Product Formulas for Periods of CM Abelian Varieties and the Function Field Analog

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Abstract

We survey Colmez’s theory and conjecture about the Faltings height and a product formula for the periods of abelian varieties with complex multiplication, along with the function field analog developed by the authors. In this analog, abelian varieties are replaced by Drinfeld modules and \( A \)-motives. We also explain the necessary background on abelian varieties, Drinfeld modules and \( A \)-motives, including their cohomology theories and comparison isomorphisms and their theory of complex multiplication.

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1 Introduction

The purpose of this survey is to give a brief introduction to abelian varieties with complex multiplication over number fields, some of their cohomology theories with comparison isomorphisms, and to explain Colmez’s conjectures \[ \text{Col93} \] on a product formula for the periods and on the Faltings height of these abelian varieties. The second purpose is to explain the function field analog of this theory. There abelian varieties are replaced by Drinfeld modules \[ \text{Dri76, Gos96} \] and their higher dimensional generalizations, so-called \( A \)-motives. So we give a brief introduction to Drinfeld modules and \( A \)-motives with complex multiplication, some of their cohomology theories with comparison isomorphisms, and explain the conjecture \[ \text{HST18} \] of the authors on periods of these \( A \)-motives.

1.1. We begin with a review of product formulas for global fields. For a rational number \( \alpha \in \mathbb{Q}^\times \), all of its absolute values \( |\alpha|_v \) are linked by the product formula \( \prod_v |\alpha|_v = 1 \) where only finitely many factors are different from 1. Here \( v \) runs through the set \( \mathcal{P} \) of places of \( \mathbb{Q} \) consisting of all prime numbers \( p \) together with \( \infty \), and the \( p \)-adic absolute values \( |\cdot|_p \) are normalized such that \( |p|_p = p^{-1} \). This product formula extends to number fields, i.e. finite extensions of \( \mathbb{Q} \), as follows. Let \( \mathbb{Q}_p^{\text{alg}} \) be the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \), and if \( p \) is a prime number let \( \mathbb{Q}_p \) be the completion of \( \mathbb{Q} \) with respect to \( |\cdot|_p \) and let \( \mathbb{Q}_p^{\text{alg}} \) be an algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic absolute value \( |\cdot|_p \) extends canonically to \( \mathbb{Q}_p^{\text{alg}} \). We denote by \( |\cdot|_\infty \) the usual absolute value on \( \mathbb{C} \). In addition to the embedding \( \mathbb{Q}_p^{\text{alg}} \subset \mathbb{C} \) we fix once and for all an embedding of \( \mathbb{Q}_p^{\text{alg}} \) in \( \mathbb{Q}_p^{\text{alg}} \) for every \( p \) and consider the induced absolute value \( |\cdot|_p \) on \( \mathbb{Q}_p^{\text{alg}} \). For a finite field extension \( K \) of \( \mathbb{Q} \) we set \( H_K := \text{Hom}_\mathbb{Q}(K, \mathbb{Q}_p^{\text{alg}}) \). Then the product formula \[ \text{Lau94} \] Chapter V, §1, bottom of page 99] for \( 0 \neq \alpha \in K \) can be written as

\[
\prod_{p \in \mathcal{P}} \prod_{\eta \in H_K} |\eta(\alpha)|_p = 1. \tag{1.1}
\]

1.2. The product formula also holds for function fields. More precisely, let \( Q \) be a finitely generated field of transcendence degree one over the finite field \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). Let \( \mathbb{F}_q := \{ \alpha \in Q \mid \alpha \text{ is algebraic over } \mathbb{F}_p \} \subset Q \) be the field of constants, see \[ \text{VS06} \] Definition 2.1.3], which is a finite field with \( q \) elements. Then \( Q \) is the field of rational functions on a smooth, projective curve \( C \) over \( \mathbb{F}_q \) by \[ \text{Liu02} \] Chapter 7.3, Proposition 3.13] which is geometrically irreducible by \[ \text{Gro65} \] 4.3.1 and Proposition 4.5.9c]. Every closed point \( v \) of \( C \) is called a place. We denote its residue field by \( \mathbb{F}_v \) and set \( q_v := \# \mathbb{F}_v = q^{[\mathbb{F}_q : \mathbb{F}_v]} \). The local ring \( \mathcal{O}_{C,v} \) is a discrete valuation ring by \[ \text{Sai86} \] Proposition 1.1]. We denote the corresponding valuation also by \( v \) and the corresponding absolute value on \( Q \) by \( |\cdot|_v \). Both are normalized such that \( v(z_v) = 1 \) and \( |z_v|_v = q_v^{-1} \) for a uniformizing parameter \( z_v \in Q \) at \( v \). Then every \( \alpha \in Q \setminus \{ 0 \} \) satisfies \( \prod_v |\alpha|_v = 1 \) where again only finitely many factors are different from 1, see \[ \text{Cas67} \] Chapter II, §12, Theorem]. This can be reinterpreted in terms of divisors on \( C \). Namely, since \( |\alpha|_v = q_v^{-v(\alpha)} \) we have \( -\log \prod_v |\alpha|_v = \sum_v v(\alpha) \cdot [\mathbb{F}_v : \mathbb{F}_q] \cdot \log q = 0 \), because \( \sum_v v(\alpha) \cdot [\mathbb{F}_v : \mathbb{F}_q] \) is the degree of the principal divisor of \( \alpha \) which is zero, see \[ \text{VS06} \] Corollary 3.2.9].
Let $Q^{al}$ be a fixed algebraic closure of $Q$. For every place $v$ of $Q$ let $Q_v$ be the completion of $Q$ with respect to $|.|_v$ and let $Q^{al}_v$ be an algebraic closure of $Q_v$. The $v$-adic absolute value $|.|_v$ extends canonically to $Q^{al}_v$. We fix once and for all an embedding of $Q^{al}$ in $Q^{al}_v$ for every $v$ and consider the induced absolute value $|.|_v$ on $Q^{al}$. For a finite field extension $K$ of $Q$ we set $H_K \colonequals \text{Hom}_Q(K, Q^{al})$. Then by transformations of equations as in [Lan94] Chapter V, §1, bottom of page 99) the product formula [Cas67] Chapter II, §12, Theorem] for $0 \neq a \in K$ can be written as
\begin{equation}
\prod_{v} \prod_{\eta \in H_K} |\eta(a)|_v = 1.
\end{equation}

### 1.3

In [Col93] P. Colmez considers product formulas for periods of abelian varieties. Let $X$ be an abelian variety defined over a number field $K$ with complex multiplication by the ring of integers in a CM-field $E$ and of CM-type $\Phi$, see Section 6 for explanations. Assume that $K$ contains $\psi(E)$ for every $\psi \in H_E$. For a $\psi \in H_E$ let $\omega_\psi \in H^1_{dR}(X, K)$ be a non-zero cohomology class such that $b^\ast \omega_\psi = \psi(b) \cdot \omega_\psi$ for all $b \in E$, see Section 1.3. For every embedding $\eta: K \hookrightarrow Q^{al}$ let $X^\eta \colonequals X \otimes_{K, \eta} (K, \eta(K))$ and $\omega^\eta_\psi \in H^1_{dR}(X^\eta, \eta(K))$ be deduced from $X$ and $\omega_\psi$ by base extension. Let $(u_\eta)_\eta \in \prod_{\eta \in H_K} H_1(X^\eta(C), \mathbb{Z})$ be a family of cycles compatible with complex conjugation, see Section 1.1. Let $v$ be a place of $Q$. If $v = \infty$ the de Rham isomorphism between Betti and de Rham cohomology (Theorem 1.4) yields a complex number $\int_{u_\eta} \omega^\eta_\psi$ and its absolute value $|\int_{u_\eta} \omega^\eta_\psi|_v \in \mathbb{R}$.

### 1.4

There is a beautiful analog to the theory of elliptic curves and abelian varieties in the “Arithmetic of function fields”. Namely, Drinfeld [Dr76] invented the analog of elliptic curves under the name “elliptic modules”. These are today called Drinfeld modules, see Section 9. Since then, the arithmetic of function fields has evolved into an equally rich parallel world to the arithmetic of number fields. As higher dimensional generalizations of Drinfeld modules and analogs of abelian varieties, Anderson [And86] has defined abelian t-modules and the dual notion of t-motives, which are a kind of “global Dieudonné-modules” for abelian t-modules, see Remark 9.3. They can be slightly generalized to A-motives as follows. In the notation of §1.2 let $\mathcal{O}$ be a fixed closed point on $C$ and let $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C) = \{a \in A; v(a) \geq 0 \text{ for all } v \neq \infty\}$. Let $K \subseteq Q^{al}$ be a finite field extension of $Q$. We write $A_K := A \otimes_{\mathbb{Z}} K$ and consider the endomorphism $\sigma^*: \text{id}_A \otimes \text{Frob}_{K, b} \subseteq A_K$, where $\text{Frob}_{K, b}(x) = b^q x$ for $b \in K$. For an $A_K$-module $M$ we set $\sigma^* M := M \otimes_{A_K, \sigma} A_K$ and for a homomorphism $f: M \rightarrow N$ of $A_K$-modules we set $\sigma^* f := f \otimes \text{id}_{A_K} : \sigma^* M \rightarrow \sigma^* N$. Let $\gamma: A \hookrightarrow K$ be the inclusion $A \subseteq Q \subseteq K$, and set $\mathcal{J} := (a + 1 - 1 - \gamma(a)) A \subseteq A$. Then $\gamma$ can be recovered as the homomorphism $A \rightarrow A_K/\mathcal{J} = K$.

**Definition 1.5.** An (effective) $A$-motive of rank $r$ and dimension $d$ over $K$ is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free $A_K$-module $M$ of rank $r$ and an $A_K$-homomorphism $\tau_M: \sigma^* M \rightarrow M$ such that
\begin{itemize}
  \item[(a)] $\dim_K(\text{coker} \tau_M) = d$.
  \item[(b)] $(a - \gamma(a))^d \cdot \text{coker} \tau_M = 0$ for all $a \in A$.
\end{itemize}

We write $rk \underline{M} := r$ and $dim \underline{M} := d$.

A-motives possess cohomology realizations in analogy with abelian varieties, see Section 13.

### 1.6

Let $\underline{M}$ be a uniformizable A-motive defined over a finite extension $K \subseteq Q^{al}$ of $Q$ with complex multiplication by the ring of integers in a CM-algebra $E$ and of CM-type $\Phi$, see Sections 13.1 and 14 for explanations. Assume that $K$ contains $\psi(E)$ for every $\psi \in H_E := \text{Hom}_Q(E, Q^{al})$. For a $\psi \in H_E$ let $\omega_\psi \in H^1_{dR}(M, K[z - \zeta])$ be a non-zero cohomology class such that $b^\ast \omega_\psi = \psi(b) \cdot \omega_\psi$ for all $b \in E$, see Section 13.3. For every embedding $\eta: K \hookrightarrow Q^{al}$, let $M^{\eta} := M \otimes_{A_K, \eta} K$ and $\omega^{\eta}_\psi \in H^1_{dR}(M^{\eta}, \eta(K)[z - \zeta])$ be deduced from $M$ and $\omega_\psi$ by base extension. Let $(u_\eta)_\eta \in \prod_{\eta \in H_K} H_1(Betti(M^{\eta}, A))$ be a family of cycles, see Section 13.1. Let $v$ be a place of $Q$. If $v = \infty$ the comparison isomorphism between Betti and de Rham cohomology (Theorem 13.17) yields an element $\int_{u_\eta} \omega^{\eta}_\psi$ in the
completion $\mathbb{C}_\infty$ of $Q^{alg}_{\infty}$ with respect to $| \cdot |_\infty$ and its absolute value $| \int_{u_n} \omega^\eta_v |_\infty \in \mathbb{R}$. If $v$ corresponds to a maximal ideal of $A$, the period isomorphism between $v$-adic and de Rham cohomology (Theorem 14.12) gives a period $\int_{u_n} \omega^\eta_v$ in the analog $\mathbb{C}_v((z_v - \xi_v))$ of Fontaine’s $p$-adic period field $\mathbb{B}_{p,\text{dR}}$ and an absolute value $| \int_{u_n} \omega^\eta_v |_v \in \mathbb{R}$, see Definition 14.14. We consider the product $\prod_v \prod_{\eta \in H_K} | \int_{u_n} \omega^\eta_v |_v$ and (after some modifications analogous to Colmez’s which we explain in Section 16) we conjecture that this product evaluates to 1; see Conjecture 16.4 for the precise formulation. In [HS18] we have computed $\prod_{\eta \in H_K} | \int_{u_n} \omega^\eta_v |_v$ at all finite places $v \neq \infty$ in terms of the local factor at $v$ of the Artin $L$-series associated with an Artin character $a^0_{E,\psi,\Phi} : \text{Gal}(Q^{alg}/Q) \to \mathbb{C}$ that only depends on $E$, $\psi$ and $\Phi$ but not on $M$ and $v$; see Theorem 16.2.

This survey contains no new results. It summarizes material from various sources. But all shortcomings of the exposition are solely due to the authors. We describe the content of the individual sections of this survey. In Part I we first define elliptic curves and abelian varieties and discuss their torsion points in Section 2. Section 3 is concerned with simple and semi-simple abelian varieties and their endomorphism rings. In Section 4 we review the singular (co-)homology, Tate modules and the étale (co-)homology, and the de Rham (co-)homology of abelian varieties and period isomorphisms between these (co-)homologies. The period isomorphism between étale and de Rham (co-)homology is explained in Section 5. It is based on the concept of $p$-divisible groups, which we also review in this section. The definition of complex multiplication of abelian varieties, of CM-fields, CM-algebras and CM-types is explained in Section 6. A short review of the Faltings height of an abelian variety fills Section 7. Finally, in Section 8 we discuss Colmez’s conjecture alluded to in §1.3 above.

In Part II we discuss the analog of Colmez’s theory in the “Arithmetic of function fields”. We define Drinfeld modules and $A$-motives in Section 9 and isogenies and semi-simplicity in Section 10 where we also describe the endomorphism rings of semi-simple $A$-motives. The analytic theory of Drinfeld modules via lattices is explained in Section 11. Section 12 is devoted to torsion points and Tate modules of Drinfeld modules. In Section 13 we review the singular (co-)homology, Tate modules and the étale (co-)homology, and the de Rham (co-)homology of $A$-motives and period isomorphisms between these (co-)homologies. The period isomorphism between étale and de Rham (co-)homology is explained in Section 14. It is based on the concept of $z$-divisible local Anderson modules and local shtukas, which we also review in this section. In Section 15 we introduce the concept of complex multiplication of $A$-motives and of their CM-types. Then in Section 16 we present the theory of the authors on the product formula for periods of $A$-motives analogous to Colmez’s conjecture. In the last Section 17 we compute an interesting example for this product formula where $Q$ and $C$ have genus 1.

Contents

1 Introduction ................................................................................................................................. 1

I Abelian Varieties and Elliptic Curves .......................................................................................... 4

2 Basic Definitions............................................................................................................................ 4

3 Semi-simple Abelian Varieties ....................................................................................................... 6

4 Cohomology .................................................................................................................................... 7

4.1 Singular Cohomology .................................................................................................................. 7

4.2 Étale Cohomology ....................................................................................................................... 8

4.3 De Rham Cohomology ................................................................................................................ 9

5 $p$-divisible Groups and the $p$-adic Period Isomorphism ............................................................. 11

6 Complex Multiplication ............................................................................................................... 13

7 The Faltings Height of an Abelian Variety .................................................................................... 15

8 Colmez’s Conjecture on Periods of CM Abelian Varieties .......................................................... 15

II Drinfeld Modules and $A$-motives ............................................................................................... 19
2 Basic Definitions

Notation 2.1. As usual we denote by $\mathbb{Q}$ and $\mathbb{R}$ the fields of rational and real numbers, respectively, by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}_0$, respectively $\mathbb{N}_{\geq 0}$ the set of non-negative, respectively positive integers. By a place of $\mathbb{Q}$ we mean either $\infty$ or a maximal ideal $\mathfrak{p} \subset \mathbb{Z}$ for a prime number $p \in \mathbb{N}_{> 0}$. It defines a normalized absolute value $|.|_v : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given for $v = \infty$ by the usual absolute value $|x|_\infty = x$ if $x \geq 0$ and $|x|_\infty = -x$ if $x \leq 0$, and for $v = (p)$ by the $p$-adic absolute value $|x|_v := |x|_p = p^{-v_p(x)}$ where $v_p(x) = n$ if $x = p^n \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $p \nmid ab$. Let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ with respect to the valuation $v$, that is $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Q}_v = \mathbb{Q}_p$ for $v = (p)$. Let $\mathbb{Q}^{alg}_v$ be a fixed algebraic closure of $\mathbb{Q}_v$ and let $\mathbb{C}_v$ be the completion of $\mathbb{Q}^{alg}_v$ with respect to the canonical extension of the absolute value $|.|_v$ to $\mathbb{Q}^{alg}_v$. Note that $\mathbb{C}_v$ is algebraically closed. It equals the field of complex numbers $\mathbb{C}$ when $v = \infty$, and is usually denoted $\mathbb{C}_p$ when $v = (p)$. We also fix an algebraic closure $\mathbb{Q}^{alg}$ of $\mathbb{Q}$ and an embedding $\mathbb{Q}^{alg} \hookrightarrow \mathbb{Q}_v^{alg}$ for every place $v$ of $\mathbb{Q}$. We let $\mathcal{O}_v$ be the ring of integers of $\mathbb{C}_v$.

Definition 2.2. Let $K$ be an arbitrary field, let $K^{alg}$ be a fixed algebraic closure and let $K^{sep}$ be the separable closure of $K$ in $K^{alg}$, and $\mathcal{G}_K := \text{Gal}(K^{sep}/K)$. We mean by a (smooth) group variety over $K$ an irreducible smooth separated scheme $G$ of finite type over $K$ with a group law $\text{mult} : G \times_K G \to G$, an inverse map $\text{inv} : G \to G$ and a $K$-rational point $0 \in G(K)$, the identity element, such that $\text{mult}$ and $\text{inv}$ are morphisms of varieties satisfying the usual axioms, see [Mum70, Chapter III, § 11]. A morphism of group varieties is a morphism of varieties which is also a homomorphism of groups.

For a group variety $G$ over $K$, let $\text{Lie}(G) = T_0 G$ be the tangent space to $G$ at the identity element $0$. It is also called the Lie algebra of $G$. For every endomorphism $f$ of $G$ we let $\text{Lie}(f)$ be the induced endomorphism of $\text{Lie}(G)$.

Definition 2.3. An elliptic curve over a field $K$ is a smooth projective curve $E$ of genus $1$, together with a distinguished point $0 \in E(K)$. Every such can be written as a smooth projective plane curve which is the zero locus of an equation
\[
Y^2 Z + a_1 X Y Z + a_3 Y^2 Z^2 = X^3 + a_2 X^2 Z + a_4 X Z^2 + a_6 Z^3 \quad \text{with } a_i \in K \quad (2.1)
\]
and with distinguished point $0 = (0 : 1 : 0)$. It carries a group law making it into a commutative group variety with identity element $0$ (see [Sil86, Hus04]).
Let $E$ be an elliptic curve over $\mathbb{C}$. Then $E(\mathbb{C})$ inherits a complex structure as a sub-manifold of $\mathbb{P}^2(\mathbb{C})$. It is a complex manifold (because $E$ is nonsingular) and compact (because it is closed in the compact space $\mathbb{P}^2(\mathbb{C})$). It is connected and carries a commutative group structure. Therefore, $E$ is a compact connected complex Lie group of dimension 1. Let $T_0E(\mathbb{C})$ be the tangent space of $E(\mathbb{C})$ at the identity element 0. It is also called the Lie algebra of $E(\mathbb{C})$ and denoted $\text{Lie} E$. Then there is a unique homomorphism

$$\exp : T_0E(\mathbb{C}) \to E(\mathbb{C})$$

of complex manifolds such that, for each $v \in T_0E(\mathbb{C})$, $z \mapsto \exp(zv)$ is the one parameter subgroup $[\varphi] f_v : \mathbb{C} \to E(\mathbb{C})$ corresponding to $v$. The differential of $\exp$ at 0 is the identity map

$$T_0E(\mathbb{C}) \to T_0E(\mathbb{C}),$$

and the map $\exp$ is surjective, and its kernel is a lattice $\Lambda = \Lambda(E)$ in the complex vector space $T_0E(\mathbb{C})$. So $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ as a complex Lie group (for more details see [Mum70, Chapter I, §1]).

Now we explain how one associates an elliptic curve with a lattice. Let $\Lambda$ be a lattice in $\mathbb{C}$, that is, a discrete $\mathbb{Z}$-module $\Lambda \subset \mathbb{C}$ which is free of rank 2. With $\Lambda$, we associate its Weierstrass $\wp$-function

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}. \quad (2.2)$$

Then $\wp_\Lambda(z)$ is $\Lambda$-invariant and meromorphic on $\mathbb{C}$ with poles of order 2 at all $\omega \in \Lambda$. It satisfies the equation

$$\wp'_\Lambda(z)^2 = 4\wp^3_\Lambda(z) - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda) \quad (2.3)$$

where $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$, and

$$G_k(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^k},$$

is the Eisenstein series of the lattice $\Lambda$ for $k > 2$ even. $g_2$ and $g_3$ satisfy the relation

$$\Delta := g_2^3 - 27g_3^2 \neq 0. \quad (2.4)$$

This means $(\wp_\Lambda(z), \wp'_\Lambda(z)) \in \mathbb{C}^2$ for $z \notin \Lambda$ is a point on the smooth affine curve $E^\text{aff}_\Lambda$ (since $\Delta \neq 0$) with equation

$$Y^2 = 4X^3 - g_2X - g_3 \quad (2.5)$$

and $(\wp_\Lambda(z) : \wp'_\Lambda(z) : 1) \in \mathbb{P}^2(\mathbb{C})$ for all $z \in \mathbb{C}$ is a point on the projective model of the above curve with equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3. \quad (2.6)$$

The above yields a biholomorphic isomorphism of the complex torus $\mathbb{C}/\Lambda$ with $E(\mathbb{C})$, well-defined through its restriction to $\mathbb{C}/\Lambda$ by $z \mapsto (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1)$. Note that $E(\mathbb{C})$ inherits a group structure from $\mathbb{C}/\Lambda$, which may however be defined in purely algebraic terms on the algebraic curve $E_\Lambda$, and which turns $E_\Lambda$ into an elliptic curve. This is the elliptic curve associated with the lattice $\Lambda$. In fact, each elliptic curve $E$ over $\mathbb{C}$ has the form $E = E_\Lambda$ for some lattice $\Lambda$ as above, and two such, $E_\Lambda$ and $E_{\Lambda'}$, are isomorphic as elliptic curves (i.e., as algebraic curves through some isomorphism preserving the group structures) if and only if $\Lambda'$ and $\Lambda$ are homothetic, that is, $\Lambda' = c\Lambda$ for some $c \in \mathbb{C}^\times$.

**Definition 2.4.** An **abelian variety** over a field $K$ is a smooth projective connected group variety. The group law is automatically commutative; see [Mum70] Chapter II, §4, Corollary 2. Abelian varieties are higher-dimensional generalizations of elliptic curves, which in turn are abelian varieties of dimension 1.

A **homomorphism** $f : X \to Y$ between abelian varieties over $K$ is a morphism of varieties over $K$ which is compatible with the group structure. The abelian group of homomorphisms $f : X \to Y$ over $K$ is denoted $\text{Hom}_K(X,Y)$ and we write $\text{End}_K(X) = \text{Hom}_K(X,X)$. We also write $\text{QHom}_K(X,Y) = \text{Hom}_K(X,Y) \otimes \mathbb{Q}$ and

---

1 For a complex Lie group $G$, a one parameter subgroup of $G$ is a holomorphic homomorphism $f : \mathbb{C} \to G$. In complex analysis one proves that for every tangent vector $v$ to $G$ at $e$, there is a unique one-parameter subgroup $f_v : \mathbb{C} \to G$ such that $f_v(0) = e$ and $(df_v)(1) = v$, see [Hoc65] pp. 79 and 195.
Theorem 3.1. For any \( \Lambda \subset \mathbb{C} \) arises from an abelian variety, that is, the quotient \( \mathbb{C}/\Lambda \) admits a Riemannian form \( g \), see [Mil08, Remark 6.5]. This means that \( f \) becomes invertible in \( \text{QHom}_K(X,Y) \), in the sense that \( f^{-1} = g \otimes \frac{1}{n} \in \text{QHom}_K(Y,X) \) is its inverse.

Remark 2.5. (a) Let \( X \) and \( Y \) be abelian varieties over \( K \). If \( X \) and \( Y \) are isogenous over \( K \) via an isogeny \( f \), then

\[
\text{QEnd}_K(X) \cong \text{QHom}_K(X,Y) \cong \text{QEnd}_K(Y), \quad h \mapsto f \circ h \mapsto f \circ h \circ f^{-1}.
\]

More precisely, \( \text{QHom}_K(X,Y) \) is a free right \( \text{QEnd}_K(X) \)-module of rank 1 and a free left \( \text{QEnd}_K(Y) \)-module of rank 1. If \( X \) and \( Y \) are not isogenous then \( \text{QHom}_K(X,Y) = (0) \).

(b) The homomorphism \( [m] \) is an isogeny of degree \( m^{2g} \). It is always étale when \( K \) has characteristic zero, and when \( K \) has characteristic \( p > 0 \) it is étale if and only if \( p \) does not divide \( m \), see [Mum70, Chapter II, § 6].

(c) The kernel \( X[m] := \ker([m] : X \to X) \) is a finite group scheme over \( K \) of order \( m^{2g} \).

Definition 2.6. Let \( X \) be an abelian variety and let \( m \in \mathbb{Z} \) with \( m \geq 1 \). The \( m \)-torsion subgroup of \( X \), denoted by \( X[m](K^{alg}) \), is the subgroup of points of \( X(K^{alg}) \) of order \( m \),

\[
X[m](K^{alg}) = \{ P \in X(K^{alg}) : [m]P = 0 \}.
\]

It equals the group of \( K^{alg} \)-valued points of the finite group scheme \( X[m] \).

Remark 2.7. For any \( m \) not divisible by the characteristic of \( K \), \( X[m](K^{alg}) \) has order \( m^{2g} \) and is contained in \( X(K^{pp}) \). Since this is also true for any \( n \) dividing \( m \), \( X[m](K^{alg}) \) must be a free \( \mathbb{Z}/m\mathbb{Z} \)-module of rank \( 2g \).

Finally, if \( X \) is an abelian variety over \( \mathbb{C} \) of dimension \( g \), then \( X(\mathbb{C}) \) is isomorphic to a complex torus \( \mathbb{C}^g/\Lambda \),

\[
X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda
\]

for some lattice \( \Lambda = \Lambda(X) \subset \mathbb{C}^g \) under an isomorphism of complex manifolds which preserves the group structures. Here \( \Lambda \subset \mathbb{C}^g \) is a discrete \( \mathbb{Z} \)-submodule which is free of rank \( 2g \). However, when \( g > 1 \), not every lattice \( \Lambda \subset \mathbb{C}^g \) arises from an abelian variety, that is, that \( \mathbb{Q}^g/\Lambda \subset \mathbb{C}^g \) by an arbitrary lattice \( \Lambda \) does not always arise from an abelian variety. There is a criterion on \( \Lambda \) for when \( \mathbb{C}^g/\Lambda \) is an algebraic (hence abelian) variety, namely, that \( (\mathbb{C}^g,\Lambda) \) admits a Riemannian form\(^3\), see [Mum70, Chapter I, § 3].

3 Semi-simple Abelian Varieties

Theorem 3.1. For two abelian varieties \( X \) and \( Y \) over a field \( K \) the \( \mathbb{Z} \)-module \( \text{Hom}_K(X,Y) \) is finite projective of rank \( \leq (2 \dim X) \cdot (2 \dim Y) \).

Proof. See for example [Mum70, Chapter IV, § 19, Corollary 1].

Definition 3.2. Let \( X \) be an abelian variety over \( K \). Then \( X \) is called

(a) **simple over \( K \)** if \( X \) is non-trivial and there does not exist an abelian subvariety \( Y \subset X \) over \( K \) other than \( (0) \) and \( X \).

\(^2\)For a complex torus \( V/\Lambda \) where \( V \) is a complex vector space and \( \Lambda \) is a full lattice in \( V \), a skew-symmetric form \( F : \Lambda \times \Lambda \to \mathbb{Z} \), that is \( F(u, v) = -F(v, u) \), extended to a skew-symmetric \( \mathbb{R} \)-bilinear form \( F_k : V \times V \to \mathbb{R} \) is a Riemannian form if

\[
F_k(\mathbb{R}) = F_k(v, w) = F_k(v, w) = \Im(H(v, w)) \text{ is positive definite.}
\]
(b) semisimple over $K$ if $X$ is isogenous over $K$ to a direct product of simple abelian varieties, i.e. $X \cong_K X_1 \times \cdots \times X_n$ with $X_i$ simple.

Remark 3.3. The Theorem of Poincaré and Weil [Mil08, Proposition 9.1] states that any abelian variety is semisimple over $K$. More precisely, for any abelian variety $X$ over $K$, there are simple abelian subvarieties $X_1, \cdots, X_n \subset X$ such that the map $X_1 \times \cdots \times X_n \to X$, $(a_1, \cdots, a_n) \mapsto a_1 + \cdots + a_n$ is an isogeny. The proof of this is analogous with a standard proof for the semisimplicity of a representation of a finite group $G$ on a finite-dimensional vector space over $\mathbb{Q}$, see [Mil08, Remark 9.2].

Let $X$ be a simple abelian variety, and let $0 \neq f \in \text{End}_K(X)$. Then $f$ is an isogeny, because by the simplicity of $X$, the image of $f$ equals $X$ and the connected component of $\ker f$ equals $\{0\}$, as both are abelian subvarieties. So $f$ is surjective with finite kernel. From this it follows that $\mathbb{Q}\text{End}_K(X)$ is a division algebra or equivalently a skew-field, i.e., a ring, possibly non commutative, in which every nonzero element has an inverse.

Remark 3.4. Let $X$ be a simple abelian variety over $K$, and let $D = \mathbb{Q}\text{End}_K(X)$. Then $\mathbb{Q}\text{End}_K(X^n) = M_n(D)$ is the ring of $n \times n$ matrices with coefficients in $D$.

Now consider an arbitrary abelian variety $X$. Then $X$ is isogenous over $K$ to a product $X_1^{n_1} \times \cdots \times X_r^{n_r}$, where each $X_i$ is simple, and $X_j$ is not isogenous to $X_j$ for $i \neq j$ over $K$. The above remarks show that

$$\mathbb{Q}\text{End}_K(X) \cong \prod M_{n_i}(D_i), \quad D_i = \mathbb{Q}\text{End}_K(X_i).$$

Since $\text{End}_K(X)$ is a free $\mathbb{Z}$-module of finite rank $\leq (2 \dim X)^2$ we know that $\mathbb{Q}\text{End}_K(X)$ is a finite dimensional $\mathbb{Q}$-algebra.

In the following we recall a few facts about semi-simple algebras. Let $Q$ be a field, let $B$ be a semisimple $Q$-algebra of finite dimension, and let $B = \prod B_i$ be its decomposition into a product of simple algebras $B_i$. A simple $Q$-algebra is isomorphic to a matrix algebra over a division $Q$-algebra. The center of each $B_i$ is a field $F_i$, and each degree $[B_i : F_i]$ is a square. The reduced degree of $B$ over $Q$ is defined to be

$$[B : Q]_{\text{red}} = \sum_i [B_i : F_i]^{1/2}[F_i : Q].$$

For any field $Q'$ containing $Q$, 

$$[B : Q] = [B \otimes_Q Q' : Q'] \quad \text{and} \quad [B : Q]_{\text{red}} = [B \otimes_Q Q' : Q']_{\text{red}}.$$

Proposition 3.5 ([Mil06, Proposition 1.2]). Let $B$ be a semi-simple $Q$-algebra which is finite dimensional over $Q$. For any faithful $B$-module $M$, 

$$\dim_Q M \geq [B : Q]_{\text{red}};$$

and there exists a faithful module for which equality holds if and only if the simple factors of $B$ are matrix algebras over their centers.

Proposition 3.6 ([Mil06, Proposition 1.3]). Let $\text{char}(Q) = 0$ and let $B$ be a semisimple $Q$-algebra. Every maximal étale $Q$-subalgebra of $B$ has degree $[B : Q]_{\text{red}}$ over $Q$. Here we mean by an étale $Q$-algebra a finite product of finite separable field extensions of $Q$.

