler to describe: they are (again, up to half-twist for $A_n$, $D_{n\text{ odd}}$, $E_6$) precisely the $\check{V}_{k=1}^\lambda$ for $k \geq 1$ [FL].

### 2.4. The main calculation.

We now apply Proposition [2.7] to compute the $\{\mathcal{H}_k(j)\}$ for the Hermitian variations $\check{V}_\lambda$ (over $X$ or $D$) with $E(\lambda)$ odd. Of course, we can work with (its summands) the complex variations $\check{V}_\lambda^C$, and do the computation at one point $\varphi_0 \in D$. Write

$$W^0(k, j) := \left\{ w \in W^0(k) \left| \frac{1}{2}(E(w \cdot \lambda) - 1) = j \right. \right\}.$$  

**Proposition 2.11.** For the $\check{V}_\lambda^C$, $\mathcal{H}_k(j)|_{\varphi_0} \cong \bigoplus_{w \in W^0(k, j)} V_0^{w \cdot \lambda}$.

---

3 this is only relevant for $A_n$, cf. §4
Proof. First, we note that
\begin{equation}
\oplus_j \mathcal{H}^k(j)|_{\varphi_0} \cong H^k(n, V^\lambda)
\end{equation}
follows from the identification of the Lie algebra cohomology complex \((\wedge^n n^\vee \otimes V^\lambda, d)\) with the associated graded \((\oplus_j Gr_j F, C^*, \nabla)\) under the isomorphism \(n^\vee \cong \Omega^*_{\mathbb{D}, \varphi_0}\). To see this identification, extend \(v \in (V^\lambda)^{j,-j-1}\) to a section of \((V^\lambda)^{j,-j-1}\) by \(\tilde{v} = \tilde{\rho}^\lambda(g).v\), and write (for \(X \in n\)) \((\nabla_X \tilde{v})|_{\varphi_0} = \lim_{t \to 0} \frac{1}{t}(e^{t\tilde{d}^\lambda(X)}v - v) = d\tilde{\rho}^\lambda(X).v (= "X(v)"").

Next, we compare Hodge gradings on the two sides of \((2.14)\). For \(X^* \in n^\vee\) and \(v \in Gr_j F^\lambda = (V^\lambda)^{j-1,-j}\), we have \(E(X^*) = 2X^*\) and \(E(v) = (2j - 1)v \implies E(X^* \otimes v) = (2j + 1)X^* \otimes v\). So the image of \(\mathcal{H}^k(j)|_{\varphi_0}\) in \(H^k(n, V^\lambda) = \bigoplus_{w \in W^\lambda(k)} V_{w^\lambda}\) consists of all the weight spaces with weights \(\xi\) for which \(E(\xi) = 2j + 1\). Since \(\mathfrak{g}^0\) commutes with \(E\), these are just the \(V_{w^\lambda}\) for which \(E(w \cdot \lambda) = 2j + 1\). The result follows. \(\square\)

Obviously \(W^0(0) = \{\text{id}\}\), and always \(E(\lambda) \leq -1\); so immediately we have \(\mathcal{H}^0(j) = \{0\}\) for \(j \geq 0\). We claim that \(W^0(1) = \{s\}\), where \(s\) is the reflection in the distinguished simple root \(\sigma_1\). Indeed, elements of \(W^0(1)\) have length one, so must be the reflection in one simple root. But the reflection \(w_i\) in \(\sigma_i \neq 1\) doesn’t satisfy \(w_i(\Delta^+) \supset \Delta^+_i\) (as \(\sigma_i \neq w_i(\Delta^+)\)), whereas \(s(\Delta^+_i) \subset \Delta^+\) since \(s(\sigma_1) = -\sigma_1\) and \(|s(\Delta^+) \cap \Delta^-| = 1\). So we have the basic invariant
\begin{equation}
\mu(\lambda) := \frac{1}{2} (E(s \cdot \lambda) - 1)
\end{equation}
which depends upon the choice of \(\sigma_1\) (hence \(D\) and \(X\)). Propositions \(2.3\) and \(2.11\) yield at once the

**Theorem 2.12.** If \(\mu(\lambda), \mu(\tau(\lambda)) < 0\), then \(\text{NF}_{\mathcal{F}}(j^* \mathcal{V}) = \{0\}\) for any étale neighborhood \(j : \mathcal{T} \to X\) and any variation \(\mathcal{V}\) on \(X\) with the canonical Hermitian variation \(V^\lambda_{\mathbb{R}}\) as underlying \(\mathbb{R}\)-VHS. (In particular, we have \(AJ(Z) = 0\) if \(X, Z\) are as in \((2.4)\), with \(\mathcal{F} = X\)).

One can of course replace \(X\) by a Zariski open subset in these statements.

**Remark 2.13.** If \(\mu(\lambda) = \mu(\tau(\lambda)) = 0\), then (arguing as in \([Pa]\)) one can show that M. Saito’s canonical MHS on \(H^1(X, \mathcal{V})\) is of pure type \((0,0)\). (Naturally, \(H^1(X, \mathcal{V})\) could still be \(\{0\}\).)

