Geometric proof of the
Grobman-Hartman Theorem

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May 28, 2014

Abstract
We give geometric proofs for Grobman-Hartman theorem for diffeomorphisms and ODEs. Proofs use covering relations and cone conditions for maps and isolating segments and cone condition for ODEs. We prove also the Hölder condition for the conjugating homeomorphisms.

2000 Mathematics Subject Classification. 37C15, 37D05
Key words and phrases. Grobman-Hartman theorem; Hölder regularity; covering relation; isolating segment; cone condition

1 Introduction

The goal of this paper is to give a geometric proof of the Grobman-Hartman theorem for diffeomorphisms and ODEs in finite dimension. By 'the geometric proof' we understand here the proof which works in the phasespace of the system under consideration and uses concepts of qualitative geometric nature. This differs from the standard functional analysis proof [Pa, Pu, BV], which is now a textbook proof (see for example [A, C99, PdM, Ze]), which studies the conjugacy problem in some abstract Banach space of maps. The original proof by P. Hartman [H1, H2, H3] also belongs to this category, but it lacks the elegance and the simplicity of the contemporary approach, because to solve the conjugacy problem Hartman required first to introduce new coordinates which straighten the invariant manifolds of the hyperbolic fixed point. The standard functional analysis proof, whose idea apparently comes from paper by Moser [M], in a current form is a straightforward application of the Banach contraction principle. The whole effort is to chose the correct Banach space and a contraction, whose fixed point will give us the conjugacy.

The original proof by Grobman [G1, G2] (for ODEs only) is not geometric in the above sense, while it works in the phase space and it is rather purely analytical, in fact the conjugacy in the Grobman-Hartman theorem is given explicitly in terms of some infinite integrals involving the fundamental matrix of

\footnote{Research has been supported by Polish National Science Centre grant 2011/03B/ST1/04780}
solution of the linear hyperbolic ODE and the perturbation. The proof appears rather to be analytical than conceptual.

In this paper we would like to give a geometric proof, which matches the conceptual elegance of the standard functional analysis proof. The geometric idea behind our approach can be seen a shadowing of $\epsilon$-pseudo orbit, with $\epsilon$ not small. This is accomplished using covering relations and cone condition $\text{ZGi}$ $\text{ZCC}$ in case of diffeomorphisms and for ODEs the notion of the isolating segment $\text{S1 S2 S3 SW WZ}$ and the cone conditions has been used. Let us comment about the relation between our proofs of the theorem for maps and for ODEs. The standard approach would be to derive the ODE case from the map case, by considering the time shift by one time unite and then arguing that we can obtain from it the conjugacy for all times (see [H1 Pa Pu PdM]). Here we provide the proof for ODEs which is independent from the map case in order to illustrate the power of the concept the isolating segment with the aim to obtain a clean ODE-type proof. For a clean ODE-type proof using the functional analysis type arguments see [CS].

It is up to the reader to judge whether our proof matches the elegance of the standard functional analysis proof, we would like to point here to the beginning of the Section 2.4 which contains the outline of the proof for the map case, and clearly shows what is the basic idea and as we hope make it clear that the details are not hard to fill-in.

We believe that the conceptual geometric proof, while may be not shorter and works only in finite dimension, has one fundamental advantage. Namely, it clearly follows our geometric intuition about the nature of the phenomenon under study (the underlying hyperbolicity). So we can avoid a situation when we think about the problem one way, but we prove the theorem using a totally different technique, which apparently has little in common with our geometrical insight into the problem.

To show that our geometric approach has the same analytical power as the standard functional analysis method we prove the Hölder regularity of the conjugacy in the Grobman-Hartman theorem and we obtain basically the same estimate for the Hölder exponent as in the work by Barreira and Valls [BV], which apparently is the only published proof of this fact (see [BV] and references given there).

The content of our work can be described as follows. Section 2 contains the geometric proof of the global version of the Grobman-Hartman theorem. In Section 3 we show the Hölder regularity of the conjugacy in the Grobman-Hartman theorem. Section 4 contains a geometric proof of the Grobman-Gartman theorem for flows, which is independent from the proof for maps.

At the end of this paper we included two appendices, which contains relevant definitions and theorems about the covering relations and the isolating segments.
1.1 Notation

If \( A \in \mathbb{R}^{d_1 \times d_2} \) is matrix, then by \( A^t \) we will denote its transpose. By \( B(x,r) \) we will denote the open ball centered at \( x \) and radius \( r \). For maps depending on some parameters \( h : P \times X \to X \) by \( h_p : X \to X \) we will denote the map \( h_p(x) = h(p,x) \).

1.2 The norms used in this paper

In this note we will work on \( \mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^s \). According to this decomposition we will often represent points \( z \in \mathbb{R}^n \) as \( z = (x, y) \), where \( x \in \mathbb{R}^u \) and \( y \in \mathbb{R}^s \). On \( \mathbb{R}^n \) we assume the standard scalar product \( (u,v) = \sum_i u_i v_i \). This scalar product induces the norm on \( \mathbb{R}^u \) and \( \mathbb{R}^s \). We will use the following norm on \( \mathbb{R}^n \), \( \| (x,y) \|_{\text{max}} = \max(\|x\|, \|y\|) \) and we will usually drop the subscript max.

2 Global version of the Grobman-Hartman theorem for maps

In this section we will give a geometric proof of the following theorem.

**Theorem 1** Assume that \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a linear isomorphism, of the following form

\[
A(x, y) = (A_u x, A_s y),
\]

where \( n = u + s \), \( A_u : \mathbb{R}^u \to \mathbb{R}^u \) and \( A_s : \mathbb{R}^s \to \mathbb{R}^s \) are linear isomorphisms such that

\[
\|A_u x\| \geq c_u \|x\|, \quad c_u > 1, \quad \forall x \in \mathbb{R}^u \tag{2}
\]

\[
\|A_s y\| \leq c_s \|y\|, \quad 0 < c_s < 1, \quad \forall y \in \mathbb{R}^s \tag{3}
\]

Assume that \( h : \mathbb{R}^n \to \mathbb{R}^n \) is of class \( C^k \), \( k \geq 1 \), is such that

\[
\|h(x)\| \leq M, \forall x \in \mathbb{R}^n \tag{4}
\]

\[
\|Dh(x)\| \leq \epsilon, \forall x \in \mathbb{R}^n \tag{5}
\]

Under the above assumptions there exists \( \epsilon_0 = \epsilon_0(A) > 0 \), such that if \( \epsilon < \epsilon_0(A) \), then there exists a homomorphism \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
\rho \circ A = (A + h) \circ \rho. \tag{6}
\]

**Comment:** Observe that there is no bound on \( M \), we also do not assume that \( h(0) = 0 \).

Before the proof of Theorem 1 we need first to develop some technical tools. The basic steps and constructions used in the proof are given in Section 2.4. We invite the reader to jump first to this section to see the overall picture of the proof and then consult other more technical sections when necessary.

We will use the following notation: \( g_\lambda = A + \lambda h \) for \( \lambda \in [0, 1] \). We will write also \( g = g_1 \).
2.1 \( g_\lambda \) are homeomorphisms

The following lemma can be found for example in [Ze] as the Proposition II.2.

**Lemma 2** There exists \( \epsilon_1(A) = \frac{1}{\|A^{-1}\|} > 0 \), such that if \( \epsilon \leq \epsilon_1(A) \), then \( g_\lambda \) is a homeomorphism and \( g_\lambda^{-1} \) is Lipschitz.

This lemma is proved in [Ze], using the contraction mapping principle. Here, for the sake of completeness, we will outline a more topological proof, which fits better with other arguments used in the paper.

**Proof:**

The surjectivity of \( g_\lambda \) follows from the following observation: a bounded continuous perturbation a linear isomorphism is a surjection - the proof is based on the local Brouwer degree (see for example Appendix in [ZGi] for the definition and properties).

Namely, for fixed \( y \in \mathbb{R}^n \) we consider equation \( y = g_\lambda(x) \), which is equivalent to \( x + \lambda A^{-1} h(x) = A^{-1} y = \tilde{y} \). Let us define a map

\[
F_\lambda(x) = x + \lambda A^{-1} h(x) - \tilde{y}. \tag{7}
\]

Observe that for \( \|x - \tilde{y}\| > \|A^{-1}\| M \), then \( F_\lambda(x) \neq 0 \). This shows that \( \text{deg}(F_\lambda, B(\tilde{y}, \|A^{-1}\| M), 0) \) the local Brouwer degree of \( F_\lambda \) on the set \( B(\tilde{y}, \|A^{-1}\| M) \) at 0 is defined and

\[
\text{deg}(F_\lambda, B(\tilde{y}, \|A^{-1}\| M), 0) = \text{deg}(F_0, B(\tilde{y}, \|A^{-1}\| M), 0). \tag{8}
\]

But for \( \lambda = 0 \) we have \( F_0(x) = x - \tilde{y} \). Hence \( \text{deg}(F_0, B(\tilde{y}, \|A^{-1}\| M), 0) = 1 \). Therefore \( F_\lambda(x) = 0 \) has solution for any \( \tilde{y} \in \mathbb{R}^n \).

