RELAXATION FOR OPTIMAL DESIGN PROBLEMS WITH NON-STANDARD GROWTH

Ana Cristina Barroso
Departamento de Matemática and CMAF-CIO
Faculdade de Ciências da Universidade de Lisboa
Campo Grande, Edifício C6, Piso 1
1749-016 Lisboa, Portugal
acbarroso@ciencias.ulisboa.pt

and

Elvira Zappale
Dipartimento di Ingegneria Industriale
Università degli Studi di Salerno
Via Giovanni Paolo II, 132
84084 Fisciano (SA), Italy
and CIMA (Universidade de Évora)
ezappale@unisa.it

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This paper is dedicated to the memory of our good friend and colleague Graça Carita, with whom the second author had the privilege of collaborating on many previous occasions. Graça joined us at the early stages of this work and tragically passed away on September 26, 2016.

Abstract

In this paper we investigate the possibility of obtaining a measure representation for functionals arising in the context of optimal design problems under non-standard growth conditions and perimeter penalization. Applications to modelling of strings are also provided.

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1 Introduction

Given two conductive materials present in a container Ω, and prescribing the volume fraction of each one, the optimal design problem, originally studied by Murat and Tartar and by Kohn and Strang in [33, 32, 30], consists of identifying the minimal energy configuration of the mixture. As emphasized in [33] (see, for instance, the one dimensional model case in Proposition 4 and Remark 7 therein), this problem might not have a solution. In the works of Ambrosio and Buttazzo [4] and Kohn and Lin [27] this difficulty is overcome by the introduction, in the energy functional to be minimized, of a term which penalizes the perimeter of the sets where the mixture equals one of the conductive materials, thus also eliminating the case where the two materials are finely mixed.
In order to localize (1.2) we set, for every open set \( U \),
\[
 u \in W^{1,1}(\Omega; \mathbb{R}^d), \chi_E \in BV(\Omega; \{0,1\}), \quad u = u_0 \text{ on } \partial \Omega,
\]
where the densities \( W_i, i = 1, 2 \), are continuous functions such that there exist positive constants \( \alpha \) and \( \beta \) for which
\[
\alpha|\xi| \leq W_i(\xi) \leq \beta(1 + |\xi|), \quad \forall \xi \in \mathbb{R}^{d \times N}.
\]
Since no convexity assumptions were placed on \( W_i \), they considered the relaxed localized energy arising from the above problem
\[
\mathcal{F}_{OD} \chi, u; A \) := \inf \left\{ \liminf_{n \to +\infty} \left[ \int_{\Omega} \chi_n(x)W_1(\nabla u_n(x)) + (1 - \chi_n(x))W_2(\nabla u_n(x)) \, dx + |D\chi_n|(A) \right] : \right.
\]
\[
 u_n \in W^{1,1}(A; \mathbb{R}^d), \chi_n \in BV(A; \{0,1\}),
\]
\[
u_n \rightharpoonup u \text{ in } L^1(A; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(A; \{0,1\}) \right\},
\]
and they showed that \( \mathcal{F}_{OD} \chi, u; \cdot \) is the trace on the open subsets of \( \Omega \) of a finite Radon measure. The characterisation of this measure was provided by obtaining an integral representation for \( \mathcal{F}_{OD} \chi, u; A \), where \( A \) is an open subset of \( \Omega \).

The case of non-convex \( W_i \) with superlinear growth was addressed in the context of thin films in [12].

Our aim in this paper is to study the above optimal design problem within the context of non-standard growth conditions. We take \( \Omega \) to be a bounded, open subset of \( \mathbb{R}^N \) and we let
\[
1 < p \leq q < \frac{Np}{N-1}.
\]
If \( N = 1 \) we let \( 1 < p \leq q < +\infty \). Let \( F : BV(\Omega; \{0,1\}) \times W^{1,p}(\Omega; \mathbb{R}^d) \to [0, +\infty) \) be given by
\[
F(\chi, u) := \int_{\Omega} \chi(x)W_1(\nabla u(x)) + (1 - \chi(x))W_2(\nabla u(x)) \, dx + |D\chi|(\Omega)
\]
(1.2)
where \( W_i : \mathbb{R}^{d \times N} \to \mathbb{R}, i = 1, 2 \), are continuous functions satisfying the following growth condition
\[
\exists \beta > 0 : 0 \leq W_i(\xi) \leq \beta(1 + |\xi|^p), \quad \forall \xi \in \mathbb{R}^{d \times N}.
\] (1.3)
To simplify the notation, in the sequel, we let \( \chi_n \in BV(A; \{0,1\}) \times W^{1,p}(A; \mathbb{R}^d) \),
\[
F(\chi, u; A) := \int_{A} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(A),
\] (1.5)
and we define the relaxed functionals
\[
F(\chi, u; A) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}(A; \mathbb{R}^d), \chi_n \in BV(A; \{0,1\}) ,
\]
\[
u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(A; \{0,1\}) \right\},
\] (1.6)
and
\[
\mathcal{F}_{loc} \chi, u; A) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}_{loc}(A; \mathbb{R}^d), \chi_n \in BV(A; \{0,1\}) ,
\]
\[
u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(A; \{0,1\}) \right\}.
\] (1.7)
Notice that in these functionals there is a gap between the space of admissible macroscopic fields \( u \in W^{1,p}(A;\mathbb{R}^d) \) and the smaller space \( W^{1,q}(A;\mathbb{R}^d) \) where the growth hypothesis (1.3) ensures boundedness of the energy. Thus, assuming also that the following coercivity condition holds

\[ \exists \alpha > 0 : W_1(\xi) \geq \alpha |\xi|^p, \quad \forall \xi \in \mathbb{R}^{d \times N}, \quad (1.8) \]

sequences of deformations \( u_n \in W^{1,q}(A;\mathbb{R}^d) \) that have bounded energy will be weakly compact in \( W^{1,p}(A;\mathbb{R}^d) \), but not necessarily in \( W^{1,q}(A;\mathbb{R}^d) \), so it may be possible to energetically approach functions \( u \in W^{1,p} \setminus W^{1,q} \) and the aim of the above relaxed functionals is to provide the effective energy associated to \( u \).

We recall that a sequence \( \chi_n \to \chi \) in \( BV \) weak * if and only if \( \chi_n \) is bounded in \( BV \) and \( \chi_n \to \chi \) in the strong topology of \( L^1 \). Due to this fact, and to the expression of the energy (1.2), in the above functionals we could also have taken \( \chi_n \to \chi \) in \( L^1(A;\{0,1\}) \), obtaining in each case the same infimum. However, replacing the \( W^{1,p} \) weak convergence of the sequence \( u_n \) by its strong convergence in \( L^p \) does not yield the same infimum, unless a coercivity condition of the type (1.8) is considered.

We will also consider the case \( p = 1 \) and \( 1 \leq q < \frac{N}{N-1} \) (1 \( < q < +\infty \) when \( N = 1 \)), so we define

\[ \mathcal{F}_1(\chi, u; A) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}(A;\mathbb{R}^d), \chi_n \in BV(A;\{0,1\}) \right\}, \quad (1.9) \]

and, analogously, by replacing \( W^{1,q} \) by \( W^{1,q}_{\text{loc}} \), \( \mathcal{F}_{1,\text{loc}}(\chi, u; A) \). Contrary to what happens with regard to the topology used for the convergence of the sequence \( \chi_n \), as in the case \( p > 1 \), taking \( L^1 \) strong or \( BV \) weak * convergence for the sequence \( u_n \) does not give the same infimum, unless a coercivity condition of the type (1.8) is considered.

Similar functionals were considered by Soneji in [35] and [36] in the case where there is no dependence on the field \( \chi \).

Our goal in this paper is to investigate whether the functionals (1.6), (1.7), and their variants when \( p = 1 \), can be represented by certain Radon measures defined on the open subsets of \( \Omega \) and, if so, whether a characterisation of these measures can be obtained. As it turns out a (strong) measure representation is only true for (1.7) (see Theorem 4.3 where some information on the corresponding measure is provided under some convexity assumptions), whereas (1.6) only admits a weak measure representation (cf. Definition 2.11 and Theorem 4.1). In the one dimensional case we provide a characterisation of these measures (see Proposition 4.8).

In the case independent of the field \( \chi \), and when \( p = q \), it is well known that

\[ \mathcal{F}(u; A) = \inf \left\{ \liminf_{n \to +\infty} \int_\Omega f(\nabla u_n(x)) \, dx : u_n \in W^{1,q}(A;\mathbb{R}^d), u_n \to u \text{ in } W^{1,p}(A;\mathbb{R}^d) \right\} \]

\[ = \int_\Omega Qf(\nabla u(x)) \, dx, \]

where \( Qf \) denotes the quasiconvex envelope of \( f \) (see Definition 2.8). If \( q > \frac{Np}{N-1} \) it may happen that \( \mathcal{F}(u; \Omega) = 0 \) (see [3]), and if \( q = \frac{Np}{N-1} \) then \( \mathcal{F}(u; \cdot) \) may not even be subadditive (see [3]).

Assuming the above growth and coercivity hypotheses on the density \( f \),

\[ \alpha |\xi|^p \leq f(\xi) \leq \beta (1 + |\xi|^q), \quad (1.10) \]

and relying on the existence of a trace-preserving linear operator from \( W^{1,p} \) to \( W^{1,q} \) which improves the integrability of \( u \) and \( \nabla u \) (see Lemma 3.2), it was shown by Fonseca and Malý in [20] that the functionals

\[ \mathcal{F}_{q,p}(u; A) := \inf \left\{ \liminf_{n \to +\infty} \int_\Omega f(\nabla u_n(x)) \, dx : u_n \in W^{1,q}(A;\mathbb{R}^d), u_n \to u \text{ in } W^{1,p}(A;\mathbb{R}^d) \right\} \]

and \( \mathcal{F}_{q,p}^{\text{loc}}(u; A) \) (defined as above just replacing \( W^{1,q} \) by \( W^{1,q}_{\text{loc}} \)) when finite, admit a weak measure representation in the first case, and a strong measure representation in the second. In the latter case,
denoting by $\mu_a$ the density of the corresponding measure $\mu$ with respect to the Lebesgue measure, they also showed that

$$\mu_a(x_0) \geq Qf(\nabla u(x_0)), \ \forall u \in W^{1,p}(\Omega; \mathbb{R}^d), \ \text{a.e. } x_0 \in \Omega.$$ 

The reverse inequality was obtained in [9], thus fully identifying the bulk part of the measure $\mu$.

We also refer to Acerbi, Bouchitté and Fonseca [1] where the case of inhomogeneous densities $h(x, \xi)$ is treated. Assuming convexity of $h$ with respect to the second variable, as well as a growth condition of the type (1.10), it is shown that

$$F_{\text{loc}}^{q,p}(u; A) = \int_A h(x, \nabla u(x)) \, dx + \mu_s(u; A),$$

where $\mu_s(u, \cdot)$ is a non-negative Radon measure, singular with respect to the Lebesgue measure.

Finally, we mention the results of Coscia and Mucci [14] and Mucci [32]. In their work, the authors consider relaxation and integral representation of integral functionals of the type $\int_\Omega h(x, \nabla u(x)) \, dx$, when $h$ satisfies

$$|\xi|^{p(x)} \leq h(x, \xi) \leq C(1 + |\xi|^{p(x)}), \quad (1.11)$$

so that the integrability exponent $p(x)$ of the admissible fields depends in a continuous or regular piecewise continuous way on the location in the body (see [13] assumptions (2.9) and (2.1) and Definitions 2.1 and 2.2, respectively). We point out that the bulk energy density in (1.2) can be seen as satisfying a generalization of (1.11) to the case of a discontinuous exponent $p(x)$, we refer to Remark 4.5 for more details.

The paper is organized as follows. In Section 2 we set the notation and we provide some definitions and results which will be used throughout the paper, whereas in Section 3 we prove some auxiliary results. The main representation theorems are proved in Section 4, where we provide a partial characterisation of the measures which represent (1.6) and (1.7) in the convex case and a full characterisation in the one dimensional setting, we also obtain a sufficient condition for lower semicontinuity of (1.2). Finally, in Section 5 we give an application to a 3D-1D dimension reduction problem.

