Microscopic entropy of the charged BTZ black hole

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Abstract

The charged BTZ black hole is characterized by a power-law curvature singularity generated by the electric charge of the hole. The curvature singularity produces \( \ln r \) terms in the asymptotic expansion of the gravitational field and divergent contributions to the boundary terms. We show that these boundary deformations can be generated by the action of the conformal group in two dimensions and that an appropriate renormalization procedure allows for the definition of finite boundary charges. In the semiclassical regime the central charge of the dual CFT turns out to be that calculated by Brown and Henneaux, whereas the charge associated with time translation is given by the renormalized black hole mass. We then show that the Cardy formula reproduces exactly the Bekenstein–Hawking entropy of the charged BTZ black hole.

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1. Introduction

The discovery of the existence of black hole solutions in three spacetime dimensions by Ba\~nados, Teitelboim and Zanelli (BTZ) \cite{1, 2} (for a review see \cite{3}) represents one of the main recent advances for low-dimensional gravity theories. Owing to its simplicity and to the fact that it can be formulated as a Chern–Simons theory, 3D gravity has become paradigmatic for understanding general features of gravity, and in particular its relationship with gauge field theories, in any spacetime dimensions.

The realization of the existence of three-dimensional (3D) black holes not only deepened our understanding of 3D gravity but also became a central key for recent developments in gravity, gauge and string theory.

In this context an important role is played by the notion of asymptotic symmetry. This notion was applied with success some time ago to asymptotically 3D anti-de Sitter (AdS\textsubscript{3}) spacetimes to show that the asymptotic symmetry group (ASG) of AdS\textsubscript{3} is the conformal group...
in two dimensions [4]. This fact represents the first evidence of the existence of an anti-de Sitter/conformal field theory (AdS/CFT) correspondence and was later used by Strominger to explain the Bekenstein–Hawking entropy of the BTZ black hole in terms of the degeneracy of states of the boundary CFT generated by the asymptotic metric deformations [5]. Moreover, the Chern–Simons formulation of 3D gravity allowed us to give a nice physical interpretation of the degrees of freedom whose degeneracy should account for the Bekenstein–Hawking entropy of the BTZ black hole [6–8].

Nowadays, the best-known example of the AdS/CFT correspondence [9, 10] is represented by bulk 3D gravity whose dual is a two-dimensional (2D) conformal field theory (CFT). The BTZ black hole fits nicely in the AdS/CFT framework and can be interpreted as excitation of the AdS3 background, which is dual to thermal excitations of the boundary CFT. The BTZ black hole continues to play a key role in recent investigations aiming to improve our understanding of 3D gravity and of the general feature of the gravitational interaction [11].

A characterizing feature of the BTZ black hole (at least in its uncharged form) is the absence of curvature singularities. The scalar curvature is well behaved (and constant) throughout the whole 3D spacetime. This feature is shared by other low-dimensional examples such as 2D AdS black holes (see, e.g., [12]), for which also the microscopic entropy could be calculated [13, 14] using the method proposed in [5].

The absence of curvature singularities makes the BTZ black hole very different from its higher-dimensional cousins such as the 4D Schwarzschild black hole. The AdS3/CFT2 correspondence has been used with success in several cases to investigate the microscopical behavior of higher-dimensional black holes or extended objects whose near horizon geometry has an AdS3 factor. Well-known examples are the D1–D5 system and wrapped or intersecting branes [15, 16]. Some of these systems involve curvature singularities but they usually require supersymmetry. For different reasons, it would be of interest to consider the pure 3D not supersymmetric case in which the BTZ black hole emerges as solution 3D gravity coupled to matter fields.

On the other hand, one can try to consider low-dimensional black holes with curvature singularities generated by matter sources. But, in general the presence of these sources generates a gravitational field which asymptotically falls off less rapidly than the AdS term producing divergent boundary terms [17].

In this paper we consider the alternative case in which the curvature singularity is not generated by mass sources but by charges of the matter fields. Because matter fields fall off more rapidly than the gravitational field, we expect the divergent boundary contributions to be much milder and removable by an appropriate renormalization procedure. An example of this behavior, which we discuss in detail in this paper, is the electrically charged BTZ black hole. This charged 3D black hole solution is of particular interest also because of its possible relationship with the 2D Maxwell-dilaton gravity theory recently investigated in [18] as an example of the AdS2/CFT1 correspondence.

