Three dimensional resonating valence bond liquids and their excitations

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We show that there are two types of RVB liquid phases present in three-dimensional quantum dimer models, corresponding to the deconfining phases of $U(1)$ and $Z_2$ gauge theories in $d = 3 + 1$. The former is found on the bipartite cubic lattice and is the generalization of the critical point in the square lattice quantum dimer model found originally by Rokhsar and Kivelson. The latter exists on the non-bipartite face-centred cubic lattice and generalizes the RVB phase found earlier by us on the triangular lattice. We discuss the excitation spectrum and the nature of the ordering in both cases. Both phases exhibit gapped spinons. In the $U(1)$ case we find a collective, linearly dispersing, transverse excitation, which is the photon of the low energy Maxwell Lagrangian and we identify the ordering as quantum order in Wen’s sense. In the $Z_2$ case all collective excitations are gapped and, as in $d = 2$, the low energy description of this topologically ordered state is the purely topological BF action. As a byproduct of this analysis, we unearth a further gapless excitation, the pîñôn, in the square lattice quantum dimer model at its critical point.

I. INTRODUCTION

Fractionalised phases, i.e. phases where low energy excitations exhibit fractional quantum numbers, have played an important role in condensed matter physics over the past two decades. Probably the two most salient examples are provided by the quantum Hall effect, where the charge of a quasiparticle can be a rational fraction of the fundamental electronic charge, and by Anderson’s suggestion of spin-charge separation being at the root of high-temperature superconductivity.

This latter proposal has spawned a set of theories collectively known as resonating valence bond (RVB) theories. In its short-range incarnation, RVB theory uses dimers as its basic degree of freedom, which are cartoons for a pair of adjacent electrons forming a singlet (‘valence’) bond. At zero doping, when all sites are singly occupied, all electrons are involved in a singlet bond with a neighbour. The Hilbert space of this theory is thus given by the classical dimer coverings of the lattice. Resonance moves between dimer coverings provide a quantum dynamics for the dimers.

The original hope was that the resulting quantum dimer model would have a liquid phase, the RVB liquid, hypothesized as an alternative to Neel order in antiferromagnets by Anderson and Fazekas in the early seventies. This state would exhibit fractionalised $S = 1/2$ spinons and as a consequence holes doped into it would undergo spin-charge separation and the resulting charged spinless quasiparticles (‘holons’) would Bose condense to form a superconductor.

However, it turns out that the square lattice quantum dimer model appropriate for the cuprates exhibits only solid phases with the exception of one critical point. These properties are now well-understood within the framework provided by the height representation for dimer coverings. By contrast, for dimer models on non-bipartite lattices, the present authors have demonstrated that an RVB liquid phase is possible. The different outcomes on bipartite and non-bipartite lattices can be rationalised, following the ideas of Fradkin and Shenker on lattice gauge theories via the absence (presence) of a deconfined phase in $U(1)$ ($Z_2$) gauge theories in $2+1$ dimensions. We note that the RVB phase on non-bipartite lattices is best characterized as a topologically ordered phase whose low energy theory is the purely topological BF theory.

This understanding of the situation in $2+1$ dimensions has led us, in the present work, to investigate the behaviour of quantum dimer models in $d = 3+1$, specifically on the bipartite cubic lattice and the non-bipartite face-centred cubic lattice. We find that both models exhibit RVB phases with deconfined spinons, which correspond to the Coulomb phase of the $U(1)$ gauge theory (ordinary electromagnetism) and the deconfined phase of the $Z_2$ gauge theory, respectively. The FCC lattice RVB liquid is topologically ordered. It has a gap to all excitations and its ground state degeneracies and topological interactions between spinons and vortex loops are encoded in the topological $3+1$ dimensional BF action. By contrast, the cubic lattice RVB phase supports a gapless, transverse collective mode. This can be identified as the photon of the low energy theory, which is now a non-topological Maxwell theory. Consequently, we identify the ordering of this phase as quantum order in Wen’s sense, reserving the term topological order for cases where the low energy theory is purely topological. As a byproduct of this investigation, we have also revisited the theory of the critical point of the square lattice dimer model where we find, in addition to the resonons which are the analogs of our photons in $d = 2$, a further gapless excitation, the pîñôn, which signals the incipient crystalline order.

We note that our main results were announced in Ref. 10.

