NEW APPLICATIONS OF GRADED LIE ALGEBRAS TO LIE ALGEBRAS, GENERALIZED LIE ALGEBRAS AND COHOMOLOGY

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ABSTRACT. We give new applications of graded Lie algebras to: identities of standard polynomials, deformation theory of quadratic Lie algebras, cyclic cohomology of quadratic Lie algebras, 2k-Lie algebras, generalized Poisson brackets and so on.

0. INTRODUCTION

Graded Lie algebras (gla) are commonly used in many areas of Mathematics and Physics. One of the reasons is that they offer a very convenient framework for the development of theories such as Cohomology Theory, Deformation Theory, among others, very often avoiding heavy computations. But the fundamental reason, developed with the work of M. Gerstenhaber and others, is that the gla notion allows to endow with a structure, objects that a priori had none, providing a new and efficient material to study these objects. The aim of this paper is to give some new applications of classical well-known gla related to Deformation Theory.

Let us start with some notations: g will be a complex vector space, \( \bigwedge g \) the Grassmann algebra of g, that is, the algebra of skew multilinear forms on g, with the wedge product. When g is finite-dimensional, one has \( \bigwedge g = \text{Ext}(g^*) \), where \( \text{Ext}(g^*) \) denotes the exterior algebra of the dual space \( g^* \). However, when g is not finite dimensional, the strict inclusion \( \text{Ext}(g^*) \subset \bigwedge g \) holds. A quadratic vector space is a vector space endowed with a non degenerate symmetric bilinear form. In the case of a quadratic Lie algebra this bilinear form has to be invariant. A theory of finite dimensional quadratic Lie algebras, based on the notion of double extension, was developed in [8, 18] following Kac’s arguments [13]. In this paper, we shall present another interpretation based on the concept of super Poisson bracket.

The gla we shall use here are:

1. Gerstenhaber’s graded Lie algebras \( \mathcal{M}(g) \), related to associative algebra structures on g (see Section 1).

2. Gerstenhaber-Nijenhuis’s graded Lie algebras \( \mathcal{M}_a(g) \), related to Lie algebra structures on g (see Section 1).

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The graded Lie algebra $\mathcal{D}(\mathfrak{g})$ of derivations of the Grassmann algebra $\wedge \mathfrak{g}$ (often called $W(n)$ when $n = \dim \mathfrak{g}$, see Section 2).

Assuming that $\mathfrak{g}$ is finite dimensional, the graded Lie algebra $\mathcal{W}(\mathfrak{g})$ of skew symmetric polynomial multivectors on $\mathfrak{g}^*$ with the Schouten bracket (see Section 3).

Given a quadratic finite dimensional space $\mathfrak{g}$, the super Poisson graded Lie algebra structure on the Grassmann algebra $\wedge \mathfrak{g}$ (see Section 4) and the superalgebra $\mathcal{H}(\mathfrak{g})$ of Hamiltonian derivations of $\wedge \mathfrak{g}$.

For (1) and (2), we refer to [11], for (3) to [20], for (4) to Koszul’s presentation [17] (though [11] could be convenient as well). For (5), though it is a known algebra, we have no references, probably because of the lack of applications up to now (we shall show, e.g. in Sections 5 to 8, that there are some natural and interesting ones). Since we want to fix our conventions and notations and since we do not wish to address the present work to experts only, we give an introduction to all the above $\mathfrak{g}la$, recalling the main properties that will be used all along this paper.

Section 1 is a review of $\mathcal{M}(\mathfrak{g})$ and $\mathcal{M}_a(\mathfrak{g})$. We conclude the Section with a notion of Generalized Lie Algebras structures, that we call $2k$-Lie algebras, namely the elements $F$ in $\mathcal{M}_{2k}^a(\mathfrak{g})$ that satisfy $[F,F]_a = 0$. Such structures are introduced in [9] and many other papers (e.g. [3]), under various names.

In Section 2, we recall how to go from $\mathcal{M}_a(\mathfrak{g})$ to $\mathcal{D}(\mathfrak{g})$, an operation that can be translated as going from a structure to its cohomology, as we shall now explain. The argument is given by (2.1.1): there exists a one to one gla homomorphism from $\mathcal{M}_a(\mathfrak{g})[1]$ to $\mathcal{D}(\mathfrak{g})$, which turns out to be an isomorphism when $\mathfrak{g}$ is finite dimensional. So given a $2k$-Lie algebra structure on $\mathfrak{g}$, there is an associated derivation $D$ of $\wedge \mathfrak{g}$, and the (generalized) Jacobi identity $[F,F]_a = 0$ is equivalent to $D^2 = 0$, so that $D$ defines a cohomology complex (2.3). This is well known for Lie algebras since the corresponding complex is the Chevalley complex of trivial cohomology. The existence of a cohomology complex for a $2k$-Lie algebra was pointed out (without the gla interpretation), e.g. in [3]. We then recall the definition and properties of the Schouten bracket for a finite dimensional $\mathfrak{g}$. As in [3], we define a Generalized Poisson Bracket (GPB) as an element $W$ of $\mathcal{W}^{2k}(\mathfrak{g})$ satisfying $[W,W]_S = 0$ (2.4.1), the obvious generalization of the classical definition of a Poisson bracket. We show that there exists a one to one gla homomorphism from $\mathcal{D}(\mathfrak{g})$ into $\mathcal{W}(\mathfrak{g})[1]$ (2.5.1), so that any $2k$-Lie algebra structure on $\mathfrak{g}$ has an associated GPB, generalizing the classical Lie-Kostant-Kirillov bracket associated to a Lie algebra.

We apply the results of Sections 1 and 2 to standard polynomials $\mathcal{A}_k$ ($k \geq 0$) on an associative algebra $\mathfrak{g}$, appear in the PI algebras theory (see [12]) and also in cohomology theory (for instance, the cohomology of $\mathfrak{gl}(n)$ is $\text{Ext}[a_1,a_3,\ldots,a_{2n-1}]$ where $a_k = \text{Tr}(\mathcal{A}_k)$, and the cohomology of $\mathfrak{gl}(\infty)$ is $\text{Ext}[a_1,a_3,\ldots]$ [10]). We show that there exist two different structures on the space $\mathcal{A} = \text{span}\{\mathcal{A}_k \mid k \geq 0\}$, both with interesting consequences. The first one comes from the Gerstenhaber bracket of $\mathcal{M}_a(\mathfrak{g})$: we compute explicitly
and it results that $\mathcal{A}$ is a subalgebra of the $\text{gla}$ $\mathcal{M}_a(\mathfrak{g})$ (3.2.1). Since $[\mathcal{A}_k, \mathcal{A}_k]_a = 0$, any even standard polynomial define a $2k$-Lie algebra structure on $\mathfrak{g}$ (3.2.2). Moreover, $\mathcal{A}_2$ is a coboundary (an invariant one) of the adjoint cohomology of the Lie algebra $\mathcal{A}_2$ defined by the associative algebra $\mathfrak{g}$. The second product, denoted by $\times$, is associative and is defined on $\mathcal{M}_a(\mathfrak{g})$ using both the wedge product on $\bigwedge \mathfrak{g}$ and the initial product on $\mathfrak{g}$ (a priori, non commutative). So, it is rather a surprise to find that $\mathcal{A}$ is an Abelian algebra for $\times$, and in fact a very simple one, since $\mathcal{A}_k = (\mathcal{A}_2) \otimes_k, \forall k$ (3.3.2). For instance, for $\mathfrak{g} = \text{gl}(n), \mathcal{A}$ with its $\times$-product is isomorphic to $\mathbb{C}[x]/x^{2n}$, since $\mathcal{A}_k = 0, \forall k \geq 2n$ (the Amitsur-Levitzki theorem [1, 15]). From identities (3.3.2), one deduces some classical well-known identities of standard polynomials (e.g. $\mathcal{A}_2 = (\mathcal{A}_2)^{\otimes k}, \forall k$, usually proved by hand). When $\mathfrak{g}$ has a trace, we prove that $\text{Tr}(F \cdot G) = 0$, for all $F, G \in \mathcal{M}_a(\mathfrak{g})$ (3.3.5), and then (keeping the notation $a_k = \text{Tr}(\mathcal{A}_k)$), that $a_{2k} = 0$, $\forall k > 0$, and that $a_{2k+1}$ is an invariant Lie algebra cocycle (3.3.6). To conclude Section 3, we compute the cohomology of the Lie algebra $\mathfrak{g}$ of finite rank operators in an infinite dimensional space. Obviously, $\text{gl}(\infty) \subset \mathfrak{g}$, but this inclusion is strict. Our result is $H^*(\mathfrak{g}) = \text{Ext}[a_1, a_3, \ldots]$ (3.3.11), so the above inclusion induces an isomorphism in cohomology.

The first part of Section 4 is devoted to the construction of the super Poisson bracket defined on $\bigwedge \mathfrak{g}$, when $\mathfrak{g}$ is a finite dimensional quadratic vector space. We follow a deformation argument as in [14]: the Clifford algebra $\text{Cliff}(\mathfrak{g}^*)$ can be seen as a quantization of the algebra $\bigwedge \mathfrak{g}$ of skew polynomials, similarly to the classical Moyal quantization of polynomials by the Weyl algebra. In 4.1, we introduce some formulas for the construction of the Clifford algebra that are convenient since they easily provide a transparent explicit formula for the deformed product (4.1.1), with leading term the super Poisson bracket, explicitly computed in (4.2.1). The relation with the superalgebra $\mathcal{H}(n)$ [20] is given in (4.1.3), and a Moyal type formula is obtained (4.2.2) (an equivalent formula without the super Poisson bracket can be found in [14]). In the second part of Section 4, we use the $\text{gla}$ $\bigwedge \mathfrak{g}$ and the super Poisson bracket to study quadratic Lie algebras. We obtain that quadratic Lie algebra structures on $\mathfrak{g}$ with bilinear form $B$ are in one to one correspondence with elements $I$ in $\bigwedge^3 \mathfrak{g}$ satisfying $\{I, I\} = 0$; more precisely, $I(X, Y, Z) = B([X, Y], Z), \forall X, Y, Z \in \mathfrak{g}$, and the differential $\partial$ of $\bigwedge \mathfrak{g}$ is $\partial = -\frac{1}{2} \text{ad}_p(I)$ (4.5.1, 4.6.1). We prove that any quadratic deformation of a quadratic Lie algebra is equivalent to a deformation with unchanged invariant bilinear form (4.4.1), and finally, we propose a $\text{gla}$ framework well adapted to the deformation theory of quadratic Lie algebras (4.6.2).

We use the results of Section 4 in Section 5 to give a complete description of finite dimensional elementary quadratic Lie algebras, i.e. those with decomposable associated element $I$ in $\bigwedge^3 \mathfrak{g}$ (5.1.3). We first give a simple characterization (5.1.4): a non Abelian quadratic Lie algebra $\mathfrak{g}$ is elementary if and only if $\text{dim}(\mathfrak{g}, \mathfrak{g}) = 3$. We then show that any non Abelian quadratic Lie algebra reduces, up to a central factor, to a quadratic Lie algebra with totally isotropic center (5.2.1); the property of being elementary is preserved under the reduction. This reduces the problem of finding all elementary non Abelian quadratic Lie algebras to algebras of dimension 3 to 6 (5.2.2), that we completely describe in (5.3) and (5.3.2). Some remarks: as
we show in (5.1.5), if \( \mathfrak{g} \) is an elementary quadratic Lie algebra, all coadjoint orbits have dimension at most 2. Now, a classification of Lie algebras whose coadjoint orbits are of dimension less than 2, is given in [2], and the proof, using classical Lie algebra theory is not at all trivial. With some effort, one could probably find directly in the classification of [2], which algebras are quadratic and which are not. Our geometric flavored proof is completely different, using essentially elementary properties of quadratic forms.

In Section 6, we study cyclic cohomology of quadratic Lie algebras. Given a quadratic vector space \( \mathfrak{g} \), we use \( \mathfrak{g} \)-valued cochains (rather than \( \mathfrak{g}^* \)-valued, by analogy to the associative case [7]) to define cyclic cochains (6.2.1) (both notions are equivalent when \( \mathfrak{g} \) is finite dimensional). Thanks to this definition, we can use the Gerstenhaber bracket of \( \mathcal{M}_a(\mathfrak{g}) \) and we show that cyclic cochains are well behaved with respect to this bracket: the space \( \mathcal{C}_c(\mathfrak{g}) \) of cyclic cochains is a subalgebra of the \( \mathfrak{g}la \cdot \mathcal{M}_a(\mathfrak{g}) \) (6.2.2) and if \( \mathfrak{g} \) is a Lie algebra, \( \mathcal{C}_c(\mathfrak{g}) \) is a subcomplex of the adjoint cohomology complex \( \mathcal{M}_a(\mathfrak{g}) \) (6.3.1); we define the cyclic cohomology \( H^*_{c}(\mathfrak{g}) \) as the cohomology of this subcomplex (6.3.2). There is a natural one to one map from \( \mathcal{C}_c(\mathfrak{g}) \) into \( \wedge\mathfrak{g} = \wedge\mathfrak{g}/\mathbb{C} \) (6.2.2) which induces a map from \( H^*_{c}(\mathfrak{g}) \) into \( H^*_{\wedge}(\mathfrak{g}) = H^*(\mathfrak{g})/\mathbb{C} \). When \( \mathfrak{g} \) is finite dimensional, \( \wedge\mathfrak{g} \) is a \( \mathfrak{g}la \) for the (quotient) Poisson bracket, isomorphic to \( \mathcal{H}(\mathfrak{g}) \), and there is an induced \( \mathfrak{g}la \) structure on \( H^*_{\wedge}(\mathfrak{g}) \). We show that there is a \( \mathfrak{g}la \) isomorphism from \( \mathcal{C}_c(\mathfrak{g}) \) onto \( \wedge\mathfrak{g} \) (6.3.4), and from \( H^*_{c}(\mathfrak{g}) \) onto \( H^*_{\wedge}(\mathfrak{g}) \) (6.4.7). We also introduce a wedge product on \( \mathcal{C}_c(\mathfrak{g}) \), and on \( H^*_{c}(\mathfrak{g}) \) (6.4.4, 6.4.5) which proves to be useful to describe \( H^*_{c}(\mathfrak{g}) \) (6.4.7). When \( \mathfrak{g} \) is not finite dimensional, the isomorphism between \( H^*_{c}(\mathfrak{g}) \) and \( H^*_{\wedge}(\mathfrak{g}) \) is no longer true: we give an example where the natural map is neither one to one, nor onto (6.4.10, 6.4.11). So the cyclic cohomology \( H^*_{c}(\mathfrak{g}) \) can have its own life, independently of the reduced cohomology \( H^*_{\wedge}(\mathfrak{g}) \).

Section 7 starts with the study of invariant cyclic cochains in the case of a finite dimensional quadratic Lie algebra. We first prove that any invariant cyclic cochain is a cocycle (7.1.2). When \( \mathfrak{g} \) is reductive, we demonstrate that the inclusion of invariants cyclic cochains into cocycles induces an isomorphism in cohomology (7.1.2), so that \( H^*_{c}(\mathfrak{g}) \cong \mathcal{C}_c(\mathfrak{g})^0 \). Assuming that \( \mathfrak{g} \) is a semisimple Lie algebra, we prove:

\[
\text{If } I, I' \in (\wedge \mathfrak{g})^0, \text{ then } \{I, I'\} = 0. \tag{7.2.1}
\]

As a corollary, when \( \mathfrak{g} \) is semisimple, the Gerstenhaber bracket induces the null bracket on \( H^*_{c}(\mathfrak{g}) \). Applying the preceding results, we give a complete description of the super Poisson bracket in \( (\wedge \mathfrak{g})^0 \), and of the \( \mathfrak{g}la \) \( H^*_{c}(\mathfrak{g}) \), when \( \mathfrak{g} = \mathfrak{gl}(n) \) (7.2.6).

We develop in Section 8, the theory of quadratic 2k-Lie algebra structures on a semisimple Lie algebra \( \mathfrak{g} \), in relation with cyclic cochains (8.1.3). This is a direct generalization of the case of quadratic Lie algebras studied in Section 4. We show that any invariant even cyclic cochain \( F \) defines a quadratic 2k-Lie algebra (8.2.1) and that \( (\wedge \mathfrak{g})^0 = H^*(\mathfrak{g}) \) is contained in \( H^*(F) \). Finally, we give an interpretation of some interesting examples given in [3] of 2k-Lie algebras in terms of the techniques developed in the present paper, pointing out where these examples come from.
Finally, we give some examples of quadratic 2-Lie algebra structures on $gl(n)$ (8.3.1).

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1. $M(g)$, $M_n(g)$ and 2-Lie algebra structures

This Section is essentially a review, except 1.3. For more details, see [11] and [19].

Let $g$ be a complex vector space. We denote by $M(g)$ the space of multilinear mappings from $g$ to $g$. The space $M(g)$ is graded as follows:

$$M(g) = \sum_{k \geq 0} M^k(g)$$

where $M^0(g) = g$, $M^k(g) = \{ F : g^k \to g \mid F \text{ k-linear} \}$, for $k \geq 1$.

1.1. The theory of associative algebra structures on $g$ can be described in a graded Lie algebra framework [11, 19]: first, consider $M(g)$ with shifted grading $M^k[1] = M^{k+1}(g)$ and denote it $M[1]$. Then define a graded Lie bracket on $M[1]$ as follows: for all $F \in M^p[1]$, $G \in M^q[1]$, then $[F, G] \in M^{p+q}[1]$ with

$$[F, G](X_1, \ldots, X_{p+q+1}) :=
\begin{align*}
(-1)^{pq} & \sum_{j=1}^{p+1} (-1)^{q(j-1)} F(X_1, \ldots, X_{j-1}, G(X_j, \ldots, X_{j+q}), X_{j+q+1}, \ldots, X_{p+q+1}) \\
& \quad - \sum_{j=1}^{q+1} (-1)^{p(j-1)} G(X_1, \ldots, X_{j-1}, F(X_j, \ldots, X_{j+p}), X_{j+p+1}, \ldots, X_{p+q+1}).
\end{align*}$$

for $X_1, \ldots, X_{p+q+1} \in g$.

