Limit theory for the Gilbert graph

Matthias Reitzner, Matthias Schulte and Christoph Thäle

Abstract

Consider a Poisson point process within a convex set in a Euclidean space. Two points are connected by an edge if their distance is bounded by a prescribed distance parameter. The resulting random geometric graph is called Gilbert graph. The behaviour of this graph is investigated as the intensity of the underlying Poisson point process tends to infinity and the distance parameter goes to zero. The asymptotic expectation and the covariance structure of a class of length-power functionals are investigated and a number of central and non-central limit theorems are derived using the recent Malliavin-Stein technique, which provides explicit error bounds in most cases. Moreover, large deviation inequalities are provided using Talagrand’s concentration inequality for the convex distance.

Keywords. Central limit theorem, covariogram, Gilbert graph, large deviation inequality, Malliavin-Stein method, Poisson point process, random geometric graph, stochastic geometry, Talagrand’s convex distance.

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1 Introduction

Let $\eta_t$ be the restriction to a compact convex observation window $W \subset \mathbb{R}^d$ with volume $V(W) > 0$ of a Poisson point process of constant intensity $t > 0$ in $\mathbb{R}^d$ and let $(\delta_t : t > 0)$ be a sequence of positive real numbers such that $\delta_t \to 0$, as $t \to \infty$. A random graph $G(\eta_t, \delta_t)$ is defined by taking the points of $\eta_t$ as vertices and by connecting two distinct points $x, y \in \eta_t$ by an edge if and only if

$$0 < ||x - y|| \leq \delta_t,$$

where $||x - y||$ stands for the Euclidean distance between $x$ and $y$ (in particular, such a graph cannot have loops). The resulting graph $G(\eta_t, \delta_t)$ is called Gilbert graph, random geometric graph or distance graph, and in the special cases $d = 1$ and $d = 2$ also interval and disc graph, respectively. It has been introduced by Gilbert [11] in the same year in which Erdős and Rényi [8] have introduced their celebrated model for random graphs. As opposed to the Erdős-Rényi random graph, which is a purely combinatorial object, the Gilbert graph is a random geometric graph, because in its construction the relative position of the points in space plays an important role. The Gilbert graph is the maybe most natural construction of a random geometric graph and we refer to the seminal book by Penrose [20] and the references cited therein.

The aim of the present paper is to investigate functionals related to the edge lengths of the Gilbert graph by using recently developed tools, namely the Malliavin-Stein method and a version of Talagrand’s large deviation
inequality for Poisson point processes. The length-power functionals \( L_t^{(α)} \) of interest are defined by

\[
L_t^{(α)} = \frac{1}{2} \sum_{(x,y) \in \eta^2_{t,δ}} \mathbf{1}(\|x - y\| \leq δ_t) \|x - y\|^α,
\]

where \( α \in \mathbb{R} \) and \( \eta^2_{t,δ} \) stands for the set of all pairs of distinct points of \( η_t \). The cases \( α = 0 \) and \( α = 1 \) are of particular importance. Namely, \( L_t^{(0)} \) is the number of edges of \( G(η_t, δ_t) \) and \( L_t^{(1)} \) is its total edge length of the Gilbert graph. In our analysis, we focus on the asymptotic behaviour of \( L_t^{(α)} \), as \( t \to \infty \) and \( δ_t \to 0 \). For this situation we compute the asymptotic expectation of \( L_t^{(α)} \) and asymptotic covariance of \( L_t^{(α_1)} \) and \( L_t^{(α_2)} \) for \( α_1, α_2 \in \mathbb{R} \) and derive univariate and multivariate central limit theorems with explicit rates of convergence as well as a compound Poisson limit theorem. Our main tool to prove the central limit theorems is the recently developed Malliavin-Stein method for Poisson functionals, see [27] [28]. Our investigations of the behaviour of \( L_t^{(α)} \) are completed by large deviation inequalities, which are based on Talagrand’s convex distance and its relative for Poisson point processes introduced in [22]. A further goal of this paper is to understand the behaviour of the single edge lengths. In this context, we will show that the collection of all edge lengths converges, after a suitable re-scaling, to a Poisson point process on the real line. The proofs rest upon recent findings in [27] [28].

For the asymptotic behaviour of the Gilbert graph the interplay between the intensity \( t \) and the distance parameter \( δ_t \) plays a crucial rôle. Clearly, the number of vertices of \( G(η_t, δ_t) \) is just the cardinality of \( η_t \), which has expectation \( t V(W) \) by the definition of a Poisson point process. In addition, the number of edges satisfies the approximation

\[
E L_t^{(0)} \approx \frac{κ_d}{2} t^2 δ_t^d V(W)
\]

as \( t \to \infty \), where \( ≈ \) means that the quotient of the left and the right hand side tends to 1, as \( t \to \infty \), see Section 2 for details. Heuristically, this means that the degree of a typical vertex, i.e. the number of edges emanating from this vertex, is approximately of order \( κ_d t δ_t^d \) (this can be made precise by investigating the degree of a typical point of the point process). This heuristic observation about the degree of a typical vertex leads naturally to three different asymptotic regimes as in Penrose’s book [20]. These are

- the **sparse regime**, where we assume that \( \lim_{t \to \infty} t δ_t^d = 0 \), implying that the degree of a typical vertex tends to zero,
- the **thermodynamic regime**, where we assume that \( \lim_{t \to \infty} t δ_t^d = c \in (0, \infty) \), implying that the mean degree of a typical vertex is asymptotically constant,
- the **dense regime**, where we have \( \lim_{t \to \infty} t δ_t^d = \infty \), which means that the degree of a typical vertex of the Gilbert graph tends to infinity.

The Gilbert graph can be also constructed as an infinite graph with respect to all points of a stationary Poisson point process in \( \mathbb{R}^d \). This construction is most convenient, for example, for investigating percolative properties. In our context, focussing on length-power functionals, one would consider those edges and those parts of edges that are within the observation window \( W \). Although this approach is different from the one described above and used in this paper, the difference arises from boundary effects, which are asymptotically negligible. We also remark that many of our results can be extended to more general notions of distances \( d(x, y) \) and arbitrary bounded measurable functions \( f \), leading to results for \( \sum_{(x,y) \in \eta^2_{t,δ}} \mathbf{1}(d(x, y) \leq δ_t) f(x, y) \).

Yet to keep the exposition short and concise we have decided to concentrate on the Euclidean case and on \( f(x, y) = \|y - x\|^α \).
There is a vast literature on the Gilbert graph. The developments until 2003 are summarized in Penrose’s book [20]. More recent developments, which are relevant in our context are – among others – due to Bourguin and Peccati [3], Lachièze-Rey and Peccati [15, 16] and Reitzner and Schulte [23] or Schulte and Thäle [27]. Important investigations not touched in this paper concern subgraph counting statistics (a far reaching generalization of the concept of $L^{(0)}_t$), which is at the core of Penrose’s book [20], the construction of random simplicial complexes using the Gilbert graph, see e.g. [5] and [13, 14], and questions about percolation in the thermodynamic regime, see e.g. [2, 3, 20]. For investigations concerning the limit theory for similar random graphs we refer e.g. to Penrose and Yukich [21] and more generally to Chapter 11 in the recent book by Haenggi [12].

This paper is organized as follows. In Section 2 the asymptotic expectations as well as asymptotic variances and covariances for functionals $L^{(0)}_t$ are derived, while Section 3 concerns their limiting distributions. The asymptotic behaviour of the point process of the $\alpha$-probability space; expectation, variance and covariance of random variables Section 5 contains large deviation inequalities for $L^{(0)}_t$ using the Gilbert graph, see e.g. [2, 9, 20]. For investigations concerning the limit theory for similar random graphs we refer e.g. to Penrose and Yukich [21] and questions about percolation in the thermodynamic regime, see e.g. [2, 3, 20].

Notation. In this paper we frequently use the following notation. By $(\Omega, \mathcal{F}, \mathbb{P})$ we mean our underlying probability space; expectation, variance and covariance of random variables $X$ and $Y$ with respect to $\mathbb{P}$ are denoted by $\mathbb{E}X$, $\text{Var}X$ and $\text{Cov}(X,Y)$, respectively. We also write $1(\cdot)$ for the indicator function.

Let $\lambda(A)$ stand for the Lebesgue measure on $\mathbb{R}^d$, where $d \geq 1$ is a fixed integer. For a compact and convex set $W \subset \mathbb{R}^d$, $V(W) := \lambda(W)$ and $S(W)$ are the volume and the surface area of $W$, respectively. A $d$-dimensional ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by $B_d(x, r)$ and for a non-negative integer $j$, $\kappa_j$ stands for the volume of the $j$-dimensional unit ball $B_j(0, 1)$. The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1}$.

In this paper we shall use the Landau notation. That is, for $g, h : \mathbb{R} \to \mathbb{R}$ we write (by slight abuse of notation) $g = o(h)$ if $\lim_{t \to \infty} g(t)/h(t) = 0$, $g = O(h)$ if $\lim_{t \to \infty} g(t)/h(t) = c \in \mathbb{R}$ and $g = \Theta(h)$ if $g = O(h)$ and $h = O(g)$.

2 Expectation and covariance structure

To fix the set-up, let $N(W)$ be the space of finite counting measures $\eta = \sum_{i=1}^n \delta_{x_i}$, where $x_1, \ldots, x_n \in W$ are distinct points and where $\delta_{x_i}$ stands for the unit-mass Dirac measure concentrated at $x \in W$. The space $N(W)$ is endowed with the $\sigma$-field $\mathcal{N}(W)$ generated by the evaluation mappings $T_A : N(W) \to \mathbb{R}, \eta \mapsto \eta(A)$ for Borel sets $A \subset W$, see [24, Chapter 3.1]. Alternatively, one can think of $N(W)$ as the set of all finite point configurations of distinct points from $W$. This can be achieved by identifying the measure $\eta$ with its support, which forms a closed subset of $W$, cf. [24, Lemma 3.1.4]. For $\eta \in N(W)$ and a Borel set $A \subset \mathbb{R}^d$, $\eta(A)$ is the number of points of $\eta$ falling in $A$ and $\eta \cap A$ stands for the restricted point configuration $\{x_1, \ldots, x_n\} \cap A$. Due to the geometrical flavour of the Gilbert graph, in most cases we will think of $\eta$ as a set or configuration of points in $W$.

A random measure $\eta_t$, i.e. a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(N(W), \mathcal{N}(W))$, is called a (homogeneous) Poisson point process with intensity $t > 0$ if $P(\eta_t(A) = 0) = \exp(-t\lambda(A))$ for any Borel set $A \subset W$. It should be noted that under these circumstances Rényi’s theorem [24, Corollary 3.2.2] implies that $\eta_t(A)$ is Poisson distributed with mean $t\lambda(A)$ and that for disjoint Borel sets $A_1, \ldots, A_m \subset W$, $m \in \mathbb{N}$, the random variables $\eta_t(A_1), \ldots, \eta_t(A_m)$ are independent. Alternatively, one can think of $\eta_t$ as a random set of $\eta_t(W)$ random points, which are independently placed within $W$ according to the uniform distribution.

