Coarse graining and a new strategy for renormalization

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We present the natural arguments for the rationality of a recently proposed simple approach for renormalization which is based solving differential equations. The renormalization group equation is also derived in a natural way and recognized as a decoupling theorem of the UV modes that underlie a QFT. This new strategy has direct implications to the scheme dependence problem.

I. INTRODUCTION

Recently, we proposed a new approach for calculating radiative corrections without introducing any form of regulator and any form of removal of UV divergence [1–3], which is a differential equation approach with ambiguities to be fixed through rational boundary conditions. In this simple approach, many complicated aspects associated with conventional regularizations (e.g., the subtle definition of Dirac matrices $\{\gamma^5, \gamma^\mu\}$ and metric tensor $g^{\mu\nu}$ in dimensional regularization; the notorious power law divergences in cutoff regularization, and so on) simply do not show up [3]. It is especially efficient in the nonperturbative contexts where conventional regularization and/or subtraction schemes often make it very hard to extract physical information from the calculated quantities, as the proposed approach can dramatically reduce the difficulty in extracting physical information [4]. It is also applied in Ref. [2] to massless $\lambda \phi^4$ to discuss the problem of nontrivial symmetry breaking solution in various regularization and renormalization prescriptions.

In this short report, we shall: (1) to present the natural rationality of the simple approach in Sec. II; (2) sketch a simple derivation of the renormalization group equation as a natural decoupling theorem of the underlying short distance modes in Sec. III. The final section is devoted to discussion and summary. Part of these arguments have been available in the e-print form [1].

II. UNDERLYING THEORY AND FINITENESS OF QFT

Our starting point is the well known point of view that the conventional QFT should be replaced by a complete quantum theory of everything (QTOE) with correct high energy details. The low energy physics are defined by the coarse grained low energy sectors of QTOE with the extremely short distance processes integrated out. The high energy modes’ contributions are physically suppressed by certain physical mechanism defined in QTOE (unknown to us) rather than ‘cut off’ by hand. This understanding naturally motivates the presence of a set of parameters (denoted as $\{\sigma\}$) to characterize the high energy modes’ contributions in the coarse grained objects. Technically, it is these constants and the way they appear in the generating functional that suppress the high energy modes while keep the ‘effective’ quanta dominant. For this coarse graining or emergence scenario to be effective, the magnitude of the parameters in energy unit must be such that $\sup \{\Lambda_{QFT}\} \ll \inf \{\sigma\}$ with $\Lambda_{QFT}$ representing a general dimensional parameter (momenta or masses) in the QFT in under consideration.

The preceding magnitude order analysis automatically activates a limit operation with respect to $\{\sigma\}$ on the coarse grained amplitudes for describing ‘low’ energy processes, which will be denoted as $L_{(\sigma)} (\equiv \lim_{\sigma \to 0}$ in length unit). Then the coarse grained vacuum functional in the presence of the external sources for low energy processes reads

$$Z \left( J \left( x \right) | \{ \hat{c} \} \right) \equiv L_{(\sigma)} Z \left( J \left( x \right) | \{ \sigma \} \right) \equiv L_{(\sigma)} \int D \Phi \left( x \right) \{ \sigma \} \exp \left[ \frac{i}{\hbar} S \left( \Phi \left( x \right) \{ \sigma \} \right) ; \{ \sigma \} \right] \right), \quad (1)$$

where the $\{\sigma\}$ dependence of a functional indicates that they are coarse grained objects well defined in QTOE. The appearance of the constants $\{\hat{c}\}$ (including $\hat{\mu}$) in the RHS of Eq.(1) implies that the order of functional integration and $L_{(\sigma)}$ can not be trivially exchanged, otherwise we would get the ill defined QFT’s or divergences, i.e.,

$$L_{(\sigma)} \int D \Phi \left( x \right) \{ \sigma \} \exp \left[ \frac{i}{\hbar} S \left( \Phi \left( x \right) \{ \sigma \} \right) ; \{ \sigma \} \right] \neq \int D \Phi \left( x \right) \exp \left[ \frac{i}{\hbar} S \left( \Phi \left( x \right) \right) \right] \right), \quad (2)$$

with $S \left( \Phi \left( x \right) \right) \equiv L_{(\sigma)} S \left( \Phi \left( x \right) \{ \sigma \} ; \{ \sigma \} \right) J$ and $\Phi \left( x \right) \equiv L_{(\sigma)} \Phi \left( x \right) \{ \sigma \}$.

