On algebraic fiber spaces

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An algebraic fiber space is a relative version of an algebraic variety. We prove some basic topological and analytical properties of algebraic fiber spaces. We start with reviewing a result by Abramovich and Karu on the standard toroidal models of algebraic fiber spaces in §1. We can eliminate the singularities of the fibers in a topological sense by performing the real oriented blowing-up on the toroidal model (§2). The rest of the paper concerns the Hodge theory of the degenerate fibers. In §3, we define the weight filtration on the cohomology with coefficients in \( \mathbb{Z} \). We prove the logarithmic and relative version of the Poincaré lemma in §4. Finally we prove that the logarithmic de Rham complex gives rise to a cohomological mixed Hodge complex in §5 (Theorem 5.2). As a corollary, we prove that certain spectral sequences degenerate.

We consider only varieties and morphisms defined over \( \mathbb{C} \). The topology is the classical (or Euclidean) topology instead of the Zariski topology unless stated otherwise.

1 Weak semistable model

An algebraic variety in this paper is a reduced and irreducible scheme of finite type over Spec \( \mathbb{C} \). An algebraic fiber space is a relative version of an algebraic variety; it is a morphism \( f : X \to Y \) of algebraic varieties which is generically surjective and such that the geometric generic fiber is reduced and irreducible. We look for a standard model of an algebraic fiber space in the category of toroidal varieties.
Definition 1.1. A toric variety \((V, D)\) is a pair consisting of a normal algebraic variety and a Zariski closed subset such that an algebraic torus \(T \cong (\mathbb{C}^*)^n\) acts on \(V\) with an open orbit \(V \setminus D\).

A toroidal variety is a pair \((X, B)\) consisting of an algebraic variety and a Zariski closed subset which is locally analytically isomorphic to toric varieties in the following sense: for each point \(x \in X\), there exists a toric variety \((V_x, D_x)\) with a fixed point \(x'\), called a local model at \(x\), and open neighborhoods \(U_x\) and \(U'_x\) of \(x \in X\) and \(x' \in V_x\) in the classical topology such that \((U_x, B \cap U_x)\) is isomorphic to \((U'_x, D_x \cap U'_x)\).

A toroidal variety \((X, B)\) is called strict if any irreducible component of \(B\) is normal.

A strict toroidal variety \((X, B)\) is called a smooth toroidal variety if \(X\) is smooth and \(B\) is a simple normal crossing divisor. Namely, a local model of a smooth toroidal variety has the form \((\mathbb{C}^n, \text{div}(x_1 \ldots x_n'))\), where \((x_1, \ldots, x_n)\) are the coordinates.

A strict toroidal variety \((X, B)\) is called a quasi-smooth toroidal variety if its local model is a quotient of a smooth local model by a finite abelian group action which is fixed point free on \(X \setminus B\). Namely, a local model has the form \((\mathbb{C}^n/G, \text{div}(x_1 \ldots x_n')/G)\), where \(G\) is a finite abelian subgroup of \(GL(n, \mathbb{C})\) which acts on \(\mathbb{C}^n\) diagonally on the coordinates \((x_1, \ldots, x_n')\).

Let \((V, D)\) and \((W, E)\) be toric varieties with the actions of algebraic tori \(T\) and \(S\). A toric morphism \(g : (V, D) \to (W, E)\) is a morphism \(g : V \to W\) of algebraic varieties which is compatible with a homomorphism \(g_0 : T \to S\) of algebraic groups.

Let \((X, B)\) and \((Y, C)\) be toroidal varieties. A toroidal morphism \(f : (X, B) \to (Y, C)\) is a morphism \(f : X \to Y\) of algebraic varieties such that \(f(X \setminus B) \subset Y \setminus C\) and that for any point \(x \in X\) and any local model \((W_y, E_y)\) at \(y = f(x) \in Y\), there exists a local model \((V_x, D_x)\) at \(x\) and a toric morphism \(g : (V_x, D_x) \to (W_y, E_y)\) which is locally analytically isomorphic to \(f\).

The resolution theorem of singularities by Hironaka implies that there is always a smooth birational model for any algebraic variety; for any complete algebraic variety \(X\), there exists a birational morphism \(\mu : Y \to X\) from a smooth projective variety.

As for the relative version of the resolution theorem, one cannot expect that we have a birational model which is a smooth morphism. Indeed, we cannot eliminate singular fibers. Instead of the smooth model, Abramovich
and Karu obtained a toroidal model by using the method of de Jong on the moduli space of stable curves.

**Theorem 1.2 (Abramovich-Karu [1]).** Let \( f_0 : X_0 \to Y_0 \) be a surjective morphism of complete algebraic varieties whose geometric generic fiber is irreducible. Let \( B_0 \) and \( C_0 \) be Zariski closed subsets of \( X_0 \) and \( Y_0 \) such that \( f_0(X_0 \setminus B_0) \subset Y_0 \setminus C_0 \). Then there exist a quasi-smooth projective toroidal variety \((X, B)\), a smooth projective toroidal variety \((Y, C)\), birational morphisms \( \mu : X \to X_0 \) and \( \nu : Y \to Y_0 \), and a toroidal and equi-dimensional morphism \( f : (X, B) \to (Y, C) \) such that \( \mu(X \setminus B) \subset X_0 \setminus B_0 \), \( \nu(Y \setminus C) \subset Y_0 \setminus C_0 \) and \( \nu \circ f = f_0 \circ \mu \).

Let \( f : (X, B) \to (Y, C) \) be a toroidal and equi-dimensional morphism of quasi-smooth toroidal varieties. We can describe \( f \) explicitly by using local coordinates as follows. Let us fix \( x \in X \) and \( y = f(x) \in Y \). Let \( n = \dim X \) and \( m = \dim Y \). We have local models of \( X \) and \( Y \): there are integers \( 0 \leq n' \leq n \) and \( 0 \leq m' \leq m \), finite abelian groups \( G \) and \( H \) which act diagonally on the first \( n' \) and \( m' \) coordinates of the polydisks \( \Delta^n = \{ (x_1, \ldots, x_n) | |x_i| < 1 \} \) and \( \Delta^m = \{ (y_1, \ldots, y_m) | |y_j| < 1 \} \), and open neighborhoods \( U \) and \( V \) of \( x \in X \) and \( y \in Y \) in the classical topology such that \( (U, B \cap U) \cong (\Delta^n/G, \text{div}(x_1 \ldots x_{n'})/G) \) and \( (V, C \cap V) \cong (\Delta^m/H, \text{div}(y_1 \ldots y_{m'})/H) \). We may assume that the fixed locus of each element of \( G \) and \( H \) except the identities has codimension at least 2. Then the morphism \( f \) induces a morphism \( \tilde{f} : \Delta^n \to \Delta^m \). The fact that \( f \) is toroidal and equi-dimensional means the following: there are integers \( 0 = t_0 < t_1 < t_2 < \cdots < t_{m'} \leq n' \) and \( 1 \leq l_i \ (i = 1, \ldots, t_{m'}) \) such that

\[
\tilde{f}^*(y_j) = \prod_{k=1}^{l_j-t_{j-1}} x_{t_{j-1}+k}^{t_{j}-t_{j-1}}
\]

for \( j = 1, \ldots, m' \) and \( \tilde{f}^*(y_j) = x_{n'-m'+j} \) for \( j = m'+1, \ldots, m \). Indeed, since \( f \) is toroidal, it is expressed by monomials by some local coordinates. The equi-dimensionality implies that the sets of indices \( i \) of the \( x_i \) on the right hand side for different \( j \)'s are disjoint.