4 Cohomology

4.1 Singular Cohomology

Let $X$ be an abelian variety of dimension $g$ over $\mathbb{C}$. Let $V$ be the tangent space of $X$ at the identity element and let $\Lambda$ be the kernel of the exponential map $\exp : V \to X$. Now the space $V \cong \mathbb{C}^g$ is simply connected, and $\exp : V \to X$ is a covering map, therefore it realizes $V$ as the universal covering space of $X$, and so $\pi_1(X)$ is its group of covering transformations, which is $\Lambda$. In particular, it is abelian. As for any good topological space we obtain for the singular cohomology of $X$

$$H^1(X, \mathbb{Z}) \cong \text{Hom}_{\text{groups}}(\pi_1(X), \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}).$$
Since we have seen that \( X \) is a complex torus of dimension \( g \), it is as a real Lie group isomorphic to \((\mathbb{R}/\mathbb{Z})^{2g} = (S^1)^{2g}\), where \( S^1 \) is the circle group. We claim that for all \( r \in \mathbb{N}_{>0} \)

\[
\bigwedge^n H^1(X, \mathbb{Z}) \cong H^r(X, \mathbb{Z})
\]

under the natural map defined by the cup product. Indeed, by the Künneth formula if the above map is an isomorphism for spaces \( X_1 \) and \( X_2 \) with finitely generated cohomologies, then it is an isomorphism for \( X_1 \times X_2 \). Since it is an isomorphism for \( S^1 \) for all \( r \geq 0 \), where the module is \((0)\) for \( r \geq 2 \), the result for \( X \) follows.

Since \( X \) is compact and orientable and \( H^r(X, \mathbb{Z}) \) is torsion free, the duality theorems gives us for the singular homology of \( X \)

\[
H_r(X, \mathbb{Z}) \cong H^r(X, \mathbb{Z})^\vee
\]

and in particular \( H_1(X, \mathbb{Z}) = \Lambda \).

### 4.2 Étale Cohomology

We follow [Mil86 §15]. Let \( X \) be an abelian variety of dimension \( g \) over a separably closed field \( K \), and let \( \ell \) be a prime different from \( \text{char}(K) \). Recall that, for any \( m \) not divisible by the characteristic of \( K \), \( X[m](K^{\text{sep}}) \) has order \( m^{2g} \). Define the \( \ell \)-adic Tate module of \( X \) as

\[
T_\ell(X) = \varprojlim \left( X[\ell^n](K^{\text{sep}}), [\ell] \right).
\]

It follows that \( T_\ell(X) \) is a free \( \mathbb{Z}_\ell \)-module of rank \( 2g \). There is a continuous action of \( \mathcal{G}_K \) on this module.

Let \( X \) and \( Y \) be two abelian varieties over \( K \). A homomorphism \( f : X \to Y \) induces a homomorphism \( X[\ell^n] \to Y[\ell^n] \), and hence a homomorphism

\[
T_\ell(f) : T_\ell(X) \to T_\ell(Y), \quad (a_1, a_2, \cdots) \mapsto (f(a_1), f(a_2), \cdots).
\]

Therefore, \( T_\ell \) is a functor from abelian varieties to \( \mathbb{Z}_\ell \)-modules. It is easy to see that for any prime \( \ell \neq \text{char}(K) \), the natural map

\[
\text{Hom}_K(X, Y) \to \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y))
\]

is injective. From this one obtains that the \( \mathbb{Z} \)-algebra \( \text{Hom}_K(X, Y) \) of morphisms \( X \to Y \) of group varieties is torsion-free. The following theorem was conjectured by Tate [Tat66] and proved by him for finite fields \( K \). It was proved by Zarhin [Zar75] for fields of positive characteristic and by Faltings [Fal83, Fal84] for fields of characteristic zero.

**Theorem 4.1** (Tate conjecture for abelian varieties). Let \( X \) and \( Y \) be two abelian varieties over a finitely generated field \( K \) and let \( \ell \) be a prime different from the characteristic of \( K \). Then the natural map

\[
\text{Hom}_K(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y)), \quad f \otimes a \mapsto a \cdot T_\ell(f)
\]

is an isomorphism of \( \mathbb{Z}_\ell \)-modules.

Now we denote by \( \pi_1^{\text{ét}}(X, 0) \) the étale fundamental group, then

\[
H_1^{\text{ét}}(X, \mathbb{Z}_\ell) \cong \text{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, 0), \mathbb{Z}_\ell).
\]

For each \( n \) the map \([\ell^n] : X \to X\) is a finite étale covering of \( X \) with group of covering transformations \( X[\ell^n](K^{\text{sep}}) \). By definition \( \pi_1^{\text{ét}}(X, 0) \) classifies such coverings, and therefore there is a canonical epimorphism \( \pi_1^{\text{ét}}(X, 0) \to X[\ell^n] \).

On passing to the inverse limit, we get an epimorphism \( \pi_1^{\text{ét}}(X, 0) \to T_\ell(X) \), and consequently an injection

\[
\text{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), \mathbb{Z}_\ell) \hookrightarrow H_1^{\text{ét}}(X, \mathbb{Z}_\ell).
\]

which actually is an isomorphism, see [Mil86 Theorem 15.1]. So we obtain for the first étale homology group of \( X \)

\[
H_{1,\text{ét}}(X, \mathbb{Z}_\ell) = T_\ell(X) \quad \text{and} \quad H_{1,\text{ét}}(X, \mathbb{Q}_\ell) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,
\]

and for the first étale cohomology group of \( X \)

\[
H_{1,\text{ét}}^\vee(X, \mathbb{Z}_\ell) = H_{1,\text{ét}}(X, \mathbb{Z}_\ell)^\vee \quad \text{and} \quad H_{1,\text{ét}}^\vee(X, \mathbb{Q}_\ell) = H_{1,\text{ét}}(X, \mathbb{Q}_\ell)^\vee.
\]
The cup product pairings define isomorphisms

\[ H_{r,\text{ét}}(X, \mathbb{Z}_\ell) \cong \bigwedge^r H_1(\mathbb{Z}, \mathbb{Z}_\ell) \quad \text{and} \quad H^r_{\text{ét}}(X, \mathbb{Q}_\ell) \cong \bigwedge^r H^1_{\text{ét}}(X, \mathbb{Q}_\ell). \]

Now, over the field \( K = \mathbb{C} \) the choice of an isomorphism \( X(\mathbb{C}) \cong \mathbb{C}^d / \Lambda \) determines \( X[m](\mathbb{C}) \cong m^{-1} \Lambda / \Lambda \). Then

\[ T_\ell(X) = \lim(X[\ell^n](\mathbb{C}), [\ell]) \cong \lim(\ell^{-n} \Lambda \mod \ell^n) \]

\[ \cong \Lambda \otimes_{\mathbb{Z}} \lim(\ell^n / \ell^m), \text{ because } \Lambda \text{ is a } \mathbb{Z}\text{-module} \]

\[ \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell. \]

Taking also duals and exterior powers, we can summarize the results as a

**Theorem 4.2.** For every abelian variety \( X \) over \( \mathbb{C} \) there are canonical comparison isomorphisms between singular and étale (co-)homology

\[ H^r_{\text{ét}}(X, \mathbb{Z}_\ell) \cong H^r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \quad \text{and} \quad H_{r,\text{ét}}(X, \mathbb{Z}_\ell) \cong H_r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell. \]

**Example 4.3.** Also for the multiplicative group scheme \( \mathbb{G}_m := \mathbb{G}_{m, \mathbb{Q}} = \text{Spec } \mathbb{Q}[x, x^{-1}] \) there is a period isomorphism between \( H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \) and \( H_{1,\text{ét}}(\mathbb{G}_m, \mathbb{Z}_\ell) \). Namely, \( H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \cdot u \), where \( u: [0, 1] \to \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \) is the cycle given by \( u(s) = \exp(2\pi is) \). Also let \( \varepsilon_\ell^{(n)} := \exp(2\pi i / \ell^n) \in \mathbb{Q}^{1k} \subset \mathbb{C} \). It is a primitive \( \ell^n \)-th root of unity with \( \varepsilon_\ell^{(n + 1)} \varepsilon_\ell^{(n)} = \varepsilon_\ell \) for all \( n \). Let \( \varepsilon_\ell := (\varepsilon_\ell^{(n)})_{n \in \mathbb{N}} \in T_\ell \mathbb{G}_m \). Then \( H_{1,\text{ét}}(\mathbb{G}_m, \mathbb{Z}_\ell) = T_\ell \mathbb{G}_m = \varepsilon_\ell^{\mathbb{Z}_\ell} \) and the comparison isomorphism

\[ H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong H_{1,\text{ét}}(\mathbb{G}_m, \mathbb{Z}_\ell) \]

sends \( u \) to \( \varepsilon_\ell \). This can be seen from the exact sequence \( 0 \to Z = \pi_1(\mathbb{C}^\times) \to \mathbb{C} \xrightarrow{\exp(2\pi i \cdot \cdot \cdot )} \mathbb{C}^\times \to 0 \) and the induced comparison isomorphism \( \pi_1(\mathbb{C}^\times) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong T_\ell \mathbb{G}_m, 1 \mapsto (\exp(2\pi i / \ell^n))_{n \in \mathbb{N}}. \)

### 4.3 De Rham Cohomology

We will now explain the construction of the Dolbeault complex associated with \( X \) which is an analogue of the de Rham complex for complex manifolds. Let \( X \) be an abelian variety over \( \mathbb{C} \).

Let \( \mathcal{E}^n = \bigoplus_{p+q=n} \mathcal{E}^{p,q} \) be the sheaf of \( C^\infty \) complex valued differential \( n \)-forms, where \( \mathcal{E}^{p,q} \) is the sheaf of \( C^\infty \) complex valued differential forms of type \( (p, q) \). In terms of local coordinates, let \( (z_1, \ldots, z_g) \) be a holomorphic coordinate system. First we decompose the complex coordinates into their real and imaginary parts: \( z_j = x_j + iy_j \) for each \( j \). Letting \( dz_j = dx_j + idy_j, \ d\bar{z}_j = dx_j - idy_j, \) one sees that any differential 1-form with complex coefficients can be written uniquely as a sum

\[ \sum_{j=1}^{n} \left( f_j dz_j + g_j d\bar{z}_j \right), \]

for \( \mathbb{C} \)-valued \( C^\infty \)-functions \( f_j \) and \( g_j \). Let \( \mathcal{E}^{1,0} \) be the sheaf of \( C^\infty \) complex valued differential 1-forms where all \( g_j \) are zero, and let \( \mathcal{E}^{0,1} \) be the sheaf of \( C^\infty \) complex valued differential 1-forms where all \( f_j \) are zero. Then the space \( \mathcal{E}^{p,q} \) of type \( (p, q) \)-forms is defined by taking linear combinations of the wedge products of \( p \) elements from \( \mathcal{E}^{1,0} \) and \( q \) elements from \( \mathcal{E}^{0,1} \). Symbolically,

\[ \mathcal{E}^{p,q} = \bigwedge^p \mathcal{E}^{1,0} \wedge \bigwedge^q \mathcal{E}^{0,1} \]

In particular for each \( n \) and each \( p \) and \( q \) with \( p + q = n \), there are canonical projection maps which we denote by

\[ \pi^{(p,q)} : \mathcal{E}^n \to \mathcal{E}^{p,q}. \]

The exterior derivative defines a map \( d : \mathcal{E}^n \to \mathcal{E}^{n+1} \) i.e. if \( \varphi \in \mathcal{E}^{p,q} \) then \( d(\varphi) \in \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} \). Using \( d \) and the projections maps, it is possible to define the operators:

\[ \partial = \pi^{p+1,q} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}, \quad \bar{\partial} = \pi^{p,q+1} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1} \]

9
In terms of local coordinates \( z = (z_1, \cdots z_g) \) we can write \( \varphi \in \mathcal{C}^{p,q} \) as

\[
\varphi = \sum_{#I=p, #J=q} f_{IJ} \, dz_I \wedge d\bar{z}_J \in \mathcal{C}^{p,q}
\]

where \( I \) and \( J \) are multi-indices and \( dz_I = \bigwedge_{i \in I} dz_i \) and \( d\bar{z}_I = \bigwedge_{i \in I} d\bar{z}_i \). Then

\[
\partial \varphi = \sum_{#I=p, #J=q} \sum_i \frac{\partial f_{IJ}}{\partial z_i} \, dz_i \wedge d\bar{z}_I \wedge d\bar{z}_J \quad \text{and} \quad \bar{\partial} \varphi = \sum_{#I=p, #J=q} \sum_i \frac{\partial f_{IJ}}{\partial \bar{z}_i} \, d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J.
\]

It is not difficult to see the following properties:

\[
\begin{align*}
\partial^2 &= \partial \partial + \bar{\partial} \bar{\partial} = 0, \\
\bar{\partial}^2 &= \bar{\partial} \partial + \partial \bar{\partial} = 0.
\end{align*}
\]

Then the Poincaré lemma gives that the complex

\[
0 \to \mathbb{C} \to \mathcal{C}^0 \xrightarrow{d} \mathcal{C}^1 \xrightarrow{d} \cdots
\]

is a fine resolution of the constant sheaf \( \mathbb{C} \). It is called the de Rham resolution. We define the de Rham cohomology as the cohomology of this complex i.e.

\[
H^p_{\text{dR}}(X, \mathbb{C}) = \frac{\{\text{global } n\text{-forms } \varphi \in \mathcal{C}^n(X) \text{ on } X \text{ which are } d\text{-closed}, \text{ i.e. } d\varphi = 0\}}{\{d\psi : \text{where } \psi \in \mathcal{C}^{n-1}(X) \text{ is a global } (n-1)\text{-form on } X\}}.
\]

Let \( V = T_0X \) be the tangent space to \( X \) at \( 0 \) (regarded as a complex vector space). Let \( T = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) be the complex cotangent space to \( X \) at \( 0 \) and \( \mathbb{T} = \text{Hom}_\mathbb{C}^\text{antilinear}(V, \mathbb{C}) \). Then from linear algebra

\[
\text{Hom}_\mathbb{R}(V, \mathbb{C}) \cong \text{Hom}_\mathbb{C}(V, \mathbb{C}) \oplus \text{Hom}_\mathbb{C}^\text{antilinear}(V, \mathbb{C}) \quad \text{i.e. } \text{Hom}_\mathbb{R}(V, \mathbb{C}) \cong T \oplus \mathbb{T},
\]

and

\[
\bigwedge^r \text{Hom}_\mathbb{R}(V, \mathbb{C}) \cong \bigoplus_{p+q=r} \bigwedge^p T \otimes \bigwedge^q \mathbb{T}.
\]

By translation under the group law on \( X \) every complex co-vector \( \varphi \in \bigwedge^p T \otimes \bigwedge^q \mathbb{T} \) extends to a unique translation invariant \( \omega_{\varphi} \in \mathcal{C}^{p,q} \), and therefore every complex co-vector \( \varphi \in \bigwedge^r \text{Hom}_\mathbb{R}(V, \mathbb{C}) \) extends to a unique translation invariant form \( \omega_{\varphi} \) belonging to \( \mathcal{C}^n \). For all \( d \)-closed \( n \)-forms \( \omega \), there is unique translation invariant \( \omega_{\varphi} \) for \( \varphi \in \bigwedge^n \text{Hom}_\mathbb{R}(V, \mathbb{C}) \), such that

\[
\omega - \omega_{\varphi} = d\eta, \text{ for some } (n-1)\text{-form } \eta.
\]

Therefore, \( H^p_{\text{dR}}(X, \mathbb{C}) \cong \bigwedge^r \text{Hom}_\mathbb{R}(V, \mathbb{C}) \), and has the decomposition

\[
H^r_{\text{dR}}(X, \mathbb{C}) \cong \bigoplus_{p+q=r} \bigwedge^p T \otimes \bigwedge^q \mathbb{T}.
\]

For the sheaf \( \mathcal{O}^p := \mathcal{C}^{p,0} \) of holomorphic \( p \)-forms on \( X \) we know from [Mum70 Chapter I, §1, Theorem] that

\[
H^q(X, \mathcal{O}_X) \cong \bigwedge^q \mathbb{T} \quad \text{and} \quad H^q(X, \mathcal{O}^p) \cong \bigwedge^p T \otimes \bigwedge^q \mathbb{T},
\]

so

\[
H^r_{\text{dR}}(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X), \quad \text{where } H^{p,q}(X) := H^q(X, \mathcal{O}^p).
\]

This is the famous Hodge decomposition.

Now we obtain the de Rham isomorphism

\[
H^1(X, \mathbb{C}) = H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \text{Hom}_\mathbb{R}(V, \mathbb{C}) \cong H^1_{\text{dR}}(X, \mathbb{C}).
\]

Then, \( H^p_{\text{dR}}(X, \mathbb{C}) \cong H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \). Note that complex conjugation on the right tensor factor of the target defines a conjugate-linear automorphism of \( H^p_{\text{dR}}(X, \mathbb{C}) \). For more details see [Mum70 Chapter I, §1]. Taking also exterior powers, we can summarize the results as a
**Theorem 4.4** (De Rham isomorphism). For every abelian variety $X$ over $\mathbb{C}$ there are canonical comparison isomorphisms between singular and de Rham cohomology

$$H^\text{dR}_i(X, \mathbb{C}) \cong H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$ 

**Example 4.5.** Also for the multiplicative group scheme $G_m := \mathbb{G}_{m,\mathbb{C}} = \text{Spec } \mathbb{C}[x, x^{-1}]$ there is a de Rham isomorphism between $H^1(G_m(\mathbb{C}), \mathbb{Z})$ and $H^1_\text{dR}(G_m, \mathbb{C}) = \mathbb{C}[\pi]$. As in Example 4.3 the singular homology $H^1(G_m(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \cdot u$, where $u: [0, 1] \to G_m(\mathbb{C}) = \mathbb{C}^\times$ is the cycle given by $u(s) = \exp(2\pi is)$. The de Rham isomorphism is given as the pairing

$$H^1(G_m, \mathbb{Z}) \times H^1_\text{dR}(G_m, \mathbb{C}) \to \mathbb{C}, \quad (nu, \omega) \mapsto n \int_0^1 u \omega, \quad (u, \frac{d\omega}{du}) \mapsto \int_0^1 \frac{d\omega}{du} = 2\pi i.$$

The corresponding isomorphism $H^1(G_m, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1_\text{dR}(G_m, \mathbb{C})$ sends the generator of $H^1(G_m, \mathbb{Z})$, which is dual to $u$, to $(2\pi i)^{-1} \frac{d\omega}{du}$.

## 5 $p$-divisible Groups and the $p$-adic Period Isomorphism

Let $R$ be a commutative ring. Let $p$ be a prime number, and $h$ an integer $\geq 0$. A $p$-divisible group $G$ over $R$ of height $h$ is an inductive system

$$(G_n, i_n), \quad n \geq 0$$

where

(a) $G_n$ is a finite flat commutative group scheme of finite presentation over $R$ of order $p^n h$,

(b) for each $n \geq 0$,

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{[p^n]} G_{n+1}$$

is exact (i.e., $G_n$ can be identified via $i_n$ with the kernel of multiplication by $p^n$ in $G_{n+1}$).

These axioms for ordinary abelian groups would imply

$$G_n \cong (\mathbb{Z}/p^n\mathbb{Z})^h \quad \text{and} \quad G = \varprojlim G_n = (\mathbb{Q}_p/\mathbb{Z}_p)^h.$$ 

A homomorphism $f : G \to H$ of $p$-divisible groups is defined in the obvious way: if $G = (G_n, i_n)$, $H = (H_n, i_n)$ then $f$ is a system of homomorphisms $f_n : G_n \to H_n$ of group schemes over $R$, satisfying $i_n \circ f_n = f_{n+1} \circ i_n$ for all $n \geq 1$.

**Example 5.1.** We can associate a $p$-divisible group with any commutative group variety $G$ over a field $K$:

Define $G[m]$ as the kernel of multiplication by $m$. Then $(G[p^n], i_n)$ is a $p$-divisible group, where $i_n$ denotes the obvious inclusion. This $p$-divisible group is sometimes denoted $G[p^\infty]$.

(a) If $G = X$ is an abelian variety, then the height of $G[p^\infty]$ is $2\dim X$.

(b) If $G = \mathbb{G}_m$ is the multiplicative group scheme, then $G_m[p^\infty] = \mathfrak{m}_p := (\mathfrak{m}_p^n, i_n)$ with height 1. Here

$$\mathfrak{m}_p^n = \text{Spec } K[x]/(x^{np} - 1)$$

is the group scheme of $p^n$-th roots of unity.

Let us see how $p$-divisible groups generalize Tate modules. Suppose $p \neq \text{char}(K)$. Then for a $p$-divisible group $(G_n, i_n)$ of height $h$ over $K$ each $G_n$ is a finite étale group scheme over $K$ and each $M_n := G_n(K^{\text{sep}})$ is a discrete $\mathcal{G}_K$-module of size $p^{nh}$ annihilated by $p^n$ and $M_{n+1}[p^n] = M_n$. It follows that $M_n \cong (\mathbb{Z}/p^n\mathbb{Z})^h$. We can form two kinds of limits:

(i) the direct limit $M_\infty = \varinjlim M_n$ is $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ with a continuous $\mathcal{G}_K$-action for the discrete topology, and

(ii) multiplication by $p$ on $M_{n+1}$ provides a quotient map $M_{n+1} \to M_n$ of discrete $\mathcal{G}_K$-modules yielding an inverse limit $T_p(M) = \varprojlim M_n$ that is a finite free $\mathbb{Z}_p$-module of rank $h$ equipped with a continuous action of $\mathcal{G}_K$ for the $p$-adic topology.

We can recover the direct system $(M_n, i_n)$ from both limits, namely $M_n = M_\infty[p^n]$ and $M_n = T_p(M)/(p^n)$. The viewpoint of $M_\infty$ explains the $p$-divisible aspect of the situation (since multiplication by $p$ is surjective on $(\mathbb{Q}_p/\mathbb{Z}_p)^h$), whereas $T_p(M)$ has a nicer $\mathbb{Z}_p$-module structure. Since the étale group scheme $G_n$ is uniquely determined by the $\mathcal{G}_K$-module $M_n$, this proves:
Proposition 5.2. If $K$ is a field with $p \neq \text{char}(K)$, then the functor $G \to T_p(G)$ is an equivalence from the category of $p$-divisible groups over $K$ to the category of finite free $\mathbb{Z}_p$-modules with continuous $\mathcal{G}_K$-action.

On the other hand let $K$ be a finite extension of $\mathbb{Q}_p$, and let $X$ be an abelian variety over $K$. Assume that $X$ has good reduction, i.e. there exists a smooth projective commutative group scheme $\mathcal{X}$ over $\mathcal{O}_K$ with $X \cong \mathcal{X} \otimes \mathcal{O}_K$. Then $X[p^n]$ admits an integral model $\mathcal{G}_n := \mathcal{X}[p^n]$ with $\mathcal{G}_n = \mathcal{G}_{n+1}[p^n]$ for all $n \geq 1$ and $\mathcal{G} := (\mathcal{G}_n, i_n)$ is a $p$-divisible group over $\mathcal{O}_K$ with $\mathcal{G}_K := \mathcal{G} \otimes \mathcal{O}_K K \cong \mathbb{X}[p^n]$.

Now due to Tate [<Laum97>], we know that if $\mathcal{G}$ and $\mathcal{H}$ are $p$-divisible groups over $\mathcal{O}_K$ then

$$\text{Hom}_{\mathcal{O}_K} (\mathcal{G}, \mathcal{H}) \cong \text{Hom}_K (\mathcal{G}_K, \mathcal{H}_K).$$

Remark 5.3. $p$-divisible groups over a perfect field $k$ of characteristic $p$ have a description via semi-linear algebra by their Dieudonné module. The latter is a finite free module $M$ over the ring $W(k)$ of $p$-typical Witt-vectors over $k$, equipped with a Frobenius semi-linear morphism $F : M \to M$, called Frobenius, and a $\text{Frob}_p^{-1}$-semi-linear morphism $V : M \to M$, called Verschiebung, satisfying $FV = p = VF$.

This was generalized by Fontaine [<Fon77>] to $p$-divisible groups $G$ over the ring of integers $\mathcal{O}_K$ of a finite field extension $K$ of $\mathbb{Q}_p$. Those $p$-divisible groups are described by the Dieudonné module of the special fiber $G \otimes \mathcal{O}_K$ together with a decreasing exhaustive and separated filtration $\text{Fil}^0$ on $D_K = D \otimes_K K$ satisfying $\text{Fil}^0(D_K) = D_K$, $\text{Fil}^1(D_K) = 0$.

Notation 5.4. Let $\mathcal{O}_{\mathcal{C}_p} := \lim \left( \mathcal{O}_{\mathcal{C}_p}, \text{Frob}_p \right) = \{ x = (x^{(n)})_{n \in \mathbb{N}} \in (\mathcal{O}_{\mathcal{C}_p})^{\mathbb{N}} : (x^{(n+1)}) = x^{(n)} \} \}$ and $\text{A}_{\mathcal{inf}} := W(\mathcal{O}_{\mathcal{C}_p})$ the ring of Witt vectors. Every element of $\text{A}_{\mathcal{inf}}$ can be written in the form $\sum_{i \geq 0} \lambda_i x^i$ where $\lambda_i$ denotes the Teichmüller lift of the element $x = (x^{(n)})_{n \in \mathbb{N}} \in \mathcal{O}_{\mathcal{C}_p}$. Let $\Theta : \text{A}_{\mathcal{inf}} \to \mathbb{C}_p$ be the morphism sending $\sum_{i \geq 0} \lambda_i x^i$ to $\sum_{i \geq 0} \lambda_i (x^{(i)})$. The de Rham period ring $\mathbb{B}_{p, \text{dR}}^+$ is the completion of $\text{A}_{\mathcal{inf}} \left[ \frac{1}{p} \right]$ at the maximal ideal ker $\Theta$ and $\mathbb{B}_{p, \text{dR}} := \text{Frac}(\mathbb{B}_{p, \text{dR}}^+)$ is the field of $p$-adic periods. The de Rham period ring $\mathbb{B}_{p, \text{dR}}^+$ is a complete discrete valuation ring with residue field $\mathbb{C}_p$ and maximal ideal ker $\Theta$. Note that a generator of ker $\Theta \subset \text{A}_{\mathcal{inf}}$ is given by $[p^i] - p \in \text{A}_{\mathcal{inf}}$, where $p^i = (p, p^{1/p}, p^{1/p^2}, \cdots) \in \mathcal{O}_{\mathcal{C}_p}$. Any other generator is of the form $([p^i] - p) \cdot u$ for $u \in \text{A}_{\mathcal{inf}}$. For more details see [<Fon77>]. There is a filtration on $\mathbb{B}_{p, \text{dR}}$ defined by putting $\mathbb{B}_{p, \text{dR}}^{[i]} = ([p^i] - p)^i \cdot \mathbb{B}_{p, \text{dR}}^+$ for $i \in \mathbb{Z}$, and we define $v_p(x)$ for $x \in \mathbb{B}_{p, \text{dR}} \setminus \{ 0 \}$ by $v_p(x) = i$ if $x \in \mathbb{B}_{p, \text{dR}}^{[i]} \setminus \mathbb{B}_{p, \text{dR}}^{[i+1]}$. For $x \in \mathbb{B}_{p, \text{dR}} \setminus \{ 0 \}$, the quantity

$$v_p(x) := v_p (\Theta (x \cdot ([p^i] - p)^{-v_p(x)})) \in \mathbb{Q}$$

(5.1)

does not depend on the choice of the generator $[p^i] - p$ of $\text{A}_{\mathcal{inf}} \cap \text{ker} \Theta$. Indeed, if we replace the generator $[p^i] - p$ by another generator $([p^i] - p) \cdot u$ with $u \in \text{A}_{\mathcal{inf}}$, because then $v_p (\Theta (x \cdot ([p^i] - p) \cdot u)^{-v_p(x)}) = v_p (\Theta (x \cdot ([p^i] - p)^{-v_p(x)})) + v_p(\Theta(u)) - v_p(x) = v_p (\Theta (x \cdot ([p^i] - p)^{-v_p(x)}))$ as $\Theta(u) \in \mathcal{O}_{\mathcal{C}_p}$. If $x$ and $y$ are two elements of $\mathbb{B}_{p, \text{dR}}$, then $v_p(xy) = v_p(x) + v_p(y)$, and hence $v_p(xy) = v_p(x) + v_p(y)$. But note that $v_p$ does not satisfy the triangle inequality.

Finally, if $K \subset \mathcal{C}_p$ is a finite field extension of $\mathbb{Q}_p$, then there is an action of $\mathcal{G}_K$ on $\mathbb{B}_{p, \text{dR}}$ which respects the filtration, and $(\mathbb{B}_{p, \text{dR}})^{\mathcal{G}_K} = K$. Also note that there does not exist a lift of the absolute Frobenius $\varphi_p$ on $\mathbb{B}_{p, \text{dR}}$.

The $p$-adic period isomorphism is provided by the following theorem which was proved by Fontaine and Messing [<FM87>] using the associated $p$-divisible group.

Theorem 5.5. Let $K_p \subset \mathbb{Q}_p^{\text{alg}}$ be a finite extension of $\mathbb{Q}_p$ and let $X$ be an abelian variety over $K_p$. Then there is a period isomorphism from $p$-adic Hodge theory

$$h_{p, \text{dR}} : H^1_{\text{dR}} (X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{p, \text{dR}} \cong H^1_{\text{dR}} (X, K_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{p, \text{dR}},$$

which is $\mathcal{G}_K$-equivariant and compatible with the filtrations, where on the left hand side, $\mathcal{G}_K$ acts on both factors and the filtration is induced only by $\mathbb{B}_{p, \text{dR}}$, and on the right hand side $\mathcal{G}_K$ acts only on $\mathbb{B}_{p, \text{dR}}$, and the filtration is induced by the Hodge filtration on $H^1_{\text{dR}} (X, K_p)$ and the filtration on $\mathbb{B}_{p, \text{dR}}$.

It was conjectured by Fontaine [<Fon82], A.6] and proved by Faltings [<Fal79] Theorem 8.1], Niziol [<Niz98], and Tsuji [<Tsu99], that the theorem also holds for arbitrary smooth proper schemes over $K_p$.

Example 5.6. Also for the multiplicative group scheme $\mathbb{G}_m := \mathbb{G}_m, \mathbb{Q}_p = \text{Spec} \mathbb{Q}_p [x, x^{-1}]$ there is a period isomorphism between $H^1_{\text{dR}} (\mathbb{G}_m, \mathbb{Z}_p)$ and $H^1_{\text{dR}} (\mathbb{G}_m, \mathbb{Q}_p) = \mathbb{Q}_p^{\mathcal{G}_m}$, see Example 4.5. As in Example 4.3 let $\hat{\varepsilon}_p^{(n)} \in \mathbb{Q}_p^{\text{alg}} \subset \mathbb{Q}_p^{\text{alg}}$ be a primitive $p^n$-th root of unity with $\varepsilon_p^{(n+1)} \rho = \varepsilon_p^{(n)}$, such that $\varepsilon_p = (\varepsilon_p^{(n)})_n \in \mathcal{O}_{\mathcal{C}_p}$. Then...
and an arbitrary abelian variety has complex multiplication if and only if each simple isogeny factor does.

Remark 6.4. This inclusion is an equality.

By Proposition 3.6 this definition is equivalent to the statement that \( \text{QEnd}_\mathbb{C} \) of \( \mathbb{Q} \)-endomorphisms has complex multiplication if and only if \( \text{QEnd}_\mathbb{C} \) of degree 2 is a division algebra, and \( \text{QEnd}_\mathbb{C} \) is the choice of one element from each such pair. More formally:

Definition 6.2. A complex multiplication (CM) type on an abelian variety \( X \) is a subset \( \Phi \) such that

\[
\text{Hom}_\mathbb{Q}(E, \mathbb{C}) = \Phi \sqcup i\Phi \quad \text{(disjoint union)}.
\]

Here \( i\Phi := \{ i \circ \varphi \mid \varphi \in \Phi \} \).

Let \( X \) be an abelian variety over the complex numbers \( \mathbb{C} \). We have seen that \( \text{QEnd}_\mathbb{C}(X) \) is a semi-simple \( \mathbb{Q} \)-algebra which acts faithfully on the (2 dim \( X \))-dimensional \( \mathbb{Q} \)-vector space \( H_1(X, \mathbb{Q}) \). Therefore, by Proposition 6.3,

\[
2 \dim X \geq \text{QEnd}_\mathbb{C}(X) : \mathbb{Q}, \text{red}
\]

and when equality holds, \( \text{QEnd}_\mathbb{C}(X) \) is a product of matrix algebras over fields.

