ON A VON NEUMANN ALGEBRA WHICH IS A COMPLEMENTED SUBSPACE

ERIK CHRISTENSEN AND LIGUANG WANG

Abstract. Let $M$ be a von Neumann algebra of type II$_1$ which is also a complemented subspace of $B(H)$. We establish an algebraic criterion, which ensures that $M$ is an injective von Neumann algebra. As a corollary we show that if $M$ is a complemented factor of type II$_1$ on a Hilbert space $H$, then $M$ is injective if its fundamental group is non-trivial.

1. Introduction

In the early works [12]-[14] by Murray and von Neumann, they realized that there is a certain sort of rings of operators which to a large extent behave like the algebras $M_n(C)$ consisting of all complex $n \times n$ matrices, except that the natural dimension function now has the image $[0,1]$ instead of the set $\{0,1,\ldots,n\}$. Today rings of operators are called von Neumann algebras and the ones with a continuous dimension function with values in $[0,1]$ are called von Neumann algebras of type II$_1$. Factors are von Neumann algebras whose centers consist of scalar multiples of the identity. Finite-dimensional factors are (isomorphic to) full matrix algebras. Infinite-dimensional factors admitting a positive and bounded trace are called factors of type II$_1$. Murray and von Neumann also realized that there are at least two non-isomorphic factors of type II$_1$, namely the free group factor $L(F_2)$ and the hyperfinite type II$_1$ factor $R$, where $L(F_2)$ is the von Neumann algebra obtained by taking the ultraweak closure of the left regular representation of the non-abelian free group $F_2$ on 2 generators, while the hyperfinite type II$_1$ factor $R$ was constructed as the ultraweak closure of an increasing sequence of full matrix algebras.

The hyperfinite type II$_1$ factor $R$ is easily proven to be an injective object in the category of C*-algebras and completely positive mappings (see Paulsen’s book [15]), and it turns out [23] that a C*-algebra $A$ acting on a Hilbert space $H$ is injective if and only if there exists a projection $\Pi : B(H) \to A$ of norm 1. Such a projection turns out to be completely positive and $A$-modular, in the sense that $\Pi(AT) = A\Pi(T)$ and $\Pi(TA) = \Pi(T)A$ for all $T$ in $B(H)$ and $A$ in $A$. The fundamental work by Connes [5] showed among other things, that all injective type II$_1$ factors on a separable Hilbert space are isomorphic to the hyperfinite type II$_1$ factor $R$. Based on parts of this work, Connes gave in [6] a

Date: today.

2000 Mathematics Subject Classification. 46L10, 46L50.

Key words and phrases. Type II$_1$ factor; Fundamental group; Hyperfinite type II$_1$ factor; Injective von Neumann algebra; Complemented subspace.

Partially supported by NSFC (11371222) and NSF of Shandong Province (ZR2012AM024).
cohomological characterization of the injective type II\textsubscript{1} factor. The module Connes used as coefficients for the cohomology consists of bounded mappings on $\mathcal{B}(\mathcal{H})$ which are modular for the commutant $\mathcal{M}'$. This module was then studied by Bunce and Paschke in \cite{1} and they proved that if a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is the image of a bounded $\mathcal{M}$-module projection $\Pi : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$, then $\mathcal{M}$ is injective. It is then a natural question to ask if a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, which is a complemented subspace of $\mathcal{B}(\mathcal{H})$, must be injective. For such an algebra, there exists a bounded projection $\Pi : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$ and it was proven - by quite different methods - by Pisier in \cite{18} and by the first named author and Sinclair in \cite{2}, that if the projection $\Pi$ is completely bounded (see \cite{15}), then $\mathcal{M}$ is an injective von Neumann algebra. In Corollary 4.6 of \cite{10}, which was published before the just named result, Haagerup and Pisier showed that if a von Neumann algebra $\mathcal{M}$ is isomorphic to $M_n(\mathcal{M})$ for some natural number $n \geq 2$ and also is a complemented subspace then there exists a completely bounded projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$. It then follows that if a type II\textsubscript{1} factor $\mathcal{M}$ is a complemented subspace and its fundamental group contains an integer $n \geq 2$ then it is injective.

