TOWARD FREE RESOLUTIONS OVER SCROLLS

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ABSTRACT. Let $R = \mathbb{k}[x]/I$ where $I$ is the defining ideal of a rational normal $k$-scroll. We compute the Betti numbers of the ground field $\mathbb{k}$ as a module over $R$. For $k = 2$, we give the minimal free resolution of $\mathbb{k}$ over $R$.

1. INTRODUCTION

Free resolutions are a mainstay in commutative algebra, as they contain a wealth of information about the object resolved. A free resolution is an extended presentation: if a module is given by generators and relations, the resolution records also relations among the relations, relations among the relations of the relations, and so on. In the special case of modules over the polynomial ring over a field, Hilbert’s Syzygy Theorem guarantees that this process always terminates. Furthermore, there are algorithms to compute resolutions over polynomial rings, that are implemented in computer algebra systems. In some special cases where additional structure is present, such as for monomial ideals in polynomial rings, free resolutions can be given combinatorially. Free resolutions over polynomial rings have been the focus of intense study; over more general rings however, free resolutions are typically infinite, and are consequently harder to work with.

If $R$ is a standard graded $\mathbb{k}$-algebra, where $\mathbb{k}$ is a field, it is important to understand the resolution of $\mathbb{k}$ as an $R$-module. One reason is that, for any $R$-module $M$, the rank of the $i$th free module in a minimal free resolution of $M$, called its $i$th Betti number, equals $\dim_{\mathbb{k}} \text{Tor}^R_i(\mathbb{k}, M)$, which can be computed from a free resolution of $\mathbb{k}$. Such a ring $R$ is Koszul if $\mathbb{k}$ has a linear free resolution over $R$, that is, if the entries of the differentials in a resolution of $\mathbb{k}$ as an $R$-module are linear forms. The Koszul property has received much attention in combinatorial settings. An early result [5] states that if $R = \mathbb{k}[x_1, \ldots, x_n]/I$, where $I$ is generated by monomials of degree two, then $R$ is Koszul. By a degeneration argument, if $R = \mathbb{k}[x_1, \ldots, x_n]/J$ where $J$ has a quadratic initial ideal, then $R$ is Koszul. For semigroup rings, a characterization of the Koszul property is an open problem, see [8] for a survey of known results on resolutions over semigroup rings.

In many cases of rings that are known to be Koszul, the resolution of the residue field is not explicitly known. For semigroup rings, we are aware only of resolutions over the rings associated to rational normal curves [6]. In fact, [6] gives the minimal free resolution for any monomial ideal in this case.

In this article, we consider the next class of examples after rational normal curves, namely rational normal scrolls. We compute the Betti numbers of the residue field (Theorem 3.1), and for 2-scrolls, we give its minimal free resolution.

We illustrate our results in an example. Consider $R = \mathbb{k}[x_1, \ldots, x_6]/I$, where $I$ is the ideal of $2 \times 2$ minors of the matrix

$$\begin{bmatrix}
  x_1 & x_2 & x_4 & x_5 \\
  x_2 & x_3 & x_5 & x_6
\end{bmatrix}.$$

The ideal $I$ gives the defining equations of the rational normal scroll $S(2, 2)$. In this case, the minimal free resolution of $\mathbb{k}$ over $R$ is

$$\ldots \to R^{64} \xrightarrow{\partial_{63}} \ldots \xrightarrow{\partial_6} R^{64} \xrightarrow{\partial_5} R^{21} \xrightarrow{\partial_2} R^6 \xrightarrow{[x_1 x_2 \ldots x_6]} R \to \mathbb{k} \to 0$$

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The matrices giving the differentials $\partial_i$ are highly structured. Throughout this article, we adopt the following notations: $0^{p \times q}$ denotes a zero matrix of size $p \times q$; where it causes no confusion, zero blocks or entries of a matrix are indicated by $0$ or simply left empty; $\mathbb{1}_\ell$ is the $\ell \times \ell$ identity matrix; direct sum of matrices denotes concatenation of blocks along the main diagonal (with off-diagonal blocks equal to zero). With these conventions,

$$\partial_2 = \begin{bmatrix}
\varphi_0 & 0 & x_4 & x_4 & x_5 & x_6 \\
0 & \varphi_0 & -x_1 & -x_2 & -x_3 & -x_3
\end{bmatrix},$$

where $\varphi_0 = \begin{bmatrix} x_2 & x_3 & x_5 & x_6 \\ -x_1 & -x_2 & -x_4 & -x_5 \end{bmatrix}$;

$$\partial_3 = \begin{bmatrix}
\varphi_0^4 & x_4 \cdot \mathbb{1}_8 & -x_3 \cdot \mathbb{1}_8 \\
-x_0 & -x_0 & -x_0 & -x_0
\end{bmatrix},$$

where $\varphi_1 = \begin{bmatrix} x_2 & x_3 & x_5 & x_6 \\ -x_1 & -x_2 & -x_4 & -x_5 \\ 0 & 0 & 0 & -x_1 & -x_2 & -x_4 & -x_5 \\ 0 & 0 & 0 & 0 & -x_3 \end{bmatrix}$; and for $i \geq 4$,

$$\partial_i = \begin{bmatrix}
\varphi_0^4 & x_4 \cdot \mathbb{1}_{8,3^{i-3}} & -x_3 \cdot \mathbb{1}_{8,3^{i-3}} \\
-x_1 & -x_0 & -x_0 & -x_0
\end{bmatrix},$$

where

$$\varphi_2 = \begin{bmatrix}
\varphi_1 & x_1 & x_2 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & 0 \\
-x_2 & -x_3 & -x_5 & -x_6 & -x_2 & -x_3 & -x_5 & -x_6 \\
x_1 & x_2 & x_4 & x_5 & x_1 & x_2 & x_4 & x_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \in \mathbb{R}^{12 \times 36},$$

and $\varphi_i = \varphi_{i-1} \oplus \varphi_{i-2}^3 \oplus \varphi_{i-1}$ for $i \geq 3$.

**Outline.** This article is organized as follows. Section 2 contains necessary background. In Section 3 we compute the Betti numbers of rational normal $k$-scrolls. Section 4 is devoted to constructing the minimal resolution of $\mathbb{C}^2$ over a $2$-scroll.

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2. Preliminaries

We work in \( n = \sum_{i=1}^{k} m_i \) variables, and denote the polynomial ring by \( S = \mathbb{k}[x_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq m_i] \). The rational normal \( k \)-scroll \( S(m_1-1, m_2-1, \ldots, m_k-1) \) is the variety in \( \mathbb{P}^{n-1} \) defined by the ideal \( I_2(M) \) of \( 2 \times 2 \) minors of the \( 2 \times (n-k) \) matrix

\[
M = \begin{bmatrix}
  x_{1,1} & \cdots & x_{1,m_1-1} & x_{2,1} & \cdots & x_{2,m_2-1} & \cdots & x_{k,1} & \cdots & x_{k,m_k-1} \\
  x_{1,2} & \cdots & x_{1,m_1} & x_{2,2} & \cdots & x_{2,m_2} & \cdots & x_{k,2} & \cdots & x_{k,m_k}
\end{bmatrix}.
\]

Throughout this article, we often forego writing “rational normal” and call \( S(m_1-1, m_2-1, \ldots, m_k-1) \) a \( k \)-scroll and \( S(m-1, n-m-1) \) a scroll.

