COMPLETENESS PROPERTIES OF THE OPEN-POINT AND BI-POINT-OPEN TOPOLOGIES ON \( C(X) \)

ANUBHA JINDAL, R. A. MCCOY, S. KUNDU, AND VARUN JINDAL

Abstract. This paper studies various completeness properties of the open-point and bi-point-open topologies on the space \( C(X) \) of all real-valued continuous functions on a Tychonoff space \( X \). The properties range from complete metrizability to the Baire space property.

1. Introduction

The set \( C(X) \) of all real-valued continuous functions on a Tychonoff space \( X \) has a number of natural topologies. One important topology on \( C(X) \) is a set-open topology, introduced by Arens and Dugundji in \( \cite{2} \). Among the set-open topologies on \( C(X) \), the point-open and compact-open topologies are most useful and frequently studied from different perspectives. The point-open topology \( p \) is also known as the topology of pointwise convergence. The study of pointwise convergence of sequences of functions is as old as the calculus. The topological properties of \( C^p(X) \), the space \( C(X) \) equipped with the topology \( p \), have been studied particularly in \( \cite{3}, \cite{12}, \cite{18}, \cite{19} \) and \( \cite{20} \).

In the definition of a set-open topology on \( C(X) \), we use a certain family of subsets of \( X \) and open subsets of \( \mathbb{R} \). Occasionally, there have been attempts, such as in \( \cite{8} \), to define a new type of set-open topology on \( C(X) \). But even these attempts did not help to move much away from the traditional way of defining the set-open topologies on \( C(X) \). So in \( \cite{9} \), by adopting a radically different approach, two new kinds of topologies called the open-point and bi-point-open topologies on \( C(X) \) have been defined. One main reason for adopting such a different approach is to ensure that both \( X \) and \( \mathbb{R} \) play equally significant roles in the construction of topologies on \( C(X) \). This gives a function space where the functions get more involved in the behavior of the topology defined on \( C(X) \).

The point-open topology on \( C(X) \) has a subbase consisting of sets of the form

\[
[x, V]^+ = \{ f \in C(X) : f(x) \in V \},
\]

where \( x \in X \) and \( V \) is an open subset of \( \mathbb{R} \). The open-point topology on \( C(X) \) has a subbase consisting of sets of the form

\[
[U, r]^- = \{ f \in C(X) : f^{-1}(r) \cap U \neq \emptyset \},
\]

where \( U \) is an open subset of \( X \) and \( r \in \mathbb{R} \). The open-point topology on \( C(X) \) is denoted by \( h \) and the space \( C(X) \) equipped with the open-point topology \( h \) is denoted by \( C_h(X) \). The term “\( h \)” comes from the word “horizontal” because, in the case of \( C_h(\mathbb{R}) \) a subbasic...
open set can be viewed as the set of functions in \( C(\mathbb{R}) \) whose graphs pass through some given horizontal open segment in \( \mathbb{R} \times \mathbb{R} \), as opposed to a subbasic open set in \( C_p(\mathbb{R}) \) which consists of the set of functions in \( C(\mathbb{R}) \) whose graphs pass through some given vertical open segment in \( \mathbb{R} \times \mathbb{R} \).

The bi-point-open topology on \( C(X) \) is the join of the point-open topology \( p \) and the open-point topology \( h \). In other words, it is the topology having subbasic open sets of both kinds: \([x, V]^+\) and \([U, r]^-\), where \( x \in X \) and \( V \) is an open subset of \( \mathbb{R} \), while \( U \) is an open subset of \( X \) and \( r \in \mathbb{R} \). The bi-point-open topology on the space \( C(X) \) is denoted by \( ph \) and the space \( C(X) \) equipped with the bi-point-open topology \( ph \) is denoted by \( C_{ph}(X) \). One can also view the bi-point-open topology on \( C(X) \) as the weak topology on \( C(X) \) generated by the identity maps \( id_1 : C(X) \to C_p(X) \) and \( id_2 : C(X) \to C_h(X) \).

