Polynomial-Time Preprocessing for Weighted Problems Beyond Additive Goal Functions

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Abstract

Kernelization is the fundamental notion for polynomial-time preprocessing with performance guarantees in parameterized algorithmics. When preprocessing weighted problems, the need of shrinking weights might arise. Marx and Végh [ACM Trans. Algorithms 2015] and Etscheid et al. [J. Comput. Syst. Sci. 2017] used a technique due to Frank and Tardos [Combinatorica 1987] that we refer to as losing-weight technique to obtain kernels of polynomial size for weighted problems. While the mentioned earlier works focus on problems with additive goal functions, we focus on a broader class of goal functions. We lift the losing-weight technique to what we call linearizable goal functions, which also contain non-additive functions. We apply the lifted technique to five exemplary problems, thereby improving two results from the literature by proving polynomial kernels.

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1 Introduction

In the early eighties, Grötschel et al. [16] employed the famous ellipsoid method by Khachiyan [17, 18] for solving the Weighted Independent Set (WIS) problem: Given an undirected graph \( G = (V, E) \) with vertex weights \( w: V \to \mathbb{Q}_+ \), find a set \( U \subseteq V \) such that \( U \) is an independent set and maximizes \( \sum_{v \in U} w(v) \). Grötschel et al. [10] proved WIS to be solvable in polynomial time on perfect graphs. The running time of their algorithm, however, depends on the length of the encoding of the maximum vertex weight in the input. This led to the following question: Is WIS on perfect graphs solvable in polynomial time with the running time being independent of the maximum vertex weight in the input?

In their seminal work, Frank and Tardos [15] affirmatively answered this question by developing a (what we call) losing-weight technique. Their technique employs a “preprocessing algorithm” that, exemplified for WIS, does the following:

**Example 1.1 (Weighted Independent Set).** Compute in time polynomial in the number \( n \) of graph vertices some vertex weights \( \hat{w} \) with small values, that is,

(a) where the encoding length of the maximum value of \( \hat{w} \) is upper-bounded by a polynomial function in \( n \), such that the quality of all solutions and non-solutions is preserved, that is,

(b) for every two (independent) sets \( U, U' \subseteq V \) we have that \( \sum_{v \in U} w(v) \geq \sum_{v \in U'} w(v) \) if and only if \( \sum_{v \in U} \hat{w}(v) \geq \sum_{v \in U'} \hat{w}(v) \).

Hence, when first applying the losing-weight technique and then the algorithm of Grötschel et al. [10], then WIS on perfect graphs is solved in polynomial time independent of the maximum input vertex weight. The preprocessing algorithm makes use of the simultaneous Diophantine approximation algorithm due to Lenstra et al. [20].

To the best of our knowledge, Frank and Tardos’ [15] technique was used the first time in parameterized algorithmics in the work of Fellows et al. [14]. However, Fellows et al. [14] employed the technique to obtain fixed-parameter algorithms running in polynomial space. Marx and Végh [21] first observed the connection of the losing-weight technique with polynomial-time data preprocessing in parameterized algorithmics, namely polynomial kernelization. Marx and Végh [21] proved a polynomial kernel utilizing the losing-weight technique for Minimum-Cost Edge-Connectivity Augmentation by One, where, given an undirected \((k-1)\)-edge-connected graph \( G = (V, E) \), edge set \( E^* \), two weight functions \( w: E^* \to \mathbb{N} \) and \( c: E^* \to \mathbb{R}_+ \cup \{+\infty\} \), and two integers \( k, p \in \mathbb{N} \), the task is to find a set \( F \subseteq E^* \) with \( \sum_{c \in F} w(c) \leq p \) such that the graph \((V, E \cup F)\) is \( k\)-edge-connected and \( \sum_{c \in F} c(c) \) is minimized. Interestingly, their kernelization first increases the size of the instance and then introduces additional edge weights. Marx and Végh [21] find that “[... this technique seems to be an essential tool for kernelization of problems involving costs.” Subsequently, Etscheid et al. [12] used the technique to prove polynomial kernels for several weighted problems, supporting Marx and Végh’s statement. Notably, van Bevern et al. [7] applied the losing-weight technique to kernelize an unweighted problem. While such an application seems new, the idea of first introducing weights to an unweighted problem and then shrinking them can also be found in the work of Marx and Végh [21]. Notably, in all but one problem studied by the three papers mentioned above, the goal functions are additive set functions \((f(A \cup B) = f(A) + f(B))\). We show how to apply the technique to obtain polynomial kernels for problems with non-additive goal functions, like the maximum or minimum function. The only problem of Etscheid et al. [12] whose goal function is not an additive set function is Integer Polynomial Programming, where they use a different (enumerative) approach than we propose in this work.
Our Contribution. We lift Frank and Tardos’ losing-weight technique to problems whose goal functions are not additive set functions. We define a family of functions (that we refer to as linearizable functions) amenable for our lifted technique and provide some tools to easily recognize them. We exemplify the lifted technique using problems from graph theory, operations research, and computational social choice. Thereby, we improve two results known from the literature. We settle an open problem on the one hand and on the other hand we prove a theorem that has been claimed in the literature but has not yet been proven.

Structure of this Work. In Section 2.1 we introduce some notation and definitions (Section 2.1) and give a brief introduction into the losing-weight technique for polynomial kernelization (Section 2.2). In Section 3, we present applications of the the losing-weight technique for two specific problems with non-additive goal functions. In Section 4, we introduce linearizable functions and prove our framework of applying the losing-weight technique for these functions. In Section 5, we give examples of problems with linearizable goal functions and show how our framework applies. We conclude in Section 6. Due to the space constraints, several details and proofs (marked with *) are deferred to the appendix.

2 Preliminaries and the Losing Weight Technique

In Section 2.1 we first introduce necessary notation and concepts. Then, in Section 2.2 we briefly recall and discuss the result of Frank and Tardos [15], which forms the central basis of our work.

2.1 Notation & Definitions

An n-dimensional vector \( x \in S^n \) for some set \( S \) is interpreted as a column vector, and we denote by \( x^\top \) its transposition (i.e., \( x^\top \) is a row vector). The \( \ell_1 \)-norm of a vector \( x \in \mathbb{R}^n \) is \( \|x\|_1 := \sum_{i=1}^n |x_i| \). The \( \ell_\infty \)-norm (also known as max-norm) of \( x \) is \( \|x\|_\infty := \max_{i \in \{1,\ldots,n\}} |x_i| \).

For a number \( x \in \mathbb{Q} \), \( \text{sign}(x) = 1 \) if \( x > 0 \), \( \text{sign}(x) = 0 \) if \( x = 0 \), and \( \text{sign}(x) = -1 \) if \( x < 0 \).

Let \( \Sigma \) be a finite alphabet. A set \( L \subseteq \Sigma^* \times \mathbb{N} \) is called a parameterized problem. A problem kernelization for a parameterized problem \( L \subseteq \Sigma^* \times \mathbb{N} \) is an algorithm that, given an instance \( (x,k) \), computes in polynomial time an instance \( (x',k') \) such that (i) \( (x,k) \in L \) if and only if \( (x',k') \in L \), and (ii) \( |x'| + k' \leq f(k) \) for some computable function \( f \) only depending on \( k \). We call \( f \) the size of the problem kernel \( (x',k') \). If \( f \in \mathbb{O}(1) \), we call the problem kernel polynomial.

2.2 Losing Weight Technique

The preprocessing algorithm of Frank and Tardos [15] provides the following result central for this section.

\[ \text{Theorem 2.1 ([15] Section 3).} \]

On inputs \( w \in \mathbb{Q}^d \) and integer \( N \), one can compute in time polynomial in the encoding length of \( w \) and \( N \) a vector \( \hat{w} \in \mathbb{Z}^d \) with

(i) \( \|\hat{w}\|_1 \leq 2^{|d|} N^d (d+2) \) such that
(ii) \( \text{sign}(w^\top b) = \text{sign}(\hat{w}^\top b) \) for all \( b \in \mathbb{Z}^d \) with \( \|b\|_1 \leq N - 1 \).

