REAL-VALUED MEASURABLE CARDINALS AND SEQUENTIALLY CONTINUOUS HOMOMORPHISMS

VLADIMIR V. USPENSKIJ

Abstract. A. V. Arkhangel’skii asked in 1981 if the variety \( \mathfrak{V} \) of topological groups generated by free topological groups on metrizable spaces coincides with the class of all topological groups. We show that if there exists a real-valued measurable cardinal then the variety \( \mathfrak{V} \) is a proper subclass of the class of all topological groups. A topological group \( G \) is called \( g \)-sequential if for any topological group \( H \) any sequentially continuous homomorphism \( G \to H \) is continuous. We introduce the concept of a \( g \)-sequential cardinal and prove that a locally compact group is \( g \)-sequential if and only if its local weight is not a \( g \)-sequential cardinal. The product of a family of non-trivial \( g \)-sequential topological groups is \( g \)-sequential if and only if the cardinal of this family is not \( g \)-sequential. Suppose \( G \) is either the unitary group of a Hilbert space or the group of all self-homeomorphisms of a Tikhonov cube. Then \( G \) is \( g \)-sequential if and only if its weight is not a \( g \)-sequential cardinal. Every compact group of Ulam-measurable cardinality admits a strictly finer countably compact group topology.

1. Introduction

In 1981 A.V.Arkhangel’skii [1, p. 171, Remark f] asked if

(A) the family of free topological groups on metrizable spaces generates (by means of the operations of taking products, subgroups, and factor groups) the class of all topological groups.

Assuming the existence of real-valued measurable cardinals, we answer this question in the negative (Theorem 2.13). Actually it suffices to assume the existence of smaller cardinals, which we call \( g \)-sequential (Definition 1.3).

Definition 1.1. [24, 25, 26, 27]. A class of topological groups is called a variety if it is closed under arbitrary products, subgroups, and topological quotient groups.

Our results do not depend on whether we assume or not that all topological groups are Hausdorff. Unless otherwise stated, we assume that all groups are Hausdorff, but
in Section 2 it will be more convenient to consider also non-Hausdorff group topologies. Throughout the paper we denote by \( \mathcal{V} \) the variety of topological groups generated by free topological groups on metrizable spaces. Thus (A) is the assertion that \( \mathcal{V} \) coincides with the class of all topological groups. Recall that the (Markov) free topological group \( F(X) \) on a Tikhonov space \( X \) is characterized by the following property: \( X \) is a subspace of \( F(X) \), and every continuous mapping \( f \) from \( X \) to a topological group \( G \) extends uniquely to a continuous homomorphism \( \bar{f} : F(X) \to G \).

The present paper was inspired by the manuscript [28] by S. Morris, P. Nickolas, V. Pestov and S. Svetlichny, where an approach was suggested to a positive solution of Arkhangel’skiı’s question. We discuss the ideas of [28] below in Section 4. It follows from the results of [28] and from our arguments that if there are no sequential cardinals \( \mathfrak{g} \) and if the assertion (F) below is true, then Arkhangel’skiı’s question has a positive answer. Moreover, in this case every topological group is a quotient of a closed subgroup of the free topological group on a metrizable space, as conjectured in [28].

(F) If \( X \subset Y \) are Tikhonov spaces and the fine uniformity \( \mathcal{U}_Y \) on \( Y \) induces the fine uniformity \( \mathcal{U}_X \) on \( X \), then the natural injection \( F(X) \to F(Y) \) between free topological groups is a topological embedding.

This assertion is contained in [33, 34]. However, the arguments in [33, 34] are hard to follow, and some specialists doubt that the proof of (F) is complete. We therefore prefer a cautious approach and consider the statement

(M) if there are no \( g \)-sequential cardinals, then (A) holds

as a conditional result that is true if (F) is.

Another way to prove (M) could be first to establish the following:

(U) For every Tikhonov space \( X \) the free topological group \( F(X) \) admits a topologically faithful unitary representation, that is, is isomorphic to a subgroup of the unitary group \( U(H) \) of a (non-separable) Hilbert space.

Here \( U(H) \) is equipped with the strong operator topology that it inherits from the product \( H^H \). The conjecture (U) can be viewed as a version of the following problem due to A. Kechris: is every Polish group a quotient of a closed subgroup of the unitary group of a separable Hilbert space? Indeed, it was noted in [39, Proposition 4.2] that this question has a positive answer if the free group on the space of irrationals admits a topologically faithful unitary representation. If (U) is true, then so is (M) (this follows from Theorem 4.10 below). It was proved in [39] that for every Tikhonov space \( X \) the free Abelian group \( A(X) \) admits a topologically faithful unitary representation. It follows that if there are no \( g \)-sequential cardinals, all Abelian topological groups belong to the variety \( \mathcal{V} \).

An ultrafilter on a set \( A \) is \( \kappa \)-complete if it is closed under intersections of families of cardinality \( < \kappa \). Let us say that a cardinal \( \text{Card} \) \((A)\) is Ulam-measurable if there exists an \( \omega_1 \)-complete free ultrafilter on \( A \). (A free ultrafilter is the same as a nonprincipal ultrafilter.) A cardinal is Ulam-measurable if and only if it greater than or equal to the
first measurable cardinal. As usual [19, Definition 10.3], an uncountable cardinal \( \kappa \) is measurable if there exists a \( \kappa \)-complete free ultrafilter on \( \kappa \).

The variety \( \mathfrak{V} \) consists of quotients of subgroups of products of free topological groups on metrizable spaces. The following definition, introduced in [28], plays a crucial role in the study of this variety.

**Definition 1.2 ([28]).** A topological group \( G \) is \( g \)-sequential if one of the following three equivalent properties holds:

1. for any topological group \( H \), any sequentially continuous homomorphism \( f : G \to H \) is continuous;
2. \( G \) admits no strictly finer group topology with the same convergent sequences;
3. \( G \) is isomorphic to a quotient group of the free topological group of a metrizable space.

The equivalence of the conditions 1 and 2 is clear, and their equivalence to the condition 3 is Theorem 3.7 in [28] (see also Section 4 below). It follows that the variety \( \mathfrak{V} \) can be described as the variety generated by all \( g \)-sequential groups. Hence the following assertion (G) implies that Arhangel’skii’s question has a positive answer, that is, the variety \( \mathfrak{V} \) coincides with the class of all topological groups.

(G) Every topological group is isomorphic to a subgroup of a \( g \)-sequential topological group.

Our main result, Theorem 2.13, implies that (G) is incompatible with the existence of real-valued measurable cardinals, so one cannot expect to prove (G) in ZFC. It is not clear if (G) is consistent. Assuming there are no \( g \)-sequential cardinals, we prove that for every Hilbert space \( H \) the unitary group \( U(H) \) is \( g \)-sequential, and that the same is true for the group \( \text{Aut } I^\tau \) of all self-homeomorphisms of a Tikhonov cube \( I^\tau \) (Theorem 4.11). Every topological group with a countable base is isomorphic to a subgroup of \( \text{Aut } I^\omega \) [35], but it is an open problem if a similar assertion holds for uncountable cardinals, that is, if every group of weight \( \tau \) is isomorphic to a subgroup of \( \text{Aut } I^\tau \). If it is true, then Theorem 4.11 implies that (G) holds under the assumption that there are no \( g \)-sequential cardinals.

We say that the local weight of a topological group \( G \) is \( \leq \kappa \) if the neutral element of \( G \) has a neighborhood of weight \( \leq \kappa \). If \( G \) is a locally compact group of non-Ulam-measurable local weight, then every sequentially continuous homomorphism \( f : G \to H \) to a locally compact group \( H \) is continuous [40]. It is natural to ask what locally compact groups are \( g \)-sequential, in other words, what locally compact groups do not admit strictly finer group topologies with the same convergent sequences. Some results in this direction were obtained in [9], [6]. We prove that a locally compact group \( G \) is \( g \)-sequential if and only if its local weight is not \( g \)-sequential (Theorem 3.14).

A slight modification of our main construction of refinements of group topologies yields the following result (Theorem 3.16): every locally compact group \( G \) of non-Ulam-measurable local weight admits a strictly finer group topology which agrees with the
VLADIMIR V. USPENSKIJ

original one on every set of non-Ulam-measurable cardinality. If $G$ is compact in the
original topology, it follows that in the new topology it is countably compact (moreover,
$\kappa$-compact for every non-Ulam-measurable cardinal $\kappa$). By a theorem of A. V. Arhangel’-
ski˘ı [2], a necessary condition for a compact group $G$ to admit a strictly finer countably
compact group topology is that the weight of $G$ be an Ulam-measurable cardinal. The-
orem 3.16 shows that this condition is also sufficient. Under the assumption that $G$ is
either Abelian or connected this was proved in [9].

The second part of Theorem 3.16 says that the result of [40] mentioned above can
be reversed: if $G$ is a locally compact group of Ulam-measurable local weight, then there
exists a sequentially continuous discontinuous homomorphism $f : G \to H$ of $G$ to a locally
compact group $H$.

Theorem 3.14 implies that $g$-sequential cardinals can be defined as follows: a cardinal
$\tau$ is $g$-sequential if the compact group $2^\tau$ is not $g$-sequential (here 2 denotes a discrete
group consisting of two points). We accept another property as the definition. For a set
$E$ let $P(E)$ denote the set of all subsets of $E$.

Definition 1.3. A subadditive measure on a set $E$ is a function $p : P(E) \to [0,1]$ such
that:

1. $p(A \cup B) \leq p(A) + p(B)$ for every $A, B \in P(E)$ with $A \cap B = \emptyset$;
2. for any decreasing sequence $A_1 \supset A_2 \supset \cdots \in P(E)$ with $\bigcap A_n = \emptyset$ we have
\[ \lim_{n \to \infty} p(A_n) = 0. \]

A cardinal $\tau = \text{Card } (E)$ is $g$-sequential if there is a subadditive measure $p$ on $E$ such that
$p(E) = 1$ and $p(F) = 0$ for all finite subsets $F \subset E$.

Every subadditive measure is monotone (that is, $A \subset B$ implies $p(A) \leq p(B)$), so
the assumption that $A$ and $B$ are disjoint can be omitted in the condition 1. A subadditive
measure $p$ on $E$ such that $p(A \cup B) = p(A) + p(B)$ for every $A, B \in P(E)$ with $A \cap B = \emptyset$ is a $\sigma$-additive measure in the usual sense. A cardinal $\text{Card } (E)$ is real-valued measurable
if there is a $\sigma$-additive measure $p : P(E) \to [0,1]$ such that $p(E) = 1$ and $p(F) = 0$ for all
finite subsets $F \subset E$. It follows that real-valued measurable cardinals are $g$-sequential.

Definition 1.4. A topological space $X$ is $f$-sequential if every sequentially continuous
real-valued function $f : X \to \mathbb{R}$ is continuous.

This concept was considered in [28] under another name. Recall that a cardinal $\tau$ is
sequential [7] if the space $2^\tau$ is not $f$-sequential. Every $f$-sequential group is $g$-sequential.
Applying this remark to the group $2^\tau$, we see that every $g$-sequential cardinal is sequential.
Another way to prove this is to note that every subadditive measure $p : P(E) \to [0,1]$ is
sequentially continuous on $P(E)$, which is identified with the compact space $2^E$ (Proposi-
tion 2.3). If $p(E) = 1$ and $p(F) = 0$ for all finite subsets $F \subset E$, then $p$ is not continuous.

Under Martin’s Axiom, for any cardinal ‘Ulam-measurable’ is equivalent to ‘sequential’
and to ‘real-valued measurable’ [7, theorem 2.2] and hence also to ‘$g$-sequential’. It is not
clear if it can be proved without additional set-theoretic assumptions that $g$-sequential
cardinals coincide with either real-valued measurable cardinals or with sequential cardinals.

In Section 2 we use the concept of a subadditive measure to introduce the main tool for the proof of Theorem 2.13: for any topological space \((X, \mathcal{T})\) we define a canonical refinement \(\mathcal{T}_g\) of the topology \(\mathcal{T}\) such that convergent sequences in \((X, \mathcal{T})\) remain the same as in \((X, \mathcal{T}_g)\). We then prove Theorem 2.13: if \(\text{Card}(E)\) is a \(g\)-sequential cardinal, the topological group \(\text{Sym}(E)\) of all permutations of \(E\) does not belong to the variety \(\mathcal{W}\), hence the existence of \(g\)-sequential cardinals implies that the variety \(\mathcal{W}\) is proper. In Section 3 we study refinements of locally compact group topologies and prove Theorems 3.14 and 3.16 mentioned above. We also prove that the product of a family of \(g\)-sequential groups is \(g\)-sequential provided that the cardinality of the family is not \(g\)-sequential (Theorem 3.2). In Section 4 we discuss, following [28], a possible way to prove Arhangel’skii’s conjecture (A) under some additional assumptions. The arguments of [28] depend on the fact that some products of Banach spaces are \(f\)-sequential. It is known that the product of a family \(\{X_\alpha : \alpha \in A\}\) of separable metric spaces is \(f\)-sequential if \(\text{Card}(A)\) is non-sequential [7, Theorem 1.5]. We prove (Theorem 4.4) that this remains true for arbitrary metric (or, more generally, bi-sequential) spaces. We then prove Theorem 4.11, which answers the question of when the groups \(U(H)\) and \(\text{Aut}_I\) are \(g\)-sequential and reduces the conjecture (G) to the problem whether the group \(\text{Aut}_I\) is a universal topological group of weight \(\tau\).

2. If there are large cardinals, the variety \(\mathcal{W}\) is proper

For any topological space \((X, \mathcal{T})\) we define in a canonical way a refinement \(\mathcal{T}_g\) of the topology \(\mathcal{T}\) so that convergent sequences in \(X\) are the same for \(\mathcal{T}\) and for \(\mathcal{T}_g\) (Definition 2.1).

