Uniqueness for the inverse problem of determining the piecewise continuous coefficients in a wave equation from the interior measurements

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Abstract. In this paper, we discuss the inverse problem of determining the piecewise continuous coefficients in a wave equation from the interior measurements. It is proved that the coefficients can be uniquely determined in the domain where the shear wave reaches.

Keywords: uniqueness, piecewise continuous coefficient, wave equation

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1. Introduction
The Elastography is a well-developed technique (see [2], [3], [11]– [14], [16] and [17]), which can directly visualize and quantitatively measure propagating acoustic strain waves in tissue-like materials subjected to harmonic mechanical axial extension. The interior displacement is measured on a fine grid of points using ultrasound or magnetic resonance imaging. The elastography problem then is to construct high resolution images of tissue stiffness characteristics from the measured displacement. This high resolution is expected for two reasons: (1). interior measurements are used instead of boundary measurements; (2). the shear wave speed can be substantially more than double in abnormal stiff-tissue. Some elegant reconstruction algorithms for transient elastography have also been proposed, among which we refer to [9] etc.

In [12], Mclaughlin et al discussed the unique identifiability for the transient elastography problem. The uniqueness results for the inverse problem are that the elastic parameters can be uniquely determined by a single interior time-dependent scalar or vector displacement measurement. In the scalar shear displacement case, the mathematical model is the scalar wave equation. The authors consider the case that the medium is isotropic and the relevant elastic properties are the density $\rho \in C^0(\Omega)$, the Lamé parameter $\mu \in C^1(\Omega)$.

From the practical point of view, it is more reasonable to consider the case that the elastic properties $\rho, \mu$ are piecewise continuous in different subregion of $\Omega$. In this paper, we will consider the unique identifiability of the piecewise continuous Lamé coefficients from the single interior measurements. It is shown that the piecewise continuous coefficients can be uniquely determined from the interior measurements.
2. Formulation of the inverse problem

Let \( \Omega \subset \mathbb{R}^n (n = 2, 3) \) be an open connected domain of \( C^2 \)-class boundary, \( D \subset \Omega \) be an open subset of piecewise \( C^2 \)-class boundary, and \( T > 0 \) be fixed. We consider the boundary value problem for wave equation:

\[
\begin{align*}
\nabla \cdot [\mu(x) \nabla u(x, t)] &= \rho(x)u_t(x, t) \quad \text{in } \Omega \times (0, T) \\
\frac{\partial u}{\partial \nu} &= \mu(x) \nabla u(x, t) \cdot N = g(x, t) \quad \text{on } \partial \Omega \times (0, T) \\
u(x, 0) &= u_t(x, 0) = 0 \quad \text{in } \Omega
\end{align*}
\]

(2.1)

Where \( N \) is the unit outward normal to \( \partial \Omega \),

\[
\begin{align*}
\mu(x) &= \begin{cases} 
\tilde{\mu}(x) \in C^1(\bar{\Omega} \setminus D) \\
\hat{\mu}(x) \in C^1(D)
\end{cases} \\
\rho(x) &= \begin{cases} 
\tilde{\rho}(x) \in C^0(\bar{\Omega} \setminus D) \\
\hat{\rho}(x) \in C^0(D)
\end{cases} \\
\lambda(x), \mu(x), \rho(x) \geq \alpha_0 > 0
\end{align*}
\]

(2.2)

Substituting Equation (2.2) into the Equation (2.1), we have

\[
\begin{align*}
\nabla \cdot [(\hat{\tilde{\mu}}(x)\chi(\Omega \setminus D) + \hat{\mu}(x)\chi(D))]\nabla u &= (\hat{\tilde{\rho}}(x)\chi(\Omega \setminus D) + \hat{\rho}(x)\chi(D))u_t \quad \text{in } \Omega \times (0, T) \\
\frac{\partial u}{\partial \nu} &= \hat{\tilde{\mu}}(x)\nabla u \cdot N = g \quad \text{on } \partial \Omega \times (0, T) \\
u(x, 0) &= u_t(x, 0) = 0 \quad \text{in } \Omega
\end{align*}
\]

(2.3)

By the results in [4] or [10], we know that there exists a unique solution \( u \in C^2((0, T); H^1(\Omega)) \) for \( g \in C^2((0, T); H^{3/2}(\partial \Omega)) \).