Recall our description of the losing-weight technique in Section 1 for the example of Weighted Independent Set. We briefly explain how Example 1.1(a) and (b) from the description there relate to Theorem 2.1. While the correspondence between Example 1.1(a) and Theorem 2.1(i) is immediate, that (b) corresponds to Theorem 2.1(ii) is less obvious: Let \( u = \)
Poly-Time Preprocessing for Weighted Problems Beyond Additive Goal Functions

$(u_v)_{v \in V}, u' = (u'_v)_{v \in V} \in \{0, 1\}^{|V|}$ be the vectors representing the sets $U$ and $U'$, respectively, that is, $u_v = 1$ if and only if $v \in U$ (analogously for $u'$ and $U'$). Then $(v \in U, v_w = u^T w$ and $\sum_{v \in U'} w_v = u'^T w$. By this, observe that the “if-and-only-if” statement in Example 1.1(b) can be rewritten as $u^T w - u'^T w \geq 0 \iff u^T \hat{w} - u'^T \hat{w} \geq 0$. With $w := u - u'$ (note that $\|b\|_1 \leq 2|V|$) the correspondence to Theorem 2.1(ii) now becomes clear.

Next, we provide some first observations concerning Theorem 2.1. First, observe that the sign of each entry of the weight vector can be maintained.

**Observation 2.2.** For $N \geq 2$, Theorem 2.1 implies that $\text{sign}(u^T \hat{e}_i) = \text{sign}(\hat{w}^T \hat{e}_i)$ for each $i \in \{1, \ldots, d\}$, where $\hat{e}_i \in \mathbb{R}^d$ is the vector that has 1 in the $i$-th entry and zeroes in all others. Thus, $\text{sign}(w_i) = \text{sign}(\hat{w}_i)$ for each $i \in \{1, \ldots, d\}$.

Moreover, we observe that the order between weights can also be maintained.

**Observation 2.3.** For $N \geq 3$, Theorem 2.1 implies that $\text{sign}(u^T (\hat{e}_i - \hat{e}_j)) = \text{sign}(\hat{w}^T (\hat{e}_i - \hat{e}_j))$ for each $i, j \in \{1, \ldots, d\}$. Thus, $w_i - w_j \geq 0 \iff \hat{w}_i - \hat{w}_j \geq 0$ for each $i \in \{1, \ldots, d\}$.

Theorem 2.1 also works for decision rather than optimization problems. Indeed, the application to decision problems is a direct corollary, first stated by Marx and Végh [21, Remark 3.15] and then formalized by Etscheid et al. [12], by observing that the value given along in the description of the decision problem can be “attached” to the weight vector.

**Corollary 2.4 ([12]).** Given $(w, k) \in \mathbb{Q}^{d+1}$ with $w \in \mathbb{Q}^d$ and $N \in \mathbb{N}$, one can compute in time polynomial in the encoding length of $w$, $k$, and $N$ a vector $(\hat{w}, \hat{k}) \in \mathbb{Z}^{d+1}$ with $\hat{w} \in \mathbb{Z}^d$ and $\hat{k} \in \mathbb{Z}$ such that

- $(i)$ $\|\hat{w}\|_\infty, |\hat{k}| \leq 2^{d(d+1)^3}N^{(d+1)(d+3)}$
- $(ii)$ $w^T b \leq k \iff \hat{w}^T b \leq \hat{k}$ for all $b \in \mathbb{Z}^d$ with $\|b\|_1 \leq N - 2$.

Whenever we are facing a weighted problem with an additive goal function, that is, for example finding some set $S$ such that $\sum_{s \in S} w(s)$ is minimized (or maximized), the application of Theorem 2.1 is often immediate. So it is for the well-known Knapsack problem, as first proven by Etscheid et al. [12], yet solving a long-standing open question [10].

**Example 2.5.** Recall the Knapsack problem: Given a set $X = \{1, \ldots, n\}$ of $n$ items with weights $w: X \to \mathbb{Q}$ and values $v: X \to \mathbb{Q}$, and rational numbers $k, \ell \in \mathbb{Q}$, the question is whether there is a subset $S \subseteq X$ of items such that $\sum_{i \in S} w(i) \leq k$ and $\sum_{i \in S} v(i) \geq \ell$. Let $w$ and $v$ be interpreted as $n$-dimensional vectors such that $w_i = w(i)$ and $v_i = v(i)$. A direct application of Corollary 2.4 to each of $(w, k)$ and $(v, p)$ with $d = n$ and $N = n + 1$ yields a kernel of size polynomial in $n$.

One may wonder whether an application as outlined in Example 2.5 works also for non-additive goal functions or goal functions that are not initially of the form $w^T b$ (recall that $b$ in Theorem 2.1 represents all solution candidates): If the goal function is not initially of the form $w^T b$, an application of Theorem 2.1 is possibly not directly immediate. Yet we show applications to problems with non-additive goal functions in the next section. Our core idea is as follows. In Example 1.1 (WIS) and Example 2.5 (Knapsack) the used vector-representation has a one-to-one correspondence to solutions: Any solution to WIS or Knapsack is a set of vertices or items, respectively. This set can clearly be represented with a binary vector and every binary vector represents a solution. In our applications this is not the case: We still require that every “solution” can be represented as vector. However, not every vector will represent a solution. This is fine as Theorem 2.1(ii) holds for all specified vectors $b$, and thus, also for all vectors that do represent solutions.

In the rest of this work we describe a general framework and some applications. The details of our framework are somewhat technical; thus, we start with two concrete case studies exposing the general ideas used in the framework.
3 Two case studies with non-additive goal functions

In this section, we exemplify the ideas used in our framework to be formally introduced in Section 3.1. In Section 3.1, we show that some non-additive goal functions can be handled by simple scaling. In Section 3.2, we show how to deal with the maximum function as part of the goal function.

3.1 The Case of Small Set Expansion

Consider the following graph partitioning problem that received attention in the context of bicriteria approximation [3] and the unique games conjecture [22].

Problem (Small Set Expansion (SSE)).

**Input:** An undirected graph \( G = (V, E) \) with edge weights \( w:E \to \mathbb{Q}_+ \).

**Question:** Find a non-empty subset \( S \subseteq V \) of size at most \( |S| \leq n/2 \) that minimizes

\[
\frac{1}{|S|} \sum_{e \in (S,V \setminus S)} w(e),
\]

where \((S,V \setminus S)\) denotes the set of all edges with exactly one endpoint in \( S \).

The goal function’s value for a vertex set \( S \) can be represented by \( w^\top s \) for a fractional vector \( s \in \{0,1/|S|\}^{|E|} \), where an entry of \( s \) is nonzero if and only if the corresponding edge is in the edge cut \((S,V \setminus S)\). Yet fractional numbers are not captured by Theorem 2.1. However, with a scaling argument we can derive an analogue to Theorem 2.1 dealing with fractional numbers. Let

\[
Q_r := \left\{ \frac{p}{q} \mid p \in \{0, \ldots, r\}, q \in \{1, \ldots, r\} \right\}.
\]

The analogue is as follows.

**Proposition 3.1.** On input \( w \in Q_d \) and integer \( r \in \mathbb{N} \), one can compute in time polynomial in the encoding length of \( w \) and \( r \) a vector \( \hat{w} \in \mathbb{Z}_d \) with

1. \( \|\hat{w}\|_\infty \leq 2^{d^2} (r^2 \cdot d + 1)^r \cdot d^{d+2} \)
2. \( \text{sign}(w^\top b) = \text{sign}(\hat{w}^\top b) \) for all \( b \in Q_d \).

