SYNDETIC SUBMEASURES AND PARTITIONS OF $G$-SPACES AND GROUPS

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Abstract. We prove that for every $k \in \mathbb{N}$ each countable infinite group $G$ admits a partition $G = A \cup B$ into two sets which are $k$-meager in the sense that for every $k$-element subset $K \subset G$ the sets $KA$ and $KB$ are not thick. The proof is based on the fact that $G$ possesses a syndetic submeasure, i.e., a left-invariant submeasure $\mu : \mathcal{P}(G) \to [0,1]$ such that for each $\varepsilon > \frac{1}{|G|}$ and subset $A \subset G$ with $\mu(A) < 1$ there is a set $B \subset G \setminus A$ such that $\mu(B) < \varepsilon$ and $FB = G$ for some finite subset $F \subset G$.

In this paper we continue the studies [11]–[13] of combinatorial properties of partitions of $G$-spaces and groups.

By a $G$-space we understand a non-empty set $X$ endowed with a left action of a group $G$. The image of a point $x \in X$ under the action of an element $g \in G$ is denoted by $gx$. For two subsets $F \subset G$ and $A \subset X$ we put $FA = \{fa : f \in F, a \in A\} \subset X$.

1. Prethick sets in partitions of $G$-spaces

A subset $A$ of a $G$-space $X$ is called

- large if $FA = X$ for some finite subset $F \subset G$;
- thick if for each finite subset $F \subset G$ there is a point $x \in X$ with $Fx \subset A$;
- prethick if $KA$ is thick for some finite set $K \subset G$.

Now we insert number parameters in these definitions. Let $k, m \in \mathbb{N}$. A subset $A$ of a $G$-space $X$ is called

- $m$-large if $FA = X$ for some subset $F \subset G$ of cardinality $|F| \leq m$;
- $m$-thick if for each finite subset $F \subset G$ of cardinality $|F| \leq m$ there is a point $x \in X$ with $Fx \subset A$;
- $(k,m)$-prethick if $KA$ is $m$-thick for some set $K \subset G$ of cardinality $|K| \leq k$;
- $k$-prethick if $KA$ is thick for some set $K \subset G$ of cardinality $|K| \leq k$;
- $k$-meager if $A$ is not $k$-prethick (i.e., $KA$ is not thick for any subset $K \subset G$ of cardinality $|K| \leq k$).

In the dynamical terminology [6, 4.38], large subsets are called syndetic and prethick subsets are called piecewise syndetic. We note also that these notions can be defined in much more general context of balleans [11], [13].

The following proposition is well-known [6, 4.41], [9, 1.3], [11, 11.2].

Proposition 1.1. For any finite partition $X = A_1 \cup \cdots \cup A_n$ of a $G$-space $X$ one of the cells $A_i$ is prethick and hence $k$-prethick for some $k \in \mathbb{N}$.

For finite groups the number $k$ in this proposition can be bounded from above by $n(\ln(\frac{|G|}{n})+1)$. We consider each group $G$ as a $G$-space endowed the natural left action of $G$.

Proposition 1.2. Let $G$ be a finite group and $n, k \in \mathbb{N}$ be numbers such that $k \geq n \cdot \left(\ln(\frac{|G|}{n}) + 1\right)$. For any $n$-partition $G = A_1 \cup \cdots \cup A_n$ of $G$ one of the cells $A_i$ is $k$-large and hence $k$-prethick.

Proof. One of the cells $A_i$ of the partition has cardinality $|A_i| \geq \frac{|G|}{n}$. Then by [15] or [2, 3.2], there is a subset $B \subset G$ of cardinality $|B| \leq \frac{|G|}{|A_i|}(\ln |A_i| + 1) \leq n(\ln(\frac{|G|}{n}) + 1) \leq k$ such that $G = BA_i$. It follows that the set $A_i$ is $k$-large and hence $k$-prethick. \hfill \Box

For $G$-spaces we have the following quantitative version of Proposition 1.1.
Proposition 1.3. Let \( m, n \in \mathbb{N} \). For any \( n \)-partition \( X = A_1 \cup \cdots \cup A_n \) of a \( G \)-space \( X \) one of the cells \( A_i \) is \((m^{n-1}, m)\)-prethick in \( X \).

Proof. For \( n = 1 \) the proposition is trivial. Assume that it has been proved for some \( n \) and take any partition \( X = A_0 \cup \cdots \cup A_n \) of \( X \) into \((n + 1)\) pieces. If the cell \( A_0 \) is \((1, m)\)-prethick, then we are done. If not, then there is a set \( F \subseteq X \) of cardinality \(|F| \leq m \) such that \( Fx \not\subseteq A_0 \) for all \( x \in X \). This implies that \( x \in F^{-1}(A_0 \cup \cdots \cup A_n) \) and then by the inductive assumption, there is an index \( 1 \leq i \leq n \) such that the set \( F^{-1}A_i \) is \((m^{n-1}, m)\)-prethick. The latter means that there is a subset \( E \subseteq G \) of cardinality \(|E| \leq m^{n-1} \) such that \( EF^{-1}A_i \) is \( m \)-thick. Since \(|EF^{-1}| \leq |E| \cdot |F| \leq m^{n-1}m = m^n \), the set \( A_i \) is \((m^n, m)\)-prethick. \( \square \)

Looking at Proposition 1.3 it is natural to ask what happens for \( n = \omega \). Is there any hope to find for every \( n \in \mathbb{N} \) a finite number \( k_n \) such that for each \( n \)-partition \( X = A_1 \cup \cdots \cup A_n \) some cell \( A_i \) of the partition is \( k_n\)-prethick? In fact, \( G \)-spaces with this property do exist.

