Abstract. We would like to present a general principle for the shrinking target problem in a topological dynamical system. More precisely, let \((X,d)\) be a compact metric space and \(T : X \to X\) a continuous transformation on \(X\). For any integer valued sequence \(\{a_n\}\) and \(y \in X\), define
\[
E_y(\{a_n\}) = \bigcap_{\delta > 0} \{ x \in X : T^n x \in B_{a_n}(y, \delta), \text{ for infinitely often } n \in \mathbb{N} \},
\]
the set of points whose orbit can well approximate a given point infinitely often, where \(B_{\alpha}(x,r)\) denotes the Bowen-ball. It is shown that
\[
h_{\text{top}}(E_y(\{a_n\}),T) = \frac{1}{1 + a} h_{\text{top}}(X,T), \quad \text{with } a = \liminf_{n \to \infty} \frac{a_n}{n},
\]
if the system \((X,T)\) has the specification property. Here \(h_{\text{top}}\) denotes the topological entropy. An example is also given to indicate that the specification property required in the above result cannot be weakened even to almost specification.

1. Introduction. The analysis of Diophantine properties in a dynamical system is concerned with the distribution of orbits. More precisely, in a measure-theoretic dynamical system \((X,\mathcal{B},\mu,T)\) with a compatible metric \(d\), one concerns the size, in the sense of measure and fractal dimensions, of the following limsup set
\[
W_y(\phi) := \left\{ x \in X : T^n x \in B(\mu, \phi(n)), \text{ for infinitely often } n \in \mathbb{N} \right\},
\]
where \(y\) is a fixed point and \(\phi\) is a positive function with \(\phi(n) \to 0\) as \(n \to \infty\) (here \(\phi\) can also be a function depending both on \(n\) and \(x\)). This is the set of points whose orbit can well approximate a given point \(y\) infinitely often. This is called the shrinking target problems [12] or dynamical Diophantine approximation [9, 16] in the literature. We use \(i.o.\) to denote \(\text{infinitely often}\) for brevity in the following.
Due to its close connections with the classic Diophantine approximation (for example, when $T$ is the Gauss map [12] or irrational rotation [14]), it has recently been attracted an increased interest in the literature.

On the size of $W_y(\phi)$ in the sense of a measure, one is referred to the work of Chernov & Kleinbock [7], Galatolo [10], Galatolo & Kim [11] and related references. Besides the measure of $W_y(\phi)$, its dimensional theory is also extensively studied. This is initiated by Hill & Velani who studied the dimension of $W_y(\phi)$ for expanding rational maps acting on its Julia set [12, 13]. For infinite iterated function systems or some concrete systems without finite Markov partitions, one is referred to the works [5, 15, 22, 24, 20, 27, 29] etc.

It should be mentioned that in most of the expanding systems cited above, the dimension of $W_y(\phi)$ is usually given by a unified formula. For example, by taking

$$\phi(n,x) = e^{-S_n f(x)},$$

where $S_n f(x) := f(x) + \cdots + f(T^{n-1}(x))$, in [5, 12, 13, 15, 20, 27], the dimension of $W_y(\phi)$ is always given as

$$\inf \left\{ s \geq 0 : P(T,-s(\log |T'| + f)) \leq 0 \right\}, \quad (1)$$

where $P$ is the pressure function. So, it is believed that for “well behaved” systems there should be a general principle for the structure of $W_y(\phi)$ (see [28] for a survey and some possible conjectures).

As far as a topological dynamical system is concerned, instead of using the classic metric, one always uses the Bowen-metric to measure the behavior of the orbits. So, we present another version of the shrinking target problem in topological dynamical system as follows.

Let $(X,T)$ be a topological dynamical system, i.e. $X$ is a compact metric space with a metric $d$, and $T : X \to X$ is a continuous transformation. The Bowen metric $d_n$ with $n \geq 1$, defined by

$$d_n(x,y) = \sup \{ d(T^i x, T^i y) : 0 \leq i < n \},$$

is used to measure the closeness of $x$ and $y$. For any $\delta > 0$, the Bowen ball is defined as

$$B_n(y,\delta) := \{ x \in X : d_n(x,y) < \delta \}.$$

For a non-integer value $\xi$ or $\xi = 0$, we also define the Bowen metric $d_{\xi}$ as $d_{\lfloor \xi \rfloor + 1}$ and the Bowen ball $B_{\xi}$ as $B_{\lfloor \xi \rfloor + 1}$, where $\lfloor \cdot \rfloor$ denotes the integer part.

For any $a \geq 0$ and $y \in X$, define a topological version of the shrinking target problem as

$$E_y(a) := \bigcap_{\delta > 0} \left\{ x \in X : T^n x \in B_{an}(y,\delta), \ i.o. \ n \in \mathbb{N} \right\}.$$

More generally, one can also replace $a \cdot n$ by a general sequence $\{a_n\}_{n\geq 1}$.

Analogous to the main concerns of the classic shrinking target problem, in this paper, we also intend to find a general principle for the topological entropy about the above topological version of the shrinking target problem in systems as general as possible.

Before going further, let us predict what a general principle should be for the topological entropy of $E_y(a)$, denoted by $h_{top}(E_y(a),T)$. 
1.1. **A possible principle for the entropy of** $E_y(a)$. Firstly, because of the limsup nature of $E_y(a)$, namely, for any $\delta > 0$,

$$E_y(a) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n} B_{an}(y, \delta),$$

it is not difficult to find an upper bound of the topological entropy of $E_y(a)$, i.e.

$$h_{\text{top}}(E_y(a), T) \leq \frac{1}{1+a} h_{\text{top}}(X, T). \quad (2)$$

For the details see Section 3.

Secondly, if there DO exist a general principle, it should include some “nice” systems. If a system is of zero topological entropy, so is $E_y(a)$ by the formula (2) and there is nothing to do. Thus one needs only focus on the systems with positive entropy. In such a sense, full shift space over finite alphabets should be a “nice” candidate. For this concrete system, it can be proved that the equality in (2) holds. So, one could conjecture that

$$\text{In very general systems, } h_{\text{top}}(E_y(a), T) = \frac{1}{1+a} h_{\text{top}}(X, T). \quad (3)$$

Thirdly, one crucial reason convinces us of the validity of (3) is a comparison between the situation here with that of the mass transference principle developed by Beresnevich & Velani [2]. We cite their result only in 1-dimensional case. A positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a dimension function if $f(r)$ decreases to 0 as $r \to 0$.