**Proof.** Apply Theorem 2.1 with \( N = r! \cdot r \cdot d + 1 \) to obtain a vector \( \hat{w} \in \mathbb{Z}_d \) with

\[
\|\hat{w}\|_\infty \leq 2^{d^2} N^{d(d+2)} = 2^{d^2} (r! \cdot r \cdot d + 1)^{d(d+2)} \leq 2^{d^2} (r^2 \cdot d + 1)^{r \cdot d(d+2)}
\]

such that \( \text{sign}(w^\top b) = \text{sign}(\hat{w}^\top b) \) for all \( b \in \mathbb{Z}_d \) with \( \|b\|_1 \leq N - 1 \). Let \( b^* \in Q_d \). We have

\[
\text{sign}(w^\top b^*) = \text{sign}(\hat{w}^\top b^*) \iff \text{sign}(r! \cdot w^\top b^*) = \text{sign}(r! \cdot \hat{w}^\top b^*) \\
\iff \text{sign}(w^\top (r! \cdot b^*)) = \text{sign}(\hat{w}^\top (r! \cdot b^*)) \\
\iff \text{sign}(w^\top b^*) = \text{sign}(\hat{w}^\top b^*),
\]

where for \( b' = r! \cdot b^* \) it holds true that \( b' \in \mathbb{Z}_d \) and \( \|b'\|_1 \leq r! \cdot r \cdot d = N - 1 \).

Now, with Proposition 3.1, we get the following.

**Lemma 3.2.** On any input instance \((G = (V, E), w)\) of SSE with \( n := |V| \) and \( m := |E| \), one can compute in time polynomial in \(|(G, w)|\) an instance \((G, \hat{w})\) of SSE such that

1. \( \|\hat{w}\|_\infty \leq 2^{4m^2} \cdot (n^4 \cdot m + 1)^{n^2 m(m+2)} \) and
Poly-Time Preprocessing for Weighted Problems Beyond Additive Goal Functions

(ii) a set \( S \subseteq V \) is an optimal solution for \((G, w)\) if and only if \( S \) is an optimal solution for \((G, \hat{w})\).

Proof. Without loss of generality, consider the edges of \( G \) as enumerated \( E = \{e_1, \ldots, e_m\} \) and the weight functions \( w, \hat{w} \) as vectors in \( \mathbb{N}^m \) such that \( w_i = w(e_i) \) and \( \hat{w}_i = \hat{w}(e_i) \) for all \( i \in \{1, \ldots, m\} \). Apply Proposition 3.3 with \( d = m \) and \( r = n^2 \). Let \( S \subseteq V \) and let \( s \in \{0,1/|S|\}^m \) be the vector corresponding to the edges in the cut \((S, V \setminus S)\), that is, \( s_i \neq 0 \) if and only if \( e_i \in (S, V \setminus S) \). Let \( S' \subseteq V \) be another set, and let \( s' \in \{0,1/|S'|\}^m \) with \( s'_i \neq 0 \) if and only if \( e_i \in (S', V \setminus S') \). Let \( b := s - s' \). Note that

\[
|s_i - s'_i| = \frac{|S'|s_i - |S|s'_i|}{|S|} \in \left\{ 0, \frac{|S'| - |S|}{|S|}, \frac{1}{|S'|}, \frac{1}{|S|} \right\},
\]

and hence \( b_i \in \mathbb{Q}_{\pm 1} \). We thus get

\[
|s \top w - (s') \top w| \leq 0 \iff (s - s') \top w \leq 0
\]

\[
\iff (s - s') \top \hat{w} \leq 0 \iff s \top \hat{w} - (s') \top \hat{w} \leq 0.
\]

From Lemma 3.2, we immediately obtain the following kernel.

\[ \textbf{Proposition 3.3.} \text{ SSE admits a polynomial kernel with respect to the number of vertices.} \]

3.2 The Case of Min-Power Symmetric Connectivity

Consider the following NP-hard optimization problem from survivable network design [9, 1].

\[ \textbf{Problem \textit{(MIN-POWER SYMMETRIC CONNECTIVITY (MPSC)).}} \]

\textit{Input:} A connected undirected graph \( G = (V,E) \) and edge weights \( w : E \rightarrow \mathbb{N} \).

\textit{Task:} Find a connected spanning subgraph \( T = (V,F) \) of \( G \) that minimizes

\[
\sum_{v \in V} \max_{S \subseteq V \atop \{u,v\} \in F} w(\{u,v\}).
\]

Applying Theorem 2.1 to the goal function (3) is not obvious: Let \( E = \{e_1, \ldots, e_m\} \) be enumerated and the weight function \( w \) be represented as a vector in \( \mathbb{N}^m \) such that \( w_i = w(e_i) \). Let \( b \in \{0,1\}^m \) be the vector representing the edge set \( F \) of a solution \( T = (V,F) \), that is, \( b_i = 1 \) if and only if \( e_i \in F \). Then, the value \( w \top b \) is not equal to \( \sum_{v \in V} \max_{\{u,v\} \in F} w(\{u,v\}) \). See Figure 1(a) for an example.

However, we can circumvent this issue (arising from the max-function in the goal function) and still apply Theorem 2.1. To this end, observe that we only need to find a correct representation of a solution. An edge \( e \in F \) contributes its weight to the maximum in the sum, that is, either zero, one, or two times. Hence, a solution can be represented as vector \( b \in \{0,1,2\}^m \) with \( b_i = x \) if \( e_i \) appears \( x \in \{0,1,2\} \) times in the maximum in the sum of cost function concerning \( T = (V,F) \). See Figure 1(b) for another example.

Notably, this change of the representation of a solution only changes the domain of the vector \( b \), and hence the value of \( N \) in the application of Theorem 2.1 by a factor of two. Eventually, we obtain the following.

\[ \textbf{Lemma 3.4.} \text{ On any input instance } (G = (V,E),w) \text{ of MPSC with } m := |E|, \text{ one can compute in time polynomial in } |(G,w)| \text{ an instance } (G,\hat{w}) \text{ of MPSC such that} \]

(i) \( \|\hat{w}\|_\infty \leq 2^{4m^2} \cdot (2m + 2)^{m(m+2)} \) and

(ii) a connected subgraph \( T = (V,F) \) of \( G \) is an optimal solution for \((G,w)\) if and only if \( T \) is an optimal solution for \((G,\hat{w})\).
The feedback edge number is the smallest number of edges that must be removed to make a graph acyclic. 

In previous work we developed a partial kernel, that is, an algorithm that maps any instance of MPSC to an equivalent instance where the number of vertices and edges, yet not necessarily the edge weights, are polynomially upper-bounded in the feedback edge number of the input graph [3] [4]. Finding a polynomial kernel regarding the feedback edge number was an open problem. Given the partial kernel, Proposition 5.8 yields the following answer.

\[ \sum_{\{u,v\} \in F} \hat{w}(|u,v|) = s^T \hat{w} \quad \text{and} \quad \sum_{\{u,v\} \in F'} \hat{w}(|u,v|) = s'^T \hat{w}. \]

Due to Lemma 3.4 we immediately obtain a polynomial kernel.

\[ \text{Figure 1 Illustrative example for MPSC and the application of Theorem 2.1} \]

(a) depicts an edge-weighted undirected example graph, and (b) shows its incidence matrix, the vector \( w \) of edge-weights, and the vector \( b \) representing the solution indicated by thick edges in (a).

