NON-UNIFORM CONTINUOUS DEPENDENCE ON INITIAL DATA OF SOLUTIONS TO THE EULER-POINCARÉ SYSTEM

JINLU LI, LI DAI, AND WEIPENG ZHU

Abstract. In this paper, we investigate the continuous dependence on initial data of solutions to the Euler-Poincaré system. By constructing a sequence approximate solutions and calculating the error terms, we show that the data-to-solution map is not uniformly continuous in Sobolev space $H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$.

1. Introduction and main result

In the paper, we consider the following Cauchy problem of the Euler-Poincaré system:

$$\begin{cases}
\partial_t m + u \cdot \nabla m + (\nabla u)^T m + (\text{div} u) m = 0, \\
m = (1 - \Delta) u, \\
u(0, x) = u_0,
\end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $u = (u_1, u_2, \cdots, u_d)$ denotes the velocity of the fluid, $m = (m_1, m_2, \cdots, m_d)$ represents the momentum. The notation $(\nabla u)^T$ represents the transpose of the matrix $\nabla u$.

The system (1.1) is the classical Camassa-Holm (CH) equation for $d = 1$, while it is also called the Euler-Poincaré equations in the higher dimensional case $d \geq 1$. The Camassa-Holm equation can be regarded as a shallow water wave equation [4, 5, 16]. It is completely integrable [4, 8], has a bi-Hamiltonian structure [7, 21], and admits exact peaked solitons of the form $ce^{-|x-ct|}, c > 0$, which are orbitally stable [18]. We have to say that the peaked solitons present the characteristic for the travelling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf., [10, 14, 15, 31] and references therein. The local well-posedness and ill-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in [11, 12, 19, 20, 22, 28, 30]. The existence and finite time blow-up of strong solutions to the CH equation was shown in [9, 11, 12, 13]. For the existence and uniqueness of global weak solutions to the CH equation, we refer the reader to see [17, 32]. The global conservative and dissipative solutions of CH equation were studied in [2, 3]. The non-uniform dependence on initial data for the CH equation was discussed in [23, 24].

For the Euler-Poincaré system, it was first theoretically studied by Chae and Liu in the pioneering work [6]. The authors obtained the local well-posedness in Hilbert spaces $m_0 \in H^{s+\frac{d}{2}}, s \geq 2$ and also gave a blow-up criterion, zero α limit and the Liouville type theorem. In [27], Li, Yu and Zhai showed that the solution to (1.1) with a large class of smooth initial data blows up in finite time or exists globally in

2010 Mathematics Subject Classification. 35Q35.
Key words and phrases. The Euler-Poincaré system, Non-uniform continuous dependence.
time, which reveals the nonlinear depletion mechanism hidden in the Euler-Poincaré system. By applying Littlewood-Paley theory, the local existence and uniqueness in Besov spaces $B^s_{2,r}$, $s > \max\{\frac{4}{p}, 1 + \frac{d}{p}\}$ and $s = 1 + \frac{d}{p}$, $1 \leq p \leq 2d$, $r = 1$, were established by Yan and Yin [33]. Lately, Li and Yin [29] proved that the corresponding solution is continuous dependence for the initial data in Besov spaces. Zhao, Yang and Li [34] showed that the solution map of the periodic Euler-Poincaré system is not uniformly continuous in $B^s_{2,r}$, $s > 1 + \frac{d}{2}$.

In this paper, inspired by [25, 26], we will show that the solution map of (1.1) is not uniformly continuous in dependence in Sobolev space $H^s$. Compared with the Camassa-Holm type equation in one dimension, we need choose the suitable approximate solutions and calculate the more error terms.

According to [33], we can transform (1.1) into the following form:

$$\partial_t u + u \cdot \nabla u = P(u, u) := -(1 - \Delta)^{-1}\text{div}Q(u, u) - (1 - \Delta)^{-1}R(u, u), \quad (1.2)$$

where

$$Q(u, v) = \nabla u \nabla v + \nabla u(\nabla v)^T - (\nabla u)^T \nabla v - (\text{div} u) \nabla v + \frac{1}{2} I(\nabla u : \nabla v),$$

$$R(u, v) = (\text{div} u) v + u \cdot \nabla v.$$

Then, we have the following result.