Definition 6.3. An abelian variety \( X \) over a subfield \( K \subset \mathbb{Q} \) is said to have complex multiplication (or be of CM-type, or be a complex abelian variety) over \( K \) if

\[
2 \dim X = \text{QEnd}_K(X) : \mathbb{Q}, \text{red}.
\]

By Proposition 6.6 this definition is equivalent to the statement that \( \text{QEnd}_K(X) \) contains an étale \( \mathbb{Q} \)-subalgebra of degree 2 over \( \mathbb{Q} \). Indeed, if the latter holds then 2 dim \( X \) is less or equal to the degree of a maximal étale \( \mathbb{Q} \)-subalgebra. By Proposition 5.6 the latter degree equals \( \text{QEnd}_K(X) : \mathbb{Q}, \text{red} \). The inequality \( \text{QEnd}_K(X) : \mathbb{Q}, \text{red} \leq 2 \dim X \) proves the claim.

Note that when \( X \) is a CM abelian variety over a field \( K \subset \mathbb{Q} \) then \( \text{QEnd}_K(X) \subset \text{QEnd}_\mathbb{C}(X) \) implies that this inclusion is an equality.

Remark 6.4. Let \( X \approx_K \prod_i X_i^{n_i} \) be the decomposition of \( X \) (up to isogeny) into a product of isotypic abelian varieties over \( K \). Then \( D_i = \text{QEnd}_K(X_i) \) is a division algebra, and \( \text{QEnd}_K(X) \cong \prod M_{n_i}(D_i) \) is the decomposition of \( \text{QEnd}_K(X) \) into a product of simple \( \mathbb{Q} \)-algebras. From the above definition and Proposition 6.3 we see that \( X \) has complex multiplication if and only if \( D_i \) is a commutative field of degree 2 for all \( i \). In particular, a simple abelian variety \( X \) has complex multiplication if and only if \( \text{QEnd}_K(X) \) is a field of degree 2 over \( \mathbb{Q} \), and an arbitrary abelian variety has complex multiplication if and only if each simple isogeny factor does.
Let $X$ be an abelian variety over $\mathbb{C}$. An endomorphism $\alpha$ of $X$ defines an endomorphism of the vector space $H_1(X, \mathbb{Q})$ of dimension $2\dim X$ over $\mathbb{Q}$. Therefore, the characteristic polynomial $P_{\alpha}$ of $\alpha$ is defined as

$$P_{\alpha}(T) := \det (\alpha - T| H_1(X, \mathbb{Q})).$$

It is monic, of degree $2\dim X$, and has coefficients in $\mathbb{Z}$. More generally, we define the characteristic polynomial of any element of $\text{QEnd}(X)$ by the same formula.

Consider an endomorphism $\alpha$ of an abelian variety $X$ over $\mathbb{C}$, and write $X = \mathbb{C}^g/\Lambda$ with $\Lambda = H_1(X, \mathbb{Z})$. If $\alpha$ is an isogeny, then $\alpha : \Lambda \to \Lambda$ is injective, and it defines an isomorphism

$$\ker(\alpha) = \alpha^{-1}\Lambda/\Lambda \hookrightarrow \Lambda/\alpha\Lambda.$$

Therefore, for an isogeny $\alpha : X \to X$

$$\deg \alpha = \# \ker(\alpha) = \left| \det (\alpha| H_1(X, \mathbb{Q})) \right| = |P_{\alpha}(0)|.$$

More generally, for any integer $r$ we have $\deg(\alpha - r) = \left| \det (\alpha - r| H_1(X, \mathbb{Q})) \right| = |P_{\alpha}(r)|$; compare [CSS6, Chap 5 § 12].

For the convenience of the reader we reproduce the proof from [Mil06] of the following results.

**Lemma 6.5** ([Mil06, Lemma 3.7]). Let $F$ be a subfield of $\text{QEnd}(X)$, where $X$ is an abelian variety over $\mathbb{C}$. If $F$ has a real prime, then $[F : \mathbb{Q}]$ divides $\dim X$.

**Proof.** First note that $H_1(X, \mathbb{Q})$ is a vector space over $F$ of dimension $m := 2\dim X/[F : \mathbb{Q}]$. So for any $\alpha \in \text{End}(X) \cap F$, the characteristic polynomial $P_{\alpha}(T)$ is the $m$-th-power of the characteristic polynomial of $\alpha$ in $F/\mathbb{Q}$. In particular,

$$\text{Norm}_{F/\mathbb{Q}}(\alpha)^m = \deg \alpha \geq 0.$$

However, if $F$ has a real prime, then from the weak approximation theorem $\alpha$ can be chosen to be large and negative at that prime and close to 1 at the remaining primes so that $\text{Norm}_{F/\mathbb{Q}}(\alpha) < 0$. This gives a contradiction unless $m$ is even.

For the next proposition recall the definition of a Rosati involution on $\text{QEnd}_K(X)$. Let $\lambda$ be a polarization on $X$, that is, $\lambda$ is an isogeny $X \to X^\vee$. It has an inverse in $\text{QHom}_K(X^\vee, X)$. The Rosati involution on $\text{QEnd}_K(X)$ corresponding to $\lambda$ is

$$\alpha \mapsto \alpha^\dagger = \lambda^{-1} \circ \alpha^\vee \circ \lambda$$

(6.1)

**Proposition 6.6** ([Mil06, Proposition 3.6]). (a) A simple abelian variety $X$ has complex multiplication if and only if $\text{QEnd}(X)$ is a CM-field of degree $2\dim X$ over $\mathbb{Q}$.

(b) An isotypic abelian variety $X$ has complex multiplication if and only if $\text{QEnd}(X)$ contains a field of degree $2\dim X$ over $\mathbb{Q}$ (which can be chosen to be a CM-field invariant under some Rosati involution).

(c) An abelian variety $X$ has complex multiplication if and only if $\text{QEnd}(X)$ contains an étale $\mathbb{Q}$-algebra $E$ (which can be chosen to be a CM-algebra invariant under some Rosati involution) of degree $2\dim X$ over $\mathbb{Q}$. In this case $H_1(X, \mathbb{Q})$ is free over $E$ of rank 1.

**Proof.** (a) $\text{QEnd}_K(X)$ is a field extension of $\mathbb{Q}$ of degree $2\dim X$ by Remark 5.4. We know that it is either totally real or CM because it is stable under the Rosati involutions (6.1). Now Lemma 6.5 shows that $\text{QEnd}_K(X)$ is a CM-field.

For (b) and (c) see [Mil06, Proposition 3.6].

**Definition 6.7.** Let $X$ be an abelian variety with complex multiplication, so that $\text{QEnd}(X)$ contains a CM-algebra $E$ for which $H_1(X, \mathbb{Q})$ is a free $E$-module of rank 1, and let $\Phi$ be the set of homomorphisms $E \to \mathbb{C}$ occurring in the representation of $E$ on $T_0(X)$, i.e., $T_0(X) \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi$ where $\mathbb{C}_\varphi$ is the one-dimensional $\mathbb{C}$-vector space on which $\varphi(a)$ acts as $\varphi(a)$. Then, because

$$H_1(X, \mathbb{R}) \cong T_0(X) \oplus \overline{T_0(X)},$$

(6.2)

$\Phi$ is a CM-type on $E$, and we say that, $X$ together with the injective homomorphism $E \to \text{QEnd}(X)$ is of CM-type $(E, \Phi)$. 

14
Let \( e \) be a basis vector for \( H_1(X, \mathbb{Q}) \) as an \( E \)-module, and let \( a \) be the \( \mathcal{O}_E \)-lattice in \( E \) such that \( ae = H_1(X, \mathbb{Z}) \). Under the above isomorphism

\[
H_1(X, \mathbb{R}) \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi \oplus \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi, \\
e \otimes 1 \quad \mapsto (\cdots, e_\varphi, \cdots; \cdots; e_{i_0\varphi}, \cdots)
\]

where each \( e_\varphi \) is a \( \mathbb{C} \)-basis for \( \mathbb{C}_\varphi \). The \( e_\varphi \) determine an isomorphism

\[
T_0(X) \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi.
\]

Next we state two important results on abelian varieties with complex multiplication from [ST61] and [ST68] which we will need later.

**Proposition 6.8.** [ST61] Prop 26 §12.4] Let \( X \) be an abelian variety over \( K = K^{\text{sep}} \) with complex multiplication, then there exists an abelian variety isogenous to \( X \) defined over a field which is a finite extension of \( \mathbb{Q} \) or of a finite field.

**Theorem 6.9.** [ST68] Thm 6] Let \( X \) be an abelian variety over a finite extension \( K/\mathbb{Q} \) with complex multiplication, then there exists a finite extension \( L/K \) such that \( X \) has good reduction at every place of \( \mathcal{O}_L \).

### 7 The Faltings Height of an Abelian Variety

We recall the definition of the *Faltings height* of an abelian variety introduced by Faltings [CS86] Chapter 2, § 3. Let \( K \) be a number field, \( \mathcal{O}_K \) the ring of integers in \( K \). We define a metrized line bundle on \( \text{Spec}(\mathcal{O}_K) \) to be a projective \( \mathcal{O}_K \)-module \( P \) of rank 1, together with norms \( \| \cdot \|_v \) on \( P \otimes_{\mathcal{O}_K} K_v \) for all infinite places \( v \) of \( K \), where \( K_v \) denotes the completion of \( K \) at \( v \). We define \( \varepsilon_v = 1 \) or 2 according to whether \( K_v \cong \mathbb{R} \) or \( K_v \cong \mathbb{C} \). The degree of the metrized line bundle is defined as

\[
\deg(P, \| \cdot \|) = \log(\#(P/\mathcal{O}_K \cdot x)) - \sum_{v \mid \infty} \varepsilon_v \log \| x \|_v,
\]

where \( x \) is a nonzero element of \( P \) and the sum runs over all infinite places of \( K \). The right-hand side is of course independent of \( x \) because of the product formula (1.1).

Let now \( X \) be an abelian variety of dimension \( g \) over \( K \), and let \( \mathcal{X} \) be the relative identity component of the Néron model of \( X \) over \( \mathcal{O}_K \). Assume that \( \mathcal{X} \) is semi-abelian, i.e. a smooth algebraic group \( \phi : \mathcal{X} \to \text{Spec}(\mathcal{O}_K) \), whose fibers are connected of dimension \( g \), and are extensions of an abelian variety by a torus. Let \( s : \mathcal{X} \to \text{Spec}(\mathcal{O}_K) \) be the zero section. Let \( \omega_{\mathcal{X}/\mathcal{O}_K} = s^*(\Omega^g_{\mathcal{X}/\mathcal{O}_K}), \omega_{\mathcal{X}/\mathcal{O}_K} \) is a line bundle on \( \mathcal{O}_K \). The metrics at the infinite places are given by

\[
\| \alpha \|^2_v := \frac{1}{(2\pi)^g} \int_{X_v(\mathbb{C})} |\alpha \wedge \bar{\alpha}| \quad \text{for} \quad \alpha \in \omega(X_v) = \Gamma(X_v, \Omega^g_{\mathcal{X}_v}),
\]

where \( X_v \) denotes the base change of \( X \) under the map \( K \to K_v \). Then Faltings [CS86] Chapter 2, § 3] defines a moduli-theoretic height as follows.

**Definition 7.1.** The *Faltings height* \( h_{\text{Fal}}(X) \) of \( X \) is defined as

\[
h_{\text{Fal}}(X) := \frac{1}{[K : Q]} \deg(\omega_{\mathcal{X}/\mathcal{O}_K}, \| \cdot \|). \tag{7.1}
\]

It is easy to check that \( h_{\text{Fal}}(X) \) is invariant under extension of the ground field.

### 8 Colmez’s Conjecture on Periods of CM Abelian Varieties

In [Col93] P. Colmez considers product formulas for periods of abelian varieties in the following

\[
\text{End}_F(X) \otimes \mathbb{Q} = \mathbb{Q} \left[ \sum_{\varphi \in \Phi} a_\varphi E_\varphi \right],
\]

where each \( E_\varphi \) is an \( \mathbb{F}_v \)-basis for \( \mathbb{C}_\varphi \).
Situation 8.1. Let $X$ be an abelian variety defined over a number field $K$ equipped by the ring of integers $\mathcal{O}_E$ in a CM-field $E$ and of CM-type $(E, \Phi)$. Let $H_E := \text{Hom}_\mathbb{Q}(E, \mathbb{Q}^{\text{alg}})$ be the set of all ring homomorphisms $E \hookrightarrow \mathbb{Q}^{\text{alg}}$ and assume that $K$ contains $\psi(E)$ for every $\psi \in H_E$. By Theorem 4.3, we may assume moreover, that $K$ is a finite Galois extension of $\mathbb{Q}$ and that $X$ has good reduction at every prime of $\mathcal{O}_K$. For a $\psi \in H_E$ let $\omega_\psi \in H^1_{\text{dR}}(X, K)$ be a non-zero cohomology class such that $b^*\omega_\psi = \psi(b) \cdot \omega_\psi$ for all $b \in E$. For every embedding $\eta : K \hookrightarrow \mathbb{Q}^{\text{alg}}$, let $X^K := X \otimes_{K, \eta} K$ and $\omega_\psi^K \in H^1_{\text{dR}}(X^K, K)$ be deduced from $X$ and $\omega_\psi$ by base extension. Let $(u_\eta)_{\eta} \in \prod_{\eta \in H_K} H_1(X^K(C), \mathbb{Z})$ be a family of cycles compatible with complex conjugation $c$, that is $u_\eta = c(u_\eta)$. Let $v$ be a place of $\mathbb{Q}$.

If $v = \infty$ the de Rham (Theorem 4.4) between Betti and de Rham cohomology yields a pairing

$$\langle \cdot, \cdot \rangle : H_1(X^K(C), \mathbb{Z}) \times H^1_{\text{dR}}(X^K, K) \to \mathbb{C}, \quad (u_\eta, \omega_\psi^K) \mapsto \langle u_\eta, \omega_\psi^K \rangle_{\infty}.$$ 

We define the complex absolute value $|\langle u_\eta, \omega_\psi^K \rangle|_{\infty} := |\langle u_\eta, \omega_\psi^K \rangle|_{\mathbb{R}} \in \mathbb{R}$.

If $v$ corresponds to a prime number $p \in \mathbb{Z}$, the comparison isomorphism $H^1(X^K(C), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong H^1_{\text{et}}(X^K, \mathbb{Z}_p)$ together with the comparison isomorphism from $p$-adic Hodge theory (Theorems 4.4 and 5.5) yield a pairing

$$\langle \cdot, \cdot \rangle_p : H_1(X^K(C), \mathbb{Z}) \times H^1_{\text{et}}(X^K, K) \to \mathbb{B}_{p, \text{dR}}, \quad (u_\eta, \omega_\psi^K) \mapsto \langle u_\eta, \omega_\psi^K \rangle_p.$$ 

We define the absolute value $|\langle u_\eta, \omega_\psi^K \rangle_p| := |\langle u_\eta, \omega_\psi^K \rangle|_{\mathbb{R}} \in \mathbb{R}$, where the “valuation” $v_p$ on $\mathbb{B}_{p, \text{dR}}$ was defined in (5.1). Let $\mathcal{O}_K$ now consider the product $\prod_{\eta \in H_K} |\langle u_\eta, \omega_\psi^K \rangle|_{v}$, or equivalently (modulo $1/\#H_K$) times its logarithm

$$\frac{1}{\#H_K} \sum_{\eta \in H_K} \sum_{v \in \mathfrak{v}_K} \log |\langle u_\eta, \omega_\psi^K \rangle|_v = \frac{1}{\#H_K} \sum_{\eta \in H_K} \log |\langle u_\eta, \omega_\psi^K \rangle|_{\infty} - \frac{1}{\#H_K} \sum_{v \neq v_p \in \mathfrak{v}_K} \sum_{\eta \in H_K} v_p (\langle u_\eta, \omega_\psi^K \rangle_p) \log p. \quad (8.1)$$

The right sum over all $v = v_p$ does not converge. Namely, Colmez [Col93 Theorem II.1.1] proves the following Theorem 8.3 below. To formulate the theorem we need a

Definition 8.2. In this definition we denote by $Q$ the function field from the introduction or the field $\mathbb{Q}$, and by $\mathcal{O}_E$ the completion of $Q$ at a place $v \neq \infty$. The case $Q = \mathbb{Q}$ is relevant in the present section, and the other case will be relevant in Section 10. For $F = Q$ or $F = \mathcal{O}_E$, let $F^{\text{sep}}$ be the separable closure of $F$ in $F^{\text{alg}}$ and let $\mathcal{G}_F := \text{Gal}(F^{\text{sep}}/F)$. Let $C(\mathcal{G}_F, \mathbb{Q})$ be the $\mathbb{Q}$-vector space of locally constant functions $\alpha : \mathcal{G}_F \to \mathbb{Q}$ and let $C^0(\mathcal{G}_F, \mathbb{Q}) := \text{Hom}_{\mathbb{Q}}(\mathcal{G}_F, \mathbb{Q})$ be the subspace of those functions which are constant on conjugacy classes, that is, which satisfy $a(h^{-1}gh) = \alpha(g)$ for all $g, h \in \mathcal{G}_F$. Then the $\mathbb{C}$-vector space $C^0(\mathcal{G}_F, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is spanned by the traces of representations $\rho : \mathcal{G}_F \to \text{GL}_n(\mathbb{C})$ with open kernel for varying $n$ by [Ser77 § 2.5, Theorem 6]. Via the fixed embedding $Q^{\text{sep}} \hookrightarrow Q_{\text{sep}}$ we consider the induced inclusion $\mathcal{G}_Q \subset \mathcal{G}_F$ and morphism $C(\mathcal{G}_F, \mathbb{Q}) \to C(\mathcal{G}_Q, \mathbb{Q})$. If $\chi$ is a trace of a representation $\rho : \mathcal{G}_Q \to \text{GL}_n(\mathbb{C})$ with open kernel we let $L(\chi, s) := \prod_{v \neq \infty} L_v(\chi, s)$, respectively $L^{\infty}(\chi, s) := \prod_{v \neq \infty} L_v(\chi, s)$ be the Artin $L$-function of $\rho$ with, respectively without the factor at $\infty$. Note that the latter factor involves the Gamma-function if $Q = \mathbb{Q}$. These $L$-functions only depend on $\chi$ and converge for all $s \in \mathbb{C}$ with $Re(s) > 1$; see [Lang84 Chapter XII, § 2] for $Q = \mathbb{Q}$ and [Ros02, pp. 126ff] for the function field case. We also let $q_v$ be the cardinality of the residue field of $Q_v$ (this means $q_v = p$ if $Q = \mathbb{Q}$ and $Q_v = \mathbb{F}_q$) and we set

$$Z^\infty(\chi, s) := \frac{d}{ds} \frac{L^{\infty}(\chi, s)}{L^{\infty}(\chi, s)} = - \sum_{v \neq \infty} Z_v(\chi, s) \log q_v \quad \text{with} \quad (8.2)$$

$$Z_v(\chi, s) := \frac{d}{ds} \frac{L_v(\chi, s)}{L_v(\chi, s) \cdot \log q_v} = \frac{d_q}{dq} \frac{L_v(\chi, s)}{q_v \cdot L_v(\chi, s)} \quad (8.3).$$

Moreover, we let $f_\chi$ be the Artin conductor of $\chi$. If $Q = \mathbb{Q}$, it is a positive integer $f_\chi = \prod_{v \neq \infty} p_{\text{Art}, v}(\chi) \in \mathbb{Z}$, and if $Q$ is the function field of the curve $C$ it is an effective divisor $f_\chi = \sum_{v} \mu_{\text{Art}, v}(\chi) \cdot [v]$ on $C$; see [Ser77 Chapter VI, §§ 2.3], where $\mu_{\text{Art}, v}(\chi)$ is denoted $f(\chi, v)$. In particular, only finitely many values $\mu_{\text{Art}, v}(\chi)$ are non-zero. We set

$$\mu^{\infty}_{\text{Art}}(\chi) := \log(f_\chi) = \sum_{v \neq \infty} \mu_{\text{Art}, v}(\chi) \log q_v \quad \text{if } Q = \mathbb{Q}, \quad \text{respectively} \quad (8.4)$$

$$\mu_{\text{Art}}(\chi) := \deg(f_\chi) \log q = \sum_{v \neq \infty} \mu_{\text{Art}, v}(\chi) [F_v : F_q] \log q = \sum_{v \neq \infty} \mu_{\text{Art}, v}(\chi) \log q_v \quad \text{and} \quad (8.5)$$

$$\mu^{\infty}_{\text{Art}}(\chi) := \sum_{v \neq \infty} \mu_{\text{Art}, v}(\chi) \log q_v \quad \text{if } Q \text{ is a function field}. \quad (8.5)$$
By linearity we extend $Z^\infty(\ldots, s)$ and $\mu_\infty^{\text{Art}}$ to all $a \in C^0(\mathcal{G}_Q, \mathbb{Q})$ and $Z_v(\ldots, s)$ and $\mu_{\text{Art}, v}$ to all $a \in C^0(\mathcal{G}_{Q_v}, \mathbb{Q})$. The map $Z_v(\ldots, s)$ takes values in $\mathbb{Q}(q_v^{-s})$.

For our CM-type $(E, \Phi)$ and for every $\psi \in H_E$ we define the functions

$$a_{E, \psi, \Phi}: \mathcal{G}_Q \to \mathbb{Z}, \quad g \mapsto \begin{cases} 1 & \text{when } g \psi \in \Phi \\ 0 & \text{when } g \psi \notin \Phi \end{cases} \quad \text{ and}$$

$$a_{E, \psi, \Phi}^0: \mathcal{G}_Q \to \mathbb{Q}, \quad g \mapsto \frac{1}{\#H_K} \sum_{\eta \in H_K} a_{E, \eta \psi, \eta \Phi}(g) = \frac{\#\{\eta \in H_K: \eta^{-1} g \psi \in \Phi\}}{\#H_K} \tag{8.6}$$

which factor through $\text{Gal}(K/\mathbb{Q})$ by our assumption that $\psi(E) \subset K$ for all $\psi \in H_E$. In particular, $a_{E, \psi, \Phi} \in C(\mathcal{G}_Q, \mathbb{Q})$ and $a_{E, \psi, \Phi}^0 \in C^0(\mathcal{G}_Q, \mathbb{Q})$ is independent of $K$.

We also define integers $v_p(\omega_p^0)$ which are all zero except for finitely many. Let $K_p$ be the $p$-adic completion of $K \subset \mathbb{Q}_{\text{alg}} \subset \mathbb{Q}_p^{\text{alg}} \subset \mathbb{C}_p$ and let $X^\eta$ be an abelian scheme over $\mathcal{O}_{K_p}$ with $X^\eta \otimes \mathcal{O}_{K_p} K_p \cong X^\eta \otimes K K_p$. Then there is an element $x \in K_p^*$, unique up to multiplication by $\mathcal{O}_{K_p}^*$, such that $x^{-1} \omega_p^0$ is an $\mathcal{O}_{K_p}$-generator of the free $\mathcal{O}_{K_p}$-module of rank one

$$H^0(\mathcal{X}^\eta, \mathcal{O}_{K_p}): = \{ \omega \in H^1_{\text{dR}}(\mathcal{X}^\eta, \mathcal{O}_{K_p}): b^* \omega = \eta \psi(b) \cdot \omega \ \forall b \in \mathcal{O}_E \},$$

and we set

$$v_p(\omega_p^0) := v_p(x) \in \mathbb{Z}. \tag{8.7}$$

This value does not depend on the choice of the model $\mathcal{X}^\eta$ with good reduction, because all such models are isomorphic over $\mathcal{O}_{K_p}$. Now Colmez [Col93] Theorem II.1.1 computed the terms in $(8.1)$ as follows.

**Theorem 8.3.** If the image of $u_\eta$ in $H_1(X^\eta(\mathbb{C}), \mathbb{Q}_p) = H_{1, \text{et}}(X^\eta, \mathbb{Z}_p)$ is a generator of the $\mathcal{O}_E \otimes \mathbb{Z}_p$-module $H_{1, \text{et}}(X^\eta, \mathbb{Z}_p) = T_p X^\eta$, then

$$-\frac{1}{\#H_K} \sum_{\eta \in H_K} v_p(\omega_p^0, u_\eta) = Z_p(a_{E, \psi, \Phi}^0, v) - \mu_{\text{Art}, p}(a_{E, \psi, \Phi}^0) + \frac{1}{\#H_K} \sum_{\eta \in H_K} v_p(\omega_p^0). \tag{8.8}$$

Since $-\mu_{\text{Art}, p}(a_{E, \psi, \Phi}^0) + \frac{1}{\#H_K} \sum_{\eta \in H_K} v_p(\omega_p^0)$ vanishes for all but finitely many primes $p$ and $\sum_p Z_p(a_{E, \psi, \Phi}^0)$ diverges, the sum $(8.1)$ diverges. Colmez [Col93] Convention 0 assigns to this divergent sum a value by the following

**Convention 8.4.** Let $(x_p)_{p < \infty}$ be a tuple of complex numbers indexed by the prime numbers $p$ in $\mathbb{Z}$. We will give a sense to the (divergent) series $\Sigma = \sum_{p < \infty} x_p$ in the following situation. We suppose that there exists an element $a \in C^0(\mathcal{G}_Q, \mathbb{Q})$ such that $x_p = -Z_p(a, 1) \log p$ for all $p$ except for finitely many. Then we let $a^* := a(g^{-1})$. We further assume that $Z^\infty(a^*, s)$ does not have a pole at $s = 0$, and we define the limit of the series $\sum_{p < \infty} x_p$ as

$$\Sigma := -Z^\infty(a^*, 0) - \mu_{\text{Art}}(a) + \sum_{p < \infty} (x_p + Z_p(a, 1) \log p) \tag{8.9}$$

inspired by the functional equation relating $L(a, s)$ with $L(a^*, 1 - s)$ deprived of the terms at $\infty$.

**Example 8.5.** The convention allows to prove the product formula for the multiplicative group $G_m := \mathbb{G}_{m, \mathbb{Q}} = \text{Spec } \mathbb{Q}[x, x^{-1}]$. Namely, for the generator $\omega = \frac{dx}{x}$ of $H_1_{\text{et}}(G_m, \mathbb{Q}) = \mathbb{Q} : \omega$ and for the cycle $u: [0, 1] \to G_m(\mathbb{C}) = \mathbb{C}^\times$ given by $u(s) = \exp(2\pi is)$ with $H_1(G_m(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \cdot u$, we have computed in Examples 4.3, 4.5 and 5.6

$$\langle \omega, u \rangle_{\infty} = 2\pi i$$

and

$$\log |\langle \omega, u \rangle|_{\infty} = \log(2\pi),$$

$$\langle \omega, u \rangle_p = t_p$$

and

$$\log |\langle \omega, u \rangle|_p = \log |t_p|_p = -\frac{\log p}{p-1} = -Z_p(1, 1) \log p,$$

where $1(g) = 1$ for every $g \in \mathcal{G}_Q$. So Convention 8.4 implies $\sum_{p < \infty} \log |\langle \omega, u \rangle|_p = -\zeta(0) = -\log(2\pi)$ for the Riemann Zeta-function $\zeta$, and $\sum_v \log |\langle \omega, u \rangle|_v = 0$. Therefore $\prod_v |\langle \omega, u \rangle|_v = 1$. 

17
The Convention 8.4 and the Theorem 8.3 allow us to give to the divergent sum (8.1) a convergent interpretation. In order to remove the dependency on the chosen cycles \((u_\eta)_\eta \in \prod_{\eta \in H_K} H_1(X^\eta(\mathbb{C}), \mathbb{Z})\), Colmez considers the value

\[
\langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v := \left( t_v \cdot \frac{\langle \omega^\eta_{E,\psi}, u_\eta \rangle_v}{\langle \omega^\eta_{cE,\psi}, u_\eta \rangle_v} \right)^{\frac{1}{2}},
\]

where \(t_\infty = 2\pi i\) and for \(v = v_p \neq \infty\), \(t_v = t_p\) is the \(p\)-adic analog of \(2\pi i\) from Examples 5.6 and 8.5. Note that \(\Phi \cup c\Phi = H_E\) implies \(a_{E,\psi,\Phi}^0 = a_{E,\psi,cE,\psi,\Phi}^0 = 1\), and hence \(Z_p(a_{E,\psi,\Phi}^0, 1) + Z_p(a_{E,\psi,cE,\psi,\Phi}^0, 1) = Z_p(1, 1)\) and \(\mu_{Art,p}(a_{E,\psi,\Phi}^0) + \mu_{Art,p}(a_{E,\psi,cE,\psi,\Phi}^0) = \mu_{Art,p}(1) = 0\). Therefore, Theorem 8.3 implies

\[
\frac{1}{\# H_K} \sum_{\eta \in H_K} v_p(\langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v) = \frac{1}{2} \left( Z_p(1, 1) + Z_p(a_{E,\psi,\Phi}^0, 1) - \mu_{Art,p}(a_{E,\psi,\Phi}^0) + \frac{1}{\# H_K} \sum_{\eta \in H_K} v_p(\omega^\eta_{cE,\psi}) \right)
\]

\[
- Z_p(a_{E,\psi,cE,\psi,\Phi}^0, 1) - \mu_{Art,p}(a_{E,\psi,cE,\psi,\Phi}^0) - \frac{1}{\# H_K} \sum_{\eta \in H_K} v_p(\omega^\eta_{cE,\psi}) \right).
\]

Using Convention 8.4 one thus obtains

\[
\frac{1}{\# H_K} \sum \sum_{\eta \in H_K} \log \left| \langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v \right|_v \quad (8.11)
\]

\[
= -Z^\infty((a_{E,\psi,\Phi}^0)^*, 0) + \frac{1}{\# H_K} \sum_{\eta \in H_K} \left( \log \left| \langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v \right|_v \right) - \frac{1}{2} \sum_{p<\infty} \left( v_p(\omega^\eta_{cE,\psi}) - v_p(\omega^\eta_{E,\psi}) \right) \log p
\]

Colmez formulated the following

**Conjecture 8.6 (Col93 Conjecture 0.1).** The sum (8.11) is zero, or equivalently the product formula holds:

\[
\prod_v \prod_{\eta \in H_K} \left| \langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v \right|_v = 1.
\]

He then proved

**Lemma 8.7 (Col93 Lemme II.2.9).** In Situation 8.1 the value

\[
ht(E, \psi, \Phi) := \frac{1}{\# H_K} \sum_{\eta \in H_K} \left( \log \left| \langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v \right|_v \right) - \frac{1}{2} \sum_{p<\infty} \left( v_p(\omega^\eta_{cE,\psi}) - v_p(\omega^\eta_{E,\psi}) \right) \log p \quad (8.12)
\]

only depends on \(E, \psi\) and \(\Phi\) and not on the choice of \(X, \omega_\psi, u_\eta\) and \(K\).

Colmez also relates the product formula to the Faltings height, see Definition 7.1

**Theorem 8.8 (Col93 Théorème II.2.10(ii)).** In Situation 8.1 the Faltings height \(ht_{Fal}(X)\) of \(X\) satisfies

\[
ht_{Fal}(X) = -\sum_{\psi \in \Phi} ht(E, \psi, \Phi) - \frac{1}{2} \mu_{Art}(a_{E,\psi,\Phi}^0) \quad (8.13)
\]

This immediately implies the following

**Corollary 8.9.** In Situation 8.1 the following assertions are equivalent.

(a) \(ht(E, \psi, \Phi) = Z^\infty((a_{E,\psi,\Phi}^0)^*, 0)\).

(b) The product formula holds, that is, the expression (8.11) is zero and \(\prod_v \prod_{\eta \in H_K} \left| \langle \omega^\eta_{E,\psi}, \omega^\eta_{cE,\psi}, u_\eta \rangle_v \right|_v = 1\).

If (a) and (b) hold for all \(\psi \in \Phi\) then \(ht_{Fal}(X) = -\sum_{\psi \in \Phi} Z^\infty((a_{E,\psi,\Phi}^0)^*, 0) - \frac{1}{2} \mu_{Art}(a_{E,\psi,\Phi}^0)\).

