ON THE SPECTRUM OF THE NEUMANN PROBLEM FOR LAPLACE EQUATION IN A DOMAIN WITH A NARROW SLIT

RUSTEM R. GADYL’SHIN† AND ARLEN M. IL’IN‡

Abstract. The Neumann problem in two-dimensional domain with a narrow slit is studied. The width of the slit is a small parameter. The complete asymptotic expansion for the eigenvalue of the perturbed problem converging to a simple eigenvalue of the limiting problem is constructed by means of the method of the matched asymptotic expansions. It is shown that the regular perturbation theory can formally be applied in a natural way up to terms of order \( \varepsilon^2 \). However, the result obtained in that way is false. The correct result can be obtained only by means of inner asymptotic expansion.

Bibliography: 8 titles.

Key words. singular perturbation, asymptotics, eigenvalues, Neumann problem.

AMS subject classifications. Primary 35C20; Secondary 35J25.

Introduction. The Neumann problem in a two-dimensional domain with a narrow slit is considered; it is called the perturbed problem in what follows. The slit’s width is a small parameter \( \varepsilon \). In the paper, we construct the complete asymptotics expansions for an eigenvalue converging to a simple eigenvalue of the limiting problem. The limiting problem is the Neumann problem in the domain without the segment which the slit shrinks to. The perturbed problem is singular; the asymptotics series in power of the small parameter for the eigenfunction is valid everywhere far from the endpoints of the segment and it fails near them. Besides, the coefficients of the outer expansion have increasing singularities near the segment’s endpoints. Moreover, below (in § 3) we shall show that the regular perturbation theory can be formally realized in a natural way up to a quantity \( \varepsilon^2 \). However, it turns out that the results obtained in such a way are not valid. Only using inner asymptotic expansion allows us to get correct results. In this paper, the construction of the asymptotics for an eigenvalue and of the uniform asymptotics for an eigenfunction is carried out by the method of matched asymptotics expansions [1]-[4].

1. Statement of the problem and formulation of the results. Let \( \Omega \) be a bounded simply connected domain in \( \mathbb{R}^2 \) with infinitely differentiable boundary \( \Gamma \), \( \omega_0 \) be the interval \((0, 1)\) in the axis \( Ox_1 \), \( \bar{\omega}_0 \subset \Omega \), \( \omega_\varepsilon = \{x: 0 < x_1 < 1, \varepsilon g_-(x_1) < x_2 < \varepsilon g_+(x_1)\} \), where \( 0 < \varepsilon \ll 1 \), \( g_\pm \in C^\infty(\omega_0) \), \( \pm g_\pm > 0 \). We assume that in a neighbourhood of the endpoints of the slit \( \omega_\varepsilon \) its boundary lies on the parabolas, i.e.,

\[
g_\pm(t) = \pm g^{-1/2} \text{ as } t < t_0, \quad g_\pm(t) = \pm g^+(1-t)^{1/2} \text{ as } t > 1 - t_0, \quad g^+ > 0,
\]

where \( t_0 > 0 \) is some fixed number. We denote \( \Omega_\delta = \Omega \setminus \overline{\omega_\delta} \), \( \delta \geq 0 \), \( \gamma_\varepsilon = \partial \omega_\varepsilon \); \( \gamma_0 \) is the cut \( \{x: 0 < x_1 < 1, x_2 = 0\} \) on the plain interpreted as double-sided, \( \Gamma_\delta = \Gamma \cup \gamma_\delta \). Under the notation introduced the limiting and perturbed problems can be written in the uniform way

\[
-\Delta \phi_\delta = \lambda_\delta \phi_\delta, \quad \phi_\delta \in \Omega_\delta, \quad \frac{\partial}{\partial \nu} \phi_\delta = 0, \quad \phi_\delta \in \Gamma_\delta,
\]  

(1.1)

*This work was supported by RFBR grants nos. 99-01-01143 and 99-01-00139.
†Bashkir State Pedagogical University, Ufa, Russia, (gadylshin@bspu.ru).
‡Institute of Mathematics and Mechanics of the Russian academy of Sciences, Ekaterinburg, Russia (iam@imm.uran.ru)
where \( \nu \) is the outer normal, \( \delta = 0 \) corresponds to the limiting problem, and \( \delta = \varepsilon > 0 \) does to the perturbed problem. It is convenient to consider the solutions of both the perturbed and limiting problem in the class of generalized solutions in Sobolev space \( H_1 \). We use the notation \( H_m(Q) \) for the Sobolev space of functions on \( Q \) whose derivatives of order less than or equal to \( m \) are square integrable.

Note that the Neumann boundary condition on the “outer” boundary \( \Gamma \) are chosen for the sake of unambiguousness and it is not principal for proofs used in paper. The only specific (but not principal) consequence of this choice is that the minimal perturbed eigenvalue equals zero.

We denote by \( \Sigma_\delta \) the set of the eigenvalues of the problem (1.1). In the second section, we shall prove the following statement.

**Theorem 1.1.** a) If \( K \) is any compact set in the complex plane such that \( K \cap \Sigma_0 = \emptyset \), then \( K \cap \Sigma_\varepsilon = \emptyset \) for all sufficiently small \( \varepsilon \);

b) If the multiplicity of \( \lambda_0 \in \Sigma_0 \) equals \( N \), then \( N \) eigenvalues of the perturbed problem (with multiplicities taken into account) converge to \( \lambda_0 \).

We denote by \( S_-(t) \) and \( S_+(t) \) the circles of radius \( t \) and centers at the points \( O_-(0,0) \) and \( O_+(1,0) \), respectively. For the sake of brevity we shall use the following notations \( x- = x, x_+ = ((1-x_1),x_2) \), \( (r_\pm, \theta_\pm) \) are associated polar coordinates. Below it will be shown that the limiting eigenfunction \( \phi_0 \) normalized in \( H_0(\Omega_0) \) and associated with simple eigenvalue \( \lambda_0 \) has the asymptotics (as \( r_\pm \to 0 \))

\[
\phi_0(x) = \phi_0(O_{\pm}) + d_{\pm} r_\pm^{1/2} \cos \left( \frac{\theta_\pm}{2} \right) + O(r_\pm).
\]

The main statement of the paper reads as follows.

**Theorem 1.2.** The asymptotics for the eigenvalue \( \lambda_\varepsilon \) of the perturbed problem converging to a simple eigenvalue \( \lambda_0 \) of the limiting problem and the asymptotics for the associated eigenfunction have the form

\[
\lambda_\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j \lambda_j, \quad (1.2)
\]

\[
\phi_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j(x), \quad x \in \Omega_\varepsilon \setminus (S_+(\varepsilon) \cup S_-(\varepsilon)), \quad (1.3)
\]

\[
\phi_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j v_+^j \left( \frac{x_+}{(g_{\pm}^{1/2})} \right), \quad x \in S_\pm(2\varepsilon), \quad (1.4)
\]

\[
\lambda_1 = \lambda_0 - \int_0^1 \left( g_+(x_1) \phi_0^2(x_1, +0) - g_-(x_1) \phi_0^2(x_1, -0) \right) dx_1 \quad (1.5)
\]

\[
\lambda_2 = \frac{\pi}{8} \left( (d_+g^+) + (d_-g^-) \right) + \tilde{\lambda}, \quad (1.6)
\]
\[ v_0^\pm(\xi) \equiv \phi_0(O_\pm), \quad v_1^\pm(\xi) = d_\pm g^\pm \text{Re} \left( \xi_1 + i\xi_2 - \frac{1}{4} \right)^{1/2} + \phi_1(O_\pm), \quad (1.7) \]

\[ \phi_2(x) = \frac{1}{8} \left( d_+ (g^+) \psi_+(x) + d_- (g^-) \psi_-(x) \right) + \tilde{\phi}(x), \quad (1.8) \]

where \( \phi_1, \tilde{\phi}, \) and \( \psi_\pm \) are the functions satisfying the statements of Lemmas 3.2, 3.3, and 4.4, respectively, \( \tilde{\lambda} \) is the constant determined by the equality (3.10), \( \xi = (\xi_1, \xi_2) \), and \( i \) is the imaginary unit.

The sections 3–8 are devoted to the construction and justification of the asymptotics (2.2)–(1.8) (i.e., to the complete proof of the Theorem 1.2). In the third section, the coefficients \( \lambda_1 \) and \( \phi_1 \) are defined by the regular theory of perturbation. In the fourth section, on the basis of the method of matched asymptotics expansions the coefficients \( \lambda_2, \phi_2 \), and two first couples of the coefficients \( v_0^\pm, v_1^\pm \) for the inner expansion in the neighbourhood of the slit’s endpoints are determined. In the fifth and sixth sections, we construct the complete outer and inner expansions (1.4) and (1.3), respectively. In the seventh section, it is shown that they can be matched. In the eighth section, the formally constructed asymptotics are justified what completes the proofs of Theorem 1.2. In the concluding ninth, section we discuss the cases of other boundary conditions on the boundary of the slit.

2. Proof of the Theorem 1.1. For a set \( Q \) we denote by \((\bullet, \bullet)_Q\) the scalar product in \( H_0(Q) \). The solution of the boundary value problem

\[ -\Delta u_\delta = \lambda u_\delta + f_\delta, \quad x \in \Omega_\delta, \quad \frac{\partial}{\partial n} \phi_\delta = 0, \quad x \in \Gamma_\delta, \quad (2.1) \]

where \( f_\delta \in H_0(\Omega_\delta) \) is the element \( u_\delta \in H_1(\Omega_\delta) \) satisfying the integral identity

\[ (\nabla u_\delta, \nabla v)_{\Omega_\delta} = (\lambda u_\delta + f_\delta, v)_{\Omega_\delta} \quad (2.2) \]

for each \( v \in H_1(\Omega_\delta) \). Hereinafter, the function in \( H_0(\Omega_\varepsilon) \) are assumed to be continued by zero inside \( \omega_\varepsilon \) and they and the functions in \( H_0(\Omega_0) \) are identified with the elements of \( H_0(\Omega) \). By \( \| \bullet \|_{m,Q} \) we denote the \( H_m(Q) \)-norm.

Beforehand we prove an auxiliary lemma being a convenient variant of well-known embedding theorems.