The injectivity is obtained as follows

\[
\|g_\lambda(z_1) - g_\lambda(z_2)\| = \|A z_1 + \lambda h(z_1) - (A z_2 + \lambda h(z_2))\| \geq \|A(z_1) - A(z_2)\| - \lambda \|h(z_1) - h(z_2)\| \geq \frac{1}{\|A^{-1}\|} \|z_1 - z_2\| - \epsilon \|z_1 - z_2\| = \left( \frac{1}{\|A^{-1}\|} - \epsilon \right) \|z_1 - z_2\|. \tag{9}
\]

From the above formula it follows also that

\[
\|z_1 - z_2\| \geq \left( \frac{1}{\|A^{-1}\|} - \epsilon \right) \|g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)\|. \tag{10}
\]

Therefore

\[
\|g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)\| \leq \left( \frac{1}{\|A^{-1}\|} - \epsilon \right)^{-1} \|z_1 - z_2\|. \tag{11}
\]

\[\blacksquare\]
2.2 Cone conditions

In this section we will establish the cone condition for \( g_\lambda \) using the approach from [ZCC], where the cones are defined in terms of a quadratic form.

Let \( Q(x, y) = (x, x) - (y, y) \) be a quadratic form \( \mathbb{R}^n \). Our goal is show the following cone condition: for sufficiently small \( \eta > 0 \) holds

\[
Q(Az_1 - Az_2) > (1 + \eta)Q(z_1 - z_2), \quad z_1, z_2 \in \mathbb{R}^n, z_1 \neq z_2.
\] (11)

This will be established in Lemma 4.

By \( Q \) we will also denote a matrix, such that \( Q(z) = z^tQz \). In our case \( Q = \begin{bmatrix} I_u & 0 \\ 0 & -I_s \end{bmatrix} \), where \( I_u \in \mathbb{R}^{u \times u} \) and \( I_s \in \mathbb{R}^{s \times s} \) are the identity matrices.

**Lemma 3** For \( 0 \leq \eta \leq \min(c_u^2 - 1, 1 - c_s^2) \) matrix \( A^tQA - (1 + \eta)Q \) is positive definite.

**Proof:** Easy computations show that

\[
A^tQA = \begin{pmatrix} A_u^tA_u & 0 \\ 0 & A_s^tA_s \end{pmatrix}.
\]

Hence for any \( z = (x, y) \in \mathbb{R}^u \times \mathbb{R}^s \setminus \{0\} \) holds

\[
z^t(A^tQA - (1 + \eta)Q)z = x^tA_u^tA_ux - (1 + \eta)x^2 + (1 + \eta)y^2 - y^tA_s^tA_sy = (A_u^tA_ux - (1 + \eta)x^2 + (1 + \eta)y^2 - (A_s^tA_sy) \geq (c_u^2 - 1 - \eta)x^2 + (1 - \eta - c_s^2)y^2 > 0,
\]

if \( c_u^2 - 1 > \eta \) and \( 1 - c_s^2 > \eta \). \( \blacksquare \)

**Lemma 4** There exists \( \epsilon_0(A) > 0 \), such that if \( 0 \leq \epsilon < \epsilon_0(A) \), then there exists \( \eta \in (0, 1) \) such that for any \( \lambda \in [0, 1] \) the following cone condition holds

\[
Q(g_\lambda(z_1) - g_\lambda(z_2)) > (1 + \eta)Q(z_1 - z_2), \quad \forall z_1, z_2 \in \mathbb{R}^n, z_1 \neq z_2.
\] (12)

**Proof:** We have

\[
Q(g_\lambda(z_1) - g_\lambda(z_2)) = (z_1 - z_2)^t(D(z_1, z_2)^tQD(z_1, z_2))(z_1 - z_2),
\]

\[
D(z_1, z_2) = \int_0^1 Dg_\lambda(t(z_1 - z_2) + z_2)dt
\]

Let

\[
C(z_1, z_2) = \int_0^1 Dh(t(z_1 - z_2) + z_2)dt,
\] (13)

then

\[
D(z_1, z_2) = A + \lambda C(z_1, z_2).
\] (14)

Observe that \( \|C(z_1, z_2)\| \leq \epsilon \).
From Lemma 3 it follows that $A^TQA - (1 \pm \eta)Q$ is positively defined for sufficiently small $\eta$. Let us fix such $\eta$.

Since being a positively defined symmetric matrix is an open condition, hence there exists $\epsilon_0(A) > 0$ be such that the matrix

$$(A + \lambda C)^TQ(A + \lambda C) - (1 \pm \eta)Q$$

is positive definite for any $\lambda \in [0, 1]$ and $C \in \mathbb{R}^{n \times n}$ satisfying $\|C\| \leq \epsilon_0$. ■

From Lemma 2 it follows that for any $\lambda \in [0, 1]$ and any point $z$ we can define a full orbit for $g_\lambda$ through this point, i.e. $g_\lambda^k(z)$ makes sense for any $k \in \mathbb{Z}$.

Lemma 5 Assume that $\epsilon < \min(\epsilon_0(A), \epsilon_1(A))$ from Lemmas 4 and 2. Let $\lambda \in [0, 1]$. If $z_1, z_2 \in \mathbb{R}^n$ and $\beta$ are such that

$$\|g_\lambda^k(z_1) - g_\lambda^k(z_2)\| \leq \beta, \quad \forall k \in \mathbb{Z},$$

then $z_1 = z_2$.

Proof: The proof is by the contradiction. Assume that $z_1 \neq z_2$. Either $Q(z_1 - z_2) \geq 0$ or $Q(z_1 - z_2) < 0$.

Let us consider first case $Q(z_1 - z_2) \geq 0$. By the cone condition (Lemma 4) we obtain for any $k > 0$

$$Q(g_\lambda^k(z_1) - g_\lambda^k(z_2)) \geq Q(z_1 - z_2) \geq 0$$

$$||\pi_x(g_\lambda^k(z_1) - g_\lambda^k(z_2))|| \geq Q(g_\lambda^k(z_1) - g_\lambda^k(z_2)) > (1 + \eta)^{k-1}Q(g_\lambda(z_1) - g_\lambda(z_2)).$$

Therefore $g_\lambda^k(z_1) - g_\lambda^k(z_2)$ is unbounded. This contradicts (10).

Now we consider the case $Q(z_1 - z_2) < 0$. The cone condition (Lemma 4) applied to the inverse map gives for any $k > 0$

$$Q(z_1 - z_2) > (1 - \eta)Q(g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)) > (1 - \eta)^kQ(g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)).$$

Therefore we obtain

$$-Q(g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)) > \frac{1}{(1 - \eta)^k}(-Q(z_1 - z_2)).$$

Therefore $g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)$ is unbounded. This contradicts (10). ■

2.3 Covering relations

We assume that the reader is familiar with the notion of h-set and covering relation [ZG3]. For the convenience of the reader we recall these notions in Appendix 5.

For any $z \in \mathbb{R}^n$, $\alpha > 0$ we define an h-set (with a natural structure $N(z, \alpha) = z + B_\alpha(0, \alpha) \times B_\alpha(0, \alpha)$.

The following theorem follows immediately from Theorem 22 in Appendix 5.
Theorem 6 Assume that we have a bi-infinite chain of covering relations
\[ N_i \xrightarrow{f} N_{i+1}, \quad i \in \mathbb{Z}. \] (18)

Then there exists a sequence \( \{z_i\}_{i \in \mathbb{Z}} \) such that \( z_i \in N_i \) and \( f(z_i) = z_{i+1} \).

The following lemma plays the crucial role in the construction of \( \rho \) from Theorem 1.

