Bayes factors for longitudinal model assessment via power posteriors

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Abstract
Bayes factor, defined as the ratio of the marginal likelihood functions of two competing models, is the natural Bayesian procedure for model selection. Marginal likelihoods are usually computationally demanding and complex. This scenario is particularly cumbersome in linear mixed models (LMMs) because marginal likelihood functions involve integrals of large dimensions determined by the number of parameters and the number of random effects, which in turn increase with the number of individuals in the sample. The power posterior is an attractive proposal in the context of the Markov chain Monte Carlo algorithms that allows expressing marginal likelihoods as one-dimensional integrals over the unit range. This paper explores the use of power posteriors in LMMs and discusses their behaviour through two simulation studies and a real data set on European sardine landings in the Mediterranean Sea.

1 Introduction

Model selection is a key issue in parametric statistics that has generated a large scientific literature. Several proposals have been made from different perspectives to carry out model comparison. This is a multifaceted topic with a high philosophical content that will undoubtedly continue to improve scientific knowledge in the future.

Bayesian methodology uses a conception of probability that allows assigning probability distributions to any kind of quantity with uncertainty, in particular parameters, hyperparameters, and probabilistic models. The posterior distributions of these quantities are the basis of the natural Bayesian procedure for model selection through the posterior distribution for each of the candidate models. When comparing two competing models, the ratio of their corresponding posterior distributions (posterior odds) is obtained through the product of the prior odds and the Bayes factor, the latter defined
in terms of the ratio of the marginal likelihood of each of the two models (Kass and Raftery, 1995; Berger and Pericchi, 1996). This is a conceptually simple and powerful procedure with a key limitation in that it is indeterminate when working with improper prior distributions, i.e., those distributions that do not integrate to unity.

Longitudinal data are observations of one or more variables measured over time on each of the individuals in the study. They include observations between and within individuals that allow the assessment of general patterns of the target population as well as specific individual characteristics. They are multivariate, clustered, and repeated measures data. Data among individuals are commonly assumed to be independent, whilst the repeated measurements within each subject are correlated (Hedeker and Gibbons, 2006). Linear mixed effects models (LMMs) (Laird and Ware, 1982; Pinheiro and Bates, 2000) constitute a flexible and powerful tool for the analysis of longitudinal data within the normal distribution framework.

The computation of marginal likelihoods for assessing Bayes factors can be demanding and complex. In fact, this situation is exacerbated in models such as LMMs. The marginal likelihood function for these models involves the integral with respect to the prior distribution of the conditional likelihood function that depends on both the parameters and the random effects. The dimension of this integral increases with the size of the random effects set that, in turn, increases with the number of individuals in the sample.

The challenge in calculating the marginal likelihoods in these models has produced a large literature and a fruitful scientific debate, which in turn has generated different proposals for its computation. All of them have interesting properties, but none of them has managed to close the issue definitively, mainly due to computational problems. Although it is not our aim to present in this paper an exhaustive list of all of the proposals, we would like to comment very briefly on some of the most popular ones. One of the first procedures to estimate marginal likelihoods was through the Laplace’s method (Tierney and Kadane, 1986). Subsequently, Newton and Raftery (1994) expressed marginal likelihoods through the posterior harmonic mean of the likelihood, a simple but computationally unstable approach (Raftery et al., 2007). Chib (1995) and Chib and Jeliazkov (2001) developed an algorithm which computes marginal likelihoods from the Markov chain Monte Carlo (MCMC) outputs by making use of additional iterations. Importance sampling ideas have been proposed for estimating the evidence such as annealed importance sampling (Neal, 2001) or bridge sampling (Meng and Wong, 1996). Nested sampling proposed by Skilling (2006) represents marginal likelihoods in terms of one-dimensional integral over [0, 1]. This is an interesting approach based on simulated values from the prior distribution subject to constraints in the conditional likelihood but it involves many challenges when working with multidimensional models and prior distributions poorly informative.

The power posterior (Lartillot and Philippe, 2006; Friel and Pettitt, 2008, and Friel et al., 2014)) is a proposal developed in the context of MCMC algorithms in which the logarithm of the marginal likelihood is evaluated numerically through the path sampling algorithm of Gelman and Meng (1998), also called thermodynamic in-
tegration, to compute ratios of normalising constants.

