ABOUT THE QWEP CONJECTURE

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ABSTRACT. This is a detailed survey on the QWEP conjecture and Connes’ embedding problem. Most of contents are taken from Kirchberg’s paper [Invent. Math. 112 (1993)].

1. Introduction

Following Kirchberg [Ki1], we prove that several important conjectures arising in several branches of operator algebras are in fact equivalent.

Theorem 1.1. The following conjectures are equivalent.

(i) We have $C^*F_{\infty} \otimes_{\max} C^*F_{\infty} = C^*F_{\infty} \otimes_{\min} C^*F_{\infty}$.

(ii) The predual of any (separable) von Neumann algebra is finitely representable in the trace class $S_1$.

(iii) Any separable II$_1$-factor is a subfactor of the ultrapower $R^\omega$ of the hyperfinite II$_1$-factor $R$.

Let $A$ and $B$ be $C^*$-algebras and consider the problem to introduce a norm on their algebraic tensor product $A \otimes B$. Although there are presumably many $C^*$-norms on the $*$-algebra $A \otimes B$, there are two distinguished norms; the minimal one and the maximal one. (A rather surprising theorem of Takesaki [Ta1] states that the minimal tensor norm is indeed the smallest $C^*$-norm on $A \otimes B$.) These $C^*$-norms have nice functorial properties; both tensor norms tensorize completely positive maps and persist in passing to the second dual, and moreover, the minimal tensor norm is injective while the maximal tensor norm is projective. Thus, it is natural to ask whether or not the minimal (resp. maximal) tensor norm is the unique $C^*$-norm with these functoriality. Since injective (resp. projective) $C^*$-norm comes from a fixed $C^*$-norm on $B(\ell_2) \otimes B(\ell_2)$ (resp. $C^*F_{\infty} \otimes C^*F_{\infty}$), this problem is equivalent to asking whether there is only one $C^*$-norm on $B(\ell_2) \otimes B(\ell_2)$ (resp. $C^*F_{\infty} \otimes C^*F_{\infty}$). Probably led by this consideration (cf. [Ki4]), Kirchberg conjectured that the answer would be yes for both cases. However, the problem

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on $\mathcal{B}(\ell_2) \otimes \mathcal{B}(\ell_2)$ was solved negatively by Junge and Pisier [JP]. The problem on $C^*F_\infty \otimes C^*F_\infty$ remains outstanding.

The conjecture (ii) seems old (cf. [HP]). Since several analytical structures of Banach spaces depend only on those of finite dimensional subspaces, it is desirable to know whether any noncommutative $L_1$-spaces are finitely representable in the trace class $S_1$. Many problems arising in operator spaces and noncommutative $L_p$ spaces are also connected to finite representability (as an operator space, e.g. [Ju], [EJR], [JR]).

The problem (iii) was casually raised by Connes [Co] and is drawing increased attention in recent years in connection with free probability theory. Haagerup [Ha3] showed a remarkable theorem that invariant subspaces exist for a large class of operators in II$_1$-factors that are embeddable in $R^\omega$. Thus, a positive answer would nicely complement Haagerup’s theorem. The problem (iii) is equivalent to that microstates always exist. Microstates are used to define free entropy $\chi$ introduced by Voiculescu [Vo1]. Besides its own interest, free entropy has a number of important applications to theory of von Neumann algebras. Voiculescu [Vo2] later introduced another free entropy $\chi^*$ and asked the “unification problem” whether they actually coincide. A negative answer to (iii) would imply this is not the case. See Voiculescu’s survey [Vo3] for details.

The problem (iii) may well be connected to geometric group theory in deep and important ways. A positive answer would imply that all countable discrete groups are hyperlinear and hence would refute the famous “theorem” of Gromov claiming that a proposition which holds for all countable discrete groups is either trivial or false. Here, we say a group is hyperlinear if it is embeddable into the unitary group $U(R^\omega)$ of $R^\omega$. The class of hyperlinear groups is closed under several natural operations and contains all amenable groups and all residually finite groups. Since many exotic groups (or “monsters”), such as periodic simple groups with Kazhdan’s property (T) and non-uniformly embeddable groups of Gromov [Gr1], [Gr2], are arising as limits of hyperbolic groups, it is particularly interesting to know whether all hyperbolic groups are hyperlinear. In particular, if such simple property (T) groups or non-uniformly embeddable groups are not hyperlinear, then there would exist non-hyperlinear hyperbolic groups. Whether all hyperbolic groups are residually finite (and hence hyperlinear) is one of the major open problem in geometric group theory.

It is not completely understood for which $C^*$-algebra $A$ the semigroup $\text{Ext}(A)$ is actually a group. Kirchberg [Ki1] showed the Ext semigroup of the cone (or the suspension) over $A$ is a group if and only if $A$ has the LLP (local lifting property). Since a positive answer would imply that exact and non-nuclear $C^*$-algebras cannot have the LLP, it would follow $\text{Ext}(\text{cone}(A))$ (and $\text{Ext}(S(A))$) never a group when $A$ is exact and non-nuclear.
Proof of Theorem 1.1. The equivalence (i) ⇔ (ii) follows from Propositions 3.19, 4.1 and Corollary 5.3. The implication (ii) ⇒ (iii) follows from Corollary 6.2. Now, we assume (iii) and prove (i). The assumption (iii) is equivalent to that all extremal traces (and a fortiori all traces) of $C^*F_\infty$ satisfy the conditions in Theorem 6.1. This implies that all separable von Neumann algebras of type II$_1$ are QWEP (cf. Corollary 6.2). We complete the proof by Proposition 4.1 and the well-known Takesaki’s theorem [Ta2] that any separable von Neumann algebra is *-isomorphic to a cp complemented von Neumann subalgebra in a separable semifinite von Neumann algebra. □

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Notations. Unless otherwise stated, all $C^*$-algebras in this paper are assumed unital (except for ideals) and maps between them are assumed linear. We will write $A, B, C, \ldots$ for unital $C^*$-algebras, $J, \ldots$ for ideals (which always mean closed two-sided ideals) in $C^*$-algebras, $M, N, \ldots$ for von Neumann algebras, $R$ for the hyperfinite II$_1$-factor, $F$ (resp. $F_\infty$) for a free group (resp. on countably many generators), $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{K}(\mathcal{H})$) for the $C^*$-algebra of all bounded linear (resp. compact) operators on a Hilbert space $\mathcal{H}$, $\mathbb{M}_n$ for the $n$ by $n$ full matrix algebra.

2. Preliminary Background

In this section, we collect some basic results on positive maps. We will omit proofs if they are found in standard references.

For $C^*$-algebras $A$ and $B$, we denote by $A \otimes B$ (resp. $A \otimes_{\min} B$, $A \otimes_{\max} B$) the algebraic (resp. minimal, maximal) tensor product. For von Neumann algebras $M$ and $N$, we denote by $M \bar{\otimes} N$ the von Neumann tensor product. For the definitions and basic properties of these tensor products, we refer the reader to Chapter IV in Takesaki’s book [Ta1] and Appendix (T) in Wegge-Olsen’s book [We].

A general reference for the positive maps is Sections 2 and 3 in Paulsen’s book [Pa1]. For simplicity, we only deal with unital $C^*$-algebras and unital maps, although this restriction is inessential.

Lemma 2.1. A unital map between unital $C^*$-algebras is positive if and only if it is contractive. A positive map from a commutative $C^*$-algebra into a (possibly noncommutative) $C^*$-algebra is automatically cp.

For the proof, see Corollary 2.9, Proposition 3.6 and Theorem 3.11 in [Pa1]. The following two theorems are fundamental. The first one is due to Arveson [Ar1] for cp maps and Wittstock for cb maps. Recall that a unital self-adjoint subspace of a $C^*$-algebra is called an operator system. This theorem means that $\mathbb{B}(\mathcal{H})$ is
in the category of operator systems with ucp maps (resp. operator spaces with complete contractions).

**Theorem 2.2.** Let $X \subset B$ be an operator space. Then any cb map $\varphi : X \to \mathcal{B}(\mathcal{H})$ extends to a cb map $\tilde{\varphi} : B \to \mathcal{B}(\mathcal{H})$ with $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. In particular, a ucp map from an operator system into $\mathcal{B}(\mathcal{H})$ extends to a ucp map.

The second one is the Stinespring dilation theorem.

**Theorem 2.3.** Let $\varphi : A \to \mathcal{B}(\mathcal{H})$ be a ucp map. Then, there is a representation $\pi : A \to \mathcal{B}(\hat{\mathcal{H}})$ and an isometry $V : \mathcal{H} \to \hat{\mathcal{H}}$ such that $\pi(A)V\mathcal{H}$ is dense in $\hat{\mathcal{H}}$ and that $\varphi(a) = V^*\pi(a)V$ for $a \in A$. Moreover, the triple $(\pi, \hat{\mathcal{H}}, V)$ is unique up to unitary equivalence.

**Corollary 2.4.** If $\varphi : A \to B$ is a ucp map, then $\varphi \otimes \text{id}_C : A \otimes C \to B \otimes C$ is continuous w.r.t. the minimal (resp. maximal) tensor norms.

**Proof.** We only prove this corollary for the maximal tensor product. Take a faithful representation $B \otimes_{\text{max}} C \subset \mathcal{B}(\mathcal{H})$ and let $(\pi, \hat{\mathcal{H}}, V)$ be the Stinespring triplet for $\varphi : A \to \mathcal{B}(\mathcal{H})$. We claim that for $x \in C$, the operator $\rho(x)$ defined by

$$\rho(x) : \pi(A)V\mathcal{H} \ni \sum_j \pi(a_j)V\xi_j \longmapsto \sum_j \pi(a_j)Vx\xi_j \in \hat{\mathcal{H}}$$

is well-defined and $\|\rho(x)\| \leq \|x\|$. Indeed, putting $\xi = [\xi_1, \ldots, \xi_n]^T \in \ell_2^n \otimes \mathcal{H}$, we have

$$\|\sum_j \pi(a_j)Vx\xi_j\|^2 = \sum_{i,j} (\varphi(a_i^*a_j)x\xi_i, x\xi_j)$$

$$= ((1 \otimes x)[\varphi(a_i^*a_j)](1 \otimes x)\xi, \xi)$$

$$\leq \|x\|^2([\varphi(a_i^*a_j)]\xi, \xi) = \|x\|^2\|\sum_j \pi(a_j)V\xi_j\|^2,$$

where we used the fact that $[\varphi(a_i^*a_j)]_{i,j} \in \mathcal{M}_n(B)$ is positive and commutes with $1 \otimes x$ for $x \in C \subset B'$. Since $\pi(A)V\mathcal{H} \subset \hat{\mathcal{H}}$ is dense, $\rho$ gives rise to a representation of $C$ on $\hat{\mathcal{H}}$ whose range commutes with $\pi(A)$. Hence, $\pi \otimes \rho$ is a representation of $A \otimes_{\text{max}} C$ such that $(\varphi \otimes \text{id}_C)(a \otimes x) = V^*(\pi \otimes \rho)(a \otimes x)V$ for $a \in A$ and $x \in C$. This completes the proof. The above construction is taken from [Ar]. □

The Jordan product $\circ$ on a $C^*$-algebra $A$ is defined as $a \circ b = (ab + ba)/2$ for $a$ and $b$ in $A$. This equips $A$ with a commutative non-associative $*$-algebra structure. A self-adjoint subspace in a $C^*$-algebra which is closed under the Jordan product will be simply called a Jordan algebra. A positive linear map between Jordan algebras is called a Jordan morphism if it preserves the Jordan product. Jordan morphisms are necessarily contractive (cf. Lemma 2.1).

The following Schwarz type inequalities are due to Kadison [Ka] for positive maps and Choi [Ch1] for $\text{cp}$ maps.
Corollary 2.5. Let \( \varphi : A \to \mathbb{B}(\mathcal{H}) \) be a ucp (resp. unital positive) map. Then, we have \( \varphi(a^*a) \geq \varphi(a)^*\varphi(a) \) (resp. \( \varphi(a^*a) \geq \varphi(a)^*\varphi(a) \)) for every \( a \in A \).

Proof. First, let \( \varphi \) be a ucp map and \( (\pi, \hat{\mathcal{H}}, V) \) be the Stinespring triplet. Then, we have
\[
\varphi(a^*a) - \varphi(a)^*\varphi(a) = V^*\pi(a)^*(1 - VV^*)\pi(a)V \geq 0
\]
for every \( a \in A \). This proves the case where \( \varphi \) is cp.

Now, let \( \varphi \) be a unital positive map. It follows from the cp case that \( \varphi(b^2) \geq \varphi(b)^2 \) for every self-adjoint element \( b \) in \( A \) since the restriction of \( \varphi \) to the commutative \( C^* \)-subalgebra generated by \( b \) is automatically cp. Hence, denoting by \( Re a = (a^* + a)/2 \) and \( Im a = i(a^* - a)/2 \), we have
\[
\varphi(a^*a) = \varphi((Re a)^2 + (Im a)^2) \geq \varphi(Re a)^2 + \varphi(Im a)^2 = \varphi(a)^*\varphi(a)
\]
for every \( a \in A \).

\( \square \)

Corollary 2.6. Let \( \varphi : A \to \mathbb{B}(\mathcal{H}) \) be a ucp (resp. unital positive) map. For \( a \in A \), we have
\[
\varphi(a^*a) = \varphi(a)^*\varphi(a) \Rightarrow \varphi(xa) = \varphi(x)\varphi(a) \text{ for all } x \in A
\]
\[
\varphi(aa^*) = \varphi(a)^*\varphi(a) \Rightarrow \varphi(ax) = \varphi(a)\varphi(x) \text{ for all } x \in A
\]
(resp. \( \varphi(a^*a) = \varphi(a)^*\varphi(a) \Rightarrow \varphi(x \circ a) = \varphi(x) \circ \varphi(a) \text{ for all } x \in A \)).

