Two-Phase Flow in Porous Media: Scaling of Steady-State Effective Permeability

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(Dated: April 19, 2012)

PACS numbers: 47.56.+r, 89.75.Da, 47.61.Jd, 47.55.Ca

I. INTRODUCTION

A system consisting of two immiscible fluids flowing simultaneously through a porous medium has provided a complex challenge for both scientists and engineers [1, 2]. An improved understanding of two-phase flow in porous media is of considerable importance for such applications as enhanced oil recovery and groundwater management. Also, attention has recently been directed towards the reduced efficiency of polymer electrolyte membrane (PEM) fuel cells caused by produced water flooding the pore space of the PEM’s gas diffusion and catalyst layers [3].

Simulations of two-phase flow in porous media may be done by modeling the porous medium as a network [4–8] or by simulating fluid dynamics directly on porous media reconstructions from microstructure images [9]. The advantage of the latter approach is a more realistic description of the pore space. Network modeling, on the other hand, has the advantage of allowing a large range of simulation parameters on reasonably large systems, even when the viscous pressure field needs to be re-solved for each timestep.

Most work on two-phase flow in porous media, both numerical and experimental, has focused on displacement processes [10]. Relative permeabilities, used in reservoir modeling, can be measured under such un-steady conditions. Another approach is to maintain constant external control parameters, such as total flow-rate and average saturation, and run the simulation until macroscopic quantities such as relative permeabilities only fluctuate around mean values. These are steady-state values, which in general will be different from the un-steady values. An open question is whether there exists some range of control parameters within which the steady-state values are unique, independent of initial conditions.

Network steady-state simulations have considered various aspects such as the dynamics of disconnected ganglia of oil, the phase diagram in the regime where both phases are mobile, the transition where one of the phases becomes immobilized and diffusion on non-wetting clusters [11–15].

Experimental work on steady-state with simultaneous flow of both phases has been done in two dimensions using chamber-throat networks etched in glass [19–23] and Hele-Shaw cells [24, 25]. A recent experiment performed with three-dimensional bead packings [26] is also of interest to this work. Recently, microfluidic devices have been used as model porous media, achieving unprecedented control over the pore-scale geometry, but so far they have only been used to study drainage [27].

When studying steady-state two parameters are usually kept under control. Flow is then maintained until all measurable quantities only fluctuate around some mean value. One control parameter is the capillary number Ca, a dimensionless quantity which parametrizes the relative strength of viscous and capillary pressure drops. It is proportional to the total flow-rate. The other parameter which controls the steady-state may be either the non-wetting saturation S or the fractional flow of the non-wetting phase F. The pressure drop required to drive flow may be measured, as well as relative permeabilities. The sum of relative permeabilities is the effective permeability \( \kappa_{eff} \). This is typically found to be less than unity under steady-state conditions, which signifies that the mixture of two phases results in a lower permeability than if only a single phase is present.

Tallakstad et al. [24, 25] have reported that the pressure drop \( \Delta P \) driving flow of air and a water-glycerol solution under steady-state conditions in a two-dimensional model porous medium shows a power law dependence with respect to the capillary number Ca,

\[
\Delta P \sim Ca^\beta.
\]

They keep fractional flow fixed at \( F = 7/15 \), due to injecting the two phases through 15 syringes of equal flow-rates, 7 of which inject air. In the following this will be referred to as the Oslo experiment.

Rassi et al. [26] have also reported a power law similar to Eq. 1. Their experiment uses three-dimensional bead packings partially saturated with air under flowing water conditions. The partially saturated state is achieved

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by initial simultaneous flow of both two phases, combining them at a tee junction beneath the bead pack. In the following this experiment will be referred to as the Bozeman experiment.

Using a two-dimensional network simulation this work investigates whether power law behavior similar to Eq. 1 is found when saturation is kept fixed, and whether it extends beyond the limit of large viscosity ratios used in the experiments. A preliminary report has been published in [28].

II. THE MODEL

A. General

The porous medium is modeled as a network of connected capillaries. A capillary corresponds to the throat connecting two pores, within which one or more menisci may be present. With regards to the pressure drop across a meniscus, the throats are modeled as having an hour glass shape, while the flow equations are solved with permeabilities corresponding to cylindrical tubes. This is a matter of convenience, and reflects an interest in the network-scale dynamics rather than a realistic description at the pore scale.

The network considered in this work is a square lattice inclined at 45° relative to the main direction of flow. Boundary conditions are bi-periodic, such that the net-torus inclined at 45° relative to the main direction of flow. Boundary conditions are bi-periodic, such that the net-

A modified version of the Young-Laplace relation is used to obtain the capillary pressure drop

\[ p_c(x) = \frac{2\sigma}{r} \left( 1 - \cos\left(\frac{2\pi x}{\ell}\right) \right), \]  

where \( x \) is the position of the meniscus in the interval \([0, \ell]\) and \( \sigma \) is the surface tension.

Disregarding wetting films and considering flow to be laminar, the Hagen-Poiseille permeability for cylindrical tubes gives the flow-rate \( q \) by the Washburn equation,

\[ q = -\frac{\pi \tau^4}{84 \mu} \left( \Delta p + \sum p_c \right), \]

where \( \mu \) is the viscosity of the phase contained in the throat and \( \Delta p \) is the difference between the pressures of the two nodes connected by the throat. If the viscosities of the two phases are different, the viscosity \( \mu \) of Eq. 3 is replaced by the volume-weighted effective viscosity averaged over the length of the throat. The two phases are considered to be incompressible.

