Black holes with supertranslation field and Israel theorem

Mikhail Z. Iofa
Skobeltsyn Institute of Nuclear Physics
Lomonosov Moscow State University
Moscow, 119991, Russia

Abstract

Black hole solution of the Einstein equations containing supertranslation field is studied in the context of the Israel theorem. Axial-symmetric solution diffeomorphic to the Schwarzschild solution is discussed. The solution apparently violates the Israel theorem. The solution is transformed to a form with the horizon located at the sphere $r = 2M$, where $M$ is the mass of black hole. Violation of the theorem is shown to be due to singular behavior of solution at the horizon. Violation of the assumption of finiteness of the square of the Riemann tensor, crucial for validity of the theorem, results in changes in the Israel inequalities. We examine step by step the resulting changes in the proof of the theorem, and show how the changes allow for existence of a vacuum solution not reducible to the Schwarzschild space-time.

1 Introduction

The metric of final stationary space-time of rotating black hole resulting from axisymmetric gravitational collapse is diffeomorphic to the Kerr metric [1, 2, 3] (the uniqueness theorem for stationary axisymmetric vacuum solution). Because diffeomorphisms contain also supertranslations (angular-dependent time translations at the null infinity) [4], metric of the final state of collapse, in general, contains also supertranslation field associated with supertranslations.

The uniqueness theorem for static vacuum solutions of the Einstein equations with regular event horizon is given by the Israel theorem which states that any static vacuum asymptotically flat solution satisfying certain regularity conditions (precise formulation is below) is the Schwarzschild metric [5, 6]. Physically, this means that no static asymmetric perturbation by sources within the horizon can preserve regularity of the event horizon. Examples of such sources are quadrupole [7], magnetic dipole fields [8] inside a black hole, or small perturbations of the Schwarzschild black hole [9].

Vacuum solution of the Einstein equations diffeomorphic to the Schwarzschild metric and containing supertranslation field was constructed in [10]. Because the metric contains supertranslation field, it is physically different from the Schwarzschild metric and cannot be transformed to the Schwarzschild metric by a diffeomorphism not containing supertranslation field. Although the metric with supertranslation field has much resemblance to the Schwarzschild solution, it violates the Israel theorem. Loosely speaking, supertranslation field acts as an inner non-spherical source destroying the regularity of the Kretschmann scalar (square of the Riemann tensor) at the horizon, but this source is not a perturbation, but a part of solution.

In this note we consider a black hole metric containing supertranslation field diffeomorphic to the Schwarzschild metric [10] and discuss which conditions of the Israel theorem are valid and which are violated by the solution. It is shown that the crucial difference from the Schwarzschild metric is that in the case of the metric with supertranslation field the Kretschmann scalar is divergent at the horizon. It follows that the Israel inequalities are modified so that they do not imply that the
metric with supertranslation field reduces to the Schwarzschild metric. Step by step, we trace how is formed a gap allowing black hole with supertranslation field to evade the Israel theorem.

2 Conditions of the Israel theorem

Israel theorem refers to static vacuum solutions of the Einstein equations. In a static space-time which admits a Killing vector $\xi$ the line element can be locally reduced to a form

$$ds^2 = g_{\alpha\beta}(x^1, x^2, x^3)dx^\alpha dx^\beta - V^2(x^1, x^2, x^3)dt^2,$$

where $|V^2| = \xi \cdot \xi$, Greek indices run 1-3.

The Israel theorem states that the only metric satisfying conditions listed below is the Schwarzschild solution.

Conditions are: Let $\Sigma$ be a hypersurface $t = const$, maximally extended so that square $V$ of the Killing vector $\xi$ is negative $\xi \cdot \xi < 0$. It is assumed that the 3D hypersurface $\Sigma$ is regular and non-compact.

1. The metric has the following asymptotic form
   
   $g_{\alpha\beta} = \delta_{\alpha\beta} + O(r^{-1}), \quad \partial_\gamma g_{\alpha\beta} = O(r^{-2}), \quad r^2 = g_{\alpha\beta}x^\alpha x^\beta \to \infty,$
   
   $V = 1 - M/2 + \eta, \quad \eta = O(r^{-2}), \quad \partial_\alpha \eta = O(r^{-3}), \quad \partial_\alpha \partial_\beta \eta = O(r^{-4}),$
   
   and the surfaces $V(x) = const > 0$ are connected closed regular 2D surfaces.
   
