1. Introduction

A Leibniz algebra, $L$, is an algebra with multiplication defined by the (left) Leibniz identity

$$a(bc) = (ab)c + b(ac).$$

This multiplication need not be antisymmetric. If this multiplication is, in fact, antisymmetric, then $L$ is a Lie algebra. Leibniz algebras have been studied at length in [12], [3], [2], [10], and other works. In this paper, we consider $L$ to be a finite-dimensional Leibniz algebra over an arbitrary field unless otherwise stated.

The Frattini subalgebra of $L$, denoted $F(L)$, is the intersection of all maximal subalgebras of $L$. Note that $F(L)$ need not be an ideal of $L$. Thus, we consider the Frattini ideal of $L$, denoted $\Phi(L)$, which is the largest ideal of $L$ contained in the Frattini subalgebra of $L$. These structures have been studied at length in [5]. It is often especially useful to consider a Leibniz algebra whose Frattini ideal is 0. A particularly well-known Lie algebra result that carries over to the Leibniz case is the following: The Frattini ideal of $L/\Phi(L)$ is 0 for any Leibniz algebra $L$. 

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Let $\mathcal{F}$ be a subalgebra closed formation of solvable Leibniz algebras (see [3] and [10] for an introduction to Leibniz algebras and [4] for results on formations). Examples of such formations are the classes of nilpotent, supersolvable, and strongly solvable Leibniz algebras. Let $L$ be a solvable Leibniz algebra that is minimally not in $\mathcal{F}$ with $\phi(L) = 0$. If $A$ is a minimal ideal in $L$, then $A$ is complemented in $L$ by a subalgebra. We claim that there is a unique minimal ideal of $L$. Suppose that $A$ and $B$ are minimal ideals of $L$. They are complemented by subalgebras $H$ and $K$, respectively. $H$ and $K$ are both in $\mathcal{F}$ by the minimality of $L$. Then $L = L/(A \cap B)$, which is in $\mathcal{F}$ since $\mathcal{F}$ is a formation. This contradicts the assumption that $L$ does not belong to $\mathcal{F}$. Hence, $L$ has a unique minimal ideal, $A$. Thus, $\text{Soc}(L) = A$. Suppose that $L$ is not Lie.

Let $N(L)$ denote the nilradical of $L$. Since $\phi(L) = 0$, $N(L) = A = Z_L(A) \subset \text{Leib}(L)$ [5]. Furthermore, consider the mapping $L: x \mapsto L_x|_A$. $L$ is a homomorphism from $L$ into the derivation algebra of $A$. $L$ is the right centralizer of $A$ (since $A \subset \text{Leib}(L)$), and $A = Z_L(A)$. Hence, $A$ is the kernel of $L$. Hence, $H = L/A$ is Lie, and $A = \text{Leib}(L)$.

We construct a companion Lie algebra, $C$, for this Leibniz algebra, $L = A + H$, where $A$ is the unique minimal ideal in $L$, and $H$ is the subalgebra of $L$ complementing $A$. Let $C$ be the vector space direct sum of $A$ and $H$ with the product $[a+h, b+k] = [h, b] - [k, a] + [h, k]$, where the products on the right-hand side of this equation are the same as in $L$. It is verified that $C$ is a Lie algebra. Note that such correspondences have been considered previously [1]. There the reverse construction is developed, obtaining a Leibniz algebra from a Lie algebra. Their conditions and purposes are different, but the construction is the inverse of ours.

**Lemma 2.1.** Let $\mathcal{F}$ be a formation of solvable Leibniz algebras. Let $L$ be minimally not in $\mathcal{F}$, and suppose that $\phi(L) = 0$. Then $L = A + H$, where $A$ is the unique minimal ideal in $L$.

If $L$ is not a Lie algebra, then $A = \text{Leib}(L)$ is self-centralizing in $L$, and $H$ is a Lie algebra. Then the algebra $C$ constructed as $C = A + H$ with product $[a+h, b+k] = [h, b] - [k, a] + [h, k]$, where the right-hand-side products are the same as in $L$, is a Lie algebra. We call $C$ the companion Lie algebra of $L$. 