Flow is driven by a pressure drop applied across a cut through the network. The cut is invisible to the local flow field, so it does not cause any boundary effects. A constant volumetric flow-rate \( Q \) is achieved by solving the pressure field for two applied pressure drops \( \Delta P \) and extrapolating linearly to the pressure drop which gives the desired \( Q \) using the relationship

\[ Q = Q_0 + Q_c = -\frac{\kappa_0}{\mu_{\text{eff}}} \frac{\phi A}{L_y} \Delta P + Q_c, \]

where the first term is the single-phase flow-rate \( Q_0 \) corresponding to Darcy’s law and \( Q_c \) is a flow-rate induced by the capillary pressure drops, i.e., it is the flow-rate the system would have if the external pressure drop was set to zero. Under steady-state conditions \( Q_c \) is always in the opposite direction of \( Q \), i.e., the net effect of the capillary pressure drops is to oppose flow. In the Darcy term \( Q_0, L_y \) is the length of the sample in the direction of flow, \( A \) is the cross-sectional area of the sample and \( \phi \) is its porosity. \( \kappa_0 \) is the single-phase absolute permeability of the porous medium and \( \mu_{\text{eff}} \) is the volume-weighted effective viscosity averaged over the entire system.

After solving the pressure field, the simulation is iterated forward in time. The throat with largest flow-rate is identified and the timestep is set by allowing any menisci in this throat to advance 10% of the throat length. Most throats will have flow-rates orders of magnitude slower than this, giving a smooth motion of menisci through the hour glass shaped throats. The 10% rule is chosen by trial and error as being as large as possible without affecting the results of the simulation. Timestepping is done by the Euler method, which is only first order accurate. The numerical error may be estimated by comparing the difference between driving power and heat dissipation, which should average out to zero in the steady-state. It is found that the error is negligible for \( Ca > 10^{-3} \), but grows to a few percent for \( Ca = 10^{-4} \). For \( Ca \) lower than this the Euler method is therefore not reliable.

Since the network is a closed system (mapped onto a torus) the total volume of the two phases is conserved, i.e., any given run of the simulation maintains a constant saturation \( S \) which is determined by the initialization conditions.

When a meniscus reaches the end of a throat, new menisci are created in throats flowing out of the connecting pore. In order to prevent the number of menisci from growing without bounds some scheme is needed to merge nearby pairs of menisci. Menisci within a certain distance from the ends of throats may be merged, mimicking fluid coalescence within the pore volumes. The closest pair of menisci may be merged whenever some maximum allowed number of menisci within a throat is reached. A minimum separation between menisci may also be enforced, merging pairs of menisci when they come too close. Whenever pairs of menisci are merged, this is done in such a way that the merging does not contribute to fractional flow. All these schemes require some coalescence parameter. All have been implemented and tested. It is found that while fractional flow depends on the merging scheme and coalescence parameter, the power law exponent does not. Only the range of validity...
of the power law is affected, as it ceases to be valid when fractional flow becomes either 0 or 1.

For the results given here, pairs of menisci less than 10% of the length of a throat apart are merged.

A more realistic description would need to carefully consider mechanisms for snap-off and coalescence. As a detailed description of flow at the pore scale is not the primary interest of this work, the simpler approach is considered sufficient.

An impression of the model is provided by Fig. 1. Connected strands of large non-wetting saturations can be seen (dark) amidst patches of high wetting saturations (light). Internal flow-rates span several orders of magnitude. Channels of both high and low flow-rates are clearly visible. There are channels within channels, forming a complex pattern. For the choice of parameters and initialization shown in Fig. 1, channels are dynamic. At any given time flow-rates outside fast channels are relatively slow, but when menisci block old channels the blocks are by-passed by new channels, and no spatial region remains stagnant indefinitely.

B. Definitions

The main parameter of the model is the relative strength of capillary and viscous forces, parametrized by the capillary number

\[
Ca = \frac{\mu_{\text{eff}} Q}{\sigma \phi A}.
\]

Larger values of \(Ca\) correspond to faster flow, or lower surface tension.

The viscosity ratio \(M\) is defined as

\[
M = \frac{\mu_{\text{nw}}}{\mu_{\text{w}}},
\]

where subscripts denote the non-wetting and wetting phases. In this work a viscosity ratio \(M = 1\) (viscosity matched fluids) and \(M = 10^{-4}\) (gas/liquid) as in the experiment has been explored.

The effective permeability \(\kappa_{\text{eff}}\) is defined by

\[
\kappa_{\text{eff}} = \frac{Q}{Q_0}.
\]

which gives

\[
Q = \kappa_{\text{eff}} Q_0 = -\frac{\kappa_{\text{eff}} \kappa_0}{\mu_{\text{eff}}} \frac{\phi A}{L_y} \Delta P.
\]

Eq. 8 resembles Darcy’s law, with \(\kappa_{\text{eff}} \kappa_0\) substituted for the absolute permeability. It should be kept in mind that \(\kappa_{\text{eff}}\) depends on \(Q\) and \(S\), so Eq. 8 should not be interpreted as a linear relation between \(Q\) and \(\Delta P\).

A power law dependence between effective permeability and capillary number is expressed as

\[
\kappa_{\text{eff}} \sim Ca^\gamma,
\]

which is meant to be valid within some range for \(Ca\).

Given Eq. 9 and considering that \(Q \sim Ca\) when all other parameters are fixed, the scaling of \(\Delta P\) can be expressed as

\[
\Delta P \sim \kappa_{\text{eff}}^{-1} Q \sim Ca^{1-\gamma},
\]

which together with the definition in Eq. 1 gives the relation \(\gamma + \beta = 1\). Eqs. 1 and 10 are valid when \(Q \sim Ca\), i.e., all parameters in \(Ca\) except \(Q\) are kept constant. This is the case in the Oslo and Bozeman experiments. Eq. 10 relates two dimensionless quantities and is valid regardless of the manner by which \(Ca\) is changed.

C. Initialization and steady-state

The reached state is not necessarily independent of how the initial conditions are chosen – different initial distributions of the two phases may lead to different states. For this reason it is important to compare different initialization schemes.

