An algebraic proof of Deligne’s regularity criterion

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Abstract. Deligne’s regularity criterion for an integrable connection $\nabla$ on a smooth complex algebraic variety $X$ says that $\nabla$ is regular along the irreducible divisors at infinity in some fixed normal compactification of $X$ if and only if the restriction of $\nabla$ to every smooth curve on $X$ is regular (i.e. has only regular singularities at infinity). The “only if” part is the difficult implication. Deligne’s proof is transcendental, and uses Hironaka’s resolution of singularities. We give here an elementary and purely algebraic proof of this implication: it is, as far as we know, the first algebraic proof of Deligne’s regularity criterion.

Introduction.

0.1. Let $X$ be a smooth connected algebraic variety over a field $K$ of characteristic 0. Let $\mathcal{E}$ be a coherent module on $X$ endowed with an integrable connection $\nabla$. When $X$ is a curve, the dichotomy between regular and irregular singularities (at infinity) goes back to the 19th century. The connection $\nabla$ is said to be regular if all singularities (at infinity) are regular, see Manin’s classical paper [Ma]. This is obviously a birational notion.

In higher dimension, one may consider a normal compactification $\bar{X}$ of $X$, look at generic points $P$ of irreducible components of $\partial X = \bar{X} \setminus X$ of codimension one in $\bar{X}$, and tensor $\mathcal{E}$ with the discrete valuation ring $\mathcal{O}_{\bar{X}, P}$. The notion of regularity of $\nabla$ at $P$ is then the familiar one.

In order to obtain a birational notion of regularity, one is then led to say that $\nabla$ is regular if for any $(X’, P)$ as before, $\nabla$ is regular at $P$.

This definition (which is one of the equivalent definitions taken by Deligne in his Lecture Notes [De, II.4.5]) is rather forbidding, since it requires to consider all divisorial valuations of $K(X)$ at the same time. Fortunately, it actually suffices
to consider only one normal compactification $\bar{X}$: in fact, according to Deligne, one has the following characterizations of regularity:

**Theorem 0.1.** [De, II.4.4, 4.6] The following are equivalent:

i) $\nabla$ is regular,

ii) for some normal compactification $\bar{X}$, $\nabla$ is regular at (the generic points of) all irreducible components of $\partial X$ of codimension one in $\bar{X}$,

iii) for any smooth curve $C$ and any locally closed embedding $h : C \to X$, $h^*\nabla$ is regular,

iv) for any smooth $Y$ and any morphism $f : Y \to X$, $f^*\nabla$ is regular,

v) for some dominant morphism $f : Y \to X$ with $Y$ smooth, $f^*\nabla$ is regular.

The difficult implication is $ii) \Rightarrow iii)$. The implication $iii) \Rightarrow i)$ is comparatively easy to establish (cf. e.g. [AB, I. 3.4.7]), and the other implications follow very easily from these two. The difficulty with $ii) \Rightarrow iii)$ arises when the closure of $C$ in $\bar{X}$ does not meet $\partial X$ transversally. A closely related difficulty, with $ii) \Rightarrow i)$, is to show that $\nabla$ remains regular at the exceptional divisor when one blows up a subvariety of $\partial X$.

**0.2.** In order to prove the difficult implication $ii) \Rightarrow iii)$, Deligne relied in his Lecture Notes upon a statement about stability of regularity under specialization. This statement [De, II.1.23] is actually false, as B. Malgrange pointed out.

Deligne later circulated a corrigendum, using a transcendental argument to settle the problem (assuming, as one may, that $K = \mathbb{C}$), and using the existence of smooth compactifications $\bar{X}$ such that $\partial X$ is a divisor with strict normal crossings (Hironaka). Over such a $\bar{X}$, the analytic connection $\nabla_{an}$ attached to $\nabla$ extends to a connection with logarithmic poles $\nabla_{an}^\log$ along $\partial X$, which is uniquely algebraizable. Its restriction to $X$ is an algebraic integrable connection $\nabla_{reg}$, whose analytification $\nabla_{reg}^{an}$ is canonically isomorphic to $\nabla_{an}^{an}$. The condition that $\nabla$ is regular amounts to saying that this isomorphism is algebraizable.