When \( k = 1 \), \( S(n-1) \) is a rational normal curve, that is, the variety defined by \( 2 \times 2 \) the minors of the matrix

\[
\begin{bmatrix}
  x_1 & x_2 & \cdots & x_{n-1} \\
  x_2 & x_3 & \cdots & x_n
\end{bmatrix}.
\]

2.1. Koszul algebras. Let \( A = \bigoplus_{i \geq 0} A_i \) be a standard graded \( \mathbb{k} \)-algebra, and let \( \beta_i^A(\mathbb{k}) \) be the \( i \)th Betti number of \( \mathbb{k} \) as an \( A \)-module. We consider the Poincaré series \( P_A(t) \) of \( A \), and its Hilbert series \( \text{Hilb}(A; t) \), defined as follows

\[
P_A(t) = \sum_{i \geq 0} \beta_i^A(\mathbb{k}) \cdot t^i \quad \text{and} \quad \text{Hilb}(A; t) = \sum_{i \geq 0} \dim_{\mathbb{k}} A_i \cdot t^i.
\]

When \( A \) is a Koszul ring, there is a strong relationship between these two series, that is useful later on. The following result can be taken as a definition.

**Theorem 2.1.** (cf. [4, Definition-Theorem 1]) A graded algebra \( A \) is Koszul if and only if the following equivalent conditions are satisfied:

1. the minimal graded \( A \)-resolution of \( \mathbb{k} \) is linear.
2. \( \text{Hilb}(A; -t) P_A(t) = 1 \).

As we mentioned in the introduction, the rings that are studied in this article are Koszul.

**Theorem 2.2.** For \( M \) as in \((1)\), \( R = S/I_2(M) \) is a Koszul ring.

**Proof.** By \([1\text{ Theorem 2.2}]\), a sufficient condition for a quotient \( \mathbb{k}[x_1, \ldots, x_n]/I \) to be Koszul is the existence of a homogeneous quadratic Gröbner basis for \( I \). It follows that \( R \) is Koszul, since the \( 2 \times 2 \) minors of \( M \) form a Gröbner basis for \( I_2(M) \) with respect to a reverse lexicographic ordering (see \([7\text{ Lemma 2.2}]\)).

2.2. Semigroup Rings. Let \( \mathcal{A} = \{ \gamma_1, \ldots, \gamma_n \} \subseteq \mathbb{N}^d \). We also use \( \mathcal{A} \) to denote the \( d \times n \) matrix with columns \( \gamma_1, \ldots, \gamma_n \). We assume that \( d \leq n \) and rank \( \mathcal{A} = d \). The configuration (or matrix) \( \mathcal{A} \) induces a map

\[
\mathbb{k}[x_1, \ldots, x_n] \rightarrow \mathbb{k}[t_1, \ldots, t_d]
\]

\[x_i \mapsto t_i^{\gamma_i} = t_1^{\gamma_{i,1}} \cdots t_d^{\gamma_{i,d}}.
\]

The kernel \( I_\mathcal{A} = \langle x^u - x^v \mid \mathcal{A}u = \mathcal{A}v \rangle \) of this map is a prime binomial ideal called the toric ideal associated to \( \mathcal{A} \). The semigroup ring associated to \( \mathcal{A} \) is

\[
\mathbb{k}[t^{\gamma_1}, t^{\gamma_2}, \ldots, t^{\gamma_n}] \cong \mathbb{k}[x_1, \ldots, x_n]/I_\mathcal{A}.
\]
By [10] Lemma 2.1, \( I_2(M) = I_A \), where \( A \) is the \((k + 1) \times n\) matrix
\[
A = \begin{bmatrix}
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & & & & & & & & & & & \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & m_1 - 1 & 0 & 1 & \cdots & m_2 - 1 & \cdots & 0 & 1 & \cdots & m_k - 1
\end{bmatrix}, \tag{2}
\]
so that \( R = S/I_2(M) \) is a semigroup ring.

3. Betti Numbers of \( \mathbb{k} \) over \( k \)-scrolls

Our first main theorem gives the Betti numbers of the field \( \mathbb{k} \) over \( R = S/I_2(M) \), where \( M \) is as in [1]

**Theorem 3.1.** Let \( I_2(M) \) define the rational normal \( k \)-scroll \( S(m_1 - 1, \ldots, m_k - 1) \). If \( R = S/I_2(M) \), then the \( i \)th Betti number of \( \mathbb{k} \) as an \( R \)-module is
\[
\beta^R_i(\mathbb{k}) = \sum_{j=0}^{i} \binom{k+1}{j} (n-k-1)^{i-j}.
\]
In particular \( \beta^R_{k+r}(\mathbb{k}) = (n-k-1)^{r-1}(n-k)^{k+1} \) for \( r \geq 0 \).

Because \( R \) is Koszul, Theorem 2.1 implies that we can obtain the Poincaré series of \( R \) by inverting its Hilbert series. Since Hilbert series are preserved under Gröbner degeneration, it is enough to compute the Hilbert series of \( S \) in \( \text{in}_<(I_2(M)) \) for \( < \) a monomial order in \( S \). This task is easiest if we are fortunate enough that our ideal has a squarefree initial ideal. The next result states that this is indeed the case for scrolls.

**Theorem 3.2.** Let \( < \) be the lexicographic monomial order on \( S \) given by \( x_{1,1} > x_{1,2} > \cdots > x_{1,m_1} > x_{2,1} > \cdots > x_{k,m_k} \), then
\[
\text{in}_<(I_2(M)) = \langle x_{i,j}x_{i,\ell} \mid |j - \ell| \geq 2 \rangle + \langle x_{i,j}x_{r,s} \mid 1 \leq i < r \leq k; 1 \leq j < m_i, 1 < s < m_r \rangle, \tag{3}
\]
that is, \( \text{in}_<(I_2(M)) \) is generated by the products of variables on the main diagonals of \( M \). In particular, \( \text{in}_<(I_2(M)) \) is a squarefree monomial ideal.

Denote by \( D \) the ideal on the right hand side of (3). To prove Theorem 3.2 we begin by pinpointing which monomials are not in \( D \).