**Theorem 1.1** ([2]). Let $f$ be a dimension function such that $f(r)/r$ is monotonic increasing as $r \to 0$. Assume that for any ball $B \subset \mathbb{R}$,

$$\mathcal{H}^f \left\{ x \in B : |x - x_n| < f(r_n), \text{ i.o. } n \in \mathbb{N} \right\} = \mathcal{H}^f(B).$$

Then for any ball $B \subset \mathbb{R}$

$$\mathcal{H}^f \left\{ x \in B : |x - x_n| < r_n, \text{ i.o. } n \in \mathbb{N} \right\} = \mathcal{H}^f(B).$$

Here $\mathcal{H}^f$ denotes the $f$-Hausdorff measure.

Loosely speaking, the mass transference principle says that, from a full measure limsup set defined by a sequence of balls to the limsup set defined by shrinking the balls to smaller ones, the size of the shrunk limsup set can be determined by just transferring the dimension function.

Now let us turn back to the set $E_y(a)$ we are interested in. Recall that

$$E_y(0) = \bigcap_{\delta > 0} \left\{ x \in X : T^n x \in B(y, \delta), \text{ i.o. } n \in \mathbb{N} \right\}.$$

If the system $(X, T)$ is ergodic with respect to some probability measure $\mu$, $E_y(0)$ is a full measure limsup set. From $E_y(0)$ to $E_y(a)$, we also just shrink the ball $B(y, \delta)$ to the ball $B_{an}(y, \delta)$. So, the situation here is very much similar to that of the mass transference principle. Moreover, the conjectured formula of $h_{\text{top}}(E_y(a), T)$ (3) is very similar to Jarník’s theorem in classic Diophantine approximation and that of Stratmann & Urbaniński [24] for the Diophantine analysis of rational parabolic maps, which now is a consequence of the mass transference principle [1].
All these reasons together compel us to believe that the formula (3) should be valid in a very large scope of systems and we call it the entropy transference principle. So, the task of the current work is to see to which extent it can be true.

1.2. Main result. We will show that the entropy transference principle holds if the system \((X, T)\) satisfies the specification properties. An example is also given to indicate that the specification property cannot be weakened even to almost specification.

**Theorem 1.2.** Let \((X, d)\) be a compact metric space and \(T : X \to X\) a continuous transformation. Assume that \((X, T)\) has the specification property, then for any \(y \in X\),

\[
h_{\text{top}}(E_y(a), T) = \frac{1}{1 + a} h_{\text{top}}(X, T),
\]

for any \(a \geq 0\) where \(h_{\text{top}}\) denotes the topological entropy.

With minor notational modifications in proving Theorem 1.2, we can also get the following a little generalized form. Let \(\{y_n\}_{n \geq 1}\) be a sequence of elements in \(X\) and \(\{a_n\}_{n \geq 1}\) a sequence of positive integers. Define

\[
E_{\{y_n\}}(\{a_n\}) := \bigcap_{\delta > 0} \left\{ x \in X : T^n x \in B_{a_n}(y_n, \delta), \ i.o. \ n \in \mathbb{N} \right\}.
\]

**Theorem 1.3.** Assume that \((X, T)\) has the specification property. Then

\[
h_{\text{top}}(E_{\{y_n\}}(\{a_n\}_{n \geq 1}), T) = \frac{1}{1 + a} h_{\text{top}}(X, T), \quad \text{with} \quad a = \liminf_{n \to \infty} \frac{a_n}{n}.
\]

As proved by Blokh [3], specification property (see Definition 2.1) is equivalent with the topological mixing for continuous interval transformations, so it is satisfied for a large scale of systems. We list some (see, for example, [23]) for which our results can be applied directly.

- The full shift and the topological mixing subshifts of finite type.
- The continuous topologically mixing transformations of the interval.
- If \(X\) is the \(n\)-dimensional torus and \(T\) an automorphism of \(X\) induced by a matrix from \(\text{SL}(n, \mathbb{Z})\) whose eigenvalues are off the unit circle.

Admittedly specification is a fairly strong condition. One would like to enlarge the framework for which the main theorems are still valid. But the following example shows that, as far as the general form in Theorem 1.3 is concerned, one cannot go so far. The example comes from the symbolic dynamics of the \(\beta\)-shift \((S_\beta, \sigma)\) which is known to have the almost specification property [19, 26], but no specification property as far as a general \(\beta\) is concerned [21].

**Theorem 1.4.** There exist \(\beta > 1, y \in S_\beta\) and an integer sequence \(\{a_n\}_{n \geq 1}\), such that

\[
h_{\text{top}}(E_y(\{a_n\}_{n \geq 1}), \sigma) < \frac{1}{1 + a} h_{\text{top}}(S_\beta, \sigma), \quad \text{where} \quad a = \liminf_{n \to \infty} \frac{a_n}{n}.
\]

Theorem 1.4 implies that one cannot extend the main theorems in full to systems even with a slightly weaker specification property. This shows to some extent the optimality of the specification assumption and is what we announced in the beginning of this part. This paper is organized as follows. In Section 2, we recall the definitions of the specification property and topological entropy. The next two sections are devoted to the upper bound and the lower bound of \(h_{\text{top}}(E_y(a), T)\) respectively. The proof of Theorem 1.4 will be given in the last section.
2. Preliminaries. In this section, we recall the definitions of the specification property and the topological entropy.

The following two simple facts about the Bowen-metric \( d_n \) will be used frequently:

- \( d_n \) is indeed a metric, so the triangular inequality works;
- \( d_n(x,y) \geq d_m(x,y) \) whenever \( n \geq m \).

2.1. Specification. Here we give the formal definition of the specification property.

**Definition 2.1.** A continuous transformation \( T : X \to X \) satisfies the specification property if for any \( \epsilon > 0 \) there exists an integer \( m = m(\epsilon) \) such that for any finitely many intervals of integers \( I_j = [a_j, b_j] \subset \mathbb{N}, \ j = 1, \cdots, k \), with

\[
\text{dist}(I_i, I_j) \geq m(\epsilon), i \neq j,
\]

and any \( x_1, \cdots, x_k \) in \( X \) there exists a point \( x \in X \) such that

\[
d_{n_j}(T^{a_j}x, x_j) < \epsilon, \quad \text{where } n_j = b_j - a_j, \text{ for every } j = 1, \cdots, k.
\]

In other words, there is a point whose orbit will be \( \epsilon \)-close to that of \( x_j \) among the times in \( I_j \) for \( 1 \leq j \leq k \) or

\[
\bigcap_{j=1}^{k} T^{-a_j}B_{n_j}(x_j, \epsilon) \neq \emptyset.
\]

Loosely speaking, the specification property ensures the existence of a point whose orbit can be \( \epsilon \)-close to that of \( x_1, \cdots, x_k \) at each time stage \( \{I_j : 1 \leq j \leq k\} \) after a time gap of length \( m(\epsilon) \). For a detailed exploration of the specification property, one is referred to the works of Blokh [3], Buzzi [6] and the related references.