It is well known that type I von Neumann algebras are injective and the results on completely bounded mappings yield quite easily that if a properly infinite von Neumann algebra is a complemented subspace, then it is an injective von Neumann algebra. For a von Neumann algebra $\mathcal{M}$ which is a complemented subspace of $\mathcal{B}(\mathcal{H})$, the question of injectivity is then reduced to the case of von Neumann algebra of type II\textsubscript{1}. Some partial results, which all show injectivity, exist (\cite{3}, \cite{4}, \cite{10}, \cite{16}, \cite{17}) and in this article we will add some more, which are based on properties of certain $\ast$-endomorphisms on $\mathcal{M}$.

We show that if a von Neumann algebra $\mathcal{M}$ of type II\textsubscript{1} on a Hilbert space $\mathcal{H}$ is a complemented subspace of $\mathcal{B}(\mathcal{H})$ and $\Phi : \mathcal{M} \to \mathcal{M}$ is an injective normal $\ast$-endomorphism which has the property that for the center-valued trace $\text{Tr}$ of $\mathcal{M}$ onto the center of $\mathcal{M}$, we have $\|\text{Tr}(\Phi(I))\| < 1$, then $\mathcal{M}$ is injective. This result may be applied to the factor case and we show that if $\mathcal{M}$ is a factor of type II\textsubscript{1} on a Hilbert space $\mathcal{H}$ and there is a bounded projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$, then $\mathcal{M}$ is injective, if the fundamental group of $\mathcal{M}$ is non-trivial.

The proof of the theorem is based on a trick which shows the existence of a completely bounded projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$. The methods used in the trick are very much inspired by some matrix constructions, which were used by Pisier in his studies on the similarity degree. Examples of such constructions may be found in the article \cite{19}.

2. Complemented von Neumann algebras and endomorphisms

In this section $\mathcal{M}$ will always denote a von Neumann algebra of type II\textsubscript{1} on a Hilbert space $\mathcal{H}$ and $\Phi : \mathcal{M} \to \mathcal{M}$ a normal and injective $\ast$-endomorphism. We will define the projection $E$ in $\mathcal{M}$ by $E := \Phi(I)$ and let $\text{Tr} : \mathcal{M} \to \mathcal{M} \cap \mathcal{M}'$ denote the unique center-valued trace of $\mathcal{M}$. For a linear mapping $\Gamma$ between operator algebras, we will use the notation $\Gamma_n := \Gamma \otimes \text{id}_{M_n(\mathbb{C})}$ and $\|\Gamma\|_n := \|\Gamma_n\|$.

It is our aim to prove the following theorem.
Theorem 1. Suppose \( \Pi : \mathcal{B}(\mathcal{H}) \to \mathcal{M} \) is a bounded projection and \( \Phi : \mathcal{M} \to \mathcal{M} \) is a normal injective *-homomorphism. If \( \|\text{Tr}(\Phi(I))\| < 1 \), then \( \mathcal{M} \) is injective.

We will base the proof of Theorem 1 on some steps which we formulate as individual lemmas.

Lemma 2. Let \( n \) be a natural number. Without loss of generality we may assume that \( \|\text{Tr}(E)\| < \frac{1}{n} \) and that there exists a surjective isometry \( V : \mathcal{H} \to E(\mathcal{H}) \) such that for all \( T \in \mathcal{M} \) we have \( \Phi(T) = VTV^* \).

Proof. Since \( \Phi \) is faithful and \( E \neq 0 \), we have \( 0 < \alpha := \|\text{Tr}(E)\| < 1 \). Since \( \Phi \) is a normal isomorphism onto its image, say \( \mathcal{N} \), we get that \( E \) is the unit of \( \mathcal{N} \) and for \( F := \Phi(E) \) we get \( \text{Tr}_\mathcal{N}(F) \leq \alpha E \). Hence \( \text{Tr}_\mathcal{M}(F) \leq \alpha^2 I \). This process may be iterated and we see that there exists a natural number \( k \) such that \( \|\text{Tr}(\Phi^k(I))\| < \frac{1}{n} \). A replacement of \( \Phi \) by \( \Phi^k \) proves the first claim.

