CLASS NUMBERS AND SELF-CONJUGATE 7-CORES

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Abstract. We investigate $sc_7(n)$, the number of self-conjugate 7-core partitions of size $n$. It turns out that $sc_7(n) = 0$ for $n \equiv 7 \pmod{8}$. For $n \equiv 1, 3, 5 \pmod{8}$, with $n \neq 5 \pmod{7}$, we find that $sc_7(n)$ is essentially a Hurwitz class number. Using recent work of Gao and Qin, we show that

$$sc_7(n) = 2^{-\varepsilon(n)-1} \cdot H(-D_n),$$

where $-D_n := -4^{\varepsilon(n)}(7n + 14)$ and $\varepsilon(n) := \frac{1}{2} \cdot (1 + (-1)^{\frac{n-3}{2}})$. This fact implies several corollaries which are of interest. For example, if $-D_n$ is a fundamental discriminant and $p \notin \{2, 7\}$ is a prime with $\text{ord}_p(-D_n) \leq 1$, then for every positive integer $k$ we have

$$(1) \quad sc_7((n + 2)p^{2k} - 2) = sc_7(n) \cdot \left(1 + \frac{p^{k+1} - p}{p - 1} - \frac{p^k - 1}{p - 1} \cdot \left(\frac{-D_n}{p}\right)\right),$$

where $\left(\frac{-D_n}{p}\right)$ is the Legendre symbol.

1. Introduction and statement of results

A partition of a non-negative integer $n$ is any nonincreasing sequence of positive integers which sum to $n$. The partition function $p(n)$, which counts the number of partitions of $n$, has been studied extensively in number theory. In particular, Ramanujan proved that

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7},$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$

Partitions also play a significant role in representation theory (for example, see [10]). Indeed, partitions of size $n$ are used to define Young tableaux, and their combinatorial properties encode the representation theory of the symmetric group $S_n$. Moreover, the $t$-core partitions of size $n$ play an important role in number theory (for example, see [6, 7, 12]) and the modular representation theory of $S_n$ and $A_n$ (for example, see Chapter 2 of [10], and [5, 7]). Recall that a partition is a $t$-core if none of the hook numbers of its Ferrers-Young diagram are multiples of $t$. If $p$ is prime, then the existence of a $p$-core of size $n$ is equivalent to the existence of a defect 0 $p$-block for both $S_n$ and $A_n$. 

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For positive integers \( t \), we let \( c_t(n) \) denote the number of \( t \)-core partitions of size \( n \). If \( t \in \{2, 3\} \), then it is well-known that \( c_t(n) = 0 \) for almost all \( n \in \mathbb{N} \). However, if \( t \geq 4 \), then \( c_t(n) > 0 \) for every positive integer \( n \) (see [7]).

The case where \( t = 4 \) is particularly interesting, as these partitions arise naturally in algebraic number theory. As usual, let \( H(-D) \) denote the discriminant \(-D < 0\) Hurwitz class number. For the discriminants considered here, \( H(-D) \) is the number of inequivalent (not necessarily primitive) positive definite binary quadratic forms with discriminant \(-D < 0\). In particular, if \(-D \not\in \{-3, -4\} \) is a fundamental discriminant, then \( H(-D) \) is the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-D}) \). Sze and the first author proved (see Theorem 2 of [12]) that

\[
c_4(n) = \frac{1}{2} H(-32n - 20).
\]

Furthermore, for primes \( p \) and \( N \in \mathbb{N} \) with \( \text{ord}_p(N) \leq 1 \), they proved (see Corollary 2 of [12]) for positive integers \( k \) that

\[
c_4\left(\frac{Np^{2k} - 5}{8}\right) = c_4\left(\frac{N - 5}{8}\right) \cdot \left(1 + \frac{p^{k+1} - p}{p - 1} - \frac{p^k - 1}{p - 1} \cdot \left(\frac{-N}{p}\right)\right),
\]

where \( \left(\frac{-N}{p}\right) \) is the Legendre symbol. These formulas implied earlier conjectures of Hirschhorn and Sellers [9].

Further relationships between integer partitions and class numbers are expected to be extremely rare. In this note we find one more instance where \( t \)-cores and class numbers are intimately related. To this end, we let \( s_{c_t}(n) \) denote number of the self-conjugate \( t \)-core partitions of size \( n \). These are \( t \)-core partitions which are symmetric with respect to the operation which switches the rows and columns of a Ferrers-Young diagram.

