TENSOR PRODUCTS OF COMPLEMENTARY SERIES OF $SO(n, 1)$ AND BILINEAR INTERTWINING DIFFERENTIAL OPERATORS

GENKAI ZHANG

ABSTRACT. We give a different construction of the bilinear intertwining differential operators on tensor product of spherical principal series representations of the Lie group $O(n + 1, 1)$ found earlier by Ovsienko and Redou [12]. Using these bilinear differential intertwining operators we prove further that there are finitely many complementary series $\pi_\gamma$ appearing in the tensor product $\pi_\alpha \otimes \pi_\beta$ of two complementary series $\pi_\alpha$ and $\pi_\beta$. For the group $SO(2, 1)$ this was proved earlier by Repka.

1. INTRODUCTION

The study of bilinear differential operators is of natural interests in representation theory of Lie groups and in quantization. The most studied case might be the Rankin-Cohen brackets on tensor products of holomorphic discrete series of $SL(2, \mathbb{R})$, which yield also the decomposition of the tensor products in the unitary sense. There exist further formal sums of the brackets producing associative products, or quantizations; see e.g. [7]. These operators are also bounded operators acting on the tensor product of holomorphic discrete series can be abstractly understood as they are both highest weight representations and the decomposition can be treated algebraically, more precisely the intertwining operators can be constructed using Lie algebra computations. Generally it is not expected the existence of bounded intertwining differential operators on tensor products of unitary principal series representations as they contain functions with only $L^2$-conditions. On the other hand the complementary series representations are defined roughly as certain spaces of distributions with some Sobolev type differentiability conditions, and it might be still possible to construct differential and bounded intertwining operators on their tensor products. Indeed we shall prove this is the case for the real orthogonal group $O(n, 1)$.

To start with we may just consider formal bilinear intertwining operators acting on tensor product of the smooth principal series. Indeed in this setup Ovsienko and Redou [12] have found a family of invariant bilinear differential operators on tensor product of spherical principal series representations $\pi_\alpha, \alpha \in \mathbb{C}$, of the conformal group $O(n + 1, 1)$; the representations considered there are defined on spaces of smooth functions on $\mathbb{R}^n$ and are viewed as conformal densities. They found the operators by using an Ansatz expressing the operators as polynomials of the Laplacian operators $L_x, L_y$, and the inner product $\nabla_x \cdot \nabla_y$. The same operators are obtained in [4] as residues of a family of integral bilinear intertwining operators, which is based on some earlier work on trilinear form [3 5 10].

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In the present paper we shall give a direct construction of those operators. We apply the operators to prove that there are finitely many discrete components of complementary series \( \pi_{\alpha+\beta+2N} \) in the tensor product \( \pi_\alpha \otimes \pi_\beta \), \( \alpha, \beta > 0 \), of complementary series when the parameters \( \alpha \) and \( \beta \) are relatively small (in our parametrization). We describe briefly our idea of the construction and the proof of the appearance of the discrete components.

Let \( P = MAN \) be the maximal parabolic subgroup of \( G = O(n + 1, 1) \) where \( A = \exp(\mathfrak{a}) = \mathbb{R}^+ \) is the split Cartan subgroup, and \( \pi_\nu \) be the smoothly induced spherical principal series of \( G \) for \( \nu \in \mathfrak{a}^* = \mathbb{C} \) with the normalization that the \( L^2 \)-induced series on \( L^2(K/M) \) for \( \nu \in \rho + i\mathbb{R} \) are the unitary principal series. \( \pi_\nu \) can be realized on a space of smooth functions on \( N = \mathbb{R}^n \). The Knapp-Stein operator \( J_\nu \) intertwines \( \pi_\nu \) with \( \pi_{\tilde{\nu}} \), \( \tilde{\nu} = 2\rho - \nu \), as meromorphic continuation of certain integral operator with kernels \( |x - y|^\nu \). Our construction of bilinear differential operators is based on similar ideas in our construction [13] of intertwining differential operators on holomorphic discrete series on Hermitian symmetric spaces. In that case we first construct some integral intertwining operators and prove then they are differential operators by using reproducing property of holomorphic functions. In the present case reproducing kernel property is replaced by the simple fact that \( J_\nu J_\nu = \text{Id} \). More precisely we construct first some integral intertwining operator \( T_{N,\alpha,\beta} : \pi_\alpha \otimes \pi_\beta \to \pi_{\alpha+\beta+2N} \) on the dual representation \( \pi_\alpha \otimes \pi_\beta \). This composed with the Knapp-Stein intertwining operator \( J_\alpha \otimes J_\beta : \pi_\alpha \otimes \pi_\beta \to \pi_\alpha \otimes \pi_\beta \), turns out to be a differential operator \( D_N(\alpha, \beta) := T_{N,\alpha,\beta}(J_\alpha \otimes J_\beta) : \pi_\alpha \otimes \pi_\beta \to \pi_{\alpha+\beta+2N} \). We prove this by showing that \( T_{N,\alpha,\beta}(J_\alpha \otimes J_\beta) \) can be obtained by differentiating of the identity \( f = (J_\alpha \otimes J_\beta)(J_\alpha \otimes J_\beta)f \) evaluated at the diagonal. In the simplest case we have realized the diagonal restriction \( R : \pi_\alpha \otimes \pi_\beta \to \pi_{\alpha+\beta}, f(x, y) = f(x, x) \) as the product \( R = (R(J_\alpha \otimes J_\beta))(J_\alpha \otimes J_\beta) \) with each factor being an (analytic continuation of) integral intertwining operators.