In order to achieve a desired saturation the throats are initially filled with only one phase, after which the other phase is inserted in randomly chosen throats until the desired saturation is reached. Two schemes for inserting a phase into a throat are considered here. First, the chosen throat may be filled completely. Second, a bubble of some small, fixed volume and consisting of two menisci may be inserted at some random position within the throat. Several bubbles may then exist in a single throat. If they overlap, they are merged. In the first scheme there are no capillary pressure drops initially, as these vanish at the ends of the throats. In the second scheme the total capillary pressure drop across the system averages out to zero, but there are finite, random capillary pressure drops on the scale of the pore throats. These two schemes will be referred to as large bubble and small bubble initialization, respectively.

After the desired saturation is reached, two schemes are considered for the approach to steady-state. First, the simulation may be run with a surface tension which is initially low, and then gradually increased until the desired \(Ca\) is reached. Second, the simulation may be run with constant parameters throughout. This will be referred to as gradual and abrupt initialization, respectively.

After a sufficient number of timesteps the system always settles into a dynamic steady-state, unless a spanning cluster of one of the phases is formed, in which case the spanning cluster carries all flow and the system ceases to fluctuate. This will be referred to as the spanning cluster transition. The dynamic steady-state is characterized by persistent fluctuations around average values with no perceivable drift. It is observed that the range of parameters which lead to a dynamic steady-state coincides with those which gives flow of both phases.
III. RESULTS

A. Two states

It is found that there are two possible states, one state with stagnant clusters and one without.

The stagnant cluster state is reached by large bubble and abrupt initialization, for a narrow range of saturations $S \approx 0.2$. The stagnant clusters are spatial regions with slow flow-rates and slowly changing saturation distributions. The locations of the stagnant clusters are static throughout the simulations. The stagnant clusters form after only a very small number of timesteps and their locations depend on the initial distribution of phases. In this sense, the stagnant cluster state is a state with memory of initial conditions. The meaning of this is discussed in Sec. [IV.B]. The range of saturations which lead to the stagnant cluster state is affected by the choice of merging scheme and coalescence parameter.

Reaching the stagnant cluster state requires particular choices for initialization scheme and parameters. By contrast, the state without stagnant clusters is reached by all initialization schemes and parameters, except those that produce the stagnant cluster state or a spanning cluster. No memory of initial conditions is observed. There are structures: channels of flow and connected strands of the non-wetting phase, but these do not appear to have any particular length-scale. All throats and menisci appear to affect and be affected by the overall flow. This is in contrast with the stagnant cluster state, where the configurations within stagnant clusters appear to be insulated from the overall flow pattern. Because all menisci participate in forming the pattern of flow, the state without stagnant clusters will be referred to as the foam state.

Starting from the stagnant cluster state, the stagnant clusters will mobilize if $Ca$ is increased above some threshold. If $Ca$ is then returned to its original value, the foam state results. The stagnant cluster state can thus transition to the foam state given a perturbation. The opposite transition is never observed. This leads to identifying the stagnant cluster state as metastable. The foam state, on the other hand, is robust with regards to changes in $Ca$ - after a period of increased or decreased $Ca$ the system returns to the same state.
B. Power law exponents

Fig. 2 shows simulation results for \( \kappa_{\text{eff}} \) in the foam state, for a broad range of \( \text{Ca} \) and \( M = 1 \). Three values of \( S \) are shown. Four simulation runs underlie each data point. Each run has independent realizations of the network disorder and initial phase distribution. The simulation is ended when the initial transient is judged to occupy no more than the initial one-third of the run; \( \kappa_{\text{eff}} \) is then averaged over the last third. The variation of the averaged \( \kappa_{\text{eff}} \) between independent runs is quite small. If the variations were plotted as error bars on Fig. 2 they would be smaller than the point representations.

A power law is fitted by eye, with a transition at some upper threshold value for \( \text{Ca} \), around which the system has a smooth but distinct transition. The threshold value is denoted \( \text{Ca}_{\text{up}} \). The fit for \( \kappa_{\text{eff}} \) is given by

\[
\kappa_{\text{eff}} = \begin{cases} 
\left( \frac{\text{Ca}}{\text{Ca}_{\text{up}}} \right)^\gamma & \text{Ca} < \text{Ca}_{\text{up}}, \\
1 & \text{Ca} > \text{Ca}_{\text{up}}.
\end{cases}
\]  
(11)

From Fig. 2 a distinct bending is observed with respect to the fit. A possible cause of the bending may be the numerical error caused by the Euler timestepping method, which becomes more serious at low \( \text{Ca} \). The bending may also signal some cross-over behavior between two regimes with different scaling exponents. Also, if the power law is governed by some critical point at a finite \( \text{Ca} \), omitting this will cause an apparent bending near the critical point. A further discussion of a possible modification of the power law is given in Sec. IV C.

Fig. 2 shows simulation results for \( \kappa_{\text{eff}} \) in the stagnant cluster state. Some bending exists in the limit of very small \( \text{Ca} \). It is in this limit that error bars begin to become substantial. For a range of values of \( \text{Ca} \) exceeding two decades, good power law behavior is observed.

Tab. I reports the best fit values for \( \gamma \) and \( \text{Ca}_{\text{up}} \) for the foam and stagnant cluster states, for \( M = 1 \) and \( M = 10^{-4} \) and various values of \( S \). In the foam state the best fit for the exponent \( \gamma \) depends on \( S \), and varies in the range of \( 0.67 - 0.80 \).