Example 1.4. Let \( X \) be an infinite set endowed with the natural action of the group \( G = S_X \) of all bijections of \( X \). Then each subset \( A \subseteq X \) of cardinality \(|A| = |X| \) is 2-large, which implies that for each finite partition \( X = A_1 \cup \cdots \cup A_n \) one of the cells \( A_i \) has cardinality \(|A_i| = |X| \) and hence is 2-large and 2-prethick.

The action of the normal subgroup \( FS_X \subseteq S_X \) consisting of all bijections \( f : X \to X \) with finite support \( \text{supp}(f) = \{ x \in X : f(x) \neq x \} \) has a similar property.

Example 1.5. Let \( X \) be an infinite set endowed with the natural action of the group \( G = FS_X \) of all finitely supported bijections of \( X \). Then each infinite subset \( A \subseteq X \) is thick, which implies that for each finite partition \( X = A_1 \cup \cdots \cup A_n \) one of the cells \( A_i \) is infinite and hence is thick and 1-prethick.

However the \( G \)-spaces described in Examples 1.3 and 1.4 are rather pathological. In the next section we shall show that each \( G \)-space admitting a syndetic submeasure for every \( k \in \mathbb{N} \) can be covered by two \( k \)-meager (and hence not \( k \)-prethick) subsets. In Section 2, using syndetic submeasures we shall prove that each countable infinite group admits a partition into two \( k \)-meager subsets for every \( k \in \mathbb{N} \).

2. Syndetic submeasures on \( G \)-spaces

A function \( \mu : \mathcal{P}(X) \to [0, 1] \) defined on the family of all subsets of a \( G \)-space \( X \) is called

- \( G \)-invariant if \( \mu(gA) = \mu(A) \) for each \( g \in G \) and a subset \( A \subseteq X \);
- monotone if \( \mu(A) \leq \mu(B) \) for any subsets \( A \subseteq B \subseteq X \);
- subadditive if \( \mu(A \cup B) \leq \mu(A) + \mu(B) \) for any sets \( A, B \subseteq X \);
- additive if \( \mu(A \cup B) = \mu(A) + \mu(B) \) for any disjoint sets \( A, B \subseteq X \);
- a submeasure if \( \mu \) is monotone, subadditive, and \( \mu(\emptyset) = 0, \mu(X) = 1 \);
- a measure if \( \mu \) is an additive submeasure;
- a syndetic submeasure if \( \mu \) is a \( G \)-invariant submeasure such that for each subset \( A \subseteq X \) with \( \mu(A) < 1 \) and each \( \varepsilon > 1/|X| \) there is a large subset \( L \subseteq X \setminus A \) of submeasure \( \mu(L) < \varepsilon \).

In this definition we assume that \( 1/|X| = 0 \) if the \( G \)-space \( X \) is infinite.

Proposition 2.1. A finite \( G \)-space \( X \) possesses a syndetic submeasure if and only if \( X \) is transitive.

Proof. If \( X \) is transitive, then the counting measure \( \mu : \mathcal{P}(X) \to [0, 1], \mu : A \mapsto |A|/|X| \), is syndetic.

Now assume conversely that a finite \( G \)-space \( X \) admits a syndetic submeasure \( \mu : \mathcal{P}(X) \to [0, 1] \). If \( X \) is a singleton, then \( X \) is transitive. So, we assume that \( X \) contains more than one point. Since the empty set \( A = \emptyset \) has submeasure \( \mu(A) = 0 < 1 \), for the number \( \varepsilon = \frac{|X|}{|X| - 1} = \frac{1}{|X| - 1} \) there is a large subset \( L \subseteq X \setminus A \subseteq X \) of submeasure \( \mu(L) < \varepsilon \). It follows that \( L \), being large in \( X \), has non-empty intersection with each orbit \( Gx \), \( x \in X \). Replacing \( L \) by a smaller subset we can assume that \( L \) meets each orbit in exactly one point. For every point \( x \in L \) we can find a finite subset \( F_x \subseteq G \) of cardinality \(|Gx| - 1 \) such that \( F_x x = Gx \setminus \{ x \} \). Then the set \( F = \{ 1 \} \cup \bigcup_{x \in L} F_x \) has cardinality \( |F| = 1 + \sum_{x \in L} |Gx| - 1 = 1 - |L| + \sum_{x \in L} |Gx| = 1 - |L| + |X| \) and \( FL = X \). By the subadditivity and the \( G \)-invariance of the submeasure \( \mu \), we get

\[
1 = \mu(FL) \leq |F| \cdot \mu(L) < |F| \cdot \varepsilon = \frac{|F|}{|X| - 1} = \frac{1 - |L| + X}{|X| - 1},
\]

which implies \(|L| = 1\). This means that \( X \) has exactly one orbit and hence is transitive. \( \square \)
For $G$-spaces admitting a syndetic submeasure we have the following result completing Propositions 1.1–1.3.

