**Abstract.** Let \( \mu \) be an even measure on the real line \( \mathbb{R} \) such that
\[
c_1 \int_{\mathbb{R}} |f|^2 \, dx \leq \int_{\mathbb{R}} |f|^2 \, d\mu \leq c_2 \int_{\mathbb{R}} |f|^2 \, dx
\]
for all functions \( f \) in the Paley-Wiener space \( \text{PW}_a \). We prove that \( \mu \) is the spectral measure for the unique Hamiltonian \( H = \left( \begin{array}{cc} w_0 & 0 \\ 0 & w_1 \end{array} \right) \) on \([0, a] \) generated by a weight \( w \) from the Muckenhoupt class \( A_2 \). As a consequence of this result, we construct Krein’s orthogonal entire functions with respect to \( \mu \) and prove that every positive, bounded, invertible Wiener-Hopf operator on \([0, a] \) with real symbol admits triangular factorization.

1. **Introduction**

The classical Paley-Wiener space \( \text{PW}_a \) consists of entire functions of exponential type at most \( a \) square summable on the real line, \( \mathbb{R} \). A measure \( \mu \) on \( \mathbb{R} \) is called a sampling measure for the space \( \text{PW}_a \) if there exist positive constants \( c_1, c_2 \) such that
\[
c_1 \int_{\mathbb{R}} |f|^2 \, dx \leq \int_{\mathbb{R}} |f|^2 \, d\mu \leq c_2 \int_{\mathbb{R}} |f|^2 \, dx, \quad f \in \text{PW}_a.
\]

Let \( \mathcal{H} \) be a regular Hamiltonian on \([0, a] \), that is, \( \mathcal{H} \) is a mapping from \([0, a] \) to the set of \( 2 \times 2 \) non-negative matrices with real entries such that \( \text{trace } \mathcal{H} \) is a positive non-vanishing function in \( L^1(0, a] \). Denote by \( \Theta_{\mathcal{H}} = \Theta_{\mathcal{H}(r, z)} \) solution of the following Cauchy problem:
\[
JX' = z \mathcal{H}(r) X, \quad X : [0, a] \to \mathbb{C}^2, \quad X(0) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad z \in \mathbb{C}.
\]

It is known from a general theory of canonical Hamiltonian systems that for every measure \( \mu \) satisfying (1) there exists a regular Hamiltonian \( \mathcal{H} \) with \( \int_0^a \sqrt{\text{det } \mathcal{H}} = a \) such that \( \mu \) is a spectral measure for problem (2). The latter means that the Weyl-Titchmarsh transform
\[
W_{\mathcal{H}, a} : X \mapsto \frac{1}{\sqrt{\pi}} \int_0^a \langle \mathcal{H}(r) X(r), \Theta_{\mathcal{H}(r, \bar{z})} \rangle_{\mathbb{C}^2} \, dr, \quad z \in \mathbb{C},
\]

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SAMPLING MEASURES, MUCKENHOUPT HAMILTONIANS, AND TRIANGULAR FACTORIZATION

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generated by solution $\Theta_H$ of Cauchy problem (2) maps isometrically the space
$$L^2(H, \mu) = \left\{ X : [0, a] \rightarrow \mathbb{C}^2 : \|X\|_{L^2(H, \mu)}^2 = \int_0^a \langle H(r)X(r), X^*(r) \rangle dz \, dr < \infty \right\} / \mathcal{K}(H),$$
into the space $L^2(\mu)$. A general problem in the inverse spectral theory is to translate properties of a spectral measure $\mu$ into properties of the Hamiltonian $H$ it generates.

Two essentially different cases of the above problem attracted much attention. If $\mu$ is a “small perturbation” of the Lebesgue measure on $\mathbb{R}$ (in the sense that the Fourier transform of $\mu$ restricted to the interval $[-a, a]$ differs from the point mass measure $\delta_0$ concentrated at 0 by a function in $L^1([-a, a])$), the I. M. Gelfand–B. M. Levitan approach \[6, 12\] gives a quite precise information on relation between $\mu$ and $H$. On the other hand, if $\mu$ is arbitrary measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{d\mu(t)}{1 + t^2} < \infty$, the theory of M. G. Krein \[8\] (for even measures $\mu$) and L. de Branges \[13\] (for all $\mu$) implies the existence of a unique Hamiltonian $H \in L^1_{\text{loc}}[0, \infty)$ such that $\mu$ is the spectral measure for $H$. However, it is not known how translate even simple properties of a Hamiltonian $H$ (e.g., membership in $L^p$ class for some $p > 1$) to the properties of its spectral measure $\mu$ and vice versa. In this paper we consider a “median” situation (spectral measures with sampling property \[11\] for the Paley-Wiener space $PW_a$) and use both Gelfand-Levitan and Krein-de Branges theories.

A measure $\mu$ on $\mathbb{R}$ is called even if $\mu(S) = \mu(-S)$ for every Borel set $S \subset \mathbb{R}$. A function $w > 0$ belongs to the Muckenhoupt class $A_2[0, a]$ if the supremum of products $(\frac{1}{|I|} \int_I w) \cdot (\frac{1}{|I|} \int_I \frac{1}{w})$ over all intervals $I \subset [0, a]$ is finite. Here is the main result of the paper.

**Theorem 1.** Let $\mu$ be an even sampling measure for $PW_a$. Then $\mu$ is the spectral measure for problem \[2\] corresponding to the unique Hamiltonian $H = \begin{pmatrix} w & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ generated by a weight $w \in A_2[0, a]$.

The Hamiltonian $H$ in Theorem 1 could be recovered from the spectral measure $\mu$ by means of the following simple formula:
$$w(r) = \pi \frac{\partial}{\partial r} \|T_{\mu,r}^{-1} \sin r \|_{L^2(\mu)}^2, \quad \sin_r = \frac{\sin rx}{\pi x}, \quad r \in [0, a],$$
where $T_{\mu,r}$ is the truncated Toeplitz operator on $PW$, with symbol $\mu$ defined by
$$(T_{\mu,r}f)(z) = \int_{\mathbb{R}} f(x) \frac{\sin r(x - z)}{\pi(x - z)} \, d\mu(x), \quad z \in \mathbb{C}. \quad (4)$$
A nontrivial fact is that the continuous increasing function $r \mapsto \|T_{\mu,r}^{-1} \sin_r \|_{L^2(\mu)}^2$ is absolutely continuous and its derivative $w/\pi$ does not vanish on a set of positive Lebesgue measure. In the proof of Theorem 1 we first obtain an estimate for the “$A_2^\infty$-norm” of $w$ in terms of $c_1$, $c_2$ assuming above properties of $w$; then use an approximation argument based on a description of positive truncated Toeplitz operators on $PW$ and $L^p$-summability of weights $w \in A_2[0, a]$ for some $p > 1$.

Section 5 in \[2\] contains an example of a diagonal Hamiltonian $H$ on $[0, 1]$ such that both $H$, $H^{-1}$ are uniformly bounded on $[0, 1]$, but the spectral measures of the corresponding problem \[2\] fail to have sampling property. This shows that $A_2[0, a]$
Theorem 1 yields two results of independent interest.

Given a measure $\mu$ satisfying (1) and a number $r \in [0, 2a]$, denote by $(\text{PW}_{[0,r]}, \mu)$ the Paley-Wiener space of functions from $L^2(\mathbb{R})$ with Fourier spectrum in $[0, r]$ equipped with the inner product taken from $L^2(\mu)$.

**Theorem 2.** Let $\mu$ be an even sampling measure for the space $\text{PW}_a$. Then there exists a family of entire functions $\{P_t\}_{t \in [0, 2a]}$ such that $\mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^r f(t)P_t(z)\, dt$ is the unitary operator from $L^2[0, r]$ to $(\text{PW}_{[0,r]}, \mu)$ for every $r \in [0, 2a]$.

In the case where $\mu$ is a “small perturbation” of the Lebesgue measure (see discussion above), the functions $P_t$ in Corollary 2 coincide with orthogonal entire functions constructed by M. G. Krein in [10]. S. A. Denisov provides an extensive treatment of the subject, collecting many old and new results in paper [5].

The second application of Theorem 1 concerns the classical factorization problem for positive invertible operators. Let $\mathcal{H}$ be a separable Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Consider a complete chain $\mathcal{N}$ of subspaces in $\mathcal{H}$ and denote by $A_{\mathcal{N}} = \{A \in B(\mathcal{H}) : AE \subset E, E \in \mathcal{N}\}$ the nest algebra of upper-triangular operators with respect to $\mathcal{N}$. In sixties, I. C. Gohberg and M. G. Krein proved (see Theorem 6.2 in Chapter 4 of [7]) that every positive invertible operator $T$ on $\mathcal{H}$ of the form $T = I - K$ with $K$ in Macaev ideal $S_\omega$ admits the triangular factorization $T = A^*A$, where $A = I - K_A$ is an invertible operator on $\mathcal{H}$ such that $K_A \in S_\omega \cap A_{\mathcal{N}}$. Famous theorem by D. R. Larson [11] says that every positive invertible operator $T$ admits triangular factorization $T = A^*A$ with $A, A^{-1} \in A_{\mathcal{N}}$ if and only if the chain $\mathcal{N}$ is countable. Moreover, given $0 < \varepsilon < 1$, the non-factorable operator $T$ can be chosen so that $K = I - T$ is a compact operator with $\|K\| < \varepsilon$.

We consider the problem of triangular factorization for Wiener-Hopf convolution operators. Let $\psi \in \mathcal{S}'$ be a tempered distribution on $\mathbb{R}$ and let $0 < a \leq \infty$. The Wiener-Hopf operator $W_\psi$ on $L^2[0, a)$ with symbol $\psi$ is densely defined by

$$(W_\psi f)(y) = \langle \psi, s_y f \rangle_{\mathcal{S}'}, \quad y \in (0, a), \quad s_y f : x \mapsto f(x - y),$$

on smooth functions $f$ with compact support in $(0, a)$. In the case where $\psi \in L^1(\mathbb{R})$ we have more familiar definition, $W_\psi : f \mapsto \int_0^a \psi(x-y)f(x)\, dx$. As a following result shows, Wiener-Hopf operators with real symbols are always factorable.