**Lemma 3.3.** Suppose \( x^u \notin D \).

a) If there exists \( i \) such that \( x^u \) contains two variables with first index \( i \) with nonzero exponents, then \( u \) is of the form
\[
u = (0 \ldots 0 a_1|0 \ldots 0 a_2| \ldots |0 \ldots 0 a_{i-1}|0 \ldots 0 a_{i,\ell} a_{i,\ell+1} 0 \ldots 0|a_{i+1} 0 \ldots 0| \ldots |a_k 0 \ldots 0)
\]
b) Otherwise, \( u \) is of the form
\[
u = (0 \ldots 0 a_1|0 \ldots 0 a_2| \ldots |0 \ldots 0 a_{i-1}|a_i 0 \ldots 0|a_{i+1} 0 \ldots 0| \ldots |a_k 0 \ldots 0)
\]

**Proof.** The lemma follows from these observations.

i) If \( x^u \) contains the variables \( x_{i,j}, x_{i,\ell} \) with \( j < \ell \) both with nonzero exponent, then \( \ell = j + 1 \). Consequently, \( x^u \) cannot contain 3 variables from the same block with nonzero exponent.

ii) If \( x^u \) contains variables \( x_{i,j}, x_{r,s} \) with \( i < r \) and \( j < m_i \), both with nonzero exponent, then \( s = 1 \). \( \square \)
The following result is used to show that $D$ is equal to $\text{in}_{\prec} I_2(M)$.

**Proposition 3.4.** Let $A$ be as in (2) (so that $I_2(M) = I_A$). If $x^u \not\in D$, $x^u \succ x^v$, and $Au = Av$, then $u = v$.

**Proof.** In Lemma 3.3, case b) is a special case of a) where $\ell = 1$ and $a_{i,\ell+1} = 0$, so we may assume $u$ satisfies case a). We also assume $u \neq 0$, and write $v = (b_{1,1}, b_{1,2}, \ldots, b_{k,m_k})$.

Suppose $a_1 \neq 0$. Since $x^u \succ x^v$, the monomial $x^v$ cannot contain any variable greater than $x_{1,m_1}$. Then, as $Au = Av$, $x^u$ and $x^v$ must contain the same power of $x_{1,m_1}$. The same argument implies that $x^u$ and $x^v$ contain the same powers of all variables up to and including $x_{i-1,m_{i-1}}$.

Now again, since $x^u \succ x^v$ lexicographically, $a_{i,\ell} \geq b_{i,\ell}$ and $b_{i,0} = \ldots = b_{i,\ell-1} = 0$. As $Au = Av$, we have $a_{i,\ell} + a_{i,\ell+1} = b_{i,\ell} + b_{i,\ell+1} + \ldots + b_{i,m_i}$. But if $a_{i,\ell} > b_{i,\ell}$, then $(Au)_{k+1} > (Av)_{k+1}$. This implies that $a_{i,\ell} = b_{i,\ell}$, and similarly $a_{i,\ell+1} = b_{i,\ell+1}$, so $b_{i,\ell} = 0$ for $i \geq \ell + 2$.

To finish the proof, note that $(Au)_{k+1} = (m_1 - 1)a_1 + \ldots + (m_{i-1} - 1)a_{i-1} + (\ell - 1)a_{i,\ell} + \ell a_{i,\ell+1} = (m_1 - 1)b_{1,m_1} + \ldots + (m_{i-1} - 1)b_{i-1,m_{i-1}} + (\ell - 1)b_{i,\ell} + \ell b_{i,\ell+1}$. Because $Au = Av$, this implies that $b_{j,t} = 0$ for $j > i$ and $t > 1$. Again, using $Au = Av$, we conclude that $b_{j,1} = a_j$ for all $j > i$. □

We are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Since $I_2(M)$ is $A$-homogeneous, its initial ideal is generated by the initial forms of $A$-homogeneous elements of $I_2(M)$. If $P \in I_2(M)$ is $A$-homogeneous and $\text{in}_{\prec} P \not\in D$, then $P$ has one term by Proposition 3.4. But since $I_2(M)$ is a toric ideal, it contains no monomials, so that such a $P$ cannot belong to $I_2(M)$. We conclude that if $P$ is $A$-homogeneous and $P \in I_2(M)$, then $\text{in}_{\prec} P \in D$. □

With a squarefree initial ideal in hand, we now turn to Stanley–Reisner theory. Let $\Delta$ be the simplicial complex on the vertex set $\{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq m_i\}$ whose Stanley–Reisner ideal is $D = \text{in}_{\prec} I_2(M)$. By definition, this means that $D$ is generated by monomials whose index sets correspond to nonfaces of $\Delta$. It follows from Lemma 3.3 that $\Delta$ is the simplicial complex whose maximal faces are

\[
\{(1,m_1), (2,m_2), \ldots, (i,m_{i-1}), (i,j), (i+1,1), \ldots, (k,1)\} \text{ for } 1 \leq i \leq k, \ 1 \leq j \leq m_i-1,
\]

in particular, $\Delta$ is pure of dimension $k$. Figure 1 illustrates this simplicial complex in an example.

**Figure 1.** The simplicial complex $\Delta$ for $S(3, 2)$

It is well known that the Hilbert series of a Stanley–Reisner ring can be given in terms of the face numbers of the corresponding simplicial complex. Explicitly,

\[
\text{Hilb}(S/D; t) = \frac{1}{(1-t)^{k+1}} \sum_{d=0}^{k+1} f_d t^d (1-t)^{k+1-d},
\]

where $f_d$ is the number of $d$-dimensional faces of $\Delta$. We now compute these face numbers.

**Proposition 3.5.** If $\Delta$ is the simplicial complex whose Stanley–Reisner ideal is $D$, then $f_d = \binom{k}{d} n - d \binom{k+1}{d+1}$ for $d \geq -1$. In particular, the face numbers of $\Delta$ depend only on $k$ and $n$, and not on $m_1, \ldots, m_k$. 
Proof. We prove this by induction on $k$. Note that, by construction, $f_0 = n$, regardless of the value of $k$.

If $k = 1$, $\Delta$ has $f_1 = n - 1$ one-dimensional faces, namely $\{(1, i), (1, i + 1)\}$ for $i = 1, \ldots, n - 1$ (cf [9 Theorem 3.9]).

For the inductive step, let $\Delta$ be the complex associated to $S(m_1 - 1, \ldots, m_k - 1)$ and $\Delta'$ be the complex associated to $S(m_1 - 1, \ldots, m_{k+1} - 1)$. The complex $\Delta$ is naturally a subcomplex of $\Delta'$. We assume that $f_d(\Delta) = \binom{k}{d}(m_1 + \ldots + m_k) - d\binom{k+1}{d+1}$. Using the description of the facets of $\Delta$ from (4) (and the corresponding description for the facets of $\Delta'$) we see that the $d$-dimensional faces of $\Delta'$ are:

- $f_d(\Delta)$ $d$-dimensional faces of $\Delta$,
- $f_{d-1}(\Delta)$ faces of the form $\tau \cup \{(k+1, 1)\}$, where $\tau$ is a $(d-1)$-dimensional face of $\Delta$,
- $\binom{k}{d}(m_{k+1} - 1)$ faces with $d$ vertices from the set $\{(i, m_i) \mid 1 \leq i \leq k\}$ and one vertex from $\{(k+1, j) \mid 2 \leq j \leq m_{k+1}\}$, and
- $\binom{k}{d-1}(m_{k+1} - 1)$ faces with $d-1$ vertices from $\{(i, m_i) \mid 1 \leq i \leq k\}$ union an element of $\{\{(k+1, j), (k+1, j+1)\} \mid 1 \leq j \leq m_{k+1} - 1\}$.

