Stabilization rates for the damped wave equation with Hölder-regular damping

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Abstract

We study the decay rate of the energy of solutions to the damped wave equation in a setup where the geometric control condition is violated. We consider damping coefficients which are 0 on a strip and vanish like polynomials, \( x^\beta \). We prove that the semigroup cannot be stable at rate faster than \( 1/t^{(\beta + 2)/(\beta + 3)} \) by producing quasimodes of the associated stationary damped wave equation. We also prove that the semigroup is stable at rate at least as fast as \( 1/t^{(\beta + 2)/(\beta + 4)} \). These two results establish an explicit relation between the rate of vanishing of the damping and rate of decay of solutions. Our result generalizes one of Nonnemacher in which the damping is an indicator function on a strip.

1 Introduction

Let \( M = (M,g) \) be a Riemannian manifold. Fix some \( W \in L^\infty(M), W \geq 0 \). We study the asymptotic behavior as \( t \to \infty \) of solutions to the damped wave equation

\[
\begin{cases}
\partial_t^2 u - \Delta u + W(x) \partial_t u = 0 & \text{in } \mathbb{R}^+ \times M \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M \\
u|_{\partial M} = 0 & \text{if } \partial M \neq \emptyset.
\end{cases}
\]

The quantity of particular interest is the energy

\[
E(u, t) = \frac{1}{2} \left( \|\nabla u(\cdot, t)\|_{L^2}^2 + \|\partial_t u(\cdot, t)\|_{L^2}^2 \right).
\]

In this paper we will work on the square \( M = [-b, b] \times [-b, b] \) or torus \( M = T^2 \), which we parametrize by \((x, y)\). We will discuss the square in detail and then show how those results extend to the torus.

For some fixed \( \beta \geq 0 \) and \( a, \sigma > 0 \), such that \( a + \sigma < b \), we study damping \( W \in L^\infty(M) \) of the form

\[
W(x, y) = \begin{cases}
0 & |x| < a \\
(|x| - a)^\beta & a < |x| < a + \sigma \\
c(|x|) & a + \sigma < |x| < b.
\end{cases}
\]
In particular note that the damping is invariant in the $y$ direction.

Remark. The particular form of $c(x)$ does not affect our result so long as $c(x) \in L^\infty(a + \sigma, b)$. However choosing a $c$ such that $W$ is smooth for $|x| > a$ and $W = C$ for $|x| > a + \sigma$ is perhaps the most interesting case in the context of existing results.

Definition 1. Let $f(t)$ be a function such that $f(t) \to 0$ as $t \to \infty$. We say that (1) is stable at rate $f(t)$ if there exists a constant $C > 0$ such that for all $(u_0, u_1) \in H^2(M) \times H^1(M)$ if $u$ solves (1) with $(u_0, u_1)$ as Cauchy data then

$$E(u, t) \leq C f(t)^2 \left( ||u_0||^2_{H^2(M)} + ||u_1||^2_{H^1(M)} \right)$$

for all $t > 0$.

Our main result is

Theorem 1.1. For all $\varepsilon > 0$, with $W$ as in (3) the equation (1) cannot be stable at rate

$$t^{-\frac{2+\varepsilon}{5+\varepsilon}}.$$

More precisely in this situation Lemma 4.6 of [AL+14] and Proposition 3 of [BD08] show that $m_1(t) \geq C/(1 + t)^{\frac{2+\varepsilon}{5+\varepsilon}}$ for some $C > 0$, where we use the notation of [BD08] where $m_1(t)$ denotes the best decay rate.

We also show that

Theorem 1.2. For $W$ as in (3) with $\beta > 0$ the equation (1) is stable at rate

$$t^{-\frac{2}{5+\varepsilon}}.$$

The decay of energy of the damped wave equation is a well studied question. The strongest possible decay is uniform stabilization, which is defined as the existence of $F(t) \to 0$ as $t \to \infty$ such that, for all $u$ solving (1) with initial data in $H^1(M) \times L^2(M)$

$$E(u, t) \leq F(t) E(u, 0), \quad t \geq 0.$$

It was established by [RT75] that uniform stabilization occurs with $F = C e^{-\kappa t}$ for some $\kappa, C > 0$, when $\partial M = \emptyset$, $W \in C^0(M)$ and $\{W > 0\}$ satisfies the geometric control condition (GCC). We recall that a set $U$ satisfies the GCC if there exists $T > 0$ such that for every geodesic $\gamma$ on $M$ of length $T$, $\gamma \cap U \neq \emptyset$. This result was extended to the case $M \neq \emptyset$ by [BLR92]. The reverse implication, that uniform stabilization with any $F$ implies $\{W > 0\}$ satisfies the GCC, was shown in [BG97]. This in turn guarantees that when uniform stabilization occurs one can always let $F = C e^{-\kappa t}$ for some $\kappa, C > 0$. For a finer discussion of when uniform stabilization occurs for $L^\infty$ damping see [BG18].

A natural next question to ask is what occurs when the GCC does not hold for $\{W > 0\}$. Because of the necessity of the GCC for uniform stabilization, as soon as it does not hold we must adjust the kind of decay we hope for. The next best thing is the stability defined in
Definition 1, which comes from [Leb96] and requires initial data with an additional spatial derivative.

In [Leb96], the author showed that the energy of a solution to (1) decays at least logarithmically \((f(t) = 1/\log(2 + t))\) as soon as the damping \(W(x) \geq c > 0\) on some open, nonempty set. Moreover the author gave explicit examples of domains on which this is the exact decay rate, in particular when \(M = S^2\) and \(\{W > 0\}\) does not intersect a neighborhood of the equator. For related work see also [Bur98] and [LR97].

In the case of the square when the damped region contains a vertical strip, [LR05] established a decay rate of \(f(t) = (\log(t)/t)^{1/2}\). This was expanded to the case of partially rectangular domains when \(\{W > 0\}\) contains a neighborhood of the nonrectangular part in [BH07]. Additionally in [BH07] a relation between vanishing rate of the damping and decay rate for the damped wave equation was established.

These results were refined by [AL⁺14]. The authors established a decay rate of \(f(t) = 1/t^{1/2}\) for the damped wave equation in a more general setting, so long as the associated Schrödinger equation is controllable. This includes the case of not identically vanishing damping on the 2 dimensional square (or torus) as a consequence of [Jaf90] (resp. [Mac10], [BZ12]).

Continuing in the case of the 2 dimensional square [AL⁺14] also show that the system can not be stable at rate \(f(t) = 1/t^{1+\varepsilon}\) for any \(\varepsilon > 0\), when \(\text{supp}(W)\) does not satisfy GCC, (this condition is referred to as the GCC being strongly violated). They further show the existence of a smooth damping coefficient which strongly violates the GCC for which the energy decays at rate \(f(t) = 1/t^{1-\varepsilon}\) for any \(\varepsilon > 0\).