At the large \( \text{Ca} \) limit of the law (\( \text{Ca} \gtrsim \text{Ca}_{\text{up}} \)) the permeability approaches the single-phase permeability of the system, \( \kappa_{\text{eff}} = 1 \). The existence of this viscous transition is not surprising; it indicates the point where capillary pressure drops become perturbations of a dominating field of viscous pressure drops. The transition has been investigated for hysteresis, which has not been found. Its smooth, reversible behavior is reminiscent of a second order phase transition. For \( \text{Ca} > \text{Ca}_{\text{up}} \), \( F = S \) for all values of \( S \). A precise estimation of \( \text{Ca}_{\text{up}} \) has not been attempted. The results presented in Tab. I are those resulting from the fit according to Eq. 11.

The power law also has a limit for low \( \text{Ca} \). When \( \text{Ca} \) is decreased below some threshold \( \text{Ca}_{\text{lo}} \), a spanning cluster forms. At this spanning cluster transition the saturation distribution and flow pattern becomes static. Decreasing \( \text{Ca} \) further after the formation of a spanning cluster does not change \( \kappa_{\text{eff}} \). Independent runs at \( \text{Ca} \) beneath the spanning cluster transition display significant hysteresis, and the concept of a reproducible state is lost. Due to the hysteresis effects, the spanning cluster transition is reminiscent of a first order phase transition. A further discussion of this is given in I.

\( \text{Ca}_{\text{lo}} \) depends on \( S \). For the purpose of this work, the transition is avoided by choosing \( S \) in the range of \( 0.1 - 0.4 \). For larger or smaller values of \( S \) the spanning cluster transition would appear within the range of \( \text{Ca} \) shown in Fig. 2.

No size-dependence has been found for \( \kappa_{\text{eff}} \). In addition to the results for 32x64 systems in Tab. I, identical values have been found for 16x32 systems. Also, sev-
eral representative points have been checked for larger systems, up to 256x512, with indistinguishable results.

IV. DISCUSSION

A. The viscosity ratio

It is somewhat surprising that \( \kappa_{\text{eff}} \) has almost no dependence on the viscosity ratio M. The simulations display a strong dependence on M whenever transients are considered, in agreement with established experimental results. To the author’s knowledge, no experiment has studied a dependence on M under steady-state conditions.

The reason for the lack of sensitivity to M shown by the simulations may be attributed to the lack of large single-phase clusters under steady-state conditions. If the distribution of menisci causes the system to resemble a foam, no flow of one phase is possible without causing the other phase to flow in nearly equal measure. Then, only the volume-weighted effective viscosity \( \mu_{\text{eff}} \) matters; the viscosity ratio M does not.

In the high Ca limit this explanation seems plausible. It is in this limit that the picture of a foam with small variations in the saturation distribution is most appropriate. However, when Ca is reduced the distribution of saturation becomes more heterogeneous, and a large viscosity contrast could be expected to become more important. The absence of an increased M dependence at low Ca indicates that the relative importance of capillary pressure drops relative to viscous pressure drops, which increases with decreasing Ca, out-weighs the increased importance of M.

In other words, the scaling of \( \kappa_{\text{eff}} \) with Ca is dominated by the effects of capillary pressure drops, with only a modest contribution from heterogeneities in the viscosity distribution.

The viscosity ratio has a significant importance in at least one respect. The spanning cluster transition at \( C_{\text{a,lo}} \) is sensitive to M. For M \( \gg 1 \) (more-viscous non-wetting fluid), \( C_{\text{a,lo}} \) is between \( 10^{-2} \) and \( 10^{-3} \) for all values of S. This does not allow for a power law spanning much more than a single decade. For this reason the case of M \( \gg 1 \) is not considered in this work.

B. The metastable state

How can the stagnant cluster state be claimed to be a state, given that it only appears for a particular initialization scheme and the location of stagnant clusters depends on the initial distribution of phases? While the state retains memory of the initial distribution of phases for a long time, possibly indefinitely, the macroscopic averages are consistent from realization to realization. The fluctuations of effective permeability within a single run and the fluctuations of run averages between independent runs are of the same order. This is true both when the individual runs are on identical networks with different initial phase distributions and when the individual runs have different realizations of the network disorder. The variation of macroscopic averages between different simulation runs are all consistent with these fluctuations. In this sense, the macroscopic averages are not sensitive to the initial phase distribution.

In other words, the stagnant cluster state has well-defined properties despite displaying hysteresis. For this reason, it is referred to as a state.

It should be noted that a stagnant cluster does not consist of a single phase. Due to the way the simulations are initialized, both phases will in general exist within the cluster. Therefore, there will be some capillary pressure drops within. These pressure drops do not seem to be organized, they are random and cancel out. This indicates that the perimeter of the stagnant cluster is formed early in the simulation, before the menisci within the cluster organize themselves in response to the flow.

If S is larger than 0.3 the stagnant clusters are always mobilized, eventually resulting in the foam state, even for the case of large bubble and abrupt initialization. This is due to the larger pressure drop required to obtain the desired flow-rate at these saturations. The larger pressure drop prevents stagnant clusters from forming, and the foam state results. For a discussion on the dependence of \( \Delta P \) on S, see [29].

C. The scaling of the foam state

A recent paper [30] explores a power law which is different from Eq. [9]. The simulator used in their paper is similar to the one used here. They explore a power law on the form \( (\Delta P - \Delta P_c) \sim C_{\alpha}^{\beta} \) and find \( \beta \approx 0.5 \). \( \Delta P_c \) is a threshold pressure drop, beneath which the authors of [30] state that no flow occurs. The introduction of this new power law is in part motivated by the wandering value of \( \beta \) reported in [28].

The primary purpose of this work is to consider the power law given by Eq. [9]. It corresponds to the power laws reported by the Oslo and Bozeman experiments. However, the bending observed in Fig. 2 does indicate that a power law of a different form should also be considered.