Notice that when $X \in M^{-1}[1] = g$, then $[X, G]$ is defined by:

$$[X, G](X_1, \ldots, X_q) = - \sum_{j=1}^{q+1} (-1)^{j-1} G(X_1, \ldots, X_{j-1}, X, X_j, \ldots, X_q).$$

Notice also that when $F$ and $G$ are in $M^0[1] = \text{End}(g)$, then $[F, G]$ is the usual bracket of the two linear maps $F$ and $G$.

Now, suppose that $F \in M^1[1]$ defines a product on $g$ by:

$$X \cdot Y = F(X, Y), \forall X, Y \in g.$$  

This product is associative if and only if $[F, F] = 0$. In this case, the derivation $\text{ad}(F)$ of the graded Lie algebra $M[1]$ satisfies $(\text{ad}(F))^2 = 0$, so it defines a complex on $M(g)$ which turns out to be the Homchil cohomology complex of the associative algebra defined by $F$ [11].
1.2. In the remaining of the paper, we use $\mathfrak{S}_{p,q}$ to denote the set of all $(p,q)$-unshuffles, that is, elements $\sigma$ in the permutation group $\mathfrak{S}_{p+q}$ satisfying $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

The theory of Lie algebra structures on $\mathfrak{g}$ can also be described in a graded Lie algebra framework [11, 19]. First, let $\mathcal{M}_a = \mathcal{M}_a(\mathfrak{g})$ be the space of skew symmetric elements in $\mathcal{M}(\mathfrak{g})$. One has $\mathcal{M}_a = \sum_{k\geq 0} \mathcal{M}_a^k$ with $\mathcal{M}_a^0 = \mathfrak{g}$ and $\mathcal{M}_a^1 = \text{End}(\mathfrak{g})$.

Then consider $\mathcal{M}_a$ with shifted grading denoted by $\mathcal{M}_a[1]$, and define a graded Lie bracket as follows: for all $F \in \mathcal{M}_a^p[1], G \in \mathcal{M}_a^q[1]$, then $[F, G]_a \in \mathcal{M}_a^{p+q}[1]$ with

$$[F, G]_a (X_1, \ldots, X_{p+q+1}) := \left(-1\right)^{pq} \sum_{\sigma \in \mathfrak{S}_{p+1,q}} \varepsilon(\sigma) F(G(X_{\sigma(1)}, \ldots, X_{\sigma(q+1)}), X_{\sigma(q+2)}, \ldots, X_{\sigma(p+q+1)}) - \sum_{\sigma \in \mathfrak{S}_{p+1,q}} \varepsilon(\sigma) G(F(X_{\sigma(1)}, \ldots, X_{\sigma(p+1)}), X_{\sigma(p+2)}, \ldots, X_{\sigma(p+q+1)})$$

for $X_1, \ldots, X_{p+q+1} \in \mathfrak{g}$.

Notice that when $X \in \mathcal{M}_a^{-1}[1] = \mathfrak{g}$, then

$$[X, G]_a (X_1, \ldots, X_q) = -G(X, X_1, \ldots, X_q) \quad (= -\text{ad}_X(G)(X_1, \ldots, X_q)).$$

Moreover, when $F, G \in \mathcal{M}_a^0[1] = \text{End}(\mathfrak{g})$, then $[F, G]_a$ is the usual bracket of the linear maps $F$ and $G$.

Now, any $F \in \mathcal{M}_a[1]$ defines a bracket on $\mathfrak{g}$ by

$$[X, Y] = F(X, Y), \forall X, Y \in \mathfrak{g}.$$  

The Jacobi identity is satisfied if and only if $[F, F]_a = 0$. In this case, the derivation $\text{ad}(F)$ of the graded Lie algebra $\mathcal{M}_a[1]$ satisfies $(\text{ad}(F))^2 = 0$, so it defines a complex on $\mathcal{M}_a$ which turns out to be the Chevalley cohomology complex with coefficients in the adjoint representation, of the Lie algebra structure defined by $F$.

At this point, let us quickly explain the relations between the two brackets defined in 1.1 and 1.2. First, define the skew symmetrization map $A : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}_a(\mathfrak{g})$:

$$A(F)(X_1, \ldots, X_k) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) F(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$$

with $F \in \mathcal{M}^k(\mathfrak{g})$ and $X_1, \ldots, X_k \in \mathfrak{g}$. One has:

**Proposition 1.2.1.** For all $F, G \in \mathcal{M}(\mathfrak{g})$, $A([F, G]) = [A(F), A(G)]_a$.

Obviously, when $F \in \mathcal{M}^1[1]$ induces an associative product on $\mathfrak{g}$, then $A(F)$ induces a Lie algebra structure on $\mathfrak{g}$. However one should notice that from Proposition 1.2.1, Lie algebra structures of type $A(F)$ can be obtained from a “product” $F$ on $\mathfrak{g}$ satisfying other conditions than associativity, for instance:

**Proposition 1.2.2.** Let $F \in \mathcal{M}^1[1]$ such that there exists $\tau \in \mathfrak{S}_3$ satisfying $\tau [F, F] = -\varepsilon(\tau) [F, F]$. Then $A(F)$ defines a Lie algebra structure on $\mathfrak{g}$. 

1.3. Let us introduce a concept of generalized Lie algebra structures on $g$:

**Definition 1.3.1.** An element $F \in \mathcal{M}^{2k-1}_a[1]$ is a $2k$-Lie algebra structure on $g$ if

$$[F, F]_a = 0.$$

We shall often use a bracket notation: for $X_1, \ldots, X_{2k} \in g$,

$$[X_1, \ldots, X_{2k}] = F(X_1, \ldots, X_{2k}).$$

The identity $[F, F]_a = 0$ can be seen as a generalized Jacobi identity (see [9, 3]).

Given a $2k$-Lie algebra structure $F$ on $g$, $\text{ad}(F)$ is an odd derivation of $M^{1}_a$ and satisfies $(\text{ad}(F))^2 = 0$, so there is an associated cohomology defined by

$$\ker(\text{ad}(F))/\text{Im}(\text{ad}(F)),$$

which can be interpreted as a generalization of the Chevalley complex of 1.2.

2. $\mathcal{D}(g)$, $\mathcal{W}(g)$, COHOMOLOGY OF 2k-LIE ALGEBRAS AND GPB

In this Section, with exception made to 2.3 and 2.5, we recall classical material needed in the paper.

2.1. We denote by $\mathcal{D} = \mathcal{D}(g)$ the space of (graded) derivations of $\bigwedge g$. The space $\mathcal{D}$ is graded by $\mathcal{D} = \sum_{n=1}^{N} \mathcal{D}^n$ with $D \in \mathcal{D}^d$ if $D(\bigwedge^p g) \subset \bigwedge^{p+d} g$, for all $p$, and has a graded Lie algebra structure with the bracket defined by:

$$[D, D'] = D \circ D' - (-1)^{dd'} D' \circ D, \forall \ D, D' \in \mathcal{D}^d.$$

We denote by $t_X, X \in g$, the elements of $\mathcal{D}^{-1}$ defined by

$$t_X(\Omega)(Y_1, \ldots, Y_k) := \Omega(X, Y_1, \ldots, Y_k), \forall \, \Omega \in \bigwedge^{k+1} g, X, Y_1, \ldots, Y_k \in g \, (k \geq 0),$$

and $t_X(1) = 0$. When $g$ is finite dimensional, given a basis $\{X_1, \ldots, X_n\}$ and its dual basis $\{\omega_1, \ldots, \omega_n\}$, any element $D \in \mathcal{D}$ can be written in a unique way:

$$D = \sum_{r=1}^{n} D_r \wedge t_{X_r}$$

where $D_r = D(\omega_r)$. Moreover, $\mathcal{D}$ is a simple Lie superalgebra (often denoted by $W(n)$, see [20]) and there exists an obvious vector space isomorphism $D : \mathcal{M}_a[1] \rightarrow \mathcal{D}$ defined as $D(\Omega \otimes X) = -\Omega \wedge t_X, \forall \, \Omega \in \bigwedge g, X \in g$ which turns out to be a gla isomorphism.

Since we do not want to restrict ourselves to the finite dimensional case, we give a proof of the following result:

**Proposition 2.1.1.** There exists a one to one gla homomorphism $D : \mathcal{M}_a[1] \rightarrow \mathcal{D}$ such that

$$D(\Omega \otimes X) = -\Omega \wedge t_X, \forall \, \Omega \in \bigwedge g, X \in g.$$

When $g$ is finite dimensional, $D$ is an isomorphism.
Proof. Given a basis \{X_r \mid r \in \mathcal{B}\} of \mathfrak{g}, and the forms \omega_r, r \in \mathcal{B}, defined by \omega_r(X_r) = \delta_{r,s}, \forall r, s, for F \in \mathcal{M}_a^k \text{ let } D(F) = -\sum_{r \in \mathcal{B}} F(\omega_r) \wedge t_X. \text{ It is easy to see that though its indexes set is infinite, this sum applied to an element } \Omega \in \wedge^w \mathfrak{g} \text{ gives: }

\begin{align*}
D(F)(\Omega)(Y_1, \ldots, Y_{k+w-1}) &= -\sum_{\sigma \in \mathcal{S}_{k+w-1}} e(\sigma)\Omega(F(Y_{\sigma(1)}, \ldots, Y_{\sigma(k)}), Y_{\sigma(k+1)}, \ldots, Y_{\sigma(k+w-1)}),
\end{align*}

for all \(Y_1, \ldots, Y_{k+w-1} \in \mathfrak{g}\). It results that our definition of \(D\) does not depend on the basis of \(\mathfrak{g}\), and that \(D(F \otimes X) = -A \wedge t_X, A \in \wedge^k \mathfrak{g}, X \in \mathfrak{g}\). Keeping in mind the remark about the sum defining \(D\), we compute for \(G \in \mathcal{M}_a^k\):

\[\begin{align*}
[D(F), D(G)] &= \\
&= \sum_{r,s} \left( t F(\omega_r) \wedge t_X (t G(\omega_s)) - (-1)^{(k+1)(k'+1)} G(\omega_r) \wedge t_X (t F(\omega_s)) \right) \wedge t_X.
\end{align*}\]

By a direct computation:

\[\begin{align*}
\sum_r \left( t F(\omega_r) \wedge t G(\omega_r) - (-1)^{(k+1)(k'+1)} G(\omega_r) \wedge t_X (t F(\omega_r)) \right) (Y_1, \ldots, Y_{k+k'-1}) &= \omega_r \left( \sum_{\sigma \in \mathcal{S}_{k+k'-1}} \epsilon(\sigma)G(F(Y_{\sigma(1)}, \ldots, Y_{\sigma(k)}), Y_{\sigma(k+1)}, \ldots, Y_{\sigma(k+k'-1)}) \right) - \\
&- (-1)^{(k+1)(k'+1)} \sum_{\sigma \in \mathcal{S}_{k+k'-1}} \epsilon(\sigma)F(G(Y_{\sigma(1)}, \ldots, Y_{\sigma(k')}), Y_{\sigma(k'+1)}, \ldots, Y_{\sigma(k+k'-1)})) \\
&= -[F, G](\omega_r)(Y_1, \ldots, Y_{k+k'-1}),
\end{align*}\]

Hence:

\[\begin{align*}
[D(F), D(G)] &= -\sum_s t[F, G](\omega_s) \wedge t_X = D([F, G])
\end{align*}\]

In the sequel, given \(F \in \mathcal{M}_a(\mathfrak{g})\), we denote by \(D_F\) the associated derivation of \(\wedge \mathfrak{g}\). If \(\mathfrak{g}\) is finite dimensional, for \(D \in \mathcal{D}\), we denote by \(F_D\) the associated element in \(\mathcal{M}_a(\mathfrak{g})\). Here are some examples:

**Example 2.1.2.** If \(T \in \text{End}(\mathfrak{g}) = \mathcal{M}_a^0[1]\), then

\[D_T(\Omega)(Y_1, \ldots, Y_p) = -\sum_{i=1}^p \Omega(Y_1, \ldots, Y_{i-1}, T(Y_i), Y_{i+1}, \ldots, Y_p)\]

for all \(\Omega \in \wedge^p \mathfrak{g}, Y_1, \ldots, Y_p \in \mathfrak{g}\).

**Example 2.1.3.** If \(F \in \mathcal{M}_a^1[1]\), then

\[D_F(\Omega)(Y_1, \ldots, Y_{p+1}) = \sum_{i<j} (-1)^{j+i} \Omega(F(Y_i, Y_j), Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_{p+1})\]

for all \(\Omega \in \wedge^p \mathfrak{g}, Y_1, \ldots, Y_{p+1} \in \mathfrak{g}\).
2.2. Let $F$ be a Lie algebra structure on $\mathfrak{g}$, then $F \in \mathcal{M}_a^1[1]$ and $[F,F]_a = 0$. Let $\partial = D_F$, then $[\partial, \partial] = 0$ gives $\partial^2 = 0$ and formula (II) shows that the associated complex in the Grassmann algebra $\wedge \mathfrak{g}$ is exactly the Chevalley cohomology complex of trivial cohomology of $\mathfrak{g}$. One defines $\theta_X$:

$$\theta_X = [i_X, \partial] = D_{\text{ad}(X)}.$$

If $\{X_r \mid r \in \mathcal{R}\}$ is a basis of $\mathfrak{g}$, consider the forms $\omega_r$, $r \in \mathcal{R}$, defined by $\omega_r(X_s) = \delta_{rs}, \forall r,s$. The map $\theta$ defines a Lie algebra representation of $\mathfrak{g}$ in $\wedge \mathfrak{g}$ and one has:

$$\partial = \frac{1}{2} \sum_{r \in \mathcal{R}} \omega_r \wedge \theta_{X_r}. \quad (\text{III})$$

Let us precise that this formula is well-known when $\mathfrak{g}$ is finite dimensional (see [16]), and that a proof in the infinite dimensional case is given in the proof of Lemma 3.3.7 of the present paper. In any case, a very important consequence of formula (III) is that any invariant in $(\wedge \mathfrak{g})^0$ is a cocycle.

2.3. Let us now check how 2.2 can be extended to $2k$-Lie algebra structures on $\mathfrak{g}$. Let $F \in \mathcal{M}_{2k-1}^1[1]$. Assume that $[F,F]_a = 0$, and let

$$[Y_1, \ldots, Y_{2k}] := F(Y_1, \ldots, Y_{2k}) \quad (\text{IV})$$

for $Y_1, \ldots, Y_{2k} \in \mathfrak{g}$. Denote by $D = D_F$ the associated derivation of $\wedge \mathfrak{g}$. Using Proposition 2.1.1, one concludes $D^2 = 0$, so one can define an associated cohomology $H^*(F) = \ker(D)/\operatorname{Im}(D)$. One has

$$D\omega(Y_1, \ldots, Y_{2k}) = -\omega([Y_1, \ldots, Y_{2k}])$$

for $\omega \in \mathfrak{g}^*$, $Y_1, \ldots, Y_{2k} \in \mathfrak{g}$. We shall come back to cohomology of $2k$-Lie algebras in Section 8.

In the remaining of this Section, we will assume that $\mathfrak{g}$ is a finite dimensional vector space with $\dim \mathfrak{g} = n$. We state next some properties of the Schouten bracket. For more details, we refer to [17].

2.4. Let $\mathcal{W} = \mathcal{W}(\mathfrak{g}) = \mathcal{P} \otimes \wedge \mathfrak{g}$, graded by $\mathcal{W}^p = \mathcal{P} \otimes \wedge^p \mathfrak{g}$, where $\mathcal{P}$ is the symmetric algebra of $\mathfrak{g}^*$. Elements of $\mathcal{W}$ act as skew symmetric multivectors on $\mathcal{P}$ as follows: for $\Omega \in \wedge^p \mathfrak{g}$, $P \in \mathcal{P}$, $f_1, \ldots, f_p \in \mathcal{P}$,

$$(P \otimes \Omega)(f_1, \ldots, f_p) = P(\varphi) \Omega((df_1)\varphi, \ldots, (df_p)\varphi).$$

For instance, if $\{X_1, \ldots, X_p\}$ is a basis of $\mathfrak{g}$ and $\{\omega_1, \ldots, \omega_p\}$ its dual basis, one has for all $i = 1, \ldots, p$:

$$\omega_i(f) = \frac{\partial f}{\partial X_i}, \forall f \in \mathcal{P}.$$

There is a natural $\wedge$-product on $\mathcal{W}$, defined by: for all $P, P' \in \mathcal{P}$, $\Omega, \Omega' \in \wedge \mathfrak{g}$:

$$(P \otimes \Omega) \wedge (P' \otimes \Omega') = PP' \otimes (\Omega \wedge \Omega').$$

Each $f \in \mathcal{P}$ defines a derivation $t_f$ of degree $-1$ of $\mathcal{W}$ by:

$$t_f(P \otimes \Omega)(f_1, \ldots, f_{p-1}) = P(\varphi) t_{(df_p)\varphi}((df_1)\varphi, \ldots, (df_{p-1})\varphi) = P(\varphi) \Omega((df_1)\varphi, (df_{p-1})\varphi).$$
For instance, if $V \in \mathcal{W}^1$, one has $I_V(V) = V(f)$. There is a graded Lie bracket on $\mathcal{W}[1]$ called the Schouten bracket, and defined by: for all $W, W' \in \mathcal{W}^p[1], \mathcal{W}' \in \mathcal{W}^q[1]$, then $[W, W']_S \in \mathcal{W}^{p+q}[1]$ with

$$[W, W']_S (f_1, \ldots, f_{p+q+1}) = (-1)^{pq} \sum_{\sigma \in \mathcal{S}_{p+1, q}} \epsilon(\sigma) W(W'(f_{\sigma(1)}, \ldots, f_{\sigma(q+1)}), f_{\sigma(q+2)}, \ldots, f_{\sigma(p+q+1)})$$

$$- \sum_{\sigma \in \mathcal{S}_{p+1, q}} \epsilon(\sigma) W'(W(f_{\sigma(1)}, \ldots, f_{\sigma(p+1)}), f_{\sigma(p+2)}, \ldots, f_{\sigma(p+q+1)})$$

for $f_1, \ldots, f_{p+q+1} \in \mathcal{D}$.