**Theorem 1.1.** Let $d \geq 2$ and $s > 1 + \frac{d}{2}$. The data-to-solution map for the Euler-Poincaré system (1.1) is not uniformly continuous from any bounded subset in $H^s$ into $C([0, T]; H^s(\mathbb{R}^d))$. That is, there exists two sequences of solutions $u^n$ and $v^n$ such that

$$||u^n||_{H^s} + ||v^n||_{H^s(\mathbb{R}^d)} \lesssim 1, \quad \lim_{n \to \infty} ||u^n - v^n||_{H^s(\mathbb{R}^d)} = 0,$$

$$\lim \inf_{n \to \infty} ||u^n(t) - v^n(t)||_{H^s(\mathbb{R}^d)} \gtrsim |\sin t|, \quad t \in [0, T].$$

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in the sequel. In Section 3, we give the proof of our main theorem.

**Notations.** Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. The symbol $A \lesssim B$ means that there is a uniform positive constant $c$ independent of $A$ and $B$ such that $A \leq cB$. Here

$$(\nabla u)_{i,j} = \partial_{x_i} u_j, \quad (u \cdot \nabla v)_i = \sum_{k=1}^{d} u_k \partial_{x_k} u_j, \quad (\nabla u \nabla v)_{ij} = \sum_{k=1}^{d} \partial_{x_i} u_k \partial_{x_k} v_j,$$

$$\nabla u : \nabla v = \sum_{i,j=1}^{d} \partial_{x_i} u_j \partial_{x_i} v_j.$$

2. **Littlewood-Paley Analysis**

In this section, we will recall some facts about the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties. For more details, the readers can refer to [1].

There exists a couple of smooth functions $(\chi, \varphi)$ valued in $[0, 1]$, such that $\chi$ is supported in the ball $B \triangleq \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\}$, and $\varphi$ is supported in the ring.
Lemma 2.4. \( \{ \xi \in \mathbb{R}^d : \frac{d}{2} \leq |\xi| \leq \frac{d}{2} \} \). Moreover,

\[ \forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \]

\[ \forall 0 \neq \xi \in \mathbb{R}^d, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \]

\[ |j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}) \cap \text{Supp } \varphi(2^{-j'}) = \emptyset, \]

\[ j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j}) = \emptyset, \]

Then, we can define the nonhomogeneous dyadic blocks \( \Delta_j \) and nonhomogeneous low frequency cut-off operator \( S_j \) as follows:

\[ \Delta_j u = 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \]

\[ \Delta_j u = \varphi(2^{-j}D) u = \mathcal{F}^{-1}(\varphi(2^{-j}) \mathcal{F} u), \text{ if } j \geq 0, \]

\[ S_j u = \sum_{j' = -\infty}^{j-1} \Delta_j u. \]

**Definition 2.1.** ([1]) Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}^d) \) consists of all tempered distribution \( u \) such that

\[ \|u\|_{B^s_{p,r}(\mathbb{R}^d)} \triangleq \left( \sum_{j \in \mathbb{Z}} \|\Delta_j u\|_{L^p(\mathbb{R}^d)}^r \right)^{\frac{1}{r}} < \infty. \]

**Remark 2.2.** ([1]) When \( p = r = 2 \), we have \( B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d) \). Here, \( H^s(\mathbb{R}^d) \) is the standard Sobolev space with the norm

\[ \|u\|_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi. \]

For any \( u \in B^s_{2,2}(\mathbb{R}^d) \), there holds

\[ c_0 \|u\|_{H^s(\mathbb{R}^d)} \leq \|u\|_{B^s_{2,2}(\mathbb{R}^d)} \leq C_0 \|u\|_{H^s(\mathbb{R}^d)}. \]

If \( s > \frac{d}{2} \), we also have \( \|u\|_{L^\infty(\mathbb{R}^d)} \lesssim \|u\|_{B^s_{2,2}(\mathbb{R}^d)}. \)

Then, we have the following product laws.

**Lemma 2.3.** ([1]) Let \( d \geq 2 \) and \( s > \frac{d}{2} \). Then there exists a constant \( C = C(d, s) \) such that

\[ \|uv\|_{B^s_{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{L^\infty(\mathbb{R}^d)} \|v\|_{B^s_{2,2}(\mathbb{R}^d)} + \|v\|_{L^\infty(\mathbb{R}^d)} \|u\|_{B^s_{2,2}(\mathbb{R}^d)}), \]

\[ \|uv\|_{B^s_{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{B^s_{2,2}(\mathbb{R}^d)} \|v\|_{B^s_{2,2}(\mathbb{R}^d)}), \quad \frac{1}{2} \leq \|uv\|_{L^2(\mathbb{R}^d)} \leq C(\|u\|_{B^s_{2,2}(\mathbb{R}^d)} \|v\|_{B^s_{2,2}(\mathbb{R}^d)}). \]

