Decoherence induced by squeezing control errors in optical and ion trap holonomic quantum computations

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We study decoherence induced by stochastic squeezing control errors considering the particular implementation of Hadamard gate on optical and ion trap holonomic quantum computers. We analytically obtain both the purity of the final state and the fidelity for Hadamard gate when the control noise is modeled by Ornstein-Uhlenbeck stochastic process. We demonstrate the purity and the fidelity oscillations depending on the choice of the initial superimposed state. We derive a linear formulae connecting the gate fidelity and the purity of the final state.

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I. INTRODUCTION

Holonomic quantum computations exploiting non-abelian geometrical phases [1] was primarily proposed in the Ref. 2 and developed further in the Ref. 3. Many implementations of holonomic quantum computers (HQC) have been proposed. Particularly, the realization of HQC within quantum optics was suggested (optical HQC) [4]. Laser beams in a non-linear Kerr medium were exploited for this purpose. Two different sets of control devices can be used in this case. The first one considered in this paper consists of one- and two-mode displacing and squeezing devices. The second one includes SU(2) interferometers. As well trapped ions with the excited state connected to a triple degenerate subspace (four level Λ-system) can be used to implement HQC [5]. Another approach to HQC exploiting squeezing and displacement of the trapped ions vibrational modes was suggested in the Ref. 6. This implementation of HQC is mathematically similar to the first embodiment of the optical HQC [4] and thus it is also considered in this work. Particularly, expressions for the adiabatic connection and holonomies are the same in these cases. Another proposed implementation of HQC was the HQC with neutral atoms in cavity QED [7]. The computational subspace was spanned by the dark states of the atom and the fidelity oscillations depending on the choice of the initial superimposed state. We derive a linear formulae connecting the gate fidelity and the purity of the final state.

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II. HADAMARD GATE IMPLEMENTATION

Optical and ion trap implementations of HQC are mathematically equivalent since the corresponding holonomies are the same (compare the Refs. [4] and [6]). Therefore we can consider both HQC models simultaneously.

The laser beams in the nonlinear Kerr medium are explored in order to perform holonomic quantum computation in the framework of the optical setup. The corresponding interaction Hamiltonian describing a single beam in the medium is

\[ H_I = \hbar Xa^\dagger a (a^\dagger a - 1), \]

where \( a \) and \( a^\dagger \) are the annihilation and creation opera-
tors of the photons respectively, $X$ is a constant proportional to the third order nonlinear susceptibility of the medium. The degenerate computational subspace of the single qubit is spanned on the photon Fock states $|0\rangle$ and $|1\rangle$. More details one can find in the Ref. [1].

In order to implement the same holonomic quantum computational scheme in the framework of the ion trap setup one has to deal with the two-level trapped ion placed in the common node of two standing electromagnetic waves with frequencies $\omega_0-\omega_z$ and $\omega_0+\omega_z$ as well as being affected by the traveling wave with the frequency $\omega_0$. Here $\omega_0$ denotes the frequency corresponding to the transition between the two ion levels being exploited, $\omega_z$ is the frequency of the ion’s harmonic oscillations along the $z$ axis of the linear Paul trap. The basis qubit states are $|g\rangle \otimes |0\rangle$ and $\left((|g\rangle \otimes |1\rangle - |e\rangle \otimes |0\rangle\right)/\sqrt{2}$. Here $|g\rangle$ and $|e\rangle$ are the ground and excited internal states of the ion, $|0\rangle$ and $|1\rangle$ are the two lowest vibrational Fock states of the ion in the trap. More details can be found in the Ref. [2].

One-qubit gates are given as a sequence of single mode squeezing and displacing operations [3, 4]:

$$U(\eta, \nu) = D(\eta)S(\nu),$$

where

$$S(\nu) = \exp(\nu a^\dagger a),$$

$$D(\eta) = \exp(\eta a^\dagger - \eta^* a)$$

(5)

describe single mode squeezing and displacing operators respectively, $\nu = r_1 e^{i \theta_1}$ and $\eta = x + iy$ are corresponding complex control parameters, $a$ and $a^\dagger$ are annihilation and creation operators. The asterisk denotes complex conjugation. The expressions for the adiabatic connection and the curvature tensor can be found in the Refs. [3, 4, 5]. Following our previous Letter [21] we consider Hadamard gate

$$H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(6)

implemented when two rectangular loops belonging to the planes $(x, r_1) |_{\theta_1=0}$ and $(y, r_1) |_{\theta_1=0}$ are passed. Namely,

$$-i H_0 = \Gamma(C_{I\ell}) |_{\Sigma_{I\ell}=\pi/2} \Gamma(C_I) |_{\Sigma_{I}=\pi/4},$$

(7)

where the holonomies are

$$\Gamma(C_I) = \exp(-i \sigma_y \Sigma_I), \quad \Sigma_I = \int_{S(C_I)} dx dr_1 2 e^{-2 r_1},$$

$$\Gamma(C_{I\ell}) = \exp(-i \sigma_x \Sigma_{I\ell}), \quad \Sigma_{I\ell} = \int_{S(C_{I\ell})} dy dr_1 2 e^{2 r_1},$$