Moreover, the subspace \( C = \{ a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a), \varphi(aa^*) = \varphi(a)^*\varphi(a) \} \) (resp. \( C = \{ a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a) \} \)) is a \( C^* \)-subalgebra (resp. a Jordan subalgebra) of \( A \).

Proof. We denote by \( \cdot \) the usual product (resp. the Jordan product) throughout this proof. Let \( a \in A \) be so that \( \varphi(a^*a) = \varphi(a)^*\varphi(a) \) and \( a \in A \) be arbitrary. By Corollary 2.5, we have for any \( t \in \mathbb{R} \),
\[
0 \leq \varphi((ta + x)^* \cdot (ta + x)) - \varphi((ta + x)^* \cdot (ta + x)) = \varphi((ta + x)^* \cdot (ta + x)) + \varphi((ta + x)^* \cdot (ta + x)) - \varphi((ta + x)^* \cdot (ta + x))
\]
\[
\quad = t\{ \varphi(a^* \cdot x + x^* \cdot a) - \varphi(a)^* \cdot \varphi(x) - \varphi(x)^* \cdot \varphi(a) \} + \varphi((ta + x)^* \cdot (ta + x)) - \varphi((ta + x)^* \cdot (ta + x))
\]
Since \( t \in \mathbb{R} \) is arbitrary, it follows that \( \varphi(a^* \cdot x + x^* \cdot a) = \varphi(a)^* \cdot \varphi(x) + \varphi(x)^* \cdot \varphi(a) \).
Replacing \( x \) with \( ix \), we obtain \( \varphi(a^* \cdot x - x^* \cdot a) = \varphi(a)^* \cdot \varphi(x) - \varphi(x)^* \cdot \varphi(a) \).
Combining them, we obtain \( \varphi(x^* \cdot a) = \varphi(x)^* \cdot \varphi(a) \) for all \( x \in A \). This proves the first half.

We only prove that \( C \) is a Jordan subalgebra for unital positive \( \varphi \). It is clear that \( C \) is a self-adjoint subspace of \( A \). Hence, it is sufficient to show that \( C_{sa} = \{ a \in C : a = a^* \} \) is closed under the Jordan product. Since the usual product and the Jordan product coincide on a commutative subalgebra, we have \( a^2 \in C_{sa} \) provided that \( a \in C_{sa} \). Now, the claim that \( a \circ b \in C_{sa} \) for any \( a, b \in C_{sa} \) follows from the first half of this corollary and the equation
\[
(a \circ b)^2 = a \circ (b \circ (a \circ b)) + \frac{1}{2}(a^2 \circ b) \circ b - \frac{1}{2}a^2 \circ b^2.
\]
This completes the proof. \hfill \square

**Definition 2.7.** We say the subalgebra $C$ in Corollary 2.6 is the *multiplicative domain* for $\varphi$.

By Corollary 2.3, $a \in A$ is in the multiplicative domain for $\varphi$ provided that $\|a\| = 1$ and $\varphi(a)$ is a unitary. Therefore, if $\varphi : A \to B$ is a ucp (resp. unital positive) map which maps the closed unit ball of $A$ onto the closed unit ball of $B$, then $\varphi$ is surjective on the multiplicative domain for $\varphi$.

The following important corollary will be used frequently without mention.

**Corollary 2.8.** Let $A \subset B$ and let $\varphi : B \to \mathbb{B}(H)$ be a ucp map such that the restriction of $\varphi$ to $A$ is a $\ast$-homomorphism $\pi$, then $\varphi$ is an $A$-bimodule map, i.e., $\varphi(axb) = \pi(a)\varphi(x)\pi(b)$ for $a,b \in A$ and $x \in B$.

A map $\varphi$ on $B$ is called a *projection* if $\varphi^2 = \varphi$. The following theorem is due to Tomiyama [To]. A simple proof is found in Section 9 in Strătilă’s book [St] and Chapter IX in Takesaki’s book [Ta2].

**Theorem 2.9.** For $A \subset B$ and a projection $\varphi$ from $B$ onto $A$, the following are equivalent.

(i) The map $\varphi$ is a conditional expectation, i.e., the map $\varphi$ is an $A$-bimodule map; $\varphi(axb) = a\varphi(x)b$ for $a,b \in A$ and $x \in B$.

(ii) The map $\varphi$ is cp.

(iii) The map $\varphi$ is contractive.

We need the structure theorem for Jordan morphisms due to Størmer [HS].

**Theorem 2.10.** Let $A$ be a $C^*$-algebra and $\theta : A \to \mathbb{B}(H)$ be a Jordan morphism. Then, there is a central projection $p$ in the von Neumann algebra $M = \theta(A)''$ generated by $\theta(A)$ such that the map $A \ni a \mapsto \theta(a)p \in Mp$ (resp. $A \ni a \mapsto \theta(a)(1-p) \in M(1-p)$) is a $\ast$-homomorphism (resp. $\ast$-antihomomorphism).

We review some operator space theory. We refer the reader to the books of Effros and Ruan [ER2] and of Pisier [Pi2]. Many important results in operator space theory are related to the dual operator space structure introduced by Blecher-Paulsen and Effros-Ruan [Bl], [BP], [ER1].

Let $X \subset \mathbb{B}(H)$ be an operator space and let $X^*$ be its dual Banach space. For $x = [x_{ij}]_{i,j} \in M_n(X)$, we define $\theta_x : X^* \ni f \mapsto [f(x_{ij})]_{i,j} \in M_n$. Let $\Omega$ be the union of the closed unit balls of $M_n(X)$ ($n = 1, 2, \ldots$). We introduce an operator space structure on $X^*$ by the isometric inclusion

$$\Theta : X^* \ni f \mapsto (\theta_x(f))_{x \in \Omega} \in \prod_{x \in \Omega} M_n(x).$$

Unless otherwise stated, we always assume that the dual space $X^*$ is equipped with this operator space structure. (We note that $M_n(X^{**}) = M_n(X)^{**}$ always
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It holds isometrically, but \( M_n(X^*) = M_n(X)^* \) is not isometric. It is not hard to see \( X^* \otimes_{\min} M_n = M_n(X^*) = \text{CB}(X, M_n) \) isometrically. It follows that for operator spaces \( X \) and \( Y \), there is a canonical isometric inclusion defined by

\[
X^* \otimes_{\min} Y \ni \sum_k f_k \otimes y_k \mapsto \sum_k f_k(\cdot) y_k \in \text{CB}(X,Y).
\]

The predual \( M_* \) of a von Neumann algebra \( M \) is equipped with the operator space structure induced from that of \( M^* \). It is proved in [Bl] that \( M = (M_*)^* \) completely isometrically as one should expect.

3. WEP AND LLP

Definition 3.1. We say a \( C^* \)-subalgebra \( A \) in \( B \) is cp complemented (resp. weakly cp complemented) in \( B \) if there is a a ucp map \( \varphi : B \to A \) (resp. \( \varphi : B \to A^{**} \)) such that \( \varphi|_A = \text{id}_A \).

We say a \( C^* \)-algebra \( B \) has the WEP (weak expectation property), if it is weakly cp complemented in \( \mathbb{B}(\mathcal{H}) \) for a faithful representation \( B \subset \mathbb{B}(\mathcal{H}) \).

Since \( \mathbb{B}(\mathcal{H}) \) is injective, the definition of the WEP does not depend on choice of faithful representation of \( B \). We say a \( C^* \)-algebra is QWEP if it is a quotient of a \( C^* \)-algebra with the WEP. The QWEP conjecture states that all \( C^* \)-algebras are QWEP.

If \( M \subset N \) are von Neumann algebras with a faithful normal trace \( \tau \) on \( N \), then there is a unique trace preserving conditional expectation \( \Phi \) from \( N \) onto \( M \) defined by the relation \( \tau(a\Phi(x)) = \tau(ax) \) for \( a \in M \) and \( x \in M \). In particular, \( M \) is cp complemented in \( N \). Indeed, this follows from the fact that linear functionals on \( M \) of the form \( \tau(a \cdot) \) with \( a \in M \) are dense in \( M_* \). Let \( \Delta \) be a subgroup of \( \Gamma \). Then, the \( C^* \)-subalgebra in the full \( C^* \)-algebra \( C^* \Gamma \) generated by \( \Delta \) is naturally \( * \)-isomorphic to \( C^* \Delta \). Indeed, this follows from existence of induction. Moreover, \( C^* \Delta \) is cp complemented in \( C^* \Gamma \) by the conditional expectation \( \varphi \) defined by \( \varphi(s) = 0 \) for \( s \in \Gamma \setminus \Delta \).

Lemma 3.2. For \( C^* \)-algebras \( A \subset B \), the following are equivalent.

(i) The \( C^* \)-algebra \( A \) is weakly cp complemented in \( B \).

(ii) The second dual \( A^{**} \) is cp complemented in \( B^{**} \).

(iii) For any finite dimensional subspace \( E \subset B \) and any \( \varepsilon > 0 \), there exists a map \( \varphi : E \to A \) such that \( ||\varphi|| \leq 1 + \varepsilon \) and \( \varphi|_{E \cap A} = \text{id}_{E \cap A} \).

In particular, if \( A \) is weakly cp complemented in \( B \) and \( B \) has the WEP, then so does \( A \). If \( (A_i)_{i \in I} \) is a family of \( C^* \)-algebras with the WEP, then \( \prod A_i \) has the WEP.

Proof. The equivalence (i) \( \Leftrightarrow \) (ii) is a consequence of the fact that any bounded linear map \( \varphi \) from a Banach space \( X \) into a dual Banach space \( Y = (Y_*)^* \) uniquely extends to a weak*-continuous map \( \tilde{\varphi} \) from \( X^{**} \) into \( Y \). The implication (i) \( \Rightarrow \) (iii)
follows from the principle of local reflexivity for Banach spaces. To prove (iii) \( \Rightarrow \) (ii), let \( I \) be the set of pairs \( (E, \varepsilon) \), where \( E \subset B \) is finite dimensional and \( \varepsilon > 0 \). The set \( I \) is directed by the order relation \( (E_1, \varepsilon_1) \leq (E_2, \varepsilon_2) \) if and only if \( E_1 \subset E_2 \) and \( \varepsilon_1 \geq \varepsilon_2 \). By the condition (iii), for each \( i = (E, \varepsilon) \in I \), there exists a map \( \varphi_i : B \to A^{**} \) such that \( \|\varphi_i|_E\| \leq 1 + \varepsilon \) and \( \varphi_i|_{E \cap A} = \text{id}_{A^{**}} \) of the net \( \{\varphi_i\}_{i \in I} \), in the point-weak* topology, satisfies that \( \|\varphi\| \leq 1 \) and \( \varphi|_A = \text{id}_A \). It follows from Theorem 2.9 that the weak*-continuous extension \( \hat{\varphi} \) of \( \varphi \) on \( B^{**} \) is a conditional expectation from \( B^{**} \) onto \( A^{**} \).

Finally, we observe that the condition (iii) is stable under direct product. \( \square \)

The notion of WEP was introduced by Lance [La1] where he showed in the following theorem, which in particular shows nuclear \( C^* \)-algebras have the WEP.

**Theorem 3.3.** Let \( A \subset B \). Then, \( A \) is weakly cp complemented in \( B \) if and only if \( A \otimes_{\text{max}} C \subset B \otimes_{\text{max}} C \) isometrically for any \( C^* \)-algebra \( C \) (resp. for \( C = C^* F_\infty \) or \( C = A^{\text{op}} \)).

**Proof.** We note that the canonical map \( A \otimes_{\text{max}} C \subset A^{**} \otimes_{\text{max}} C \) is always isometric and the canonical map \( \iota \otimes \text{id}_C : A \otimes_{\text{max}} C \to B \otimes_{\text{max}} C \) is always contractive. Hence, the “only if” part follows from Corollary 2.4. To prove “if” part, let \( A \subset B \) be the universal representation, i.e., \( A' = A^{**} \), and let \( C \) be a \( C^* \)-algebra which has a representation \( \pi : C \to B(\mathcal{H}) \) with \( \pi(C)' \) isometrically into \( A' \). By the assumption, the representation \( A \otimes_{\text{max}} C \) on \( B(\mathcal{H}) \) extends to a ucp map \( \Phi : B \otimes_{\text{max}} C \to B(\mathcal{H}) \). Then, the map \( \varphi : B \ni b \mapsto \Phi(b \otimes 1) \in B(\mathcal{H}) \) is a ucp extension of \( \text{id}_A \). Since \( \Phi \) is a \( C \)-bimodule map (cf. Corollary 2.8), we have \( \varphi(b)\pi(x) = \Phi(b \otimes x) = \pi(x)\varphi(b) \) for \( b \in B \) and \( x \in C \), i.e., \( \varphi(b) \in \pi(C)' = A^{**} \). This completes the proof. \( \square \)

The following proposition reduces many problems to that for separable \( C^* \)-algebras.

**Proposition 3.4.** Let \( B \) be a (non-separable) \( C^* \)-algebra and let \( X \subset B \) be a separable subspace. Then, there is a separable \( C^* \)-subalgebra \( A \) which contains \( X \) and is weakly cp complemented in \( B \).

