Non-BPS Solutions of the Noncommutative $CP^1$
Model in 2+1 Dimensions

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Abstract

We find non-BPS solutions of the noncommutative $CP^1$ model in 2+1 dimensions. These solutions correspond to soliton anti-soliton configurations. We show that the one-soliton one-anti-soliton solution is unstable when the distance between the soliton and the anti-soliton is small. We also construct time-dependent solutions and other types of solutions.
1 Introduction

Noncommutative field theories naturally arise as low-energy descriptions of string theory (for a review see [1]). Non-perturbative dynamics of string theory was investigated through the study of solitons and instantons in noncommutative gauge theories (see e.g. [2]). In four-dimensional Yang-Mills theory, there exist instantons even in the $U(1)$ case [3]. Non-trivial solutions are also known in noncommutative scalar theories [4]. The scattering of noncommutative solitons was studied in Yang-Mills theory [5] and in scalar theories [6, 7].

In lower dimensions, nonlinear sigma models exhibit many similarities with four-dimensional Yang-Mills theory. In 2+1 dimensions, nonlinear sigma models possess soliton solutions. The BPS solitons of the $CP^N$ model were extended to a noncommutative space [8]. The low-energy dynamics of the BPS solitons in the noncommutative $CP^1$ model was investigated [9]. In noncommutative integrable sigma models, time-dependent solutions were written down explicitly [10] and the scattering of solitons and anti-solitons was studied [11].

In this paper, we consider the noncommutative $CP^1$ model in 2+1 dimensions. In the commutative $CP^N$ model, general static solutions are known [12]. For $N \geq 2$, there exist non-BPS solutions in addition to the BPS solutions. On the other hand, in the commutative $CP^1$ model no non-BPS solutions exist. We construct non-BPS solutions of the noncommutative $CP^1$ model and study their dynamics. These solutions represent the co-existence of solitons and anti-solitons.

This paper is organized as follows. In the following section, we summarize the properties of the noncommutative $CP^N$ model. In section 3, we construct non-BPS solutions of the noncommutative $CP^1$ model and investigate the stability of the solutions. We further construct time-dependent solutions by boosting static solutions. In section 4, other types of solutions are presented. Finally, in section 5 we discuss future problems.

2 The Noncommutative $CP^N$ Model in 2+1 Dimensions

We consider the $CP^N$ model on a (2+1)-dimensional noncommutative spacetime [8] whose spatial coordinates obey the commutation relation

$$[\hat{z}, \hat{\bar{z}}] = \theta > 0,$$

(2.1)

where $\hat{z} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$, $\hat{\bar{z}} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$. Since (2.1) is the commutation relation of creation and annihilation operators, we can identify the field with an operator in the Fock space
of (2.1). The Lagrangian of the model is given by

$$L = 2\pi \theta \text{Tr}(|D_t \Phi|^2 - |D_z \Phi|^2 - |D_{\bar{z}} \Phi|^2),$$

(2.2)

where $\Phi$ is a $(N+1)$-component complex vector with the constraint $\Phi^\dagger \Phi = 1$, and $\text{Tr}$ denotes the trace over the Fock space. The covariant derivative is defined by

$$D_a \Phi = \partial_a \Phi - i \Phi A_a, \quad A_a = -i \Phi^\dagger \partial_a \Phi, \quad a = t, z, \bar{z},$$

(2.3)

where

$$\partial_z = -\theta^{-1}[\hat{z}, ], \quad \partial_{\bar{z}} = \theta^{-1}[\hat{\bar{z}}, ].$$

(2.4)

The model has a global $SU(N)$ symmetry and a local $U(1)$ symmetry, $\Phi \rightarrow \Phi g, \ g \in U(1)$.

The energy of a static configuration is given by

$$E = 2\pi \theta \text{Tr}(|D_z \Phi|^2 + |D_{\bar{z}} \Phi|^2) \geq 2\pi |Q|,$$

(2.5)

where

$$Q = \theta \text{Tr}(|D_z \Phi|^2 - |D_{\bar{z}} \Phi|^2)$$

(2.6)

is the topological charge. The configuration which saturates the energy bound satisfies the BPS soliton equation

$$D_{\bar{z}} \Phi = 0,$$

(2.7)

or the BPS anti-soliton equation

$$D_z \Phi = 0.$$

(2.8)

The BPS (anti-)soliton solution has the positive (negative) topological charge.

