Disagreement and polarization in group formation

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Abstract

Agreement is the rare exception while disagreement is the universal. To study how opinion disagreement or even polarization emerges, we proposed an agent-based model to mimic a noisy group formation process by recruiting new members who hold binary opposite opinions on different issues. We examine the long-term effects on the proportions of group members who hold different opinions to see how much disagreement there is. We discover that disagreement always arises regardless of the infinitesimal level of noise. In addition, opinions tend to be polarized as group size grows. More importantly, we find the proportions of group members with different opinions are closely correlated with the eigenvalues and eigenvectors of the transition probability matrix. By constructing social networks, our work can be extended to study social fragmentation and community structure in the future.

Keywords: group formation, multi-dimensional opinion, disagreement, polarization

1. Introduction

Social groups naturally and continuously emerge within the society, which can be thought of as a vast social network [1, 2]. As primary functional components of our society, nearly all human activities take place in one or more types of social groups [2, 3], including working, learning, talking, etc. An online social group, for instance, is created to enable people to communicate with one another efficiently [4, 5]. The underlying mechanism of group formation has thus attracted significant interests from various disciplines, leading to a huge body of work on modeling group formation. Natural scientists [6, 7, 8, 9] are especially interested in how the various growth mechanisms affect the network structures. In contrast,
social scientists [10, 2, 11, 12] are principally concerned with how links are activated by people’s payoff, and the condition under which an efficient network structure can emerge, i.e. strategic group formation.

The popularity of opinion dynamics has also increased due to the fact that interactions within a social group would inevitably lead to changes in people’s opinions. Existing models in economics and sociology largely are based on a connected social network, and people regularly update their opinions in some patterns of social learning processes [13, 14, 15]. Another well-known area of research is the exploration of opinion dynamics using traditional models derived from physics and mathematics, such as the Ising model [16, 17] and the voter model[18, 19]. In these models, opinions are often represented as binary variables, and opinion updating is governed by very simple rules [20, 21, 22]. These powerful models can yield many fascinating insights into the study of opinion agreement/disagreement or polarization.¹

However, most scholars find: (i) agreement is the rare exception, particularly when it comes to important topics that have been the subject of centuries-long dispute [24, 25, 26]. (ii) Opinions have become increasingly polarized, i.e. there are half-to-half population with opposite opinions. Indeed, shared opinions are frequent catalyst for the creation of social organizations [1, 2, 27] as evidenced by political parties, online groups, student unions, etc, but a large group often gradually loses opinion alignment. To this end, we created an agent-based model to imitate a social group formation to examine how opinion disagreement and even polarization arises. Our work is a generalization of the baseline model that was first presented in a very young literature [28], which our model nests for $d = 1$. In our model, a small group of individuals endowed with the same $d$-dimensional opinions, gradually grows by recruiting new candidates. We suppose that each candidate will hold a opinion vector drawn from the opinion vector space. The candidates is accepted/rejected based on positive/negative judgement received from some group member. We assume that the judgement from any group member can be wrong to some extent, which is at odds with homophily [29] or assortative mixing theory [30, 1, 31]. We refer to it as noise, which has

¹An indicator of opinion polarization in our model is the percentage of a population that supports each viewpoint. The more equally divided the opinions, the more polarized those opinions become [23].
two aspects of interpretations\(^2\): (i) the group member is sceptical of the candidate’s opinion orientation; (ii) the candidate conceals its opinion orientation.

Our study aims to examine the long-term effects on the proportions of group members who hold different opinions to see how much disagreement there is. We discover that disagreement always arises regardless of the infinitesimal level of noise. Opinion polarization or multi-polarization is realized in an infinite-size social group. More importantly, we find analytically that the probabilities of holding different opinions at each time, are determined by a non-linear combination of the transition probability matrix and the identity matrix. Furthermore, the proportions of group members holding different opinions are closely correlated with the eigenvalues and eigenvectors of the transition probability matrix.

