Derived brackets and sh Leibniz algebras

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Abstract

We will give a generalized framework for derived bracket construction. It will be shown that a deformation differential on a Leibniz algebra provides a strong homotopy (sh) Leibniz algebra structure by derived bracket construction. A relationship between the three concepts, i.e., homotopy algebra theory, deformation theory and derived bracket construction, will be discussed. We will prove that the derived bracket construction is a map from the equivalence classes of deformation theory to the one of sh Leibniz algebras.

1 Introduction.

Let \((V, d, [, ])\) be a differential graded (dg) vector space, or a complex equipped with a binary bracket product. It is called a dg Leibniz algebra, or sometimes called a dg Loday algebra, if the differential is a derivation with respect to the bracket product and the bracket product satisfies a graded Leibniz identity. When the bracket is skewsymmetric, or graded commutative, the Leibniz identity is equivalent with a Jacobi identity. Hence a (dg) Leibniz algebra is considered as a noncommutative version of classical (dg) Lie algebra.

Let \((V, d, [, ])\) be a dg Leibniz algebra. We define a modified bracket by \([x, y]_d := \pm [dx, y]\), where \(\pm\) is an appropriate sign and \(x, y \in V\). In Kosmann-Schwarzbach [5], it was shown that the new bracket also satisfies a Leibniz identity. This modified bracket is called a derived bracket. (The original idea of derived bracket was given by Koszul, cf. [18]). The derived brackets play important roles in modern analytical mechanics (cf. [6], Roytenberg [16]). For instance, a Poisson bracket on a smooth manifold is given as a derived bracket \(\{f, g\} := [df, g]\), where \(f, g\) are smooth functions, \([, ]\) is a Schouten-Nijenhuis bracket and \(d\) is a coboundary operator of Poisson cohomology. It is known that the Schouten-Nijenhuis bracket is also

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a derived bracket of a certain graded Poisson bracket. Namely, there is a hierarchy of derived brackets. This hierarchy is closely related with a hierarchy of various Hamiltonian formalisms (classical Hamiltonian-, BV-, AKSZ-formalism and so on).

In general, even if a first bracket is Lie, the derived bracket is not skewsymmetric, and, in the case of $dd \neq 0$, the derived bracket has a Leibniz anomaly. Usually, this anomaly is controlled by some cocycle conditions. It is well-known that a certain collection of derived brackets becomes a strong homotopy Lie (sh Lie- or $L_\infty$-) algebra structure, under some good assumptions (see [15]). In Voronov [19], he introduced a new notion, derived bracket up to projection (so-called higher derived bracket). It was shown that a collection of Voronov’s derived brackets also generates a strong homotopy Lie algebra. In Vallejo [18], he researched an $n$-ary derived bracket of differential forms, along Koszul’s original theory. He gave a necessary and sufficient condition for an $n$-ary derived bracket becomes a Nambu-Lie bracket.

As a generalized framework for derived bracket construction, we will consider sh Leibniz algebras ($Leibniz_\infty$-algebras or sh Loday algebras or $Loday_{\infty}$-algebras). It is a homotopy version of Leibniz algebra, and it is considered as a noncommutative version of sh Lie algebra. An explicit construction of sh Leibniz algebras was given by Ammar and Poncin [1]. One can find a geometric example of sh Leibniz algebra in [17]. We will prove that a deformation differential of dg Leibniz algebra induces a sh Leibniz algebra structure by an extended derived bracket construction (Theorem 3.4 below). This result is considered as a complete version of the classical derived bracket construction in [5] [15] [18]. The theorem is followed from more general result, Lemma 4.4 below, as a corollary.

In Section 5, a relationship between homotopy algebra theory and deformation theory will be cleared. In Proposition 5.1, we will prove that if two deformation differentials are equivalent via a gauge transformation, then the induced sh Leibniz algebras are also equivalent, in other words, the derived bracket construction is a gauge invariance.

In Section 6, we will give a proof of Lemma 4.4.

Remark. In Loday and collaborators works [2, 9, 10, 11], they study right Leibniz algebras. In the following, we study the left version, or opposite Leibniz algebras. Hence we should translate their results to the left version.

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2 Preliminaries

2.1 Notations and Assumptions

In the following, we assume that the characteristic of a ground field $\mathbb{K}$ is zero and that a tensor product is defined over the field, $\otimes := \otimes_{\mathbb{K}}$. The mathematics of graded linear algebra is due to Koszul sign convention. For instance, a linear map $f \otimes g : V \otimes V \to V \otimes V$ satisfies, for any $x \otimes y \in V \otimes V$,

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),$$

where $|g|$ and $|x|$ are degrees of $g$ and $x$. We will use a degree shifting operator, which is denoted by $s (s^{-1})$, with degree +1 (−1). The Koszul sign convention for shifting operators is, for instance,

$$s \otimes s = (s \otimes 1)(1 \otimes s) = -(1 \otimes s)(s \otimes 1).$$

We assume that a graded vector space is a complex. We say a square zero derivation a differential.

