A MIRROR SYMMETRIC SOLUTION TO THE QUANTUM TODA LATTICE

KONSTANZE RIETSCH

Abstract. We give a representation-theoretic proof of a conjecture from [28] providing integral formulas for solutions to the quantum Toda lattice in general type. This result generalizes work of Givental for SL_n/B and can be interpreted as a kind of mirror theorem for the full flag variety G/B. We also prove the existence of a totally positive critical point of the ‘superpotential’ in every mirror fiber.

1. Introduction

In [28] we introduced a conjectural ‘mirror datum’ (Z_P, ω, F_P) associated to a general flag variety G/P, and used it to recover the quantum cohomology ring qH^*(G/P)_{(q)} with quantum parameters inverted in its presentation due to Dale Peterson.

The goal of this paper is to show, in the full flag variety case, that associated integrals

\[ S_{\Gamma}(h) = \int_{[\Gamma, h]} e^{\mathcal{F}_B} \omega_h, \]

defined in terms of the mirror datum (Z_B, ω, F_B) of G/B, and with integration cycles satisfying certain technical conditions, are annihilated by the quantum Toda Hamiltonian associated to G^\vee [13]. By Kim [10], the quantum Toda equations of G^\vee are the ‘quantum differential equations’ of G/B in the sense of Givental, whose symbols recover the relations in the quantum cohomology ring. See also [1]. Therefore this result is in a sense ‘quantization’ of [28].

In type A integral solutions of the form (1.1) to the quantum Toda lattice were obtained earlier by Givental [6], in what he called a ‘mirror theorem’ for SL_n/B, using ingenious and very explicit coordinates. The general mirror family introduced in [28] was inspired by this construction and is such that one recovers Givental’s mirror family by restricting to a certain open subset of Z_B and considering a special choice of coordinates there.

Givental’s construction of solutions to the quantum Toda lattice was also recently revisited by Gerasimov, Kharchev, Lebedev and Oblezin [4] (GKLO), who gave a new proof of Givental’s type A result using Kostant’s Whittaker model. Their proof has some features in common with our construction [28] but still relied in an essential way on the use of Givental’s special coordinates, and hence didn’t readily generalize to other types.

Date: October 25, 2008.

2000 Mathematics Subject Classification. 14N35, 14M17, 57T15.

Key words and phrases. Flag varieties, quantum cohomology, mirror symmetry.

The author is supported by EPSRC advanced fellowship EP/S071395/1.
In this paper we use a combination of methods from [4] and [28] to prove that the integrals (1.1) provide solutions to the quantum Toda lattice in general type. We also describe explicitly two distinguished choices for the families of integration cycles. The one focused on in [4] is a non-compact one, which we think of in the general setting as running through the ‘totally negative parts’ of the mirror fibers. The other distinguished family consists of cycles that are compact. We also describe explicitly two distinguished choices for the families of integration.

The other distinguished family consists of cycles that are compact. As running through the ‘totally negative parts’ of the mirror fibers.

We refer to [29, 11] for background on algebraic groups and basic representation theory. Let $G$ be a simple, simply connected algebraic group over $\mathbb{C}$ of rank $n$ with split real form. We fix opposite Borel subgroups $B = B_-$ and $B_+$ with unipotent radicals $U_-$ and $U_+$, respectively. Assume that $B_+$ and $B_-$ are also defined over $\mathbb{R}$ and the maximal torus $T = B_+ \cap B_-$ is split. Let $W = N_G(T)/T$ denote the Weyl group.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We denote by $\mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_-, \mathfrak{u}_+, \mathfrak{h}$ the Lie algebras of $B_-, B_+, U_-, U_+$ and $T$, respectively. Let $\mathfrak{h}_\mathbb{R}$ be the real form of $\mathfrak{h}$ and $\mathfrak{g}_\mathbb{R}$ the real form of $\mathfrak{g}$.

We set $I = \{1, \ldots, n\}$ and choose $\{\alpha_i \mid i \in I\}$ to be the set of simple roots associated to the positive Borel $B_+$. We may view the $\alpha_i$ as elements of $\mathfrak{h}^*$ or as characters on $T$. The coroot associated to $\alpha_i$ is denoted $\alpha_i^\vee$ and gives a 1-parameter subgroup of $T$ denoted $s \mapsto s^{\alpha_i^\vee}$.

2. Notation and Preliminaries

We refer to [29, 11] for background on algebraic groups and basic representation theory. Let $G$ be a simple, simply connected algebraic group over $\mathbb{C}$ of rank $n$ with split real form. We fix opposite Borel subgroups $B = B_-$ and $B_+$ with unipotent radicals $U_-$ and $U_+$, respectively. Assume that $B_+$ and $B_-$ are also defined over $\mathbb{R}$ and the maximal torus $T = B_+ \cap B_-$ is split. Let $W = N_G(T)/T$ denote the Weyl group.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We denote by $\mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_-, \mathfrak{u}_+, \mathfrak{h}$ the Lie algebras of $B_-, B_+, U_-, U_+$ and $T$, respectively. Let $\mathfrak{h}_\mathbb{R}$ be the real form of $\mathfrak{h}$ and $\mathfrak{g}_\mathbb{R}$ the real form of $\mathfrak{g}$.

We set $I = \{1, \ldots, n\}$ and choose $\{\alpha_i \mid i \in I\}$ to be the set of simple roots associated to the positive Borel $B_+$. We may view the $\alpha_i$ as elements of $\mathfrak{h}^*$ or as characters on $T$. The coroot associated to $\alpha_i$ is denoted $\alpha_i^\vee$ and gives a 1-parameter subgroup of $T$ denoted $s \mapsto s^{\alpha_i^\vee}$.

Corresponding to the positive and negative simple roots we have Chevalley generators $e_i, f_i \in \mathfrak{g}$ and one parameter subgroups,

$$x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i), \quad t \in \mathbb{C},$$

in $U_+$ and $U_-$, respectively. Let

$$s_i = x_i(-1)y_i(1)x_i(-1).$$

This element represents a simple reflection $s_i$ in the Weyl group $W$.

For $w \in W$, a representative $\dot{w} \in G$ is defined by $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_m}$, where $s_{i_1}s_{i_2}\cdots s_{i_m}$ is a (any) reduced expression for $w$. The length $m$ of a reduced expression for $w$ is denoted by $\ell(w)$.

Let $\langle , \rangle$ be the $W$-invariant inner product on $\mathfrak{h}^*$ such that $\langle \alpha, \alpha \rangle = 2$ for any long root $\alpha$. We also denote the corresponding inner product on $\mathfrak{h}$ in the same way, by $\langle , \rangle$.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{Z}(\mathfrak{g})$ its center, and let

$$\gamma : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}^*]^W,$$

be the Harish-Chandra homomorphism. Define $c_2 \in \mathcal{Z}(\mathfrak{g})$ by $\gamma(c_2)(a) = \langle a, a \rangle$ for $a \in \mathfrak{h}^*$.

For any integral dominant weight $\lambda$ we have an irreducible representation $V(\lambda)$ of $G$. In each $V(\lambda)$ let us fix a highest weight vector $v_\lambda$ Then for any $v \in V(\lambda)$
and extremal weight vector \( \dot{w} \cdot v^+ \) we have the coefficient \( \langle v, \dot{w} \cdot v^+ \rangle \in \mathbb{C} \) defined by
\[
v = \langle v, \dot{w} \cdot v^+ \rangle \dot{w} \cdot v^+ + \text{other weight space summands}.
\]
We define \( v^- := \dot{w}_0 \cdot v^+ \). The most important choices for \( \lambda \) are the fundamental weights \( \omega_i \), where \( i \in I \), and \( \rho := \sum_{i \in I} \omega_i \).

Let \( B \) be the set of Borel subgroups in \( G \) with the conjugation action,
\[
g \cdot B := gBg^{-1} \quad g \in G, \ B \in B.
\]
Then we may identify \( B \) with the flag variety \( G/B \) in the usual way, by sending \( gB \) to \( g \cdot B \).

The intersection of two opposed big Bruhat cells
\[
R_{1,w_0} := (B_+ B_- \cap B_- \dot{w}_0 B_-)/B_-,
\]
which can also be written more symmetrically as
\[
R_{1,w_0} = B_+ \cdot B_- \cap B_- \cdot B_+.
\]
\( R_{1,w_0} \) is an open subset of \( G/B_- \) whose complement is an anti-canonical divisor. Note also that any line bundle on \( G/B_- \) becomes trivial when restricted to \( R_{1,w_0} \), since \( R_{1,w_0} \) lies inside the big cell \( B_+ B_- / B_- \cong \mathbb{C}^N \).

### 3. The Toda lattice

The classical Toda lattice is an integrable system with Hamiltonian
\[
H(x_i, p_i) = \frac{1}{2} \sum p_i^2 + \sum e^{x_i - x_{i+1}},
\]
which was solved by Moser [22] in the 1970’s. This integrable system was generalized to arbitrary \( G \) by Kostant [15], using phase space \( T^*(\mathfrak{t}_{\text{ad}}) \cong T_{\text{ad}} \times \mathfrak{h}^* \), the cotangent bundle to the adjoint torus \( T_{\text{ad}} \), and Hamiltonian
\[
H(t, h^*) = \frac{1}{2} \langle h^*, h^* \rangle + \sum \alpha_i(t),
\]
where the simple root \( \alpha_i \) is understood as a character on \( T_{\text{ad}} \). Kostant then solved this system using a carefully chosen embedding of the phase space into the dual space \( \mathfrak{g}^* \) of \( \mathfrak{g} \), given by
\[
(t, h^*) \mapsto F + h^* + \sum_{i \in I} \alpha_i(t) f_i^*.
\]
Here \( e_i^*, f_i^* \in \mathfrak{g}^* \) are defined to take value 1 on \( e_i \) and \( f_i \), respectively, and vanish on all other weight spaces of \( \mathfrak{g} \). Also \( F = \sum e_i^* \). Note that \( \mathfrak{h}^*, \mathfrak{b}^* \) are subspaces of \( \mathfrak{g}^* \).