Lemma 7 There exists \( \hat{\alpha} = \hat{\alpha}(A, M) = \max \left( \frac{2M}{R^2}, \frac{2M}{1-R} \right) \) such that for any \( \alpha > \hat{\alpha}, \lambda_1, \lambda_2 \in [0, 1] \) and \( z \in \mathbb{R}^n \) holds
\[ N(z, \alpha) \xrightarrow{A + \lambda_2 h} N((A + \lambda_2 h)(z), \alpha). \] (19)

Proof:
Let us fix \( z \in \mathbb{R}^n \) and let us define the homotopy \( H : [0, 1] \times \overline{B}_u(0, \alpha) \times \overline{B}_u(0, \alpha) \rightarrow \mathbb{R}^n \) as follows
\[ H_t((x, y)) = (A_x x, (1-t)A_y y) + (1-t)\lambda_1 h(z + (r, y)) + (A + t\lambda_2 h)(z) \] (20)

We have
\[ H_0(x, y) = A(z + (x, y)) + \lambda_1 h(z + (x, y)) = (A + \lambda_1 h)(z + (x, y)) \]
\[ H_1(x, y) = (A + \lambda_2 h)(z) + (A_x, 0). \]

For the proof it is enough to show the following conditions for all \( t, \lambda_1, \lambda_2 \in [0, 1] \)
\[ \|\pi_x(H_t((x, y)) - (A + \lambda_2 h)(z))\| > \alpha, \quad (x, y) \in (\partial B_u(0, \alpha)) \times \overline{B}_u(0, \alpha), \] (21)
\[ \|\pi_y(H_t((x, y)) - (A + \lambda_2 h)(z))\| < \alpha, \quad (x, y) \in \overline{B}_u(0, \alpha) \times \overline{B}_u(0, \alpha). \] (22)

First we establish (21). We have
\[ \|\pi_x(H_t((x, y)) - (A + \lambda_2 h)(z))\| = \]
\[ \|\pi_x((A_x x, (1-t)A_y y) + (1-t)\lambda_1 h(z + (x, y)) + (A + t\lambda_2 h)(z) - (A + \lambda_2 h)(z))\| = \]
\[ \|A_x x + (1-t)\lambda_1 \pi_x h(z + (x, y)) + (t-1)\lambda_2 \pi_x h(z)\| \geq \]
\[ \|A_x x - \|h(z + r)\| - \|h(z)\| \| \geq c_u \alpha - 2M. \]

Hence (21) holds if the following inequality is satisfied
\[ (c_u - 1)\alpha > 2M. \] (23)

Now we deal with (22). We have
\[ \|\pi_y(H_t((x, y)) - (A + \lambda_2 h)(z))\| = \]
\[ \|(1-t)A_y y + (1-t)\lambda_1 \pi_y h(z + (x, y)) + (t-1)\lambda_2 \pi_y h(z)\| \leq \]
\[ \|A_y y\| + \|h(z + r)\| + \|h(z)\| \leq c_u \alpha + 2M. \]
Hence (22) holds if the following inequality is satisfied
\[(1 - cs)\alpha > 2M.\] (24)

Hence it is enough to take \(\hat{\alpha} = \max\left(\frac{2M}{c_u-1}, \frac{2M}{1-c_s}\right)\). \(\blacksquare\)

2.4 The proof of Thm. 1

We define map \(\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n\) and a candidate for its inverse \(\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n\) as follows:

1. Let us fix \(\alpha > \hat{\alpha}\), where \(\hat{\alpha}\) is obtained in Lemma 7,

2. for \(z \in \mathbb{R}^n\), from Lemma 7 with \(\lambda_1 = 1\) and \(\lambda_2 = 0\) we have a bi-infinite chain of covering relations

\[
\ldots \xrightarrow{g} N(A^{-2}z, \alpha) \xrightarrow{g} N(A^{-1}z, \alpha) \xrightarrow{g} N(Az, \alpha) \xrightarrow{g} N(A^2z, \alpha) \xrightarrow{g} N(A^3z, \alpha) \xrightarrow{g} \ldots \] (25)

3. from Theorem 6 and Lemma 5 it follows that the chain of covering relations (25) defines a unique point, which we will denote by \(\rho(z)\), such that

\[g^k(\rho(z)) \in N(A^kz, \alpha) \quad k \in \mathbb{Z}.\] (26)

4. for \(z \in \mathbb{R}^n\), from Lemma 7 with \(\lambda_1 = 0\) and \(\lambda_2 = 1\) we have a bi-infinite chain of covering relations

\[
\ldots \xrightarrow{A} N(g^{-2}z, \alpha) \xrightarrow{A} N(g^{-1}z, \alpha) \xrightarrow{A} N(\alpha, \alpha) \xrightarrow{A} N(gz, \alpha) \xrightarrow{A} N(g^2z, \alpha) \xrightarrow{A} N(g^3z, \alpha) \xrightarrow{A} \ldots \] (27)

5. from Theorem 6 and Lemma 5 it follows that the chain of covering relations (27) defines a unique point, which we will denote by \(\sigma(z)\), such that

\[A^k(\sigma(z)) \in N(g^kz, \alpha) \quad k \in \mathbb{Z}.\] (28)

In fact we have the following statement related to the definition of \(\rho(z)\), which shows that it does not depend on \(\alpha\)

**Lemma 8** Assume \(\hat{\alpha} < \beta\), that

Let \(z \in \mathbb{R}^n\). If \(z_1\) is such that

\[g^k(z_1) \in N(A^kz, \beta), \quad k \in \mathbb{Z},\] (29)

then \(z_1 = \rho(z)\).
Proof: Observe that from (26) and (29) it follows that
\[ \| g^k(z_1) - g^k(\rho(z)) \| \leq \alpha + \beta. \] (30)

The assertion follows from Lemma 5.

We continue with the proof of Thm. 1. From the definition of \( \rho \) and \( \sigma \) we immediately conclude that \( \rho \circ A = g \circ \rho \) and \( \sigma \circ g = A \circ \sigma \).

We will show that \( \sigma \circ \rho = Id \) and \( \rho \circ \sigma = Id \).

Let us consider first \( \sigma \circ \rho \). Let us fix \( z \in \mathbb{R}^n \), then for any \( k \in \mathbb{Z} \) holds
\[ \| g^k(\rho(z)) - A^k(z) \| \leq \alpha, \] (31)
\[ \| A^k(\sigma(\rho(z))) - g^k(\rho(z)) \| \leq \alpha. \] (32)
Hence
\[ \| A^k(\sigma(\rho(z))) - A^k(z) \| \leq 2\alpha, \quad k \in \mathbb{Z}. \] (33)

From Lemma 5 it follows that \( z = \sigma(\rho(z)) \).

The proof of \( \rho \circ \sigma = Id \) is analogous.

Now we show that \( \rho \) is continuous. (In Section 3 we prove stronger statement. We show that \( \rho \) satisfies the Hölder condition.) Assume that \( z_j \to \bar{z} \), we will show that the sequence \( \{\rho(z_j)\}_{j \in \mathbb{N}} \) is bounded and each its converging subsequence converges to \( \rho(\bar{z}) \).

We can assume that \( \| z_j - \bar{z} \|_m < \alpha \). Then, since \( \| \rho(z_j) - z_j \| < \alpha \) we obtain
\[ \| \rho(z_j) - \bar{z} \| < 2\alpha. \] (34)
Hence \( \{\rho(z_j)\}_{j \in \mathbb{N}} \) is bounded.

Now let us take a convergent subsequence, which we will again index by \( j \), hence \( z_j \to \bar{z} \) and \( \rho(z_j) \to w \) for \( j \to \infty \), where \( w \in \mathbb{R}^n \). We will show that \( w = \rho(\bar{z}) \). This implies that \( \rho(z_j) \to \rho(\bar{z}) \).

Let us fix \( k \in \mathbb{Z} \). From the continuity of \( z \mapsto A^kz \) it follows, that there exists \( j_0 \) such for \( j \geq j_0 \) holds
\[ \| A^k z_j - A^k \bar{z} \| < \alpha. \] (35)
Since by the definition of \( \rho \) we have
\[ g^k(\rho(z_j)) \in N(A^k z_j, \alpha) \] (36)
implies that
\[ \| g^k(\rho(z_j)) - A^k \bar{z} \| \leq 2\alpha. \] (37)
By passing to the limit with \( j \) we obtain
\[ \| g^k(w) - A^k \bar{z} \| \leq 2\alpha. \] (38)
Since (38) holds for all \( k \in \mathbb{Z} \), then by Lemma 8 \( w = \rho(\bar{z}) \).