**Proof.** Without loss of generality, we consider the edges \( E = \{ e_1, \ldots, e_m \} \) of \( G \) as enumerated and the weight functions \( w, \hat{w} \) as (column) vectors in \( \mathbb{N}^m \) such that \( w_i = w(e_i) \) and \( \hat{w}_i = \hat{w}(e_i) \) for all \( i \in \{1, \ldots, m\} \). We apply Theorem 2.1 with \( d = m \) and \( N = 2m + 2 \) to the weight vector \( w \). From Theorem 2.1 it immediately follows that (i) holds, that is, \( \|\hat{w}\|_\infty \leq 2^{m+1} \cdot (2m + 1)^{m(m+2)} \). Moreover, recall that \( \hat{w}_i > 0 \) for all \( i \in \{1, \ldots, m\} \) due to Observation 2.3. Next, we prove that also (ii) holds true, that is, a connected graph \( T = (V, F) \) is an optimal solution for \((G, \hat{w})\) if and only if \( T \) is an optimal solution for \((G, \hat{w})\).

Let \( T = (V, F) \) be a connected subgraph of \( G \) and let \( s \in \{0, 1, 2\}^m \) be an \( m \)-dimensional vector that represents for each edge \( e_i \) the number \( s_i \) of appearances in the sum (3) (note that \( s_i > 0 \) \( \iff \ e_i \in F \)) such that \( \sum_{e \in V} \max_{\{u,v\} \in F} w(|{u,v}|) = s^T w \). For a connected subgraph \( T' = (V, F') \) of \( G \), let \( s' \in \{0, 1, 2\}^m \) be derived analogously to \( s \) such that the cost of \( T' \) is \( (s')^T w \). Define \( b := s - s' \). Note that for each \( i \in \{1, \ldots, m\} \) it holds true that \( -2 \leq b_i \leq 2 \), and hence \( \|b\|_1 \leq 2m + 1 = N - 1 \). Moreover, due to Theorem 2.1 we have \( \text{sign}(b^T w) = \text{sign}(b^T \hat{w}) \), or, equivalently,

\[ s^T w - (s')^T w \leq 0 \iff (s - s')^T \hat{w} \leq 0 \iff s^T \hat{w} - (s')^T \hat{w} \leq 0. \]

Finally, note that due to Observation 2.3 the goal function’s values of both \( T \) and \( T' \) are still correctly represented by \( s \) and \( s' \) given \( \hat{w} \), that is,

\[ \sum_{\{u,v\} \in F} \hat{w}(|{u,v}|) = s^T \hat{w} \quad \text{and} \quad \sum_{\{u,v\} \in F'} \hat{w}(|{u,v}|) = s'^T \hat{w}. \]

The feedback edge number is the smallest number of edges that must be removed to make a graph acyclic.

\[ \text{Figure 1 Illustrative example for MPSC and the application of Theorem 2.1} \]

(a) depicts an edge-weighted undirected example graph, and (b) shows its incidence matrix, the vector \( w \) of edge-weights, and the vector \( b \) representing the solution indicated by thick edges in (a).
Corollary 3.6. MPSC admits a polynomial kernel with respect to the feedback edge number of the input graph.

4 Linearizable Functions

In this section, we first provide our central framework for the case of rational functions. Then, we discuss integer functions which are more restrictive but allow for better bounds.

4.1 Q-linearizable Functions

The cases of SSE and MPSC show that problems with non-additive goal functions still allow an application of the losing-weight technique. A natural question is what characterizes these goal functions. Both of our cases have in common that for any weight vector \( w \), the goal function’s value for every solution \( s \) can be represented as \( b^T_s \cdot w \) with \( b_s \) being a vector associated with \( s \). Moreover, to apply the losing-weight technique, we also need that if we change the weight vector to a “smaller” weight vector \( \hat{w} \), then the goal function’s value is still represented for solution \( s \) as \( b^T_s \cdot \hat{w} \) and vice versa (for this we needed Observation 2.3 in the proof of Lemma 3.4). That is, we want that the value of solution \( s \) for \( w \) is \( b^T_s \cdot w \) if and only if the value of solution \( s \) for \( \hat{w} \) is \( b^T_s \cdot \hat{w} \). What we described so far is captured in the following (recall Equation (2) for the definition of \( Q_r \)).

Definition 4.1. A function \( f : L \times \mathbb{Q}^d \to \mathbb{Q} \) with \( L \subseteq \Sigma^* \) is \( \alpha \)-linearizable for some \( \alpha \in \mathbb{N} \) if for all \( w \in \mathbb{Q}^d \) and for all \( x \in L \) it holds that

(A) there exists \( b_x \in \mathbb{Q}^d_{\alpha} \) such that \( f(x, w) = b^T_x \cdot w \) and

(B) for all \( w' \in \{ w'' \in \mathbb{Q}^d \mid \forall \beta \in \mathbb{Q}^d, \text{sign}(\beta \cdot w') = \text{sign}(\beta \cdot w'') \} \) it holds that \( f(x, w) = b^T_x \cdot w \iff f(x, w') = b^T_x \cdot w' \).

Intuitively, an \( \alpha \)-linearizable goal function of a problem maps a solution (contained in the set \( L \)) together with a weight vector to its value. For a fixed weight vector, by (A) in Definition 4.1, this value can be expressed for every solution as the product of some vector representing the solution and the weight vector. Moreover, by (B) in Definition 4.1, this representation of the solution is robust against small changes to the weight vector.

The following, while possibly intuitively clear, holds true:

Observation 4.2. If \( f \) is \( \alpha \)-linearizable, then \( f \) is \( \alpha' \)-linearizable for all \( \alpha' \geq \alpha \).

We prove next that the losing-weight technique can be adapted to work for problems with \( \alpha \)-linearizable goal functions, that is, for any weight vector we can compute in time polynomial in the input size a “smaller” weight vector such that (the quality of) solutions are preserved.

Theorem 4.3. Let \( f : L \times \mathbb{Q}^d \to \mathbb{Q} \) with \( L \subseteq \Sigma^* \) be an \( \alpha \)-linearizable function, and let \( w \in \mathbb{Q}^d \). Then we can compute in time polynomial in the encoding length of \( w \) and \( \alpha \) a vector \( \hat{w} \in \mathbb{Z}^d \) such that

(i) \( \|\hat{w}\|_{\infty} \leq 2^{2d^2}(4\alpha^4 \cdot d + 1)^{2\alpha^3 \cdot d(d+2)} \), and

(ii) \( f(x, w) \geq f(y, w) \iff f(x, \hat{w}) \geq f(y, \hat{w}) \) for all \( x, y \in L \).

Proof. Apply Proposition 3.1 with \( r = 2\alpha^2 \) to obtain the vector \( \hat{w} \). Since \( f \) is \( \alpha \)-linearizable, by Definition 4.1(A) we know that there are \( b_x, b_y \in \mathbb{Q}^d_{\alpha} \) such that \( f(x, w) = b^T_x \cdot w \) and
we know that the function

where the last equivalence follows from the fact that for \( \beta \in \mathbb{Q}_{2^\alpha} \)
we have \( \text{sign}(\beta^\top \hat{w}) = \text{sign}(\beta^\top \hat{w}) \), and hence from Definition 4.1 we get
\( f(x, \hat{w}) = b_x^\top \hat{w} \) and \( f(y, \hat{w}) = b_y^\top \hat{w} \). \( \blacksquare \)

Intuitively, Theorem 4.3 yields the following. If we know that our goal function is \( \alpha \)-linearizable, then we can employ the losing-weight technique where the encoding length of the computed weight vector is polynomially upper-bounded in \( \alpha \) and the dimension \( d \). To easily employ Theorem 4.3 we only need to determine whether our goal function is \( \alpha \)-linearizable, and, in particular, determine \( \alpha \). In fact, in what follows, we show that \( \alpha \)-linearizable functions are functionally \textit{composable}. If a function is composed of \( \alpha \)-linearizable functions, then it is \( \alpha' \)-linearizable for some \( \alpha' \). This allows for recognizing whether a function is \( \alpha \)-linearizable by only looking at the functions it is composed of. In the following we outline several of these functional compositions, and exemplify its usage on \textsc{Min-Power Symmetric Connectivity} (MPSC) and \textsc{Small Set Expansion} (SSE).