Let \(\mathcal{S} = \{U_\alpha : \alpha \in A\}\) be a family of open sets in \(X\), and let \(p\) be a subadditive measure on the index set \(A\) (Definition 1.3). For any \(\varepsilon > 0\) let \(W(\mathcal{S}, p, \varepsilon)\) be the set of all \(x \in X\) such that the set \(\{\alpha \in A : x \notin U_\alpha\}\) has \(p\)-measure \(< \varepsilon\). If \(A(\varepsilon)\) denotes the collection of all \(B \subset A\) with \(p(A \setminus B) < \varepsilon\), then

\[
W(\mathcal{S}, p, \varepsilon) = \bigcup_{B \in A(\varepsilon)} \bigcap_{\alpha \in B} U_\alpha.
\]

**Definition 2.1.** For a topological space \((X, \mathcal{T})\) let \(\mathcal{T}_g\) be the topology generated by the collection \(\mathcal{B}\) of the sets \(W(\mathcal{S}, p, \varepsilon)\) for all possible choices of \(\mathcal{S}, p\) and \(\varepsilon\). We say that \(\mathcal{T}_g\) is the \(g\)-modification of \(\mathcal{T}\), and denote by \(X_g\) the space \(X\) equipped with the topology \(\mathcal{T}_g\).

It is easy to see that \(\mathcal{B}\) is actually a base for \(\mathcal{T}_g\). A similar construction was used in [12].

**Proposition 2.2.** (a) The operation of \(g\)-modification preserves continuous maps: if \(f : X \to Y\) is continuous, then \(f : X_g \to Y_g\) is also continuous; (b) the operation of \(g\)-modification is compatible with subspaces: if \(Y\) is a subspace of \(X\), then \(Y_g\) is a subspace of \(X_g\).
The proof is straightforward.

Before we prove that spaces $X$ and $X_g$ have the same convergent sequences, let us note that subadditive measures are nothing else as sequentially continuous seminorms on groups of the form $2^E$. Recall that a non-negative real-valued function on a group $G$ is a seminorm if $p(e) = 0$ (here $e$ is the neutral element) and $p(xy^{-1}) \leq p(x) + p(y)$ for all $x, y \in G$. Identifying $P(E)$, the set of subsets of $E$, with the compact group $2^E$, we see that every sequentially continuous seminorm $p : 2^E \to [0,1]$ is a subadditive measure on $E$. Conversely, let $p$ be a subadditive measure on $E$. Then $p$ is a seminorm on the compact group $2^E$. To prove that $p$ is sequentially continuous, it suffices to check that $p$ is sequentially continuous at the unity, or that $\lim p(A_n) = 0$ for any sequence $\{A_n\}$ of subsets of $E$ which converges to the empty set. The last assumption means that $\bigcap B_n = \emptyset$, where $B_n = \bigcup_{k \geq n} A_k$, so the definition of a subadditive measure implies that $\lim p(B_n) = 0$. Since $p(A_n) \leq p(B_n)$, it follows that $\lim p(A_n) = 0$. We thus have established

**Proposition 2.3.** A function $p : P(E) \to [0,1]$ is a subadditive measure on $E$ in the sense of Definition 1.3 if and only if $p$ is a sequentially continuous seminorm on the compact group $2^E$. \hfill $\square$

**Proposition 2.4.** The topology $\mathcal{T}_g$ is finer than $\mathcal{T}$. Convergent sequences in $X$ for the topologies $\mathcal{T}$ and $\mathcal{T}_g$ are the same, so the identity map $X \to X_g$ is sequentially continuous.

**Proof.** If $\mathcal{S} = \{U_\alpha : \alpha \in A\} \subset \mathcal{T}$ and $p$ is an atomic measure of full mass 1 concentrated at a point $\alpha \in A$, then $W(\mathcal{S},p,\varepsilon) = U_\alpha$. It follows that $\mathcal{T}_g$ is finer than $\mathcal{T}$. Suppose that a sequence $\{x_n\}$ converges to $x$ in $(X,\mathcal{T})$. We must show that every neighbourhood $W$ of $x$ of the form $W = W(\mathcal{S},p,\varepsilon)$, where $\mathcal{S} = \{U_\alpha : \alpha \in A\}$ and $p$ is a subadditive measure on $A$, contains all but finitely many $x_n$’s. Let $A_n$ be the set of all $\alpha \in A$ such that $x_k \notin U_\alpha$ for some $k > n$, and let $B$ be the set of all $\alpha \in A$ such that $x \notin U_\alpha$. The sequence of $A_n$’s is decreasing and $\bigcap A_n \subset B$. Since $x \in W$, the definition of the set $W(\mathcal{S},p,\varepsilon)$ implies that $p(B) < \varepsilon$. Since $p$ is sequentially continuous (Proposition 1.2), we have $\lim p(A_n) = p(\bigcap A_n) \leq p(B) < \varepsilon$. Pick $N$ so that $p(A_N) < \varepsilon$. Then $x_k \in W$ for every $k > N$. \hfill $\square$

**Proposition 2.5.** The operation of $g$-modification preserves finite products: $(X \times Y)_g = X_g \times Y_g$ for any spaces $X$ and $Y$.

**Proof.** Proposition 2.2(a) implies that the projections $(X \times Y)_g \to X_g$ and $(X \times Y)_g \to Y_g$ are continuous, hence the topology of $(X \times Y)_g$ is finer than that of $X_g \times Y_g$. To prove the converse, suppose that $z = (x,y) \in X \times Y$, and let $W = W(\mathcal{S},p,\varepsilon)$ be a neighbourhood of $z$ in $(X \times Y)_g$. We must find $U$ open in $X_g$ and $V$ open in $Y_g$ such that $z \in U \times V \subset W$. Let $\mathcal{S} = \{O_\alpha : \alpha \in A\}$. Without loss of generality we may assume that $z \in \bigcap \mathcal{S}$. Otherwise let $B$ be the set of $\alpha \in A$ such that $z \notin O_\alpha$. Since $z \in W$, we have $p(B) < \varepsilon$. Let $\varepsilon' = \varepsilon - p(B)$, let $A' = A \setminus B$, and let $p'$ be the restriction of the subadditive measure $p$ to $A'$. Then for $\mathcal{S}' = \{O_\alpha : \alpha \in A'\}$ we have $z \in \bigcap \mathcal{S}' \subset W(\mathcal{S}',p',\varepsilon') \subset W.$
So assume that \( z \in \bigcap \mathcal{S} \). For every \( \alpha \in A \) pick open sets \( U_\alpha \subset X \) and \( V_\alpha \subset Y \) so that \( z \in U_\alpha \times V_\alpha \subset O_\alpha \). Let \( \mathcal{P}_1 = \{ U_\alpha : \alpha \in A \} \) and \( \mathcal{P}_2 = \{ V_\alpha : \alpha \in A \} \). Let \( U = W(\mathcal{P}_1, p, \varepsilon/2) \) and \( V = W(\mathcal{P}_2, p, \varepsilon/2) \). Then \( U \) is open in \( X_g \), \( V \) is open in \( Y_g \) and \( z \in U \times V \subset W \). □

**Proposition 2.6.** The operation of \( g \)-modification preserves topological groups: if \( X \) is a topological group, then so is \( X_g \).

**Proof.** We must show that the multiplication \( X_g \times X_g \to X_g \) and the inversion \( X_g \to X_g \) are continuous. This follows from Propositions 2.2(a) and 2.5. □

Plainly the same argument can be applied to any other class of topological algebras.

**Definition 2.7.** A topological space \((X, \mathcal{T})\) is \( g \)-stable if \( \mathcal{T} = \mathcal{T}_g \).

It follows from Propositions 2.4 and 2.6 that \( g \)-sequential groups are \( g \)-stable. Any subspace of a \( g \)-stable space is \( g \)-stable (Proposition 2.2(b)).

**Proposition 2.8.** The property of being \( g \)-stable is preserved by quotient maps.

**Proof.** Let \( f : X \to Y \) be a quotient map. Then \( Y \) admits no strictly finer topology for which \( f \) remains continuous. If \( X \) is \( g \)-stable, that is \( X = X_g \), then \( f : X \to Y_g \) is continuous (Proposition 2.2(a)). It follows that \( Y = Y_g \). □

Let \( \mathfrak{C} \) be the smallest class of topological spaces which contains all metrizable spaces (or just a convergent sequence) and is closed under arbitrary disjoint sums, finite products and quotient maps. M. Hušek asked if the class \( \mathfrak{C} \) contains all topological spaces. This problem is investigated in [12], where it is proved that if there is a real-valued measurable cardinal, then the answer is no. Since \( g \)-stable spaces are plainly preserved by sums, it follows from Propositions 2.5 and 2.8 that all spaces in \( \mathfrak{C} \) are \( g \)-stable. If there is a \( g \)-sequential cardinal, then not every space is \( g \)-stable (and vice versa). Hence in the result of Dow and Watson quoted above ‘real-valued measurable’ can be replaced by ‘\( g \)-sequential’.

A subset \( A \) of a partially ordered set \((L, \leq)\) is directed if for any \( a, b \in A \) there is \( c \in A \) with \( a \leq c \) and \( b \leq c \).

**Definition 2.9.** Let \( L_1, L_2 \) be complete lattices. An increasing map \( h : L_1 \to L_2 \) is lattice-continuous if for any directed subset \( \Gamma \subset L_1 \) we have \( h(\sup \Gamma) = \sup \{ h(\gamma) : \gamma \in \Gamma \} \).

For a group \( G \) let \( L_G \) be set of all (not necessarily Hausdorff) group topologies on \( G \). The set \( L_G \) is a complete lattice. For an onto homomorphism \( f : G \to H \) let \( f_* : L_G \to L_H \) be the map which assigns to each \( \mathcal{T} \in L_G \) the quotient topology of \( \mathcal{T} \) on \( H \).

**Proposition 2.10.** For any onto homomorphism \( f : G \to H \) of groups the map \( f_* : L_G \to L_H \) is lattice-continuous.

**Proof.** Let \( \Gamma \subset L_G \) be a directed set of group topologies on \( G \). Let \( \mathcal{T} = f_*(\sup \Gamma) \) and \( \mathcal{T}' = \sup \{ f_*(\mathcal{P}) : \mathcal{P} \in \Gamma \} \). We must show that \( \mathcal{T} = \mathcal{T}' \). Since \( f_* \) is increasing, it is clear that \( \mathcal{T}' \subset \mathcal{T} \). To prove the reverse inclusion, it suffices to show that every \( \mathcal{T} \)-neighbourhood \( V \) of the unity \( e_H \) in \( H \) is a \( \mathcal{T}' \)-neighbourhood. Let \( U = f^{-1}(V) \). Then \( U \)
is a neighbourhood of the unity $e_G$ in $G$ relative to the topology $\sup \Gamma$ and hence, since $\Gamma$ is directed, also relative to some $\mathcal{P} \in \Gamma$. The map $f : (G, \mathcal{P}) \to (H, f_\ast(\mathcal{P}))$, being a quotient homomorphism of topological groups, is open. It follows that $V = f(U)$ is a neighbourhood of $e_H$ relative to $f_\ast(\mathcal{P})$ and hence also relative to $\mathcal{T}'$. □

**Proposition 2.11.** Let $\mathfrak{W}$ be the class of all topological groups $(G, \mathcal{T})$ with the following property:

The topology $\mathcal{T}$ can be represented as the least upper bound of a set of $g$-stable (not necessarily Hausdorff) group topologies on $G$.

Then $\mathfrak{W}$ is a variety of topological groups containing all $g$-sequential topological groups.

**Proof.** It is clear that the class $\mathfrak{W}$ is closed under arbitrary products and subgroups. The remark after Definition 2.7 shows that all $g$-sequential groups are in $\mathfrak{W}$. It remains to prove that the class $\mathfrak{W}$ is closed under quotients. Note that the least upper bound $\mathcal{T} = \sup(\mathcal{T}_1, \mathcal{T}_2)$ of two $g$-stable topologies on a set $X$ is $g$-stable. This follows from Propositions 2.2(b) and 2.5, since the space $(X, \mathcal{T})$ is homeomorphic to a subspace of the product $(X, \mathcal{T}_1) \times (X, \mathcal{T}_2)$. Consequently, a topological group $(G, \mathcal{T})$ is in $\mathfrak{W}$ if and only if the topology $\mathcal{T}$ can be represented as the least upper bound of a directed set of $g$-stable group topologies on $G$. In virtue of Propositions 2.8 and 2.10 the last property is preserved by quotient homomorphisms. □

For a set $E$ we denote by $\text{Sym}(E)$ the topological group of all permutations of $E$, equipped with the topology of pointwise convergence ($E$ being considered as a discrete space). For any $F \subset E$ let $H_F$ be the group of all permutations of $E$ which leave fixed every point in $F$. The family $\{H_F : F \subset E, \ F \text{ finite}\}$ is a base at the unity of $\text{Sym}(E)$. If $F$ is a singleton $\{x\}$, we write $H_x$ instead of $H_{\{x\}}$.

**Proposition 2.12.** If $\text{Card}(E)$ is a $g$-sequential cardinal, the topological group $\text{Sym}(E)$ does not belong to the variety $\mathfrak{W}$ defined in Proposition 2.11.

