PERVERSE SHEAVES ON AFFINE FLAGS AND LANGLANDS DUAL GROUP

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Though he goeth on his way weeping that beareth the measure of seed, he shall come home with joy, bearing his sheaves.

Psalm 126 "Shir ha-maalot" (Song of Ascendance)

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A.5. Thus $D = D(\mathcal{F}^{-}) = D(\mathfrak{g}^{-})$ be the constructible derived category of l-adic sheaves ($l \neq \text{char}(k)$; see [D], 1.1.2; [BD], 2.2.14–2.2.18; and also [G], §A.2) on $\mathcal{F}^{-}$, $\mathfrak{g}^{-}$ are direct limits of projective varieties with transition maps being closed embeddings, and $\mathcal{F}(k) = G(F)/I$, $\mathfrak{g}(k) = G(F)/G(O)$.

Let $D = D(\mathcal{F}^{-}) = D(\mathfrak{g}^{-})$ be the equivariant derived categories (cf. [BL]). Let $\mathcal{P} \subset D$, $\mathcal{P}(\mathfrak{g}) \subset D(\mathfrak{g})$, $\mathcal{P}(\mathfrak{g}) \subset D(\mathfrak{g})$ be the subcategories of perverse sheaves.

By $\ast$ we denote the convolution; thus $\ast$ provides $D_{G(O)}(\mathfrak{g}(\mathfrak{r}))$, $D_{I}(\mathcal{F}^{-})$, with a monoidal structure, and defines a “right” action of $D_{I}(\mathcal{F}^{-})$ on $D(\mathcal{F}^{-})$.

Let $G$ be the Langlands dual group over the field $\overline{\mathbb{Q}}_{F}$, and $\text{Rep}(G')$ be its category of representations.

Recall that according to a result of Lusztig (see also [G] for an alternative proof and generalization) $\mathcal{P}_{G(O)}(\mathfrak{g}(\mathfrak{r})) \subset D_{G(O)}(\mathfrak{g}(\mathfrak{r}))$ is a monoidal subcategory. Moreover, $\mathcal{P}_{G(O)}(\mathfrak{g}(\mathfrak{r}))$ is equipped with a commutativity constraint and a fiber functor, and we have (for $k$ algebraically closed) an equivalence of Tannakian categories $\mathcal{P}_{G(O)}(\mathfrak{g}(\mathfrak{r})) \cong \text{Rep}(G')$. This Theorem is known as the geometric Satake isomorphism; see [L], [G], [MV] and [BD]. As the name suggests, this result is a geometric, or categorical, counterpart of the classical Satake isomorphism $\text{K}(\text{Rep}(G')) \cong H_{\text{sph}}^{\text{rep}}$, where $H_{\text{sph}}$ is the spherical Hecke algebra, and $K$ stands for the Grothendieck group. Here the word “geometric” means that, following the Grothendieck “sheaf-function” correspondence principle, one replaces the space of functions on the set of $F_{\mathfrak{p}}$-points of a scheme by the category of $l$-adic complexes (or perverse sheaves) on this scheme (or on its base change to an algebraically closed field).
In this and subsequent paper we extend the geometric Satake isomorphism to a description of various categories of \( \ell \)-adic sheaves on \( \mathcal{F} \ell \) in terms of \( G^\ell \). These results have found several applications to representation theory: to cells in affine Weyl groups and bases in the Grothendieck groups of equivariant coherent sheaves \([B3]\), to cohomology of tilting modules over quantum groups at a root of unity \([B4]\); and also to Lusztig’s conjectures on nonrestricted representations of modular representations of \( g^\ell \) (in preparation, see announcement in \([B5]\)). We also think that they are closely related to some aspects of the recent work \([GW]\) which discusses tamely ramified geometric Langlands duality from the point of view of Yang-Mills theory.

The possibility to realize the affine Hecke algebra \( H \) and the “anti-spherical” module over it as Grothendieck groups of (equivariant) coherent sheaves on varieties related to \( G^\ell \) plays a crucial role in the proof of classification of irreducible representations of \( H \), which a particular case of the local Langlands conjecture, \([KL]\), see also \([CG]\). Thus one may hope that the “categorification” of these realizations proposed here can contribute to the geometric Langlands program. Let us point out that existence of (some variant of) such a categorification was proposed as a conjecture by V. Ginzburg (see Introduction to \([CG]\)).

1.1.1. Let us now describe some known statements about spaces of functions on \( G(F) \), whose geometric counterparts will be provided in the paper.

Set \( k = \mathbb{F}_p^\ell \), and let \( H = \mathbb{C}[I \backslash G(F)/I] \) be the Iwahori-Matsumoto Hecke algebra. Let \( T_w \) be the standard basis of \( H \); here \( w \) runs over the extended affine Weyl group \( W \). Let \( \Lambda \subset W \) be the coweight lattice of \( G \), and \( \Lambda^+ \subset \Lambda \) be the semigroup of dominant coweights. Let \( A \subset H \) be the commutative subalgebra generated by the elements \( T_\lambda, \lambda \in \Lambda^+ \) and their inverses (see e.g. \([L0]\), beginning of §7). Thus \( A \) has a basis \( \theta_\lambda, \lambda \in \Lambda \), where \( \theta_\lambda \) are defined by the conditions

\[
\theta_\lambda = q^{-\ell(\lambda)/2}T_\lambda \quad \text{for} \quad \lambda \in \Lambda^+, \quad \theta_{\lambda + \mu} = \theta_\lambda \cdot \theta_\mu.
\]

Recall that the anti-spherical (right\(^1\)) module \( M_{asp} \) over \( H \) is defined as the induction from the sign representation of the finite Hecke algebra \( H_f \subset H \). One can also describe \( M_{asp} \) as follows. Let \( C_w \) be the Kazhdan-Lusztig basis of \( H \); let \( W_f \subset W \) be the finite Weyl group, and \( W_f \subset W \) be the set of minimal length representatives of, respectively, left and two-sided cosets of \( W_f \) in \( W \). Then

\[
M_{asp} \cong H/(C_w, w \notin W_f).
\]

Notice that \( M_{asp} \) is free of rank 1 over the subalgebra \( A \).

Another important realization of \( M_{asp} \) is in terms of the Whittaker model. Let \( N \subset G \) be a maximal unipotent, and \( \Psi : N(F) \to \mathbb{C} \) be a generic character. Then

\[
M_{asp} \cong (\text{ind}_{N(F)}^{G(F)}(\Psi))^f;
\]

here the right hand side is identified with the space of Whittaker functions on \( G(F)/I \).

The group \( N(F) \) is not compact; because of this there is no straightforward definition of the category of Whittaker sheaves on \( \mathcal{F} \ell \) (the geometric counterpart of the right hand side of \((2)\)). Following \([FGV]\) one can provide such a definition using Drinfeld’s compactification of the moduli space of \( B \)-bundles on a curve. However, the following technically simpler (though probably less suited for generalizations) approach suffices for our purposes.

Let \( I_u \subset G(F) \) be the pro-\( p \) radical of an Iwahori subgroup, and \( \psi : I_u \to \mathbb{C} \) be a generic character (the definition is recalled below). Then one can use Lemma\(^2\) below to show that

\[
M_{asp} \cong \left(\text{ind}_{I_u}^{G(F)}(\psi)\right)^f;
\]

\(^1H \) has a canonical anti-involution coming from the map \( g \mapsto g^{-1}, \ g \in G(F) \); thus the categories of left and right modules are canonically identified. We define \( M_{asp} \) as a right \( H \)-module to make some notations more natural: \( M_{asp} \) is realized in the space of functions on \( G(F)/I \) where \( H \) acts naturally on the right.
and moreover, the arising isomorphism between the right hand sides of (2) and (3) is compatible with the standard bases consisting of functions supported on one two-sided coset.

We call the right hand side of (3) the Iwahori-Whittaker module. It is easy to define the category of Iwahori-Whittaker sheaves on $\mathcal{F}_z$. It can probably be shown to be equivalent to the category of Whittaker sheaves on $\mathcal{F}_\ell$ (where the latter is defined following [FGV]); this is not pursued in this paper (see, however, Theorem 9 below).

The methods of this paper can be used also to describe in a similar fashion geometric counterparts of the algebra $H$; this will be addressed in a future publication (see announcement in [B5]).

Below we will define a triangulated monoidal category $D(A)$ which is a geometric counterpart of the commutative algebra $A$; and abelian categories $\mathcal{F}_p, \mathcal{P}_G^\gamma$ which are geometric counterparts of the right hand sides of (1), (3) respectively; we will then describe $D(A)$, and the derived categories $D^b(\mathcal{F}_p), D^b(\mathcal{P}_G^\gamma)$, in terms of the Langlands dual group.

1.1.2. We now recall the realizations of $H$, $A$, $M_{asp}$ in terms of $G^\ast$, whose categorical counterparts will be given below.

Let $g^\ast$ be the Lie algebra of $G^\ast$. Let $B = G^\ast / B^\ast$ be the flag variety; $\mathcal{N}$ be the nilpotent cone of $g^\ast$, and $\pi_{Sph} : \mathcal{N} = T^\ast((G^\ast / B^\ast) \to \mathcal{N}$ be the Springer map. Let $St = \mathcal{N} \times \mathcal{N} \tilde{\mathcal{N}}$ be the Steinberg variety of triples.

For a scheme $X$ equipped with an action of an algebraic group $H$ we will let $\text{Coh}^H(X)$ be the category of $H$-equivariant coherent sheaves; we will write $D^H(X)$, or $D(X)$ if the group is unambiguous, for the bounded derived category $D^b(\text{Coh}^H(X))$. We will denote the Grothendieck group of either abelian or triangulated category $\mathcal{C}$ by $K(\mathcal{C})$.

Let $\mathbb{H}$ be the affine Hecke algebra of $G$. Thus $\mathbb{H}$ is an algebra over $\mathbb{Z}[v, v^{-1}]$, and $H = \mathbb{H} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{C}$, where the map $\mathbb{Z}[v, v^{-1}] \to \mathbb{C}$ sends $v$ to $q^{1/2}$. One can define a subalgebra $A \subset \mathbb{H}$, and a module $M_{asp}$ such that $A = A \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{C}$, $M_{asp} = M_{asp} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{C}$.

Then we have an isomorphism (see e.g. [CG] or [L2])

$$\mathbb{H} \cong K(D^{G^\ast \times \mathcal{G}_m}(St));$$

where the algebra structure on the right hand side is provided by convolution. Moreover, under this isomorphism the subalgebra $A$ is identified with the image of $\delta_* : K(D^{G^\ast \times \mathcal{G}_m}(\tilde{\mathcal{N}})) \to K(D^{G^\ast \times \mathcal{G}_m}(St))$, where $\delta : \mathcal{N} \hookrightarrow St$ is the diagonal embedding; notice that $\delta_*$ is a homomorphism where the algebra structure on $K(D^{G^\ast \times \mathcal{G}_m}(\tilde{\mathcal{N}}))$ is defined by $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}]$. The module $M_{asp}$ is identified with the $K(D^{G^\ast \times \mathcal{G}_m}(St))$ module $K(D^{G^\ast \times \mathcal{G}_m}(\mathcal{N}))$.

1.1.3. Informal summary. Our method relies heavily on [G] which provides a categorical counterpart of the description of the center $Z(H)$ of the affine Hecke algebra $H$. According to a well known result of Bernstein [L0] we have $Z(H) \cong H_{sph}$, thus by Satake isomorphism we have $Z(H) \cong K(Rep(G^\ast))$. In [G] Gaitsgory uses geometric Satake isomorphism and nearby cycles functor to define a central functor $Z$ from $Rep(G^\ast)$ to $D_I(\mathcal{F}_\ell)$ (the notion of a central functor is recalled below).

The present paper can be informally summarized as follows. We upgrade Gaitsgory’s functor $Z$ to a functor from $D^G(\tilde{\mathcal{N}})$, which is then shown to induce an equivalence with the Iwahori-Whittaker category, by linking various ingredients in the definition of $D^G(\tilde{\mathcal{N}})$ to relevant structures on the perverse sheaves side. To make this more precise recall that $\tilde{\mathcal{N}} = \{(b, x) \mid b \in B, x \in rad(b))$, where $B$ is identified with the set of Borel subalgebras in $g^\ast$ and $rad$ stands for the nilpotent radical. We show that, in the appropriate formal sense, the tensor functor from $Rep(G^\ast)$ to $D_I(\mathcal{F}_\ell)$ corresponds to the fact that in the dual side we deal with the category
of $G^-$ equivariant coherent sheaves on some algebraic variety. The element $x \in \mathfrak{g}^-$ in the description of $\tilde{N}$ arises from the logarithm of monodromy acting on the nearby cycles sheaf $\mathcal{Z}(V)$, $V \in \text{Rep}(G^\vee)$ by Tannakian formalism. Finally, the "flag" $b \in B$ corresponds to a filtration on $\mathcal{Z}(V)$, $V \in \text{Rep}(G^\vee)$ by Wakimoto sheaves, see Theorem 4. Wakimoto sheaves categorify elements $\theta_\lambda \in A \subset H$, and Theorem 4 is a categorification of the fact that $Z(H) \subset A$, thus an element in $Z(H)$ is a linear combination of $\theta_\lambda$, $\lambda \in \Lambda$. On the other hand, Theorem 4 is equivalent to the computation of cohomology of the so-called semi-infinite orbits with coefficients in these sheaves, and is closely related to Mirković and Vilonen’s computation of corresponding cohomology for spherical sheaves; see section 3.6 for further comments.

We finish the introduction by pointing out another result on the structure of central sheaves of $[G]$, Theorem 7 proved below. It says that the objects of the Iwahori-Whittaker category cooked out of central sheaves are tilting. This result is inspired by the “Koszul duality” yoga of [BGS], see Remark 12.

Finally, let us make a standard remark that all the results and proofs of the paper work in the alternative setup where the finite characteristic base field $k$ is replaced by $\mathbb{C}$, the field of coefficients $\mathbb{Q}$ is also replaced by $\mathbb{C}$, and the category of $l$-adic constructible sheaves by the category of $D$-modules (in the part of the paper where neither Artin-Schreier sheaf, nor weights are used one can work with constructible sheaves in the classical topology).

1.2. More notations. From now on we fix $k = \mathbb{F}_p$.

The convolution diagram will be written as $\mathcal{F}\ell \times \mathcal{F}\ell \to \mathcal{F}\ell$. If $X \subset \mathcal{F}\ell$, $Y \subset \mathcal{F}\ell$ are subschemes, and $Y$ is $I$ invariant, then we get a subscheme $\overline{X \times Y} \subset \mathcal{F}\ell \times \mathcal{F}\ell$. For $\mathcal{F} \in D(\mathcal{F}\ell)$, $\mathcal{G} \in D_I(\mathcal{F}\ell)$ we get an object $\mathcal{F} \boxtimes \mathcal{G}$ (twisted product) of the category of $l$-adic complexes on $\mathcal{F}\ell \times \mathcal{F}\ell$.

Let $\kappa : \Lambda \to I W$ be the bijection such that $\kappa(\lambda) \in W_f \cdot \lambda$.

All derived categories below will be bounded derived categories, notation $D$ will be used instead of a more traditional $D^b$, unless stated otherwise.

We now describe the results of the paper.

1.3. The monoidal functor. If $X$ is smooth, then $D(X)$ is a tensor category under the (derived) tensor product of coherent sheaves. The first result of the paper (see section 3) is construction of a monoidal functor

\[ D^{G^-}(\tilde{N}) \to D_I. \]

In fact we will do a little bit more. We will define (in section 3.6, 3.7) a full subcategory $A \subset \mathcal{P}_I$ which will turn out to be closed under convolution. Then the homotopy category $\text{Hot}(A)$, of finite complexes of objects in $A$ inherits a monoidal structure. Further, let $D(A)$ denote the quotient of the triangulated category $\text{Hot}(A)$ by the subcategory of acyclic complexes. Then $D(A)$ is also a monoidal category. We have the obvious functor $D(A) \to D^b(\mathcal{P}_I)$.

We will construct a monoidal functor

\[ F : D^{G^-}(\tilde{N}) \to D(A) \]

One can use the argument of [BGS], 1.3 to define a natural functor $D^b(\mathcal{P}_I) \to D_I$; thus we can define (4) as the composition $D^{G^-}(\tilde{N}) \to D(A) \to D^b(\mathcal{P}_I) \to D_I$. It is then easy to see that it comes with a natural monoidal structure.

Below we will not use (4), only (5).
1.4. Compatibility with Frobenius. Let $q : \tilde{N} \to \tilde{N}$ be the map sending a pair $(b, x) \in \tilde{N} \subseteq B \times x$ to $(b, qx)$.

Let also $Fr = Fr_q$ be the geometric Frobenius; recall that for a (ind)scheme $X$ over $\mathbb{F}_q$, $Fr$ induces an autoequivalence of the (derived) category of $l$-adic sheaves on $X_{\mathbb{F}_q}$, $X \mapsto Fr^*(X)$, see [D].

**Proposition 1.** There exists a natural isomorphism of functors
\begin{equation}
Fr^* \circ F \cong F \circ q^*
\end{equation}

**Remark 1.** Let the multiplicative group $G_m$ act on $\tilde{N}$ by $t : (b, x) \mapsto (b, t^{-2}x)$. Then for $\mathcal{F}, \mathcal{G} \in D^{G} \times G_m(\tilde{N})$ the vector space $\text{Hom}_{D^{G} \times G_m}(\mathcal{F}, \mathcal{G})$ carries an action of $G_m$, hence a grading. The Proposition provides isomorphisms $Fr^*(\mathcal{F}) \cong \mathcal{F}$, $Fr^*(\mathcal{G}) \cong \mathcal{G}$; the map $\text{Hom}_{D^{G} \times G_m}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{D}(\mathcal{F}(\mathcal{G}), F(\mathcal{G}))$ carries the degree $n$ component into a subspace where Frobenius acts with weight $n$.