Then for all $P, P' \in \mathcal{D}$, $\Omega, \Omega' \in \wedge^{p+1} g$, $\Omega' \in \wedge^{q+1} g$:

$$[P \otimes \Omega, P' \otimes \Omega']_S = (-1)^{pq} P \otimes (\Omega' \wedge t_1 P'(\Omega)) - P' \otimes (\Omega \wedge t_1 P(\Omega')).$$

As a particular case, one has $[\Omega, \Omega']_S = 0$, for all $\Omega, \Omega' \in \wedge g$.

Let $W \in \mathcal{W}^1[1]$, then $W$ defines a Poisson bracket on $\mathcal{D}$ by $\{P, P'\} = W(P, P')$ if and only if $[W, W]_S = 0$. More generally, as proposed in [3], one can define Generalized Poisson Brackets (GPB) as follows:

**Definition 2.4.1.** An element $W \in \mathcal{W}^{2k-1}[1]$ is a GPB if $[W, W]_S = 0$.

(see [3] where these structures are introduced and applications are proposed).

2.5. Let us now show that 2k-Lie algebras have associated GPB, exactly as Lie algebras have associated Poisson brackets. This will be a consequence of the following construction: define a map $V : \mathcal{D} = \mathcal{D}(g) \to \mathcal{W}$ by $V_D = V(D) := -X \otimes \Omega$ for $D = \Omega \wedge t_1 X$ with $\Omega \in \wedge g, X \in g$. Then, it is easy to check that:

**Proposition 2.5.1.** One has $V_{[D, D']} = [V_D, V_{D'}]_S$, for $D, D' \in \mathcal{D}$. Moreover $V$ is a one to one graded Lie algebras homomorphism from $\mathcal{D}$ into $\mathcal{W}[1]$.

For example, given a 2k-Lie algebra structure $F$ on $g$, denoted by $[Y_1, \ldots, Y_{2k}] = F(Y_1, \ldots, Y_{2k}), \forall Y_1, \ldots, Y_{2k} \in g$, let $D$ be the associated derivation (see Proposition 2.1.1) in $\mathcal{D}$. Then one has:

$$V_D(f_1, \ldots, f_{2k})_\varphi = \langle \varphi \rangle[(df_1)_\varphi, \ldots, (df_{2k})_\varphi],$$

and since $[F, F]_a = 0$ by (2.1.1), one has $[D, D] = 0$. Using Proposition 2.5.1 above, $[V_D, V_{D'}]_S = 0$, so $V_D$ defines a GPB on $\mathcal{D}$.

Finally, using 2.1 and Proposition 2.5.1, one deduces an inclusion of the simple Lie superalgebra $W(n)$ into the graded Lie algebra $\mathcal{W}[1]$, endowed with the Schouten bracket which provides a natural realization of $W(n)$.

3. APPLICATION TO IDENTITIES OF STANDARD POLYNOMIALS, AND COHOMOLOGY

In this Section, $g$ denotes an associative algebra, with product $m$. We also use the notation: $X, Y = m(X, Y), \forall X, Y \in g$. We assume that $m$ has a unit $l_m$, but this is not really necessary.
3.1. We first define the iterated \( m_k \) \((k \geq 0)\) of \( m \) as:

\[
m_0 = 1, \quad m_1 = \text{Id}_g, \quad m_2 = m, \ldots, \quad m_k(Y_1, \ldots, Y_k) = Y_1 \ldots Y_k, \quad \forall Y_1, \ldots, Y_k \in g, \ldots
\]

It is easy to check that:

**Proposition 3.1.1.** For all \( k, k' \geq 0 \), one has:

\[
\begin{align*}
[m_{2k}, m_{2k'}] &= 0, \\
[m_{2k}, m_{2k'+1}] &= (2k - 1) m_{2k+2k'}, \\
[m_{2k+1}, m_{2k'+1}] &= 2(k - k') m_{2k+2k'+1}.
\end{align*}
\]

Hence the space generated by \( \{m_k, k \geq 0\} \) is a subalgebra of the \( \text{gla } \mathcal{M}(g) \) of Section 1.

3.2. Now define the standard polynomials \( \mathcal{A}_k \) \((k \geq 0)\) on \( g \) as:

\[
\mathcal{A}_k := A(m_k)
\]

Using Propositions 1.2.1 and 3.1.1, one immediately obtains:

**Proposition 3.2.1.** For all \( k, k' \geq 0 \), one has:

\[
\begin{align*}
\mathcal{A}_{2k}, \mathcal{A}_{2k'} &= 0, \\
\mathcal{A}_{2k}, \mathcal{A}_{2k+1} &= (2k - 1) \mathcal{A}_{2k+2k'}, \\
\mathcal{A}_{2k+1}, \mathcal{A}_{2k'+1} &= 2(k - k') \mathcal{A}_{2k+2k'+1}.
\end{align*}
\]

Let \( \mathcal{A} \) be the subspace generated by \( \{\mathcal{A}_k, k \geq 0\} \). Hence \( \mathcal{A} \) is a subalgebra of the graded Lie algebra \( \mathcal{M}(g) \) of Section 1. The standard polynomial \( \mathcal{A}_2 \) is the Lie algebra structure on \( g \) associated to \( m \). Since \( [\mathcal{A}_{2k}, \mathcal{A}_{2k}] = 0, \forall k \), we conclude:

**Proposition 3.2.2.** The standard polynomials \( \mathcal{A}_{2k}, k \geq 1 \) define \( 2k \)-Lie algebra structures on \( g \).

Remark that \( \mathcal{A}_k \) is a \( g \)-invariant map from \( g^k \) to \( g \) for the Lie algebra structure. Moreover the standard polynomial \( \mathcal{A}_{2k} \) is a coboundary of the adjoint representation of the Lie algebra \( g \) whereas \( [\mathcal{A}_2, \mathcal{A}_{2k-1}] = \mathcal{A}_{2k} \).

3.3. Let us now define an associative product on \( \mathcal{A} \). First consider an associative product \( \circ \) on \( \mathcal{M}(g) \):

\[(F \circ G)(Y_1, \ldots, Y_{p+q}) = F(Y_1, \ldots, Y_p), G(Y_{p+1}, \ldots, Y_{p+q}), \]

for all \( F \in \mathcal{M}(g), G \in \mathcal{M}(g), Y_1, \ldots, Y_{p+q} \in g \).

Then define an associative product \( \times \) on \( \mathcal{M}(g) \) by:

\[(F \times G)(Y_1, \ldots, Y_{p+q}) = \sum_{\sigma \in S_{p+q}} c(\sigma) F(Y_{\sigma(1)}, \ldots, Y_{\sigma(p)}), G(Y_{\sigma(p+1)}, \ldots, Y_{\sigma(p+q)}), \]

for all \( F \in \mathcal{M}(g), G \in \mathcal{M}(g), Y_1, \ldots, Y_{p+q} \in g \). By a straightforward computation, one has:

**Proposition 3.3.1.** For all \( F, G \in \mathcal{M}(g) \), \( A(F \circ G) = A(F) \times A(G) \).

It is obvious that \( m_k = m_1 \circ \cdots \circ m_1 \), so:
Corollary 3.3.2. \( \mathcal{A}_k = \underbrace{\mathcal{A}_1 \times \cdots \times \mathcal{A}_1}_k \) for all \( k \geq 1 \) and \( \mathcal{A}_k \times \mathcal{A}_\ell = \mathcal{A}_\ell \times \mathcal{A}_k = \mathcal{A}_{k+\ell} \), for all \( k, \ell \geq 0 \).

As a consequence, \( \mathcal{A} \) is a commutative algebra for the \( \times \)-product.

Any element \( Z \in g \) defines a super derivation \( t_Z \) of degree \(-1\) of the \( \times \)-product of \( \mathcal{M}_a(g) \) by: for all \( F \in \mathcal{M}_a^p(g) \), \( Y_1, \ldots, Y_{p-1} \in g \),

\[
t_Z(F)(Y_1, \ldots, Y_{p-1}) := F(Z, Y_1, \ldots, Y_{p-1}),
\]

Denote by \( Z(g) \) the center of the algebra \( g \). If \( Z \in Z(g) \), one has \( t_Z(\mathcal{A}_2) = 0 \).

Hence using Corollary 3.3.2 and the derivation property of \( t_Z \), we deduce:

**Proposition 3.3.3.** Assume that \( Z \in Z(g) \). Then for all \( k \),

\[
t_Z(\mathcal{A}_{2k}) = 0 \text{ and } t_Z(\mathcal{A}_{2k+1}) = Z \cdot \mathcal{A}_{2k}.
\]

This Proposition expresses classical identities on standard polynomials, generally written in the case \( Z = 1_m \).

Let us now assume that \( g \) is equipped with a trace, that is, a linear form \( \text{Tr}: g \to \mathbb{C} \) satisfying:

\[
\text{Tr}(X.Y) = \text{Tr}(Y.X), \ \forall X, Y \in g.
\]

Let \( \wedge g \) be the Grassmann algebra of \( g \). We extend the trace \( \text{Tr} \) to a map \( \text{Tr}: \mathcal{M}_a(g) \to \wedge g \) defined by:

\[
\text{Tr}(F)(Y_1, \ldots, Y_p) = \text{Tr}(F(Y_1, \ldots, Y_p)),
\]

for all \( F \in \mathcal{M}_a^p(g) \), \( Y_1, \ldots, Y_p \in g \).

**Proposition 3.3.4.** One has \( \text{Tr}(F \times G) = (-1)^{pq} \text{Tr}(G \times F) \), for all \( F \in \mathcal{M}_a^p(g) \), \( G \in \mathcal{M}_a^q(g) \).

**Proof.** Let \( F \in \mathcal{M}_a^p(g) \), \( G \in \mathcal{M}_a^q(g) \), \( Y_1, \ldots, Y_{p+q} \in g \):

\[
\text{Tr}(F \times G)(Y_1, \ldots, Y_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p+q}} \varepsilon(\sigma) \text{Tr}(F(Y_{\sigma(1)}, \ldots, Y_{\sigma(p)}) \cdot G(Y_{\sigma(p+1)}, \ldots, Y_{\sigma(p+q)}))
\]

\[
= \sum_{\sigma \in \mathcal{S}_{p+q}} \varepsilon(\sigma) \text{Tr}(G(Y_{\sigma(p+1)}, \ldots, Y_{\sigma(p+q)}) \cdot F(Y_{\sigma(1)}, \ldots, Y_{\sigma(p)}))
\]

Given \( \sigma \in \mathcal{S}_{p+q} \), define \( \tau \in \mathcal{S}_{q,p} \) as \( \tau(1) = \sigma(p+1), \ldots, \tau(q) = \sigma(p+q) \) and \( \tau(q+1) = \sigma(1), \ldots, \tau(q+p) = \sigma(p) \). Then one has \( \varepsilon(\tau) = (-1)^{pq} \varepsilon(\sigma) \), so:

\[
\text{Tr}(F \times G)(Y_1, \ldots, Y_{p+q}) = (-1)^{pq} \sum_{\tau \in \mathcal{S}_{q,p}} \varepsilon(\tau) \text{Tr}(G(Y_{\tau(1)}, \ldots, Y_{\tau(q)}) \cdot F(Y_{\tau(q+1)}, \ldots, Y_{\tau(p+q)}))
\]

\[
= (-1)^{pq} \text{Tr}(G \times F)(Y_1, \ldots, Y_{p+q})
\]

\( \square \)
Hence our extension of the trace has, in fact, the properties of a $\wedge g$-valued super trace on the graded algebra $(\mathcal{M}_a(g), \times)$. Denoting the super bracket associated to the $\times$-product on $\mathcal{M}_a(g)$ by:

$$[F, G]_\times = F \times G - (-1)^{p_1} G \times F, \forall F \in \mathcal{M}_a^p(g), G \in \mathcal{M}_a^q(g),$$

one obtains

**Corollary 3.3.5.** $\text{Tr}([F, G]_\times) = 0$.

**Proposition 3.3.6.** One has $\text{Tr}(\mathcal{A}^{2k}) = 0 (k \geq 1)$ and $\text{Tr}(\mathcal{A}^{2k+1}) (k \geq 0)$ is an invariant cocycle for the (trivial) cohomology of the Lie algebra $g$.

**Proof.** For the first claim, use $[\mathcal{A}_1, \mathcal{A}_{2k-1}]_\times = 2 \mathcal{A}_{2k}$ and apply the Corollary above. For the second, we remark that $\mathcal{A}^{2k+1}$ is a $g$-invariant map from $\mathcal{M}_a^{2k+1}$ into $g$, so $\text{Tr}(\mathcal{A}^{2k+1}) \in (\wedge g)^9$ and therefore a cocycle by the following classical Lemma. $\square$

**Lemma 3.3.7.** Let $h$ be a Lie algebra. Then any invariant cochain in $(\wedge h)^9$ is a cocycle.

**Proof.** If $h$ is finite dimensional, the result is well known ([16]) and is a direct consequence of the formula $\partial = \frac{1}{2} \sum_{i=1}^{n} \partial_i \wedge \theta_X$, where $\partial$ is the differential, $\{X_1, \ldots, X_n\}$ a basis of $h$ and $\{\omega_1, \ldots, \omega_n\}$ its dual basis.

For the sake of completeness, we give a proof in the general case, let $\{X_i | i \in I\}$ be a basis of $h$, and $\{\omega_i | i \in I\}$ be the forms defined by $\omega_i(X_j) = \delta_{ij}$, $\forall i, j$. We claim that the formula $\partial = \frac{1}{2} \sum_{i \in I} \omega_i \wedge \theta_{X_i}$ is still valid. To prove this, let $D = \frac{1}{2} \sum_{i \in I} \omega_i \wedge \theta_{X_i}$. Though its indexes set is infinite, this sum exists since for $\Omega \in \wedge^p h$ and $Y_1, \ldots, Y_{p+1} \in h$, one has:

$$\frac{1}{2} \sum_{i \in I} \omega_i \wedge \theta_{X_i}(\Omega)(Y_1, \ldots, Y_{p+1}) = \frac{1}{2} \sum_{j=1}^{p+1} (-1)^{j+1} \sum_{i \in I} \omega_i(Y_j) \theta_{X_i}(\Omega)(Y_1, \ldots, \hat{Y}_j, \ldots, Y_{p+1})$$

Then

$$D(\Omega)(Y_1, \ldots, Y_{p+1}) = -\frac{1}{2} \sum_{j=1}^{p+1} (-1)^{j+1} \times$$

$$\left( \sum_{k=1}^{j-1} \Omega(Y_1, \ldots, \hat{Y}_j, \ldots, Y_{p+1}) + \sum_{k=j+1}^{p+1} \Omega(Y_1, \ldots, \hat{\hat{Y}}_j, \ldots, \hat{Y}_{k}, \ldots, Y_{p+1}) \right)$$

$$= \frac{1}{2} \sum_{j=1}^{p+1} (-1)^{j+1} \left( \sum_{k<j} (-1)^{k+1} \Omega(Y_j, Y_k, Y_1, \ldots, \hat{Y}_k, \ldots, Y_{p+1}) + \right)$$

$$\left( \sum_{j<k} (-1)^{j-1} \Omega(Y_j, Y_k, Y_1, \ldots, \hat{Y}_j, \ldots, \hat{Y}_k, \ldots, Y_{p+1}) \right)$$

$$= \sum_{j<k} (-1)^{j+k} \Omega(Y_j, Y_k, Y_1, \ldots, \hat{Y}_j, \ldots, \hat{Y}_k, \ldots, Y_{p+1}) = \partial(\Omega)(Y_1, \ldots, Y_{p+1})$$

$\square$
Now recall the well-known formula (e.g. [15]):

**Proposition 3.3.8.** $\text{Tr}(\mathcal{A}_{2k+1}(Y_1, \ldots, Y_{2k+1})) = (2k+1)\text{Tr}(\mathcal{A}_{2k}(Y_1, \ldots, Y_{2k}))Y_{2k+1},$

for all $Y_1, \ldots, Y_{2k+1} \in g.$

This formula will be reinterpreted in Section 6 in terms of cyclic cohomology of the Lie algebra $g$: $\mathcal{A}_{2k}$ is a cocycle of the adjoint action (actually a coboundary since $[\mathcal{A}_2, \mathcal{A}_{2k-1}] = \mathcal{A}_{2k}$), and Proposition 3.3.8 tells that it is a cyclic cocycle, as will be defined in Section 6.

**Example 3.3.9.** Assume that $g = \mathfrak{gl}(n)$. Then $H^*(g)$ can be completely described in terms of standard polynomials (see e.g. [15] or [10]):

\[ H^*(g) = \text{Ext}[\text{Tr}(\mathcal{A}_1), \text{Tr}(\mathcal{A}_3), \ldots, \text{Tr}(\mathcal{A}_{2n-1})]. \]

Moreover, by the Amitsur-Levitzki theorem ([1, 15]):

\[ \mathcal{A}_k = 0, \text{ if } k \geq 2n. \]

So $\dim \mathcal{A} = 2n$. For the $\times$-product, $\mathcal{A} \simeq \mathbb{C}[X]/X^{2n}$. For the graded bracket, the structure of $\mathcal{A}$ is given by formula (IV), $\mathcal{A}_2$ is the Lie algebra structure on $g$ and the standard polynomials $\mathcal{A}_4, \ldots, \mathcal{A}_{2n-2}$ define 2k-Lie algebra structures on $g$ by Proposition 3.2.1.