**Proof.** (a) We first show that the group $G = \text{Sym}(E)$ is not $g$-stable. Let $p$ be a subadditive measure on $E$ which witnesses that $\text{Card}(E)$ is $g$-sequential, that is, $p(E) = 1$ and $p(F) = 0$ for every finite subset $F \subset E$. For every $x \in E$ let $H_x$, as above, be the stability subgroup at $x$. Denote the family $\{H_x : x \in E\}$ by $\mathcal{S}$. The set $W = W(\mathcal{S}, p, 1)$, defined as in the beginning of this Section, consists of all $f \in G$ with the following property: the set of points $x \in E$ which are moved by $f$ has $p$-measure $< 1$. The set $W$ is not a neighbourhood of the unity in $G$. Indeed, let $F$ be a finite subset of $E$. Pick $f \in G$ so that $F$ is the set of all $f$-fixed points in $E$. Then $f \notin W$, since $f$ moves every point in $E \setminus F$ and $p(E \setminus F) = 1$. Thus $f \in H_F \setminus W$, so $H_F$ is not a subset of $W$. Since the subgroups $H_F$ form a base at the unity in $G$, it follows that $W$ is not open in $G$. On the other hand, $W$ is open in $G_g$ (Definition 2.1). Thus $G$ is not $g$-stable.

(b) The proof of Proposition 2.11 shows that a topological group is in the variety $\mathfrak{W}$ if and only if its topology can be written as $\sup \Gamma$ for some directed family $\Gamma$ of $g$-stable group topologies. In virtue of the part (a) of the proof, the topology $\mathcal{T}$ of $G$ is not $g$-stable. Hence to prove that $G \notin \mathfrak{W}$ it suffices to show that for any directed family $\Gamma$ of
group topologies on $G$ such that $T = \sup \Gamma$ we have $T \in \Gamma$. Pick a point $x \in E$. Since $\Gamma$ is directed, there is a topology $\mathcal{P} \in \Gamma$ such that the subgroup $H_x$ is a $\mathcal{P}$-neighbourhood of the identity. Since the subgroup $H_y$ is conjugate to $H_x$ for every $y \in E$ and since $\mathcal{P}$ is a group topology, $H_y$ is a $\mathcal{P}$-neighbourhood of the identity for every $y \in E$, and the same is true for every subgroup $H_F = \bigcap\{H_y : y \in F\}$, where $F$ is a finite subset of $E$. It follows that $T = \mathcal{P}$.

Let $\mathfrak{V}$ be the variety defined in Proposition 2.11. We noted in Section 1 that the variety $\mathfrak{V}$ is generated by $g$-sequential groups, hence Proposition 2.11 implies that $\mathfrak{V} \subset \mathfrak{W}$. It follows from Proposition 2.12 that the group $\text{Sym}(E)$ is not in $\mathfrak{V}$ if $\text{Card}(E)$ is $g$-sequential. We have thus arrived at our main result:

**Theorem 2.13.** Assume there exist $g$-sequential cardinals. Then the variety $\mathfrak{V}$ generated by free topological groups of metrizable spaces is a proper subclass of the class of all topological groups. For example, if $\text{Card}(E)$ is a $g$-sequential cardinal, the topological group $\text{Sym}(E)$ does not belong to $\mathfrak{V}$.

Salvador Hernandez raised the following question: is $g$-modification an idempotent operation? In other words, is it true that $(X_g)_g = X_g$ for every space $X$? Equivalently, is it true that $X_g$ is $g$-stable? The answer turns out to be positive (Proposition 2.14).

We need some preparations. Suppose $B = \bigcup_{\alpha \in A} B_\alpha$ is a disjoint union, $p$ is a subadditive measure on $A$, and for every $\alpha \in A$ a subadditive measure $q_\alpha$ on $B_\alpha$ is given. Then one can define a subadditive measure $\mu = \bigvee_p q_\alpha$ on $B$ as follows: if $E \subset B$, then

$$\mu(E) = \inf\{\varepsilon > 0 : p(\{\alpha \in A : q_\alpha(E \cap B_\alpha) \geq \varepsilon\}) \leq \varepsilon\}.$$ 

Note that $\mu(E) < \varepsilon$ implies $p(\{\alpha \in A : q_\alpha(E \cap B_\alpha) \geq \varepsilon\}) < \varepsilon$ implies $\mu(E) \leq \varepsilon$.

Let us check that $\mu$ is a subadditive measure. Suppose $E_1, E_2 \subset B$ and $\mu(E_i) < \varepsilon_i$, $i = 1, 2$. Let $A_i = \{\alpha \in A : q_\alpha(E_i \cap B_\alpha) \geq \varepsilon_i\}$. Then $p(A_i) < \varepsilon_i$. If $\alpha \in A$ is such that $q_\alpha((E_1 \cup E_2) \cap B_\alpha) \geq \varepsilon_1 + \varepsilon_2$, then $\alpha \in A_1 \cup A_2$, hence

$$p(\{\alpha \in A : q_\alpha((E_1 \cup E_2) \cap B_\alpha) \geq \varepsilon_1 + \varepsilon_2\}) \leq p(A_1) + p(A_2) < \varepsilon_1 + \varepsilon_2.$$ 

It follows that $\mu(E_1 \cup E_2) \leq \varepsilon_1 + \varepsilon_2$.

We also have to check that $\mu(E_n) \to 0$ for every decreasing sequence $(E_n)$ of subsets of $B$ such that $\bigcap E_n = \emptyset$. Let $\varepsilon > 0$ be given. Put

$$A_n = \{\alpha \in A : q_\alpha(E_n \cap B_\alpha) \geq \varepsilon\}.$$ 

The sets $A_n$ decrease and have empty intersection, therefore $p(A_n) < \varepsilon$ for $n$ large enough. For such $n$ we have $\mu(E_n) \leq \varepsilon$.

Now suppose that $f_\alpha > 0$ is given for every $\alpha \in A$. Let $\varepsilon > 0$. We claim that

$$\exists \delta > 0 \forall E \subset B (\mu(E) < \delta \implies p(\{\alpha : q_\alpha(E \cap B_\alpha) \geq f_\alpha\} < \varepsilon).$$

Indeed, let $A_\delta = \{\alpha : f_\alpha < \delta\}$. Then $p(A_\delta) \to 0$ as $\delta \to 0$. Pick $\delta < \varepsilon/2$ so that $p(A_\delta) < \varepsilon/2$. We claim that $\delta$ has the required property. Let $E \subset B$ be such that $\mu(E) < \delta$. We must show that the set $C = \{\alpha : q_\alpha(E \cap B_\alpha) \geq f_\alpha\}$ has $p$-measure $< \varepsilon$.
Let $C_1 = C \cap A_\delta$ and $C_2 = C \setminus C_1$. Then $C_2 \subseteq \{ \alpha : q_\alpha(E \cap B_\alpha) \geq \delta \}$, and the latter set has $p$-measure $< \delta$ since $\mu(E) < \delta$. Thus $C = C_1 \cup C_2$ is covered by two sets of $p$-measure $< \varepsilon/2$. It follows that $p(C) < \varepsilon$.

**Proposition 2.14.** The $g$-modification is an idempotent operation: $(X_g)_g = X_g$ for every $X$. In other words, $X_g$ is $g$-stable.

**Proof.** Let $x_0 \in X$. A basic open set in $(X_g)_g$ that contains $x_0$ has the form $W(S, p, \varepsilon)$, where $S = \{ U_\alpha : \alpha \in A \}$ is a family of open sets in $X_g$, and $p$ is a subadditive measure on the index set $A$. We may assume that each $U_\alpha$ is a basic open set of the form $U_\alpha = W(G_\alpha, q_\alpha, f_\alpha)$ for some family $G_\alpha = \{ G_\beta : \beta \in B_\alpha \}$ of open sets in $X$, where $q_\alpha$ is a subadditive measure on the index set $B_\alpha$ and $f_\alpha > 0$. We also may assume that $x_0 \in G_\beta$ for all indices $\beta$ (discard those $G_\beta$ for which this does not hold). Consider the subadditive measure $\mu = \bigvee_p q_\alpha$ on $B = \bigcup B_\alpha$ constructed above, and pick $\delta > 0$ as in Eq. (2.7). We claim that

$$x_0 \in W(G, \mu, \delta) \subset W(S, p, \varepsilon),$$

where $G = \{ G_\beta : \beta \in B \}$. This will prove that the topology of $X_g$ is finer than (and hence equal to) the topology of $(X_g)_g$.

We have

$$W(G, \mu, \delta) = \bigcup_{E} \bigcap_{\beta \in B \setminus E} G_\beta,$$

where $E$ runs over all subsets of $B$ such that $\mu(E) < \delta$. Pick such a set $E \subset B$. Put $C = \{ \alpha : q_\alpha(E \cap B_\alpha) \geq f_\alpha \}$. By the choice of $\delta$, we have $p(C) < \varepsilon$, and

$$\bigcap_{\beta \in B \setminus E} G_\beta = \bigcap_{\alpha \in A} \bigcap_{\beta \in B_\alpha \setminus E} G_\beta \subset \bigcup_{\alpha \in A \setminus C} \bigcap_{\beta \in B_\alpha \setminus E} G_\beta.$$

If $\alpha \in A \setminus C$, then $q_\alpha(E \cap B_\alpha) < f_\alpha$, hence $\bigcup_{\beta \in B_\alpha \setminus E} G_\beta \subset W(G_\alpha, q_\alpha, f_\alpha) = U_\alpha$, and we can continue the previous displayed formula:

$$\bigcup_{\alpha \in A \setminus C} U_\alpha \subset W(S, p, \varepsilon).$$

This proves the inclusion $W(G, \mu, \delta) \subset W(S, p, \varepsilon)$. \hfill $\square$

### 3. Refinements of locally compact group topologies

In this Section we prove that a locally compact group $G$ is $g$-sequential if and only if its local weight is not a $g$-sequential cardinal (Theorem 3.14). We also prove that the product of a family of non-trivial $g$-sequential groups is $g$-sequential if and only if the cardinality of the family is not $g$-sequential (Theorem 3.2 and Proposition 3.3).

If a compact group $G$ admits a strictly finer countably compact group topology, then the weight of $G$ is an Ulam-measurable cardinal [2]. The converse of this result of A. V. Arhangel’skii was proved by W. W. Comfort and D. Remus [9] under the assumption that $G$ is either Abelian or connected. We prove that this assumption is superfluous (Theorem 3.16).
Proposition 3.1. A topological group $G$ is $g$-sequential if and only if every sequentially continuous seminorm on $G$ is continuous.

Proof. The topology of every topological group is defined by the set of all continuous seminorms. If a topological group $(G, T)$ admits a strictly finer group topology $T'$, there is a $T'$-continuous seminorm $p$ on $G$ which is not $T$-continuous. If convergent sequences in $G$ are the same for $T$ and $T'$, then $p$ is sequentially continuous. Conversely, suppose $p$ is sequentially continuous but not continuous. Then the same is true for every seminorm $p_g$ defined by $p_g(x) = p(gxg^{-1})$, $g, x \in G$. The family of all seminorms of the form $p_g$ is invariant under inner automorphisms and hence defines a group topology $T'$ on $G$. The identity map $(G, T) \to (G, T')$ is sequentially continuous but not continuous, thus $(G, T)$ is not $g$-sequential.

If $\tau$ is a $g$-sequential cardinal, the compact group $2^\tau$ is not $g$-sequential. This follows from Propositions 2.3 and 3.1. Actually for such a cardinal $\tau$ the space $2^\tau$ is not even $g$-stable. To see this, one can either repeat the part (a) of the proof of Proposition 2.12 or use the result itself: since the group $\text{Sym} (E)$ is homeomorphic to a subspace of $2^\tau$ and $\text{Sym} (E)$ is not $g$-stable, it follows that $2^\tau$ is not $g$-stable neither. We now show that if $\tau$ is not $g$-sequential, then the group $2^\tau$ is $g$-sequential. This is a special case of a more general theorem:

Theorem 3.2. Let $\{G_\alpha : \alpha \in A\}$ be a family of $g$-sequential topological groups. If the cardinal $\text{Card} (A)$ is not $g$-sequential, then the product $G = \prod_{\alpha \in A} G_\alpha$ is a $g$-sequential topological group.

Proof. In virtue of Proposition 3.1 it suffices to show that every sequentially continuous seminorm $p$ on $G$ is continuous. Clearly we may assume that $p$ is bounded by 1. For every $B \subset A$ let $G_B$ be the group $\prod_{\alpha \in B} G_\alpha$. We identify $G_B$ with the subgroup of $G$ consisting of all $\{g_\alpha\}$ such that $g_\alpha$ is the unity of $G_\alpha$ for each $\alpha \in A \setminus B$. Let $\mathcal{N}(G)$ be the set of all neighbourhoods of unity in $G$. Define a function $q : P(A) \to \mathbb{R}$ by

$$q(B) = \lim_{x \to e, x \in G_B} \sup p(x) = \inf \{ \sup \{ p(x) : x \in U \} : U \in \mathcal{N}(G_B) \}.$$ 

In other words, $q(B)$ is the oscillation of the restriction of $p$ to $G_B$ at the unity. Let us check that $q$ is a subadditive measure on $A$. Suppose $B$ and $C$ are disjoint subsets of $A$. For a given $\varepsilon > 0$ pick $U \in \mathcal{N}(G_B)$ and $V \in \mathcal{N}(G_C)$ so that $p(x) < q(B) + \varepsilon$ for every $x \in U$ and $p(y) < q(C) + \varepsilon$ for every $y \in V$. Let $W = UV$ (the product in the group $G$). Then $W \in \mathcal{N}(G_{B \cup C})$. If $z \in W$, then $z = xy$ for some $x \in U$ and $y \in V$, so we have $p(z) \leq p(x) + p(y) < q(B) + q(C) + 2\varepsilon$. It follows that $q(B \cup C) \leq q(B) + q(C) + 2\varepsilon$ and hence $q(B \cup C) \leq q(B) + q(C)$, since $\varepsilon$ was arbitrary. Thus the condition 1 of Definition 1.3 is verified. To verify the second condition, suppose that $A_1 \supset A_2 \supset \ldots$ is a decreasing sequence of subsets of $A$ with empty intersection. We must show that $\lim q(A_n) = 0$. Let $a_n = \sup \{ p(x) : x \in G_{A_n} \}$. Any sequence $\{x_n\}$ such that $x_n \in G_{A_n}$ converges to the unity. Since $p$ is sequentially continuous, we have $\lim p(x_n) = 0$. It follows that $\lim a_n = 0$. Since $q(A_n) \leq a_n$, we have $\lim q(A_n) = 0$. 