It follows that the equivalences $\mathcal{I}_D$, $\mathcal{I}_{D^\infty}$ defined below also satisfy this property.

**Remark 2.** In fact, a slight modification of our argument provides a monoidal functor from $D^{G} \times G_m(\tilde{N})$ to the derived category of mixed $l$-adic sheaves on $\mathcal{I}_{D^\infty}$. We expect that it induces an equivalence between $D^{G} \times G_m(\tilde{N})$ and $D(\mathcal{P}_{\mathcal{I}} \mathcal{P}_{\mathcal{I}})$, $D(\mathcal{P}_{\mathcal{I}} \mathcal{P}_{\mathcal{I}})$, where $D(\mathcal{P}_{\mathcal{I}} \mathcal{P}_{\mathcal{I}})$, $D(\mathcal{P}_{\mathcal{I}} \mathcal{P}_{\mathcal{I}})$ are mixed versions of the categories $D(\mathcal{P}_{\mathcal{I}})$, $D(\mathcal{P}_{\mathcal{I}})$ in the sense of [BGS]. (We warn the reader that the mixed category is not the category of all mixed sheaves with an appropriate equivariance condition; a necessary condition for a perverse sheaf to lie in the mixed category is that its associated graded with respect to the weight filtration is semisimple cf [BGS], 4.4).

1.5. Anti-spherical quotient category. Recall that $I$ orbits on $\mathcal{F}$ (the so-called Schubert cells) are parameterized by $W$; for $w \in W$ let $j_w : \mathcal{F}_{\mathcal{W}} \to \mathcal{F}$ be the embedding of the corresponding Schubert cell. We let $L_w = j_{w!}(\mathbb{Q}[\ell(w)])$, $w \in W$ be the irreducible objects of $\mathcal{P}_I$, and $j_{w^!} = j_w!(\mathbb{Q}[\ell(w)])$, $j_{w^*} = j_w^!(\mathbb{Q}[\ell(w)])$ be the standard and costandard objects. For an abelian category $\mathcal{A}$, and a set $S$ of irreducible objects of $\mathcal{A}$ let $\langle S \rangle$ denote the full abelian subcategory of objects obtained from elements of $S$ by extensions. Define the Serre quotient category of $\mathcal{P}_I$ by $\mathcal{P}_I / \langle L_w \mid w \notin \mathcal{I}_D \rangle$.

Let $pr_I : \mathcal{P}_I \to \mathcal{I}_D$ be the projection functor.

**Theorem 1.** The functor $\mathcal{I}_D := pr_I \circ F$ is an equivalence
\begin{equation}
\mathcal{I}_D : D^{G} \times G_m(\tilde{N}) \xrightarrow{\sim} D(\mathcal{I}_D).
\end{equation}

1.6. Iwahori-Whittaker category. Let $B = T \cdot N$, $B_\infty = T \cdot N_\infty$ be opposite Borel subgroups, and assume that $I, I_\infty \subseteq G, I \supseteq B_\bar{O}, I_\infty \supseteq B_\bar{O}$ for Iwahori group schemes $I, I_\infty$. Let $I_\mu \subset I_\infty$ be the pro-unipotent radical.

Let also $N_F^- \subset G_F$ be the group ind-scheme, $N_F^-(k) = N^-(F)$. For a simple root $\alpha$ let $u_\alpha : N_F^- \to \mathbb{G}_a$ be the corresponding homomorphism. We define $\Psi : N_F^- \to \mathbb{G}_a$ by $\Psi(n) = \text{Res}((\sum u_\alpha(n))$. We also define $\psi_I : I_\mu \to \mathbb{G}_a$ by $\psi_I(g^{-}\bar{g}^{\mu}) = \Psi(g^{-})$ for $g^{-} \in I_\mu \cap N_F^-$.

Let $D_{\mathcal{W}}$, $\mathcal{P}_{\mathcal{W}}$ be, respectively, the $I_\mu^\infty$, $\psi$-equivariant derived category of $l$-adic sheaves on $\mathcal{F}_I$, and the subcategory of perverse sheaves therein. Since the group scheme $I$ is pro-unipotent, it follows that the forgetful functor $D_{\mathcal{W}}$ to $D(\mathcal{F}_I)$ is a full embedding, thus $D_{\mathcal{W}} \subset D(\mathcal{F}_I)$, $\mathcal{P}_{\mathcal{W}} \subset \mathcal{P}(\mathcal{F}_I)$ are full subcategories.

Let us mention the following

\footnote{We thank V. Ginzburg for pointing out this difficulty to us.}
Lemma 1. We have a natural equivalence
\[ D_{\mathcal{J}W} \cong D^b(\mathcal{P}_{\mathcal{J}W}). \]

Proof is parallel to that of Lemma 4.4.6, and Corollary 3.3.2 in [BGS].

We have an injection \( W \hookrightarrow \mathcal{F}\ell, w \mapsto w\mathbf{1} \); for each one of the (ind) group schemes \( N_{\mathcal{F}}, I_{\mathcal{F}}, I \) the image of this map is a set of representatives for the orbits of the group on \( \mathcal{F}\ell. \) For \( w \in W \) let \( \mathcal{F}\ell^w \) (respectively, \( \Sigma^w \)) be the corresponding orbit of \( I_{\mathcal{F}} \) (respectively, \( N_{\mathcal{F}} \)), and \( i_w : \mathcal{F}\ell^w \hookrightarrow \mathcal{F}\ell, t_w : \Sigma^w \hookrightarrow \mathcal{F}\ell \) be the embeddings.

The proof of the next Lemma is left to the reader.

Lemma 2. a) For \( w \in W \) the following are equivalent

i) \( w \in \mathfrak{F}. \)

ii) \( \text{Stab}_{\mathcal{F}\ell^w}(w\mathbf{1}) \subset \text{Ker}(\psi). \)

iii) \( \text{Stab}_{\mathcal{F}\ell^w}(w\mathbf{1}) \subset \text{Ker}(\psi). \)

b) For \( w \) satisfying the equivalent conditions in (a) the \( \mathcal{F}\ell^w \) is contained in one \( N_{\mathcal{F}}\)-orbit.

If \( w \in \mathfrak{F} \) then there exist unique maps \( \psi_w : \mathcal{F}\ell_w \to \mathcal{G}_{\mathcal{F}}, \Psi_w : \Sigma_w \to \mathcal{G}_{\mathcal{F}} \) defined by \( \psi_w(g \cdot w\mathbf{1}) = \psi(g), \Psi_w(n \cdot w\mathbf{1}) = \Psi(n). \) Define \( \Delta_w, \nabla_w \in \mathcal{P}_{\mathcal{J}W} \) by \( \Delta_w = i_w(\psi^w_\star(\mathcal{A}\mathcal{S})[\ell(w)]), \nabla_w = i_w(\psi^w_\star(\mathcal{A}\mathcal{S})[\ell(w)]) \) where \( \mathcal{A}\mathcal{S} \) in the Artin-Schreier sheaf.

Define the functor \( \text{Av}_\psi : D_I \to D_{\mathcal{J}W}, \) by \( \mathcal{F} \mapsto \Delta_0 * \mathcal{F}. \)

Theorem 2. The functor \( \text{Av}_\psi|_{\mathcal{P}_I} \) induces an equivalence \( \mathcal{P}_I \cong \mathcal{P}_{\mathcal{J}W}. \)

Define the functor \( F_{\mathcal{J}W} : D^{G}\hat{\mathfrak{N}} \to D^b(\mathcal{P}_{\mathcal{J}W}) = D_{\mathcal{J}W} \) by \( \mathcal{F} \mapsto \text{Av}_\psi \circ F(\mathcal{F}). \)

In view of Theorem 2 and Lemma 1, Theorem 1 is equivalent to the following

Theorem 3. The functor \( F_{\mathcal{J}W} \) provides an equivalence \( D^{G}(\hat{\mathfrak{N}}) \cong D_{\mathcal{J}W}. \)

2. Comparison of anti-spherical and Whittaker categories

In this section we will prove a result in the direction of Theorem 2. The proof of the Theorem will be finished in section 4.5 after the proof of Theorem 3.

Proposition 2. a) We have

\[ \text{Av}_\psi : \mathcal{P}_I \mapsto \mathcal{P}_{\mathcal{J}W}; \]

thus \( \text{Av}_\psi|_{\mathcal{P}_I} \) induces an exact functor \( \mathcal{P}_I \to \mathcal{P}_{\mathcal{J}W}. \) This functor factors through \( \mathfrak{F}\ell. \)

b) The functor \( \mathfrak{F}\ell \to \mathcal{P}_{\mathcal{J}W} \) induced by \( \text{Av}_\psi \) is a full embedding.

Set \( \delta_e = j_e = j_e^1 \) where \( e \in W \) is the identity element.

Let \( W' \subset W \) be the subgroup generated by simple reflections (non-extended affine Weyl group). Thus \( \bigcup_{w \in W'} \mathcal{F}\ell_w \) is a connected component of \( \mathcal{F}\ell. \)

Lemma 3. a) For \( w \in W' \) we have nonzero morphisms

\[ \delta_e \to j_w!; \]

\[ j_w^* \to \delta_e, \]

whose (co)kernel does not contain \( \delta_e \) in its Jordan-Hoelder series.

b) If \( w = w_1 w_2 \in W, w_2 \in W' \) and \( \ell(w) = \ell(w_1) + \ell(w_2) \) then

\[ \dim \text{Hom}(j_w!, j_w) = 1 = \dim \text{Hom}(j_w^*, j_w^*); \]

and a nonzero map \( j_w! \to j_w! \) (respectively, \( j_w^* \to j_w^* \)) is injective (respectively, surjective).
Proof. We prove the statements concerning \( j_w! \), the ones concerning \( j_{w*} \) are obtained by duality.

For a simple reflection \( s_\alpha \in W \) we have an exact sequence of perverse sheaves on the projective line \( \mathcal{F}_{/s_\alpha} \)

\[
0 \to \delta_\alpha \to j_{s_\alpha!} \to L_{s_\alpha} \to 0.
\]

If \( u \in W \) is such that \( \ell(u \cdot s_\alpha) > \ell(u) \) consider the convolution of \( j_{u!} \) with \( \mathcal{F}_{/s_\alpha} \); it is an exact triangle

\[
\delta_\alpha \to j_{u * s_\alpha!} \to j_{u!} \cdot L_{s_\alpha}.
\]

Notice that \( j_{u!} \cdot L_{s_\alpha} = \pi_\alpha^* \pi_{\alpha*}(j_{u!})[1] \), where \( \pi_\alpha : \mathcal{F}_{\ell} \to \mathcal{F}_{\ell}(\alpha) \) is the projection to the partial affine flag variety \( \mathcal{F}_{\ell}(\alpha) = G_{\mathbb{F}}/I_\alpha \) for the minimal parahoric \( I_\alpha \) corresponding to \( \alpha \). Since \( \pi_\alpha \circ j_u \) is a locally closed affine embedding (because \( \ell(u \cdot s_\alpha) > \ell(u) \)), we see that \( \pi_{\alpha*}(j_{u!}) \), and hence \( \pi_\alpha^* \pi_{\alpha*}(j_{u!})[1] \) are perverse sheaves. Thus the exact triangle (9) is in fact an exact sequence of perverse sheaves. Also all irreducible subquotients of \( j_{u!} \cdot L_{s_\alpha} \) are of the form \( \pi_\alpha^*(L[1]) \) for a perverse sheaf \( L \) on \( \mathcal{F}_{\ell}(\alpha) \); thus none of them is isomorphic to \( \delta_\alpha \). This implies (a) by induction in \( \ell(u) \).

Since \( j_{w2!} \) is invertible under convolution (see Lemma 8(b) below) we have

\[
\text{Hom}(j_{w1!}, j_{w2!}) = \text{Hom}(j_{w1!}, j_{w1!} \cdot j_{w2!}) = \text{Hom}(\delta_\alpha, j_{w2!}),
\]

thus the first statement in (b) follows from (a). Finally, the exact sequence (9) implies by induction in \( \ell(w_2) \) existence of an injective map \( j_{w1!} \to j_{w!} \).

Lemma 4. a) We have

\[
A_{\psi}(L_w) = 0 \iff w \not\in \mathcal{F} W.
\]

b) We have \( \Delta_\psi \cong \nabla \). c) For \( w = w_f \cdot w', w_f \in W_f, w' \in \mathcal{F} W \) we have

\[
A_{\psi}(j_{w!}) \cong \Delta_{w'},
\]

\[
A_{\psi}(j_{w*}) \cong \nabla_{w'}.
\]

Proof. If \( w \in \mathcal{F} W \), then the convolution map \( \mathcal{F} \times \mathcal{F} \to \mathcal{F} \) restricted to the generic point of the support of \( \Delta_0 \boxtimes L_w \) is an isomorphism. Hence \( \Delta_0 \cdot L_w \neq 0 \) for \( w \in \mathcal{F} W \). On the other hand, for \( w \not\in \mathcal{F} W \) there exists a simple root \( \alpha \neq \alpha_0 \) such that \( L_w \) is equivariant with respect to the corresponding minimal parahoric subgroup \( I_\alpha \) (here \( \alpha_0 \) denotes the affine simple root). Then the functor \( \mathcal{F} \to \mathcal{F} \cdot L_w \) factors through the functor \( \pi_{\alpha*} \) (recall that \( \pi_{\alpha*} : \mathcal{F} \to \mathcal{F}_{\ell}(\alpha) \) is the projection to the corresponding partial affine flag variety). However, \( \pi_{\alpha*}(\Delta_0) = 0 \) because the character \( \psi_f \) is nontrivial on \( Stab_{I_\alpha}(x) \) for any \( x \) in the image of the support of \( \Delta_0 \) under \( \pi_{\alpha*} \). This proves (a).

(b) is clear because \( \psi_f \) is nontrivial on \( Stab_{I_\alpha}(x) \) for any \( x \in \mathcal{F} W - \mathcal{F} \).

In view of (b) it suffices to prove the first equality in (c); the second one then follows by duality. The equality is clear when \( w \in \mathcal{F} W \), because in this case the convolution map restricted to the support of \( \Delta_0 \boxtimes j_{w!} \) is an isomorphism over \( \mathcal{F} \), while the * restriction of \( \Delta_0 \boxtimes j_{w!} \) to the preimage of the complement of \( \mathcal{F} \) is zero. Let now \( w \) be arbitrary; we have \( w = w_f \cdot w' \) for some \( w_f \in W_f, w' \in \mathcal{F} W \), where \( \ell(w) = \ell(w_f) + \ell(w') \). Then Lemma 8 and part (a) of this Lemma imply that \( \Delta_\psi \cdot j_{w!} \cong \Delta_\psi \cdot \delta_\psi = \Delta_\psi \). Thus we have

\[
\Delta_\psi \cdot j_{w!} \cong \Delta_\psi \cdot j_{w!} \cdot j_{w!} \cong \Delta_{w'},
\]

For an algebraic group \( H \) and a subgroup \( H' \subset H \) (or more generally, for group schemes of possibly infinite type, such that the quotient \( H/H' \) is of finite type) let \( \Gamma_H^H \) be the * induction
functor from $H'$-equivariant to $H$-equivariant sheaves; recall that it is defined by $\Gamma_H^F(\mathcal{F}) = a_* \left( \underline{\mathcal{G}}_H \otimes_H \mathcal{F} \right)$, where $a : H \times_H' X \to X$ is the action map (cf \cite{BM}).

Define the functor $Av_I : D_{jW} \to D_I$ by $Av_I = \Gamma_{I \cap I^-}^F$.

**Lemma 5.** We have $H^{p,0}(Av_I(\Delta_0)[\ell(w)]) \cong j_{w_0!}$, where $w_0 \in W_f$ is the longest element, and superscript $p$ refers to the $t$-structure of perverse sheaves.

**Proof.** It suffices to construct an exact triangle

$$j_{w_0!} \to Av_I(\Delta_0)[\ell(w)] \to C$$

such that $C \in D^{p,>0}$. The definition of $Av_I$ implies that

$$\text{Hom}_{D_I}(X, Av_I(\Delta_0)[\ell(w)]) = \text{Hom}_D(\text{Forg}(X), (\Delta_0)[\ell(w)])$$

where $D$ is the derived category of $l$-adic sheaves on $\mathcal{F}l$, and $\text{Forg} : D_I \to D$ is the forgetful functor (notice that $D_{I \cap I^-}$ is a full subcategory in $D$ because $I \cap I^- = \text{unipotent}$). The proof of Lemma \ref{4}(a) shows that

$$(10) \quad \text{Hom}^*(L_w, \Delta_0) = 0$$

for $w \in W_f$, $w \neq e$ (and more generally for $w \notin J W f$). Also it is clear that

$$\text{Hom}_{D_I}(\delta_\varepsilon, Av_I(\Delta_0)[\ell(w)]) \cong \underline{\mathbb{Q}}l.$$ 

Now Lemma \ref{3}(a) implies that

$$\text{Hom}_{D_I}(j_{w_0!}, Av_I(\Delta_0)[\ell(w)]) \cong \underline{\mathbb{Q}}l$$

for $w \in W_f$. Moreover, the composition of nonzero arrows

$$j_{w_0!} \to j_{w_0!} \to Av_I(\Delta_0)[\ell(w)]$$

is nonzero, because the composition $\delta_\varepsilon \to j_{w_0!} \to j_{w_0!}$ is nonzero by \ref{3}(b). Hence, if $C = \text{cone}(j_{w_0!} \to Av_I(\Delta_0)[\ell(w)])$, then using again Lemma \ref{3}(b) we see that for all $w \in W_f$

$$\text{Hom}(j_{w_0!}, C[i]) = 0$$

for $i \leq 0$, which implies that $C \in D^{p,>0}$ according to the definition of perverse $t$-structure. \Box

**Remark 3.**\footnote{This Remark is included here following the referee’s suggestion.} A slightly different proof of the Lemma can be given as follows. It is not hard to describe $Av_{I_0}(\Delta_0)$, where $I_0 \subset I$ is the pro-unipotent radical (details will appear in \cite{BM}). This is obviously an object in the category of $I_0$-equivariant sheaves supported on $G/B \subset \mathcal{F}l$, which is identified with category $O$ for $G$. Then $Av_{I_0}(\Delta_0) \cong \Xi[\ell(w)]$, where $\Xi$ is the maximal projective in category $O$ (a projective cover of the irreducible Verma module). Lemma \ref{3} can then be deduced from the fact that the subobject $j_{w_0!} \subset \Xi$ is the maximal $B$-equivariant subobject in the $B$-monodromic object $\Xi$.