In an appendix to [AL⁺14], Nonnenmacher shows that when the damping is the indicator function on a strip on the square or torus the system cannot be stable at rate \(f(t) = 1/t^{1+\varepsilon}\) for any \(\varepsilon > 0\). [Sta17] showed the complementary result to establish \(f(t) = 1/t^{2/3}\) as the exact rate of decay when damping is a strip on the square or torus. The difference in behavior between the smooth and discontinuous damping led the authors of [AL⁺14] to pose the problem of establishing an explicit relation between the vanishing rate of the damping and the decay rate.

An explicit relation was established by [LL17] in a slightly different setting. The authors study the case of a general manifold in which the damping is supported everywhere but a flat subtorus. In the 2 dimensional case this is an example of the GCC not holding but also not being strongly violated. The damping is required to be invariant along this subtorus and must satisfy \(W(x) \leq C|x|^\beta\) near where it vanishes. When this is the case the authors show that the system cannot be stable at rate \(f(t) = 1/t^{\alpha+\varepsilon}\), for any \(\varepsilon > 0\). They also show that if the vanishing behavior of the damping is further limited to \(C_1^{-1}|x|^\beta \leq W(x) \leq C_1|x|^\beta\) the system is stable at exactly the rate \(f(t) = 1/t^\beta\) (See also [BZ15]).

Note that in [LL17] decreasing \(\beta\) corresponds to faster vanishing (i.e. less regular damping) which produces faster decay, which is counter to the behavior exhibited in [AL⁺14], [BH07] and our own result, namely that faster vanishing (i.e. less regular damping) pro-
duces slower decay. However the dynamics in 2 dimensions in the two situations are different, with only one undamped orbit in the former as opposed to a whole family in the latter.

Our paper provides a partial answer to the problem posed by [AL+14]. We establish an explicit relation between the rate of vanishing of the damping and the stability rate of the system, in a case where the GCC is strongly violated on the square or $T^2$. Our work also extends that of Nonnenmacher in the appendix to [AL+14], which agrees with our first theorem when $\beta = 0$. Our two results provide further evidence for the fact, discussed in [AL+14], [LL17], [BH07], that once the support of the damping is fixed the rate of vanishing of the damping is the most significant feature when determining the decay rate.

We note that our second result improves that of Theorem 1.2 of [BH07], which gives a decay rate of $f(t) = \frac{1}{t^{\beta/(\beta+4)}}$ (see also [AL+14] Theorem 2.6). We also note that there is a gap between our two results. Closing this gap would be an interesting area for further work.

In the next section we outline the proof of Theorem 1.1. Sections 3, 4 and 5 contain the details of the proof. Section 6 contains the proof of Theorem 1.2.

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2 Outline of Proof

To prove Theorem 1.1 we rely on the following result from [AL+14] (Proposition 2.4) and [BT10] which relates energy decay to resolvent estimates of the stationary damped equation.

**Proposition 2.1.** Fix $\alpha$, if there exist sequences $\{q_j\} \in \mathbb{C}, \{u_j\} \in H^2(M) \cap H^1_0(M)$, (or $H^2(M)$ if $\partial M = \emptyset$) with

$$\|\Delta u_j + iq_jW(x)u_j - q_j^2u_j\|_{L^2(M)}^2 \leq \frac{C}{|\text{Re}(q_j)|^{2/\alpha}} \left( \|u_j\|^2_{H^1(M)} + |q_j|^2 \|u_j\|^2_{L^2(M)} \right)$$

as $j \to \infty$, \hspace{1cm} (4)

and

$$|q_j| \to \infty, \quad |\text{Im}(q_j)| \leq \frac{C}{|\text{Re}(q_j)|^{1/\alpha}}$$

as $j \to \infty$, \hspace{1cm} (5)

then for all $\varepsilon > 0$ the system (1) is not stable at rate $1/t^{\alpha+\varepsilon}$.

**Remark.** Although Proposition 2.1 has $\|u_j\|^2_{H^1}$ on the right hand side of (4) the quasimodes $u_j$ we produce will satisfy a sharper estimate with a better power of $\text{Re}(q_j)$,
We will eventually specify \( h \)ters. Let \( q \)satisfy (8) with \( \beta \) be a bounded parameter with either \( l \). Namely

\[
\text{Re}(q) \quad \text{and} \quad \text{Im}(q)
\]

be the relative size of \( \text{Re}(q) \) and \( \text{Im}(q) \) of the \( q_j \) for which we have this estimate.

Note that producing sequences \( q_j \) and \( u_j \) which satisfy the hypotheses of this proposition with \( \alpha = \frac{4 + 2}{\pi^2 \pi^2} \) proves Theorem 1.1

We will make two simplifications before proceeding. First we will reduce the problem to obtaining quasimodes of an ordinary differential equation on \([0, b] \). We will then further restrict our attention to the same equation on \([0, b] \). After making these simplifications we will introduce three key parameters and the complex absorbing potential problem on \((0, \infty) \), solutions of which we will use to produce our desired quasimodes.

For the first simplification note that for any sequence of integers \( m_j \) if \( \tilde{u}_j \) is a sequence of functions on \([-b, b] \) which satisfy

\[
\begin{align*}
\left\| -\Delta u_j + iq_j W(x)u_j - q_j^2 u_j \right\|_{L^2(M)}^2 &\leq \frac{C}{\text{Re}(q_j)^2} \| u_j \|_{L^2(M)}^2.
\end{align*}
\]

(6)

We do not obtain a stronger result using Proposition 2.1 because of the relative size of \( \text{Re}(q_j) \) and \( \text{Im}(q_j) \) of the \( q_j \) for which we have this estimate.

The second simplification we make is to limit our attention to \([0, b] \) from \([-b, b] \). Since our damping is even, if we find integers \( m_j \) and functions \( \tilde{u}_j \) on \([0, b] \) which satisfy

\[
\begin{align*}
\left\| -\partial_x^2 \tilde{u}_j + iq_j W \tilde{u}_j + \left( \frac{4x^2m_j^2}{b^2} - q_j^2 \right) \tilde{u}_j \right\|_{L^2(-b, b)}^2 &\leq \frac{C}{\text{Re}(q_j)^2} \| \tilde{u}_j \|_{L^2(-b, b)}^2 \quad \text{as} \quad j \to \infty.
\end{align*}
\]

(7)

then \( u_j(x, y) = \tilde{u}_j(x) \sin \left( \frac{2\pi m_j b}{b} \right) \) satisfy (6). Therefore it is enough for us to find functions which satisfy (7) with \( q_j \) which satisfy (5).

