Mild Solutions for a Class of Fractional SPDEs and Their Sample Paths

by

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Abstract. We introduce a notion of mild solution for a class of non-autonomous parabolic stochastic partial differential equations defined on a bounded open subset $D \subset \mathbb{R}^d$ and driven by an infinite-dimensional fractional noise. We prove the existence of such a solution, establish its relation with the variational solution introduced in [31] and the Hölder continuity of its sample paths when we consider it as an $L^2(D)$-valued stochastic process. When $h$ is an affine function, we also prove uniqueness. An immediate consequence of our results is the indistinguishability of mild and variational solutions in the case of uniqueness.

Keywords: Fractional Brownian motion, stochastic partial differential equation, Green’s function, sample path regularity.

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1 Introduction and Outline

In the last decades, the interest in fractional Brownian motion, first introduced in [20] and referred to as fBm in the sequel, has increased enormously as one important ingredient of fractal models in the sciences. The paper [25] has been one of the keystones that has attracted the attention of part of the probabilistic community to this challenging object. Some of the research on fBm has significantly influenced the present state of the art of Gaussian processes (see for instance [5], [29], [33], just to mention a few). An important aspect of the study of fBm lies in the domain of stochastic analysis. Since this process is neither a semimartingale nor a Markov process, Itô’s theory does not apply. For values of the Hurst parameter $H$ greater than $\frac{1}{2}$ -the regular case- integrals of Young’s type and fractional calculus techniques have been considered ([39], [40]). However, for $H$ less than $\frac{1}{2}$ this approach fails. The integral representation of fBm as a Volterra integral with respect to the standard Brownian motion has been successfully exploited in setting up a stochastic calculus where classical tools of Gaussian processes along with fractional and Malliavin calculus are combined. A pioneering work in this context is [10], then [2], [6], [9] and also [15]. Since then, there have been many contributions to the subject. Let us refer to [28] for enlightening contents and a pretty complete list of references. Rough path analysis (see [24]) provides a new approach somehow related to Young’s approach.

One of the main reasons for developing a stochastic calculus based on fBm is mathematical modeling. The theory of ordinary and partial differential equations driven by a fractional noise is nowadays a very active field of research. Some of the motivations come from a number of applications in engineering, biophysics and mathematical finance; to refer only to a few, let us mention [12], [21], [35]. There are also purely mathematical motivations. Problems studied so far range from the existence, the uniqueness, the regularity and the long-time behaviour of solutions to large deviations, support theorems and the analysis of the law of the solutions using Malliavin calculus. Without aiming to be exhaustive, let us refer to [3], [14], [17], [18], [19], [24], [26], [27], [30], [31], [34] and [38]) for a reduced sample of published work.

This paper aims to pursue the investigations of [31], where the authors develop an existence and uniqueness theory of variational solutions for a class of non-autonomous semilinear partial differential equations driven by an infinite-dimensional multiplicative fractional noise through the construction and the convergence of a suitable Faedo-Galerkin scheme.

As is the case for deterministic partial differential equations, a recurrent
difficulty is the necessity to decide \textit{ab initio} what solution concept is relevant, since there are several \textit{a priori} non-equivalent possibilities to choose from. Thus, while in [31] two notions of \textit{variational solution} that are subsequently proved to be indistinguishable are introduced, the focus in [17] or [26] is rather on the idea of \textit{mild solution}, namely, a solution which can be expressed as a nonlinear integral equation that involves the linear propagator of the theory without any reference to specific classes of test functions. Consequently, this leaves entirely open the question of knowing whether the variational and mild notions are in some sense equivalent, and indeed we are not aware of any connections between them thus far in this context. For equations of the type considered in this article but driven by standard Wiener processes, this issue was addressed in [36]. In [11] a similar question was analyzed for a class of very general SPDEs driven by a finite-dimensional Brownian motion.

In this article we consider the same class of equations as in [31]. We develop an existence and uniqueness theory of mild solutions and prove the indistinguishability of variational and mild solutions. We also prove the Hölder continuity of their sample paths.

Before defining the class of problems we shall investigate, let us fix the notation. All the functional spaces we introduce are real and we use the standard notations for the usual Banach spaces of differentiable functions, of Hölder continuous functions, of Lebesgue integrable functions and for the related scales of Sobolev spaces defined on regions of Euclidean space used for instance in [1]. For \(d \in \mathbb{N}^+\) let \(D \subset \mathbb{R}^d\) be an open and bounded set whose boundary \(\partial D\) is of class \(C^{2+\beta}\) for some \(\beta \in (0, 1)\) (see, for instance, [13] and [22] for a definition of this and related concepts). We will denote by \((., .)_2\) the standard inner product in \(L^2(D)\), by \((., .)_{\mathbb{R}^d}\) the Euclidean inner product in \(\mathbb{R}^d\) and by \(|.|\) the associated Euclidean norm.

Let \((\lambda_i)_{i \in \mathbb{N}^+}\) be any sequence of positive real numbers such that \(\sum_{i=1}^{+\infty} \lambda_i < +\infty\) and \((e_i)_{i \in \mathbb{N}^+}\) an orthonormal basis of \(L^2(D)\) such that \(\sup_{i \in \mathbb{N}^+} \|e_i\|_{\infty} < +\infty\) (the existence of such a basis follows from [32]). We then define the linear, self-adjoint, positive, non-degenerate trace-class operator \(C\) in \(L^2(D)\) by \(Ce_i = \lambda_i e_i\) for each \(i\). In the sequel we write \(\left( (B_i^H(t))_{i \in \mathbb{R}^+} \right)_{i \in \mathbb{N}^+}\) for a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter \(H \in (0, 1)\), defined on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and starting at the origin. We introduce
the \( L^2(D) \)-valued fractional Wiener process \( (W^H(\cdot, t))_{t \in \mathbb{R}^+} \) by setting

\[
W^H(\cdot, t) := \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} e_i(.) B^H_i(t),
\]

where the series converges a.s. in the strong topology of \( L^2(D) \), by virtue of the basic properties of the \( B^H_i(t) \)'s and the fact that \( C \) is trace-class.

Let \( T > 0, \alpha \in (1 - H, \frac{1}{2}) \) and \( (F(t), t \in [0, T]) \) be a stochastic process taking values in the space of linear bounded operators on \( L^2(D) \). Assume that

\[
\sup_{i \in \mathbb{N}^+} \| F(s)e_i \|_{\alpha, 1} < +\infty,
\]

where for a function \( f : [0, T] \rightarrow L^2(D) \),

\[
\| f \|_{\alpha, 1} = \int_0^T \left( \frac{\| f(s) \|_2}{s^\alpha} + \int_s^T \frac{\| f(s) - f(r) \|_2}{(s - r)^{\alpha+1}} dr \right) ds.
\]

Following [26] we can define a pathwise generalized Stieltjes integral (see also [39])

\[
\int_0^T F(s)W^H(ds) := \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \int_0^T F(s)e_i B^H_i(ds),
\]

which satisfies the property

\[
\left\| \int_0^T F(s)W^H(ds) \right\|_2 \leq \sup_{i \in \mathbb{N}^+} \| F(s)e_i \|_{\alpha, 1} \left( \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \Lambda_\alpha(B^H_i) \right).
\]

Here \( \Lambda_\alpha(B^H_i) \) is a positive random variable defined in terms of a Weyl derivative (see [26], Equation (2.4)), satisfying \( \sup_{i \in \mathbb{N}^+} E(\Lambda_\alpha(B^H_i)) < +\infty \), as is proved in [30], Lemma 7.5. Consequently, if \( \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \Lambda_\alpha(B^H_i) < +\infty \), the random variable

\[
r^H_\alpha := \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \Lambda_\alpha(B^H_i).
\]

is finite, a.s., and then

\[
\left\| \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \int_0^T F(s)e_i B^H_i(ds) \right\|_2 \leq r^H_\alpha \sup_{i \in \mathbb{N}^+} \| F(s)e_i \|_{\alpha, 1}.
\]
Next, we introduce the class of real, parabolic, initial-boundary value problems formally given by

\[ du(x,t) = \left( \text{div}(k(x,t) \nabla u(x,t)) + g(u(x,t)) \right) dt + h(u(x,t)) W^H(x,dt), \]

\[ (x,t) \in D \times (0,T], \]

\[ u(x,0) = \varphi(x), \quad x \in \overline{D}, \]

\[ \frac{\partial u(x,t)}{\partial n(k)} = 0, \quad (x,t) \in \partial D \times (0,T], \]  \tag{1.4}

where the last relation stands for the conormal derivative of \( u \) relative to the matrix-valued field \( k \).

In the next section we shall give a rigorous meaning to such a formal expression and for this, we shall use the pathwise integral described before.

In the sequel we write \( n(x) \) for the unit outer normal vector at \( x \in \partial D \) and introduce the following set of assumptions:

(C) The square root \( C^{1/2} \) of the covariance operator is trace-class, that is, we have \( \sum_{i=1}^{+\infty} \lambda_i \frac{1}{2} < +\infty \).

(K\( \beta,\beta' \)) The entries of \( k \) satisfy \( k_{i,j}(.) = k_{j,i}(.) \) for all \( i,j \in \{1,...,d\} \) and there exists a constant \( \beta' \in (\frac{1}{2},1] \) such that \( k_{i,j} \in C^{\beta,\beta'}(\overline{D} \times [0,T]) \) for each \( i,j \). In addition, we have \( k_{i,j,x_l} := \frac{\partial k_{i,j}}{\partial x_l} \in C^{\beta,\beta'}(\overline{D} \times [0,T]) \) for each \( i,j,l \) and there exists a constant \( k \in \mathbb{R}_+^* \) such that the inequality \( (k(x,t)q,q)_{2^d} \geq k \|q\|^2 \) holds for all \( q \in \mathbb{R}^d \) and all \( (x,t) \in \overline{D} \times [0,T] \). Finally, we have

\[ (x,t) \mapsto \sum_{i=1}^{d} k_{i,j}(x,t)n_i(x) \in C^{1+\beta,\frac{1+\beta}{2}}(\partial D \times [0,T]) \]

for each \( j \) and the conormal vector-field \( (x,t) \mapsto n(k)(x,t) := k(x,t)n(x) \) is outward pointing, nowhere tangent to \( \partial D \) for every \( t \).

(L) The functions \( g,h : \mathbb{R} \mapsto \mathbb{R} \) are Lipschitz continuous.

(I) The initial condition satisfies \( \varphi \in C^{2+\beta}(\overline{D}) \) and the conormal boundary condition relative to \( k \).

Finally, we consider the following assumption which also appears in [31]:

(II\(_{\gamma}\)) The derivative \( h' \) is Hölder continuous with exponent \( \gamma \in (0,1] \) and bounded; moreover, the Hurst parameter satisfies \( H \in \left(\frac{1}{1+\gamma},1\right) \).
Notice that if the derivative $h'$ is Lipschitz continuous this amounts to assuming $H \in \left(\frac{1}{2}, 1\right)$.