Colmez [Col93 Conjecture II.2.11] conjectures that statements (a) and (b) of Corollary 8.9 hold for all \(E, \psi, \Phi\). There are various partial results in this direction. The first is due to Colmez himself who was able to prove the following theorem up to a rational multiple of \(\log 2\), which was then removed by Ous.
Theorem 8.10 ([Col93 Théorème 0.5], [Obu13 Theorem 4.9]). If $E$ is abelian over $\mathbb{Q}$, then the product formula holds true for every $\psi, \Phi$, and hence

$$ht_{\text{Fal}}(X) = -\sum_{\psi \in \Phi} Z^\infty((a^0_{E,\psi,\Phi})^*, 0) - \frac{1}{2} \mu_{\text{Art}}(a^0_{E,\psi,\Phi}).$$ (8.14)

There has been much further work and progress on Colmez’s conjecture by many people. For example, Yang [Yan13] proved it for a large class of CM fields of degree $|E : \mathbb{Q}| = 4$, including the first known cases when $E/\mathbb{Q}$ is non-abelian. Let us also mention the most recent results by Andreatta, Goren, Howard, Madapusi Pera [AGHMP18], Yuan, Zhang [YZ18] and Barquero-Sanchez, Masri [BSM18].

Theorem 8.11 ([AGHMP18 Theorem A], [YZ18 Theorem 1.1]). For every CM-field $E$ Colmez’s conjecture holds true on average over all CM-types $\Phi$, that is

$$\sum_{\Phi} \sum_{\psi \in \Phi} ht(E, \psi, \Phi) = \sum_{\Phi} \sum_{\psi \in \Phi} Z^\infty((a^0_{E,\psi,\Phi})^*, 0).$$

As a consequence of Theorem 8.11 Barquero-Sanchez and Masri [BSM18] Theorem 1.1] proved that for any fixed totally real number field $F$ of degree $|F : \mathbb{Q}| \geq 3$ there are infinitely many effective, “positive density” sets of CM extensions $E/F$ such that $E/\mathbb{Q}$ is non-abelian and Colmez’s conjecture (8.14) on the Faltings height holds true for $E$ and any $\Phi$. Moreover, they prove

Theorem 8.12 ([BSM18 Theorem 1.4]). In Situation 8.1 if the Galois closure of $E$ has degree $2^{\dim X} \cdot (\dim X)!$ over $\mathbb{Q}$, then

$$ht_{\text{Fal}}(X) = -\sum_{\psi \in \Phi} Z^\infty((a^0_{E,\psi,\Phi})^*, 0) - \frac{1}{2} \mu_{\text{Art}}(a^0_{E,\psi,\Phi}).$$

As another consequence of Theorem 8.11 Tsimerman [Tsi18] and Gao [Gao17] prove the André-Oort-Conjecture for mixed Shimura varieties of abelian type:

Theorem 8.13 ([Tsi18 Theorem 1.3], [Gao17 Theorem 13.6]). Let $S$ be a mixed Shimura variety of abelian type over $\mathbb{C}$. Let $S' \subset S$ be an irreducible closed subvariety which contains a Zariski dense subset of special points of $S$. Then $S'$ is a special subvariety.

Part II

Drinfeld Modules and $A$-motives

9 Basic Definitions

Following the general philosophy about similarities between number fields and function fields, we now transfer the contents of Part I to characteristic $p$. Here Drinfeld modules replace elliptic curves and $A$-motives replace abelian varieties. We follow the expositions in [Gos96 Ch.4], [Tha04 Ch.2] and begin with the the analog of Notation 2.1

Notation 9.1. Let $\mathbb{F}_q$ be a finite field with $q$ elements and characteristic $p$. Let $C$ be a smooth projective, geometrically irreducible curve over $\mathbb{F}_q$ with function field $Q = \mathbb{F}_q(C)$. Let $\infty \in C$ be a fixed closed point and let $A := \Gamma(C \setminus \{\infty\}, O_C)$ be the $\mathbb{F}_q$-algebra of those rational functions on $C$ which are regular outside $\infty$. Let $v_\infty$ be the valuation associated with the prime $\infty$.

By a place of $C$ we mean a closed point $v \in C$. So either $v = \infty$ or $v$ is a maximal ideal of $A$. It defines a normalized valuation on $Q$ which we also denote by $v$, respectively by $v_\infty$ and which takes the value $v(z_v) = 1$ on a uniformizing parameter $z_v \in Q$ at $v$. We now fix such a uniformizer $z_v$ at every $v$ and if $v = \infty$ we abbreviate $z_\infty$ to $z$.

We denote the residue field of $v$ by $\mathbb{F}_v$, its degree over $\mathbb{F}_q$ by $d_v = [\mathbb{F}_v : \mathbb{F}_q]$ and its cardinality by $q_v := \#\mathbb{F}_v = q^{d_v}$. Thus if $a \in A \setminus \mathbb{F}_q$ then $v_\infty(a) < 0$, because $\mathbb{F}_q$ is the field of constants in $Q$ as $C$ is geometrically irreducible, see [Gro63 4.3.1 and Proposition 4.5.9c]). The ring $A$ and its fraction field $Q$ play the role of $\mathbb{Z}$ and $\mathbb{Q}$ in the arithmetic of function fields.

Let $Q_v$ be the completion of $Q$ with respect to the valuation $v$ and let $A_v \subset Q_v$ be the valuation ring of $v$. Then there is a canonical isomorphism $A_v \cong \mathbb{F}_v[[z_v]]$. Let $Q_v^{alg}$ be a fixed algebraic closure of $Q_v$, and let $C_v$ be the completion of $Q_v^{alg}$ with respect to the canonical extension of $v$. We also use $v$ to denote this extension to
$Q_v^{\text{alg}}$ and thus to $\mathbb{C}_v$. However, we denote the image of $z_v$ in $\mathbb{C}_v$ by $\zeta_v$ and abbreviate $\zeta_v$ to $\zeta$. Note that $\mathbb{C}_v$ is algebraically closed. On $\mathbb{C}_v$ and all its subrings we consider the normalized absolute value $| \cdot |_v : \mathbb{C}_v \to \mathbb{R}_{\geq 0}$ given by $|x|_v = q_v^{-v(x)}$. We let $\mathcal{O}_{\mathbb{C}_v} = \{ x \in \mathbb{C}_v : |x|_v \leq 1 \}$ be the valuation ring of $\mathbb{C}_v$. We also fix an algebraic closure $Q^{\text{alg}}$ of $Q$ and an embedding $Q^{\text{alg}} \hookrightarrow Q^{\text{alg}}$ for every place $v$ of $Q$.

Let $K$ be a field extension of $\mathbb{F}_q$ and fix an $\mathbb{F}_q$-morphism $\gamma : A \to K$. We will call the pair $(K, \gamma : A \to K)$ an $A$-field. The prime ideal $\ker(\gamma) \subset A$ is called the $A$-characteristic of $K$ and is denoted $A\text{-char}(K, \gamma)$ or simply $A\text{-char}(K)$. If $A\text{-char}(K) = 0$ we say $K$ has generic $A$-characteristic. Then $\gamma$ is injective and $K$ is via $\gamma$ a field extension of $Q$. If $A\text{-char}(K) = v \subset A$ is a maximal ideal, we say that $A\text{-char}(K)$ is finite and $K$ has finite $A$-characteristic $v$. Then $K$ is via $\gamma$ a field extension of $\mathbb{F}_q$.

Let $G_{a,K} = \text{Spec}(K[X])$ be the additive group scheme over $K$ and let $\tau \in \text{End}_K(G_{a,K})$ be the $q$-th power Frobenius endomorphism given by $\tau^*(X) = X^q$. Also every $b \in K$ induces an endomorphism $\psi_b \in \text{End}_K(G_{a,K})$ given by $\psi_b^*(X) = bX$. These endomorphisms satisfy $\tau \circ \psi_b = \psi_{\tau b} \circ \tau$. Then the ring $\text{End}_{K,\mathbb{F}_q}(G_{a,K})$ of $\mathbb{F}_q$-linear endomorphisms of group schemes over $K$ equals the non-commutative polynomial ring over $K$ in $\tau$:

$$K\{\tau\} := \{ \sum_{i=0}^n b_i \tau^i : n \in \mathbb{N}_0, b_i \in K \} \quad \text{with} \quad \tau b = b^q \tau.$$

For $n \in \mathbb{N}_0$, $\tau^i = \sum_{i=0}^n b_i \tau^i$ we set $\text{deg}_\tau(\sum_{i=0}^n b_i \tau^i) = \max \{ i : b_i \neq 0 \}$.

**Definition 9.2.** Let $(K, \gamma : A \to K)$ be an $A$-field. A Drinfeld $A$-module over $K$ is a pair $G = (G, \varphi)$ with $G \cong G_{a,K}$ and $\varphi$ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to \text{End}_{K,\mathbb{F}_q}(G) \cong K\{\tau\}$, $a \mapsto \varphi(a)$, such that

(a) $\text{Lie}(\varphi(a)) = \gamma(a)$ i.e. $(a - \gamma(a)) \cdot \text{Lie}(G) = 0$ in $K$ for all $a \in A$.

(b) There exists an $a \in A$ such that $\varphi(a) \in K\{\tau\} \setminus K$, i.e. $\varphi(a) \neq \gamma(a) \cdot \tau^0$ i.e. $\text{deg}_\tau(\varphi(a)) > 0$.

Then there is an integer $r > 0$ such that $\text{deg}_\tau(\varphi(a)) = -r \text{deg}_\tau(\varphi(a))$ for every $a \in A$, see [Gos96] § 4.5. It is called the rank of $(G, \varphi)$ and is denoted $\text{rk}(G, \varphi)$ or $\text{rk}_\varphi$. Also sometimes a Drinfeld $A$-module $G = (G, \varphi)$ is simply denoted by $\varphi$.

A morphism between Drinfeld $A$-modules $(G, \varphi)$ and $(G', \varphi')$ over $K$ is a homomorphism $f : G \to G'$ of group schemes such that $\varphi'_a \circ f = f \circ \varphi_a$ for every $a \in A$. We denote the set of morphisms between $G$ and $G'$ by $\text{Hom}_K(G, G')$ and we write $\text{End}_K(G) := \text{Hom}_K(G, G)$.

In particular, for every $c \in A$ the commutation $\varphi(c) \circ \varphi(a) = \varphi(ac) = \varphi(c \circ a) = \varphi(c) \circ \varphi(a)$ implies that $\varphi(c) \in \text{End}_K(G)$. Thus $\text{End}_{K,\mathbb{F}_q}(G)$ is an $A$-algebra via $A \to \text{End}_{K,\mathbb{F}_q}(G)$, $c \mapsto \varphi(c)$ and $\text{Hom}_{K,\mathbb{F}_q}(G, G')$ is an $A$-module. So we may also define $\text{QHom}_{K,\mathbb{F}_q}(G, G') := \text{Hom}_{K,\mathbb{F}_q}(G, G') \otimes_A Q$ and write $\text{QEnd}_{K,\mathbb{F}_q}(G) := \text{QHom}_{K,\mathbb{F}_q}(G, G) = \text{End}_{K,\mathbb{F}_q}(G) \otimes_A Q$.

**Remark 9.3.** Drinfeld $A$-modules possess higher dimensional generalizations, which are called abelian Anderson $A$-modules, see [Har17] Definition 1.2. They were originally defined by Anderson [And86] for $A = \mathbb{F}_q[t]$ under the name abelian $t$-modules. These are group schemes which carry an action of the ring $A$ subject to certain conditions. Abelian Anderson $A$-modules are the function field analogs of abelian varieties. Although Anderson worked over a field, abelian Anderson $A$-modules also exist naturally over arbitrary $A$-algebras as base rings. They possess an (anti-)equivalent description by semi-linear algebra objects called $A$-motives, which we will define next. Through the work of Drinfeld and Anderson it was realized very early on that a Drinfeld module or abelian Anderson $A$-module over a field is completely described by its $A$-motive. The same is true over an arbitrary $A$-algebra $R$, as is shown for example in [Har17]. So in a way the situation in function field arithmetic is much better than in the arithmetic of abelian varieties (which only have a local $p$-adic semi-linear algebra description via the Dieudonné module of the associated $p$-divisible group, see Remark 5.3) the $A$-motive is a “global” Dieudonné module which integrates the “local” Dieudonné modules for every prime in a single object. We will return to this in Section 13 and Proposition 14.4.7.

Before we define $A$-motives we have to fix some

**Notation 9.4.** For an $A$-field $(K, \gamma)$ we write $A_K := A \otimes_{\mathbb{F}_q} K$ and set $J := (a \otimes 1 - 1 \otimes \gamma(a)) : a \in A \subset A_K$. We consider the endomorphism $\sigma^* := \text{id}_A \otimes \text{Frob}_{b^q,K}$ of $A_K$, where $\text{Frob}_{b^q,K}(b) = b^q$ for $b \in K$. For an $A_K$-module $M$ we set $\sigma^* M := M \otimes_{A_K,\sigma^*} A_K$ and we write $\sigma^m : M \to \sigma^m M$, $m \mapsto m \otimes 1$ for the natural $\sigma^*$-semi-linear map. For a homomorphism $f : M \to N$ of $A_K$-modules we set $\sigma^* f := f \otimes \text{id}_{A_K} : \sigma^* M \to \sigma^* N$. Note that the endomorphism $\sigma^*$ corresponds to a morphism of schemes

$$\sigma := \text{id}_C \otimes \text{Spec}(\text{Frob}_{b^q,K}) : C_K := C \times_{\mathbb{F}_q} \text{Spec} K \to C_K \quad (9.1)$$
which is the identity on points and on sections of $\mathcal{O}_C$ and the $q$-Frobenius on $K$. It satisfies $\sigma|_{\text{Spec} A_K} = \text{Spec}(\sigma^*)$: $\text{Spec} A_K \to \text{Spec} A_K$.

**Example 9.5.** Before we give the general definition of $A$-motives, we define the $A$-motive associated to a Drinfeld $A$-module $G = (G, \varphi)$ over $K$ as in [And86]. Namely, we set

$$M := M(G) := M(\varphi) := \text{Hom}_{K,F_q}(G,G_{a,K}),$$

where $\text{Hom}_{K,F_q}(\mathbb{F}_q, \mathbb{F}_q)$ is the group of $\mathbb{F}_q$-linear homomorphisms of group schemes over $K$. Every choice of an isomorphism $G \cong G_{a,K}$ induces an isomorphism $M(G) \cong K\{\tau\}$. We make $M(G)$ into an $A\{\tau\} = A \otimes_{\mathbb{F}_q} K\{\tau\}$ module in the fashion given below:

$$\begin{align*}
(a,m) \mapsto m \circ \varphi_a & \quad \text{for } m \in M, a \in A; \\
(b,m) \mapsto \psi_b \circ m & \quad \text{for } m \in M, b \in K; \\
(\tau,m) \mapsto \tau m = \text{Frob}_q \circ m & \quad \text{for } G_{a,K} \to G_{a,K}: m \in M.
\end{align*}$$

Since the actions of $a \in A$ and of $b \in K$ commute, i.e. $a(b \cdot m) = \psi_b \circ m \circ \varphi_a = b(a \cdot m)$, this makes $M$ into a module over $A\{\tau\} := A \otimes_{\mathbb{F}_q} K$. It is not difficult to see that $M$ is a locally free $A_{K}$-module of rank $r := \text{rk}_G$, see [Gos96, Lemma 5.4.1]. Now for $a \in A$ and $b \in K$ we have

$$\tau \circ (a \otimes b)(m) = \tau \circ (\psi_b \circ m \circ \varphi_a) = \psi_b \circ \tau \circ m \circ \varphi_a = (a \otimes b^\theta) \circ \tau m.$$ 

Since the action of $\tau$ is not $A_{K}$-linear but $\sigma^*$-semi linear, it induces an $A_{K}$-linear map $\tau_M : \sigma^* M \to M$ defined by $\tau_M(m \otimes 1) = \tau m$. There is a canonical isomorphism coker $\tau_M \cong \text{Lie} G^\vee$ given by sending $m \in M$ to Lie($m$): Lie($G$) $\to$ Lie($G_{a,K}$) $= K$, see [Gos96, 5.4.7]. This implies $\dim_K(\text{coker} \tau_M) = 1$, which can also be seen directly from $M \cong K\{\tau\}$ and $\tau_M(\sigma^* M) \cong K\{\tau\} \cdot \tau$.

The above construction motivate the definition of $A$-motives:

**Definition 9.6.** An (effective) $A$-motive of rank $r$ and dimension $d$ over $K$ is a pair $M = (M, \tau_M)$ consisting of a locally free $A_{K}$-module $M$ of rank $r$ and an $A_{K}$-homomorphism $\tau_M : \sigma^* M \to M$ such that

1. $\dim_K(\text{coker} \tau_M) = d$.
2. $(a - \gamma(a))^d \cdot \text{coker} \tau_M = 0$ for all $a \in A$.

We write $\text{rk}_M := r$ and $\dim M := d$.

A morphism between $A$-motives $(M, \tau_M)$ $\to$ $(N, \tau_N)$ over $K$ is an $A_{K}$-homomorphism $f : M \to N$ with $f \circ \tau_M = \tau_N \circ \sigma^* f$. We denote the set of morphisms between $M$ and $N$ by $\text{Hom}_K(M,N)$ and we write $\text{End}_K(M) := \text{Hom}_K(M,M)$. Since $\sigma^*(a) = a$ for all $a \in A$ and $\tau_M$ is $A_{K}$-linear, we have $a \cdot \text{id}_M \in \text{End}_K(M)$. Thus $\text{End}_K(M)$ is an $A_{K}$-algebra via $A \to \text{End}_K(M), a \mapsto a \cdot \text{id}_M$ and $\text{Hom}_K(M,N)$ is an $A_{K}$-module. So we may also define $Q\text{Hom}_K(M,N) := \text{Hom}_K(M,N) \otimes_A Q$ and write $Q\text{End}_K(M) := Q\text{Hom}_K(M,M) = \text{End}_K(M) \otimes_A Q$.

On the relation with Drinfeld $A$-modules we have the following theorem, see [And86] or [Gos96, §5.4].

**Theorem 9.7.** The contravariant functor $G \mapsto M(G)$ from Drinfeld $A$-modules to $A$-motives over $K$ is fully faithful. Its essential image consists of all $M = (M, \tau_M)$ such that $M$ is free over $K\{\tau\}$ of rank 1. The latter implies that $\dim M = 1$.

In this sense we view $A$-motives as higher dimensional generalizations of Drinfeld $A$-modules. As an illustration of the claim that $A$-motives (and abelian Anderson $A$-modules) play the role of abelian varieties, see for example [BH09] where the theory of $A$-motives over finite fields is developed in analogy with [Tat66].

**Example 9.8.** Let $C = \mathbb{F}_q^1$, and set $A = \mathbb{F}_q[t]$. Then $A_{K} = K[t]$. Let $K = \mathbb{F}_q(\theta)$ be the rational function field in the variable $\theta$ and let $\gamma : A \to K$ be given by $\gamma(t) = \theta$. The Carlitz module over $K$ is given by $G = (G_{a,K}, \varphi)$ with $\varphi : \mathbb{F}_q[t] \to K\{\tau\}$ defined by $\varphi t = \theta + \tau$. The $A$-motive associated with the Carlitz module is given by $\mathcal{C} = (\mathcal{C} = K[t], \tau_{\mathcal{C}} = t - \theta)$ and is called the Carlitz motive. Both $G$ and $\mathcal{C}$ have rank 1. As we will see in Examples 12.3 and 14.10 below, the Carlitz module is the function field analog of the multiplicative group $G_{m,K}$ from Example 4.3.
10 Isogenies and Semi-simple A-Motives

If we define the rank of an abelian variety $X$ as $\text{rk} X := 2 \cdot \text{dim} X$, see Remark 3.3 below, the analog of Theorem 3.1 is the following:

**Theorem 10.1.** For two A-motives $\mathcal{M}$ and $\mathcal{N}$ over an A-field $K$ the $A$-module $\text{Hom}_K(\mathcal{M}, \mathcal{N})$ is finite projective of rank $\leq (\text{rk} \mathcal{M}) \cdot (\text{rk} \mathcal{N})$. The same is true for Drinfeld $A$-modules over $K$.

**Proof.** For $A$-motives this was proved by Anderson [And86 Corollary 1.7.2] and for Drinfeld $A$-modules it can be found in [Gos96 Theorem 4.7.8].

**Definition 10.2.** Let $\mathcal{G} = (G, \varphi)$ and $\mathcal{G}' = (G', \varphi')$ be two Drinfeld $A$-modules over $K$. A non zero morphism $f \in \text{Hom}_K(\mathcal{G}, \mathcal{G}')$ is called an isogeny. If there is an isogeny $f : \mathcal{G} \to \mathcal{G}'$, then $\mathcal{G}$ and $\mathcal{G}'$ are isogenous.

From [Gos96 4.7.13], we know that if there is an isogeny $f : \mathcal{G} \to \mathcal{G}'$, then there exists some nonzero $a \in A$ and an isogeny $\hat{f} : \mathcal{G}' \to \mathcal{G}$ such that $\hat{f} f = \varphi_a$ and $f \hat{f} = \varphi'_a$.

In particular, if $0 \neq f \in \text{End}_K(\mathcal{G})$, then $f$ is invertible in $\text{QEnd}(\mathcal{G}) := \text{End}_K(\mathcal{G}) \otimes_A Q$, so $\text{QEnd}(\mathcal{G})$ is a finite dimensional division algebra over $Q$.

**Definition 10.3.** Let $\mathcal{M}$ and $\mathcal{N}$ be two A-motives over $K$. A morphism $f \in \text{Hom}_K(\mathcal{M}, \mathcal{N})$ is called an isogeny if $f$ is injective and $\text{coker} f$ is a finite dimensional $K$-vector space. If there exists an isogeny $f \in \text{Hom}_K(\mathcal{M}, \mathcal{N})$ then $\mathcal{M}$ and $\mathcal{N}$ are said to be isogenous over $K$ and we write $\mathcal{M} \cong_K \mathcal{N}$. This defines an equivalence relation by Remark 10.4(d) below.

**Remark 10.4.**

(a) Two Drinfeld $A$-modules are isogenous if and only if their associated $A$-motives are isogenous, see [Har17 Theorem 5.9 and Proposition 5.8].

(b) If two $A$-motives $\mathcal{M}$ and $\mathcal{N}$ are isogenous then $\text{rk} \mathcal{M} = \text{rk} \mathcal{N}$ and $\text{dim} \mathcal{M} = \text{dim} \mathcal{N}$, see [Har17 Proposition 5.8].

(c) Conversely, let $f : \mathcal{M} \to \mathcal{N}$ be a morphism of $A$-motives with $\text{rk} \mathcal{M} = \text{rk} \mathcal{N}$. Then $f$ is injective if and only if $\text{coker} f$ is a finite dimensional $K$-vector space, and in this case $f$ is an isogeny. Indeed, since $\mathcal{M}$ is locally free over $A_K$, it is contained in $\mathcal{M} \otimes_{A_K} \text{Quot}(A_K)$ where $\text{Quot}(A_K)$ denotes the fraction field of $A_K$. Since $\text{rk} \mathcal{M} = \text{rk} \mathcal{N}$ the injectivity of $f$ is equivalent to $f$ inducing an isomorphism $\mathcal{M} \otimes_{A_K} \text{Quot}(A_K) \to \mathcal{N} \otimes_{A_K} \text{Quot}(A_K)$, and this in turn is equivalent to $f$ being torsion, and hence finite.

(d) If $f : \mathcal{M} \to \mathcal{N}$ is an isogeny between $A$-motives, then there exists non-canonically an isogeny $\hat{f} : \mathcal{N} \to \mathcal{M}$ and a non-zero element $a \in A$ with $\hat{f} f = a \cdot \text{id}_\mathcal{M}$ and $f \hat{f} = a \cdot \text{id}_\mathcal{N}$ by [Har17 Corollary 5.15]

(e) Let $\mathcal{M}$ and $\mathcal{N}$ be $A$-motives over $K$. If $\mathcal{M}$ and $\mathcal{N}$ are isogenous over $K$ via an isogeny $f$, then $\text{QEnd}_K(\mathcal{M}) \cong \text{QHom}_K(\mathcal{M}, \mathcal{N}) \cong \text{QEnd}_K(\mathcal{N})$, $h \mapsto f \circ h \mapsto f \circ h \circ f^{-1}$.

More precisely, $\text{QHom}_K(\mathcal{M}, \mathcal{N})$ is a free right $\text{QEnd}_K(\mathcal{M})$-module of rank 1 and a free left $\text{QEnd}_K(\mathcal{N})$-module of rank 1. If $\mathcal{M}$ and $\mathcal{N}$ are not isogenous then $\text{QHom}_K(\mathcal{M}, \mathcal{N}) = (0)$.

**Definition 10.5.** Let $\mathcal{M}$ be an $A$-motive over $K$.

(a) An $A$-factor-motive over $K$ of $\mathcal{M}$ is an $A$-motive $\mathcal{M}'$ together with a surjective morphism $\mathcal{M} \to \mathcal{M}'$ of $A$-motives over $K$.

(b) $\mathcal{M}$ is called simple over $K$ if $\mathcal{M}$ is non trivial and $\mathcal{M}$ has no $A$-factor-motives over $K$ other than $(0)$ and $\mathcal{M}$.

(c) $\mathcal{M}$ is called semisimple over $K$ if $\mathcal{M}$ is isogenous to a direct sum of simple $A$-motives over $K$, i.e. $\mathcal{M} \cong_K \oplus_i \mathcal{M}_i$ with $\mathcal{M}_i$ simple.

**Remark 10.6.**

(a) In comparison to the analogous Definition 3.2 for abelian varieties, $A$-motives behave dually. This is due to the fact that the functor from Drinfeld $A$-modules to $A$-motives is contravariant.

(b) All Drinfeld $A$-modules are simple.

(c) But in contrast to abelian varieties (Remark 3.3) not every A-motive is semi-simple up to isogeny. This was observed in [BHO09 Examples 6.1 and 6.13].
(d) Let $M$ and $N$ be two $A$-motives over $K$ of same rank and let $M$ be simple over $K$. Then every non-zero morphism $f \in \text{Hom}_K(M,N)$ is an isogeny. Namely, the image of $f$ is a non-zero $A$-factor-motive of $M$, and hence isomorphic to $M$ via $f_1$, because $M$ is simple. So $f$ is injective and hence an isogeny by Remark 10.2(d).

In particular, if $M$ is simple over $K$ then every non-zero endomorphism $0 \neq f \in \text{End}_K(M)$ is an isogeny and therefore invertible in $Q\text{End}_K(M)$ by Remark 10.2(d). This implies that $Q\text{End}_K(M)$ is a division algebra over $Q$.

Moreover, if $M$ is semisimple over $K$ with decomposition $M \approx_K M_1 \oplus \cdots \oplus M_n$ up to isogeny into simple $A$-motives $M_i$ over $K$, then $Q\text{End}_K(M)$ decomposes into a finite direct product of full matrix algebras over the division algebras $Q\text{End}_K(M_i)$ over $Q$, compare Remark 3.3.

11 Analytic Theory of Drinfeld Modules

In this section we consider Drinfeld $A$-modules over $\mathbb{C}_\infty$, which is an $A$-field via the natural inclusion $A \subset Q \subset Q_\infty \subset \mathbb{C}_\infty$ denoted by $\gamma$.

If $G = (G_n, \mathbb{C}_\infty, \varphi)$ with $\varphi : A \rightarrow \mathbb{C}_\infty \{\tau\}$ is a Drinfeld $A$-module over $\mathbb{C}_\infty$ then there is a uniquely determined power series $\exp_G(z) = \sum_{i=0}^{\infty} e_i z^i$ with $e_i \in \mathbb{C}_\infty$, $e_0 = 1$ satisfying

$$\varphi_a(\exp_G(z)) = \exp_G(\gamma(a) \cdot z)$$

for all $a \in A$, see [Gos96 4.6.7]. It is called the exponential function of $G$. The power series $\exp_G(z)$ converges for every $z \in \mathbb{C}_\infty$ and its kernel $\Lambda(G)$ is an $A$-lattice in $\mathbb{C}_\infty$ (that is, a finitely generated projective, discrete $A$-submodule) of the same rank as the Drinfeld $A$-module $G$. Note that $\mathbb{C}_\infty$ is infinite dimensional over $Q_\infty$ and therefore contains $A$-lattices of arbitrarily high rank.

Conversely, let $\Lambda \subset \mathbb{C}_\infty$ be an $A$-lattice of rank $r$. Then the function

$$\exp_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - \frac{x}{\lambda})$$

(11.1)

converges for every $z \in \mathbb{C}_\infty$ and can be written as an everywhere convergent power series in $z$. Moreover $\exp_\Lambda : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is a surjective $\mathbb{F}_q$-linear map whose zeroes are simple and located at $\Lambda$. For more details see [Gos96 §4.2]. For $a \in A \setminus \{0\}$ we can now define the polynomial

$$\varphi^a_\Lambda(x) := \gamma(a) \cdot x \cdot \prod_{0 \neq \lambda \in \gamma(a)^{-1}\Lambda/\Lambda} (1 - \frac{x}{\exp_\Lambda(\lambda)}) \in \mathbb{C}_\infty[x].$$

(11.2)

It satisfies

$$\exp_\Lambda(\gamma(a) \cdot z) = \varphi^a_\Lambda(\exp_\Lambda(z))$$

(11.3)

and makes the following diagram with exact rows commutative

$$\begin{array}{ccc}
0 & \rightarrow & \Lambda \\
\uparrow & & \uparrow \gamma(a) \\
0 & \rightarrow & \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \rightarrow 0 \\
\downarrow & & \downarrow \exp_\Lambda \\
0 & \rightarrow & \Lambda \rightarrow \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \rightarrow 0 \\
\downarrow & & \downarrow \varphi^a_\Lambda \\
0 & \rightarrow & \Lambda \\
\end{array}$$

(11.4)

It is easy to see that

(a) $\varphi^a_\Lambda(x)$ is an $\mathbb{F}_q$-linear polynomial, i.e. $\varphi^a_\Lambda \in \mathbb{C}_\infty[\tau]$, of $\tau$-degree $\deg_\tau(\varphi^a_\Lambda) = -rd_\infty v_\infty(a)$;

(b) $\varphi^a : a \mapsto \varphi^a_\Lambda$ defines a ring homomorphism $\varphi^a : A \rightarrow \mathbb{C}_\infty$, of $\tau$-degree $\deg_\tau(\varphi^a) = -rd_\infty v_\infty(a)$.

The additive group $\mathbb{C}_\infty$, considered as the quotient $\mathbb{C}_\infty/\Lambda$ via $\exp_\Lambda$, thus carries a new structure as an $A$-module given by $z \mapsto \varphi^a_\Lambda(z)$ for $a \in A$. Therefore, for every $A$-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank $r$ we get a Drinfeld $A$-module $G^\Lambda := (G_n, \mathbb{C}_\infty, \varphi^a)$ of rank $r$ over $\mathbb{C}_\infty$.

Definition 11.1. Let $\Lambda_1$, $\Lambda_2$ be two $A$-lattices of the same rank. A morphism from $\Lambda_1 \rightarrow \Lambda_2$ is an element $c \in \mathbb{C}_\infty$, with $c\Lambda_1 \subseteq \Lambda_2$. If the ranks of $\Lambda_1$ and $\Lambda_2$ are different, then we only allow $0 \in \mathbb{C}_\infty$ to be a morphism.

Theorem 11.2 ([Gos96 Theorem 4.6.9]). The functors $G \mapsto \Lambda(G)$ and $\Lambda \mapsto G^\Lambda$ give an equivalence of categories between the category of Drinfeld $A$-modules over $\mathbb{C}_\infty$ and the category of $A$-lattices in $\mathbb{C}_\infty$. 23
Corollary 11.3. If $G$ is a Drinfeld $A$-module over a field $K$ of generic $A$-characteristic, then $\text{QEnd}_K(G)$ is a commutative field whose degree over $K$ divides $\text{rk}G$.

Proof. Since $G$ and all elements of $\text{QEnd}_K(G)$ are defined over a finitely generated subfield $K_0$ of $K$, we can choose a $Q$-embedding $K_0 \hookrightarrow C_\infty$ and it suffices to prove the corollary when $K = C_\infty$. In this case $G \cong G^\Lambda$ for an $A$-lattice $\Lambda \subset C_\infty$ of rank equal to $\text{rk}G$. By Theorem 11.2, we have isomorphisms $\text{End}_K(G) \cong \{ c \in C_\infty : c\Lambda \subset \Lambda \}$, $f \mapsto \text{Lie}(f)$ and $\text{QEnd}_K(G) \cong \{ c \in C_\infty : c(Q \cdot \Lambda) \subset Q \cdot \Lambda \}$. In particular $\text{QEnd}_K(G) \subset C_\infty$ is a commutative field. Since $Q \cdot \Lambda \subset C_\infty$ is a $Q$-vector space of dimension $\text{rk}G$ and also a $\text{QEnd}_K(G)$-vector space, the formula $\text{rk}G = \dim_Q(Q \cdot \Lambda) = [\text{QEnd}_K(G) : Q] \cdot \dim_{\text{QEnd}_K(G)}(Q \cdot \Lambda)$ tells us that $[\text{QEnd}_K(G) : Q]$ divides $\text{rk}G$. 