In \([9]\) and \([10]\), in addition to studying some basic properties of the spaces \( C_h(X) \) and \( C_{ph}(X) \), the submetrizability, metrizability, separability and some cardinal functions on \( C_h(X) \) and \( C_{ph}(X) \) have been studied. But another important family of properties, the completeness properties of \( C_h(X) \) and \( C_{ph}(X) \), is yet to be studied. In this paper, we plan to do exactly that. More precisely, we study the completeness properties such as complete metrizability, \( \check{C}ech \)-completeness, local \( \check{C}ech \)-completeness, sieve-completeness, partition-completeness, pseudocompleteness and property of being a Baire space of the spaces \( C_h(X) \) and \( C_{ph}(X) \). We see that the completeness of \( C_h(X) \) and \( C_{ph}(X) \) behave like that of \( C_p(X) \). Here we would like to recall that the completeness properties of \( C_p(X) \) were studied in \([11]\).

In this paper, we use the following conventions. The symbols \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{N} \) denote the space of real numbers, rational numbers, integers and natural numbers, respectively. For a given horizontal open segment in \( \mathbb{R} \times \mathbb{R} \), the symbol \( \{x, V\}^+ \) denotes the set of all isolated points in \( \mathbb{R} \). For other basic topological notions, one can see \([6]\).

1.2. Metrizability of \( C_h(X) \) and \( C_{ph}(X) \)

In this section, we show that a number of properties of the spaces \( C_h(X) \) and \( C_{ph}(X) \) are equivalent to their metrizability. Some of these properties will be used in the sequel. We first define these properties.

A space \( X \) is said to have countable pseudocharacter if each point in \( X \) is a \( G_\delta \)-set. A subset \( S \) of a space \( X \) is said to have countable character if there exists a sequence \( \{W_n : n \in \mathbb{N}\} \) of open subsets in \( X \) such that \( S \subseteq W_n \) for all \( n \) and if \( W \) is any open set containing \( S \), then \( W_n \subseteq W \) for some \( n \).

A space \( X \) is said to be of countable type (pointwise countable type) if each compact set (point) is contained in a compact set having countable character. Clearly, every first countable space is of pointwise countable type.

A property weaker than being a space of pointwise countable type is that of being an r-space. A space \( X \) is an r-space if each point of \( X \) has a sequence \( \{V_n : n \in \mathbb{N}\} \) of neighborhoods with the property that if \( x_n \in V_n \) for each \( n \), then the set \( \{x_n : n \in \mathbb{N}\} \) is contained in a compact subset of \( X \). Another property weaker than being an r-space is that of being a q-space. A space \( X \) is a q-space if for each point \( x \in X \), there exists a sequence \( \{U_n : n \in \mathbb{N}\} \) of neighborhoods of \( x \) such that if \( x_n \in U_n \) for each \( n \), then \( \{x_n : n \in \mathbb{N}\} \) has a cluster point.
Another property stronger than being a \(q\)-space is that of being an \(M\)-space, which can be characterized as a space that can be mapped onto a metric space by a \(\textit{quasi-perfect map}\) (a continuous closed map in which inverse images of points are countably compact).

A space \(X\) is called a \(p\)-space if there exists a sequence \((U_n)\) of families of open sets in a compactification of \(X\) such that each \(U_n\) covers \(X\) and \(\bigcap_n \bigcup \{ U \in U_n : x \in U \} \subseteq X\) for any \(x \in X\).

A metrizable space is of countable type and a space of pointwise countable type is an \(r\)-space. Also every metrizable space is a \(p\)-space and every \(p\)-space is a \(q\)-space.

In order to relate the metrizability of the spaces \(C_h(X)\) and \(C_{ph}(X)\) with the topological properties discussed above, we need the following known lemma, the proof of which is omitted.

**Lemma 2.1.** Let \(D\) be a dense subsets of a space \(X\) and \(A\) be a compact subset of \(D\). Then \(A\) has countable character in \(D\) if and only if \(A\) has countable character in \(X\).

By Theorems 3.7 and 3.8 in [9], if \(X^0\) is \(G_\delta\)-dense in \(X\), then the spaces \(C_h(X)\) and \(C_{ph}(X)\) are topological groups and hence homogeneous. A space \(X\) is called \(\textit{homogeneous}\) if for every pair of points \(x, y\) in \(X\), there exists a homeomorphism of \(X\) onto itself which carries \(x\) to \(y\). So Lemma 2.1 can be used to prove the following result.