Our paper generalises the use of power posterior for approximating marginal likelihoods into complex models which include random effects and serial correlation terms in their conditional formulations. In particular, we extend the use of power posteriors to LMMs, although it can be easily generalised to any other models with random effects, and we discuss its behaviour through two simulated studies and a real study on European sardine (Sardina pilchardus, Walbaum, 1792) landings in the Mediterranean Sea. The data is available at [https://github.com/gcalvobayarri/Bayes_factor_longitudinal_models.git](https://github.com/gcalvobayarri/Bayes_factor_longitudinal_models.git).

This paper is organised as follows. Section 2 introduces the basic Bayesian linear mixed models (BLMMs) and describes two popular generalisations based on autoregressive terms. Section 3 reviews Bayes factors and marginal likelihoods and presents the power posterior for models with random effects. Section 4 assesses the behaviour of the power posterior in two simulated longitudinal studies. The first one deals with data generated from a simple LMM model that competes with three basic models. The second databank was simulated from a model with a high level of complexity, in the sense that it includes three types of random variation: random effects, an autoregressive term and normally distributed measurement errors. This model is compared with two simpler competing models with only two of the three sources of variation discussed above. Section 5 is devoted to the selection of a Bayesian longitudinal model to analyse sardine fisheries in different countries of the Mediterranean Sea. The paper concludes with a small discussion emphasising the usefulness of the power posterior methodology in the calculation of marginal likelihoods for comparing LMMs, and highlights the most interesting results of the three studies carried out in the work. Finally, Appendix collects the full conditionals of the Gibbs sampling for all of the models in the paper.

## 2 Bayesian longitudinal linear mixed models

Let $y_i = (y_{i1}, \ldots, y_{in_i})'$ be the random vector describing the response of individual $i$, $i = 1, \ldots, N$, recorded at times $t_i = (t_{i1}, \ldots, t_{in_i})'$, and consider $y = (y_1, \ldots, y_N)'$. A BLMM for $y$ is specified via the joint probability distribution

$$f(y, \theta, \phi) = f(y | \theta, \phi) f(\phi | \theta) \pi(\theta)$$

$$= \left( \prod_{i=1}^{N} f(y_i | \theta, \phi_i) \right) \left( \prod_{i=1}^{N} f(\phi_i | \theta) \right) \pi(\theta),$$ (1)

where $f(y_i | \theta, \phi_i)$ is the conditional distribution of $y_i$ given the vector $\phi_i$ of random effects associated with individual $i$, with $\phi = (\phi_1, \ldots, \phi_N)'$, and the vector $\theta$ of parameters and hyperparameters of the model; $f(\phi_i | \theta)$ is the conditional distribution of $\phi_i$ given $\theta$; and $\pi(\theta)$ the prior distribution for $\theta$.

The simplest BLMM assumes the following conditional normal distribution for $y_i$:

$$f(y_i | \theta, \phi) = \mathcal{N}(\mu_i = X_i \beta + Z_i b_i, \Sigma_i = \sigma^2 I_{n_i}),$$ (2)
where $X_i$ and $Z_i$ are the design matrices for the fixed effects $\beta$ and the random effects $b_i$, respectively, and $\Sigma_i$ is the variance-covariance matrix defined in terms of the identity matrix $I_n_i$ of size $n_i$ and the common variance $\sigma^2$. The vector of random effects $\phi$ in (1) is here $b = (b_1, \ldots, b_N)'$ whose components are assumed conditional i.i.d. $(b_i \mid \Sigma_b) \sim \mathcal{N}(0, \Sigma_b)$. This conditional model (2) implies conditional independence not only between the observations of different individuals but also of all observations of the same individual.