We regard Drinfeld $A$-modules and particularly those of rank two as analogues of elliptic curves, where the functional equation (11.3) for $\exp_A(z)$ corresponds to the group law derived from (2.3). The point is that (2.3) defines a $Z$-module structure on the elliptic curve $C/\Lambda \cong E_\Lambda(C)$, while (11.2) and (11.3) define the above $A$-module structure on the additive group scheme $G_{a,K}$.

Definition 11.4. Let $G$ be a Drinfeld $A$-module of rank $r$ over $C_\infty$. The Betti (co-)homology realization of $G$ is defined by

$$H^1_{\text{Betti}}(G, R) := \Lambda(G) \otimes_A R \quad \text{and} \quad H^1_{\text{Betti}}(G, R) := \text{Hom}_A(\Lambda(G), R)$$

for any $A$-algebra $R$. Both are free $R$-modules of rank $r$.

12 Torsion Points and $v$-adic Cohomology of Drinfeld Modules

Definition 12.1. Let $G = (G, \varphi)$ be a Drinfeld $A$-module over an $A$-field $K$ and let $G(K^{\text{alg}})$ be the set of $K^{\text{alg}}$-valued points of $G$. For an element $a \in A$, we set

$$G[a](K^{\text{alg}}) := \varphi[a](K^{\text{alg}}) := \{ P \in G(K^{\text{alg}}) \mid \varphi_a(P) = 0 \},$$

and we call $G[a](K^{\text{alg}})$ the module of $a$-torsion points of $G = (G, \varphi)$. If $a \subseteq A$ is an ideal, we set

$$G[a](K^{\text{alg}}) := \varphi[a](K^{\text{alg}}) := \{ P \in G(K^{\text{alg}}) \mid \varphi_a(P) = 0 \text{ for all } a \in a \}.$$

The latter are the $K^{\text{alg}}$-valued points of a closed subgroup scheme $G[a]$ of $G$, which is an $A/a$-module scheme via $\tau \mapsto \varphi_a[\tau]$. If $a = (a)$ then $G[a](K^{\text{alg}}) = G[a](K^{\text{alg}})$.

Remark 12.2. We have the following observation, see [Gos96, §4.5], where we denote the $A$-characteristic of $K$ by $p = A$-char$(K) := \ker(\gamma : A \to K)$:

(a) If $a \in A$ is prime to $A$-char$(K)$, we see that the polynomial $\varphi_a$ is separable and $#G[a](K^{\text{alg}}) = (#A/(a))^{\text{rk}G}$. Since this holds for every $a \in A$ and $G[a](K^{\text{alg}})$ is an $A$-module, one obtains $G[a](K^{\text{alg}}) \cong (A/a)^{\text{rk}G}$ as $A$-modules.

(b) $#G[p](K^{\text{alg}}) = (#A/(p))^{\text{rk}G-h}$ and $G[p](K^{\text{alg}}) \cong (A/(p))^{\text{rk}G-h}$, where $h$ is the height of the Drinfeld $A$-module defined by $h := \frac{\omega(a)}{\nu_p(a)}$ for every $a \in A$, where $\omega(a)$ is the smallest integer $i \geq 0$ with $\tau^i$ occurring in $\varphi_a$, with nonzero coefficient.

Example 12.3. The Carlitz module $G = (G_{a,K}, \varphi)$ over $K = \mathbb{F}_q(\theta)$ with $\varphi_t = \theta + \tau$ from Example 9.8 has rank 1. For every $a = \sum_{i=0}^n a_i t^i$ with $a_i \in \mathbb{F}_q$ and $a_n \neq 0$, we have $\varphi_a = \sum_{i=0}^n a_i \varphi^i = \sum_{i=0}^n a_i (\theta + \tau)^i = (\sum_{i=0}^n a_i \theta^i) \cdot r^0 + \ldots + a_n \tau^n = \gamma(a) r^0 + \ldots + a_n \tau^n$. Therefore, the polynomial $\varphi_a(x) = \gamma(a) x + \ldots + a_n x^{\tau^n}$ has degree $\tau^n$ and is separable, because $\gamma(a) \neq 0$. From this it follows that $#G[a](K^{\text{alg}}) = q^n = #A/(a)$ and that $G[a](K^{\text{alg}}) \cong A/(a)$ for every $a \in A$. This illustrates that the Carlitz module is the function field analog of the multiplicative group $\mathbb{G}_m = \mathbb{G}_m, \mathbb{Q}$ from Example 14.3, which for $a \in \mathbb{N}_{>0}$ satisfies $\mathbb{G}_m[a](\mathbb{Q}^{\text{alg}}) := \ker[a](\mathbb{Q}^{\text{alg}}) = \{ x \in \mathbb{Q}^{\text{alg}} : x^a = 1 \} \cong \mathbb{Z}/(a)$.

Definition 12.4. Let $v$ be a prime ideal of $A$. Let $G = (G, \varphi)$ be a Drinfeld $A$-module over $K$ of fixed rank $r$ and define the $A_v$-module $G[v^{\infty}](K^{\text{alg}})$ := $\bigcup_{n \geq 1} G[v^n](K^{\text{alg}})$. The $A_v$-module

$$H^1_{v}(G, A_v) := T_v(G) = \text{Hom}_{A_v}(Q_v/A_v, G(K^{\text{alg}})) = \text{Hom}_{A_v}(Q_v/A_v, G[v^{\infty}](K^{\text{alg}})).$$

(12.1)
is called the $v$-adic homology realization or the $v$-adic Tate module of $G$. It carries a continuous $G_K$-action. Note that when $z = \frac{\mathfrak{a}}{\mathfrak{b}} \in Q$ is a uniformizing parameter of $A_v$ then the map $\varphi_z := \varphi_\mathfrak{b}^{-1} \circ \varphi_\mathfrak{a}: G[v^n](K^{alg}) \to G[v^{n-1}](K^{alg})$ is well defined and

$$T_v(G) \cong \lim_{\leftarrow} (G[v^n](K^{alg}), \varphi_z).$$

see for example [HK14, after Definition 4.8]. A morphism $f: G \to G'$ of Drinfeld $A$-modules gives a morphism $T_v(f): T_v(G) \to T_v(G')$ of $A_v[G_K]$-modules. If $v$ is different from the $A$-characteristic $A$-char$(K)$ of $K$, then $T_v(G)$ is isomorphic to $A_v^{rk}$. 

**Remark 12.5.** The results of this section parallel Remark 2.7 for abelian varieties. Since the $\ell$-adic Tate module of an abelian variety $X$ is isomorphic to $(\mathbb{Z}_\ell)^{2 \dim X}$, while the $v$-adic Tate module of a Drinfeld $A$-module $G$ is isomorphic to $A_v^{rk}G$, it is natural to call the number $\text{rk } X := 2 \dim X$ the rank of the abelian variety $X$, compare also Theorems 3.1 and 10.1.

There is a similar theory of Tate modules for $A$-motives which we will explain in the next section.

### 13 Cohomology Realizations and Period Maps for $A$-Motives

#### 13.1 Uniformizability and Betti Cohomology

In this section we discuss the notion of uniformizability, cohomology realizations and period maps for $A$-motives from [11] and also we generalize the results to the case $d_\infty = [F_\infty : F_q] \neq 1$. For a field extension $K$ of $F_q$ we consider the closed subscheme $\infty_K := \infty \times_{F_q} \text{Spec } K \subset C_K := C \times_{F_q} \text{Spec } K$. If $K$ contains $F_\infty$, then $\infty_K$ is the disjoint union of $d_\infty$-many $K$-rational points of $C_K$.

In order to define the notion of uniformizability for $A$-motives we have to introduce some notion of rigid analytic geometry as in [HP04]. For a general introduction to rigid analytic geometry see [BGR84].

**Notation 13.1.** With the curve $C_{C_\infty}$ and its open affine part $C'_{C_\infty} := C_{C_\infty} \setminus \infty_{C_\infty}$ one can associate by [BGR84] §9.3 rigid analytic spaces $C_{C_\infty} := (C_{C_\infty})^{rig}$ and $C'_{C_\infty} := (C'_{C_\infty})^{rig} = C_{C_\infty} \setminus \infty_{C_\infty}$. The underlying sets of $C_{C_\infty}$ and $C'_{C_\infty}$ are the sets of $C_\infty$-valued points of $C_{C_\infty}$ and $C_{C_\infty} \setminus \infty_{C_\infty}$, respectively. The endomorphism $\sigma$ of $C_{C_\infty}$ from [9.1] induces endomorphisms of $C_{C_\infty}$ and $C'_{C_\infty}$ which we denote by the same symbol $\sigma$.

Let $O_{C_\infty}$ be the valuation ring of $C_\infty$ and let $k_{C_\infty}$ be its residue field. By the valuative criterion of properness every point of $C_{C_\infty} = C_{C_\infty}(C_\infty) = C_{C_\infty}(k_{C_\infty})$ extends uniquely to an $O_{C_\infty}$-valued point of $C$ and in the reduction gives rise to a $k_{C_\infty}$-valued point of $C$. This gives us a reduction map

$$red : C_{C_\infty} = C(C_\infty) \rightarrow C(k_{C_\infty}) \quad (13.1).$$

The subscheme $\infty_{k_{C_\infty}} \subset C_{k_{C_\infty}}$ contains $d_\infty$ points. We denote them by $\{\infty_i \text{ for } i \in \mathbb{Z}/d_\infty \mathbb{Z}\}$ in such a way that the map $\sigma$ from [9.1] transports $\infty_i$ to $\infty_{i+1}$ and $(\sigma^{d_\infty})^* \text{ stabilizes each } \infty_i$. Since the curve $C_{k_{C_\infty}}$ is non-singular, [BLR85] Proposition 2.2 implies for each $i$ that the preimage $D_i$ of $\infty_i \in C_{k_{C_\infty}}$ under $red$ is an open rigid analytic unit disc in $C_{C_\infty}$ around $\infty_i$. Let $D'_i := D_i \setminus \infty_i$ be the punctured open unit disc around $\infty_i$ in $C_{C_\infty}$. Then $\sigma$ maps $D_i$ isomorphically onto $D_{i+1}$. We let $O(D_i)$ and $O(C_{C_\infty} \setminus \cup_i D_i)$ be the coordinate rings of rigid analytic functions on the spaces $D_i$ and $C_{C_\infty} \setminus \cup_i D_i$, respectively. The uniformizer $z \in O(D_i)$ is a coordinate function on the disc $D_i$ for every $i$.

**Example 13.2.** If $C = \mathbb{P}^1_{F_q}$, $A = F_q[t]$, and $[F_\infty : F_q] = 1$, we can give the following explicit description. $D_0 \subset \mathbb{P}^1(C_\infty)$ is the open unit disc around $\infty$.

$$O(C_{C_\infty} \setminus D_0) := C_\infty(t) := \left\{ \sum_{i=0}^{\infty} a_it^i, \quad a_i \in C_\infty, \quad a_i \to 0 \text{ as } i \to \infty \right\}$$

and $C_{C_\infty} \setminus D_0$ is the closed unit disc inside $C(C_\infty) \setminus \infty_{C_\infty} = C_\infty$ on which the coordinate $t$ has absolute value less or equal to 1. Also we can take $z = 1/t$ as the coordinate on the disc $D_0$, and suggestively write $D_0 = \{|z| < 1\}$.

**Definition 13.3.** For an $A$-motive $M$ over $C_\infty$, we define the $\tau$-invariants

$$\Lambda(M) := (M \otimes_{A_{C_\infty}} O(C_{C_\infty} \setminus \cup_i D_i))^* := \{ m \in M \otimes_{A_{C_\infty}} O(C_{C_\infty} \setminus \cup_i D_i) : \quad \tau_M(\sigma^mM) = m \}.$$
Definition 13.4. An A-motive $M$ is called \textit{uniformizable} (or \textit{rigid analytically trivial}) if the natural homomorphism

$$h_M : \Lambda(M) \otimes_A \mathcal{O}(C_{\infty} \cup \mathfrak{D}) \rightarrow M \otimes_{A_{\infty}} \mathcal{O}(C_{\infty} \cup \mathfrak{D}), \quad \lambda \otimes f \mapsto f \cdot \lambda$$

is an isomorphism.

Example 13.5. We keep the notation from Example 13.4. We recall that the Carlitz motive over $C_{\infty}$ is given by $\mathcal{C} = (C = C_{\infty}[t], \tau_C = t - \theta)$. We set $\tau^* := \prod_{i=0}^{\infty} (1 - \frac{t}{\theta^i}) \in \mathcal{O}(C_{\infty}) \subset \mathcal{O}(C_{\infty} \cup \mathfrak{D}_0)$ and choose an $\eta \in C_{\infty}$ with $\eta^{1-q} = -\theta$. Then we see that $\eta \tau^* \in \Lambda(\mathcal{C})$, because

$$\tau_C(\sigma^* \eta \tau^*) = (t - \theta) \cdot \frac{t}{\theta^i} \cdot \prod_{i=0}^{\infty} (1 - \frac{t}{\theta^i}) = \frac{t - \theta}{\theta} \cdot \prod_{i=1}^{\infty} (1 - \frac{t}{\theta^i}) = \eta \tau^*.$$

Since $\eta \tau^*$ has no zeroes outside $\mathfrak{D}_0$ it generates the $\mathcal{O}(C_{\infty} \cup \mathfrak{D}_0)$-module $\mathcal{C} \otimes_{A_{\infty}} \mathcal{O}(C_{\infty} \cup \mathfrak{D}_0) = \mathcal{O}(C_{\infty} \cup \mathfrak{D}_0)$ and so $h_M$ is an isomorphism and $\mathcal{C}$ is uniformizable.

The next criterion for uniformizability follows for example from [BH07, Lemma 4.2].

Lemma 13.6. Let $M$ be an A-motive of rank $r$.

(a) The homomorphism $h_M$ is injective and it satisfies $h_M \circ (\text{id}_{\Lambda(M)} \otimes \text{id}) = (\tau_M \otimes \text{id}) \circ \sigma^* h_M$.

(b) $M$ is uniformizable if and only if $rk_A \Lambda(M) = r$.

Next we state the generalization of [HJ, Proposition 3.25], which we will need to define period maps. The point $V(J) \in C_{\infty} \subset C_{\infty}$ lies in one of the discs $\mathfrak{D}_i$, because $|\gamma(a)| > 1$ for all $a \in A \subset F_q$. We normalize the indexing of the $\mathfrak{D}_i$ in such a way that $V(J) \subset \mathfrak{D}_0$. Then for any $i \in \mathbb{N}_0$, we consider the pullbacks $\sigma^* J = (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in A) \subset C_{\infty}$, and the points $V(\sigma^* J)$ of $C_{\infty}$ and $\mathcal{C}_{\infty}$. They correspond to the point $V(z - \zeta^q) \in \mathfrak{D}_i$, and have $\infty_{\infty} = \{\infty_0, \cdots, \infty_{d_i-1}\}$ accumulation points. More precisely, for each $k = 0, 1, \cdots, d_i - 1$ the point $\infty_k$ is the limit of the sequence $V(\sigma^{(k+d_i)} J) = V(z - \zeta^{q_d+\cdots})$ for $i \in \mathbb{N}_0$. Therefore, $C_{\infty} \cup \mathfrak{D}_0 \subset V(\sigma^* J)$ is an admissible open rigid analytic subspace of $C_{\infty}$.

Proposition 13.7. [HJ, Proposition 3.25] Let $M$ be a uniformizable effective A-motive over $C_{\infty}$. Then $\Lambda(M)$ equals $\{m \in M \otimes_{A_{\infty}} \mathcal{O}(C_{\infty}) : \tau_M(\sigma^* m) = m\}$ and the isomorphism $h_M$ extends to an injective homomorphism

$$h_M : \Lambda(M) \otimes_A \mathcal{O}(C_{\infty}) \rightarrow M \otimes_{A_{\infty}} \mathcal{O}(C_{\infty}), \quad \lambda \otimes f \mapsto f : \lambda$$

with $h_M \circ (\text{id}_{\Lambda(M)} \otimes \text{id}) = (\tau_M \otimes \text{id}) \circ \sigma^* h_M$. At the point $V(J)$ its cokernel satisfies $\text{coker} h_M \otimes C_{\infty} \subset V[z - \zeta]$ $\equiv M/\tau_M(\sigma^* M)$. The morphism $h_M$ is a local isomorphism away from $\cup_{i \in \mathbb{N}_0} V(\sigma^* J)$, and $\sigma^* h_M$ is a local isomorphism away from $\cup_{i \in \mathbb{N}_0} V(\sigma^* J)$.

Proof. This follows in the same way as [HJ, Proposition 3.25].

Definition 13.8. Let $M$ be an A-motive of rank $r$ over $C_{\infty}$. Anderson defined the Betti cohomology realization of $M$ by setting

$$H^1_{\text{Betti}}(M, R) := \Lambda(M) \otimes_A R$$

and

$$H_{1, \text{Betti}}(M, R) := \text{Hom}_A(\Lambda(M), R)$$

for any $A$-algebra $R$. This is most useful when $M$ is uniformizable, in which case both are locally free $R$-modules of rank equal to $rk_M$.

Example 13.9. We keep the notation from Example 13.5. There we have calculated $\Lambda(\mathcal{C})$ as the $A$-module generated by $\eta \tau^*$, so

$$H^1_{\text{Betti}}(M, A) = \eta \tau^* \cdot A$$

and

$$H_{1, \text{Betti}}(M, A) = (\eta \tau^*)^{-1} \cdot A.$$

To explain the compatibility with Definition 13.4 let $\Omega^1_{A/F_q}$ be the module of Kähler differentials of $A$ over $F_q$.

Proposition 13.10 ([And88, Corollary 2.12.1]). Let $\mathcal{C} = (G, \varphi)$ be a Drinfeld A-module over $C_{\infty}$ and let $M = \Lambda(\mathcal{C})$ be the associated A-motive. Then $M$ is uniformizable and there is a perfect pairing of $A$-modules

$$H_{1, \text{Betti}}(\mathcal{C}, A) \times H^1_{\text{Betti}}(\mathcal{M}, A) \rightarrow \Omega^1_{A/F_q}, \quad (\lambda, m) \mapsto \omega_{A, \lambda, m}$$

where $\omega_{A, \lambda, m}$ is determined by the residues $\text{Res}_{a}(a \cdot \omega_{A, \lambda, m}) = -m(\exp_{\mathcal{C}}(\text{Lie} \varphi_a(\lambda))) \in F_q$ for all $a \in Q$.  

26
13.2 $v$-adic Cohomology

**Definition 13.11.** For an $A$-field $K$ consider the $v$-adic completion $A_{v,K} := \lim\limits_{\rightarrow} A_K/v^n A_K$ of $A_K$. Let $\mathcal{M}$ be an $A$-module over $K$ and let $v \subset A$ be a maximal ideal with $v \neq A$-char($K$). Since $(A_{v,K})^\gamma \overset{\text{id}}{=} A_v$, we can define the $v$-adic cohomology realizations of $\mathcal{M}$ as the $A_v$-modules

$$H^1_v(\mathcal{M}, A_v) := (\mathcal{M} \otimes_{A_K} A_{v,K})^\gamma := \{m \in \mathcal{M} \otimes_{A_K} A_{v,K} | \tau_M(\sigma_M^m) = m\} \quad \text{and} \quad H_{1,v}(\mathcal{M}, A_v) := \text{Hom}_{A_v}(H^1_v(\mathcal{M}, A_v), A_v).$$

They are free $A_v$-modules of rank equal to $\text{rk} M$, carrying a continuous action of the Galois group $\mathcal{G}_K$ by [TW96, Proposition 6.1], and the inclusion $H^1_v(\mathcal{M}, A_v) \subset M \otimes_{A_K} A_{v,K}$ induces a canonical isomorphism of $A_{v,K}$-modules

$$H^1_v(\mathcal{M}, A_v) \otimes_{A_K} A_{v,K} \overset{\sim}{\rightarrow} M \otimes_{A_K} A_{v,K},$$

which is both $\mathcal{G}_K$ and $\tau$-equivariant, where on the right module $\mathcal{G}_K$ acts on both factors and $\tau$ is $\text{id} \otimes \sigma^*$ and on the right module $\mathcal{G}_K$ acts only on $A_v,K$ and $\tau$ is $(\tau_M \circ \sigma_M^*) \otimes \sigma^*$. One also sometimes denotes $H^1_v(\mathcal{M}, A_v)$ by $\hat{T}_v(\mathcal{M})$ and calls this the $v$-adic dual Tate module associated with $\mathcal{M}$ at $v$. We also define the $Q_v$-vector spaces with continuous $\mathcal{G}_K$-action

$$H^1_v(\mathcal{M}, Q_v) := H^1_v(\mathcal{M}, A_v) \otimes_{A_v} Q_v \quad \text{and} \quad H_{1,v}(\mathcal{M}, Q_v) := \text{Hom}_{A_v}(H^1_v(\mathcal{M}, A_v), Q_v) = H_{1,v}(\mathcal{M}, A_v) \otimes_{A_v} Q_v.$$

The analogous of the Tate conjecture is the following theorem which was proved by Taguchi [Tag95] and Tamagawa [Tam94, §2].

**Theorem 13.12** (Tate conjecture for $A$-motives). If $K$ is a finitely generated $A$-field and $v \neq A$-char($K$) then

$$\text{Hom}(\mathcal{M}, \mathcal{M}') \otimes_A A_v \overset{\sim}{\rightarrow} \text{Hom}_{A_v}[\mathcal{G}_K](H^1_v(\mathcal{M}, A_v), H^1_v(\mathcal{M}', A_v))$$

is an isomorphism of $A_v$-modules for $A$-motives $\mathcal{M}$ and $\mathcal{M}'$.

Let us explain the relation between $T_v G$ and $T_v M(\mathcal{G}) := H^1_v(\mathcal{M}(\mathcal{G}), A_v)$ for a Drinfeld $A$-module $\mathcal{G}$. The $A_v$-module $\text{Hom}_K(Q_v/A_v, F_v)$ is canonically isomorphic to the $A_v$-module $\Omega^1_A/F_v = A_v dz_v$ of continuous differential forms; see [HK14, Equation (4.5)], and therefore, it is a free $A_v$-module of rank 1. If $\mathcal{G}$ is a Drinfeld $A$-module over $K$ and $\mathcal{M} = M(\mathcal{G})$ is its associated $A$-motive, then there is a natural $\mathcal{G}_K$-equivariant perfect pairing of $A_v$-modules

$$\langle . , . \rangle : T_v \mathcal{G} \times \hat{T}_v M \longrightarrow \text{Hom}_{F_v}(Q_v/A_v, F_v) \cong \hat{\Omega}^1_{A_v/F_v}, \quad (f, m) := m \circ f. \tag{13.2}$$

which identifies $T_v \mathcal{G}$ with the contragredient $\mathcal{G}_K$-representation $\text{Hom}_{A_v}(\hat{T}_v M, \hat{\Omega}^1_{A_v/F_v})$ of $\hat{T}_v M$; see [HK14, Proposition 4.9]. Together with Theorem 9.7 and 13.12 this implies the following

**Corollary 13.13.** (Tate conjecture for Drinfeld $A$-modules.) Let $\mathcal{G}$ and $\mathcal{G}'$ be two Drinfeld $A$-modules over a finitely generated field $K$. Then the natural map

$$\text{Hom}_K(\mathcal{G}, \mathcal{G}') \otimes_A A_v \rightarrow \text{Hom}_{A_v}[\mathcal{G}_K](T_v \mathcal{G}, T_v \mathcal{G}'), \quad f \otimes a \mapsto a \cdot T_v(f)$$

is an isomorphism of $A_v$-modules.

13.3 De Rham Cohomology and Period Isomorphisms

In this subsection let $(K, \gamma)$ be an $A$-field of generic $A$-characteristic. Then $K$ is a field extension of $Q$ via $\gamma$ and we set $\zeta := \gamma(z)$. There is an identification $\lim\limits_{\leftarrow} A_K/\mathcal{J}^n = K[[z - \zeta]]$ from [H1] Lemma 1.3.

**Definition 13.14.** Let $\mathcal{M}$ be an $A$-motive over an $A$-field $K$ of generic $A$-characteristic. The de Rham realization of $\mathcal{M}$ is defined as

$$H^1_{dR}(\mathcal{M}, K[[z - \zeta]]) := \sigma^* M \otimes_{A_R} \lim\limits_{\leftarrow} A_K/\mathcal{J}^n,$$

$$H^1_{dR}(\mathcal{M}, K[[z - \zeta]]) := H^1_{dR}(\mathcal{M}, K[[z - \zeta]]) \otimes_{K[[z - \zeta]]} K[[z - \zeta]]$$

and

$$H^1_{dR}(\mathcal{M}, K) := \sigma^* M \otimes_{A_R} A_K/\mathcal{J}$$

$$= H^1_{dR}(\mathcal{M}, K[[z - \zeta]]) \otimes_{K[[z - \zeta]]} K[[z - \zeta]]/(z - \zeta).$$

27
The Hodge-Pink lattice of $\mathcal{M}$ is defined as $q^\mathcal{M} := \tau^{-1}_M(M \otimes_{A_K} \lim A_K/J^n) \subset H^1_{dR}(\mathcal{M}, K((z-\zeta)))$, and the descending Hodge-Pink filtration of $\mathcal{M}$ is defined via $p^\mathcal{M} := H^1_{dR}(\mathcal{M}, K[z-\zeta])$ and

$$F^i H^1_{dR}(\mathcal{M}, K) := (p^\mathcal{M} \cap (z-\zeta)^i q^\mathcal{M}) / ((z-\zeta)^i p^\mathcal{M} \cap (z-\zeta)^i q^\mathcal{M})$$

= image of $(\sigma^* M \cap \tau^{-1}_M(j^1 M)) \otimes_R K$ in $H^1_{dR}(\mathcal{M}, K)$;

compare also with [Gos96] §2.6]. Since $M$ is effective, we have $p^\mathcal{M} \subset q^\mathcal{M}$ with $\coker \tau_M \cong q^\mathcal{M}/p^\mathcal{M}$ and $F^0 H^1_{dR}(\mathcal{M}, K) = H^1_{dR}(\mathcal{M}, K)$. Note that the de Rham realization with Hodge-Pink lattice and filtration is a covariant functor on the category of $A$-motives over $K$ with quasi-morphisms.

**Definition 13.15.** If $\mathcal{G}$ is a Drinfeld $A$-module over an $A$-field $K$ of generic characteristic, let $\hat{\mathcal{M}} = (M, \tau_M) = \mathcal{M}(\mathcal{G})$ be the associated $A$-motive. Then the de Rham cohomology realization of $\mathcal{G}$ is defined to be

$$H^1_{dR}(\mathcal{G}, K) := \Hom_A(\Omega^1_{A/H_a}, \sigma^* M/J \cdot \sigma^* M),$$

$$H^1_{dR}(\mathcal{G}, K[z-\zeta]) := \Hom_A(\Omega^1_{A/H_a}, \sigma^* M \otimes_{A_K} K[z-\zeta]),$$

$$H^1_{dR}(\mathcal{G}, K[\zeta-\zeta]) := \Hom_{A_K}(\sigma^* M, \widehat{\Omega^1_{K[z-\zeta]/K}}) \quad \text{and}$$

$$H^1_{dR}(\mathcal{G}, K) := \Hom_{A_K}(\sigma^* M, \widehat{\Omega^1_{K[z-\zeta]/K}}) \otimes_{K[z-\zeta]} K[z-\zeta]/(z-\zeta),$$

where $\Omega^1_{A/H_a}$ is the module of Kähler differentials of $A$ over $\mathbb{F}_q$ and $\widehat{\Omega^1_{K[z-\zeta]/K}} = K[z-\zeta][d\zeta]$ is the $K[z-\zeta]$-module of continuous differentials. We define the Hodge-Pink lattices of $\mathcal{G}$ as the $K[z-\zeta]$-submodules

$$\varphi^\mathcal{G} := \Hom_A(\Omega^1_{A/H_a}, \tau^{-1}_M(M) \otimes_{A_K} K[z-\zeta]) \subset H^1_{dR}(\mathcal{G}, K((z-\zeta)))$$

and

$$\varphi^\mathcal{G} := (\tau^{-1}_M \otimes \id_{K[z-\zeta]})(\Hom_{A_K}(M, \widehat{\Omega^1_{K[z-\zeta]/K}})) \subset H^1_{dR}(\mathcal{G}, K((z-\zeta))).$$

In both cases the Hodge-Pink filtrations $F^i H^1_{dR}(\mathcal{G}, K)$ and $F^i H^1_{dR}(\mathcal{G}, K)$ of $\mathcal{G}$ are recovered as the images of $H^1_{dR}(\mathcal{G}, K[z-\zeta]) \cap (z-\zeta)^i \varphi^\mathcal{G}$ in $H^1_{dR}(\mathcal{G}, K)$ and of $H^1_{dR}(\mathcal{G}, K[z-\zeta]) \cap (z-\zeta)^i \varphi^\mathcal{G}$ in $H^1_{dR}(\mathcal{G}, K)$ like in Definition [13.14]. All these structures are compatible with the natural duality between $H^1_{dR}$ and $H^1_{dR}$.

**Remark 13.16.** It was shown in [HJ] Remark 4.45 and Lemma 5.46 that this definition coincides with the definitions given by Deligne, Anderson, Gekeler and Jing Yu, see [Gos94] Definition 2.6.1, [Gek99] §2 and [Yu90]. Moreover, it was shown in [HJ] Diagram (5.36) in the Proof of Theorem 5.40 that the dual of the sequence of $K[z-\zeta]$-modules $0 \to \varphi^\mathcal{G} \to \varphi^\mathcal{G} \to \coker \tau_M \to 0$ is isomorphic to the sequence

$$0 \to \varphi^\mathcal{G} \to H^1_{dR}(\mathcal{G}, K[z-\zeta]) \to \Lie \mathcal{G} \to 0.$$  

Since $z-\zeta = 0$ on $\Lie \mathcal{G}$ we obtain modulo $(z-\zeta)^i H^1_{dR}(\mathcal{G}, K[z-\zeta])$ the exact sequence of $K$-vector spaces

$$0 \to F^0 H^1_{dR}(\mathcal{G}, K) \to H^1_{dR}(\mathcal{G}, K) \to \Lie \mathcal{G} \to 0,$$

which is the analog of the decomposition \([0.2]\).

For a uniformizable $A$-motive $M$ over $\mathbb{C}_\infty$ the morphism $h_M$ from Proposition [13.7] induces comparison isomorphisms between the Betti and the $v$-adic, respectively the de Rham realizations as follow. Since $v \neq \infty$ the points in the closed subscheme $(\psi, \times \mathbb{P}_v, \mathbb{C}_\infty) \subset C_{\mathbb{C}_\infty}$ do not specialize to $\infty_{\mathbb{K}} < C_{\mathbb{C}_\infty}$ and so this closed subscheme lies in $C_{\mathbb{C}_\infty} \cup \cup \mathbb{D}_i$. This gives us isomorphisms $O(C_{\mathbb{C}_\infty} \cup \cup \mathbb{D}_i) \to v^n A_{\mathbb{C}_\infty}$ for all $n \in \mathbb{N}$ and $\lim v^n O(C_{\mathbb{C}_\infty} \cup \cup \mathbb{D}_i) \cong \lim A_{\mathbb{C}_\infty} / v^n A_{\mathbb{C}_\infty}$. The isomorphism $h_M$ induces a $\tau$-equivariant isomorphism $H^1_{\text{Betti}}(M, A) \otimes_A \lim v^n O(C_{\mathbb{C}_\infty} \cup \cup \mathbb{D}_i) / v^n O(C_{\mathbb{C}_\infty} \cup \cup \mathbb{D}_i) \cong M \otimes A_{\mathbb{C}_\infty} A_{\mathbb{C}_\infty}$. Taking $\tau$-invariant on both sides provides us with the isomorphism between the Betti and the $v$-adic realization

$$h_{\text{Betti}, v} : H^1_{\text{Betti}}(M, A) = H^1_{\text{Betti}}(M, A) \otimes_A A_v \cong H^1_{\text{Betti}}(M, A), \quad \lambda \otimes f \mapsto (f : \lambda \mod v^n)_{n \in \mathbb{N}}.$$
On the other hand, Proposition 13.7 implies that \( \sigma^* h_M \) is an isomorphism locally at \( \mathcal{V} \) that is

\[
\sigma^* h_M \otimes \text{id}_{C_{\infty}} \zeta : H^1_{\text{Betti}}(M, A) \otimes_{A} C_{\infty} \zeta \to \sigma^* M \otimes_{A_{\infty}} C_{\infty} \zeta.
\]

This induces an isomorphism between the Betti and the de Rham realization

\[
\begin{align*}
\text{h}_{\text{Betti}} & := \sigma^* h_M \otimes \text{id}_{C_{\infty}} \zeta : H^1_{\text{Betti}}(M, C_{\infty} \zeta) \to H^1_{\text{dR}}(M, C_{\infty} \zeta), \\
\text{h}_{\text{Betti}} & := \sigma^* h_M \mod \mathcal{J} : H^1_{\text{Betti}}(M, C_{\infty}) \to H^1_{\text{dR}}(M, C_{\infty}).
\end{align*}
\]

We summarize the above result as follows, compare [HJ, Theorem 3.39].

