The groups $G_n^2$ and Coxeter groups

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The groups $G_n^k$, which are closely related to braid groups, geometry, topology, and dynamical systems, were constructed in [1]. The word problem and the conjugacy problem for $G_n^k$ are extremely important. Already the groups $G_n^2$ of free braids (known also as groups of virtual Gauss braids [2]) are deeply non-trivial and have geometric meaning [3], [4], and an understanding of them can shed light on the groups $G_n^k$ with $k > 2$, because there are homomorphisms $G_n^k \rightarrow G_{n-1}^k$ and $G_n^k \rightarrow G_{n-1}^{k-1}$ (see [1] and [5]). The objective of this note is the construction of an explicit bijection between $G_n^2$ and the Coxeter group $C(n, 2)$, a bijection which is an isomorphism on finite-index subgroups of the two groups. This leads to an algebraic solution of the word problem in $G_n^2$. The final result is stated in the form of Theorem 3: the group $G_n^2$ can be embedded in a semidirect product of the Coxeter group $C(n, 2)$ and a permutation group. We first describe a ‘rewriting procedure’ giving a bijection between words in $C(n, 2)$ and words in $G_n^2$. A similar ‘rewriting’ algorithm was studied in [6]. The problem of an analogous representation of $G_n^k$ for $k > 2$ in terms of Coxeter groups remains open.

Let $\Gamma$ be a graph on $\binom{n}{2}$ vertices $\{b_{ij}\}$, where $i$ and $j$ range over all unordered pairs of distinct numbers in $\{1, \ldots, n\}$; two vertices are joined by an edge (of index 3) if and only if they have a single common index. The group $C(n, 2)$ is defined by the generators $b_{ij}$ and the families of generating relations

$$
(b_{ij}b_{ik})^3 = 1 \quad \forall i, j, k \in \{1, \ldots, n\}: \text{Card}\{\{i, j, k\}\} = 3; \quad (1)
$$

$$
b_{ij}b_{kl} = b_{kl}b_{ij} \quad \forall i, j, k, l \in \{1, \ldots, n\}: \text{Card}\{\{i, j, k, l\}\} = 4; \quad (2)
$$

$$
b_{ij}^2 = 1 \quad \forall i, j \in \{1, \ldots, n\}: i \neq j. \quad (3)
$$

For an integer $n > 2$ we define $G_n^2$ to be the group with $\binom{n}{2}$ generators $a_{ij}$ (for all possible unordered pairs $(i, j)$ of integers in $\{1, \ldots, n\}$) and the three families of defining relations:

$$
a_{ij}a_{ik}a_{jk} = a_{jk}a_{ik}a_{ij} \quad \forall i, j, k \in \{1, \ldots, n\}: \text{Card}\{\{i, j, k\}\} = 3; \quad (1')
$$

$$
a_{ij}a_{kl} = a_{kl}a_{ij} \quad \forall i, j, k, l \in \{1, \ldots, n\}: \text{Card}\{\{i, j, k, l\}\} = 4; \quad (2')
$$

$$
a_{ij}^2 = 1 \quad \forall i, j \in \{1, \ldots, n\}: i \neq j. \quad (3')
$$

We consider a homomorphism $l: G_n^2 \rightarrow S_n$ taking $a_{ij}$ to the transposition $(i, j)$ and a homomorphism $m: C(n, 2) \rightarrow S(n)$ taking $b_{ij}$ to $(i, j)$, and we let $C'(n, 2) = \text{Ker}(m)$ and $PG_n^2 = \text{Ker}(l)$.

Let $w = a_{i_1,1,1}a_{i_1,1,2} \cdots a_{i_k,1,1}a_{i_k,1,2}$ be a word defining an element in $G_n^2$. For $j = 1, \ldots, k$ we denote by $w_j$ the product of the first $j$ letters of $w$ and by $w_0$ the empty word. For $p = 1, \ldots, k$, we define the permutation $\sigma_p = l(w_p)^{-1}$ by setting $\sigma_0 = \text{id}$. We write $w = b_{\sigma_0(i_1,1),\sigma_0(i_1,2)}b_{\sigma_1(i_2,1),\sigma_1(i_2,2)} \cdots b_{\sigma_{k-1}(i_k,1),\sigma_{k-1}(i_k,2)}$.