   3. 4D invariant $R_{ijkl}R^{ijkl}$ is bounded on $\Sigma$.
   
   4. If the greatest lower bound of $V$ on $\Sigma$ is zero, then the geometry of the equipotential surfaces $V = \varepsilon$ in the limit $\varepsilon \to 0$ approaches a geometry corresponding to a closed regular 2-space of finite area.

3 Vacuum solution with supertranslation field

The metric of a static asymptotically flat spacetime containing supertranslation field $C(\theta, \varphi)$ constructed in [10] is

$$ds^2 = \frac{(1 - M/2 \rho_s)^2}{(1 + M/2 \rho_s)} dt^2 + (1 + M/2 \rho_s)^4 \left(dp^2 + (((\rho - E)^2 + U)\gamma_{ab} + (\rho - E)C_{ab})dz^a dz^b\right)$$

Here the variables $z^a$ are realized as angles $\theta, \varphi$ on the unit sphere with the metric $ds^2_{(2)} = \gamma_{ab}dz^a dz^b = d\theta^2 + \sin^2 \theta d\varphi^2$. We consider general axial-symmetric metrics with supertranslation field depending only on angle $\theta$. The functions $C_{ab}$ and $E, U$ depend on $\theta$ and on $C(\theta)$ and its derivatives,

$$\rho_s(\rho, C) = \sqrt{(\rho - C)^2 + C'^2(\theta)}.$$  

Here prime is derivative over $\theta$. Horizon of the metric is located at the surface $\rho_s(\rho, C) = M/2$. By the coordinate transformation

$$r(\rho, \theta) = \rho_s(\rho, C) \left(1 + \frac{M}{2\rho_s(\rho, C)}\right)^2,$$
the metric (2) is transformed to a form with horizon located at the surface $r = 2M$ [11]:

$$ds^2 = -V^2dt^2 + \frac{dr^2\bar{g}_{rr}}{V^2} + \frac{2drd\theta\bar{g}_{r\theta}}{V} + d\theta^2\bar{g}_{\theta\theta} + d\varphi^2\sin^2\theta\bar{g}_{\varphi\varphi}$$

$$= -V^2dt^2 + \frac{dr^2}{V^2(1 - b^2)} + 2drd\theta \frac{br(\sqrt{1 - b^2} - b')}{(1 - b^2)V} +$$

$$+ d\theta^2r^2\frac{(\sqrt{1 - b^2} - b')^2}{(1 - b^2)} + d\varphi^2r^2\sin^2\theta (b\cot\theta - \sqrt{1 - b^2})^2.$$  \hspace{1cm} (5)

In (5) are introduced the functions

$$V^2 = 1 - \frac{2M}{r}, \quad b = \frac{2C'(\theta)}{K}, \quad K = r - M + rV.$$  \hspace{1cm} (6)

The space-time (5) has the time-like Killing vector $\xi^i = const V$ which becomes null at the horizon. Solving the geodesic equations for null geodesics, it is possible to show that the surface $r = 2M$ is the surface of infinite redshift [11].

In the limit $r \to \infty$ the metric (5) takes a form

$$ds^2 = -dt^2 + dr^2 + 2C'(\theta)drd\theta + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$  \hspace{1cm} (7)

Here $\theta = \arctan(x_3/\rho)$, $\rho = (x_1^2 + x_2^2)^{1/2}$. We have

$$drd\theta = dx_1dx_3\frac{\rho x_3}{r^3} - dx_1dx_1\frac{x_1x_3x_3}{r^4\rho} - (1 \leftrightarrow 2)$$

All the coefficients at $dx_1dx_j$ are of order $O(1/r)$, but the expression $\partial/\partial x_1[x_i x_3/(r^3 \rho)]$ contains the term $x_i x_3/(r^3 \rho) = O(1/r \rho)$ violating condition 1 of the theorem.