The proof of continuity of \( \sigma \) is analogous.
2.5 From global to local Grobman-Hartman theorem

Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism satisfying

$$\varphi(z) = Az + h(z), \quad (39)$$

where $A \in \mathbb{R}^{n \times n}$ is a linear hyperbolic isomorphism and

$$h(0) = 0, \quad Dh(0) = 0. \quad (40)$$

Let us fix $\epsilon > 0$. There exists $\delta > 0$, such that

$$\|Dh(z)\| < \epsilon, \quad \|z\| \leq \delta. \quad (41)$$

Let $t : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that

$$t(r) = r, \quad r \leq \delta/2, \quad (42)$$
$$t(r) = w < \delta, \quad r \geq \delta, \quad (43)$$
$$t(r_1) \leq t(r_2), \quad r_1 < r_2 \quad (44)$$
$$0 < t'(r) < 1, \quad r \in [\delta/2, \delta]. \quad (45)$$

Consider now the function $R : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$R(0) = 0, \quad R(z) = \frac{t(\|z\|)}{\|z\|}z, \quad z \neq 0 \quad (46)$$

It is easy to see that

$$R(z) = z, \quad z \in \overline{B}(0, \delta/2), \quad (47)$$
$$R(\mathbb{R}^n) \subset \overline{B}(0, w), \quad (48)$$
$$\|DR\| \leq 1. \quad (49)$$

Consider now the following modification of $\varphi$ given by

$$\hat{\varphi}(z) = Az + h(R(z)). \quad (50)$$

It is easy to see that

$$\hat{\varphi}(z) = \varphi(z), \quad z \in \overline{B}(0, \delta_2) \quad (51)$$
$$\|h(R(z))\| \leq \epsilon \delta, \quad z \in \mathbb{R}^n, \quad (52)$$
$$\|D(h \circ R)(z)\| \leq \epsilon, \quad z \in \mathbb{R}^n. \quad (53)$$

It is clear that by taking $\epsilon$ and $\delta$ small enough $h \circ R$ will satisfy the smallness assumption in Theorem 1, hence we will obtain the local conjugacy, which is the Grobman-Hartman theorem.
3 Hölder regularity of \( \rho \)

In fact the proof of the continuity of \( \rho \) from Thm. 1 can be done more ‘quantitatively’. Our goal is to show the Hölder property for \( \rho \). The main result in this section is Theorem 12. The same arguments apply also to \( \sigma \).

Let \( L \) be the Lipschitz constant for \( A \) and \( A^{-1} \).

**Lemma 9** Let \( Q, A, g \) be as in the proof of Theorem 1. If \( Q(z_1 - z_2) \geq 0 \), \( z_1 \neq z_2 \). Then \( Q(g(z_1) - g(z_2)) > 0 \) and

\[
\|\pi_x g(z_1) - \pi_x g(z_2)\| > \theta_u\|\pi_x z_1 - \pi_x z_2\|,
\]

where \( \theta_u = c_u - 2c_0 > 1 \)

**Proof:** From cone condition (Lemma 4) it follows that \( Q(g(z_1) - g(z_2)) > 0 \).

Since \( Q(z_1 - z_2) \geq 0 \), hence

\[
\|\pi_x z_1 - \pi_x z_2\| \geq \|\pi_y z_1 - \pi_y z_2\|.
\]

We have

\[
\pi_x g(z_1) - \pi_x g(z_2) = \int_0^1 D\pi_x g(t(z_1 - z_2) + z_2)dt \cdot (z_1 - z_2) =
A_u\pi_x(z_1 - z_2) + \int_0^1 \frac{\partial \pi_x h}{\partial x}(t(z_1 - z_2) + z_2)dt \cdot \pi_x(z_1 - z_2) + \int_0^1 \frac{\partial \pi_x h}{\partial y}(t(z_1 - z_2) + z_2)dt \cdot \pi_y(z_1 - z_2).
\]

Hence for if \( Q(z_1 - z_2) \geq 0 \) we obtain

\[
\|\pi_x g(z_1) - \pi_x g(z_2)\| \geq c_u\|\pi_x(z_1 - z_2)\| - 2\epsilon\|\pi_x(z_1 - z_2)\|.
\]

An analogous lemma holds for the inverse map.

**Lemma 10** Let \( Q, A, g, \rho \) be as in the proof of Theorem 1. If \( Q(z_1 - z_2) \leq 0 \), \( z_1 \neq z_2 \). Then \( Q(g^{-1}(z_1) - g^{-1}(z_2)) < 0 \) and

\[
\|\pi_y g^{-1}(z_1) - \pi_y g^{-1}(z_2)\| > \theta_s\|\pi_y z_1 - \pi_y z_2\|,
\]

where \( \theta_s = \frac{1}{c_s - 2\epsilon} > 1 \)

**Proof:** From cone condition (Lemma 4) it follows that \( Q(g^{-1}(z_1) - g^{-1}(z_2)) < 0 \).

Since \( Q(z_1 - z_2) \leq 0 \), hence

\[
\|\pi_y z_1 - \pi_y z_2\| \geq \|\pi_x z_1 - \pi_x z_2\|.
\]
We have for any $z_1, z_2$

$$
\pi_y g(z_1) - \pi_y g(z_2) = \int_0^1 D\pi_y g(t(z_1 - z_2) + z_2) dt \cdot (z_1 - z_2) = 
$$
$$
A_s \pi_y (z_1 - z_2) + \int_0^1 \frac{\partial \pi_y h}{\partial x}(t(z_1 - z_2) + z_2) dt \cdot \pi_x (z_1 - z_2) + 
$$
$$
\int_0^1 \frac{\partial \pi_y h}{\partial y}(t(z_1 - z_2) + z_2) dt \cdot \pi_y (z_1 - z_2).
$$

Hence if $Q(g(z_1) - g(z_2)) \leq 0$, then $Q(z_1 - z_2) < 0$ and we have

$$
||\pi_y g(z_1) - \pi_y g(z_2)|| \leq c_\epsilon ||\pi_y(z_1 - z_2)|| + 2\epsilon ||\pi_y(z_1 - z_2)|| = (c_\epsilon + 2\epsilon) ||\pi_y(z_1 - z_2)||,
$$

which after the substitution $z_i \mapsto g^{-1}z_i$ gives for $Q(z_1 - z_2) \leq 0$ the following

$$
\|\pi_y z_1 - \pi_y z_2\| \leq (c_\epsilon + 2\epsilon) \|\pi_y (g^{-1}z_1) - g^{-1}(z_2)\|.
$$

(58)

Lemma 11 Let $Q, A, g, \rho$ be as in the proof of Theorem 7.

Let $\theta = \min(\theta_1, \theta_2)$.

Then for any $k \in \mathbb{Z}_+$ holds

$$
\|\rho(z_1) - \rho(z_2)\| \leq \frac{2\alpha}{\theta^k} + \left(\frac{L}{\theta}\right)^k \|z_1 - z_2\|.
$$

(59)

Proof: From our assumptions it follows that $\theta > 1$ and $L > \theta$.

We show the derivation for the case $Q(\rho(z_1) - \rho(z_2)) \geq 0$, the case $Q(\rho(z_1) - \rho(z_2)) \leq 0$ is analogous.

From Lemma 9 applied to $\rho(z_1)$ and $\rho(z_2)$ it follows that for any $k > 0$

$$
\|g^k(\rho(z_1)) - g^k(\rho(z_2))\| \geq \|\pi_x g^k(\rho(z_1)) - \pi_x g^k(\rho(z_2))\| \geq \theta^k \|\pi_x \rho(z_1) - \pi_x \rho(z_2)\| = \theta^k \|\rho(z_1) - \rho(z_2)\|.
$$

Now we derive an upper bound on $\|g^k(\rho(z_1)) - g^k(\rho(z_2))\|$. Since $g^k(\rho(z_1)) \in N(A^kz_1, \alpha)$ for $i = 1, 2$ we obtain

$$
\|g^k(\rho(z_1)) - g^k(\rho(z_2))\| \leq \|g^k(\rho(z_1)) - A^k z_1\| + \|A^k z_2 - A^k z_2\| + \|A^k z_2 - g^k(\rho(z_2))\| \leq \alpha + L^k \|z_1 - z_2\| + \alpha = 2\alpha + L^k \|z_1 - z_2\|.
$$

By combining the above inequalities we obtain

$$
\|\rho(z_1) - \rho(z_2)\| \leq \frac{2\alpha}{\theta^k} + \left(\frac{L}{\theta}\right)^k \|z_1 - z_2\|.
$$

(60)
Observe that from Lemma 11 we can establish easily the uniform continuity of \( \rho \). To obtain \( \| \rho(z_1) - \rho(z_2) \| < \kappa \), we first chose \( k \) big enough to have \( \frac{2^{\alpha}}{\theta^k} < \kappa/2 \).

With this \( k \) we take

\[
\| z_1 - z_2 \| \leq \frac{\kappa}{2} \left( \frac{\theta}{L} \right)^k
\]

(61)

We are now ready to prove the Hölder regularity of \( \rho \).