2 Preliminaries

In this section we fix notations and quote some definitions and results dealing with sets of finite perimeter and several notions of quasiconvexity that will be used in the sequel.

Throughout the text $\Omega \subset \mathbb{R}^N$ will denote an open, bounded set.

We will use the following notations:

- $\mathcal{O}(\Omega)$ is the family of all open subsets of $\Omega$;
- $\mathcal{M}(\Omega)$ is the set of finite Radon measures on $\Omega$;
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ stand for the $N$-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively;
- $|\mu|$ stands for the total variation of a measure $\mu \in \mathcal{M}(\Omega)$;
- the symbol $dx$ will also be used to denote integration with respect to $\mathcal{L}^N$;
- $C$ represents a generic positive constant that may change from line to line.

We start by recalling a well known result due to Ioffe [24 Theorem 1.1].

**Theorem 2.1.** Let $g : \mathbb{R}^m \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a Borel integrand such that $g(b, \cdot)$ is convex for every $b \in \mathbb{R}^m$. Then the functional

$$G(v, u) := \int_{\Omega} g(v(x), \nabla u(x)) \, dx$$

is lower semicontinuous in $L^1(\Omega; \mathbb{R}^m)_{\text{strong}} \times W^{1,1}(\Omega; \mathbb{R}^d)_{\text{weak}}$. 


In the following we give some preliminary notions related with sets of finite perimeter. For a detailed treatment we refer to [4].

A function \( w \in L^1(\Omega; \mathbb{R}^d) \) is said to be of bounded variation, and we write \( w \in BV(\Omega; \mathbb{R}^d) \), if all its first order distributional derivatives \( D_j w_i \) belong to \( M(\Omega) \) for \( 1 \leq i \leq d \) and \( 1 \leq j \leq N \).

The matrix-valued measure whose entries are \( D_j w_i \) is denoted by \( Dw \) and \( |Dw| \) stands for its total variation. We observe that if \( w \in BV(\Omega; \mathbb{R}^d) \) then \( w \mapsto |Dw|(\Omega) \) is lower semicontinuous in \( BV(\Omega; \mathbb{R}^d) \) with respect to the \( L^{1}_{\text{loc}}(\Omega; \mathbb{R}^{d^2}) \) topology.

**Definition 2.2.** Let \( E \) be an \( L^N \)-measurable subset of \( \mathbb{R}^N \). For any open set \( \Omega \subset \mathbb{R}^N \) the perimeter of \( E \) in \( \Omega \), denoted by \( P(E; \Omega) \), is the variation of \( \chi_E \) in \( \Omega \), i.e.

\[
P(E; \Omega) := \sup \left\{ \int_E \text{div}\varphi(x) \, dx : \varphi \in C^1_c(\Omega; \mathbb{R}^d), ||\varphi||_{L^\infty} \leq 1 \right\}.
\]

We say that \( E \) is a set of finite perimeter in \( \Omega \) if \( P(E; \Omega) < +\infty \).

Recalling that if \( L^N(E \cap \Omega) \) is finite, then \( \chi_E \in L^1(\Omega) \), by [5] Proposition 3.6, it follows that \( E \) has finite perimeter in \( \Omega \) if and only if \( \chi_E \in BV(\Omega) \) and \( P(E; \Omega) \) coincides with \( |D\chi_E|(\Omega) \), the total variation in \( \Omega \) of the distributional derivative of \( \chi_E \). Moreover, a generalized Gauss-Green formula holds:

\[
\int_E \text{div}\varphi(x) \, dx = \int_{\Omega} \langle \nu_E(x), \varphi(x) \rangle \, d|D\chi_E|, \quad \forall \varphi \in C^1_c(\Omega; \mathbb{R}^d),
\]

where \( D\chi_E = \nu_E |D\chi_E| \) is the polar decomposition of \( D\chi_E \).

We also recall that, when dealing with sets of finite measure, a sequence of sets \( E_n \) converges to \( E \) in measure in \( \Omega \) if \( L^N(\Omega \cap (E_n\Delta E)) \) converges to 0 as \( n \to +\infty \), where \( \Delta \) stands for the symmetric difference. This convergence is equivalent to \( L^1(\Omega) \) convergence of the characteristic functions of the corresponding sets.

In order to compare several notions of quasiconvexity, we start by recalling the one introduced by Morrey.

**Definition 2.3.** A Borel measurable and locally bounded function \( f : \mathbb{R}^{d \times N} \to \mathbb{R} \) is said to be quasiconvex if

\[
f(\xi) \leq \frac{1}{\mathcal{L}^N(D)} \int_D f(\xi + \nabla \varphi(x)) \, dx,
\]

for every bounded, open set \( D \subset \mathbb{R}^N \), for every \( \xi \in \mathbb{R}^{d \times N} \) and for every \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^d) \).

**Remark 2.4.** We recall that if \( 2 \geq p \geq 1 \) holds for a certain set \( D \), then it holds for any bounded, open set in \( \mathbb{R}^N \). Notice also that, in the above definition, the value \( +\infty \) is excluded from the range of \( f \).

The following notion of \( W^{1,p}\)-quasiconvexity was introduced by Ball and Murat in [5].

**Definition 2.5.** Let \( p \in [1, +\infty] \) and let \( f : \mathbb{R}^{d \times N} \to (-\infty, +\infty] \) be Borel measurable and bounded below. \( f \) is said to be \( W^{1,p}\)-quasiconvex if

\[
f(\xi) \leq \frac{1}{\mathcal{L}^N(D)} \int_D f(\xi + \nabla \varphi(x)) \, dx,
\]

for every bounded, open set \( D \subset \mathbb{R}^N \), with \( \mathcal{L}^N(\partial D) = 0 \), for every \( \xi \in \mathbb{R}^{d \times N} \) and for every \( \varphi \in W^{1,p}_0(D; \mathbb{R}^d) \).

**Remark 2.6.** Definition 2.5 coincides with quasiconvexity for finite functions \( f \) and when \( p = +\infty \).

If \( f \) is \( W^{1,p} \)-quasiconvex, then it is \( W^{1,q} \)-quasiconvex for all \( q \) with \( p \leq q \leq +\infty \), so \( W^{1,1} \)-quasiconvexity is the strongest notion and \( W^{1,\infty} \)-quasiconvexity is the weakest notion.

The following facts were established in [5] Proposition 2.4. If \( f \) is bounded below, upper semicontinuous and

\[
f(\xi) \leq K(1 + |\xi|^p), \quad \text{for every } \xi \in \mathbb{R}^{d \times N},
\]
then $f$ is $W^{1,q}$-quasiconvex if and only if it is $W^{1,\infty}$-quasiconvex.

If there exist $K_1 \in (0, +\infty)$ and $K_2 \in \mathbb{R}$ such that
\[ K_1 |\xi|^p + K_2 \leq f(\xi), \quad \text{for every } \xi \in \mathbb{R}^{d \times N}, \tag{2.5} \]
for $p \in [1, +\infty)$, then $f$ is $W^{1,1}$-quasiconvex if and only if it is $W^{1,p}$-quasiconvex.

It is known that the above notions, when applied to $f(b, \cdot) := bW_1(\cdot) + (1 - b)W_2(\cdot)$, are necessary and sufficient conditions for lower semicontinuity and Jensen's inequality (2.6), when $f$ satisfies both (2.5) and (2.4) with the same exponent $p = q > 1$. The sufficiency of $W^{1,p}$-quasiconvexity for the lower semicontinuity of $F(\chi, u)$ is no longer true in the case $p < q$, see Gangbo [22] for an example.

The notion of closed $W^{1,p}$-quasiconvexity, which we recall next, was introduced by Pedregal in [34] and used by Kristensen in [31] in order to provide sufficient conditions for the lower semicontinuity of the functional

$F$.

The following result (see [21, Proposition 2.5]) will be used in the sequel.

Definition 2.8. The integrand $f : \mathbb{R}^{d \times N} \to \mathbb{R}$ is said to be homogeneous if there is a Radon measure $\mu$ such that $\langle \mu, |\cdot|^p \rangle < +\infty$, when $p < +\infty$, and with bounded support when $p = +\infty$, and such that the following Jensen’s inequality,
\[ \int_{\mathbb{R}^{d \times N}} f(\xi) \, d\mu \geq f(\overline{\mu}), \quad \text{where } \overline{\mu} := \langle \mu, \text{id} \rangle, \tag{2.6} \]
holds for all $W^{1,\infty}$-quasiconvex functions $f : \mathbb{R}^{d \times N} \to \mathbb{R}$ for which there exists a constant $c = c(f)$ such that $|f(\xi)| \leq C(1 + |\xi|^p)$.

The Young measure $\mu$ is said to be homogeneous if there is a Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\mu_x = \mu_0$ for a.e. $x \in \Omega$.

In [25, 26], the set $\mathcal{M}^p$ is shown to coincide with the homogeneous $W^{1,p}$-gradient Young measures. We recall that a Young measure $\mu$ is called a gradient Young measure if it is generated by a sequence of gradients, more precisely, $\mu$ is a $W^{1,p}$-gradient Young measure if it is generated by $\nabla u_n$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$.

We conclude by recalling the notions of weak and strong representations by means of measures.

Definition 2.10. Let $\mu$ be a Radon measure on $\overline{\Omega}$. We say that
\begin{itemize}
  \item[a)] $\mu$ (strongly) represents $\mathcal{F}(\chi, u; \cdot)$ if $\mu(A) = \mathcal{F}(\chi, u; A)$ for all open sets $A \subset \Omega$;
  \item[b)] $\mu$ weakly represents $\mathcal{F}(\chi, u; \cdot)$ if $\mu(A) \leq \mathcal{F}(\chi, u; A) \leq \mu(A)$ for all open sets $A \subset \Omega$.
\end{itemize}
3 Auxiliary Results

In this section we prove some auxiliary results which will be used in the sequel in the proofs of our representation theorems. We begin by showing that, under the coercivity assumption (1.8) the infima in (1.6) and (1.7) are attained.

Proposition 3.1. Let $f$ be as in (1.4) where $W_i$, $i = 1, 2$, satisfy the growth condition (1.3), as well as the coercivity condition (1.8). Let $F(\chi, u; \Omega)$ be as in (1.6).

i) If $F(\chi, u; \Omega) < +\infty$, then it is attained, that is, there exist sequences $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$, $\chi_n \in BV(\Omega; \{0,1\})$ such that $u_n \rightarrow u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \rightharpoonup \chi$ in $BV(\Omega; \{0,1\})$ and

$$F(\chi, u; \Omega) = \lim_{n \rightarrow +\infty} F(\chi_n, u_n; \Omega).$$

ii) If $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$, $\chi_n \in BV(\Omega; \{0,1\})$ are sequences such that $u_n \rightarrow u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \rightharpoonup \chi$ in $BV(\Omega; \{0,1\})$ and if $F(\chi_n, u_n; \Omega) < +\infty$, $\forall n$, then

$$F(\chi, u; \Omega) \leq \liminf_{n \rightarrow +\infty} F(\chi_n, u_n; \Omega).$$

iii) In the case $p = 1$, $1 \leq q < \frac{N}{N-1}$ the above results also hold for $F_{\text{loc}}$ under the assumption $W_i(\xi) \geq \alpha |\xi|$, for some $\alpha > 0$ and for all $\xi \in \mathbb{R}^{d \times N}$.

Similar statements hold for the functionals $F_{\text{loc}}$ given in (1.7) and $F_{1,\text{loc}}$.