With respect to the second condition, we remark that if we look at the amplifications of \( \mathcal{M} \) and \( \Phi \) to the algebra \( \mathcal{M} := \mathcal{M} \otimes \mathbb{C}I \) on \( \mathcal{H} \otimes l^2(\mathbb{N}) \), then \( \Phi \) is implemented by a surjective isometry \( V : \mathcal{H} \otimes l^2(\mathbb{N}) \to E(\mathcal{H}) \otimes l^2(\mathbb{N}) \). Since \( \mathcal{B}(\mathcal{H}) \) is an injective von Neumann algebra, it follows that \( \mathcal{M} \) is complemented inside \( \mathcal{B}(\mathcal{H} \otimes l^2(\mathbb{N})) \), and thus we may as well assume that \( \Phi(T) = VTV^* \) for \( T \in \mathcal{M} \), as claimed. \( \square \)

For projections \( P \) and \( Q \) in \( \mathcal{M} \), we say \( P \) is weaker than \( Q \) (and write \( P \preceq Q \)) when \( P \) is (Murray-von Neumann) equivalent to a subprojection of \( Q \) (see Definition 6.2.1 of [11]). The following lemma is probably well known, but we do not have a reference at hand.

Lemma 3. Let \( P \) and \( Q \) be projections in \( \mathcal{M} \). Then \( P \preceq Q \) if and only if \( \text{Tr}(P) \leq \text{Tr}(Q) \).

Proof. Suppose \( P \preceq Q \). Then there exists a partial isometry \( W \) in \( \mathcal{M} \) such that \( W^*W = P \) and \( WW^* \leq Q \). Since \( \text{Tr} \) is positive and a trace \( (\text{Tr}(S^*S) = \text{Tr}(SS^*)) \) for all \( S \) in \( \mathcal{M} \), we get \( \text{Tr}(P) \leq \text{Tr}(Q) \).

Suppose that \( \text{Tr}(P) \leq \text{Tr}(Q) \) and \( P \) is not weaker than \( Q \). By the Comparison Theorem (see Theorem 6.2.7 in [11]), there exists a nonzero central projection \( Z \) such that \( ZQ \preceq ZP \) and \( ZQ \sim ZP \). Hence there exists a partial isometry \( W \) in \( \mathcal{M} \) such that \( W^*W = ZQ \), \( WW^* \leq ZP \) and \( WW^* \neq ZP \). Since \( \text{Tr} \) is a faithful center-valued trace and center-modular, we get \( \text{Tr}(Z(P - Q)) \geq 0 \) and \( \text{Tr}(Z(P - Q)) \neq 0 \). On the other hand,

\[
\text{Tr}(Z(P - Q)) = Z\text{Tr}(P - Q) = Z(\text{Tr}(P) - \text{Tr}(Q)) \leq 0
\]

which is a contradiction. So \( P \preceq Q \) and the lemma follows. \( \square \)

Lemma 4. Let \( n \) be a natural number. If \( \|\text{Tr}(\Phi(I))\| = \|\text{Tr}(E)\| < \frac{1}{n} \), then there is a set \( \{E_1, E_2, \cdots, E_n\} \) of pairwise orthogonal and equivalent projections in \( \mathcal{M} \) such that \( E_1 = E \).

Proof. Let \( \{E_1, E_2, \cdots, E_k\} \) be a maximal family of pairwise orthogonal and equivalent projections in \( \mathcal{M} \) with \( E_1 = E \). Such a family must be finite since \( \mathcal{M} \) is a finite von Neumann algebra. Let

\[
F = I - E_1 - E_2 - \cdots - E_k.
\]
If \( k < n \), then
\[
\text{Tr}(F) = I - k\text{Tr}(E) \geq I - (n - 1)\text{Tr}(E) \\
\geq I - \frac{n-1}{n}I = \frac{1}{n}I \geq \text{Tr}(E).
\]
Hence by Lemma 4, \( E \not\subset F \) and the family \( \{E_1, E_2, \cdots, E_k\} \) is not maximal which is a contradiction. Therefore \( k \geq n \) and the lemma follows. \( \square \)

**Lemma 5.** Suppose \( \Pi : \mathcal{B}(H) \to \mathcal{M} \) is a bounded projection and \( \mathcal{A} \) is a finite dimensional von Neumann subalgebra of \( \mathcal{M} \), which has the same unit as \( \mathcal{M} \). Then there exists a bounded projection \( \Psi : \mathcal{B}(H) \to \mathcal{M} \) which is \( \mathcal{A} \)-modular and satisfies \( \|\Psi\| \leq \|\Pi\| \).