However, in the proof of the theorem it is used that integration is performed over two-dimensional surfaces $V = const$ with the intrinsic coordinates on the surfaces $\hat{x}^A, A = 1, 2$ which are constant along trajectories orthogonal to surfaces $V = const$. From condition 2 of the theorem it follows that in the limit $r \to \infty$ $\rho \to r^2/M$. The spacial metric takes a form

$$ds^2 = \hat{g}_{AB}d\hat{x}^Ad\hat{x}^B + \rho^2(V, \hat{x})dV^2,$$  \hspace{1cm} (8)

where $\rho^{-2} = V_{,\alpha}V^{,\alpha}$. A typical integral is

$$\int_{\Sigma} d^3\hat{x}\sqrt{\hat{g}^{(3)}}\frac{\partial}{\partial V}F(\hat{x}) = \int_{V \to 1} d^2\hat{x}\sqrt{\hat{g}^{(2)}}F(\hat{x}) - \int_{V \to 0^+} d^2\hat{x}\sqrt{\hat{g}^{(2)}}F(\hat{x}).$$  \hspace{1cm} (9)

The metric (7) can be written as

$$ds^2 = dr_1^2 + d\theta^2(r_1^2 - 2r_1C) + (r_1 - C)^2\sin^2\theta d\varphi^2$$  \hspace{1cm} (10)

where $r_1 = r + C(\theta)$. In the limit $r \to \infty$ the metric (10) approaches the standard metric on the sphere, and for $\hat{x}^A$ can be taken coordinates $\theta$ and $\varphi$. 


4 Kretschmann scalar at the horizon

Below, we turn to a problem whether the metric (5) satisfies condition 4. For a vacuum solution of the Einstein equations the Kretschmann scalar $R_{ijkl}R^{ijkl}$ can be presented in a form

$$\frac{1}{8} R_{ijkl} R^{ijkl} = V^{-2} V^{\alpha\beta} V_{\alpha\beta}. \quad (11)$$

Here the latin indices run over $0 - 3$, the Greek ones over $1 - 3$.

Separating the leading dependence on $V$ as $V \to 0$, we present the metric and its inverse in a form

$$g_{\alpha\beta} = \begin{vmatrix} g_{rr}/V^2 & g_{r\theta}/V & 0 \\ g_{r\theta}/V & g_{\theta\theta} & 0 \\ 0 & 0 & \bar{g}_{\varphi\varphi} \end{vmatrix}, \quad g^{\alpha\beta} = \begin{vmatrix} V^2 & -Vg_{r\theta}/g_{\theta\theta} & 0 \\ -Vg_{r\theta}/g_{\theta\theta} & g_{rr}/g_{\theta\theta} & 0 \\ 0 & 0 & 1/\bar{g}_{\varphi\varphi} \end{vmatrix}. \quad (12)$$

The metric components satisfy the identity

$$g_{rr}g_{\theta\theta} - g_{r\theta}^2 = \frac{\bar{g}_{\theta\theta}}{V^2} \quad (13)$$

Calculating $V_{\alpha\beta}$, we obtain

$$V_{rr} = V_{rr} - \Gamma_{rr}^r V_r = -M(2rV^2 + M) \frac{1}{r^4V^3} - \frac{1}{2} \left[ g^{rr} g_{rr,r} + g^{r\theta} (2g_{r\theta,r} - g_{r\theta}) \right] V_r =$$

$$= -\frac{2M}{V^3} - \frac{M}{2Vr^2} \left( \bar{g}_{rr,r} - \frac{2\bar{g}_{r\theta}g_{r\theta,r}}{\bar{g}_{\theta\theta}} - \frac{\bar{g}_{rr}\bar{g}_{r\theta}}{V\bar{g}_{\theta\theta}} \right),$$

$$V_{r\theta} = -\Gamma_{r\theta}^r V_r = -M \frac{1}{Vr^2} \left( \frac{1}{2} g^{rr} g_{r\theta} + g^{r\theta} g_{\theta\theta,r} \right),$$

$$V_{\theta\theta} = -\Gamma_{\theta\theta}^r V_r = -M \frac{1}{Vr^2} \left( \frac{1}{2} g^{rr} (g_{r\theta} - g_{\theta\theta,r}) + \frac{1}{2} g^{r\theta} g_{\theta\theta} \right),$$

$$V_{\varphi\varphi} = -\Gamma_{\varphi\varphi}^r V_r = \frac{M}{r^2} \left( \frac{1}{2} g^{rr} g_{\varphi\varphi} + \frac{1}{2} g^{r\theta} g_{\varphi\varphi} \right),$$

$$V_{r\varphi} = V_{\theta\varphi} = 0. \quad (14)$$

In the limit $V \to 0$, because of the identity (13), the leading terms in $V_{rr}$ which are of order $O(V^{-3})$ cancel.