**Theorem 2.2.** Let $G$ be a countable group and $X$ be an infinite $G$-space possessing a syndetic submeasure $\mu : \mathcal{P}(X) \to [0, 1]$. Then for every $k \in \mathbb{N}$ there is a partition $X = A \cup B$ of $X$ into two $k$-meager subsets.

**Proof.** Fix any $k \in \mathbb{N}$ and choose an enumeration $(K_n)_{n=1}^\infty$ of all $k$-element subsets of $G$.

Using the definition of a syndetic submeasure, we can inductively construct two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of large subsets of $X$ satisfying the following conditions for every $n \in \mathbb{N}$:

1. $A_n \subset X \setminus \bigcup_{i<n} K^{-1}_i B_i$;
2. $\mu(A_n) < \frac{1}{k^2}$;
3. $B_n \subset X \setminus \bigcup_{1 \leq i \leq n} K^{-1}_i A_i$;
4. $\mu(B_n) < \frac{1}{k^2}$.

At each step the choice of the set $A_n$ is possible as

$$\mu(\bigcup_{i<n} K^{-1}_i B_i) \leq \sum_{i<n} \sum_{x \in K^{-1}_i} \mu(xB_i) = \sum_{i<n} |K^{-1}_i| \cdot \mu(B_i) \leq \sum_{i<n} k^2 \frac{1}{k^2} < 1$$

by the subadditivity of $\mu$. By the same reason, the set $B_n$ can be chosen.

After completing the inductive construction, we get the disjoint sets $A = \bigcup_{n=1}^\infty K_n A_n$ and $B = \bigcup_{n=1}^\infty K_n B_n$.

It remains to check that the sets $A$ and $X \setminus A$ are $k$-meager. Given any $k$-element subset $K \subset G$ we need to prove that the sets $K \cap A$ and $K \setminus A$ are not thick. Find $n \in \mathbb{N}$ such that $K_n = K^{-1}$.

Since the set $K_n B_n$ is disjoint with $A$, the large set $B_n$ is disjoint with $K_n^{-1} A = KA$, which implies that $X \setminus KA$ is large and $KA$ is not thick.

Next, we show that the set $K(X \setminus A) = K^{-1}(X \setminus A)$ is not thick. We claim that $A_n \subset X \setminus K^{-1}_n(X \setminus A)$. Assuming the converse, we can find a point $a \in A_n \cap K^{-1}_n(X \setminus A)$. Then $K_n a$ intersects $X \setminus A$, which is not possible as $K_n a \subset K_n A_n \subset A$. So, the set $X \setminus K(X \setminus A) \supset A_n$ is large, which implies that $K(X \setminus A)$ is not thick. \hfill $\square$

### 3. Toposyndetic submeasures on $G$-spaces

In light of Theorem 2.2, it is important to detect $G$-spaces possessing a syndetic submeasure. We shall find such spaces among $G$-spaces possessing a toposyndetic submeasure. To define such submeasures, we need to recall some information from Measure Theory.

Let $\mu : \mathcal{P}(X) \to [0, 1]$ be a submeasure on a set $X$. A subset $A \subset X$ is called $\mu$-measurable if $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$ for each subset $B \subset X$. By (the proof of) [4, 2.1.3], the family $\mathcal{A}_\mu$ of all $\mu$-measurable subsets of $X$ is an algebra (called the measure algebra of $\mu$) and the restriction $\mu|_{\mathcal{A}_\mu}$ is additive in the sense that $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint $\mu$-measurable sets $A, B \in \mathcal{A}_\mu$.

A $G$-invariant submeasure $\mu : \mathcal{P}(X) \to [0, 1]$ on a $G$-space $X$ will be called toposyndetic if $\mathcal{A}_\mu \cap \tau$ is a base of some $G$-bounded $G$-invariant regular topology $\tau$ on $X$. The $G$-boundedness of the topology $\tau$ means that each non-empty open set $U \in \tau$ is large in $X$. The $G$-boundedness of $\tau$ implies the density of all orbits $Gx$, $x \in X$, in the topology $\tau$.

**Theorem 3.1.** If a $G$-space $X$ admits a toposyndetic submeasure, then each non-empty $G$-invariant subspace $Y \subset X$ possesses a syndetic submeasure.