Revisiting the Case of MPSC

The goal function in MPSC is composed of a sum over maxima. We prove that such a composition preserves linearizability.

\textbf{Lemma 4.4}. Let \( f : L \times \mathbb{Q}^d \to \mathbb{Q} \) and \( f' : 2^L \times \mathbb{Q}^d \to \mathbb{Q} \) be two functions where \( L \) is a set of cardinality \( n \in \mathbb{N} \). If \( f \) is \( \alpha \)-linearizable, then

(i) \( f'(X, w) = \sum_{x \in X} f(x, w) \) is \( \alpha n \alpha \)-linearizable;

(ii) \( f'(X, w) = \max_{x \in X} f(x, w) \) is \( 2\alpha \)-linearizable;

(iii) \( f'(X, w) = \min_{x \in X} f(x, w) \) is \( 2\alpha \)-linearizable.

We explain the use of our machinery for \textsc{Min-Power Symmetric Connectivity}. First observe that we can rewrite the goal function to fit our notion as follows. Let \( F_v := \{ e \in F \mid v \in e \} \) and \( F := \{ F_v \mid v \in V \} \). Then

\[ h(F, w) = \sum_{F_v \in F} g(F_v, w), \quad \text{with } g(F, w) = \max_{e \in F} w(e). \]

Clearly, with \( E = \{ e_1, \ldots, e_m \} \) the function \( f : E \times \mathbb{Q}^m \to \mathbb{Q} \), \( f(e_i, w) \mapsto w_i \) is 1-linearizable: On the one hand, we have that \( f(e_i, w) = \hat{e}_i^\top w \) (recall that \( \hat{e}_i \) denotes the unit vector with the \( i \)-th entry being one). On the other hand, for all \( w' \in \mathbb{Q}^m \) it holds true that \( f(e_i, w) = \hat{e}_i^\top w \iff f(e_i, w') = \hat{e}_i^\top w' \) (in fact, this even holds true without any conditions on \( w' \)).

Due to Lemma 4.4, we know that the function \( g(F, w) = \max_{e \in F} f(e, w) \) is 2-linearizable. Finally, due to Lemma 4.4 (where \( L = E \) and hence contains \( m \) elements), we know that the function \( h(F, w) = \sum_{F_v \in F} g(F_v, w) \) is \( 4m \)-linearizable. Employing Theorem 4.3 we get in polynomial time a vector \( \hat{w} \in \mathbb{Q}^m \) such that

\[ \| \hat{w} \|_\infty \in 2^{O(m^2)}, \]
for any two connected subgraphs $T = (V, F)$, $T' = (V, F')$ of $G$, we have

$$\sum_{v \in V, (u,v) \in F} \max_{w} w(\{u,v\}) \geq \sum_{v \in V, (u,v) \in F'} \max_{\hat{w}} \hat{w}(\{u,v\})$$

that is, optimal solutions are preserved under $\hat{w}$. Altogether, we have again proven Lemma 3.4.

Revisiting the Case of SSE

The goal function in SSE is a multiplication of a number and a sum. By Lemma 4.4, we already know that the sum preserves linearizability. However, we also need to show how linearizability is preserved under multiplication with some constant.

Lemma 4.5. Let $f, f': L \times Q^d \rightarrow Q$, where $L$ is equipped with some function $c: L \rightarrow Q_n \setminus \{0\}$ and $n \in N$. If $f$ is $\alpha$-linearizable, then $f'(x, w) = c(x) \cdot f(x, w)$ is $\alpha \cdot n$-linearizable.

We now explain the usage of our machinery for SSE. Let $E_S := (S, V \setminus S)$ for all $S \subseteq V$. Let $L = \{(S, E_S) | S \subseteq V, 1 \leq |S| \leq n/2 \}$. Let $c: L \rightarrow Q_n \setminus \{0\}, c: (S, E_S) \mapsto \frac{1}{|S|}$. Then

$$h((S, E_S), w) = \frac{1}{|S|} g((S, E_S), w), \quad \text{with} \quad g((S, E_S), w) = \sum_{e \in E_S} w(e).$$

We already know that $w: E \rightarrow Q$ is $1$-linearizable. Moreover, by Lemma 4.4, we know that $g$ is $m$-linearizable. Finally, due to Lemma 3.4, we arrive at $h$ being $n \cdot m$-linearizable. Finally employing Theorem 4.3 again proves Lemma 3.2.

4.2 $Z$-linearizable Functions

Note that in Lemma 4.4, we obtain an $\alpha'$-linearizable function from an $\alpha$-linearizable function where $\alpha'$ contains the factor $\alpha!$. This is not the case if we replace $Q_\alpha$ in Definition 4.1 by $Z_r$, where

$$Z_r := \{ \pm p \in Z \mid p \in \{0, \ldots, r\} \}.$$ 

This replacement, although more restrictive, appears to be often sufficient like in the case of MPSC, and also allows for “chaining up sums” while keeping $\alpha$ polynomially bounded. To provide formal details, we state the analogous versions of Definition 4.4, Theorem 4.3 and Lemma 4.3. The only difference is the replacement of $Q_\alpha$ with $Z_r$ and the corresponding better bounds.

Definition 4.6. A function $f: L \times Q^d \rightarrow Q$ with $L \subseteq \Sigma^*$ is $\alpha$-$Z$-linearizable, $\alpha \in N$, if for all $w \in Q^d$ and for all $x \in L$ it holds that

(A) there exists $b_x \in Z_\alpha^d$ such that $f(x, w) = b_x^T w$ and

(B) for all $w' \in \{ w'' \in Q^d | \forall \beta \in Z_\alpha^d. \text{sign}(\beta \cdot w) = \text{sign}(\beta \cdot w'') \}$ it holds that $f(x, w) = b_x^T w \iff f(x, w') = b_x^T w'$.

Theorem 4.7. Let $f: L \times Q^d \rightarrow Q$ with $L \subseteq \Sigma^*$ be an $\alpha$-$Z$-linearizable function, and let $w \in Q^d$. Then we can compute in time polynomial in the encoding length of $w$ and $\alpha$ a vector $\hat{w} \in Z^d$ such that

(i) $\|\hat{w}\|_{\infty} \leq 2^{4d^2}(2\alpha \cdot d + 1)^{d(d+2)}$, and

(ii) $f(x, w) \geq f(y, w) \iff f(x, \hat{w}) \geq f(y, \hat{w})$ for all $x, y \in L.$
Lemma 4.8. Let \( f : L \times \mathbb{Q}^d \rightarrow \mathbb{Q} \) and \( f' : 2^L \times \mathbb{Q}^d \rightarrow \mathbb{Q} \) be two functions, where \( L \) is a set of cardinality \( n \in \mathbb{N} \). If \( f \) is \( \alpha \)-Z-linearizable, then

(i) \( f'(X, w) = \sum_{x \in X} f(x, w) \) is \( n\alpha \)-Z-linearizable;
(ii) \( f'(X, w) = \max_{x \in X} f(x, w) \) is \( 2\alpha \)-Z-linearizable;
(iii) \( f'(X, w) = \min_{x \in X} f(x, w) \) is \( 2\alpha \)-Z-linearizable.

Similarly, we can derive the following analogue to Lemma 4.5.

Lemma 4.9. Let \( f, f' : L \times \mathbb{Q}^d \rightarrow \mathbb{Q} \) where \( L \) is equipped with some function \( c : L \rightarrow \mathbb{Z} \setminus \{0\} \), where \( n \in \mathbb{N} \). If \( f \) is \( \alpha \)-Z-linearizable, then \( f'(x, w) = c(x) \cdot f(x, w) \) is \( \alpha n \)-Z-linearizable.

