RELATIVE $\bar{\partial}$-COMPLEX AND ITS CURVATURE PROPERTIES

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ABSTRACT. We shall define the relative $\bar{\partial}$-complex and study the curvature properties of the associated vector bundles. As an application, we shall prove that Yamaguchi’s theory on subharmonicity of the Green operator can be seen as a curvature property of the quotient bundle. A short survey of other recent applications will also be given in this paper.

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1. Introduction

Our motivation to write this paper is to give a unified proof of recent results in [13], [32], [33] and [34] by using the relative $\bar{\partial}$-complex. Let $\pi$ be a holomorphic submersion from a complex manifold $\mathcal{X}$ to a complex manifold $B$, let $E$ be a holomorphic vector bundle over $\mathcal{X}$, let $X_t$ be the fibre at $t$ and $E_t$ be the restriction of $E$ to $X_t$. Denote by $A^{p,q}_t$ the space of smooth $E_t$-valued ($p,q$)-forms on $X_t$. We call the collection of $\bar{\partial}$-operators on fibres

$$\{\bar{\partial}_t\}_{t \in B} : \{A^{p,q}_t\}_{t \in B} \to \{A^{p,q+1}_t\}_{t \in B},$$

the relative $\bar{\partial}$-complex. Put $A^{p,q} = \{A^{p,q}_t\}_{t \in B}$, $K^{p,q} := \{\text{Ker } \bar{\partial}_t\}_{t \in B}$, $T^{p,q+1} := \{\text{Im } \bar{\partial}_t\}_{t \in B}$. Then we have the following exact sequence

$$0 \to K^{p,q} \to A^{p,q} \to T^{p,q+1} \to 0. \tag{1.1}$$

We call $K^{p,q}$ the $\bar{\partial}$ subbundle and $T^{p,q+1}$ the $\bar{\partial}$ quotient bundle. In general, they are not holomorphic vector bundles but still we can define the Chern connection on them as an operator on the space of smooth sections (see [2] and [23] for related results). Our starting point is the following result of Berndtsson (see Theorem 1.1 in [2]):

If $\pi$ a product family then one may study the curvature properties of $K^{n,0}$, $n$ is the fibre dimension, by looking at $K^{n,0}$ as a holomorphic subbundle of $A^{n,0}$.

The curvature properties of $K^{n,0}$ are crucial in Berndtsson’s complex Brunn-Minkowski theory, which also contains curvature properties of vector bundles associated to a non-trivial fibration (see [1], [2], [5], [3], [7], [9], [11], [10], [12], see also [31], [30], [24], [16], [27], [28] and [32] for other generalizations). Thus it is natural to ask: Whether the relative $\bar{\partial}$-complex can be used to prove the curvature properties of the associated vector bundles for a general fibration? We shall try to answer this question in this paper. In particular, we shall show that the relative $\bar{\partial}$-complex can be used to give a unified proof of recent results in [13], [32], [33] and [34].

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Another related theory is **Yamaguchi’s theory** on subharmonicity properties of the Green-operators (see [35], [25], [26], [22] and [33]). In the KAWA-NORDAN 2014 conference in Marseille, Levenberg asked the following question:

Is Yamaguchi’s theory a curvature property?

In this paper, we shall answer Levenberg’s question by proving that Yamaguchi’s theory can be seen as a curvature property of the quotient bundle $\mathcal{I}^{n,n}$.

2. **Family of domains in a fixed manifold**

First we shall consider variation of domains $X_t$, $t \in B$, in a fixed complex manifold $M$. We shall show how to use the curvature properties of the $\overline{\partial}$-quotient bundle to study variation of the $L^2$-minimal solution of the $\partial$-operator on fibres, see [33].

Let $(E_0, h^0)$ be a fixed holomorphic vector bundle over $(M, \omega^0)$, let $g$ be a fixed smooth $\overline{\partial}$-closed $E_0$-valued $(n, q + 1)$-form on $M$. We shall introduce the following definition:

**Definition 2.1.** If $a^t$ is the $L^2$-minimal solution of $\overline{\partial} (a^t) = g$ on $X_t$ with respect to $\omega^0$ and $h^0$ then we call $\|a^t\|^2_{\overline{\partial}}(t)$ the Green norm of $g$ on $X_t$ and denote it by $\|g\|_{G}^2(t)$.