**Proof.** Let $\mu : \mathcal{P}(X) \to [0, 1]$ be a toposyndetic submeasure on $X$ and $\tau$ be a $G$-bounded $G$-invariant Tychonoff topology on $X$ such that $\mathcal{A}_\mu \cap \tau$ is a base of the topology $\tau$.

Fix any non-empty $G$-invariant subspace $Y \subset X$. The $G$-boundedness of the topology $\tau$ implies that $Y$ is dense in the topological space $(X, \tau)$. If the regular topological space $(X, \tau)$ has an isolated point $x$, then by the $G$-boundedness of the topology $\tau$ for the open set $U = \{x\}$ there is a finite set $F \subset G$ with $X = FU \subset Gx$, which means that $X$ is a finite transitive space. By the density of $Y$ in $X$, $Y = X$ and by Proposition 2.1 $Y$ possesses a syndetic submeasure.

So, we assume that the topological space $(X, \tau)$ has no isolated points. The $G$-invariant submeasure $\mu$ induces a $G$-invariant submeasure $\lambda : \mathcal{P}(Y) \to [0, 1]$ defined by $\lambda(A) = \mu(\bar{A})$ for every subset $A \subset Y$, where $\bar{A}$ is the closure of $A$ in the topological space $(X, \tau)$. To see that the submeasure $\lambda$ is syndetic, fix any $\varepsilon < \frac{1}{|Y|} = 0$ and any subset $A \subset Y$ with $\lambda(A) < 1$. Then $\mu(\bar{A}) = \lambda(A) < 1$, which implies that $X \setminus \bar{A}$ is an
open non-empty subset of \( X \). Since \( \mathcal{A}_\mu \cap \tau \) is a base of the topology \( \tau \), there is a non-empty \( \mu \)-measurable open set \( U \subset X \setminus A \subset X \setminus A \). Since the topological space \( (X, \tau) \) has no isolated points, we can fix pairwise disjoint non-empty open sets \( U_1, \ldots, U_n \subset U \) for some integer number \( n > 1/\varepsilon \). Since \( \mathcal{A}_\mu \cap \tau \) is a base of the topology \( \tau \), we can additionally assume that these open sets \( U_1, \ldots, U_n \) are \( \mu \)-measurable, which implies that \( \sum_{i=1}^n \mu(U_i) \leq 1 \) and hence \( \mu(U_i) \leq \frac{1}{n} < \varepsilon \) for some \( i \leq n \). By the regularity of the topological space \( (X, \tau) \), the open set \( U_i \) contains the closure \( \overline{V} \) of some non-empty open set \( V \subset X \). The \( G \)-boundedness of \( X \) guarantees that \( V \) is large in \( X \) and hence \( V \cap Y \) is large in \( Y \). Also \( \lambda(V \cap Y) = \mu(V \cap Y) \leq \mu(V) \subset \mu(U_i) < \varepsilon \). This means that the submeasure \( \lambda \) on \( Y \) is syndetic. \( \square \)

Many examples of \( G \)-spaces having a toposyndetic submeasure occur among subspaces of minimal compact measure \( G \)-spaces. By a compact (measure) \( G \)-space we understand a \( G \)-space \( X \) endowed with a compact Hausdorff \( G \)-invariant topology \( \tau_X \) (and a \( G \)-invariant probability Borel \( \sigma \)-additive measure \( \lambda_X : \mathcal{B}(X) \rightarrow [0,1] \) defined on the \( \sigma \)-algebra \( \mathcal{B}(X) \) of Borel subsets of \( X \)). A compact \( G \)-space \( X \) is called minimal if each orbit \( Gx, x \in X \), is dense in \( X \).

**Theorem 3.2.** If \((X, \tau_X, \lambda_X)\) is a minimal compact measure \( G \)-space, then each non-empty \( G \)-invariant subspace \( Y \) of \( X \) possesses a (topo)syndetic submeasure.

**Proof.** By the minimality of \( X \), the \( G \)-invariant subspace \( Y \) is dense in \( X \). Let \( \tau = \{U \cap Y : U \in \tau_X\} \) be the induced topology on \( Y \). The \( G \)-invariant measure \( \lambda_X : \mathcal{B}(X) \rightarrow [0,1] \) induces a \( G \)-invariant submeasure \( \mu : \mathcal{P}(Y) \rightarrow [0,1] \) defined by the formula \( \mu(A) = \lambda_X(\overline{A}) \) for \( A \subset Y \), where \( \overline{A} \) denotes the closure of \( A \) in the compact space \( (X, \tau_X) \). To prove that the submeasure \( \mu \) is toposyndetic, it remains to prove that the topology \( \tau \) is \( G \)-bounded and \( \mathcal{A}_\mu \cap \tau \) is a base of the topology \( \tau \).

