1. INTRODUCTION

1.1. The goal of this paper is to investigate the algebraic structure of certain quantized algebras of functions associated to affine Kac-Moody Lie algebras and to describe their irreducible representations. Let $C$ be an affine Cartan matrix and $\mathfrak{g} = \mathfrak{g}(C)$ be the associated affine Lie algebra. The main object of our interest - $\mathbb{C}_q[G]$ - is a $*$-subalgebra of the dual space $\text{Hom}_k(U,k)$ generated by matrix coefficients of integrable highest weight $U$-modules, where $*$ is an involutive antiautomorphism. Similar to the finite-type case, $\mathbb{C}_q[G]$ is a triangular-type algebra whose commutativity relations can be computed using the $R$-matrix. We construct the irreducible quotient $N_w$ of an induced $\mathbb{C}_q[G]$-module and our main result contains a description of its annihilator in terms of the Weyl group element $w$. Furthermore, these simple modules satisfy a ‘Tensor-Product Theorem’ which asserts that $N_w \cong N_{i_1} \otimes \cdots \otimes N_{i_k}$, for $w = s_{i_1} \cdots s_{i_k}$ where the $N_i$ are the $\mathbb{C}_q[SL_2]$-modules considered as $\mathbb{C}_q[G]$-modules by restriction to the $i$-th node in the extended Dynkin diagram of $\mathfrak{g}$, which induces a surjective homomorphism from $\mathbb{C}_q[G]$ to $\mathbb{C}_q[SL_2]$. In fact, the modules $N_w \cong N_{w'}$ if $w' \simeq w$ in $W$. Finally, unlike the finite-type case, there is a one-dimensional $\mathbb{C}_q[G]$-module, $N_\infty$, which does not correspond to any Weyl group element. The next 3 subsections describe the historical motivations, classical Kac-Moody group theory and the finite semisimple quantum function algebras, respectively.

1.2. In most approaches to quantum group theory, the basic object, introduced independently by V. Drinfeld [D1, D2] and M. Jimbo [Ji], is the quantized enveloping algebra $U_q(\mathfrak{g})$, which can be viewed as a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The quantized function algebra $\mathbb{C}_q[G]$ is the non-commutative or quantum version of the classical function algebra $\mathbb{C}[G]$. It is a subalgebra of the dual vector space $\text{Hom}_k(U,k)$, where $k = \mathbb{C}(q)$, $q$ an indeterminate. Since the various spectra of the coordinate algebra $\mathbb{C}[G]$ of a classical Lie group $G$ contain all the geometric information about it, it is natural, from the point of view of noncommutative (algebraic) geometry, to study these quantized function algebras. A number of techniques have been developed to work with quantized function algebras of finite dimensional semisimple Lie algebras, and there are...
several interesting results and applications. Two important non-trivial examples are the discovery of their relation with $q$ special functions by L.Vaksman and Y.Soibelman in 1986 and the construction of solutions to the Zamolodchikov tetrahedra equations from their irreducible representations by D.Kazhdan, M.Kapranov, V.Voevodsky and Y.Soibelman in 1992 [K-S, K-V].

However, in the infinite dimensional case, there appears to be very little literature in this direction [J0, ND], in sharp contrast to the wealth of information on the representation theory of $U_q(g(C))$ for a Kac-Moody algebra $g(C)$ associated to a Generalized Cartan Matrix $C$ (see [L1, J1, J2]). The main point, even in the finite type case, is that whereas the irreducible representations of $U(g)$ and $U_q(g)$ are very similar, the representation theories for $C[G]$ and $C_q[G]$ are extremely different [S1]. This difference stems from the fact that the algebra $C[G]$ is commutative, hence its irreducible representations are one dimensional, and correspond to the (closed) points of $G$.

1.3. It is possible [K-P] to define a Lie group $G = G(C)$ whose Lie algebra coincides with the derived subalgebra $g' = g'(C) = [g, g]$. Denote by $B_+, B_-, N_+, N_-, H$ and $K$ the subgroups corresponding to $b_+, b_-, n_+, n_-, h$ and $\mathfrak{k}$ respectively. In [K-P], the algebra of strongly regular functions $C[G]_{s.r}$ is defined as the algebra generated by matrix coefficients of all integrable highest weight modules $L(\Lambda)$ of the Kac-Moody Lie algebra $g(C)$. Their main results are summarized below:

(i) $C[G]_{s.r.}$ is a unique factorization domain.

(ii) Let $P$ be a subgroup of $G(C)$ and $C[G]_{s.r.}^P$ be the algebra of all $f \in C[G]_{s.r.}$ such that $f(gp) = f(g)$ for all $p \in P$. Now, let $\theta_\Lambda$ be the character of $B_+$ defined by the formula $\theta_\Lambda((\exp h)n) = e^{\Lambda(h)}$, for $h \in \mathfrak{h}$, $n \in N_+$ and for any $\Lambda \in P_+$, let

$$S_\Lambda = \{ f \in C[G]_{s.r.} \mid f(gb) = \theta_\Lambda(b)f(g), \forall g \in G, b \in B_+ \}.$$ 

Then,

- (a) (Borel-Weil-type Theorem). The map $L^*(\Lambda) \rightarrow S_\Lambda$ defined by $l \mapsto c_{l,v}^\Lambda$ is a $G$-module isomorphism, where $L^*(\Lambda)$ is the graded dual of $L(\Lambda)$.
- (b) $C[G]_{s.r.}^{P_+} = \oplus_{\Lambda \in P_+} S_\Lambda$ and this algebra is isomorphic to $\oplus_{\Lambda \in P_+} L^*(\Lambda)$ as an algebra with the Cartan product: $L^*(\Lambda) L^*(\Lambda') = L^*(\Lambda + \Lambda')$, $\forall \Lambda, \Lambda' \in P_+$.
- (c) The algebra $C[G]_{s.r.}^{P_+}$ is a unique factorization domain and the coordinate ring of strongly regular functions on $\nu_\Lambda$ is integrally closed, where $\nu_\Lambda$ is the (projective) orbit of the highest weight vector $v_\Lambda$ in $L(\Lambda)$ under the action of $G$.

(iii) $\text{Specmax } C[G]_{s.r.} \setminus G$ is non-empty if $C$ is of infinite type.

(iv) $C[G]_{s.r.}$ fails to be a Hopf algebra, since it is neither closed under the comultiplication nor antipodal maps.
Remarks. (i) By using only the $U(g)$ bimodule structure on $U(g)^*$, M. Kashiwara [Kas1] defines a subalgebra $\mathbb{C}[G]$ of $U(g)^*$ satisfying certain finiteness conditions which is isomorphic to $\mathbb{C}[G]_{s.r.}$. The ‘$q$’ analogue of his definitions are given in Proposition 3.1.

(ii) C. Mokler [Mo] has established that $\mathbb{C}[G]_{s.r.}$ is really the coordinate ring $\mathbb{C}[M]$ of the monoidal completion $M$ of $G$, which is isomorphic to $\mathbb{C}[G]$ by restriction, since $M$ is a monoid containing $G$ as its group of units. He also computes $\text{Specmax } \mathbb{C}[G]_{s.r.}\setminus G$.

(iii) The books by O. Mathieu [M] and S. Kumar [Ku] examine the structure of Kac-Moody groups in detail.

1.4. When $G$ is a finite-dimensional semisimple Lie algebra, $\mathbb{C}_q[G]$ is the Hopf-dual of $U_q(g)$, equivalently defined as the algebra spanned by matrix coefficients of all finite-dimensional $U_q(g)$-modules (Peter-Weyl type theorem). It has a triangular structure and its commutativity relations are obtained using the quantum $R$-matrix. The main results in the representation theory of $\mathbb{C}_q[G]$ [S1, J1] are summarized below:

(i) The irreducible representations $V_w$ of $\mathbb{C}_q[G]$ are parameterized by Weyl group elements $w$ and are independent of the reduced presentation of $w$. The representations corresponding to the identity element in the Weyl group are 1-dimensional, while the others are infinite-dimensional [S2].

(ii) The collection of $V_w$’s is in 1-1 correspondence with the symplectic leaves of the classical Poisson-Lie group $G$ associated with $g$.

(iii) $V_w$ is the irreducible quotient of the $\mathbb{C}_q[G]$ module induced from a one-dimensional $\mathbb{C}_q[G/N]$-module, where $N$ is the nilpotent subgroup of $G$ [J1].

1.5. This paper is adapted from my Ph.D thesis [N] and I would like to thank Y.Solomon for suggesting the main result to me. I also acknowledge here that the construction of the induced modules and their annihilators follows the approach of A.Joseph in [J1] for finite dimensional semisimple Lie algebras. Finally, a special word of thanks to my advisors, Z.Lin and A.Rosenberg for their invaluable guidance.
2. PRELIMINARIES

2.1. Notations and Basic Definitions. Here I summarize the results and notation relating to Kac-Moody algebras and quantum groups. A more detailed treatment can be found in the books by Kac [K] and Lusztig [L1]. Let \( q \in \mathbb{C} \setminus 0 \) which is not a root of unity and let \( k = \mathbb{Q}(q) \subseteq \mathbb{C}(q) \).

The following concise description of a Kac-Moody Lie algebra is adapted from Lusztig [L1].

**Definition.** Let \( C = ((a_{ij})) \) be a symmetrizable \( l \times l \) generalized Cartan matrix of rank \( r \). It is defined by integers \( a_{ij} \), with \( a_{ij} \) being non-positive for \( i \neq j \) such that

\[
a_{ii} = 2, a_{ij} = 0 \Rightarrow a_{ji} = 0
\]

and let \( d_i \) be the coprime positive integers such that the matrix \(((d_ia_{ij}))\) is symmetric. Fix the index set \( I = \{1, 2, \ldots, l\} \). Denote by \( \mathfrak{g}'(C) \) the complex Lie algebra on \( 3l \) generators \( e_i, f_i, h_i, i \in I \) and the defining relations:

\[
[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij}h_j,
\]

\[
[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j,
\]

\[
(ad e_i)^{1-a_{ij}}e_j = 0, (ad f_i)^{1-a_{ij}}f_j = 0, i \neq j.
\]

Let \( \mathfrak{h}' \) be the linear span of the \( h_i, i \in I \). Choose a vector space \( \mathfrak{h}'' \) of dimension \( l - r \), with basis \( \{D_{r+1}, \ldots, D_l\} \). The associated Kac-Moody algebra, \( \mathfrak{g}(C) \) is the Lie algebra with generators \( e_i, f_i, h_i, i \in I \) and \( D_i, i = r + 1, \ldots, l \) and with defining relations those of \( \mathfrak{g}'(C) \) together with

\[
[D_i, D_j] = 0, \quad [D_i, h_j] = 0, \quad [D_i, e_j] = \delta_{ij}e_j, \quad [D_i, f_j] = -\delta_{ij}f_j.
\]

**Remark.** The direct sum \( \mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'' \) is called the Cartan subalgebra of \( \mathfrak{g}(C) \).

Let \( q_i = q^{d_i} \) and define the simple roots \( \alpha_i : \mathfrak{h} \rightarrow \mathbb{C}, i \in I \), by

\[
\alpha_i(h_j) = a_{ij}, \alpha_i(D_j) = \delta_{ij}.
\]

They are linearly independent. Let \( \pi(C) = \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \) denote the set of simple roots of \( \mathfrak{g} \). There is a non-degenerate bilinear form on \( \mathfrak{h}^* \) such that \( d_i a_{ij} = (\alpha_i, \alpha_j) \), \( i, j \in I \). Write the fundamental weights as \( \{\omega_\alpha \in \mathfrak{h}^* : \alpha \in \pi(C)\} \), satisfying \( (\omega_\alpha, \beta^\vee) = \delta_{\alpha, \beta} \), where \( \beta \in \mathfrak{h}^* \), \( \beta^\vee := 2\beta/(\beta, \beta) \). Let \( P(C) = \bigoplus_{\alpha \in \pi(C)} \mathbb{Z}_{\omega_\alpha} + \mathbb{Z}_{\pi(C)} \subseteq \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in \pi(C)\} \) and \( P_+ = \{\lambda \in P(C) | (\lambda, \alpha^\vee) \in \mathbb{N}\} \).