2.2. Topological entropy. We cite two definitions of the topological entropy, one for compact sets and the other for general sets.

2.2.1. Bowen’s definition for compact sets.

**Definition 2.2.** Let \( Z \) be a subset of \( X \). Call a subset \( S'(n, \epsilon) \) of \( Z \) an \((n, \epsilon)\)-separated set, if for any two distinct points \( x_1, x_2 \in S'(n, \epsilon) \),

\[
d_n(x_1, x_2) > \epsilon.
\]

**Definition 2.3** (Entropy for compact sets). Let \( Z \) be a compact subset of \( X \). Denote by \( S(n, \epsilon) \) an \((n, \epsilon)\)-separated sets of \( Z \) with maximal cardinality. Then the topological entropy \( h_{\text{top}}(Z, T) \) of \( Z \) is defined as

\[
h_{\text{top}}(Z, T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log |S(n, \epsilon)|}{n} = \sup_{\epsilon > 0} \limsup_{n \to \infty} \frac{\log |S(n, \epsilon)|}{n},
\]

(4)

where \( |S| \) denotes the cardinality of a finite set.

2.2.2. Definition for general sets. The generalization of the topological entropy to non-compact or non-invariant sets also goes back to Bowen [4]. Here we cite the one in [18].

Let \( Z \) be a (compact or noncompact) subset of \( X \). Call a countable collection of balls \( \Gamma = \{B_{n_i}(x_i, \epsilon)\}_{i=1} \) an \((N, \epsilon)\) cover of \( Z \) if \( Z \subset \bigcup_i B_{n_i}(x_i, \epsilon) \) and \( n_i \geq N \) for all \( i \). Let \( s \geq 0 \) and define

\[
\mathcal{M}(Z, s, N, \epsilon) = \inf \left\{ \sum_{B_{n_i}(x_i, \epsilon) \in \Gamma} e^{-n_i s} : \Gamma \text{ is an } (N, \epsilon) \text{ cover of } Z \right\},
\]
where the infimum is taken over all \((N, \epsilon)\) covers of \(Z\). The quantity \(\mathcal{M}(Z, s, N, \epsilon)\) is nondecreasing with \(N\), hence the following limit exists:

\[
\mathcal{M}(Z, s, \epsilon) = \lim_{N \to \infty} \mathcal{M}(Z, s, N, \epsilon).
\]

It is easy to show that there exists a critical value of the parameter \(s\), which we denote by \(h_{\text{top}}(Z, T, \epsilon)\), where \(\mathcal{M}(Z, s, \epsilon)\) jumps from \(+\infty\) to \(0\), i.e.,

\[
h_{\text{top}}(Z, T, \epsilon) = \inf\{s \geq 0 : \mathcal{M}(Z, s, \epsilon) = 0\} = \sup\{s \geq 0 : \mathcal{M}(Z, s, \epsilon) = \infty\}.
\]

Since \(\mathcal{M}(Z, s, \epsilon)\) is increasing with respect to \(\epsilon\), the following limit exists

\[
h_{\text{top}}(Z, T) = \lim_{\epsilon \to 0} h_{\text{top}}(Z, T, \epsilon) = \sup_{\epsilon > 0} h_{\text{top}}(Z, T, \epsilon).
\]

Call \(h_{\text{top}}(Z, T)\) the topological entropy of \(Z\). It is clear that

\[
h_{\text{top}}(Z_1, T) \leq h_{\text{top}}(Z_2, T) \quad \text{if} \quad Z_1 \subset Z_2.
\]

2.3. Entropy distribution principle. The following proposition allows one to estimate the topological entropy of a set from below.

**Proposition 1** (Entropy Distribution Principle [25]). Let \(T : X \to X\) be a continuous transformation. Suppose a set \(Z \subset X\) and a constant \(s \geq 0\) are such that for some \(\epsilon > 0\) one can find a Borel probability measure \(\mu = \mu_\epsilon\) satisfying:

- \(\mu_\epsilon(Z) > 0;\)
- \(\mu_\epsilon(B_n(x, \epsilon)) \leq C(\epsilon)e^{-ns}\) for some absolute constant \(C(\epsilon) > 0\) and every ball \(B_n(x, \epsilon)\) with \(B_n(x, \epsilon) \cap Z \neq \emptyset\).

Then \(h_{\text{top}}(Z, T) \geq s\).

**Proof.** Let \(\Gamma = \{B_n(x_i, \epsilon)\}_i\) be an \((N, \epsilon)\) cover of \(Z\). Without loss of generality, we may assume that \(B_n(x_i, \epsilon) \cap Z \neq \emptyset\) for every \(i\). Then

\[
\sum_i e^{-ns_i} \geq C(\epsilon)^{-1} \sum_i \mu_\epsilon\left(B_n(x_i, \epsilon)\right)
\]

\[
\geq C(\epsilon)^{-1} \mu_\epsilon\left(\bigcup_i B_n(x_i, \epsilon)\right) \geq C(\epsilon)^{-1} \mu_\epsilon(Z).
\]

Therefore, \(\mathcal{M}(Z, s, \epsilon) > 0\) and hence \(h_{\text{top}}(Z, T, \epsilon) \geq s\). \(\square\)

3. Proof of Theorem 1.2: Upper bound. The upper bound is valid for any compact dynamical system.

Fix \(h > h_{\text{top}}(X)\) and let \(\eta = (h - h_{\text{top}}(X))/2\). Then by the definition of \(h_{\text{top}}(X, T)\) (4), for any \(\epsilon > 0\), there exists an integer \(n_0 = n_0(\epsilon, h)\) such that for all \(n \geq n_0\),

\[
\frac{1}{n} \log \sharp S(n, \epsilon) < h - \eta.
\]

By the definition of \(E_{\gamma}(a)\), trivially, for any \(\epsilon > 0\),

\[
E_{\gamma}(a) \subset \bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ x \in X : T^n x \in B_n(y, \epsilon) \right\}.
\]