From the foregoing it follows that a deconfined phase in a valence bond dominated regime is easier to achieve in $d = 3$ than in $d = 2$, where its existence is ruled out for bipartite lattices. This encouraging fact is, however, counterbalanced by the difficulty of stabilising a valence-
bond dominated phase in comparison to a Neel phase, which becomes increasingly hard as the coordination of the lattice grows. There is also the possibility of realizing dimer models in other contexts – e.g. the Santa Barbara group has shown that multiple dimer models arise when frustrated Ising magnets are imbued with a ring-exchange dynamics. Indeed in Ref. [24] the reader can find a parallel discussion of a Coulomb phase in multidimer models on the diamond and cubic lattices. Alternatively, dimer models can arise as Bose Mott insulators, electronic Mott insulators at fractional fillings [21] and in mixed-valence systems on frustrated lattices [22].

We would be remiss if we did not note that quite generically $U(1)$ gauge theories of Heisenberg magnets should be expected to predict a Coulomb phase in $d = 3$. Specifically, a treatment of three-dimensional quantum magnets cically be expected to predict a Coulomb phase in number of bosons per flavour, supplemented by 1 electronic Mott insulators at fractional fillings and in mixed-valence systems on frustrated lattices.

The models he has considered that give rise to an emergent photon have considerable resemblance to the cubic dimer model considered in this paper. Indeed, dimers and monomers have a natural string interpretation when superposed on a reference configuration.

In the following, we first briefly summarise the known properties of the quantum dimer model which are of relevance to us. We then describe the RVB liquids, first on the simple and second on the face-centred cubic lattice. In both cases, we discuss in turn correlations, topological properties and the excitation spectrum.

II. QUANTUM DIMER MODELS: GENERALITIES

On a general lattice, allowed moves between two different hardcore dimer configurations consist of interchanging alternately occupied and empty links forming a closed loop, known as a resonance loop. If we denote the presence (absence) of a dimer by an Ising link variable, $\sigma^z = +1(-1)$, such a resonance move is generated by the kinetic operator

$$\hat{T}_0 = \left( \prod_{i=1}^{n_1/2} \sigma_{2i-1}^+ \sigma_{2i}^- \right) + h.c., \quad (2.1)$$

where the loop, denoted pictorially by $\circ$, contains $n_1$ links, and $\sigma^\pm$ are the raising and lowering operators for to $\sigma^z$. A corresponding projection operator,

$$\hat{P}_0 \equiv \hat{T}_0^2$$

(2.2)

determines whether dimers are arranged on a given loop so as to be able to carry out a resonance move; such loops we call flippable loops.

The quantum dimer Hamiltonian proposed by Rokhsar and Kivelson considers only the shortest possible resonance loops. It contains two terms, each involving one of these operators. The first carries out resonance moves with a kinetic energy $t_0$, and the second exacts an energy cost $v_0$ for a flippable loop:

$$H_{QDM} = \sum_{\circ} -t_0 \hat{T}_0 + v_0 \hat{V}_0$$

(2.3)

Some features are common to the phase diagrams of quantum dimer models on most lattices. We have depicted a generic phase diagram in Fig. [1]. For $v/t > 1$, it is favourable not to have any possible flippable loops; configurations satisfying this requirements can usually be found, and the resulting phase is called the staggered phase, a name borrowed from the appearance of one of the inert square lattice configurations. For $v/t = -\infty$, by contrast, one wants to maximise the number of flippable loops. In both cases, there are no quantum fluctuations. These phases are confining, and confinement typically exists for most values of $v/t$, with the possibility of transitions between different confining phases.

From the point of view of deconfined, liquid phases, the region $v/t \lesssim 1$ is the most interesting, because the balance between kinetic term and potential term overly favours neither high or low densities of resonance loops. The ground state wavefunction can then be spread moderately uniformly over most of the classical configuration space; if the ensemble of classical configurations is sufficiently disordered, this can then lead to an RVB liquid phase.

This statement can be made precise at the Rokhsar-Kivelson (RK) point $v/t = 1$. Define a sector as the set of dimer coverings connected by $\hat{T}_0$. Then the equal amplitude superpositions of dimer coverings in each sector is a ground state at the RK point. Hence equal time
Correlations can be computed as correlations of the uniform fugacity classical dimer model on the same lattice. These can often be obtained analytically — exactly (especially in $d \leq 2$) or at least asymptotically — or from Monte Carlo simulations.

Consequently, advances in determining classical correlations give valuable information on the quantum model. Indeed, it was the solution of the classical dimer model on the triangular lattice with its short-ranged correlations that allowed the triangular RVB liquid to be discovered.

Recently, Huse, Krauth and the present authors have discussed the classical dimer correlations on the cubic and FCC lattices. In the next section we will build on this work to establish the presence and nature of RVB liquid phases on these two lattices.

