Permanence of a predator–prey discrete system with Holling-IV functional response and distributed delays

X. Zhang, Z. Wu and T. Zhou

College of Science, Hunan Agricultural University, Changsha, Hunan 410128, People's Republic of China

ABSTRACT
A predator–prey discrete-time model with Holling-IV functional response and distributed delays is investigated in this paper. By using the comparison theorem of the difference equation and some analysis technique, some sufficient conditions are obtained for the permanence of the discrete predator–prey system. Two examples are given to illustrate the feasibility of the obtained result.

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1. Introduction

In population dynamics, a functional response of the predator to the prey is one of important factors impacted on a predator–prey system. The common functional response function, such as Holling I, Holling II and Holling III, are all monotonously non-descending in the first quadrant [11]. But for some predator–prey systems, a non-monotonic response occurs because the growth of predator may be inhibited when the prey density reaches a high level. The Holling-IV functional response function was proposed by Andrews to describe such kind of biological phenomena [1], and its the simplified form of Holling-IV function was suggested by Sokol and Howell in [18] as follows:

\[ \varphi(x) = \frac{cx}{m^2 + x^2}. \]

From then on, many authors have explored the dynamics of predator–prey systems with Holling IV-type functional responses [5, 6, 10–15, 17, 20, 21, 23, 26, 27]. For example, Ruan and Xiao considered the following predator–prey model with Holling-IV functional response [17],

\[
\begin{align*}
\frac{dx}{dt} &= rx(t) \left(1 - \frac{x(t)}{K}\right) - \frac{x(t)y(t)}{a + x^2(t)}, \\
\frac{dy}{dt} &= y(t) \left(\frac{\mu x(t)}{a + x^2(t)} - D\right),
\end{align*}
\]

(1)
where \(x(t)\) and \(y(t)\) represent prey and predator densities, respectively. They made a global qualitative analysis of the model (1) depending on all parameters and showed that the system (1) exhibits numerous kinds of bifurcation phenomena. Their analysis indicated that the predator–prey systems with non-monotonic functional response can possess more complex dynamic properties. The conclusion was also supported by the literature [23].

Experiments showed that there is a time delay between the changes in prey density and the corresponding changes in the predator growth rate [2]. It is reasonable to introduce delay in predator–prey system. Li and Fan considered the existence of a positive periodic solution for the following predator–prey model with Holling-IV functional response and delay [14],

\[
\begin{align*}
\frac{dx}{dt} &= x(t)[a(t) - b(t)x(t)] - \frac{x(t)y(t)}{m^2 + x^2(t)}, \\
\frac{dy}{dt} &= y(t) \left[ \frac{\mu(t)x(t - \tau)}{m^2 + x^2(t - \tau)} - d(t) \right].
\end{align*}
\]

Xia and Han proposed a periodic ratio-dependent predator–prey model with Holling-IV functional response and distributed delay as follows [22],

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t) \left[ a(t) - b(t) \int_{-\infty}^{t} K(t - s)x(s) \, ds - \frac{c(t)y^2(t)}{m^2y^2(t) + x^2(t)} \right], \\
\frac{dy(t)}{dt} &= y(t) \left[ -d(t) + \frac{h(t)x(t - \tau(t))y(t - \tau(t))}{m^2y^2(t - \tau(t)) + x^2(t - \tau(t))} \right].
\end{align*}
\]

Based on Mawhin’s coincidence degree, they showed that the system (3) has two positive periodic solutions under some suitable conditions.

Yang et al. [26] considered the following non-autonomous periodic Holling-type IV predator–prey system with stage-structure,

\[
\begin{align*}
\frac{dx_1}{dt} &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - \frac{p(t)x_1(t)}{k(t) + x_1^2(t)}y(t), \\
\frac{dx_2}{dt} &= c(t)x_1(t) - f(t)x_2^2(t), \\
\frac{dy}{dt} &= y(t) \left[ -g(t) + \frac{h(t)x_1(t)}{k(t) + x_1^2(t)} - q(t)y(t) \right],
\end{align*}
\]

where \(x_1(t)\), \(x_2(t)\) and \(y(t)\) represent the density of immature prey species, mature prey species and predators, respectively. The system (4) was shown to be permanent if and only if the growth of the predator by foraging minus its death rate is positive on average during the period. The condition reflects the periodic characteristics of system (4).

It is well known that discrete population models are essential forms for the modelling and analysis of population ecology system. They are more suitable for describing systems which the populations have non-overlapping generations than the continuous models. Moreover, discrete-time models can produce a much richer dynamics than the continuous-time models. Therefore, many scholars have focused on some discrete population models [4, 5, 8, 9, 15, 16, 24, 25, 27]. But, at present, there is only a few research on discrete predator–prey system with non-monotonic functional response [5, 15, 27]. Chen et al. proposed
the following prey–predator discrete-time model with Holling-IV functional response by applying the Euler forward scheme,

\[
x_1(n + 1) = x_1(n) + \delta x_1(n) \left( 1 - x_1(n) - kx_1^2(n) - \frac{x_2(n)}{m^2 + x_1^2(n)} \right),
\]

\[
x_2(n + 1) = x_2(n) + \delta x_2(n) \left( -\delta_0 - \delta_1 x_2(n) + \frac{y x_1(n)}{m^2 + x_1^2(n)} \right).
\]

By using the centre manifold theorem and bifurcation theory, they showed that the system (5) undergoes a flip bifurcation, a Hopf bifurcation and a saddle-node bifurcation [5]. Lu and Wang considered the following discrete semi-ratio-dependent predator–prey system with Holling-type IV functional response and time delay [15],

\[
x(n + 1) = x(n) \exp \left[ r_1(n) - a_{11}(n)x(n - \tau) - \frac{a_{12}(n)y(n)}{m^2 + x^2(n)} \right],
\]

\[
y(n + 1) = y(n) \exp \left[ r_2(n) - a_{21}(n)\frac{y(n)}{x(n)} \right].
\]

They proved that the system (6) is permanent and globally attractive under some appropriate conditions. Furthermore, they also obtained some sufficient conditions which guarantee the existence and global attractivity of positive periodic solution of system (6).

