Thermodynamics of Riemannian Kerr-AdS black holes in Poincaré gauge theory

M. Blagojević and B. Cvetković

Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

Abstract

A Hamiltonian approach to black hole entropy is used to study Riemannian Kerr-AdS solutions in the general, parity-violating Poincaré gauge theory. Entropy and the asymptotic charges are entirely determined by the parity-even sector of the theory, whereas the parity-odd contributions vanish. Entropy is found to be proportional to the horizon area, and the first law of black hole thermodynamics is confirmed.

1 Introduction

From the experience with general relativity (GR), we know that exact solutions play an essential role in the physical interpretation of gravitational theories [1]. In Poincaré gauge theory (PG), where both the torsion $T^i$ and the curvature $R^{ij}$ define the gravitational dynamics [2], exact solutions have been often constructed by suitably “incorporating” torsion into the known solutions of GR. In particular, an advanced technique was used by Baekler et al. [3] to construct a Kerr-AdS black hole with propagating torsion, in the sector of parity-invariant PG Lagrangians. Recently, Obukhov [4] made one more step in the same direction by extending the construction to the most general, parity-violating PG [5]. Thus, there are at least three versions of Kerr-AdS spacetimes, one in GR and the other two in PG; they possess the same metric but live in different dynamical setups.

Investigations of black hole thermodynamics have given rise to a deeper insight into both the classical and quantum nature of gravity [6]. In the 1990s, classical black hole entropy was most compactly described as the diffeomorphism Noether charge on horizon [7, 8, 9]. In a recently proposed Hamiltonian approach to black hole entropy [10], the same idea was extended to PG, where diffeomorphisms are replaced by the Poincaré gauge symmetry. It offers a unified description of the asymptotic charges (energy and angular momentum) and entropy of black holes with or without torsion, in terms of certain boundary integrals at infinity and on horizon, respectively. The approach was successfully applied to Kerr-AdS thermodynamics in GR [11], as well as to parity-invariant and general PG models [12, 13]. In the present work, we extend the Hamiltonian analysis to the class of Riemannian Kerr-AdS solutions in the general PG. A comparison to the results found in Riemannian theories with

*Email addresses: mb@ipb.ac.rs, cbranislav@ipb.ac.rs
quadratic curvature Lagrangians [8, 14], as well as in the general PG models [13], reveals how black holes react to different dynamical frameworks.

The paper is organized as follows. In section 2, we give a short account of the Hamiltonian approach to black hole thermodynamics in the context of the general, parity-violating PG. In section 3, we introduce the tetrad formulation of the Kerr-AdS geometry, needed in the Hamiltonian analysis, and discuss limitations of Boyer-Lindquist coordinates. Sections 4 and 5 contain the main results of the paper—derivation of the asymptotic charges and entropy for Riemannian Kerr-AdS black holes in PG. Section 6 is devoted to discussion.

Our basic notation is the same as in Refs. [11, 13]. The latin indices \((i, j, \ldots)\) refer to the local Lorentz frame, the greek indices \((\mu, \nu, \ldots)\) refer to the coordinate frame, \(b^i\) is the orthonormal coframe (tetrad) and \(\omega^{ij}\) is a metric compatible connection (1-forms), \(h^i\) is the dual basis (frame) such that \(h^i b^k = \delta^k_i\), and \(\eta_{ij} = (1, -1, -1, -1)\) is the local Lorentz metric. The wedge symbol in the exterior products is omitted, the volume 4-form is \(\hat{\epsilon} = b_0 b_1 b_2 b_3\), the Hodge dual of a form \(\alpha\) is denoted by \(\star \alpha\), with \(\star 1 = \hat{\epsilon}\), and the totally antisymmetric symbol \(\varepsilon_{ijmn}\) is normalized to \(\varepsilon_{0123} = +1\).

2 Black hole entropy as a boundary term

In this section, we give a short account of the Hamiltonian approach to the entropy of black holes, restricted to the class of Riemannian solutions in the general PG, see Refs. [11, 13].