The image of the phase space is in fact the translate by \( F \) of a \( \mathfrak{b}^- \)-coadjoint orbit in \( \mathfrak{b}^* \). Moreover, the Toda Hamiltonian now appears naturally as the restriction of (essentially) the Killing form, and a full set of Poisson-commuting constants of motion comes from restricting the remaining generators of \( \mathbb{C}[[\mathfrak{g}^*]]^G \).
3.1. The (Givental-)Kim presentation of $qH^*(G^\vee/B^\vee)$. In [10] Kim described the relations of the small quantum cohomology ring of $G^\vee/B^\vee$ in terms of constants of motion of the Toda lattice associated to $G$, generalizing his joint work with Givental [7] in type $\Lambda$. Formally, this goes as follows. Let $A^*$ denote the image of the embedding (3.1),

$$A^* = F + \mathfrak{h}^* + \sum (\mathbb{C} \setminus \{0\}) f_i^*,$$

and let $\phi_1, \ldots, \phi_n$ be a set of homogeneous generators for $\mathbb{C}[\mathfrak{h}^*]^W$, following Chevalley. Consider the ‘diagonalization’ map

$$\Sigma : A^* \to \mathfrak{h}^*/W.$$

That is, $\Sigma$ corresponds to

$$\mathbb{C}[\mathfrak{h}^*]^W \cong \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[A^*],$$

where the first map is given by Chevalley’s restriction theorem, and the third map comes from the inclusion $A^* \hookrightarrow \mathfrak{g}^*$. Then for the ring $qH^*(G^\vee/B^\vee)[q_1^{-1}, \ldots, q_n^{-1}]$ with quantum parameters inverted, Kim’s presentation takes the form of an isomorphism

$$qH^*(G^\vee/B^\vee, \mathbb{C})[q_1^{-1}, \ldots, q_n^{-1}] \cong \mathbb{C} \left[ A^* \times_{\mathfrak{h}^*/W} \{0\} \right] = \mathbb{C}[A^*]/(\Sigma_1, \ldots, \Sigma_n),$$

where the $\Sigma_i := \Sigma^*(\phi_i)$ are precisely Kostant’s constants of motion for the Toda lattice. In this presentation the usual Chern class generators $x_i$ of the quantum cohomology ring appear via the isomorphism $H^2(G^\vee/B^\vee) \cong (\mathfrak{h}^*)^*$ and the identification of $(\mathfrak{h}^*)^*$ with functions on the $\mathfrak{h}^*$-part of $A^*$. Explicitly, the $x_i$ and $q_i$ are identified with the coordinates on the degenerate leaf in $A^*$ defined by

$$h^* = -\sum_{i \in I} (x_1 + \ldots + x_i)\alpha_i, \quad t = \prod_{i \in I} (-q_i)^{\omega_i^\vee},$$

where $h^*$ and $t$ are as in (3.1).

3.2. Peterson’s presentation of $qH^*(G^\vee/B^\vee)$. The mirror symmetric approach to the quantum cohomology ring [25] is more closely related to an alternative presentation of $qH^*(G^\vee/B^\vee)$ due to Dale Peterson [23], see also [10, 17, 25, 27, 28]. Namely consider the closed subvariety $Y$ in $G/B_-$ defined by

$$Y = \{ gB_- \mid (g^{-1} \cdot F)|_{u_- u_-} = 0 \},$$

which is called the Peterson variety. Then in Peterson’s presentation the quantum cohomology ring appears as ring of regular functions on an open stratum of $Y$. Explicitly, in the case where we localize at the quantum parameters,

$$qH^*(G^\vee/B^\vee, \mathbb{C})[q_1^{-1}, \ldots, q_n^{-1}] \cong \mathbb{C} \left[ Y \times_{G/B_-} \mathcal{R}_{1,w_0} \right].$$

This presentation relates to Kim’s presentation by the map taking $uB_- \mapsto u^{-1} \cdot F$ for $u \in U_+ \cap B_- ^{\mathfrak{w}}_0 B_-$, which defines a morphism

$$Y \times_{G/B_-} \mathcal{R}_{1,w_0} \to A^* \times_{\mathfrak{h}^*/W} \{0\}.$$

That this is an isomorphism follows from work of Kostant, see [12].

Peterson’s presentation via his variety $Y$ has some distinct advantages over Kim’s presentation. Namely, the Peterson variety also sees the quantum cohomology rings.
of partial flag varieties $G^\vee/P^\vee$, by intersection with associated smaller Bruhat cells, and it provides presentations over the integers in general type [23].

4. The Quantum Toda lattice

The quantum Toda Hamiltonian we are interested in is the differential operator on $\mathfrak{h}$ defined by

$$H_G = \frac{1}{2} \Delta - \frac{1}{z^2} \sum_{i \in I} e^{\alpha_i},$$

where $\Delta$ be the Laplace operator associated to the $W$-invariant inner product $\langle , \rangle$ on $\mathfrak{h}$, and the $\alpha_i$ are the simple roots inside $\mathfrak{h}^*$. This can be understood as a quantization via the orbit method (for the solvable group $B_-$), which gives rise to Kostant’s ‘Whittaker model’ for the centre of the universal enveloping algebra inside $\mathcal{U}(b_-)$. The quantum Toda lattice was constructed, analysed and solved by Kostant in [13, 14].

As Kazhdan and Kostant observed [3], the operator $H_G$ can be obtained from the Laplace operator on $G$ itself by restricting to appropriate ‘Whittaker functions’. This approach is the analogue of the embedding of $T^*(T_{ad})$ into a ‘tridiagonal’ part of $\mathfrak{g}^*$ in order to realize the classical Toda Hamiltonian as coming from an invariant quadratic form. The higher Casimirs generalizing the Laplace operator provide quantum integrals of motion, in analogy with the Poisson commuting constants of motion obtained in the classical case as generators of the ring of invariants (the direct translation from quantum integrals to classical integrals of motion being given by the Harish-Chandra homomorphism).

In Section 4.1 we will review and adapt to our situation this construction of the quantum Toda lattice, as Whittaker functions will turn out to appear very naturally in the mirror model. The main goal of the present paper is to connect Kostant’s representation theoretic approach to the quantum Toda lattice with the $B$-model of $G^\vee/B^\vee$, and use it to prove the mirror conjecture from [28] for full flag varieties.

4.1. Whittaker functions and the quantum Toda Hamiltonian. Let $\chi_+ : u_+ \to \mathbb{C}$ and $\chi_- : u_- \to \mathbb{C}$ be Lie algebra homomorphisms. Then $\chi_+$ and $\chi_-$ are determined by $\chi_+(e_i) =: \chi_+^{(i)}$, and $\chi_-(f_i) =: \chi_-^{(i)}$, respectively. Namely,

$$\chi_+ = \sum_{i \in I} \chi_+^{(i)} e_i^*,
\chi_- = \sum_{i \in I} \chi_-^{(i)} f_i^*. $$

By abuse of notation we denote by $e_i^*$ also the map $U_+ \to \mathbb{C}$ which takes $u$ to the coefficient of $e_i$ in the series expansion $u = \exp(n) = 1 + \sum e_i^*(n)e_i + \ldots$, and similarly for $f_i^*$. The holomorphic characters of $U_+$ and $U_-$, respectively, corresponding to $\chi_+$ and $\chi_-$ are denoted by

$$e^{\chi_+} : U_+ \to \mathbb{C}^*,
\chi_+ : U_- \to \mathbb{C}^*. $$

If for all $i$ the coefficient $\chi_+^{(i)} \neq 0$, then $e^{\chi_+}$ and also $\chi_+$ are called non-degenerate. Analogously for $\chi_-$. 
For our purposes the ‘Whittaker functions’ will not be functions on \( G \) but functions on the universal cover of the open dense subset, \( U_+TU_- \) of \( G \). That is, they are defined on

\[
X := U_+TU_- \times \mathfrak{h}.
\]

More generally we will be interested in subsets of \( X \) of the form

\[
X_O := U_+TU_- \times O,
\]

where \( O \) is a connected open subset in \( \mathfrak{h} \) or \( \mathfrak{h}_\mathbb{R} \). We call a smooth function \( f : X_O \to \mathbb{C} \) a Whittaker function with respect to \( \chi_+ \) and \( \chi_- \) if

\[
f(u_+e^h u_-, h) = e^\chi_+ (u_+) f(e^h, h) e^{\chi_- (u_-)}.
\]

(4.2)

Clearly \( f \) on \( X_O \) is completely determined by its restriction to the Cartan component, and conversely any smooth function on \( O \) gives rise to a Whittaker function on \( X_O \).

Consider the representation of \( \mathcal{U}(\mathfrak{g}) \) on \( C^\infty(X_O) \) defined by

\[
\xi \cdot f(u_+e^h u_-, h) := \left. \frac{df}{ds} \right|_{s=0} f(\exp(-s\xi)u_+e^h u_-, h), \quad \xi \in \mathfrak{g}_\mathbb{R},
\]

where \( s \mapsto h_s \) is the lift of the torus factor of \( \exp(-s\xi)u_+e^h u_- \) to \( \mathfrak{h}_\mathbb{R} \) with \( h_0 = h \), defined for small enough \( s \).

The connection between the quantum Toda Hamiltonian and Whittaker functions can now be stated as follows, compare [3].

1. If \( f \) is a Whittaker function (for \( \chi_+ \) and \( \chi_- \)) and \( c \) is a central element in \( \mathcal{U}(\mathfrak{g}) \), then \( c \cdot f \) is again a Whittaker function. The resulting action of \( Z(\mathfrak{g}) \) on Whittaker functions on \( X_O \) factors through a representation of \( Z(\mathfrak{g}) \) on \( C^\infty(O) \), by restriction to the Cartan factor and extension.

2. Let \( c_2 \) be the degree 2 Casimir such that \( \gamma(c_2)(a) = \langle a, a \rangle \). Then the action of \( c_2 \) on \( C^\infty(O) \) as defined in (1) is given by

\[
c_2 \cdot f = e^\rho \left( \frac{1}{2} \Delta + \sum_{i \in I} \chi_+^{(i)} \chi_-^{(i)} e^{\alpha_i} \right) (e^{-\rho} f), \quad f \in C^\infty(O).
\]

Constructing solutions to the quantum Toda lattice is thereby equivalent to constructing Whittaker functions annihilated by generators \( \gamma^{-1}(\phi_i) \) of \( Z(\mathfrak{g}) \). [Explicitly, we will use \( \chi_+^{(i)} = -\frac{1}{2} \) and \( \chi_-^{(i)} = \frac{1}{2} \), to find solutions for (4.2).] The construction of appropriate Whittaker functions is done via Kostant’s theory of ‘Whittaker modules’ and their (Whittaker vector) matrix coefficients [13 14].

4.2. The \( A \)-model of \( G^\vee/B^\vee \). The quantum Toda lattice for \( G \) appears in the \( A \)-model of \( G^\vee/B^\vee \) as the natural quantization of the (Givental-)Kim presentation for the quantum cohomology. In that context it is sometimes called the quantum cohomology \( D \)-module, or the quantum differential equations of \( G^\vee/B^\vee \), see [3 10]. Officially this means that it consists of the differential operators annihilating the components of Givental’s \( J \)-function,

\[
J(t_1, \ldots, t_n) \in H^\ast(G^\vee/B^\vee, \mathbb{C}[t_1, \ldots, t_n, z^{-1}])[e^{t_1}, \ldots, e^{t_n}],
\]

which is defined in terms of descendent 2-point Gromov-Witten invariants. Therefore the Gromov-Witten theory of \( G^\vee/B^\vee \) provides a full set of solutions inside
\[ \prod_{i=1}^{n} \frac{e^{t_i} - x_i}{e^{t_i} - e^{t_n}} \] to the quantum Toda lattice for \( G \), namely by expansion of the \( J \)-function with respect to the Schubert basis.