Motivated by the above, this paper is to investigate the permanence of the following discrete predator–prey system with density dependence, distributed delay and Holling-IV functional response,

\[
x(n + 1) = x(n) \exp \left[ a(n) - b(n) \sum_{l=0}^{+\infty} K(l)x(n - l) - \frac{c(n)y(n)}{m^2 + x^2(n)} \right],
\]

\[
y(n + 1) = y(n) \exp \left[ -d(n) + \frac{h(n)x(n - \tau(n))}{m^2 + x^2(n - \tau(n))} - q(n)y(n) \right] - \tau(n) - \tau(n)
\]

for \( n \in \mathbb{Z}_0^+ \), where \( a, d : \mathbb{Z} \to \mathbb{R}, b, c, h, q : \mathbb{Z} \to \mathbb{R}^+ \) are all real sequences, and \( \tau : \mathbb{Z} \to \mathbb{Z}_0^+ \) is a nonnegative integer sequence, \( m \) is a positive real constant, \( K : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+ \) satisfies \( \sum_{l=0}^{+\infty} K(l) = 1 \), \( \mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}^+, \mathbb{R}, \mathbb{R}^+ \) denote the sets of all integers, nonnegative integers, positive integers, real numbers and positive real numbers, respectively.

The initial conditions associated with Equation (7) are of the form

\[
x(n) = \phi(n), \quad y(n) = \psi(n), \quad n \in \mathbb{Z} - \mathbb{Z}^+,
\]

where \( \phi(0) > 0, \psi(0) > 0 \), and there exists a constant \( M_0 \) such that \( 0 \leq \phi(n) \leq M_0, 0 \leq \psi(n) \leq M_0 \) for \( n \in \mathbb{Z} - \mathbb{Z}_0^+ \).

2. Preliminaries

For convenience, we will use the following notations in the discussion. Let \( \mathbb{C} \) denote the set of all bounded sequence \( f, \mathbb{C}^+ \) is the set of all \( f \in \mathbb{C} \) such that \( f > 0 \), \( \mathbb{P}_\omega \) is the set of all
A \omega \text{ periodic sequence } f : \mathbb{Z} \to \mathbb{R} \text{ such that } f(k + \omega) = f(k) \text{ for any } k \in \mathbb{Z}, C_{\omega} := \mathbb{C} \cap \mathbb{P}_{\omega}, C^+_{\omega} := C^+ \cap \mathbb{P}_{\omega}. \text{ Let }

\tilde{f} := \frac{1}{\omega} \sum_{k=0}^{\omega-1} f(k),

where \( f \in \mathbb{P}_{\omega}. \) Given \( f \in \mathbb{C}, \) we denote

\( f^M = \sup_{k \geq 0} f(k), \quad f^L = \inf_{k \geq 0} f(k). \)

And for any sequence \( f, \) we define \( \sum_{i=m}^{n} f(i) = 0 \) if \( n < m. \)

In our proof of the permanence for system (7), we need the following lemmas.

**Lemma 2.1 ([19]):** Suppose that \( f : \mathbb{Z}^+ \times [0, +\infty) \to [0, +\infty) \) and \( g : \mathbb{Z}^+ \times [0, +\infty) \to [0, +\infty) \) with \( f(n, x) \leq g(n, x)(f(n, x) \geq g(n, x)) \) for \( n \in \mathbb{Z}^+ \) and \( x \in [0, +\infty). \) Assume that \( g(n, x) \) is nondecreasing with respect to argument \( x. \) If \( \{x(n)\} \) and \( \{u(n)\} \) are solutions of

\[ x(n+1) = f(n, x(n)) \quad \text{and} \quad u(n+1) = g(n, u(n)), \]

respectively, and \( x(0) \leq u(0) (x(0) \geq u(0)), \) then

\[ x(n) \leq u(n) \quad (x(n) \geq u(n)) \]

for all \( n \geq 0. \)

Consider the following periodic difference equation:

\[ x(n+1) = x(n) \exp \{A(n) - B(n)x(n)\}, \]
\[ x(0) = x_0 > 0, \]

where \( A \in C_{\omega}, B \in C^+_{\omega}. \) We have the following Lemmas for Equation (9).

