State estimation for neural networks with random delays and stochastic communication protocol

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ABSTRACT
This paper deals with the state estimation problem for delayed neural networks under stochastic communication protocol (SCP). The time delay addressed is random and its probability distribution is known. The information between the sensors and state estimator is transmitted through constrained networks, and the SCP is introduced to determine which sensor could send data at a specific time. Moreover, a Markov chain is applied to described the SCP scheduling in networks. By using the stochastic analysis method, some delay-distribution-dependent conditions are obtained to guarantee the stability of the error dynamics with $H_{\infty}$ performance. Finally, a numerical example is provided to illustrate the effectiveness of the proposed method.

ARTICLE HISTORY
Received 1 July 2018
Accepted 2 October 2018

1. Introduction
Neural networks are a class of highly-nonlinear dynamic systems that can imitate human brain for information processing and calculation, which has been applied in various fields such as signal processing, pattern recognition, automatic control and so on. On the other hand, due to good features of high reliability, low cost, and ease of expansion, network communications are widely used in many different aspects of society. Although the network communication has strong stability and reliability, there are still plenty of network-induced problems in the actual networked system affected by the communication mechanism, communication protocol, and network bandwidth during the transmission of the network channel. The common network phenomena includes network-induced delay (Niu, Sheng, & Wang, 2016; Pang, Liu, Zhou, & Sun, 2016; Wang & Li, 2016), data packet dropout (Hu, Wang, & Liu, 2016; Sheng, Wang, & Wang, 2017; Tan, Li, & Zhang, 2015), quantization (Ma, Wang, Han, & Lam, 2018; Wan, Cao, & Guo, 2017; Zhang, Zhu, & Zheng, 2016; Zhu, Sugie, & Fujimoto, 2015), sampling (Li, Shen, & Liu, 2017; Sheng, Wang, & Zou, 2017; Wang, Wang, & Huang, 2016), communication protocols (Alsaadi, Luo, Liu, & Wang, 2016; Long & Yang, 2014; Luo, Wang, Wei, Alsaadi, & Hayat, 2016; Ugrinovskii & Fridman, 2014; Zou, Wang, & Gao, 2016a, 2016b, 2017; Zou, Wang, Gao, & Liu, 2016) and so on.

The stability of the time-delay system has always been a topic of concern and time delay is inevitable in the control system (Lu & Huang, 2014; Ma, Wang, Liu, & Alsaadi, 2017; Souza & Coutinho, 2014; Yuan, Wang, & Guo, 2018). However, in the earlier investigations, only the range of the time delay was discussed without considering the its randomness, and the stability criteria were also based only on the time-varying range (Lu & Huang, 2014; Ma, 2014; Souza & Coutinho, 2014). In fact, the time delays in neural networks are usually random in actual situations (Hirasawa, Mabu, & Hu, 2006). In these circumstances, the output signal of the node is transmitted to another node with arbitrary time delays. These time delays are random and their probabilistic characteristics can usually be obtained by statistical methods such as Poisson distribution, Gaussian distribution and so on. In this case, if some values of the time delay are very large, but the probabilities to take these values are very small, it will lead to quite conservative results when only the range of the time delay is considered (Sheng, Wang, Tian, & Alsaadi, 2016; Yue, Zhang, Tian, & Peng, 2008). Until now, only a small fraction of attention has been paid to the stability of neural networks with time-varying random delays (Bao & Cao, 2011; Meng, Lam, Du, & Gao, 2010), while considering both the range of the delay and its probability distribution. A Bernoulli variable has been introduced to characterize random delays...
(Yue, Zhang, Tian, & Chen, 2008), and several less conservative stability conditions for delayed neural networks have been deduced. By using both the probability distribution of time delay and the variation range, the problem of exponential $H_{\infty}$ filtering has been solved (Mathiyalan, Su, Shi, & Sakhivel, 2015). However, the investigation on random time-delay still needs more attention to solve practical problems more perfectly.

Due to the limitation of the network bandwidth and the hardware devices, the disorderly access of multiple nodes to the network would result in data collisions. In order to solve this problem, communication protocols are generally used in actual network communication process to coordinate the transmission sequence of the nodes, which improves the utilization of the network and makes the network communication more organized. In addition, the communication protocol can also ease the network load and reduce the frequency of network-induced problems. Up to now, the widely used network protocols include stochastic communication protocol (Alsaad et al., 2018; Long & Yang, 2014; Zou et al., 2016a, 2017), Round-Robin protocol (Luo et al., 2016; Ugrinovski & Fridman, 2014; Zou et al., 2016), and Try-Once-Discard protocol (Zou et al., 2016b). The stochastic communication protocol is an abstract model established by several types of network protocols used in industry. At present, stochastic communication protocol has been widely applied to wireless networks, satellite networks, and local area networks.