Proof. i) By definition of $F(\chi, u; \Omega)$, for each $n \in \mathbb{N}$, there exist sequences $u_{n,k} \in W^{1,q}(\Omega; \mathbb{R}^d)$ and $\chi_{n,k} \in BV(\Omega; \{0,1\})$ such that $u_{n,k} \rightarrow u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ as $k \rightarrow +\infty$, $\chi_{n,k} \rightharpoonup \chi$ in $BV(\Omega; \{0,1\})$ as $k \rightarrow +\infty$ and

$$\liminf_{k \rightarrow +\infty} F(\chi_{n,k}, u_{n,k}) < F(\chi, u; \Omega) + \frac{1}{n}.$$ 

By taking subsequences if necessary, for each $n$, we may assume that

$$\liminf_{k \rightarrow +\infty} F(\chi_{n,k}, u_{n,k}) = \lim_{k \rightarrow +\infty} F(\chi_{n,k}, u_{n,k}).$$

Hence for each $n$, there exists $k_n^1$ such that

$$F(\chi_{n,k}, u_{n,k}) < F(\chi, u; \Omega) + \frac{1}{n}, \quad \forall k \geq k_n^1. \quad (3.1)$$

Since $u_{n,k} \rightarrow u$, as $k \rightarrow +\infty$, in $L^p(\Omega; \mathbb{R}^d)$, the sequence $u_{n,k}$ is bounded in $L^p(\Omega; \mathbb{R}^d)$ so, by the metrizability of the unit ball of $L^p(\Omega; \mathbb{R}^d)$ in the weak topology, there exists a metric $d$ such that $d(u_{n,k}, u) \rightarrow 0$ as $k \rightarrow +\infty$. Thus, there exists $k_n^2$ such that

$$d(u_{n,k}, u) < \frac{1}{n}, \quad \|\chi_{n,k} - \chi\|_{L^1(\Omega; \{0,1\})} < \frac{1}{n}, \quad \forall k \geq k_n^2.$$ 

Choose an increasing sequence $k_n \in \mathbb{N}$ such that $k_n \geq \max\{k_n^1, k_n^2\}$, then $d(u_{n,k_n}, u) \rightarrow 0$, so $u_{n,k_n} \rightarrow u$ in $L^p(\Omega; \mathbb{R}^d)$, and $\chi_{n,k_n} \rightarrow \chi$ in $L^1(\Omega; \{0,1\})$, as $n \rightarrow +\infty$. By the coercivity condition (1.8) and (3.1) it follows that

$$\sup_n \left[ \alpha \int_{\Omega} |\nabla u_{n,k_n}(x)|^p dx + |D\chi_{n,k_n}\{0,1\}|(\Omega) \right] \leq \limsup_n F(\chi_{n,k_n}, u_{n,k_n}; \Omega) \leq F(\chi, u; \Omega) + 1 < +\infty$$

so $u_{n,k_n}$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^d)$ and $\chi_{n,k_n}$ is bounded in $BV(\Omega; \{0,1\})$ and therefore these sequences are admissible for $F(\chi, u; \Omega)$. Hence, by (3.1),

$$F(\chi, u; \Omega) \leq \liminf_{n \rightarrow +\infty} F(\chi_{n,k_n}, u_{n,k_n}; \Omega) \leq \limsup_{n \rightarrow +\infty} \left( F(\chi, u; \Omega) + \frac{1}{n} \right) = F(\chi, u; \Omega),$$
which, passing to another subsequence if necessary, proves i).

ii) Follows immediately from part i) and a standard diagonalization argument.

iii) The proof in the case \( p = 1 \) is similar to the case \( p > 1 \) replacing the weak topology in \( W^{1,p} \) by the weak * topology in \( BV \).

In order to prove our measure representation results we will need the following lemmas (see Lemmas 2.4 and 3.4 in [20] for the case \( p = 1 \) and Lemma 5.4 in [35] for the case \( p = 1 \)).

**Lemma 3.2.** Let \( p \) and \( q \) satisfy (1.4). Let \( V \subset \subset \Omega \), \( W \subset \subset \Omega \) be open sets such that \( \Omega = V \cup W \), and let \( v \in W^{1,q} (V; \mathbb{R}^d) \), \( w \in W^{1,q} (W; \mathbb{R}^d) \). Then, for every \( m \in \mathbb{N} \), there exist \( z \in W^{1,q} (\Omega; \mathbb{R}^d) \) and open sets \( V' \subset V \) and \( W' \subset W \) such that \( V' \cup W' = \Omega \), \( z = v \) in \( \Omega \setminus W' \), \( z = w \) in \( \Omega \setminus V' \),

\[
L^N (V' \cap W') \leq \frac{C}{m} \tag{3.2}
\]

and

\[
\|z\|_{W^{1,q}(V' \cap W')} \leq \frac{C}{m^\tau} \left( \|v\|_{W^{1,q}(V' \cap W')} + \|w\|_{W^{1,q}(V' \cap W')} + m \|w - v\|_{L^p(V' \cap W')} \right), \tag{3.3}
\]

where \( C = C (p, q, V, W) \) and \( \tau = \tau (N, p, q) > 0 \).

**Lemma 3.3.** Let \( V, W \subset \subset \Omega \) be open sets such that \( V \subset \subset \Omega \) and \( \Omega = V \cup W \). Let \( f \) be as in (1.4) where \( W_i, i = 1, 2 \), satisfy the growth condition (1.3). If \( u \in W^{1,p} (\Omega; \mathbb{R}^d) \) and \( \chi \in BV (\Omega; \{0,1\}) \), then

\[
\mathcal{F} (\chi, u; \Omega) \leq \mathcal{F} (\chi, u; V) + \mathcal{F} (\chi, u; W).
\]

The same result holds for \( \mathcal{F}_1 (\chi, u; \Omega) \).

**Proof.** Choose \( \varepsilon > 0 \) and let \( V' \subset V \), \( W' \subset W \) be such that \( \Omega = V' \cup W' \), \( V' \cap W' \subset V \cap W \) and \( V' \) and \( W' \) have \( C^1 \) boundaries so that we can apply Rellich’s compact embedding theorem. Thus, by this result and the definition of the relaxed functionals \( \mathcal{F} (\chi, u; V) \) and \( \mathcal{F} (\chi, u; W) \), there exist \( v_n \in W^{1,q} (V; \mathbb{R}^d) \), \( \chi_n \in BV (V; \{0,1\}) \) and \( \zeta_n \in BV (W; \{0,1\}) \) such that

\[
\begin{align*}
&v_n \rightharpoonup u \text{ in } W^{1,p} (V; \mathbb{R}^d), \quad w_n \rightharpoonup u \text{ in } W^{1,p} (W; \mathbb{R}^d), \\
&\chi_n \rightharpoonup \chi \text{ in } BV (V; \{0,1\}), \quad \zeta_n \rightharpoonup \chi \text{ in } BV (W; \{0,1\}), \\
&\|v_n - u\|_{L^p(V' \cap W')} \leq \frac{1}{n}, \quad \|w_n - u\|_{L^p(V' \cap W')} \leq \frac{1}{n}, \\
&\|\chi_n - \chi\|_{L^1(V' \cap W')} \leq \frac{1}{n}, \quad \|\zeta_n - \chi\|_{L^1(V' \cap W')} \leq \frac{1}{n},
\end{align*}
\]

and

\[
\begin{align*}
&\int_V f (\chi_n (x), \nabla v_n (x)) \, dx + |D\chi_n| (V) \leq \mathcal{F} (\chi, u; V) + \varepsilon, \\
&\int_W f (\zeta_n (x), \nabla w_n (x)) \, dx + |D\zeta_n| (W) \leq \mathcal{F} (\chi, u; W) + \varepsilon.
\end{align*}
\]

By virtue of Lemma 3.2 there exist open sets \( V_n \subset V' \), \( W_n \subset W' \), and functions \( z_n \in W^{1,q} (\Omega; \mathbb{R}^d) \), such that \( V_n \cup W_n = \Omega \),

\[
z_n = v_n \text{ in } \Omega \setminus W_n, \quad z_n = w_n \text{ in } \Omega \setminus V_n, \tag{3.4}
\]

and estimates of the type (3.2) and (3.3) hold.

The sequence \( z_n \) is admissible for \( \mathcal{F} (\chi, u; \Omega) \) since \( z_n \rightharpoonup u \) weakly in \( W^{1,p} (\Omega; \mathbb{R}^d) \). Indeed, by (3.2) and (3.4), \( z_n \) converges to \( u \) in measure, and by (3.3), \( z_n \) is bounded in \( W^{1,p} (\Omega; \mathbb{R}^d) \), so it follows that, for a subsequence which we do not relabel, \( z_n \rightharpoonup u \) in \( W^{1,p} (\Omega; \mathbb{R}^d) \).

We must now build a transition sequence \( \eta_n \) between \( \chi_n \) and \( \zeta_n \) in the above sets \( V_n \) and \( W_n \), in such a way that an upper bound for the total variation of \( \eta_n \) in \( V_n \cap W_n \) is obtained. In order to
connect these functions without adding more interfaces, we argue as in \[7\] (see also \[13\]). For \(\delta > 0\) small enough, consider
\[
V_\delta' := \{ x \in W' \cap V' : \text{dist}(x, \partial V') < \delta \}.
\]
Given \(x \in W\), let \(d(x) := \text{dist}(x; V')\). Since the distance function to a fixed set is Lipschitz continuous (see Exercise 1.1 in Ziemer \[37\] and \[23\]) we can apply the change of variables formula (see Theorem 2, Section 3.4.3, in Evans and Gariepy \[17\]), to obtain
\[
\int_{\partial V'_{\rho}} \left| \chi_n(x) - \zeta_n(x) \right| |\det \nabla d(x)| \, dx = \int_0^\delta \int_{d^{-1}(y)} \left| \chi_n(x) - \zeta_n(x) \right| d\mathcal{H}^{N-1}(x) \, dy
\]
and, as \(|\det \nabla d(x)|\) is bounded and \(\chi_n - \zeta_n \to 0\) in \(L^1(W' \cap V'; \mathbb{R}^d)\), it follows that, for almost every \(\rho \in [0, \delta]\),
\[
\lim_{n \to +\infty} \int_{d^{-1}(\rho)} |\chi_n(x) - \zeta_n(x)| \, d\mathcal{H}^{N-1}(x) = \lim_{n \to +\infty} \int_{\partial V'_{\rho}} |\chi_n(x) - \zeta_n(x)| \, d\mathcal{H}^{N-1}(x) = 0. \tag{3.5}
\]
Fix \(\rho_0 \in [0, \delta]\) such that \(\ref{3.5}\) holds. We observe that \(V'_{\rho_0}\) is a set with locally Lipschitz boundary since it is a level set of a Lipschitz function (see, for example, Evans and Gariepy \[17\]). Hence, for every \(n\) and \(V_n \subset V'\) and \(W_n \subset W'\), we can consider \(\chi_n, \zeta_n\) on \(\partial V_{n, \rho_0}\) in the sense of traces and define
\[
\eta_n = \begin{cases} \chi_n & \text{in } V_{n, \rho_0}, \\ \zeta_n & \text{in } W_n \setminus V_{n, \rho_0}, \end{cases}
\]
and
\[
\int_{\partial V_{n, \rho_0}} |\chi_n(x) - \zeta_n(x)| \, d\mathcal{H}^{N-1}(x) \leq \frac{C}{n},
\]
for a suitable constant \(C\).

By the choice of \(\rho_0\), \(\eta_n\) is admissible for \(\mathcal{F}(\chi, u; \Omega)\). In particular, \(\eta_n \rightharpoonup \chi\) in \(BV(\Omega; \{0, 1\})\). Using \(\ref{1.3}\), \(\ref{3.2}\) and \(\ref{3.3}\) one has
\[
\int_{V_n \cap W_n} f(\eta_n(x), \nabla z_n(x)) \, dx \leq \beta \int_{V_n \cap W_n} (1 + |\nabla z_n(x)|^q) \, dx \leq \frac{C}{n} + \frac{C}{n^{q\tau}},
\]
where \(\tau\) is as in Lemma \(\ref{3.2}\). It follows that
\[
\int_{\Omega} f(\eta_n(x), \nabla z_n(x)) \, dx + |D\eta_n|(\Omega) \leq \int_{V} f(\chi_n(x), \nabla v_n(x)) \, dx + |D\chi_n|(V) + \int_{W} f(\zeta_n(x), \nabla w_n(x)) \, dx + |D\zeta_n|(W) + C(n^{-1} + n^{-q\tau} + n^{-1}).
\]
Taking the limit in \(n\) one obtains
\[
\mathcal{F}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; V) + \mathcal{F}(\chi, u; W) + 2\varepsilon,
\]
so, letting \(\varepsilon \to 0^+\), we deduce the desired subadditivity inequality.