2.0.1. **Right inverse to $Av_\Phi$.** To prove Proposition \ref{2} we will explicitly construct a right inverse functor to $Av_\Phi$. Namely, define $F' : \mathbb{P}^W \to \mathbb{P}f_I$ by

$$F'(\mathcal{F}) = \text{proj}(H^{p,\ell(w)}(Av_I(\mathcal{F}))).$$

To motivate this definition we remark that one can easily show that $F'$ is right adjoint to $Av_\Phi$ (we neither check, nor use this fact below).

**Lemma 6.** There exists a canonical isomorphism $F' \circ Av_\Phi \cong \text{id}$. 

3This Remark is included here following the referee’s suggestion.
Proof. For \( w \in W_f \) the functor from \( D_1 \) to \( \mathcal{I} D_1 \) sending \( \mathcal{I} \) to \( pr_f(L_w * \mathcal{I}) \) is zero if \( w \neq e \) and is isomorphic to \( pr_f \) otherwise. Hence the convolution functor descends to a functor \( \mathcal{I} D^0_1 \times \mathcal{I} D^0_1 \to \mathcal{I} D^0_1 \) exact in each variable; here \( \mathcal{I} D^0_1 \) denotes the Serre quotient category of \( \mathcal{I} \)-equivariant perverse sheaves on \( G_0/\mathcal{I} \subset \mathcal{I} \ell \) by the subcategory generated by \( L_w, w \neq e \). In particular, for \( \mathcal{I} \in \mathcal{I} \mathcal{D}_1 \) we have

\[
F' \circ Av_{\mathcal{I}}(pr_f(\mathcal{I})) \cong pr_f \circ H^{p,0}(Av_f(\Delta_0[^p(w)]) * \mathcal{I}) \cong pr_f \circ H^{p,0}(j_{w!*} * \mathcal{I}) \cong pr_f(\mathcal{I}),
\]

where the last isomorphism follows from Lemma 3(a), and the previous one from Lemma 5. \( \square \)

2.0.2. Proof of Proposition 2(conclusion). For a triangulated category \( D \) and a set of objects \( S \subset Ob(D) \) we let \( \langle S \rangle \) be the set of all objects obtained from elements of \( S \) by extensions; i.e. \( \langle S \rangle \) is the smallest subset of \( D \) containing \( S \cup \{0\} \) and such that for all \( A, B \in \langle S \rangle \) and an exact triangle \( A \to C \to B \to A[1] \) we have \( C \in \langle S \rangle \).

The definition of perverse \( t \)-structure implies that

\[
\text{Ob}(\mathcal{P}_f) = (j_{w!*}[i] \mid i \geq 0) \cap (j_{w*}[i] \mid i \leq 0); \\
\text{Ob}(\mathcal{P}_f^0) = (\Delta_w[i] \mid i \geq 0) \cap (\nabla_w[i] \mid i \leq 0).
\]

Thus the first statement in Proposition 2(a) follows from Lemma 4(c). The second one is immediate from part (a) of that Lemma. Part (a) of the Proposition is proved.

We also see that \( F(L_w) \) is irreducible for \( w \in \mathcal{I}W \), because it is the image of a nonzero map \( \Delta_w \to \nabla_w \); it is clear that \( F(L_w) \not\cong F(L_{w'}) \) for \( w \neq w', w, w' \in \mathcal{I}W \), because they have different supports.

Thus the Proposition follows from Lemma 5 and the following Lemma. \( \square \)

Lemma 7. Let \( F : \mathcal{A} \to \mathcal{B} \) be an additive functor between abelian categories. Assume that

i) \( F \) is exact.

ii) Every object of \( \mathcal{A} \) has finite length, and \( F \) induces an isomorphism \( \text{Hom}(L_1, L_2) \cong \text{Hom}(F(L_1), F(L_2)) \) for any irreducible objects \( L_1, L_2 \) of \( \mathcal{A} \).

iii) There exists an additive functor \( F' : \mathcal{B} \to \mathcal{A} \) such that \( F' \circ F \cong \text{id} \).

Then \( F \) is a full embedding.

Proof. Conditions (i) and (iii) imply that \( F \) is injective on \( \text{Ext}^1 \). Indeed, let \( 0 \to X \to Y \to Z \to 0 \) be a short exact sequence in \( \mathcal{A} \). If \( F(Y) \cong F(X) \oplus F(Z) \) is the splitting of its image under \( F \), then applying \( F' \) to it we see that the original sequence is split.

Now induction in the lengths of \( X, Y \) shows that \( F \) induces an isomorphism \( \text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y)) \) for any two objects \( X, Y \in \mathcal{A} \). \( \square \)

3. Construction of the monoidal functor \( F \)

3.1. Plan of the construction. We will use a version of Serre’s description of \( \text{Coh}(\mathbb{P}^n) \) as a quotient of the category of graded modules over the symmetric algebra. We need some notations.

Let \( \hat{N} \) be the preimage of \( \hat{N} \subset \mathcal{B} \times \mathfrak{g}^- \) in \( G'/U^- \times \mathfrak{g}^- \). Thus \( \hat{N} \) is a locally closed subscheme in the affine variety \( G'/U^- \times \mathfrak{g}^- \); here \( U^- \subset G' \) is a maximal unipotent, and \( G'/U^- \) is the affine closure of the basic affine space \( G'/U^- \). We now define a closed (obviously affine) subscheme \( \hat{N}_{af} \subset G'/U^- \times \mathfrak{g}^- \) containing \( \hat{N} \) as an open subscheme (though different from the closure of \( \hat{N} \) in \( G'/U^- \times \mathfrak{g}^- \)). On \( G'/U^- \times \mathfrak{g}^- \) we have a canonical vector field \( v_{\text{aut}} \) whose value at a point \( (p, x) \) equals \( (a(x), 0) \) where \( a \) stands for the action of the Lie algebra \( \mathfrak{g}^- \) on \( G'/U^- \). The vector field \( v_{\text{aut}} \) induces a derivation of \( \mathcal{O}_{G'/U^- \times \mathfrak{g}^-} \); we let \( \hat{N}_{af} \) be the zero-set of this derivation (i.e. the defining ideal of \( \hat{N}_{af} \) is generated by the image of the derivation). It is clear
that \( \hat{N}_{af} \cap (G^* / U^* \times g^*) = \hat{N} \). We set \( \hat{O}_N = \hat{O}_{\hat{N}_{af}} \), and call this ring the multi-homogeneous coordinate ring of \( \hat{N} \). For a scheme \( S \) equipped with an action of an algebraic group \( H \) we write \( \text{Coh}^H(S) \) (or \( \text{O} \) \(-mod^H \), \( S = \text{Spec}(0) \)) for the full subcategory in \( \text{Coh}^H(S) \) consisting of objects of the form \( V \otimes \text{O}_S \), \( V \in \text{Rep}(H) \).

We now describe the plan of the construction. The first piece of data is a monoidal functor \( \tilde{F} : \text{Rep}(G^* \times T^*) \rightarrow \mathcal{P}_I \) (i.e. a monoidal functor to \( \mathcal{P}_I \) landing in \( \mathcal{P}_I \)); the main ingredient is provided by \( \mathcal{G} \).

We then explain that a certain natural endomorphism of this action (also defined in \( \mathcal{G} \)) yields an extension of \( \tilde{F} \) to a monoidal functor \( \hat{F} : \text{Coh}^{G^* \times T^*}(\hat{N}_{af}) \rightarrow \text{Coh}^{\mathcal{G}}(\mathcal{G}) \) to \( \text{Hot}(\mathcal{P}_I) \), where \( \text{Hot} \) stands for the homotopy category. Let \( \text{Acycl} \subset \text{Hot}(\text{Coh}^{G^* \times T^*}(\hat{N}_{af})) \) be the full subcategory of such complexes \( F^* \) that \( F^*|_{\hat{N}} \) is acyclic. In section 3.3 we prove certain facts about the central sheaves of \( \mathcal{G} \), and deduce from it that \( \tilde{F} \) sends \( \text{Acycl} \) to acyclic complexes. Hence \( \tilde{F} \) factors to a functor \( D^G(\hat{N}) \rightarrow D^B(\mathcal{P}_I) \rightarrow D_I \).

3.2. Central and Wakimoto sheaves: definition of the functor \( \tilde{F} \). Recall the functor \( Z : \mathcal{P}_{G_0}(\mathfrak{g}_T) \rightarrow \mathcal{P}_I \subset D_I \) constructed in \( \mathcal{G} \).

We identify \( \mathcal{P}_{G_0}(\mathfrak{g}_T) \) with \( \text{Rep}(G^*) \) by means of the geometric Satake equivalence \( S : \text{Rep}(G^*) \rightarrow \mathcal{P}_{G_0}(\mathfrak{g}_T) \). We set \( V_\lambda = S^{-1}(IC_\lambda) \) where \( IC_\lambda = j_{\lambda*} \left( \mathcal{O}(\ell(\lambda)) \right) \), and \( j_\lambda : \mathfrak{g}_T \rightarrow \mathfrak{g}_T \) is the embedding of the image \( \mathfrak{g}_T \) of \( \mathcal{F}_\ell \) under the projection \( \pi : \mathcal{F}_\ell \rightarrow \mathfrak{g}_T \); thus \( V_\lambda \) is a representation with highest weight \( \lambda \). Notice that the convolution map \( \text{supp}(IC_\lambda \boxtimes IC_\mu) \rightarrow \text{supp}(IC_\lambda * IC_\mu) = \mathcal{O}_{\lambda + \mu} \) is an isomorphism over \( \mathcal{O}_{\lambda + \mu} \); hence we have

\[
\tilde{j}_{\lambda + \mu} \circ IC_\lambda \boxtimes IC_\mu \cong \mathcal{O}(\ell(\lambda + \mu))
\]

canonical. Thus we get a canonical element \( m_{\lambda, \mu} \) in the one dimensional vector space \( \text{Hom}(IC_\lambda * IC_\mu, IC_{\lambda + \mu}) \).

We also set \( Z_\lambda = Z(V_\lambda) \).

The functor \( Z \) is monoidal, and moreover central: the latter means that for every \( V \in \text{Rep}(G^*) \) and \( F \in D_I \) there is a fixed “centrality” isomorphism \( \sigma_{V, F} : Z(V^* \boxtimes F) \cong \mathbf{1} \) satisfying some natural compatibilities (spelled out e.g. in \( \mathcal{B}_1 \), §2.1, and checked in \( \mathcal{C} \) and Gaitsgory’s Appendix to \( \mathcal{B}_1 \)). Notice that a central functor from a tensor category \( \mathcal{C} \) to a monoidal category \( \mathcal{E} \) is the same as a tensor (compatible with braiding) functor from \( \mathcal{A} \) to the center of \( \mathcal{E} \) (see e.g. \( \mathcal{K}_3 \), XIII.4).

Recall that \( j_{w!} = j_{w!}(\mathcal{O}(\ell(\ell(w)))) \), \( j_{w*} = j_{w*}(\mathcal{O}(\ell(\ell(w)))) \); and \( \delta_e = j_{e!} = j_{e*} \) is the unit object of \( D_I \) (here \( e \) is the unit element of \( W \)). The following statement is well-known.

**Lemma 8.** a) If \( w_1, w_2 \in W \) are such that \( \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \) then we have a canonical isomorphism

\[
\tag{11} j_{w_1!} * j_{w_2*} \cong j_{w_1 w_2*}.
\]
If \( w_1, w_2, w_3 \) are such that
\[
\ell(w_1 w_2 w_3) = \ell(w_1) + \ell(w_2) + \ell(w_3)
\]
then the two isomorphisms between \( j_{w_1} \circ j_{w_2} \circ j_{w_3} \) and \( j_{w_1 w_2 w_3} \) arising from (11) coincide.

b) \( j_{w^*} \) is an invertible object of the monoidal category \( D_1 \). More precisely, we have
\[
j_{w^*} \circ j_{w^{-1}} \cong \delta_c \cong j_{w^{-1}} \circ j_{w^*}.
\]

**Corollary 1.** a) The map \( \lambda \mapsto j_{\lambda^*} \) for \( \lambda \in \Lambda^+ \) extends naturally to a monoidal functor \( \text{Rep}(T) \to D_1 \).

b) The map \( V \otimes (\lambda) \mapsto \mathbb{Z}(V) \circ j_{\lambda^*} \) for \( V \in \text{Rep}(G) \), \( \lambda \in \Lambda^+ \subset \text{Rep}(T^*) \) extends naturally to a monoidal functor \( \tilde{F} : \text{Rep}(G^* \times T^*) \to D_1 \).

**Proof.** The length function on \( W \) is additive on the subsemigroup \( \Lambda^+ \), thus Lemma 8 applies to \( w_1, w_2, w_3 \in \Lambda^+ \), and implies statement (a). Then (b) follows from the central property of the functor \( \mathbb{Z} \), which yields a commutativity isomorphism \( \mathbb{Z}(V) \circ j_{\lambda^*} \cong j_{\lambda^*} \circ \mathbb{Z}(V) \) satisfying the pentagon identity.

We denote the image of \( \lambda \) under the functor defined in the Corollary by \( J_\lambda \). It follows from the definition that \( J_\lambda = j_{\lambda} \) for \( \lambda \in -\Lambda^+ \), \( J_\lambda = j_{\lambda} \) for \( \lambda \in \Lambda^+ \), and \( J_{\lambda + \mu} \cong J_\lambda \circ J_\mu \).

Following Mirković we call \( J_\lambda \) the Wakimoto sheaves. Theorem 5 below asserts that \( J_\lambda \) are actually objects of the abelian category \( \mathcal{P}_I \) (a priori they are defined as objects of the triangulated category \( D_1 \)).

### 3.3. Monodromy and "highest weight" arrows: characterization of the functor \( \tilde{F} \).

#### 3.3.1. Arrows between perverse sheaves.

Recall that the monoidal functor \( \mathbb{Z} \) comes equipped with a tensor endomorphism \( M = \{ M_V = M_{\mathbb{Z}(V)} \in \text{End}(\mathbb{Z}(V)) \} \) defined by the logarithm of monodromy (see [G], Theorem 2; we fix and use an isomorphism \( \mathbb{Q} \cong \mathbb{Q}(1) \)).

We also define an arrow \( b_\lambda : Z_\lambda \to j_{\lambda^*}, \lambda \in \Lambda^+ \). The definition is clear from the next

**Lemma 9.** For all \( \lambda \in \Lambda^+ \) the Schubert cell \( \mathcal{F}_\lambda \) is open in the support of \( Z_\lambda \); and we have a canonical isomorphism
\[
j_\lambda^*(Z_\lambda) \cong \mathbb{Q}[\ell(\lambda)].
\]

**Proof.** Recall that \( \pi \) denotes the projection \( \mathcal{F} \to \mathcal{G} \). It is immediate to see from the definition of the functor \( \mathbb{Z} \) (see [G], 2.2.3) that the support of \( Z_\lambda \) is contained in the preimage under \( \pi \) of the closure of the Schubert cell \( \mathcal{G}_\lambda \); and also that its dimension equals \( \dim \mathcal{G}_\lambda = \dim(\mathcal{F}_\lambda) \). Thus it can not contain \( \mathcal{F}_w \) for \( w > \lambda \). It contains \( \mathcal{F}_\lambda \), and we have the canonical isomorphism (12), because the support of \( \pi_*(Z_\lambda) = I\mathcal{G}_\lambda \) contains \( \mathcal{G}_\lambda \), and \( I\mathcal{G}_\lambda |_{\mathcal{F}_\lambda} = \mathbb{Q}[\ell(\lambda)] \).

#### 3.3.2. Arrows between coherent sheaves.

First, consider the variety \( g^- \) equipped with the adjoint action. Then every \( \mathcal{F} \in \text{Coh}^{G^*} (g^-) \) carries a canonical endomorphism, such that the induced endomorphism of the fiber at a point \( x \in g^- \) coincides with the action of \( x \in \text{Stab}_{g^-} (x) \) coming from the equivariant structure; we denote this endomorphism by \( N_{\mathcal{F} \text{out}} \), and abbreviate \( N_{\mathcal{F} \text{out}} = N_{\mathcal{F} \text{out}} \).

Next, consider the basic affine space \( G^*/U^- \), and its affine closure \( G^*/U^- \). We fix an isomorphism between the ring of regular functions on \( G^*/U^- \) and the ring \( \bigoplus_{\lambda^+} V_\lambda \) with multiplication given by \( \bigoplus_{\lambda^+} m_{\lambda, \mu} \) (see section 3.2 for notation). Then for \( \lambda \in \Lambda^+ \) we get a morphism in
\[
\text{Coh}_{G^* \times G^-}(G^*/U^-);
\]
\[
B_\lambda : V_\lambda \otimes O \to O_\lambda.
\]
Proposition 3. There exists a unique extension of $\tilde{F}$ to a monoidal functor $\tilde{F} : \text{Coh}^{G \times T^*}_f (\hat{N}_a) \to \mathcal{F}$ such that $\tilde{F}(N_V^{\text{taut}}) = M_{Z(V)}, \tilde{F}(B_\lambda) = b_\lambda$.

The proof of the Proposition will be given at the end of the next section after some general nonsense preparation.