**Theorem 12** Let \( \gamma = \frac{\ln \theta}{\ln L} \). There exists \( C > 0 \), such that any \( z_1, z_2 \in \mathbb{R}^n \), \( z_1 \neq z_2 \) and \( \| z_1 - z_2 \| < 1 \) holds

\[
\frac{\| \rho(z_1) - \rho(z_2) \|}{\| z_1 - z_2 \|^\gamma} \leq C,
\]

(62)

**Proof:**

Let us set \( \delta_0 = 1 \). Let us denote \( \delta = \| z_1 - z_2 \| \). For any \( \gamma > 0 \) and \( k \in \mathbb{Z}_+ \) from Lemma 11 we have

\[
\frac{\| \rho(z_1) - \rho(z_2) \|}{\| z_1 - z_2 \|^\gamma} \leq \frac{2^{\alpha} \theta^k \delta^{-\gamma}}{\theta^k} + \left( \frac{L}{\theta} \right)^k \delta^{1-\gamma}.
\]

(63)

Observe that (62) holds if there exists constants \( C_1 \) and \( C_2 \) such that for each \( 0 < \delta < \delta_0 \) there exists \( k \in \mathbb{Z}_+ \) such that the following inequalities are satisfied

\[
\frac{2^{\alpha}}{\theta^k} \delta^{-\gamma} \leq C_1,
\]

(64)

\[
\left( \frac{L}{\theta} \right)^k \delta^{1-\gamma} \leq C_2.
\]

(65)

We show that we can take

\[
C_1 = 2\alpha,
\]

(66)

\[
C_2 = \frac{L}{\theta}.
\]

(67)

The strategy is as follows: first from (64) we compute \( k \) and then we insert it to (65), which will give an inequality, which should hold for any \( 0 < \delta < \delta_0 \), this will produce bound for \( \gamma \), \( C_1 \) and \( C_2 \).

From (64) we obtain

\[
\theta^k \geq \frac{2^{\alpha} \delta^{-\gamma}}{C_1},
\]

\[
k \ln \theta \geq \ln \frac{2^{\alpha}}{C_1} - \gamma \ln \delta.
\]

(68)

Taking into account (66) we have

\[
k \ln \theta \geq -\gamma \ln \delta.
\]

(69)
We set \( k_0 = k_0(\delta) = -\frac{2}{\ln \theta} \ln \delta \). \( k_0 \) might not belong to \( \mathbb{Z} \), but \( k_0 > 0 \). We set \( k = k(\delta) = \lceil k_0 + 1 \rceil \), where \( \lceil z \rceil \) is the integer part of \( z \). With this choice of \( k \) equation (69) is satisfied. Hence also (64) holds.

Now we work on (65). Since
\[
\left( \frac{L}{\theta} \right)^k \leq \left( \frac{L}{\theta} \right)^{k_0+1},
\]
then (65) is satisfied if the following inequality holds
\[
\left( \frac{L}{\theta} \right)^{1 - \frac{\gamma \ln \delta}{\ln \theta}} \leq C_2.
\]

By taking the logarithm of both sides of the above inequality we obtain
\[
\left( 1 - \frac{\gamma}{\ln \theta} \ln \delta \right) \ln \left( \frac{L}{\theta} \right) + (1 - \gamma) \ln \delta \leq \ln C_2.
\]

Finally, after an rearrangement of terms arrive at
\[
\left( 1 - \gamma \left( 1 + \frac{\ln \frac{L}{\theta}}{\ln \theta} \right) \right) \ln \delta \leq \ln C_2 - \ln \frac{L}{\theta}.
\]

The last inequality should be satisfied for all \( \delta \leq \delta_0 = 1 \). Therefore, we need the coefficient on lhs by \( \ln \delta \) to be nonnegative and the rhs to be nonnegative. It is easy to see that rhs is nonnegative with \( C_2 \) given by (67). For the lhs observe that
\[
1 + \frac{\ln \frac{L}{\theta}}{\ln \theta} = 1 + \frac{\ln L - \ln \theta}{\ln \theta} = \frac{\ln L}{\ln \theta}.
\]

Hence we obtain
\[
1 - \gamma \frac{\ln L}{\ln \theta} \geq 0
\]
and finally
\[
\gamma \leq \frac{\ln \theta}{\ln L}.
\]

\[\square\]

**Remark 13** A closer analysis of the proof of Theorem 12 shows that the Hölder exponent can be a improved a bit. Namely we can take \( \gamma = \min \left( \frac{\ln \theta}{\ln \|A_0\|}, \frac{\ln \theta}{\ln \|A_z\|^2} \right) \).

Namely, we obtain \( \gamma = \frac{\ln \theta}{\ln \|A_0\|} \) when considering \( Q(\rho(z_1) - \rho(z_2)) \geq 0 \) and \( \gamma = \frac{\ln \theta}{\ln \|A_z\|^2} \) when \( Q(\rho(z_1) - \rho(z_2)) \leq 0 \).
3.1 Comparison with known estimates

In [BV, Theorem 1] the following estimate has been given for the Hölder exponent for the \( \rho \) and \( \rho^{-1} \) if the size of the perturbation goes to 0 (we use our notation)

\[
\alpha_0 = \min \left\{ \frac{-\ln r(A_s)}{\ln r(A_s^{-1})}, \frac{-\ln r(A_u)}{\ln r(A_u^{-1})} \right\},
\]

(70)

where \( r(A) \) denotes the spectral radius of the matrix \( A \).

Let us consider our estimate of the Hölder exponent from Remark 13. In the limit of vanishing perturbation we obtain (see Lemma 9 and 10)

\[
\theta_u = c_u, \quad \theta_s = \frac{1}{c_s}.
\]

(71)

Since from assumptions of Theorem 12 it follows that we can assume that

\[
\frac{1}{c_u} = \|A_u^{-1}\|, \quad c_s = \|A_s\|
\]

(72)

we obtain

\[
\frac{\ln \theta_u}{\ln \|A_u\|} = \frac{\ln \left( \frac{1}{\|A_u^{-1}\|} \right)}{\ln \|A_u\|} = -\frac{\ln \|A_u^{-1}\|}{\ln \|A_u\|},
\]

\[
\frac{\ln \theta_s}{\ln \|A_s^{-1}\|} = \frac{\ln \left( \frac{1}{\|A_s\|} \right)}{\ln \|A_s^{-1}\|} = -\frac{\ln \|A_s\|}{\ln \|A_s^{-1}\|}.
\]

Therefore our estimate for the Hölder exponent is

\[
\alpha_1 = \min \left\{ \frac{-\ln \|A_u^{-1}\|}{\ln \|A_u\|}, \frac{-\ln \|A_s\|}{\ln \|A_s^{-1}\|} \right\}.
\]

(73)

It differs from (70) by the exchange of the spectral radius of matrices in (70) by the norms of matrices. It is quite obvious that by playing with various norms and slightly changing the reasoning leading to Rem. 13 we can get arbitrary close to the bound given by (70). For example, if \( A_u \) and \( A_s \) are diagonalizable over \( \mathbb{R} \) if we define the scalar product so that the eigenvectors are orthogonal, then we obtain \( \|A_u^{-1}A_s\| = r(A_u^{-1}A_s) \).

To conclude, we claim that we were able to reproduce the Hölder exponent from [BV].

4 Grobman-Hartman Theorem for ODEs

We would like to make a reasoning, which will not reduce the proof to the map case, but rather we prefer its clean ODE version.

In such approach, the chain of covering relations along the full orbit will be replaced by an isolating segment along the orbit of a fixed diameter in the
extended phase space. The cone conditions for maps have also its natural analog, we will demand that
\[ \frac{d}{dt} Q(\varphi(t, z_1) - \varphi(t, z_2)) > 0. \] (74)

Here is a global version of Grobman-Hartman Theorem for ODEs, which is similar in spirit to Theorem 1. As in the map case we express our assumptions in 'good' coordinates.

**Theorem 14** Assume that \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a linear map of the following form
\[ A(x, y) = (A_u x, A_s y) \] (75)
where \( n = u + s \), \( A_u : \mathbb{R}^u \to \mathbb{R}^u \) and \( A_s : \mathbb{R}^s \to \mathbb{R}^s \) are linear maps such that
\[ (x, A_u x) \geq c_u \|x\|^2, \quad c_u > 0, \quad \forall x \in \mathbb{R}^u \] (76)
\[ (y, A_s y) \leq -c_s \|y\|^2, \quad c_s > 0, \quad \forall y \in \mathbb{R}^s. \] (77)

Assume that \( h : \mathbb{R}^n \to \mathbb{R}^n \) is of class \( C^k, \quad k \geq 1 \), is such that
\[ \|h(x)\| \leq M, \quad \forall x \in \mathbb{R}^n \] (78)
\[ \|Dh(x)\| \leq \epsilon, \quad \forall x \in \mathbb{R}^n. \] (79)

Let \( \varphi \) be the (local) dynamical system induced by
\[ z' = Az + h(z). \] (80)

Under the above assumptions there exists \( \epsilon_0 = \epsilon_0(A) > 0 \), such that if \( \epsilon < \epsilon_0(A) \), then there exists a homomorphism \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) such that for any \( t \in \mathbb{R} \) holds
\[ \rho(\exp(At)) = \varphi(t, \rho(z)). \] (81)

In the sequel for \( \lambda \in [0, 1] \) by \( \varphi^\lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) we will denote the dynamical system induced by
\[ z' = f^\lambda(z) := Az + \lambda h(z). \] (82)

Before the proof of Theorem 14 we need first to develop some technical tools. The basic steps and constructions used in the proof are given in Section 4.4. We invite the reader to jump first to this section to see the overall picture of the proof and then consult other more technical sections when necessary.

