A better approach to discuss medical science and engineering data with a modified Lehmann Type – II model [version 1; peer review: awaiting peer review]

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Abstract
Background: Modeling with the complex random phenomena that are frequently observed in reliability engineering, hydrology, ecology, medical science, and agricultural sciences was once thought to be an enigma. Scientists and practitioners agree that an appropriate but simple model is the best choice for this investigation. To address these issues, scientists have previously discussed a variety of bounded and unbounded, simple to complex lifetime models.

Methods: We discussed a modified Lehmann type II (ML-II) model as a better approach to modeling bathtub-shaped and asymmetric random phenomena. A number of complementary mathematical and reliability measures were developed and discussed. Furthermore, explicit expressions for the moments, quantile function, and order statistics were developed. Then, we discussed the various shapes of the density and reliability functions over various model parameter choices. The maximum likelihood estimation (MLE) method was used to estimate the unknown model parameters, and a simulation study was carried out to evaluate the MLEs' asymptotic behavior.

Results: We demonstrated ML-II’s dominance over well-known competitors by modeling anxiety in women and electronic data.

Keywords
Power Function Distribution; Lehmann Type I, II Distribution; Failure Rate Function; Moments; Maximum Likelihood; Order Statistics; Quantile; Rényi Entropy.
Introduction

Over the last two decades, researchers’ increasing interest in the development of new models has explored the remarkable characteristics of the baseline model. As a result, new models open new avenues for theoretical and applied researchers to address real-world problems, allowing them to fit asymmetric and complex random phenomena more proficiently and adequately. As a result, several modifications, extensions, and generalizations have been developed and discussed in the literature, with the Lehmann type – I (L – I) and Lehmann type – II (L – II) models being among the simplest and most useful. The simple exponentiation of any arbitrary baseline model is given by L – I.

\[ F(x;\alpha,\xi) = G^\alpha(x;\xi) \]

where \( 0 < x < 1 \), and \( \alpha > 0 \) is a shape parameter.

Gupta et al.\(^7\) are credited with the use of L–I on exponential distributions. On the other hand, Cordeiro et al.\(^3\) established the L–II–G class of distributions and developed a dual transformation of L–I, which is given by

\[ F(x;\alpha,\xi) = 1 - (1 - G(x;\xi))^\alpha \]

where \( G(x;\xi) \) is cumulative distribution function (CDF) of the arbitrary baseline model, based on the parametric vector \( \xi \) with \( \alpha > 0 \) as a shape parameter.

The closed-form feature of L–II allows one to derive and study its numerous properties, and in the literature, both approaches (L–I and L–II) have been extensively used in favor of the power function (PF) model, to study the unexplored characteristics of the classical and modified models. Recently, Arshad et al.\(^1,5\) developed bathtub-shaped failure rate and PF models followed by L–II and L–I families, respectively, and explored their applications in engineering data. Awodutire et al.\(^6\) generalized the half-logistic via L–II class, and Akilandeswari et al.\(^7\) proposed the Laplace L–I reliability growth model and discussed its application in the early detection of software failure based on time between failure observations.

The PF is a special case of the beta distribution in distribution theory, and its significance can be evaluated using statistical tests such as the likelihood ratio test. The PF’s simplicity and utility has compelled researchers to investigate its further generalizations and applications in various fields of science. For this, we recommend that the reader look at Dallas’s illustrious work.\(^8\) He discovered an intriguing relationship between PF and Pareto models when the inverse transformation of the Pareto variable explored the PF.\(^8\) Meanwhile, Meniconi and Barry\(^9\) discovered the PF as a best-fit model on electronic component data. Characterization is based on the independence of record values and order statistics, with lower record values attributed to Chang\(^10\) and Tavangar,\(^11\) respectively. Cordeiro and Brito\(^12\) created the beta version of the PF and discussed its use in petroleum reservoir and milk production data. The PF was characterized by Ahsanullah et al.\(^13,14\) using lower record values. Zaka et al.\(^15\) discussed techniques for estimating PF parameters such as least square (LS), relative least square (RLS), and ridge regression (RR). Tahir et al.\(^16\) generalized the PF via the Weibull-G class and applied it to bathtub-shaped data. Shahzad et al.\(^16\) used the techniques of L-, TL-, LL-, and LH moments to calculate the PF moments. Haq et al.\(^17\) generalized the PF via the QRMTM-G class and investigated its application in two-lifetime data. Okorie et al.\(^18\) generalized the PF via the Marshall-Olkin-G class (Marshall and Olkin\(^19\)) and investigated its application in data on anxiety in women and evaporation. Usman et al.\(^20\) proposed an exponentiated version of transmuted PF and investigated its application in biological and engineering data. Hassan et al.\(^21\) generalized the PF by following the odd exponential-G class (Tahir et al.\(^22\)) and discussed its application in three-lifetime data. Zaka et al.\(^23\) developed a new reflected PF and investigated its application in medical sciences data, while Al-Mutairi\(^24\) discussed the weighted PF via the QRMTM G-class and investigated its application in the engineering sector.

Modified Lehmann type II model

We developed a potentiated lifetime model known as the modified Lehmann type II (ML–II) model. It is constrained by the interval (0, 1). The ML–II is extremely well suited to modeling asymmetric and bathtub-shaped phenomena. By including a scale parameter (\( \alpha > 0 \)), in the baseline model, it begins to outperform its competitors in terms of fit and robustness of the tail weight/skewness of the density function.

The ML–II is said to follow a random variable X if its associated CDF and corresponding probability density function (PDF) are given by

\[ F_{ML-II}(x;\alpha,\beta) = 1 - \left( \frac{1 - x}{1 + \alpha x} \right)^\beta \quad (1) \]
and

\[ f_{ML-II}(x; \alpha, \beta) = \frac{\beta(\alpha + 1)(1-x)^{\beta-1}}{(1+\alpha x)^{\beta+1}} \]  

(2)

where \(0 < x < 1\), and \(\alpha > -1, \beta > 0\) are the scale and shape parameters, respectively. For \(\alpha = 0\), the ML-II reduces to the L-II (baseline model).

Balogun et al. \(^{25,26}\) has developed the generalized version and G-class of ML-II (Equation (1)), respectively, and explored their applications in multidisciplinary areas of science.

We had the following objectives:

(i) to develop a two-parameter model with an approach that had not been studied and discussed in the past;

(ii) For the new model to have attractive closed-form features for CDF, PDF, and a likelihood function that is simple to interpret;

(iii) PDF and HRF to hold J, reversed-J, and bathtub shapes;

(iv) To provide comparative results and a better fit than competing models.

This article is divided into the following sections: ‘Linear representation’ presents a mixture representation as well as numerous structural and reliability measures. ‘Estimation’ includes the estimation of model parameters using the maximum likelihood estimation (MLE) method, as well as a simulation study. ‘Application’ discusses real-world applications, and the final section summarizes the conclusion.