III. CUBIC LATTICE: AN RVB $U(1)$ LIQUID

The general structure of the phase diagram on the cubic lattice is indeed as sketched in Fig. For $v/t > 1$, the ground states are the non-flippable (‘staggered’) configurations, such as the ones obtained by appropriately stacking the two-dimensional staggered configurations. For $v/t$ large and negative, there are six maximally flippable columnar configurations, again obtained from stacking planar columnar configurations. Apart from these two crystalline phases we have been able to establish the existence of a liquid, RVB phase for $v/t \gtrless 1$. This is a “Coulomb” phase, characterised by algebraically decaying dimer (“magnetic field”) correlations and whose spectrum contains a linearly dispersing transverse collective excitation (“photon”), a gapped topological defect (“electric monopole”) in addition to Coulombically interacting spinons (“magnetic charges”). We now sketch the derivation of these results.

A. RK Point

We begin with the properties of the RK point, $v/t = 1$. The classical dimer model on the cubic lattice was studied in Ref. It was shown there that a useful parametrization of the dimer configurations is in the language of solenoidal magnetic fields. Briefly, on the bipartite cubic lattice, one can think of a dimer as a magnetic flux, $B$ of strength $5/6 = 1 - 1/z$ pointing from sublattice A to sublattice B; here, $z$ is the coordination of the lattice. Identifying an unoccupied bond with flux $-1/6 = -1/z$, one has $\nabla \cdot B = 0$ — magnetic charges are excluded.

Upon local coarse-graining, analogous to what is done for models with height representations in two-dimensions, long wavelength configurations acquire an entropic weight quadratic in $B$ and this allows computation of the long distance correlations of the classical model. These are purely dipolar,

$$\langle B_i(x)B_j(0)\rangle = \frac{1}{K} \frac{3x_ix_j - r^2 \delta_{ij}}{r^5} \quad (3.1)$$

where $K$ can be determined via Monte Carlo simulations. The language of magnetic fields also allows a characterization of the sectors. For a cube with periodic boundary conditions, the flux through any surface that wraps around it is invariant under local dimer moves and is invariant under lattice translations of the surface. In particular, if we let $\Sigma_i$ be planes perpendicular to the cubic unit vectors $e_i$, the fluxes $\Phi_i^B$ through them are the invariants that characterize a given sector of the dimer model. This flux is also known as a $U(1)$ winding number. For an $L^3$ cube these range in magnitude from 0 to $L^2/2$. The form of the correlations given above is the average over all sectors, but also holds individually for all sectors with $\Phi^B = |\sum \Phi_i^B e_i| \sim O(\sqrt{L})$, which dominate the sum at large $L$. Finally, we note that the “staggered” configurations maximize $\Phi^B$, or more accurately, the average magnetic field strength.

At the RK point equal amplitude superpositions in all magnetic flux sectors constitute degenerate ground states. As we will see below, for $v/t \lesssim 1$ this degeneracy is lifted and only the sectors with $\Phi^B \sim o(\sqrt{L})$ will be important in the low energy analysis — so we will not need more classical information than given in Eq. above. For this reason it is sufficient to focus on the zero flux sector at the RK point for now.

As the exact ground state wavefunction is known, we can use the single mode approximation (SMA) for dimer models, pioneered by RK in their original paper to get a good description of the excitation spectrum. We remind the reader that the SMA will provide a rigorous upper bound on the excited state energy at each momentum. As RK’s explanation was rather compressed and their treatment incomplete in one important respect in the $d = 2$ case, we will provide a fuller account between this section and the appendices.

Single Mode Approximation: Let $|0\rangle$ denote the ground state of the dimer model under consideration. As above, we define $\sigma^\pm(x)$ as the Pauli spin operator the eigenvalues $\pm 1$ of which correspond to presence and absence of a dimer on the link at location $x$. Here, we have added the ‘direction’ vector $\hat{r}$ for clarity to distinguish between dimers pointing in the different possible directions.

Next, we define the Fourier transform of the dimer density operator,

$$\hat{\sigma}^\pm(q) = \sum \sigma^\pm(x) \exp(iq \cdot x) \quad (3.2)$$

Acting with this operator on the ground state yields a state

$$|q, \hat{r}\rangle \equiv \hat{\sigma}^\pm(q) |0\rangle \quad (3.3)$$

with nonzero momentum $q$ so that

$$\langle 0 |q, \hat{r}\rangle = 0 \quad (3.4)$$
Provided that

\[ \langle \mathbf{q}, \mathbf{r} | \mathbf{q}, \mathbf{r} \rangle \neq 0, \]

this excitation has a variational energy of

\[ E(\mathbf{q}, \mathbf{r}) \leq \frac{\langle 0 | \hat{\sigma}^z(-\mathbf{q})_0, [H_{QDM}, \hat{\sigma}^z(\mathbf{q})] | 0 \rangle}{\langle 0 | \hat{\sigma}^z(-\mathbf{q})_0, \hat{\sigma}^z(\mathbf{q})_0 | 0 \rangle} = \frac{f(\mathbf{q})}{s(\mathbf{q})}, \]

where the numerator and denominator in the final line are the oscillator strength \( f(\mathbf{q}) \) – and structure factor \( s(\mathbf{q}) \). These can be evaluated as ground state expectation values.