**4.1 \( \varphi^\lambda \) is a global dynamical system**

**Lemma 15** For every \( (t, z) \in \mathbb{R} \times \mathbb{R}^n \) \( \varphi^\lambda(t, z) \) is defined.
Proof: Observe that
\[ \| f_\lambda(z) \| \leq (\| A \| + \varepsilon) \| z \| + M. \] (83)
From this using the Gronwall inequality we obtain the following estimate
\[ \| x(t) \| \leq \| x(0) \| e^{(\| A \| + \varepsilon) \cdot |t|} + \frac{M}{\varepsilon} \left( e^{(\| A \| + \varepsilon) \cdot |t|} - 1 \right). \] (84)
In particular this implies that \( \varphi^\lambda(t, z) \) is defined. □

4.2 Isolating segment

We assume that the reader is familiar with the notion of the isolating segment for an ode. It has its origin in the Conley index theory [C] and was developed in papers by Roman Srzednicki and his coworkers [S1, S2, S3, SW, WZ].

Roughly speaking, an isolating segment for a (non-autonomous) ode is the set in the extended phasespace, whose boundaries are sections of the vector field. The precise definition can be found Appendix 6.

Lemma 16 There exists \( \hat{\alpha} = \max \left( \frac{2M}{\varepsilon u}, \frac{2M}{\varepsilon v} \right) \), such that for \( \alpha > \hat{\alpha} \) and for any \( \lambda_1, \lambda_2 \in [0, 1] \) and \( z_0 \in \mathbb{R}^n \) the set \( N_{\lambda_1}(z_0, \alpha) = \{(t, (x, y)) \mid (x - \pi_x \varphi^{\lambda_1}(t, z_0))^2 \leq \alpha^2, \ (y - \pi_y \varphi^{\lambda_1}(t, z_0))^2 \leq \alpha^2 \} \)
with
\[ N^{-}_{\lambda_1}(z_0, \alpha) = \{(t, (x, y)) \in N_{\lambda_1}(z_0, \alpha) \mid (x - \pi_x \varphi^{\lambda_1}(t, z_0))^2 = \alpha^2 \}, \quad (85) \]
\[ N^{+}_{\lambda_1}(z_0, \alpha) = \{(t, (x, y)) \in N_{\lambda_1}(z_0, \alpha) \mid (y - \pi_y \varphi^{\lambda_1}(t, z_0))^2 = \alpha^2 \}. \quad (86) \]
is an isolating segment for \( \varphi^{\lambda_2} \).

Proof: Let us introduce the following notation
\[ L^{-}(t, x, y) = (x - \pi_x \varphi^{\lambda_1}(t, z_0))^2 - \alpha^2, \]
\[ L^{+}(t, x, y) = (y - \pi_y \varphi^{\lambda_1}(t, z_0))^2 - \alpha^2. \]
The outside normal vector field to \( N^{-}_{\lambda_1}(z_0, \alpha) \) is given by \( \nabla L^{-} \). We have
\[ \frac{\partial L^{-}}{\partial t}(t, x, y) = -2(x - \pi_x \varphi^{\lambda_1}(t, z_0)) \cdot \pi_x f^{\lambda_1}(\varphi^{\lambda_1}(t, z_0)) = -2(x - \pi_x \varphi^{\lambda_1}(t, z_0)) (A u \varphi^{\lambda_1}(t, z_0) + \lambda_1 \pi_x h(\varphi^{\lambda_1}(t, z_0))) \]
\[ \frac{\partial L^{-}}{\partial x}(t, x, y) = 2(x - \pi_x \varphi^{\lambda_1}(t, z_0)), \]
\[ \frac{\partial L^{-}}{\partial y}(t, x, y) = 0. \]
We verify the exit condition by checking that for \((t, z) \in N^-_{\alpha_1}(z_0, \alpha)\) holds
\[\nabla L^- (t, z) \cdot (1, f^{\lambda_2}(t, z)) > 0.\]
We have for \((t, (x, y)) \in N^-_{\alpha_1}(z_0, \alpha)\)
\[
\frac{1}{2} \nabla L^- (t, z) \cdot (1, f^{\lambda_2}(t, z)) = \]
\[-(x - \pi_x \varphi^{\lambda_1}(t, z_0)) \cdot (A_x \varphi^{\lambda_1}(t, z_0) + \lambda_1 \pi_x h(\varphi^{\lambda_1}(t, z_0))) + \]
\[(x - \pi_x \varphi^{\lambda_1}(t, z_0)) \cdot (A_x + \lambda_2 \pi_x h(x, y)) = \]
\[(x - \pi_x \varphi^{\lambda_1}(t, z_0)) \cdot (A_x - \pi_x \varphi^{\lambda_1}(t, z_0)) + \]
\[(x - \pi_x \varphi^{\lambda_1}(t, z_0)) \cdot (-\lambda_1 \pi_x h(\varphi^{\lambda_1}(t, z_0)) + \lambda_2 \pi_x h(x, y)) \geq c_u \alpha^2 - 2 \alpha M = \alpha (c_u \alpha - 2 M).\]
We see that it is enough to take \(\hat{\alpha} > \frac{2M}{c_u}\).

For the verification of the entry condition we will show that for \((t, z) \in N^+_{\alpha_1}(z_0, \alpha)\) holds \(\nabla L^+(t, z) \cdot (1, f^{\lambda_2}(t, z)) < 0\).
The outside normal vector field to \(N^+_{\alpha_1}(z_0, \alpha)\) is given by \(\nabla L^+\). We have
\[
\frac{\partial L^+}{\partial t}(t, x, y) = -2(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot \pi_y f^{\lambda_1}(\varphi^{\lambda_1}(t, z_0)) = \]
\[-2(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot (A_y \varphi^{\lambda_1}(t, z_0) + \lambda_1 \pi_y h(\varphi^{\lambda_1}(t, z_0))) \]
\[
\frac{\partial L^+}{\partial x}(t, x, y) = 0,\]
\[
\frac{\partial L^+}{\partial y}(t, x, y) = 2(y - \pi_y \varphi^{\lambda_1}(t, z_0)).\]
We have for \((t, (x, y)) \in N^+_{\alpha_1}(z_0, \alpha)\)
\[
\frac{1}{2} \nabla L^+(t, z) \cdot (1, f^{\lambda_2}(t, z)) = \]
\[-(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot (A_y \varphi^{\lambda_1}(t, z_0) + \lambda_1 \pi_y h(\varphi^{\lambda_1}(t, z_0))) + \]
\[(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot (A_y + \lambda_2 \pi_y h(x, y)) = \]
\[(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot (A_y - \pi_y \varphi^{\lambda_1}(t, z_0)) + \]
\[(y - \pi_y \varphi^{\lambda_1}(t, z_0)) \cdot (-\lambda_1 \pi_y h(\varphi^{\lambda_1}(t, z_0)) + \lambda_2 \pi_y h(x, y)) \leq -c_s \alpha^2 + 2 \alpha M = \alpha (c_s \alpha + 2M).\]
We see that it is enough to take \(\hat{\alpha} > \frac{2M}{c_u}\).

The following theorem can be obtained using the ideas from the Wazewski Retract Theorem [Wa] (see also [C]).

**Theorem 17** Let \(\alpha > \hat{\alpha}\). Then for any \(\lambda_1, \lambda_2 \in [0, 1]\) and \(z_0 \in \mathbb{R}^n\), there exists \(z_1 \in \mathbb{R}^n\), such that for \(t \in \mathbb{R}\) holds
\[
\varphi^{\lambda_2}(t, z_1) \in \varphi^{\lambda_1}(t, z_0) + B_\alpha(0, \alpha) \times B_\alpha(0, \alpha). \quad (89)
\]

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**Proof:** We will show that for any \( T > 0 \) there exists \( z_T \in z_0 + \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha) \) such that

\[
\varphi^{\lambda_2}(t, z_T) \in \varphi^{\lambda_1}(t, z_0) + \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha), \quad t \in [-T, T]. \tag{90}
\]

Observe that once (90) is established by choosing a convergent subsequence from \( z_n \to \bar{z} \) for \( n \in \mathbb{Z}_+ \) we obtain an orbit for \( \varphi^{\lambda_2} \) satisfying (89).

