ABSTRACT

The interplanetary magnetic fluctuation spectrum obeys a Kolmogorovian power law at scales above the proton inertial length and gyroradius which is well regarded as an inertial range. Below these scales, a power-law index around $-2.5$ is often measured and associated with nonlinear dispersive processes. Recent observations reveal a third region at scales below the electron inertial length. This region is characterized by a steeper spectrum that some refer to as the dissipation range. We investigate this range of scales in the electron magnetohydrodynamic approximation and derive an exact and universal law for a third-order structure function. This law can predict a magnetic fluctuation spectrum with an index of $-11/3$ which is in agreement with the observed spectrum at the smallest scales. We conclude on the possible existence of a third turbulence regime in the solar wind instead of a dissipation range as recently postulated.

Key words: magnetic fields – magnetohydrodynamics (MHD) – solar wind – turbulence

1. INTRODUCTION

Turbulence plays a central role in a wide range of astrophysical plasmas. Examples are given by the solar wind (Matthaeus et al. 1999), the interstellar (Scalo & Elmegreen 2004), galactic, and even intergalactic media (Govoni et al. 2006). In the solar wind, turbulence evolves freely and is not perturbed by in situ diagnostics; therefore, it provides an ideal laboratory for studying high Reynolds number plasma turbulence. This unique situation allows us to investigate, for example, the origin of anisotropy (see, e.g., Klein et al. 1993; Galtier et al. 2000; Alexakis et al. 2007; Bigot et al. 2008), to evaluate the mean energy dissipation rate (MacBride et al. 2008; Carbone et al. 2009), to detect multiscale intermittency (Kiyani et al. 2009), or to analyze different regimes of turbulence characterized by a steepening of the magnetic field fluctuations spectrum with a power-law index going from $-5/3$ at frequencies lower than $1$ Hz, to indices lying around $-2.5$ at higher frequencies (see, e.g., Smith et al. 2006).

The spectral break near 1 Hz has been a subject of intensive studies and controversies in the last decades. It was first interpreted as the onset of dissipation caused, for example, by kinetic Alfvén wave damping (Leamon et al. 1998). Then, it was demonstrated that the wave damping rate usually increases very strongly with wavenumbers and should lead to a strong cutoff in the power spectra rather than a steepened power law (Li et al. 2001). In the meantime, there are some indications that the fluctuations are accompanied by a bias of the polarization suggesting the presence of right-hand polarized, outward propagating waves (Goldstein et al. 1994). Also it was proposed (Stawicki et al. 2001) that Alfvén fluctuations—left circularly polarized—are suppressed by proton cyclotron damping and that the high-frequency power-law spectra are likely to consist of whistler fluctuations (Matthaeus et al. 2008). It is currently believed that the steepening of the spectra at 1 Hz is mainly due to nonlinear dispersive processes that range from kinetic Alfvén waves (Hasegawa & Chen 1976; Howes et al. 2008), electromagnetic ion-cyclotron Alfvén waves (Gary et al. 2008), and/or electron whistler waves (Ghosh et al. 1996; Galtier 2006; Galtier & Buchlin 2007; Narita & Gary 2010) in the framework of Hall magnetohydrodynamics (MHD) or simply electron MHD.

The most recent solar-wind observations made with the high-resolution magnetic field data of the Cluster spacecraft (Sahraoui et al. 2009; Alexandrova et al. 2009) reveal the presence of a third region—called dissipation range—at scales smaller than $d_e$ and characterized by even steeper magnetic fluctuation spectra with a power-law index around $-3.8$. These spectra, which have been observed for only half a decade, are interpreted as either a power law (Sahraoui et al. 2009) or an exponential law (Alexandrova et al. 2009). Although the theoretical interpretation of such a regime is still open (Matthaeus et al. 2008), a recent theoretical analysis shows that a kinetic Alfvén wave cascade subject to collisionless damping cannot reach electron scales in the solar wind at 1 AU (Podesta et al. 2010). The direct consequence is that the spectra observed must be supported by another type of wave mode. It is noteworthy that this new regime at electron scales gives rise to the same controversy as the steepening found two decades ago around 1 Hz which naturally brings up the following question: have we really found the dissipation scale of the solar-wind plasma or is it the onset of a new turbulence regime?