**Example 3.3.10.** More generally, let $V$ be an infinite dimensional vector space. Let $g$ be the space of finite rank linear maps. So $g$ is an ideal of the associative algebra $\text{End}(V)$. There is a vector spaces isomorphism $g \simeq V^* \otimes V$ defined by $(\omega \otimes v)(v') = \omega(v')v$, for all $v, v' \in V, \omega \in V^*$. So we can define the trace $\text{Tr}(X)$ when $X \in g$ by $\text{Tr}(\omega \otimes v) := \omega(v)$, for all $\omega \in V^*, v \in V$. It is easy to check that $\text{Tr}([X, Y]) = 0$, for all $X, Y \in g$, so the preceding results apply. Moreover, the symmetric bilinear form $B$ defined on $g$ by $B(X, Y) = \text{Tr}(XY)$ is non degenerate and invariant, therefore $g$ is a quadratic Lie algebra. Since $\mathfrak{gl}(n) \subset g, \forall n$, by Example 3.3.9, we can conclude that

\[ \text{Ext}[\text{Tr}(\mathcal{A}_1), \text{Tr}(\mathcal{A}_3), \ldots, \text{Tr}(\mathcal{A}_{2n-1}), \ldots] \subset H^*(g) \]

**Proposition 3.3.11.** Let $a_{2n+1} = \text{Tr}(\mathcal{A}_{2n+1})$. Then

\[ H^*(g) = \text{Ext}[a_1, a_3, \ldots, a_{2n+1}, \ldots]. \]

**Proof.** Recall that for any Lie algebra $\mathfrak{h}$, there is an isomorphism $H^k(\mathfrak{h}) \simeq H^k(\mathfrak{h})^*$, induced by the restriction $\Omega \in Z^k(\mathfrak{h}) \mapsto \Omega|_{Z_k(\mathfrak{h})}$ where $H_k(\mathfrak{h})$ is the homology of $\mathfrak{h}$ defined as $H_k(\mathfrak{h}) = Z_k(\mathfrak{h})/B_k(\mathfrak{h})$ (with $Z_k(\mathfrak{h})$ the cycles and $B_k(\mathfrak{h})$ the boundaries).

Let us define $\mathcal{S} = \{ S = (W, W') \mid W, W' \text{ complementary subspaces of } V \text{ with } \dim(W) < \infty \}$ and for $S = (W, W') \in \mathcal{S}$, $g_S = \{ X \in g \mid X(V) \subset W, X(W') = \{0\} \}$. Then $g_S$ is a subalgebra of the (associative or Lie) algebra $g$ and one has $g_S \simeq \mathfrak{gl}(\dim(W))$. It is easy to check that given $X_1, \ldots, X_r \in g$, there exists $S \in \mathcal{S}$ such that $X_i \in g_S, \forall i = 1, \ldots, r$. It results that, if $c \in \text{Ext}^k(g)$, there exists $S$ such that $c \in \text{Ext}^k(g_S)$, so that $\text{Ext}^k(g) = \bigcup_{S \in \mathcal{S}} \text{Ext}^k(g_S).

Set $\mathcal{S}^k = \text{Ext}[a_1, a_3, \ldots, a_{2n+1}, \ldots] \subset H^*(g)$ and $\mathcal{S}^k = \mathcal{S} \cap H^k(g)$. Then $\dim(\mathcal{S}^k) = \# I_k$ with $I_k = \{ (i_j) \in \{0, 1\}^N \mid \sum_{j \in \mathbb{N}}(2j+1)i_j = k \}$. We fix a basis $\{ \Omega_i \mid i \in I_k \}$ of $\mathcal{S}^k$. 
Given \( c \in Z_k(\mathfrak{g}) \), denote by \( \tau \) its class in \( H_k(\mathfrak{g}) \). Let us assume that \( \Omega_i(\tau) = 0 \), \( \forall i \in I_k \). Take \( S \in \mathcal{S} \) such that \( c \in \text{Ext}^k(\mathfrak{g}_S) \), then by (3.3.9), \( \{ \Omega_i \mid i \in I_k \} \) generates \( H^k(\mathfrak{g}_S) = H_k(\mathfrak{g}_S)^* \) and since \( c \in Z_k(\mathfrak{g}_S) \), it results that \( c \in B_k(\mathfrak{g}_S) \subset B_k(\mathfrak{g}) \), therefore \( \tau = 0 \). So, \( \{ \Omega_i \mid i \in I_k \} \) is free in \( H_k(\mathfrak{g})^* \) and \( \cap_{i \in k} \ker(\Omega_i) = \{ 0 \} \). It results that \( \dim(H_k(\mathfrak{g})) = \# I_k \). Since \( H^k(\mathfrak{g}) = H_k(\mathfrak{g})^* \), one has \( \dim(H^k(\mathfrak{g})) = \# I_k \) and since \( \mathfrak{g} \subset H^k(\mathfrak{g}) \), one obtains \( \mathfrak{g}^k = H^k(\mathfrak{g}) \).

**Remark 3.3.12.** From \( H^1(\mathfrak{g}) = \mathbb{C} \text{ Tr} \), we deduce that \( [\mathfrak{g}, \mathfrak{g}] = \ker(\text{Tr}) \). From \( H^2(\mathfrak{g}) = \{ 0 \} \), we deduce that \( \mathfrak{g} \) has no (non trivial) central extension.

### 4. Super Poisson Brackets and Quadratic Lie Algebras

The canonical Poisson bracket on \( \mathbb{R}^{2n} \) appears as the leading term of a quantization of the algebra of polynomial functions by the Weyl algebra, the so-called *Moyal product*. We will develop a similar formalism, replacing polynomials (i.e. commuting variables) by skew multilinear forms (i.e. skew commuting variables) and the Weyl algebra by the Clifford algebra. The leading term of the deformation will be the *super Poisson bracket*.

#### 4.1. Definition of the Clifford Algebra

Let us give a definition of the Clifford algebra that is well adapted to the realization of this algebra as a deformation of the exterior algebra. Denote by \( \mathcal{C}_t \), \( t \in \mathbb{C} \), the associative algebra with basis \( \{ e_I, I \in \mathbb{Z}_+^n \} \) and product defined by

\[
e_I * e_J = (-1)^{\Omega(IJ)} |I| |J| \begin{pmatrix} e_I + J \end{pmatrix}_{(2)}
\]

where \( \Omega \) is the bilinear form associated to the matrix \( (a_{ij})_{i,j=1}^n \) with \( a_{ij} = 1 \) if \( i > j \) and 0 otherwise.

Take \( I_i = (j_k) \in \mathbb{Z}_+^n \), with \( j_i = 1 \) and 0 otherwise. Set \( e_I = e_{I_i}, i = 1, \ldots, n \) and \( V = \text{span} \{ e_1, \ldots, e_n \} \). When \( t = 0 \), one obtains \( \mathcal{C}_0 = \text{Ext}(V) \). When \( t \neq 0 \), \( \mathcal{C}_t \) is the *Clifford algebra*. The following relations hold:

\[
e_i^2 = t, \forall i, \quad e_i * e_j + e_j * e_i = 0, i \neq j,
\]

\[
e_i * e_1 * \cdots * e_p = e_i \wedge e_1 \wedge \cdots \wedge e_p, \text{ if } i_1 < i_2 < \cdots < i_p
\]

So that \( \mathcal{C}_t \) is the quotient algebra of the tensor algebra \( T(V) \) by the relations:

\[
v \otimes v = t. B(v,v), 1, v \in V,
\]

where \( B \) is the bilinear form \( B(e_i, e_j) = \delta_{ij} \), for all \( i, j \), and we recover the usual definition of the Clifford algebra.

But, we are mainly interested in realizing \( \mathcal{C}_t \) as a deformation of \( \text{Ext}(V) \). Using:

\[
t^k = 0^k + t \delta_{k,1} + t^2 \delta_{k,2} + \cdots.
\]

this deformation becomes transparent:

**Proposition 4.1.1.** One has

\[
e_i * e_j = e_i \wedge e_j + \sum_{k=1}^n t^k D_k(e_i, e_j)
\]

where \( D_k(e_i, e_j) = \delta_{|I|, k} (-1)^{\Omega(IJ)} e_{I+J} \).

Symmetry properties of the coefficients are resumed in:

**Proposition 4.1.2.** For all \( \Omega \in \text{Ext}^w(V), \ \Omega' \in \text{Ext}^{w'}(V) \),

\[
D_j(\Omega, \Omega') = (-1)^j (-1)^{w'w} D_j(\Omega', \Omega).
\]

We insist on the fact that \( \mathcal{C}_r \) is not a \( \mathbb{Z} \)-graded, but only a \( \mathbb{Z}_2 \)-graded algebra. The associated Lie superalgebra has bracket:

\[
[g, \bigwedge \Omega]_s = 2 \sum_{p \geq 0} i^{2p+1} D_{2p+1}(\Omega, \Omega').
\]

**Definition 4.1.3.** We define the *super Poisson bracket* on \( \text{Ext}(V) \) by:

\[
\{ \Omega, \Omega' \} = 2 \, D_1(\Omega, \Omega'), \ \forall \ \Omega, \Omega' \in \text{Ext}(V).
\]

Since \([.,.], \) satisfies the super Jacobi identity, so does \( \{.,.\} \). Moreover, since \( \text{ad}_i(\Omega) \) is derivation of the \( \mathcal{C}_r \) product, \( \text{ad}_p(\Omega) := \{\Omega, .\} \) is a derivation of the \( \wedge \) product (actually of degree \( (w - 2) \) if \( \Omega \in \text{Ext}^w(V) \)).

Finally, by a straightforward computation, one gets:

\[
(V) \quad \{v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_q\} = 2 (-1)^{p+1} \times \sum_{i=1}^{p} (-1)^{i+j} B(v_i, w_j) \, v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_q,
\]

for all \( v_1, \ldots, v_p, w_1, \ldots, w_q \in V \).

Comparing with the formulas given in [20], we conclude that the Lie superalgebra \( \text{Ext}(V)/\mathbb{C} \) is isomorphic to the simple Lie superalgebra \( H(n) \). Notice that \( \text{Ext}(V)/\mathbb{C} \simeq \text{ad}_p(\text{Ext}(V)) \subset \text{Der}(\text{Ext}(V)) = \mathfrak{g}(V^*), \) so we obtain the classical inclusion \( H(n) \subset W(n) \) ([20]).

4.2. Let us modify slightly the formalism in 4.1 in order to apply it to Lie algebras deformation theory. We begin with a \( n \)-dimensional vector space \( g \) and we set \( V = g^* \). We assume that \( g \) is a quadratic space with bilinear form \( B \). Denote by \( \{X_1, \ldots, X_n\} \) an orthonormal basis of \( g \) and by \( \{\omega_1, \ldots, \omega_n\} \) the dual basis; we define \( B \) on \( g^* \) by \( B(\omega_i, \omega_j) = \delta_{ij} \). Applying the construction in 4.1 with \( e_i = \omega_i, \ i = 1, \ldots, n \), we get a super Poisson bracket on \( \wedge g \) and it is easy to check that:

**Proposition 4.2.1.** For all \( \Omega \in \wedge^w g, \ \Omega' \in \wedge^w g, \) one has

\[
\{\Omega, \Omega'\} = 2 (-1)^{w+1} \sum_{j=1}^{n} t_{X_j}(\Omega) \wedge t_{X_j}(\Omega').
\]

This formula is valid in any orthonormal basis of \( g \) and it is enough for our purpose in Section 4, but a general formula can be found in Lemma 5.3.1. There is a Moyal type formula which gives the Clifford product in terms of the super Poisson bracket: let \( m_\alpha \) be the product from \( \wedge g \otimes \wedge g \rightarrow \wedge g \), and define \( \mathcal{F}: \wedge g \otimes \wedge g \rightarrow \wedge g \otimes \wedge g \) by

\[
\mathcal{F}(\Omega \otimes \Omega') = (-1)^w \sum_{j=1}^{n} t_{X_j}(\Omega) \otimes t_{X_j}(\Omega')
\]

for all \( \Omega \in \wedge^w g, \ \Omega' \in \wedge g \). Then:
**Proposition 4.2.2.**

$$\Omega \star \Omega' = m_\wedge \circ \exp(-tD)(\Omega \otimes \Omega').$$

**Proof.** As in the beginning of 4.2, let $e_i = \omega_i$ and let $\partial_i = t \omega_i$, $i = 1, \ldots, n$. As in 4.1, for $I = (i_1, \ldots, i_n) \in \mathbb{Z}_2^n$, let $e_I = e_{i_1}^{1} \wedge \cdots \wedge e_{i_n}^{n}$ and $\partial_I = \partial_{i_1}^{1} \circ \cdots \circ \partial_{i_n}^{n}$. For $J = (j_1, \ldots, j_n) \in \mathbb{Z}_2^n$, let $J^I = j_1^{i_1} \cdots j_n^{i_n}$. One has $\partial_I(e_J) = (-1)^{\Omega(I,J)} \sum_{i} \partial^i_{I} e_{i+J}$. Since all $\partial_i \otimes \partial_i$ commute, and $\partial_i^2 = 0$, one has:

$$\left( \sum_i \partial_i \otimes \partial_i \right)^k = k! \sum_{|I|=k} \partial_I \otimes \partial_I$$

For $k > 0$, one has:

$$m_\wedge \circ D^k(e_R \otimes e_S) = (-1)^{|I|} \left( -1 \right)^{|I| - \sum_{i} \partial_i \otimes \partial_i \left( e_R \otimes e_S \right) = (-1)^{|I|} \left( -1 \right)^{k(k-1)} k! \sum_{|I|=k} (-1)^{\Omega(I,R)} (-1)^{\Omega(I,S)} \delta_{RS}^{I} S^{I} e_{I+R} \wedge e_{I+S} \right.$$}

This vanishes, except if $k = |RS|$, and in that case, the only remaining term in the sum is when $I = RS$. We compute this term:

$$m_\wedge \circ D^k(e_R \otimes e_S) = (-1)^{|I|} \left( -1 \right)^{k(k-1)} k! (-1)^{\Omega(RS,R) + \Omega(RS,S)} (-1)^{\Omega(RS+R,RS+S)} e_{R+S} \right.$$}

But one has $\Omega(A,B) + \Omega(B,A) = |A||B| - |AB|$, so:

$$\Omega(RS,R) + \Omega(RS,S) = k|R| - k, \quad \text{and} \quad \Omega(RS,RS) = \frac{k(k-1)}{2}.$$}

So finally, we have proved that

$$m_\wedge \circ D^k e_{R \wedge S} = (-1)^k k! \left( -1 \right)^{\Omega(RS,S)} \delta_{RS\wedge}^{I} S^{I} e_{R+S} \right.$$}

On the other hand, by Proposition 4.1.1, one has

$$e_R \star e_S = e_R \wedge e_S + (-1)^{\Omega(R,S)} \sum_{k=1}^{n} \delta_{RS\wedge}^{I} S^{I} e_{R+S} \right.$$}

so the result follows.

**Remark 4.2.3.** An equivalent formula is given in [14], but without the use of super Poisson bracket.

4.3. A derivation $D \in D$ is Hamiltonian if it belongs to $\text{ad}_p(\wedge \mathfrak{g})$. Actually, the space of Hamiltonian derivations is a subalgebra of $D$, that we denote by $H(\mathfrak{g})$, which is isomorphic to $\wedge Q \mathfrak{g} = \wedge \mathfrak{g}/\mathbb{C}$ and therefore, by (V), isomorphic to the simple Lie superalgebra $\tilde{H}(n)$. Here a simple characterization of Hamiltonian derivations:
Proposition 4.3.1. A derivation $D = \sum D_r \wedge t_X$ is Hamiltonian if and only if $t_X(D_r) + t_X(D_r) = 0, \forall r,s$.

Proof. When the condition is satisfied, one has $D = \text{ad}_p(\Omega)$ where $\Omega = \frac{1}{2^n} \sum D_r \wedge \omega_r$ and $w = \text{deg}(D) + 2$. □

Remark 4.3.2. A Hamiltonian derivation is a derivation of the $\wedge$-product and also of the super Poisson bracket.

In fact, one has:

Proposition 4.3.3. Let $D \in \mathcal{D}$. Then $D$ is Hamiltonian if and only if $D$ is a derivation of the super Poisson bracket.

Proof. Let $D = \sum D_r \wedge t_X$, with $D \in \wedge^d \mathfrak{g}$, then $D_r = D(\omega_r)$. Since $\{\omega_r, \omega_s\} \in \mathbb{C}$, assuming that $D$ is a derivation of the super Poisson bracket, one has:

$$0 = D(\{\omega_r, \omega_s\}) = 2 (-1)^{d+1} (t_X(D_r) + t_X(D_r))$$

and the result follows by Proposition 4.3.1. □

4.4. We now want to apply super Poisson brackets to the theory of quadratic Lie algebras, in a deformation framework that we will quickly set up. Given a quadratic Lie algebra $(\mathfrak{g}_0, B_0)$ with bilinear form $B_0$ and product $[\cdot,\cdot]$, a deformation $(\mathfrak{g}_t, B_t)$ of $(\mathfrak{g}_0, B_0)$ is:

1. a deformation $g_t$ of $g$ in the usual sense, so:

$$[X,Y]_t = [X,Y] + tC_1(X,Y) + \ldots, \forall X,Y \in \mathfrak{g},$$

2. a formal bilinear form $B_t = B_0 + tB_1 + \ldots$ such that

$$B_t([X,Y]_t, Z) = -B_t(Y, [X, Z]_t), \forall X,Y,Z \in \mathfrak{g}.$$

Two deformations $(\mathfrak{g}_t, B_t)$ and $(\mathfrak{g}'_t, B'_t)$ with respective brackets $[\cdot,\cdot]_t$ and $[\cdot,\cdot]'_t$ are equivalent if there exists $T_t = \text{Id} + tT_1 + \ldots$ such that:

$$[X,Y]'_t = T_t^{-1}([T_t(X), T_t(Y)]) \quad \text{and} \quad B'_t(X,Y) = B_t(T_t(X), T_t(Y)), \forall X,Y \in \mathfrak{g}.$$

Proposition 4.4.1. Any deformation $(\mathfrak{g}_t, B_t)$ of $(\mathfrak{g}_0, B_0)$ is equivalent to a deformation with unchanged bilinear form.