**Proof.** Let \( C = C^* F_\infty \) and let \( (B_i)_{i \in I} \) be the directed set of all separable \( C^* \)-subalgebras in \( B \) ordered by inclusions. We claim that \( \|x\|_{B \otimes_{\text{max}} C} = \lim_{i \to \infty} \|x\|_{B_i \otimes_{\text{max}} C} \) for every \( x \in B \otimes C \). Note that the limit in R.H.S. always exists since the net is decreasing and that the inequality \( \leq \) is clear. To prove the converse inequality, suppose that \( \|x\|_{B_i \otimes_{\text{max}} C} \geq 1 \) for all \( i \in I \). It follows that for every \( i \in I \), there are commuting representations \( \pi_i : B_i \to B(\mathcal{H}_i) \) and \( \rho_i : C \to B(\mathcal{H}_i) \) such that \( \|(\pi_i \cdot \rho_i)(x)\|_{B(\mathcal{H}_i)} \geq 1 \). Let \( \omega \) be a nontrivial ultrafilter on the directed set \( I \) and let \( \mathcal{H}_\omega \) be the ultraproduct Hilbert space of \( (\mathcal{H}_i)_{i \in I} \). Then, the nets \( (\pi_i)_{i \in I} \) and \( (\rho_i)_{i \in I} \) give rise to commuting representations \( \pi_\omega : B \to B(\mathcal{H}_\omega) \) and \( \rho_\omega : C \to B(\mathcal{H}_\omega) \). Therefore, we have \( \|x\|_{B \otimes_{\text{max}} C} \geq \|(\pi_\omega \cdot \rho_\omega)(x)\|_{B(\mathcal{H}_\omega)} \geq 1 \), which proves the claim.
Let $A_0 \subset B$ be a separable $C^*$-subalgebra containing $X$. Using the first part of this proof recursively, we can find an increasing sequence $(A_n)_{n=0}^\infty$ of separable $C^*$-subalgebras in $B$ such that $\|x\|_{B \otimes_{\max} C} = \|x\|_{A_n \otimes_{\max} C}$ for every $n \in \mathbb{N}$ and $x \in A_{n-1} \otimes C$. Let $A$ be the closure of $\bigcup_{n=1}^\infty A_n$. Then, $A$ is a separable $C^*$-subalgebra such that $A \otimes_{\max} C \subset B \otimes_{\max} C$ isometrically. Hence, by Theorem 3.8, $A$ is weakly cp complemented in $B$.

**Definition 3.5.** Let $\varphi: A \to B/J$ be a ucp map. We say $\varphi$ is ucp liftable if there is a ucp lifting $\psi: A \to B$, i.e., there is a ucp map $\psi$ such that $\varphi = \pi \psi$ for the quotient $\pi$ from $B$ onto $B/J$. We say $\varphi$ is locally ucp liftable if for any finite dimensional operator system $E$ in $A$, there is a ucp lifting $\psi: E \to B$.

We say $A$ has the LP (lifting property) (resp. the LLP (local lifting property)) if any ucp map from $A$ into any quotient $C^*$-algebra $B/J$ is (resp. locally) ucp liftable.

The following Effros-Haagerup lifting theorem [EH] characterizes those ucp maps which are locally ucp liftable.

**Theorem 3.6.** A ucp map $\varphi: A \to B/J$ is locally ucp liftable if and only if $\varphi \otimes \text{id}: A \otimes \mathcal{B}(\ell_2) \to (B \otimes_{\min} \mathcal{B}(\ell_2))/(J \otimes_{\min} \mathcal{B}(\ell_2))$ is continuous w.r.t. the minimal tensor norm on $A \otimes \mathcal{B}(\ell_2)$.

Existence of completely contractive local lifting immediately follows from operator space duality explained in Section 2, but making the lifting completely positive requires a technical lemma. See [EH] or [Wa] for the proof of the following lemma.

**Lemma 3.7.** Let $\varphi: E \to B/J$ be a ucp map from a finite dimensional operator system into a quotient $C^*$-algebra. Suppose that for any $\varepsilon > 0$, there is a lifting $\psi: E \to B$ with $\|\psi\|_{\text{cb}} \leq 1 + \varepsilon$. Then, $\varphi$ is ucp liftable.

The $C^*$-algebra $\mathcal{B}(\ell_2)$ is universal in the sense that it contains all separable $C^*$-algebras and is injective. The other universal thing is the full $C^*$-algebra $C^*\mathbb{F}_\infty$ of the free group $\mathbb{F}_\infty$ on countably many generators since any separable $C^*$-algebra is a quotient of it and it has the LP.

**Theorem 3.8.** The full $C^*$-algebra $C^*\mathbb{F}$ of a countable free group $\mathbb{F}$ has the LP.

**Proof.** This proof is taken from [Ki2]. We first show that a $*$-homomorphism $\theta: C^*\mathbb{F} \to B/J$ is ucp liftable. To this end, let $x_1, x_2, \ldots \in B$ be a contractive liftings of $\theta(s_1), \theta(s_2), \ldots$, where $s_1, s_2, \ldots$ are the free generators of $\mathbb{F}$. Then, each $x_n$ dilates to a unitary

$$\hat{x}_n = \begin{bmatrix} x_n & (1 - x_n x_n^*)^{1/2} \\ (1 - x_n x_n^*)^{1/2} & -x_n^* \end{bmatrix} \in M_2(B).$$

By universality, there is a $*$-homomorphism $\rho: C^*\mathbb{F} \to M_2(B)$ with $\rho(s_n) = \hat{x}_n$. It is not hard to see that the $(1,1)$-corner of $\rho$ is a desired lifting of $\theta$. 


Now, let $\varphi : C^*F \to B/J$ be a ucp map. Since $F$ is countable, we may assume $B/J$ is separable. By the Kasparov-Stinespring dilation theorem \[La2\], there is a $*$-homomorphism $\theta : C^*F \to \mathcal{M}(\mathbb{K} \otimes_{\min} B/J)$ such that $\varphi(a) = \theta(a)_{11}$ for $a \in C^*F$, where $x_{11}$ is the $(1,1)$-entry of $x \in \mathcal{M}(\mathbb{K} \otimes_{\min} B/J)$. By the noncommutative Tietze extension theorem, the surjective $*$-homomorphism $\pi$ from $\mathbb{K} \otimes_{\min} B$ onto $\mathbb{K} \otimes_{\min} B/J$ extends to a surjective $*$-homomorphism $\tilde{\pi}$ between their multiplier algebras. Hence, by the first part of this proof, there is a ucp map $\rho : C^*F \to \mathcal{M}(\mathbb{K} \otimes_{\min} B)$ such that $\theta = \tilde{\pi}\rho$. The ucp map $\psi : C^*F \ni a \mapsto \rho(a)_{11} \in B$ is a desired lifting of $\varphi$. \[ \square \]

The above proof seems unreasonably involved. Hence it would be interesting to find another proof. (The LLP, rather than the LP, is sufficient in most of places and an independent proof of the LLP for $C^*F_\infty$ will be given in Theorem 3.14.)

We note that a map $\varphi$ from a discrete group $\Gamma$ into a $C^*$-algebra $A$ extends to a ucp map on the full $C^*$-algebra $C^*\Gamma$ if and only if it is unital and positive definite, i.e., $[\varphi(s_i^{-1}s_j)]_{i,j}$ is positive in $M_n(A)$ for any $n$ and any $s_1, \ldots, s_n \in \Gamma$.

Since countable noncommutative free groups are isomorphic to subgroups of each other, their distinction in our story is very minor. The full $C^*$-algebras of free groups (on uncountable generators) do have the LLP. It seems they do not have the LP, but it is not known (even for the free group of continuous cardinality). There is a (non-separable) $C^*$-algebra which has the LLP but not the LP. Indeed, the commutative $C^*$-algebra $\ell_\infty/c_0$ is such an example as there is no bounded linear lifting from $\ell_\infty/c_0$ into $\ell_\infty$. (Since $\ell_\infty/c_0$ contains a $C^*$-subalgebra $*$-isomorphic to $c_0(I)$ with an uncountable index set $I$, it cannot be embedded (as a Banach space) into $\ell_\infty$ which has a separable predual.) We will see that the LP and the LLP are equivalent for separable $C^*$-algebras provided that the QWEP conjecture is true.

**Corollary 3.9.** Let $A$ be a separable $C^*$-algebra and let $J \triangleleft C^*F_\infty$ be such that $A = C^*F_\infty/J$. Then, $A$ has the LP (resp. the LLP) if $\text{id}_A$ is (resp. locally) ucp liftable.

There is a criteria when a locally ucp liftable map has a global lifting. This is due to Arveson \[Ar2\] and its variant is due to Effros and Haagerup \[EH\]. The condition (ii) in this lemma is satisfied whenever $J$ is nuclear. The separability condition is essential as the quotient from $\ell_\infty$ onto $\ell_\infty/c_0$ is not globally liftable.

**Lemma 3.10.** Let $p : A \to B/J$ be a ucp map. Then, $\varphi$ is ucp liftable provided that $A$ is separable and either (i) or (ii) holds.

(i) The ucp map $\varphi$ can be approximated by ucp liftable maps in the point-norm topology.

(ii) The ucp map $\varphi$ is locally ucp liftable, and moreover for any finite dimensional operator systems $E \subset F \subset A$, any $\varepsilon > 0$ and for any cp map $\theta : E \to J$, there is a cp map $\hat{\theta} : F \to J$ with $\|\hat{\theta}|_E - \theta\| < \varepsilon$. 
Proof. Let $\pi$ be the quotient map from $B$ onto $B/J$ and let $E_1 \subset E_2 \subset \cdots \subset A$ be an increasing sequence of finite dimensional operator system with dense union. We first deal with the case (ii). Let $\psi_n : E_n \to B$ be ucp maps such that $\|\varphi(a) - \pi\psi_n(a)\| < 2^{-n}\|a\|$ for $a \in E_n$. We claim that there is a sequence of ucp maps $\tilde{\psi}_n : E_n \to B$ such that $\pi\tilde{\psi}_n = \pi\psi_n$ and $\|\psi_{n+1}(a) - \tilde{\psi}_n(a)\| < 2^{-n+2}\|a\|$ for $n \in \mathbb{N}$ and $a \in E_n$.

We proceed by induction. Suppose that $\tilde{\psi}_n$ is already constructed. Take $\varepsilon > 0$ sufficiently small. Since $\|\pi(\psi_{n+1}(a) - \tilde{\psi}_n(a))\| < 2^{-n+1}\|a\|$ for $a \in E_n$, we can find a quasicentral approximate unit $e \in J$ with $0 \leq e \leq 1$ such that

$$
\|(1 - e)^{1/2}(\psi_{n+1}(a) - \tilde{\psi}_n(a))(1 - e)^{1/2}\| < 2^{-n+1}\|a\|
$$

and moreover

$$
\|\tilde{\psi}_n(a) - (e^{1/2}\tilde{\psi}_n(a)e^{1/2} + (1 - e)^{1/2}\tilde{\psi}_n(a)(1 - e)^{1/2})\| < \varepsilon\|a\|
$$

for $a \in E_n$ (cf. [Ar2]). By the condition (ii), there is a cp map $\tilde{\theta} : E_{n+1} \to J$ such that $\|\tilde{\theta}(a) - e^{1/2}\tilde{\psi}_n(a)e^{1/2}\| < \varepsilon\|a\|$. We define $\tilde{\psi}_{n+1}$ by

$$
\tilde{\psi}_{n+1}(a) = b^{-1/2}(\tilde{\theta}(a) + (1 - e)^{1/2}\psi_{n+1}(a)(1 - e)^{1/2})b^{-1/2},
$$

where $b = \tilde{\theta}(1) + 1 - e \in B$. Since $\|b - 1\| < \varepsilon$, we have

$$
\|\tilde{\psi}_{n+1}(a) - \tilde{\psi}_n(a)\| < (f(\varepsilon) + 2^{-n+1})\|a\|
$$

for $a \in E_n$, where $f$ is a continuous function with $f(0) = 0$. This proves the claim.

The desired ucp lifting $\psi : A \to B$ is now obtained by letting (for $a \in \bigcup E_n$)

$$
\psi(a) = \lim_{n \to \infty} \tilde{\psi}_n(a).
$$

The proof for the case (i) is similar but easier than the case (ii). Indeed, $\psi_n$ and $\tilde{\psi}_n$ in the above proof are defined globally on $A$ (hence no need to choose $\tilde{\theta}$).

The following is the Choi-Effros lifting theorem [CE].

**Corollary 3.11.** Let $\varphi : A \to B/J$ be a ucp map. Then, $\varphi$ is ucp liftable provided that $A$ is separable and $\varphi$ is nuclear, i.e., there is a sequence of ucp maps $\beta_n : A \to \mathbb{M}_{k(n)}$ and $\alpha_n : \mathbb{M}_{k(n)} \to B/J$ such that $\alpha_n\beta_n$ converges to $\varphi$ pointwise.

In particular, a separable nuclear $C^*$-algebra has the LP.

**Proof.** It is well-known and not too hard to show that a map $\alpha$ from $\mathbb{M}_k$ into a $C^*$-algebra $C$ is cp if and only if $[\alpha(e_{ij})]_{ij} \in \mathbb{M}_k(C)$ is positive. It follows that any ucp map $\alpha : \mathbb{M}_k \to B/J$ is ucp liftable. The conclusion follows from this fact and Lemma 3.10.

The following is due to Kirchberg [Ki1].