Remark 4. One can show that any arrow in $\text{Coh}^{G \times T^*}_f (\hat{N}_a)$ can be obtained from the arrows $B_\lambda, N_V$ and identity arrows by taking tensor products and direct summands. This implies the uniqueness statement in the Proposition. We will give a slightly different argument in the next section.

The only geometric statement needed for the proof of Proposition 3 is the next

Lemma 10. a) For $\lambda, \mu \in \Lambda^+$ we have $\text{Hom}(Z_\mu, J_\lambda) = 0$ unless $\lambda \preceq \mu$.

b) For $\lambda, \mu \in \Lambda^+$ the following diagram is commutative

$$
\begin{array}{ccc}
Z_{\lambda+\mu} & \xrightarrow{z_{(m_{\lambda, \mu})}} & Z_\lambda \ast Z_\mu \\
\downarrow b_{\lambda+\mu} & & \downarrow b_\lambda \ast b_\mu \\
\tilde{j}_{\lambda+\mu} & \xrightarrow{\tilde{j}_{\lambda} \ast \tilde{j}_\mu} & j_\lambda \ast j_\mu \\
\end{array}
$$

where the lower horizontal isomorphism comes from Lemma 8(a).

c) We have $b_\lambda \circ M_{Z_\lambda} = 0$.

Proof. a) As was said in the proof of the previous Lemma, the support of $Z_\mu$ is contained in the preimage under $\pi$ of the closure of the Schubert cell $\mathfrak{S}_\mu$. It is well known that $\mathcal{F}_\ell_\lambda$ is contained in this set iff $\lambda \preceq \mu$.

b) $\pi_*$ induces an isomorphism of one dimensional vector spaces

$$
\text{Hom}(Z_\lambda, j_{\lambda*}) \xrightarrow{\sim} \text{Hom}(IC_{\lambda}, \tilde{j}_{\lambda*}),
$$

where $\tilde{j}_{\lambda*} = j_{\lambda*}(\mathbb{Q}(\ell(\lambda)))$. Thus it suffices to check that applying $\pi_*$ to the above diagram we get a commutative one. This follows from the definition, and the canonical isomorphism $\pi_* \circ Z \cong \text{id}_{\text{Coh}(O)}$.

c) It suffices to see that

$$
\tilde{j}_{\lambda*}(M_{Z_\lambda}) \in \text{End}(j_{\lambda*}^*(Z_\lambda)) = 0,
$$

This follows from nilpotency of $M_{Z_\lambda}$ and Lemma 4 which shows that $\text{End}(j_{\lambda*}^*(Z_\lambda))$ is one dimensional. $\blacksquare$

3.4. Tannakian and Drinfeld-Plucker formalism. Notice that the arrows $B_\lambda$ introduced in section 3.3.2 satisfy the so-called Plucker relations, i.e.

$$
B_\lambda \otimes B_\mu = B_{\lambda+\mu} \circ (m_{\lambda, \mu} \otimes \text{id}_O).
$$

Lemma 11. Let $A$ be a commutative algebra with a $G$ action.

a) Let $N$ be a tensor endomorphism of the functor $V \mapsto A \otimes V$; thus $N$ is a collection of $G$-invariant endomorphisms $N_V \in \text{End}_A (A \otimes V), V \in \text{Rep}(G)$, functorial in $V$ and such that

$$
N_{V_1} \otimes 1 + 1 \otimes N_{V_2} = N_{V_1 \otimes V_2}
$$

for all $V_1, V_2 \in \text{Rep}(G)$; Then there exists a unique element $x_N \in g^* \otimes A$, such that $N_V$ coincides with the action of $x$ in $A$.

Also, there is a unique $G$-equivariant homomorphism $\phi : \mathcal{O}_{\theta^*} \to A$ such that $N_V = \phi_* (N_V^{\text{taut}}) = \text{id}_A \otimes \mathcal{O}_e \cdot N_V^{\text{taut}}$. 


b) Assume that $A$ is equipped with a $\Lambda$ grading compatible with the $G^*$ action (in other words, an action of $T^*$ commuting with the $G^*$ action is given); and suppose that for every $\lambda \in \Lambda^+$ we are given a $G^*$-equivariant morphism $b_\lambda : V_\lambda \otimes \Lambda \to A(\lambda)$ satisfying the Plucker relations (with $B_\lambda$ replaced by $b_\lambda$, and $O$ replaced by $A$). Then there exists a unique $G^* \times T^*$-equivariant homomorphism $\phi : \mathcal{O}(G^*/U^*) = \mathcal{O}(G^*/U^*) \to A$ such that $b_\lambda = \phi_\lambda(B_\lambda)^{\text{def}} = \text{id}_A \otimes \phi_\lambda \cdot B_\lambda$.

c) Let $A, b_\lambda$ be as in (b), and $N$ be as in (a). Assume that

$$b_\lambda \circ N_{V_\lambda} = 0$$

for all $\lambda$. Then the homomorphism $\mathcal{O}(G^*/U^* \times g^*) \to A$ provided by $(a, b)$ factors through $\hat{\mathcal{O}}_{N'}$.

Proof. The first statement in (a) is well-known.

The second statement in (a) is a restatement of the first one. More precisely, a homomorphism $\mathcal{O}_{g^*} \to A$ is specified by an element of $\text{Hom}(\mathcal{O}((g^*)^*, A) = \mathcal{O} \otimes A$, and it is straightforward to see that $x \in g^* \otimes A$ satisfies the conditions of the first statement in (a) iff the corresponding homomorphism $\mathcal{O}_{g^*} \to A$ satisfies the conditions of the second one.

To check (b) recall that $\mathcal{O}(G^*/U^*) \cong \bigoplus_{\Lambda^+} V_{\lambda + 1}$. Then the requirement on $\phi$ is equivalent to

$$\phi|_{V_\lambda} = b_\lambda|_{V_\lambda \otimes 1}.$$ 

Thus uniqueness of $\phi$ is clear. The Plucker relations ensure that the map $\phi : \mathcal{O}(G^*/U^*) \to A$ defined by (15) is indeed a homomorphism, which shows existence.

c) is immediate from the definition of $\hat{N}_{af}$. □

**Proposition 4.** Let $\mathcal{C}$ be an additive monoidal category.

a) Let $F : \text{Rep}(G^*) \to \mathcal{C}$ be a monoidal functor.

Let $N = \{N_V\}$ be a tensor endomorphism of $F$, such that the image under $F$ of the commutativity isomorphism in $\text{Rep}(G^*)$ is functorial with respect to $N$. Then there exists a unique extension of $F$ to a monoidal functor $\hat{F} : \text{Coh}_{fr}(g^*) \to \mathcal{C}$ such that $N_V = \hat{F}(N_V^{\text{aut}})$ for all $V \in \text{Rep}(G^*)$ (here $G^*$ acts on $g^*$ by the adjoint action).

b) Let $F : G^* \times T^* \to \mathcal{C}$ be a monoidal functor. Suppose that for each $\lambda \in \Lambda^+$ we are given transformations $b_\lambda : F(\lambda) \to F(V_\lambda)$ satisfying the Plucker relations, i.e. such that

$$b_\lambda \otimes b_\mu = b_{\lambda + \mu} \circ F(m_{\lambda, \mu}).$$

Assume that the image of the commutativity isomorphism under $F$ is functorial with respect to $b_\lambda$. Then there exists a unique extension of $F$ to a monoidal functor $\hat{F} : \text{Coh}_{fr}^{G^* \times T^*}(G^*/U^*) \to \mathcal{C}$ such that $b_\lambda = \hat{F}(B_{\lambda})$.

c) Let $F$, $b_\lambda$ be as in (b), and $N \in \text{End}(F|_{\text{Rep}(G^*)})$ be as in (a). Assume that

$$b_\lambda \circ N_{V_\lambda} = 0$$

for all $\lambda$. Then $(a, b)$ provide an extension of $F$ to a monoidal functor $\text{Coh}_{fr}^{G^* \times T^*}(G^*/U^* \times g^*) \to \mathcal{C}$, which factors through $\hat{\mathcal{O}}_{N_G} \otimes \text{mod}_{fr}^{G^* \times T^*}$.

Proof. Let $H$ stand for $G^*$ if we are in the situation of (a), and for $G^* \times T^*$ if we are in the situation of either (b) or (c).

First we claim that without loss of generality we can assume that $\mathcal{C}$ is a tensor category, and $F$ is a tensor functor. More precisely, we claim that it is possible to factor $F$ as a composition $F = F' \circ F'$, where $F'$ is a tensor functor from $\text{Rep}(H)$ to a tensor category $\mathcal{C}'$, and $F'' : \mathcal{C}' \to \mathcal{C}$ is a monoidal functor; moreover, in the situation of (a) there exists a tensor endomorphism $N'$ of $F'$ satisfying the conditions of (a), and such that $N = F''(N')$; and similarly for (b) and (c).
Namely, we can define $\mathfrak{C}'$ as follows. We set $\text{Ob}(\mathfrak{C}') = \text{Ob}(\text{Rep}(H))$, and $\text{Hom}_{\mathfrak{C}'}(V, W) \subset \text{Hom}_{\mathfrak{C}}(F(V), F(W))$ consists of such elements $\phi$ that for all $U \in \text{Rep}(H)$ the diagram
\[
\begin{array}{ccc}
F(U) \otimes F(V) & \xrightarrow{id \otimes \phi} & F(U) \otimes F(W) \\
\downarrow & & \downarrow \\
F(V) \otimes F(U) & \xrightarrow{\phi \otimes id} & F(W) \otimes F(U)
\end{array}
\]
is commutative for all $U \in \text{Rep}(H)$; here the vertical arrows are images under $F$ of the commutativity isomorphism. The pentagon identity implies that $\mathfrak{C}'$ is indeed a tensor category; the definition of $F'$, $F''$, $N'$, $b'_\lambda$ is clear. So from now on we will assume that $\mathfrak{C}$, $F$ are tensor.

We will use underlined symbols to denote representations of $H$, and the corresponding unadorned symbol will denote the underlying vector space.

Let $\underline{\mathfrak{O}}$ be the module of regular functions on $H$ where $H$ acts by left translations; thus $\underline{\mathfrak{O}}$ is an ind-object of $\text{Rep}(H)$, $\underline{\mathfrak{O}} = \bigoplus_{\underline{\lambda} \in \text{IrrRep}(H)} V^* \otimes V_{\underline{\lambda}}$, where $\text{IrrRep}(H)$ is a set of representatives for isomorphism classes of irreducible representations of $H$. Thus $\underline{\mathfrak{O}}$ is a commutative ring ind-object in $\text{Rep}(H)$. Set
\[
A = \text{Hom}_{\mathfrak{C}}(1_{\mathfrak{C}}, F(\underline{\mathfrak{O}})) = \bigoplus_{\underline{\lambda} \in \text{IrrRep}(H)} V^* \otimes \text{Hom}_{\mathfrak{C}}(1_{\mathfrak{C}}, F(V)) ;
\]
here (and below) we use the same notation for a functor on a category, and the induced functor on the category of ind-objects. Then $A$ is an associative algebra equipped with an $H$ action. Commutativity of $\mathfrak{O}$ and tensor property of $F$ show that $A$ is commutative; and the functor $\Phi : X \mapsto \text{Hom}_{\mathfrak{C}}(1, X \otimes F(\underline{\mathfrak{O}}))$ is a tensor functor from the full image of $F$ to $A \mod H$. It is easy to see from the definitions that $\Phi$ induces an isomorphism
\[
\text{Hom}_{\mathfrak{C}}(1_{\mathfrak{C}}, F(\underline{\mathfrak{O}})) \sim A = \text{Hom}_{A \mod H} (\Phi(1_{\mathfrak{C}}), \Phi(F(\underline{\mathfrak{O}}))) .
\]
Since $H$ is reductive, $\text{Rep}(H)$ is semisimple, and every irreducible $V \in \text{Rep}(H)$ is a direct summand of $\underline{\mathfrak{O}}$; hence for all $V$ we have an isomorphism induced by $\Phi$
\[
\text{Hom}_{\mathfrak{C}}(1_{\mathfrak{C}}, F(V)) \sim \text{Hom}_{A \mod H} (\Phi(1_{\mathfrak{C}}), \Phi(F(V))) \cong \text{Hom}_{A \mod H} (A, V \otimes A).
\]
Since $\text{Rep}(H)$ is rigid, we see that $\Phi$ is a full embedding. Thus we can assume that $\mathfrak{C} = A \mod H$ for a commutative algebra with an $H$-action, and $F : V \mapsto V \otimes A$. In this case the statements of the Proposition reduce to that of Lemma[11] \[\square\]

3.4.1. **Proof of Proposition [5] and definition of the functor $\hat{F}$.** The Proposition follows from Lemma[11] in view of Proposition[4].

We now extend $\hat{F}$ to the homotopy category $\text{Hot}(\text{Coh}_{fr}^{G} \times T^{-}(\hat{N}_a))$.

3.5. **Proof of Proposition [11]** We will construct an isomorphism
\[
Fr \circ \hat{F} \sim \hat{F} \circ q^* ;
\]
the claim about $F$ follows (once we show that $F$ exists).

Uniqueness part of Proposition[4] shows that we will be done if we construct an isomorphism of monoidal functors
\[
\phi : Fr \circ \hat{F} \sim \hat{F}
\]
such that
\[
\phi(Fr^*(B_\lambda)) = B_\lambda, \\
\phi(Fr^*(M_Z(V))) = q^{-1} \cdot M_Z(V).
\]
(17)
An isomorphism $\phi$ induces a structure of a Weil sheaf on $J_\lambda$, $Z_\mu$, and it is clearly uniquely determined by this structure. For $\lambda \in \Lambda^+$ we fix the Weil structure on $J_\lambda = j_\lambda$, so that the resulting Weil sheaf is $j_\lambda\overline{\langle \ell(\nu) \rangle}(\overline{\langle \ell(\mu) \rangle})$. We also require that the isomorphism $J_\lambda \ast J_\mu \cong J_{\lambda + \mu}$ lifts to an isomorphism of Weil sheaves; this fixes the Weil structure on $J_\lambda$ for all $\lambda$.

Let us now define the Weil sheaf which provides the desired isomorphism $Fr^*(Z_\lambda) \cong Z_\lambda$. The functor $\mathcal{Z} : \mathcal{P}_{G_0}(\mathcal{O}) \to \mathcal{P}_I(\mathcal{O})$ is actually defined as a functor between the categories of Weil sheaves (if one fixes the splitting of the surjection $Gal(\mathcal{O}_q((t))) \to Gal(\mathcal{O}_q)$, cf. the footnote on p. 263 in [4]). Then Weil sheaf in question is defined to be $Z^W_{\lambda} := \mathcal{Z}(IC^W_{\lambda})$ where $IC^W_{\lambda} = j_{\lambda+\mu}\left(\overline{\langle \ell(\lambda) \rangle}(\overline{\langle \ell(\mu) \rangle})\right)$.

These requirements clearly define the tensor isomorphism $\phi$ uniquely. Verification of existence of $\phi$ reduces to checking that the isomorphism

$$Fr \circ S \cong S,$$

providing $IC_\lambda$ with the Weil structure isomorphic to $IC^W_{\lambda}$ is tensor; the rest then follows from $\mathcal{Z}$ being tensor. Existence of a tensor structure on (18) will be clear if we show that the convolution $IC^W_{\lambda} \ast IC^W_{\mu}$ is isomorphic to a direct sum of Weil sheaves $IC^W_{\nu}$. We now prove this.

Notice that Frobenius acts on the total cohomology $H^\bullet(IC^W_{\lambda})$ by a diagonalizable automorphism with eigenvalues $q^{n/2}$, $n \in \mathbb{Z}$; this follows e.g. from [3], §4.4. The functor of total cohomology on $\mathcal{P}_{G_0}(\mathcal{O})$ carries a tensor structure (see [5], [6]); the latter is readily seen to be compatible with the Frobenius action. Thus the action of Frobenius on $H^\bullet(IC^W_{\lambda} \ast IC^W_{\mu})$ is diagonalizable with eigenvalues $q^{n/2}$. This implies the desired statement, because we know that $IC^W_{\lambda} \ast IC^W_{\mu} \cong \oplus IC^W_{\nu}$, and the action of Frobenius on cohomology determines the isomorphism class of a Weil sheaf which is geometrically isomorphic to a direct sum of $IC^W_{\nu}$, $\nu \in \Lambda^+$. It remains to check (17). The first equality in (17) is clear from the definition. The second one follows from the fact that for an $l$-adic sheaf $\mathcal{F}$ the logarithm of monodromy on nearby cycles is a morphism of Weil sheaves $\Psi(\mathcal{F}) \to \Psi(\mathcal{F})(-1)$. 

3.6. Filtration of central sheaves by Wakimoto sheaves. The property of central sheaves proved in this section is a geometric counterpart of Bernstein’s description of the center $Z_H$ of the Iwahori-Matsumoto Hecke algebra $\mathcal{H}$, which says that $Z_H = C[\theta_\lambda]^H$; moreover, the map $K(Rep(G)) \to Z_H$ sends the class of representation $V$ to its character $\chi_V \in k[\lambda] = k[\theta_\lambda]$ (see e.g. [9], Theorem 8.1). Bernstein presentation for $H$ (in particular, the elements $\theta_\lambda$) can be easily described in terms of their action in the space of $I$-invariant vectors in the universal principal series representation $C_\nu(G(F))/(N(F) \cdot T(O))$, and thus in terms of their integrals over $N(F)$-orbits in $G(F)$. Quite similarly, the property of central sheaves proved in this section is related to computation of compactly supported cohomology of their restrictions to $N_F$-orbits.

The next Theorem, which is the main result of this section, contains two close statements. Statement (a) will be used later; statement (b) is included for completeness.

Recall that the $N_F$-orbits on $\mathcal{F} \ell$ are parameterized by $W$; and $i_w : S_w \hookrightarrow \mathcal{F} \ell$ denotes the embedding of an orbit.

