Incomplete Analytic Hierarchy Process with Minimum Ordinal Violations

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Abstract

Pairwise comparisons offer a natural way of expressing the preferences of the decision makers. Complete and incomplete pairwise comparison matrices have been applied in multi-criteria decision making, as well as in scoring and ranking. Although ordinal information is crucial in both theory and practice, there is a bias in the literature, cardinal models dominate. Purely ordinal models usually lead to non-unique, therefore a dual approach that takes ordinal and cardinal data into consideration, is needed. We allow the priority of ordinal information compared to cardinal one. The incomplete (sparse) logarithmic least squares method is extended by constraints on ordinal consistency. Due to the equivalence of the logarithmic least squares problem on multiplicative pairwise comparison matrices and the least squares problem on additive matrices, the results are widely applicable.

Keywords: pairwise comparison matrix, incomplete, logarithmic least squares, minimum ordinal violations

1. Introduction

Pairwise comparisons are applied in decision theory and decision support, preference modelling, multi-criteria decision making, voting, ranking, scoring and estimating subjective probabilities of future events. We focus on multiplicative or reciprocal ($W_{ij} = 1/W_{ji}$) pairwise comparison matrices, where...
the elements are chosen from a ratio scale, usually composed by the values 1/9, 1/8, ..., 1/2, 1, 2, ..., 8, 9. The use of such matrices has been popular due to the Analytic Hierarchy Process [31], see [22, 24, 33, 34, 36]) for a wide variety of applications. Another important and relevant class of decision problems involves incomplete (sparse) pairwise comparison matrices [9, 19, 23].

The logarithmic least squares problem [12, 16, 17, 30],
\[
\min_{x \in \mathbb{R}^n_+} \sum_{i,j : W_{ij} \text{ is known}} \left( \ln(W_{ij}) - \ln\left( \frac{x_i}{x_j} \right) \right)^2.
\] (1)

originally defined for complete matrices, is extended to the incomplete case in a natural way: taking only the known elements into consideration [25, 35]. The incomplete/sparse logarithmic least squares problem has been applied for weighting criteria [6], ranking (tennis players [7], chess teams [14] and Go players [10]).

Pairwise comparison matrices introduced above are also known as multiplicative pairwise comparison matrices. The element-wise logarithm of a multiplicative pairwise comparison matrix is called an additive (skew-symmetric) pairwise comparison matrix [2, 4]. The least squares (LS) minimization problem for an additive pairwise comparison matrix \( B = [B_{ij}]_{i,j=1...n} \)
\[
\min_{y \in \mathbb{R}^n} \sum_{i,j : B_{ij} \text{ is known}} (B_{ij} - y_i + y_j)^2.
\] (2)

has many applications [2, 3, 4, 26], see also [11] Section 8.1] and [37] Section 2.

When a pairwise comparison matrix is filled in, the decision maker performs two steps, an ordinal and a cardinal one at a time. The ordinal information is that a given criterion/alternative is more important/better compared to another one, or the preference holds in the opposite way, or they are indifferent. Notably, typical decision makers provide ordinal information effortlessly; however, the collection of cardinal data might be more challenging. In fact, the latter collection process often requires additional explanation, e.g., what two times more important/better means in practice. The models and solutions proposed in our paper are motivated by this observation, namely,
that ordinal information is primal, while the cardinal one is secondary in a wide class of decision problems.

The outline of the paper is as follows. Notations and preliminaries are given in Section 2. Minimum Violations (MV) [21, p. 213] also known as the Number of Judgment Reversals (NJR) in [1, p. 217] measures the ordinal discrepancy between a pairwise comparison matrix and a weight vector, i.e., whether the matrix elements \((W_{ij} >, =, < 1)\) contradict the pair of coordinates of the weight vector \((w_i >, =, < w_j)\) or not. This concept is generalized to incomplete matrices by introducing the Overall Ordinal Satisfaction Index, and its combination with the logarithmic least squares in Section 3. The models, written for multiplicative pairwise comparison matrices, transform in a natural way to the additive case. The proposed methods are presented on numerical examples in Section 4, while Section 5 collects some conclusive remarks and future work directions.

