The Large-$N_c$ Renormalization Group

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Abstract

In this talk we review how effective theories of mesons and baryons become exactly soluble in the large-$N_c$ limit. We start with a generic hadron Lagrangian constrained only by certain well-known large-$N_c$ selection rules. The bare vertices of the theory are dressed by an infinite class of UV divergent Feynman diagrams at leading order in $1/N_c$. We show how all these leading-order diagrams can be summed exactly using semiclassical techniques. The saddle-point field configuration is reminiscent of the chiral bag: hedgehog pions outside a sphere of radius $\Lambda^{-1}$ ($\Lambda$ being the UV cutoff of the effective theory) matched onto nucleon degrees of freedom for $r \leq \Lambda^{-1}$. The effect of this pion cloud is to renormalize the bare nucleon mass, nucleon-$\Delta$ hyperfine mass splitting, and Yukawa couplings of the theory. The corresponding large-$N_c$ renormalization group equations for these parameters are presented, and solved explicitly in a series of simple models. We explain under what conditions the Skyrmion emerges as a UV fixed-point of the RG flow as $\Lambda \to \infty$.

Introduction. The large-$N_c$ limit of QCD is thought to retain all the important dynamical features of the realistic case, such as confinement, chiral symmetry breaking and asymptotic freedom, while at the same time offering considerable simplifications for the effective theory of hadrons. The effective theory of mesons becomes semiclassical in this limit and Witten has argued that large-$N_c$ baryons should be identified with chiral soliton solutions of the corresponding meson field equations.

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The Skyrme model \cite{3, 4} provides the simplest possible realization of this idea and yields a surprisingly accurate picture of baryon properties. In addition there are several model-independent features of the chiral soliton approach, such as the $I = J$ rotor spectrum for the baryons, and the “proportionality rule” (see rules 2-4 below), which can be regarded as pure large-$N_c$ predictions.

More recently, a seemingly orthogonal approach to large-$N_c$ baryons has been promoted by Dashen, Jenkins and Manohar \cite{5}. These authors start from an effective Lagrangian with explicit fields for baryons as well as mesons and then obtain powerful constraints on the allowed spectrum and couplings of the theory by demanding self-consistency as $N_c \to \infty$. Pleasingly, these consistency conditions include all the model-independent predictions of the chiral soliton approach mentioned above. Despite this agreement, there appear to be several fundamental differences between these two pictures of large-$N_c$ baryons. First, the nucleon is represented as a point-like spinor field in one and a semiclassical extended object in the other. Second, in the effective Lagrangian picture, baryon number is an ordinary $U(1)$ Noether charge while in the soliton picture it is identified with a topological charge which does not correspond to any symmetry of the Lagrangian. And third, the number of free parameters is far greater in the effective Lagrangian approach; for example the nucleon mass is arbitrary in this case but is a fixed function of the mesonic couplings in the soliton approach, and likewise for the Yukawa coupling(s).

And yet, since both approaches purport to describe QCD in the low- and medium-energy regimes, must they not be equivalent to one another? In this talk, which is based on our recent papers \cite{6, 7}, we will explain precisely how this equivalence comes about—at least to leading order in $1/N_c$. (We are optimistic that the equivalence continues to hold order by order in the $1/N_c$ expansion.) A cartoon of our research program may be seen in Fig. 1. Starting with a large-$N_c$-compatible effective Lagrangian, we will use semiclassical techniques to demonstrate equivalence to a (quasi) chiral bag, with the role of the bag radius being played by the inverse UV cutoff $\Lambda^{-1}$ which regulates the divergent meson-baryon Feynman diagrams in the effective theory. It is then natural to ask, under what circumstances can the “continuum limit” $\Lambda \to \infty$ be taken, corresponding to the limit of zero bag radius? The answer to this question is fairly intricate, and is discussed in detail in Ref. \cite{7}. Under certain favorable circumstances, the UV limit exists—and is in fact a Skyrmion/soliton model. We will review the formulation of the so-called large-$N_c$ renormalization group, which is the tool for exploring this question (for a comparison with the Cheshire Cat Principle, see Ref. \cite{8}). This talk, therefore, covers two of the three arrows depicted in Fig. 1 (similar ideas are arrived at by Manohar \cite{8}).
Fig. 1: Three types of large-$N_c$ models of the strong interactions, and the relationships between them. This talk examines two of the three arrows, Effective QFT $\rightarrow$ “Chiral bags,” and “Chiral bags” $\rightarrow$ Skyrmions. The third relation, Skyrmions $\rightarrow$ Effective QFT, is examined in the following talk [9].

The third arrow in Fig. 1 concerns how, starting with a soliton model, one bootstraps one’s way in the backwards direction to an effective Lagrangian; this is the topic of our talk which immediately follows this one [9].

**Large-$N_c$ hadron models.** We study generic 2-flavor relativistic hadron Lagrangians that conserve $C$, $P$, $T$, and isospin, and are further restricted *only* by these five large-$N_c$ consistency conditions:

1: Straightforward quark-gluon counting arguments show that $n$-meson vertices $\sim N_c^{1-\frac{4}{n}}$, as do $n$-meson 2-baryon vertices [4, 10, 11]. Thus, baryon masses ($n = 0$) and Yukawa couplings ($n = 1$) grow like $N_c$ and $\sqrt{N_c}$, respectively.