**Lemma 2.1.** Let a function \( w \in H_1(\Omega_\varepsilon) \), \( \Pi(\alpha) \) be a rectangle \( \{ x : -\alpha \leq x_1 \leq 1 + \alpha, \ x_2 \leq \alpha \} \), \( \Pi(\alpha, \varepsilon) = \Pi(\alpha) \cap \Omega_\varepsilon \), \( Q(\alpha) = \Pi(2\alpha) \setminus \Pi(\alpha) \). If parameters \( \alpha \) and \( \varepsilon \) are such that \( \Pi(2\alpha) \subset \Omega \) and \( \omega_\varepsilon \subset \Pi_\alpha \), then for all sufficiently small \( \varepsilon > 0 \) the estimate

\[ \| w \|_{0, \Pi(\alpha, \varepsilon)}^2 \leq 2 \| w \|_{0, Q(\alpha)}^2 + 4\alpha^2 \| w \|_{1, \Omega_\varepsilon}^2 \quad (2.3) \]

holds.

**Proof.** At first let us consider the values \( x_2 \geq 0 \). We set \( \Pi^+(\alpha) = \Pi(\alpha) \cap \{ x : x_2 > 0 \}, \Pi^+(\alpha, \varepsilon) = \Pi(\alpha, \varepsilon) \cap \{ x : x_2 > 0 \}, Q_+^+(\alpha) = \Pi_+^+(2\alpha) \setminus \Pi_+^+(\alpha) \). Due to the density of embedding \( C^\infty(\overline{\Omega_\varepsilon}) \) in \( H_1(\Omega_\varepsilon) \) we may suppose that \( w \in C^\infty(\overline{\Omega_\varepsilon}) \). Then for each \( x_1 \in [-\alpha, 1 + \alpha] \) there exists a point \( z \in [\alpha, 2\alpha] \) such that

\[ |w(x_1, z)| \leq \frac{1}{\alpha} \int_\alpha^{2\alpha} |w(x_1, \eta)| d\eta. \]
Hence, for each point \((x_1, x_2) \in \Pi_+(\alpha, \varepsilon)\)

\[
|w(x_1, x_2)| \leq \frac{1}{\alpha} \int_0^{2\alpha} |w(x_1, \eta)|d\eta + \int_{x_2}^z \frac{\partial w}{\partial \eta}(x_1, \eta)d\eta,
\]

\[
|w(x_1, x_2)|^2 \leq \frac{2}{\alpha} \int_0^{2\alpha} |w(x_1, \eta)|^2d\eta + 4\alpha \int_{x_2}^z |\nabla w(x_1, \eta)|^2d\eta.
\tag{2.4}
\]

Let \(\bar{g}_+(x_1)\) be the function that equals \(g_+(x_1)\) for \(0 < x_1 < 1\) and vanishes for \(-\alpha \leq x_1 \leq 0\) and for \(1 \leq x_1 \leq 1 + \alpha\). If we integrate inequality (2.4) with respect to \(x_2\) from \(\varepsilon \bar{g}_+(x_1)\) to \(\alpha\), and after that we integrate the inequality obtained with respect to \(x_1\) from \(-\alpha\) to \(1 + \alpha\), and we take into account that similar estimates are true for \(x_2 \leq 0\), then we get estimate (2.3).

**Lemma 2.2.** Let condition a) of Theorem 1.1 hold. Then

a) the statement a) of Theorem 1.1 is valid;
b) for each \(\lambda \in K\), \(f_\varepsilon \in H_0(\Omega_\varepsilon)\) the solutions of the boundary value problem (2.1) satisfy the uniform estimate

\[
\|u_\varepsilon\|_{1, \Omega_\varepsilon} \leq C\|f_\varepsilon\|_{0, \Omega}\tag{2.5}
\]

**Proof.** It is easily seen that the equality (2.2) yields the a priori uniform estimate

\[
\|u_\varepsilon\|_{1, \Omega_\varepsilon} \leq C_1 (\|u_\varepsilon\|_{0, \Omega} + \|f_\varepsilon\|_{0, \Omega})\tag{2.6}
\]

We prove the item b) by arguing by contradiction. Suppose that there exist sequences \(\lambda^{(n)}\) and \(\varepsilon_n\) such that the inequalities

\[
\|u_{\varepsilon_n}\|_{1, \Omega_\varepsilon} \geq n\|f_\varepsilon\|_{0, \Omega}\quad \tag{2.7}
\]

hold for \(\lambda = \lambda^{(n)}\) and some \(f_\varepsilon\). Without loss of generality, we may assume that \(\|u_{\varepsilon_n}\|_{0, \Omega} = 1\), \(u_{\varepsilon_n} \to u_0\) weakly in \(H_0(\Omega)\), \(\varepsilon_n \to \varepsilon_0\) and \(\lambda^{(n)} \to \lambda^{(0)} \in K\) such that \(\varepsilon_n \to 0\). We also assume that \(\varepsilon_0 = 0\). We shall prove both the items a) and b) simultaneously. Indeed, if the item a) is wrong, then there exist sequences of eigenfunctions \(u_{\varepsilon_n}\) and eigenvalues \(\lambda^{(n)} \to \lambda^{(0)} \in K\) such that \(\varepsilon_n \to 0\). Obviously, inequality (2.7) is correct for the eigenfunctions. If the item a) is valid, \(\varepsilon_0 \neq 0\) and it is sufficiently small, then \(\lambda^{(0)}\) is not an eigenvalue of the problem for \(\varepsilon = \varepsilon_0\). Then the uniform estimate (2.5) follows from the well-known a priori estimates for the solutions of the elliptic equations in a domain with a smooth boundary. Thus, \(\varepsilon_n \to 0\).

From (2.6) and (2.7) it follows that

\[
\|u_{\varepsilon_n}\|_{1, \Omega_\varepsilon} \leq C_2\tag{2.8}
\]

Observe that estimates (2.7) and (2.8) yield the convergence to zero of \(f_\varepsilon\) in \(H_0(\Omega)\)-norm. In the proofs of this and next lemmas we denote by \(M\) any compact set \(M \subset \overline{\Omega}\) separated from the segment \(\omega_0\). We select the subsequence from the sequence \(u_{\varepsilon_n}\) which converges to the function \(u_0\) in \(H_1\)-norm on \(M\). For this subsequence we use the former notation \(u_{\varepsilon_n}\). The existence of this subsequence follows from the convergence to zero of \(f_\varepsilon\) in \(H_0(\Omega)\)-norm and from the well-known a priori estimates.
for the solutions of an elliptic equation in \( M \). Note that \( u_0 \in H^1(\Omega_0) \) due to the uniform boundedness (2.8).

Let us show that \( u_0 \) is a solution of the limiting problem, i.e., for each \( v \in H^1(\Omega_0) \) the identity

\[
(\nabla u_0, \nabla v)_{\Omega_0} = (\lambda^{(0)} u_0, v)_{\Omega_0}
\]

holds. By definition, the identity

\[
(\nabla u_{\varepsilon_n}, \nabla v)_{\Omega_{\varepsilon_n}} = (\lambda^{(n)} u_{\varepsilon_n} + f_{\varepsilon_n}, v)_{\Omega_{\varepsilon_n}}
\]

holds. By the weak convergence \( u_{\varepsilon_n} \rightarrow u_0 \) in \( H_0(\Omega) \) we deduce the convergence

\[
(\lambda^{(n)} u_{\varepsilon_n} + f_{\varepsilon_n}, v)_{\Omega_{\varepsilon_n}} \rightarrow (\lambda^{(0)} u_0, v)_{\Omega_0}.
\]

For all \( \mu > \varepsilon \) we have the obvious equality

\[
(\nabla u_{\varepsilon}, \nabla v)_{\Omega_\varepsilon} - (\nabla u_0, \nabla v)_{\Omega_0} = \left( (\nabla (u_{\varepsilon} - u_0), \nabla v)_{\Omega_\varepsilon} + (\nabla (u_{\varepsilon} - u_0), \nabla v)_{\Omega_{\varepsilon_n} \setminus \Omega_n} - (\nabla u_0, \nabla v)_{\Omega_0 \setminus \Omega_n} \right).
\]

From (2.12), the arbitrariness in choosing \( \mu \), the estimate (2.8), and the convergence \( u_{\varepsilon_n} \) in \( H^1 \)-norm on each compact set \( M \) we derive the convergence

\[
(\nabla u_{\varepsilon_n}, \nabla v)_{\Omega_{\varepsilon_n}} \rightarrow (\nabla u_0, \nabla v)_{\Omega_0}.
\]

Assertions (2.10), (2.11), and (2.13) yield (2.9).

Let us show that \( u_0 \neq 0 \). Suppose contrary, i.e., that \( u_{\varepsilon_n} \rightarrow 0 \) in \( H^1 \)-norm on each compact set \( M \). Then from this assumption and estimate (2.3), the arbitrariness in choosing \( \alpha \) and estimate (2.3) we deduce a convergence \( u_{\varepsilon_n} \rightarrow 0 \) in \( H_0(\Omega) \), what contradicts the normalization of \( u_{\varepsilon_n} \) in \( H_0(\Omega) \). Hence, \( u_0 \) is a nontrivial solution of the limiting problem

\[-\Delta u_0 = \lambda^{(0)} u_0, \quad x \in \Omega_0, \quad \frac{\partial}{\partial \nu} u_0 = 0, \quad x \in \Gamma_0,
\]

what is impossible since \( \lambda^{(0)} \notin \Sigma_0 \). The latter contradiction proves the lemma. \( \blacksquare \)

**Lemma 2.3.** Let \( K \) be the compact set described in the formulation of Lemma 2.2 and \( f_{\varepsilon} \rightarrow f_0 \) in \( H_0(\Omega) \) as \( \varepsilon \rightarrow 0 \). Then the solution of the perturbed problem (2.4) converges to the solution of the limiting problem as \( \varepsilon \rightarrow 0 \) in \( H^1 \)-norm on each compact set \( M \subset \overline{\Omega} \) separated from the segment \( \omega_\varepsilon \) and in \( H_0(\Omega) \)-norm uniformly on \( \lambda \in K \).