In terms of Feynman diagram algorithm, this is (for a one loop divergent diagram in QFT),
with \( f_{\Gamma}(Q,(p),(m)) \) being the integrand of this diagram defined in conventional QFT. The loop momentum, external momenta and masses are denoted respectively by \( Q, (p) \) and \( (m) \).

In principle we could not evaluate the generating functional or the Feynman amplitudes without knowing the exact dependence upon \( \sigma \). However, we can determine each one loop amplitude (ill defined in QFT) \( L_{(\sigma)} \Gamma ((p),(m); \sigma) \) up to an appropriate polynomial of momenta and masses with finite but undetermined coefficients as long as we accept that the QTOE version of the loop diagram exists.

**THEOREM.** A one loop amplitude \( \Gamma \) defined in QTOE (ill defined in the conventional QFTs) satisfies the following kind of natural differential equation,

\[
(\partial_p)^{\omega_T+1} L_{(\sigma)} \Gamma ((p),(m); \sigma) = \int d^D Q (\partial_p)^{\omega_T+1} f_{\Gamma}(Q,(p),(m)) \equiv \Gamma^{(\omega_T)}((p),(m))
\]

with \( \omega_T \) being the superficial divergence degree or scaling dimension of such a diagram.

**Proof:** Since QTOE is completely well defined, then in any dimension \( D \) of spacetime we have [1]

\[
(\partial_p)^{\omega_T+1} L_{(\sigma)} \Gamma ((p),(m); \sigma) = L_{(\sigma)} \int d^D Q (\partial_p)^{\omega_T+1} f_{\Gamma}(Q,(p),(m) | \sigma) = \int d^D Q (\partial_p)^{\omega_T+1} f_{\Gamma}(Q,(p),(m)) \equiv \Gamma^{(\omega_T)}((p),(m)) \quad Q.E.D.
\]

Similar differential equations also hold with \( \partial_p \) replaced by \( \partial_m \). The key observation here is that differentiating a Feynman amplitude with respect to external parameters lower the divergence degrees of the amplitude [5].

The solutions to such differential equations are easy to obtain as

\[
\Gamma ((p),(m); \sigma) \equiv L_{(\sigma)} \Gamma ((p),(m); \sigma) \equiv \left( \int_p \right)^{\omega_T+1} \Gamma^{(\omega_T)}((p),(m))
\]

\[
= \left( \int_p \right)^{\omega_T+1} \int d^D Q (\partial_p)^{\omega_T+1} f_{\Gamma}(Q,(p),(m))
\]

with the symbol \( \equiv \) indicating that the two sides are equal up to certain integration constants in a polynomial of momenta and masses of power \( \omega_T \). To determine the integration constants (which is definitely defined as \( \{ \bar{e} \} \) in QTOE from the limit operation \( L_{(\sigma)} \)) we need 'boundary conditions' like symmetries, sum rules and finally experimental data, which parallels the procedure of choosing renormalization conditions. For later convenience we note that there must be a dimensional constant characterizing the typical length or energy of the QFT under consideration and we denote it as \( \bar{\mu} \). Eq.(4) or (5) is just our general recipe for evaluating the Feynman amplitudes that dispenses the notorious divergences and the associated subtraction. This recipe works in the same way for multiloop diagrams, for details please refer to Ref. [1]. The guideline is to insert a pair of \( \left( \int_p \right)^{\omega_T+1} \) and \( (\partial_p)^{\omega_T+1} \) to the two sides of each divergent loop integration as \( L_{(\sigma)} \) crosses the loop integration from the left until the \( L_{(\sigma)} \) is finally removed from all loops in the diagram. For convergent loops \( L_{(\sigma)} \) can safely cross the loop integrations. However, by defining that \( (\partial)^n \equiv (\int)^{|n|} \), \( (\int)^{n} \equiv (\partial)^{|n|} \), for \( n < 0 \), \( (\int)^{0} = (\partial)^{0} = 1 \), for \( n = 0 \) and noting that \( (\int)^{n} \times (\partial)^{n} = (\partial)^{|n|} \times (\int)^{|n|} = 1 \) for \( n < 0 \) we can also put a convergent loop into the form of Eq.(5) with now \( \omega_T \) denoting the negative scale dimension of the convergent loop diagram.