Suppose that for \(\mathcal{V}\) as above we have \(\mu(\lambda) \geq 0\), so that \(E^{1-\mu(\lambda), \mu(\lambda)} = \mathcal{H}^1_{\mathcal{V}}(\mu(\lambda)) \neq \{0\}\). To conclude the stronger result that \(E^{1-\mu(\lambda), \mu(\lambda)}\) hence \(\mathcal{H}^1_{\mathcal{V}}(P^0 C^*)\) is nonzero, we would need to compute the nonlinear \(d_t : E^{1-\mu(\lambda), \mu(\lambda)} \to E^{1-\mu(\lambda)-\ell, \mu(\lambda)+\ell}_t\). The present methods are only of use here if they show that all \(\mathcal{H}^j_{\mathcal{V}}(\mu(\lambda)) = \{0\}\) for \(j > \mu(\lambda)\). But this is asking.
too much: even in the key case \( D = \text{III}_3, \lambda = \omega_3, \mu(\lambda) = 0 \) where we have a nonzero geometric normal function coming from the Ceresa cycle, we compute that \( \mathcal{H}^2(1) \neq \{0\} \). So with regard to predicting normal functions it is probably more useful to stick to \( \mathcal{H}^1(1) \) and make the provisional

**Definition 2.14.** The complex (resp. real) variation \( V^\lambda \) (resp. \( \tilde{V}_R^\lambda \)) has an **infinitesimal normal function** if \( \mu(\lambda) \geq 0 \).

3. Analysis of (mostly) tube domain cases

In this section we study the real variations \( \tilde{V}_R^\lambda \) arising from dominant integral \( \lambda \) with \( E(\lambda) \) an odd integer. Though we (more or less) carry this out for all the domains in the table 2.3 the results are of interest mainly when \( D \) is of tube type (\( \text{I}_{p.p}, \text{II}_{2m}, \text{III}_n, \text{IV}_m \), or \( \text{EVIII} \)). As above, we have \( E(\sigma_1) - 2, E(\sigma_{j\neq1}) = 0 \), and write \( \lambda = \sum_{i=1}^n m_i \omega_i \); note that \( s \) sends \( \sigma_1 \mapsto -\sigma_1 \) and fixes all \( \omega_j \neq 1 \). To streamline the discussion of examples, we make the

**Definition 3.1.** A family \( \mathcal{A} \xrightarrow{s} \mathcal{S} \) of abelian varieties admits no reduced normal functions if \( \text{NF}_{\mathcal{S}}(\mathcal{V}) = \{0\} \) for every \( r \in \mathbb{N} \) and irreducible VHS \( \mathcal{V} \subset R^{2r-1} \pi_* Q_{\mathcal{A}}(r) \otimes \mathcal{O}_{\mathcal{S}} \) of level \( > 1 \).

In all the cases below where we obtain such an assertion, one knows that all level-one real sub-VHS are in fact defined over \( \mathbb{Q} \).

3.1. Case \( \text{III}_n \). This is the case analyzed by Nori [No] (\( \mathcal{H}^1(1) \)) and Fakhruddin [Fa] (\( \mathcal{H}^k(1/j) \) for \( k > 1 \)). We have

\[
\Sigma = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}, \\
\Omega = \{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + \cdots + e_n\},
\]

and so \( I = n \implies E(\lambda) = -\sum_{i=1}^n im_i \); the only level-one variation is thus \( V_R^\omega \) (which arises from \( H^1 \) of the universal abelian variety). There are no complex or quaternionic irreps, and (taking \( E(\lambda) \) odd) we have \( s(\omega_n) = 2\omega_{n-1} - \omega_n \implies \mu(\lambda) = -\frac{1}{2} \sum_{i<n} im_i + (1 - \frac{n}{2})m_n + \frac{1}{2} \).

So for \( \ell(V^\lambda) = -E(\lambda) > 1 \) odd, \( \mu(\lambda) \geq 0 \) for only

\[
\begin{align*}
n &= 1 \text{ and } \lambda = a\omega_1 \ (a > 1), \\
n &= 2 \text{ and } \lambda = \omega_1 + a\omega_2 \ (a \geq 1), \\
n &= 3 \text{ and } \lambda = \omega_3;
\end{align*}
\]
in particular, no étale pullback of the universal abelian variety \( \mathcal{A}^{III_s} \rightarrow A_g = Sp_{2g}(\mathbb{Z})/III_s \) admits reduced normal functions for \( g \geq 4 \). These are the results of Nori [No].

### 3.2. Case EVIII.

We have \( I = 7 \) and

\[ \Sigma = \left\{ \frac{1}{2}(e_8 - e_7 - \cdots - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, \ldots, e_6 - e_5 \right\}; \]

\( \Omega \) is written in terms of the \( \{ \sigma_i \} \) in [Kn] p. 687]. This yields

\[ E(\lambda) = -2m_1 - 3m_2 - 4m_3 - 6m_4 - 5m_5 - 4m_6 - 3m_7, \]

\[ E(\sigma(\omega_7)) = -1, \text{ and} \]

\[ \mu(\lambda) = -m_1 - \frac{3}{2}m_2 - 2m_3 - 3m_4 - \frac{5}{2}m_5 - 2m_6 - \frac{1}{2}m_7 + \frac{1}{2}. \]

There are no complex or quaternionic irreps, and \( \mu(\lambda) \neq 0 \) (with \( E(\lambda) \) odd) for

\[ \lambda = \omega_7. \]

### 3.3. Case EIII.

Here \( I=1 \), \( \Omega \) is expressed in terms of

\[ \Sigma = \left\{ \frac{1}{2}(e_8 - e_7 - \cdots - e_2 + e_1), e_1 + e_2, e_2 - e_1, \ldots, e_5 - e_4 \right\} \]

in [Kn] p. 687, and \( \tau = (16)(35) \) on both the \( \{ \sigma_i \} \) and \( \{ \omega_i \} \). Since

\[ E(\lambda) = -\frac{2}{3}(4m_1 + 5m_3 + 4m_5 + 2m_6) - 2m_2 - 4m_4, \]

we find that \( E(\lambda) \in \mathbb{Z} \iff 3|(4m_1 + 5m_3 + 4m_5 + 2m_6) \iff E(\lambda) \in 2\mathbb{Z} \). So there are no odd-level VHS without half-twists, and further analysis is deferred to \$4$.