**Proof.** In view of estimate (2.3) the proof of the convergence in \( H^1 \)-norm for each fixed \( \lambda \) on each compact set \( M \) reproduces the proof of previous lemma. From this convergence and estimate (2.3) we obtain the convergence in \( H_0(\Omega) \) for each fixed \( \lambda \). In its turn, the latter convergence and estimate (2.3) imply uniform convergences in required norms in obvious way. \( \blacksquare \)

**Proof of Theorem 1.1.** Recall that the validity of the item a) of Theorem 1.1 was shown in Lemma 2.2. Thus, it remains to prove the item b). We denote by \( \Lambda(\mu) \) a closed circle in \( \lambda \)-plane with radius \( \mu \) and center at \( \lambda_0 \). We choose \( \mu \) sufficiently small to satisfy \( \Lambda(\mu) \cap \Sigma_0 = \{ \lambda_0 \} \). The existence of such \( \mu \) follows from the item a) of Theorem 1.1. Let \( \phi_0 \) be an eigenfunction associated with \( \lambda_0 \). We set \( f_0 = \phi_0, f_\varepsilon = f_0 \) in \( \Omega_\varepsilon \) and \( f_\varepsilon = 0 \) in \( \omega_\varepsilon \). Then by Lemma 2.3

\[
\int_{\partial \Lambda(\mu)} u_\varepsilon(\bullet, \lambda) d\lambda \rightarrow \int_{\partial \Lambda(\mu)} u_0(\bullet, \lambda) d\lambda \neq 0, \quad \varepsilon \rightarrow 0.
\]
This assertion by the arbitrariness in choosing \( \mu \) yields the existence of an eigenvalue \( \lambda_\varepsilon \) of the perturbed problem converging to \( \lambda_0 \).

Let us show that the total multiplicity of the eigenvalues of the perturbed problem converging to \( \lambda_0 \) equals \( N \). We choose sequence \( \varepsilon_n \to 0 \) such that for \( \varepsilon = \varepsilon_n \) there exist \( L \) eigenfunctions \( \{ \phi^{(i)}_\varepsilon \}_{i=1}^L \) associated with eigenvalues converging to \( \lambda_0 \). There is no loss of generality in assuming that they are orthonormalized in \( H_0(\Omega) \). By analogy with the proofs Lemmas 2.2, 2.3 it is easy to show the existence of the subsequence for which \( \phi^{(i)}_\varepsilon \to \phi^{(j)}_0 \not= 0 \) in \( H_0(\Omega) \), where \( \phi^{(j)}_0 \) are orthonormalized in \( H_0(\Omega) \) eigenfunctions of the limiting problem associated with \( \lambda_0 \). Since the multiplicity of \( \lambda_0 \) equals \( N \), we have an inequality \( L \le N \). Suppose that \( L < N \). Then there exists an eigenfunction \( \phi^{(L+1)}_0 \) of the limiting problem orthogonal to \( \phi^{(j)}_0 \) for \( j \le L \). We set

\[
   f_\varepsilon = \phi^{(L+1)}_0 - \sum_{i=1}^L \langle \phi^{(L+1)}_0, \phi^{(i)}_\varepsilon \rangle_{\Omega} \phi^{(i)}_\varepsilon, \quad x \in \Omega_\varepsilon, \quad f_\varepsilon = 0, \quad x \in \omega_\varepsilon.
\]

By definition,

\[
   (f_\varepsilon, \phi^{(i)}_\varepsilon)_\Omega = 0, \quad i \le L, \quad f_\varepsilon \to f_0 = \phi^{(L+1)}_0, \quad \varepsilon \to 0.
\] (2.14)

Lemma 2.3 and assertions (2.14) yield

\[
   0 = \int_{\partial \Lambda(\mu)} u_\varepsilon(\bullet, \lambda) d\lambda \to \int_{\partial \Lambda(\mu)} u_0(\bullet, \lambda) d\lambda \neq 0, \quad \varepsilon \to 0.
\]

Due to contradiction obtained, \( L = N \).

**Lemma 2.4.** Let \( \lambda_0 \) be a simple eigenvalue of the limiting problem. Then an eigenfunction \( \phi_\varepsilon \) associated with \( \lambda_\varepsilon \to \lambda_0 \) converges to an eigenfunction \( \phi_0 \) of the limiting problem in \( H_0(\Omega) \).

**Proof.** Let \( f_0 = \phi_0, \quad f_\varepsilon = f_0 \) in \( \Omega_\varepsilon \), and \( f_\varepsilon = 0 \) in \( \omega_\varepsilon \). Then by Lemma 2.3 and Theorem 1.1, we get the convergence as \( \varepsilon \to 0 \)

\[
   \alpha_\varepsilon \phi_\varepsilon = \int_{\partial \Lambda(\mu)} u_\varepsilon(\bullet, \lambda) d\lambda \to \int_{\partial \Lambda(\mu)} u_0(\bullet, \lambda) d\lambda = \alpha_0 \phi_0 \neq 0.
\]

Hence, \( \phi_\varepsilon \to \phi_0 \).

In justification of the asymptotics for the eigenelements constructed in the next sections we need uniform on \( \lambda \) and \( \varepsilon \) estimates for the solutions of the perturbed problem for \( \lambda \) close to a simple eigenvalue of the limiting problem. By analogy with Lemmas 2.2, 2.3 one can prove

**Lemma 2.5.** Let \( \lambda_0 \) be a simple eigenvalue of the limiting problem, \( u_\varepsilon \) be the solution of (2.4) for \( \lambda = \lambda_\varepsilon \), \( \phi_\varepsilon \) be an associated eigenfunction and \( (u_\varepsilon, \phi_\varepsilon)_\Omega = 0 \). Then the estimate

\[
   \| u_\varepsilon \|_{1, \Omega_\varepsilon} \le C \| f_\varepsilon \|_{0, \Omega},
\]

holds, where the constant \( C \) does not depend on \( \varepsilon \).

**Lemma 2.6.** For \( \lambda \) close to a simple eigenvalue \( \lambda_0 \) for the solution of the problem (2.4) the uniform estimates

\[
   \| \lambda_\varepsilon - \lambda \| (u_\varepsilon, \phi_\varepsilon)_\Omega \le C \| f_\varepsilon \|_{0, \Omega},
\]

\[
   \| u_\varepsilon - (u_\varepsilon, \phi_\varepsilon)\Omega \phi_\varepsilon \|_{1, \Omega_\varepsilon} \le C_1 (\| f_\varepsilon \|_{0, \Omega} + |\lambda - \lambda_\varepsilon||u_\varepsilon\|_{0, \Omega}), \quad (2.15)
\]

\[
   \| u_\varepsilon - (u_\varepsilon, \phi_\varepsilon)\Omega \phi_\varepsilon \|_{1, \Omega_\varepsilon} \le C_2 |\lambda - \lambda_\varepsilon|^{1/2} u_\varepsilon\|_{0, \Omega}, \quad (2.16)
\]
where \( \phi_e \) is an eigenfunction of the perturbed problem normalized in \( H_0(\Omega) \).

Proof. Substituting \( v = \phi_e \) into assertion [2.2], bearing in mind the equality \((\nabla \phi_e, \nabla u_e)_{\Omega} = \lambda_e (\phi_e, u_e)_{\Omega}\) and the fact that \( \phi_e \) and \( \lambda_e \) are real-valued, we obtain estimate (2.13). In its turn, applying Lemma 2.5 to the function \( u_e - (u_e, \phi_e)_{H_0} \phi_e \), we obtain estimate (2.16). \( \square \)

3. Regular perturbation theory. Hereinafter, we deal with a simple eigenvalue \( \lambda_0 \), we shall not fix this fact additionally in what follows. Lemma 2.4 implies that the eigenfunction \( \phi_e \) converges to \( \phi_0 \). For this reason, the leading term of the asymptotics for \( \phi_e \) is \( \phi_0 \). It is naturally to seek the asymptotics for \( \lambda_e \) and \( \phi_e \) as \([1.2]\) and \([1.3]\).

The boundary condition on \( \gamma_e \) in \([1.1]\) has the form
\[
\varepsilon g_{\pm}^\prime(x_1) \frac{\partial \phi_e}{\partial x_2}(x_1, \varepsilon g_{\pm}(x_1)) - \frac{\partial \phi_e}{\partial x_2}(x_1, \varepsilon g_{\pm}(x_1)) = 0. 
\]

Substituting \([1.2]\) and \([1.3]\) into \([1.4]\), due to \([3.1]\) we arrive at the boundary value problems for the coefficients \( \phi_j \):
as \( r \to 0 \) and \( x_1 \to 0 \) and similar expansions as \((1 - x_1)^2 + x_2^2 \to 0\). Further, let \( u \in H_1(\Omega_0) \) be a solution of the boundary value problem

\[
-(\Delta + \lambda)u = f, \quad x \in \Omega_0, \quad \frac{\partial u}{\partial \nu} = 0, \quad x \in \Gamma,
\]

\[
\frac{\partial^2 u}{\partial x_2^2} = h_\pm, \quad x_1 \in \omega_0, \quad x_2 = \pm 0.
\]

Then \( u \in C^\infty(\Omega_0 \setminus \{O_+; O_+\}) \) has the asymptotics of the form (3.8) in the vicinity of the endpoints of the cut \( \gamma_0 \).

Hereinafter, the asymptotics series are assumed to be infinitely differentiable with respect to the variables \( x_1 \) and \( x_2 \).

**COROLLARY.** The eigenfunction \( \phi_0 \) has the expansion of the form (3.3) in the vicinity of the endpoints of the cut \( \gamma_0 \), where \( \Psi_0 = \phi_0(O_+) \), \( \Psi_1(\theta) = d_\pm \cos(\theta/2) \) in the vicinity of the left and right endpoints.

**LEMMA 3.2.** There exists a function \( \phi_1 \in H_1(\Omega_0) \cap C^\infty(\overline{\Omega_0} \setminus \{O_+; O_+\}) \), having asymptotics (3.3) that is a solution of the boundary value problem (3.3) for \( \lambda_1 \) determined by the equality (1.5) and is orthogonal to \( \phi_0 \) in \( H_0(\Omega_0) \).

