Reduction of quantum systems with arbitrary first class constraints and Hecke algebras

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Abstract

We propose a method for reduction of quantum systems with arbitrary first class constraints. An appropriate mathematical setting for the problem is homology of associative algebras. For every such an algebra $A$ and its subalgebra $B$ with an augmentation $\varepsilon$ there exists a cohomological complex which is a generalization of the BRST one. Its cohomology is an associative graded algebra $H^k(A, B)$ which we call the Hecke algebra of the triple $(A, B, \varepsilon)$. It acts in the cohomology space $H^*(B, V)$ for every left $A$– module $V$. In particular the zeroth graded component $H^0(A, B)$ acts in the space of $B$– invariants of $V$ and provides the reduction of the quantum system.

Introduction

The purpose of this paper is to generalize the well–known BRST quantization procedure to arbitrary associative algebras. An appropriate mathematical setting for the BRST cohomology of Lie algebras [3] was proposed by Kostant.

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and Sternberg in [8]. Their observation was that given a Hamiltonian action of a Lie group on a Poisson manifold one can construct a super–Poisson complex whose zeroth cohomology is the algebra of functions on the reduced space over zero value of the corresponding moment map [1], [10]. This complex admits a quantization and the zeroth cohomology of the quantum complex may be treated as a quantization of the classical reduced space. In that case the quantum counterparts of matrix elements of the moment map form a Lie algebra and represent a system of the first–class constraints for the quantum reduction.

Our approach to the BRST cohomology differs from the one described above. We start with the quantum complex directly. It turns out that the quantum BRST cohomology may be defined using the language of homological algebra, resolutions, etc.. The particular complex proposed in [8] corresponds to the standard resolution of the ground field. As usual, in homological algebra different choices of resolutions lead to the same homology. This allows us to generalize the BRST cohomology to arbitrary systems of the first–class constraints.

The BRST quantization is not unique in the following sense. One can always define the BRST cohomology related to the usual cohomology of representation spaces of the quantum system. However under some technique assumptions there exists another version of the BRST reduction related to the semiinfinite cohomology of the representation spaces [6]. In the case of Lie algebras it requires a normal ordering in the differential [8]. In the present paper we only discuss the usual BRST cohomology. The latter one will be explained in a subsequent paper.

1 Endomorphisms of complexes

Let $A$ be an associative ring with unit, $X$ be a graded complex of left $A$–modules equipped with a differential of degree -1. Recall the definition of the complex $Y = End_A(X)$ [8].

By the definition $Y$ is a $\mathbb{Z}$ graded complex

$$Y = \bigoplus_{n=-\infty}^{\infty} Y^n$$

with graded components defined as
\[ Y^n = \prod_{p+q=n} Y^{p,q}, \quad (2) \]

where

\[ Y^{p,q} = \text{Hom}_A(X^p, X^{-q}). \quad (3) \]

Clearly \( Y \) is a subalgebra in the full algebra of \( A \)-endomorphisms of \( X \). It is easy to see that \( Y \) is closed with respect to the multiplication given by composition of endomorphisms. Thus it is a graded associative algebra. We emphasize that \( Y \) is not a bigraded space.

Introduce a differential on \( Y \) of degree +1 as follows

\[ (df)^{p,q} = (-1)^{p+q}fp - 1, q \circ d + f^{p,q-1}, \quad f = \{ f^{p,q} \}, f^{p,q} \in Y^{p,q}. \quad (4) \]

If \( f \) is homogeneous then

\[ df = d \circ f - (-1)^{\text{deg}(f)} f \circ d. \quad (5) \]

So that \( d \) is the supercommutator by \( d \)

We shall consider also the partial differentials \( d' \) and \( d'' \) on \( Y \):

\[ (d'f)(x) = (-1)^{p+q+1} f(dx), \quad x \in X^{p+1}; \]
\[ (d''f)(x) = df(x), \quad x \in X^p. \quad (6) \]

It is easy to check that

\[ d'^2 = d''^2 = d'd'' + d''d' = 0 \quad (7) \]

These conditions ensure that \( d'^2 = 0 \).

The following property of \( d \) is crucial for the subsequent considerations.

**Lemma 1** \( d \) is a superderivation of \( Y \).

**Proof.** Let \( f \) and \( g \) be homogeneous elements of \( Y \). Then \( \text{deg}(fg) = \text{deg}(f) + \text{deg}(g) \) and (3) yields:
The proof follows.