2.2 Unshuffle permutations

Let $(x_1, ..., x_n)$ be a sentence composed of $n$-words. By definition, an $(i, n - i)$-unshuffle permutation is

$$(x_{\sigma(1)}, ..., x_{\sigma(i)}, x_{\sigma(i+1)}, ..., x_{\sigma(n)}),$$

where $\sigma \in S_n$ such that

$$\sigma(1) < ... < \sigma(i), \quad \sigma(i + 1) < ... < \sigma(n).$$

In next section, we will use partial unshuffle permutations. Namely, for a given sentence $(x_1, ..., x_n)$,

$$(x_{\tau(1)}, ..., x_{\tau(i)}, x_{\tau(i+1)}, ..., x_{\tau(k)}, x_{k+1}, ..., x_n),$$

where $\tau$ is an $(i, k - i)$-unshuffle permutation.

2.3 Leibniz algebras and derived brackets

Let $(V, d, [,])$ be a differential graded (dg) vector space, or a complex equipped with a binary bracket product. We assume that the degree of differential is +1 and the one of bracket is 0. The space is called a dg Leibniz algebra or sometimes called
a dg Loday algebra, if \( d \) is a graded derivation with respect to \([ , ]\) and the bracket satisfies a graded Leibniz identity,

\[
\begin{align*}
    d[x, y] &= [dx, y] + (-1)^{|x|}[x, dy], \\
    [x, [y, z]] &= [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],
\end{align*}
\]

where \( x, y, z \in V \), \(| \cdot |\) means the degree of element. A dg Lie algebra is a special Leibniz algebra such that the bracket is graded commutative, or skewsymmetric. In this sense, (dg) Leibniz algebras are considered as noncommutative version of (dg) Lie algebras.

In the following, we denote \((-1)^{|x|}\) by simply \((-1)^x\), without miss reading.

We recall classical derived bracket construction in [5, 6]. Define a new bracket product on the shifted space \( sV \) by

\[
    [sx, sy]_d := (-1)^{x}s[dx, y].
\]

This bracket is called a derived bracket on \( sV \). The sign \((-1)^x\) is given, via the Koszul sign convention, by the identity,

\[
    [sx, sy]_d = s[\cdot, \cdot](s^{-1} \otimes s^{-1})(sds^{-1} \otimes 1)(sx \otimes sy).
\]

We recall standard two propositions.

- The derived bracket also satisfies the graded Leibniz identity,

\[
    [sx, [sy, sz]] = [[sx, sy], sz] + (-1)^{(x+1)(y+1)}[sy, [sx, sz]].
\]

We consider the case of dg Lie algebra.

- Let \((V, d, [ , ])\) be a dg Lie algebra and let \( g(\subset V) \) an abelian, or trivial subalgebra of the Lie algebra. If the derived bracket is closed on \( s g \), then it is still Lie on \( s g \), because for any \( x, y \in g \),

\[
\begin{align*}
    (-1)^x[dx, y] &= (-1)^x\left(d[x, y] - (-1)^x[x, dy]\right) \\
               &= -[dx, y] \\
               &= (-1)^{x(y+1)}[dy, x] \\
               &= (-1)^{(x+1)(y+1)}(-1)^y[dy, x].
\end{align*}
\]

In next section, we will give a generalized version of the two propositions.
3 Main results

Let $V$ be a graded vector space and let $l_i : V^\otimes i \to V$ be an $i$-ary multilinear map with degree $2 - i$, for any $i \geq 1$.

**Definition 3.1.** ([1]) The system $(V, l_1, l_2, ...)$ is called sh Leibniz algebra, when (1) below holds.

$$\sum_{i+j=\text{Const}} \sum_{k \geq j} \chi(\sigma)(-1)^{(k+1-j)(j-1)}(-1)^j(x_{\sigma(1)} + \ldots + x_{\sigma(k-j)})$$

$$l_i(x_{\sigma(1)}, ..., x_{\sigma(k-j)}, l_j(x_{\sigma(k+1-j)}, ..., x_{\sigma(k-1)}, x_k), x_{k+1}, ..., x_{i+j-1}) = 0,$$

where $(x_1, ..., x_{i+j-1}) \in V^\otimes (i+j-1)$, $\sigma$ is $(k-j, j-1)$-unshuffle, $\chi(\sigma)$ is an anti-Koszul sign, $\chi(\sigma) := \text{sgn}(\sigma)\epsilon(\sigma)$.

Sh Lie algebras are special examples of sh Leibniz algebras such that all $l_i$ ($i \geq 2$) skewsymetric.

Let $(V, \delta, [\cdot, \cdot])$ be a dg Leibniz algebra with differential $\delta$, $|\delta| := +1$. We assume a deformation of differential,

$$d = \delta + t\delta_1 + t^2\delta_2 + ...$$

Here $d$ is a differential on $V[[t]]$, which is a Leibniz algebra of formal series with coefficients in $V$. The square zero condition $dd = 0$ is equivalent with

$$\sum_{i+j=\text{Const}} \delta_i\delta_j = 0.$$  

(2)

We define an $i$-ary bracket product by

$$[x_1, ..., x_i] := \ldots ([x_1, x_2], x_3], ..., x_i].$$

It is well-known that the $i$-ary bracket satisfies an $i$-ary Leibniz identity, so-called Nambu-Leibniz identity (cf. [2]). We denote by $N_i$ the $i$-ary bracket,

$$N_i(x_1, ..., x_i) := [x_1, ..., x_i].$$

**Definition 3.2.** An $i(\geq 1)$-ary derived bracket on $sV$:

$$[sx_1, ..., sx_i]_d := (-1)^\frac{i(i-1)(i-2)}{2}s \circ N_i \circ s^{-1}(i) \circ (s_{i-1}s^{-1} \otimes 1),$$

(3)

where $s^{-1}(i) = s^{-1} \otimes \ldots \otimes s^{-1}$, $1 = 1 \otimes \ldots \otimes 1$. 