**Proof.** It follows from Corollary to Lemma 3.1 that the right hand side of equation (3.3) satisfies all assumptions of Lemma 3.1. By the equation in (3.2),

\[
\left( g_{\pm} \frac{\partial}{\partial x_1} - g_{\pm} \frac{\partial^2}{\partial x_2^2} \right) \phi_0 \bigg|_{x_2 = \pm 0} = \left( \frac{\partial}{\partial x_1} \left( g_{\pm} \frac{\partial}{\partial x_1} \right) + \lambda_0 \right) \phi_0 \bigg|_{x_2 = \pm 0}. \tag{3.7}
\]

From (3.7) and Corollary to Lemma 3.1 we deduce that the right hand side of the boundary condition in (3.3) satisfies the hypothesis of Lemma 3.1. Therefore, for each \( \lambda \) distinct from an eigenvalue there exists a solution of a boundary value problem (3.3) in the form \( \phi_1 = u + w \), we obtain the following boundary value problem for the function \( w \)

\[
-(\Delta + \lambda_0)w = (\lambda_0 - \lambda)u + \lambda_1 \phi_0, \quad x \in \Omega_0, \quad \frac{\partial w}{\partial \nu} = 0, \quad x \in \Gamma_0. \tag{3.8}
\]

The necessary and sufficient solvability condition for (3.3) is the orthogonality in \( H_0(\Omega) \) of the right hand side and \( \phi_0 \); we achieve it by a suitable choice of the constant \( \lambda_1 \). Let us assume that \( \lambda_1 \) is chosen exactly in this way. Since the right hand side of the equation in (3.8) belongs to \( H_1(\Omega_0) \cap C^\infty(\overline{\Omega_0} \setminus \{O_+; O_+\}) \), it follows that, by Lemma 3.1, the function \( \phi_1 \) satisfies the statements of the lemma being proved. Integrating by parts the left hand side of the equality

\[
-(\Delta + \lambda_0)\phi_1, \phi_0)_\Omega = \lambda_1 (\phi_0, \phi_0)_\Omega = \lambda_1
\]

and taking into account (3.7), we have

\[
\lambda_1 = \frac{1}{\Omega_0} \left[ \phi_0(x_1, +0) \left( \frac{d}{dx_1} \left( g_+(x_1) \frac{d}{dx_1} + \lambda_0 \right) \phi_0(x_1, +0) \right. \right.
\]

\[
- \left. \phi_0(x_1, -0) \left( \frac{d}{dx_1} \left( g_-(x_1) \frac{d}{dx_1} + \lambda_0 \right) \phi_0(x_1, -0) \right) \right] dx_1.
\]
In its turn, integrating by parts the right hand side of latter equality one can obtain relation (3.3). Since the function $\phi_i$ is defined up to the term $a \phi_0$, for definiteness we choose the constant $a$ on the basis of the orthogonality condition $(\phi_1, \phi_0)_{\Omega} = 0$.

Observe that by Corollary to Lemma 3.1 and Lemma 3.2 the right hand side in equation (3.4) satisfies all assumptions of Lemma 3.1, and by (3.2) and (3.3)

$$
\left( g_\pm \frac{\partial}{\partial x_1} - g_\pm \frac{\partial^2}{\partial x_2^2} \right) \phi_1 \bigg|_{x_2 = \pm 0} + \left( g_\pm g_\pm \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{1}{2} g_\pm^2 \frac{\partial^3}{\partial x_2^3} \right) \phi_0 \bigg|_{x_2 = \pm 0} = \left( \frac{\partial}{\partial x_1} (g_\pm \frac{\partial}{\partial x_1}) + \lambda_0 \right) \phi_1 \bigg|_{x_2 = \pm 0}.
$$

Hence, by Lemma 3.2 the boundary condition in (3.4) satisfies the hypothesis of Lemma 3.3. Reproducing the proof of Lemma 3.2 and taking into account the equality (3.9), we get the validity of the following analog of Lemma 3.2.

**LEMMA 3.3.** There exists a function $\tilde{\phi} \in H_1(\Omega_0) \cap C^\infty(\Omega_0 \setminus \{O_-; O_+\})$, having asymptotics (3.1), and being a solution of the boundary value problem (3.4) for $\lambda_2 = \lambda$, defined by the equality

$$
\lambda_2 = \frac{1}{\lambda_1} \int_0^1 \left( g_+ (x_1) \phi_0^2 (x_1, +0) - g_- (x_1) \phi_0^2 (x_1, -0) \right) dx_1
$$

$$
+ \lambda_0 \int_0^1 \left( g_+ (x_1) \phi_0 (x_1, +0) \phi_1 (x_1, +0) - g_- (x_1) \phi_0 (x_1, -0) \phi_0 (x_1, -0) \right) dx_1
$$

$$
- \int_0^1 \left( g_+ (x_1) \frac{d}{dx_1} \phi_0 (x_1, +0) \frac{d}{dx_1} \phi_1 (x_1, +0) \right)
$$

$$
- g_- (x_1) \frac{d}{dx_1} \phi_0 (x_1, -0) \frac{d}{dx_1} \phi_1 (x_1, -0) \right) dx_1.
$$

**REMARK 3.1.** Below, the construction (and justification) of complete asymptotics expansions for the eigenelements will imply that the coefficients $\phi$ and $\lambda_1$ obtained above are correct. From the formal point of view, by Lemma 3.3 it can be set $\phi_2 = \phi$ and $\lambda_2 = \lambda$. However, as we shall show below, the values $\lambda_2 = \lambda$ and $\phi_2 = \phi$ are wrong.

4. **Construction of the second terms of the asymptotics by the method of matched asymptotics expansions.** In order to define correct terms $\lambda_2$ and $\phi_2$ one should use inner asymptotics expansions near the endpoints of the segment. The form of this expansions is defined by already constructed $\phi_0$ and $\phi_1$ (in accordance with the method of matched asymptotics expansions). By Lemmas 3.1, 3.2, and Corollary to Lemma 3.1 the asymptotics

$$
\phi_0 (x) = \phi_0^+ + d_\pm r_\pm^{1/2} \cos \left( \frac{\theta_\pm}{2} \right) + O(r_\pm), \quad \phi_1 (x) = \phi_1^+ + O \left( r_\pm^{1/2} \right),
$$

hold at the endpoints of the cut, where $\phi_i^\pm = \phi_i (O_\pm)$.

In the vicinity of the endpoints of the slit, we seek asymptotics for the solution in the form of a power series in $\varepsilon$ whose coefficients are functions depending on the
scaled (inner) variables $\xi = x (g^{-})^{-2} \varepsilon^{-2}$ in the vicinity of the left endpoint and $\xi_1 = (1 - x_1) (g^{+})^{-2} \varepsilon^{-2}$ in the vicinity of the right endpoint.

Rewriting asymptotics for $\phi_j$ at the ends of the slit in terms of the inner variables, we obtain that

$$\phi_0(x) + \varepsilon \phi_1(x) = \phi_0^\pm + \varepsilon \left( d_{\pm} g^\pm \rho^{1/2} \cos(\theta/2) - \phi_1^\pm \right) + O(\varepsilon^2),$$

where $(\rho, \theta)$ are polar coordinates in the plane $\xi = (\xi_1, \xi_2)$.

Equality (4.2) suggests that in the vicinity of the endpoints of the slit the eigenfunction expressed in terms of the inner variables must have the form

$$\phi_\varepsilon(x) = v_0^\pm(\xi) + \varepsilon v_1^\pm(\xi) + O(\varepsilon^2).$$

The boundary value problems for $v_j^\pm$ are obtained in the standard way [3], [7]. We substitute (4.3) and (1.2) into (1.1) and pass to the inner variables in the equation and boundary conditions bearing in mind that near the ends the equation of the slit has the form $\xi_1 = \xi_2^2$. Equating the coefficients of the least powers of $\varepsilon$, we get the following boundary value problems

$$\Delta v_j^\pm = 0, \quad \xi \in \Pi, \quad \frac{\partial v_j^\pm}{\partial \nu} = 0, \quad \xi \in \partial \Pi,$$

where $\Pi = \{\xi : \xi_1 < \xi_2^2\}$.

Due to relation (4.2) (and the ideology of the method of matched asymptotics expansions) the solutions of (4.4) must have asymptotics

$$v_0^\pm(\xi) = \phi_0^\pm + o(1), \quad v_1^\pm(\xi) = d_{\pm} g^\pm \rho^{1/2} \cos(\theta/2) + \phi_1^\pm + o(1), \quad \rho \to \infty.$$ (4.5)

It is easy to see that the functions (4.7), where the cut is made along the ray $(1/4, \infty)$ of the real axis, are the solutions of the problem (4.4), having asymptotics (4.5). Moreover, from (4.7) it follows that

$$v_1^\pm(\xi) = d_{\pm} g^\pm \rho^{1/2} \cos(\theta/2) + \phi_1^\pm - \frac{1}{8} d_{\pm} g^\pm \rho^{-1/2} \cos(\theta/2) + O(\rho^{-3/2}), \quad \rho \to \infty.$$ (4.6)

Rewriting now the asymptotics for $v_0^\pm + \varepsilon v_1^\pm$ at infinity in terms of the outer variables $x$, from (4.7) and (4.4) we deduce the leading terms of the asymptotics at the endpoints of the slit for the coefficient $\phi_2$ of the series (4.3)

$$\phi_2(x) = -\frac{1}{8} d_- (g^-)^2 r^{-1/2} \cos(\theta/2) + O(1), \quad r \to 0$$ (4.7)

at the left endpoint and a similar form holds at the right endpoint.

**Remark 4.1.** From the asymptotics (4.7) it follows that $\phi_2$ does not belong to the class $H_1(\Omega_0)$ (in the general case $|d_+| + |d_-| \neq 0$). For this very reason the formally consistent regular second terms mentioned in Remark 3.1 lead one to wrong values of the required quantities.

Let us proceed to the construction of the correct terms $\phi_2$ and $\lambda_2$ of the expansions (4.2) and (4.3). To this end, one should modify $\tilde{\phi}$ and $\tilde{\lambda}$ constructed in Lemma 3.3 by taking into account the asymptotics (4.7). In other words, we should add the singular term $\text{const} r^{-1/2} \cos(\theta/2)$ to the function $\tilde{\phi}(x)$ near the left endpoint of the slit and

10

R.R. Gadyl’shin AND A. M. Il’in
On the spectrum of the Neumann problem...

In order that the function \( \phi_2 \) remains the solution of the problem (3.4) we must add an additional term belonging to \( H_1(\Omega_0) \).

We use the notation \( \chi(t) \) for the infinitely differentiable cut-off function equal to one for \( t < c \) and to zero for \( t > 2c \), where \( c < 1/2 \) is sufficiently small number so that the closed circles of radius \( 2c \) with centers at \((0,0)\) and \((1,0)\) lie in \( \Omega \).