A Brief Summary. We introduced \( \alpha \)-linearizable functions (Definition 4.1). Due to Lemmas 4.4 and 4.5, we can efficiently recognize special types of \( \alpha \)-linearizable functions by simply looking at how a given function is composed. Further, we proved that if a problem has an \( \alpha \)-linearizable goal function, then the losing-weight technique applies (Theorem 4.3). Thus, we can wrap up our framework into the following recipe: Consider a weighted problem that seeks for some set that maximizes/minimizes some goal function \( f \).

Step 1: Find a representation of \( f \) such that \( f \) can be decomposed into linearizable functions.
Step 2: Recursively employ Lemmas 4.4 and 4.5 to determine \( \alpha \) such that \( f \) is \( \alpha \)-linearizable.
Step 3: Apply Theorem 4.3 with \( \alpha \) to obtain a smaller weight vector that preserves equivalence among (optimal) solutions.

We showed that any combination of sums, maxima, minima, and multiplication with constant factors preserves linearizability. Finding compositions of further functions preserving linearizability remains for future work.

5 Further Applications

In this section, we provide further problems with linearizable goal functions and demonstrate how our framework applies to them. Herein, one problem is from computational geometry, one from computational social choice, and one is a graph-labeling problem.

5.1 Uncapacitated Facility Location

Our next example is the Uncapacitated Facility Location Problem, also known as the Simple Plant Location Problem, which is well studied in operations research [19, Section 3.4], yet also in the context of parameterized complexity and data reduction [5, 13].

Problem (Uncapacitated Facility Location Problem (UFLP)).

Input: A set \( \mathcal{C} \) of \( n \) clients, a set \( \mathcal{F} \) of \( m \) facilities, facility opening costs \( f : \mathcal{F} \rightarrow \mathbb{Q}_{\geq 0} \), and client service costs \( c : \mathcal{F} \times \mathcal{C} \rightarrow \mathbb{Q}_{\geq 0} \).

Task: Find a subset \( \mathcal{F}' \subseteq \mathcal{F} \) that minimizes

\[
\sum_{i \in \mathcal{F}'} f(i) + \sum_{j \in \mathcal{C}} \min_{i \in \mathcal{F}'} c(i, j). \tag{4}
\]

By showing that the goal function (4) is linearizable, we prove a problem kernel with size polynomial in the number of clients and facilities, thus complementing a kernelization result of Fellows and Fernau [13], who showed a problem kernel with size exponential in a given upper bound on the optimum (which may be arbitrarily larger than \( n + m \)).

Lemma 5.1. There is an algorithm that, on any input instance \((\mathcal{C}, \mathcal{F}, f, c)\) of UFLP, computes in time polynomial in \( |(\mathcal{C}, \mathcal{F}, f, c)| \) an instance \((\mathcal{C}, \mathcal{F}, \bar{f}, \bar{c})\) of UFLP such that
Poly-Time Preprocessing for Weighted Problems Beyond Additive Goal Functions

(i) \( \| \vec{f} \|_\infty + \| \vec{c} \|_\infty \leq 2^{4(nm+m)^3} (4(2n + m)(nm + m) + 1)^{(nm+m)(nm+m+2)} \) and

(ii) any subset \( \mathcal{F}' \subseteq \mathcal{F} \) forms an optimal solution for \( (\mathcal{C}, \mathcal{F}, f, c) \) if and only if \( \mathcal{F}' \) forms an optimal solution for \( (\mathcal{C}, \mathcal{F}, f, c) \).

We can apply Lemma 5.1 also for the Metric Uncapacitated Facility Location Problem (MUFLP), which requires the cost function to satisfy the triangle inequality \( c(i,j) \leq c(i,j') + c(i',j) \) for all \( i, j \in \mathcal{C} \) and \( j', j \in \mathcal{F} \), since in such cases, the resulting instance \( (\mathcal{C}, \mathcal{F}, f, c) \) also satisfies the triangle inequality and is thus also an instance of MUFLP. This follows from the following easy observation:

\( \textbf{Observation 5.2.} \) For \( N \geq 5 \), Theorem 2.1 implies that \( \text{sign}(w^\top (\vec{c}_i + \vec{c}_j + \vec{c}_k - \vec{c}_\ell)) = \text{sign}(\hat{w}^\top (\hat{c}_i + \hat{c}_j + \hat{c}_k - \hat{c}_\ell)) \) for each \( i, j, k, \ell \in \{1, \ldots, d\} \).

The consequence of Observation 5.2 is that if Theorem 2.1 is applied to a vector \( w \) encoding a metric (i.e., the entries of \( w \) are pairwise distances of some points), then the resulting vector \( \hat{w} \) also encodes a metric in the same way. As Theorems 4.3 and 4.7 are built on Theorem 2.1, applications of these propositions preserve metrics as well.

Overall, we obtain the following kernels.

\( \textbf{Proposition 5.3.} \) UFLP and MUFLP each admit a polynomial kernel with respect to the number of vertices.

5.2 Chamberlin-Courant Committee with Cardinal Utilities

Another exemplary application is the following problem from computational social choice [8].

\( \textbf{Problem (Chamberlin-Courant Committee with Cardinal Utilities (C^4U)).} \)

\textit{Input:} A set \( V \) of voters, a set \( A \) of alternatives, an integer \( k \), and an utility function \( u : V \times A \to \mathbb{Q}_{\geq 0} \).

\textit{Task:} Find a subset \( A' \subseteq A \) of size at most \( k \) that maximizes

\[ \max_{v \in V} \max_{a \in A'} u(v, a). \quad (5) \]

We will show that the goal function (5) is linearizable.

\( \textbf{Lemma 5.4.} \) There is an algorithm that, on any input instance \((V, A, k, u)\) of \( C^4U \) with \( n := |V| \) and \( m := |A| \), computes in time polynomial in \(|V, A, k, u|\) an instance \((V, A, k, \bar{u})\) of \( C^4U \) such that

(i) \( \| \bar{u} \|_\infty \leq 2^{4((nm)^3)} + (4n^2 + 1)^{nm(nm + 2)} \) and

(ii) any subset \( A' \subseteq A \) forms an optimal solution for \((V, A, k, u)\) if and only if \( A' \) forms an optimal solution for \((V, A, k, \bar{u})\).

\textbf{Proof.} Observe that the following restatement of the goal function is true:

\[ \max_{v \in V} \max_{a \in A'} u(v, a) = \max_{v \in V} \max_{(v, a) \in \{v\} \times A'} u(v, a). \]

Clearly, \( u : V \times A \to \mathbb{Q}_{\geq 0} \) is 1-Z-linearizable. Note that the weight vector representing \( u \) is of dimension \( d = nm \). By Lemma 4.8(iii), we know that \( g(v, A') = \max_{(v, a) \in \{v\} \times A'} u(v, a) \) is 2-Z-linearizable. Finally, by Lemma 4.8(iv), \( h(V, A') = \sum_{v \in V} g(v, A') \) is \( 2n \)-Z-linearizable. By Theorem 4.7 with \( a = 2n \) and \( d = nm \), the claim follows.

Lemma 5.4 yields the following problem kernel.

\( \textbf{Proposition 5.5.} \) \( C^4U \) admits a polynomial kernel with respect to the number of voters and alternatives.
5.3 Power Vertex Cover

Angel et al. [2] claim to have shown a polynomial-size problem kernel for the following NP-hard problem parameterized by the number of vertices that are assigned non-zero values in a solution:

Problem (Power Vertex Cover (PVC)).

Input: An undirected graph \( G = (V, E) \) with edge weights \( w : E \rightarrow \mathbb{Q}_{\geq 0} \).