We emphasize that the above expressions are correct provided the magnitude order sup \( \{ |p|, m, \bar{\mu} \} \ll \inf \{ \sigma \} \) is satisfied, no matter how large the mass or momentum is. It is clear that no subtraction is necessary, no infinite counterterms and bare parameters is present except finite 'bare' parameters—the tree parameters in Lagrangian. It is also evident that our strategy is obviously applicable to any interactions (fields with any spin) in any spacetime, even for nonlocal interactions, as our deduction does not need any specifics about interaction. Even the Lorentz invariance and other symmetry status are not needed at all, as long as the whole dynamics are consistently defined.

Among the integration constants (which will be denoted as \( \{ C \} \) in contrast to \( \{ \bar{e} \} \)), there must be a dimensional scale to balance the dimensions in the logarithmic function of momenta (which will be denoted as \( \mu_{\text{int}} \) that corresponds to \( \bar{\mu} \)). The integration constants \( \{ C \} \) span a space in which the QTOE prediction \( \{ \bar{e} \} \) just lies on one point of this space. Obviously, the QTOE definition of the Lagrangian constants and the 'loop' constants \( \{ \bar{e} \} \) should be scheme and scale invariant [6,7]. This may accentuate and accelerate the extraction of physical parameters out of renormalization scheme and scale dependent parametrization [7], namely, once we fixed the Lagrangian parameters in some physical way, we can in principle systematically extract \( \{ \bar{e} \} \) from experiments. Thus inequivalent choices of \( \{ C \} \)
would correspond to different physics. (Note that here the words 'bare parameters' does not mean no interaction. The 'bare' or tree parameters in QFT in fact characterize the 'elementary' quantum dynamics of the low energy processes in the lagrangian level.)

For perturbative Feynman diagram representation, our differential equation approach is similar to the celebrated BPHZ algorithm [8]. However, we must point out that: 1) in practice one must first specify a regularization scheme (that is, introducing certain kind of artificiality in the computation) before BPHZ is implemented; 2) the subtraction procedure in BPHZ could only lead to special set of constants that solve the differential equations for relevant Feynman amplitudes; 3) the BPHZ program ends up with the introduction of infinite bare quantities while there is no room for such infinite quantities at all if one adopts the underlying theory standpoint; 4) the application of BPHZ (and other conventional programs) in nonperturbative circumstances is rather involved that might preclude any useful (or trustworthy) predictions, while the differential equation approach makes the calculation easier and the physical predictions more accessible [4]. We think our differential equation approach (and the underlying theory scenario) generalizes, refines or improves various conventional renormalization programs in a natural way. Moreover, we could get rid of the various shortcomings in conventional programs due to inevitable introduction of a regularization scheme, an artificial substitute for the true short distance physics. This drawback is especially troublesome in nonperturbative applications [9].

**III. RENORMALIZATION GROUP EQUATION AND DECOUPLING OF UNDERLYING MODES**

From the preceding discussions on the constants \( \{ \bar{c} \} \), we can parametrize them in such a way that \( \{ \bar{c} \} = (\bar{\mu}, [c^0]) \), \( \dim\{ c^0 \} = 0, \partial_g c^0 = \partial_\mu c^0 = 0, \forall g \neq 0 \), i.e., we parametrize all the dimensional constants in \( \{ \bar{c} \} \) as \( \{ \bar{\mu}c^0 \} \) with \( \{ c^0 \} \) dimensionless constants. This is legitimate as \( \{ \bar{c} \} \) are all of the same order as \( \{ g \} \), otherwise they should belong to the same set of the underlying constants \( \{ \sigma \} \) and vanish from the explicit formulation of QFTs.