Starting with $\eta_t$, we thus define the Gilbert graph for distances parameters $(\delta_t)_{t \geq 0}$ as the random graph with vertex set $\eta_t$ and where an edge between two distinct points of $\eta_t$ is constructed if and only if their Euclidean distance is at most $\delta_t$. Recall now the definition of the length-power functional $I^{(0)}_t$ from [14].
We begin by investigating the expectation of $L_t^{(\alpha)}$. To state it, let $g_W(y) = V(W \cap (W + y))$, $y \in \mathbb{R}^d$, be the so-called covariogram of $W$. This functional is well known in convex geometry and has a long history. In particular, we refer to the recent breakthrough by Averkov and Bianchi [1] regarding the famous covariogram problem, the work of Galerne [10], and the references cited therein.

**Theorem 2.1.** For any $\alpha > -d$ we have

$$
\frac{d\kappa_d}{2(\alpha + d)} t^2 \delta_t^{\alpha+d} V(W) - \frac{\kappa_{d-1}}{2(\alpha + d + 1)} t^2 \delta_t^{\alpha+d+1} S(W) \leq E L_t^{(\alpha)} \leq \frac{d\kappa_d}{2(\alpha + d)} t^2 \delta_t^{\alpha+d} V(W). 
$$

(2.1)

In particular, it holds that

$$
\mathbb{E} L_t^{(\alpha)} = \frac{t^2}{2} \int_{B^d(0, \delta_t)} \|y\|^\alpha g_W(y) \, dy = \frac{d\kappa_d}{2(\alpha + d)} t^2 \delta_t^{\alpha+d} V(W)(1 + O(\delta_t)).
$$

(2.2)

**Remark 2.2.** Theorem [2.1] especially shows that the number of edges of the Gilbert graph is of order $t^2 \delta_t^d$, whereas its total edge length is of order $t^2 \delta_t^{d+1}$.

Before proving the result, let us recall one of the main tools of our analysis, the multivariate Mecke formula for Poisson processes. In our set-up, it says that

$$
\mathbb{E} \sum_{(x_1, \ldots, x_k) \in \eta_{t,x}} f(x_1, \ldots, x_k, \eta_t) = t^k \int_W \cdots \int_W \mathbb{E} f(x_1, \ldots, x_k, \eta_t + \delta x_1 + \ldots + \delta x_k) \, dx_1 \ldots dx_k,
$$

(2.3)

where $k \geq 1$ is a fixed integer, $f : W^k \times \mathbb{N}(W) \to \mathbb{R}$ is a non-negative measurable function and $\eta_{t,x}$ is the set of all $k$-tuples of distinct points of $\eta_t$, cf. [24 Corollary 3.2.3]. If $f$ is only a function on $W^k$ and does not depend on $\eta_t$, which will often be the case in the sequel, the expectation on the right-hand side can be omitted.

**Proof of Theorem 2.1.** We apply the multivariate Mecke formula (2.3) with $k = 2$ and $f(x, y) = \mathbf{1}(|x - y| \leq \delta_t)||x - y||^\alpha$ there to obtain

$$
\mathbb{E} L_t^{(\alpha)} = \frac{1}{2} \mathbb{E} \sum_{(x,y) \in \eta_{t,x}^2} \mathbf{1}(|x - y| \leq \delta_t)||x - y||^\alpha
$$

$$
= \frac{t^2}{2} \int_W \int_W \mathbf{1}(|x - y| \leq \delta_t)||x - y||^\alpha \, dx \, dy
$$

$$
= \frac{t^2}{2} \int_{\mathbb{R}^d} \mathbf{1}(|y| \leq \delta_t)||y||^\alpha \int_{\mathbb{R}^d} \mathbf{1}(x \in W, x - y \in W) \, dx \, dy
$$

$$
= \frac{t^2}{2} \int_{\mathbb{R}^d} \mathbf{1}(|y| \leq \delta_t)||y||^\alpha g_W(y) \, dy
$$

$$
= \frac{t^2}{2} \int_{B^d(0, \delta_t)} ||y||^\alpha g_W(y) \, dy,
$$

4
which gives the first part of (2.2). Transformation into spherical coordinates yields

$$\mathcal{E}L_i^{(\alpha)} = \frac{t^2}{2} \int_0^{\delta_\varepsilon} \|y\|^\alpha g_W(y) \, dy = \frac{t^2}{2} \int_0^{\delta_\varepsilon} r^{\alpha+d-1} \int_{S^{d-1}} g_W(ru) \, du \, dr,$$

(2.4)

where $du$ stands for the infinitesimal element of the spherical Lebesgue measure.

Now, fix $u \in S^{d-1}$ and make a Taylor expansion of $g_W(ru)$ at $r = 0$. This gives

$$g_W(ru) = g_W(0) + r \frac{d}{dr} g_W(ru) \bigg|_{r=0} + o(r)$$

for any $r > 0$. According to [10] Theorem 14 (ii), the derivative of $g_W(ru)$ at $r = 0$ exists and is finite. By integration of (2.5) with respect to $u$ and the facts that $g_W(0) = V(W)$ and that

$$\int_{S^{d-1}} \frac{d}{dr} g_W(ru) \bigg|_{r=0} \, du = -\kappa_{d-1} S(W)$$

by [10] Theorem 14 (iii), we infer the relation

$$\int_{S^{d-1}} g_W(ru) \, du = d\kappa_d V(W) - \kappa_{d-1} S(W) r + o(r), \quad \text{for} \quad r > 0,$$

(2.6)

which remains valid for all so-called sets $W$ of finite perimeter. Furthermore, for given $u \in S^{d-1}$, $g_W(ru)$ is a Lipschitz function in $r$ whose Lipschitz constant coincides with the $(d-1)$-dimensional volume $V_{d-1}(W|u^\perp)$ of the orthogonal projection of $W$ onto the hyperplane $u^\perp$ orthogonal to $u$, see [10] Theorem 13.3. In particular, this implies that $V(W) \geq g_W(ru) \geq V(W) - V_{d-1}(W|u^\perp) r$ for all $r > 0$. Thus, by the well-known Cauchy’s surface area formula from integral geometry [24] Equation (6.12) we obtain

$$d\kappa_d V(W) \geq \int_{S^{d-1}} g_W(ru) \, du \geq d\kappa_d V(W) - \kappa_{d-1} S(W) r,$$

(2.7)

which proves that the $o(r)$-term in (2.6) is positive and bounded by $\kappa_{d-1} S(W) r$. Combining (2.4) and (2.6), (2.2) follows by integration with respect to $r$:

$$\mathcal{E}L_i^{(\alpha)} = \frac{t^2}{2} \int_0^{\delta_\varepsilon} \left( d\kappa_d V(W) r^{\alpha+d-1} - \kappa_{d-1} S(W) r^{\alpha+d} + o(r^{\alpha+d}) \right) \, dr.$$

(2.8)

Analogously, (2.1) is immediate from (2.7).

\[ \square \]

After having investigated the first-moment behaviour of $L_i^{(\alpha)}$, we now investigate the covariance structure of these functionals for different values of $\alpha \in \mathbb{R}$.

**Theorem 2.3.** For $\alpha, \beta > -d$ such that $\alpha + \beta > -d$ we have the inequality

$$\left( \sigma_1 t^2 \delta_1^{\alpha+\beta+d} + \sigma_2 t^3 \delta_1^{\alpha+\beta+2d} \right) (V(W) - S(K) \delta_t) \leq C(L_i^{(\alpha)}, L_i^{(\beta)}) \leq \left( \sigma_1 t^2 \delta_1^{\alpha+\beta+d} + \sigma_2 t^3 \delta_1^{\alpha+\beta+2d} \right) V(W)$$

(2.8)
Define \( \overline{L}_i^{(\alpha_i)} = L_i^{(\alpha_i)}/\max\{t^\alpha_i, t^{\alpha_i+\alpha/2}, t^{\alpha_i+\beta} \} \) with distinct \( \alpha_i > -d/2 \) for \( i = 1, \ldots, m \). Then the random vector \( (\overline{L}_1^{(\alpha_1)}, \ldots, \overline{L}_i^{(\alpha_m)}) \) has the asymptotic covariance matrix

\[
\Sigma := \lim_{t \to \infty} \left( C(\overline{L}_i^{(\alpha_i)}, \overline{L}_j^{(\alpha_j)}) \right)_{i,j=1}^m = \begin{cases} 
\Sigma_1 & : \lim_{t \to \infty} t \delta_t^d = 0 \\
\Sigma_1 + c \Sigma_2 & : \lim_{t \to \infty} t \delta_t^d = c \in (0, 1] \\
\frac{1}{c} \Sigma_1 + \Sigma_2 & : \lim_{t \to \infty} t \delta_t^d = c \in (1, \infty) \\
\Sigma_2 & : \lim_{t \to \infty} t \delta_t^d = \infty,
\end{cases}
\]

(2.9)

with the matrices \( \Sigma_1 \) and \( \Sigma_2 \) defined as

\[
\Sigma_1 = \frac{d \kappa_d}{2} V(W) \left( \frac{1}{\alpha_i + \alpha_j + d} \right)_{i,j=1}^m \quad \text{and} \quad \Sigma_2 = d^2 \kappa_d^2 V(W) \left( \frac{1}{(\alpha_i + d)(\alpha_j + d)} \right)_{i,j=1}^m.
\]

In particular, for \( \alpha > -d/2 \) one has the variance asymptotics

\[
V L_i^{(\alpha)}(\beta) = \left( \frac{d \kappa_d}{2 (2\alpha + d)} t^{2\alpha+d} + \frac{d^2 \kappa_d^2}{(\alpha + d)^2} t^{3\alpha+2d} \right) V(W)(1 + O(\delta_t)).
\]

Proof. By definition, we have

\[
L_t^{(\alpha)} = \left( \frac{1}{2} \sum_{(x_1,y_1) \in \eta_t^{\alpha}} 1(\|x-y\| \leq \delta_t) \|x-y\|^\alpha \right) \left( \frac{1}{2} \sum_{(x_2,y_2) \in \eta_t^{\alpha}} 1(\|x-y\| \leq \delta_t) \|x-y\|^\beta \right).
\]