**Proposition 2.2.** If \(X^0\) is \(G_\delta\)-dense in \(X\), then \(C_\tau(X)\), where \(\tau = h, ph\), is of pointwise countable type if and only if \(C_\tau(X)\), where \(\tau = h, ph\), has a dense subspace of pointwise countable type.

**Theorem 2.3.** If \(X^0\) is \(G_\delta\)-dense in \(X\), then the following are equivalent.

\(a\) \(C_h(X)\) is metrizable.

\(b\) \(C_h(X)\) is first countable.

\(c\) \(C_h(X)\) has countable pseudocharacter.

\(d\) \(X\) has a countable \(\pi\)-base.

\(e\) \(X\) is a countable discrete space.

\(f\) \(C_\tau(X)\) is metrizable.

\(g\) \(C_h(X)\) is of countable type.

\(h\) \(C_h(X)\) is of pointwise countable type.

\(i\) \(C_h(X)\) has a dense subspace of pointwise countable type.

\(j\) \(C_h(X)\) is an \(r\)-space.

\(k\) \(C_h(X)\) is an \(M\)-space.

\(l\) \(C_h(X)\) is a \(p\)-space.

\(m\) \(C_h(X)\) is a \(q\)-space.

**Proof.** The equivalences \((a) \iff (b) \iff (c) \iff (f)\) follow from Theorem 4.6 in [9].

\((b) \implies (c)\). Immediate.

\((c) \implies (d)\). This follows from Proposition 4.5 in [9].

\((a) \implies (g) \implies (h) \implies (j) \implies (m)\), \((a) \implies (k) \implies (m)\) and \((a) \implies (l) \implies (m)\) follow from the previous discussion. Also \((h) \implies (i)\) follows from Proposition 2.2.

\((m) \implies (e)\). Since \(X^0\) is dense in \(X\), the constant zero function \(0_X\) has a sequence \(\{ B_n : n \in \mathbb{N} \}\) of basic neighborhoods of the form \(B_n = \{ x^n_1, 0 \} \cap \ldots \cap \{ x^n_m, 0 \} \) which satisfies the definition of \(q\)-space at \(0_X\). Suppose that there exists \(x_0 \in X^0 \setminus \bigcup \{ A_n : n \in \mathbb{N} \}\), where \(A_n = \{ x^n_1, \ldots, x^n_m \}\). Then for each \(n \in \mathbb{N}\), there is a \(g_n \in C(X)\) such that \(g_n(x_0) = n\) and \(g_n(x) = 0\) for each \(x \in A_n\). Each \(g_n \in B_n\). But \(\{ g_n : n \in \mathbb{N} \}\) does not cluster in \(C_h(X)\), because for any \(g \in C(X)\), \([ \{ x_0 \}, g(x_0) \] \(^{-}\) is a neighborhood of \(g\) in \(C_h(X)\) which contains at
most one member of the sequence \((g_n)_{n \in \mathbb{N}}\). Hence \(X^0\) is countable. Since \(X^0\) is \(G_\delta\)-dense, \(X\) is a countable discrete space. \(\square\)

**Theorem 2.4.** If \(X^0\) is \(G_\delta\)-dense in \(X\), then the following are equivalent.

(a) \(C_{ph}(X)\) is metrizable.
(b) \(C_{ph}(X)\) is first countable.
(c) \(C_{ph}(X)\) has countable pseudocharacter.
(d) \(X\) is separable.
(e) \(X\) is a countable discrete space.
(f) \(C_p(X)\) is metrizable.
(g) \(C_{ph}(X)\) is of countable type.
(h) \(C_{ph}(X)\) is of pointwise countable type.
(i) \(C_{ph}(X)\) has a dense subspace of pointwise countable type.
(j) \(C_{ph}(X)\) is an \(r\)-space.
(k) \(C_{ph}(X)\) is an \(M\)-space.
(l) \(C_{ph}(X)\) is a \(p\)-space.
(m) \(C_{ph}(X)\) is a \(q\)-space.

**Proof.** The equivalences (a) \(\Leftrightarrow\) (b) \(\Leftrightarrow\) (c) \(\Leftrightarrow\) (f) follow from Theorem 4.10 in [9] and Theorem 2.3.

(b) \(\Rightarrow\) (c). Immediate.

(c) \(\Rightarrow\) (d). This follows from Proposition 4.5 in [9].

