Slow Fermions in Quantum Critical Metals

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We study the low-energy behavior of metals coupled to gapless bosons. This problem arises in several contexts in modern condensed matter physics; we focus on the theory of metals near continuous quantum phase transitions (where the boson is the order parameter). In the vicinity of \(d = 3\) spatial dimensions, the upper critical dimension of the theory, the ratio of fermion and boson speeds, \(v/c\), acts as an additional control parameter, enabling us to access IR fixed points where this ratio vanishes. This limit corresponds to a non-Fermi liquid coupled to bosons with critical exponents governed by the Wilson-Fisher fixed point.

Theories of quantum critical points in metals form a central pillar of the broader study of non-Fermi-liquid behavior in quantum materials.\(^{10,13}\) Specifically, near a quantum phase transition to a broken symmetry state that preserves translational symmetry (e.g. a ferromagnetic or electron nematic state), the integrity of quasiparticle excitations on the entire Fermi surface is destroyed due to the scattering of electrons off the soft bosonic fluctuations\(^{16,24}\) associated with the order parameter. However, the ultimate low energy behavior of such systems remains a matter of debate, following the demonstration\(^{22}\) that the standard approach\(^{23,26}\) breaks down, even in a suitable large \(N\) limit where it was previously thought to be exact.

In the present paper we study this problem using a Wilsonian renormalization group (RG) procedure in spatial dimension \(d = 3 - \epsilon\) and in the limit in which the collective mode velocity, \(c\), is larger than the Fermi velocity, \(v\). In the context of Fermi liquid theory, there is a sharp distinction between collective modes with \(c/v > 1\) (e.g. zero sound), which lie outside the particle-hole continuum and so are undamped, and those with \(c/v < 1\), which are typically overdamped\(^{22}\). This same distinction applies to the initial RG flows in quantum critical metals. Moreover, (as we will see) since under renormalization, \(v\) decrease rapidly with decreasing energy, if \(c/v > 1\) in the ultraviolet (UV), this inequality is increasingly well satisfied at lower energies.

For small \(\epsilon\), we find a perturbatively accessible fixed-point in which the critical exponents are governed by the usual Wilson-Fisher fixed-point, but the Fermi liquid is destroyed and the Fermi velocity tends to zero (the effective mass diverges). Identical fixed-point properties were obtained previously in a particular large \(N\) limit of the problem in which the order parameter field of the present analysis is replaced by an \(N \times N\) matrix field coupled to \(N\) flavors of fermions\(^{20,21}\). However, this fixed-point is distinct from the more usual (and still unsolved) large \(N\) limit in which a single scalar field is coupled to \(N\) flavors of fermions. Here, we will not need to take either large \(N\) limit, though such a parameter can provide additional control to our calculations.

If the bare coupling to the collective modes is not too strong, for \(c/v > 1\) this fixed point governs the behavior of the system over a range of energies and temperatures. We identify several possible instabilities that might destroy due to the scattering of electrons off the soft bosonic fluctuations\(^{16,24}\). Identical fixed-point properties are obtained previously in a particular large \(N\) limit in which a single scalar field is coupled to \(N\) flavors of fermions. Here, we will not need to take either large \(N\) limit, though such a parameter can provide additional control to our calculations.

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The coupling between the two fields is a generalized Yukawa interaction, best written in momentum space as

\[
S_{\psi, \phi} = \int d^{d+1}k \frac{d^{d+1}q}{(2\pi)^{2d+1}} g(k, q) \bar{\psi}_\sigma(k) \psi_\sigma(k + q) \phi(q) \tag{2}
\]

where the measure \(d^{d+1}k\) includes both frequency and \(d\)-dimensional momenta, and repeated spin indices \(\sigma\) are
summed. The contraction \( \bar{\psi}\psi \) implicitly includes any spin matrices that should be included if e.g. \( \phi \) is a ferromagnetic order parameter. Note that the Yukawa coupling is parametrized both by the momentum state of the initial fermion \( k \) as well as the momentum transfer \( q \). At low energies, the Yukawa coupling takes the form

\[
g(k, q) = g(k_F, 0) + \cdots
\]  

(3)

where the ellipsis denotes irrelevant corrections. The symmetries of \( g \) depend on the particular form of broken symmetry that characterizes the proximate ordered state (in which \( \langle \phi \rangle \neq 0 \)). An example of considerable interest is that of Ising nematic order in a tetragonal crystal, where \( \phi \), and hence \( g \) as well, are odd under rotation by \( \pi/2 \): \( g(k_F) \sim \cos(k_F^x) - \cos(k_F^y) \). Therefore, there are “cold spots” on the Fermi surface where \( g \to 0 \) and the fermions do not couple to the order parameter.

