When is star formation episodic? A delay differential equation ‘negative feedback’ model

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ABSTRACT

We introduce a differential equation for star formation in galaxies that incorporates negative feedback with a delay. When the feedback is instantaneous, solutions approach a self-limiting equilibrium state. When there is a delay, even though the feedback is negative, the solutions can exhibit cyclic and episodic solutions. We find that periodic or episodic star formation only occurs when two conditions are satisfied. First the delay time-scale must exceed a cloud consumption time-scale. Secondly, the feedback must be strong. This statement is quantitatively equivalent to requiring that the time-scale to approach equilibrium be greater than approximately twice the cloud consumption time-scale. The period of oscillations predicted is approximately four times the delay time-scale. The amplitude of the oscillations increases with both feedback strength and delay time.

We discuss applications of the delay differential equation (DDE) model to star formation in galaxies using the cloud density as a variable. The DDE model is most applicable to systems that recycle gas and only slowly remove gas from the system. We propose likely delay mechanisms based on the requirement that the delay time is related to the observationally estimated time between episodic events. The proposed delay time-scale accounting for episodic star formation in galaxy centres on periods similar to $P \sim 10$ Myr, irregular galaxies with $P \sim 100$ Myr, and the Milky Way disc with $P \sim 2$ Gyr, could be that for exciting turbulence following creation of massive stars, that for gas pushed into the halo to return and interact with the disc and that for spiral density wave evolution, respectively.

Key words: ISM: evolution – galaxies: ISM.

1 INTRODUCTION

Gas present in a galaxy fuels star formation or nuclear black hole growth. However, both star formation and active galactic nuclei then release energy and momentum into the interstellar medium (ISM). Consequently the activity can suppress subsequent star formation. The process in which part of the output of a system is returned to its input and influences its further output is termed ‘feedback’. Early studies showed that when feedback by radiative heating is taken into account during gas accretion on to a central mass, steady solutions may not exist (Ostriker et al. 1976) and the feedback process can cause oscillations or periodic bursts of accretion (Cowie, Ostriker & Stark 1978; Scalo & Struck-Marcell 1986; Parravano 1996; Kamaya 2005). Simulations taking into account feedback processes illustrate that gas flows and star formation in galaxies can exhibit episodic or cyclic behaviour (Dong, Lin & Murray 2003; Pelupessy, van der Werf & Icke 2004; Ciotti & Ostriker 2007; Stinson et al. 2007).

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and has not explored when episodic rather than a steady rate of star formation is expected (though see Scalo & Struck-Marcell 1986; Parravano 1996; Dong et al. 2003; Kamaya 2005).

As gas flows involving energy input, heating and cooling are complex, there is no simple way to predict when behaviour is episodic or cyclic. However, it is possible that average quantities can be estimated for these flows and relations based on these quantities can be used to classify their behaviour. Delay differential equations (DDEs) can exhibit solutions that asymptotically approach a self-limiting equilibrium state and those that are periodic, even when feedback is negative. Consequently these equations can be used to differentiate between these two behaviours. DDEs have been used to model biological systems with delayed negative feedback (e.g. Gurney, Blythe & Nisbet 1980; Wazewska-Czyzewska & Lasota 1988; Kulemnic, Ladas & Sficas 1989; Györi & Ladas 1991) but have seldom been applied to astrophysical systems. Previous work using coupled differential equations for stars and the different phases of the ISM found that star formation is episodic only when the delay time between cloud formation and destruction exceeds the cloud collision or condensation time-scale (Scalo & Struck-Marcell 1986; Parravano 1996). In this paper, using a DDE, we determine when cyclic or periodic behaviour is exhibited by the solutions rather than a smooth decay to a self-regulated steady state. We apply the theory to star-forming galaxies, identifying delay mechanisms that could account for episodic accretion events inferred from observations.

2 \textbf{ONE-DIMENSIONAL FEEDBACK MODELS}

We begin by considering a galactic disc model with cloud surface density \( \Sigma(t) \) in units of mass per unit area that depends on time, \( t \). This density could represent the total disc gas, or the gas in self-gravitating clouds, or the gas in molecular form, depending upon the setting. The gas density available for star formation decreases when clouds are dispersed following star formation. Conversely, the gas density increases during accretion, coagulation or cooling, all of which can enhance star formation. We therefore write

\[ \dot{\Sigma}(t) = g(\Sigma, t) - h(\Sigma, t), \]

where \( \Sigma = d\Sigma/dt \). Here the function \( g(\Sigma, t) \) is the accretion or cloud formation rate. The star formation rate should depend on the cloud density variable, \( \Sigma \) and this should determine the rate that clouds are dispersed or disrupted. A Schmidt-type law (Schmidt 1959; Kennicutt 1998) relates the cloud dispersal rate to the disc density variable with the function \( h(\Sigma) \). In principle, the accretion rate also depends on time in a non-trivial manner. For example, it could depend on the previous cloud surface density and associated star formation rate. We do not expect feedback to be instantaneous as it takes millions of years for a burst of star formation to produce type II supernovae, and winds and supernova remnants require time to evacuate gas or induce turbulence in a gas disc. Following a burst of star formation, accretion on to the disc would not resume until heated, evacuated or dispersed gas has had time to cool and reform into clouds.