Common generalisations of (2) when the number of observations per individual is large include temporal elements that account for serial correlation (Diggle et al., 2002), frequently in terms of autoregressive processes in the conditional mean $\mu_i$ or in the conditional variance-covariance matrix $\Sigma_i$. In this sense, we consider two different modelling approaches based on a first order autoregressive (AR(1)) process. The first proposal introduces an $AR(1)$ term in the random measurement (Chi and Reinsel, 1989; Hedeker and Gibbons, 2006) which, as a result, reformulates the $(j, l)$ element of $\Sigma_i$ in (2) as follows:

$$
\text{Cov}(y_{ij}, y_{il} \mid \theta, \phi_i) = \frac{\sigma^2}{1 - \rho^2} \rho^{t_{ij} - t_{il}},
$$

where $\rho$ is the coefficient of the autoregressive term of a stationary $AR(1)$. Because of the stationarity condition, $\rho$ is also equal to the conditional autocorrelation at lag 1. More generally, $\rho^h$ is the conditional autocorrelation at lag $h$ or $-h$, with $h \geq 0$. This model maintains the conditional independence between individuals but not between the observations of the same individual. Furthermore, a reformulation of this model is possible by conditioning the response of individual $i$ in the $j$-th measurement, $y_{ij}$ on their $(j-1)$-th measurement as follows

$$
f(y_{ij} \mid y_{ij-1}, \theta, \phi) = \mathcal{N}(\mu_{ij} + \rho(y_{ij-1} - \mu_{ij-1}), \sigma^2), \ j = 2, \ldots, t_{in_i}
$$

being $f(y_{i1} \mid \theta, \phi) = \mathcal{N}(\mu_{i1}, \sigma^2/(1 - \rho^2))$.

The second proposal introduces a latent autoregressive element $w_i = (w_i(t_{i1}), \ldots, w_i(t_{in_i}))'$ in the conditional mean $\mu_i$ of (2) as follows:

$$
\mu_i = X_i/\beta + Z_i/\beta + w_i,
$$

where each $w_i(t_{ij})$, $j = 2, \ldots, n_i$, is a realisation at time $t_{ij}$ from a Gaussian process with mean $\rho w_i(t_{ij-1})$ and variance $\sigma_w^2$, where $(w_i(t_{i1}) \mid \rho, \sigma_w) \sim \mathcal{N}(0, \sigma_w^2/(1 - \rho^2))$ (Diggle et al., 2002), that is $w_i$ is a vector of time correlated noises. This model augments the vector of random effects associated with individual $i$ to $\phi_i = (b_i, w_i)$. The two elements in $\phi_i$ are conditionally independent given $\theta$.

The full specification of the Bayesian model is completed with the elicitation of a prior distribution $\pi(\theta)$ for $\theta$. This includes parameters in $\beta$ and $\Sigma_i$ as well as hyper-parameters in the covariance-matrix $\Sigma_b$ and in the autoregressive terms.

Let $y_{obs}$ denote the vector of data. On the basis of the information provided by $y_{obs}$, we can calculate the posterior distribution of $(\theta, \phi)$. This is the most important element of the Bayesian inferential process from which we can derive posterior
distributions for relevant outcomes of the problem. It is obtained through the Bayes’ theorem according to

\[ \pi(\theta, \phi | y_{obs}) \propto f(y_{obs} | \theta, \phi) f(\phi | \theta) \pi(\theta), \]  

(6)

where \( f(y_{obs} | \theta, \phi) \) represents now the likelihood function of \((\theta, \phi)\) given \(y_{obs}\) which is expressed as the product of each individual’s contribution to the likelihood:

\[ f(y_{obs} | \theta, \phi) = \prod_{i=1}^{N} f(y_{obs_i} | \theta, \phi_i) = \prod_{i=1}^{N} \exp\left( -\frac{1}{2} (y_{obs_i} - \mu_i)' \Sigma_i (y_{obs_i} - \mu_i) \right) \frac{1}{\sqrt{(2\pi)^n |\Sigma_i|}}. \]  

(7)

Note that the posterior distribution \( \pi(\theta, \phi | y_{obs}) \) in (6) involves \( \theta \) and \( \phi \) together. We have chosen to express the prior information on them as \( f(\phi | \theta) \pi(\theta) \) and not as \( \pi(\theta, \phi) \). This is a philosophically debatable topic and it is not our intention to enter into this issue here. We have simply chosen that expression because we recognise it as the most standard one.

