Dissipative dynamics of superconducting hybrid qubit systems

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Abstract. We perform a theoretical study of composed superconducting qubit systems for the case of a coupled qubit configuration based on a hybrid qubit circuit made of both charge and phase qubits, which are coupled via a $\sigma_x \otimes \sigma_z$ interaction. We compute the system’s eigen-energies in terms of the qubit transition frequencies and the strength of the inter-qubit coupling, and describe the sensitivity of the energy crossing/anti-crossing features to such coupling. We compute the hybrid system’s dissipative dynamics for the cases of i) collective and ii) independent decoherence, whereby the system interacts with one common and two different baths of harmonic oscillators, respectively. The calculations have been performed within the Bloch-Redfield formalism and we report the solutions for the populations and the coherences of the system’s reduced density matrix. The dephasing and relaxation rates are explicitly calculated as a function of the heat bath temperature.

1. Introduction
A quantum computer uses quantum two-level systems or quantum bits (qubits) which must be essentially interacting systems weakly or, even more desirable, completely decoupled from dissipative environments [1]. The low temperature superconductors, like aluminum, are good candidates for this purpose. For this reason, in the last few years, interest in the study of superconducting-qubit systems has grown [2, 3]. In these kind of systems, the circuit must be cooled to a temperature at which the thermal energy $\beta^{-1} \equiv k_BT$ is less than the transition energy $\hbar \omega_{0,1}$ between the qubit basis states, say |0⟩ and |1⟩. Another important requirement is that $\hbar \omega_{0,1} << \Delta$, where $\Delta$ is the energy gap of the superconducting material [4]. The basic elements used in the preparation of superconducting qubits are the Josephson junctions, which are one of the few electronic systems which, besides being nonlinear, also exhibit the property of a very weak dissipation [5]. Quantum information processors require a qubit coherent dynamics in order to allow the construction of quantum logic gates and circuits [1]. The hybrid systems have had an impact in the development of these circuits because they can bind the scales of atomic motion to macroscopic degrees of freedom [6]. Some solid-state systems, such as the so-called superconducting qubits, allow the implementation of such hybrid systems in their respective circuits which can be connected in different ways, in the form of simple circuit elements [7].

2. Model
Hybrid quantum circuit. — A generic superconducting qubit can be modeled by the Hamiltonian

$$\hat{H}_{qb} = -\frac{1}{2} \epsilon \hat{\sigma}_z - \frac{1}{2} \Delta \hat{\sigma}_x,$$

(1)
where \( \epsilon \) and \( \Delta \) denote the longitudinal and transversal parameters of the corresponding qubit, and \( \sigma_i \) are the usual Pauli matrices. Charge and flux qubits are two different types of superconducting qubits for quantum computing. In short, the charge qubit has the advantage of a more flexible controllability via external parameters: it can be conveniently controlled by a voltage gate or an applied magnetic flux. These external control parameters appear in the longitudinal (\( \sigma_z \)) and transverse (\( \sigma_x \)) terms of the circuit’s reduced Hamiltonian. For the flux qubit, the longitudinal term can be controlled by the applied magnetic flux, but it is hard to control the transversal term via an external parameter.

The hybrid quantum circuit we study in this work is a system that couples a charge and a flux qubit, as schematically shown in Figure 1, with an effective interaction due to a Josephson junction that binds them. This system has been proposed in order to control the transversal term of the flux qubit with the charge qubit [7]. A calculation of the explicit Hamiltonian for the hybrid qubit system reads

\[
\hat{H}_{2q} = \epsilon_1 (V_q) \sigma_z^{(1)} - \Delta_1 \sigma_x^{(1)} + \epsilon_2 (\phi_2) \sigma_z^{(2)} - \Delta_2 \sigma_x^{(2)} + \chi \sigma_z^{(1)} \sigma_z^{(2)},
\]

where \( \epsilon_1 \) and \( \Delta_1 \) are the longitudinal and transversal parameters of the charge qubit, and likewise \( \epsilon_2 \) and \( \Delta_2 \) for the flux qubit. The interaction between the qubits gives rise to an effective \( \sigma_x \otimes \sigma_z \) geometric term [7] with strength \( \chi \).