**Lemma 2.4.** ([1]) Let \( d \geq 2 \) and \( s > \frac{d}{2} - 1 \). Assume that \( f_0 \in B^s_{2,2}(\mathbb{R}^d) \) and \( \nabla v \in L^1_t(B^s_{2,2}(\mathbb{R}^d) \cap B^s_{2,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \). If \( f \in C([0, T]; B^s_{2,2}(\mathbb{R}^d)) \) solves the following linear transport equation:

\[ \begin{aligned}
\partial_t f + v \cdot \nabla f &= F, \\
\quad f(0, x) &= f_0,
\end{aligned} \]

then there exists a positive constant \( C = C(s) \) such that

\[ \|f\|_{B^s_{2,2}(\mathbb{R}^d)} \leq \|f_0\|_{B^s_{2,2}(\mathbb{R}^d)} + C \int_0^t \|V'(\tau)\|_{B^s_{2,2}(\mathbb{R}^d)} d\tau + \int_0^t \|F(\tau)\|_{B^s_{2,2}(\mathbb{R}^d)} d\tau, \]
or
\[
\|f\|_{B^1_2(\mathbb{R}^d)} \leq e^{CV(t)} \left( \|f_0\|_{B^1_2(\mathbb{R}^d)} + \int_0^t e^{-CV(\tau)} \|f(\tau)\|_{B^1_2(\mathbb{R}^d)} d\tau \right),
\]
where
\[
V(t) = \begin{cases} 
\int_0^t \|\nabla v\|_{B^{s-1}_2(\mathbb{R}^d)} \, d\tau, & s > \frac{d}{2}, \\
\int_0^t \|\nabla v\|_{B^s_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)} \, d\tau, & s \leq \frac{d}{2}.
\end{cases}
\]

Lemma 2.5. ([25]) Let \( \phi \in S(\mathbb{R}) \), \( \delta > 0 \) and \( s \geq 0 \). Then we have for all \( \alpha \in \mathbb{R} \),
\[
\lim_{n \to \infty} n^{-\frac{2s}{2s-1}} \|\phi(x/n)\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}} \|\phi\|_{L^2(\mathbb{R})},
\]
\[
\lim_{n \to \infty} n^{-\frac{2s}{2s-1}} \|\phi(x/n)\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}} \|\phi\|_{L^2(\mathbb{R})}.
\]

3. Non-uniform continuous dependence

In this section, we will give the proof of our main theorem. Firstly, motivated by [25, 26], we can construct a sequence approximate solutions where the last \( d - 1 \) component is 0. Lately, we consider the difference about approximate solution and actual solution and also show that this distance is decaying. Finally, by the precious steps, we can conclude that the the solution map is not uniformly continuous. In order to state our main result, we first recall the following local-in-time existence of strong solutions to (1.1) in [33]:

Lemma 3.1. ([33]) For \( s > 1 + \frac{d}{2} \), \( d \geq 2 \), and initial data \( u_0 \in B^{s-1}_2(\mathbb{R}^d) \), there exists a time \( T = T(s, d, \|u_0\|_{B^{s-1}_2(\mathbb{R}^d)}) > 0 \) such that the system (1.1) have a unique solution \( u \in C([0, T]; B^{s}_2(\mathbb{R}^d)) \). Moreover, for all \( t \in [0, T] \), there holds
\[
\|u(t)\|_{B^s_2(\mathbb{R}^d)} \leq \frac{\|u_0\|_{B^{s-1}_2(\mathbb{R}^d)}}{1 - C_s T \|u_0\|_{B^{s-1}_2(\mathbb{R}^d)}}.
\]

Corollary 3.2. Let \( s > 1 + \frac{d}{2} \), \( d \geq 2 \). Let \( u \in C([0, T]; H^s) \) be the solution of the system (1.1). Then, we have for all \( t \in [0, T] \),
\[
\|u(t)\|_{B^s_2(\mathbb{R}^d)} \leq \|u_0\|_{B^{s-1}_2(\mathbb{R}^d)} e^{\int_0^t \|\phi(\tau)\|_{B^s_2(\mathbb{R}^d)} d\tau},
\]
and
\[
\|u(t)\|_{B^s_2(\mathbb{R}^d)} \leq \|u_0\|_{B^{s+1}_2(\mathbb{R}^d)} e^{\int_0^t \|\phi(\tau)\|_{B^{s+1}_2(\mathbb{R}^d)} d\tau}.
\]

Proof. The results can easily deduce from Lemma 2.4 and Gronwall’s inequality. Here, we omit it.