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\[
\begin{align*}
\left\| -\partial_x^2 \tilde{u}_j + iq_j W \tilde{u}_j + \left( \frac{4x^2m_j^2}{b^2} - q_j^2 \right) \tilde{u}_j \right\|_{L^2(0, b)}^2 &\leq \frac{C}{\text{Re}(q_j)^2} \| \tilde{u}_j \|_{L^2(0, b)}^2 \quad \text{as} \quad j \to \infty.
\end{align*}
\]

(8)

we can extend the \( \tilde{u}_j \) to \(-b \leq x < 0 \) by setting \( \tilde{u}_j(-x) = -\tilde{u}_j(x) \) (or \( \tilde{u}_j(-x) = \tilde{u}_j(x) \) resp.) and the resulting functions satisfy (7). Therefore it is enough for us to find functions which satisfy (8) with \( q_j \) which satisfy (5).

Before we introduce the complex absorbing potential we introduce three new parameters. Let \( h \in (0, 1) \) be a small parameter which will be sent to 0 and have \( \frac{b^2}{4\pi^2 m_j^2} \in \mathbb{N} \). Let \( l \) be a bounded parameter with either \( l \in \mathbb{Z} \) or \( l + \frac{1}{2} \in \mathbb{Z} \) and otherwise free in relation to \( h \). We define

\[
\lambda = \frac{\pi lh}{a} + C_h h^{(\beta+4)/(\beta+2)} \quad C_h = O_h(1) \in \mathbb{C}.
\]

We will eventually specify \( C_h \) more completely in Section 4.
Now we introduce the complex absorbing potential problem on \((0, \infty)\)
\[
\begin{align*}
0 &= -\hbar^2 \partial_x^2 v + i(x-a)^\beta v - \lambda^2 v \\
v(0) &= 0 \text{ or } v'(0) = 0.
\end{align*}
\]
(10)

In order to relate this to (8) we make an ansatz for relations between the parameters. If \(v_j\) are a sequence of solutions of (10) for some \(h_j, l_j, \lambda_j\), we define \(q_j\) and \(m_j\) as follows
\[
m_j = \frac{b}{2\pi h_j^2} \in \mathbb{N}
\]
(11)
\[
q_j = \frac{1}{h_j^2} + \frac{\lambda_j^2}{2} = \frac{1}{h_j^2} + \frac{\pi^2 l_j^2 h_j^2}{a^2} + \frac{2C_j \pi l_j h_j^{(2\beta+6)/(\beta+2)} + C_j^2 h_j^{(2\beta+8)/(\beta+2)}}{h_j^2}.
\]

Note that in this regime
\[
\text{Re}(q_j) = \frac{1}{h_j^2} + O(h_j^2) \quad \text{Im}(q_j) = \frac{2\text{Im}(C_j) \pi l_j h_j^{(2\beta+6)/(\beta+2)} + O(h_j^{(2\beta+8)/(\beta+2)})}{\text{Re}(q_j)^{(\beta+3)/(\beta+2)}}
\]
as \(j \to \infty\). As we will see shortly, solutions of (10) in this regime satisfy the inequality in (8) but not necessarily the boundary condition at \(x = b\). In order to ensure they do we multiply these solutions by a cutoff function which is 0 in a neighborhood of \(b\). We will see the resulting functions still satisfy the inequality in (8) as the solutions of (10) in this regime have rapid decay on the support of the potential (see Lemma 3.1), which is exactly where errors introduced by the cutoff function appear.

Fix \(\delta > 0\) such that \(a + \sigma < b - 2\delta\); we define \(\phi \in C^\infty(0, \infty)\) to satisfy
\[
\phi(x) = \begin{cases} 
1 & x < b - 2\delta \\
0 & b - \delta < x.
\end{cases}
\]
(12)

Proposition 2.2. Fix \(M > 0\), let \(\{v_j\} \in H^2(0, \infty)\) be a sequence of solutions of (10) with eigenvalues
\[
\lambda_j = \frac{\pi l_j h_j}{a} + C_j h_j^{(\beta+4)/(\beta+2)}, \quad C_j = O(1) \in \mathbb{C},
\]
where \(|l_j| \leq M\) and \(h_j \to 0\) as \(j \to \infty\) and \(1/h_j^2 \in \mathbb{Z}\). Set
\[
u_j(x) = \phi(x)v_j(x).
\]
Then for \(j\) large enough so that \(h_j < \sigma^{\beta/2}\) the functions \(\nu_j\) with \(q_j, m_j\) as defined in (11) satisfy (8) and (5).
Remark. We only need to have $v_j$ solve (10) on $(0,b)$ but because of the rescaling we use to produce these solutions it is more convenient to work on $(0,\infty)$.

It remains to be seen that we can indeed find solutions to the complex absorbing potential problem with eigenvalues of this form.

**Theorem 2.3.** Fix $l \in \mathbb{Z}$, (or $l + \frac{1}{2} \in \mathbb{Z}$), there exists $h_0 > 0$ such that for all $h \in (0,h_0)$, there exists $v \in H^2(0,\infty) \cap H^1_0(0,\infty)$ (resp. $H^2(0,\infty)$) a solution of (10) with $v(0) = 0$ (resp. $v'(0) = 0$) with $\lambda$ as in (9).

Using Theorem 2.3 we obtain a sequence $\{v_j\}$ of solutions of (10) which satisfy the hypotheses of Proposition 2.2 which in turn produces sequences which satisfy the hypotheses of Propositions 2.1 which in turn proves Theorem 1.1.

Remark. It is straightforward to extend these results to the case $M = \mathbb{T}^2$. We parametrize $\mathbb{T}^2$ by $[-b,b] \times [-b,b]$ with parallel edges identified. Thus it is enough to show that the quasimodes we produced on the square satisfy periodic boundary conditions and are thus functions on the torus. Our quasimodes are of the form

$$u(x,y) = v_j(x)\phi(x)\sin\left(\frac{2\pi m_j y}{b}\right),$$

so it is straightforward to see they satisfy periodic boundary conditions in $y$ and $x$ (as $u(x,y) \equiv 0$ for $|x - b| < \delta$ and $|b + x| < \delta$).

We will prove Proposition 2.2 in Section 3, we will then prove Theorem 2.3 in Section 4. We finally prove a necessary estimate in Section 5.

## 3 Proof of Proposition 2.2

We begin by stating an estimate necessary for the proof.