For \( \nu \in (0, 2\rho) \) the \((\mathfrak{g}, K)\)-module of induced representation can be unitarized and we get unitary spherical representation of \( G \), the complementary series. In the non-compact picture it is certain space of distributions with their Fourier transform being in a weighted \( L^2 \)-space on \( \mathbb{R}^n \). We prove that for relative small parameters \( \alpha, \beta \) the bilinear differential operators \( D_N(\alpha, \beta) \) for a finitely many \( N \) are bounded and thus \( \pi_{\alpha+\beta+2N} \) appear in the tensor product \( \pi_\alpha \otimes \pi_\beta \). We use the characterization of complementary series in terms of Fourier transform. In the case of \( n = 2 \) with \( \mathfrak{so}(2, 1) = \mathfrak{sl}(2, \mathbb{R}) \) we give a different and straightforward proof for the appearance of \( \pi_{\alpha+\beta} \) for small \( \alpha, \beta \) in \( \pi_\alpha \otimes \pi_\beta \) proved earlier by Repka [15, 14]. Here \( \pi_{\alpha+\beta} \) is the only possible discrete component the tensor product. To our knowledge the tensor product of complementary series of real groups has been studied earlier only for this case; see [1] for the case \( \mathfrak{sl}(2, k) \) of a local field \( k \). Theses results combined with the general theory of Burger-Li-Sarnak [2] have also found applications in automorphic forms [6].

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2. **Spherical representations of rank one group** \( G \)

We fix notation and recall some known results on induced representations of \( G \) and the Knapp-Stein intertwining operator. We shall use non-compact realizations of the representations. There has been much study on these representations, and we shall be rather brief and recall the precise the results we need. Most of the technical formulas can be found e.g. in [11], [8] where the general case of rank one groups is studied.

Let \( G = O(n + 1, 1) \) be the group of linear transformations on \( \mathbb{R}^{n+2} \) preserving the quadratic form \( |x_1|^2 + \cdots + |x_{n+1}|^2 - |x_{n+2}|^2 \). The known theory on induced representations are mostly for simply connected group, i.e. \( \text{Spin}(n+1, 1) \) in our case. However for simplicity of the exposition we shall work with \( G \) instead. Elements in \( g \in G \) will be written as \((n+2) \times (n+2)\) block matrices

\[
g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

where \( a, b, c, d \) are of size \((n+1) \times (n+1), (n+1) \times 1, 1 \times (n+1), 1 \times 1\), respectively.

Let \( K = O(n+1) \times \mathbb{Z}_2 \subset G \) be the subgroup with \( b = 0, c = 0, d = \pm 1 \) in the above matrix form. Let \( g = k + p \) be the corresponding Cartan decomposition. We fix \( H_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) in \( p \) and let \( \mathfrak{a} = \mathfrak{R} H_0 \subset \mathfrak{p} \). Then \( \mathfrak{a} \) is a maximal abelian subspace of \( \mathfrak{p} \). The root space decomposition of \( g \) under \( H_0 \) is

\[
\mathfrak{g} = \mathfrak{n}_- + (\mathfrak{a} + \mathfrak{m}) + \mathfrak{n}_1
\]

with roots \( \pm 1, 0 \). Here \( \mathfrak{m} \subset \mathfrak{k} \) is the zero root space in \( \mathfrak{k} \). Thus \( \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \) is a maximal parabolic subalgebra of \( \mathfrak{g} \). Then \( \mathfrak{n}^- = \mathbb{R}^n \) is abelian. We denote \( \rho = \frac{n}{2} \), viewed also as the linear functional \( H \to \frac{n}{2} \).