**Corollary 3.12.** Let $\varphi : A \to B/J$ be a ucp map. If $A$ is a separable $C^*$-algebra with the LLP and $B$ is QWEP, then $\varphi$ is ucp liftable.
We have 

$$B_I = \{(b_i)_{i \in I} \in \prod_{i \in I} B : \text{strong }^*\text{-lim } b_i \text{ exists in } B^{**}\}$$

and let \( \pi : B_I \to B^{**} \) be the map which takes \((b_i)_{i \in I}\) to its limit. Since the adjoint-operation and product is jointly strong \(^*\)-continuous on bounded sets, \(B_I\) is a \(C^*\)-algebra and \(\pi\) is a \(*\)-homomorphism. Choosing the directed set \(I\) appropriately, we may assume that \(\pi\) is surjective. Since \(A\) has the LLP, there is a ucp map \(\psi : F \to B_I\) such that \(\theta|_F = \pi \psi\). It follows that there is a net of ucp maps \(\psi_i : F \to B\) such that the net \(\psi_i|_F\) converges to \(\theta\) in the pointwise weak topology. Taking convex combinations and multiplying by an approximate unit, we find a desired cp map \(\tilde{\theta} : F \to J\).

The converse is also true. For the proof, see [Oz1].

**Proposition 3.13.** A separable \(C^*\)-algebra \(A\) has the LLP if and only if any ucp map from \(A\) into the Calkin algebra \(\mathbb{B}(\ell_2)/\mathbb{K}(\ell_2)\) is ucp liftable.

It might be the case that any ucp map from a separable \(C^*\)-algebra into the Calkin algebra \(\mathbb{B}(\ell_2)/\mathbb{K}(\ell_2)\) dilates to a \(*\)-homomorphism into \(\mathcal{M}_2(\mathbb{B}(\ell_2)/\mathbb{K}(\ell_2))\). If it is the case, then it would imply that a separable \(C^*\)-algebra has the LLP if and only if \(\text{Ext}(A)\) is a group. Recall that the (unitized) cone of a \(C^*\)-algebra \(A\) is defined to be \(\text{cone}(A) = \{f \in C([0, 1], A) : f(0) \in \mathbb{C}1\}\). By an ingenious argument on the cone, Kirchberg [Ki1] proved that a separable \(C^*\)-algebra \(A\) has the LLP if and only if \(\text{Ext}(\text{cone}(A))\) is a group.

Kirchberg [Ki2][Ki1] proved the following important theorem and its corollary.

**Theorem 3.14.** We have \(C^*\mathbb{F}_\infty \otimes_{\min} \mathbb{B}(\ell_2) = C^*\mathbb{F}_\infty \otimes_{\max} \mathbb{B}(\ell_2)\).

We will present an elegant and simple proof of Pisier [Pi1]. To this end, let \(E_n\) be the \(n\)-dimensional operator space in \(C^*\mathbb{F}_\infty\) spanned by \(1 = U_0, U_1, \ldots, U_{n-1}\), where \(U_1, U_2, \ldots\) are the canonical generators of \(C^*\mathbb{F}_\infty\). It is rather obvious that \(E_n\) is canonically isometric to \(\ell_1^n\), or equivalently, \(\|\sum_{k=0}^{n-1} \alpha_k U_k\| = \sum_{k=0}^{n-1} |\alpha_k|\) for every \((\alpha_k)_{k=0}^{n-1} \in \mathbb{C}\). This gives rise to the canonical one-to-one correspondence between an element \(z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes \mathbb{B}(\ell_2)\) and a map

\[\tilde{z} : \ell_1^n \ni (\alpha_k)_{k=0}^{n-1} \mapsto \sum_{k=0}^{n-1} \alpha_k x_k \in \mathbb{B}(\ell_2)\].

**Lemma 3.15.** The above operator space \(E_n\) is canonically completely isometrically isomorphic to the dual operator space \(\ell_1^n = (\ell_1^n)^*\), or equivalently, \(\|z\|_{\min} = \|\tilde{z}\|_{cb}\) for every \(z \in E_n \otimes \mathbb{B}(\ell_2)\).
Proof. Since \((U_k)_{k=0}^{n-1} \in E_n \otimes_{\min} \ell_\infty\) is contractive and \(z = (\id_{E_n} \otimes \tilde{z})(U_k)_{k=0}^{n-1}\), we have \(\|z\|_{\min} \leq \|	ilde{z}\|_{\cb}\). To prove the converse inequality, we give ourselves contractions \(a_0, \ldots, a_{n-1} \in \mathcal{B}(\mathcal{H})\). Let \(\hat{a}_k \in M_2(\mathcal{B}(\mathcal{H}))\) be their unitary dilations (cf. the proof of Theorem 3.8). It follows that the map \(\varphi\) extends to a \(\|n\) for every \(H\). Hence, by the Stinespring-type theorem for cb maps, there are a Hilbert space \(a\) the contraction \((\hat{a}_k)_{k=0}^{n-1} \in \mathcal{B}(\mathcal{H}) \otimes_{\min} \ell_\infty\). Then, we obtain the following.

Theorem 3.14. Combining this theorem with Theorems 3.3, 3.6 and Corollary 3.9, we obtain the following.

Thanks to Lemma 3.16, it suffices to show that the formal identity from \(E_n\) is contractive and \(z = (\id_{E_n} \otimes \tilde{z})(U_k)_{k=0}^{n-1}\). We give ourselves \(z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes \mathcal{B}(\ell_2)\) with \(\|z\|_{\min} = 1\). By Lemma 3.13 the corresponding map \(\tilde{z}: \ell_n^\infty \to \mathcal{B}(\ell_2)\) is completely contractive. Hence, by the Stinespring-type theorem for cb maps, there are a Hilbert space \(H\), a representation \(\pi: \ell_n^\infty \to \mathcal{B}(H)\) and contractions \(V, W \in \mathcal{B}(\ell_2, H)\) such that \(\tilde{z}(f) = V^* \pi(f) W\) for \(f \in \ell_n^\infty\). We may assume that \(H = \ell_2\). Then, \(a_k = \pi(\delta_k)V\) and \(b_k = \pi(\delta_k)W\) are in \(\mathcal{B}(\ell_2)\), and satisfy \(x_k = a_k^* b_k\) for \(k = 0, \ldots, n-1\) and \(\sum_{k=0}^{n-1} a_k^* a_k \leq 1\). It follows that

\[
\|\sum_{k=0}^{n-1} U_k \otimes x_k\|_{C^*\mathcal{F}_\infty \otimes_{\max} \mathcal{B}(\ell_2)} = \|\sum_{k=0}^{n-1} (1 \otimes a_k)^* (U_k \otimes b_k)\|_{C^*\mathcal{F}_\infty \otimes_{\max} \mathcal{B}(\ell_2)} \leq \|\sum_{k=0}^{n-1} (1 \otimes a_k)^* (1 \otimes a_k)\|_{\max}^{1/2} \|\sum_{k=0}^{n-1} (U_k \otimes b_k)^* (U_k \otimes b_k)\|_{\max}^{1/2} \leq 1.
\]

This shows that the formal identity from \(E_n \otimes_{\min} \mathcal{B}(\ell_2)\) into \(C^*\mathcal{F}_\infty \otimes_{\max} \mathcal{B}(\ell_2)\) is contractive. Since \(\mathcal{B}(\ell_2)\) is stable, it is also completely contractive.

We remark that the fact that \(C^*\mathcal{F}_\infty\) has the LLP was not used in the proof of Theorem 3.14. Combining this theorem with Theorems 3.3, 3.6 and Corollary 3.9, we obtain the following.
Corollary 3.17. For $A$ and $B$, we have the following.

(i) $A \otimes_{\text{min}} B = A \otimes_{\text{max}} B$ if $A$ has the LLP and $B$ has the WEP.
(ii) $A \otimes_{\text{min}} \mathbb{B}(\ell_2) = A \otimes_{\text{max}} \mathbb{B}(\ell_2)$ if and only if $A$ has the LLP.
(iii) $C^*F_\infty \otimes_{\text{min}} B = C^*F_\infty \otimes_{\text{max}} B$ if and only if $B$ has the WEP.

We remark that both the WEP and the LLP (and the LP for separable $C^*$-algebras) are stable under taking a tensor product with a nuclear $C^*$-algebra, and taking a crossed product by an amenable group. This fact easily follows from the above corollary (or the completely positive approximation property).

Corollary 3.18. Let $A_i$ ($i = 1, 2$) be $C^*$-algebras and let $\pi_i: C^*F \to A_i$ be $*$-homomorphisms. If at least one of $A_i$’s is QWEP, then the $*$-homomorphism $\pi_1 \otimes \pi_2: C^*F \otimes C^*F \to A_1 \otimes_{\text{max}} A_2$ is continuous w.r.t. the minimal tensor product.

Proof. Thanks to Proposition 3.14 we may assume that $A_i$’s are separable and the free group $F$ is countable. Suppose that $A_2 = B/J$ and $B$ has the WEP. Since $C^*F$ has the LLP, $\pi_2$ lifts to a ucp map $\psi_2: C^*F \to B$. It follows that

$$\pi_1 \otimes \pi_2: C^*F \otimes_{\text{min}} C^*F \xrightarrow{\text{id} \otimes \psi_2} C^*F \otimes_{\text{min}} B = C^*F \otimes_{\text{max}} B \xrightarrow{\pi_1 \otimes Q} A_1 \otimes_{\text{max}} A_2$$

is continuous. □

Choi [Ch2] proved that the $C^*$-algebra $C^*F_\infty$ is residually finite dimensional, i.e., it has a faithful family of finite dimensional representations. It follows that $C^*F_\infty$ has a faithful trace. Kirchberg [Ki1] observed that his conjecture is equivalent to the above for $F_\infty \times F_\infty$. We note that Bekka [Be2] proved that the full $C^*$-algebra of a residually finite group need not be residually finite dimensional (e.g. $SL_3(\mathbb{Z})$).

Proposition 3.19. The following conjectures are equivalent.

(i) We have $C^*F_\infty \otimes_{\text{min}} C^*F_\infty = C^*F_\infty \otimes_{\text{max}} C^*F_\infty$.
(ii) The full $C^*$-algebra $C^*(F_\infty \times F_\infty)$ is residually finite dimensional.
(iii) The full $C^*$-algebra $C^*(F_\infty \times F_\infty)$ has a faithful trace.
(iv) The $C^*$-algebra $C^*F_\infty$ has the WEP.
(v) All separable $C^*$-algebras are QWEP.
(vi) The LLP implies the WEP.

Proof. We note that $C^*(F_\infty \times F_\infty) = C^*F_\infty \otimes_{\text{max}} C^*F_\infty$ canonically. The implications (i)⇒(ii)⇒(iii) are trivial and (iii)⇒(i) follows from Lemma 3.15 below.

The implications (i)⇒(iv)⇒(v) and (vi)⇒(i) follow from Corollary 3.17. For (v)⇒(vi), we prove that a $C^*$-algebra $A$ with the LLP and QWEP has the WEP. Let $A \subset \mathbb{B}(H)$ and let $\pi$ be a quotient onto $A$ from a $C^*$-algebra $B$ with the WEP. Fix a finite dimensional operator system $E \subset A$, and let $\psi_E: E \to B$ be a ucp lifting. Since $B$ has the WEP, $\psi_E$ extends to a ucp map $\bar{\psi}_E: \mathbb{B}(H) \to B^{**}$. It follows that $\psi_E = \pi^{**} \bar{\psi}_E$ is a ucp map which coincides with the identity on $E$. Any cluster point of the net of ucp maps $\varphi_E: \mathbb{B}(H) \to A^{**}$ in the pointwise weak* topology is a desired weak expectation. This completes the proof. □
Lemma 3.20. Any trace on the maximal tensor product $A_1 \otimes_{\text{max}} A_2$ factors through the minimal tensor product $A_1 \otimes_{\text{min}} A_2$.

Proof. It suffices to show the assertion for extremal traces. It is well-known and not too hard to see that a trace is extremal if and only if its GNS representation generates a finite factor. Let $\tau$ be an extremal trace on $A_1 \otimes_{\text{max}} A_2$ with the GNS representation $\pi$ and let $\pi_i: A_i \to M := \pi(A_1 \otimes_{\text{max}} A_2)^\tau$ be the restriction of $\pi$ to $A_i$. It follows that $M_i := \pi_i(A_i)^\tau$ are commuting von Neumann subalgebras of $M$, which have to be factors. By uniqueness of the trace on a finite factor, we have $\tau_M(a_1a_2) = \tau_{M_i}(a_1)\tau_{M_j}(a_2)$ for $a_i \in \pi_i(A_i)$, which means that $M = M_1 \otimes M_2$. □

Recall the notations used in the proof of Theorem 3.14. According to Lemma 3.16, $C^*F_{\infty} \otimes_{\text{max}} C^*F_{\infty} = C^*F_{\infty} \otimes_{\text{min}} C^*F_{\infty}$ if and only if the formal identity $\theta_n$ from $E_n \otimes_{\text{min}} E_n$ into $C^*F_{\infty} \otimes_{\text{max}} C^*F_{\infty}$ is completely contractive for every/some $n \geq 3$. It follows from a Grothendieck-type factorization theorem of Junge [Jn], Paulsen [Pa2] and Pisier [Pi2] that $\|\theta_n\| \leq 2$ for all $n \in \mathbb{N}$. However, no nontrivial estimates of $\|\theta_n\|_{cb}$ is known. It is known that the $C^*$-subalgebra in $C^*F_{\infty} \otimes_{\text{min}} C^*F_{\infty}$ generated by its “diagonal” $\{s \otimes s : s \in F_{\infty}\}$ can be canonically *-isomorphic to $C^*F_{\infty}$. Indeed, since $C^*F_{\infty}$ is QWEP (cf. the remark at the end of Section 4), the *-homomorphism $C^*F_{\infty} \otimes_{\text{min}} C^*F_{\infty} \to C^*F_{\infty} \otimes_{\text{max}} C^*F_{\infty}$ is continuous thanks to Corollary 3.18. The claim now follows from Pisier’s observation [Pi2] that the “diagonal” in $C^*\Lambda \otimes_{\text{max}} C^*\Lambda$ is canonically *-isomorphic to $C^*\Lambda$ for any discrete group $\Gamma$.

