Complexity, Combinatorial Positivity, and Newton Polytopes

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Based on joint work with:

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- and -
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**Poorly understood issue:** Why are some decision problems have fast algorithms and others seem to need costly search?

Multiplication is easy:

\[ 90912135295978188784406583026004374858926083103 
28358720428512168960411528640933367824950788367 
956756806141 \times 814385925911004526572780912628442 
93358778990021676278832009141724293243601330041 
16702003240828777970252499 \]
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93358778990021676278832009141724293243601330041 \\
1670200324082877797970252499
\end{align*}
\]

Factoring seems hard. RSA $30,000$ challenge:

\[
\begin{align*}
74037563479561712828046796097429573142593188889 \\
23128908493623263897276503402826627689199641962 \\
51178439958943305021275853701189680982867331732 \\
73108930900552505116877063299072396380786710086 \\
096962537934650563796359
\end{align*}
\]

Solved in 2012.
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∴ I now give a brief summary of complexity theory:

- **NP**: LP ($\exists x \geq 0, Ax=b$?)
- **coNP**: Primes
- **P**: LP and Primes!
- **NP-complete**: Graph coloring

Famous theoretical computer science problems relevant to us:

- $P \overset{?}{=} NP$
- $NP \overset{?}{=} coNP$
- $NP \cap coNP \overset{?}{=} P$
In algebraic combinatorics and combinatorial representation theory we often study:

\[
F_\diamond = \sum_{\alpha} c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \text{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n]
\]

**Example 1:** \( \diamond = \lambda \implies F_\diamond = s_\lambda \) (Schur), \( c_{\alpha,\lambda} = K_{\lambda,\alpha} = \text{Kostka coeff.} \)

**Example 2:** \( \diamond = G = (V, E) \implies F_\diamond = \chi_G \) (Stanley’s chromatic symmetric polynomial), \( c_{\alpha,G} = \#\text{proper colorings of } G \text{ with } \alpha_i\text{-many colors } i \)

**Example 3:** \( \diamond = w \in S_\infty \implies F_\diamond = S_w \) (Schubert polynomial).

More later.
The decision problem we care about: Nonvanishing

**Nonvanishing**: What is the complexity of deciding \( c_{\alpha,\diamond} \neq 0 \) as measured in the length of the input \((\alpha,\diamond)\) assuming arithmetic takes constant time?

- In general **undecidable**: Gödel incompleteness '31, Turing’s halting problem '36.
- Our cases of interest have combinatorial positivity:
  \( \exists \) rule for \( c_{\alpha,\diamond} \in \mathbb{Z}_{\geq 0} \implies \) Nonvanishing\((F_{\diamond})\) \(\in\) NP.
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- Our cases of interest have combinatorial positivity:
  \[
  \exists \text{ rule for } c_{\alpha, \Diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_{\Diamond}) \in \text{NP}.
  \]

**Warning**: Standard combinatorics might not be manifestly in NP.

Ex. Does this SSYT certify Kostka coeff. \( K_{\lambda, \mu} \neq 0 \) where 
\[
\lambda = (10^{100}, 10^{100}) \quad \text{and} \quad \mu = (0^{20}, 4, 3, 2, 1, 2, 1, 0^6, 2, \ldots) ?
\]

This is a complexity rationale for Gelfand-Tsetlin polytopes.
Evidently, nonvanishing concerns the Newton polytope,

\[ \text{Newton}(F_\diamond) = \text{conv}\{\alpha : c_{\alpha, \diamond} \neq 0\} \subseteq \mathbb{R}^n. \]

**Definition:** (Monical-Tokcan-Y.) \( F_\diamond \) has saturated Newton polytope (S.N.P.) if \( \beta \in \text{Newton}(F_\diamond) \iff c_{\beta, \diamond} \neq 0 \)

- Many polynomials in algebraic comb. have this property.
- Application: A. Woo-Y. solves a complexity problem of D. Grigoriev-G. Koshevoy.
- Further work: subsets of \{A. Fink, J. Huh, R. Liu, J. Matherne, K. Mészáros, A. St. Dizier\}.
- Numerous open problems remain. For example:

**Fact:** (MTY) \( \Delta_n := \prod_{1 \leq i < j \leq n}(x_i - x_j)^2 \) is S.N.P. \( \iff n \leq 4. \)

**Conjecture:** (MTY) Fix \( k \), \( \exists n \) such that \( \Delta_n^k \) is not S.N.P.
Observation 1: S.N.P. $\Rightarrow$ nonvanishing($F_{\Diamond}$) is equivalent to checking membership of a lattice point in Newton($F_{\Diamond}$).

Observation 1': S.N.P. + “efficient” halfspace description of Newton($F_{\Diamond}$) $\implies$ nonvanishing($F_{\Diamond}$) $\in$ coNP.

$\therefore$ in many cases nonvanishing($F_{\Diamond}$) $\in$ NP $\cap$ coNP.
Nonvanishing and NP

**Example 1’**: $s_{\lambda}$ has S.N.P. \( \text{Newton}(s_{\lambda}) = P_{\lambda} \) (the permutahedron). Nonvanishing \( (s_{\lambda}) \in P \) by dominance order (Rado’s theorem).

**Example 2’**: \( \chi_G \) does not have S.N.P..

\( \chi_G \) coloring \( \in \) NP-complete \( \implies \) Nonvanishing \( (\chi_G) \in \) NP-complete.

\( \therefore \) nonvanishing hits the extremes of NP.
Example 1': $s_\lambda$ has S.N.P. Newton($s_\lambda$) = $P_\lambda$ (the permutahedron). Nonvanishing($s_\lambda$) $\in$ P by dominance order (Rado’s theorem).