**Remark 1:** If the above $\overline{\partial}$-equation has no $L^2$-solution then we say the Green norm of $g$ is infinite on $X_t$.

**Relation with Green operator:** Since the $L^2$-minimal solution $a^t$ of $\overline{\partial} (a^t) = g$ can be written as $a^t = (\overline{\partial}^*)^n G^t g$, where each $G^t$ is the Green-operator with respect to the $\overline{\partial}$-Laplacian on $X_t$. Thus we have

\[ \|g\|_{G}^2(t) = \|a^t\|^2 = ((\overline{\partial}^*)^n G^t g, a^t) = (G^t g, g). \]

That is why we call the $L^2$-norm of the minimal solution of the $\overline{\partial}$-equation the Green-norm. See [1] for the relation between the Green-norm and the Robin constant.

**Relation with the quotient norm:** Since the minimal solution is just the minimal lift w.r.t. the relative $\overline{\partial}$-complex: $\overline{\partial} : \mathcal{A}^{p,q}_t \rightarrow \mathcal{I}^{p,q+1}_t$, we know that the Green norm is just the quotient norm.

Now we know that one may use curvature formula of the quotient bundle to study variation of the Green norm. The latter is known as **Yamaguchi’s theory**, thus Yamaguchi’s theory can be seen as a curvature property of the quotient bundle, this answers the previous question raised by Levenberg. Now we have another formulation of Yamaguchi’s theory:

**Theorem 2.1** (Developed by Yamaguchi-Maitani-Levenberg-Kim, etc). Under some convexity and curvature assumptions, $-\|g\|^2_{\overline{\partial}}$ is plurisubharmonic if $g$ is a holomorphic section of the quotient bundle $\mathcal{I}^{n,n}$, where $n$ is the fibre dimension.

Our first result is a generalization of Yamaguchi’s theory to the $(n, q + 1)$-case. More precisely, we shall prove the following theorem (see the main theorem in [33]):

**Theorem 2.2.** ($\mathcal{I}^{n,q+1}$-version of Yamaguchi’s theorem) Let $g$ be a $\overline{\partial}$-closed $E^0$-valued $(n, q + 1)$-form on $M$. Assume that the total space $\mathcal{X} := \{(z, t) \in M \times B : z \in X_t\}$ is Stein, $\omega^0$ is Kähler and

\[ i\Theta(E_0, h^0) \wedge (\omega^0)^q \geq 0, \]
on $M$. Assume further $g$ has \textbf{compact support in each fibre $X_t$}. Then $-||g||^2_{\bar{\partial}}$ is plurisubharmonic on $B$.

\textbf{Main tools in the proof:} Hörmander’s $L^2$-theory \cite{20} and the curvature formula for the $\bar{\partial}$ quotient bundle $\mathcal{I}^{n,q+1}$. By a generalized version of Berndtsson’s approximation process (see \cite{1} or \cite{33} for details), it suffices to prove the following \textbf{product case} with a non-product weight $\psi$ (a real smooth function on $M \times B$):

\textbf{Theorem 2.3} (Product case with non-product weight $\psi$). Assume that $X = X_0 \times B$, where $X_0$ is a smoothly bounded strictly pseudoconvex domain in $M$. Assume that the pull back to the total space, say $\omega$, of $\omega^0$ is Kähler and

$$i\Theta(E, h) \wedge \omega^q > 0, \quad E := E_0 \times B, \quad h := e^{-\psi}(h^0 \times B).$$

Assume further that $g$ has \textbf{compact support} in $X_0$ and $\psi$ \textbf{does not depend on $t$} on $\text{Supp}(g) \times B$. Then $-||g||^2_{\bar{\partial}}$ is plurisubharmonic on $B$.