The following is proved by Pisier [Pi1] and Boca [Bo].

Proposition 3.21. The LLP (resp. the LP for separable $C^*$-algebras) is stable under a full free product.

By modifying Pisier’s proof [Pi1], one can prove that the LLP is stable under a full amalgamated free product over a finite dimensional $C^*$-subalgebra. Indeed, if $A_i$ $(i = 1, 2)$ are $C^*$-algebras with a common $C^*$-subalgebra $B$, then the linear span of $A_1A_2$ in the full amalgamated free product $A_1 \ast_B A_2$ is canonically completely isometrically isomorphic to the relative Haagerup tensor product $A_1 \otimes_B^h A_2 := A_1 \otimes h A_2/N_B$, where $N_B = \overline{\text{span}\{a_1x \otimes a_2 - a_1 \otimes xa_2 : a_i \in A_i, x \in B\}}$. Hence if $B$ is finite dimensional, then there is a completely contractive lifting from $A_1 \otimes_B^h A_2$ into $A_1 \otimes h A_2$. It follows from Lemma 3.16 that the canonical quotient map from $A_1 \ast_B A_2$ onto $A_1 \ast_B A_2$ is locally ucp liftable whenever $B$ is finite dimensional. It is not known whether the full amalgamated free product (over a finite dimensional $C^*$-subalgebra) preserves the LP.

The LLP (resp. the LP) also passes to a (resp. separable) weakly cp complemented $C^*$-subalgebra. It is not known whether the LLP (resp. the LP) is stable under the maximal tensor product, or equivalently whether the full $C^*$-algebra $C^*(F_\infty \times F_\infty)$ has the LLP (resp. the LP). It seems related to the QWEP conjecture, but we have found no logical connection. Although it seems full group
$C^*$-algebras rarely have the LLP, there is no example of groups whose full $C^*$-algebra is known to fail the LLP. It was observed in [Oz3] that there is a group whose full $C^*$-algebra does not have the LLP. We note that the full $C^*$-algebra $C^*_\Gamma$ has the LLP if and only if any positive definite function from $\Gamma$ into the Calkin algebra $\mathbb{B}(\ell_2)/\mathbb{K}(\ell_2)$ has a positive definite lifting.

4. Permanence Properties of the QWEP

Following Kirchberg [Ki1], we study the permanence properties of the QWEP.

**Proposition 4.1.** We have the following.

(i) If $A_i$ is QWEP for all $i \in I$, then so is $\prod_{i \in I} A_i$.
(ii) If $A \subset B$ is weakly cp complemented and $B$ is QWEP, then so is $A$.
(iii) Let $(A_i)_{i \in I}$ be an increasing net of (possibly non-unital) $C^*$-subalgebras in $A$ (resp. $M$) whose union is dense in norm (resp. weak*) topology. If all $A_i$ are QWEP, then so is $A$ (resp. $M$).
(iv) A $C^*$-algebra $A$ is QWEP if and only if the second dual $A^{**}$ is QWEP.
(v) If $A$ is QWEP and $B$ is nuclear, then $A \otimes_{\min} B$ is QWEP. If $M$ and $N$ are QWEP, then so is $M \bar{\otimes} N$.
(vi) If $A$ (resp. $M$) is QWEP and $\alpha$ is an action of amenable group $\Gamma$, then $\Gamma \rtimes_\alpha A$ (resp. $\Gamma \rtimes_\alpha M$) is QWEP.
(vii) The commutant $M'$ is QWEP if and only if $M$ is QWEP.
(viii) Let $M = \int^\oplus M(\gamma) d\gamma$ be the direct integral of separable von Neumann algebras. Then, $M$ is QWEP if and only if $M(\gamma)$ are QWEP for a.e. $\gamma$.

**Proof.** Ad(i): This follows from Lemma 3.2.

Ad(ii): Let $J$ be an ideal in $C$ with the WEP such that $B = C/J$ and let $\pi: C \to B$ be the quotient map. Then, the $C^*$-subalgebra $\pi^{-1}(A)$ is weakly cp complemented in $C$ and thus has the WEP. Indeed, this follows from Lemma 3.2 and the fact

$$\pi^{-1}(A)^{**} = J^{**} \oplus A^{**} \subset J^{**} \oplus B^{**} = C^{**}.$$ 

Hence, $A$ is a quotient of $\pi^{-1}(A)$ which has the WEP.

Ad(iii): Let $M$ be the weak*-closure of $A = \bigcup A_i$. We will prove that $M$ is QWEP. (Then the QWEP property of $A$ follows from (ii) by considering the case where $M = A^{**}$.) By the Kaplansky density theorem, $\text{Ball}(A)$ is strong*-dense in $\text{Ball}(M)$. Hence, amplifying the directed set $I$ if necessary, we may assume that for any $x \in \text{Ball}(M)$, there is a net $(a_i)_{i \in I} \in \text{Ball}(\prod_{i \in I} A_i)$ such that $x = \text{strong}^*-\lim_{i \in I} a_i$. We denote by $C = \prod_{i \in I} A_i$. The $C^*$-algebra $C$ is QWEP by (i). Let

$$B = \{(a_i)_{i \in I} \in C : \text{strong}^*-\lim_{i \in I} a_i \text{ exists in } M\}$$
and let \( \pi: B \to M \) be the map which takes \((a_i)_{i \in I}\) to its limit. Since the adjoint-operation and product is jointly strong*-continuous on bounded sets, \( B \) is a C*-subalgebra of \( C \) and \( \pi \) is a surjective *-homomorphism onto \( M \). We claim that \( B \) is weakly cp complemented in \( C \). Let \((e_j)_j\) be an increasing approximate unit for \( J = \ker \pi \) and let \( e = \lim_j e_j \in C^{**} \). We note that \( e \) is in the center of \( B^{**} \) and \( B^{**} = B^{**} e \oplus B^{**} (1 - e) = J^{**} \oplus M^{**} \). Since \( J \) is hereditary in \( C \) (i.e., \( e_i x e_i \in J \) for all \( x \in C \)), we have \( e C^{**} = J^{**} \).

For a fixed free ultrafilter \( \omega \) on \( I \), the map defined by

\[
\tilde{\pi}: C \ni (a_i)_{i \in I} \mapsto \text{weak*}-\lim_{i \in \omega} a_i \in M
\]

is a ucp extension of \( \pi \). This ucp extension gives rise to a conditional expectation from \((1 - e)C^{**} (1 - e)\) onto \( B^{**} (1 - e) = M^{**} \). It follows that

\[
B^{**} = J^{**} \oplus M^{**} \subset e C^{**} e \oplus (1 - e) C^{**} (1 - e)
\]

is cp complemented. This proves our claim and we are done by (ii).

**Ad(iv):** This follows from (ii) and (iii).

**Note.** We are not going to use (v) to (vii) of this proposition and the proof requires some results which will be proved later.

**Ad(v):** The first assertion follows from the fact that the tensor product of a WEP C*-algebra with a nuclear C*-algebra again has the WEP. The second assertion follows from Corollary 5.3 (or Corollary 6.2) and the following result of Tomiyama [10]. If \( \varphi: M_1 \to M_2 \) is a (not necessarily weak*-continuous) ucp map between von Neumann algebras, then the map \( \varphi \otimes \text{id}_N: M_1 \otimes N \to M_2 \otimes N \) extends to a ucp map \( \varphi \otimes \text{id}_N: M_1 \oplus N \to M_2 \oplus N \). Indeed, this follows from the fact that \((M_2)_s \oplus N_s\) is dense in \( M_2 \oplus N \).

**Ad(vi):** The assertion for von Neumann algebras follows from that for C*-algebras. Let \((A, \Gamma, \alpha)\) be a C*-dynamical system with \( \Gamma \) amenable. We claim that there are nets of ucp maps \( \psi_t: \Gamma \times A \to \mathbb{M}_{n(t)}(A) \) and \( \varphi_t: \mathbb{M}_{n(t)}(A) \to \Gamma \times A \) such that \( \varphi_t \psi_t \) converges to \( \text{id}_\Gamma \times \alpha \) pointwisely. Once this is proved, then the QWEP property of \( \Gamma \times A \) follows from that of \( A \) (cf. Corollary 5.3).

Take a faithful representation \( \pi: A \to \mathbb{B}(\mathcal{H}) \) with a unitary action \( u \) of \( \Gamma \) on \( \mathcal{H} \) which implements the action \( \alpha \). i.e., \( \pi(\alpha_s(a)) = \text{Ad} u_s(\pi(a)) \) for \( a \in A \). Then, we have \( \Gamma \times A \subset \mathbb{B}(\ell_2 \Gamma \otimes \mathcal{H}) \) where \( \Gamma \ni s \mapsto \lambda_s \otimes 1 \in \mathbb{B}(\ell_2 \Gamma \otimes \mathcal{H}) \) and \( A \ni a \mapsto \tilde{\pi}(a) \in \mathbb{B}(\ell_2 \Gamma \otimes \mathcal{H}) \) with \( \tilde{\pi}(a)(\delta_s \otimes \xi) = \delta_s \otimes \alpha_{s^{-1}}(a) \xi \). We fix a finite subset \( F \subset \Gamma \). The compression of \( \mathbb{B}(\ell_2 \Gamma) \) to \( \mathbb{B}(\ell_2 F) = \mathbb{M}_F \) defines a ucp map \( \varphi_F: \Gamma \times A \to \mathbb{M}_F(A) \).

Let \( \varphi_F: \mathbb{M}_F(A) \to \mathbb{B}(\ell_2 \Gamma \otimes \mathcal{H}) \) be the ucp map defined by \( \varphi_F(x) = V_F^*(1 \otimes x)V_F \), where the isometry \( V_F \) is given by

\[
V_F: \ell_2 \Gamma \otimes \mathcal{H} \ni \delta_s \otimes \xi \longmapsto \frac{1}{|F|} \sum_{t \in F} \delta_{t^{-1}s} \otimes \delta_t \otimes u_{t^{-1}s} \xi \in \ell_2 \Gamma \otimes \ell_2 F \otimes \mathcal{H}.
\]

A direct computation shows that \( \varphi_F(e_{s,t} \otimes a) = \frac{1}{|F|}(\lambda_s \otimes 1)\tilde{\pi}(\alpha_s(a)) \). Therefore, \( \varphi_F \) maps into \( \Gamma \times A \) and \( \varphi_F \varphi_F(\lambda_s \tilde{\pi}(a)) = \frac{|F|}{|F|}\lambda_s \tilde{\pi}(a) \). This proves the claim.
Ad(vii): This follows from standard representation theory of von Neumann algebras.

Ad(viii): Suppose $M$ is QWEP and let $A$ be a separable weak*‐dense $C^*$‐subalgebra which is weakly cp complemented in $M$ (cf. Proposition 3.3). Then, $A$ is QWEP by (ii) in this proposition. Since almost all $M(\gamma)$ arise as the weak∗‐closure of representations of $A$, they are QWEP. Conversely, suppose $M(\gamma)$ are QWEP for a.e. $\gamma$ and let $\pi$ (resp. $\pi'$) be a ∗‐homomorphism from $C^*F_\infty$ onto a weak∗‐dense $C^*$‐subalgebra of $M$ (resp. $M'$). Since $\pi_\gamma \cdot \pi'_\gamma$ is continuous w.r.t. the minimal tensor product for a.e. $\gamma$ (cf. Corollary 3.18), the representation $\pi \cdot \pi' = \int_\gamma^\oplus \pi_\gamma \cdot \pi'_\gamma \ d\gamma$ is also continuous w.r.t. the minimal tensor product. If $C^*F_\infty \subset B(H)$, then one can show that $\pi : C^*F_\infty \rightarrow M$ extends to a ucp map $\tilde{\pi} : B(H) \rightarrow M$ (cf. the proof of Theorem 3.3). The QWEP property of $M$ now follows from Corollary 5.3. □

It is not known whether the minimal tensor product of QWEP $C^*$-algebras is again QWEP, or equivalently whether $B(\ell_2) \otimes_{min} B(\ell_2)$ is QWEP or not. We note that this $C^*$-algebra fails the WEP as shown in [Oz2].

Haagerup [Ha1] showed that there is a sequence of (complete) contractions $\varphi_n : LF_2 \rightarrow C_{red}^*F_2 = : A$ such that $\lim_n \varphi_n(a) = a$ for $a \in A$. Let $\varphi : LF_2 \rightarrow A^{**}$ be a cluster point of the sequence $\varphi_n$. Then, $\varphi$ is a unital (complete) contraction such that $\varphi|_A = id_A$. This shows that $C_{red}F_2$ is weakly cp complemented in $LF_2$ (cf. Theorem 2.3). (It follows that $C_{red}F_2$ is weakly cp complemented in any $C^*$-superalgebra with a trace.) Since $LF_2$ is QWEP (cf. Proposition 6.3), the reduced group $C^*$-algebra $C_{red}^*\Gamma$ of, say, $\Gamma = SL_3(\mathbb{Z})$ is QWEP or not.