(d) \(\Rightarrow\) (c). This follows from Theorem 4.8 in [10].

(d) \(\Rightarrow\) (e). If \(X\) is separable, then \(X^0\) is countable. Consequently, \(X\) is a countable discrete space.

(a) \(\Rightarrow\) (g) \(\Rightarrow\) (h) \(\Rightarrow\) (j) \(\Rightarrow\) (m), (a) \(\Rightarrow\) (k) \(\Rightarrow\) (m) and (a) \(\Rightarrow\) (l) \(\Rightarrow\) (m) follow from the previous discussion. Also (h) \(\Leftrightarrow\) (i) follows from Proposition 2.2.

(m) \(\Rightarrow\) (d). Let \(\{B_n : n \in \mathbb{N}\}\) be a sequence of basic neighborhoods of \(0_X\) in \(C_{ph}(X)\) which satisfies the definition of \(q\)-space at \(0_X\). Without loss of generality, we can assume that each \(B_n\) is of the form \([y^n_1, V^n_1]^+ \cap \ldots \cap [y^n_{m_n}, V^n_{m_n}]^+ \cap [U^n_1, 0]^+ \cap \ldots \cap [U^n_{m_n}, 0]^+\). Choose \(x^n_i \in U^n_i\) for \(1 \leq i \leq r_n\). Suppose that there exists \(x_0 \in X \setminus \cup\{A_n : n \in \mathbb{N}\}\), where \(A_n = \{y^n_1, \ldots, y^n_{m_n}, x^n_1, \ldots, x^n_{r_n}\}\). Then for each \(n \in \mathbb{N}\), there is a \(g_n \in C(X)\) such that \(g_n(x_0) = n\) and \(g_n(x) = 0\) for each \(x \in A_n\). Each \(g_n \in B_n\). But \(\{g_n : n \in \mathbb{N}\}\) does not cluster in \(C_{ph}(X)\), because for any \(g \in C(X)\), \([x_0, g(x_0) - \frac{1}{n}, g(x_0) + \frac{1}{n}]^+\) is a neighborhood of \(g\) in \(C_{ph}(X)\) which contains at most one member of the sequence \((g_n)_{n \in \mathbb{N}}\). Hence \(X\) is countable. \(\square\)

### 3. Completeness properties of \(C_h(X)\) and \(C_{ph}(X)\)

In this section, we study various kinds of completeness of \(C_h(X)\) and \(C_{ph}(X)\). In particular, we look at the complete metrizability of the spaces \(C_h(X)\) and \(C_{ph}(X)\) in a wider setting, more precisely, in relation to several other completeness properties. We first find when these spaces are pseudocomplete and Baire.

A quasi-regular space \(X\) is called pseudocomplete if it has a sequence of \(\pi\)-bases \(\{B_n : n \in \mathbb{N}\}\) such that whenever \(B_n \in B_n\) for each \(n\) and \(B_{n+1} \subseteq B_n\), then \(\cap \{B_n : n \in \mathbb{N}\} = \emptyset\) ([17]). In [1], it has been shown that a pseudocomplete space is a Baire space.

The following results help us to find a necessary condition for \(C_h(X)\) and \(C_{ph}(X)\) to be Baire spaces.
Proposition 3.1. A nonempty basic open set \([U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) in \(C_h(X)\) is dense in \(C_h(X)\) if and only if \(U_i\) is an infinite subset of \(X\) for each \(i \in \{1,\ldots,n\}\).

Proof. Suppose that \([U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) is a nonempty basic open set in \(C_h(X)\) such that \(U_i\) is an infinite subset of \(X\) for each \(1 \leq i \leq n\). Let \(G = [V_1,t_1]^{-} \cap \ldots \cap [V_m,t_m]^{-}\) be any nonempty basic open set in \(C_h(X)\). As \(G \neq \emptyset\), there exists \(f \in G\) such that \(f(x_i) = t_i\), for some \(x_i \in V_i\) and \(1 \leq i \leq m\). Since \(U_i\) is an infinite subset of \(X\), we can choose \(y_i \in U_i\) for each \(1 \leq i \leq n\) such that \(y_i \neq y_j\) for \(i \neq j\) and \(\{y_1,\ldots,y_n\} \cap \{x_1,\ldots,x_m\} = \emptyset\). Let \(S = \{x_1,\ldots,x_m,y_1,\ldots,y_n\}\). Since \(S\) is finite and \(X\) is Tychonoff, there exists \(h \in C(X)\) such that \(h(z_k) = f(z_k)\) for each \(k \in \{1,\ldots,p\}\) and \(h(y_j) = r_j\) for each \(j \in \{1,\ldots,n\}\). Thus \(h \in [U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) is dense in \(C_h(X)\).