Our inverse problem is to uniquely identify the \( \mu, \rho \) from the interior measurements \( u|_{\Omega \times (0, T)} \).

3. Main result

Our main result can be stated as follows:

**Theorem 3.1.** Suppose that \( \rho_j, \mu_j \) for \( j = 1, 2 \) satisfy Equation (2.2). For \( j = 1, 2 \), if \( u \in C^2((0, T); H^1(\Omega)) \) is a common solution of the equations

\[
\begin{align*}
\nabla \cdot [(\hat{\tilde{\mu}}_j(x)\chi(\Omega \setminus D) + \hat{\mu}_j(x)\chi(D))]\nabla u &= (\hat{\tilde{\rho}}_j(x)\chi(\Omega \setminus D) + \hat{\rho}_j(x)\chi(D))u_t \quad \text{in } \Omega \times (0, T) \\
\frac{\partial u}{\partial \nu} &= \hat{\tilde{\mu}}_j(x)\nabla u \cdot N = g \quad \text{on } \partial \Omega \times (0, T) \\
u(x, 0) &= u_t(x, 0) = 0 \quad \text{in } \Omega
\end{align*}
\]

(3.1)

then we have \((\mu_1, \rho_1) = (\mu_2, \rho_2)\) in \( \Omega \setminus \Omega_E \), where \( \Omega_E := \bigcup \{V \subset \Omega \text{ is an open set satisfying } \|u\|_{L^2(V \times (0, T))} = 0\} \).

**Remark.** The domain \( \Omega_E \) defined in Theorem 3.1 is the subset of \( \Omega \) where the shear wave has not yet reached in the time \((0, T)\).
4. Proof of the main result
Firstly, we present and prove the following lemmas:

Lemma 4.1. The boundary value problem Equation (2.3) is equivalent the following transmission problem:

\[
\begin{align*}
\nabla \cdot (\hat{\mu}(x)\nabla u) &= \hat{\rho}(x)u_{tt} \quad \text{in} \quad \Omega \setminus \bar{D} \times (0, T) \\
\nabla \cdot (\bar{\mu}(x)\nabla u) &= \bar{\rho}(x)u_{tt} \quad \text{in} \quad D \times (0, T) \\
u|_+ &= u|_- \\
\frac{\partial u}{\partial \nu}|_- &= \frac{\partial u}{\partial \nu}|_+ \quad \text{on} \quad \partial D \times (0, T) \\
\frac{\partial u}{\partial \nu}|_{\partial \Omega} &= g \quad \text{on} \quad \partial \Omega \times (0, T) \\
u(x, 0) &= u_t(x, 0) = 0, \quad \text{in} \quad \Omega
\end{align*}
\]

with \( u \in C^2((0, T); H^1(\Omega)) \cap C^2((0, T); H^2(\Omega \setminus D)) \cap C^2((0, T); H^2(D)) \).

Proof. For any function \( \varphi \in C_0^\infty(\Omega) \), and assume that

\[
\frac{\partial u}{\partial \nu} := \tilde{\mu}(x)\nabla u \cdot N
\]