Since each $G_\alpha$ is g-sequential, the restriction of $p$ to each $G_\alpha$ is continuous (Proposition 3.1). It follows that $q(B) = 0$ if $B \subset A$ is a singleton. Since Card $(A)$ is not g-sequential, every subadditive measure on $A$ which is zero on singletons is zero on $A$. Hence $q(A) = 0$. This means that $p$ is continuous at the unity and hence everywhere, as required. □

The condition that Card $(A)$ be not g-sequential is also necessary for the product $G = \prod_{\alpha \in A} G_\alpha$ to be g-sequential, provided that all groups $G_\alpha$ are non-trivial (we call a group trivial if it consists of one point):

**Proposition 3.3.** Let $\{G_\alpha : \alpha \in A\}$ be a family of non-trivial topological groups. If the cardinal Card $(A)$ is g-sequential, then the product $G = \prod_{\alpha \in A} G_\alpha$ is not a g-sequential topological group.

*Proof.* Let $\tau = \text{Card} (A)$. We have observed that the space $2^\tau$ is not g-stable. The space $G$ contains a topological copy of $2^\tau$ and hence is not g-stable either (Proposition 2.2(b)). The remark after Definition 2.7 shows that the group $G$ is not g-sequential. □

Our next aim is Theorem 3.14, which states that a locally compact group $G$ is g-sequential if and only if its local weight is not a g-sequential cardinal. The weight $w(X)$ of a space $X$ is defined by $w(X) = \min\{|B| : B \text{ is a base for } X\}$, and the local weight of a topological group $G$ is the cardinal $\min\{w(U) : U \in \mathcal{N}(G)\}$, where $\mathcal{N}(G)$ is the set of neighborhoods of the neutral element. The local weight of a locally compact group $G$ equals its character $\chi(G) = \min\{|B| : B \text{ is a base at the unity}\}$.

We shall need a few known facts concerning topology of locally compact groups. For the reader’s convenience, we sketch the proofs.

**Proposition 3.4.** If $G$ is a locally compact group and $\tau = \chi(G)$, then $G$ contains a topological copy of the cube $2^\tau$.

*Proof.* First assume that $G$ is compact. Then $G$ is dyadic (= an image of a Cantor cube; see [37] for an easy proof of this theorem, due to Ivanovskii and Kuzminov, and for its generalizations). The $\pi$-character of a group equals its character [1], so the $\pi$-character of $G$ equals $\tau$. Shapiro’s theorem [31] implies that $G$ can be mapped onto the Tikhonov cube $I^\tau$. By a theorem of Efimov [13], a dyadic compact space $X$ contains a copy of $2^\tau$ if and only if $X$ can be mapped onto $I^\tau$ (or, equivalently, if and only if $X$ is not the union of countably many closed subspaces of weight < $\tau$), and the proposition follows. See [30] for another proof.

The general case reduces to the case of a compact group. Indeed, let $K$ be a compact $G_\delta$-subgroup of $G$. Then $\chi(K) = \tau$, so $K$ contains a copy of $2^\tau$. □

**Proposition 3.5.** Let $G$ be a locally compact topological group. If the local weight of $G$ is a g-sequential cardinal, the group $G$ is not g-sequential.

*Proof.* Let $\tau = \chi(G)$. The group $G$ contains a topological copy of the cube $2^\tau$ (Proposition 3.4), so we can apply the same argument as in the proof of Proposition 3.3. □
Definition 3.6. We say that a map \( f : X \to Y \) between topological spaces \( X \) and \( Y \) has the lifting property for convergent sequences, or simply lifting property, if for any point \( x \in X \) and any convergent sequence \( y_1, y_2, \ldots \in Y \) with the limit \( y = f(x) \) there exists a sequence \( x_1, x_2, \ldots \) converging to \( x \) and such that \( f(x_n) = y_n \) for every \( n = 1, 2, \ldots \).

It is easy to see that an open map defined on a metrizable space has the lifting property. More generally, every open map with a metrizable kernel has this property. We say that a map \( f : X \to Y \) has metrizable kernel if there exists a map \( g : X \to M \) to a metrizable space \( M \) such that the map \( (f, g) : X \to Y \times M \) is a homeomorphic embedding.

Proposition 3.7. Let \( G \) be a locally compact group, \( H \) its closed subgroup. Then the quotient map \( q : G \to G/H \) has the lifting property for convergent sequences.

Proof. An open map \( f : X \to Y \) has the lifting property if \( X \) can be covered by open sets \( U \) such that the restriction \( f|U : U \to f(U) \) has this property. It follows that without loss of generality we may assume that \( G \) is \( \sigma \)-compact (= the union of countably many compact sets). We represent the map \( q : G \to G/H \) as a limit projection of an inverse system. Let \( K = \{ K_\alpha : \alpha < \tau \} \) be the family of all compact \( G_\delta \) subgroups of \( G \), where \( \tau = w(G) \). For every \( \alpha < \tau \) let \( N_\alpha = H \cap \bigcap_{\beta < \alpha} K_\beta \) and \( X_\alpha = G/N_\alpha \). For \( \alpha < \beta \) a natural quotient map \( p_\alpha^\beta : X_\beta \to X_\alpha \) is defined, so we get an inverse system \( S = \{ X_\alpha, p_\alpha^\beta \} \) with open bonding maps. Every neighbourhood of unity in \( G \) contains a subgroup \( K \in K \) [17, theorem 8.7], so the inverse limit of \( S \) can be identified with \( G \), and \( q \) can be identified with the projection \( \lim S \to X_0 \). The system \( S \) is continuous, in the sense that for every limit \( \alpha \) the space \( X_\alpha \) is the inverse limit of its predecessors. Every map \( p_\alpha^\alpha \) has metrizable kernel. Indeed, \( G/K_\alpha \) is metrizable, and the natural map

\[
X_{\alpha+1} = G/(N_\alpha \cap K_\alpha) \to X_\alpha \times (G/K_\alpha)
\]

is a homeomorphic embedding. Hence every map \( p_\alpha^\alpha \) has the lifting property for convergent sequences. Given a convergent sequence in \( G/H = X_0 \), we can lift it in \( X_\alpha \) by recursion, using the continuity of \( S \) at limit steps. In the end we get a required lifting in \( G \).

Actually this proof shows more: the quotient map \( q : G \to G/H \) is 0-soft. A map \( p : X \to Y \) is called \( n \)-soft [32], where \( n \) is an integer, if for any paracompact space \( Z \) with \( \dim Z \leq n \), any closed subspace \( A \subset Z \) and any commutative diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow i & & \downarrow p \\
Z & \xrightarrow{g} & Y,
\end{array}
\]

where \( i : A \to Z \) is the inclusion, there exists a map \( h : Z \to X \) which is compatible with the diagram in the sense that \( hi = f \) and \( ph = g \). Clearly 0-softness implies the lifting property for convergent sequences: just take for \( Z \) the one-point compactification of a
countable discrete space. It follows from Michael’s selection theorem for 0-dimensional spaces [21] that any open map with a completely metrizable kernel is 0-soft. (A map \( f : X \to Y \) has completely metrizable kernel if there exists a complete metric space \( M \) and a map \( g : X \to M \) such that the map \( (f, g) : X \to Y \times M \) is a homeomorphic embedding.)

The maps \( p^\alpha+1 \) from the proof of Proposition 3.7 have completely metrizable kernel, since metrizable locally compact spaces are completely metrizable. The 0-softness of quotient maps \( G \to G/H \) for (locally) compact groups immediately implies the Ivanovski˘ı—Kuzminov theorem mentioned above, which says that compact groups are dyadic (see [37]). Indeed, if \( X \) is a compact space such that the constant map of \( X \) to a singleton is 0-soft (such spaces are called Dugundji-compact [29], [16]), then \( X \) is dyadic: there exists an onto map \( f : A \to X \), where \( A \) is a closed subspace of a Cantor cube \( Z = 2^\tau \), and the assumption on \( X \) implies that \( f \) has an extension \( h : Z \to X \).

**Proposition 3.8.** Let \( H \) be a closed normal subgroup of a topological group \( G \). Suppose that the quotient map \( q : G \to G/H \) has the lifting property for convergent sequences (Definition 3.6). If \( H \) and \( G/H \) are \( g \)-sequential, then \( G \) is also \( g \)-sequential.

**Proof.** Let \( T \) be the original topology of \( G \), and let \( T' \) be a finer group topology with the same convergent sequences. We must prove that \( T = T' \). Since \( H \) is \( g \)-sequential, the topologies \( T \) and \( T' \) agree on \( H \). Since \( q : G \to G/H \) has the lifting property, the quotient topologies \( T/H \) and \( T'/H \) on \( G/H \) have the same convergent sequences. Since the group \( (G/H, T/H) \) is \( g \)-sequential, the topologies \( T/H \) and \( T'/H \) are equal. Now Lemma 1 of [11] implies that \( T = T' \). \( \square \)

Combining Proposition 3.8 with Proposition 3.7, we get the following corollary:

**Corollary 3.9.** If a locally compact group \( G \) contains a closed normal subgroup \( H \) such that both \( H \) and \( G/H \) are \( g \)-sequential, then \( G \) is \( g \)-sequential. \( \square \)

This corollary permits to reduce the proof of Theorem 3.14 to the case of a compact group and to consider separately the cases of connected groups and of zero-dimensional groups.

**Proposition 3.10.** Let \( G \) be a compact group of weight \( \tau \). If \( G \) is either Abelian or connected, there exists a continuous surjective homomorphism \( \prod_{i \in I} K_i \to G \), where \( \{K_i : i \in I\} \) is a family of compact metric groups such that \(|I| = \tau\).

**Proof.** Consider first the case when \( G \) is Abelian. The groups \( K_i \) that we are looking for will also be Abelian. By the Pontryagin duality, the assertion is equivalent to the following: every Abelian group \( A \) is a subgroup of an Abelian group \( B \) such that \(|B| = |A| \) and \( B \) is the direct sum of a family of countable groups. This is clear: embed \( A \) into a divisible group \( B \) of the same cardinality [15, theorem 24.1 and proposition 26.2] and note that every divisible Abelian group is the direct sum of a family of countable groups [15, theorem 23.1]. Now consider the case when \( G \) is connected. Then \( G \) is a homomorphic image of the product of its center \( Z \) and its commutator group \( D \) [18, Theorem 9.24]. We
have just proved that $Z$ is an image of the product of a family of compact metric groups, and the same is true for $D$, since $D$ is an image of the product of a family of semisimple compact Lie groups [18, Theorem 9.19].

The case of zero-dimensional compact groups is more complicated. It seems that the easiest way to cope with it is to invoke the notion of the free zero-dimensional compact group. Let $X$ be a zero-dimensional compact space. Then we can define the free zero-dimensional compact group $C_0(X)$ on $X$ which is characterized by the following properties: $C_0(X)$ is a zero-dimensional compact group, $X$ is a subspace of $C_0(X)$, and every continuous map $f : X \to G$, where $G$ is a zero-dimensional compact group, extends uniquely to a continuous homomorphism $\bar{f} : C_0(X) \to G$. It suffices if the last property holds for all finite groups $G$, since every zero-dimensional compact group is a subgroup of a product of finite groups. Hence the group $C_0(X)$ can be constructed as follows. Let $\mathcal{F}$ be the family of all continuous maps $f : X \to G$ of finite discrete groups (considered up to an isomorphism). Let $K = \prod \{G_f : f \in \mathcal{F}\}$. There is a natural embedding $j : X \to K$ defined by $j(x) = \{f(x) : f \in \mathcal{F}\}$. The group $C_0(X)$ can be identified with the subgroup of $K$ generated by $X$. Since $w(K) = \text{Card} (\mathcal{F}) = w(X)$, it follows that the weight of the group $C_0(X)$ equals the weight of $X$. In particular, the group $C_0(X)$ is metrizable if $X$ is metrizable.

**Proposition 3.11.** Let $G$ be a zero-dimensional compact group of weight $\tau$. Let $A$ be a set of cardinality $\tau$. One can assign to every subset $B \subset A$ a closed normal subgroup $G_B$ so that

1. $G_{B \cup C} = G_B G_C$ for all $B, C \subset A$;
2. if $B_1 \supset B_2 \supset \ldots$ is a decreasing sequence of subsets of $A$ such that $\bigcap B_n = \emptyset$, then the sequence of subgroups $G_{B_1}, G_{B_2}, \ldots$ converges to the unity (in the sense that every $U \in \mathcal{N}(G)$ contains all but finitely many of them);
3. if $B \subset A$ and $A \setminus B$ is countable, there exists a closed metrizable subgroup $M$ of $G$ such that $G = G_B M$.

**Proof.** The property of $G$ that we must prove is preserved by homomorphisms, so we may replace $G$ by any group which admits a homomorphism onto $G$. Let $X = 2^A$. Then $G$ is homeomorphic to $X$ [17] theorem 9.15] (for our purposes it would suffice to know that $G$ is dyadic), so there exists a continuous homomorphism of the free zero-dimensional group $C_0(X)$ onto $G$. Thus we may confine ourselves to the case $G = C_0(X)$. For every $B \subset A$ let $X_B = 2^B$, and let $p_B : X \to X_B$ be the natural projection. The map $p_B$ extends to a homomorphism $C_0(p_B) : G \to C_0(X_B)$. Let $K_B$ be the kernel of this homomorphism, and let $G_B = K_{A \setminus B}$. We show that the assignment $B \mapsto G_B$ is as required.