Define linear maps \( s_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \), called the simple reflections, by

\[
s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i.
\]
The Weyl group \( W \) of \( g \) is the subgroup of \( GL(h^*) \) generated by \( s_1, \ldots, s_n \). The action of \( W \) preserves the bilinear form \( (, \) on \( h^* \).

Introduce \( \rho \in h^* \) by \( (\rho, \alpha_i^\vee) = a_{ii}/2 \), \( i \in I \). Note that \( \rho \) is not unique if \( \det \ C = 0 \) and we pick any solution. Let \( e \) denote the identity element in the Weyl group \( W \) and \( P_{++} = \rho + P_+ = \{ \lambda \in P_+ \mid \text{Stab}_W \lambda = \{ e \} \} \).

Define \( n_i = \langle \pi^i, \lambda \rangle \) where \( \pi^i \) is given by the antipode \( S \). Let \( \Gamma = \{ \lambda \in Z\Phi \otimes Z \mathbb{Q} \mid (\lambda, \mu) \in Z, \forall \mu \in \sum_{\beta \in \pi(C)} Z\omega_\beta \} \), where \( Z\Phi = \sum_{\alpha \in \Phi} Z\alpha \).

**Definition.** Let \( U = U_q(g) \) be the Hopf algebra over \( k \) with generators \( \langle E_\alpha, F_\alpha, K_\lambda \rangle \ (\alpha \in \pi(C), \lambda \in \text{Hom}_Z(\Gamma, Z)) \) and defining relations:

\[
K_\lambda K_\mu = K_{\lambda+\mu}, \\
K_\lambda E_\beta K_\lambda^{-1} = q^{(\lambda, \beta)} E_\beta, \\
K_\lambda F_\beta K_\lambda^{-1} = q^{-(\lambda, \beta)} F_\beta,
\]

along with the quantum Serre relations: for \( i \neq j \),

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} (E_i)^k (E_j)(E_i)^{1-a_{ij}-k} = 0, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_j} (F_j)^k (F_i)(F_j)^{1-a_{ij}-k} = 0.
\]

Let \( i \in I \). We can identify \( \alpha_i \in \text{Hom}_Z(\Gamma, Z) \) by \( \alpha_i(\beta) = (\beta, \alpha_i^\vee) \). Denoting \( E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_i} \) by \( E_i, F_i, K_i \) respectively, the Hopf algebra structure on \( U \) is given by the antipode \( S \), comultiplication \( \Delta \), and counit \( \epsilon \), which are defined on the generators via:

(i) \( S(E_i) = -K_i^{-1} E_i, \ S(F_i) = -F_i K_i, \ S(K_i) = K_i^{-1} \)

(ii) \( \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \ \Delta(K_i) = K_i \otimes K_i. \)

(iii) \( \epsilon(E_i) = \epsilon(F_i) = 0, \ \epsilon(K_i) = \epsilon(K_i^{-1}) = 1. \)

**Remark.** \( U \) is called the **simply connected form** of the quantum enveloping algebra. For each \( i \in I \), the subalgebra \( U_i \) of \( U \) generated by the \( E_i, F_i \) and \( K_i \) is isomorphic to \( U_q(sl_2) \) as a Hopf algebra. Denote by \( U_{\leq 0} \) (resp. \( U_{>0} \)) the subalgebra of \( U \) generated by the \( E_i \) and \( K_i \), \( i \in I \) (resp. \( E_i, i \in I \)). Similarly define \( U_{\geq 0} \) and \( U_{<0} \) by replacing \( E_i \) with \( F_i \).
2.2. $\ast$-structures. Classically the $\ast$-structure on the Lie algebra $\mathfrak{g}$, with generators $\{e_i, f_i, h_i\}$, is given by a Lie algebra anti-automorphism interchanging $e_i$ with $f_i$ and preserving the $h_i$. Then the (real) compact form of $\mathfrak{g}$ is $\mathfrak{k} = \{x \in \mathfrak{g} \mid x^* = -x\}$. Recall that this is the Lie algebra of the compact subgroup $K < G$, where $G$ is the Lie group of the Lie algebra $\mathfrak{g}$. Let $U^0$ denote the group algebra of the multiplicative group $T$ in $U$ generated by $\{K_\alpha : \alpha \in \pi(C)\}$.

**Definition.** Let $\omega$ be the Cartan involution on $U$. It is an algebra automorphism given by

$$\omega(E_i) = -F_i, \ \omega(F_i) = -E_i, \ \omega(K_i) = K_i^{-1}, \ \omega(q) = q.$$ 

**Definition.** Let $A$ be any complex associative algebra. A $\ast$-structure on $A$ is an antilinear, involutive, algebra anti-automorphism and coalgebra automorphism:

$$(cx)^\ast = \bar{c}(x^\ast), \ (x^\ast)^\ast = x, \ (xy)^\ast = y^\ast x^\ast$$

for any $c \in \mathbb{C}$, $x, y \in A$, where $\bar{x}$ is the complex conjugate of $x$.

$U$ can be given a $\ast$-structure by setting $\ast = S \circ \omega$. The dual vector space $U^\ast$ (with a different meaning of $\ast$!) then becomes a $\ast$-algebra via $l^\ast(u) = l(S(u)^\ast)$, i.e. under the transpose of the automorphism $\ast \circ S = \omega$ on $U$ where $l \in U^\ast$ and $u \in U$. The following (equivalent) identities are easily checked:

$$S(u)^\ast = S^{-1}(u^\ast), S \circ \omega \circ S = \omega.$$ 

2.3. Integrable U-modules and their duals. By $U$-modules we will mean left $U$-modules, unless otherwise specified. For any $U$-module $M$ over $k$, define a $U$-module structure (resp. right $U$-module structure) on the dual vector space $\text{Hom}_k(M, k)$ via $(uf)(m) = f(\gamma(u)m)$ (resp.) $(fu)(m) = f(um)$ where $\gamma$ is an anti-endomorphism of $U$, $u \in U$ and $m \in M$. If we take $\gamma = S$, the antipode of $U$, the resulting module is denoted $M^\ast$ and if we take $\gamma = \omega \circ S$, we denote it $M^\circ$. Further, if $M'$ is any right $U$-module, then $\text{Hom}_k(M', k)$ is a left $U$-module, via $(uf)(m) = f(mu)$.

Now fix $\phi : U \rightarrow \text{End}(M)$, $M = L(\Lambda)$, highest weight, integrable $U$-module of highest weight $\Lambda \in P_+$, with highest weight vector $v_\Lambda$. Let $L(\Lambda)_{\lambda}^\ast = (L(\Lambda)_{-\lambda})^\ast$ and $L^\ast(\Lambda) = \bigoplus_{\lambda \in \Omega(\Lambda)} L(\Lambda)_{\lambda}^\ast$. $L^\ast(\Lambda)$ is a lowest weight integrable module with lowest weight vector $l_{-\Lambda}$.

**Definition.** If we give the vector space $L(\Lambda)$ a left $U$-module structure by replacing $\phi$ by $\phi^\ast = \phi \circ \omega$, it becomes the graded dual, denoted $L(\Lambda)^\sharp$.

**Remarks.**

(i) As $U$-modules, $L(\Lambda)^\sharp \simeq L^\ast(\Lambda)$, since the map taking $v_\Lambda$ to $l_{-\Lambda}$ extends to a $U$-module isomorphism from $L(\Lambda)^\sharp$ to $L^\ast(\Lambda)$. 

\[\text{\textbf{References}}\]
(ii) \( L(\Lambda)^\sharp = L(\Lambda)^* \) if and only if \( L(\Lambda) \) is finite dimensional.

**Definition.** Let \( M_1 \) and \( M_2 \) be \( U \)-modules. A bilinear form \( \langle , \rangle : M_1 \otimes M_2 \rightarrow k \) is said to be \( U \)-invariant if \( \langle uv_1, v_2 \rangle = \langle v_1, S(u)v_2 \rangle \forall u \in U \) and \( v_i \in M_i, i = 1, 2 \).

There exists a unique \( U \)-invariant bilinear form on \( L^*(\Lambda) \otimes L(\Lambda) \) satisfying
\[
\langle l_{-\Lambda}, v_\Lambda \rangle = 1,
\]
where \( l_{-\Lambda} \) is a fixed lowest weight vector for \( L(\Lambda)^\sharp \). It is nondegenerate. Using \( \langle , \rangle \) we may regard \( L(\Lambda)^\sharp \) as a subspace of the dual vector space \( L(\Lambda)^* \). Note that if a statement holds for \( L(\Lambda) \) (resp. \( L(\Lambda)^* \)) then a similar one holds for \( L(\Lambda)^\sharp \) (resp \( L(\Lambda)^\circ \)) using \( \omega \) (resp \( \omega \circ S \)). Finally, observe that if \( C \) is of finite type, then \( L(\Lambda) \) is finite dimensional as a vector space and it follows that \( L(\Lambda)^\circ \simeq L(-w_0\Lambda) \) and \( L(\Lambda)^\sharp \simeq L(-w_0\Lambda)^* \), as \( U \)-modules, where \( w_0 \) is the Weyl group element of maximal length.
3. QUANTIZED FUNCTION ALGEBRAS OF A KAC-MOODY LIE ALGEBRA

3.1. Definition of $A = \mathbb{C}_q[G]$. Since $U$ is a coalgebra, its vector space dual $U^*$ is naturally an algebra.

**Definition**. The matrix coefficient $C_{l,v}^\Lambda$ of an integrable highest weight (type 1) $U_q(g)$-module $L(\Lambda)$ is defined via:

$$C_{l,v}^\Lambda(u) := \langle l, \rho_\Lambda(u)v \rangle,$$

for $l \in L(\Lambda)^\sharp$, $v \in L(\Lambda)$ and $u \in U_qg$.

Let $F := \bigoplus_{\Lambda \in P_+} \Lambda \in L(\Lambda)^\sharp \otimes L(\Lambda)$. Then the map $l \otimes v \mapsto C_{l,v}^\Lambda$ extends to a vector space map $F \rightarrow \text{Hom}_k(U,k)$. Let $R$ denote its image. It carries a multiplicative structure via:

$$(C_{l,v}^\Lambda C_{l',v'}^\Lambda')(u) = (l \otimes l')(\phi_\Lambda \otimes \phi_\Lambda') \Delta(u)(v \otimes v').$$

This map is well defined since the tensor product of two integrable highest weight modules decomposes as the direct sum of integrable highest weight modules.

**Definition**. Let $R$ be the subspace of $U^* = \text{Hom}_k(U,k)$ spanned by matrix coefficients $C_{l,v}^\Lambda$ with $\Lambda \in P_+$.

**Remark**. This is the quantum analog of the algebra of strongly regular functions on the group $G$ associated to the derived subalgebra $g' = [g, g]$, $G$ being an infinite-dimensional affine algebraic group of Shafaravich type [K-P]. It is important to note that, unless $C$ is of finite type, $R$ is not a Hopf algebra, since it is not closed under the comultiplication (resp. antipodal) map dual to the multiplication (resp. antipodal) map in $U$. The Hopf Dual of $U$, denoted $U^*$, is defined [J1] as $U^* := \{ \xi \in U^* \mid \xi(I) = 0 \text{ for some two-sided ideal } I \text{ in } U \}$. It is a fact that $U^*$ consists of matrix coefficients of finite dimensional $U$-modules and is properly contained in $R$ in the infinite type case. $U^*$ coincides with $R$ if $C$ is of finite type and is commonly referred to in the literature as the Quantum Coordinate Algebra of $G$ [APW].