A direct connection between Givental’s \( J \)-function, albeit in the \( G \)-equivariant case, and Whittaker modules (inside completed Verma modules) was made by A. Braverman [1].

4.3. The \( B \)-model of \( G^\vee / B^\vee \). A conjectural \( B \)-model to \( G^\vee / B^\vee \) was constructed in [28]. In this \( B \)-model there is a family (parameterized by \( \mathfrak{h} \)) of holomorphic volume forms on \( \mathcal{R}_{1,w_0} \), giving rise to ‘period integrals’ varying over the family. The quantum Toda equations are supposed to describe the variation of these periods as \( h \in \mathfrak{h} \) varies.

For \( SL_n \) the mirrors, written down in explicit coordinates by Givental [6], can be embedded into \( \mathcal{R}_{1,w_0} \) in such a way that his \( B \)-model reappears as a restriction to an open subset of our construction from [28]. The connection between Givental’s mirrors and Kostant’s Whittaker modules in type \( A \) was worked out by GKLO [4].

Details of the general construction of a \( B \)-model for \( G^\vee / B^\vee \) follow in Section 6.

In preparation we recall the definition of our basic holomorphic \( N \)-form on \( \mathcal{R}_{1,w_0} \) and a particular variant of it coming from GKLO.

5. Two regular \( N \)-forms on \( \mathcal{R}_{1,w_0} \)

The following is a special case of Proposition 7.2 in [28].

**Proposition 5.1.** Let \( \mathfrak{i} = (i_1, \ldots, i_N) \in I^N \) correspond to a reduced expression \( s_{i_1}s_{i_2}\ldots s_{i_N} \) of \( w_0 \) in \( W \). There is a unique holomorphic \( N \)-form \( \omega_i \) on \( \mathcal{R}_{1,w_0} \) such that the restriction of \( \omega_i \) to the open subset
\[ \mathcal{R}_1 = \{ x_{i_1}(a_1)\cdots x_{i_N}(a_N)B_- \mid a_i \in \mathbb{C}^* \} \]
in \( \mathcal{R}_{1,w_0} \) is given by
\[ \frac{da_1}{a_1} \wedge \frac{da_2}{a_2} \wedge \cdots \wedge \frac{da_N}{a_N}. \]

If \( \mathfrak{j} \) is another reduced expression of \( w_0 \), and is related to \( \mathfrak{i} \) by a single braid relation of length \( m \), then
\[ \omega_j = (-1)^{m+1} \omega_i. \]

In particular the form \( \omega_i \) is independent of the reduced expression \( \mathfrak{i} \) up to sign. □

We note that the subset \( \mathcal{R}_1 \) in \( \mathcal{R}_{1,w_0} \) is the open stratum in a stratification introduced by Deodhar [219].

In the following we assume a reduced expression \( \mathfrak{i} \) has been chosen, and suppress the subscript \( \mathfrak{i} \), referring to the \( N \)-form defined in Proposition 5.1 simply as \( \omega \). The variant of our form \( \omega \) defined by

\[ \omega_{GKLO}(uB_-) := \langle u \cdot v_\rho^-, v_\rho^+ \rangle \omega(uB_-), \]

is a Lie-theoretic version of the \( N \)-form introduced by [4] in the type \( A \) case.

**Proposition 5.2.** For any \( g \in G \) denote by \( \kappa_g : G/B_- \to G/B_- \) the map of left translation, \( \kappa_g(g'B_-) = gg'B_- \). Let \( \omega \) and \( \omega_{GKLO} \) be viewed as rational \( N \)-forms
on $G/B_-$. Then we have the identities

$$
\kappa^*_g\omega(G/B_-) = \frac{\langle u \cdot v^-, v^+ \rangle}{\langle gu \cdot v^-, v^+ \rangle} \omega(G/B_-),
$$

(5.2)

$$
\kappa^*_g\omega_{GKLO}(uB_-) = \frac{1}{\langle gu \cdot v^-, v^+ \rangle} \omega_{GKLO}(uB_-).
$$

(5.3)

Remark 5.3. Proposition 5.2 implies the following identities.

1. For $h \in \mathfrak{h}$ and Chevalley generators $f_i, e_i$ the volume form $\omega$ transforms according to

$$
\kappa^*_{\exp(h)}\omega = \omega,
$$

$$
\kappa^*_{y_i(s)}\omega = (1 + e_i(u)s)^{-1} \omega,
$$

$$
\kappa^*_{x_i(s)}\omega = \left(1 + \frac{\langle u \cdot v^-, f_i \cdot v^+ \rangle}{\langle u \cdot v^-, v^+ \rangle} s\right)^{-1} \omega.
$$

(5.4)

2. The alternative volume form $\omega_{GKLO}$ is $U_+\text{-invariant}$. In particular it is well-defined on the entire big cell $U_+B_-/B_-$. However it is not $T\text{-invariant}$, satisfying instead

$$
\kappa^*_{\exp(h)}\omega_{GKLO} = e^{2\rho(h)} \omega_{GKLO}.
$$

Proof. It is easy to see that the formulas (5.2) and (5.3) are equivalent to one another. Now suppose $g_1, g_2 \in G$ are such that the identity (5.3) holds. We claim that

$$
\kappa^*_{g_2} \left(\kappa^*_{g_1} \omega_{GKLO}\right)(uB_-) = \frac{1}{\langle g_1 g_2 u \cdot v^-, v^+ \rangle} \omega_{GKLO}(uB_-).
$$

(5.4)

If so, then the formula (5.3) follows also for $g = g_1 g_2$, and therefore we need to check it (or equivalently (5.2)) only on a generating subset of $G$.

To prove the claim compute

$$
\kappa^*_{g_2} \left(\kappa^*_{g_1} \omega_{GKLO}\right)(uB_-) = \kappa^*_{g_2} \left(\frac{1}{\langle g_1 u \cdot v^-, v^+ \rangle} \omega_{GKLO}\right)(uB_-) =
$$

$$
\kappa^*_{g_2} \left(uB_- \mapsto \frac{1}{\langle g_1 u \cdot v^-, v^+ \rangle} \right) \frac{1}{\langle g_2 u \cdot v^-, v^+ \rangle} \omega_{GKLO}(uB_-).
$$

(5.5)

To apply $\kappa^*_{g_2}$ above we need a factorization

$$
g_2u = u_{g_2} b_{g_2}
$$

where $u_{g_2} \in U_+$ and $b_{g_2} \in B_-$. Then

$$
\kappa^*_{g_2} \left(uB_- \mapsto \frac{1}{\langle g_1 u \cdot v^-, v^+ \rangle} \right)(uB_-) = \frac{1}{\langle g_1 u_{g_2} \cdot v^-, v^+ \rangle^2}
$$

$$
= \frac{1}{\langle g_1 g_2 u b_{g_2}^{-1} \cdot v^-, v^+ \rangle^2} = \frac{1}{\langle g_1 g_2 u \cdot v^-, v^+ \rangle^2} \frac{1}{\rho(b_{g_2})^2}
$$

$$
= \frac{1}{\langle g_2 u \cdot v^-, v^+ \rangle^2} \frac{1}{\rho(b_{g_2})^2} = \frac{1}{\langle g_2 u \cdot v^-, v^+ \rangle^2} \rho(b_{g_2})^2.
$$

(5.6)
The claim now follows from the combination of (5.5) and (5.6). To finish the proof it suffices to check the identities from Remark 5.3. Let us choose a reduced expression $i$ of $w_0$ such that $\omega$ restricted to

$$\mathcal{R}_i = \{ x_{i_1}(a_1) \ldots x_{i_N}(a_N)B_- \mid a_i \neq 0 \}$$

takes the form

$$\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_N}{a_N}.$$ 

Note that it is clear from the definition that it is invariant under translation by elements of $T$. In fact $\mathcal{R}_i$ is itself a much bigger torus, $\mathcal{R}_i \cong (\mathbb{C}^*)^N$, on which $\omega$ is the standard invariant $N$-form. We are therefore left with two kinds of transformations to consider, $\kappa_{y_i(s)}$ and $\kappa_{x_i(s)}$.

1. To work out the coordinate transformation corresponding to $\kappa_{y_i(s)}$ we note that $y_i(s)x_j(a) = x_j(a)y_i(s)$ for $i \neq j$ and

$$y_i(s)x_j(a) = x_j(a) \left( \frac{a}{1 + as} \right) \left( \frac{1}{1 + as} \right)^{\alpha^\vee_j} y_i(s) \left( \frac{s}{1 + as} \right).$$

Suppose $1 \leq l_1 < \ldots < l_m \leq N$ are the indices for which $i_{l_j} = i$. Applying the above identities repeatedly we obtain

$$y_i(s)x_{i_1}(a_1) \ldots x_{i_N}(a_N) = x_{i_1}(a_1) \ldots x_{i_{l_1}}(a_{l_1}) \left( \frac{a_{l_1}'}{a_{l_1}} \right) x_{i_{l_1}+1}(a_{l_1+1}) \ldots$$

$$\ldots x_{i_{l_2}}(a_{l_2}') \left( \frac{a_{l_2}'}{a_{l_2}} \right) \ldots x_{i_{l_m}}(a_{l_m}') \left( \frac{a_{l_m}'}{a_{l_m}} \right) x_{i_N}(a_N)y_i(s) \left( \frac{s}{1 + (a_{l_1} + \ldots + a_{l_m})s} \right),$$

where

$$a_{l_j}' = \frac{a_{l_j}(1 + (a_{l_1} + \ldots + a_{l_{j-1}})s)}{1 + (a_{l_1} + \ldots + a_{l_j}s)}.$$ 

Since $\omega$ restricted to $\mathcal{R}_i$ is invariant under the action of the 'big' torus, that is $\mathcal{R}_i$ itself, we may disregard the factors $\left( \frac{a_{l_j}'}{a_{l_j}} \right)^{\alpha^\vee_i}$. Thus $\omega$ transforms under $\kappa_{y_i(s)}$ as under the coordinate transformation

$$(a_1, \ldots, a_N) \mapsto (a_1', \ldots, a_N')$$

where $a_j' := a_j$ if $j \notin \{l_1, \ldots, l_m\}$. Since this coordinate transformation is lower triangular its Jacobian is easily computed to be

$$\det \left( \frac{\partial a_j'}{\partial a_k} \right)_{j,k} = \prod_{j=1}^m \left( \frac{1 + (a_{l_1} + \ldots + a_{l_{j-1}})s}{1 + (a_{l_1} + \ldots + a_{l_j}s)} \right)^2 = \left( \frac{1}{1 + (a_{l_1} + \ldots + a_{l_m})s} \right)^2.$$ 

Note also the telescopic product identity

$$\frac{1}{\prod_{j=1}^N a_j'} = (1 + (a_{l_1} + \ldots + a_{l_m})s) \frac{1}{\prod_{j=1}^N a_j}.$$
Therefore we obtain
\[
\frac{da'_1 \wedge da'_2 \wedge \cdots \wedge da'_N}{a'_1} = \frac{1}{\prod_{j=1}^{N} a'_j} \left( \frac{1}{1 + (a_{l_1} + \ldots + a_{l_m})s} \right)^2 \frac{da_1 \wedge da_2 \wedge \cdots \wedge da_N}{a_1} = \frac{1}{1 + (a_{l_1} + \ldots + a_{l_m})s} \omega.
\]
Clearly \(a_{l_1} + \ldots + a_{l_m}\) is nothing other than \(e^*_i(u)\) for \(u = x_{i_1}(a_1) \ldots x_{i_N}(a_N)\), confirming the identity from Remark 5.3.