5. Finite Representability in the Trace Class

Following Kirchberg [Ki1], we study the relation between the QWEP property and finite representability.

**Definition 5.1.** The Banach-Mazur distance $d$ between two Banach spaces $E$ and $F$ is defined by

$$d(E, F) = \inf \{\|\theta\| : \theta : E \rightarrow F \text{ a continuous linear isomorphism}\}.$$ 

We put $d(E, F) = \infty$ when $E$ is not isomorphic to $F$. We say a Banach space $X$ is finitely representable in a Banach space $Y$ if for every finite dimensional subspace $E$ of $X$, one has $\inf\{d(E, F) : F \subset Y\} = 1$.

Similarly, the cb Banach-Mazur distance $d_{cb}$ and os finite representability are defined by just replacing the usual norm with the cb norm in the above definitions.

For a Banach space $X$, we denote by $\text{Ball}(X)$ (resp. $\overline{\text{Ball}(X)}$) the open (resp. closed) unit ball of $X$. A map $\varphi : X \rightarrow Y$ is called a metric surjection if it maps $\text{Ball}(X)$ onto $\text{Ball}(Y)$, i.e., $X/\ker \varphi = Y$ isometrically. If $Y$ embeds into $X$
isometrically, then the transpose of the isometric inclusion is a weak*-continuous metric surjection which maps $\text{Ball}(X^*)$ onto $\text{Ball}(Y^*)$.

**Proposition 5.2.** For von Neumann algebras $M$ and $N$, we have the following.

(i) If $M_*$ embeds into $N_*$ isometrically, then there is a weak*-continuous unital metric surjection $\varphi$ from $N$ onto $M$.

(ii) If there is a weak*-continuous unital metric surjection $\varphi$ from $N$ onto $M$, then there are projections $p$ and $q$ in $N$ such that $M$ is *-isomorphic to a (normally) $cp$ complemented von Neumann subalgebra of $pNp \oplus (qNq)^{\text{op}}$.

**Proof.** Ad(i): Let $\varphi_0 : N \to M$ be the transpose of an isometric embedding of $M_*$ into $N_*$. Let $v$ be an extreme point of the weak*-closed face $\{ x \in N : \|x\| \leq 1 \} \cap \text{Ball}(N)$. It follows from a standard argument (cf. [Sa]) that $v$ is a partial isometry and $\varphi_0(v^*v) = \varphi_0(v)$. Hence, the map $\varphi : N \to M$ defined by $\varphi(x) = \varphi_0(vx)$ is the desired unital metric surjection.

Ad(ii): Let $\varphi$ be a weak*-continuous unital metric surjection from $N$ onto $M$. Let $e \in N$ be the support projection of $\varphi$, i.e., $1 - e = \text{sup}(a \in N : 0 \leq a \leq 1$ and $\varphi(a) = 0)$. (We note that $a \geq 0$ and $\varphi(a) = 0$ implies that $\varphi$ is zero on the support projection of $a$.) It follows that $\varphi(a) = \varphi(eae)$ for every $a \in N$ and that $\varphi(a) \neq 0$ for all positive non-zero element $a \in eNe$. Let $C \subseteq eNe$ be the Jordan multiplicative domain for $\varphi|_{eNe}$, i.e.,

$$C = \{ a \in eNe : \varphi(a^*a) = \varphi(a^*) \circ \varphi(a) \} = \{ a \in eNe : \varphi(a \circ x) = \varphi(a) \circ \varphi(x) \text{ for all } x \in eNe \}.$$ 

By Corollary 2.6 and the following remark, $C$ is a weak*-closed Jordan subalgebra in $eNe$ and $\varphi|_C$ is a normal Jordan isomorphism from $C$ onto $M$. Hence, $\theta = (\varphi|_C)^{-1}$ is a normal Jordan isomorphism from $M$ onto $C$ with $\varphi \theta = \text{id}_M$. By Theorem 2.10 there is a projection $p$ in $C' \cap eNe$ with $q := e - p$ such that the map $\theta_1 : M \ni x \mapsto \theta_1(x)p \in pNp$ (resp. $\theta_2 : M \ni x \mapsto \theta_2(x)q \in qNq$) is a *-homomorphism (resp. *-antihomomorphism). It follows that the map

$$\tilde{\theta} : M \ni x \mapsto (\theta_1(x), \theta_2(x)^{\text{op}}) \in pNp \oplus (qNq)^{\text{op}}$$

defines a normal unital injective *-homomorphism and the map

$$\tilde{\varphi} : pNp \oplus (qNq)^{\text{op}} \ni (a, b^{\text{op}}) \mapsto \varphi(a + b) \in M$$

defines a unital positive map such that $\tilde{\varphi} \tilde{\theta} = \text{id}_M$. Therefore, the map $\tilde{\theta} \tilde{\varphi}$ is a weak*-continuous contractive projection onto the von Neumann subalgebra $\theta(M)$ in $pNp \oplus (qNq)^{\text{op}}$. We complete the proof by Theorem 2.9. \[\square\]

**Corollary 5.3.** For a von Neumann algebra $M$, the following are equivalent.

(i) The von Neumann algebra $M$ is QWEP.

(ii) There are a Hilbert space $\mathcal{H}$ and a ucp map $\varphi : \mathcal{B}(\mathcal{H}) \to M$ which maps $\text{Ball}(\mathcal{B}(\mathcal{H}))$ onto $\text{Ball}(M)$. 
(iii) The predual $M_*$ of $M$ is (os) finitely representable in the trace class $S_1$.
(iv) There is a QWEP C*-algebra $A$ and a contraction $\varphi: A \to M$ such that $\varphi(\text{Ball}(A))$ is weak*-dense in $\text{Ball}(M)$.

Moreover, we can choose the Hilbert space $H$ in (ii) separable when so is $M$.

Proof. Ad(i)⇒(ii): Let $B \subset \mathbb{B}(H)$ be a C*-algebra with the WEP and let $\pi$ be a surjective *-homomorphism from $B$ onto $M$. Since $B$ has the WEP, there is a ucp map $\psi: \mathbb{B}(H) \to B^\ast$ with $\psi|_B = \text{id}_B$. It follows that the composition $\varphi = \hat{\pi} \psi$ of $\psi$ and the normal extension $\hat{\pi}: B^\ast \to M$ of $\pi$ is a ucp map such that $\varphi(\text{Ball}(\mathbb{B}(H))) = \text{Ball}(M)$. If $M$ is separable, then there is a separable C*-subalgebra $A$ in $\mathbb{B}(H)$ such that $\varphi(\text{Ball}(A))$ is weak*-dense in $\text{Ball}(M)$. Let $\theta: A \to \mathbb{B}(\ell_2)$ be a faithful representation and let $\sigma: \mathbb{B}(\ell_2) \to \mathbb{B}(H)$ be a ucp extension of $\theta^{-1}$. It follows that $\varphi_0 = \varphi \sigma$ is a ucp map such that $\varphi_0(\text{Ball}(\mathbb{B}(\ell_2)))$ is weak*-dense in $\text{Ball}(M)$. Fixing a free ultrafilter $\omega$ on $\mathbb{N}$, we define $\Phi: \prod_{n \in \mathbb{N}} \mathbb{B}(\ell_2) \to M$ by

$$\Phi((a_n)_{n \in \mathbb{N}}) = \text{weak*-lim}_{n \in \omega} \varphi_0(a_n) \in M.$$  

Since $M_*$ is separable, the ucp map $\Phi$ maps $\text{Ball}(\prod_{n \in \mathbb{N}} \mathbb{B}(\ell_2))$ onto $\text{Ball}(M)$.

Ad(ii)⇒(iii): A metric surjection between C*-algebras which is ucp is automatically a complete metric surjection (cf. Corollary 2.6). If $\varphi: \mathbb{B}(H) \to M$ is a ucp metric surjection, then $\varphi^*: M^* \to \mathbb{B}(H)^*$ is completely isometric. By the principle of local reflexivity, $\mathbb{B}(H)^*$ is finitely representable in $\mathbb{B}(H)_* = S_1(\mathbb{H})$. The operator space analogue of the principle of local reflexivity for $S_1(\mathbb{H})$ is a deep theorem of Junge (see [12] for a more general result).

Ad(iii)⇒(iv): Let $(E_i)_{i \in I}$ be an increasing net of finite dimensional subspaces in $M_*$ with $\bigcup E_i = M_*$. By the assumption, there is an embedding $\psi_i: E_i \to S_1$ for each $i \in I$ such that $\|\psi_i\| \leq 1 + (\dim E_i)^{-1}$ and $\|\psi_i^{-1}\| \leq 1$. Fixing a free ultrafilter $\omega$ on $I$, we define a contraction $\varphi: \prod_{i \in I} \mathbb{B}(\ell_2) \to M$ by

$$\langle \varphi((a_i)_{i \in I}), f \rangle = \lim_{i \in \omega} \langle x_i, \psi_i(f) \rangle$$

for $(a_i)_{i \in I} \in \prod_{i \in I} \mathbb{B}(\ell_2)$ and $f \in M_*$ (since $f \in E_i$ eventually, the limit in the above definition makes sense). We give ourselves an arbitrary $x \in \text{Ball}(M)$. For each $i \in I$, let $a_i \in \mathbb{B}(\ell_2) = S_1^*$ be a Hahn-Banach extension of $x\psi_i^{-1} \in E_i(E_i)^*$. It follows that $a = (a_i)_{i \in I} \in \text{Ball}(\prod_{i \in I} \mathbb{B}(\ell_2))$ is such that $\varphi(a) = x$.

Ad(iv)⇒(i): By the Kaplansky density theorem, there is a directed set $I$ such that for any $x \in \text{Ball}(M)$, there is a net $(a_i)_{i \in I}$ in $\text{Ball}(A)$ such that $x = \text{weak*}-\lim_{i \in I} \varphi(a_i)$. It follows that the map

$$\prod_{i \in I} A \ni (a_i)_{i \in I} \mapsto \text{weak*}-\lim_{i \in \omega} \varphi(a_i) \in M$$

defines a metric surjection from $\prod_{i \in I} A$ onto $M$. Taking the transpose, we obtain an isometric embedding of $M_*$ into $(\prod_{i \in I} A)^*$. By Proposition 5.2, $M$ is
For a trace \( \tau \) which is continuous w.r.t. the maximal tensor norm. It follows that the condition (ii) gives rise to a representation \( \pi \) and let \( \tau \) be the faithful trace on \( R^\omega \). It turns out that \( N_\omega = \{ a \in \prod R : \tau_\omega(a^*a) = 0 \} \) is a maximal ideal in \( \prod R \) and \( R^\omega = (\prod R)/N_\omega \) (as a C*-algebra) is a II\(_1\)-factor which is not separable.

Let \( A \) be a C*-algebra and \( \tau \) be a trace on \( A \) with the GNS-triplet \((\pi_\tau, \mathcal{H}_\tau, \xi_\tau)\). We denote by \( \pi^*_\tau \) the representation of the conjugate C*-algebra \( \bar{A} \) of \( A \) on \( \mathcal{H}_\tau \) defined by \( \pi^*_\tau(b)\pi_\tau(a)\xi_\tau = \pi_\tau(ab^*)\xi_\tau \) for \( a, b \in A \). (The conjugate C*-algebra \( \bar{A} \) is \(*\)-isomorphic to the opposite C*-algebra \( A^{op} \) via \( \bar{a} \mapsto (a^*)^{op} \in A^{op} \).) This gives rise to a representation \( \sigma_\tau \) of \( A \otimes \bar{A} \) on \( \mathcal{H}_\tau \) given by

\[
\sigma_\tau : A \otimes \bar{A} \ni \sum_k a_k \otimes \bar{b}_k \longmapsto \sum_k \pi_\tau(a_k)\pi^*_\tau(b_k) \in \mathbb{B}(\mathcal{H}_\tau)
\]

which is continuous w.r.t. the maximal tensor norm. It follows that the linear functional \( \mu_\tau \) on \( A \otimes \bar{A} \) given by

\[
\mu_\tau : A \otimes \bar{A} \ni \sum_k a_k \otimes \bar{b}_k \longmapsto \tau(\sum_k a_k\bar{b}_k^*) \in \mathbb{C}
\]

is also continuous w.r.t. the maximal tensor norm. Connes showed that a II\(_1\)-factor \((A, \tau)\) is injective iff \( \sigma_\tau \) (or \( \mu_\tau \)) is continuous w.r.t. the minimal tensor norm on \( A \otimes \bar{A} \). The following theorem of Kirchberg generalizes Connes' characterization.

**Theorem 6.1.** For a trace \( \tau \) on a C*-algebra \( A \) in \( \mathbb{B}(\mathcal{H}) \), the following are equivalent.

(i) The trace \( \tau \) extends to an \( A \)-central state \( \varphi \) on \( \mathbb{B}(\mathcal{H}) \), i.e., the trace \( \tau \) extends to a state \( \varphi \) on \( \mathbb{B}(\mathcal{H}) \) such that \( \varphi(ax) = \varphi(xa) \) for every \( a \in A \) and \( x \in \mathbb{B}(\mathcal{H}) \).

(ii) There is a net of ucp maps \( \theta_i : A \to \mathbb{M}_{n(i)} \) such that \( \tau(a) = \lim_i \text{tr}_{n(i)}(\theta_i(a)) \) and \( \lim_i \text{tr}_{n(i)}(\theta_i(ab^*) - \theta_i(a)\theta_i(b)^*) = 0 \) for every \( a, b \) in \( A \).