Denote \( M, A, N, N^- = \mathbb{R}^n \) the corresponding subgroups with Lie algebras \( \mathfrak{m}, \mathfrak{a}, \mathfrak{n}, \mathfrak{n}_- \). Then \( M = O(n) \times \mathbb{Z}_2 \) and \( MAN \) is a maximal parabolic subgroup of \( G \). The Weyl group will be identified with \( \{1, w\} \),

\[
w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

For \( \nu \in \mathbb{C} \) let \( \pi_{\nu}^\infty \) be the induced smooth representation of \( G \) from the character \( e^{-\nu} : me^{tH_0}n \in P = MAN \mapsto e^{-\nu t} \) consisting of \( C^\infty \)-functions \( f \) on \( G \) such that

\[
f(gme^{tH_0}n) = e^{-\nu t} f(g), e^{tH_0}mn \in MAN.
\]

In particular \( f \) are determined by their restriction on \( K \) and are identified as smooth functions on \( K/M \) by the \( M \)-invariance. We have \( \pi_{\nu}^\infty = C^\infty(K/M) = C^\infty(S) \) as vector spaces. Restricting the smooth functions in \( \pi_{\nu}^\infty \) to \( N^- \) results in an injective map to a subspace of \( C^\infty(N^-) = C^\infty(\mathbb{R}^n) \). We shall fix this realization of \( \pi_{\nu}^\infty \).
The representation \(\pi_\nu(g), g \in G, \nu \in (\rho + i\mathbb{R})\) is already unitary for the natural unitary norm in \(L^2(K/M)\). However for \(\nu \in (0, 2\rho)\) a different \(g\)-invariant inner product on the space of \(K\)-finite vectors can be defined and completed to a unitary representation of \(G\), the complementary series; see [11]. The representation shall also be denote by \(\pi_\nu\) for the real parameter \(\nu \in (0, 2\rho)\).

The transfer from compact picture on \(C^\infty(S)\) to the non-compact picture is done via the Cayley transform \(c\). We present its formula, but shall not need it in the sequel. The transform \(c\) maps \(\mathbb{R}^n\) to \(S \setminus \{e_1\}\),

\[
c : \mathbb{R}^n \to S, \quad x \mapsto s = (s_1, s') = \left(\frac{|x|^2 - 1}{|x|^2 + 1}, \frac{2x}{|x|^2 + 1}\right).
\]

The representation space \(\pi_\nu^\infty\) consists now of smooth functions \(f\) on \(\mathbb{R}^n\) of the form

\[
f(x) = (1 + |x|^2)^{-\nu}F(s) = \left(\frac{1 + s_1}{2}\right)^\nu F(s), \quad F \in C^\infty(S).
\]

The explicit formulas for \(\pi_\nu(g)\) can be found in [11] in the compact picture and in [16] for the non-compact picture. For our purpose here it is enough to know that the action of \(\pi_\nu(g)\) for \(g \in MAN^-\) and \(g = w\), as they generate the whole group. These are given by

\[
\pi_\nu(g)f(x) = e^{-\nu}f(e^\nu m^{-1}(x - x_0)), \quad (m, e^\nu H, x_0) = me^\nu x_0 \in MAN^- \quad \text{and}
\]

\[
\pi_\nu(w)f(x) = \|x\|^{-2\nu}f\left(-\frac{x}{\|x\|^2}\right).
\]

Note also that the Jacobians of \(g = (m, e^\nu H, x_0)\) and of \(w\) acting on \(N^- = \mathbb{R}^n\) are given by

\[
J_g(x) = e^{tn}, \quad J_w(x) = \frac{1}{|x|^{2n}}.
\]

The Knapp-Stein intertwining operator \(J_\nu\) is given by

\[
J_\nu f(x) = \int_{\mathbb{R}^n} K_\nu(x, y)f(y)dy,
\]

where

\[
K_\nu(x, y) = C_\nu \frac{1}{|x - y|^{2\nu}}, \quad C_\nu = \frac{\Gamma\left(\rho - \frac{\nu}{2}\right)\Gamma\left(\rho - \frac{\nu}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\rho - \nu\right)} = \frac{2^{1-2\rho+1}\sqrt{\pi}\Gamma(2\rho - \nu)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\rho - \nu\right)}.
\]

The normalization is chosen here so that in the compact picture

\[
J_\nu 1 = 1.
\]