(2) To apply \(\kappa^*_{x_i(s)}\) to \(\omega\) let us assume without loss of generality that the reduced expression \(i\) of \(w_0\) begins with \(i_1 = i\). Then \(\kappa^*_{x_i(s)}\) corresponds to the coordinate transformation
\[
(a_1, a_2, \ldots, a_N) \mapsto (a_1 + s, a_2, \ldots, a_N),
\]
and therefore
\[
\kappa^*_{x_i(s)} \omega = \frac{a_1}{a_1 + s} \omega.
\]
It is easy to see that for \(u = x_{i_1}(a_1) \ldots x_{i_N}(a_N)\) we have indeed
\[
\left(1 + \frac{\langle u \cdot v_p^-, f_i \cdot v_p^+ \rangle}{\langle u \cdot v_p^-, v_p^+ \rangle} s \right)^{-1} = \left(1 + \frac{1}{a_1 s} \right)^{-1} = \frac{a_1}{a_1 + s}.
\]

6. The Mirror Family to \(G^\vee/B^\vee\)

In this section we will review the ingredients of mirror symmetry for the full flag variety \(G^\vee/B^\vee\) following [28]. We will consider from the start the variant of the mirror family over \(\mathfrak{h}\), rather than the family over \(T\). Also, for convenience, we have chosen \(G\) simply connected, while in [28] the group on the mirror symmetric side was adjoint. The mirror family over \(\mathfrak{h}\) is however unaffected by this change. Finally, we replace the mirror family from [28] by its translate by \(\dot{w}_0\), which will turn out to be more natural in connection with the Whittaker functions we are constructing.

6.1. Let
\[
Z := \{ (g, h) \in U_+TU_- \times \mathfrak{h} \mid g \cdot B_+ = B_- \}.
\]
\(Z\) is viewed as a family of varieties via the map \(\text{pr}_2 : Z \to \mathfrak{h}\) projecting onto the second factor. For \(h \in \mathfrak{h}\) let us write
\[
Z_h := U_+ e^h U_- \cap B_- \dot{w}_0 = \{ g \in U_+ e^h U_- \mid g \cdot B_+ = B_- \}.
\]
\(Z_h\) may be identified with the fiber \(\text{pr}_2^{-1}(h)\) in \(Z\). We record the following basic properties of the family \(Z\).

(1) Fix \(h \in \mathfrak{h}\). Then the fiber \(Z_h\) is isomorphic to the intersection of opposite big cells, \(R_{1,w_0}\), via the map
\[
\beta_h : Z_h \longrightarrow R_{1,w_0}
\]
\(ue^h u^{-1} \mapsto u^{-1} \cdot B_- = e^h u^{-1} \cdot B_+\).

In particular \(Z_h\) is smooth of dimension \(N = \dim_\mathbb{C}(G/B)\).
The isomorphisms from (1) can be combined to give a trivialization
\[(6.2) \quad \beta : Z \xrightarrow{\sim} \mathcal{R}_{1,w_0} \times \mathfrak{h},\]
where \((ue^h \bar{u}^{-1}, h) \mapsto (u^{-1} \cdot B_-, h)\).

Note that in particular
\[Z_0 = U_+ U_- \cap B_- w_0 = \{u \bar{u}^{-1} \in U_+ U_- \mid u^{-1} \cdot B_- = \bar{u}^{-1} \cdot B_+\},\]
and we have the isomorphism
\[(6.3) \quad \beta_0 : Z_0 \xrightarrow{\sim} \mathcal{R}_{1,w_0}, \quad u \bar{u}^{-1} \mapsto u^{-1} \cdot B_- = \bar{u}^{-1} \cdot B_+\).

6.2. We define a function \(\mathcal{F}\) on \(Z\) (the ‘superpotential’) as follows.
\[(6.4) \quad \mathcal{F}(ue^h \bar{u}^{-1}, h; z) = \frac{1}{z} \left(\sum_{i \in I} e_i^*(u) + \sum_{i \in I} f_i^*(\bar{u})\right),\]
where we may think of \(z\) as a positive parameter \(z \in \mathbb{R}_{>0}\). The restriction of \(\mathcal{F}\) to a fiber \(Z_h\) of the mirror family (or rather \(Z_h \times \mathbb{R}_{>0}\)) is denoted by \(\mathcal{F}_h\).

Remark 6.1. Note that for fixed \(z\) the function \(e^{\mathcal{F}(\cdot ; z)} : Z \rightarrow \mathbb{C}\) is the restriction to \(Z\) of the ‘trivial’ Whittaker function,
\[\mathcal{W}_0 : U_+ TU_- \times \mathfrak{h} \rightarrow \mathbb{C}, \quad (u_+ e^h u_-, h) \rightarrow e^{\chi_+(u_+)} e^{\chi_-(u_-)},\]
corresponding to characters \(\chi_+, \chi_-\) defined by \(\chi_+(e_i) = \frac{1}{2}\) and \(\chi_-(f_i) = -\frac{1}{2}\), see also Section 6.1.

6.3. Let \(i\) be a reduced expression for \(w_0\) and \(\omega_i\) the \(N\)-form on \(\mathcal{R}_{1,w_0}\) defined in Proposition 5.1. Denote by \(\omega_Z\) or \(\omega_{i,Z}\) the pullback of \(\omega_i\) to \(Z\) along the map
\[pr_1 \circ \beta : Z \rightarrow \mathcal{R}_{1,w_0}, \quad (ue^h \bar{u}^{-1}, h) \mapsto u^{-1} \cdot B_- ,\]
and write \(\omega_h\) or \(\omega_{i,h}\) for the pullback of \(\omega_Z\) to the fiber \(Z_h\).

Note that the non-canonical choice of a reduced expression \(i\) will affect at most the sign of \(\omega_Z\). In some special cases we will consider later on, this sign will cancel out against a sign coming from the choice of orientation for the integration cycle, making for a canonical solution to the quantum Toda lattice.

The mirror datum to \(G^\vee / B^\vee\) is now made up of the three ingredients introduced above: the family \(Z \rightarrow \mathfrak{h}\), the regular function \(\mathcal{F}\), and the holomorphic \(N\)-form \(\omega_Z\) on \(Z\). We may denote it compactly as \((Z, \omega_Z, \mathcal{F})\).

6.4. A translation action on \(Z\). We can transfer the natural action of the additive group of \(\mathfrak{h}\) on the product \(\mathfrak{h} \times \mathcal{R}_{1,w_0}\), given by \(h \cdot (h', g \cdot B_-) := (h' + h, g \cdot B_-)\), to \(Z\) using the trivialization \(\beta\) from (6.2).

Lemma 6.2. Let \(h \in \mathfrak{h}\) and \((h', g) \in Z\). The translation action of \(h\) on \((h', g)\) takes the form
\[h \cdot (h', g) = (h + h', ge^h)\].
Moreover β

7.1. The compact cycles \( R \) and work in Proof.

We have that \( 12 KONSTANZE RIETSCH \)

Lemma 7.2. Note that \( \Gamma(1) \)

|N

associated lies inside \( \Gamma(1) \)

\( N \)

We may identify \( \Gamma(1) \)

on each \( S \) factor. The resulting family of integration contours is denoted \( \Gamma_{i,h}^{(1)} \), or just \( \Gamma^{(1)} \) if the choice of reduced expression is clear or irrelevant.

Lemma 7.2. The cycle \( [\Gamma_{i,h}^{(1)}] \) defines a nonzero element of \( H_N(Z_h, \mathbb{Z}) \). Moreover, if two reduced expressions \( i \) and \( j \) are related by a braid relation of length \( m \) then

\[
[\Gamma_{i,h}^{(1)}] = (-1)^{m+1} [\Gamma_{j,h}^{(1)}].
\]

In particular, the cycle \( [\Gamma_{i,h}^{(1)}] \) is independent of the reduced expression \( i \) up to sign.

Proof. We may identify \( Z_h \) with \( R_{1,u_0} \) by \( \beta_h \), and \( \Gamma_{i,h}^{(1)} \) with

\[
\Gamma_i := \{ x_{i_1} \cdots x_{i_N} (a_{i_1}) \cdots a_{i_N} \cdot B_\cdot \mid \|a_j\| = 1 \text{ for all } j \}.
\]

and work in \( R_{1,u_0} \). By Section 5 we have a well defined \( N \)-form \( \omega \) on \( R_{1,u_0} \). It follows from

\[
\int_{[\Gamma_i]} \omega = (2\pi i)^N
\]

that \( [\Gamma_i] \) defines a nontrivial homology class in \( H_N(R_{1,u_0}, \mathbb{Z}) \).

Now \( \Gamma_i \) lies in the open coordinate patch \( R_i \) (see Proposition 5.1) while \( \Gamma_j \) lies inside \( R_j \). Note that \( R_j \) is isomorphic to \( (\mathbb{C}^*)^N \) and therefore \( [\Gamma_i] \) generates \( H_N(R_i, \mathbb{Z}) \). Rescaling the \( S^1 \) factors in \( \Gamma_j \) we may replace this cycle by one that is homologous but lies in the intersection \( R_i \cap R_j \), and therefore also defines a cycle \( [\Gamma_j] \) in \( H_N(R_i, \mathbb{Z}) \). It follows that \( [\Gamma_j] \) must be a multiple of \( [\Gamma_i] \) in \( H_N(R_i, \mathbb{Z}) \). Hence \( [\Gamma_j] \) is a multiple of \( [\Gamma_i] \) in \( H_N(R_{1,u_0}, \mathbb{Z}) \). However we have by Proposition 5.1 that

\[
\omega = (-1)^{m+1} \omega_j.
\]

Therefore

\[
\int_{[\Gamma_j]} \omega = (-1)^{m+1} \int_{[\Gamma_j]} \omega_j = (-1)^{m+1} (2\pi i)^N,
\]

which implies \( [\Gamma_j] = (-1)^{m+1} [\Gamma_i] \). □
7.2. Total positivity and the non-compact integration cycles $\Gamma^{(w_0)}$.