To formulate these results, for a positive odd integer \( n \) we define the negative discriminant

\[
-D_n := \begin{cases} 
-28n - 56 & \text{if } n \equiv 1 \pmod{4}, \\
-7n - 14 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

**Theorem 1.** If \( n \not\equiv 5 \pmod{7} \) is a positive odd integer, then we have

\[
s_{c_7}(n) = \begin{cases} 
\frac{1}{4} H(-D_n) & \text{if } n \equiv 1 \pmod{4} \\
\frac{1}{2} H(-D_n) & \text{if } n \equiv 3 \pmod{8} \\
0 & \text{if } n \equiv 7 \pmod{8}.
\end{cases}
\]

**Example.** Here we consider the case where \( n = 9 \). According to Theorem 1, we have that \( s_{c_7}(9) = H(-308)/4 \). One readily finds that there are eight equivalence classes of discriminant \(-308\) binary quadratic forms. The reduced forms representing these classes are:

\[
2X^2 + 2XY + 39Y^2, \quad 3X^2 - 2XY + 26Y^2, \quad 3X^2 + 2XY + 26Y^2, \quad 6X^2 - 2XY + 13Y^2, \quad 6X^2 + 2XY + 13Y^2, \quad 7X^2 + 11Y^2, \quad 9X^2 - 4XY + 9Y^2, \quad 9X^2 + 4XY + 9Y^2.
\]
There are fourteen 7-core partitions of \( n = 9 \). However, only two of them are self-conjugate. They are (subscripts are the hook numbers):

\[
\begin{align*}
\bullet_5 \bullet_4 \bullet_3 & \quad \bullet_9 \bullet_4 \bullet_3 \bullet_2 \bullet_1 \\
\bullet_4 \bullet_3 \bullet_2 & \quad \bullet_4 \\
\bullet_3 \bullet_2 \bullet_1 & \quad \bullet_3 \\
\bullet_2 & \quad \bullet_2 \\
\bullet_1 & \quad .
\end{align*}
\]

This example illustrates the conclusion that \( \text{sc}_7(9) = 2 = H(-308)/4 \).

**Example.** For \( n = 25 \), we have that \(-D_n = -756 = -3^2 \cdot 84\) and \( H(-756) = 16 \). Therefore, Theorem 1 implies that \( \text{sc}_7(25) = H(-756)/4 = 4 \).

Theorem 1 implies simple short finite formulas for \( \text{sc}_7(n) \). To this end, we recall the standard Kronecker character for a discriminant \( D \). Define the Kronecker character \( \chi_D(n) \) for positive integers \( n \) by

\[
\chi_D(n) = \left( \frac{D}{n} \right) \prod \left( \frac{D}{p_i} \right)^{a_i},
\]

where \( n = \prod p_i^{a_i} \) and \( \left( \frac{D}{p} \right) \) is the Legendre symbol when \( p \) is an odd prime and

\[
\left( \frac{D}{2} \right) = \begin{cases} 
0 & \text{if } D \text{ is even} \\
(-1)^{(D^2-1)/8} & \text{if } D \text{ is odd.}
\end{cases}
\]

**Corollary 2.** If \( n \equiv \not 5 \pmod{7} \) is a non-negative odd integer for which \(-D_n \) is a fundamental discriminant, then

\[
\text{sc}_7(n) = \begin{cases} 
-\frac{1}{4D_n} \sum_{m=1}^{D_n} \left( \frac{-D_n}{m} \right) m & \text{if } n \equiv 1 \pmod{4}, \\
-\frac{1}{2D_n} \sum_{m=1}^{D_n} \left( \frac{-D_n}{m} \right) m, & \text{if } n \equiv 3 \pmod{8}, \\
0 & \text{if } n \equiv 7 \pmod{8}.
\end{cases}
\]

**Example.** If \( n = 11 \), then \(-D_{11} = -91\). We then find that

\[
\text{sc}_7(11) = -\frac{1}{182} \sum_{m=1}^{91} \left( \frac{-91}{m} \right) m = 1.
\]

Hurwitz class numbers enjoy a host of multiplicative properties which can be formulated in terms of the Möbius function \( \mu(d) \) and the divisor function \( \sigma_1(n) := \sum_{1 \leq d \mid n} d \). These imply the following simple corollary which is analogous to (2).

**Corollary 3.** If \( n \equiv \not 5 \pmod{7} \) is a positive odd integer, and \(-D_n \) is a fundamental discriminant, then for all odd integers \( f \) coprime to 7, we have

\[
\text{sc}_7((n + 2)f^2 - 2) = \text{sc}_7(n) \sum_{1 \leq d \mid f} \mu(d) \left( \frac{-D_n}{d} \right) \sigma_1(f/d).
\]
Remark. The cases where \( f = p^k \) is a prime power coincide with (1).