**Theorem 3.** Let $0 < a \leq \infty$. Every positive, bounded, and invertible Wiener-Hopf operator $W_\psi$ on $L^2[0, a)$ with real symbol $\psi \in \mathcal{S}'$ admits triangular factorization: $W_\psi = A^*A$, where $A$ is a bounded invertible operator such that $AL^2[0, r] = L^2[0, r]$ for every $r \in [0, a]$.

Wiener-Hopf operators $W_\psi$ in Theorem 3 admit triangular factorizations in the reverse order $W_\psi = AA^*$ as well. Relation of absolute continuity of aforementioned function $r \mapsto \|T_{\mu,r}^{-1}\text{sinc} \| \mathcal{T}_2(\mu)$ to triangular factorization problems has been previously found in different terms by L. A. Sakhnovich, see Theorem 4.2 in [16]. On the other hand, Theorem 3 contradicts Theorem 4.1 from another work [17] by the same author. See discussion in Section 5.
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2. Integration over simplex and the Muckenhoupt class $A_2$

Let $w$ be a positive function on an interval $[0, a]$. We associate to $w$ the quantity

$$\|w\|_{A_2[0,a]} = \sup_{I \subset [0,a]} \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \cdot \left( \frac{1}{|I|} \int_I \frac{1}{w(x)} \, dx \right),$$

where $I$ runs over all subintervals of $[0, a]$. Note that $\| \cdot \|_{A_2[0,a]}$ is not a norm in the standard sense, but we will use this convenient notation. The Muckenhoupt class $A_2[0,a]$ consists of functions $w > 0$ such that $\|w\|_{A_2[0,a]} < \infty$. In this section we present a special integral condition for a weight $w$ to belong to the $A_2[0,a]$ class.

Let $\varphi$ be a real-valued function on the interval $[0, a]$. For a real $0 < t < a$ and an integer $n \geq 1$ define the mapping

$$G_{\varphi,n} : x \mapsto \sum_{k=1}^{n} (-1)^{n+k} \varphi(x_k), \quad x \in K_{t,n},$$

on simplex $K_{t,n} = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), \ t \geq x_1 \geq \ldots \geq x_n \geq 0\}$. Let $m_n$ denote the usual Lebesgue measure on $\mathbb{R}^n$.

Next proposition will be used in the proof of Theorem 1.

**Proposition 2.1.** Let $\varphi$ be a function on $[0, a]$ such that $e^{|\varphi|} \in L^1[0,a]$. Assume that for every $r \in [0, a]$ and every integer $n \geq 1$ we have

$$\frac{1}{a_n(r)} \int_0^r e^{(-1)^n \varphi(t)} \left( \int_{K_{t,n}} e^{G_{\varphi,n}(x)} dm_n(x) \right)^2 \, dt \leq b_2, \quad (6)$$

$$b_1 \leq \frac{1}{r} \int_0^r e^{\varphi(t)} \, dt \leq b_2; \quad (7)$$

where $b_1, b_2$ are positive constants, and $a_n(r) = r^{2n+1}(2n+1)^{-1}(n!)^{-2}$. Then the function $w = e^\varphi$ belongs to $A_2[0,a]$ and $\|w\|_{A_2[0,a]} \leq 2^{28}(b_2 + b_1^{-2}b_2)^{14}$.

We first prove several preliminary estimates.

**Lemma 2.1.** Let $\varphi$ be a function as in Proposition 2.1. Then for every $r \in [0, a]$ and $b = 2(b_2 + b_1^{-2}b_2)$ we have

$$\frac{1}{r} \int_0^r |\varphi(t)| \, dt \leq \log b, \quad \frac{1}{r} \int_0^r e^{|\varphi(t)|} \, dt \leq b. \quad (8)$$

Consequently, for every decreasing differentiable function $k \geq 0$ on $[0, r]$ satisfying $\int_0^r k(t) \, dt = 1$ and $k(r) = 0$ we have $\int_0^r |\varphi(t)||k(t)| \, dt \leq \log b$.

**Proof.** Clearly, the first estimate in (8) follows from the second one and the Jensen’s inequality for convex function $e^x$. Taking $n = 1$ in (6), we obtain

$$\frac{3}{r^2} \int_0^r e^{-\varphi(t)} \left( \int_0^t e^{\varphi(t_1)} \, dt_1 \right)^2 \, dt \leq b_2.$$
From (7) we know that $\frac{1}{r} \int_0^r e^{\varphi(t)} \, dt_1 \geq b_1$ for all $t \in [0, r]$. It follows that

$$b_1^{-2} b_2 \geq \frac{3}{r^3} \int_0^r e^{-\varphi(t)} t^2 \, dt \geq \frac{1}{r} \int_{r/2}^r e^{-\varphi(t)} \, dt.$$  

Using the other side estimate $\frac{1}{r} \int_0^r e^{\varphi(t)} \, dt \leq b_2$ and inequality $e^{|x|} \leq e^x + e^{-x}$, we see that

$$\frac{2}{r} \int_{r/2}^r e^{\varphi(t)} \, dt \leq b_2 + b_1^{-2} b_2$$

for all $r \in [0, a]$. Then (8) follows from

$$\frac{1}{r} \int_0^r e^{\varphi(t)} \, dt = \frac{1}{r} \left( \sum_{k=0}^\infty |I_{r,k}| \cdot \frac{1}{|I_{r,k}|} \int_{I_{r,k}} e^{\varphi(t)} \, dt \right) \leq b,$$

where $I_{r,k} = [2^{-k-1} r, 2^{-k} r]$. Now if $k$ is a function on $[0, r] \subset [0, a]$ as in the statement, we have

$$\int_0^r |\varphi(t)| k(t) \, dt = - \int_0^r |\varphi(t)| \int_0^t \chi_{[t,r]}(s) k'(s) \, ds \, dt$$

$$= - \int_0^r k'(s) \int_0^r \chi_{[0, s]}(t) |\varphi(t)| \, dt \, ds$$

$$\leq - \log b \int_0^r k'(s) s \, ds = \log b.$$

This completes the proof. \hfill \Box

For $n \geq 1$ introduce the intervals $I_{t,n} = [\delta_n t, t]$, where $\delta_n = 1 - \frac{1}{n+1}$ if $n$ is odd, and $\delta_n = 1 - \frac{1}{n}$ if $n$ is even. In particular, $I_{t,n} = I_{t,n+1}$ for every odd $n$. Set

$$[\varphi]_{t,n} = 2(-1)^{n+1} \int_{K_{t,n}} G_{\varphi,n}(x) \, dm_{t,n}(x),$$

where $m_{t,n} = \frac{m}{n}$, $m_n$ is the scalar multiple of the Lebesgue measure $m_n$ on $\mathbb{R}^n$ normalized so that $m_{t,n}(K_{t,n}) = 1$.

**Lemma 2.2.** For $r \in [0, a]$ and odd $n \geq 1$ we have $|[\varphi]_{t,n} - [\varphi]_{t,n+1}| < 6 \log b$, where $b$ is the constant from Lemma 2.1.

**Proof.** Arguing by induction, it is easy check that for all $n \geq 1$ and $\tau \in [0, a]$ we have

$$[\varphi]_{\tau,n} = \int_0^\tau \varphi(s) k_{\tau,n}(s) \, ds, \quad k_{\tau,n}(s) = \frac{2n}{\tau^n} (2s - \tau)^{n-1}.$$  

For odd (correspondingly, even) integers $n$ the kernels $k_{\tau,n}$ are even (correspondingly, odd) functions with respect to the point $\tau/2$. As $n$ tends to infinity, the kernels $k_{\tau,n}$ tend to zero uniformly on every closed interval in $(0, \tau)$. We also have

$$\int_0^\tau |k_{\tau,n}(s)| \, ds = 2, \quad \sup_{s \in [\tau/2, \tau]} |k_{\tau,n}(s) - k_{\tau,n+1}(s)| \leq \frac{2}{\tau}. \quad (9)$$
Now take an odd integer $n \geq 1$ and note that $\delta_n = \delta_{n+1} = \frac{1}{n+1}$. Setting $\tau = \delta_n r$, we obtain
\[
|\phi|_{\tau,n} - |\phi|_{\tau,n+1} | \leq \int_0^{\tau/2} |\phi(s)k_{\tau,n}(s)| \, ds + \int_0^{\tau/2} |\phi(s)k_{\tau,n+1}(s)| \, ds + \int_{\tau/2}^{\tau} |\phi(s)| \cdot |k_{\tau,n}(s) - k_{\tau,n+1}(s)| \, ds.
\]
By Lemma 2.1 for functions $\phi$, $k = \frac{1}{2}|k_{\tau,n}|$, and $k = \frac{1}{2}|k_{\tau,n+1}|$ on $[0, \frac{\tau}{2}]$, the sum of first two integrals is bounded from above by $4\log b$. To show that the last integral does not exceed $2\log b$, use (8) and the second estimate in (9).