Adding these together and applying the inductive hypothesis yields $f_d(\Delta') = \binom{k+1}{d}(m_1 + \ldots + m_{k+1}) - d\binom{k+2}{d+2}$, as we wanted. \hfill $\Box$

The following result gives the Hilbert series of $R$; the proof is a straightforward, if hefty, bullying of binomial coefficients.

**Proposition 3.6.** $\text{Hilb}(R; -t) = \text{Hilb}(S/D; -t) = \frac{1-(n-k-1)t}{(1+t)(k+1)}$ \hfill $\Box$

We are finally ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Since $R$ is a Koszul ring, it follows from Proposition 3.6 that the Poincaré series of $R$ is

$$P_R(t) = \frac{1}{\text{Hilb}(R; -t)} = \frac{(1+t)^{k+1}}{1-(n-k-1)t} = \sum_{i=0}^{\infty} \left[ \sum_{j=0}^{i} \binom{k+1}{j} (n-k-1)^{i-j} \right] t^i.$$  

For the last equality, we use $(1 + t)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} t^i$ and $\frac{1}{1-(n-k-1)t} = \sum_{i=0}^{\infty} (n-k-1)^i t^i$. We conclude that $R_k^i = \sum_{j=0}^{i} \binom{k+1}{j} (n-k-1)^{i-j}$. The special formula for $R_{k+1}(k)$ follows from the simplification of this sum when $\binom{k+1}{j}$ becomes 0. \hfill $\Box$

4. **The Resolution of $kR$ for $k = 2$**

One of the difficulties when dealing with infinite free resolutions and unbounded Betti numbers is to give an explicit presentation for the differentials. In the case $k = 2$, the combinatorics of the ring $R$ ensure a strong block structure that makes giving explicit matrices achievable.

**Notation.** In the case $k = 2$, we write $S(m - 1, n - m - 1)$ instead of $S(m_1 - 1, m_2 - 1)$, and forego double indexing to replace $x_{1,j}$ by $x_j$ and $x_{2,j}$ by $x_{m+j}$. Finally, we denote $p = n - m$.

With this new notation, the matrix (1) is replaced by the $2 \times (n-2)$ matrix

$$M = \begin{bmatrix} x_1 & x_2 & \ldots & x_{m-1} & x_{m+1} & x_{m+2} & \ldots & x_{n-1} \\ x_2 & x_3 & \ldots & x_m & x_{m+2} & x_{m+3} & \ldots & x_n \end{bmatrix}.$$
and the ideal $I_2(M)$ is the toric ideal $I_A$ associated to the $3 \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & \ldots & m-1 & 0 & 1 & \ldots & p-1 \end{bmatrix}.$$  

Our ultimate goal is to construct the minimal free resolution of $I_A$ over $R$ as an $R$-module. Our point of departure is the short exact sequence

$$0 \rightarrow \langle x_1, \ldots, x_m \rangle \cap \langle x_{m+1}, \ldots, x_n \rangle \rightarrow \langle x_1, \ldots, x_m \rangle \oplus \langle x_{m+1}, \ldots, x_n \rangle \rightarrow \langle x_1, \ldots, x_n \rangle \rightarrow 0.$$  

(5)

We construct free resolutions $(F_*(I_1), \partial_{I_1})$, $(F_*(I_2), \partial_{I_2})$, and $(F_*, \partial_{J,i})$ of the ideals $I_1 = \langle x_1, \ldots, x_m \rangle$, $I_2 = \langle x_{m+1}, \ldots, x_n \rangle$ and $J = I_1 \cap I_2$ respectively. We then combine these resolutions via mapping cone to make a resolution of $I = \langle x_1, \ldots, x_n \rangle$. Augmenting the resolution of $I$ to be a resolution of $\mathbb{k} = R/m$ results in a shift of one step, and minimality is assured by the previous Betti number computations. We obtain the resolution

$$F_* : \ldots \rightarrow \frac{R^{(n-2)(n-3)^2}}{\partial_{I_2}} \rightarrow \frac{R^{(n-2)(n-3)^3}}{\partial_{I_1}} \rightarrow \frac{R^{(n-3)^3}}{\partial_{J,i}} \rightarrow \frac{R^{n^2-3n+3}}{\partial_{J,i}} \rightarrow \frac{R^n}{\partial_{J,i}} \rightarrow \frac{R}{\partial_{J,i}} \rightarrow \mathbb{k} \rightarrow 0.$$  

4.1. The Differentials of $F_*$. Our first objective is to explicitly describe the differentials $\partial_i$ of $F_*$. These differentials are induced by a mapping cone. More precisely,

$$\partial_1 = [x_1 \ x_2 \ \cdots \ x_n], \quad \partial_2 = \begin{bmatrix} \frac{\partial_{I_1,1}}{\partial_{I_2,1}} & \alpha_0 \\ \partial_{J,i} & \alpha_{i+1} \end{bmatrix} \quad \text{for all } i \geq 2.$$  

The maps $\alpha$ are the chain maps from $F_*(J)$ to $F_*(I_1) \oplus F_*(I_2)$, which are:

$$\alpha_0 = \begin{bmatrix} x_{m+1} & 0 & \ldots & 0 & 0 \\ 0 & x_{m+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & x_{m+1} & x_{m+2} \\ -x_1 & -x_2 & \ldots & -x_m & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & -x_{m+1} & -x_{m+2} & x_{m+3} & \ldots & x_n \end{bmatrix} \in R^{n \times (n-1)}$$

$$\alpha_i = \begin{bmatrix} x_{m+1} & 1 & (m-1)(n-2)(n-3)^i-1 & 0 \\ 0 & -x_m & 1 & (p-1)(n-2)(n-3)^i-1 \\ 0 & 0 & \ldots & -x_{m+1} & -x_{m+2} & x_{m+3} & \ldots & x_n \end{bmatrix} \in R^{(n-2)(n-3)^i-1 \times (n-2)^2(n-3)^i-1}$$

The constituent resolutions $F_*(I_1)$, $F_*(I_2)$, and $F_*(J)$ have highly structured differentials, the building blocks of which are now given:

$$\varphi_0 = \begin{bmatrix} x_2 & x_3 & \ldots & x_m & x_{m+2} & \ldots & x_n \\ -x_1 & -x_2 & \ldots & -x_{m+1} & -x_{m+2} & \ldots & -x_{n-1} \end{bmatrix} \in R^{2 \times (n-2)}$$

$$\Phi_d = \begin{bmatrix} \varphi_0 & 0 \times (n-2) & \ldots & \ldots & 0 \times (n-2) \\ 0 \times (n-2) & \varphi_0 & 0 \times (n-2) & \ldots & 0 \times (n-2) \\ \vdots & \vdots & \varphi_0 & 0 \times (n-2) & \ldots & 0 \times (n-2) \\ 0 \times (n-2) & \ldots & \ldots & \varphi_0 & 0 \times (n-2) \end{bmatrix} \in R^{d \times (d-1)(n-2)}$$
Note that $\Phi_d$ is very sparse and consists of block components, but is not a block diagonal matrix. The structure of $\Phi_d$ is illustrated in Figure 2 with gray squares denoting non-zero entries, and empty squares denoting 0. These nonzero $\varphi_0$-blocks appear $d - 1$ times.