We can now turn to the richer remaining classical cases.

### 3.4. Case II\(_n\). With \( I = n \) (note that \( n \geq 4 \)),

\[ \Sigma = \{ e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n \}; \]

\[ \Omega = \{ e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-2}, \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n), \frac{1}{2}(e_1 + \cdots + e_n) \} \]

we obtain

\[ E(\lambda) = -\sum_{i=1}^{n-2} im_i - \frac{n-2}{2}m_{n-1} - \frac{n}{2}m_n. \]

Note that \( \tau \) swaps \( \sigma_{n-1} \leftrightarrow \sigma_n \) and \( \omega_{n-1} \leftrightarrow \omega_n \) if \( n \) is odd, and is trivial for \( n \) even. The only complex irreps are therefore the \( V^\lambda \) with \( n \) odd and \( m_{n-1} \neq m_n \). The quaternionic ones are the \( V^\lambda \) with

\[ \sum_{i=1}^{n-2} im_i + m_{n-1} \text{ odd \, (n even), or} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n-2} im_i \text{ odd} \\ m_{n-1} = m_n \quad \text{ (n odd).} \end{array} \right. \]

We have \( \ell(V^\lambda) = 1 \) for \( \lambda = \omega_1, \omega_3 \, (n = 4) \) and \( \lambda = \omega_1 \, (n > 4) \); these \( V^\lambda \) are all quaternionic (i.e. \( V^\lambda = (V^\lambda)^{\oplus 2} \) over \( \mathbb{C} \)).
Definition 3.2. By a universal quaternionic abelian variety, we shall mean any family $A_{\mathbb{II} n} \rightarrow \Gamma \backslash \Pi_n$ of abelian $2n$-folds whose $H^1$ recovers $\tilde{V}_{\omega}$. Such families admit an embedding of a definite rational quaternion algebra $\mathcal{Q}$ into $\text{End}(A)_{\mathbb{Q}}$. (The Mumford-Tate group $G$ is a $\mathbb{Q}$-form of $G_{\mathbb{R}}$ and so, by considering its fixed 2-tensors, $G$ is a $\mathbb{Q}$-form of $\mathbb{H}$. See [vGV] for more details on quaternionic abelian varieties.) There are natural embeddings $\mathbb{II} n \hookrightarrow \mathbb{III} 2n$ which yield countably many “quaternionic subfamilies” of $A_{\mathbb{III} 2n}$.

Now from $s(\omega_n) = \omega_{n-2} - \omega_n$ we find

$$\mu(\lambda) = -\frac{1}{4} \sum_{i=1}^{n-2} im_i - \frac{n-2}{4} m_{n-1} - \frac{n-4}{4} m_n + \frac{1}{2}. $$

Imposing $-E(\lambda)$ odd $> 1$, $n \geq 4$, and $\mu(\lambda) \geq 0$ yields

$$\lambda = \omega_1 + a\omega_4, \omega_3 + a\omega_4 \quad (n = 4, a > 0)$$

which is quaternionic, and

$$\lambda = \omega_6 \quad (n = 6)$$

which is real.

Theorem 3.3. No étale pullback of a universal quaternionic abelian variety of (relative) dimension $2n$ admits reduced normal functions outside the case $n = 4$.

Proof. It remains to deal with $n = 6$. The point is that only $V_{\omega_5 + \omega_6}$ and $V_{2\omega_5}, V_{2\omega_6}$ occur in $H^*_rel(\mathcal{A}_{\mathbb{III}}) \cong \Lambda^* (V_{\omega_1} \oplus V_{\omega_1})$; the half-spin variations $V_{\omega_5}, V_{\omega_6}$ do not. (See [FH] for the equivalent fact on representations of $SO(12)$.)

What is special about the quaternionic 8-folds that might yield $\overline{AJ}$-nontrivial elements of the Griffiths group? According to [vGV], there exist families $\mathcal{A}_{\mathbb{II} 4}$ whose general member arises as a quaternionic Prym variety, associated to a certain $8:1$ unramified cover of a general genus 3 curve (plausible as $\dim \mathcal{M}_3 = \dim \Pi_4 = 6$). Since $\bigwedge^3 V_{\omega_1} = V_{\omega_3 + \omega_4}$, the (non-Calabi-Yau) variation $\bigwedge^3_{\mathbb{R}}$ occurs in $H^3$ of $\mathcal{A}_{\mathbb{II} 4}$, and it seems reasonable to expect that one can construct $\overline{AJ}$-nontrivial 1-cycles from the Abel-Prym image of the genus 17 cover curves. For $n = 6$, it seems to be an open problem to give a simple motivic construction of $V_{\mathbb{R}}^6$, so we cannot speculate about cycles in this case.