Conversely, let \(H = [U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) be a nonempty basic dense open set in \(C_h(X)\). Suppose that for some \(i \in \{1,\ldots,n\}\), \(U_i\) is finite. Then \(U_i \subseteq X^0\). Let \(U_i = \{x_1,\ldots,x_m\}\). Take \(r \in \mathbb{R} \setminus \{r_1,\ldots,r_n\}\) and \(G = [x_1,r]^{-} \cap \ldots \cap [x_m,r]^{-}\). Then \(G\) is a nonempty open subset of \(X\) such that \(H \cap G = \emptyset\). So we arrive at a contradiction. \(\square\)

Corollary 3.2. If \(X\) is a space without isolated points, then every nonempty open set in \(C_h(X)\) is dense in \(C_h(X)\).

Proposition 3.3. A nonempty open set of the form \([U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) in \(C_{ph}(X)\) is dense in \(C_{ph}(X)\) if and only if \(U_i\) is an infinite subset of \(X\) for each \(i \in \{1,\ldots,n\}\).

Proof. Suppose that \([U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) is a nonempty open set in \(C_{ph}(X)\) such that \(U_i\) is an infinite subset of \(X\) for each \(1 \leq i \leq n\). Let \(G = [z_1,H_1]^+ \cap \ldots \cap [z_p,H_p]^+ \cap [V_1,t_1]^{-} \cap \ldots \cap [V_m,t_m]^{-}\) be any nonempty basic open set in \(C_{ph}(X)\). As \(G \neq \emptyset\), there exists \(f \in G\) such that \(f(z_j) \in H_j\) for \(1 \leq j \leq p\) and \(f(x_i) = t_i\), for some \(x_i \in V_i\) and \(1 \leq i \leq m\). Since \(U_i\) is infinite, we can choose \(y_j \in U_j\) for \(1 \leq j \leq n\) such that \(y_i \neq y_j\) and \(\{y_1,\ldots,y_n\} \cap \{x_1,\ldots,x_m\} = \emptyset\). Let \(S = \{x_1,\ldots,x_m,y_1,\ldots,y_n\}\). Since \(S\) is finite and \(X\) is Tychonoff, there exists \(h \in C(X)\) such that \(h(z_k) = f(z_k)\) for each \(k \in \{1,\ldots,p\}\) and \(h(y_j) = r_j\) for each \(j \in \{1,\ldots,n\}\). Thus \(h \in [U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) is dense in \(C_{ph}(X)\).

Conversely, let \(H = [U_1,r_1]^{-} \cap \ldots \cap [U_n,r_n]^{-}\) be a nonempty dense open set in \(C_{ph}(X)\). This implies that \(H\) is also a nonempty dense open set in \(C_h(X)\). Then Proposition 3.1 implies that for each \(1 \leq i \leq n\), \(U_i\) is an infinite subset of \(X\). \(\square\)

Theorem 3.4. Suppose that \(X\) has a non-isolated point \(x_0\) such that \(x_0\) is contained in a compact set having countable character. Then \(C_h(X)\) and \(C_{ph}(X)\) are of first category, and hence are not Baire spaces.

Proof. Let \(B\) be a compact set in \(X\) such that \(x_0 \in A\) and \(A\) has a countable character. Let \(B\) be a countable family of open sets in \(X\) such that every member of \(B\) contains \(A\) and if \(U\) is an open set in \(X\) containing \(A\), then there exists a \(B \in B\) such that \(B \subseteq U\). Since \(x_0\) is a non-isolated point, for every \(U \in B\), \(U\) is an infinite subset of \(X\). So by Proposition 3.1 for every \(U \in B\) and \(r \in \mathbb{R}\), \([U,r]^{-}\) is an open dense subset of \(C_h(X)\). Thus \(D = \{(U,q)^{-} : U \in B, q \in \mathbb{Q}\}\) is a countable collection of closed and nowhere dense subsets of \(C_h(X)\). In order to prove that \(C_h(X)\) is of first category, we show that \(C(X) = \bigcup D\).