**0.3.** In our book [AB], we have revisited these mostly well-known results about regularity, with the aim of offering purely algebraic, and as elementary as possible, proofs. In particular, we claimed to give such a proof of the above implication $ii) \Rightarrow iii)$ (cf. [AB, I.5.4]). We used the fact that $\nabla$ extends to a logarithmic connection outside a divisor of $\partial X$ (a closed subset of codimension at least 2 in $\bar{X}$) and extended it by direct image to the whole of $\bar{X}$. We claimed that its restriction to the closure of $h(C)$ is a logarithmic connection, relying on a certain lemma on differentials with logarithmic poles (lemma 5.5).
This lemma [AB, I.5.5] is actually false\(^{(1)}\) - in the case of a divisor with non-normal crossings - as J. Bernstein pointed out to us.

0.4. The aim of the present paper is twofold:

1) to analyse in detail the nature of the error in [AB, I.5.5], following Bernstein (sections 3 and 4),

2) to supply an elementary\(^{(2)}\) and purely algebraic proof of Deligne’s regularity criterion, in the following slightly refined form:

**Proposition 0.2.** (cf. [AB, I.5.4]). Let \(X'\) be an algebraic variety over a field \(K\) of characteristic 0. Let \(X\) be a smooth open subset of \(X'\), with complement \(\partial X = X' \setminus X\). We fix a closed point \(P \in \partial X\) such that for each irreducible component \(Z\) of \(\partial X\) passing through \(P\) whose local ring \(\mathcal{O}_{X',Z}\) in \(X'\) is of dimension 1, that local ring is a discrete valuation ring.

Let \(C' \xrightarrow{h} X'\) be a morphism from a smooth connected \(K\)-curve \(C'\) such that \(h(C') \not\subseteq \partial X\), and let \(C\) be the open subset \(h^{-1}(X)\) of \(C'\). We assume there is a closed point \(Q \in C'\) with \(h(Q) = P\).

Let \((\mathcal{E}, \nabla)\) be a coherent sheaf with integrable connection on \(X/K\), with generic fiber the \(\kappa(X)/K\)-differential module \((E, \nabla) := (\mathcal{E}, \nabla)_{\eta_X}\). We assume that, for each irreducible component \(Z\) of \(\partial X\) passing through \(P\) whose local ring \(\mathcal{O}_{X',Z}\) in \(X'\) is of dimension 1, \((E, \nabla)\) is regular at the corresponding divisorial valuation \(v_Z\) of \(\kappa(X)\).

Then \((\mathcal{E}, \nabla)|_C\) is regular at \(Q\).

Our proof begins with a reduction to the case of an affine open neighborhood \(X'\) of \(P = O\) in \(\mathbb{A}^2_K\) (section 1). In that situation, we proceed with a corrected version of the wrong lemma [AB, I.5.5], based on the study of a certain filtration on a logarithmic De Rham complex (section 2).

**Acknowledgements.** Professor Bernstein in a letter of September 26, 2003, indicated to us a crucial (counter-)example on which to test a possible solution of the problem. We reproduce a simplified version of his (counter-)example below. His remarks were of great help to us. We thank him heartily.

We are also much indebted to Maurizio Cailotto for the key log geometry argument in lemma 2.1.

\(^{(1)}\) That error - which we shall analyse in detail below - questions the validity of the proof of proposition 5.4 loc. cit.. The argument sketched in the subsequent remark 5.6, as an alternative proof of (5.4), also suffers of the same objection. Remark 6.5.6 of loc. cit., suggested a third possible proof of (5.4). That does not work either, since the reference to (3.4.4) is not appropriate, because we do not have a morphism of models.

\(^{(2)}\) not using resolution of singularities beyond the classical case of surfaces
1. Reduction to the two-dimensional case.

Our first aim is to reduce to the case when $X'$ is a smooth surface. This is done in the next two lemmas. This reduction serves two purposes:
- the embedded resolution of $\partial X$ becomes elementary,
- any reflexive coherent module on such an $X'$ is locally free.

**Lemma 1.1.** Let $X$ be a smooth open dense subvariety of a quasi-projective irreducible $K$-variety $X' \subset P^N_K$ of dimension $n > 2$, and let $P$ be a closed point of $\partial X = X' \setminus X$. Let $Z_1, \ldots, Z_r$ be the distinct irreducible components of codimension one of $\partial X$ passing through $P$. We assume that for any $i = 1, \ldots, r$ the local ring $O_{X',Z_i}$ is a DVR. Let $C'$ be a simple closed $K$-curve in $X'$ through $P$, $C' \not\subset \partial X$. Let $C = C' \cap X$. For any sufficiently large $d$, there exists an irreducible complete intersection $Y \subset P^N_K$ of multidegree $(d, \ldots, d)$ $(n-2)$ entries such that:

(i) $Y$ contains $C'$ and cuts $X$ transversally at $\eta_C$;

(ii) $Y \cap X'$ is an irreducible surface and $Y$ cuts $X \setminus C$ transversally;

(iii) in a neighborhood of $P$, $Y$ cuts each $Z_i \setminus \{P\}$ transversally and does not cut any irreducible component of $\partial X$ of codimension $> 1$ in $X'$, nor the singular locus of $\bigcup_i Z_i$, except in $P$.