\footnote{In all the $D_n$ cases, $V_{\omega_{n-1}}$ and $V_{\omega_n}$ identify with spin$^-$ and spin$^+$ (with order depending on parity of $n$).}
Definition 3.4. A universal Weil abelian variety is a family \( \tilde{\mathcal{V}} \) into \( \mathbb{C} \) or complex, and that the sole level \( \omega \) (or \( \omega_1 + \omega_2 \) if \( p = 1 \)). This yields

\[
\begin{align*}
\mathcal{E}(\lambda) &= \frac{2}{n+1} \left( \sum_{i=1}^{p} (n + 1 - p)im_i + \sum_{i=p+1}^{n} p(n + 1 - i)m_i \right) \\
\mathcal{E}(s \cdot \lambda) &= 2 + 2m_p + \mathcal{E}(\lambda), \quad \mu(\lambda) = \frac{1}{2} + m_p + \frac{1}{2} \mathcal{E}(\lambda)
\end{align*}
\]

and for \( n \) odd there are quite a few (mostly complex) representations with \( \mathcal{E}(\lambda) \) an odd integer and \( \mu(\lambda) \geq 0 \). For example, \( p = 2, n = 7 \) produces \( \tilde{\mathcal{V}}^{\omega_6} = \mathcal{V}^{\omega_2} \oplus \mathcal{V}^{\omega_6} \) and \( \tilde{\mathcal{V}}^{2\omega_7} = \mathcal{V}^{2\omega_7} \oplus \mathcal{V}^{2\omega_1} \), which both have level 3 (and \( \mu(2\omega_7) = \mu(\omega_6) = \mu(\omega_2) = 0 \)). We have not carried out an exhaustive search, because the resulting variations seem obscure and the situation drastically improves with the introduction of half-twists (§4).

So restricting to \( p = \frac{n+1}{2} \) (odd), i.e. the case \( \mathcal{I}_{p,p} \) \( (p \geq 2) \), we have

\[
\begin{align*}
\mathcal{E}(\lambda) &= -\sum_{i=1}^{p} im_i - \sum_{i=p+1}^{2p-1} (2p - i)m_i \\
\mu(\lambda) &= -\frac{1}{2} \sum_{i<p} im_i - \left( \frac{n}{2} - 1 \right) m_p - \sum_{i>p} (p - \frac{1}{2}) m_i + \frac{1}{2}.
\end{align*}
\]

We remark that all representations are either real \( (m_j = m_{n+1-j} \forall j) \) or complex, and that the sole level 1 variation is \( \tilde{\mathcal{V}}^{\omega_1} = \mathcal{V}^{\omega_1} \oplus \mathcal{V}^{\omega_2} \).

Definition 3.4. A universal Weil abelian variety is a family \( \mathcal{A}^{I_{p,p}} \rightarrow \Gamma \backslash \mathcal{I}_{p,p} \) of abelian 2p-folds (of dimension \( p^2 \)) whose \( H^1 \) recovers \( \tilde{\mathcal{V}}^{\omega_1} \).

Such families admit an embedding of an imaginary quadratic field into \( \text{End}(\mathcal{A})_{\mathbb{Q}} \), and produce countably many subfamilies of \( \mathcal{A}^{\text{III}_{2p}} \).

Theorem 3.5. No étale pullback of a universal Weil abelian variety of (relative) dimension 2p admits reduced normal functions outside the cases \( p = 2, 3 \).

Proof. For \( p > 3 \), there are no solutions to \( \mu(\lambda) \geq 0 \) (with the usual constraints). \( \square \)

Indeed, the only solutions are

\[
\lambda = \omega_1 + a\omega_2, \quad \omega_3 + a\omega_2 \quad (p = 2)
\]

which are conjugate complex, and

\[
\lambda = \omega_3 \quad (p = 3)
\]

which is real (with \( \mathcal{V}^{\omega_3} \) of Calabi-Yau type). One easily shows that \( \tilde{\mathcal{V}}^{\omega_1+\omega_2} \subset H^3_{\text{rel}}(\mathcal{A}^{\text{III}_{2}}) \) and \( \tilde{\mathcal{V}}^{\omega_3} \subset H^3_{\text{rel}}(\mathcal{A}^{\text{III}_{3}}) \), and these are the two
cases where there exist families $\mathcal{M}^H_{p,p}$ with generic member arising from a generalized (3 to 1) Prym construction (with base curves of genera $g = 3$ resp. 4). This was used by Schoen \cite{Sc} to prove the Hodge Conjecture for the corresponding Weil 4- and 6-folds.

At least in the $g = 4$ ($p = 3$) case, one can easily show that the Ceresa-type cycle arising from the Abel-Prym curve generates a non-trivial reduced normal function. (This is in contrast to the more standard 2:1 Prym setting, where such cycles are algebraically equivalent to zero.) This is accomplished by degenerating the genus 4 curve with a genus 10 3:1 unramified cover, to a union $C \cup E$ ($C$ of genus 3, $E$ elliptic, meeting at a single node) with cover (each $C_i \cong C$ meeting $E \cong E$ once). The cyclic automorphism $\alpha$ permutes the $\{C_i\}$, and the specialized Prym variety $A$ is the quotient of $J(E) \oplus \bigoplus_{i=1}^3 J(C_i)$ by the invariant part $J(E) \oplus J(C) \Delta$ ($\Delta$ = diagonal). We have $H^1(A) \cong V^{\omega_1} \oplus V^{\omega_3}$, and $H^3(A)^\alpha \cong V^{\omega_3} \oplus V^{\omega_1}$. Writing $\{\omega_i\}_{i=1,2,3} \subset \Omega^1(J(C_i))$ for bases, the invariant form

$$\omega := (\omega_1 + \zeta_3 \omega_2 + \bar{\zeta}_3 \omega_3) \wedge (\omega_1 + \zeta_3 \omega_2 + \bar{\zeta}_2 \omega_3) \wedge (\omega_3 + \zeta_3 \omega_2 + \bar{\zeta}_3 \omega_3)$$

is the pullback of a class from $H^{3,0}(A)^\alpha$. The projection of the Ceresa cycle $C_1^+ - C_1^- = \partial \Gamma$ (for $C_1$) to $A$ has $\int_{\Gamma} \omega = \int_{\Gamma} \omega_1 \wedge \omega_1 \wedge \omega_1$ generically nonzero (otherwise $\overline{\mathcal{A}_J}$ would not detect the usual generic Ceresa cycle). But the degeneration of the Prym-Ceresa cycle is the sum of the projections of the $C_i^+ - C_i^-$ to $A$, and (by invariance under $\alpha$) each has the same nonzero image under the composition of $\overline{\mathcal{A}_J}$ with the projection $J(H^3(A)^\vee) \to J((H^3(A)^\alpha)^\vee)$.