The charged BTZ black hole we consider in this paper is characterized by a power-law curvature singularity generated by the electric charge of the hole. The curvature singularity generates ln r terms in the asymptotic expansion of the gravitational field, which give divergent contributions to the boundary terms. We will show that these boundary deformations can be generated by the action of the conformal group and that an appropriate renormalization procedure allows for the definition of finite boundary charges. The central charge of the dual CFT turns out to be the same as that calculated in [4], whereas the charge associated with time translation is given by the renormalized mass. We then show that the Cardy formula reproduces exactly the Bekenstein–Hawking entropy of the charged BTZ black hole.
2. The charged BTZ black hole

The BTZ black hole solutions [1, 2] in \((2 + 1)\)-spacetime dimensions are derived from a three-dimensional theory of gravity

\[ I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R + 2\Lambda) \]  

where \(G\) is the 3D Newton constant and \(\Lambda = \frac{1}{\ell^2} > 0\) is the cosmological constant. We use units where \(G\) and \(\ell\) have both the dimension of a length\(^3\). The corresponding line element in Schwarzschild coordinates is

\[ ds^2 = -f(r) \, dt^2 + f^{-1} \, dr^2 + r^2 \left( d\theta - \frac{4GJ}{r^2} \, dt \right)^2, \]  

with the metric function

\[ f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}. \]  

where \(M\) is the Arnowitt–Deser–Misner (ADM) mass, \(J\) is the angular momentum (spin) of the BTZ black hole and \(-\infty < t < +\infty, 0 \leq r < +\infty, 0 \leq \theta < 2\pi\). The outer and inner horizons, i.e. \(r_+\) (henceforth simply black hole horizon) and \(r_-\) respectively, concerning the positive mass black hole spectrum with spin \((J \neq 0)\) of the line element, (2) are given by

\[ r_\pm^2 = 4G\ell^2 \left( M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}} \right). \]  

In addition to the BTZ solutions described above, it was also shown in [1, 19] that charged black hole solutions similar to (2) exist. These are solutions following from the action [19, 20]

\[ I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R + 2\Lambda - 4\pi G F_{\mu\nu} F^{\mu\nu} \right). \]  

The Einstein equations are given by

\[ G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}, \]  

where \(T_{\mu\nu}\) is the energy–momentum tensor of the electromagnetic (EM) field:

\[ T_{\mu\nu} = F_{\rho\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F^2. \]  

The electrically charged black hole solutions of the equations (6) take the form (2), but with

\[ f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} - 8\pi G Q^2 \ln \left( \frac{r}{l} \right), \]  

whereas the \(U(1)\) Maxwell field is given by

\[ F_{\mu\nu} = \frac{Q}{r}, \]  

where \(Q\) is the electric charge. Although these solutions for \(r \to \infty\) are asymptotically AdS, they have a power-law curvature singularity at \(r = 0\), where \(R \sim \frac{8\pi G Q^2}{r^2}\). This \(r \to 0\) behavior of the charged BTZ black hole has to be compared with that of the uncharged one, for which \(r = 0\) represents just a singularity of the causal structure. For \(r > \ell\), the charged black hole is described by the Penrose diagram as usual [21].

\[ ^3 \text{Note that often in the literature units are chosen such that} \ G \text{ is dimensionless,} \ 8G = 1. \]
In the present paper we will consider for simplicity only the non-rotating case (i.e. we will set $J = 0$); however our results can easily be extended to the charged, rotating BTZ black hole.

In the $J = 0$ case the black hole has two, one or no horizons, depending on whether
\[ \Delta = 8GM - 4\pi G Q^2 [1 - 2 \ln(2Q \sqrt{\pi G})] \] (10)
is greater than, equal to or less than zero, respectively. The Hawking temperature $T_H$ of the black hole horizon is
\[ T_H = \frac{r_+}{2\pi l^2} - \frac{2GQ^2}{r_+}. \] (11)
According to the Bekenstein–Hawking formula, the thermodynamic entropy of a black hole is proportional to the area $A$ of the outer event horizon,
\[ S = \frac{\pi r_+}{G} \sqrt{2GM + 2\pi GQ^2 \ln \frac{r_+}{l}}. \] (12)