**Definition**. Letting $U_i^{\geq 0} := U \geq 0 U_i$ denote the quantum parabolic algebra, define

(i) $C = \{J \mid J$ is a 2-sided ideal of $U_i^{\geq 0}$, $\dim(U_i^{\geq 0}/J) < \infty$, and $U^0$ acts semisimply on $(U_i^{\geq 0}/J)$ with weights in $P(\pi), \forall i \in I \}$,

(ii) $F = \bigoplus_{\Lambda \in P_+} \{ \phi \in \text{Hom}_k(U,k) \mid \phi$ is $U_i^{\geq 0}$-finite under left and right multiplication for all $i \in I$ and $\phi$ is a weight vector of weight $\Lambda$ under the left action of $U^0 \}$.

The proof of the next proposition is straightforward.
Proposition. Let $\phi \in \text{Hom}_k(U, k)$. Then the following statements are equivalent:

(i) $\phi \in R$
(ii) $\phi(J) = 0$ for some $J \in C$.
(iii) $\phi \in F$.

Recall (Section 2.2) that $\text{Hom}_k(U, k)$ is a $\star$-algebra, with the involution $\star$ defined via $l^\star(u) = l(S(\omega(S(u)))) = l(\omega(u))$, for any $l \in \text{Hom}_k(U, k)$, $u \in U$.

Definition. Define the quantized function algebra of a Kac-Moody Lie algebra, $C_q[G] := \langle R, \star(R) \rangle$ as the minimal $\star$-subalgebra of $\text{Hom}_k(U, k)$ generated by elements of $R$, where $\star$ is the transpose of the involution $\omega$ on $U$, and denote it by $A$.

Remark. If we interchange $U^>0$ and $U^<0$ in the definition of highest weight module to define a lowest weight module, it follows from Proposition 3.2 that the image of a matrix coefficient of a highest weight $U$-module, under $\star$, is a matrix coefficient of a lowest weight $U$-module. Thus $A = C_q[G]$ can be viewed as the analogue of the algebra of holomorphic and anti-holomorphic functions of a complex variable.

3.2. U-bimodule structure. The elements $C_{i,v}^\Lambda$ of $R$ are written as $C_{-\lambda,i;\mu,j}^\Lambda$ if $l \in L(\Lambda)_\lambda$ is the $i$th basis vector $1 \leq i \leq \dim(L(\Lambda))$ and $v \in L(\Lambda)_\mu$ is the $j$th basis vector $1 \leq j \leq \dim(L(\Lambda)_\mu)$. It is a common practice to omit the indices $i$ and $j$, since all formulae involved are expected to hold irrespective of their choice.

Proposition. Let $u \in U$. The following relations express the $U$-bimodule structure of $A$.

$$uC_{i,v}^\Lambda = C_{i,uv}^\Lambda, \quad C_{i,v}^\Lambda u = C_{lu,v}^\Lambda, \quad u(C_{i,v}^\Lambda)^* = (C_{i,\omega(u)v}^\Lambda)^*, \quad (C_{i,v}^\Lambda)^* u = (C_{lu,u}^\Lambda,v)^*.$$ 

Proof. Use the definition of the $U$-bimodule structure on $\text{Hom}_k(U, k)$ and the right $U$-module structure on $\text{Hom}_k(V, k)$, for any left $U$-module $V$. For example, for any $x \in U$, we have that

$$uC_{i,v}^\Lambda(x) = C_{i,v}^\Lambda(xu) = l(\rho_\Lambda(xu)v) = C_{i,u,v}^\Lambda(x),$$

and

$$(C_{i,v}^\Lambda u)(x) = C_{i,v}^\Lambda(ux) = l(\rho_\Lambda(ux)v) = lu(\rho_\Lambda(x)v) = C_{lu,v}^\Lambda(x).$$
3.3. Triangular-Type Structure.

**Definition.** Let \( A_+ \) denote the subspace of \( A \) spanned by the \( \{ C_{\mu,j}^\Lambda \in P_+, \mu \in \Omega(\Lambda) \} \), and \( A_- = \ast(A_+) \). Similarly define \( A_{++} \) (resp. \( A_{--} \)) by replacing \( P_+ \) by \( P_{++} \) in the definition of \( A_+ \) (resp. \( A_- \)).

**Remark.** We define the set of left \( U^0 \)-invariants of \( A \) as the set \( \{ \xi \in A \mid x\xi = \epsilon(x)\xi, \forall x \in U^0 \} \). Similarly define the left \( U^{<0} \)-invariants as \( \{ \xi \in A \mid y\xi = \epsilon(y)\xi, \forall y \in U^{<0} \} \). Then it is clear that \( A_+ \) (resp. \( A_- \)) is just the set of invariants of \( A \) with respect to the left action of \( U^0 \) (resp. \( U^{<0} \)) and may be viewed as the quantum function algebras on \( G/N^+ \) (resp. \( G/N^- \)).

**Definition.** For \( \Lambda \in P_+ \), define the subspaces \( L^+(\Lambda)^\ast \) (resp. \( L^-(\Lambda)^\ast \)) of \( A_+ \) (resp. \( A_- \)) as
\[
L^+(\Lambda)^\ast = \{ C_{\xi,\Lambda}^\Lambda/\xi \in L(\Lambda)^\sharp \},
\]
\[
L^-(\Lambda)^\ast = \{ (C_{\xi,\Lambda}^\Lambda)^\ast/\xi \in L(\Lambda)^\sharp \}.
\]

**Remark.** As a right \( U \)-module, \( A_+ \) is a direct sum of the \( L^+(\Lambda)^\ast \) which is isomorphic to \( L(\Lambda)^4 \). Similarly, \( A_- \) is a direct sum of the \( L^+(\Lambda)^\ast \) which is isomorphic to \( L(\Lambda)^4 \). Further, these modules satisfy the Cartan multiplication rule:
\[
L^+(\Lambda)^\ast L^+(\Lambda')^\ast = L^+(\Lambda + \Lambda')^\ast, \forall \Lambda, \Lambda' \in P_+.
\]

Thus, we have the following isomorphisms of \( k \)-vector spaces:
\[
A_+ \simeq \oplus_{\Lambda \in P_+} L(\Lambda)^\sharp
\]
\[
A_- \simeq \oplus_{\Lambda' \in P_-} L(\Lambda')^0.
\]

We are ready to prove the important “triangular-type” structure Theorem for \( A \):

**Theorem.** The multiplication map \( \Delta^* |_{A_+ \otimes A_-} : A_+ \otimes A_- \rightarrow A \) is an isomorphism of \( U - U \)-bimodules.

**Proof.** Using the \( U - U \)-bimodule structure on \( U^\ast \) and the fact that \( \Delta \) is an algebra homomorphism on \( U \), it is easily seen that \( \Delta^* : U^\ast \otimes U^\ast \rightarrow U^\ast \) is a \( U - U \)-bimodule morphism. Thus it suffices to prove that \( \Delta^* |_{A_+ \otimes A_-} \) is bijective. Fix \( \Lambda, \Lambda' \in P_+ \). Let \( m = \Delta^* |_{L^+(\Lambda)^\ast \otimes L^-(\Lambda')^\ast} \).

(1) \( m \) is injective.

Let \( u_{\Lambda} \) and \( u_{\Lambda'} \) be the highest weight vectors of \( L(\Lambda) \) and \( L(\Lambda') \) respectively and let \( \{ \xi_i \}, \{ \xi'_i \} \) be bases for \( L(\Lambda)^\sharp \), \( L(\Lambda')^\sharp \) respectively.
Remark. It easily follows that the multiplication map from $A_- \otimes A_+$ to $A$ is also bijective.

Lemma. $A$ is a domain.
Proof. Let \( fg = 0 \) for some non-zero elements \( f, g \in A \). Assume that \( f \) and \( g \) are weight vectors under the \( U^0 \cdot U^0 \) action. Since the action of \( U^{>0} \cdot U^{<0} \) on \( U^0 \cdot U^0 \) weight vectors in \( A \) is by locally nilpotent skew derivations, one may assume that \( f \) and \( g \) are \( U^{\geq 0} \cdot U^{\leq 0} \) invariant. Then there exist \( \Lambda, \Lambda' \in P_+ \) such that \( f = C^\Lambda_{\Lambda,\Lambda} \) and \( g = C_{\Lambda',\Lambda'}^{\Lambda} \). Consequently, \( (fg)(1) = f(1)g(1) \), which gives the required contradiction. \( \square \)

3.4. Commutativity Relations. Define \( V^{>0} = \text{Ker}(\epsilon|_{U^{>0}}) \) and \( V^{<0} = \text{Ker}(\epsilon|_{U^{<0}}) \). For \( a_\mu \in U^{>0}_\mu \), \( b_{-\nu} \in U^{<0}_{-\nu} \), the following formulae are easily checked:

\[
\Delta(a_\mu) = a_\mu \otimes 1 + K_\mu \otimes a_\mu \mod V^{>0} \otimes V^{>0}
\]

\[
\Delta(b_{-\nu}) = 1 \otimes b_{-\nu} + b_{-\nu} \otimes K_{-\nu} \mod V^{<0} \otimes V^{<0}.
\]

Lemma. The commutation relations between elements of \( A_+ \) and \( A_- \) are given by:

(i) \( \forall \Lambda, \Lambda' \in P_+ \), there exist constants \( a_\gamma \) such that

\[
(C^\Lambda_{-\lambda,\Lambda'})^*(C^\Lambda_{-\mu,\nu}) = q^{(\nu,\Lambda')-(\lambda,\mu)}(C^\Lambda_{-\mu,\nu})(C^\Lambda_{-\lambda,\Lambda'})^* + \sum_\gamma a_\gamma(C^\Lambda_{l_\gamma,v,\Lambda})(C^\Lambda_{l_\gamma,v,\Lambda'})^*
\]

for \( l_\gamma \in (l_\mu U^\mu)_{\mu-\gamma} \) and \( l_\gamma' \in (l_\Lambda U^\Lambda)_{\lambda-\gamma'} \).

(ii) Let \( J_{\Lambda}(\mu, \nu) \) be the smallest 2-sided ideal of \( A \) containing the elements

\[
\{C^\Lambda_{-\gamma,\Lambda'}}_{\gamma \in \Omega(\Lambda), 1 \leq i \leq \dim(L(\Lambda)_\gamma)} \text{ such that } \gamma < \mu. \text{ Then the following relation holds in } A/J_{\Lambda}(\mu, \nu), \text{ for any } \mu, \nu \in \Omega(\Lambda) \text{ and } \lambda \in \Omega(\Lambda'):
\]

\[
C^\Lambda_{\lambda,\Lambda} C^\Lambda_{-\mu,\nu} = q^{(\nu,\Lambda')-(\lambda,\mu)} C^\Lambda_{-\mu,\nu} C^\Lambda_{\lambda,\Lambda'},
\]

where we use the same symbols for elements of \( A \) as for their images in \( A/J_{\Lambda}(\mu, \nu) \) under the canonical projection map.

