Understanding the Planck blackbody spectrum and Landau diamagnetism within classical electromagnetism

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Abstract
Electromagnetism is a relativistic theory, and one must exercise care in coupling this theory with nonrelativistic classical mechanics and with nonrelativistic classical statistical mechanics. Indeed historically, both the blackbody radiation spectrum and diamagnetism within classical theory have been misunderstood because of two crucial failures: (1) the neglect of classical electromagnetic zero-point radiation, and (2) the use of erroneous combinations of nonrelativistic mechanics with relativistic electrodynamics. Here we review the treatment of classical blackbody radiation, and show that the presence of Lorentz-invariant classical electromagnetic zero-point radiation can explain both the Planck blackbody spectrum and Landau diamagnetism at thermal equilibrium within classical electromagnetic theory. The analysis requires that relativistic electromagnetism is joined appropriately with simple nonrelativistic mechanical systems which can be regarded as the zero-velocity limits of relativistic systems, and that nonrelativistic classical statistical mechanics is applied only in the low-frequency limit when zero-point energy makes no contribution.

Keywords: blackbody radiation, Planck spectrum, Landau diamagnetism, classical zero-point radiation

1. Introduction

1.1. The Rayleigh–Jeans versus the Planck spectrum for blackbody radiation

When a physicist is asked for the theoretical difference between the Rayleigh–Jeans spectrum and the Planck spectrum for blackbody radiation, the response is likely to be that the first is
the result of classical physics while the second is the result of quantum physics. This is the point of view presented in the textbooks of modern physics [1]. This response may have represented the best understanding of nature in the early years of the 20th century. However, it does not represent accurate physics today. Today we are aware that any description of nature in terms of classical physics must include classical electromagnetic zero-point radiation and must recognise the demands of special relativity. Here we review the treatment of classical blackbody radiation, and note that when the two missing aspects are included accurately, then classical physics predicts not merely the low-frequency Rayleigh–Jeans spectrum but indeed the full Planck spectrum for thermal radiation.

1.2. Diamagnetism within classical theory

Diamagnetism represents a second phenomenon which involves the same misunderstandings as are involved in classical blackbody radiation. Current electromagnetism textbooks provide examples of diamagnetic behaviour for a single nonrelativistic particle in a magnetic field, but then often state that diamagnetism is not a phenomenon of classical physics [2]. The presence or absence of diamagnetism within classical physics is controversial, partly because of a failure to distinguish what is meant by ‘classical physics’. Classical physics includes two incompatible theories, both nonrelativistic particle mechanics as well as relativistic classical electrodynamics. The Bohr–van Leeuwen theorem applies the classical statistical mechanics of nonrelativistic particle mechanics to the behaviour of charges in an external magnetic field, and concludes that diamagnetism does not exist [3]. The Bohr–van Leeuwen analysis does not include the magnetic energy of interacting particle fields, since magnetic field energy is not part of nonrelativistic particle mechanics. On the other hand, if one includes the magnetic field energy of the particles (even at the level of the Darwin Lagrangian), then diamagnetism can appear in classical physics, [4] though the use of ideas of nonrelativistic classical statistical mechanics may be inappropriate. In the present article, we treat single-particle diamagnetism within the classical electrodynamics of a very-low velocity particle in connection with classical blackbody radiation. The analysis gives Landau diamagnetism along with the Planck spectrum when classical zero-point radiation is introduced and nonrelativistic classical statistical mechanics is coupled correctly with relativistic classical electromagnetism.

1.3. Coupling nonrelativistic and relativistic physics in elastic particle collisions

A sense of the erroneous results arising from the mismatch of nonrelativistic physics and relativistic physics can be obtained by treating the elastic point collision between two massive particles. The collision should be treated relativistically using energy \( E = (p^2c^2 + m^2c^4)^{1/2} \) and momentum \( p = mv \) with \( \gamma = (1 - v^2/c^2)^{-1/2} \) for each particle. If we use relativistic physics consistently, then the elastic collision can be treated using conservation of total relativistic energy and momentum in any inertial frame, and then, using Lorentz transformations, the results for the collision can be transferred to any other inertial frame. The results are independent of the inertial frame in which the relativistic conservation laws were applied. We note that the relativistic system centre of energy \( X_{\text{CM}} = (U_1 \mathbf{r}_1 + U_2 \mathbf{r}_2)/(U_1 + U_2) \), where \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) refer to the particle positions, will move with constant velocity during the collision in every inertial frame [5].

However, now imagine that we combine the use of nonrelativistic mechanics for the first particle with relativistic mechanics for the second. Thus for the first particle, we have energy \( U_1 = p v / (2m) \) and momentum \( p_1 = m_1 v_1 \), while for the second particle, we have energy \( U_2 = (p_2^2c^2 + m_2^2c^4)^{1/2} \) and momentum \( p_2 = m_2 \gamma_2 v_2 \). Once again we can use energy and
momentum conservation to solve for the velocities of the particles after the collision. However, the results for the final velocities will depend upon the specific inertial frame in which energy and momentum conservation was applied.

The situation is visualised most easily for the elastic collision of particles of equal rest mass \( m_1 = m_2 \). In the fully relativistic collision, the particles exchange energy and momentum in every inertial frame, and the relativistic centre of energy moves with constant velocity despite the collision. However, in the situation involving the coupling between nonrelativistic mechanics for the first particle and relativistic mechanics for the second, the use of energy and momentum conservation will not lead to exchange of energy and momentum between the particles. In the centre of momentum frame where \( 0 = p_1 + p_2 = mv_1 + m\gamma_2 v_2 \), both the centre of energy and the centre of mass move with nonzero velocity; the centre of energy has velocity \( V_{\text{CofE}} = [(mv_1^2/2)v_1 + (m\gamma_2 c^2)v_2]/[(mv_1^2/2) + (m\gamma_2 c^2)] \) and the centre of mass has velocity \( V_{\text{CofM}} = (mv_1 + mv_2)/(m + m) \). On collision, each particle retains its own energy, having a final velocity whose magnitude is unchanged from its initial velocity, but the sign of the velocity is reversed. In this situation, both the centre of energy and the centre of mass have their velocities reversed and so do not retain their values on collision.

In fully relativistic physics, the centre-of-energy conservation law always holds; if no external forces are present, the centre of energy always moves with constant velocity \([5]\). In nonrelativistic mechanics, the centre-of-mass conservation law always holds; if no external forces are present, the centre of mass moves with constant velocity. However, in our simple collision example, we see that if we mix relativistic and nonrelativistic physics, then neither conservation law holds; the theory is neither relativistic nor nonrelativistic. For the mixed relativistic-nonrelativistic situation, the conservation laws of energy and momentum can still be used, but the results will depend explicitly upon the inertial frame in which the energy and momentum conservation laws were applied, and the results are valid in neither nonrelativistic nor relativistic theory.