We fix a total ordering $\leq$ on the group $\Lambda$ compatible with the group structure and with the standard partial ordering (i.e. $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots).

**Theorem 4.** a) For $V \in Rep(G)$ the sheaf $\mathcal{Z}(V)$ has a unique filtration indexed by $(\Lambda, \leq)$ such that the associated graded $gr_\nu(\mathcal{Z}(V)) = \mathcal{Z}(V)_{<\nu}/\mathcal{Z}(V)_{<\nu}$ is of the form

$$gr_\nu(\mathcal{Z}(V)) \cong J_\nu \otimes W^\nu$$
for some vector space $W^\nu_V$. The functor

$$
\Phi : V \mapsto \bigoplus_{\nu} W^\nu_V = \bigoplus_{\nu} \text{Hom}(J_{\nu}, \text{gr}_\nu(Z(V))
$$

is a tensor functor from $\text{Rep}(G^\vee)$ to the category of $\Lambda$-graded vector spaces (obviously equivalent to $\text{Rep}(T^\vee)$). $\Phi$ is isomorphic the restriction functor $\text{Rep}(G^\vee) \to \text{Rep}(T^\vee)$; in particular, $\dim W^\nu_V$ equals the multiplicity of the weight $\nu$ in $V$.

b) The space $H^i_{\nu}(i_{w*}(Z(V)))$ vanishes unless $w = \nu \in \Lambda$, $i = \ell(\nu)$; in which case we have

$$
H^{\ell(\nu)}(Z(V)) \cong W^\nu_V
$$

where $W^\nu_V$ is as in (a).

Remark 5. The sheaf $V_\mu \otimes O_{\tilde{Z}}$ carries a filtration with subquotients being sums of line bundles (this filtration is actually a pull-back of a filtration on $V_\mu \otimes O_{G^\vee/B^\vee}$). It will be clear from the construction of the functor $F$ that the filtration of Theorem 3(a) is the image of this filtration under $F$.

Remark 6. In [MV] Mirković and Vilonen prove a result similar to part (b) of the above Theorem; namely, they compute the compactly supported cohomology of $P$ coefficient in an irreducible object of $\mathcal{P}_{G_0}(\mathfrak{g}r)$. One can show that the two results are actually equivalent.

The proof of the Theorem occupies the rest of this section.

3.6.1. (a) implies (b). The last statement in the next Lemma yields the implication (a) $\Rightarrow$ (b).

Lemma 12. For $\lambda \in \Lambda$, and $X \in D_I$ we have

$$
H^\bullet(i^\lambda_{X, w} J_{\lambda} * X) = H^\bullet(i^\lambda_{w*}(X))(\lambda, 2\rho).
$$

In particular, $H^i(i^\lambda_{w*}(J_{\lambda})) = 0$ unless $w = \lambda$, $i = \ell(\lambda)$, in which case it has dimension one.

Proof. It is clear that if (19) holds for $\lambda_1$, $\lambda_2$ then it also holds for $\lambda_1 - \lambda_2$. Thus we can assume without loss of generality that $\lambda \in \Lambda^+$.

For $w \in W$ let $\tilde{w}$ be a representative of the coset $w \in \text{Norm}(T(O))/T(O)$, where $\text{Norm}(T(O))$ is the normalizer of $T(O)$. It follows from the definitions that for $X \in D_I$ we have

$$
j_{w*} * X \cong \Gamma^\lambda|_{I \cap w^{-1} I \cap \tilde{w}*}(X)[\ell(w)].
$$

(Notations for the induction functor $\Gamma$ were recalled before before Lemma 5 above.) It is clear that for $\lambda \in \Lambda$

$$
H^\bullet(i^\lambda_{X, w} \tilde{\lambda}_{w}(X)) = H^\bullet(i^\lambda_{w*}(X))
$$

since $\tilde{\lambda}(\mathcal{S}_{w}) = \mathcal{S}_{\lambda-w}$. Also it is not difficult to check that for $\lambda \in \Lambda^+$ we have $\tilde{\lambda} I \tilde{\lambda}^{-1} \supset I \cap B^w_F$. Then the triangular decomposition $I = I \cap N_F \cdot I \cap B^w_F$ yields an isomorphism

$$
\Gamma^\lambda|_{I \cap \tilde{\lambda} I \tilde{\lambda}^{-1}} = \Gamma^\lambda|_{I \cap \tilde{\lambda} I \tilde{\lambda}^{-1} \cap N_F}.
$$

The induction functor $\Gamma^\lambda_{H'}$ commutes with the $!$ restriction to an $H$-invariant subvariety; when $H$, $H'$ are unipotent it also does not change the total cohomology. Applying this observation to $H = I \cap N_F$, $H' = I \cap \tilde{\lambda} I \tilde{\lambda}^{-1} \cap N_F$ and the subvariety $\mathcal{S}_{\lambda-w} \subset \mathcal{F}_\ell$ we get the statement. $\square$
3.6.2. **Uniqueness of the filtration.** Uniqueness of the filtration follows from the following

**Lemma 13.** We have $\text{Hom}^*(J_\lambda, J_\mu) = 0$ unless $\lambda \preceq \mu$; and $\text{Hom}^*(J_\lambda, J_\lambda) = \mathbb{Q}_I$.

**Proof.** Pick $\nu$ such that $\nu + \lambda, \nu + \mu \in \Lambda^+$. Since the functor of convolution with $J_\nu$ is invertible we have

$$\text{Hom}^*(J_\lambda, J_\mu) = \text{Hom}^*(J_{\nu + \lambda}, J_{\nu + \mu}) = \text{Hom}^*(j_{\nu + \lambda*}, j_{\nu + \mu*}).$$

The latter space can be nonzero only if $\mathcal{F}_{\nu + \lambda}$ lies in the closure of $\mathcal{F}_{\nu + \mu}$, which is known to be equivalent to $\lambda \preceq \mu$. \qed

3.6.3. **Existence of the filtration.** We will say that an object $X \in \mathcal{P}_f$ is **convolution exact** if $X * L \in \mathcal{P}_f$ for all $L \in \mathcal{P}_f$. We will say that $X$ is **central** if $X * L \cong L * X$ for all $X \in \mathcal{P}_f$.

It will be convenient to extend the definition of $J_\lambda$ to all $w \in W$ by setting $J_w = J_{\lambda} * j_{w!*}$ for $w = \lambda \cdot w_f$, $\lambda \in \Lambda$, $w_f \in W_f$.

The next result is proved in the Appendix.

**Theorem 5.** a) The objects $J_w \in \mathcal{D}_f$ actually lie in $\mathcal{P}_f$.
   b) $\mathcal{F}_{\nu *} J_w$ is open in the support of $J_w$, and $j_{w*}(J_w) \cong \mathcal{Q}(\ell(w))$.

The next Proposition obviously implies the existence of the filtration.

**Proposition 5.** a) Any convolution exact object of $\mathcal{P}_f$ has a filtration whose subquotients are Wakimoto sheaves $J_w$.
   b) If $X$ is also central then only $J_w$ with $w \in \Lambda$ appear in the filtration of (a).

**Remark 7.** Statement (a) of the Theorem can be compared to the following result due (to the best of our knowledge) to Mirković (unpublished): every convolution exact sheaf on the finite dimensional flag variety $G/B$ which is smooth along the Schubert stratification is tilting, i.e. has a filtration with subquotients $j_{w!}$, and also a filtration with subquotients $j_{w*}$.

We sketch a proof of Mirković’s result for the sake of completeness. Let $\mathcal{F}$ be a convolution exact perverse sheaf on $G/B$ as above. We have to check that $\text{Ext}^{>0}(j_{w!}, \mathcal{F}) = 0 = \text{Ext}^{>0}((\mathcal{F}, j_{w*})$.

We check the first equality, the other one is similar. Since $\mathcal{F}$ is convolution exact, the convolution $\mathcal{F} * j_{w!}$ is a perverse sheaf, thus it lies in the full subcategory generated by the objects $j_{w!}[d]$, $w \in W$, $d \geq 0$ under extensions. Thus $\mathcal{F} = \mathcal{F} * j_{w0!*} * j_{w0*}$ lies in the full subcategory generated by $j_{w!} * j_{w0*}[d] = j_{w0!*} [d]$, $d \geq 0$, which implies the needed Ext vanishing.

The central sheaves $Z_\lambda$ (for $\lambda \neq 0$) provide examples of convolution exact objects of $\mathcal{P}_f$ which are not tilting (see, however, Theorem 7 and Remark 10 below).

The proof of the Proposition will be given after some auxiliary Lemmas.

**Lemma 14.** a) We have $J_\lambda * J_w \cong J_{\lambda \cdot w}$.

b) If $w \in \Lambda^+, W_f$ then $J_w = j_{w!*}$. If $w \in (-\Lambda^{++}) \cdot W_f$ then $J_w = j_{w!}$; here $\Lambda^{++}$ is the set of strictly dominant weights.

**Proof.** (a) is immediate from the definitions. To prove (b) we observe that for $w_f \in W_f$, $\lambda \in \Lambda^+, \mu \in \Lambda^{++}$ we have

$$\ell(\lambda \cdot w_f) = \ell(\lambda) + \ell(w_f) \Rightarrow j_{\lambda \cdot w_f!*} = j_{\lambda!*} * j_{w_f!*} = J_{\lambda} * J_{w_f} = J_{\lambda \cdot w_f};$$

$$\ell((-\mu) \cdot w_f) = \ell(-\mu) - \ell(w_f) \Rightarrow j_{-\mu \cdot w_f!} = j_{-\mu!} * j_{w_f!*} = J_{-\mu} * J_{w_f} = J_{-\mu \cdot w_f}. \qed$$
3.6.4. Perverse sheaves on stratified spaces. We now recall some facts about perverse sheaves on stratified spaces.

Let $X = \bigcup_{s \in S} X_s$ be a stratified scheme over a field; thus $X_s \subset X$ are locally closed smooth subschemes. We assume for simplicity of notations that the embeddings $j_s : X_s \hookrightarrow X$ are affine, and that $j_{s*} j_{s!}(\mathbb{Q})$ has constant cohomology sheaves for all $u, s \in S$. We abbreviate $j_{ss*} = j_{ss!}(\mathbb{Q}(\dim X_s))$, $j_{ss!} = j_{ss!}(\mathbb{Q}(\dim X_s))$. Let $D$ be the derived category of constructible sheaves on $X$, and $(D^{<0}, D^{>0})$ be the perverse $t$-structure, and $\mathcal{P}$ be its heart (the category of perverse sheaves). The following statement is standard.

Claim 1. For $\mathcal{F} \in D$ set $S^*_\mathcal{F} = \{s \in S \mid j^*_s(\mathcal{F}) \neq 0\}$; $S^*_\mathcal{F} = \{s \in S \mid j^*_s(\mathcal{F}) \neq 0\}$. We have

a) If $\mathcal{F} \in D^{\leq 0}$ and cohomology of $j^*_s(\mathcal{F})$ are constant sheaves for all $s \in S$, then $\mathcal{F} \in (j_i^*[i] \mid i \geq 0, s \in S^*_\mathcal{F})$ (cf section 2.0.2 for notations).

b) If $\mathcal{F} \in D^{>0}$ and $\mathcal{F} \in (j_i^*[i] \mid i \geq 0, s \in S)$ then $\mathcal{F}$ is a perverse sheaf; moreover, $\mathcal{F}$ carries a filtration with subquotients isomorphic to $j_{ss*}$, $s \in S^*_\mathcal{F}$.

Proof. (a) is equivalent to saying that $H^k(j^*_s(\mathcal{F})) = 0$ for $k > -\dim(X_s)$, and is constant otherwise. Here the first condition is the definition of the perverse $t$-structure, and the second one was imposed as an assumption.

The assumptions of (b) imply that $H^k(j^*_s(\mathcal{F})) = 0$ for $k > -\dim(X_s)$, and is constant otherwise. However, $H^k(j^*_s(\mathcal{F})) = 0$ for $k < -\dim(X_s)$ by the definition of the perverse $t$-structure. Hence $j^*_s(\mathcal{F}) \cong (\mathbb{Q}(\dim X_s))^\oplus a$, which implies the conclusion of (b).

For $X \in D_I$ set

$$W^*_X = \{w \in W \mid j^*_w(X) \neq 0\};$$
$$W^I_X = \{w \in W \mid j^*_w(X) \neq 0\}.$$ 

Lemma 15. For $X \in D_I$ there exists a finite subset $S \subset W$, such that for all $w \in W$ we have

$$W^I_{j_{w*}X}, W^*_{j_{w!}wX} \subset w \cdot S;$$
$$W^I_{X \circ j_{w*}}, W^*_{X \circ j_{w!}} \subset S \cdot w.$$

Proof. Proper base change shows that any point $x \in \mathcal{H}I$ such that the stalk of $j_{w!}X$ at $x$ is nonzero lies in the convolution of sets $\mathcal{H}I_w$ and $\text{Supp}(X)$ (i.e. in the image of $\mathcal{H}I_w \times \text{Supp}(X)$ under the convolution map). Thus to prove the first of the four statements it is enough to show the corresponding estimate for convolution of sets; the other three statements follow in a similar way. Thus for a fixed $I$-invariant $\mathcal{G} \subset \mathcal{H}I$ we have to show that for some $S \subset W$, the convolution of sets $\mathcal{G} \circ \mathcal{H}I_w$ (respectively, $\mathcal{H}I_w \circ \mathcal{G}$) is contained in $\bigcup_{v \in v \circ w} \mathcal{H}I_w$ (respectively, $\bigcup_{w \in w \circ S} \mathcal{H}I_w$). Without loss of generality we can assume that $\mathcal{G} = \mathcal{H}I_v$ for some $v \in W$. The claim easily follows by induction in $\ell(v)$. □

Proof of Proposition 14 a) Let $X \in D_I$ be convolution exact, and let $S$ be as in Lemma 15. We can write $S$ as

$$S = \{\lambda_i w_i\},$$

$w_i \in W, \lambda_i \in \Lambda$. Choose $\nu_0 \in -\Lambda^+$ such that $\nu_0 + \lambda_i \in -\Lambda^+\Lambda$; thus $j_{(\nu_0,s)} = J_{(\nu_0,s)}$ for $s \in S$. Since $j_{-\nu_0} X = J_{-\nu_0} X \in \mathcal{P}$ we see by Lemma 15 Claim 1(a) that

$$J_{-\nu_0} X \in \langle J_{(-\nu_0,s)[i]} \mid i \geq 0, s \in S \rangle = \langle J_{(-\nu_0,s)[i]} \mid i \geq 0, s \in S \rangle.$$

Hence

$$J_{\nu} X \in \langle J_{\nu,s}[i] \mid i \geq 0, s \in S \rangle$$
for all $\nu \in \Lambda$. In particular, choosing $\nu \in \Lambda^+$ such that $\nu + \lambda_i \in \Lambda^+$ we see that
\[ J_\nu \ast X \in \langle j_{\nu + s} | i \geq 0, s \in S \rangle. \]
Since $J_\nu \ast X \in \mathcal{P}_I$ this implies statement (a) by Claim (1).

(b) We can choose $\lambda \in \Lambda^+$ such that $\lambda \cdot S \subset \Lambda^{++} \cdot W_f$, $S \cdot \lambda \subset W_f \cdot \Lambda^+$. Then we see that
\[ W^+_X \ast J_\lambda \subset W_f \cdot \Lambda^{++} \cap \Lambda^{++} \cdot W_f = \Lambda^{++}, \]
which implies statement (b). \(\square\)

3.6.5. Construction of tensor structure on $\Phi$. Let $\mathcal{A} \subset \mathcal{P}_I$ be the full subcategory of sheaves which admit a filtration whose subquotients are Wakimoto sheaves $J_\lambda$ (which makes sense by Theorem 5(a)). Since $J_\lambda \ast J_\mu = J_{\lambda + \mu}$ we see that $\mathcal{A}$ is a monoidal subcategory of $D_I$.

Let $gr\mathcal{A} \subset \mathcal{A}$ be the subcategory whose objects are sums of sheaves $J_\lambda$, and morphisms are direct sums of isomorphisms $J_\lambda \rightarrow J_\mu$ and zero arrows. Thus $\mathcal{A}$, $gr\mathcal{A}$ are monoidal subcategories in $D_I$, and $gr\mathcal{A}$ is obviously equivalent to $Rep(T^*)$. Since $Ext^1(J_\lambda, J_\mu) = 0$ for $\mu \neq \lambda$ (in particular, for $\mu \geq \lambda$) every object $X \in \mathcal{A}$ actually admits a filtration $(X_{<\nu})$ indexed by $(\Lambda, \leq)$ such that $gr_\nu(X) = X_{<\nu}/X_{\leq \nu}$ is the sum of several copies of $J_\nu$. (Recall that $\geq$ is some complete order on $A$ compatible with the standard partial order). Since $Hom(J_\lambda, J_\mu) = 0$ for $\lambda \nleq \mu$, in particular, for $\mu > \lambda$, such filtration is unique. Thus taking the associated graded is a well defined functor $gr : \mathcal{A} \rightarrow gr\mathcal{A}$.

The next statement is an equivalent form of Theorem 5(a).

**Theorem 6.** The functor $gr \circ Z : Rep(G^*) \rightarrow gr\mathcal{A} \cong Rep(T^*)$ is tensor, and is isomorphic to the functor of restriction to a maximal torus.

The proof of the Theorem will be given at the end of the subsection.

**Proposition 6.** a) The functor $gr : \mathcal{A} \rightarrow gr\mathcal{A}$ has a natural monoidal structure.

b) The composition $gr \circ Z : Rep(G^*) \rightarrow gr\mathcal{A}$ has a natural structure of a central functor (see §3 for the definition of a central functor).