\textit{Proof.} Since the \textbf{Green norm is just the quotient norm}, it is enough to estimate the curvature of the quotient bundle $\mathcal{I}^{n,q+1}$.

Let us assume that $B$ is \textbf{one dimensional}. Denote by $\Theta_{tt}$ the curvature operators on $\mathcal{A}^{n,q}$, then we have

$$\Theta_{tt} = [D_t, \bar{\partial}_t] = \psi_{tt}, \quad D_t := \partial/\partial t - \psi_t, \quad \bar{\partial}_t := \partial/\partial \bar{t}.$$ 

Since $\mathcal{I}^{n,q+1} = \mathcal{A}^{n,q}/\mathcal{K}^{n,q}$ is the quotient bundle of $\mathcal{A}^{n,q}$, we have

$$(\Theta_{tt}^\mathcal{I} g, g)_\mathcal{I} = (\psi_{tt} a, a) + ||P(a)||^2, \quad a : t \mapsto a^t \in \mathcal{A}^{n,q},$$

where $\Theta_{tt}^\mathcal{I}$ is the curvature operator on $\mathcal{I}^{n,q+1}$, each $a^t$ is the $L^2$-\textbf{minimal solution} of $\bar{\partial}^\mathcal{I} (\cdot) = g$ and $P$ denotes the \textbf{orthogonal projection} to $\mathcal{K}^{n,q}$. Notice that \textbf{Hamilton’s theory} (see \cite{18} and \cite{19}) implies that $a$ is a smooth section.

Since $g$ does not depend on $t$ and has \textbf{compact support} in each fibre, we know that there exists a \textbf{fixed} smooth $L^2$-solution, say $u$, of $\bar{\partial}u = g$ on $X_0$. Since $u$ is fixed, we have $u_t = 0$, thus $u$ is a \textbf{holomorphic section} of $\mathcal{A}^{n,q}$. Since $g$ is the image of $u$ under the $\bar{\partial}$-\textbf{quotient map}, we know that $g$ is a \textbf{holomorphic section} of $\mathcal{I}^{n,q+1}$, thus we have

$$||g||^2_{\bar{\partial}} = ||D_t a||^2 - (\Theta_{tt}^\mathcal{I} g, g)_\mathcal{I} \leq ||D_t a||^2 - (\psi_{tt} a, a).$$

We shall use \textbf{Hörmander’s $L^2$-theory} to control the norm of $D_t a$. Notice that each $D_t a$ is the $L^2$-\textbf{minimal solution} of

$$\bar{\partial}^\mathcal{I} (\cdot) = \bar{\partial} D_t a = [\bar{\partial}^\mathcal{I}, D_t] a + D_t g = [\bar{\partial}^\mathcal{I}, D_t] a = -\bar{\partial}^\mathcal{I} \psi_t \wedge a,$$

the third equality follows from $D_t g = g_t - \psi_t g \equiv 0$ since $\psi$ \textbf{does not depend on $t$} on $\text{Supp}(g) \times B$. By \textbf{Hörmander’s $\bar{\partial}$-$L^2$-estimate}, we have

$$||D_t a||^2 \leq (Q^{-1}\bar{\partial}^\mathcal{I} \psi_t \wedge a, \bar{\partial}^\mathcal{I} \psi_t \wedge a),$$

where

$$Q := [i\Theta(E_0, e^{-\psi^0} h^0), \Lambda_{\omega^0}].$$

Thus

$$||g||^2_{\bar{\partial}} \leq (Q^{-1} \bar{\partial}^\mathcal{I} \psi_t \wedge a, \bar{\partial}^\mathcal{I} \psi_t \wedge a) - (\psi_{tt} a, a).$$

By a direct computation, we know that $i\Theta(E, h) \wedge \omega^q > 0$ implies

$$(Q^{-1} \bar{\partial}^\mathcal{I} \psi_t \wedge a, \bar{\partial}^\mathcal{I} \psi_t \wedge a) \leq (\psi_{tt} a, a).$$