In particular, there exists a quasi-projective irreducible neighborhood $X'_2$ of $P$ in $Y \cap X'$, containing an open subset of $C'$, such that $X_2 := X'_2 \cap X$ is smooth, the distinct irreducible components of codimension one of $\partial X_2 = X'_2 \setminus X_2$ passing through $P$, are precisely $Z_1 \cap X'_2, \ldots, Z_r \cap X'_2$, and for any $i = 1, \ldots, r$, the local ring $O_{X'_2,Z_i \cap X'_2}$ is a DVR.

**Proof.** This is standard, but, for lack of reference, we give some detail. Replacing $X'$ by its closure in $P^N$, we may assume that $X'$ (and $C'$) is a closed subvariety of $P^N$. Except at the very end of this proof, $n = \dim X'$ will be allowed to take the value $n = 2$. Let $\pi : B \to P^N$ be the blowing-up of $P^N$ centered at $C'$, with exceptional divisor $D$.

We set $\mathcal{I}_{C'} = \ker(O_{P^N} \to O_{C'})$. Then $(\pi^{-1}\mathcal{I}_{C'}) \cdot O_B = O(-D)$ and, for $d > 0$, $\pi^*(O_{P^N}(d)) \otimes O(-D)$ is very ample [Ha, II.7.10.3]: a basis of sections defines an embedding of $B$ into $P^M$. Since $\pi_*O_B = O_{P^N}$ [Ha, proof of III.11.4], one has $\pi_*\pi^{-1}\mathcal{I}_{C'} = \mathcal{I}_{C'}$, as ideals of $O_{P^N}$ and therefore $\pi_*O(-D) = \mathcal{I}_{C'}$. From the projection formula we deduce

$$\pi_*(\pi^*(O_{P^N}(d)) \otimes O(-D)) \cong O_{P^N}(d) \otimes \mathcal{I}_{C'} \cong \mathcal{I}_{C'}(d) .$$

So, one has

$$\ker(H^0(P^N, O_{P^N}(d)) \to H^0(C', O_{C'}(d))) = \ker(H^0(P^N, \mathcal{I}_{C'}(d)) = H^0(B, \pi^*(O_{P^N}(d)) \otimes O(-D)) .$$
and the linear system of hypersurfaces of degree $d$ in $\mathbb{P}^N$ containing $C'$ gives rise to a locally closed embedding

$$\mathbb{P}^N \setminus C' \hookrightarrow \mathbb{P}^M = \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_{C'}(d))),$$

with Zariski closure $B$. The canonical bijection between hyperplanes $\mathcal{H}$ of $\mathbb{P}^M$ and hypersurfaces $H$ of degree $d$ in $\mathbb{P}^N$ containing $C'$, is such that the intersection $\mathcal{H} \cap (\mathbb{P}^N \setminus C')$ (in $\mathbb{P}^M$) equals $H \setminus C'$. So, the intersection of $X' \setminus C'$ with a general complete intersection $Y$ of multidegree $(d, \ldots, d)$ ($1 \leq s \leq n - 1$ entries) in $\mathbb{P}^N$ containing $\eta_C$, is the intersection of $X' \setminus C'$ with a general linear subvariety $\mathcal{Y}$ of codimension $s$ in $\mathbb{P}^M$. By [S.G.A. 4, Exp. XI, Thm. 2.1 (i)], $Y$ cuts $X \setminus C$ (resp. the smooth part of $(\cup_i Z_i) \setminus (C' \cap (\cup_i Z_i))$) transversally and intersects properly any irreducible component of $\partial X \setminus (C' \cap \partial X)$. Since $s < n$, Bertini’s theorem shows that the intersection of $\mathcal{Y}$ with the strict inverse image of $X'$ in $\mathbb{P}^M$ is irreducible. On the other hand, since $\eta_C$ is a simple point of $X$, it is well-known that a general complete intersection of $s$ hypersurfaces of degree $d$ in $\mathbb{P}^N$ containing $\eta_C$, intersects $X$ transversally at this point.