Proof. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ with respect to $B_0$. By a Gram-Schmidt type strategy, one can construct $\{e_1(t), \ldots, e_n(t)\}$ such that:

$$e_\ell(t) = \lambda_1(t)e_1(t) + \cdots + \lambda_{\ell-1}(t)e_{\ell-1}(t) + e_\ell, \forall \ell \leq n,$$

with $\lambda_j(t) \in t \mathbb{C}[[t]]$, and $B_t(e_\ell(t), e_m(t)) = 0$, for all $\ell, m \leq n$. Since $[B_t(e_\ell(t), e_m(t))]_{t=0} = B_0(e_\ell, e_m) = 1$, $\forall \ell \leq n$, $B_t(e_\ell(t), e_\ell(t))$ is invertible, and

$$e_\ell'(t) = \frac{1}{B_t(e_\ell(t), e_\ell(t))} e_\ell(t)$$

does satisfy $B_t(e_\ell'(t), e'_m(t)) = \delta_{nm}, \forall \ell, m$.

Now if we define $T_t$ by $T_t(e_\ell) = e_\ell'(t), \forall \ell \leq n$, and a new deformation

$$[X,Y]_t = T_t([T_t(X), T_t(Y)]_t), \forall X,Y \in \mathfrak{g},$$

does satisfy $B_t([X,Y]'_t, Z) = -B_t(Y, [X, Z]'_t), \forall X,Y,Z \in \mathfrak{g}$, so:

$$[X,Y]'_t = T_t^{-1}([T_t(X), T_t(Y)]) \quad \text{and} \quad B'_t(X,Y) = B_t(T_t(X), T_t(Y)), \forall X,Y \in \mathfrak{g}.$$
with bilinear form $B'(X, Y) = B(T_t(X), T_t(Y)) = B_0(X, Y), \forall X, Y \in g$, we obtain a deformation that is equivalent to the initial one. □

So if one want to study quadratic Lie algebras in terms of deformation theory, one can restrict to quadratic Lie algebras with a specified bilinear form, and that is what we shall do next.

4.5. The constructions made in the beginning of this section can now be applied as follows: given a finite dimensional quadratic Lie algebra $g$ with bilinear form $B$, let $\partial$ be the corresponding derivation of $\bigwedge g$ (i.e. the differential of the trivial cohomology complex of $g$, see 2.2), we define:

$$I(X, Y, Z) := B([X, Y], Z), \forall X, Y, Z \in g$$

Then one has:

**Proposition 4.5.1.**

1. $I \in (\bigwedge^3 g)^g$
2. $\partial = -\frac{1}{2} \text{ad}_P(I)$.
3. $\{I, I\} = 0$.

**Proof.** The assertion (1) is obvious. To show (2), let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of $g$ and $\{\omega_1, \ldots, \omega_n\}$ the dual basis. Then for all $Y, Z \in g$:

$$-\frac{1}{2} \text{ad}_P(I) (\omega_i)(Y, Z) = -\left(\sum_j \iota_{X_j}(I) \wedge \iota_{X_j}(\omega_i)\right)(Y, Z) = -B([X_i, Y], Z) = -B(X_i, [Y, Z]) = -\omega_i([Y, Z]) = \partial \omega_i(Y, Z)$$

Hence, $\partial = -\frac{1}{2} \text{ad}_P(I)$.

Finally $\text{ad}_P(\{I, I\}) = [\text{ad}_P(I), \text{ad}_P(I)] = 4[\partial, \partial] = 8\partial^2 = 0$. So $\{I, I\} = 0$ and that proves (3). □

Note that $\partial, \iota_X$ and $\theta_X = [\iota_X, \partial], \forall X \in g$ are all Hamiltonian derivations.

4.6. Conversely, assume that $g$ is (only) a finite dimensional quadratic vector space. Fix $I \in \bigwedge^3 g$ and define $\partial = -\frac{1}{2} \text{ad}_P(I)$. Then the formula

$$\text{ad}_P (\{\Omega, \Omega'\}) = [\text{ad}_P(\Omega), \text{ad}_P(\Omega')], \forall \Omega, \Omega' \in \bigwedge g$$

leads to

(VI) $[\partial, \partial] = 0$ if and only if $\{I, I\} = 0$.

Let $F = F_\partial$ be the structure on $g$ associated to $\partial$ (see 2.1 and 2.2), then from (VI), it follows:

**Proposition 4.6.1.** $F$ is a Lie algebra structure if and only if $\{I, I\} = 0$. In that case, with the notation $[X, Y] = F(X, Y)$, one has:

$$I(X, Y, Z) = B([X, Y], Z), \forall X, Y, Z \in g$$

the form $B$ is invariant and $g$ is a quadratic Lie algebra.
Proof. We have to prove that if $F$ is a Lie algebra structure, then $I(X,Y,Z) = B([X,Y],Z), \forall X,Y,Z \in \mathfrak{g}$.

Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis, then $\partial = -\sum k t_{X_k}(I) \wedge t_{X_k}$, so $F = \sum k t_{X_k}(I) \otimes X_k$, and therefore $B([X_i,X_j],X_k) = t_{X_i}(I)(X_i,X_j) = I(X_i,X_j,X_k)$, for all $i, j, k$.

\[ \square \]

Remark 4.6.2. Using 4.4, 4.5 and 4.6, it appears that $\wedge \mathfrak{g}[2]$ with super Poisson bracket is a gla associated to deformation theory of finite dimensional quadratic Lie algebras: by 4.4, one can assume that $B$ does not change, then quadratic Lie algebra structures with the same $B$ are in one to one correspondence with elements $I \in \wedge^3 \mathfrak{g}$ such that $\{I,I\} = 0$ (4.5, 4.6). An equivalent description can be given in terms of Hamiltonian derivations, i.e. of the gla $\mathcal{H}(\mathfrak{g}) = \text{ad}_F(\wedge \mathfrak{g}) \simeq \wedge \mathfrak{g}/C = \wedge Q \mathfrak{g}$.

Let us note that in this picture, one has to redefine equivalence: a priori, one might think that equivalence should be defined as Lie algebras isomorphism keeping $B$ fixed. But this is too restrictive, since $[\ldots]$ and $\lambda(t)[\ldots]$, with $\lambda(t) = 1+t(\ldots)$ will not be equivalent in that sense as they should be. So one has rather to work with the notion of a conformal equivalence, i.e. an equivalence defined by a Lie algebras isomorphism $T(t) = \text{Id}+t(\ldots)$ satisfying $B(T_t(X),T_t(Y)) = \mu(t)B(X,Y)$, with $\mu(t) = 1+t(\ldots)$. This will change the corresponding gla : one can consider the subalgebra $\mathcal{R} \oplus \mathcal{H}(\mathfrak{g})$ of $\mathcal{H}(\mathfrak{g})$ (where $R = \sum_i \omega_i \wedge t_{X_i}$ is the super radial vector field), rather than $\mathcal{H}(\mathfrak{g})$. Hence, there are some adaptations to carry out, which will not be developed here since they are somewhat standard. Let us only indicate that in this framework, if $(\mathfrak{g}_0,B_0)$ is the initial quadratic Lie algebra with associated $I_0 \in \wedge^3(\mathfrak{g})$, then the first obstruction to triviality of a quadratic deformation will lie in $H^3(\mathfrak{g})/C I_0$. For instance, if $\mathfrak{g}_0$ is semisimple, it is shown in [16] that $H^3(\mathfrak{g}_0)$ and the space of symmetric invariant bilinear forms on $\mathfrak{g}_0$ are isomorphic, the isomorphism being $B \mapsto I_0$ where $I_0(X,Y,Z) = B([X,Y],Z), \forall X,Y,Z \in \mathfrak{g}$. It results that when $\mathfrak{g}_0$ is simple, it is rigid in quadratic deformation theory.

5. Elementary quadratic Lie algebras

5.1. First, let us recall two results:

**Proposition 5.1.1.** Let $V$ be a finite dimensional vector space and $I$ a $k$-form in $\wedge^k V$. Denote by $V(I)$ the orthogonal subspace in $V^*$ of the subspace $\{X \in V \mid t_X(I) = 0\}$. Then $\dim(V(I)) \geq k$ and if $I$ is non zero, $I$ is decomposable if and only if $\dim V(I) = k$. In this case, if $\{\omega_1, \ldots, \omega_k\}$ is a basis of $V(I)$, one has $I = \alpha \omega_1 \wedge \cdots \wedge \omega_k$, for some $\alpha \in \mathbb{C}$ ([4]).

**Proposition 5.1.2.** Let $V$ be a finite dimensional quadratic vector space with a non degenerate symmetric bilinear form $B$. For a subspace $W$ of $V$, denote by $W^\perp$ its orthogonal subspace in $V$ with respect to $B$ and $W^{\perp*}$ its orthogonal in $V^*$. Let $\phi$ be the isomorphism from $V$ onto $V^*$ induced by $B$. Then $\phi|_{W^\perp}$ is an isomorphism from $W^\perp$ onto $W^{\perp*}$, so $\dim(W^{\perp}) = \dim(V) - \dim(W)$. One has $V = W \oplus W^\perp$ if and only if $W \cap W^\perp = \{0\}$ and in this case the restriction of $B$ to $W$ or $W^\perp$ is non degenerate.
Proposition 5.1.4. \( \text{Proof.} \) \( \text{orbits have dimension at most 2.} \)

Remark that the obvious identity \( Z(g)^\perp = [g, g] \) holds here. As a consequence,

**Proposition 5.1.5.** Let \( g \) be a non Abelian quadratic Lie algebra. Then \( \dim([g, g]) \geq 3. \) Moreover, \( g \) is elementary if and only if the equality holds.

**Proof.** Since \( Z(g)^\perp = [g, g] \) and \( V_0 = Z(g)^\perp, \) the result follows directly from Proposition 5.1.1.

**Corollary 5.1.6.** Let \( g \) be an elementary quadratic Lie algebra. Then all codjoint orbits have dimension at most 2.

**Proof.** Let \( \omega \in g^* \) and \( X_\omega \in g \) such that \( \omega = \phi(X_\omega). \) Then \( \text{ad}(g)(\omega) = \phi([g, X_\omega]) \subset \phi([g, g]), \) so \( \dim(\text{ad}(g)(\omega)) \leq 3 \) and since all coadjoint orbits have even dimension, the result is proved.

**Remark 5.1.6.** Suppose that \( g \) is (only) a finite dimensional quadratic vector space and let \( I \) be a decomposable 3-vector in \( \wedge^3 g. \) Then it is easy to check that \( \{ I, I \} = 0 \) for the super Poisson bracket. So by Proposition 4.6.1 there is a quadratic elementary Lie algebra structure on \( g \) such that \( I(X, Y, Z) = B([X, Y], Z), \forall X, Y, Z \in g. \)

In the sequel, we will classify all elementary non Abelian quadratic Lie algebras. Our proof is a two steps one: first, in 5.2, we demonstrate a result on quadratic Lie algebras that reduces the classification problem to small dimensions, namely between 3 and 6. Then in 5.3, we proceed by classifying these small dimensional elementary quadratic algebras. Explicit commutations in a canonical basis with respect to \( B \) are computed as well.

5.2. Here is the reduction result on quadratic Lie algebras:

**Proposition 5.2.1.** Let \( g \) be a non Abelian quadratic Lie algebra with bilinear form \( B. \) Then there exist a central ideal \( 3 \) and an ideal \( l \neq \{0\} \) such that:

1. One has \( g = 3 \oplus l, \) and \( l \) and \( 3 \) are orthogonal with respect to \( B. \)
2. The ideals \( 3 \) and \( l \) are quadratic (with bilinear forms induced by the restriction of \( B \)) and \( l \) is non Abelian. Moreover, \( l \) is elementary if and only if \( g \) is elementary.
3. The center \( Z(l) \) is totally isotropic and one has \( \dim(Z(l)) \leq \frac{1}{2} \dim(l) \leq \dim([l, l]). \)

**Proof.** Let \( z_0 = Z(g) \cap [g, g]. \) Fix any subspace \( z \) such that \( Z(g) = z_0 \oplus z. \) Since \( Z(g)^\perp = [g, g], \) one has \( B(z_0, z) = \{0\} \) and \( z \cap z^\perp = \{0\}. \) It results from Proposition 5.1.2 that \( g = 3 \oplus l \) where \( l = z^\perp. \)

Since \( B([g, g], z) = \{0\}, \) one has \( [g, g] \subset l. \) It is easy to check that \( Z(l) = z_0 \) and \( [l, l] = [g, g] = Z(g)^\perp \) so \( Z(l) \) is totally isotropic; moreover the restriction of \( B \) to \( 3 \) is...
Lemma 5.3.1. Let $V$ be a quadratic vector space with bilinear form $B$. Define $B$ as in Proposition 5.2.2, to the case of non zero elementary quadratic Lie algebra. We shall now finish the classification of non Abelian elementary quadratic Lie algebra, by using the following Lemma:

Corollary 5.2.2. Let $l$ be an elementary non zero quadratic Lie algebra such that $Z(l)$ is totally isotropic. Then one has

$$3 \leq \dim(l) \leq 6.$$  

Proof. Use Propositions 5.2.1(3) and 5.1.4. □

5.3. We shall now finish the classification of non Abelian elementary quadratic Lie algebras. This classification is reduced, by Proposition 5.2.1 and Corollary 5.2.2, to the case of non zero elementary quadratic Lie algebra with a totally isotropic center $Z(l)$. Applying Proposition 5.2.2 one has $3 \leq \dim(l) \leq 6$. Note that if $\dim(l) = 3$, one has $l = [l, l]$ (Proposition 5.1.4), so $l \simeq s(2)$ and $B$ is the Killing form up to a scalar. So we have to consider $\dim(l) \geq 4$ (therefore $\dim(Z(l)) \geq 1$).

We need the following Lemma:

Lemma 5.3.1. Let $V$ be a quadratic vector space with bilinear form $B$. Define $B$ on $V^*$ by $B(\omega, \omega') := B(\phi^{-1}(\omega), \phi^{-1}(\omega'))$, $\forall \omega, \omega' \in V^*$ ($\phi$ as in Proposition 5.1.2). Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of $V^*$, $\{X_1, \ldots, X_n\}$ its dual basis and $\{Y_1, \ldots, Y_n\}$ the basis of $\mathfrak{g}$ defined by $Y_i = \phi^{-1}(\omega_i)$. Then the super Poisson bracket on $\wedge \mathfrak{g}$ is given by

$$\{\Omega, \Omega'\} = 2 (-1)^{w+1} \sum_{i,j} B(Y_i, Y_j) t_{X_i}(\Omega) \wedge t_{X_j}(\Omega'), \quad \Omega \in \wedge^w \mathfrak{g}, \Omega' \in \wedge \mathfrak{g}.$$  

Proof. Using Proposition 4.2.1, one has

$$\{\Omega, \Omega'\} = 2 (-1)^{w+1} \sum_{i,j} \alpha_{ij} t_{X_i}(\Omega) \wedge t_{X_j}(\Omega'),$$

$$\Omega \in \wedge^w \mathfrak{g}, \Omega' \in \wedge \mathfrak{g} \text{ and } \alpha_{ij} = \frac{1}{2} \{\omega_i, \omega_j\}. \quad \text{But from 4.1, one has} \quad \{\omega_i, \omega_j\} = 2B(\omega_i, \omega_j) = 2B(Y_i, Y_j). \quad \square$$

Proposition 5.3.2. Let $l$ be an elementary quadratic Lie algebra with non zero totally isotropic center $Z(l)$. Then:

1. If $\dim(l) = 6$, there exists a basis $\{Z_1, Z_2, Z_3, X_1, X_2, X_3\}$ of $l$ such that:
   (i) $\{Z_1, Z_2, Z_3\}$ is a basis of $Z(l)$.
   (ii) $B(Z_i, Z_j) = B(X_i, X_j) = 0$, $B(Z_i, X_j) = \delta_{ij}$, $\forall i, j$.
   (iii) $[X_1, X_2] = Z_3$, $[X_2, X_3] = Z_1$, $[X_3, X_1] = Z_2$ and the other brackets vanish.

2. If $\dim(l) = 5$, there exists a basis $\{Z_1, Z_2, X_1, X_2, T\}$ of $l$ such that:
   (i) $\{Z_1, Z_2\}$ is a basis of $Z(l)$.
   (ii) $B(Z_i, Z_j) = B(X_i, X_j) = 0$, $B(Z_i, X_j) = \delta_{ij}$, $\forall i, j$, $B(T, Z_i) = B(T, X_i) = 0$, $B(T, T) = 1$.
   (iii) $[X_1, T] = -Z_2$, $[X_2, T] = Z_1$, $[X_1, X_2] = T$ and the other brackets vanish.
If $\dim(l) = 4$, then $\dim(Z(l)) = 1$ and there exist totally isotropic subspaces $i$ with basis $\{Z, P\}$ and $i'$ with basis $\{X, Q\}$ such that $Z(l) \subset i \subset [l, i']$. $l = i \oplus i'$ and:

(i) $Z(l) = C Z, B(Z, X) = B(P, Q) = 1, B(Z, Q) = B(P, X) = 0$.
(ii) $[X, P] = P, [X, Q] = -Q, [P, Q] = Z$ and the other brackets vanish.