We have to distinguish three cases. The first case arises if the points of the two pairs \( (x_1, y_1) \) and \( (x_2, y_2) \) are all distinct. The second case arises if one of the points of the first pair is identical with one of the points in the second pair. Finally, in the third case both pairs are comprised of the same points of \( \eta_t \). Taking additionally into account multiple counting, we thus arrive at

\[
L_t^{(\alpha)} L_t^{(\beta)} = \frac{1}{4} \sum_{(x_1, x_2, x_3, x_4) \in \eta_t^{\alpha}} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha 1(\|x_3 - x_4\| \leq \delta_t) \|x_3 - x_4\|^\beta \\
+ \sum_{(x_1, x_2, x_3) \in \eta_t^{\alpha}} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha 1(\|x_1 - x_3\| \leq \delta_t) \|x_1 - x_3\|^\beta \\
+ \frac{1}{2} \sum_{(x_1, x_2) \in \eta_t^{\alpha}} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha \|x_1 - x_2\|^{\beta}. 
\]
Applying the multivariate Mecke formula (2.3) to each of the three sums yields

\[
\begin{align*}
    \mathbb{E}L^{(\alpha)}_t L^{(\beta)}_t &= \frac{t^4}{4} \int_W \int_W \int_W \int_W 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha 1(\|x_3 - x_4\| \leq \delta_t) \|x_3 - x_4\|^\beta dx_1 dx_2 dx_3 dx_4 \\
    &+ t^3 \int_W \int_W \int_W 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha 1(\|x_3 - x_3\| \leq \delta_t) \|x_1 - x_3\|^\beta dx_1 dx_2 dx_3 \\
    &+ \frac{t^2}{2} \int_W \int_W 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^\alpha \|x_1 - x_2\|^\beta dx_1 dx_2.
\end{align*}
\]

The first term is just the product of \(\mathbb{E}L^{(\alpha)}_t\) and \(\mathbb{E}L^{(\beta)}_t\) as can be seen from the proof of Theorem 2.1 and we thus get

\[
C(L^{(\alpha)}_t, L^{(\beta)}_t) = t^3 \int_W \int_W 1(\|y - x_1\| \leq \delta_t) \|y - x_1\|^\alpha dx_1 \int_W 1(\|y - x_2\| \leq \delta_t) \|y - x_2\|^\beta dx_2 dy \\
+ \frac{t^2}{2} \int_W \int_W 1(\|y - y\| \leq \delta_t) \|y - y\|^\alpha \|x - y\|^\beta dx_1 dy.
\]

For a point \(y \in W\) such that \(B^d(y, \delta_t) \subset W\) and some \(\gamma > -d\), we see by transforming into spherical coordinates that

\[
\int_W 1(\|y - z\| \leq \delta_t) \|y - z\|^\gamma dz = \int_{B^d(y, \delta_t)} 1(\|y - z\| \leq \delta_t) \|y - z\|^\gamma dz = d \kappa_d \delta_t^\delta \int_0^\gamma r^{\gamma - d - 1} dr = \frac{d \kappa_d}{\gamma + d} \delta_t^{\gamma + d}.
\]

For points \(y \in W\) with \(B^d(y, \delta_t) \nsubseteq W\), one has the inequality

\[
0 \leq \int_W 1(\|y - z\| \leq \delta_t) \|y - z\|^\gamma dz \leq \frac{d \kappa_d}{\gamma + d} \delta_t^{\gamma + d}.
\]

Denote by \(W_{-\delta_t} = \{w \in W : B^d(w, \delta_t) \subset W\}\) the (possibly empty) inner parallel set of \(W\) and observe that

\[
V(W_{-\delta_t}) \geq V(W) - S(W) \delta_t^d.
\]

(2.11)

Now we obtain

\[
\left(\frac{d \kappa_d}{2(\alpha + \beta + d)} t^2 \delta_t^{\alpha + \beta + d} + \frac{d^2 \kappa_d^2}{(\alpha + d)(\beta + d)} t^3 \delta_t^{\alpha + 2\beta + 2d}\right) V(W_{-\delta_t}) \leq C(L^{(\alpha)}_t, L^{(\beta)}_t) \leq \left(\frac{d \kappa_d}{2(\alpha + \beta + d)} t^2 \delta_t^{\alpha + \beta + d} + \frac{d^2 \kappa_d^2}{(\alpha + d)(\beta + d)} t^3 \delta_t^{\alpha + 2\beta + 2d}\right) V(W),
\]

which together with (2.11) yields (2.8). The form of the asymptotic covariance matrix in (2.9) is now a direct consequence.

**Remark 2.4.** As it can be seen from the proof, the covariance \(C(L^{(\alpha)}_t, L^{(\beta)}_t)\) is not finite if \(\alpha \leq -d, \beta \leq -d\) or \(\alpha + \beta \leq -d\).
We finally discuss definiteness properties of the asymptotic covariance matrix $\Sigma$, which has been defined in Theorem 2.3.

**Proposition 2.5.** For $m \geq 2$ the asymptotic covariance matrix $\Sigma$ given in (2.9) is positive definite in the sparse and in the thermodynamic regime, while it is singular in the dense regime.

*Proof.* The matrix $\Sigma_2$ is only of rank 1 so that (2.9) implies that the asymptotic covariance matrix is singular in the dense regime for $m \geq 2$. In order to show the positive definiteness for the remaining regimes, we prove that, for any $(a_1, \ldots, a_m) \neq 0$,

$$\lim_{t \to \infty} \mathbb{V} \sum_{i=1}^{m} a_i \tilde{L}_i^{(a_i)} > 0.$$ 

We can assume that $\lim t\delta_t^d = c$ with $c \in [0, \infty)$. By the same arguments as in the proof of Theorem 2.3, we obtain that

$$\lim_{t \to \infty} \mathbb{V} \sum_{i=1}^{m} a_i \tilde{L}_i^{(a_i)} = \lim_{t \to \infty} \left[ t^3 \int \frac{1}{W} \left( \int \mathbb{I}(\|x - y\| \leq \delta_t) \sum_{i=1}^{m} a_i \|x - y\|^{\alpha_i} / \max\{t\delta_t^{\alpha_i + d/2}, t^{3/2}\delta_t^{\alpha_i + d}\} \, dx \right)^2 \, dy 

+ \frac{t^2}{2} \int \frac{1}{W} \left( \sum_{i=1}^{m} a_i \|x - y\|^{\alpha_i} / \max\{t\delta_t^{\alpha_i + d/2}, t^{3/2}\delta_t^{\alpha_i + d}\} \right)^2 \, dx \, dy \right].$$

Since the first term is non-negative, we have, for any $u \in (0, 1)$,

$$\lim_{t \to \infty} \mathbb{V} \sum_{i=1}^{m} a_i \tilde{L}_i^{(a_i)} \geq \lim_{t \to \infty} \frac{1}{2 \max\{\delta_t^d, t\delta_t^d\}} \int_{W} \frac{1}{B^d(y, u\delta_t)} \int \mathbb{I}(\|x - y\| \leq \delta_t) \sum_{i,j=1}^{m} a_i a_j (\|x - y\| / \delta_t)^{\alpha_i + \alpha_j} \, dx \, dy 

= \lim_{t \to \infty} \frac{d\kappa_d V_d(W)}{2 \max\{1, t\delta_t^d\}} \sum_{i,j=1}^{m} a_i a_j \frac{1}{\alpha_i + \alpha_j + d} u^{\alpha_i + \alpha_j + d} 

= \frac{d\kappa_d V_d(W)}{2 \max\{1, c\}} \sum_{i,j=1}^{m} a_i a_j \frac{1}{\alpha_i + \alpha_j + d} u^{\alpha_i + \alpha_j + d}.$$ 

Since the powers $\alpha_1, \ldots, \alpha_m$ are distinct, there is an $i_0 \in \{1, \ldots, m\}$ such that $a_{i_0} \neq 0$ and $\alpha_{i_0} < \alpha_i$ for all $i \in \{1, \ldots, m\}\{i_0\}$ with $a_i \neq 0$. This means that the sum on the right-hand side behaves as $a_{i_0}^2 u^{2\alpha_{i_0} + d}/(2\alpha_{i_0} + d)$ as $u \to 0$. Consequently, the right-hand side is positive for sufficiently small $u$, which concludes the proof. \qed

### 3 Distributional limit theorems

After having investigated first- and second-order properties of the functionals $L_t^{(\alpha)}$ in the previous section, we turn now to distributional limit theorems. More precisely, we state a univariate and multivariate central limit theorem for the regime, where $t^2 \delta_t^d \to \infty$ as $t \to \infty$, and show a limit theorem with a limiting compound Poisson distribution if $t^2 \delta_t^d \to c \in (0, \infty)$ as $t \to \infty$. We start with bounds for the normal approximation, whose proofs make use of the so-called Malliavin-Stein method and are postponed to the end of the section.
Theorem 3.1. Let $\alpha > -d/2$ and let $N$ be a standard Gaussian random variable. Then there is a constant $C > 0$ only depending on $\alpha$ and $W$ such that

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{L_t^{(\alpha)}}{\sqrt{\text{Var}(L_t^{(\alpha)})}} \leq x \right) - \mathbb{P} \left( N \leq x \right) \right| \leq C t^{-\frac{d}{2}} \max \{ 1, (t \delta_i^d)^{-\frac{1}{2}} \}
$$

for $t \geq 1$.

For $\alpha = 0$, $L_t^{(0)}$ counts the number of edges of the Gilbert graph, which is a special case of a so-called subgraph counting. In this case, Theorem 3.1 extends [20, Theorem 3.9], which does not provide a rate of convergence.

Remark 3.2. An alternative way to measure the distance between the distributions of two random variables $Y$ and $Z$ is the so-called Wasserstein distance. It is defined as

$$d_W(Y, Z) = \sup_{h \in \text{Lip}(1)} |\mathbb{E} h(Y) - \mathbb{E} h(Z)|,$$

where Lip(1) stands for the set of all functions $h : \mathbb{R} \rightarrow \mathbb{R}$ with a Lipschitz constant bounded by 1. The Wasserstein distance between the standardization of $L_t^{(\alpha)}$ and a standard Gaussian random variable has been considered in [23] and for the special case of edge counting also in [15].

In order to compare the distributions of two $m$-dimensional random vectors $Y, Z$, we will use the so-called $d_3$-distance that, similarly to the Wasserstein distance discussed in Remark 3.2, is defined as

$$d_3(Y, Z) = \sup_{g \in \mathcal{H}_m} |\mathbb{E} g(Y) - \mathbb{E} g(Z)|.$$

Here, $\mathcal{H}_m$ stands for the class of all thrice continuously differentiable functions $g : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$\max_{1 \leq i_1, i_2 \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^2 g}{\partial x_{i_1} \partial x_{i_2}}(x) \right| \leq 1 \quad \text{and} \quad \max_{1 \leq i_1, i_2, i_3 \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^3 g}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}}(x) \right| \leq 1.$$

Notice that Lip(1) as well as $\mathcal{H}_m$ are convergence determining classes, that is, convergence in Wasserstein distance or $d_3$-distance implies weak convergence of the involved laws, respectively.