Given a subset $B \subset A$, let us say that a map $f$ defined on $X$ is $B$-nice if for any $x, y \in X$ with $p_B(x) = p_B(y)$ we have $f(x) = f(y)$. Our proof leans on the following Assertion:

Let $q : G \to H$ be a continuous homomorphism, and let $B$ be a subset of $A$. The kernel of $q$ contains the subgroup $K_B$ if and only if the restriction of $q$ to $X$ is $B$-nice.
This is clear, since the kernel of \( q \) contains \( K_B \) if and only if \( q \) can be written as the composition \( q = q'C_0(p_B) \) for some homomorphism \( q' : C_0(X_B) \to H \), and a map \( f : X \to H \) is \( B \)-nice if and only if \( f \) can be written as the composition \( f = f'p_B \) for some \( f' : X_B \to H \).

We must check that

1. \( K_{B \cap C} = K_B K_C \) for all \( B, C \subset A \);
2. if \( B_1 \subset B_2 \subset \ldots \) is an increasing sequence of subsets of \( A \) such that \( \bigcup B_n = A \), then the sequence of subgroups \( K_{B_1}, K_{B_2}, \ldots \) converges to the unity;
3. if \( B \subset A \) and \( B \) is countable, there exists a closed metrizable subgroup \( M \) of \( G \) such that \( G = K_B M \).

We check 1. It suffices to prove that for any homomorphism \( q : G \to H \) the kernel of \( q \) contains \( K_{B \cap C} \) if and only if it contains \( K_B \) and \( K_C \). In virtue of the Assertion, this is equivalent to the following: a map \( f : X \to H \) is \((B \cap C)\) nice if and only if it is \( B \)-nice and \( C \)-nice. The last assertion is obviously true: if, say, \( f \) is \( B \)-nice and \( C \)-nice, then for any \( x, y \in X \) such that \( p_{B \cap C}(x) = p_{B \cap C}(y) \) we can find \( z \in X \) so that \( p_B(z) = p_B(x) \) and \( p_C(z) = p_C(y) \), and then \( f(x) = f(z) = f(y) \), which means that \( f \) is \((B \cap C)\)-nice.

We check 2. Since \( G \) is a zero-dimensional compact group, the kernels of homomorphisms of \( G \) to finite groups form a base at the unity. Hence it suffices to prove the following: if \( q : G \to H \) is a homomorphism to a finite group \( H \) and \( B_1 \subset B_2 \subset \ldots \) is an increasing sequence of subsets of \( A \) such that \( \bigcup B_n = A \), then the subgroup \( K_{B_n} \) is contained in the kernel of \( q \) for some \( n \) (and hence also for all larger indices). Equivalently (see the Assertion), we must prove that every continuous map \( f : X \to H \) is \( B_n \)-nice for some \( n \). Each fiber of \( f \) is a closed-and-open subset of \( X \), hence the union of finitely many basic closed-and-open sets. It follows that \( f \) is \( C \)-nice for some finite set \( C \subset A \). Since \( \bigcup B_n = A \), we have \( C \subset B_n \) if \( n \) is sufficiently large, and then \( f \) is \( B_n \)-nice, as required.

We check 3. Let \( B \) be a countable subset of \( A \). The map \( p_B : X \to X_B \) has a right inverse \( s : X_B \to X \), hence the homomorphism \( C_0(p_B) : G \to C_0(X_B) \) also has a right inverse \( C_0(s) : C_0(X_B) \to G \). It follows that \( G \) is the semidirect product of the kernel \( K_B \) of \( p_B \) and the range of \( C_0(s) \), say \( M \), which is isomorphic to \( C_0(X_B) \). In the paragraph preceding the proposition that we are proving we noted that the functor \( C_0 \) preserves metrizability. It follows that the group \( C_0(X_B) \) is metrizable, and so is the group \( M \). □

Proposition 3.11 allows to apply the argument used in the proof of Theorem 3.2 to zero-dimensional compact groups. We need two more propositions for the proof of Theorem 3.14.

**Proposition 3.12.** Let \( p \) be a subadditive measure on a set \( A \) (Definition 1.3). If the cardinal \( \text{Card}(A) \) is not \( g \)-sequential, there exists a countable subset \( B \subset A \) such that \( p(A \setminus B) = 0 \).

*Proof.* Since the group \( 2^A \) is \( g \)-sequential (Theorem 3.2), it follows from Propositions 2.3 and 3.1 that \( p \) is continuous on \( 2^A \). Hence the set \( p^{-1}(0) \) is of the type \( G_\delta \). This implies what we need. □
Proposition 3.13. Let $G = HK$ be a compact group, where $H$ and $K$ are closed subgroups. If $U \in \mathcal{N}(H)$ and $V \in \mathcal{N}(K)$, then $UV \in \mathcal{N}(G)$.

Proof. Let $P = H \times K$. We must show that the map $f : P \to G$ defined by $f(x, y) = xy$ is open. Let $L = H \cap K$. Consider the right action of $L$ on $P$, defined by $(x, y).g = (xg, g^{-1}y)$. The map $f$ can be identified with the canonical map $P \to P/L$ of $P$ onto the orbit space $P/L$. Since the latter map is open, so is $f$. \hfill $\Box$

We are now ready to prove Theorem 3.14, which shows that whether a locally compact group is $g$-sequential or not depends only on its local weight.

Theorem 3.14. A locally compact group $G$ is $g$-sequential if and only if the local weight of $G$ is not a $g$-sequential cardinal.

Proof. The implication in one direction was established in Proposition 3.5. Now suppose that the local weight of $G$ is not a $g$-sequential cardinal. We must show that the group $G$ is $g$-sequential. Clearly a group is $g$-sequential if it contains an open $g$-sequential subgroup, so we may assume that $G$ is $\sigma$-compact. In this case $G$ contains a compact normal subgroup $K$ such that the quotient group $G/K$ is metrizable (and hence $g$-sequential) [17, Theorem 8.7]. Corollary 3.9 implies that we may confine ourselves to the case when $G$ is compact. Let $C$ be the connected component of the unity in $G$. The quotient group $G/C$ is zero-dimensional, and another application of Corollary 3.9 shows that we may assume that $G$ is either connected or zero-dimensional. In the first case there exists a surjective homomorphism $P \to G$, where $P$ is the product of a family $\gamma$ of compact metric groups such that $\text{Card}(\gamma) = w(G)$ (Proposition 3.10). In virtue of Theorem 3.2, $P$ is $g$-sequential, hence so is $G$. Now consider the case when $G$ is zero-dimensional. Let $A$ be a set of cardinality $w(G)$, and let $B \to G_B$ be an assignment with the properties described in Proposition 3.11. In order to prove that every sequentially continuous seminorm $p$ on $G$ is continuous (Proposition 3.1), we can apply the same construction as in the proof of Theorem 3.2. Just as in that proof, define a function $q : P(A) \to \mathbb{R}$ by $q(B) = \inf\{\sup\{p(x) : x \in U\} : U \in \mathcal{N}(G_B)\}$. Then $q$ is a subadditive measure on $A$. To prove that $q(B \cup C) \leq q(B) + q(C)$, use the property 1 of Proposition 3.11 and Proposition 3.13. To prove that $\lim q(B_n) = 0$ whenever $B_1 \supset B_2 \supset \ldots$ and $\bigcap B_n = \emptyset$, use the property 2 of Proposition 3.11. Since the cardinal $\text{Card}(A) = w(G)$ is not $g$-sequential, Proposition 3.12 implies that $q(B) = 0$ for some subset $B \subset A$ such that $A \setminus B$ is countable. Pick a closed metrizable subgroup $M \subset G$ so that $G = G_BM$ (property 3 of Proposition 3.11). The restriction of $p$ to $G_B$ is continuous since $q(B) = 0$, and the restriction of $p$ to $M$ is continuous since $M$ is metrizable. The argument used to prove the inequality $q(B \cup C) \leq q(B) + q(C)$, based upon Proposition 3.13, shows that $p$ is continuous on $G$. \hfill $\Box$

Theorem 3.14 answers the question of what locally compact groups admit a strictly finer group topology with the same convergent sequences. We now show that a locally compact group admits a strictly finer group topology which agrees with the original one.
on every countable set if and only if the local weight of $G$ is Ulam-measurable. As observed by A. V. Arhangel’ski˘ı [2], in one direction this follows from the results of [10]. If $(G, T)$ is a locally compact group and $T'$ is a strictly finer group topology which agrees with $T$ on countable sets, then the group $(G, T')$ is locally countably compact, hence its completion $H$ is locally compact. Since the natural homomorphism of $G$ to $H$ is sequentially continuous but not continuous, the local weight of $(G, T)$ must be Ulam-measurable [40]. We now establish the converse (Theorem 3.16).

The definition of the $g$-modification $T_g$ of a topology $T$ on a set $X$ (Definition 2.1) depended on the notion of a subadditive measure. We get the definition of the $m$-modification $T_m$ of a topology $T$ if we replace subadditive measures by two-valued measures. In other words, the topology $T_m$ can be described as follows. Let $S = \{U_\alpha : \alpha \in A\}$ be a family of $T$-open sets in $X$, and let $p$ be an $\omega_1$-complete ultrafilter on the index set $A$. Let

$$W(S, p) = \bigcup_{B \in p} \bigcap_{\alpha \in B} U_\alpha.$$  

be the set of all $x \in X$ such that the set $\{\alpha \in A : x \in U_\alpha\}$ is in $p$. The sets $W(S, p)$ form a base for the topology $T_m$.

All properties of the $g$-modification $T_g$ of a topology $T$ established in Propositions 2.2 and 2.4–2.6 hold also for the $m$-modification $T_m$. Note that $T \subset T_m \subset T_g$.

**Proposition 3.15.** Let $(X, T)$ be a topological space. The topologies $T$ and $T_m$ agree on every subset of $X$ of non-Ulam-measurable cardinality.

**Proof.** It suffices to prove that $T_m = T$ if $\text{Card}(X)$ is non-Ulam-measurable. Let $W(S, p)$ be a basic open set for $(X, T_m)$, where $S = \{U_\alpha : \alpha \in A\}$ is a family of open sets in $(X, T)$ and $p$ is an $\omega_1$-complete ultrafilter on $A$. Since $\text{Card}(X)$ is non-Ulam-measurable, the cardinal $\text{Card}(T) \leq 2^{\text{Card}(X)}$ is also non-Ulam-measurable. It follows that there is $B \in p$ and $U \in T$ such that $U_\alpha = U$ for every $\alpha \in B$. Then $W(S, p) = U$. Thus $T_m = T$. □

Let us say that a space $(X, T)$ is $m$-stable if $T = T_m$. Proposition 3.15 means that every space of non-Ulam-measurable cardinality is $m$-stable. On the other hand, it is easy to see that the cube $2^\tau$ is not $m$-stable if $\tau$ is a measurable cardinal.

**Theorem 3.16.** Let $G$ be a locally compact group of Ulam-measurable local weight. Then

1. $G$ admits a strictly finer group topology which agrees with the original one on every set of non-Ulam-measurable cardinality;
2. there exists a locally compact group $H$ and a sequentially continuous homomorphism $G \to H$ which is not continuous.

**Proof.** The group $G$ contains a topological copy of the cube $2^\tau$, where $\tau = \chi(G)$ (Proposition 3.4), hence $G$ is not $m$-stable. It follows that the $m$-modification $T_m$ of the original topology $T$ of $G$ is as required. We noted above that the second part of the theorem follows from the first: we can take for $H$ the completion of $(G, T_m)$. □
Corollary 3.17. Every compact group of Ulam-measurable weight admits a strictly finer countably compact group topology.

This answers Question 5.4(a) from [9].

4. ON SUBGROUPS OF $g$-SEQUENTIAL GROUPS

We now return to Arhangel’skiǐ’s problem: is the variety $\mathfrak{U}$ generated by free topological groups on metrizable spaces proper? If there exist real-measurable cardinals, the answer is no (Theorem 2.13). The following approach to a possible positive solution was suggested in [28].

To prove that the variety $\mathfrak{U}$ coincides with the class of all topological groups, it suffices to show that $\mathfrak{U}$ contains the free topological group $F(X)$ for every Tikhonov space $X$. Embed $X$ in a product $P$ of metric spaces so that every continuous pseudometric on $X$ extends to a continuous pseudometric on $P$. Assume that the following assertions are true:

(H) Any product of metric spaces is $f$-sequential (Definition 1.4).

(F) If $Y \subset Z$ and every continuous pseudometric on $Y$ extends to a continuous pseudometric on $Z$, then the natural map $F(Y) \to F(Z)$ is a topological embedding.

The assertion (F) was formulated in the Introduction in an equivalent form involving fine uniformities. It is easy to see that the free topological group $F(Y)$ is $g$-sequential if and only if the space $Y$ is $f$-sequential, hence (H) implies that the group $F(P)$ is $g$-sequential. Now (F) implies that $F(X)$ is a subgroup of a $g$-sequential group and hence $F(X) \in \mathfrak{U}$.