In order to find solutions of (2.7) and (2.8), it is convenient to introduce a $(N+1)$-component complex vector $W$ and the projection operator $P$ as

$$\Phi = W \frac{1}{\sqrt{W^\dagger W}},$$

(2.9)

$$P = W \frac{1}{W^\dagger W^\dagger}.$$  

(2.10)

$P$ satisfies the relations

$$PW = W, \quad P^\dagger = P, \quad P^2 = P.$$  

(2.11)

In terms of $W$ and $P$ (2.7) can be written as

$$(1 - P)\partial_z W = 0.$$  

(2.12)
The general solution of the BPS soliton equation \((2.12)\) is given by
\[
W = W_0(\hat{z})\Delta(\hat{z}, \hat{\bar{z}}),
\] (2.13)
where the components of \(W_0(\hat{z})\) is polynomials of \(\hat{z}\) and \(\Delta(\hat{z}, \hat{\bar{z}})\) is an arbitrary invertible scalar function. The highest degree of the components of \(W_0(\hat{z})\) is equal to the topological charge. The anti-soliton solution has the same form as \((2.13)\) with the components of \(W_0\) being polynomials of \(\hat{\bar{z}}\).

In the \(CP^N\) model some gauge transformations become singular after the noncommutative extension. In the commutative \(CP^1\) model \(W_1 = \left(\begin{array}{c} \mu z^{-1} \\ 1 \end{array}\right)\), where \(\mu\) is a parameter, is gauge-equivalent to the BPS soliton solution \(W_2 = \left(\begin{array}{c} \mu \\ z \end{array}\right)\), but this is not true in the noncommutative case. In the noncommutative case \(\hat{z}\) is not invertible because \(\hat{z}\) has a zero eigen value. However we can define the right inverse element \(\hat{z}^{-1} = \hat{\bar{z}}^{-1} = 1 - |0\rangle\langle 0|\). Since \(\tilde{W} = \left(\begin{array}{c} \mu z^{-1} \\ 1 \end{array}\right)\) is not of the form \((2.13)\), \(\tilde{W}\) is not a BPS solution. Moreover, \(\tilde{W}\) is not a solution of the equation of motion. In section 4 we use \(\tilde{W}\) to construct a solution.

In terms of \(P\) the Lagrangian is written as
\[
L = \pi \theta \text{Tr}(\partial_t P \partial_t P - 2\partial_{\hat{z}} P \partial_\hat{z} P),
\] (2.14)
where \(\text{Tr}\) consists of the trace over the Fock space and that over the \((N + 1) \times (N + 1)\) matrix indices. The equation of motion is
\[
[\partial_t^2 P - 2\partial_{\hat{z}} \partial_\hat{z} P, P] = 0.
\] (2.15)
For a static configuration this equation is written as
\[
[[\hat{z}, [\hat{z}, P]], P] = 0,
\] (2.16)
where we have used \((2.4)\). The BPS equations \((2.7)\) and \((2.8)\) are written as
\[
(1 - P)\hat{z}P = 0,
\] (2.17)
\[
(1 - P)\hat{\bar{z}}P = 0.
\] (2.18)
Rewriting \((2.16)\) as
\[
[[\hat{z}, (1 - P)\hat{z}P] + [\hat{\bar{z}}, P\hat{\bar{z}}(1 - P)] = 0,
\] (2.19)
we see that solutions of \((2.17)\) satisfy \((2.16)\). Similarly solutions of \((2.18)\) satisfy \((2.16)\).
3 Non-BPS Solutions of the Noncommutative $CP^1$ Model

In this section we consider non-BPS solutions of the noncommutative $CP^1$ model, i.e. solutions of (2.16) which satisfy neither (2.17) nor (2.18). These solutions correspond to soliton anti-soliton configurations. We investigate the stability of the one-soliton one-anti-soliton solution. We also construct the boosted solution.