We begin by recalling some geometric aspects of PG. The gravitational field is described by two independent dynamical variables, the tetrad \(b^i\) and the metric compatible spin connection \(\omega^{ij}\) (1-forms), which are associated to the translation and the Lorentz subgroups of the Poincaré group, respectively. The corresponding field strengths are the torsion \(T^i = db^i + \omega^i_k b^k\) and the curvature \(R^{ij} := d\omega^{ij} + \omega^j_k \omega^{ki}\) (2-forms), and the underlying spacetime structure is characterized by a Riemann–Cartan geometry.

The general PG dynamics is determined by a Lagrangian \(L_G(b^i, T^i, R^{ij})\) which is at most quadratic in the field strengths and contains both even and odd parity modes. In this work, we are interested in the class of vacuum solutions with vanishing torsion. Their dynamics is effectively described by a simplified Lagrangian without torsion invariants,

\[
L_G = -(a_0 R + \bar{a}_0 X + 2A_0) + \frac{1}{2} \sum_{n=1}^{6} \left[ \star \left( b^{(n)} R_{ij} \right) + \bar{b}^{(n)} R_{ij} \right].
\]

(2.1)

Here, \((a_0, A_0, b_n)\) and \((\bar{a}_0, \bar{b}_n)\) are the Lagrangian parameters in the parity even and odd sectors, respectively. \(\star R = \star (b_i b_j) R^{ij}\) is the Einstein-Hilbert and \(\star X = (b_i b_j) R^{ij}\) the Holst term [13], and \((n) R^{ij}\) are irreducible parts of the curvature [10]; for \(T^i = 0\), only those for \(n = (1, 4, 6)\) are nonvanishing. The gravitational field equations are derived by varying \(L_G\) with respect to \(b^i\) and \(\omega^{ij}\). The status of Riemannian solutions in the framework of PG was clarified by Obukhov [5]:

\[\triangle\] Any solution of GR with a cosmological constant is a torsion-free solution of the general PG field equations.
In particular, this is true for Kerr-AdS solutions, the subject of the present work.

In the Hamiltonian approach, the asymptotic charges are defined as certain boundary terms at spatial infinity, which make the associated canonical gauge generator $G$ differentiable \[16\]. By extending this construction, one can naturally introduce entropy as a boundary term on horizon \[10\]. Consider a stationary, axisymmetric black hole, and let $\Sigma$ be a spatial section of spacetime whose boundary consists of two components, one at infinity and the other at horizon, $\partial \Sigma = S_\infty \cup S_H$. The asymptotic charges and black hole entropy are defined by the variational equations for the respective boundary terms $\Gamma_\infty$ and $\Gamma_H$:

$$
\delta \Gamma_\infty = \oint_{S_\infty} \delta B(\xi), \quad \delta \Gamma_H = \oint_{S_H} \delta B(\xi),
$$

(2.2a)

$$
\delta B(\xi) := \frac{1}{2}(\xi \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta \omega^{ij}(\xi \mathbf{1} H_{ij}).
$$

(2.2b)

where $\xi$ is a Killing vector ($\partial_t$ or $\partial_\phi$ on $S_\infty$, and a linear combination thereof on $S_H$, such that $\xi^2 = 0$), and $H_{ij}$ is the covariant momentum determined by the Lagrangian (2.1),

$$
H_{ij} := \frac{\partial L_G}{\partial R^{ij}} = -2a_0^*(b_i b_j) - 2\bar{a}_0(b_i \bar{b}_j) + 2 \sum_{n=1,4,6} \left[ \epsilon \left( b_n^{(n)} R_{ij} \right) + \bar{b}_n^{(n)} R_{ij} \right],
$$

(2.3)

The operation $\delta$ is assumed to be in accordance with the following two rules:

(r1) The variation $\delta \Gamma_\infty$ is defined by varying parameters of the black hole state over the boundary $S_\infty$, leaving the background configuration fixed.

(r2) The variation $\delta \Gamma_H$ is defined by varying the parameters on horizon, but keeping surface gravity constant over the horizon, in accordance with the zeroth law.