Notice that in the case \(p = 1\) there is no need to consider the auxiliary sets \(V'\) and \(W'\) since, by definition of \(\mathcal{F}_1\), we have strong convergence in \(L^1(\Omega; \mathbb{R}^d)\) of \(v_n\) and \(w_n\) to \(u\). In this case, to show admissibility of \(z_n\) for \(\mathcal{F}_1(\chi, u; \Omega)\) we conclude first that \(z_n\) is bounded in \(W^{1,1}(\Omega; \mathbb{R}^d)\) and then use the embedding theorem to obtain \(z_n \rightharpoonup u\) in \(L^1(\Omega; \mathbb{R}^d)\).

**Remark 3.4.** A similar result holds for \(\mathcal{F}_{loc}(\chi, u; \cdot)\) and \(\mathcal{F}_{1, loc}(\chi, u; \cdot)\), with an analogous proof. Indeed the proof relies on Lemma \(\ref{3.2}\) which can still be applied for \(\mathcal{F}_{loc}\) and \(\mathcal{F}_{1, loc}\), and a careful glueing argument for characteristic functions as was seen above.
4 Main Results

Assume that $f$ is defined by (1.4) with $W_i, i = 1, 2$, satisfying (1.3). For every open set $A \subset \Omega \subset \subset \mathbb{R}^N$ and every $(\chi, u) \in BV(A; \{0, 1\}) \times W^{1,p}(A; \mathbb{R}^d)$ let $F$ be as in (1.5) and consider the relaxed functional $\mathcal{F}(\chi, u; A)$ in (1.6). Then the following weak representation result holds.

**Theorem 4.1.** Let $f$ be given by (1.4) and satisfy (1.3), let $p, q$ be as in (1.7) and let $\chi \in BV(\Omega; \{0, 1\})$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$. If $\mathcal{F}(\chi, u; \Omega) < +\infty$, then there exists a non-negative Radon measure $\mu$ on $\overline{\Omega}$ which weakly represents $\mathcal{F}(\chi, u; \cdot)$. Likewise for $\mathcal{F}_1(\chi, u; \cdot)$.

**Proof.** Step 1. We assume first that the coercivity condition (1.8) holds. Thus, by Proposition 3.1, let $(\chi_n, u_n) \in BV(\Omega; \{0, 1\}) \times W^{1,q}(\Omega; \mathbb{R}^d)$ be a realizing sequence for $\mathcal{F}(\chi, u; \Omega)$, that is, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \rightharpoonup \chi$ in $BV(\Omega; \{0, 1\})$ and

$$\lim_{n \to +\infty} F(\chi_n, u_n; \Omega) = \mathcal{F}(\chi, u; \Omega). \quad (4.1)$$

Let $\mu_n = f(\chi_n(\cdot), \nabla u_n(\cdot)) \mathcal{L}^N |\Omega| + |D\chi_n|(\Omega)$ and extend this sequence of measures outside of $\Omega$ by setting, for any Borel set $E \subset \mathbb{R}^N$,

$$\lambda_n(E) = \mu_n(E \cap \Omega).$$

Passing, if necessary, to a subsequence, we can assume that there exists a non-negative Radon measure $\mu$ (depending on $\chi$ and $u$) on $\overline{\Omega}$ such that $\lambda_n \rightharpoonup \mu$ in the sense of measures in $\overline{\Omega}$. Let $\phi_k \in C_0(\overline{\Omega})$ be an increasing sequence of functions such that $0 \leq \phi_k \leq 1$ and $\phi_k(x) \to 1$ a.e. in $\overline{\Omega}$. Then, by Fatou’s Lemma and (1.1), we have

$$\mu(\overline{\Omega}) = \int_{\overline{\Omega}} \liminf_{k \to +\infty} \phi_k(x) \, d\mu \leq \liminf_{k \to +\infty} \int_{\overline{\Omega}} \phi_k(x) \, d\mu = \liminf_{k \to +\infty} \lim_{n \to +\infty} \left( \int_{\Omega} \phi_k(x) f(\chi_n(x), \nabla u_n(x)) \, dx + \int_{\Omega} \phi_k(x) \, d|D\chi_n| \right) \leq \lim_{n \to +\infty} \left( \int_{\Omega} f(\chi_n(x), \nabla u_n(x)) \, dx + |D\chi_n|(\Omega) \right) = \mathcal{F}(\chi, u; \Omega),$$

so that

$$\mu(\overline{\Omega}) \leq \mathcal{F}(\chi, u; \Omega). \quad (4.2)$$

On the other hand, by the upper semicontinuity of weak * convergence of measures on compact sets, for every open set $V \subset \Omega$, it follows that

$$\mathcal{F}(\chi, u; V) \leq \liminf_{n \to +\infty} F(\chi_n, u_n; V) = \liminf_{n \to +\infty} \mu_n(V) \leq \limsup_{n \to +\infty} \mu_n(\overline{V}) \leq \mu(\overline{V}),$$

which proves the upper bound inequality. To prove the lower bound inequality, we start by considering an open set $V \subset \subset \Omega$ and $\varepsilon > 0$. Then we can consider an open set $Z \subset \subset V$ such that

$$\mu(V) - \mu(Z) < \varepsilon.$$

By (1.2), (1.3) and Lemma 3.3 we have

$$\mu(V) \leq \mu(Z) + \varepsilon = \mu(\overline{\Omega} \setminus Z) + \varepsilon \leq \mathcal{F}(\chi, u; \Omega) - \mathcal{F}(\chi, u; \Omega \setminus Z) + \varepsilon \leq \mathcal{F}(\chi, u; V) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we obtain

$$\mu(V) \leq \mathcal{F}(\chi, u; V),$$

whenever $V$ is an open set such that $V \subset \subset \Omega$. For a general open subset $V \subset \Omega$ we have

$$\mu(V) = \sup \{ \mu(O) : O \subset \subset V \} \leq \sup \{ \mathcal{F}(\chi, u; O) : O \subset \subset V \} \leq \mathcal{F}(\chi, u; V),$$

and this concludes the proof of the theorem under the coercivity assumption (1.8).
Step 2. We now remove the coercivity requirement. For each $\varepsilon > 0$ we define
\[
F_{\varepsilon}(\chi, u; \Omega) := F(\chi, u; \Omega) + \varepsilon \int_{\Omega} |\nabla u(x)|^p \, dx
\]
and we let $\mathcal{F}_{\varepsilon}(\chi, u; \Omega)$ denote its relaxed functional given by
\[
\mathcal{F}_{\varepsilon}(\chi, u; \Omega) := \inf \left\{ \liminf_{n \to +\infty} F_{\varepsilon}(\chi_n, u_n; \Omega) : u_n \in W^{1,q}(\Omega; \mathbb{R}^d), \chi_n \in BV(\Omega; \{0,1\}), \right.
\]
\[
\left. \quad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^d), \chi_n \overset{a}{\rightharpoonup} \chi \text{ in } BV(\Omega; \{0,1\}) \right\}.
\]
By the previous step we know that there exists a measure $\mu_\varepsilon$ which weakly represents $\mathcal{F}_{\varepsilon}$. So, by (4.2), we have
\[
\mu_\varepsilon(\Omega) \leq \mathcal{F}_{\varepsilon}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; \Omega) + \varepsilon \sup_n \|u_n\|_{W^{1,p}(\Omega; \mathbb{R}^d)} \leq C,
\]
where $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$ is an admissible sequence for $\mathcal{F}_{\varepsilon}(\chi, u; \Omega)$. Thus, up to a subsequence, which we do not relabel, $\mu_\varepsilon$ converges weakly * to a finite, non-negative, Radon measure $\mu$. Given an open set $U \subset \Omega$, by (4.3) it follows that
\[
\mathcal{F}(\chi, u; U) \leq \mathcal{F}_{\varepsilon}(\chi, u; U) \leq \mu_\varepsilon(U),
\]
which, passing to the weak * limit, yields
\[
\mathcal{F}(\chi, u; U) \leq \mu(U).
\]
In order to prove the reverse inequality, let $\varepsilon' > 0$, and, by definition of $\mathcal{F}$, choose $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$ and $\chi_n \in BV(\Omega; \{0,1\})$, such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \overset{a}{\rightharpoonup} \chi$ and
\[
\int_U f(\chi_n(x), \nabla u_n(x)) \, dx + |D\chi_n|(U) \leq \mathcal{F}(\chi, u; U) + \varepsilon',
\]
for all $n \in \mathbb{N}$. Then, for a sufficiently large $k$, we have
\[
\int_U \left( f(\chi_n(x), \nabla u_n(x)) + \varepsilon_k |\nabla u_n(x)|^p \right) \, dx + |D\chi_n|(U) \leq \mathcal{F}(\chi, u; U) + 2\varepsilon',
\]
and hence
\[
\mu_{\varepsilon_k}(U) \leq \mathcal{F}_{\varepsilon_k}(\chi, u; U) \leq \mathcal{F}(\chi, u; U) + 2\varepsilon'.
\]
Thus the result is proved by passing first to the weak * limit as $\varepsilon_k \to 0^+$ and then to the limit as $\varepsilon' \to 0^+$.

The proof of the weak representation for $\mathcal{F}_1(\chi, u; \cdot)$ is analogous, replacing the weak topology of $W^{1,p}(\Omega; \mathbb{R}^d)$ by the $BV(\Omega; \mathbb{R}^d)$ weak * topology for the convergence of the sequence $u_n$. \hfill \Box

Remark 4.2. Let $f$ be given by (1.4) satisfying (1.9), let $p$, $q$ be as in (1.1) and let $\chi \in BV(\Omega; \{0,1\})$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$. Let $\mathcal{F}$ be as in (1.6) and $\mu$ be a Radon measure on $\Omega$ which weakly represents $\mathcal{F}(\chi, u; \cdot)$. Arguing as in [20, Lemma 3.6, Corollary 3.7 and Remark 3.8], and using Lemma 3.3 the following facts hold.

i) For every $U$ open subset of $\Omega$,
\[
\mu(U) = \mathcal{F}(\chi, u; U),
\]
provided that
\[
\inf_K \{ \mathcal{F}(\chi, u; U \setminus K) : K \subset U, K \text{ compact} \} = 0. \tag{4.4}
\]
Likewise for $\mathcal{F}_{\text{loc}}$, $\mathcal{F}_1$ and $\mathcal{F}_{1,\text{loc}}$.\hfill \hfill
ii) \( \mu \) represents \( \mathcal{F} \) if and only if there exists a Radon measure \( \nu \) such that
\[
\mathcal{F}(\chi, u; U) \leq \nu(U),
\]
for every open subset \( U \) of \( \Omega \). Likewise for \( \mathcal{F}_{\text{loc}}, \mathcal{F}_1 \) and \( \mathcal{F}_{1,\text{loc}} \).

In particular, if \( \chi \inBV(\Omega; \{0,1\}) \) and \( u \in W^{1,q}(\Omega; \mathbb{R}^d) \), then we can consider
\[
\nu(U) := \int_U Qf(\chi(x), \nabla u(x)) \, dx + |D\chi|(U),
\]
where \( Qf \) denotes the usual quasiconvex envelope of \( f \) in the last variable. Following [20] Corollary 4.3 and exploiting standard results about quasiconvex envelopes (cf. [15], [19, Theorem 8.4.1]), we have that
\[
\mathcal{F}(\chi, u; U) \leq \int_U Qf(\chi(x), \nabla u(x)) \, dx + |D\chi|(\Omega). \tag{4.6}
\]
Indeed, it suffices to fix \( \chi \inBV(\Omega; \{0,1\}) \) and consider \( u_n \in W^{1,q}(\Omega; \mathbb{R}^d) \) such that \( u_n \rightharpoonup u \) in \( W^{1,q}(\Omega; \mathbb{R}^d) \) and
\[
\lim_{n \to +\infty} \int_\Omega f(\chi(x), \nabla u_n(x)) \, dx = \int_\Omega Qf(\chi(x), \nabla u(x)) \, dx.
\]
Inequality (4.6) also holds for \( \mathcal{F}_1(\chi, u; \Omega) \).