In reality, the internal states of systems cannot be measured directly in most cases. In addition, due to the various influences of natural factors and human factors, interferences and noises are ubiquitous in the work process of the system (Ma, Wang, Han, & Liu, 2017; Ma, Wang, Lam, & Kyriakoulis, 2017; Yuan, Yuan, Wang, Guo, & Yang, 2017). Based on inaccurate measurements, it is difficult to estimate the states of system (Yuan, Wang, Zhang, & Dong, 2018). During the past few decades, a lot of results have been derived for state estimation of neural networks (Kan, Wang, & Shu, 2013; Shen, Zhu, & Zhang, 2017; Yang, Dong, & Wang, 2016). Many design methods of the state estimator for linear systems have been proposed and successfully applied to neural networks, such as Wiener filtering, Kalman filtering and so on. However, up to now, few investigations have been made on the state estimation of neural networks with random time-delay restricted by network stochastic communication protocol. In fact, it is unreasonable to consider the range of time-delay without considering its distribution probability since the random time-delay is ubiquitous in neural networks. Besides, there are usually multiple sensor nodes due to the special nature of the neural network system's structure. When multiple nodes access the network at the same time, the data cannot be sent normally because of the collisions. Therefore, it is very important to study the neural network state estimation problem with random time-delay under the stochastic communication protocol.

Notations Throughout this paper, $\mathbb{R}$ (respectively, $\mathbb{N}^+$) is the set of all real numbers (respectively, non-negative integers). $\mathbb{R}^n$ is the set of all real $n$-dimensional vectors and $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices. $A > 0$ (respectively, $A \geq 0$) is a real symmetric positive definite (respectively, positive semi-definite) matrix. $A^T$ denotes the transpose of a matrix $A$. $[a : b]$ is a set involving all integers between $a$ and $b$. $\mathbb{E} \{x\}$ stands for the mathematical expectation of $x$. $\text{diag} \{ \cdots \}$ is a block-diagonal matrix. The asterisk $*$ in a matrix is used to denote the term that is induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2. Problem formulation and preliminaries

In this section, some preliminaries related to $H_{\infty}$ state estimation of neural network are introduced. Besides, the problem formulation is given in this part. Consider the following neural network with time-delay:

$$
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Gf(x(k)) + Hg(x(k - d(k))) + C\nu(k) \\
\dot{y}(k) &= D\dot{x}(k) + Ev(k) \\
z(k) &= Fx(k) \\
x(e) &= \varphi(e), \quad -d_m \leq e \leq 0
\end{align*}
$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector associated with $n$ neurons, $y(k) \in \mathbb{R}^{n_y}$ is the measurement output, $z(k) \in \mathbb{R}^{n_z}$ is the neural signal to be estimated and $\omega(k)$ is a one-dimensional zero-mean Gaussian white noise sequence. $\nu(k) \in \mathbb{R}^{n_{\nu}}$ is the disturbance input which belongs to $L_2([0, \infty), \mathbb{R}^{n_{\nu}}_+)$. $d(k) \in [d_m : d_M] \quad (0 \leq d_m \leq d_M)$ is the time-varying random delay. $\varphi(e), -d_m < e \leq 0$ is the initial condition. $A, G, H, C, \bar{A}, \bar{B}, D, E$ and $F$ are known real constant matrices with appropriate dimensions.

The neuron activation functions $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]^T$, $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), \ldots, g_n(x_n(k))]^T$ are continuous, and satisfy $f(0) = 0$, $g(0) = 0$ and the following sector-bounded conditions:

$$
\begin{align*}
[f(x) - f(y) - \Phi_f(x - y)]^T[f(x) - f(y) - \Psi_f(x - y)] &\leq 0 \\
[g(x) - g(y) - \Phi_g(x - y)]^T[g(x) - g(y) - \Psi_g(x - y)] &\leq 0
\end{align*}
$$

for all $x, y \in \mathbb{R}^n$, where $\Phi_f, \Psi_f, \Phi_g, \Psi_g$ are real matrices with appropriate dimensions.
2.1. Random delay