3.1 Soliton Anti-Soliton Solution

We consider the $2 \times 2$ projection operator of the diagonal form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{3.1}$$

If $P$ satisfies the equation of motion (2.16), $P_1$ and $P_2$ also do. We take $P_1$ and $P_2$ to be solutions of (2.17) and (2.18) respectively and hence we have

$$\begin{align*}
(1 - P_1) \hat{z} P_1 &= 0, \tag{3.2} \\
(1 - P_2) \hat{z} P_2 &= 0. \tag{3.3}
\end{align*}$$

In this case $P$ satisfies the equation of motion but not the BPS equations. In the commutative case there exist only the trivial solutions of (3.2) and (3.3), $P_{1,2} = 0, 1$, but in the noncommutative case non-trivial solutions are known. The solution of (3.2) is given by [6, 10, 13]

$$P_1 = \sum_{i,j=1}^{r} |z^i \rangle h_{ij}^{-1} \langle z^j |, \tag{3.4}$$

where

$$\begin{align*}
|z^i \rangle &= e^{\theta^{-1}(z^i \bar{z} - z^{\dagger} \bar{z})} |0\rangle, \tag{3.5} \\
h^{ij} &= \langle z^i | z^j \rangle, \quad h^{-1}_{ij} h^{jk} = \delta^k_i. \tag{3.6}
\end{align*}$$

Since $1 - P_2$ satisfies the BPS soliton equation (2.17), $P_2$ can be written as

$$P_2 = 1 - \sum_{k,l=1}^{s} |\tilde{z}^k \rangle \tilde{h}_{kl}^{-1} \langle \tilde{z}^l |. \tag{3.7}$$

Therefore $P$ takes the following form

$$P = \begin{pmatrix} \sum_{i,j=1}^{r} |z^i \rangle h_{ij}^{-1} \langle z^j | & 0 \\ 0 & 1 - \sum_{k,l=1}^{s} |\tilde{z}^k \rangle \tilde{h}_{kl}^{-1} \langle \tilde{z}^l | \end{pmatrix}. \tag{3.8}$$
To see that $z^i$ ($i = 1, \ldots, r$) and $\tilde{z}^k$ ($k = 1, \ldots, s$) are interpreted as positions of solitons and anti-solitons respectively, we consider the large $|z^i|$ and $|\tilde{z}^k|$ limits. Taking the $|z^i| \to \infty$ limit and considering the finite region on the $z$-plane, (3.8) reduces to

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 - \sum_{k,l=1}^s |\tilde{z}^k| \tilde{h}_{kl}^{-1} \langle \tilde{z}^l | \end{pmatrix}.$$  (3.9)

This is the BPS anti-soliton solution. On the other hand, in the limit of $|\tilde{z}^k| \to \infty$, (3.8) reduces to

$$P = \begin{pmatrix} \sum_{i,j=1}^r |z^i| h_{ij}^{-1} \langle z^j | \\ 0 & 1 \end{pmatrix}. $$ (3.10)

This is the BPS soliton solution. Thus we can interpret the non-BPS solution (3.8) as the configuration which contains $r$ solitons at $z = z^1, \ldots, z^r$ and $s$ anti-solitons at $z = \tilde{z}^1, \ldots, \tilde{z}^s$. The topological charge and the energy of the solution (3.8) are the sums of the contributions of $P_1$ and $P_2$. Since the contributions of $P_1$ and $P_2$ to the topological charge are $r$ and $-s$ respectively, we obtain $Q = r - s$ and $E = 2\pi(r + s)$.