Problem (1.4) is identical to the initial-boundary value problem investigated in [31], up to Hypotheses (K$_{\beta', \beta'}$) which imply Hypotheses (K) of that article. This immediately entails the existence of what is called there a variational solution of type II for (1.4), henceforth simply coined variational solution. With (K$_{\beta', \beta'}$) we have the existence and regularity properties of the Green function associated with the differential operator governing (1.4). We shall give more details on this in the next section.

We organize this article in the following way. In Section 2 we first recall the notion of variational solution and introduce a notion of mild solution for (1.4) by means of a family of evolution operators in $L^2(D)$ generated by the corresponding deterministic Green’s function. We then proceed by stating our main results concerning the existence, uniqueness, and Hölder regularity of the mild solution along with its indistinguishability from the variational solution when $h$ is an affine function. The section ends with a discussion about the results and methods of their proofs. These are gathered in Section 3.

2 Statement and Discussion of the Results

In the remaining part of this article we write $H^1(D \times (0, T))$ for the isotropic Sobolev space on the cylinder $D \times (0, T)$, which consists of all functions $v \in L^2(D \times (0, T))$ that possess distributional derivatives $v_x, v_{\tau} \in L^2(D \times (0, T))$. The set of all $v \in H^1(D \times (0, T))$ which do not depend on the time variable identifies with $H^1(D)$, the usual Sobolev space on $D$ whose norm we denote by $\| \cdot \|_{1, 2}$.

For $0 < \alpha < 1$ we introduce the Banach space $B^{\alpha, 2}(0, T; L^2(D))$ of all Lebesgue-measurable mappings $u : [0, T] \mapsto L^2(D)$ endowed with the norm

$$
\|u\|_{\alpha, 2, T}^2 := \left(\sup_{t \in [0, T]} \|u(t)\|_2 \right)^2 + \int_0^T dt \left(\int_0^t d\tau \frac{\|u(t) - u(\tau)\|_2}{(t - \tau)^{\alpha + 1}}\right)^2 < +\infty.
$$

(2.1)

Notice that $\| \cdot \|_{\alpha, 1} \leq c \| \cdot \|_{\alpha, 2, T}$, and also that the spaces $B^{\alpha, 2}(0, T; L^2(D))$ decrease when $\alpha$ increases.

We recall the following notion introduced in [31], in which the function $x \mapsto v(x, t) \in L^2(D)$ is interpreted as the Sobolev trace of $v \in H^1(D \times (0, T))$ on the corresponding hyperplane.
Definition 2.1 Fix $H \in (\frac{1}{2}, 1)$ and let $\alpha \in (1 - H, \frac{1}{2})$. We assume that conditions (C), (L) are satisfied and that the initial condition $\varphi$ belongs to $L^2(D)$. In addition we suppose that the symmetric matrix valued function $k$ satisfies
\[
\|kq\|^2 \leq (k(x, t)q, q)_{\mathbb{R}^d} \leq |q|^2,
\]
for any $q \in \mathbb{R}^d$ and some positive constants $\underline{k}$, $\overline{k}$ independent of $x$ and $t$.

Under these conditions, the $L^2(D)$-valued random field $(u_V(. , t))_{t \in [0,T]}$ defined and measurable on $(\Omega, \mathcal{F}, \mathbb{P})$ is a variational solution to Problem (1.4) if:

(1) $u_V \in L^2(0, T; H^1(D)) \cap B^\alpha, 2(0, T; L^2(D))$ a.s., which means that
\[
\int_0^T dt \|u_V(., t)\|_{1,2}^2 = \int_0^T dt \left(\|u_V(., t)\|_2^2 + \|\nabla u_V(., t)\|_2^2\right) < +\infty
\]
and $\|u_V\|_{\alpha, 2, T} < +\infty$ hold a.s.

(2) The integral relation
\[
\int_D dx v(x, t)u_V(x, t) = \int_D dx v(x, 0)\varphi(x) + \int_0^t d\tau \int_D dx v_\tau(x, \tau)u_V(x, \tau)
\]
\[
- \int_0^t d\tau \int_D dx (\nabla v(x, \tau), k(x, \tau)\nabla u_V(x, \tau))_{\mathbb{R}^d}
\]
\[
+ \int_0^t d\tau \int_D dx v(x, \tau)g(u_V(x, \tau))
\]
\[
+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (v(., \tau), h(u_V(., \tau))e_i)_{2} B_i^H(d\tau).
\]
holds a.s. for every $v \in H^1(D \times (0, T))$ and every $t \in [0, T]$.

With the standing hypotheses we easily infer that each term in (2.2) is finite a.s. In particular, Hypothesis (C) and the fact that $h$ is Lipschitz continuous, along with (1.3) imply the absolute convergence, a.s., of the series of the last term in (2.2).

Let $G : \overline{D} \times [0, T] \times \overline{D} \times [0, T] \setminus \{s, t \in [0, T] : s \geq t\} \to \mathbb{R}$ be the parabolic Green’s function associated with the principal part of (1.4). Assume that $(K_{\delta, \delta'})$ holds; it is well-known that $G$ is a continuous function, twice continuously differentiable in $x$, once continuously differentiable in $t$. For every $(y, s) \in D \times (0, T]$, it is also a classical solution to the linear initial-boundary value problem
\[
\partial_t G(x, t; y, s) = \text{div}(k(x, t)\nabla_x G(x, t; y, s)), \quad (x, t) \in D \times (0, T],
\]
\[
\frac{\partial G(x, t; y, s)}{\partial n(k)} = 0, \quad (x, t) \in \partial D \times (0, T),
\]
with
\[
\int_D dy G(., s; y, s) \varphi(y) := \lim_{t \searrow s} \int_D dy G(., t; y, s) \varphi(y) = \varphi(.,)
\]
and satisfies the heat kernel estimates
\[
|\partial^\mu \partial^\nu G(x, t; y, s)| \leq c(t-s)^{-\frac{d+|\mu|+2\nu}{2}} \exp \left[ -c \frac{|x-y|^2}{t-s} \right]
\]
for \( \mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d, \nu \in \mathbb{N} \) and \( |\mu| + 2\nu \leq 2 \), with \( |\mu| = \sum_{j=1}^d \mu_j \) (see, for instance, [13] or [22]). In particular, for \( |\mu| = \nu = 0 \) we have
\[
|G(x, t; y, s)| \leq c(t-s)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{t-s} \right].
\]
We shall refer to (2.5) as the Gaussian property of \( G \).

We can now define the notion of mild solution for (1.4).

**Definition 2.2** Fix \( H \in (\frac{1}{2}, 1) \) and let \( \alpha \in (1 - H, \frac{1}{2}) \). Assume that the hypotheses (C), (K\( _\beta,\beta' \)), (L) hold and that the initial condition \( \varphi \) is bounded.

Under these assumptions, the \( L^2(D) \)-valued random field \( (u_M(., t))_{t \in [0, T]} \) defined and measurable on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a mild solution to Problem [L] if the following two conditions are satisfied:

1. \( u_M \in L^2(0, T; H^1(D)) \cap B^{\alpha,2}(0, T; L^2(D)) \) a.s.
2. The relation
\[
u_M(., t) = \int_D dy \ G(., t; y, 0) \varphi(y) + \int_0^t d\tau \int_D dy \ G(., t; y, \tau) g (u_M(y, \tau))
+ \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \int_0^t \left( \int_D dy \ G(., t; y, \tau) h (u_M(y, \tau)) e_i(y) \right) B_i^H(d\tau)
\]
holds a.s. for every \( t \in [0, T] \) as an equality in \( L^2(D) \).

We shall prove in Lemma 3.2 that with the standing assumptions, each term in (2.6) indeed defines a \( L^2(D) \)-valued stochastic process.

The main results of this article are gathered in the next theorem.
Theorem 2.3 Assume that Hypotheses (C), (K$_{\beta,\beta'}$), (L), (I) and (H$_{\gamma}$) hold; then the following statements are valid:

(a) Fix $H \in \left(\frac{1}{\gamma+1}, 1\right)$ and let $\alpha \in \left(1 - H, \frac{\gamma}{\gamma+1}\right)$. Then Problem (1.4) possesses a variational solution $u_V$; moreover, every such variational solution is a mild solution $u_M$ to (1.4). More precisely, for every $t \in [0, T]$, $u_V(., t) = u_M(., t)$ a.s. in $L^2(D)$.

(b) Fix $H \in \left(\frac{1}{\gamma+1} \lor \frac{d+1}{d+2}, 1\right)$ and then $\alpha \in \left(1 - H, \frac{\gamma}{\gamma+1} \land \frac{1}{d+2}\right)$. Assume in addition that $h$ is an affine function. Then $u_V$ is the unique variational solution to (1.4), while $u_M$ is its unique mild solution.

(c) Let $H$ and $\alpha$ be as in part (b). Then every mild solution $u_M$ to Problem (1.4) is Hölder continuous with respect to the time variable. More precisely, there exists a positive, a.s. finite random variable $R^H_{\alpha}$ such that the estimate

$$\|u_M(., t) - u_M(., s)\|_2 \leq R^H_{\alpha} |t - s|^\theta \left(1 + \|u_M\|_{\alpha, 2, T}\right) \quad (2.7)$$

holds a.s. for all $s, t \in [0, T]$ and every $\theta \in \left(0, (\frac{1}{2} - \alpha) \land \frac{\beta}{2}\right)$.

Remarks

1. The existence of a mild solution will be proved by reference to the existence of a variational solution. This is in contrast with the method of [26], in which the authors prove the existence of mild solutions for a class of autonomous, parabolic, fractional stochastic initial-boundary value problems by means of Schauder’s fixed point theorem. Their method thus requires the construction of a continuous map operating in a compact and convex set of a suitable functional space. If $h$ is an affine function, the arguments of the proof of Statement (b) (see (3.46)) show that a similar approach might be possible for our equation. To the best of our knowledge, there exists as yet no such direct way to prove the existence of mild solutions to (1.4) for a non affine $h$.

2. If $h$ is not affine, the question of uniqueness remains unsettled. In fact, uniqueness could be proved if we were able to extend the inequality (3.46) to any Lipschitz function $h$. This does not seem to be a trivial point, due to the form of the second term in the right-hand side of (2.1).
3. If $h$ is an affine function, Theorem 2.3 establishes the complete indistinguishability of mild and variational solutions, although we do not know whether this property still holds for a general $h$.

4. If $h$ is a constant function the setting of the problems and their proofs become much simpler. Indeed, in Definitions 2.1 and 2.2 the space $B^{\alpha,2}(0, T; L^2(D))$ can be replaced by the larger one $L^\infty(0, T; L^2(D))$, consisting of Lebesgue-measurable mappings $u : [0, T] \to L^2(D)$ such that $\sup_{t \in [0, T]} \|u(t)\|_2 < \infty$. This can be checked by going through the proofs of [31] and Lemma 3.2 of Section 3. Moreover, the range of values of $\theta$ in statement (c) of Theorem 2.3 can be extended to the interval $\left(0, \frac{\alpha}{2}\right)$. This can be easily checked by going through the proof of Proposition 3.12 by checking first that the right-hand side of the inequalities (3.53), (3.54) can be replaced by $c(t-s)^{\delta}(s-\tau)^{-\delta}$ and $c(t-s)^{\frac{\delta}{2}}(s-\tau)^{-\frac{\delta}{2}}(\tau-\sigma)^{1-\delta}$, respectively.