When there exist finite solutions for $\Gamma_\infty$ and $\Gamma_H$ ($\delta$-integrability), they are interpreted as thermodynamic charges. Their values are strongly correlated to the adopted boundary conditions.

The boundary terms $\Gamma_\infty$ and $\Gamma_H$ in (2.2) are introduced as apriori independent objects. However, if the canonical generator $G$ defining local symmetries of a black hole is differentiable, the corresponding boundary term $\Gamma$ is not needed, it vanishes. Thus, assuming the boundary $S_H$ to have the opposite orientation with respect to $S_\infty$, we have

$$
\delta \Gamma := \delta \Gamma_\infty - \delta \Gamma_H = 0,
$$

(2.4)

which is nothing but the first law of black hole thermodynamics.

As follows from Eq. (2.3), the covariant momentum $H_{ij}$ consists of two independent parts, defined by the even and odd parity sectors of $L_G$. Consequently, each of the boundary terms $\Gamma_\infty$ and $\Gamma_H$ can be represented as the sum of two parts with opposite parities.
3 Kerr-AdS geometry

3.1 Tetrad formalism

The Kerr-AdS metric is a solution of GR with a cosmological constant. In Boyer-Lindquist coordinates \((t, r, \theta, \varphi)\), it can be formulated in terms of the orthonormal tetrad [11, 12]

\[
\begin{align*}
  b^0 &= N \left( dt + \frac{a}{\alpha} \sin^2 \theta \, d\varphi \right), \\
  b^1 &= \frac{dr}{N}, \\
  b^2 &= P d\theta, \\
  b^3 &= \frac{\sin \theta}{P} \left( a dt + \frac{(r^2 + a^2)}{\alpha} d\varphi \right),
\end{align*}
\]

(3.1a)

where

\[
\begin{align*}
  N &= \sqrt{\Delta/\rho^2}, \\
  \rho^2 &= r^2 + a^2 \cos^2 \theta, \\
  \Delta &= (r^2 + a^2)(1 + \lambda r^2) - 2mr, \\
  \alpha &= 1 - \lambda a^2, \\
  P &= \sqrt{\rho^2/f}, \\
  f &= 1 - \lambda a^2 \cos^2 \theta.
\end{align*}
\]

(3.1b)

Here, \(0 \leq \theta < \pi\) and \(0 \leq \varphi < 2\pi\), \(m\) and \(a\) are parameters of the solution, and \(\lambda\) is determined by the PG field equations, \(3a_0\lambda = -A_0\).

The metric \(ds^2 = \eta_{ij} b^i \otimes b^j\), which is stationary and axially symmetric, admits the Killing vectors \(\partial_t\) and \(\partial_\varphi\). Many metric-related characteristics of geometry play an essential role in black hole thermodynamics, such as the location of the outer horizon \(r = r_+\), the horizon area \(A_H\), the angular velocity \(\omega_+\) and the surface gravity \(\kappa\),

\[
\begin{align*}
  \Delta(r_+) &\equiv (r_+^2 + a^2)(1 + \lambda r_+^2) - 2mr_+ = 0, \\
  A_H &= \int_{r_+} b^2 b^3 = 4\pi \frac{r_+^2 + a^2}{\alpha}, \\
  \omega &= \frac{g_\varphi}{g_{\varphi\varphi}}, \\
  \omega_+ &= \omega|_{r_+} = \frac{a\alpha}{r_+^2 + a^2}, \\
  \kappa &= \frac{[\partial \Delta]_{r_+}}{2(r_+^2 + a^2)}.
\end{align*}
\]

(3.2a)

The quantities \(\omega_+\) and \(\kappa\) are constant over the horizon, and for large \(r\), \(\omega \sim -\lambda a\).