Theorem 3.3. Let $L_t = (L_t^{(\alpha_1)}, \ldots, L_t^{(\alpha_m)})$ with $L_t^{(\alpha_i)} = (L_t^{(\alpha_i)} - \mathbb{E} L_t^{(\alpha_i)})/\max \{ t^{\delta_i^{\alpha_i}+d/2}, t^{3/2} \delta_i^{\alpha_i}+d \}$ and $\alpha_i > -d/2, i = 1, \ldots, m$, and let $\mathcal{N}(\Sigma)$ be an $m$-dimensional centred Gaussian random vector with the covariance matrix $\Sigma$ as defined at (2.9). Then there are constants $C_1, C_2, C_3 > 0$ only depending on $\alpha_i, i = 1, \ldots, m$, and $W$ such that

$$d_3(L_t, \mathcal{N}(\Sigma)) \leq C_1 \delta_t + C_2 \overline{R}_t + C_3 t^{-\frac{d}{2}} \max \{ 1, (t \delta_i^d)^{-\frac{1}{2}} \}$$

for any $t \geq 1$, where

$$\overline{R}_t = \begin{cases} t^{-1} \delta_i^{-d} & : \lim_{t \to \infty} t \delta_i^d = \infty, \\ 0 & : \lim_{t \to \infty} t \delta_i^d = c \in (0, \infty), \\ t \delta_i^d & : \lim_{t \to \infty} t \delta_i^d = 0. \end{cases}$$
Remark 3.4. It should be mentioned that if $\alpha_i + d/2$ for all $i = 1, \ldots, m$ is uniformly bounded away from zero, by some $K > 0$ say, then the constant $C$ in Theorem 3.1 and the constants $C_1, C_2, C_3$ in Theorem 3.3 can be chosen in such a way that they only depend on $K$ and $W$.

Observe that the bounds in Theorem 3.1 and Theorem 3.3 both imply convergence in distribution of $L_t^{(\alpha)}$ to $N$ or $L_t$ to $N(\Sigma)$, respectively, if $t^2 \delta_t^d \to \infty$ as $t \to \infty$.

Next, for $\alpha, \beta > 0$, let $L_t^{(\alpha)}$ be chosen in such a way that they only depend on $K$ and $W$.

Remark 3.6. Note that due to the assumption that $t^2 \delta_t^d \to 0$ as $t \to \infty$, the number of edges and hence also $L_t^{(\alpha)}$ for any $\alpha > -d$ converges in distribution to the random variable, which is identically zero. Before proving Theorems 3.4 and 3.5 we shall discuss the remaining case in which $t^2 \delta_t^d \to c \in (0, \infty)$, as $t \to \infty$. It has been shown in Theorem 4.12 in [15] that in this case, $L_t^{(0)}$ converges in distribution to a Poisson random variable, and in Theorem 5.1 in [17] even a rate of convergence is derived. Our next result shows a similar asymptotic behaviour for $\alpha > 0$. In that case, the limiting random variable has a compound Poisson distribution. The proof is postponed to Section 4.

Theorem 3.5. Assume that $t^2 \delta_t^d \to c \in (0, \infty)$, as $t \to \infty$, let $\alpha > 0$, and let $Z$ be a random variable having the same distribution as $\sum_{i=1}^Y X_i$, where $Y$ is a Poisson distributed random variable with mean $\kappa_d c V(W)/2$ and $X_1, \ldots, X_Y$ are independent random variables with probability density

$$u \mapsto \frac{d}{\alpha c} u^{d/\alpha - 1} \mathbf{1}_{[0, c^d/\alpha]}(u).$$

(3.1)

Then $t^{2\alpha/d} L_t^{(\alpha)}$ converges in distribution to $Z$, as $t \to \infty$.

Remark 3.6. Note that due to the assumption that $t^2 \delta_t^d \to c \in (0, \infty)$, as $t \to \infty$, also $L_t^{(\alpha)}/\delta_t^d$ converges to a compound Poisson random variable.

Proof of Theorem 3.1 and Theorem 3.3 For $\alpha > -d/2$ define functions $f_1^{(\alpha)} : W \to \mathbb{R}$ and $f_2^{(\alpha)} : W^2 \to \mathbb{R}$ by

$$f_1^{(\alpha)}(x) = t \int_W \mathbf{1}(\|x - y\| \leq \delta_t) \|x - y\|^\alpha \, dy$$

and

$$f_2^{(\alpha)}(x, y) = \frac{1}{2} \mathbf{1}(\|x - y\| \leq \delta_t) \|x - y\|^\alpha.$$

Next, for $\alpha, \beta > -d/2$ define the quantities $M_1^{(\alpha, \beta)}$, $M_2^{(\alpha, \beta)}$, and $M_3^{(\alpha, \beta)}$ by

$$M_1^{(\alpha, \beta)} = t \int_W f_1^{(\alpha)}(x)^2 f_1^{(\beta)}(x)^2 \, dx,$$

$$M_2^{(\alpha, \beta)} = 8t^3 \int_W \int_W \int_W f_1^{(\alpha)}(x_1) f_1^{(\alpha)}(x_2) f_2^{(\beta)}(x_1, x_3) f_2^{(\beta)}(x_2, x_3) \, dx_1 \, dx_2 \, dx_3$$

$$+ 4t^2 \int_W \int_W f_1^{(\alpha)}(x_1) f_1^{(\alpha)}(x_2) f_2^{(\beta)}(x_1, x_2) f_2^{(\beta)}(x_1, x_2) \, dx_1 \, dx_2.$$
and

\[
M_{22}^{(\alpha,\beta)} = 48t^4 \int \int \int \int f_2^{(\alpha)}(x_1, x_2) f_2^{(\beta)}(x_3, x_4) f_2^{(\beta)}(x_4, x_1) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \\
+ 96t^3 \int \int \int \int f_2^{(\alpha)}(x_1, x_2) f_2^{(\alpha)}(x_1, x_3) f_2^{(\beta)}(x_3, x_4) f_2^{(\beta)}(x_4, x_2) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \\
+ 8t^2 \int \int \int \int f_2^{(\alpha)}(x_1, x_2)^2 f_2^{(\beta)}(x_1, x_2)^2 \, dx_1 \, dx_2.
\]

We can now recall bounds for the Kolmogorov distance between the standardization of a Gaussian random variable \( \mathcal{N} \) and the distance between \( \mathbf{L}_i \) and and centred Gaussian random vector \( \mathbf{N}(\Sigma) \) with covariance matrix \( \Sigma \) defined at (2.9). First, it follows from Theorem 4.2 in [26] that

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{L_i^{(\alpha)} - \mathbb{E}L_i^{(\alpha)}}{\sqrt{V L_i^{(\alpha)}}} \leq x \right) - \mathbb{P}(N \leq x) \right| \leq 621 \frac{\sqrt{M_{11}^{(\alpha,\alpha)}} + 2\sqrt{M_{12}^{(\alpha,\alpha)}} + \sqrt{M_{22}^{(\alpha,\alpha)}}}{VL_i^{(\alpha)}}. \quad (3.2)
\]

A similar bound with a different constant for the Wasserstein distance was derived in [23]. For the multivariate case we have

\[
d_3(\mathbf{L}_i, \mathbf{N}(\Sigma)) \leq \frac{1}{2} \sum_{i,j=1}^{m} \left| \Sigma_{i,j} - \mathcal{C}(\bar{L}_i^{(\alpha)}, \bar{L}_j^{(\alpha)}) \right| \\
+ 4\sqrt{2m} \left( \sum_{i=1}^{m} \sqrt{V \bar{L}_i^{(\alpha)}} + 1 \right) \sum_{i,j=1}^{m} \frac{\sqrt{M_{11}^{(\alpha,\alpha),j}} + 2\sqrt{M_{12}^{(\alpha,\alpha),j}} + \sqrt{M_{22}^{(\alpha,\alpha),j}}}{\max\{t^2 \delta_i^t, t^3 \delta_i^t \delta_i^2 d \} \delta_i^{\alpha + \alpha}}, \quad (3.3)
\]

which can be deduced from Theorem 4.2 in [19] in a similar way as the bound for the Wasserstein distance in [23] is obtained from Theorem 3.1 in [18] (for an exact proof we refer to the PhD thesis [25]). Thus, in order to show Theorem 3.1 and Theorem 3.3 we need to evaluate upper bounds for the deterministic integrals in \( M_{11}^{(\alpha,\beta)}, M_{12}^{(\alpha,\beta)} \) and \( M_{22}^{(\alpha,\beta)} \), \( \alpha, \beta > -d/2 \), and to control the convergence of the covariance matrix of \( \mathbf{L}_i \) to \( \Sigma \).

By the same arguments as in the proof of Theorem 2.3, we have

\[
f_1^{(\alpha)}(x) = t \int_W 1(\|x - y\| \leq \delta_i) \|x - y\|^\alpha \, dy \leq \frac{d \kappa_d^i}{\alpha + d} \delta_i^{\alpha + d}.
\]

Hence,

\[
M_{11}^{(\alpha,\beta)} \leq \frac{d^4 \kappa_d^4 V(W)}{(\alpha + d)^2 (\beta + d)^2} t^5 \delta_i^{2\alpha + 2\beta + 4d}
\]

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and, similarly,

\[
M_{12}^{(\alpha,\beta)} \leq 2t^3 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\|, \|x_2 - x_3\| \leq \delta_t) \|x_1 - x_2\|^{\alpha} \|x_2 - x_3\|^{\beta} dx_1 dx_2 dx_3 \\
+ t^2 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^{2\beta} dx_1 dx_2 \\
\leq \frac{2d^4 \kappa_d^4 V(W)}{(\alpha + d)^2} t^5 \delta_t^{2\alpha+2\beta+4d} + \frac{d^3 \kappa_d^3 V(W)}{(\alpha + d)^2(2\beta + d)} t^4 \delta_t^{2\alpha+2\beta+3d}.
\]