3. Asymptotic symmetries

It is a well-known fact that the asymptotic symmetry group (ASG) of AdS$_3$, i.e. the group that leaves invariant the asymptotic form of the metric, is the conformal group in two spacetime dimensions [4]. This fact supports the AdS$_3$/CFT$_2$ correspondence [9, 10] and has been used to calculate the microscopical entropy of the BTZ black hole [5]. In order to determine the ASG one has first to fix boundary conditions for the fields at $r = \infty$, then to find the Killing vector fields preserving them are
\[ \chi_t = l(\epsilon^+ (x^+) + \epsilon^- (x^-)) + \frac{l^3}{2r^2} (\partial_x^2 \epsilon^+ + \partial_x^2 \epsilon^-) + O \left( \frac{1}{r^3} \right), \]
\[ \chi^\theta = \epsilon^+ (x^+) - \epsilon^- (x^-) - \frac{l^2}{2r^2} (\partial_x^2 \epsilon^+ - \partial_x^2 \epsilon^-) + O \left( \frac{1}{r^3} \right), \]
\[ \chi^r = - r (\partial_x^+ \epsilon^+ + \partial_x^- \epsilon^-) + O \left( \frac{1}{r^2} \right). \] (14)
where $\epsilon^+ (x^+)$ and $\epsilon^- (x^-)$ are arbitrary functions of the light-cone coordinates $x^+ = (t/l) \pm \theta$ and $\partial_x^\pm = \partial/\partial x^\pm$. The generators $L_n (\bar{L}_n)$ of the diffeomorphisms with $\epsilon^+ \neq 0 (\epsilon^- \neq 0)$ obey the Virasoro algebra
\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n0}, \]
\[ [\bar{L}_m, \bar{L}_n] = (m - n) \bar{L}_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n0}, \]
\[ [L_m, \bar{L}_n] = 0. \] (15)
where $c$ is the central charge. In the semiclassical regime $c \gg 1$, an explicit computation of $c$ gives [4]

$$c = \frac{3l}{2G}. \quad (16)$$

The previous construction, in principle, should work for every 3D geometry which is asymptotically AdS. However, it is not difficult to realize that it works well only for the uncharged BTZ black hole (2). In its implementation to the charged case one runs into two main problems. First, the boundary conditions (13) do not allow for the term in equation (8) describing boundary deformations behaving as $\ln r$. One could relax the boundary conditions by allowing for such terms, but this will produce divergent boundary terms. Second, if the black hole is charged we must also provide boundary conditions at $r \to \infty$ for the electromagnetic field. In view of equation (9), simple-minded boundary conditions would require

$$F_{tr} = O\left(\frac{1}{r}\right), \quad F_{t\theta} = O\left(\frac{1}{r^2}\right), \quad F_{r\theta} = O\left(\frac{1}{r}\right).$$

(19)

Note that we are using very weak boundary conditions for the EM field. We allow for deformations of the EM field which are of the same order of the classical background solution (9). Although the boundary conditions are left invariant under the action of the ASG the classical solution (9) is not. Thus, we use a broader notion of asymptotic symmetry, in which
the classical background solution for matter fields (but not that for the gravitational field) may change under the action of the ASG. This broader notion of ASG remains self-consistent because, as we will see in detail in the following section, the contribution of the matter fields to the boundary terms generating the boundary charges vanishes.

4. Boundary charges and statistical entropy

In the previous section we have shown that, choosing suitable boundary conditions, the deformations of the charged BTZ black hole can be generated by the action of the conformal group in 2D. However, the weakening of the boundary conditions with respect to the uncharged case is potentially dangerous because it can result in divergences of the charges associated with the generators of the conformal algebra.

In the case of the uncharged BTZ black hole, these charges can be calculated using a canonical realization of the ASG [4, 22, 23]. Alternatively, one can use a Lagrangian formalism and work out the stress–energy tensor for the boundary CFT [24]. The relevant information we are interested in is represented by the charge \( I_0 = I_0 \) associated with the Virasoro operators \( L_0 \) and \( \bar{L}_0 \) (we consider the spinless case) and by the central charge \( c \) appearing in algebra (15). The information about \( I_0 \) and \( c \) is encoded in the boundary stress–energy tensor \( \Theta_{\pm\pm} \).

Passing to consider the charged BTZ black hole, we have to worry about both the contribution to \( \Theta_{\pm\pm} \) coming from the EM field and about divergent terms originating from the \( \ln r \) terms in equation (17). On general grounds, the contribution of matter fields is expected to fall off for \( r \to \infty \) more rapidly than those coming from the gravitational terms and from the cosmological constant. Thus, as anticipated in the previous section, the EM part of the action gives a vanishing contribution to \( \Theta_{\pm\pm} \). This can explicitly be shown by working out explicitly the surface term \( I^{(EM)}_{\text{bound}} \) one has to add to the action (5) in order to make functional derivatives with respect to the EM potential vector \( A_\mu \) well defined. One has

\[
\delta I^{(EM)}_{\text{bound}} \propto \int d^2 x \sqrt{-g} N_\mu F^{\mu\nu} \delta A_\nu,
\]

where \( N_\mu \) is a unit vector normal to the boundary. Using the boundary conditions (17) and (19) one finds \( \delta I^{(EM)}_{\text{bound}} = O(1/r) \), giving a vanishing contribution when the boundary is pushed to \( r \to \infty \). The same result can be reached considering the Hamiltonian. In this case the variation of the EM part of the Hamiltonian gives the boundary term

\[
\delta H^{(EM)}_{\text{bound}} \propto \int d\theta A_\tau \delta \pi^\tau N^\nu,
\]

where \( \pi^\tau \) denote the conjugate momenta to \( A_\tau \).