Where \( N \) is the unit outward normal to \( \partial \Omega \), we compute

\[
\begin{align*}
\int_\Omega \nabla \cdot (\hat{\mu}(x)\chi(\Omega \setminus D) + \bar{\mu}(x)\chi(D))\nabla u\varphi \, dx \\
&= - \int_{\Omega \setminus D} \hat{\mu}(x)\nabla u \cdot \nabla \varphi \, dx - \int_D \bar{\mu}(x)\nabla u \cdot \nabla \varphi \, dx \\
&= \int_{\Omega \setminus D} \nabla \cdot [\hat{\mu}(x)\nabla u] \varphi \, dx \quad + \quad \int_D \frac{\partial u}{\partial \nu} \varphi \, ds \\
&\quad + \int_D \nabla \cdot [\bar{\mu}(x)\nabla u] \varphi \, dx - \int_D \frac{\partial u}{\partial \nu} \varphi \, ds \\
&= \int_{\Omega \setminus D} (\hat{\rho}(x)\chi(\Omega \setminus D) + \bar{\rho}(x)\chi(D))u_{tt}\varphi \, dx \\
&\quad + \int_D \tilde{\rho}(x)u_{tt}\varphi \, dx + \int_D \bar{\rho}(x)u_{tt}\varphi \, dx.
\end{align*}
\]

This completed the proof. \( \blacksquare \)

Lemma 4.2 (Finite propagation speed). Assume that \( \mu, \rho \) satisfy Equation (2.2), \( u \in C^2((0, T); H^1(\Omega)) \cap C^2((0, T); H^2(\Omega \setminus D)) \cap C^2((0, T); H^2(D)) \) is the solution of equation Equation (2.3), and if for any time \( t_0 \in [0, T) \), \( u(\cdot, t_0) = u_t(\cdot, t_0) = 0 \) in a open ball \( B_c(x_0) \subset \Omega \), then we have \( u = 0 \) a.e. in the space-time cone \( \bigcup_{0<s<c} C_s \ (C_s = C_s(x_0, t_0, \epsilon, c) := B_{c-CS}(x_0) \times \{ t = t_0 + s \} ) \), where \( c := \sup_{x \in B_c(x_0)} \sqrt{\mu(x)/\rho(x)} \).

Proof. For any fixed \( t_0 \in [0, T) \) and assume that

\[
u(\cdot, t_0) = u_t(\cdot, t_0) = 0, \quad \text{in} \quad B_c(x_0)
\]

we need to prove that \( u(x, t) = 0 \) a.e. in \( \bigcup_{0<s<c} C_s \), where \( (C_s = C_s(x_0, t_0, \epsilon, c) := B_{c-CS}(x_0) \times \{ t = t_0 + s \} ) \). Define the elastic energy contained in \( C_s \)

\[
e(s) := \frac{1}{2} \int_{C_s} \left\{ \rho|u_t|^2 + \mu|\nabla u|^2 \right\} \, dx,
\]
we will show \( e(s) = 0 \) for any \( s \in (0, e/c) \).

For any fixed \( s \in (0, e/c) \), define \( \Lambda(s) := \bigcup_{0 < \tau < s} C_\tau \), \( \hat{\Omega}(s) := (\Omega \setminus D) \times (t_0, t_0 + s) \) and \( \hat{\Omega}(s) := D \times (t_0, t_0 + s) \), followed by Equation (2.3), we have

\[
0 = \int_{\Lambda(s)} u_t \cdot \{(\hat{\rho} \chi(\Omega \setminus D) + \hat{\rho} \chi(D))u_t - \nabla \cdot [(\hat{\mu} \chi(\Omega \setminus D) + \hat{\mu} \chi(D))\nabla u]\} \, dx \, dt \\
= \int_{\Lambda(s) \cap \hat{\Omega}(s)} u_t \cdot \{\hat{\rho}u_{tt} - \nabla \cdot [\hat{\mu}\nabla u]\} \, dx \, dt \\
+ \int_{\Lambda(s) \cap \hat{\Omega}(s)} u_t \cdot \{\hat{\rho}u_{tt} - \nabla \cdot [\hat{\mu}\nabla u]\} \, dx \, dt \\
= \frac{1}{2} \int_{\Lambda(s) \cap \hat{\Omega}(s)} \{(\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2)_t - 2\nabla \cdot (\hat{\mu}u_t \nabla u)\} \, dx \, dt \\
+ \frac{1}{2} \int_{\Lambda(s) \cap \hat{\Omega}(s)} \{(\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2)_t - 2\nabla \cdot (\hat{\mu}u_t \nabla u)\} \, dx \, dt .
\]