**Linear representation**

The linear representations of CDF and PDF make the calculations much easier than the traditional integral computation for determining the mathematical properties. We consider the binomial expansion for this.

\[
(1-z)^\beta = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i, |z| < 1
\]

\[
F_{ML-II}(x; \alpha, \beta) = 1 - (1-x)^\beta(1+\alpha x)^{-\beta}
\]

\[
F_{ML-II}(x; \alpha, \beta) = 1 - \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{\beta}{j} (ax)^j
\]

Hence, the mixture representation of CDF is given as

\[
F_{ML-II}(x; \alpha, \beta) = \sum_{i,j=0}^{\infty} \phi_{ij} x^i x^j
\]

PDF is given as follows

\[
f_{ML-II}(x; \alpha, \beta) = \beta(1+\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} x^i \sum_{j=0}^{\infty} \binom{\beta+1}{j} (ax)^j
\]

or

\[
f_{ML-II}(x; \alpha, \beta) = \beta(1+\alpha) \sum_{i,j=0}^{\infty} \phi_{ij} x^i x^j
\]
where \( \phi_{ij} = \binom{\beta}{i} \binom{-\beta}{j} (-1)^j \alpha^i, \alpha > 1, \beta > 0, \eta_{ij} = \alpha^i (-1)^j \binom{\beta - 1}{i} \binom{\beta + 1}{j}, \alpha > 1, \beta > 0 \)

Reliability characteristics

The reverse hazard rate function plays an important role in reliability engineering by analyzing and predicting the lifespan of a component. Notable contributions include the survival function \( S(x) \), hazard rate function \( h(x) \), cumulative hazard rate function \( h_c(x) \), reverse hazard rate function \( h_r(x) \), Mills ratio \( M(x) \), and odd function \( O(x) \).

The reliability function can be defined as the probability that a component will survive at time \( x \). It is defined analytically as \( S(x) = 1 - F(x) \). The survival/reliability function of \( X \) is denoted by

\[
S_{\text{ML-II}}(x; \alpha, \beta) = \left( \frac{1-x}{1+\alpha x} \right)^\beta
\]

(3)

Terms such as “failure rate function”, “hazard rate function”, and “force of mortality” are frequently discussed in the literature. These terms are used to describe the failure rate of a component over a specific time period (say \( x \)). It is mathematically defined as \( h(x) = f(x)/R(x) \).

The failure rate function of \( X \) is denoted by

\[
h_{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(\alpha + 1)}{(1 - x)(1 + \alpha x)}
\]

(4)

The reverse hazard rate function is defined by \( h_r(x) = f(x)/F(x) \). The reverse hazard rate function of \( X \) is given by

\[
h_{r-\text{ML-II}}(x; \alpha, \beta) = -\log \left( \frac{1-x}{1+\alpha x} \right)^\beta
\]

The mechanical components/parts of some systems are frequently assumed to follow the bathtub-shaped failure rate phenomenon. To discuss the significance of the MLII, several well-established and useful reliability measures are available in the literature. One of them is the cumulative hazard rate function, which is defined as \( h_c(x) = -\log(S(x)) \). The cumulative hazard rate function of \( X \) is denoted by

\[
h_{c-\text{ML-II}}(x; \alpha, \beta) = -\log \left( \frac{1-x}{1+\alpha x} \right)^\beta
\]

Mills ratio is defined by \( M(x) = S(x)/f(x) \). Mills ratio of \( X \) is given by

\[
M_{\text{ML-II}}(x; \alpha, \beta) = \frac{(1+\alpha x)(1-x)^\beta}{\beta(\alpha + 1)(1-x)^{\beta+1}}
\]

The odd function is defined by \( O(x) = F(x)/S(x) \). The odd function of \( X \) is given by

\[
O_{\text{ML-II}}(x; \alpha, \beta) = \frac{(1+\alpha x)^\beta - (1-x)^\beta}{(1-x)^\beta}
\]

As mentioned above, we can obtain the linear expression for reliability characteristics. In terms of linear expression, the reliability and failure rate functions of \( X \) are given by

\[
S^\ast_{\text{ML-II}}(x; \alpha, \beta) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{-\beta}{j} (\alpha x)^j
\]

and

\[
h^\ast_{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(1+\alpha) \sum_{i=0}^{\infty} (-1)^i \binom{\beta - 1}{i} x^i \sum_{j=0}^{\infty} \binom{\beta + 1}{j} (\alpha x)^j}{\sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} x^i \sum_{j=0}^{\infty} \binom{-\beta}{j} (\alpha x)^j}
\]
Limiting behavior
Propositions 1 and 2 discuss the limiting behavior of the ML-II's cumulative distribution (CDF), density (PDF), reliability ($S(x)$), and failure rate ($h(x)$) functions for $x \to 0$ and $x \to 1$.

**Proposition-1.** The limiting behavior of the CDF, PDF, $S(x)$, and $h(x)$ of the ML-II at $x \to 0$ is given below.

\[
\begin{align*}
F_{ML-II}(0; \alpha, \beta) & \sim 0 \\
\int_{ML-II}(0; \alpha, \beta) & \sim \beta(1 + \alpha) \\
S_{ML-II}(0; \alpha, \beta) & \sim 1 \\
h_{ML-II}(0; \alpha, \beta) & \sim \beta(1 + \alpha)
\end{align*}
\]

**Proposition-2.** The limiting behavior of CDF, PDF, $S(x)$, and $h(x)$ of the ML-II at $x \to 1$ is given as follows, respectively.

\[
\begin{align*}
F_{ML-II}(1; \alpha, \beta) & = 1 \\
\int_{ML-II}(1; \alpha, \beta) & = 0 \\
S_{ML-II}(1; \alpha, \beta) & = 1 \\
h_{ML-II}(1; \alpha, \beta) & = 0
\end{align*}
\]

Shapes of density and failure rate functions
The possible shapes of the ML-II's density and failure rate functions are sketched over various model parameter choices shown in Figures 1 and 2. Figure 1 depicts the J, reverse-J, and bathtub shapes of the density function, while Figure 2 depicts the U, bathtub, and reverse-J shapes of the failure rate function.