The most important consequence of the lemma is

**Theorem 2** The homology space $H^*(Y)$ inherits a multiplicative structure from $Y$. Thus $H^*(Y)$ is a graded associative algebra.

Proof. First, the product of two cocycles is a cocycle. For if $f$ and $g$ are homogeneous and $df = dg = 0$ then

$$
d (fg) = d \circ fg = 0.
$$

(8)

Now we have to show that the product of homology classes is well defined. It is suffices to verify that the product of a homogeneous cocycle with a homogeneous coboundary is homologous to zero. For instance consider the product $f d h$. Then (5) gives

$$
f d h = f \circ (d \circ h + (-1)^{\text{deg}(h)} h \circ d) = (-1)^{\text{deg}(f)} d \circ (f h).
$$

(9)

This completes the proof.

One of the principal statements of homological algebra says that homotopically equivalent complexes have the same homology. In particular the vector space $H^*(Y)$ depends only on the homotopy class of the complex $X$. It turns out that the same is true for the algebraic structure of $H^*(Y)$. Indeed we have the following

**Theorem 3** Let $X, X'$ be two homotopically equivalent graded complexes of left $A$-modules. Then

$$
H^*(Y) \simeq H^*(Y')
$$

as graded associative algebras.
Proof. Let $F : X \to X', F' : X' \to X$ be two maps of the complexes such that

\begin{align}
F'F - \text{id}_X &= d_Xs + sd_X, \quad s : X \to X, \\
FF' - \text{id}_{X'} &= d_{X'}s' + s'd_{X'}, \quad s : X' \to X', \\
&\quad s \in Y^{-1}, \quad s' \in Y'^{-1}.
\end{align} \tag{12}

Consider the induced mappings of the complexes $Y, Y'$:

\begin{align}
FF'^* : Y &\to Y', \\
FF'^*f &= F \circ f \circ F', f \in Y; \\
F'F^* : Y' &\to Y, \\
F'F^*g &= F' \circ g \circ F', g \in Y'.
\end{align} \tag{13}

Their compositions are homotopic to the identity maps of $Y$ and $Y'$ (see [4], Chap.4 for a general statement about equivalences of functors). But it means that $FF'^*$ is inverse to $F'F^*$ when restricted to homology. Thus $H^*(Y)$ is isomorphic to $H^*(Y')$ as a vector space. We have to show that the restrictions of $FF'^*$ and $F'F^*$ to the homologies are homomorphisms of algebras.

Let $f$ and $g$ be homogeneous elements of $Y$ and $d_Xf = d_Xg = 0$. By the definition of the induced maps we have

\[ FF'^*(fg) = F \circ f \circ F'. \tag{14} \]

On the other hand

\[ FF'^*(f)FF'^*(g) = F \circ f \circ F'F \circ g \circ F' = \]

\[ F \circ f(id_X + d_Xs + sd_X)g \circ F'. \tag{15} \]

Now recall that $f$ and $g$ are cocycles in $Y$. By (5) they supercommute with $d_X$:

\[ d_X \circ f = (-1)^{\deg(f)}f \circ d_X. \tag{16} \]

Using (16) and the fact that $F$ and $F'$ are morphisms of complexes we can rewrite (15) as follows
\[ F \circ f(id_X + d_X s + s d_X) g \circ F' = F \circ f g \circ F' + (-1)^{deg_f} d_X \circ F \circ f s g \circ F' + (-1)^{deg_f} F \circ f s g \circ F' \circ d_X. = (17) \]

\[ F \circ f g \circ F' + (-1)^{deg_f} d_X(F \circ f s g \circ F'). \]

Finally observe that by (17) \( F F'^*(f g) \) and \( F F'^*(f) F F'^*(g) \) belong to the same homology class in \( H^*(Y') \). This completes the proof.

2 Hecke algebras

Let \( A \) be an associative algebra over a ring \( K \) with unit, \( B \) be its subalgebra with an augmentation \( \varepsilon : B \to K \) (\( \varepsilon \) is a homomorphism of \( K \)-algebras).

Let \( X \) be a projective resolution of the left \( B \)-module \( K \). Then the complex

\[ A \otimes_B X \quad (18) \]

has the natural structure of a left \( A \)-module. We can apply theorem 2 to define a graded associative algebra

\[ Hk^*(A, B) = H^*(End_A(A \otimes_B X)). \quad (19) \]

Observe that all \( B \)-projective resolutions of \( K \) are homotopically equivalent. Hence by theorem 3 \( Hk^*(A, B) \) does not depend on the resolution \( X \). We shall call it the Hecke algebra of the triple \( (A, B, \varepsilon) \).