From the vacuum Einstein equations it follows that $V$ is harmonic function

$$g^{\alpha\beta} V_{\alpha\beta} = 0. \quad (15)$$

Making use of (15) and (13), rearranging the terms, we present $V_{\alpha\beta} V^{\alpha\beta}$ in a form

$$V_{\alpha\beta} V^{\alpha\beta} = 2(V_{rr} V_{\theta\theta} - V_{r\theta}^2)((g^{r\theta})^2 - g^{rr} g^{\theta\theta}) + (V_{r\varphi} g^{\varphi\varphi})^2. \quad (16)$$

Using (13), we have

$$(g^{r\theta})^2 - g^{rr} g^{\theta\theta} = V^2 \frac{g^{r\theta} - \bar{g}_{r\theta} g^{\theta\theta}}{\bar{g}_{\theta\theta}} = \frac{V^2}{\bar{g}_{\theta\theta}}. \quad (17)$$

Because the combination $V_{rr} V_{\theta\theta} - V_{r\theta}^2$ is the difference of two terms, it is necessary to verify, if in the difference there is a cancellation of the leading terms in the limit $V \to 0$. In this limit
\( r \approx 2M(1 + V^2) \) and \( K \approx M(1 + 2V) \). All metric components except \( \bar{\gamma}_{\varphi\varphi} \) depend on \( \theta \) through \( b = 2C'(\theta)/K \). The \( r \)-derivative of \( b \) is \( b_r = -b(1 + V + M/rV)/K^2 \), and
\[
(-2V)b_r \approx \frac{2C'(\theta)}{M^2}(1 + 2V)
\]  
(18)
as \( V \to 0 \). Derivative of \( b \) over \( \theta \) is \( \partial_\theta b = b' \).

To estimate \( V_{;rr} \), we consider only the second term in (14) which we rewrite as
\[
V_{;rr} = \frac{M}{2V^2r^2} \left( \frac{1}{2}(-2V)\bar{g}_{rr,r} + \frac{\bar{g}_{r\theta}(-2V)\bar{g}_{\theta r}}{\bar{g}_{\theta\theta}} - \frac{\bar{g}_{rr,\theta}\bar{g}_{\theta\theta}}{\bar{g}_{\theta\theta}} \right).
\]  
(19)It is seen that in the limit \( V \to 0 \) this term is of order \( O(V^{-2}) \).

Next, we consider \( V_{;\theta\theta} \). Transforming the combination in brackets, substituting the explicit expressions for the metric components, and using (18), we obtain
\[
VV_{;\theta\theta}\big|_{V \to 0} = -\frac{M}{2r^2} \left( 2\bar{g}_{\theta\theta,\theta} - V\bar{g}_{\theta\theta,r} \right) \bigg|_{V \to 0} = \frac{M}{2r^2} \left[ \bar{g}_{\theta\theta}\ln \frac{\bar{g}_{\theta\theta}^2}{\bar{g}_{\theta\theta}} + \frac{1}{2}(-2V)\bar{g}_{\theta\theta,r} \right] = -\frac{M}{2r^2} \left[ \frac{rb(\sqrt{1-b^2} - b')}{1-b^2} \times \frac{2b'}{b_1-b^2} + \frac{1}{2}(-2V)\partial_r \frac{r^2(\sqrt{1-b^2} - b')^2}{1-b^2} \right]\bigg|_{V \to 0} = 0,
\]  
(20)i.e. \( V_{;\theta\theta} = O(V^0) \).

Calculating \( V_{;r\theta} \), we have
\[
VV_{;r\theta}\big|_{V \to 0} = -\frac{M}{2r^2} \left( \bar{g}_{r\theta,\theta} - V\bar{g}_{r\theta,r} \right) \bigg|_{V \to 0} = -\frac{M}{2r^2} \left[ \partial_\theta \left( \frac{1}{1-b^2} \right) + \frac{1}{2r(\sqrt{1-b^2} - b')} \frac{b}{1-b^2} \right](21)
i.e. \( V_{;r\theta} = O(V^0) \).