**Theorem 4.3.** Let \( p, q \) satisfy (1.1), let \( f \) be defined as in (1.4), satisfying (1.3). Let \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \), then there exists a non-negative finite Radon measure \( \lambda \) on \( \Omega \) which strongly represents \( \mathcal{F}_{\text{loc}}(\chi, u; \cdot) \). The same holds in the case \( p = 1, 1 \leq q < \frac{N}{N-1} \) for the functional \( \mathcal{F}_{1,\text{loc}}(\chi, u; \cdot) \).

Moreover, if
\[
f(b, \cdot) \text{ is convex for every } b \in \{0,1\},
\]
then, for every open subset \( U \subset \Omega \), and every \( \chi \inBV(\Omega; \{0,1\}) \) and \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \),
\[
\mathcal{F}_{\text{loc}}(\chi, u; U) = \int_U f(\chi(x), \nabla u(x)) \, dx + |D\chi|(U) + \nu^s(\chi, u; U),
\]
where \( \nu^s \) is a non-negative Radon measure singular with respect to the Lebesgue measure.

**Proof.** By Theorem 4.1 we can find a Radon measure \( \lambda \) in \( \overline{\Omega} \) such that
\[
\lambda(U) \leq \mathcal{F}_{\text{loc}}(\chi, u; U) \leq \lambda(\overline{U}),
\]
for every open set \( U \subset \Omega \). We want to prove that
\[
\lambda(U) \geq \mathcal{F}_{\text{loc}}(\chi, u; U).
\]
Consider an increasing sequence of open, bounded, smooth sets \( U_h \subset \subset U \), \( h \in \mathbb{N} \), such that \( \overline{U}_h \subset \subset U_{h+1} \) and \( U = \bigcup_{h=1}^\infty U_h \). By definition of \( \mathcal{F}_{\text{loc}} \), for \( h \geq 3 \), we can find two sequences \( u_{h,n} \in W^{1,q}_{\text{loc}}(U_h \setminus \overline{U}_{h-2}; \mathbb{R}^d) \) and \( \chi_{h,n} \in BV(U_h \setminus \overline{U}_{h-2}; \{0,1\}) \) such that
\[
u(U), \quad \nabla u_{h,n}(x)) \, dx + |D\chi_{h,n}|(U_h \setminus \overline{U}_{h-2}) \leq \mathcal{F}_{\text{loc}}(\chi, u; (U_h \setminus \overline{U}_{h-2})); \quad 2^{-h}.
\]
Up to the extraction of subsequences we may assume that the above convergences, as \( n \to +\infty \), also hold a.e. in \( U_h \setminus \overline{U}_{h-2} \), and that
\[
\|u_{h,n} - u\|_{L^p(U_h \setminus \overline{U}_{h-2}; \mathbb{R}^d)} \leq 2^{-h} 
\]
and
\[
\|\chi_{h,n} - \chi\|_{L^1(U_h \setminus \overline{U}_{h-2}; \{0,1\})} \leq 2^{-h}.
\]
for some \( \alpha_h \) to be determined later. By Lemma 3.2 we can connect \( u_{h,n} \) with \( u_{h+1,n} \) across \( U_h \setminus \overline{U}_{h-1} \). Hence, there exist \( V_{h,n}^+ \) and \( V_{h+1,n}^+ \) such that \( V_{h,n}^+ \subset U_h \setminus \overline{U}_{h-2}, V_{h+1,n}^- \subset U_{h+1} \setminus \overline{U}_{h-1}, U_{h+1} \setminus \overline{U}_{h-2} = V_{h,n}^+ \cup V_{h+1,n}^- \),

\[
\mathcal{L}^N(V_{h,n}^+ \cap V_{h+1,n}^-) \leq C_h 2^{-h-n} \alpha_h^{-1},
\]

and there exist \( z_{h,n} \in W^{1,q}(U_{h+1} \setminus \overline{U}_{h-2}; \mathbb{R}^d) \), and \( \eta_{h,n} \in \mathcal{BV}(U_{h+1} \setminus \overline{U}_{h-2}; \{0,1\}) \) such that \( z_{h,n} = u_{h,n}, \eta_{h,n} = \chi_{h,n} \in (U_{h+1} \setminus \overline{U}_{h-2}) \setminus V_{h,n}^- \), and \( z_{h,n} = u_{h+1,n} \), \( \eta_{h,n} = \chi_{h+1,n} \) in \( (U_{h+1} \setminus \overline{U}_{h-2}) \setminus V_{h+1,n}^+ \).

Indeed, an argument entirely similar to the one exploited in Lemma 3.3 leads us to define \( \eta_{h,n} = \begin{cases} \chi_{h,n} & \text{in } (U_{h+1} \setminus \overline{U}_{h-2}) \setminus V_{h,n}^-, \chi_{h+1,n} & \text{in } (U_{h+1} \setminus \overline{U}_{h-2}) \setminus V_{h+1,n}^+ \end{cases} \) in such a way that the transition between \( \chi_{h,n} \) and \( \chi_{h+1,n} \) occurs along a curve denoted by \( \gamma_{h,n} \) satisfying

\[
\int_{\gamma_{h,n}} |\chi_{h,n}(x) - \chi_{h+1,n}(x)| \, d\mathcal{H}^{N-1}(x) \leq C_h 2^{-h-n} \alpha_h^{-1}.
\]

This choice is possible since in \( V_{h+1,n}^- \cap V_{h,n}^+ \) both \( \chi_{h+1,n} \) and \( \chi_{h,n} \) converge strongly in \( L^1 \) to \( \chi \) and a formula analogous to (3.5) holds.

Also

\[
\int_{V_{h,n}^+ \cap V_{h+1,n}^-} f(\eta_{h,n}(x), \nabla z_{h,n}(x)) \, dx \leq C \int_{V_{h,n}^+ \cap V_{h+1,n}^-} (1 + |\nabla z_{h,n}(x)|^q) \, dx \\
\leq C C_h 2^{-h-n} \alpha_h^{-1} + C_h \alpha_h^{-\tau q} 2^{-\tau q (n+h)},
\]

where \( \tau \) is as in Lemma 3.2 and \( C_h \) takes into account the dependence on \( h \). Next we specify the choice of \( \alpha_h \) so that \( \alpha_h^{-\tau q} C_h \leq 1 \).

Let \( z_n \in W^{1,q}(\Omega \setminus \overline{U}_1; \mathbb{R}^d) \) be given by \( z_n = z_{h,n} \), in \( V_{h,n}^+ \cap V_{h+1,n}^- \), and \( z_n = u_{h+1,n} \), in \( (U_{h+1} \setminus \overline{U}_{h-1}) \setminus (V_{h,n}^+ \cup V_{h+1,n}^-) \), and let

\[
\eta_n := \begin{cases} \eta_{h,n} & \text{in } V_{h,n}^+ \cap V_{h+1,n}^-, \chi_{h+1,n} & \text{in } U_{h+1} \setminus \overline{U}_{h-1} \setminus (V_{h,n}^+ \cup V_{h+1,n}^-) \end{cases}.
\]

In this way \( \eta_n \in \mathcal{BV}(U \setminus \overline{U}_1; \{0,1\}) \) and \( \eta_n \ast \chi \in \mathcal{BV}(U \setminus \overline{U}_1; \{0,1\}) \). Fix \( k \in \mathbb{N}, k \geq 2 \), then

\[
\int_{U \setminus \overline{U}_k} f(\eta_n(x), \nabla z_n(x)) \, dx + |D\eta_n|(U \setminus \overline{U}_k) \\
\leq \sum_{h=k+1}^{+\infty} \left( \int_{U_h \setminus \overline{U}_{h-1}} f(\eta_n(x), \nabla z_n(x)) \, dx \right) + |D\eta_n|(U_h \setminus \overline{U}_{h-1}) \\
\leq \sum_{h=k+1}^{+\infty} \left\{ \int_{U_{h+1} \setminus \overline{U}_h} f(\chi_{h+1,n}(x), \nabla u_{h+1,n}(x)) \, dx + |D\chi_{h+1,n}|(U_{h+1} \setminus \overline{U}_h) \right. \\
+ \int_{U_h \setminus \overline{U}_{h-1}} f(\chi_{h,n}(x), \nabla u_{h,n}(x)) \, dx + |D\chi_{h,n}|(U_h \setminus \overline{U}_{h-1}) \\
\left. + \int_{V_{h,n}^+ \cap V_{h+1,n}^-} f(\eta_{h,n}(x), \nabla z_{h,n}(x)) \, dx + \int_{\gamma_{h,n}} |\chi_{h,n}(x) - \chi_{h+1,n}(x)| \, d\mathcal{H}^{N-1}(x) \right\} \\
\leq \sum_{h=k}^{+\infty} \left( 2F_{\text{loc}}(\chi; (U_{h+1} \setminus \overline{U}_h)) + 2^{-h-1} \right) + \sum_{h=k+1}^{+\infty} 2^{-q(n+h)} + \sum_{h=k+1}^{+\infty} 2^{-(n+h)} \alpha_h^{-1} C_h \\
\leq \sum_{h=k}^{+\infty} \left( 2\mu(U_{h+2} \setminus U_{h-1}) + 2^{-k} + C 2^{-q(n+k)} + C 2^{-(n+k)} \right) \\
\leq 6\lambda(U \setminus U_{h-1}) + 2^{-k} + C 2^{-(q+1)(n+k)}.
The argument used in Lemma 3.3 ensures that \( z_n \to u \) in \( W^{1,p}(U \setminus U_k; \mathbb{R}^d) \). Since we also have \( \eta_n \rightharpoonup^* \chi \) in \( BV(U \setminus U_k; \{0,1\}) \), it follows that

\[
\mathcal{F}_{loc}(\chi, u; U \setminus U_k) \leq 6\lambda(U \setminus U_{k-1}) + C2^{-k(1+q\tau)}.
\]

Hence \( \text{(4.4)} \) is verified and by Remark 4.2 we can conclude that

\[
\lambda(U) = \mathcal{F}_{loc}(\chi, u; U).
\]

The proof of the strong representation for \( \mathcal{F}_{1,loc}(\chi, u; \cdot) \) is analogous, replacing the weak topology of \( W^{1,p}(\Omega; \mathbb{R}^d) \) by the \( BV(\Omega; \mathbb{R}^d) \) weak * topology for the convergence of the sequence \( u_n \).

Concerning the last part of the statement, for \( f \) convex, we start by observing that by (4.7), Ioffe’s Theorem 2.1 (see also [5, Theorem 5.8]) and the superadditivity of the liminf, the functional

\[
\int_{\Omega} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(\Omega)
\]

is lower semicontinuous with respect to the \( L^1 \) strong convergence for \( \chi \) and the \( W^{1,p} \) weak convergence for \( u \). Indeed, for every \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \) and \( \chi \in BV(\Omega; \{0,1\}) \), and for every \( u_n \in W^{1,q}_{loc}(\Omega; \mathbb{R}^d) \) and \( \chi_n \in BV(\Omega; \{0,1\}) \), such that \( u_n \to u \) in \( W^{1,p}(\Omega; \mathbb{R}^d) \) and \( \chi_n \to \chi \) in \( L^1(\Omega; \{0,1\}) \), it follows that

\[
\liminf_{n \to +\infty} \left( \int_{\Omega} f(\chi_n(x), \nabla u_n(x)) \, dx + |D\chi_n|(\Omega) \right) \geq \int_{\Omega} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(\Omega).
\]

Thus,

\[
\int_{\Omega} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(\Omega) \leq \mathcal{F}_{loc}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; \Omega), \tag{4.8}
\]

and a similar result holds in any open subset \( U \) of \( \Omega \).