2. The Model

The model can be defined as follows. A group of people hold binary opposite opinions on \(d\) different issues. Thus each group member’s opinions can be denoted by a \(d\)-dimensional opinion vector \(o\) with elements \(\{-1, +1\}\). The whole opinion vector space can be represented by \(o \in \{-1, +1\}^d\) \((x = 1, \ldots, 2^d)\). At time \(t = 0\), we assume that opinion vectors of \(N_0\) group members are \(+1\)^\(d\). At time \(t > 0\), opinion vector of a candidate \(t\) with \(o_y\) is drawn at random from the opinion vector space. Then the candidate will be judged by some group member \(i\) with opinion vector \(o_x\) who is chosen uniformly (UC) or by preferential attachment (PA). The opinion difference \(K_{xy}\) between \(o_x\) and \(o_y\) can be represented by their inner product, i.e. \(K_{xy} = o_xo_y\). Let the probability of positive judgement be \(M_{xy} = (1/2 + (1/2 - \eta)K_{xy}/d)^{2^{d-1}},\) and negative judgement otherwise. Here \(M\) is the \(2^d \times 2^d\) transition probability matrix with entries \(M_{xy}\), and \(\eta \in [0, 0.5]\) is a control parameter that represents the level of noise. We assume that a candidate can be admitted if the judgement is positive. The group growth ceases until a given group size \(N\) is reached. We mainly investigate the mean proportions of group members holding different opinions, which can be denoted by \(\overline{p}_x\) \((x = 1, \ldots, 2^d)\).

\(^2\)For the interpretation of noise parameter \(\eta\), please refer to Appendix A.
3. The Benchmark: Disagreement with \( d = 1 \) Dimensional Opinion

To assess the effects of opinion dimension on disagreement, a natural benchmark is, of course, the lowest dimensional opinion people can hold: \( d = 1 \). In the original paper by [28], the focus is group cohesion, which can be translated to the proportion of group members holding the opinion \( o_x = +1 \) in our model. Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_{2^d}\} \) be the eigenvalues of the transition probability matrix \( M \) in the decreasing order. Let \( \{q_1, q_2, \ldots, q_{2^d}\} \) be its corresponding eigenvectors, normalized such that its components sum to one, i.e. \( Mq_i = \lambda_i q_i (i = 1, \ldots, 2^d) \).

**Proposition 1 (Benchmark).** Under UC or PA case with \( d = 1 \) dimensional opinion, the proportions of group members holding the opinion \( o_x \in \{+1, -1\} \) which are closely correlated with the eigenvalues (i.e. \( \lambda_i (i = 1, 2) \)) and its corresponding eigenvectors (i.e. \( q_i (i = 1, 2) \)) of the transition probability matrix \( M \), can be given by

\[
p(N, N_0, \eta) = \begin{cases} 
\frac{1}{2} \left( 1 + \frac{f(\lambda_2)}{f(\lambda_1)} \right), & o_x = \{+1\}, \\
\frac{1}{2} \left( 1 - \frac{f(\lambda_2)}{f(\lambda_1)} \right), & o_x = \{-1\},
\end{cases}
\]

(1)

where

\[
f(\lambda_i) = \frac{\Gamma(N + \lambda_i)}{\Gamma(N_0 + \lambda_i)}
\]

or

\[
f(\lambda_i) = 1 + \frac{\lambda_i}{\lambda_i + 1}(N_0 - 1) \left( g(\lambda_i) - 1 \right)
\]

with

\[
g(\lambda_i) = \frac{\Gamma \left( \frac{1 + \lambda_i}{2} + N + \frac{1}{2}N_0(N_0 - 3) \right) \Gamma \left( \frac{1}{2}N_0(N_0 - 1) \right)}{\Gamma \left( N + \frac{1}{2}N_0(N_0 - 3) \right) \Gamma \left( \frac{1 + \lambda_i}{2} + \frac{1}{2}N_0(N_0 - 1) \right)}.
\]

Moreover, opinion polarization emerges, i.e. \( p^* = \{1/2, 1/2\} \) as group size \( N \) grows to \( \infty \).