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Remark that \([d]_d = s\delta_0 s^{-1}\).

It is clear that the degree of \(i\)-ary derived bracket is \(2 - i\) on \(sV\). We see an explicit expression of derived brackets.

**Proposition 3.3.** The derived bracket has the following form on \(V\),

\[
(\pm)[\delta_{i-1}x_1,\ldots,x_i] = s^{-1}[sx_1,\ldots,sx_i]_d,
\]

where

\[
\pm = \begin{cases} 
(-1)^{x_1+x_3+\ldots+x_{2n+1}} & i = \text{even}, \\
(-1)^{x_2+x_4+\ldots+x_{2n}} & i = \text{odd}.
\end{cases}
\]

**Proof.** \([sx_1,\ldots,sx_i]_d = (-1)^{(i-1)(i-2)/2} s \circ N_i \circ s^{-1}(i) \circ (s\delta s^{-1} \otimes 1)(sx_1 \otimes \ldots \otimes sx_i) =
\]

\[
= (\pm)(-1)^{(i-1)(i-2)/2} s \circ N_i \circ s^{-1}(i) \circ (s\delta s^{-1} \otimes 1) \circ s(i)(x_1 \otimes \ldots \otimes x_i) =
\]

\[
= (\pm)(-1)^{(i-1)(i-2)/2} s \circ N_i \circ s^{-1}(i) \circ (s\delta \otimes s(i-1))(x_1 \otimes \ldots \otimes x_i) =
\]

\[
= (\pm)(-1)^{(i-1)(i-2)/2}(-1)^{(i-1)/2} s \circ N_i \circ s^{-1}(i) \circ s(i)(\delta x_1 \otimes \ldots \otimes x_i) =
\]

\[
= (\pm)(-1)^{(i-1)(i-2)/2}(-1)^{(i-1)/2}(-1)^{(i-1)/2} s \circ N_i(\delta x_1 \otimes \ldots \otimes x_i) =
\]

\[
= (\pm)s[\delta x_1,\ldots,x_i],
\]

where \(s(i)\) is defined by the same manner as \(s^{-1}(i)\). \(\square\)

The main result of this note is as follows.

**Theorem 3.4.** The system \((sV,[\cdot]_d,[\cdot,\cdot]_d,\ldots)\) becomes a sh Leibniz algebra.

We will give a proof of the theorem in next section.

**Corollary 3.5.** In Theorem 3.4, if \(V\) is a dg Lie algebra and if \(g \subset V\) is an abelian subalgebra and if \(sg\) is a subalgebra of the induced sh Leibniz algebra, then \(sg\) becomes a sh Lie algebra.

**Example 3.6.** (Deformation theory, cf [3]) Let \((V,\delta_0,[\cdot,\cdot])\) be a dg Lie algebra with a Maurer-Cartan (MC) element \(\theta(t) := t\theta_1 + t^2\theta_2 + \ldots\), which is a solution of

\[
\delta_0\theta(t) + \frac{1}{2}[\theta(t),\theta(t)] = 0.
\]

We put \(\delta_i(-) := [\theta_i,-]\) for any \(i \geq 1\). Then the MC equation implies the condition (2). Thus an algebraic deformation theory provides a sh Leibniz algebra structure via the derived bracket construction.
4 Proof of Theorem 3.4

The theorem is followed from more general result (Lemma 4.4 below). We need to recall an alternative definition of sh Leibniz algebras.

4.1 Bar/coalgebra construction

It is well-known that sh Leibniz algebra structures are equivalent with codifferentials on the cofree nilpotent (Koszul-)dual-Leibniz coalgebra. See [4, 7, 8, 11, 14], for general theory of homotopy algebras.

First we recall the notion of dual-Leibniz coalgebra. By definition, a dual-Leibniz coalgebra is a (graded) vector space equipped with a comultiplication, $\Delta$, satisfying the identity below.

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta + (\sigma \otimes 1)(\Delta \otimes 1)\Delta$$

where $\sigma(\neq 1) \in S_2$. Let $V$ be a graded vector space. We put,

$$\bar{TV} := V \oplus V^2 \oplus V^3 \oplus ....$$

Proposition 4.1. ([1]) Define a comultiplication, $\Delta : \bar{TV} \rightarrow \bar{TV} \otimes \bar{TV}$, by

$$\Delta(x_1, \ldots, x_{n+1}) := \sum_{i \geq 1} \sum_{\sigma} \epsilon(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}) \otimes (x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}, x_{n+1}),$$

where $\epsilon(\sigma)$ is a Koszul sign, $\sigma$ is $(i, n - i)$-unshuffle and $(x_1, \ldots, x_{n+1}) \in V^{\otimes (n+1)} \subset \bar{TV}$. Then $(\bar{TV}, \Delta)$ becomes a cofree nilpotent dual-Leibniz coalgebra.