By the Cauchy-Schwarz inequality and the same arguments as above, the term \(M_{22}^{(\alpha,\beta)}\) can be estimated as follows:

\[
M_{22}^{(\alpha,\beta)} \leq 3t^4 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\|, \|x_2 - x_3\|, \|x_3 - x_4\|, \|x_4 - x_1\| \leq \delta_t) \\
\|x_1 - x_2\|^{\alpha} \|x_2 - x_3\|^{\alpha} \|x_3 - x_4\|^{\beta} \|x_4 - x_1\|^{\beta} dx_1 dx_2 dx_3 dx_4 \\
+ 6t^4 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\|, \|x_1 - x_3\|, \|x_2 - x_3\| \leq \delta_t) \|x_1 - x_2\|^{\alpha} \|x_1 - x_3\|^{\alpha+\beta} \|x_2 - x_3\|^{\beta} dx_1 dx_2 dx_3 \\
+ \frac{1}{2} t^2 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^{2(\alpha+\beta)} dx_1 dx_2 \\
\leq 3t^4 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\|, \|x_2 - x_3\| \leq \delta_t) \left(\int \frac{1}{W} 1(\|x_4 - x_1\| \leq \delta_t) \|x_4 - x_1\|^{2\beta} dx_4\right)^{1/2} \\
\left(\int \frac{1}{W} 1(\|x_4 - x_3\| \leq \delta_t) \|x_4 - x_3\|^{2\beta} dx_4\right)^{1/2} \|x_1 - x_2\|^{\alpha} \|x_2 - x_3\|^{\alpha} dx_1 dx_2 dx_3 \\
+ 6t^4 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_3\| \leq \delta_t) \|x_1 - x_3\|^{\alpha+\beta} \left(\int \frac{1}{W} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^{2\alpha} dx_2\right)^{1/2} \\
\left(\int \frac{1}{W} 1(\|x_2 - x_3\| \leq \delta_t) \|x_2 - x_3\|^{2\beta} dx_2\right)^{1/2} dx_1 dx_3 \\
+ \frac{1}{2} t^2 \int \frac{d^2 \kappa_d}{(\alpha + d)^2} t^2 \delta_t^{2(\alpha+d)} 1(\|x_1 - x_2\| \leq \delta_t) \|x_1 - x_2\|^{2(\alpha+\beta)} dx_1 dx_2 \\
\leq \frac{3d^4 \kappa_d^4 V(W)}{(\alpha + d)^2} t^4 \delta_t^{2\alpha+2\beta+3d} + \frac{6d^3 \kappa_d^3 V(W)}{2(\alpha + d)^2} \delta_t^{2\alpha+2\beta+2d} + \frac{d \kappa_d V(W)}{2(\alpha + d)^2} \delta_t^{2\alpha+2\beta+d}.
\]

It is now easy to see that there are constants \(c_1, c_2, c_3, c_4, c_5, c_6 > 0\) only depending on \(\alpha, \beta\) and \(W\) such that

\[
\sqrt{M_{11}^{(\alpha,\beta)}} / \max\{t^2 \delta_t^2, t^3 \delta_t^2\} \delta_t^{\alpha+\beta} \leq c_1 (t^{5/2} \delta_t^{2d}) / \max\{t^2 \delta_t^d, t^3 \delta_t^{2d}\} = c_1 t^{-1/2} \min\{t \delta_t^d, 1\},
\]

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Theorem 4.1. If $\alpha > 0$ and $t^2 \delta^d_t \to \infty$, as $t \to \infty$, the point process $t^{2\alpha/d} \xi_t^{(\alpha)}$ converges in distribution to a Poisson point process on $\mathbb{R}_+$ with intensity measure

$$\nu(B) = \frac{d \delta^d_t}{2\alpha} V(W) \int_B u^{(d/\alpha)-1} \, du, \quad B \subset \mathbb{R}_+ \text{ Borel.}$$
Proof. We will make use of the theory developed in [27], in particular Theorem 1.1 ibidem. According to that result, the desired point process convergence follows once we have shown that i)

$$\lim_{t \to \infty} a_t(u) := \lim_{t \to \infty} \frac{1}{2} \sum_{(x,y) \in \eta_t \cap \partial B} \mathbb{1}(\|x - y\|^\alpha \leq \min\{\delta_t^\alpha, ut^{-2\alpha/d}\}) = \frac{K_\alpha}{2} V(W) u^{d/\alpha},$$  \hspace{2cm} (4.1)

and that ii)

$$\lim_{t \to \infty} r_t(u) := \lim_{t \to \infty} \sup_{y \in W} \lambda\{x \in W : \|x - y\|^\alpha \leq \min\{\delta_t^\alpha, ut^{-2\alpha/d}\}\} = 0 \hspace{2cm} (4.2)$$

for all \(u > 0\). Since \(t^2 \delta_t^\alpha \to \infty\) as \(t \to \infty\), for every \(u > 0\) there exists some \(t_0 \geq 1\) such that \(\min\{\delta_t^\alpha, ut^{-2\alpha/d}\} = ut^{-2\alpha/d}\) for all \(t \geq t_0\). To show i) we apply the multivariate Mecke formula [28] to see that for all \(t \geq t_0\),

$$\frac{1}{2} \mathbb{E} \sum_{(x,y) \in \eta_t \cap \partial B} \mathbb{1}(\|x - y\|^\alpha \leq \min\{\delta_t^\alpha, ut^{-2\alpha/d}\}) = \frac{t^2}{2} \int W \int W \mathbb{1}(\|x - y\|^\alpha \leq ut^{-2\alpha/d}) \, dx \, dy \leq \frac{t^2}{2} \int W \int W \mathbb{1}(\|x\| \leq u^{1/\alpha} t^{-2/d}) \, dx \, dy \leq \frac{t^2}{2} V(W \cap B^d(y, u^{1/\alpha} t^{-2/d})) \, dy.$$  \hspace{2cm} (4.1)

Now, Fubini’s theorem gives

$$\frac{t^2}{2} \int_{\mathbb{R}^d} V(W \cap B^d(y, u^{1/\alpha} t^{-2/d})) \, dy = \frac{t^2}{2} V(W) \kappa_d(u^{1/\alpha} t^{-2/d})^d = \frac{K_\alpha}{2} V(W) u^{d/\alpha}.$$  \hspace{2cm} (4.2)

Denoting by \(V_j()\) the \(j\)-th intrinsic volume, we deduce from Steiner’s formula [24, Equation (14.5)] that

$$\frac{t^2}{2} \int_{W} V(W \cap B^d(y, u^{1/\alpha} t^{-2/d})) \, dy \leq \frac{t^2}{2} \kappa_d(u^{1/\alpha} t^{-2/d})^d \sum_{j=0}^{d-1} \kappa_{d-j} V_j(W)(u^{1/\alpha} t^{-2/d})^{(d-j)} \leq \bar{c} t^{-2/d},$$

with a constant \(\bar{c} > 0\) depending on \(W\), \(\alpha\) and \(u\), where the right-hand side tends to zero as \(t \to \infty\). To see ii) one writes, for \(t \geq t_0\),

$$t \sup_{y \in W} \lambda\{x \in W : \|x - y\|^\alpha \leq \min\{\delta_t^\alpha, ut^{-2\alpha/d}\}\} = t \sup_{y \in W} \int_W \mathbb{1}(\|x - y\|^\alpha \leq ut^{-2\alpha/d}) \, dx \leq t \kappa_d(u^{1/\alpha} t^{-2/d})^d = \kappa_d u^{d/\alpha} t^{-1},$$

whose limit is again zero, as \(t \to \infty\). This yields the desired result. \(\square\)
**Remark 4.2.** Application of the main result from [27] not only gives convergence of $t^{2a/d}S_m^{(a)}$ to a Poisson process, but also delivers rates of convergence for the order statistics, i.e., the finite dimensional distributions. In particular, for $u > 0$ there exists some $t_0 \geq 1$ and a constant $C$ only depending on $W$, $u$ and $(\delta_t)_{t \geq 1}$ such that

$$
\left| \mathbb{P}(t^{2a/d}S_m^{(a)} > u) - \exp\left(\frac{K_d}{2} V(W) u^{d/a}\right) \right| \leq C t^{- \min\{2/d, 1/2\}}
$$

for all $t \geq t_0$ and $m \in \mathbb{N}$. In particular, the $\alpha$-power of the minimal edge length converges, as $t \to \infty$, in distribution to a Weibull random variable with distribution function $1 - \exp\left(-\frac{\alpha}{d} V(W) u^{\frac{d}{2}}\right)$.

We next consider the regime where the number of edges of the Gilbert graph stays asymptotically constant in the mean, i.e., there is a constant $0 < c < \infty$ such that $\lim_{t \to \infty} t^2 \delta_t^d = c$. In this case we cannot build upon the theory from [27], which was tailored to applications developed there. Instead, we use the extension presented as Proposition 2 in [28].

**Theorem 4.3.** If $\alpha > 0$ and $\lim_{t \to \infty} t^2 \delta_t^d = c \in (0, \infty)$, the point process $t^{2a/d}S^{(a)}_t$ converges in distribution to a Poisson point process on $\mathbb{R}_+$ of intensity measure

$$
\nu(B) = \frac{dK_d}{2\alpha} V(W) \int_{B \cap [0, c^{\alpha/d}]} u^{(d/a) - 1} \, du, \quad B \subset \mathbb{R}_+ \text{ Borel}.
$$

**Proof.** Using the same arguments as in the proof of Theorem 4.1, one sees that

$$
\lim_{t \to \infty} \frac{1}{2} \sum_{(x, y) \in [0, t]^2, x \neq y} 1(\|x - y\|^\alpha \leq \min\{\delta_t^{(a)} u^{2\alpha/d}\}) = \lim_{t \to \infty} \frac{t^2}{2} \int_W V(W \cap B^d(y, \min\{\delta_t, u^{1/\alpha t^{-2/d}}\})) \, dy
$$

$$
= \lim_{t \to \infty} \frac{t^2}{2} V(W) \kappa_d \min\{\delta_t^d, u^{d/a}t^{-2}\}
$$

$$
= \frac{K_d}{2} V(W) \min\{u^{d/a}, c\}
$$

and that

$$
\lim_{t \to \infty} t \sup_{y \in W} \lambda\{x \in W : \|x - y\|^\alpha \leq \min\{\delta_t^{(a)} u^{2\alpha/d}\}\} \leq \lim_{t \to \infty} \kappa_d \delta_t^d t^{-1} = 0
$$

for $u \geq 0$. This shows that $a_t(u)$ defined at (4.1) converges to $\frac{K_d}{2} V(W) \min\{u^{d/a}, c\}$ and that $r_t(u)$ from (4.2) tends to 0, as $t \to \infty$. Application of Proposition 2 in [28] gives now the desired result.

**Remark 4.4.** Similar to the case discussed in Remark 4.2, Proposition 2 in [28] also provides limiting distributions and rates of convergence for the single order statistics.