Before introducing complicated functions for the accretion rate, we first consider the simplest case that lacking any feedback, \( g(x, t) = A \), corresponding to a constant accretion or cloud formation rate. The above differential equation can be written as

\[ \dot{x} = f(x) = A - B x^\alpha, \]

where we have replaced \( \Sigma \) with the variable \( x \), use a Schmidt-type law (Schmidt 1959) for the cloud destruction or dispersal rate with positive power index \( \alpha \), and \( A \) and \( B \) are positive constants. By setting \( dx/dt = 0 \) and solving for \( x \) we find a fixed point, \( x_* \), corresponding to the self-regulated steady-state or equilibrium value at \( x_* = (A/B)^{1/\alpha} \). We can assess the nature of solutions by taking the derivative of the right-hand side with respect to \( x \); or \( df/dx = -B x^{\alpha-1} \). This derivative is always negative and is smoothly decreasing function implying that solutions always smoothly (asymptotically) approach the equilibrium state solution on a time-scale determined by the inverse of this derivative. There are no oscillating or divergent solutions.

2.1 \textbf{Instantaneous feedback}

The case of instantaneous feedback can be modelled with the assumption that the accretion rate is affected by the current star formation rate, which in turn is set by the density of the disc. We expect that feedback would occur by reducing the quantity of gas available for star formation in the disc when the star formation rate is high. Since the gas quantity available to form stars is reduced by the star formation process, the feedback is negative. We can describe this situation with an accretion rate \( g(x) = AG(x) \), where \( G(x) \) is a function that approaches unity when \( x \) is small and there is no feedback, and drops to zero when \( x \) is large, star formation is vigorous and the energy arising from it has prevented further accretion or cloud formation. A simple form for the function \( G \) that satisfies our requirements is \( G(x) = e^{-x/C} \) for which \( C > 0 \). The parameter \( C \) depends on the cloud density and associated star formation rate that is effective at cutting off accretion or cloud formation.

The evolution of the disc density is then described by

\[ \dot{x} = f(x) = g(x) - h(x) = A e^{-x/C} - B x^\alpha. \]

The equilibrium state can be found by solving \( \dot{x} = 0 \) for \( x \) and satisfies

\[ x_* = \frac{A}{B} e^{-x_*/C}. \]

The derivative of \( f \) is

\[ \frac{df}{dx} = -B x^{\alpha-1} - AC^{-1} e^{-x/C}. \]

Since \( A, B, C, \alpha > 0 \) the derivative is always negative and solutions always smoothly approach the equilibrium state. Because the feedback is negative there is no instability, and no periodic or cyclic solutions exist. Solutions to this equation resemble those that do not oscillate shown in Fig. 1.

It is useful to define two time-scales, a consumption time-scale dependent only on the second term of equation (2) and evaluated at \( x_* \):

\[ t_{\text{con}} = \frac{x_*}{h(x_*)} = \frac{1}{B x_*^{\alpha-1}}. \]

and the time-scale to approach equilibrium, \( t_{\text{eq}} \), that depends on the derivative of \( f \) evaluated at \( x_* \):

\[ t_{\text{eq}} = \left( \frac{df}{dx} \right)^{-1} |_{x_*} = t_{\text{con}} (\alpha + x_*/C)^{-1}. \]

It is also useful to quantify the strength of the feedback near the equilibrium point from the sensitivity of the accretion term \( g(x) \)
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(i) The solutions lack oscillations. After some time period, solutions smoothly or asymptotically approach the stable equilibrium state.

(ii) The solutions exhibit oscillations about an equilibrium state but asymptotically approach that state.

(iii) The solutions oscillate and are attracted to a periodic function or cycle.

When oscillating solutions are present, the oscillation period is approximately four times the delay time-scale or $P \sim 4\tau$. Roughly speaking, this follows by considering the equation $x(t) = -\alpha x(t - \tau)$ that has the solution $x(t) = \sin(\omega t)$ with a period of $4\tau$ if $\omega \tau = \pi/2$. From Fig. 1 we see that the actual period displayed by the oscillating solutions is approximately $5\tau$. This is broadly consistent with the approximate estimate for the period of $4\tau$.