2. Notation and Preliminaries

2.1. General Notation

We denote vectors via boldface letters, while matrices are shown with uppercase letters. We use \(A_{ij}\) to address the \((i, j)\)-th entry of a matrix \(A\) and \(x_i\) for the \(i\)-th entry of a vector \(x\). Moreover, we write \(1_n\) and \(0_n\) to denote a vector with \(n\) components, all equal to one and zero, respectively; similarly, we use \(1_{n \times m}\) and \(0_{n \times m}\) to denote \(n \times m\) matrices all equal to one and zero, respectively. We denote by \(I_n\) the \(n \times n\) identity matrix. We express by \(\exp(x)\) and \(\ln(x)\) the component-wise exponentiation or logarithm of the vector \(x\), i.e., a vector such that \(\exp(x)_i = e^{x_i}\) and \(\ln(x)_i = \ln(x_i)\), respectively.

2.2. Graph Theory

Let \(G = \{V, E\}\) be a graph with \(n\) nodes \(V = \{v_1, \ldots, v_n\}\) and \(e\) edges \(E \subseteq V \times V \setminus \{(v_i, v_j) \mid v_i \in V\}\), where \((v_i, v_j) \in E\) captures the existence of a link from node \(v_i\) to node \(v_j\). A graph is said to be undirected if \((v_i, v_j) \in E\) whenever \((v_j, v_i) \in E\), and is said to be directed otherwise. In the following we will consider only undirected graphs. A graph is connected if for each pair of nodes \(v_i, v_j\) there is a path over \(G\) that connects them. Let the neighborhood \(\mathcal{N}_i\) of a node \(v_i\) be the set of nodes \(v_j\) that are connected to \(v_i\) via an edge \((v_i, v_j)\) in \(E\). The degree \(d_i\) of a node \(v_i\) is the number of its
incoming edges\footnote{Over undirected graphs, for each node $v_i$ the number of its incoming and outgoing edges coincide.}, i.e., $d_i = |\mathcal{N}_i|$. The weighted adjacency matrix $A$ of a graph $G = \{V, E\}$ with $n$ nodes is the $n \times n$ matrix such that $A_{ij} > 0$ if $(v_j, v_i) \in E$ and $A_{ij} = 0$ otherwise. The weighted Laplacian matrix associated to a graph $G$, described by a weighted adjacency matrix $A$ is the $n \times n$ matrix $L(A)$, having the following structure.

\[
L_{ij}(A) = \begin{cases} 
-A_{ij} & \text{if } i \neq j \\
\sum_{j \in \mathcal{N}_i} A_{ij}, & \text{if } i = j.
\end{cases}
\]

It is well known that $L(A)$ has an eigenvalue equal to zero, and that, in the case of undirected graphs, the multiplicity of such an eigenvalue corresponds to the number of connected components of $G$\footnote{Over undirected graphs, for each node $v_i$ the number of its incoming and outgoing edges coincide.}. Therefore, the eigenvalue zero has multiplicity one if and only if the graph is connected.

2.3. Incomplete Analytic Hierarchy Process

In this subsection we review the Analytic Hierarchy Process (AHP) problem when the available information is incomplete. Specifically, we review the problem and discuss the Logarithmic Least Squares approach for solving it.

Let us consider a set of $n$ alternatives, and suppose that each alternative is characterized by an unknown utility or value $w_i > 0$. Within the AHP problem, the aim is to compute an estimate of the unknown utilities, based on information on relative preferences. In the incomplete information case, we are given a value $w_{ij} = \epsilon_{ij} w_i / w_j$ for selected pairs of alternatives $i, j$; such a piece of information corresponds to an estimate of the ratio $w_i / w_j$, where $\epsilon_{ij} > 0$ is a multiplicative perturbation that represents the estimation error. Moreover, for all available $w_{ij}$, we assume that $w_{ji} = w_{ij}^{-1} = \epsilon_{ij}^{-1} w_j / w_i$, i.e., the available terms $w_{ij}$ and $w_{ji}$ are always consistent and satisfy $w_{ij} w_{ji} = 1$.