Quantile, mode, skewness, and kurtosis
The concept of quantile function was introduced by Hyndman and Fan. Inverting the CDF yields the $p^{th}$ quantile function of ML-II (Equation (1)). The quantile function is defined as follows: $p = F(x_p) = P(X \leq x_p), \ p \in (0, 1)$. Then, the quantile function of $X$ is given by

\[
x_{p-ML-II} = \frac{1 - (1 - p)^{1/\beta}}{1 + \alpha(1 - p)^{1/\beta}}, \alpha > -1, \beta > 0
\]

Put $p = 0.25, 0.50, \text{and} 0.75$ in Equation (3) to get the first quartile, median, and third quartile of $X$. To generate random numbers in the future, we will assume that the CDF in Equation (1) follows a uniform distribution $u = U. (0, 1)$.
The mode of $X$ is calculated by taking the first derivative of PDF (Equation (2)) and equating it to zero, as shown by

$$f'_\text{ML-II}(x; \alpha, \beta) = \frac{\beta(\alpha + 1)(1-x)^{\beta-1}(2ax - (1+\alpha)\beta - \alpha + 1)}{(1-x)^2} = 0$$

As a result, a simplified form of the mode is given by

$$\text{Mode}_{\text{ML-II}} = \frac{(\alpha + 1)(\beta + 1) - 1}{2\alpha}, \alpha > -1, \beta > 0$$

Measures of skewness and kurtosis based on quartiles and octiles are less sensitive to outliers and work well against models, but they are deficient in moments.

| Bowley’s measure of skewness | Moor’s measure of kurtosis |
|-----------------------------|---------------------------|
| $S_B = \frac{Q_3 - Q_1}{Q_4 - Q_2}$ | $K_M = \frac{Q_5 - Q_3}{Q_4 - Q_2}$ |

According to Figures 3 and 4, the skewness and kurtosis plots of the ML-II may be positively skewed.

**Moments and associated measures**

Moments play a significant role in distribution theory, where they are used to discuss the various characteristics and important features of the probability model.

Figure 2. Failure rate function.

Figure 3. Skewness.
Theorem 1. Let $X \sim \text{ML-II}(\tau, \alpha, \beta)$, with $\alpha > -1, \beta > 0$, then the $r$-th ordinary moment (say $\mu'_r$) of $X$ is given by

$$\mu'_r = E \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

where $E = \beta(1+\alpha), D_i = \alpha \left( -\frac{\beta-1}{i} \right)$

Proof: $r$-th ordinary moment can be written by following Equation (2) as

$$\mu'_r = \beta(1+\alpha) \int_{0}^{1} x^{r}(1-x)^{\beta-1}(1+\alpha x)^{-(\beta+1)} dx$$

$$\mu'_r = \alpha \beta \gamma(1+\alpha) \sum_{i=0}^{\infty} D_i \int_{0}^{1} x^{r+i}(1-x)^{\beta-1} dx$$

As a result, the above integral reduces to the $r$-th moment, which is given by

$$\mu'_r = E \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

Moment generating function $M_X(t)$ is defined as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$M_X(t) = E \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$

Characteristic function is defined as $\varphi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$\varphi_X(t) = E \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=0}^{\infty} D_i B(r+i+1, \beta)$$
The factorial generating function of $X$ is defined as $F_x(t) = E(1 + t)^x = E(e^{t \ln(1+x)}) = \sum_{r=0}^{\infty} \frac{(\ln(1+x))^r}{r!} \mu'_r$. It is obtained by following Equation (6) and is given by

$$F_{x-ML-II}(t) = \sum_{r=0}^{\infty} \frac{(\ln(1+t))^r}{r!} \sum_{i=0}^{\infty} D_i B(r + i + 1, \beta)$$

The Fisher index $\frac{\text{Var}(X)}{\mu(X)}$ may play a supportive role in the discussion of variability in $X$ and it is given by

$$\text{Fisher index}_{ML-II} = \frac{\sum_{i=0}^{\infty} D_i B(3 + i, \beta) - (\sum_{i=0}^{\infty} D_i B(2 + i, \beta))^2}{\sum_{i=0}^{\infty} D_i B(2 + i, \beta)}$$

For the negative moments of $X$, substitute $r$ by $-w$ in Equation (6) and it is given by

$$\mu'_{-w-ML-II} = \sum_{i=0}^{\infty} D_i B(-w + i + 1, \beta)$$

Furthermore, for fractional positive and fractional negative moments of $X$, substitute $r$ by $\left(\frac{\alpha}{\beta}\right)$ and $\left(\frac{-\alpha}{\beta}\right)$ in Equation (6), respectively.

The Mellin transformation is well-known in statistics as a product distribution as well as a quotient for independent random variables. The Mellin transformation is presented by $M_x(m) = E(x^{m-1}) = \int x^{m-1} f(x) dx$. The Mellin transformation of $X$ is given by

$$M_{x-ML-II}(m) = \sum_{i=0}^{\infty} D_i B(m + i, \beta).$$

where $B(x; \alpha, \beta) = \int_0^1 t^\alpha (1 - t)^{\beta-1} dt$ is the beta function, $E = \beta(1 + \alpha), D_h = \alpha \left(\begin{array}{c} -\beta - 1 \\ i \end{array}\right)$ and $\alpha > -1, \beta > 0$.

One may perhaps further determine the well-established statistics such as skewness ($\tau_1 = \mu'_3 / \mu'_2^2$), and kurtosis ($\tau_2 = \mu'_4 / \mu'_2$), of $X$ by integrating Equation (6). A well-established relationship between the central moments ($\mu_x$) and cumulants ($K_x$) of $X$ may easily be defined by ordinary moments by $\mu_x = \sum_k \left(\begin{array}{c} k \\ k \end{array}\right) (-1) \left(\begin{array}{c} 1 \\ i \end{array}\right) \mu'_k$. Hence, the first four cumulants can be calculated by $K_1 = \mu'_1, K_2 = \mu'_2 - \mu'_1^2, K_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu'_1^3, \text{and } K_4 = 4\mu'_3 \mu'_1 - 3\mu'_2^2 + 12\mu'_2 \mu'_1^2 - 6\mu'_1^4$.

Some numerical results of the first four ordinary moments ($\mu'_1, \mu'_2, \mu'_3, \mu'_4$), mode (a value that appears frequently in data), $\sigma^2$ = variance (a measure of dispersion), $\tau_1$ = skewness (measure of asymmetry), and $\tau_2$ = kurtosis (a measure to discuss the heaviness of the distribution tails) for some chosen parameters are presented in Table 1 for $S-I$ ($\alpha = 0.5, \beta = 0.9$), $S-II$ ($\alpha = 0.9, \beta = 1.5$), $S-III$ ($\alpha = 1.2, \beta = 1.9$), $S-IV$ ($\alpha = 3.2, \beta = 0.9$), and $S-V$ ($\alpha = 0.2, \beta = 3.9$). Note that the results of

| $\mu'_k$ | $S-I$ | $S-II$ | $S-III$ | $S-IV$ | $S-V$ |
|-----------|-------|--------|---------|--------|-------|
| $\mu'_1$  | 0.460 | 0.295  | 0.224   | 0.302  | 0.181 |
| $\mu'_2$  | 0.297 | 0.143  | 0.089   | 0.163  | 0.056 |
| $\mu'_3$  | 0.221 | 0.086  | 0.046   | 0.110  | 0.023 |
| $\mu'_4$  | 0.176 | 0.058  | 0.028   | 0.067  | 0.011 |
| Mode      | 1.850 | 2.083  | 2.241   | 1.091  | 12.200|
| $\sigma^2$| 0.029 | 0.044  | 0.025   | 0.067  | 0.002 |
| $\tau_1$  | 0.010 | 0.100  | 0.492   | 0.028  | 1.292 |
| $\tau_2$  | 0.119 | 0.598  | 0.981   | 0.483  | -0.076|
moments and variance decrease gradually, while skewness falls between 0 and 1, mode increases, and kurtosis can be negative subject to the model parameter combinations.