The proof of Proposition 1 can be given as follows. With \( d = 1 \) dimensional opinion, we can have \( M = \left( \begin{smallmatrix} 1-\eta & \eta \\ \eta & 1-\eta \end{smallmatrix} \right) \). Then we can easily get its eigenvalues, i.e. \( \{\lambda_1, \lambda_2\} = \{1, 1 - 2\eta\} \), and the corresponding eigenvectors, i.e. \( q_1 = (1/\sqrt{2}, 1/\sqrt{2})^T, q_2 = (-1/\sqrt{2}, 1/\sqrt{2})^T \). By Proposition 2, we can get \( p(N, N_0, \eta) \) under UC or PA case as this proposition shows.
This proposition implies that, with \( d = 1 \) dimensional opinion, the mean proportions of group members who hold the positive and negative opinion, i.e. \( \overline{p_1}, \overline{p_2} \) are symmetric along the line \( \overline{p} = 1/2 \) (illustrated in Fig. 1(A,E) and Fig. 2(A,E)). For a larger group, e.g. \( N = 1000 \), as the level of noise \( \eta \) grows to 0.5, \( \overline{p_1} \) decreases but \( \overline{p_2} \) increases, and both converge to \( p^* = \{1/2, 1/2\} \), i.e. a random growing group (Fig. 1(A) and Fig. 2(A)). In addition, in the limit of group size \( N \), both converge to \( p^* \) (Fig. 1(E) and Fig. 2(E)). These indicate that, the increase of judgement noise \( \eta \) or group size \( N \) could push a social group towards opinion polarization.

Besides above, Proposition 1 can give us many other deeper insights than the work from the original paper [28]. First, the values of \( \overline{p_1} \) (i.e. group cohesion), \( \overline{p_2} \) are both closely correlated with the eigenvalues of matrix \( M \), i.e. \( \lambda_1 \) and \( \lambda_2 \). Second, the evolution of \( \overline{p} \) is solely determined by as a general form of the ratio of eigenvalue function, i.e. \( \frac{f(\lambda_2)}{f(\lambda_1)} \). Since the increase of either noise \( \eta \) or group size \( N \) can result in enlarging the gap of this ratio, the gap between \( \overline{p_1} \) and \( \overline{p_2} \) is bound to shrink and gradually converges to zero.

4. Results

We now study how the proportions \( p \) of group members holding different opinions \( o_x \in \{+1, -1\}^d \) are affected by opinion dimension \( d \).

4.1. The Results under UC Case with \( d \) Dimensional Opinion

Let us first consider the case where the candidate is judged by one group member who is chosen uniformly (UC). The probabilities that the admitted member \( i \) holds opinions \( o_x \) can be denoted by \( P_x(i) \). In addition, the probabilities that the positive judgement from group member \( i \) on candidate \( t \) who holds opinions \( o_y \), can be denoted by \( W_y(i) \). Then we can have

\[
W_y(i) = \sum_{x=1}^{2^d} M_{yx} P_x(i).
\]

Since the candidate \( t + 1 \) holding opinions \( o_y \) will be judged by any group member \( i \) (i.e. \( i < t \)) holding opinion vector \( o_x \) at random, we can have

\[
P_y(t + 1) = \frac{N_0}{t + N_0} W_y(0) + \frac{1}{t + N_0} \sum_{\tau=1}^{t} W_y(\tau).
\]
Eq. 3 can be easily arranged as

\[(t + N_0)P_y(t + 1) = N_0W_y(0) + \sum_{\tau=1}^{t} W_y(\tau).\]  \(4\)

By replacing \(t\) by \(t - 1\) in Eq. 4, we can get

\[(t - 1 + N_0)P_y(t) = N_0W_y(0) + \sum_{\tau=1}^{t-1} W_y(\tau).\]  \(5\)

By subtracting Eq. 4 from Eq. 5, we can have

\[(N_0 + t)P_y(t + 1) - (N_0 + t - 1)P_y(t) = \sum_{x=1}^{2^d} M_{yx}P_x(t).\]  \(6\)

From Eq. 7, we can easily write the following matrix form

\[(N_0 + t)P(t + 1) - (N_0 + t - 1)P(t) = MP(t),\]  \(7\)

where \(P(t)\) is a vector containing elements \(\{P_1(t), \ldots, P_{2^d}(t)\}\).

Then by Eq. 7, we can obtain a recursive equation for \(P(t)\) with the initial condition \(P(0) = \{1, 0, \ldots, 0\}\) as the following form

\[P(t + 1) = B(t)P(t),\]  \(8\)

where \(B(t) = \left(\alpha(t)M + (1 - \alpha(t))I\right), \alpha(t) = 1/(t + N_0)\) and identity matrix \(I\).