We now apply the previous considerations for $n > 2$ and $s = n - 2$ to get properties (i), (ii) and (iii). The last assertion is clear.

We now reconsider the situation in the statement of the proposition. We apply lemma 1 taking as $C'$ the closure of the image of $h$ in $X'$, with reduced induced structure. The closed embedding $X'_2 \hookrightarrow X'$, induces, for sufficiently small open neighborhoods $U_i$ of $\eta_{Z_i}$ in $X'$, a closed embedding of smooth $K$-models $(U_i \cap X'_2, U_i \cap Z_i \cap X'_2) \hookrightarrow (U_i, U_i \cap Z_i)$. By [AB, I.3.4.4], $(\mathcal{E}, \nabla)|_{X'_2}$ is regular along every component of codimension $1$ of $\partial X_2 = X'_2 \setminus X_2$, containing $P$. This reduces our proposition to the case where $X'$ is a projective surface. We next show that we may further assume without loss of generality that $P$ a smooth point of $X'$, and in fact that $X'$ is an open neighborhood of the origin $P = O$ in $\mathbb{A}^2_K$.

**Lemma 1.2.** In the notation of lemma 1.1, let us further assume that $X'$ is a closed irreducible subvariety of $\mathbb{P}^N_K$, of dimension $2$. There exists a finite morphism $g : X' \longrightarrow \mathbb{P}^2_K$ whose branch locus $B \subset \mathbb{P}^2_K$ does not contain the image of $\eta_{C'}$, and for any irreducible component $W$ of $\partial X$ of dimension $1$ with $P \notin W$, $g(P) \notin g(W)$.

**Proof.** We consider the Grassmannian $G := G(N - 3, \mathbb{P}^N) \cong G(2, (\mathbb{P}^N)^\vee)$ (resp. $G(N - 2, \mathbb{P}^N) \cong G(1, (\mathbb{P}^N)^\vee)$) of linear subvarieties of $\mathbb{P}^N$ of codimension $3$ (resp. $2$). Let $F \subset G(N - 2, \mathbb{P}^N) \times G(N - 3, \mathbb{P}^N)$ be the incidence subvariety (locus of $(\mathcal{L}, \alpha)$ such that $\mathcal{L}$ contains $\alpha$). There is a natural smooth projective fibration $p : F \longrightarrow G$ with fiber $\mathbb{P}^2$, where $p^{-1}(\alpha)$, for a linear subvariety $\alpha$ of $\mathbb{P}^N$ of codimension $3$, is the projective plane of linear subvarieties of codimension $2$ in $\mathbb{P}^N$, passing through $\alpha$. The $\alpha$’s which do not intersect $X'$, form an open dense subset $U$ of $G$. For any $\alpha \in U$, the points of $p^{-1}(\alpha)$ corresponding to
a 2-codimensional linear subvariety \( L \) of \( \mathbb{P}^N \) which fails to intersect \( X' \) only at smooth points and there transversally (resp. passing through \( P \), resp. which has a non-empty intersection with \( C' \) (resp. \( W \), for any \( W \) as in the statement)), form a Zariski closed subset \( B_\alpha \) (resp. \( P_\alpha \), resp. \( C_\alpha \) (resp. \( W_\alpha \))) in \( p^{-1}(\alpha) \). All of \( B_\alpha, C_\alpha \) and \( W_\alpha \) have dimension 1, while \( P_\alpha \) is a point. The curves \( C_\alpha \) and \( W_\alpha \) and the closed set \( P_\alpha \) are irreducible. Our problem is solved if we find \( \alpha \) such that \( C_\alpha \) is not a component of \( B_\alpha \), and \( P_\alpha \not\in W_\alpha \). We consider the algebraic subset \( H_B \) (resp. \( H_P \), resp. \( H_C \), resp. \( H_W \)) in \( F \) defined as the closure in \( F \) of \( \cup_{\alpha \in U} B_\alpha \) (resp. \( \cup_{\alpha \in U} \{ P_\alpha \} \), resp. \( \cup_{\alpha \in U} C_\alpha \), resp. \( \cup_{\alpha \in U} W_\alpha \)). The hypersurface \( H_C \) is then irreducible, as well as the algebraic sets \( H_P \) and \( H_W \). It is clear that \( G \cong H_P \not\subset H_W \), since there are linear 2-codimensional subvarieties \( L \) of \( \mathbb{P}^N \) passing through \( P \) which avoid \( W \). So, for \( \alpha \) in a dense open subset of \( G \), \( P_\alpha \not\in W_\alpha \). It will then suffice to prove that \( H_B \) does not contain \( H_C \). But the curve \( C' \) is simple in \( X' \), meaning that \( C' \) is a closed integral \( K \)-subscheme of dimension 1 of \( X' \), and that \( \eta_{C'} \) is a simple point of \( X' \). So, let \( R \) be a closed point of \( C' \), simple in \( C' \) and in \( X' \). Then there exist linear 2-codimensional subvarieties \( L \) of \( \mathbb{P}^N \), intersecting \( X' \) transversally in \( R \). This shows that \( H_B \) does not contain \( H_C \).