Thus $||g||^2_{\bar{\partial}} \leq 0$. The proof is complete. \qed
If we consider the subbundle $\mathcal{K}_{n,q}$ and use Hörmander’s $L^2$-theory to control the second fundamental form as Berndtsson did for the $q = 0$ case (see the proof of Theorem 1.1 in [2]) then we can prove the following theorem in [33]:

**Theorem 2.4.** $(n,q)$-version of Berndtsson’s theorem] Let $v$ be a fixed smooth $E_0$-valued $(n,q)$-form with compact support in each fibre. Assume that $\omega^0$ is Kähler, the total space is Stein and $i\Theta(E_0,h^0) \wedge (\omega^0)^q \geq 0$, then $\log ||P(t)|| : t \mapsto ||P(t)||$ is plurisubharmonic on $B$, where each $P(t)$ denotes the Bergman projection of $v$ to $\ker \overline{\partial} = \mathcal{K}_{n,q}$.

Relation with the Ohsawa-Takegoshi theorem: If $q = 0$ then by Berndtsson-Lempert-Blocki’s method [11], the above theorem (with $v$ a general current, see section 5.3 in [32] or [33]) can be used to prove Blocki-Guan-Zhou’s sharp version of the Ohsawa-Takegoshi theorem (see [29], [15], [14] and [17], to cite just a few).

3. Proper Kähler fibration

Now let us consider the case that $\pi : \mathcal{X} \to B$ is a proper fibration. We shall prove that:

**Theorem 3.1** (Generalized Berndtsson-Mourougane-Takayama’s theorem). Assume that $\omega$ is Kähler and $i\Theta(E,h) \wedge \omega^q \geq 0$. Assume further that $\dim H^{n,q}(E_t)$ is a constant. Then

- $\mathcal{K}_{n,q}$ is Nakano semipositive if $q = 0$ or $c_{jk}(\omega) \equiv 0$;
- $P^t \pi_* \mathcal{O}(\mathcal{K}_{X/B} \otimes E) \simeq \mathcal{K}_{n,q}/\mathcal{T}^{n,q}$ is Griffiths semipositive,

where $c_{jk}(\omega) := \langle V_j, V_k \rangle_\omega$ and each $V_j$ denotes the horizontal lift of $\partial/\partial t^j$ with respect to $\omega$.

**Remark:** In case $E$ is Nakano-semipositive, Mourougane-Takayama [28] proved that $\mathcal{K}_{n,q}/\mathcal{T}^{n,q}$ is Nakano semipositive (with different metric, i.e. not the quotient metric).

**Proof.** Recall in the product case, the Chern connection on $\mathcal{A}^{n,q}$ is defined by $\overline{\partial}_\nu = \partial/\partial t^j$ and $\nu_j = \partial/\partial t^j - \psi_j$. But now we have to use the generalized Lie derivatives to define the Chern connection on $\mathcal{A}^{n,q}$, i.e.,

\[
D_{\nu_j} u := [\partial^E, \delta_{V_j}] u, \quad \overline{\partial}_{\nu_j} u = [\overline{\partial}, \delta_{V_j}] u, \quad \Theta_{j\overline{k}} := [D_{\nu_j}, \partial_{\nu_k}],
\]

where $\partial^E := \overline{\partial} + \partial^E$ denotes the Chern connection on $E$, each $V_j$ is the horizontal lift of $\partial/\partial t^j$ with respect to $\omega$ and $u$ is the representative of a smooth section $u : t \mapsto u^t$ of $\mathcal{A}^{n,q}$, i.e. $u|_{X_t} = u^t$. Since $\mathcal{K}_{n,q}$ is the subbundle of $\mathcal{A}^{n,q}$, we have

\[
(\Theta_{j\overline{k}} u, v) = (\Theta_{j\overline{k}} u, v) - (P^t D_{\nu_j} u, P^t D_{\nu_k} v).
\]

By Theorem 2.5 in [13], we have

\[
(\Theta_{j\overline{k}} u, v) = ([L_j, L_{\overline{k}}] u, v) - (\partial V_j|_{X_t}^t u, \partial V_{\overline{k}}|_{X_t}^t v) + (\overline{\partial} V_j|_{X_t}^t u, \overline{\partial} V_{\overline{k}}|_{X_t}^t v).
\]