Now, we give the details of the proof to our theorem.

Proof of the main theorem. Set \( 0 < \delta < \frac{1}{2} \) and let \( \phi \) be a \( C_0(\mathbb{R}) \) function such that
\[
\phi(x) = \begin{cases} 1, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}
\]
Moreover, let \( \Phi \) be a \( C_0(\mathbb{R}) \) function such that \( \Phi(x) \phi(x) = \phi(x) \). Firstly, we choose the velocity having the following form:
\[
u^{\omega,n} = u^{1,\omega,n}(t, x) + u^{h,\omega,n}(t, x), \quad \omega \in \{\pm 1\}, \ x \in \mathbb{R}^d, \ t \in \mathbb{R},
\]
where \( u^{h,\omega,n} \) is the high frequency term

\[
u^{h,\omega,n}(t, x) = \left( \phi^{h,\omega,n}(t, x), 0, \ldots, 0 \right),
\]

with

\[
\phi^{h,\omega,n}(t, x) = n^{-\frac{4}{3} - s} \phi(\frac{x_1}{n^\delta}) \sin(nx_1 - \omega t) \phi(x_2) \cdots \phi(x_d), \quad n \in \mathbb{Z}.
\]

To choose the suitable low frequency term \( u^{l,\omega,n} \), we let \( u^{l,\omega,n} \) satisfy the following initial value problem:

\[
\begin{aligned}
\partial_t u^{l,\omega,n} + u^{l,\omega,n} \cdot \nabla u^{l,\omega,n} &= P(u^{l,\omega,n}, u^{l,\omega,n}), \\
u^{l,\omega,n}(0, x) &= (\omega n^{-1} \Phi(\frac{x_1}{n^\delta}) \Phi(x_2) \cdots \Phi(x_d), 0, \ldots, 0).
\end{aligned}
\] (3.1)

By the well-posedness result (see Lemma 3.1), \( u^{l,\omega,n} \) belong to \( C([0, T]; B^2_{2,2}) \) and have lifespan \( T \simeq 1 \). In order to simplify the notation, we set \( \tilde{\phi}(x_2, \ldots, x_d) = \phi(x_2) \cdots \phi(x_d) \) and \( \tilde{\Phi}(x_2, \ldots, x_d) = \Phi(x_2) \cdots \Phi(x_d) \). Thus, we can find that the constructional solution \( u^{w,n} \) satisfies the following equations:

\[
\begin{aligned}
\partial_t u^{w,n} + u^{w,n} \cdot \nabla u^{w,n} &= P(u^{w,n}, u^{w,n}) + \partial_t u^{h,n} + u^{h,n} \cdot \nabla u^{h,n}, \\
u^{w,n}(0, x) &= (\omega n^{-1} \Phi(\frac{x_1}{n^\delta}) \Phi(\tilde{x}_2, \ldots, \tilde{x}_d), 0, \ldots, 0).
\end{aligned}
\]

with initial data

\[
u^{w,n}_0 := u^{w,n}(0, x) = (n^{-\frac{4}{3} - s} \phi(\frac{x_1}{n^\delta}) \sin(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) - \omega n^{-1} \Phi(\frac{x_1}{n^\delta}) \tilde{\Phi}(x_2, \ldots, x_d), 0, \ldots, 0).
\]

Let us consider the actual solution \( u_{\omega,n} \) which is the solution of (1.2) with the same initial data \( u^{w,n}_0 \). Then, \( u_{\omega,n} \) satisfies

\[
\partial_t u_{\omega,n} + u_{\omega,n} \cdot \nabla u_{\omega,n} = P(u_{\omega,n}, u_{\omega,n}),
\]

with initial data

\[
u_{\omega,n}(0, x) = (n^{-\frac{4}{3} - s} \phi(\frac{x_1}{n^\delta}) \sin(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) - \omega n^{-1} \Phi(\frac{x_1}{n^\delta}) \tilde{\Phi}(x_2, \ldots, x_d), 0, \ldots, 0).
\]