Let $\text{Coder}(\bar{TV})$ be the space of coderivations with respect to the coalgebra structure, i.e., $D^c \in \text{Coder}(\bar{TV})$ is satisfying,

$$\Delta D^c = (D^c \otimes 1)\Delta + (1 \otimes D^c)\Delta.$$

We recall a well-known proposition.

Proposition 4.2. ([1]) $\text{Coder}(\bar{TV}) \cong \text{Hom}(\bar{TV}, V)$.

For our aim, an explicit formula of the isomorphism is needed. Let $f : V^\otimes i \rightarrow V$ be an $i$-ary linear map. It is the one of generators in $\text{Hom}(\bar{TV}, V)$. The coderivation associated with $f$ is defined by $f(V^{\otimes n < i}) := 0$ and

$$f^c(x_1, \ldots, x_{n\geq i}) := \sum_{k \geq i} \sum_{\sigma} \epsilon(\sigma)(-1)^{|f|}(x_{\sigma(1)} + \ldots + x_{\sigma(k-i)})$$

$$\cdot (x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, f(x_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_n), \quad (5)$$
where $\sigma$ is $(k - i, i - 1)$-unshuffle. The inverse of $f \mapsto f^c$ is the restriction.

The space of coderivations has a canonical Lie bracket of commutator. If $f, g$ are $i$-ary, $j$-ary multilinear maps respectively, then the Lie bracket $[f^c, g^c]$ is also the associated coderivation with an $(i + j - 1)$-ary map which is denoted by $\{f, g\}$, namely, $[f^c, g^c] = \{f, g\}^c$. Since the mapping $f \mapsto f^c$ is an isomorphism, $\{f, g\}$ is also a Lie bracket. Thus Coder($\bar{T}V$) is identified with Hom($\bar{T}V, V$) as a Lie algebra. Via the isomorphism, a sh Leibniz structure $\{l_i\}_{i \in \mathbb{N}}$ corresponds with a collection of coderivations $\{\partial_i\}_{i \in \mathbb{N}}$. The following proposition provides an alternative definition of sh Leibniz algebras.

**Proposition 4.3.** ([1]) Let $sV$ be a (shifted) graded vector space with a collection of $i$-ary multilinear maps with degree $2 - i$, $\{l_i\}_{i \in \mathbb{N}}$. Remark that $l_i$ is an element in Hom($\bar{T}sV, sV$). We consider the shifted maps,

\[ \partial_i := s^{-1} \circ l_i \circ (s \otimes \ldots \otimes s), \]

for any $i$. Since $\bar{T}V \cong \bar{T}s^{-1}(sV)$, $\partial_i$ is an element in Hom($\bar{T}V, V$) and thus it is in Coder($\bar{T}V$). The degree of $\partial_i$ is $+1$ for any $i$. Define a coderivation as a perturbation,

\[ \partial := \partial_1 + \partial_2 + \ldots. \]

The system $(sV, l_1, l_2, \ldots)$ is a sh Leibniz algebra if and only if

\[ \frac{1}{2} [\partial, \partial] = 0 \]

or equivalently, $\partial \partial = 0$. (More correctly, $[\partial^c, \partial^c] = \{\partial, \partial\}^c = 0$.)

In the following, we will identify the Lie algebra Coder($\bar{T}V$) with Hom($\bar{T}V, V$). Hence we omit the subscript “$c$” from $f^c$.

### 4.2 Key Lemma

We consider the Leibniz algebra $(V, [,])$. Let Der($V$) be the space of derivations with respect to the Leibniz bracket. We define a collection of maps, for any $i \geq 1$,

\[ \text{Der}(V) \rightarrow \text{Coder}(\bar{T}V), \quad D \mapsto N_i D, \]

where $N_i D$ is defined, up to the isomorphism Hom($\bar{T}V, V$) $\cong$ Coder($\bar{T}V$), as an $i$-ary multiplication,

\[ N_i D(x_1, \ldots, x_i) := [D(x_1), x_2, \ldots, x_i]. \]

Remark that $N_1 D = D$. The theorem is a corollary of the key lemma:
Lemma 4.4. For any derivations \( D, D' \in \text{Der}(V) \) and for any \( i, j \geq 1 \), the following identity holds.

\[
N_{i+j-1}[D, D'] = [N_i D, N_j D'],
\]

where the brackets are both Lie bracket of graded commutator.