Consider the algebra \( \mathcal{A}_X = \{A \subset X : \lambda_X(\partial A) = 0\} \) consisting of subsets \( A \subset X \) whose boundary \( \partial A \) in \( X \) have measure \( \lambda_X(\partial A) = 0 \), and let \( \mathcal{A}_Y = \{A \cap Y : A \in \mathcal{A}_X\} \). It can be shown that each set \( A \subset \mathcal{A}_Y \) is \( \mu \)-measurable and \( \mathcal{A}_Y \cap \tau \subset \mathcal{A}_\mu \cap \tau \) is a base of the topology \( \tau \). The \( G \)-boundedness of the topology \( \tau \) on \( Y \) is proved in the following lemma. Therefore, \( \mu \) is a toposyndetic submeasure on \( X \). By the proof of Theorem 3.2, the submeasure \( \mu \) is syndetic. \( \square \)

**Lemma 3.3.** For each minimal compact \( G \)-space \( X \), the induced topology on each \( G \)-invariant subspace \( Y \subset X \) is \( G \)-bounded.

**Proof.** To show that the induced topology on \( Y \) is \( G \)-bounded, fix any non-empty open subset \( U \subset Y \). Find an open set \( \tilde{U} \subset X \) such that \( \tilde{U} \cap Y = U \). By the regularity of the compact Hausdorff space \( X \), there is a non-empty open subset \( V \subset X \) with \( \overline{V} \subset \tilde{U} \).

By a classical Birkhoff theorem in Topological Dynamics (see e.g. Theorem 19.26 [6]), the minimal compact \( G \)-space \( X \) contains a uniformly recurrent point \( y \in X \). The uniform recurrence of \( y \) means that for each open neighborhood \( O_y \subset X \) of \( y \) the set \( \{g \in G : gy \in O_y\} \) is large in \( G \). By the density of the orbit \( G \) there is \( s \in G \) with \( sy \in V \). Then \( s^{-1}V \) is a neighborhood of \( y \) and by the uniform recurrence of \( y \), the set \( L = \{g \in G : gy \in s^{-1}V\} \) is large in \( G \). Consequently, we can find a finite subset \( F \subset G \) such that \( G = FL \). Then \( G \subset FS^{-1}V \subset FS^{-1} \) which implies that the open set \( FS^{-1}V \) is dense in \( X \). Consequently, \( X = FS^{-1}V \subset FS^{-1} \) and \( Y = FS^{-1}(Y \cap U) \subset FS^{-1}U \), witnessing that the topology of \( Y \) is \( G \)-bounded. \( \square \)

### 4. Groups possessing a toposyndetic submeasure

In this section we shall detect groups possessing a toposyndetic submeasure. Each group \( G \) will be considered as a \( G \)-space endowed with the natural left action of the group \( G \). A group \( G \) is called amenable if it admits an \( G \)-invariant additive measure \( \mu : \mathcal{P}(G) \rightarrow [0,1] \).

We shall say that a \( G \)-space \( X \) has a free orbit if for some \( x \in X \) the map \( \alpha_x : G \rightarrow X, \alpha_x : g \mapsto gx \), is injective.

**Theorem 4.1.** A group \( G \) admits a toposyndetic submeasure if one of the following conditions holds:

1. there is a minimal compact measure \( G \)-space \( X \) with a free orbit;
2. \( G \) is a subgroup of a compact topological group;
3. \( G \) is countable;
4. \( G \) is amenable.
Proof. 1. Assume that \((X, \tau_X, \lambda_X)\) is a minimal compact measure G-space with a free orbit. In this case there is a point \(x \in X\) for which the map \(\alpha_x : G \to Gx \subset X, \alpha_x : g \mapsto gx\), is injective. This map allows us to define a Tychonoff G-invariant topology 
\[
\tau = \{\alpha_x^{-1}(U) : U \in \tau_X\}
\]
on the group \(G\). By Lemma 3.3 the topology \(\tau\) is G-bounded.

Since the orbit \(Gx\) is dense in \(X\) (which follows from the minimality of \(X\)), the formula
\[
\mu(A) = \lambda(\overline{Ax}) \quad \text{for} \quad A \subset G
\]
determines a G-invariant submeasure on \(G\). Observe that \(B = \{U \in \tau_X : \lambda(U) = \lambda(\overline{U})\}\) is a base of the topology \(\tau_X\) on \(X\) and \(A = \{\alpha_x^{-1}(U) : U \in B\}\) is a base of the topology \(\tau\) on \(G\). It can be verified that each set \(A \in \mathcal{A}\) is \(\mu\)-measurable, which implies that \(\mathcal{A}_\mu \cap \tau \supset A\) is a base of the topology \(\tau\). This means that the submeasure \(\mu\) is toposyndetic.

2. The second statement follows immediately from the first statement and the well-known fact [5, §449] stating that for each countable group \(G\) there is a compact minimal measure G-space with a free orbit.

3. The third statement follows from the first one and a recent deep result of B. Weiss [16] stating that for any amenable group \(G\), each compact G-space \(X\) possesses a G-invariant probability Borel measure. \(\square\)

Problem 4.2. Is the class of groups admitting a toposyndetic submeasure hereditary with respect to taking subgroups?

Problem 4.3. Has every group a toposyndetic submeasure?

Problem 4.4. Has the group \(S_X\) of all bijections of an infinite set \(X\) a toposyndetic submeasure?