Rescaling every dimensional parameters in a general vertex function \( \Gamma^{(n)}((p), (g); \{ \bar{c} \}) \) (we denote masses and couplings collectively as \( \{ g \} \)) that is well defined in QTOE, we have

\[
\{ \bar{\sigma} \partial_\sigma + \Sigma_d g \partial_g + \bar{\mu} \partial_\mu - \bar{d}_T^{(n)} \} \bar{\Gamma}^{(n)}((p), (g); \{ \bar{c} \}) = 0. \tag{6}
\]

with \( \partial_g \) denoting the mass dimensions of the associated constants. Since all the constants \( \{ \bar{c} \} \) only appear in the local parts of 1PI vertices, then \( \partial_g c \bar{c} \partial_g \bar{\mu} \) induces the insertion of all the vertex operators \( \{ O \} \), i.e., \( \Sigma_{\{ O \}} \partial_\sigma \bar{I}_O \bar{\mu} \partial_\mu \bar{\Gamma}^{(n)}((p), (g); \{ \bar{\mu}, [c^0] \}) \).

\[
\partial_\sigma \bar{\Gamma}^{(n)}((p), (g); \{ \bar{\mu}, [c^0] \}) = \Sigma_{\{ O \}} \partial_\sigma \bar{I}_O \partial_\sigma \bar{\Gamma}^{(n)}((p), (g); \{ \bar{\mu}, [c^0] \}). \tag{7}
\]

This is just the general form of renormalization group equation (RGE) in our approach. Close investigation of the solutions of Eq. (4) in terms of masses will show that the anomalous dimension \( \delta_\sigma \) of a vertex operator \( O \) must be functions of dimensionless tree couplings \( [g^0] \) and \( [c^0] \), i.e., \( \delta_\sigma = \delta_\sigma ([g^0], [c^0]) \) [10]. The insertion of all the Lagrangian operators with couplings \( \{ g \} \) can be realized by \( g \partial_g \) (for mass, it is \( m^2 \partial_m \), \( k = 1 \) (fermion), 2 (boson)), i.e., \( \Sigma_{\{ O \}} \partial_\sigma \bar{I}_O = \Sigma \delta_\sigma g \partial_g \Sigma \delta_\sigma \bar{I}_O \partial_\sigma + \Sigma_{\{ O \}} \partial_\sigma \bar{I}_O \), with \( \phi \) and \( \bar{O} \) denoting respectively the 'elementary' fields in Lagrangian and the operators not defined in Lagrangian. (Here we use \( \partial_\phi \partial_\phi \phi \) to refer to the kinetic vertex for both fermionic and bosonic fields of any spin for simplicity, this does not affect the following deduction as the kinetic terms must be quadratic in the field operators.) Apparently \( \Sigma_{\{ O \}} \partial_\sigma \bar{I}_O \) is absent in renormalizable theories, while for unrenormalizable models, there will be infinitely many \( \bar{O} \) operators. The insertion of the kinetic operator \( \delta_\sigma \bar{I}_O \partial_\phi \partial_\phi \phi \) will induce a rescaling of the field operator \( \phi \) by amount \( \frac{\partial_\sigma}{\Phi} \). Thus in renormalizable theories, we obtain that

\[
\{ \bar{\mu} \partial_\mu - \Sigma \delta_\sigma g \partial_g - \bar{d}_T^{(n)} \} \bar{\Gamma}^{(n)}((p), \{ g \}; \{ \bar{\mu}, [c^0] \}) = 0. \tag{8}
\]