By the well-posedness result (see Lemma 3.1), the solution \( u_{\omega,n} \) belong to \( C([0, T]; B^2_{2,2}) \) and have common lifespan \( T \simeq 1 \). For the estimates of \( u_{\omega,n} \), we get from Corollary 3.2 that

\[
||u_{\omega,n}||_{L^\infty_T(B^{2\gamma}_{2,2}(\mathbb{R}^d))} \lesssim n^{-1 + \frac{1}{2\gamma}}, \quad ||u_{\omega,n}||_{L^\infty_T(B^{2\gamma}_{2,2}(\mathbb{R}^d))} \lesssim 1, \quad ||u_{\omega,n}||_{L^\infty_T(B^{2\gamma+1}_{2,2}(\mathbb{R}^d))} \lesssim n. \quad (3.2)
\]

Next, considering the difference \( v = u_{\omega,n} - u^{w,n} \), we observe that \( v \) satisfy

\[
\partial_t v + u_{\omega,n} \cdot \nabla v + v \cdot \nabla u^{w,n} = P(u_{\omega,n}, v) + P(v, u^{w,n}) - E^{w,n} - F^{w,n},
\]

with initial data \( v_0 = 0 \).

Now, we shall estimate the difference \( v \) in the Sobolev \( B^2_{2,2} \) norm. We hope that the decay of \( ||v||_{B^2_{2,2}} \) is less than \( n^{-1} \). According to Lemmas 2.3-2.4, we have for all
Collecting the following estimates

\[ \|u(t)\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \leq C \int_0^t \|v\|_{B_{2,2}^{-1}} \left( \|u_n^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} + \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \right) \, dt + C \int_0^t \left( \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} + \|F_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \right) \, dt. \]  

(3.3)

Hence, we shall estimate the terms \(E_{n}^\omega\) and \(F_{n}^\omega\) in Sobolev \(B_{2,2}^{-1}\) norm. To obtain the desired result, we need to estimate the terms \(u_{n}^\omega\) and \(u_{n}^\omega\). By Lemma 2.5 and Corollary 3.2, we have for any \(r \geq 0\) and \(t \in [0,T]\),

\[ \begin{align*}
\|u_{n}^\omega(t)\|_{B_{2,2}^1(\mathbb{R}^d)} & \lesssim \|u_{n}^\omega(t)\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-1+\frac{3}{2}} \\
n^{-2}, \quad \|\nabla u_{n}^\omega(t)\|_{L^\infty(\mathbb{R}^d)} & \lesssim n^{-2}.
\end{align*} \]

(3.4)

Collecting the following estimates

\[ \|u_{n}^\omega\cdot \nabla u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}, \]

\[ \|u_{n}^\omega\cdot \nabla u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|\nabla u_{n}^\omega\|_{L^\infty(\mathbb{R}^d)} \lesssim n^{-2}, \]

\[ \|Q(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|\nabla u_{n}^\omega\|_{L^\infty(\mathbb{R}^d)} \lesssim n^{-2}, \]

\[ \|Q(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}, \]

\[ \|Q(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}, \]

\[ \|R(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2}, \]

\[ \|R(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}, \]

\[ \|R(u_{n}^\omega, u_{n}^\omega)\|_{B_{2,2}^{-2}(\mathbb{R}^d)} \lesssim \|u_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \|u_{n}^\omega\|_{B_{2,2}^1(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}, \]

we can infer that

\[ \|F_{n}^\omega\|_{B_{2,2}^{-1}(\mathbb{R}^d)} \lesssim n^{-2+\frac{3}{2}}. \]  

(3.5)

For the term \(E_{n}^\omega\), we can estimate it by component. Direct calculation shows that \((E_{n}^\omega)_{i} = 0\) for \(i = 2, \cdots, d\). According to the definition of \(u_{n}^\omega(0, x)\), we can write the first component of \(\partial_{t}u_{n}^\omega\) and \(u_{n}^\omega \cdot \nabla u_{n}^\omega\) in the form

\[ \begin{align*}
\left( \partial_{t}u_{n}^\omega \right)_{1}(t, x) & = -\omega n^{-\delta+s} \phi(\frac{x_1}{n^s}) \cos(nx_1 - \omega t) \tilde{\phi}(x_2, \cdots, x_d) \\
& = -n^{-s+1+\frac{3}{2}} u_{1}^\omega(0, x) \phi(\frac{x_1}{n^s}) \cos(nx_1 - \omega t) \tilde{\phi}(x_2, \cdots, x_d),
\end{align*} \]

where \(\phi\) is a suitable function.
and
\[
\left( u^{l,\omega,n} \cdot \nabla u^{h,\omega,n} \right)_1 (t, x) = n^{-s+1-\frac{1}{2}\delta} u_1^{l,\omega,n}(t, x) \phi \left( \frac{x_1}{n^\delta} \right) \cos(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) \\
+ n^{-s-\frac{1}{2}\delta} u_1^{l,\omega,n}(t, x) \phi' \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) \\
+ n^{-\frac{1}{2}\delta - s} \phi \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t) \sum_{i=2}^d u_i^{l,\omega,n}(t, x) \partial_i \tilde{\phi}(x_2, \ldots, x_d).
\]