**Lemma 4.1.** There exist functions \( \psi_{\pm}(x) = \chi(r_{\pm})r_{\pm}^{-1/2}\cos(\theta_{\pm}/2) + \widetilde{\psi}_{\pm}(x) \),

where \( \widetilde{\psi}_{\pm} \in H_1(\Omega_0) \cap C^\infty(\overline{\Omega_0} \setminus \{O_-; O_+\}) \), that are solutions of a boundary value problem

\[-(\Delta + \lambda_0)\psi_{\pm} = \lambda_{\pm}\phi_0, \quad x \in \Omega_0, \quad \frac{\partial \psi_{\pm}}{\partial \nu} = 0, \quad x \in \Gamma_0\]

for

\[\lambda_{\pm} = -\pi d_{\pm}\]

**Proof.** Let us seek \( \widetilde{\psi}_{\pm} \) in the form

\[\widetilde{\psi}_{\pm}(x) = -\frac{\lambda_0}{2}\chi(r_{\pm})r_{\pm}^{3/2}\cos(\theta_{\pm}/2) + \hat{\psi}_{\pm}(x)\]

Substituting the expression (4.8) into (4.9), we arrive at the following problem for \( \hat{\psi}_{\pm} \):

\[-(\Delta + \lambda_0)\hat{\psi}_{\pm} = \lambda_{\pm}\phi_0 + f_{\pm}, \quad x \in \Omega_0, \quad \frac{\partial \hat{\psi}_{\pm}}{\partial \nu} = 0, \quad x \in \Gamma_0\]

where \( f_{\pm} \in H_1(\Omega_0) \cap C^\infty(\overline{\Omega_0} \setminus \{O_-; O_+\}) \) have the asymptotics (3.5). A sufficient (and necessary) condition for solvability of (4.11) in \( H_1(\Omega_0) \) is the equality \( \lambda_{\pm} = -(f_{\pm}, \phi_0)_{\Omega_0} \).

It remains to get the relations (4.10). We denote \( B_{\pm}(t) = \Omega_0 \setminus S_{\pm}(t) \). The functions \( \widetilde{\psi}_{\pm} \) satisfy the statements of Lemma 3.1; integrating by parts the left hand sides of the equalities

\[-((\Delta + \lambda_0)\psi_{\pm}, \phi_0)_{B_{\pm}(t)} = \lambda_{\pm}(\phi_0, \phi_0)_{B_{\pm}(t)}\]

taking into account (4.8) and the asymptotics \( \phi_0(x) \) and passing to the limit as \( t \to 0 \), we obtain the relations (4.10).

In view of Lemmas 4.1 and 3.3 the function \( \phi_2 \) defined by the equality (4.8) is a solution of the boundary value problem (3.4) for \( \lambda_2 \) defined by the equality (1.6).

Thus, the coefficients \( \lambda_2, \phi_2, v_0^\pm, v_1^\pm \) of the series (1.1)–(1.3) satisfying the statement of Theorem 1.2 have been constructed.

5. **Inner expansion.** We shall seek the complete inner expansion for the eigenfunction in the form (1.4). Since the following construction of the coefficients of the inner expansions is the same for both endpoints of the slit, then, firstly, we consider only expansions at the left endpoint, and, secondly, to avoid cumbersome expressions we omit the superscripts “±” where possible. Substituting (1.2) and (1.4) into (1.1),

a similar term near the right endpoint of the slit. However, this is not sufficient.

We use the notation \( \chi(t) \) for the infinitely differentiable cut-off function equal to one for \( t < c \) and to zero for \( t > 2c \), where \( c < 1/2 \) is sufficiently small number so that the closed circles of radius \( 2c \) with centers at \((0,0)\) and \((1,0)\) lie in \( \Omega \).
passing to the inner variables and writing down the equality of the same power of $\varepsilon$, one can obtain the following recursive system of the boundary value problems

$$-\Delta v_n = \sum_{k=0}^{n-4} \lambda_k v_{n-k-4}, \quad \xi \in \Pi, \quad \frac{\partial v_n}{\partial \nu} = 0, \quad \xi \in \partial \Pi. \quad (5.1)$$

We denote by $\zeta_1$ and $\zeta_2$ the real and imaginary parts of the complex variable $w = \sqrt{\xi_1 - \frac{1}{4} + i\xi_2}$, $i$ is the imaginary unit. In variables $\zeta = (\zeta_1, \zeta_2)$ the boundary value problem (5.1) becomes simpler

$$-\Delta \zeta v_n = |\zeta|^2 \sum_{k=0}^{n-4} \lambda_k v_{n-k-4}, \quad \zeta_2 > \frac{1}{2}, \quad (5.2)$$

$$\frac{\partial v_n}{\partial \zeta_2} = 0, \quad \zeta_2 = \frac{1}{2}. \quad (5.3)$$

**Lemma 5.1.** For each natural $k$

a) the boundary value problem

$$\Delta v = 0, \quad \zeta_2 > \frac{1}{2}, \quad \frac{\partial v}{\partial \zeta_2} = \zeta_1^k, \quad \zeta_2 = \frac{1}{2}$$

has a solution of the form

$$v(\zeta) = \sum_{j=0}^{[k/2]} \alpha_j \text{Im} w^{k+1-2j},$$

where $\alpha_j$ are some explicitly calculated constants, $\alpha_0 = \frac{1}{k+1}$;

b) there exists a solution of a boundary value problem

$$\Delta \zeta Y_{k+1} = 0, \quad \zeta_2 > \frac{1}{2}, \quad \frac{\partial Y_{k+1}}{\partial \zeta_2} = 0, \quad \zeta_2 = \frac{1}{2}$$

that can be represented in the form

$$Y_n(\zeta) = \text{Re} w^n + \sum_{j=0}^{n-2} \beta_j \text{Im} w^{n-1-2j},$$

where $\beta_j$ are some explicitly calculated constants. The leading term of the asymptotics for the function $X_n(\xi) = Y_n(\zeta_1(\xi), \zeta_2(\xi))$ as $\xi \to \infty$ has the form $\rho^{n/2} \cos(n \theta/2)$.

**Proof.** The validity of the item a) follows from the equality

$$\left. \frac{\partial}{\partial \zeta_2} \text{Im} w^n \right|_{\zeta_2 = \frac{1}{2}} = n \sum_{j=0}^{[k/2]-1} C_{n-1}^{2j} \zeta_1^{n-1-2j} \left( -\frac{1}{4} \right)^j.$$

in the obvious way. In its turn, the item a) and an equality

$$\left. \frac{\partial}{\partial \zeta_2} \text{Re} w^n \right|_{\zeta_2 = \frac{1}{2}} = -\frac{n}{2} \sum_{j=0}^{[k/2]-1} C_{n-1}^{2j+1} \zeta_1^{n-2-2j} \left( -\frac{1}{4} \right)^j.$$
yield the validity of the item b). \( \square \)

For the sake of uniformity of notations, we set \( X_0 = Y_0 = 1 \). Note that under these notations the functions \( v_k^\pm \) constructed above have the form \( v_0^\pm = \phi_0^\pm X_0 \), \( v_i^\pm = d_i^\pm g_i^\pm X_1 + \phi_i^\pm X_0 \).

**Lemma 5.2.** Let \( 0 \leq n < \infty \), \( 0 \leq k \leq n \), and \( \{a_k^{(n)}\} \), \( \lambda_n \) be arbitrary sequences of real numbers. Then the system of the boundary value problems (5.2), (5.3) has the system of solutions represented in the form

\[
v_n(\zeta) = \sum_{i=0}^{n} a_i^{(n)} Y_i(\zeta) + \bar{v}_n(\zeta), \tag{5.4}
\]

\[
\bar{v}_n(\zeta) = \sum_{k=1}^{[\frac{n}{2}]} \|w\|^{4k} \left( \sum_{j=0}^{n-4k} \alpha_{n,k,j} \text{Re} w^j + \sum_{j=2}^{n-4k} \beta_{n,k,j} \text{Im} w^{j-1} \right) + \sum_{j=2}^{n} \beta_{n,0,j} \text{Im} w^{j-1} \tag{5.5}
\]

where the constants \( \alpha_{n,k,j} \) and \( \beta_{n,k,j} \) do not depend on \( \lambda_m \) as \( m > n - 4k - j \) and on \( a_{s}^{(m)} \) as \( m > n - 4k - j + s \).

**Proof.** The proof is carried out by induction. For \( n \leq 3 \) equations (5.2) are homogeneous and, in view of the item b) of Lemma 5.1, there exist solutions of the form (5.3) with \( \bar{v}_n \equiv 0 \). For \( n \geq 4 \) equations (5.2) are inhomogeneous and in order to construct their solutions one has to bear in mind that

\[
\Delta \left( |w|^{4k} \text{Im} w^j \right) = 4\text{Im} \frac{\partial^2}{\partial w \partial \bar{w}} \left( w^{2k} w^{2k+j} \right) = 8k(2k+j)|w|^{4k-2} \text{Im} w^j
\]

and a similar equality holds for \( |w|^{4k} \text{Re} w^j \). For this reason, it is easy to construct the solution of the inhomogeneous solution (5.2) of the form (5.4), moreover, without the last sum in the representation for \( \bar{v}_n \). To eliminate the discrepancy appeared in the boundary condition (5.3) one should use the item b) of Lemma 5.1, what imply the appearance of the last sum in the representation for \( \bar{v}_n \). The independence of the constants \( \alpha_{n,k,j} \) and \( \beta_{n,k,j} \) on \( a_{s}^{(m)} \) and \( \lambda_m \) for the changing of their indexes in ranges mentioned in the lemma follows from the algorithm of construction \( v_n \) which has been adduced. \( \square \)

We denote \( z = \xi_1 - \frac{1}{2} + i\xi_2 \), where \( i \) is an imaginary unit. By definition, \( z = w^2 \), what and Lemma 5.2 imply

**Lemma 5.3.** Let \( 0 \leq n < \infty \), \( 0 \leq k \leq n \), and \( \{a_k^{(n)}\} \), \( \lambda_n \) be arbitrary sequences of real numbers. Then the system of the boundary value problems (5.4) has the system of solutions represented in the form

\[
v_n(\xi) = \sum_{i=0}^{n} a_i^{(n)} X_i(\xi) + \bar{v}_n(\xi),
\]

\[
\bar{v}_n(\xi) = \sum_{k=1}^{[\frac{n}{2}]} |z|^{2k} \left( \sum_{j=0}^{n-4k} \alpha_{n,k,j} z^{j/2} + \sum_{j=2}^{n-4k} \beta_{n,k,j} z^{(j-1)/2} \right) + \sum_{j=2}^{n} \beta_{n,0,j} z^{(j-1)/2}, \tag{5.6}
\]
where the constants \( \alpha_{n,k,j} \) and \( \beta_{n,k,j} \) do not depend on \( \lambda_m \) as \( m > n - 4k - j \) and on \( a_s^{(m)} \) as \( m > n - 4k - j + s \).