**Lemma 3.1.** Let $v \in H^1(0,\infty)$ be a solution of (10) with eigenvalue $\lambda = O(h)$ and let $\phi$ be as in (12). Fix $s \in \mathbb{R}$ then for $h < \beta/2$ for all $N$ there exists $C_{N,s} > 0$ such that

$$||\phi v||^2_{H^s(a+\sigma,b)} \leq C_{N,s} h^N ||\phi v||^2_{L^2(0,b)}. \quad (13)$$

This will be proved in Section 5 using the semiclassical ellipticity of $-h^2 \frac{\partial^2}{\partial x^2} + i(x-a)\beta^+ - \lambda^2$ on $(a + \sigma/4, b)$.

**Proof of Proposition 2.2.** We have a sequence $v_j$ of solutions of

$$0 = -h_j^2 \frac{\partial^2}{\partial x^2} v_j + i(x-a)\beta^+ v_j - \lambda_j^2 v_j, \quad x \in (0,\infty),$$

with

$$\lambda_j = \frac{\pi l h_j}{a} + O\left(h_j^{(\beta+4)/2}\right), \quad h_j \to 0 \text{ as } j \to \infty.$$
It is clear that \( u_j = \phi v_j \) has \( u_j(b) = \phi(b)v_j(b) = 0 \). Recalling (11) and the subsequent discussion \( q_j, m_j \) satisfy (5). It remains to be seen that \( u_j \) satisfies the inequality in (8). By (11) and (10) \( u_j \) satisfies
\[
-\partial_x^2 u_j + iq_j W(x) u_j + \left( \frac{4\pi^2 m_j^2}{a^2} - q_j^2 \right) u_j
= \phi \left( \frac{\lambda^2}{2} (x-a)^4 v_j - \frac{\lambda^4}{4} v_j \right) - \phi'' v_j - 2\phi' v'_j + iq_j \left( W(x) - (x-a)^2 \right) \phi v_j.
\]
Thus
\[
\left\| -\partial_x^2 u_j + iq_j W(x) u_j + \left( \frac{4\pi^2 m_j^2}{a^2} - q_j^2 \right) u_j \right\|^2_{L^2(0,b)} \leq \frac{\lambda^2}{4} \left\| (x-a)^4 u_j \right\|^2_{L^2(0,b)} + \frac{\lambda^4}{16} \left\| \phi'' v_j \right\|^2_{L^2(0,b)} + 4 \left\| \phi' v'_j \right\|^2_{L^2(0,b)} + |q_j|^2 \left\| (W(x) - (x-a)^2) \phi v_j \right\|^2_{L^2(0,b)}.
\]
Since \( b - \delta > a + \sigma \), by Lemma 3.1 for any \( N > 0 \)
\[
\left\| \phi'' v_j \right\|^2_{L^2(0,b)} + \left\| \phi' v'_j \right\|^2_{L^2(0,b)} \leq \frac{C}{h^2} \left\| \phi v_j \right\|^2_{H^2(0,b)} \leq C_N h^N \left\| \phi v_j \right\|^2_{L^2(0,b)}.
\]
Furthermore by Lemma 3.1 for any \( N > 0 \)
\[
\left\| (W - (x-a)^2) \phi v_j \right\|^2_{L^2(0,b)} \leq C \left\| \phi v_j \right\|^2_{L^2(0,b)} d\sigma \leq C_N h^N \left\| \phi v_j \right\|^2_{L^2(0,b)}.
\]
Therefore
\[
\left\| -\partial_x^2 u_j + iq_j (x-a)^4 u_j + \left( \frac{4\pi^2 m_j^2}{a^2} - q_j^2 \right) u_j \right\|^2_{L^2(0,b)} \leq \frac{\lambda^2}{4} \left\| u_j \right\|^2_{L^2(0,b)} + \frac{\lambda^4}{16} \left\| u_j \right\|^2_{L^2(0,b)} + C_N h^N \left\| u_j \right\|^2_{L^2(0,b)}
\]
\[
\leq \frac{C}{|\text{Re}(q_j)|^2} \left\| u_j \right\|^2_{L^2(0,b)}.
\]
Where we used that \( |\text{Re}(q_j)| = 1/h^2 + O(h^2) \) and \( \lambda_j = O(h) \).
\[\square\]

We now show that there are solutions of (10) with the desired eigenvalues.

4 Proof of Theorem 2.3

From this point on we focus on solutions of (10) on \( (0, \infty) \).
In order to produce solutions to (10) with the desired eigenvalues we will solve it on \([0,a)\) and \((a,\infty)\) separately. That is given solutions \(v_l, v_r \in H^2\) of (10) on \([0,a)\) and \((a,\infty)\) respectively, with the same values of \(\lambda\) and \(h\), if there exists \(B \in \mathbb{C}\) such that
\[
\begin{aligned}
v_l(a) &= B v_r(a) \\
v'_l(a) &= B v'_r(a),
\end{aligned}
\]
then
\[
v(x) = \begin{cases} v_l(x) & x < a \\ B v_r(x) & a < x, \end{cases}
\]
solves (10) on \([0,\infty)\) with the same \(\lambda\) and \(h\) and \(v \in H^2(0, \infty) \cap H^1_0(0, \infty)\) (or \(H^2(0, \infty)\) if \(v'_l(0) = 0\)). We will refer to equations (14) as the compatibility condition.

We will explicitly solve (10) on \([0,a)\). We will then use a rescaling of the equation on \((a,\infty)\) and the implicit function theorem to show that the compatibility condition can be satisfied when \(\lambda\) is of the form (9) and \(h\) is small enough.

On \([0,a)\) (10) is solved by
\[
v_l(x) = e^{i\lambda(x-a)/h} + \text{Ref}(\lambda)e^{-i\lambda(x-a)/h},
\]
where we choose \(\text{Ref}(\lambda)\) to ensure the boundary condition at 0 is satisfied. That is
\[
\text{Ref}(\lambda) = \begin{cases} -e^{-2i\lambda\alpha/h} & v(0) = 0, \\ e^{-2i\lambda\alpha/h} & v'(0) = 0. \end{cases}
\]
We will work through the proof with \(v(0) = 0\) in detail and then summarize how it changes for \(v'(0) = 0\).

We now rescale the equation on \((a,\infty)\). If \(F\) solves
\[
\begin{aligned}
0 &= -F''(x) + \left(ix^\beta - \frac{\lambda^2}{k^2/\beta + 2}\right) F(x) \\ F'(0) &= 1,
\end{aligned}
\]
then \(v_r(x) = F(h^{-2/(\beta+2)}(x-a))\) solves (10) on \((a,\infty)\) with \(v'_r(a) = h^{-2/(\beta+2)}\). This follows immediately from the definition of \(F\) and (10).