**Remark 4.5.** We notice that if $t^2 \delta_t^d \to 0$ as $t \to \infty$, the probability that there are no edges tends to one. This is an immediate consequence of Theorem 2.1 and together with Markov’s inequality we see that the rate for this convergence is at least $t^2 \delta_t^d$. The limiting point process in this case might be interpreted as the empty point process or configuration.
Proof of Theorem 3.5. We denote by $\xi^{(\alpha)}$ the limiting Poisson point process of Theorem 4.3 and by $N(R)$ the space of all locally finite integer-valued counting measures on $R$. For every bounded continuous function $f : R \to R$ let us define $g : N(R) \to R$ by

$$g(\zeta) = f\left(\sum_{x \in \zeta \cap [0,c^{\alpha/d}+1]} x + \sum_{x \in \zeta \cap (c^{\alpha/d}+1,2c^{\alpha/d}+2]} (2c^{\alpha/d} + 2 - x)\right),$$

which is also bounded and continuous in the so-called vague topology on $N(W)$ (this is the topology whose associated Borel $\sigma$-field coincides with $N(W)$, see [24] Notes to Chapter 3.1). Now the point process convergence in Theorem 4.3 implies that

$$Eg(t^{2\alpha/d}\xi^{(\alpha)}_{t^{\delta}}) \to Eg(\xi^{(\alpha)}), \quad \text{as} \quad t \to \infty. \quad (4.3)$$

Due to the assumption that $\lim_{t \to \infty} t^{2\delta_0} \delta_0^{\alpha} = c$, there exists an intensity parameter $t_0 > 0$ such that $t^{2\alpha/d}\delta_0^{\alpha} \leq c^{\alpha/d} + 1$ for all $t \geq t_0$. From now on we shall assume that $t \geq t_0$. Then,

$$g(t^{2\alpha/d}\xi^{(\alpha)}_{t^{\delta}}) = f\left(t^{2\alpha/d}\frac{1}{2} \sum_{(x_1,x_2) \in \eta^{(\alpha)}_{t^{\delta}}} 1(\|x_1 - x_2\| \leq \delta_0, \|x_1 - x_2\|^\alpha)\right) = f(t^{2\alpha/d}L^{(\alpha)}_{t^{\delta}}).$$

On the other hand, we can think of $\xi^{(\alpha)}$ as a collection of $N := \xi^{(\alpha)}(R)$ independent and identically distributed random variables $X_1, \ldots, X_N$ with density defined in (3.1). Since $N$ is Poisson distributed with mean $\frac{2\alpha}{\delta_0}eV(W)$, we see that $\sum_{i=0}^{N} X_i$ has the same compound Poisson distribution as $Z$ in Theorem 3.5. Hence, it follows from (4.3) that

$$Ef(t^{2\alpha/d}L^{(\alpha)}_{t^{\delta}}) \to Ef(Z) \quad \text{as} \quad t \to \infty,$$

which completes the proof.

5 Large deviation inequalities

This section concerns large deviation inequalities for the functionals $L^{(\alpha)}_{t^{\delta}}$. In a recent paper by Eichelsbacher, Raič and Schreiber [7], large deviation inequalities for stabilizing Poisson functionals were derived. But due the dependence on $t$ and $\delta_0$ in the present paper, this result can be only applied in the thermodynamic regime, where $\delta_0 = \delta t^{-1/d}$ with some fixed $\delta > 0$. In this case we deduce the following large deviation inequality (the proof is postponed to the end of Subsection 5.2).

Proposition 5.1. For $\alpha \geq 0$ there is a constant $c > 0$ depending on $\alpha$, $W$ and $\delta_0$ such that

$$P(|L^{(\alpha)}_{t^{\delta}} - EL^{(\alpha)}_{t^{\delta}}| \geq u) \leq \exp\left(-c \min\left\{t^{(2\alpha-d)/d}u^2, t^{\alpha/(3d)}u^{1/3}, t^{(3\alpha-d)/(4d)}u^{3/4}\right\}\right) \quad (5.1)$$

for $u \geq 0$.

Part of the investigations in this section was motivated by a recent application of the Gilbert graph to the study of so-called empty triangles in [3]. In this context it was important to work with a binomial process instead of a Poisson process. For this reason we also include a section on large deviation inequalities in the case of an underlying binomial point process (this was left as an open problem in [7]). Thereafter we will deal with the Poisson case. Throughout this section we assume that $\alpha \geq 0$. 

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5.1 The binomial process and the convex distance

Let us denote by \( W \subset \mathbb{R}^d \) a compact convex set with volume \( V(W) > 0 \) and let \( (\delta_n)_{n \in \mathbb{N}} \) be a sequence of positive real numbers such that \( \delta_n \to 0 \), as \( n \to \infty \). Fix some integer \( n \geq 2 \), let \( X_1, \ldots, X_n \) be independent and uniformly distributed random points in \( W \) and consider the random set \( \xi = \{X_1, \ldots, X_n\} \). The Gilbert graph with vertex set \( \xi \) has an edge between two points if their distance is at most \( \delta_n \). Similar to (1.1), we define for \( \alpha \in \mathbb{R} \) the functional

\[
L_n^{(\alpha)}(\xi) = \frac{1}{2} \sum_{(X_i, X_j) \in \xi^2} 1(\|X_i - X_j\| \leq \delta_n)\|X_i - X_j\|^\alpha,
\]

where the sum ranges over all pairs \((X_i, X_j)\) of distinct points of \( \xi \). For these random variables we can deduce the following large deviation inequalities.

Theorem 5.2. Let \( \alpha \geq 0 \), let \( m_n \) be the median of \( L_n^{(\alpha)}(\xi) \) and define

\[
x^*(u, n) = \inf_{s > 0} \frac{\ln(n)V(W)/(nk_d\delta_n^d) + e^s - 1}{2s} + \sqrt{\frac{V(W)^2u^2}{8n^2k_d^2\delta_n^d\alpha(u + m_n)s} + \left(\frac{\ln(n)V(W)/(nk_d\delta_n^d) + e^s - 1}{2s}\right)^2}.
\]

Then, for \( u \geq 0 \),

\[
\mathbb{P}(|L_n^{(\alpha)}(\xi) - m_n| \geq u) \leq 8 \exp\left(-u^2 \frac{1}{8n^2k_d^2\delta_n^d\alpha(u + m_n)}\right)\left(\frac{V(W)^2u^2}{8n^2k_d^2\delta_n^d\alpha(u + m_n)}\right).
\]

and, in particular,

\[
\mathbb{P}(|L_n^{(\alpha)}(\xi) - m_n| \geq u) \leq 8 \exp\left(-\frac{1}{2} \min\left\{\frac{u^2}{8\ln(n)\delta_n^d + 2nk_d\delta_n^d\alpha/\sqrt{2V(W)}(u + m_n)}, \frac{u}{\sqrt{8\ln(n)\delta_n^d(u + m_n)}}\right\}\right).
\]

For the proof of Theorem 5.2, a local version of \( L_n^{(\alpha)}(\xi) \) plays an important rôle, namely

\[
L_n^{(\alpha)}(\xi; X_i) = \sum_{X_j \in \xi \setminus \{X_i\}} 1(\|X_i - X_j\| \leq \delta_n)\|X_i - X_j\|^\alpha.
\]

Thus, we may write \( L_n^{(\alpha)}(\xi) = \frac{1}{n} \sum_{i=1}^n L_n^{(\alpha)}(\xi; X_i) \). Observe that conditioned on \( X_i \), \( L_n^{(0)}(\xi; X_i) \) is a binomial random variable with distribution \( \text{Bin}(n - 1, p(X_i)) \) with

\[
p(X_i) = \mathbb{P}(\|X_i - X_j\| \leq \delta_n | X_i) = \frac{V(B^d(X_i, \delta_n) \cap W)}{V(W)},
\]

and thus

\[
\mathbb{E}[L_n^{(0)}(\xi; X_i) | X_i] = (n - 1)p(X_i) \leq n\frac{k_d\delta_n^d}{V(W)} =: E.
\]

Let \( X \sim \text{Bin}(m, p) \) be a binomial random variable with parameters \( m \in \mathbb{N} \) and \( p \in (0, 1) \). Then the exponential Markov inequality and the elementary inequality \( 1 + x \leq e^x, x \geq 0 \), imply the Chernoff bound

\[
\mathbb{P}(X \geq y) \leq \inf_{s \geq 0} \mathbb{E}e^{sX - sy} = \inf_{s \geq 0} \left(1 + p(e^s - 1)\right)^m e^{-sy} \leq \inf_{s \geq 0} e^{mp(e^s - 1) - sy}.
\]
for \( y \geq 0 \). At this point, one can compute the infimum on the right-hand side explicitly, but the representation above is more appropriate for our purpose. Applying the previous inequality with \( y = B \geq 0 \) to the conditionally binomial random variable \( L_n^{(0)}(\xi; X_i) \), we see that

\[
P(L_n^{(0)}(\xi; X_i) \geq B) = \mathbb{E} \mathbb{E}[\mathbb{I}\{L_n^{(0)}(\xi; X_i) \geq B\} \mid X_i] \leq \mathbb{E} \inf_{s \geq 0} e^{(n-1)p(X_i)(e^s - 1) - sB} \leq \inf_{s \geq 0} e^{E(e^s - 1) - sB}.
\]

This implies

\[
P(\exists i : L_n^{(0)}(\xi; X_i) \geq B) \leq n \inf_{s \geq 0} e^{E(e^s - 1) - sB}.
\]

(5.4)

Assume that an additional point set \( \zeta = \{y_1, \ldots, y_n\} \) with points \( y_1, \ldots, y_n \in W \) is given, where \( y_1, \ldots, y_n \) might have some points in common with \( \xi \). Each edge between points in \( \xi \) is either also an edge between points in \( \zeta \) if both endpoints are contained in \( \xi \cap \zeta \), or is counted at least once in some \( L_n^{(\alpha)}(\xi; X_i) \) for the endpoint \( X_i \in \xi \setminus \zeta \). Thus,

\[
L_n^{(\alpha)}(\xi) \leq L_n^{(\alpha)}(\zeta) + \sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i) \mathbb{I}(X_i \notin \zeta)
\]

\[
\leq L_n^{(\alpha)}(\zeta) + \sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i) \mathbb{I}(X_i \neq y_i).
\]

(5.5)

To prove a deviation inequality, we use Talagrand’s convex distance. For \( \xi = \{X_1, \ldots, X_n\} \) with \( X_1, \ldots, X_n \in W \) and \( A \subset W^n \) it is defined as

\[
d_T(\xi, A) = \max_{u \in S^{n-1}} \min_{\xi \in A} \sum_{i=1}^{n} u_i \mathbb{I}(X_i \neq y_i)
\]

where, as usual, \( u_i, i \in \{1, \ldots, n\} \), are the coordinates of \( u \in S^{n-1} \), cf. [6, Definition 11.1]. We fix a suitable vector

\[
u(\xi) = (u_1(\xi), \ldots, u_n(\xi)) = \frac{1}{\sqrt{\sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i)^2}}(L_n^{(\alpha)}(\xi; X_1), \ldots, L_n^{(\alpha)}(\xi; X_n)) \in S^{n-1}
\]

(put \( \nu \equiv 0 \) if \( \sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i)^2 = 0 \) ) and obtain together with (5.5) that

\[
d_T(\xi, A) = \max_{u \in S^{n-1}} \min_{\xi \in A} \sum_{i=1}^{n} u_i \mathbb{I}(X_i \neq y_i)
\]

\[
\geq \min_{\xi \in A} \sum_{i=1}^{n} u_i(\xi) \mathbb{I}(X_i \neq y_i)
\]

\[
\geq \min_{\xi \in A} \frac{1}{\sqrt{\sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i)^2}} \left(L_n^{(\alpha)}(\xi) - L_n^{(\alpha)}(\zeta)\right)
\]