**Perturbative considerations:** Below \( d = 3 \) spatial dimensions, the scalar self-coupling and the Yukawa interactions are relevant; even weak interactions produce a large effect on the low energy physics. Conversely, this means that the theory (neglecting the weak four-Fermi interactions, whose effects are already well-known) enjoys a weakly-coupled UV fixed point: the theory at high energies is just a Landau Fermi liquid nearly decoupled from a free, critical scalar field, and the interactions can be treated perturbatively. As one moves from the UV into the IR, the couplings flow toward non-trivial values. Since \( d = 3 \) is the upper critical dimension for all couplings, at small \( \epsilon \) one can follow this flow by computing the logarithmic divergences of the theory in \( d = 3 \) and thereby obtaining the RG equations. However, the effect of Landau damping could present an obstacle to continuing this flow arbitrarily in the IR because in diagrams with closed fermion loops, the Yukawa coupling can effectively act like a relevant coupling even in \( d = 3 \). The one-loop boson self-energy of Fig. 1 generates a contribution to the boson self-energy which in the long-wavelength limit, and for real frequencies is

\[
\Pi_{d=3}(q_0, q) = \frac{g^2 k_F^2}{2\pi^2 v} \left[ 1 + \frac{q_0}{2vq} \log \frac{q_0 - vq}{q_0 + vq} \right]
\]

(4)

When \( |q_0/v| \leq 1 \) the self-energy is real, while in the opposite limit it has an imaginary part. Physically, this follows from kinematic constraints on the boson decay into fermion pairs, and so it is true in any \( d \). (Note that in the generalized theory with an \( N \times N \) matrix boson coupled to \( N \) fermions (as in Ref. 22), the Landau damping is \( 1/N \) suppressed, i.e., Eq. 4 is multiplied by \( 1/N \); in this way complete parametric control can be obtained.) In practice, we will perform computations after continuing to Euclidean space-time, \( q_0 \to iq_0 \).

One can think of Eq. 4 as a non-local “mass-like” term in the limit \( |q_0/v| \leq 1 \). In particular, it takes the form \( \Pi(q_0, q) = g^2 k_F^2 F(q_0/vq) \), where \( F \) is a real dimensionless function. At sufficiently low energy, this therefore becomes a large effect. If \( F \) were just a constant, the physics of this term would be well-understood: \( \Pi(q_0, q) \) would be just a mass term that, at criticality, would be cancelled by a local counterterm. For the actual function \( F \) in Eq. 4 its implication for the IR dynamics of the boson is less clear.

Our primary observation is that at \( v \ll c \), a great simplification occurs, because \( F \to 0 \) as \( v \to 0 \). There is a simple diagrammatic argument why this occurs. Consider the Feynman integral corresponding to Figure 1

\[
\Pi(q_0, q) = \frac{g^2}{(2\pi)^4} \int \frac{d\omega d\ell k^2 d\cos \theta}{(i\omega - v\ell)(i\omega + q_0 - v(\ell + q \cos \theta))}
\]

We can change variables \( \ell \to \ell/|v| \) and pull the \( 1/|v| \) from the integration measure out front. Then, it is easy to see that at \( v \to 0 \), all poles in \( \omega \) are always on the same side of the real axis, and therefore the integral vanishes. Moreover, the rescaled integral is invariant under \( v \to -v \), since this can be compensated for by changing integration variables \( \cos \theta \to -\cos \theta \). This argument holds in any \( d \), since the integration measure always takes the form given above, multiplied by an even function of \( \cos \theta \). Therefore \( \Pi(q_0, q) \) vanishes like \( O(v) \) at small \( v \). Furthermore, it is clear that this argument applies equally well to any diagram with a single closed fermion loop and any number of scalar external legs.

This demonstrates that when \( v/c \to 0 \), the effect of the Yukawa coupling on the boson propagator vanishes, simplifying the use of RG to study the deep IR of the theory. One must ask, though, what happens when \( v/c \) is merely small but not vanishing. In this limit, the leading small \( v \) contribution to \( \Pi(q_0, q) \) is the non-local term

\[
\Pi(q_0, q) \sim v \frac{g^2 k_F^2 q^2}{2\pi^2 q_0^2} + O(v^2),
\]

(5)

so one might worry that the the neglect of this term is justified only when \( v \) is finely tuned to be identically zero. Fortunately, as we describe below, the RG flow of the theory drives the velocity to a fixed point at \( v/c = 0 \), so that all one needs is for the theory in the UV to start out in the basin of attraction of this fixed point.