**Theorem 13.17.** If \( M \) is a uniformizable \( A \)-motive over \( C_{\infty} \) there are canonical comparison isomorphisms, sometimes also called period isomorphisms

\[
\begin{align*}
\text{h}_{\text{Betti}}_{\nu} & : H^1_{\text{Betti}}(M, A_v) = H^1_{\text{Betti}}(M, A) \otimes_{A} A_v \to H^1_{\nu}(M, A_v), \lambda \otimes f \mapsto (f \cdot \lambda \mod \nu^n)_{n \in \mathbb{N}} \\
\text{and}
\end{align*}
\]

\[
\begin{align*}
\text{h}_{\text{Betti}}_{\text{dR}} & := \sigma^* h_M \otimes \text{id}_{C_{\infty}} \zeta : H^1_{\text{Betti}}(M, C_{\infty} \zeta) \to H^1_{\text{dR}}(M, C_{\infty} \zeta), \\
\text{h}_{\text{Betti}}_{\text{dR}} & := \sigma^* h_M \mod \mathcal{J} : H^1_{\text{Betti}}(M, C_{\infty}) \to H^1_{\text{dR}}(M, C_{\infty}).
\end{align*}
\]

All these cohomology realizations and period isomorphisms are functorial in \( M \) and by [HJ, Theorem 5.49] compatible with the functor from Drinfeld \( A \)-modules to \( A \)-motives, Proposition 13.10 and the pairing [13.2].

**Example 13.18.** For the Carlitz motive \( C = (C = F_q(\theta)[t], \tau_{C} = t - \theta) \) from Example 13.3 the period isomorphism \( h_{\text{Betti}, \text{dR}} \) is given as follows. By Example 13.5 the generator \( \eta \) of \( H^1_{\text{Betti}}(C, C_{\infty}) = A \cdot \eta \) is sent under \( h_{\text{Betti}, \text{dR}} \) to the element \( \sigma^*(\eta\zeta) |_{t = \theta} = \theta^q \prod_{i = 1}^{\infty} (1 - \theta^{1-q^i}) |_{C_{\infty}} \) which has absolute value \( |\eta|_{C_{\infty}} = |\theta|_{C_{\infty}}^{1/(1-q)} = q^{-q/(q-1)} \). This element is the analog of the period \((2\pi i)^{-1}\) from Example 1.5 because the Carlitz module and Carlitz motive are the analogs of the multiplicative group \( \mathbb{G}_m \), see Example 13.3.

14 Local Shtukas and the \( \nu \)-adic Period Isomorphism

We next describe the function field analog of \( p \)-divisible groups.

**Notation 14.1.** We fix a place \( \nu \neq \infty \) of \( Q \). Let \( K \subset Q_{\text{alg}} \) be an \( A \)-field which is a finite extension of \( Q \) via \( \gamma \). Under the fixed embedding \( Q_{\text{alg}} \to C_v \), let \( L \) be the \( v \)-adic completion of \( K \subset C_v \). Let \( R \) be the valuation ring of \( L \), let \( \pi_L \) be a uniformizing parameter of \( R \) and let \( \kappa \) be the residue field of \( R \). Then \( R = \kappa[\pi_L] \) and \( L = \kappa((\pi_L)) \). The homomorphism \( \gamma : A \to K \) extends by continuity to \( \gamma : A_v \to \gamma(L) \) and factors through \( \gamma : A_v \to R \) with \( \zeta = \gamma(v) \in \pi_L R \setminus \{0\} \). Let \( R[z_v] \) be the power series ring in the variable \( z_v \) over \( R \) and \( \delta^* \) the endomorphism of \( R[z_v] \) with \( \delta^*(z_v) = z_v \) and \( \delta^*(b) = b^{q^v} \) for \( b \in R \), where \( q_v = \#F_v \). For an \( R[z_v] \)-module \( M \) we let \( \delta^* M := M \otimes_{R[z_v]} \delta^* R[z_v] \) as well as \( M(\frac{1}{z_v - \zeta}) := M \otimes_{R[z_v]} R[z_v]([z_v - \zeta]) \) and \( M(\frac{1}{z_v}) := M \otimes_{R[z_v]} R[z_v]([\frac{1}{z_v}]) \).

We obtain a canonical embedding \( A_R : A \otimes_{q_v} R \to R[z_v] \) by mapping \( z_v \otimes 1 \mapsto z_v \) and \( 1 \otimes \zeta_v \mapsto \zeta_v \).

The function field analog of \( p \)-divisible groups is given by the following

**Definition 14.2.** A \( z_v \)-\textit{divisible local Anderson module over} \( R \) is a sheaf of \( F_q[z_v] \)-modules \( G \) on the big fpf-site of \( \text{Spec} \; R \) such that

(a) \( G \) is \( z_v \)-\textit{torsion}, that is \( G = \lim_{\to} G[z_v^n] \),

(b) \( G \) is \( z_v \)-\textit{divisible}, that is \( z_v \to G \) is an epimorphism,

(c) For every \( n \) the \( F_q \)-module \( G[z_v^n] \) is representable by a finite locally free strict \( F_q \)-module scheme over \( R \) in the sense of Faltings (see [Fal02] or [HS18, Definition 4.7]), and

(d) locally on \( \text{Spec} \; R \) there exists an integer \( d \in \mathbb{Z}_{\geq 0} \), such that \( (z_v - \zeta_v)^d = 0 \) on \( \omega_G := \lim \omega_G[z_v^n] \) and \( \omega_G[z_v^n] := \varepsilon^* \Omega^1_{G[z_v^n]/\text{Spec} \; R} \) for the unit section \( \varepsilon \) of \( G[z_v^n] \) over \( R \).
Example 14.3. Let $G = (G, \varphi)$ be a Drinfeld $A$-module over $R$ which is defined as in Definition [12.4] by replacing $K$ by $R$. By [Har14, Theorem 6.6] the torsion module $\hat{G}[v^n]$ is a finite locally free strict $\mathcal{O}_v$-module scheme and the inductive limit $\hat{G}[v^\infty] := \varinjlim\hat{G}[v^n]$ is a $z_v$-divisible local Anderson module over $R$ for which one can take $d = 1$ in Definition [12.4].

Similarly to Remark [5.8], divisible local Anderson modules have a description by semi-linear algebra. It is given by local $\hat{a}_v^*$-shtukas.

Definition 14.4. A local $\hat{a}_v^*$-shtuka of rank $r$ over $R$ is a pair $\hat{M} = (\hat{M}, \tau_\hat{M})$ consisting of a free $R[\hat{z}_v]\hat{M}$-module $M$ of rank $r$, and an isomorphism $\tau_\hat{M} : \hat{a}_v^*M[\frac{1}{\hat{z}_v}] \xrightarrow{\sim} \hat{M}[\frac{1}{\hat{z}_v}]$. It is effective if $\tau_\hat{M}(\hat{a}_v^*\hat{M}) \subset \hat{M}$. We write $rk\hat{M}$ for the rank of $\hat{M}$.

A morphism of local shtukas $f : (\hat{M}, \tau_\hat{M}) \to (\hat{N}, \tau_\hat{N})$ over $R$ is a morphism of the underlying modules $f : \hat{M} \to \hat{N}$ which satisfies $\tau_\hat{N} \circ \hat{a}_v^*f = f \circ \tau_\hat{M}$. We denote the $A_v$-module of homomorphisms $f : \hat{M} \to \hat{N}$ by $\text{Hom}_R(\hat{M}, \hat{N})$ and write $\text{End}_R(\hat{M}) = \text{Hom}_R(\hat{M}, \hat{M})$.

A quasi-morphism between local shtukas $f : (\hat{M}, \tau_\hat{M}) \to (\hat{N}, \tau_\hat{N})$ over $R$ is a morphism of $R[\hat{z}_v]\hat{M}$-modules $f : \hat{M}[\frac{1}{\hat{z}_v}] \to \hat{N}[\frac{1}{\hat{z}_v}]$ with $\tau_\hat{N} \circ \hat{a}_v^*f = f \circ \tau_\hat{M}$. It is called a quasi-isogeny if it is an isomorphism of $R[\hat{z}_v]\hat{M}$-modules. We denote the $Q_v$-vector space of quasi-morphisms from $\hat{M}$ to $\hat{N}$ by $\text{QHom}_R(\hat{M}, \hat{N})$ and write $\text{QEnd}_R(\hat{M}) = \text{QHom}_R(\hat{M}, \hat{M})$.

Note that $\text{Hom}_R(\hat{M}, \hat{N})$ is a finite free $A_v$-module of rank at most $rk\hat{M} \cdot rk\hat{N}$ by [HK14, Corollary 4.5] and $\text{QHom}_R(\hat{M}, \hat{N}) = \text{Hom}_R(\hat{M}, \hat{N}) \otimes_{A_v} Q_v$. Also every quasi-isogeny $f : \hat{M} \to \hat{N}$ induces an isomorphism of $Q_v$-algebras $\text{QEnd}_R(\hat{M}) \to \text{QHom}_R(\hat{N})$, $g \mapsto fgf^{-1}$, similarly to Remark [5.9].

The analog of the ("local") Dieudonné functor from Remark [5.9] is given by the following

Theorem 14.5 ([HST18, Theorem 8.3]). There is an anti-equivalence between the category of $z_v$-divisible local Anderson modules over $R$ and the category of local $\hat{a}_v^*$-shtukas over $R$ given by the contravariant functor $\text{QEnd}_R : \text{QHom}_R(\hat{M}, \hat{N}) \to \text{QEnd}_R(\hat{M}, \hat{N})$, where $\text{QEnd}_R(\hat{M}, \hat{N}) := (\text{Hom}_R(\hat{M}, \hat{N}), \text{End}_R(\hat{M}), \text{QEnd}_R(\hat{M}))$ and $\text{QEnd}_R(\hat{M}, \hat{N})$ is provided by the relative $q_v$-Frobenius of the additive group scheme $G_{\mathcal{O}_v, R}$ over $R$ like in [7.4].

It turns out that like with abelian Anderson $A$-modules, one can dispense with the notions of $z_v$-divisible local Anderson modules, because their equivalent description by local $\hat{a}_v^*$-shtukas can be obtained purely from $A$-motives as in the following

Example 14.6. Let $\hat{M} = (M, \tau_M)$ be an $A$-motive over $K$ and assume that it has good reduction, that is, there exist a pair $(\hat{M}, \tau_\hat{M})$ consisting of a locally free module $\hat{M}$ over $A_K := \hat{A} \otimes_{\mathcal{O}_K} R$ of finite rank and a morphism $\tau_{A_K} : \sigma \hat{M} \to \hat{M}$ of $A_K$-modules whose cokernel is annihilated by a power of the ideal $J := (a \otimes 1 \otimes \gamma(a) : a \in A) \subset A_K$. We call $(\hat{M}, \tau_\hat{M})$ a $K$-motive over $R$ and it is called an $A$-motive over $R$ and a good model of $\hat{M}$.

We consider the $v$-adic completions $A_{v, R}$ of $A_K$ and $(\hat{M}, \tau_\hat{M}) \otimes_{A_K} A_{v, R} := (\hat{M} \otimes_{A_K} A_{v, R}, \tau_{A_{v, R}} \otimes 1)$ of $(\hat{M}, \tau_\hat{M})$. We let $d_v := [\mathbb{F}_v : \mathbb{F}_q]$ and discuss the two cases $d_v = 1$ and $d_v > 1$ separately. If $d_v = 1$, and hence $q_v = q$ and $\hat{a}_v^* = \sigma^*$, we have $A_{v, R} = R[\hat{z}_v]$, and $(\hat{M} \otimes_{A_K} A_{v, R})$ is an effective local $\hat{a}_v^*$-shtuka over $\text{Spec} \ R$ which we denote by $\hat{M}_v(\hat{M})$ and call the local $\hat{a}_v^*$-shtuka at $v$ associated with $\hat{M}$.

If $d_v = 1$, the situation is more complicated, because $\mathbb{F}_v \otimes_{\mathbb{F}_q} R$ and $A_{v, R}$ fail to be integral domains. Namely,

$$\mathbb{F}_v \otimes_{\mathbb{F}_q} R = \prod_{i \in \mathbb{Z}/d_v} \mathbb{F}_v \otimes_{\mathbb{F}_q} R / ((a \otimes 1 \otimes \gamma(a)) : a \in \mathbb{F}_v)$$

and $\sigma^*$ transports the $i$-th factor to the $(i + 1)$-th factor. In particular $\hat{a}_v^*$ stabilizes each factor. Denote by $a_i$ the ideal of $A_{v, R}$ generated by $\{a \otimes 1 \otimes \gamma(a) : a \in \mathbb{F}_v\}$. Then

$$A_{v, R} = \prod_{i \in \mathbb{Z}/d_v} A_{v, R} / a_i$$

Note that each factor is isomorphic to $R[\hat{z}_v]$ and the ideals $a_i$ correspond precisely to the places $v_i$ of $\mathbb{F}_q$ lying above $v$. The ideal $J$ decomposes as follows $J \cdot A_{v, R} / a_i = (z_v - \zeta_v)$ and $J \cdot A_{v, R} / a_0 = (1)$ for $i \neq 0$. We define the local $\hat{a}_v^*$-shtuka at $v$ associated with $\hat{M}$ as $\hat{M}(\hat{M}) := (\hat{M}, \tau_\hat{M}) := (\hat{M} \otimes_{A_K} A_{v, R} / a_0, (\tau_\hat{M} \otimes 1)^{d_v})$, where
Definition 14.11. \( \tau_{M} \) defines \( \sigma \circ \tau_{M} \circ \ldots \circ \sigma^{(d_{v} - 1)} \circ \tau_{M} \). Of course if \( d_{v} = 1 \) we get back the definition of \( \hat{M}_{v}(\mathcal{M}) \) given above. Also note that \( \mathcal{M}/\tau_{M}(\sigma^{*} \mathcal{M}) = \hat{M}_{v}(\mathcal{M})/\tau_{M}(\hat{M}_{v}(\mathcal{M})) \).

The local shtuka \( \hat{M}_{v}(\mathcal{M}) \) allows to recover \( \mathcal{M} \otimes_{R} A_{r, R} \) via the isomorphism

\[
\bigoplus_{i=0}^{d_{v} - 1} (\tau_{M} \otimes 1)^{i} \mod \alpha_{i} : \bigoplus_{i=0}^{d_{v} - 1} \sigma^{i\varphi}(\mathcal{M} \otimes_{R} A_{r, R}/\alpha_{i}), \quad (\tau_{M} \otimes 1)^{d_{v}} \bigoplus_{i \neq 0} \text{id} \cong \mathcal{M} \otimes_{R} A_{r, R},
\]

because for \( i \neq 0 \) the equality \( J \cdot A_{r, R}/\alpha_{i} = (1) \) implies that \( \tau_{M} \otimes 1 \) is an isomorphism modulo \( \alpha_{i} \); see [BH11] Propositions 8.8 and 8.5 for more details.

Proposition 14.7 ([Har17] Theorem 7.6). Let \( \mathcal{G} = (G, \varphi) \) be a Drinfeld A-module over \( R \) and let \( \mathcal{G}[v^{\infty}] := \lim_{\longrightarrow} \mathcal{G}[v^{n}] \) be its \( v \)-divisible local Anderson module over \( R \) from Example 14.3. Let \( \hat{M}(\mathcal{G}) \) be the associated A-motive over \( R \) and let \( \hat{M}_{v}(\mathcal{G}[v^{\infty}]) \) be the associated local \( \hat{M}_{v}(\mathcal{M}) \) over \( R \). Then \( \hat{M}_{v}(\mathcal{G}[v^{\infty}]) \) is canonically isomorphic to the local \( \hat{M}_{v}(\mathcal{M}) \) from Example 14.4.

Example 14.8. It was shown in [HK13] Example 2.7 that the local \( \hat{M}_{v}(\mathcal{M}) \) at \( v \) associated with the Carlitz motive \( \mathcal{G} = (\mathcal{C} = \mathbb{F}_{q}[\theta][\tau_{C} = t - \theta]) \) Example 14.8 equals \( \hat{M}_{v}(\mathcal{C}) = (\mathbb{F}_{v}[\mathbb{C}][\mathbb{Z}], \tau_{M} = (z_{v} - \zeta_{v})) \). Here \( L = \mathbb{F}_{v}(\mathbb{C}) \) and \( R = \mathcal{O}_{L} = \mathbb{F}_{v}[\mathbb{C}] \).

Next we define the \( v \)-adic realization and the de Rham realization of the local shtuka \( \hat{M}_{v}(\mathcal{M}) \) over \( R \). Since \( \tau_{M} \) induces an isomorphism \( \tau_{M} : \hat{M}_{v}(\mathcal{M}) \otimes_{R[z_{v}]} L(z_{v}) \cong \hat{M}_{v}(\mathcal{M}) \otimes_{R[z_{v}]} L(z_{v}) \), we can think of \( \hat{M}_{v}(\mathcal{M}) \) as an \( \mathbb{E} \)-local shtuka over \( L \).

Definition 14.9. The \( v \)-adic realization \( H^{1}_{v}(\hat{M}_{v}(\mathcal{M}), A_{v}) \) of a local \( \hat{M}_{v}(\mathcal{M}) \) is the \( \mathbb{E}_{L} \)-module of \( \tau \)-invariants

\[
H^{1}_{v}(\hat{M}_{v}(\mathcal{M}), A_{v}) := \{ \mathcal{M} \otimes_{R[z_{v}]} L^{\exp}[z_{v}] : \tau_{M}(\hat{M}_{v}(\mathcal{M}) m) = m \},
\]

where we set \( \hat{M}_{v}(\mathcal{M}) := m = m \otimes 1 \in \hat{M}_{v}(\mathcal{M}) \otimes_{R[z_{v}]} \mathcal{A}_{v}[z_{v}] \) and calls this the dual Tate module of \( \mathcal{M} \). By [HK13] Proposition 4.2 it is a free \( A_{r} \)-module of the same rank as \( \mathcal{M} \). We also write \( H^{1}_{v}(\hat{M}_{v}(\mathcal{M}), B) := H^{1}_{v}(\hat{M}_{v}(\mathcal{M}), A_{v}) \otimes_{A_{v}, B} B \) for an \( A_{v} \)-algebra \( B \).

Example 14.10. We describe the \( v \)-adic realization \( H^{1}_{v}(\hat{M}_{v}(\mathcal{M}), A_{v}) \) of the local \( \hat{M}_{v}(\mathcal{M}) \) of the Carlitz module from Example 14.8 by using its local shtuka \( \hat{M}_{v}(\mathcal{C}) = (\mathbb{F}_{v}[\mathbb{C}][\mathbb{Z}], \tau_{M} = (z_{v} - \zeta_{v})) \) at \( v \) computed there. For all \( i \in \mathbb{N} \) and \( \ell_{i} \in L^{\exp} \) be solutions of the equations \( \ell_{0}^{v-1} = -\zeta_{v} \) and \( \ell_{0} + \zeta_{v} \ell_{i} = \ell_{i-1} \). This implies \( |\ell_{i}| = |\zeta_{v}|^{v^{-1}/(q^{-1})} < 1 \).

Define the power series \( \ell^{v} = \sum_{i=0}^{\infty} \ell_{i} z_{v}^{i} \in \mathcal{O}_{L^{\exp}}[z_{v}] \). It satisfies \( \hat{\sigma}_{v}^{*}(\ell^{v}) = (z_{v} - \zeta_{v}) \ell^{v} \), but depends on the choice of the \( \ell \). A different choice yields a different power series \( \hat{\ell}^{v} \) which satisfies \( \hat{\ell}^{v} = u \ell^{v} \) for a unit \( u \in (L^{\exp}[z_{v}])[z_{v}] \). Define \( \hat{\sigma}_{v}^{*}(u) = \hat{\sigma}_{v}^{*}(\ell_{i}) = \ell_{i}^{v} \), because \( \hat{\sigma}_{v}^{*}(u) = \ell_{i}^{v} = \ell_{i}^{v} \), \( \ell_{i}^{v} = u \ell^{v} \). The field extension \( \mathbb{F}_{v}(\mathbb{C})[\ell_{i} : i \in \mathbb{N}] \) of \( \mathbb{F}_{v}(\mathbb{C})[\ell_{i}] \) is the function field analog of the cyclotomic tower \( \mathbb{F}_{v}(\mathbb{T}) : i \in \mathbb{N} \); see [Har09] §1.3 and §3.4. There is an isomorphism of topological groups called the \( v \)-adic cyclotomic character

\[
\chi_{v} : \text{Gal}(\mathbb{F}_{v}(\mathbb{C}))(\ell_{i} : i \in \mathbb{N})/\mathbb{F}_{v}(\mathbb{C}) \cong A_{v}^{*},
\]

which satisfies \( g(\ell_{i}) := \sum_{i=0}^{\infty} g(\ell_{i}) z_{v}^{i} = \chi_{v}(g) \cdot \ell^{v} \) in \( L^{\exp}[z_{v}] \) for \( g \) in the Galois group. It is independent of the choice of the \( \ell \). The \( v \)-adic (dual) Tate module \( \hat{T}_{v}(\hat{M}_{v}(\mathcal{C}), A_{v}) \) of \( \hat{M}_{v}(\mathcal{C}) \) and \( \mathcal{C} \) is generated by \( \ell_{i}^{v} \) on which the Galois group acts by the inverse of the \( v \)-adic cyclotomic character. The reader should compare this to Example 14.6.

Definition 14.11. Let \( \hat{M}_{v}(\mathcal{M}) \) be a local \( \hat{M}_{v}(\mathcal{M}) \) over \( R \). We define the de Rham realizations of \( \hat{M}_{v}(\mathcal{M}) \) as

\[
H^{1}_{\text{dr}}(\hat{M}_{v}(\mathcal{M}), R) := \hat{\sigma}_{v}^{*} M/(z_{v} - \zeta_{v}) \hat{M}_{v}(\mathcal{M}) \otimes_{R[z_{v}]} L(z_{v} - \zeta_{v}),
\]

and

\[
H^{1}_{\text{dr}}(\hat{M}_{v}(\mathcal{M}), L) := \hat{\sigma}_{v}^{*} M \otimes_{R[z_{v}]} L[z_{v} - \zeta_{v}] = H^{1}_{\text{dr}}(\hat{M}_{v}(\mathcal{M}), L[z_{v} - \zeta_{v}]) \otimes_{L[z_{v} - \zeta_{v}]} L(z_{v} - \zeta_{v}).
\]
It carries the Hodge-Pink lattice \( q^\ast_M := \tau_M^{-1}(M \otimes_{R[z_\nu]} L[z_\nu - \zeta_v]) \subset H^1_{dR}(\hat{M}, L[z_\nu - \zeta_v]) \) \( \bigcup_{\nu} \{ 1 \} \) \( \zeta_v \in \mathbb{Z}_p \).

If \( M = (M, \tau_M) \) is an \( A \)-motive over \( L \) with good model \( A \) and \( \hat{M} = \hat{M}(L) \) is the local \( g \) associated with \( A \) and \( d_\nu = [F_\nu : F_\nu] \) as in Example \ref{compare} the map

\[
\sigma^* \tau_{M, d_\nu}^{-1} = \sigma^* \tau_M \circ \sigma^* \tau_M \circ \cdots \circ \sigma^* \tau_M : \sigma^* M \otimes_{A_R} A_{v, R}/a_0 \cong \sigma^* M \otimes_{A_R} A_{v, R}/a_0
\]
is an isomorphism, because \( \tau_M \) is an isomorphism over \( A_{v, R}/a_0 \). Therefore, it defines canonical isomorphisms of the de Rham realizations

\[
\sigma^* \tau_{M, d_\nu}^{-1} : H^1_{dR}(\hat{M}, L[z_\nu - \zeta_v]) \cong H^1_{dR}(\hat{M}, L[z_\nu - \zeta_v]) \quad \text{and}
\]

\[
\sigma^* \tau_{M, d_\nu}^{-1} : H^1_{dR}(\hat{M}, L) \cong H^1_{dR}(\hat{M}, L),
\]
which are compatible with the Hodge-Pink lattices and the Hodge-Pink filtrations.

The \( \nu \)-adic period isomorphism for an \( A \)-motive \( M \) over a field \( K \subset Q_{\nu}^{alg} \) is provided by the following theorem by using the local \( \hat{\sigma}^* \)-shtuka \( \hat{M} := \hat{M}_q(M) \).

**Theorem 14.12 (HK14 Theorem 4.14).** If \( M \) is a local \( \hat{\sigma}^* \)-shtuka over \( R \) then there is a canonical comparison isomorphism

\[
h_{v, dR} : H^1_{\nu}(\hat{M}, Q_\nu) \otimes_{Q_\nu} C_v((z_\nu - \zeta_v)) \cong H^1_{dR}(\hat{M}, L((z_\nu - \zeta_v))) \otimes_{L((z_\nu - \zeta_v))} C_v((z_\nu - \zeta_v))
\]

If \( M \) is an \( A \)-motive over \( L \) (which does not need to have good reduction) then there is a canonical comparison isomorphism

\[
h_{v, dR} : H^1_{\nu}(\hat{M}, Q_\nu) \otimes_{Q_\nu} C_v((z_\nu - \zeta_v)) \cong H^1_{dR}(\hat{M}, L((z_\nu - \zeta_v))) \otimes_{L((z_\nu - \zeta_v))} C_v((z_\nu - \zeta_v))
\]

(14.1)

Both isomorphisms are equivariant for the action of \( J_L \), where on the source this group acts on both factors of the tensor product and on the target it acts only on \( C_v((z_\nu - \zeta_v)) \).

In comparison with the \( p \)-adic comparison isomorphism for an abelian variety over a finite extension of \( Q_p \) from Theorem \ref{compare} the ring \( C_v((z_\nu - \zeta_v)) \) is the function field analog of \( B_{p, dR} \).

**Example 14.14.** For the Carltiz motive \( C = (C = F_q(\theta)[t], \tau_C = t - \theta) \) from Example \ref{compare} we have \( H^1_{\nu}(C, Q_\nu) = Q_\nu \cdot (\ell^+_v)^{-1} = Q_\nu \) and \( H^1_{dR}(C, F_q(\theta)[z_\nu - \zeta_v]) = F_q(\theta)[z_\nu - \zeta_v] = p \), see Example \ref{compare}. The Hodge-Pink lattice \( q = (z_\nu - \zeta_v)^{-1} \) and the Hodge filtration satisfies \( F^1 = H^1_{dR}(C, F_q(\theta)) \) \( F^2 = (0) \). With respect to the bases \( (\ell^+)^{-1} \) of \( H^1_{\nu}(C, Q_\nu) \) and \( 1 \) of \( H^1_{dR}(C, F_q(\theta)[z_\nu - \zeta_v]) \) the comparison isomorphism \( h_{v, dR} \) is given by the \( \nu \)-adic Carlitz period \( (z_\nu - \zeta_v)^{-1}(\ell^+_v)^{-1} = \hat{\sigma}^*(\ell^+_v)^{-1} \). It has a pole of order one at \( z_\nu = \zeta_v \) because \( \ell^+_v \in F_q((z_\nu - \zeta_v)^{1/\nu}) \). So \( h_{v, dR} : H^1_{dR}(C, Q_\nu) \otimes_{Q_\nu} C_v((z_\nu - \zeta_v)) = (z_\nu - \zeta_v)^{-1}C_v[[z_\nu - \zeta_v]] = \sum \otimes_K[z_\nu - \zeta_v] C_v[[z_\nu - \zeta_v]] \).

**Definition 14.14.** On the power series ring \( O_{C_v[[z_\nu - \zeta_v]]} \) we consider the \( O\)-embedding \( O_{C_v[[z_\nu - \zeta_v]]} \hookrightarrow C_v[[z_\nu - \zeta_v]] \) given by \( z_\nu \mapsto z_\nu = \zeta_v + (z_\nu - \zeta_v) \). Let \( \Theta : C_v[[z_\nu - \zeta_v]] \to C_v[[z_\nu - \zeta_v]] \) be the residue map. Then \( O_{C_v[[z_\nu - \zeta_v]]} \cap \ker \Theta \) is a principal ideal of \( O_v[[z_\nu - \zeta_v]] \) generated by \( z_\nu - \zeta_v \). Any other generator is of the form \( (z_\nu - \zeta_v) \cdot u \) with \( u \in O_{C_v[[z_\nu - \zeta_v]]} \).

On \( C_v((z_\nu - \zeta_v)) \) we define a valuation \( \hat{v} \) by

\[
\hat{v} : \sum_{i=-N}^{\infty} b_i(z_\nu - \zeta_v)^i \mapsto \min \{ i \mid b_i \neq 0 \}.
\]

and we extend the valuation \( v \) on \( C_v \) to \( C_v((z_\nu - \zeta_v)) \) by

\[
v(f) := v(\Theta(f \cdot (z_\nu - \zeta_v)^{-v(f)})).
\]

(14.2)

If \( f \) and \( g \) are two elements of \( C_v((z_\nu - \zeta_v)) \), then \( \hat{v}(fg) = \hat{v}(f) + \hat{v}(g) \), and hence \( v(fg) = v(f) + v(g) \). But note that \( v \) does not satisfy the triangle inequality. The valuation \( v(f) \) is unchanged, if we replace the generator \( z_\nu - \zeta_v \) of \( O_{C_v[[z_\nu - \zeta_v]]} \cap \ker \Theta \) by another generator \( (z_\nu - \zeta_v) \cdot u \) with \( u \in O_{C_v[[z_\nu - \zeta_v]]} \), because then

\[
\hat{v}(f) = v(\Theta(f \cdot (z_\nu - \zeta_v) \cdot u)^{-v(f)}) = v(\Theta(f \cdot (z_\nu - \zeta_v)^{-v(f)}) + v(\Theta(u)) = v(\Theta(f \cdot (z_\nu - \zeta_v)^{-v(f)}) \quad \text{for} \quad f \in O_{C_v[[z_\nu - \zeta_v]]}.
\]

**Example 14.15.** The inverse \( (z_\nu - \zeta_v)(\ell^+_v) \) satisfies \( \hat{v}(\ell^+_v) = v_p(\ell^+_v) = v_p((z_\nu - \zeta_v)(\ell^+_v)) = v_p((z_\nu - \zeta_v)(\ell^+_v)) = v_p(\sum_{i=0}^\infty \ell^+_v) = v_p(\ell^+_v) = \frac{1}{q_{\nu}^{-1}} \), see Example \ref{compare}. The reader should compare this to Example \ref{compare}.
15 Complex Multiplication

Definition 15.1. Let \( M \) be an A-motive over an A-field \( K \). If \( \text{QEnd}_K(M) \) contains a commutative semi-simple \( Q \)-algebra \( E \) of dimension \( \dim_Q E = \text{rk} \) \( M \), then we call \( M \) a CM A-motive over \( K \) and we say that \( M \) has complex multiplication by \( E \) over \( K \).

Here semi-simple means that \( E \) is a product of fields. Note that we do not assume that \( E \) is itself a field. By \cite[Theorem 4.2.5]{Sch09} any CM A-motive \( M \) is semi-simple. We know from \cite[Theorem 4.4.7]{Sch09} if \( M \) is simple, uniformizable then \( \text{dim}_Q \text{QEnd}_K(M) \leq \text{rk} \) \( M \) and if in addition \( M \) has complex multiplication by \( E \), then \( E = \text{QEnd}_K(M) \) is a field.