(5.6)

(for convenience, we interpret \( 0/0 \) as \( 0 \)). If we assume that \( L_n^{(0)}(\xi; X_i) \), the number of edges emanating from \( X_i \), is bounded by some \( B > 0 \) for all \( i \in \{1, \ldots, n\} \), we have

\[
\sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i)^2 \leq B \delta_n \sum_{i=1}^{n} L_n^{(\alpha)}(\xi; X_i) = 2B \delta_n L_n^{(\alpha)}(\xi),
\]

(5.7)
and it follows from the previous inequality that

$$d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \min_{\xi \in A} \frac{L_n^{(\alpha)}(\xi) - L_n^{(\alpha)}(\xi^*)}{\sqrt{L_n^{(\alpha)}(\xi)}}.$$  

It was proved by Talagrand in [29] that $d_T$ satisfies a large deviation inequality (see also Theorem 11.1 in [6]). Namely, for $A \subset W^n$ we have

$$\mathbb{P}(A) \mathbb{P}(d_T(\xi, A) \geq s) \leq \exp\left(-\frac{s^2}{4}\right). \quad (5.6)$$

Let us now make the relation between $d_T$ and $L_n^{(\alpha)}(\xi)$ more explicit. First, define $A = \{\xi \in W^n : L_n^{(\alpha)}(\xi) \leq m_n\}$, where $m_n$ is the median of $L_n^{(\alpha)}(\xi)$. By definition of the median, we have $\mathbb{P}(A) \geq \frac{1}{2}$. Since the function $u \mapsto u/\sqrt{u + m_n}$ is increasing, $L_n^{(\alpha)}(\xi) \geq u + m_n$ and $d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \frac{L_n^{(\alpha)}(\xi) - m_n}{\sqrt{L_n^{(\alpha)}(\xi)}}$ imply that $d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \frac{u}{\sqrt{u + m_n}}$. Together with (5.4) and (5.6), this yields

$$\mathbb{P}(L_n^{(\alpha)}(\xi) \geq u + m_n) \leq \mathbb{P}(L_n^{(\alpha)}(\xi) \geq u + m_n, \forall i : L_n^{(0)}(\xi; X_i) \leq B) + \inf_{s \geq 0} e^{E(e^{s-1})-sB}$$

$$\leq \mathbb{P}(d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \frac{u}{\sqrt{u + m_n}} + n \inf_{s \geq 0} e^{E(e^{s-1})-sB}$$

$$\leq 2 \exp\left(-\frac{u^2}{8B\delta_n^\alpha(u + m_n)}\right) + n \inf_{s \geq 0} e^{E(e^{s-1})-sB}. \quad (5.7)$$

Similarly, for $u \geq 0$, putting $A = \{\xi : L_n^{(\alpha)}(\xi) \leq m_n - u\}$, the monotonicity of $(u - a)/\sqrt{u}$ together with the assumption that $L_n^{(\alpha)}(\xi) \geq m_n$ imply

$$\frac{L_n^{(\alpha)}(\xi) - (m_n - u)}{\sqrt{L_n^{(\alpha)}(\xi)}} \geq \frac{m_n - (m_n - u)}{\sqrt{m_n}}.$$  

In this case we have

$$d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \min_{\xi \in A} \frac{L_n^{(\alpha)}(\xi) - L_n^{(\alpha)}(\xi)}{\sqrt{L_n^{(\alpha)}(\xi)}} \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \frac{u}{\sqrt{m_n}},$$

which, again in view of (5.4) and (5.6), yields

$$\frac{1}{2} \leq \mathbb{P}(L_n^{(\alpha)}(\xi) \geq m_n) \leq \mathbb{P}(L_n^{(\alpha)}(\xi) \geq m_n, \forall i : L_n^{(0)}(\xi; X_i) \leq B) + \inf_{s \geq 0} e^{E(e^{s-1})-sB}$$

$$\leq \mathbb{P}(d_T(\xi, A) \geq \frac{1}{\sqrt{2B\delta_n^\alpha}} \frac{u}{\sqrt{m_n}} + n \inf_{s \geq 0} e^{E(e^{s-1})-sB}$$

$$\leq \frac{1}{\mathbb{P}(L_n^{(\alpha)}(\xi) \leq m_n - u)} \exp\left(-\frac{u^2}{8B\delta_n^\alpha m_n}\right) + n \inf_{s \geq 0} e^{E(e^{s-1})-sB},$$

and we deduce that

$$\mathbb{P}(L_n^{(\alpha)}(\xi) \leq m_n - u) \leq 2 \exp\left(-\frac{u^2}{8B\delta_n^\alpha m_n}\right) + 2n \inf_{s \geq 0} e^{E(e^{s-1})-sB}. \quad (5.8)$$

Combining (5.7) and (5.8) implies the following large deviation inequality for $L_n^{(\alpha)}(\xi)$:
Proposition 5.3. For $\alpha \geq 0$ and $u \geq 0$ we have
\[
P(|L_n^{(\alpha)}(\xi) - m_n| \geq u) \leq \inf_{s > 0, x > 0} 4 \exp \left( -\frac{V(W)u^2}{8n\kappa_d \delta_n^{\alpha+\alpha} (u + m_n)x} \right) + 4n \exp \left( (e^s - 1 - sx)n\kappa_d \delta_n^{d}/V(W) \right), \tag{5.9}
\]
where $m_n$ is the median of $L_n^{(\alpha)}(\xi)$.

Proof of Theorem 5.2. For fixed $s > 0$, the right-hand side of (5.9) becomes up to a constant factor minimal if both summands are equal, i.e., if
\[
\frac{V(W)u^2}{8n\kappa_d \delta_n^{\alpha+\alpha} (u + m_n)x} = \ln(n) + \frac{n\kappa_d \delta_n^{d}}{V(W)}(e^s - 1 - sx).
\]
This means that
\[
x = \frac{\ln(n)V(W)/(n\kappa_d \delta_n^{d}) + e^s - 1}{2s} + \sqrt{\frac{V(W)^2u^2}{8n^2\kappa_d^2\delta_n^{2d+\alpha}(u + m_n)s}} + \left( \frac{\ln(n)V(W)/(n\kappa_d \delta_n^{d}) + e^s - 1}{2s} \right)^2,
\]
since the negative solution of the quadratic equation does not satisfy the required assumption $x > 0$. Choosing now $s$ in such a way that $x$ becomes minimal, leads to (5.2). For $s = 1$ we obtain that
\[
x \leq 2 \max \left\{ \frac{\ln(n)V(W)}{n\kappa_d \delta_n^{d}}, 2, \sqrt{\frac{V(W)^2u^2}{8n^2\kappa_d^2\delta_n^{2d+\alpha}(u + m_n)}} \right\},
\]
which yields (5.3). \qed

From the previous computations leading to Proposition 5.3 we can deduce the following bound for the sparse regime $n^2 \delta_n^{d} = C > 0$, which has been applied in [3].

Theorem 5.4. Assume $C > 0$, $B > 0$ and choose the sequence $(\delta_n)_{n \in \mathbb{N}}$ such that $n^2 \delta_n^{d} = C$ for all $n \in \mathbb{N}$. Then there is a constant $c = c(B, C, W, \alpha) > 0$ only depending on $B, C, W$ and $\alpha$ such that
\[
P(L_n^{(\alpha)}(\xi) \geq 9B^2 \delta_n^{\alpha} \ln n) \leq cn^{-B+1}.
\]

Proof. Choosing $u = 9B^2 \delta_n^{\alpha} \ln n - m_n$ in (5.7), we see that
\[
P(L_n^{(\alpha)}(\xi) \geq u + m_n) \leq 2 \exp \left( -\frac{(9B^2 \delta_n^{\alpha} \ln n - m_n)^2}{8B^2 \delta_n^{\alpha} 9B^2 \delta_n^{\alpha} \ln n} \right) + n \inf_{s \geq 0} e^{E(e^s - 1) - sB}. \tag{5.10}
\]
Since $m_n \leq 2E_L^{(\alpha)}(\xi) \leq 2Cn^2 \delta_n^{d+\alpha} = 2C_n^2 \delta_n^{\alpha}$ with a constant $c_2 > 0$, we see that
\[
9B^2 \delta_n^{\alpha} \ln n - m_n \geq \sqrt{72B^2 \delta_n^{\alpha} \ln n}
\]
for sufficiently large $n$. Choosing $s = \ln n$ we find
\[
n \inf_{s \geq 0} e^{E(e^s - 1) - sB} \leq e^{E(n-1)n^{-B+1}}.
\]
Now, $n^2 \delta_n^{d} = C$ and $E = n\kappa_d \delta_n^{d}/V(W)$ imply that $e^{E(n-1)}$ is uniformly bounded in $n$. Combining the two previous inequalities with (5.10) concludes the proof. \qed
5.2 The Poisson process and the convex distance

In this subsection we consider the case of an underlying homogeneous Poisson process of intensity \( t > 0 \) within a compact convex observation window \( W \) and prove the following large deviation inequality for \( L_t^{(\alpha)} \).

**Theorem 5.5.** Let \( \alpha \geq 0 \), let \( m_t \) be the median of \( L_t^{(\alpha)} \) and define

\[
x^*(u, t) = \inf_{s > 0} \left[ \frac{\ln(tV(W))}{(tk_d\delta_t^d) + e^s - 1} + \frac{u^2}{8t^2k_d^2d^{2+\alpha}(u + m_t)s} + \left( \frac{\ln(tV(W))}{(tk_d\delta_t^d) + e^s - 1} \right)^2 \right].
\]

Then, for \( u \geq 0 \),

\[
P(|L_t^{(\alpha)} - m_t| \geq u) \leq 8 \exp \left( -\frac{u^2}{8tk_d\delta_t^d + x^*(u, t)(u + m_t)} \right) \tag{5.11}
\]

and, in particular,

\[
P(|L_t^{(\alpha)} - m_t| \geq u) \leq 8 \exp \left( -\frac{1}{2} \min \left\{ \frac{u^2}{8(\ln(tV(W))\delta_t^d + 2tk_d\delta_t^d + \alpha)(u + m_t)}, \frac{u}{\sqrt{8}\delta_t^d(u + m_t)} \right\} \right). \tag{5.12}
\]

**Remark 5.6.** In the thermodynamic regime we can compare our bound with that of Eichelsbacher, Raic and Schreiber stated in Proposition 5.1. The second inequality in Theorem 5.5 delivers up to a constant factor the exponential exponent

\[
\min \left\{ \frac{u^2}{t^{2\alpha/d}}, \frac{u^{\alpha/d}}{t^{1/4}}, \sqrt{u} t^{\alpha/(2d)} \right\},
\]

whereas Proposition 5.1 and \( m_t \leq 2E L_t^{(\alpha)} \) yield

\[
\min \left\{ \frac{u^2}{t^{2\alpha/d}}, \frac{u^{3/2} t^{3\alpha/(4d)}}{t^{1/4}}, u^{1/3} t^{\alpha/(3d)} \right\}.
\]

This means that for \( u \) such that \( u^2 \) dominates the minimum, the re-scaling with respect to the intensity parameter \( t \) in the result of Eichelsbacher, Raic and Schreiber is better than ours. On the other hand, our worst \( u \)-exponent is 1/2, whereas it is 1/3 in Proposition 5.1. Moreover, one should notice that Proposition 5.1 deals with an inequality for \( L_t^{(\alpha)} - E L_t^{(\alpha)} \), whereas in our result \( E L_t^{(\alpha)} \) is replaced by the median \( m_t \).