5. By using the factorization method, and under a different set of assumptions on the range of admissible values of $H$ and $\alpha$, we can obtain a different range of values for the Hölder exponent which in general do not provide as good an estimate as (2.7) does. We deal with this question in Proposition 3.14. The factorization method has been introduced in [8] and since then extensively used for the analysis of the sample paths of solutions to parabolic stochastic partial differential equations (see, for instance, [36]).

3 Proofs of the Results

In what follows we write $c$ for all the irrelevant deterministic constants that occur in the various estimates. We begin by recalling that the uniformly elliptic partial differential operator with conormal boundary conditions in the principal part of (1.4) admits a self-adjoint, positive realization $A(t) := -\text{div}(k(\cdot,t)\nabla)$ in $L^2(D)$ on the domain

$$D(A(t)) = \{ v \in H^2(D) : (\nabla v(x), k(x,t)n(x))_{\mathbb{R}^d} = 0, \ (x,t) \in \partial D \times [0,T] \}$$

(see, for instance, [23]). An important consequence of this property is that the parabolic Green’s function $G$ is also, for every $(x,t) \in D \times (0,T)$ with $t > s$, a classical solution to the linear boundary value problem

$$\partial_t G(x,t;y,s) = -\text{div}(k(y,s)\nabla_y G(x,t;y,s)), \quad (y,s) \in D \times (0,T),$$
\[ \frac{\partial G(x,t;y,s)}{\partial n(k)} = 0, \quad (y,s) \in \partial D \times (0,T) , \] (3.2)
dual to (2.3) (see, for instance, [13] or [16]); this means that along with (2.4) we also have
\[ |\partial_\mu^\nu G(x,t;y,s)| \leq c(t-s)^{-\frac{d+|\mu|+2|\nu|}{2}} \exp \left[ -c \frac{|x-y|^2}{t-s} \right] \] (3.3)
for $|\mu| + 2|\nu| \leq 2$. We now use these facts to prove in the next lemma estimates for $G$, which we shall invoke repeatedly in the sequel to analyze various singular integrals. For the sake of clarity we list those inequalities by their chronological order of appearance in the proofs below.

**Lemma 3.1** Assume that Hypothesis $(K_{\beta,\beta'})$ holds. Then, for all $x, y \in D$ and for every $\delta \in \left( \frac{d}{d+2}, 1 \right)$ we have the following inequalities.

(i) For all $t, \tau, \sigma \in [0,T]$ with $t > \tau > \sigma$ and some $t^* \in (\sigma, \tau),
\begin{align*}
|G(x,t;y,\tau) - G(x,t;y,\sigma)| &\leq c(t-\tau)^{-\delta} (\tau-\sigma)^\delta (t-t^*)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{t-t^*} \right].
\end{align*}
(3.4)

(ii) For all $t, s, \tau \in [0,T]$ with $t > s > \tau$ and some $\tau^* \in (s,t),
\begin{align*}
|G(x,t;y,\tau) - G(x,s;y,\tau)| &\leq c(t-s)^\delta (s-\tau)^{-\delta} (\tau^*-\tau)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{\tau^*-\tau} \right].
\end{align*}
(3.5)

and
\begin{align*}
|G(x,t;y,\tau) - G(x,s;y,\tau)|^\delta &\leq c(t-s)^\delta (s-\tau)^{-\delta} (\tau^*-\tau)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{\tau^*-\tau} \right].
\end{align*}
(3.6)

(iii) For all $t, s, \tau, \sigma \in [0,T]$ with $t > s > \tau > \sigma,
\begin{align*}
|G(x,t;y,\tau) - G(x,t;y,\sigma)|^{1-\delta} &\leq c(\tau-\sigma)^{1-\delta} (s-\tau)^{-\frac{d+2}{2}(1-\delta)} \quad (3.7)
\end{align*}
uniformly in $t$. 

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Proof. By applying successively (2.5), the mean-value theorem for $G$ and (3.3) with $|\mu| = 0$ and $\nu = 1$ we may write

\[
|G(x; t; y, \tau) - G(x; t; y, \sigma)| \\
\leq (|G(x; t; y, \tau)| + |G(x; t; y, \sigma)|)^{1-\delta} |G(x; t; y, \tau) - G(x; t; y, \sigma)|^\delta \\
\leq c \left( (t - \tau)^{-\frac{d}{2}} + (t - \sigma)^{-\frac{d}{2}} \right)^{1-\delta} (\tau - \sigma)^\delta |G_{t^*}(x; t; y, t^*)|^\delta \\
\leq c (t - \tau)^{-\frac{d}{2}(1-\delta)} (t - t^*)^{-\frac{d+2}{2}\delta + \frac{d}{2}} (\tau - \sigma)^\delta (t - t^*)^{-\frac{d}{2}} \exp \left[ -c \frac{|x - y|^2}{t - t^*} \right] \\
\leq c (t - \tau)^{-\delta} (\tau - \sigma)^\delta (t - t^*)^{-\frac{d}{2}} \exp \left[ -c \frac{|x - y|^2}{t - t^*} \right]
\]

for some $t^* \in (\sigma, \tau)$, since $-\frac{d+2}{2}\delta + \frac{d}{2} < 0$ and $-\frac{d}{2}(1-\delta) - \frac{d+2}{2}\delta + \frac{d}{2} = -\delta$. This proves (3.4). Up to some minor but important changes, the remaining inequalities can all be proved in a similar way.

Estimate (3.4) now allows us to prove that our notion of mild solution in Definition 3.2 is indeed well-defined; to this end for arbitrary mappings $\varphi$ and $u$ defined on $D$ and $D \times [0, T]$, respectively, we introduce the functions $A(\varphi), B(u), C(u) : D \times [0, T] \rightarrow \mathbb{R}$ by

\[
A(\varphi)(x; t) := \int_D dy \, G(x; t; y, 0) \varphi(y), \\
B(u)(x; t) := \int_0^t d\tau \int_D dy \, G(x; t; y, \tau) g(u(y, \tau)), \\
C(u)(x; t) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left( \int_D dy \, G(x; t; y, \tau) h(u(y, \tau)) e_i(y) \right) B_i^H(d\tau),
\]

and prove the following result.

Lemma 3.2 The hypotheses are the same as in Definition 2.2. Then, for every $u \in \mathcal{B}^{a,2}(0, T; L^2(D))$ we have $A(\varphi)(., t), B(u)(., t) \in L^2(D)$, and also $C(u)(., t) \in L^2(D)$ a.s., for every $t \in [0, T]$.

Proof. The assertion is evident for $A(\varphi)(., t)$, since $\varphi$ is bounded and (2.5) holds. As for $B(u)(., t)$, we infer from the Gaussian property of $G$ that the measure $d\tau dy |G(x; t; y, \tau)|$ is finite on $[0, T] \times D$ uniformly in $(x, t) \in$
$D \times [0,T]$, so that by using successively Schwarz inequality with respect to this measure along with Hypothesis (L) for $g$ we obtain

$$|B(u)(x,t)| \leq \int_0^t d\tau \int_D d y |G(x,t;y,\tau)g(u(y,\tau))|$$

$$\leq c \left( \int_0^t d\tau \int_D d y |G(x,t;y,\tau)| \left( 1 + |u(y,\tau)| \right)^{\frac{3}{2}} \right)^{\frac{1}{2}}$$

for every $x \in D$. We then get the inequalities

$$\|B(u)(.,t)\|_2^2 = \int_D dx \left| \int_0^t d\tau \int_D d y G(x,t;\tau)g(u(\tau)) \right|^2$$

$$\leq c \int_0^t d\tau \int_D d y \left( 1 + |u(y,\tau)|^2 \right) \leq c \left( 1 + \int_0^t d\tau \|u(\tau)\|_2^2 \right) < +\infty.$$
\[
\begin{align*}
&\leq c \int_D dy \left| G(x, t; y, \tau) \right| |u(y, \tau) - u(y, \sigma)|^2 \\
&\quad + c \int_D dy \left| G(x, t; y, \tau) - G(x, t; y, \sigma) \right| \left( 1 + |u(y, \sigma)|^2 \right) \\
&\leq c \int_D dy \left| G(x, t; y, \tau) \right| |u(y, \tau) - u(y, \sigma)|^2 \\
&\quad + c(t - \tau)^{-\delta} (\tau - \sigma)^{\delta} \int_D dy (t - t^*)^{-\frac{d}{2}} \exp \left[ -c \frac{|x - y|^2}{t - t^*} \right] \left( 1 + |u(y, \sigma)|^2 \right)
\end{align*}
\]

for some \( t^* \in (\sigma, \tau) \) and for every \( \delta \in \left(\frac{d}{d+2}, 1\right) \). This is achieved by using Schwarz inequality with respect to the finite measures \( dy \left| G(x, t; y, \tau) \right| \) and \( dy \left| G(x, t; y, \tau) - G(x, t; y, \sigma) \right| \) on \( D \), respectively, along with (3.14). We then integrate the preceding estimate with respect to \( x \in D \) and apply the Gaussian property of \( G \) to eventually obtain

\[
\sup_{i \in \mathbb{N}^+} \| f_{i,t}(u)(., \tau) - f_{i,t}(u)(., \sigma) \|_2 \\
\leq c \left( \| u(., \tau) - u(., \sigma) \|_2 + (t - \tau)^{-\frac{\delta}{2}} (\tau - \sigma)^{\frac{\delta}{2}} (1 + \| u(., \sigma) \|_2) \right). \quad (3.14)
\]

Therefore, by applying (123) we have

\[
\sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \left\| \int_0^t f_{i,t}(u)(., \tau) B_i^H (d\tau) \right\|_2 \\
\leq r_\alpha^H \sup_{i \in \mathbb{N}^+} \int_0^t d\tau \left( \frac{\| f_{i,t}(u)(., \tau) \|_2}{\tau^\alpha} + \int_0^\tau d\sigma \frac{\| f_{i,t}(u)(., \tau) - f_{i,t}(u)(., \sigma) \|_2}{(\tau - \sigma)^{\alpha+1}} \right) \\
\leq c r_\alpha^H \left( 1 + \int_0^t d\tau \frac{\| u(., \tau) \|_2}{\tau^\alpha} + \int_0^t d\tau \int_0^\tau d\sigma \frac{\| u(., \tau) - u(., \sigma) \|_2}{(\tau - \sigma)^{\alpha+1}} \right) \\
\quad + \int_0^t d\tau (t - \tau)^{-\frac{\delta}{2}} \int_0^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (1 + \| u(., \sigma) \|_2) \quad (3.15)
\]
a.s.