Conversely, the \( \ln r \) terms appearing in the asymptotic expansion (17) give divergent contributions to the surface term. This fact has already been noted in [19], where a renormalization procedure was also proposed. One encloses the system in a circle of radius \( r_0 \) and, in the limit \( r \to \infty \), one also takes \( r_0 \to \infty \) keeping the ratio \( r/r_0 = 1 \).
This renormalization procedure can easily be implemented to define a renormalized black hole mass \( M_0(r_0) \), which has to be interpreted as the total energy (electromagnetic and gravitational) inside the circle of radius \( r_0 \). We have just to write the metric function (8) as

\[
M_0(r_0) = M + \pi Q^2 \ln \left( \frac{r_0}{r} \right).
\]

Taking now the limit \( r, r_0 \to \infty \), keeping \( r/r_0 = 1 \), the third term in \( f(r) \) vanishes, leaving just the renormalized mass term. Moreover, because the total energy of the system cannot depend on the value of \( r_0 \) we can take \( r_0 = r_+ \), so that the total energy is just \( M_0(r_+) \), the renormalized mass evaluated on the outer horizon. The same renormalization procedure can easily be implemented for the boundary deformations in equation (17). We just define renormalized deformations

\[
\gamma^{(R)}(R) = \gamma_{\pm \pm} + \Gamma_{\pm \pm} \ln \left( \frac{r_0}{r} \right),
\]

and similarly for \( \gamma_{++, \pm \pm}, \gamma_{++', \pm \pm}, \gamma_{++r}, \gamma_{++r} \). In the limit \( r, r_0 \to \infty \), with \( r/r_0 = 1 \), the ln\( (r/r_0) \) term in equation (25) vanishes and we are left with boundary conditions which have exactly the same form as those for the uncharged BTZ black hole but with the boundary fields \( \gamma_{\mu \nu} \) replaced by the renormalized boundary deformations (24). It follows immediately that the stress–energy tensor for the boundary CFT dual to the charged BTZ black hole is

\[
\Theta_{\pm \pm} = \frac{1}{4G} \gamma^{(R)}_{\pm \pm},
\]

with \( \gamma^{(R)}_{\pm \pm} \) given by equation (24). One can also check that the field equations (6) imply that \( \gamma^{(R)}_{\pm \pm} \) have to be chiral functions of \( x^\pm \), respectively.

The central charge of the 2D CFT can be calculated using the anomalous transformation law for \( \gamma^{(R)}_{\pm \pm} \) under the conformal transformations generated by (18),

\[
\delta_{\epsilon} \gamma^{(R)}_{\pm \pm} = 2(\epsilon^- \partial^- \epsilon^\pm + 2\partial^- \epsilon^\pm) \gamma^{(R)}_{\pm \pm} - l^2 \partial^\pm \epsilon^\pm.
\]

As expected, it turns out that the central charge is given by equation (16). The charge associated with time translations, \( l_0 = l_0 \), can be calculated using equation (26). One obtains

\[
l_0 = \frac{1}{2} l \left( \frac{M_0(r_+)}{r_+} \right),
\]

where \( M_0 \) is the renormalized black hole mass (23).

In the semiclassical regime of large black hole mass, the existence of an AdS\(_3\)/CFT\(_2\) correspondence implies that the number of excitations of the AdS\(_3\) vacuum with mass \( M \) and charge \( Q \) should be counted by the asymptotic growth of the number of states in the CFT [25]:

\[
S = 4\pi \sqrt{\frac{cl_0}{6}}.
\]

Using equations (16), (28) and (23) we get

\[
S = 4\pi l \sqrt{\frac{M_0}{8G}} = \frac{\pi l}{2G} \sqrt{8GM + 8\pi G Q^2 \ln \left( \frac{r_0}{l} \right)},
\]

which matches exactly the Bekenstein–Hawking entropy of the charged BTZ black hole (12). In this paper we have shown that the Bekenstein–Hawking entropy of the charged BTZ black hole can be exactly reproduced by counting states of the CFT generated by deformations
of the AdS$_3$ boundary. The difficulties related to the presence of a curvature singularity have been circumvented by using a renormalization procedure. Our result shows that the notion of asymptotic symmetry and related machinery can be successfully used to give a microscopic meaning to the thermodynamical entropy of black holes also in the presence of curvature singularities. In particular, this result could be very important for the generalization to the higher-dimensional case of low-dimensional gravity methods for calculating the statistical entropy of black holes.

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