Suppose that there exists \(f \in C(X)\) such that \(f \notin \bigcup D\). This implies that \(f \notin [U,q]^{-}\) for every \(U \in B\) and \(q \in \mathbb{Q}\). Thus \(\mathbb{Q} \subseteq f(U)\) for all \(U \in B\). Since \(A\) is compact in \(X\), there exists an open interval \((a,b)\) in \(\mathbb{R}\) such that \(f(A) \subseteq (a,b)\). Now \(A\) has countable character,
Theorem 3.8. respectively.

C this case situation when Cψ R discrete space). It is easy to see that φ Cψ Define R Suppose that Lemma 3.7. If X is a space of pointwise countable type, and either Ch(X) or Cph(X) is a Baire space, then X is discrete.

In order to show that the converse of Corollary 3.5 is also true, we need the following results.

Proposition 3.6. For any space X, the following are equivalent.

(a) The space Cp(X) < Ch(X).
(b) The space Cph(X) = Ch(X).
(c) X is a discrete space.

Proof. (a) ⇒ (b). It is immediate.
(b) ⇒ (c). Suppose that X is not discrete. Then X has some non-isolated point x0. Let 0X be the constant zero-function on X. Now [x0, (−1, 1)]+ is an open neighborhood of 0X in Cph(X). To show that [x0, (−1, 1)]+ is not a neighborhood of 0X in Ch(X), let

\[ B = [U_1, 0]^- \cap \ldots \cap [U_n, 0]^- \]

be any basic neighborhood of 0X in Ch(X). Since x0 is a non-isolated point, there exists \(x_i \in U_i \setminus \{x_0\}\) for each 1 ≤ i ≤ n. As X is a Tychonoff space, there exists a continuous function \(g : X \to [0, 1]\) such that \(g(x_0) = 1\) and \(g(x) = 0\) for all \(x \in \{x_1, \ldots, x_n\}\). Then \(g \in B\) but \(g \notin [x_0, (−1, 1)]^+\). So [x0, (−1, 1)]+ cannot be open in Ch(X).

(c) ⇒ (a). Suppose that X is discrete. Take any subbasic open set H in Cp(X), where

\[ H = \{x, V\}^+ = \{f \in C(X) : f(x) \in V\} \]

for some \(x \in X\) and some open set V in \(\mathbb{R}\). Since \(\{x\}\) is open in X, for any \(f \in [x, V]^+\), we have \(f \in [x, f(x)]^- \subseteq [x, V]^+\). Hence \([x, V]^+\) is open in Ch(X).

Lemma 3.7. If X is a discrete space, then Ch(X) is homeomorphic to the product of \(|X|\) many copies of the space \(\mathbb{R}\) with discrete topology.

Proof. Suppose that \(X = \{x_i : i \in I\}\) and let \(\mathbb{R}_d\) denote the set \(\mathbb{R}\) with discrete topology. Define \(\psi : C_\tau(X) \to \mathbb{R}_d^{\{x_i : i \in I\}}\) by \(\psi(f) = (f(x_i))_{i \in I}\) for each \(f \in C(X)\), and define \(\phi : \mathbb{R}_d^{\{x_i : i \in I\}} \to C_\tau(X)\) by \(\phi((r_i)_{i \in I}) = f\), where \(f(x_i) = r_i\) for each \(i \in I\) (note that \(f \in C(X)\) since X is a discrete space). It is easy to see that \(\phi \circ \psi\) is the identity map on Ch(X) and \(\psi \circ \phi\) is the identity map on \(\mathbb{R}_d^{\{x_i : i \in I\}}\). So \(\psi\) is a bijection. Since X is a discrete space, it is easy to prove that the functions \(\psi\) and \(\phi\) are continuous.

When X is of pointwise countable type, the next result completely characterizes the situation when \(C_\tau(X) (\tau = h, ph)\) is either pseudocomplete or a Baire space. In fact, in this case \(C_\tau(X) (\tau = h, ph)\) is pseudocomplete or a Baire space if and only if \(C_p(X)\) is so respectively.