By Proposition 4.2 in [32], we know that

\[
[L_j, L_{\overline{k}}] = [d^E, \delta_{[V_j, V_{\overline{k}}]}] + \Theta(E, h)(V_j, \overline{V}_{\overline{k}}).
\]

Moreover, by Lemma 6.1 in [32], we know that

\[
\delta_{[V_j, V_{\overline{k}}]} = (\partial^E c_{jk}(\omega))^* - (\overline{\partial} c_{jk}(\omega))^*.
\]
Notice that $\alpha^* = *\alpha^*$ if $\alpha$ is an one-form, by a direct computation, we have
\begin{align}
(3.5) \quad \left( [d^E, \delta_{[V_jV_k]}] u \right) v &= (c_{j\bar{k}}(\omega) u, \overline{\partial} (\overline{\partial})^* v) - (c_{j\bar{k}}(\omega) u, \partial E_i (\partial E_i)^* v) \\
&\quad + (c_{j\bar{k}}(\omega)(\partial E_i)^* u, (\partial E_i)^* v) - (c_{j\bar{k}}(\omega)(\overline{\partial})^* u, (\overline{\partial})^* v) \\
&\quad + (c_{j\bar{k}}(\omega)(\partial E_i)^* \partial E_i u, v) - (c_{j\bar{k}}(\omega)(\overline{\partial})^* \overline{\partial} u, v) \\
&\quad + (c_{j\bar{k}}(\omega)\overline{\partial} u, \overline{\partial} v) - (c_{j\bar{k}}(\omega)\partial E_i u, \partial E_i v).
\end{align}

Thus if $u$ is an $(n,q)$-form and $c_{j\bar{k}}(\omega) \equiv 0$ then we have
\begin{align}
(3.6) \quad (\Theta_{j\bar{k}} u, v) &= (\Theta(E, h)(V_j, \overline{\partial} V_k) u, v) + (\overline{\partial} V_j |_{X_i \cup} u, \overline{\partial} V_k |_{X_i \cup} v).
\end{align}

Now let us control the second fundamental form $(P^1 D_{j} u, P^1 D_{k} v)$. Since each $P^1 D_j u$ is the $L^2$-minimal solution of
\begin{align}
(3.7) \quad \overline{\partial} (\cdot) = \overline{\partial} D_{j} u,
\end{align}

similar as the product case, one may also use the Hörmander’s $L^2$-theory to control the norm of $P^1 D_j u$. By definition, we have
\begin{align}
(3.8) \quad D_{j} u = [\overline{\partial} E, \delta_{V_j}] u.
\end{align}

Since
\begin{align}
(3.9) \quad [\overline{\partial}, [\overline{\partial} E, \delta_{V_j}]] + [\overline{\partial} E, [\overline{\partial}, \delta_{V_j}]] + [\delta_{V_j}, [\overline{\partial}, \partial E]] = 0,
\end{align}

we have
\begin{align}
(3.10) \quad \overline{\partial} D_{j} u &= -[\partial E_i, \overline{\partial} V_j |_{X_i}] u - (V_j \Theta(E, h)) |_{X_i} \wedge u.
\end{align}

If each $\partial E_i u_j = 0$ then we know that $a := -\sum P^1 D_j u_j$ is the $L^2$-minimal solution of
\begin{align}
(3.11) \quad \overline{\partial} (\cdot) = \partial E_i b + c,
\end{align}

where
\begin{align}
(3.12) \quad b := \sum \overline{\partial} V_j |_{X_i \cup} u_j, \quad c := \sum (V_j \Theta(E, h)) |_{X_i} \wedge u_j.
\end{align}