For a given tetrad field, one can calculate the Riemannian connection 1-form \(\omega^{ij}\) by

\[
\omega^{ij} = \frac{1}{2} \left[ h^i \mathbf{J} db^j - h^j \mathbf{J} db^i - \left( h^i \mathbf{J} (h^j \mathbf{J} db^m) \right) b_m \right].
\]

(3.3)

Then, the curvature is defined by \(R^{ij} = d\omega^{ij} + \omega^k \omega^{kj}\), and it has only two nonvanishing irreducible parts, \((^1)R^{ij}\) and \((^6)R^{ij}\). The first part, also known as the Weyl curvature, \(W^{ij} \equiv (^1)R^{ij}\), is given by [11]

\[
\begin{align*}
  W_{01} &= 2Cb^0 b^1 + 2Db^2 b^3, \\
  W_{12} &= Cb^1 b^2 - Db^0 b^3, \\
  W_{02} &= -Cb^0 b^2 + Db^1 b^3, \\
  W_{13} &= Cb^1 b^3 + Db^0 b^2, \\
  W_{03} &= -Cb^0 b^3 - Db^1 b^2, \\
  W_{23} &= -2Cb^2 b^3 + 2Db^0 b^1.
\end{align*}
\]

(3.4a)
where the coefficients $C$ and $D$ are the same as in Kerr spacetime \[17\],

$$C := \frac{mr}{\rho^6} (r^2 - 3a^2 \cos^2 \theta), \quad D := \frac{ma \cos \theta}{\rho^6} (3r^2 - a^2 \cos^2 \theta).$$ \hspace{1cm} (3.4b)

The second irreducible part is $(6) R^{ij} = \lambda b^i b^j$. 

For $m = 0$, we have $W^{ij} \equiv R^{ij} - (6) R^{ij} = 0$, and the curvature corresponds to the background, AdS configuration, $R^{ij} = \lambda b^i b^j$. The quadratic, even and odd curvature invariants,

$$W^{ij} W_{ij} = 24(C^2 - D^2) \hat{\epsilon}, \quad W^{ij} W_{ij} = -48(CD) \hat{\epsilon},$$ \hspace{1cm} (3.5)

are singular at $\rho^2 = 0$ (a ring at $r = 0, \theta = \pi/2$).

### 3.2 Limitations of Boyer-Lindquist coordinates

The background configuration for $m = 0$ is described by the AdS geometry but in somewhat non-standard coordinates, in which metric components depend on the parameter $a$. Hence, one cannot clearly distinguish whether the variation $\delta a$ is related to the background or to the genuine black hole configuration. To avoid the variation of the AdS background, we introduce an improved version of the rule (r1).

(r1') When $\delta \Gamma_\infty$ is calculated for Kerr-AdS black holes in Boyer-Lindquist coordinates, first apply $\delta$ to all $a$’s, then remove those $\delta a$ terms that survive the limit $m = 0$, as they are associated to the background configuration.

However, this is not sufficient to make the Boyer-Lindquist coordinates well defined. Namely, as shown by Henneaux and Teitelboim \[17\], see also Carter \[18\], the metric in these coordinates does not have a proper, asymptotically AdS behavior, needed for the canonical identification of asymptotic charges. To avoid the problem, they introduced a suitable change of coordinates which brings the metric to the standard, asymptotically AdS form. In our variational approach (2.2), the problem shows up as $\delta$-nonintegrability of the asymptotic charges. It can be resolved by using the reduced form of the Carter-Henneaux-Teitelboim coordinate transformations, see for instance \[11\],

$$T = t, \quad \phi = \varphi - \lambda at.$$ \hspace{1cm} (3.6)

This change of coordinates and the improved rule (r1') ensure the new asymptotic charges, $\delta E_{T} := \delta \Gamma_\infty[\partial_T]$ and $\delta E_{\phi} := \delta \Gamma_\infty[\partial_{\phi}]$, to be well defined. In addition to that,

$$\delta E_{T} = \delta E_{t} + \lambda a \delta E_{\varphi}, \quad \delta E_{\phi} = \delta E_{\varphi}.$$ \hspace{1cm} (3.7)

Black hole entropy is determined by $\delta \Gamma_{H} [\xi]$, where $\xi = \partial_T - \Omega_+ \partial_\phi$ and

$$\Omega := \frac{g_{T\phi}}{g_{\phi\phi}} = \omega + \lambda a, \quad \Omega_+ = \Omega \bigg|_{r_+}.$$ \hspace{1cm} (3.8)

For large $r$, the new angular velocity $\Omega$ vanishes, as expected \[9\]. It turns out that the expression $\delta \Gamma_{H} [\xi]$ is invariant under the coordinate transformations (3.6).

