Multidimensional Golay-Rudin-Shapiro structures

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Abstract. We extend the standard two-stage construction of the double-sided Golay-Rudin-Shapiro sequence to higher dimensions. We explicitly present the two-dimensional structure which is a convenient paradigm for all natural dimensions. We show also the three-dimensional four-symbol first-stage structure which we call proto-GRS and the final two-symbol Golay-Rudin-Shapiro structure. They may serve as models of disordered equicomposition alloys with some short range order of four and two components, respectively. Finally we show the essential features of the three- and four-dimensional structures.

1. Introduction
The Golay-Rudin-Shapiro sequence (GRS) was invented independently by Marcel Golay [1], Walter Rudin [2] and Harold Shapiro [3]. One of its purposes was to simulate white noise in transmission lines. Its Fourier spectrum is absolutely continuous. We here intend to generalize GRS to arbitrary dimensions. Thus, for instance, in three dimensions it might serve as a model of a disordered substitutional AB alloy such as β-brass while its four-symbol forerunner may serve as a model for an equicomposition four-component alloy with some short range order such as e.g. CoCrFeNi.

Each term of the original Golay-Rudin–Shapiro sequence is either +1 or −1. The nth term of the sequence, a(n), is defined by the rules

\[ a(n) = (-1)^{b(n)}, \quad b(n) = \sum_{i=0}^{k} \varepsilon_i \varepsilon_{i+1}, \]

where \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k \) are the digits in the binary expansion of \( n \). Thus \( b(n) \) counts the number of (possibly overlapping) occurrences of the substring 11 in the binary expansion of \( n \), and \( a(n) \) is +1 if \( b(n) \) is even and −1 if \( b(n) \) is odd [4].

2. Two dimensions
The 2D version of GRS is a convenient paradigm for GRS in all natural dimensions. The standard way to construct a 1D double-sided GRS is by a two-stage substitution [4-6]. We extend the method to higher dimensions. In 2D the first-stage substitution becomes:

\[ 0 \rightarrow 1 \quad 3, \quad 1 \rightarrow 1 \quad 0, \quad 2 \rightarrow 2 \quad 3, \quad 3 \rightarrow 2 \quad 0. \]

In order to fill the entire 2D lattice \( \mathbb{Z}^2 \) rather than only a single quadrant we start with the seed

\[ 2 \quad 3 \]
\[ 0 \quad 1. \]
We apply the square of substitution (2), that is to say, we apply the substitution twice. Thus we produce a 2D four-symbol structure which we call proto-GRS. Its generation 2 is shown in Fig. 1. In order to enhance visibility and intuition we replace the alphabet of digits \( \mathcal{A}_4 = \{0, 1, 2, 3\} \) by the color alphabet \( \mathcal{C}_4 = \{ \square, \triangle, \diamond, \heartsuit \} \). We also use the terms "digit and "color" interchangeably.

**Figure 1.** Two-dimensional proto-GRS structure – generation 2.

Now we apply the mapping

\[
\begin{align*}
\square, \triangle & \rightarrow \square; \\
\square, \diamond & \rightarrow \diamond
\end{align*}
\]

and obtain our final product – generation 2 of 2D GRS shown in Fig. 2.

**Figure 2.** Two-dimensional GRS structure – generation 2.

### 3. Three dimensions and higher

In order to construct the three-dimensional GRS structure we again start with the four-symbol alphabet \( \mathcal{A}_4 \) or \( \mathcal{C}_4 \). We extend the first-stage substitution (2) to 3D as shown in Fig. 3 as well as the seed shown in Fig. 4.

**Figure 3.** Three-dimensional substitution.

**Figure 4.** Three-dimensional seed.

We want to cover the whole 3D lattice \( \mathbb{Z}^3 \). Thus we start with the 3D seed and apply the substitution squared to obtain the 3D proto-GRS. The size of its generation \( g \) is \( 2^{3(g+1)} \) unit cubes. Hence, in 3D, even the representation of generation 2 consists of eight \( 8 \times 8 \) matrices.
The four-color 3D proto-GRS is shown in Fig. 5 while the 3D black-and-white GRS is shown in Fig. 6.

![Figure 5. Three-dimensional four-color proto-GRS – generation 2.](image)

![Figure 6. Three-dimensional black-and-white GRS – generation 2.](image)

In principle, construction of a GRS structure in arbitrary dimension presents no problem. However, its representation becomes rather impractical. Therefore, dealing with the fourth dimension we show only the 4D seed and the 4D substitution. For clarity and to save space we represent those by a transparent 2D isometric projection of a 4D cube; moreover, we show only its vertices (Figs. 7 and 8). Figure 9 shows aspects of the hulls of the second generation of 4D proto-GRS and 4D GRS.

![Figure 7. Four-dimensional seed in isometric 2D projection.](image)

![Figure 8. Four-dimensional substitution in isometric 2D projection.](image)
Figure 9. Hulls of the second generation of 4D proto-GRS (left) and 4D GRS (right).

4. Spectral properties
The two-symbol GRS in any dimension inherits the spectral properties of the 1D sequence since it consists of 1D rows and columns. This has been rigorously proven by Barbé and von Haeseler [7] (also cf. [8] and [9]). As for the four-symbol proto-GRS we conjecture that that is true as well but for the time being we are not sure and hence we work on it.

5. Conclusions and outlook
We have generalized the one-dimensional Golay-Rudin-Shapiro sequence to arbitrary natural dimension, shown explicitly the 2D and 3D versions and outlined the essential features of the 4D version. In all dimensions there is some short-range order. Following this report we intend to study the symbolic complexity of the GRS structures. Eventually we shall present alternatives to the standard two-stage construction.

References
[1] Golay M J E 1949 *J. Opt. Soc. Amer.* 39 437-444
[2] Rudin W 1959 *Proc. Amer. Math. Soc.* 10 855-859
[3] Shapiro H S 1951 *Extremal problems for polynomials and power series*, Master’s thesis (MIT, Cambridge MA)
[4] Queffélec M 1995 *Substitution dynamical systems – spectral analysis*, LNM 1294, 2nd. ed. (Springer Verlag, Berlin)
[5] Allouche J P and Shallit J 2003 *Automatic Sequences: Theory, Applications, Generalizations*, p. 104 (Cambridge University Press)
[6] Baake M and Grimm U 2013 *Aperiodic Order. Volume 1: A Mathematical Invitation*, p. 78 (Cambridge University Press)
[7] Barbé A and von Haeseler F 2003 *J. Phys. A: Math. Gen.* 38 2599-2622
[8] Frank N P 2003 *Ergod. Th. & Dynam. Syst.* 23 519-532
[9] Chan L, Grimm U and Short I 2018 *Indag. Math.* 29 1072-1086