By Hörmander’s $L^2$-theory, $i\Theta(E_t, h_t) \wedge (\omega^t)^{q} \geq 0$ implies that
\begin{align}
(3.13) \quad ||a||^2 \leq ||b||^2 + \lim_{\varepsilon \to 0} (Q_{\varepsilon}^{-1} c, c), \quad Q_{\varepsilon} := [i\Theta(E_t, h_t), \Lambda_{\omega^t}] + \varepsilon.
\end{align}

Thus we have
\begin{align}
(3.14) \quad \sum (\Theta_{j\bar{k}} u_j, u_k) \geq \sum (\Theta(E, h)(V_j, \overline{\partial} V_k) u_j, u_k) - \lim_{\varepsilon \to 0} (Q_{\varepsilon}^{-1} c, c).
\end{align}

By Lemma 3.10 in [34], we know that the right hand side is non-negative. Similar proof works for other parts of this theorem, please see [34] for the details.

By a similar argument, one may also prove the following:

**Theorem 3.2** (Yamaguchi’s theorem for a proper Kähler fibration). Assume that $\omega$ is Kähler, $c_{j\bar{k}}(\omega) \equiv 0$ and $i\Theta(E, h) \wedge \omega^q \geq 0$. Assume further that the dimension of $H^{n,q}(E_t)$ is a constant. If $g$ is a holomorphic section of the quotient bundle $\mathcal{D}^{n,q+1}$, and
\begin{align}
(3.15) \quad (D_{j} g)(t) = ([\partial E, \delta_{V_j}] g)|_{X_i} \equiv 0.
\end{align}

Then $-||g||^2_G$ is a smooth plurisubharmonic function on $B$. 

\[ \square \]
4. Twisted version of Griffiths’ theorem

We will give a short account of a recent joint work with Berndtsson and Paun. We shall show how to look at it by using the relative $\bar{\partial}$-complex. Let $E$ be a holomorphic vector bundle over the total space $X$. By Theorem 2.3 in [13] or [34], if $\dim H^{p,q}(E_t)$ does not depend on $t$ then one may look at

$$\mathcal{H}^{p,q} := R^t\pi_*\mathcal{O}(\wedge^p T^*_X/B \otimes E),$$

as the holomorphic quotient bundle $\mathcal{K}^{p,q}/\mathcal{T}^{p,q}$. Notice that the quotient norm of a class $[u^t]$ in $\mathcal{K}_t^{p,q}/\mathcal{T}_t^{p,q}$ is equal to

$$||[u^t]|| := \inf\{||u^t + v^t|| : v^t \in \mathcal{T}_t^{p,q}\}. \tag{4.1}$$

By the Hodge theory, we know that

$$\inf\{||u^t + v^t|| : v^t \in \mathcal{T}_t^{p,q}\} = ||\mathbb{H}u^t||,$$

where $\mathbb{H}u^t$ denotes the $\bar{\partial}$-harmonic part of $u^t$. We shall also write

$$\mathbb{H}^\perp u^t := u^t - \mathbb{H}u^t. \tag{4.3}$$

Let us denote by $\Theta_{jk}^H$ the curvature operators on $\mathcal{H}^{p,q}$. By the curvature formula for the quotient bundle, we have:

**Theorem 4.1.** Assume that the total space $X$ is Kähler. Assume further that $\dim H^{p,q}(E_t)$ does not depend on $t$. Then we have

$$\Theta_{jk}^H[u, v] = (\Theta_{jk}^K\mathbb{H}u, \mathbb{H}v) + \left(\mathbb{H}^\perp(\bar{\partial}_k\mathbb{H}u), \mathbb{H}^\perp(\bar{\partial}_j\mathbb{H}v)\right), \tag{4.4}$$

where $[u], [v]$ are smooth sections of $\mathcal{H}^{p,q}$.

Now let us consider the following special cases:

- **A:** $\omega$ is Kähler and $\Theta(E, h) \equiv 0$;
- **B:** $p + q = n$, $E$ is a line bundle and $i\Theta(E, h) = \pm \omega$.