Finally, for \( V_{;\varphi\varphi} \) we obtain
\[
VV_{;\varphi\varphi}\big|_{V \to 0} = \frac{M}{2r^2} \left( g^{rr}g_{\varphi\varphi,r} + g^{r\theta}g_{\varphi\varphi,\theta} \right) = \frac{M}{2r^2} \left[ V^2\partial_r(r^2(b\cos \theta - \sqrt{1-b^2}\sin \theta)^2) - \frac{b}{2r(\sqrt{1-b^2} - b')} \partial_\theta(r^2(b\cos \theta - \sqrt{1-b^2}\sin \theta)^2) \right]\bigg|_{V \to 0} = O(V),
\]  
(22)i.e. \( V_{;\varphi\varphi} = O(V^0) \).

To conclude, we have
\[
V_{;rr} = O(V^{-2}), \quad V_{;r\theta} = O(V^0), \quad V_{;\theta\theta} = O(V^0), \quad V_{;\varphi\varphi} = O(V^0),
\]  
(23)and in the combination \( V_{;rr}V_{;\theta\theta} - V_{;r\theta}^2 \) there is no cancellation of the leading terms.

From (16) and (17) it follows that
\[
V_{;\alpha\beta}V^{;\alpha\beta} = O(V^0),
\]  
(24)and
\[
R_{ijkl}R^{ijkl} = O(V^{-2}).
\]  
(25)
To compare, in the case of the pure Schwarzschild metric
\[
V^{-2}V_{;\alpha\beta}V^{;\alpha\beta} = \frac{1}{V^2} \left[ (V_{;rr}g^{rr})^2 + (V_{;\theta\theta}g^{\theta\theta})^2 + (V_{;\varphi\varphi}g^{\varphi\varphi})^2 \right] = \frac{6M^2}{r^6},
\]  
(26)i.e. the Kretschmann scalar is regular at \( V = 0 \).
5 Israel inequalities

Let us find what are the (technical) consequences of the fact that (25) violates the condition of the theorem to be finite in \( \Sigma \). In [5] was obtained the relation (11) in coordinates \( \hat{x}, V \) was obtained in a form:

\[
\frac{1}{8} R_{ijkl} R^{ijkl} = \frac{1}{(V \rho)^2} \left[ K_{AB} K^{AB} + 2 \rho^{-2} \rho_{,A} \rho^{,A} + \rho^{-4}(\partial \rho/\partial V)^2 \right],
\]

where

\[
K_{AB} = \frac{1}{2\rho} \frac{\partial g_{AB}}{\partial V}
\]

is the external curvature. In [5] it was proved that, if a metric satisfies the assumptions of the theorem, then \( \rho \) approaches a regular nonzero limit \( \rho_0 = \lim_{V \to 0} \rho \). In this case from (27) it follows that \( K_{AB} K^{AB} = O((\rho_0 V)^2) \) in the limit \( V \to 0 \). From these results and from the relation

\[
R^{(2)} = K_{AB} K^{AB} - K^2 - 2K/\rho V,
\]

where

\[
K = (1/\rho^2)(\partial \rho/\partial V),
\]

it follows that

\[
R^{(2)}|_{V \to 0} = -\frac{2}{\rho_0} \lim_{V \to 0} K/V.
\]

In the case of the metric (5) the explicit expression for \( \rho^{-2} \) is

\[
\rho^{-2} = V_\rho V^\rho = \left( \frac{M}{r^2} \right)^2 = \frac{(1 - V^2)^4}{(16M^2)},
\]

and we have \( \rho \to 4M, \quad \rho_{,A} = 0, \quad \rho^{-2}(\partial \rho/\partial V) \to 0 \) in the limit \( V \to 0 \). Thus, in contrast to the case of a metric satisfying the assumptions of the theorem, from (27) we obtain

\[
K_{AB} K^{AB} = O(V^0).
\]

In the case of the metric (5), in the limit \( V = 0 \), we have \( K = 2V/r \). In distinction to (30), from (29) we obtain

\[
-2/\rho_0 \lim_{V \to 0} (K/V) = \lim_{V \to 0} [R^{(2)} - K_{AB} K^{AB}].
\]

The extra term \( K_{AB} K^{AB} \) in the r.h.s of the equation (33) changes the topological result for the integral

\[
-2/\rho_0 \lim_{V \to 0} \int d^2 \hat{x} \sqrt{\hat{g}^{(2)}} K/V = \lim_{V \to 0} \int d^2 \hat{x} \sqrt{\hat{g}^{(2)}} R^{(2)} = -8\pi
\]

in the case \( K_{AB} K^{AB} \) vanishes.

