String theory phenomenology and quantum many–body systems

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The main idea in the present work is the definition of an experimental proposal for the detection of the number of extra–compact dimensions contained as a core feature in String Theory. This goal will be achieved as a consequence of the fact that the density of states of a bosonic gas does depend upon the number and geometry of the involved space–like dimensions. In particular our idea concerns the detection of the discontinuity of the specific heat at the condensation temperature as a function of the number of particles present in the gas. It will be shown that the corresponding function between these two variables defines a segment of a straight line whose slope depends upon the number of extra–compact dimensions. Resorting to some experiments in the detection of the specific heat of a rubidium condensate the feasibility of this proposal using this kind of atom is also analyzed.

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INTRODUCTION

Nowadays it is widely accepted that modern physics has an unsolved problem, i.e., the so–called quantum gravity puzzle, namely, there is no quantum version of gravity [1]. This issue has been addressed for more than seven decades [2]. There is an element in this context that has to be mentioned, namely, the path in this direction has no guidance, at all, from any kind of experiments, a fact stemming from the disparity between our current technology and the order of magnitude of the effects predicted by the corresponding models [3]. Of course, this situation cannot be considered an advantage in this quest.

The number of ideas in the pursuit of a quantum theory of gravity grows from day to day and includes String Theory, Loop Quantum Gravity, Non–Commutative Spacetimes, etc. [1]. In the context of String Theory the predicted new effects lie completely outside the testable zone for our technology, a situation consequence of the fact that their order of magnitude is determined by the Planck scale, an apparently very pessimistic scenario. At this point it has to be commented that the proposed experiments involve always a single quantum system, i.e., a single quantum particle suffers the effects of the new features predicted by the involved model [4].

The topic of ultra–cold quantum gases is currently one of the hottest spots in modern physics [5] and the work in this realm embodies many possibilities, they range from superfluidity, low–dimensional systems, etc. The question concerning its possibilities in quantum gravity phenomenology is an issue that has to be addresses thoroughly and its limitations and ensuing options have to be understood. Here we deal with one of the possible windows that ultra–cold systems could offer us in connection with gravitation. In the present work we address the issue of the number of extra compact dimensions from the perspective of a many–body system at ultra–cold temperatures. In other words, we do not seek the effects upon a single quantum particle of some new theoretical element contained in String Theory. The main idea is to exploit some features contained in the physics of ultra–cold systems which depend strongly upon the number of space–like dimensions. More precisely, some thermodynamical quantities, for instance, specific heat, energy, do show a clear dependence upon the number of spacelike dimensions of the corresponding model [6].

The system to be analyzed will be a gas containing \( N \) bosonic particles, which interact pairwise in repulsive way. The corresponding properties of this model will be studied in the region of ultra–cold temperatures, i.e., about the corresponding condensation temperature. An anisotropic harmonic oscillator will be present, as is usual in Bose–Einstein Condensation experiments (BEC) [7], and the ensuing thermodynamics will be deduced assuming that \( l \) compact dimensions are also present, the order of magnitude of these dimensions will be considered equal or smaller than Planck’s length \( l_P \) [8]. This element is one of the core features of String Theory [9]. The effects of the trap upon the compact dimensions can be neglected. Indeed, for a gas, with atoms of mass \( m \) and frequency of the trap given by \( \omega \), the characteristic length of this system is \( \sqrt{\hbar/(m\omega)} \), since, by hypothesis, \( l_P \ll \sqrt{\hbar/(m\omega)} \), then in any of the compact dimensions the presence of the trap reduces to a constant potential \( V_0 \). In other words, concerning the extra part of the geometry, the particles behave as a particles in a box in which a constant potential is present, this last comment implies that the density of states (as function of the energy) of this situation is the same as in the case in which \( V_0 = 0 \) [10]. The energy of this system will be calculated and the ensuing specific heat will be deduced. As is known, even for an ideal bosonic gas in a box, the specific heat shows a discontinuity at the critical temper-
ature, such that this jump is a function of the number of spacelike dimensions \[.\] This last trait will be exploited in the present work. Indeed, we will deduce the discontinuity of our bosonic gas as function of the number of atoms present in the gas, assuming \( l \) compact dimensions and \( s \) non-compact ones. It will be shown that the functional dependence between discontinuity and number of particles defines a segment of a straight line whose slope is a function of \( s \) and \( l \).

