COUNTEREXAMPLE TO STRONG DIAMAGNETISM
FOR THE MAGNETIC ROBIN LAPLACIAN

AYMAN KACHMAR AND MIKAEL P. SUNDQVIST

ABSTRACT. We determine a counterexample to strong diamagnetism for the Laplace operator in the unit disc with a uniform magnetic field and Robin boundary condition. The example follows from the accurate asymptotics of the lowest eigenvalue when the Robin parameter tends to $-\infty$.

1. INTRODUCTION

1.1. Magnetic Robin Laplacian. We denote by $\Omega = \{ x \in \mathbb{R}^2 : |x| < 1 \}$ the open unit disk and by $\Gamma = \partial \Omega = \{ x \in \mathbb{R}^2 : |x| = 1 \}$ its boundary. We study the lowest eigenvalue of the magnetic Robin Laplacian in $L^2(\Omega)$,

$$\mathcal{P}_\gamma^b = -((\nabla - ib\mathbf{A}_0)^2, \quad (1.1)$$

with domain

$$D(\mathcal{P}_\gamma^b) = \{ u \in H^2_0(\Omega) : \nu \cdot (\nabla u - ib\mathbf{A}_0)u + \gamma u = 0 \text{ on } \partial \Omega \}. \quad (1.2)$$

Here $\nu$ is the unit outward normal vector of $\Gamma$, $\gamma < 0$ the Robin parameter and $b > 0$ is the intensity of the applied magnetic field. The vector field $\mathbf{A}_0$ generates the unit magnetic field and is defined as follows

$$\mathbf{A}_0(x_1, x_2) = \frac{1}{2}(-x_2, x_1). \quad (1.3)$$

To be more precise, the operator $\mathcal{P}_\gamma^b$ is defined as the Friedrichs extension, starting from the quadratic form $[8, \text{Ch. 4}]$,

$$H^1(\Omega) \ni u \mapsto Q^b_{\gamma}(u) := \int_\Omega |(\nabla - ib\mathbf{A}_0)u(x)|^2 \, dx + \gamma \int_{\Gamma} |u(x)|^2 \, ds(x). \quad (1.4)$$

1.2. Main result. The operator $\mathcal{P}_\gamma^b$ has a compact resolvent, and thus its spectrum consists of an increasing sequence of eigenvalues. We are interested in examining the asymptotics of the principal eigenvalue

$$\lambda_1(b, \gamma) = \inf_{u \in H^1_0(\Omega)} \frac{Q^b_{\gamma}(u)}{\|u\|^2_{L^2(\Omega)}} \quad (1.5)$$

when $b > 0$ is fixed and the Robin parameter $\gamma$ tends to $-\infty$.

**Theorem 1.1.** Let $b > 0$. Then, as $\gamma \to -\infty$,

$$\lambda_1(b, \gamma) = -\gamma^2 + \gamma + \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right)^2 - \frac{1}{2} + o(1).$$

2010 Mathematics Subject Classification. Primary 35P15, 47A10, 47F05.

Key words and phrases. Magnetic Laplacian, Robin boundary condition, eigenvalues, diamagnetic inequalities.
The first two terms in the asymptotic expansion given in Theorem 1.1 are well known after many contributions (see [15, 16, 17] for the case \( b = 0 \) and [12] for the case \( b > 0 \)); however, the third correction term is new for the disc geometry for \( b > 0 \). The recent contribution [11, Thm. 1.5] shows that Theorem 1.1 continues to hold in the case \( b = 0 \).

1.3. Lack of strong diamagnetism. The celebrated diamagnetic inequality yields

\[
\lambda_1(b, \gamma) \geq \lambda(0, \gamma).
\]

By using Theorem 1.1 we can quantify the diamagnetic inequality as follows

\[
\lambda_1(b, \gamma) - \lambda_1(0, \gamma) \sim e(b) := \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right)^2.
\]  

Connected to the diamagnetic inequality is the property of strong diamagnetism [1]; this is whether the function \( b \mapsto \lambda_1(b, \gamma) \) is monotone increasing on some interval \([b_0, +\infty) \subset \mathbb{R}_+\).

As consequence of Theorem 1.1 we obtain a counterexample to strong diamagnetism.

**Corollary 1.2.** There exists \( \gamma_0 < 0 \) such that, for all \( \gamma \in (-\infty, \gamma_0] \), the function \( b \mapsto \lambda_1(b, \gamma) \) is not monotone increasing.

Besides its mathematical interest, the question of strong diamagnetism has applications to Physics, particularly in the context of superconductivity [2]. In the case of a simply connected domain subject to a uniform applied magnetic field and Neumann boundary condition \( (\gamma = 0) \), strong diamagnetism holds [3, 4]. Counter examples of strong diamagnetism exist for uniform magnetic fields in non-simply connected domains, or for non-uniform magnetic fields in simply connected domains [6, 9]. Interestingly, the Robin boundary condition has the unique feature where strong diamagnetism fails for the disc (which is a simply connected domain) even when it is subject to a uniform applied magnetic field.