3. Coupling to the bath

In order to study the system’s dissipative dynamics, we modeled the dissipative environment as either one or two baths of harmonic oscillators which are coupled to the \( \sigma_z \) components of each qubit. The first case considers each qubit coupled to its own bath of oscillators in the form \( \hat{\sigma}_z^{(i)} \hat{X}^i \), where \( \hat{X}^i \) (\( i = 1, 2 \)) denotes the bath coordinates. The full interacting Hamiltonian reads

\[
\hat{H}_{2B} = \sum_{i=1,2} \left( -\frac{1}{2} \epsilon_i \hat{\sigma}_z^{(i)} - \frac{1}{2} \Delta_i \hat{\sigma}_x^{(i)} + \frac{1}{2} \hat{\sigma}_z^{(i)} \hat{X}^i \right) + \chi \sigma_x^{(1)} \sigma_z^{(2)} + \hat{H}_{B1} + \hat{H}_{B2},
\]

where \( \hat{H}_{Bi} \) (\( i = 1, 2 \)) is the Hamiltonian of the \( i \)-th bath. In the second case, we consider that the qubits are coupled to a common bath. Hence, the full system’s Hamiltonian reads

\[
\hat{H}_B = \sum_{i=1,2} \left( -\frac{1}{2} \epsilon_i \hat{\sigma}_z^{(i)} - \frac{1}{2} \Delta_i \hat{\sigma}_x^{(i)} + \frac{1}{2} \left( \hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)} \right) \hat{X} + \chi \sigma_x^{(1)} \sigma_z^{(2)} \right) + \hat{H}_B,
\]

where \( \hat{X} \) is the bath coordinate and \( \hat{H}_B \) represents the Hamiltonian of the common bath.

**Bloch-Redfield formalism.** — This formalism is derived from a projector operator approach and provides an important tool for finding a set of coupled master equations which describe the dynamics of the reduced density matrix for a given system in contact with a dissipative environment. The Liouville equation for the total density operator \( \rho_T \) (of the whole system) is given by
\[ i\hbar \frac{d}{dt} \rho_{\text{R}}(t) = [\hat{H}(t), \rho_{\text{R}}(t)] \]. The corresponding reduced density matrix obeys, in the Born-Markov limit, the so-called Redfield equation [8]:

\[ \dot{\rho} = -i\omega_{nm}\rho_{nm}(t) - \sum_{kl} R_{nmkl}\rho_{kl}(t), \quad (5) \]

where the first term represents the reversible motion in terms of the transition energies \( h\omega_{nm} = E_n - E_m \), and the second term describes the system’s relaxation. The Redfield relaxation tensor \( R_{nmkl} \) includes the dissipative effects of the system-environment coupling:

\[ R_{nmkl} = \delta_{lm} \sum_r \Gamma_{nrrk}^{(+)} + \delta_{nk} \sum_r \Gamma_{lrrm}^{(-)} - \Gamma_{lmnk}^{(+)} - \Gamma_{lmnk}^{(-)}, \quad (6) \]

and its components are given by the ‘golden rule’ expressions

\[ \Gamma_{lmnk}^{(+)} = \bar{\hbar}^{-2} \int_0^\infty dt e^{-i\omega_{nl}t} \langle \hat{H}_{l,lm}(t)\hat{H}_{n,nk}(0) \rangle, \]

\[ \Gamma_{lmnk}^{(-)} = \bar{\hbar}^{-2} \int_0^\infty dt e^{-i\omega_{nl}t} \langle \hat{H}_{l,lm}(0)\hat{H}_{n,nk}(t) \rangle. \]

Here, \( \hat{H}_{l,lm}(t) = \exp(i\hat{H}_{l,lm}/\hbar)H_I \exp(-i\hat{H}_{l,lm}/\hbar) \) is the interaction term in the interaction picture, and the bracket denotes the thermal average over the bath’s degrees of freedom.