2: The 2-flavor baryon spectrum of large-$N_c$ QCD consists of an infinite tower of positive parity states with $I = J = 1/2, 3/2, 5/2 \ldots$. To leading order these
states are degenerate, with mass $M_{\text{bare}} \sim N_c$ \cite{1, 2, 4-12}. (There are similar degeneracies amongst the mesons that need not concern us here.)

3: Hyperfine baryon mass splittings have the form $J(J + 1)/2I_{\text{bare}}$ where $I_{\text{bare}} \sim N_c$ \cite{2, 4-13}.

4: Yukawa couplings are constrained to obey the “proportionality rule” \cite{1, 4, 12, 14, 15}, which fixes the interaction strength of a given meson with each member of the baryon tower as a multiple of one overall coupling constant (e.g., $g_{\pi NN}/g_{\pi N \Delta} = 3/2$).

5: Finally, the allowed couplings of mesons to the baryon tower must obey the $I_t = J_t$ rule \cite{14, 13, 16, 14}; e.g., the $\rho$ meson must be tensor-coupled to the nucleon while the $\omega$ meson is vector-coupled at leading order in $1/N_c$, in good agreement with phenomenology.

A concrete effective Lagrangian that embodies these selection rules, and which is useful to keep in mind in the ensuing discussion, is the following, consisting of baryons and pions only:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 - \frac{m_\pi^2}{2} \vec{\pi}^2 - V(\vec{\pi}) + \bar{N}(i\gamma - M_N)N - g_{\pi}^{\text{bare}}\partial_\mu \pi^a \bar{N}\gamma^5 \gamma^\mu \tau^a N + (\text{higher-spin baryons})$$

(0.1)

Here $V$ is a general pion potential including quartic and higher vertices subject to rule 1; also $M_N = M_{\text{bare}} + 3/8I_{\text{bare}}$ including the hyperfine splitting, as per rules 2-3.

The pseudovector form of the $\pi N$ coupling is determined by the $I_t = J_t$ rule while the proportionality rule fixes the corresponding pion couplings to the higher-spin baryons. The optional incorporation of additional meson species will be discussed below.

Summing the leading-order graphs. Let us review how one power counts effective meson-baryon Feynman graphs as per the $1/N_c$ expansion. The $N_c$ dependence of coupling constants described in rule 1 above trivially suffices to identify the leading-order meson-baryon Feynman graphs for any given physical process. Thus, purely mesonic processes are dominated by meson tree graphs, which vanish as $N_c \to \infty$; each meson loop costs a factor of $1/N_c$. Similarly, meson-baryon processes are dominated by those graphs which become meson trees if the baryon lines are removed.
Fig. 2: (a) The bare meson-baryon coupling, which we shall refer to generically as a "Yukawa coupling." Henceforth, directed lines are baryons, undirected lines are mesons. Internal baryon lines must be summed over all allowed states in the $I = J$ tower. (b) A typical multi-loop dressing of (a) that contributes at leading order, $N_c^{1/2}$, as it contains no purely mesonic loops. (c) A systematic counting of the diagrams such as (b). The shaded blob contains only tree-level meson branchings. There are $n!$ distinct "tanglings" of the attachments of the shaded blob to the baryon line. (d) A typical dressing such as (b), augmented by additional baryon self-energy and vertex corrections, all of which also contribute at leading order.

To illustrate the uncontrolled complexity of such graphs, look at a typical multi-
loop correction, Fig. 2b, to the bare Yukawa coupling shown in Fig. 2a. Since, by
design, the graph in Fig. 2b contains no loops formed purely from mesonic legs, this
graph scales like $\sqrt{N_c}$ just like the bare vertex. This is easily checked by multiplying
together all the vertex constants and ignoring propagators entirely, since both meson
and baryon propagators $\sim N_c^0$. Hence, in order to calculate the dressed Yukawa
vertex to leading order, one must sum this infinite set of diagrams, shown somewhat
schematically in Fig. 2c. One must also sum all multiple insertions of the baryon
self-energy corrections and additional vertex corrections as illustrated in Fig. 2d, as
these too contribute at leading order $[10, 13]$. Since many of the loop integrations in
these diagrams are UV divergent, it is necessary to regulate the theory with a UV
cutoff $\Lambda$.

It is clear that the naive perturbative graph-by-graph method is of no use at
all in large $N_c$ beyond the soft-pion regime; this is because arbitrarily complicated
radiative corrections such as Fig. 2b-d contribute to the renormalized Yukawa coupling
at leading order in the $1/N_c$ expansion. In order to make progress, one had better
sum all such graphs, all at once. Pleasingly, such a summation is in fact possible,
if one restricts one’s attention to leading order in $1/N_c$. This can be seen either
from momentum-space or position-space Feynman rules, following Refs. [17] or [6],
respectively. The arguments are a little more transparent in position space. From
the position-space Feynman rules for (0.1), the sum of all the graphs such as Fig. 2b
(i.e., Fig. 2c) is formally given by:

$$\sum_{n=1}^{\infty} \int \left( \prod_{i=1}^{n} d^4 w_i d^4 z_i \right) Blob_n(x; w_1, \ldots, w_n) \sum_{\rho \in S_n} \prod_{i=1}^{n} g(w_i, z_{\rho(i)}) \cdot \mathcal{V}(z_{\rho(i)}) \prod_{i=1}^{n-1} G(z_i, z_{i+1})$$