**Lemma 2.2 ([7]):** Equation (9) has at least one periodic solution \( x^* \) if \( B \in C^+_{\omega}, A \in C_{\omega} \) and \( \bar{a} > 0, \) moreover, the following properties hold:

(a) \( x^* \) is positive \( \omega \)-periodic;
(b) \( x^* \) is constant if \( A/B \) is constant, in this case, \( x^* = A/B. \)

**Lemma 2.3:** Assume that \( B \in C^+_{\omega}, A \in C_{\omega} \) and \( \bar{A} > 0, \) and \( x^* \) are a positive periodic solution of Equation (9). Then, for any solution \( x(n) \) of Equation (9), one has

\[ \limsup_{n \to \infty} x(n) \leq D, \]

where \( D = (x^*(n))^M \exp\{[B(n)x^*(n)]^M\}. \)

**Proof:** Set \( w(n) = \ln x(n) - \ln x^*(n), \) then we have from Equation (9)

\[ w(n+1) - w(n) = -B(n)x^*(n)[\exp(w(n)) - 1]. \]
Obviously,
\[
\limsup_{n \to \infty} x(n) = \limsup_{n \to \infty} x^*(n) \exp(w(n)). \tag{12}
\]

First, we assume that \(w(n)\) does not oscillate about zero, then \(w(n)\) either will be eventually positive or eventually negative. If the latter holds, we have that
\[
x(n) = x^*(n) \exp(w(n)) < x^*(n) \leq (x^*(n))^M.
\]

Either if the former holds, then from Equation (11), we have \(w(n + 1) < w(n)\), which means that \(w(n)\) is eventually decreasing, also in terms of its positivity \((w(n) > 0)\), we know that \(\lim_{n \to \infty} w(n)\) exists. Then, Equation (11) yields \(\lim_{n \to \infty} w(n) = 0\). And combining it with Equation (12), we have
\[
\limsup_{n \to \infty} x(n) = \limsup_{n \to \infty} x^*(n) \leq (x^*(n))^M.
\]

If \(w(n)\) oscillates about zero, by Equation (11), we know that \(w(n) > 0\) implies \(w(n + 1) \leq w(n)\). Thus, if we let \(\{w(n_k)\}\) be a subsequence of \(\{w(n)\}\), where \(w(n_k)\) is the first element of the positive semi-cycle of \(\{w(n)\}\), then \(\limsup_{n \to \infty} w(n) = \limsup_{k \to \infty} w(n_k)\).

Combining
\[
w(n_k) = w(n_k - 1) - B(n_k - 1)x^*(n_k - 1)[\exp(w(n_k - 1)) - 1]
\]
with \(w(n_k - 1) < 0\), we have
\[
w(n_k) \leq B(n_k - 1)x^*(n_k - 1)[1 - \exp(w(n_k - 1))]
\leq B(n_k - 1)x^*(n_k - 1)
\leq [B(n)x^*(n)]^M.
\]

Therefore,
\[
\limsup_{k \to \infty} w(n_k) \leq [B(n)x^*(n)]^M.
\]

Then, we have \(\limsup_{n \to \infty} x(n) \leq D\). This completes the proof.

**Lemma 2.4:** Assume that \(B \in \mathbb{C}^{+\omega}, A \in \mathbb{C}_\omega\) and \(\bar{A} > 0\), \(x(n)\) and \(x^*\) are any solution and a positive periodic solution of Equation (9), respectively, and \(\limsup_{n \to \infty} x(n) \leq D\). Then, one has
\[
\liminf_{n \to \infty} x(n) \geq d_1, \tag{13}
\]
where \(d_1 = [x^*(n)]^L \exp\{[B(n)x^*(n)]^L - D[B(n)]^M\}\).

**Proof:** Set \(w(n) = \ln x(n) - \ln x^*(n)\), then we have from Equation (9)
\[
w(n + 1) - w(n) = -B(n)x^*(n)[\exp(w(n)) - 1]. \tag{14}
\]

Obviously,
\[
\liminf_{n \to \infty} x(n) = \liminf_{n \to \infty} x^*(n) \exp(w(n)). \tag{15}
\]

First, we assume that \(w(n)\) does not oscillate about zero, then \(w(n)\) either will be eventually positive or eventually negative. If the latter holds, then, by Equation (14), we have
Therefore, \( \lim_{n \to \infty} x(n) = \lim_{n \to \infty} x^*(n) \geq (x^*(n))^L. \)

If the former holds, that is \( u(n) > v(n) \), we have
\[
x(n) = x^* \exp w(n) > x^*(n) \geq (x^*(n))^L.
\]

Now we suppose that \( w(n) \) oscillates about zero, by Equation (14), we know that \( w(n) < 0 \) implies \( w(n+1) \geq w(n) \). Thus, if we let \( \{w(n_k)\} \) be a subsequence of \( \{w(n)\} \), where \( w(n_k) \) is the first element of the negative semi-cycle of \( \{w(n)\} \), then \( \liminf_{n \to \infty} w(n) = \liminf_{k \to \infty} w(n_k) \).

Combining
\[
w(n_k) = w(n_k - 1) - B(n_k - 1)x^*(n_k - 1)[\exp(w(n_k - 1)) - 1]
\]
with \( w(n_k - 1) > 0 \), we have
\[
w(n_k) > -B(n_k - 1)x^*(n_k - 1)[\exp(w(n_k - 1)) - 1]
\]
\[
= -B(n_k - 1)[x^*(n_k - 1) \exp(w(n_k - 1)) - x^*(n_k - 1)]
\]
\[
= B(n_k - 1)x^*(n_k - 1) - B(n_k - 1)x^*(n_k - 1) \exp(w(n_k - 1))
\]
\[
= B(n_k - 1)x^*(n_k - 1) - B(n_k - 1)x(n_k - 1)
\]
\[
\geq [B(n)x^*(n)]^L - [B(n)]^MD.
\]

Therefore,
\[
\liminf_{k \to \infty} w(n_k) \geq [B(n)x^*(n)]^L - D[B(n)]^M.
\]

Then, we have from Equation (15)
\[
\liminf_{n \to \infty} x(n) = \limsup_{n \to \infty} x^*(n) \exp(w(n))
\]
\[
\geq (x^*(n))^L \exp\{[B(n)x^*(n)]^L - D[B(n)]^M\}.
\]

Therefore, \( \liminf_{n \to \infty} x(n) \geq d_1 \), where
\[
d_1 = [x^*(n)]^L \exp\{[B(n)x^*(n)]^L - D[B(n)]^M\}.
\]

This completes the proof.