Applying the divergence theorem on the boundaries \( \partial(\Lambda(s) \cap \hat{\Omega}(s)) := (C_0 \cap (\Omega \setminus D)) \cup (C_\tau \cap (\Omega \setminus D)) \cup (\bigcup_{0 < \tau < s} \partial C_\tau) \cup \partial D \) and \( \partial(\Lambda(s) \cap \hat{\Omega}(s)) := (C_0 \cap D) \cup (C_\tau \cap D) \cup \partial D \), we have

\[
0 = \int_{\partial(\Lambda(s) \cap \hat{\Omega}(s))} \left\{(\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2)_t \hat{N}_t - 2\hat{\mu}u_t \nabla u \cdot \hat{N}_x \right\} ds_{x,t} \\
+ \int_{\partial(\Lambda(s) \cap \hat{\Omega}(s))} \left\{(\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2)_t \hat{N}_t - 2\hat{\mu}u_t \nabla u \cdot \hat{N}_x \right\} ds_{x,t} 
\]  

(4.3)

where \( (\hat{N}_x, \hat{N}_t) \) is the unit outward normal to \( \partial(\Lambda(s) \cap \hat{\Omega}(s)) \), and \( (\hat{N}_x, \hat{N}_t) \) is the unit outward normal to \( \partial(\Lambda(s) \cap \hat{\Omega}(s)) \). Since on \( \partial D \), we have \( \hat{N}_t = \hat{N}_x = 0, \hat{N}_x = -N, \hat{N}_x = N \) and

\[
\left\{ \begin{array}{ll}
(\hat{\mu}u_t \nabla u \cdot \hat{N}_x = u_t(-\frac{\partial u}{\partial v})_+ = u_t(\frac{\partial u}{\partial v})_+ = (\hat{\mu}u_t \nabla u \cdot \hat{N}_x) & \text{on } \partial D \\
\frac{\partial u}{\partial v}_+ = \frac{\partial u}{\partial v}_- & \end{array} \right\
\]

We also have

\[
(\hat{N}_x, \hat{N}_t) := \begin{cases} (0, 1) & \text{on } C_\tau \cap (\Omega \setminus D) \\
(0, -1) & \text{on } C_0 \cap (\Omega \setminus D) \\
\frac{1}{\sqrt{1 + c^2}} \frac{x - x_0}{|x - x_0|} c & \text{on } \hat{L} := \bigcup_{0 < \tau < s} \partial C_\tau \\
\end{cases}
\]

(4.3)

\[
(\hat{N}_x, \hat{N}_t) := \begin{cases} (0, 1) & \text{on } C_\tau \cap D \\
(0, -1) & \text{on } C_0 \cap D \\
\end{cases}
\]

So the equations Equation (4.3) can be rewritten as

\[
e(s) - e(0) = \frac{-c}{2\sqrt{1 + c^2}} \int_{\hat{L}} \left\{\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2 - \frac{2}{c}(\hat{\mu}u_t \nabla u \cdot \frac{x - x_0}{|x - x_0|}) \right\} dx_s, t 
\]

(4.4)

Using the Cauchy - Schwartz inequality and the fact that \( \sqrt{\hat{\mu}/\rho} \leq c \), the integrand of the right side of Equation (4.4) can be rewritten as

\[
\hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2 - \frac{2}{c}(\hat{\mu}u_t \nabla u \cdot \frac{x - x_0}{|x - x_0|}) \\
\geq \hat{\rho}|u_t|^2 + \hat{\mu}|
abla u|^2 - \frac{2\hat{\mu}}{c}|
abla u||u_t| \geq \hat{\mu} \left( \sqrt{\frac{\hat{\rho}}{\hat{\mu}}|u_t|^2 - |
abla u|^2} \right)^2 \geq 0
\]
Thus we have \( e(s) - e(0) \leq 0 \). Since \( e(s) \geq 0 \) and \( e(0) = 0 \), \( 0 \leq e(s) \leq e(0) = 0 \), which implies \( e(s) = 0 \) for all \( s \in (0, \epsilon/c) \).