Task: Find an assignment \( \mu : V \rightarrow \mathbb{Q}_{\geq 0} \) such that for each edge \( e = \{v, w\} \in E \) it holds true that \( \max \{\mu(v), \mu(w)\} \geq w(e) \) and that minimizes \( \sum_{v \in V} \mu(v) \).

However, actually Angel et al. [2] only proved a partial kernel, since the edge weights in their kernel can remain arbitrarily large. We prove that we can shrink the weights, and hence obtain a true polynomial kernel. In fact, our application of the losing-weight technique for PVC relies on the following.

Observation 5.6. If \( \mu \) is an optimal solution, then for every \( v \in V \) we have \( \mu(v) \in \{w(e) \mid e \in E\} \cup \{0\} \).

Lemma 5.7. There is an algorithm that, on any input instance \( I = (G = (V, E), w) \) of PVC with \( n := |V| \) and \( m := |E| \), computes in time polynomial in \(|I|\) an instance \( I' = (G = (V, E), \hat{w}) \) of PVC such that

(i) \( \|\hat{w}\|_{\infty} \leq 2^{nm^2} \cdot (2nm + 1)^{(m+2)} \) and

(ii) any assignment \( \mu : V \rightarrow E \cup \{\emptyset\} \) forms an optimal solution for \( I \) if and only if \( \mu \) forms an optimal solution for \( I' \).

Proof. By Observation 5.6 we can restate the problem as follows: Find an assignment \( \mu : V \rightarrow E \cup \{\emptyset\} \) such that for every \( e = \{u, v\} \) we have \( w(e) \leq \max \{w(\mu(u)), w(\mu(v))\} \) and that minimizes \( \sum_{v \in V} w(\mu(v)) \), where \( w(\emptyset) = 0 \). Let \( f(v, w) = w(e) \) if \( \mu(v) = e \) and \( 0 \) if \( \mu(v) = \emptyset \).

Clearly, \( f \) is 1-Z-linearizable. Hence, \( g(V, w) = \sum_{v \in V} f(v, w) \) is \( n \)-Z-linearizable. Theorem 4.7 now yields the desired statement with \( \alpha = n \) and \( d = m \).

Using Lemma 5.7 and the partial kernel of Angel et al. [2] we obtain the following kernels.

Proposition 5.8. PVC admits a polynomial kernel with respect to the number of non-zero values in a solution.

6 Concluding Remarks

The losing-weight technique due to Frank and Tardos [15] emerges as a key ingredient for obtaining polynomial kernelization for weighted parameterized problems. While Marx and Végh [21] and Etscheid et al. [12] proved the usefulness of the technique for several problems with additive goal functions, we proved the technique to be applicable to problems with non-additive goal functions, like the MIN-POWER SYMMETRIC CONNECTIVITY problem.

As Etscheid et al. [12] already pointed out, one direction for future work could be to improve on the upper bound in Theorem 2.1. In this direction Eisenbrandt et al. [11] recently proved a stronger upper bound, yet unconstructively. Another direction, seemingly not addressed so far, aims on the running time. Note that Frank and Tardos [15] state no explicit running time of their algorithm, and Lenstra et al. [20] Proposition 1.26 state that their simultaneous Diophantine approximation algorithm, which forms a subroutine in Frank and Tardos’ technique, runs in \( O(d^6(\log(\|w\|_{\infty}))^{O(1)}) \) time. Hence, we put forward the following: Can Theorem 2.1 be executed in quadratic, or even linear time? We remark that van Bevern et al. [6] give a linear-time variant Theorem 2.1 for approximate kernelizations.
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A Additional Material for Section 4

A.1 Proof of Observation 4.2

Proof. We have to prove that for all $w \in \mathbb{Q}_d$, \([A]\) and \([B]\) for $\alpha'$ hold true. For \([A]\) for $\alpha'$, observe that for all $x \in L$, there exists $b_x \in \mathbb{Q}_d \subseteq \mathbb{Q}_{d_{n0\alpha}}$ such that $f(x, w) = b_x^T w$, and hence \([A]\) for $\alpha'$ follows from \([A]\) for $\alpha$. For \([B]\) for $\alpha'$, let $w' \in \mathbb{Q}_d$ such that $\text{sign}(\beta^T w) = \text{sign}(\beta^T w')$ for all $\beta \in \mathbb{Q}_{d_{n0\alpha}}$. Then, for $w'$ we also have that $\text{sign}(\beta^T w) = \text{sign}(\beta^T w')$ for all $\beta \in \mathbb{Q}_{d_{n0\alpha}}$ as $\mathbb{Q}_{d_{n0\alpha}} \subseteq \mathbb{Q}_{d_{n0\alpha}}$. Said differently, we have that $\{ w' \in \mathbb{Q}_d | \forall \beta \in \mathbb{Q}_{d_{n0\alpha}}: \text{sign}(\beta^T w) = \text{sign}(\beta^T w') \} \subseteq \{ w' \in \mathbb{Q}_d | \forall \beta \in \mathbb{Q}_{d_{n0\alpha}}: \text{sign}(\beta^T w) = \text{sign}(\beta^T w') \}$. By \([B]\) for $\alpha$ we have that $f(x, w) = b_x^T w$ if and only if $f(x, w) = b_x^T w'$, and hence \([B]\) for $\alpha'$ follows. □

A.2 Proof of Lemma 4.4

Proof. \([\text{1}]\): Since $f$ is $\alpha$-linearizable, by Definition 4.1 \([\text{A}]\) for $f$ we know that $f(x, w) = b_x^T w$ with $b_x \in \mathbb{Q}_{d_{n0\alpha}}$ for all $x \in L$. Hence, we have

$$f'(X, w) = \sum_{x \in X} f(x, w) = \sum_{x \in X} b_x^T w = (\sum_{x \in X} b_x)^T w = b_X^T w,$$

where $b_X \in \mathbb{Q}_{d_{n0\alpha}}$. That is, \([\text{A}]\) holds for $f'$. To prove Definition 4.1 \([\text{B}]\) for $f'$, let $\hat{w} \in \mathbb{Q}_d$ such that $\text{sign}(\beta^T \hat{w}) = \text{sign}(\beta^T \hat{w})$ for all $\beta \in \mathbb{Q}_{d_{n0\alpha}}$. Note that due to \([\text{B}]\) for $f$, we have

$$f(x, w) = b_x^T w \iff f(x, \hat{w}) = b_x^T \hat{w}, \quad (6)$$

since $\text{sign}(\beta^T w) = \text{sign}(\beta^T \hat{w})$ holds for all $\beta \in \mathbb{Q}_{d_{n0\alpha}}$. It follows that

$$f'(X, w) = b_X^T w \iff \sum_{x \in X} f(x, w) = \sum_{x \in X} b_x^T w \iff \sum_{x \in X} f(x, \hat{w}) = \sum_{x \in X} b_x^T \hat{w} \iff f'(X, \hat{w}) = b_X^T \hat{w},$$

and hence \([\text{B}]\) follows.