**Corollary 1.2** results from the following statement. Given a positive real number \( A \), there exist \( \gamma_0 < 0 \) and \( A < b_1 < b_2 < b_3 \) such that, for all \( \gamma \in (-\infty, \gamma_0] \),

\[
\lambda_1(b_1, \gamma) < \lambda_1(b_2, \gamma) \& \lambda_1(b_2, \gamma) > \lambda_1(b_3, \gamma).
\]

We can simply select the constants \( b_i \) as follows

\[
b_1 = 2n_0, \quad b_2 = 2n_0 + 1, \quad b_3 = 2n_0 + \frac{3}{2},
\]

where \( n_0 \) is the smallest natural number satisfying \( n_0 > A \); the conclusion then follows from Theorem 1.1.

Using the periodicity of the function \( b \mapsto e(b) \), given a natural number \( N \), we can select \( \gamma_1 < 0 \) such that

\[
b_{1,i} := b_1 + i < b_{2,i} := b_2 + i < b_{3,i} := b_3 + i
\]

with the following two inequalities

\[
\lambda_1(b_{1,i}, \gamma) < \lambda_1(b_{2,i}, \gamma), \quad \lambda_1(b_{2,i}, \gamma) > \lambda_1(b_{3,i}, \gamma),
\]

holding for all \( \gamma \leq \gamma_1 \) and \( i \in \{1, 2, \cdots, N\} \).
1.4. The Little–Parks effect. When cooled below a certain critical temperature, a normal conductor becomes a superconductor and losess electrical resistance. The Little–Parks experiment displays oscillations in the critical temperature of a superconductor as the applied magnetic field varies. Typically, the superconducting sample used in the experiment is a thin ring. From a mathematical perspective, the trivial normal solution of the Ginzburg–Landau equations changes back and forth from stable to unstable states.

Using a model for a superconductor with enhanced surface [5, 14], we can use Theorem 1.1 to estimate the critical temperature as a function of the applied magnetic field, consistent with the Little–Parks experiment. The novelty in our situation is that, unlike the Little–Parks experiment, the superconducting sample is a disc subject to a uniform magnetic field.

The model we study is a variant of the Ginzburg–Landau energy by adding a (negative) surface energy term (accounting for the enhanced surface). The surface term can be derived naturally starting from a Ginzburg–Landau model for two adjust superconductors [13, Thm. 1.2].

Following the presentation in [7, Sec. 3], we introduce the functional

\[
\mathcal{E}^{\text{phys}}(u, a) = \frac{\hbar^2}{2m\ell} \int_{\partial \Omega} |u|^2 d\sigma(x) + \int_{\tilde{\Omega}} \left( \frac{1}{2m} \left( \hbar \nabla - i \frac{2e}{c} a \right) u \right)^2 + \alpha(T)|u|^2 + \frac{\beta}{2} |u|^4 + \frac{1}{8\pi} |\text{curl} a - H|^2 \right) d\tilde{x}.
\]

(1.7)

Here \((u, a)\) describes the superconducting properties \(\equiv (0, H A_0)\) signifies the normal state, where \(A_0\) is introduced in (1.3); \(T\) denotes the temperature; \(\hbar, e, c, m, \beta\) are positive constants; \(H \geq 0\) measures the intensity of the applied magnetic field and \(\ell < 0\) models the enhanced surface. The disc \(\tilde{\Omega} = \{\tilde{x} \in \mathbb{R}^2 : \|\tilde{x}\| < \tilde{R}\}\) is the horizontal cross section of the superconducting sample. The magnetic permeability \(\mu_0\) in \(\tilde{\Omega}\) is assumed uniform, so
we take $\mu_0 = 1$. For later use, we introduce the Ginzburg–Landau parameter

$$\kappa = \sqrt{\frac{m^2\beta c^2}{8\pi e^2\hbar}}.$$  \hspace{1cm} (1.8)

Among all the parameters in (1.7), only $\alpha(T)$ depends on the temperature $T$; consequently, $\kappa$ is temperature independent. The expression of $\alpha(T)$ is given via the following relation

$$\alpha(T) = \frac{\hbar^2}{2m\xi_0^2} \left( \frac{T}{T_c} - 1 \right),$$  \hspace{1cm} (1.9)

where $\xi_0 > 0$ is a temperature independent parameter, the coherence length at zero temperature. The parameter $T_c$ is the critical temperature of the superconductor occupying $\Omega$ in the absence of a magnetic field (i.e. when $H = 0$).