The inaccurate combing of relativistic electromagnetism with nonrelativistic mechanical systems is involved in the erroneous ideas regarding the blackbody spectrum and regarding diamagnetism within classical physics in the early years of the 20th century. Today, the erroneous ideas are still presented in the textbooks of modern physics \([1]\).

1.4. Consistent classical derivation of the Planck spectrum and of diamagnetism

In addition to giving us a warning about the inconsistency of mixing nonrelativistic and relativistic physics when describing nature, the situation involving particle collisions also suggests an accurate way of using such mixtures. Nonrelativistic mechanics is the low-velocity limit of relativistic mechanics; the relativistic particle kinetic energy and momentum go over to the nonrelativistic values in the inertial frame where a particle has very small velocity \( v/c \ll 1 \). For the two-particle collision mentioned above, we can go to the inertial frame in which the first particle (which is treated nonrelativistically) has zero-initial velocity. Then the motion of the second relativistic particle will be accurately described provided that the first particle has very small velocity also after the collision, since at small velocity nonrelativistic mechanics represents an accurate approximation to relativistic mechanics. The final velocity of the first particle will indeed be small provided that its rest mass is taken to the large-mass limit.

This combination, involving use of the inertial frame in which a particle always has very small velocity and also requiring that we go to the limit of large particle mass, is a valid prescription for combing a nonrelativistic classical mechanical particle with a relativistic
classical system. In this case, we can regard the nonrelativistic particle system as the valid limit of a relativistic particle system. In this article, we will reexamine the discussions of the blackbody radiation spectrum and of single-particle diamagnetism within classical physics. We will indicate how the treatments are modified by the inclusion of classical electromagnetic zero-point radiation and the accurate coupling between nonrelativistic classical mechanics and relativistic classical electromagnetic theory. The accurate treatments will indeed yield correct Landau diamagnetic behaviour and the Planck spectrum.

2. Blackbody radiation within classical theory

2.1. Scattering calculations

2.1.1. Stability under scattering and the importance of relativity. Blackbody radiation is the equilibrium spectrum of random radiation within an enclosure whose walls are held at a constant temperature $T$. Now radiation within a reflecting-walled cavity cannot bring itself to equilibrium. Rather, there must be some scattering system which redistributes the energy among the various normal modes of the cavity and so brings about the spectrum of thermal equilibrium. Once in equilibrium, the spectrum will be unchanged by the presence of a scatterer.

During the 20th century, there were several calculations using nonrelativistic nonlinear scatterers to determine the theoretical equilibrium spectrum for thermal radiation within classical theory. And all these nonrelativistic nonlinear scattering calculations led to the Rayleigh–Jeans spectrum [6–8]. However, all these nonrelativistic scattering calculations are inaccurate precisely because they attempt to couple a nonrelativistic classical mechanical system with relativistic classical electromagnetism. There is one scattering calculation which uses the relativistic expressions for the momentum of the scattering charged particle; however, the calculation limits itself to nonrelativistic potential functions which exclude the Coulomb potential, and so is not a relativistic calculation [8]. Indeed, these nonrelativistic nonlinear scatterers push the Lorentz-invariant spectrum of classical zero-point radiation toward the Rayleigh–Jeans spectrum. Only fully relativistic scatterers have the qualitative features which will leave the zero-point radiation spectrum invariant and will allow equilibrium at the Planck spectrum [9].

More recently there have been fully relativistic treatments of classical thermal radiation in a relativistic accelerating frame (a Rindler frame), and these treatments indeed lead to the Planck spectrum for thermal equilibrium within classical physics [10]. However, the analysis in a relativistic accelerating frame involves sophistication well beyond that of familiar nonrelativistic mechanics and basic electromagnetism. Therefore in this article, we show how to combine accurately the simpler nonrelativistic mechanics with electromagnetism.

2.1.2. Harmonic oscillator systems as relativistic scattering systems for small velocity. Using classical electromagnetism, we can form a one-dimensional harmonic oscillator by trapping a bead of charge $e$ and mass $m$ on a frictionless wire between two charges $q$ (of the same sign as $e$) which are held by external forces at a separation $2d$. Due to electrostatic forces, the particle $e$ will undergo small oscillations at frequency $\omega_0 = (4eq/(md^3))^{1/2}$, and the forces of constraint do no work in the inertial frame in which the charges $q$ are at rest. Thus in the small-oscillation limit where the velocity of the charge $e$ vanishes $v \rightarrow 0$, we can regard this oscillator as the limit of a relativistic system.

Indeed, a nonrelativistic classical mechanical particle in a potential will always seek the lowest point in the potential, and small oscillations are always harmonic oscillations. The
nonrelativistic nonlinear scattering systems which were used to derive the Rayleigh–Jeans spectrum \([7, 8]\) can never be considered as relativistic since the analysis depended on the nonlinear nature of the scatterer, and the scattering associated with the nonlinearity disappears in the limit of small particle velocity \(v \rightarrow 0\), and therefore small spatial excursion.

Now even a charged particle undergoing strictly harmonic oscillator motion at frequency \(\omega_0\) with finite amplitude \(x_0\) will radiate at all multiples of the fundamental frequency \(\omega_0\) with the time-average power per unit solid angle radiated in the \(n\)th harmonic given by \([11]\)

\[
\frac{dP_n}{d\Omega} = \frac{e^2c^2\beta^2}{2\pi^2\lambda_0^2} n^2 \tan^2 \theta J_n^2(n\beta \cos \theta),
\]

where \(\beta = \omega_0x_0/c\) involves the maximum speed \(v_{\text{max}} = \omega_0x_0\) of the oscillating particle. Thus a harmonic oscillator of finite amplitude can act as a radiation scatterer transferring energy from one frequency to another. However, we notice that the power radiated into each harmonic depends on the ratio \(\beta\) of the maximum particle speed \(v_{\text{max}}\) to the speed of light \(c\), \(\beta = v_{\text{max}}/c = \omega_0x_0/c\). In the nonrelativistic limit \(v_{\text{max}}/c \rightarrow 0\), all the radiation is emitted at the fundamental frequency \(\omega_0\), and the scatterer no longer transfers energy from one frequency to another.

Thus the presence of a very small linear dipole harmonic oscillator within a large reflecting-walled cavity filled with radiation will send radiation into new directions and so tend to make the radiation isotropic, \([12]\) but it will not redistribute the radiation among the various frequencies. A linear dipole harmonic oscillator in the small-velocity limit does not enforce any spectrum of random radiation. In particular, a dipole harmonic oscillator in the zero-velocity limit does not alter the Lorentz-invariant spectrum of classical electromagnetic zero-point radiation.

