Gorenstein algebras presented by quadrics

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Lincoln, October 15, 2011

Set-up

\[ R = k[x_1, \ldots, x_r], \quad k \text{ a field} \]
\[ A = R/I = \oplus_{j=1}^{e} [A]_j, \quad \text{a graded artinian Gorenstein algebra} \]

Hilbert series of \( H_A(z) := \sum_{j=0}^{e} \dim_k [A]_j z^j := \sum_{j=0}^{e} h_j z^j \)

- **h-vector or Hilbert function** of \( A \) is \( h = (h_0, h_1, \ldots, h_e) \)
- **Symmetry**: \( h_i = h_{e-i} \), \( e \) the socle degree
- **WLOG** \([I]_1 = 0\), so \( h_1 = r \), the codimension

Two scenarios:
(A) \( I \) contains a regular sequence of \( r \) quadrics.
(B) The minimal generators of \( I \) all have degree 2.

**Question:** What are the possible Hilbert functions of \( A \), assuming condition (A) or (B)?

Basic restrictions

**Lemma**

Assume \( A \) contains a regular sequence of \( r \) quadrics. Then

(a) \( h_2 \leq \binom{r+1}{2} - r = \binom{r}{2} \).
(b) \( e \leq r \) and \( e = r \) if and only if \( A \) is a complete intersection of \( r \) quadrics. In this case

\[ h = (1, r, \binom{r}{2}, \binom{r}{3}, \ldots, \binom{r}{r-3}, \binom{r}{r-2}, r, 1) \]

(c) (Kunz) If \( e < r \), then \( I \) has at least \( r + 2 \) minimal generators.

Constructions

**Proposition**

If \( A \) and \( B \) are artinian Gorenstein algebras presented by quadrics, then so is \( A \otimes_k B \), and its Hilbert series is \( H_A(z) \cdot H_B(z) \).

**Corollary 1.**

If \((h_0, \ldots, h_e)\) is the \( h \)-vector of a Gorenstein algebra presented by quadrics, then so is the \( h \)-vector \((h_0, h_0 + h_1, h_1 + h_2, \ldots, h_{e-1} + h_e, h_e)\).

**Example**

(i) \( e = 2 \): (Sally) For each \( r \geq 1 \), \( h = (1, r, 1) \) is the \( h \)-vector of a Gorenstein algebra presented by quadrics.
(ii) \( e = 3 \): For each \( r \geq 1 \), \( h = (1, r, r, 1) \) is the \( h \)-vector of a Gorenstein algebra presented by quadrics.
**Constructions**

**Corollary 2.**
Fix $r \geq 2$. Then, for each $e$ such that $2 \leq e \leq r$, there is an artinian Gorenstein algebra presented by quadrics of codimension $h_1 = r$ with socle degree $e$.

Recall: $e = r$ iff $I$ is a complete intersection.

**Corollary 3.**
Fix $r \geq 2$. Then, for each $e$ such that $4 \leq e \leq r - 2$, there are at least two artinian Gorenstein algebras presented by quadrics with socle degree $e$ and $h_1 = r$ that have different values of $h_2$.

When $e = r - 1$?

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**Submaximal socle degree**

**Theorem**
Assume that $e = r - 1 \geq 4$ and that $I$ contains a regular sequence of $r$ quadrics. Then:

(a) $h_2$ must be either \( \binom{r}{2} - 2 \), \( \binom{r}{2} - 1 \), or \( \binom{r}{2} \), and all of these possibilities do occur.

(b) If $R/I$ is presented by quadrics, then $h_2 = \binom{r}{2} - 2$.

(c) If $h_2 = \binom{r}{2} - 2$, then $R/I$ is presented by quadrics, and the entire Hilbert function of $R/I$ is uniquely determined. It is

\[
h_j = \binom{r - 1}{j} + \binom{r - 3}{j - 1}.
\]

(d) For $r \geq 7$, if $h_2 = \binom{r}{2} - 1$ or \( \binom{r}{2} \), then the Hilbert function of $R/I$ is not uniquely determined, at least if $\text{char } k = 0$.

For $r = 6$, $h = (1, 6, h_2, 6, 1)$

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**Injectivity conjecture**

**Injectivity Conjecture**
Assume $R/I$ is presented by quadrics, has socle degree $e \geq 3$, and $\text{char } k \neq 2$. Let $L \in R$ be a general linear form. Then the multiplication $\times L : [R/I]_1 \to [R/I]_2$ is injective.

Assumption on the characteristic is necessary:
If $I = (x_1^2, \ldots, x_r^2)$ and $\text{char } k = 2$, then $L^2 \in I$ for every linear form $L$.

**Socle Lemma (Huneke, Ulrich)**
Let $M \neq 0$ be a finitely generated graded $R$-module. Assume $\text{char } k = 0$. Let $L \in [R]_1$ be a general linear form, and consider the exact sequence

$$0 \to K \to M(-1) \xrightarrow{L} M \to C \to 0.$$  

If $K \neq 0$, then the initial degrees satisfy $a(K) > a(Soc(C))$.

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**Submaximal socle degree**

**Proposition**
Assume that $\text{char } k = 0$. Then, for any complete intersection $I = (Q_1, \ldots, Q_r)$ of quadrics, the Injectivity Conjecture is true.