We denote by $u_i \in R^{(m-2)(n-2) \times (n-2)}$ and $v_i \in R^{(p-2)(n-2) \times (n-2)}$ the following matrices, which are almost entirely composed of zeros save for a single row that equals the first row or second row of $M$, respectively. More precisely,

$$u_i = \begin{bmatrix} x_1 & \cdots & x_{m-1} & x_{m+1} & \cdots & x_{n-1} \\ \text{0} & \cdots & \text{0} & \text{0} & \cdots & \text{0} \end{bmatrix}$$

and

$$v_i = \begin{bmatrix} -x_2 & \cdots & -x_{m} & -x_{m+2} & \cdots & -x_{n} \\ \text{0} & \cdots & \text{0} & \text{0} & \cdots & \text{0} \end{bmatrix}.$$

Despite the length of the exponents, these matrices are simple: $u_i$ is the $(m - 2)(n - 2) \times (n - 2)$ matrix with the top row of $M$ in the $(i(n - 2) + m)$-th row, and $v_i$ is the $(p - 2)(n - 2) \times (n - 2)$ matrix with the negative of the bottom row of $M$ in the $(i(n - 2) + m - 1)$-st row.

Finally, we introduce the following notation:

- $\varphi_1 = \begin{bmatrix} x_{m+1} & 0 & \cdots & 0 \\ 0 & x_{m+1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \text{0} & \cdots & \text{0} & x_m \\ -x_1 & -x_2 & \cdots & -x_{m-1} & -x_{m} & \text{0} & \cdots & \text{0} \\ \text{0} & -x_{m} & \cdots & \text{0} \\ \text{0} & \cdots & \text{0} & -x_{m} \\ \oplus \varphi_{1-1} & u_0 & u_1 & \cdots & u_{m-3} & \text{0} & \text{0} & \text{0} \end{bmatrix} \in R^{(n-2)(n-2)(n-3)}$

- $\varphi_2 = \begin{bmatrix} \varphi_{2-1} & u_0 & u_1 & \cdots & u_{m-3} & \text{0} & \text{0} & \text{0} \\ \text{0} & \text{0} & -\Phi_{n-2} & v_0 & v_1 & \cdots & v_{p-3} \end{bmatrix} \oplus \varphi_{1-1} \oplus \varphi_{2-2} \oplus \varphi_{(p-2)} \in R^{(n-2)(n-3)^{i-1} \times (n-2)(n-3)^i} \quad \text{for } i \geq 3$

The presentation for $\varphi_2$ is perhaps deceiving; the brunt of the matrix is a direct sum of $\varphi_1$’s. It is only (most of) the middle $(n - 3)(n - 3)$ columns that have additional entries above or below the middle $-\Phi_{n-2}$.

Using these $\varphi$’s, we construct resolutions of $J$, $I_1$ and $I_2$. The ideal $J$ has resolution over $R$ as shown below:

$$\mathcal{F}_*(J) : \cdots \xrightarrow{\partial_{J,4}} R^{(n-2)^2(n-3)^2} \xrightarrow{\partial_{J,3}} R^{(n-2)^2(n-3)} \xrightarrow{\partial_{J,2}} R^{(n-2)^2} \xrightarrow{\partial_{J,1}} R^{n-1} \xrightarrow{\partial_{J,0}} J \to 0$$
We start by checking that we are working with complexes. Indeed the minimal free resolution of $\mathbb{R}/CZ$ is

$$\partial_{J,0} = \begin{bmatrix} x_1 x_{m+1} & x_2 x_{m+1} & \cdots & x_m x_{m+1} & x_m x_{m+2} & x_m x_{m+3} & \cdots & x_m x_n \end{bmatrix} \in R^{1 \times (n-1)}$$

$$\partial_{J,1} = \Phi_{n-1} \in R^{(n-1) \times (n-2)^2}$$

$$\partial_{J,i} = \varphi_{i-1}^{(n-2)} \in R^{(n-2)^2(n-3)^{i-2} \times (n-2)(n-3)^{i-1}} \text{ for } i \geq 2$$

The ideal $I_1$ has resolution over $R$ as shown below:

$$\mathcal{F}_\bullet(I_1) : \cdots \xrightarrow{\partial_{1,4}} R^{(m-1)(n-2)(n-3)^2} \xrightarrow{\partial_{1,3}} R^{(m-1)(n-2)(n-3)} \xrightarrow{\partial_{1,2}} R^{(m-1)(n-2)} \xrightarrow{\partial_{1,1}} R^m \xrightarrow{\partial_{1,0}} I_1 \to 0$$

where

$$\partial_{1,0} = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \in R^{1 \times m}; \quad \partial_{1,1} = \Phi_m \in R^{m \times (m-1)(n-2)}$$

$$\partial_{1,i} = \varphi_{i-1}^{(m-1)} \in R^{(m-1)(n-2)(n-3)^{i-2} \times (m-1)(n-2)(n-3)^{i-1}} \text{ for } i \geq 2$$

The ideal $I_2$ has resolution over $R$ as shown below:

$$\mathcal{F}_\bullet(I_2) : \cdots \xrightarrow{\partial_{2,4}} R^{(p-1)(n-2)(n-3)^2} \xrightarrow{\partial_{2,3}} R^{(p-1)(n-2)(n-3)} \xrightarrow{\partial_{2,2}} R^{(p-1)(n-2)} \xrightarrow{\partial_{2,1}} R^p \xrightarrow{\partial_{2,0}} I_2 \to 0$$

where

$$\partial_{2,0} = \begin{bmatrix} x_{m+1} & x_{m+2} & \cdots & x_n \end{bmatrix} \in R^{1 \times p}; \quad \partial_{2,1} = \Phi_p \in R^{p \times (p-1)(n-2)}$$

$$\partial_{2,i} = \varphi_{i-1}^{(p-1)} \in R^{(p-1)(n-2)(n-3)^{i-2} \times (p-1)(n-2)(n-3)^{i-1}} \text{ for } i \geq 2$$

Our main result is as follows.