**Proof.** Let \( G = U(\mathcal{A}) \) be the group of all unitary operators in \( \mathcal{A} \). Then \( G \) is a compact group and there is a Haar probability measure \( \mu \) on \( G \). For \( S \in \mathcal{B}(H) \), we define
\[
\Psi(S) = \int_{U_1 \in G} \int_{U_2 \in G} U_1\Pi(U_1^*SU_2)U_2^*d\mu(U_1)d\mu(U_2).
\]
Then for \( T \in \mathcal{M} \), we have \( \Psi(T) = T \) and for \( U_1, U_2 \in G, S \in \mathcal{B}(H) \),
\[
\Psi(U_1SU_2) = U_1\Psi(S)U_2
\]
and the lemma follows. \( \square \)

**Lemma 6.** If \( \|\text{Tr}(E)\| < \frac{1}{n} \), then there exists a projection \( \Gamma \) of \( \mathcal{B}(H) \) onto \( \mathcal{M} \) such that \( \|\Gamma\|_n \leq \|\Pi\| \).

**Proof.** It follows from Lemma 4 that there exists a set \( \{E_1, E_2, \cdots, E_n\} \) of pairwise orthogonal and equivalent projections in \( \mathcal{M} \) such that \( E = E_1 \). We may supplement this set to a set of matrix units \( \{E_{ij} : 1 \leq i, j \leq n\} \) for which \( E_{ii} = E_i \) and define \( F = I - E_1 - E_2 - \cdots - E_n \). Then the algebra \( \mathcal{A} \) defined as
\[
\mathcal{A} = \text{span}\{F \cup \{E_{ij} : 1 \leq i, j \leq n\}\}
\]
is a finite dimensional von Neumann subalgebra of \( \mathcal{M} \) having the same unit as \( \mathcal{M} \), and by Lemma 5 we have a projection \( \Psi : \mathcal{B}(H) \to \mathcal{M} \) such that \( \Psi \) is \( \mathcal{A} \)-modular and \( \|\Psi\| \leq \|\Pi\| \). Let \( \mathcal{L} \) be the von Neumann subalgebra of \( \mathcal{M} \) generated by \( \Phi(\mathcal{M}) \) and \( \mathcal{A} \). Then there is a projection \( \Omega \) of norm 1 from \( \mathcal{M} \) onto \( \mathcal{L} \). Based on this we can construct a projection \( \Gamma : \mathcal{B}(H) \to \mathcal{M} \) by
\[
\Gamma(T) = V^*\Omega(\Psi(VTV^*))V.
\]
In order to estimate the norm \( \|\Gamma\|_n \), we construct an isometry \( W \) of \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) (\( n \)-times) onto \( (E_{11} + E_{22} + \cdots + E_{nn})\mathcal{H} \) by defining a row matrix
\[
W = [E_{11}V \ E_{21}V \ \cdots \ \ E_{n1}V]
\]
in \( M_{1 \times n}(\mathcal{B}(H)) \). Then for any \( X = [X_{ij}]_{i,j=1}^n \in M_n(\mathcal{B}(H)) \), we have
\[
WT_n(X)W^* \in (E_{11} + E_{22} + \cdots + E_{nn})\mathcal{B}(H)(E_{11} + E_{22} + \cdots + E_{nn}).
\]
Note $\Gamma_n(X) = [V^*\Omega(\Psi(VX_{ij}V^*))V]_{i,j=1}^n$ and then by the $\mathcal{A}$-modularity of $\Omega$ and $\Psi$, we get

$$WT_n(X)W^* = \sum_{i,j=1}^n E_{ij}\Omega(\Psi(VX_{ij}V^*))E_{ij}$$

$$= \Omega(\Psi(\sum_{i,j=1}^n E_{ij}VX_{ij}V^*E_{ij}))$$

$$= \Omega(\Psi(W[X_{ij}]_{i,j=1}^n W^*))$$

Hence

$$||\Gamma_n(X)|| = ||WT_n([X_{ij}]_{i,j=1}^n)W^*||$$

$$= ||\Omega(\Psi(W[X_{ij}]_{i,j=1}^n W^*))||$$

$$\leq ||\Psi|| ||[X_{ij}]_{i,j=1}^n||$$

$$\leq ||\Pi|| ||X||.$$

This completes the proof of the lemma. $\square$

**Proof of Theorem 1.** From the lemmas above we see that for each $n \in \mathbb{N}$, there exists a projection $\Gamma_n$ of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$ such that

$$||\Gamma_n||_n = ||\Gamma_n|| \leq ||\Pi||.$$  

Since $\{\Gamma_n : n \in \mathbb{N}\}$ is a bounded sequence of projections of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$, it has a subnet that converges pointwise ultraweakly to a linear mapping $\Theta$ of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$. Hence $\Theta$ is a completely bounded projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$ with $||\Theta||_{cb} \leq ||\Pi||$, and it follows from [2] or [18] that $\mathcal{M}$ is an injective von Neumann algebra. This completes the proof. $\square$