In what follows, we will use the notation $\text{PG}^+$ and $\text{PG}^-$ for the even and odd parity sectors of $\text{PG}$, respectively.
4 Thermodynamic charges in PG$^+$

To simplify technical exposition of our analysis of Kerr-AdS thermodynamics, we first analyse Eqs. (2.2) in the PG$^+$ sector, leaving PG$^-$ for the next section.

The PG$^+$ sector is effectively described by the Lagrangian

$$L_G^+ = -(a_0 R + 2A) + \frac{1}{2} R^{ij}(b_1 W_{ij} + b_6^{(6)} R_{ij}),$$

(4.1a)

and the corresponding covariant momentum is

$$H_{ij} = -2 a_0^* (b_i b_j) + 2^*(b_1 W_{ij} + b_6^{(6)} R_{ij})$$

$$= -2(a_0 - \lambda b_6)^* (b_i b_j) + 2 b_1^* W_{ij}.$$  

(4.1b)

The expressions for angular momentum, energy and entropy, produced by the first term in (4.1b), are of the GR form, but with $a_0 \rightarrow (a_0 - \lambda b_6)$, see [11]:

$$\delta E^{GR}_\phi = 16\pi (a_0 - \lambda b_6) \delta \left( \frac{ma}{\alpha^2} \right),$$

(4.2a)

$$\delta E^{GR}_T = 16\pi (a_0 - \lambda b_6) \delta \left( \frac{ma}{\alpha^2} \right),$$

(4.2b)

$$\delta \Gamma^{GR}_H = T \delta S^{GR}, \quad S^{GR} = 16\pi (a_0 - \lambda b_6) \frac{A_H}{4},$$

(4.2c)

where $T = \kappa/2\pi$. Hence, to obtain the complete result, one needs to calculate only an additional contributions from the Weyl curvature term $H_{ij}^W := 2 b_1^* W_{ij}$.

In the calculations that follow, the integration over the boundaries is implicit.

4.1 Asymptotic charges

Angular momentum. We start with the additional contribution to angular momentum, determined by the $W$-reduced relation (2.2) with $\xi = \partial_\phi = \partial_\varphi$,

$$\delta E^W_\varphi := \delta \Gamma_\infty [\partial_\varphi] = \frac{1}{2} \omega^{ij}_\varphi H^W_{ij} + \frac{1}{2} \delta \omega^{ij} H^W_{ij\varphi}.$$  

(4.3)

Here, there are only two nonvanishing terms,

$$\omega^{01}_\varphi \delta H^W_{01} + \delta \omega^{01} H^W_{01\varphi} = 4 \lambda b_1 \delta \left( \frac{ma}{\alpha^2} \right) \sin^3 \theta d\theta d\varphi,$$

$$\omega^{13}_\varphi \delta H^W_{13} + \delta \omega^{13} H^W_{13\varphi} = 2 \lambda b_1 \delta \left( \frac{ma}{\alpha^2} \right) \sin^3 \theta d\theta d\varphi.$$

After completing the integration, one obtains

$$\delta E^W_\varphi = 16\pi \lambda b_1 \delta \left( \frac{ma}{\alpha^2} \right).$$

(4.4)

Summing up this expression with the GR-like term (4.2a), the complete PG$^+$ contribution to angular momentum takes the form

$$\delta E_\phi = 16\pi A_0 \delta \left( \frac{ma}{\alpha^2} \right), \quad A_0 := (a_0 - \lambda b_6) + \lambda b_1.$$  

(4.5)
Energy. Consider now the $W$-contribution to energy, defined by the variational equation (2.2) with $\xi = \partial_T = \partial_t + \lambda a \partial_\phi$,

\[
\begin{align*}
\delta E^W_T := & \delta \Gamma^W_{\infty}[\partial_T] = \delta E^W_t + \lambda a \delta E^W_\phi, \quad (4.6a) \\
\delta E^W_t = & \frac{1}{2} \omega^{ij}_t \delta H^W_{ij} + \frac{1}{2} \delta \omega^{ij} H^W_{ijt}. \quad (4.6b)
\end{align*}
\]