Therefore, the term \((E^{\omega,n})_1\) can be written as
\[
\left( \partial_t u^{h,\omega,n} + u^{l,\omega,n} \cdot \nabla u^{h,\omega,n} \right)_1 (t, x) = n^{-s+1-\frac{1}{2}\delta} [u_1^{l,\omega,n}(t, x) - u_1^{l,\omega,n}(0, x)] \phi \left( \frac{x_1}{n^\delta} \right) \cos(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) \\
- n^{-s-\frac{1}{2}\delta} u_1^{l,\omega,n}(t, x) \phi' \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t) \tilde{\phi}(x_2, \ldots, x_d) \\
+ n^{-\frac{1}{2}\delta - s} \phi \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t) \sum_{i=2}^d u_i^{l,\omega,n}(t, x) \partial_i \tilde{\phi}(x_2, \ldots, x_d) \\
:= E^{\omega,n}_{1,1}(t) + E^{\omega,n}_{1,2}(t) + E^{\omega,n}_{1,3}(t).
\]

By Lemmas 2.3-2.5, we can estimate the last two terms as follows:
\[
||E^{\omega,n}_{1,2}(t)||_{B^{-1}_{2,2}(\mathbb{R}^d)} \lesssim n^{-s-\frac{1}{2}\delta} ||u_1^{l,\omega,n}(t, x)||_{B^{1}_{2,2}(\mathbb{R}^d)} \\
\times ||\phi' \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t)||_{B^{-1}_{2,2}(\mathbb{R}^d)} ||\tilde{\phi}||_{B^{-1}_{2,2}(\mathbb{R}^{d-1})} \\
\lesssim n^{-s-\frac{1}{2}\delta} \cdot n^{-1-\frac{1}{2}\delta} \cdot n^{-1+\frac{1}{2}\delta} \lesssim n^{-2},
\]
and
\[
||E^{\omega,n}_{1,3}(t)||_{B^{1}_{2,2}(\mathbb{R}^d)} \lesssim n^{-s-\frac{1}{2}\delta} ||u^{l,\omega,n}(t, x)||_{B^{1}_{2,2}(\mathbb{R}^d)} \\
\times ||\phi \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t)||_{B^{1}_{2,2}(\mathbb{R}^d)} ||\tilde{\phi}||_{B^{1}_{2,2}(\mathbb{R}^{d-1})} \\
\lesssim n^{-s-\frac{1}{2}\delta} \cdot n^{-1+\frac{1}{2}\delta} \cdot n^{-1+\frac{1}{2}\delta} \lesssim n^{-2+\frac{1}{2}\delta}.
\]