**Theorem 4.1.** $\mathcal{F}_\bullet$ constructed above is the minimal free resolution of $\mathbb{k}$ over $R$.

### 4.2. Outline of the proof of Theorem 4.1

Most of the work goes to showing that $\mathcal{F}_\bullet(I_1)$, $\mathcal{F}_\bullet(I_2)$ and $\mathcal{F}_\bullet(J)$ are free resolutions of $I_1$, $I_2$ and $J$ respectively. The matrices considered in these three cases have very similar structure, and the details in proving exactness are virtually identical. Thus, we give only the proof that $\mathcal{F}_\bullet(J)$ is a resolution. Exactness of $\mathcal{F}_\bullet(J)$ is shown in Subsection 4.3 using ideas from 2.

What remains is to provide the map of complexes $\alpha : \mathcal{F}_\bullet(J) \to \mathcal{F}_\bullet(I_1) \oplus \mathcal{F}_\bullet(I_2)$ lifting the inclusion $J \to I_1 \oplus I_2$ from the short exact sequence 5. This is done in Subsection 4.4.

Once $\alpha$ is constructed, the mapping cone procedure ensures that $\mathcal{F}_\bullet$ is exact, and thus a free resolution of $\mathbb{k}$. That it is the minimal free resolution of $\mathbb{k}$ follows by inspection, or by Theorem 3.1.

### 4.3. $\mathcal{F}_\bullet(J)$ is exact

We need generators for $J = \langle x_1, \ldots, x_m \rangle \cap \langle x_{m+1}, \ldots, x_n \rangle$. Clearly,

$$J = \left\langle x_1 x_{m+1}, x_2 x_{m+1}, \ldots, x_m x_{m+1}, x_1 x_{m+2}, \ldots, x_2 x_{m+2}, \ldots, x_m x_{m+2}, \ldots, x_1 x_n, \ldots, x_2 x_n, \ldots, x_m x_n \right\rangle$$

However, many of these monomials are equal in $R$; in fact, $x_i x_j = x_k x_\ell$ if $i + j = k + \ell$, as long as $j \neq m$ and $k \neq m + 1$. This means that, in the above arrangement, all monomials on the same skew-diagonal are the same, for example, $x_3 x_{m+1} = x_2 x_{m+2} = x_1 x_{m+3}$. Consequently,

$$J = \langle x_1 x_{m+1}, x_2 x_{m+1}, \ldots, x_m x_{m+1}, x_m x_{m+2}, x_m x_{m+3}, \ldots, x_m x_n \rangle$$

We start by checking that we are working with complexes.
Proposition 4.2. \( \mathcal{F}_{\bullet}(J), \mathcal{F}_{\bullet}(I_1) \) and \( \mathcal{F}_{\bullet}(I_2) \) are complexes.

Proof. This is a straightforward, if tedious, calculation. A key observation is that \( \varphi_i \circ \varphi_{i+1} = 0 \) for all \( i \). This follows, as each of these compositions has entries that are either 0 or binomials in \( I_2(M) \). Given the direct sum structure of the differentials, this is enough to show our proposed differentials compose to zero.

Our next goal is to show that \( \mathcal{F}_{\bullet}(J) \) is exact. (The same argument, with minor modifications, shows the same for \( \mathcal{F}_{\bullet}(I_1) \) and \( \mathcal{F}_{\bullet}(I_2) \).) We need some notation.

Definition 4.3. Let \( A \) be a noetherian commutative ring. Let \( f : F \to G \) be a map of free \( A \)-modules, which is represented by a matrix with entries in \( A \). The rank of \( f \) is the size of the largest nonvanishing minor of this matrix. If \( f \) has rank \( r \), we use \( I(f) \) to denote the ideal generated by the \( r \times r \) minors of (the matrix representing) \( f \).

The following results are used to prove exactness.

Lemma 4.4. [3] Lemma 20.10] Let \( A \) be a commutative noetherian ring. A complex \( F \xrightarrow{\varphi} G \xrightarrow{\psi} H \) of free \( A \)-modules with \( I(f) = I(g) = A \) is exact iff \( \text{rank } f + \text{rank } g = \text{rank } G \).

Lemma 4.5 (Sylvester’s Rank Inequality). If \( U \) and \( V \) are matrices with entries in a field, where \( U \) is \( r \times s \) and \( V \) is \( s \times t \), then

\[
\text{rank } U + \text{rank } V - s \leq \text{rank } UV.
\]

By Lemma 4.4, it is important to know the ranks of the differentials of \( \mathcal{F}_{\bullet}(J) \). Due to the block structure, we must first address the matrices \( \varphi_i \).

Proposition 4.6. The rank of \( \Phi_d \) is \( d - 1 \) for all \( d \geq 2 \) and the rank \( \varphi_i \) is \( (n - 3)^i \) for all \( i \geq 0 \).

Proof. Because \( R \) is a domain, dependences among rows of a matrix over \( R \) can be read off from the vanishing of minors. In fact, the rank of a matrix over \( R \) equals the rank of that matrix over the field of fractions of \( R \). In this proof, we work over the field of fractions of \( R \), which gives us access to Lemma 4.5.

It is clear that \( \text{rank } \varphi_0 = 1 \), as all \( 2 \times 2 \) minors of \( \varphi_0 \) are exactly the same as the minors of \( M \), which belong to \( I_2(M) \).

Next we must show that \( \text{rank } \varphi_1 = n - 3 \). We know that the rank of \( \varphi_1 \) is at least \( n - 3 \), as the minor of size \( n - 3 \) corresponding to rows 2, 3, \ldots, \( n - 2 \) and columns 1, 1 + (\( n - 2 \), 1 + 2(\( n - 2 \)), \ldots, 1 + (n - 4)(n - 2) equals \( -1^{n-3}x_1 \neq 0 \). On the other hand, by Lemma 4.5, \( \text{rank } \varphi_0 + \text{rank } \varphi_1 - (n - 2) \leq \text{rank } (\varphi_0 \circ \varphi_1) = 0 \), so \( \text{rank } \varphi_1 \leq n - 3 \). Consequently \( \text{rank } \varphi_1 = n - 3 \).

To compute \( \text{rank } \varphi_2 \), we consider the minor of size \( (n - 3)^2 \) corresponding to rows \( \{ s + t(n - 2) \mid 2 \leq s \leq n - 2, 0 \leq t \leq n - 4 \} \) and columns 1, (\( n - 2 \) + 1, 2(\( n - 2 \) + 1), \ldots, ((n - 3)^2 - 1)(n - 2) + 1 which equals \( x_1^{(n-3)^2} \neq 0 \), so that \( \text{rank } \varphi_2 \geq (n - 3)^2 \). Again by Lemma 4.5 and because \( \varphi_1 \circ \varphi_2 = 0 \), we know that \( \text{rank } \varphi_2 \leq (n - 2)(n - 3) - \text{rank } \varphi_1 = (n - 2)(n - 3) - (n - 3) = (n - 3)^2 \). We conclude that \( \text{rank } \varphi_2 = (n - 3)^2 \).