In neural network (1), it is assumed that the random delay $d(k)$ is bounded and its probability distribution can be derived. Suppose that $d(k)$ takes values in $[d_1^m : d_1^M]$, or $[d_2^m : d_2^M]$, or ... , or $[d_N^m : d_N^M]$ with $d_m \leq d_1^m < d_2^m < d_2^m < \ldots < d_N^m < d_N^M$, and

\[
\begin{align*}
\text{Prob}(d(k) \in [d_1^m : d_1^M]) &= \tilde{a}_1, \\
\text{Prob}(d(k) \in [d_2^m : d_2^M]) &= \tilde{a}_2, \ldots \\
\text{Prob}(d(k) \in [d_N^m : d_N^M]) &= \tilde{a}_N,
\end{align*}
\]

where $0 \leq \tilde{a}_s \leq 1, s = 1, 2, \ldots, N$ and $\sum_{s=1}^{N} \tilde{a}_s = 1$.

Define the following mapping functions and stochastic variables

\[
\begin{align*}
d_1(k) &= \begin{cases} 
(d(k), & d(k) \in [d_1^m : d_1^M] \\
\tilde{d}_1^m, & \text{else}
\end{cases} \\
d_2(k) &= \begin{cases} 
(d(k), & d(k) \in [d_2^m : d_2^M] \\
\tilde{d}_2^m, & \text{else}
\end{cases} \\
d_N(k) &= \begin{cases} 
(d(k), & d(k) \in [d_N^m : d_N^M] \\
\tilde{d}_N^m, & \text{else}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\alpha_1(k) &= \begin{cases} 
1, & d(k) \in [d_1^m : d_1^M] \\
0, & \text{else}
\end{cases} \\
\alpha_2(k) &= \begin{cases} 
1, & d(k) \in [d_2^m : d_2^M] \\
0, & \text{else}
\end{cases} \\
\alpha_N(k) &= \begin{cases} 
1, & d(k) \in [d_N^m : d_N^M] \\
0, & \text{else}
\end{cases}
\end{align*}
\]

According to (3), we have \(\text{Prob}(\alpha_s(k) = 1) = E[\alpha_s(k)] = \tilde{a}_s, s = 1, 2, \ldots, N\). Then, the system (1) can be rewritten as

\[
x(k + 1) = Ax(k) + Gf(x(k)) + \sum_{s=1}^{N} \alpha_s(k)Hg(x(k) - d_s(k))) + Cu(k) + \tilde{A}x(k) + \sum_{s=1}^{N} \alpha_s(k)\tilde{B}x(k) - d_s(k)
\]

\[
y(k) = Dx(k) + Ev(k)
\]

\[
z(k) = Fx(k)
\]

\[
x(e) = \psi(e), \quad -d_M \leq e \leq 0.
\]

(4)

2.2. Stochastic communication protocol (SCP)

The measurement output is \(y(k) = \begin{bmatrix} y_1^T(k) & y_2^T(k) & \ldots & y_{n_y}^T(k) \end{bmatrix}^T\). In this paper, assume that only one measurement signal \(y_i(k)(i = 1, 2, \ldots, n_y)\) is transmitted at time \(k\). Let \(y_{\sigma(k)}(k)\) denote the signal sent to the network at time \(k\). \(\sigma(k)\) can be regarded as a stochastic process which could be modelled by a Markov chain:

\[
\text{Prob}[\sigma(k + 1) = j | \sigma(k) = i] = p_{ij}^k, \quad i, j = 1, 2, \ldots, n_y
\]

where \(p_{ij}^k \geq 0\) is the transition probability from \(i\) to \(j\) at time instant \(k\) and \(\sum_{j=1}^{n_y} p_{ij}^k = 1(i = 1, 2, \ldots, n_y)\).

Let \(\tilde{y}(k) = \begin{bmatrix} \tilde{y}_1^T(k) & \tilde{y}_2^T(k) & \ldots & \tilde{y}_{n_y}^T(k) \end{bmatrix}^T\) be the measurement output after transmitted, where

\[
\tilde{y}_{\sigma(k)}(k) = \begin{cases} 
y_i(k), & \text{if } i = \sigma(k) \\
\tilde{y}_{i}(k - 1), & \text{otherwise}
\end{cases}
\]

Then we have

\[
\tilde{y}(k) = \Phi_{\sigma(k)}y(k) + (I - \Phi_{\sigma(k)})\tilde{y}(k - 1)
\]

(7)

where \(\Phi_{\sigma(k)} = \text{diag} [\tilde{\sigma}_{\sigma(k)}^1, \tilde{\sigma}_{\sigma(k)}^2, \ldots, \tilde{\sigma}_{\sigma(k)}^{n_y}]\), and \(\tilde{\sigma}_{\sigma(k)}^s \equiv \delta(a - b)\) in which \(\delta(\cdot)\) is the Kronecker delta function.