5. Groups possessing a syndetic submeasure

In this section we shall detect groups possessing a syndetic submeasure. By Theorem 3.1 the class of such groups contains all groups possessing a toposyndetic submeasure, in particular, all countable groups.

**Theorem 5.1.** A group \(G\) possesses a syndetic submeasure if one of the following conditions is satisfied:

1. there is an infinite transitive G-space possessing a syndetic submeasure;
2. there is an infinite minimal compact measure G-space;
3. \(G\) admits a homomorphism onto an infinite group possessing a (topo)syndetic submeasure;
4. \(G\) admits a homomorphism onto a countable infinite group;
5. \(G\) contains an amenable infinite normal subgroup.

*Proof. 1.* Assume that \(X\) is an infinite transitive G-space possessing a syndetic submeasure \(\lambda : \mathcal{P}(X) \to [0,1]\). Fix any point \(x \in X\) and consider the map \(\alpha_x : G \to X, \alpha_x : g \mapsto gx\), which is surjective (by the transitivity of the G-space \(X\)). One can check that the syndetic submeasure \(\lambda\) on \(X\) induces a syndetic submeasure \(\mu : \mathcal{P}(G) \to [0,1]\) defined by \(\mu(A) = \lambda(\alpha_x(A)) = \lambda_X(Ax)\) for \(A \subset G\).

2. Let \((X, \tau_X, \mu_X)\) be an infinite minimal compact measure G-space. By the minimality, the orbit \(Gx\) of any point \(x \in X\) is dense in \((X, \tau_X)\). Then the formula \(\mu(A) = \mu_X(\overline{Ax})\), \(A \subset X\), determines a G-invariant submeasure \(\mu : \mathcal{P}(G) \to [0,1]\) on the group \(G\). We claim that the submeasure \(\mu\) is syndetic. Given any \(\varepsilon > \frac{1}{|X|}\) and a set \(A \subset G\) with \(\mu(A) < 1\), we should find a large set \(L \subset G\setminus A\) with \(\mu(L) < \varepsilon\). Since \(\mu_X(\overline{Ax}) = \mu(A) < 1\), the closed subset \(\overline{Ax}\) is not equal to \(X\). By the minimality, the infinite compact G-space \((X, \tau_X)\) has no isolated points, which allows us to find an open non-empty set \(U \subset X\setminus \overline{Ax}\) such that \(\mu_X(\overline{U}) < \varepsilon\). By Lemma 3.3 the topology \(\tau_X\) is G-bounded, which implies that the set \(U \subset X\) is large in \(X\) and hence \(V = \alpha_x^{-1}(U) \subset X\setminus A\) is large in \(G\) and has submeasure \(\mu(V) \leq \mu_X(\overline{U}) < \varepsilon\).

3. The third statement follows from the first statement and Theorem 3.1.

4. The fourth statement follows from the third statement and Theorem 4.1(3).

5. Suppose that the group \(G\) contains a normal infinite amenable subgroup \(H\). Denote by \(P_\omega(H)\) the set of finitely supported probability measures on \(H\). Each measure \(\mu \in P_\omega(H)\) can be written as a convex
combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures concentrated at points $x_i$ of $H$. This allows us to identify $P_\omega(H)$ with a convex subset of the Banach space $\ell_1(H)$ endowed with the norm $\|f\| = \sum_{x \in H} |f(x)|$.

We claim that the function

$$\sigma_H : \mathcal{P}(G) \to [0, 1], \quad \sigma_H : A \mapsto \inf_{\mu \in P_\omega(H)} \sup_{y \in G} \mu(Ay),$$

is a syndetic left-invariant submeasure on $G$.

First we prove that $\sigma_H$ is left-invariant. Given any $x \in G$ and $A \subset G$, it suffices to check that $\sigma_H(xA) \leq \sigma_H(A) + \varepsilon$ for every $\varepsilon > 0$. The definition of $\sigma_H$ guarantees that $\sigma_H$ is right-invariant. Consequently, $\sigma_H(xA) = \sigma_H(xAx^{-1})$. By the definition of $\sigma_H(A)$, there is a finitely supported probability measure $\mu \in P_\omega(H)$ such that $\sup_{y \in G} \mu(Ay) < \sigma_H(A) + \varepsilon$. Write $\mu$ as a convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures concentrated at points $a_1, \ldots, a_n \in H$. Since $H$ is a normal subgroup of $G$, the probability measure $\mu' = \sum_{i=1}^{n} \alpha_i \delta_{xa_i x^{-1}}$ belongs to $P_\omega(H)$. Taking into account that for every $y \in G$

$$\mu'(xAx^{-1}y) = \mu'(xAx^{-1}yx^{-1}) = \mu(Ax^{-1}yx),$$

we conclude that

$$\sigma_H(xAx^{-1}) \leq \sup_{y \in G} \mu'(xAx^{-1}y) \leq \sup_{y \in G} \mu(Ax^{-1}yx) < \sigma_H(A) + \varepsilon.$$

So, $\sigma_H$ is left-invariant.