Now consider \( A \) as a left \( A \)-module and a right \( B \)-module via multiplication. In this way \( A \) becomes an \( A \otimes B^{opp} \)-left module. Let \( X' \) be a projective resolution of the module. The complex

\[ X' \otimes_B K \quad (20) \]

is a left \( A \)-module. Therefore there exists an associative algebra

\[ \hat{H}k^*(A, B) = H^*(End_A(X' \otimes_B K)) \quad (21) \]

independent of the resolution \( X' \).
Theorem 4 \( H^k(A, B) \) is isomorphic to \( \hat{H}^k(A, B) \) as a graded associative algebra.

Proof. We shall use the standard bar resolutions for computation of \( \hat{H}^k(A, B) \) and \( H^k(A, B) \) \[^9\], \[^4\]. Consider the complex \( B \otimes T(I(B)) \otimes B \), where \( I(B) = B/K \) and \( T \) denotes the tensor algebra of the vector space. Elements of \( B \otimes T(I(B)) \otimes B \) are usually written as \( a[a_1, \ldots, a_s]a' \). The differential is given by

\[
da[a_1, \ldots, a_s]a' = aa_1[a_2, \ldots, a_s]a' + \sum_{k=1}^{s-1} (-1)^k a[a_1, \ldots, a_k a_{k+1}, \ldots, a_s]a' + (-1)^s a[a_1, \ldots, a_{s-1}]a_s a'.
\]

Then \( B \otimes T(I(B)) \otimes B \otimes B K = B \otimes T(I(B)) \otimes K \) is a free resolution of the left \( B \)-module \( K \). And \( A \otimes A B \otimes T(I(B)) \otimes B = A \otimes T(I(B)) \otimes B \) is a free resolution of \( A \) as a right \( B \)-module. The complex \( A \otimes T(I(B)) \otimes B \) is also a left free \( A \)-module via the left multiplication by elements from \( A \). Hence this is an \( A \otimes B^{opp} \)-free resolution of \( A \).

Thus the complex \( \text{End}_A(A \otimes B \otimes T(I(B)) \otimes K) = \text{End}_A(A \otimes T(I(B)) \otimes K) \) for computation of \( H^k(A, B) \) is canonically isomorphic to the complex \( \text{End}_A(A \otimes T(I(B)) \otimes B \otimes B K) = \text{End}_A(A \otimes T(I(B)) \otimes K) \) for computation of \( \hat{H}^k(A, B) \). This establishes the isomorphism of the algebras.

3 Action in homology and cohomology spaces

Recall that for every left \( B \)-module \( V \) the cohomology modules are defined to be

\[
H^*(V) = \text{Ext}_B^*(K, V) = H^*(\text{Hom}_B(X, V)),
\]  \(^{23}\)

where \( X \) is a projective resolution of \( K \). While for every right \( B \)-module \( W \) one can define the homology modules

\[
H_*(W) = \text{Tor}_B^*(W, K) = H_*(W \otimes_B X).
\]  \(^{24}\)

Now observe that for every right \( A \)-module \( V \) the complex \(^{23}\) for calculation its cohomology as a right \( B \)-module may be represented as follows:
\( \text{Hom}_B(X, V) = \text{Hom}_A(A \otimes_B X, V). \) \hspace{1cm} (25)

Endow the space \( \text{Hom}_A(A \otimes_B X, V) \) with a right \( \text{End}_A(A \otimes_B X) \)–action:

\[
\text{Hom}_A(A \otimes_B X, V) \times \text{End}_A(A \otimes_B X) \to \text{Hom}_A(A \otimes_B X, V), \\
\varphi \times f \mapsto \varphi \circ f, \\
\varphi \in \text{Hom}_A(A \otimes_B X, V), f \in \text{End}_A(A \otimes_B X). \hspace{1cm} (26)
\]

The action is well defined since \( f \) commutes with the left \( A \)–action. Clearly this action respects the gradings, i.e., it is an action of the graded associative algebra on the graded module.