With the assumption Equation (2.2), we have the \( L^2 \)-estimate for \( u_t \) in the space-time cone \( \Lambda(\epsilon/c) = \bigcup_{0 < s < \epsilon/c} C_s \)

\[
\|u_t\|_2^2(\Lambda(\epsilon/c)) \leq \int_{\Lambda(\epsilon/c)} \frac{\rho}{\alpha_0} |u_t|^2 \, dx \, dt \leq \frac{2}{\alpha_0} \int_0^{\epsilon/c} e(s) \, ds = 0
\]

Which implies \( u_t = 0 \) in the space-time cone. Added the condition

\[
u\big((\cdot,t_0) = u_t(\cdot,t_0) = 0, \text{ in } B_\epsilon(x_0)\]

We have \( u(x,t) = 0 \) a.e. in \( \Lambda(\epsilon/c) \).

**Lemma 4.3.** Let \( M \subset \mathbb{R}^n \) and \( \bar{U} = (U_1, \ldots, U_m) \in [H^1_{loc}(M)]^m \) is the solution to

\[\Delta \bar{U} + B(\nabla \bar{U}) + V(\bar{U}) = 0 \quad (4.5)\]

in the distributional sense, where the operators are given by

\[
[B(\nabla \bar{U})]_k := \sum_{i=1}^n \sum_{j=1}^m a_{ij}^k(x) \frac{\partial U_j}{\partial x_i} \quad [V(\bar{U})]_k := \sum_{j=1}^m b_{j}^k(x) \bar{U}_j \quad (4.6)
\]

Then \( \bar{U} = 0 \) in an open subset of \( M \) implies \( \bar{U} = 0 \) in \( M \).

**Proof.** See [15].

Based on this Lemma, we have:

**Lemma 4.4.** Let \( M \subset \mathbb{R}^n \) be an open set and \( \rho_j, \mu_j \) satisfy \( \rho_j(x) \in C(M), \mu_j(x) \in C^1(M), \mu_j(x), \rho_j(x) \geq \alpha_0 > 0 \) for \( j = 1, 2 \). Let \( u(x,t) \in C^2((0,T);H^2_{loc}(M)) \) be a common solution for \( j = 1, 2 \)

\[\nabla \cdot [\mu_j(x) \nabla u] = \rho_j(x) u_{tt} \text{ in } M \times (0,T) \quad (4.7)\]

\(|\langle \frac{\mu_j}{\rho_j} \rangle| \geq \beta_0 > 0 \) in \( M \) is also given with \( \langle F \rangle := F_1 - F_2 \). Then for any \( t_0 \in (0,T), u(\cdot, t_0) = 0 \) in an open subset of \( M \) implies \( u(\cdot, t_0) = 0 \) in \( M \).

**Proof.** For any \( t_0 \in (0,T) \) and \( |\langle \frac{\mu_j}{\rho_j} \rangle| \geq \beta_0 > 0 \) in \( M \), we have

\[
\Delta u = 0 \quad (4.8)
\]

Thus \( u(\cdot, t_0) \) satisfies \( \Delta u + B(\nabla u) + V(u) = 0 \) in \( M \), where the coefficients of the operators satisfy \( a_{ij}^k, b_j^k \in L^\infty(M) \).

By Lemma 4.3, we have the conclusion.

The proof is completed.

By the similar technique used in the proofs in [1], we have

**Lemma 4.5.** Let \( M \subset \mathbb{R}^n \) be an open set with piecewise \( C^2 \)-class boundary. \( \rho_j, \mu_j \) satisfy the assumptions in Lemma 4.4 for \( j = 1, 2 \). Let \( u(x,t) \in C^2((0,T);H^2(M)) \) be a common solution for \( j = 1, 2 \) to Equation (4.7). If for any \( t_0 \in (0,T), \) it is satisfied that