More precisely, our experimental proposal is the following one: Take a bosonic gas with \( N \) particles and measure the specific heat above and below the condensation temperature and calculate the corresponding discontinuity. Repeat this procedure for different values of \( N \). The resulting graph will be the segment of a straight line whose slope will contain information of the number of extra compact dimensions, of course, also of the non-compact ones. A realistic proposal shall include the pairwise interaction present in any dilute bosonic gas \[.\] In order to have a clear interpretation of some thermodynamic parameters, for instance, the chemical potential at or below the condensation temperature, the analysis will be done assuming that the pairwise interaction, here codified in the scattering length \( a \), can be handled in the context of the variational approach \[.\]. This condition leads us to an inequality relating \( N, a \) and \( \sqrt{\hbar/(m\omega)} \), i.e., \( Na < \sqrt{\hbar/(m\omega)} \). The present status in the experimental part tells us that most of the situations satisfy the opposite condition \((Na > \sqrt{\hbar/(m\omega)})\), called the Thomas–Fermi limit \[.\], nevertheless, there are experiments carried out within the validity regime of the variational approach \[.\], i.e., our mathematical restriction does not imply an unrealistic proposal.

**EXTRA–DIMENSIONS AND DENSITY OF STATES FOR A BOSONIC GAS**

As mentioned before our model is defined by \( s \) non-compact dimensions and \( l \) compact ones such that the latter have a size denoted by \( R_j \), with \( j = 1, 2, \ldots, l \), and all of them have an order of magnitude similar to Planck length, i.e., \( R_j \sim L_p \); this is one of the main assumptions of String Theory \[.\].

In this geometry we consider bosonic particles with mass \( m \) trapped by a harmonic potential that has non-vanishing frequencies (here denoted by \( \omega_i \), \( i = 1, 2, \ldots, s \)) only in the non-compact space-like dimensions, in other words, the trap has no effects upon the compact part of the geometry. The energy of a particle in this situation will be determined by its contribution from the non-compact dimensions plus the part of the compact ones. Clearly, we have

\[
\epsilon = \hbar[\omega_1(n_1 + 1/2) + \ldots + \omega_s(n_s + 1/2)] + \frac{\pi^2\hbar^2}{2m}[\frac{q_1^2}{R_1^2} + \ldots + \frac{q_s^2}{R_s^2}],
\]

(1)

In this last expression \( n_i \) and \( q_j \) denote the quantum numbers associated to the harmonic oscillator potential and the case of a particle immersed in a box of size \( R_j \), respectively. In other words, the energy of a particle is the sum of the contributions stemming from the harmonic trap and that emerging from the fact that in the extra–dimensions the particle shows the behavior of a free particle \[.\].

A main element is the density of states of a single particle immersed in the aforementioned geometry, with it we may deduce the thermodynamics of our system \[.\]. Let us address the issue of the number of states, here denoted by \( G(\epsilon) \), whose energy is equal or smaller than a certain value \( \epsilon \). The answer to this question is, in the semiclassical approximation \[.\], proportional to the volume of a figure with two independent contributions, namely, one stemming from an hyper–ellipsoid in \( l \) dimensions and the usual case for a harmonic trap in \( s \) non-compact dimensions \[.\]. The analysis of the compact dimensions is equal to the case of a gas immersed in an \( l \)-dimensional box, and the deduction of the corresponding number of states appears in any text on Statistical Mechanics \[.\].