**Lemma 16.** Let $D, \otimes$ be a triangulated monoidal category (where $\otimes$ is triangulated in each variable), and $\mathcal{A} \subset D$ be a heart of a $t$-structure. Let $A, B \in \mathcal{A}$ be objects with filtrations $(A_{\leq i}, B_{\leq i})$. Assume that $gr(A) \otimes gr(B) \in \mathcal{A}$. Then $A_{\leq i} \otimes B_{\leq j} \in \mathcal{A}$; and we have a natural isomorphism
\[ (20) \quad gr(A \otimes B) \cong gr(A) \otimes gr(B), \]
where $gr(A \otimes B)$ is the associated graded with respect to the tensor product filtration $(A \otimes B)_{\leq k} = \sum_i A_{\leq i} \otimes B_{\leq k-i}$. For a third filtered object $C \in \mathcal{A}$ the isomorphism (20) is compatible with the associativity isomorphism.

**Proof.** The first statement is obvious. To see the second one notice that for all $i, j$ the morphism $A_{\leq i} \otimes B_{\leq j} \rightarrow (A \otimes B)_{\leq i + j}$ factors through an arrow $s_{i,j} : gr_i(A) \otimes gr_j(B) \rightarrow gr_{i+j}(A \otimes B)$. Also the image of $(A \otimes B)_{\leq i + j}$ in $(A/(A_{<i})) \otimes (B/B_{<j})$ equals $gr_i(A) \otimes gr_j(B)$ which induces an arrow $\sigma_{i,j} : gr_i(A) \otimes gr_j(B) \rightarrow gr(A \otimes B)$. It is clear that $\sigma_{i,j} \circ s_{i,j} = id$, and that
\[ s = \sum_{i,j} s_{i,j} : gr(A) \otimes gr(B) \rightarrow gr(A \otimes B) \]
is surjective. Hence $s$ is an isomorphism. Compatibility with associativity is clear. \(\square\)
Lemma 17. Let $F : \mathcal{T} \to \mathcal{C}$ be a central functor from a tensor category $\mathcal{T}$ to a monoidal category $\mathcal{C}$. Let $G : \mathcal{C} \to \mathcal{C}'$ be a monoidal functor to another monoidal category $\mathcal{C}'$. Assume that $G$ admits a right inverse, i.e. there exists a monoidal functor $G' : \mathcal{C}' \to \mathcal{C}$ such that $G \circ G' \cong id$. Then $G \circ F$ is naturally a central functor.

Proof. Let $\sigma_{X,Y} : F(X) \otimes Y \to Y \otimes F(X)$, $X, Y \in \mathcal{T}$, $Y, Y \in \mathcal{C}$ be the centrality isomorphism for $F$. Define the centrality isomorphism for $G \circ F$ by $\sigma'_{X,Y} = G(\sigma_{X,G(Y)})$, $X \in \mathcal{T}$, $Y \in \mathcal{C}'$. Then $\sigma'$ provides $G \circ F$ with a structure of a central functor, because all the required diagrams commute being images of commutative diagrams in $\mathcal{C}$.

Proof of Proposition 7. (a) is immediate from Lemma 16. (b) follows from Lemma 17 by setting $\mathcal{T} = \text{Rep}(G'$), $\mathcal{C} = A$, $\mathcal{C}' = \text{gr}A$, $G = \text{gr}$; $G'$ is the embedding $\text{gr}A \hookrightarrow A$.

To prove Theorem 6 we need another

Lemma 18. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be abelian rigid tensor categories, and $F : \mathcal{T}_1 \to \mathcal{T}_2$, $\{\sigma_{X,Y} : F(X) \otimes Y \cong Y \otimes F(X)\}$ be an additive map $f : F(V) \to U$ such that for all $X \in \mathcal{T}_2$ the following diagram is commutative

$$
F(V) \otimes X \xrightarrow{\sigma_{V,X}} X \otimes F(V) \\
\downarrow \quad \quad \quad \downarrow \\
U \otimes X \xrightarrow{\sigma'_{U,X}} X \otimes U,
$$

where $\sigma'_{U,X}$ denotes the commutativity isomorphism in $\mathcal{T}_2$. Then $\sigma_{X,U} = C_{F(X),U}$ for all $X \in \mathcal{T}_1$.

Proof. By the definition of a central functor we have

$$
\sigma_{V,F(X)} \circ \sigma_{X,F(V)} = F(C^\mathcal{T}_1_{V,X} \circ C^\mathcal{T}_1_{X,Y}) = \text{id}_{F(V) \otimes F(X)},
$$

where $C^\mathcal{T}_1$ is the commutativity isomorphism in $\mathcal{T}_1$. The morphism $C_{U,X} \circ \sigma_{X,U}$ is a quotient of $\sigma_{V,F(X)} \circ \sigma_{X,F(V)}$ (recall that tensor product in a rigid abelian tensor category is exact in each variable); hence $C_{U,X} \circ \sigma_{X,U} = \text{id}_{F(X) \otimes U}$, and $\sigma_{X,U} = C_{X,U}'$. □

3.6. Proof of Theorem 6. By Proposition 6 the functor $\text{gr} \circ \mathcal{Z} : \text{Rep}(G') \to \text{gr}A \cong \text{Rep}(T')$ is central. We need to check that it is in fact tensor. It suffices to check that

$$
\sigma_{V,A} = C_{\text{gr}(A)}^{\text{gr}(\mathcal{Z}(V)),A},
$$

for $V \in \text{Rep}(G')$, $A \in \text{gr}(A)$; the tensor property would then follow from the definition of a central functor. Lemma 1(b) implies that conditions of Lemma 15 hold for $\mathcal{T}_1 = \text{Rep}(G')$, $\mathcal{T}_2 = \text{gr}A$, $V = V_\lambda$, $U = J_\lambda$, $\lambda \in \Lambda^+$. Hence (21) holds for $A = J_\lambda$, $\lambda \in \Lambda^+$. If (21) holds for some (rigid) object $A$ then its validity for another object $A'$ is equivalent to its validity for $A \otimes A'$. Thus (21) holds for all $V$, $J_\lambda$, $\lambda \in \Lambda$; and hence holds always.

Thus $\text{gr} \circ \mathcal{Z}$ comes from a homomorphism of algebraic groups $T^+ \to G^+$. This homomorphism is injective, because for every $\lambda \in \Lambda^+$ the character $\lambda$ is a direct summand in $\text{gr} \circ \mathcal{Z}(V_\lambda)$ by Lemma 6. Hence the image of $T^-$ in $G^-$ under the above homomorphism is indeed a maximal torus.

This establishes Theorem 6 and thus also Theorem 6. □

3.7. Factoring $\hat{F}$ to $F$. Let $\partial \hat{N} \subset \hat{N}_{af}$ be the complement to $\hat{N}$. We will show that $\hat{F}$ yields a functor $D^G(N) \to D(A)$ by checking that it sends all complexes whose cohomology is supported on $\partial \hat{N}$ to acyclic complexes. This will be deduced from the existence of a filtration on $Z_\lambda$ constructed in the previous section (recall that the definition of $\hat{F}$ only relied on Lemmas
Notice that $\partial N_{x,f}$ contains the support of the cokernel of the morphism $B_\lambda$ for any $\lambda \in \Lambda^+$, and equals this support if $\lambda \in \Lambda^{++}$ (see [3.3.2] for notation).

For a morphism $\phi : V \to L$ in a tensor category over a characteristic zero field and $d \in \mathbb{Z}_{>0}$ one can form the Koszul complex $0 \to \Lambda^d(V) \to \Lambda^{d-1}(V) \otimes L \to \cdots \to \Lambda^i(V) \otimes \text{Sym}^{d-i}(L) \to \cdots \to \text{Sym}^d(L) \to 0$. In the examples below some exterior power of $V$ vanishes, and we will let $d$ be the maximal integer such that $\Lambda^d(V) \neq 0$, the resulting complex will be called the Koszul complex associated to $\phi$.

Let $K_\lambda \in \text{Kom}(\text{Coh}_{fr}^{G \times T} (\hat{N}))$ denote the Koszul complex associated to $B_\lambda$. Thus

$$K_\lambda = (0 \to \mathcal{O} = \Lambda^d(V_\lambda) \otimes \mathcal{O} \to \Lambda^{d-1}(V_\lambda) \otimes \mathcal{O}(\lambda) \to \cdots \otimes \mathcal{O}((d-1)\lambda) \otimes V_\lambda \to \mathcal{O}(d\lambda) \to 0)$$

The key step is the following

**Lemma 19.** We have $\hat{F}(K_\lambda) = 0$ for all $\lambda \in \Lambda^+$. 

**Proof.** We keep the notations of the previous section. Thus $\hat{F}(K_\lambda)$ is a complex of objects of $A$. To see that $\hat{F}(K_\lambda)$ is acyclic it is enough to see that $gr(\hat{F}(K_\lambda)) \in \text{Kom}(\text{gr} A)$ is acyclic. The latter is a complex in $\text{gr} A \cong \text{Rep}(T)$. Since the differential in $K_\lambda$ is obtained from the arrow $B_\lambda$ by tensoring with $V_\lambda$ and taking direct summands, Theorem 8 together with Proposition 6(a) show that $gr(K) \in \text{Kom}(\text{gr} A) \cong \text{Kom}(\text{Rep}(T))$ is identified with the Koszul complex associated to the non-zero map $V_\lambda|_{T} \to \lambda$ in $\text{Rep}(T)$. Since the latter complex is acyclic, we get the statement. □

Now the definition of $F$ follows from the next

**Lemma 20.** Let $\text{Hot}_0(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f})) \subset \text{Hot}(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$ be the thick subcategory of complexes whose cohomology is supported on $\hat{N}_{x,f}$.

a) Any $\mathcal{F} \in \text{Hot}_0(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f})) \subset \text{Hot}(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$ is a direct summand in $\mathcal{F} \otimes K_\lambda$ for some $\lambda$.

b) The functor of restriction to $\hat{N}$ provides an equivalence

$$\text{Hot}(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f})) / \text{Hot}_0(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f})) \cong D^b(\text{Coh}_{z}^{G \times T} (\hat{N})) \cong D^b(\hat{N}).$$

**Proof.** a) It is clear that $\text{Hot}(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$ is identified with a full subcategory in $D^b(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$. For any $\mathcal{F} \in \text{Hot}_0(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$ the corresponding object of $D^b(\text{Coh}_{fr}^{G \times T} (\hat{N}_{x,f}))$ can be represented by a finite complex of coherent sheaves set-theoretically supported on $\hat{N}_{x,f}$. This is clear by the following well-known fact (cf. e.g. [122], Lemma 3(b)): for an algebraic variety $X$ and a closed subvariety $Z \subset X$ the tautological functor provides an equivalence $D^b(\text{Coh}_{Z}^{G} (X)) \cong D^b_{Z}(\text{Coh}(X))$, where $\text{Coh}_{Z}^{G} (X) \subset \text{Coh}(X)$ is the full subcategory of sheaves set-theoretically supported on $Z$, and $D^b_{Z}(\text{Coh}(X)) \subset D^b(\text{Coh}(X))$ is the full subcategory of complexes whose cohomology sheaves lie in $\text{Coh}_{Z}^{G} (X)$.

If $\mathcal{C}$ is a finite complex of coherent sheaves set-theoretically supported on $\hat{N}_{x,f}$, then it is scheme-theoretically supported on some nilpotent neighborhood of $\hat{N}$. For some $\lambda \in \Lambda^+$ the restriction of $B_\lambda$ to this neighborhood vanishes, thus we have $B_\lambda \otimes \text{id}_C = 0$. Hence

$$\mathcal{K}_\lambda \otimes C \cong \bigoplus_i \mathcal{F} \otimes \Lambda^i(V_\lambda) \otimes \mathcal{O}((d-i)\lambda),$$

which implies (a).
b) It suffices to check that the image of the functor (22) generates \( D^G(\tilde{\mathcal{N}}) \) as a triangulated category; and that is a full embedding. Here the first statement follows from Lemma 21(a) below; and the second one is a particular case of the following general statement. \( \square \)

**Sublemma 1.** Let \( A \) be a finitely generated commutative algebra graded by \( \mathbb{Z}^N \), and let \( X \) be the corresponding multi-Proj scheme. Let \( D^{fr}_A \) be the homotopy category of free \( \mathbb{Z}^N \)-graded \( A \)-modules, and \( D^{fr,0}_A \) be the full subcategory of complexes whose localization to \( D^h(\text{Coh}_X) \) is zero. Then \( D^{fr}_A/D^{fr,0}_A \) is identified with a full subcategory in \( D^b(\text{Coh}_X) \).

Same is true for the categories of \( H \)-equivariant sheaves/modules, where \( H \) is a reductive algebraic group acting on \( A \).

**Proof.** For any finite complex \( C \in D^{fr}_A \), and any \( \lambda_0 \in \mathbb{Z}^N \) there exists \( C' \in D^{fr}_A \), and a morphism \( f : C' \to C \), such that \( \text{cone}(f) \in D^{fr,0}_A \), and \( (C')^i \) is a sum of modules of the form \( A \otimes V(\lambda), \lambda_0 - \lambda \in \mathbb{Z}^N \). (To see this pick a \((\tilde{H},-\text{invariant})\) subspace \( V \in A_\mu \), for \( \mu \) large, such that \( A/V \cdot A \) is supported on the complement to the cone over \( X \), then consider the Koszul complex

\[
K = (0 \to A(-\mu_1) \to A(-(d-1)\mu) \otimes V^* \to \cdots \to \Lambda^d(V^*) \otimes A \to 0)
\]

placed in degrees from \(-d\) to 0. We have \( K \otimes C \in D^{fr,0}_A \), and the complex \( \text{Ker}(K \otimes \Lambda^d(V) \otimes C \to C)[-1] \) has the required form.

Now given \( B \in D^{fr}_A \) we can find \( \lambda_0 \) such that \( \text{Hom}_{D^A}(A(\lambda) \otimes V, B) \to \text{Hom}_{D^A(\text{Coh}_X)}(O(\lambda) \otimes V, \mathcal{L}(B)) \) whenever \( \lambda_0 - \lambda \in \mathbb{Z}^n_+ \). \( \square \)

4. Proof of Theorem 1

4.1. Intermediate statements. We will deduce the Theorem from the next two statements.

Recall that \( \kappa \) denotes the bijection \( \Lambda \to \mathcal{J}W \). For \( \mathcal{F} \in D_{\mathcal{J}W} \), and \( \mu \in \Lambda \) set \( \text{Stalk}_\mu(\mathcal{F}) = i_\ast^\mathcal{F}[-\dim \mathcal{J}^{\kappa(\mu)}], \text{Costalk}_\mu(\mathcal{F}) = i^\ast_\mu(\mathcal{F})[\dim \mathcal{J}^{\kappa(\mu)}] \) for \( x \in \mathcal{J}^{\kappa(\mu)} \); these are objects of the derived category of vector spaces defined up to an isomorphism.

**Proposition 7.** For \( k \) algebraically closed we have

\[ \text{Stalk}_\mu(F_{\mathcal{J}W}(V)) \cong \underline{\text{Stalk}}_\mu^\mathcal{F} \cong \text{Costalk}_\mu(F_{\mathcal{J}W}(V)), \]

where \( \mu \in V \) is the multiplicity of the weight \( \mu \) in \( V \).

**Proposition 8.** For \( V \in \text{Rep}(G^+) \), \( \mu \in \Lambda^+ \) the map

\[
(23) \quad \text{Hom}_{DG^+}(\mathcal{F}_0)(V \otimes O(\mu)) \to \text{Hom}(F_{\mathcal{J}W}(V \otimes O), F_{\mathcal{J}W}(O(\mu)))
\]

is injective.

4.2. Proof of Theorem 3. We now deduce the Theorem from Propositions 7, 8.

**Lemma 21.** a) The objects \( O(\lambda), \lambda \in \Lambda \) generate \( D^G(\tilde{\mathcal{N}}) \) as a triangulated category.

b) The objects \( O(-\lambda) \otimes V_\mu, \lambda, \mu \in \Lambda^+ \) generate \( D^G(\tilde{\mathcal{N}}) \) as a triangulated category.

**Proof.** (a) Since \( \tilde{\mathcal{N}} \) is smooth every object of \( D^G(\tilde{\mathcal{N}}) \) is represented by a finite complex of \( G \)-equivariant vector bundles. We now claim that every such vector bundle is filtered by line bundles \( O(\lambda) \). Let \( \mathcal{E} \) be such a vector bundle. It is enough to show that there exists a \( G \)-equivariant injection of vector bundles \( O(\lambda) \hookrightarrow \mathcal{E} \).

We have an equivalence \( \text{Coh}^G(\tilde{\mathcal{N}}) \cong \text{Coh}^B(\mathcal{N}), \mathcal{E} \mapsto \mathcal{E}|_{\mathcal{N}^\ast} \). Let \( M = \Gamma(\mathcal{E}|_{\mathcal{N}^\ast}) \); then the data of an injection \( O(\lambda) \hookrightarrow \mathcal{E} \) is equivalent to the data of an element \( v \in M \) such that \( B^\ast \) acts on
v by the character λ, and v projects to a nonzero element in the coivariants $M/(n^*)^n M$. It is easy to see that if λ is a lowest weight of $T^- \subset B^-$ in $M$ (which necessarily exists, because the set of weights of $M$ is readily seen to be bounded below) then every v of weight $λ$ satisfies these requirements.

(b) In view of statement (a) it is enough to show that for all $λ$ the line bundle $O(λ)$ lies in the triangulated category generated by the restriction of $F = 4.3$. Proof of Proposition 8: category $v$

Proof of Theorem 3. 

Lemma 22. The objects $F_{3\mathcal{W}}(O(λ))$, $λ \in Λ$ generate $D_{3\mathcal{W}}$ as a triangulated category.

Proof. Theorem 5(b) implies that the support of $F_{3\mathcal{W}}(O(λ))$ is contained in the closure of $\mathcal{F}_{\ell_k(λ)}$.