Proof. (i) Let \( \Delta_+ \) denote the set of positive roots of the Lie algebra \( \mathfrak{g} = \mathfrak{g}(C) \). The following expression for the universal quasi-\( R \)-matrix for \( U_q(\mathfrak{g}) \) can be found in [K-T]:

\[
R = \prod_{\alpha \in \Delta_+} \exp_{q^{-2}}(C_\alpha(q) E_{\alpha} \otimes F_{\alpha}) q^{t_0},
\]

where \( t_0 = \sum_{i,j} d_{ij} h_i \otimes h_j \) and \( ((d_{ij})) \) is an inverse matrix for the symmetrical Cartan matrix \( C \) if \( C \) is not degenerate. In the case of degenerated \( C \), we extend it to a non-degenerated matrix and then take an inverse to this extended matrix. Here \( C_\alpha(q) \) are certain constants and the \( q \)-exponential is defined as

\[
\exp_q(x) = \sum_{k=0}^{\infty} q^{k(k+1)/2} x^k /[k]_q!.
\]
It has been established, \[ G \] that in the case of irreducible integrable highest weight \( U \)-modules \( L \) and \( L' \), the term \( q^\mu \) is the operator which acts as the scalar \( q^{(\lambda,\mu)} \) on the subspace \( L_\lambda \otimes L'_\mu \). It follows that \((1 \otimes \omega)q^\mu \) acts as the scalar \( q^{(\lambda,-\nu)} \), while \((1 \otimes \omega)q^{-\mu} \) acts as the scalar \( q^{-(\lambda,\nu)} \), on the subspace \( L_\lambda \otimes L'_\mu \).

For any \( x \in U \), one has:
\[
(C_{\lambda,A}^\Lambda)(x) = (l \otimes l)(x) = \langle l \otimes l, (1 \otimes \omega)\Delta(x)v_\lambda \otimes v_\nu \rangle
\]
\[
= \langle l \otimes l, (1 \otimes \omega)\Delta'(x)v_\lambda \otimes v_\nu \rangle + \sum_{\gamma} a_{\gamma} \langle l \otimes l, (1 \otimes \omega)\Delta(x)v_\lambda \otimes v_\nu \rangle,
\]
as required.

(ii) is proved in a similar way.

Remark. Note that the remaining commutativity relations in \( A \) can be derived easily, by using the involution \( \star \) on the relations (i) and (ii) above, along with Theorem 3.3.

Definition. Let \( P_0 = \{ \Lambda \in P_+, \text{ such that } \Lambda(h_i) = 0 \text{ for } i \in I \} \), \( P_\perp = P_+ \setminus P_0 \), \( A_0 = \oplus_{\Lambda \in P_0} C_{\Lambda,A}^\Lambda \), and \( A_\perp = \oplus_{\Lambda \in P_\perp} C_{\Lambda,A}^\Lambda \).

We shall henceforth assume the following:

Assumption A0 The integrable \( U \)-module \( L(\Lambda) \) is finite dimensional \( \iff \Lambda \in P_0 \)

This is equivalent to requiring that every connected component of the Dynkin diagram is of infinite type. Recall that the level of a highest weight module is constant. Then the Lemma implies that the elements in \( A_\perp \) are central in \( A \). The proof of the next proposition is the same as for Lemma 6.2.1 in [Kas1].

Proposition. The subspace \( A_\perp \) is a 2-sided ideal of \( A \).

Hence we have a surjection \( f : A \rightarrow A/A_\perp \cong A_0 \). Then, \( \epsilon_A \circ f \) gives a one-dimensional \( A \)-module, denoted \( N_\infty \), with kernel \( A_\perp \).

3.5. Filtration and Gradings on \( A \). In the case where the Cartan matrix \( C \) is of finite type, the quantum function algebra \( \mathbb{C}_q[G] \) can be filtered (see 1.4.8, 9.2.4, 10.1.4 in [J1]). We now construct a similar filtration in the infinite type case.
Recall (Section 3.3) that $A_- \simeq \bigoplus_{A' \in P_+} L(A')^\circ$. Consider $A_-$ as a right $U$-module via the antipodal map $S$ on $U$ (see Section 2.3) and write $(D_{-\Lambda, \Lambda}^\Lambda)^\ast$ for $(C_{-\Lambda, \Lambda}^\Lambda)^\ast$. Let $F^i(A_-)$ be the subspaces of $A_-$ defined inductively via:

$$F^0(A_-) = \oplus_{A \in P_+} k(D_{-\Lambda, \Lambda}^\Lambda)^\ast.$$ 

$$F^i(A_-) = F^{i-1}(A_-) + \sum_{\alpha \in \pi(C)} F^{i-1}(A_-)E_\alpha.$$ 

Then each $F^i(A_-)$ is clearly a $U^0-U^0$-stable subspace of $A_-$. Recall the fact that $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$ and $(D_{-\Lambda, \Lambda}^\Lambda)^\ast U^+ = L^-(\Lambda)^\ast$ (since $l_\Lambda U^+ = L(\Lambda)^\circ$), for any $\Lambda \in P_+$. It follows that $F^i(A_-)$ is a $\mathbb{Z}_{\geq 0}$-filtration of $A_-$. Let us now show that $(F^iA_-)A_+$ forms a filtration of $A$.

Let us fix $\Lambda, \Lambda' \in P_+$. Then 3.4(ii) gives, by using induction on $i$, that for any $C_{-\Lambda, \Lambda'}^\Lambda$ in $A_+$, and $(D_{-\Lambda, \Lambda}^\Lambda)^\ast \in F^i(A_-)$,

$$(D_{-\Lambda, \Lambda}^\Lambda)^\ast C_{-\Lambda, \Lambda'}^\Lambda = q(\Lambda, \Lambda') - (\Lambda', \Lambda) C_{-\Lambda, \Lambda'}^\Lambda (D_{-\Lambda, \Lambda}^\Lambda)^\ast \mod(F^iA_-)A_+.$$

Thus, in particular,

$$(F^i(A_-))A_+ = A_+(F^i(A_-)) \mod(F^{i-1}(A_-))A_+.$$

We conclude, by Theorem (3.3), that these subspaces form a filtration of $A$.

3.6. **The Prime Spectrum of $A$.** Let $P$ be an ideal in $A$. We say that $P$ is prime in $A$ if the following condition holds: for any ideals $I_1, I_2$ in $A$, $I_1I_2 \subset P$ implies $I_1 \subset P$ or $I_2 \subset P$.

**Definition.** For $\Lambda \in P_+$, define

$$C^-_P(\Lambda) = \{-\nu \in \Omega(\Lambda) \mid C^-_{-\Lambda, \Lambda} \notin P\} \quad C^+_P(\Lambda) = \{-\nu \in \Omega(\Lambda) \mid (C^-_{-\Lambda, \Lambda})^\ast \notin P\}.$$ 

If $C^-_P(\Lambda) = \emptyset$, set $D^-_P(\Lambda) = \emptyset$. If $C^+_P(\Lambda) \neq \emptyset$ let $D^+_P(\Lambda)$ denote the set of maximal elements of $C^+_P(\Lambda)$; define $D^-_P(\Lambda)$ similarly. Fix $\vec{w} = (w_+, w_-) \in W \times W$. Define $B(w_+, w_-)$ as the set of all prime ideals $P$ in $A$ for which $D^+_P(\Lambda) = w_+ \Lambda$ and $D^-_P(\Lambda) = w_- \Lambda$, $\forall \Lambda \in P_+$.

**Definition.** Let $J^\Lambda_\Lambda(\lambda, \mu)$ be the 2-sided ideal in $A$ generated by elements $C^-_{-\Lambda, \Lambda'}^\Lambda$ with either $\Lambda' \leq \lambda$ and $\mu' > \mu$ or $\Lambda' < \lambda$ and $\mu' \geq \mu$, while $J^-_\Lambda(\lambda, \mu) = (J^\Lambda_\Lambda(\lambda, \mu))^\ast \subset A$.

**Assumption (A1):** The collection $\Omega$ of primes $P$ in $A$ for which $D^+_P(\Lambda_i) \neq \emptyset$ is nonempty. Let $\text{Spec}_\Omega(A)$ denote the part of the prime spectrum of $A$ that lies in $\Omega$.

**Lemma.** **Assuming (A1) holds, one has**

$$\text{Spec}_\Omega(A) = \bigsqcup_{(w_+, w_-) \in W \times W} B(w_+, w_-).$$
Proof. Assuming that $D^+_P(\Lambda_i) \neq \emptyset$, pick $\eta_i \in D^+_P(\Lambda_i)$ for $i = 1, 2$. It follows from the definition of $D^+_P(\Lambda_i)$ that $C^+_{\eta_i, \Lambda_i} \not\in P$ and $C^+_{\eta_i', \Lambda_i} \in P$, $\forall \eta_i < \eta_i$. Thus, $J^+_{\Lambda_i}(\eta_i, \Lambda_i) \subseteq P$. The commutation relations imply that the image $\overline{\gamma}$ of $C^+_{\eta_i, \Lambda_i}$ in $A/P$ is normal, meaning that

$$
\overline{c_1} \overline{c_2} = \overline{c_1} \overline{c_2}.
$$

But $\overline{\gamma} \neq 0$ by definition of $D^+_P(\Lambda_i)$ and, since $P$ is prime, $\overline{\gamma}$ is a non-zero divisor. Now the commutation relations imply that

$$
\overline{c_1} \overline{c_2} = q^{2((\lambda_1, \lambda_2) - (\eta_1, \eta_2))} \overline{c_1} \overline{c_2}.
$$

If we interchange the roles of $\overline{c_1}$ and $\overline{c_2}$, we get the same exponent in $q$ which must therefore be zero. In other words, $(\Lambda_1, \Lambda_2) = (\eta_1, \eta_2)$. The following Proposition is well-known (see Lemma A.1.17 in [J1]):

**Proposition.** Take $\Lambda_1, \Lambda_2 \in P_+$. Then $(\lambda_1, \lambda_2) \leq (\Lambda_1, \Lambda_2)$ $\forall \lambda_i \in \Omega(\Lambda_i)$ and if $\Lambda_2$ is regular equality implies $\lambda_1 = w\lambda_1$ for some $w \in W$. Then if $\Lambda_1$ is also regular, $\lambda_2 = w\lambda_2$.

For some $\lambda \in P_{++}$, it follows from the preceding Proposition that $D^+_P(\Lambda) = w\Lambda$ for some $w \in W$. Thus $w\omega_i \in C^+_P(\omega_i)$ for all $i$. Assume that $w\omega_i \not\in D^+_P(\omega_i)$ for some $i$. Then there exists $\xi \in \mathbb{N}\pi(C) \setminus \{0\}$ such that $w\omega_i - \xi \in D^+_P(\omega_i)$. Let $c = C^+_{w\omega_i - \xi, \omega_i}$. Then $c \not\in P$ and its image $\overline{\xi}$ in $A/P$ is normal and hence a non-zero divisor. Therefore $ccw\Lambda \notin P$. This gives that $w(\omega_i + \Lambda) - \xi \in D^+_P(\omega_i + \Lambda)$. Yet $\omega_i + \Lambda$ is regular so this contradicts the previous result. Suppose that there exists $\Lambda' \in P_+$ such that $D^+_P(\Lambda') \not\supseteq \{w\Lambda'\}$. Then, using the preceding Proposition as above, it follows that there exists $w' \in W$ such that $w'\Lambda' \in D^+_P(\Lambda') \setminus \{w\Lambda'\}$. Again, this forces $w\Lambda + w'\Lambda' \in C^+_P(\Lambda + \Lambda')$. Consequently, $w\Lambda + w'\Lambda' \geq w(\Lambda + \Lambda')$ and so $w'\Lambda' \geq w\Lambda'$. This contradicts the fact that $w'\Lambda'$ and $w\Lambda'$ are distinct minimal elements of $C^+_P(\Lambda')$ which proves that $D^+_P(\Lambda) = w_+\Lambda$, with $w = w_+$. The second case can be proved in a similar fashion, by the use of the involution $\ast$ on $A$. \qed
4. HIGHEST WEIGHT MODULES

4.1. Characters.

**Definition.** Let $\chi : A_+ \to k^*$ be an algebra homomorphism and $k_\chi$ denote the corresponding $A_+$-module with generator $1_\chi$. $\chi$ is called a *character* on $A_+$.