Now we try to find a cover of \(\{x \in X : T^n x \in B_n(y, \epsilon)\}\) for every \(n\). The maximality of \(S(n, \epsilon)\) implies that \(\{B_n(x_1, \epsilon) : x_1 \in S(n, \epsilon)\}\) is a cover of \(X\). As a consequence,

\[
\left\{ x \in X : T^n x \in B_n(y, \epsilon) \right\} \subset \bigcup_{x_1 \in S(n, \epsilon)} \left\{ x \in B_n(x_1, \epsilon) : T^n x \in B_n(y, \epsilon) \right\}.
\]
For each \( x_1 \in S(n, \epsilon) \), if
\[
\left\{ x \in B_n(x_1, \epsilon) : T^n x \in B_\alpha n(y, \epsilon) \right\} \neq \emptyset,
\]
choose a point \( \tilde{x}_1 \) from it. Then it is easy to see that
\[
\left\{ x \in B_n(x_1, \epsilon) : T^n x \in B_\alpha n(y, \epsilon) \right\} \subset B_{(a+1)n}(\tilde{x}_1, 2\epsilon).
\]
This shows
\[
\left\{ x \in X : T^n x \in B_\alpha n(y, \epsilon) \right\} \subset \bigcup_{x_1 \in S(n, \epsilon)} B_{(a+1)n}(\tilde{x}_1, 2\epsilon).
\]
Hence, for any \( N \geq 1 \), we get an \((N, 2\epsilon)\) cover of \( E_y(a) \) as:
\[
E_y(a) \subset \bigcup_{n=N}^{\infty} \bigcup_{x_1 \in S(n, \epsilon)} B_{(a+1)n}(\tilde{x}_1, 2\epsilon).
\]
Thus, for any \( s \geq h/(a + 1) \),
\[
M(E_y(a), s, 2\epsilon) \leq \liminf_{N \to \infty} \sum_{n \geq N} \sum_{x_1 \in S(n, \epsilon)} e^{-(a+1)ns} \leq \liminf_{N \to \infty} \sum_{n \geq N} e^{n(h-\eta)} e^{-(a+1)ns} = 0.
\]
This implies
\[
h_{top}(E_y(a), T, 2\epsilon) \leq \frac{h}{1 + a}.
\]
Letting \( \epsilon \to 0 \) and then letting \( h \to h_{top}(X, T) \), we obtain that
\[
h_{top}(E_y(a), T) \leq \frac{1}{1 + a} h_{top}(X, T).
\]

4. **Proof of Theorem 1.2: Lower bound.** We will apply the Entropy Distribution Principle (Proposition 1) to estimate the entropy of \( E_y(a) \) from below. For such a purpose, for some \( \epsilon > 0 \), we will construct a subset \( E_{a, \epsilon} \subset E_y(a) \), a probability measure supported on \( E_{a, \epsilon} \) and then check that the conditions in Proposition 1 are fulfilled.

From now on, we fix \( h < h_{top}(X, T) \) and let \( \eta = \frac{1}{2}(h_{top}(X, T) - h) \). Then by (4) on the topological entropy of \( X \), we choose an \( \epsilon_0 > 0 \) such that for all \( \epsilon < \epsilon_0 \),
\[
\limsup_{n \to \infty} \frac{1}{n} \log \# S(n, \epsilon) > h + \eta.
\]
Fix an \( \epsilon < \epsilon_0/5 \).
4.1. Cantor subset $E_{a,\epsilon}$. The idea for the Cantor set construction is more or less similar to those used in [25] and [8]. At first, we fix some notation.

- $m_k = m(\epsilon/2^k)$: the time gap appearing in the specification property.
- $\{n_k\}_{k \geq 1}$: a largely sparse sequence of integers and for all $k \geq 1$,
\[ 2S(n_k, 5\epsilon) \geq \exp \{ (h + \eta)n_k \}, \text{ and } n_k \geq \exp \{ k + m_k \}. \tag{6} \]
- $\ell_k, t_k$: two sequences of integers defined recursively as follows. Choose $\ell_1 \gg n_2 + m_2$ and let $t_1 = (n_1 + m_1)\ell_1$; for $k \geq 2$, choose
\[ \ell_k \geq 2^k \max \{ t_{k-1}, n_{k+1} + 2m_{k+1} \} \tag{7} \]
and let
\[ t_k := (1 + a)t_{k-1} + m_k + (n_k + m_k)\ell_k. \]
- For each $k \geq 1$,
\[ \mathcal{A}_k := \left\{ (x^{(1)}_1, \cdots, x^{(k)}_i) : x^{(k)}_i \in S(n_k, 5\epsilon), 1 \leq i \leq \ell_k \right\}. \]

Note that we choose $\ell_k$ and $t_k$ correspondingly after the choices of the sequences $\{n_k\}$ and $\{m_k\}$, so they can be chosen sufficiently large compared with any given term in $\{n_k\}_{k \geq 1}$ and $\{m_k\}_{k \geq 1}$ for our use. This is what was done in the third item. So, by these choices presented above, we can assume that for each $k \geq 1$,
\[ \frac{n_k}{m_k} + n_k \geq 1 - \eta, \quad \frac{t_{k-1}}{\ell_k} \leq \eta, \quad \frac{n_{k+1} + 2m_{k+1}}{t_k} \leq \eta, \quad \frac{\ell_k n_k}{t_k} \geq 1 - \eta. \tag{8} \]

The subset $E_{a,\epsilon}$ to be constructed is a tree-like Cantor set. Recall that
\[ E_y(a) := \bigcap_{\delta > 0} \left\{ x \in X : T^n x \in B_{an}(y, \delta), \text{ i.o. } n \in \mathbb{N} \right\}. \]

So a general principle in constructing $E_{a,\epsilon}$ is to realize the events
\[ T^n x \in B_{an}(y, \epsilon_n) \tag{9} \]
for infinitely many times with $\epsilon_n \to 0$ as $n \to \infty$. Now we construct the tree-like Cantor set $E_{a,\epsilon}$ level by level inductively.

**LEVEL 1.** From the root, sprout out a collection of branches, which makes up the first level of $E_{a,\epsilon}$. More precisely, we construct a collection of well separated Bowen-balls for which the event (9) is realized for $\epsilon_1 = \epsilon$ at the time $t_1$. The detail is given as follows.

For each vector $x^{(1)} = (x^{(1)}_1, \cdots, x^{(1)}_{\ell_1}) \in A_1$, applying the specification property to $\epsilon/2$, $x^{(1)}$ and $y$, we get a nonempty set:
\[ \left( \bigcap_{i=1}^{\ell_1} T^{-(i-1)(n_1 + m_1)}B_{n_1}(x^{(1)}_i, \frac{\epsilon}{2}) \right) \cap T^{-t_1} \left( B_{n_1}(y, \frac{\epsilon}{2}) \right). \tag{10} \]

Loosely speaking, this is the set of points with the following properties:
- its orbit is close to that of $x^{(1)}_1$ from the time $(i-1)(n_1 + m_1) + 1$ to $(i-1)(n_1 + m_1) + n_1$ and after a time gap $m_1$, is close to that of $x^{(1)}_i$ for all $1 \leq i < \ell_1$;
- then after one more time gap $m_1$, i.e. at the time $t_1$, enters the Bowen ball centered at $y$. 