In this argument we tacitly used the equivalence of the conditions 1–3 of Definition 1.2, which is Theorem 3.7 in [28]. For the reader’s convenience we reproduce the proof. If $X$ is metrizable (or just $f$-sequential), then every sequentially continuous homomorphism $F(X) \to G$ is continuous, since its restriction to $X$ is continuous. Thus free groups on metrizable spaces have the property 1 of Definition 1.2, and so do their quotient groups, since the property 1 is preserved by quotients. This means that $3 \Rightarrow 1$. The implication $1 \Rightarrow 2$ is clear. To prove that $2 \Rightarrow 3$, note that every topological space $X$ can be represented as the image of a metric space $M$ under a continuous map $f : M \to X$ such that every convergent sequence in $X$ is the image of a convergent sequence in $M$; just take for $M$ the disjoint sum of sufficiently many convergent sequences. If $G$ is a topological group and $f : M \to G$ is as above, then the homomorphism $\bar{f} : F(M) \to G$ which extends $f$ is quotient if and only if $G$ does not admit a strictly finer group topology with the same convergent sequences. Hence $2 \Rightarrow 3$.

Now let us discuss the assertions (H) and (F). If there are sequential cardinals, then (H) cannot be true: the product of a family $\gamma$ of non-trivial metric spaces is not $f$-sequential if $\text{Card}(\gamma)$ is sequential. We prove that if there are no sequential cardinals, then (H) is true. More generally, the product of a family $\gamma$ of metric spaces is $f$-sequential if $\text{Card}(\gamma)$ is not sequential (Theorem 4.4). For products of separable metric spaces this was proved in [7, theorem 1.5]. In the non-separable case we apply the technique of [38].
As far as (F) is concerned, this is the main result of [33, 34]. Whether or not one shares the opinion of V. Pestov that “Sipacheva’s proofs gradually became commonly recognized as correct” [10, p. 108], it would be desirable to find an independent proof of the fact that under some additional set-theoretic assumptions every free topological group $F(X)$ is a subgroup of a $g$-sequential group. To prove that the variety $\mathfrak{W}$ coincides with the class of all topological groups, it would actually suffice to prove the last assertion only for some very special spaces $X$. Call a topological space $X$ ultrasimple if it is either (1) discrete or (2) contains exactly one non-isolated point $p$, and the trace of the filter of neighbourhoods of $p$ on $X \setminus \{p\}$ is an ultrafilter. Call a space simple if it is a disjoint sum of ultrasimple spaces. It is easy to see that every topological space is an image of a simple space under a quotient map. It follows that a variety $\mathfrak{W}$ of topological groups contains all topological groups if and only if $F(X) \in \mathfrak{W}$ for every simple space $X$. Thus to prove that the variety $\mathfrak{W}$ contains all topological groups it would suffice to show that the group $F(X)$ is a subgroup of a $g$-sequential group for every simple space $X$. This observation motivates the following question:

**Question 1.** Let $X$ be a simple space. Is the group $F(X)$ isomorphic to a topological subgroup of $F(M)$ for some metric space $M$?

Theorem 2.13 implies that the answer is no if there are large cardinals, but it is not clear if the answer can be consistently yes. The question seems to be closely related to the problem considered in [12].

The question remains open whether every topological group is a subgroup of a $g$-sequential group (assuming there are no $g$-sequential cardinals). We obtain some partial results in this direction. We prove that the group $\text{Aut} I^\tau$ of all self-homeomorphisms of a Tikhonov cube $I^\tau$ is $g$-sequential if and only if the cardinal $\tau$ is not $g$-sequential (Theorem 4.11). Similarly, the unitary group $U(H)$ of a Hilbert space $H$ is $g$-sequential if and only if the weight of $H$ is not $g$-sequential. These results do not answer the above-mentioned question, since not every topological group can be embedded in a unitary group $U(H)$, and it is not known whether every topological group can be embedded in a group of the form $\text{Aut} I^\tau$.

We use the techniques of [38] to prove Theorems 4.3 and 4.10. Our proofs depend on the notion of the functional tightness of a topological space, due to A.V. Arhangel’skii [3]. A function $f : X \to Y$ is $\omega$-continuous if for every countable subset $A \subset X$ the restriction $f|A : A \to Y$ is continuous. A space $X$ has countable functional tightness if every $\omega$-continuous function $f : X \to \mathbb{R}$ is continuous. Recall that a space $X$ has countable tightness if for any $x \in X$ and $A \subset X$ such that $x \in \bar{A}$ (where the bar denotes the closure) there is a countable $B \subset A$ such that $x \in B$. Clearly every space of countable tightness has countable functional tightness. The converse is not true. For example, a power $\mathbb{R}^\tau$ of the real line has countable tightness if and only if $\tau = \omega$, and it has countable functional tightness if and only if the cardinal $\tau$ is non-Ulam-measurable [38]. The last assertion is a special case of the main result of [38]: the function space $C_p(X)$ has countable functional tightness if and only if $X$ is realcompact. Here $C_p(X)$ denotes the space of continuous
functions on \( X \), endowed with the topology of pointwise convergence. Recall that the space \( C_p(X) \) has countable tightness if and only if all finite powers of \( X \) are Lindelöf [4, theorem 4.1.2]. In particular, the space \( C_p(X) \) has countable tightness if \( X \) is compact.

The proof of the main theorem in [38] was based on a lemma which we reproduce here for convenience. For a subset \( A \) of a topological space \( X \) denote by \([A]_\omega\) the \( \omega \)-closure of \( A \), that is, the set \( \bigcup \{ \bar{B} : \bar{B} \subset A \text{ and } B \text{ is countable} \} \).

**Lemma 4.1** ([38]). Let \( Y \) be a space of countable functional tightness, and let \( p : Y \to X \) be a surjective continuous map. Suppose there is a base \( \mathcal{B} \) for \( Y \) such that for every \( U \in \mathcal{B} \) there is an open set \( V \) in \( X \) such that

\[
(*) \quad p(U) \subset V \subset [p(U)]_\omega.
\]

Then \( X \) has countable functional tightness.

**Proof.** We must show that every \( \omega \)-continuous function \( f : X \to \mathbb{R} \) is continuous. Let \( x \in X \), and let \( O \) be an open interval in \( \mathbb{R} \) containing \( f(x) \). Pick \( y \in Y \) such that \( p(y) = x \). The function \( f_p : Y \to \mathbb{R} \) is \( \omega \)-continuous and hence continuous, so there is \( U \in \mathcal{B} \) such that \( y \in U \) and \( f_p(U) \subset O \). Since \( f \) is \( \omega \)-continuous, it maps the \( \omega \)-closure \( [p(U)]_\omega \) of the set \( p(U) \) into \( O \). Pick an open \( V \) in \( X \) so that \( p(U) \subset V \subset [p(U)]_\omega \). Then \( V \) is a neighbourhood of \( x \) and \( f(V) \subset O \). Thus \( f \) is continuous. \( \square \)

This lemma leads to the following sufficient condition for a space to have countable functional tightness.

**Proposition 4.2.** Let \( X \) be a topological space, and let \( \{ d_\alpha : \alpha \in A \} \) be a family of pseudometrics on \( X \) generating the topology of \( \mathcal{T} \) of \( X \). Assume that for any two points \( x, y \in X \) and any finite set \( F \) of free ultrafilters on \( A \) there exists a sequence \( y_0, y_1, \ldots \in X \) such that \( \lim n \in \omega \) the set

\[
\{ \alpha \in A : d_\alpha(y_n, x) = 0 \}
\]

is in the filter \( \bigcap F \). Then \( X \) has countable functional tightness.

**Proof.** In virtue of Lemma 4.1 it suffices to construct a finer topology \( \mathcal{T}' \) on \( X \) such that the space \( Y = (X, \mathcal{T}') \) has countable tightness and the identity map \( Y \to X \) has the property described in the lemma. We may assume that the pseudometrics \( d_\alpha \) are \( \leq 1 \); otherwise replace \( d_\alpha \) by \( \min(d_\alpha, 1) \). Let \( \beta A \) be the set of all ultrafilters on \( A \), considered as a compact space (the Čech—Stone compactification of the discrete space \( A \)). We identify \( A \) with its image under the natural map \( A \to \beta A \). For each ultrafilter \( p \in \beta A \) define a pseudometric \( d_p \) on \( X \) as the \( p \)-limit of pseudometrics \( d_\alpha \). In other words, if \( x, y \in X \), then \( d_p(x, y) \) is determined by the requirement that the function \( p \mapsto d_p(x, y) \) must be continuous on \( \beta A \). Let \( \mathcal{T}' \) be the topology on \( X \) generated by all pseudometrics \( d_p, p \in \beta A \), and let \( Y = (X, \mathcal{T}') \). Let us check that the identity map \( Y \to X \) satisfies the condition of the lemma.

For every finite set \( K \subset \beta A \) let \( d_K = \max \{ d_p : p \in K \} \). The collection \( \mathcal{B} \) of all open \( d_K \)-balls, where \( K \) runs over the set of all finite subsets of \( \beta A \), is a base for \( Y \). Thus it...
suffices to check that for every $U \in \mathcal{B}$ there exists an open $V$ in $X$ such that the condition (*) of the lemma holds.

Let $U \in \mathcal{B}$. Pick a finite set $K \subset \beta A$, a point $x \in X$ and $\varepsilon > 0$ so that $U = \{z \in X : d_K(z, x) < \varepsilon\}$. Let $L = K \cap A$, and let $V = \{z \in X : d_L(z, x) < \varepsilon\}$. Then $V$ is open in $X$. We show that $V$ is as required: $U \subset V \subset [U]_\omega$, where $[U]_\omega$ denotes the $\omega$-closure of $U$ in $X$.

The inclusion $U \subset V$ is obvious. To prove that $V \subset [U]_\omega$, it suffices to show that every $y \in V$ is the limit of a sequence $y_0, y_1, \cdots \in U$. Let $F = K \setminus L$ be the set of all free ultrafilters in $K$, and let $s = \bigcap F$. By the assumption, there is a sequence $y_0, y_1, \cdots \in X$ converging to $y$ and such that for any $n \in \omega$ the set $\{\alpha \in A : d_\alpha(y_n, x) = 0\}$ belongs to $s$. Then $d_\alpha(y_n, x) = 0$ for every ultrafilter $\alpha \in F$, hence $d_\alpha(y_n, x) = 0$ and $d_K(y_n, x) = d_L(y_n, x)$. Since $y = \lim y_n \in V$, we have $d_K(y_n, x) = d_L(y_n, x) < \varepsilon$ if $n$ is sufficiently large. This means that $y_n \in U$. Thus $V \subset [U]_\omega$.

In virtue of Lemma 4.11, the proof will be complete if we show that the space $Y$ has countable functional tightness. We prove more: the tightness of $Y$ is countable. Let $x \in X$. Consider the map $f_x : X \to C_p(\beta A)$, defined by $f_x(y)(q) = d_q(x, y)$. It follows from the definition of the topology $T'$ of the space $Y$ that $x$ is in the $T'$-closure of a subset $E$ of $X$ if and only if the zero function $f_x(x)$ is in the closure of the set $f_x(E)$ in the space $C_p(\beta A)$. Since the tightness of $C_p(\beta A)$ is countable, it follows that the tightness of $Y$ is also countable. 

We now apply Proposition 4.2 to show that certain spaces have countable functional tightness. Recall that a space $X$ is bi-sequential [22] if any ultrafilter on $X$ converging to a point $x \in X$ contains a filter with a countable basis converging to the same point. All first-countable spaces are bi-sequential. An onto map $f : X \to Y$ is bi-quotient [22] if for any $y \in Y$ and any cover $\mathcal{W}$ of $f^{-1}(y)$ by open subsets of $X$ there exists a finite subfamily $\mathcal{E} \subset \mathcal{W}$ such that $y$ belongs to the interior of $f(\bigcup \mathcal{E})$. Bi-sequential spaces are precisely the images of metric spaces under bi-quotient maps [23]. Bi-quotient maps are preserved by arbitrary products [22] (see [36] for a similar assertion concerning so-called tri-quotient maps).

**Theorem 4.3.** Let $\{X_\alpha : \alpha \in A\}$ be a family of bi-sequential spaces, and let $X = \prod_{\alpha \in A} X_\alpha$.

1. If $\text{Card}(A)$ is non-Ulam-measurable, the product $X$ has countable functional tightness.
2. If $\text{Card}(A)$ is non-sequential, the product $X$ is $f$-sequential.

**Proof.** The theorem reduces to the case when all spaces $X_\alpha$ are metrizable. Indeed, for every $\alpha \in A$ there exist a metric space $Y_\alpha$ and a bi-quotient onto map $f_\alpha : Y_\alpha \to X_\alpha$ [23]. The product $\prod_\alpha f_\alpha : \prod_\alpha Y_\alpha \to \prod_\alpha X_\alpha$ is bi-quotient [22] and hence quotient. Since spaces of countable tightness and $f$-sequential spaces are preserved by quotient maps, it suffices to show that the product $\prod_\alpha Y_\alpha$ is of countable tightness or $f$-sequential, respectively.
So assume that the spaces $X_\alpha$ are metrizable. For every $\alpha \in A$ let $\rho_\alpha$ be a compatible metric on $X_\alpha$. Let $d_\alpha$ be the pseudometric on $X$ defined by $d_\alpha(\{x_\alpha\}, \{y_\alpha\}) = \rho_\alpha(x_\alpha, y_\alpha)$. The family $D = \{d_\alpha : \alpha \in A\}$ generates the topology of $X$. We show that if Card $(A)$ is non-Ulam-measurable, then the family $D$ satisfies the condition of Proposition 4.2.

Let $x, y \in X$, and let $F$ be a finite set of free ultrafilters on $A$. Since the cardinal Card $(A)$ is assumed to be non-Ulam-measurable, free ultrafilters on $A$ are not $\omega_1$-complete. Hence there exists a decreasing sequence $B_0 \supseteq B_1 \supseteq \cdots \in \bigcap F$ of subsets of $A$ such that $\bigcap B_n = \emptyset$. Define $y_n \in X$ by $y_{n,\alpha} = y_\alpha$ if $\alpha \in A \setminus B_n$ and $y_{n,\alpha} = x_\alpha$ if $\alpha \in B_n$. Since $\bigcap B_n = \emptyset$, the sequence $\{y_n\}$ converges to $y$. For every $n \in \omega$ the set $\{\alpha \in A : d_\alpha(y_{n,\alpha}, x_\alpha) = 0\}$ contains $B_n$ and hence belongs to $\bigcap F$. Proposition 4.2 implies that $X$ has countable functional tightness.