### 3.2 Stability

We analyze the stability of the solution which contains one soliton at $z$ and one anti-soliton at $z = 0$

$$P = \begin{pmatrix} |z\rangle \langle z| & 0 \\ 0 & 1 - |0\rangle \langle 0| \end{pmatrix}. $$ (3.11)

We connect this solution to the vacuum solution

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} $$ (3.12)

by a path

$$P_\phi = \begin{pmatrix} \sin^2 \phi |z\rangle \langle z| & \sin \phi \cos \phi |z\rangle \langle 0| \\ \sin \phi \cos \phi |0\rangle \langle z| & 1 - \sin^2 \phi |0\rangle \langle 0| \end{pmatrix}, \quad \phi \in \left[0, \frac{\pi}{2}\right], $$ (3.13)

which gives $P_0$ at $\phi = 0$ and $P$ at $\phi = \frac{\pi}{2}$. The energy of the configuration (3.13) is

$$E = 2\pi \theta \text{Tr}(\partial_\phi P_\phi \partial_\phi P_\phi) = 4\pi \sin^2 \phi \left(1 + \frac{\tilde{z}z}{\theta} \cos^2 \phi \right). $$ (3.14)

Differentiating this with respect to $\phi$ twice we obtain

$$\frac{\partial^2 E}{\partial \phi^2} \bigg|_{\phi = \frac{\pi}{2}} = 8\pi \left(\frac{\tilde{z}z}{\theta} - 1\right). $$ (3.15)
The energy (3.14) has a minimum at $\phi = 0$. When $\bar{z}z < \theta$ the energy (3.14) has a local maximum at $\phi = \frac{\pi}{2}$ and monotonically decreases to zero at $\phi = 0$. In this case the solution (3.11) is unstable and the soliton anti-soliton pair annihilates. When $\bar{z}z > \theta$ the energy (3.14) has a local minimum at $\phi = \frac{\pi}{2}$ and therefore the solution (3.11) is metastable in this parameter space. We do not know whether the solution is unstable under fluctuations in other directions.

### 3.3 Time-Dependent Solution

Time-dependent solutions can be obtained by boosting static solutions $P(\hat{z}, \hat{\bar{z}})$. The solution moving in the $x(=\sqrt{2}\text{Re}z)$-direction with the velocity $v$ is given by

$$P_v(\hat{z}, \hat{\bar{z}}, t) = P(\hat{z}', \hat{\bar{z}}'), \tag{3.16}$$

which satisfies the equation of motion

$$[\partial_t^2 P_v - 2\partial_{\bar{z}}\partial_z P_v, P_v] = 0, \tag{3.17}$$

where $\hat{z}' = \frac{1}{\sqrt{2}}(\hat{x}' + i\hat{y}')$, $\hat{\bar{z}}' = \frac{1}{\sqrt{2}}(\hat{x}' - i\hat{y}')$ are given by the Lorentz transformation

$$\hat{t}' = \frac{t - v\hat{x}}{\sqrt{1-v^2}}, \quad \hat{x}' = \frac{\hat{x} - vt}{\sqrt{1-v^2}}, \quad \hat{y}' = \hat{y}. \tag{3.18}$$

The spatial coordinates $\hat{z}'$, $\hat{\bar{z}}'$ obey

$$[\hat{z}', \hat{\bar{z}}'] = \theta', \quad \theta' = \frac{\theta}{\sqrt{1-v^2}}. \tag{3.19}$$

The Lorentz symmetry is explicitly broken by the noncommutativity. The moving solution is obtained by the boost accompanied by rescaling of the noncommutative parameter $[14]$. For the solution of the diagonal form (3.1), $P_1$ and $P_2$ can be boosted with the different velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. We introduce the coordinates $\hat{z}_1(\hat{z}_2)$, $\hat{x}_1(\hat{x}_2)$ given by the Lorentz transformation with the velocity $\vec{v}_1(\vec{v}_2)$. These coordinates obey the same commutation relation as (3.19)

$$[\hat{z}_a, \hat{\bar{z}}_a] = \theta_a, \quad \theta_a = \frac{\theta}{\sqrt{1-v_a^2}}, \quad a = 1, 2. \tag{3.20}$$

Boosting the solution (3.8) we obtain the time-dependent solution

$$P_1\tilde{v}_2 = \begin{pmatrix} \sum_{i,j=1}^r |z_i^j\rangle h_{1,ij}^{-1} \langle z_i^j| & 0 \\ 0 & 1 - \sum_{k,l=1}^s |z_2^k\rangle h_{2,kl}^{-1} \langle z_2^k| \end{pmatrix}, \tag{3.21}$$
where
\[ |z_i^a⟩ = e^{θ_a^{-1}(z_i^a \hat{z}_a - \hat{z}_a z_i^a)} |0_a⟩, \]
\[ \hat{z}_a |0_a⟩ = 0, \quad ⟨0_a |0_a⟩ = 1, \]
\[ h_{ij}^a = ⟨z_i^a |z_j^a⟩, \quad a = 1, 2. \quad (3.22) \]