Here the time gap $m_1$ comes from the specification property.

Choose one and only one point from the above nonempty set and denote it by $z_1(x^{(1)})$. Let

$$
\mathcal{L}_1 = \left\{ z_1 = z_1(x^{(1)}) : x^{(1)} \in \mathcal{A}_1 \right\}.
$$

Then define the first level of the Cantor set as

$$
\mathcal{F}_1 = \left\{ B_{(1+a)t_1}(z_1, \epsilon/2) : z_1 \in \mathcal{L}_1 \right\},
$$

where $\overline{B}$ denotes the closure of a set $B$.

It is not difficult to see that we have the following facts:

**Lemma 4.1.**

- For each $x \in B_{(1+a)t_1}(z_1, \epsilon/2)$ with $z_1 \in \mathcal{L}_1$,

  $$
  T^{t_1}(x) \in B_{at_1}(y, \epsilon).
  $$

- For any two distinct vectors $x^{(1)}$ and $y^{(1)}$ in $\mathcal{A}_1$, their corresponding points $z_1(x^{(1)})$ and $z_1(y^{(1)})$ in $\mathcal{L}_1$ are $(t_1, 4\epsilon)$-separated (i.e. the Bowen-balls in $\mathcal{F}_1$ are well separated). This implies that

  $$
  \sharp \mathcal{L}_1 = \sharp \mathcal{A}_1 = \left( \sharp S(n_1, 5\epsilon) \right)^{\ell_1}.
  $$

**Proof.** By (10) and the range of $x$, we have

$$
\frac{d_{at_1}(T^{t_1}z_1, y)}{2} < \frac{\epsilon}{2}, \quad \text{and} \quad \frac{d_{at_1}(T^{t_1}x, T^{t_1}z_1)}{2} \leq \frac{\epsilon}{2}.
$$

Then the first assertion follows.

Now we check the second one. Assume that $x_i^{(1)} \neq y_i^{(1)}$ for some $1 \leq i \leq \ell_1$. On one hand, since both of them are in $S(n_1, 5\epsilon)$, $x_i^{(1)}$ and $y_i^{(1)}$ are $(n_1, 5\epsilon)$-separated. On the other hand, by the definition of $z_1$, we know that

$$
T^{(i-1)(n_1+m_1)}(z_1(x^{(1)})) \in B_{n_1}(x_i^{(1)}, \frac{\epsilon}{2}), \quad T^{(i-1)(n_1+m_1)}(z_1(y^{(1)})) \in B_{n_1}(y_i^{(1)}, \frac{\epsilon}{2}).
$$

Combining these two facts together, we have

$$
\frac{d_{n_1}(T^{(i-1)(n_1+m_1)}(z_1(x^{(1)})), T^{(i-1)(n_1+m_1)}(z_1(y^{(1)})))}{2} \geq 4\epsilon.
$$

This shows the claim. $\square$

Assume that the $(k-1)$th level of the Cantor set has been well constructed. Then we have the followings in hand: The $(k-1)$th level is of the form:

$$
\mathcal{F}_{k-1} = \left\{ B_{(1+a)t_{k-1}}(z_{k-1}, \epsilon/2^{k-1}) : z_{k-1} \in \mathcal{L}_{k-1} \right\},
$$

where

(i). any two elements in $\mathcal{L}_{k-1}$ are

$$
(t_{k-1}, 4\epsilon - \frac{\epsilon}{2} - \cdots - \frac{\epsilon}{2^{k-2}})-\text{separated},
$$

hence $(t_{k-1}, 3\epsilon)$-separated. Moreover,

$$
\sharp \mathcal{L}_{k-1} = \prod_{j=1}^{k-1} \left( \sharp S(n_j, 5\epsilon) \right)^{\ell_j};
$$

(ii). for each $x \in B_{(1+a)t_{k-1}}(z_{k-1}, \epsilon/2^{k-1})$ with $z_{k-1} \in \mathcal{L}_{k-1}$, one has

$$
T^{t_{k-1}}(x) \in B_{at_{k-1}}(y, \frac{\epsilon}{2^{k-2}}).
$$
Now we are in the position to construct the $k$th level:

**Level $k$.** Each element in $F_{k−1}$ will sprout out a collection of new branches. More precisely, for each element $B ∈ F_{k−1}$, we construct a collection of well separated Bowen-balls inside $B$ for which the event (9) is realized for $ε_n = ϵ/2^{k−1}$ at the time $n = t_k$.

For each $z_{k−1} ∈ L_{k−1}$ and $x^{(k)} = (x_1^{(k)}, \ldots, x_{t_k}^{(k)}) ∈ A_k$, applying the specification property to $ϵ/2^k$, $z_{k−1}$, $x^{(k)}$ and $y$, we get a nonempty set:

\[
\left(B_{(1+a)t_{k−1}}(z_{k−1}, \frac{ε}{2^k}) \right) \cap \left(T^{−(1+a)t_{k−1}−m_k} \bigcap_{i=1}^{t_k} T^{−(i−1)(n_k+m_k)} B_{n_k}(x_i^{(k)}, \frac{ε}{2^k}) \right) \cap T^{−t_k} \left(B_{at_k}(y, \frac{ε}{2^k}) \right).
\]

Loosely speaking, this is the set of points with the following properties:

- at first, its orbit is close to that of $z_{k−1}$ until the time $(1 + a)t_{k−1}$,
- second (after a time gap $m_k$), is close to that of $x_i^{(k)}$ from the time $(1 + a)t_{k−1} + m_k + (i − 1)(n_k + m_k) + 1$ to $(1 + a)t_{k−1} + m_k + (i − 1)(n_k + m_k) + n_k$ and after a time gap $m_k$, is close to that of $x_i^{(k)}$ for all $1 ≤ i < t_k$;
- at last, after one more time gap $m_k$, i.e. at the time $t_k$, enters the Bowen ball centered at $y$.

Here the time gap $m_k$ comes from the specification property.

Choose one and only one point from the above nonempty set and denote it by $z_k(z_{k−1}, x^{(k)})$.

Let $L_k = \{ z_k = z_k(z_{k−1}, x^{(k)}) : z_{k−1} ∈ L_{k−1}, x^{(k)} ∈ A_k \}$.