3 Bayes factors and power posteriors

The marginal or predictive density \( m_k(y) \) for \( y \) when considering a BLMM model \( \mathcal{M}_k \) with parameter and hyperparameter \( \theta_k \) and random effects \( \phi_k \) is defined as

\[ m_k(y) = \int f(y, \theta_k, \phi_k) \, d(\theta_k, \phi_k) \]

\[ = \int f(y | \theta_k, \phi_k) \, f(\phi_k | \theta_k) \, \pi(\theta_k) \, d(\theta_k, \phi_k). \]  

(8)

This distribution evaluated on the data, \( m_k(y_{obs}) \), is known by different terms in the literature such as marginal likelihood (Newton and Raftery, 1994), predictive distribution (Gelfand and Dey, 1994), marginal probability (Kass and Raftery, 1995), predictive probability (Lartillot and Philippe, 2006) or evidence (Friel et al., 2014). It can be interpreted as the support provided by the data in favour of model \( \mathcal{M}_k \).

The main tool used for choosing between two models, \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), in the Bayesian methodology is the Bayes factor (Kass and Raftery, 1995; Berger and Pericchi, 1996) of model \( \mathcal{M}_1 \) against model \( \mathcal{M}_2 \). It is defined as follows

\[ B_{12} = \frac{m_1(y_{obs})}{m_2(y_{obs})}, \]

and measures the strength with which the data support \( \mathcal{M}_1 \) with regard to \( \mathcal{M}_2 \).

In the following, we will generalise the power posterior to the case of longitudinal models involving not only parameters and hyperparameters \( \theta \) but also random effects \( \phi \). Accordingly, we define the power posterior of \((\theta, \phi)\) as

\[ \pi_\tau(\theta, \phi | y_{obs}) \propto f(y_{obs} | \theta, \phi) f(\phi | \theta) \tau(\theta), \]  

(9)
where \( \tau \in [0, 1] \) is an auxiliary temperature variable that modulates the effect of the likelihood. Let the “power marginal likelihood” be

\[
m(\mathbf{y}_{\text{obs}} \mid \tau) = \int f(\mathbf{y}_{\text{obs}} \mid \theta, \phi)^\tau f(\phi \mid \theta) \pi(\theta) \, d(\theta, \phi);
\]

(10)

when \( \tau = 0 \), \( m(\mathbf{y}_{\text{obs}} \mid \tau = 0) = \int f(\phi \mid \theta) \pi(\theta) \, d(\theta, \phi) = 1 \), whereas when \( \tau = 1 \), \( m(\mathbf{y}_{\text{obs}} \mid \tau = 1) \) equals the marginal likelihood \( m(\mathbf{y}_{\text{obs}}) \), our target quantity. Moreover, derived from the expression (10) we obtain the following result:

\[
E_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau)}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) = \int \log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi)) \pi_\tau(\theta, \phi \mid \mathbf{y}_{\text{obs}}) \, d(\theta, \phi)
\]

\[
= \frac{1}{m(\mathbf{y}_{\text{obs}} \mid \tau)} \int \log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi)) \, f(\mathbf{y}_{\text{obs}} \mid \theta, \phi)^\tau f(\phi \mid \theta) \pi(\theta) \, d(\theta, \phi)
\]

\[
= \frac{1}{m(\mathbf{y}_{\text{obs}} \mid \tau)} \int \left[ \frac{d}{d\tau} f(\mathbf{y}_{\text{obs}} \mid \theta, \phi)^\tau f(\phi \mid \theta) \pi(\theta) \right] \, d(\theta, \phi)
\]

\[
= \frac{1}{m(\mathbf{y}_{\text{obs}} \mid \tau)} \frac{d}{d\tau} m(\mathbf{y}_{\text{obs}} \mid \tau)
\]

\[
= \frac{d}{d\tau} \log (m(\mathbf{y}_{\text{obs}} \mid \tau)).
\]

As a consequence, we have

\[
\int_0^1 E_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau)}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) \, d\tau = \log \left( \frac{m(\mathbf{y}_{\text{obs}} \mid \tau = 1)}{m(\mathbf{y}_{\text{obs}} \mid \tau = 0)} \right) = \log (m(\mathbf{y}_{\text{obs}})).
\]

(11)

This expression in the log-scale by means of the expectations provides numerical stability to the estimation of the marginal likelihood [Friel and Pettitt 2008].