There are three nonvanishing contributions to $\delta E^W_t$, defined by $(i, j) = (0, 1), (1, 2)$ and $(1, 3)$. A direct calculation yields the result

\[
\delta E^W_t = 16\pi\lambda b_1 \left[ \delta \left( \frac{m}{\alpha} \right) + \frac{m}{2} \delta \left( \frac{1}{\alpha} \right) \right],
\]

which is not $\delta$-integrable. Such an inconsistency of Boyer-Lindquist coordinates was noted also in GR [11]. Transition to the well-behaved $(T, \phi)$ coordinates via (4.6a) yields

\[
\delta E^W_T = 16\pi\lambda b_1 \delta \left( \frac{m}{\alpha^2} \right).
\]

Then, adding the GR-like term (4.2b) yields the complete PG$^+$ contribution to energy,

\[
\delta E_T = 16\pi A_0 \delta \left( \frac{m}{\alpha^2} \right). \quad (4.9)
\]

4.2 Entropy and the first law

For $\xi = \partial_T - \Omega_+ \partial_\phi$, the $W$-contribution to entropy is given by

\[
\delta \Gamma^W_H[\xi] := \frac{1}{2} \omega^{ij}_\xi \delta H^W_{ij} + \frac{1}{2} \delta \omega^{ij} H^W_{ij\xi}, \quad (4.10)
\]

where $\omega^{ij}_\xi := \xi_j \omega^{ij}$ and similarly for $H^W_{ij\xi}$. It contains only one nonvanishing contribution,

\[
\begin{align*}
\delta \Gamma^W_H[\xi] = & \omega^{01}_\xi \delta H^W_{01} = 4b_1\kappa \delta \left( Cb^2 b^3 \right) \\
= & 8\pi b_1\kappa \delta \left( \frac{1 + \lambda r^2_+}{\alpha} \right) = 8\pi \lambda b_1\kappa \delta \left( \frac{r^2_+ + a^2}{\alpha} \right) \quad (4.11a)
\end{align*}
\]

In (4.11a), we used $\omega^{01}_\xi = -\kappa$, and in (4.11b), the first term is obtained by integration, and the last equality follows from the identity $1 + \lambda r^2_+ = \alpha + \lambda(r^2_+ + a^2)$. Summing the above result with (4.2c), one obtains the complete PG$^+$ expression for entropy,

\[
\delta \Gamma_H[\xi] = 8\pi A_0\kappa \delta \left( \frac{A_H}{4\pi} \right) = T\delta S, \quad S := 16\pi A_0 \frac{A_H}{4}. \quad (4.12)
\]

Each of the Kerr-AdS thermodynamic charges $(E_\phi, E_T, S)$ in the PG$^+$ sector can be obtained from the corresponding GR expression by $a_0 \to A_0$. Hence, the first law is automatically satisfied,

\[
\delta E_T - \Omega_+ \delta E_\phi = T\delta S. \quad (4.13)
\]
5 Thermodynamic charges in PG−

The analysis starts with the effective Lagrangian

\[ L_{G^-} = -\bar{a}_0^* X + \frac{1}{2} R^{ij}(\bar{b}_1 W_{ij} + \bar{b}_6 (6) R_{ij}) , \]  

and the corresponding covariant momentum

\[ H_{ij} = -2\bar{a}_0 (b_i b_j) + 2 (\bar{b}_1 W_{ij} + \bar{b}_6 (6) R_{ij}) \]
\[ = -2(\bar{a}_0 - \lambda \bar{b}_6) (b_i b_j) + 2 \bar{b}_1 W_{ij} . \]  

1. The PG− expression for angular momentum is defined by

\[ \delta E_\varphi := \delta \Gamma_\infty [\partial_\varphi] = \frac{1}{2} \omega^{ij} \varphi H_{ij} + \frac{1}{2} \delta \omega^{ij} H_{ij}. \]  