Proof. We show the case of \( i = 1 \). The general case will be shown in Section 6. Assume \( i = 1 \). For any \((x_1, \ldots, x_j) \in V^\otimes j\), \( N_j[D, D'] = \)

\[
[[D, D'](x_1), \ldots, x_j] = [DD'(x_1), \ldots, x_j] - (-1)^{DD'}[D'D(x_1), \ldots, x_j].
\]  

(6)

We consider the first term in (6), which has the following form.

\[
[DD'(x_1), \ldots, x_j] = [...DD'(x_1), x_2], \ldots, x_j].
\]

By the derivation rule of \( D \), it is modified with

\[
 [...D[D'(x_1), x_2], \ldots, x_j] - (-1)^{DD'}[D'D(x_1), Dx_2], \ldots, x_j]
\]

Thus (6) is equal with

\[
[[D, D'](x_1), \ldots, x_j] = [...D[D'(x_1), x_2], \ldots, x_j]
\]

\[
- (-1)^{DD'}[D'D(x_1), \ldots, x_j].
\]

(7)

Similar way, the first term in (7) is modified with

\[
 [...D[D'(x_1), x_2], \ldots, x_j] = [...D[[D'(x_1), x_2], x_3], \ldots, x_j] - (-1)^{DD'}[D'D(x_1), Dx_2, Dx_3], \ldots, x_j].
\]

We repeat this modification. Finally, we obtain

\[
[[D, D'](x_1), \ldots, x_j] = D[D', \ldots, x_j] - (-1)^{DD'} \sum_{k=1}^{j} (-1)^{D(x_1+\ldots+x_{k-1})} [D'D(x_1), \ldots, x_{k-1}, Dx_k, x_{k+1}, \ldots, x_j],
\]

(8)

which is equal with \( N_j[D, D'] = [N_1 D, N_j D'] \). \( \square \)

The derived brackets are elements in \( \text{Hom}(\bar{T}sV, sV) \). Thus they correspond with coderivations in \( \text{Coder}(\bar{T}V) \), via the maps,

\( \text{Hom}(\bar{T}sV, sV)^{\text{shift}} \sim \text{Hom}(\bar{T}V, V) \cong \text{Coder}(\bar{T}V) \).

Lemma 4.5. Let \( \partial_i \) be the coderivation associated with the \( i \)-ary derived bracket. For any \( i \geq 1 \), it has the following form.

\[
\partial_i = N_i \delta_{i-1}.
\]
Proof. Up to the identification \( \text{Hom}(\bar{T}V, V) \cong \text{Coder}(\bar{T}V) \), \( \partial_i \) is defined by
\[
\partial_i := s^{-1} \circ [\ldots i\text{-ary} \ldots]_d \circ (s \otimes \ldots \otimes s).
\]

We directly have
\[
\partial_i := s^{-1} \circ [\ldots i\text{-ary} \ldots]_d \circ (s \otimes \ldots \otimes s)
\]
\[
= (-1)^{(i-1)(i-2)/2} N_i \circ (s^{-1} \otimes \ldots \otimes s^{-1}) \circ (s \delta_{i-1} \otimes s \otimes \ldots \otimes s)
\]
\[
= (-1)^{(i-1)(i-2)/2} N_i \circ (\delta_{i-1} \otimes s^{-1} \otimes \ldots \otimes s^{-1}) \circ (1 \otimes s \otimes \ldots \otimes s)
\]
\[
= N_i \delta_{i-1}.
\]

\[
\square
\]

We give a proof of Theorem 3.4 here.

Proof. By Lemma 4.5, the deformation derivation \( d = \delta_0 + t\partial_1 + t^2\partial_2 + \ldots \) corresponds with a perturbation,
\[
\partial := \partial_1 + \partial_2 + \partial_3 + \ldots
\]

By Lemmas 4.4 the deformation condition \( [d, d]/2 = 0 \) corresponds with the homotopy algebra condition,
\[
\sum_{i+j=\text{Const}} [\partial_i, \partial_j] = \sum_{i+j=\text{Const}} [N_i \delta_{i-1}, N_j \delta_{j-1}] = N_{i+j-1} \sum_{i+j=\text{Const}} [\delta_{i-1}, \delta_{j-1}] = 0.
\]

\[
\square
\]

Remark 4.6. (On Lemma 4.4) We consider the case of \( \partial_1 \neq 2 = 0 \). In the case, the sh Leibniz algebra is the usual Leibniz algebra. We put \( CL^n(V) := \text{Hom}(\bar{T}^nV, V) \) and \( b(-) := [\partial_2, -] \). Then \( (CL^*(V), b) \) is a complex of Leibniz cohomology ([9]). The key Lemma implies that \( \text{Der}(V) \) provides a subcomplex,
\[
N_i \text{Der}(V) \subset CL^i(V),
\]

because \( [\partial_2, N_iD] = N_{i+1} [\delta_1, D] \).

5 Deformation theory

Finally, we discuss a relationship between deformation theory and sh Leibniz algebras. The deformation differential, \( d = \delta_0 + t\delta_1 + \ldots \), is considered as a differential on \( V[[t]] \) which is a Leibniz algebra of formal series with coefficients in \( V \). Let
Let $th \in \text{Der}(V[[t]])$ be a derivation with degree 0. Then an equivalence deformation is defined by

$$d' := \exp(X_{th})(d),$$

where $X_{th} := [\cdot, th]$. By a standard argument, $d'$ is also a differential, which is the formal sum of $\delta'_i$,

$$\begin{align*}
\delta'_0 &= \delta_0, \\
\delta'_1 &= \delta_1 + [\delta_0, h], \\
\delta'_2 &= \delta_2 + [\delta_1, h] + \frac{1}{2!}[[\delta_0, h], h], \\
\vdots & \vdots \\
\delta'_i &= \sum_{n=0}^{i} \frac{1}{(i-n)!} X_h^{i-n}(\delta_n).
\end{align*}$$