**Incomplete moments**

Lower incomplete (LI) and upper incomplete moments are the two types of incomplete moments. LI moments are defined by

$$\Phi_\text{LI}(t) = \int_0^t x f(x) dx$$

The LI moments of X are given as

$$\Phi_\text{ML-II}(t) = E \sum_{i=0}^{\infty} D_i B_i (r + i + 1, \beta)$$

(7)

The residual life function of random variable X, \( R(t) = (X - t/X > t) \), is the likelihood that a component whose life says \( x \), survives in the different time intervals at \( t \geq 0 \). Analytically, it can be written as follows:

$$S_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{S(s + t)}{S(t)}$$

Residual life function of X

$$S_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{(1 - (x + t))^{(1 + \alpha t)^\beta}}{(1 + \alpha(x + t))^{(1 - t)^\beta}}$$

with the associated CDF is given as follows

$$F_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{(1 + \alpha(x + t))^{(1 - t)^\beta}}{(1 + (x + t))^{(1 - t)^\beta}}$$

The mean residual life function of X is given by

$$E(S_{R(t) \text{-ML-II}}(x; \alpha, \beta)) = \frac{1}{S(t)} \left( \mu_t - \int_0^t x f(x) dx \right) - t, t \geq 0$$

$$E(S_{R(t) \text{-ML-II}}(x; \alpha, \beta)) = \frac{(1 + \alpha t)^\beta}{(1 - t)^\beta} \left( E \sum_{i=0}^{\infty} D_i (B(2 + i, \beta) - B_i (2 + i, \beta)) \right) - t$$

Further, the reverse residual life can be defined as \( R(t) = t - X/X \leq t \)

$$S_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{F(t - x)}{F(t)}, t \geq 0$$

Reverse residual life function of X

$$S_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{(1 - (t - x))^{(1 + \alpha t)^\beta}}{(1 + \alpha(t - x))^{(1 - t)^\beta}}$$

with the associated CDF is given as follows

$$F_{R(t) \text{-ML-II}}(x; \alpha, \beta) = \frac{(1 + \alpha(t - x))^{(1 - t)^\beta}}{(1 + \alpha(t - x))^{(1 - t)^\beta}}$$

The mean reversed residual life function/mean waiting time is given by

$$E(S_{R(t) \text{-ML-II}}(x; \alpha, \beta)) = t - \frac{1}{F(t)} \int_0^t x f(x) dx, t \geq 0$$
The corresponding order statistics. The random variables \( B(x; \alpha, \beta) = \int_0^x (1-t)^{\beta-1} dt \) is the beta function, \( E = \beta(1+\alpha), D_i = \alpha \left( \frac{-\beta - 1}{i} \right) \), and \( \alpha > -1, \beta > 0 \).

**Order statistics**

Order statistics and its moments play an important role in reliability analysis and life testing of a component in quality control. Let \( X_1, X_2, X_3, \ldots, X_n \) be a random sample of size \( n \) following the ML-II model and \( \{X_{(1)} < X_{(2)} < X_{(3)} < \ldots < X_{(n)}\} \) be the corresponding order statistics. The random variables \( X_{(i)} \), \( X_{(1)} \), and \( X_{(n)} \) are the \( i \)-th, minimum, and maximum order statistics of \( X \), respectively.

The PDF of the \( i \)-th order statistic is given by

\[
f_{(i,n)}(x) = \frac{1}{B(i,n-i+1)!} (F(x))^{i-1} (1-F(x))^{n-i} f(x),
\]

\( i = 1, 2, 3, \ldots, n. \)

By incorporating Equations (1) and (2), the PDF of the \( i \)-th order statistics is given by

\[
f_{(i,n)}^{\text{ML-II}}(x; \alpha, \beta) = \frac{\beta(\alpha + 1)}{B(i,n-i+1)!} \left( 1 - \left( \frac{1-x}{1+\alpha x} \right)^{\beta} \right) \left( \frac{1-x}{1+\alpha x} \right)^{i-1} \left( \frac{1-x}{1+\alpha x} \right)^{n-i} \left( \frac{(1-x)^{\beta-1}}{(1+\alpha x)^{\beta+1}} \right)
\]

Equation (8) can be written as

\[
f_{(i,n)}^{\text{ML-II}}(x; \alpha, \beta) = \frac{1}{B(i,n-i+1)!} (\beta(\alpha + 1)) \sum_{i,j=0}^{\infty} G_{ij} (-1)^i a^j x^j (1-x)^{2\beta+i-1}
\]

Straightforward computation of Equation (9) leads to the \( w \)-th moment order statistics of \( X \) and it is given by

\[
E(X^n) = \mu_{w-\text{OS-ML-II}} = \frac{1}{B(i,n-i+1)!} (\beta(\alpha + 1)) \sum_{i,j=0}^{\infty} G_{ij} (-1)^i a^j B(j+1,2\beta+i)
\]

where \( G_{ij} = \left( \frac{\beta}{i} \right) \left( \frac{1}{j} \right) \left( -a\beta + \alpha + i \right) \). \( \alpha > -1, \beta > 0 \). Further, the minimum and maximum order statistics of \( X \) follow directly from Equation (8) with \( i = 1 \) and \( i = n \), respectively.

**Stress – strength reliability**

Let \( X_1 \) and \( X_2 \) represent a component's strength and stress, respectively, following the same univariate distribution. The inadequacy or effectiveness of a component is dependent on whether \( X_2 > X_1 \) and \( X_2 < X_1 \), respectively. Stress – strength reliability can be written as \( R = P(X_2 < X_1) \).

**Theorem 2.** Let \( X_1 \sim \text{ML-II} (x; \alpha, \beta_1) \) and \( X_2 \sim \text{ML-II} (x; \alpha, \beta_2) \) be independent ML-II distributed random variables; then the reliability \( R \) is defined as \( \frac{\beta_1}{\beta_1 + \beta_2} \).