2. Companion Lie Algebra
3. Strongly Solvable Leibniz Algebras

As in the case of Lie algebras, a Leibniz algebra $L$ is called strongly solvable if $L^2$ is nilpotent. Suppose $\mathcal{F}$ is the formation of strongly solvable Leibniz algebras with $L$, $C$, $A$, and $H$ as in the previous section. Let $x \in L$, and let $x$ also denote the corresponding element in $C$. Let $L_x$ denote left multiplication of $L$ by $x$, and let $\text{ad } x$ denote left multiplication of $C$ by $x$ under the above bracket. If $x$ is in $A$, then $L_x$ and $\text{ad } x$ are both nilpotent, although perhaps not equal. Suppose that $L$ is a $\phi$-free, solvable, minimally non strongly solvable Leibniz algebra. We claim that $C$ also has these properties.

If $x$ is in $H^2$, then $L_x$ and $\text{ad } x$ are essentially the same, although acting on $L$ and $C$, respectively, and are either both nilpotent or both non-nilpotent. If all $x \in H^2$ have $\text{ad } x$ nilpotent, then left multiplication by each element in the Lie set $H^2 \cup A$ is nilpotent on both $L^2$ and $C^2$. Thus, $L^2$ is nilpotent if and only if $C^2$ is nilpotent; hence, $C$ is not strongly solvable.

$A$ is self-centralizing in $L$, and hence, $A$ is also self-centralizing in $C$. Thus, $A$ is the unique minimal ideal in $C$. Then $H$ is a maximal subalgebra in $C$ and thus, contains $\phi(C)$. We claim that $\phi(C) = 0$. Suppose not. $H$ is a maximal subalgebra of $C$, and hence, $\phi(C) \subset H$. $A$ is the unique minimal ideal in $C$, and hence, $A$ is contained in $\phi(C)$, which is a contradiction. Thus, the claim holds.

We also claim that any proper subalgebra, $K$, of $C$ is strongly solvable. Suppose that $C = A + K$. Then $K \cap A$ is equal to either $A$ or $0$ since $A$ is a minimal ideal of $C$.

In the former case, $A$ is contained in $K$, and then $K$ is equal to $C$, which contradicts $K$ being a proper subalgebra of $C$. In the latter case, $K$ is isomorphic to $H$, which is strongly solvable in $L$, and thus, in $C$. Thus, $K$ is strongly solvable.

Now suppose that $C \neq A + K$. We will show that $A + K$ is strongly solvable, and hence, that $K$ is also strongly solvable. Thus, we assume that $A \subseteq K$.

Then $K = K \cap (H + A) = (K \cap H) + A$, which is a subalgebra of $L$ by the correspondence between $L$ and $C$. Then $K^2 = (K \cap H)^2 + A$ in Leibniz algebra $L$, and analogously, $K^2 = (K \cap H)^2 + A$ in Lie algebra $C$. In $L$, both summands of $K^2$ are nilpotent Lie sets, and likewise, the corresponding summands are nilpotent in $C$. Thus, $K^2$ is nilpotent as a subalgebra of $C$. Hence, $K$ is strongly solvable.
solvable for \( K \) any proper subalgebra of \( C \). Since \( C \) is itself not strongly solvable, \( C \) is a solvable, minimally non-strongly solvable Lie algebra.

Such Lie algebras have been described in Theorem 3.1 of [7].

**Theorem 3.1.** Let \( L \) be a solvable \( \phi \)-free minimally non-strongly solvable Leibniz algebra over field \( \mathbb{F} \). Then \( \mathbb{F} \) has characteristic \( p > 0 \), and \( L = A + H \) is a semidirect sum, where

1. \( A \) is the unique minimal ideal of \( L \),
2. \( \dim A \geq 2 \),
3. \( A^2 = 0 \), and
4. either \( H = M \oplus \mathbb{F}x \), where \( M \) is a minimal abelian ideal of \( H \), or \( H \) is the three-dimensional Heisenberg algebra.