The collection $\{\delta'_i\}_{i \in \mathbb{N}}$ also induces a sh Leibniz algebra structure $\partial' = \sum \partial'_i$. From Lemmas 4.4, 4.5, we have

$$\partial'_{i+1} = N_i \delta'_i = \sum_{n=0}^{i} \frac{1}{(i-n)!} X_h^{i-n}(\partial_{n+1}).$$

Thus we obtain

$$\partial' = \exp(X_{Nh})(\partial),$$

which implies an equivalency of $\partial$ and $\partial'$. We consider a general case. Let $h(t) := th_1 + t^2h_2 + \ldots$ be a derivation on the Leibniz algebra $V[[t]]$ with degree $|h(t)| := 0$. By definition, a gauge transformation on deformation differentials is the transformation,

$$d' := \exp(X_{h(t)})(d). \quad (9)$$

**Proposition 5.1.** (I) If two deformation differentials are equivalent, or related via the gauge transformation, then the induced sh Leibniz algebra structures are also so, i.e., the codifferential $\partial'$ which is induced by $d'$ is related with $\partial$ via the transformation,

$$\partial' = \exp(X_{Nh})(\partial), \quad (10)$$

where $Nh$ is a well-defined infinite sum,

$$Nh := N_2h_1 + N_3h_2 + \ldots + N_{i+1}h_i + \ldots$$

(II) An integral of $Nh$,

$$e^{Nh} := 1 + Nh + \frac{1}{2!}(Nh)^2 + \ldots$$
is a dg coalgebra isomorphism between \((\bar{T}V, \partial)\) and \((\bar{T}V, \partial')\), namely, (11) and (12) below hold.

\[
\partial' = e^{-Nh} \cdot \partial \cdot e^{Nh}, \tag{11}
\]
\[
\Delta e^{Nh} = (e^{Nh} \otimes e^{Nh}) \Delta. \tag{12}
\]

The notion of sh Leibniz algebra homomorphism is defined as a map satisfying (11) and (12). Thus (II) says that \(e^{Nh}\) is a sh Leibniz algebra isomorphism.

**Proof.** (I) From (9) we have

\[
\delta_{n+1}' = \delta_n + \sum_{n=i+j} [\delta_i, h_j] + \frac{1}{2!} \sum_{n=i+j+k} [[\delta_i, h_j], h_k] + ....
\]

Thus \(\delta_{n+1}' = N_{n+1} \delta'_n\) =

\[
N_{n+1} \delta_n + \sum_{n=i+j} N_{n+1} [\delta_i, h_j] + \frac{1}{2!} \sum_{n=i+j+k} N_{n+1} [[\delta_i, h_j], h_k] + ... =
\]
\[
\partial_{n+1} + \sum_{n=i+j} [\delta_{i+1}, N_{j+1} h_j] + \frac{1}{2!} \sum_{n=i+j+k} [[\delta_{i+1}, N_{j+1} h_j], N_{k+1} h_k] + ....
\]

This gives (10).

(II) The integral \(e^{Nh}\) is well-defined as an isomorphism on \(\bar{T}V\), because \(e^{Nh}\) is finite on \(V \otimes^n\) for any \(n\). For instance, on \(V \otimes^3\),

\[
e^{Nh} \equiv 1 + (N_2 h_1 + N_3 h_2) + \frac{1}{2} (N_2 h_1)^2.
\]

By a direct computation, we have

\[
\exp(X_{Nh})(\partial) = e^{-Nh} \cdot \partial \cdot e^{Nh}.
\]

Thus (11) holds. Since \(Nh\) is coderivation, \(e^{Nh}\) satisfies (12). The proof is completed. \(\square\)

## 6 Proof of Lemma 4.4

**Claim 6.1.** Let \(f : V \otimes^i \to V\) be an \(i\)-ary linear map. It is identified with the coderivation, recall (5). We put

\[
f^{(k)}(x_1, \ldots, x_n) := \sum_{\sigma} \epsilon(\sigma)(-1)^{\lfloor f(x_{\sigma(1)} + \ldots + x_{\sigma(k-i)})\rfloor}
\]
\[
(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, f(x_{\sigma(k+1-i)}), \ldots, x_{\sigma(k-1)}, x_k, x_{k+1}, \ldots, x_n).
\]
Then we have the decomposition of coderivation,

\[ f^c = \sum_{k \geq i} f^{(k)}. \]

In Section 4.2, we showed the lemma under the assumption of \( i = 1 \). By induction, we assume the identity of the lemma and prove the case of \( i + 1 \):

\[ N_{i+j}[D, D'] = [N_{i+1}D, N_jD']. \]

We put \( x := (x_1, \ldots, x_{i+j-1}) \). From the definition of \( ND \), we have

\[ N_{i+j}[D, D'](x, x_{i+j}) = [N_{i+j-1}[D, D'](x), x_{i+j}]. \]