Then \(J_\nu\) is a \(G\)-intertwining operator

\[
J_\nu : \pi_\rho^\infty \to \pi_\nu^\infty, \quad \nu := 2\rho - \nu
\]

for \(\nu << 0\). It has holomorphic continuation to the whole complex plane, and in particular holomorphic and non-zero in the two symmetric strips around \(\Re \nu = \rho\),

\[
\{\nu; 0 < \Re \nu < \rho\}, \quad \{\nu; \rho < \Re \nu < 2\rho\}.
\]
The formal intertwining property can be proved by using the following transformation rule of $K_\nu$,

$$K_\nu(gz, gw) = (cz + d)^{-\nu} K_\nu(z, w)(cw + d)^{-\nu} = J_g(z)^{\nu} K_\nu(z, w) J_g(w)^{-\nu}$$

where $J_g$ is the Jacobian of the action of $g \in G$ on $N^- = \mathbb{R}^n$. The holomorphic continuation can also be done using the identity (3.2) below. The smooth case is also consequence of the general theory of intertwining operators [17].

The inner product

$$(f_1, f_2)_\nu = (J_\nu f_1, f_2)_{L^2(\mathbb{R}^n)}$$

for $f_1, f_2 \in C_0^\infty(\mathbb{R}^n)$ is a pre-Hilbert norm, and is invariant under $g \in G$ sufficiently close to the identity (depending on $f_1, f_2$). We shall use its description using Fourier transform $f \mapsto \mathcal{F} f$. The space $\pi_\nu$ is the completion of $C_0^\infty(\mathbb{R}^{n-1})$ with the equivalent norm

$$\|f\|_\nu^2 = \int_{\mathbb{R}^n} |\mathcal{F} f(\xi)|^2 |\xi|^{-2\nu} d\xi = \| \mathcal{F} f(\cdot) \|_{L^2(\mathbb{R}^n)}^2,$$

for $0 < \nu < 2\rho$. See e.g. [16].

3. INVARIANT BILINEAR DIFFERENTIAL OPERATORS FOR GENERAL SPHERICAL SERIES REPRESENTATIONS

We denote $\pi_\alpha^\infty \otimes \pi_\beta^\infty$ the induced smooth representation of $G \times G$ from the parabolic subgroup $P \times P$ and the character $e^{-\alpha} \times e^{-\beta}$. The group $G$ is viewed as the diagonal subgroup of $G \times G$.

**Theorem 3.1.** For any $N \geq 0$ there exists a $G$-intertwining differential operator $\mathcal{D}_{N,\alpha,\beta}$ of degree $2N$ meromorphic in $(\alpha, \beta) \in \mathbb{C}^2$,

$$\mathcal{D}_{N,\alpha,\beta} : \pi_\alpha^\infty \otimes \pi_\beta^\infty \rightarrow \pi_{\alpha+\beta+2N}^\infty.$$  

The only possible poles of $\mathcal{D}_{N,\alpha,\beta}$ appear when $\alpha$ or $\beta \in \Lambda_N$, where

$$\Lambda_N = \{0, -1, -N + 1\} \cup (\rho - 1 + \{0, -1, -N + 2\}).$$

The proof will be divided into a few elementary Lemmas.

Let $S_{\alpha,\beta,N}(x; y, z, w)$ be the kernel

$$S_{\alpha,\beta,N}(x; y, z, w) = \left( \frac{|(x - z) - (y - w)|^2}{|x - z|^2 |y - w|^2} \right)^N \frac{1}{|x - z|^{2\alpha} |y - w|^{2\beta}},$$

and write for simplicity

$$S_{\alpha,\beta,N}(x; z, w) = S_{\alpha,\beta,N}(x; x, z, w), \quad S_{\alpha,\beta,N}(x, y) = S_{\alpha,\beta,N}(x, y; 0, 0)$$

**Lemma 3.2.** The integral operator

$$T_N f(x) = T_{N,\alpha,\beta} f(x) := C_\alpha C_\beta \int_{\mathbb{R}^{2n}} S_{\alpha,\beta,N}(x; z, w) f(z, w) dz dw$$
defines an intertwining operator
\[ \pi_\alpha^\infty \otimes \pi_\beta^\infty \to \pi_{\alpha+\beta+2N}^\infty. \]