(0.2)

$Blob_n$ denotes the shaded blob in Fig. 2c; it contains the complete set of tree-level
meson branchings only, with no purely mesonic loops. The sum over permutations
$\rho \in S_n$ counts the $n!$ possible tanglings of the $n$ meson lines which connect the
baryon line to the blob. All isospin and spin indices have been suppressed in (0.2);
intermediate baryons are assumed to be summed over all allowed values of $I = J$. $g_{ab}$
and $G_J$ are the position-space meson and spin-$J$ baryon propagators, respectively.
Finally $\mathcal{V}_{J, J'}$ is the appropriate pseudovector Yukawa vertex factor connecting the
meson to an incoming (iso)spin $J$ and outgoing (iso)spin $J'$ baryon, according to the
proportionality rule 4 listed earlier.

Passing to the large-$N_c$ limit, we can exploit two important simplifications to
this expression. First, the baryons become very massive and can be treated nonrel-
ativistically. For forward time-ordering, $z_0 < z'_0$, the baryon propagator $G(z, z')$
can be replaced by its nonrelativistic counterpart $G_{NR}(z, z') + \mathcal{O}(1/N_c)$. As usual, the
reversed or “Z-graph” time ordering \( z'_0 < z_0 \) contributes effective pointlike vertices in which two or more mesons couple to the baryon at a single point. However these effective vertices turn out to be down by two additional factors of \( 1/N_c \) since they are proportional to \( (\vec{\sigma} \cdot \vec{p} / M_{\text{bare}})^2 \), and we may neglect them.

A second simplification comes from anticipating the “moral of our story,” namely the equivalence to Skyrmion models, and borrowing from the Skyrme-model literature a useful construct, namely the \( SU(2) \) collective coordinate basis for the \( I = J \) baryons \[4, 15, 18\]. This is the basis denoted \( |A\rangle \), related to the more familiar spin-isospin basis \( |I = J, i_z, s_z\rangle \) via the overlap formula

\[
\langle I = J, i_z, s_z | A \rangle = (2J + 1)^{1/2} D^{(J)}_{\vec{z}_0, \vec{z}_0'}(A^\dagger) \cdot (-)^{J - s_z} \tag{0.3}
\]

with \( D^{(J)}(A) \) a standard Wigner \( D \)-matrix. In the \( |A\rangle \) basis, the baryon propagator can be expressed as a quantum mechanical path integral over two collective coordinates: \( \vec{X} \), representing the position of the center of the baryon, and \( A \), describing its \( SU(2) \) (iso)orientation,

\[
G_{NR}^{A,A'}(z, z') = \theta(z'_0 - z_0) \int_{\vec{X}(z_0) = \vec{z}}^{\vec{X}(z'_0) = \vec{z}'} \mathcal{D}X(t) \int_{A(z_0) = A}^{A(z'_0) = A'} \mathcal{D}A(t) \times \exp \left( i \int_{z_0}^{z'_0} dt M_{\text{bare}} + \frac{1}{2} M_{\text{bare}} \dot{X}^2 + I_{\text{bare}} \text{Tr} \dot{A}^\dagger \dot{A} \right) \tag{0.4}
\]

The path integration over \( A(t) \) can be performed using the beautiful result of Schulman for free motion on the \( SU(2) \) group manifold \[19\],

\[
\int_{A(z_0) = A}^{A(z'_0) = A'} \mathcal{D}A(t) \exp i \int_{z_0}^{z'_0} dt I_{\text{bare}} \text{Tr} \dot{A}^\dagger \dot{A} = \sum_{J=1/2, 3/2, \ldots}^{J} \sum_{i_z, s_z = -J} J(J+1) \frac{1}{2I_{\text{bare}}} \langle J, i_z, s_z | A \rangle \tag{0.5}
\]

yielding the conventional nonrelativistic propagator for an infinite tower of particles with masses \( M_{\text{bare}}(J) = M_{\text{bare}} + J(J+1)/2I_{\text{bare}} \) as required by the large-\( N_c \) selection rules.

For our present purposes, the greatest advantage of the \( |A\rangle \) basis is that the Yukawa vertex factor \( \mathcal{Y} \) for the pion-baryon coupling becomes diagonal \[4, 15, 18\]:

\[
\mathcal{Y}_{A,A'}^\pi(z) = -3g_{\pi \text{bare}} D^{(J)}_{ab}(A) \delta(A - A') \frac{\partial}{\partial z_b} \tag{0.6}
\]

Substituting for \( G \) and \( \mathcal{Y} \) in \( (0.2) \), we find that this property allows us to perform the sum over all tanglings of the meson lines trivially (the product over temporal step-functions in Eq. \( (0.4) \) summing to unity). Interchanging the order of path integration
and the product over baryon legs, one obtains

\[
\int \mathcal{D}X(t)\mathcal{D}A(t) \sum_{n=1}^{\infty} \int \left( \prod_{i=1}^{n} d^4w_i d^4z_i \right) B\text{lob}_n(x; w_1, \ldots, w_n) \\
\times \prod_{i=1}^{n} g(w_i, z_i) \cdot \mathcal{Y}(z_i) \delta_{3\Lambda}^3(z_i - X(z_i^0)) \exp iS_{\text{baryon}}[X, A]
\]

(0.7)

where \(S_{\text{baryon}}\) is short for the exponent of (0.4). We have reduced the problem to a sum of tree diagrams (namely, \(\text{Blob}_n\)) for the pions interacting with the baryon collective coordinates through a \(\delta\)-function source. The only remaining manifestation of the UV cutoff \(\Lambda\) is that this \(\delta\)-function should be smeared out over a radius \(\sim \Lambda^{-1}\), as denoted by \(\delta_{3\Lambda}\) in (0.7), which we assume still preserves rotational invariance.