**Theorem 5** The action \((26)\) gives rise to a right action

\[
H^*(V) \times Hk^*(A, B) \to H^*(V), \\
H^n(V) \times Hk^m(A, B) \to H^{n+m}(V). \hspace{1cm} (27)
\]

**Proof.** Let \( \varphi \in \text{Hom}_A(A \otimes_B X, V) \) and \( d\varphi = \varphi \circ d = 0 \). Let also \( f \in \text{End}_A(A \otimes_B X) \) be a homogeneous cocycle. By \((18)\) \( \varphi \circ f \) is a cocycle in \( \text{Hom}_A(A \otimes_B X, V) \). Indeed

\[
d(\varphi \circ f) = \varphi \circ f \circ d = (-1)^{\deg(f)} \varphi \circ d \circ f = 0. \hspace{1cm} (28)
\]

Then we need to show that the action does not depend on the choice of the representative \( f \) in the homology class \([f]\), that is \( \varphi \circ df \) is homologous to zero for every homogeneous \( g \in \text{End}_A(A \otimes_B X) \). This is a direct consequence of the definitions:

\[
\varphi \circ dg = \varphi \circ (d \circ g - (-1)^{\deg(g)} g \circ d) = -(-1)^{\deg(g)} d(\varphi \circ g), \hspace{1cm} (29)
\]

since \( \varphi \circ d = 0 \). Finally let us check that the action is independent of the representative in the homology class \([\varphi]\). For \( \psi \in \text{Hom}_A(A \otimes_B X, V) \) \( d\psi \circ f \) is always homologous to zero:

\[
d\psi \circ f = \psi \circ d \circ f = (-1)^{\deg(f)} \psi \circ f \circ d = (-1)^{\deg(f)} d(\psi \circ f). \hspace{1cm} (30)
\]
This concludes the proof.
Similarly for every right $A$-module $W$ one can equip the homology module $H_*(W)$ with a structure of left $H^k(A,B)$-module. First the complex $W \otimes_B X = W \otimes_A A \otimes_B X$ has a natural structure of left $\text{End}_A(A \otimes_B X)$-module:

\begin{align*}
\text{End}_A(A \otimes_B X) \times W \otimes_A A \otimes_B X &\to W \otimes_A A \otimes_B X, \\
f \times w \otimes x &\mapsto w \otimes f(x), \\
w \otimes x &\in W \otimes_A (A \otimes_B X), f \in \text{End}_A(A \otimes_B X).
\end{align*}

Observe that according to the convention of section 1 elements of $\text{End}_A^n(A \otimes_B X)$ have degree -1 as operators in the graded space $W \otimes_A A \otimes_B X$:

\begin{align*}
\text{End}_A^n(A \otimes_B X) \times W \otimes_A A \otimes_B X_m &\to W \otimes_A A \otimes_B X_{m-n}.
\end{align*}

The following assertion is an analogue of theorem 5 for homology.

**Theorem 6** The action \[ (31) \] gives rise to a left action

\begin{align*}
H^k(A,B)^* \times H_*(W) &\to H_*(W), \\
H^k(A,B)^n \times H_m(W) &\to H_{m-n}(W).
\end{align*}

4 Structure of the Hecke algebras

In this section we investigate the Hecke algebras under some technique assumptions. The main theorem here is

**Theorem 7** Assume that

\begin{equation}
\text{Tor}_n^B(A,K) = 0 \text{ for } n > 0.
\end{equation}

Then

\begin{equation}
H^k(A,B) = \text{Ext}_A^n(A \otimes_B K, A \otimes_B K) = \text{Ext}_B^n(K, A \otimes_B K).
\end{equation}

In particular

\begin{align*}
H^k(A,B) &= 0, n < 0; \\
H^k(0,A,B) &= \text{Hom}_B(K, A \otimes_B K).
\end{align*}
Proof. Equip the complex $Y = End_A(A \otimes T(I(B)) \otimes K)$, which we used in theorem \[ for computation of $Hk^*(A, B)$, with the first filtration as follows:

$$F^k Y = \sum_{n=-\infty}^{\infty} \prod_{p+q=n, p \geq k} Y^{p,q}. \quad (37)$$

The associated graded complex with respect to the filtration is the double direct sum

$$GrY = \sum_{p,q=-\infty}^{\infty} Y^{p,q}. \quad (38)$$

One can show that the filtration is regular and the second term of the corresponding spectral sequence is

$$E_2^{p,q} = H^p_d(H^q_{d'}(GrY)), \quad (39)$$

where $H^*_d$ and $H^*_{d'}$ denote the homologies of the complex with respect to the partial differentials \[.\]