From the assumption of induction, we have

\[
N_{i+j}[D, D'](x, x_{i+j}) = \left[ [N_iD, N_jD'](x), x_{i+j} \right] = \left[ N_iD \circ N_jD'(x), x_{i+j} \right] - (-1)^{DD'} [N_jD' \circ N_iD(x), x_{i+j}].
\]

We use the decomposition above for \( N_jD' \),

\[ N_jD' = \sum_{k \geq j} N_j^{(k)}D'. \]

We have

\[ N_{i+j}[D, D'](x, x_{i+j}) = \sum_{k=j}^{i+j-1} [N_iD \circ N_j^{(k)}D'(x), x_{i+j}] - (-1)^{DD'} [N_jD' \circ N_iD(x), x_{i+j}]. \]

We obtain

\[
\sum_{k=j}^{i+j-1} [N_iD \circ N_j^{(k)}D'(x), x_{i+j}] = \sum_{k=j}^{i+j-1} N_{i+1}D \circ N_j^{(k)}D'(x, x_{i+j})
\]

\[ = N_{i+1}D \circ N_jD'(x, x_{i+j}) - N_{i+1}D \circ N_j^{(i+j)}D'(x, x_{i+j}). \]

because the coderivation preserves the position of the most right component \( x_{i+j} \).

So it suffices to show that

\[ -(-1)^{DD'} [N_jD' \circ N_iD(x), x_{i+j}] =
\]

\[ N_{i+1}D \circ N_j^{(i+j)}D'(x, x_{i+j}) - (-1)^{DD'} N_jD' \circ N_{i+1}D(x, x_{i+j}). \quad (13) \]

We need a lemma.

**Lemma 6.2.** For any elements \( A, B, y_1, \ldots, y_n \in V \),

\[
[A, B, y_1, \ldots, y_n] = (-1)^{AB}[B, [A, y_1, \ldots, y_n]] + \sum_{a=1}^{n} (-1)^{B(y_1+\ldots+y_{a-1})}[A, y_1, \ldots, y_{a-1}, [B, y_a], y_{a+1}, \ldots, y_n].
\]
Proof. Immediately.

We show (13). By the definition of coderivation,

\[ N_i D(x) = \sum_{k \geq i} \sum_{\sigma} E(\sigma, k-i)(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}), [Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k], x_{k+1}, \ldots, x_{i+j-1}], \]

where

\[ E(\sigma, *) := \epsilon(\sigma)(-1)^D(x_{\sigma(1)} + \ldots + x_{\sigma(*)}). \]

We have \((-1)^{DD'} [N_j D' \circ N_i D(x), x_{i+j}] = \)

\[ -(-1)^{DD'} \sum_{k \geq i} \sum_{\sigma} E(\sigma, k-i)\{[D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}], [Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k], x_{k+1}, \ldots, x_{i+j}]\}, \]

(14)

where \([I, \ldots, F] = [[I, \ldots, -], \ldots, F] is used. We put \(A := [D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}]\) and 

\(B := [Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k].\) From Lemma 6.2, we have

(14) = (15) + (16),

where

\[ -(-1)^{DD'} \sum_{k \geq i} \sum_{\sigma} E(\sigma, k-i)E_1[Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k, [D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, x_{k+1}, \ldots, x_{i+j}]\} \]

(15)

\[ -(-1)^{DD'} \sum_{k \geq i} \sum_{\sigma} \sum_{a=1}^{i-j-k} E(\sigma, k-i)E_2 [D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, x_{k+1}, \ldots, x_{k+a-1}, [Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k, x_{k+a}], x_{k+a+1}, \ldots, x_{i+j}], \]

(16)

where \(E_1\) and \(E_2\) are appropriate signs given by the manner in the lemma above.

(I) We show the identity,

(15) = \(N_{i+1} D \circ N_j^{(i+j)} D'(x, x_{i+j}).\)

We replace \(\sigma\) in (15) with an unshuffle permutation \(\tau\) along the table,

| \(\sigma(k+1-i)\) | \(\ldots\) | \(\sigma(k-1)\) | \(k\) | \(\sigma(1)\) | \(\ldots\) | \(\sigma(k-i)\) |
|-------------------|---------|----------------|-----|----------------|---------|----------------|
| \(\tau(1)\)       | \(\ldots\) | \(\tau(i-1)\) | \(\tau(i)\) | \(\tau(i+1)\) | \(\ldots\) | \(\tau(k)\) |
Then Koszul sign is also replaced with $\epsilon(\tau)$ below.

$$\epsilon(\tau) = \epsilon(\sigma)(-1)^{x_\sigma(1)+\ldots+x_\sigma(k-i)+x_k}.$$  

We have $E(\sigma, k-i)E_1 =$

$$= -\epsilon(\sigma)(-1)^{D(x_\sigma(1)+\ldots+x_\sigma(k-i))(-1)^{AB}}$$

$$= -\epsilon(\sigma)(-1)^{D(x_\sigma(1)+\ldots+x_\sigma(k-i))(-1)^{x_\sigma(1)+\ldots+x_\sigma(k-i)+x_k+D}}$$

$$= -\epsilon(\sigma)(-1)^{x_\sigma(1)+\ldots+x_\sigma(k-i)+x_k}(1)^{D(x_\sigma(1)+\ldots+x_\sigma(k-i)+x_k+D)D'}$$

$$= -\epsilon(\epsilon)(-1)^{D'(x_\sigma(k+1-i)+x_k+D)D'}$$

$$= -\epsilon(\epsilon)(-1)^{D'(x_\sigma(1)+\ldots+x_\sigma(i)+x_\tau(i))}D' = -E'(\tau, i)(-1)^{DD'}.$$  