In the case of the metric (5) the volume elements expressed in coordinates \( x^\alpha \) and \( V, \hat{x}^A \) are

\[
dr d\theta d\varphi \sqrt{(g_{rr} g_{\theta \theta} - g_{r \theta}) g_{\varphi \varphi}} = dr d\theta d\varphi \sqrt{g_{\theta \theta} g_{\varphi \varphi} V^{-2}} =
\]

\[
dV d^2 \hat{x} \sqrt{(\hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^2) g_{VV}} = (\partial V/\partial r) dr d^2 \hat{x} \sqrt{(\hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^2)(V(\partial V/\partial r))^{-2}},
\]

(34)
from which it follows that $d\theta d\varphi \sqrt{g^{(2)}} = d^2\hat{x} \sqrt{\hat{g}^{(2)}}$. In the first line of (34) we have used the relation (13), and in the second line we have substituted $V = (1 - 2M/r)^{1/2}$.

Multiplying (28) by $g^{AB}$ one obtains the identity

$$\frac{\partial}{\partial V} \left( \frac{\sqrt{\hat{g}^{(2)}}}{\rho} \right) = 0.$$  \hspace{1cm} (35)

Integrating (35) over $\Sigma$, one obtains

$$S_0/\rho_0 = 4\pi M.$$ \hspace{1cm} (36)

Here, we defined $S_0 \equiv \lim_{V \to 0} \int \sqrt{\hat{g}^{(2)}} d^2\hat{x}$ and have used that $\lim_{V \to 1} \int \rho^{-1} \sqrt{\hat{g}^{(2)}} d^2\hat{x} = \lim_{r \to \infty} \int \rho^{-1} \sqrt{g^{(2)}} d\theta d\varphi = (4\pi r^2)(M/r^2)$.

• To prove the theorem, Israel used the identities following from the Einstein equations. In the following, we discuss what changes in the identities appear in the case of the metric (5) as compared to the metrics satisfying the assumptions of the theorem.

Integrating the first identity

$$\frac{\partial}{\partial V} \left( \sqrt{\hat{g}^{(2)}} \frac{K}{\rho} \right) = -\frac{\sqrt{\hat{g}^{(2)}}}{V} \left[ \nabla^2 (\rho^{1/2}) + \frac{1}{2} \rho^{-3/2} \rho, A \rho ; A + \rho^{1/2} (K_{AB} K^{AB} - K^2/2) \right]$$ \hspace{1cm} (37)

over $\Sigma$, one obtains

$$\int_{V \to 1} d\theta d\varphi \sqrt{g^{(2)}} 2\sqrt{M} - \int_{V \to 0} d^2\hat{x} \sqrt{\hat{g}^{(2)}} \rho_0 \left( -\frac{\rho_0}{2} \right) (R^{(2)} - K_{AB} K^{AB}) =$$

$$= 8\pi \sqrt{M} - \frac{\sqrt{\rho_0}}{2} \left( 8\pi + \int_{V \to 0} d^2\hat{x} \sqrt{\hat{g}^{(2)}} K_{AB} K^{AB} \right) < 0.$$ \hspace{1cm} (38)

Here we have used that the integral of the expression in the r.h.s of (37) is negative. In the l.h.s. of (38) we have used that

$$\sqrt{\hat{g}^{(2)}} K d^2\hat{x} \bigg|_{V \to 1} \simeq \sqrt{\hat{g}^{(2)}} \frac{M}{r^2} d^2\theta d\varphi \frac{2}{r} \bigg|_{r \to \infty}. $$

We rewrite the inequality (38) in a form

$$4M \leq \rho_0 (1 + k)^2,$$ \hspace{1cm} (39)

where

$$k = \frac{1}{8\pi} \int_{V \to 0} d^2\hat{x} \sqrt{\hat{g}^{(2)}} K_{AB} K^{AB}.$$ 