**Proof.** Recall the group \( G \) is generated by \( P \) and \( w \) as a consequence of the Bruhat decomposition \([9\text{ Theorem 1.4, Ch. IX}]. \) The formal intertwining property follows directly from a change of variables \((x, y) \mapsto (gx, gy)\) for \( g \in P \) and \( g = w \) along with the formula \((2.3)\) for the Jacobians. To prove the meromorphic continuation in \( \alpha \) and \( \beta \) we observe that changing \((x, y)\) to \((x - z, y - z)\) we need only to prove that the integral
\[ \int_{\mathbb{R}^{2n}} \frac{1}{|x - y|^{2N}} f(x, y) dx dy \]
is meromorphic in \((\alpha, \beta)\). But this is just up to normalization constants the integral \((J_{\alpha+N} \otimes J_{\beta+N})(F)\), \( F(x, y) = |x - y|^{2N} f(x, y) \) and thus has the continuation. \( \square \)

In the compact-realization this operator is
\[ T_N f(x) = \int_{S \times S} \left( \frac{1 - \langle z, w \rangle}{(1 - \langle x, z \rangle)(1 - \langle x, w \rangle)} \right)^N \frac{C_\alpha C_\beta}{(1 - \langle x, z \rangle)(1 - \langle x, w \rangle)} f(z, w) dz dw. \]
That the integral is well-defined for \( \alpha, \beta << 0 \) can also be easily deduced from this formula.

First we need some known Bernstein-Sato type identities for the Laplacian operator \( \mathcal{L} = \partial_1^2 + \cdots + \partial_n^2 \) acting on \(|x|^{-2\alpha} \). Recall the Pochammer symbol defined by \((\alpha)_j = \alpha(\alpha + 1) \cdots (\alpha + j - 1)\).

**Lemma 3.3.** The following differentiation formula holds
\[ (3.2) \quad \mathcal{L}^j |x|^{-2\alpha} = 2^j (\alpha)_j (\alpha + 1 - \rho)_j |x|^{-2(\alpha+j)}, \quad x \neq 0. \]

We define a family of differential operators of constant coefficients on \( C^\infty(\mathbb{R}^{2n}) \) by \( M_{\alpha,\beta,0} = I, M_{\alpha,\beta,1} = \nabla_x \cdot \nabla_y, \) and
\[ M_{N+1,\alpha,\beta} = (\nabla_x \cdot \nabla_y) M_{N,\alpha,\beta} - \frac{N(n - 1 - 3N - 2\alpha - 2\beta)}{(\alpha + 1 - \rho)(\beta + 1 - \rho)} M_{N-1,\alpha+1,\beta+1} \mathcal{L}_x \mathcal{L}_y \]
It follows from the construction that the only possible poles of \( M_{N,\alpha,\beta}, N \geq 2, \) appear when \( \alpha \) or \( \beta \) is in
\[ \{ \rho - j; j = 1, \cdots, N - 1 \} \]

**Lemma 3.4.** The following formula holds for all \((\alpha, \beta) \in \mathbb{C}^2 \) and \( m \in \mathbb{N}, \)
\[ (3.3) \quad \mathcal{M}_{m,\alpha,\beta} S_{\alpha,\beta}(x, y; z, w) = 2^{2m} (\alpha)_m (\beta)_m \left( \frac{\langle x - z, y - w \rangle}{|x - z|^2 |y - w|^2} \right)^m S_{\alpha,\beta}(x, y; z, w) \]