In order to explore this possibility, more general power law is considered in the following, on the form

\[
(\kappa_{\text{eff}} - \kappa_{\text{eff,ext}}) \sim (C_{\alpha} - C_{\alpha,\text{cr}})^{\gamma_1}, \tag{12}
\]

where \( C_{\alpha,\text{lo}} < C_{\alpha} < C_{\alpha,\text{ upp}} \). \( C_{\alpha,\text{cr}} \) and \( \kappa_{\text{eff,ext}} \equiv \kappa_{\text{eff}}(C_{\alpha,\text{cr}}) \) may now be interpreted as critical values for the control and order parameter, respectively. In the following they will be treated as independent parameters in an optimization problem, aiming to obtain the best possible fit to the power law.

Part of the motivation for considering Eq. [12] is to see whether a dependence of \( C_{\alpha,\text{cr}} \) on S can remove the appar-
ent dependence of the exponent on $S$. If an $S$-dependent
critical value is ignored this can be misinterpreted as an $S$-dependent exponent [31].

In addition to the simulation results reported for the foam state in Tab. I, simulations without disorder are considered here. Fig. 4 shows results from simulations with $M = 1$, $S = 0.2$ and $S = 0.4$, identical to those reported in Fig. 2 except that all throat radii are set equal to $r = 0.2f$. With no disorder, the exponent $\gamma$ increases to 0.82 and 0.85 and less bending is observed. As before, the power laws have been fitted by eye.

The critical values $Ca_{cr}$ and $\kappa_{eff,cr}$ are determined by minimizing the error of the best fit for the power law. Instead of fitting by eye, the best fit is now found using a least squares method, considering only the data points within the interval $10^{-4} < Ca < 10^{-2}$. The error is minimized with respect to variations of the two critical values by means of simulated annealing.

Tab. II reports the results from this procedure. Fig. 5 displays the results and power law fits for the same values of $M$ and $S$ as Fig. 2.

The values found for $Ca_{cr}$ are beneath the range of $Ca$ which are accessible to the simulator at this time. Therefore, the results reported in Tab. II may not be reliable, as they are extrapolated by an assumption of a power law on the form of Eq. 12 and can not be verified independently of this.

For the case of no disorder, the datasets give reasonable fits to a power law without the introduction of critical values. When an attempt is made to estimate the critical values, they are found to be orders of magnitude smaller than in the disordered case. This makes extrapolating their values highly inaccurate, therefore they have simply been set to zero.

The conclusion that can be drawn from this analysis, is that a power law on the form of Eq. 12 can explain the observed bending with respect to Eq. 9. This does not remove the dependence of the exponent on $S$. The estimated values for $Ca_{cr}$ are of the same order of magnitude as $Ca_{lo}$. Hysteresis and numerical inaccuracies prevent the hypothesis $Ca_{lo} = Ca_{cr}$ from being explored further. It has also been found that the exponent is sensitive to disorder, as are the hypothesized critical values. Further work should investigate whether this remains true in three dimensions.

![Figure 4](image1.png)

FIG. 4. (Color online) Scaling of $\kappa_{eff}$ with $Ca$ for saturations 0.2 (triangles) and 0.4 (squares). The porous medium was initialized without disorder. Otherwise the simulations are identical to those in Fig. 2 except that all throat radii are set equal to $r = 0.2f$. With no disorder, the exponent $\gamma$ increases to 0.82 and 0.85 and less bending is observed. As before, the power laws have been fitted by eye.

![Figure 5](image2.png)

FIG. 5. (Color online) Scaling of $\kappa_{eff} - \kappa_{eff,cr}$ with $Ca - Ca_{cr}$ for saturations 0.1 (circles), 0.2 (triangles) and 0.4 (squares). Straight lines are power law fits. $M = 1$.

### TABLE II

| $M$ | $S$ | $\gamma_1$ | $Ca_{cr}$ | $\kappa_{eff,cr}$ |
|-----|-----|------------|-----------|-------------------|
|     |     |            |           |                   |
| **Disorder** |     |            |           |                   |
| 1   | 0.1 | 0.65       | $7.0 \times 10^{-5}$ | $1.3 \times 10^{-2}$ |
| 1   | 0.2 | 0.71       | $5.9 \times 10^{-5}$ | $4.4 \times 10^{-3}$ |
| 1   | 0.3 | 0.75       | $6.1 \times 10^{-5}$ | $3.1 \times 10^{-3}$ |
| 1   | 0.4 | 0.77       | $4.0 \times 10^{-5}$ | $9.8 \times 10^{-4}$ |
| $10^{-4}$ | 0.1 | 0.66       | $7.7 \times 10^{-5}$ | $1.4 \times 10^{-2}$ |
| $10^{-4}$ | 0.2 | 0.71       | $6.1 \times 10^{-5}$ | $5.1 \times 10^{-3}$ |
| $10^{-4}$ | 0.3 | 0.75       | $5.2 \times 10^{-5}$ | $2.4 \times 10^{-3}$ |
| $10^{-4}$ | 0.4 | 0.77       | $3.9 \times 10^{-5}$ | $1.6 \times 10^{-3}$ |
| **No disorder** |     |            |           |                   |
| 1   | 0.2 | 0.81       | 0          | 0.0               |
| 1   | 0.4 | 0.85       | 0          | 0.0               |
D. Comparison with experiments – $\kappa_{\text{eff}}(\text{Ca})$

The Oslo experiment \cite{24, 27} found an exponent $\beta = 0.54 \pm 0.08$, corresponding to $\gamma = 0.46 \pm 0.07$. Their result covers roughly two decades in Ca. Their definition of Ca is different from Eq. \ref{eq:5} by the definition used here their experiments are carried out at Ca roughly between $3 \cdot 10^{-6}$ and $2 \cdot 10^{-4}$.

They use a two-dimensional Hele-Shaw cell: a monolayer of glass beads sandwiched between transparent plates. At the inlet 15 evenly spaced syringes inject the two phases with equal flow-rate through each syringe. The two phases are injected through every other syringe, with 7 syringes injecting air, fixing fractional flow to $F = 7/15$. The other syringes inject a water-glycerol solution which will simply be referred to as water in the following.