**RG Flows:** The one-loop logarithmic divergences and resulting \( \beta \) functions of the theory were computed in Refs. 20,21, where one sees that \( v \to 0 \) at low energies. Here we will focus on the parameters \( g \) and \( v \) since...
the other couplings of the theory do not appear in their one-loop $\beta$ functions.

Let us write the renormalization scale $\mu$ of the theory in terms of an initial UV scale $\Lambda$ as $\mu \equiv e^{-t}\Lambda$. The fermions do not affect the RG equations for the purely bosonic part of the Lagrangian, $L_{\phi}$, so they run exactly as in the Wilson-Fisher model:

$$\frac{d\phi}{dt} = 0, \quad \frac{d\lambda_{\phi}}{dt} = -\beta_{\lambda_{\phi}} = \lambda_{\phi} - \frac{3\lambda_{\phi}^2}{16\pi^2} + O(\lambda_{\phi}^3). \tag{6}$$

The RG equations for the fermion velocity $v$ and the Yukawa coupling $g$ then take the form

$$\frac{dv}{dt} = -\beta_v = \frac{g^2}{(2\pi c)^2} S(v, w, ...), + O(\lambda_{\phi}^2, g^2),$$

$$\frac{dg}{dt} = -\beta_g = g \left( \frac{\epsilon}{2} - 2 \frac{g^2}{4\pi^2 c^2 (c + |v|)} \right) + O(\lambda_{\phi}^2, g^2),$$

$$\frac{dw}{dt} = -w|1 + O(g^2)| \tag{7}$$

So long as $v$ is large enough that the higher order terms in the fermion dispersion can be ignored (i.e. for $v \gg w\mu$), we can replace $S(v, w, ...)$ $\approx$ $S(v, 0, ...)$ $= \text{sign}(v)$. However, the exact beta function must be analytic; as $v$ tends to 0, we eventually reach a scale at which the higher order (dangerously irrelevant) terms in the dispersion cannot be neglected, with the result that $\beta_v \to 0$ as $v \to 0$.

For small $\epsilon$, these RG equations have a fixed point in the perturbative regime at which the higher order terms in the beta functions are negligible: $\lambda_{\phi}^* = \epsilon(16\pi^2/3)$, $g^* = \sqrt{\epsilon} \sqrt{2\pi^2 c}$, and $v^* = w^* = 0$. In the idealized model with $w = 0$, the fixed point describes the properties of the transition down to the deep IR, as long as $v$ flows to zero at a scale higher than the scale $\mu_{LD}$ where Landau damping becomes important. We will define $\mu_{LD}$ as the scale where the self-energy $\Pi(\mu_{LD}, g)$ becomes larger than the tree-level kinetic term $g_0^2 + c^2 g^2$. This statement is somewhat ambiguous since it depends on the ratio $x \equiv q_0/cq$. If we look “on-shell” where $x \sim 1$, the self-energy correction is increasingly unimportant the further we proceed into the IR in a large region of parameter space. To be more precise, we can define $\mu_{LD}$ to be the solution of the implicit equation

$$\frac{x^2 |\Pi(\mu_{LD}, \mu_{LD}/c)|}{\mu_{LD}^2 (1 + x^2)} = 1, \tag{8}$$

with $x = 1$, i.e. $\mu_{LD}$ is the highest scale where the loop correction to the boson propagator is as large as the tree-level propagator. In practice, we will adopt a strictly more conservative definition of $\mu_{LD}$, which chooses $x$ to maximize the ratio in (8). It is important to note that we use the running value of the fermion velocity. With this definition, we find that $v$ flows to zero at a scale $\mu_{LD} > \mu_{LD}$ for a wide range of modestly small values of $v_0/c$, and at perturbatively small values of the coupling $g_0$, where we have defined the bare parameters $v_0 \equiv v(0)$ and $g_0 \equiv g(0)$. Since both $\mu_{*}$ and $\mu_{LD}$ depend on $g_0$ and $v_0$, this condition is satisfied only for some range of parameters, as shown in Fig. 3. In general, for fixed $g_0$, we expect this condition to be satisfied for small enough $v_0/c$, as shown by the upper diagonal boundary-line in the figure. In computing this line we have taken the cutoff, i.e. the Fermi energy, to be $\Lambda = v_0 k_F$, which is problematic for very small $v_0$; if the Fermi energy is proportional to $v_0$, we do not know what happens when $v_0$ is small, because with such a low Fermi energy the Landau damping term is dominant already at the UV cut-off. If the Fermi energy is independent of $v_0$, then the small $v_0$ region is not a problem.