In this paper, we investigate the turbulence regime at scales smaller than the electron inertial length $d_e$ through the electron MHD approximation. The assumption of homogeneity and isotropy will be made to derive an exact and universal law for third-order structure functions. We show that this law corresponds to a magnetic fluctuation isotropic spectrum in $k^{-11/3}$ compatible with the solar-wind measurements. Although the assumption of isotropy is in apparent contradiction with the observations, it is claimed that the method used is a powerful way to obtain a first estimate of the anisotropic spectrum. Indeed, the main source of anisotropy is the presence of a large-scale magnetic field which reduces the nonlinear transfer along its direction. Then, the most relevant spectral scaling is the transverse one for which the spectral index corresponds to the isotropic case if arguments based on the critical balance condition are used. It is only in the asymptotic limit of wave turbulence—for which anisotropy is strong—that the spectral index for transverse fluctuations is (slightly) modified (see the review by Galtier 2009a). Finally, we conclude the paper on the possible existence of a third inertial range for solar-wind turbulence instead of a dissipation range as recently postulated.

2 Also at Institut universitaire de France.
2. ELECTRON MAGNETOHYDRODYNAMICS

Electron MHD provides a fluid description of the plasma behavior on length scales smaller than the ion inertial length $d_i$ and on timescales of the order of, or shorter than, the ion cyclotron period (Kinsep et al. 1990). In this case, ions do not have time to respond because of their heavy mass and merely provide a neutralizing background. Then, the plasma dynamics is governed by electron flows and their self-consistent magnetic field. This model has attracted a lot of interest because of its potential applications in fast switches, Z-pinch devices, impulsive magnetospheric/solar corona reconnection, and ionospheric phenomena (see, e.g., Bhattacharjee 2004; Chacon et al. 2007).

The inviscid three-dimensional electron MHD equations can be written in SI units as (Biskamp et al. 1996)

\[ \partial_t (1 - d_i^2 \Delta) \mathbf{B} = -d_i \nabla \times [\mathbf{J} \times (1 - d_i^2 \Delta) \mathbf{B}], \]  

(1)

where $\mathbf{B}$ is the magnetic field normalized to a velocity $(\mathbf{B} \rightarrow \sqrt{\mu_0} \mathbf{B})$ and $\mathbf{J} = \nabla \times \mathbf{B}$ is the normalized current density. Under the limit of electron MHD, we remind that the current density is proportional to the electron velocity. This equation has two invariants (Biskamp et al. 1999) which are the total energy

\[ E = \frac{1}{2} \int (B^2 + d_i^2 J^2) \, dx, \]  

(2)

and the generalized helicity

\[ H = \int (A - d_i^2 J) \cdot (B - d_i^2 \Delta B) \, dx, \]  

(3)

with $A$ being the normalized magnetic potential.

Equation (1) is often used when $d_i \rightarrow 0$, namely for scales between $d_i$ and $d_e$, for which a $k^{-11/3}$-isotropic magnetic energy spectrum is found numerically and heuristically (Biskamp et al. 1996, 1999; Daugast et al. 2000a). This result is compatible with a rigorous derivation of a universal law for third-order correlation tensors (Galtier 2008a, 2008b). A steeper magnetic spectrum in $k^{-11/3}$ may also be found when the kinetic energy overtake the magnetic energy (Galtier & Buchlin 2007). Such a situation—generally not discussed in the literature—can only be observed when the full Hall MHD system is considered. Note that this $-11/3$ power-law index, valid for length scales larger than $d_e$, has a different origin from the one derived in the present paper which is applicable for scales shorter than $d_e$.

The behavior at scales shorter than $d_e$ has attracted much less attention (Biskamp et al. 1996; Daugast et al. 2000a). This regime corresponds to the limit $d_e \rightarrow +\infty$ for which we have

\[ \frac{1}{d_i} \partial_t \mathbf{J} = -\nabla \times [\nabla \times \mathbf{J} - \partial_t (\nabla \times \mathbf{J})], \]  

(4)

where $\Phi$ is an unknown function. The second term in the right-hand side may be seen as a gauge: actually, an analysis performed directly on the generalized Ohm’s law shows that it corresponds to an electron pressure. Note that the form of Equation (4) is well adapted to the problem under consideration since we are going to assume isotropy which means we will not consider any background (large-scale) magnetic field $\mathbf{B}_0$.