Let us now examine more closely the singular integrals in the above terms. On the one hand, we may write

\[
\int_0^t d\tau \frac{\| u(., \tau) \|_2}{\tau^\alpha} + \int_0^t d\tau \int_0^\tau d\sigma \frac{\| u(., \tau) - u(., \sigma) \|_2}{(\tau - \sigma)^{\alpha+1}} \leq c \| u \|_{a, 2, T}, \quad (3.16)
\]

by using Schwarz inequality relative to the measure \( d\tau \) on \((0, t)\) in the last two integrals along with (2.1). On the other hand, in (3.15) the exponent
\(\delta\) can be taken arbitrarily close to 1; consequently our range of values of \(\alpha\) allows the condition \(2\alpha < \delta\) to be satisfied. Thus we can integrate the singularities of the time increments in the last line of (3.15) and get the bound
\[
\int_0^t d\tau (t-\tau)^{-\frac{\delta}{2}} \int_0^\tau d\sigma (\tau-\sigma)^{\frac{\delta}{2}-\alpha-1} (1 + \|u(.,\sigma)\|_2) \leq c \left(1 + \sup_{t\in[0,T]} \|u(.,t)\|_2\right).
\]
(3.17)

Therefore, we can substitute (3.16), (3.17) into (3.15) to obtain (3.12).

In order to relate the notions of variational and mild solution, we recall that the self-adjoint operator \(A(t) = -\text{div}(k(.,t)\nabla)\) defined on (3.1) generates the family of evolution operators \(U(t,s)_{0\leq s\leq t\leq T}\) in \(L^2(D)\) given by
\[
U(t,s)v = \begin{cases} v, & \text{if } s = t, \\ \int_D dy \ G(.,t;y,s)v(y), & \text{if } t > s, \end{cases}
\]
(3.18)

and that each such \(U(t,s)\) is itself self-adjoint (see, for instance, [37]), which means that the symmetry property
\[
G(x,t;y,s) = G(y,t;x,s)
\]
(3.19)
holds for every \((x,t;y,s) \in \overline{D} \times [0,T] \times \overline{D} \times [0,T] \setminus \{s,t \in [0,T] : s \geq t\}\).

**Proof of Statement (a) of Theorem 2.3**

The existence of a variational solution \(u_V\) was proved in Theorem of [31].

In order to prove that every variational solution is mild, we follow the same approach as in Theorem 2 of [36]. For the sake of completeness, we sketch the main ideas.

We shall check that the \(L^2(D)\)-valued stochastic process
\[
u_V(.,t) - \int_D dy \ G(.,t;y,0)\varphi(y) - \int_0^t d\tau \int_D dy \ G(.,t;y,\tau)g (u_V(y,\tau))
- \sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \int_0^t \left( \int_D dy \ G(.,t;y,\tau)h (u_V(y,\tau)e_i(y)) \right) B_i^H (d\tau)
\]
is a.s. orthogonal for every \(t \in [0,T]\) to the dense subspace \(C^2_0(D)\) consisting of all twice continuously differentiable functions with compact support in
D. To this end, for every $v \in C^2_0(D)$ and all $s, t \in [0, T]$ with $t \geq s$ we define $v^t(\cdot, s) := U(t, s)v$, that is,

$$v^t(x, s) = \begin{cases} v(x), & \text{if } s = t, \\ \int_D dy G(y, t; x, s)v(y), & \text{if } t > s, \end{cases}$$

for every $x \in D$ by taking (3.18) and (3.19) into account. It then follows from (3.2), (3.19) and Gauss’ divergence theorem that $v^t \in H^1(D \times (0, T))$, and that for every $t \in [0, T]$, the relation

$$\int_0^t d\tau \int_D dx v^t(x, \tau) u_V(x, \tau) = \int_0^t d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_V(x, \tau)) \in \mathbb{R}$$

(3.21)

holds a.s. Therefore, we may take (3.20) as a test function in (2.2), which, as a consequence of (3.21), leads to the relation

$$(v, u_V(\cdot, t))_2 = (v^t(\cdot, 0), \varphi)_2 + \int_0^t d\tau (v^t(\cdot, \tau), g(u_V(\cdot, \tau)))_2$$

$$+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (v^t(\cdot, \tau), h(u_V(\cdot, \tau))e_i)_2 B_i^H(d\tau),$$

valid a.s. for every $t \in [0, T]$. After some rearrangements, the substitution of (3.20) into the right-hand side of the preceding expression then leads to the equality

$$(v, u_V(\cdot, t))_2 = \left(v, \int_D dy G(\cdot, t; y, 0) \varphi(y)\right)_2$$

$$+ \left(v, \int_0^t d\tau \int_D dy G(\cdot, t; y, \tau) g(u_V(y, \tau))\right)_2$$

$$+ \left(v, \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(\int_D dy G(\cdot, t; y, \tau) h(u_V(y, \tau)) e_i(y)\right) B_i^H(d\tau)\right)_2,$$

which holds for every $t \in [0, T]$ a.s. and every $v \in C^2_0(D)$, thereby leading to the desired orthogonality property. 

$\blacksquare$
Proof of Statement (b) of Theorem 2.3

Under the standing assumptions, we already know from [31] that the variational solution is unique. Moreover, we have just proved that every variational solution is also a mild solution. Hence, it suffices to prove that uniqueness holds within the class of mild solutions. To this end, let us write $u_M$ and $\tilde{u}_M$ for any two such solutions corresponding to the same initial condition $\varphi$; from (2.6) and (3.8)-(3.10) we have

$$
\|u_M(\cdot, t) - \tilde{u}_M(\cdot, t)\|_2 \leq \|B(u_M)(\cdot, t) - B(\tilde{u}_M)(\cdot, t)\|_2 + \|C(u_M)(\cdot, t) - C(\tilde{u}_M)(\cdot, t)\|_2
$$

(3.22)
a.s. for every $t \in [0, T]$.

We proceed by estimating both terms on the right-hand side of (3.22). Since $g$ is Lipschitz, we have

$$
\|B(u_M)(\cdot, t) - B(\tilde{u}_M)(\cdot, t)\|_2^2 \leq c \int_0^t d\tau \|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau)\|_2^2
$$

(3.23)
a.s. for every $t \in [0, T]$.

In order to analyze the second term or the right-hand side of (3.22) we will need the following preliminary result.

Lemma 3.3 The hypotheses are the same as in part (a) of Theorem 2.3 and let the $f_{i,t}(u)$’s be the functions given by (3.11). Then, the estimate

$$
\sup_{(i,t) \in \mathbb{N}^+ \times [0,T]} \|f_{i,t}(u_M)(\cdot, \tau) - f_{i,t}(\tilde{u}_M)(\cdot, \tau)\|_2 \leq c \|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau)\|_2
$$

(3.24)
holds a.s. for every $\tau \in [0, t)$.

Moreover, if $h$ is an affine function we have

$$
\sup_{i \in \mathbb{N}^+} \|f_{i,t}(u_M)(\cdot, \tau) - f_{i,t}(\tilde{u}_M)(\cdot, \tau) - f_{i,t}(u_M)(\cdot, \sigma) + f_{i,t}(\tilde{u}_M)(\cdot, \sigma)\|_2
\leq c(t - \tau)^{-\delta}(\tau - \sigma)^\delta \|u_M(\cdot, \sigma) - \tilde{u}_M(\cdot, \sigma)\|_2
+ c \|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau) - u_M(\cdot, \sigma) + \tilde{u}_M(\cdot, \sigma)\|_2
$$

(3.25)
a.s. for all $t, \tau, \sigma \in [0, T]$ with $t > \tau > \sigma$ and every $\delta \in \left(\frac{d}{d+2}, 1\right)$.

Proof. Up to minor modifications, we can prove (3.24) as we argued in the proof of (3.13).
For the proof of (3.25) we first write
\[
\| f_i,t(u_M)(., \tau) - f_i,t(\tilde{u}_M)(., \tau) - f_i,t(u_M)(., \sigma) + f_i,t(\tilde{u}_M)(., \sigma) \|^2_2 \\
\leq 2 \left( F^1(i, t, \tau, \sigma) + F^2(i, t, \tau, \sigma) \right),
\]
with
\[
F^1(i, t, \tau, \sigma) = \int_D dx \left| \int_D dy \; e_i(y) (G(x, t; y, \tau) - (G(x, t; y, \sigma)) (u_M(y, \tau) - \tilde{u}_M(y, \tau)) \right|^2,
\]
\[
F^2(i, t, \tau, \sigma) = \int_D dx \left| \int_D dy \; e_i(y)G(x, t; y, \sigma) (u_M(y, \tau) - \tilde{u}_M(y, \tau) - u_M(y, \sigma) + \tilde{u}(y, \sigma)) \right|^2.
\]
From the Gaussian property of \( G \) we clearly see that \( F^2(i, t, \tau, \sigma) \) is bounded above by the square of the last term of (3.25). Moreover, by applying first (3.4) and then Schwarz inequality we obtain
\[
F^1(i, t, \tau, \sigma) \leq c(t - \tau)^{-2\delta} (\tau - \sigma)^{2\delta} \| u_M(., \sigma) - \tilde{u}_M(., \sigma) \|^2_2.
\]
Hence (3.25) is proved.

The preceding result now leads to the following estimate for the second term on the right-hand side of (3.22).

**Lemma 3.4** The hypotheses are those of Theorem 2.3 part (b). Then we have
\[
\| C(u_M)(., t) - C(\tilde{u}_M)(., t) \|_2 \\
\leq c r^H \alpha \left( \int_0^t \! \! d\tau \left( \frac{1}{\tau^\alpha} + \frac{1}{(t - \tau)^\alpha} \right) \| u_M(., \tau) - \tilde{u}_M(., \tau) \|_2 \\
+ \int_0^t \! \! d\tau \int_0^\tau \! \! d\sigma \frac{\| u_M(., \tau) - \tilde{u}_M(., \tau) - u_M(., \sigma) + \tilde{u}(., \sigma) \|_2}{(\tau - \sigma)^{\alpha + 1}} \right) \tag{3.26}
\]
a.s. for every \( t \in [0, T] \).

**Proof.** From (3.10), (3.11), and by using (1.3), (3.24), (3.25), we have
\[
\| C(u_M)(., t) - C(\tilde{u}_M)(., t) \|_2
\]

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\[
\leq c r_{\alpha}^H \left( \int_0^t d\tau \frac{\|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau)\|_2}{\tau^\alpha} \right) \\
+ \int_0^t d\tau \int_0^\tau d\sigma (t - \tau)^{-\delta} (\tau - \sigma)^{\delta - \alpha - 1} \|u_M(\cdot, \sigma) - \tilde{u}_M(\cdot, \sigma)\|_2 \\
+ \int_0^t d\tau \int_0^\tau d\sigma \frac{\|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau) - u_M(\cdot, \sigma) + \tilde{u}_M(\cdot, \sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \right) 
\]  
(3.27)

a.s. for every \( t \in [0, T] \).