We point out that, while traditional AHP approaches\cite{31,13,5} require knowledge on every pair of alternative, in the partial information setting we are able to estimate the vector $\mathbf{w} = [w_1, \ldots, w_n]^T$ of the utilities, knowing just a subset of the perturbed ratios. Specifically, let us consider a graph $G = \{V, E\}$ with $|V| = n$ nodes; in this view, each alternative $i$ is associated to a node $v_i \in V$, while the knowledge of $w_{ij}$ corresponds to an edge $(v_i, v_j) \in E$. Clearly, since we assume to know $w_{ji}$ whenever we know $w_{ij}$,
the graph $G$ is undirected. Let $W$ be the $n \times n$ matrix such that $W_{ij} = w_{ij}$ for all $(v_i, v_j) \in E$ and $W_{ij} = 0$ if it holds $(v_i, v_j) \notin E$.

Notice that, in the AHP literature, there is no universal consent on how to estimate the utilities in the presence of perturbations (see for instance the debate in [18, 32] for the original AHP problem). This is true also in the incomplete information case, see, for instance, [9, 29, 27]. While the debate is still open, we point out that the logarithmic least-squares approach appears particularly appealing, since it focuses on error minimization.

For these reasons, we conclude this section by reviewing the Sparse Logarithmic Least Squares (SLLS) Method [9, 27], which represents an extension of the classical Logarithmic Least Squares (LLS) Method developed in [13, 5] for solving the AHP problem in the complete information case.

2.4. Logarithmic Least Squares approach to AHP

Within the SLLS algorithm, the aim is to find a logarithmic least-squares approximation $w^*$ to the unknown utility vector $w$, i.e., to find the vector that solves

$$w^* = \arg \min_{w \in \mathbb{R}_+^n} \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \left( \ln(W_{ij}) - \ln \left( \frac{x_i}{x_j} \right) \right)^2 \right\}. \quad (3)$$

An effective strategy to solve the above problem is to operate the substitution $y = \ln(x)$, where $\ln(\cdot)$ is the component-wise logarithm, so that Eq. (3) can be rearranged as

$$w^* = \exp \left( \arg \min_{y \in \mathbb{R}^n} \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \left( \ln(W_{ij}) - y_i + y_j \right)^2 \right\} \right), \quad (4)$$

where $\exp(\cdot)$ is the component-wise exponential. Let us define

$$\kappa(y) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} \left( \ln(W_{ij}) - y_i + y_j \right)^2;$$

because of the substitution $y = \ln(x)$, the problem becomes convex and unconstrained, and its global minimum is in the form $w^* = \exp(y^*)$, where $y^*$ satisfies

$$\frac{\partial \kappa(y)}{\partial y_i} \bigg|_{y=y^*} = \sum_{j \in N_i} \left( \ln(W_{ij}) - y_i^* + y_j^* \right) = 0, \quad \forall i = 1, \ldots, n.$$
Let us consider the $n \times n$ matrix $P$ such that $P_{ij} = \ln(W_{ij})$ if $W_{ij} > 0$ and $P_{ij} = 0$, otherwise; we can express the above conditions in a compact form as

$$L(A)y^* = P1_n,$$  

(5)

where $L(A)$ is the Laplacian matrix associated to the graph $G$, considering an adjacency matrix $A$ with unitary weights, i.e., $A_{ij} \in \{0, 1\}$. Notice that, since for hypothesis $G$ is undirected and connected, the Laplacian matrix $L(A)$ has rank $n - 1$ [20]. Therefore, a possible way to calculate a vector $y^*$ that satisfies the above equation is to fix one arbitrary component of $y^*$ and then solve a reduced size system by simply inverting the resulting nonsingular $(n - 1) \times (n - 1)$ matrix.