In sum, the massive baryon has become a translating, (iso)rotating, smeared point-source for the pion field, the effect of which can be found by solving the appropriate classical Euler-Lagrange equation for a configuration we call \(\bar{\pi}_{cl}(x; [X], [A])\):

\[
\left(\Box + m_{\pi}^2\right)\bar{\pi}_{cl}^a + \frac{\partial V}{\partial \pi_{cl}^a} = 3g_{\text{bare}} D_{ai}^{(1)}(A(t)) \frac{\partial}{\partial x^i} \delta_{3\Lambda}^3(x - X(t))
\]

(0.8)

It is easily checked (Fig. 3) that the order-by-order perturbative solution of Eq. (0.8) generates precisely the sum of graphs appearing in (0.7).

**Fig. 3:** The graphical perturbative solution to Eq. (0.8) as a sum of tree-level one-point functions terminating in the effective Yukawa vertex.
However, we still have not accounted for the baryon self-energy and meson-baryon vertex corrections, highlighted in Fig. 2d. The key to summing these is to notice that if the baryon line were erased, they would be disconnected vacuum corrections, and that to leading order in $1/N_c$ only the subset of such corrections that are meson trees are important. As usual for vacuum corrections, they exponentiate. Furthermore, by similar semi-classical reasoning as used above, they are correctly accounted for by evaluating the mesonic plus Yukawa pieces of the action (call this sum $S_{\text{eff}}$) on $\vec{\pi}_c$. This is illustrated in Fig. 4.

![Diagram](https://example.com/diagram)

**Fig. 4:** Diagrammatic representation of $S_{\text{eff}}(\vec{\pi}_c)$. When combined with the expansion depicted in Fig. 3, $\exp i S_{\text{eff}}$ combinatorically correctly accounts for all the leading-order baryon self-energy and vertex corrections highlighted in Fig. 2d.

The final leading-order result for the complete sum of graphs contributing to the dressed pion-baryon vertex is then:

$$\int\mathcal{D}X(t)\mathcal{D}A(t) \pi^a_{\text{cl}}(x; [X], [A]) \exp i(S_{\text{baryon}} + S_{\text{eff}}[\vec{\pi}_c, X(t), A(t)])$$

The blob has been eliminated from the problem, replaced simply by $\vec{\pi}_c$. For this to have happened, two conditions needed to be met. First, the blob could be expressed as the complete sum of tree graphs, an obvious consequence of large-$N_c$ since purely mesonic loops are suppressed by $1/N_c$. Second, and more subtly, there needed to exist a baryon basis (the $\langle A \rangle$ basis) in which the Yukawa source function $Y$ is *diagonal*. The importance of this second “diagonality” condition was first emphasized by
Gervais and Sakita in their classic work on large $N_c$ \cite{12}. Only if both conditions are met is a semiclassical summation of the relevant Feynman graphs possible.

**Solving the classical field equation.** Now that we have reduced our problem to Eq. (0.8), one may ask, how does one actually solve such an equation? Again, we borrow Skyrme-model techniques, relating it to the analogous equation for the static pion cloud, $\vec{\pi}_{\text{stat}}(x)$, surrounding a fixed baryon source ($X(t) \equiv 0$, $A(t) \equiv 1$):

$$
(-\nabla^2 + m_{\pi}^2)\pi_{\text{stat}}^a + \frac{\partial V}{\partial \pi_{\text{stat}}^a} = 3g_{\pi}^{\text{bare}} \frac{\partial}{\partial x^a} \delta_\Lambda(x) \tag{0.10}
$$

The solution will generically have the hedgehog form familiar from the Skyrme model:

$$
\pi_{\text{stat}}^a(x) = (f_\pi x^a/2r)F(r) \text{ where } r = |x|. \text{ The profile function } F(r) \text{ is found, in turn, by solving the induced nonlinear radial ODE. While its detailed form depends sensitively on the potential } V(\vec{\pi}), \text{ its asymptotic behavior for large } r \text{ is fixed by the linearized field equation,}
$$

$$
F(r) \rightarrow \mathcal{A} \left(\frac{m_{\pi}}{r} + \frac{1}{r^2}\right) e^{-m_{\pi}r} \tag{0.11}
$$

where the constant $\mathcal{A}$ must, in the end, be extracted numerically. The solution to (0.8) is then simply given, up to $1/N_c$ corrections, by translating and (iso)rotating $\vec{\pi}_{\text{stat}}$:

$$
\pi_{\text{cl}}^a(x; [X], [A]) = D_{ab}^{(1)}(A(t)) \pi_{\text{stat}}^b(x - X(t)) \tag{0.12}
$$

The additional collective coordinate dependence carried by $\vec{\pi}_{\text{cl}}$ versus $\vec{\pi}_{\text{stat}}$ is precisely that required for overall isospin, angular momentum and 4-momentum conservation, as is easily checked \cite{20}.