Either \( L \) is Lie, or if not, then \( H \) is Lie, \( A = \text{Leib}(L) \), and \( AL = 0 \).

Proof. If \( L \) is a Lie algebra, then the result is Theorem 3.1 of [7]. If \( L \) is not Lie, then the companion Lie algebra, \( C \), satisfies the conditions and conclusions of this theorem. Thus, \( L \) does also by Lemma 2.1

[Proof]

In [7], Bowman, Towers, and Varea prove the Lie algebra version of the next result from Theorem 3.1. We obtain it from [8] instead.

**Proposition 3.2.** Let \( L \) be a solvable Leibniz algebra that is minimally non-strongly solvable. Then \( L \) is two-generated.

Proof. If \( L \) is not two-generated, then all two-generated subalgebras are strongly solvable. Then the result follows from Corollary 1 of Theorem 2 of [8].

**Proposition 3.3.** Let \( L \) be a solvable Leibniz algebra with each two-generated proper subalgebra strongly solvable. Then

1. \( L \) is either strongly solvable or two-generated, and
(2) Every proper subalgebra of \( L \) is strongly solvable.

Proof. The proof of (1) follows the proof of [3,2].

We now prove (2). Let \( S \) be a minimally non-strongly solvable subalgebra of \( L \). If \( S \) is two-generated, then \( S \) is strongly solvable by hypothesis. Hence, \( S \) is not two-generated. Thus, all two-generated subalgebras of \( S \) are strongly solvable, and \( S \) is strongly solvable also. \( \square \)

4. SUPERSolvable Leibniz Algebras

A Leibniz algebra \( L \) is called supersolvable if there exists a chain of ideals \( 0 = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_{n-1} \subset L_n = L \), where \( L_i \) is an \( i \)-dimensional ideal of \( L \). In this section, supersolvability will be considered in the same manner as was strong solvability in the previous section. Let \( L \) be a solvable, \( \phi \)-free, minimally non-supersolvable Leibniz algebra. Supersolvable Leibniz algebras form a formation; hence, Lemma 2.1 applies. Then \( L = A + H \) as in Lemma 2.1. Now, if \( L \) is Lie, then the structure of \( L \) has been determined in [11] and set forth in [7]. We state the result as

Theorem 4.1. Let \( L \) be a solvable, minimally non-supersolvable Lie algebra which is \( \phi \)-free. Then the candidates for \( L \) are:

1. If \( L \) is strongly solvable, \( L = A \oplus \langle x \rangle \), where \( A \) is the unique minimal ideal of \( L \) and \( \dim A > 1 \).

2. If \( L \) is not strongly solvable, then \( F \) is of characteristic \( p > 0 \), \( L \) has unique minimal ideal \( A \) with basis \( \{e_1, \ldots, e_p\} \), and one of the following holds:

   a. \( L = A + \langle x, y \rangle \) with antisymmetric multiplication \( xe_i = e_{i+1} \), \( ye_i = (\alpha + i)e_i \), with indices mod \( p \), \( yx = x \), and for all \( a \in F \), \( a = tp - t \) for some \( t \in F \), or

   b. \( L = A + \langle x, y, z \rangle \) with anti-symmetric multiplication \( xe_i = e_{i+1} \), \( ye_i = (i + 1)e_{i-1} \), \( ze_i = e_i \), with indices mod \( p \), \( yx = z \), \( xz = yz = 0 \), and \( F \) is perfect when \( p = 2 \).
The list of algebras in Theorem 4.1 are the possible minimally non-supersolvable Lie algebras, however, it is important to note that not all algebras of these forms have this property. However, if $F$ is algebraically closed, the algebras in (2) are minimally non-supersolvable, while the algebra in (1) must be supersolvable. Hence, all minimally non-supersolvable Lie algebras over an algebraically closed field are as in (2).