3. UNIVERSAL LAW FOR $R < D_E$

In the following, we shall derive an exact and universal law for third-order structure functions for homogeneous three-dimensional isotropic electron MHD turbulence at scales smaller than $d_e$ and discuss the implications in terms of the magnetic fluctuation spectrum. After simple manipulations in Equation (4) we obtain for the $i$th component

\[ \frac{1}{d_i} \partial_t J_i = J_i \partial_t \Phi + \partial_i (\Phi + J^2 / 2), \]  

(5)

where the Einstein notation is used. Note that we also have $\partial_i J_i = 0$. The second term in the right-hand side is similar to pressure whereas the first term exhibits a difference with the usual advection term encountered in Navier–Stokes equations. Actually, the Navier–Stokes equations may be recovered when the electron velocity is used instead of the current density or when the generalized Ohm’s law is directly used. We made the choice to use the well-known electron MHD Equations (1) and (5) mainly because we shall compare eventually the new prediction with the previous one for scales larger than $d_e$ (Galtier 2008a).

It is straightforward to derive a universal law for a homogeneous and isotropic electron MHD turbulence. First we introduce the second-order correlation tensor:

\[ R_{ij}(r) \equiv \langle J_i(x) J_j(x') \rangle = \langle J_i J_j' \rangle, \]  

(6)

where $x' = x + r$. We obtain

\[ \frac{1}{d_i} \partial_t R_{ij} = \langle J_i \partial_t J_j' \rangle - \langle J_i \partial_j (\Phi + J^2 / 2) \rangle + \langle J_j' \partial_t J_i \rangle - \langle J_j' \partial_j (\Phi + J^2 / 2) \rangle. \]  

(7)

After simple manipulations where we use the homogeneity and the divergence-free condition we obtain

\[ \frac{1}{d_i} \partial_t R_{ij} = \partial_i \langle (J_i J_j') - (J_j' J_i) \rangle. \]  

(8)

Note that the pressure-type contributions are removed as usual for isotropic turbulence (Batchelor 1953). When only the diagonal part of the energy tensor is retained we have

\[ \frac{1}{d_i} \partial_t R_{ii} = -2 \partial_i \langle J_i J_i' \rangle = -2 \partial_i \langle J_i J_i' \rangle. \]  

(9)

At this level of analysis, it is necessary to say a word about the small-scale dissipation and large-scale forcing terms which have been neglected so far. The dissipation is a linear term which is seen as a sink for the energy. Since we are interested in a universal behavior of turbulence we are in a situation where the scales considered are supposed to be much larger than the dissipation scales: in other words, we are deep inside the inertial range where the dissipation has no effect. The forcing term is assumed to be at the largest scales and acts as a constant source of energy for the system. Formally, the introduction of a small-scale dissipation $\mathcal{D}$ and a large-scale force $\mathcal{F}$ leads to the expression

\[ \frac{1}{2} d_i^2 \partial_t R_{ii} = -d_i^2 \nabla \cdot \langle J_i (J_i') \rangle + \mathcal{F} + \mathcal{D}. \]  

(11)

An exact relation may be derived for third-order structure functions by assuming the following assumptions specific to fully developed turbulence (Kolmogorov 1941; Frisch 1995; Politano & Pouquet 1998). First, we take the long time limit for
which a stationary state is reached with a finite mean energy dissipation rate per unit mass. Second, we consider the infinite magnetic Reynolds number limit for which the mean energy dissipation rate per unit mass tends to a finite positive limit, $\epsilon^T$ (see, e.g., Biskamp et al. 1996). By noting that

$$R_{ij} = \langle J^2 \rangle - \frac{1}{2} \langle \delta J_i \delta J_j \rangle,$$

(12)

where $\delta J \equiv \mathbf{J}(\mathbf{x} + \mathbf{r}) - \mathbf{J}(\mathbf{x})$, we obtain

$$\frac{1}{2} \partial_x^2 R_{ii} = \partial_i \left( \frac{1}{2} \partial_x^2 J^2 \right) - \frac{1}{2} \partial_x \partial_i (\delta J_i \delta J_j),$$