2.2. Thermodynamic analysis

2.2.1. Radiation normal modes. Blackbody radiation can be explored not only in terms of stability under scattering, but also in terms of thermodynamics. Within classical physics, thermal radiation corresponds to a solution of the homogeneous Maxwell equations involving standing electromagnetic waves in an enclosure. Choosing for simplicity a rectangular conducting-walled cavity of dimensions \(a \times b \times d\), the radiation inside can be written as a sum over the radiation normal modes with vanishing scalar potential \(\Phi\) and with vector potential \(A\) given by (see, for example, \([2]\), problem 9.40 on p 435, or \([13]\) or \([14]\))

\[
A(x, y, z, t) = \sum_{l,m,n=0}^{\infty} \sum_{l,m,n=0}^{2} \hat{\epsilon}_{lmn}^{(\lambda)} \left( \frac{2\pi}{abcd} \right)^{1/2} \left( \hat{\epsilon}_{lmn}^{(\lambda)} \cos \left( \frac{lx}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \sin \left( \frac{n\pi z}{d} \right) \right)
+ \hat{\epsilon}_{lmn}^{(\lambda)} \sin \left( \frac{lx}{a} \right) \left( \frac{m\pi y}{b} \right) \cos \left( \frac{n\pi z}{d} \right)
+ \hat{\epsilon}_{lmn}^{(\lambda)} \sin \left( \frac{lx}{a} \right) \left( \frac{m\pi y}{b} \right) \cos \left( \frac{n\pi z}{d} \right),
\]

where \(\hat{\epsilon}_{lmn}^{(\lambda)}\) with \(\lambda = 1, 2\) are the mutually orthogonal unit vectors satisfying \(\epsilon_{l+} + \epsilon_{m+} + \epsilon_{n+} = 0\), where \(\rho_{lmn,\lambda}\) is the time-varying amplitude of the mode, and where the frequency of the mode is given by \(\omega_{lmn} = c\pi (l^2/a^2 + m^2/b^2 + n^2/d^2)^{1/2}\). The radiation energy in the box is given by \(E = \frac{1}{18\pi} \int \int \int dx dy dz (E^2 + B^2)\) where \(E = -\nabla \Phi - (1/c)\partial A/\partial t\) and \(B = \nabla \times A\), so that \([13, 14]\)
Thus the energy of thermal radiation in a cavity can be expressed as a sum over the energies of the normal modes of oscillation, with each mode taking the form of a harmonic oscillator

\[ E = (1/2)(\dot{q}^2 + \omega^2 q^2). \]  

2.2.2. Thermodynamics of the simple harmonic oscillator. Now the thermodynamics of a harmonic oscillator takes a particularly simple form because the system has only two thermodynamic variables \( T \) and \( \omega \) [15]. In thermal equilibrium with a bath, the average oscillator energy is denoted by \( U = \langle E \rangle = (J) \omega \) (where \( J \) is the action variable), and satisfies \( dQ = dU + dW \) with the entropy \( S \) satisfying \( dS = dQ/T. \) Now since \( J \) is an adiabatic invariant, [16] the work done by the system is given by \( dW = -J \omega d\omega = -U(\omega)d\omega. \) Combining these equations, we have \( dS = dQ/T = [dU - (U/\omega)d\omega]/T. \) Writing the differentials in terms of \( T \) and \( \omega, \) we have \( dS = (\partial S/\partial T)dT + (\partial S/\partial \omega)d\omega \) and \( dU = (U/\partial T)dT + (U/\partial \omega)d\omega. \) Therefore \( \partial S/\partial T = (U/\partial T)/T \) and \( \partial S/\partial \omega = [(U/\partial \omega) - (U/\omega)]/T. \) Now equating the mixed second partial derivatives \( \partial^2 S/\partial T\partial \omega = \partial^2 S/\partial \omega^2, T, \) we have \( (\partial^2 U/\partial \omega^2)T = -[(U/\partial \omega) - (U/\omega)]/T \) or \( 0 = (\partial U/\partial T)/(\omega T) - [(U/\omega) - (U/\omega)]/T^2. \) The general solution of this equation is [15]

\[ U(\omega, T) = \omega f(\omega/T), \]  

where \( f(\omega/T) \) is an unknown function. When applied to thermal radiation, the result obtained here, purely from thermodynamics, corresponds to the familiar Wien displacement law of classical physics.

2.2.3. Classical zero-point radiation. The energy expression (5) for an electromagnetic radiation mode (or for a harmonic oscillator) in thermal equilibrium allows two limits which make the energy independent from one of its two thermodynamics variables. When the temperature \( T \) becomes very large, \( T \gg \omega, \) so that the argument of the function \( f(\omega/T) \) is small, the average energy \( U \) of the mode becomes independent of \( \omega \) provided \( f(\omega/T) = \text{const}_1 \times T/\omega \) so that

\[ U(\omega, T) = \omega f(\omega/T) \rightarrow \omega \times \text{const}_1 \times T/\omega = \text{const}_1 \times T \text{ for } \omega/T \ll 1. \]  

This is the familiar high-temperature limit where we expect to recover the equipartition limit giving the Rayleigh–Jeans spectrum. Therefore we choose this constant as \( \text{const}_1 = k_B \) corresponding to Boltzmann’s constant. With this choice, our thermal radiation now goes over to the Rayleigh–Jeans limit for high temperature or low frequency

\[ U(\omega, T) = \omega f(\omega/T) \rightarrow k_B T \text{ for } \omega/T \ll 1. \]  

In the other limit of small temperature, \( T \ll \omega, \) the dependence on temperature is eliminated provided \( f(\omega/T) \rightarrow \text{const}_2, \) so that

\[ U(\omega, T) = \omega f(\omega/T) \rightarrow \text{const}_2 \times \omega \text{ for } \omega/T \gg 1. \]  

If the second constant does not vanish, then there exists random, temperature-independent radiation present in the system. Since we are describing nature using classical theory, this
random radiation which exists at temperature $T = 0$ is classical electromagnetic zero-point radiation. We emphasize that the possibility of classical electromagnetic zero-point radiation is not a new postulate, but rather is an integral part of classical electromagnetic theory [17]. Classical electromagnetism allows the presence of homogeneous solutions of Maxwell’s equations, and the actual presence of the radiation must be determined by experimental measurements.

We also emphasize that thermodynamics allows classical zero-point radiation within classical physics. The physicists of the early 20th century were not familiar with the idea of classical zero-point radiation, and so they assumed that $\text{const}_2 = 0$ which excluded the possibility of classical zero-point radiation. In his monograph on classical electron theory, Lorentz [18] makes the explicit assumption that there is no radiation present at $T = 0$. Today, we know that the exclusion of classical zero-point radiation is an error. Experiments involving the (Casimir) forces [19] between two uncharged conducting parallel plates show that valid classical electromagnetic theory must assume that there is electromagnetic zero-point radiation [20, 21]. By comparing theoretical calculations with experiments, one finds that the scale constant for classical zero-point radiation appearing in equation (8) must take the value $\text{const}_2 = 1.05 \times 10^{-34}$ J s. However, this value corresponds to the value of a familiar constant in physics; it corresponds to the value $\hbar/2$ where $\hbar$ is Planck’s constant. Thus in order to account for the experimentally observed Casimir forces between parallel plates, the scale of classical zero-point radiation must be such that $\text{const}_2 = h/2$, and for each normal mode, the average energy becomes

$$U(\omega, T) = \omega f(\omega/T) \rightarrow U(\omega, 0) = (\hbar/2)\omega \quad \text{for} \ T \rightarrow 0.$$ (9)

At this point, we have the high-temperature and low-temperature asymptotic limits of the function $U(\omega, T)$. The full blackbody radiation spectrum represents the interpolation between these limits. In an earlier article, we suggested that, using an entropy-related function, the Planck spectrum could be obtained as the smoothest possible interpolation between the high-temperature and low-temperature asymptotic forms [15]. Here we will show that nonrelativistic classical statistical mechanics, when accurately applied, will demand exactly this same interpolation given by the Planck formula.