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Theorem 1. If two words $w$ and $w'$ define equal elements of $G_n^2$, then the words $\bar{w}$ and $\bar{w}'$ generate equal elements of $C(n,2)$. Moreover, $l(w) = m(\bar{w})^{-1}$.

Proof. Three cases can be verified immediately: when $w'$ is connected with $w$ by a relation of the form $(1')$, a relation of the form $(2')$, or a relation of the form $(3')$. In each of these cases, in going from $\bar{w}$ to $\bar{w}'$ we obtain a corresponding relation of the group $C(n,2)$. The last assertion of the theorem is proved by induction: we begin with the transposition $(i_1,1, i_1,2)$, which is the same for the words $w$ and $\bar{w}$, and then we use the fact that $(ab)^{-1} = a^{-1}(ab^{-1}a^{-1})$.

We denote the map $PG_n^2 \rightarrow C(n,2)$, $w \mapsto \bar{w}$, constructed above by $co$; it is clear that $C'(n,2) = \text{Im}(PG_n^2)$. The inverse map is constructed similarly: we just take account of the fact that if $\bar{w} = co(w)$, then $l(w) = m(\bar{w})^{-1}$. Therefore, if we know $\bar{w}$, then we know all the permutations $\sigma_k(w)$ and $l(w)$ for the word $w$ that we are going to construct. Since for $w \in PG_n^2$ the permutation $l(w)$ is the identity, we have $(\bar{w}_1\bar{w}_2) = \bar{w}_1\bar{w}_2$ for $w_1, w_2 \in PG_n^2$. This leads to the following theorem.

Theorem 2. The map $co: w \mapsto \bar{w}$ is an isomorphism $PG_n^2 \rightarrow C'(n,2)$.

For Coxeter groups there is an algorithm of descent (for example, see [7]), which looks as follows. Let the Coxeter group $W$ be given by a system of generators-reflections $S = \{s_i\}$ and the relations $s_i^2 = 1$ and $(s_is_j)^{m_{ij}} = 1$. We have $s_is_j \cdots = s_js_i \cdots$ ($m_{ij}$ factors on the left- and right-hand sides). This relation of exchange is an elementary equivalence which does not change the length of a word. The relation $s_is_j = 1$ can be understood as an elementary equivalence changing the length by 2. A word $w$ in the generators $s_i$ is said to be reduced if its length is minimal among the lengths of all the words representing the same element of $W$. If $w$ is equivalent to a reduced word $w'$, then we can go from $w$ to $w'$ using only elementary transformations not increasing the length. In particular, if both the words $w$ and $w'$ are reduced, then we can go from $w$ to $w'$ by means of a sequence of exchanges. The rewriting algorithm is given by the following assertion.

Corollary 1. For a word $w$ in the alphabet $\{a_{ij}\}$ and a reduced word $w'$ equivalent to $w$ in $G_n^2$ one can pass from $w$ to $w'$ by a sequence of transformations $(2')$, exchanges of type $(1')$, and deletions of consecutive identical letters $a_{ij}a_{ij} \rightarrow \varnothing$. If the words $w$ and $w'$ are reduced, then one can pass from $w$ to $w'$ using only $(1')$ and $(2')$.

We consider the semidirect products $G_n^2 \rtimes S_n$ and $C(n,2) \rtimes S_n$, where $S_n$ acts on the generators $a_{ij}$ and $b_{ij}$ by permutations of indices.

Theorem 3. There exists an isomorphism between the semidirect products $C(n,2) \rtimes S_n$ and $G_n^2 \rtimes S_n$ that takes $(b_{ij}, 1)$ to $(a_{ij}, (i,j))$ for every transposition $(i,j)$ and preserves the pair $(1, \sigma)$. Thus, $G_n^2$ is isomorphic to the normal subgroup of $C(n,2) \rtimes S_n$ complementary to $S_n$.

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