By using [5], Abramovich and Karu obtained a model with reduced fibers:

**Corollary 1.3.** Let \((X, B)\) be a quasi-smooth projective toroidal variety, \((Y, C)\) a smooth projective toroidal variety, and \( f : (X, B) \to (Y, C) \) a toroidal and equi-dimensional morphism as in Theorem 1.2. Then there exists
a finite and surjective morphism $\pi_Y : (Y', C') \to (Y, C)$ from a smooth projective toroidal variety such that $\pi^{-1}(C) = C'$ and the following holds: if we set $X'$ to be the normalization of the fiber product $X \times_Y Y'$ and $B' = \pi_X^{-1}(B)$ for the induced morphism $\pi_X : X' \to X$, then $(X', B')$ is a quasi-smooth projective toroidal variety and the induced morphism $f' : (X', B') \to (Y', C')$ is a toroidal and equi-dimensional morphism whose fibers are reduced.

The morphism $f'$ is called a weak semistable reduction of $f_0$. We note that the fibers of $f'$ are reduced if and only if all the exponents $l_i$ are equal to 1 in the corresponding local description (1.1) for $f'$.

2 Real oriented blowing-up

It has been known that the general fiber of a semistable degeneration is topologically homeomorphic to the real oriented blowing-up of the singular fiber (e.g. [9]). [6] used this knowledge to put a $\mathbb{Z}$-structure on a certain cohomological mixed Hodge complex. The real oriented blowing-up is a special case of the associated logarithmic topological space to a logarithmic complex space defined by [4].

**Definition 2.1.** The real oriented blowing-up of a quasi-smooth toroidal variety $(X, B)$ is a real analytic morphism from a real analytic manifold with boundary to a complex variety $\rho_X : X^\# \to X$ defined by the following recipe:

(0) If there is no boundary, then $\rho_X$ is the identity: if $X = \Delta = \{z \in \mathbb{C} | |z| < 1\}$ and $B = \emptyset$, then $X^\# = X$ and $\rho_X = \text{Id}$.

(1) This is the basic case: if $X = \Delta$ and $B = \{0\}$, then $X^\# = [0, 1) \times S^1$ and $\rho_X(r, \theta) = re^{i\theta}$.

(2) The real oriented blowing-up of a product is the product of the real oriented blowing-ups: if $X = X_1 \times X_2$ and $B = p_1^*B_1 + p_2^*B_2$, then $X^\# = X_1^\# \times X_2^\#$ and $\rho_X = \rho_{X_1} \times \rho_{X_2}$.

(3) The real oriented blowing-up of a quotient of a smooth toroidal variety by a diagonal action of a finite abelian group which is free on the complement of the boundary divisor is the quotient of the real oriented blowing-up: if $X = X_1/G$ and $B = B_1/G$, then $X^\# = X_1^\# / G$ and $\rho_X = \rho_{X_1} / G$. We note that the action of $G$ on $X_1^\#$ is free so that $X^\#$ has no singularities.

(4) We can glue together the real oriented blowing-ups of local models: if $X = \bigcup_i X_i$ and $B = \bigcup_i B_i$, then $X^\# = \bigcup_i X_i^\#$ and $\rho_X = \bigcup_i \rho_{X_i}$.
Proposition 2.2. Let \((X, B)\) be a quasi-smooth toroidal variety. Then the real oriented blowing-up \(X^\#\) is homeomorphic to the complement of an \(\epsilon\) neighborhood of the boundary \(B\) in \(X\) with respect to some metric for sufficiently small \(\epsilon\).

The real oriented blowing-up is functorial:

Proposition 2.3. Let \(f : (X, B) \to (Y, C)\) be a toroidal morphism between quasi-smooth toroidal varieties. Let \(\rho_X : X^\# \to X\) and \(\rho_Y : Y^\# \to Y\) be the real oriented blowing-ups. Then a morphism of real analytic varieties with boundaries \(f^\# : X^\# \to Y^\#\) is induced so that the following diagram is commutative:

\[
\begin{array}{ccc}
X^\# & \xrightarrow{\rho_X^\#} & X \times_Y Y^# \\
\downarrow f^\# & & \downarrow f \\
Y^\# & \xrightarrow{\rho_Y^\#} & Y
\end{array}
\]

By using the real oriented blowing-up, we can eliminate the singularities of fibers topologically:

Theorem 2.4. Let \(f : (X, B) \to (Y, C)\) be a proper surjective toroidal and equi-dimensional morphism of quasi-smooth toroidal varieties. Then the induced morphism \(f^\# : X^\# \to Y^\#\) is locally topologically trivial in the following sense: each point \(y' \in Y^\#\) has an open neighborhood \(V'\) such that \((f^\#)^{-1}(V')\) is homeomorphic to \(V' \times (f^\#)^{-1}(y')\) over \(V'\).

Proof. We use the local description explained after Theorem 1.2. We write \(\rho_X^*(x_i) = r_i e^{\sqrt{-1} \theta_i}\) \((1 \leq i \leq n')\) and \(\rho_Y^*(y_j) = s_j e^{\sqrt{-1} \phi_j}\) \((1 \leq i \leq m')\). Then the map \(f^\#\) is described by the following formulas:

\[
(\tilde{f}^\#)^* s_j = \prod_{k=1}^{t_j-t_{j-1}} r_{t_{j-1}+k}^{l_{t_{j-1}+k}}
\]

\[
(\tilde{f}^\#)^* \phi_j = \sum_{k=1}^{t_j-t_{j-1}} l_{t_{j-1}+k} \theta_{t_{j-1}+k}
\]

for \(j = 1, \ldots, m'\) and \((\tilde{f}^\#)^*(y_j) = x_{n'-m'+j}\) for \(j = m' + 1, \ldots, m\). We note that the actions of \(G\) and \(H\) are trivial on the \(r_i\) and \(s_j\) while those on the \(\theta_i\)
and $\phi_j$ are translations. Since
\[
\{(r_1, \ldots, r_t) | t_i \in [0, 1), \prod_i t_i = c\} \cong [0, 1)^{t-1}
\]
for any $c \in [0, 1)$, the fibers of $f^# : U^# \to V^#$ are homeomorphic to
\[
[0, 1)^{n'-m'} \times (S^1)^{n'-m'} \times (D^2)^{n-n'-m'+m'}.
\]
The restriction of $f^#$ on $U^# \cap (f^#)^{-1}(V')$ for sufficiently small $V'$ is homeomorphic to the first projection of $V' \times (f^#)^{-1}(y')$ to $V'$. By gluing together, we obtain our result.