Furthermore, by swapping each integration variable for the other in the second term on the right-hand side and by using Fubini’s theorem we may write

\[
\int_0^t d\tau \int_0^\tau d\sigma (t - \tau)^{-\delta} (\tau - \sigma)^{\delta - \alpha - 1} \|u_M(\cdot, \sigma) - \tilde{u}_M(\cdot, \sigma)\|_2 \\
= \int_0^t d\tau \|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau)\|_2 \int_\tau^t d\sigma (t - \sigma)^{-\delta} (\sigma - \tau)^{\delta - \alpha - 1} \\
= c \int_0^t d\tau \frac{\|u_M(\cdot, \tau) - \tilde{u}_M(\cdot, \tau)\|_2}{(t - \tau)^\alpha},
\]

after having evaluated the singular integral explicitly in terms of Euler’s Beta function; this is possible since we can choose \( \delta \in \left( \frac{d}{d + 2}, 1 \right) \) such that \( \alpha < \delta \). The substitution of the preceding expression into (3.27) then proves (3.26). \( \blacksquare \)

In what follows, we write \( R \) for all the irrelevant a.s. finite and positive random variables that appear in the different estimates, unless we specify these variables otherwise. The preceding inequalities then lead to a crucial estimate for \( z_M := u_M - \tilde{u}_M \) with respect to the norm in \( B^{\alpha,2}(0, t; L^2(D)) \).

**Lemma 3.5** We assume the same hypotheses as in part (b) of Theorem 2.3. Then we have

\[
\|z_M\|_2^2 \leq R \left( \int_0^t d\tau \sup_{\sigma \in [0, \tau]} \|z_M(\cdot, \sigma)\|_2^2 \\
+ \int_0^t d\tau \left( \int_0^\tau d\sigma \frac{\|z_M(\cdot, \tau) - z_M(\cdot, \sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \right)^2 \right)
\]

a.s. for every \( t \in [0, T] \).
Proof. We apply Schwarz inequality relative to the measure $d\tau$ on $(0,t)$ to both integrals on the right-hand side of (3.26). This leads to

$$\|C(u_M)(.,t) - C(\tilde{u}_M)(.,t)\|_2^2 \leq R \left( \int_0^t d\tau \|z_M(.,\tau)\|_2^2 + \int_0^t d\tau \left( \int_0^\tau d\sigma \frac{\|z_M(.,\tau) - z_M(.,\sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \right)^2 \right)$$

(3.28)
a.s. for every $t \in [0,T]$. This estimate along with (3.22), (3.23) yields the result.

As a consequence of the preceding Lemma we obtain

$$\|z_M\|_{\alpha,2,t}^2 \leq R \left( \int_0^t d\tau \sup_{\sigma \in [0,\tau]} \|z_M(.,\sigma)\|_2^2 + \int_0^t d\tau \left( \int_0^\tau d\sigma \frac{\|z_M(.,\tau) - z_M(.,\sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \right)^2 \right),$$

(3.29)

by the very definition of the norm $\|\cdot\|_{\alpha,2,t}^2$.

We proceed by analyzing further the second term on the right-hand side of (3.29), so as to eventually obtain an inequality of Gronwall type for $\|z_M\|_{\alpha,2,t}^2$.

First we introduce some notation. For $0 \leq \tau < s \leq t \leq T$, we set

$$f_{i,t,s}(u_M)(.,\tau) := f_{i,t}(u_M)(.,\tau) - f_{i,s}(u_M)(.,\tau),$$

(3.30)

where the $f_{i,t}(u_M)$'s are given by (3.11).

By reference to (2.6), we may write

$$z_M(.,\tau) - z_M(.,\sigma)$$

$$= \int_\sigma^\tau d\rho \int_D dy G(.,\tau; y, \rho) (g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho)))$$

$$+ \int_0^\sigma d\rho \int_D dy (G(.,\tau; y, \rho) - G(.,\sigma; y, \rho)) (g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho)))$$

$$+ \sum_{i=1}^{+\infty} \lambda_i^{1/2} \int_\sigma^\tau (f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho)) B^H_i(d\rho)$$

$$+ \sum_{i=1}^{+\infty} \lambda_i^{1/2} \int_0^\sigma (f_{i,\tau,\sigma}(u_M)(.,\rho) - f_{i,\tau,\sigma}(\tilde{u}_M)(.,\rho)) B^H_i(d\rho)$$

(3.31)
for all \( \sigma, \tau \in [0, t] \) with \( \tau > \sigma \).

Our next goal is to estimate the \( L^2(D) \)-norm of each contribution on the right-hand side of (3.31). Regarding the first two terms we have the following result.

**Lemma 3.6** The hypotheses are the same as in part (a) of Theorem 2.3; then we have

\[
\left\| \int_\sigma^\tau d\rho \int_D dy \ G(., \tau; y, \rho) \left( g(u_M(y, \rho)) - g(\bar{u}_M(y, \rho)) \right) \right\|_2 \\
\leq c (\tau - \sigma)^{\frac{1}{2}} \left( \int_\sigma^\tau d\rho \|z_M(., \rho)\|_2^2 \right)^{\frac{1}{2}}
\]

(3.32)

and

\[
\left\| \int_0^\sigma d\rho \int_D dy \ (G(., \tau; y, \rho) - G(., \sigma; y, \rho)) \left( g(u_M(y, \rho)) - g(\bar{u}_M(y, \rho)) \right) \right\|_2 \\
\leq c (\tau - \sigma)^{\frac{1}{2}} \left( \int_0^\sigma d\rho (\sigma - \rho)^{-\delta} \|z_M(., \rho)\|_2^2 \right)^{\frac{1}{2}}
\]

(3.33)

a.s. for all \( \sigma, \tau \in [0, t] \) with \( \tau > \sigma \) and every \( \delta \in \left( \frac{d}{d+2}, 1 \right) \).

**Proof:** The inequality (3.32) follows by applying Schwarz inequality and using the Gaussian property along with assumption (L). As for (3.33), we first apply Schwarz inequality with respect to the measure on \([0, \sigma] \times D\) given by \( |G(x, \tau; y, \rho) - G(x, \sigma; y, \rho)|d\rho \ dy \) and then (3.5).

Next, we turn to the analysis of the third term on the right-hand side of (3.31).

**Lemma 3.7** With the same hypotheses as in part (b) of Theorem 2.3, we have

\[
\sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \left\| \int_\sigma^\tau (f_{i, \tau}(u_M)(., \rho) - f_{i, \tau}(\bar{u}_M)(., \rho)) B_i^H (d\rho) \right\|_2 \\
\leq R \left( \int_\sigma^\tau d\rho \left( \frac{1}{(\rho - \sigma)^\alpha} + \frac{1}{(\tau - \rho)^\alpha} \right) \|z_M(., \rho)\|_2 \\
+ \int_\sigma^\tau d\rho \int_\sigma^\rho d\varsigma \frac{\|z_M(., \rho) - z_M(., \varsigma)\|_2}{(\rho - \varsigma)^{\alpha+1}} \right)
\]

a.s. for all \( \sigma, \tau \in [0, t] \) with \( \tau > \sigma \).
Proof. In terms of the variables $\tau, \rho$ and $\varsigma$, inequalities (3.24), (3.25) of Lemma 3.3 now read

$$\sup_{(i,\tau) \in \mathbb{N}^+ \times [0,T]} \| f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho) \|_2 \leq c \| z_M(.,\rho) \|_2$$

(3.34)

and

$$\sup_{i \in \mathbb{N}^+} \| f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho) - f_{i,\tau}(u_M)(.,\varsigma) + f_{i,\tau}(\tilde{u}_M)(.,\varsigma) \|_2$$

$$\leq c(\tau - \rho)^{-\delta}(\rho - \varsigma)^\delta \| z_M(.,\varsigma) \|_2 + c \| z_M(.,\rho) - z_M(.,\varsigma) \|_2,$$

(3.35)

respectively. Thus, by an extended version of (1.3) for indefinite generalized Stieltjes integrals (see Proposition 4.1 in [30]),

$$\sum_{i=1}^{+\infty} \lambda_i^\frac{1}{2} \left\| \int_{\sigma}^{\tau} (f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho)) B_i^H(\rho) \right\|_2$$

$$\leq r_H \sup_{i \in \mathbb{N}^+} \left( \left( \int_{\sigma}^{\tau} d\rho \| f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho) \|_2 \right)$$

$$+ \int_{\sigma}^{\tau} d\rho \int_{\sigma}^{\rho} \frac{d\varsigma}{(\rho - \varsigma)^{\alpha+1}}$$

$$\times \| f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho) - f_{i,\tau}(u_M)(.,\varsigma) + f_{i,\tau}(\tilde{u}_M)(.,\varsigma) \|_2 \right)$$

$$\leq R \left( \int_{\sigma}^{\tau} d\rho \frac{\| z_M(.,\rho) \|_2}{(\rho - \sigma)^\alpha} + \int_{\sigma}^{\tau} d\rho (\tau - \rho)^{-\delta} \int_{\sigma}^{\rho} d\varsigma (\rho - \varsigma)^{\delta-\alpha-1} \| z_M(.,\varsigma) \|_2$$

$$+ \int_{\sigma}^{\tau} d\rho \int_{\sigma}^{\rho} d\varsigma \frac{\| z_M(.,\rho) - z_M(.,\varsigma) \|_2}{(\rho - \varsigma)^{\alpha+1}} \right)$$

a.s. for all $\sigma, \tau \in [0,t]$ with $\tau > \sigma$ and every $\delta \in \left( \frac{d}{d+2}, 1 \right)$. But the second term on the right-hand side is equal to

$$c \int_{\sigma}^{\tau} d\rho (\tau - \rho)^{-\alpha} \| z_M(.,\rho) \|_2,$$

as can be easily checked by applying Fubini’s theorem and by evaluating the resulting inner integral in terms of Euler’s Beta function. ■

As for the analysis of the fourth term on the right-hand side of (3.31) we need some preparatory results. In particular we shall use the estimate for time increments of the Green function, valid for any $\delta \in \left( \frac{d}{d+2}, 1 \right)$:

$$|G(x, t; y, \tau) - G(x, s; y, \tau) - G(x, t; y, \sigma) + G(x, s; y, \sigma)|$$

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\[ \left| G(x,t;y,\tau) - G(x,s;y,\tau) \right|^{\delta} + \left| G(x,t;y,\sigma) - G(x,s;y,\sigma) \right|^{\delta} \leq (t-s)^{\delta} (s-\tau)^{-\frac{d+2}{2}+\frac{d}{2}} (\tau-\sigma)^{-\frac{d+2}{2}(1-\delta)} \times \left| \frac{G(x,t;y,\tau)}{\tau* - \tau} \right| \exp \left[ -c \frac{|x-y|^2}{\tau* - \tau} \right] + \left( \sigma* - \sigma \right)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{\sigma* - \sigma} \right] \right) \]

(3.36)

with \( \tau^*, \sigma^* \in (s,t) \), which follows from (3.6)–(3.7).