To prove (90) we will use the ideas from the Wazewski Retract Theorem \cite{Wa}.

Let us fix \( T > 0 \). We define map \( h : [0, 2T] \times \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha) \to \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha) \) as follows. Let \( \tau : N_{\lambda_1}(z_0, \alpha) \to \mathbb{R} \cup \{ \infty \} \) be the exit time function from isolating segment \( N_{\lambda_1}(z_0, \alpha) \) for the process \( \varphi^{\lambda_2} \). From the properties of the isolating segments (see Appendix 6) it follows that this function is continuous.

Map \( h(s, \cdot) \) does the following: in the coordinate frame with moving origin given by \( \varphi^{\lambda_1}(s - T, z_0) \) to a point \( z \) we assign \( \varphi^{\lambda_2}(s, z) \) if \( s \) is smaller than the exit time, or the exit point (all in the moving coordinate frame).

The precise definition is as follows: let

\[
i(z) = z + \varphi^{\lambda_1}(-T, z_0), \quad \tau(z) = \tau(-T, i(z)) \tag{91}
\]

then

\[
h(s, z) = \begin{cases} 
\varphi^{\lambda_2}(s, i(z)) - \varphi^{\lambda_1}(s - T, z_0), & \text{if } s \geq \tau(z), \\
\varphi^{\lambda_2}(\tau(z), i(z)) - \varphi^{\lambda_1}(\tau(z) - T, z_0) & \text{otherwise}. 
\end{cases} \tag{92}
\]

To prove (90) it is enough to show that there exist \( z \) such that

\[
\tau(-T, z + \varphi^{\lambda_1}(-T, z_0)) < 2T. \tag{93}
\]

We will reason by the contradiction. In such case \( h \) will have the following property

\[
h(2T, z) \in (\partial B_u(0, \alpha)) \times \overline{B}_s(0, \alpha) \quad \forall z \in \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha). \tag{94}
\]

This together with the following obvious properties of \( h \)

\[
h(0, z) = z, \quad \forall z \in \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha) \tag{95}
\]

\[
h(s, z) = z, \quad \forall s \in [0, 2T], \quad \forall z \in (\partial B_u(0, \alpha)) \times \overline{B}_s(0, \alpha), \tag{96}
\]

means that \( h \) is the deformation retraction of \( \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha) \) onto \( (\partial B_u(0, \alpha)) \times \overline{B}_s(0, \alpha) \). This is not possible because the homology groups of both spaces are different, hence (93) is true for some \( z \).

This finishes the proof.
4.3 Cone condition

The cone condition for ODE is treated using the methods from [ZCC] and the cones are defined in terms of a quadratic form.

Let $Q(x, y) = (x, x) - (y, y)$ be a quadratic form on $\mathbb{R}^n$.

By $Q$ we will also denote a matrix, such that $Q(z) = z^tQz$. In our case $Q = \begin{bmatrix} I_u & 0 \\ 0 & -I_s \end{bmatrix}$, where $I_u \in \mathbb{R}^{u \times u}$ and $I_s \in \mathbb{R}^{s \times s}$ are the identity matrices.

Lemma 18 There exists $\epsilon_0 = \epsilon_0(A) > 0$ such that if $\epsilon < \epsilon_0$, then there exists $\eta > 0$ such that for $\lambda \in [0, 1]$ holds the following cone condition

$$\frac{d}{dt}Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) \geq \pm \eta Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)), \quad \forall z_1, z_2 \in \mathbb{R}^n. \quad (97)$$

Proof: It is enough to consider (97) for $t = 0$. We have

$$\frac{d}{dt}Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2))_{t=0} =$$

$$(f^\lambda(z_1) - f^\lambda(z_2))^tQ(z_1 - z_2) + (z_1 - z_2)^tQ(f^\lambda(z_1) - f^\lambda(z_2)) =$$

$$(z_1 - z_2)^t(D(z_1, z_2)^tQ + QD(z_1, z_2))(z_1 - z_2),$$

where

$$D(z_1, z_2) = \int_0^1 Df^\lambda(z_2 + t(z_1 - z_2))dt = A + \lambda \int_0^1 Dh(z_2 + t(z_1 - z_2))dt$$

We set

$$C(z_1, z_2) = \int_0^1 Dh(z_2 + t(z_1 - z_2))dt,$$

hence

$$D(z_1, z_2) = A + \lambda C(z_1, z_2), \quad \|C(z_1, z_2)\| \leq \epsilon. \quad (98)$$

It is enough to prove that $D^tQ + QD$ is positive definite. Observe first that $A^tQ + QA$ is positive definite. Indeed, we have for any $z = (x, y) \in \mathbb{R}^n$

$$v^t(A^tQ + QA)v = v^t \begin{pmatrix} A_u^t + A_u & 0 \\ 0 & -(A_s^t + A_s) \end{pmatrix} \cdot v =$$

$$x^t(A_u^t + A_u)x - y^t(A_s^t + A_s)y = 2(x, A_u x) - 2(y, A_s y) \geq 2c_u x^2 + 2c_s y^2 \geq 2 \min(c_u, c_s) \|v\|^2.$$

Since being a positive definite is an open property we see that the desired $\eta > 0$ and $\epsilon_0 > 0$ exist.

Lemma 19 Assume that $\epsilon < \epsilon_0$ as in Lemma 18. Let $\lambda \in [0, 1]$.

Assume that for some $z_1, z_2 \in \mathbb{R}^n$ there exists $\beta$, such that for all $t \in \mathbb{R}$ holds

$$\|\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)\| \leq \beta. \quad (99)$$

Then $z_1 = z_2$. 

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Proof:

Observe that from our assumption it follows that there exists \( \beta_1 \), such that
\[
|Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2))| \leq \beta_1, \quad \forall t \in \mathbb{R}.
\] (100)

We consider two cases: \( Q(z_1 - z_2) \geq 0 \) and \( Q(z_1 - z_2) < 0 \).

Consider first \( Q(z_1 - z_2) \geq 0 \). From Lemma 18 it follows that for all \( t > 0 \) holds
\[
Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) > 0 \quad \text{and for any } t_0, t > 0 \text{ holds}
\]
\[
Q(\varphi^\lambda(t + t_0, z_1) - \varphi^\lambda(t + t_0, z_2)) \geq \exp(\eta t)Q(\varphi^\lambda(t_0, z_1) - \varphi^\lambda(t_0, z_2)).
\] (101)

This is in a contradiction with (100).

Now we consider case \( Q(z_1 - z_2) < 0 \). It is easy to see that \( Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) < 0 \) for \( t < 0 \).

From cone condition (Lemma 18) it follows that
\[
Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) < \exp(-\eta t)Q(z_1 - z_2), \quad t < 0.
\] (102)

Hence
\[
|Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2))| > \exp(\eta |t|)|Q(z_1 - z_2)|, \quad t < 0.
\] (103)

This is in a contradiction with (100).

This finishes the proof.

\[ \blacksquare \]

4.4 Proof of Theorem 14.

The proof follows the pattern of the proof of Theorem 1. Below we will just list the basic steps of the proof.

We define map \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) and a candidate for its inverse \( \sigma : \mathbb{R}^n \to \mathbb{R}^n \) as follows:

1. let us fix \( \alpha > \hat{\alpha} \), where \( \hat{\alpha} \) is obtained in Lemma 16;

2. for \( z \in \mathbb{R}^n \), from Lemma 16 with \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \) we have an isolating segment \( N_0(z, \alpha) \) for \( \varphi^1 \);

3. from Theorem 17 and Lemma 19 it follows that \( N_0(z, \alpha) \) defines a unique point, which we will denote by \( \rho(z) \), such that
\[
\varphi^1(t, \rho(z)) \in B(\varphi^0(t, z), \alpha) \quad t \in \mathbb{R}.
\] (104)

4. for \( z \in \mathbb{R}^n \), from Lemma 16 with \( \lambda_1 = 0 \) and \( \lambda_2 = 1 \) we have an isolating segment \( N_1(z, \alpha) \) for \( \varphi^0 \);

5. from Theorem 17 and Lemma 19 it follows that the isolating segment \( N_1(z, \alpha) \) defines a unique point, which we will denote by \( \sigma(z) \), such that
\[
\varphi^0(t, \sigma(z)) \in B(\varphi^1(t, z), \alpha) \quad t \in \mathbb{R}.
\] (105)

The details of the proof are basically the same as in the proof of the map case and are left to the reader.
5 Appendix. h-set and Covering relations

The goal of this section is present the notions of the h-set and the covering relation, and to state the theorem about the existence of point realizing the chain of covering relations.