(8)

and $S(C_{I\ell})$ are the regions in the planes $(x, r_1) |_{\theta_1=0}$ and $(y, r_1) |_{\theta_1=0}$ enclosed by the rectangular loops $C_I$ and $C_{I\ell}$ respectively. The sides of the rectangles $C_I$ and $C_{I\ell}$ are parallel to the coordinate axes. For the loop $C_I$ these sides are given by the lines $r_1 = 0$, $x = b_x$, $r_1 = d_x$ and $x = a_x$, where the length of the rectangle’s sides parallel to the $x$ axis is $l_x = b_x - a_x$. In the Ref. [21] it was shown that

$$d_x = -\frac{1}{2} \ln \left(1 - \frac{\pi}{4l_x}\right), \quad l_x > \frac{\pi}{4}$$

(9)

In the same way the rectangle $C_{I\ell}$ is composed of the lines $r_1 = 0$, $y = b_y$, $r_1 = d_y$ and $y = a_y$, where [21]:

$$d_y = \frac{1}{2} \ln \left(1 + \frac{\pi}{2l_y}\right), \quad l_y = b_y - a_y.$$  

(10)

Proposed Hadamard gate implementation is not a unique one. The same gate can be realized by passing another loops in the control manifold. Our choice is motivated by the simplicity of the loops.

III. DECOHERENCE INDUCED BY STOCHASTIC SQUEEZING CONTROL ERRORS

We restrict ourselves by the consideration of the squeezing control errors only. Moreover, we can neglect the fluctuations of the squeezing control parameter when $r_1 = 0$. Thus to take into account random squeezing control errors we have to replace $d_x$ by $d_x + \delta r_x(x)$ and $d_y$ by $d_y + \delta r_y(y)$, where $\delta r_x(x)$ and $\delta r_y(y)$ are two independent Ornstein-Uhlenbeck stochastic processes. Making this substitution into the Eqs. (3) instead of the formulae [11] we obtain the following expression for the perturbed Hadamard gate, see also [21]:

$$-i H = -\frac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha) (\sin \beta + i \sigma_x \cos \beta) -$$

$$-\frac{i}{\sqrt{2}} (\cos \alpha + \sin \alpha) (\sigma_y \cos \beta - \sigma_x \sin \beta),$$

(11)

where

$$\alpha = e^{-2d_x} \int_{a_x}^{b_x} dx \left(1 - e^{-2\delta r_x}\right),$$

$$\beta = e^{2d_y} \int_{a_y}^{b_y} dy \left(e^{2\delta r_y} - 1\right).$$

(12)

Let the qubit initially to be in the pure superimposed state $|\psi_0\rangle = c_0 |0\rangle + c_1 |1\rangle$, where amplitudes $c_0$ and $c_1$ obey the normalization constrain $|c_0|^2 + |c_1|^2 = 1$. For the fixed noise realization the final qubit state will be pure. However, it will differ from the desired one. In the real experiment we do not follow the random fluctuations of the control parameters (nevertheless in principle we can do it). In this situation quantum mechanics prescribes us to describe the final state of the system by the
density matrix and represent the state as a mixture of all possible final states weighted with the probabilities of the corresponding noise realizations. Following this strategy we find the density matrix of the final state for a given noise implementation and than average over the stochastic processes. 

Using the Eqs. (12)-(13) and the properties of Ornstein-Uhlenbeck stochastic process (see Ref. [30]):

Here the introduced parameter \( \gamma \) is defined as \( \gamma = \alpha - \pi/4 \) and the asterix denotes the complex conjugate quantities.

We assume that the noise \( \delta r_x \) has variance \( \bar{\delta}_x \) and a lorentzian spectrum with the bandwidth \( \Gamma_x \). The fluctuations \( \delta r_y \) have the variance \( \bar{\delta}_y \) and bandwidth \( \Gamma_y \). Using the Eqs. (13) and the properties of Ornstein-Uhlenbeck stochastic process (see Ref. [30]):

Here we introduced the following denotations:

we average the density matrix \( \rho \) over the stochastic fluctuations of the squeezing control parameters \( \delta r_x \) and \( \delta r_y \). The line over the random quantities means the averaging operation. We assume that \( \delta r_{x,y} \ll 1 \) and restricted ourselves by the first non-vanishing terms depending on \( \delta r_x \) or \( \delta r_y \). As the result of straightforward but a bit lengthy calculations we analytically obtain the elements of the averaged density matrix \( \overline{\rho} \):

In order to quantify decoherence strength we exploit the purity of the final state. It is defined as the trace of the squared density matrix. Purity equals to 1 for pure states and less than 1 otherwise. Using the Eqs. (13) it is easy to check that for a fixed noise realization the purity

we use the following operators:

Here we introduced the following denotations:

Using the Eqs. (16) we obtain the purity of the final state in the case of the stochastic squeezing control errors. The result of the lengthy but straightforward calculations is