Lemma 3.8 The hypotheses are the same as in part (b) of Theorem 2.3 and the \( f_{i,\tau,\sigma}^*(u) \)'s are the functions given by (3.31). Then, the estimates

\[
\sup_{i \in \mathbb{N}^+} \left\| f_{i,\tau,\sigma}^*(u_M)(\cdot,\rho) - f_{i,\tau,\sigma}^*(\tilde{u}_M)(\cdot,\rho) \right\|_2 \leq c(\tau-\sigma)^{\frac{d}{2}} (\sigma-\rho)^{-\frac{d}{2}} \| z_M(\cdot,\rho) \|_2
\]

(3.37)

and

\[
\sup_{i \in \mathbb{N}^+} \left\| f_{i,\tau,\sigma}^*(u_M)(\cdot,\rho) - f_{i,\tau,\sigma}^*(\tilde{u}_M)(\cdot,\rho) - f_{i,\tau,\sigma}^*(u_M)(\cdot,\varsigma) + f_{i,\tau,\sigma}^*(\tilde{u}_M)(\cdot,\varsigma) \right\|_2 \leq c(\tau-\sigma)^{\frac{d}{2}} \left( (\sigma-\rho)^{-\frac{d}{2}} (\rho-\varsigma)^{\frac{d}{2}(1-\delta)} \| z_M(\cdot,\varsigma) \|_2 \right. \]

\[ + \left. (\sigma-\rho)^{-\frac{d}{2}} \| z_M(\cdot,\rho) - z_M(\cdot,\varsigma) \|_2 \right) \]

(3.38)

hold a.s. for all \( \tau, \sigma, \rho, \varsigma \in [0,T] \) with \( \tau > \sigma > \rho > \varsigma \) and every \( \delta \in \left( \frac{d}{d+2}, 1 \right) \).

Proof. It follows from the same type of arguments as those outlined in the proof of Lemma 3.3. For the proof of (3.37) the key estimate is (3.5). For (3.38), we also apply (3.5) along with (3.36).

\[ \blacksquare \]

The last relevant \( L^2(D) \)-estimate regarding (3.31) is then the following.
Lemma 3.9 The hypotheses are the same as in part (b) of Theorem 2.3. Then we have

\[
\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^\sigma \left( f_{i,\tau,\sigma}^*(u_M)(\cdot,\rho) - f_{i,\tau,\sigma}^*(\tilde{u}_M)(\cdot,\rho) \right) B_i^H(d\rho) \right\|_2 \\
\leq R(\tau - \sigma)^{\frac{d}{2}} \left( \int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \left( \frac{1}{\rho^\alpha} + \frac{1}{(\sigma - \rho)^\alpha} \right) \left\| z_M(\cdot,\rho) \right\|_2 \\
+ \int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \int_0^\rho d\varsigma \frac{\left\| z_M(\cdot,\varsigma) - z_M(\cdot,\rho) \right\|_2}{(\rho - \varsigma)^{\alpha + 1}} \right)
\]

a.s. for all $\sigma, \tau \in [0, t]$ with $\tau > \sigma$ and every $\delta \in \left( \frac{d}{d+2}, 1 - 2\alpha \right)$.

Proof. By applying (1.3), together with (3.37), (3.38), we get

\[
\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^\sigma \left( f_{i,\tau,\sigma}^*(u_M)(\cdot,\rho) - f_{i,\tau,\sigma}^*(\tilde{u}_M)(\cdot,\rho) \right) B_i^H(d\rho) \right\|_2 \\
\leq R(\tau - \sigma)^{\frac{d}{2}} \left( \int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \left( \frac{1}{\rho^\alpha} + \frac{1}{(\sigma - \rho)^\alpha} \right) \left\| z_M(\cdot,\rho) \right\|_2 \\
+ \int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \int_0^\rho d\varsigma \frac{\left\| z_M(\cdot,\varsigma) - z_M(\cdot,\rho) \right\|_2}{(\rho - \varsigma)^{\alpha + 1}} \right)
\]

a.s. for all $\sigma, \tau \in [0, t]$ with $\tau > \sigma$. But for every $\delta \in \left( \frac{d}{d+2}, 1 - 2\alpha \right)$, we have

\[
\int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \int_0^\rho d\varsigma \frac{\left\| z_M(\cdot,\varsigma) - z_M(\cdot,\rho) \right\|_2}{(\rho - \varsigma)^{\alpha + 1}} = c \int_0^\sigma d\rho(\sigma - \rho)^{-\frac{d}{2}} \left\| z_M(\cdot,\rho) \right\|_2.
\]

This yields the result.

\[\blacksquare\]

Let us go back to the inequality (3.29). Owing to (3.31) and by using the estimates (3.32), (3.33) together with Lemmas 3.7 and 3.9, we have

\[
\| z_M \|_{0,2,t}^2 \leq R \int_0^t d\tau \left( \sup_{\rho \in [0,\tau]} \| z_M(\cdot,\rho) \|_2^2 + \sum_{k=1}^6 [I_k(\tau)]^2 \right) \tag{3.39}
\]

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where

\[
I_1(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{1+\alpha/2}} \left( \int_\sigma^\tau d\rho \|z_M(\cdot, \rho)\|_2^2 \right)^{\frac{1}{2}},
\]

\[
I_2(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{\frac{3}{2}+\alpha+1}} \left( \int_\sigma^\tau d\rho (\sigma - \rho)^{-\delta} \|z_M(\cdot, \rho)\|_2^2 \right)^{\frac{1}{2}},
\]

\[
I_3(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{\alpha+1}} \int_\sigma^\tau d\rho \left( \frac{1}{(\rho - \sigma)^\alpha} + \frac{1}{(\tau - \rho)^\alpha} \right) \|z_M(\cdot, \rho)\|_2,
\]

\[
I_4(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{\alpha+1}} \left( \int_\sigma^\tau d\rho \int_\rho^\tau d\xi \|z_M(\cdot, \rho) - z_M(\cdot, \xi)\|_2 \right),
\]

\[
I_5(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{\alpha+1}} \int_\sigma^\tau \frac{d\rho}{(\sigma - \rho)^{1+\frac{1}{2}} \alpha} \left( \frac{1}{\rho^\alpha} + \frac{1}{(\sigma - \rho)^\alpha} \right) \|z_M(\cdot, \rho)\|_2,
\]

\[
I_6(\tau) = \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^{\alpha+1}} \int_\sigma^\tau \frac{d\rho}{(\sigma - \rho)^{1+\frac{1}{2}} \alpha} \int_\rho^\tau \frac{d\xi}{(\xi - \rho)^{1+\frac{1}{2}} \alpha} \|z_M(\cdot, \rho) - z_M(\cdot, \xi)\|_2.
\]

Set \( T_k(t) = \int_0^t d\tau \ [I_k(\tau)]^2, \ k = 1, \ldots, 6. \)

The function \( \sigma \mapsto (\tau - \sigma)^{-\frac{1}{2} - \alpha} \) is integrable on \((0, \tau)\) for \(\alpha \in (0, \frac{1}{2})\). Thus we have

\[
T_1(t) \leq c \int_0^t d\tau \|z_M(\cdot, \tau)\|_2^2. \tag{3.40}
\]

Since we can choose \( \delta > 2\alpha \), we have that \( \sigma \mapsto (\tau - \sigma)^{-\alpha - 1 + \frac{\delta}{2}} \) is integrable on \((0, \tau)\). Then, applying Schwarz inequality with respect to the measure given by \( d\sigma(\tau - \sigma)^{-\alpha - 1 + \frac{\delta}{2}} \), we obtain

\[
T_2(t) \leq c \int_0^t d\tau \int_0^\tau d\sigma (\tau - \sigma)^{-\alpha - 1 + \frac{\delta}{2}} \left( \int_0^\sigma d\rho (\sigma - \rho)^{-\delta} \|z_M(\cdot, \rho)\|_2^2 \right)
\]

\[
\leq c \int_0^t d\tau \left( \sup_{0 \leq \rho \leq \tau} \|z_M(\cdot, \rho)\|_2^2 \right) \left( \int_0^\tau d\sigma(\tau - \sigma)^{-\alpha - 1 + \frac{\delta}{2}} \sigma^{1-\delta} \right)
\]

\[
\leq c \int_0^t d\tau \left( \sup_{0 \leq \rho \leq \tau} \|z_M(\cdot, \rho)\|_2^2 \right), \tag{3.41}
\]

where in the last inequality we have used that \( \alpha + \frac{\delta}{2} < 1 \) along with the definition of Euler’s Beta function.

By integrating one obtains

\[
\int_\sigma^\tau d\rho \left( \frac{1}{(\rho - \sigma)^\alpha} + \frac{1}{(\tau - \rho)^\alpha} \right) = \frac{2(\tau - \sigma)^{1-\alpha}}{1-\alpha}.
\]
Moreover, the function $\sigma \mapsto (\tau - \sigma)^{-2\alpha}$ is integrable on $(0, \tau)$. Consequently,

$$T_3(t) \leq c \int_0^t d\tau \left( \sup_{\rho \in [0, \tau]} \|z_M(\cdot, \rho)\|^2 \right) \int_0^\tau d\sigma \ (\tau - \sigma)^{-2\alpha} \leq c \int_0^t d\tau \left( \sup_{\rho \in [0, \tau]} \|z_M(\cdot, \rho)\|^2 \right). \tag{3.42}$$

For any $\tau \in (0, t)$, set

$$I_\tau = \int_0^\tau d\sigma \ (\tau - \sigma)^{-\alpha - 1 + \frac{\eta}{2}} \left( \int_0^\sigma d\rho \ (\rho - \sigma)^{\frac{-\eta}{2}} \left( \frac{1}{\rho^\alpha} + \frac{1}{(\sigma - \rho)^\alpha} \right) \right).$$

It is a simple exercise to check that for $\alpha + \frac{\eta}{2} < 1$, $\sup_{\tau \in [0, t]} I_\tau < +\infty$. Since

$$T_5(t) \leq \int_0^t d\tau \ T_4^2 \left( \sup_{\rho \in [0, \tau]} \|z_M(\cdot, \rho)\|^2 \right),$$

we conclude that

$$T_5(t) \leq c \int_0^t d\tau \left( \sup_{\rho \in [0, \tau]} \|z_M(\cdot, \rho)\|^2 \right). \tag{3.43}$$

Fix $\eta \in (0, 1)$ so that $\sigma \mapsto (\tau - \sigma)^{-\eta}$ is integrable on $(0, \tau)$. Applying Schwarz inequality first with respect to the measure $d\sigma(\tau - \sigma)^{-\eta}$, and then with respect to the Lebesgue measure on the interval $(\sigma, \tau)$ yields

$$T_4(t) = \int_0^t d\tau \left( \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^\eta} (\tau - \sigma)^{-\alpha - 1 + \eta} \times \left( \int_\sigma^\tau d\rho \int_\sigma^\rho d\xi \left\| z_M(\cdot, \rho) - z_M(\cdot, \xi) \right\| \right)^2 \right) \leq c \int_0^t d\tau \int_0^\tau \frac{d\sigma}{(\tau - \sigma)^\eta} (\tau - \sigma)^{-2\alpha - 2 + 2\eta} \times \left( \int_\sigma^\tau d\rho \int_\sigma^\rho d\xi \left\| z_M(\cdot, \rho) - z_M(\cdot, \xi) \right\| \right)^2 \leq c \int_0^t d\tau \int_0^\tau d\sigma \ (\tau - \sigma)^{-\eta - 2\alpha - 1} \times \int_\sigma^\tau d\rho \left( \int_\sigma^\rho d\xi \left\| z_M(\cdot, \rho) - z_M(\cdot, \xi) \right\| \right)^2.$$
By choosing $\eta > 2\alpha$, the function $\sigma \mapsto (\tau - \sigma)^{\eta - 2\alpha - 1}$ is integrable on $(0, \tau)$. Thus, from the preceding inequalities we obtain

$$
T_4(t) \leq c \int_0^t d\tau \int_0^\tau d\rho \left( \int_\rho^\tau d\xi \frac{||z_M(\cdot, \rho) - z_M(\cdot, \xi)||_2}{(\rho - \xi)^{\alpha + 1}} \right)^2 \\
\leq c \int_0^t d\tau \ ||z_M||_{2, \alpha, \tau}^2. \quad (3.44)
$$