Vector $y^*$ can also be written as the arithmetic mean of vectors calculated from the spanning trees of the graph of comparisons, corresponding to the incomplete additive pairwise comparison matrix $\ln W$ [8]. Finally, it is worth mentioning that, when the graph $G$ is connected, the differential equation

$$\dot{y}(t) = -Ly(t) + P1_n$$

asymptotically converges to $y^*$ (see [28]), and represents yet another way to compute it. Notably, the latter approach is typically used by the control system community for formation control of mobile robots, since the computations are distributed in nature and can be performed cooperatively by different mobile robots. Therefore, such an approach appears particularly appealing in a distributed computing setting.

3. Incomplete AHP with Minimum Ordinal Violations

In this section, we develop a novel framework, namely Logarithmic Least Squares with Minimum Ordinal Violations (LLS-MOV). Specifically, let us consider a situation where we are given an incomplete matrix $W$ for $n$ alternatives, corresponding to a connected undirected graph $G$ with $n$ nodes. In view of the developments in this paper, it is convenient to provide the following definitions.

**Definition 1 (Pairwise ordinal preference).** A pairwise ordinal preference for a pair of alternatives $i, j$ is expressed by the pair $x_{ij}, x_{ji} \in \{0, 1\}$, where

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ preferred to } j \\ 0 & \text{otherwise} \end{cases}$$
and it holds
\[ x_{ij} + x_{ji} \leq 1. \]  
(6)

Notice that the condition in Eq. (6) guarantees to avoid inconsistent situations where the \( i \)-th alternative is preferred to the \( j \)-th one and the \( j \)-th one is preferred to the \( i \)-th one. Moreover, we point out that Eq. (6) allows situations where \( x_{ij} = 0 \) and \( x_{ji} = 0 \), i.e., where neither \( i \) nor \( j \) is preferred.

Minimum Violations (MV) [21, p. 213] also known as the Number of Judgment Reversals (NJR) in [11, p. 217] were defined to check whether relations \( W_{ij} > 1 \) and \( x_i > x_j \) are fulfilled together (equalities are also taken into account with weight \( 1/2 \)). We now adopt the idea, with the modification of that indifferences are not considered, and generalize to incomplete matrices.

**Definition 2 (Ordinal Satisfaction Index for \( i, j \)).** Let us consider a pair of alternatives \( i, j \) such that \( W_{ij} > 0 \). Moreover, suppose that an ordinal pairwise preference, expressed as the pair \( x_{ij}, x_{ji} \in \{0, 1\} \), is defined for \( i, j \). The ordinal satisfaction index \( \sigma_{ij}(x_{ij}, x_{ji}) \) for the pair \( i, j \) is

\[
\sigma_{ij}(x_{ij}, x_{ji}) = \begin{cases} 
1 & \text{if } W_{ij} > 1 \text{ and } x_{ij} = 1 \\
1 & \text{if } W_{ij} < 1 \text{ and } x_{ji} = 1 \\
-1 & \text{if } W_{ij} > 1 \text{ and } x_{ji} = 1 \\
-1 & \text{if } W_{ij} < 1 \text{ and } x_{ij} = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

**Definition 3 (Overall Ordinal Satisfaction Index).** Suppose that an ordinal pairwise preference, expressed in terms of the pair \( x_{ij}, x_{ji} \in \{0, 1\} \), is defined for all pairs \( i, j \) of alternatives such that \( W_{ij} > 0 \). The overall ordinal satisfaction index \( \sigma(\{x_{ij}\}) \) is defined as

\[
\sigma(\{x_{ij}\}) = \sum_{(v_i, v_j) \in E, i<j} \sigma_{ij}(x_{ij}, x_{ji}).
\]

In this paper, we aim at finding a ranking vector for the AHP problem in the incomplete information setting while maximizing the Overall Ordinal Satisfaction Index. This is done by solving two cascading subproblems. As a first step, we aim at specifying pairwise ordinal preferences \( \{x_{ij}^*\} \) that minimize the Overall Ordinal Satisfaction Index and satisfy transitivity, in
that if $i$ is preferred to $k$ and $k$ is preferred to $j$ then $i$ is preferred to $j$. Once the first subproblem is solved, we aim at finding a ranking vector $w^*$ via the logarithmic least-squares approach, with the constraint that $w^*_i > w^*_j$ whenever $x^*_{ij} > 0$, i.e., the ranking vector is consistent with the pairwise ordinal preferences $\{x^*_{ij}\}$ obtained as a result of the first subproblem. Let us now characterize the two subproblems.