**Definition 7.3** (Lusztig [18]). Inside the real points of a (split) real algebraic group $G$ and each of its related varieties $X = T, U_+, U_-$ and $G/B_-$, there is a semi-algebraic subset $X^{>0}$ called the totally positive part, which is defined as follows.

The totally positive part of $T$ is the precisely the subset of $T$ for which all characters take values in $\mathbb{R}_{>0}$. Equivalently, if we consider the real points of $T$ (isomorphic to $(\mathbb{R}^*)^n$), then $T^{>0}$ is the connected component of the identity.

For $U_+$ and $U_-$ the totally positive parts are given by
\[
U_+^{>0} := \{x_{i_1}(a_1)\ldots x_{i_N}(a_N) \mid a_i \in \mathbb{R}_{>0}\},
\]
\[
U_-^{>0} := \{y_{i_1}(a_1)\ldots y_{i_N}(a_N) \mid a_i \in \mathbb{R}_{>0}\},
\]
where $i = (i_1, \ldots, i_N)$ is a (any) reduced expression of $w_0$. And one puts these together to build
\[
G^{>0} := U_+^{>0} U_-^{>0} = U_-^{>0} T_0 U_+^{>0},
\]
where a proof of the last identity may be found in [18].

The totally positive part of the flag variety $B$ is
\[
B^{>0} := U_+^{>0} \cdot B_- = U_-^{>0} \cdot B_+.
\]
Again the last identity is proved in [18]. Note that the totally positive part of $B$ lies in $\mathcal{R}_{1,w_0}$. We may also denote it by $\mathcal{R}^{>0}_{1,w_0}$.

These definitions generalize the classical notion of total positivity inside $GL_n$ developed by Polya and Schoenberg among others. The introduction of a theory of total positivity to flag varieties is due to Lusztig, even in type $A$.

7.2.1. Let $h \in \mathfrak{h}_\mathbb{R}$, and note that $\exp(\mathfrak{h}_\mathbb{R}) = T^{>0}$. We can now use the trivialization $\beta$ to pull back the totally positive part in $\mathcal{R}_{1,w_0}$ to the mirror fibers $Z_h$. We will refrain from calling this the totally positive part of $Z_h$, which instead will be defined differently in Section 11.

Let
\[
\Gamma_h^{(w_0)} := \{g \in Z_h \mid \beta_h(g) \in B^{>0}\} = \{u e^h \bar{u}^{-1} \in Z_h \mid u^{-1} \in U_+^{>0}\} = \{u e^h \bar{u}^{-1} \in Z_h \mid \bar{u}^{-1} \in U_-^{>0}\},
\]
where the final equality uses that $e^h \in T^{>0}$. We will see in Section 11 that $\mathcal{F}_h$ always has a critical point in $\Gamma_h^{(w_0)}$.

For any choice of reduced expression $i$ of $w_0$ we obtain a parameterization,
\[
\mathbb{R}^N_{>0} \xrightarrow{\sim} \Gamma_h^{(w_0)}(a_1, \ldots, a_N) \mapsto \beta_h^{-1}(x_{i_1}(a_1)\ldots x_{i_N}(a_N)) \cdot B_-,
\]
which gives rise to an orientation on $\Gamma_h^{(w_0)}$. We denote by $[\Gamma_h^{(w_0)}]$ the oriented real (semi-algebraic) manifold inside $Z_h$ obtained in this way, and by $\Gamma^{(w_0)} = \Gamma_1^{(w_0)} = ([\Gamma_h^{(w_0)}])_{h \in \mathfrak{h}_\mathbb{R}}$, the corresponding family over $\mathfrak{h}_\mathbb{R}$.

By Proposition 5.1, if $j$ is a reduced expression obtained from $i$ by a braid relation of length $m$, then the corresponding (rational subtraction-free) coordinate transformation $(a_1, \ldots, a_N) \mapsto (a'_1, \ldots, a'_N)$ reverses orientations precisely if $m$ is even. Therefore the orientation of $[\Gamma_{i,h}^{(w_0)}]$ depends on $i$ in the same way.
7.3. Conditions on more general families. Let $\mathcal{O}$ be a connected open subset of $\mathfrak{h}_{\mathbb{R}}$ or $\mathfrak{h}$. Let $\Gamma = ([\Gamma_h])_{h \in \mathcal{O}}$ be a continuous family of real, possibly non-compact, $N$-dimensional semi-algebraic cycles $[\Gamma_h]$ in $Z_h$, for which $\text{Re}(\mathcal{F}_h) \to -\infty$ in any non-compact direction of $\Gamma_h$, for any fixed $z > 0$. In this case

$$
\int_{[\Gamma_h]} e^{\mathcal{F}} \omega_h
$$

is well-defined and absolutely convergent, therefore differentiable in $h$.

We will also assume that $0 \in \mathcal{O}$, as we want $[\Gamma_h]$ to be equivalent to the translate $[h \cdot \Gamma_0]$ defined using Section 6.4 for small $h$. We remark that the formal setting for the cycles $[\Gamma_h]$ should be the $N$-th rapid decay homology group associated to an irregular rank one connection, $\nabla(f) = df - f \, dF_h(z)$, defined by $\mathcal{F}_h$ on $\mathcal{R}_{1,w_0}$. This homology group was defined by Bloch and Esnault in dimension 1 and more recently generalized by Hien [8] to arbitrary dimension using work of T. Mochizuki [21]. The case of exponential connections is also treated in the earlier work of Hien and Roucairol [9].

We note that in certain cases we can start with a cycle $[\Gamma_0]$ in $Z_0$ and extend it automatically to a family over all of $\mathfrak{h}_{\mathbb{R}}$ with the above decay properties. Namely, suppose $[\Gamma_0]$ has the property that in any non-compact direction of $\Gamma_0$ we have $\text{Re}(\mathcal{F}_0) \to -\infty$, and all individual summands $\text{Re}(f_i^*(\bar{u}))$ and $\text{Re}(e_i^*(u))$ are bounded from above. This means that no summand of $\text{Re}(\mathcal{F}_0)$ can tend to $+\infty$. In that case we can use the translation action to define

$$
\Gamma_h := h \cdot \Gamma_0 = \{ u\bar{u}^{-1}e^h \in Z_h \mid u\bar{u}^{-1} \in \Gamma_0 \},
$$

for $h \in \mathfrak{h}_{\mathbb{R}}$. The claim that $[\Gamma_h]$ again has the same decay behavior for $\text{Re}(\mathcal{F}_h)$ as $[\Gamma_0]$ for $\text{Re}(\mathcal{F}_0)$ follows from the simple observation that

$$
\mathcal{F}_h(u\bar{u}^{-1}e^h) = \frac{1}{z} \left( \sum_{i \in I} e_i^*(u) + \sum_{i \in I} f_i^*(e^{-h}\bar{u}e^h) \right) = \frac{1}{z} \left( \sum_{i \in I} e_i^*(u) + \sum_{i \in I} e_i^{\alpha_i(h)} f_i^*(\bar{u}) \right),
$$

and that $e_i^{\alpha_i(h)} > 0$ (since $h \in \mathfrak{h}_{\mathbb{R}}$).

Remark 7.4. The families of cycles $\Gamma(1)$ and $\Gamma(w_0)$ are obtained from $[\Gamma_0^{(1)}]$ and $[\Gamma_0^{(w_0)}]$ in this way.

For the proof of Proposition 10.3 we require another technical condition on $\Gamma_0$ (identified with its image in $G/B_\beta$ under $\beta_0$). Namely for small $s > 0$ we assume that for the intersection with $\mathcal{R}_{1,w_0}$ of the closures of $\exp(s f_i) \cdot \Gamma_0$ and $\exp(s e_i) \cdot \Gamma_0$, respectively, the real part of $\mathcal{F}_0$ tends to $-\infty$ in any unbounded direction. This is satisfied for the two specific families introduced above. The $e_i$ and $f_i$ could also be replaced by another set of generators of $\mathfrak{g}$, to make the condition weaker.

Another approach to constructing integration cycles, due to Givental, is recalled in Section 7.3.

8. Statement of the main theorem

Let $\mathcal{O}$ be a connected open subset either of $\mathfrak{h}_{\mathbb{R}}$ or of $\mathfrak{h}$ that contains 0. Let $\Gamma = ([\Gamma_h])_{h \in \mathcal{O}}$ be a family of real, possibly non-compact, $N$-dimensional semi-algebraic cycles in $Z_h$, as described in Section 7.3.
We fix a reduced expression $i$ of $w_0$ and let $\omega_h = \omega_{i,h}$ be the $N$-form on $Z_h$ defined in Section 6.3. Then let
\begin{align}
S_{\Gamma}(h,z) := \int_{[\Gamma_{h}]} e^{F} \omega_h,
\end{align}
for $h \in \mathcal{O}$ and $z \in \mathbb{R}_{>0}$. For example
\begin{align}
S_{\Gamma}^{(1)}(h,z) = \int_{[\Gamma_{1,k}^{(1)}]} e^{F} \omega_{1,h} \quad \text{and} \quad S_{\Gamma}^{(w_0)}(h,z) = \int_{[\Gamma_{1,k}^{(w_0)}]} e^{F} \omega_{1,h},
\end{align}
using the integration cycles defined in Section 7.

In general, the sign of $S_{\Gamma}(h,z)$ depends on the reduced expression $i$ used to define $\omega_h$. However the special solutions $S_{\Gamma}^{(1)}$ and $S_{\Gamma}^{(w_0)}$, where we have chosen the orientation of the integration cycle concurrently, are independent of $i$. The integration cycles $\Gamma^{(w_0)}$ are the ones focused on in type A by GKLO [4]. Note also that $S_{\Gamma}^{(1)}(h,z)$ extends to a global holomorphic function on $\mathfrak{h}_\mathbb{C} \times \mathbb{C}^*$.

The following result was conjectured in [28].

**Theorem 8.1.** The integrals (8.1) are solutions to the quantum Toda lattice. In particular they are annihilated by the quantum Toda Hamiltonian (8.1).