We seek the renormalized on-shell $\pi N$ interaction, to leading order in $1/N_c$. It is defined in the usual way as the on-shell residue of the LSZ amputation of the full set of graphs that are summed implicitly by Eq. (0.9). Formally, this amputation is identical to the procedure one follows in the Skyrme model \cite{20}. In particular, the physically correct analytic structure of the one-point function follows from the $1/N_c$ corrections to $\vec{\pi}_{\text{cl}}$ which describe its response to the rotation of the source. (The specifics of this response, involving an interesting small distortion away from the hedgehog ansatz \cite{20}, need not concern us here; see the following talk \cite{13}.\) Thanks to the (iso)vector nature of the hedgehog, the resulting S-matrix element for one-pion emission defines a renormalized on-shell pseudovector interaction of the pion with the baryon tower, **identical** to the bare interaction in (0.1), except for the coupling constant renormalization $g_{\pi}^{\text{bare}} \rightarrow g_{\pi}^{\text{ren}}$. Again as in the Skyrme model, this latter quantity is determined by the asymptotics of $\vec{\pi}_{\text{stat}}$, Eq. (0.11), and is explicitly given by \cite{1, 23} $g_{\pi}^{\text{ren}} = (2/3)\pi f_\pi A$. Thus the proportionality and $I_t = J_t$ rules for the pion-baryon coupling remain true at the renormalized level, as claimed.
Furthermore, the result of evaluating $S_{\text{eff}}[\vec{\pi}_{\text{cl}}]$ is just an additive renormalization of the bare parameters of $S_{\text{baryon}}$, due to the meson cloud:

$$S_{\text{baryon}} + S_{\text{eff}}[\vec{\pi}_{\text{cl}}, X, A] = \int dt \left( M_{\text{ren}} + \frac{1}{2} M_{\text{ren}} \dot{X}^2 + I_{\text{ren}} \text{Tr} \dot{A}^\dagger \dot{A} \right)$$

(0.13)

where

$$M_{\text{ren}} = M_{\text{bare}} + \int d^3x (\nabla \pi_{\text{cl}}^a)^2, \quad I_{\text{ren}} = I_{\text{bare}} + \frac{2}{3} \int d^3x \pi_{\text{cl}}^2$$

(0.14)

It follows that $M_{\text{ren}}(J) = M_{\text{ren}} + J(J+1)/2I_{\text{ren}}$ and so the form of the hyperfine baryon mass splitting is likewise preserved by renormalization. The self-consistency of large-$N_c$ effective models, as evidenced by the last two paragraphs, is one of the most striking features of the large-$N_c$ approach; namely, that selection rules implemented at the bare level survive the all-loops renormalization process.

The generalization of the above analysis to models including several species of mesons involves solving the coupled classical radial ODE’s for all the meson fields, using generalized hedgehog ansatze familiar from vector-meson-augmented Skyrme models. A particularly rich meson model might include, in addition to the pion, the tensor-coupled $\rho$, i.e., $g_{\rho}^{\text{bare}} \partial_\mu \vec{\rho} \cdot \vec{N} \sigma^{\mu\nu} \pi N$, the vector-coupled $\omega$, i.e., $g_{\omega}^{\text{bare}} \omega_\mu \bar{N} \gamma^\mu N$, and/or the “$\sigma$-meson,” which couples simply as $g_{\sigma}^{\text{bare}} \sigma \bar{N} N$. Again, on shell, the form of these particular couplings survives renormalization.

**Large-$N_c$ Renormalization Group.** We have described an explicit numerical procedure for calculating the renormalized Yukawa couplings, baryon masses and hyperfine mass splittings, to leading order in $1/N_c$, directly from the classical meson cloud surrounding the baryon. Since the $\delta$-function source on the right-hand side of Eq. (0.8) is smeared out over a characteristic length $\Lambda^{-1}$, these quantities depend explicitly on $\Lambda$. In order to hold the physical, renormalized masses and couplings fixed, it is necessary to vary simultaneously both $\Lambda$ and the corresponding bare quantities. This procedure defines an RG flow for $M_{\text{bare}}(\Lambda)$, $I_{\text{bare}}(\Lambda)$ and $g_{\pi,\rho,\omega,\sigma}^{\text{bare}}(\Lambda)$, valid to all orders in the loop expansion but strictly to leading order in $1/N_c$. We term this flow the large-$N_c$ Renormalization Group, and devote the rest of this talk to illustrating its solutions.

It is particularly interesting to ask whether this flow has a UV fixed point; this would correspond to a continuum limit for the theory. Unfortunately we are not able to prove any general theorems about the RG flow for large-$N_c$ effective theories. Nevertheless, in [7], we were able to carry out the program outlined above explicitly in a series of simple but physically relevant models of pions with a pseudovector coupling to the $I = J$ baryon tower. The models are distinguished from one another only by the choice of the purely mesonic Lagrangian denoted $L_{\text{meson}}$. For details we refer the
interested reader to [8], and here content ourselves with a summary of the salient results.