**WLP Conjecture**
Assume that $\text{char } k = 0$. Then every artinian Gorenstein algebra presented by quadrics and of socle degree $e \geq 3$ has the Weak Lefschetz Property (WLP), that is, for some linear form $L$, the multiplication

$$\times L : [A]_i \to [A]_{i+1}$$  

has maximal rank for all $i$ (i.e. is injective or surjective).

Note: The Gorenstein assumption can not be dropped. For example, if the ideal $I$ is generated by squares of $r + 1$ general linear forms in an even number of variables, $r$, then $R/I$ does not have the WLP (Migliore, Miró-Roig, N., 2010).
Injectivity conjecture

Corollary
Any complete intersection of at most 4 quadrics has the WLP, provided \( \text{char } k = 0. \)

Corollary
Let \( I \) be a complete intersection of \( r \geq 5 \) quadrics, and assume \( \text{char } k = 0. \) Then, for a general linear form \( L \), the multiplication \( \times L : [R/I]_2 \to [R/I]_3 \) has at most a 1-dimensional kernel.

For \( r = 5 \), the \( h \)-vector is \((1, 5, 10, 10, 5, 1)\).

Persistence

Proposition
Assume that \( A \) is an artinian Gorenstein algebra for which the multiplication by a general linear form on \( A \) from degree 1 to degree 2 is an isomorphism (so \( h_2 = r \)). Then \( h_i = r \) for all \( i = 1, 2, \ldots, e - 1. \) Furthermore, if \( A \) is presented by quadrics then \( e = 3. \)

Corollary
Let \( A \) be an artinian Gorenstein algebra presented by quadrics. Assume that \( r \geq 3 \) and that the Injectivity Conjecture is true for \( A \). Then the following are equivalent:
1. \( h_2 = r; \)
2. \( e = 3; \)
3. The \( h \)-vector is \((1, r, r, 1)\).

Injectivity conjecture

Lemma
Assume that \( R/I \) is presented by quadrics and that the Injectivity Conjecture is true for \( R/I \). Let \( Q_1, \ldots, Q_{h-1} \in R \) be \( h - 1 \) general quadrics, where \( h = \dim_k [R/I]_2 \), and consider the ideal \( J = (I, Q_1, \ldots, Q_{h-1}) \). Then \( R/J \) is Gorenstein with Hilbert function \((1, r, 1)\).

Idea of Proof: To show \( \text{Soc } R/J \] \( = 0, \) that is, for each \( \ell \in [R]_1, \ell \cdot [R]_1 \) is not in \((I, Q_1, \ldots, Q_{h-1})\). Let

\[ \Lambda = \{ \text{hyperplanes in } \mathbb{P}([R]_2) \text{ containing } [I]_2 \} \]

\[ \Sigma = \{ \text{linear subvarieties } \mathbb{P}(\ell \cdot [R]_1) \mid \ell \in [R]_1 \} \]

\[ I = \{ (A, H) \in \Sigma \times \Lambda \mid A \subset H \} \]

Consider the projections \( \phi_1 \), \( \phi_2 \), \( \phi_3 \).

The Injectivity Conjecture provides \( \dim I = h - 2 \), whereas \( \dim \Lambda = h - 1. \)

Persistence

Theorem
Let \( A \) be presented by quadrics and of socle degree \( e \geq 4. \) Assume that \( h_2 = r < 4e - 6 \) and \( \text{char } k = 0. \) Then multiplication by a general linear form on \( A \) from degree 1 to degree 2 must be an isomorphism.

Proof uses the theory of generic initial ideals.

Note: \( h = (1, 21, 21, 20, 21, 21, 1) \) is the \( h \)-vector of a Gorenstein algebra.
Proposition

For $r \leq 5$, the following are the only $h$-vectors of artinian Gorenstein algebras presented by quadrics.

| $r$ | $h$-vectors |
|-----|-------------|
| 2   | $(1, 2, 1)$  |
| 3   | $(1, 3, 1), (1, 3, 3, 1)$ |
| 4   | $(1, 4, 1), (1, 4, 4, 1), (1, 4, 6, 4, 1)$ |
| 5   | $(1, 5, 1), (1, 5, 5, 1), (1, 5, 8, 5, 1), (1, 5, 10, 10, 5, 1)$ |

Remark

(i) $h_2 \leq 12$, $h_2 \neq 6$.

(ii) If char $k > 0$ is small, then $h_2 = 12$ is possible.

Theorem (Davis, Okun, 2001)

If $I$ is a squarefree, monomial Gorenstein ideal generated by quadrics with $h$-vector $(1, r, h_2, r, 1)$, then $h_2 \geq 2r - 2$. 

Proposition

For $r = 6$, the following are $h$-vectors of artinian Gorenstein algebras presented by quadrics and of socle degree $e$.

| $e$ | $h$-vectors |
|-----|-------------|
| 2   | $(1, 6, 1)$  |
| 3   | $(1, 6, 6, 1)$ |
| 4   | $(1, 6, h_2, 6, 1)$, where $h_2 \in \{10, 11\}$ |
| 5   | $(1, 6, 13, 13, 6, 1)$ |
| 6   | $(1, 6, 15, 20, 15, 6, 1)$ |

Remark

(i) $h_2 \leq 12$, $h_2 \neq 6$.

(ii) If char $k > 0$ is small, then $h_2 = 12$ is possible.