2.3. State estimator

In this paper, we will investigate the problem of \(H_\infty\) state estimation for a class of neural networks with time-varying random delays under the stochastic communication protocol. The state estimator is of the form

\[
\hat{x}(k + 1) = \tilde{A}\hat{x}(k) + Gf(x(k)) + \sum_{i=1}^{N} \tilde{a}_sHg(\hat{x}(k) - d_s(k)) + L_{\sigma(k)}(\tilde{y}(k) - D\hat{x}(k))
\]

\[
\hat{z}(k) = F\hat{x}(k)
\]

(8)

where \(\hat{x}(k)\) is the estimator of the neuron state, and \(L_{\sigma(k)} \in \mathbb{R}^{n_{x} \times n_y}\) is the estimator gain matrix to be designed.

Setting \(\hat{z}(k) = [\hat{x}(k) \hat{z}(k) \hat{y}(k - 1)]^T\) and \(\hat{z}(k) = z(k) - \hat{z}(k)\), the neural system with the estimator (8) is obtained as follows:

\[
\begin{align*}
\hat{z}(k + 1) &= A_{\sigma(k)}\hat{z}(k) + Gf(\hat{z}(k)) + \eta\mathcal{H}g_{\eta}(\hat{z}(k)) \\
&\quad + \tilde{\eta}(k)\mathcal{H}g_{\eta}(\hat{z}(k)) + C_{\sigma(k)}v(k) \\
&\quad + [\tilde{A}\hat{z}(k) + \eta\mathcal{H}g_{\eta}(\hat{z}(k)) + \tilde{B}\hat{z}(k)]\hat{z}(k) + \omega(k)
\end{align*}
\]

\[
\hat{z}(k) = F\hat{z}(k)
\]

(9)

where

\[
A_{\sigma(k)} = \begin{bmatrix} A & 0 & 0 \\
L_{\sigma(k)} & A - L_{\sigma(k)} & L_{\sigma(k)}(I - \Phi_{\sigma(k)}) \\
\Phi_{\sigma(k)}D & 0 & 0
\end{bmatrix},
\]

\[
G = \begin{bmatrix} G & 0 & 0 \\
0 & G & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H & 0 & 0 \\
0 & H & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
The system (9) is said to be stochastically stable if the following requirements are satisfied:

(i) The system (9) with \( v(k) = 0 \) is stochastically stable.
(ii) Under the zero-initial condition, for any \( v(k) \in L_2[0, \infty) \), the estimator error \( \tilde{z}(k) \) satisfies

\[
\sum_{k=0}^{\infty} \mathbb{E}\{||\tilde{z}(k)||^2\} < \phi^2 \sum_{k=0}^{\infty} \mathbb{E}\{||v(k)||^2\} \tag{10}
\]

where \( \phi > 0 \) is a given disturbance attenuation level.

**Lemma 2.1 (Schur Complement):** For a given matrix

\[
S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0,
\]

is equivalent to any one of the following conditions:

(i) \( S_{22} < 0, S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0 \);
(ii) \( S_{11} < 0, S_{22} - S_{21}^T S_{12}^{-1} S_{12} < 0 \).

**Remark 2.1:** In this paper, we consider a kind of random delays, which is characterized by introducing a sequence of Bernoulli stochastic variables. A general class of delayed stochastic neural networks is studied, and the system state and disturbance input are both subject to noises. The neural network model is comprehensive to describe the practical phenomena more precisely. The nonlinear description in (2) is quite general that includes the usual Lipschitz condition as a special case, and it provides a vector-based sector-bounded condition that would facilitate the mathematical analysis on the dynamic behaviours of neural networks.