**Proof of Proposition 2.1**. Take an odd integer $n \geq 1$. Since the integrand in (6) is positive, we have
\[
b_2 \geq \frac{1}{a_n(r)} \int_{\delta_n r}^{r} e^{-\phi(t)} \left( \int_{K_{t,n}} e^{G_{\phi,n}(x)} \, dm_n(x) \right)^2 \, dt,
\]
\[
\geq \frac{1}{a_n(r)} \left( \int_{\delta_n r}^{r} e^{-\phi(t)} \, dt \right) \cdot \left( \int_{K_{\delta_n r,n}} e^{G_{\phi,n}(x)} \, dm_n(x) \right)^2.
\]
By Jensen’s inequality,
\[
\int_{K_{\delta_n r,n}} e^{G_{\phi,n}(x)} \, dm_n(x) \geq \left( \frac{\delta_n r}{n!} \right)^n \exp \left( \frac{|\phi|_{\delta_n r,n}}{2} \right).
\]
For all $n \geq 1$ we have
\[
\frac{1}{a_n(r)} \left( \frac{(\delta_n r)^n}{n!} \right)^2 = \frac{(2n+1)(n!)^2}{r^{2n+1}} \cdot \frac{r^{2n}}{(n!)^2} \geq \frac{n+1}{32r} = \frac{1}{32|I_{n,r}|}.
\]
We now see that
\[
\frac{1}{|I_{r,n}|} \int_{I_{r,n}} \exp \left( -\phi(t) + |\phi|_{\delta_n r,n} \right) \, dt \leq 32b_2.
\]
(10)
Analogously, for the even integer $n+1$ we have
\[
\frac{1}{|I_{r,n+1}|} \int_{I_{r,n+1}} \exp \left( \phi(t) - |\phi|_{\delta_n r,n} \right) \, dt \leq 32b_2.
\]
(11)
Recall that $I_{r,n+1} = I_{r,n}$. Applying Lemma 2.2, we obtain
\[
\frac{1}{|I_{r,n}|} \int_{I_{r,n}} \exp \left( |\phi(t) - |\phi|_{\delta_n r,n} \right) \, dt \leq 32b_2 e^{6\log b} \leq 32b^7,
\]
(11)
where $b$ is the constant from Lemma 2.1. Using inequality $e^{|x|} \leq e^x + e^{-x}$, we get from (10) and (11) the estimate
\[
\frac{1}{|I|} \int_{I} e^{|\phi(t) - c_I|} \, dt \leq 64b^7
\]
(12)
for all intervals $I$ of the form $I = [(1 - \frac{1}{n+1})r, r]$, where $r \in [0, a]$, and integer $n \geq 1$ is odd. Here $c_I$ is a constant depending on $I$ (in fact, $c_I = |\phi|_{\delta_n r,n}$ works, but from now on the particular choice of $c_I$ plays no role). Formula (8) gives (12) with $c_I = 0$ for intervals of the form $I = [0, t]$.
Next, observe that each interval \( J \subset [0, a] \) is contained in an interval \( I \) satisfying (12) and such that \( |I| \leq 2|J| \). Indeed, let \( t \) be the right point of \( J \). If \( |J| \geq |t|/2 \), take \( I = [0, t] \). In the case \( |J| < |t|/2 \) find an odd number \( n \geq 1 \) such that \( I_{t,n+2} \subset J \subset I_{t,n} \) and take \( I = I_{t,n} \). Fix this interval \( I \) and the corresponding constant \( c_I \) form (12). We have

\[
\left( \frac{1}{|J|} \int_J e^\varphi \, dt \right) \cdot \left( \frac{1}{|J|} \int_J e^{-\varphi} \, dt \right) \leq \left( \frac{2}{|I|} \int_I e^\varphi \, dt \right) \cdot \left( \frac{2}{|I|} \int_I e^{-\varphi} \, dt \right) \leq \left( \frac{2}{|I|} \int_I e^{\varphi - c_I} \, dt \right) \cdot \left( \frac{2}{|I|} \int_I e^{-\varphi + c_I} \, dt \right) \leq (2h)^4.
\]

Since interval \( J \) is arbitrary, this shows that function \( w = e^\varphi \) belongs to the Muckenhoupt class \( A_2[0, a] \) and \( \|w\|_{A_2[0, a]} \leq (2h)^4 = 2^{28}(b_2 + b_1^{-2}b_2)^4 \). \( \square \)

3. Proof of Theorem 1

As it was mentioned in the Introduction, we will use an approximation argument in the proof of Theorem 1. To have a stable approximation, we need a result describing positive truncated Toeplitz operators on \( PW_a \).

3.1. Preliminaries on truncated Toeplitz operators. Let \( \mu \geq 0 \) be a measure on the real line \( \mathbb{R} \) such that \( \|f\|_{L^2(\mu)}^2 \leq c\|f\|_{L^2(\mathbb{R})}^2 \) for all functions \( f \in PW_{[0,a]} \). Define the truncated Toeplitz operator \( A_{\mu,a} \) on \( PW_{[0,a]} \) by the sesquilinear form

\[
(A_{\mu,a}f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f\overline{g} \, d\mu, \quad f, g \in PW_{[0,a]}, \tag{13}
\]

In the case where \( \mu = u \, dm \) is absolutely continuous with respect to the Lebesgue measure \( m \) on \( \mathbb{R} \) and has density \( u \), the operator \( A_{\mu,a} \) coincides with the projection of the standard Toeplitz operator \( T_u \) on the Hardy space \( H^2 \) to the subspace \( PW_{[0,a]} \). This explains the name “truncated Toeplitz” for the operator \( A_{\mu,a} \).

It is well-known (see, e.g., Section 6.1 in [14]) that the operator

\[
V : h \mapsto \frac{1}{\sqrt{\pi}} \frac{1}{z + i} h \left( \frac{z - i}{z + i} \right), \quad z \in \mathbb{C}_+ \tag{14}
\]

maps unitarily the Hardy space \( H^2(\mathbb{D}) \) in the open unit disk \( \mathbb{D} = \{ \xi \in \mathbb{C} : |\xi| < 1 \} \) onto the Hardy space \( H^2 \) in the upper half-plane \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \). Moreover, for every \( a > 0 \) we have \( V K_{\theta_a} = PW_{[0,a]} \), where \( \theta_a = \exp(a\frac{-i\pi}{2}) \) is the inner function in \( \mathbb{D} \) and \( K_{\theta_a} \) is the orthogonal complement in \( H^2(\mathbb{D}) \) to the subspace \( \theta_a H^2(\mathbb{D}) \). As we will see in a moment, the truncated Toeplitz operators defined by (14) are unitarily equivalent to truncated Toeplitz operators on the shift-coinvariant subspace \( K_{\theta_a} \) of \( H^2(\mathbb{D}) \). See D. Sarason’s paper [15] for basic properties of truncated Toeplitz operators on general coinvariant subspaces of \( H^2(\mathbb{D}) \).

We also will deal with the operators \( T_{\mu,a} \) on the space \( PW_a \) defined by the same sesquilinear form

\[
(T_{\mu,a}f, g) = \int_{\mathbb{R}} f\overline{g} \, d\mu, \quad f, g \in PW_a.
\]

It is easy to see that this definition agrees with formula (4). By construction, we have \( T_{\mu,a} = V_a^{-1} A_{\mu,a} V_a \), where \( V_a : PW_a \to PW_{[0,2a]} \) is the unitary operator taking a function \( f \) into \( e^{iaz} f \).
Lemma 3.1. Let $T$ be a positive bounded operator on $\text{PW}_{[0,a]}$ satisfying relation
\begin{equation}
(Tf,f)_{L^2(\mathbb{R})} = (\tilde{\mu} \tilde{T} f, \tilde{T} f)_{L^2(\mathbb{R})}
\end{equation}
for all functions $f \in \text{PW}_{[0,a]}$ such that $f(-i) = 0$. Then there exists a positive measure $\mu$ on $\mathbb{R}$ such that $T = A_{\mu,a}$. Similarly if $T$ is a positive bounded operator on $\text{PW}_a$ satisfying \(15\) for all $f \in \text{PW}_a$ such that $f(-i) = 0$, then $T = T_{\mu,a}$ for a positive measure $\mu$ on $\mathbb{R}$.

**Proof.** Let $\theta_a$, $K_{\theta_a}$, and $V : K_{\theta_a} \to \text{PW}_{[0,a]}$ be defined as above. Consider the operator $\tilde{T} = V^{-1}TV$ on $K_{\theta_a}$ unitarily equivalent to the operator $T$ on $\text{PW}_{[0,a]}$. Recall that the inner product in $K_{\theta_a}$ is inherited from the space $L^2(\mathbb{T})$ on the unit circle $\mathbb{T} = \{ \xi \in \mathbb{C} \colon |\xi| = 1 \}$. Assumption \(15\) means that
\begin{equation}
(\tilde{T}h,h)_{L^2(\mathbb{T})} = (\tilde{T}\xi h,\xi h)_{L^2(\mathbb{T})}
\end{equation}
for every function $h \in K_{\theta_a}$ such that $\xi h \in K_{\theta_a}$. Indeed, $(V \xi h)(z) = \frac{\xi}{z}T(\xi h)(z)$ and hence $V(\xi h) \in \text{PW}_{[0,a]}$ if and only if $Vh(-i) = 0$. Theorem 2.1 in [15] says that a bounded operator $\tilde{T}$ on $K_{\theta_a}$ (or on any other coinvariant subspace $K_{\theta_a}^2$ of the Hardy space $H^2(\mathbb{D})$) satisfying \(16\) is a truncated Toeplitz operator on $K_{\theta_a}$. By Theorem 2.1 in [15], for every positive bounded truncated Toeplitz operator $T$ on $K_{\theta_a}$ there exists a finite positive measure $\mu$ on $\mathbb{T}$ such that $\tilde{\mu}(\{1\}) = 0$ and
\begin{equation}
(\tilde{T}h,h)_{L^2(\mathbb{T})} = \int_{\mathbb{T}} |h|^2 \tilde{\mu}
\end{equation}
for all continuous functions $h$ in $K_{\theta_a}$. Changing variables in the last integral, we find a positive measure $\mu$ on $\mathbb{R}$ such that
\begin{equation}
\int_{\mathbb{T}} |h|^2 \tilde{\mu} = \int_{\mathbb{R}} |f|^2 d\mu, \quad f = Vh.
\end{equation}
It follows that $(Tf,f) = (\tilde{T}h,h)_{L^2(\mathbb{T})} = (A_{\mu,a} f, f)_{L^2(\mathbb{R})}$ for a dense set of functions $f$ in $\text{PW}_{[0,a]}$. Since $T$ is continuous, we have $T = A_{\mu,a}$. The second part of the Lemma is a direct consequence of relation $T_{\mu,a} = V^{-1}A_{\mu,2a}V_a$. \hfill $\square$

3.2. Preliminaries on canonical Hamiltonian systems. Let $\mathcal{H}$ be a Hamiltonian on $[0,a]$ with trace $\mathcal{H} \in L^2[0,a]$. Assume that there is no interval $(r_1,r_2) \subset [0,a]$ such that $\mathcal{H}(t)$ is a constant matrix of rank one for all points $t \in (r_1,r_2)$. For $r \in [0,a]$ we will denote by $\mathcal{B}(\mathcal{H},r)$ the de Branges space generated by $\mathcal{H}$ on $[0,r]$, that is,
\begin{equation}
\mathcal{B}(\mathcal{H},r) = \mathcal{W}_{\mathcal{H},r} L^2(\mathcal{H},r) = \left\{ \text{entire } f : f = \mathcal{W}_{\mathcal{H},r} X, \quad X \in L^2(\mathcal{H},r) \right\},
\end{equation}
where the Weyl-Titchmarsh transform $\mathcal{W}_{\mathcal{H},r}$ is defined in \(3\) for $a = r$. The space $\mathcal{B}(\mathcal{H},r)$ is actually the Hilbert space with respect to the inner product $(f,g)_{\mathcal{B}(\mathcal{H},r)} = (f,g)_{L^2(\mu)}$, where $\mu$ is any spectral measure for problem \(2\). We refer the reader to paper \[2\] for the summary of results on direct and inverse spectral theory of canonical Hamiltonian systems and de Branges spaces of entire functions. The readers interested in proofs or in a more detailed account may find necessary information in Chapter 2 of classical book [4] by L. de Brange or its recent exposition \[13\] by R. Romanov.
Lemma 3.2. Let \( \mu \) be an even measure on \( \mathbb{R} \) of the form \( \mu = cm + \nu \), where \( c > 0 \) and \( \nu \) is a finite positive measure on \( \mathbb{R} \) with compact support. Then there exists an infinitely smooth diagonal Hamiltonian \( H \) on \([0, +\infty)\) such that \( \det H(r) = 1 \) for all \( r \geq 0 \), and \( \mu \) is the spectral measure for \( H \).