(13)

where the first term in the right-hand side is the time variation of energy. Therefore, in the stationary state both terms in the right-hand side are equal to zero. Since dissipation effects are negligible, we only have to include the mean energy injection rate per unit mass $\epsilon J$. It is important to remind that the energy (an inviscid invariant) is built directly from the current density, hence the name $\epsilon J$. As it will be shown below, this remark turns out to be fundamental for the prediction of the magnetic fluctuation spectrum. The insertion of the previous statements into Equation (11) leads to

$$d_i d_x^2 \nabla \cdot \langle \mathbf{J}(\mathbf{J}, \mathbf{j}') \rangle = \epsilon J.$$

(14)

The introduction of structure functions for the current density eventually gives

$$d_i d_x^2 \nabla \cdot \langle \delta \mathbf{J}(\delta \mathbf{J})^2 \rangle = 4 \epsilon J.$$

(15)

An integration of Equation (15) over a full sphere of radius $r$ (since isotropy is assumed) and the application of the divergence theorem finally give the universal and exact law for three-dimensional homogeneous isotropic electron MHD turbulence for scales smaller than $d_e$; it gives

$$d_i d_x^2 \langle \delta J_i (\delta \mathbf{J})^2 \rangle = \frac{3}{4} \epsilon J r,$$

(16)

where $L$ means the longitudinal component of the vector, i.e., the one along the direction $r$. Note the positive sign in the right-hand side which is compatible with the negative sign in front of the nonlinear term of Equation (4). The most remarkable aspect of this law is that it not only provides a linear scaling for third-order structure functions within the inertial range of length scales, but it also fixes the value of the numerical factor appearing in front of the scaling relation.

4. EXTENSION TO $R > D_E$

For scales larger than $d_e$ (but still smaller than $d_i$) the universal law for three-dimensional isotropic electron MHD turbulence takes the form (Galtier 2008a)

$$d_i \langle (\mathbf{J} \times \mathbf{B}) \times \mathbf{B} \rangle_{L} = -\frac{1}{2} \epsilon^T r,$$

(17)

where $L$ still means the longitudinal component. This law may also be written as (Galtier 2009b)

$$d_i \langle (\mathbf{J} \times \mathbf{B}) \times \delta \mathbf{B} \rangle_{L} = -\frac{1}{2} \epsilon^T r,$$

(18)

3 It is also possible to consider a system without external forcing (Landau & Lifchitz 1989). In this case, we have to deal with the decay problem for which the time derivative of the energy is equal (up to a sign) to the mean energy dissipation rate per unit mass $\epsilon J$. It is still possible to assume the time independence of the second term in the right-hand side of Equation (13) and to finally recover the same relation as in Equation (15).

where $\overline{X} = \langle X + X' \rangle / 2$. Both universal laws (16) and (18) may be gathered by noting that in this case the forcing scale is pushed at scales much larger than $d_e$. One needs to consider the following expression in the stationary state and in the inertial range

$$\frac{1}{2} (R_{ei}^2 R_{ji} + \partial_i \tilde{R}_{ij}) = \mathcal{N} L + \mathcal{F} + \mathcal{D} = \mathcal{N} L + \epsilon^T,$$

(19)

where $\tilde{R}_{ij} = \left\langle B_i B_j ^* \right\rangle$, $\mathcal{N} L$ is the nonlinear contribution, and $\epsilon^T$ is the mean total energy injection rate per unit mass. Then, one obtains the general law for three-dimensional isotropic electron MHD turbulence (with $r < d_e$)

$$4 d_i \langle (\mathbf{J} \times \mathbf{B}) \times \delta \mathbf{B} \rangle_{L} - d_i d_x^2 \langle (\delta \mathbf{J})^2 \delta J_l \rangle = -\frac{3}{4} \epsilon^T r.$$

(20)

This universal law conserves the linearity in $r$ but emphasizes the role of each term according to the scale considered. The importance of each term is given by the current density which involves a derivative: at small scales (scales smaller than $d_e$) the contribution of the current density will be more pronounced than at large scales (scales greater than $d_e$).