### 3.6. Cases $IV_{2n-2}$ and $IV_{2n-1}$

Beginning with $IV_{2n-2}$ (the $D_n$ case), we note that $n \geq 4$, $I = 1$, $\Sigma$ and $\Omega$ are as in §3.4, and $E(\lambda) (= E(\tau(\lambda))) = -2(m_1 + \cdots + m_{n-2}) - (m_{n-1} + m_n)$.

$V^\lambda$ is complex iff $n$ is odd and $m_{n-1} \neq m_n$, and quaternionic iff $4|n$ and $m_{n-1} + m_n$ is odd. So the level-one real Hermitian VHS are ($\otimes \mathbb{C}$)

$$\begin{align*}
(n \text{ odd}) & \quad \check{V}^{\omega_n} = V^{\omega_n} \oplus V^{\omega_{n-1}} \\
(4|n) & \quad \check{V}^{\omega_n} = (V^{\omega_n})^{\otimes 2}, \quad \check{V}^{\omega_{n-1}} = (V^{\omega_{n-1}})^{\otimes 2} \\
(4|n + 2) & \quad \check{V}^{\omega_n} = V^{\omega_n}, \quad \check{V}^{\omega_{n-1}} = V^{\omega_{n-1}}.
\end{align*}$$

From $s(\omega_1) = \omega_2 - \omega_1$, we define

$$\mu(\lambda) = \frac{1}{2} - m_2 - \cdots - m_{n-2} - \frac{1}{2}m_{n-1} - \frac{1}{2}m_n,$$

so that $-E(\lambda) > 1$ odd and $\mu(\lambda) \geq 0 \implies \lambda = a\omega_1 + \omega_{n-1}$ or $a\omega_1 + \omega_n$ ($a > 0$),

which works for arbitrary $n$!
A similar (but simpler) story unfolds in the $B_n$ case $IV_{2n-1}$, where (with $n \geq 3$, $I = 1$)

$$\Sigma = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\},$$
$$\Omega = \{e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}, \frac{1}{2}(e_1 + \cdots + e_n)\},$$
$$E(\lambda) = -2(m_1 + \cdots + m_{n-1}) - m_n,$$

and (with no complex irreps, as $\tau = \text{id}$) the only quaternionic irreps are the $V^{(2k+1)\omega_n}$ for $\left[\frac{n-1}{2}\right]$ odd. The level one Hermitian $\mathbb{R}$-VHS are thus $(\otimes \mathbb{C})$ the

$$\tilde{V}^\omega_{\omega_n} = (\tilde{V}^\omega_{\omega_n})^\otimes 2 \quad (n = 3, 4; 7, 8; \ldots) \quad \text{or} \quad V^\omega_{\omega_n} \quad (\text{otherwise}).$$

We have $s(\omega_1) = \omega_2 - \omega_1$, hence

$$\mu(\lambda) = \frac{1}{2} - m_2 - \cdots - m_{n-1} - \frac{1}{2}m_n,$$

so that the variations associated to

$$\lambda = a\omega_1 + \omega_n \quad (a > 0)$$

are (for any $n$) the ones with infinitesimal reduced normal functions.

**Definition 3.6.** A universal spin abelian variety is an abelian family $\mathscr{A} \to \Gamma\backslash IV_m$ with $H^1$ (over $\mathbb{R}$) a number of copies of $\tilde{V}^\omega_{\omega_n-1}$ and $\tilde{V}^\omega_{\omega_n}$ ($m = 2n - 2$ even) resp. $V^\omega_{\omega_n}$ ($m = 2n - 1$ odd).

The minimum possible (relative) dimension of $\mathscr{A}$ is clearly $2^n$ ($m$ and $\left[\frac{n-1}{2}\right]$ odd), $2^{n-1}$ ($m$ and $\left[\frac{n-1}{2}\right]$ odd; $m$ even and $4 \nmid n + 2$), or $2^{n-2}$ ($m$ even and $4|n + 2$). However, the main natural source of spin abelian varieties is the Kuga-Satake construction (cf. [vG2]), which produces varieties of much higher dimension: for $m \leq 19$, one has families of $K3$ surfaces $\mathcal{X}$ with $H^2_{tr}(\mathcal{X}) \cong V^\omega_{\omega_1}$, and Clifford algebras produce an embedding $H^2_{tr}(\mathcal{X}) \hookrightarrow H^1(\mathscr{A})^\otimes 2$ with

$$H^1(\mathscr{A}) \cong \begin{cases} (V^\omega_{\omega_{n-1}})^\otimes 2^{n-1} \oplus (V^\omega_{\omega_n})^\otimes 2^{n-1} & (m \text{ even}) \\ (V^\omega_{\omega_n})^\otimes 2^n & (m \text{ odd}) \end{cases}.$$ 

So the situation is in marked contrast to those encountered above.

**Proposition 3.7.** The relative cohomology of any Kuga-Satake family of spin abelian varieties over $IV_m$ ($any m \geq 7$) admits infinitesimal (reduced) normal functions.