For any value \( \tau \in [0, 1] \), an estimate of the expectation \( E_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau)}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) \) and an estimate of the variance \( V_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau)}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) \) are obtained by using the MCMC draws generated from the power posterior \( \pi_\tau(\theta, \phi | \mathbf{y}_{\text{obs}}) \). Running \( M + 1 \) MCMC algorithms with a temperature ladder such as \( 0 = \tau_0 < \tau_1 < \ldots < \tau_M = 1 \) [Friel and Pettitt 2008] provides an estimate of \( \log m(\mathbf{y}_{\text{obs}}) \) by applying the trapezoidal rule, which in our case will remain as follows:

\[
\log (m(\mathbf{y}_{\text{obs}})) \approx \sum_{m=0}^{M-1} (\tau_{m+1} - \tau_m) \times
\]

\[
\times \frac{1}{2} \left( E_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau_m)}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) + E_{(\theta, \phi | \mathbf{y}_{\text{obs}}, \tau_{m+1})}(\log (f(\mathbf{y}_{\text{obs}} \mid \theta, \phi))) \right)
\]

(12)

The temperature ladder from \( \tau = 0 \) to \( \tau = 1 \) defines a path from the prior distribution to the posterior distribution, hence the name of path sampling [Gelman and Meng 1998].
4 Simulation studies

We explore below the behaviour of the power posterior for approximating the evidence in Bayesian longitudinal models. Two different simulation studies are conducted to evaluate the capability of the power posterior to identify the model which generated the data within a set of potential competitors. In the first study, we work with a small data set generated from a simple model with random effects and normally distributed measurement errors. In the second case, we have a larger dataset generated from a model with more complexity including random effects, measurement errors and an autoregressive term.

4.1 Study 1: a balanced longitudinal data set

We assume a scenario based on a small balanced set of simulated longitudinal data generated from a simple model, approximate the evidence of this model for the simulated data, and compare it with the evidence of three other competing longitudinal models for the same data.

We consider the LMM \( (2) \) with design matrices \( X_i = Z_i = 1 \), where 1 is a \( n_i \)-vector whose components are all the unity 1. We represent by \( \beta_0 \) the common intercept and by \( b_0i \) the individual random intercepts, which are conditionally normally distributed, \( (b_0i \mid \sigma^2_0) \sim N(0, \sigma^2_0) \). We set the values \( \beta_0 = 2, \sigma = 0.5, \) and \( \sigma_0 = 1.5 \) for the parameters and hyperparameter of the model, and generate data for \( n_i = 10 \) response values corresponding to \( N = 5 \) individuals in a complete balanced design for times \( \{0, 1, \ldots, 9\} \) (see Figure 1).

Four competing longitudinal models \( M_1, M_2, M_3 \) and \( M_4 \), ordered from least to most complex, are taken into account for analysing the simulated data, i.e.,

\[
M_1 : f(y_{it} \mid \theta_1) = N(\beta_0, \sigma^2),
M_2 : f(y_{it} \mid \theta_2, \phi_2) = N(\beta_0 + b_{0i}, \sigma^2),
M_3 : f(y_{it} \mid \theta_3, \phi_3) = N(\beta_0 + b_{1i}t, \sigma^2),
M_4 : f(y_{it} \mid \theta_4, \phi_4) = N(\beta_0 + b_{0i} + b_{1i}t, \sigma^2). \tag{13}
\]

The model that generated the data is \( M_2 \). It includes a random effect associated with the intercept of each individual \( (b_{0i}) \) and a normal measurement error. Model \( M_1 \), with only normal measurement errors, is the simplest model. \( M_3 \) and \( M_4 \) includes normal measurement errors, \( M_3 \) also includes random slopes associated to individuals \( (b_{1i}) \), and \( M_4 \) considers random intercepts and random slopes.

All four models assume conditional independence between individuals and within observations from the same individual as well as homogeneity of variances. Individual random effects \( b_{0i} \) and random slopes \( b_{1i} \) are mutually independent and conditionally normally distributed as \( (b_{0i} \mid \sigma^2_0) \sim N(0, \sigma^2_0) \) and \( (b_{1i} \mid \sigma^2_1) \sim N(0, \sigma^2_1) \).