From Lemma 5.3 it follows that for any \( \lambda_\varepsilon \) having power asymptotics with arbitrary coefficients \( \lambda_j \), the series

\[
v_{\pm}(x; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n v_n \left( \frac{x_{\pm}}{(g_{\pm} \varepsilon)^2} \right),
\]

whose coefficients satisfy the statements of Lemma 5.3 are asymptotic solutions of (1.1) near the endpoints of the slit. In order to define the true values of the constants \( \lambda_j \) and \( a_s^{(m)} \) one must consider the outer expansions in the vicinity of the endpoints of the slit and to compare (match) it with the inner expansion constructed.

6. Outer expansion. We seek the asymptotics for the eigenfunction outside a neighbourhood of the endpoints of the slit (the outer expansion) in the form of the series (1.3). Observe that the asymptotics expansion (1.3) (similarly to the asymptotics expansions (1.4)) corresponds to the eigenfunction \( \phi_{\varepsilon} \) with "lax" normalization \( \| \phi_{\varepsilon} \|_{0, \Omega} = 1 + o(1) \) as \( \varepsilon \to 0 \).

The boundary value problems for the coefficients of the series (1.3) are obtained in the standard way. We substitute the series (1.2) and (1.3) into (1.1) and then we write down the equalities of the same power of \( \varepsilon \) and formally pass to limit as \( \varepsilon \to 0 \).

As a result, we get the following recursive system of the boundary value problems

\[
-\Delta \phi_n = \sum_{j=0}^{n} \lambda_j \phi_{n-j}, \quad x \in \Omega_0, \quad \frac{\partial \phi_n}{\partial \nu} = 0, \quad x \in \Gamma,
\]

\[
\frac{\partial}{\partial x_2} \phi_n(x_1, \pm 0) = -\sum_{j=1}^{n} \frac{1}{j!} (g_{\pm} (x_1))^j \frac{\partial^{j+1} \phi_{n-j}}{\partial x_2^{j+1}}(x_1, \pm 0) + g_{\pm}^{'} (x_1) \sum_{j=0}^{n-1} \frac{1}{j!} (g_{\pm} (x_1))^j \frac{\partial^{j+1} \phi_{n-j-1}}{\partial x_2^{j+1} \partial x_1}(x_1, \pm 0), \quad x_1 \in \omega_0.
\]

(6.1)

The aim of this section is to study the solvability of the problems of the form (6.1) in a class of singular solutions.

Let \( j \) be any half-integer, \( H_j(x) \) be homogeneous functions of order \( j \) belonging to \( C^\infty (\mathbb{R}^2 \setminus l_+) \), where \( l_+ \) is the semiaxis \( x_2 = 0, \ x_1 \geq 0 \). We denote by \( \tilde{H}_m \) the set of the series of the form

\[
H(x) = \sum_{j=-m}^{\infty} H_{j/2}(x).
\]

(6.2)

We call the terms of negative order the singular part of the series.

Similarly, we denote by \( \tilde{A}_m \) the set of the series of the form

\[
h(x) = \sum_{j=-m}^{\infty} \alpha_j x_1^{j/2}.
\]

**Definition.** The scalar sequence

\[
\left\{ b_j = \frac{1}{2\pi^{1/2}} \int_{0}^{2\pi} H_{j/2}(x) \cos \left( \frac{j \theta}{2} \right) d\theta \right\}_{j=0}^{\infty}
\]
is called a harmonic sequence of the series \( \{ b_j \} \).

**Lemma 6.1.** Let the series \( H \in \mathcal{H}_0 \) have the zero harmonic series and be a formal asymptotic solution as \( r \to 0 \) of the boundary value problem

\[
(\Delta + \lambda_0)H = 0, \quad x \in \mathbb{R}^2 \setminus \mathbb{L}^+_-,
\]

\[
\frac{\partial}{\partial x_2} H(x_1, \pm 0) = 0, \quad x_1 > 0.
\]

Then \( H = 0 \).

In proving this lemma one should bear in mind that the terms of the series belonging to \( \mathcal{H}_0 \) have the form \( r^{j/2} \Phi_j(\theta) \), \( j \geq 0 \). After the substitution in the equation we get an ordinary differential equation for \( \Phi_j \). The explicit form of the solutions of these equations, the boundary conditions as \( \theta = 0, \theta = 2\pi \) and the fact that \( b_j \) equals zero yield the statement of the lemma.

**Corollary.** Let \( F \in \mathcal{H}_m \), \( h_\pm \in \mathcal{A}_m \), and let the series \( H^{(1)} \), \( H^{(2)} \in \mathcal{H}_m \) be formal asymptotic solutions as \( r \to 0 \) of a boundary value problem

\[
(\Delta + \lambda_0)H = F, \quad x \in \mathbb{R}^2 \setminus \mathbb{L}^+_-,
\]

\[
\frac{\partial}{\partial x_2} H(x_1, \pm 0) = h_\pm(x_1), \quad x_1 > 0,
\]

and let these series have the same harmonic sequences and \( H^{(1)} - H^{(2)} \in \mathcal{H}_0 \). Then \( H^{(1)} = H^{(2)} \).

We denote by \( \mathcal{H}_m \) the subset of the functions in \( C^\infty(\Omega_0 \setminus \{ O_+; O_- \}) \) whose asymptotics at the points \( O_\pm \) belong to the class \( \mathcal{H}_m \) (with respect to the coordinate systems \( x_+ \) and \( x_- \)). Similarly, let \( \mathcal{A}_m \) be the subset of functions in \( C^\infty(\omega) \) with asymptotic behaviour at the endpoints of the slit described by functions in the class \( \mathcal{A}_m \).

We consider a boundary value problem

\[
(\Delta + \lambda_0)u = F + \lambda \phi_0, \quad x \in \Omega_0,
\]

\[
\frac{\partial}{\partial x_2} u(x_1, \pm 0) = h_\pm(x_1), \quad x_1 \in \omega_0,
\]

(6.3)

where \( F \in \mathcal{H}_m \), \( h_\pm \in \mathcal{A}_m \), \( m \in \mathbb{N} \).

**Lemma 6.2.** Suppose that two classes of series \( H_\pm(x; \{ b_j \}_{j=0}^\infty, \lambda) \) belong to \( \mathcal{H}_m \) for each value of the parameters \( b_j \) and \( \lambda \) and have the following properties

a) the series \( \{ H_\pm(x; \{ b_j \}_{j=0}^\infty, \lambda) \} \) are asymptotics solutions of the problem (6.3) for \( x_+ \to 0 \);

b) \( \{ b_j \}_{j=0}^\infty \) is a harmonic sequence of the series \( H_\pm(x; \{ b_j \}_{j=0}^\infty, \lambda) \);

c) \( H_\pm(x; \{ b_j \}_{j=0}^\infty, \lambda) = H_\pm(x; \{ \tilde{b}_j \}_{j=0}^\infty, \lambda) \in \mathcal{H}_0 \) for each sequences \( \{ b_j \}_{j=0}^\infty \) and \( \{ \tilde{b}_j \}_{j=0}^\infty \).

Then there exist numbers \( \lambda \) and \( \{ b_j \}_{j=0}^\infty \) and a function \( u \in \mathcal{H}_m \) such that \( u \) is a solution of the boundary value problem (6.3) and it has asymptotics coinciding with \( H_\pm(x; \{ b_j \}_{j=0}^\infty, \lambda) \) as \( x_+ \to 0 \).

**Proof.** This statement is proved by arguments similar to those used in the proof of Lemma 4.1. We seek the solution of the boundary value problem (6.3) in the form

\[
u(x) = u_N(x) = \chi(r_+) H_N^+(x_+) + \chi(r_-) H_N^-(x_-) + U_N(x),
\]

(6.4)

where \( H_N^\pm \) are partial sums (to the powers \( r^j \), inclusive) of the series \( H_\pm(x; b_j = 0)_{j=0}^\infty, 0 \). Substituting (6.4) into (6.3), we deduce a boundary value problem for \( u_N \):

\[
(\Delta + \lambda_0)u_N = F_N + \lambda \phi_0, \quad x \in \Omega_0,
\]

\[
\frac{\partial}{\partial x_2} u_N(x_1, \pm 0) = h_\pm(x_1), \quad x_1 \in \omega_0.
\]

(6.5)
Here the function \( F_N \in C^{N-2}(\Omega_0) \) (where, recall, the cut \( \gamma_0 \) is interpreted as double-sided) has zeroes of order \( r_N^\pm \) at the endpoints of the slit, and the functions \( h_N^\pm, h_N^\pm \in C^{N-1}(\Omega_0) \) have zeroes of order \( x_N^1 \) and \( (1-x_N) \) at the corresponding endpoints of the interval \( \omega_0 \). Therefore, there exists a constant \( \lambda = \lambda(N) \) for which the boundary value problem (6.3) is solvable in the functional class \( H_1(\Omega_0) \). On the other hand, substituting \( u_{N_1} - u_{N_2} \in H_1(\Omega_0) \) into (6.3), one can easily see that, firstly, \( \lambda \) does not depend on \( N \), and, secondly, the functions \( u_N \) are the same for different values of \( N \) up to the term equal to the eigenfunction. Due to the arbitrariness in choosing \( N \), the results of [6] and Lemma 3.1 we conclude that there exists a solution \( u \in \mathcal{H}_m \) of the boundary value problem (6.3) (for some constant \( \lambda \), having asymptotics at the endpoints of the slit of the form

\[
u(x) = \tilde{H}_\pm(x_\pm), \quad x_\pm \to 0,
\]

where \( \tilde{H}_\pm \) differs from \( \hat{H}_\pm \) by a term in \( \hat{H}_0 \).

Let \( \{b_n^\pm\} \) be a harmonic sequence of the series \( \tilde{H}_\pm \). Form the Corollary to Lemma 6.1 it follows that the series \( \tilde{H}_\pm(x_\pm) \) and \( H_\pm(x_\pm; \{b_n^\pm\}_{j=0}^\infty, \lambda) \) are the same.

**7. Matching the expansions.** We introduce re-expansion operators \( \mathcal{M}^\pm \) on the formal series of the type

\[
V(\xi; \varepsilon) = \sum_{n=0}^\infty \varepsilon^n V_n(\xi)
\]

by the following standard procedure. Coefficients of the series \( V \) are replaced by their asymptotics at infinity and then we pass to the variables \( x_\pm = (\varepsilon y^\pm)^2 \xi \). The formal double series obtained is called the value of \( \mathcal{M}^\pm(V(\xi; \varepsilon)) \).