To facilitate classification of solutions, we transform equation (7) into dimensionless form. We let a dimensionless density variable $y = x/x_*$ and time variable $\tilde{T} = t/t_{con}$ with the deletion or consumption time-scale defined in equation (4). Using these new variables, equation (7) becomes

$$\frac{dy}{d\tilde{T}} = e^{(1 - \alpha)(\tilde{\tau} - T)s} - y^\alpha,$$

where the dimensionless parameters are

$$\tilde{\tau} = \frac{\tau}{t_{con}}, \quad S \equiv x_*/C,$$

and we have used equation (6) for the feedback strength $S$. For a given exponent, $\alpha$, equation (8) only depends on two parameters, $\tilde{\tau}$ and $S$, thus solutions of this equation can be classified based on estimates for these two ratios alone.

In the case of power index $\alpha = 1$, the above DDE equation (7) is the same as the model used to describe survival of red blood cells by Wazewska-Czyzewska & Lasota (1988). Initially positive solutions of the dimensionless version of equation (8) oscillate about the equilibrium value, $y_0 = 1$, if and only if

$$\tilde{\tau} S e^{(\tilde{\tau} - 1)} > 1,$$

(Kulenovic & Ladas 1987; Győri & Ladas 1991). The equilibrium value is a global attractor (solutions approach this value) when

$$S(1 - e^{-\tilde{T}}) < \ln 2$$

(Kulenovic et al. 1989; Győri & Ladas 1991). If this condition is not satisfied, a periodic oscillating attractor may exist.

A more generalized oscillation criterion that can be used when $\alpha \neq 1$ is

$$M_1 \equiv \tilde{\tau} S e^{(\tilde{\tau} - 1)} > 1,$$

where we have defined $M_1$ as a parameter describing the nature of the DDE. We have derived this criterion in Appendix A using the procedures rigorously described by Győri & Ladas (1991) and following the example given for the Lasota–Wazewska model. By analogy with equation (11) we guess that $y_*$ is a global attractor when

$$M_2 \equiv S(1 - e^{-x_T}) < \ln 2.$$  

The parameters $M_1, M_2 > 1$ when

$$\tilde{\tau} = \tau/t_{con} \gtrsim 1 \quad \text{and} \quad S \gtrsim 1.$$  

(12)

Using equation (5) we find that the second of these conditions is equivalent to a constraint on the time-scale to reach equilibrium

$$t_{con}/t_{eq} \gtrsim 1 + \alpha.$$
Thus when two conditions are satisfied we predict periodic solutions.

(i) The delay time-scale exceeds the consumption time-scale.

(ii) Feedback is strong and effective at shutting off accretion or cloud formation near the equilibrium density. Equivalently the time-scale to approach equilibrium exceeds the consumption time-scale by a factor similar to 2.

The first of these conditions is similar to that found by Scalo & Struck-Marcell (1986) for two coupled DDEs based on the Oort model for molecular cloud collisions. Previous work has not found an additional requirement for episodic behaviour dependent on the feedback strength.

We consider how the amplitude of oscillations depends on the parameters. We describe the amplitude of oscillations as the ratio of the maximum value divided by the minimum value of the solution has decayed either to an equilibrium value or a periodic cycle.

Figure 2. Amplitude contours are shown for solutions of equation (7) after the system has converged to a cycle. Oscillation amplitudes are shown in Fig. 2 as a function of $\tau$ and strength $S$, for index $\alpha = 1$ and 1.5. The conditions for periodic behaviour estimated in equation (12) are consistent with the lowest contours in Fig. 2 that separate between solutions that decay to the equilibrium state and those that are periodic. As expected from the form of $M_1, M_2$, this division occurs at a higher strength when $\alpha = 1.5$ than for $\alpha = 1.0$.

The further away from the line dividing asymptotically decaying solutions from those with periodic solutions, the larger the oscillations about the equilibrium value. The amplitude of the oscillations does not depend on the initial conditions but rather on the parameters defining the differential equation. Since $x$ cannot cross zero when large oscillations are present, the periodic solutions are less symmetric or less like sinusoids but exhibit spikes followed by longer periods of low periods of accretion when the amplitudes are high (see Fig. 1). This follows as the accretion rate depends on the exponential of $x$ so when $x$ is high, it can take a long time for the system to recover from a previous episode of star formation.