3.1. Pairwise Ordinal Preferences with Maximum Ordinal Satisfaction

Let us define the $n \times n$ matrix $S$ as the matrix such that $S_{ij} = \text{sign}(\ln(W_{ij}))$ if $W_{ij} > 0$ and $S_{ij} = 0$ if $W_{ij} = 0$. Moreover, for all pairs $i, j$ of alternatives, let us introduce Boolean variables $x_{ij}, x_{ji} \in \{0, 1\}$ representing pairwise ordinal preferences, and let us denote by $\{x_{ij}\}$ the set of all $x_{ij}$ for $i, j \in \{1, \ldots, n\}, i \neq j$. We aim at finding a set $\{x^*_{ij}\}$ that solves the following problem.

**Problem 1.** Find the set $\{x^*_{ij}\} \subset \mathbb{R}$ that solves

$$
\arg\max_{\{x_{ij}\} \subset \mathbb{R}} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} S_{ij} x_{ij}
$$

subject to

$$
\begin{align*}
x_{ij} + x_{ji} &\leq 1, \quad \forall i, j \\
x_{ij} &\geq x_{ik} x_{kj}, \quad \forall i, j, k \\
x_{ij} &\in \{0, 1\}, \quad \forall i, j.
\end{align*}
$$

(7)

Notice that the first constraint is required for $x_{ij}, x_{ji}$ to represent a pairwise ordinal preference, as discussed in Definition 1. Moreover, the constraint $x_{ij} \geq x_{ik} x_{kj}$ models the requirement that the ordinal ranking encoded by the variables $\{x_{ij}\}$ is transitive. In other words, if the $i$-th alternative is preferred to the $k$-th one and the $k$-th one is preferred to the $j$-th one, then alternative $i$ must be preferred to alternative $j$.

Let us denote the objective function of the above problem by $\ell(\{x_{ij}\})$. We point out that the objective function can be rearranged as

$$
\ell(\{x_{ij}\}) = \sum_{(v_i, v_j) \in E, i < j} (S_{ij} x_{ij} + S_{ji} x_{ji}),
$$

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and that, by construction, it holds $S_{ij}x_{ij} + S_{ji}x_{ji} = \sigma_{ij}(x_{ij}, x_{ji})$; therefore, we conclude that $\ell(\{x_{ij}\}) = \sigma(\{x_{ij}\})$, i.e., Problem 1 aims at maximizing the overall ordinal satisfaction index corresponding to $\{x_{ij}\}$.

Before discussing the second problem, let us rearrange Problem 1 as an ILP formulation; this is done in a rather straightforward way by transforming each nonlinear constraint $x_{ij} \geq x_{ik}x_{kj}$ into a set of linear constraints featuring additional Boolean variables $\{z_{ijk}\}$, as shown below.

**Problem 2.** Find the sets $\{x_{ij}^*\} \subset \mathbb{R}$ and $\{z_{ijk}^*\} \subset \mathbb{R}$ that solve

$$\arg \max_{\{x_{ij}\} \subset \mathbb{R}, \{z_{ijk}\} \subset \mathbb{R}} \sum_{i=1}^{n} \sum_{j \in N_i} S_{ij}x_{ij}$$

subject to

$$\begin{cases} x_{ij} + x_{ji} \leq 1, & \forall i, j \\ z_{ijk} \geq x_{ik} + x_{kj} - 1, & \forall i, j, k \\ z_{ijk} \leq x_{ik}, & \forall i, j, k \\ z_{ijk} \leq x_{kj}, & \forall i, j, k \\ x_{ij} \in \{0, 1\}, & \forall i, j \\ z_{ijk} \in \{0, 1\}, & \forall i, j, k. \end{cases}$$

Notice that Problem 1 features $O(n^2)$ variables and $O(n^2)$ constraints; conversely, since each constraint $x_{ij} \geq x_{ik}x_{kj}$ is replaced by $O(n^2)$ variables and $O(n^2)$ constraints, the ILP formulation in Problem 2 requires of $O(n^3)$ variables and constraints.