Now observe that at the same time the complex $A \otimes T(I(B)) \otimes K$ is a complex for calculation of $\text{Tor}^B_n(A, K)$ because $A \otimes T(I(B)) \otimes B$ is a free resolution of $A$ as a right $B$–module. It is also free as a left $A$–module. Therefore the functor $\text{Hom}_A(A \otimes T(I(B)) \otimes K, \cdot)$ is exact. By the assumption $H^*(A \otimes T(I(B)) \otimes K) = \text{Tor}^B_0(A, K) = A \otimes_B K$. Using the last two observations we can calculate the cohomology of the complex $GrY$ with respect to the differential $d''$:

$$H^p_{d''}(GrY) = H^p_{d'}(\text{Hom}_A(A \otimes T(I(B)) \otimes K, A \otimes T(I(B)) \otimes K) = \text{Hom}_A(A \otimes T(I(B)) \otimes K, A \otimes_B K). \quad (40)$$

Here $\text{Hom}_A$ should be thought of as the direct sum of the double graded components. Now \[[3] provides that the spectral sequence \[[39] degenerates at the second term. Moreover

$$E_2^{p,*} = H^p_d(H^0_{d'}(GrY)) = H^p_d(A \otimes T(I(B)) \otimes K, A \otimes_B K). \quad (41)$$

But the complex $A \otimes T(I(B)) \otimes K$ may be regarded as a free resolution of the left $A$–module $A \otimes_B K$. Therefore
Finally by theorem 5.12 [4] we have:

\[ H_k^n(A, B) = H^n(Y) = \text{Ext}^n_A(A \otimes_B K, A \otimes_B K). \]  

Since \( \text{Tor}^B_n(A, K) = 0 \) for \( n > 0 \) we can apply Shapiro lemma to simplify the last expression:

\[ \text{Ext}^n_A(A \otimes_B K, A \otimes_B K) = \text{Ext}^n_B(K, A \otimes_B K). \]  

This completes the proof.

Remark 1 In particular the conditions of the theorem are satisfied if \( A \) is projective as a right \( B \)-module. For instance suppose that there exists a subspace \( N \subset A \) such that the multiplication in \( A \) provides an isomorphism of the vector spaces \( A \cong N \otimes B \). Then \( A \) is a free right \( B \)-module.

5 Comparison with the BRST complex

Let \( g \) be a Lie algebra over a field \( K \). For simplicity we suppose that \( g \) is finite–dimensional. However the arguments presented below remain to be true, with some technique modifications, for an arbitrary Lie algebra. We shall apply the construction of section 2 in the following situation.

Let \( A \) be an associative algebra over \( K \) and \( B = U(g) \) be its subalgebra. Note that \( U(g) \) is naturally augmented. Consider the \( U(g) \)-free resolution of the left \( U(g) \)-module \( K \) as follows:

\[
X = U(g) \otimes \Lambda(g), \\
d(u \otimes x_1 \wedge \ldots \wedge x_n) = \sum_{i=1}^n (-1)^{i+1} u.x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n + \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n, \tag{45}
\]

where the symbol \( \hat{x}_i \) indicates that \( x_i \) is to be omitted. Then

\[ A \otimes_{U(g)} X \cong A \otimes \Lambda(g) \tag{46} \]

is a complex with a differential given by the operator
\[ d = \sum_i e_i \otimes e_i^* - \sum_{i,j} 1 \otimes [e_i, e_j] e_i^* e_j^*, \tag{47} \]

Here \( e_i \) is a linear basis of \( g \), \( e_i^* \) is the dual basis; \( e_i \otimes 1 \) being regarded as the operator of right multiplication in \( A \) and \( 1 \otimes e_i, 1 \otimes e_i^* \) as the operators of exterior and inner multiplications in \( \Lambda(g) \) respectively.

Now observe that

\[ \text{End}_A(A \otimes \Lambda(g)) = \text{Hom}_K(\Lambda(g), A^{\text{opp}} \otimes \Lambda(g)) = A^{\text{opp}} \otimes \text{End}_K(\Lambda(g)) = A^{\text{opp}} \otimes C(g + g^*), \tag{48} \]

where \( C(g + g^*) \) is the Clifford algebra of the space \( g + g^* \). Under this identification \( A^{\text{opp}} \) acts on \( A \otimes \Lambda(g) \) by the multiplications in \( A \) from the right and the Clifford algebra acts by the exterior and inner multiplications in \( \Lambda(g) \). This allows to consider the differential (47) as an element of the complex \( A^{\text{opp}} \otimes C(g + g^*) \).