2.3. Use of nonrelativistic classical statistical mechanics

2.3.1. The traditional treatment in modern physics. The discussions of the blackbody radiation spectrum within classical physics which appear in textbooks do not involve the scattering of radiation by nonrelativistic nonlinear oscillators, or the introduction of classical zero-point radiation allowed by thermodynamics, but rather involve the application of nonrelativistic classical statistical mechanics. Classical statistical mechanics was developed for nonrelativistic classical particle mechanics before the ideas of special relativity and does not allow the idea of classical zero-point energy. In this article, we will always refer to classical statistical mechanics as nonrelativistic classical statistical mechanics in order to emphasize its nonrelativistic character. In the textbooks of modern physics, the energy equipartition theorem of nonrelativistic classical statistical mechanics is applied directly to each normal mode of the classical radiation field [1]. The result is the Rayleigh–Jeans spectrum. However, this spectrum is unjustified since it presents the application of a result of nonrelativistic classical statistical mechanics to a relativistic radiation system.
2.3.2. Dipole harmonic oscillator in thermal radiation. In contrast to the blackbody analysis favoured by the textbooks of modern physics, the derivation favoured by Planck [22] involved first deriving the connection between the average energy of a harmonic oscillator of frequency $\omega_0$ and the average energy per normal mode of the radiation spectrum at frequency $\omega$. Here we will repeat the traditional calculation, because we will need the result, and because some later calculations will proceed in analogy with it.

For thermal radiation in a large enclosure, one may treat the radiation not only as a sum over normal modes as in equation (2), but also, alternatively, as a sum over plane waves with periodic boundary conditions. Thus we can take the scalar potential $F_T$ to vanish and the vector potential as

$$\mathbf{A}_T(\mathbf{r}, t) = \sum_k \sum_{\lambda=1}^2 e \hat{\mathbf{e}}(k, \lambda) \left( \frac{8\pi U(\omega, T)}{V} \right)^{1/2} \sin[\mathbf{k} \cdot \mathbf{r} - \omega t + \theta(k, \lambda)],$$

(10)

where the wave vectors $\mathbf{k}$ correspond to

$$k = \frac{1}{a} \left( l \hat{\mathbf{e}}_x + m \hat{\mathbf{e}}_y + n \hat{\mathbf{e}}_z \right)$$

with $l, m, n$ running over all positive and negative integers, $a$ is a length such that $a^3 = V$, and the two mutually orthogonal polarisation vectors $\hat{\mathbf{e}}(k, \lambda), \lambda = 1, 2$, are orthogonal to the wave vectors $\mathbf{k}$. Since thermal radiation is isotropic in the inertial frame of its container, the amplitude of $U(\omega, T)$ depends only on the frequency $\omega = c |k| = ck$, and the constants are chosen so that $U(\omega, T)$ is the energy per normal mode appropriate for the thermal radiation spectrum in classical physics. In order to describe the randomness of the radiation, the phases $\theta(k, \lambda)$ are chosen as random variables uniformly distributed on $[0, 2\pi]$, independently distributed for each $k$ and $\lambda$.

When thermal radiation falls on a small dipole harmonic oscillator, modelled as a particle of charge $e$ and mass $m$ at the end of a small spring of spring constant $\kappa$ located at the origin of coordinates and oriented along the $z$-axis (so that the oscillator frequency satisfies $\omega_0^2 = \kappa/m$), the equation of motion becomes (see, for example, [2] p 420)

$$m\ddot{z} = -m\omega_0^2 z + m\tau \ddot{z}_c + e\mathbf{E}_T(0, t),$$

(11)

where $-m\omega_0^2 z$ represents the spring restoring force, $m\tau \ddot{z}_c$ is the radiation damping force with $\tau = 2e^2/(3mc^3)$, and $e\mathbf{E}_T(0, t)$ is the driving force of the random radiation.

For thermal radiation as given in equation (10) and the oscillator located at the coordinate origin, the steady-state solution of equation (11) is

$$z(t) = \sum_k \sum_{\lambda=1}^2 \frac{e \hat{\mathbf{e}}(k, \lambda)}{m} \left( \frac{2\pi U(\omega, T)}{V} \right)^{1/2} \left\{ \exp[-i\omega t + i\theta(k, \lambda)] \right\} \left\{ -\omega^2 + \omega_0^2 - i\tau \omega \right\} + \text{cc},$$

(12)

where ‘cc’ stands for the complex conjugate of the first quantity in the curly bracket.

In thermal radiation, the mean displacement of the oscillator and the mean velocity are both zero $\langle z(t) \rangle = 0$, $\langle \dot{z}(t) \rangle = 0$, but the mean squares are nonzero. We can find the mean-square displacement by averaging over time or averaging over the random phases at a fixed time. Since the random phases $\theta(k, \lambda)$ are distributed randomly and independently for each mode, we have the averages

$$\langle \exp \{ i[\omega t + \theta(k, \lambda)] \} \exp \{ i[\omega' t + \theta(k', \lambda')] \} \rangle = 0$$

(13)

and

$$\langle \exp \{ i[\omega t + \theta(k, \lambda)] \} \exp \{-i[\omega' t + \theta(k', \lambda')] \} \rangle = \delta_{kk'} \delta_{\lambda\lambda'}$$