The rank computations for the remaining maps \( \varphi_i \) follow easily from the block structure: \( \text{rank } \varphi_i = (m - 2)(n - 3)^{i-1} + (n - 3)(n - 3)^{i-2} + (p - 2)(n - 3)^{i-1} = (n - 3)^i \) for any \( i \geq 2 \).

In the case of \( \Phi_d \), we consider the minor corresponding to rows 2, 3, \ldots, \( d \) and columns 1, 1 + (\( n - 2 \), 1 + 2(\( n - 2 \)), \ldots, 1 + (d - 2)(n - 2) which equals \( -1^{d-1}x_1^{d-1} \), so that \( \text{rank } \Phi_d \geq d - 1 \). On the other hand,
because \( \varphi_0 \circ \varphi_1 = 0, \Phi_d \circ \bigoplus_{d-1} \varphi_1 = 0 \). Therefore rank \( \Phi_d \leq dp - 1)(n - 2) - (d - 1) \) rank \( \varphi_1 = d - 1 \), and in fact rank \( \Phi_d = d - 1 \).

The ranks of the differentials of \( F_\bullet(J) \) can be computed directly from Proposition 4.6.

**Corollary 4.7.** The ranks of the \( \partial_{J,i} \) are:

1. \( \text{rank } \partial_{J,1} = n - 2 \)
2. \( \text{rank } \partial_{J,2} = (n - 2)(n - 3) \)
3. \( \text{rank } \partial_{J,i} = (n - 2)(n - 3)^{i-1} \) for all \( i \geq 2 \)

In order to apply Lemma 4.4, we need more information regarding the ideals of maximal nonvanishing minors of the matrices involved.

**Proposition 4.8.** For \( i \in [n] \), we have \( x_i^{p-1} \in I(\Phi_p) \) and \( x_i^{(n-3)^j} \in I(\varphi_j) \) for all \( j \geq 0 \).

**Proof.** Because we are considering the ideals generated by the minors, we can ignore signs in our computations. It is clear that \( x_i \in I(\varphi_0) \) for all \( i \in [n] \). To see that any \( x_i^{n-3} \in I(\varphi_1) \) for any \( i \in [n] \), we can consider the minors corresponding to the rows \( r_{i,1} \) and columns \( c_{i,1} \) listed below.

| \( x_i^{n-3} \in I(\varphi_1) \) | rows \( r_{i,1} \) | columns \( c_{i,1} \) |
|---|---|---|
| \( 1 \leq i \leq m - 1 \) | \( 2, 3, \ldots, n - 2 \) | \( i + j(n - 2) \) for \( 0 \leq j \leq n - 4 \) |
| \( i = m \) | \( 1, 2, \ldots, m - 2, m, \ldots, n - 2 \) | \( (m - 1) + j(n - 2) \) for \( 0 \leq j \leq m - 3 \) |
| \( i = m + 1 \) | \( 1, 2, \ldots, m - 1, m + 1, \ldots, n - 2 \) | \( \ell + (m - 2)(n - 2) \) for \( m \leq \ell \leq n - 2 \) |
| \( m + 2 \leq i \leq n \) | \( 1, 2, \ldots, n - 3 \) | \( i - 2 + j(n - 2) \) for \( 0 \leq j \leq n - 4 \) |

The proposed submatrices of \( \varphi_1 \) whose rows and columns listed above are strictly triangular, so the minors are easily computed.

We can make a similar table with recipes for the appropriate minors in \( \varphi_2 \), given below.

| \( x_i^{(n-3)^2} \in I(\varphi_2) \) | rows \( r_{i,2} \) | columns \( c_{i,2} \) |
|---|---|---|
| \( 1 \leq i \leq m - 1 \) | \( s + t(n - 2) \) for \( s \in r_{i,1} \) and \( 0 \leq t \leq n - 4 \) | \( i + j(n - 2) \) for \( 0 \leq j \leq (n - 4)(n - 2) \) |
| \( i = m \) | \( s + t(n - 2) \) for \( s \in r_{i,1} \) and \( 0 \leq t \leq n - 4 \) \( \setminus (m - 1)(n - 2) \setminus (m - 2)(n - 2) + 1 \) | \( a + j(n - 2)(n - 3) \) for \( a \in c_{i,1} \) and \( 0 \leq j \leq (n - 4) \) but \( j \neq m - 2 \) |
| \( i = m + 1 \) | \( s + t(n - 2) \) for \( s \in r_{i,1} \) and \( 0 \leq t \leq n - 4 \) \( \setminus (m - 2)(n - 2) + 1 \setminus (m - 2)(n - 2) \setminus (m - 2)(n - 2) + 1 \) | \( a + j(n - 2)(n - 3) \) for \( a \in c_{i,1} \) and \( 0 \leq j \leq (n - 4) \) but \( j \neq m - 2 \) |
| \( m + 2 \leq i \leq n \) | \( s + t(n - 2) \) for \( s \in r_{i,1} \) and \( 0 \leq t \leq n - 4 \) | \( (i - 2) + j(n - 2) \) for \( 0 \leq j \leq (n - 4)(n - 2) \) |

The block structures of the successive \( \varphi_j \)'s combined with the previous two statements is enough to see that \( x_i^{(n-3)^j} \in I(\varphi_j) \).

Finally, use the minors whose columns and rows are given below to obtain \( x_i^{d-1} \in I(\Phi_d) \).
$x_i^{d-1} \in I(\Phi_d)$

| rows | columns |
|------|---------|
| $1 \leq i \leq m - 1$ | $i + j(n - 2)$ for $0 \leq j \leq d - 2$ |
| $i = m$ | $m - 1 + j(n - 2)$ for $0 \leq j \leq d - 2$ |
| $i = m + 1$ | $m + j(n - 2)$ for $0 \leq j \leq d - 2$ |
| $m + 2 \leq i \leq n$ | $i - 2 + j(n - 2)$ for $0 \leq j \leq d - 2$ |

We are now ready to give the main result in this subsection.

**Theorem 4.9.** The complexes $\mathcal{F}_*(J)$, $\mathcal{F}_*(I_1)$ and $\mathcal{F}_*(I_2)$ are exact.

**Proof.** We only provide details for $\mathcal{F}_*(J)$. We show that we have exactness after localizing at any prime ideal of $R$, from which exactness over $R$ follows. If $q$ is any prime ideal in $R$, we denote by $\partial_{J,i,q}$ the localized map induced by $\partial_{J,i}$ over $R_q$. The (unique) graded maximal ideal of $R$ is $m = \langle x_1, \ldots, x_n \rangle$.