**Proof:** Reliability \( R \) is defined as

\[
R = \int f_1(x) F_2(x) dx
\]

Reliability of \( X \) is given by

\[
R_{\text{ML-II}} = \int \left( \frac{\beta_1(\alpha+1)(1-x)^{\beta_1-1}}{(1+\alpha x)^{\beta_1+1}} \right) \left( 1 - \left( \frac{1-x}{1+\alpha x} \right)^{\beta_2} \right) dx.
\]
Hence the above integral reduces \( R \) in terms of \( \beta_1 \) and \( \beta_2 \), refers to the stress-strength reliability of the ML-II, and it is given by

\[
R_{\text{ML-II}} = \frac{\beta_2}{\beta_1 + \beta_2}, \beta_1, \beta_2 > 0
\]

### Entropy

There are a number of schools of thought about defining entropy measures. Entropy can be the quantity of disorderedness, randomness, diversity, or sometimes an uncertainty in a system.

The Rényi\(^{30} \) entropy of \( X \) is defined by

\[
I_{\delta}(X) = \frac{1}{1 - \delta} \log \int_0^1 f^\delta(x) \, dx, \; \delta > 0 \text{ and } \delta \neq 1
\]

First, we simplify \( f(x) \) in terms of \( f^\delta(x) \) by considering Equation (2)

\[
f^\delta_{\text{ML-II}}(X; \alpha, \beta, \delta) = \beta^\delta(\alpha + 1)^\delta (1 - X)^{\delta(\beta - 1)} (1 + \alpha X)^{-\delta(\beta + 1)}
\]

by applying the binomial expansion to this equation, we get

\[
f^\delta_{\text{ML-II}}(X; \alpha, \beta, \delta) = \beta^\delta(\alpha + 1)^\delta \sum_{i=0}^{\infty} \left( \frac{-\delta(\beta + 1)}{i} \right) (\alpha X)^i (1 - X)^{\delta(\beta - 1)}
\]

and by placing this information in \( I_{\delta}(X) \), we get

\[
I_{\delta-\text{ML-II}}(X; \alpha, \beta, \delta) = \frac{1}{1 - \delta} \log \left( \beta^\delta(\alpha + 1)^\delta \sum_{i=0}^{\infty} \left( \frac{-\delta(\beta + 1)}{i} \right) (\alpha X)^i (1 - X)^{\delta(\beta - 1)} \right)
\]

hence, by integrating the last equation we obtain the reduced form of the Rényi entropy of \( X \) and it is given by

\[
I_{\delta-\text{ML-II}}(X; \alpha, \beta, \delta) = \frac{1}{1 - \delta} \log \left( \beta^\delta(\alpha + 1)^\delta \sum_{i=0}^{\infty} A_i B(\delta\beta - 1) + 1, i + 1 \right)
\]

\[
(11)
\]

where \( A_i = \left( \frac{-\delta(\beta + 1)}{i} \right) (\alpha X)^i, \alpha > -1, \beta > 0. \)

A generalization of the Boltzmann-Gibbs entropy is the \( \eta \) – entropy, although in physics, it is referred to as the Tsallis entropy. The Tsallis\(^{31} \) entropy/\( \eta \) – entropy is described by

\[
H_\eta(X) = \frac{1}{\eta - 1} \left( 1 - \int_0^1 f^\eta(x) \, dx \right), \; \eta > 0 \text{ and } \eta \neq 1.
\]

The Tsallis Entropy of \( X \) is given by

\[
H_\eta(\text{ML-II}) = \frac{1}{\eta - 1} \left( 1 - \beta^\eta(\alpha + 1)^\eta \sum_{i=0}^{\infty} C_i B(\eta\beta - 1) + 1, i + 1 \right)
\]

where \( C_i = \left( \frac{-\eta(\beta + 1)}{i} \right) (\alpha X)^i, \alpha > -1, \beta > 0. \)

Havrda and Charvat\(^{32} \) introduced \( \omega \) – entropy measure. It is presented by

\[
H_\omega(X) = \frac{1}{\omega - 1} \left( \int_0^1 f^\omega(x) \, dx - 1 \right), \; \omega > 0 \text{ and } \omega \neq 1.
\]
The Havrda and Charvat entropy of $X$ is given by

$$H_{\omega}^{\text{Havrda-Charvat}}(X; \alpha, \beta, \omega) = \frac{1}{2^\omega - 1} \left( \int_0^1 f^\omega(x)dx \right) - 1,$$

where $F_i = \left( -\omega(\beta+1) \right) a^i, a > -1, \beta > 0.$

Arimoto\(^{33}\) generalized the work of\(^{32}\) by introducing $\varepsilon$-entropy measure. It is presented by

$$H_{\varepsilon}(X) = \frac{1}{2^\varepsilon - 1} \left( \int_0^1 f^\varepsilon(x)dx \right) - 1, \varepsilon > 0 \text{ and } \varepsilon \neq 1.$$

The Arimoto entropy of $X$ is given by

$$H_{\omega}^{\text{Arimoto}}(X; \alpha, \beta, \varepsilon) = \frac{1}{2^\varepsilon - 1} \left[ \left( \frac{\beta^\varepsilon(\alpha+1)^2}{2} \right) \times \sum_{i=0}^{\infty} G_i B_i \left( \frac{1}{\beta} (\beta-1) + 1, i+1 \right) \right]^{-\varepsilon},$$

where $G_i = \left( -\frac{1}{\varepsilon} (\beta+1) \right) a^i, a > -1, \beta > 0.$

Boekee and Lubba\(^{34}\) developed the $\tau$-entropy measure. It is presented by

$$H_{\tau}(X) = \frac{\tau}{\tau - 1} \left( 1 - \left( \int_0^1 f^{\tau-1}(x)dx \right)^{\frac{1}{\tau}} \right), \tau > 0 \text{ and } \tau \neq 1.$$

Boekee and Lubba entropy of $X$ is given by

$$H_{\tau}^{\text{Boekee-Lubba}}(X; \alpha, \beta, \tau) = \frac{\tau}{\tau - 1} \left( 1 - \left( \frac{\beta^{\tau-1}(\alpha+1)^{\tau-1}}{\sum_{i=0}^{\infty} I_i B_i (\tau - 1)(\beta - 1) + 1, i+1} \right) \right)^{\frac{1}{\tau}},$$

where $I_i = \left( -(\tau - 1)(\beta+1) \right) a^i, a > -1, \beta > 0.$

Mathai and Haubold\(^{35}\) generalized the classical Shannon entropy known as $\zeta$-entropy. It is presented by

$$H_{\zeta}(X) = \frac{1}{\zeta - 1} \left( \int_0^1 f^{2-\zeta}(x)dx - 1 \right), \zeta > 0 \text{ and } \zeta \neq 1.$$