Proof. The result is a kind of folklore. Since the Fourier transform of \( \frac{1}{c} \nu \) is a smooth (in fact, analytic) function, one can use the classical Gelfand-Levitan approach to find a smooth diagonal potential \( Q \) on \([0, a]\) such that \( m + \frac{1}{c} \nu \) is the spectral measure for the Dirac system \( JY' + QY = zY \) corresponding to the boundary condition \( Y(0) = (\frac{1}{c})\). Then rewrite system \( JY' + QY = zY \) as a canonical Hamiltonian system \( JX' = zHX \) setting \( X = M^{-1}Y, \ H = M^*M \), where \( M \) is the matrix solution of equation \( JM' = -QM, \ M(0) = (\frac{1}{c} \ 0) \). Observe that \( \det H = 1 \) almost everywhere on \([0, a]\) and \( m + \frac{1}{c} \nu \) is the spectral measure for system \( JX' = zHX \) at \( (\frac{1}{c})\). To obtain the Hamiltonian on \([0, a]\) corresponding to the spectral measure \( \mu \), put \( H = (\frac{c}{0} \ 0 \ 1) \ H \). Another (in a sense, equivalent) way of proving Lemma 3.2 is the application of Theorem 5.1 from [20]. □

Define type \( B(H, r) = \sup \{ \text{type}(f), f \in B(H, r) \} \) to be the maximal exponential type of entire functions in de Branges space \( B(H, r) \). The following remarkable formula of Krein [9] and de Brange (Theorem X in [3])

\[
\text{type} B(H, r) = \int_{0}^{r} \sqrt{\det H(t)} \, dt,
\]

represents the maximal exponential type of functions in \( B(H, r) \) in terms of the Hamiltonian \( H \). Section 6 in [15] contains an elegant self-contained proof of this result.

Lemma 3.3. Let \( H \) be a Hamiltonian on an interval \([0, a]\) such that its spectral measure \( \mu \) satisfies (14). Assume that \( \det H(r) = 1 \) for almost all \( r \in [0, a] \). Then for all \( r \in [0, a] \) we have \( B(H, r) = (\text{PW}_r, \mu) \).

Proof. Let \( r \in [0, a] \) and let \( \varepsilon > 0 \) be such that \( r \in [\varepsilon, a - \varepsilon] \). Then the Hilbert space \( (\text{PW}_{r-\varepsilon}, \mu) \) of entire functions satisfies an axiomatic description of de Branges spaces (Theorem 23 in [4]) and the embedding \( (\text{PW}_{r-\varepsilon}, \mu) \subset L^2(\mu) \) is isometric. Since \( \mu \) is a spectral measure for \( H \), the embedding \( B(H, r) \subset L^2(\mu) \) is isometric as well. Applying de Branges chain theorem (Theorem 35 in [4]), we see that either \( (\text{PW}_{r-\varepsilon}, \mu) \subset B(H, r) \) or \( B(H, r) \subset (\text{PW}_{r-\varepsilon}, \mu) \). Since \( \det H = 1 \) almost everywhere on \([0, a] \), formula (17) implies the second alternative. Analogously, one can show that \( (\text{PW}_{r-\varepsilon}, \mu) \subset B(H, r) \). Since this holds for every small number \( \varepsilon \) and \( \mu \) is sampling, we have \( B(H, r) = (\text{PW}_r, \mu) \). Finally, for \( r = a \) we have

\[
B(H, a) = \bigcup_{0 < r < a} B(H, r) = (\text{PW}_a, \mu),
\]

where the completion is taken with respect to the norm inherited from \( L^2(\mu) \). □

Let \( \Theta_H \) be the absolutely continuous solution of Cauchy problem (2) on \([0, a]\), and denote \( \Theta_{\overline{H}} = (\Theta_H, (\frac{1}{c})) \), \( \Theta_{\overline{H}} = (\Theta_H, (\frac{1}{c})) \). The reproducing kernel \( k_{B(H, r), \lambda} \) at a point \( \lambda \in \mathbb{C} \) of the Hilbert space of entire functions \( B(H, r) \) has the form

\[
k_{B(H, r), \lambda} = \frac{1}{\pi} \frac{\Theta_{\overline{H}}^+(r, z)\Theta_{\overline{H}}^-(r, \lambda) - \Theta_{\overline{H}}^-(r, z)\Theta_{\overline{H}}^+(r, \lambda)}{z - \lambda}, \quad z \in \mathbb{C}.
\]

(18)
The Paley-Wiener space $\text{PW}_r$ is the de Branges space $\mathcal{B}(\mathcal{H}_0, r)$ for the Hamiltonian $\mathcal{H}_0 = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$. The reproducing kernel of $\text{PW}_r$ at $\lambda \in \mathbb{C}$ will be denoted by $\text{sinc}_{r, \lambda}$:

$$\text{sinc}_{r, \lambda} = \frac{\sin r(z - \lambda)}{\pi(z - \lambda)}, \quad z \in \mathbb{C}.$$  

Using integration by parts and equation (2), it is easy to show that for each $\lambda \in \mathbb{C}$ we have

$$\mathcal{W}_{\mathcal{H}_r, \lambda} \Theta_{\mathcal{H}}(\cdot, \lambda) = \sqrt{T} k_{\mathcal{B}(\mathcal{H}_r), \lambda}, \quad \mathcal{W}_{\mathcal{H}_0, \lambda} \Theta_{\mathcal{H}_0}(\cdot, \lambda) = \sqrt{T} \text{sinc}_{r, \lambda},$$

where $\Theta_{\mathcal{H}}(\cdot, \lambda)$ denotes the mapping $t \mapsto \Theta_{\mathcal{H}}(t, \lambda)$ and $\Theta_{\mathcal{H}_0}(\cdot, \lambda)$ is defined analogously.

Next assertion is Lemma 4.2 in [2].

**Lemma 3.4.** Let $\mu$ be a sampling measure for $\text{PW}_a$ and let $r \in [0, a]$. The reproducing kernel of the space $(\text{PW}_r, \mu)$ at $\lambda \in \mathbb{C}$ equals $T_{\mu, r, \lambda}$.

**Proof.** For every function $f$ in $(\text{PW}_r, \mu) \subset \text{PW}_r$ and every $\lambda \in \mathbb{C}$ we have

$$f(\lambda) = (f, \text{sinc}_{a, \lambda})_{L^2(\mathbb{R})} = (f, T_{\mu, r, \lambda} \text{sinc}_{r, \lambda})_{L^2(\mu)},$$

where we used the fact that $c_1 I \leq T_{\mu, a, \lambda} \leq c_2 I$ on $\text{PW}_a$ and hence $T_{\mu, r, \lambda}$ is bounded and invertible on $\text{PW}_r$. \hfill \square

**Lemma 3.5.** Let $\varphi$ be a function on $[0, a]$ such that $e^{\varphi}$ belongs to the Muckenhoupt class $A_2[0, a]$ and $\|w\|_{A_2[0,2]} \leq 2^{28} c_{14}$, where $c = c_1^{-1} + c_2 c_1^{-1}$. We also have \[\frac{1}{a} \int_0^a (w + \frac{1}{w}) \, dx \leq 4c.\]

**Proof.** Let us obtain estimates (16), (17) for the function $\varphi$ as it was suggested in Proposition 3.2 of [2]. Take $r \in [0, a]$. Set $\mathcal{H}_0 = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ and consider the corresponding Weyl-Titchmarsh transforms

$$\mathcal{W}_{\mathcal{H}_0, r} : L^2(\mathcal{H}_0, r) \to \mathcal{B}(\mathcal{H}_0, r), \quad \mathcal{W}_{\mathcal{H}, r} : L^2(\mathcal{H}, r) \to \mathcal{B}(\mathcal{H}, r).$$

We have $\mathcal{B}(\mathcal{H}_0, r) = \text{PW}_r$ and $\mathcal{B}(\mathcal{H}, r) = (\text{PW}_r, \mu)$, see Lemma 3.3. Since $\mu$ satisfies (1), the spaces $\text{PW}_r$, $(\text{PW}_r, \mu)$ coincide as sets and

$$c_2^{-1} \|f\|_{L^2(\mathbb{R})}^2 \leq \|T_{\mu, r} f\|_{L^2(\mu)}^2 \leq c_1^{-1} \|f\|_{L^2(\mathbb{R})}^2$$

for every function $f \in \text{PW}_r$. Hence, the operator $T = \mathcal{W}_{\mathcal{H}_0, r}^{-1} T_{\mu, r}^{-1} \mathcal{W}_{\mathcal{H}_0, r}$ from $L^2(\mathcal{H}_0, r)$ to $L^2(\mathcal{H}, r)$ is correctly defined, bounded, and invertible. Moreover,

$$c_2^{-1} \|X\|_{L^2(\mathcal{H}_0, r)}^2 \leq \|T X\|_{L^2(\mathcal{H}, r)}^2 \leq c_1^{-1} \|X\|_{L^2(\mathcal{H}_0, r)}^2$$  \hspace{1cm} (19)

for every $X \in L^2(\mathcal{H}_0)$. Next, by Lemma 3.4 for each $z \in \mathbb{C}$ we have

$$T \Theta_{\mathcal{H}_0}(\cdot, z) = \mathcal{W}_{\mathcal{H}_0, r}^{-1} (\sqrt{T} T_{\mu, r} \text{sinc}_{r, z}) = \Theta_{\mathcal{H}}(\cdot, z).$$

For $z = 0$ and all $t \in [0, r]$ we have $\Theta_{\mathcal{H}}(t, 0) = \Theta_{\mathcal{H}_0}(t, 0) = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$, hence

$$c_2^{-1} \| (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \|_{L^2(\mathcal{H}_0, r)}^2 \leq \| (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \|_{L^2(\mathcal{H}_0, r)}^2 \leq c_1^{-1} \| (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \|_{L^2(\mathcal{H}_0, r)}^2.$$  

This relation is inequality (17) for the function $\varphi$ and constants $b_1 = c_2^{-1}$, $b_2 = c_1^{-1}$.  