For the sake of brevity we use the notations \( \mathcal{H}_m = \tilde{H}_m \), \( \mathcal{H}_m = \mathcal{H}_0 \). From Lemma 6.1 and the definition of the re-expansion operators it follows

**LEMMA 7.1.** Let all assumptions of Lemma 5.3 hold and the coefficients of the series

\[
v(\xi; \varepsilon) = \sum_{n=0}^\infty \varepsilon^n v_n(\xi)
\]

satisfy the statement of Lemma 5.3. Then the representation

\[
\mathcal{M}^\pm(v(\xi; \varepsilon)) = \sum_{j=0}^\infty \varepsilon^j \Phi_j^\pm(x_\pm)
\]

is true. The series \( \Phi_j^\pm = H_{j-1} \) are formal asymptotic solutions (as \( x_\pm \to 0 \)) of the recursive system of boundary value problems (6.3), where the functions \( \Phi_j \) in the right hand sides of the equations and boundary conditions are replaced by \( \Phi_j^\pm \).

A harmonic sequence \( \{b_n^\pm\}_{i=0}^\infty \) of a series \( \Phi_j^\pm \) has the form \( b_n^{(m)} = a_i^{(n+i)} (g^\pm)^{-i} \).

The series \( \Phi_j^\pm \) do not depend on \( a_i^{(n+i)} \) and \( \lambda_m \) for \( m > n \), and their singular parts also do not depend on \( a_i^{(n+i)} \) and \( \lambda_n \).

We denote by \( v^\pm(\xi; \varepsilon) \) the series \( \{b_n^\pm\} \).

**THEOREM 7.2.** There exist series \( \{b_n^\pm\}_{i=0}^\infty \) having the following properties

a) \( \Phi_j \in \mathcal{H}_{j-1} \) are the solutions of (6.3).
b) $v_n^\pm$ are the solutions of the boundary value problems (5.1) and they can be represented in the form (5.6) (with the indexes "±" added);

c) $\mathcal{M}^\pm(v^\pm(\xi; \varepsilon)) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j(x)$ as $x_\pm \to 0$;

d) the coefficients $\lambda_1, \lambda_2, \phi_1, \phi_2, v_0^\pm$ and $v_1^\pm$ satisfy the statements of theorem 7.2.

Proof. Let $v_n^\pm(\xi)$ satisfy the statements of Lemma 7.3 with constants $\lambda_j, a_s^{(m)} = \tilde{a}_s^{(m)}$ undefined yet. Then, in view of Lemma 7.1,

$$\mathcal{M}^\pm(v^\pm(\xi; \varepsilon)) = \sum_{j=0}^{\infty} \varepsilon^j \tilde{\Phi}_j^\pm(x_\pm).$$

For $n \leq 2$ we denote by $\tilde{\Phi}_n^\pm(x_\pm)$ the asymptotics as $x_\pm \to 0$ of the solutions $\phi_n$ of the boundary value problems (5.3)–(5.4), those are defined above, and we denote by $\{\hat{b}_i^{(n)}\}_{n=0}^{\infty}$ their harmonic sequences. In construction of the coefficients $v_n^\pm$, we set $\tilde{a}_i^{(n+1)} = \hat{b}_i^{(n)}(g^\pm)^i$, and we define $\lambda_i$ in accordance with the formulae 1.3 and 1.4. Lemma 7.1 and Corollary to Lemma 6.1 imply that $\Phi_0^\pm = \tilde{\Phi}_0^\pm$, $\Phi_1^\pm = \tilde{\Phi}_1^\pm$. Note that having defined $a_i^{(n+i)}$ we determine $v_1^\pm$ and three leading harmonics for other $v_j^\pm$. From the structure (5.6) of the functions $v^\pm$ (more precisely, from the form of $v_1^\pm$) it follows that $\Phi_2^\pm - \tilde{\Phi}_2^\pm \in H_0$. Thus, by Corollary to Lemma 6.1 we obtain $\Phi_2^\pm = \tilde{\Phi}_2^\pm$. It should be stressed that having defined $v_2^\pm$, due to Lemma 7.1 we determined singular parts of the series $\Phi_3^\pm \in H_2$.

In next step due to Lemma 6.2 by singular parts of the asymptotics series $\Phi_3^\pm$ we define the function $\phi_3 \in H_2$ which is a solution of (1.1) for some value of $\lambda_3$ and whose asymptotics as $x_\pm \to 0$ coincide with $\Phi_3^\pm$ for some $\tilde{a}_3^{(3+i)} = \hat{b}_3^{(3)}(g^\pm)^i$, etc.

The only fact following from the items a) and b) of Theorem 7.2 is the series (1.3) and (1.4) are asymptotic solutions of the problem (1.1) for $r_+ < \varepsilon$, $r_- > \varepsilon$, and the series (1.2) and (1.3) are asymptotic solutions of the problem (1.1) for $r_\pm < 2\varepsilon$. The key condition of matching is determined by the item c) of the theorem proved.

8. Justification of the asymptotics. We use the notations $\lambda_{\varepsilon,N}, \phi_{\varepsilon,N}(x)$ and $v_{\varepsilon,N}^\pm(x_\pm(\varepsilon g^\pm)^{-2})$ for the partial sums of the series (1.2)–(1.3). Further, we set

$$\Phi_{\varepsilon,N}(x) = \left(1 - \chi(r_+ \varepsilon^{-1})\right) \left(1 - \chi(r_- \varepsilon^{-1})\right) \phi_{\varepsilon,N}(x) +$$

$$+ \chi(r_+ \varepsilon^{-1}) v_{\varepsilon,N}^+(x_+ (\varepsilon g^+)^{-2}) + \chi(r_- \varepsilon^{-1}) v_{\varepsilon,N}^-(x_- (\varepsilon g^-)^{-2}).$$



**Lemma 8.1.** Suppose that the series (1.2)–(1.3) satisfy the statements of Theorem 7.2. Then the function $\Phi_{\varepsilon,N}(x)$ is a solution of a boundary value problem

$$-(\Delta + \lambda_{\varepsilon,N}) \Phi_{\varepsilon,N} = f_{\varepsilon,N}, \quad x \in \Omega_{\varepsilon},$$

$$\frac{\partial}{\partial \nu} \Phi_{\varepsilon,N} = 0, \quad x \in \Gamma, \quad \frac{\partial}{\partial \nu} \Phi_{\varepsilon,N} = h_{\varepsilon,N}, \quad x \in \gamma_{\varepsilon},$$

where $\|f_{\varepsilon,N}\|_{0, \Omega} \leq C \varepsilon^M$, $h_{\varepsilon,N} = O(\varepsilon^M)$ in the norm of $C^1(\gamma_{\varepsilon})$, and $M \to \infty$ as $N \to \infty$.

**Proof.** The statement of the lemma being a standard implication of the items a)–c) of Theorem 7.2 (see, for instance, [6]), we give a brief proof. Substituting $\Phi_{\varepsilon,N}$
in the left hand sides of (8.1), we get that a homogeneous boundary condition on \( \Gamma \) holds and the functions \( f_{\varepsilon,N}, h_{\varepsilon,N} \) can be represented in the form

\[
f_{\varepsilon,N} = \sum_{i=1}^{N} f^{(i)}_{\varepsilon,N}, \quad h_{\varepsilon,N} = \sum_{i=1}^{N} h^{(i)}_{\varepsilon,N},
\]

where

\[
f^{(1)}_{\varepsilon,N} = -(1 - \chi(r_+ \varepsilon^{-1})) (1 - \chi(r_- \varepsilon^{-1})) \sum_{n=1}^{N} \sum_{k=N+1-n}^{N} \varepsilon^{k+n} \lambda_n \phi_k,
\]

\[
f^{(2)}_{\varepsilon,N} = -\sum_{n=1}^{N} \sum_{k=N-3-n}^{N} \varepsilon^{k+n} \lambda_n \left( \chi(r_- \varepsilon^{-1}) v_k^- + \chi(r_+ \varepsilon^{-1}) v_k^+ \right),
\]

\[
f^{(3)}_{\varepsilon,N} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^-(x - (\varepsilon g)^{-2}) \right) \frac{\partial}{\partial x_i} \chi(r_- \varepsilon^{-1})
\]

\[
+ \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^+(x + (\varepsilon g)^{+2}) \right) \frac{\partial}{\partial x_i} \chi(r_+ \varepsilon^{-1})
\]

\[
+ \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^-(x - (\varepsilon g)^{-2}) \right) \Delta \chi(r_- \varepsilon^{-1})
\]

\[
+ \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^+(x + (\varepsilon g)^{+2}) \right) \Delta \chi(r_+ \varepsilon^{-1}),
\]

\[
h^{(1)}_{\varepsilon,N} = (1 - \chi(r_+ \varepsilon^{-1})) (1 - \chi(r_- \varepsilon^{-1})) \frac{\partial}{\partial \nu} \phi_{\varepsilon,N}(x),
\]

\[
h^{(2)}_{\varepsilon,N} = \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^-(x - (\varepsilon g)^{-2}) \right) \frac{\partial}{\partial \nu} \chi(r_- \varepsilon^{-1})
\]

\[
+ \left( \phi_{\varepsilon,N}(x) - v_{\varepsilon,N}^+(x + (\varepsilon g)^{+2}) \right) \frac{\partial}{\partial \nu} \chi(r_+ \varepsilon^{-1}).
\]

In view of the item a) of Theorem 7.2, the functions \( f^{(1)}_{\varepsilon,N} \) and \( h^{(1)}_{\varepsilon,N} \) have the norms of order \( O(\varepsilon^{M_1}) \) in \( H_0(\Omega) \) and in \( C^1(\gamma_\varepsilon) \), respectively, and \( M_1 \to \infty \) as \( N \to \infty \). Similarly, by the item b) we deduce that \( f^{(2)}_{\varepsilon,N} \) has a norm of order \( O(\varepsilon^{M_2}) \) in \( C^1(\gamma_\varepsilon) \), and \( M_2 \to \infty \) as \( N \to \infty \). Finally, the item c) of Theorem 7.2 implies that norms of \( f^{(3)}_{\varepsilon,N} \) and \( h^{(2)}_{\varepsilon,N} \) have order \( \varepsilon^{M_3} \) in \( H_0(\Omega) \) and \( C^1(\gamma_\varepsilon) \), respectively, and \( M_3 \to \infty \) as \( N \to \infty \). These facts completes the proof for \( M = \min\{M_1; M_2; M_3\} \). \( \square \)

The boundary condition on \( \gamma_\varepsilon \) in (8.1) being inhomogeneous, we can not apply Lemma 2.6 directly to (8.1) in order to justify the asymptotics. For this reason, beforehand we shall prove two auxiliary statements.