By Fubini’s theorem and evaluations based upon Euler’s Beta function, we have

$$
T_5(t) = \int_0^t d\tau \left( \int_0^\tau d\rho \left( \int_\rho^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (\sigma - \rho)^{-\frac{\delta}{2}} \right) \right) \\
\times \int_0^\tau d\xi \frac{||z_M(\cdot, \rho) - z_M(\cdot, \xi)||_2}{(\rho - \xi)^{\alpha + 1}} \right)^2 \\
\leq c \int_0^t d\tau \left( \int_0^\tau d\rho \left( \int_\rho^\tau d\xi \frac{||z_M(\cdot, \rho) - z_M(\cdot, \xi)||_2}{(\rho - \xi)^{\alpha + 1}} \right)^2 \right) \\
\leq c \int_0^t d\tau \ ||z_M||_{2, \alpha, \tau}^2. \quad (3.45)
$$

Finally, inequalities (3.39) to (3.45) imply

$$
||z_M||_{2, \alpha, t} \leq R \int_0^t d\tau \ ||z_M||_{2, \alpha, \tau}^2 \quad (3.46)
$$

a.s. By Gronwall’s lemma, this clearly implies the uniqueness of the mild solution. Now the proof of part (b) of Theorem 2.3 is complete. \(\blacksquare\)

**Proof of Statement (c) of Theorem 2.3**

We investigate each of the functions (3.8)–(3.10) separately.

**Proposition 3.10** Assume that Hypotheses $(K_{\beta, \nu})$ and $(I)$ hold. Then, there exists $c \in (0, +\infty)$ such that the estimate

$$
||A(\varphi)(., t) - A(\varphi)(., s)||_2 \leq c |t - s|^{\theta'} \quad (3.47)
$$

holds for all $s, t \in [0, T]$ and every $\theta' \in \left(0, \frac{\beta}{2}\right]$. 

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Proof. Relation (3.8) defines a classical solution to (1.4) when $g = h = 0$, so that the standard regularity theory for linear parabolic equations gives $(x, t) \rightarrow A(\varphi)(x, t) \in C^{\beta, 2}(\overline{D} \times [0, T])$ (see, for instance, [13]), from which (3.47) follows immediately.

Regarding (3.9) we have the following result.

**Proposition 3.11** Assume that the same hypotheses as in Theorem 2.3 (a) hold and let $u_M$ be any mild solution to (1.4). Then, there exists $c \in (0, +\infty)$ such that the estimate

$$
\|B(u_M)(., t) - B(u_M)(., s)\|_2 \leq c |t - s|^\theta^p \left(1 + \sup_{t \in [0, T]} \|u_M(., t)\|_2\right) \quad (3.48)
$$

holds a.s. for all $s, t \in [0, T]$ and every $\theta^p \in (0, \frac{1}{2})$.

**Proof.** Without restricting the generality, we may assume that $t > s$. We have

$$
B(u_M)(., t) - B(u_M)(., s) = \int_s^t d\tau \int_D dy \ G(., t; y, \tau)g(u_M(y, \tau)) + \int_0^s d\tau \int_D dy \ (G(., t; y, \tau) - G(., s; y, \tau)) \ g(u_M(y, \tau)), \quad (3.49)
$$

and remark that in order to keep track of the increment $t - s$ we can estimate the first term on the right-hand side of (3.49) by using the same kind of arguments as we did in the first part of the proof of Lemma 3.2. For every $x \in D$ this gives

$$
\int_s^t d\tau \int_D dy \ |G(x, t; y, \tau)g(u_M(y, \tau))| \\
\leq c(t - s)^{\frac{1}{2}} \left(\int_s^t d\tau \int_D dy \ |G(x, t; y, \tau)| \left(1 + |u_M(y, \tau)|^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},
$$

so that we eventually obtain

$$
\left\|\int_s^t d\tau \int_D dy \ G(., t; y, \tau)g(u_M(y, \tau))\right\|_2 \leq c(t - s)^{\frac{1}{2}} \left(1 + \sup_{t \in [0, T]} \|u_M(., t)\|_2\right) \quad (3.50)
$$

a.s. for all $s, t \in [0, T]$ with $t > s$. In a similar manner, we can keep track of the increment $t - s$ in the second term on the right-hand side of (3.49) by using (3.5). We thus have

$$
\left\|\int_0^s d\tau \int_D dy \ (G(., t; y, \tau) - G(., s; y, \tau)) \ g(u_M(y, \tau))\right\|_2
$$

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\[ \leq c \int_0^s d\tau \int_D dy \int_D dx |G(x,t;y,\tau) - G(x,s;y,\tau)| \left( 1 + |u_M(y,\tau)|^2 \right) \]
\[ \leq c(t-s)^{\delta} \int_0^s d\tau (s-\tau)^{-\delta} \left( 1 + \|u_M(.,\tau)\|_2^2 \right) \]
\[ \leq c(t-s)^{\delta} \left( 1 + \sup_{t \in [0,T]} \|u_M(.,t)\|_2^2 \right) \tag{3.51} \]

for every \( \delta \in \left( \frac{d}{d+2}, 1 \right) \), a.s. for all \( s, t \in [0,T] \) with \( t > s \). This last relation holds a fortiori for each \( \delta \in (0, 1) \), so that (3.50) and (3.51) indeed prove (3.48).

As for the stochastic term (3.10), we have the following.

**Proposition 3.12** Assume the same hypotheses as in Theorem 2.3 (c), and let \( u_M \) be any mild solution to (1.4). Then, there exists \( c \in (0, +\infty) \) such that the estimate

\[ \|C(u_M)(.,t) - C(u_M)(.,s)\|_2 \leq c r_H^H |t-s|^{\theta'''} \left( 1 + \|u_M\|_{\alpha,2,T} \right) \tag{3.52} \]

holds a.s. for all \( s, t \in [0,T] \) and every \( \theta''' \in (0, \frac{1}{2} - \alpha) \).

The proof of Proposition 3.12 is more complicated than that of Proposition 3.11. We begin with a preparatory result whose proof is based on inequalities (3.5)–(3.7).

**Lemma 3.13** With the same hypotheses as in part (a) of Theorem 2.3, the estimates

\[ \sup_{i \in \mathbb{N}^+} \| f_{i,t,s}^*(u_M)(.,\tau) \|_2 \leq c (t-s)^{\frac{\delta}{2}} (s-\tau)^{-\frac{\delta}{2}} \left( 1 + \sup_{t \in [0,T]} \|u_M(.,t)\|_2 \right) \tag{3.53} \]

and

\[ \sup_{i \in \mathbb{N}^+} \| f_{i,t,s}^*(u_M)(.,\tau) - f_{i,t,s}^*(u_M)(.,\sigma) \|_2 \]
\[ \leq c (t-s)^{\frac{\delta}{2}} (s-\tau)^{-\frac{\delta}{2}} \|u_M(.,\tau) - u_M(.,\sigma)\|_2 \]
\[ + c (t-s)^{\frac{\delta}{2}} (s-\tau)^{-\frac{\delta}{2}} (\tau - \sigma)^{\frac{1}{2}(1-\delta)} \left( 1 + \sup_{t \in [0,T]} \|u_M(.,t)\|_2 \right) \tag{3.54} \]

hold a.s. for every \( \delta \in \left( \frac{d}{d+2}, 1 \right) \) and for all \( \sigma, \tau \in [0,s] \) with \( \tau > \sigma \) in (3.54).
Proof. The proof of (3.53) is analogous to that of (3.14) and is thereby omitted. As for (3.54), by using Schwarz inequality relative to the measures $dy |G(x, t; y, \tau) - G(x, s; y, \tau)|$ and

$$dy |G(x, t; y, \tau) - G(x, s; y, \tau) - G(x, t; y, \sigma) + G(x, s; y, \sigma)|$$
on D along with Hypothesis (L) for $h$, we get

$$\|f^*_i,t,s(u_M)(., \tau) - f^*_i,t,s(u_M)(., \sigma)\|^2_2 \leq c \int_D dx \int_D dy |G(x, t; y, \tau) - G(x, s; y, \tau)| |u_M(y, \tau) - u_M(y, \sigma)|^2$$

$$+ \int_D dx \int_D dy |G(x, t; y, \tau) - G(x, s; y, \tau) - G(x, t; y, \sigma) + G(x, s; y, \sigma)|$$

$$\times \left(1 + |u_M(y, \sigma)|^2\right)$$

$$\leq c (t - s)^{\delta} (s - \tau)^{-\delta} \|u_M(., \tau) - u_M(., \sigma)\|^2_2$$

$$\leq c (t - s)^{\delta} (s - \tau)^{-1} (\tau - \sigma)^{1-\delta} \left(1 + \sup_{t \in [0,T]} \|u_M(., t)\|^2_2\right), \tag{3.55}$$

a.s. for all $s, t, \sigma, \tau \in [0, T]$ with $t \geq s > \tau > \sigma$ and every $\delta \in \left(\frac{d}{d+2}, 1\right)$, as a consequence of (3.5), (3.36) and the Gaussian property. \hfill \blacksquare

Proof of Proposition 3.12. For $t > s$ we write

$$C(u_M)(., t) - C(u_M)(., s) = \sum_{i=1}^{+\infty} \lambda^\frac{1}{2} \int_s^t f_{i,t}(u_M)(., \tau)B^H_i(d\tau)$$

$$+ \sum_{i=1}^{+\infty} \lambda^\frac{1}{2} \int_0^s f^*_i,t,s(u_M)(., \tau)B^H_i(d\tau). \tag{3.56}$$

In order to estimate the first term on the right-hand side of (3.56), we can start by using inequalities (3.13) and (3.14) to obtain

$$\sum_{i=1}^{+\infty} \lambda^\frac{1}{2} \left\|\int_s^t f_{i,t}(u_M)(., \tau)B^H_i(d\tau)\|_2$$

$$\leq c t^H \rho^H \left(\int_s^t d\tau \frac{d\tau}{(\tau - s)^\alpha} + \int_s^t d\tau \frac{\|u_M(., \tau)\|^2_2}{(\tau - s)^\alpha} + \int_s^t \int_0^\tau d\sigma \frac{\|u_M(., \tau) - u_M(., \sigma)\|^2_2}{(\tau - \sigma)^{\alpha+1}} \right)$$

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\[
\int_t^s d\tau (t - \tau)^{-\frac{\delta}{\alpha}} \int_s^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (1 + \|u_M(., \sigma)\|_2)
\]

(3.57)

a.s. for every \(s, t \in [0, T]\) with \(t > s\) and each \(\delta \in \left(\frac{d}{d+2}, 1\right)\).