**Remark.** This rescaling is necessary; in order to show the compatibility condition can be satisfied for all \(h\) in a neighborhood of 0 we will apply the implicit function theorem and so must be able to set \(h = 0\). This can not be done in a satisfactory way with solutions of (10); however for \(\lambda\) of the form (9), (15) is well defined at \(h = 0\) as
\[
\left.\frac{\lambda^2}{h^{2\beta/(\beta+2)}}\right|_{h=0} = \left.\frac{1}{h^{2\beta/(\beta+2)}}Ch^2 + O(h^{(2\beta+6)/(\beta+2)})\right|_{h=0} = 0.
\]
Before proceeding we mention that we will establish the existence and uniqueness of $H^2(0, \infty)$ solutions to (15) in Lemma 4.1 and we will show $F(0, \mu, h)$ is uniformly bounded away from 0 in Lemma 4.2.

Now we introduce $\mu \in \mathbb{C}$, which we will solve for implicitly in terms of $h$. We use it to clarify the dependence of $\lambda$ on $h$. That is we refine our definition of $\lambda$ to

$$\lambda = \frac{\pi l h}{a} + A_1 h^{(\beta+4)/(\beta+2)} + \mu h^{(\beta+4)/(\beta+2)},$$

where $l \in \mathbb{Z},$

$$A_1 = \frac{\pi l F_0}{a^2},$$

and $F_0$ is the Dirichlet data of the $H^2$ solution of (15) when $\mu = h = 0$. Note that this expression for $\lambda$ agrees with our definition of $\lambda$ in [9] with $C_h = A_1 + \mu$.

Now take $v_r(x) = F(h^{-2/(\beta+2)}(x - a))$, where $F$ is the $H^2$ solution of (15) with the above $\lambda$. Recalling the explicit form of $v_r(x)$, the compatibility condition becomes

$$\frac{i \lambda}{\mu}(1 - \text{Ref}(\lambda)) = Bh^{-2/(\beta+2)}$$

$$1 + \text{Ref}(\lambda) = BF(0, \mu, h),$$

where $F(0, \mu, h)$ denotes the Dirichlet data of $F$ and is written to emphasize its dependence on $\mu$ and $h$. Divide the top equation by the bottom

$$\left(\frac{\pi i a}{1 + A_1 i a h^{2/(\beta+2)} + i \mu h^{2/(\beta+2)}}\right) \left(1 + \exp(-2\pi i l - 2A_1 i a h^{2/(\beta+2)} - 2ia \mu h^{2/(\beta+2)})\right) F(0, \mu, h)$$

$$= h^{-2/(\beta+2)} \left(1 - \exp(-2\pi i l - 2A_1 i a h^{2/(\beta+2)} - 2ia \mu h^{2/(\beta+2)})\right).$$

Now Taylor expand the exponentials for small $h$

$$\left(\frac{\pi i a}{1 + A_1 i a h^{2/(\beta+2)} + i \mu h^{2/(\beta+2)}}\right) \left(2 - 2A_1 i a h^{2/(\beta+2)} - 2ia \mu h^{2/(\beta+2)} + g(h)\right) F(0, \mu, h)$$

$$= h^{-2/(\beta+2)} \left(2A_1 i a h^{2/(\beta+2)} + 2ia \mu h^{2/(\beta+2)} - g(h)\right),$$

where $g(h)$ is the remainder term from the Taylor expansion with $g(h) = O(h^{4/(\beta+2)})$.

So in order to prove Theorem 2.3 it is enough to establish that for all $h$ near 0 $\in [0, \infty)$ there exists $\mu_h \in \mathbb{C}$ such that the following function has a zero at $(h, \mu_h)$;

$$G(\mu, h) = \left(\frac{\pi i a}{1 + A_1 i a h^{2/(\beta+2)} + i \mu h^{2/(\beta+2)}}\right) \left(2 - 2A_1 i a h^{2/(\beta+2)} - 2ia \mu h^{2/(\beta+2)} + g(h)\right) F(0, \mu, h)$$

$$- 2A_1 i a - 2ia \mu + g(h) h^{-2/(\beta+2)}.$$

To do so we apply the implicit function theorem with weak regularity hypotheses to solve for $\mu = \mu(h)$. We recall the implicit function theorem as stated in Theorem 9.3 of [LS09] can be applied if there exists some $h_0$ such that
1. \(G(0,0) = 0\)
2. \(G\) is continuous on \([0,1] \times [0,h_0]\)
3. \(D_\mu G\) exists and is continuous on \([0,1] \times [0,h_0]\)
4. \(D_\mu G(0,0)\) is invertible.

We see immediately that \(G(0,0) = 0\).

In order to show that the other hypotheses are satisfied we must first show some properties of solutions to the rescaled equation (15). For convenience we will write the spectral parameter as \(\eta = \frac{\lambda^2}{h^{2\beta/(\beta+2)}}\). We will show that when \(\eta\) is small enough there exists a unique \(H^2\) solution of (15) for which \(F(0)\) is holomorphic in \(\eta\). We will then use this holomorphy to show \(F(0)\) is bounded away from 0 and \(\infty\) on the same neighborhood. We will then use these results to show that the hypotheses of the implicit function theorem are satisfied.

Let \(\sigma_N(-\partial_x^2 + x^{\beta})\) and \(\sigma_D(-\partial_x^2 + x^{\beta})\) be the spectrum of \(-\partial_x^2 + x^{\beta}\) on \((0,\infty)\) with Neumann and Dirichlet boundary conditions respectively, and let

\[
\tilde{\lambda}_1 = \min \left( \inf \sigma_N(-\partial_x^2 + x^{\beta}), \inf \sigma_D(-\partial_x^2 + x^{\beta}) \right).
\]

Since both spectra are discrete we know that \(\tilde{\lambda}_1\) is the minimum of the lowest eigenvalues of the two spectra. We also know that neither spectrum contains 0, so \(\tilde{\lambda}_1 > 0\).

By the variational principle for the spectrum of self adjoint operators

\[
\tilde{\lambda}_1 \left\| u \right\|_{L^2(0,\infty)}^2 \leq \left\| u' \right\|_{L^2(0,\infty)}^2 + \left\| x^{\beta/2} u \right\|_{L^2(0,\infty)}^2,
\]

for all \(u \in H^1(0,\infty)\) (as \(H_0^1(0,\infty) \subset H^1(0,\infty)\)).

**Lemma 4.1.** For \(|\eta| < \tilde{\lambda}_1\), there exists a unique \(H^2(0,\infty)\) solution of

\[
\begin{aligned}
0 &= -F''(x) + ix^{\beta} F(x) - \eta F(x) \\
F'(0) &= 1.
\end{aligned}
\]

Furthermore the value of this function at \(x = 0, F(0,\eta)\) is holomorphic in \(\eta\).