We will consider a type II$_1$ von Neumann algebra $\mathcal{M}$ but now also assume that $\mathcal{M}$ is a factor on a Hilbert space $\mathcal{H}$ which is a complemented subspace of $\mathcal{B}(\mathcal{H})$. We will present four results of the type ”If $\mathcal{M}$ has a certain property, then $\mathcal{M}$ is injective”. Only the first corollary is new, the second and the third appeared in [4], [17], [18] and the fourth in [10]. We have included new proofs here, because these results follow easily from our theorem. We remind the readers of the four properties we want to look at.

**Definition 7.** Let $\mathcal{M}$ be a factor of type II$_1$.

(a) Let $\mathcal{R}$ denote the hyperfinite type II$_1$ factor. Then $\mathcal{M}$ is said to be a McDuff factor if $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}$.

(b) $\mathcal{M}$ is said to be non-prime if $\mathcal{M}$ is isomorphic to the tensor product $\mathcal{F} \otimes \mathcal{F}$ of two type II$_1$ factors $\mathcal{F}$ and $\mathcal{F}$.

In Kadison and Ringrose’s book (Exercise 13.4.6 of [11]), the authors studied the fundamental group $\mathcal{F}(\mathcal{M}) \subseteq (\mathbb{R}^+, \times)$ of a type II$_1$ factor $\mathcal{M}$. We say that the fundamental group of $\mathcal{M}$ is nontrivial if $\mathcal{F}(\mathcal{M}) \neq \{1\}$.

As usual we will let $\mathcal{L}(\mathbb{F}_n)$ denote the von Neumann algebra generated by the left regular representation of the non-abelian free group $\mathbb{F}_n$ on $n$ generators $(2 \leq n \leq \infty)$. Let $\mathcal{L}(\mathbb{F}_r)$
denote the interpolated free group factors for \( r \in (1, +\infty] \). We refer to the articles [8] and [21] for the constructions and properties of the interpolated free group factors. We have obtained the following corollaries.

**Corollary 8.** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \) on a Hilbert space \( \mathcal{H} \). If the fundamental group \( \mathcal{F}(\mathcal{M}) \neq \{1\} \) and there is a bounded projection \( \Pi \) of \( \mathcal{B}(\mathcal{H}) \) onto \( \mathcal{M} \), then \( \mathcal{M} \) is injective.

**Proof.** Suppose \( \tau \) is the normal faithful tracial state of \( \mathcal{M} \). Since \( \mathcal{F}(\mathcal{M}) \neq \{1\} \), there exists a \( t \in \mathcal{F}(\mathcal{M}) \), \( 0 < t < 1 \). Thus there exists a nonzero projection \( E \) in \( \mathcal{M} \) such that \( \tau(E) = t \) and \( \mathcal{M} \cong E \mathcal{M} E \). Let \( \Phi : \mathcal{M} \to E \mathcal{M} E \) be a normal \( * \)-isomorphism. Then the result follows from Theorem [1]. \( \square \)

**Corollary 9.** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \) on a Hilbert space \( \mathcal{H} \). If \( \mathcal{M} \) is McDuff and there is a bounded projection \( \Pi \) of \( \mathcal{B}(\mathcal{H}) \) onto \( \mathcal{M} \), then \( \mathcal{M} \) is injective.

**Proof.** If \( \mathcal{M} \) is McDuff, then \( \mathcal{M} \cong \mathcal{M} \mathcal{B} \mathcal{M} \) which in turn implies that \( \mathcal{F}(\mathcal{M}) = \mathbb{R}^+ \) and the result follows from Corollary [8]. \( \square \)

**Corollary 10.** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \) on a Hilbert space \( \mathcal{H} \). If \( \mathcal{M} \) is non-prime and there is a bounded projection \( \Pi \) from \( \mathcal{B}(\mathcal{H}) \) onto \( \mathcal{M} \), then \( \mathcal{M} \) is an injective factor.