Now define, for any $\Lambda \in P_+$ and any character $\chi$, the set

$$C_\chi^+(\Lambda) = \{-\nu \in \Omega(\Lambda)/\chi(C_{-\nu,\Lambda}) \neq 0\}.$$  

If $C_\chi^+(\Lambda) = \emptyset$, or if $C_\chi^+(\Lambda)$ is not bounded above, set $D_\chi^+(\Lambda) = \emptyset$. Otherwise let $D_\chi^+(\Lambda)$ denote the set of maximal elements of $C_\chi^+(\Lambda)$. Similarly define,

$$C_\chi^-(\Lambda) = \{-\nu \in \Omega(\Lambda)/\chi((C_{-\nu,\Lambda})^*) \neq 0\}$$

and let $D_\chi^-(\Lambda)$ denote the set of maximal elements of $C_\chi^-(\Lambda)$.

Consider the following assumption:

**A.2.** Let $\chi : A_+ \to k^*$ be a non-zero character such that $D_\chi^+(\Lambda) \neq \emptyset$, $\forall \Lambda \in P_+$.

**Lemma.** Assume $\chi$ satisfies A.2. If $\chi(A_{++}) \neq 0$, then there exists $w \in W$ such that $C_\chi^+(\Lambda) = w\Lambda$. Further, $w$ is independent of $\Lambda$.

**Proof.** The proof of this Lemma is similar to Lemma (3.6). Assume $D_\chi^+(\Lambda_1) \neq 0$ and $C_\chi^+(\Lambda_2) \neq 0$ and pick $\lambda_1 \in D_\chi^+(\Lambda_1)$ and $\lambda_2 \in C_\chi^+(\Lambda_2)$. Let $c_i = C_{-\lambda_i,\Lambda_i}^\Lambda$, for $i = 1, 2$.

Then, it follows by definition that $\chi(J_{\lambda_i}(\lambda_1, \mu_1)) = 0$, and so 3.4(ii) gives that

$$\chi(c_2)\chi(c_1) = q^{(\Lambda_1, \Lambda_2) - (\lambda_1, \lambda_2)}\chi(c_1)\chi(c_2).$$

But $\chi(c_i)$ is a non-zero scalar, so the exponent of $q$ must be zero. Then, Proposition (3.6) implies that $\lambda_i = w\Lambda_i$ for a unique $w \in W$. If $\Lambda$ is regular, $D_\chi^+(\Lambda) \neq \emptyset$ implies $C_\chi^+(\omega_i) \neq 0$ for all $i$ by the Cartan multiplication rule. \qed

**Remark.** Fix $w \in W$ and let $E(\Lambda, w)$ be the $w\Lambda$ weight subspace of $L^+(\Lambda)^*$ considered as a right $U^0$-module. Define $I_w := \oplus_{\Lambda \in P_+} L^+(\Lambda)^{*w\Lambda}$, where $L^+(\Lambda)^{*w\Lambda}$ denotes the unique $U^0$-stable complement in $L^+(\Lambda)^*$ of $E(\Lambda, w)$. Using the left $U^0$-action gives $L^+(\Lambda)^{*w\Lambda}L^+(\Lambda')^{*w\Lambda'} \subset L^+(\Lambda + \Lambda')^{*w(\Lambda + \Lambda')}$ for all $\Lambda, \Lambda' \in P_+$ and so the sum $I_w$ is a 2-sided ideal of $R^+$. Thus, each $l$-tuple $\chi_w = \{\chi_{w,i}\}_{i \in I} \in k^l$ can be viewed as the unique character on $A_+$ satisfying $\text{Ker}\chi_{w,i} \supset I_w$ and $\chi_w(C_{-w_i,\omega_i}^\Lambda) = \chi_{w,i}$ for all $i$. Further, $\chi_w$ does not vanish on $A_{++}$ if and only if $\chi_{w,i} \neq 0$ for all $i$. The Lemma implies all characters with this non-vanishing property are so obtained.
4.2. The Induced Highest Weight Modules and their Irreducible Quotients $N(w)$. Let $k(q)v_\chi$ denote the one-dimensional $A_+$-module corresponding to each character $\chi$ on $A_+$, and $V(\chi) := A \otimes_{A_+} k(q)v_\chi$ the induced $A$-module. Here $A$ acts on itself by left multiplication and $A_+$ acts on $A$ by right multiplication.

Definition. We say that an $A$-module $V$ is a Highest Weight Module with Highest Weight $\chi$ if there exists a $v \in V$ such that $fv = \chi(f)v, \forall f \in A_+$, and $V = Av$.

Write $V(w)$ for $V(\chi_w)$, where $w$ is given by Lemma (4.1) and $\chi(C_{-\omega_i,w_i}) = \chi_{w,i}, \forall i \in I$. The highest weight vector is denoted $v_w$.

Lemma. The $\mathbb{Z}_{\geq 0}$ graded $A_+$-module $gr_{F}V(w)$ is a direct sum of one-dimensional modules with characters lying in the set of $l$-tuples $q^{(\omega_i,w_i^{-1}\Lambda)}\chi_{w,i}$, where $\Lambda \in P_+$ and $\eta \in \Omega(L(\Lambda))$. Further, the module with character $\chi_{w,i}$ occurs with the multiplicity of one.

Proof. If $(C_{-\eta,\Lambda}^\Lambda)^* \in F^iA_-$, then

$$(C_{-\eta,\Lambda}^\Lambda)^*C_{-\xi,N}^{\Lambda'} = q^{(\Lambda',\Lambda)-\xi,\eta}C_{-\xi,N}^{\Lambda'}(C_{-\eta,\Lambda}^\Lambda)^* \mod (F^iA_-)A_+.$$

Let $x \in gr_{F}V(w)$ be such that $x = F^{m-1}(A_-)v_w + y \in F^m(A_-)v_w/F^{m-1}(A_-)v_w$ for some $m \in \mathbb{Z}$ with $y = (C_{-\eta,\Lambda}^\Lambda)^*v_w$. Then

$$C_{-\lambda,A}^{\Lambda'}y = q^{(\lambda,\eta)-\lambda,\lambda'}(C_{-\eta,\Lambda}^\Lambda)^*C_{-\lambda,A}^{\Lambda'}v_w \mod F^{m-1}(A_-)A_+v_w.$$

Thus,

$$C_{-\lambda,A}^{\Lambda'}x = F^{m-1}(A_-)v_w + q^{(\lambda,\eta)-\lambda,\lambda'}(C_{-\eta,\Lambda}^\Lambda)^*C_{-\lambda,A}^{\Lambda'}v_w.$$

Take $\Lambda' = \omega_i$ and $\lambda = w\omega_i$, together with the fact that $gr_{F}V(w) = \bigoplus_{m \in \mathbb{Z}} gr_{F}V(w)$ where $gr_{F}V(w) = F^m(A_-)v_w/F^{m-1}(A_-)v_w$ to complete the proof for the first part of the Lemma. For the last part, note that the exponent in $q$ vanishes for all $i$ implies $\eta = w\Lambda$. Yet, the corresponding element $(C_{w\Lambda,\Lambda}^\Lambda)^*$ acts on $v_w$ by a scalar. □

We are now ready to prove the following important theorem.
Theorem. The following statements are true:

(i) The induced module \( V(\chi) \) is a Highest Weight Module.
(ii) Every highest weight module \( H \) of highest weight \( \chi \) is an image of \( V(\chi) \) under a surjective
\( A \)-module homomorphism \( \psi : V(\chi) \rightarrow H \).
(iii) \( V(w) \) has a unique maximal proper submodule \( V' \) and a unique irreducible quotient \( N(w) \).
\( N(w) \) is the unique, simple \( A \)-module generated by a one-dimensional \( A_+ \)-module with character \( \chi_w \).

Proof. (i) Define \( v_\chi = 1 \otimes 1_\chi \). Then \( A_+v_\chi = \chi(A_+)v_\chi \) which proves (i).

(ii) The proof of this statement is standard.

(iii) Define \( V' \) as the sum of all proper submodules of \( V(w) \). In other words, \( V' \) is the sum of
d all submodules of \( V(w) \) which do not contain the highest weight vector \( v_\chi \). The preceding Lemma
implies that no proper \( A \)-submodule of \( V(w) \) contains a copy of the one-dimensional \( A_+ \)-module
with character \( \chi_w \). Hence the sum of any two proper \( A \)-submodules is again proper. It follows that
\( V' \) is the maximal proper submodule of \( V(w) \). Let \( N := N(w) = V(w)/V' \). \( N \) is clearly irreducible.
It unique since \( V' \) is unique. As \( N \) is cyclic, the last part of the statement is easily established.

Remark. If we define \( J = J_+ + J_- \), where \( J_+ = \sum_{\Lambda \in P_+} J_\Lambda(\Lambda, \Lambda) \) and \( J_- = \ast(J_+) \), then each
\( l \)-tuple \( \chi = \{ \chi_i \}_{i \in I} \) can be viewed as the character on \( A \) satisfying \( \text{Ker} \chi \supset J \) and \( \chi(C_{-\omega_i, \omega_i}) = \chi_i \).
Let \( N(\chi) \) denote the corresponding one-dimensional \( A \)-module. The preceding Lemma implies that
\( N(\chi) \cong N(\chi_{e,i}) \) given \( \chi_i = \chi_{e,i} \) for all \( i \), where \( e \) is the identity element of the Weyl group. More
generally, for \( w \in W \) define \( N'(w) \) to be \( N(w) \) when \( \chi_{w,i} = 1 \) for all \( i \). Then, the preceding
Lemma implies that
\[
N(w) \cong N'(w) \otimes N(\chi) \cong N(\chi) \otimes N'(w).
\]

4.3. Annihilators of \( N(w) \). Our next goal is to describe the annihilator \( J(w) \) of \( N(w) \) in \( A \). A
(left) primitive ideal of \( A \) is the annihilator of a (left) simple \( A \)-module. Let \( \text{Prim}(A) \) denote the
collection of all primitive ideals in \( A \). It is well-known ((4.4.1) in [J1]) that the \( U \)-module \( L(\Lambda)_{wA} \)
is 1-dimensional.

Definition. For \( w \in W \), \( \Lambda \in P_+ \), let \( u_{w\Lambda} \) denote the weight vector in \( L(\Lambda) \) of weight \( w\Lambda \).
Recall the quantum Demazure module \( L^+_w(\Lambda) = U_{\geq 0}u_{w\Lambda} \) and let \( L^+_w(\Lambda)^\perp = \{ C_{\xi, \Lambda} \mid \xi(\xi(L_w^+(\Lambda))) = 0 \}
for \( \xi \in L(\Lambda)^J \} \) be the orthogonal complement of \( L^+_w(\Lambda) \) in \( L(\Lambda)^J \) identified with (a subspace of)
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$L^+(\Lambda)$. Similarly, let $L_w^-(\Lambda) = \ast(L_w^+(\Lambda)^\perp) \subseteq L^-(\Lambda)^\ast$. Now define

$$Q_w^+ = \sum_{\Lambda \in P_+} L_w^+(\Lambda)^\perp,$$

$$Q_w^- = \sum_{\Lambda \in P_+} L_w^-(\Lambda)^\perp.$$ 

For $\bar{w} = (w_+, w_-) \in W \times W$, let $Q_{(w_+, w_-)}$ be the minimal 2-sided ideal of $A$ containing $Q_{w_+}^+A_-$ and $A_+Q_{w_-}^-$. Finally, define

$$J_w^+ = \sum_{\Lambda \in P_+} J_w^+(w, \Lambda, \Lambda)$$

and

$$J_w^- = \ast(J_w^+).$$

For any set $S \subset A$, let $\ll S \gg$ denotes the ideal in $A$ generated by the elements in $S$. Denote $Q_{(w, w)}$ by $Q_w$ and note that

$$J_w^t(w, \Lambda, \Lambda) = \ll C_{w, \Lambda}^\Lambda(x, \Lambda) \mid x \in L(\Lambda)^t_\Lambda, \Lambda \ll w, \Lambda \gg \subseteq \ll L_w^+(\Lambda)^\perp \gg.$$ 

This implies that, as ideals in $A$, $\ll Q_{(w, w)}^t \gg \supseteq J_w^t$. Let $J(w) = \text{Ann}_A(N(w))$ and write the elements $C_{w, \Lambda}^\Lambda$ as $C_{w, \Lambda}$ for any $w$ and $\Lambda$.