For two finite counting measures \( \xi \) and \( \nu \) we define the (set-)difference \( \xi \setminus \nu \) by

\[
\xi \setminus \nu = \sum_{x \in W} \max\{\xi(x) - \nu(x), 0\} \delta_x,
\]

which is a finite counting measure again. For a finite counting measure \( \xi \) and \( x \in \xi \) let us define

\[
L_t^{(\alpha)}(x; \xi) = \sum_{y \in \xi} 1(\|x - y\| \leq \delta_t)\|x - y\|^\alpha,
\]

and

\[
L_t^{(\alpha)}(\xi) = \frac{1}{2} \sum_{x \in \xi} L_t^{(\alpha)}(x; \xi) = \frac{1}{2} \sum_{x \in \xi} L_t^{(\alpha)}(x; \xi),
\]

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which means that $L_t^{(\alpha)}(\eta_t)$ has the same distribution as $L_t^{(\alpha)}$.

Observe that for fixed $x \in W$, which is independent of $\eta_t$, $L_t^{(0)}(x; \eta_t \cup \{x\}) = \eta_t(B^d(x, \delta_t))$ is a Poisson distributed random variable with mean

$$E(x) := \mathbb{E}_{\eta_t}(B^d(x, \delta_t)) = tV(B^d(x, \delta_t) \cap W) \leq t \kappa_d \delta_t^d =: E.$$  \hfill (5.13)

Mecke’s formula (2.3) gives, for $B > 0$,

$$\mathbb{P}(\exists x \in \eta_t : L_t^{(0)}(x; \eta_t) > B) \leq E \sum_{x \in \eta_t} \mathbb{1}(L_t^{(0)}(x; \eta_t) > B)$$

$$= t \int_W \mathbb{P}(L_t^{(0)}(x; \eta_t) > B) \, dx$$

$$\leq tV(W) \inf_{s \geq 0} e^{E(e^s - 1) - sB},$$

where in the last line we used the Chernoff bound for the Poisson distribution, namely

$$\mathbb{P}(\eta_t(B^d(x, \delta_t)) > B) \leq \inf_{s \geq 0} e^{E(e^s - 1) - sB}.$$  

Indeed, if $X$ is a Poisson random variable with mean $a > 0$ and $y \geq 0$, we have that

$$\mathbb{P}(X \geq y) = \inf_{s \geq 0} \mathbb{E} e^{sX - sy} = \inf_{s \geq 0} e^{a(e^s - 1) - sy}.$$  

Assume now that besides of $\eta_t$ a second point set $\zeta \in \mathbb{N}(W)$ is chosen, which might have a non-trivial intersection with $\eta_t$. Similar to the binomial case considered in the previous subsection, we find

$$L_t^{(\alpha)}(\eta_t) \leq L_t^{(\alpha)}(\zeta) + \sum_{x \in \eta_t} L_t^{(\alpha)}(x; \eta_t) \mathbb{1}(x \notin \zeta).$$  \hfill (5.14)

To present our deviation inequality we use an analogue of Talagrand’s convex distance for Poisson point processes, which has been introduced in [22]. For $\xi \in \mathbb{N}(W)$ and $A \subset \mathbb{N}(W)$ it is given by

$$d_T^\pi(\xi, A) = \max_{\|u\|_{2,\xi} \leq 1} \min_{\xi \in A} \sum_{x \in (\xi \setminus \zeta)} u(x),$$

where $u : W \to \mathbb{R}_+$ is a non-negative measurable function and $\|u\|_{2,\xi}^2 := \sum_{x \in \xi} u(x)^2$. For $x \in W$, let us put

$$u(x) = \frac{1}{\|L_t^{(\alpha)}(\cdot; \eta_t)\|_{2,\eta_t}} L_t^{(\alpha)}(x; \eta_t)$$

(recall that we interpret $0/0$ as $0$), which gives

$$\|u\|_{2,\eta_t}^2 = \frac{1}{\|L_t^{(\alpha)}(\cdot; \eta_t)\|_{2,\eta_t}^2} \sum_{x \in \eta_t} L_t^{(\alpha)}(x; \eta_t)^2 = 1,$$
and rewrite $L_t^{(a)}$, using (5.14), in terms of the convex distance as follows:

$$
\begin{align*}
\mathcal{d}_T^n(\eta, A) &= \max_{\|u\|_{2, \eta} \leq 1} \min_{\zeta \in A} \sum_{x \in (\eta, \zeta)} u(x) \\
&\geq \min_{\zeta \in A} \frac{1}{\|L_t^{(a)}(\cdot; \eta)\|_{2, \eta}} \sum_{x \in (\eta, \zeta)} L_t^{(a)}(x; \eta) \\
&\geq \frac{1}{\|L_t^{(a)}(\cdot; \eta)\|_{2, \eta}} \left( L_t^{(a)}(\eta) - L_t^{(a)}(\zeta) \right). 
\end{align*}
$$

(5.15)

We now assume that $L_t^{(0)}(x; \eta) \leq B$ for all $x$, in which case

$$
\|L_t^{(a)}(\cdot; \eta)\|^2_{2, \eta} = \sum_{x \in \eta} L_t^{(a)}(x; \eta)^2 \leq B \delta_t^a \sum_{x \in \eta} L_t^{(a)}(x; \eta) = 2B \delta_t^a L_t^{(a)}(\eta).
$$

In view of (5.15) this gives

$$
\mathcal{d}_T^n(\eta, A) \geq \frac{1}{\sqrt{2B \delta_t^a}} \min_{\zeta \in A} \frac{L_t^{(a)}(\eta) - L_t^{(a)}(\zeta)}{\sqrt{L_t^{(a)}(\eta)}}.
$$

(5.16)

Inequality (5.16) links $\mathcal{d}_T^n$ to $L_t^{(a)}$. We are now in the position to recall the main result from [22], in which Talagrand’s large deviation inequality for the binomial processes was extended to finite Poisson processes. Namely, for $A \subset \mathbf{N}(W)$ and $s \geq 0$ we have that

$$
\mathbb{P}(A) \mathbb{P}(\mathcal{d}_T^n(\eta, A) \geq s) \leq \exp \left( -\frac{s^2}{4} \right).
$$

Exactly as in the binomial case, this yields a large deviation inequality for the length-power functionals $L_t^{(a)}$ of the Gilbert graph.

**Proposition 5.7.** For $\alpha \geq 0$ and $u \geq 0$ we have

$$
\mathbb{P}(\|L_t^{(a)} - m_t\| \geq u) \leq \inf_{s \geq 0, x > 0} 4 \exp \left( -\frac{u^2}{8t \kappa_d \delta_t^{2+\alpha}(u + m_t)x} \right) + 4tV(W) \exp \left( t \kappa_d \delta_t^d(e^s - 1 - sx) \right),
$$

(5.17)

where $m_t$ is the median of $L_t^{(a)}$.

**Proof of Theorem 5.7** Note that the right-hand side of (5.17) is minimized up to a constant factor if both summands are of the same order, i.e., if

$$
-\frac{u^2}{8t \kappa_d \delta_t^{2+\alpha}(u + m_t)x} = \ln(tV(W)) + t \kappa_d \delta_t^d(e^s - 1 - sx).
$$

This is equivalent to say that

$$
x = \frac{\ln(tV(W))}{(t \kappa_d \delta_t^d) + e^s - 1}{2s} + \sqrt{\frac{u^2}{8t \kappa_d \delta_t^{2+\alpha}(u + m_t)x}} + \left( \frac{\ln(tV(W))/(t \kappa_d \delta_t^d) + e^s - 1}{2s} \right)^2,
$$

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since the negative solution of the quadratic equation above does not satisfy \( x > 0 \), as required. Now, choosing \( s > 0 \) in such a way that \( x \) is minimized leads to (5.11). For \( s = 1 \) we obtain

\[
x \leq 2 \max \left\{ \frac{\ln(tV(W))}{t\kappa_d\delta_t^d} + 2 \sqrt{\frac{u^2}{8\kappa_d^2t^{2d+\alpha}(u + m_t)}} \right\},
\]

which yields (5.12).

Proof of Proposition 5.1. For \( x \in \mathbb{R}^d \) and a finite counting measure \( \nu \) we define the score function

\[
\xi(x, \nu) = \frac{1}{2} \sum_{y \in \nu} 1(||x - y|| \leq \hat{\delta})||x - y||^\alpha
\]

and define \( \xi_t(x, \nu) \) as

\[
\xi_t(x, \nu) = \xi(t^{1/d}x, t^{1/d}\nu)
\]

for \( t \geq 1 \) (to simplify comparison with [7] we use the same notation as in that paper, which should not be confused with the notation used at the beginning of this subsection). Hence, we can rewrite \( L_t^{(\alpha)} \) as

\[
L_t^{(\alpha)} = t^{-\alpha/d} \sum_{x \in \eta_t} \xi_t(x, \eta_t).
\]

Note that \( \xi_t \) has \( t^{-1/d} \hat{\delta} \) as its so-called radius of stabilization (see [7]) and that the number of points of \( \eta_t \) which affect the value \( \xi_t(x, \eta_t) \) is Poisson distributed for all \( t > 0 \). Consequently, for

\[
t^{\alpha/d}L_t^{(\alpha)} = \sum_{x \in \eta_t} \xi_t(x, \eta_t)
\]

the conditions of Theorem 1.3 in [7] are satisfied with \( \alpha = 1 \) and \( \beta = 0 \) there. This implies that there are constants \( C_1, C_2, C_3 > 0 \), such that

\[
P(t^{\alpha/d}|L_t^{(\alpha)} - E[L_t^{(\alpha)}]| \geq x) \leq \exp \left(-\min \left\{ C_1 \frac{x^2}{t^{2\alpha/d}V[L_t^{(\alpha)}]}, C_2 x^{1/3}, C_3 x^{3/4} t^{-1/4} \right\} \right)
\]

for all \( x \geq 0 \). Choosing \( x = t^{\alpha/d}u \) and Theorem 2.3 conclude the proof. \( \square \)

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