We will express the functional in (1.7) in temperature independent units, and introduce some notation

$$\tilde{x} = Rx, \quad \tilde{\Omega} = R\Omega, \quad u(\tilde{x}) = \sqrt{\frac{|\alpha(0)|}{\beta}} \psi(Rx),$$

$$a(\tilde{x}) = \frac{ch}{2e\xi_0^2} RA(Rx), \quad b = \frac{ch}{2e\xi_0^2} RH, \quad \gamma = \frac{R}{\ell}.$$

The functional in (1.7) becomes

$$\mathcal{E}^{phys}(u, a) = \frac{|\alpha(0)|\hbar^2}{2m\beta} \mathcal{E}(\psi, A),$$

where

$$\mathcal{E}(\psi, A) = \gamma \int_{\Omega} |\psi|^2 d\sigma(x) + \int_{\Omega} \left( |(\nabla - iA)\psi|^2 - \frac{R^2}{\xi_0^2} \mu(T)|\psi|^2 + \frac{R^2}{2\xi_0^2} |\psi|^4 + \kappa^2 |\text{curl} A - b|^2 \right) dx,$$  \hspace{1cm} (1.10)

and

$$\mu(T) := 1 - \frac{T}{T_c}.$$  \hspace{1cm} (1.11)

The functional in (1.10) is defined on the space

$$\mathcal{H} = H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2).$$

Clearly, the normal solution $(0, bA_0)$ is a critical point of the functional in (1.10); it is said to be stable if it is a local minimizer. Using the direct method of the calculus of variations, we can prove that a minimizer $(\psi_*, A_*) \in \mathcal{H}$ of $\mathcal{E}$ exists (cf. [7, Sec. 3]).

Recall the eigenvalue $\lambda(b, \gamma)$ introduced in (1.5). Linearizing the functional in (1.10) near the normal state $(0, bA_0)$, we get the following:

- If $\lambda(b, \gamma) < \frac{R^2}{\xi_0^2} \mu(T)$, then the normal state is not stable and the global minimizer $(\psi_*, A_*)$ is non-trivial in the sense that $\psi_* \neq 0$;
- If $\lambda(b, \gamma) > \frac{R^2}{\xi_0^2} \mu(T)$, then the normal state is a local minimizer.
Consequently, we introduce the critical temperature \( T_c(b) \), in the non-zero magnetic field \( b \), as the solution of the equation

\[
\lambda(b, \gamma) = \frac{R^2}{\xi_0^2} \mu(T) .
\]

Thanks to (1.11) we find that

\[
T_c(b) = \left( 1 - \frac{\xi_0^2}{R^2} \lambda(b, \gamma) \right) T_{c_0} .
\] (1.12)

Using Theorem [1.1] we can estimate \( T_c(b) \) as \( \gamma \to -\infty \); we find

\[
T_c(b) = \left[ 1 - \frac{\xi_0^2}{R^2} \left( -\gamma^2 + \gamma + e(b) - \frac{1}{\gamma} \right) \right] T_{c_0} + o(1) ,
\] (1.13)

where \( e(b) \) is introduced in (1.6). It is worth noticing that

- \( T_c(b) > T_{c_0} \);
- Up to approximation errors, \( T_c(b) \) is a periodic function of \( b \), which is consistent with the Little–Parks effect;
- For \( T < T_c(b) \), the global minimizer of \( E \) is non-trivial (in the sense \( \psi_{s} \neq 0 \) ); while for \( T > T_c(b) \), the normal solution is a local minimizer of \( E \).

Acknowledgments. This work started when A. Kachmar visited the mathematics department in Lund University. The research of A. Kachmar is supported by a grant from the Lebanese university within the project “Analytical and numerical aspects of the Ginzburg–Landau model”.

2. Proof of Theorem [1.1]

2.1. Outline. The proof consists of several reductions to operators that are easier to handle. In the first step we change parameter to have a semi-classical parameter. We then observe that we have localization close to the boundary, make a Fourier decomposition, and express the interesting operators and quadratic forms in suitable coordinates. Some effective operators appear, and we expand their eigenvalues in terms of the semi-classical parameter.

Since we are not looking at the large magnetic field limit, the terms in the potential that appears in polar coordinates is easier to handle since the angular momentum and magnetic field strength do not compete against each other.

2.2. Translation of Theorem [1.1] into a semi-classical statement. It is convenient to work in a semi-classical framework. We do so by introducing the semi-classical parameter \( h = \gamma^{-2} \). Then \( h \to 0_+ \) when \( \gamma \to -\infty \), and the quadratic form \( Q^h \) can be written as

\[
Q^h(u) = h^{-2} \left( \int_{\Omega} |(h \nabla - ibhA_0)u(x)|^2 dx - h^{3/2} \int_{\partial \Omega} |u(x)|^2 ds(x) \right).
\]

Consequently, we get the \( h \)-dependent self-adjoint operator

\[
L^h = -(h \nabla - ibhA_0)^2 ,
\] (2.1)
with domain
\[ D(L_h^b) = \{ u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u - h^{1/2}u = 0 \text{ on } \partial \Omega \}. \] (2.2)

The spectra of the operators \( P_\gamma^b \) and \( L_h^b \) are related as
\[ \sigma(P_\gamma^b) = h^{-2}\sigma(L_h^b). \]

Let \( \mu_1(h, b) \) be the principal eigenvalue of the operator \( L_h^b \). Theorem 1.1 can be rephrased as follows.