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial \nu} = 0 \\
u(\cdot, t_0) = 0
\end{array} \right. \quad \text{on } S := \partial M \cap B_0 \neq \emptyset
\]

where \( B_0 \) is an open ball whose center is a point of \( \partial M \). Thus we have \( u(\cdot, t_0) = 0 \) in \( M \).
**Proof.** By the assumptions that
\[ \begin{cases} u(\cdot, t_0) = 0 \\ \frac{\partial u(\cdot, t_0)}{\partial \nu} = 0 \end{cases} \quad \text{on } S := \partial M \cap B_0 \neq \emptyset \]  
we can define
\[ \tilde{u}(\cdot, t_0) = \begin{cases} u(\cdot, t_0) & \text{on } M \\ 0 & \text{on } M_0 \setminus M \end{cases} \]
on $M_0 := M \cup B_0$. Also we extend $\rho_j, \mu_j$ and they satisfy $\rho_j(x) \in C(M_0), \mu_j(x) \in \mathcal{C}^1(M_0), \rho_j(x), \mu_j(x) \geq \alpha_0 > 0$ and $|\frac{\mu_j}{\rho_j}| \geq \beta_0 > 0$ in $M_0$ for $j = 1, 2$.

It is obvious that $\tilde{u}(\cdot, t_0) \in \mathcal{H}^2_{loc}(\Omega_0)$ and $u(\cdot, t_0)$ in $M$ satisfies
\[ Lu := \langle \frac{\mu}{\rho} \rangle \Delta u + \langle \nabla \mu \rangle \cdot \nabla u = 0. \]

For $\psi \in C_0^\infty(M_0)$, we have
\[ \int_{M_0} \tilde{u} \cdot L \psi \, dx = \int_M u \cdot L \psi \, dx = \int_M L u \cdot \psi \, dx + \int_{\partial M \cap S} [\frac{\partial \psi}{\partial \nu} \cdot u - \frac{\partial u}{\partial \nu} \cdot \psi] \, ds + \int_S [\frac{\partial \psi}{\partial \nu} \cdot u - \frac{\partial u}{\partial \nu} \cdot \psi] \, ds = 0 \]

Thus for any $t_0 \in (0, T)$, $\tilde{u}(\cdot, t_0)$ satisfies $L\tilde{u} = 0$ in $M_0$ in the distributional sense. Since $\tilde{u}(\cdot, t_0) = 0$ on $M_0 \setminus M \subset M_0$, Lemma 4.4 implies $\tilde{u}(\cdot, t_0) = 0$ in $M_0$.

**Lemma 4.6 (Unique continuation).** Let $M$ be an open subset of $\Omega$ with piecewise $C^2$-class boundary and $\rho_j, \mu_j$ satisfy Equation (2.2) in $\Omega$ for $j = 1, 2$. $|\langle \frac{\mu_j}{\rho_j} \rangle| \geq \beta_0 > 0$ with $\langle \mathcal{F} \rangle := \mathcal{F}_1 - \mathcal{F}_2$ is satisfied in $M$. Let $u \in C^2((0, T); H^1(\Omega)) \cap C^2((0, T); H^2(\Omega \setminus D)) \cap \mathcal{C}^2((0, T); H^2(D))$ be a common solution for $j = 1, 2$ to
\[ \begin{cases} \nabla \cdot [(\hat{\rho}_j(x) \chi(\Omega \setminus D) + \hat{\mu}_j(x) \chi(D)) \nabla u] \\ = (\hat{\rho}_j(x) \chi(\Omega \setminus D) + \hat{\mu}_j(x) \chi(D)) \nu_t \end{cases} \quad \text{in } \Omega \times [0, T) \]

Thus for any $t_0 \in (0, T)$, $u(\cdot, t_0) = 0$ in an open subset $Q \subset M$ implies $u(\cdot, t_0) = 0$ in $M$.