9. A $\mathcal{U}(\mathfrak{g})$-module structure on $\mathcal{H}(\mathfrak{hol}(Z_0))$.

We consider the restriction of the complex line bundle $L_{-\rho} = G \times_{B_-} \mathbb{C}_{-\rho}$ to the intersection of opposite big cells $\mathcal{R}_{1,w_0}$. Since $\mathcal{R}_{1,w_0}$ is open in $G/B_-$, the representation of $G$ on the space of sections induces a representation of $\mathfrak{g}$ on $\Gamma_{\mathfrak{hol}}(L_{-\rho}|\mathcal{R}_{1,w_0})$, the space of holomorphic sections of the restricted line bundle. Moreover, since $\mathcal{R}_{1,w_0}$ is preserved by $T$, we have a corresponding representation of $T$ on $\Gamma_{\mathfrak{hol}}(L_{-\rho}|\mathcal{R}_{1,w_0})$ which is compatible with the $\mathfrak{g}$-module structure. Explicitly, let us set
\begin{align}
M_{-\rho} := \Gamma_{\mathfrak{hol}}(L_{-\rho}|\mathcal{R}_{1,w_0})
\end{align}
\begin{align}
= \{ \tilde{f} : (U_+ \cap B_- \cap \mathfrak{h}_0 B_-) \to \mathbb{C} \mid \tilde{f} \text{ holomorphic}, \tilde{f}(gb) = \tilde{f}(g)\rho(b), \ \forall b \in B_- \}.
\end{align}

The actions of $\mathfrak{g}$ and $T$ on $M_{-\rho}$ are given by
\begin{align}
(X \cdot \tilde{f})(g) &:= \frac{d}{ds} \bigg|_{s=0} \tilde{f}(\exp(-sX)g), \\
t \cdot \tilde{f}(g) &:= \tilde{f}(t^{-1}g),
\end{align}
for $X \in \mathfrak{g}$ and $t \in T$. Note that the $\mathcal{U}(\mathfrak{g})$-module $M_{-\rho}$ has zero infinitesimal character, see [13] Proposition 5.1.

The restriction of $\tilde{f} \in M_{-\rho}$ to $U_+ \cap B_- \cap \mathfrak{h}_0 B_-$ defines an isomorphism,
\begin{align}
M_{-\rho} \xrightarrow{\sim} \mathcal{H}(\mathfrak{hol}(\mathcal{R}_{1,w_0})),
\end{align}
\begin{align}
\tilde{f} \mapsto \left( \frac{du \cdot B_- \mapsto \tilde{f}(u)}{u \in U_+ \cap B_- \cap \mathfrak{h}_0 B_-} \right),
\end{align}
identifying $M_{-\rho}$ with holomorphic functions on $\mathcal{R}_{1,w_0}$. The actions of $\mathfrak{g}$ and $T$ on $M_{-\rho}$ thereby carry over to representations on $\mathcal{H}(\mathfrak{hol}(\mathcal{R}_{1,w_0}))$.

Consider now the zero fiber, $Z_0$, of our mirror family. We obtain a $\mathfrak{g}$-module and compatible $T$-module structure on $\mathcal{H}(\mathfrak{hol}(Z_0))$ via the isomorphism $\beta_0 : Z_0 \to \mathcal{R}_{1,w_0}$ from (6.3). By construction, this representation of $\mathfrak{g}$ on $\mathcal{H}(\mathfrak{hol}(Z_0))$ extends to a representation of $\mathcal{U}(\mathfrak{g})$ with zero infinitesimal character.
9.1. A $u_+-$Whittaker vector. Let $\chi : u_+ \to \mathbb{C}$ be the 1-dimensional representation defined by $\chi(e_i) = \frac{1}{z}$ for all $i \in I$, and consider the corresponding holomorphic character $e^x$ on $U_+$. Let $\psi_+ \in \mathcal{H}ol(Z_0)$ be defined by

$$\psi_+(u \bar{u}^{-1}) := e^{\chi(u)}$$

and let $\bar{\psi}_+ \in M_{-\rho}$ denote the section of $L_{-\rho}|_{Z_0}$ associated to $\psi_+$.

**Lemma 9.1.** $\psi_+$ is a $u_+-$Whittaker vector in $\mathcal{H}ol(Z_0)$ with character $\chi$. That is,

$$e_i \cdot \psi_+ = \frac{1}{z} \psi_+, \quad \text{for all } i \in I.$$

**Proof.** From the definitions we see

$$(e_j \cdot \psi_+)(u \bar{u}^{-1}) = (e_j \cdot \bar{\psi}_+)(u^{-1}) = \frac{d}{ds} \Big|_{s=0} \bar{\psi}_+(\exp(-se_j)u^{-1})$$

$$= \frac{d}{ds} \Big|_{s=0} \exp \left( \frac{1}{z} \sum_{i \in I} e_i^*(u \exp(se_j)) \right) = \frac{d}{ds} \Big|_{s=0} \exp \left( \frac{1}{z} \sum_{i \in I} e_i^*(u) + s \right)$$

$$= \frac{1}{z} \psi_+(u \bar{u}^{-1}).$$

Definition 9.2 ([14]). A $U(g)$-module with a cyclic Whittaker vector is called a Whittaker module.

Definition 9.3. Let $V_+$ be the $U(g)$-submodule of $\mathcal{H}ol(Z_0)$ generated by $\psi_+$. By [14, Theorem 3.6.2] $V_+$ is an irreducible Whittaker module, see also [20]. Note that $V_+$ no longer has an action of $T$.

9.2. A $u_--$Whittaker vector. Let $\bar{\chi} : u_- \to \mathbb{C}$ be the 1-dimensional representation defined by $\bar{\chi}(f_i) = \frac{1}{z}$ for all $i \in I$, and consider the corresponding holomorphic character $e^{\bar{x}}$ on $U_-$. Let $\psi_- \in \mathcal{H}ol(Z_0)$ be defined by

$$\psi_-(u \bar{u}^{-1}) := \frac{1}{(u^{-1} \cdot v^-_\rho, v^+_\rho)} e^{\bar{\chi}(\bar{u})}.$$

The functions $\psi_+$ and $\psi_-$ are Lie-theoretic analogues of the functions introduced in terms of Givental coordinates in [4].

**Lemma 9.4.** $\psi_- \in \mathcal{H}ol(Z_0)$ is a $u_--$Whittaker vector with character $\bar{\chi}$. That is,

$$f_i \cdot \psi_- = \frac{1}{z} \psi_-.$$

In the following lemma we collect some identities used in the proof of the Lemma 9.4.

**Lemma 9.5.** Suppose $u \in U_+, \bar{u} \in U_-$ are given such that $u \bar{u}^{-1}$ lies in $Z_0$. Consider a fixed Chevalley generator $f_i$.

(1) For $s \in \mathbb{C}$ such that $1 + se_i^*(u) \neq 0$ we have the identity

$$uy_i(s) = b_{(s)}u(s),$$
where \( b_{(s)} \in B_- \) and \( u_{(s)} \in U_+ \) are given explicitly by
\[
\begin{align*}
b_{(s)} &= (1 + se_i^+ (u))^\alpha \gamma y_i (s(1 + se_i^+ (u))) \\
u_{(s)} &= y_i (-s(1 + se_i^+ (u))) \left( \frac{1}{1 + se_i^+ (u)} \right)^{\alpha \gamma} u y_i (s).
\end{align*}
\]

(2) Let \( s \) be as above, and define \( \bar{u}_{(s)} \) by
\[
\bar{u}_{(s)}^{-1} \cdot B^+ = u_{(s)}^{-1} \cdot B^-.
\]

Then \( \bar{u}_{(s)} = \bar{u} y_i (s) \). The element \( u_{(s)} \bar{u}_{(s)}^{-1} \) in \( Z_0 \) is given by
\[
y_i (-s(1 + se_i^+ (u))) \left( \frac{1}{1 + se_i^+ (u)} \right)^{\alpha \gamma} u \bar{u}^{-1}.
\]

The proof of the Lemma is straightforward.

**Proof of Lemma 9.6** To analyze the action of \( f_i \) on \( \psi_- \) we use Lemma 9.5 with all its notations. Let \( \tilde{\psi}_- \) denote the element of \( M_{\rho} \) associated to \( \psi_- \).
\[
(f_i \cdot \tilde{\psi}_-)(u \bar{u}^{-1}) = (f_i \cdot \tilde{\psi}_-)(u^{-1}) = \frac{d}{ds} \bigg|_{s=0} \tilde{\psi}_- (\exp(-sf_i) u^{-1})
\]
\[
= \frac{d}{ds} \bigg|_{s=0} \tilde{\psi}_- (u_{(s)}^{-1} \bar{u}_{(s)}^{-1}) = \frac{d}{ds} \bigg|_{s=0} \tilde{\psi}_- (u_{(s)}^{-1}) \rho (b_{(s)})^{-1} = \frac{d}{ds} \bigg|_{s=0} \tilde{\psi}_- (u_{(s)}^{-1}) \frac{1}{1 + se_i^+ (u)}
\]
\[
\frac{1}{u_{(s)}^{-1} \cdot v_{\rho}^+} e^{\tilde{\chi}(u_{(s)})} = \frac{1 + se_i^+ (u)}{\langle u_{(s)}^{-1} \cdot v_{\rho}^+ \rangle} e^{\tilde{\chi}(u_{(s)})} e^{\frac{1}{2} s}
\]
using the formulas for \( u_{(s)} \) and \( \bar{u}_{(s)} \) from Lemma 9.5. Therefore we get
\[
(f_i \cdot \tilde{\psi}_-)(u \bar{u}^{-1}) = \frac{1}{u^{-1} \cdot v_{\rho}^+} e^{\tilde{\chi}(u)} e^{\frac{1}{2} s} = \frac{1}{z} \tilde{\psi}_- (u \bar{u}^{-1}).
\]
\[\square\]

**Definition 9.6.** Let \( V_- \) be the \( \mathcal{U}(g) \) module in \( \mathcal{H}(Z_0) \) generated by \( \psi_- \). This is another irreducible Whittaker module.