Let \( M \) be an A-motive over \( K \) with complex multiplication through \( E \) and let \( O_E \) be the integral closure of \( A \) in \( E \). If \( E = \prod E_i \) is a product of finite field extensions of \( Q \), then \( O_E = \prod O_{E_i} \), where \( O_{E_i} \) is the integral closure of \( A \) in \( E_i \). By \cite[Theorem 6.3.6]{Sch09} there exists an A-motive \( M' \) isogenous to \( M \) such that \( O_E \subseteq \text{End}_K(M') \). So for all aspects which only depend on the isogeny class of \( M \) we can assume that \( O_E \subseteq \text{End}_K(M) \). Then \( M \) is a locally free module over the ring \( O_E \otimes_{\mathbb{F}_q} K \) and

\[
M = \bigoplus_i (M \otimes_{O_E} O_{E_i}).
\]

Since \( O_E \rightarrow \text{End}_K(M) \) is injective, \( M \otimes_{O_E} O_{E_i} \) is a locally free module over the ring \( O_{E_i} \otimes_{\mathbb{F}_q} K \) of rank \( \geq 1 \), because otherwise \( O_{E_i} \) acts as 0 on \( M \), which is a contradiction. Now the estimate

\[
\text{rk}_{A_K} M = \sum_i \text{rk}_{A_K} (M \otimes_{O_E} O_{E_i}) = \sum_i \text{rk}_{(O_{E_i} \otimes_{\mathbb{F}_q} L)} (M \otimes_{O_E} O_{E_i}) \cdot [E_i : Q] \\
\geq \sum_i [E_i : Q] = [E : Q] = \text{rk}_{A_K} M
\]

shows that \( \text{rk}_{(O_{E_i} \otimes_{\mathbb{F}_q} L)}(M \otimes_{O_E} O_{E_i}) = 1 \) for all \( i \). Therefore, \( M \) is a locally free module over \( O_E \otimes_{\mathbb{F}_q} K \) of rank 1. Thus we have the following proposition.

Proposition 15.2. \cite[Proposition 3.3.5]{Sch09} Let \( M = (M, \tau_M) \) be an A-motive over \( K \) with complex multiplication \( E \) such that \( O_E \subseteq \text{End}_K(M) \), then

(a) \( M \) is a locally free \( O_E \otimes_{\mathbb{F}_q} K \)-module of rank 1.

(b) \( \tau_M : \sigma^* M \rightarrow M \) is an \( O_E \otimes_{\mathbb{F}_q} K \)-linear injection.

Theorem 15.3 \cite[Theorem 6.3.6]{Sch09}. Let \( M \) be an A-motive over an A-field \( K \) with complex multiplication \( E \) such that \( O_E \subseteq \text{End}_K(M) \), then \( M \) is already defined over a finite separable extension \( L \) of the A-field \( \text{Quot}(A/\text{A-char}(K)) \) which is \( Q \) or a finite field i.e. \( M \cong M_L \otimes_L K \) for an A-motive \( M_L \) over \( L \).

Theorem 15.4 \cite[Section 3.6]{Pez09}. If \( M \) is an A-motive defined over a finite extension \( K/Q \) with complex multiplication, then there exist a finite separable extension \( L/K \) such that \( M \) has good reduction at every prime of \( O_L \).

Definition 15.5. A CM-type is a pair \((E, (d_\psi)_{\psi \in H_E})\) consisting of a finite dimensional, semi-simple, commutative \( Q \)-algebra \( E \) and a tuple of integers \((d_\psi)_{\psi \in H_E}\) indexed by \( H_E := \text{Hom}_Q(E, \mathbb{Q}^{alg}) \).

An isomorphism \( f : (E, (d_\psi)_{\psi \in H_E}) \rightarrow (E', (d'_\psi)_{\psi \in H_{E'}}) \) of CM-types is an isomorphism \( f : E \rightarrow E' \) of \( Q \)-algebras with \( d_{\psi \circ f} = d'_\psi \) for all \( \psi \in H_{E'} \).

Remark 15.6. The analog of a classical CM-type \((E, \Phi)\) as in Definition \ref{CM-type} would be a tuple \((d_\psi)_{\psi \in H_E}\) for which \( d_\psi \in \{0, 1\} \). Then one can set \( \Phi := \{\psi \in H_E : d_\psi = 1\} \) and has \( d_\psi = 1 \) for all \( \psi \in \Phi \) and \( d_\psi = 0 \) for all \( \psi \in H_E \setminus \Phi \). But note, that we need a more flexible definition of CM-type here, due to the construction of the CM-type of a CM A-motive in Definition \ref{CM-type} below.

To prepare for this construction let \( z \in Q \) be a uniformizer at \( \infty \) and denote by \( \zeta \) the image of \( z \) in \( \mathbb{Q}^{alg} \) under the natural inclusion \( Q \subset \mathbb{Q}^{alg} \). We consider the power series ring \( \mathbb{Q}^{alg}[[z - \zeta]] \) over \( \mathbb{Q}^{alg} \) in the “variable” \( z - \zeta \) as a \( Q \)-algebra via \( z \mapsto \zeta + (z - \zeta) \). Let \( E \) be a finite dimensional, semi-simple, commutative \( Q \)-algebra. Then by \cite[Lemma A.3]{HS18} there is a decomposition

\[
E \otimes Q \mathbb{Q}^{alg}[[z - \zeta]] = \prod_{\psi \in H_E} \mathbb{Q}^{alg}[[y_\psi - \psi(y_\psi)]]
\]
where \( y_\psi \) is a uniformizer at a place of \( E \) such that \( \psi(y_\psi) \neq 0 \). By \([14]\) Lemma 1.5 the factors are obtained as

the completion of \( \mathcal{O}_E \otimes_A A_{Q^{alg}} = \mathcal{O}_E \otimes_{\mathbb{F}_q} Q^{alg} \) along the kernels \( (a \otimes 1 - 1 \otimes \psi(a) : a \in \mathcal{O}_E) \) of the homomorphisms 

\( \psi \otimes 1_{Q^{alg}} : \mathcal{O}_E \otimes_{\mathbb{F}_q} Q^{alg} \to Q^{alg} \) for \( \psi \in H_E \). If \( (E, (d_\psi)_{\psi \in H_E}) \) is a CM-type, then there is a finite free \( Q^{alg}(z - \zeta) \)-submodule

\[
q := \bigoplus_{\psi \in H_E} (y_\psi - \psi(y_\psi))^{-d_\psi} \cdot Q^{alg}(y_\psi - \psi(y_\psi)) = E \otimes Q Q^{alg}(z - \zeta) \tag{15.2}
\]

with \( q \cdot Q^{alg}(z - \zeta) = E \otimes Q Q^{alg}(z - \zeta) \). Conversely, every such \( Q^{alg}(z - \zeta) \)-submodule \( q \) uniquely determines a tuple \( (d_\psi)_{\psi \in H_E} \) of integers satisfying \((15.2)\). So we could equivalently call \((E, q)\) a “CM-type”.

\[\text{Definition 15.7.}\]

Let \( M \) be an \( A \)-motive over a finite field extension \( K \subset Q^{alg} \) of \( Q \) with complex multiplication through \( E \). We assume that \( K \) contains \( \psi(E) \) for all \( \psi \in H_E \). Then the decomposition \((15.1)\) exists already with \( Q^{alg} \) replaced by \( K \).

The \( E \otimes Q K [z - \zeta] \)-module \( H_{dR}^1(M, K[z - \zeta]) \) is finite free of rank 1, and correspondingly decomposes into eigenspaces

\[
H^0(M, K[y_\psi - \psi(y_\psi)]) := H^1_{dR}(M, K[z - \zeta]) \otimes_{E \otimes Q K[z - \zeta]} K[y_\psi - \psi(y_\psi)]
\]

each of which is free of rank 1 over \( K[y_\psi - \psi(y_\psi)] \). There are integers \( d_\psi \) such that the Hodge-Pink lattice from Definition \((14.1)\) is \( q^M := \bigoplus_{\psi} (y_\psi - \psi(y_\psi))^{-d_\psi} H^0(M, K[y_\psi - \psi(y_\psi)]) \).

The tuple \( (d_\psi)_{\psi \in H_E} \) is the CM-type of \( M \).

Since \( \text{coker } \tau_M \otimes q^M \otimes H_{dR}^1(M, K[z - \zeta]) \) we see that \( d_\psi \) is the dimension over \( K \) of the generalized \( \psi \)-eigenspace of the action of \( E \) on \( \text{coker } \tau_M \).

If we fix an isomorphism \( \alpha : H_{dR}^1(M, K[z - \zeta]) \to E \otimes Q K[z - \zeta] \), then the CM-type of \( M \) can equivalently be described as \((E, \alpha(q^M))\).

\[\text{Example 15.8.}\]

Let \( G \) be a Drinfeld A-module over an \( A \)-field \( K \) of generic \( A \)-characteristic, such that \( M := M(G) \) has CM by \( E \). Then Theorem \((15.3)\) and Corollary \((13.3)\) imply that \( E = QEnd_K(M) = QEnd_K(G) \) is a (commutative) field extension of \( Q \) of degree equal to \( \text{rk } G \).

Since \( \text{coker } \tau_M \cong (\text{Lie } G)^\vee \) is 1-dimensional, the CM-type of \( G \) is \((E, (d_\psi)_{\psi \in H_E}) \) with \( d_{\psi_0} = 1 \) for one \( \psi_0 \in H_E \) and \( d_\psi = 0 \) for all \( \psi \neq \psi_0 \). Thus as \( E \)-modules, sequence \((13.3)\) takes the form

\[
0 \to \bigoplus_{\psi \neq \psi_0} K_{\psi} \to H_{1, dR}(G, K) \to K_{\psi_0} \to 0
\]

where \( K_{\psi} \) denotes the 1-dimensional \( K \)-vector space on which \( E \) acts via \( \psi \). In particular \( \text{Lie } G = K_{\psi_0} \), and hence \((13.3)\) is analogous to the decomposition \((6.3)\).

16 The Analog of Colmez’s Conjecture for CM \( A \)-Motives

In \([14, 15]\) the authors have formulated the analog of Colmez’s conjecture (Section \(8\)) for periods of CM \( A \)-motives. We consider the following

\[\text{Situation 16.1.}\]

Let \( M \) be a uniformizable \( A \)-motive over a finite extension \( K \subset Q^{alg} \) of \( Q \) with complex multiplication of CM-type \((E, (d_\psi)_{\psi \in H_E})\), in the sense of Definition \((15.5)\) such that \( E \) is a product of separable field extensions of \( Q \) and \( M \) has complex multiplication by the ring of integers \( \mathcal{O}_E \) of \( E \). As an abbreviation we denote the CM-Type of \( M \) by \((E, \Phi)\) with \( \Phi = (d_\psi)_{\psi \in H_E} \).

Let \( H_E : = \text{Hom}_Q(E, Q^{alg}) \) be the set of all \( Q \)-homomorphisms \( E \to Q^{alg} \) and assume that \( K \) contains \( \psi(E) \) for every \( \psi \in H_E \). By Theorems \((15.3)\) and \((15.4)\) we may assume moreover, that \( K \) is a finite Galois extension of \( Q \) and that \( M \) has good reduction at every prime of \( K \).

For a \( \psi \in H_E \) let \( \omega_\psi \) be a generator of the \( K[y_\psi - \psi(y_\psi)] \)-module \( H^0(M, K[y_\psi - \psi(y_\psi)]) \). The image of \( \omega_\psi \) in \( H_{dR}^0(M, K) \) is non-zero and satisfies \( a^* \omega_\psi = \psi(a) \cdot \omega_\psi \) for all \( a \in E \). For every embedding \( \eta : K \to Q^{alg} \), let \( M^{\eta} := M \otimes_{K, \eta} K \) and \( \omega^{\eta}_\psi \in H^0(M^{\eta}, K[y_{\psi^{\eta}} - \eta(\psi(y_{\psi^{\eta}}))]) \) be deduced from \( M \) and \( \omega_\psi \) by base extension, and let \( u_{\eta} \in H_1(M^{\eta}, Q) = \text{Hom}_A(H_{\text{Betti}}(M^{\eta}, A), Q) \) be an \( E \)-generator.

If \( v \to \infty \) the comparison isomorphism \((13.5)\) from Theorem \((13.1)\) between Betti and de Rham cohomology yields a pairing

\[
\langle \cdot, \cdot \rangle_{\infty} : H_{1, \text{Betti}}(M^{\eta}, Q) \times H_{dR}^0(M^{\eta}, K) \to C_{\infty} , \tag{13.5}
\]

\[
(u_{\eta}, \omega^{\eta}_\psi) \mapsto \langle u_{\eta}, \omega^{\eta}_\psi \rangle_{\infty} := u_{\eta} \otimes \text{id}_{C_{\infty}}(h_{\text{Betti}}^{-1}(\omega^{\eta}_\psi \mod z - \zeta)) .
\]
We define the absolute value $|f_{u, n} \omega^n_v|_v := |(u_n, \omega_v^n)\rangle = q^v \in \mathbb{R}$.

If $v \subset A$ is a maximal ideal, the comparison isomorphism $h_{\text{Betti}, v}$ from (13.4) in Theorem 13.17 between Betti and $v$-adic cohomology together with the comparison isomorphism $h_{\text{v, dR}, v}$ between $v$-adic and de Rham cohomology from (14.1) in Theorem 14.12 yield a pairing

$$
\langle \cdot, \cdot \rangle_v : \ H_{\text{Betti}}(M^n, Q) \times H_{\text{dR}}(M^n, K) \to \mathbb{C}_v((z_v - \zeta_v)),
$$

$$(u_n, \omega^n_v) \mapsto \langle u_n, \omega^n_v \rangle_v := u_n \odot \text{id}_{\mathbb{C}_v((z_v - \zeta_v))}(h_{\text{Betti}, v}^{-1} h_{\text{dR}, v}^{-1}(\omega^n_v)).$$

We define the absolute value $|f_{u, n} \omega^n_v|_v := |\langle u_n, \omega^n_v \rangle_v| := q^v |(u_n, \omega_v^n)\rangle \in \mathbb{R}$, where the “valuation” $v$ on $\mathbb{C}_v((z_v - \zeta_v))$ was defined in (14.2) in Definition 14.13.

In analogy with Section 8 we now consider the product $\prod_{v \in H_K} |f_{u, n} \omega^n_v|_v$ over all places $v$ of $Q$, or equivalently $\frac{1}{\|H_K\|}$ times its logarithm

$$
\frac{1}{\#H_K} \sum_{v \in H_K} \log |f_{u, n} \omega^n_v|_v = \frac{1}{\#H_K} \sum_{v \in H_K} \log |f_{u, n} \omega^n_v|_\infty - \frac{1}{\#H_K} \sum_{v \in \mathbb{Q}^\times \setminus H_K} \sum_{\eta \in H_K} v(f_{u, n} \omega^n_v) \log q_{v}.
$$

Again the right sum over all $v \neq \infty$ does not converge. Namely, we prove in [HS18, Theorem 1.3] the following Theorem 16.2 below. To formulate the theorem we recall Definition 8.2. For our CM-type $(E, \Phi)$ and for every $\psi \in H_E$ we define the functions

$$
a_{E, \psi, \Phi} : \mathcal{H}_Q \to \mathbb{Z}, \ g \mapsto d_{g, \psi} \quad \text{and}$$

$$a_{E, \psi, \Phi}^0 : \mathcal{H}_Q \to \mathbb{Q}, \ g \mapsto \frac{1}{\#H_K} \sum_{\eta \in H_K} d_{\eta^{-1} g, \psi},$$

which factor through $\text{Gal}(K/Q)$ by our assumption that $\psi(E) \subset K$ for all $\psi \in H_E$. In particular, $a_{E, \psi, \Phi} \in \mathcal{C}(\mathcal{H}_Q, Q)$ and $a_{E, \psi, \Phi}^0 \in \mathcal{C}(\mathcal{H}_Q, Q)$ is independent of $K$.

We also define integers $v(\omega^n_v)$ and $v_{\psi}(u_n)$ for all $v \neq \infty$ which are all zero except for finitely many. Let $O_{E_v} := O_E \otimes_A A_v$ and let $c \in E_v := E \otimes_Q Q_v$ be such that $c^{-1}u_n$ is an $O_{E_v}$-generator of $H_{\text{Betti}}(M^n, A) \otimes_A A_v$, which exists because $O_{E_v}$ is a product of discrete valuation rings. Then $c$ is unique up to multiplication by an element of $O_{E_v}$ and we set

$$v_{\psi}(u_n) := v(\psi(c)) \in \mathbb{Z},$$

where we extend $\psi \in H_E$ by continuity to $\eta \psi : E_v \to Q_{v}^{\text{alg}}$. Also let $K_v$ be the $v$-adic completion of $K \subset Q_v^{\text{alg}} \subset Q_v^{\text{al}} \subset \mathbb{C}_v$ and let $M^n_v = (M^n, \tau_{M^n})$ be an $A$-motive over $O_{K_v}$ with good reduction and $M^n_v \otimes_{O_{K_v}} K_v \cong M^n \otimes_K K_v$; see Example 14.6. Then there is an element $x \in K_v^\times$ unique up to multiplication by $O_{K_v}$, such that $x^{-1}\omega_v^n \mod y_{v, \psi} - \eta \psi(y_{v, \psi})$ is an $O_{K_v}$-generator of the free $O_{K_v}$-module of rank one

$$H^{\eta \psi}(M^n, O_{K_v}) := \{ \omega \in H^{1}_{\text{dR}}(M^n, O_{K_v}) : \sigma^* M^n \otimes_{O_{K_v}} \gamma \otimes \text{id}_{O_{K_v}} \mathcal{O}_{K_v} : [b]^* \omega = \eta \psi(b) \cdot \omega \ \forall \ b \in O_{E_v} \},$$

and we set

$$v(\omega^n_v) := v(x) \in \mathbb{Z}.$$ (16.5)

This value only depends on the image of $\omega^n_v$ in $H^{1}_{\text{dR}}(M^n, K)$. It also does not depend on the choice of the model $M^n$ with good reduction, because all such models are isomorphic over $O_{K_v}$ by [Gar03, Proposition 2.13(ii)]. Now in [HS18, Theorem 1.3] we computed the terms in (16.1) as follows.

**Theorem 16.2.**

$$\frac{1}{\#H_K} \sum_{v \in H_K} v(f_{u, n} \omega^n_v) = Z_v(a_{E, \psi, \Phi}^0, 1) - \mu_{\text{Art}, v}(a_{E, \psi, \Phi}^0) - v(\delta_{\psi(E)/Q}) \frac{1}{\#H_K} \sum_{\eta \in H_K} (v(\omega^n_v) + v_{\psi}(u_n)),$$

where $\delta_{\psi(E)/Q}$ is the discriminant of the field extension $\psi(E)/Q$.

Since $-\mu_{\text{Art}, v}(a_{E, \psi, \Phi}^0) \frac{v(\delta_{\psi(E)/Q})}{\#H_K} \sum_{\eta \in H_K} (v(\omega^n_v) + v_{\psi}(u_n))$ vanishes for all but finitely many places $v$ and $\sum_{v \neq \infty} Z_v(a_{E, \psi, \Phi}^0, 1)$ diverges, the sum (16.1) diverges. But as in Section 8 we can assign to this divergent sum a value by the following...
**Convention 16.3.** Let \((x_v)_{v \neq \infty}\) be a tuple of complex numbers indexed by the finite places \(v\) of \(Q\). We will give a sense to the (divergent) series \(\Sigma \mathcal{Z} = \sum_{v \neq \infty} x_v\) in the following situation. We suppose that there exists an element \(a \in \mathcal{C}(\mathbb{Q}, \mathbb{Q})\) such that \(x_v = -Z_v(a, 1) \log q_v\) for all \(v\) except for finitely many. Then we let \(a^* \in \mathcal{C}(\mathbb{Q}, \mathbb{Q})\) be defined by \(a^*(g) := a(g^{-1})\). We further assume that \(Z_v(a^*, s)\) does not have a pole at \(s = 0\), and we define the limit of the series \(\sum_{v \neq \infty} x_v\) as

\[
\Sigma := -Z_v(a^*, 0) - \mu_{\text{Art}}(a) + \sum_{v \neq \infty} \left( x_v + Z_v(a, 1) \log q_v \right)
\]

(16.6)

inspired by Weil’s \([Wei48\), p. 82] functional equation

\[
Z(\chi, 1 - s) = -Z(\chi^*, s) - (2 \cdot \text{genus}(C) - 2) \chi(1) \log q - \mu_{\text{Art}}(\chi)
\]

deprived of the summands at \(\infty\), where the genus term is considered as belonging to \(\infty\).

Convention 16.3 and Theorem 16.2 allow us to give to the divergent sum (16.1) the convergent interpretation

\[
\sum_{v \neq \infty} (a_0^0 E, \psi, \eta) = 0 + \sum_{v \neq \infty} \log \#(\mathcal{A}_v / \mathcal{O}_v) - \sum_{v \neq \infty} \left( v(\omega_v^\eta) + v(\psi u_\eta) \right) \log q_v.
\]

(16.7)

We can thus make the following

**Conjecture 16.4.** The sum (16.7) is zero, or equivalently the product formula holds:

\[
\prod_v \prod_{\eta \in H_K} \langle \omega, \eta \rangle_v = 1
\]

**Example 16.5.** Similarly to Example 8.3, the convention allows to prove the product formula for the Carlitz motive \(\mathcal{C} = (\mathcal{C} = \mathbb{F}_q[t] / \mathbb{F}_{q^r})\) from Example 9.8. We let \(u \in H_{1, \text{Betti}}(\mathcal{C}, A)\) be the generator which is dual to \(\eta \in H_{1, \text{Betti}}(\mathcal{C}, A)\) and we let \(\omega = 1 \in H_{1, \text{Art}}(\mathcal{C}, C_{\infty})\). Then we have computed in Examples 13.18, 14.13 and 14.15 that

\[
\langle \omega, u \rangle_{\infty} = \eta^{-q} \prod_{i=1}^{\infty} (1 - q^{1-i})^{-1} \quad \text{and} \quad \log \langle \omega, u \rangle_{\infty} = \log(q^{q/2(q-1)}) = - \frac{q}{q-1} \log q,
\]

\[
\langle \omega, u \rangle_v = \hat{d}_v^* (e_v^+), \quad \text{and} \quad \log \langle \omega, u \rangle_v = -v(\hat{d}_v^*(e_v^+)) \log q_v = - \frac{q}{q-1} = -Z_v(1, 1) \log q_v,
\]

where \(1(g) = 1\) for every \(g \in \mathbb{Q}_v\). So Convention 16.3 implies \(\sum_{v \neq \infty} \log \langle \omega, u \rangle_v = -\frac{\zeta_A'(0)}{\zeta_A(0)} = - \frac{q}{q-1} \log q\) for the Riemann Zeta-function

\[
\zeta_A(s) := \prod_{v \neq \infty} (1 - (\#\mathcal{F}_v)^{-1})^{-1} = \prod_{v \neq \infty} (1 - q_v^{-s})^{-1} = \frac{1}{1 - q^{1-s}}.
\]

We conclude \(\sum_v \log \langle \omega, u \rangle_v = 0\) and \(\prod_v \langle \omega, u \rangle_v = 1\).

In Section 17 we will discuss a very interesting example where \(C\) and \(Q\) have genus 1.
17 Example

We verify Conjecture [15.3] for an $A$-motive $M$ of rank 1 where the curve $C$ has genus 1. This example was studied in detail by Green and Papanikolas [GP10]. Our exposition follows [GP10]. It is a beautiful exercise in computing with elliptic curves.

17.1. Let $C$ be an elliptic curve over $\mathbb{F}_q$, given by the (non-homogeneous) Weierstraß equation

$$F := F(t, y) := y^2 + a_1 ty + a_3 y - t^3 - a_2 t^2 - a_4 t - a_6,$$

in the variables $t = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, compare (2.1). Let $\infty \in V(Z^3 : F) \subset \mathbb{F}_q^2$ be the $\mathbb{F}_q$-rational point with $(X : Y : Z) = (0 : 1 : 0)$ at which $t$ and $y$ have pole order given by

$$v_{\infty}(t) = -2, \quad v_{\infty}(y) = -3.$$

We have $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C) = \mathbb{F}_q[t, y]/(F(t, y))$. For any field extension $L$ of $\mathbb{F}_q$ there is exactly one point $\infty_L$ on $C_L$ above $\infty$, because $\infty$ is $\mathbb{F}_q$-rational. To shorten the notation we sometimes denote the point $\infty_L$ again by $\infty$.

We consider a second copy of the ring $A$ given by $\mathbb{F}_q[\theta, \varepsilon]/(F(\theta, \varepsilon))$ in the variables $\theta$ and $\varepsilon$, and its fraction field $\mathbb{F}_q(\theta, \varepsilon)$. This is the function field of a second copy of the elliptic curve $C$, which we denote by $X_0$ and which has coordinates $\theta$ and $\varepsilon$. That is $\mathbb{F}_q(\theta, \varepsilon) = \mathbb{F}_q(X_0)$. Let $\gamma : A \to \mathbb{F}_q(\theta, \varepsilon)$ be given by $\gamma(t) = \theta$ and $\gamma(y) = \varepsilon$. This makes $\mathbb{F}_q(\theta, \varepsilon)$ into an $A$-field. We use the isomorphism $\gamma : Q \xrightarrow{\sim} \mathbb{F}_q(\theta, \varepsilon)$ to embed $\mathbb{F}_q(\theta, \varepsilon)$ canonically into $\mathbb{C}_v$ for all places $v$ of $Q$. We note that

$$\Xi = V(t - \theta, y - \varepsilon) = V(J)$$

for the ideal $J := (a \otimes 1 - 1 \otimes \gamma(a) : a \in A) = (t - \theta, y - \varepsilon)$

is an $\mathbb{F}_q(\theta, \varepsilon)$-rational point of $C$. Furthermore, $\Xi \in C(\mathbb{F}_q(\theta, \varepsilon)) \subset C(\mathbb{C}_v)$ specializes to $\infty \in C(\mathbb{C}_v)$ under the reduction map $\text{red} : C(\mathbb{C}_v) \to C(\mathbb{C}_v)$ from [13.1]. Recall the rigid analytic space $\mathcal{C} := \mathcal{C}_{\mathbb{C}_v} = (C_{\mathbb{C}_v})^{\text{rig}}$ and the disc $\mathcal{D} \subset \mathcal{C}$, which is defined in Notation [13.1] as the preimage in $\mathcal{C} = C(\mathbb{C}_v)$ of $\infty \in C(\mathbb{C}_v)$. This disc $\mathcal{D}$ is the formal group of the elliptic curve $C_{\mathbb{C}_v}$ over $\mathbb{C}_v$, see $[Si86]$ Example IV.3.1.3, where this formal group is denoted $\hat{C}(\mathbb{C}_v)$ for the maximal ideal $\mathfrak{m}_{\infty} \subset \mathcal{O}_{\mathbb{C}_v}$.

For any field extension $L$ of $\mathbb{F}_q$ the relative $q$-Frobenius isogeny $\text{Fr}_{q, C_L/L} : C_L \to C_L$ of $C_L$ over $L$ is given on $\text{Spec} A_L \subset C_L$ by the $L$-homomorphism $\text{Fr}_{q, C_L/L} : A_L \to A_L$, $t \mapsto t^q$, $y \mapsto y^q$. For any point $P \in C_L(L)$ we denote by $P^{(1)} := \text{Fr}_{q, C_L/L}(P) \in C_L(L)$ the image of $P$. The composition $\sigma \circ \text{Fr}_{q, C_L/L} = \text{Fr}_{q, C_L/L} \circ \sigma$ with the morphism $\sigma : C_L \to C_L$ from [11.1] equals the absolute $q$-Frobenius on $C_L$, which is the identity on points and the $q$-power map on the structure sheaf. For example, the morphism $\text{Fr}_{q, C/\mathbb{F}_q}$ sends $\Xi$ to $\Xi^{(1)} = \text{Fr}_{q, C/\mathbb{F}_q}(\Xi) = V(t - \theta^q, y - \varepsilon^q)$.

The isogeny $1 - \text{Fr}_{q, C/\mathbb{F}_q} : C \to C$ is separable by [Si86] Corollary III.5.5 and it induces an isomorphism of formal groups $1 - \text{Fr}_{q, C/\mathbb{F}_q} : \hat{C}(\mathbb{m}_{\infty}) \to \hat{C}(\mathbb{m}_{\infty})$ by [Si86] Corollary IV.4.3 and Lemma IV.2.4. Therefore, we can pick a unique point $V \in \hat{C}(\mathbb{m}_{\infty}) \subset \mathbb{O}_{\mathbb{C}_v}$ so that under the group law of $C$

$$(1 - \text{Fr}_{q, C/\mathbb{F}_q})(V) = V - V^{(1)} = \Xi,$$

and moreover, $(1 - \text{Fr}_{q, C/\mathbb{F}_q})^{-1}(\Xi) = \{V + P | P \in C(\mathbb{F}_q)\}$.

If we set $V = V(t - \alpha, y - \beta)$ with $\alpha, \beta \in C_{\mathbb{C}_v}$ then $K := \mathbb{F}_q(\theta, \varepsilon)(\alpha, \beta) = \mathbb{F}_q(\alpha, \beta) \subset \mathbb{C}_v$ is the Hilbert class field of $\mathbb{F}_q(\theta, \varepsilon)$ by [GP10] Proposition 3.3. We view $K$ as the function field of a third copy of the elliptic curve $C$, which we denote by $X_1$ and which has coordinates $\alpha$ and $\beta$. The inclusion of fields $\mathbb{F}_q(\theta, \varepsilon) \subset K$ corresponds to a morphism $X_1 \to X_0$ which is equal to the morphism $1 - \text{Fr}_{q, C/\mathbb{F}_q} : C \to C$ under the identifications $X_1 = C = X_0$. In particular, the set $X_1(\mathcal{D})$ equals the preimage of $\infty = (0 : 1 : 0)$ in $X_0$ under this map. This set consists of the points with $\alpha, \beta \in \mathbb{F}_q$ together with the point $P = \infty \in X_1$ where $\alpha$ and $\beta$ have poles of order 2 and 3 respectively. It follows that $X_1 \setminus X_1(\mathcal{D}) = \text{Spec} \mathcal{O}_K$ for the integral closure $\mathcal{O}_K$ of $A$ in $K$.

17.2. Now by [17.1] and the definition of the group law on $C$, see [Si86] §III.2, the $K$-valued points $V^{(1)} = V(t - \alpha^q, y - \beta^q)$ and $-V = V(t - \alpha, y + \beta + a_1 \alpha + a_3)$ and $\Xi$ in $C(K)$ are collinear. We take $m$ to be the slope of the line connecting them:

$$m = \frac{\varepsilon - \beta^q}{\theta - \alpha^q} = \frac{\varepsilon + \beta + a_1 \alpha + a_3}{\theta - \alpha} = \frac{\beta^q + \beta + a_1 \alpha + a_3}{\alpha^q - \alpha} \in K. \quad (17.2)$$

With respect to the valuation $v_{\infty}$ on $K \subset C_{\mathbb{C}_v}$ we compute $v_{\infty}(\theta) = v_{\infty}(\alpha) = -2$ and $v_{\infty}(\varepsilon) = v_{\infty}(\beta) = -3$, and hence obtain $v_{\infty}(m) = v_{\infty}(\frac{\beta^q + \beta + a_1 \alpha + a_3}{\alpha^q - \alpha}) = -q$. We extend this to the following.
Lemma 17.3. Let \( P \in X_1 \) be a closed point. Then the element \( m \in K \) has a pole at \( P \) if and only if \( P \in X_1(F_q) = X_1 \setminus \text{Spec} O_K \). In particular, \( m \in O_K \). Moreover, for the normalized valuation \( v_P \) corresponding to \( P \) we have
\[
v_P(m) = \begin{cases} -1 & \text{when } P \in X_1(F_q), P \neq \infty_1, \\ -q & \text{when } P = \infty_1. \end{cases}
\]

Proof. This can be proved by computing a uniformizing parameter at \( P \), but we use the following different strategy. The element \( m \in K \) was defined as the slope of the line through \( V(1) \), \( -V \) and \( \Xi \). This also holds over \( X_1 \) for the canonical extensions of \( V(1) \), \(-V\) and \( \Xi \) to \( X_1 \)-valued points of \( C \times_{\mathbb{F}_q} X_1 \). We now specialize to the residue field \( L := \kappa(P) \). If \( |m| = \infty \) then \( \frac{1}{m}(P) = 0 \) then on the elliptic curve \( C \otimes_{\mathbb{F}_q} L \) the line through \( V(1) \), \(-V\) and \( \Xi \) contains the neutral element \( \Xi_L \), so \( V(1) = \Xi_L \) or \( -V = \Xi_L \) or \( \Xi = \Xi_L \). If \( V(1) = \Xi_L \) or \( -V = \Xi_L \) then \( V = \Xi_L \), because \( \Xi_L = \Xi_L \) and this is the only point in \( \mathbb{F}_q \mathbb{C}_{\infty L}(\Xi_L) \). From \( V = V(t - \alpha, y - \beta) \) it follows that \( P = \infty_1 \in X_1(F_q) \). In this case \( v_P(\theta) = v_P(\alpha) = -2 \) and \( v_P(\varepsilon) = v_P(\beta) = -3 \), and we obtain \( v_P(m) = v_P(\frac{t - \alpha}{\theta - \alpha}) = -q \) as above. If \( \Xi_L = \Xi = V - V(1) \) and \( V \neq \Xi_L \), then \( V(1) = V = V(t - \alpha, y - \beta) \) lies in \( C(F_q) \). Thus \( \alpha, \beta \in \mathbb{F}_q \) and \( P \in X_1(F_q) \). In this case \( v_P(\alpha), v_P(\beta) \geq 0 \), and \( \Xi = V(t - \theta, y - \varepsilon) = \Xi_L \) implies \( v_P(\theta) = -2 \) and \( v_P(\varepsilon) = -3 \). We obtain \( v_P(m) = v_P(\frac{t - \alpha}{\theta - \alpha}) = -1 \). Conversely, if \( P \in X_1(F_q) \), then \( V = V(t - \alpha, y - \beta) \in C(F_q) \) and \( \Xi = V - V(1) \) and \( \Xi = \Xi_L \) and so the line through \( V(1) \), \(-V \) and \( \Xi \) has slope \( m = \infty \).