If it now so happens that the density \( \hat{\sigma}^z(\mathbf{q}_0) \) is a conserved quantity, then \( [H_{QDM}, \hat{\sigma}^z(\mathbf{q}_0)] = 0 \), and hence \( E(\mathbf{q}_0, \mathbf{r}) = 0 \). From the orthogonality relation \( (\hat{\sigma}^z, \hat{\sigma}^z) = 0 \) it then follows that this is a variational upper bound on the energy of the ‘excited’ state \( |\mathbf{q}_0, \mathbf{r}\rangle \). Similarly, the behaviour of \( f(\mathbf{q}) \) near \( \mathbf{q}_0 \) can be used to determine a bound on the dispersion of the soft excitations. Further, and again only if \( (\hat{\sigma}^z, \hat{\sigma}^z) \) holds, a smooth behaviour of \( f(\mathbf{q}) \) while \( s(\mathbf{q}) \) diverges, can also be used to infer gapless excitations. Indeed, this is the classic signature of incipient order.

RK identified a conserved quantity for the square lattice, which generalises to the cubic case as the density of dimers pointing in a given direction \( (\hat{\mathbf{r}} = \hat{x}, \text{say}) \) at wavevector \( \mathbf{q}_0 = (q_x, \pi, \pi) \). This follows from the fact that the quantum dynamics always creates and destroys pairs of neighboring dimers. This implies that the oscillator strength, \( f(\mathbf{q}_0) \), vanishes. In fact, at a wavevector \( \mathbf{q}_0 + \mathbf{k} \), one finds

\[ f(\mathbf{k}) \propto (\mathbf{k} \times \mathbf{r})^2, \]

as outlined in Appendix A. This form holds for all values of \( -\infty < v/t \leq 1 \).

This, however, does not lead to lines of zero energy excitations because the structure factor also vanishes for all \( \mathbf{q}_0 \) with \( q_x \neq \pi \) – Eq. (3.3) is not satisfied. In particular, at the RK point, for momentum \( \mathbf{q} = (\pi, \pi, \pi) + \mathbf{k} \), it has the form of a transverse projector, as one can see by Fourier transforming Eq. (3.4)

\[ s(\mathbf{k}) \propto \frac{k_x^2 + k_y^2}{k^2} = \frac{k_z^2}{k^2}. \]

This implies that only transverse excitations are generated by \( \hat{\sigma}^z \), a fact that traces back to the absence of monopoles, \( \nabla \cdot \mathbf{B} = 0 \). The transverse nature of such excitations has already been noted by Hastings. It is important to note that these are the only gapless excitations for the cubic lattice. By contrast, the square lattice also exhibits gapless excitations near \( (\pi, 0) \) and \( (0, \pi) \), where there is a divergence of, respectively, \( s_z \) and \( s_y \), signalling the immediate proximity of a crystalline phase. This is consistent with the identification of the RK point as a critical point terminating a crystalline phase. These excitations, which we christened pi0ns, are discussed further in Appendix B. The absence of pi0ns in \( d = 3 \) signals that the RK point does not abut a crystalline phase but instead terminates an RVB phase. We will use this ‘soft pi0n theorem’ again in the analysis of the FCC lattice.

B. Effective field theory and RVB phase

The above RK point results can now be used to constrain a long wavelength field theory valid in the proximity of this point. From it, we will be able to deduce the existence and properties of the neighboring RVB phase.