### 3. Main results

**Theorem 3.1:** Assume that estimator gain \( L_{\sigma(k)} \) is given. The system (9) is stochastically stable with a prescribed \( H_\infty \) performance \( \phi > 0 \) if there exist positive definite matrices \( P_{\sigma(k)} = \text{diag}(P_{\sigma(k)}^1, P_{\sigma(k)}^2, P_{\sigma(k)}^3) > 0 \), \( Q_5 > 0 \), and positive constant scalars \( \epsilon_{\sigma(k)} > 0, \delta_{\sigma(k),s} > 0 \), where \( \sigma(k) = 1, 2, \ldots, n_y, s = 1, 2, \ldots, N \) such that the following LMI holds:

\[
\tilde{\Omega} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ \Omega_{21} & \Omega_{22} & 0 & \Omega_{24} & 0 \\ \Omega_{31} & 0 & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{45} \\ \Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} \end{bmatrix} < 0 \tag{11}
\]

where

\[
\begin{align*}
\Omega_{11} &= \Omega_{11} + F^T F, \\
\Omega_{12} &= A^T_{\sigma(k)} \tilde{P}_r A_{\sigma(k)} - P_i + A^T \tilde{P}_r \tilde{A} + \sum_{s=1}^{N} (d_s^M - d_s^m + 1) Q_s - \epsilon_{\sigma(k)} \mu_f, \\
\Omega_{13} &= A^T_{\sigma(k)} \tilde{P}_r \eta^T B, \\
\Omega_{14} &= \frac{A^T_{\sigma(k)} \tilde{P}_r \eta^T \eta B}{2}, \\
\Omega_{15} &= \frac{A^T_{\sigma(k)} \tilde{P}_r C_{\sigma(k)}}{2}, \\
\Omega_{21} &= B^T \eta^T \tilde{P}_r \eta \B, \\
\Omega_{22} &= B^T \eta^T \tilde{P}_r \eta \B - \text{diag}(Q_1, Q_2, \ldots, Q_N) - \mu_{g_2}, \\
\Omega_{24} &= \frac{\mu_{g_1}}{2}, \\
\Omega_{31} &= G^T \tilde{P}_r G - \epsilon_{\sigma(k)} I, \\
\Omega_{33} &= G^T \tilde{P}_r G, \\
\Omega_{34} &= G^T \tilde{P}_r \eta H, \\
\Omega_{35} &= G^T \tilde{P}_r C_{\sigma(k)}, \\
\Omega_{44} &= H^T \eta^T \tilde{P}_r \eta H + H^T \tilde{P}_r (k) \tilde{P}_r (k) \H - \gamma_{\sigma(k),s}, \\
\Omega_{45} &= H^T \eta^T \tilde{P}_r C_{\sigma(k)}, \\
\Omega_{55} &= \Omega_{55} - \phi^2 I, \\
\end{align*}
\]
\[
\mu_1 = \begin{bmatrix}
\Phi_g + \Psi_g \delta_{\sigma(k),1} l & 0 \\
0 & (\Phi_g + \Psi_g) \delta_{\sigma(k),2} l \\
0 & 0 \\
0 & 0 \\
(\Phi_g + \Psi_g) \delta_{\sigma(k),3} l & 0 \\
0 & (\Phi_g + \Psi_g) \delta_{\sigma(k),N} l 
\end{bmatrix},
\]
\[
\mu_2 = \frac{1}{2} \begin{bmatrix}
\Phi_g^T \Phi_g + \Psi_g^T \Phi_g \delta_{\sigma(k),1} l & 0 \\
0 & (\Phi_g^T \Phi_g + \Psi_g^T \Phi_g) \delta_{\sigma(k),2} l \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}.
\]

**Proof:** Choose the following Lyapunov functional \( V(k) \)
\[
V(k) = V_1(k) + \sum_{s=1}^{N} V_s(k)
\]
where
\[
V_1(k) = \zeta^T(k) P_{\sigma(k)} \zeta(k),
\]
\[
V_s(k) = \sum_{l=k-d_s(k)}^{k-1} \zeta^T(l) Q_s \zeta(l)
\]
\[
+ \sum_{l=-d_0}^{-1} \sum_{t=k-l}^{k-1} \zeta^T(t) Q_s \zeta(t), \quad s = 1, 2, \ldots, N.
\]