In the situation of the proposition with \( X' \) a projective surface, we apply lemma 1.2 to the simple curve which is the closure of the image of \( h \) in \( X' \), equipped with the reduced induced structure. We obtain an étale covering

\[
g : V := X' \setminus (g^{-1}(B) \cup g^{-1}(g(\partial X))) \longrightarrow \mathbb{P}^2_K \setminus (B \cup g(\partial X))
\]

and push forward \( (\mathcal{E}, \nabla) \) via \( g \). By [AB, I.3.4.4], the connection \( g_*((\mathcal{E}, \nabla)|_V) \) has a regular singularity at the generic points of the 1-dimensional components of \( B \) passing through \( g(P) \), and at \( g(\eta_{Z_1}), \ldots, g(\eta_{Z_r}) \). But these are all the generic points of the 1-dimensional irreducible components of \( B \cup g(\partial X) \) passing through \( g(P) \). Therefore, the assumptions of the proposition are satisfied by the new choice \( X := \mathbb{P}^2_K \setminus (B \cup g(\partial X)) \), \( X' = \mathbb{P}^2_K \), \( (\mathcal{E}, \nabla) \) being replaced by \( g_*((\mathcal{E}, \nabla)|_V) \), \( P \) by \( g(P) \), and \( h \) by \( g \circ h \).

Therefore we are reduced to the case when \( X' \) is a smooth surface, and even an affine open subset of the affine plane.

2. The case of an open subset of \( \mathbb{A}^2_K \).

**Lemma 2.1.** Let \( X' = \text{Spec} A \), be an affine open \( K \)-neighborhood of the origin \( O \) in \( \mathbb{A}^2_K \) and set \( U := X' \setminus \{ O \} \). Let \( g_1, \ldots, g_r \) be non-associated irreducible elements of \( A \), \( Z_i = V(g_i) \), for \( i = 1, \ldots, r \), and \( Z := \cup_i Z_i \). We assume that \( U \cap Z \) is a normal crossing divisor in \( U \).

Let \( C' \xrightarrow{h} X' \) be a morphism from a smooth connected \( K \)-curve \( C' \) and assume there is a closed point \( Q \in C' \) with \( h(Q) = O \), while, for \( C := C' \setminus \{ Q \} \), \( h(C) \subset X \).
If $(\mathcal{E}, \nabla)$ is an integrable connection on $X = X' \setminus Z$, regular at $\eta_1, \ldots, \eta_r$, then $(\mathcal{E}, \nabla)|_C$ is regular at $Q$.

Proof.
We are presently working on this section!

3. Example (after J. Bernstein).

We consider the situation of lemma 2.1. Let $(u, v)$ be affine coordinates on $X' = \mathbb{A}^2$, $Z = V(uv(u + v))$. Let $\pi : \tilde{X} \longrightarrow X'$ be the blowing-up of $X'$ centered at $O$. We use a local coordinate $t = \frac{v}{u}$ on $\tilde{X}$.

3.1. $\omega^1_{(X', Z)} \subseteq j_*\Omega^1_U(\log Z \cap U)$

Consider the 1-form

$\alpha = \frac{du}{u} + \frac{dv}{v} - 2\frac{d(u + v)}{u + v} \in \Gamma(X, j_*\Omega^1_U(\log Z \cap U))$