Furthermore, it is shown in [41, 4.1.2, Lemma 11] that the restriction of $J_λ$ to the $G_O$-orbit of $\mathcal{F}_\ell_λ$ coincides with the restriction of the standard sheaf $j_{\lambda}$. Thus Lemma 11(c) shows that the restriction of $F_{3\mathcal{W}}(O(λ)) = Av_0(J_λ)$ to $\mathcal{F}_{\ell_λ}(λ)$ has rank 1. This implies the Lemma. □

4.2.1. Proof of Theorem 5 We first check that $F_{3\mathcal{W}}$ is a full embedding, i.e. that the map

$$Hom^*_{\mathcal{G}}(\mathcal{F}, \mathcal{O}) \rightarrow Hom^*_{\mathcal{G}}(\mathcal{F}_{3\mathcal{W}}(\mathcal{F}), F_{3\mathcal{W}}(\mathcal{O}))$$

is an isomorphism.

It is known e.g. by results of [KLT] that for $λ \in \Lambda^+$ and $V \in Rep(G^*)$ we have

$$Ext^i_{\mathcal{O}}(V \otimes O(λ), O(λ)) = Hom_{\mathcal{G}}(V, H^i(\tilde{N}, O(λ))) = 0$$

for $i \neq 0$; and

$$\dim Hom_{\mathcal{O}}(V \otimes O, O(λ)) = |λ : V|$$

The latter also equals

$$\dim Hom(F_{3\mathcal{W}}(V \otimes O), F_{3\mathcal{W}}(O(λ))) = \dim Stalk_λ(F_{3\mathcal{W}}(V \otimes O))$$

by Proposition 1. Thus Propositions 1, 8 imply that (24) is an isomorphism for $\mathcal{F} = V \otimes O$, $\mathcal{O} = O(λ)$, $λ \in \Lambda^+$. Since $F_{3\mathcal{W}}(\mathcal{F} \otimes O(λ)) = F_{3\mathcal{W}}(\mathcal{F}) \ast J_λ$, and the functor $\mathcal{F} \mapsto \mathcal{F} \otimes O(λ)$ is invertible we see that it is also an isomorphism for $\mathcal{F} = V \otimes O(\lambda)$, $\mathcal{O} = O$. Hence by Lemma 21(b) it is an isomorphism for $\mathcal{O} = O$ and all $\mathcal{F}$. Again twisting by $O(λ)$ we deduce that it is an isomorphism for all $\mathcal{F}$ and $\mathcal{O} = O(λ)$. Hence this is also true for all $\mathcal{F}$, $\mathcal{O}$ by Lemma 21(a).

We proved that $F_{3\mathcal{W}}$ is a full embedding. It is then essentially surjective by Lemma 22. □

4.3. Proof of Proposition 8 category $\mathcal{P}_0$ and the regular orbit. Let $\mathcal{N}_0 \subset \mathcal{N}$ be the open $G^*$ orbit, and $N_0 \subset N_0$ be an element. We introduce yet another auxiliary category $\mathcal{P}_0^0$; this category can be identified with $Coh_{\mathcal{G}}^0(N_0) = Rep(ZG^*(N_0))$.

Let $D_{i}^{\mathcal{P}_0} \subset D_i$ be the thick subcategory generated by $L_w$, $\ell(w) \neq 0$; and let $D_i^{\mathcal{P}_0}$ be the quotient category. Let also $\mathcal{P}_0 \subset \mathcal{P}_1$ be the image of $\mathcal{P}_1$. Thus $\mathcal{P}_1 = \mathcal{P}_1/(\mathcal{P}_1 \cap D_i^{\mathcal{P}_0})$ is an abelian category; it has one irreducible object $L_e$ if $G$ is simply connected; in general the number of irreducible objects in $\mathcal{P}_0$ equals $\#\pi_1(G)$.

The convolution of any $X \in D_i^{\mathcal{P}_0}$ with any object of $D_i$ lies in $D_i^{\mathcal{P}_0}$; thus the convolution induces a monoidal structure on $D_i^{\mathcal{P}_0}$ (which we denote by the same symbol). Moreover, the abelian subcategory $\mathcal{P}_0 \subset D_i$ is monoidal.

Let $F_0 : Rep(G^*) \rightarrow \mathcal{P}_0$ be the composition of the projection functor $\mathcal{P}_1 \rightarrow \mathcal{P}_1$ to $\mathcal{P}_0$ with $\mathcal{Z}$. Then $F_0$ respects the monoidal structure. Let $\mathcal{F}$ denote the full subcategory of $\mathcal{P}_0$ consisting of all
subquotients of $F_0(V)$ for some $V \in \text{Rep}(G^\circ)$ (in fact, one can show that $\mathcal{I} = P_0^I$, but we neither use nor prove this here). Also $\mathcal{M}$ induces a tensor endomorphism of the functor $F_0$.

Let us recall that for any subgroup $H \subset Z_{G^\circ}(N_0)$ the restriction functor $\text{res}^G_H$ carries a canonical tensor endomorphism induced by $N_0$.

**Lemma 23.** There exists a subgroup $H \subset Z_{G^\circ}(N_0)$, and an equivalence of monoidal categories $\mathcal{I} \cong \text{Rep}(H)$, which intertwines $F_0$ with the restriction functor $\text{res}^G_H$, and sends the tensor endomorphism $\mathcal{M}$ into the endomorphism induced by $N_0$.

**Proof.** See [B1].

**Remark 8.** It is not difficult to deduce from the results of the present paper that in fact $H = Z_{G^\circ}(N_0)$; and also that $\mathcal{I} = P_0^I$.

**Remark 9.** Lemma 23 is a particular case of a more general result proved in [B1], which relates representations of a centralizer of any nilpotent in $g^\circ$ to a two-sided cell in $W$. This result is deduced from some "non-elementary" Theorems of Lusztig about (asymptotic) Hecke algebras, see [L1] (which rely on the theory of character sheaves). However, the particular case used in Lemma 23 depends only on the elementary particular case of Lusztig’s Theorems, which deal with the maximal cell and the regular nilpotent orbit (the corresponding fact about the Hecke algebra amounts to the computation of the action of the center of the affine Hecke algebra in the Steinberg representation).

For $V_1, V_2 \in \text{Rep}(G^\circ)$ Lemma 23 yields an injective map

$$\text{Hom}_{Z_{G^\circ}(N_0)}(V_1, V_2) \to \text{Hom}_{Z_{G^\circ}(N_0)}(F_0(V_1), F_0(V_2)).$$

(25)

On the other hand, we have

$$\text{Hom}_{Z_{G^\circ}(N_0)}(V_1, V_2) = \text{Hom}_{\text{Coh}_{Z_{G^\circ}(N_0)}}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O}) = \text{Hom}_{\text{Coh}_{Z_{G^\circ}(N_0)}}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O}),$$

thus the functor $F$ induces another map

$$\text{Hom}_{Z_{G^\circ}(N_0)}(V_1, V_2) \to \text{Hom}_{Z_{G^\circ}(N_0)}(2\mathcal{O}(V_1), 2\mathcal{O}(V_2)) \to \text{Hom}_{Z_{G^\circ}(N_0)}(F_0(V_1), F_0(V_2)).$$

(26)

**Lemma 24.** The map (26) coincides with the composition in (25).

**Proof.** This follows from the uniqueness statement in Proposition 4(a), since both (25) and (26) send the “tautological” tensor endomorphism to the logarithm of monodromy endomorphism $\mathcal{M}$. □

**Corollary 2.** The composed map (26) is injective. □

4.3.1. **Proof of Proposition 8** We first claim that for any $\lambda \in \Lambda^+$ the sheaf $\mathcal{O}(\lambda) \in \text{Coh}_{Z_{G^\circ}(N)}$ can be realized as a subsheaf in $V \otimes \mathcal{O}_{\mathcal{N}}$ for some $V \in \text{Rep}(G^\circ)$. Indeed, for a simple coroot (root of $G^\circ$) $\alpha \in \Lambda$ let us denote by $D_\alpha \subset \mathcal{N}$ the $G^\circ$-invariant divisor $T^*\left(G^\circ/P_\alpha^\circ \times G^\circ/P_\alpha^\circ, G^\circ/B^\circ\right)$, where $P_\alpha^\circ \subset G^\circ$ is the corresponding minimal parabolic. Then it is easy to see that

$$\mathcal{O}_{\mathcal{N}}(-\alpha) \cong \mathcal{O}(D_\alpha);$$

thus we have an injective map of sheaves $\mathcal{O} \hookrightarrow \mathcal{O}(-\alpha)$. Taking tensor products we get also injections $\mathcal{O}(\lambda) \hookrightarrow \mathcal{O}(\lambda - 2n\rho)$. For large $n$ we have $\lambda - n\rho \in -\Lambda^+$, so we get an injection

$$\mathcal{O}(\lambda - n\rho) \hookrightarrow V_{w_0(\lambda - n\rho) \otimes \mathcal{O}}.$$

Thus it suffices to see that the map

$$\text{Hom}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O}) \to \text{Hom}(F_{\mathcal{J}W}(V_1 \otimes \mathcal{O}), F_{\mathcal{J}W}(V_2 \otimes \mathcal{O})).$$

(27)
is into. The functor $Av : \mathcal{P}_I \to \mathcal{P}_{IW}$ is exact by Proposition 2(a); by Lemma 4(a) it does not kill $L_w$ for $w \in W$, in particular for $\ell(w) = 0$. Hence it does not kill any morphism whose image in $\mathcal{P}_I^0$ is nonzero. The statement now follows from Corollary 2.

4.4. Proof of Proposition 7. We first prove the following

**Theorem 7.** Stalk$_\mu(F_{IW}(V))$, Costalk$_\mu(F_{IW}(V))$ are concentrated in homological degree 0 for all $\mu \in \Lambda^+$, $V \in \text{Rep}(\hat{G})$.

**Remark 10.** Theorem 7 says that $F_{IW}(V)$ is a tilting object of $\mathcal{P}_{IW}$.

**Remark 11.** We do not know whether the following strengthening of Theorem 7 is true: “for every convolution exact object $\mathcal{F}$ of $\mathcal{P}_I$ the sheaf $\Delta_0 \ast \mathcal{F}$ is tilting” (cf. Remark 7 in section 3.6.2).

**Remark 12.** Recall that the parabolic-singular Koszul duality is an equivalence between the mixed versions of parabolic and singular categories $\mathcal{O}$, see [BGS]. An appropriate version of this equivalence (see [BG]) sends irreducible objects into tilting ones. The parabolic category $\mathcal{O}$ is equivalent to the category of perverse sheaves on the partial flag variety $G/P$. Using (a variation of) the result of [MS] one can realize the singular category as an appropriate category of Whittaker sheaves. One can try to generalize this picture by replacing $G$ by the loop group $G_F$, and $P$ by the maximal parahoric $G_O$. Thus we are led to the conjecture that there exists an equivalence between the mixed versions of $D_{IW}$ and the category of Iwahori monodromic sheaves on the affine Grassmanian. In fact, this conjecture can be derived from a combination of the results of this paper and those of [ABG], or by adapting the method of [BGS]; see also discussion in [B4, 1.2].

In view of some formal properties of this duality (in particular, the fact that the central sheaves are Koszul self-dual), the statement of Theorem 7 is Koszul dual to the statement that the $\pi(V_{\text{Weil}})$ is simple of weight 0; the latter statement is clear from the definition of $Z$ together with the fact that nearby cycles commute with proper direct image, cf. [G], Theorem 1(d).

The Theorem will be deduced from the following two statements.

**Lemma 25.** If Theorem 7 holds for two representations $V_1, V_2$, then it holds for $V = V_1 \otimes V_2$.

**Lemma 26.** Theorem 7 holds if $V = V_\lambda$, where $\lambda \in \Lambda^+$ is either minuscule or quasi-minuscule (i.e. is the short dominant root).

4.4.1. Proof of Theorem 7. If $G$ is not adjoint let $V$ be the sum of its minuscule irreducible representations; otherwise let $V$ be the quasi-minuscule representation. Lemma 26 shows that the statement of the Theorem holds for $V$. However, it is easy to see $V$ is a faithful representation; hence it induces a surjective map from functions on $End(V)$ to functions on $G$. Since $V$ is self-dual, any irreducible representation of $G$ is a direct summand of $V^{\otimes n}$ for some $n$. Thus the Theorem follows by Lemma 26.

**Remark 13.** The trick of reduction to the special case of a (quasi-)minuscule representation was also (independently) used in [NP].

4.4.2. Proof of Proposition 7. The Proposition follows from Theorem 7 and the following Lemma.

**Lemma 27.** The Euler characteristic $\sum(-1)^i \dim H^i(\text{Stalk}_\mu F_{IW}(V \otimes O))$ equals $[\mu : V]$, the multiplicity of the weight $\mu$ in $V$.

---

This fact was suggested to us by M. Finkelberg.
**Proof.** For a triangulated category $D$ we will denote its Grothendieck group by $K(D)$, and for $X \in D$ will let $[X] \in K(D)$ be its class. We have an isomorphism $K(D) \cong \mathbb{Z}[W]$. Thus filtration with subquotients isomorphic to $\Delta_\mu$ carries a filtration with subquotients of the form

$$[Z(V)] = \bigoplus_{\mu} [\mu : V] \cdot [J_\mu] = \bigoplus_{\mu} [\mu : V] \cdot [J_\mu],$$

where $s$ is a simple reflection. It follows that $[J_\lambda] = [J_\lambda!]$ for all $\lambda \in \Lambda$. Thus Theorem 4 implies that

$$[F_{\mathcal{W}}(V \otimes 0)] = \bigoplus_{\mu} [\mu : V] \cdot [\Delta_0 \ast j_\mu] = \bigoplus_{\mu} [\mu : V] \cdot [\Delta_\mu],$$

which implies the statement of the Lemma. \qed

4.4.3. **Proof of Lemma 2**. We will consider the condition on stalks; the one on costalks is treated similarly. Notice that this condition is equivalent to saying that $F_{\mathcal{W}}(V)$ carries a filtration with subquotients isomorphic to $\Delta_\mu$. If this is the case for $V = V_1$, then $F_{\mathcal{W}}(V_1 \otimes V_2)$ carries a filtration with subquotients of the form

$$(28) \quad \Delta_\mu \ast Z(V_2) \cong \Delta_0 \ast Z(V_2) \ast j_\mu!,$$

where the central property of the sheaf $Z(V_2)$ is used. The Lemma will be proven if we show the corresponding statement for the stalk of the sheaf $\mathcal{F}$ appearing in either side of (28). $\mathcal{F}$ is a perverse sheaf by exactness of convolution with $Z(V_2)$; hence $\text{Stalk}_\mu(\mathcal{F}) \in D^{>0}$ by the definition of a perverse sheaf. The opposite estimate follows from the assumption that $\Delta_0 \ast Z(V_2)$ has a filtration with subquotients $\Delta_\nu$, and the following \qed

**Sublemma 2.** For all $\lambda, \nu \in \Lambda$, $w \in W$ we have

$$\text{Stalk}_\lambda(\Delta_\nu \ast j_\omega!) \in D^{>0}.$$

**Proof.** The statement is equivalent to

$$\Delta_\nu \ast j_\omega! \in \langle \Delta_\mu[i] , \ i \leq 0, \mu \in \Lambda \rangle$$

(cf. sections 2.0.2, 3.6.4 for notations). Since $\Delta_\lambda \cong \Delta_0 \ast j_{\mu!}$ for any $u \in W_f \cdot \lambda$ by Lemma 4(c), the latter follows from the following statement (see e.g. [BeBe]),

$$(29) \quad j_{\omega,1!} \ast j_{\omega,2!} \in \langle j_{\omega,1!}[i] , \ i \leq 0, w \in W \rangle.$$

To verify (29) we can assume that $w_2 = s$ is a simple reflection. If $\ell(w_1 \cdot s) > \ell(w_1)$, then $j_{\omega,1!} \ast j_{s!} \cong j_{\omega,1,s!}$, so (29) is clear. If $\ell(w_1 \cdot s) < \ell(w_1)$, then we have an exact triangle

$$j_{\omega,1!} \oplus j_{\omega,1!}[-1] \to j_{\omega,1!} \ast j_{s!} \to j_{\omega,1,s!},$$

which shows (29) in this case also. \qed
4.4.4. **Proof of Lemma [28]**

**Lemma 28.** For $w_f \in W_f$, $V \in \text{Rep}(G^*)$ we have an isomorphism

$$\text{Stalk}_\lambda(F_{2W}(V \otimes \mathcal{O})) \cong \text{Stalk}_{w_f(\lambda)}(F_{2W}(V \otimes \mathcal{O})).$$

**Proof.** Let $s \in W_f$ be a simple reflection. Then $\Delta_0 \ast L_s = 0$ by Lemma 31a above. By the central property of $\mathcal{Z}(V)$ we have also

$$F_{2W}(V \otimes \mathcal{O}) \ast L_s = \Delta_0 \ast L_s \ast \mathcal{Z}(V) = 0.$$

For $X \in D_{2W}$, and $\lambda \in \Lambda$ such that $s(\lambda) \preceq \lambda$ it is easy to construct an exact triangle

$$\text{Stalk}_\lambda(X)[-1] \to \text{Stalk}_\lambda(X \ast L_s) \to \text{Stalk}_{s(\lambda)}(X) \to \text{Stalk}_\lambda(X).$$

Thus $\text{Stalk}_\lambda(X) \cong \text{Stalk}_{s(\lambda)}(X)$ provided that $X \ast L_s = 0$. This proves the statement of the Lemma for $w_f = s$, and hence for all $w_f \in W_f$. □

**Lemma 29.** For $V \in \text{Rep}(G^*)$ let $d_V = \dim V^{N_0}$, where $N_0$ is a regular nilpotent element. Then we have

$$\dim \text{Hom}(\Delta_0 \ast \mathcal{Z}(V), \Delta_0) \leq d_V. \tag{30}$$

The proof of the Lemma will rely on the following result of D. Gaitsgory (unpublished); we reproduce the proof in the Appendix.