\[\Box\]

**Corollary 2.5.** $R^p(f^#)_*\mathbb{Z}_{X^#}$ is a locally constant sheaf on $Y^#$ for any $p \geq 0$.

**Proposition 2.6.** Let $(X, B)$ be a quasi-smooth toroidal variety and let $B = \sum_{i=1}^N B_i$ be the irreducible decomposition. Let $E$ be a connected component of the intersection $\bigcap_{i=1}^l B_i$ of codimension $l$, and let $G = \sum_{i=l+1}^N G_i$ for $G_i = B_i \cap E$. Then $(E, G)$ is a quasi-smooth toroidal variety. Let $\rho_X : X^# \to X$ and $\rho_E : E^# \to E$ be the real oriented blowing-ups. Then $\rho_X$ induces a map $\rho^E_X : (\rho^{-1}_X E) \to E^#$ which is an $l$-times fiber product of oriented $S^1$-bundles.

**Proof.** Since the boundary $B$ has no self-intersection, $E$ is normal, hence $(E, G)$ is toroidal and quasi-smooth. $E \setminus G$ is smooth and the normal bundle $N_{(E \setminus G)/(X \setminus G)}$ is the direct product of line bundles. Since $E^#$ is homeomorphic to the complement of an $\epsilon$ neighborhood of $G$ in $E$, we obtain our assertion. \[\Box\]

**Corollary 2.7.**
\[
R^p(\rho_X)_*\mathbb{Z}_{X^#} \cong \bigwedge_p^{\oplus N} \mathbb{Z}_{B_i}
\]
\[
R^p(\rho_X)_*\mathbb{Z}_{\rho^{-1}_X E} \cong \bigwedge_p^{\oplus N} (\mathbb{Z}_E \oplus \bigoplus_{i=l+1}^N \mathbb{Z}_{G_i})
\]
where the exterior products are taken respectively as $\mathbb{Z}_X$-modules and $\mathbb{Z}_E$-modules.

In the case $p = 0$, the above formula means that $(\rho_X)_*\mathbb{Z}_{X^#} \cong \mathbb{Z}_X$ and $(\rho_X)_*\mathbb{Z}_{\rho^{-1}_X E} \cong \mathbb{Z}_E$. We note that $G_i$ may be empty or reducible.
3 Weight filtration

We put a weight filtration on a complex on singular fibers. The definition of the filtration is natural thanks to the geometric construction of the real oriented blowing-up. One can compare with rather complicated earlier definitions in [10] and [3].

In this section, we denote by \( f : (X, B) \to (Y, C) \) a weak semistable reduction as in Corollary 13.3. We fix \( y \in Y \) and let \( E_1, \ldots, E_l \) be irreducible components of the fiber \( E = f^{-1}(y) \). We have \( \dim E_k = n - m \) for any \( k \).

For any combination of integers \( 1 \leq i_0 < \cdots < i_t \leq l \), we define \( E_{i_0, \ldots, i_t} = \bigcap_{k=0}^{t} E_{i_k} \). We note that \( \dim E_{i_0, \ldots, i_t} \) may be larger or smaller than \( n - m - t \).

For example, if a local model \( f : \Delta^4 \to \Delta^2 \) is given by \( f^*(y_1) = x_1x_2 \) and \( f^*(y_2) = x_3x_4 \) with \( y = (0, 0) \), then the irreducible components of \( E \) are \( E_1 = \{ x_1 = x_3 = 0 \} \), \( E_2 = \{ x_1 = x_4 = 0 \} \), \( E_3 = \{ x_2 = x_3 = 0 \} \) and \( E_4 = \{ x_2 = x_4 = 0 \} \) so that \( E_{14} = E_{23} = E_{234} = E_{134} = E_{124} = E_{123} = E_{1234} \).

We call each \( E_{i_0, \ldots, i_t} \) a stratum of \( E \). Let \( G_{i_0, \ldots, i_t} \) be the union of all the strata which are properly contained in \( E_{i_0, \ldots, i_t} \). Then the pair

\[
(E_{i_0, \ldots, i_t}, G_{i_0, \ldots, i_t})
\]

is again a quasi-smooth toroidal variety. Let \( \rho_{i_0, \ldots, i_t} : E_{i_0, \ldots, i_t}^\# \to E_{i_0, \ldots, i_t} \) be the real oriented blowing-up.

Let \( E[0] \) be the disjoint union of all the \( E_{i_0, \ldots, i_t} \). Then we have an exact sequence

\[
0 \to \mathbb{Z}_E \to \mathbb{Z}_{E[0]} \to \mathbb{Z}_{E[1]} \to \cdots \to \mathbb{Z}_{E[t]} \to \cdots
\]

where we identified the sheaves \( \mathbb{Z}_{E[t]} \) with their direct image sheaves on \( E \) and the differentials are given by the alternate sums of the restriction maps.

If the number of irreducible components of \( C \) which contain \( y \) is \( m' \), then \( \rho_Y^{-1}(y) \) is homeomorphic to \((S^1)^{m'}\). Let \( D = \rho_X^{-1}(E) = (f^\#)^{-1}\rho_Y^{-1}(y) \). We have natural maps \( \rho_X : D \to E \) and \( f^\# : D \to \rho_Y^{-1}(y) \). Corresponding to the irreducible decomposition of \( E \), we have \( D = \bigcup_{i=1}^l D_i \) for \( D_i = \rho_X^{-1}(E_i) \).

We write \( D_{i_0, \ldots, i_t} = \bigcap_{k=0}^{t} D_{i_k} \) and \( D[0] = \bigcup_{1 \leq i_0 < \cdots < i_t \leq l} D_{i_0, \ldots, i_t} \). Then we have similarly an exact sequence

\[
0 \to \mathbb{Z}_D \to \mathbb{Z}_{D[0]} \to \mathbb{Z}_{D[1]} \to \cdots \to \mathbb{Z}_{D[t]} \to \cdots
\]

We fix a point \( \bar{y} \in \rho_Y^{-1}(y) \) and let \( \bar{D} = (f^\#)^{-1}(\bar{y}) \). We set \( \bar{D}_{i_0, \ldots, i_t} = \bar{D} \cap D_{i_0, \ldots, i_t} \) and \( \bar{D}[0] = \bigcup_{1 \leq i_0 < \cdots < i_t \leq l} \bar{D}_{i_0, \ldots, i_t} \).
Proposition 3.1. Let \( t' = \dim E - \dim E_{i_0, \ldots, i_t} \). Then \( D_{i_0, \ldots, i_t} \) is homeomorphic to the direct product \( \bar{D}_{i_0, \ldots, i_t} \times \rho_Y^{-1}(y) \), and \( f^\# \) corresponds to the second projection. Moreover, \( D_{i_0, \ldots, i_t} \) is a \( t' \)-times fiber product of \( S^1 \)-bundles over \( E_{i_0, \ldots, i_t} \).