At the fixed point, the fermions are formally dispersionless, which is a singular situation in which the for-
nally irrelevant terms (including $v$ and $w$ and higher power terms in the dispersion) cannot be safely neglected upon approach to the fixed point. So long as $v \gg w \mu$, the leading irrelevant operator is $v$, and it is thus possible to ignore the effects of $w$ and all higher order terms in the fermion dispersion. However, it is a peculiar feature of this problem that $v$ ultimately flows toward zero so fast that there is always an emergent low energy scale, $\mu_w = \Lambda e^{-1/\epsilon}$, at which the higher order terms become important, i.e., where $v(t_w) = w(t_w)\mu_w$. We can estimate $\mu_w$ by adopting the approximation $S(v) = \text{sign}(v)$ (valid where $\mu \gg \mu_w$ and computing the scale at which $v \to 0$: this gives $\mu_w \sim \Lambda \exp[-v_0c^2/\bar{g}^2]$ where $\bar{g}^2$ is an appropriate average value of $g$ which depends on both $v_0/c$ and $g_0$, but always lies between $g_0$ and $g^*$. In the weak coupling limit $\mu_w$ is exponentially small. New physics can emerge at energy scales less than $\mu_w$; one likely implication is the existence of a Lifshitz transition (i.e., a change in the topology of the Fermi surface) close enough to criticality that energy scales smaller than $\mu_w$ are significant. We will explore the fermionic properties in this regime in a future study.

**Higher-Point Correlators:** So far, we have analyzed the requirement that there is no breakdown of perturbation theory due to non-local terms in the boson two-point function, but it is important to make sure that there is no earlier breakdown due to higher-point functions. For instance, the local term $\lambda \psi^4$ is marginal, so the four-point diagram with a fermion loop becomes essentially a relevant effect and will indeed create a scale where perturbation theory breaks down. We already presented a diagrammatic argument above eqn. [5] that all such diagrams vanish at $v = 0$, so we need only show that this additional scale of breakdown is lower than the scale arising from the two-point function.

This turns out to be straightforward for any $n$-point function. The closed fermion loop produces a dimensionful factor $k_F^{2n-4}$ corresponding to the area of the Fermi surface, which is dimensionally compensated for by factors of the external momentum. So far, this is identical to the factor of the two-point diagram. However, the $n$-point function has $n$ factors of the coupling $g$ rather than two, so the dimensionless prefactor controlling the size of the breakdown scale is parametrically

$$g^n \frac{k_F^{2n-4}}{E^{2n-4}} \sim 1. \quad (9)$$

Because of these extra factors of the coupling, for perturbative $g$ the breakdown scale associated with $n$-point functions is strictly lower than that associated with the two-point function.

**Discussion:** The central physical insight underlying the present analysis is that for $v/c \ll 1$, the fermions cannot respond to the rapidly propagating collective modes. In the case of critical order parameter fluctuations, this means that they exhibit the same Wilson-Fisher universal properties as they would in the absence of coupling to fermions. Conversely, the singular forward scattering interactions between fermions induced by the exchange of gapless bosons cause a spectacular breakdown of Fermi-liquid theory: instead of well defined quasiparticles, the fermion fields develop an anomalous dimension, which to leading order in the $\epsilon$ expansion is $\gamma_\psi = \epsilon/8$. Moreover, the associated fermion mass renormalization at criticality implies that $\psi$ itself is a running coupling constant, so that if $v/c < 1$ at the bare level, it will tend increasingly to renormalize to smaller values at lower energies, meaning that this simple physical argument becomes increasingly accurate at lower energies. The possible role, if any, of off-shell bosons, which are damped, will be explored in a future publication.

Conversely, if the bare ratio $v/c \gg 1$, then Landau damping likely plays an important role at low energies. As best we understand, it is still unclear what the behavior of such a system is at low energy. It seems plausible that it is governed by a different fixed point than the one explored here, perhaps one that emphasizes the role of Landau damping. In physical terms, this would imply that there is more than one possible universality class for the quantum critical phenomena associated with the same symmetry breaking transition in a metallic system. A well established precedent for this exists, at least in $d = 3$.

In a future publication, we will explore some of the experimentally accessible signatures of this new class of metallic quantum critical points. However, a few of the more obvious points are worth mentioning: The specific heat at the critical point exhibits the non-Fermi liquid power law, $C \sim T^{1-\gamma_\psi}$ (becoming $\sim T \log(T)$ as $\epsilon \to 0$). Where the coupling function $g(K_F, 0)$ has nodes on the Fermi surface (as it does in the case of an Ising nematic order parameter), even at criticality, well defined quasiparticles survive at these points ("cold spots"). Away from criticality, well defined quasiparticles are recovered along the entire Fermi surface, but with an effective mass that diverges as the critical point is approached.

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