The Mathai and Haubold entropy of $X$ is given by

$$H_{\zeta}^{\text{Mathai-Haubold}}(X; \alpha, \beta, \zeta) = \frac{1}{\zeta - 1} \left( \frac{1 - \beta^{2-\zeta}(\alpha+1)^{\zeta-1}}{\sum_{i=0}^{\infty} J_i B_i (2 - \zeta)(\beta - 1) + 1, i+1} \right),$$

where $J_i = \left( -(2-\zeta)(\beta+1) \right) a^i, a > -1, \beta > 0.$

Table 2 presents the flexible behavior of the entropy measures for some chosen model parameters for S-VI ($\alpha = \text{1.1}, \beta = \text{2.1}$), S-VII ($\alpha = \text{1.1}, \beta = \text{1.5}$), and S-VIII ($\alpha = \text{2.1}, \beta = \text{3.5}$).
In this section, we estimate the parameters of the ML-II by following the method of MLE as this method provides the maximum information about the unknown model parameters. Let \(X_1, X_2, X_3, \ldots, X_n\) be a random sample of size \(n\) from the ML-II, then the likelihood function \(L(\vartheta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta)\) of \(X\) is given by

\[
L_{\text{ML-II}}(\vartheta) = (\beta (\alpha + 1))^n \prod_{i=1}^{n} \frac{(1-x_i)^{\beta-1}}{(1+\alpha x_i)^{\beta+1}}
\]

The log-likelihood function \(\log L(\vartheta) = l(\vartheta)\) of \(X\) is

\[
l_{\text{ML-II}}(\vartheta) = n(\log \beta + \log(1+\alpha)) + (\beta - 1) \sum_{i=0}^{n} \log(1 - x_i) - (\beta + 1) \sum_{i=0}^{n} \log(1 + \alpha x_i)
\]

(12)

Now, we are concerned about obtaining the MLEs of the ML-II. For this, first, we maximize the Equation (12) and second, we calculate the partial derivatives w.r.t. the unknown parameters \((\alpha, \beta)\) and equate to zero, respectively. The score vector components are given by

\[
U(\vartheta) = \frac{\partial l}{\partial \vartheta} = \left( \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta} \right)^T
\]
Partial derivatives w.r.t. $\alpha, \beta$ are given as follows, respectively

$$\frac{\partial l_{ML-II}}{\partial \alpha} = \frac{n}{1 + \alpha} - (\beta + 1) \sum_{i=0}^{n} \left( \frac{x_i}{1 + \alpha x_i} \right)$$

$$\frac{\partial l_{ML-II}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=0}^{n} \log(1 - x_i) - \sum_{i=0}^{n} \log(1 + \alpha x_i)$$

The last two non-linear equations do not provide the analytical solution for the MLEs and the optimum value of $\alpha$ and $\beta$. The Newton-Raphson (an appropriate) algorithm plays a supportive role in these kinds of ML estimates. Numerical solutions and estimates of the parameters are calculated using R version 3.6.2 (statistical software).

**Simulation**

In this sub-section, to observe the asymptotic performance of MLE’s $\hat{\varphi} = (\hat{\alpha}, \hat{\beta})$, we discuss the following algorithm.

Step - 1: A random sample $x_1, x_2, x_3, ..., x_n$ of sizes $n = 150, 200, 250, 300, 350, 400, 450, 500$ from Equation (5).

Step - 2: The required results are obtained based on the different combinations of the model parameters placed in S-IX ($\alpha = 3.2, \beta = 2.5$), S-X ($\alpha = 0.2, \beta = 0.9$), and S-XI ($\alpha = 0.2, \beta = 1.5$).

Step - 3: Results of mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) are calculated via *nlmib* in R. These results are presented in Tables 3 to 8.

Step - 4: Each sample is replicated 1000 times.

Step - 5: A gradual decrease with the increase in sample size is observed in mean, biases, MSEs, and Var.

Step - 6: CPs of all the parameters $\varphi = (\alpha, \beta)$ are approximately 0.975, approaching the nominal value, and AW decreases when sample sizes increases.

Furthermore, the frequent use of the measures in the development of average estimate (AE), bias, MSE, CP, AW, are given as follows:

$$AE(\varphi) = \frac{1}{N} \sum_{i=1}^{N} \hat{\varphi}, Var(\varphi) = \frac{1}{N} \sum_{i=1}^{N} (\varphi - \bar{\varphi})^2, Bias(\varphi) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\varphi}_i - \varphi)$$

$$MSE(\varphi) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\varphi}_i - \varphi)^2, CP(\varphi) = \frac{1}{N} \sum_{i=1}^{N} I(\hat{\varphi}_i - 0.975se_{\hat{\varphi}}, \hat{\varphi}_i + 0.975se_{\hat{\varphi}}), and$$

$$AW(\varphi) = \frac{1}{N} \sum_{i=1}^{N} |(\hat{\varphi}_i + 0.975) - (\hat{\varphi}_i - 0.975)|$$

**Table 3.** Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of ($\alpha$) for S-IX.

| Sample | Mean  | Var.  | Bias   | MSE   | CP    | AW    |
|--------|-------|-------|--------|-------|-------|-------|
| 150    | 3.126 | 2.822 | -0.074 | 1.681 | 0.912 | 6.383 |
| 200    | 3.157 | 2.262 | -0.043 | 1.505 | 0.915 | 5.527 |
| 250    | 3.163 | 1.823 | -0.036 | 1.351 | 0.915 | 4.931 |
| 300    | 3.142 | 1.442 | -0.058 | 1.202 | 0.919 | 4.469 |
| 350    | 3.155 | 1.211 | -0.045 | 1.101 | 0.928 | 4.137 |
| 350    | 3.155 | 1.109 | -0.046 | 1.054 | 0.922 | 3.867 |
| 350    | 3.155 | 0.956 | -0.045 | 0.979 | 0.929 | 3.942 |
| 350    | 3.174 | 0.843 | -0.026 | 0.919 | 0.929 | 3.463 |
In this section, we explore the application of ML-II in medical science and engineering data. For this, we consider two lifetime datasets. The first dataset was originally reported by Smithson and Verkuilen. They studied a group of 166 healthy women's anxiety performance, i.e., outside of a pathological clinical setting, from Townsville, Queensland, Australia; data points are 0.01, 0.17, 0.01, 0.05, 0.09, 0.41, 0.05, 0.01, 0.13, 0.01, 0.01, 0.05, 0.17, 0.01, 0.09, 0.01, 0.05, 0.09, 0.09, 0.05, 0.01, 0.01, 0.29, 0.01, 0.01, 0.01, 0.01, 0.01, 0.09, 0.37, 0.05, 0.01, 0.05, 0.29, 0.09.