(14)
which gives
\[
\langle z^2 \rangle = \sum_{\mathbf{k} \lambda=1}^{2} c_{\mathbf{k} \lambda}^2 \left( \frac{2\pi U(\omega, T)}{V} \right) \frac{2e^2}{m^2 [(-\omega^2 + \omega_0^2)^2 + (\tau \omega^3)^2]},
\]
\[
(15)
\]
\[
\langle z^2 \rangle = \sum_{\mathbf{k} \lambda=1}^{2} c_{\mathbf{k} \lambda}^2 \left( \frac{2\pi U(\omega, T)}{V} \right) \frac{2e^2 \omega^2}{m^2 [(-\omega^2 + \omega_0^2)^2 + (\tau \omega^3)^2]},
\]
\[
(16)
\]
and the average energy of the oscillator
\[
\langle \mathcal{E}(\omega_0, T) \rangle = \sum_{\mathbf{k} \lambda=1}^{2} c_{\mathbf{k} \lambda}^2 \left( \frac{2\pi U(\omega, T)}{V} \right) \frac{e^2 (\omega^2 + \omega_0^2)}{m [(-\omega^2 + \omega_0^2)^2 + (\tau \omega^3)^2]}.
\]
\[
(17)
\]
For a large box of thermal radiation, the normal modes are very closely spaced, and therefore the sum over normal modes can be replaced by an integral, \[ \int \frac{1}{2\pi} \frac{d^3 k}{\sqrt{a^2 + b^2}} = \frac{\pi}{ab}. \]
\[
(19)
\]
which is sharply peaked at \[ \omega = \omega_0. \] In this case, we can integrate over all angles so that \[ c_{\mathbf{k} \lambda}^2 \] contributes a factor of \[ 1/3 \] for each polarisation, and then approximate the integral over \[ k = \omega/e \] by extending the lower limit to minus infinity, setting \[ \omega = \omega_0 \] in every term except for \[ (-\omega^2 + \omega_0^2) \approx 2\omega_0(\omega_0 - \omega) \], and using the definite integral
\[
\langle \mathcal{E}(\omega_0, T) \rangle = U(\omega_0, T).
\]
\[
(20)
\]
Thus the average energy \[ \langle \mathcal{E}(\omega_0, T) \rangle \] of the oscillator with resonant frequency \[ \omega_0 \] is the same as the average energy \[ U(\omega_0, T) \] of the radiation normal mode at the same frequency.

2.3.3. Failure of the view from the beginning of the 20th century. Now the physicists of the early 20th century did not appreciate the idea of classical zero-point radiation nor the importance of special relativity. They assumed that they could apply nonrelativistic classical statistical mechanics to the dipole oscillator motion. Therefore they suggested that the average linear oscillator energy was \[ \langle \mathcal{E}(\omega_0, T) \rangle = k_B T \], and that classical physics required that the corresponding radiation mode must have average energy \[ U(\omega_0, T) = k_B T \]. In other words, they arrived at the Rayleigh–Jeans spectrum because of the use of nonrelativistic classical statistical mechanics for all mechanical oscillators, not merely for the lowest frequency oscillators.

Nonrelativistic statistical mechanics can not tolerate the idea of zero-point energy, because within nonrelativistic statistical mechanics, all random energy is thermal energy associated with temperature \[ T \]. Thus when attempting to couple relativistic classical electromagnetism with ideas of nonrelativistic classical statistical mechanics, it is not sufficient simply to take the limit of low velocity for the oscillator to bring compliance with relativity; we must also require that there is no contribution from zero-point radiation. Thus we can apply nonrelativistic statistical mechanics only in the limit that the velocity of the oscillator goes to the zero-velocity limit and also that the zero-point radiation for the oscillator...
is very small compared to $k_B T$. $(1/2)\hbar \omega_0 \ll k_B T$. This situation indeed corresponds to the low-frequency section of the blackbody spectrum.

### 2.3.4. Large-mass limit of a harmonic oscillator

According to equations (9) and (20), a mechanical oscillator in classical zero-point radiation will acquire an average mechanical energy

$$\langle \mathcal{E}(\omega_0, 0) \rangle = (1/2)\hbar \omega_0 = (1/2)m v_{\text{max}}^2,$$

where $v_{\text{max}}$ is the maximum velocity of the oscillator. However, a harmonic oscillator can be regarded as the limit of a relativistic system only when $v_{\text{max}}$ becomes very small, $v_{\text{max}} \ll c$. In order to make $v_{\text{max}}$ very small for fixed average energy $\langle \mathcal{E}(\omega_0, 0) \rangle$, we need to take the particle mass $m$ as very large. However, for mechanical systems, large mass $m$ with fixed spring constant $\kappa$ means that the oscillation frequency $\omega = (\kappa/m)^{1/2} \to 0$ as $m \to \infty$. Thus in the large-$m$ limit required to fit with relativity, the mechanical zero-point energy vanishes along with the oscillation frequency, $\langle \mathcal{E}(\omega_0, 0) \rangle = (1/2)\hbar \omega_0 \to 0$ as $\omega_0 \to 0$ for $m \to \infty$ and $\kappa$ fixed. In this low-frequency limit, the zero-point energy becomes ever smaller so that for any nonzero temperature $(1/2)\hbar \omega_0 \ll k_B T$, and we recover the Rayleigh–Jeans spectrum, which is indeed found for the low-frequency radiation modes.

### 3. Single-particle diamagnetism in classical physics

#### 3.1. Inclusion of a uniform magnetic field

##### 3.1.1. Cyclotron motion allows a special large-mass limit

In order to combine simple nonrelativistic mechanical systems with relativistic electromagnetism, we must take the large-mass limit so as to approximate a valid relativistic mechanical system. At the same time, we wish to maintain the oscillation frequency $\omega_0$ unchanged so as to see the influence of the electromagnetic zero-point energy on the mechanical system. Clearly we need some additional variable which can be taken large in order to compensate for the large-mass limit. One system in which this occurs is the motion of a free charged particle in a uniform magnetic field $B$. The relativistic equation for cyclotron motion is $m v^2/r = e(v/c)B$. In the nonrelativistic limit of small velocity, this becomes (see, for example, [2] section 5.1.2, pp 212–125) $m v^2/r = e(v/c)B$, giving the frequency of rotation

$$\omega_B = (v/r) = eB/(mc).$$

Thus we can maintain the cyclotron frequency $\omega_B$ as constant while increasing $m$ to reach the nonrelativistic limit provided that we increase the magnetic field $B$. For fixed total energy, the velocity $v$ of the charge becomes smaller as the mass $m$ is increased while the frequency $\omega_B$ is held constant. Here indeed we have a simple nonrelativistic particle system which can be regarded as the limit of a relativistic system while maintaining the zero-point energy contribution $(1/2)\hbar \omega_B$.

#### 3.1.2. Absence of diamagnetism in nonrelativistic classical statistical mechanics

Diamagnetism is an equilibrium thermodynamic condition which does not exist within nonrelativistic classical physics when we deal with charges in an external magnetic field but neglect the magnetic field energy of the charges themselves. This is the content of the Bohr–van Leeuwen theorem [3]. Of course, electromagnetism (except for electrostatics) does not exist within nonrelativistic classical physics since electromagnetism (beyond electrostatics) is a relativistic theory. Although single-particle diamagnetic behaviour is discussed with
examples in textbooks of classical electromagnetism, there is often a disclaimer noting that the application of nonrelativistic classical statistical mechanics eliminates diamagnetic behaviour \[2\]. Within classical physics, single-particle diamagnetism, just like the Planck spectrum, depends upon the inclusion of classical electromagnetic zero-point radiation, something which is incompatible with nonrelativistic classical statistical mechanics. The presence of single-particle diamagnetism in connection with classical zero-point radiation has appeared in the research literature \[24\]; however, none of the earlier treatments adequately addresses the appropriateness of nonrelativistic classical statistical mechanics within a relativistic classical theory. Here, our discussion, which includes classical zero-point radiation, will depend upon simple nonrelativistic systems which can be regarded as the low-velocity limits of relativistic systems.