Initial conditions consist of a fully water-saturated medium. Initially, pockets of air grow at the injection points. Eventually, the pockets become detached from the injection points and travel through the medium until they escape at the far end. It is observed that smaller pockets are more likely to be trapped. The pockets of air are embedded within a spanning cluster of water. In the steady-state no drift is observed in the average global pressure drop, and the distribution of air pockets appears to be translationally invariant in a statistical sense.

The Oslo experiment explores values of Ca and $S$ which, interpreted in terms of the simulations, places them below the spanning cluster transition. In the simulations only one of the phases flow after Ca is reduced below $C_{\text{a}}$. In the Oslo experiment both phases flow despite the existence of a spanning cluster, due to the simultaneous injection. Also, below the spanning cluster transition, hysteresis is expected. This suggests that the experiment only sees simultaneous flow of air and water due to forcing air to flow by the conditions at the inlets. This will in the following be referred to as forced fractional flow, which should be distinguished from the case of free fractional flow. Free fractional flow is the case in the simulations here when Ca > $C_{\text{a}}$.

There are substantial differences between the Oslo experiment and the simulations reported here. However, the Oslo experiment shares some similarities with the stagnant cluster state. First, the stagnant clusters are characterized by low, nearly vanishing flow-rates surrounded by channels of fast flow, held in place by a perimeter of capillary pressure drops. The trapped pockets of air in the Oslo experiment must be held in place by a similar mechanism. Second, the stagnant clusters are formed by initial conditions with few capillary pressure drops and develop until they reach a size where they are by-passed by flow without being mobilized. The pockets of air in the Oslo experiment are formed at the inlets, where they are allowed to grow until they reach a size where flow of the surrounding water forces them to either mobilize or fragment. In both cases, clusters with a characteristic size are produced.

Could the setup of the Oslo experiment obtain the foam state? Two approaches may be suggested. First, large values of Ca could be explored, possibly obtained by adding surfactants to reduce surface tension. This would also allow an investigation of the transition at Ca$_{\text{up}}$. Second, instead of injecting the two phases at alternating syringes, the two phases could be mixed in tee junctions prior to injection through the syringes. This would prevent the build-up of large air pockets at the inlets, possibly resulting in a different state.

In \cite{24, 27} a scaling argument is presented for $\beta = \gamma = 0.5$. A close relative of this argument is presented in Sec. V, where the comparison between the Oslo experiment and the stagnant cluster state is revisited.

The Bozeman experiment \cite{20} found an exponent $\beta$ in the range of 0.30 – 0.45, corresponding to $\gamma$ in the range of 0.55 – 0.70. Their result covers a little more than a single decade in Ca. Their definition of Ca is similar to the one used here, and is roughly in the range between $10^{-4}$ and $10^{-3}$. Their definition of Ca uses the viscosity of water rather than the volume-weighted average of the two phases, however the water saturation does not change dramatically except for at the largest Ca. The different definitions do not seriously affect the exponents.

They use a three-dimensional bead pack, where water floods the sample at gradually increasing flow-rates after an initial preparation with simultaneous flow of air and water. The initial preparation results in a partially saturated state. As the sample is flooded by water, air escapes until the only pockets of air left within the sample are immobile. Even at the largest flow-rates the pattern of flow is found to be different from that of the fully water saturated sample, meaning that there are always some pockets of air remaining. These pockets are suggested to be quite small, as they do not show up on MR images. Repeated floodings of a single sample gives comparable results, suggesting that the method produces a reproducible state.

The partially saturated state is achieved by a procedure which mixes the two phases before they are injected, as opposed to the Oslo experiment. This could prevent the formation of stagnant clusters. At the end of the initial preparation only water flows. The saturation decreases gradually until the spanning cluster transition is reached, at which all air is trapped. When the flow-rate is increased, some air mobilizes temporarily until the spanning cluster transition is reached again. If the experiment resembles the foam state as it approaches the spanning cluster transition, the partially saturated state will have an effective permeability near that of the foam state at similar $S$ and Ca.

The best estimate for $\beta$ provided in \cite{20} is 0.33. This gives $\gamma \approx 0.67$, which could suggest that their experiment has indeed seen evidence of the foam state. However, there are significant differences between the Bozeman experiment and the simulations. The simulations are two-dimensional and saturation is kept constant under steady-state conditions, as opposed to gradually deceas-
ing while the sample is flooded. In Sec. [VD], it is argued that the stagnant cluster exponent $\gamma \approx 0.5$ should be valid for both two and three-dimensional porous media. This is further indication the the Bozeman experiment does not produce the stagnant cluster state, which leaves the foam state. It is not known whether the foam state exponents change in three dimensions.

E. Comparison with experiments $- S(Ca)$

Both the Oslo and Bozeman experiments have reported decreasing $S$ with increasing $Ca$. In both cases it is found that the low Ca limit is $S \approx 0.5$.

The Bozeman experiment finds that $S$ strongly decreases for $Ca \geq 10^{-3}$. Because water is the only injected phase it is not surprising that $S$ decreases for larger Ca, this agrees with the spanning cluster transition moving to lower $S$ for larger Ca. This also suggests that the Bozeman experiment transitions into the regime of free fractional flow. If the sharp decrease is caused by a transition from the regime of forced to free fractional flow, the sharp decrease can be explained as the disappearance of hysteresis effects.

In the Oslo experiment the decrease of $S$ is harder to explain. As Ca is increased, the characteristic size of trapped air pockets decreases. The number of air pockets must somehow be governed by the mechanisms which govern flow of air. More air pockets should allow more air to flow. The decrease of $S$ with increasing Ca suggests that the number of air pockets needed to maintain a fixed $F$ does not grow fast enough to compensate the reduced size of the air pockets.