The integral to be calculated reads

\[
G(\epsilon) = \frac{1}{2^l} \int_0^{\epsilon_1} dn_1 \int_0^{\epsilon_2} dn_2 \ldots \int_0^{\epsilon_s} dn_s \int_0^{\hat{Q}} Q^{l-1} dr.
\]

(2)

Here we have that \( \hat{Q} = \frac{2m\hbar^2}{\pi} \epsilon \), the radius (in the \( q \)-space) of the hyper–sphere, see below the definition of this hyper–sphere in terms of the hyper–ellipsoid. The term \( 1/2^l \) stems from the fact that concerning the compact dimensions only that part of the geometry (in which all the quantum numbers \( q_j \) are non–negative) is to be taken into account \[.\]. In addition, we have resorted to hyper–spherical coordinates in which the volume of the hyper–ellipsoid equals that of a hyper–sphere whose radius is the geometric average of the semi–axis of the hyper–ellipsoid, namely,

\[
\hat{R} = [R_1 R_2 \ldots R_l]^{1/l}
\]

(3)

Furthermore, the term \( S_l \) appears as a consequence of the integration in hyper–spherical coordinates of the angle parameters \[.\]

\[
S_l = \frac{2\pi^{l/2}}{\Gamma(l/2)}
\]

(4)
Here $\Gamma$ denotes Euler gamma function. Clearly, as is known for the case of an ideal gas in an $l$-dimensional box \cite{14}, $Q^2$ defines in the space of the quantum numbers $q_j$ the radius of our hyper-sphere. For our case, the functional dependence of $R$ with $\epsilon$ and the involved quantum numbers is provided by

$$q_1^2 + \ldots + q_l^2 = \frac{2mR^2}{\pi^2\hbar^2}(\epsilon - \hbar[\omega_1n_1 + \ldots + \omega_sn_s])$$  \hspace{1cm} (5)

Finally, we have that

$$c_s = -\frac{1}{\hbar\omega_s}(\epsilon - \hbar[\omega_1n_1 + \ldots + \omega_{(s-1)}n_{(s-1)}])$$  \hspace{1cm} (6)

$$c_1 = \frac{\epsilon}{\hbar\omega_1}. \hspace{1cm} (7)$$

With these previous remarks we may now calculate \cite{23} (here $\tilde{\omega}$ denotes the geometric average of the frequencies)

$$G(\epsilon) = \frac{\pi^{l/2}}{2^l\Gamma(l/2 + s + 1)} \left(\frac{2mR^2}{\pi^2\hbar^2}\right)^{l/2} \left(\frac{\epsilon}{\hbar\tilde{\omega}}\right)^s. \hspace{1cm} (8)$$

The density of states is provided by $\Omega(\epsilon) = dG(\epsilon)/d\epsilon$

$$\Omega(\epsilon) = \frac{\pi^{l/2}}{2^l\Gamma(l/2 + s + 1)} \left(\frac{2mR^2}{\pi^2\hbar^2}\right)^{l/2} \frac{\epsilon^{s-1}}{(\hbar\tilde{\omega})^s}. \hspace{1cm} (9)$$

At this point let us consider some limit cases. For instance, if we fix $l = 0$ and $s = 3$ our last expression reduces to

$$\Omega(\epsilon) = \frac{1}{2} \frac{\epsilon^2}{(\hbar\omega_3)^3}. \hspace{1cm} (10)$$

Clearly, we recover an already known result \cite{13}, the density of states of a bosonic gas trapped by an anisotropic harmonic oscillator. On the other hand, if we consider the case of gas trapped in a three-dimensional box, we end up with the case which, in connection with the density of states, is equal to the case of a gas in a three-dimensional compact geometry. In other words, the conditions $l = 3$ and $s = 0$ shall imply the deduction of a gas as trapped in a three-dimensional box. Expression \cite{14} leads us to confirm our conjecture \cite{14}

$$\Omega(\epsilon) = \frac{\pi^{3/2}}{\Gamma(3/2)} \left(\frac{m\tilde{eR}^2}{2\pi^2\hbar^2}\right)^{3/2}. \hspace{1cm} (11)$$

Our result also states that if we start with a three-dimensional BEC (assuming the absence of non-compact dimensions) then the density of states is provided by \cite{10}. If we now restrict the motion of the particles along any of the available axis, then the resulting density of states is provided by setting $s = 2$ and $l = 1$, it is not the case of two non-compact dimensions \cite{10} in which $s = 2$ and $l = 0$. Of course, \cite{19} contains the purely two-dimensional situation, we must fix $s = 2$ and $l = 0$. In other words, restricting the motion along a non-compact dimension is not equivalent to the case of a BEC in which the universe has one less dimension. The existence of restricted motion along any axis entails that the energy eigenvalues are the same as those stemming from a compact dimension with the same size, of course, they impinge upon the density of states, unavoidably.