Theorem 3.8. If X is of pointwise countable type, then the following are equivalent.

(a) Ch(X) is a Baire space.
(b) Cph(X) is a Baire space.
(c) X is discrete.
(d) Ch(X) is pseudocomplete.
(e) Cph(X) is pseudocomplete.
(f) $C_p(X)$ is pseudocomplete.
(g) $C_p(X)$ is a Baire space.

Proof. (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c). These follow from Corollary 3.5.
(c) $\Rightarrow$ (d) and (c) $\Rightarrow$ (e). These follow from Proposition 3.6, Lemma 3.7 and the fact that every completely metrizable space is pseudocomplete and arbitrary product of pseudocomplete spaces is pseudocomplete (see [17]).
(d) $\Rightarrow$ (a), (e) $\Rightarrow$ (b) and (f) $\Rightarrow$ (g). These are true because every pseudocomplete space is a Baire space (see [17]).
(c) $\Rightarrow$ (f). If $X$ is discrete, then $C_p(X) = \mathbb{R}^X$. But an arbitrary product of pseudocomplete spaces is pseudocomplete.
(g) $\Rightarrow$ (c). This follows from Corollary I.3.6 in [3].

Now we study the complete metrizability of the spaces $C_h(X)$ and $C_{ph}(X)$ in relation to several other completeness properties. We first recall the definitions of various kinds of completeness. For the rest of this section all spaces are assumed to be Tychonoff.

A space $X$ is called Čech-complete if $X$ is a $G_δ$-set in $βX$, where $βX$ is the Stone-Čech compactification of $X$. A space $X$ is called locally Čech-complete if every point $x ∈ X$ has a Čech-complete neighborhood. Clearly, every Čech-complete space is locally Čech-complete.

In order to deal with sieve-completeness, partition-completeness, one needs to recall the definitions of these concepts from [15]. The central idea of all these concepts is that of a complete sequence of subsets of $X$.

Let $F$ and $U$ be two collections of subsets of $X$. Then $F$ is said to be controlled by $U$ if for each $U ∈ U$, there exists some $F ∈ F$ such that $F ⊆ U$. A sequence $(U_n)$ of subsets of $X$ is said to be complete if every filter base $F$ on $X$ which is controlled by $(U_n)$ clusters at some $x ∈ X$. A sequence $(U_n)$ of collections of subsets of $X$ is called complete if $(U_n)$ is a complete sequence of subsets of $X$ whenever $U_n ∈ U_n$ for all $n$. It has been shown in Theorem 2.8 of [7] that the following statements are equivalent for a Tychonoff space $X$: (a) $X$ is a $G_δ$-subset of any Hausdorff space in which it is densely embedded; (b) $X$ has a complete sequence of open covers; and (c) $X$ is Čech-complete. From this result, it easily follows that a Tychonoff space $X$ is Čech-complete if and only if $X$ is a $G_δ$-subset of any Tychonoff space in which it is densely embedded.

For the definitions of sieve, sieve-completeness and partition-completeness, see [4], [15], and [16]. The term “sieve-complete” is due to Michael [13], but the sieve-complete spaces were studied earlier under different names: as $λ_0$-spaces by Wicke in [21], as spaces satisfying condition $K$ by Wicke and Worrel Jr. in [22] and as monotonically Čech-complete spaces by Chaber, Coban and Nagami in [5]. Every space with a complete sequence of open covers is sieve-complete; the converse is generally false, but it is true in paracompact spaces, see Remark 3.9 in [5] and Theorem 3.2 in [13]. So a Čech-complete space is sieve-complete and a paracompact sieve-complete space is Čech-complete. Also every sieve-complete space is partition-complete.

**Theorem 3.9.** For any space $X$, the following are equivalent.

(a) $C_h(X)$ is completely metrizable.
(b) $C_h(X)$ is Čech-complete.
(c) $C_h(X)$ is locally Čech-complete.
(d) $C_h(X)$ is sieve-complete.
Theorem 3.10. For any space \( X \), the following are equivalent.