4. Results and Discussion

System’s eigen-energies.— These are worked out in the singlet/triplet basis states. In this basis, the Hamiltonian \( \hat{H}_{2q} \) of the two-qubit system assumes the matrix form:

\[ \hat{H}_{2q} = -\frac{1}{2} \left( \begin{array}{cccc} \epsilon & \eta + \chi & 0 & -\chi \\ \eta + \chi & 0 & \eta - \chi & 0 \\ 0 & \eta - \chi & -\epsilon & -\chi \\ -\chi & 0 & -\chi & 0 \end{array} \right), \quad (8) \]

where we have introduced \( \epsilon = \epsilon_1 + \epsilon_2 \), \( \Delta \epsilon = \epsilon_1 - \epsilon_2 \), \( \eta = \Delta \eta \sqrt{\epsilon_1 \epsilon_2} \), \( \Delta \eta = \Delta \eta_1 - \Delta \eta_2 \), and considered qubits with the same transition energies \( \epsilon_1 = \epsilon_2 \) and transversal coupling \( \Delta \eta_1 = \Delta \eta_2 \). Thus, \( \Delta \epsilon = \Delta \eta = 0 \).

In Figure 2, we present the system’s energies for three different scenarios, in which two of the parameters \( \epsilon, \eta, \chi \) have been fixed and the third one is varied. In Figure 2(a) we have varied the longitudinal parameter \( \epsilon \), showing a crossing between the energy of the singlet \( E_1 \) and one of the energies of the triplet \( E_3 \). In Figure 2(b) the influence of the transversal parameter \( \eta \) has been plotted. We notice the appearance of the same crossing observed in Figure 2(a), between the singlet and the \( E_3 \)-triplet state, which indicates that these states do not interact. Figure 2(c) shows the dependence on the coupling term \( \chi \). Now, a crossing between the energy of the singlet state and that of the triplet state \( E_4 \) takes place. Note that for large values of \( \chi \), there is a degeneracy between the singlet state \( E_1 \) and the triplet state \( E_3 \). This feature means that for large positive values of the parameter \( \chi \), a large ferromagnetic coupling is present between the qubits.

Dissipative dynamics and temperature effects.— The solution of the Redfield equation \( \dot{\rho}_{nm} = -\sum_{kl} R_{nmkl}\rho_{kl}(t) \) allows to analyse the system’s reduced dynamics, for which explicit expressions have been found for the components of Equation (7), for both types of coupling to the environment. In the case of independent coupling we get

\[ \Gamma_{lmnk}^{(+)} = \Gamma_{lmnk}^{(-)} = \frac{1}{4\hbar} \left[ \sigma_{z,lm}^{(1)}\sigma_{z,nk}^{(1)}\alpha_1 + \sigma_{z,lm}^{(2)}\sigma_{z,nk}^{(2)}\alpha_2 \right], \quad (9) \]
Figure 2. Energy spectrum of the coupled hybrid two-qubit system. (a) $\chi = 10 \text{ GHz}$, $\eta = 1 \text{ GHz}$, and $\epsilon$ is varied, (b) $\chi = 1 \text{ GHz}$, $\epsilon = 1 \text{ GHz}$ ($\eta$ is varied), and (c) $\epsilon = \eta = 1 \text{ GHz}$ ($\chi$ is varied). In all the graphs, $E_1$ is the singlet state and $E_2, E_3,$ and $E_4$ are the triplet states.