Our first example consists simply of free massless pions,

$$\mathcal{L}_{\text{meson}} = \frac{1}{2}(\partial_{\mu} \vec{\pi})^2$$  \hspace{1cm} (0.15)

coupled derivatively to the $I = J$ baryon tower. In the hedgehog ansatz, the static Euler-Lagrange equation (0.10) becomes

$$F'' + \frac{2}{r} F' - \frac{2}{r^2} F = 6 f_\pi^{-1} g_{\pi \text{bare}}(\Lambda) \frac{\partial}{\partial r} \delta(r)$$  \hspace{1cm} (0.16)

This being a linear equation, it is trivially solved using the method of Green’s functions:

$$F(r) = \frac{6 f_\pi^{-1} g_{\pi \text{bare}}(\Lambda)}{2\pi f_\pi r^2} \int_0^\infty dr' \frac{1}{r'} G(r, r') \frac{\partial}{\partial r'} \delta(r')$$  \hspace{1cm} (0.17)

where the Green’s function that is well behaved at both $r = 0$ and $r = \infty$ is

$$G(r, r') = -\frac{r_<}{3r_>^2}$$  \hspace{1cm} (0.18)

where $r_< = \min[r, r']$ and $r_> = \max[r, r']$.

The renormalized Yukawa coupling $g_{\pi \text{ren}}$ is extracted from the large-distance behavior of $F$ as per equation (0.11). With the mild (and relaxable) assumption that $\delta(r')$ has compact support, Eqs (0.17) and (0.18) imply

$$F(r) \to \frac{3 g_{\pi \text{bare}}(\Lambda)}{2\pi f_\pi r^2}$$  \hspace{1cm} (0.19)

Comparing (0.19) and (0.11), we deduce

$$g_{\pi \text{bare}}(\Lambda) = g_{\pi \text{ren}}$$  \hspace{1cm} (0.20)

for all $\Lambda$, admittedly not a surprising result for free field theory, but a reassuring sanity check on our formalism. The simplest modification to the Lagrangian (0.15) is to add a pion mass term. In that case a similar analysis yields;

$$g_{\pi \text{bare}}(\Lambda) = g_{\pi \text{ren}} \cdot \left(1 + O(m_\pi^2/\Lambda^2)\right)$$  \hspace{1cm} (0.21)

In either variation, massless or massive, the “continuum limit” $\Lambda \to \infty$ can be safely taken, and the “ultraviolet fixed point” that emerges is just what one started with: a theory of free pions derivatively coupled to the baryon tower.

For our second example, consider the nonlinear $\sigma$ model for pions,

$$\mathcal{L}_{\text{meson}} = \frac{f_\pi^2}{16} \text{Tr} \partial_{\mu} U \partial^{\mu} U^\dagger$$  \hspace{1cm} (0.22)
where $U = \exp(2i\vec{\tau} \cdot \vec{f}/f_\pi)$ again augmented by the bare pseudovector Yukawa coupling. The static Euler-Lagrange equation now works out to

$$F'' + \frac{2}{r} F' - \frac{1}{r^2} \sin 2F = 6f_\pi^{-1}g_{\pi}^{\text{bare}}(\Lambda) \frac{\partial}{\partial r}\delta_\Lambda(r)$$

(0.23)

Solving this nonlinear equation for $F(r)$ requires that we specify a smearing of the source. For convenience, we follow [8], and choose a radial step-function

$$\delta_\Lambda(r) = \frac{3\Lambda^3}{4\pi} \theta(\Lambda^{-1} - r)$$

(0.24)

which is properly normalized to unit volume. The technical advantage, which we exploit presently, is that the right-hand side of (0.23) is now proportional to a true $\delta$-function, since

$$\frac{\partial}{\partial r}\delta_\Lambda(r) = -\frac{3\Lambda^3}{4\pi} \delta(r - \Lambda^{-1})$$

(0.25)

There is also a conceptual advantage: the right-hand side of (0.25) means that the baryon and meson degrees of freedom only interact at the “bag radius” $\Lambda^{-1}$ which sharpens the analogy to the traditional chiral bag (a topic we shall return to at the end of this talk).

With this convenient choice of regulator, the prescription for satisfying (0.23) is transparent: First solve the homogeneous version of equation (0.23) for $r < \Lambda^{-1}$ (“region I”) and for $r > \Lambda^{-1}$ (“region II’); next, match the solutions in these two regions, $F_I(r)$ and $F_{II}(r)$, at the point $r = \Lambda^{-1}$; and finally, read off $g_{\pi}^{\text{bare}}(\Lambda)$ from the slope discontinuity,

$$g_{\pi}^{\text{bare}}(\Lambda) = \frac{2}{9} \pi f_\pi \Lambda^{-3} \left(F'_I(\Lambda^{-1}) - F'_{II}(\Lambda^{-1})\right)$$

(0.26)

In Ref. [7] we implemented this prescription numerically and determined the behaviour of $g_{\pi}^{\text{bare}}$ as $\Lambda$ is varied with $g_{\pi}^{\text{ren}}$ held fixed at its experimental value. We discovered a critical value of the cutoff, $\Lambda_c \simeq 340$ MeV above which there is no real solution, corresponding to a critical bag radius $\Lambda^{-1}_c \simeq .6$ fm. In this case, therefore, there exists an obstacle to taking a continuum limit $\Lambda \rightarrow \infty$.