**Proof.** By invariance we can assume \( z = w = 0 \). We prove the identity using induction. It is trivially true for \( m = 0 \). Assuming the identity holds for \( 0 \leq m \leq N \) for all \( \alpha, \beta \) we perform the differentiation \( \nabla_x \cdot \nabla_y \mathcal{M}_{N,\alpha,\beta} S_{\alpha,\beta}(x, y) \) on the identity with \( m = N \). We have
\[ (3.4) \quad \nabla_x \cdot \nabla_y \mathcal{M}_{N,\alpha,\beta} S_{\alpha,\beta}(x, y) = 2^{2N} (\alpha)_N (\beta)_N (I + II) \]
a sum of two terms, with the first term
\[ I = 2^{2N}(\alpha)N(\beta)N 2^{2(\alpha + N)(\beta + N)} \left( \frac{(x, y)}{|x^2y^2|} \right)^{N+1} S_{\alpha, \beta}(x, y) \]
\[ = 2^{(N+1)}(\alpha)N+1(\beta)N+1 \left( \frac{(x, y)}{|x^2y^2|} \right)^{N+1} S_{\alpha, \beta}(x, y) \]
being the RHS of (3.3) for \( m = N + 1 \), and
\[ II = N(n - 1 - 3N - 2\alpha - 2\beta) \left( \frac{(x, y)}{|x^2y^2|} \right)^{N-1} S_{\alpha+1, \beta+1}(x, y). \]
We treat the second term using the induction hypothesis for \( m = N - 1 \) with \((\alpha, \beta)\) being replaced by \((\alpha + 1, \beta + 1)\).
\[ 2^{(N+1)}(\alpha+1)N-1(\beta+1)N-1 \left( \frac{(x, y)}{|x^2y^2|} \right)^{N-1} S_{\alpha+1, \beta+1}(x, y) = M_{N-1, \alpha+1, \beta+1} S_{\alpha+1, \beta+1}(x, y), \]
which is furthermore
\[ \frac{1}{2^\alpha(\alpha + \rho - 1)\beta(\beta + \rho - 1)} M_{N-1, \alpha+1, \beta+1} L_x L_y S_{\alpha, \beta}(x, y). \]
Rewriting (3.4) we find
\[ (\nabla_x \cdot \nabla_y M_{N, \alpha, \beta} - \frac{N(n - 1 - 3N - 2\alpha - 2\beta)}{(\alpha + \rho - 1)(\beta + \rho - 1)} M_{N-1, \alpha+1, \beta+1} L_x L_y) S_{\alpha, \beta}(x, y), \]
which is \( M_{N+1, \alpha, \beta} S_{\alpha, \beta}(x, y) \) by the definition. This finishes the proof.

Combining the two Lemmas we have
\[ M_{k, \alpha+j, \beta+i} L_x^j L_y^i \frac{1}{|x^2|y^2} = c_{i, j, k}(\alpha, \beta) \left( \frac{(x, y)}{|x^2y^2|} \right)^k \frac{1}{|x^2y^2|/|2^\beta+2i|}, \]
where
\[ c_{i, j, k}(\alpha, \beta) = 2^{2k+2j+2i}(\alpha)_{j+k}(\alpha + 1 - \rho)_{j+i+k}(\beta + 1 - \rho)_i. \]
Here we have used the fact that
\[ (\gamma)_j(\gamma + j)_k = (\gamma)_j+k. \]
By translation invariance we have
\[ M_{k, \alpha+j, \beta+i} L_x^j L_y^i \frac{1}{|x-z|^{2\alpha}|y-w|^{2\beta}}, \]
\[ = c_{i, j, k}(\alpha, \beta) \left( \frac{(x-z, y-w)}{|x-z|^{2\alpha}|y-w|^{2\beta}} \right)^k \frac{1}{|x-z|^{2\alpha+2j}|y-w|^{2\beta+2i}}, \]
We prove now Theorem 3.1

Proof. The operator
\[ T_{N, \alpha, \beta}(J_{\alpha} \otimes J_{\beta}) : \pi_{\alpha}^\infty \otimes \pi_{\beta}^\infty \to \pi_{\alpha+\beta+2N}^\infty. \]
is an intertwining operator by Lemma 3.2. We prove it is a differential operator. The idea is to differentiate the identity \( f = (J_\alpha \otimes J_\beta)(J_\bar{\alpha} \otimes J_\bar{\beta})f \). We shall perform formal computations on the integral first and justify them in the end. Let \( f \in \pi^\infty_\alpha \otimes \pi^\infty_\beta \) and \( g = J_\bar{\alpha} \otimes J_\bar{\beta}f \). We denote

\[
\mathcal{E}_{N,\alpha,\beta}f(z, w) = \sum_{i+j+k=N} \varepsilon_{i,j,k}(\alpha, \beta) \mathcal{M}_{k,\alpha+j,\beta+i} \mathcal{L}_x \mathcal{L}_y f(x, y)
\]

and

\[
(3.7) \quad \mathcal{D}_{N,\alpha,\beta}f(x) = \mathcal{E}_{N}(\alpha, \beta)f|_{x=y},
\]

for \( f \in C^\infty(\mathbb{R}^{2n}) \), where

\[
\varepsilon_{i,j,k}(\alpha, \beta) := \left( \begin{array}{c} N \\ i, j, k \end{array} \right) \frac{(-2)^k}{c_{i,j,k}(\alpha, \beta)}.
\]

We claim that

\[
(3.8) \quad \mathcal{D}_{N,\alpha,\beta}f = T_{N,\alpha,\beta}(J_\alpha \otimes J_\beta)f, \quad f \in \pi^\infty_\alpha \otimes \pi^\infty_\beta
\]

proving the intertwining property of the differential operator \( \mathcal{D}_{N,\alpha,\beta} \).