Next, we consider an inequality following from the identity

$$\frac{\partial}{\partial V} \left[ \sqrt{\hat{g}^{(2)}} \rho \left( K V + \frac{4}{\rho} \right) \right] = -\sqrt{\hat{g}^{(2)}} V \left[ \nabla^2 (\ln \rho) + \rho, A \rho ; A + 2 K_{AB} K^{AB} - K^2 - R^{(2)} \right].$$ \hspace{1cm} (40)
Integrating (40) over $\Sigma$, we have
\[
\int_{V \to 1} d^2\hat{x} \sqrt{\hat{g}^{(2)}} \frac{1}{\rho} \left( K + \frac{4}{\rho} \right) - \int_{V \to 0} d^2\hat{x} \sqrt{\hat{g}^{(2)}} \frac{4}{\rho_0^2} = \]
\[
= - \int d^2\hat{x} \sqrt{\hat{g}^{(2)}} \int_0^1 dV V \left[ \nabla^2 (\ln \rho) + \rho_{,A} \rho_{,A} + 2K_{AB}K^{AB} - K^2 \right] + 4\pi, \tag{41}\]
where it was used that
\[
\int d^2\hat{x} \sqrt{\hat{g}^{(2)}} \int_0^1 dV VR^{(2)} = -4\pi.
\]
The integral in the r.h.s of (41) is negative, the first integral in l.h.s vanishes leaving
\[
4\pi \leq \frac{4S_0}{\rho_0^2} \tag{42}
\]
Combining the relation (36) and the inequalities (39) and (42), we obtain the inequality
\[
\frac{4M}{(1+k)^2} \leq \rho_0 \leq 4M \tag{43}
\]
If a metric satisfies the assumptions of the theorem, in the inequality (43) stands $k = 0$. Then from the inequalities (42) and (43) it follows that $\rho_0 = 4M$, and the metric is Schwarzschild. In the case of the metric (5), a gap between the limits in (43) allows for $\rho_0 \neq 4M$.

6 Conclusion

To conclude, we have shown that the metric with supertranslation field (5) violates the assumptions of the theorem, because the Kretschmann scalar is singular at the horizon. From this, it follows that the Israel inequalities are modified leaving a gap for the metric (5) to differ from the Schwarzschild metric. Violation of condition 1 of the theorem on behavior of the metric at the Euclidean infinity appears to be not crucial, because in the proof it was only used that the surface integral over the sphere of radius $r \to \infty$ tends to $4\pi r^2$, and this is the case of the metric (5).

Acknowledgments

This work was partially supported by the Ministry of Science and Higher Education of Russian Federation under the project 01201255504.

References

[1] B. Carter, *Axisymmetric black hole has only two degrees of freedom*, Phys. Rev. Lett. 26, 331 (1971).

[2] D. C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. 34, 905 (1975).

[3] S. Chandrasekhar *The mathematical theory of black holes*, Oxford University Press, (1983).
[4] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theories*, arXiv:1703.05448.

[5] W. Israel, *Event Horizons in Static Vacuum Space-Times*, Phys. Rev. **164**, 1776 (1967).

[6] W. Israel, *Event Horizons in Static Electrovac Space-Times*, Comm. Math. Phys. **8**, 245 (1968).

[7] A. G. Doroshkevich, Ya. B. Zeldovich and I. D. Novikov, *Perturbations in an Anisotropic Homogeneous Universe* Soviet Phys JETP **22**, 122 (1966).

[8] V. L. Ginzburg, *On magnetic fields of collapsing masses*, Soviet Phys. Doklady **9**, 329 (1964).

[9] T. Regge and J. A. Wheeler, *Stability of a Schwarzschild Singularity*, Phys. Rev. **108**, 1063 (1957).

[10] G. Compere and J. Long, *Classical static final state of collapse with supertranslation memory*, Class. Quant. Grav. **33**, 195001 (2016), arXiv:1602.05197.

[11] M. Z. Iofa, *Near-horizon symmetries of the Schwarzschild black hole with supertranslation field*, Phys. Rev. D **99**, 064052 (2019), arXiv:1801.03328.