5. MAGNETIC FLUCTUATIONS SPECTRUM

The universal law (Equation (16)) gives a precise description of electron MHD turbulence at scales smaller than $d_e$. It also provides the possibility of predicting the form of the magnetic fluctuation spectrum and then a comparison with observations. Dimensionally, relation (16) corresponds to

$$d_i d_x^2 J^3 \sim \epsilon^T r,$$

(21)

which means an energy spectrum in

$$E^J (k) \sim \left( \frac{\epsilon^T}{d_i d_x^2} \right)^{2/3} k^{-5/3},$$

(22)

compatible with the dynamical Equation (5) and direct numerical simulations (Biskamp et al. 1996). Since the current density and the magnetic field satisfy the equation $\mathbf{J} = \nabla \times \mathbf{B}$, we obtain the scaling relation

$$d_i d_x^2 B^3 \sim \epsilon^T r.$$
6. CONCLUSION

The turbulence regime at scales smaller than the electron inertial length $d_e$ has been investigated through the approximation of electron MHD. A new universal and exact law has been established in terms of structure functions for the current density. This law leads to the prediction of a $k^{-11/3}$ power-law spectrum for the magnetic field fluctuations compatible with the most recent observations made with Cluster. It is proposed that electron MHD turbulence provides a valuable first-order approximation for the solar-wind dynamics in particular below the length scale $d_e$. It also provides the first prediction for the magnetic fluctuation spectrum at these length scales. The possibility of getting a turbulence regime at electron scales questions the origin of dissipation in the solar wind and more generally in space plasmas. In previous analyses (Sahraoui et al. 2009; Alexandrova et al. 2009), it was suggested that the range of scales where the heating occurs was discovered but it was also confessed that the characteristics of turbulence in the vicinity of the kinetic plasma scales are not well known either experimentally or theoretically and are a matter of debate (see also Podesta et al. 2010). It is believed that the present theoretical prediction will help significantly in such a debate.

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APPENDIX

EXTENSION TO TWO-DIMENSIONAL OR SLAB TURBULENCE

We now discuss the extension of the exact prediction (16) to the non-isotropic case. The simplest situation is when a background magnetic field $B_0$ is applied to the plasma flow for which the dynamics becomes statistically axisymmetric. We will adopt the viewpoint of MacBride et al. (2008) and assume by simplicity that electron MHD turbulence is either two dimensional or a slab. If we define the energy flux vector $F$ as (see relation (15))

$$F \equiv d_e^2 \nabla \cdot (\delta J(\delta J)^T), \quad (A1)$$

then we obtain the general formulation for isotropic electron MHD turbulence

$$F(r) = \frac{4}{D} \varepsilon J r, \quad (A2)$$

where $D$ is the space dimension. Two-dimensional turbulence means that $F$ and $B_0$ are perpendicular whereas they are parallel for slab (or one-dimensional) turbulence. Under these simplifications, it is straightforward to derive the following predictions:

$$F^{2D}(r) = 2 \varepsilon J V_{SW} \tau \sin \theta, \quad (A3)$$

and

$$F^{slab}(r) = 4 \varepsilon J V_{SW} \tau \cos \theta, \quad (A4)$$

where $\tau$ is the time, and $\theta$ is the angle between the mean magnetic field and the solar-wind velocity $V_{SW}$. Note that to obtain these predictions, the Taylor hypothesis has been used (MacBride et al. 2008). It is believed that such relations could be useful for analyzing solar-wind turbulence and to evaluate the perpendicular and parallel mean energy dissipation rates per unit mass. Note that a more sophisticated approach has been recently used to describe axisymmetric electron MHD turbulence for length scales larger than $d_e$ (Galtier 2009b). Under the assumption of critical balance, it leads to a vectorial relation for the energy flux with a dependence on both $r$ and $\theta$.