**Proof.** Suppose \( \mathcal{M} = \mathcal{I} \mathcal{E} \mathcal{T} \) where \( \mathcal{I} \) and \( \mathcal{T} \) are two \( \text{II}_1 \) factors. Then \( \mathcal{I} \) contains a copy \( \mathcal{B} \) of the hyperfinite \( \text{II}_1 \) factor with the same unit as \( \mathcal{I} \). Hence \( \mathcal{M} \supseteq \mathcal{I} \mathcal{E} \mathcal{T} \) and since \( \mathcal{M} \) is a \( \text{II}_1 \) factor, the von Neumann algebra \( \mathcal{I} \mathcal{E} \mathcal{T} \) is complemented in \( \mathcal{M} \) and then also complemented in \( \mathcal{B}(\mathcal{H}) \). On the other hand, \( \mathcal{I} \mathcal{E} \mathcal{T} \) is clearly a McDuff factor and then it is injective by Corollary [9]. Since \( \mathcal{C} \mathcal{I} \mathcal{E} \mathcal{T} \) is a \( \text{II}_1 \) subfactor of the injective \( \text{II}_1 \) factor \( \mathcal{I} \mathcal{E} \mathcal{T} \), we see that \( \mathcal{T} \) is injective and by symmetry \( \mathcal{I} \) is injective. Hence \( \mathcal{M} = \mathcal{I} \mathcal{E} \mathcal{T} \) is also injective and the corollary follows. \( \square \)

The last corollary is due to Haagerup and Pisier and was presented in Corollary 4.7 of [10]. We present a proof which is based on Voiculescu’s formula

\[
\mathcal{L}(\mathbb{F}_{k+1}) \cong \mathcal{L}(\mathbb{F}_{n^k+1}) \otimes M_n(\mathbb{C}) \quad (k, n \in \mathbb{N})
\]

and its generalization to the interpolated free group factors

\[
\mathcal{L}(\mathbb{F}_r) \cong \mathcal{L}(\mathbb{F}_{(r-1)^2+1}) \quad (1 < r \leq \infty, 0 < t < \infty)
\]

The latter formula was obtained independently by Dykema [8] and Radulescu [21].

**Corollary 11.** No interpolated free group factor is isomorphic to a complemented von Neumann algebra of type \( \text{II}_1 \).

**Proof.** Let \( \mathcal{L}(\mathbb{F}_r) \) be the interpolated free group factor for \( r \) \((1 < r \leq \infty)\). Suppose \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) is a complemented type \( \text{II}_1 \) von Neumann algebra which is isomorphic to \( \mathcal{L}(\mathbb{F}_r) \). Since \( \mathcal{L}(\mathbb{F}_r)^{\frac{1}{2}} \cong \mathcal{L}(\mathbb{F}_{4(r-1)+1}) \) and \( \mathcal{L}(\mathbb{F}_r) \subseteq \mathcal{L}(\mathbb{F}_{4(r-1)+1}) \), there is a injective \( * \)-isomorphism from \( \mathcal{M} \) into \( \mathcal{M}_{\frac{1}{2}} \). It follows from Theorem [1] that \( \mathcal{M} \) is injective which is a contradiction and the corollary follows. \( \square \)
Remark 12. There exist factors of type $\Pi_1$ with trivial fundamental groups that are not complemented subspaces.

It follows from Popa’s work [20] that for each finite $n \geq 2$, there exists free, ergodic, measure-preserving actions $\sigma$ of $\mathbb{F}_n$ on $L^\infty([0,1],\mu)$, where $\mu$ is the Lebesgue measure on $[0,1]$, such that the crossed product

$$\mathcal{M} = L^\infty([0,1],\mu) \rtimes \sigma \mathbb{F}_n$$

is a factor of type $\Pi_1$ with trivial fundamental group, $\mathcal{F} (\mathcal{M}) = \{1\}$.

By construction, $\mathcal{M}$ contains a von Neumann subalgebra $\mathcal{N}$ which is isomorphic to $\mathcal{L}(\mathbb{F}_n)$. Let $E : \mathcal{M} \to \mathcal{N}$ be the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ (see [11] and [23]). If there is a bounded projection $\Pi$ of $B(\mathcal{H})$ onto $\mathcal{M}$, then $E \circ \Pi$ would be a bounded projection of $B(\mathcal{H})$ onto $\mathcal{N}$ which contradicts with Corollary [11] (or Corollary 4.7 in [10]). Hence there is no bounded projection from $B(\mathcal{H})$ onto $\mathcal{M}$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, COPENHAGEN, DENMARK
E-mail address: E-mail: echris@math.ku.dk

SCHOOL OF MATHEMATICAL SCIENCES, QUFU NORMAL UNIVERSITY, QUFU, 273165, CHINA
E-mail address: E-mail: wangliguang0510@163.com