Proof. We assume first that \( t = t' = 0 \). Then we have \( \bar{D}_{i_0} = E_{i_0}^\# \). By Proposition 2.6, \( \rho_X \) induces an \((S^1)^{m'}\)-bundle \( \rho'_{i_0} : D_{i_0} \to E_{i_0}^\# \), and we have a map

\[
\rho'_{i_0} \times f^\# : D_{i_0} \to E_{i_0}^\# \times \rho_Y^{-1}(y).
\]

Since the fiber of \( f \) is reduced, we may take the homeomorphism to the \( \epsilon \) neighborhood as in Proposition 2.6 in a suitable way to conclude that \( \rho'_{i_0} \times f^\# \) is bijective and a homeomorphism.

In the general case, if we restrict \( \rho'_{i_0} \times f^\# \) to \( D_{i_0, \ldots, i_t} \), we obtain a homeomorphism

\[
D_{i_0, \ldots, i_t} \to \rho_{i_0}^{-1}(E_{i_0, \ldots, i_t}) \times \rho_Y^{-1}(y).
\]

By applying Proposition 2.6 again to \( E_{i_0, \ldots, i_t} \subset E_i \), we conclude the proof. \( \square \)

Definition 3.2. We define the weight filtration \( W_q \) on the complex of sheaves \( R(\rho_X^1)_* \mathbb{Z}D \) on \( E \times \rho_Y^{-1}(y) \) by

\[
W_q(R(\rho_X^1)_* \mathbb{Z}D) = (\tau_{\leq q}(R(\rho_X^1)_* \mathbb{Z}D^{|0|}) \to \tau_{\leq q+1}(R(\rho_X^1)_* \mathbb{Z}D^{|1|}) \to \cdots \to \tau_{\leq q+t}(R(\rho_X^1)_* \mathbb{Z}D^{|t|}) \to \cdots)
\]

in the derived category of complexes of sheaves on \( E \times \rho_Y^{-1}(y) \), where \( \tau \) denotes the truncation (cf. 1.4.6 of \[2\]).

If we take the successive quotients with respect to this filtration, then we can check that all the differentials of the resolution become trivial:

Proposition 3.3.

\[
Gr^W_q(R(\rho_X^1)_* \mathbb{Z}D) \cong R^q(\rho_X^1)_* \mathbb{Z}D^{|0|}[-q] \oplus R^{q+1}(\rho_X^1)_* \mathbb{Z}D^{|1|}[-q - 2] \oplus \cdots \oplus R^{q+t}(\rho_X^1)_* \mathbb{Z}D^{|t|}[-q - 2t] \oplus \cdots
\]
Corollary 3.4. The local monodromies of $R^p(f^\#)_*Z_{X^\#}$ on $Y^\#$ are unipotent for any $p \geq 0$.

Proof. We consider the restriction of the locally constant sheaf $R^p(f^\#)_*Z_{X^\#}$ on $Y^\#$. Since $f^\# = \text{pr}_2 \circ \rho_X^1$, the weight filtration induces a spectral sequence

$$E_{p,q}^{1} = R^{p+q}(\text{pr}_2)_*(\text{Gr}^W_{-p}(R(\rho_X^1)_*Z_D)) \Rightarrow R^{p+q}(f^\#)_*Z_{X^\#}.$$

By Proposition 3.1, $R^{p}(\rho_X^1)_*Z_D[t]$ is a constant sheaf for any $p, t$. Hence we have our result.

4 Relative log de Rham complex

Let $(X, B)$ be a quasi-smooth toroidal variety of dimension $n$. Then the sheaf of logarithmic 1-forms $\Omega^1_X(\log B)$ is a locally free sheaf of rank $n$. Indeed, if the local model of $(X, B)$ at $x \in X$ is the quotient of type $(\Delta^n/G, B_n'/G)$ with $B_n' = \text{div}(x_1 \ldots x_{n'})$, then the action of $G$ on the basis $dx_i/x_i$ ($1 \leq i \leq n'$) and $dx_i$ ($n' < i \leq n$) of the sheaf $\Omega^1_X(\log B_n')$ is trivial. The sheaf of logarithmic $p$-forms $\Omega^p_X(\log B) = \bigwedge^p \Omega^1_X(\log B)$ is a locally free sheaf of rank $\binom{n}{p}$.

Let $f : (X, B) \to (Y, C)$ be a toroidal and equi-dimensional morphism of quasi-smooth toroidal varieties. Let $\dim X = n$ and $\dim Y = m$. Then

$$\Omega^1_{X/Y}(\log) = \Omega^1_X(\log B)/f^*\Omega^1_Y(\log C)$$

is a locally free sheaf of rank $n - m$.

Following [10] and [4], we define the structure sheaf $\mathcal{O}_{X^\#}$ of the real oriented blowing-up by adding the logarithms of the coordinates:

**Definition 4.1.** (0) If there is no boundary, then $\mathcal{O}_{X^\#} = \mathcal{O}_X$.

(1) If $(X, B) = (\Delta, \{0\})$, then

$$\mathcal{O}_{X^\#} = \sum_{k \in \mathbb{Z}_{\geq 0}} \rho_X^{-1}(\mathcal{O}_X)(\log x)^k$$

where $\rho_X^{-1}$ denotes the inverse image for the sheaves of abelian groups, and $x$ is the coordinate. The symbol $\log x$ is identified with a local section of $\rho_X^{-1}(\mathcal{O}_X)$ on $X \setminus B$. We note that the multivalued function $\log x$ on $X$ becomes locally
single valued on \( X^\# \). Therefore, the stalk of \( \mathcal{O}_{X^\#} \) at \((r, \theta)\) is isomorphic to \( \mathcal{O}_{X,x} \) if \( r \neq 0 \) and to \( \mathcal{O}_{X,0} \otimes \mathbb{Z}[t] \) if \( r = 0 \), where \( t \) is an independent variable corresponding to \( \log x \). We note that a section of \( \mathcal{O}_{X^\#} \) is a finite polynomial on \( \log x \) instead of an infinite power series.

2) If \((X, B) = (X_1 \times X_2, p_1^*B_1 + p_2^*B_2)\), then
\[
\mathcal{O}_{X^\#} = (pr_1^{-1}(\mathcal{O}_{X_1^\#}) \otimes_{\rho_{X_1}(\mathcal{O}_{X_1})} \rho_X^{-1}(\mathcal{O}_X)) \\
\otimes_{\rho_X^{-1}(\mathcal{O}_X)} (pr_2^{-1}(\mathcal{O}_{X_2^\#}) \otimes_{\rho_{X_1}(\mathcal{O}_{X_2})} \rho_X^{-1}(\mathcal{O}_X)).
\]

For example, if \( X = \Delta^n \) with \( B = \text{div}(x_1 \ldots x_n) \), then we have
\[
\mathcal{O}_{X^\#} = \sum_{k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}} \rho_X^{-1}(\mathcal{O}_X) \prod_{i=1}^n (\log x_i)^{k_i}.
\]

In this case, the stalk of \( \mathcal{O}_{X^\#} \) at any point over the origin of \( X \) is isomorphic to \( \mathcal{O}_{X,0} \otimes \mathbb{Z}[t_1, \ldots, t_{n'}] \), where the \( t_i \) are independent variables corresponding to the \( \log x_i \).