Here, there are only two nonvanishing terms,

\[ \omega^{23} \varphi H_{23} + \delta \omega^{23} H_{23} = \delta [\omega^{23} \varphi (H_{23}) \theta \varphi] d\theta d\varphi , \]
\[ \omega^{02} \varphi H_{02} + \delta \omega^{02} H_{02} = \delta [\omega^{02} \varphi (H_{02}) \theta \varphi] d\theta d\varphi . \]

For each of these terms, the integration over \( \theta \) yields an expression of the general form

\[ I = \int_0^\pi d\theta f(\cos^2 \theta) \cos \theta \sin \theta , \]  

where the change of variables \( x = \cos \theta \) implies \( I = 0 \). Hence, the complete angular momentum stemming from the PG− sector vanishes,

\[ \delta E_\varphi = \delta E_\varphi = 0 . \]  

2. The PG− contribution to energy is determined by the relations

\[ \delta E_T = \delta E_t + \lambda a \delta E_\varphi = \delta E_t , \]
\[ \delta E_t := \delta \Gamma_\infty [\partial_t] = \frac{1}{2} \omega^{ij} t \delta H_{ij} + \frac{1}{2} \delta \omega^{ij} H_{ij}. \]  

There are seven nontrivial contributions to \( \delta E_t \),

\[ \omega^{01} t \delta H_{01} , \quad \omega^{23} t \delta H_{23} , \quad \delta \omega^{23} H_{23} , \]  
\[ \delta \omega^{0c} H_{0ct} , \quad \delta \omega^{1c} H_{1ct} , \quad c = 2, 3 . \]  

A straightforward calculation shows that they are all of the general form (5.3), so that

\[ \delta E_T = \delta E_t = 0 . \]  

3. Consider the variational expression for \( \delta \Gamma_H[\xi] \) with \( \xi = \partial_T - \Omega_+ \partial_\varphi \),

\[ \delta \Gamma_H[\xi] = \frac{1}{2} \omega^{ij} \xi H_{ij} + \frac{1}{2} \delta \omega^{ij} H_{ij} \xi . \]  

There is only one nontrivial term on the right-hand side,

\[ \delta \Gamma_H[\xi] = \omega^{01} \xi \delta H_{01} = -4\bar{b}_1 \kappa \delta (D^2 b^3) . \]  

Again, the integral on the right-hand side is of the general type (5.3), hence, the PG− contribution to black hole entropy vanishes,

\[ \delta \Gamma_H[\xi] = T \delta S = 0 . \]  

8
6 Concluding remarks

In the present paper, we analyzed thermodynamic properties of Riemannian Kerr-AdS solutions in the context of general, parity-violating PG models, using the Hamiltonian approach proposed in [10]. Black hole entropy and asymptotic charges are completely determined by the contributions stemming from the PG\(^+\) sector, whereas those from the PG\(^-\) sector vanish. The general form of the thermodynamic charges guarantees that the first law is automatically satisfied.

Using the identity \(W_{ij} = R_{ij} - \lambda b_ib_j\), one can rewrite the complete covariant momentum \((4.1b) + (5.1b)\) in an equivalent form as

\[
H_{ij} = -2A_0^*(b_ib_j) + 2b_1^*R_{ij} - 2\bar{A}_0(b_ib_j) + 2\bar{b}_1R_{ij}.
\] (6.1)

where \(\bar{A}_0 = (\bar{a}_0 - \lambda\bar{b}_6) + \lambda\bar{b}_1\). The last terms in the upper and lower line are associated to the Euler and Pontryagin topological invariants, \(R^{ij\star}R_{ij}\) and \(R^{ij}R_{ij}\), respectively, in the Lagrangian. Our calculations show that these two terms produce vanishing contributions to the thermodynamic charges. Note also that the Holst term is not a topological invariant, but it also has no impact on the Kerr-AdS thermodynamic charges. These conclusions are in agreement with those obtained by Jacobson and Mohd [14] in their analysis of the tetrad form of higher curvature gravity.