Now let \( \partial^n_0 \Theta_H(\cdot, 0) \) denote the derivative of order \( n \) of the mapping \( z \mapsto \Theta_H(\cdot, z) \) from \( \mathbb{C} \) to \( L^2(\mathcal{H}, r) \) at the point \( z = 0 \). Then \( T \partial^n_0 \Theta_H(\cdot, 0) = \partial^n_0 \Theta_H(\cdot, 0) \) for all integers \( n \geq 1 \). The right inequality in (19) yields
\[
\| \partial^n_0 \Theta_H(\cdot, 0) \|_{L^2(\mathcal{H}, r)}^2 \leq 2c_1^{-1} \| \partial^n_0 \Theta_{H_0}(\cdot, 0) \|_{L^2(\mathcal{H}_0, r)}^2.
\] (20)

From equation (2) we obtain
\[
\partial^n_0 \Theta_H(t, 0) = n! \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} J^* \mathcal{H}(t_1)J^* \mathcal{H}(t_2) \cdots J^* \mathcal{H}(t_n) (0) \, dt_n \cdots dt_1,
\] (21)
for all \( t \in [0, r] \) and \( n \geq 1 \). Observe that
\[
J^* \mathcal{H}(t_1)J^* \mathcal{H}(t_2) \cdots J^* \mathcal{H}(t_n) (0) = \begin{cases} \left( (1) \frac{-n \pm 2}{2} \exp(\phi_{x,n}(t)) \right), & n \text{ is odd,} \\ \left( (-1)^n \frac{-n}{2} \exp(\phi_{x,n}(t)) \right), & n \text{ is even,} \end{cases}
\]
where \( t = (t_1, \ldots, t_n) \) is a point in simplex \( K_{t,n} \), and \( \phi_{x,n} \) is defined on \( K_{t,n} \) by formula (3). Substitute this representation of \( J^* \mathcal{H}(t_1)J^* \mathcal{H}(t_2) \cdots J^* \mathcal{H}(t_n) (0) \) to (21). Then (21), (22), and (20) give us inequality (1) for all \( n \geq 1 \) and all \( r \in [0, a] \).

It remains to use Proposition 2.1 to see that \( w \in A_2[0, a] \) and \( \| w \|_{A_2[0, a]} \leq 2^{28} c_{14}^1 \). The estimate \( \frac{1}{a} \int_0^a (w + \frac{1}{w}) \, dx \leq 4c \) follows from Lemma 2.1. \( \square \)

3.3. Proof of Theorem 1 Let \( \mu \) be a measure on \( \mathbb{R} \) such that estimate (1) holds for some \( a > 0 \). Consider the truncated Toeplitz operator \( T_{\mu,a} = T_\mu \) on \( PW_a \). We have \( c_1 I \leq T_\mu \leq c_2 I \), where \( I \) stands for the identity operator on \( PW_a \). The operator \( T_\mu - c_1 I \) satisfies assumptions of Lemma 3.1. Hence, there exists a measure \( \nu \geq 0 \) on \( \mathbb{R} \) such that \( T_\nu = T_\mu - c_1 I \). One can suppose that \( \nu \) is even (otherwise consider the measure \( \hat{\nu} \) such that \( \hat{\nu}(S) = \frac{1}{2}(\nu(S) + \nu(-S)) \), and note that \( T_\nu = T_\nu \)).

Define a sequence of measures \( \mu_j \) by \( \mu_j = c_1 m + \chi_j \nu \), where \( m \) is the Lebesgue measure on \( \mathbb{R} \), and \( \chi_j \) denotes the indicator function of the interval \( [-j, j] \). For every \( j \geq 1 \) the measure \( \mu_j \) is even and satisfies relation (1) with the same constants \( c_1, c_2 \). Indeed, \( T_{\mu_j} = c_1 I + T_{\chi_j \nu} \) and
\[
c_1 I \leq c_1 I + T_{\chi_j \nu} \leq c_1 I + T_\nu = T_\mu \leq c_2 I.
\]

By Lemma 3.2 and Lemma 3.5 for every \( j \) there exists a smooth function \( w_j > 0 \) on the interval \( [0, a] \) such that \( \| w_j \|_{A_2[0, a]} \leq 2^{28} c_{14}^1 \), \( c = c_1^{-1} + c_2 c_1^{-1} \), and \( \mu_j \) is the spectral measure for the Hamiltonian \( \mathcal{H}_j = \left( \begin{smallmatrix} w_j & 0 \\ 0 & w_j \end{smallmatrix} \right) \) on \([0, a] \). We also have \( \frac{1}{a} \int_0^a (w_j + \frac{1}{w_j}) \, dx \leq 4c \) for all \( j \geq 1 \). This allows us to use “a reverse Hölder inequality” for weights in \( A_2[0, a] \). It says that for every \( C_1 > 0 \) there exist \( p > 1 \) and \( C_2 > 0 \) such that for all \( h \in A_2[0, a] \) with \( \| h \|_{A_2[0, a]} \leq C_1 \) we have
\[
\frac{1}{a} \int_0^a h(x)^p \, dx \leq C_2 \left( \frac{1}{a} \int_0^a h(x) \, dx \right)^p.
\]

Explicit relations between \( C_1, C_2, \) and \( p \) can be found in (19). From here we see that sequences \( \{ w_j \}_{j \geq 1}, \{ \frac{1}{w_j} \}_{j \geq 1} \) are uniformly bounded in \( L^p[0, a] \) for some \( p > 1 \). Hence we can find subsequences \( w_{j_k}, \frac{1}{w_{j_k}} \) converging weakly in \( L^p[0, a] \) to functions \( w, v \), respectively. To simplify notations, let the sequences \( \{ w_j \}_{j \geq 1}, \{ \frac{1}{w_j} \}_{j \geq 1} \) themselves be weakly convergent. Let us show that \( v = w^{-1} \) almost everywhere...
on the interval \([0, a]\]. This is not always the case for arbitrary weakly convergent sequences in \(L^p[0, a]\).

For \(z \in \mathbb{C}\) denote by \(\Theta_j(\cdot, z)\) solution of equation (2) for the Hamiltonian \(H_j\).

Integrating (2), we get

\[
J\Theta_j(r, z) - \left( \frac{1}{0} \right) = z \int_0^r H_j(t)\Theta_j(t, z) \, dt. \tag{23}
\]

Then for every \(j \geq 1\) and \(r, r' \in [0, a]\) we have the estimates

\[
\|\Theta_j(r, z)\|_{C^2} \leq \exp \left( |z| \int_0^a \|H_j(t)\| \, dt \right),
\]

\[
\|\Theta_j(r, z) - \Theta_j(r', z)\|_{C^2} \leq |z| |r - r'|^{\frac{1}{p}} \left( \int_0^a \|H_j(t)\|_p \cdot \|\Theta_j(t, z)\|_{C^2}^2 \, dt \right)^{\frac{1}{p}},
\]

showing that functions \(\Theta_j(\cdot, z)\) are uniformly bounded and equicontinuous on \([0, a]\).

Therefore, there is a subsequence of the sequence \(\Theta_j(\cdot, z)\) converging uniformly on \([0, a]\) to a function \(\Theta(\cdot, z)\). As before, we suppose that the sequence \(\Theta_j(\cdot, z)\) itself is uniformly convergent on \([0, a]\). It is clear that the limit function \(\Theta\) satisfies equation (23) for the Hamiltonian \(H = (w, 0)\). Hence, it satisfies equation (2) for \(H\).

Fix a number \(r \in [0, a]\). For every \(\lambda \) and \(z \in \mathbb{C}\) we have

\[
k_{B(H, r), \lambda}(z) = \lim_{j \to \infty} k_{B(H_j, r), \lambda}(z) = \lim_{j \to \infty} (T_{\mu, r}^{-1} \cdot \text{sinc}_{r, \lambda})(z) = (T_{\mu, r}^{-1} \cdot \text{sinc}_{r, \lambda})(z). \tag{24}
\]

Indeed, the first equality above follows from formula (13) and convergence of \(\Theta_j\) to \(\Theta\) on \([0, a]\) when a spectral parameter (\(\lambda\) or \(z\)) is fixed. Lemma 3.3 and Lemma 3.3 give us the second equality. Finally, using the fact that the operators \(T_{\mu, r}\) tend to \(T_{\mu, r}\) in the strong operator topology, we obtain the last equality in (24).