Since \(M\) is a real symmetric matrix (from the description of our model), it can be decomposed as \(M = QAQ^T\). Here \(\Lambda\) is a diagonal matrix with the decreasing order of eigenvalues, i.e. \(\{\lambda_1, \lambda_2, \ldots, \lambda_{2^d}\}\), and \(Q\) is composed of the corresponding eigenvectors as columns, i.e. \(\{q_1, q_2, \ldots, q_{2^d}\}\).

Then \(B(t)\) can be decomposed as

\[B(t) = Q\left(\alpha(t)\Lambda + (1 - \alpha(t))I\right)Q^T.\]  \(9\)

According to Eqs. 8 and 9, we can have

\[P(t) = Q \left[ \prod_{i=1}^{t-1} \left(\alpha(i)\Lambda + (1 - \alpha(i))I\right) \right] Q^T P(1),\]  \(10\)

where \(P(1) = QAQ^TP(0)\).
Due to the fact that, mean proportions of group members holding different opinions can be computed by $p(N, N_0, \eta, d) = \sum_{t=0}^{N-N_0} p(t)$, we can have the following form by Eq. 10 as

$$p(N, N_0, \eta, d) = QDQ^TP(0),$$

(11)

where $D$ is a diagonal matrix with

$$\frac{\Gamma(\lambda_y + N)\Gamma(1 + N_0)}{\Gamma(N + 1)\Gamma(\lambda_y + N_0)} (y = 1, \ldots, 2^d).$$

Eq. 11 can be further decomposed as

$$p(N, N_0, \eta, d) = \sum_{y=1}^{2^d} \frac{\Gamma(\lambda_y + N)\Gamma(1 + N_0)}{\Gamma(N + 1)\Gamma(\lambda_y + N_0)} q_y q_y^TP(0).$$

(12)

Since $M$ is a row/column stochastic matrix, the largest eigenvalue must be $\lambda_1 = 1$, and its corresponding eigenvector is $q_1 = (1/\sqrt{2^d}, \ldots, 1/\sqrt{2^d})$.

So from Eq. 12, we can have

$$p(N, N_0, \eta, d) = q_1 q_1^TP(0) + \sum_{i=2}^{2^d} \frac{f(\lambda_i)}{f(\lambda_1)} q_i q_i^TP(0),$$

(13)

where

$$f(\lambda_i) = \frac{\Gamma(N + \lambda_i)}{\Gamma(N_0 + \lambda_i)} (i = 1, \ldots, 2^d).$$

4.2. The Results under PA Case with d Dimensional Opinion

Let us next consider the case where the candidate is judged by one group member who is chosen by preferential attachment (PA). We assume that the group member and the candidate both accumulate a social capital when the group member makes a positive judgement on the candidate (i.e. the candidate is admitted into the social group). Additionally, each founder member has $k_i(0) = N_0 - 1$ social capital at time $t = 0$ (when $N_0=1$, we set $k_i(0) = 1$). With PA mechanism, the evolution on social capitals for group member $i$ can be expressed as

$$k_i(t + 1) = k_i(t) + \frac{k_i(t)}{2t + N_0(N_0 - 1)},$$

(14)

\footnotetext{3}{We can treat the social capital as the node degree of a social network which is formed by preferential attachment mechanism in the Barabási-Albert model [6].}
Figure 1: Mean proportions $\bar{p}$ of group members holding different opinions under UC case. (A-D) The relation between $\bar{p}$ and the level of noise $\eta$. Settings: $N = 1000$, $N_0 = 1$ and $d = \{1, 2, 3, 4\}$ with 1000 iterations. (E-H) The relation between $\bar{p}$ and the group size $N$. Settings: $\eta = 0.1$, $N_0 = 1$ and $d = \{1, 2, 3, 4\}$ with 1000 iterations. The solid lines are analytical results, and the points are simulation results. The error bars show twice the standard error of the mean.
which is a recursive equation for \( k_i(t) \) using continuum approximation.

In Eq. 14, the evolution of social capitals for founder members, i.e. \( k_0(t) \) and other members, i.e. \( k_i(t)(i > 0) \) should be solved separately. For notation convenience, we denote \( L := 1/2N_0(N_0 - 1), g(t) := \Gamma(1/2 + L + t)/\Gamma(L + t) \).