The number of excited states available to the gas is given by $(z = \exp(\mu\beta))$, here $\mu$ is the chemical potential and $\beta = 1/(kT)$, with $\kappa$ Boltzmann’s constant \cite{14}

$$N_e = \int_0^\infty \frac{\Omega(\epsilon)}{z-1 \exp(\beta\epsilon)} \ d\epsilon. \hspace{1cm} (12)$$

The energy reads

$$E = \int_0^\infty \frac{\epsilon\Omega(\epsilon)}{z-1 \exp(\beta\epsilon)} \ d\epsilon. \hspace{1cm} (13)$$

We may now proceed to calculate the condensation temperature and the energy of the system. As usual, the critical temperature is deduced imposing the condition that the number of particles equals $N_e$ and that the chemical potential at the critical temperature ($\mu_c$) equals the smallest of the energy eigenvalues of our system \cite{13}. In the general case there is no known answer to the question concerning the eigenvalues of those particles in a BEC \cite{17}, but in the variational approach the presence of a non-vanishing pairwise interaction entails only a re-scaling of the frequencies of the trap \cite{11}, the order parameter, after the introduction of a non-vanishing scattering length, continues to have the structure determined by a harmonic oscillator, but now the presence of a repulsive interaction ($a > 0$) entails that it becomes larger. Under this assumption, we have that the chemical potential at the critical temperature, or below it, reads $\mu_c = (\hbar/2)[\tilde{\omega}_x + \tilde{\omega}_y + \tilde{\omega}_z]$, where now these effective frequencies are provided by $(\tilde{L} = (L_xL_yL_z)^{1/3}, \tilde{\omega} = (\tilde{\omega}_x\tilde{\omega}_y\tilde{\omega}_z)^{1/3})$, with $L_j = \sqrt{\frac{\hbar}{m\tilde{\omega}_j}}$

$$\sqrt{\frac{\hbar}{m\tilde{\omega}_j}} = \left(\frac{2}{\pi}\right)^{1/10} \left(\frac{Na}{L}\right)^{1/5} \tilde{\omega}_j \tilde{L}. \hspace{1cm} (14)$$

The variational approach allows us to have an analytical result for $\mu_c$. This last expression is valid only if $Na < L_j$.

We may now deduce the condensation temperature, here $g_\epsilon(x)$ are the so-called Bose functions \cite{14}
\[ N = \frac{\pi^{1/2}}{2^l} \left( \frac{2mkT \tilde{R}^2}{\pi^2 \hbar^2} \right)^{l/2} \left( \frac{\kappa T_c}{\hbar \omega} \right)^s g_\nu(x). \] (15)

In this last expression we have that, concerning the Bose function, \( \nu = s + 1/2 \) and \( x = \frac{\mu_c}{\kappa T_c} \). In order to have an analytical expression for the critical temperature let us recall that our expressions (starting with the deduction of the density of states) are valid in the semi–classical limit \((1/2)\hbar(\omega_z + \omega_x + \omega_y) < \kappa T_c \). This last comment allows us to approximate our Bose function \( g_\nu(x) = \sum_{l=1}^{\infty} \frac{x^l}{l!} \) as follows \((\zeta(x) \text{ denotes the Riemann’s zeta–function [14]}\)   

\[ N = \frac{\pi^{1/2}}{2^l} \left( \frac{2mkT \tilde{R}^2}{\pi^2 \hbar^2} \right)^{l/2} \zeta(s + 1/2) \left( \frac{\kappa T_c}{\hbar \omega} \right)^s \left( 1 + \frac{\zeta(s + 1 + l/2)}{\zeta(s + l/2)} \frac{\mu_c}{\kappa T_c} \right). \] (16)

We now proceed to calculate the energy of the system. Below the critical temperature the chemical potential is a constant [15], therefore, we deduce the energy in two parts.