Now suppose that $L$ is not a Lie algebra. Then $L$ is as in Lemma 2.1; $L = A \varoplus H$, where $A = \text{Leib}(L)$, and $H$ is as in the lemma. Take $C$ to be the companion Lie algebra defined in Lemma 2.1. $A$ is the minimal ideal in $C$, and $C$ is $\phi$-free.

In Leibniz algebra $L$, $H$ is supersolvable by assumption, and $A$ is abelian. However, $A$ is irreducible under the action of $H$ acting on the left. Since $L$ is not supersolvable by hypothesis, the dimension of $A$ is greater than 1. The left action of $H$ on $A$ is the same in $C$ as in $L$, and hence, $\dim A$ is greater than one and $C$ is not supersolvable.

We show that all proper subalgebras of $C$ are supersolvable, and hence, that $C$ is a solvable, minimally non-supersolvable Lie algebra. Let $K$ be a proper subalgebra of $C$. If $C = A + K$, then either $A \cap K = 0$ or $A \cap K = A$ since $A$ is also a minimal ideal in $C$. In the former case, $K$ is isomorphic to $H$ and hence, is supersolvable. In the latter case, $K = C$, which is a contradiction.

Now suppose that $A + K$ is a proper subalgebra of $C$. We show that $A + K$ is supersolvable, and hence, that $K$ is also supersolvable. We may assume that $A \subset K$. Then as above, $K = (K \cap H) + A$. Note that $K \cap H$ is the same in $L$ and $C$.

In $L$, the action of $K \cap H$ on $A$ is simultaneously triangularizable. The left action of $K \cap H$ on $A$ in $C$ is the same as in $L$, and hence, $K$ is supersolvable in Lie algebra $C$. Then $C$ is minimal non-supersolvable and is as in Theorem 4.1. Hence, we have the following theorem.

**Theorem 4.2.** Let $L$ be a solvable, $\phi$-free, minimally non-supersolvable Leibniz algebra. If $L$ is a Lie algebra, then $L$ is as in Theorem 4.1. If $L$ is not Lie, then $L$ is as in Theorem 4.1 with the added conditions that $A = \text{Leib}(L)$ and $AL = 0$. 6
Corollary 4.3. If \( L \) is as in Theorem 4.2 and \( F \) is algebraically closed, then the solvable, minimally non-supersolvable Leibniz algebras are precisely the algebras at characteristic \( p \) such that either

(a.) \( L = A + B \) where \( A = ((e_1, \ldots, e_p)) \) is the unique minimal ideal which is abelian, \( B = \langle x, y \rangle \) with \( ye_i = -ie_i \) and \( xe_i = -e_{i+1} \) (indices mod \( p \)), \( yx = x \), and either \( A = \text{Leib}(L) \) or \( L \) is Lie, or

(b.) \( L = A + B \) where \( A = ((e_1, \ldots, e_p)) \) is the unique minimal ideal which is abelian, \( B \) is the Heisenberg Lie algebra with basis \( \{x, y, z\} \) and multiplication \( ye_i = (i+1)e_{i-1}, xe_i = -e_{i+1}, ze_i = e_i, yx = z \), and either \( A = \text{Leib}(L) \) or \( L \) is Lie. Again, the index action is mod \( p \).

Several results similar to Theorem 3.5 in [7] are now obtained in the Leibniz algebra case. Note that if Leibniz algebra \( L \) is solvable and minimally non-supersolvable, then all two-generated proper subalgebras are supersolvable. If \( L \) is not two-generated, then all two-generated subalgebras are supersolvable, and hence \( L \) is supersolvable from Theorem 4 of [8], a contradiction. Thus, we have the following.

Theorem 4.4. If \( L \) is solvable and minimal non-supersolvable, then \( L \) is two-generated.

Theorem 4.5. If every two-generated proper subalgebra of \( L \) is supersolvable, \( L \) itself is not supersolvable, and \( L \) is \( \phi \)-free, then \( L \) has the structure of one of the algebras in Theorem 4.2.