with \( \bar{\delta}_g \equiv \delta_\sigma - \Sigma_{\{ g \}} \delta_\sigma \). Since \( (g) \) and \( \{ \bar{\mu}, [c^0] \} \) should be uniquely determined by QTOE, the variation in Eq. (8) should be understood as the change due to the global rescaling of everything. Thus by introducing a natural set of scale co-moving (or 'running') parameters basing on Coleman’s bacteria analogue [11], we finally arrive at the standard form of RGE which replaces Eq. (8)

\[
\{ \mu \partial_\mu - \Sigma \delta_\sigma g \partial_g - \bar{d}_T^{(n)} \} \Gamma^{(n)}((p), (\mu); \{ \mu, [c^0] \}) = 0. \tag{9}
\]

with \( \delta_\sigma g \equiv \delta_\sigma [g^0 (\mu; [g^0]), [c^0]] \). \( g (\mu; [g^0]), \mu = \bar{\mu}, \mu : \max [\mu] \ll \inf \{ \sigma \} \). Now we see that the 'running' of the parameters is closely related to the rescaling procedure of \( \bar{\mu} \) whose appearance is
naturally guaranteed in QTOE by the low energy limit operation, the mystery atmosphere around the dimensional transmutation phenomenon is therefore removed.

Inserting Eq. (9) back into Eq. (6) we will get the full scaling law due to Callan-Symanzik [12]

$$\{s \partial_s + \Sigma \delta \bar{g} \partial_{\bar{g}} + \delta_{\Gamma(n)} - d_{\Gamma(n)}\} \Gamma^{(n)}((s \bar{p}), (\bar{g}) ; \{\bar{\mu}, [\bar{c}^0]\}) = -i \Gamma^{(n)}_{\Omega}(0, (s \bar{p}), (\bar{g}) ; \{\bar{\mu}, [\bar{c}^0]\}),$$

(10)

where

$$s \partial_s \bar{g} (s \bar{p}; (g)) = \bar{g} (s \bar{p}; (g)) \delta_\bar{g} \left( [\bar{g}^0(s \bar{p}; [\bar{g}^0])], [\bar{c}^0] \right), \quad \bar{g} (s \bar{p}; (g)) |_{s = 1} = g,$$

$$i \Gamma^{(n)}_{\Omega}(0, (s \bar{p}), (\bar{g}) ; \{\bar{\mu}, [\bar{c}^0]\}) = \Sigma d_{\bar{g}} \bar{g} \bar{g} \cdot \Gamma^{(n)}((s \bar{p}), (\bar{g}) ; \{\bar{\mu}, [\bar{c}^0]\}),$$

(11)

(12)

with \(\Theta\) being the trace of the energy tensor of the theory. Of course in reality we are forced to replace \(\{\bar{c}\}\) with \(\{C\} = \{\mu_{\text{int}}, [C^0]\}^1\), but in principle we can start with tree parameters and determine \(\{C\}\) by confronting our calculations with experimental data as mentioned above.

One might oppose that the QTOE is never seen. Our answer is that the present QFT or field equations has been verified only within a limited region of the phase space. As a matter of fact, it is no harm to start with a postulated high energy modes. This is easy to see: since the constants indefinite momentum integration and

$$\{s \partial_s + \Sigma \delta \bar{g} \partial_{\bar{g}} + \delta_{\Gamma(n)} - d_{\Gamma(n)}\} \Gamma^{(n)}((s \bar{p}), (\bar{g}) ; \{\bar{\mu}, [\bar{c}^0]\}) = 0$$

(13)

with the \(\Sigma d_{\bar{g}} \bar{g} \partial_{\bar{g}}\) or \(\bar{\mu} \partial_{\bar{\mu}}\) replaced by \(\Sigma d_s \partial_s\). This is obviously a normal scaling law in QTOE. However, as the constants \(\{\sigma\}\) are vanishingly small the high energy modes become ‘formally’ decoupled while their contributions in the scaling law persist and appear as ‘anomalies’ in terms of the ‘tree’ parameters \(\{g\}\), i.e., the terms \(\{\Sigma \delta \bar{g} \partial_{\bar{g}} + \delta_{\Gamma(n)}\} \Gamma^{(n)}(\ldots)\) in Eq. (9), which are subsumed into \(\bar{\mu} \partial_{\bar{\mu}}\), a coarse grained way to reproduce the underlying structures’ contributions according to Eq. (9),

$$L_{\{\sigma\}} \left( \Sigma d_s \partial_s \Gamma^{(n)}(\ldots) \right) = \{\Sigma \delta \bar{g} \partial_{\bar{g}} + \delta_{\Gamma(n)}\} \Gamma^{(n)}(\ldots) = \bar{\mu} \partial_{\bar{\mu}} \Gamma^{(n)}(\ldots).$$