The binomial expansion of \( S(x, y; z, w) \) reads as follows

\[
S(x, y; z, w) = \left( \frac{|x-z|^2 + |y-w|^2 - 2 \langle x-z, y-w \rangle}{|x-z|^2|y-w|^2} \right)^N \frac{1}{|x-z|^{2\alpha}|y-w|^{2\beta}}
\]

\[
= \sum_{i+j+k=N} \left( \begin{array}{c} N \\ i, j, k \end{array} \right) (-2)^k \left( \frac{\langle x-z, y-w \rangle}{|x-z|^2|y-w|^2} \right)^k \frac{1}{|x-z|^{2i+2\alpha}|y-w|^{2j+2\beta}}
\]

Summing the formula (3.6) over \((i, j, k)\) we have then

\[
\mathcal{E}_{N}(\alpha, \beta) \frac{1}{|x-z|^{2\alpha}|y-w|^{2\beta}} = S(x, y; z, w)
\]

which further implies that

\[
(3.9) \quad \mathcal{D}_{N,\alpha,\beta} \frac{1}{|x-z|^{2\alpha}|y-w|^{2\beta}} = S(x, x; z, w) = S(x; z, w)
\]

The identity \( f = (J_\alpha \otimes J_\beta)(J_\bar{\alpha} \otimes J_\bar{\beta})f = (J_\alpha \otimes J_\beta)g \) reads

\[
f(x, y) = (J_\alpha \otimes J_\beta)g = C_\alpha C_\beta \int_{\mathbb{R}^{2n}} \frac{1}{|x-z|^{2\alpha}|y-w|^{2\beta}} g(z, w) dz dw.
\]

We perform the differentiation \( \mathcal{D}_{N,\alpha,\beta} \) on this identity and find

\[
\mathcal{D}_{N,\alpha,\beta}f(x) = C_\alpha C_\beta \int_{\mathbb{R}^{2n}} S(x; z, w) g(z, w) dz dw = T_N g(x) = T_N J_\bar{\alpha} \otimes J_\bar{\beta} f(x).
\]

Finally the differentiation under integral sign can be justified by taking first \( \alpha, \beta << 0 \) and \( \alpha \notin \mathbb{Z}_-, \beta \notin \mathbb{Z}_-, \) with \( \bar{\alpha} >> 0, \bar{\beta} >> 0 \) and Lemma 2.1 implies that all integrals involved are absolutely convergent. \( \square \)
4. Discrete components in the tensor product decomposition

We apply the intertwining operators $D_N = D_{N, \alpha, \beta}$ to the study of appearance of discrete components in the tensor product $\pi_\alpha \otimes \pi_\beta$ of complementary series. For $\alpha, \beta \in (0, \rho)$ the tensor product $\pi_\alpha \otimes \pi_\beta$ in the non-compact picture is the completion of $C_0^\infty(\mathbb{R}^{2n})$ with the norm

$$\|f\|_\alpha^2 := \int_{\mathbb{R}^{2n}} |\mathcal{F} f(\xi, \eta)|^2 |\xi|^{n-2\alpha} |\eta|^{n-2\beta} d\xi d\eta,$$

in view of (2.10).

**Theorem 4.1.** Suppose $\alpha > 0, \beta > 0$ and $N \in \mathbb{N}$ satisfy $0 < \alpha < \rho, 0 < \beta < \rho, \alpha + \beta + 2N < \rho$. Then the intertwining operator $D_{N, \alpha, \beta}$ is a non-zero bounded intertwining operator $\pi_\alpha \otimes \pi_\beta \to \pi_{\alpha + \beta + 2N}$, and thus $\pi_{\alpha + \beta + 2N}$ appears in the tensor product $\pi_\alpha \otimes \pi_\beta$ as an irreducible component.

**Proof.** Note that for $\alpha, \beta$ and $N$ as above we have that the operator $D_N$ is well-defined, and $\pi_\alpha, \pi_\beta$ and $\pi_{\alpha + \beta + 2N}$ are unitary representations. Let $f \in C_0^\infty(\mathbb{R}^{2n}) \subset \pi_\alpha \otimes \pi_\beta$. We claim that

$$\|D_N f\|_{\alpha + \beta + 2N}^2 \leq C \|f\|_{\alpha \otimes \beta}^2.$$

Thus $D_N$ defines a non-zero intertwining operator from $\pi_\alpha \otimes \pi_\beta$ into $\pi_{\alpha + \beta + 2N}$, proving our theorem. Using Fourier inversion we have