The following two lemmas are special cases of [3, Lemmas 2.2 and 2.3] when \( \theta = 1 \), respectively.

**Lemma 2.5:** Assume that \( x(n) \) satisfies
\[
x(n + 1) \leq x(n) \exp\{a(n) - b(n)x(n)\} \quad \forall n \geq n_0,
\]
where \( \{a(n)\} \) and \( \{b(n)\} \) are positive sequences, \( x(n_0) > 0 \), and \( n_0 \in \mathbb{N} \). Then, one has
\[
\limsup_{n \to \infty} x(n) \leq D,
\]
where \( D = 1/b^L \exp(a^M - 1). \)
Lemma 2.6: Assume that \( x(n) \) satisfies
\[
    x(n+1) \geq x(n) \exp\{a(n) - b(n)x(n)\} \quad \forall n \geq n_0,
\]
where \( \{a(n)\} \) and \( \{b(n)\} \) are positive sequences, \( x(n_0) > 0 \), and \( n_0 \in \mathbb{N} \). Also, \( \limsup_{n \to \infty} x(n) \leq D \) and \( b^MD/a^l > 1 \), Then, one has
\[
    \liminf_{n \to \infty} x(n) \geq d,
\]
where \( d = a^l/b^M \exp(a^l - b^M D) \).

3. Permanence

We will discuss the permanent of system (7) in the following two cases: (a) positive coefficients; (b) periodic coefficients, respectively. To this end, we suppose that the coefficients satisfy the assumptions (H1) or (H2) as follows.

(H1) \( a, b, c, d, h, q, \tau \in \mathbb{C}^+ \).
(H2) \( a, d \in \mathbb{C}_0^+, c, b, h, q, \tau \in \mathbb{C}_0^+ \).

Proposition 3.1: Suppose (H1) holds, or (H2) and \( \tilde{a} > 0 \) hold. Then, for any positive solution \( \{x(n), y(n)\} \) of system (7), there exists a positive constant \( M_1 \) such that \( \limsup_{n \to \infty} x(n) \leq M_1 \).

Proof: From the first equation of system (7), we have
\[
    x(n+1) \leq x(n) \exp\left[a(n) - b(n) \sum_{l=0}^{+\infty} K(l)x(n-l)\right]. \tag{20}
\]
Set \( u(n) = \ln x(n) \), then
\[
    u(n+1) - u(n) \leq a(n) - b(n) \sum_{l=0}^{+\infty} K(l) \exp(u(n-l)). \tag{21}
\]
Summing at both sides of inequality (21), we obtain
\[
    \sum_{i=n-l}^{n-1} (u(i+1) - u(i)) \leq \sum_{i=n-l}^{n-1} \left[a(i) - b(i) \sum_{l=0}^{+\infty} K(l) \exp(u(i-l))\right] \leq \sum_{i=n-l}^{n-1} a(i).
\]
That is \( u(n) - u(n-l) \leq \sum_{i=n-l}^{n-1} a(i) \), then \( u(n-l) \geq u(n) - \sum_{i=n-l}^{n-1} a(i) \). So we obtain
\[
    \exp(u(n-l)) \geq \exp(u(n)) \exp\left[-\sum_{i=n-l}^{n-1} a(i)\right],
\]
where \( l = 0, 1, 2, \ldots \). Multiplied by \( K(l) \) and summing at both sides,
\[
    \sum_{l=0}^{+\infty} K(l) \exp(u(n-l)) \geq \exp(u(n)) \sum_{l=0}^{+\infty} K(l) \exp\left[-\sum_{i=n-l}^{n-1} a(i)\right].
\]
That is
\[ \sum_{l=0}^{+\infty} K(l) x(n - l) \geq x(n) \sum_{l=0}^{+\infty} K(l) \exp \left( - \sum_{i=n-l}^{n-1} a(i) \right). \]

Then, from Equation (20), we obtain
\[ x(n + 1) \leq x(n) \exp \left\{ a(n) - b(n) x(n) \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} a(i) \right] \right\}. \quad (22) \]

If the assumption \((H1)\) holds, then \(b(n) \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} a(i) \right] \in \mathbb{C}^+, \ a \in \mathbb{C}^+. \) From Lemma 2.5, we have \(\limsup_{n \to \infty} x(n) \leq M_1,\) where
\[ M_1 = \frac{1}{b(n) \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} a(i) \right]} \exp (a^M - 1). \quad (23) \]

If the assumption \((H2)\) and \(\bar{a} > 0\) hold, then \(b(n) \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} a(i) \right] \in \mathbb{C}_\omega^+, \ a \in \mathbb{C}_\omega, \) and \(\bar{a} > 0.\) We consider the following auxiliary equation:
\[ z(n + 1) = z(n) \exp \left\{ a(n) - b(n) z(n) \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} a(i) \right] \right\}. \quad (24) \]