Example. If \( n = 11 \), then we have that \(-D_{11} = -91\). Now suppose that \( f = 15 \). Corollary 3 implies that
\[
\text{sc}_7(2923) = \text{sc}_7(11) \sum_{1 \leq d \mid 15} \mu(d) \left( \frac{-91}{d} \right) \sigma_1(15/d).
\]
By direct calculation, since \( \text{sc}_7(11) = 1 \), the right hand side of this expression equals
\[
\sigma_1(15) + \sigma_1(5) - \sigma_1(3) - \sigma_1(1) = 25.
\]
Using the \( q \)-series identities in the next section (for example (4)), one can indeed check directly that \( \text{sc}_7(2923) = 25 \).

2. Proofs

Here we prove Theorem 1 and Corollaries 2 and 3.

Proof of Theorem 1. If \( t \) is a positive odd integer, then the generating function for \( \text{sc}_t(n) \) (see (2) of [8]) is
\[
\sum_{n=0}^{\infty} \text{sc}_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{2tn})^{\frac{t-1}{2}}(1 + q^{2n-1})}{(1 + q^{tn})(1 + q^{(2n-1)\tau})}.
\]
Therefore, in terms of Dedekind’s eta-function \( \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n) \), where \( q := e^{2\pi i \tau} \), we have that
\[
S(\tau) := \sum_{n=0}^{\infty} \text{sc}_7(n)q^{n+2} = \frac{\eta(2\tau)^2 \eta(14\tau) \eta(7\tau) \eta(28\tau)}{\eta(4\tau) \eta(\tau)}.
\]
In particular, by the standard theory of modular forms (for example, see Chapter 1.4 of [11]), it follows that \( S(\tau) \) is a holomorphic modular form of weight \( \frac{3}{2} \) on \( \Gamma_0(28) \) with Nebentypus \( \chi_7(n) := (\frac{7}{n}) \). In terms of theta functions, we have that\footnote{This corrects the statement of Theorem 9 of [1].}
\[
S(\tau) = \frac{1}{14} \Theta_1(\tau) - \frac{1}{7} \Theta_2(\tau) + \frac{1}{14} \Theta_3(\tau),
\]
where
\[
\Theta_1(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_1(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{x^2+y^2+2z^2-yz},
\]
\[
\Theta_2(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_2(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{x^2+4y^2+8z^2-4yz}
\]
and
\[
\Theta_3(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_3(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{2x^2+2y^2+3z^2+2yz+2xz+2xy}.
\]
As a result, we get that
\[
\text{sc}_7(n) = \frac{1}{14} R(Q_1; n + 2) - \frac{1}{7} R(Q_2; n + 2) + \frac{1}{14} R(Q_3; n + 2),
\]
where \( R(Q; n + 2) \) is the number of integral representations of \( n + 2 \) by \( Q_4(x, y, z) \).

To prove the theorem, we must relate these three theta functions to the Eisenstein series in the space \( M_\frac{3}{2}(28, \chi_7) \). Thankfully, Gao and Qin [4] have already carried out these calculations. They produce three Eisenstein series which form a basis of this space (see Theorem 3.2 of [4]). These series are given by their Fourier expansions:

\[
\begin{align*}
g_1(\tau) &= 1 + 2\pi\sqrt{7} \sum_{n=1}^{\infty} \lambda(7n, 28) \alpha(7n) \left( A(7, 7n) - \frac{1}{7} \right) \sqrt{nq^n}, \\
\frac{g_2(\tau)}{49} &= 2 \pi\sqrt{7} \sum_{n=1}^{\infty} \lambda(7n, 28) \alpha(7n) \sqrt{nq^n}, \\
\frac{g_3(\tau)}{2} &= 2 \pi\sqrt{7} \sum_{n=1}^{\infty} \lambda(7n, 28) \left( A(7, 7n) - \frac{1}{7} \right) \sqrt{nq^n}.
\end{align*}
\]

The quantities \( \alpha(m), A(p, m), \lambda(7m, 28), h_p(m) \) and \( h'_p(m) \) are defined by

\[
\alpha(m) := \begin{cases} 
3 \cdot 2^{\frac{1 + h_p(m)}{2}} & \text{if } h_p(m) \text{ is odd,} \\
3 \cdot 2^{1 - \frac{h_p(m)}{2}} & \text{if } h_p(m) \text{ is even and } h'_p(m) \equiv 1 \pmod{4}, \\
0 & \text{if } h_p(m) \text{ is even and } h'_p(m) \equiv 3 \pmod{8}, \\
2^{1 - \frac{h_p(m)}{2}} & \text{if } h_p(m) \text{ is even and } h'_p(m) \equiv 7 \pmod{8},
\end{cases}
\]

\[
A(p, m) := \begin{cases} 
p^{-1} - (1 + p) p^{\frac{3 + h_p(m)}{2}} & \text{if } h_p(m) \text{ is odd,} \\
p^{-1} - 2p^{-1} p^{\frac{3 + h_p(m)}{2}} & \text{if } h_p(m) \text{ is even and } \left( \frac{-h'_p(m)}{p} \right) = -1, \\
p^{-1} & \text{if } h_p(m) \text{ is even and } \left( \frac{-h'_p(m)}{p} \right) = 1,
\end{cases}
\]