From (24) we see that Hilbert spaces of entire functions \(B(H, r)\), \((\text{PW}_r, \mu)\) have the same reproducing kernels. Hence \(B(H, r) = (\text{PW}_r, \mu)\) and formula (17) implies

\[
r = \int_0^r \sqrt{\det H(t)} \, dt, \quad r \in [0, a].
\]

It follows that \(\det H = 1\) almost everywhere on \([0, a]\), that is, \(v = w^{-1}\). Next, from the direct spectral theory we know that the family \(\{\Theta(\cdot, \lambda)\}_{\lambda \in \mathbb{C}}\) is complete in \(L^2(H, a)\) and \(W_{H, a}\Theta(\cdot, \lambda) = k_{B(H, a), \lambda}(\cdot)\) for every \(\lambda \in \mathbb{C}\), where \(W_{H, a}\) denotes the Weyl-Titchmarsh transform associated to \(H\). Using (24) again, we get

\[
(\Theta(\cdot, \lambda), \Theta(\cdot, z))_{L^2(H, a)} = \pi k_{B(H, a), \lambda}(z) = \pi (T_{\mu, a}^{-1} \cdot \text{sinc}_{a, \lambda})_{L^2(\mathbb{R})}
= \pi (T_{\mu, a}^{-1} \cdot \text{sinc}_{a, \lambda}^{-1} \cdot \text{sinc}_{a, z})_{L^2(\mu)} = (W_{H, a}\Theta(\cdot, \lambda), W_{H, a}\Theta(\cdot, z))_{L^2(\mu)}.
\]

Hence, the operator \(W_{H, a}\) acts isometrically from \(L^2(H, a)\) to \(L^2(\mu)\) and \(\mu\) is a spectral measure for \(H\). In particular, we can apply Lemma 3.3 to \(H, \mu\), and conclude that the function \(w = e^\varphi\) is in \(A_2[0, a]\) and \(\|w\|_{A_2[0, a]} \leq 2^{28} c^{14}\). Uniqueness of the Hamiltonian \(H\) follows immediately from formula (24):

\[
\int_0^r w(t) \, dt = \int_0^r \langle H(t) \left( \frac{1}{0} \right), \left( \frac{1}{0} \right) \rangle \, dt = \pi k_{B(H, a), \lambda}(0) = \pi \|T_{\mu, a}^{-1} \cdot \text{sinc}_{r, 0}\|_{L^2(\mu)}, \tag{25}
\]

where the right hand side is completely determined by \(\mu\), while the left hand side determines \(H\). 

\(\square\)
Differentiating formula (25), we obtain the following corollary.

**Corollary 1.** The Hamiltonian $\mathcal{H} = \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix}$ in Theorem 2 could be recovered from $\mu$ by means of the following formula: $w(r) = \pi \frac{\partial}{\partial r} \|T_{\mu,r}^{-1}\|_{\mathcal{L}_2(\mu)}, r \in [0,a]$.

4. **Proof of Theorem 2 and Theorem 3**

Let us first show that Theorem 2 does not follow from a general theory of canonical Hamiltonian systems. Consider the simplest case where the Hamiltonian $\mathcal{H}$ coincides with the identity matrix $(\frac{1}{0} 0)$ on $[0,a]$. We claim that there is no unitary operator $U : L^2(\mathcal{H},a) \to \text{PW}_{[0,2a]}$ such that $UL^2(\mathcal{H},r) = \text{PW}_{[0,2r]}$ for all $r \in [0,a]$. Indeed, existence of such a unitary operator yields the existence of another unitary operator $\bar{U} : L^2[-a,a] \to L^2[0,2a]$ such that $\bar{U}L^2[-r,r] = L^2[0,2r]$ for all $r \in [0,a]$. For every $r_1 > r_2 \geq 0$ let $\chi_{[r_1,r_2]}$ denote the indicator function of the interval $[r_1,r_2]$. Put $g = U\chi_{[0,a]}$ and consider decomposition $g = f_r + h_r$, where $f_r = \bar{U}\chi_{[0,r]}$, $h_r = \bar{U}\chi_{[r,a]}$, $r \in [0,a]$. Since $\bar{U}L^2[-r,r] = L^2[0,2r]$ by our assumption, the function $f_r$ is supported on $[0,2r]$. Note also that the function $h_r$ is orthogonal to all functions from $L^2[0,2r]$ and hence it is supported on $[2r,2a]$. From here we see that $f_r = \chi_{[0,2r]}g$ for all $r \in [0,a]$. Next, unitarity of the operator $\bar{U}$ implies that

$$\int_0^{2r} |g(t)|^2 dt = \int_0^{2r} |f_r(t)|^2 dt = \int_{-a}^a |\chi_{[0,r]}(t)|^2 dt = r, \quad r \in [0,a].$$

It follows that $|g(t)|^2 = 1/2$ for almost all $t \in [0,2a]$. In particular, the linear span of functions $f_r \in UL^2([0,a])$, $r \in [0,a]$, is dense in $L^2[0,2a]$. This contradicts to the fact that $\bar{U}$ is a unitary operator from $L^2[0,a]$ to $L^2[0,2a]$. Thus, the Weyl-Titchmarsh transform $W_{\mathcal{H},a}$ from formula (24) can not be used to construct the operator $\mathcal{F}_\mu$ from Theorem 2 by means of superposition with some simple unitary operators like shifts, reflections, etc.

The main point that helps in proof of Theorem 2 is the fact that Hamiltonian $\mathcal{H}$ generated by an even sampling measure for the Paley-Wiener space $\text{PW}_a$ must have rank two almost everywhere on its domain of definition. It is an open question if this is true for general (not necessarily even) sampling measures for $\text{PW}_a$. See also Proposition 5.1 in Section 5 for more details.

**Proof of Theorem 2.** Fix an even sampling measure $\mu$ and construct the Hamiltonians $\mathcal{H}_j$, $\mathcal{H}$, on $[0,a]$ as in the proof of Theorem 1. Put $\varphi_j = \log w_j$ and $\varphi = \log w$, where $w_j$, $w$ are the functions generating $\mathcal{H}_j$, $\mathcal{H}$. Recall that $w_j$ tend to $w$ weakly in $L^p[0,a]$ for some $p > 1$ and the same is true for $w_j^{-1}$ and $w^{-1}$. Let $\Theta_j, \Theta$ be the solutions of system (2) generated by Hamiltonians $\mathcal{H}_j$, $\mathcal{H}$, correspondingly. As we have seen, the functions $\Theta_j(z) = \begin{pmatrix} \Theta_j^+ \\ \Theta_j^- \end{pmatrix}$ converge uniformly to $\Theta(z) = \begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix}$ on the interval $[0,a]$ when $z \in \mathbb{C}$ is fixed. For $r \in [0,a]$, define entire functions $P_{2r,j}$...
and $P_{2r,j}^*$ by
\[
P_{2r,j}^*: z \mapsto e^{irz} \left( e^{-\varphi_j(r)/2} \Theta_j^+(r, z) - ie^{-\varphi_j(r)/2} \Theta_j^-(r, z) \right),
\]
and let $P_{2r}, P_{2r}^*$ be defined similarly with $\varphi_j$ replaced by $\varphi$. These functions satisfy
the Krein system of differential equations:
\[
\begin{aligned}
P_{r,j}'(z) &= iz P_{r,j}(z) + \frac{\varphi_j'(r/2)}{4} P_{r,j}^*(z), \\
P_{r,j}^*(z) &= \frac{\varphi_j'(r/2)}{4} P_{r,j}(z), \\
P_{0,j}(z) &= e^{\varphi_j(0)}, \\
P_{0,j}^*(z) &= e^{-\varphi_j(0)},
\end{aligned}
\tag{26}
\]
where $\varphi_j'(r/2)$ is the value of smooth function $\varphi_j'$ at $r/2$. From system (26), we obtain
by integration by parts (see Lemma 9.1 in [5]) the Christoffel-Darboux formula:
\[
\int_0^r P_{t,j}(z) P_{t,j}(\lambda) dt = i \frac{P_{r,j}^*(z) P_{r,j}^*(\lambda) - P_{r,j}(z) P_{r,j}(\lambda)}{z - \lambda}.
\]
The right-hand side could be rewritten in the form
\[
\ldots = 2e^{i\pi (z-\lambda)} \Theta_j^+(\frac{z}{2}, z) \Theta_j^-(\frac{z}{2}, \lambda) - \Theta_j^-(\frac{z}{2}, z) \Theta_j^+(\frac{z}{2}, \lambda)
\]
which tends to $2\pi k_{r,\lambda}(z)$, the scalar multiple of the reproducing kernel $k_{r,\lambda}$ at $\lambda$ of
the Hilbert space $e^{i\pi B(H, \frac{r}{2})} = (PW_{[0, r]}^2, \mu)$, see formula (18). On the other hand,
for every pair $z, \lambda \in \mathbb{C}$ we have
\[
P_{t,j}(z) P_{t,j}(\lambda) = e^{i\frac{\pi}{2} (z-\lambda)} \left( e^{\varphi_j'(\frac{z}{2})} \Theta_j^+(\frac{z}{2}, z) \Theta_j^+(\frac{z}{2}, \lambda) + e^{-\varphi_j'(\frac{z}{2})} \Theta_j^-(\frac{z}{2}, z) \Theta_j^-(\frac{z}{2}, \lambda) + i \Theta_j^+(\frac{z}{2}, z) \Theta_j^-(\frac{z}{2}, \lambda) - i \Theta_j^-(\frac{z}{2}, z) \Theta_j^+(\frac{z}{2}, \lambda) \right).
\]
Since functions $e^{\varphi_j}$, $e^{-\varphi_j}$ converge weakly in $L^p[0,a]$ to functions $e^\varphi$, $e^{-\varphi}$, cor-
respondingly, we see that
\[
\int_0^r P_t(z) \overline{P_t(\lambda)} dt = \lim_{j \to \infty} \int_0^r P_{t,j}(z) \overline{P_{t,j}(\lambda)} dt = 2\pi k_{r,\lambda}(z)
\tag{27}
\]
for every $r \in [0, 2a]$. Let $\chi_r$ be the indicator function of the interval $[0, r]$. Denote
by $L$ the set of all finite linear combinations of functions $t \mapsto \chi_r(t) P_t(z)$ on $[0, 2a]$, where
$z \in \mathbb{C}$ and $r \in [0, 2a]$. The linear manifold $L$ is dense in $L^2[0, 2a]$. Indeed,
for every function $g \in L^2[0, 2a]$ orthogonal to $L$ we have
\[
0 = \int_0^{2a} g(t) \chi_r(t) P_t(0) dt = \int_0^r g(t) e^{\varphi(t/2)} dt, \quad r \in [0, 2a],
\]
yielding $g = 0$ in $L^2[0, 2a]$. Formula (27) also shows that a nontrivial finite linear
combination of functions $\chi_r(t) P_t(z)$ cannot vanish almost everywhere on $[0, 2a]$. Consider
the operator $\mathcal{F}_\mu : L^2[0, 2a] \to (PW_{[0,2a]}^2, \mu)$ densely defined on $L$ by
\[
\mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^{2a} f(t) P_t(z) dt, \quad z \in \mathbb{C}.
\]
The operator $F_\mu$ takes the function $t \mapsto \chi_r(t)\overline{P_t(\lambda)}$ on $[0,2a]$ into $\sqrt{2\pi}k_{r,\lambda}$, see formula (27). Moreover, for every $r_1, r_2 \in [0,2a]$ we have

$$\left(\mathcal{F}_\mu \chi_{r_1} \overline{P_t(\lambda)}, \mathcal{F}_\mu \chi_{r_2} \overline{P_t(\lambda)}\right)_{L^2(\mu)} = 2\pi(k_{r_1,\lambda}, k_{r_2,\lambda}) = 2\pi k_{r,\lambda}(z),$$

where $r = \min(r_1, r_2)$. This shows that $F_\mu$ is an isometry on $L$. Since the linear span of the set $\{k_{2a,\lambda}, \lambda \in \mathbb{C}\}$ is complete in $(PW_{[0,2a]}, \mu)$, the operator $F_\mu$ is unitary. It is also clear from the definition that $F_\mu$ maps $L^2[0, r]$ onto $(PW_{[0,r]}, \mu)$ for every $r \in [0, 2a]$.