3 APPLICATIONS OF THE ONE-DIMENSIONAL MODEL TO GALAXIES

Our DDE model for delayed feedback is appropriate if the mean gas density averaged over long periods of time is nearly constant. This follows because the form we have for the accretion or cloud formation rate does not change, though it does depend on the past disc density. The DDE model is best applied to systems that recycle gas and only slowly remove gas from the system. Star formation laws illustrate that star formation is inefficient. For example, Kennicutt (1998) found that star formation rates in nearby galaxies could be described with $\Sigma \sim \epsilon \Sigma^2$, where $\Omega$ is the angular rotation rate and the efficiency is low, $\epsilon \sim 0.017$ (Kennicutt 1998). This suggests that we should not adopt as our defining variable the total density in molecular and atomic gas but rather that in molecular clouds or self-gravitating clouds as adopted in explanations for the Schmidt–Kennicutt star formation law and observational studies of molecular gas in galaxies (Gao & Solomon 2004; Wu et al. 2005; Blitz & Rosolowsky 2006; Krumholz et al. 2006; Robertson & Kravtsov 2008). In this case the DDE tracks cloud formation and cloud disruption following star formation. The accretion term of the DDE describes how cloud formation is depressed by star formation associated with the past cloud density.

Molecular clouds are estimated to last $t_c \sim 2–3 \times 10^7$ yr and are disrupted following star formation (Blitz et al. 2007). Theoretical work suggests that clouds disperse after a few times their free-fall or dynamical time-scale (Krumholz & McKee 2005) so lifetimes of star-forming clouds could be shorter in denser environments (Wada & Norman 2007), such as circumnuclear discs. The depletion term in equation (7) has $B = t_c^{-1} \times$ index $\alpha = 1$, so the consumption time-scale in our model is the mean cloud lifetime, $t_{\text{con}} \sim t_c$.

3.1 Possible delay time-scales

For star formation to be maintained in a disc, clouds must constantly reform. Enhanced turbulence in the disc should reduce the star formation rate (e.g. Struck & Smith 1999; Silk 2001; de Avillez & Breitschwerdt 2004; Li et al. 2006). Turbulence increases the disc thickness reducing the mean density and increasing the mean free-fall time-scale (cf. Silk 2001; Krumholz & McKee 2005; Dib et al. 2006; Joung & Mac Low 2006; McKee & Ostriker 2007; Robertson & Kravtsov 2008). The primary energy source for the turbulence is expected to be from supernovae, though differential rotation, gravitational and magnetic instabilities and stellar outflows could also play a role (Silk 2001; Kim, Ostriker & Stone 2003; Quillen et al. 2005; de Avillez & Breitschwerdt 2007; Piontek & Ostriker 2007). Thus a delay time for feedback is the sum of the time for massive stars to move off the main sequence and produce supernovae (a few $\times 10^6$ yr), the time-scale for the supernova remnants
to reach their maximum size (of the order of $10^7$ yr but depending on the ambient pressure and density), and the time-scale for them to be mixed into the disc (e.g. Dib et al. 2006). This last time-scale is a turbulence mixing time-scale, $t_{\text{mix}} \sim h/\sigma$, that depends on the gas disc thickness, $h$, and gas velocity dispersion, $\sigma$. The mixing time-scale is similar to a few $\times 10^7$ yr in the solar neighbourhood. Thus the delay time for a reduction in the rate of molecular cloud formation by turbulence in discs such as the Milky Way is a few $\times 10$ Myr and dominated by the time-scale for mixing and supernova remnant expansion. [The time-scale could be shorter if star formation is triggered by the rapid collapse of the evacuated region ($\sim 2$ Myr) shortly after the hot gas escapes the disc.] Both turbulent mixing time-scales and supernova remnant expansion time-scales should be longer in the outskirts of galaxies and in irregular or dwarf galaxies where the densities and pressures are lower. In contrast, on the scales of circumnuclear discs ($\sim 10$ pc), mixing and supernova remnant expansion time-scales should be shorter due to higher densities and pressures and larger velocity dispersions.

In spiral galaxies, molecular cloud formation occurs primarily in spiral arms with their formation being triggered on a time-scale related to the spiral density wave pattern rather than on a time-scale related to turbulent mixing of supernova remnants (e.g. Elmegreen 2007). A possible longer delay time-scale is that for spiral density waves to evolve (e.g. Clarke & Gittins 2006). When the Toomre $Q$ parameter is greater than 1.5, spiral structure is suppressed. Here $Q = \kappa/\pi G \Sigma_1$ where $\kappa$ is the epicyclic frequency and $G$ is the gravitational constant. The $Q$ parameter is related to the gas free-fall time-scale (e.g. Krumholz & McKee 2005; McKee & Ostriker 2007) and so its value can be discussed in terms of a self-regulated star formation model. Spiral density waves are expected to grow on a time-scale of a few rotation periods (Sellwood & Carlberg 1984; Clarke & Gittins 2006; Vorobyov & Theis 2006). Star formation not only influences the gaseous velocity dispersion but lowers the mean stellar velocity dispersion and increases the stellar mass density. Hence the current strength of spiral structure (set by $Q$) may depend on the star formation rate a few galactic rotation periods ago. In this setting the cloud formation rate would be forced by spiral arms sweeping through the disc with an oscillation period dependent on the spiral pattern speed and amplitude dependent on the strength of spiral structure. This amplitude would be the quantity that experiences the delayed feedback.