**Proof.** Assume that $\tilde{M} = M \cap (\Omega \setminus D)$ and $\tilde{M} = M \cap D$. Without loss generality, we assume that $\tilde{M} \neq \emptyset$ and $Q \subset \tilde{M}$. Since for any $t_0 \in (0, T)$, $u(\cdot, t_0) = 0$ in $Q \subset M$, Lemma 4.5 implies
\[ u(\cdot, t_0) = 0 \quad \text{on } \tilde{M} \]

We also have
\[ u(\cdot, t_0) = 0, \frac{\partial u(\cdot, t_0)}{\partial \nu} = 0 \quad \text{on } \partial \tilde{M} \cap \partial D \]
The equations Equation (4.1) implies
\[ u(\cdot, t_0) = 0, \frac{\partial u(\cdot, t_0)}{\partial \nu} = 0 \quad \text{on } \partial \tilde{M} \cap \partial (\Omega \setminus D) \subset \partial \tilde{M} \]

Applying Lemma 4.4, we have $u(\cdot, t_0) = 0$ in $\tilde{M}$.

Therefore, we have $u(\cdot, t_0) = 0$ in $M \subset \Omega$ for any $t_0 \in (0, T)$.

The proof is completed.
Now we can complete the proof of Theorem 3.1:

**Proof of Theorem 3.1:**

1) Firstly, we will show that $\mu_1/\rho_1 = \mu_2/\rho_2$ in $\Omega \setminus \Omega_E$ under the assumptions of Theorem 3.1. We can express $\Omega$ by the union of disjoint subsets $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$, where

$$
\Omega^0 := \{ x \in \Omega : \mu_1/\rho_1 = \mu_2/\rho_2 \} \quad \Omega^\pm := \{ x \in \Omega : \mu_1/\rho_1 \gtrless \mu_2/\rho_2 \}
$$

If we can prove $(\Omega^+ \cup \Omega^-) \subset \Omega_E$, i.e. $\Omega \setminus \Omega_E \subset \Omega_0$, the result can be obtained immediately. We will show that $(\Omega^+ \cup \Omega^-) \subset \Omega_E$. For any fixed point $x_0 \in \Omega^+$, there exist an open ball $B_c(x_0) \subset \Omega^+$. By the continuity of $\mu$, $\rho$ and the condition $\alpha_1, \alpha_2 > 0$, we have

$$
\alpha_1 \leq \mu_1/\rho_1 - \mu_2/\rho_2 \leq \alpha_2 \quad \text{on } B_c(x_0)
$$

(4.11)

i) “Shrink”: By the initial condition

$$
u(x, 0) = u_t(x, 0) = 0 \quad \text{on } \Omega
$$

and Lemma 4.2, we have $u(x, t) = 0$ a.e. in space-time cone $\bigcup_{0 < s < \epsilon/c} C_s$, where $C_s = C_s(x_0, 0, \epsilon, c) := B_{c - s}(x_0) \times \{ t = s \}$ with the condition $c = \sup_{x \in B_c(x_0)} \sqrt{\mu(x)/\rho(x)}$.

ii) “Spread”: By Equation (4.11), we have $\langle \mu/\rho \rangle \geq \alpha_1 > 0$ in $B_c(x_0)$. Thus Lemma 4.6 implies $u(\cdot, t_0) = 0$ in $B_c(x_0) \times \{ t = t_0 \}$ for any $t_0 \in [0, \epsilon/c]$.

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**Figure. 1.** Schematic diagram of “shrink - spread” procedure

Therefore, we have $u(x, t) = 0$ in $B_c(x_0) \times [0, \epsilon/c]$ which implies $u(\cdot, \epsilon/2c) = u_t(\cdot, \epsilon/2c) = 0$ in $B_c(x_0) \times \{ t = \epsilon/2c \}$. Similar to the step i) and ii), we have $u(x, t) = 0$ in $B_c(x_0) \times [\epsilon/2c, 3\epsilon/2c]$.

Iterating such “shrink - spread” procedures (Figure. 1), we have $u \equiv 0$ in $B_c(x_0) \times (0, T)$. Imlying $x_0 \in B_c(x_0) \subset \Omega_E$, we have $\Omega^+ \subset \Omega_E$. Similarly we have $\Omega^- \subset \Omega_E$. Which implies that $\Omega \setminus \Omega_E \subset \Omega^0$.

2) Secondly, we will prove that $(\mu_1, \rho_1) = (\mu_2, \rho_2)$ in $\Omega \setminus \Omega_E$.