The (little-known!) fact that a non-linear radial ODE in three dimensions has no solution with a point-like $\delta$-function source strongly suggests that any continuum limit of the RG flow necessarily involves the coupling $g_{\pi}^{\text{bare}}$, which multiplies the source, tending to zero as $\Lambda \rightarrow \infty$. In this case the resulting fixed-point field configuration should be a solution of the homogeneous meson field equation (ie with the source term set to zero). It is therefore natural to conjecture that the model only has a continuum limit if the homogeneous meson field equation admits a chiral soliton solution. In this
light, the occurrence of a critical cutoff for the non-linear $\sigma$-model coupled to large-$N_c$ baryons has an obvious explanation: the mesonic sector of this model, which consists of a single two-derivative term, does not support a soliton solution because of Derrick’s theorem \[21\]. Hence there can be no continuum limit.

The simplest way to remedy this problem is to augment the mesonic sector of the model by adding the Skyrme term. In this case $\mathcal{L}_{\text{meson}}$ is just the Lagrangian density of the standard two-term Skyrme model.

$$\mathcal{L}_{\text{meson}} = \frac{f_\pi^2}{16} \text{Tr} \partial_\mu U^\dagger \partial^\mu U + \frac{1}{32e^2} \text{Tr}\left[ U^\dagger \partial_\mu U , U^\dagger \partial_\nu U \right]^2$$ \hspace{1cm} (0.27)

In this case the radial ODE becomes

$$F'' + \frac{2}{r} F' - \frac{\sin 2F}{r^2} \left( 1 + \frac{4 \sin^2 F}{e^2 f_\pi^2 r^2} - \frac{4F'^2}{e^2 f_\pi^2} \right) = 6f^{-1}_\pi g^\text{bare}_\pi(\Lambda) \frac{\partial}{\partial r} \delta_\Lambda(r)$$ \hspace{1cm} (0.28)

while the derivative matching condition (0.26) remains unchanged. We know that the homogeneous pion field equation obtained by setting the RHS of (0.28) to zero has a soliton solution; the Skyrmion. Hence a naive conjecture is that the Skyrmion arises as the UV fixed point of the RG flow in these models. However, as stated, this naive conjecture cannot be right! The reason is the mismatch in the number of free parameters between soliton/Skyrmion theories and effective Lagrangian theories, mentioned earlier. In particular, in the latter the Yukawa coupling is independent of the mesonic parameters, whereas in the Skyrme model the asymptotic behaviour of the Skyrmion at large $r$ fixes a unique value for $g^\text{ren}_\pi$ originally determined numerically by Adkins, Nappi and Witten \[4\]:

$$g^\text{ren}_\pi \equiv g^\text{ANW}_\pi \simeq 18.0 \frac{e^2}{f_\pi}$$ \hspace{1cm} (0.29)

Our analysis \[7\] of the equation (0.28) resolves this conundrum: it reveals that the Skyrmion emerges as a UV fixed point only when $g^\text{ren}_\pi$ is fine-tuned to obey (0.29). If $g^\text{ren}_\pi \neq g^\text{ANW}_\pi$ then there again exists a critical value of the UV cutoff beyond which the RG flow does not go. The obstacle in this case is somewhat different to that which occurs when only the two-derivative term is included. There the solution of equation (0.23) actually ceases to exist above the critical cutoff. Here the solution still exists but is no longer locally stable and is thus no longer an appropriate saddle-point. We refer the reader to \[7\] for further details.

In summary we have shown how a generic, large-$N_c$ consistent, effective theory of hadrons can be solved exactly as $N_c \to \infty$ using semi-classical methods. We should stress that this solubility is precisely a consequence of the large-$N_c$ selection rules incorporated in the bare Lagrangian; for example, without the proportionality
rule, which implies that the baryon-pion vertex is diagonal in the $A$ basis, it would *not* have been possible to sum over all possible spins and isospins of the internal baryon lines with a single saddle-point field configuration. Correspondingly, the hedgehog structure of the resulting pion cloud ensures that the large-$N_c$ selection rules emerge unscathed at the renormalized level; an important check on the self-consistency of large-$N_c$ effective theories.

We also showed how the large-$N_c$ RG equations for the flow of the bare Lagrangian parameters can be determined exactly at leading order, and we exhibited their solutions in a series of specific models. Our results for the Skyrme Lagrangian *coupled to explicit baryon fields* show how the Skyrmion emerges naturally as the continuum limit of the dressed large-$N_c$ baryon. From our construction, it is easy to see how the apparent differences, discussed in the Introduction, between the effective Lagrangian and chiral soliton treatment of large-$N_c$ baryons are resolved. Although the bare nucleon in the former approach is a point-like field, it is dressed at leading order by an infinite set of Feynman diagrams which sum up to give a semiclassical pion cloud which coincides with the Skyrmion at large distances. As the continuum limit is taken the explicit baryon number carried by the bare nucleon is completely screened by the topological charge of the Skyrmion in a manner familiar from the chiral bag model [22]. Finally the puzzle about the number of free parameters in the two approaches has an obvious solution; the continuum limit only exists when the renormalized baryon parameters are fine tuned to obey Skyrme-model relations. In the language of the renormalization group, the corresponding terms in the bare Lagrangian (e.g., the Yukawa couplings) are *irrelevant operators*. In contrast, the mesonic self-couplings in the Lagrangian which determine the soliton solution are not renormalized at leading order in $1/N_c$ because purely mesonic loops are subleading. Hence they correspond to the *marginal operators* of the large-$N_c$ renormalization group which dominate in the continuum limit.