---

5We haven’t ruled out that for some “minimal” spin abelian varieties one might have a vanishing result analogous to the above ones, except for $IV_m$ with $m = 7, 8, 9; 13, 15; \text{etc.}$
Proof. We only need to show that (say) $V_1 \omega_n$ occurs in $H^3(\mathcal{X}) = \wedge^3 \mathcal{H}^1(\mathcal{X})$. This is done by considering the decomposition of $(V_1 \omega_n)^{\otimes 2}$ or $V_1 \omega_n \otimes V_1 \omega_{n-1}$; e.g. for $m$ odd,

$$(V_1 \omega_n)^{\otimes 2} \cong V_2 \omega_n \oplus V_1 \omega_{n-1} \oplus \cdots \oplus V_1 \omega_1 \oplus 1$$

$$\implies (V_1 \omega_n)^{\otimes 3} \supseteq V_1 \omega_1 \otimes V_1 \omega_n \supset V_1 \omega_1 + \omega_n.$$ (There are enough copies of $V_1 \omega_n$ that we can consider tensor rather than wedge powers.)

Finally, we remark that the triality isomorphism for $D_4$ exhibits the universal quaternionic abelian 8-folds as “minimal” spin 8-folds, by equating $\tilde{V}_1 \omega_4 \mathbb{R} \to IV_6$ and $\tilde{V}_1 \omega_1 \mathbb{R} \to II_4$. Specialization under $IV_5 \hookrightarrow IV_6$ shrinks the Mumford-Tate group to $\text{Spin}(2,5)$ and gives geometric realizations of $\tilde{V}_1 \omega_3 \mathbb{R} \to IV_5$ (as noticed by [vGV]). All these cases admit infinitesimal normal functions.

4. Half-twists and non-tube cases

In this section we consider a slight generalization of the homogeneous variations described in §2.3, by enlarging our simple $G_{\mathbb{C}}$ to $\tilde{G}_{\mathbb{C}} = \mathbb{G}_m \cdot G_{\mathbb{C}}$ (or $G_\mathbb{R}$ to $U(1) \cdot G(\mathbb{R})$) and taking $\tilde{E}$ to have a component in the abelian factor of $\tilde{g}$. This allows us to shift the Hodge grading of a complex summand (e.g. to make it integral), which is necessary in order to study the cohomology of abelian varieties of generalized Weil type and to obtain all Calabi-Yau variations. We restrict our investigation to these examples and proceed with a minimum of formality.

Given an irrep $V^\lambda$ of $g$ and $E \in \mathfrak{t}$ as before, we take $\tilde{E} = (E, 1) \in g \oplus \mathbb{C} = \tilde{g}$, and define representations of $\tilde{g}$ by

$$V^\lambda \left\{ \frac{a}{2} \right\}_{p,-p-1} := \left( V^\lambda \right)^{p+\frac{a}{2},-p-\frac{a}{2}-1}_{p,-p-1}.$$ We write

$$\tilde{V}^\lambda \left\{ \frac{a}{2} \right\} := V^\lambda \left\{ \frac{a}{2} \right\} \oplus V^{\tau(\lambda)} \left\{ -\frac{a}{2} \right\},$$

whether or not $V^\lambda$ or $V^\lambda \left\{ \frac{a}{2} \right\}$ is complex; this is a variant of van Geemen’s half-twist [vG1] that preserves the weight. As a real Hodge structure, (4.1) has level

$$\ell \left( \tilde{V}^\lambda \left\{ \frac{a}{2} \right\} \right) = \max \left\{ -E(\lambda) + a, -E(\tau(\lambda)) - a \right\}.$$ When (4.2) is odd, define

$$\mu(\lambda, a) := \frac{1}{2} \left\{ E(s \cdot \lambda) - a - 1 \right\}.$$
Proposition 4.1. Assume (4.2) is odd. If either \( \mu(\lambda, a) \) or \( \mu(\tau(\lambda), -a) \) is \( \geq 0 \), then the corresponding weight-\((-1)\) \( \mathbb{R} \)-VHS \( \hat{V}_p^a \{ \frac{a}{2} \} \) over \( X = \Gamma \backslash D \) has a nonzero \( H^1(j) \) (\( j \geq 0 \)). (As above, we shall say it has an infinitesimal normal function.) If both invariants are negative, any underlying \( \mathbb{Q} \)-VHS admits no nontrivial reduced normal function on an \( \acute{e}tale \) neighborhood of \( X \).

4.1. Case \( I_{p,n-p+1} \) (bis). As before, we assume \( p \leq \left\lfloor \frac{n+1}{2} \right\rfloor \). Referring to (3.1), the only choices that make (2.10) equal 1 are \( \lambda = \omega_i \) or \( \omega_n \), unless \( p = 1 \) (in which case all \( \omega_i \) work). We shall focus attention on the variations arising in the odd exterior powers of

(4.4) \( \mathcal{H}_{p,n-p+1} := \hat{V}_p^a \{ \frac{n-p+1}{2(n+1)} \} \) (i.e. \( a = \frac{2p-n-1}{n+1} \))

over \( X = \Gamma \backslash I_{p,n-p+1} \). Notice that (4.4) has level one, with \( h(V^a \{ \frac{a}{2} \}) = (p,n-p+1) \) and \( h(V^\omega \{ \frac{2}{2} \}) = (n-p+1,p) \) for its two complex summands. (These are both the Hodge numbers and the signatures of the relevant Hermitian forms.) Our focus on these cases is motivated by the

Definition 4.2. (cf. [Iz]) A universal \( k \)-Weil abelian \( (n+1) \)-fold is an abelian variety \( \mathcal{A} \rightarrow X \) whose \( H^1 \) recovers \( \mathcal{H}_{p,n-p+1} \) with \( n-p+1 = p+k \).