REFERENCES

Alexakis, A., Bigot, B., Politano, H., & Galtier, S. 2007, Phys. Rev. E, 76, 056313
Alexandrova, O., et al. 2009, Phys. Rev. Lett., 103, 165003
Batchelor, G. K. 1953, The Theory of Homogeneous Turbulence (Cambridge: Cambridge Univ. Press)
Bhattacharjee, A. 2004, ARA&A, 42, 365
Bigot, B., Galtier, S., & Politano, H. 2008, Phys. Rev. E, 78, 066301
Biskamp, D., Schwarz, E., & Drake, J. F. 1996, Phys. Rev. Lett., 76, 1264
Biskamp, D., et al. 1999, Phys. Plasmas, 6, 751
Carbone, V., et al. 2009, Phys. Rev. Lett., 103, 061102
Chacon, L., Simakov, A. N., & Zocco, A. 2007, Phys. Rev. Lett., 99, 235001
Dastgeer, S., Das, A., & Kaw, P. 2000a, Phys. Plasmas, 7, 1366
Dastgeer, S., Das, A., Kaw, P., & Diamond, P. H. 2000b, Phys. Plasmas, 7, 751
Frisch, U. 1995, Turbulence (Cambridge: Cambridge Univ. Press)
Galtier, S. 2006, J. Plasma Phys., 72, 721
Galtier, S. 2008a, Phys. Rev. E, 77, 015302
Galtier, S. 2008b, J. Geophys. Res., 113, A01102
Galtier, S. 2009a, Nonlinear Process. Geophys., 16, 83
Galtier, S. 2009b, Phys. Plasmas, 16, 112310
Galtier, S., & Buchlin, E. 2007, ApJ, 656, 560
Galtier, S., Nazarenko, S. V., Newell, A. C., & Pouquet, A. 2000, J. Plasma Phys., 63, 447
Gary, S. P., Saito, S., & Li, H. 2008, Geophys. Res. Lett., 35, L02104
Ghosh, S., Siregar, E., Roberts, D. A., & Goldstein, M. L. 1996, J. Geophys. Res., 101, 2499
Goldstein, M. L., Roberts, D. A., & Fitch, C. A. 1994, J. Geophys. Res., 99, 11519
Govoni, F., et al. 2006, A&A, 460, 425
Hasegawa, A., & Chen, L. 1976, Phys. Rev. Lett., 36, 1362
Howes, G. G., et al. 2008, J. Geophys. Res., 113, A05103
Kimpen, A. S., Chukbar, K. V., & Yan'kov, V. V. 1990, Reviews of Plasma Physics, Vol. 16 (New York: Consultants Bureau)
Kiyani, K. H., et al. 2009, Phys. Rev. Lett., 103, 075006
Klein, L., Bruno, R., Bavassano, B., & Rosenbauer, H. 1993, J. Geophys. Res., 98, 17461
Kolmogorov, A. N. 1941, Dokl. Akad. Nauk SSSR, 32, 16
Kolmogorov, A. N. 1941, Dokl. Akad. Nauk SSSR, 32, 16
Landau, L., & Lifchitz, E. 1989, Mécanique des Fluides (2nd ed.; Moscow: Editions Mir)
Leamon, R. J., Smith, C. W., Ness, N. F., Matthaeus, W. H., & Wong, H. K. 1998, J. Geophys. Res., 103, 4775
Li, H., Gary, S. P., & Stawicki, P. 2001, Geophys. Res. Lett., 28, 1347
MacBride, B. T., Smith, C. W., & Forman, M. A. 2008, ApJ, 679, 1644
Matthaeus, W. H., Servidio, S., & Dmitruk, P. 2008, Phys. Rev. Lett., 101, 149501
Matthaeus, W. H., Zank, G. P., Smith, C. W., & Oughton, S. 1999, Phys. Rev. Lett., 82, 3444
Narita, Y., & Gary, S. P. 2010, Ann. Geophys., 28, 597
Podesta, J. J., Borovsky, J. E., & Gary, S. P. 2010, ApJ, 712, 685
Politano, H., & Pouquet, A. 1998, Phys. Rev. E, 57, R21
Sahraoui, F., Goldstein, M. L., Robert, P., & Khotyaintsev, Yu. V. 2009, Phys. Rev. Lett., 102, 231102
Scalo, J., & Elmegreen, B. G. 2004, ARA&A, 42, 275
Smith, C. W., Hamilton, K., Vasquez, B. J., & Leamon, R. J. 2006, ApJ, 645, L85
Stawicki, O., Gary, P. S., & Li, H. 2001, J. Geophys. Res., 106, 8273