Then the $k$th level $F_k$ is defined as

\[
F_k = \left\{ B_{(1+a)t_k}(z_k, \frac{ε}{2^k}) : z_k ∈ L_k \right\}.
\]

Note that for each $z_k = z_k(z_{k−1}, x^{(k)})$ in $L_k$ for some $z_{k−1} ∈ L_{k−1}$ and $x^{(k)} ∈ A_k$, by (12) that $d((1+a)t_k−1(z_k, z_{k−1}) < ε/2^k$, we have

\[
B_{(1+a)t_k}(z_k, \frac{ε}{2^k}) ⊂ B_{(1+a)t_k−1}(z_{k−1}, \frac{ε}{2^k−1}).
\]

Thus

\[
\bigcup_{B ∈ F_k} B ⊂ \bigcup_{B ∈ F_{k−1}} B.
\]

Still we have the followings:

**Lemma 4.2.**

- For each $x ∈ B_{(1+a)t_k}(z_k, \frac{ε}{2^k})$ and $z_k ∈ L_k$,

\[
T^{t_k}(x) ∈ B_{at_k}(y, \frac{ε}{2^k−1}).
\]

- For any two pairs $(z_{k−1}, x^{(k)})$ and $(z'_{k−1}, y^{(k)})$ with $z ∈ L_{k−1}$ and $x^{(k)}, y^{(k)} ∈ A_k$, their corresponding points $z_k = z_k(z_{k−1}, x^{(k)})$ and $z'_k = z_k(z'_{k−1}, y^{(k)})$ in $L_k$ are

\[
( t_k, 4ε − \frac{ε}{2} − \cdots − \frac{ε}{2^{k−1}} ) - separated,
\]

hence $(t_k, 3ε)$-separated. This also implies that

\[
\sharp L_k = \sharp L_{k−1} \cdot \sharp A_k = \prod_{j=1}^{k} \left( \sharp S(n_j, 5ε) \right)^{t_j}.
\]
Proof. The assertion (14) can be proved with the same argument as (11):
\[ d_{a,k}(T^{t_k} z_k, y) < \frac{\epsilon}{2^k}, \quad d_{a,k}(T^{t_k} x, T^{t_k} z_k) \leq \frac{\epsilon}{2^k}. \]

We check the second one in detail.

(1) If \( x^{(k)}_i \neq y^{(k)}_i \) for some \( 1 \leq i \leq \ell_k \). With the same argument as in Lemma 4.1, we deduce that \( z_k \) and \( z'_k \) are \((t_k, 4\epsilon)\)-separated.

(2) If \( z_{k-1} \neq z'_{k-1} \). On one hand, by induction, \( z_{k-1} \) and \( z'_{k-1} \) are \((t_{k-1}, 4\epsilon - \epsilon - \cdots - \frac{\epsilon}{2^{k-2}})\)-separated. On the other hand,
\[ z_k \in B_{t_{k-1}}(z_{k-1}, \frac{\epsilon}{2^k}), \quad z'_k \in B_{t_{k-1}}(z'_{k-1}, \frac{\epsilon}{2^k}). \]

As a result,
\[ d_{t_k}(z_k, z'_k) \geq d_{t_{k-1}}(z_k, z'_k) \geq \left( 4\epsilon - \frac{\epsilon}{2} - \cdots - \frac{\epsilon}{2^{k-2}} \right) - 2 \cdot \frac{\epsilon}{2^k}. \]

By (13), the elements in \( \{F_k\}_{k \geq 1} \) are nested. Now we define the desired Cantor set as
\[ E_{a,\epsilon} = \bigcap_{k \geq 1} \bigcup_{B \in F_k} B = \bigcap_{k \geq 1} \bigcup_{z_k \in \mathcal{L}_k} \overline{B(1+a)t_k}(z_k, \epsilon/2^k). \]

By (14), we have that
\[ E_{a,\epsilon} \subset E_{\eta}(a). \]

4.2. Measure distribution. Now we construct a probability measure \( \mu_\epsilon \) supported on \( E_{a,\epsilon} \).

For each \( k \geq 1 \), \( z_k \in \mathcal{L}_k \) and \( \overline{B(1+a)t_k}(z_k, \epsilon/2^k) \in \mathbb{F}_k \), we define
\[ \mu_\epsilon \left( \overline{B(1+a)t_k}(z_k, \epsilon/2^k) \right) = \frac{1}{\sharp \mathcal{L}_k}, \]
i.e., the mass is uniformly distributed among the balls in \( \mathbb{F}_k \). Once the set function \( \mu_\epsilon \) satisfies the Kolomogrov’s consistency condition, it can be uniquely extended into a probability measure supported on \( E_{a,\epsilon} \). For any \( k \geq 1 \), fix an element in \( \mathbb{F}_{k-1} \), say
\[ C := \overline{B(1+a)t_{k-1}}(z_{k-1}, \epsilon/2^{k-1}) \in \mathbb{F}_{k-1}. \]

Consider the elements in \( \mathbb{F}_k \) which are contained in \( C \). By the construction of \( \mathbb{F}_k \), it follows that for all \( z_k = z_k(z_{k-1}, x^{(k)}) \) with \( x^{(k)} \in \mathcal{A}_k \), the corresponding balls
\[ \overline{B(1+a)t_k}(z_k, \epsilon/2^k) \subset C \quad \text{(see (13)).} \]

As a result,
\[ \sum_{B \in \mathbb{F}_k : B \subset C} \mu_\epsilon(B) = \sum_{x^{(k)} \in \mathcal{A}_k} \mu_\epsilon \left( \overline{B(1+a)t_k}(z_k(z_{k-1}, x^{(k)}), \epsilon/2^k) \right) = \sharp \mathcal{A}_k \cdot \frac{1}{\sharp \mathcal{L}_k} = \frac{1}{\sharp \mathcal{L}_{k-1}} = \mu_\epsilon(C). \]

This shows that \( \mu \) satisfies Kolomogrov’s consistency condition.
4.3. Hölder exponent of the measure. We will use the entropy distribution principle (Proposition 1) to conclude the result. So, we estimate the $\mu_\epsilon$-measure of a Bowen-ball $B_n(x, \epsilon)$ to look for the Hölder exponent of $\mu_\epsilon$.

Let $k$ be the integer such that

\[(1 + a)t_k + m_{k+1} \leq n < (1 + a)t_{k+1} + m_{k+2}.
\]

We distinguish two cases according to the range of $n$.