Comparing the results obtained here to those describing Kerr-AdS solutions with a nonvanishing torsion [13], one can conclude that they are characterized by different characteristic constants \(A_0\) and \(a_1\), respectively,

\[
A_0 \equiv a_0 + \lambda(b_1 - b_6), \quad a_1 \equiv a_0 - \lambda(b_4 + b_6).
\] (6.2)

This difference can be understood as a consequence of different dynamical settings in the two cases, or, more specifically, as an effect of torsion on the Riemann–Cartan connection.

References

[1] J. B. Griffiths and J. Podolský, Exact Space-Times in Einstein’s General Relativity (Cambridge University Press, Cambridge, 2009).

[2] M. Blagojević and F. W. Hehl (eds.), Gauge Theories of Gravitation, A Reader with Commentaries (Imperial College Press, London, 2013).

[3] J. D. McCrea, P. Baekler, and M. Gürses, A Kerr-like solution of the Poincaré gauge field equations, Nuovo Cimento 99 B (1987) 171–177.

P. Baekler, M. Gürses, F. W. Hehl, and J. D. McCrea, The exterior gravitational field of a charged spinning source in the Poincaré gauge theory: a Kerr-Newmann metric with dynamic torsion, Phys. Lett. A 128 (1988) 245–250.

[4] Y. N. Obukhov, Exact solutions in Poincaré gauge gravity theory, Universe 5 (2019) 127 (13 pages), https://doi.org/10.3390/universe5050127 [arXiv:1905.11906]
[5] Yu. N. Obukhov, Poincaré gauge gravity: An overview, Int. J. Geom. Meth. Mod. Phys 15 (2018) suppl. 01, 1840005 (23 pages) [arXiv:1805.07385].

[6] T. A. Jacobson, Introductory lectures on black hole thermodynamics, 1996 (38 pages), http://www.physics.umd.edu/grt/taj/776b/lectures.pdf.

[7] R. Wald, Black hole entropy is the Noether charge, Phys. Rev. D 48 (1993) R3427–R3431 [arXiv:gr-qc/9307038]; The thermodynamics of black holes, Living Rev. Relativity 4, 6 (2001) doi:10.12942/lrr–2001–6 (44 pages) [arXiv:gr-qc/9912119].

[8] T. Jacobson and R. C. Myers, Black hole entropy and higher curvature interactions, Phys. Rev. Lett. 70 (1993) 3684–3687 [arXiv:hep-th/9305016].

[9] G. W. Gibbons, M. J. Perry, and C. N. Pope, The first law of thermodynamics for Kerr–anti–de Sitter black holea, Class. Quant. Grav. 22 (2005) 1503–1526 [arXiv:hep-th/0408217].

[10] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D 99 (2019) 104058 (12 pages) [arXiv:1903.02263].

[11] M. Blagojević and B. Cvetković, Entropy in general relativity: Kerr-AdS black hole, Phys. Rev D 101 (2020) 084023 (7 pages) [arXiv:2002.05029].

[12] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Kerr-AdS solution, Phys. Rev. D 102 (2020) 064034 (11 pages) [arXiv:2007.10721].

[13] M. Blagojević and B. Cvetković, Thermodynamics of Kerr-AdS black holes in general Poincaré gauge theory [arXiv:2012.10471].

[14] T. Jacobson and A. Mohd, Black hole entropy and Lorentz-diffeomorphism Noether charge, Phys. Rev. D 92 (2015) 124010 (8 pages) [arXiv:1507.01054].

[15] S. Holst, Barbero’s Hamiltonian derived from a generalized Hilbert-Palatini action, Phys. Rev. D 53 (1996) 5966–5969 [arXiv:gr-qc/9511026].

[16] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y) 88 (1974) 286–318.

[17] M. Henneaux and C. Teitelboim, Asymptotically anti-de Sitter spaces, Commun. Math. Phys. 98 (1985) 391–424.

[18] B. Carter, Black hole equilibrium states, in Black holes, 1972 Les Houches Lectures, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973) pp. 58–214.