**Lemma. (A)** $J(w) \in B(w, w)$.

**Proof.** Since $J(w)$ is primitive by definition, and primitive ideals are prime (see 3.1.5 in [Dix]), it follows from Lemma 3.6 using assumption $A1$, that we can assume $J(w) \in B(w_+, w_-)$. It suffices to prove that $w_+ = w_- = w$. Writing $C_{w, \Lambda}^\Lambda = C_{w, \Lambda}$, relation 3.4(ii) implies that $\forall \Lambda \in P_+$, the image $C_{w, \Lambda}^\Lambda \in A/J(w)$ of $C_{w, \Lambda}^\Lambda$ under the canonical projection map is normal and nonzero. Thus $0 \neq C_{w, \Lambda}^\Lambda(A/J(w))v_w = (A/J(w))C_{w, \Lambda}^\Lambda v_w$. Then, by Lemma 4.1 (uniqueness of maximal element), this forces $w_+ = w$. Now we prove the second case. By definition of $w_-$, one has that $J(w) \supset \ast(J_w^t(w, -\Lambda, \Lambda))$, for any $\Lambda \in P_+$. Let $d_w^\Lambda = (C_{w, \Lambda}^\Lambda)^\ast$. Then 3.4(ii) implies $J_w^t(d_w^\Lambda w, v) = 0$ and $C_{w, \Lambda}^\Lambda d_w^\Lambda v_w = q(\Lambda, w^{-1}w, \Lambda - \Lambda) d_w^\Lambda v_w$. If $(\Lambda', w^{-1}w - \Lambda - \Lambda) = 0$, then $w = w_-$, by Proposition 3.6, and we’re done. Otherwise, assume, if possible that $(\Lambda', w^{-1}w - \Lambda - \Lambda) \neq 0$. Lemma 4.2 implies that $Ad_w^\Lambda v_w = A_+d_w^\Lambda v_w$ admits no copy of the $A_+$-module $kv_w$ and hence is 0 in $N(w)$. That is, $d_w^\Lambda v_w = 0$. Yet by 3.4, the image of $d_w^\Lambda$ in $A/J(w)$ is normal and non-zero. This gives the required contradiction and the Lemma is proved. \qed

**Lemma. (B)** Every $P \in B(w_+, w_-)$ contains $Q_{(w_+, w_-)}$.
Proof. Given \( \Lambda \in P_+ \), \( \xi \in L(\Lambda)_{\eta} \) such that \( C^{\Lambda}_{\xi,\Lambda} \not\in P \) and \( C^{\Lambda}_{\xi,a,\Lambda} \in P \), \( \forall a \in U_+^+ \), then claim \( \eta = w_+ \Lambda \). Indeed, if \( \nu \neq w_+ \nu \), then there exists \( a_\zeta \in \text{Ker}(\epsilon |_{U_+^+}) \) of weight \( \zeta = \nu - w_+ \nu \) such that \( \xi_\nu \cdot a_\zeta = \xi_{w_+ \nu} \). But

\[
C^{w_+ \nu,\nu}C^{\Lambda}_{-\eta,\Lambda} = q^{-\nu,\nu}(\nu,\nu)C^{\Lambda}_{-\eta,\Lambda}C^{w_+ \nu,\nu}.
\]

Acting on the right by \( a_\zeta \) on both sides of this equation gives that

\[
C^{w_+ \nu,\nu}C^{\Lambda}_{-\eta,\Lambda} = q^{-(w_+ \nu,\nu)+(\nu,\nu)}C^{\Lambda}_{-\eta,\Lambda}C^{w_+ \nu,\nu} \mod P.
\]

Yet \( P \supset J_{w_+}^+ \), so the commutation relations (3.4) give that

\[
C_{-\eta,\Lambda}C^{w_+ \nu,\nu} = q^{(\Lambda,\nu)-\nu,\nu}C^{w_+ \nu,\nu} \mod P.
\]

Now \( C^{w_+ \nu,\nu} \not\in P \) implies that it has a normal image which is a non-zero divisor. Thus \( (w_+ \nu, \eta) = (\Lambda, \nu) \) and therefore \( w_+ \eta = \Lambda \) and the claim is proved. Similarly, it can be shown that given \( \Lambda \in P_+ \), \( \xi \in L(\Lambda)_{\eta} \) such that \( (C^\Lambda_{\xi,\Lambda})^* \not\in P \) and \( (C^\Lambda_{\xi,a,\Lambda})^* \in P \), \( \forall a \in U_+^+ \), then \( \eta = w_+ \Lambda \). Now suppose, on the contrary, that \( P \not\supset Q_{w_+,\ldots,w_-} \). Assume that \( C^{\Lambda}_{\xi,\Lambda} \not\in P \) and yet \( C^{\Lambda}_{\xi,a,\Lambda} \in P \), with \( \xi \in L(\Lambda)_{\eta} \) and \( \eta \) maximal with respect to this property. Since \( Q_{w_+}^+ \) is stable for the right action of \( U_+^+ \), it follows that \( C^{\Lambda}_{\xi,a,\Lambda} \in P \) for all \( a \in U_+^+ \). This forces \( \eta = w_+ \Lambda \), contradicting that \( C^{\Lambda}_{-w_+ \Lambda,\Lambda} \not\in Q_{w_+}^+ \). The second case can be proved in a similar fashion.

An immediate consequence of these Lemmas is the following important Theorem:

**Theorem.** \( J(w) \supset Q_w \)

### 4.4. Correspondence with Schubert cells

Fix \( \Lambda \in P_+, w \in W \) of length \( l(w) \) and recall (Section 4.3) that the ideal generated by \( Q_w^+ \) in \( A \) contains \( J_w^+ \supset J(w, \Lambda) \) where \( J(w, \Lambda) = J_{w}^+(w\Lambda, \Lambda) \) is the 2-sided ideal in \( A \) generated by \( \{C^{\Lambda}_{\mu,\Lambda} \text{ with } \mu \leq w\Lambda\} \) and \( J_w^+ = \sum_{A \in P_+} J(w, A) \). As in [K-P], a Schubert Cell is defined as \( S_w = B_+ w B_+ / B_+ \) in \( G / B_+ \), with \( \dim S_w = 2l(w) \).

**Definition.** Let \( \phi : A \rightarrow \text{End}(V) \) be an irreducible representation of \( A \). We say that \( (V, \phi) \) corresponds to the Schubert Cell \( S_w \) if the following two conditions are satisfied:

1. \( \phi(Q_w^+) = 0 \), \( \forall \Lambda \in P_+ \).
2. \( \phi(C^{\Lambda}_{-w_+ \Lambda,\Lambda}) \neq 0 \), \( \forall \Lambda \in P_+ \).

**Remarks.**

(i) Some reasons for this terminology are discussed now. Classically, a Flag Manifold \( F_\Lambda \) is the orbit in the projective space \( \mathbb{P}(L(\Lambda)) \) of a highest weight vector. Given \( \Lambda \in P_+ \), the quantized algebra of functions on the Flag Manifold \( F_\Lambda \), also called the Representation ring, is the subalgebra of \( A \) generated by the matrix coefficients of the form \( \{C^{\Lambda}_{-\mu,\Lambda}, \forall \mu, i\} \). Denote it as
While observing that it is not a $\star$-subalgebra of $A$, its algebra structure can be described explicitly via the Cartan multiplication rule (projection on highest weight component of the tensor product $L(m\Lambda)^{\dagger} \otimes L(n\Lambda)^{\dagger} \rightarrow L((m+n)\Lambda)^{\dagger}$, since $A[F_{\Lambda}] \simeq \bigoplus_{k=0}^{\infty} L(k\Lambda)^{\dagger}$ as $U$-modules). Classically, (see Section 2.6 in [K-P]) the Schubert varieties $S_{w}^{\Lambda}$ are the closures in the projective space $\mathbb{P}(L(\Lambda))$ of the $B_+^+$-orbits of the extremal vector of weight $w\Lambda$, which is unique up to a scalar multiple. If $\Lambda \in P_{++}$ is strictly dominant, then $S_{w}^{\Lambda} \simeq S_{w}$ is the closure of a Schubert cell. Now, for $\Lambda \in P_+$, $w \in W$, let $\hat{Q}(w, \Lambda)$ be the 2-sided ideal in $A[F_{\Lambda}]$ generated by the matrix coefficients $C_{\mu,i}^{\Lambda}$ such that $\mu \nleq w\Lambda$. Then the quotient algebra $A[F_{\Lambda}]/\hat{Q}(w, \Lambda)$ is called the quantized algebra of functions on the Schubert variety $S_{w}^{\Lambda}$ and is denoted by $A[S_{w}^{\Lambda}]$. Note that $\hat{Q}(w, \Lambda) \subseteq L^+_w(\Lambda)^\perp$, with equality in the $sl_2$ case. The motivation for the definition of an irreducible representation corresponding to a Schubert cell is then apparent from Theorem 4.3, since such a representation induces a representation of $A[S_{w}^{\Lambda}]$ for all $\Lambda \in P_+$ and may thus be viewed as being “supported” on these quantum Schubert varieties.

(ii) In the affine case, the module $N_{\infty}$ (Proposition 3.4) does not correspond to any Schubert cell, since its annihilator contains all of $A_{-\Lambda}$. At present, we know of no other irreducible representations other than these. It would be nice if one could establish that these modules do indeed exhaust all the irreducible representations of $A$. 
5. WEIGHT SPACE DECOMPOSITIONS

We will assume that $0 < q < 1$ for this Section (this condition is necessary for unitarizability).

**Definition.** Recall the algebra $A = C[G]_q$ carries an involution $\ast$. An $A$-module $V$ is said to be *unitarizable* if it admits a positive-definite Hermitian form $\langle \cdot , \cdot \rangle : V \times V \rightarrow k$, such that

$$\langle av_1, v_2 \rangle = \langle v_1, a^* v_2 \rangle,$$

for any $a \in A, v_1, v_2 \in V$.

5.1. Weight Space Decompositions of $N(w)$. The goal now is to prove the Tensor Product Theorem for the structure of the irreducible $A$-modules corresponding to the Schubert cells.

**Proposition.** (A) For any $\Lambda \in P_+$ the following identity holds:

$$\sum_{\mu,i} (C_{-\mu,i;\Lambda})^\ast C_{-\mu,i;\Lambda} = 1,$$

where the sum is taken over all weights $\mu$ of $L(\Lambda)$ and $1 \leq i \leq \dim(L(\Lambda))$.