Example 2': $\chi_G$ does not have S.N.P.

$\chi_G$ coloring $\in$ NP-complete $\implies$ Nonvanishing($\chi_G$) $\in$ NP-complete.

$\therefore$ nonvanishing hits the extremes of NP.

Question: What about the nonextremes?

- Many problems suspected of being NP-intermediate: e.g., graph isomorphism, factorization
- Ladner’s theorem: $P \neq NP \implies$ NP-intermediate $\neq \emptyset$
- Problems in $NP \cap coNP$ are suspects for NP-intermediate since

$$\text{coNP} \cap \text{NP-complete} \neq \emptyset \implies \text{NP} = \text{coNP}!$$

- This is why factorization is not expected to be NP-complete.
**Conjecture 1:** [Stanley ’95] If $G$ is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then $\chi_G$ is Schur positive.

**Conjecture 2:** [C. Monical ’18] If $\chi_G$ is Schur positive, then it is SNP.

**Conjecture 1+2:** If $G$ is claw-free then $\chi_G$ is SNP.

**Theorem:** (Holyer ’81) Coloring of claw-free $G$ is NP-complete.

**Corollary:** nonvanishing($\chi_{\text{claw-free}G}$) $\in$ NP-complete.

**Proposition:** (Adve-Robichaux-Y. ’18) Conjecture 1+2 and a halfspace description of Newton($\chi_{\text{clawfree}G}$) $\implies$ NP = coNP

Suggests a new complexity-theoretic rationale for the study of $\chi_G$. 
In many cases of algebraic combinatorics, \( \{F_\diamond \} \) has combinatorial positivity and SNP. If one also has an efficient halfspace description of Newton\((F_\diamond)\), then nonvanishing\((F_\diamond) \in \text{NP} \cap \text{coNP}\).

Three *plausible* outcomes of such a study:

(I) **Unknown**: it is an open problem to find additional problems that are in \( \text{NP} \cap \text{coNP} \) that are not *known* to be in P.

(II) **P**: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) **NP-complete**: (conjecturally) implies \( \text{NP} \cong \text{coNP} \) with “\( \cong \)”.

Your favorite polynomial family to think about this way?

My favorite is Schubert polynomials. Initially Adve, Robichaux and I got to outcome (I), but then achieved outcome (II).
\( B \) acts on \( GL_n/B \) with \textit{finitely many orbits}, the Schubert cells, whose closures \( X_w, w \in S_n \) are the \textbf{Schubert varieties}.

Lascoux and Schützenberger’s (1982) main idea in \textit{type A} (after Bernstein-Gelfand-Gelfand):

- Pick \( \mathcal{G}_w = x_1^{n-1}x_2^{n-2} \cdots x_{n-1} \) as an especially nice representative of the class of a point
- Apply \textit{Newton’s divided difference operator}

\[
\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}},
\]

...to recursively define all other \( \mathcal{G}_w \) using weak Bruhat order.

This starts the theory of \textit{Schubert polynomials}. 

There are many combinatorial rules that establish that $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$. However, none of these prove nonvanishing $(\mathcal{S}_w) \in P$ since they involve exponential search.

**Theorem A:** (Adve-Robichaux-Y. '18) $c_{\alpha,w}$ is $\#P$-complete.

∴ no poly. time algorithm to compute $c_{\alpha,w}$ exists unless $P = NP$.

Counting is hard, nonvanishing is easy:

**Theorem B:** (Adve-Robichaux-Y. '18) nonvanishing $(\mathcal{S}_w) \in P$

**Analogy:** Computing the permanent of a 0, 1-matrix is $\#P$-complete but nonzeroness is easy (Edmonds-Karp matching algorithm).
A tableau rule for nonvanishing

Fillings of the Rothe diagram of 31524:

\[
\begin{array}{ccc}
1 & 1 & \bar{2} \\
\bar{2} & 2 & \bar{1} \\
1 & \bar{1} & \bar{3}
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & \bar{2} \\
\bar{2} & 1 & \bar{3} \\
1 & \bar{1} & \bar{3}
\end{array}
\]

\[
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1 & 1 & \bar{2} \\
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1 & \bar{1} & \bar{3}
\end{array}
\]

**Theorem C:** (Adve-Robichaux-Y. ’18)

\[c_{\alpha,w} \neq 0 \iff \text{Tab}(w, \alpha) \neq \emptyset.\]
The *Schubitope* $S_D$ was introduced by Monical-Tokcan-Y. for any $D \subseteq [n]^2$.
We give a generalization of tableau of Theorem C to any $D$.
Then introduce a new polytope $T_D$ whose integer points biject with tableaux.
Integer linear programming is hard but $T_D$ is totally unimodular. Now use LPfeasibility $\in P$.
Link to Schubert polynomials we use:

**Conjecture** (MTY) For $D = D(w)$, $S_D = \text{Newton}(\mathcal{G}_w)$ and $\mathcal{G}_w$ is S.N.P.

**Theorem** (Fink-Mészáros-St. Dizier ’18): The above conjecture is true.

- NP and #$P$ proof via transition.
In this talk we described an *algebraic* combinatorics paradigm for complexity on theoretical computer science.

Conversely, complexity gives some new perspectives on algebraic combinatorics (Stanley’s chromatic symmetric polynomials).

In our main example, we obtain new results about Schubert polynomials and the Schubitope.

More $F_\diamond$’s in algebraic combinatorics deserve analysis of $\text{Newton}(F_\diamond)$ and $\text{Nonvanishing}(F_\diamond)$.