As in the (0-)Weil case, \( \text{End}(\mathcal{A})_\mathbb{Q} \) contains an imaginary quadratic field; but the generic Mumford-Tate group is \( SU(p,p+k) \) instead of \( SU(p,p) \). For the \( H^{2d+1}_{\text{prim}} \) of \( \mathcal{A} \), an easy argument with Young diagrams shows that

(4.5) \( \bigwedge^{2d+1} \mathcal{H}_{p,n-p+1} \cong \bigoplus_{j=0}^{2d+1} \bigwedge^{2d-1} \mathcal{H}_{p,d-p+1} \bigwedge^{\omega_j,\omega_{n-2d+j}} \{ \frac{(n-2p+1)(2d-2j+1)}{2(n+1)} \} \)

for \( 1 \leq d \leq \frac{n}{2} \). There are no terms of level one in (4.5), except at \( j = 0, 2d+1 \) when \( 2d+1 = n+1 \) in case \( I_{p,p+1} \) and when \( 2d+1 = n \) in case \( I_{p,p} \). For all the other terms on the right-hand side of (3.3), we compute using (3.1)

(4.6) \( \mu(\omega_j, \omega_{n-2d+j}, \frac{(n-2p+1)(2n-2j+1)}{n+1}) = \)

\[
\begin{cases}
  j - d - p, & p < j \\
  1 - d, & p = j \text{ or } n - 2d + j \\
  -d, & j < p < n - 2d + j \\
  p + d - j - n, & n - 2d + j < p
\end{cases}
\]

\(^6\)where \( \omega_0 + \omega_{n-2d} \) [resp. \( \omega_{2d+1} + \omega_0 \)] means \( \omega_{n-2d} \) [resp. \( \omega_{2d+1} \)]
Since \( p + d - j - n \geq 0 \) (with \( d \leq \frac{n}{2}, p \leq \frac{n+1}{2} \)) boils down to the level one cases we identified, the last two entries can be ignored. We have \( 1 - d \geq 0 \) and \( p = j \) or \( n - 2d + j \) in the cases \( (d, p) = (1, 1), (1, 2), (1, 3) \); and \( j - d - p \geq 0 \) for some \( j \) implies \( 2d + 1 \geq 2p - 1 \). Hence we conclude that the variations of level \( > 1 \) in \( H^\text{odd} \) of \( \mathcal{A} \) admitting infinitesimal normal functions therefore lie in

\[
\begin{align*}
s &\quad \text{degrees } 2p - 1 \text{ thru } 2n - 2p + 3, & \text{if } 3 < p < \frac{n+1}{2}. \\
&\quad \text{no degrees}, & \text{if } 3 < p = \frac{n+1}{2}.
\end{align*}
\]

Note that \( p = \frac{n+1}{2} \) \( (k = 0) \) is the Weil abelian setting dispensed with in \( \S 3.5 \). Omitting this case, we can restate our conclusion as follows:

**Theorem 4.3.** For \( k \leq n - 7 \), no étale pullback of a universal \( k \)-Weil abelian \((n+1)\)-fold admits reduced normal functions arising from cycles of dimension less than \( \frac{n-k-1}{2} \) or codimension less than \( \frac{n-k+1}{2} \).

As to why (or whether) greater disparity in the signature of the Hermitian form should lead to more \( AJ \)-nontrivial cycles in the Griffiths group, we can say nothing yet.

### 4.2. Comments on the Hermitian Calabi-Yau VHS.

For the tube domain cases, the CY variations are nothing but the \( V_{\omega}^{k,\omega} \) (described for \( k = 1 \) by Gross [Gro]). In the remaining cases (after [SZ], [FL]), where \( V^{k,\omega} \) is complex, we choose the shift in \( \tilde{V}^{k,\omega} \{ \frac{a}{2} \} \) to minimize the (odd) level while having the CY property.

Writing \( \epsilon(m) = 1 \) (m even) resp. 2 (m odd), to accomplish this we need (cf. (2.10), (4.2))

\[
\ell \left( \tilde{V}^{k,\omega} \left\{ \frac{a}{2} \right\} \right) = \ell \left( V^{k,\omega} \right) + \epsilon \left( \ell \left( V^{k,\omega} \right) \right) \quad \Rightarrow \\
\quad a = \frac{1}{2} \left( E(k, \omega_1) - E(k, \omega_1) \right) + \epsilon \left( \frac{1}{2} \left( E(k, \omega_1) + E(k, \omega_1) \right) \right).
\]

This yields\( a = \frac{2kp^2}{n+1} - kp + \epsilon(kp) \) (for \( I_{p,n-p+1} \)), \( a = -\frac{k}{3} + \epsilon(mk) \) (for \( \Pi_{2m+1} \)), and \( a = -\frac{2k}{3} + 1 \) (for EIII)\( \Box \) Plugging into (4.3) gives

\[
\mu(k, \omega_1, a) = k - \left\lfloor \frac{kp+1}{2} \right\rfloor, \quad 1 - \left\lfloor \frac{mk+1}{2} \right\rfloor, \quad \text{resp.} \quad 1 - k,
\]

which is non-negative when \( p = 1 \) or 2 \( (I_{p,n-p+1,1}, (m, k) = (2, 1) \) \( (\Pi_{2m+1}) \), resp. \( k = 1 \) (EIII). This includes all the level 3 cases, but also two infinite series of examples (since \( k \) is arbitrary) for \( I_{1,n} \) and \( I_{2,n-1} \). Putting this together with the results in \( \S 3 \), we have the

\footnote{For \( k = 1 \), this gives the complex examples in [FL], where we note that the sign on our half-twist is the opposite of theirs. Note that [op. cit.] omits the \( (p, k) = (1, 2) \) cases \( I_{1,n} (\sigma_1; 2 \omega_1) \left\{ \frac{3-n}{2(n+1)} \right\} \), which also have level 3.}
Proposition 4.4. Amongst the minimal-level Calabi-Yau variations over irreducible Hermitian symmetric domains other than $I_{1,n}$ or $I_{2,n-1}$, only those of level three admit infinitesimal normal functions.