Table 4. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-IX.

| Sample | Mean  | Var.  | Bias  | MSE  | CP   | AW   |
|--------|-------|-------|-------|------|------|------|
| 150    | 2.964 | 1.664 | 0.464 | 1.370| 0.950| 4.491|
| 200    | 2.835 | 1.066 | 0.334 | 1.085| 0.936| 3.397|
| 250    | 2.759 | 0.743 | 0.259 | 0.900| 0.942| 2.809|
| 300    | 2.713 | 0.520 | 0.213 | 0.752| 0.956| 2.431|
| 350    | 2.669 | 0.382 | 0.169 | 0.641| 0.951| 2.148|
| 400    | 2.652 | 0.319 | 0.152 | 0.585| 0.950| 1.974|
| 450    | 2.632 | 0.257 | 0.133 | 0.524| 0.954| 1.823|
| 500    | 2.607 | 0.209 | 0.107 | 0.471| 0.953| 1.693|

Table 5. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (α) for S-X.

| Sample | Mean  | Var.  | Bias  | MSE  | CP   | AW   |
|--------|-------|-------|-------|------|------|------|
| 150    | 0.255 | 0.083 | 0.055 | 0.293| 1.000| 1.432|
| 200    | 0.247 | 0.069 | 0.048 | 0.267| 0.998| 1.218|
| 250    | 0.236 | 0.058 | 0.036 | 0.243| 0.996| 1.070|
| 300    | 0.223 | 0.046 | 0.023 | 0.216| 0.997| 0.962|
| 350    | 0.218 | 0.040 | 0.018 | 0.201| 0.996| 0.883|
| 400    | 0.216 | 0.037 | 0.015 | 0.193| 0.990| 0.824|
| 450    | 0.211 | 0.033 | 0.011 | 0.182| 0.994| 0.772|
| 500    | 0.211 | 0.031 | 0.011 | 0.175| 0.992| 0.730|

Table 6. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-X.

| Sample | Mean  | Var.  | Bias  | RMSE | CP  | AW  |
|--------|-------|-------|-------|------|-----|-----|
| 150    | 0.906 | 0.013 | 0.006 | 0.115| 0.970| 0.587|
| 200    | 0.904 | 0.011 | 0.004 | 0.104| 0.954| 0.499|
| 250    | 0.906 | 0.009 | 0.006 | 0.095| 0.957| 0.442|
| 300    | 0.906 | 0.008 | 0.006 | 0.088| 0.969| 0.401|
| 350    | 0.906 | 0.007 | 0.006 | 0.081| 0.969| 0.369|
| 400    | 0.905 | 0.006 | 0.005 | 0.078| 0.968| 0.345|
| 450    | 0.906 | 0.005 | 0.005 | 0.075| 0.971| 0.324|
| 500    | 0.9050| 0.005 | 0.005 | 0.072| 0.960| 0.306|

Application
In this section, we explore the application of ML-II in medical science and engineering data. For this, we consider two lifetime datasets. The first dataset was originally reported by Smithson and Verkuilen. They studied a group of 166 healthy women's anxiety performance, i.e., outside of a pathological clinical setting, from Townsville, Queensland, Australia; data points are 0.01, 0.17, 0.01, 0.05, 0.09, 0.41, 0.05, 0.01, 0.13, 0.01, 0.05, 0.17, 0.01, 0.09, 0.01, 0.05, 0.09, 0.09, 0.05, 0.01, 0.01, 0.29, 0.01, 0.01, 0.01, 0.01, 0.01, 0.09, 0.37, 0.05, 0.01, 0.05, 0.29, 0.09,
The second dataset was recently reported by Rahman et al. They studied the lifetime (in days) of 30 electronic devices and the dataset was as follows: 0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982, 0.990.

The ML-II is compared with its competing models (presented in Table 9) based on a series of criteria, namely, -Log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Kolmogorov Smirnov test (K-S) test statistics. Table 10 presents a set of descriptive statistics and Tables 11 and 12 present the parameter estimates and standard errors (in parenthesis) along the goodness-of-fit test. As seen in Tables 11 and 12, the performance of the ML-II abundantly satisfies the criteria of a better fit model. Consequently, we declare that the ML-II is a better fit among all competing models on the anxiety in women data. Moreover, Figures 5 and 6 present the empirically fitted plots comprising PDF (a), CDF (b), Kaplan-Meier Survival (c), Probability-Probability (P-P) (d), Box (e), and total test time (TTT) (f) plots (see Aarset38), which confirm the close fit to the data as well.

Table 7. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (α) for S-XI.

| Sample | Mean  | Var.  | Bias  | MSE  | CP   | AW   |
|--------|-------|-------|-------|------|------|------|
| 150    | 0.263 | 0.097 | 0.064 | 0.319| 1.00 | 1.633|
| 200    | 0.254 | 0.082 | 0.054 | 0.291| 0.998| 1.382|
| 250    | 0.243 | 0.068 | 0.043 | 0.264| 0.995| 1.211|
| 300    | 0.228 | 0.055 | 0.028 | 0.235| 0.998| 1.087|
| 350    | 0.222 | 0.047 | 0.022 | 0.219| 0.996| 0.996|
| 400    | 0.219 | 0.044 | 0.019 | 0.211| 0.994| 0.928|
| 450    | 0.214 | 0.039 | 0.014 | 0.198| 0.993| 0.869|
| 500    | 0.213 | 0.036 | 0.013 | 0.191| 0.988| 0.821|

Table 8. Mean, variance (Var.), bias, mean square error (MSE), coverage probability (CP), and average width (AW) of (β) for S-XI.

| Sample | Mean  | Var.  | Bias  | MSE  | CP   | AW   |
|--------|-------|-------|-------|------|------|------|
| 150    | 1.509 | 0.052 | 0.009 | 0.227| 0.957| 1.335|
| 200    | 1.508 | 0.045 | 0.008 | 0.208| 0.945| 1.125|
| 250    | 1.511 | 0.037 | 0.011 | 0.194| 0.954| 0.992|
| 300    | 1.513 | 0.032 | 0.013 | 0.180| 0.963| 0.898|
| 350    | 1.513 | 0.028 | 0.013 | 0.168| 0.965| 0.824|
| 400    | 1.512 | 0.026 | 0.013 | 0.163| 0.960| 0.768|
| 450    | 1.514 | 0.024 | 0.015 | 0.156| 0.966| 0.722|
| 500    | 1.512 | 0.022 | 0.012 | 0.149| 0.958| 0.680|
Table 9. Competitive models Cumulative distribution functions (cdfs). Kum = Kumaraswamy distribution; L-I = Lehmann type I distribution; L-II = Lehmann type II distribution; T-Kum = transmuted Kumaraswamy distribution; Mostafa type-II distribution; T-GPW = transmuted generalized power Weibull distribution; T-Lin = transmuted Lindley distribution.

