CONTINUITY OF SPECTRAL RADIUS AND TYPE I $C^*$-ALGEBRAS

TATIANA SHULMAN

Abstract. It is shown that the spectral radius is continuous on a $C^*$-algebra if and only if the $C^*$-algebra is type I. This answers a question of V. Shulman and Yu. Turovskii [10]. It is shown also that the closure of nilpotents in a $C^*$-algebra contains an element with non-zero spectrum if and only if the $C^*$-algebra is not type I.

1. Introduction

Let $H$ be a Hilbert space and $B(H)$ the $C^*$-algebra of all bounded operators on $H$. As is well-known, the spectral radius is not a continuous function on $B(H)$, it is only upper semi-continuous (see [6], Solutions 103 and 104). However it is not hard to prove that on the $C^*$-algebra $K(H)$ of all compact operators it is continuous. Thus there arises a question on which $C^*$-algebras the spectral radius function is continuous. In [10] V. Shulman and Yu. Turovskii proved that it is continuous on any type I $C^*$-algebra and asked if the converse is true ([10], Problem 9.20). In this short paper we answer this question in the affirmative.

Theorem A. Let $A$ be a $C^*$-algebra. The spectral radius is a continuous function on $A$ if and only if $A$ is type I.

Theorem A adds one more (analytical) characterization to numerous characterizations of type I $C^*$-algebras.

One of famous questions of P. Halmos asks: ”What is the closure of the nilpotent operators on a complex, separable, infinite-dimensional Hilbert space?” It was completely answered in [1] (see also [12]). In particular the answer shows that in the closure of nilpotent operators there are many operators with non-zero spectrum. In the recent work [11] P. Skoufranis considered similar question for $C^*$-algebras with focus on normal limits of nilpotents. His results show in particular that in many $C^*$-algebras (for example in all UHF-algebras and all purely infinite $C^*$-algebras) the closure of nilpotents contains elements with non-zero spectrum. Here we give a characterization of such $C^*$-algebras.

Theorem B. The closure of nilpotents in a $C^*$-algebra $A$ contains an element with non-zero spectrum if and only if $A$ is not type I.

2. Preliminaries

2.1. AF-telescopes and projectivity. Let $A = \bigcup A_n$ be an inductive limit of a sequence of $C^*$-algebras $A_1 \subset A_2 \subset \ldots \subset A$
with injective connecting maps. In [8] the \textit{mapping telescope} of \((A_n)\) was defined as the \(C^*\)-algebra
\[ T(A) = \{ f \in C_0((0, \infty), A) \mid t \leq n \Rightarrow f(t) \in A_n \}. \]
Clearly the mapping telescope depends on the sequence \((A_n)\), but nevertheless we will use the notation \(T(A)\). When each \(A_n\) is finite-dimensional, the \(C^*\)-algebra \(A\) is AF, and \(T(A)\) is called an \textit{AF-telescope}. In particular, we denote by \(T(M_{2\infty})\) the mapping telescope corresponding to the inductive sequence
\[ M_2 \subset M_4 \subset \ldots \subset M_{2^n} \subset \ldots \subset M_{2\infty} \]
where \(M_{2^n}\) is identified with a subalgebra of \(M_{2^{n+1}}\) by the map \(a \mapsto a \oplus a\). Recall that \(M_{2\infty}\) is referred to as the CAR algebra ([4]).

Recall that a \(C^*\)-algebra \(A\) is \textit{projective} ([2], [7]) if for any \(C^*\)-algebras \(B\) and \(C\) with surjective \(*\)-homomorphism \(q : B \rightarrow C\), any \(*\)-homomorphism \(\phi : A \rightarrow C\) lifts to a \(*\)-homomorphism \(\psi : A \rightarrow B\) such that \(q \circ \psi = \phi\). In other words, we have the following commutative diagram.

\[
\begin{array}{c}
\psi \\
\downarrow \phi \\
A \quad \rightarrow \quad C
\end{array}
\]

In [8] Loring and Pedersen proved that AF-telescopes are projective. This fact will be crucial for the proof of Theorem B. Some other applications of projectivity of AF-telescopes can be found in the recent work of Kristin Courtney and the author [3].

2.2. Type I \(C^*\)-algebras. Let \(H\) be a Hilbert space. A \(C^*\)-algebra \(A\) is \textit{type I} (or, equivalently, GCR) if
\[ K(H) \subseteq \pi(A) \]
for any irreducible representation \((\pi, H)\) of \(A\).

A \(C^*\)-algebra \(D\) is a \textit{subquotient} of a \(C^*\)-algebra \(A\) if there is a \(C^*\)-subalgebra \(B\) of \(A\) and a surjective \(*\)-homomorphism \(q : B \rightarrow D\).

We will need the following property (in fact a characterization) of non-type I \(C^*\)-algebras, which is due to Glimm for separable case [4] and Sakai for the general case [9].