Along the trajectory of system (9), we calculate the expectation of the difference of \( V(k) \)
\[
\mathbb{E}[V(k+1) - V(k)] = \mathbb{E} \left\{ V_1(k+1) + \sum_{s=1}^{N} V_s(k+1) - V_1(k) - \sum_{s=1}^{N} V_s(k) \right\}
\]
\[
= \mathbb{E} \left\{ \Delta V_1(k) + \sum_{s=1}^{N} \Delta V_s(k) \right\}, \quad (12)
\]
and
\[
\mathbb{E}[\Delta V_1(k)] = \mathbb{E}[\zeta^T(k+1) P_{\sigma(k+1)} \zeta(k+1)]
\]
\[
- \zeta^T(k) P_{\sigma(k)} \zeta(k) \sigma(k) = \|l\|
\]
\[
= \mathbb{E}[\zeta^T(k+1) \tilde{P} \zeta(k+1) - \zeta^T(k) P_{\sigma(k)} \zeta(k) | \sigma(k) = \|l\|] = \mathbb{E}[\zeta^T(k) A_{\sigma(k)}^T \tilde{P} A_{\sigma(k)} \zeta(k) - \zeta^T(k) P_{\sigma(k)} \zeta(k)]
\]
\[
+ 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\|
\]
\[
+ \zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2f^T(k) \zeta(k) \sigma(k) \|l\|
\]
\[
= \mathbb{E}[\Delta V_1(k)] = \mathbb{E}[\zeta^T(k+1) P_{\sigma(k+1)} \zeta(k+1)]
\]
\[
- \zeta^T(k) P_{\sigma(k)} \zeta(k) \sigma(k) = \|l\|
\]
\[
= \mathbb{E}[\zeta^T(k+1) \tilde{P} \zeta(k+1) - \zeta^T(k) P_{\sigma(k)} \zeta(k) | \sigma(k) = \|l\|] = \mathbb{E}[\zeta^T(k) A_{\sigma(k)}^T \tilde{P} A_{\sigma(k)} \zeta(k) - \zeta^T(k) P_{\sigma(k)} \zeta(k)]
\]
\[
+ 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\|
\]
\[
+ \zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2f^T(k) \zeta(k) \sigma(k) \|l\|
\]
\[
= \mathbb{E}[\Delta V_1(k)] = \mathbb{E}[\zeta^T(k+1) P_{\sigma(k+1)} \zeta(k+1)]
\]
\[
- \zeta^T(k) P_{\sigma(k)} \zeta(k) \sigma(k) = \|l\|
\]
\[
= \mathbb{E}[\zeta^T(k+1) \tilde{P} \zeta(k+1) - \zeta^T(k) P_{\sigma(k)} \zeta(k) | \sigma(k) = \|l\|] = \mathbb{E}[\zeta^T(k) A_{\sigma(k)}^T \tilde{P} A_{\sigma(k)} \zeta(k) - \zeta^T(k) P_{\sigma(k)} \zeta(k)]
\]
\[
+ 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2\zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\|
\]
\[
+ \zeta^T(k) A_{\sigma(k)}^T \tilde{P} \zeta(k) \sigma(k) \|l\| + 2f^T(k) \zeta(k) \sigma(k) \|l\|
\]
\[
= \mathbb{E}[\Delta V_1(k)] = \mathbb{E}[\zeta^T(k+1) P_{\sigma(k+1)} \zeta(k+1)]
\]
\[
- \zeta^T(k) P_{\sigma(k)} \zeta(k) \sigma(k) = \|l\|
\]
There exists $\epsilon_{\sigma(k)} \geq 0, \delta_{\sigma(k),s} \geq 0$ such that
\[
e_{\sigma(k)}(f^T(\zeta(k))f(\zeta(k)) - f^T(\zeta(k))(\Phi_f + \Psi_f)(\zeta(k)) + \zeta^T(k)\mu_{\xi} (\zeta(k)) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) + \zeta^T(k) - d_q(k)))g(\zeta(k) - d_q(k))) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) + \zeta^T(k) - d_q(k)))g(\zeta(k) - d_q(k))). (18)
\]
Setting $\tilde{\zeta}(k) = [\zeta(k) \xi_N(k) f(\zeta(k)) g_N(\zeta(k)) \nu(k)]^T$, then we have
\[
\mathbb{E}[V(k + 1) - V(k)] \leq \mathbb{E}\left[\Delta V_1(k) + \sum_{i=1}^{N} \Delta V_2(k)\right] \
- \epsilon_{\sigma(k)}f^T(\zeta(k))f(\zeta(k)) \
- f^T(\zeta(k))(\Phi_f + \Psi_f)(\zeta(k)) \
+ \zeta^T(k)\mu_{\xi} (\zeta(k)) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) \
g^T(\zeta(k) - d_q(k)))g(\zeta(k) - d_q(k))) + \zeta^T(k) - d_q(k)))g(\zeta(k) - d_q(k))). (19)
\]
Adding the zero term
\[
\tilde{z}^T(k)\tilde{z}(k) - \phi^2 v^T(k) v(k) - [\tilde{z}^T(k)\tilde{z}(k) - \phi^2 v^T(k) v(k)]
\]
to both sides of (19) yields
\[
\mathbb{E}[\Delta V(k)] \leq \mathbb{E}[\xi^T(k)\Omega \xi(k) + \tilde{z}^T(k)\tilde{z}(k) - \phi^2 v^T(k) v(k)] \
- [\tilde{z}^T(k)\tilde{z}(k) - \phi^2 v^T(k) v(k)]
\]
Summing up (20) on both sides from 0 to $n$ with respect to $k$, one gets
\[
\mathbb{E}[V(n + 1) - V(0)] \leq \mathbb{E}\left[\sum_{k=0}^{n} \xi^T(k)\tilde{\Omega}_T(k)\right] \
- \mathbb{E}\left[\sum_{k=0}^{n} \tilde{z}^T(k)\tilde{z}(k) - \phi^2 v^T(k) v(k)\right] (21)
\]
where $\tilde{\Omega}$ is defined in (11). Letting $n \to \infty$ and considering the zero-initial condition, it can be obtained from (11) and (21) that
\[
\mathbb{E}\left[\sum_{k=0}^{n} \xi^T(k)\tilde{\Omega}_T(k)\right] \leq 0 (22)
\]
which is equivalent to (10), and the proof is now complete.