The solution for the founder members, can be given by

\[
k_0(t) = \frac{1 + 2L \cdot g(t)}{N_0 \cdot g(0)}.
\] (15)

For other members, instead, the solution is

\[
k_i(t) = \frac{g(t)}{g(i)}(i > 0).
\] (16)

Analogously to Eq. 3, we can have the similar form about \( P_y(t) \) as

\[
P_y(t + 1) = \frac{N_0k_0(t)}{2t + 2L}W_y(0) + \sum_{\tau=1}^{t} \frac{k_{\tau}(t)}{2t + 2L}W_y(\tau).
\] (17)

By plugging Eqs. 15 and 16 into Eq. 17, we can get

\[
\frac{2t + 2L}{g(t)}P_y(t + 1) = W_y(0) \frac{1 + 2L}{g(0)} + \sum_{\tau=1}^{t} \frac{W_y(\tau)}{g(\tau)}.
\] (18)

By replacing \( t \) by \( t - 1 \) in Eq. 18, we can get

\[
\frac{2(t - 1) + 2L}{g(t - 1)}P_y(t) = W_y(0) \frac{1 + 2L}{g(0)} + \sum_{\tau=1}^{t-1} \frac{W_y(\tau)}{g(\tau)}.
\] (19)

By substracting Eq. 18 from Eq. 19, we can have the following form

\[
\frac{2t + 2L}{g(t)}P_y(t + 1) - \frac{2(t - 1) + 2L}{g(t - 1)}P_y(t) = \sum_{x=1}^{2d} M_{yx}P_x(t) \frac{2t}{g(t)}.
\] (20)

Then we can easily write the following matrix form of Eq. 20 as

\[
\frac{2t + 2L}{g(t)}P(t + 1) - \frac{2(t - 1) + 2L}{g(t - 1)}P(t) = MP(t) \frac{2t}{g(t)}.
\] (21)

Analogously to Eq. 8, Eq. 21 can be rearranged as the following similar form

\[
P(t + 1) = B'(t)P(t),
\] (22)
where $B'(t) = \left( \alpha'(t)M + (1 - \alpha'(t))I \right)$, $\alpha'(t) = 1/(2t + 2L)$ and identity matrix $I$.

Analogously to Eq. 9, $B'(t)$ can be decomposed as

$$B'(t) = Q\left( \alpha'(t)A + (1 - \alpha'(t))I \right)Q^T. \quad (23)$$

Analogously to Eq. 10, we can have

$$P(t) = Q \left[ \prod_{i=1}^{t-1} \left( \alpha'(i)A + (1 - \alpha'(i))I \right) \right] Q^TP(1), \quad (24)$$

where $P(1) = QAQ^TP(0)$.

Due to $p(N, N_0, \eta, d) = \sum_{i=0}^{N-N_0} P(t)$ and Eq. 24, we can have

$$p(N, N_0, \eta, d) = q_1q_1^TP(0) + \sum_{i=2}^{2^d} \frac{f(\lambda_i)}{f(\lambda_1)}q_iq_i^TP(0), \quad (25)$$

where

$$f(\lambda_i) = 1 + \frac{\lambda_i}{\lambda_i + 1}(N_0 - 1)\left( g(\lambda_i) - 1 \right) \quad (i = 1, \ldots, 2^d)$$

with

$$g(\lambda_i) = \frac{\Gamma\left( \frac{1+\lambda_i}{2} + N + \frac{1}{2}N_0(N_0 - 3) \right)}{\Gamma\left( N + \frac{1}{2}N_0(N_0 - 3) \right)} \frac{\Gamma\left( \frac{1}{2}N_0(N_0 - 1) \right)}{\Gamma\left( \frac{1+\lambda_i}{2} + \frac{1}{2}N_0(N_0 - 1) \right)}.$$

10
Figure 2: Mean proportions $p$ of group members holding different opinions under PA case. (A-D) The relation between $p$ and the level of noise $\eta$. Settings: $N = 1000$, $N_0 = 1$ and $d = \{1, 2, 3, 4\}$ with 1000 iterations. (E-H) The relation between $p$ and the group size $N$. Settings: $\eta = 0.1$, $N_0 = 1$ and $d = \{1, 2, 3, 4\}$ with 1000 iterations. The solid lines are analytical results, and the points are simulation results. The error bars show twice the standard error of the mean.
4.3. The Key Results