**Proof of Theorem 3** At first, consider a positive bounded invertible operator $W_\psi$ with real symbol $\psi \in S'$ on a finite interval $[0, a)$. Let $\mathcal{F}$ denote the unitary Fourier transform on $L^2(\mathbb{R})$. Take a smooth function $h$ with support in $(0, a)$ and put $f = \mathcal{F}f$. Consider the operator $W_\psi = \mathcal{F}W_\psi \mathcal{F}^{-1}$ on $PW_{[0,a]}$. We have $(W_\psi h, \hat{h})_{L^2(\mathbb{R})} = \langle \hat{\psi}, \hat{\psi} \rangle_{S'}$, where $\hat{\psi}$ is the Fourier transform of the tempered distribution $\psi$. It follows that

$$(\hat{W_\psi f}, f)_{L^2(\mathbb{R})} = (\hat{W_\psi f}, f)_{L^2(\mathbb{R})}$$

on a dense subset of the set $Z_{-1} = \{f \in PW_{[0,a]} : f(\sqrt{-1}) = 0\}$. Since $\hat{W_\psi}$ is bounded on $PW_{[0,a]}$, we have the last identity for all $f \in Z_{-1}$. Hence, the operator $W_\psi$ satisfies assumptions of Lemma 3.1 and we can find a positive Borel measure $\mu$ on $\mathbb{R}$ such that

$$\langle \hat{W_\psi f}, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} fg \ d\mu$$

for all $f, g \in PW_{[0,a]}$. As in the proof of Theorem 1 we can assume that the measure $\mu$ is even. Indeed, since $\psi$ is real, we have $(\hat{W_\psi f}, f) = (\hat{W_\psi f^*}, f^*)$ for arbitrary $f \in PW_{[0,a]}$ and its reflection $f^* : x \mapsto f(-x)$. By the assumption, the operator $\hat{W_\psi}$ is positive, bounded and invertible on $PW_{[0,a]}$. Hence the measure $\mu$ satisfies (1) for some $c_1, c_2$ and $a/2$ in place of $a$. By Theorem 2 there is a unitary operator $F_\mu : L^2([0, a]) \to (PW_{[0,a]}, \mu)$ such that $F_\mu : L^2([0, r]) = (PW_{[0,r]}, \mu)$ for every $r \in [0, a]$. Identifying Hilbert spaces $(PW_{[0,a]}, \mu)$ and $PW_{[0,a]}$ as sets, we can define the operator $A = F_\mu^{-1} F$ on $L^2([0, a])$. By construction, the operator $A$ is bounded and invertible and $A L^2([0, r]) = L^2([0, r])$ for every $r \in [0, a]$. We also have

$$(W_\psi h, h)_{L^2([0,a])} = \int_{\mathbb{R}} \|\hat{h}\|^2 \ d\mu = (\mathcal{F} h, \mathcal{F} h)_{L^2(\mu)} = (F_\mu^{-1} \mathcal{F} h, F_\mu^{-1} \mathcal{F} h)_{L^2([0,a])}$$

(28)

for all smooth functions $h$ with support in $(0, a)$. It follows that the operator $W_\psi$ admits the triangular factorization $W_\psi = A^* A$.

It remains to consider the case where $W_\psi$ is a positive bounded invertible Wiener-Hopf operator on $L^2[0, \infty)$ with real symbol $\psi \in S'$. It is known (see Section 4.2.7 in [19]) that in this case the Fourier transform of the distribution $\psi$ coincides with a function $\sigma$ on $\mathbb{R}$ such that $c_1 \leq \sigma(x) \leq c_2$ for some positive constants $c_1, c_2$ and almost all $x \in \mathbb{R}$. In particular, the measure $\mu = \sigma dm$ is sampling for all Paley-Wiener spaces $PW_{[0,r]}$, $r > 0$. Since $\psi$ is real, the function $\sigma$ is even. For every $r > 1$ we can use Theorem 1 and find a Hamiltonian $\mathcal{H}_r$ on $[0, r]$ such that $\det \mathcal{H}_r(t) = 1$ for almost all $t \in [0, r]$ and $\mu$ is the spectral measure for $\mathcal{H}_r$. Since the Hamiltonian $\mathcal{H}$ in Theorem 1 is defined uniquely, we have $\mathcal{H}_r(t) = H_{\mu'}(t)$ for
almost all \( t \in [0, \min(r, r')] \). This shows that there is the Hamiltonian \( \mathcal{H} \) on \([0, \infty)\) such that \( \det \mathcal{H} = 1 \) almost everywhere and \( \mu \) is the spectral measure for \( \mathcal{H} \). In particular, we can define a family of entire functions \( \{P_t\}_{t \geq 0} \) such that the mapping

\[ \mathcal{F}_\mu : f \mapsto \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)P_t(z) \, dt \quad (29) \]

sends unitarily the space \( L^2[0, r] \) onto the space \((\text{PW}_{[0, r]}, \mu)\) for every \( r > 0 \), see the proof of Theorem \([2]\). Let \( H^2_\mu(\mathbb{C}_+) \) be the weighted Hardy space with the inner product \((f, g)_{H^2_\mu(\mathbb{C}_+)} = (f, g)_{L^2(\mu)}\). Since \( c_1 \leq \sigma \leq c_2 \) on \( \mathbb{R} \), the space \( H^2_\mu(\mathbb{C}_+) \) coincides as a set with the standard Hardy space \( H^2(\mathbb{C}_+) = \mathcal{F}L^2(0, \infty) \). Define the unitary operator \( \mathcal{F}_\mu \) from \( L^2(0, \infty) \) to \( H^2_\mu(\mathbb{C}_+) \) by formula (29) with \( r = \infty \) on the dense set of compactly supported bounded functions in \( L^2[0, \infty) \). Then the operator \( A = \mathcal{F}_\mu^{-1}\mathcal{F} \) on \( L^2[0, \infty) \) is bounded and invertible. Moreover, \( AL^2[0, r] = L^2[0, r] \) for every \( r \geq 0 \), and \( W_\psi = A^*A \), see formula (28).

**Remark.** It can be shown that positive bounded invertible Wiener-Hopf operators \( W_\psi \) on \( L^2[0, a] \) with real symbols \( \psi \in S' \) admit triangular factorisation in the reverse order, \( W_\psi = AA^* \). In the case \( a = \infty \) the classical Wiener-Hopf factorization works: one can take \( A = \mathcal{F}^{-1}T_{\varphi_\sigma} \mathcal{F} \), where \( T_{\varphi_\sigma} \) is the Toeplitz operator on \( H^2(\mathbb{C}_+) \) with analytic symbol \( \varphi_\sigma \) such that \(|\varphi_\sigma|^2 = \sigma = \mathcal{F}\psi \). If \( a > 0 \) is finite, then we can use Theorem \([3]\) to find left triangular factorization \( W_\psi = A^*A \) and then put \( A = C_a A C_a \), where \( C_a : f \mapsto \int (a - x) \) is the conjugate-linear isometry on \( L^2[0, a] \). Since \( C_a W_\psi C_a = W_\psi \) for the self-adjoint Wiener-Hopf operator \( W_\psi \) on \( L^2[0, a] \), and \( C_a^2 = I \), we have \( W_\psi = AA^* \). It is also clear that the operator \( A \) is upper-triangular.

5. **Appendix. Two results by L. A. Sakhnovich**

In paper \([17]\) L. A. Sakhnovich proved (see Theorem 4.1 and Remark 4.1 in \([17]\)) that positive bounded invertible Wiener-Hopf operator

\[ T : f \mapsto f - \mu \int_0^\infty f(t) \frac{\sin \pi(t - x)}{\pi(t - x)} \, dt, \quad f \in L^2(0, \infty), \quad 0 < \mu < 1, \quad (30) \]

densely defined on \( L^2[0, \infty) \) does not admit triangular factorization \( T = A^*A \), where a bounded invertible operator \( A \) on \( L^2[0, \infty) \) is such that \( AL^2[0, r] = L^2[0, r] \) for every \( r \geq 0 \). Clearly, this assertion contradicts Theorem \([3]\). Let us point out an error in its proof.

The argument in \([17]\) crucially uses the following claim. Let \( \chi_{[-\pi, \pi]} \) be the indicator function of the interval \([-\pi, \pi]\). Formulas (4.1) - (4.4) in \([17]\) for \( n = 0 \) and \( a_0 = \pi \) determine the function \( \sigma' : x \mapsto \frac{1}{\pi i}(1 - \mu \cdot \chi_{[-\pi, \pi]}(x)) \) on \( \mathbb{R} \). The function

\[ \Pi(z) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{(z - t)(1 + t^2)} \log \sigma'(t) \, dt \right) \]

from formula (4.10) of \([17]\) (see also formula (4.12) therein) is claimed to satisfy the following relation (formula (4.18) in \([17]\)):

\[ \lim_{y \to +0} \Pi(\pm iy) = \sqrt{1 - \mu}. \]

However, this fact is false. Indeed, we have

\[ \frac{1}{\pi i} \frac{1 + tz}{(z - t)(1 + t^2)} = -\frac{1}{\pi i} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right). \]
and hence $\sqrt{2\pi}\Pi(z)$ is the outer function in $\mathbb{C}_+$ whose absolute value on $\mathbb{R}$ coincides with $(\sigma')^{-1/2}$ almost everywhere on $\mathbb{R}$. Since $(\sigma')^{-1/2}$ is regular (in fact, constant) near the origin, we have

$$\lim_{y \to +0} \Pi(iy) = \frac{1}{\sqrt{2\pi}}(\sigma')^{-1/2}(0) = \frac{1}{\sqrt{1-\mu}}.$$ 

We also would like to note that the last relation agrees well with the first identity in formula (4.19) from [17].