The second part of the theorem reduces to the first. Assume that the cardinal Card $(A)$ is not sequential. We must prove that every sequentially continuous function $f : X \to \mathbb{R}$ is continuous. Since every countable subset of $X$ lies in a product $\prod Y_\alpha$, where $Y_\alpha$ is a countable subset of $X_\alpha$ for every $\alpha \in A$, we may assume without loss of generality that all spaces $X_\alpha$ are separable (or even countable), and in this case the result is known [7, Theorem 1.5]. For completeness we sketch the proof.

For each $\alpha \in A$ pick a point $z_\alpha \in X_\alpha$, and let $S$ be the $\Sigma$-product of the family $\{X_\alpha\}$ with the base-point $\{z_\alpha\}$,

$$S = \{\{x_\alpha\} \in X : x_\alpha = z_\alpha \text{ for all but countably many } \alpha \in A\}.$$ 

Then $S$ is a Fréchet space, so every sequentially continuous function $f : X \to \mathbb{R}$ is continuous on $S$, and hence the restriction of $f$ to $S$ depends on countably many coordinates [5]. The assumption that the cardinal $\tau = \text{Card } (A)$ is not sequential means that any sequentially continuous function on the cube $2^\tau$ is continuous. Since $X$ is covered by copies $F$ of this cube such that the intersection $F \cap S$ is dense in $F$, it follows that $f$ depends on countably many coordinates and hence is continuous. \hfill $\square$

Sequentially continuous functions on products were first studied by S. Mazur [20], who proved that the first sequential cardinal (if it exists) is weakly inaccessible (that is, regular limit). Theorem 4.3 implies that if there are no sequential cardinals, then every Tikhonov space is a subspace of an $f$-sequential Tikhonov space. Conversely, if there exists a sequential cardinal $m$, then the cube $I^m$ cannot be embedded in an $f$-sequential Tikhonov space $X$. Indeed, if $I^m \subset X$, then there is a retraction $r : X \to I^m$. Let $f : I^m \to \mathbb{R}$ be a sequentially continuous discontinuous function. Then the composition $fr : X \to \mathbb{R}$ is sequentially continuous but not continuous, so $X$ is not $f$-sequential.

We now establish a counterpart of Theorem 4.3 for topological groups (Theorem 4.10). We need some preparations.

A homomorphism $f : G \to H$ of topological groups splits if there exists a homomorphism $s : H \to G$ which is right inverse to $f$, that is $fs = \text{id}_H$. In this case $G$ is a semidirect product of $H$ and the kernel of $f$. An inverse system $S = \{X_\alpha, \rho^\alpha_\alpha : \alpha, \beta < \tau\}$
of topological spaces is continuous if for every limit ordinal $\alpha < \tau$ the space $X_\alpha$ is naturally homeomorphic to the inverse limit of the system $\{X_\beta : \beta < \alpha\}$.

**Definition 4.4.** Let $\{H_\alpha : \alpha < \tau\}$ be a well-ordered family of topological groups. We say that a topological group $G$ is an **iterated semidirect product** of the groups $H_\alpha$, and write $G = \bigotimes_{\alpha<\tau} H_\alpha$, if there exists a continuous inverse system

$$\{e\} = G_0 \leftarrow \ldots \leftarrow G_\alpha \leftarrow G_{\alpha+1} \leftarrow \ldots \leftarrow G$$

of topological groups $G_\alpha$ and homomorphisms $p_\alpha^\beta : G_\beta \to G_\alpha$ for $\alpha < \beta$ such that $G = \varprojlim G_\alpha$ and for every $\alpha < \tau$ the homomorphism $p_\alpha^{\alpha+1}$ splits, and its kernel is isomorphic to $H_\alpha$.

In the notation of Definition 4.4, if we pick for every $\alpha < \tau$ a homomorphism $s_\alpha : G_\alpha \to G_{\alpha+1}$ which is right inverse to $p_\alpha^{\alpha+1}$, then each $H_\alpha$ can be identified with a subgroup of $G$, and $G$ can be identified with the product $\prod H_\alpha$ as a topological space but not as a group. If $H'_\alpha$ is a closed subgroup of $H_\alpha$ for every $\alpha < \tau$, the product $\prod H'_\alpha$ is a subgroup of $G$ if and only if $H'_\alpha$ normalizes $H'_\beta$ whenever $\alpha < \beta$. If $B$ is a set of ordinals $< \tau$, let $G_B = \prod H'_\alpha$, where $H'_\alpha = H_\alpha$ for $\alpha \in B$ and $H'_\alpha = \{e\}$ otherwise. Then $G_B$ is a subgroup of $G$.

Iterated semidirect products appear naturally as groups of self-homeomorphisms of products of compact spaces. For a compact space $K$ let $\text{Aut} \ K$ be the group of all self-homeomorphisms of $K$, equipped with the compact-open topology.

**Definition 4.5.** Let $X$ and $Y$ be compact spaces, and let $p : X \times Y \to X$ be the projection. We say that a self-homeomorphism $g \in \text{Aut} \ (X \times Y)$ of the product $Z = X \times Y$ is $X$-special if there exists a self-homeomorphism $h \in \text{Aut} \ X$ such that $p(gz) = h(pz)$ for every $z \in Z$.

A self-homeomorphism $g \in \text{Aut} \ Z$ is $X$-special if and only if it permutes fibers of $p$.

For a topological group $G$ and a compact space $X$ let $C(X, G)$ be the topological group of all continuous maps from $X$ to $G$, endowed with the compact-open topology.

**Lemma 4.6.** Let $X$ and $Y$ be compact spaces. The topological group $G$ of all $X$-special self-homeomorphisms of $X \times Y$ is the topological semidirect product of the groups $\text{Aut} \ X$ and $C(X, \text{Aut} \ Y)$. The natural homomorphism $G \to \text{Aut} \ X$ has a natural right inverse.

**Proof.** The kernel $K$ of the natural homomorphism $q : G \to \text{Aut} \ X$ consists of all $g \in \text{Aut} \ (X \times Y)$ which leave invariant each fiber of the projection $X \times Y \to X$. Every such $g$ defines a map $X \to \text{Aut} \ Y$ in a natural way. The exponential law for function spaces [14] Theorem 3.4.8] implies that we thus obtain a homeomorphism between $K$ and $C(X, \text{Aut} \ Y)$. On the other hand, the homomorphism $q$ has a natural right inverse homomorphism $s : \text{Aut} \ X \to G$, defined by $s(f)(x, y) = (f(x), y)$.

Now let $\{Y_\alpha : \alpha < \lambda\}$ be a well-ordered family of compact spaces. Let $Y = \prod_{\alpha<\lambda} Y_\alpha$, and for every $\alpha \leq \lambda$ let $X_\alpha = \prod_{\beta<\alpha} Y_\alpha$. 
**Definition 4.7.** Let the notation be as above. We say that a self-homeomorphism $g \in \text{Aut } Y$ is *special* if it is $X_\alpha$-special for each $\alpha < \lambda$ (if $Y$ is considered as the product $X_\alpha \times \prod_{\beta \geq \alpha} Y_\beta$).

Let $G$ be the group of all special $g \in \text{Aut } Y$. For every $\alpha < \lambda$ let $G_\alpha$ be the group of all special $g \in \text{Aut } X_\alpha$ (this notion is well-defined, since $X_\alpha$ is the product of a well-ordered family of compact spaces). There are natural homomorphisms $G \to G_\alpha$ and $G_\beta \to G_\alpha$ for $\alpha < \beta < \lambda$. We thus obtain an inverse system of the groups $G$ with the limit $G$. Lemma 4.6 shows that each $G_{\alpha+1}$ is the topological semidirect product of $G_\alpha$ and $C(X_\alpha, \text{Aut } Y_\alpha)$, and the homomorphism $G_{\alpha+1} \to G_\alpha$ has a natural right inverse. This proves the following:

**Lemma 4.8.** With the notation as above, the group $G$ of all special self-homeomorphisms of $Y$ is an iterated semidirect product of the groups $C(X_\alpha, \text{Aut } Y_\alpha)$, $\alpha < \lambda$. Each of the groups $C(X_\alpha, \text{Aut } Y_\alpha)$ can be identified with a subgroup of $G$ in a canonical way. □

Let $X$ and $Y$ be compact spaces and $Z = X \times Y$. We observed in the proof of Lemma 4.6 that there is a natural embedding $\text{Aut } X \to \text{Aut } Z$, which sends every $f \in \text{Aut } X$ to $f \times \text{id}_Y$. In particular, if $G = \text{Aut } I^A$, then for any subset $B \subset A$ there is a natural embedding $\text{Aut } I^B \to G$. Let $G_B$ be the range of this embedding. The subgroup $G_B$ of $G$ consists precisely of self-homeomorphisms of $I^A$ which preserve the $\alpha$-coordinate for every $\alpha \in A \setminus B$.

**Lemma 4.9.** Let $A$ be a countable set and $G = \text{Aut } I^A$. For a subset $B \subset A$ let $G_B$, as above, be the image of $\text{Aut } I^B$ under the natural embedding $\text{Aut } I^B \to G$. Let $B_0 \subset B_1 \subset \ldots$ be an increasing sequence of infinite subsets of $A$ such that $\bigcup B_n = A$. Then for any compact space $K$ the union of the subgroups $C(K, G_{B_n})$ is dense in $C(K, G)$. □

**Proof.** For every $n \in \omega$ pick a bijection $f_n : A \to B_n$ so that every $x \in A$ is fixed by $f_n$ for all but finitely many $n \in \omega$. Let $r_n : G \to G_{B_n}$ be the isomorphism induced by $f_n$, and let $\tilde{r}_n : C(K, G) \to C(K, G_{B_n})$ be the isomorphism induced by $r_n$. It is easy to see that the sequence $r_0, r_1, \ldots$ of self-maps of $G$ converges to the identity map of $G$ uniformly on compact sets (use the fact that compact subsets of $G$ are equicontinuous). It follows that the sequence $\tilde{r}_0, \tilde{r}_1, \ldots$ of self-maps of $C(K, G)$ converges to the identity map pointwise, hence the union of the groups $C(K, G_{B_n}) = \tilde{r}_n(C(K, G))$ is dense in $C(K, G)$. □

For a Hilbert space $H$ we denote by $U(H)$ the group of all unitary operators in $H$, equipped with the strong operator topology (= the topology of pointwise convergence). As in Section 2, $\text{Sym } E$ is the topological group of all permutations of a discrete space $E$. Note that $w(U(H)) = w(H)$, $w(\text{Aut } I^\tau) = \tau$ and $w(\text{Sym } E) = \text{Card } (E)$.

**Theorem 4.10.** Let $G$ be one of the following topological groups:

(a) the group $\text{Aut } I^\tau$ of all self-homeomorphisms of a Hilbert cube $I^\tau$;

(b) the group $\text{Sym } E$ of all permutations of a set $E$;

(c) the unitary group $U(H)$ of a Hilbert space $H$;
Then:

1. the group \( G \) has countable functional tightness if and only if its weight is non-Ulam-measurable;
2. the group \( G \) is \( g \)-sequential if and only if its weight is not a \( g \)-sequential cardinal.

**Proof.** We use the same method as in Theorem 4.3 based on Proposition 4.2. Let \( \tau \) be the weight of \( G \) \((\tau = \text{Card}(E)\) in the case \((b)\) and \( \tau = w(H)\) in the case \((c)\)), and let \( A \) be a set of cardinality \( \tau \). We construct a family \( D = \{d_\alpha : \alpha \in A\} \) of left-invariant pseudometrics on \( G \) generating the topology of \( G \) as follows. Let \( f, g \in G \).

(a) If \( G = \text{Aut} \, I^\tau \), we assume that \( A = \tau \). The group \( G \) acts on the right by isometries on the Banach space \( B = C(I^\tau) \). For \( \alpha \in A \) let \( b_\alpha \in B \) be the projection to the \( \alpha \)th coordinate, \( b_\alpha : I^A \to I = [0, 1] \). Let \( d_\alpha(f, g) = \|b_\alpha f^{-1} - b_\alpha g^{-1}\| \).

(b) If \( G = \text{Sym} \, E \), we may assume that \( A = E \). For \( x \in E \) let \( d_x(f, g) = 0 \) if \( f(x) = g(x) \) and \( d_x(f, g) = 1 \) otherwise.

(c) If \( G = U(H) \), let \( \{e_\alpha : \alpha \in A\} \) be an orthonormal basis for \( H \). For \( \alpha \in A \) let \( d_\alpha(f, g) = \|f(e_\alpha) - g(e_\alpha)\| \).

In all three cases the family \( D = \{d_\alpha : \alpha \in A\} \) generates the topology \( T \) of \( G \). Assume that \( \tau \) is non-Ulam-measurable. We show that the family \( D \) satisfies the conditions of Proposition 4.2.

Let \( f, g \in G \), and let \( F \) be a finite set of free ultrafilters on \( A \). We must show that there is a sequence \( g_0, g_1, \ldots \in G \) converging to \( g \) such that for every \( n \in \omega \) the set \( \{\alpha \in A : d_\alpha(g_n, f) = 0\} \) is in \( p = \bigcap F \). Since the metrics \( d_\alpha \) are left-invariant, we have \( d_\alpha(g_n, f) = d_\alpha(f^{-1}g_n, e) \), where \( e \) is the unity of \( G \). A sequence \( g_0, g_1, \ldots \) converges to \( g \) if and only if the sequence \( f^{-1}g_0, f^{-1}g_1, \ldots \) converges to \( f^{-1}g \). It follows from these remarks that without loss of generality we may assume that \( f = e \).