10. A Whittaker functional on \( V_+ \)

Consider again the holomorphic character \( e^{\tilde{\chi}} \) on \( U_- \) given by
\[
\tilde{u} \mapsto e^{\tilde{\chi}(\tilde{u})} = e^{\frac{1}{2} \sum f_i^* (\tilde{u})}.
\]

Suppose we have fixed a cycle \([\Gamma_0] \) in \( Z_0 \) as in Section 7.3 Then we define a linear map \( \Psi_{[-\rho]}^{[\Gamma_0]} : V_+ \to \mathbb{C} \) by
\[
\Psi_{[-\rho]}^{[\Gamma_0]} (f) := \int_{[\Gamma_0]} e^{\tilde{\chi}(\tilde{u})} f \omega.
\]

**Remark 10.1.** Recall how the integration cycles were chosen in Section 7.3 for \( e^x \) to have exponential decay in any non-compact direction. Now \( \Psi_{[-\rho]}^{[\Gamma_0]} (\psi_+) = \int_{[\Gamma_0]} e^{\tilde{\chi}(\tilde{u})} \psi_+ \omega = \int_{[\Gamma_0]} e^x \omega \). Since repeated actions by elements of \( g \) on the \( \psi_+ \) produce only rational amplitude factors which do not affect convergence, \( \Psi_{[-\rho]}^{[\Gamma_0]} \) is defined on all of \( V_+ \).
Let us consider $\Psi_{[\Gamma_0]}$ as an element of the $\mathcal{U}(\mathfrak{g})$-module $V^*_+$ dual to $V_+$.

**Proposition 10.2.** For all $i \in I$, we have

$$f_i \cdot \Psi_- = \frac{1}{z} \Psi_-.$$  

That is, $\Psi_-$ is a $u_-$-Whittaker vector in $V^*_+$.

**10.1. A bilinear pairing.** To prove Proposition 10.2 we will construct a pairing between the Whittaker modules $V_+$ and $V_-$, such that the Whittaker vector $\psi_-$ becomes identified with the Whittaker functional $\Psi_-$ from Theorem 10.2. This pairing essentially generalizes one introduced in [4].

Let $[\Gamma_0]$ be a middle-dimensional cycle in $Z_0$ as above. Consider integrals of the form

$$\int_{[\Gamma_0]} \phi \psi \omega_{GKLO},$$

where $\phi, \psi$ are holomorphic functions on $Z_0$ and $\omega_{GKLO}$ is the volume form from (5.1), pulled back to $Z_0$ via the isomorphism $\beta_0 : Z_0 \to \mathcal{R}_{1,w_0}$.

**Proposition 10.3.** Let $\phi \in V_-$ and $\psi \in V_+$.

1. The formula

$$\langle \phi, \psi \rangle_{[\Gamma_0]} := \int_{[\Gamma_0]} \phi \psi \omega_{GKLO}$$

defines a bilinear pairing, $\langle \phi, \psi \rangle_{[\Gamma_0]} : V_- \times V_+ \to \mathbb{C}$.

2. For any $X \in \mathfrak{g}$,

$$\langle X \cdot \phi, \psi \rangle_{[\Gamma_0]} + \langle \phi, X \cdot \psi \rangle_{[\Gamma_0]} = 0.$$

**Proof.** The pairing $\langle , \rangle_{[\Gamma_0]}$ is well-defined for the same reason as $\Psi_{[\Gamma_0]}$, see Remark 10.1. Let us now prove (2). We transfer the integrals from $Z_0$ to $\mathcal{R}_{1,w_0}$ via $\beta_0$, but keeping the notation the same. For a function $f$ on $\mathcal{R}_{1,w_0}$ identified with a function on $x \in U_+ \cap B_- \dot{w}_0 B_-$, we denote by $\tilde{f}$ the corresponding element of $M_{-\rho}$ given by

$$\tilde{f}(xb_-) = f(x)\rho(b_-), \quad \text{for } b_- \in B_-,$$

and by $\bar{f}$ the usual function

$$\bar{f}(xb_-) = f(x), \quad \text{for } b_- \in B_-$$

on $(U_+ \cap B_- \dot{w}_0 B_-)B_-$ given by $f$.

The proposition relies on the observation that $\omega_{GKLO}$ ‘compensates’ for the weight $-\rho$ twists coming from the representation on $M_{-\rho}$. Namely, on the level of $N$-forms we claim that

$$\kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO})$$

where on the left hand side we have the local action of $\exp(sX)$ on sections of $L_{-\rho}$, and on the right hand side we have the pull-back of forms, as in Proposition 5.2.

To prove (10.1), write $\exp(-sX)x = x(s)b_-$ for $x(s) \in U_+$ and $b_- \in B_-$. Then

$$\kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO}) = \kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO}).$$

To prove (10.1), write $\exp(-sX)x = x(s)b_-$ for $x(s) \in U_+$ and $b_- \in B_-$. Then

$$\kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO}) = \phi(x(s)b_-)\psi(x(s)b_-) = \phi(x(s))\psi(x(s))\rho(b_-)^2.$$
On the other hand we have

\begin{equation}
\kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO})(x) = \tilde{\phi}(\exp(-sX)x)\tilde{\psi}(\exp(-sX)x)\kappa^*_{\exp(-sX)}\omega_{GKLO}
\end{equation}

\begin{equation}
= \tilde{\phi}(\exp(-sX)x)\tilde{\psi}(\exp(-sX)x)\frac{1}{(\exp(-sX)x \cdot v_{-\rho}, v_{-\rho})^2} \omega_{GKLO}
\end{equation}

\begin{equation}
= \phi(x(s))\psi(x(s)) \frac{1}{(x(s)b_\cdot v_{-\rho}, v_{-\rho})^2} \omega_{GKLO} = \phi(x(s))\psi(x(s)) \rho(b_-)^2 \omega_{GKLO},
\end{equation}

using Proposition 5.2 and that \(x(s) \in U_+\). This completes the proof of the identity (10.1).

If \(\Gamma_0\) is compact, then \(\phi \psi \omega_{GKLO}\) being holomorphic implies that the left hand side of

\begin{equation}
\int_{[\Gamma_0]} \kappa^*_{\exp(-sX)}(\phi \psi \omega_{GKLO}) = \int_{[\Gamma_0]} (\exp(sX) \cdot \phi)(\exp(sX) \cdot \psi) \omega_{GKLO}
\end{equation}

is constant for \(s\) small. Therefore the differential \(\frac{d}{ds}\big|_{s=0}\) vanishes, implying (2) in this case,

\begin{equation}
\frac{d}{ds}\bigg|_{s=0} \int_{[\Gamma_0]} (\exp(sX) \cdot \phi)(\exp(sX) \cdot \psi) \omega_{GKLO}
\end{equation}

\begin{equation}
= \int_{[\Gamma_0]} (X \cdot \phi) \psi \omega_{GKLO} + \int_{[\Gamma_0]} \phi (X \cdot \psi) \omega_{GKLO} = 0.
\end{equation}

For more general \([\Gamma_0]\) as in Section 7.3, we consider \(X\) to be a generator, such as \(e_i\) or \(f_i\), so that (10.2) converges for small \(s > 0\). Then we can use the exponential decay condition and an argument as in [9, Section 2] for the ‘limit Stokes formula’, to see that the differential of (10.2) vanishes.

**Proof of Proposition 9.4.** For any \(\psi \in V_+\) we have

\begin{equation}
\Psi^{[\Gamma_0]}(\psi) = (\psi_-, -f_i \cdot \psi)_{[\Gamma_0]} = (f_i \cdot \psi_-, \psi)_{[\Gamma_0]} = \frac{1}{2} (\psi_- \cdot \psi)_{[\Gamma_0]} = \frac{1}{z} \Psi^{[\Gamma_0]}(\psi),
\end{equation}

by Proposition 10.3 and then Lemma 9.4.

**Proof of Theorem 8.1** Consider the ‘matrix coefficient’

\begin{equation}
g \mapsto \Psi^+_{\Gamma}(g^{-1} \cdot \psi_+).
\end{equation}

Strictly speaking only \(U(g)\) acts on \(V_+\), so the above definition doesn’t make sense for general \(g \in G\). However, for \([\Gamma_0]\) a member of a family of cycles \([\Gamma_h]_{h \in \mathcal{O}}\) as in Section 7.3 this particular matrix coefficient turns out to be well defined on the corresponding \(X_0\) (compare Section 4.1), giving

\begin{equation}
X_0 := U_+TU_- \times T \to \mathcal{O}
\end{equation}

\begin{equation}
(u_+e^h u_-, h) \mapsto \Psi^+(u_-^{-1} e^{-h} u_+^{-1} \cdot \psi_+) = e^{\chi_-(-u_-)} \Psi^-(e^{-h} \cdot \psi_+) e^{\chi_+(u_+)}
\end{equation}

where \(\chi^{(i)}_+ = -\frac{1}{2}, \chi^{(i)}_- = \frac{1}{2}\), for all \(i\), and we compute \(\Psi^+(e^{-h} \cdot \psi_+)\) below. In this way (10.3) gives rise to a Whittaker function, which we denote by \(\mathcal{W}_\Gamma\).

Since \(V_+\), and with it \(V_+^*\), have zero infinitesimal character, \(\mathcal{W}_\Gamma\) is annihilated by the Casimir generators in \(Z(g)\). By Section 4.1 (2), therefore,

\begin{equation}
e^{-\rho} \mathcal{W}_\Gamma|_\mathcal{O} \quad : \quad h \mapsto e^{-\rho(h)} \Psi^+(e^{-h} \cdot \psi_+)
\end{equation}

is a solution to the quantum Toda lattice (4.1).
To compute this solution, suppose $h \in \mathcal{O}$. Then
\begin{equation}
(10.4) \quad e^{-\rho(h)} \Psi_-(e^{-h} \cdot \psi_+) = e^{-\rho(h)} \int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})}(e^{-h} \cdot \psi_+)(u\bar{u}^{-1})\omega_0
\end{equation}
\begin{align*}
&= e^{-\rho(h)} \int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})}(e^{-h} \cdot \psi_+)(u^{-1})\omega_0 = e^{-\rho(h)} \int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})} \psi_+(e^h u^{-1})\omega_0 \\
&= e^{-\rho(h)} \int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})} \psi_+(e^h u^{-1} e^{-h})\rho(e^h)\omega_0 = \int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})} \psi_+(e^h u^{-1} e^{-h})\omega_0
\end{align*}
Now we can use Proposition 10.3 (2) to rewrite the integral
\begin{equation}
\int_{[\Gamma_0]} e^{\tilde{x}(\bar{u})} e^{(e^h u e^{-h})}\omega_0 = \int_{[\Gamma_0]} e^{x(u)} e^{(e^h u e^{-h})}\omega_0,
\end{equation}
to get precisely
\begin{equation}
(10.5) \quad e^{-\rho(h)} \Psi_-(e^{-h} \cdot \psi_+) = \int_{[\Gamma_h]} e^{x(u) + \tilde{x}(\bar{u})} \omega_h = \int_{[\Gamma_h]} e^{\frac{1}{2}(\sum c^*_i(u) + \sum f^*_i(\bar{u}))} \omega_h,
\end{equation}
where $[\Gamma_h]$ is the translate of $[\Gamma_0]$, as in Section 7.3. This completes the proof of the theorem.