By the Hodge theory, we know that in both cases, the space of $\bar{\partial}$-harmonic $E_t$-valued $(p, q)$-forms is equal to the space of $\bar{\partial}E_t$-harmonic $E_t$-valued $(p, q)$-forms. Thus

$$\langle d^E\mathbb{H}u \rangle_{X_t} \equiv 0, \tag{4.5}$$

which implies that

$$\langle [d^E, \delta_{\{V_j, V_k\}}]\mathbb{H}u, \mathbb{H}v \rangle \equiv 0. \tag{4.6}$$

Thus by (4.3), we have

$$\langle [L_j, L_k]\mathbb{H}u, \mathbb{H}v \rangle \equiv (\Theta(E, h)(V_j, \bar{V}_k)u, v). \tag{4.7}$$

Assume further that $B$ is one dimensional. Put

$$\Theta_{ij}^H := \Theta_{ij}^K, \bar{\partial}_t := \bar{\partial}_{ij}, \ D_t := D_{ij}, \ V := V_1, \ A := (\Theta(E, h)(V, \bar{V})u, u). \tag{4.8}$$

By (3.2), (3.1) and Theorem 4.1, we have

$$\langle \Theta_{ij}^H[u], [u] \rangle = A + ||\bar{\partial}V|_{X_t, \mathbb{H}u^t}\mathbb{H}u||^2 - ||\bar{\partial}V|_{X_t, \mathbb{H}u^t}\mathbb{H}u||^2 + ||\mathbb{H}^\perp(\bar{\partial}_t\mathbb{H}u)||^2 - ||P^\perp(D_t\mathbb{H}u)||^2. \tag{4.9}$$

We shall use the following proposition in [13]:

**Proposition 4.2.** Assume that **A** or **B** is true. Then $P^\perp(D_t\mathbb{H}u)$ is the $L^2$-minimal solution of $\bar{\partial}(\cdot) = -\partial^E(\bar{\partial}V|_{X_t, \mathbb{H}u^t})$ and $\mathbb{H}^\perp(\bar{\partial}_t\mathbb{H}u)$ is the $L^2$-minimal solution of $\partial^{E_t}(\cdot) = -\bar{\partial}(\partial^E|_{X_t, \mathbb{H}u^t})$. 
Let us denote by $\square$ the $\overline{\partial}$-Laplace and denote by $\Box$ the $\partial^{E_h}$-Laplace. By the Bochner-Kodaira-Nakano formula, in case A, we have $\square \alpha = \Box \alpha$, in case B, we have $\square \alpha = \Box \alpha + \alpha$ if $i \Theta(E,h) = \pm \omega$. Thus (4.12), Proposition 4.2 and page 15 in [5] together imply the following twisted version of Griffiths’ theorem in [13]:

**Theorem 4.3** (Twisted version of Griffiths’ theorem). Assume that $\omega$ is Kähler and $\dim H^{p,q}(E_t)$ does not depend on $t$. Assume further that $B$ is one dimensional. If $\Theta(E,h) \equiv 0$ then we have the following Griffiths formula:

$$
(\Theta_{It}^{H}_t[u], [u]) = ||H(\overline{\partial}V|_{X_t, \square} \mathbb{H}u)||^2 - ||H(\overline{\partial}V|_{X_t, \Box} \mathbb{H}u)||^2.
$$

If $p + q = n$, $E$ is a line bundle and $i \Theta(E,h) = \omega$ then

$$
(\Theta_{It}^{H}_t[u], [u]) \geq (|V|^2 \mathbb{H}u, \mathbb{H}u) + ||H(\overline{\partial}V|_{X_t, \square} \mathbb{H}u)||^2 - ||H(\overline{\partial}V|_{X_t, \Box} \mathbb{H}u)||^2.
$$

If $p + q = n$, $E$ is a line bundle and $i \Theta(E,h) = -\omega$ then

$$
(\Theta_{It}^{H}_t[u], [u]) \leq -(|V|^2 \mathbb{H}u, \mathbb{H}u) + ||H(\overline{\partial}V|_{X_t, \square} \mathbb{H}u)||^2 - ||H(\overline{\partial}V|_{X_t, \Box} \mathbb{H}u)||^2.
$$

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