Firstly, we take \( T > T_c \), then the corresponding integration renders

\[ E = (l/2 + s) \frac{\pi^{1/2}}{2^l} \left( \frac{2mkT \tilde{R}^2}{\pi^2 \hbar^2} \right)^{l/2} \zeta(s + 1 + l/2)(z) \left( \frac{\kappa T_c}{\hbar \omega} \right)^s \kappa T. \] (17)

Secondly, for \( T < T_c \), we conclude

\[ E = (l/2 + s) \frac{\pi^{1/2}}{2^l} \left( \frac{2mkT \tilde{R}^2}{\pi^2 \hbar^2} \right)^{l/2} \zeta(s + 1 + l/2)(z) \left( \frac{\kappa T_c}{\hbar \omega} \right)^s \kappa T \left( 1 + \frac{\zeta(s + 1 + l/2)}{\zeta(s + l/2)} \frac{\mu_c}{\kappa T_c} \right). \] (18)

A fleeting glimpse at these last two expressions allows us to notice that they depend upon the number of non–compact or compact. Indeed, \( \zeta(3) \) is a constant featured in string theory, and \( \zeta(2) \) is the Riemann’s zeta–function at 2.

In the context of BEC it is known that the specific heat shows a discontinuity at the critical temperature [15], a fact present even in the liquefaction of helium, and referred as a \( \lambda \) transition [18].

This last two expressions allow us to deduce the specific heats (one above the critical temperature and, the second one, below it) at constant \( N \) and constant \( \tilde{\omega} \), \( C_\omega = \frac{\partial E}{\partial T} \). Note that \( C_\omega \) is independent of the number of non–compact dimensions, either non–compact or compact.

\[ \Delta C_\omega = \frac{E}{T} \left( (s + 1 + l/2) \right) - \left( (s + l/2) \right) \frac{g_{(s+l/2)}(z)^2}{g_{(s+l/2-1)}(z)g_{(s+l/2+1)}(z)} \right), \] \( T > T_c \). (19)

On the other hand

\[ C_\omega = \frac{\pi^{1/2}}{2^l} \left( \frac{2mkT \tilde{R}^2}{\pi^2 \hbar^2} \right)^{l/2} \left( \frac{\kappa T_c}{\hbar \omega} \right)^s \left( l/2 + s \right) \left( l/2 + s + 1 \right) \left( \zeta(l/2 + s + 1) \kappa T \right) \left( 1 + \frac{\zeta(s + 1/2)}{\zeta(s + l/2 + 1) \kappa T} \right), \] \( T < T_c \). (20)

We recover the situation for the BEC in an anisotropic harmonic trap [15].

The discontinuity is defined as

\[ \Delta C_\omega = \lim_{T \to T_c^+} C_\omega - \lim_{T \to T_c^-} C_\omega \] (21)

In order to simplify our expression we resort to (15) and find

\[ \Delta C_\omega = -N \kappa \left( s + 1 \right)^2 \left[ \frac{\zeta(s + l/2)}{\zeta(s + l/2 - 1)} \right] \frac{\mu_c}{(s + l/2) \kappa T_c}. \] (22)

**DISCUSSION**

This last expression is our main result and it shows an explicit dependence upon the number of non–compact dimensions. For the particular case \( s = 3 \) and \( l = 0 \) we recover the situation for the BEC in an anisotropic harmonic trap [15].

\[ \Delta C_\omega = -9N \kappa \left( \frac{3}{\zeta(3)} - \frac{\mu_c}{3 \kappa T_c} \right). \] (23)

Furthermore, imposing \( s = 0 \) and \( l = 3 \) we recover the case of a gas in a three-dimensional box, in which (for this case \( \mu_c = 0 \) no discontinuity exists [14].

\[ \Delta C_\omega = 0. \] (24)

If we set \( s = 3 \) and \( l = 6 \) (the case of smallest number of extra–dimensions allowed by String Theory [9]), then

\[ \Delta C_\omega = -36N \kappa \left( 0.98 - \frac{\mu_c}{6 \kappa T_c} \right). \] (25)

The experimental realm tell us that [11] (assuming \( N \sim 10^3 \))
\[
\frac{\mu_c}{\kappa T_c} \sim N^{-1/3} \Rightarrow \frac{\mu_c}{6\kappa T_c} \sim 10^{-2}.
\] (26)

These last remarks entail that \(\Delta C_\omega\) is (in the roughest approximation) a linear function of \(N\), and the error in this model is of 1 percent. The magnitude of the error just derived matches the experimental uncertainty mentioned in the literature in connection with the detection of some properties of a BEC of rubidium [19].