11. Total positivity and critical points of $F_h$

Let us fix $z > 0$. By the mirror symmetric construction of the quantum cohomology ring of $G^t/B^t$ proved in [28], the critical points of the $F_h = F_h(\cdot ; z)$ (for varying $h$) sweep out the Peterson variety
\[ Y_B = Y \times_{G/B} \mathcal{R}_1,w_0, \]
with $h$ determining the values of the quantum parameters. Explicitly, $g \in Z_h$ is a critical point of $F_h$ precisely if $g \cdot B_- \in Y_B^t$ with $q_t(g \cdot B_-) = e^{\alpha_\gamma(h) \rho}$. On the other hand by [16] the quantum cohomology ring of the full flag variety $G^t/B^t$ is semisimple for generic choice of quantum parameters. This implies that $F_h$ has precisely $\dim H^*(G^t/B^t) = |W|$ critical points, all non-degenerate, for generic $h$.

Following Givental [9], the critical points are directly related to integration cycles. Namely to any non-degenerate critical point $p$ in $Z_h$, Givental associates a ‘descending gradient cycle’ for $\text{Re}(F)$. In this way, one may obtain $|W|$ cycles in a generic mirror fiber $Z_h$, which should (conjecturally) provide a basis of the appropriate rapid decay homology group in the sense of Hien [8].

**Definition 11.1.** Consider the isomorphism $\delta_h : Z_h \to \mathcal{R}_{1,w_0}$ given by $g \mapsto g \cdot B_-$. Let $Z_{h}^{>0} := \delta_h^{-1}(\mathcal{R}_{1,w_0}^{>0})$.

**Lemma 11.2.** Suppose $u\bar{u}^{-1} \in Z_0^{>0}$. Then $u \in U_+^{>0}$ and $\bar{u} \in U_-^{>0}$.

**Proof.** It is clear from the definitions that $u \in U_+^{>0}$. The rest of the lemma follows from $u^{-1} \cdot B_- = \bar{u}^{-1} \cdot B_+$, together with Lusztig’s result [18] that
\[ U_+^{>0} \cdot B_- = U_-^{>0} \cdot B_+, \]
applied to the opposite pinning, where $\bar{x}_i(t) = \exp(-te_i)$ and $\bar{y}_i(t) = \exp(-tf_i)$. \qed
Proposition 11.3. For any \( h \in \mathfrak{h}_R \) and \( M \in \mathbb{R}_{>0} \), consider the set

\[
\mathcal{M}_{h,M} := \left\{ g \in Z_h^> \mid \mathcal{F}(g; z) \leq \frac{1}{z} M \right\}.
\]

If \( M \) is sufficiently large then \( \mathcal{M}_{h,M} \) is a nonempty, compact subset of \( Z_h^> \). In particular the restriction of \( \mathcal{F}_h \) to \( Z_h^> \) attains a minimum.

Note that by definition \( \mathcal{M}_{h,M} \) is independent of the positive scalar \( z \).

Proof. If \( u e^h \bar{u}^{-1} \in \mathcal{M}_{h,M} \), then by Lemma 11.2, we have \( u \in U_z^0 \) and \( \bar{u} \in U_z^0 \). So we can fix a reduced expression \( i \) of \( w_0 \) and write

\[
\begin{align*}
\quad u &= x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_N}(a_N), \\
\bar{u} &= y_{i_1}(b_1)y_{i_2}(b_2)\cdots y_{i_N}(b_N),
\end{align*}
\]

for positive \( a_i, b_i \) with \( i \in I \). Using the \( a_i \) and \( b_i \) as coordinates for \( u \) and \( \bar{u} \), respectively, we may define

\[
\mathcal{N}_{h,M} := \{ u e^h \bar{u}^{-1} \in Z_h^> \mid a_i \leq M, b_i \leq M \quad \forall i \in I \}.
\]

Since \( \mathcal{F}(ue^h \bar{u}^{-1}; z) = \frac{1}{z} (\sum a_i + \sum b_i) \), it is clear that \( \mathcal{M}_{h,M} \) is a closed subset of \( \mathcal{N}_{h,M} \). It suffices therefore to show the following claim.

Claim: There exists an \( m < M \) such that \( \mathcal{N}_{h,M} \) is a subset of the compact set

\[
\mathcal{N}_{h,M}^m := \{ u e^h \bar{u}^{-1} \in Z_h^> \mid m \leq a_i \leq M \quad \forall i \in I \} \cong [m, M]^N.
\]

Suppose indirectly that we have an index \( i \) and a sequence \( u(s)e^h \bar{u}^{-1}(s) \) in \( \mathcal{N}_{h,M} \) for which the coordinate \( a_i \to 0 \) as \( s \to \infty \). We may assume that \( u^{-1}(s) \cdot B_- \) converges, passing to a subsequence if necessary. Then it follows that

\[
\lim_{s \to \infty} (u^{-1}(s) \cdot B_-) \in B_- \bar{w} \cdot B_-
\]

for some \( w < w_0 \). On the other hand using \( u^{-1}(s) \cdot B_- = e^h \bar{u}^{-1}(s) \cdot B_+ \) we see that

\[
\lim_{s \to \infty} (u^{-1}(s) \cdot B_-) = e^h \lim_{s \to \infty} (\bar{u}^{-1}(s) \cdot B_) \in B_- \cdot B_+ = B_- w_0 \cdot B_-.
\]

Namely, this last limit cannot leave the big cell \( B_- \cdot B_+ \) because the \( b_i \) coordinates of \( \bar{u}(s) \) are bounded from above by \( M \).

Therefore we have arrived at a contradiction and the Claim is proved. \( \square \)

Corollary 11.4. For every \( h \in \mathfrak{h}_R \) the function \( \mathcal{F}_h \) has a totally positive critical point.

The totally positive critical point is provided by the minimum of \( \mathcal{F}_h \) on \( Z_h^> \). For type \( A \) this result was proved already in [26], where moreover it was shown that the totally positive critical point is unique, and this was used to describe the totally nonnegative part of the Peterson variety. One would hope to find uniqueness also in the general case, but this is still conjecture at the moment.

The same proof with the negative pinning also gives the following

Corollary 11.5. For every \( h \in \mathfrak{h}_R \) the function \( \mathcal{F}_h \) has a critical point in \( \Gamma_h^{(w_0)} \).
This critical point is given by a maximum of $F_h$ on $\Gamma^{(w_0)}_h$. We might call it the totally negative critical point, and it is a special feature of the full flag variety that a symmetry between totally positive and totally negative critical points exists, compare \[23\] and Lemma 5.3 in \[28\].

Assuming the conjecture about uniqueness of the totally positive/negative critical points in every fiber over $h \in \frak{h}_R$, it seems natural to think of the family $\Gamma^{(w_0)}_h$ of integration cycles as associated to the family of totally negative critical points, and the family $\Gamma^{(1)}_h$ of integration cycles as associated to the family of totally positive critical points (in both cases via Givental’s construction). For $SL_2$ this is exactly the case, by direct calculation.

**References**

1. A. Braverman, *Instanton counting via affine Lie algebras I: Equivariant J-functions of (affine) flag manifolds and Whittaker vectors*, arxiv.org/abs/math/0401409 (2004).
2. V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math. (1985), no. 79, 499–511.
3. P. Etingof, *Whittaker functions on quantum groups and q-deformed Toda operators*, Amer. Math. Soc. Transl. Ser. 2 194 (1999), 9–25.
4. A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, *On a Gauss-Givental representation of quantum Toda chain wave function*, Int. Math. Res. Not. (2006), no. Article ID 96489, 23 pages.
5. A. Givental, *Equivariant Gromov-Witten invariants*, IMRN 13 (1996), 613–663.
6. , *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, American Mathematical Society Translations Ser 2., AMS, 1997.
7. A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. 168 (1995), 609–641.
8. M. Hien, *Periods of flat algebraic connections*, arxiv.org/abs/0803.3463 (2008).
9. M Hien and C. Roucairol, *Integral representations for solutions of exponential Gauss-Manin systems*, Bulletin de la Société Mathématique de France 136 (2008), no. fascicule 4, 505–532.
10. B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Annals of Mathematics 149 (1999), 129–148.
11. A. Knapp, *Lie groups beyond and introduction*, Progress in Mathematics, vol. 140, Birkhäuser, Boston, 1996.
12. B. Kostant, *Lie group representations on polynomial rings*, American J. Math. 86 (1963), 327–404.
13. , *Quantization and representation theory*, Proc. Oxford Conference on Group Theory and Physics, Oxford, 1977, pp. 287–316.
14. , *On Whittaker vectors and representation theory*, Inventiones Math. (1978), 101–184.
15. , *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. 34 (1979), no. 3, 195–338.
16. , *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight $\rho$*, Selecta Math. (N.S.) 2 (1996), 43–91.
17. , *Quantum cohomology of the flag manifold as an algebra of rational functions on an unipotent algebraic group*, Deformation Theory and Symplectic Geometry (D. Sternheimer et al, ed.), Kluwer, 1997, pp. 157–175.
18. G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry: in honor of Bertram Kostant (G. I. Lehrer, ed.), Progress in Mathematics, vol. 123, Birkhaeuser, Boston, 1994, pp. 531–568.
19. R. J. Marsh and K. Rietsch, *Parametrizations of flag varieties*, Representation Theory 8 (2004), 212–242.
20. D. Milicic and W. Soergel, *Twisted Harish-Chandra sheaves and Whittaker modules*, (1995).
21. T. Mochizuki, *Wild harmonic bundles and wild pure twistor d-modules*, arXiv:0803.1344 [math.DG] (2008), 1–352.
22. J. Moser, *Finitely many mass points on the line under the influence of an exponential potential-an integrable system*, Dynamical systems, theory and applications, Springer Lecture Notes in Phys., vol. 38, Springer, Berlin, 1975, pp. 467–497.

23. D. Peterson, *Quantum cohomology of G/P*, Lecture Course, M.I.T., Spring Term, 1997.

24. K. Rietsch, *Quantum cohomology of Grassmannians and total positivity*, Duke Math. J. 113 (2001), no. 3, 521–551.

25. , *Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties*, J. Amer. Math. Soc. 16 (2003), 363–392.

26. , *A mirror construction for the totally nonnegative part of the Peterson variety*, Nagoya Mathematical Journal (2006), 105–142.

27. , *Errata to: "totally positive Toeplitz matrices and quantum cohomology of partial flag varieties"*, J. Amer. Math. Soc. 21 (2008), 611–614.

28. , *A mirror symmetric construction for qH∗ P (G/P)(q)*, Advances in Mathematics 217 (2008), 2401–2442.

29. T. A. Springer, *Linear algebraic groups, second edition*, Progress in Mathematics, vol. 9, Birkhäuser, Boston, 1998.

King’s College London, UK

E-mail address: konstanze.rietsch@kcl.ac.uk