When \( r = 1, s = 1 \) the solution (3.21) represents a soliton and an anti-soliton moving independently with arbitrary velocities. In the case of \( r > 1 (s > 1) \) the energy density of the solution (3.21) may have multiple peaks, but this solution is not a time-dependent multi-(anti-)soliton configuration where (anti-)soliton peaks display relative motion because all (anti-)soliton peaks move with a common velocity.

4 Other Solutions

4.1 Other Non-BPS Solutions

We can construct other non-BPS solutions of the form (3.1) by taking the diagonal elements \( P_1 \) and \( P_2 \) to be solutions of the equation of motion which do not satisfy the BPS equations. The operator \( \sum_{i=1}^k |n_i⟩⟨n_i| (0 ≤ n_1 < \ldots < n_k) \) satisfies the equation of motion (2.16) but not the BPS equations (2.17) nor (2.18) except for the case of \( n_i = i - 1, i = 1, \ldots, k \) where this operator is the solution of the BPS equation (2.17).

For example, choosing \( P_1 = |n⟩⟨n| \) (\( n > 0 \)) we have the non-BPS solution
\[ P = \begin{pmatrix} |n⟩⟨n| & 0 \\ 0 & 1 \end{pmatrix}, \quad n > 0. \quad (4.1) \]

This configuration has the topological charge \( Q = 1 \) and the energy \( E = 2π(2n + 1) \).

We can also consider the non-BPS solution
\[ P = \begin{pmatrix} |n⟩⟨n| & 0 \\ 0 & |m⟩⟨m| \end{pmatrix}, \quad n > 0, \quad m ≥ 0. \quad (4.2) \]

For this solution we cannot find \( W \) from (2.10). This is the solution of (2.10) but not a solution of the equation of motion derived from the Lagrangian (2.2).

4.2 Finite Size Solution

We have constructed the solutions which have the zero size in the commutative limit. In this subsection we consider the finite size solution which has a parameter \( μ \) related to
the size of the solution. As mentioned in section 2, $\tilde{W} = \left( \begin{array}{c} \mu \hat{z} - 1 \\ 1 \end{array} \right)$ is not a solution. The projection operator

$$\tilde{P} = \tilde{W} \frac{1}{W^\dagger W} \tilde{W}^\dagger = \left( \begin{array}{cc} \hat{z} \frac{\mu^2}{\mu^2 + \hat{z}^2 + \theta} & \hat{z} \frac{1}{\mu^2 + \hat{z}^2 + \theta} \\ \frac{\mu}{\mu^2 + \hat{z}^2 + \theta} & \frac{1}{\mu^2 + \hat{z}^2 + \theta} \end{array} \right).$$

(4.3)

does not satisfy (2.16), (2.17) nor (2.18). We show that we can construct the solution by adding the correction to $\tilde{P}$. We consider the projection operator

$$P = \tilde{P} + \frac{1}{\mu^2 + \theta} \left( \begin{array}{cc} \theta |1\rangle \langle 1| & -\mu \sqrt{\theta} |1\rangle \langle 0| \\ -\mu \sqrt{\theta} |0\rangle \langle 1| & \mu^2 |0\rangle \langle 0| \end{array} \right).$$

(4.4)

After a little algebra we obtain

$$(1 - P) \hat{z} P = \left( \begin{array}{cc} \sqrt{\theta} |0\rangle \langle 1| & 0 \\ 0 & 0 \end{array} \right).$$

(4.5)