**Proof.** First let \(\psi \in C_0^\infty(0,\infty)\) with \(\psi'(0) = 1, \psi(0) = 0\). Define \(Q(\eta, \psi)\) as

\[
Q(\eta, \psi) := \psi'' - ix^{\beta} \psi + \eta \psi.
\]
Now let $J$ solve
\begin{align}
-J'' + ix^\beta J - \eta J &= Q \\tag{18} \\
J'(0) &= 0,
\end{align}
and note that $F = \psi + J$ solves (17). We will apply the Lax-Milgram theorem to show the existence of solutions to (18).

Let
\[ H = H^1(0, \infty) \cap x^{-\beta/2}L^2(0, \infty), \]
and define the norm
\[ ||u||^2_H = ||u||^2_{H^1(0, \infty)} + \left| \left| \left| x^{\beta/2}u \right| \right| \right|_{L^2(0, \infty)}^2, \]
noting that $H$ is a Hilbert space with this norm.

We define the sesquilinear form $B : H \times H \rightarrow \mathbb{C}$
\[ B[u, v] = \int_0^\infty u'\bar{v}' + ix^\beta u\bar{v} - \eta u\bar{v}. \]
For any $u, v \in H$
\[ |B[u, v]| \leq \int |u'||v'| + x^\beta |u||v| + |\eta||u||v| \leq C ||u||_H ||v||_H. \]

Furthermore for $u \in H \subset H^1$
\[ |B[u, u]| \geq \int |u'|^2 + x^\beta |u|^2 dx \geq \left( 1 - \frac{|\eta|}{\tilde{\lambda}_1} \right) \int |u'|^2 + x^\beta |u|^2 dx \geq C ||u||^2_H. \]

Therefore by Lax-Milgram for any $Q \in H$ there exists a unique $J \in H$ such that
\[ B[J, v] = \int Q\bar{v}dx, \]
for all $v \in H$.

Therefore there exists an $F \in H$ solving (17) given by $F = J + \psi$.

Now to show that $F(0, \eta)$ is holomorphic in $\eta$ we restate the result of our application of Lax-Milgram. We have shown that when $|\eta| < \tilde{\lambda}_1$, for all $Q \in H$ there exists a unique $J \in H$ such that
\[ \left| \left| \left( -\partial_x^2 + ix^\beta - \eta \right)^{-1}Q \right| \left| \right|_H = ||J||_H \leq C ||Q||_H. \]

Therefore $(-\partial_x^2 + ix^\beta - \eta)$ is bijective with a bounded inverse for $|\eta| < \tilde{\lambda}_1$. Thus the resolvent $(-\partial_x^2 + ix^\beta - \eta)^{-1}$ exists for $|\eta| < \tilde{\lambda}_1$ and by standard Banach algebra results it is holomorphic in $\eta$ there as well.

Now recall that the trace operator $T : H^1(0, \infty) \rightarrow \mathbb{R}$ is linear and continuous on $H$. Thus $TJ = J(0, \eta)$ is also holomorphic in $\eta$. Recall that $F(0, \eta) = J(0, \eta) + \psi(0) = J(0, \eta)$, so $F(0, \eta)$ is holomorphic in $\eta$. 

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To see that $F$ is unique assume otherwise, so there exists $F_1 \in H^1(0, \infty)$ which also solves (17) with $F_1(0) \neq F(0)$. Set $F_2 = F_1 - F$, then $F_2'(0) = 0$, $F_2 \in H^1(0, \infty)$ and

$$0 = -\partial_x^2 F_2 + i x^\beta F_2 - \eta F_2.$$ 

Multiply both sides by $\overline{F}_2$ and integrate then integrate by parts

$$0 = \int_0^\infty |\partial_x F_2|^2 + i x^\beta |F_2|^2 - \eta |F_2|^2 \, dx.$$ 

Then

$$0 \geq \int |\partial_x F_2|^2 + x^\beta |F_2|^2 - |\eta||F_2|^2 \, dx \geq \left(1 - \frac{|\eta|}{\tilde{\lambda}_1}\right) \int |\partial_x F_2|^2 + x^\beta |F_2|^2 \, dx > 0,$$

since $F_2 \in H^1(0, \infty)$. The inequality is strict since $|\eta| < \tilde{\lambda}_1$, which is a contradiction.

To establish the stated regularity of $F$ note that the equation (17) is elliptic and the left hand side is 0 so in fact $F \in H^\infty(0, \infty)$ and thus $F \in H^2(0, \infty)$ in particular.

Now we show that $F(0, \eta)$ is bounded away from 0 and $\infty$ in order to justify a division we made earlier.

**Lemma 4.2.** For $|\eta| \leq \frac{\tilde{\lambda}_1}{2}$ there exists $C > 0$ such that

$$1/C \leq |F(0, \eta)| \leq C.$$

**Proof.** The upper bound follows immediately by the holomorphy of $F(0, \eta)$ and the boundedness of $\eta$.

To see $|F(0, \eta)| > 1/C$ it is enough to show that $F(0, \eta) \neq 0$ since $F$ is continuous and we are working on a compact set. Assume otherwise, so $F(0, \eta_0) = 0$ for some $\eta_0 \in \mathbb{C}$, $|\eta_0| \leq \tilde{\lambda}_1/2$. The proof is essentially the same as that of the uniqueness of $F$. Multiply both sides of (17) by $\overline{F}$ and integrate and integrate by parts

$$0 = \int_0^\infty |\partial_x F|^2 + i x^\beta |F|^2 - \eta_0 |F|^2 \, dx.$$ 

Then

$$0 \geq \int |\partial_x F|^2 + x^\beta |F|^2 - |\eta_0||F|^2 \, dx \geq \left(1 - \frac{|\eta_0|}{\lambda_1}\right) \int |\partial_x F|^2 + x^\beta |F|^2 \, dx > 0,$$

since $F(\cdot, \eta_0) \in H^1_0(0, \infty)$. We note that the inequality is strict since $|\eta_0| < \tilde{\lambda}_1$, which is a contradiction.

Now that we have shown these properties of $F$ we will show that the other hypotheses of the implicit function theorem are satisfied.
Lemma 4.3. There exists $h_0 > 0$ such that $G$ as defined in (16) is continuous and $\frac{\partial}{\partial \mu} G$ exists and is continuous on $\{\mu \in \mathbb{C}; |\mu| < 1\} \times [0, h_0)$.

Proof. To see that $G$ is continuous it will be enough to show that $F(0, \mu, h)$ is continuous as the other terms in $G(\mu, h)$ are clearly continuous in $\mu$ and $h$. Similarly to see that $\frac{\partial}{\partial \mu} G$ exists and is continuous it is enough to see that $\frac{\partial}{\partial \mu} F(0, \mu, h)$ exists and is continuous in $\mu$ and $h$.