3.1.3. Isotropic dipole oscillator in a magnetic field. In order to discuss diamagnetism, we consider a three-dimensional harmonic-oscillator potential \( V(r) = (1/2)\kappa r^2 = (1/2)\kappa (x^2 + y^2 + z^2) \) in which there is a nonrelativistic particle of charge \( e \) and mass \( m \), in the presence of a uniform magnetic field \( \mathbf{B} \) along the \( z \)-direction \( \mathbf{B} = \hat{k}B \). Taking \( \omega_0 = (\kappa/m)^{1/2} \), the nonrelativistic equation of motion for the particle is

\[
\mathbf{m}\ddot{r} = -m\omega_0^2\mathbf{r} + e(\hat{r}/c) \times \mathbf{B} + m\tau\dot{\mathbf{r}} + e\mathbf{E}_T(0, t),
\]

where \(-m\omega_0^2\mathbf{r}\) is the force due to the harmonic-oscillator potential, \(e(\hat{r}/c) \times \mathbf{B}\) is the Lorentz force of the magnetic field, \(m\tau\dot{\mathbf{r}}\) is the radiation damping force, and \(e\mathbf{E}_T(0, t)\) is the driving force due to the electric field of the thermal radiation taken in the dipole approximation. After dividing through by the mass \(m\), the vector equation \(23\) can be rewritten as three component equations

\[
\begin{align*}
\ddot{x} &= -\omega_0^2 x + 2\omega_L \dot{x} + \tau \dot{x} + (e/m)E_{Tz}, \\
\ddot{y} &= -\omega_0^2 y - 2\omega_L \dot{y} + \tau \dot{y} + (e/m)E_{Ty}, \\
\ddot{z} &= -\omega_0^2 z + \tau \dot{z} + (e/m)E_{Tz},
\end{align*}
\]

where

\[
\omega_L = \omega B/2 = eB/(2mc).
\]

Here we have a system of three linear differential equations. The steady-state solution for equation \(26\) was given earlier in equation \(12\). The first two equations \(24\) and \(25\) are coupled linear differential equations with steady-state solutions

\[
\begin{align*}
x &= \sum_{k,\lambda=1}^2 \frac{e}{m} \left( \frac{2\pi U(\omega, T)}{V} \right)^{1/2} \left\{ \frac{(C\epsilon_x - i2\omega_L\epsilon_x)\exp[-i\omega t + i\theta(k, \lambda)]}{C^2 - (2\omega_L)^2} + cc \right\}, \\
y &= \sum_{k,\lambda=1}^2 \frac{e}{m} \left( \frac{2\pi U(\omega, T)}{V} \right)^{1/2} \left\{ \frac{(C\epsilon_y + i2\omega_L\epsilon_y)\exp[-i\omega t + i\theta(k, \lambda)]}{C^2 - (2\omega_L)^2} + cc \right\},
\end{align*}
\]

where

\[
C = -\omega^2 + \omega_0^2 - i\tau\omega^3.
\]

3.1.4. System magnetic moment. Because we cannot apply nonrelativistic classical statistical mechanics to a system where zero-point energy is involved, we will consider not the system energy but rather the magnetic moment \(\mathbf{M}_{\text{dia}}\) of our diamagnetic system; later we
will compare this diamagnetic magnetic moment with that of a different (paramagnetic) system where nonrelativistic classical statistical mechanics can be legitimately applied. Symmetry for our diamagnetic system dictates that only a \( z \)-component is possible for the magnetic moment. The particle angular momentum \( \mathbf{L} \) has \( L_z = m(x\dot{y} - y\dot{x}) \) so that from \( M = [e/(2mc)]\mathbf{L} \), we have (see [11] p 187)

\[
\langle M_{z,\text{diam}} \rangle = \frac{e}{2mc} \langle L_z \rangle = \frac{e}{2c} \langle xy - yx \rangle.
\]  

(31)

We can differentiate to obtain the time derivatives and then take the averages in either time or over the random phases as in equations (13) and (14) so as to obtain

\[
\langle xy \rangle = \frac{e^2}{m^2} \sum_k \sum_{\lambda=1}^2 \left( \frac{2\pi U(\omega, T)}{V} \right) \left( \epsilon_\lambda^2 + \epsilon_\lambda^2 \right) \left( 2(\omega^2 + \omega_0^2) \right) \frac{[2(-\omega^2 + \omega_0^2)]}{|A_+|^2|A_-|^2},
\]  

(32)

where

\[
|A_+|^2 = (-\omega^2 + \omega_0^2 + 2\omega_L^2)^2 + (\tau\omega)^2
\]  

(33)

and

\[
|A_-|^2 = (-\omega^2 + \omega_0^2 - 2\omega_L^2)^2 + (\tau\omega)^2.
\]  

(34)

Following the pattern taking us from equation (17) to (18), we assume that the normal modes are closely spaced; we replace the summation over \( k \) by an integral, integrate over angles, and sum over polarisations to obtain

\[
\langle xy \rangle = \frac{e^2}{m^2} \int_{\omega=0}^{\infty} \frac{d\omega}{c^3} \omega^2 \left( \frac{2\pi U(\omega, T)}{V} \right) \left( \epsilon_\lambda^2 + \epsilon_\lambda^2 \right) \left( 2(\omega^2 + \omega_0^2) \right) \frac{[2(-\omega^2 + \omega_0^2)]}{|A_+|^2|A_-|^2},
\]  

(35)

Now for positive \( \omega \), the quantity \( |A_+|^2 \) takes its minimum value when \( -\omega^2 + \omega_0^2 + 2\omega_L^2 = 0 \) or

\[
\omega = \omega_+ = (\omega_0^2 + \omega_L^2)^{1/2} + \omega_L
\]  

(36)

and the quantity \( |A_-|^2 \) takes its minimum value when \( -\omega^2 + \omega_0^2 - 2\omega_L = 0 \) or

\[
\omega = \omega_- = (\omega_0^2 + \omega_L^2)^{1/2} - \omega_L.
\]  

(37)

If the quantities \( \tau\omega_+ \) and \( \tau\omega_- \) are small, \( \tau\omega_+ \ll 1, \tau\omega_- \ll 1 \), corresponding to small radiation damping, the integral in equation (35) is sharply peaked at \( \omega_+ \) and \( \omega_- \). Thus we will evaluate the integral in the approximation of two resonances, one at \( \omega_+ \) and one at \( \omega_- \). We replace every appearance of the frequency \( \omega \) by \( \omega_+ \) or by \( \omega_- \), except in \( |A_+|^2 \) and \( |A_-|^2 \) where the combination \( \omega - \omega_+ \) or \( \omega = \omega_+ \) appears, so that

\[
|A_+|^2 \approx 4(\omega_0^2 + \omega_L^2)(\omega - \omega_+)^2 + (\tau\omega_+^2)^2
\]  