An easy computation shows that

$uv(u + v)\alpha = (u - v)(udv - vdu)$

Now set $\omega = \frac{\alpha}{u - v}$. To show that $\omega \in \Gamma(X, j_*\Omega^1_U(\log Z \cap U))$, we use the criterion of Deligne [De, I.2.2.1]. Consider the behavior of the form $\omega$ on $U$. Formula (3.1.2) implies that $uv(u + v)\omega = udv - vdu$. Hence the form $\omega$ is regular on $X \setminus Z$. In a neighborhood of the variety $Z \cap U$ the form $\omega$ is proportional to the form $\alpha$ which by construction has simple poles along $Z \cap U$. This shows that the form $\omega$ has simple poles along $Z \cap U$ on $U$. On the other hand, $d\omega = -\frac{du \wedge dv}{uv(u + v)}$ also has only simple poles along $Z \cap U$. We conclude that $\omega$ is a section of $j_*\Omega^1_U(\log Z \cap U)$. Notice that $\omega$ is not a section of $\omega^1_{(X', Z)}$, since $\pi^*\omega = \frac{dt}{ut(1 + t)}$, so that $d\pi^*\omega = -\frac{du \wedge dt}{u^2 t(1 + t)}$ has a double pole along the exceptional divisor, and is not therefore a section of $\Omega^1_X(\log \tilde{Z})$. A more precise computation shows that $\Gamma(U, \Omega^1_U(\log Z \cap U))$ is the free $K[u, v]$-module generated by $\omega$ and $\frac{du}{u}$ (or by $\omega$ and $\frac{dv}{v}$).

On the other hand, the closed exact immersion of the log scheme $(X', Z)$ in the smooth (over $K$) log scheme $(\mathbb{A}^3, V(xyz))$, $(u, v) \longmapsto (u, v, u + v)$, shows [IL1, II.2.5.(a)] that $\omega^1_{(X', Z)}$ is the free $K[u, v]$-module generated by $d\log(u)$, $d\log(v)$, $d\log(u + v)$, subject to the single relation

$u \log(u) + v \log(v) = (u + v) \log(u + v)$.

3.2. A counterexample to [AB, I.5.5].

Let now $C'$ be the curve described in parametric form by $u = s$, $v = \nu s^2$, for generic $\nu \in K$. 

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The restriction of the form $\omega$ (which is a section of $j_*\Omega_U^1(\log Z \cap U)$) to the curve $C'$ equals $\frac{ds}{s^2}(1 + O(s))$ and thus is not of logarithmic type at $s = 0$.

This is a counter-example to [AB, I.5.5].

A consequence of the previous construction is that the restriction to $C'$ of the direct image by $j$ of an integrable connection with logarithmic poles along $Z \cap U$ is not necessarily of logarithmic type at $s = 0$.

Explicitly, let $\mathcal{E} = \mathcal{O}_U^2$ be equipped with the logarithmic connection $d + G$, where

$$G = \begin{pmatrix} 0 & 0 \\ \omega & \frac{du}{2u} + \frac{dv}{2v} \end{pmatrix}.$$ 

Since $dG + G \wedge G = 0$, the connection is integrable. What we said before about $\omega$ shows however that the connection given on $j_*\mathcal{E} = \mathcal{O}_X^2$, by the same formula is not logarithmic. The matrix of the pulled-back connection on $\mathcal{O}_{C'}^2$ has a double pole at $s = 0$.

To show that this connection is nevertheless regular, we follow the proof of lemma 3 above. The pulled-back connection on $\tilde{\mathcal{E}} = \mathcal{O}_{\tilde{X}}^2$ is locally expressed as $d + H$, where

$$H = \begin{pmatrix} 0 \\ \frac{dt}{ut(1+t)} & \frac{du}{u} + \frac{dt}{2t} \end{pmatrix}.$$ 

The restriction of this connection to the subsheaf $\mathcal{F} = u\mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\tilde{X}}$ of $\tilde{\mathcal{E}}$, meaning, in matrix notation,

$$\mathcal{F} = \{ \begin{pmatrix} uf \\ g \end{pmatrix} \mid f, g \in \mathcal{O}_{\tilde{X}} \},$$ 

is therefore logarithmic. We now pull-back $(\mathcal{F}, d + H)$ via the strict inverse image of $C'$, namely $\tilde{h} : C' \hookrightarrow \tilde{X}$, $s \mapsto (u, t) = (s, \nu s)$, and we get that

$$\tilde{h}^*\mathcal{F} = \{ \begin{pmatrix} sf \\ g \end{pmatrix} \mid f, g \in \mathcal{O}_{C'} \},$$ 

is stable under $d + H(s)$, where

$$H(s) = \begin{pmatrix} 0 \\ \frac{ds}{s^4(1+\nu s^2)} & \frac{3ds}{2s} \end{pmatrix}.$$
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