In this manner, the large-$N_c$ Renormalization Group in the UV limit $\Lambda \rightarrow \infty$ serves as the long-sought connection between effective Lagrangians and and Skyrmion models of the baryon (Fig. 1 again). But as alluded to earlier, for fixed, finite $\Lambda$, the picture we have arrived at of the meson-dressed large-$N_c$ baryon is highly reminiscent of yet a third class of phenomenological models, the chiral bag models [22]. These, too, are hybrid descriptions of the dressed baryon, in which explicit quark (rather than nucleon) degrees of freedom inside a bag of radius $R = \Lambda^{-1}$ are matched onto an effective theory of hedgehog pions outside the bag. Even this presumably important distinction between ‘nucleon’ versus ‘quark’ degrees of freedom inside the bag
disappears as $N_c \to \infty$. For, in this limit, the $N_c$ quarks may be treated in Hartree approximation, and their individual wave functions effectively condense into a common mean-field wave function, which we may identify with the “wave function of the nucleon.” Outside the bag, the analogy is closer still: the pion field configuration is again determined by solving a nonlinear field equation coupled to a static source at $r = R$. The only significant difference between our composite meson-dressed large-$N_c$ baryon and the traditional chiral bag is this: our composite baryon follows solely from large $N_c$ and has nothing whatsoever to do with chiral symmetry! It is for this reason that we referred to it as a “chiral bag” in Fig. 1, being careful to retain the quotation marks to avoid confusion with the traditional chiral bag.

The link in the opposite direction, from Skyrmions to effective Lagrangians, is presented in the talk to follow [9].

References

[1] G. ’t Hooft, Nucl. Phys. B72 (1974) 461 and B75 (1974) 461; G. Veneziano, Nucl. Phys. B117 (1976) 519.

[2] E. Witten, Nucl. Phys. B160 (1979) 57.

[3] T. H. R. Skyrme, Proc Roy. Soc. A260 (1961) 127.

[4] G. Adkins, C. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.

[5] R. Dashen and A. V. Manohar, Phys. Lett. B315 (1993) 425 and 438; R. Dashen, E. Jenkins and A. V. Manohar, Phys. Rev. D49 (1994) 4713; E. Jenkins and A. V. Manohar, Phys. Lett. B335 (1994) 452.

[6] N. Dorey, J. Hughes and M. P. Mattis, Phys. Rev. Lett. 73 (1994) 1211.

[7] N. Dorey and M. P. Mattis, From effective Lagrangians, to chiral bags, to Skyrmions with the large-$N_c$ renormalization group, [hep-ph/9412373].

[8] A. Manohar, Phys. Lett. B336 (1994) 502.

2This observation was originally made by Witten (Ref. [2], Secs. 5 and 9), and exploited by Gervais and Sakita (Ref. [12], Sec. V). These two papers are highly recommended background reading, as they too are concerned primarily with the semiclassical nature of large-$N_c$. In particular Gervais and Sakita were the first to study chiral-bag-type structures in this limit, although not from our effective hadron Lagrangian starting point.
[9] N. Dorey and M. P. Mattis, *The rotationally improved Skyrmion, or "RISKY,"* to appear in the Proceedings of the 1995 International Workshop on Nuclear & Particle Physics: Chiral Dynamics in Hadrons & Nuclei.

[10] M. A. Luty and J. March-Russell, *Nucl. Phys.* **B426** (1994) 71; M. A. Luty, [hep-ph/9405271](http://arxiv.org/abs/hep-ph/9405271).

[11] C. Carone, H. Georgi, L. Kaplan and D. Morin, *Phys. Rev.* **D50** (1994) 5793; C. Carone, H. Georgi and S. Osofsky, *Phys. Lett.* **B322** (1994) 227.

[12] J. Gervais and B. Sakita, *Phys. Rev.* **D30** (1984) 1795.

[13] E. Jenkins, *Phys. Lett.* **B315** (1993) 441.

[14] M. P. Mattis and M. Mukerjee, *Phys. Rev. Lett.* **61** (1988) 1344.

[15] M. P. Mattis and E. Braaten, *Phys. Rev.* **D39** (1989) 2737.

[16] M. P. Mattis, *Phys. Rev.* **D39** (1989) 994 and *Phys. Rev. Lett.* **63** (1989) 1455.

[17] P. Arnold and M. P. Mattis *Phys. Rev. Lett.* **65** (1989) 8311.

[18] A. V. Manohar, *Nucl. Phys.* **B248** (1984) 19.

[19] L. S. Schulman *Phys. Rev.* **176** (1968) 1558.

[20] N. Dorey, J. Hughes and M. P. Mattis, *Phys. Rev.* **D50** (1994) 5816.

[21] G. H. Derrick, *J. Math. Phys.* **5** (1964) 1252; R. Hobart, *Proc. Roy. Soc. London* **82** (1963) 201.

[22] M. Rho, A. S. Goldhaber and G. E. Brown, *Phys. Rev. Lett.* **51** (1983) 747; J. Goldstone and R. L. Jaffe, *Phys. Rev. Lett.* **51** (1983) 1518. An excellent recent review of chiral bags is M. Rho, *Phys. Rep.* **240** (1994) 1.