\([\text{2}]\): Since $f$ is $\alpha$-linearizable, by \([\text{A}]\) for $f$ we know that $f(x, w) = b_x^T w$ with $b_x \in \mathbb{Q}_{d_{n0\alpha}}$ for all $x \in L$. Hence, we have

$$f'(X, w) = \max_{x \in X} f(x, w) = \max_{x \in X} b_x^T w = b_X^T w,$$

where $x' = \arg \max_{x \in X} b_x^T w$ and $b_X \in \mathbb{Q}_{d_{2\alpha}}$. That is, \([\text{A}]\) holds for $f'$. To prove that \([\text{B}]\) holds for $f'$, let $\hat{w} \in \mathbb{Q}_d$ such that $\text{sign}(\beta^T \hat{w}) = \text{sign}(\beta^T \hat{w})$ for all $\beta \in \mathbb{Q}_{d_{2\alpha}}$. Note that due to \([\text{B}]\) for $f$, we have

$$f(x, w) = b_x^T w \iff f(x, \hat{w}) = b_x^T \hat{w}, \quad (7)$$

since $\text{sign}(\beta^T w) = \text{sign}(\beta^T \hat{w})$ holds for all $\beta \in \mathbb{Q}_{d_{2\alpha}}$. Moreover, for $y \in X$ let $f(y, w) = b_y^T w$. Let $b := b_x - b_y$ and note that $b \in \mathbb{Q}_{d_{2\alpha}}$. Hence, by the choice of $\hat{w}$, it holds true that $b^T \hat{w} \geq 0$ if and only if $b^T \hat{w} \geq 0$. Thus, with \([\text{7}]\) and by the choice of $x'$ we have $f(x', w) \geq f(y, w) \iff f(x', \hat{w}) \geq f(y, \hat{w})$ for all $y \in L$, and hence

$$\max_{x \in X} f(x, w) = b_{x'}^T w \iff \max_{x \in X} f(x, \hat{w}) = b_{x'}^T \hat{w}. \quad (8)$$
It follows that
\[ f'(X, w) = b^T X w \iff \max_{x \in X} f(x, w) = b^T x w \]
and hence (B) follows.

(iii): Follows analogously to (ii).

\[ f'(X, w) = b^T x w \iff f'(X, w) = b^T x \hat{w}, \]

\[ f'(X, w) = b^T x w \iff f'(X, w) = b^T x \hat{w}, \]

A.3 Proof of Lemma 4.5

Proof. Since \( f \) is \( \alpha \)-linearizable, by (A) for \( f \) we know that \( f(x, w) = b^T x w \) with \( b^T x \in Q^d_\alpha \) for all \( x \in L \). Hence, we have
\[ f'(x, w) = c(x) f(x, w) = c(x) \cdot b^T x w = (c(x) \cdot b^T x) w = b^T x w, \]
where \( b^T x \in Q^d_\alpha \) proving (A). Let \( \hat{w} \in Q^d \) such that \( \text{sign}(\beta \hat{w}) = \text{sign}(\beta \hat{w}) \) for all \( \beta \in Q^d_\alpha \).

Note that we have \( f(x, w) = (b^T x/c(x))^T w \iff f(x, \hat{w}) = (b^T x/c(x))^T \hat{w} \). It follows that
\[ f'(x, w) = b^T x w \iff f(x, w) = (b^T x/c(x))^T w \]
\[ \iff f(x, \hat{w}) = (b^T x/c(x))^T \hat{w} \]
\[ \iff f'(x, \hat{w}) = b^T x \hat{w}, \]
and hence (B) follows.

A.4 Proof of Theorem 4.7

Proof. Apply Theorem 2.1 with \( N = 2ad + 1 \) to obtain the vector \( \hat{w} \). Since \( f \) is \( \alpha \)-linearizable, by Definition 4.6(A) we know that there are \( b_x, b_y \in Z^d_\alpha \) such that \( f(x, w) = b^T x w \) and \( f(y, w) = b^T y w \) for every \( x, y \in L \). Moreover, for \( b := b_x - b_y \), we have \( b \in Z^d_\alpha \). Moreover, we have that
\[ f(x, w) - f(y, w) \geq 0 \iff (b_x - b_y)^T w \geq 0 \]
\[ \iff (b_x - b_y)^T \hat{w} \geq 0 \]
\[ \iff f(x, \hat{w}) - f(y, \hat{w}) \geq 0, \]
where the last equivalence follows from the fact that for \( \hat{w} \), by Theorem 2.1 for all \( \beta \in Z^d_\alpha \) we have \( \text{sign}(\beta \hat{w}) = \text{sign}(\beta \hat{w}) \), and hence from Definition 4.6(B) we get \( f(x, \hat{w}) = b^T \hat{w} \) and \( f(y, \hat{w}) = b^T \hat{w} \).

A.5 Proof of Lemma 4.8

Proof. The proofs of (i) and (iii) are analogous to those of Lemma 4.4(i) and (iii) and hence omitted. We prove (ii). Since \( f \) is \( \alpha \)-linearizable, by Definition 4.6(A) for \( f \) we know that \( f(x, w) = b^T x w \) with \( b^T x \in Z^d_\alpha \) for all \( x \in L' \). Hence, we have
\[ f'(X, w) = \sum_{x \in X} f(x, w) = \sum_{x \in X} b^T x w = (\sum_{x \in X} b^T x) w = b^T L_\alpha X w, \]
where \( b^T L_\alpha X w \). That is, (A) holds for \( f' \). To prove Definition 4.6(B) for \( f' \), let \( \hat{w} \in Q^d \) such that \( \text{sign}(\beta \hat{w}) = \text{sign}(\beta \hat{w}) \) for all \( \beta \in Z^d_\alpha \). Note that due to (B) for \( f \), we have
\[ f(x, w) = b^T x w \iff f(x, \hat{w}) = b^T \hat{w} \]
\[ (9) \]
since \( \text{sign}(\beta^T w) = \text{sign}(\beta^T \hat{w}) \) holds for all \( \beta \in \mathbb{Z}^d \subseteq \mathbb{Z}_{\text{na}}^d \). It follows that

\[
f'(X, w) = b_X^T w \iff \sum_{x \in X} f(x, w) = \sum_{x \in X} b'_x^T w \iff \sum_{x \in X} f(x, \hat{w}) = \sum_{x \in X} b'_x^T \hat{w} \iff f'(X, \hat{w}) = b_X^T \hat{w},
\]

and hence (B) follows.

\[\square\]

A.6 Proof of Lemma 5.1

**Proof.** First, observe that \( f \) and \( c \) are \( 1 \)-\( \mathbb{Z} \)-linearizable as they can be represented as \( e_i^T w \), where \( e_i \) denotes the unit vector with the \( i \)th entry being one and \( w = (f(1), \ldots, f(m), c(1, 1), \ldots, c(m, n)) \) denotes a weight vector that contains all possible opening and serving costs. Since the goal function is composed of a sum of two sums, we will first analyze each of the sums individually and then analyze the outer sum. Observe that \( \sum_{i \in I'} f(i) \) is \( m \)-\( \mathbb{Z} \)-linearizable by Lemma 4.8(i) as \( |I'| \leq m \). Similarly, since \( \min_{i \in I'} c(i, j) \) is \( 2 \)-\( \mathbb{Z} \)-linearizable by Lemma 4.8(iii), it follows from Lemma 4.8(i) that \( \sum_{j \in \mathcal{C}} \min_{i \in I'} c(i, j) \) is \( 2n \)-\( \mathbb{Z} \)-linearizable as \( |\mathcal{C}| = n \). Next we define

\[
f'(\ell, \mathcal{C}, I') := \begin{cases} \sum_{i \in I'} f(i) & \text{if } \ell = 1, \\ \sum_{j \in \mathcal{C}} \min_{i \in I'} c(i, j) & \text{if } \ell = 2. \end{cases}
\]

Observe that \( f' \) is \((2n + m)\)-\( \mathbb{Z} \)-linearizable as it is \((2n + m)\)-\( \mathbb{Z} \)-linearizable in each of the two cases by Observation 4.2 (notice that an equivalent version for \( \mathbb{Z} \)-linearizable does trivially also hold). We use \( f' \) to show that the goal function (which can be represented as \( \sum_{\ell \in \{1, 2\}} f'\ell, \mathcal{C}, I') \) is \( 2 \cdot (2n + m) \)-\( \mathbb{Z} \)-linearizable by Lemma 4.8(i). Finally, notice that Theorem 4.7 now yields the desired statement with \( \alpha = 2(2n + m) \) and \( d = nm + m \). \[\square\]