**Theorem 8.** There exists an element $\mathfrak{M} = \mathfrak{M}_\mathfrak{r}$ of the center of $\mathfrak{P}_1$ such that

i) $\mathfrak{M}_{\mathcal{Z}(V)} = \mathcal{M}_V$ for $V \in \text{Rep}(G^*)$.

ii) $\mathfrak{M}_L = 0$ for any irreducible object $L \in \mathfrak{P}_1$.

**Proof of Lemma 29.** We have

$$\text{Hom}(\Delta_0 \ast \mathcal{Z}(V), \Delta_0) = \text{Hom}_{\mathfrak{P}_1}(\mathcal{Z}(V), L_\mathcal{L}) = \text{Hom}_{\mathfrak{P}_1}(\mathcal{Z}(V), L_\mathcal{M}) \to \text{Hom}_{\mathfrak{P}_1}(\mathcal{Z}(V), L_\mathfrak{M}),$$

where the first equality follows from Proposition 2b, and the second one from Theorem 8. But Lemma 23 implies that $\mathcal{Z}(V)_{\mathfrak{M}} \mod \mathfrak{P}^{\mathfrak{r}0}$ has length $d_V$, which shows that $\dim \text{Hom}_{\mathfrak{P}_1}(\mathcal{Z}(V), L_\mathfrak{M}) \leq d_V$. □

**Proof of Lemma 29 (conclusion).** It follows from Lemma 9 that $\mathfrak{H}^\lambda$ is open in the support of $F_{2W}(V_\lambda)$, and

$$\text{Stalk}_{w_0(\lambda)}(F_{2W}(V_\lambda \otimes \mathcal{O})) \cong \overline{\mathfrak{H}^\lambda}.$$

By Lemma 28 we conclude that

$$\text{Stalk}_{w_0(\lambda)}(F_{2W}(V_\lambda \otimes \mathcal{O})) \cong \overline{\mathfrak{H}^\lambda} \quad \text{for} \quad w_f \in W_f. \tag{31}$$

If $\lambda$ is minuscule then (31) implies the statement of the Proposition, because in this case the support of $F_{2W}(V_\lambda \otimes \mathcal{O})$ contains $\mathfrak{H}^\mu$ iff $\mu \in W_f(\lambda)$.

Assume now that $\lambda$ is quasi-minuscule, i.e. $\lambda$ is the short dominant coroot. Then the support of $F_{2W}(V_\lambda \otimes \mathcal{O})$ contains $\mathfrak{H}^\mu$ iff either $\mu \in W_f(\lambda)$ or $\mu = 0$. The first case is treated by (31), so it remains to consider the case $\mu = 0$.

We first claim that $\text{Stalk}_0(\Delta_0 \ast \mathcal{Z}(V))$ can only be concentrated in degrees 0 and $-1$. This follows from the exact triangle

$$j_*j^*(\Delta_0 \ast \mathcal{Z}(V)) \to \Delta_0 \ast \mathcal{Z}(V) \to i_*i^*(\Delta_0 \ast \mathcal{Z}(V)),$$

where $i$ is the embedding of $G/B \hookrightarrow \mathfrak{H}$, and $j$ is the embedding of its complement. (31) shows that $j^*(\Delta_0 \ast \mathcal{Z}(V))$ carries a filtration whose subquotients are of the form $j^*(\Delta_\mu)$. Since $j_*j^*(\Delta_\mu) = \Delta_\mu$ for $\mu \neq 0$ we see that $j_*j^*(\Delta_0 \ast \mathcal{Z}(V))$ is a perverse sheaf, hence $i_*i^*(\Delta_0 \ast \mathcal{Z}(V))$ is concentrated in homological degrees 0 and $-1$. 

To finish the proof it now suffices to check that
\begin{equation}
\dim H^0(\text{Stalk}_0(\Delta_0 * \mathbb{Z}(V))) \leq \sum (-1)^i \dim H^i(\text{Stalk}_0(\Delta_0 * \mathbb{Z}(V))).
\end{equation}

Here the vector space in the left hand side is dual to the vector space $\text{Hom}(F_{\mathfrak{g}}(V \otimes \mathcal{O}), \Delta_0)$, so its dimension is estimated by Lemma 29. The right hand side of (32) is computed in Lemma 27. To see (32) it remains to notice that for $\lambda$ quasi-minuscule we have
\[ [0 : V_\lambda] = \dim V^h_\lambda = \dim V^{N_0}_\lambda, \]
where $h$ is the semisimple element in a regular $s\ell(2)$ triple $e = (N_0, h, f)$; here the first (respectively, the second) equality is true because every non-zero weight of $V_\lambda$ is a root, hence is non-zero (respectively, even) on the Cartan of a principle $S\ell(2)$.

4.5. **Proof of Theorem 2** In view of Proposition 2, it suffices to check that $Av_{\Psi} : \mathcal{F}_I \to \mathcal{F}_{\mathfrak{g}}$ is essentially surjective. By Theorem 3, any $X \in \mathcal{F}_{\mathfrak{g}}$ is isomorphic to $F_{\mathfrak{g}}(\mathcal{F}) = pr_f(F(\mathcal{F}))$ for some $\mathcal{F} \in D(\mathcal{N})$. Since $Av_{\Psi}$ is exact by Proposition 2(a) we see that $X \cong Av_{\Psi}(F(\mathcal{F})) \cong Av_{\Psi}(H^{0,p}(F(\mathcal{F})))$, which shows essential surjectivity of $Av_{\Psi}$.

4.6. **Application: Whittaker integrals of central sheaves.** For $\mathcal{F} \in D$ set $Wh^i_w(\mathcal{F}) = H^i_w(\Psi_w(\mathcal{S} \otimes \mathcal{F}))$. If $\mathcal{F}$ is endowed with a Weil structure then $Wh^i_w(\mathcal{F})$ carries an action of Frobenius.

**Theorem 9.** Let $\lambda \in \Lambda^+$, $w = \kappa(\mu) \in \ell W$. Recall that $[\mu : V_\lambda]$ denote the multiplicity of the weight $\mu$ in $V_\lambda$. Then we have
\[ Wh^i_w(Z_\lambda) = 0 \quad \text{for} \quad i \neq \dim(\mathcal{F} \ell^{w}); \]
\[ \dim Wh^i_w(Z_\lambda) = [\mu : V_\lambda]. \]

Moreover, we have
\[ Tr(Fr, Wh^i_w(Z_\lambda)) = Q_{\lambda, \mu}(q^{1/2}); \]
here the polynomial $Q_{\lambda, \mu}(t)$ is defined by
\begin{equation}
Q_{\lambda, \mu} = Q_{\lambda, w(\mu)} \quad \text{for} \quad w \in W_f; \tag{33}
\end{equation}
\begin{equation}
Q_{\lambda, \mu} = t^{\ell(\lambda) + \ell(w)}P_{\lambda, \mu}(t^2), \tag{34}
\end{equation}
where $P_{\lambda, \mu}$ is the Kazhdan-Lusztig polynomial (the $q$-analogue of weight multiplicity), see [L].

We will abuse notations by writing $\nabla_w$ for the Weil sheaf $i_w^* \psi_w^*(\mathcal{S} \otimes \ell(w))(\frac{\ell(w)}{2})$.

**Lemma 30.** a) For $\mathcal{F} \in D_f$ we have a canonical isomorphism
\begin{equation}
Wh^i_w(Av_{\Psi}(\mathcal{F})) \cong Wh^i_w(F(\mathcal{F})) \left( \frac{\ell(w)}{2} \right). \tag{35}
\end{equation}
If $\mathcal{F}$ is equipped with a Weil structure, the isomorphism is compatible with the Frobenius action.

b) For $\mathcal{F} \in D_{\mathfrak{g}}$ we have
\begin{equation}
Wh^i_w(\mathcal{F}) = Hom\left( \mathcal{F}, \nabla_w[i + \ell(w)] \left( \frac{\ell(w)}{2} \right) \right); \tag{36}
\end{equation}
for $\mathcal{F}$ equipped with the Weil structure the isomorphism is compatible with the Frobenius action.
Proof. It is easy to see that \( \mathcal{F}^\ell \) is an orbit of the group \( I^- \cap N^-_F \). It follows that

\[
Av_q(\mathcal{F}) \cong Av^l_{I^- \cap N^-_F} [\ell(w)](\frac{\ell(w\mu)}{2})
\]

where

\[
(37) \quad Av^l_{I^- \cap N^-_F, \psi} = \alpha_1(\psi^*_0_{I^- \cap N^-_F/U}(A\mathcal{S}) \otimes pr^l(\mathcal{F})),
\]

and \( pr: I^- \cap N^-_F / U \times supp(\mathcal{F}) \to supp(\mathcal{F}), a: I^- \cap N^-_F / U \times supp(\mathcal{F}) \to \mathcal{F} \ell \) are the projection to the second factor, and the action map respectively, while \( U \subset I^- \cap N^-_F / U \) is some open subgroup which stabilizes all \( x \in supp(\mathcal{F}) \). Proper base change shows that

\[
Wh^i_w(\mathcal{F}) = Wh^i_w(a(\psi^*_0_{I^- \cap N^-_F/U}(A\mathcal{S}) \otimes pr^l(\mathcal{F})))
\]

for any group subscheme \( H \subset N^-_F \), in evident notations. This proves (a).

To prove (b) notice that Lemma 2 implies that \( Wh^i_w(\Delta w^\tau) = 0 \) unless \( w = w', i = \dim \mathcal{F}^\ell_w \); and

\[
Wh^i_\dim \mathcal{F}^\ell_w (\Delta w) = \overline{\mu}i(-\frac{\dim \mathcal{F}^\ell_w}{2}).
\]

It follows that

\[
Wh^*_w(\mathcal{F}) = Wh^*_w(i_w^*i^*_w(\mathcal{F})) = Hom^*(\mathcal{F}, \nabla_w)[-\dim \mathcal{F}^\ell_w](\overline{\dim \mathcal{F}^\ell_w}^\ell).
\]

Proof of Theorem 9 (sketch). The first two statements of the Theorem follow from Lemma 30 (b), Theorem 7 and Lemma 27.

The equality (33) follows from

\[
Wh^*_\mu(Z_{\lambda} \ast L_s) = 0
\]

which yields an isomorphism

\[
Wh^*_\mu(Z_{\lambda}) \cong Wh^*_{s(\mu)}(Z_{\lambda})(\frac{1}{2})
\]

where \( s \in W_f \) is a simple reflection such that \( s(\mu) < \mu \).

Let us prove (34). Using Lemma 30 and equivalence of Theorem 3 we see that

\[
Wh^*_{\dim \mathcal{F}_\ell}(Z_{\lambda}) \cong Hom(V_\lambda \otimes \mathcal{O}, \mathcal{O}(\mu)),
\]

where the action of Frobenius on the left hand side corresponds to \( q^{(-\ell(\mu)-\ell(w_0))/2} \) times the action of the automorphism induces by the dilatation by \( q \).

By the well-known interpretation (due to Hesselink) of \( P_{\lambda, \mu} \) in terms of the character of the space of functions on \( N \) (see e.g. [15], Lemma 6.1) the trace of the latter automorphism equals \( q^{(\ell(\lambda)+\ell(w_0))}/2P_{\lambda, \mu}(q) \). □

5. Appendix

by ROMAN BEZRUKAVNIKOV and IVAN MIRKOVIĆ

5.1. Sketch of proof of Theorem 5. It is easy to see that the morphisms \( m_{\ell}^w : \mathcal{F}_w \times \mathcal{F}_\ell \to \mathcal{F}_\ell \)

\( m_{\ell}^w : \mathcal{F}_\ell \times \mathcal{F}_w \to \mathcal{F}_\ell \) (restrictions of the convolution map) are affine (notations of section 1.2).

Hence for \( w = (\lambda)^{-1} \cdot \mu \cdot w_f, \lambda, \mu \in \Lambda^+, w_f \in W_f \) we have

\[
J_w = j_{-\lambda} \ast j_{\mu \cdot w_f} = (m_{\ell}^{-\lambda})(j_{-\lambda} \bigotimes j_{\mu \cdot w_f}(\mathcal{F}_w \times \mathcal{F}_\ell)) \in D_{\geq 0}^{\mathcal{F}_\ell}
\]

\[\footnote{Partly supported by an NSF grant.}\]
because ! direct image under an affine morphism is left exact in the perverse $t$-structure (see [BBD], 4.1.2). On the other hand

$$J_w = (m_\mu^w)^* (j_{-\lambda})! \otimes j_\mu^w \circ [\mathcal{F}_w] \in D^{p' \leq 0}$$

because $*$ direct image under an affine morphism is right exact in the perverse $t$-structure (see [BBD], 4.1.1). The two observations together give statement (a).

In view of (a), statement (b) will be proven if we check that the Euler characteristic of the stalk $Eu\mu_w(J_w) = \sum (-1)^f \dim H^f(j_\mu^w(J_w))$ is zero for $w' \neq w$, and is nonzero for $w' = w$. (Indeed, we have $Eu\mu_w(J_w) \neq 0$ whenever $\mathcal{F}_w$ is open in the support of $J_w$.) In the proof of Lemma 27 we have seen the equality $[J_w] = [J_{w_1}]$ in the Grothendieck group $K(D_1)$. It implies that $Eu\mu_w(J_w) = (-1)^f (w) \delta_{w,w'}$.

5.2. Sketch of proof of Theorem 8. Let $X$ be a scheme with a $\mathbb{G}_m$ action. Recall the notion of a monodromic constructible complex on $X$, and the monodromy action of the tame fundamental group of $\mathbb{G}_m$ on such a sheaf, see [VI]. In fact, the definition of loc. cit. works in the set up when $X$ is a cone over a base scheme $S$, i.e. a closed $\mathbb{G}_m$-invariant subscheme in $\mathbb{A}^n_S$.

For an arbitrary $X$ we will say that a constructible complex $\mathcal{F}$ on $X$ is monodromic if $j \mu^*(\mathcal{F})$ is monodromic in the sense of [VI], where $a : \mathbb{G}_m \times X \to X$ is the action map, and $j : \mathbb{G}_m \times X \to \mathbb{A}^1 \times X$ is the embedding. If $\mathcal{F}$ is a perverse sheaf, then we have $End(\mathcal{F}) \to End(j \mu^*(\mathcal{F}))$; so the monodromy action on $j \mu^*(\mathcal{F})$ introduced in [VI] yields an action on $\mathcal{F}$, which we also call the $\mathbb{G}_m$-monodromy action. If this action is unipotent, it defines the logarithm of monodromy operator $\mathcal{F} \to \mathcal{F}(-1)$, see e.g. [D], §1.7.2, which we denote by $\mathfrak{M}^\circ_{\mathcal{F}}$.

Let us now describe the element $\mathfrak{M}$. Recall that the pro-algebraic group $Aut(O)$ of automorphisms of $O$ acts on $\mathcal{F}$. All $I$-orbits on $\mathcal{F}$ are $Aut(O)$ invariant. This implies that $L_w$ are $Aut(O)$ equivariant; in particular, they are equivariant with respect to the subgroup of dilations $\mathbb{G}_m \subset Aut(O)$. Hence any $\mathcal{F} \in \mathcal{F}_I$ is monodromic with respect to this action, and the monodromy action on $\mathcal{F}$ is unipotent.

We set $\mathfrak{M}_{\mathcal{F}} = -\mathfrak{M}^\circ_{\mathcal{F}}$.

Then property (ii) is clear. To establish (i) we recall that the functor $Z(V)$ is defined as the nearby cycles of a certain sheaf $\mathcal{F}_V$ on the space $\mathcal{F}_X$ defined in terms of a global curve $X$. We can assume that $X = \mathbb{A}^1$, so that there is an action of $\mathbb{G}_m \subset Aut(O)$ on $X$ compatible with the action on $O$. It also induces an action of $\mathbb{G}_m$ on $\mathcal{F}_X$. Then it is easy to see that the sheaf $\mathcal{F}_V$ is equivariant with respect to this action of $\mathbb{G}_m$. Now property (i) follows from the following general

Claim 2. Let $X$ be a variety with a $\mathbb{G}_m$-action, and $f : X \to \mathbb{A}^1$ be a function, such that $f(tx) = tf(x)$ for $x \in X$, $t \in \mathbb{G}_m$. If $\mathcal{F}$ is a $\mathbb{G}_m$ equivariant perverse sheaf on $X$, then the nearby cycles complex $\Psi_f(\mathcal{F})$ is $\mathbb{G}_m$-monodromic. Moreover, the monodromy action on $\Psi_f(\mathcal{F})$ as on the nearby cycles sheaf factors through the tame fundamental group of $\mathbb{G}_m$; the resulting action of this tame fundamental group on $\mathcal{F}$ is opposite to the $\mathbb{G}_m$-monodromy action.

Proof. This is a restatement of [VI], Proposition 7.1(a). More precisely, the set up of loc. cit. is as follows. One considers a constructible complex $\tilde{\mathcal{F}} = pr_1^*(\mathcal{F})$ on $X \times \mathbb{G}_m$ where $pr_1 : \mathbb{G}_m \times X \to X$ is the projection; and a function $\tilde{f}$ on $\mathbb{G}_m \times X$ given by $\tilde{f}(t,x) = tf(x)$. It is then proved that the $\mathbb{G}_m$ monodromy action on $\Psi_f(pr_1^*(\mathcal{F}))$ is opposite the canonical monodromy acting on the nearby cycles sheaf. Under our assumptions $pr_1^*(\mathcal{F}) \cong \mathcal{F}(\tilde{f})$, $\tilde{f} = a^*(f)$, where $a : \mathbb{G}_m \times X \to X$ is the action map. Thus the statement of loc. cit. implies the Claim. □
Remark 14. This Claim implies that the action of monodromy on the functor $Z$ is unipotent (because the $G_m$ monodromy is obviously unipotent, as it is functorial, and is trivial on irreducible perverse sheaves). This fact was proved in [G] by a different argument.

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