From Lemma 2.2, Equation (24) exists a positive periodic solution \(z^*(n).\) Then, by Lemma 2.3, we have \(\limsup_{n \to \infty} z(n) \leq M_1,\) where
\[ M_1 = (z^*(n))^M \exp \left\{ b(n) \sum_{l=0}^{+\infty} K(l) \exp \left( - \sum_{i=n-l}^{n-1} a(i) \right) z^*(n) \right\}. \quad (25) \]

Therefore, we obtain \(\limsup_{n \to \infty} x(n) \leq \limsup_{n \to \infty} z(n) \leq M_1\) by Lemma 2.1. This completes the proof. \[ \square \]

**Proposition 3.2:** Suppose \((H1)\) and \(-d(n) + h(n)/(2m) > 0\) hold, or \((H2)\) and \(-\bar{a} + \bar{h}/(2m) > 0\) hold. Then, for any positive solution \((x(n), y(n))\) of Equation (7), there exists a positive constant \(M_2\) such that \(\limsup_{n \to \infty} y(n) \leq M_2.\)

**Proof:** From the second equation of system (7), we have
\[ y(n + 1) \leq y(n) \exp \left[ -d(n) + \frac{h(n)}{2m} - q(n)y(n) \right]. \quad (26) \]

If the assumption \((H1)\) and \(-d(n) + h(n)/(2m) > 0\) hold, by Lemma 2.5, we have \(\limsup_{n \to \infty} y(n) \leq M_2,\) where
\[ M_2 = \frac{1}{[q(n)]^L} \exp \left\{ -d(n) + \frac{h(n)}{2m} \right\}^M - 1 \quad (27) \]
If the assumption (H2) and $-\bar{d} + \bar{h}/2m > 0$ hold, we consider the following auxiliary equation:
\[
u(n + 1) = \nu(n) \exp \left[ -d(n) + \frac{h(n)}{2m} - q(n)\nu(n) \right]. \tag{28}\]

By Lemma 2.2, Equation (28) has at least one positive $\omega$-periodic solution, denoted as $\nu^*(n)$. According to Lemma 2.3, we have $\limsup_{n \to \infty} \nu(n) \leq M_2$, where
\[
M_2 = (\nu^*(n))^M \exp \left\{ [q(n)\nu^*(n)]^M \right\}. \tag{29}\]

Therefore, according to Lemma 2.1, we obtain $\limsup_{n \to \infty} y(n) \leq \limsup_{n \to \infty} \nu(n) \leq M_2$. This completes the proof. \[\square\]

To the constants $M_1$ and $M_2$ in Propositions 3.1 and 3.2, we introduce the following conditions.

(H3) For any fixed integer $n > 0$,
\[
G(n) = \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} \left( a(i) - (M_1 + \epsilon)b(i) - \frac{M_2 c(i)}{m^2} \right) \right] < +\infty. \tag{30}\]

(H4) $(b(n)G(n))^M M_1 > [a(n) - M_2 c(n)/m^2]^L > 0$.

(H5) $\bar{a} - M_2 \bar{c}/m^2 > 0$.

**Proposition 3.3:** Suppose (H1), (H3), (H4) and $-d(n) + h(n)/(2m) > 0$, or (H2), (H3), (H5) and $-\bar{d} + \bar{h}/(2m) > 0$ hold. Then, for any positive solution $(x(n), y(n))$ of Equation (7), there exists a positive constant $m_1$ such that $\liminf_{n \to \infty} x(n) \geq m_1$.

**Proof:** From assumptions (H3)–(H5), there exists a number $\epsilon > 0$ small enough such that
\[
G(n, \epsilon) = \sum_{l=0}^{+\infty} K(l) \exp \left[ - \sum_{i=n-l}^{n-1} \left( a(i) - (M_1 + \epsilon)b(i) - \frac{(M_2 + \epsilon c(i)}{m^2} \right) \right] < +\infty, \tag{30}\]

$(b(n)G(n, \epsilon))^M M_1 > \left[ a(n) - \frac{(M_2 + \epsilon)c(n)}{m^2} \right]^L > 0$ \tag{31}\]

and
\[
\bar{a} - \frac{(M_2 + \epsilon)\bar{c}}{m^2} > 0. \tag{32}\]