**Theorem 2.1.** Let \( b > 0 \). Then, as \( h \to 0^+ \),
\[ \mu_1(h, b) = -h - h^{3/2} + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right) - \frac{1}{2} \right) h^2 + o(h^2). \]

2.3. **Reduction to a thin ring.** Our aim is to work in (a variant of) polar coordinates. However, if we change directly to polar coordinates we will get some illusive problems at the origin with negative powers of \( r \) coordinates. However, if we change directly to polar coordinates we will get some illusive problems at the origin with negative powers of \( r \).

For small \( h \), the ground states of the operator \( L_h^b \) are localized near the boundary of \( \Omega \) (Proposition 2.3 below). This will allow us to work in an annulus instead of the disk. Before we give the localization result we show that there exists a sufficiently small eigenvalue for small \( h \).

**Lemma 2.2.** Let \( b > 0 \). Then there exists \( h_0 \in (0, 1) \) such that for all \( h \in (0, h_0) \)
\[ \mu_1(h, b) \leq -h - \frac{1}{2} h^{3/2}. \]

**Proof.** Let \( u(x) = c \exp(h^{-1/2}(|x| - 1)) \), where \( c \) is chosen so that \( u \) becomes normalized in \( L^2(\Omega) \). A direct calculation gives
\[ \frac{\hbar^2 Q_\gamma^b(u)}{\|u\|^2} = -h - h^{3/2} + O(h^2). \]

By changing the coefficient in front of \( h^{3/2} \) we get the existence of \( h_0 \in (0, 1) \) such that the claimed inequality holds for \( h \in (0, h_0) \). \( \square \)

**Proposition 2.3** (localization of ground states). Let \( M \in (-1, 0) \). For all \( \alpha < \sqrt{-M} \), there exist constants \( C > 0 \) and \( h_0 \in (0, 1) \) such that, if \( u_h \) is a normalized ground state of \( L_h^b \) with eigenvalue bounded above by \( Mh \), then, for all \( h \in (0, h_0) \),
\[ \int_{\Omega} \left( |u_h(x)|^2 + h \|\nabla - ibA_0\|u_h(x)|^2 \right) \exp \left( \frac{2\alpha \text{dist}(x, \partial \Omega)}{h^{1/2}} \right) dx \leq C. \]

The proof of Proposition 2.3 is similar to the one of [10] Thm. 5.1], and we leave out the details.

As a consequence of the concentration properties of the ground states, we can approximate the principal eigenvalue \( \mu_1(h, b) \) by a ground state energy \( \tilde{\mu}(h, b, \rho) \) that we describe next.

Let \( \rho \in (0, \frac{1}{2}) \) and consider the annulus \( \Omega_h = \{ x \in \mathbb{R}^2 : 1 - h^\rho < |x| < 1 \} \). We introduce the quadratic form
\[ q_h^b(\rho)(u) = h^2 \int_{\Omega_h} |(\nabla - ibA_0)u(x)|^2 dx - h^{3/2} \int_{|x|=1} |u(x)|^2 ds \] (2.3)
defined on functions in $H^1(\Omega_h)$ with trace zero on the inner part of the boundary $\Gamma_{in} := \{x \in \mathbb{R}^2 : |x| = 1 - h^\rho\}$. This quadratic form is related to a self-adjoint operator with mixed boundary conditions. Its lowest eigenvalue $\mu_1(h, b, \rho)$ is given by

$$\mu_1(h, b, \rho) = \inf_{u \in H^1(\Omega_h)} \frac{\mathcal{L}^b_h(u)}{|u(x)|^2}$$  \hspace{1cm} (2.4)$$

where the infimum is taken over all $u$ in the domain of the quadratic form (i.e. $u \in H^1(\Omega_h)$ with $u = 0$ on $\Gamma_{in}$).

**Lemma 2.4.** Assume that $b > 0$ and $\rho \in (0, \frac{1}{2})$. Then there exists $h_0 \in (0, 1)$ such that, for all $h \in (0, h_0)$,

$$\mu_1(h, b) = \mu_1(h, b, \rho) + \mathcal{O}\left(\exp(-h^{\rho-\frac{1}{2}})\right).$$  \hspace{1cm} (2.5)$$

**Proof.** The inequality $\mu_1(h, b) \leq \mu_1(h, b, \rho)$ is not asymptotic. If $\tilde{u}$ is a function minimizing the quotient in (2.4), then we can extend it by zero inside the annulus. Inserting the new function into the quadratic form for $\mathcal{L}^b_h$, we find that $\mu_1(h, b) \leq \mu_1(h, b, \rho)$.

To get a bound in the opposite direction, we cut off (smoothly) the eigenfunction corresponding to $\mu_1(h, b)$, since it does not satisfy the correct boundary condition if $|x| = 1 - h^\rho$. Thanks to Proposition 2.3 (with the choice $\alpha = \frac{1}{2}$) the error introduced is exponentially small. \hfill \Box

In light of (2.5), we finish the proof of Theorem 2.1 once we prove that

$$\mu(h, b, \rho) = -h - h^{3/2} + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2}\right) h^2 + \mathcal{O}(h^2).$$  \hspace{1cm} (2.6)$$

The $m$ in the right-hand side stands for the quantized angular momentum. Our next step is to make a Fourier expansion that will reduce our study to the study of an infinite family (parametrized by $m \in \mathbb{Z}$) of ordinary differential operators.