3) If \((X, B) = (X_1/G, B_1/G)\), then \( \mathcal{O}_{X^\#} = (\mathcal{O}_{X^\#})^G \).

4) We can glue together the structure sheaves of local models.

**Lemma 4.2.** Let \((X, B)\) be a quasi-smooth toroidal variety, \( x \in X \) a point, and let \( n' \) be the number of irreducible components \( B_i \) of \( B \) which contain \( x \). Then \( \mathcal{O}_{X^\#} \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \rho_X^{-1}(\mathcal{C}_x) \) is a locally constant sheaf \( L \) on \( \rho_X^{-1}(x) \cong (S^1)^{n'} \) whose fibers are isomorphic to \( \mathbb{C}[t_1, \ldots, t_{m'}] \), where the \( t_j \) are independent variables corresponding to the logarithms of the local equations of the \( B_i \) at \( x \). The monodromies \( M_i \) of \( L \) around the loop in \( \rho_X^{-1}(x) \) corresponding to the \( B_i \) are given by the following formula
\[
M_i(t_j) = t_j + 2\delta_{ij} \pi \sqrt{-1}.
\]

**Definition 4.3.** We define the log de Rham complex on the real blowing-up by
\[
\Omega^p_{X^\#} = \rho_X^{-1}(\Omega^p_X(\log B)) \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^\#}
\]
for \( p \geq 0 \). The differential of the complex \( \Omega^\bullet_{X^\#} \) is defined by the rule: \( d \log x = dx/x \). The relative log de Rham complex is defined by
\[
\Omega^p_{X^\#/Y^\#} = \rho_X^{-1}(\Omega^p_{X/Y}(\log)) \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^\#}.
\]
The differential of the complex \( \Omega^\bullet_{X^\#/Y^\#} \) is induced from that of \( \Omega^\bullet_{X^\#} \).
If \( B = \emptyset \), then the Poincaré lemma says that
\[
\mathbb{C}_X \cong \Omega^\bullet_X
= (\mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^p_X \to \ldots)
\]
in the derived category of sheaves on \( X \). We have the following logarithmic version:

**Theorem 4.4.** (1)

\[
\mathbb{C}_X \# \cong \Omega^\bullet_{X \#}
\]
in the derived category of sheaves on \( X \# \).

(2)

\[
R(\rho_X)_* \Omega^p_{X \#} \cong \Omega^p_X(\log B)
\]
in the derived category of sheaves on \( X \).

(3)

\[
R(\rho_X)_* \mathbb{C}_{X \#} \cong \Omega^\bullet_X(\log B)
\]
in the derived category of sheaves on \( X \).

**Proof.** (1) This is a special case of Theorem 3.6 of [4]. For example, we check the case where \((X, B) = (\Delta, \{0\})\). Let \( h = \sum h_k (\log x)^k \) for \( h_k \in \mathcal{O}_{X,0} \). If
\[
dh = dx/x \sum_k (xh'_k + (k + 1)h_{k+1})(\log x)^k = 0
\]
then we have \( xh'_k + (k + 1)h_{k+1} = 0 \) for all \( k \). Since \( h_k = 0 \) for \( k \gg 0 \), we conclude that \( h \in \mathbb{C} \) by the descending induction on \( k \).

On the other hand, we can solve the equations
\[
xg'_k + (k + 1)g_{k+1} = h_k
\]
by the descending induction on \( k \) to find \( g = \sum g_k (\log x)^k \) such that \( hdx/x = dg \).

(2) We check the case where \((X, B) = (\Delta, \{0\})\) and \( p = 0 \), that is, \((\rho_X)_* \mathcal{O}_{X \#} = \mathcal{O}_X \) and \( R^1(\rho_X)_* \mathcal{O}_{X \#} = 0 \). The case \( p > 0 \) follows from the projection formula, and the general case is similarly proved.

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The first equality follows from the fact that \( \log x \) is multi-valued on \( X \) near 0. We have an exact sequence

\[
0 \to \rho_X^{-1}(\mathcal{O}_X) \to \mathcal{O}_X^\# \to \bar{L} \otimes \mathcal{O}_{X,0} \to 0
\]

where \( \bar{L} = L/\mathbb{C} \) is a locally constant sheaf on \( \rho_X^{-1}(0) \cong S^1 \) such that the stalk \( \bar{L}_0 \) at \( 0 \in S^1 \) has a \( \mathbb{C} \)-basis \( \{e_k\}_{k \in \mathbb{Z}_{>0}} \) corresponding to the \( (\log x)^k \) with the monodromy action given by

\[
M(e_k) = \sum_{j=1}^{k} \binom{k}{j} (2\pi \sqrt{-1})^{k-j} e_j.
\]

Then \( H^0(S^1, \bar{L}) = \mathbb{C} e_1 \) and \( H^1(S^1, \bar{L}) = 0 \). Hence

\[
(\rho_X)_*\rho_X^{-1}(\mathcal{O}_X) \cong (\rho_X)_*\mathcal{O}_X^#
\]

and \( R^1(\rho_X)_*\mathcal{O}_X^# = 0 \).

(3) is a consequence of (1) and (2). This follows also from Proposition 3.1.8 of [2], because we have

\[
R(\rho_X)_*\mathbb{C}_X^# \cong Rj_*\mathbb{C}_{X\setminus B}
\]

for the inclusion \( j : X \setminus B \to X \) by Proposition 2.2.

In the relative setting, if \( B = C = \emptyset \), then we have \( f^{-1}(\mathcal{O}_Y) \cong \Omega_{X/Y}^\bullet \) in the derived category of sheaves on \( X \). The logarithmic version is the following:

**Theorem 4.5.** (1)

\[
(f^#)^{-1}(\mathcal{O}_Y^#) \cong \Omega_{X^#/Y^#}^\bullet
\]

in the derived category of sheaves on \( X^# \).

(2)

\[
R(\rho_X^1)_*\Omega_{X^#/Y^#}^\bullet \cong (pr_1)^{-1}(\Omega_{X/Y}^\bullet (\log)) \otimes_{(pr_2)^{-1}\rho_Y^{-1}(\mathcal{O}_Y)} (pr_2)^{-1}(\mathcal{O}_Y^#)
\]

in the derived category of sheaves on \( X \times_Y Y^# \).