Thus we see that the final state purity \( I < 1 \) and the stochastic squeezing control errors induce decoherence and lead the final state to be a mixture of pure states.

\[ I_0 = \text{tr} \rho^2 \]
We can simplify the expression (17) if we exploit the following parametrization for the amplitudes \( c_0 \) and \( c_1 \): 

\[
c_0 = e^{i\xi} \cos \varphi, \quad c_1 = e^{i\chi} \sin \varphi,
\]

and assume that \( \xi - \chi = \pi n \), where \( n \) is an integer. In this rather general case formula (17) reduces to 

\[
I = 1 - \frac{16\delta_x}{\Gamma_x} F_x - \frac{16\delta_y}{\Gamma_y} F_y \sin^2 2\varphi.
\]

We see that the final state purity oscillates depending on the choice of the initial qubit state. The purity decreases when the initial qubit state is a superposition of the basis states. For example, it has its minimum when \( |\psi_0\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2} \). In contrary, the purity has its maximum if the initial qubit state \( |\psi_0\rangle \) is proportional to one of the basis states \( |0\rangle \) or \( |1\rangle \). In this case the errors made in the \((y, r_1)|_{a_1=0}\) plane are eliminated.

\section*{IV. HADAMARD GATE FIDELITY}

Now we obtain the fidelity of the non-ideal Hadamard gate. In the case when there are no control errors \((\delta r_x = \delta r_y = 0)\) the density matrix \( \rho^{(0)} \) of the final (pure) state has the following form:

\[
\rho^{(0)} \equiv H_0 |\psi_0\rangle \langle \psi_0| H_0 = \frac{1}{2} \left( \begin{array}{cc} |c_0|^2 & c_0 c_1^* + c_0 c_1 \cos \theta \frac{I_y - c_0 c_1^* - c_0 c_1}{1 - c_0^2 c_1^2 - c_0 c_1^2} 
\end{array} \right).
\]

The non-ideal Hadamard gate fidelity \( F \equiv tr(\rho^{(0)}\pi) \) under the same assumptions as in the Eq. (15) is given by the expression

\[
F = 1 - \frac{32\delta_y}{\Gamma_y} F_y |c_0|^2 |c_1|^2 - \frac{8\delta_x}{\Gamma_x} F_x \left( c_0^2 + c_1^2 \right) \left( c_0^2 + c_1^2 \right).
\]

From this expression we can conclude that when the initial state vector is proportional either to \( |0\rangle \) or \( |1\rangle \) the contribution of the control errors made in the \((y, r_1)|_{a_1=0}\) plane can be neglected at the accepted approximation degree. In our previous work we obtained that \( 1 - F \sim \delta_x^2 \). Thus our previous result concerning the cancellation of the squeezing control errors up to the fourth order on their magnitude is reproduced as it should be (remind that \( \delta_x \) has the order of \( (\delta r_x)^2 \)).

We can simplify the expression (21) exploiting the parametrization for the amplitudes \( c_0 \) and \( c_1 \) and assuming \( \xi - \chi = \pi n \), where \( n \) is an integer. Under these assumptions the gate fidelity is given by

\[
F = 1 - \frac{8\delta_x}{\Gamma_x} F_x - \frac{8\delta_y}{\Gamma_y} F_y \sin^2 2\varphi.
\]

It is evident that the gate fidelity oscillates depending on the choice of the initial qubit state. It is less for the superimposed initial states than for the basis ones. Namely, the fidelity has its maximum when the initial qubit state \( |\psi_0\rangle \) is proportional either to \( |0\rangle \) or \( |1\rangle \). In this case the errors made in the \((y, r_1)|_{a_1=0}\) plane are eliminated. As well the fidelity has its minimum when the initial state vector is equal to \((|0\rangle \pm |1\rangle) / \sqrt{2}\). Moreover from Eqs. (17) and (21) we find a simple linear formulae connecting the purity of the final state and the gate fidelity:

\[
I = 2F - 1.
\]

This expression demonstrates the close connection between these quantities defining the decoherence strength and the gate stability.

\section*{V. CONCLUSIONS}

We considered optical and ion trap HQC proposed the Refs. [4] and [5] respectively. Regarding the particular implementation of Hadamard gate we have studied decoherence induced by stochastic squeezing control errors. Ornstein-Uhlenbeck stochastic process was exploited to model random fluctuations of the squeezing control parameter. We have analytically obtained the purity of the final qubit state and calculated the fidelity of the non-ideal Hadamard gate. It was shown that the stochastic squeezing control errors reduce the final state into a mixture of pure states and, thus, induce decoherence. We have shown that the final state purity oscillates depending on the choice of the initial qubit state. The purity decreases when the initial qubit state is a superposition of the basis states. For example, it has its minimum when \( |\psi_0\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2} \). In contrary, the purity has its maximum when the initial qubit state is proportional to one of the basis states \( |0\rangle \) or \( |1\rangle \). In this case the control errors made in the \((y, r_1)|_{a_1=0}\) plane are eliminated. The same conclusions can be made for the gate fidelity. Simple linear formulae connecting the gate fidelity and the purity of the final state was derived.
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