Proposition 2 (Key Result). Under UC or PA case with $d$ dimensional opinion, the proportions of group members holding $o_x \in \{+1, -1\}^d (x = 1, \ldots, 2^d)$ which are closely correlated with the eigenvalues (i.e. $\{\lambda_1, \ldots, \lambda_{2^d}\}$) and its corresponding eigenvectors (i.e. $\{q_1, \ldots, q_{2^d}\}$) of the transition probability matrix $M$, can be given by

$$p(N, N_0, \eta, d) = q_1 q_1^T P(0) + \sum_{i=2}^{2^d} \frac{f(\lambda_i)}{f(\lambda_1)} q_i q_i^T P(0),$$

where

$$f(\lambda_i) = \frac{\Gamma(N + \lambda_i)}{\Gamma(N_0 + \lambda_i)} \quad (i = 1, \ldots, 2^d)$$

or

$$f(\lambda_i) = 1 + \frac{\lambda_i}{\lambda_i + 1} (N_0 - 1) \left( g(\lambda_i) - 1 \right) \quad (i = 1, \ldots, 2^d)$$

with

$$g(\lambda_i) = \frac{\Gamma \left( \frac{1}{2} + \frac{1}{2} N + \frac{1}{2} N_0 (N_0 - 3) \right)}{\Gamma \left( N + \frac{1}{2} N_0 (N_0 - 3) \right)} \Gamma \left( \frac{1}{2} + \frac{1}{2} N_0 (N_0 - 1) \right).$$

Moreover, opinion multi-polarization emerges, i.e. $p^* = \{1/2^d, \ldots, 1/2^d\}$ as group size $N$ grows to $\infty$.

Proposition 2 (illustrated in Fig. 1 and Fig. 2) implies that, the proportions of group members holding different opinions are largely affected by the opinion dimension $d$. When the opinion dimension $d$ is an odd value, e.g. $d = 1, 3$ (shown in Fig. 1(A,C,E,G) and Fig. 2(A,C,E,G)), there always exists growth or decrease for all proportions $p$ as the level of noise or group size grows. This is due to the fact that the absolute value of opinion difference between $o_x = \{+1\}^d$ held by the founder member with $o_y$ held by the candidate, i.e. $|K_{xy}|$, is always not zero. But an even value of $d$, e.g. $d = 2, 4$ produces some values of $|K_{xy}| = 0$ between $o_x = \{+1\}^d$ held by the founder member and the opinion vector $o_y$ which are composed of one-half of positive/negative opinions. As such these candidates holding these corresponding opinion vectors are at random admitted into the social group, so their mean proportions won’t be affected by noise $\eta$ or group size $N$, and always stay on $1/2^d$ (shown in Fig. 1(B,D,F,H) and Fig. 2(B,D,F,H)). In addition, the proportions of group members who hold opinion vector consisting of the same proportion of positive/negative
opinions, are always symmetric along $p = 1/2^d$. Similarly to $d = 1$ dimensional opinion, both the growth of judgement noise $\eta$ (under larger group size) and group size $N$ push a social group towards opinion multi-polarization, i.e. $p^* = \{1/2^d, \ldots, 1/2^d\}$.

5. Conclusion and Discussion

To the best of our knowledge, it is the first model to study how disagreement or even polarization emerges throughout the group formation process. The proportions of group members who hold various opinions has a strong correlation with the transition probability matrix. We show that the social group will be divided into smaller groups with various viewpoints with the infinitesimal level of judgement noise, i.e. opinion disagreement always exist. Additionally, opinions in a larger group tend to be more polarized.

Our work can be extended to many aspects. First of all, the admission of candidate can be determined by the aggregation of judgements from several group members, such as majority judgement (the number of positive judgements should be greater than negative ones) and consensus judgement (all judgements should be positive). Second, opinion dynamics could be introduced along with the group formation process. Third, different opinions held by candidates can be not uniform. Finally, we can construct social networks by our model to explore social fragmentation and community structure in the future.

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Appendix A. Different Interpretations for Noise Parameter $\eta$

With respect to the noise parameter $\eta$, there are in fact two different interpretations. In a nutshell, these interpretations are (i) group member *scepticizes* opinions held by candidate with probability $\eta_1$, and (ii) candidate *conceals* its opinions with probability $\eta_2$.