**Proof.** Define $x_{\Lambda} \in A_- \otimes A_+$ as the element corresponding to the left-hand-side of the above sum. Using the formulae (Section 3.4):

$$\Delta(a_\mu) = a_\mu \otimes 1 + K_\mu \otimes a_\mu \mod V^{>0} \otimes V^{>0},$$

$$\Delta(b_{-\nu}) = 1 \otimes b_{-\nu} + b_{-\nu} \otimes K_{-\nu} \mod V^{<0} \otimes V^{<0},$$

along with the triangular structure of $U$, one has that $\Delta^\ast(x_{\Lambda}(u)) = e(u), \forall u \in U$. $x_{\Lambda}$ is therefore a bi-invariant element of $A$ and so it equals 1. \[\square\]

Returning now to the irreducible $A$-modules $N(w)$ constructed in Section 4.2, we let $B_w$ be the $\ast$-subalgebra in $\text{End}_k(N(w)) = \text{Hom}_k(N(w), N(w))$ generated by the operators corresponding to $\{C_{w,\Lambda;\Lambda}, \Lambda \in P_+\}$. One easily checks that it is commutative, due to the fact that $\text{Ann} N(w) \supseteq Q_w \supseteq J_w^+ + J_w^-$. It follows that the elements $C_{w,\Lambda}$ and $(C_{w,\Lambda})^\ast$ have normal images $\overline{C_{w,\Lambda}}$ and $\overline{d_{w,\Lambda}} = (\overline{C_{w,\Lambda}})^\ast$ in $A/\text{Ann} N(w)$, which means that

$$\overline{C_{w,\Lambda}} \overline{d_{w,\Lambda}} = \overline{d_{w,\Lambda}} \overline{C_{w,\Lambda}}.$$

**Definition.** The *weight space* $N(w)_\gamma$, of weight $\gamma \in N\pi(C)$, is defined as the subspace of $N(w)$ in which the commuting elements $\overline{C_{w,\Lambda}}$ and $\overline{d_{w,\Lambda}} = (\overline{C_{w,\Lambda}})^\ast$ act by the scalar $q^{(\gamma,\Lambda)}$, for any $\Lambda \in P_+$. 
Remark. Make the important observation that a highest weight \(A\)-module has highest weight \(\gamma = 0\), hence \(q^{(\gamma,\Lambda)} = 1\). That is, \(\chi = 1\), where \(\chi\) is the highest weight of the induced module \(V(\chi)\). More generally (see Proposition (B)) the weight \(\gamma'\) of \(N(w)\) corresponds to a weight \(\chi'\) of \(V(\chi)\) via \(\chi'(C^\Lambda_{-w\Lambda,\Lambda}) = q^{(\gamma',\Lambda)}\), since \((C^\Lambda_{-w\Lambda,\Lambda})^*v_{\gamma'} = C^\Lambda_{-w\Lambda,\Lambda}v_{\gamma'} = q^{(\gamma',\Lambda)}v_{\gamma'}\), for any \(v_{\gamma'} \in N(w)_{\gamma'}, \Lambda \in P_+\).

Let \(\Omega(w)\) denote the collection of all \(\gamma\) for which \(N(w)_{\gamma} \neq 0\). Then, for any \(\Lambda, \Lambda' \in P_+\), and any \(\eta \in \Omega(\Lambda)\), the commutation relations (3.4) imply that

\[
\frac{(C^\Lambda_{-\eta,\Lambda})^*C_{w\Lambda'}}{q^{(\Lambda',\Lambda)} - (\eta, w\Lambda')C_{w\Lambda'}} = \frac{(C^\Lambda_{-\eta,\Lambda})^*d_{w\Lambda'}}{q^{(\Lambda',\Lambda)} - (\eta, w\Lambda')d_{w\Lambda'}}.
\]

Proposition. (B) Let \(N = N(w)\) be a simple module corresponding to a Schubert cell \(S_w\). Then \(N = \bigoplus_{\gamma \in \Omega(w)} N_{\gamma}\).

Proof. Recall that \(N(w) = A_-v_w\). Then it holds that \(\gamma \in \Omega(w)\) satisfies \(\gamma \leq 0\). Indeed, since

\[
C_{w\Lambda'}(C^\Lambda_{-\eta,\Lambda})^*v_w = q^{(\Lambda', w^{-1}\eta - \Lambda)}(C^\Lambda_{-\eta,\Lambda})^*v_w,
\]

and also,

\[
d_{w\Lambda'}(C^\Lambda_{-\eta,\Lambda})^*v_w = q^{(\eta, w\Lambda') - (\Lambda', \Lambda)}(C^\Lambda_{-\eta,\Lambda})^*d_{w\Lambda'}v_w = q^{(w^{-1}\eta - \Lambda, \Lambda')}(C^\Lambda_{-\eta,\Lambda})^*v_w,
\]

we have that \((C^\Lambda_{-\eta,\Lambda})^*v_w\) belongs to the subspace \(N(w)_{\gamma}\) with \(\gamma = w^{-1}\eta - \Lambda\). Thus,

\[
\Omega(w) = \bigcup_{\Lambda \in P_+} (w^{-1}\Omega(L_w^-(\Lambda)) - \Lambda).
\]

Now \(\gamma = 0\) only when \(\eta = w\Lambda\), in which case \((C^\Lambda_{-w\Lambda,\Lambda})^*v_w = v_w\) and so \(N_0 = kv_w\). For any other \(d \in A_-\), it follows as above that \(dN_0 \subseteq N_{\gamma'}\), with \(\gamma' < \gamma\). Theorem (4.3) implies that \(\text{Ann}(N) \supseteq Q_w\), so \(A_-\) can be replaced by \(A_-/Q_w\). But, since \(L(\Lambda)^2/L_w^-(\Lambda) \simeq L_w^-(*), A_-/Q_w\) can be identified with \(\bigoplus_{\Lambda \in P_+} L_w^-(\Lambda)^*\), which, in turn, are spanned by \(\{C^\Lambda_{\xi,\Lambda}\}^*, \xi \in L_w^-(\Lambda)^*\).

Since, \(\Omega(L_w^-(\Lambda)^*) = \Omega(L_w^-(\Lambda))\), the assertion follows from the formulae immediately preceding this Proposition. \(\square\)

Proposition. (C) \(\forall \Lambda \in P_+\), and for any \(\chi\) as in (4.2) it holds that \(|\chi(C^\Lambda_{-w\Lambda,\Lambda})| = 1\).

Proof. This is an immediate consequence of Proposition (A) and the fact that if \(v \in N_{\chi}\), then \(C^\Lambda_{-\mu,\Lambda}v = 0, \forall \mu \in \Omega(\Lambda) \setminus \{w\Lambda\}\).  \(\square\)
6. TENSOR PRODUCT THEOREM

Continue to assume that $0 < q < 1$ for this Section.

6.1. Elementary Modules. Let $\alpha_i$ be a simple root of the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(C)$. Define $\psi_i : U_q(sl(2, \mathbb{C})) \to U_q(\mathfrak{g})$ as the canonical embedding of Hopf algebras given by $E \mapsto E_i, F \mapsto F_i, K \mapsto K_i$. Consider the restriction of the dual morphism $\psi_i^* : A \to k_q[SL_2]$, which is surjective, since every finite dimensional $U_i = U_q(sl(2, \mathbb{C})$ module occurs in some integrable $U$-module. We see that, given an irreducible representation $\phi$ of $k_q[SL_2]$, we get an irreducible representation $\phi_i = \phi \psi_i^*$ of $A$. The corresponding $A$-module is called elementary, and is isomorphic to $N(s_i)$, by 4.2(iii) (see 10.1.4 in [J1]). It will be denoted simply as $N(i)$.

6.2. Proof of the Tensor Product Theorem. Consider the following Lemma:

Lemma. Let $i \in I, w \in W$ such that $l(s_iw) > l(w)$. Then

(i) $\text{Ann}_A(N(i) \otimes N(w)) \supseteq Q^+_{s_iw}$, for all $a \in N(i), b \in N(w)$.

(ii) $C^\Lambda_{-s_iw, \Lambda} (a \otimes b) = C^\Lambda_{-s_iw, \Lambda, a} (a) \otimes C^\Lambda_{-w, \Lambda, b}$.

Proof. (i) The hypothesis $l(s_iw) > l(w)$ implies that $L_{s_iw}^+(\Lambda)$ is $U_i$-stable, since $L_{s_iw}^+(\Lambda) = U_i L_{s_iw}^+(\Lambda) \Rightarrow U_i L_{s_iw}^+(\Lambda) = L_{s_iw}^+(\Lambda)$. Using the non-degenerate Shapovalov form on $L(\Lambda)$, identify $L_{s_iw}^+(\Lambda)^\perp$ in $L(\Lambda)^\mathfrak{g}$ with a $U_i$-stable subspace of $L(\Lambda)$ complementing $L_{s_iw}^+(\Lambda)$. Fix a basis $\{u_j\}_{j \in \mathbb{Z}_{\geq 0}}$ for $L(\Lambda)$ which is compatible with the inclusion $L_{w}^+(\Lambda) \subseteq L_{s_iw}^+(\Lambda)$, i.e. with the choice of bases $\{u_j\}_{j=1}^r$ for $L_{w}^+(\Lambda)$, $\{u_j\}_{j=1}^r$ for $L_{s_iw}^+(\Lambda)$ (see [Lak]) and $\{u_j\}_{j=r}^t$ for $L_{w}^+(\Lambda)$.

Let $\{l_j\}$ be the dual basis for $L(\Lambda)^\mathfrak{g}$. Let $\xi \in L(\Lambda)^\mathfrak{g}, a \in N(i), b \in N(w)$. Then

$$C^\Lambda_{\xi, \Lambda} (a \otimes b) = \sum_{j \in \mathbb{Z}_{\geq 0}} C^\Lambda_{\xi, a_j} (a) \otimes C^\Lambda_{l_j, \Lambda} (b),$$

where

$$\Delta(C^\Lambda_{\xi, \Lambda}) = \sum_{j \in \mathbb{Z}_{\geq 0}} C^\Lambda_{\xi, a_j} \otimes C^\Lambda_{l_j, \Lambda}.$$

Now, by Theorem 4.3, we have that $\text{Ann}_A N(w) \supseteq Q^+_{w}$. It follows that $C^\Lambda_{l_j, \Lambda} (b) = 0$, for any $j > s$. Clearly $C^\Lambda_{\xi, u_j} (a) = \psi_i^* (C^\Lambda_{\xi, a_j}) (a)$. For $\xi \in L_{s_iw}^+(\Lambda)^\perp$ and $v \in L_{s_iw}^+(\Lambda)$, we have that $C^\Lambda_{\xi, v} (x) = \xi (xv) = 0$ if $x \in U_i$. So, in particular, $C^\Lambda_{\xi, u_j} (a) = 0 \ \forall j \leq r$ and $\xi \in L_{s_iw}^+(\Lambda)^\perp$, which proves (i).

(ii) Let $\xi \in L(\Lambda)^\mathfrak{g}$ of weight $-s_iw\Lambda$. It is clear that the only non-zero contributions to the right hand side of the expression

$$C^\Lambda_{\xi, \Lambda} (a \otimes b) = \sum_{j \in \mathbb{Z}_{\geq 0}} C^\Lambda_{\xi, a_j} (a) \otimes C^\Lambda_{l_j, \Lambda} (b)$$
such that $f_{\xi;u_j}(x) \neq 0$ for some $x \in U_1$.

Further, one can assume $u_j$ to be weight vectors with $u_s = u_{w\Lambda}$. But the $U_i$-module generated by $\xi$ has weights $\{-s_i w\Lambda, -s_i w\Lambda - \alpha_i, ..., -w\Lambda\}$. The definition of $L_{w\Lambda}^+(\Lambda)$ then implies that the only non-zero term corresponds to $j = s$. So $u_j = u_{w\Lambda}$ whereas $l_j = -w\Lambda$, which proves $(ii)$.

We are now ready for our main result, the Tensor Product Theorem:

**Theorem.** $N := N(i) \otimes N(w)$ is unitarizable, irreducible, and isomorphic to $N(s_i w)$.

**Proof.** Observe that $N(i)$ is unitarizable, by the results of 5.1 and 6.1. The tensor product of any 2 unitarizable $A$-modules $M_1$ and $M_2$ is also a unitarizable $A$-module under

$$\langle v_1 \otimes v_2, v_3 \otimes v_4 \rangle = \langle v_1, v_3 \rangle_{M_1} \langle v_2, v_4 \rangle_{M_2}.$$

Thus, it follows using induction that $N$ is unitarizable. We shall prove the irreducibility using induction on the length of $w$.