Proof:

(1) Assuming that $\dim(l) = 6$, one has $\dim(Z(l)) = 3$, so $Z(l) = [l, i'] = Z(l)^{±}$.

Using [4], there is a totally isotropic subspace $l'$ such that $l = Z(l) \oplus l'$.

With the notation of Proposition 5.1.2, since $\phi|_l$ is an isomorphism from $l'$ onto $Z(l)^{∗}$, we can find a basis $\{Z_1, Z_2, Z_3\}$ of $Z(l)$ and a basis $\{X_1, X_2, X_3\}$ of $l'$ such that $B(Z_i, X_j) = \delta_{ij}$. Then

$$Z(l)^{±} = \text{span}\{X_1^r, X_2^r, X_3^r\} = \text{span}\{\phi(Z_1), \phi(Z_2), \phi(Z_3)\}.$$

Let $I_l = B([X, Y], Z), \forall X, Y, Z \in l$. Since $V_i = Z(l)^{±}$, it results from Proposition 5.1.1 that $I_l = \alpha X^r_1 \land X^r_2 \land X^r_3, \alpha \in C$. Replacing $X_1$ by $\frac{1}{\alpha}X_1$ and $Z_1$ by $\alpha Z_1$, we can assume that $\alpha = 1$. Using Proposition 4.5.1 and Lemma 5.3.1, one obtains $\partial = -\frac{1}{2} \text{ad}_P(I) = -\sum_{i=1}^{3} i x_i (X^r_i \land X^r_j \land X^r_l) \land i z_r$, so by 2.2 and 2.1, $[X, Y] = \sum_{i=1}^{3} i x_i (X^r_i \land X^r_j \land X^r_l)(X, Y) Z_i, \forall X, Y \in l$ and the commutation rules follow.

(2) Assuming $\dim(l) = 5$, one has $\dim(Z(l)) = 2$. Using [4], there is a totally isotropic subspace $l'$ and a one-dimensional subspace $l''$ such that $l = Z(l) \oplus l' \oplus l''$ and $B(Z(l) \oplus l', l'') = \{0\}$. Then one can find a basis $\{Z_1, Z_2\}$ of $Z(l)$, a basis $\{X_1, X_2\}$ of $l$ and a basis $\{T\}$ of $l''$ such that $B(Z_i, X_j) = \delta_{ij}, \forall i, j$ and $B(T, T) = 1$. Therefore

$$Z(l)^{±} = \text{span}\{X_1^r, X_2^r, T^r\} = \text{span}\{\phi(Z_1), \phi(Z_2), \phi(T)\}.$$

So $I_l = \alpha X^r_1 \land X^r_2 \land T^r, \alpha \in C$. Replacing $X_1$ by $\frac{1}{\alpha}X_1$ and $Z_1$ by $\alpha Z_1$, we can assume that $\alpha = 1$. By Proposition 4.5.1 and Lemma 5.3.1, one obtains $\partial = -\frac{1}{2} \text{ad}_P(I) = -\sum_{i=1}^{3} i x_i (X^r_i \land X^r_j \land T^r) \land i z_r, -i r(X^r_i \land X^r_j \land T^r) \land r, \text{so by} 2.2$ and 2.1, $[X, Y] = \sum_{i=1}^{3} i x_i (X^r_i \land X^r_j \land T^r)(X, Y) Z_i + i r(X^r_i \land X^r_j \land T^r) T, \forall X, Y \in l$ and the commutation rules follow.

(3) Assuming $\dim(l) = 4$, one has $\dim(Z(l)) = 1$. Using [4], there is a totally isotropic 2-dimensional subspace $i$ such that $Z(l) \subset i$. Since $Z(l)^{±} = [l, i']$, one has $i \subset [l, i']$. Using [4] once more, there exists a totally isotropic $i'$ such that $l = i \oplus i'$. Let us write $i = \text{span}\{Z, P\}, i' = \text{span}\{X, Q\}$ with $Z(l) = C Z$ and $B(Z, X) = B(P, Q) = 1, B(Z, Q) = B(P, X) = 0$. Therefore

$$Z(l)^{±} = \text{span}\{P^r, Q^r, X^r\} = \text{span}\{\phi(Q), \phi(P), \phi(Z)\}.$$

So $I_l = \alpha P^r \land Q^r \land X^r, \alpha \in C$. Replacing $P$ by $\frac{1}{\alpha}P$ and $Q$ by $\alpha Q$, we can assume that $\alpha = 1$. Using Proposition 4.5.1, Lemma 5.3.1, 2.2 and 2.1 as above, one finds $[A, B] = [i r(P^r \land Q^r \land X^r), Q + i q(P^r \land Q^r \land X^r) P + i x(P^r \land Q^r \land X^r) Z(A, B), \forall A, B \in l$ and the commutation rules follow.

As a final remark, the brackets in (1), (2) and (3) do satisfy Jacobi identity thanks to Remark 5.1.6. □
Remark 5.3.3. In the Proposition above, cases (1) and (2) are nilpotent Lie algebras and case (3) is a solvable, non-nilpotent Lie algebra, with derived algebra the Heisenberg algebra.

6. Cyclic cochains and cohomology of quadratic Lie algebras

6.1. First we need some notations: \( g \) will be a \( n \)-dimensional quadratic vector space with bilinear form \( B \) and \( \Lambda^g = \sum_{k \geq 1} \Lambda^k g \), which is an associative algebra without unit. If \( g \) is a quadratic Lie algebra, we denote by \( F_0 \) its bracket (i.e. \( F_0(X,Y) = [X,Y], X, Y \in g \)), by \( \partial = D(F_0) \) (see 2.2) the differential of \( \Lambda g \), by \( H^*(g) \) the corresponding cohomology, and by \( H^+_*(g) \) the restricted cohomology, i.e. \( H^+_*(g) = \sum_{k \geq 1} H^k(g) \) which is an algebra without unit (for the induced wedge product).

When \( g \) is a \( n \)-dimensional quadratic vector space, \( \Lambda g \) is a \( gla \) for the super Poisson bracket with grading \( \Lambda g[2] \). Denote by \( \Lambda_Q g \) the quotient \( gla \Lambda_Q g = \Lambda g/C \), and by \( [... ]_Q \) its bracket. The map \( ad_P: \Lambda g \to \mathcal{D}(g) \) is a \( gla \) homomorphism, we define the \( gla \mathcal{H}(g) \) of Hamiltonian derivations to be the image \( \mathcal{H}(g) = ad_P(\Lambda g) \), as in 4.3. There is an obvious \( gla \) isomorphism from \( \Lambda_Q g \) onto \( \mathcal{H}(g) \), and since \( \Lambda_Q g \simeq \tilde{H}(n) \) (see 4.3), the \( gla \Lambda_Q g, \mathcal{H}(g) \) and \( \tilde{H}(n) \) are isomorphic. Moreover, if \( g \) is a quadratic Lie algebra, since \( \partial \) is Hamiltonian (see Proposition 4.5.1), the super Poisson bracket induces a \( gla \) structure on \( H^*(g) \) and also on \( H^*_Q(g) = H^*(g)/C \).

6.2. Given \( C \in \mathcal{M}_\ell^k(g) \) (see 1.2), we define \( \hat{C} \) by:

- if \( k = 0, C \in g \), \( \hat{C}(Y) := B(C,Y), \forall Y \in g \).
- if \( k > 0, \hat{C}(Y_1,\ldots,Y_{k+1}) := B(C(Y_1,\ldots,Y_k),Y_{k+1}), \forall Y_1,\ldots,Y_{k+1} \in g \).

Definition 6.2.1. \( C \) is a cyclic cochain if

\[
\hat{C}(Y_1,\ldots,Y_{k+1}) = (-1)^k\hat{C}(Y_{k+1},Y_1,\ldots,Y_k), \forall Y_1,\ldots,Y_{k+1} \in g.
\]

We denote by \( \mathcal{C}_c(g) \) the space of cyclic cochains.

Proposition 6.2.2.

1. \( C \) is a cyclic cochain if and only if \( \hat{C} \in \Lambda g \). The map \( \Theta \) from \( \mathcal{C}_c(g) \) into \( \Lambda g \) defined by \( \Theta(C) = \hat{C} \), is one to one.

2. When \( g \) is finite dimensional, the map \( \Theta: \mathcal{C}_c(g) \to \Lambda g \) is an isomorphism.

3. \( \mathcal{C}_c(g) \) is a subalgebra of the \( gla \mathcal{M}_\ell(g) \)

Proof:

1. Let \( \tau \) be the cycle \( \tau = (1 2 \ldots k + 1) \in S_{k+1} \). Given \( \sigma \in S_{k+1} \), let \( \ell = \sigma^{-1}(k+1), \) then \( \sigma' = \sigma \circ \tau^\ell \in S_k \). If \( C \) is cyclic, one has \( \tau^{-1}\hat{C} = \varepsilon(\tau)\hat{C} \). So \( \sigma\hat{C} = (\sigma' \circ \tau^{-\ell})\hat{C} = \varepsilon(\sigma)\hat{C} \) and therefore \( \hat{C} \in \Lambda g \). Since \( B \) is non degenerate, \( \Theta \) is clearly one to one.

2. Given \( \Omega \in \Lambda_{k+1}^g \), define \( D \in \mathcal{M}_\ell^k(g) \) by \( \Omega(Y_1,\ldots,Y_k,Y) = B(D(Y_1,\ldots,Y_k,Y),Y), \forall Y_1,\ldots,Y_k,Y \in g \). Then \( \Omega = \hat{D} \).
(3) Let $F \in \mathcal{M}_a^a(g)$, and $G \in \mathcal{M}_a^a(g)$, from (1) we have to prove that:

$$B([F, G]_a(Y_1, \ldots, Y_{p+q-1}, Y_{p+q}) = B([F, G]_a(Y_1, \ldots, Y_{p+q-2}, Y_{p+q}), Y_{p+q-1}),$$

for all $Y_1, \ldots, Y_{p+q} \in g$. Using the formulas of 1.1, we can write the left hand side as a sum of four terms, $B([F, G]_a(Y_1, \ldots, Y_{p+q-1}, Y_{p+q}) = \alpha + \beta + \gamma + \delta$ where:

$$\alpha = (-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

$$\beta = (-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

$$\gamma = - \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

$$\delta = - \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

In $\alpha$, we can commute, up to a sign, $Y_{p+q-1}$ and $Y_{p+q}$. In $\delta$, we commute, up to a sign, $F(Y_{\sigma(1)}, \ldots, Y_{\sigma(p-1)}, Y_{p+q-1})$ and $Y_{p+q}$ to obtain:

$$\delta = \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

Now commute, up to a sign, $G(Y_{p+q}, Y_{\sigma(p+1)}, \ldots, Y_{\sigma(q-1)}, Y_{p+q-1})$ and $Y_{p+q-1}$ to obtain:

$$\delta = - \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

$$\delta = \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

$$\delta = - \sum_{\sigma \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\sigma)$$

Let $Z_i = Y_i$, $i = 1, \ldots, p + q - 2$ and $Z_{p+q-1} = Y_{p+q}$, then:

$$F(G(Y_{\sigma(p+1)}, \ldots, Y_{\sigma(p+q-1)}, Y_{p+q}), Y_{\sigma(1)}, \ldots, Y_{\sigma(p-1)})$$

$$= F(G(Z_{\sigma(p+1)}, \ldots, Z_{\sigma(p+q-1)}, Z_{p+q-1}), Z_{\sigma(1)}, \ldots, Z_{\sigma(p-1)})$$

$$= F(G(Z_{\tau(1)}, \ldots, Z_{\tau(q)}), Z_{\tau(q+1)}, \ldots, Z_{\tau(p+q-1)})$$

where $\tau(1) = \sigma(p+1), \ldots, \tau(q-1) = \sigma(p+q-1), \tau(q) = p+q-1 = \sigma(p), \tau(q+1) = \sigma(1), \ldots, \tau(q+p-1) = \sigma(p-1)$. Comparing the inversions of $\tau$ with the inversions of $\sigma$, it is easy to check that $\epsilon(\tau) = (-1)^{(p-1)(q-1)}(-1)^{(p-1)(q-1)}\epsilon(\sigma)$.

Finally

$$\delta = -(-1)^{(p-1)(q-1)} \sum_{\tau \in \mathcal{S}_{p-1} \cap \mathcal{S}_{q-1}} \epsilon(\tau)$$

$$B(F(G(Z_{\tau(1)}, \ldots, Z_{\tau(q)}), Z_{\tau(q+1)}, \ldots, Z_{\tau(p+q-1)}), Y_{p+q})$$
Then
\[ \alpha + \delta = -(-1)^{(p-1)(q-1)} \sum_{\tau \in \Omega_{p-1}} \varepsilon(\tau) \]
\[ B(F(G(Z_{\tau(1)}), \ldots, Z_{\tau(q)}), Z_{\tau(q+1)}, \ldots, Z_{\tau(p+q-1)}, Y_{p+q-1}) \]

Using similar arguments to compute \( \beta + \gamma \), one obtains the required identity.

\[ \square \]

**Remark 6.2.3.** When \( \mathfrak{g} \) is finite dimensional, there is a direct proof of (6.2.2)(3) (avoiding computations) that we shall give in the proof of Proposition 6.4.1, in Remark 6.4.2.

6.3. We assume now that \( \mathfrak{g} \) is a quadratic Lie algebra.

**Proposition 6.3.1.** \( (\mathcal{C}_c(\mathfrak{g}), d) \) is a subcomplex of the adjoint cohomology complex \( (\mathcal{M}_{\mathfrak{g}}(\mathfrak{g}), d) \) of \( \mathfrak{g} \).

**Proof.** It is enough to check that \( d(\mathcal{C}_c(\mathfrak{g})) \subseteq \mathcal{C}_c(\mathfrak{g}) \), but this is obvious from Proposition 6.2.2(3) because \( d = \text{ad}(F_0) \) and \( F_0 \in \mathcal{C}_c(\mathfrak{g}) \) since \( \mathfrak{g} \) is quadratic.

**Definition 6.3.2.** The cohomology of the complex \( (\mathcal{C}_c(\mathfrak{g}), d) \) is called the cyclic cohomology of \( \mathfrak{g} \), and denoted by \( H^*_c(\mathfrak{g}) \).

**Remark 6.3.3.** Since \( d = \text{ad}(F_0) \), the Gerstenhaber bracket induces a gla structure on \( H^*_c(\mathfrak{g}) \).

**Proposition 6.3.4.** The map \( \Theta : \mathcal{C}_c(\mathfrak{g}) \to \wedge^+ \mathfrak{g} \) is a homomorphism of complexes. Moreover, \( \Theta \) induces a map \( \Theta^* : H^*_c(\mathfrak{g}) \to H^*_c(\mathfrak{g}) \), which is an isomorphism when \( \mathfrak{g} \) is finite dimensional.

**Proof.** By an easy computation, one has \( \Theta \circ d = \partial \circ \Theta \) and the two first claims follow. For the third claim, use Proposition 6.2.2.

**Example 6.3.5.** Assume that \( \mathfrak{g} \) is the Lie algebra associated to an associative algebra with a trace such that the bilinear form \( \text{Tr}(XY) := XY, \forall X, Y \in \mathfrak{g} \) is non-degenerate (e.g. \( \mathfrak{g} \) is the Lie algebra of finite rank operators on a given vector space, see Examples 3.3.9 and 3.3.10). Consider the standard polynomials \( \mathcal{A}_k \), for \( k \geq 0 \) if \( \mathfrak{g} \) had a unit, or for \( k > 0 \), if \( \mathfrak{g} \) has no unit. Since \( [\mathcal{A}_2, \mathcal{A}_2]_a = 0 \) by Proposition 3.2.1, each \( \mathcal{A}_2k \) is a cocycle, then by Proposition 3.3.8, it is a cyclic cocycle, and one has \( \Theta(\mathcal{A}_2k) = \frac{1}{2k+1} \text{Tr}(\mathcal{A}_22k+1) \).

6.4. We assume now that \( \mathfrak{g} \) is a \( n \)-dimensional quadratic vector space. Using the super Poisson bracket, we shall now go further into the structure of \( \mathcal{C}_c(\mathfrak{g}) \). We need to renormalize the map \( \Theta \), defining \( \Phi := -\frac{1}{2} \Theta \). We denote by \( \mu \) the canonical map from \( \wedge \mathfrak{g} \) onto \( \wedge Q \mathfrak{g} \), and by \( \Psi \) the map \( \Psi = \mu \circ \Phi \) from \( \mathcal{C}_c(\mathfrak{g}) \) into \( \wedge Q \mathfrak{g} \).

**Proposition 6.4.1.**

1. If \( C \in \mathcal{C}_c(\mathfrak{g}) \), one has \( D(C) = \text{ad}_0(\Phi(C)) \).
2. The restriction map \( \mathcal{H} = D|_{\mathcal{C}_c(\mathfrak{g})} \) is a gla isomorphism from \( \mathcal{C}_c(\mathfrak{g})[1] \) onto \( \mathcal{H}(\mathfrak{g}) \).
(3) \( \Psi \) is a \textit{gla} isomorphism from \( \mathcal{C}_c(\mathfrak{g})[1] \) onto \( \bigwedge \mathfrak{g}[2] \).

\textbf{Proof.} Fix an orthonormal basis \( \{X_1, \ldots, X_n\} \) of \( \mathfrak{g} \) and \( \{\omega_1, \ldots, \omega_n\} \) the dual basis. Given \( C \in \mathcal{C}_c(\mathfrak{g}) \), \( Y_1, \ldots, Y_p \in \mathfrak{g} \),

\[
\text{ad}_\mathfrak{p}(\Phi(C))(\omega_k)(Y_1, \ldots, Y_p) = 2(-1)^p \left( \sum_{r=1}^n t_{X_r}(\Phi(C)) \wedge t_{X_r}(\omega_r) \right)(Y_1, \ldots, Y_p)
\]

\[
= (-1)^{p+1} B(C(X_k, Y_1, \ldots, Y_{p-1}), Y_p)
\]

\[
= B(C(Y_1, \ldots, Y_{p-1}, X_k), Y_p)
\]

\[
= -B(C(Y_1, \ldots, Y_{p-1}, Y_p), X_k)
\]

\[
= -\omega_k(C(Y_1, \ldots, Y_p)) = -D(C)(\omega_k)(Y_1, \ldots, Y_p)
\]

by a formula given in 2.3, and this proves (1). From (1), we deduce that \( D \) maps \( \mathcal{C}_c(\mathfrak{g}) \) into \( \mathcal{H}(\mathfrak{g}) \).