To this end we follow Ref. 19 and solve the constraint \( \nabla \cdot \mathbf{B} = 0 \) by writing \( \mathbf{B} = \nabla \times \mathbf{A} \) where \( \mathbf{A} \) lives on the dual cubic lattice. As this representation comes with a local gauge invariance, we will pick the gauge \( \nabla \cdot \mathbf{A} = 0 \) to have a faithful representation of the dimer states. If we now think of the time evolution generated by the kinetic energy, this will involve flipping plaquettes. It is straightforward to see that such a flip represents an (essentially) spatially local change in \( \mathbf{A} \). This fact and the RK point properties allow us to write down the action.

\[ S = \int d^3x dt (\partial_t \mathbf{A})^2 - \rho_2 (\nabla \times \mathbf{A})^2 - \rho_4 (\nabla \times \nabla \times \mathbf{A}^2), \]
\[ \nabla \cdot \mathbf{B} = \pm 1 \text{ on sublattice A/B. Hence they carry a magnetic charge. As the Maxwell action is invariant under the duality } E \leftrightarrow B, \text{ we could also recast the spinons as electrically charged particles. Either way, they interact via an inverse square force law and are therefore deconfined. At the RK point the vanishing of } \rho_2 \text{ results in the force vanishing altogether.} \]

In addition to the photon, there is a topological defect of interest in the model with dimers alone. This is the electric monopole, being the magnetic monopole that is present in lattice \( U(1) \) gauge theories in our dual labelling of the fields. To see how it is constructed, consider solving the Gauss’s law constraint \( \nabla \cdot \mathbf{E} = 0 \) that arises in the passage from (3.10) by writing \( \mathbf{E} = \nabla \times \mathbf{P} \). As \( \mathbf{E} \) lives on the dual lattice, \( \mathbf{P} \) lives again on the direct lattice. [In quantizing the theory we take \( \mathbf{P} \) and \( \mathbf{B} \) to be the conjugate variables on a link.] Following the standard construction of the monopole, one can pick a classical \( \mathbf{P} \) such that \( \mathbf{E} \) has the form \( \mathbf{E} \propto \mathbf{P} / r^2 \) at long distances from a specified point of the dual lattice while a Dirac string of plaquettes ensures that \( \nabla \cdot \mathbf{E} = 0 \) holds everywhere. The monopole state then can be written down in the dimer representation as

\[ e^{i \int \mathcal{D}^2 \mathbf{P} \mathbf{B}} |0\rangle \]  

(3.11)

where \(|0\rangle\) is again the ground state. Requiring that the resonance energy cost along the string does not diverge with system size quantizes the monopole flux. The monopole generalizes the construction of the vortex at the RK point in \( d = 3 \) and is gapped as befits a deconfined phase. It was also noted in that analysis that the binding of vortices to spinons could alter their statistics. Likewise in \( d = 3 \) the binding of electric monopoles to bosonic spinons will lead to fermionic dyons. The monopoles also interact via an inverse square force.

C. Fractionalization and quantum order

RVB liquid phases are fractionalized in that they support \( S = 1/2 \) spinons. In cases where there are no gapless excitations in the spectrum, e.g. the RVB phase on the triangular lattice and the quantum Hall liquids, the fractionalization is of a piece with a ground state degeneracy sensitive to ground state topology and a topological interaction between various gapped excitations—a complex of properties named topological order by Wen. In its pristine form, topological order is unaccompanied by any symmetry breaking. Most succinctly, the low energy theory for topologically ordered systems is a topological field theory, the Chern-Simons theory for the quantum Hall states and the BF theory for the \( Z_2 \) RVB state. We will see in the next section that the FCC RVB phase is indeed topologically ordered in this sense.

However, the situation with regard to the \( U(1) \) phase discussed in this section is different. The low energy theory is the Maxwell theory, which is not topological. Concomitantly, the gapless photon makes the distinction between ground states and excited states fuzzy. While at the RK point there is an exact degeneracy between different magnetic flux sectors, elsewhere in the RVB phase, this degeneracy is lifted by the \( B^2 \) term in the action, the coefficient \( \rho_2 \) of which no longer vanishes. The splitting thus generated is \( L^3 \times \Phi^2 / L^4 = \Phi^2 / L \), which vanishes—only algebraically—in the thermodynamic limit provided \( \Phi^2 = o(\sqrt{L}) \). For \( \Phi^2 = O(1) \), the splitting has the same magnitude as the excitation energy of a photon; this is no coincidence if one thinks of a photon as imposing a flux modulated at the photon’s wavelength, so that locally a sector with a photon present looks like it belonged to a sector with a slightly different \( \Phi^2 \).

As a matter of principle, one can identify the minimum energy states in the nonzero \( \Phi^2 \) sectors as degenerate ground states while reserving the term excitation for excited states in a given sector. Indeed, at this level the symmetry between \( \mathbf{E} \) and \( \mathbf{B} \) implies that there is another set of states with a net electric flux that also have an energy of \( O(1/L) \) and should be identified as members of the ground state multiplet. These are the analogs, in \( d = 3 \) of the height shift mode discussed by Henley for the RK point — in \( d = 2 \) the time derivative of \( h \) is the electric field. Nevertheless, such degeneracies would be hard to detect against the background of excitations with similar energies and so their utility appears to be limited.