### 2.4. Reduction to Fiber Operators

We recall that $b > 0$ and $\rho \in (0, \frac{1}{2})$ are considered to be fixed constants. In polar coordinates ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$) the quadratic form $\mathcal{L}^b_h$ reads

$$h^2 \left( \int_0^{2\pi} \int_{1-h^\rho}^{1} \left( \partial_r u \right)^2 + \frac{1}{r^2} \left( |\partial_\theta - i \frac{b}{2} r^2|^2 u \right)^2 \right) r \, dr \, d\theta - h^{-1/2} \int_0^{2\pi} |u|^2 \, d\theta.$$  \hspace{1cm} (2.6)$$

Next, we use the completeness of the orthonormal family $\{e^{im\theta} / \sqrt{2\pi}\}_{m \in \mathbb{Z}}$ in $L^2([0, 2\pi])$, and write

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} u_m(r) \frac{e^{im\theta}}{\sqrt{2\pi}}.$$  \hspace{1cm} (2.6)$$

Here we assume that each $u_m$ belongs to $L^2((1 - h^\rho, 1), r \, dr)$. We are led to study the family of quadratic forms

$$u_m \mapsto h^2 \left( \int_{1-h^\rho}^{1} \left( |u'_m(r)|^2 + \frac{1}{r^2} \left( m - \frac{b}{2} r^2 \right) |u_m(r)|^2 \right) r \, dr - h^{-1/2} |u_m(1)|^2 \right).$$  \hspace{1cm} (2.6)$$
Since we have localization to the outer circle, it is convenient to work with the variable $\tau = h^{-1/2}(1 - r)$, the scaled distance from $|x| = 1$. We write $\tilde{u}_m(\tau) = u_m(\tau)$ and denote by

$$\delta := h^{\rho - \frac{1}{2}},$$

the upper limit of $\tau$. The relevant quadratic forms to study is

$$\tilde{u}_m \mapsto \int_0^\delta \left\{ |\tilde{u}'_m(\tau)|^2 + h(1 - h^{1/2}\tau)^{-2} \left| \left( m - \frac{b}{2}(1 - h^{1/2}\tau)^2 \right) \tilde{u}_m \right|^2 \right\} (1 - h^{1/2}\tau) d\tau - |\tilde{u}_m(0)|^2$$

The differential operator that corresponds to this quadratic form acts as

$$\mathcal{H}^{h,\rho}_{m,h} = - \frac{d^2}{d\tau^2} + \frac{h^{1/2}}{1 - h^{1/2}\tau} \frac{d}{d\tau} + \frac{h}{(1 - h^{1/2}\tau)^2} \left( m - \frac{b}{2}(1 - h^{1/2}\tau)^2 \right)^2. \quad (2.8)$$

With domain

$$D(\mathcal{H}_{\beta,h}) = \{ u \in H^2((0,\delta)) : u'(0) = -u(0) \text{ and } u(\delta) = 0 \}. \quad (2.9)$$

$\mathcal{H}^{h,\rho}_{m,h}$ becomes self-adjoint in the weighted space $L^2((0,\delta),(1 - h^{1/2}\tau) d\tau)$. We denote the smallest eigenvalue of $\mathcal{H}^{h,\rho}_{m,h}$ by $\lambda_1(\mathcal{H}^{h,\rho}_{m,h})$. From the completeness and orthogonality of the family $\{ e^{im\phi} \}_{m \in \mathbb{Z}}$, it follows that

$$\tilde{\mu}(h, b, \rho) = h \inf_{m \in \mathbb{Z}} \lambda_1(\mathcal{H}^{h,\rho}_{m,h}). \quad (2.10)$$

To take advantage of this equality we need information about $\lambda_1(\mathcal{H}^{h,\rho}_{m,h})$. We will get the information needed by comparing with simpler operators. In fact, we will first compare with the weighted Laplace obtained by ignoring the third term in the right-hand side of (2.8). To do this, we first look at the simpler operator obtained by ignoring also the second term.

### 2.5. A 1D Laplacian

The spectrum of the operator $-\frac{d^2}{d\tau^2}$ in $L^2(\mathbb{R}_+)$ with domain $\{ u \in H^2(\mathbb{R}_+) : u'(0) = -u(0) \}$ is explicitly known (see [10]). It consists of the simple eigenvalue $-1$ together with the interval $[0, +\infty)$. A normalized eigenfunction corresponding to the eigenvalue $-1$ is given by

$$u_0(\tau) = \sqrt{2} \exp(-\tau). \quad (2.11)$$

### 2.6. A weighted 1D Laplacian

Let $\rho \in \left( \frac{1}{4}, \frac{1}{2} \right)$ be a fixed constant. In the sequel, the parameter $h \in (0, 1)$ varies so that $h^{\frac{1}{2} - \rho} < \frac{1}{3}$. We recall that $\delta = h^{\rho - \frac{1}{2}}$ and note that $\delta \to +\infty$ when $h \to 0_+$. In the weighted space $L^2((0,\delta),(1 - h^{1/2}\tau) d\tau)$, we introduce the self-adjoint operator,