References

[Ce] G. Ceresa, *C is not equivalent to $C^-$ in its Jacobian*, Ann. of Math. (2) 117 (1983), no. 2, 285-291.

[CNP] A. Collino, J. Naranjo, and G. Pirola, *The Fano normal function*, J. Math. Pures Appl. (9) 98 (2012), no. 3, 346-366.

[Fa] N. Fakhruddin, *Algebraic cycles on generic abelian varieties*, Compositio Math. 100 (1996), no. 1, 101-119.

[FL] R. Friedman and R. Laza, *Semialgebraic horizontal subvarieties of Calabi-Yau type*, Duke Math. J. 162 (2013), no. 12, 2077-2148.

[FH] W. Fulton and J. Harris, “Representation Theory: A First Course”, Springer-Verlag, New York, 1991.

[vG1] B. van Geemen, *Half-twists of Hodge structures of CM-type*, J. Math. Soc. Japan 53 (2001), no. 4, 813-833.

[vG2] ———, *Kuga-Satake varieties and the Hodge conjecture*, in “The arithmetic and geometry of algebraic cycles (Banff, AB, 1998),” 51-82, NATO Sci. Ser. C Math. Phys. Sci. 548, Kluwer Acad. Publ., Dordrecht, 2000.

[vGV] B. van Geemen and A. Verra, *Quaternionic Pryms and Hodge classes*, Topology 42 (2003), no. 1, 35-53.

[GGK] M. Green, P. Griffiths and M. Kerr, “Mumford-Tate groups and domains: their geometry and arithmetic”, Annals of Math. Studies 183, Princeton Univ. Press, Princeton, NJ, 2012.

[Gre] M. Green, *Griffiths’s infinitesimal invariant and the Abel-Jacobi map*, J. Differ. Geom. 29 (1989), 545-555.

[GG] M. Green and P. Griffiths, *Algebraic cycles and singularities of normal functions*, in “Algebraic cycles and motives (Vol. I)”, 206-263, London Math. Soc. Lect. Not. Ser. 343, Cambridge Univ. Press, Cambridge, 2007.

[Gro] B. Gross, *A remark on tube domains*, Math. Res. Lett. 1 (1994), no. 1, 1-9.

[Ha] R. Hain, *Torelli groups and geometry of moduli spaces of curves*, in “Current topics in complex algebraic geometry (Clemens and Kollar, eds.)”, 97-143, MSRI Publications, No. 28, Cambridge Univ. Press, 1995.

[Iz] E. Izadi, *Some remarks on the Hodge conjecture for abelian varieties*, Ann. Mat. Pura Appl. (4) 189 (2010), no. 3, 487-495.

[KP] M. Kerr and G. Pearlstein, *An exponential history of functions with logarithmic growth*, in “Topology of stratified spaces”, 281-374, Math. Sci. Res. Inst. Publ. 58, Cambridge Univ. Press, 2011.

[Kn] A. Knapp, “Lie groups: beyond an introduction (2nd ed.)”, Progress in Math. 140, Birkhäuser, Boston, 2002.

[Ko] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) 74 (1961), 329-387.

[LZ] R. Laza and Z. Zhang, *Classical period domains*, to appear in “Recent Advances in Hodge Theory (Kerr and Pearlstein, eds.)”, Cambridge Univ. Press, 2015.
[MO] B. Moonen and F. Oort, The Torelli locus and special subvarieties, in “Handbook of moduli (vol. II)”, 549-594, Adv. Lect. Math. (ALM) 25, Intl. Press, Somerville, MA, 2013.

[No] M. Nori, Algebraic cycles and Hodge-theoretic connectivity, Invent. Math. 111(1993), no. 2, 349-373.

[Ra] M. Ragunathan, Cohomology of arithmetic subgroups of algebraic groups: I, Ann. of Math. 86 (1967), 409-424.

[Ro] C. Robles, Schubert varieties as variations of Hodge structure, Selecta Math. (N.S.) 20 (2014), no. 3, 719-768.

[Sc] C. Schoen, Hodge classes on self-products of a variety with an automorphism, Compositio Math. 65 (1988), no. 1, 3-32; Addendum, Compositio Math. 114 (1998), no. 3, 329-336.

[SZ] M. Sheng and K. Zuo, Polarized variations of Hodge structures of Calabi-Yau type and characteristic subvarieties over bounded symmetric domains, Math. Ann. 348 (2010), no. 1, 211-236.

DEPARTMENT OF MATHEMATICS, CAMPUS BOX 1146
WASHINGTON UNIVERSITY IN ST. LOUIS
ST. LOUIS, MO 63130, USA

e-mail: ryan.keast@gmail.com

DEPARTMENT OF MATHEMATICS, CAMPUS BOX 1146
WASHINGTON UNIVERSITY IN ST. LOUIS
ST. LOUIS, MO 63130, USA

e-mail: matkerr@math.wustl.edu