where \( h_p(m) \) is the non-negative integer for which \( p^{h_p(m)} \mid m \) and \( h'_p(m) := \frac{m}{p^{h_p(m)}} \), and

\[
\lambda(7m, 28) = \begin{cases} 
\frac{49}{4\pi\sqrt{7m}} \cdot H(-7m), & \text{if } m \equiv 5 \pmod{8}, \\
\frac{49}{12\pi\sqrt{7m}} \cdot H(-7m), & \text{otherwise.}
\end{cases}
\]

The formula for \( \alpha(m) \), when \( h'_p(m) \equiv 3, 7 \pmod{8} \), corrects a typographical error in [4].

Since the three Eisenstein series form a basis of this space, it is trivial to deduce that

\[
\Theta_1(\tau) = g_1(\tau) - 3g_2(\tau), \quad \Theta_2(\tau) = g_1(\tau) - \frac{3}{2}g_3(\tau), \quad \Theta_3(\tau) = g_1(\tau) + 14g_2(\tau).
\]

Using \( h_7(7n) = 1, h'_7(7n) = n, h_2(7n) = 0 \) and \( h'_2(7n) = 7n \) for \( (n, 14) = 1 \), we find that

\[
A(7, 7n) = \frac{1}{7} - \frac{8}{49},
\]

where \( A(7, 7n) \) is the non-negative integer for which \( p^{h_p(m)} \mid m \) and \( h'_p(m) := \frac{m}{p^{h_p(m)}} \).
and
\[
\alpha(7n) := \begin{cases} 
\frac{3}{2} & \text{if } n \equiv 3 \pmod{4}, \\
1 & \text{if } n \equiv 1 \pmod{8}, \\
0 & \text{if } n \equiv 5 \pmod{8}.
\end{cases}
\]

Combining these facts, for positive integers \(n\) we deduce that
\[
R(Q_1; n) = 2\pi \sqrt{7n} \lambda(7n, 28) \left( A(7, 7n) - \frac{1}{7} \right) (\alpha(7n) - 3),
\]
\[
R(Q_2; n) = 2\pi \sqrt{7n} \lambda(7n, 28) \left( A(7, 7n) - \frac{1}{7} \right) (\alpha(7n) - \frac{3}{2}),
\]
and
\[
R(Q_3; n) = 2\pi \sqrt{7n} \lambda(7n, 28) \alpha(7n) \left( A(7, 7n) - \frac{1}{7} + \frac{14}{49} \right).
\]

Finally, these expressions, for positive odd \(n \not\equiv 5 \pmod{7}\), simplify to
\[
R(Q_1; n + 2) = \begin{cases} 
2H(-D_n) & \text{if } n \equiv 1 \pmod{4}, \\
8H(-D_n) & \text{if } n \equiv 3 \pmod{8}, \\
4H(-D_n) & \text{if } n \equiv 7 \pmod{8},
\end{cases}
\]
\[
R(Q_2; n + 2) = \begin{cases} 
0 & \text{if } n \equiv 1 \pmod{4}, \\
2H(-D_n) & \text{if } n \equiv 3 \pmod{8}, \\
2H(-D_n) & \text{if } n \equiv 7 \pmod{8},
\end{cases}
\]
\[
R(Q_3; n + 2) = \begin{cases} 
\frac{3}{2}H(-D_n) & \text{if } n \equiv 1 \pmod{4}, \\
3H(-D_n) & \text{if } n \equiv 3 \pmod{8}, \\
0 & \text{if } n \equiv 7 \pmod{8}.
\end{cases}
\]

The claimed formulas now follow from (5). \(\square\)

**Proof of Corollary 2.** If \(-D < 0\) is a fundamental discriminant, then it is well-known (for example, see (7.29) of [3]) that
\[
H(-D) = -\frac{|O_K^\times|}{2D} \sum_{m=1}^{D} \left( \frac{-D}{m} \right) m,
\]
where \(K = \mathbb{Q}(\sqrt{-D})\) and \(O_K^\times\) denotes the units in its corresponding ring of integers. The claim now follows from Theorem 1. \(\square\)

**Proof of Corollary 3.** If \(-D < 0\) is a fundamental discriminant of an imaginary quadratic field, then for every integer \(f\) it is known (p. 273, [2]) that
\[
H(-Df^2) = -\frac{H(-D)}{w(-D)} \sum_{1 \leq d | f} \mu(d) \left( \frac{-D}{d} \right) \sigma_1(f/d),
\]
where $w(-D)$ denotes half of the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$. The claim now follows from Theorem 1.

□

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