We recall that $F(0, \mu, h)$ is the Dirichlet data for the $L^2$ solution of (17) with spectral parameter

$$
\eta = \frac{\lambda^2}{h^{2\beta/(\beta+2)}} = \frac{\pi^2 l^2 h^{4/(\beta+2)} + (2A_1 \pi l + 2 \pi l \mu) h^{6/(\beta+2)}}{a} + (A_1^2 + \mu^2 + A_1 \mu) h^{8/(\beta+2)}.
$$

By Lemma 4.1 the Dirichlet data for the $L^2$ solution is holomorphic in $|\eta| < \tilde{\lambda}_1$. This $\eta$ is a sum of functions which are jointly continuous in $\mu$ and $h$ and continuously differentiable in $\mu$, therefore $F(0, \mu, h)$ is as well.

Furthermore since $|\mu| < 1$ and there is a positive power of $h$ in each term of $\eta$ there exists some $h_0$ such that $|\eta| < \tilde{\lambda}_1/2$ for $|\mu| < 1$ and $h \in [0, h_0)$.

Lemma 4.4. With $G$ defined as in (16)

$$
\left. \frac{\partial}{\partial \mu} G(\mu, h) \right|_{h=0, \mu=0} \neq 0.
$$

Proof. Note

$$
G(\mu, 0) = \frac{2\pi l i}{a} F(0, \mu, 0) - 2A_1 a i - 2 a i \mu
$$

so

$$
\left. \frac{\partial}{\partial \mu} G(\mu, 0) \right|_{h=0, \mu=0} = \frac{2\pi l i}{a} \left( \left. \frac{\partial}{\partial \mu} F(0, \mu, 0) \right|_{h=0, \mu=0} \right) - 2 ai.
$$

When $h = 0$ the equation $F$ solves is $0 = -F''(x) + ix^\beta F(x)$. Therefore when $h = 0$ there is no dependence on $\mu$ so $\frac{\partial}{\partial \mu} F(0, \mu, h = 0) = 0$. Thus

$$
\left. \frac{\partial}{\partial \mu} G(\mu = 0, h = 0) \right|_{h=0, \mu=0} = -2 ai \neq 0.
$$

4.1 Case $u'(0) = 0$

We now discuss how these proofs change when $u'(0) = 0$. When this is the case

$$
\text{Ref}(\lambda) = e^{-2i\lambda a/h}.
$$
Because of this we take $l + 1/2 \in \mathbb{Z}$ rather than $l \in \mathbb{Z}$. This changes the specific steps taken when going from the compatibility condition to the definition of $G$ in (16) but the eventual definition of $G$ is the same. The specific form of $l$ is otherwise not used so the remaining proofs in this section hold unchanged.

5 Proof of Lemma 3.1

Proof. First note that $P_h = -\hbar^2 \partial_x^2 + i(x-a)^\beta_+ - \lambda^2$ is semiclassically elliptic on $(a+\sigma/4,b)$ as its principal symbol is

$$|\xi|^2 + i(x-a)^\beta_+.$$ 

Recall that we required $a + \sigma < b - 2\delta$ and in (12) that $\phi$ satisfies

$$\phi(x) = \begin{cases} 1 & x < b - 2\delta \\ 0 & b - \delta < x. \end{cases}$$

We now define $\psi \in C^\infty(0,b)$ satisfying

$$\psi = \begin{cases} 0 & x < a + \sigma/2 \\ 1 & a + \sigma < x < b, \end{cases}$$

so that $\phi\psi \in \Psi^0_\hbar(0,b)$ with $WF_\hbar(\phi\psi) \subset ell_\hbar(P_h)$.

If $v$ is a solution of $P_h v = 0$, by standard semiclassical elliptic estimates (see for instance [Zwo12] Theorem 7.1) there exists $\chi \in C^\infty_0(0,b)$ such that for all $s \in \mathbb{R}$

$$||\phi\psi v||_{H^s_\hbar(a+\sigma/2,b)} \leq O(h^\infty) ||\chi v||_{L^2(0,b)}.$$ 

In particular

$$||\phi v||_{H^s_\hbar(a+\sigma,b)} \leq O(h^\infty) ||v||_{L^2(0,b)}.$$ 

It remains to show that

$$||v||_{L^2(0,b)} \lesssim ||\phi v||_{L^2(0,b)}.$$ 

To proceed we multiply both sides of (10) by $\bar{v}$ then integrate and integrate by parts

$$0 = \hbar^2 \int_0^\infty |\partial_x v|^2 dx + i \int_0^\infty (x-a)^\beta_+ |v|^2 dx - \lambda^2 \int_0^\infty |v|^2 dx.$$ 

Take the imaginary part of both sides and rearrange

$$\int_0^\infty (x-a)^\beta_+ |v|^2 dx = \text{Im}(\lambda^2) \int_0^\infty |v|^2 dx \leq O(h^2) \int_0^\infty |v|^2 dx.$$ 

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Furthermore
\[ \int_{a+\sigma}^{\infty} \sigma^\beta |v|^2 \, dx \leq \int_{a+\sigma}^{\infty} (x-a)^\beta_+ |v|^2 \, dx \leq \int_{0}^{\infty} (x-a)^\beta_+ |v|^2 \, dx. \]

So
\[ \int_{a+\sigma}^{\infty} |v|^2 \, dx \leq \frac{h^2}{\sigma^\beta} \int_{0}^{\infty} |v|^2 \, dx. \]

Add \( \int_{0}^{\infty} |v|^2 \, dx \) to both sides and rearrange
\[ \left( 1 - \frac{h^2}{\sigma^\beta} \right) \int_{0}^{\infty} |v|^2 \, dx \leq \int_{a+\sigma}^{\infty} |v|^2 \, dx. \]

Notice that \((b-2\delta, b) \subset (0, \infty)\) and \((0, a+\sigma) \subset (0, b-2\delta)\) so
\[ \left( 1 - \frac{h^2}{\sigma^\beta} \right) \int_{b-2\delta}^{b} |v|^2 \, dx \leq \left( 1 - \frac{h^2}{\sigma^\beta} \right) \int_{0}^{\infty} |v|^2 \, dx \leq \int_{0}^{a+\sigma} |v|^2 \, dx \leq \int_{0}^{b-2\delta} |v|^2 \, dx. \]

Therefore for \( h < \sigma^{\beta/2} \)
\[ \int_{0}^{b} |v|^2 \, dx \leq \left( 1 + \frac{\sigma^\beta}{\sigma^\beta - h^2} \right) \int_{0}^{b-2\delta} |v| \, dx \leq C \int_{0}^{b} |\phi v|^2 \, dx. \]

6 Proof of Theorem 1.2

In this section we establish a rate of decay of the energy by adapting and improving Section 3 of [BH07] for our particular setup. That result and our result rely heavily on an observability result by Burq and Zworski (Proposition 6.1) in [BZ04].