The second part of this section concerns factorization problem for truncated Toeplitz operators generated by general sampling measures for the space $PW_a$ not necessarily symmetric with respect to the origin. The result is equivalent to Theorem 4.2 in [16]. The proof below seems to be a bit more straightforward than the original one, possibly, because we consider the one-dimensional situation.

**Proposition 5.1.** Let $\mathcal{H}$ be a Hamiltonian on $[0, \ell]$ such that $\int_0^\ell \text{trace } \mathcal{H}(r) < \infty$, and let $\mu$ be a spectral measure for problem (2). Set $a = \int_0^\ell \sqrt{\text{det } \mathcal{H}(r)} \, dr$. Assume that $\mu$ satisfies (1). The following assertions are equivalent:

(a) $\text{det } \mathcal{H} > 0$ almost everywhere on $[0, \ell]$;
(b) there exists a unitary operator $V_\mu : PW_a \to (PW_a, \mu)$ such that for every $r \in [0, a]$ we have $V_\mu PW_r = (PW_r, \mu)$.
(c) there exists a bounded invertible operator $A$ on $PW_a$ such that $T_{\mu, a} = A^*A$ and for every $r \in [0, a]$ we have $APW_r = PW_r$.

Given a Hamiltonian $\mathcal{H}$ on $[0, \ell]$ such that $a = \int_0^\ell \sqrt{\text{det } \mathcal{H}(t)} \, dt > 0$, we define continuous from the left function $\xi_{\mathcal{H}}$ from $[0, a]$ to $[0, \ell]$ by

$$r = \int_0^{\xi(r)} \frac{1}{\sqrt{\text{det } \mathcal{H}(r)}} \, dt, \quad r \in [0, a].$$

This function is continuous if and only if there are no interval $(r_1, r_2) \subseteq [0, \ell]$ such that $\text{det } \mathcal{H}(t) = 0$ for almost all $t \in (r_1, r_2)$. The function $\xi_{\mathcal{H}}$ is absolutely continuous if and only if $\text{det } \mathcal{H}(t) > 0$ for almost all $t \in [0, a]$, see Exercise 13 in Chapter IX of [13].

**Proof of Proposition 5.1** (a) $\Rightarrow$ (b). Since $\text{det } \mathcal{H} > 0$ almost everywhere on the interval $[0, \ell]$, the function $\xi = \xi_{\mathcal{H}}$ is absolutely continuous and

$$\xi'(r) = \frac{1}{\sqrt{\text{det } \mathcal{H}(\xi(r))}}$$

for almost all $r \in [0, a]$. Consider the Hamiltonian $\tilde{\mathcal{H}} : r \mapsto \xi'(r)\mathcal{H}(\xi(r))$ on the interval $[0, a]$. We have $\text{det } \tilde{\mathcal{H}} = 1$ and $\Theta_{\tilde{H}}(r, z) = \Theta_{\mathcal{H}}(\xi(r), z)$ on $[0, a]$. Changing variable in (3), we see that $\mathcal{B}(\tilde{\mathcal{H}}, r) = \mathcal{B}(\mathcal{H}, \xi(r))$ for every $r \in [0, a]$, hence $\mu$ is the spectral measure for $\tilde{\mathcal{H}}$. Consider the Weyl-Titchmarsh transforms generated by Hamiltonians $\mathcal{H}$ and $\mathcal{H}_0 = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, correspondingly,

$$\mathcal{W}_{\tilde{\mathcal{H}}, a} : L^2(\tilde{\mathcal{H}}, a) \to B(\tilde{\mathcal{H}}, a), \quad \mathcal{W}_{\mathcal{H}_0, a} : L^2(\mathcal{H}_0, a) \to PW_a.$$ 

Define the operator $V_\mu : PW_a \to B(\tilde{\mathcal{H}}, a)$ by $V_\mu = \mathcal{W}_{\tilde{\mathcal{H}}, a} \mathcal{M}_{\tilde{\mathcal{H}}, a}^{-1/2} \mathcal{W}_{\mathcal{H}_0, a}$, where $\mathcal{M}_{\tilde{\mathcal{H}}, a} : L^2(\mathcal{H}_0, a) \to L^2(\tilde{\mathcal{H}}, a)$ is the multiplication operator by $\tilde{\mathcal{H}}^{-1/2}$, that is, $\mathcal{M}_{\tilde{\mathcal{H}}, a} : X \mapsto \tilde{\mathcal{H}}^{-1/2}X$. Since $\mathcal{M}_{\tilde{\mathcal{H}}, a}$ is unitary, the operator $V_\mu$ is unitary as
well. It is also clear that $V_{\mu} PW_r = B(\hat{H}, r)$ for every $r \in [0, a]$. Using Lemma 3.3 we see that $B(\hat{H}, r) = (PW_r, \mu)$, as required.

(b) $\Rightarrow$ (a). We will show that the function $\xi = \xi_\mu$ is absolutely continuous. Let $\chi_r$ be the indicator function of an interval $[0, r]$. For every $r \in [0, a]$ consider the functions $X_{\xi(r)} = \chi_{\xi(r)} \left( \frac{1}{r} \right)$, $Y_{\xi(r)} = \chi_{\xi(r)} \left( \frac{\nu}{r} \right)$ in $L^2(\mathcal{H}, \xi(r))$. A straightforward modification of Lemma 3.3 gives $B(\mathcal{H}, \xi(r)) = (PW_r, \mu)$ for all $r \in [0, a]$. Put

$$X_0^r = W_{\mathcal{H}, \mu, a}^{-1} V_{\mu, a}^{-1} W_{\mathcal{H}, \mu, a} X_{\xi(r)}, \quad Y_0^r = W_{\mathcal{H}, \mu, a}^{-1} V_{\mu, a}^{-1} W_{\mathcal{H}, \mu, a} Y_{\xi(r)}.$$ 

Since $V_{\mu}$ is isometric and $V_{\mu} PW_r = (PW_r, \mu)$, we have $\mathcal{P}_{\mu, r} V_{\mu} = V_{\mu} \mathcal{P}_{r}$, where $\mathcal{P}_r$, $\mathcal{P}_{\mu, r}$ are the orthogonal projections on $PW_a$, $(PW_a, \mu)$, with ranges $PW_r$, $(PW_r, \mu)$, respectively. It follows that $X_0^r = \chi_r X_0^a$ and $Y_0^r = \chi_r Y_0^a$. Using the fact that the operators $W_{\mathcal{H}, \mu, a}, W_{\mathcal{H}, \mu, a}$ are unitary, we obtain

$$\int_0^{\xi(r)} \text{trace } \mathcal{H}(t) \, dt = \int_0^{\xi(r)} \left( \langle \mathcal{H}(t) \left( \frac{1}{r} \right), \left( \frac{1}{r} \right) \rangle + \langle \mathcal{H}(t) \left( \frac{\nu}{r} \right), \left( \frac{\nu}{r} \right) \rangle \right) \, dt,$$

$$= \|X_{\xi(r)}\|_{L^2(\mathcal{H}, \ell)}^2 + \|Y_{\xi(r)}\|_{L^2(\mathcal{H}, \ell)}^2,$$

$$= \|X_0^r\|_{L^2(\mathcal{H}, \mu, a)}^2 + \|Y_0^r\|_{L^2(\mathcal{H}, \mu, a)}^2,$$

$$= \int_0^r \left( \|X_0^a(t)\|_{L^2}^2 + \|Y_0^a(t)\|_{L^2}^2 \right) \, dt. \quad (31)$$

The above equalities hold for all $r \in [0, a]$. Let us define the function $\kappa$ on $[0, \ell]$ by

$$\kappa(s) = \int_0^s \text{trace } \mathcal{H}(t) \, dt, \quad s \in [0, \ell].$$

Then $\kappa$ is an absolutely continuous function with positive derivative almost everywhere on $[0, \ell]$, hence the inverse mapping $\kappa^{-1}$ is also absolutely continuous and has positive derivative. On the other hand, formula (31) shows that $\kappa(\xi)$ is an absolutely continuous function. It follows that the superposition $\xi = \kappa^{-1}(\kappa(\xi))$ is absolutely continuous and hence det $\mathcal{H} > 0$ almost everywhere on $[0, \ell]$.

(b) $\Rightarrow$ (c). Since $\mu$ satisfies (1), the identical embedding $j : PW_a \rightarrow (PW_a, \mu)$ is a bounded and invertible operator. Define $A = V_{\mu}^{-1} j$. Then for all $f, g$ in $PW_a$ we have

$$(A^* A f, g) = (V_{\mu}^{-1} j f, V_{\mu}^{-1} j g)_{L^2(\mathbb{R})} = (j f, j g)_{L^2(\mu)} = \int_{\mathbb{R}} f \overline{g} \, d\mu = (\mathcal{T}_{\mu, a} f, g), \quad (32)$$

by the unitarity of the operator $V_{\mu}$. It follows that $T_{\mu, a} = A^* A$. By construction, the operator $A$ is invertible. We also have $APW_r = PW_r$ for all $r \in [0, a]$, hence $A$ is upper-triangular.

(c) $\Rightarrow$ (b). Assume that $T_{\mu, a}$ admits a left triangular factorization $T_{\mu, a} = A^* A$. Define the operator $V_{\mu} : PW_a \rightarrow (PW_{\mu, a}, \mu)$ by $V_{\mu} = j A^{-1}$, where $j$ is the embedding from $PW_a$ to $(PW_a, \mu)$. Then $V_{\mu} PW_r = (PW_r, \mu)$ for every $r \in [0, a]$ and

$$(V_{\mu} f, V_{\mu} g)_{L^2(\mu)} = ((A^{-1})^* j^* A^{-1} f, g)_{L^2(\mathbb{R})} = ((A^*)^{-1} T_{\mu, a}^{-1} f, g)_{L^2(\mathbb{R})} = (f, g)_{L^2(\mathbb{R})},$$

where we used the identity $T_{\mu, a} = j^* j$, see (32). Since $A$ and $j$ are invertible, $V_{\mu}$ is a unitary operator. 

\[\square\]
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