$$f(x, y) = C \int_{\mathbb{R}^{2n}} e^{i(x, \xi) + i(y, \eta)} \mathcal{F} f(\xi, \eta) d\xi d\eta$$

where $C$ is a normalization constant. We write the differential operator $\mathcal{E}_N(\alpha, \beta)$ in the proof of Theorem 3.1 as $Q(\mathcal{L}_x, \mathcal{L}_y, \nabla_x \cdot \nabla_y)$ where $Q$ is a homogeneous polynomial of three variables of degree $N$. Thus $D_N f(x) = Q(\mathcal{L}_x, \mathcal{L}_y, \nabla_x \cdot \nabla_y) f(x, y)|_{x=y}$. Its action on the inversion formula results in

$$D_N f(x) = C \int_{\mathbb{R}^{2n}} e^{i(x, \xi) + i(y, \eta)} Q(-|\xi|^2, -|\eta|^2, -\langle \xi, \eta \rangle) \mathcal{F} f(\xi, \eta) d\xi d\eta$$

$$= C \int_{\mathbb{R}^n} e^{i(x, \zeta)} \int_{\mathbb{R}^n} Q(-|\zeta - \eta|^2, -|\eta|^2, -\langle \zeta - \eta, \eta \rangle) \mathcal{F} f(\zeta - \eta, \eta) d\eta d\zeta.$$

That is

$$\mathcal{F}(D_N f)(\zeta) = C \int_{\mathbb{R}^n} Q(-|\zeta - \eta|^2, -|\eta|^2, -\langle \zeta - \eta, \eta \rangle) \mathcal{F} f(\zeta - \eta, \eta) d\eta,$$

and furthermore

$$|\mathcal{F}(D_N f)(\zeta)|^2 \leq A(\zeta) \int_{\mathbb{R}^n} |\mathcal{F} f(\zeta - \eta, \eta)|^2 |\zeta - \eta|^{2\alpha} |\eta|^{2\beta} d\eta,$$

with

$$A(\zeta) := C \int_{\mathbb{R}^n} |Q(-|\zeta - \eta|^2, -|\eta|^2, -\langle \zeta - \eta, \eta \rangle)|^2 |\zeta - \eta|^{2\alpha} |\eta|^{2\beta} d\eta.$$
To estimate the integral $A(\zeta)$ we write $\zeta = |\zeta|u$, $|u| = 1$, and perform the change of variables $\eta = |\zeta|v$. It is

$$A(\zeta) = C|\zeta|^{4N - 2\tilde{\alpha} - 2\tilde{\beta} + n}\int_{\mathbb{R}^n} |Q(-|u - v|^2, -|v|^2, -(u - v, v)))|^2|u - v|^{-2\tilde{\alpha}}|v|^{-2\tilde{\beta}}dv$$

and the integral is convergent since it is locally integrable near $v = 0$, and $v = u$ for $2\tilde{\alpha}, 2\tilde{\beta} < n$ and is integrable at infinity since the integrand is dominated by

$$(1 + |v|^2)^{-(\tilde{\alpha} + \tilde{\beta} - 2N)}$$

with $\tilde{\alpha} + \tilde{\beta} - 2N = n + (n - \alpha - \beta - 2N) < n$. Thus

$$|F(D_N f)(\zeta)|^2|\zeta|^{-4N + 2\tilde{\alpha} + 2\tilde{\beta} - n} \leq C \int_{\mathbb{R}^n} |F f(\zeta - \eta, \eta)|^2|\zeta - \eta|^{2\tilde{\alpha}}|\eta|^{2\tilde{\beta}}d\eta,$$

and its integration over $\zeta$ gives

$$\int_{\mathbb{R}^n} |F(D_N f)(\zeta)|^2|\zeta|^{-4N + 2\tilde{\alpha} + 2\tilde{\beta} - n}d\zeta \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F f(\zeta - \eta, \eta)|^2|\zeta - \eta|^{2\tilde{\alpha}}|\eta|^{2\tilde{\beta}}d\eta = C\|f\|^{2}_{\alpha \otimes \beta}$$

whereas the LHS is precisely $\|D_N f\|^2_{\alpha + \beta + 2N}$. This finishes the proof. $\square$

When $n = 1$ then $N = 0$ and the theorem states that $\pi_{\alpha + \beta}$ appears in the tensor product $\pi_{\alpha} \otimes \pi_{\beta}$ if $\alpha + \beta < 1$. This has been proved earlier in [15].

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Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden

E-mail address: genkai@chalmers.se