| Abbreviations | Model | Parameter/ variable range | Reference |
|---------------|-------|---------------------------|-----------|
| Kum           | $G_1(x; \alpha, \beta) = 1 - (1 - x^\beta)^\alpha$ | $\alpha, \beta > 0$, $0 < x < 1$ | Kumaraswamy<sup>39</sup> |
| L-I           | $G_{II}(x; \alpha) = x^\alpha$ | $\alpha > 0$, $0 < x < 1$ | Lehmann<sup>1</sup> |
| L-II          | $G_{III}(x; \alpha) = 1 - (1 - x^\alpha)^\alpha$ | $\alpha > 0$, $0 < x < 1$ | |
| T-Kum         | $G_{IV}(x; \alpha, \beta, \epsilon) = (1 + \epsilon) \left(1 - (1 - x^\beta)^\alpha\right) - \epsilon \left(1 - (1 - x^\beta)^\alpha\right)^2$ | $\alpha, \beta > 0$, $|\epsilon| \leq 1$, $x > 0$ | Khan et al.<sup>40</sup> |
| MT-II         | $G_v(x; \alpha) = \exp\left(x^\alpha \log 2\right) - 1$ | $\alpha > 0$, $0 < x < 1$ | Muhammad<sup>41</sup> |
| T-GPW         | $G_{VI}(x; \alpha, \beta, \epsilon) = (1 + \epsilon) \left(1 - e^{(1-(1+x^\alpha))^\beta}\right) - \epsilon \left(1 - e^{(1-(1+x^\alpha))^\beta}\right)^2$ | $\alpha, \beta > 0$, $|\epsilon| \leq 1$, $x > 0$ | Khan<sup>42</sup> |
| T-Lin         | $G_{VII}(x; \alpha) = (1 + \epsilon) \left(1 - \frac{1+\epsilon x}{1+\epsilon} e^{(-\alpha x)}\right) - \epsilon \left(1 - \frac{1+\epsilon x}{1+\epsilon} e^{(-\alpha x)}\right)^2$ | $\alpha > 0$, $|\epsilon| \leq 1$, $x > 0$ | Khan et al.<sup>43</sup> |

Table 10. Descriptive statistics.

| Data           | Minimum | 1<sup>st</sup> Quart | Mean | 3<sup>rd</sup> Quart | Skewness | Kurtosis | 95% C.I. | Maximum |
|----------------|---------|----------------------|------|----------------------|----------|----------|---------|---------|
| Women’s anxiety| 0.010   | 0.010                | 0.091| 0.130                | 2.212    | 7.959    | (0.071,0.112) | 0.690   |
| Electronic devices | 0.020 | 0.143                | 0.494| 0.892                | 0.062    | 1.312    | (0.353,0.634) | 0.990   |

Table 11. Maximum likelihood estimations with standard errors (in parenthesis) and goodness of fit for the women’s anxiety data.

| Model   | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\epsilon}$ | -LL          | AIC          | BIC          | HQIC       | K-S      |
|---------|-----------------|----------------|-------------------|--------------|--------------|--------------|-----------|----------|
| ML-II   | 27.793 (7.857)  | 0.988 (0.154) | –                 | -258.657     | -513.315     | -507.091     | -510.788  | 0.276    |
| T-GPW   | 0.780 (0.046)   | 5.898 (1.005) | 0.453 (0.213)     | -250.605     | -495.210     | -485.874     | -491.420  | 0.292    |
| T-Kum   | 0.703 (0.048)   | 3.805 (0.688) | 0.569 (0.178)     | -247.915     | -489.831     | -480.495     | -486.041  | 0.290    |
| Kum     | 0.642 (0.045)   | 4.473 (0.582) | –                 | -244.314     | -484.629     | -478.405     | -482.103  | 0.287    |
| T-Lin   | 9.434 (0.960)   | –              | 0.555 (0.124)     | -239.783     | -475.565     | -469.341     | -473.039  | 0.376    |
| L-II    | 9.082 (0.705)   | –              | –                 | -218.538     | -435.077     | -431.965     | -433.814  | 0.412    |
| L-I     | 0.299 (0.023)   | –              | –                 | -187.727     | -373.454     | -370.342     | -372.191  | 0.270    |
Table 12. Maximum likelihood estimations (MLEs) with standard errors (in parenthesis) and goodness of fit for the lifetime (in days) of 30 electronic devices data.

| Model   | $\hat{a}$  | $\hat{b}$  | $\hat{c}$ | -LL   | AIC   | BIC   | HQIC  | K-S   |
|---------|------------|------------|------------|-------|-------|-------|-------|-------|
| ML-II   | 4.852      | 0.428      |            | −4.401| −4.802| −2.000| −3.906| 0.128 |
| T-Kum   | 0.609      | 0.585      |            | −3.534| −1.069| 3.134 | 0.276 | 0.409 |
| Kum     | 0.587      | 0.611      |            | −3.502| −3.005| −0.203| −2.108| 0.385 |
| L-II    | 0.780      |            |            | −1.00 | −0.007| 1.393 | 0.440 | 0.029 |
| L-I     | 0.816      |            |            | −0.659| 0.680 | 2.081 | 1.128 | 0.189 |
| MT-II   | 0.700      |            |            | −1.056| −0.113| 1.287 | 0.334 | 0.253 |

Figure 5. Fitted Probability density function (PDF) (a), Cumulative distribution function (CDF) (b), Kaplan-Meier survival (c), Probability – Probability (PP) (d), Box (e), and Total Test Time (TTT) (f) Plots for women’s anxiety data.
Conclusion

We developed a potentiated lifetime model that exhibited the bathtub-shaped failure rate and addressed the most efficient and consistent results over complex random phenomena in this article. We called the proposed model a modified Lehmann type II (ML-II) model because it was a modified version of the Lehmann type II model. Several structural and reliability measures were developed and discussed. We used the maximum likelihood estimation method to estimate model parameters, and we also ran a simulation study to investigate the asymptotic performance of the MLEs. Data on anxiety in women from Smithson and Verkuilen and data on 30 electronic devices from Rahman et al., were modeled to reveal the dominance ML-II over its competitors. It is hoped that in the future, the ML-II will be regarded as a superior alternative to the baseline model.

Data availability

Figshare: two_dataset.doc, https://doi.org/10.6084/m9.figshare.14903142.

This project contains the following extended data:

- Smithson and Verkuilen dataset: anxiety test scores from a group of 166 healthy women
- Rahman et al., dataset: lifetime (in days) of 30 electronic devices

The authors of this work sought and obtained consent to use the dataset from the respective authors. The dataset from Smithson and Verkuilen was originally published on Smithson's website and saved under Example 2. Rahman et al.'s dataset is available in their publication. The dataset was published in an open repository with the authors' permission.

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Competing interests

No competing interests were disclosed.
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