For the term \(E^{\omega,n}_{1,1}(t)\), by Lemmas 2.3-2.5, we can compute it as
\[
||E^{\omega,n}_{1,1}(t)||_{B^{1}_{2,2}(\mathbb{R}^d)} \lesssim n^{-s+1-\frac{1}{2}\delta} ||u^{l,\omega,n}(t) - u^{l,\omega,n}(0)||_{B^{1}_{2,2}(\mathbb{R}^d)} \\
\times ||\phi' \left( \frac{x_1}{n^\delta} \right) \sin(nx_1 - \omega t)||_{B^{1}_{2,2}(\mathbb{R}^d)} ||\tilde{\phi}||_{B^{1}_{2,2}(\mathbb{R}^{d-1})} \\
\lesssim n^{-s+1-\frac{1}{2}\delta} \cdot n^{-1+\frac{1}{2}\delta} ||u^{l,\omega,n}(t) - u^{l,\omega,n}(0)||_{B^{1}_{2,2}(\mathbb{R}^d)} \\
\lesssim ||u^{l,\omega,n}(t) - u^{l,\omega,n}(0)||_{B^{1}_{2,2}(\mathbb{R}^d)}.
\]
Since \(u^{l,\omega,n}\) is the solution of \((3.1)\), then we can estimate the integral term above by
\[
||u^{l,\omega,n}(t) - u^{l,\omega,n}(0)||_{B^2_{2,2}(\mathbb{R}^d)}
\leq \int_0^t ||\partial_x u^{l,\omega,n}(\tau)||_{B^2_{2,2}(\mathbb{R}^d)}d\tau
\leq \int_0^t \left(||u^{l,\omega,n} \cdot \nabla u^{l,\omega,n}||_{B^2_{2,2}(\mathbb{R}^d)} + ||P(u^{l,\omega,n}, u^{l,\omega,n})||_{B^2_{2,2}(\mathbb{R}^d)}\right)d\tau
\lesssim ||u^{l,\omega,n}||_{B^2_{2,2}(\mathbb{R}^d)} ||u^{l,\omega,n}||_{B^{2+1}_{2,2}(\mathbb{R}^d)} \lesssim n^{-1+\frac{1}{2}\delta} \cdot n^{-1+\frac{1}{2}\delta} \lesssim n^{-2+\delta}.
\] Combining these estimates \((3.6)-(3.9)\), we get
\[
||E^{\omega,n}||_{B^{2-1}_{2,2}(\mathbb{R}^d)} \lesssim n^{-2+\delta}.
\] Plugging \((3.2), (3.5)\) and \((3.10)\) into \((3.3)\), we have for all \(t \in [0, T]\),
\[
||v(t)||_{B^{2-1}_{2,2}(\mathbb{R}^d)} \leq C \int_0^t ||v||_{B^{2-1}_{2,2}(\mathbb{R}^d)}d\tau + Cn^{-2+\delta}.
\] This along with Growall’s inequality yields
\[
||v||_{L^\infty(T;B^{2-1}_{2,2}(\mathbb{R}^d))} \leq Cn^{-2+\delta}.
\] Noticing that
\[
||v||_{L^\infty(T;B^{2+1}_{2,2}(\mathbb{R}^d))} \lesssim ||u^{\omega,n}||_{L^\infty(T;B^{2+1}_{2,2}(\mathbb{R}^d))} + ||u^{\omega,n}||_{L^\infty(T;B^{2+1}_{2,2}(\mathbb{R}^d))} \lesssim n,
\] and using the interpolation inequality and \((3.11)\), we have
\[
||v||_{L^\infty(T;B^{2}_{2,2}(\mathbb{R}^d))} \leq ||v||_{L^\infty(T;B^{2}_{2,2}(\mathbb{R}^d))}^{\frac{1}{2}} ||v||_{L^\infty(T;B^{2+1}_{2,2}(\mathbb{R}^d))}^{\frac{1}{2}} \leq Cn^{-\frac{1}{2}+\frac{1}{2}\delta}.
\] Combining \((3.4)\) and \((3.12)\), then there exists some positive constant \(c_0\) such that
\[
||u_{1,n}(t) - u_{-1,n}(t)||_{B^{2}_{2,2}(\mathbb{R}^d)}
\geq ||u^{1,n}(t) - u^{-1,n}(t)||_{B^{2}_{2,2}(\mathbb{R}^d)} - C\varepsilon_n
\geq ||u^{h,1,n}(t) - u^{-h,-1,n}(t)||_{B^{2}_{2,2}(\mathbb{R}^d)} - C\varepsilon'_n
\geq 2|\sin t| \cdot ||n^{-\frac{1}{2}+\delta} \phi \left(\frac{x_1}{n^\delta}\right) cos(nx_1) \tilde{\phi}(x_2, \cdots, x_d)||_{B^{2}_{2,2}(\mathbb{R}^d)} - C\varepsilon'_n
\geq c_0 |\sin t| \cdot ||n^{-\frac{1}{2}+\delta} \phi \left(\frac{x_1}{n^\delta}\right) cos(nx_1) ||_{B^{2}_{2,2}(\mathbb{R}^d)} ||\tilde{\phi}(x_2, \cdots, x_d)||^{\frac{1}{2}}_{B^{2}_{2,2}(\mathbb{R}^{d-1})} - C\varepsilon'_n,
\]
where
\[
\varepsilon_n = n^{-\frac{1}{2}+\frac{1}{2}\delta}, \quad \varepsilon'_n = n^{-\frac{1}{2}+\frac{1}{2}\delta} + n^{\frac{1}{2}\delta-1}.
\] Letting \(n\) go to \(\infty\), we can show that
\[
\lim_{n \to \infty} ||u_{1,n}(t) - u_{-1,n}(t)||_{B^{2}_{2,2}(\mathbb{R}^d)} \gtrsim |\sin t|.
\] Notice that \(u^{h,1,n}(0, x) = u^{h,-1,n}(0, x)\), we get from Lemma 2.5 that
\[
||u_{1,n}(0, x) - u_{-1,n}(0, x)||_{B^{2}_{2,2}(\mathbb{R}^d)}
= ||u^{l,1,n}(0, x) - u^{l,-1,n}(0, x)||_{B^{2}_{2,2}(\mathbb{R}^d)} \leq Cn^{-1+\frac{1}{2}\delta} \to 0, \quad n \to \infty.
\] Then \((3.13)\) together with \((3.14)\) complete the proof of Theorem 1.1.