(38)

and

\[
|A_-|^2 \approx 4(\omega_0^2 + \omega_L^2)(\omega - \omega_-)^2 + (\tau\omega_-^2)^2.
\]  

(39)

Now treating each resonant term separately, we extend the integrals over \( \omega \) from \(-\infty\) to \(+\infty\) to obtain
\[\langle xy \rangle = \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{m^2} \frac{e^2}{c^3} \frac{8}{3\pi} U(\omega, T) \left( \frac{2\omega^2_L}{\omega^2_0 + \omega^2_L}(\omega^2_0 + \omega^2_L)^2 \right) \left( \frac{-\omega^2_0 + \omega^2_L}{4(\omega^2_0 + \omega^2_L)(\omega - \omega_0)^2 + (\tau \omega^2)^2} \right) \]

\[+ \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{m^2} \frac{e^2}{c^3} \frac{8}{3\pi} U(\omega, T) \left( \frac{2\omega^2_L}{\omega^2_0 + \omega^2_L}(\omega^2_0 + \omega^2_L)^2 \right) \left( \frac{-\omega^2_0 + \omega^2_L}{4(\omega^2_0 + \omega^2_L)(\omega - \omega_0)^2 + (\tau \omega^2)^2} \right).\]

Using the integral in equation (19), we have

\[\langle xy \rangle = \frac{e^2}{m^2} \frac{8}{c^3} \frac{2\omega^2_L}{3\pi} U(\omega, T) \left( \frac{-\omega^2_0 + \omega^2_L}{-4\omega_L \omega_L} \right) \frac{\pi}{2(\omega^2_0 + \omega^2_L)} \left( \frac{-\omega^2_0 + \omega^2_L}{4(\omega^2_0 + \omega^2_L)^2} \right) \left( \frac{-\omega^2_0 + \omega^2_L}{4(\omega^2_0 + \omega^2_L)(\omega - \omega_0)^2 + (\tau \omega^2)^2} \right)\]

\[= -\frac{1}{2m} \frac{U(\omega, T)}{(\omega^2_0 + \omega^2_L)^{1/2}} + \frac{1}{2m} \frac{U(\omega, T)}{(\omega^2_0 + \omega^2_L)^{1/2}},\]

where we have noted that \((-\omega^2_0 + \omega^2_L) = -2\omega_L \omega_\perp\) and \((-\omega^2_0 + \omega^2_L) = +2\omega_L \omega_\perp\). We can evaluate the average \((-\langle xy \rangle\)) in a similar fashion and find that it is equal to \((\langle xy \rangle)\).

### 3.1.5. Result for the magnetic moment—single-particle diamagnetism.

Thus combining \((\langle xy \rangle)\) and \((-\langle xy \rangle)\), we find that the magnetic moment in the presence of a magnetic field is given by

\[\langle M_{\text{dia}} \rangle = \frac{e}{2c} \langle xy \rangle - \langle yx \rangle = -\frac{e}{2mc(\omega^2_0 + \omega^2_L)^{1/2}} [U(\omega, T) - U(\omega, T)].\]

In our analysis thus far, we have arrived at only the asymptotic limits for the thermal radiation energy \(U(\omega, T)\), which are given in equations (7) and (9). In the high-temperature limit where \(k_B T > \hbar \omega\) for all frequencies of interest, we recover the results of nonrelativistic classical statistical mechanics where the energy \(U(\omega, T)\) becomes the energy equipartition value \(U(\omega, T) \rightarrow k_B T\) for every frequency, so that the magnetic moment expression in equation (42) vanishes

\[\langle M_{\text{dia}} \rangle \rightarrow -\frac{e}{2mc(\omega^2_0 + \omega^2_L)^{1/2}} [k_B T - k_B T] = 0 \text{ for energy equipartition.}\]

This result agrees with the Bohr–van Leeuwen theorem for the absence of diamagnetism in nonrelativistic classical statistical mechanics which ignores the magnetic interaction energy of particles.

On the other hand, the low temperature limit indeed shows diamagnetic behaviour arising from zero-point radiation since \(U(\omega, 0) = (1/2)\hbar \omega\). We find that in this low-temperature limit equation (42) becomes

\[\langle M_{\text{dia}} \rangle \rightarrow -\frac{e}{2mc(\omega^2_0 + \omega^2_L)^{1/2}} \left[ \frac{1}{2} \hbar \omega_\perp - \frac{1}{2} \hbar \omega_\perp \right] = -\frac{e \hbar \omega_L}{2mc(\omega^2_0 + \omega^2_L)^{1/2}}\]

\[= -\left( \frac{e}{2mc} \right)^2 \frac{\hbar B}{(\omega^2_0 + \omega^2_L)^{1/2}} \text{ for } T = 0,\]

where we have inserted \(\omega_L = eB/(2mc)\) in the numerator. Thus we find that the average angular momentum and magnetic moment do not vanish in the low temperature limit. Single-particle diamagnetism as an equilibrium thermodynamic property within classical physics depends on the existence of Lorentz-invariant classical zero-point radiation.
3.1.6. Free-particle diamagnetism. The diamagnetic behaviour of our system at zero temperature becomes even more striking if we take the confining harmonic oscillator potential as extremely weak, \( \omega_L \ll c \). In this case, equation (44) gives the system magnetic moment

\[
\langle M_z, \text{dia} \rangle = -\frac{e\hbar \omega_L}{2mc(\omega_0^2 + \omega_L^2)^{1/2}} \rightarrow \frac{e\omega_L}{2mc|\omega_L|} \hbar = \frac{|e|\hbar}{2mc} \quad \text{for } \omega_0 \rightarrow 0, \ T \rightarrow 0. \tag{45}
\]

The absolute values arise because the direction of rotation \( \omega_B = cB/(mc) \) involves in cyclotron motion reverses sign with the sign of the charge. The magnetic moment is always such as to give diamagnetic behaviour.

It is also curious to see that the angular momentum \( \langle \mathbf{L} \rangle = (2mc/e) \langle M_z \rangle \) of the free charge takes the values

\[
\langle \mathbf{L} \rangle = -\frac{e}{|e|} \hbar = \mp \hbar, \tag{46}
\]

where the sign of the charge determines the plus or minus sign and the direction of the angular momentum is determined by the direction of the magnetic field \([25]\).