$$\mathcal{H}_h = - \frac{d^2}{d\tau^2} + \frac{h^{1/2}}{1 - h^{1/2}\tau} \frac{d}{d\tau}, \quad (2.12)$$

with domain

$$D(\mathcal{H}_h) = \{ u \in H^2((0,\delta)) : u'(0) = -u(0) \text{ and } u(\delta) = 0 \}. \quad (2.13)$$
The operator $\mathcal{H}_h$ is defined starting from the closed quadratic form

$$q_h(u) = \int_0^\delta |u'(\tau)|^2(1 - h^{1/2}\tau)\,d\tau - |u(0)|^2.$$ 

The increasing sequence of the eigenvalues of $\mathcal{H}_h$ (counting multiplicities) is denoted by $(\lambda_n(\mathcal{H}_h))_{n\in\mathbb{N}}$. In [10] Lem. 4.4 & Prop. 4.5 it is proved that

$$\lambda_1(\mathcal{H}_h) = -1 - h^{1/2} + o(h^{1/2}) \quad \text{and} \quad \lambda_2(\mathcal{H}_h) \geq \mathcal{O}(h^\rho) \quad (h \to 0_+). \quad (2.14)$$

We are going to refine the expansion of $\lambda_1(\mathcal{H}_h)$ by determining the term of order $h$.

**Lemma 2.5.** Assume that $\rho \in (\frac{1}{4}, \frac{1}{2})$. Then, as $h \to 0_+$,

$$\lambda_1(\mathcal{H}_h) = -1 - h^{1/2} - \frac{1}{2} h + o(h).$$

**Proof.** According to (2.14) there is a spectral gap of constant order, so it suffices to construct a trial state that will give an energy estimate of sufficient accuracy. To do this we expand the operator $\mathcal{H}_h$ formally in $h$ as

$$\mathcal{H}_h = -\frac{d^2}{d\tau^2} + h^{1/2} \frac{d}{d\tau} + \frac{h\tau}{2} \frac{d}{d\tau} + \mathcal{O}(h^{\frac{1}{2} + 2\rho}) \frac{d}{d\tau},$$

and note that, for $\rho \in (\frac{1}{4}, \frac{1}{2})$, $h^{\frac{1}{2} + 2\rho} = o(h)$ as $h \to 0_+$.

We work on the half-line $\mathbb{R}_+$ and construct functions $u_0, u_1, u_2$ and coefficients $\mu_0, \mu_1, \mu_2$ such that

$$(-\frac{d^2}{d\tau^2} - \mu_0)u_0 = 0,$$

$$(-\frac{d^2}{d\tau^2} - \mu_0)u_1 = -u_0' + \mu_1 u_0,$$

$$(-\frac{d^2}{d\tau^2} - \mu_0)u_2 = -u_1' + \mu_1 u_1 - \tau u_0' + \mu_2 u_0,$$

and $u_i'(0) = -u_i(0)$ for $i \in \{0, 1, 2\}$.

The natural choice is then to choose $u_0$ the eigenfunction in [2.11] and $\mu_0$ the corresponding eigenvalue. Then we choose $\mu_1$ so that $-u_0' + \mu_1 u_0$ is orthogonal to $u_0$; after that we can determine $u_1$ since we can invert the operator $-\frac{d^2}{d\tau^2} - \mu_0$ on the orthogonal complement of $u_0$. Finally, we select $\mu_2$ so that $-u_1' + \mu_1 u_1 - \tau u_0' + \mu_2 u_0$ is orthogonal to $u_0$ which allows us eventually to determine $u_2$. In that way we obtain

$$\mu_0 = -1, \quad u_0(\tau) = \sqrt{2} \exp(-\tau),$$

$$\mu_1 = -1, \quad u_1(\tau) = 0,$$

$$\mu_2 = -\frac{1}{2}, \quad u_2(\tau) = \left(-\frac{d^2}{d\tau^2} + 1\right)^{-1} \left[\left(\tau - \frac{1}{2}\right)u_0(\tau)\right] = \left(\frac{\tau^2}{4} - \frac{1}{8}\right)u_0(\tau).$$

Now, consider the function,

$$f(\tau) = \chi\left(\frac{\tau}{\delta}\right) \left(u_0(\tau) + h^{1/2}u_1(\tau) + hu_2(\tau)\right),$$

where $\chi \in C^\infty_c([0, \infty))$ satisfies,

$$0 \leq \chi \leq 1 \text{ in } [0, \infty), \quad \chi = 1 \text{ in } [0, 1/2) \quad \text{and} \quad \chi = 0 \text{ in } [1/2, +\infty).$$
The function \( f \) is in the domain of the operator \( \mathcal{H}_h \), and by construction it is almost normalized in the weighted Hilbert space, having a norm of size \( 1 + O(h) \). Moreover, a straightforward estimate shows that
\[
\| (\mathcal{H}_h - (\mu_0 + \mu_1 h^{1/2} + \mu_2 h)) f \|_{L^2((0,\delta);(1-h^{1/2}) \, dr)} = o(h).
\]
The spectral theorem and (2.14) now completes the proof of Lemma 2.5.