Assumption (H5) implies $\bar{a} > 0$. By Propositions 3.1 and 3.2, we have $\limsup_{n \to \infty} x(n) \leq M_1$ and $\limsup_{n \to \infty} y(n) \leq M_2$. Therefore, for the If, there exists a positive integer $N_0 > 0$
such that
\[ x(n) < M_1 + \varepsilon, y(n) < M_2 + \varepsilon \]
for all \( n > N_0 \). From the first equation of system (7), we have
\[ x(n + 1) \geq x(n) \exp \left[ a(n) - (M_1 + \varepsilon) b(n) - \frac{(M_2 + \varepsilon) c(n)}{m^2} \right] \] (33)
for \( n > N_0 \). Set \( u(n) = \ln x(n) \), then
\[ u(n + 1) - u(n) \geq a(n) - (M_1 + \varepsilon) b(n) - \frac{(M_2 + \varepsilon) c(n)}{m^2} \] (34)
which yields
\[ \sum_{i=n-l}^{n-1} (u(i+1) - u(i)) \geq \sum_{i=n-l}^{n-1} \left[ a(i) - (M_1 + \varepsilon) b(i) - \frac{(M_2 + \varepsilon) c(i)}{m^2} \right]. \]
That is \( u(n) - u(n-l) \geq \sum_{i=n-l}^{n-1} [a(i) - (M_1 + \varepsilon) b(i) - (M_2 + \varepsilon) c(i)/m^2] \), then
\[ u(n-l) \leq u(n) - \sum_{i=n-l}^{n-1} \left[ a(i) - (M_1 + \varepsilon) b(i) - \frac{(M_2 + \varepsilon) c(i)}{m^2} \right]. \]
So we obtain
\[ \exp(u(n-l)) \leq \exp(u(n)) \exp \left\{ - \sum_{i=n-l}^{n-1} \left[ a(i) - (M_1 + \varepsilon) b(i) - \frac{(M_2 + \varepsilon) c(i)}{m^2} \right] \right\}, \]
where \( l = 0, 1, 2, \ldots \). From Equation (30), multiplied by \( K(l) \) and summing at both sides,
\[ \sum_{l=0}^{+\infty} K(l) \exp(u(n-l)) \leq \exp(u(n)) \sum_{l=0}^{+\infty} K(l) \exp \left\{ - \sum_{i=n-l}^{n-1} \left[ a(i) - (M_1 + \varepsilon) b(i) - \frac{(M_2 + \varepsilon) c(i)}{m^2} \right] \right\}. \]
Then, we obtain that
\[ \sum_{l=0}^{+\infty} K(l)x(n-l) \leq G(n, \varepsilon)x(n). \]
Substituting it into Equation (33), we obtain
\[ x(n + 1) \geq x(n) \exp \left\{ a(n) - \frac{(M_2 + \varepsilon) c(n)}{m^2} - b(n)G(n, \varepsilon)x(n) \right\}. \] (35)
If the assumptions (H1), (H3) and (H4) hold, inequalities (30) and (31) hold. By Lemma 2.6, we have $\liminf_{n \to \infty} x(n) \geq m_1(\varepsilon)$, where
\[
m_1(\varepsilon) = \frac{[a(n) - (M_2 + \varepsilon)c(n)/m^2]^L}{[b(n)G(n, \varepsilon)]^M} \exp \left\{ \left[ a(n) - \frac{(M_2 + \varepsilon)c(n)}{m^2} \right]^L - [b(n)G(n, \varepsilon)]^M \right\}.
\]

Let $\varepsilon \to 0$, we have
\[
\liminf_{n \to \infty} x(n) \geq m_1,
\]
where
\[
m_1 = \frac{[a(n) - (M_2c(n)/m^2)]^L}{[b(n)G(n)]^M} \exp \left\{ \left[ a(n) - \frac{M_2c(n)}{m^2} \right]^L - [b(n)G(n)]^M \right\}.
\] (36)

If the assumptions (H2), (H3) and (H5) hold, then inequalities (30) and (32) hold. We consider the following auxiliary equation:
\[
v(n + 1) = v(n) \exp \left\{ a(n) - \frac{(M_2 + \varepsilon)c(n)}{m^2} - b(n)G(n, \varepsilon)v(n) \right\}.
\] (37)

From Equation (32), we have $\bar{a}(n) - (M_2 + \varepsilon)c(n)/m^2 = \bar{a} - (M_2 + \varepsilon)\bar{c}/m^2 > 0$. According to Lemma 2.2, we know that Equation (37) has at least one positive $\omega$-periodic solution $v^*(n, \varepsilon)$. By Lemma 2.1 and Proposition 3.1, $\limsup_{n \to \infty} v(n) \leq \limsup_{n \to \infty} x(n) \leq M_1$. So, by Lemma 2.4, $\liminf_{n \to \infty} v(n) \geq m_1(\varepsilon)$, where
\[
m_1(\varepsilon) = (v^*(n, \varepsilon))^L \exp\{[b(n)G(n, \varepsilon)v^*(n, \varepsilon)]^L - M_1[b(n)G(n, \varepsilon)]^M\}.
\]

Therefore, by Lemma 2.1, we obtain that
\[
\liminf_{n \to \infty} x(n) \geq \liminf_{n \to \infty} v(n) \geq m_1.
\]

Let $\varepsilon \to 0$, we have
\[
\liminf_{n \to \infty} x(n) \geq m_1,
\]
where
\[
m_1 = (v^*(n, 0))^L \exp\{[b(n)G(n)v^*(n, 0)]^L - M_1[b(n)G(n)]^M\}.
\] (38)

This completes the proof. ■

To the constant $M_1$, $m_1$ in Propositions 3.1 and 3.3, we introduce the following assumptions (H6) and (H7).

(H6) $q^M M_2 > [-d(n) + h(n)m_1/(m^2 + M_1^2)]^L > 0$.
(H7) $-\tilde{d} + \tilde{h}m_1/(m^2 + M_1^2) > 0.$
Proposition 3.4: Suppose that (H1), (H3), (H4) and (H6) hold, or (H2), (H3), (H5) and (H7) hold, then for any positive solution \((x(n), y(n))\) of Equation (7), there exists a positive constant \(m_2\) such that \(\liminf_{n \to \infty} y(n) \geq m_2\).