(3)

\[
R(\rho_X^1)_*(f^#)^{-1}(\mathcal{O}_Y^#) \cong (pr_1)^{-1}(\Omega_{X/Y}^\bullet (\log)) \otimes_{(pr_2)^{-1}\rho_Y^{-1}(\mathcal{O}_Y)} (pr_2)^{-1}(\mathcal{O}_Y^#)
\]

in the derived category of sheaves on \( X \times_Y Y^# \).
Proof. (1) We check the case where \((X, B) = (\Delta^2, \text{div}(x_1 x_2)), (Y, C) = (\Delta, \{0\})\), and \(f^*(y) = x_1 x_2\). The general case is similar. Let

\[
h = \sum_{k_1, k_2} h_{k_1, k_2} (\log x_1)^{k_1} (\log x_2)^{k_2}
\]

for \(h_{k_1, k_2} \in \mathcal{O}_{X, 0}\). If

\[
dh = dx_1/x_1 \sum_{k_1, k_2} (x_1 h_{k_1, k_2, x_1} - x_2 h_{k_1, k_2, x_2} + (k_1 + 1)h_{k_1+1, k_2} - (k_2 + 1)h_{k_1, k_2+1}) (\log x_1)^{k_1} (\log x_2)^{k_2} = 0
\]

in \(\Omega^1_{X^#/Y^#}\), then we have

\[
x_1 h_{k_1, k_2, x_1} - x_2 h_{k_1, k_2, x_2} + (k_1 + 1)h_{k_1+1, k_2} - (k_2 + 1)h_{k_1, k_2+1} = 0
\]

for any \(k_1, k_2\). Since \(h_{k_1, k_2} = 0\) for \(k_1 \gg 0\) or \(k_2 \gg 0\), we prove by the descending induction on \((k_1, k_2)\) that there exist \(h_k \in \mathcal{O}_{X, 0}\) for \(k \in \mathbb{Z}_{\geq 0}\) such that \(h_{k_1, k_2} = (k_1 + k_2)h_{k_1, k_2}\), and if we expand \(h_k\) in a power series \(h_k = \sum_{l_1, l_2} a_{k, l_1, l_2} x_1^{l_1} x_2^{l_2}\), then we have \(a_{k, l_1, l_2} = 0\) unless \(l_1 = l_2\). Therefore, \(h \in \mathcal{O}_{Y^#, 0}\).

On the other hand, we can solve the equations

\[
h_{k_1, k_2} = x_1 g_{k_1, k_2, x_1} - x_2 g_{k_1, k_2, x_2} + (k_1 + 1)g_{k_1+1, k_2} - (k_2 + 1)g_{k_1, k_2+1}
\]

by the descending induction on \((k_1, k_2)\) to find

\[
g = \sum_{k_1, k_2} g_{k_1, k_2} (\log x_1)^{k_1} (\log x_2)^{k_2}
\]

such that \(h dx_1/x_1 = dg\) in \(\Omega^1_{X^#/Y^#}\).

(2) We check the case where \((X, B) = (\Delta^2, \text{div}(x_1 x_2)), (Y, C) = (\Delta, \{0\}), f^*(y) = x_1 x_2\) and \(p = 0\), that is

\[
(\rho_X^1)_* \mathcal{O}_{X^#} \cong (\text{pr}_1)^{-1}(\mathcal{O}_X) \otimes_{(\text{pr}_2)^{-1} \rho_Y^{-1}(\mathcal{O}_Y)} (\text{pr}_2)^{-1}(\mathcal{O}_{Y^#})
\]

and \(R^1(\rho_X)_* \mathcal{O}_{X^#} = 0\). The general case is similar.

We have \(\rho_X^{-1}(0) \cong S^1\) and \(\rho_Y^{-1}(0) \cong S^1\). The map \(\rho_X : \rho_X^{-1}(0) \to \rho_Y^{-1}(0)\) is given by \(\rho_X^1(\theta_1, \theta_2) = \theta_1 + \theta_2\).
Let \( h = \sum_{k_1, k_2} h_{k_1, k_2} (\log x_1)^{k_1} (\log x_2)^{k_2} \) be a local section of \( \mathcal{O}_x \), where \( h_{k_1, k_2} \in \mathcal{O}_{X, 0} \). The monodromies of the multi-valued functions \( \log x_1 \) and \( \log x_2 \) along the fibers of \( \rho_X^1 \) which are homeomorphic to \( S^1 \)'s are given by

\[
\log x_1 \mapsto \log x_1 + 2\pi \sqrt{-1}, \quad \log x_2 \mapsto \log x_2 - 2\pi \sqrt{-1}.
\]

The function \( h \) is single valued along these \( S^1 \)'s if and only if there exist \( h_{k_1, k_2} \in \mathcal{O}_x \) such that

\[
h_{k_1, k_2} = (k_1^{k_1} + k_2^{k_2}) h_{k_1 + k_2}.
\]

Therefore we have the first equality.

We have an exact sequence

\[
0 \to \rho_X^{-1}(\mathcal{O}_X) \otimes (f^*)^{-1}(\mathcal{O}_Y) \otimes (f^*)^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \to 0
\]

de where \( L_X \) and \( L_Y \) are the locally constant sheaves on \( \rho_X^{-1}(0) \cong (S^1)^2 \) and \( \rho_Y^{-1}(0) \cong S^1 \) such that the stalks \( L_{X, (0,0)} \) and \( L_{Y, 0} \) at \( (0,0) \in (S^1)^2 \) and \( 0 \in S^1 \) have \( \mathbb{C} \)-bases \( \{ e_{k_1, k_2} \}_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \) and \( \{ e_k \}_{k \in \mathbb{Z}_{\geq 0}} \) whose the monodromies \( M_1, M_2 \) around the first and second factors and \( M \) are given by

\[
M_1(e_{k_1, k_2}) = \sum_{j=0}^{k_1} \binom{k_1}{j} (2\pi \sqrt{-1})^{k_1-j} e_{j, k_2}
\]

\[
M_2(e_{k_1, k_2}) = \sum_{j=0}^{k_2} \binom{k_2}{j} (2\pi \sqrt{-1})^{k_2-j} e_{k_1, j}
\]

\[
M(e_k) = \sum_{j=0}^{k} \binom{k}{j} (2\pi \sqrt{-1})^{k-j} e_j
\]

and the homomorphism \( (f^*)^{-1}L_Y \to L_X \) is given by

\[
(f^*)^{-1}e_k \mapsto \sum_{j=0}^{k} \binom{k}{j} e_{j, k-j}.
\]