Furthermore, we have

\[
\int_t^s d\tau (t - \tau)^{\frac{\delta}{2}} \int_s^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (1 + \|u_M(., \sigma)\|_2)
\]

\[
\leq c \left( (t - s)^{1 - \alpha} \left( 1 + \|u_M\|_{\alpha, 2, T} \right) + (t - s)^{\frac{\delta}{2}} \|u_M\|_{\alpha, 2, T} \right)
\]

\leq c(t - s)^{\frac{\delta}{2}} \left( 1 + \|u_M\|_{\alpha, 2, T} \right)
\]

(3.58)

since \(\alpha < \frac{1}{2}\). Moreover,

\[
\int_t^s d\tau (t - \tau)^{-\frac{\delta}{\alpha}} \int_s^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (1 + \|u_M(., \sigma)\|_2)
\]

\[
\leq c \left( 1 + \sup_{t \in [0, T]} \|u_M(., t)\|_2 \right) \int_t^s d\tau (t - \tau)^{-\frac{\delta}{\alpha}} \int_s^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1}
\]

\[
\leq c(t - s)^{1 - \alpha} \left( 1 + \sup_{t \in [0, T]} \|u_M(., t)\|_2 \right),
\]

(3.59)

by virtue of the convergence of the integral, which can be expressed in terms of Euler’s Beta function since \(\alpha < \frac{3}{4}\). The substitution of (3.58) and (3.59) into (3.57) then leads to the inequality

\[
\sum_{i=1}^{+\infty} \frac{\delta}{\alpha} \left\| \int_s^t f_{i,t}(u_M)(., \tau) B_i^H (d\tau) \right\|_2 \leq c H^H \left( t - s \right)^{\frac{\delta}{2}} \left( 1 + \|u_M\|_{\alpha, 2, T} \right)
\]

(3.60)

a.s. for every \(s, t \in [0, T]\) with \(t > s\).

It remains to estimate the second term on the right-hand side of (3.56). From (1.3) with \(T\) replaced by \(s\), we have

\[
\sum_{i=1}^{+\infty} \frac{\delta}{\alpha} \left\| \int_0^s f_{i,t,s}(u_M)(., \tau) B_i^H (d\tau) \right\|_2 \leq r_H^s
\]

\[
\times \sup_{i \in \mathbb{N}^+} \int_0^s d\tau \left( \frac{\left\| f_{i,t,s}(u_M)(., \tau) \right\|_2}{\tau^\alpha} + \int_0^\tau d\sigma \frac{\left\| f_{i,t,s}(u_M)(., \tau) - f_{i,t,s}(u_M)(., \sigma) \right\|_2}{(\tau - \sigma)^{\alpha + 1}} \right).
\]

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By substituting (3.53) and (3.54) we obtain

\[\sum_{i=1}^{+\infty} \lambda_i^2 \left\| \int_0^s f_{i,t,s}(u_M)(\cdot, \tau) B_i^H(d\tau) \right\|_2 \]

\[\leq c r_H^\alpha (t-s)^{\frac{d}{2}} \left( \int_0^s d\tau (s-\tau)^{-\frac{d}{2}} \int_0^\tau d\sigma (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} \left( 1 + \sup_{t \in [0,T]} \|u_M(\cdot, t)\|_2 \right) \right) \]

\[+ \int_0^s d\tau (s-\tau)^{-\frac{d}{2}} \int_0^\tau d\sigma (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} \left( 1 + \sup_{t \in [0,T]} \|u_M(\cdot, t)\|_2 \right) \]

\[\leq c r_H^\alpha (t-s)^{\frac{d}{2}} \left( 1 + \|u_M\|_{\alpha,2,T} \right) \]

\[\times \left( 1 + \int_0^s d\tau (s-\tau)^{-\frac{d}{2}} \int_0^\tau d\sigma (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} \left( 1 + \sup_{t \in [0,T]} \|u_M(\cdot, t)\|_2 \right) \right) \]

(3.61)

where we have got the last estimate using Schwarz inequality with respect to the measure $d\tau$ on $(0, s)$ along with (2.1) in the first two integrals on the right-hand side.

By imposing the additional restriction $\delta < 1 - 2\alpha$, we have

\[\int_0^s d\tau (s-\tau)^{-\frac{d}{2}} \int_0^\tau d\sigma (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} < +\infty.\]

Thus, we have proved that

\[\sum_{i=1}^{+\infty} \lambda_i^2 \left\| \int_0^s f_{i,t,s}(u_M)(\cdot, \tau) B_i^H(d\tau) \right\|_2 \leq c r_H^\alpha (t-s)^{\frac{d}{2}} \left( 1 + \|u_M\|_{\alpha,2,T} \right) \]

(3.62)

a.s. for all $s, t \in [0, T]$ with $t > s$ and every $\delta \in \left( \frac{d}{d+2}, 1 - 2\alpha \right)$. The existence of this restricted interval of values of $\delta$ is possible by our choice of $\alpha$. Relations (3.56), (3.60) and (3.62) clearly yield (3.52) with $\theta'' = \frac{d}{2} \in (0, \frac{1}{2} - \alpha)$.

It is immediate that Propositions 3.10 to 3.12 imply statement (c) of Theorem 2.3. Notice that $R_H^\alpha = c(1 + r_H^\alpha)$, with $r_H^\alpha$ defined in (1.2).

Finally, we give an alternate to the result proved before, as mentioned in Section 2, Remark 5.
Proposition 3.14 The assumptions are as in Theorem 2.3 part (a). Then,

\[ \|C(u_M)(., t) - C(u_M)(., s)\|_2 \leq R|t - s|^\theta^m (1 + \|u_M\|_{\alpha, 2, T}) \] (3.63)

holds a.s. for all \( s, t \in [0, T] \) and every \( \theta^m \in \left(0, \frac{2}{d+2} \wedge \frac{1}{2}\right) \). Consequently,

\[ \|u_M(., t) - u_M(., s)\|_2 \leq R|t - s|^\theta (1 + \|u_M\|_{\alpha, 2, T}), \] (3.64)

a.s. for all \( s, t \in [0, T] \) and each \( \theta \in \left(0, \frac{2}{d+2} \wedge \beta_2\right) \).

**Proof.** We use the factorization method we alluded to in Section 2. For this we express \( C(u_M)(., t) \) in terms of the auxiliary \( L^2(D)\)-valued process

\[ Y_\varepsilon(u_M)(., t) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (t - \tau)^{-\varepsilon} f_{i,t}(u_M)(., \tau)B_i^H(d\tau) \]

defined for every \( \varepsilon \in (0, \frac{1}{2}) \). In fact, by repeated applications of Fubini’s theorem and by using the fundamental property \( U(t, \tau)U(\tau, \sigma) = U(t, \sigma) \) for the evolution operators defined in (3.18) we obtain

\[ C(u_M)(., t) = \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t f_{i,t}(u_M)(., \tau)B_i^H(d\tau) \]

\[ = \frac{\sin(\varepsilon \pi)}{\pi} \int_0^t d\tau (t - \tau)^{-\varepsilon - 1} \int_D dyG(., t; y, \tau)Y_\varepsilon(u_M)(y, \tau) \] (3.65)

for every \( t \in [0, T] \) a.s.

Next we prove that a.s.,

\[ \sup_{t \in [0, T]} \|Y_\varepsilon(u_M)(., t)\|_2 \leq R (1 + \|u_M\|_{\alpha, 2, T}). \] (3.66)

Indeed, according with (1.3),

\[ \|Y_\varepsilon(u_M)(., t)\|_2 \leq r_\alpha^H \sup_{i \in N^+} \int_0^t d\tau (t - \tau)^{-\varepsilon} \]

\[ \times \left\{ \frac{\|f_{i,t}(u)(., \tau)\|_2}{\tau^\alpha} + \int_0^\tau d\sigma \frac{\|f_{i,t}(u)(., \tau) - f_{i,t}(u)(., \sigma)\|_2}{(\tau - \sigma)^{\alpha+1}} \right\}. \]

From the estimate (3.13) and the definition of the Beta function, we have

\[ \int_0^t d\tau (t - \tau)^{-\varepsilon} \frac{\|f_{i,t}(u)(., \tau)\|_2}{\tau^\alpha} \leq c \left(1 + \sup_{\tau \in [0, t]} \|u(., \tau)\|_2\right). \]
Since $\varepsilon \in (0, \frac{1}{2})$, applying Schwarz’s inequality yields
\[
\int_0^t d\tau (t - \tau)^{-\varepsilon} \int_0^\tau \frac{\|u(., \tau) - u(., \sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \leq \left[ \int_0^t d\tau (t - \tau)^{-2\varepsilon} \left( \int_0^\tau d\sigma \frac{\|u(., \tau) - u(., \sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \right)^2 \right]^{\frac{1}{2}} \leq c\|u\|_{\alpha, 2, t}.
\]
Moreover, for any $\delta \in (0, \frac{1}{2})$ such that $\frac{\delta}{2} - \alpha > 0$, we have
\[
\int_0^t d\tau (t - \tau)^{-\varepsilon - \frac{\delta}{2}} \int_0^\tau d\sigma (\tau - \sigma)^{\frac{\delta}{2} - \alpha - 1} (1 + \|u(., \sigma)\|_2) \leq c \left( 1 + \sup_{\sigma \in [0, t]} \|u(., \sigma)\|_2 \right).
\]
By virtue of (3.14) the two above estimates imply
\[
\int_0^t d\tau \int_0^\tau d\sigma (t - \tau)^{-\varepsilon} \frac{\|f_{i, t}(u)(., \tau) - f_{i, t}(u)(., \sigma)\|_2}{(\tau - \sigma)^{\alpha + 1}} \leq c \left( 1 + \|u\|_{\alpha, 2, t} \right).
\]
This ends the proof of (3.66).

We can now proceed by estimating the time increments of $C(u_M)$ using (3.65) and (3.66) rather than with the expressions of Proposition 3.12. For this we follow the arguments of the proof of (66) in Proposition 6 of [36] to see that, by choosing $\theta''' \in \left(0, \frac{2}{\alpha + 2} \wedge \frac{1}{2}\right)$ with the additional restriction $\varepsilon \in \left(\theta''', \frac{2}{\alpha + 2} \wedge \frac{1}{2}\right)$, we obtain
\[
\|C(u_M)(., t) - C(u_M)(., s)\|_2 \leq c \left( \left\| \int_s^t d\tau (t - \tau)^{\varepsilon - 1} \int_D dy G(., t; y, \tau) Y_\varepsilon(u_M)(y, \tau) \right\|_2 + \left\| \int_0^t d\tau \int_D dy \left( (t - \tau)^{\varepsilon - 1} G(., t; y, \tau) - (s - \tau)^{\varepsilon - 1} G(., s; y, \tau) \right) Y_\varepsilon(u_M)(y, \tau) \right\|_2 \right) \leq R \left( |t - s|^{\theta'''} + |t - s|^{\theta''''} \right) (1 + \|u_M\|_{\alpha, 2, T}) \leq R |t - s|^{\theta''''} (1 + \|u_M\|_{\alpha, 2, T}),
\]
proving (3.63).

Finally, this estimate along with those established in Propositions 3.10 and 3.11 provide (3.64) and finish the proof of the proposition.

\[\blacksquare\]
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