Next, we prove that $\sigma_H$ is subadditive. Given two subsets $A, B \subset G$, it suffices to check that $\sigma_H(A \cup B) \leq \sigma_H(A) + \sigma_H(B) + 3\varepsilon$ for every $\varepsilon > 0$. By the definition of the numbers $\sigma_H(A)$ and $\sigma_H(B)$, there are finitely supported probability measures $\mu_A, \mu_B \in P_\omega(H)$ such that $\sup_{y \in G} \mu_A(Ay) < \sigma_H(A) + \varepsilon$ and $\sup_{y \in G} \mu_B(By) < \sigma_H(By) + \varepsilon$. By Emerson's characterization of amenability [3, 1.7], for the probability measures $\mu_A$ and $\mu_B$ there are probability measures $\mu'_A, \mu'_B \in P_\omega(H)$ such that

$$\sup_{C \subset H} |\mu_A * \mu'_A(C) - \mu_B * \mu'_B(C)| \leq \|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \varepsilon. $$

Write the measures $\mu_A, \mu_B, \mu'_A$ and $\mu'_B$ as convex combinations of Dirac measures:

$$\mu_A = \sum_i \alpha_i \delta_{x_i}, \quad \mu'_A = \sum_j \alpha'_i \delta_{x'_i}, \quad \mu_B = \sum_i \beta_i \delta_{y_i}, \quad \mu'_B = \sum_j \beta'_i \delta_{y'_i}. $$

Then $\mu_A * \mu'_A = \sum_{i,j} \alpha_i \alpha'_j \delta_{x_i x'_j}$ and $\mu_B * \mu'_B = \sum_{i,j} \beta_i \beta'_j \delta_{y_i y'_j}$. For every $y \in G$ we get

$$\mu_A * \mu'_A(Ay) = \sum_{i,j} \alpha_i \alpha'_j \delta_{x_i x'_j}(Ay) = \sum_j \alpha'_j \sum_i \alpha_i \delta_{x_i}(Ay(x'_j)^{-1}) =$$

$$= \sum_j \alpha'_j \mu_A(Ay(x'_j)^{-1}) \leq \sum_j \alpha'_j \sup_{z \in G} \mu_A(Az) = \sup_{z \in G} \mu_A(Az) < \sigma_H(A) + \varepsilon. $$

By analogy we can prove that $\mu_B * \mu'_B(By) \leq \sigma_H(B) + \varepsilon$. Now consider the measure $\nu = \mu_A * \mu'_A$ and observe that for every $y \in B$ we get

$$\nu(By) = \mu_A * \mu'_A(By) \leq \mu_B * \mu'_B(By) + \|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \sigma_H(B) + \varepsilon + \varepsilon. $$

Then

$$\sigma_H(A \cup B) \leq \sup_{y \in G} \nu(A \cup B)y \leq \sup_{y \in G} \nu(Ay) + \sup_{y \in G} \nu(By) < \sigma_H(A) + \varepsilon + \sigma_H(B) + 2\varepsilon = \sigma_H(A) + \sigma_H(B) + 3\varepsilon,$$

which proves the subadditivity of $\sigma_H$.

Finally we prove that the left-invariant submeasure $\sigma_H$ on $G$ is syndetic. Fix a subset $A \subset G$ of submeasure $\sigma_H(A) < 1$ and take an arbitrary $\varepsilon > 0$. Since $\sigma_H(A) < 1$, there is a finitely supported measure $\mu \in P_\omega(H)$ such that $\sup_{y \in G} \mu(Ay) < 1$. Write $\mu$ as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures. We can assume that each coefficient $\alpha_i$ is positive. Then the finite set $F = \{x_1, \ldots, x_n\}$ coincides with the support $\text{supp}(\mu)$ of the measure $\mu$.

It follows that for every $y \in G$ we get $\mu(Ay) < 1$ and hence $F = \text{supp}(\mu) \nsubseteq Ay$. This ensures that the set $Fy^{-1}$ meets the complement $X \setminus A$ and hence $y^{-1} \in F^{-1}(G \setminus A)$. So, $G = F^{-1}(G \setminus A)$ and the set $X \setminus A$ is large in $G$. Now take any finite subset $E \subset H$ of cardinality $|E| > 1/\varepsilon$. Using Zorn's Lemma, choose a maximal subset $B \subset G \setminus A$ which is $E$-separated in the sense that $Ex \cap Ey = \emptyset$ for any distinct points $x, y \in B$. 
The maximality of the set $B$ guarantees that for each $x \in G \setminus A$ the set $Ex$ meets $EB$, which implies that $G \setminus A \subset E^{-1}EB$ and $G = F^{-1}(G \setminus A) = F^{-1}E^{-1}EB$. This means that the set $B$ is large in $G$. We claim that $|E^{-1} \cap By| \leq 1$ for each $y \in G$. Assume conversely that $E^{-1} \cap By$ contains two distinct points $b$ and $b'$ with $b, b' \in B$. Then $b'b^{-1} = b'y(by)^{-1} \in E^{-1}E$ and hence $Eb' \cap Eb \neq \emptyset$, which is not possible as $B$ is $E$-separated. Now consider the uniformly distributed probability measure $\nu = \frac{1}{|E|} \sum_{x \in E^{-1}} \delta_x \in P(\omega(H))$ and observe that $\sigma_H(B) \leq \sup_{y \in G} \nu(By) \leq \frac{|E^{-1} \cap By|}{|E|} \leq \frac{1}{|E|} < \varepsilon$, which means that the submeasure $\sigma_H$ is syndetic.