The experimental proposal is the following one. Measure the discontinuity in the specific heat of the condensate as a function of the number of particles. The graph has to be the segment of a straight line, whose slope shall contain information about the number of non-compact dimensions. Any deviation from the slope of \(b = -9\zeta(3)/\zeta(2)\) has to be considered as information hinting to the possibility of \(l \neq 0\).

The feasibility of the present idea depends upon several experimental aspects, one of them concerns the fact that the uncertainty in the detection of the specific heat (here denoted by \(\delta C_\omega\)) has to be smaller than \(\Delta C_\omega\), otherwise the sought effect would be hidden within the experimental error.

For \(N = 10^3\) the uncertainty in the detection of the specific heat depends upon the quantity of mass involved and reads

\[
\delta C_\omega = 12.3 \times 10^{-4} \text{J/(gm}^\circ\text{K).}
\] (27)

We take a BEC of rubidium, namely, \(a \sim 10^{-9}\text{m and } \sqrt{\frac{\hbar}{m\omega}} \sim 10^{-3}\text{m} \quad [21, 22].\) Additionally, our approach requires the fulfillment of the condition \(N a < \sqrt{\frac{\hbar}{m\omega}}\), notice that \(N \sim 10^3\) is a viable value, hence the uncertainty related to this situation is

\[
\delta C_\omega = 2.75 \times 10^{-3}\kappa.
\] (28)

For \(s = 3\) and \(l = 6\), and the previous values for rubidium, we have

\[
\Delta C_\omega = -3.5 \times 10^4\kappa.
\] (29)

Clearly, \(|\Delta C_\omega| \gg \delta C_\omega\). Since the experimental uncertainty is much smaller than the effect we conclude that we have a feasible proposal. There is a work in which the specific heat of a rubidium gas has been detected [19], of course, it shows the mentioned discontinuity. Nevertheless, it is noteworthy to add that the value of the uncertainty related to the detection in this parameter is not reported; moreover, the experiment was carried out for just one value of \(N\).

The present model considers the presence of a non-vanishing pairwise interaction, codified in the scattering length \(a \neq 0\), in the context of the validity regime of a variational calculation [11]. This condition implies the fulfillment of

\[
Na < \sqrt{\frac{\hbar}{m\omega}}.
\] (30)

This imposes an upper bound for the number of particles that can be used without violating the requirements of the variational approach. Since our proposal defines \(N\) as the independent variable we seek the largest possible interval for it. This can be done resorting to Feshbach resonances [23]. Indeed, the interaction in a BEC can be manipulated when the total energy of a pair of colliding atoms equals the energy of a quasi–bond state of a molecule, a fact that leads to the resonant formation of the latter case. In other words, it implies a change in the corresponding scattering length. In the very particular case of rubidium [24] this option opens up the possibility of going from \(N = 80\) up to \(N = 10^4\). Clearly, the restriction imposed upon \(N\) by the mathematical conditions defining the perturbative approach (see (30)) has to be fulfilled, nevertheless, we may have a larger interval for \(N\) with the use of Feshbach resonances.

Summing up, we have put forward an experimental proposal, using bosonic particles, which allows us to determine if our universe includes extra non-compact dimensions. This has been obtained resorting to the quantum effects of a many body system such that the discontinuity of the specific heat has to be measured as a function of the number of particles and the sought information will be encoded in the slope of the corresponding graph.

It has to be stressed the difference between the present proposal and the usual phenomenology in found in the literature of String Theory [4], namely, the current experimental proposals focus on a single quantum particle and the effects upon it of the new physics due to the corresponding model. The difference here is that we address the issue of a many–body quantum system and exploit the collective effects that depend upon the number of spacelike dimensions. In this sense, the present model advocates the topic of phenomenology of String Theory resorting to quantum system containing many particles, such as ultra–cold bosonic gases.

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