To prove Theorem 1.2 we again rely on Proposition 2.4 of [AL14] (see also [BT10]) which we state a variant.

**Proposition 6.1.** If there exists \( C > 0 \) and \( q_0 \geq 0 \) such that
\[ ||(-\Delta + iqW(x) - q^2)^{-1}||_{L^2 \rightarrow L^2} \leq C|q|^\frac{1}{\beta-1} \]  \hspace{1cm} (19)
for all \( q \in \mathbb{R}, |q| \geq q_0 \) then (11) is stable at rate \( 1/t^\alpha \).

**Proof of Theorem 1.2.** Consider \( u \in H^2(M) \) solving
\[ (-\Delta + iqW - q^2)u = f \in L^2(M), \quad u|_{\partial M} = 0, \quad q \gg 1. \]  \hspace{1cm} (20)
Let $0 \leq \chi \in C_0^\infty(\mathbb{R})$ be a cutoff function with $\chi = 0$ for $|x| \geq 2$ and $\chi = 1$ for $|x| \leq 1$. Now let $\chi_q = \chi(q^7 x)$ with $\gamma = \frac{\beta}{\beta^2 + 2}$. Note that $\chi_q$ vanishes for $|x| > a + \sigma/4$ and on the support of $\chi_q(x)$

\[ W(x) \sim \frac{1}{q^\gamma}, \quad q \gg 1. \quad (21) \]

**Remark.** The proof in [BH07] uses an analogous setup with $\gamma = 1$. The key change we make is to set $\gamma = \frac{\beta}{\beta^2 + 2}$. The rest of our argument is similar to the proof in [BH07] but we detail it for the convenience of the reader and to explain why this value of $\gamma$ is ideal.

The function $\chi_q u$ still vanishes on $\partial M$ (or if $\partial M = \emptyset$ it still satisfies the periodicity condition) and satisfies on $M$

\[ (-\Delta - q^2)\chi_q u = \chi_q f + \chi_q'' u - 2\partial_x(\chi_q' u) - iqW(x)\chi_q u. \quad (22) \]

We apply Proposition 6.1 of [BZ04] to this equation choosing the control region $\omega_x = [a + \sigma/4, a + \sigma/2]$ to obtain

\[
\|\chi_q u\|_{L^2}^2 \leq C \left( \|\chi_q f + \chi_q'' u - 2\partial_x(\chi_q' u) - iqW(x)\chi_q u\|_{H^{-1}_{x}L^q(M)}^2 + \|\chi_q u\|_{L^2}^2 \right) \\
\leq C \left( \|\chi_q f\|_{L^2}^2 + \|\chi_q'' u\|_{L^2}^2 + \|\chi_q' u\|_{L^2}^2 + \|qW\chi_q u\|_{L^2}^2 \right). \quad (23) 
\]

We emphasize that $\chi_q$ vanishes on $\omega$, which allowed us to drop that term. We now estimate the remaining terms on the right hand side. Using the exact form $W(x) = (|x| - a)_+$ we obtain the following bound on the derivative of $\chi_q$,

\[ |\chi_q'| = |q^7 \chi_q'(q^7 W(x))W'(x)| \leq C q^{\gamma/\beta}, \quad (24) \]

and similarly

\[ |\chi_q''| \leq C q^{2\gamma/\beta}. \]

Note that on the support of $\chi_q'$ and $\chi_q''$ the damping $W$ is smooth, so this computation is valid for all $\beta > 0$.

Now write

\[ \chi_q' u = \frac{\chi_q' W^{1/2} u}{W^{1/2}}, \]

then using (24) and (21)

\[ \left| \frac{\chi_q'}{W^{1/2}} \right| \leq q^{(1/2 + 1/\beta)}, \]

and consequently

\[ \|\chi_q' u\|_{L^2}^2 \leq C q^{(1+2/\beta)} \left\| W^{1/2} u \right\|_{L^2}^2. \quad (25) \]

We estimate the $L^2$ norm of $\chi_q'' u$ in a similar way

\[ \|\chi_q'' u\|_{L^2}^2 \leq O(1) q^{(1+4/\beta)} \left\| W^{1/2} u \right\|_{L^2}^2. \quad (26) \]
Finally a similar argument shows
\[ \|qW\chi_q u\|_{L^2}^2 \leq O(1)q^{2-\gamma}\left\| W^{1/2}u\right\|_{L^2}^2. \] (27)
The smaller these terms are the stronger the resolvent estimate is. Because of this we would like to minimize
\[ \max\{2 - \gamma, \gamma(1 + 4/\beta), \gamma(1 + 2/\beta)\}. \]
This is attained when
\[ 2 - \gamma = \gamma(1 + 4/(\beta)), \]
i.e. \( \gamma = \beta/(\beta + 2) \). Therefore (25), (26), (27) along with (23) give
\[ \|\chi_q u\|_{L^2}^2 \leq O(1) \left( \|f\|_{L^2}^2 + q^{(\beta+4)/(\beta+2)} \|W^{1/2}u\|_{L^2}^2 \right). \]
Now note that pairing (20) with \( \bar{u} \)
\[ \left|\lambda\right| \int W|u|^2 \, dx \leq \|f\|_{L^2} \|u\|_{L^2}. \] (28)
Therefore
\[ \|\chi_q u\|_{L^2}^2 \leq O(1) \left( \|f\|_{L^2}^2 + q^{2/(\beta+2)} \|f\|_{L^2} \|u\|_{L^2} \right). \]
It remains to control the \( L^2 \) norm of \( (1 - \chi_q) u \). To do so we remark that \( 1 - \chi_q \) is supported in the set where \( W \geq 1/q^\gamma \). Using (28) again
\[ \| (1 - \chi_q) u \|_{L^2}^2 \leq q^\gamma \int (1 - \chi_q)W|u|^2 \, dx \leq q^{\gamma-1} \|f\|_{L^2} \|u\|_{L^2}. \]
Therefore
\[ \|u\|_{L^2} \leq O(1) \left( \|f\|_{L^2} + q^{1/(\beta+2)} \|f\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2} \right) \]
and thus we obtain
\[ \|u\|_{L^2} \leq O(1)q^{2/(\beta+2)} \|f\|_{L^2}, \quad q \in \mathbb{R}, \quad |q| \gg 1, \]
which along with Proposition 6.1 gives stability at the stated rate.

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