**Remark 5.2.** For an infinite amenable group $G$ and the subgroup $H = G$ the syndetic submeasure $\sigma_H$ (used in the proof of Theorem 5.1(5)) coincides with the right Solecki submeasure $R$ introduced in [14] and studied in [1].

Theorem 5.1(5) implies:

**Corollary 5.3.** The group $S_X$ of bijections of any set $X$ possesses a syndetic submeasure.

**Proof.** If $X$ is finite, then the finite group $S_X$ has a syndetic submeasure according to proposition 2.1. So, we assume that the set $X$ is infinite. Observe that the subgroup $\text{FS}_X$ of finitely supported permutations of $X$ is locally finite and hence amenable. By Theorem 5.1(5) the group $S_X$ admits a syndetic submeasure as it contains the infinite amenable normal subgroup $\text{FS}_X$.

**Problem 5.4.** Has every group a syndetic submeasure?

**Problem 5.5.** Has the quotient group $S_\omega/\text{FS}_\omega$ a syndetic submeasure?

6. **Partitions of groups into $k$-meager pieces**

Now we return to the problem of partitioning groups into $k$-meager pieces, which was posed and partly resolved in [12]. Combining Theorems 2.2 and 5.1(5), we get:

**Theorem 6.1.** Each countable infinite group $G$ for every $k \in \mathbb{N}$ admits a partition into two $k$-meager subsets.

This theorem admits a self-generalization.

**Corollary 6.2.** If a group $G$ has a countable infinite quotient group, then for every $k \in \mathbb{N}$ the group $G$ admits a partition into two $k$-meager subsets.

**Proof.** Let $h : G \to H$ be a homomorphism of $G$ onto a countable infinite group $H$. By Theorem 6.1 for every $k \in \mathbb{N}$ the countable group $H$ admits a partition $H = A \cup B$ into two $k$-meager subsets. Then $G = h^{-1}(A) \cup h^{-1}(B)$ is a partition of the group $G$ into two $k$-meager subsets.

**Problem 6.3.** Is it true that each infinite group $G$ for every $k \in \mathbb{N}$ admits a partition into two $k$-meager sets?

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References

[1] T. Banakh, The Solecki submeasures and densities on groups, preprint (http://arxiv.org/abs/1211.0717).
[2] B. Bollobás, S. Janson, O. Riordan, On covering by translates of a set, Random Structures Algorithms 38:1-2 (2011), 33–67.
[3] W. Emerson, Characterizations of amenable groups, Trans. Amer. Math. Soc. 241 (1978), 183–194.
[4] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.
[5] D. Fremlin, Measure Theory, V.4, Torres Fremlin, Colchester, 2006.
[6] N. Hindman, D. Strauss, Algebra in the Stone-Čech compactification: Theory and Applications, Walter de Grueter, Berlin, New York, 2011.
[7] I. Protasov, Partition of groups into large subsets, Math. Notes, 73 (2003), 271–281.
[8] I. Protasov, Small systems of generators of groups, Math. Notes, 76 (2004), 420–426.
[9] I. Protasov, Selective survey on Subset Combinatorics of Groups, Ukr. Math. Bull., 7(2010), 220-257.
[10] I. Protasov, Partition of groups into thin subsets, Algebra Disc. Math., 11(2011), 88–92.
[11] I. Protasov, T. Banakh, Ball Structures and Colorings of Groups and Graphs, Math. Stud. Monogr. Ser., Vol. 11, VNTL Publishers, Lviv, 2003.
[12] I. Protasov, S. Slobodyanuk, Prethick subsets in partitions of groups, Algebra Discr. Math. 14:2 (2012), 267–275.
[13] I. Protasov, M. Zarichnyi, General Asymptology, Math. Stud. Monogr. Ser., Vol. 12, VNTL Publishers, Lviv, 2007
[14] S. Solecki, Size of subsets of groups and Haar null sets, Geom. Funct. Anal. 15 (2005), 246–273.
[15] G. Weinstein, Minimal complementary sets, Trans. Amer. Math. Soc., 212 (1975), 131–137.
[16] B. Weiss, Minimal models for free actions, in: Dynamical Systems and Group Actions (L. Bowen, R. Grigorchuk, Ya. Vorobets eds.), Contemp. Math. 567, Amer. Math. Soc. Providence, RI, (2012), 249–264.

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