Proof: If the condition (H6) holds, there exists a constant \(\varepsilon > 0\) small enough such that
\[
q^M M_2 > \left[ -d(n) + \frac{h(n)(m_1 - \varepsilon)}{m^2 + (M_1 + \varepsilon)^2} \right]^L > 0. \tag{39}
\]
Or if the condition (H7) holds, there also exists a small enough constant \(\varepsilon > 0\) such that
\[
-\tilde{d} + \frac{\tilde{h}(m_1 - \varepsilon)}{m^2 + (M_1 + \varepsilon)^2} > 0. \tag{40}
\]
If assumption (H7) holds, then we have
\[
-\tilde{d} + \frac{\tilde{h} m_1}{2mM_1} \geq -\tilde{d} + \frac{\tilde{h} m_1}{m^2 + M_1^2} > 0.
\]
Similarly, if (H6) holds, then \(-d(n) + h(n)/(2m) > 0\).

According to Propositions 3.1 and 3.3, for the above constant \(\varepsilon\), there exists an integer \(N_1 > t^M\) large enough such that
\[
m_1 - \varepsilon \leq x(n) \leq M_1 + \varepsilon,
\]
when \(n > N_1\). Therefore, from the second equation of system (7), we have
\[
y(n + 1) \geq y(n) \exp \left[ -d(n) + \frac{h(n)(m_1 - \varepsilon)}{m^2 + (M_1 + \varepsilon)^2} - q(n)y(n) \right]. \tag{41}
\]
If the assumptions (H1) and (H6) hold, then inequality (39) holds, and

\[
-d(n) + \frac{h(n)}{2m} \geq -d(n) + \frac{h(n)m_1}{m^2 + M_1^2} > 0.
\]

By Proposition 3.2, we have that \(\limsup_{n \to \infty} y(n) \leq M_2\). By Lemma 2.6, we have \(\liminf_{n \to \infty} y(n) \geq m_2(\varepsilon)\), where
\[
m_2(\varepsilon) = \frac{[-d(n) + (h(n)(m_1 - \varepsilon))/((m + \varepsilon)^2 + (M_1 + \varepsilon)^2))]^L}{[q(n)]^M} \exp \left\{ \left[ -d(n) + \frac{h(n)(m_1 - \varepsilon)}{(m + \varepsilon)^2 + (M_1 + \varepsilon)^2} \right]^L - [q(n)]^M M_2 \right\}. \]

Let \(\varepsilon \to 0\), it follows that \(\liminf_{n \to \infty} y(n) \geq m_2\), where
\[
m_2 = \lim_{\varepsilon \to 0} m_2(\varepsilon) = \frac{[-d(n) + (h(n)m_1/(m^2 + M_1^2))]^L}{[q(n)]^M} \times \exp \left\{ \left[ -d(n) + \frac{h(n)m_1}{m^2 + M_1^2} \right]^L - [q(n)]^M M_w \right\}. \tag{42}
\]
If the assumptions (H2) and (H7) hold, then inequality (40) holds. We consider the following auxiliary equation:

\[ w(n + 1) = w(n) \exp \left[ -d(n) + \frac{h(n)(m_1 - \varepsilon)}{m^2 + (M_1 + \varepsilon)^2} - q(n)w(n) \right]. \tag{43} \]

According to Lemma 2.2, Equation (43) has at least one positive \( \omega \)-periodic solution, denoted as \( w^*(n) \). Then, by Lemma 2.1, we have \( \limsup_{n \to \infty} w(n) \leq \limsup_{n \to \infty} y(n) \leq M_2 \). Hence, by Lemma 2.4, \( \liminf_{n \to \infty} w(n) \geq m_2 \), where

\[ m_2 = (w^*(n))^L \exp \left\{ [q(n)w^*]^L - M_2[q(n)]^M \right\}. \]

Therefore, by Lemma 2.1, we obtain that

\[ \liminf_{n \to \infty} y(n) \geq \liminf_{n \to \infty} w(n) \geq m_2. \]

This completes the proof. \( \blacksquare \)

**Theorem 3.5:** Suppose (H1), (H3), (H4) and (H6) hold, or (H2), (H3), (H5) and (H7) hold. Then, system (7) is permanent.

**Proof:** From Propositions 3.1 to 3.4, there exist positive constants \( m_1, m_2, M_1 \) and \( M_2 \) such that

\[ 0 < m_1 \leq \liminf_{n \to \infty} x(n) \leq M_1, \quad 0 < m_2 \leq \liminf_{n \to \infty} y(n) \leq M_2. \]

Therefore, system (7) is permanent. \( \blacksquare \)

**4. An example**

**Example 1:** In the system (7), let \( a(n) = 0.9 + 0.01 \sin(\sqrt{5}\pi n), b(n) = 1.4 + 0.1 \cos(\sqrt{2}\pi n), c(n) = 0.05 + 0.01 \sin(\sqrt{3}\pi n), d(n) = 0.01 + 0.01 \cos(\sqrt{5}\pi n), h(n) = 4.5 + 0.5 \sin(\sqrt{\pi}n), q(n) = 0.7 + 0.1 \cos(\frac{\pi}{2}n), \tau(n) = 3 \) and \( m = 4 \). Obviously, they are positive periodic sequences. And let the time delay kernel sequence \( K(n) = (1 - \exp(-1)) \exp(-n) \), which satisfies \( \sum_{n=0}^{+\infty} K(n) = 1 \).