### 3.2. LLS ranking with prescribed pairwise ordinal preferences

Let us assume we solved the first subproblem, thus obtaining optimal pairwise ordinal relations $\{x_{ij}^*\}$. Within the second subproblem, our aim is to find a utility vector $w^* = \exp(y^*)$, where $y^*$ solves the following problem.
Problem 3. Find $\mathbf{y}^* \in \mathbb{R}^n$ that solves

$$\arg\min_{\mathbf{y} \in \mathbb{R}^n} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} (\ln(W_{ij}) - y_i + y_j)^2$$

subject to

$$\begin{cases} y_i \geq y_j, & \forall i, j, i \neq j \text{ s.t. } x_{ij}^* = 1. \end{cases}$$

The above problem is essentially the classical logarithmic least-squares problem discussed in Section 2.4, with an additional constraint guaranteeing that the ranking vector satisfies $w_i \geq w_j$ whenever $x_{ij}^* = 1$.

Let us conclude the section by providing a necessary and sufficient global optimality condition for Problem 3. To this end, let us consider a graph $G' = \{V, E'\}$, where $V = \{v_1, \ldots, v_n\}$ represents the $n$ alternatives and $(v_i, v_j) \in E'$ whenever $x_{ij}^* > 0$. Clearly, since in Problem 1 we enforce $x_{ij}^* + x_{ji}^* \leq 1$, the graph $G'$ is directed and not necessarily connected, and it holds $E' \subseteq E$. In the following, we denote the neighborhood of a node $v_i$ over $G'$ by $\mathcal{M}_i$.

Theorem 1. Let us consider the AHP problem with incomplete information and let us assume that the graph $G$ corresponding to the ratio matrix $W$ is connected. Let $\{x_{ij}^*\}$ be the optimal solution to Problem 1 and let $G'$ be the corresponding directed graph. Moreover, let $\Lambda^*$ be an $n \times n$ matrix such that $\Lambda_{ij}^* = 0$ whenever $x_{ij}^* = 0$ and such that

$$\omega(\Lambda^*) = \Lambda \odot (Q^T(\Lambda^*) - Q(\Lambda^*) + Z^T - Z) = 0_{n \times n}, \quad (10)$$

where $\odot$ is the element-wise product between matrices of the same dimension,

$$Q(\Lambda^*) = \frac{1}{2} L^\dagger(A)(\Lambda^* - \Lambda^{*T})1_n1_n^T \quad (11)$$

and

$$Z = L^\dagger(A)r1_n^T, \quad (12)$$

where $L^\dagger(A)$ is the pseudo-inverse of the Laplacian matrix $L(A)$ corresponding to the graph $G$ and associated to the adjacency matrix $A$ with unitary weights. The global optimal solution $\mathbf{y}^*$ of Problem 3 satisfies

$$2L\mathbf{y}^*(A) = (\Lambda^* - \Lambda^{*T})1_n + 2\mathbf{r}. \quad (13)$$
Proof 1. Notice that, by construction, the problem is convex and has linear inequality constraints. Therefore, KKT first order conditions represent necessary and sufficient optimality conditions.\footnote{When the objective function is convex and the constraints are linear there is no need to check for constraint qualification conditions such as the Slater’s condition, see, for instance, \cite{boyd94} and reference therein for details.} Let us define $\Lambda$ as the $n \times n$ matrix collecting the Lagrangian multipliers $\Lambda_{ij}$ corresponding to the constraint associated to the pair $i,j$ of alternatives. Note that $\Lambda$ has the same structure as $G'$, i.e., $\Lambda_{ij} = 0$ whenever $(v_i, v_j) \in E'$. The Lagrangian function associated to the problem at hand is:

$$L(y, \Lambda) = \sum_{i=1}^{n} \sum_{j \in N_i} (\ln(W_{ij}) - y_i + y_j)^2 + \sum_{i=1}^{n} \sum_{j \in M_i} \Lambda_{ij}(y_j - y_i)$$