In other words, we have

\[
(M_1 - M_2)(e_{k_1, k_2}) = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} (-1)^{j_2-j_1} \binom{k_1}{j_1} \binom{k_2}{j_2} (2\pi \sqrt{-1})^{k_1+k_2-j_1-j_2} e_{j_1, j_2}
\]

\[
(M_1 - M_2)((f^*)^{-1}e_k) = (f^*)^{-1}e_k.
\]
We have
\[
(\rho_X^1)_*(f^#)^{-1}L_Y \cong (\rho_X^1)_*L_X \cong L_Y
\]
\[
R^1(\rho_X^1)_*(f^#)^{-1}L_Y \cong L_Y, \quad R^1(\rho_X^1)_*L_X = 0
\]
hence
\[
(\rho_X^1)_*(L_X/f^#)^{-1}L_Y \cong L_Y, \quad R^1(\rho_X^1)_*(L_X/f^#)^{-1}L_Y \cong 0.
\]
Since
\[
(\rho_X^1)_*\rho_Y^{-1}(\mathcal{O}_X) \cong (pr_1)^{-1}(\mathcal{O}_X)
\]
\[
R^1(\rho_X^1)_*\rho_Y^{-1}(\mathcal{O}_X) \cong \mathcal{C}(\rho_Y)^{-1}(0) \otimes \mathcal{O}_{X,0}
\]
\[
\mathcal{O}_X^\#|_{\rho_Y^{-1}(0)} \cong L_Y \otimes \mathcal{O}_{Y,0}
\]
we have
\[
(\rho_X^1)_*(\rho_Y^{-1}(\mathcal{O}_X) \otimes (f^#)^{-1}\rho_Y^{-1}(\mathcal{O}_Y) (f^#)^{-1}(\mathcal{O}_Y^\#))
\]
\[
\cong (pr_1)^{-1}(\mathcal{O}_X) \otimes (pr_2)^{-1}\rho_Y^{-1}(\mathcal{O}_Y) (pr_2)^{-1}(\mathcal{O}_Y^\#)
\]
\[
R^1(\rho_X^1)_*(\rho_Y^{-1}(\mathcal{O}_X) \otimes (f^#)^{-1}\rho_Y^{-1}(\mathcal{O}_Y) (f^#)^{-1}(\mathcal{O}_Y^\#))
\]
\[
\cong L_Y \otimes \mathcal{O}_{X,0}.
\]
Therefore, we have the desired result.

(3) is obtained by combining (1) and (2).

We assume that \( f \) is weakly semistable in the rest of this section, and use the notation of §3. Let \( m' \) be the number of irreducible components of \( C \) which contain \( y \in Y \), and let \( y_1, \ldots, y_{m'} \) be the corresponding local coordinates. We fix \( \bar{y} \in \rho_Y^{-1}(y) \) and denote by \( \rho_D \) and \( \rho_{D_{i_0,\ldots,i_t}} \) the restrictions of \( \rho_X \) to \( \bar{D} \) and \( \bar{D}_{i_0,\ldots,i_t} \).

**Definition 4.6.** We define the structure sheaves of \( \bar{D} \) and \( \bar{D}_{i_0,\ldots,i_t} \) by the following formula:

\[
\mathcal{O}_D = (\mathcal{O}_X^\# \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \rho_Y^{-1}(\mathcal{O}_E))/I
\]
\[
\mathcal{O}_{D_{i_0,\ldots,i_t}} = (\mathcal{O}_X^\# \otimes_{\rho_X^{-1}(\mathcal{O}_X)} \rho_Y^{-1}(\mathcal{O}_{E_{i_0,\ldots,i_t}}))/I_{i_0,\ldots,i_t}
\]

where \( I \) and \( I_{i_0,\ldots,i_t} \) are ideals generated by \( \log y_j - c_i \) \((j = 1, \ldots, m')\) for some constants \( c_i \in \mathbb{C} \) corresponding to the choice of \( \bar{y} \in \rho_Y^{-1}(y) \). We define

\[
\Omega^{p}_{E/\mathcal{C}}(\log) = \Omega^{p}_{X/Y}(\log) \otimes_{\mathcal{O}_X} \mathcal{O}_E
\]
and define the log de Rham complex on the fiber by

\[ \Omega^p_D = \rho_D^{-1}(\Omega^p_{E/\mathcal{C}}(\log)) \otimes \rho_D^{-1}(\mathcal{O}_E) \mathcal{O}_{\bar{D}} = \Omega^p_{X^\# / Y^\#} \otimes \mathcal{O}_{X^\#} \mathcal{O}_{\bar{D}} \]
\[ \Omega^p_{D_{i_0,\ldots,i_t}} = \rho_D^{-1}(\Omega^p_{E/\mathcal{C}}(\log)) \otimes \rho_D^{-1}(\mathcal{O}_E) \mathcal{O}_{D_{i_0,\ldots,i_t}} = \Omega^p_{X^\# / Y^\#} \otimes \mathcal{O}_{X^\#} \mathcal{O}_{D_{i_0,\ldots,i_t}} \]

for \( p \geq 0 \).

We note that there is no ideal sheaf of \( D \) corresponding to \( I \) because of the monodromies. The following is easy:

**Lemma 4.7.** Let \( t' = \dim E - \dim E_{i_0,\ldots,i_t} \) as in Proposition 3.1.

1. If \( t = t' = 0 \), then
   \[ \Omega^p_{E/\mathcal{C}}(\log) \otimes \mathcal{O}_E \mathcal{O}_{E_{i_0}} \cong \Omega^p_{E_{i_0}}(\log G_{i_0}). \]

2. If \( t' > 0 \), then
   \[ 0 \to \Omega^1_{E_{i_0,\ldots,i_t}}(\log G_{i_0,\ldots,i_t}) \to \Omega^1_{E/\mathcal{C}}(\log) \otimes \mathcal{O}_E \mathcal{O}_{E_{i_0,\ldots,i_t}} \to 0 \]

where the arrow next to the last is the residue homomorphism.

The following is similar to Theorem 4.5:

**Theorem 4.8.** (1)

\[ \mathbb{C}_{\bar{D}} \cong \Omega^\bullet_{\bar{D}} \]
\[ \mathbb{C}_{D_{i_0,\ldots,i_t}} \cong \Omega^\bullet_{D_{i_0,\ldots,i_t}} \]

in the derived category of sheaves on \( \bar{D} \).

(2)

\[ R(\rho_{\bar{D}})_* \Omega^p_{\bar{D}} \cong \Omega^p_{E/\mathcal{C}}(\log) \]
\[ R(\rho_{D_{i_0,\ldots,i_t}})_* \Omega^p_{D_{i_0,\ldots,i_t}} \cong \Omega^p_{E/\mathcal{C}}(\log) \otimes \mathcal{O}_E \mathcal{O}_{E_{i_0,\ldots,i_t}} \]

in the derived category of sheaves on \( E \).

(3)

\[ R(\rho_{\bar{D}})_* \mathbb{C}_{\bar{D}} \cong \Omega^\bullet_{E/\mathcal{C}}(\log) \]
\[ R(\rho_{D_{i_0,\ldots,i_t}})_* \mathbb{C}_{D_{i_0,\ldots,i_t}} \cong \Omega^\bullet_{E/\mathcal{C}}(\log) \otimes \mathcal{O}_E \mathcal{O}_{E_{i_0,\ldots,i_t}} \]

in the derived category of sheaves on \( E \).
5 Cohomological mixed Hodge complex

We prove the main result (Theorem 5.2) of this paper. We assume that $f$ is weakly semistable in this section, and use the notation of §3. We shall construct a cohomological mixed Hodge complex on $E = f^{-1}(y)$.