Thus (15) is equal with

$$\sum_{k \geq i} \sum_{\tau} E'(\tau, i)[Dx_\tau(1), \ldots, x_\tau(i-1), x_\tau(i)=k, [D'x_\tau(1), \ldots, x_\tau(k), x_{k+1}, \ldots, x_{i+j}]],$$

where $\tau$ is $(i, k-i)$-unshuffle such that $\tau(i) = k$.

Claim 6.3. (15)$' = (15)$

$$\sum_{\nu} E'(\nu, i)[Dx_\nu(i_1), \ldots, x_\nu(i_{j-1}), x_\nu(i), [D'x_\nu(i_1), \ldots, x_\nu(i_{j-1}), x_{j}]],$$

where $\nu$ is $(i, j-1)$-unshuffle.

**Proof.** Assume (15)$''$. Since $\nu$ is $(i, j-1)$-unshuffle, $\nu(i) \geq i$. We put $k = \nu(i)$. Then $k+1, k+2, \ldots, i+j-1$ are elements in $\{\nu(i+1), \ldots, \nu(i+j-1)\}$. Thus $\nu(i+j-1) = i+j-1$, $\nu(i+j-2) = i+j-2$, $\nu(k+1) = k+1$. Replace $\nu$ with $\tau$. This replacement preserves the order of variables. Thus $E'(\tau, i) = E'(\nu, i)$.  

Since (15)$'' = N_{i+1}D \circ N_j^{(i+j)}D'(x, x_{i+j})$, we obtain

$$(15) = N_{i+1}D \circ N_j^{(i+j)}D'(x, x_{i+j}).$$

(II) We show the identity,

$$ (16) = -(-1)^{DD'} N_jD' \circ N_{i+1}D(x, x_{i+j}).$$

We replace $\sigma$ in (16) with an unshuffle permutation $\tau$, along the table,

| $\sigma(1)$ | $\ldots$ | $\sigma(k-i)$ | $k+1$ | $\ldots$ | $k+a-1$ |
|------------|---------|---------------|-------|---------|---------|
| $\tau(1)$  | $\ldots$ | $\tau(k-i)$  | $\tau(k+1-i)$ | $\ldots$ | $\tau(k+a-1-i)$ |
| $\sigma(k+1-i)$ | $\ldots$ | $\sigma(k-1)$ | $k$    |         |         |
| $\tau(k+a-i)$ | $\ldots$ | $\tau(k+a-2)$ | $\tau(k+a-1)$ |       |         |


Then the Koszul sign is also replaced with $\epsilon(\tau)$,

$$
\epsilon(\tau) = \epsilon(\sigma)(-1)^{(x_{\sigma(k+1-i)} + \ldots + x_{\sigma(k-1)} + x_k)(x_{k+1} + \ldots + x_{k+a-1})}.
$$

We have $E(\sigma, k - i)E_2 =

$$ = \epsilon(\sigma)(-1)^{D(x_{\sigma(1)} + \ldots + x_{\sigma(k-i)})}(x_{\sigma(k+1-i)} + \ldots + x_{\sigma(k-1)} + x_k + D)(x_{k+1} + \ldots + x_{k+a-1})$$

$$ = \epsilon(\sigma)(-1)^{D(x_{\sigma(1)} + \ldots + x_{\sigma(k-i)})}(x_{\sigma(k+1-i)} + \ldots + x_{\sigma(k-1)} + x_k + D)(x_{k+1} + \ldots + x_{k+a-1})$$

$$ = \epsilon(\tau)(-1)^{D(x_{\tau(1)} + \ldots + x_{\tau(k+a-1-i)})} = E(\tau, k + a - 1 - i),$$

We put $m := k + a - 1$, $E(\tau, k + a - 1 - i) = E(\tau, m - i)$. We easily obtain (16) =

$$ - (-1)^{DD'} \sum_{k \geq i} \sum_{\sigma} \sum_{a=1}^{i+j-k} E(\sigma, k - i)E_2$$

$$D'(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, x_{k+1}, \ldots, x_{k+a-1}, [D x_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k, x_{k+a}], x_{k+a+1}, \ldots, x_{i+j}] =$$

$$-(-1)^{DD'} \sum_{m \geq i} \sum_{\tau} E(\tau, m-i)[D' x_{\tau(1)}, \ldots, x_{\tau(m-i)}, [D x_{\tau(m+1-i)}, \ldots, x_{\tau(m)}, x_{m+1}], x_{m+2}, \ldots, x_{i+j}] =$$

$$-(-1)^{DD'} N_j D' \circ N_{i+1} D(x, x_{i+j}). \quad (17)$$

In (17), first assume the right-hand side, then one can easily verify the left-hand side, because $i \leq \tau(m) \leq i + j - 1$. The proof is completed.

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