**Theorem 3.2:** The state estimator design problem is solvable if there exist positive definite matrices $P_{\sigma(k)} = \text{diag}\{p^2_{\sigma(k)}, p^3_{\sigma(k)}\}$ such that $0, Q_s > 0, positive constant scalars $\epsilon_{\sigma(k)} > 0, \delta_{\sigma(k),s} > 0$ and a matrix $J_{\sigma(k)}$, with $\sigma(k) = 1, 2, \ldots, n_s, s = 1, 2, \ldots, N$, such that the following LMI holds:
\[
\Pi = \begin{bmatrix}
\Lambda & \tilde{\Theta}_1 & \tilde{\Theta}_2 & \tilde{\Theta}_3 & \tilde{\Theta}_4 \\
* & \tilde{\Phi}_i & 0 & 0 & 0 \\
* & * & \tilde{\Phi}_i & 0 & 0 \\
* & * & * & \tilde{\Phi}_i & 0 \\
* & * & * & * & \tilde{\Phi}_i 
\end{bmatrix} < 0 (23)
\]
where
\[
\tilde{\Theta}_1 = [\tilde{P}_i^T A_{\sigma(k)} + \tilde{P}_i^T A_{\sigma(k)}^2 0 \tilde{P}_i^T G \tilde{P}_i^T \eta \mathcal{H} \tilde{P}_i^T C_{\sigma(k)}], \\
\tilde{\Theta}_2 = [\tilde{P}_i^T A_{\sigma(k)} + \tilde{P}_i^T A_{\sigma(k)}^2 0 \tilde{P}_i^T \eta \mathcal{H} \tilde{P}_i^T C_{\sigma(k)}], \\
\tilde{\Theta}_3 = [0 \tilde{P}_i^T \tilde{\eta}(k) \tilde{B} 0 0 0]^T, \\
\tilde{\Theta}_4 = [0 0 0 \tilde{P}_i^T \tilde{\eta}(k) \mathcal{H} 0]^T, \\
A_{\sigma(k)} = A_{\sigma(k)} + \tilde{L}_{\sigma(k)} A_{\sigma(k)} \\
C_{\sigma(k)} = C_{\sigma(k)} + \tilde{L}_{\sigma(k)} C_{\sigma(k)} \\
A_{\sigma}^1 = \begin{bmatrix}
A & 0 & 0 \\
0 & A & 0 \\
[\Phi_{\sigma(k)}D & 0 & I - \Phi_{\sigma(k)}]
\end{bmatrix}, \\
A_{\sigma}^2 = \begin{bmatrix}
\Phi_{\sigma(k)}D & 0 & I - \Phi_{\sigma(k)}
\end{bmatrix}, \\
C_{\sigma}^1 = \begin{bmatrix}
C \\
0 \\
[\Phi_{\sigma(k)}E]
\end{bmatrix}, \\
C_{\sigma}^2 = \begin{bmatrix}
[\Phi_{\sigma(k)}E]
\end{bmatrix}.
By applying Lemma 2.1, we can see that the inequality (23) can be determined by the desired estimator gain in (8). Moreover, if the aforementioned inequality is feasible, the estimator gains can be derived by (24), which completes the proof. 