On the $\mathbb{Z}$-level, we put a weight filtration $W_q$ on the complex of sheaves $R(\rho_X)_*\mathbb{Z}_D$ on $E$ as in Definition 3.2 by

$$W_q(R(\rho_X)_*\mathbb{Z}_D) = (\tau_{\leq q}(R(\rho_X)_*\mathbb{Z}_{D[0]}) \to \tau_{\leq q+1}(R(\rho_X)_*\mathbb{Z}_{D[1]}) \to \cdots \to \tau_{\leq q+t}(R(\rho_X)_*\mathbb{Z}_{D[t]}) \to \cdots),$$

in the derived category of complexes of sheaves on $E$.

We have an exact sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_E[0] \to \mathcal{O}_E[1] \to \cdots \to \mathcal{O}_E[t] \to \cdots$$

as before, where we identified the sheaves $\mathcal{O}_E[t]$ with their direct image sheaves on $E$.

On the $\mathbb{C}$-level, we define the weight filtration and the Hodge filtration on the complex $\Omega^\bullet_{E/\mathcal{C}}(\log) = \Omega^\bullet_{X/Y}(\log) \otimes_{\mathcal{O}_X} \mathcal{O}_E$.

**Definition 5.1.** The weight filtration $\{W_q\}$ on $\Omega^\bullet_{E/\mathcal{C}}(\log)$ is defined by

$$W_q(\Omega^\bullet_{E/\mathcal{C}}(\log)) = (W_q(\Omega^\bullet_{E/\mathcal{C}}(\log) \otimes \mathcal{O}_E[0]) \to W_{q+1}(\Omega^\bullet_{E/\mathcal{C}}(log) \otimes \mathcal{O}_E[1]) \to \cdots \to W_{q+t}(\Omega^\bullet_{E/\mathcal{C}}(log) \otimes \mathcal{O}_E[t]) \to \cdots),$$

where $W$ on the right hand side is the filtration defined by the order of log poles at $\mathbb{A}^3$ 3.1.5.

The Hodge filtration $\{F^p\}$ on $\Omega^\bullet_{X/Y}(\log)$ is defined by

$$F^p(\Omega^\bullet_{X/Y}(\log)) = \sigma_{\geq p}(\Omega^\bullet_{X/Y}(\log))$$

where $\sigma_{\geq p}$ denotes the stupid filtration (cf. 1.4.7 of 2). We consider its restriction $\{F^p\}$ on the fiber $E$:

$$F^p(\Omega^\bullet_{E/\mathcal{C}}(log)) = \sigma_{\geq p}(\Omega^\bullet_{E/\mathcal{C}}(log)).$$

The following is the main result of this paper:
Theorem 5.2.
\[\{(R(\rho_X)_*\mathcal{C}_D, W), (\Omega^\bullet_{E/\mathbb{C}}(\log), W, F)\}\]
is a cohomological mixed Hodge complex on \(E\) (\cite{2} 8.1.6).

The proof is reduced to Propositions 5.3 and 5.6.

Proposition 5.3.
\[\{(R(\rho_X)_*\mathcal{C}_D, W) \cong (\Omega^\bullet_{E/\mathbb{C}}(\log), W)\}\]
in the derived category of filtered complexes of sheaves on \(E\).

Proof. Since the truncation is a canonical functor, this is a consequence of Theorem 4.8 (3) and \cite{2} 3.1.8. \qed

The following is similar to Proposition 3.3:

Lemma 5.4.
\[\text{Gr}^W_q(\Omega^\bullet_{E/\mathbb{C}}(\log)) \cong \text{Gr}^W_q(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes \mathcal{O}_E[0]) \oplus \text{Gr}^W_{q+1}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes \mathcal{O}_E[1])[-1] \oplus \cdots \oplus \text{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes \mathcal{O}_E[t])[-t] \oplus \cdots\]

Lemma 5.5. Let \((X, B)\) and \((E, G)\) be as in Proposition 2.6. Then the Poincaré residue induces the following isomorphisms of filtered complexes:
\[\text{Gr}^W_q(\Omega_X^\bullet(\log B)), F) \cong (\Omega^\bullet_B[1, \ldots, q] \otimes \mathcal{O}_B[1], F[q], F[q]) \]
\[= (\bigoplus_{1 \leq i_1 < \cdots < i_q \leq N} \Omega_{B, i_1, \ldots, i_q}[-q], F[q], F[q])\]
\[\text{Gr}^W_q(\Omega_X^\bullet(\log B) \otimes \mathcal{O}_B[1], F) \cong (\bigoplus_{t=0}^q (\Omega^\bullet_{B[t]}(\log B)[q-t] \otimes \mathcal{O}_B[1], F[q-t], F[q-t])\).

We note that \(G[0] = E\) by definition.

Proof. The first formula is \cite{2} 3.1.5.2. The second is similar. \qed

Proposition 5.6.
\[\text{Gr}^W_q(\Omega^\bullet_{E/\mathbb{C}}(\log)), F)\]
is a cohomological Hodge complex of weight \(q\) on \(E\) (\cite{3} 8.1.2).

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Proof. We have to prove that

\[(H^n(E, \text{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes O_{E[0]}[-t]), F) \cong (H^{n-t}(E, \text{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes O_{E[0]})), F)\]

is a Hodge structure of weight \(n + q\):

\[H^{n-t}(E, \text{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes O_{E[0]})) = \sum_p F^p \cap \bar{F}^{n+q-p} \cap \bar{F}^{n-q-t}.\]

By Lemmas 4.7 and 5.5, we can write

\[(\text{Gr}^W_{q+t}(\Omega^\bullet_{E/\mathbb{C}}(\log) \otimes O_{E[0]}), F) \cong (\sum G \Omega^\bullet_{G[-q-t]}, F[-q-t])\]

for some set of the strata \(G\). Thus the problem is reduced to prove

\[H^{n-q-2t}(E, \Omega^\bullet_{E/\mathbb{C}}) = \sum_p F^{p-q-t} \cap \bar{F}^{n-p-t}.\]

But this is the usual Hodge theorem on \(G\).

By \[2\] Scholie 8.1.9, we obtain the following corollary (cf. \[3\]):

**Corollary 5.7.** (1) The spectral sequence

\[W_{E_1}^{p,q} = H^{p+q}(E, \text{Gr}^W_p(R(\rho_X)_*, \mathbb{C}_D)) \Rightarrow H^{p+q}(\bar{D}, \mathbb{C})\]

degenerates at \(E_2\).

(2) The spectral sequence

\[F_{E_1}^{p,q} = H^q(E, \Omega^p_{E/\mathbb{C}}(\log)) \Rightarrow H^{p+q}(\bar{D}, \mathbb{C})\]

degenerates at \(E_1\).

Combining with Corollary 2.5, we obtain the following (\[7\] and \[8\]):

**Corollary 5.8.** \(R^q f_* \Omega^p_{X/Y}(\log)\) is a locally free sheaf for any \(p, q\).
References

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