Acknowledgements. This work was partially supported by NSFC (No.11801090).
References

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.

[2] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal., 183 (2007), 215-239.

[3] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl., 5 (2007), 1-27.

[4] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., 71 (1993), 1661-1664.

[5] R. Camassa, D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech., 31 (1994), 1-33.

[6] D. Chae and J. Liu, *Blow-up, zero α limit and the Liouville type theorem for the Euler-Poincaré equations*, Comm. Math. Phys., 314 (2012), 671-687.

[7] A. Constantin, *The Hamiltonian structure of the Camassa-Holm equation*, Exposition. Math., 15 (1997), 53-85.

[8] A. Constantin, *On the scattering problem for the Camassa-Holm equation*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), 953-970.

[9] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble), 50 (2000), 321-362.

[10] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math., 166 (2006), 523-535.

[11] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), 303-328.

[12] A. Constantin and J. Escher, *Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math., 51 (1998), 475-504.

[13] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., 181 (1998), 229-243.

[14] A. Constantin and J. Escher, *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc., 44 (2007), 423-431.

[15] A. Constantin and J. Escher, *Analyticity of periodic traveling free surface water waves with vorticity*, Ann. of Math., 173 (2011), 559-568.

[16] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal., 192 (2009), 165-186.

[17] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, Comm. Math. Phys., 211 (2000), 45-61.

[18] A. Constantin and W. A. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., 53 (2000), 603-610.

[19] D. Chae and J. Liu, *Blow-up, zero α limit and the Liouville type theorem for the Euler-Poincaré equations*, Comm. Math. Phys., 314 (2012), 671-687.

[20] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math., 166 (2006), 523-535.

[21] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), 303-328.

[22] A. Constantin and J. Escher, *Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math., 51 (1998), 475-504.

[23] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., 181 (1998), 229-243.

[24] A. Constantin and J. Escher, *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc., 44 (2007), 423-431.

[25] A. Constantin and J. Escher, *Analyticity of periodic traveling free surface water waves with vorticity*, Ann. of Math., 173 (2011), 559-568.

[26] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal., 192 (2009), 165-186.

[27] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, Comm. Math. Phys., 211 (2000), 45-61.

[28] A. Constantin and W. A. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., 53 (2000), 603-610.

[29] D. Chae and J. Liu, *Blow-up, zero α limit and the Liouville type theorem for the Euler-Poincaré equations*, Comm. Math. Phys., 314 (2012), 671-687.

[30] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math., 166 (2006), 523-535.
[27] D. Li, X. Yu and Z. Zhai, On the Euler-Poincare equation with non-zero dispersion, Arch. Ration. Mech. Anal., 210 (2013), 955-974.

[28] J. Li and Z. Yin, Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces, J. Differential Equations, 261 (2016), 6125-6143.

[29] J. Li and Z. Yin, Well-posedness and analytic solutions of the two-component Euler-Poincaré system, Monatsh. Math., 183 (2017), 509–537.

[30] G. Rodríguez-Blanco, On the Cauchy problem for the Camassa-Holm equation, Nonlinear Anal., 46 (2001), 309-327.

[31] J. F. Toland, Stokes waves, Topol. Methods Nonlinear Anal., 7 (1996), 1-48.

[32] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math., 53 (2000), 1411-1433.

[33] K. Yan and Z. Yin, On the initial value problem for higher dimensional Camassa-Holm equations, Discrete Contin. Dyn. Syst., 35 (2015), 1327-1358.

[34] Y. Zhao, M. Yang and Y. Li, Non-uniform dependence for the periodic higher dimensional Camassa-Holm equations, J. Math. Anal. Appl., 461 (2018), 59-73.

School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
E-mail address: lijl29@mail2.sysu.edu.cn

School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
E-mail address: daili221726@163.com

School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, China
E-mail address: mathzwp2010@163.com