It is easy to see that the canonical \( \mathbb{Z} \) grading of the complex \( A^{\text{opp}} \otimes C(g + g^*) \) coincides mod 2 with the \( \mathbb{Z}_2 \) grading inherited from the Clifford algebra. Therefore according to (47) the differential \( d \) is given by the supercommutator in \( A^{\text{opp}} \otimes C(g + g^*) \) by element (47). This establishes

**Theorem 8** The complex \( (\text{End}_A(A \otimes U(g)) \otimes X, d) \) is isomorphic to the BRST one \( A^{\text{opp}} \otimes C(g + g^*) \) with the differential being the supercommutator by element (47).

### 6 Relation to the quantum reduction

The results of the previous section imply that if \( K \) is the field of complex numbers \( \mathbb{C} \) then \( Hk^0(A, U(g))^{\text{opp}} \) may be thought of as a result of the quantum reduction in \( A \) with \( U(g) \) being a system of the first-class constraints [8]. We shall show that this treatment remains to be true in the general situation of section 2.

Suppose that \( A \) is a quantization of a classical system, that is, \( A \) is included into a family of associative algebras \( A_h \) parametrized by a complex number \( h \) such that for different \( h \) \( A_h \) are isomorphic as vector spaces, \( A_0 \) is commutative and the formula
\{a, b\} = \lim_{h \to 0} \frac{ab - ba}{h}

defines a Poisson algebra structure on \(A_0\). The classical limit of \(B\) is a Poisson subalgebra \(B_0\) in \(A_0\) with a character \(\varepsilon_0 : B_0 \to \mathbb{C}\). Let \(J_0\) be the ideal in \(A_0\) generated by the kernel of the map \(\varepsilon_0\). Then the classical reduced Poisson algebra coincides with the subspace of Poisson \(B_0\)-invariants in the quotient \(A_0/J_0\). In typical situations \(A_0\) is the Poisson algebra of functions on a Poisson manifold. In this case the scheme of the reduction was suggested by Direc in [5].

Now assume that the conditions of theorem [1] are satisfied. Then the algebra \(Hk^0(A, B)_{\text{opp}}\) is isomorphic to the algebra of \(B\)-invariants in the quotient \(A/J\) where \(J\) is the left ideal in \(A\) generated by the kernel of the augmentation map \(\varepsilon\). Thus the classical limit of \(Hk^0(A, B)_{\text{opp}}\) is exactly the reduced Poisson algebra defined above.

If theorem [1] does not hold the algebra \(Hk^0(A, B)_{\text{opp}}\) can be still treated as a quantization of the classical reduced space in the following sense.

Recall the scheme of the Dirac quantum reduction [5]. Suppose again that we are given a quantum system with first-class constraints, that is an associative algebra \(A\) over \(\mathbb{C}\) together with a representation \(V\) and its subalgebra \(B\) equipped with a character \(\varepsilon : B \to \mathbb{C}\). According to Dirac the space of the physical states for the reduced system is the space of \(B\)-invariants in \(V\):

\[ V^B = \{ v \in V : bv = \varepsilon(b)v \text{ for every } b \in B \}, \quad (49) \]

that is the zeroth cohomology space of \(V\) as a left \(B\)-module \(H^0(V) = V^B = \text{Hom}_B(\mathbb{C}, V)\). And the algebra of observables \(A^B_V\) of the reduced system is formed by operators \(a \in A\) such that

\[ [b, a]v = 0 \text{ for every } b \in B, v \in V^B. \quad (50) \]

This condition is equivalent to

\[ bav = \varepsilon(b)av. \quad (51) \]

It means that the space \(V^B\) is invariant with respect to the action of \(A^B_V\). Clearly, such operators form an algebra.
From the other side we have an action \( Hk^0(A, B)^{opp} \) in the space \( V^B \). From the definition of the action it is clear that only \((0, 0)\)-bidegree components give a nontrivial contribution to the action. They may be represented by elements from \( A \). Moreover

**Theorem 9** The algebra \( Hk^0(A, B)^{opp} \) satisfies condition (50) for every representation \( V \) of \( A \). So it may be regarded as a universal Dirac reduction of the physical system.

The statement of the theorem follows directly from theorem [3]. Namely, condition (50) ensures that the action of the algebra \( Hk^0(A, B)^{opp} \) in the space of invariants is well-defined.

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