In order to prove the opposite inequality, let \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \) and \( \chi \in BV(\Omega; \{0,1\}) \). By definition of \( \mathcal{F}_{loc} \), for every open subset \( U \subset \Omega \), we have that

\[
\mathcal{F}_{loc}(\chi, u; U) \leq \liminf_{n \to +\infty} \int_{U} f(\chi(x), \nabla u_n(x)) \, dx + |D\chi|(U), \tag{4.9}
\]

for every sequence \( u_n \in W^{1,q}_{loc}(U; \mathbb{R}^d) \), such that \( u_n \rightharpoonup u \) in \( W^{1,p}(U; \mathbb{R}^d) \).

On the other hand, [11] Theorem 1.1 guarantees the existence of a measure \( \nu^s(u, \chi; \cdot) \), singular with respect to the Lebesgue measure, and a sequence \( \pi_n \in W^{1,q}_{loc}(U; \mathbb{R}^d) \) such that \( \pi_n \rightharpoonup u \) in \( W^{1,p}(U; \mathbb{R}^d) \) and

\[
\limsup_{n \to +\infty} \int_{U} f(\chi(x), \nabla \pi_n(x)) \, dx \leq \int_{U} f(\chi(x), \nabla u(x)) \, dx + \nu^s(\chi, u; U).
\]

This, together with (4.9), ensures that

\[
\mathcal{F}_{loc}(\chi, u; U) \leq \int_{U} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(U) + \nu^s(\chi, u; U) \tag{4.10}
\]

and concludes the proof of the upper bound.

Hence, \( \text{[10]} \) applied in an open set \( U \subset \Omega \), \( \text{[13]} \) and the first part of this theorem yield the result.

\textbf{Remark 4.4.} \hspace{1em} i) We recall that, as observed by Acerbi and Dal Maso in [3], the exponents considered in the previous result cannot be improved, as neither \( \mathcal{F} \) nor \( \mathcal{F}_{loc} \) admit any weak representation if \( q = \frac{Np}{N-1} \).

ii) Furthermore, it was shown in [20] Remark 3.3, when there is no dependence on the \( \chi \) variable, that, in general, the measure representation for \( \mathcal{F} \) is only weak, while [20] Theorem 3.1 ensures that \( \mathcal{F}_{loc} \) admits a strong representation.
iii) For the reader’s convenience we also recall that if $U \subset \subset V \subset \Omega$, then
\[
F_{\text{loc}}(\chi, u; U) \leq F(\chi, u; U) \leq F_{\text{loc}}(\chi, u; V),
\]
thus the measures $\lambda$ and $\mu$ which represent $F_{\text{loc}}$ and $F$, respectively, are such that $\lambda = \mu(\Omega)$. Notice that parts ii) and iii) of this remark also hold in the case $p = 1$.

In the convex case, we point out that, according to iii) the measure $\mu(\chi, u; \cdot) := f(\chi, \nabla u)[L^N + |D\chi|(\cdot) + \nu^s(\chi, u; \cdot)$ weakly represents $F(\chi, u; \cdot)$.

**Remark 4.5.** Observe that the functionals in [14, Theorem 6.2] are related to $F$ and $F_{\text{loc}}$ when $f$ is convex. Indeed, in [14] the authors consider relaxation and integral representation of integral functionals of the type $\int \Omega h(x, \nabla u(x)) \, dx$, when $h$ satisfies
\[
|\xi|^p(x) \leq h(x, \xi) \leq C(1 + |\xi|^p(x)), \tag{4.11}
\]
so that the integrability exponent $p(x)$ of the admissible fields depends in a continuous or regular piecewise continuous way on the location in the body (see [14] assumptions (2.9) and (2.1) and Definitions 2.1 and 2.2, respectively).

The bulk energy density in (1.2), i.e. $h(x, \xi) := \chi(x)W_1(\xi) + (1 - \chi(x))W_2(\xi)$, can be seen as satisfying a generalization of (1.11), allowing for a wide discontinuity in the exponent $p(x)$ and prohibiting the use of the variable exponent space $W^{1,p(x)}$. On the other hand, the functional spaces and the convergences involved in (1.2), (1.9) and (1.7) are not as in [14] Section 6). Indeed, in (1.9) and (1.7) it is required that the approximating sequences $u_n$ are more regular in the whole domain, but converge to $u$ in a weaker sense than as expected from the coercivity condition in (1.11). The convergence stated in [14] Section 6) is $L^1$ strong, but the coercivity condition in (1.11) might provide different bounds for $\nabla u_n$ in different subsets of $\Omega$. When $\chi$ is fixed, there are many more test sequences than in (1.9) or (1.7), thus avoiding the appearance of concentration of energy as in [32, Example 1.15], which is based on the example given in [58, page 467].

We also underline that in [32] Theorems 1.8, 1.9 and Corollary 1.10] a measure representation was obtained for an energy similar to $F$ in [14] Section 6], but with the approximating sequences in $C^1 \cap W^{1,p(x)}$. Like in our case, under the same regularity assumptions on $h(x, \xi)$ and on $p(x)$ as in [14], the author is able to obtain a representation in terms of a Lebesgue integral plus a singular measure.

**Remark 4.6.** Let $0 \leq \theta \leq 1$ and let $F$ be as in (1.2). It is easy to see that the representation given in the second part of Theorem [4.3] also holds for the functional $F_{\text{volume}}$ defined by
\[
F_{\text{volume}}(\chi, u) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; \Omega) : u_n \in W^{1,q}_{\text{loc}}(\Omega; \mathbb{R}^d), \chi_n \in BV(\Omega; \{0, 1\}), u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(\Omega; \{0, 1\}), \frac{1}{|\Omega|} \int_{\Omega} \chi_n(x) \, dx = \theta \right\}.
\]
Indeed, the lower bound inequality is obtained as in the second part of the proof of Theorem [4.3] observing that it suffices to consider sequences $\chi_n$ whose integral in $\Omega$ amounts to $\theta$. Regarding the upper bound inequality, the same argument as in the last part of the proof applies, taking the recovery sequence $\chi_n$ identically equal to $\chi$ with
\[
\frac{1}{|\Omega|} \int_{\Omega} \chi(x) \, dx = \theta.
\]

### 4.1 A sufficient condition for lower semicontinuity

Our aim in this subsection is to present suitable assumptions on the energy densities $W_i$ in (1.3) in order to guarantee lower semicontinuity of the functional $F$ in (1.2). Also, in the spirit of [31], it is possible to consider functionals related to $F$ and $F_{\text{loc}}$ in (1.6) and (1.7), respectively, but with the infima taken with respect to different admissible sequences. We introduce one such functional and compare it with the previous ones.

An argument entirely similar to the one in [31, Lemma 3.1] allows us to prove the following.
Proposition 4.7. Let $F$ be as in (1.2), where $f$ is as in (1.4) with $W_i$, $i = 1, 2$ satisfying the lower bound in (1.3). Assume also that $f(b, \cdot)$ is closed $W^{1,p}$-quasiconvex, for every $b \in \{0, 1\}$. Then
\[
F(\chi, u) \leq \liminf_{n \to +\infty} F(\chi_n, u_n),
\]
for every $\chi_n \in BV(\{0, 1\})$ weakly * converging to $\chi$ in BV$(\Omega; \{0, 1\})$ and every $u_n \in W^{1,p}(\Omega; \mathbb{R}^d)$ weakly converging to $u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$.

Proof. By the lower semicontinuity of the total variation $|D\chi|$ and the superadditivity of the liminf, it suffices to consider the asymptotic behaviour of
\[
\int_{\Omega} f(\chi_n(x), \nabla u_n(x)) \, dx
\]
when $u_n \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and $\chi_n \rightharpoonup \chi$ in $L^1(\{0, 1\})$. Let $\mu$ be the Young measure generated by $(\chi_n, \nabla u_n)$. By Proposition 2.8, $\mu = \delta_{\chi(x)} \otimes \nu_x$, where $\nu$ is the gradient Young measure generated by $\nabla u_n$. By [21] Theorem 2.2, and arguing as in the first part of the proof of Theorem 3.7 therein, we have
\[
\liminf_{n \to +\infty} \int_{\Omega} f(\chi_n(x), \nabla u_n(x)) \, dx \geq \int_{\Omega} \int_{\mathbb{R}^d} f(a, \xi) \, d(\delta_{\chi(x)} \otimes \nu_x)(a, \xi) \, dx \geq \int_{\Omega} \int_{\mathbb{R}^N} f(\chi(x), \xi) \, d\nu_x(\xi) \, dx.
\]
The proof will be complete provided we guarantee that
\[
\int_{\mathbb{R}^N} f(\chi(x), \xi) \, d\nu_x(\xi) \geq f(\chi(x), \nabla u(x)) \text{ for a.e. } x \in \Omega,
\]
and this is a consequence of [31] Lemma 3.1.

As in [31] Corollary 1.2], one could define for $f$ as in (1.4) and satisfying (1.3), the following functional
\[
\tilde{I}(\chi, u; \Omega) := \inf \left\{ \liminf_{n \to +\infty} \int_{\Omega} f(\chi_n(x), \nabla u_n(x)) \, dx + |D\chi_n| (\Omega) \right\},
\]
where the infimum is taken over all sequences $u_n \in W^{1,p}(\Omega; \mathbb{R}^d)$, converging to $u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^d)$, and all sequences $\chi_n \in BV(\{0, 1\})$, converging weakly * to $\chi \in BV(\{0, 1\})$.

Clearly by Proposition 4.7, denoting by $\hat{f}(b, \cdot)$ the closed $W^{1,p}$-quasiconvex envelope of $f(b, \cdot)$ (see Definition 2.8), we have that
\[
\int_{\Omega} \hat{f}(\chi(x), \nabla u(x)) \, dx + |D\chi| (\Omega) \leq \tilde{I}(\chi, u; \Omega) \leq \mathcal{F}_{\text{loc}}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; \Omega),
\]
when they are all finite.

Notice that, in general, we are unable to compute these functionals, in the sense of providing an explicit representation of them. However, in the one dimensional case, since $f^{**}(b, \cdot) = \hat{f}(b, \cdot)$ for every $b \in \{0, 1\}$, the four functionals in (1.12) coincide. Indeed, by definition, any convex function is closed $W^{1,p}$-quasiconvex since Jensen’s inequality holds for any probability measure. Conversely, any closed $W^{1,p}$-quasiconvex function is $W^{1,p}$-quasiconvex in light of [31] Corollary 3.4 and [6] Corollary 3.2, and the latter notion is known to be equivalent to convexity in the scalar case.

4.2 The one dimensional case

Theorem 1.3 can be improved in the one dimensional case. In fact, in this case, no singular measure appears, which does not contradict the example given in [38] page 467, since the latter relies on the fact that functions in $W^{1,p}(\Omega; \mathbb{R}^d)$, with $\Omega \subset \mathbb{R}^2$ and $1 < p < 2$, are not necessarily continuous. In fact, the next result also generalizes to the optimal design context the result of Ben Belgacem [6] Theorem 4.1, since we do not assume convexity in the original density $f(b, \cdot)$.
Proposition 4.8. Let $I$ be an open interval in $\mathbb{R}$, let $f$ be given by \([1,4]\) with $W_i$, $i = 1, 2$, satisfying \([1,3]\) and let $p$, $q$ be such that $1 < p \leq q$. Let $\chi \in BV(I; \{0, 1\})$ and $u \in W^{1,p}(I; \mathbb{R}^d)$. Then,

$$
\mathcal{F}(\chi, u; I) = \int_I f^*(\chi(x), u'(x)) \, dx + |D\chi|(I).
$$

Proof. The result is true when $p = q$, see [19, Theorem 8.4.1], so in what follows we assume that $p < q$.