A third candidate for a delay time-scale is that for material driven out of the disc to return and stir the disc. This could be influenced by a cooling time-scale for hot and low-density gas in the galactic halo. This time-scale would be longer than the local disc turbulent mixing time-scale and would be of the order of $10^9$ yr. It may be related to the 100–200 Myr relaxation time-scale exhibited by simulations (de Avillez & Breitschwerdt 2004; Joung & Mac Low 2006; Stinson et al. 2007) but could also depend on the dark matter halo mass or density (as discussed in these works).

In summary, the relevant consumption time-scale is the molecular cloud lifetime of the order of 10 Myr but could be shorter in denser environments. For delay time-scales we have three primary candidates. (1) The time-scale for supernovae to enhance disc turbulence (a few $\times 10$ Myr but longer at lower densities and pressures). (2) The time-scale for gas heated up and moved into the halo to cool back into and stir the disc (of the order of $10^7$ yr). (3) The time-scale for spiral arms to evolve (a few times the rotation period). Future work may identify delay times associated with other processes such as magnetogravitational instabilities, or internally generated stellar outflows. The delay time-scale associated with disc turbulence may not exceed the cloud consumption time-scale. However, delay time-scales associated with larger scale turbulence and cooling in the halo and spiral arm evolution are likely to exceed the cloud consumption time-scale.

### 3.2 Delay mechanisms as suggested by observations

We now put these time-scales in context with observations keeping in mind that the DDE equation (7) displays episodic bursts only when the delay time-scale is longer than the consumption time-scale.

The survey by Rocha-Pinto et al. (2000a) reveals that star formation in the solar neighbourhood experienced three bursts each separated by about 3 Gyr. A delay time-scale of one quarter of this or about 0.8 Gyr would be required to predict this periodicity with the DDE of equation (7). As spiral structure is responsible for molecular cloud formation in the solar neighbourhood a possible delay mechanism is the time-scale for spiral arms to evolve. The time 0.8 Gyr corresponds to three rotation periods at the solar circle. Clarke & Gittins (2006) have previously proposed that variations in spiral arm strength could affect the star formation rate. Here we couple the gas and stars, relying on feedback and a delay time but involving the same principle, that the spiral density waves are a strong trigger for star formation.

Surveys of galaxy centres have revealed that most late-type and elliptical galaxies harbour circumnuclear star clusters (Böker et al. 2002; Christopher et al. 2005; Koda et al. 2005; Coté et al. 2006) and have experienced star formation in their nuclei in the past few to 100 Myr (Veilleux et al. 1994; Bland-Hawthorn & Cohen 2003; Quillen et al. 2006; Walcher et al. 2006; Cecil et al. 2001). The sizes of these star clusters ranges from tens to a few hundred pc and gas densities of $10^5$–$10^6$ $M_\odot$ pc$^{-2}$. Since the gas densities are high, cloud lifetimes should be shorter than that for molecular clouds in the Milky Way’s disc or Local Group galaxies. Supernova remnant expansion and turbulent mixing time-scales may be shorter than in the solar neighbourhood due to higher pressures. However, the time-scale for stars to evolve must be similar in both settings. We expect episodic star formation with a period similar to a few $\times 10^7$ yr (set by stellar evolution of massive stars). This behaviour would only occur when the time-scale for excitation of turbulence in the disc, depending on the time-scale for stars to produce winds, is longer than the lifetime of the star-forming self-gravitating clouds.

Studies of irregular dwarf galaxies have revealed that they have complex star formation histories experiencing separated bursts of star formation separated by a hundred Myr to Gyr (e.g. Tosi et al. 1991; Dohm-Palmer et al. 2002; Dolphin et al. 2003; Skillman 2005; Young et al. 2007; Dellenbusch et al. 2008). Recent simulations (Pelupessy et al. 2004; Kamaya 2005; Stinson et al. 2007) have illustrated periodic bursts of star formation separated by 200–400 Myr. The simulations do not display strong spiral structure. The spiral structure mediated model proposed by Clarke & Gittins (2006) can account for bursts of star formation in dwarf galaxies; however, this model cannot account for the bursts seen in these simulations as they lack spiral structure. According to our model, the delay time-scale must be one quarter of the time between bursts or 50–100 Myr. The supernova remnant expansion time-scale for the galaxy simulated by Pelupessy et al. (2004) is similar to that of a supernova in the solar neighbourhood as the ISM pressures are similar. Likewise turbulent mixing time-scales are similar. Hence the long inferred delay time-scale must involve longer time-scales such as for cooling of
material in the haloes of these galaxies and interactions between this cooling material and the disc.