(i). Assume that

\[(1 + a)t_k + m_{k+1} \leq n < t_{k+1} = (1 + a)t_k + m_{k+1} + \ell_{k+1}(n_{k+1} + m_{k+1}).
\]

Let $0 \leq i < \ell_{k+1}$ be the integer such that

\[(1 + a)t_k + m_{k+1} + i(m_{k+1} + n_{k+1}) \leq n < (1 + a)t_k + m_{k+1} + i(1 + 1)(m_{k+1} + n_{k+1}).
\]

Recall that

\[\mathcal{A}_{k+1} = \left\{ (x_1^{(k+1)}, \ldots, x_{k+1}^{(k+1)}) : x_i^{(k+1)} \in S(n_{k+1}, 5\epsilon), 1 \leq i \leq \ell_{k+1} \right\}.
\]

Assume that there exists

\[z_{k+1} = z_{k+1}(z_k, x_1^{(k+1)}, \ldots, x_{k+1}^{(k+1)}) \in L_{k+1}
\]

such that

\[B_{(1+a)t_{k+1}}(z_{k+1}(z_k, x^{(k+1)}), \epsilon/2^{k+1}) \cap B_n(x, \epsilon) \neq \emptyset.
\]

(Otherwise $\mu_\epsilon(B_n(x, \epsilon)) = 0$ since $\mu_\epsilon$ is supported on $F_{k+1}$.)

We claim that for any other $z'_{k+1} = z_{k+1}(z_k', y_1^{(k+1)}, \ldots, y_{\ell_{k+1}}^{(k+1)}) \in L_{k+1}$, if

\[B_{(1+a)t_{k+1}}(z_{k+1}(z_k', y^{(k+1)}), \epsilon/2^{k+1}) \cap B_n(x, \epsilon) \neq \emptyset,
\]

we must have

\[z_k = z_k', \quad x_1^{(k+1)} = y_1^{(k+1)}, \quad \ldots, \quad x_i^{(k+1)} = y_i^{(k+1)}.
\]

On one hand, since $n \leq (1 + a)t_{k+1}$, by (15), we have

\[d_n(z_{k+1}, x) \leq \epsilon + \epsilon/2^{k+1}.
\]

This is also true for $z_{k+1}'$. Thus

\[d_n(z_{k+1}, z_{k+1}') \leq d_n(z_{k+1}, x) + d_n(z_{k+1}', x) < 2\epsilon + \frac{\epsilon}{2^{k+1}}.
\]

(16)

On the other hand, by the definition of $z_{k+1}$ (see (12)) and using $n \geq t_k$, we know that

\[d_n(z_{k+1}(z_k, x^{(k+1)}), z_{k+1}(z_k', y^{(k+1)})) \geq d_{t_k}(z_k, z_k') - 2\epsilon/2^{k+1};
\]

(17)

and using $n \geq (1 + a)t_k + m_{k+1} + i(m_{k+1} + n_{k+1})$, we know that

\[d_n(z_{k+1}(z_k, x^{(k+1)}), z_{k+1}(z_k', y^{(k+1)})) \geq d_{n_{k+1}}(x_j^{(k+1)}, y_j^{(k+1)}) - 2\epsilon/2^{k+1}.
\]

(18)

for all $1 \leq j \leq i$. Recall that different points in $L_k$ are $(t_k, 3\epsilon)$-separated (Lemma 4.2), and different points in $S(n_{k+1}, 5\epsilon)$ are $(n_{k+1}, 5\epsilon)$-separated. Thus comparing (16) with (17) and (18), we arrive at the claim (otherwise we would get a contradiction).
As a result, it follows that
\[
B_n(x, \epsilon) \cap F_{k+1} \subset \bigcup_{y_{k+1}^{(1)}, \ldots y_{k+1}^{(i)} \in S(n_{k+1}, 5\epsilon)} B_{(1+a)t_{k+1}}(z_{k+1}', \frac{\epsilon}{2k+1}),
\]
where \(z_{k+1}' = z_{k+1}(z_k, x_1^{(k+1)}, \ldots, x_i^{(k+1)}, y_i^{(k+1)}, \ldots, y_{\ell_k}^{(k+1)})\).

As a consequence, the measure of \(B_n(x, \epsilon)\) can be estimated as
\[
\mu_{\epsilon}(B_n(x, \epsilon)) \leq \left( S(n_{k+1}, 5\epsilon) \right)^{\ell_{k+1} - i} \frac{1}{\#L_{k+1}} \cdot \left( \frac{1}{\#S(n_{k+1}, 5\epsilon)} \right)^i.
\]

By using the formulae (6) and (8), we have
\[
\log \frac{\#L_k}{\#L_{k+1}} + i \log \frac{\#S(n_{k+1}, 5\epsilon)}{\#S(n_{k+1}, 5\epsilon)} \geq h \cdot \min \left( \frac{\ell_1 n_1 + \cdots + \ell_k n_k + in_{k+1}}{(1 + a)t_k + m_{k+1} + (i+1)(m_{k+1} + n_{k+1})}, \frac{n_{k+1}}{m_{k+1} + n_{k+1}} \right)
\]
\[
\geq \frac{h(1 - \eta)}{1 + a + \eta} : = s.
\]

Thus we have
\[
\mu_{\epsilon}(B_n(x, \epsilon)) \leq e^{-ns}.
\]

(ii). Assume that \(t_{k+1} \leq n < (1+a)t_{k+1} + m_{k+2}\).

In this case, since the elements in \(L_{k+1}\) are \((t_{k+1}, 3\epsilon)\)-separated, the Bowen-ball \(B_n(x, \epsilon)\) can intersect at most one element in \(F_{k+1}\). Thus, we have
\[
\mu_{\epsilon}(B_n(x, \epsilon)) \leq \frac{1}{\#L_{k+1}}.
\]

Similar estimation, by using the formulae (6) and (8) again, gives that
\[
\mu_{\epsilon}(B_n(x, \epsilon)) \leq e^{-ns}.
\]

Finally, by using the entropy distribution principle (Proposition 1), we conclude that
\[
h_{\text{top}}(E_y(a), T) \geq \frac{1}{1 + a} h_{\text{top}}(X, T).
\]

For the lower bound of the topological entropy of \(E_{\{y_n\}}(\{a_n\}_{n \geq 1})\), the argument can be carried out with only minor notational modifications on the proof for the lower bound of \(h_{\text{top}}(E_y(a), T)\).

Recall that
\[
a = \liminf_{n \to \infty} \frac{a_n}{n},
\]
then for any \(\tilde{a} > a\), there exists infinitely many \(n \in \mathbb{N}\) such that
\[
a_n \leq \tilde{a} \cdot n.
\]

Now we follow the lines in the proof of Theorem 1.2 to point out the changes.
Then the dynamics of the $\beta \Sigma$ first, we recall the dynamics of the $\beta 5$.

where, for $n 2468$ CHAO MA, BAOWEI WANG AND JUN WU

with a weaker specification property called almost specification we also conclude that we can check the validity of the conditions in the entropy distribution principle. So, $\beta$ base or an infinite series where $\lfloor \cdot \rfloor$ $\beta$ $\Sigma$

define the metric $D$

Let $D$

Fix a real number $\beta > 1$. The $\beta$-transformation $T_\beta : [0, 1] \to [0, 1]$ is defined as

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \tag{19}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Then every $x \in [0, 1]$ can be expressed as a finite or an infinite series

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots$$

where, for $n \geq 1$, $\varepsilon_n(x, \beta) = \lfloor \beta T_{\beta}^{n-1} x \rfloor$ is called the $n$-th digit of $x$ (with respect to base $\beta$).