(a) \( C_h(X) \) is completely metrizable.
(b) \( C_{ph}(X) \) is \( Č \)ech-complete.
(c) \( C_{ph}(X) \) is locally \( Č \)ech-complete.
(d) \( C_{ph}(X) \) is sieve-complete.
(e) \( C_{ph}(X) \) is an open continuous image of a paracompact \( Č \)ech-complete space.
(f) \( C_{ph}(X) \) is an open continuous image of a \( Č \)ech-complete space.
(g) \( C_{ph}(X) \) is partition-complete.
(h) \( C_p(X) \) is completely metrizable.
(i) \( C_p(X) \) is \( Č \)ech-complete.
(j) \( X \) is a countable discrete space.

Proof. (a) \( \Rightarrow \) (b). It follows from the fact that every completely metrizable space is \( Č \)ech-complete.
(b) \( \Rightarrow \) (c), (b) \( \Rightarrow \) (d) and (d) \( \Rightarrow \) (g) follows from the previous discussion. Also implications (a) \( \Rightarrow \) (c) \( \Rightarrow \) (f) are immediate. Note (c) \( \Rightarrow \) (f), see 3.12.19 (d), page 237 in [5].
(f) \( \Rightarrow \) (a). A \( Č \)ech-complete space is of pointwise countable type and the property of being pointwise countable type is preserved by open continuous maps. Hence \( C_h(X) \) is of pointwise countable type and consequently by Theorem 2.3, \( C_h(X) \) is metrizable and hence \( C_h(X) \) is paracompact. But a paracompact open image of a \( Č \)ech-complete space is \( Č \)ech-complete (see 5.5.8 (b), page 341 in [6]). Hence \( C_h(X) \) is \( Č \)ech-complete. But a metrizable and \( Č \)ech-complete space is completely metrizable.
(g) \( \Rightarrow \) (a). If \( C_h(X) \) is partition-complete, then by Propositions 4.4 and 4.7 in [15], \( C_h(X) \) contains a dense \( Č \)ech-complete subspace. But a \( Č \)ech-complete space is of pointwise countable type. Hence by Theorems 2.3, \( C_h(X) \) is metrizable. But by Theorem 1.5 of [14] and Proposition 2.1 in [15], a metrizable space is completely metrizable if and only if it is partition-complete. Hence \( C_h(X) \) is completely metrizable.
(a) \( \Rightarrow \) (j). If \( C_h(X) \) is completely metrizable, then Theorem 3.7 in [9] implies that \( X^0 \) is \( G_{δ} \)-dense in \( X \). Hence by Theorem 2.3, \( X \) is a countable discrete space.
(j) \( \Rightarrow \) (a). Since a countable product of completely metrizable spaces is completely metrizable, Lemma 3.7 implies that \( C_h(X) \) is completely metrizable.
(h) \( \leftrightarrow \) (i) \( \leftrightarrow \) (j). This follows from Theorem 8.6 in [11].

The next theorem can be proved in a manner similar to Theorem 3.9 except we need to use here Theorem 2.3 in place of Theorem 2.5.

Theorem 3.10. For any space \( X \), the following are equivalent.

(a) \( C_{ph}(X) \) is completely metrizable.
(b) \( C_{ph}(X) \) is \( Č \)ech-complete.
(c) \( C_{ph}(X) \) is locally \( Č \)ech-complete.
(d) \( C_{ph}(X) \) is sieve-complete.
(e) \( C_{ph}(X) \) is an open continuous image of a paracompact \( Č \)ech-complete space.
(f) \( C_{ph}(X) \) is an open continuous image of a \( Č \)ech-complete space.
(g) \( C_{ph}(X) \) is partition-complete.
(h) \( C_p(X) \) is completely metrizable.
(i) \( C_p(X) \) is \( Č \)ech-complete.
(j) \( X \) is a countable discrete space.

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ANUBHA JINDAL: DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY DELHI, NEW DELHI 110016, INDIA.
E-mail address: jindalanubha217@gmail.com

R. A. MCCOY: DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG VA 24061-0123, U.S.A.
E-mail address: mccoy@math.vt.edu

S. KUNDU: DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY DELHI, NEW DELHI 110016, INDIA.
E-mail address: skundu@maths.iitd.ac.in

VARUN JINDAL: DEPARTMENT OF MATHEMATICS, MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY JAIPUR, JAIPUR, RAJASTHAN 302017, INDIA.
E-mail address: vjindal.maths@mnit.ac.in