LEMMA 8.2. Let \( u \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon}), \frac{\partial}{\partial \nu} u > 0 \) on \( \gamma_\varepsilon \), \( \Delta u < 0 \) in \( \Omega_\varepsilon \) and \( u \geq 0 \) on \( \Gamma \). Then \( u \geq 0 \) in \( \overline{\Omega_\varepsilon} \).

Proof. Since \( \frac{\partial}{\partial \nu} u > 0 \) on \( \gamma_\varepsilon \), we conclude that the minimum of \( u \) lies outside \( \gamma_\varepsilon \), and, as \( \Delta u < 0 \) in \( \Omega_\varepsilon \), then it can not lie in \( \Omega_\varepsilon \). Hence, the minimum lies in \( \Gamma \), where, by conditions, \( u \geq 0 \). \( \square \)

LEMMA 8.3. Let \( U \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon}), \)

\[
\max_{\Gamma} |U| + \max_{\gamma_\varepsilon} \left| \frac{\partial U}{\partial \nu} \right| + \sup_{\Omega_\varepsilon} |\Delta U| = m.
\]

Then \( |U| < Cm\varepsilon^{-1} \), where \( C \) is some constant independent on \( U \).
Proof. Recall that near the left endpoint of the slit (for \(0 \leq x_1 < t_0\), where \(t_0\) is some positive constant) the equation of \(\gamma_\varepsilon\) has the form \(\varepsilon^2 x_1 = (g^-)^{-2} x_2^2\). Similarly, near the right endpoint of the slit (for \(0 \leq 1 - x_1 < t_0\)) its equation reads as follows \(\varepsilon^2 (1 - x_1) = (g^+)^{-2} x_2^2\). For \(t_0 \leq x_1 \leq 1 - t_0\) the equation of \(\gamma_\varepsilon\) has the form \(x_2 = \varepsilon g_\pm(x_1)\). Note that \(d = \min_{0 \leq x_1 \leq 1 - t_0} |g_\pm(x_1)| > 0\) and \(\pm g_\pm(x_1) > 0\) as \(t_0 \leq x_1 \leq 1 - t_0\). We set \(V(x) = (x_1 - \frac{1}{2})^2 + \alpha^2 x_2^2\), where \(\alpha > 0\) is some constant.

Then for \(0 \leq x_1 < t_0\) one can check

\[
\frac{\partial V}{\partial \nu} \bigg|_{x_2 = \pm g_\pm} = \mp \left( 1 + \varepsilon^2 (g_\pm'(x_1))^2 \right)^{-1/2} \left( \varepsilon g_\pm'(x_1) (1 - 2x_1) + 2\alpha^2 \varepsilon g_\pm(x_1) \right)
\]

\[
= \mp 2\varepsilon \left( 1 + \varepsilon^2 (g_\pm'(x_1))^2 \right)^{-1/2} \left( \alpha^2 g_\pm(x_1) - \left( x_1 - \frac{1}{2} \right) g_\pm'(x_1) \right) < -\varepsilon\alpha^2 d
\]

for \(\varepsilon\) sufficiently small and \(\alpha\) chosen appropriately. In its turn, for \(0 \leq x_1 < t_0\) we have

\[
\frac{\partial V}{\partial \nu} \bigg|_{x_1 = \left( \frac{\varepsilon}{\varepsilon g_\pm(x_1)} \right)^2} = \left( \varepsilon^4 + 4 (g^-)^{-4} x_2^2 \right)^{-1/2} \left( \varepsilon^2 \frac{\partial V}{\partial x_1} - 2\frac{x_2}{(g^-)^2} \frac{\partial V}{\partial x_2} \right) \bigg|_{x_1 = \left( \frac{\varepsilon}{\varepsilon g_\pm(x_1)} \right)^2}
\]

\[
= \varepsilon^{-1} \left( \varepsilon^4 + 4 (g^-)^{-2} x_1 \right)^{-1/2} \left( \varepsilon^2 (2x_1 - 1) - 4\varepsilon^2 \alpha^2 x_1 \right)
\]

\[
= - \varepsilon \left( \varepsilon^4 + 4 (g^-)^{-2} x_1 \right)^{-1/2} (1 - 2x_1 + 4\alpha^2 x_1) < \varepsilon C_- < 0.
\]

Similarly, for \(1 - t_0 < x_1 < 1\) we deduce that

\[
\frac{\partial V}{\partial \nu} \bigg|_{1 - t_1 = \left( \frac{\varepsilon}{\varepsilon g_\pm(x_1)} \right)^2} < \varepsilon C_+ < 0.
\]

We set \(C = \max |C_\pm|\), \(W(x) = R - V(x)\), where \(R > 1 + \max_{\Omega} |V(x)|\). Then the estimates (8.3)–(8.4) imply that the functions \(C\varepsilon^{-1} W \pm U\) satisfy the hypothesis of Lemma 8.2. In its turn, Lemma 8.2 yields the correctness of the statement being proved.

Proof of Theorem 8.4. Let \(\chi_\Gamma \in C_\infty(\Omega)\) be a function equal to one outside some neighbourhood of \(\Gamma\), \(U_{\varepsilon,N} \in C_\infty(\Omega_\varepsilon)\) be a harmonic function satisfying boundary condition

\[
U_{\varepsilon,N} = 0, \quad x \in \Gamma, \quad \frac{\partial}{\partial \nu} U_{\varepsilon,N} = h_{\varepsilon,N}, \quad x \in \gamma_\varepsilon,
\]

\(\Phi_{\varepsilon,N} = \chi_\Gamma U_{\varepsilon,N}\). Then from Lemmas 8.1–8.3 and well-known a priori estimates it follows that

\[
\frac{\partial}{\partial \nu} \Phi_{\varepsilon,N} = 0, \quad x \in \Gamma, \quad \frac{\partial}{\partial \nu} \Phi_{\varepsilon,N} = h_{\varepsilon,N}, \quad x \in \gamma_\varepsilon,
\]

\[
\| \Phi_{\varepsilon,N} \|_{1,\Omega_\varepsilon} + \| \Delta \Phi_{\varepsilon,N} \|_{0,\Omega_\varepsilon} \leq C_N \varepsilon^{M-1}.
\]

We set \(\Phi_{\varepsilon,N} = \Phi_{\varepsilon,N} - \Phi_{\varepsilon,N}\). By Lemma 8.4 and relations (8.5), we obtain that the function \(\Phi_{\varepsilon,N}\) is a solution of a boundary value problem

\[
-\Delta \Phi_{\varepsilon,N} = \lambda_{\varepsilon,N} \Phi_{\varepsilon,N} + F_{\varepsilon,N}, \quad x \in \Omega_\varepsilon, \quad \frac{\partial}{\partial \nu} \Phi_{\varepsilon,N} = 0, \quad x \in \partial \Omega_\varepsilon.
\]

On the spectrum of the Neumann problem . . .
where
\[ \| F_{\varepsilon,N} \|_{0,\Omega} \leq C_N \varepsilon^M. \]  

(8.7)

Since \( \Phi_{\varepsilon,N} \to \phi_0 \) \( H_0(\Omega_0) \), then from (8.5) it follows that
\[ \hat{\Phi}_{\varepsilon,N} \to \phi_0 \) \( H_0(\Omega_0). \]

(8.8)

In view of (8.6)–(8.8), Lemma 2.4, the estimate (2.16), and the arbitrariness in choosing \( N \) we conclude that the series (1.2) constructed coincides with the asymptotics of the eigenvalue \( \lambda_\varepsilon \). In their turn, the estimates (2.16) and (8.8) imply that
\[ \hat{\Phi}_{\varepsilon,N}(x) = \alpha_{\varepsilon,N} \phi_\varepsilon(x) + \tilde{\phi}_{\varepsilon,N}(x), \]
\[ \| \tilde{\phi}_{\varepsilon,N} \|_{0,\Omega} = O(\varepsilon^M), \quad |\alpha_{\varepsilon,N}| \to 1, \quad \varepsilon \to 0. \]

(8.9)

Finally, due to (8.5), (8.9) and the arbitrariness in choosing \( N \) we deduce that the asymptotic series (1.3), (1.4) are the asymptotics of the eigenfunction \( \phi_\varepsilon \) associated with the eigenvalue converging to \( \lambda_0 \).

9. Concluding remarks. The asymptotics of eigenvalues in the case of Dirichlet boundary condition on \( \gamma_\varepsilon \) was considered in [8]. There it was shown that if the boundary of the slit lies on square parabolas near the endpoints, then the asymptotics of the eigenelements have the power character (1.2)–(1.4). However, for the case the equation of the endpoints of the slit have the form of square parabolas only “in principal” it was shown in [8] that the asymptotics of \( \lambda_\varepsilon \) and \( \phi_\varepsilon \) contain also powers of \( \ln \varepsilon \).

It can be shown that this effect takes place in the case of Neumann boundary condition. It also can established that for the Robin boundary condition on \( \gamma_\varepsilon \) the powers of logarithms appear in asymptotic expansions even in the case when the boundary of slit lie on parabolas near the endpoints.

REFERENCES

[1] Van-Dike, Perturbation methods in fluid mechanics, Academic Press, New-York, 1964.
[2] A. H. Nayfeh, Perturbation Methods, John Wiley, New York, 1973.
[3] A. M. Il’in, Matching of asymptotic expansions of solutions of boundary value problems, Nauka, Moscow, 1989; English transl., Amer. Math. Soc., Providence, RI, 1992.
[4] W. Eckhaus, Eckhaus W. Matched Asymptotic Expansions and Singular Perturbations, North Holland, Amsterdam, 1973.
[5] O. A. Ladyzhenskaya, The boundary value problems of mathematical physics, Nauka, Moscow, 1973; English transl., Springer-Verlag, New York, 1985.
[6] V. A. Kondrat’ev, Boundary value problems for elliptic equations in domain with conical or angular points, Trudy Moskov., Mat., Oshch., 16 (1967), pp. 209–292; English transl. in Trans. Moscow Math. soc. 16 (1967), pp. 227-314.
[7] A. M. Il’in, A boundary value problem for the elliptic equation of second order in a domain with narrow slit. 1. The two-dimensional case, Mat. Sb., 99 (1976), pp. 513–537; English transl. in Math USSR-Sb. 28 (1976), pp. 459-480.
[8] R. R. Gadyl’shin and A. M. Il’in, Asymptotic behaviour for the eigenvalue of the Dirichlet problem in a domain with a narrow slit, Mat. Sb., 189 (1998), pp. 25-48; English transl. Sb.: Math. 189 (1998) pp. 503–526.