In all three of these cases, it is likely that the delay time-scale exceeds the cloud consumption time-scale, one of the conditions for the DDE to exhibit cyclic solutions. We base our choices for the likely delay mechanism on the requirement that the delay time is related to the observational inferred time-scale between episodic events. Thus we suspect that the relevant delay time-scale accounting for episodic star formation in galaxy centres, irregular galaxies and the Milky Way disc could be that for exciting turbulence following creating of massive stars, that for gas pushed into the halo to return and interact with the disc and that for spiral density wave evolution, respectively. In all three cases, the total supply of gas is consumed only slowly leaving a reservoir for ongoing star formation. Since the feedback is delayed on a time-scale that exceeds the cloud consumption time-scale, recurrent and periodic star formation events could occur even though the feedback is negative.

3.3 Is the feedback strong enough?

We now discuss the second requirement for cyclic solutions that feedback be effective at reducing the formation rate of molecular clouds. We have characterized the feedback strength, $S$, with a parameter defined in equation (6) that describes the change in cloud formation rate caused by a change in cloud density. Only when $S \gtrsim 1$ are the solutions to the DDE periodic in behaviour. Consequently we need to estimate the change in the cloud formation rate (or star formation rate) caused by a small change in the mean gas density.

There are few studies that have considered the time-scale for cloud formation (q.v. Padoan et al. 2006). More commonly, a density spectrum resulting from turbulence has been used to predict the number of clouds above a critical density. The star formation rate is estimated from this gas fraction divided by the dynamical time-scale at that density (Elmegreen 2002; Kravtsov 2003; Krumholz & McKee 2005; Wada & Norman 2007). A nearly universal property of isothermal turbulent media in experimental and numerical simulation studies is that the cloud densities have a lognormal density distribution (Pumir 1994; Warhaft 2000; Padoan & Nordlund 2002).

We adopt this distribution\(^1\) to estimate the strength parameter $S$ in equation (6).

Stars are born primarily in the densest clumps that form as a result of turbulence within the ISM. The disc velocity dispersion is predicted to be proportional to the square root of the supernova rate (Dib et al. 2006). So the mean gas density should depend on the square root of the star formation rate. The star formation rate is estimated from the fraction of material in the densest clumps or that above a critical density. (Elmegreen 2002; Padoan & Nordlund 2002; Kravtsov 2003; Krumholz & McKee 2005; Wada & Norman 2007). The fraction of the mass with a density, $\rho$, larger than a threshold, $\rho_c$,

$$f_c = \int_{\rho_c}^{\rho_0} \frac{\rho p(\rho) d\rho}{\rho_0 p(\rho) d\rho},$$

where the normalized probability density function

$$p(u) = \left(\frac{2\pi \Delta^2}{\Delta^2}ight)^{-1/2} \exp \left[ -0.5 \left( \frac{\ln u - \ln u_0}{\Delta^2} \right)^2 \right] \frac{d\ln u}{du}.$$

\(^1\) While there is no theoretical basis for this distribution, R. Sutherland (personal communication) points out that it is a natural consequence of a turbulent cascade with multiplicative rather than additive random phases due to folding and stretching within the medium.

Here $u = \rho/\bar{\rho}$ is the density in units of the mean density and $\ln u_0$ is the mean of the normal distribution. The mean and dispersion of the normal distribution depend on the Mach number on the largest scale and are in the range 1–5 (Padoan & Nordlund 2002).

After integrating, we estimate $f_c \propto \text{erfc}[2(\ln u_{crit} - \Delta^2)/(2\Delta^2)]$ (based on equation 20 by Krumholz & McKee 2005), where the critical density ratio $u_{crit} = \rho_c/\bar{\rho}$ have used a complementary error function and assumed that the critical density ratio exceeds the mean by more than a few dispersion lengths $\Delta$. In the large asymptotic limit this becomes $f_c \sim e^{-(\ln u_{crit}/\Delta^2)}$. A change in the density ratio $u_{crit}$ leads to a change in the cloud fraction

$$S = \left| \frac{df_c}{du} \right|_{u_{crit}} \sim \frac{2\ln u_{crit}}{\Delta^2}. \tag{15}$$

The above ratio, equivalent to the strength parameter defined in equation (6), tells us how large a change in the fraction of clouds above the critical density is caused by a fractional change in the mean density. The density ratio $u_{crit}$ is estimated to be in the range of $10^{-4}$–$10^0$ (Elmegreen 2002; Krumholz & McKee 2005). For $\Delta = 2.4$ (Elmegreen 2002; Padoan & Nordlund 2002) and $u_{crit} = 10^0$, the above fraction $S \sim 4$. We expect the condition strength $S \gtrsim 1$ for our model is satisfied but that the strength is also not extremely large. For delay times exceeding the gas consumption time-scale by a moderate factor with $S \sim 4$ we would predict solutions with moderate amplitude oscillations (see Fig. 2b).