Consider first the case \((a)\), when \( G = \text{Aut} \, I^\tau \). Call a subset \( B \subset A \) special if \( g \) is \( I^B \)-special in the sense of Definition 4.5. In virtue of Shchepin’s theorem [32], \( A \) is covered by countable special subsets. Since special subsets of \( A \) are preserved by unions, it follows that the set \( A \) can be partitioned into countable infinite sets \( A_\gamma, \gamma < \tau \), so that each \( C_\gamma = \bigcup_{\beta < \gamma} A_\beta \) is special. Let \( Y_\gamma = I^A_\gamma \) and \( X_\gamma = I^{C_\gamma} = \prod_{\beta < \gamma} Y_\gamma \). Then \( I^A_\gamma \) is identified with the product \( Y = \prod_{\gamma < \tau} Y_\gamma \), and \( g \) is a special self-homeomorphism of \( Y \) in the sense of Definition 4.7.

Lemma 4.8 implies that \( g \) belongs to a subgroup \( P \) of \( G \) isomorphic to an iterated semidirect product \( \bigotimes_{\gamma < \tau} C(X_\gamma, \text{Aut} \, Y_\gamma) \), thus \( g \) can be identified with an element \( \{h_\gamma\} \) of the product \( \prod_{\gamma < \tau} C(X_\gamma, \text{Aut} \, Y_\gamma) \). Since \( \text{Card}(A) \) is non-Ulam-measurable, there is a sequence \( B_0 \supset B_1 \supset \cdots \in \bigcap F \) such that \( \bigcap B_n = \emptyset \). We may also assume that \( A_\gamma \setminus B_0 \) is infinite for every \( \gamma < \tau \).

It suffices to construct a sequence \( g_0, g_1, \ldots \in P \) such that \( \lim g_n = g \) and for every \( n \in \omega \) the set \( \{\alpha \in A : d_\alpha(g_n, e) = 0\} \) contains \( B_n \). The last condition means that \( g_n \) preserves the \( \alpha \)th coordinate \( b_\alpha \) for every \( \alpha \in B_0 \). Given \( \gamma < \tau \) and \( \alpha \in A_\gamma \), let \( H_\alpha \subset \text{Aut} \, Y_\gamma \) be the group of all self-homeomorphisms of \( Y_\alpha \) which preserve the \( \alpha \)th coordinate. Applying Lemma 4.9 to the increasing sequence \( \{A_\gamma \setminus B_n : n \in \omega\} \) of infinite subsets of \( A_\gamma \), we
see that for every $\gamma < \tau$ there is a sequence $h_{\gamma,1}, h_{\gamma,2}, \ldots \in C(X_{\gamma}, \text{Aut}\, Y_{\gamma})$ converging to $h_{\gamma}$ such that $h_{\gamma,n} \in C(X_{\gamma}, \bigcap_{a \in B_n} H_a)$ for every $n \in \omega$. Let $g_n = \{h_{\gamma,n} : \gamma < \tau\}$. Then each $g_n$ can be regarded as an element of $P = \bigotimes C(X_{\gamma}, \text{Aut}\, Y_{\gamma}) \subset G$, and the sequence $g_0, g_1, \ldots$ is as required.

The cases (b) and (c) are similar but simpler. As above, let $B_0 \supset B_1 \supset \cdots \in p$ be a sequence of subsets of $A$ such that $\bigcap B_n = \emptyset$. It suffices to construct a sequence $g_0, g_1, \ldots \in G$ such that $g = \lim g_n$ and $g_n(x) = x$ for every $x \in B_n$ in the case (a) (in this case we assume, as above, that $A = E$) or $g(e_\alpha) = e_\alpha$ for every $\alpha \in B_n$ in the case (b). In the case (a), when $G = \text{Sym} (E)$, the set $E$ can be partitioned into countable $g$-invariant subsets $E_i, i \in I$. The product $\prod_i \text{Sym} (E_i)$ can be identified with the subgroup $P$ of $G$ consisting of all $h \in G$ which leave each set $E_i$ invariant. Since $g \in P$, we can identify $g$ with an element of the product $\prod_i \text{Sym} (E_i)$ and then construct the sequence $g_0, g_1, \ldots$ coordinate-wise. The problem is thus reduced to the case when $E$ is countable. Then $G$ is metrizable, and it is clear that for any sequence $B_0 \supset B_1 \supset \cdots$ of subsets of $E$ such that $\bigcap B_n = \emptyset$ the subgroup

$$S = \{h \in G : \text{the set of fixed points of } h \text{ contains some } B_n\}$$

is dense in $G$, since it contains the dense subgroup of all permutations with finite support. Hence there is a sequence $g_0, g_1, \ldots \in S$ converging to $g$, as required. In the case (b), when $G = U(H)$, the set $A$ can be partitioned into countable subsets $A_i, i \in I$, so that for every $i \in I$ the linear subspace $H_i$ of the Hilbert space $H$, spanned by the vectors $e_\alpha, \alpha \in A_i$, is $g$-invariant. The group $P = \prod_i U(H_i)$, regarded as a subgroup of $G$, contains $g$. As above, this reduces the problem to the case of separable $H$. The subgroup of all $h \in G$ which leave fixed all but finitely many vectors of the basis $\{e_\alpha : \alpha \in A\}$ is dense in $G$. As above, this completes the argument.

If $\tau$ is measurable, the group $G$ is not $m$-stable (Section 3) in all three cases (a), (b), (c), since $G$ contains a copy of the cube $2^\tau$. Hence the identity map of $G$ to its $m$-modification is $\omega$-continuous (Proposition 3.15) but not continuous. Thus $G$ does not have countable functional tightness. Similarly, if $\tau$ is $g$-sequential, then $G$ is not $g$-stable and moreover not $g$-sequential.

Assume $\tau$ is not $g$-sequential. We must prove that every sequentially continuous homomorphism $f : G \to H$ is continuous. Since we already know that $G$ has countable functional tightness, it suffices to prove that $f$ is $\omega$-continuous. We first consider the cases (b) and (c). We noted above that every $g \in G$ is contained in a subgroup $P$ of $G$ which is isomorphic to a product of separable metric groups. It is easy to see that every countable subset of $G$ also is contained in such a subgroup $P$. Theorem 3.2 implies that $P$ is $g$-sequential, hence the restriction of $f$ to $P$ is continuous. It follows that $f$ is $\omega$-continuous. In the case (a), every countable subset of $G$ is contained in a subgroup $P$ which is an iterated semidirect product of metrizable groups of the form $C(X, \text{Aut}\, Y)$, where $X$ and $Y$ are compact and $Y$ is metrizable. Thus it suffices to prove the following generalization of Theorem 3.2: if $\tau$ is not $g$-sequential, then any iterated semidirect
product \( P = \bigotimes_{\alpha < \tau} H_\alpha \) of \( g \)-sequential groups \( H_\alpha \) is \( g \)-sequential. We noted after Definition 4.4 that for any set \( B \) of ordinals \( < \tau \) the iterated semidirect product \( \bigotimes_{\alpha \in B} H_\alpha \) can be regarded as a subgroup of \( P \), hence the proof of Theorem 3.2 can be applied without any changes. \( \square \)

5. Acknowledgement

The present paper was inspired by ideas of V. Pestov who kindly sent to me the unpublished manuscript \[28\]. I am indebted to V. Pestov for many helpful discussions, and I regret that he declined my offer to be listed as a coauthor of the paper.

References

[1] A. V. Arhangel’skiĭ, Classes of topological groups, Russ. Math. Surveys 36:3 (1981), 151–174.
[2] A. V. Arhangel’skiĭ, On countably compact topologies on compact groups and on dyadic compacta, Topology Appl. 57:2-3 (1994), 163–181.
[3] A. V. Arhangel’skiĭ, Functional tightness, Q-spaces and \( \tau \)-embeddings, Comment. Math. Univ. Carol. 24 (1983), 105–120.
[4] A. V. Arhangel’skiĭ, Structure and classification of topological spaces and cardinal invariants, Russian Math. Surveys 33:6 (1978), 33–96.
[5] A. V. Arhangel’skiĭ, On mappings of everywhere dense subsets of topological products, Dokl. Akad. Nauk SSSR, 197:4 (1971), 750–753.
[6] A. V. Arhangel’skiĭ, W. Just, G. Plebanek, Sequential continuity on dyadic compacta and topological groups, Comment. Math. Univ. Carol. 37 (1996), No. 4, 775–790.
[7] M. Ya. Antonovskiĭ, D. V. Chudnovskiĭ, Some questions of general topology and Tikhonov semifields. II, Russian Math. Surveys 31:3 (1976), 69–128.
[8] W. W. Comfort, D. Remus, Pseudocompact refinements of compact group topologies, Math. Z. 215 (1994), 337–346.
[9] W. W. Comfort, D. Remus, Compact groups of Ulam-measurable cardinality: partial converses to theorems of Arhangel’skiĭ and Varopoulos, Math. Japonica 39 (2) (1994), 203–210
[10] W. W. Comfort, K.-H. Hofmann and D. Remus, Topological groups and semigroups, Recent Progress in General Topology, M. Hušek and J. van Mill, editors. Elsevier Science Publishers, 1992, 315–347.
[11] S. Dierolf, U. Schwanengel, Examples of locally compact non-compact minimal topological groups, Pacif. J. Math. 82 (1979), 349–355.
[12] A. Dow, S. Watson, A subvariety of TOP, Trans. Amer. Math. Soc. 337 (1993), 825–837.
[13] B. A. Efimov, Mappings and embeddings of dyadic spaces, Math. USSR Sb. 32 (1977), 45–57.
[14] R. Engelking, General Topology, 2nd edition. Heldermann Verlag, Berlin, 1989.
[15] L. Fuchs, Infinite Abelian groups. Vol. 1, Academic Press, 1970.
[16] R. Haydon, On a problem of Pelczyński: Milutin spaces, Dugundji spaces and \( AE(0\text{-dim}) \), Studia Math. 52 (1974), 23–31.
[17] E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. 1, 2nd edition. Springer-Verlag, 1979.
[18] K. H. Hofmann, S. A. Morris, The structure of compact groups, 3rd edition, De Gruyter, 2013.
[19] Th. Jech, Set Theory, The Third Millennium Edition, Revised and Expanded, Springer, 2006.
[20] S. Mazur, On continuous mappings of Cartesian products, Fund. Math. 39 (1952), 229–238.
[21] E. Michael, Selected selection theorems Amer. Math. Monthly 63 (1956), 233–238.
[22] E. Michael, Bi-quotient maps and cartesian products of quotient maps, Ann. Inst. Fourier (Grenoble) 18 (1968), 287–302.
[23] E. Michael, A quintuple quotient quest, Gen. Topology Appl. 2 (1972), 91–138.
[24] S. A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. 1 (1969), 145–160.
[25] S. A. Morris, *Varieties of topological groups. II*, Bull. Austral. Math. Soc. 2 (1970), 1–13.
[26] S. A. Morris, *Varieties of topological groups and left adjoint functor*, J. Austral. Math. Soc. 16 (1973), 220–227.
[27] S. A. Morris, *Varieties of topological groups. A survey*, Colloq. Math. 46, (1982), 147–165.
[28] S. A. Morris, P. Nickolas, V. G. Pestov, and S. Svetlichny, *The variety generated by free topological groups on metrizable spaces*, manuscript.
[29] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. 58 (1968).
[30] D. Shakhmatov, *A direct proof that every infinite compact group $G$ contains $\{0,1\}^{w(G)}$*, Annals of the New York Academy of Sciences 728:1 (1994), 276–283.
[31] B. E. Shapirovskii, *Maps onto Tikhonov cubes*, Russian Math. Surveys 35:3 (1980), 145–156.
[32] E. V. Shchepin, *Topology of limit spaces of uncountable inverse spectra*, Russian Math. Surveys 31:5 (1976), 155–191.
[33] O. V. Sipacheva, *Zero-dimensionality and completeness in free topological groups I, II*, Serdica 15 (1989), 119–140 and 141–154 (in Russian).
[34] O. V. Sipacheva, *Free topological groups of spaces and their subspaces*, Topology Appl. 101:3 (2000), 181–212.
[35] V. V. Uspenskij, *A universal topological group with a countable base*, Functional Analysis and its Applications 20 (1986), 160–161.
[36] V. V. Uspenskij, *Tri-quotient maps are preserved by infinite products*, Proc. Amer. Math. Soc. 123 (1995), No. 11, 3567–3574.
[37] V. V. Uspenskij, *Why compact groups are dyadic*, General Topology and its Relations to Modern Analysis and Algebra VI: Proc. of the 6th Prague Topological Symposium 1986, Frolík Z. (ed.), Berlin: Heldermann Verlag, 1988, 601–610.
[38] V. V. Uspenskij, *A characterization of realcompactness in terms of the topology of pointwise convergence on the function space*, Comment. Math. Univ. Carol. 24 (1983), 121–126.
[39] V. V. Uspenskij, *Unitary representability of free abelian topological groups*, Applied General Topology 9 (2008), No. 2, 197–204. arXiv: [math.RT/0604253](http://arxiv.org/abs/math.RT/0604253)
[40] N. Th. Varopoulos, *A theorem on the continuity of homomorphisms of locally compact groups*, Proc. Cambridge Phil. Soc. 60 (1964), 449–463.

Department of Mathematics, 321 Morton Hall, Ohio University, Athens, Ohio 45701, USA

Email address: uspenski@ohio.edu