Proof. Let \( S \) be a proper subalgebra of \( L \). If \( S \) is two-generated, then it is supersolvable by assumption. If \( S \) is non two-generated, then every two-generated subalgebra of \( S \) is supersolvable by assumption. Since supersolvability is two-recognizable [8], \( S \) is also supersolvable. Thus, all proper subalgebras of \( L \) are supersolvable, and \( L \) is minimally non-supersolvable. Hence, Theorem 4.2 applies. \( \square \)

5. Triangulable Leibniz Algebras

A Leibniz algebra \( L \) over a field \( F \) is triangulable on \( L \)-module \( M \) if when \( F \) is extended to \( K \), the algebraic closure of \( F, K \otimes M \) admits a basis such that the representing matrices of \( K \otimes L \) are upper triangular. \( L \) is said to be nil on \( M \) if left multiplication on \( M \) by each \( x \in L \) is nilpotent.
Then L acts nilpotently on M \([6]\). There is a maximal ideal of L that acts nilpotently on M. This ideal is denoted by \(\text{nil}(L)\). If \(L^2\) is contained in \(\text{nil}(L)\), then the same holds in the algebra and module over \(K\). By Theorem 1 of \([9]\), in the algebraically closed case, L is triangulable on M if \(L^2\) is contained in \(\text{nil}(L)\). Hence, in the general case, L is triangulable on M if \(L^2\) is in \(\text{nil}(L)\). Note that these definitions still apply in the special case that L is a subalgebra of M. We state them in this context.

**Proposition 5.1.** A subalgebra L of Leibniz algebra M is triangulable on M if and only if \(L^2\) is contained in \(\text{nil}(L)\). If L is an ideal of M, then L is triangulable on M if and only if \(L^2\) is nilpotent. M is triangulable on itself if and only if M is strongly solvable.

The following two propositions follow exactly as in their Lie algebra versions, with left multiplication replacing \(\text{ad}\) in the proofs as shown in Lemma 4.1 and Lemma 4.5 of \([7]\).

**Proposition 5.2.** Let L be a Leibniz algebra, and let S and T be subalgebras of L that are nil on S such that S is contained in the normalizer of T. Then \(S + T\) is nil on L.

**Proposition 5.3.** If S is a subalgebra of L such that \(\phi(L)\) is contained in S, then \(\text{nil}(S/\phi(L)) = \text{nil}(S)/\phi(L)\).

**Proof.** Clearly \(\phi(L)\) is contained in \(\text{nil}(L)\) since \(\phi(L)\) is a nilpotent ideal in L. Let \(\text{nil}(S/\phi(L)) = J/\phi(L)\). Clearly \(\text{nil}(S)\) is contained in J. Let \(x \in J\). Now \(L_x\) acts nilpotently on \(L/\phi(L)\), and \(L = \phi(L) + L_0(x)\), the Fitting null component of \(L_x\) acting on L. Since \(L_0(x)\) is a subalgebra of L supplementing the Frattini ideal, \(L_0(x)\) is equal to L. This holds for all such \(x\), and hence, J is contained in \(\text{nil}(S)\), and the result holds. \(\square\)

It is known that a solvable Leibniz algebra is triangulable on itself if all two-generated subalgebras of L are triangulable \([9]\). In \([7]\), Lie algebras all of whose proper two-generated subalgebras are triangulable are similarly investigated. Our purpose is to find Leibniz algebra analogues to this and related ideas.
Theorem 5.4. Let $L$ be a solvable Leibniz algebra such that each two-generated proper subalgebra is triangulable on $L$. Then $L$ is triangulable.

Proof. Each two-generated proper subalgebra of $L$ is strongly solvable. Then by Proposition 3.3, every proper subalgebra of $L$ is strongly solvable, as is each proper subalgebra of $L^* = L/\phi(L)$.