Following standard KKT theory, a necessary and sufficient optimality condition for $y^*$ to be the global optimum is that there is $\Lambda^*$ such that

$$\begin{align}
\frac{\partial L(y, \Lambda)}{\partial y_i} &= 0, \quad \forall i \in \{1, \ldots, n\}, \\
\Lambda^*_{ij}(y^*_j - y^*_i) &= 0, \quad \forall i,j \text{ s.t. } x^*_{ij}, \\
\Lambda^*_{ij} &\geq 0, \quad \forall i,j \text{ s.t. } x^*_{ij}.
\end{align} \quad (14)$$

Note that, for all $i \in \{1, \ldots, n\}$ the first of the above condition corresponds to

$$2 \sum_{j \in N_i} (y^*_i - y^*_j) = \sum_{j \in M_i} \Lambda^*_{ij} - \sum_{i,j \in M_i} \Lambda^*_{ji} + 2 \sum_{j \in N_i} \ln(W_{ij}). \quad (15)$$

Let us denote $\sum_{j \in N_i} \ln(W_{ij})$ by $r_i$ and let $r = [r_1, \ldots, r_n]^T$. Stacking Eq. \ref{eq:15} for all $i \in \{1, \ldots, n\}$, we get Eq. \ref{eq:13}, and since $G$ is connected, by construction $L$ has rank $n-1$ and the above equation is satisfied by

$$y^* = \frac{1}{2} L^\dagger(\Lambda^* - \Lambda^*^T)1_n + L^\dagger r. \quad (16)$$

Putting the second condition in Eq. \ref{eq:14} in matrix form, we get

$$\Lambda \odot (-y^*1_n^T + 1_n y^{*T}) = 0_{n \times n}, \quad (17)$$
where \( \odot \) is the element-wise product between matrices of the same dimension. By plugging Eq. (16) into Eq. (17), we have that \( \Lambda^* \) satisfies Eq. (10), which is the thesis. \( \square \)

4. Experimental Results

4.1. An Illustrative Example

In order to validate the LLS-MOV methodology, let us consider the example in [15, Example 3.4], for which the LLS approach is known to yield an ordering \( \mathbf{w}^{LLS} \) that corresponds to inconsistent pairwise ordinal preferences \( \{x_{ij}^{LLS}\} \), in that \( W_{12} > 1 \) but \( w_{1}^{LLS} < w_{2}^{LLS} \). Specifically, the example encompasses 7 alternatives and the graph underlying the available ratios is given in Figure 1a, while the weight matrix \( W \) is given by

\[ W = \begin{pmatrix} 3 & 3 & \cdots & 3 \\ 1 & 2 & \cdots & 7 \end{pmatrix} \]

3The example in [15] is given for generic coefficients \( b, 1/b \), and we choose the case \( b = 2 \).
Figure 1b shows the ranking $w^{LLS}$ and $w^*$ obtained via the logarithmic least-squares approach and via the LLS-MOV approach, with blue and orange bars, respectively. Notice that the ranking $w^{LLS}$ results in one violation of the linear ordering, since $W_{12} > 1$ but $w_{1}^{LLS} < w_{2}^{LLS}$; in other words, it holds $\sigma_{12}(x_{12}^{LLS}, x_{21}^{LLS}) = -1$, where $\{x_{ij}^{LLS}\}$ are the pairwise ordinal preferences induced by $w^{LLS}$. Notice that the LLS approach yields $\sigma(\{x_{ij}^{LLS}\}) = 11$, while the objective function of Problem 3 computed for $\log(w_{ij}^{LLS})$ is equal to 1.6963 and we observe that the LLS fails to achieve a feasible solution to Problem 3.