Corollary 4.7 provides the ranks of the maps $\partial_{J,i}$ over $R$. Because $R$ is a domain, $I(\partial_{J,i})$ contains exclusively non-zero divisors for all $i$. This means that, when localizing, the rank of $\partial_{J,i}$ does not change. Furthermore, localization at any prime ideal $q \neq m$ yields $I(\partial_{J,i,q}) = R_q$, because each $I(\partial_{J,i})$ contains some power of every $x_i$, by Proposition 4.8. By Lemma 4.4, this proves that $\mathcal{F}_*(J)$ is exact after localization at any prime ideal $q \neq m$.

We conclude that, if $\mathcal{F}_*(J)$ has a nonzero homology module, it is only supported at the graded maximal ideal $m$, and therefore has depth 0. Our goal now is to derive a contradiction.

We localize at $m$, and use $F_i$ to denote the free $R_m$-modules appearing in the localization of $\mathcal{F}_*(J)$. Use $B_i \subseteq C_i \subseteq F_i$ to denote the $i$-cycles and $i$-boundaries, and $H_i = C_i/B_i$. The ring $R$ is a semigroup ring corresponding to a saturated (normal) semigroup, and is therefore Cohen–Macaulay by Hochster’s theorem. Since $\dim R = 3$, it follows that $R_m$ has depth 3. Consequently all the free modules over $R_m$ also have depth 3, in particular the $F_i$. Any submodules of the free modules $F_i$ must have depth at least 3, so we have depth$(C_i) \geq 3$ and depth$(B_i) \geq 3$. From the exact sequence

$$0 \to B_i \to C_i \to H_i \to 0$$

it follows that depth$(H_i) \geq \min \{\text{depth}(C_i), \text{depth}(B_i) - 1\}$ (see [3] Corollary 18.6.a]), so that depth$(H_i) \geq 2$. This contradicts that depth$(H_i) = 0$.

Therefore localizations of $\mathcal{F}_*(J)$ at all prime ideals (now including $m$) are exact, and consequently $\mathcal{F}_*$ is exact.

**4.4. The Mapping Cone.** We recall that we have an exact sequence

$$0 \to J \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} I_1 \oplus I_2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} m \to 0.$$

The relevant result for us is that, if we have resolutions of $J$ and $I_1 \oplus I_2$, the inclusion $J \to I_1 \oplus I_2$ can be lifted to a map of complexes between the corresponding resolutions, and the associated mapping cone is a resolution of $m$. The definition of the mapping cone of a map of complexes is given below; we refer to the appendix of [3] for more information.

**Definition 4.10.** If $\alpha : F \to G$ is a map of complexes, and we write $f_i, g_i$ for the differentials of $F$ and $G$ respectively, then the mapping cone $M(\alpha)$ of $\alpha$ is the complex such that $M(\alpha)_{i+1} = F_i \oplus G_{i+1}$ where the
We first check that the differential $\partial_{i+1}$ is shown:

$$
\begin{align*}
F_i & \xrightarrow{-f_i} F_{i-1} \\
\oplus & \xrightarrow{\alpha_i} \oplus \\
G_{i+1} & \xrightarrow{g_{i+1}} G_i
\end{align*}
$$

that is, $\partial_{i+1}(a, b) = (-f_i(a), g_{i+1}(b) + \alpha_i(a))$.

We now construct the map of complexes that lifts the inclusion $J \to I_1 \oplus I_2$.

**Proposition 4.11.** The map of complexes $\alpha : \mathcal{F}_* (J) \to \mathcal{F}_* (I_1) \oplus \mathcal{F}_* (I_2)$ is given by

$$
\alpha_0 = \begin{bmatrix}
x_{m+1} & 0 & \ldots & 0 \\
0 & x_{m+1} & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ldots & x_{m+1}
\end{bmatrix}
$$

for $i \geq 1$.

**Proof.** We first check that $\left[ \begin{array}{c} 1 \\ -1 \end{array} \right] \partial_{J,0} = (\partial_{I_{1,0}} \oplus \partial_{I_{2,0}}) \alpha_0$. We compute both sides explicitly:

$$
\left[ \begin{array}{c} 1 \\ -1 \end{array} \right] \partial_{J,0} = \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] \begin{bmatrix}
x_1x_{m+1} & x_2x_{m+1} & \ldots & x_1 x_{m+1} \\
x_1 x_{m+1} & x_2 x_{m+1} & \ldots & x_2 x_{m+1} \\
\ddots & \ddots & \ddots & \ddots \\
x_1 x_{m+1} & x_2 x_{m+1} & \ldots & x_{m+1} x_{m+1}
\end{bmatrix}
$$

and

$$(\partial_{I_{1,0}} \oplus \partial_{I_{2,0}}) \alpha_0 =
$$

$$
\begin{bmatrix}
x_{m+1} & 0 & \ldots & 0 \\
0 & x_{m+1} & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ldots & x_{m+1} \\
\end{bmatrix}
$$
Now we can check if $\alpha_0 \partial_{f,1} = (\partial_{I_1,1} \oplus \partial_{I_2,1}) \alpha_1$. Without further ado:

$$\alpha_0 \partial_{f,1} = \begin{bmatrix}
    x_{m+1} & 0 & \ldots & 0 \\
    0 & x_{m+1} & \ldots & 0 \\
    0 & \ddots & \ddots & 0 \\
    0 & 0 & \ldots & x_{m+1} \\
    -x_1 & -x_2 & \ldots & -x_m \\
    0 & 0 & \ldots & 0 \\
    -x_m & 0 & \ldots & 0 \\
    0 & 0 & \ddots & 0 \\
    0 & 0 & \ldots & -x_m \\
\end{bmatrix} \Phi_{n-1}$$

where $x_{m+1} \varphi_0$ appears $m - 1$ times and $-x_m \varphi_0$ appears $p - 1$ times. However, because of the diagonal structure of $\alpha_1$, this is clearly the same as $(\Phi_m \oplus \Phi_p) \alpha_1 = (\partial_{I_1,1} \oplus \partial_{I_2,1}) \alpha_1.$

For remaining $i \geq 2$, $\partial_{f,i} = \partial_{I_1,i} \oplus \partial_{I_2,i}$, and all the $\alpha_i$ are diagonal matrices, so the products are easily verified to be equal. \qed

**Proof of Theorem 4.1** Since $F_\bullet(J)$ is a resolution of $J$ and $F_\bullet(I_1) \oplus F_\bullet(I_2)$ resolves $I_1 \oplus I_2$, the mapping cone of $\alpha : F_\bullet(J) \to F_\bullet(I_1) \oplus F_\bullet(I_2)$ is a resolution of $m$. Augmenting the resolution to be a resolution of $k = R/m$ results in a shift of one step, and so we finally have the resolution $F_\bullet$. Comparing the rank of the free modules in each step to the Betti numbers computed in Theorem 3.1 we conclude that $F_\bullet$ is not only exact, but minimal. \qed

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