At the $Ca_{up}$ transition seen in the simulations, fractional flow curves have a distinct transition from a non-trivial dependence on $S$ and $Ca$, to $F = S$ for all values of $S$ and all values of $Ca > Ca_{up}$. This means that if the Oslo experiment were to increase Ca further, they should at some point observe a reversal in the $S(Ca)$ trend. $S$ should increase to an ultimate value of $S = F = 7/15$ above the $Ca_{up}$ transition. The value of $Ca$ which gives a minimum value of $S$ most likely coincides with the transition from forced to free fractional flow.

V. SCALING ARGUMENT FOR $\beta = \gamma = 1/2$

In [24, 25], an argument is presented for why $\beta = \gamma = 1/2$. It will be referred to as the Oslo argument in the following.

An argument which is closely related to the Oslo argument is given in Secs. [VA] and [VB]. The two arguments lead to the same scaling exponent. Secs. [VC] and [VD] compare the two arguments and discuss their applicability.

A. Postulates

Three postulates are stated. Given that these are justified, $\beta = \gamma = 1/2$ results.

The physical picture is that of two phases somehow distributed through and injected into a two-dimensional porous medium, initially with few capillary pressure drops. Eventually menisci with large capillary pressure drops develop, causing some regions to resist flow. The imposed flow is then carried in regions that remain open to flow. These two regions are the stagnant cluster volume and the dissipative volume, respectively. The stagnant clusters are held in place by a perimeter of capillary pressure drops and are as large as they can be without mobilizing. The dissipative volume consists of a connected network of channels spanning the system.

Postulate 1. The volume $V$ of the sample can be divided into two sub-volumes,

$$V = V_{\text{diss}} + V_{\text{stag}},$$

(13)

where the dissipative volume $V_{\text{diss}}$ satisfies

$$D \approx D_{\text{diss}} = \sum_{\text{diss}} d \sim \sum_{\text{diss}} u |\nabla P| \sim V_{\text{diss}} u^2.$$  

(14)

$D$ is the total dissipation within the sample, $d$ is the dissipation within a throat, the sums run over the throats within the dissipative volume, $u$ is the mean velocity within the dissipative volume and $\nabla P$ is the mean pressure gradient within this volume. Separating $d$ into the product of the mean values of $u$ and $\nabla P$ relies on flow within the dissipative volume respecting Darcy’s law and being reasonably homogeneous, or fluctuations not being correlated.

Postulate 2. Consider the balance of pressure gradients along the stagnant cluster perimeters. Everywhere along the perimeter there is a pressure balance $p_a - p_b \approx p_c$, where $p_a$ and $p_b$ are the pressures on either side of the cluster perimeter and $p_c$ is the pressure drop caused by the meniscus on the perimeter. $p_c$ scales with the surface tension $\sigma$, which gives

$$\nabla P \cdot l^* \propto \sigma,$$  

(15)

where $l^*$ is the length of the path that connects the extreme points of the stagnant cluster, i.e., from the point of highest to the point of lowest surrounding pressure. $l^*$ is a characteristic length scale of the stagnant clusters, and should be comparable with both their mean width $l_x$ and mean length $l_y$.

Postulate 3. It is necessary to relate $l^*$ to $V_{\text{diss}}$. Assume that the mean width of the flowing channels does not scale with Ca. Both experiments and simulations observe that channels are only a single pore-width wide, supporting the assumption. Also, assume that the aspect ratio of the stagnant clusters $l_y/l_x$ does not scale with Ca, such that

$$l^* \approx (l_x + l_y) \sim l_x \sim l_y.$$  

(16)
Smaller Ca gives larger clusters, but the aspect ratio does not necessarily change. In experiments, clusters that mobilize are observed to be elongated along the direction of flow, while the smaller, stagnant clusters have \( l_x \approx l_y \). In the simulations, the stagnant clusters appear slightly elongated in the direction of flow, and larger clusters seem to be more elongated. No Ca dependence is observed.

The dissipation can also be expressed by using Eq. 8

\[
D = \Delta P \cdot Q \sim \frac{Q^2}{\kappa_{\text{eff}}},
\]

from which the scaling is obtained as

\[
\kappa_{\text{eff}} \sim \text{Ca}^{1/2} \Rightarrow \gamma = 1/2.
\]

Other scaling results that can be read out of the preceding equations are

\[
\Delta P \sim \nabla P \sim (l^*)^{-1} \sim \text{Ca}^{1/2} \Rightarrow \beta = 1/2.
\]

In the last scaling laws, dimensional quantities are related to a dimensionless number. It should be understood that this is done for \( Q \sim \text{Ca} \), which is only valid when all other dimensional quantities are kept constant. Eq. 26 is valid regardless.

C. Comparison with the Oslo argument

The Oslo argument is very similar to the argument given here. The assumptions of postulates 1, 2 and 3 are made by both. However, there are three significant differences between the arguments.

First, Eq. 16 of \([23]\) gives some length scale \( l^* \) (not identically defined as the \( l^* \) used here, but similar) as being proportional to the ratio of capillary pressure drop to overall viscous pressure drop. This is similar to postulate 2. The argument then ignores the capillary pressure drop, giving \( l^* \propto \Delta P^{-1} \). However, the capillary pressure drop should not be ignored since the interest is in scaling with Ca, which contains the surface tension. Since the Oslo argument uses \( \text{Ca} \sim Q \) their conclusion is not affected by this, however it reduces the generality of the argument, which is best expressed in terms of dimensionless quantities.

Second, Eqs. 16 and 17 of \([23]\) contain the ratio \( \kappa / \tau \), which is meant to be the pressure gradient surrounding a stagnant cluster. Taking \( L \) as the system length \( L_y \) ignores the tortuosity of flow paths as they wrap around stagnant clusters. This tortuosity could in principle scale with \( \text{Ca} \). However, the assumptions of postulate 3, which the Oslo argument also make, mean that \( \tau \) does not scale with \( \text{Ca} \). This is an important point within the scaling argument and should not be ignored. If \( \tau \) scaled with \( \text{Ca} \), the conclusion \( \Delta P \sim \text{Ca}^{1/2} \) would no longer be valid.