5.1 h-sets and covering relations

**Definition 1** [ZGa, Definition 1] An h-set, $N$, is a quadruple $(|N|, u(N), s(N), c_N)$ such that

- $|N|$ is a compact subset of $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \ldots\}$ are such that $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that $c_N(|N|) = \overline{B_{u(N)} \times B_{s(N)}}$.

We set

$$\dim(N) := n,$$
$$N_c := \overline{B_{u(N)} \times B_{s(N)}},$$
$$N_c^- := \partial B_{u(N)} \times \overline{B_{s(N)}},$$
$$N_c^+ := \overline{B_{u(N)}} \times \partial B_{s(N)},$$
$$N^- := c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).$$

Hence an h-set, $N$, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript $c$ refers to the new coordinates given by homeomorphism $c_N$. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use $N$ to denote both the h-sets and its support.

Sometimes we will call $N^-$ the exit set of $N$ and $N^+$ the entry set of $N$.

**Definition 2** [ZGa, Definition 6] Assume that $N, M$ are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let $w$ be a nonzero integer. We say that

$$N \xrightarrow{f,w} M$$

($N$ f-covers $M$ with degree $w$) iff the following conditions are satisfied

1. there exists a continuous homotopy $h : [0, 1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true
   - $h_0 = f_c$,
   - $h([0, 1], N_c^-) \cap M_c = \emptyset$,
   - $h([0, 1], N_c^+) \cap M_c^+ = \emptyset$. 

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2. If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that

$$h_1(p, q) = (A(p), 0), \text{ for } p \in \overline{B}_u(0, 1) \text{ and } q \in \overline{B}_u(0, 1).$$

Moreover, we require that

$$A(\partial B_u(0, 1)) \subset \mathbb{R}^u \setminus \overline{B}_u(0, 1).$$

We will call condition (107) the exit condition and condition (108) will be called the entry condition.

Note that in the case $u = 0$, if $N \xRightarrow{f,w} M$, then $f(N) \subset \text{int } M$ and $w = 1$.

**Remark 20** If the map $A$ in condition 2 of Def. 2 is a linear map, then condition (110) implies, that

$$\text{deg}(A, \overline{B}_u(0, 1), 0) = \pm 1.$$ 

Hence condition (4) is in this situation automatically fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not interested in the value of $w$ in the symbol $N \xRightarrow{f,w} M$ and we will often drop it and write $N \xRightarrow{f} M$, instead. Sometimes we may even drop the symbol $f$ and write $N \xRightarrow{} M$.

### 5.2 Main theorem about chains of covering relations

**Theorem 21** (Thm. 9) [ZGi] Assume $N_i$, $i = 0, \ldots, k$, $N_k = N_0$ are $h$-sets and for each $i = 1, \ldots, k$ we have

$$N_{i-1} \xRightarrow{f_{i,w}} N_i. \quad (111)$$

Then there exists a point $x \in \text{int } N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \text{int } N_i, \quad i = 1, \ldots, k \quad (112)$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x \quad (113)$$

We point the reader to [ZGi] for the proof.

The following result follows from Theorem 21.

**Theorem 22** Let $N_i$, $i \in \mathbb{Z}$ be $h$-sets. Assume that for each $i \in \mathbb{Z}$ we have

$$N_{i-1} \xRightarrow{f_{i,w}} N_i. \quad (114)$$

Then there exists a sequence $\{x_i\}_{i \in \mathbb{Z}}$, such that $x_i \in \text{int } N_i$ and

$$f_i(x_{i-1}) = x_i, \quad \forall i \in \mathbb{Z}. \quad (115)$$

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Proof: For any $k \in \mathbb{Z}_+$ let us consider a closed loop of covering relations

$$ N_{-k} \xrightarrow{f_{-k+1}} N_{-k+1} \xrightarrow{f_{-k+2}} \ldots \xrightarrow{f_{-1}} N_{k-1} \xrightarrow{f_k} N_k \xrightarrow{A_k} N_{-k}, $$

where $A_k$ is some artificial map such that $N_k \xrightarrow{A_k} N_{-k}$.

From Theorem 21 it follows that there exists a finite sequence $\{x_i^k\}_{i=-k,\ldots,k}$ such that

$$ x_i^k \in \text{int} N_i, \quad f_i(x_{i-1}^k) = x_i^k, \quad i = -k + 1, \ldots, k. \quad (116) \quad (117) $$

Since $N_i$ are compact, it is easy to construct a desired sequence, by taking suitable subsequences.

\[ \square \]

5.3 Natural structure of h-set

Observe that all the conditions appearing in the definition of the covering relation are expressed in 'internal' coordinates $e_N$ and $e_M$. Also the homotopy is defined in terms of these coordinates. This sometimes makes the matter and the notation look a bit cumbersome. With this in mind we introduce the notion of a 'natural' structure on h-set.

Definition 3 We will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_1) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an h-set with a natural structure given by:

$$ u(N) = u, \quad s(N) = s, \quad c_N(x, y) = \left( \frac{x-x_0}{r_1}, \frac{y-y_0}{r_2} \right). $$

6 Appendix. Isolating segments for ODEs

Let us consider the differential equation

$$ \dot{x} = f(t, x) \quad (118) $$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$. Let $x(t_0, x_0; \cdot)$ be the solution of (118) such that $x(t_0, x_0; t_0) = x_0$ we put

$$ \varphi(t_0, t)(x_0) = x(t_0, x_0; t + \tau). \quad (119) $$

The range of $\tau$ for which $\varphi(t_0, \tau)(x_0)$ might depend on $(t_0, x_0)$. $\varphi$ defines a local flow $\Phi$ on $\mathbb{R} \times \mathbb{R}^n$ by the formula

$$ \Phi_t(\sigma, x) = (\sigma + t, \varphi(\sigma, t)(x)). \quad (120) $$

In the sequel we will often call the first coordinate in the extended phase space $\mathbb{R} \times \mathbb{R}^n$ the time.
We use the following notation: by $\pi_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the projections and for a subset $Z \subset \mathbb{R} \times \mathbb{R}^n$ and $t \in \mathbb{R}$ we put
\[ Z_t = \{ x \in \mathbb{R}^n : (t, x) \in Z \}. \]  
(121)

Now we are going to state the definition of an isolating segment for (118), which is a modification of the notion of a periodic isolating segment over $[0, T]$ or $T$-periodic isolating segment in $\text{S1 S2 S3 SW WZ}$.

**Definition 4** Let $(W, W^-) \subset \mathbb{R} \times \mathbb{R}^n$ be a pair of subsets. We call $W$ an isolating segment for (118) (or $\varphi$) if:

(i) $(W, W^-) \cap ([a, b] \times \mathbb{R}^n)$ is a pair of compact sets

(ii) for every $\sigma \in \mathbb{R}$, $x \in \partial W$ there exists $\delta > 0$ such that for all $t \in (0, \delta)$ $\varphi(\sigma, t)(x) \notin W_{\sigma+t}$ or $\varphi(\sigma, t)(x) \in \text{int} W_{\sigma+t}$.

(iii)
\[ W^- = \{ (\sigma, x) \in W : \exists \delta > 0 \ \forall t \in (0, \delta) \ \varphi(\sigma, t)(x) \notin W_{\sigma+t} \}, \]
\[ W^+ := \text{cl} (\partial W \setminus W^-) \]

(iv) for all $(\sigma, x) \in W^+$ there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ holds
\[ \varphi(\sigma, -t)(x) \notin W_{\sigma-t} \]  
(122)

(v) there exists $\eta > 0$ such that for all $x \in W^-$ there exists $t > 0$ such that for all $\tau \in (0, t]$ $\Phi_\tau(x) \notin W$ and $\rho(\Phi_\tau(x), W) > \eta$

Roughly speaking, $W^-$ and $W^+$ are sections for (118), through which trajectories leave and enter the segment $W$, respectively.

**Definition 5** For the isolating segment $W$ we define the exit time function $\tau_{W, \varphi}$
\[ \tau_{W, \varphi} : W \ni (t_0, x_0) \mapsto \sup \{ t \geq 0 : \forall s \in [0, t] (t_0 + s, \varphi(t_0, s)(x_0)) \in W \} \in [0, \infty] \]

By the Ważewski Retract Theorem [Wa] the map $\tau_{W, \varphi}$ is continuous (compare [C]).

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