Let us now consider the result of the LLS-MOV. Specifically, by solving Problem 2, we obtain a pairwise ordering $\{x_{ij}^*\}$ that can be summarized by the $n \times n$ matrix $X^*$ such that $X^*_{ij} = x_{ij}^*$, i.e.,

$$X^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Notice that $X^* = \text{triu}(\text{sign}(W))$, where $\text{triu}(\cdot)$ it the upper triangular part; in other words, $X^*$ is in perfect accordance with the pairwise ordinal preferences induced by $W$. Moreover, it holds $\sigma(\{x_{ij}^*\}) = 11$. Let us now consider the solution of Problem 3 where the constraints depend on the above choice for $\{x_{ij}^*\}$; the resulting ranking is shown in Figure 1b. Notice that $w_1^* > w_2^*$, although the difference between $w_1^*$ and $w_2^*$ is extremely small, being $w_1^* - w_2^* = 2.4688 \times 10^{-9}$. Notably, the objective function of Problem 3 computed over $\log(w^*)$, is equal to 1.6976, i.e., an increase of just $+0.0766\%$ with respect to the results obtained for $w^{LLS}$. 

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4.2. Performance vs Log-normal Perturbations

Let us now consider the instance depicted in Figure 2 where \( n = 10 \) alternatives are considered and \( |E| = 34 \) pairwise preference information \( W_{ij} \) are given (the figure shows \( 2|E| \) links, including the reciprocal entries). In the figure, we consider nominal ratios that are not affected by perturbations, and we show the numerical value of \( W_{ij} \) along the corresponding link \((v_i, v_j)\).

In order to evaluate the performance of the LLS-MOV approach, we consider log-normal multiplicative perturbations \( \epsilon_{ij} \) for all \( W_{ij} \) (we set \( \epsilon_{ji} = \epsilon_{ij}^{-1} \)) and we plot with a red dotted line the value of the objective function against the magnitude of the standard-deviation of the logarithm of the perturbations. Moreover, for comparison, we plot the value of the objective function of the standard sparse LLS approach with a blue solid line. For each choice of the standard deviation of the perturbation we report the results in terms of average and standard-deviation, considering 100 runs. According to the figure, the objective function within the proposed LLS-MOV approach is comparable with that of the standard Sparse LLS methodology when the standard deviation of the perturbation is below 0.3, while the divide grows for larger perturbations (when the standard deviation of the perturbation is 0.3 there is an increase of +15.24% in the objective function with respect to the Sparse LLS one, while for a standard deviation equal to 1 we have +25.25%). Overall, the results suggest that LLS-MOV is quite effective, and that the cost of enforcing the additional constraints is limited, at least when the perturbations are limited.

5. Conclusions

In this paper we develop a novel approach to reconstruct the ranking of a set of alternatives based on incomplete pairwise comparisons. Specifically, the proposed approach emphasizes the satisfaction of the ordinal preferences encoded by the available information over the cardinal ones. This is done by considering two cascading optimization problems: first, we aim at finding an ordinal ranking that maximizes the accordance with the available information, then we seek a cardinal ranking via the logarithmic least-squares approach, with the additional constraint that the previously chosen ordinal ranking is satisfied. Simulations show that the proposed approach is able to generate rankings that are not in contrast with the available information, while traditional logarithmic least-squares approach may fail. Moreover, at least when the data inconsistencies/perturbations are limited, the results are
Figure 2: Instance considered in the simulations reported in Figure 3.
Figure 3: Comparison of the value of the objective function within the standard Sparse LLS (blue solid line) and the proposed approach (red dotted line), plotted against the standard deviation of the logarithm of log-normal multiplicative perturbations affecting the entries $W_{ij}$. For each choice of the standard deviation we report the results in terms of average and standard-deviation considering 100 runs.
comparable with the classical logarithmic least-squares approach in terms of objective function. Future work will aim at applying the proposed methodology to real-world situations, as well as considering a distributed computing setting.

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