Consequently, we follow Wen in identifying the \( U(1) \) RVB liquid as an instance of “quantum order” wherein the masslessness of the photon is attributed to the rigidity of the low energy gauge structure.

IV. FACE-CENTRED CUBIC LATTICE: AN RVB \( Z_2 \) LIQUID

The behavior of the FCC lattice quantum dimer model parallels that of the triangular lattice problem in \( d = 2 \). Like the triangular lattice, this non-bipartite lattice does not admit a representation by solenoidal fields. Instead, it exhibits topological sectors, not connected under a local dynamics, that are labelled by an Ising variable for each periodic direction on the lattice. This we call a winding parity to distinguish it from the \( U(1) \) winding numbers \( \Phi^B \). The winding parity is defined in an analogous manner, by counting whether the number of dimers cutting the planes \( \Sigma_i \) is even or odd.

To the right of the RK point, there exists the standard non-flippable phase (Fig. 1); a non-flippable configuration can again be obtained by appropriately stacking staggered square lattice configurations. The identification of the phase for large negative \( v/t \) remains an open problem.
A. RK Point and RVB Phase

For the RK point on the 3-torus there are 8 exactly degenerate ground states which exhibit exactly the same equal time correlations in the thermodynamic limit. More generally, there are $2^p$ degenerate states for a system with $p$ independent non-contractible loops. The classical analysis in Ref. [14] has shown that the correlations decay exponentially to zero, with an extremely short correlation length of $\xi = 0.4$ nearest neighbour distances.

Turning now to collective excitations within the SMA, we note that there are conserved quantities, the number of dimers in the $xy$ planes at wavevector $q_0 = (0, 0, \pi)$ and its symmetry related counterparts. Nevertheless, the structure factor vanishes as well at these wavevectors, quadratically by the analyticity in momentum space guaranteed by the short range of the correlations in real space.

To see this, consider grouping all the sites of a finite FCC lattice with periodic boundary conditions in the $z$-direction according to their $z$-coordinate. Let $N_z$ be the number of sites with a given $z$-coordinate. Now, for a given dimer configuration, define the number of dimers linking a pair of sites with neighbouring $z$-coordinates by $\Delta_{z,z+1}$. Then the number of dimers in the $xy$ plane, $n$, at coordinate $z$ is simply $2n(z) = N_z - \Delta_{z,z+1} - \Delta_{z-1,z}$. Therefore, for any $k_z = q_z - \pi \neq 0$,

$$\hat{n}(q_z) \sim -\sum z \Delta_{z,z+1} \exp(iq_zz) [2 \exp(iq_z/2) \cos q_z]$$

$$\sim i(q_z - \pi) \tilde{\Delta}(q_z), \quad (4.1)$$

where the last term in parentheses was expanded in $k_z$; at $q_z = \pi$, it vanishes linearly, yielding a quadratically vanishing structure factor.

From the absence of any significant crystalline correlations or gapless excitations at the RK point (no soft spinons) it is safe to conclude that for $v/t < 1$ there is a finite RVB phase with liquid dimer correlations, an 8-fold ground state degeneracy on a 3-torus and a gap to all excitations. Indeed the first and third pieces are mutually consistent. The vanishing of the oscillator strength at $q_0$ holds everywhere, while the structure factor is also zero exactly at $q_0$. If the correlations remain liquid, the structure factor will vanish analytically at $q_0$ and hence the gap will persist. Evidently, matters will be different when a solid phase is reached.

B. Topological order

The excitations of the FCC RVB phase include the ubiquitous spinons and gapped vortex (vison) loops. The topological interaction between them is described by the $3+1$ dimensional BF action which encodes the phase factor of $e^{i n_1}$ for a spinon trajectory that links $n_1$ times with the vortex loop. Quantization of the BF action on closed manifolds recovers the ground state degeneracy discussed above and the tunneling of the spinons and vortex lines lifts the ground state degeneracy by an amount of $O(e^{-L})$ and $O(e^{-L^2})$ for a system of linear dimension $L$.

V. SUMMARY

In this paper, we have established the existence of two types of RVB liquids in $d = 3$, which occur on the simple and face-centred cubic lattices. As in $d = 2$, the possible RVB phases in quantum dimer models correspond to the known deconfined phases in compact gauge theories and exhibit the same topological scaling limits. In addition to their direct physical interest, they allow attractively simple illustrations of concepts such as topological and quantum order.

Remarkably, the program outlined by Rokhsar and Kivelson in their quest for the square lattice RVB liquid can be carried through for the simple cubic lattice in toto. The ability to perform calculations in a classical framework at the Rokhsar-Kivelson point to access properties of both ground and excited states has been invaluable for our study.