Thus the judgement process in one round can be divided into two parts. Let us consider the case in which group member and candidate are holding the same opinion orientation {$+1$} on the same object from $d$ different objects. Then candidate has probability $\eta_2$ to conceal its opinion orientation as $-1$, but $1 - \eta_2$ not. Analogously, there exists probability $\eta_1$ that group member is sceptical that candidate’s opinion orientation is $+1$, but $1 - \eta_1$ is $-1$. To this end, the probability of negative judgement would be $(1 - \eta_1)\eta_2 + \eta_1(1 - \eta_2)$. Without loss of generality, absence of the scepticism of group member, i.e. $\eta_1 = 0$, $\eta = \eta_2$ is satisfied. Similarly, without concealing of candidate, i.e. $\eta_2 = 0$, $\eta = \eta_1$ is also satisfied. In another way around, the noise parameter $\eta$ in our model has included all these two interpretations.

Appendix B. The Rationality of Matrix $M$

Under $d > 1$ case the judgement from a group member on a candidate can be separated along each opinion dimension, but by same rule as $d = 1$ case [28]. Furthermore, the probability of the final positive/negative judgement is related with the proportion of positive/negative judgements along each opinion dimension.

Here is an example of $d = 2$ case, the probability of positive judgement from group member with opinions $(+1,+1)$ to candidate with opinions $(+1,+1)$, can be calculated as

$$
(1 - \eta)^2 + \binom{2}{1} \frac{1}{2} \eta(1 - \eta).
$$

(B.1)
In Eq. B.1, the first term \((1 - \eta)^2\) can be seen as the probability that two positive judgements are given along these two opinion dimensions. Since these two judgements are both positive, the final judgement must be positive. The second term can be interpreted as the probability that along these two opinion dimensions, one judgement is positive and the other negative. Since the proportion of positive judgements is \(1/2\), the probability of final judgement will be \(\frac{1}{2} \eta (1 - \eta)\). Notably, the number of the same scenarios can be given by \(\binom{d}{1}\). Finally, if two negative judgements are given, then the final judgement cannot be positive.

Analogously, positive judgements from \((+1, +1)\) to \((+1, -1)\), \((+1, +1)\) to \((-1, +1)\) and \((+1, +1)\) to \((-1, -1)\), can be given by \((1 - \eta)\eta + \frac{1}{2}(1 - \eta)^2 + \frac{1}{2}\eta^2\), \((1 - \eta)\eta + \frac{1}{2}(1 - \eta)^2 + \frac{1}{2}\eta^2\) and \(\eta^2 + \frac{1}{2}\eta (1 - \eta) + \frac{1}{2}\eta(1 - \eta)\) respectively. We can obtain the similar results under any \(d > 1\) case. By reduction, we indeed have the form \(\left(\frac{1}{2} + \frac{N_0}{d} \left(\frac{1}{2} - \eta\right)\right)/2^{d-1}\) when all probabilities are normalized. This also fully proves the rationality of the form of the transition probability matrix \(M\).

Appendix C. The Evolution of Probabilities with Different Opinions

Corollary 1 (Evolution). Under UC or PA case with \(d\) dimensional opinion, the evolution of probabilities for admitted group members holding different opinions which is determined by a convex combination of the transition probability matrix \(M\) and identity matrix \(I\), can be given by

\[
P(t + 1) = B(t)P(t)
\]

where \(B(t) = \left(\alpha(t)M + (1 - \alpha(t))I\right)\), \(\alpha(t) = 1/(t + N_0)\) or \(\alpha(t) = 1/(2t + N_0(N_0 - 1))\).

Here \(B(t)\) is a convex combination of transition probability matrix \(M\) and identity matrix \(I\). It is worth noting that, the value of \(\alpha(t)\) under UC case is always smaller than that under PA case when \(t > 1\). Technically, the above recursive process weighs less on matrix \(M\) of \(B(t)\) under UC case than that under PA case. This implies that, since \(P(0) = (1, 0, \ldots, 0)\), \(\Pi\) for UC case should be always lower than that for PA case (compared between the results shown in Fig. 1 and Fig. 2).