4. Derivation of the Planck spectrum

4.1. Use of the diamagnetic system

4.1.1. Large-mass limit for the diamagnetic system. In our analysis thus far, the values for the average thermal energy per normal mode \( U(\omega, T) \) and magnetic moment \( \langle M_z \rangle \) at a general temperature \( T \) are unknown because the spectrum of thermal radiation has not yet been obtained. So far, we know only the asymptotic high-temperature and low-temperature limits. In order to obtain the full Planck spectrum within classical physics, we will make use of nonrelativistic classical statistical mechanics applied in the appropriate limits.

First of all, at finite temperature \( T \), we go to the nonrelativistic limit for our charged particle in the harmonic potential in a magnetic field. Thus we can take the mass \( m \) of our charge as very large so that the particle velocity is small, \( v/c \ll 1 \). Then the frequency \( \omega_0 = (\kappa/m)^{1/2} \) associated with the confining potential becomes very small so that \( k_B T \gg \hbar \omega_0 \), and we find the energy equipartition behaviour associated with all frequencies, except \( \omega_B = cB/(mc) = 2\omega_L \) which we hold constant by increasing the magnetic field \( B \) to offset the large mass \( m \). In this limit where \( \omega_+ \rightarrow \omega_B \) and \( \omega_- \rightarrow 0 \), we find the magnetic moment at finite temperature in equation (42) becomes

\[
\langle M_z, \text{dia} \rangle = -\frac{e}{mc\omega_B} [U(\omega_B, T) - k_B T], \tag{47}
\]

where the crucial energy function \( U(\omega_B, T) \) corresponding to the average energy of a normal mode at frequency \( \omega_B \) and temperature \( T \) is still unknown.

4.1.2. Paramagnetic behaviour at temperature \( T \). In order to obtain \( U(\omega_B, T) \), we will compare paramagnetic and diamagnetic behaviour at temperature \( T \). We consider a three-dimensional paramagnetic rotator of magnetic moment \( \mu \) and moment of inertia \( I \) which rotates in the presence of a magnetic field \( \mathbf{B} \) along the \( z \)-axis. In the limit that the moment of inertia \( I \) is taken very large \( I \rightarrow \infty \), the rotator system will be that of a free nonrelativistic rotator, rotating at arbitrarily low frequency so that the zero-point radiation will make no contribution to the energy. Thus we can apply nonrelativistic classical statistical mechanics. In this case, the canonical phase space variables are the angular momentum \( \mathbf{L} = I \mathbf{\omega} \) and the
angles, $\theta, \phi$. The partition function associated with the kinetic energy of rotation can be treated separately from that associated with the energy of orientation so that

$$Z_\theta = \int_0^\pi d\theta \sin \theta \exp \left[ \frac{(-\mu B \cos \theta)}{k_B T} \right] = \left( \frac{2k_B T}{\mu B} \right) \sinh \left( \frac{\mu B}{k_B T} \right)$$  \hspace{1cm} (48)

The average magnetic moment $\langle M_{\text{para}} \rangle$ is given by

$$\langle M_{\text{para}} \rangle = \frac{1}{B} \frac{\partial (\ln Z_\theta)}{\partial (1/k_B T)} = \mu \left[ \coth \left( \frac{\mu B}{k_B T} \right) - \left( \frac{k_B T}{\mu B} \right) \right].$$  \hspace{1cm} (49)

The hyperbolic cotangent function has the expansion for small argument $x \ll 1$ given by $\coth x = 1/x + x/3 - x^3/45 + \ldots$, while for large $x \gg 1$, $\coth(x) \to 1$. Therefore the magnetic moment $\langle M_{\text{para}} \rangle$ of this nonrelativistic paramagnetic system has the asymptotic limits

$$\langle M_{\text{para}} \rangle \to \mu \text{ for } T \to 0$$  \hspace{1cm} (50)

and

$$\langle M_{\text{para}} \rangle \to 0 \text{ for } T \to \infty.$$  \hspace{1cm} (51)

### 4.1.3. Obtaining the Planck spectrum.

The asymptotic limits of the magnetic moment for the nonrelativistic paramagnetic rotator are analogous to those we have found for the diamagnetic magnetic moment of our nonrelativistic free particle in a magnetic field. Indeed, we can imagine a thermodynamic system consisting of our paramagnetic rotator and our diamagnetic free particle in a uniform magnetic field and taken sufficiently far apart that the magnetic interaction between them is negligible. Now the magnitude of the paramagnetic magnetic moment $\mu$ in equation (49) is arbitrary. Let us choose this magnitude $\mu$ equal to the magnitude of the free particle diamagnetic moment in equation (45) so that $\mu = \hbar/(2mc)$. Then, from equations (47) and (49), the total magnetic moment for the system is

$$\langle M_{\text{para}} \rangle + \langle M_{\text{dia}} \rangle = \frac{\hbar}{2mc} \left[ \coth \left( \frac{\hbar \omega_B}{2k_B T} \right) - \frac{2k_B T}{\hbar \omega_B} \right] - \frac{e}{mc} \left[ \frac{1}{\omega_B} U \left( \frac{\omega_B}{k_B T} \right) - \frac{k_B T}{\omega_B} \right]$$

which has the asymptotic limits

$$\langle M_{\text{para}} \rangle + \langle M_{\text{dia}} \rangle \to 0 \text{ for } T \to 0,$$  \hspace{1cm} (53)

$$\langle M_{\text{para}} \rangle + \langle M_{\text{dia}} \rangle \to 0 \text{ for } T \to \infty.$$  \hspace{1cm} (54)

There is no hidden structure in our magnetic moment system so that the only allowed interpolation between the two limits is the vanishing of the total magnetic moment at all temperatures. But then from the vanishing of the second line of equation (52), we must have the energy per normal mode of the blackbody radiation spectrum as

$$U(\omega, T) = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) = \frac{\hbar \omega}{\exp(\hbar \omega/k_B T) - 1} = \frac{1}{2} \frac{\hbar \omega}{2}$$  \hspace{1cm} (55)
which is exactly the Planck spectrum. This spectrum agrees with both the old thermal radiation measurements of Lummer and Pringheim and the recent Casimir force measurements. If we insert this spectrum into the right-hand side of equation (42) for the magnetic moment, we have the results of Landau diamagnetism. (See, for example, Pathria in [3], pp 206–209.)

5. Closing summary

In this article, we show that classical physics which includes classical electromagnetic zero-point radiation and uses relativity appropriately indeed predicts both the Planck spectrum of blackbody radiation and the presence of Landau diamagnetism at thermodynamic equilibrium. Appropriate use of relativity requires that nonrelativistic systems be considered only in the limit of zero particle velocity. In addition, classical statistical mechanics (which is a nonrelativistic theory which cannot tolerate zero-point energy) can be applied only in the zero-velocity limit for situations where the zero-point energy makes no contribution.

The conclusions of this manuscript are unsettling to many physicists. Although the basic analysis has been in the research literature for a number of years, the ideas have never entered the textbook literature. The existence of classical electromagnetic zero-point radiation and the importance of relativity were ideas unfamiliar to the physicists of the early 20th century. These ideas are still unfamiliar today.

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