where the $\alpha_i$’s correspond to the bath coupling parameters for the specific cases of Ohmic spectral densities $J_i(\omega) = \alpha_i \hbar / (1 + \omega^2 / \omega_c^2)$, and $\sigma_{z,ij} \equiv \langle E_i | \sigma_z | E_j \rangle$. For the case of collective coupling, $J(\omega) = \alpha \hbar / (1 + \omega^2 / \omega_c^2)$, and the rates become

$$\Gamma_{lmnk}^{(+)} = \Gamma_{lmnk}^{(-)} = \frac{\alpha}{4\beta \hbar} \left[ \sigma_{z,lm}^{(1)} \sigma_{z,nk}^{(1)} + \sigma_{z,lm}^{(1)} \sigma_{z,nk}^{(2)} + \sigma_{z,lm}^{(2)} \sigma_{z,nk}^{(1)} + \sigma_{z,lm}^{(2)} \sigma_{z,nk}^{(2)} \right]. \tag{10}$$

Figure 3. Populations dynamics, $\rho_{ii}(t)$. The density matrix calculation has been performed for the initial conditions: a) pure state $|E_1\rangle$, and b) maximally entangled Bell state $|\Phi^+\rangle$. Lower graphs show the relaxation rates (see main text) as a function of the bath temperature for the cases of c) independent and d) collective coupling to the environment. $\alpha_i = \alpha = 10^{-3}$.

The reduced density matrix has been computed from the system of differential equations (5). Figure 3 shows the time dependence of the system’s populations $\rho_{ii}$ for a coupling to two
Figure 4. Decoherence rates as a function of the reservoir temperature for: a) independent and b) collective bath coupling. $\alpha = \alpha = 10^{-3}$.

different kinds of environments: i) independent and ii) collective or common baths. Figures 3(a) and 3(b) correspond to the initial conditions $|E_1\rangle$, and $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, respectively.

The population $\rho_{11}$ has a slightly faster decay in the case of collective coupling for the initial condition $|E_1\rangle$, whereas this behaviour is reversed for the case of the entangled initial condition $|\Phi^+\rangle$. Thus, the dissipative population dynamics is very sensitive to initial state preparations.

The behaviour of the relaxation rates $\Gamma^{i/c}_{nn} = Re(\mathcal{R}^{i/c}_{nmm})$ as a function of the bath temperature is shown in Figures 3(c) and (d), for the initial conditions $|E_i\rangle$. The superscripts $i$, and $c$ stand for independent (Fig. 3(c)) and collective (Fig. 3(d)) couplings respectively. In agreement with graph 3(a), this shows that the rate $\Gamma_{11}$ tends to a higher value for the case of collective coupling.

In Figure 4, we have plotted the dephasing (decoherence) rates $\Gamma^{i/c}_{nm} = Re(\mathcal{R}^{i/c}_{nmnm})$, as a function of the bath temperature, for the cases of a) independent and b) collective coupling. We note that, in both cases, the decoherence rates are of the same order of magnitude. Furthermore, it is observed that $\Gamma_{21}$ is the minimum dephasing rate, while $\Gamma_{31}$ displays the higher rate.

By comparing the decoherence and the relaxation rates obtained, we conclude that both types of dissipation mechanisms occur on the same time scale. This implies a long decoherence time, which is a key requirement in the construction of quantum devices that are expected to work as quantum information processors.

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References
[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum information (Cambridge)
[2] Makhlin Y, Schö n G and Shnirman A 1999 Nature 398 305
[3] Nakamura Y, Pashkin Y A and Tsai J S 1999 Nature 398 786
[4] de Gennes P G 1966 Superconductivity of Metals and Alloys (New York: W H Benjamin)
[5] Ibach H and Luth H 1995 Solid State Physics (Springer)
[6] Neeley M et al. 2008 Nature Phys. 4 523
[7] Xiao-Ling H et al. 2007 Phys. Rev. B 76 024517
[8] Weiss U 1999 Quantum Dissipative Systems (World Scientific)
[9] Makhlin Y, Schö n G and Shnirman A 2001 Rev. Mod. Phys 73 357