\textbf{Theorem 1.} (Glimm, Sakai) If a \(C^*\)-algebra is non-type I, then it has a subquotient isomorphic to the CAR algebra.

3. Proof of Theorems A and B

We start by constructing a sequence of nilpotent elements in the CAR-algebra \(M_{2\infty}\) that converges to an element with positive spectral radius.

Let \(\epsilon_n = 1/2^n, n \geq 0\). Define matrices \(A_n \in M_{2^n}\) as follows:
\[ A_1 = \begin{pmatrix} 0 & 0 \\ \epsilon_0 & 0 \end{pmatrix} \]
\[ A_2 = \begin{pmatrix} 0 & \epsilon_0 & 0 \\ \epsilon_0 & 0 & \epsilon_1 \\ \epsilon_1 & 0 & 0 \end{pmatrix} \]
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$$A_3 = \begin{pmatrix}
0 & 0 & 0 \\
\epsilon_0 & 0 & 0 \\
\epsilon_1 & \epsilon_0 & 0 \\
\epsilon_2 & 0 & 0 \\
\epsilon_1 & 0 & 0 \\
\epsilon_0 & 0 & 0
\end{pmatrix}$$

and so on. In other words, to obtain $A_{k+1}$ we take $A_k \oplus A_k$ and put $\epsilon_k$ in the middle of the first diagonal under the main diagonal.

Then in the first diagonal under the main one $A_n$ has the first $2^n - 1$ elements of the so-called "the abacaba order" sequence

$$\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_0, \epsilon_1, \epsilon_0, \epsilon_3, \ldots$$

(1)

(here each second term equals $\epsilon_0$, each second of the remaining terms equals $\epsilon_1$, and so on) and all the other entries are zeros.

Below we will identify matrices from $M_{2n}$ with their images in $M_{2n+1}$ and in $M_{2\infty}$.

The construction of the sequence of elements of $M(2^\infty)$ above was inspired by Kaku-tani’s construction of a sequence of nilpotent operators in his remarkable proof of discontinuity of spectral radius on $B(H)$ (see [6], Solution 104).

**Lemma 2.** The sequence $A_n \in M_{2\infty}$, $n \in \mathbb{N}$, converges to some $A \in M_{2\infty}$ such that $\rho(A) > 0$.

**Proof.** We have

$$\|A_1 - A_2\| = \| \begin{pmatrix} 0 & 0 \\ \epsilon_0 & 0 \\ 0 & \epsilon_0 \\ \epsilon_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \epsilon_0 & 0 \\ 0 & \epsilon_0 \\ 0 & 0 \end{pmatrix} \| = \| \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \epsilon_0 \\ 0 & 0 \end{pmatrix} \| = \epsilon_1;$$

$$\|A_2 - A_3\| = \| \begin{pmatrix} 0 & 0 & 0 \\ \epsilon_0 & 0 & 0 \\ \epsilon_1 & 0 & \epsilon_0 \\ 0 & \epsilon_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \epsilon_0 & 0 & 0 \\ \epsilon_1 & 0 & \epsilon_0 \\ 0 & \epsilon_0 & 0 \end{pmatrix} \| = \| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 \\ \epsilon_0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \end{pmatrix} \| = \epsilon_2;$$

and, similarly,

$$\|A_n - A_{n+1}\| = \epsilon_n.$$

Hence there is $A \in M(2^\infty)$ such that

$$A_n \to A.$$  \hspace{1cm} (2)

Let us denote by $\alpha_i$ the $i$-th element of the sequence (1). Then for each $n \in \mathbb{N}$

$$A_n = \begin{pmatrix}
0 & 0 & 0 \\
\alpha_0 & 0 & 0 \\
\alpha_1 & 0 & 0 \\
& \ddots & \ddots \\
& & \alpha_{2^n-2} & 0
\end{pmatrix}.$$
The corresponding basis vectors will be denoted by \( e_i \)'s.

Fix \( k \). Then for any \( n \) such that \( 2^n > k \) we have

\[
A^k_n e_i = \alpha_{i+k-2} \alpha_{i+k-3} \ldots \alpha_{i-1} e_{i+k},
\]

when \( i \leq 2^n - k \) and \( A^k_n e_i = 0 \), when \( i > 2^n - k \). Hence

\[
\|A^k_n\| = \max_{1 \leq i \leq 2^n - k} |\alpha_{i+k-2} \alpha_{i+k-3} \ldots \alpha_{i-1}|.
\]

Then

\[
\|A^k_n\|^{1/k} = \max_{1 \leq i \leq 2^n - k} |\alpha_{i+k-2} \alpha_{i+k-3} \ldots \alpha_{i-1}|^{1/k} \geq |\alpha_{k-1} \alpha_{k-2} \ldots \alpha_0|^{1/k}
\]

for any \( n \in \mathbb{N} \). It was proved by Kakutani [see [6], Solution 104] that

\[
\lim \inf_k |\alpha_{k-1} \alpha_{k-2} \ldots \alpha_0|^{1/k} > 0.
\]

By (2), (3) and (4)

\[
\rho(A) = \lim_k \|A^n\|^{1/k} = \lim_k \lim_n \|A^n_n\|^{1/k} \geq \lim_k \lim_n |\alpha_{k-1} \alpha_{k-2} \ldots \alpha_0|^{1/k} > 0.
\]

\[\square\]

**Remark 3.** There exist other constructions than the one above, which demonstrate discontinuity of spectral radius in \( M(2^\infty) \) (see [11]). However the one above is useful for the proof of Theorem B.