If $L$ is not triangulable, then $L$ is not strongly solvable, and neither is $L^*$. Thus, Theorem 3.1 applies, $L^* = A + B$ as in the theorem, and $(L^*)^2 = A + B^2$. $B$ is two-generated. Hence $B$ is triangulable on $L^*$. Thus, $B^2$ acts nilpotently on $L^*$ and hence, on $A$. Thus, $(L^*)^2$ is nilpotent and $L^*$ is strongly solvable, a contradiction. □

It is interesting to note that $L$ is triangulable if and only if it is strongly solvable. If all two-generated subalgebras are triangulable on $L$, then $L$ is triangulable (Theorem 5.4). However, if all two-generated subalgebras are strongly solvable, this does not guarantee that $L$ is triangulable since a subalgebra can be strongly solvable without being triangulable on the algebra. The algebras in Theorem 3.1 are of this type.

Theorem 5.5. If $L$ is minimally non-triangulable (every proper subalgebra of $L$ is triangulable on $L$, but $L$ itself is not), then $L$ is two-generated and $L/\phi(L)$ is simple.

Proof. By Theorem 5.4, $L$ is not solvable. If $L^* = L/\phi(L)$ is triangulable, then $(L^*)^2$ is nilpotent and $L^*$ is solvable, a contradiction. Therefore, $L^*$ is not triangulable, and hence not strongly solvable.

Consider a proper subalgebra $S$ of $L$ and the corresponding proper subalgebra $S^* = S/\phi(L)$ of $L^*$. $S$ is triangulable on $L$, hence $S^2$ is contained in $\text{nil}(S)$ and $(S^*)^2$ is contained in $\text{nil}(S^*)$. Hence, $S^*$ is strongly solvable.

Note that the nilradical of $L^*$ is not 0. Since $\phi(L^*) = 0$, $\text{nil}(L^*)$ is complemented in $L^*$ by a subalgebra $T$ [5], which is strongly solvable. Hence, $L^* = \text{nil}(L^*) + T$ is solvable, a contradiction. Thus, $L^*$ is semisimple. If $L^*$ were to contain a proper ideal, that ideal would be strongly solvable and hence solvable, a contradiction. Thus, $L^*$ contains no proper ideals, and $L^*$ is simple. □
REFERENCES

[1] S. Abdykassymova and A. S. Dzhumadil’daev. Leibniz algebras in characteristic p. *C.R. Acad. Sci. Paris Sr. I Math.*, 332:1047–1052, 2001.

[2] Sh. A. Ayupov and B. A. Omirov. On Leibniz algebras. *Algebra and Operator Theory*, pages 1–12, 1998.

[3] D. Barnes. Some theorems on Leibniz algebras. *Communications in Algebra*, 39:2463–2472, 2011.

[4] D. Barnes. Schunck classes of soluble Leibniz algebras. *Communications in Algebra*, 41:4046–4065, 2013.

[5] C. Batten Ray, L. Bosko-Dunbar, A. Hedges, J. T. Hird, K. Stagg, and E. Stitzinger. A Frattini theory for Leibniz algebras. *Communications in Algebra*, 41:1547–1557, 2013.

[6] L. Bosko, A. Hedges, J. T. Hird, N. Schwartz, and K. Stagg. Jacobson’s refinement of Engel’s theorem for Leibniz algebras. *Involv*, 4:293–296, 2011.

[7] K. Bowman, D. Towers, and V. Varea. Two generated subalgebras of Lie algebras. *Linear and Multilinear Algebra*, 55:429–438, 2007.

[8] T. Burch, M. Harris, A. McAlister, E. Rogers, E. Stitzinger, and S. M. Sullivan. 2-recognizable classes of leibniz algebras. *Journal of Algebra*, 413:506–513, 2015.

[9] T. Burch and E. Stitzinger. Triangulable Leibniz algebras. *Communications in Algebra*. To appear.

[10] I. Demir, K. Misra, and E. Stitzinger. On some structures of Leibniz algebras. *Contemporary Math., Amer. Math. Soc.*, pages 41–54, 2014.

[11] A. Elduque and V. Varea. Lie algebras all of whose subalgebras are supersolvable. *Canad. Math. Soc. Conference Proceedings*, 5:209–218, 1986.

[12] J. Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign Math.*, 39:269–293, 1993.

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