To prove (2), we remark that \( \text{ad}_\mathfrak{p} \circ \Phi \) is onto by Proposition 6.2.2 (2), so \( \mathcal{H} \) is onto, one to one and a \textit{gla} homomorphism by Proposition 2.1.1 and this proves (2).

\textbf{Remark 6.4.2.} Let us give a direct proof of 6.2.2 (2): given \( C, C' \in \mathcal{C}_c(\mathfrak{g}) \), from the preceding results, we can assume that \( C = F(\text{ad}_\mathfrak{p}(\Omega)) \), \( C' = F(\text{ad}_\mathfrak{p}(\Omega')) \), with \( \Omega, \Omega' \in \bigwedge g \). Then:

\[
[C, C']_\mathfrak{g} = [F(\text{ad}_\mathfrak{p}(\Omega)), F(\text{ad}_\mathfrak{p}(\Omega'))]_\mathfrak{g} = F(\text{ad}_\mathfrak{p}(\{\Omega, \Omega'\})) \quad \square
\]

To prove (3), we use the \textit{gla} isomorphism \( v: \bigwedge \mathfrak{g} \to \mathcal{H}(\mathfrak{g}) \) defined from \( \text{ad}_\mathfrak{p}: \bigwedge \mathfrak{g} \to \mathcal{H}(\mathfrak{g}) \), so one has \( v(\mu(\Omega)) = \text{ad}_\mathfrak{p}(\Omega), \Omega \in \bigwedge g \), and then \( v(\Psi(C)) = \text{ad}_\mathfrak{p}(\Phi(C)) = H(C), \forall C \in \mathcal{C}_c(\mathfrak{g}) \), so \( \Psi = v^{-1} \circ H \). \quad \square

\textbf{Corollary 6.4.3.} The \textit{gla} \( \mathcal{C}_c(\mathfrak{g}) \) is isomorphic to \( \mathcal{H}(\mathfrak{g}) \), and to \( \tilde{H}(n) \).

Using \( \Phi \), we can pull back the \( \wedge \)-product of \( \bigwedge g \) on \( \mathcal{C}_c(\mathfrak{g}) \) defining:

\textbf{Definition 6.4.4.}

\[
C \wedge C' := \Phi^{-1}(\Phi(C) \wedge \Phi(C')), \forall C, C' \in \mathcal{C}_c(\mathfrak{g}).
\]

Hence \( \mathcal{C}_c(\mathfrak{g}) \) becomes an associative algebra (without unit), graded by \( \mathcal{C}_c(\mathfrak{g})[\mathfrak{g}][1] \).

To describe the \( \wedge \)-product of \( \mathcal{C}_c(\mathfrak{g}) \), we define a natural \( \bigwedge g \)-module structure on \( \mathcal{M}_a(\mathfrak{g}) \) by:

\[
\Omega \cdot (\alpha \otimes X) := (\Omega \wedge \alpha) \otimes X, \forall \Omega, \alpha \in \bigwedge g, X \in g
\]

\textbf{Proposition 6.4.5.} If \( C \in \mathcal{C}_c^k(\mathfrak{g}), C' \in \mathcal{C}_c^{k'}(\mathfrak{g}) \), then \( C \wedge C' \in \mathcal{C}_c^{k+k'+1}(\mathfrak{g}) \), and one has:

\[
C \wedge C' = \Phi(C) \cdot C' + (-1)^{(k+1)(k'+1)} \Phi(C') \cdot C.
\]
Proof. Let $C'' = \Phi(C) \cdot C' + (-1)^{(k+1)(k'+1)} \Phi(C') \cdot C$. Then

$$\Phi(C'')(Y_1, \ldots, Y_{k+k'+2}) = \sum_{\sigma \in \Theta, i \leq k+k' + 1, \tau(k+k'+2) = k+k'+2} (\sum_{\sigma \in \Theta} \epsilon(\sigma) \Phi(C)(Y_{\sigma(1)}, \ldots, Y_{\sigma(k+1)}) \Phi(C')((Y_{\sigma(k+2)}, \ldots, Y_{\sigma(k+k'+1)}, Y_{k+k'+2}) +$$

$$(1)^{(k+1)(k'+1)} \sum_{\sigma \in \Theta} \epsilon(\sigma) \Phi(C')(Y_{\sigma(1)}, \ldots, Y_{\sigma(k'+1)}) \Phi(C)((Y_{\sigma(k'+2)}, \ldots, Y_{\sigma(k+k'+1)}, Y_{k+k'+2}).$$

In the first term of the right hand side, for each $\sigma$ define $\tau$ by $\tau(i) = \sigma(i)$, $i \leq k+k' + 1$, and $\tau(k+k'+2) = k+k'+2$. In the second term, for each $\sigma$ define $\tau$ by $\tau(1) = \sigma(k'+2)$, $\tau(k) = \sigma(k+k'+1)$, $\tau(k+1) = k+k'+2$, $\tau(k+2) = \sigma(1)$, $\tau(k+3) = \sigma(2)$, ..., $\tau(k+k'+2) = \sigma(k'+1)$, then $\epsilon(\tau) = (1)^{(k+1)(k'+1)} \epsilon(\sigma)$, and one has:

$$\Phi(C'')(Y_1, \ldots, Y_{k+k'+2}) = \sum_{\tau \in \Theta_{k+k'+2}} \epsilon(\tau) \Phi(C)(Y_{\tau(1)}, \ldots, Y_{\tau(k+1)}) \Phi(C')((Y_{\tau(k+2)}, \ldots, Y_{\tau(k+k'+2)}) +$$

$$\sum_{\tau \in \Theta_{k+k'+2}} \epsilon(\tau) \Phi(C')(Y_{\tau(1)}, \ldots, Y_{\tau(k'+1)}) \Phi(C)((Y_{\tau(k'+2)}, \ldots, Y_{\tau(k+k'+2)}) =$$

$$\Phi(C) \wedge \Phi(C')(Y_1, \ldots, Y_{k+k'+2}).$$

One has to be careful that $\text{ad}(C)$ ($C \in \mathcal{G}_c(g)$) is generally not a derivation of the $\wedge$-product of $\mathcal{G}_c(g)$, so the following result is of interest:

**Proposition 6.4.6.** If $C \in \mathcal{G}_c^k(g)$, $C' \in \mathcal{G}_c^{k'}(g)$, $C'' \in \mathcal{G}_c(g)$, with $k \geq 1$, then:

$$\text{ad}(C)(C' \wedge C') = \text{ad}(C)(C') \wedge C'' + (-1)^{(k+1)(k'+1)} C' \wedge \text{ad}(C)(C'').$$

This means that when $C \in \mathcal{G}_c(g)[1]$, then $\text{ad}(C)$ is a derivation of degree $k$ of the graded algebra $\mathcal{G}_c(g)[-1]$ with the $\wedge$-product.

**Proof.** One has

$$\mu(\Phi([C,C'], a)) = \Psi([C,C']) = \Psi(C,C') = [\mu(\Phi(C)), \mu(\Phi(C'))].$$

Since $\text{ad}(\Phi(C)) (\wedge g) \subset \wedge g$, it follows that $\Phi(\text{ad}(C)(C')) = \text{ad}(\Phi(C)) (\wedge g)$, and the result is proved using the fact that $\text{ad}(\Phi(C))$ is a derivation of degree $k-1$ of $\wedge g$, and the definition of the $\wedge$-product of $\mathcal{G}_c(g)$.

Using the Proposition above, and $d = \text{ad}(F_0)$ with $F_0 \in \mathcal{G}_c^2(g)$, it results that the $\wedge$-product of $\mathcal{G}_c(g)$ induces a $\wedge$-product on $H^*_c(g)$ and $\phi^* = -\frac{1}{2} \theta^*$ is clearly an isomorphism of graded algebras from $H^*_c(g)$ onto $H^*_c(g)$. From the definition of the gla bracket on $H^*_c(g)$, denoting by $\mu^*$ the canonical map from $H^*(g)$ onto
$H_Q^*(g) = H^* (g) / \mathbb{C}$, the map $\Psi^* = \mu^* \circ \Phi^*$ is a \textit{gla} isomorphism from $H_c^*(g)$ onto $H_Q^*(g)$. We summarize in:

**Proposition 6.4.7.** As a graded associative algebra, $H_c^*(g)$ is isomorphic to $H_c^*(g)$ and as a \textit{gla}, $H_c^*(g)$ is isomorphic to $H_Q^*(g)$.

**Example 6.4.8.** Let $g = gl(n)$. Then $H_c^*(g) = \operatorname{Ext}^* [a_1, a_2, \ldots, a_{2n-1}]$, where $a_k = \operatorname{Tr}(\alpha_{2k})$, $k \geq 0$ (e.g. [10]). One has $\Theta (\alpha_{2k}) = \frac{1}{2k+1} \operatorname{Tr}(\alpha_{2k+1})$ (Example 6.3.5), so by Proposition 6.4.7, $H_c^*(g) = \operatorname{Ext}^* [\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}]$. The \textit{gla} bracket will be computed in Example 8.3.1.

**Remark 6.4.9.** When $g$ is not finite dimensional, the map $\Theta^*$ of Proposition 6.3.4 is no longer an isomorphism, as shown with the following example: let $V$ be an infinite dimensional vector space, and $g$ be the quadratic Lie algebra of finite rank operators of $V$, as defined in Example 3.3.10. Recall that the invariant bilinear form is $B(X,Y) = \operatorname{Tr}(XY)$, $X, Y \in g$. Notice that $B(X,Y)$ is well defined when $X \in g$ and $Y \in \operatorname{End}(V)$. Moreover, the formula $B([X,Y],Z) = -B(Y,[X,Z])$ is valid if at least one argument is in $g$. By Remark 3.3.12, $H_0^0(g) = Z(g) = \{0\}$ and $H_1^0(g) = \mathbb{C} \operatorname{Tr}$, so:

**Proposition 6.4.10.** The map $\Theta^* : H_0^0(g) \to H_1^0(g)$ is not onto.

Moreover,

**Proposition 6.4.11.** The map $\Theta^* : H_1^0(g) \to H_2^0(g)$ is not one to one.

**Proof.** Fix $U \in \operatorname{End}(V)$ such that $U \notin g \oplus \mathbb{C} \text{Id}_V$ and consider the skew symmetric derivation $D$ of $g$ defined by $D = \text{ad}(U)|_g$. The derivation $D$ is a cyclic cocycle but $D = \text{ad}(Y)$ with $Y \in g$ cannot be true because if $U' \in \operatorname{End}(V)$ commutes with $g$, then $U'$ must be a multiple of $\text{Id}_V$. So $D$ is not a cyclic coboundary. On the other hand, $\tilde{D}(X,Y) = B(D(X),Y) = \partial \omega(X,Y)$ where $\omega \in \wedge^1 g$ is defined by $\omega(X) = -B(U,X), X \in g$. Hence $\tilde{D}$ is a coboundary, and if we denote by $\overline{D}$ the class of $D$ in $H_c^*(g)$, we get $\Theta^*(\overline{D}) = 0$, and $\overline{D} \neq 0$.

7. The case of reductive and semisimple Lie algebras

7.1. Let $g$ be an $n$-dimensional quadratic Lie algebra with bilinear form $B$. We recall the natural $g$-modules structures on $\wedge g$ and $\mathcal{M}_a(g)$ defined by:

$$\theta_X (\Omega)(Y_1, \ldots, Y_p) = - \sum_i \Omega(Y_1, \ldots, [X,Y_i], \ldots, Y_p), \forall X, Y_1, \ldots, Y_p \in g, \Omega \in \wedge^p g.$$  

$$L_X (\Omega \otimes Y) = \theta_X (\Omega) \otimes Y + \Omega \otimes [X,Y], \forall X, Y \in g, \Omega \in \wedge^p g.$$  

Using the notation in 6.4, it is easy to check that

$$\Phi \circ L_X = \theta_X \circ \Phi, \forall X \in g.$$  

So one has:

**Proposition 7.1.1.** $C_c(g)$ is a $g$-submodule of the $g$-module $\mathcal{M}_a(g)$ and the isomorphism $\Phi$ (of 6.4) is a $g$-module isomorphism from $C_c(g)$ onto $\wedge^+ g$. 
It is well known that any element of \((\bigwedge g)^0\) is a cocycle, and if \(g\) is reductive, that \(H^*(g) = (\bigwedge g)^0\) [16]. Using Propositions 6.3.4, 6.4.7 and 7.1.1, we deduce:

**Proposition 7.1.2.** Any invariant cyclic cochain is a cocycle. If \(g\) is reductive, any cyclic cohomology class contains one, and only one invariant cyclic cocycle (for instance, the only invariant cyclic coboundary is 0).

Hence, when \(g\) is reductive, we can identify \(H^*_c(g)\) and \(\mathcal{H}_c(g)^0\). This identification is valid for the corresponding \(\wedge\)-products (actually isomorphic to the \(\wedge\)-product of \((\bigwedge g)^0 \simeq H^*_c(g)\)) and for the corresponding graded Lie bracket induced by the Gerstenhaber bracket (actually isomorphic to \(\mathcal{H}(g)^0\) and \((\bigwedge g)^0\)).

### 7.2

In the remaining of this Section, we assume that \(g\) is a semisimple Lie algebra with invariant bilinear form \(B\) (not necessarily the Killing form).

**Proposition 7.2.1.** If \(I\) and \(I'\) \(\in (\bigwedge g)^0\), then \(\{I, I'\} = 0\).

As a consequence of this Proposition and of Proposition 6.4.7, one has:

**Corollary 7.2.2.** The Gerstenhaber bracket induces the null bracket on \(H^*_c(g) \simeq \mathcal{H}_c(g)^0\).

To prove Proposition 7.2.1, we need several lemmas: first, let \(\mathfrak{h}\) be a Lie algebra and \(I \in (\bigwedge^{p+1} \mathfrak{h})^\mathfrak{h}\). Define a map \(\Omega: \mathfrak{h} \to \bigwedge^p \mathfrak{h}\) by \(\Omega(X) = i_X(I), \forall X \in \mathfrak{h}\). Then since \(I\) is invariant, one has:

**Lemma 7.2.3.** \(\Omega\) is a morphism of \(\mathfrak{h}\)-modules from \((\mathfrak{h}, \text{ad})\) into \((\bigwedge^p \mathfrak{h}, \theta)\).

*Proof.* For all \(X, Y\) and \(Z \in g\), we have:

\[
\theta_X(\Omega(Y)) = \theta_X(t_Y(I)) = [\theta_X, t_Y](I) + t_Y(\theta_X(I)) = t_{[X,Y]}(I) = \Omega([X,Y]).
\]

As a second argument for the proof of Proposition 7.2.1:

**Lemma 7.2.4.** Assuming that \(\mathfrak{h}\) is a perfect Lie algebra (i.e. \(\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]\)), there exists a map \(\alpha: \mathfrak{h} \to \bigwedge^{p-1} \mathfrak{h}\) such that \(\Omega = \partial \circ \alpha\) (\(\partial\) is the differential of the trivial cohomology of \(\mathfrak{h}\)). Moreover, if \(\mathfrak{h}\) is semisimple, there exist an \(\mathfrak{h}\)-homomorphism \(\alpha\) such that \(\Omega = \partial \circ \alpha\).

*Proof.* If \(X \in \mathfrak{h}\), we can find \(Z_i, T_i \in \mathfrak{h}\) such that \(X = \sum_i [Z_i, T_i]\). Then, \(\Omega(X) = \sum_i \theta_{Z_i}(\Omega(T_i))\) by Lemma 7.2.3. But \(\partial(\Omega(T_i)) = \partial(t_{T_i}(I)) = t_{T_i}(\partial(I)) = 0\) since \(I\) is an invariant. But \(\partial\) maps \(Z^p(\mathfrak{h})\) into \(B^p(\mathfrak{h})\), so \(\Omega(X) \in B^p(\mathfrak{h})\). To construct \(\alpha\), fix a section \(\sigma\) of the map \(\partial: \bigwedge^{p-1} \mathfrak{h} \to B^p(\mathfrak{h})\), i.e. \(\sigma: B^p(\mathfrak{h}) \to \bigwedge^{p-1} \mathfrak{h}\) such that \(\partial \circ \sigma = \text{Id}_{B^p(\mathfrak{h})}\) and then set \(\alpha = \sigma \circ \Omega\). When \(g\) is semisimple, one can fix a section \(\sigma\) which is a \(g\)-homomorphism.

*Proof. (of Proposition 7.2.1)*

Fix an orthonormal basis \(\{X_1, \ldots, X_n\}\) of \(g\) with respect to \(B\). Given \(I, I' \in (\bigwedge g)^0\), let \(\Omega_r = i_X(I), \Omega'_r = i_X(I')\), \(\alpha, \alpha'\) the \(g\)-homomorphisms given by Lemma 7.2.4 and finally \(\alpha_r = \alpha(X_r), \alpha'_r = \alpha'(X_r)\) so that \(\Omega_r = \partial \alpha_r\) and \(\Omega'_r = \partial \alpha'_r\). With
these notations, in order to finish the proof, we need to show that $\sum_r \Omega_r \wedge \Omega'_r = 0$. But:

$$\sum_r \Omega_r \wedge \Omega'_r = \sum_r \partial \alpha_r \wedge \partial \alpha'_r = \partial (\sum_r \alpha_r \wedge \partial \alpha'_r) = 0$$

since $\sum_r \alpha_r \wedge \partial \alpha'_r \in (\wedge g)^0$.