This implies that $P$ is not a BPS soliton solution but satisfies the equation of motion. Calculating $W$ which generates $P$, we obtain

$$W = \left( \begin{array}{c} \mu \hat{z} - 1 \\ 1 \end{array} \right) \hat{z} + c \left( \begin{array}{c} 1 \\ -\mu \hat{z} - 1 \end{array} \right) |1\rangle \langle 0| = \left( \begin{array}{c} \mu (1 - |0\rangle \langle 0|) + c |1\rangle \langle 0| \\ \hat{z} - c \frac{\mu}{\sqrt{\theta}} |0\rangle \langle 0| \end{array} \right),$$

(4.6)

where $c$ is a constant. The topological charge is

$$Q = \theta \text{Tr} \left[ \frac{1}{W^\dagger W} ((\partial_z W)^\dagger (1 - P) \partial_z W - (\partial_z W)^\dagger (1 - P) \partial_z W) \right]$$

$$= \theta \text{Tr} \left[ \frac{\mu^2}{\mu^2 + \hat{z}^2} \left( \frac{1}{\theta} |0\rangle \langle 0| + \frac{1}{\mu^2 + \hat{z}^2} - \left( \frac{1}{\theta} |1\rangle \langle 1| + \frac{1}{\mu^2 + \theta} |0\rangle \langle 0| \right) \right) \right]$$

$$= \theta \left[ \frac{1}{\theta} + \sum_{n=0}^{\infty} \frac{\mu^2}{(\mu^2 + \theta n)(\mu^2 + \theta n + \theta)} - \left( \frac{\mu^2}{\theta (\mu^2 + \theta)} + \frac{1}{\mu^2 + \theta} \right) \right]$$

$$= 1 + 1 - 1 = 1,$$

(4.7)

where we have set $c = \sqrt{\frac{\mu^2 \theta}{\mu^2 + \theta}}$ for simplicity because $Q$ is independent of $c$. The energy of this configuration is $E = 2\pi (2 + 1) = 6\pi$.

The solution (4.4) has the parameter $\mu$ of the dimension of length. To see that this parameter is related to the size of the solution, we consider the small $\mu$ and large $\mu$ limits. In the limit of $\mu \to 0$, (4.4) reduces to

$$P = \left( \begin{array}{cc} |1\rangle \langle 1| & 0 \\ 0 & 1 \end{array} \right).$$

(4.8)
This corresponds to (4.1) with \( n = 1 \). On the other hand, in the limit of \( \mu \to \infty \), (4.4) reduces to

\[
P = \begin{pmatrix} 1 - |0\rangle \langle 0| & 0 \\ 0 & |0\rangle \langle 0| \end{pmatrix}.
\]

(4.9)

This corresponds to the solution considered in section 3 and represents a soliton anti-soliton pair sitting at the origin. We can interpret the non-BPS solution (4.4) as the configuration which contains a soliton of the size \( \mu \) and a small soliton anti-soliton pair. In the large \( \mu \) limit the soliton spreads over the space and disappears, and hence only the soliton anti-soliton pair exists.

5 Discussion

In this paper, we have constructed non-BPS solutions of the noncommutative \( CP^1 \) model. These solutions can be extended to the case of \( CP^N \), \( N \geq 2 \) by embedding the solutions in the \( (N + 1) \times (N + 1) \) projection operator of the block diagonal form. In the commutative \( CP^N \) model, all the static non-BPS solutions were generated from the BPS solutions [12].

It is a future problem whether we can construct all static solutions of the noncommutative \( CP^N \) model. In noncommutative gauge theories, (non-)BPS solutions were constructed by using the solution generating transformation [2]. It might be possible to find such a transformation in the noncommutative \( CP^N \) model.

We have a comment on the relation between the noncommutative \( CP^N \) model and a scalar theory on a noncommutative space (the GMS model) [4]. Our non-BPS solutions have been constructed by using solitons in the GMS model. If one considers the \( (N + 1) \times (N + 1) \) hermitian matrix instead of the scalar field in the GMS model, one gets the noncommutative \( CP^N \) model in the low-energy limit.

Some non-Bogomol’nyi solutions of the Yang-Mills-Higgs equations were obtained by using the non-BPS solutions of the \( CP^N \) model [15]. It is interesting to see whether our non-BPS solutions can be used to construct new non-Bogomol’nyi solutions of the Yang-Mills-Higgs system on a noncommutative space [16].

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