2.7. Reducing the angular momentum. We disqualify some values of the angular momentum \( m \) from minimizing the right-hand side in (2.10).

**Proposition 2.6.** Assume that \( b > 0, \rho \in (\frac{1}{4}, \frac{1}{2}) \) and that \( h \in (0, 1) \). If \( |m| > (1 + \sqrt{2})b/2 \), then
\[
\lambda_1(\mathcal{H}_{m,h}^{b, \rho}) > \inf_{\ell \in \mathbb{Z}} \lambda_1(\mathcal{H}_{\ell,h}^{b, \rho}).
\]

**Proof.** We note that if \( m = 0 \), then we have the bound
\[
\frac{h}{(1-h^{1/2}\tau)^2} \left( m - \frac{b}{2} (1-h^{1/2}\tau)^2 \right)^2 \leq \frac{b^2}{4} h
\]
on the potential term in \( \mathcal{H}_{m,h}^{b, \rho} \). Comparing quadratic forms,
\[
\inf_{\ell \in \mathbb{Z}} \lambda_1(\mathcal{H}_{\ell,h}^{b, \rho}) \leq \lambda_1(\mathcal{H}_{0,h}^{b, \rho}) \leq \lambda_1(\mathcal{H}_h) + \frac{b^2}{4} h,
\]
where \( \lambda_1(\mathcal{H}_h) \) is the lowest eigenvalue of the operator introduced in (2.12).

We expand the square and estimate the potential term again, using the fact that \( 1 - h^\rho < 1 - h^{1/2}\tau < 1 \),
\[
\frac{1}{(1-h^{1/2}\tau)^2} \left( m - \frac{b}{2} (1-h^{1/2}\tau)^2 \right)^2 = \frac{m^2}{(1-h^{1/2}\tau)^2} - m b + \frac{b^2}{4} (1-h^{1/2}\tau)^2
\]
\[
\geq m^2 - m b = \left( m - \frac{b}{2} \right)^2 - \frac{b^2}{4}.
\]
We compare the quadratic forms and invoke (2.15) to find that
\[
\lambda_1(\mathcal{H}_{m,h}^{b, \rho}) \geq \lambda_1(\mathcal{H}_h) + h \left( \left( m - \frac{b}{2} \right)^2 - \frac{b^2}{4} \right)
\]
\[
\geq \inf_{\ell \in \mathbb{Z}} \lambda_1(\mathcal{H}_{\ell,h}^{b, \rho}) + h \left( \left( m - \frac{b}{2} \right)^2 - \frac{b^2}{2} \right).
\]
If \( |m| > (1 + \sqrt{2})b/2 \) then \( (m - b/2)^2 > b^2/2 \) and thus
\[
\lambda_1(\mathcal{H}_{m,h}^{b, \rho}) > \inf_{\ell \in \mathbb{Z}} \lambda_1(\mathcal{H}_{\ell,h}^{b, \rho}). \quad \square
\]

2.8. A family of 1D operators. Assume that \( A > 0, \rho \in (0, \frac{1}{2}) \) and \( b > 0 \) are fixed constants. Suppose that the parameters \( h \in (0, 1) \) and \( m \in \mathbb{Z} \) vary as follows
\[
h^{\frac{1}{2} - \rho} < \frac{1}{3} \quad \text{and} \quad |m| \leq A.
\]
We introduce also the following ground state energy
\[
\hat{\lambda}(b, A) = \inf_{m \in \mathbb{Z}, |m| \leq A} \lambda_1(\mathcal{H}_{m,h}^{b, \rho}). \quad (2.17)
\]
Proposition 2.7. Given $A, b > 0$ and $\rho \in (\frac{1}{4}, \frac{1}{2})$, it holds that
\[
\hat{\lambda}(b, A) = -1 + h^{1/2} + \left(\hat{\beta}(b, A) - \frac{1}{2}\right)h + o(h),
\]
where
\[
\hat{\beta}(b, A) = \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2}\right)^2.
\]

Remark 2.8. Given $b > 0$, there exists $A_0 > 0$ such that, for all $A \geq A_0$,
\[
\hat{\beta}(b, A) = \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2}\right)^2.
\]

Proof. We will write estimates that hold uniformly with respect to $(m, h)$ obeying the conditions in (2.16). A calculation shows that
\[
\frac{1}{(1 - h^{1/2}\tau)^2} \left( m - \frac{b}{2}(1 - h^{1/2}\tau)^2\right)^2 - \left( m - \frac{b}{2}\right)^2
\]
\[
= (2 - h^{1/2}\tau) \left( \frac{1}{(1 - h^{1/2}\tau)^2}m^2 - \frac{b^2}{4}\right)h^{1/2}\tau.
\]
Thus, using that $0 \leq h^{1/2}\tau \leq h^\rho$ for $\tau \in (0, \delta)$, we can write
\[
\left| \frac{1}{(1 - h^{1/2}\tau)^2} \left( m - \frac{b}{2}(1 - h^{1/2}\tau)^2\right)^2 - \left( m - \frac{b}{2}\right)^2 \right| \leq C h^\rho,
\]
where $C$ is a constant independent from $m \in [-A, A]$. Consequently, the min-max principle yields
\[
\left| \lambda_1(H_{m,h}^\rho) - \left( \lambda_1(H_h) + \left( m - \frac{b}{2}\right)^2 h\right) \right| \leq C h^{1+\rho},
\]
where $H_h$ is the operator introduced in (2.12). Now, using Lemma 2.5, we finish the proof of Proposition 2.7.