It is easy to obtain that \( a^M \approx 0.91, a^L \approx 0.89, b^M \approx 1.5, b^L \approx 1.3, c^M \approx 0.06, c^L \approx 0.04, d^M \approx 0.03, d^L \approx 0.01, h^M \approx 5, h^L \approx 4, q^M \approx 0.8000 \) and \( q^L \approx 0.65 \). From Equations (23), (27), (36), (42), we have \( M_1 \approx 0.9455, M_2 \approx 1.0468, m_1 \approx 0.8536, m_2 \approx 0.1106 \). Noted that

\[
K(l) \exp \left[ -\sum_{i=n-l}^{n-1} \left( a(i) - M_1 b(i) - \frac{M_2 c(i)}{m^2} \right) \right]
= (1 - \exp(-1)) \exp \left[ -l - \sum_{i=n-l}^{n-1} \left( a(i) - M_1 b(i) - \frac{M_2 c(i)}{m^2} \right) \right]
\leq (1 - \exp(-1)) \exp \left[ \left( M_1 b^M + \frac{M_2 c^M}{m^2} - a^L - 1 \right) l \right],
\]
hence, if \( M_1b^M + M_2c^M/m^2 - a^L - 1 < 0 \), then \( G(n) < +\infty \). In the example, \( M_1b^M + M_2c^M/m^2 - a^L - 1 \approx -0.4678 \), it follows that assumption (H3) holds. Furthermore, by calculation, we have that

\[
\left( a(n) - \frac{M_2c(n)}{m^2} \right)^L \approx 0.8861 > 0, \\
(b(n)G(n))^M_1 - \left( a(n) - \frac{M_2c(n)}{m^2} \right)^L \approx 0.0487 > 0, \\
\left( -d(n) + \frac{m_1h(n)}{m^2 + M_1^2} \right)^L \approx 0.1722 > 0, \\
q^MM_2 - \left( -d(n) + \frac{h(n)m_1}{m^2 + M_1^2} \right)^L \approx 0.6653 > 0,
\]

which imply that assumptions (H4) and (H6) are also satisfied. Therefore, by Theorem 3.5, system (7) is permanent. The result is verified by numerical simulation in Figure 1.

**Example 2:** In system (7), let the time delay kernel sequence \( K(0) = 1, K(n) = 0(n > 0) \), and coefficient \( b(n) = a(n)/Z^* \) where \( Z^* = 0.8 \), then system (7) can be rewritten as follows:

\[
x(n + 1) = x(n) \exp \left[ a(n) - \frac{a(n)}{Z^*} x(n) - \frac{c(n)y(n)}{m^2 + x^2(n)} \right], \\
y(n + 1) = y(n) \exp \left[ -d(n) + \frac{h(n)x(n - \tau(n))}{m^2 + x^2(n - \tau(n))} - q(n)y(n) \right].
\]

The auxiliary equation (24) can also be done as follows:

\[
z(n + 1) = z(n) \exp \left\{ a(n) - \frac{a(n)}{Z^*} z(n) \right\}.
\]

Let \( a(n) = 1 + 0.6 \sin(\frac{7}{5}\pi n), d(n) = 0.00002 + 0.00001 \cos(\frac{7}{5}\pi n), h(n) = 3 + \cos(\frac{7}{5}\pi n), \tau(n) = 3, m = 4 \) and \( q(n) = (h(n)/(2m) - d(n))/U^* \), where \( U^* = 0.5 \).

By Lemma 2.2, Equations (45) and (28) have a positive periodic solution \( \{z^*(n) = Z^*\}, \{u^*(n) = U^*\} \), respectively. Therefore, from Equations (25) and (29), we have \( M_1 \approx 3.8478 \) and \( M_2 \approx 3.6944 \).

Then, let \( c(n) = (m^2/M_2)(a(n) - V^*b(n)) \), where \( V^* = 0.2 \). Thus, the auxiliary equation (37) can be rewritten as follows:

\[
u(n + 1) = v(n) \exp\{V^*b(n) - b(n)v(n)\}.
\]

Obviously, Equation (46) has a positive periodic solution \( \{v^*(n) = V^*\} \). Hence, we obtain \( m_1 = 1.1665 \times 10^{-4} \) from Equation (38).

It is easy to obtain \( \bar{a} - M_2\bar{c}/m^2 \approx 0.25 > 0, -\bar{d} + hM_1/(m^2 + M_1^2) \approx 2.1406 \times 10^{-6} > 0 \), which imply that assumptions (H5) and (H7) are satisfied. Obviously, assumption (H3) holds by view of \( G(n) = 1 \). Therefore, by Theorem 3.5, system (7) is permanent. The result is verified by numerical simulation in Figure 2.
Figure 1. Numerical solutions \((x(n), y(n))\) of Example 1 with 10 pairs of random values as initial values.

Figure 2. Numerical solutions \((x(n), y(n))\) of system (44) with 10 pairs of random values as initial values.

5. Conclusions

In this paper, a predator–prey discrete-time model with discrete distributed delays and Holling-IV functional response is investigated. By using the comparison theorem of the difference equation and some analysis technique, we set up two groups of conditions to ensure the permanence of the system (7), a set of conditions, (H1), (H3), (H4) and (H6),
services the case of variable coefficients, another set of conditions, (H2), (H3), (H5) and (H7), utilizes the periodic characteristics of the system. Under the case of periodic coefficients, (H5) is weaker than (H4), and (H7) is also weaker than (H6). We do not consider the stability of the system, but Figures 1 and 2 all give a hint that the system may be the global stability. We will continue to study the stability of the system in the future work.

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