IRRATIONALITY OF THE SUM OF A p-ADIC SERIES

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Abstract. We prove that the sum of the series $\sum_{n=0}^{\infty} p^{v_p(n!)}$ is a $p$-adic irrational for all primes $p$, where $v_p(n!)$ denotes the exponent of the highest power of $p$ dividing $n!$.

1. Introduction

Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers. One of the features of $p$-adic analysis is that a series $\sum_{n=0}^{\infty} x_n$ converges in $\mathbb{Z}_p$ if and only if $x_n \to 0$ in the $p$-adic metric [2, 6]. An intriguing special case is the factorial series $\sum_{n=0}^{\infty} n!$, whose sum in $\mathbb{Z}_p$ is conjectured to be irrational [1, 4] for all primes $p$, although this remains an open problem since Schikhof’s remark in [6, p. 17].

The rate of convergence of $n!$ to zero in $\mathbb{Z}_p$ is determined by the highest power of $p$ dividing $n!$. According to Legendre’s formula [3], the exponent of this power is equal to

\[ v_p(n!) = \frac{n - s_p(n)}{p - 1} \tag{1.1} \]

where $s_p(n)$ is the sum of the digits in the base $p$ expansion of $n$.

In this note we prove that the series $\sum_{n=0}^{\infty} p^{v_p(n!)}$ converges to an irrational number $\alpha_p$ in $\mathbb{Z}_p$ for all $p$. This is shown by demonstrating that the $p$-adic expansion of $\alpha_p$ is not periodic.

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2. Periodicity and rationality

We rely on the fact that the $p$-adic expansion of a $p$-adic number $\alpha$ is periodic if and only if $\alpha$ rational, that is, $\alpha = a/b$ where $a$ and $b$ are ordinary integers. The proof of this fact is similar to the proof of the analogous assertion concerning real numbers in terms of their decimal expansions [5, p. 106].

Let us denote by $s_p(n)$ the sum of all the digits in the base $p$ expansion of a positive integer $n$. The set of numbers $s_p(n)$ where $n$ ranges from $kp$ to $(k + 1)p - 1$ will be called the $k$-th package. Thus the cardinality of each package is equal to $p$.
Proposition 2.1. The $p$-adic valuation of $n!$ is constant in each package.

Proof. This follows from (1.1), since the numbers in each package are consecutive and there is only one multiple of $p$ at the beginning of the package. □

In other words, $p^{\nu_p(n!)}$ is identical for all the numbers $n$ in any given package.

Note also that, when adding the values of $p^{\nu_p(n!)}$, there is precisely one carry occurring in each package. For instance, when $p = 3$, the first partial sums $\sum_{m=0}^{n} p^{\nu_p(m!)}$ grouped by packages are the following:

1, 2, 10, 20, 100, 110, 210, 1010, 1110, 2110, 10110, 11110, 111110, 101110, 1011110, . . .

Proposition 2.2. If $n$ is the largest number in a package, then the number of digits of the $p$-adic expansion of $\sum_{m=0}^{n} p^{\nu_p(m!)}$ is equal to $\nu_p(n!) + 2$.

Proof. This follows by induction over $k$ using the fact that there is one carry in each package. □

Theorem 2.3. The $p$-adic expansion of the sum of the series $\sum_{n=0}^{\infty} p^{\nu_p(n!)}$ in $\mathbb{Z}_p$ is not periodic.

Proof. We focus on what happens when $n$ is a power of $p$. Suppose that $n = p^r$ with $r \geq 1$ and compare the number of digits of a partial sum $\sum_{m=0}^{n-1} p^{\nu_p(m!)}$ with that of the next sum $\sum_{m=0}^{n} p^{\nu_p(m!)}$. Since $n = p^r$, we have $s_p(n) = 1$ and $s_p(n - 1) = r(p - 1)$; hence

$$v_p((n - 1)!) = \frac{p^r - 1}{p - 1} - r = v_p(n!) - r.$$

Consequently, the difference between the number of digits of $\sum_{m=0}^{n} p^{\nu_p(m!)}$ and that of the previous partial sum in the case when $n = p^r$ is precisely $r - 1$, by Proposition 2.2. This implies that the $p$-adic expansion of $\sum_{m=0}^{n} p^{\nu_p(m!)}$ starts with 1 followed by a string of $r - 2$ zeroes if $n = p^r$ with $r \geq 3$, and this string of zeroes is forever fixed in the subsequent partial sums. Since $r$ keeps increasing by 1 each time a power of $p$ is encountered, the $p$-adic expansion of $\sum_{n=0}^{\infty} p^{\nu_p(n!)}$ cannot be periodic. □

Corollary 2.4. The sum of the series $\sum_{n=0}^{\infty} p^{\nu_p(n!)}$ is irrational in $\mathbb{Z}_p$ for all primes $p$.

References

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