Third, the Oslo argument considers the case where the phase contained in stagnant clusters has a very low viscosity, and presents their argument as a limiting case. The simulations suggest that the viscosity ratio is of little importance. The argument as presented here also does not rely on a particular viscosity ratio. If there is a non-unity viscosity ratio, \( \text{Ca} \) should be defined with the volume-weighted effective viscosity of the dissipative volume rather than the total volume, otherwise the argument is unaffected by \( M \). Any discrepancy between the
simulations and the argument due to different definitions of Ca does not affect the scaling exponents, because the saturation within the dissipative volume does not scale with Ca. The same should be true for the experiment.

D. More on the scaling argument

In the Oslo experiment the largest air pockets migrate while small ones are trapped. The migration of a large air pocket could lead to fragmentation. Small pockets of air could be released into the channels of flowing water, leaving behind a now trapped air pocket. The small, flowing pockets could exit the system, or coalesce with other trapped air pockets, causing them to migrate until they fragment, such that the process repeats itself. This suggests the mechanism behind forced fractional flow: over-sized pockets of air produced at the inlets begin to migrate, then fragment, leading to a cascade of coalescence, migration and fragmentation through the system.

The migrating cluster state is a very different picture from that of the stagnant cluster state, where no migration is seen. There are clearly substantial differences between the simulations and the Oslo experiment. However, the origin of the $\beta = \gamma = 1/2$ exponent appears to be the same in both, namely the applicability of postulates 1, 2 and 3.

The argument is not meant to be exact. In the case of the Oslo experiment, the elongation of the largest air pockets is likely to be connected with the mechanism which allows fractional flow. Because the aspect ratio of stagnant clusters is important to postulate 3 the argument is likely to be modified by an improved understanding of this mechanism. The separation into two volumes, where one contains all dissipation and obeys Darcy’s law and the other is perfectly non-dissipative by a perimeter of capillary pressure drops, is also an approximation. The stagnant clusters are not perfectly non-dissipative in the simulations, and the air pockets in the Oslo experiment are not all completely trapped.

An advantage of simulations over experiments is that simulations have absolute knowledge of the system configuration. This allows tests of the scaling argument to be carried out, by testing its intermediate steps. It was found that $V_{\text{diss}}/V \sim \sqrt{Ca}$ and $u \sim \sqrt{Ca} \sim \sqrt{Q}$. In order to obtain $V_{\text{diss}}$ and $u$ it is necessary to determine a cut-off $d_{\text{tr}}$ for throat dissipations, with throats having $d > d_{\text{tr}}$ belonging to the dissipative volume. $d_{\text{tr}}$ is determined by increasing it to the point where the total dissipation contained above the threshold begins to decrease rapidly.

The values for $u$ and $V_{\text{diss}}$ obtained in this way are plotted for three values of Ca in Fig. 6. They scale as they are supposed to, in support of the scaling argument.

In three dimensions, the perimeter of a stagnant cluster is a two-dimensional surface and the dissipative volume consists of sheets of flow surrounding these perimeter surfaces. Postulates 1 and 2 do not need to be modified by this. Postulate 3 changes to the following: the sheets of flow have a thickness which does not scale with Ca and the stagnant clusters obey the scaling $l^* \sim l_z \sim l_y \sim l_z$. Given these modifications the rest of the scaling argument survives unchanged for three dimensions. The result $\beta = \gamma = 1/2$ is therefore expected to be valid for both two and three-dimensional systems.

An argument for a foam state exponent is not likely to share any similarities with the argument for the stagnant cluster state. None of the three postulates apply to the foam state. There is no separation into two distinct volumes of low and high flow-rates, as can be seen from inspecting histograms of flow-rates from the two states. No regions appear to be stable with respect to either saturation distribution or flow-rates – flowing channels fluctuate across the entire system – which means that a balance criterion similar to postulate 2 is not applicable. The absence of stable structures makes it difficult to search for a characteristic length scale, while the power law distribution of flow-rates suggest that no characteristic length scale exists.

VI. CONCLUSION

Network simulations of two-phase flow in porous media have been used to investigate the scaling of $\kappa_{\text{eff}}$ with Ca under steady-state conditions with fixed saturation. Two values of M were used, corresponding to viscosity matched fluids and gas/liquid. Two states were identified and named: the stagnant cluster state and the foam state.
The metastable stagnant cluster state scales with $\gamma = 0.50$ and $0.54$ for viscosity matched fluids and gas/liquid respectively. An argument in support of $\gamma = 1/2$ has been given. The argument applies to both the stagnant cluster state and the Oslo experiment. More work is needed to explore the scaling in the foam state.

The Oslo experiment has been interpreted in terms of the distinction between forced and free fractional flow. Their low values of Ca and observed dependence between $S$ and Ca is indication that they observe forced fractional flow, which indicates that their system is governed by the boundary conditions at the inlets. The Bozeman experiment, on the other hand, may have observed the foam state.

Upper and lower bounds have been established for the power law. At $Ca < Ca_{lo}$ a spanning cluster transition prevents power law scaling for the case of free fractional flow, though the Oslo experiment indicates that scaling persists under conditions of forced fractional flow. At $Ca > Ca_{up}$ a viscous transition results in $\kappa_{eff} \approx 1$. The spanning cluster transition and viscous transition appear to be first and second order, respectively.

Discussions with E. G. Flekkøy, A. Hansen, K. J. Målay, S. Sinha and K. T. Tallakstad are gratefully acknowledged. H. A. Knudsen is fondly remembered.

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