Let $D = \{0, 1, \cdots, \lfloor \beta \rfloor \}$ and define

$$\Sigma_\beta = \{ \varepsilon \in D^\mathbb{N} : \varepsilon \text{ is the digit sequence of the } \beta\text{-expansion of some } x \in [0, 1] \}.$$ 

Define the metric $d$ on $D^\mathbb{N}$ as

$$d(w, w') = \beta^{- \min\{n \geq 0 : w_{n+1} \neq w'_{n+1} \}}. \tag{20}$$

Then the dynamics of the $\beta$-shift is defined as $(S_\beta, \sigma)$, where $S_\beta$ is the closure of $\Sigma_\beta$ and $\sigma$ is the shift map.
The β-shift is totally determined by the β-expansion of 1. Let us now turn to the infinite β-expansion of 1. Apply the algorithm (19) to the number \( x = 1 \) to get its expansion

\[
1 = \frac{\epsilon_1(1, \beta)}{\beta} + \frac{\epsilon_2(1, \beta)}{\beta^2} + \cdots + \frac{\epsilon_n(1, \beta)}{\beta^n} + \cdots .
\]

If the above series is finite, we write its expansion the infinite

\[
\beta
\]

that

\[
\text{the definition of the metric}
\]

where \((\epsilon(w))\) denotes the periodic sequence \((w, w, w, \cdots)\). If the series is infinite, we write

\[
\epsilon^*(1, \beta) := (\epsilon^*_1(\beta), \epsilon^*_2(\beta), \cdots) = (\epsilon_1(1, \beta), \epsilon_2(1, \beta), \epsilon_3(1, \beta), \cdots).
\]

In both cases, we call \(\epsilon^*(1, \beta)\) the infinite expansion of 1.

We collect some basic properties of the β-shift (see, for example, Parry [17]). In the following, \(0^\ast\) means the word consisting of \([t]\) concatenated zeros for \(t > 0\), where \([\cdot]\) denotes the integer part.

**Proposition 2.** (1). Let \(\beta > 1\) be given. A non-negative integer sequence \((\varepsilon_1, \varepsilon_2, \cdots)\) belongs to \(S_\beta\) if and only if, for any \(k \geq 1\),

\[
(\varepsilon_k, \varepsilon_{k+1}, \cdots) \leq_{\text{lex}} (\varepsilon^*_1, \varepsilon^*_2, \cdots),
\]

where \(\leq_{\text{lex}}\) is the lexicographic order.

(2). A non-negative integer sequence \(\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots)\) is the expansion of 1 for some \(\beta > 1\) if and only if, for any \(k \geq 1\),

\[
(\varepsilon_k, \varepsilon_{k+1}, \cdots) \leq_{\text{lex}} (\varepsilon_1, \varepsilon_2, \cdots).
\]

**Proposition 3** ([21]). A beta shift \((S_\beta, \sigma)\) has the specification property if and only if the lengths of the zero strings in the beta-expansion of 1 is finite.

**Proof of Theorem 1.4.** Fix an integer sequence \(\{i_k\}_{k \geq 0}\) with \(i_0 = 0\) and \(i_{k+1} \geq e^{k(i_1+\cdots+i_k)}\) for all \(k \geq 0\). Consider the word

\[
\varepsilon^* = (1, 0^{a_{i_1}+i_1}, 1, 0^{a_{i_2}+i_2}, \cdots, 1, 0^{a_{i_k}+i_k}, \cdots).
\]

Since \(i_{k+1} > i_k\), the word \(\varepsilon^*\) satisfies the condition of the second item in Proposition 2. So, it defines a real number \(\beta\) and the digit sequence of the β-expansion of 1 is \(\varepsilon^*\). Then we get a β-shift \((S_\beta, \sigma)\) which has no specification property by Proposition 3.

Let \(y = \varepsilon^*\) and define the sequence \(\{a_n\}\) as

\[
a_{i_{k-1}+1} = \cdots = a_{i_k} = [1 + (a + 1)i_1] + \cdots + [1 + (a + 1)i_{k-1}] + [1 + ai_k]
\]

for all \(k \geq 1\). Then

\[
\liminf_{n \to \infty} \frac{a_n}{n} = a.
\]

Consider the set

\[
E_\delta(\{a_n\}_{n \geq 1}) = \bigcap_{\delta > 0} \left\{ x \in S_\beta : \sigma^n x \in B_{a_n}(y, \delta), \text{ i.o. } n \in \mathbb{N} \right\}.
\]

Fix \(0 < \eta < 1\) and let \(\ell_n = a_n + (1 - \eta)i_k\) when \(i_{k-1} < n \leq i_k\). Let \(\delta < \beta^{-1}\). By the definition of the metric \(d\) (see (20)), when \(i_{k-1} < n \leq i_k\) (so \(a_n = a_{i_k}\)), one has that

\[
z \in B_{a_n}(y, \delta) \implies (z_1, \cdots, z_{a_n}) = (y_1, \cdots, y_{a_n}) = (1, 0^{a_{i_1}+i_1}, \cdots, 1, 0^{a_{i_k}}).
\]
Moreover by the criterion whether an element is in $S_\beta$ or not (item (1) in Proposition 2), it forces that 
\[(z_1, \cdots, z_{a_n+i_k}) = (1, 0^{ai_1+i_1}, \cdots, 1, 0^{ai_k+i_k}).\]

So, for any $\delta > 0$, when $n$ is large, 
\[z \in B_{\ell_n}(y, \delta).\]

Thus it follows that 
\[E_y(\{a_n\}_{n \geq 1}) \subset \bigcap_{\delta > 0} \left\{ x \in S_\beta : \sigma^n x \in B_{\ell_n}(y, \delta), \text{i.o. } n \in \mathbb{N} \right\}.\]

Consequently 
\[h_{\text{top}}(E_y(\{a_n\}_{n \geq 1}), \sigma) \leq \frac{1}{1+a+1-\eta} h_{\text{top}}(S_\beta, \sigma) < \frac{1}{1+a} h_{\text{top}}(S_\beta, \sigma).\]

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