Since we have proved that $c_s^2 = \mu_1/\rho_1 = \mu_2/\rho_2$ in $\Omega \setminus \Omega_E$, if we can prove that $\rho_1 = \rho_2$ in $\Omega \setminus \Omega_E$, then $(\mu_1, \rho_1) = (\mu_2, \rho_2)$ in $\Omega \setminus \Omega_E$ can be obtained.

Similarly, let $\Omega$ be expressed by the union of disjoint subsets $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$, where

$$
\Omega^0 := \{ x \in \Omega : \mu_1(x) = \mu_2(x) \} \quad \Omega^\pm := \{ x \in \Omega : \mu_1(x) \gtrless \mu_2(x) \}
$$

We will show that $(\Omega^+ \cup \Omega^-) \subset \Omega_E$ which implies $\Omega \setminus \Omega_E \subset \Omega \setminus (\Omega^+ \cup \Omega^-) = \Omega^0$. 
Taking the inner product of $u_t$ with the equation subtracted by the first equation in Equation (3.1), we have

$$0 = \int_0^T \int_{\Omega^+ \cap (\Omega \setminus D)} \int_0^s \{ (\langle \dot{\rho} \rangle |u|^2 + \langle \dot{\mu} \rangle |\nabla u|^2)_t - 2 \nabla \cdot ((\langle \dot{\mu} \rangle u \nabla u)) \} \, dt \, dx \, ds$$

(4.12)

where $\partial (\Omega^+ \cap (\Omega \setminus D)) \setminus \partial D := (\partial (\Omega^+ \cap (\Omega \setminus D)) \setminus (\partial \Omega \cup \partial D)) \cup (\partial (\Omega^+ \cap (\Omega \setminus D)) \cap \partial \Omega)$. On the boundary $\partial (\Omega^+ \cap (\Omega \setminus D)) \setminus \partial D$, $\langle \dot{\mu} \rangle = 0$. On $\partial D$, the Neumann boundary condition implies $\langle u_t \langle \dot{\mu} \rangle \nabla u \rangle \cdot N = 0$. Similarly, on $\partial (\Omega^+ \cap D) \setminus \partial D$, $\langle \dot{\mu} \rangle = 0$. And on $\partial D$, $\langle u_t \langle \dot{\mu} \rangle \nabla u \rangle \cdot N = (\langle u_t \langle \dot{\mu} \rangle \nabla u \rangle \cdot N$. These results imply that

$$\int_0^T \int_{\partial (\Omega^+ \cap (\Omega \setminus D)) \setminus \partial D} (u_t \langle \dot{\mu} \rangle \nabla u) \cdot N \, dt \, dS_x$$

$$+ \int_0^T \int_{\partial (\Omega^+ \cap D) \setminus \partial D} (u_t \langle \dot{\mu} \rangle \nabla u) \cdot N \, dt \, dS_x$$

$$+ \int_0^T \int_{\partial D} [u_t \langle \dot{\mu} \rangle - \langle \dot{\mu} \rangle \nabla u] \cdot N \, dt \, dS_x = 0$$

On the other hand, we have $\langle \rho \rangle = c_\mu^2 \langle \dot{\mu} \rangle$ in $\Omega \setminus \Omega_E$. Which imply $\langle \rho \rangle > 0$ in $\Omega^+$. Thus Equation (4.13) become

$$\int_0^T \int_{\Omega^+} \int_0^s \langle \rho \rangle \{|u|^2 + c_\mu^2 |\nabla u|^2\} \, dt \, dx \, ds = 0$$

(4.14)

which implies $u_t = 0, \nabla u = 0$ in $\Omega^+ \times (0, T)$. Therefore, we prove that $u = 0$ a.e. in $\Omega^+ \times (0, T)$ with the initial condition $u(x, 0) = u_t(x, 0) = 0$, i.e. $\Omega^+ \subset \Omega_E$. 
Similarly, we can prove that $\Omega^- \subset \Omega_E$. The proof is completed.

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