The feedback strength estimate shown in equation (15) suggests that the feedback would be weaker at higher Mach number but stronger at lower mean density, if the critical density is similar in different environments. Stinson et al. (2007) found that oscillations were lower amplitude for larger simulated dwarf galaxies. Fig. 2 showing the amplitude as a function of feedback strength and delay time-scale implies that the feedback strength would be lower for the larger simulated dwarfs because they have longer delay times and because their mean gas density is higher.

Further examination of these simulations may test the hypothesis that equation (15) describes the feedback strength and is consistent with the relationship between oscillation amplitude and feedback strength predicted by the model. The above estimate for the feedback strength is indirect as we have used a steady-state star formation rate to estimate the cloud formation rate. Time-scales displayed by simulations of the density evolution and molecular cloud formation (e.g. Glover & Mac Low 2007) might allow a better and more appropriate estimate for the feedback strength. The strength we estimate above was based on a local probability density distribution but when feedback delay is very long (such as suggested in the solar neighbourhood) the cloud formation rate should be integrated azimuthally around the galaxy and across spiral arms.

4 SUMMARY AND CONCLUSION

It is now widely recognized that a detailed understanding of feedback and accretion processes is essential to progress in many fields of astrophysics and across the entire cosmological hierarchy, from galaxy clusters down to the scales of individual star-forming regions.

To progress, we will need improvements in analytic algorithms and computer power, as well as better conceptual tools for classifying complex behaviour. Some processes may indeed be episodic or cyclic, while other instances may exhibit quasi-periodic cycles on the way to fully chaotic behaviour. A deeper understanding requires that we should to some degree be able to distinguish between these very different dynamical manifestations.
Here we have introduced a simple differential equation model that captures some of the complexity exhibited by astrophysical star-forming systems with feedback. We introduce a one-dimensional DDE for the molecular cloud density that allows cloud formation to depend on the star formation rate but at a previous time. Thus current star formation only affects the cloud distribution at a future time, we denote the delay time. The feedback is negative, so in the absence of delay there are no cyclic solutions or instabilities and all solutions asymptotically approach a self-limiting value.

We illustrate that even when the feedback is negative a delay can cause cyclic or episodic behaviour. The DDE captures phenomena exhibited by astrophysical simulations of this process, including periodic solutions in some cases but not in others. The DDE allows us to classify the solutions and predict when an astrophysical system is self-limiting or likely to exhibit periodic behaviour based on time-scales that are related to physical feedback and star formation processes.

We find that periodic behaviour is likely when two conditions are met. First, the delay time-scale must exceed the cloud consumption time-scale. Secondly, the star formation must be effective at reducing the rate of formation at densities near the self-limiting or steady-state value. This is equivalent to requiring strong feedback or to requiring that the time-scale to approach equilibrium be larger than approximately twice the cloud consumption time-scale. We find that the amplitude of the oscillations is sensitive to the feedback strength and to a lesser extent on the ratio of the delay time to the consumption time-scale.

Previous studies found a condition for episodic star formation similar to our first one that the delay time exceed a time-scale representative of the non-delayed system (Scalo & Struck-Marcell 1986; Pparevano 1996). However, these works did not explore the sensitivity of the system to feedback strength, or relate the period and amplitude of oscillations to parameters describing the system. These systems used coupled DDEs of two or more variables and six or more free parameters. Our DDE is simpler than those previously adopted. To classify solutions we require only two-dimensional estimates, $x^*$ and $t_{\text{rot}}$, that can be estimated from observations, and two dimensionless ratios, the dimensionless delay time-scale $\tau$ and feedback strength parameter $S$, set by the feedback mechanism.

We focus on the molecular or self-gravitating cloud density in a galaxy as the most likely variable for the DDE. This allows recycling of gas over long periods of time as gas is recycled through clouds much faster than it is depleted by star formation. The consumption time-scale is set by the lifetime of molecular clouds. When feedback delay times are longer than this time-scale we predict episodic star formation events and with a period approximately four times the delay time-scale.