(iii) The trace \( \tau \) is liftable, i.e., there is a *-homomorphism \( \theta : A \to R^\omega \) with a ucp lifting \( \bar{\theta} : A \to \prod R \) such that \( \tau = \tau_\omega \bar{\theta} \).

(iv) The functional \( \mu_\tau \) is continuous w.r.t. the minimal tensor norm on \( A \otimes \bar{A} \).

Moreover if \( A \cap \mathbb{K}(\mathcal{H}) = \{0\} \), then one can choose the ucp maps \( \theta_i : A \to \mathbb{M}_{n(i)} \) in the condition (ii) as compressions to finite dimensional subspaces of \( \mathcal{H} \).
Proof. We will prove (i)⇒(ii)⇒(iii)⇒(iv)⇒(i) and (ii)⇒(ii')⇒(i).

Ad (i)⇒(ii): The proof is taken from [Ha2]. To prove (ii), we give ourselves a finite set \( \mathcal{F} \) of unitaries in \( A \) and \( \varepsilon > 0 \). We approximate the \( A \)-central state \( \varphi \) by \( \text{Tr}(h \cdot) \), where \( h \) is a positive trace class operator with \( \text{Tr}(h) = 1 \). By a standard approximation argument, we find such \( h \) that \( |\text{Tr}(hu) - \tau(u)| < \varepsilon \) and \( \|h - uhu^*\|_{1,\text{Tr}} < \varepsilon \) for \( u \in \mathcal{F} \). Further, we may assume that \( h \) is of finite rank and has no irrational eigenvalues; let \( p_1/q, \ldots, p_m/q (p_1, \ldots, p_m \in \mathbb{N} \) and \( q = \sum_k p_k \) be the non-zero eigenvalues of \( h \) with the corresponding eigenvectors \( \zeta_1, \ldots, \zeta_m \in \mathcal{H} \).

Put \( p = \max\{p_1, \ldots, p_m\} \) and define an isometry \( V_k : \ell_2^p \to \mathcal{H} \otimes \ell_2^p \) for each \( k \) by \( V_k \delta_i = \zeta_k \otimes \delta_i \) for \( i = 1, \ldots, p_k \). Since \( V_k \)'s have orthogonal ranges, the sum of \( V_k \)'s gives rise to an isometry \( V : \bigoplus_{k=1}^m \ell_2^p \to \mathcal{H} \otimes \ell_2^p \). Identifying \( M_q \) with \( B(\bigoplus_{k=1}^m \ell_2^p) \), we obtain a ucp map \( \theta : A \to M_q \) defined by \( \theta(a) = V^*(a \otimes 1)V \). It follows that \( \text{tr}_q \theta(a) = \text{Tr}(ha) \) for \( a \in A \) and that, by denoting \( u_{k,l} = (u \zeta_l, \zeta_k) \), we have

\[
|\text{Tr}(h^{1/2}u h^{1/2} u^*) - \text{tr}_q(\theta(u)\theta(u)^*)| = \sum_{k,l} |u_{k,l}|^2 \left( (p_k p_l)^{1/2} - \min\{p_k, p_l\} \right) / q \\
\leq \sum_{k,l} |u_{k,l}|^2 p_k^{1/2} p_l^{1/2} / q \\
\leq \left( \sum_{k,l} |u_{k,l}|^2 p_k / q \right)^{1/2} \left( \sum_{k,l} |u_{k,l}|^2 (p_k^{1/2} - p_l^{1/2})^2 / q \right)^{1/2} \\
= \|h^{1/2}u\|_{2,\text{Tr}} \|h^{1/2}u - u h^{1/2}\|_{2,\text{Tr}}.
\]

Recall the Powers-Størmer inequality [PS] that \( \|h^{1/2}u - u h^{1/2}\|_{2,\text{Tr}} \leq \|h - uhu^*\|_{1,\text{Tr}}^{1/2} \). It follows that

\[
\text{tr}_q(\theta(ua^*) - \theta(u)\theta(a^*)) = |\text{Tr}(uu^*)| - \text{Tr}(h^{1/2}u h^{1/2} u^*) + |\text{Tr}(h^{1/2}u h^{1/2} u^*) - \text{tr}_q(\theta(u)\theta(u)^*)| \\
\leq 2\|h^{1/2}u - u h^{1/2}\|_{2,\text{Tr}} \leq 2\varepsilon^{1/2}
\]

for \( u \in \mathcal{F} \). Finally, we note that

\[
|\text{tr}_q(\theta(ab^*) - \theta(a)\theta(b))| \leq \text{tr}_q(\theta(aa^*) - \theta(a)\theta(a)^*)^{1/2} \text{tr}_q(\theta(bb^*) - \theta(b)\theta(b)^*)^{1/2}
\]

for any \( a, b \in A \). If \( A \cap \mathbb{K} \mathcal{H} = \{0\} \), then by Glimm’s theorem the ucp map \( \theta : A \to M_q \) is approximated by compressions to \( q \)-dimensional subspaces in \( \mathcal{H} \). Since \( A \) is spanned by unitaries, this completes the proof of (i)⇒(ii).

Ad(ii)⇒(iii): Since \( \mu_n : M_n \otimes_{\text{min}} M_n \ni \sum_k x_k \otimes y_k \mapsto \text{tr}_n(\sum_k x_k y_k^*) \in \mathbb{C} \) are states for all \( n \), the net of states \( \mu_n(\theta_i \otimes \tilde{\theta}_i) \) on \( A \otimes_{\text{min}} \tilde{A} \) is well-defined and converges to the functional \( \mu_\tau \).

Ad(iii)⇒(iv): This follows from the fact that \( \zeta_\tau \) is cyclic for \( \sigma_\tau(A \otimes \tilde{A}) \) and the corresponding vector state \( \mu_\tau \) is continuous w.r.t. the minimal tensor product.
One direction is obvious. Let
Proof. \( \theta \) and let \( \Theta = (\theta, \tau, \psi) \) be such that
\( \tau \in F_{\infty} \). Since \( \Psi \) is an \((A \otimes_{\min} \hat{A})\)-bimodule map, we have
\( \pi^\tau_t(b) \psi(x) = \Psi(x \otimes b) = \psi(x) \pi^\tau_t(b) \) for all \( x \in \mathcal{B}(H) \) and \( b \in \hat{A} \). It follows that
\( \psi(\mathcal{B}(H)) \subset \pi^\tau_t(A') = \pi^\tau(A)' \) and \( \psi|_A = \pi^\tau_t \) (in particular, \( \psi \) is an \( A \)-bimodule map). Thus, the state \( \varphi \), given by \( \varphi(x) = (\psi(x)) \) for \( x \in \mathcal{B}(H) \), is a desired \( A \)-central extension of \( \tau \).

Ad(ii)\( \Rightarrow \) (i): The implication (ii)\( \Rightarrow \) (i’) follows from the definition of \( R^\omega \) and the fact that \( M_n \)'s are isomorphic to subfactors of \( R \) for all \( n \). Now assume
(ii’) and let \( \theta \) and \( \hat{\theta} \) be as in the condition. Since \( \prod R \) is injective, \( \hat{\theta} \) extends to a ucp map from \( \mathcal{B}(H) \) into \( \prod R \). Composing this with the quotient map \( \prod R \to R^\omega \), we obtain a ucp extension \( \psi: \mathcal{B}(H) \to R^\omega \) of \( \theta \). Since \( \psi \) is an \( A \)-bimodule map, the state \( \varphi \) given by \( \varphi = \tau_\omega \psi \) is a desired \( A \)-central extension of \( \tau \). \( \square \)

Corollary 6.2. Let \( M \) be a separable finite von Neumann algebra. Then \( M \) is QWEP if and only if \( M \) is \( * \)-isomorphic to a von Neumann subalgebra of \( R^\omega \).

Proof. Any von Neumann subalgebra of \( R^\omega \) is the range of a conditional expectation and fortiiori is QWEP. Now, assume that \( M \) is QWEP and let \( \tau \) be a normal faithful state on \( M \). Let \( \pi: C^*F_{\infty} \to M \) be a \( * \)-homomorphism with a weak*-dense range. Since \( M \) is QWEP, by Corollary 6.1, the \( * \)-homomorphism \( \pi \otimes \hat{\pi}: C^*F_{\infty} \otimes_{\min} C^*F_{\infty} \to M \otimes_{\max} M \) is continuous. It follows that Theorem 6.1 is applicable to the trace \( \tau \pi \) on \( C^*F_{\infty} \); there is a \( * \)-homomorphism \( \theta: C^*F_{\infty} \to R^\omega \) such that \( \tau \pi = \tau_\omega \theta \). This means that \( M \) is \( * \)-isomorphic to the von Neumann subalgebra generated by \( \theta(C^*F_{\infty}) \) in \( R^\omega \). \( \square \)

7. Groups with the Factorization Property

Otherwise stated, all groups denoted by the symbol \( \Gamma \) are assumed countable and discrete.

If Connes’ embedding problem has a negative answer, then it is quite natural to seek a counterexample in the group von Neumann algebras. It turns out \[ \text{Ra2}, \text{K3}, \text{Ra2} \] that embeddability of \( L\Gamma \) into \( R^\omega \) is equivalent to that of \( \Gamma \) into the unitary group \( \mathcal{U}(R^\omega) \) of \( R^\omega \). We say a group \( \Gamma \) is hyperlinear if it embeds into \( \mathcal{U}(R^\omega) \).

Proposition 7.1. A group \( \Gamma \) is hyperlinear if and only if \( L\Gamma \) is \( * \)-isomorphic to a von Neumann subalgebra of \( R^\omega \).

Proof. One direction is obvious. Let \( \theta: \Gamma \to \mathcal{U}(R^\omega) \) be an injective homomorphism and let \( \Theta = (\theta_n)_{n \in \mathbb{N}}: \Gamma \to \mathcal{U}(\prod_{n \in \mathbb{N}} R) \) be any lift, i.e., \( \pi\Theta(s) = \theta(s) \) for \( s \in \Gamma \). We note that \( \theta_n \) is asymptotically multiplicative, but no longer multiplicative. For each \( n \), we define
\[ \hat{\theta}_n: \Gamma \ni s \mapsto \begin{bmatrix} \theta_n(s) \\ \theta_n(s) \end{bmatrix} \in \mathcal{U}(\mathbb{M}_2(R \otimes R)). \]
Since $|z + z^2|/2 < 1$ for $z \in \mathbb{C}$ with $|z| \leq 1$ and $z \neq 1$, we have $\lim_{n \to \infty} |\tau\tilde{\theta}_n(s)| < 1$ for all $s \in \Gamma$ with $s \neq 1$. It follows that for an appropriately chosen sequence $k(n)$ of integers, the sequence of functions
\[
\psi_n := \tilde{\theta}_{n \otimes k(n)} : \Gamma \to \mathcal{U}(\mathcal{M}_2(R)^{\otimes k(n)}) \cong \mathcal{U}(R)
\]
is asymptotically multiplicative and $\lim_{n \in \omega} \tau\psi_n(s) = 0$ for all $s \in \Gamma$ with $s \neq 1$. This means that for the homomorphism $\psi = \lim \psi_n$, the von Neumann subalgebra of $R^\omega$ generated by $\psi(\Gamma)$ is canonically $\ast$-isomorphic to $L\Gamma$.

**Definition 7.2.** We say a group $\Gamma$ has the property (F) (the factorization property) if the trace $\tau$ on the full $C^*$-algebra $C^*\Gamma$, defined by $\tau(s) = \delta_1, s$ for $s \in \Gamma$, is liftable.

By Theorem 6.1, a group $\Gamma$ has the property (F) if and only if the representation $\sigma_\Gamma : C^*\Gamma \otimes C^*\Gamma \ni \sum_k a_k \otimes b_k \mapsto \sum_k \lambda_\Gamma(a_k)\rho_\Gamma(b_k) \in \mathbb{B}(\ell_2\Gamma)$
is continuous w.r.t. the minimal tensor norm, where $\lambda_\Gamma$ (reps. $\rho_\Gamma$) is the left (resp. right) regular representation of $\Gamma$ on $\ell_2\Gamma$. A group $\Gamma$ with the property (F) is hyperlinear, and to the best of our knowledge, all groups known to be hyperlinear satisfy the property (F). Let $\Gamma$ be a hyperlinear group and let $N$ be a normal subgroup of $\mathbb{F}_\infty$ such that $\Gamma = \mathbb{F}_\infty/N$. Then, because of the LLP of $C^*\mathbb{F}_\infty$, the trace $\tau_\Gamma$ on $C^*\mathbb{F}_\infty$, given by the characteristic function of $N$, is liftable. Hence, the study on permanence properties of liftable traces is more general than that of hyperlinearity.

If $P$ is a property of groups (e.g. being finite) then a group $\Gamma$ is said to be residually $P$ if, for any $s \in \Gamma$ with $s \neq 1$, there is a normal subgroup $\Delta$ in $\Gamma$ such that $s \notin \Delta$ and $\Gamma/\Delta$ satisfy $P$. When $\Delta$ is a subgroup of $\Gamma$, the quotient homogeneous space $\Gamma/\Delta$ is said to be amenable if there is a $\Gamma$-invariant mean on $\ell_\infty(\Gamma/\Delta)$. This is equivalent to the existence of a Følner sequence in $\Gamma/\Delta$ (see [Gl]). As it was observed by Wassermann [Wa] and Kirchberg [Ki3], residually finite groups and in particular the free groups have the property (F). This also follows from a more general result of Brown and Dykema [BD] that the property (F) is preserved under a free product (with amalgamation over a finite subgroup).