2.9. End of proof. We now have everything we need to finish the proof of Theorem 1.1. Remember that $b > 0$ was given in the theorem. We start by choosing some $\rho \in (\frac{1}{4}, \frac{1}{2})$. Then, by combining the Propositions 2.6 and 2.7 (see also Remark 2.8) with Lemma 2.5 we find that, as $h \to 0_+$,
\[
\inf_{m \in \mathbb{Z}} \lambda_1(H_{m,h}^\rho) = -1 - h^{1/2} + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2}\right) - \frac{1}{2}\right)h + o(h).
\]
From (2.10) we now conclude that, as $h \to 0_+$,
\[
\hat{\mu}(h, b, \rho) = -h - h^{3/2} + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2}\right) - \frac{1}{2}\right)h + o(h^2).
\]
As we mentioned in the end of Subsection 2.3 this was sufficient to prove Theorem 2.1 which in turn was a reformulation of Theorem 1.1.

References

[1] L. Erdős. Dia- and paramagnetism for nonhomogeneous magnetic fields. J. Math. Phys. 38, no. 3, (1997), pp. 1290–1317.
[2] S. Fournais, B. Helffer. Spectral Methods in Surface Superconductivity. Progress in Nonlinear Differential Equations and Their Applications, Vol. 77, Birkhäuser (2010).
[3] S. Fournais, B. Helffer. On the third critical field in Ginzburg–Landau theory. Comm. Math. Phys. 266, no. 1, (2006), pp. 153–196.
[4] S. Fournais, B. Helffer. Strong diamagnetism for general domains and applications. *Ann. Inst. Fourier* **57**, no. 7, (2007), pp. 2389–2400.

[5] H.K. Fink, W.C.H. Joiner. Surface nucleation and boundary conditions in superconductors. *Phys. Rev. Lett.* **23** (1969), pp. 120–123.

[6] S. Fournais, M. Persson-Sundqvist. Lack of diamagnetism and the Little–Parks effect. *Comm. Math. Phys.* **337**, no. 1, (2015), pp. 191–224.

[7] T. Giorgi, R. Smits. Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity. *Z. Angew. Math. Phys.* **57** (2006), pp. 1-22.

[8] B. Helffer. Spectral theory and its applications. Cambridge Studies in Advanced Mathematics, Vol. 139, 2013.

[9] B. Helffer, A. Kachmar. Thin domain limit and counterexamples to strong diamagnetism. arXiv:1905.06152.

[10] B. Helffer, A. Kachmar. Eigenvalues for the Robin Laplacian in domains with variable curvature. *Trans. Amer. Math. Soc.* **369**, no. 5 (2017), pp. 3283–3287.

[11] M. Khalile, T. Ourmières-Bonafos, K. Pankrashkin. Effective operators for Robin eigenvalues in domains with corners [arXiv:1809.04998].

[12] A. Kachmar. Diamagnetism versus Robin condition and concentration of ground states. *Asymptot. Anal.*, **98**, no. 4, (2016), pp. 341–375.

[13] A. Kachmar. Magnetic vortices for a Ginzburg-Landau type energy with discontinuous constrain. *ESAIM: COCV* **16** (2010), pp. 545–580.

[14] E. Monteverchi, J.O. Indekeu. Effects of confinement and surface enhancement on superconductivity. *Phys. Rev. B.* **62** (2000), pp. 14359–14372.

[15] K. Pankrashkin. On the asymptotics of the principal eigenvalue problem for a Robin problem with a large parameter in a planar domain. *Nanosystems: Physics, Chemistry, Mathematics, Mathematics.*, **4** no. 4, (2013), pp. 474–483.

[16] K. Pankrashkin, N. Popoff. Mean curvature bounds and eigenvalues of Robin Laplacians. *Calc. Var. Partial Dier. Equ.* **54**, no. 2, (2015), pp. 1947–1961.

[17] K. Pankrashkin, N. Popoff. An effective Hamiltonian for the eigenvalues asymptotics of a Robin Laplacian with a large parameter. *J. Math. Pures et Appl.*, **106** no. 4, (2016), pp. 615–650.

(Ayman Kachmar) Lebanese University, Department of Mathematics, Nabateh, Lebanon.  
*E-mail address: ayman.kashmar@gmail.com*

(Mikael Persson Sundqvist) Lund University, Department of Mathematical Sciences, Box 118, 221 00 Lund, Sweden.  
*E-mail address: mikael.persson_sundqvist@math.lth.se*