At the present time, there are no compelling constraints on either the feedback strength or the delay time, i.e. the two key parameters of the DDE model. Thus, it is difficult to apply the model rigorously although we suggest avenues for further exploration.

There is more than one candidate for the delay time and associated feedback mechanisms, in particular, the time-scale for supernovae to contribute to turbulence, the time-scale for spiral density waves to evolve, and the time-scale for material sent into the halo to return to interact with the disc. We associate these three candidate delay mechanisms with possible explanations for episodic star formation events in galaxy centres (on 10 Myr time-scales), the solar neighbourhood (on Gyr time-scales) and dwarf galaxies (on 100 Myr time-scales), respectively. Using a lognormal density distribution we estimate that feedback is likely to be strong enough that the second condition for episodic solutions can be satisfied.

Lacking currently are simulations and observational programs that constrain the time-scales and strengths of possible feedback mechanisms and their functional form. Observational studies relating star burst amplitudes and time-scales, dynamical times, mean gas densities, turbulence and deviations from star formation laws can provide evidence for feedback induced variations in star formation rates and could be used to quantify feedback strength and form. Other forms for the feedback function could be used, such as that of the Mackey–Glass model which can exhibit chaotic behaviour (Glass & Mackey 1988). By modelling with additional variables it may be possible to model these systems without delays.

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Non-linear DDEs can have oscillating solutions when an associated linearized equation does. One condition is the requirement for oscillating solutions is
\[ \alpha > 0, \]
which is desirable because star formation laws have non-unity values for this index. Non-linear DDEs can have oscillating solutions when an associated delay linear equation does. The non-linear DDE
\[ \dot{x} + \sum_{i=1}^{n} p_i f_i(x(t - \tau_i)) = 0 \]  
(A1)
can be associated with the linearized equation
\[ \ddot{y} + \sum_{i=1}^{n} p_i y(t - \tau_i) = 0 \]  
(A2)

(Kulenovic & Ladas 1987; Györi & Ladas 1991). Here \( p_i > 0, \tau_i \geq 0, \) and the functions \( f_i \) are well-behaved continuous functions. Given additional conditions on the functions, \( f_i, \) Kulenovic & Ladas (1987), Györi & Ladas (1991) proved that every solution of the non-linear equation oscillates if and only if every solution of the associated linearized equation does. One condition is the requirement that
\[ \lim_{u \to 0} \frac{f(u)}{u} = 1. \]  
(A3)

By manipulating equation (7) and requiring the above condition, we find an associated linearized equation that is similar to that used by Kulenovic & Ladas (1987), Györi & Ladas (1991) to establish when solutions oscillate for the Lasota–Wazewska model. This associated linearized equation is in the form
\[ x(t) + p_1 x(t) + p_2 x(t - \tau) = 0. \]  
(A4)

A necessary and sufficient condition for the oscillation of all solutions of this linear DDE is
\[ p_2 e^{\tau (\alpha - 1)} > 1, \]  
(A5)
as proved by Györi & Ladas (1991) in Section 2.2. Once we find the coefficients \( p_1 \) and \( p_2 \) of the associated linearized equation, we can use the above oscillation criterion to establish when oscillating solutions exist for the original non-linear DDE.

We wish to find an associated linearized equation for the differential equation (7) restated here:
\[ x(t) = A e^{\alpha(t-\tau)/C} - B x(t)^\tau, \]  
(A6)
with equilibrium solution, \( x_*, \) given by equation (3). The change of variables
\[ x(t) = x_* + Cu(t) \]  
(A7)
leads to the delay equation
\[ \dot{u}(t) = \frac{B x_*}{C} \left[ \left( 1 + \frac{C u(t)}{x_*} \right)^\alpha - 1 \right] + \frac{B x_*}{C} \left[ 1 - e^{\alpha(t-\tau)} \right] = 0. \]  
(A8)
This can be written in the form of the linearized equation (A2) with
\[ p_1 = B x_*^{-1}, \]
\[ p_2 = \frac{B x_*}{C}, \]
\[ f_2(u) = \frac{x_*}{C} \left[ \left( 1 + \frac{C u}{x_*} \right)^\alpha - 1 \right], \]
\[ f_2(u) = 1 - e^x, \]  
(A9)
where the functions \( f_1, f_2 \) satisfy the condition shown in equation (A3). The linearized equation is then in the form of equation (A4). Inserting \( p_1 \) and \( p_2 \) into equation (A5) we find that the requirement for oscillating solutions is
\[ \frac{B x_*}{C} e^{\tau (\alpha - 1)} > 1. \]  
(A10)
This is dimensionally correct and reduces to equation (10) for the oscillation criterion for the Lasota–Wazewska model when \( \alpha = 1, \) as expected.