(14)

Thus it is the ‘decoupling effects of high energy modes’ that lead to the violation of naive scaling law in terms of the QFT parameters \(\{g\}\), no divergence is involved here. Since such scaling ‘anomalies’ can be absorbed into the redefinition of the tree parameters, we see that such ‘anomalies from the underlying modes decoupling lead to finite ‘renormalization’ of the tree parameters’. One might understand this mechanism from the decoupling effects of heavy fermions upon the beta function in QCD or QED as was illustrated in Ref. [13], where \(\lim_{M \rightarrow \infty} M \partial_M \Gamma = \Delta \delta \partial_s \Gamma\). Thus the physical meaning of RGE is deepened in the new strategy, namely, RGE is an inevitable consequence of the fact that QFTs are incomplete formulations of the low energy sectors of a complete theory.

IV. REMARKS AND SUMMARY

Conventionally, one is forced to use some artificial regulators to define the UV or high energy ends of a QFT so that the loop amplitudes could be calculated. Such procedures do not automatically make the loop amplitudes finite and subsequent subtraction of divergent pieces is necessary. The subtraction leads to a residual ambiguity that is in fact encoded in RGE. The order of logic is first (1) regularization, then (2) subtraction/renormalization and finally (3) renormalization group. While in the underlying theory (QTOE) scenario, the amplitudes that are only formally defined in QFT are coarse grained ones in QTOE with high energy details integrated out or coarse grained

\[\text{Here } \mu_{\text{int}}, [C^0] \parallel \bar{\mu}, [\bar{c}^0] \text{ with } \mu_{\text{int}} \text{ standing for the dimensional constant scale that will necessarily appear in the indefinite momentum integration and } [C^0] \text{ for dimensionless constants.}\]
away. In the decoupling (or low energy) limit, the coarse grained objects or sectors are subject to certain freedom of redefinition, which is just the freedom corresponding to renormalization group. No extra procedures for regularization and subtraction are needed here except the natural coarse graining and decoupling limit operation, perhaps under a different name. In a sense, we provide a more physical and reasonable foundation for renormalization group and finite renormalization without the somewhat unnatural procedures of regularization and subtraction of infinities.

We remind that the present formulation are only valid provided the underlying modes’ typical time scale is vanishingly small in comparison with the QFT processes’ time scale. In this sense the parameters in QFT’s are some kind of collective ‘coordinates’ of the coarse grained objects. So our approach is in fact pointing towards a unified framework for quantization, coarse graining, renormalization and unification of interactions, at least in the conceptual sense.

Finally, we stress again the there is no need to introduce counter terms in the differential equation approach, one only needs to fix the ambiguities, which are especially important for the electroweak theory with spontaneous symmetry breaking whose renormalization is rather complicated [14]. Nonetheless, since we have heavily relied upon the effective theory versus underlying complete theory duality, our strategy could be readily applied to the effective field theory approach, especially to nonperturbative problems such as nucleon interactions [15]. Further applications in these perspectives will be pursued in the future.

In summary, we provided natural arguments in favor of a recently proposed simple strategy for renormalization. The key point is the existence of a complete quantum theory of everything which contains full information of high energy physics that are lacking in the present the QFT’s. From this QTOE scenario, the Callan-Symanzik equation and RGE can be derived in a natural way with the RGE recognized as a decoupling theorem of the high energy modes that underlie the present QFTs or similar quantum theories. The conceptual foundation for renormalization group is more reasonable in the QTOE scenario and is not entangled with divergence at all.

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