For $l(w) = 1$, $N(i)$ is irreducible by definition (6.1). Let $w = s_{i_1}s_{i_2}...s_{i_k}$ be a reduced expression for $w$. Suppose $N(w) \simeq N(i_1) \otimes ...N(i_k)$, and is irreducible.

Let $\{e_k\}_{k=0}^\infty$ (resp. $\{f_M\}_{M \in \mathbb{Z}^\infty(w)}$) be an orthonormal basis of $N(i)$ (resp. $N(w)$). Thus $\{e_k \otimes f_M\}$ is an orthonormal basis for $N$. Then the arguments in the proof of Lemma 6.2(ii) imply that

$$C^\Lambda_{s_i w\Lambda,\Lambda}(e_k \otimes f_M) = \psi_i^* C^\Lambda_{s_i w\Lambda,\Lambda}(e_k) \otimes C^\Lambda_{w\Lambda,\Lambda}(f_M).$$

It is easy to see (via the commutation relations) that $\text{Ker}(C^\Lambda_{w\Lambda,\Lambda})$ is an $A$-invariant subspace in $N(w)$. But $N(w)$ is irreducible, which forces this kernel to be trivial. Hence, using $C^\Lambda_{s_i w\Lambda,\Lambda}(e_k \otimes f_M) = \psi_i^* C^\Lambda_{s_i w\Lambda,\Lambda}(e_k) \otimes C^\Lambda_{w\Lambda,\Lambda}(f_M)$, Lemma 6.2 implies the following:

1. The $\ast$-algebra $B_{s_i w}$ in $\text{End}(N)$ generated by $C^\Lambda_{s_i w\Lambda,\Lambda}(\Lambda \in P_+)$ is commutative and diagonalizable under the basis $\{e_k \otimes f_M\}$.

2. The eigenvalue $\lambda^{(k,M)}_\Lambda$ of $C^\Lambda_{s_i w\Lambda,\Lambda}$ corresponding to $e_k \otimes f_M$ satisfies $|\lambda^{(k,M)}_\Lambda| < 1$ unless $k = 0, M = M_0$, where $M_0 = (0, 0, ..., 0)$.

3. $e_0 \otimes f_{M_0}$ is the only eigenvector for any $C^\Lambda_{s_i w\Lambda,\Lambda}$ with the property that $|\lambda^{(0,M_0)}_\Lambda| = 1$.

Also, all eigenvalues $\lambda^{(k,M)}_\Lambda$ are nonzero. Now, let $W' \supsetneq N$ be a non-trivial $A$-invariant subspace, such that $e_0 \otimes f_{M_0} \notin W'$. Thus, by (1), the action of $B_{s_i w}|_{W'}$ is diagonalizable. For any $v \in N$, Proposition 5.2(A) implies that

$$\langle C^\Lambda_{-\mu\Lambda}v, C^\Lambda_{-\mu\Lambda}v \rangle \leq \langle v, v \rangle.$$
Then, by Lemma (3.4), there exists a vector $v$ in $W'$ such that $fv = \chi(f)v$ for any $f \in A_+$, which can be chosen among the vectors of the basis $\{e_k \otimes f_M\}$ with $(k, M) \neq (0, M_0)$. If this is not the case, take the orthogonal complement of $W'$ which is also an $A$-module. Therefore $|\chi(C^\Lambda_{-s_iw,\Lambda})| < 1$ for some $\Lambda \in P_+$.

Calling such a $v$ in $W'$ as primitive and the corresponding homomorphism $\chi : A_+ \rightarrow \mathbb{C}$ its weight, we introduce a natural partial order $\chi_1 \prec \chi_2 \iff |\chi_1(f)| \leq |\chi_2(f)|$, for any $f \in \{C^\Lambda_{-s_iw,\Lambda}\}_{\Lambda \in P_+}$ (we can use the same notations, since the homomorphism $\chi$ has a canonical extension to $B_{s_iw}$ such that $\chi((C^\Lambda_{-s_iw,\Lambda})^\ast) = \overline{\chi(C^\Lambda_{-s_iw,\Lambda})}$). Let $\chi'$ be a maximal element in the set of primitive weights, with respect to the above order, with eigenvector $v'$. Then the commutation relations imply that any vector of the form $C^\Lambda_{-s_iw,\Lambda}v'$ is an eigenvector for $B_{s_iw}$ with weight larger than $\chi'$, which is absurd. This forces $C^\Lambda_{-\mu,j;\Lambda}v' = 0$, $\forall \lambda \neq s_iw\Lambda$. Now Proposition 5.2(A) implies that $\sum_{\mu,j}(C^\Lambda_{-\mu,j;\Lambda})^* C^\Lambda_{-\mu,j;\Lambda}v' = v'$. But all except one of the summands in this equation disappear, and so

$$(C^\Lambda_{-s_iw,\Lambda;j;\Lambda})^* C^\Lambda_{-s_iw,\Lambda;j;\Lambda}v' = v'.$$

One may assume that $\langle v', v' \rangle_{N(i) \otimes N(w)} = 1$. Thus it follows that $|\chi(C^\Lambda_{-s_iw,\Lambda})| = 1$, $\forall \Lambda \in P_+$. This is possible only if $v' = e_0 \otimes f_{M_0}$. But $e_0 \otimes f_{M_0} \notin W'$ by assumption. Thus we have a contradiction. Hence $N$ is irreducible. The fact that $N \simeq N(s_iw)$ follows at once from the preceding Lemma and Theorem 4.2(iv).

**Corollary.** The characterization $N(w) \simeq N(i_1) \otimes \ldots \otimes N(i_k)$ is independent of the choice of reduced decomposition $s_{i_1} \ldots s_{i_k}$ for $w$. 

\[\Box\]
7. OPEN QUESTIONS

(i) Compute the prime and primitive spectra of $\mathbb{C}_q[G]$ as well as the spectrum of the Abelian category $\mathbb{C}_q[G] - \text{mod}$ in the sense of A. Rosenberg [R].

(ii) Are there any solutions to the Zamolodchikov tetrahedron equations, using the intertwiners of isomorphic, irreducible $A$-modules?

(iii) Define the affine quantum Weyl group and relate it with the Dynamical Quantum Weyl groups of Varchenko and Etingof [E-V].

(iv) Compute the spectra of the quantized function algebra, $R = \mathbb{C}_q[G/B_+]$ (resp. $R_+ = \mathbb{C}_q[G/N_+]$) of the flag manifold (resp. base affine space).

(v) What are the answers to the above questions when you specialize to the roots of unity case?
References

[APW] H.H.Andersen, P. Polo and Wen Kexin, Representations of quantum algebras, *Invent. math.* **104**, 1991, 1-59.

[C] V.Chari, Integrable representations of affine Lie algebras, *Invent. Math.*, **85**, (1986), 317-335.

[C-P1] V.Chari and A.Pressley, *A Guide to Quantum Groups* Cambridge University Press **1994**.

[C-P2] V.Chari and A.Pressley, A new family of irreducible, integrable modules for affine Lie algebras, *Math. Ann.*, **277**, (1987), 543-562.

[D-L] C.De Concini, V.Lyubashenko, *Quantum function algebra at roots of 1*. Adv. Math. 108 (1994), no. 2, 205-262.

[Dix] J.Diximier, *Enveloping Algebras*, North-Holland mathematical library; v.14, **1977**.

[D1] V.Drinfeld, *Quantum Groups*, Proc. Internat. Congr. Math. (Berkeley 1986), Amer. Math. Soc., Providence, RI, 1987, pp.798-820.

[D2] V.Drinfeld, On Almost Cocommutative Hopf Algebras, *Leningrad Math. J.* 1 (1990) 321-42.

[E-V] P.Etingof, A.Varchenko, *Dynamical Weyl groups and applications*, math.QA/0011001.

[G] F.Gavarini, *The R-matrix action of untwisted affine quantum groups at roots of 1*, math.QA/9805009 v3.

[Ji] M. Jimbo, A q-difference analog of U(g) and the Yang-Baxter equation, *Lett. Math. Phys.* **11** (1986), 247-252.

[J0] A.Joseph, On the mock Peter-Weyl theorem and the Drinfeld double of a double Journal fur die reine und angewandte Mathematik, **507** (1999), 37-56.

[J1] A.Joseph, *Quantum Groups and their Primitive Ideals*, Springer-Verlag, **1995**.

[J2] A.Joseph, Quantum Kac Moody Lie Algebras, *J. Algebra*, 1999.

[K] V.Kac, *Infinite Dimensional Lie Algebras*, 3rd edition, Cambridge University Press, Cambridge, New York, **1990**.

[K-P] V.Kac and D.Peterson, Constructing Groups Associated to Infinite-Dimensional Lie Algebras, in *Infinite Dimensional Groups with Applications*, MSRI publ. 4, Springer-Verlag, 1985.

[K-V] M. Kapranov, V. Voevodsky, *2-categories and Zamolodchikov tetrahedra equations*. Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), 177–259, Proc. Sympos. Pure Math., 56, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[Kas1] M.Kashiwara, *The flag manifold of Kac-Moody Lie algebra* in Algebraic Analysis, Geometry and Number Theory, Baltimore, MD, 1988, pp. 161-190, Johns Hopkins univ Press, 1989.

[Kas2] M.Kashiwara, *On Level Zero Representations of Quantized Affine Algebras*, math.QA/0010293.

[K-S] D.Kazhdan, Y. Soibelman, *Representations of the quantized function algebra, 2-categories and Zamolodchikov tetrahedra equation*. The Gelfand Mathematical Seminars, 1990–1992, **163–171**, Birkhuser Boston, Boston, MA, 1993.

[K-T] S.Khoroshkin and V.Tolstoy, Uniqueness Theorem for universal R matrix, *Lett. Math. Phys.* **24** (1992) 231-44.

[Ku] Shrawan Kumar, *Kac-Moody groups, their Flag Varieties, and Representation Theory*, Progress in Mathematics- Volume 204, Birkhauser, 2002.

[Lak] V.Lakshmibai, Bases for Quantum Demazure Modules - II. *Proc. Symp. Pure Math.* **56** (1994) 149-168.
[L1] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, Boston, 1993.

[L2] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. in Math. 70 (1988), 237-249.

[M] O. Mathieu, *Formule de Charactère pour les algèbres de Kac Moody Générales*, Astérisque, 159-160, 1988.

[Mo] C. Mokler, *A monoid completion of a Kac-Moody group*, preprint available at http://www.math.unibas.ch/ mokler.

[N] B. Narayanan, *Representations of the quantized function algebras of Kac-Moody Lie algebras*, PhD Thesis, Kansas State University, May 2002.

[ND] Diep Ngoc Do, *Quantized Algebras of Functions on Affine Hecke Algebras*, math.QA/0101015.

[O] C. Ohn, "Classical" flag varieties for quantum groups: the standard quantum SL(n, C), math.QA/0007005.

[R] A. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Mathematics and its Applications, 330. Kluwer Academic Publishers Group, Dordrecht, 1995.

[Ru] W. Rudin, *Functional Analysis*, McGraw Hill, New York, 1975.

[S1] Y. Soibelman, The algebra of functions on a compact quantum group, and its representations, Leningrad Math. J. 2 (1991) 193-225.

[S2] L. Korogodski and Y. Soibelman, *Algebras of Functions on Quantum Groups: Part I*, Mathematical Surveys and Monographs, Volume 56, American Mathematical Society, 1998.

Department of Mathematics, University of Arizona