The lower bound follows as in the proof of the last part of Theorem [13]. Indeed, since $f(b, \cdot) \geq f^*(b, \cdot)$ for every $b \in \{0, 1\}$, then, for every $u \in W^{1,p}(I; \mathbb{R}^d)$ and $\chi \in BV(I; \{0, 1\}),$

$$
\int_I f^*(\chi(x), u'(x)) \, dx + |D\chi|(I) \leq \mathcal{F}_{lo}(\chi, u; I) \leq \mathcal{F}(\chi, u; I).
$$

To show the upper bound, let $u \in W^{1,p}(I; \mathbb{R}^d)$ and $\chi \in BV(I; \{0, 1\})$. We follow the proof in [20, Remark 4.6] and take a convolution kernel $\rho \geq 0$, with support on $[-1, 1]$ and such that $\int_\mathbb{R} \rho(x) \, dx = 1$. Given $k \in \mathbb{N}$, set $\rho_k(x) = k\rho(kx)$ and consider the usual mollification $\rho_k * u$. Since $u$ is continuous up to $\overline{\mathcal{T}}$ we can extend it to a larger interval $J \supset \overline{\mathcal{T}}$. By standard relaxation results [19, Theorem 8.4.1], and since $\chi$ is fixed, for each $k$, there exists a sequence $v_{k,n} \in W^{1,q}(I; \mathbb{R}^d)$ such that

$$
v_{k,n} \rightharpoonup \rho_k * u \text{ weakly in } W^{1,q}(I; \mathbb{R}^d) \text{ as } n \to +\infty,
$$

and

$$
\lim_{n \to +\infty} \int_I f(\chi(x), v_{k,n}(x)) \, dx = \int_I f^*(\chi(x), (\rho_k * u)'(x)) \, dx.
$$

As $p < q$, we may extract a diagonal subsequence $u_k := v_{k,n(k)}$ such that

$$
\|u_k - \rho_k * u\|_{W^{1,p}} \leq \frac{1}{k},
$$

and

$$
\left| \int_I f(\chi(x), u_k'(x)) \, dx - \int_I f^*(\chi(x), (\rho_k * u)'(x)) \, dx \right| \leq \frac{1}{k}.
$$

Therefore $u_k \to u$ in $W^{1,p}(I; \mathbb{R}^d)$ and

$$
\mathcal{F}(\chi, u; I) \leq \liminf_{k \to +\infty} \int_I f(\chi(x), u_k'(x)) \, dx + |D\chi|(I)
$$

$$
= \liminf_{k \to +\infty} \int_I f^*(\chi(x), (\rho_k * u)'(x)) \, dx + |D\chi|(I).
$$

Notice that $f^*(b, \xi) = bW_1^*(\xi) + (1 - b)W_2^*(\xi)$, since $b$ is either 0 or 1. As this function is convex in the second variable, we can exploit the fact that the measures $\mu^k$ defined by

$$
\langle \mu^k, \varphi \rangle := \int_{\mathbb{R}} \rho_k(x - y) \varphi(y) \, dy
$$

are probability measures. Since $\chi \in BV(I; \{0, 1\})$, it has finitely many discontinuity points, so we can divide the interval $I$ into a finite union of intervals $\{I_j\}_{j=1}^l$, corresponding to the largest connected sets where $\chi$ is constant. We can assume without loss of generality that each $I_j$ is open.

Hence, using Jensen’s inequality, we have

$$
\liminf_{k \to +\infty} \int_I f^*(\chi(x), (\rho_k * u)'(x)) \, dx = \liminf_{k \to +\infty} \sum_{j=1}^l \int_{I_j} f^*(\chi(x), (\rho_k * u)'(x)) \, dx
$$

$$
= \liminf_{k \to +\infty} \sum_{j=1}^l W_1^*((\rho_k * u)'(x)) \, dx = \liminf_{k \to +\infty} \sum_{j=1}^l W_1^*((\mu^k_x, u') \, dx
$$

$$
\leq \limsup_{k \to +\infty} \sum_{j=1}^l \int_{I_j} \langle \mu^k_x, W_1^*(u') \rangle \, dx = \int_I f^*(\chi(x), u'(x)) \, dx,
$$

17
where $W_{i}^{\ast \ast}$ is either $W_{1}^{\ast \ast}$ or $W_{2}^{\ast \ast}$ depending on whether $\chi$ is 1 or 0 in each $I_{j}$. From (4.15), we obtain
\[
\mathcal{F}(\chi, u; I) \leq \int_{I} f^{\ast \ast}(\chi(x), u'(x)) \, dx + |D\chi|(I),
\]
which, together with the lower bound obtained in the first part of the proof, yields
\[
\mathcal{F}(\chi, u; I) = \int_{I} f^{\ast \ast}(\chi(x), u'(x)) \, dx + |D\chi|(I).
\]

Remark 4.9. Recalling Remark 4.6, we observe that, when $\Omega$ is replaced by an interval $I$ of $\mathbb{R}$, $\mathcal{F}, \mathcal{F}_{\text{loc}}$ and $\mathcal{F}_{\text{volume}}$ coincide with the functional
\[
\mathcal{F}_{\text{volume}}(\chi, u) := \inf \left\{ \lim \inf_{n \to +\infty} F(\chi_{n}, u_{n}; I) : u_{n} \in W_{\text{loc}}^{1,p}(I; \mathbb{R}^{d}), \chi_{n} \in BV(I; \{0,1\}), u_{n} \rightharpoonup u \text{ in } W^{1,p}(I; \mathbb{R}^{d}), \chi_{n} \rightharpoonup^{*} \chi \text{ in } BV(I; \{0,1\}), \frac{1}{|I|} \int_{I} \chi_{n}(x) \, dx = \theta \right\}.
\]

Remark 4.10. We define
\[
\mathcal{G}_{1}(\chi, u; I) := \inf \left\{ \lim \inf_{n \to +\infty} F(\chi_{n}, u_{n}; I) : u_{n} \in W_{\text{loc}}^{1,q}(I; \mathbb{R}^{d}), \chi_{n} \in BV(I; \{0,1\}), u_{n} \rightharpoonup u \text{ in } W^{1,1}(I; \mathbb{R}^{d}), \chi_{n} \rightharpoonup^{*} \chi \text{ in } BV(I; \{0,1\}) \right\},
\]
and, by replacing $W^{1,q}$ by $W_{\text{loc}}^{1,q}$, $\mathcal{G}_{1,\text{loc}}(\chi, u; I)$. Arguing as in the previous proof we can obtain a representation for both $\mathcal{G}_{1}$ and $\mathcal{G}_{1,\text{loc}}$ in terms of $\int_{I} f^{\ast \ast}(\chi(x), u'(x)) \, dx + |D\chi|(I)$. Indeed, the lower bound inequality is a consequence of Theorem 2.24, whereas the upper bound inequality holds replacing the $W^{1,p}$ strong convergence in (1.13) by strong convergence in $W^{1,r}$ for some $1 < r < q$, which in turn ensures weak convergence in $W^{1,1}(I; \mathbb{R}^{d})$.

5 Applications to strings

In the sequel we apply the techniques of the second part of the proof of Theorem 4.3 to identify the optimal design of strings by means of dimension reduction, in the spirit of the models described in [18, 11], which also appear in the context of brutal damage evolution. Namely one can deduce, as a rigorous 3D-1D $\Gamma$-limit as $\varepsilon \to 0^{+}$, the optimal design of an elastic string $\Omega(\varepsilon) := B(0, \varepsilon) \times (0, l)$, with $B(0, \varepsilon) \subset \mathbb{R}^{2}$ the ball centered at 0 with radius $\varepsilon$ and $l \in \mathbb{R}^{+}$, constituted by Ogden type materials, which truly exhibit a gap between the growth and coercivity exponents in the hyperelastic density.

In the following we adopt the standard scaling (see [12] and the references quoted therein) which maps $x \equiv (x_{1}, x_{2}, x_{3}) \in \Omega(\varepsilon) \to \left( \frac{x_{1}}{\varepsilon}, \frac{x_{2}}{\varepsilon}, x_{3} \right) \in \Omega := B(0, 1) \times (0, l)$, in order to state the problem in a fixed domain (see [5, 3] below). We also denote by $\nabla_{a}u$ and $D_{a}\chi$, respectively, the partial derivatives of $u$ and $\chi$ with respect to $x_{a} \equiv (x_{1}, x_{2})$.

In the model under consideration, the sequence $\chi_{\varepsilon} \in BV(\Omega; \{0,1\})$ represents the design regions, whereas $u_{\varepsilon} \in W^{1,q}(\Omega; \mathbb{R}^{3})$ is the sequence of deformations, which are possibly clamped at the extremities of the string. Standard arguments in dimension reduction (see [2] and [12]) ensure that energy bounded sequences (see the term in square brackets of [5, 3]), converge (up to a subsequence), in the relevant topology, to fields $(\chi, u)$ such that $D_{a}\chi$ and $\nabla_{a}u$ are null, thus they can be identified, with an abuse of notation, with fields $(\chi, u) \in BV((0, l); \{0,1\}) \times W^{1,p}((0, l); \mathbb{R}^{3})$. In what follows we use this notation.

Proposition 5.1. Let $\Omega := B(0, 1) \times [0, l)$, where $B(0, 1)$ denotes the unit ball in $\mathbb{R}^{2}$ and $l \in \mathbb{R}^{+}$. Let $f : \{0, 1\} \times \mathbb{R}^{3} \to \mathbb{R}$ be a continuous function as in (1.13) and assume also that
\[
Qf(b, \cdot) = f^{\ast \ast}(b, \cdot),
\]
(5.1)
where $Q(\cdot)$ denotes the quasiconvex envelope of $f(b, \cdot)$ (see Definition 2.8). Let $1 < p \leq q < +\infty$ and assume that there exist $c, c_0, C \in \mathbb{R}^+$ such that
\[
c|x|^p - c_0 \leq f(b, x) \leq C(1 + |x|^q),
\]
for every $b \in \{0, 1\}$ and $x \in \mathbb{R}^{3\times3}$. Let
\[
F^{DR}(\chi, u) := \inf \left\{ \liminf_{\varepsilon \to 0^+} \left[ \int_{\Omega} f(\chi_\varepsilon(x), (\frac{1}{\varepsilon} \nabla_\alpha u_\varepsilon(x), \nabla_3 u_\varepsilon(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D_\alpha \chi_\varepsilon, D_3 \chi_\varepsilon \right) \right| (\Omega) \right] : \right.
\]
\[
\left. \quad u_\varepsilon \in W^{1,q}(\Omega; \mathbb{R}^3), \chi_\varepsilon \in BV(\Omega; [0, 1]), u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3), \chi_\varepsilon \rightharpoonup^* \chi \text{ in } BV(\Omega; [0, 1]) \right\}.
\]
Then
\[
F^{DR}(\chi, u) = \pi \left( \int_0^1 f^{**}(\chi(x_3), u'(x_3)) \, dx_3 + |D\chi|(0, l) \right),
\]
for every $\chi \in BV((0, l); [0, 1])$ and every $u \in W^{1,p}((0, l); \mathbb{R}^3)$ for which $F^{DR}(\chi, u)$ is finite, where
\[
f_0(b, \xi_3) := \inf_{(\xi_1, \xi_2) \in \mathbb{R}^{2\times3}} f(b, \xi_1, \xi_2, \xi_3), \quad \text{with } b \in \{0, 1\}, (\xi_1, \xi_2, \xi_3) \equiv \xi \in \mathbb{R}^{3\times3},
\]
and $f_0^{**}$ denotes its convex envelope with respect to the second variable.

We point out that the functional $F^{DR}$ in (5.3) is defined in full analogy with $F$ in (1.9), although it involves an asymptotic process which can be rigorously treated in the framework of $\Gamma$-convergence (we refer to [10] for more details on this subject). On the other hand, our proof of the integral representation [5.4] is obtained following the same strategy, based on proving a double inequality, adopted at the end of the previous section, and it is self-contained.

Notice that in the above result the limit total variation $|D\chi|$ is the counting measure. Before addressing its proof we start by proving a lemma following the ideas presented in [10] Lemma 2.3.