Remark 7.2.5. Proposition 7.2.1 can be directly deduced from a deep result of Kostant [14] about the structure of Cliff$(g)^\bullet$ seen as a deformation of $(\wedge g)^\otimes$; by the Hopf-Koszul-Samelson theorem, $(\wedge g)^\otimes$ is an exterior algebra $\text{Ext}[a_1, \ldots, a_r]$ with rank$(g) = r$ and $a_1, \ldots, a_r$ primitive (odd) invariants. Kostant shows that Cliff$(g)^\bullet$ is a Clifford algebra constructed on $a_1, \ldots, a_r$. Since the deformation from $\wedge g$ to Cliff$(g)^\bullet$ has leading term the Poisson bracket, it results that $\{a_i, a_j\} = 0$, $\forall i, j$, and then Proposition 7.2.1 follows.

Example 7.2.6. Using the results in Section 6, and Corollary 7.2.2, we will describe $H^*_s(s)$ and $H^*_s(g)$ when $s = s((n)$ and $g = g((n)$ both equipped with the bilinear form $B(X, Y) = \text{Tr}(XY), \forall X, Y$. Let $I_g$ be the identity matrix.

One has $\wedge s = \{\Omega \in \wedge g \mid \iota_g (\Omega) = 0\}$ and $\mathcal{M}_s(s) = \{F \in \mathcal{M}_s(g) \mid \iota_g (F) = 0$ and $F(g)^\otimes \subset s(F \in \mathcal{M}_s^1(g))\}$. By Propositions 3.3.3 and 3.3.6, $\mathcal{M}_s(\wedge s)$. Moreover, let $a_k = \text{Tr}(\omega_k)$ $(k \geq 0)$, then by Proposition 3.3.6, $a_{2k+1} \in (\wedge g)^\otimes, \forall k \geq 0$, and by Proposition 3.3.3, $a_{2k+1} \in (\wedge s)^\otimes, \forall k \geq 0$.

(1) It is well known that $H^*(g) = (\wedge g)^\otimes$ is the exterior algebra generated by the invariant cocycles $a_1, a_3, \ldots, a_{2n-1}$, i.e. $(\wedge g)^\otimes = \text{Ext}[a_1, a_3, \ldots, a_{2n-1}]$ and that $H^*(s) = (\wedge s)^\otimes$ is the exterior algebra generated by the invariant cocycles $a_3, a_5, \ldots, a_{2n-1}$, i.e. $(\wedge s)^\otimes = \text{Ext}[a_3, a_5, \ldots, a_{2n-1}]$ (see [14, 15, 10]).

(2) We need to compute the super Poisson bracket on $(\wedge g)^\otimes$. Note that $\{\Omega, \Omega'\} = 0, \forall \Omega, \Omega' \in (\wedge s)^\otimes$ by Proposition 7.2.1. Then, using $s^\perp = \mathbb{C} I_g$, an adapted orthonormal basis, and the formula in Proposition 4.2.1, one finds that $\{a_1, a_1\} = 2n$. Then, since any element in $(\wedge g)^\otimes$ decomposes as $\Omega + \Omega' \wedge a_1$, with $\Omega, \Omega' \in (\wedge s)^\otimes$, we have only to compute the following brackets:

$\{\Omega, \Omega' \wedge a_1\} = 0, \forall \Omega, \Omega' \in \text{Ext}^{w}[a_3, \ldots, a_{2n-1}]$,

$\{\Omega \wedge a_1, \Omega' \wedge a_1\} = 2n(-1)^w \Omega \wedge \Omega'$,

$\forall \Omega \in \text{Ext}[a_3, \ldots, a_{2n-1}], \Omega' \in \text{Ext}^{w}[a_3, \ldots, a_{2n-1}]$.

(3) Use the isomorphism $\Phi^*$ of Proposition 6.4.7 to find $H^*_s(s) = \mathcal{H}^*_{c}(s)$ and $H^*_s(g) = \mathcal{H}^*_{c}(g)^\otimes$. One has $[\omega_2, \omega_2] = 0$ by Proposition 3.2.1, so $\omega_2$ is a cocycle, obviously $g$-invariant. By Proposition 3.3.8, it is a cyclic cocycle, and $\Phi(\omega_2) = -2^{-1}a_{2k+1}$. It results that

$H^*_s(s) = \text{Ext}_{+}[\omega_2, \omega_4, \ldots, \omega_{2n-2}]$ and $H^*_s(g) = \text{Ext}_{+}[\omega_0, \omega_2, \ldots, \omega_{2n-2}]$.

(4) Now we compute the Gerstenhaber bracket. For $H^*_s(s)$, by Corollary 7.2.2, the Gerstenhaber bracket vanishes. For $H^*_s(g)$, we use the isomorphism $\Psi^*$ (see Proposition 6.4.7) combined with 7.2.6 (3) and the commutation
rules in $H^*(\mathfrak{g})$ computed in 7.2.6 (2) from which the commutation rules in $H^*_Q(\mathfrak{g}) = H^*(\mathfrak{g})/\mathbb{C}$ are deduced. Finally the result is the following:

$[F, F']_a = 0 \forall F, F' \in \text{Ext}_+[\mathfrak{g}_2, \ldots, \mathfrak{g}_{2n-2}],$

$[\mathfrak{g}_0, F]_a = 0 \forall F \in \text{Ext}_+[\mathfrak{g}_0, \mathfrak{g}_2, \ldots, \mathfrak{g}_{2n-2}],$

$[F, F' \wedge \mathfrak{g}_0]_a = 0 \forall F, F' \in \text{Ext}_+[\mathfrak{g}_2, \mathfrak{g}_4, \ldots, \mathfrak{g}_{2n-2}],$

$[\mathfrak{g}_0, F' \wedge \mathfrak{g}_0]_a = \frac{n}{2}(-1)^f F' \forall F' \in \text{Ext}_+[\mathfrak{g}_2, \mathfrak{g}_4, \ldots, \mathfrak{g}_{2n-2}],$

$[F \wedge \mathfrak{g}_0, F' \wedge \mathfrak{g}_0]_a = \frac{n}{2}(-1)^f F \wedge F',$

$\forall F \in \text{Ext}_+[\mathfrak{g}_2, \mathfrak{g}_4, \ldots, \mathfrak{g}_{2n-2}], F' \in \text{Ext}_+[\mathfrak{g}_2, \mathfrak{g}_4, \ldots, \mathfrak{g}_{2n-2}],$

Remark that for the last result, one uses: $F' \in \text{Ext}_+[\mathfrak{g}_2, \mathfrak{g}_4, \ldots, \mathfrak{g}_{2n-2}] \cap \mathfrak{g}_0^\mathfrak{g}(\mathfrak{g})$, then $p' = f' + 1 \mod 2$ and $\Phi(F') \in \wedge^{p'} \mathfrak{g}$.

8. QUADRATIC 2k-LIE ALGEBRAS AND CYCLIC COCHAINS

8.1. Let $\mathfrak{g}$ be a finite dimensional quadratic vector space with bilinear form $B$. Given $D \in \mathfrak{g}^{2k-1}$, $k \geq 1$ denote by $F = F_D$ the associated (even) structure on $\mathfrak{g}$ (see Sections 1 and 2), that we also denote by a bracket notation:

$[Y_1, \ldots, Y_{2k}] = F(Y_1, \ldots, Y_{2k}), \forall Y_1, \ldots, Y_{2k} \in \mathfrak{g}.$

Definition 8.1.1. The bilinear form $B$ is $F$-invariant (or $F$ is a quadratic structure with bilinear form $B$) if $B([Y_1, \ldots, Y_{2k-1}, Y], Z) = -B([Y, Y_1, \ldots, Y_{2k-1}, Z]), \forall Y_1, \ldots, Y_{2k-1}, Y, Z \in \mathfrak{g}.$

We introduce the linear maps $\text{ad}_{Y_1, \ldots, Y_{2k-1}} : \mathfrak{g} \to \mathfrak{g}$ by:

$\text{ad}_{Y_1, \ldots, Y_{2k-1}}(Y) = [Y_1, \ldots, Y_{2k-1}, Y], \forall Y_1, \ldots, Y_{2k-1}, Y \in \mathfrak{g}.$

It is obvious that

Proposition 8.1.2. The bilinear form $B$ is $F$-invariant if and only if $\text{ad}_{Y_1, \ldots, Y_{2k-1}} \in \mathfrak{o}(B), \forall Y_1, \ldots, Y_{2k-1} \in \mathfrak{g}.$

The next Proposition results directly from Propositions 6.2.2 and 6.4.1.

Proposition 8.1.3.

(1) $F$ is quadratic if and only if it is a cyclic cochain.

(2) $F$ is quadratic if and only if there exists $I \in \bigwedge^{2k+1} \mathfrak{g}$ such that $D = -\frac{1}{2} \text{ad}_I(I)$ and in that case, one has $I([Y_1, \ldots, Y_{2k+1}]) = B([Y_1, \ldots, Y_{2k}], Y_{2k+1}), \forall Y_1, \ldots, Y_{2k+1} \in \mathfrak{g}.$

8.2. Keeping the notations of Proposition 8.1.3, a quadratic $F$ will define a $2k$-Lie algebra structure on $\mathfrak{g}$ (namely a quadratic $2k$-Lie algebra) if and only if:

(VII) $[F, F]_a = 0 \text{ or } [D, D] = 0 \text{ or } \{I, I\} = 0.$

Examples of quadratic $2k$-Lie algebras can be directly deduced from Proposition 7.2.1: let us assume in the remaining of 8.2, that $\mathfrak{g}$ is a semisimple Lie algebra with bilinear form $B$ (not necessarily the Killing form). Then one has:
Proposition 8.2.1. Any invariant even cyclic cochain in \( \mathcal{M}_r(\mathfrak{g}) \) defines a quadratic 2\( k \)-Lie algebra on \( \mathfrak{g} \).

These examples were introduced for the first time in [3], in the case of primitive elements in \((\wedge \mathfrak{g})^g\) (we shall come back to the construction in [3] later in this section).

Let \( F \) be an invariant even cyclic cochain, denote by:
\[ [Y_1,\ldots,Y_{2k}] = F(Y_1,\ldots,Y_{2k}), \forall Y_1,\ldots,Y_{2k} \in \mathfrak{g} \]
the associated quadratic 2\( k \)-bracket on \( \mathfrak{g} \). Let us introduce, as in 8.1:
\[ I([Y_1,\ldots,Y_{2k+1}]) = B([Y_1,\ldots,Y_{2k}],Y_{2k+1}), \forall Y_1,\ldots,Y_{2k+1} \in \mathfrak{g}, \]
and the associated derivation \( D = \frac{1}{2} \text{ad}_p(I) \) of \( \wedge \mathfrak{g} \). Since \([D,D] = 2D^2 = 0\), we can define the associated cohomology on \( \wedge \mathfrak{g} \) by
\[ H^*(F) = Z(D)/B(D) \]
where \( Z(D) = \ker(D) \) and \( B(D) = \text{Im}(D) \).

The following Lemma has to be compared with Formula III of 2.1:

Proposition 8.2.2. Let \{\( X_1,\ldots,X_n \)\} be an orthonormal basis of \( \mathfrak{g} \) with respect to \( B \). Then there exist \( \beta_1,\ldots,\beta_n \in \wedge^{2k-1} \mathfrak{g} \) such that:
\[ D = \frac{1}{2} \sum_r \beta_r \wedge \theta_{X_r}. \]

Proof: Let \{\( \omega_1,\ldots,\omega_n \)\} be the dual basis of \{\( X_1,\ldots,X_n \)\}. One has \( \theta_{X_r} (\omega_s)(Y) = B([X_r,X_s],Y) \) for all \( Y \in \mathfrak{g} \). So \( \theta_{X_r} (\omega_s) = -\theta_{X_s} (\omega_r) \) for all \( r,s \). Define \( \Omega(X) = \tau_X(I), X \in \mathfrak{g} \). By Lemma 7.2.4, there exists a \( \mathfrak{g} \)-homomorphism \( \alpha : \mathfrak{g} \rightarrow \wedge^{2k-1} \mathfrak{g} \) such that \( \Omega = \partial \circ \alpha \). Define \( \alpha_r = \alpha(X_r) \), then \( \theta_{X_r} (\alpha_s) = \alpha_r([X_r,X_s]) \), so one has \( \theta_{X_r} (\alpha_s) = -\theta_{X_s} (\alpha_r) \). Define \( \Omega_{\alpha} = \alpha_r \theta_{X_r} \), the one has:
\[ D = -\frac{1}{2} \text{ad}_p(I) = -\sum_r \Omega_{\alpha_r} \wedge \theta_{X_r}. \]
So \( D(\omega_r) = -\partial \alpha_r \). Then using \( \partial = \frac{1}{2} \sum_s \omega_s \wedge \theta_{X_s} ([16]) \), one has:
\[ \partial \alpha_r = -\frac{1}{2} \sum_s \omega_s \wedge \theta_{X_s} (\alpha_s) = -\frac{1}{2} \left( \theta_{X_r} (\sum_s \omega_s \wedge \alpha_s) - \sum_s \theta_{X_r} (\omega_s) \wedge \alpha_s \right). \]
But \( \sum_s \omega_s \wedge \alpha_s \) is \( \mathfrak{g} \)-invariant, so:
\[ \partial \alpha_r = \frac{1}{2} \sum_s \theta_{X_r} (\omega_s) \wedge \alpha_s = -\frac{1}{2} \sum_s \alpha_s \wedge \theta_{X_r} (\omega_s) = \frac{1}{2} \sum_s \alpha_s \wedge \theta_{X_s} (\omega_r). \]
Therefore, since \( D \) and \( \sum_s \alpha_s \wedge \theta_{X_s} \) are derivations of \( \wedge \mathfrak{g} \), one has \( D = -\frac{1}{2} \sum_s \alpha_s \wedge \theta_{X_s} \), and setting \( \beta_r = -\alpha_r \), the Proposition is proved. \( \square \)

From Proposition 8.2.2, we deduce:

Proposition 8.2.3. One has \((\wedge \mathfrak{g})^g \subset Z(D)\).

From the fact that \( I \in (\wedge \mathfrak{g})^g \), \( D \) is a \( \mathfrak{g} \)-homomorphism of the \( \mathfrak{g} \)-module \( \wedge \mathfrak{g} \), which is semisimple. By standard arguments ([16]), one deduces:
Proposition 8.2.4. One has $(\bigwedge g)^g \subset H^*(F)$.

When $F$ is the Lie algebra structure of $g$, it is well known that $H^*(F) = (\bigwedge g)^g$ ([16]).

8.3. Let us now place the constructions in [3] in our context. We assume that $g$ is a semisimple Lie algebra of rank $r$ and fix a non degenerate symmetric bilinear form $B$ (not necessarily Killing) on $g$. Let $S(g) = \text{Sym}(g^*)$. Using Chevalley’s theorem, there exist homogeneous invariants $Q_1, \ldots, Q_r$ with $q_i = \deg(t_i)$ such that $S(g)^g = \mathbb{C}[Q_1, \ldots, Q_r]$. Let $t : S(g)^g \to (\bigwedge g)^g$ be the Cartan-Chevalley transgression operator ([5], [6]). By the Hopf-Koszul-Samelson theorem ([5], [6], [14]), one has $(\bigwedge g)^g = \text{Ext}(t(Q_1), \ldots, t(Q_r))$ and $\deg(t_i(Q_i)) = 2q_i - 1$. By (VII) and Proposition 7.2.1, any odd element $I$ in $(\bigwedge g)^g$ defines a quadratic 2$\cdot$Lie algebra structure on $g$ (and corresponding generalized Poisson bracket on $g^*$). As a particular case, this works for $t_i(Q_i), i = 1, \ldots, r$ which define a $(2q_i - 2)$-Lie algebra structure on $g$ and a GPB on $g^*$, and these are exactly the examples given in [3], though in these papers there are no citations, neither to Chevalley [6], nor to Cartan [5]. Let us insist that not only primitive invariants (as sometimes claimed in [3]), but actually all odd invariants do define 2$\cdot$Lie algebra structures on $g$ (Propositions 8.1.3 and 8.2.1).

Example 8.3.1. Using the notation and the results of Example 7.2.6, let us consider the case of $g = gl(n)$, with bilinear form $B(X,Y) = \text{Tr}(XY), \forall X, Y \in g$. Consider $C = F + F' \wedge \mathcal{A}_0$ with $F, F' \in \text{Ext}^+ \mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n-2}$. In order to have $C$ an even element of $\mathcal{M}_d(g)$, we have to assume that $F \in \text{Ext}^+ \mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n-2}$ and $F' \in \text{Ext}^+ \mathcal{A}_2, \ldots, \mathcal{A}_{2n-2}$ (see the last remark in Example 7.2.6 (4)). Moreover, we have to assume that $F$ and $F' \wedge \mathcal{A}_0$ have the same degree in $\mathcal{M}_d(g)$, say $2k$. Then, from commutation rules in 7.2.6 (4), $C$ defines a 2$\cdot$Lie algebra structure on $g$ if and only if $F' \wedge F'' = 0$. This last condition is obviously satisfied if $F''$ is decomposable. For instance, if $n \geq 3$, $\alpha \mathcal{A}_2 + \beta \mathcal{A}_0 \wedge \mathcal{A}_2, \alpha, \beta \in \mathbb{C}$, defines a 8-Lie algebra structure on $g$; if $n \geq 4$, $\alpha \mathcal{A}_4 + \beta \mathcal{A}_0 \wedge \mathcal{A}_4 \wedge \mathcal{A}_6$, $\alpha, \beta \in \mathbb{C}$, defines a 14-Lie algebra structure on $g$.

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