Finally, it will be interesting to see if such phases can be realised, for example by destabilising Heisenberg Neel states on both lattices using frustrating exchange interactions.

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APPENDIX A: CALCULATION OF OSCILLATOR STRENGTH

To calculate the oscillator strength, $f(q) = \langle \sigma^x_r(-q) \right| [H_{QDM}, \sigma^x_{-q}(q)] \left| \sigma^x_r(q) \rangle$, we need to consider the commutation properties of the dimer density operator, $\hat{\sigma}^x_r(q) \equiv \sum_{\mathbf{r}} \sigma^x_{\mathbf{r}} \exp(iq \cdot \mathbf{r})$ with the quantum dimer Hamiltonian. To make contact with the original quantum dimer paper, where the case of excitations near
\( \mathbf{Q} = (\pi, \pi) \) on the square lattice were considered, we write \( \mathbf{q} = \mathbf{Q} + \mathbf{k} \); more generally, \( 2\mathbf{Q} \) needs to be a reciprocal lattice vector.

The dimer density operator commutes with the potential term of \( H_{QDM} \), so that we only consider its commutators with the kinetic term. All that needs to be done here is to use

\[
[\sigma^\pm, \sigma^\pm] = \mp 2\sigma^\pm \tag{A1}
\]

repeatedly. This form implies that the result in an isotropic phase will always be the kinetic energy operator (and, at the RK point, the expectation value of the loop flippability) times a function of \( \mathbf{q} \), the details of which we are interested in.

The contribution of a given square plaquette with dimers in the \( \hat{\tau} \) directions at locations \( \mathbf{R} \) and \( \mathbf{R'} = \mathbf{R} + \mathbf{r} \) is given by

\[
[\hat{\sigma}^\pm(\mathbf{q}), \hat{\sigma}^\pm(\mathbf{q})] = 8\tau_0 \hat{T}_0 \exp(2\text{i}(\mathbf{q} \cdot \mathbf{R}))(1 + \cos((\mathbf{Q} + \mathbf{k} \cdot \mathbf{r})\Lambda_2)
\]

Summing this expression over all plaquettes forces \( 2Q \) to be a reciprocal lattice vector, and yields the expression proportional to the kinetic energy operator.

The dependence on \( \mathbf{k} \) and \( \mathbf{Q} \) now follows straightforwardly. For example, for the case of resonons on the square lattice, \( \mathbf{Q} = (\pi, \pi), \) and \( \mathbf{r} = \pm \hat{y} \) for \( \hat{\tau} = \hat{x} \). Thence, \( f(\mathbf{k}) \propto 1 - \cos k_y \sim k_y^2 \sim (\mathbf{k} \cdot \hat{\tau})^2 \), the original result of Rokhsar and Kivelson. This form also obtains for \( \mathbf{Q} = (0, \pi) \), whereas near \( \mathbf{Q} = (\pi, 0) \), \( f(\mathbf{k}) \propto \cos^2 (k_y/2) \) is nonzero everywhere.

The same analysis carries over directly to other lattices. On the cubic lattice, there are two transverse directions, and \( f(\mathbf{k}) \sim (\mathbf{k} \cdot \hat{\tau})^2 \) continues to hold near \( \mathbf{Q} = (\pi, \pi, \pi) \).

For the hexagonal case, where an elementary resonance involves a triplet of dimers, one finds the resonon to occur near zero wavevector, \( \mathbf{Q} = 0 \), with the same form of \( f(\mathbf{k}) \sim (\mathbf{k} \cdot \hat{\tau})^2 \).

On the triangular lattice, the same algebra yields a qualitatively different result. If we consider the triangular lattice as a square lattice decorated with diagonal bonds pointing from the bottom left to the top right corner of each square, the transverse projector form for \( \mathbf{Q} = (0, \pi) \) is replaced by \( f(\mathbf{k}) \sim k_y^2 + (k_y + k_x)^2 \), which vanishes quadratically as \( k \to 0 \) but is nonzero in all directions away from \( k = 0 \).

Finally, for the FCC lattice, where the conservation law involves different dimer directions, the algebra is more complicated but one again finds a vanishing \( f(k) \) as the wavevector approaches \( \mathbf{Q} = (0, 0, 0) \).