Remark 3.1: Several random time-delay conditions considering distribution are derived in this paper, which are more in line with the actual situation. Both the range of the time delay and its probability distribution are considered, meaning that more effective information may be obtained, less conservative results may be derived.

4. A numerical example

In this section, we present an example to illustrate the effectiveness of the proposed \( H_\infty \) controller design scheme. Consider system (1) with the following parameters:

\[
A = \begin{bmatrix} 0.4 & 0 & 0 \\ 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ 0 & 0.3 & 0 \end{bmatrix},
\]

\[
H = \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 \\ -0.3 \\ 0.1 \end{bmatrix},
\]

\[
\tilde{A} = \begin{bmatrix} 0.1 & -0.05 & 0 \\ 0.1 & 0.1 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0.05 & -0.02 & 0 \\ 0.05 & 0.1 & 0 \\ 0 & 0 & -0.01 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0.2 & 0.1 & 0.15 \\ 0.1 & 0.15 & 0.2 \\ 0.03 & 0.03 & 0.01 \end{bmatrix}, \quad E = \begin{bmatrix} 0.05 \\ 0.02 \end{bmatrix},
\]

The activation functions are taken as

\[
f(x(k)) = \begin{bmatrix} 0.4x_1(k) - \tanh(0.2x_1(k)) \\ 0.5x_1(k) - \tanh(0.4x_1(k)) \\ 0.4x_1(k) - \tanh(0.3x_1(k)) \end{bmatrix},
\]

\[
g(x(k)) = \begin{bmatrix} 0.4x_1(k) - \tanh(0.2x_1(k)) \\ 0.2x_1(k) - \tanh(0.1x_1(k)) \\ 0.3x_1(k) - \tanh(0.2x_1(k)) \end{bmatrix}
\]

It is easy to compute that the sector-bounded condition can be met with

\[
\Phi_y = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad \Psi_y = \begin{bmatrix} 0.4 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.4 \end{bmatrix},
\]

\[
\Phi_y = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad \Phi_y = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}
\]
In this example, the variation of the time-varying random delay $d(k)$ is shown in Figure 1 from which it can be calculated that $N = 4$, and the probability distribution of the delay is

\[
\begin{align*}
\text{Prob}\{d(k) \in [1 : 2]\} &= 0.4, \\
\text{Prob}\{d(k) \in [3 : 5]\} &= 0.4, \\
\text{Prob}\{d(k) \in [6 : 7]\} &= 0.1, \\
\text{Prob}\{d(k) \in [8 : 10]\} &= 0.1.
\end{align*}
\]

The $H_{\infty}$ performance level $\phi$ is taken as 0.9. By using MATLAB software with YALMIP 3.0, we can obtain the desired estimator gains as follows:

\[
\begin{align*}
L_1 &= \begin{bmatrix} 0.6745 & -0.0121 \\ 0.2690 & -0.1808 \\ 0.1476 & 0.1885 \end{bmatrix}, \\
L_2 &= \begin{bmatrix} 0.3152 & 0.8718 \\ -0.2577 & 0.2090 \\ 0.0126 & 0.0810 \end{bmatrix}.
\end{align*}
\]

The initial values are assumed to be $(x(k))_{k \in [-10, -1]} = [0, 0, 0]^T$ and $x(0) = [0.5, -0.5, 0.2]^T$. The disturbance inputs is selected as $v(k) = e^{-k/25} \sin(k)$. The simulation results are shown in Figures 2–4. Figure 2 depicts the system states and estimated states. Figure 3 plots the controlled output and its estimation, while Figure 4 shows the estimation error of the controlled output. The simulation results have confirmed that the designed estimator performs very well.

5. Conclusions

The problem of state estimation for neural networks with time-varying random delays under stochastic communication protocols has been investigate in this paper. Some sufficient conditions in the form of LMIs have been
derived to guarantee the stochastic stability of the system with $H_\infty$ performance, and the parameters of the state estimator can be obtained by solving the LMIs. Finally, a simulation example is provided to verify the effectiveness of the designed state estimator. It would be interesting to study the following future research topics: extension of the results obtained in this paper to neural networks with other network-induced phenomena such as packet dropout, quantization, fading measurements, and so on.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

This work was supported by National Natural Science Foundation of China under Grants 61773400, 61573377, Project for the Applied Basic Research of Qingdao under Grants 16-5-1-3-jch, 16-8-3-1-zhc, Fundamental Research Fund for the Central Universities of China under Grants 15CX08014A, 17CX02059, and the Research Fund for the Taishan Scholar Project of Shandong Province of China.

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