**Proposition 7.3.** A group $\Gamma$ which is residually (F) has the property (F). In particular, residually finite groups have the property (F).

Moreover, the property (F) is preserved under taking a subgroup, a supergroup with an amenable quotient, a direct product, an increasing union, and an amalgamated free product over a finite group.

**Proof.** The first assertion follows from the fact that the set of liftable traces is closed in the pointwise topology. We only prove nontrivial claims in the second.

The claim on a supergroup follows from a ucp analogue of induction. Let $\Delta \leq \Gamma$ and fix a section $\sigma : \Gamma/\Delta \to \Gamma$ so that $\Gamma = \bigcup_{x \in \Gamma/\Delta} \sigma(x)\Delta$. Let $\varphi : C^*\Delta \to \mathbb{B}(\mathcal{H})$
be a ucp map and let $F \subset \Gamma/\Delta$ be a finite subset. Then, the ucp map $\varphi$ dilates to a ucp map $\Phi^F: C^*\Gamma \to \mathbb{B}(\ell_2F \otimes \mathcal{H})$ given by

$$\Phi^F(s) = \sum_{x \in F \cap s^{-1}F} e_{sx,x} \otimes \varphi(\sigma(sx)^{-1}s\sigma(x)) \in \mathbb{B}(\ell_2F \otimes \mathcal{H}) \text{ for } s \in \Gamma.$$ 

Indeed, let $\hat{\mathcal{H}}$ be the Hilbert space arising from $\mathbb{C}\Gamma \otimes \mathcal{H}$ with the inner product $\langle \delta_t \otimes \eta, \delta_s \otimes \xi \rangle = (\varphi(s^{-1}t)\eta | \xi)$, where $\varphi$ is extended on $\Gamma$ by putting $\varphi(s) = 0$ for $s \in \Gamma \setminus \Delta$. Clearly, $\Gamma$ acts on $\hat{\mathcal{H}}$ by left translation. Since $\Phi^F$ coincides with the restriction of this unitary representation to the subspace generated by $\mathbb{C}\sigma(F) \otimes \mathcal{H}$, it is ucp. Now, we assume that $\Gamma/\Delta$ is amenable and $\Delta$ has the property (F). Then, there is a sequence of ucp maps $\varphi_n: C^*\Delta \to \mathbb{M}_{k(n)}$ such that $\lim \tr_{k(n)}(\varphi_n(s) = \delta_{1,s}$ and $\lim \tr_{k(n)}(\varphi_n(s)^*\varphi_n(s)) = 1$ for all $s \in \Delta$. It is not too hard to see that the sequence $\Phi_n^\Gamma$ on $C^*\Gamma$ satisfy the same conditions for a suitably chosen Følner sequence $F_n \subset \Gamma/\Delta$.

We turn to an amalgamated free product. Let $A_i$ ($i \in I$) be $C^*$-algebras with a common finite dimensional $C^*$-subalgebra $B$ and let $\tau_i$ be liftable traces on $A_i$ which agree on $B$. We have to show the free product trace $\tau$ on the full amalgamated free product $*_{i \in I}(A_i, B)$ is liftable. We may assume that $I = \{1, 2\}$. Since each $\tau_i$ is liftable, there is an asymptotically multiplicative and asymptotically trace preserving ucp map $\varphi_i$ from $A_i$ into a full matrix algebra $D_i$. We claim that $\varphi_i$ can be chosen exactly trace preserving and multiplicative on $B$. Indeed, by amplifying the range $D_i$ if necessary, we may assume that there is a trace preserving $*$-homomorphism $\pi_i: B \to D_i$ such that $\|\varphi_i(u) - \pi_i(u)\|_2 \approx 0$ for $u \in \mathcal{U}(B)$ (cf. [BD] for technical details). Note that $\mathcal{U}(B)$ is compact since $B$ is finite dimensional. We define a map $\varphi_i': A_i \to D_i$ by

$$\varphi_i'(a) = \int \int_{\mathcal{U}(B) \times \mathcal{U}(B)} \pi_i(u)\varphi_i(u^*av)\pi_i(v^*) \, du \, dv.$$ 

It is not too hard to see that $\varphi_i'$ is a completely positive $B$-bimodule map. By our assumption, $e_i := 1 - \varphi_i'(1) \in \pi_i(B)_+$ is close to zero in the 2-norm. Finally, take a ucp extension $\pi_i: A_i \to D_i$ of $\pi_i$ and define a ucp map $\varphi_i'': A_i \to D_i$ by $\varphi_i''(a) = \varphi_i'(a) + e_i^{1/2}\pi_i(a)e_i^{1/2}$ for $a \in A_i$. Then, $\varphi_i''$ is a small perturbation of $\varphi_i$ and agrees with $\pi_i$ on $B$.

Let $A_i^0 \subset A_i$ be the orthogonal complement of $B$ w.r.t. the trace $\tau_i$. By Boca’s theorem [BD], the $B$-bimodule map $\Phi: (A_1, B) * (A_2, B) \to (D_1, B) * (D_2, B)$, defined by $\Phi(a_1 \cdots a_n) = \varphi_i(a_1) \cdots \varphi_i(a_n)$ for $a_k \in A_i, i_1 \neq \cdots \neq i_n$, is a ucp map between the full amalgamated products. Since each $\varphi_i$ is asymptotically trace preserving and asymptotically multiplicative, so is $\Phi$. Thus, we reduced the problem to the case where $A_i$ are full matrix algebras $D_i$. This case was proved by Brown and Dykema [BD]. Indeed, under this assumption, they showed that the reduced amalgamated free product is embeddable into an interpolated free group.
factor (and a fortiori into $R^\omega$) and a $*$-homomorphism from the full amalgamated free product into $R^\omega$ is ucp liftable. □

Brown suggested a possibility that all one-relator groups have the property (F). We note that the Baumslag-Solitar groups $BS(p, q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$, which are typical examples of non-residually finite one-relator groups (at least for $1 < p < q$), are hyperlinear [Ra2] and moreover have the property (F) since they are HNN extension of the cyclic group $Z$ and hence embeddable in the semidirect products by $Z$ of the increasing union of residually finite groups. However, it is not that all groups have the property (F). Indeed, infinite simple groups with Kazhdan’s property (T) (for existence of such groups, see [Gr1]) do not have the property (F) as it was shown by Kirchberg [Ki3]. We refer the reader to de la Harpe and Valette’s book [HV] for Kazhdan’s property (T). The following proof is close to that of Kirchberg [Ki3], but uses an idea of Bekka [Wa].

**Theorem 7.4.** Let $\tau$ be a trace on the full $C^*$-algebra $C^*\Gamma$ of a group $\Gamma$ with Kazhdan’s property (T). Then, $\tau$ is liftable if and only if there is a sequence of $*$-homomorphisms $\pi_n : C^*\Gamma \to M_{k(n)}$ such that $\tau(a) = \lim_n \text{tr}_{k(n)} \pi_n(a)$.

In particular, a group $\Gamma$ with the properties (F) and (T) is residually finite.

**Proof.** Let $\hat{\Theta}$ be a set of irreducible representations of $\Gamma$ which contains exactly one representation from each equivalence class. It follows that $\Theta = \bigoplus_{\theta \in \hat{\Theta}} \theta$ is a faithful representation of $C^*\Gamma$ on the Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_\theta$. We consider the faithful representation $(\Theta \otimes \bar{\Theta})^\infty$ of $C^*\Gamma \otimes_{\text{min}} C^*\Gamma$ on $(\mathcal{H} \otimes \bar{\mathcal{H}})^\infty$. By the assumption, the state $\mu_\tau$ on $C^*\Gamma \otimes_{\text{min}} C^*\Gamma$ is continuous. Hence, it is approximated by a vector states associated with a unit vector $\xi_n \in (\mathcal{H} \otimes \bar{\mathcal{H}})^\infty$. Since $\mu_\tau(s \otimes \bar{s}) = 1$ for all $s \in \Gamma$, the sequence of unit vectors $\{\xi_n\}$ is almost invariant under $(\Theta \otimes \bar{\Theta})^\infty(\Gamma)$. (Here, we regard $(\Theta \otimes \bar{\Theta})^\infty$ as a representation of $\Gamma$.) Hence, by Kazhdan’s property (T), we may assume that $\xi_n$ are actually invariant under $(\Theta \otimes \bar{\Theta})^\infty(\Gamma)$. The well-known lemma of Schur states that for irreducible $\theta_1$ and $\theta_2$, the representation $\theta_1 \otimes \bar{\theta}_2$ has a non-zero invariant vector if and only if $\theta_1$ and $\theta_2$ are equivalent and finite dimensional. Moreover, any invariant vector for $\theta \otimes \bar{\theta}$ is a constant multiple of $\eta_\theta := d_{\theta}^{-1/2} \sum_k \xi_k \otimes \bar{\xi}_k$, where $\{\xi_k\}_k$ is an orthonormal basis of the $d_{\theta}$-dimensional Hilbert space $\mathcal{H}_\theta$. It follows that we may assume that the vector states associated with $\xi_n$ is a convex combination (with rational coefficients) of that of $\eta_\theta$’s with finite dimensional $\theta$. We obtain the conclusion by observing that $((\Theta(s) \otimes \bar{I}) \eta_\theta \mid \eta_\theta) = \text{tr}_{d_{\theta}} \theta(s)$ for all $s \in \Gamma$. □

It is unknown whether there exists a simple property (T) group $\Gamma$ which is hyperlinear. By the above theorem, the full $C^*$-algebra $C^*\Gamma$ of such a group $\Gamma$ cannot have the LLP. It is obvious that an inductive limit of hyperlinear groups is again hyperlinear. Since Gromov [Gr1] constructed infinite simple (T) groups as inductive limits (where connecting maps are surjective and not injective) of
hyperbolic groups, it is particularly interesting to know whether all hyperbolic

groups are hyperlinear (or more generally satisfy the property (F)).

8. Further Topics and Open Problems

Haagerup and Winsløw [HW1] [HW2] studied the space vN(H) of all von Neu-

mann algebras acting on a fixed separable Hilbert space H, equipped with the

Effros-Maréchal topology. Among other things, they proved that a II₁-factor

M ∈ vN(H) is embeddable into ℓ∞ if and only if it is approximated by finite
dimensional factors in vN(H). They also gave a new proof of equivalence between
the conjectures.

Rădulescu [Ra1] [Ra2] studied the space of the moments of noncommutative
monomials of degree p in a II₁-factor M and proved that it coincides with that for
the hyperfinite II₁-factor R when p ≤ 3. He also showed that the same assertion
for p = 4 is equivalent to the Connes’ embedding problem.

Brown [Br1] [Br2] studied various approximation properties of traces on C*-algebras and proved in particular that a separable II₁-factor (or any von Neumann
algebra) M ⊂ B(H) is embeddable into ℓ∞ (resp. QWEP) if and only if there are
a weak*-dense C*-algebra A in M and a ucp map ϕ : B(H) → M with ϕ|A = idA.
There is a related result of Brown and Dykema [BD] on the (interpolated) free
group factors LFs.

The following questions were raised by Kirchberg [Ki1].

Problem. Is the QWEP conjecture true?

Problem. Does there exist a non-nuclear C*-algebra with the WEP and the LLP?

What Kirchberg found is very close to an example. He found an short exact
sequence 0 → ℳ(ℓ2) → B → cone(C*_red F₂) → 0 with B ⊗_{min} B^{op} = B ⊗_{max} B^{op}.
By Theorem 3.3, the C*-algebra B has the WEP. Unfortunately, it is not known
whether B has the LLP. We remark that a C*-algebra with the WEP has the LLP
if and only if it is os finitely representable in C*_F∞ (cf. [JP]). In the remarkable
paper [HT], Haagerup and Thorbjørnsen proved that C*_red F₂ is embeddable into
\prod \mathcal{M}_n/ \bigoplus \mathcal{M}_n, which has to be weakly cp complemented (cf. the remarks at
the end of section 4). Therefore, all quasidiagonal extensions of C*_red F₂ have the WEP.
However, there are uncountably many mutually non-isomorphic such extensions,
among which at most countably many can possibly have the LLP. (The precise
statement is that the set of all 3-dimensional operator subspaces in quasidiagonal
extensions of C*_red F₂ is non-separable in the cb-distance topology (cf. [JP]).)

Problem. Does there exist a non-nuclear exact C*-algebra with the LLP?

Existence of such an example is in contradiction with the QWEP conjecture
since exact C*-algebras are locally reflexive [Ki2] and locally reflexive C*-algebras
with the WEP are necessarily nuclear [EH]. To the best of our knowledge, all
$C^*$-algebras known to have the LLP are either nuclear or equivalent to $C^*\mathbb{F}_\infty$ in the sense that they are (cp complemented) subquotients of each other. We note that a $C^*$-algebra $A$ has both the exactness and the LLP if and only if $A \otimes_{\min} (\mathcal{B}(\ell_2)/\mathbb{K}(\ell_2)) = A \otimes_{\max} (\mathcal{B}(\ell_2)/\mathbb{K}(\ell_2))$.

**Problem.** Does there exist a separable $C^*$-algebra with the LLP, but not the LP?

Again, existence of such an example is in contradiction with the QWEP conjecture by Corollary 3.12. It seems that we have no example of a locally ucp liftable map defined on a separable $C^*$-algebra which does not have a global lifting.

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