**Lemma 5.2.** Under the conditions of Proposition 5.1 the following holds
\[
F^{DR}(\chi, u) := \inf \left\{ \liminf_{\varepsilon \to 0^+} \left[ \int_{\Omega} f^{**}(\chi_\varepsilon(x), (\frac{1}{\varepsilon} \nabla_\alpha u_\varepsilon(x), \nabla_3 u_\varepsilon(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D_\alpha \chi_\varepsilon, D_3 \chi_\varepsilon \right) \right| (\Omega) \right] : \right.
\]
\[
\left. \quad u_\varepsilon \in W^{1,q}(\Omega; \mathbb{R}^3), \chi_\varepsilon \in BV(\Omega; [0, 1]), u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3), \chi_\varepsilon \rightharpoonup^* \chi \text{ in } BV(\Omega; [0, 1]) \right\},
\]
for every $(\chi, u) \in BV(\Omega; [0, 1]) \times W^{1,p}(\Omega; \mathbb{R}^3)$ for which $F^{DR}(\chi, u)$ is finite.

**Proof.** Using the arguments presented in [10] (2.2) we obtain that
\[
(Qf)_\varepsilon(b, \xi) = Q(f_\varepsilon)(b, \xi),
\]
where for any function $g : \{0, 1\} \times \mathbb{R}^{3\times3} \to [0, +\infty),$
\[
g_\varepsilon(b, \xi_1, \xi_2, \xi_3) := g \left( b, \frac{1}{\varepsilon} \xi_1, \frac{1}{\varepsilon} \xi_2, \xi_3 \right).
\]
Recall that for any function $g : \{0, 1\} \times \mathbb{R}^{3\times3} \to \mathbb{R}$ the convex envelope with respect to the second variable
\[
g^{**}(b, \xi) = \sup_{\varphi \in L^1(U; \mathbb{R}^{3\times3})} \left\{ \int_U g(b, \xi + \varphi(x)) \, dx : \int_U \varphi(x) \, dx = 0 \right\},
\]
where $U$ is a domain and $L^1(U) = 1$, coincides with the Legendre-Fenchel conjugate of $g$.

By the definition of Legendre-Fenchel conjugate with respect to the second variable, it follows that
\[
(g_\varepsilon)^*(b, \xi^*) = \left( g^* \right)_{\frac{1}{\varepsilon}}(b, \xi^*),
\]
19
for every $b \in \{0,1\}$ and $\xi^* \in \mathbb{R}^{3 \times 3}$. Thus, using \eqref{5.7}, \eqref{5.1} and \eqref{5.6} we have the following chain of equalities

\[
(f_*)^*(b, \xi) = (f^*)_*(b, \xi) = (Qf)_*(b, \xi) = Q(f_*)(b, \xi),
\]

for every $b \in \{0,1\}$ and $\xi \in \mathbb{R}^{3 \times 3}$.

Let $F_{f_*}^{DR}(\chi, u)$ be defined as $F_{\cdot}^{DR}(\chi, u)$ but replacing $f$ by $f^*$. Clearly, since $f^{**} \leq f$, it follows that $F_{f_*}^{DR} \leq F^{DR}$ so we only need to prove the opposite inequality. To this end, for every $\delta > 0$ and every $(\chi, u) \in BV(\Omega; \{0,1\}) \times W^{1,p}(\Omega; \mathbb{R}^3)$ for which $F^{DR}(\chi, u) < +\infty$, let $(\chi_\varepsilon, u_\varepsilon) \in BV(\Omega; \{0,1\}) \times W^{1,q}(\Omega; \mathbb{R}^3)$ be such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, $\chi_\varepsilon \rightharpoonup \chi$ in $BV(\Omega; \{0,1\})$ and

\[
F_{f_*}^{DR}(\chi, u) \geq \int_\Omega f^*(\chi_\varepsilon(x), (\frac{1}{\varepsilon} \nabla\alpha u_\varepsilon(x), \nabla_3 u_\varepsilon(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D\alpha \chi_\varepsilon, D_3 \chi_\varepsilon \right) \right| (\Omega) - \delta.
\]

By \cite{19} Theorem 8.4.1, there exists $u_{\varepsilon,k} \in W^{1,q}(\Omega; \mathbb{R}^3)$ such that $u_{\varepsilon,k} \rightharpoonup u_\varepsilon$ weakly in $W^{1,q}$, as $k \to +\infty$, and

\[
\int_\Omega f^*(\chi_\varepsilon(x), (\frac{1}{\varepsilon} \nabla\alpha u_{\varepsilon,k}(x), \nabla_3 u_{\varepsilon,k}(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D\alpha \chi_\varepsilon, D_3 \chi_\varepsilon \right) \right| (\Omega).
\]

Thus we can say that

\[
F_{f_*}^{DR}(\chi, u) \geq \lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \int_\Omega f(\chi_\varepsilon(x), (\frac{1}{\varepsilon} \nabla\alpha u_{\varepsilon,k}(x), \nabla_3 u_{\varepsilon,k}(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D\alpha \chi_\varepsilon, D_3 \chi_\varepsilon \right) \right| (\Omega) - \delta,
\]

and

\[
\lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \|u_{\varepsilon,k} - u\|_{L^p} = 0.
\]

The growth from below in \eqref{5.2}, the convexity of $| \cdot |^p$ and the fact that the weak topology is metrizable on bounded sets, ensure that there exist a diagonal sequence $u_{\varepsilon,k}$ and a subsequence $\chi_{\varepsilon,k}$ such that

\[
(\chi_{\varepsilon,k}, u_{\varepsilon,k}) \to (\chi, u) \text{ in } BV\text{-weak } \ast \times W^{1,p}\text{-weak}, \text{ as } k \to +\infty,
\]

the double limit in \eqref{5.9} exists, and thus

\[
F_{f_*}^{DR}(\chi, u) \geq \lim_{k \to +\infty} \int_\Omega f(\chi_{\varepsilon,k}(x), (\frac{1}{\varepsilon} \nabla\alpha u_{\varepsilon,k}(x), \nabla_3 u_{\varepsilon,k}(x))) \, dx + \left| \left( \frac{1}{\varepsilon} D\alpha \chi_{\varepsilon,k}, D_3 \chi_{\varepsilon,k} \right) \right| (\Omega) - \delta,
\]

which, in turn, implies that

\[
F_{f_*}^{DR}(\chi, u) \geq F^{DR}(\chi, u) - \delta.
\]

It suffices to let $\delta \to 0^+$ to conclude the proof.

**Proof of Proposition 5.7.** The proof of \eqref{5.1} is obtained by showing a double inequality. For what concerns the lower bound, it suffices to observe that $f_0^{**} \leq f$,

\[
c|x_1|^p - c_0 \leq f_0^{**}(b, x_3) \leq C(|x_3|^q + 1),
\]

for every $(b, x_3) \in \{0,1\} \times \mathbb{R}^3$ (see \cite{21}), and that the functional

\[
\int_\Omega f_0^{**}(\chi(x_\alpha, x_3), \nabla_3 u(x_\alpha, x_3)) \, dx
\]

is lower semicontinuous with respect to $BV\text{-weak } \ast \times W^{1,p}\text{-weak convergence by Theorem 2.1. Thus, the superadditivity of the limit inf, the fact that } \left| \left( \frac{1}{\varepsilon} D\alpha \chi_{\varepsilon}, D_3 \chi_{\varepsilon} \right) \right| (\Omega) \geq |D_3 \chi_{\varepsilon}|(\Omega) \text{ and the lower semicontinuity of the total variation, entail that}

\[
F^{DR}(\chi, u) \geq \liminf_{\varepsilon \to 0^+} \int_\Omega f_0^{**}(\chi_{\varepsilon}(x_\alpha, x_3), \nabla_3 u_{\varepsilon}(x_\alpha, x_3)) \, dx + |D_3 \chi_{\varepsilon}|(\Omega)
\]

\[
\geq \pi \left( \int_0^l f_0^{**}(\chi(x_3), u'(x_3)) \, dx_3 + |D\chi|(0, l) \right).
\]
In order to prove the upper bound, we use Lemma 5.2 and replace $f$ by $f^{**}$. We follow the proof of Proposition 3.3, first assuming that $u \in C^1([0, l]; \mathbb{R}^3)$.

Let $\chi \in BV([0, l]; \{0, 1\})$ and consider $(\varphi, \psi) \in C^1([0, l]; \mathbb{R}^3) \times C^1([0, l]; \mathbb{R}^3)$. Define $\chi(x) := \chi(x_3)$ and $w_\varepsilon(x) := u(x_3) + \varepsilon (x_1 \varphi(x_3) + x_2 \psi(x_3))$. Clearly $w_\varepsilon \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbb{R}^3)$, and $(\frac{1}{\varepsilon} \nabla_\alpha w_\varepsilon, \nabla_\beta w_\varepsilon) \rightharpoonup (\varphi(x_3), \psi(x_3), u'(x_3))$ strongly in $L^q$ (even uniformly). Thus the bound from above in [5.2] allows us to invoke the Dominated Convergence Theorem to obtain

$$\limsup_{\varepsilon \to 0^+} \int_{\Omega} f^{**}(\chi(x_3), (\frac{1}{\varepsilon} \nabla_\alpha w_\varepsilon(x), \nabla_\beta w_\varepsilon(x))) \, dx = \pi \int_{0}^{l} f^{**}(\chi(x_3), \varphi(x_3), \psi(x_3), u'(x_3)) \, dx.$$ 

The arbitrariness of $\varphi$ and $\psi$ yield

$$\mathcal{F}^{DR}(\chi, u) \leq \pi |D\chi|[0, l] + \inf_{(\varphi, \psi) \in (C^1([0, l]; \mathbb{R}^3))^2} \pi \int_{0}^{l} f^{**}(\chi(x_3), \varphi(x_3), \psi(x_3), u'(x_3)) \, dx.$$ 

The above inequality also holds for $u \in W^{1,p}([0, l]; \mathbb{R}^3)$, by taking the infimum over pairs $(\varphi, \psi) \in (L^p([0, l]; \mathbb{R}^3))^2$, using standard mollification results and the same arguments as in the proof of Proposition 4.8. Thus we conclude that, for every $\chi \in BV([0, l]; \{0, 1\})$ and $u \in W^{1,p}([0, l]; \mathbb{R}^3)$,

$$\mathcal{F}^{DR}(\chi, u) \leq \pi |D\chi|[0, l] + \inf_{(\varphi, \psi) \in (L^p([0, l]; \mathbb{R}^3))^2} \pi \int_{0}^{l} f^{**}(\chi(x_3), \varphi(x_3), \psi(x_3), u'(x_3)) \, dx.$$ 

Observe that the continuity and the coercivity of $f^{**}(b, \cdot)$, as in (5.2), entail

$$(f^{**})_0(b, \xi_3) = f^{**}_0(b, \xi_3) \text{ for all } b \in \{0, 1\} \text{ and } \xi_3 \in \mathbb{R}^3,$$ 

so, using the coercivity in (5.2), (5.10) and the measurability criterion which provides the existence of $(\tilde{\varphi}, \tilde{\psi}) \in L^p([0, l]; \mathbb{R}^3)$ such that

$$(f_0)^{(**)}(\chi(x_3), u'(x_3)) = (f^{**})_0(\chi(x_3), u'(x_3)) = f^{**}(\chi(x_3), u'(x_3), \tilde{\varphi}(x_3), \tilde{\psi}(x_3)),$$

for every $(\chi, u) \in BV([0, l]; \{0, 1\}) \times W^{1,p}([0, l]; \mathbb{R}^3)$, it follows that

$$\mathcal{F}^{DR}(\chi, u) \leq \pi \left( |D\chi|[0, l] + \int_{0}^{l} f^{**}_0(\chi(x_3), u'(x_3)) \, dx \right),$$ 

which completes the proof. \hfill \Box

Arguments similar to those used in Remark 4.10 ensure that the volume constraint $1 / |\Omega| \int_{\Omega} \chi(x) \, dx = \theta$ can also be treated in the dimension reduction problem studied in Proposition 5.1.

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