**Proof of Theorem B.** The "only if" part follows from continuity of spectral radius on type I \( C^* \)-algebras [11, Cor. 9.18]. To prove the "if" part, assume \( A \) is not type I. It will be sufficient to construct a sequence of nilpotent elements in \( A \) converging to an element with a positive spectral radius. By Theorem 1 there is a \( C^* \)-subalgebra \( B \subseteq A \) and a surjective \(*\)-homomorphism \( q : B \to M(2^\infty) \). Let \( \text{ev}_\infty : T(M(2^\infty)) \to M(2^\infty) \) be the evaluation at infinity map.

\[
A \xrightarrow{q} B \xrightarrow{\phi} M(2^\infty) \xleftarrow{\text{ev}_\infty} T(M(2^\infty))
\]

Since AF-telescopes are projective ([8]), there is \( \phi : T(M(2^\infty)) \to B \subseteq A \) such that

\[
q \circ \phi = \text{ev}_\infty.
\]

(5)

Let \( A_n \)'s and \( A \) be as in Lemma 2. For each \( t \in [k, k+1], k \in \mathbb{N} \), let \( A_t \) be the linear path connecting \( A_k \) and \( A_{k+1} \), and for each \( t \in (0, 1] \) let \( A_t \) be the linear path connecting \( 0 \) and \( A_1 \). Define \( f_n, n \in \mathbb{N} \), and \( f \) by

\[
f_n(t) = \begin{cases} A_t, & t \in [k, k+1], k \leq n \\ A_n, & t \geq n. \end{cases}
\]

and

\[
f(t) = \begin{cases} A_t, & t \in [k, k+1] \\ A, & t = \infty. \end{cases}
\]

The functions \( f_n, n \in \mathbb{N} \), are obviously continuous, and since \( A_n \to A \), \( f \) is also continuous. Thus \( f_n \) and \( f \) belong to \( T(M((2^\infty))) \). Since all \( A_n \)'s are lower-triangular, so are \( A_t \), for
each $t \in (0, \infty)$. Hence for each $n$ and for each $t$, $f_n(t)$ is nilpotent of order $2^n$. Hence for any $n \in \mathbb{N}$, $f_n$ is nilpotent. Clearly $f_n \to f$ and by Lemma [2]
\[ \rho(f) \geq \rho(f(\infty)) = \rho(A) > 0. \] (6)

Let
\[ a_n = \phi(f_n), \quad a = \phi(f). \]
Then $a_n \to a$ and each $a_n$ is nilpotent. By (5), $q(a) = f(\infty)$. From this and (6) we obtain
\[ \rho(a) \geq \rho(q(a)) = \rho(f(\infty)) > 0. \]

Proof of Theorem A. The ”if” part was proved in [10], Cor.9.18. The ”only if” part follows from Theorem B.

Corollary 4. Let $A$ be a $C^*$-algebra. The following are equivalent:
1) Each precompact set of $A$ is a point of continuity of the joint spectral radius,
2) $A$ is type I.

Proof. 2) $\Rightarrow$ 1) is proved in [10], Cor. 10.31.
1) $\Rightarrow$ 2) follows from Theorem A.

It would be interesting to characterize the class of $C^*$-algebras for which the closure of nilpotents contains a normal element. The following is an easy observation.

Proposition 5. If the closure of nilpotents in a $C^*$-algebra $A$ contains a normal element, then $A$ is not residually type I.

(By residually type I $C^*$-algebra we mean a $C^*$-algebra that has a separating family of $*$-homomorphisms into type I $C^*$-algebras).

Proof. Suppose $N \in A$ is normal, $a_n \in A$ are nilpotents and $a_n \to N$. Then, by Theorem A (in fact by its ”if” part which is proved in [10]), for any $*$-homomorphism $\rho$ to a type I $C^*$-algebra, $\rho(N)$ is quasinilpotent. Since it is also normal, we conclude that $\rho(N) = 0$. □

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DEPARTMENT OF MATHEMATICAL PHYSICS AND DIFFERENTIAL GEOMETRY, INSTITUTE OF MATHEMATICS OF POLISH ACADEMY OF SCIENCES, WARSAW