17.4. By [17.3] and [31.08] Corollary III.3.5] the divisor \([V(1)] - [V] + [\Xi] - [\infty] \) on \( C_K \) is principal. So there is a function \( f \in K(t, y) = \text{Quot}(A_K) \), called the shtuka function for \( A \) with
\[
\text{div}(f) = [V(1)] - [V] + [\Xi] - [\infty].
\]
The shtuka function \( f \) can be written as
\[
f = \frac{\nu(t, y)}{\delta(t)} = y - \varepsilon - m(t - \theta) = \frac{y + \beta + a_1 \alpha + a_3 - m(t - \alpha)}{t - \alpha} = \frac{y + \beta + a_1 \alpha + a_3}{t - \alpha} - m,
\]
for
\[
\nu := \nu(t, y) := y - \varepsilon - m \cdot (t - \theta) \in O_K[t, y] \quad \text{and} \quad \delta := \delta(t) := t - \alpha \in O_K[t, y],
\]
with divisors on \( C_K \) given by
\[
\text{div}(\nu) = [V(1)] + [-V] + [\Xi] - 3[\infty] \quad \text{and} \quad \text{div}(\delta) = [V] + [-V] - 2[\infty].
\]
The formulas [17.3] and [17.5] also hold for the Cartier divisors of \( f, \nu \) and \( \delta \) on the two dimensional scheme \( C \otimes_{\mathbb{F}_q} O_K \), because \( \nu \) and \( \delta \) do not vanish on an entire fiber of \( C \otimes_{\mathbb{F}_q} O_K \) over a closed point of \( \text{Spec} O_K \). Here we consider the \( O_K \)-valued points \( \infty : = V(\frac{1}{t}, \frac{1}{y}) \in \{ \infty \} \times_{\mathbb{F}_q} \text{Spec} O_K \) and \( V = V(t - \alpha, y - \beta) \) and \( \Xi = V(t - \theta, y - \varepsilon) \), etc. as Cartier divisors on \( C \otimes_{\mathbb{F}_q} O_K \).

17.5. We consider the invertible sheaf \( \Omega_{C_K}([V]) \) on \( C_K \) with
\[
\Gamma(\text{Spec} A_K, \Omega_{C_K}([V])) = \{ x \in \text{Quot}(A_K) : \text{ord}_P(x) \geq 0 \forall P \in C_K \setminus \{ V, \infty \} \text{ and } \text{ord}_V(x) \geq -1 \} = \{ x \in \text{Quot}(A_K) : \text{ord}_P(x) \geq 0 \forall P \neq \infty \text{ and } (t - \alpha)x, (y - \beta)x \in A_K \}.
\]
Then we compute \( \Gamma(\text{Spec} A_K, \sigma^* \Omega_{C_K}([V])) \) as the \( A_K \)-module
\[
\{ x \otimes b \in \text{Quot}(A_K) \otimes_{A_K, \sigma^*} A_K : \text{ord}_P(x) \geq 0 \forall P \neq \infty \text{ and } (t - \alpha)x, (y - \beta)x \in A_K \} = \{ x \otimes b \in \text{Quot}(A_K) \otimes_{A_K, \sigma^*} A_K : \text{ord}_P(x) \geq 0 \forall P \neq \infty \text{ and } x \otimes b(t - \alpha^3), x \otimes b(y - \beta^3) \in A_K \} = \Gamma(\text{Spec} A_K, \Omega_{C_K}([V(1)]))
\]
We define an \( A \)-motive \( M = (M, \tau_M) \) over \( K \) of rank 1 and dimension 1 as follows.
\[
M = \Gamma(\text{Spec} A_K, \Omega_{C_K}([V]))
\]
\[
\sigma^* M = \Gamma(\text{Spec} A_K, \Omega_{C_K}([V(1)]))
\]
\[
\tau_M := f : \sigma^* M \rightarrow M \otimes \Omega_{C_K}(-[\Xi]) \subset M
\]
\[
\text{coker} \tau_M \cong \Omega_{C_K}/\Omega_{C_K}(-[\Xi]) \cong K.
\]
This $A$-module corresponds to a Drinfeld $A$-module of rank 1 over $K$, which is described more explicitly in \textbf{[GPI6 §3]}. In particular, $\mathcal{M}$ is uniformizable. Moreover, $\mathcal{M}$ has CM through $\mathcal{O}_E := A$. We set $E = Q$ and then $H_E = \text{Hom}_Q(E, Q^{\text{alg}}) = \{ \text{id}_E \}$ consists of one single element $\psi = \text{id}_E$. Correspondingly we drop all occurrences of $\psi$ from the notation used in Section \textbf{[10]}. The de Rham cohomology of $\mathcal{M}$ is

$$H^1_{\text{dr}}(\mathcal{M}, K[t - \theta]) = \sigma^* M \otimes_{\mathcal{O}_{C_\infty}} \lim A_K / J^n = \Gamma(\text{Spec } A_K; \mathcal{O}_{C_\infty}(V(1))) \otimes_{\mathcal{O}_{C_\infty}} K[t - \theta] = K[t - \theta],$$

because $\lim A_K / J^n = K[t - \theta]$, and $\mathcal{O}_{C_\infty}(V(1))$ equals $\mathcal{O}_{C_\infty}$ on the neighborhood $C_\infty \setminus \{ V(1) \}$ of $\Xi$. For the unique element $\psi = \text{id}_E$ in $H_E$ we have $H^0(\mathcal{M}, K[t - \theta]) = \mathcal{H}^1_{\text{dr}}(\mathcal{M}, K[t - \theta])$ and the Hodge-Pink lattice $q^\mathcal{M} := \tau^{-1}_M (M \otimes_{\mathcal{O}_E} \lim A_K / J^n) \subset H^1_{\text{dr}}(\mathcal{M}, K[t - \theta])$ of $\mathcal{M}$ satisfies

$$q^\mathcal{M} = f^{-1} \cdot H^1_{\text{dr}}(\mathcal{M}, K[t - \theta]) = (t - \theta)^{-1} \cdot H^1_{\text{dr}}(\mathcal{M}, K[t - \theta])$$

by \textbf{[17.3]}. So according to Definition \textbf{[15.7]} the CM-type of $\mathcal{M}$ is $\Phi = (d_{\text{id}_E})$ with $d_{\text{id}_E} = 1$.

\textbf{17.6.} We will next see that $\mathcal{M}$ has a good integral model $\mathcal{M}$ over $\mathcal{O}_K$. Namely, by a similar computation as in \textbf{[17.6]} the invertible sheaf $\mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}((V))$ on $C \otimes_{\mathbb{F}_q} \mathcal{O}_K$ satisfies

$$\sigma^* \mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}((V)) = \mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}((V(1))).$$

Then the good model $\mathcal{M} = (\mathcal{M}, \tau_M)$ of $\mathcal{M}$ over $\mathcal{O}_K$ is given by

$$\mathcal{M} = \Gamma(\text{Spec } A_{\mathcal{O}_K}; \mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}((V)))$$
$$\sigma^* \mathcal{M} = \Gamma(\text{Spec } A_{\mathcal{O}_K}; \mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}((V(1))))$$
$$\tau_M := f : \sigma^* \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_{C \otimes_{\mathbb{F}_q} \mathcal{O}_K}(-[\Xi])} \subset \mathcal{M}$$
$$\text{coker } \tau_M \cong A_{\mathcal{O}_K} / A_{\mathcal{O}_K}(-[\Xi]) \cong \mathcal{O}_K.$$

\textbf{17.7.} With respect to the inclusion $K \subset \mathbb{C}_\infty$, Papanikolas and Green \textbf{[GPI6 §4]} calculate $H^1_{\text{betti}}(\mathcal{M}, A)$ as follows. They fix $(q - 1)$-st roots of $-\alpha$ and $m\theta - \varepsilon$, and set

$$\nu_{\varphi} := (m\theta - \varepsilon)^{(1-q)} \prod_{i=0}^{\infty} \left( 1 - \left( \frac{m}{m\theta - \varepsilon} \right)^{q^i} t \left( \frac{1}{m\theta - \varepsilon} \right)^{q^i} y \right),$$
$$\delta_{\varphi} := (-\alpha)^{(1-q)} \prod_{i=0}^{\infty} \left( 1 - \frac{t}{\alpha^{q^i}} \right).$$

Since $\nu_{\infty}(\alpha) = -2$ in $\mathbb{C}_\infty$, it follows that the product for $\delta_{\varphi}$ converges in $\Gamma(\mathcal{C} \setminus \{ \infty \}, \mathcal{O}_\mathcal{E})$, is invertible on $\mathcal{C} \setminus D$ and has zeroes of order 1 at $V(i)$ and $-V(i)$ for all $i \in \mathbb{N}_0$. Since $\nu_{\infty}(m) = -q$, and so $\nu_{\infty}(m\theta - \varepsilon) = -q - 2$ and $\nu_{\infty}(\frac{m}{m\theta - \varepsilon}) = 2$ it similarly follows that $\nu_{\varphi}$ converges in $\Gamma(\mathcal{C} \setminus \{ \infty \}, \mathcal{O}_\mathcal{E})$ and is invertible on $\mathcal{C} \setminus D$. Moreover, $\nu_{\varphi}$ has zeroes of order 1 at $\Xi^{(i)}$ and $-V(i)$ and $V^{1+i}$ for all $i \in \mathbb{N}_0$, because $1 - \frac{m}{m\theta - \varepsilon} \theta + \frac{1}{m\theta - \varepsilon} = 0$ and $1 - \frac{m}{m\theta - \varepsilon} \alpha - \frac{1}{m\theta - \varepsilon} (\beta + a_1 \alpha + a_3) = 0$ and $1 - \frac{m}{m\theta - \varepsilon} \alpha^q + \frac{1}{m\theta - \varepsilon} = 0$. These functions satisfy the equations

$$\nu_{\varphi} = \nu \cdot \sigma^* \nu_{\varphi} = (y - \varepsilon - m \cdot (t - \theta)) \cdot \sigma^* \nu_{\varphi} \quad \text{and} \quad \delta_{\varphi} = \delta \cdot \sigma^* \delta_{\varphi} = (t - \alpha) \cdot \sigma^* \delta_{\varphi}.$$

Thus with the corresponding $(q - 1)$-st root $\xi^{1/(q-1)}$ of $\xi = -\frac{m}{m\theta - \varepsilon} = -(m + \frac{\theta + a_1 \alpha + a_3}{\alpha})$ we set

$$\lambda_{\mathcal{M}} := \frac{\nu_{\varphi}}{\delta_{\varphi}} = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\sigma^* s_{\nu_{\varphi}^{r_i}}^{\mathcal{M}}}{s_{\delta_{\varphi}^{r_i}}} \in \Gamma(\mathcal{C} \setminus D, \mathcal{O}_\mathcal{E}^{\infty}).$$

(17.7)

Then $\tau_M(\sigma^* \lambda_{\mathcal{M}}) = f \cdot \sigma^* \lambda_{\mathcal{M}} = \lambda_M$, and $\lambda_M$ is a meromorphic function on $\mathcal{C} \setminus \{ \infty \}$ without poles or zeroes on $\mathcal{C} \setminus D$. (By looking at the product decomposition of $\lambda_M$, one even sees that it has a simple pole at $V$ and simple zeroes at $\Xi^{(i)}$ for all $i \in \mathbb{N}_0$.) So we obtain

$$H^1_{\text{betti}}(\mathcal{M}, A) = \lambda_M \cdot A.$$

(17.8)

Let $u \in H^1_{\text{betti}}(\mathcal{M}, A)$ be the generator such that $\langle u, \lambda_M \rangle = 1$. We also write $u_{\text{id}_K} := u$. 

39
17.8. We can take \( \omega := \omega_q := \sigma^* \delta^{-1} = (t - \alpha q)^{-1} \) as a generator of \( H^1_{\text{dR}}(M, K[t - \theta]) \). Then the comparison isomorphism \( h_{\text{Betti, dR}} = \sigma^* h_M \) from Theorem 13.17 sends the generator \( \lambda_M \) of \( H^1_{\text{Betti}}(M, A) \) to \( \sigma^* \lambda_M = \sigma^* (\lambda_M \delta) \cdot \omega \in H^1_{\text{dR}}(M, K[t - \theta]) \) and the comparison isomorphism \( h_{\text{Betti, dR}} = \sigma^* h_M \mod J \) from (13.3) sends the generator \( \lambda_M \) of \( H^1_{\text{Betti}}(M, A) \) to \( \sigma^* (\lambda_M \delta)(\Xi) \cdot \omega \in H^1_{\text{dR}}(M, K) \). Therefore,

\[
\langle u, h_{\text{Betti, dR}}^{-1}(\omega) \rangle = \langle u, \sigma^* (\lambda_M \delta)(\Xi)^{-1} \cdot \lambda_M \rangle = \frac{\xi^q/(q-1)}{(\sigma^* \delta)(\Xi)} \prod_{i=1}^{\infty} \left( \frac{\xi^q}{(\sigma^* f)(\Xi)} \right).
\]

To compute the absolute value of \( \langle u, h_{\text{Betti, dR}}^{-1}(\omega) \rangle \) we observe that for every \( i \in \mathbb{N}_{>0} \)

\[
\left| \frac{\xi^q}{(\sigma^* f)(\Xi)} \right| = \left| \frac{1 - \frac{\theta}{\alpha^q}}{1 - (\frac{m}{\text{ord}})^q \theta + (\frac{1}{m \theta})^q \varepsilon} \right| = 1,
\]

as well as \( v_\infty(\xi) = -q \), whence \( |\xi^q/(q-1)| = q^2/(q-1) \), and \( |(\sigma^* \delta)(\Xi)| = |(t - \alpha q)(\Xi)| = |t - \alpha|^\infty = |\alpha|^\infty = q^2 \). Thus we obtain

\[
\left| \int_u \omega \right|_\infty = \left| \langle u, h_{\text{Betti, dR}}^{-1}(\omega) \rangle \right|_\infty = q^{\frac{2}{q-1} - 2q} = q^{\frac{2}{q-1} - q} \quad \text{and}
\]

\[
\log \left| \int_u \omega \right|_\infty = \left( \frac{q}{q-1} - q \right) \log q.
\]

(17.9)

17.9. We consider the set \( H_K := \text{Hom}_q(K, \mathcal{O}_q) = \text{Gal}(K/\mathbb{F}_q(\theta, \varepsilon)) \) which actually is a group, because \( K \) is Galois over \( \mathbb{F}_q(\theta, \varepsilon) \). It is isomorphic to the group \( C(\mathbb{F}_q) \) under the map \( \eta \mapsto P_{\eta} := V - \eta(V) \). Indeed, since \( \eta(\Xi) = \Xi \in C(K) \) is fixed by \( \eta \) we see that \( \eta(V) \) still satisfies \( \eta(V) = \eta(V)(1) = \eta(V) - \eta(V)(1) = \eta(\Xi) = \Xi = V - V(1) \). Therefore, the point \( P_{\eta} = V - \eta(V) \) satisfies \( P_{\eta}(1) = P_{\eta} \), and hence \( P_{\eta} \in C(\mathbb{F}_q) \). Since the coordinates \( (\alpha, \beta) \) of \( V \) generate the field extension \( K/\mathbb{F}_q(\theta, \varepsilon) \), the map \( \eta \mapsto P_{\eta} \) is bijective. It is a group homomorphism, because \( P_{\eta \eta'} = V - \eta(\eta')(V) = V - \eta(V) + \eta'(V) = \eta(V) - \eta(V) = P_{\eta} + P_{\eta'} \), as \( P_{\eta} \in C(\mathbb{F}_q) \) is fixed by \( \eta \). In particular, \( \# H_K = \# C(\mathbb{F}_q) \).

We now fix an element \( \eta \in H_K \) with \( \eta \neq id_K \) and let the \( A \)-motive \( \mathcal{M}^q \) over \( \mathcal{O}_K \) and \( \omega^q \in H^1_{\text{dR}}(\mathcal{M}^q, K[t - \theta]) \) be deduced from \( \mathcal{M} \) and \( \omega \) by base extension. Then \( \mathcal{M}^q \) is isogenous to \( \mathcal{M} \) by the theory of complex multiplication, which was developed for Drinfeld modules by Hayes [Hay79] and for general \( A \)-motives by Pelzer [Pel09]. We give an elementary and explicit treatment for our \( \mathcal{M} \). We claim that there is an isomorphism

\[
g_{\eta} : \mathcal{M}^q \cong \mathcal{M} \otimes \mathcal{O}(-[P_{\eta}]) =: \mathcal{M}(-[P_{\eta}]),
\]

where \( \mathcal{O}(-[P_{\eta}]) \) denotes the invertible sheaf on \( \text{Spec} \mathcal{A}_K \) associated to the divisor \( -[P_{\eta}] \cdot \mathcal{A}_K \text{Spec} \mathcal{O}_K \). Namely, the \( A \)-motives \( \mathcal{M}^q \) and \( \mathcal{M}(-[P_{\eta}]) \) correspond to the invertible sheaves \( \mathcal{O}_{C_K}([\eta(V)]) = \mathcal{O}_{C_K}([V - P_{\eta}]) \) and \( \mathcal{O}_{C_K}([V]) \otimes \mathcal{O}_{C_K}(-[P_{\eta}]) = \mathcal{O}_{C_K}([V] - [P_{\eta}]) \) on \( C_K \), respectively.

By (17.3) and (17.8) Corollary III.3.5 the divisor \( [V - P_{\eta}] + [V] + [P_{\eta}] - [\infty] \) on \( C_K \) is principal and there is a function \( g_{\eta} \in K(t, \eta) \) with \( \text{Quot}(A_K) \) with

\[
\text{div}(g_{\eta}) = [V - P_{\eta}] + [V] + [P_{\eta}] - [\infty] = [V - P_{\eta}] + [-V] + [P_{\eta}] - [-V] - [\infty].
\]

It can be written explicitly as follows. By construction of the group law on \( C \), the three points \( V - P_{\eta} = \eta(V) \) and \( -V \) and \( P_{\eta} \) lie on a single line whose slope is

\[
\frac{\eta(\beta) - \eta(P_{\eta})}{\eta(\alpha) - t(P_{\eta})} = \frac{\eta(\beta) + \beta + a_1 \alpha + a_3}{\eta(\alpha) - \alpha} = \frac{\eta(P_{\eta}) + \beta + a_1 \alpha + a_3}{t(P_{\eta}) - \alpha} \in \mathcal{O}_K.
\]

This slope is a priory an element of \( K \), but we see that it lies in \( \mathcal{O}_K \) by reasoning like in Lemma 17.3. Indeed, the slope has a pole if and only if one of the points \( P_{\eta} \) or \( -V \) or \( V - P_{\eta} = \eta(V) \) equals \( \infty \). If \( P_{\eta} = \infty \), then the bijectivity of the map \( \eta \mapsto P_{\eta} \) implies \( \eta = \text{id}_K \) which was excluded. If \( V - P_{\eta} = \infty \), and hence \( V = P_{\eta} \in C(\mathbb{F}_q) \), or if \( -V = \infty \), then \( \Xi = \infty \), and so the poles of the slope do not lie in \( \text{Spec} \mathcal{O}_K \). That is, the slope lies in \( \mathcal{O}_K \) as claimed. Then we can take

\[
g_{\eta} = \frac{y - \eta(\beta) - \eta(P_{\eta}) - (t - \eta(\alpha))}{t - \alpha}.
\]

(17.12)

as an isomorphism \( \mathcal{M}^q \cong \mathcal{M} \otimes \mathcal{O}(-[P_{\eta}]) \). Here we use that formula (17.11) for the divisor of \( g_{\eta} \) also holds on \( C \otimes \mathcal{O}_K \), because both numerator and denominator of \( g_{\eta} \) lie in \( \mathcal{O}_K[t, y] \) and do not vanish on an entire fiber of \( C \otimes \mathcal{O}_K \) over a closed point of \( \text{Spec} \mathcal{O}_K \).
In order to see that \( g_\eta \) is an isomorphism of \( A \)-motives, it remains to prove that \( g_\eta \circ \eta(f) = f \circ \sigma^* g_\eta \). Since the divisor on both sides equals \([\eta(V)^{(1)}) + [P_\eta] - [V] + [\Xi] - 2[\infty] \), both sides differ by multiplication with an element of \( K^* \). Multiplying both sides with the common denominator and comparing the coefficients of \( t^\alpha y \) shows that both sides are equal as desired.

17.10. The isomorphism \( g_\eta: \mathcal{M}^\eta \xrightarrow{\sim} \mathcal{M}(-[P_\eta]) \) induces isomorphisms on (co-)homology

\[
\begin{align*}
    g_\eta &: H^1_{\text{dr}}(\mathcal{M}^\eta, O_K) \xrightarrow{\sim} H^1_{\text{dr}}(\mathcal{M}(-[P_\eta]), O_K), \\
    g_\eta &: H^1_{\text{Betti}}(\mathcal{M}^\eta, A) \xrightarrow{\sim} H^1_{\text{Betti}}(\mathcal{M}(-[P_\eta]), A), \quad \text{and} \\
    g_\eta &: H^1_{\text{Betti}}(\mathcal{M}^\eta, A) \xrightarrow{\sim} H^1_{\text{Betti}}(\mathcal{M}(-[P_\eta]), A).
\end{align*}
\]

These are compatible with the period isomorphisms \( h_{\text{Betti, dr}} \) and the pairing between \( H^1_{\text{Betti}} \) and \( H^1_{\text{Betti}} \). So we may replace \( \mathcal{M}^\eta \) by \( \mathcal{M}(-[P_\eta]) \) in the rest of our computation.

Since \( \omega = (t - \alpha^q)^{-1} \) and \( \omega \mod (t - \theta) = (\theta - \alpha^q)^{-1} \in H^1_{\text{dr}}(\mathcal{M}, O_K) \) we obtain \( \omega^n = (t - \eta(\alpha)^q)^{-1} \) and \( \omega^n \mod (t - \theta) = (\theta - \eta(\alpha)^q)^{-1} \), and we set \( \bar{\omega}^n := g_\eta(\omega^n) \in H^1_{\text{dr}}(\mathcal{M}(-[P_\eta]), K[t - \theta]) \) and \( \bar{\omega}^n \mod (t - \theta) = g_\eta(\omega^n) \mod (t - \theta) \in H^1_{\text{dr}}(\mathcal{M}(-[P_\eta]), O_K) \). By definition, \( H^1_{\text{dr}}(\mathcal{M}, O_K) := \sigma^* \mathcal{M}/J \sigma^* \mathcal{M} = \sigma^* \mathcal{M}|_\Xi \), with \( J = (t - \theta, y - \varepsilon) \) being the vanishing ideal of the \( O_K \)-valued point \( \Xi \in \mathcal{O}(O_K) \). We compute

\[
\bar{\omega}^n = \sigma^* (g_\eta) \cdot (t - \eta(\alpha)^q)^{-1}
\]

\[
= \frac{y - \eta(\beta)^q - \frac{\eta(\beta)^q + \beta^q + a_1 \alpha^q + a_3}{\eta(\alpha)^q - \alpha^q} (t - \eta(\alpha)^q)}{t - \alpha^q} \cdot (t - \eta(\alpha)^q)^{-1}
\]

\[
= \frac{y - \eta(\beta)^q - \frac{\eta(\beta)^q + \beta^q + a_1 \alpha^q + a_3}{\eta(\alpha)^q - \alpha^q} (\theta - \eta(\alpha)^q)}{t - \eta(\alpha)^q} \cdot (t - \theta)^{-1} \quad \text{and}
\]

\[
\bar{\omega}^n \mod (t - \theta) = \frac{\varepsilon - \eta(\beta)^q - \frac{\eta(\beta)^q + \beta^q + a_1 \alpha^q + a_3}{\eta(\alpha)^q - \alpha^q} (\theta - \eta(\alpha)^q)}{\eta(\beta)^q + \beta^q + a_1 \alpha^q + a_3} \cdot \omega \mod (t - \theta).
\]

The element \( \sigma^* g_\eta|_\Xi := \frac{\varepsilon - \eta(\beta)^q}{\theta - \eta(\alpha)^q} - \frac{\eta(\beta)^q + \beta^q + a_1 \alpha^q + a_3}{\eta(\alpha)^q - \alpha^q} \) has absolute value

\[
|\sigma^* g_\eta|_\infty = q^\alpha, \quad \text{and hence} \quad \log|\sigma^* g_\eta|_\infty = q \log q, \quad (17.13)
\]

because the first summand has absolute value \( q \) and is dominated by the second summand which has absolute value \( q^\alpha \).

17.11. We now compute \( v(\omega^n) \) for all places \( v \neq \infty \) of \( Q \) and for all \( \eta \in H_K \). Observe that by \( (17.3) \) the multiplication with \( t - \alpha^q \) induces an isomorphism \( \mathcal{O}_C \otimes_{\mathcal{O}_K} (V) \xrightarrow{\sim} \mathcal{O}_C \otimes_{\mathcal{O}_K} (2[\infty] - [-V]) \) and the multiplication with \( t - \alpha^q \) induces an isomorphism \( \mathcal{O}_C \otimes_{\mathcal{O}_K} (V^{(1)}) \xrightarrow{\sim} \mathcal{O}_C \otimes_{\mathcal{O}_K} (2[\infty] - [-V^{(1)})] \). We restrict this morphism to the \( O_K \)-valued point \( \Xi \), that is, we pull it back under the corresponding morphism \( h_\Xi: \text{Spec} \mathcal{O}_K \rightarrow C \otimes_{\mathbb{F}_q} \mathcal{O}_K \). To do so we first claim that \( h_\Xi \) factors through the open subscheme \( C \otimes_{\mathbb{F}_q} \mathcal{O}_K \) which is the complement of \( \{\infty\} \cup \{-V^{(1)}\} \). Indeed, the locus on \( C \otimes_{\mathbb{F}_q} \mathcal{O}_K \) where \( \Xi = -V^{(1)} \) is equal to the locus where \( V = \infty \), and the latter locus does not lie above \( \text{Spec} \mathcal{O}_K \). The same is true for the locus where \( \Xi = \infty \). We conclude that multiplication with \( t - \alpha^q \) induces an isomorphism

\[
\theta - \alpha^q: H^1_{\text{dr}}(\mathcal{M}, O_K) = h_\Xi^* \mathcal{O}_C \otimes_{\mathcal{O}_K} (V) \xrightarrow{\sim} h_\Xi^* \mathcal{O}_C \otimes_{\mathcal{O}_K} (2[\infty] - [-V]) = h_\Xi^* \mathcal{O}_C \otimes_{\mathcal{O}_K} = \mathcal{O}_K
\]

\[
\omega \mod (t - \theta) = (t - \alpha^q)^{-1} \quad \mapsto \quad 1.
\]

This shows that \( H^1_{\text{dr}}(\mathcal{M}, O_K) \) is an isomorphism \( \omega \mod (t - \theta) \), and by base extension under \( \eta \), also \( H^1_{\text{dr}}(\mathcal{M}^\eta, O_K) = \mathcal{O}_K \cdot \omega^n \mod (t - \theta) \). This yields

17.14. for every place \( v \neq \infty \) and every \( \eta \in H_K \).
17.12. We next compute $H^1_{\text{Betti}}(M(-[P_n]), A)$ for the $A$-motive $M(-[P_n]) = (\mathcal{O}_C, (V) - [P_n]), \tau = f)$. The function $\lambda_M$ from (17.7) satisfies $\tau(\sigma^*\lambda_M) = f \cdot \sigma^*\lambda_M = \lambda_M$, but it does not have a zero at $P_n$, and hence does not lie in $M(-[P_n]) \otimes_{\mathcal{O}_C} K$ and not in $H^1_{\text{Betti}}(M(-[P_n]), A)$. Instead,

$$H^1_{\text{Betti}}(M(-[P_n]), A) = \lambda_M \cdot \Gamma(\text{Spec } A, \mathcal{O}_C(-[P_n])) = \lambda_M \cdot p_\eta,$$

where $p_\eta \subset A$ is the maximal ideal defining the $\mathbb{F}_p$-valued point $P_n \in C$. Correspondingly, when we take $\tilde{u}_\eta := u \in H^1_{\text{Betti}}(M(-[P_n]), Q) = H^1_{\text{Betti}}(M, Q)$, which pairs with $\lambda_M$ to $(\tilde{u}_\eta, \lambda_M) = (u, \lambda_M) = 1$, we obtain

$$H^1_{\text{Betti}}(M(-[P_n]), A) = \tilde{u}_\eta \cdot \Gamma(\text{Spec } A, \mathcal{O}_C([P_n])) = \tilde{u}_\eta \cdot p_\eta^{-1},$$

This yields

$$v_\eta(\tilde{u}_\eta) \cdot \log q_v = \begin{cases} 0 & \text{if } v \neq p_\eta \text{ or } \eta = \text{id}_K, \\ \log q & \text{if } v = p_\eta \text{ and } \eta \neq \text{id}_K. \end{cases} \quad (17.15)$$

Also from (17.9) and (17.13) we compute the absolute value

$$\log f_{\tilde{u}_\eta} = \log \left(\frac{(1 - \sigma^*\eta \cdot \omega)}{4q\log q}\right) \eta, \quad (17.16)$$

17.13. Finally, we recall the zeta functions for the elliptic curve $C$, which are defined as the following products which converge for $s \in C$ with $\Re(s) > 1$

$$\zeta_C(s) := \prod_{v \neq \infty} (1 - (\#F_v)^{-s})^{-1} = \prod_{v \neq \infty} (1 - q_v^{-s})^{-1} = \frac{1 - (q + 1 - \#C(F_q))q^{-s} + q^{-2s}}{(1 - q^{-s})(1 - q^{-1+s})}$$

and

$$\zeta_A(s) := \prod_{v \neq \infty} (1 - (\#F_v)^{-s})^{-1} = \prod_{v \neq \infty} (1 - q_v^{-s})^{-1} = \frac{1 - (q + 1 - \#C(F_q))q^{-s} + q^{-2s}}{1 - q^{-1+s}}.$$

Since $L^\infty(1, s) = \zeta_A(s)$ we obtain

$$Z(1, 0) = \frac{\zeta_A(0)}{\zeta_A(1)} = \left(\frac{q + 1 - \#C(F_q) - 2q}{1 - (q + 1 - \#C(F_q)) + q} - \frac{q}{1 - q}\right) \log q = \left(\frac{1 - \#C(F_q) - q}{\#C(F_q)} + \frac{q}{q - 1}\right) \log q. \quad (17.17)$$

We now put everything together using Theorem 16.2 and formula (16.7) to compute

$$\frac{1}{\#H_K} \sum_{v \in H_K} \sum_{\eta \in H_K} \log f_{\tilde{u}_\eta} \omega^v = \left(\frac{q + \#C(F_q) - 1}{\#C(F_q)} - \frac{q}{q - 1}\right) \cdot \log q \quad \text{from } (17.17)$$

$$+ \frac{1}{\#C(F_q)} \left(\frac{q}{q - 1} - q\right) \cdot \log q \quad \text{from } (17.9)$$

$$+ \frac{\#C(F_q) - 1}{\#C(F_q)} \frac{q}{q - 1} \cdot \log q \quad \text{from } (17.10)$$

$$- \frac{\#C(F_q) - 1}{\#C(F_q)} \cdot \log q \quad \text{from } (17.14) \text{ and } (17.15)$$

$$= 0.$$  

Miraculously, all terms cancel and this shows that in the present example our Conjecture 16.3 holds true.

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