**APPENDIX B: SMA RESULTS IN \( d = 2 \)**

1. Resonons and pi0ns on the square lattice

For the bipartite square lattice, the dimer density \( \hat{\sigma}^\pm(\mathbf{q}) \) is a conserved quantity at \( \mathbf{q}_0(\hat{\tau}, q_\tau) = \hat{\tau} \times q_\tau \hat{\tau} \) for any value of \( q_\tau \). Thus, \( f(\mathbf{q}_0) \) vanishes. As in the cubic lattice, this does not lead to an entire line of zero-energy excitations along the Brilliouin zone edge. Rather, the resonon only exists as a transverse excitation near \( (\pi, \pi) \).

To demonstrate this, we have explicitly determined the structure factor for \( \hat{\sigma}(\mathbf{q}, \hat{\tau}) \) and \( \hat{\tau} = \hat{x} \). The calculation of the structure factor for two-dimensional models at the RK point is straightforwardly achieved using the Fermionic path integrals developed for the study classical dimer models. A detailed calculation of correlations for the triangular lattice has been presented in Ref. 32.

The result is plotted in Fig. 2. The structure factor vanishes everywhere on the line \( (q_\tau, \pi) \) except for \( \mathbf{Q} = (\pi, \pi) \). This implies that \( \langle \mathbf{q}, \hat{\tau} \rangle = 0 \) on this line away from \( (\pi, \pi) \), so that it is not a candidate for a gapless excitation.

In fact, the form of the structure factor near \( \mathbf{Q} \) is well-described by the transverse projector

\[
s_{\hat{\tau}}(\mathbf{q}) \sim \frac{k^2 - (\mathbf{k} \cdot \hat{\tau})^2}{k^2} \sim \frac{k_y^2}{k^2} \tag{B1}
\]

where \( \mathbf{q} = \mathbf{Q} + \mathbf{k} \). Together with \( f(\mathbf{k}) \sim 1 - \cos k_y \sim k_y^2 = (\mathbf{k} \cdot \hat{\tau})^2 \), this implies that there is only a single gapless excitation per dimer direction, namely at \( \mathbf{Q} = (\pi, \pi) \), which is of a transverse nature.

In Fig. 2 a peak in \( s_{\hat{\tau}}(\mathbf{q}) \) is visible at \( \mathbf{q}_0 = (\pi, 0) \). Its scaling with system size can be determined easily since the long-distance part of the correlations of the classical dimer model in real space are known exactly.\[11,25\]

\[
C_{\hat{\tau}}(\mathbf{r}) \sim (-1)^{x+y} \frac{y^2 - x^2}{r^4} + (-1)^y \frac{1}{r^2} \tag{B2}
\]

This form implies that the peak height grows logarithmically with system size, and equivalently that its height a
distance $k = q - q_1$ from the centre decays (isotropically) as $-\log k$.

Near $q_1 = (\pi, 0)$, the oscillator strength depends on $k$ simply as $f(k) \sim \cos^2(k_\|/2)$, which is a constant to leading order. Therefore, $E(q_1, \hat{x}) = 0$, and there exist further gapless excitations for each dimer direction, which – due to their location in reciprocal space – we call pîhns. The dispersion obtained within the SMA is logarithmic, but more on that below.

We mention in passing that the same analysis carries over to the hexagonal lattice, where the resonon exists at $Q = (0, 0)$ and the now inappropriately named pîhns at $(4\pi/3, 0)$.

This similarity is efficiently encoded in the language of two-dimensional height models, analysed in detail by Henley. Both resonon and pîhn can be connected back to slow modes in the height model at zero wavevector, the connection to the dimer modes being via the gradient and vertex operators, respectively. Indeed from the height analysis one learns that the structure factor at $q_1 + k$ decays in imaginary time as $e^{-k\sqrt{\tau}}$ which confirms that there are gapless excitations near $q_1$, but also shows that the SMA does not do a very good job of constructing them.

2. Absence of gapless modes on the triangular lattice

The argument that the SMA mode for on the triangular lattice is gapped at all momenta, despite the presence of a conservation law, is completely analogous to the one presented above for the FCC lattice. It again involves showing that the corresponding structure factor vanishes at least quadratically. The relevant momentum for dimers pointing in the $\hat{x}$ direction is $Q = (0, \pi)$. A direct computation of the structure factor using the results of Ref. 33 yields the same result. This SMA result was already noted in Ref. 13.

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these issues further and set $t_0 > 0$ uniformly.

In the lattices discussed in this paper the two terms are equivalent. On more complicated lattices, such as the Fisher lattice discussed in [R. Moessner and S. L. Sondhi, cond-mat/0212363], confining phases do not break lattice symmetries and hence are indistinguishable from liquids from the viewpoint of translational order.

The definition provided here works for a system with even sidelengths. More generally, one has to resort to the slightly more involved transition graph construction described in Ref. 4.