The elicitation of the subsequent prior distribution \( \pi(\theta) \) in all models is based on both prior independence among the parameters and a noninformative prior scenario: a
normal distribution for the common intercept $\pi(\beta_0) = \mathcal{N}(0, 10^2)$ and wide uniforms for the standard deviation parameters $\pi(\sigma) = \pi(\sigma_0) = \pi(\sigma_1) = \mathcal{U}(0, 10)$.

For the computation of the evidence via the power posterior in each of the four models in $\{13\}$, the temperature variable $\tau$ was discretised according to $0 = \tau_0 < \tau_1 < \cdots < \tau_{199} = 1$ with $\tau_r = (r/199)^5$ and $r = 0, \ldots, 199$. This temperature ladder ensures that a high proportion of values are close to 0, improving the convergence of the algorithm. Each power posterior is computed by the Gibbs sampling, and implemented in the R environment, version 4.0.5, (R Core Team, 2021) running a chain with 50,000 iterations for each discrete value of $\tau$. Then, we calculate an estimate of the model evidence applying the trapezoidal rule $\{12\}$. In addition, to quantify the variability of the process, we repeat the algorithm ten times per model.

Table 1: Mean (standard deviation) of the ten replicates of the approximate log evidence for each model in $\{13\}$ computed by means of the power posterior method.

| $\mathcal{M}_1$ | $\mathcal{M}_2$ | $\mathcal{M}_3$ | $\mathcal{M}_4$ |
|-----------------|-----------------|-----------------|-----------------|
| $-103.01 (0.14)$ | $-51.63 (0.11)$ | $-91.83 (0.14)$ | $-68.21 (0.32)$ |

Table 1 shows for each model the mean and the standard deviation of the ten values of the approximate logarithm of the evidence. The evidence for each of the four models is clearly ordered. The model with the highest log evidence ($-51.63$) is actually the true sampling model $\mathcal{M}_2$, followed by the complete mixed linear model $\mathcal{M}_4$ which includes a common population intercept and two types of individual random effects, $b_{0i}$ and $b_{1i}$. In contrast, the model with the lowest log marginal likelihood value ($-103.01$) is the fixed effects model $\mathcal{M}_1$. The standard deviation for $\mathcal{M}_4$ is the highest (0.32), possibly because it is the most complex of the four models. Note also that the variability associated with the replicate process is relatively low in all cases.
4.2 Study 2: an unbalanced longitudinal data set with serial correlation

We consider a scenario defined by an unbalanced set of simulated longitudinal data generated (see Figure 2) by a LMM which includes for each individual \((i, i = 1, \ldots, 10)\) a common intercept \(\beta_0\), an individual random slope \(b_{1i}\), with \((b_{1i} \mid \sigma_1) \sim \mathcal{N}(0, \sigma_1^2)\), and an autoregressive latent element \(w_i = (w_i(t_{i1}), \ldots, w_i(t_{in_i}))'\) in the conditional mean as defined in (5) with parameters \(\sigma_w^2\) and \(\rho\), and a common conditional variance \(\sigma^2\). This model, which we will call \(M_1\), can be written as:

\[
M_1 : f(y_i | \theta_1, \phi_1) = \mathcal{N}(1\beta_0 + t_i b_{1i} + w_i, 1\sigma^2).
\]  

(14)

We simulate from this model using the values \(\beta_0 = 2, \sigma_1 = 0.5, \sigma = 2, \rho = 0.8,\) and \(\sigma_w = 1.5\) for the parameters and hyperparameter of the model. Observation times of the response variable for each individual \(i, t_i = (t_{i1}, \ldots, t_{in_i})'\), were generated in a doubly random manner, i.e., both the number of observations (between 10 and 70) and all times when each observation is recorded (between 0 and 20). The total number of observations registered for the 10 individuals is 441.

We compare \(M_1\) with two alternative models \(M_2\) and \(M_3\) defined as follows:

\[
M_2 : f(y_i | \theta_2, \phi_2) = \mathcal{N}(1\beta_0 + 1b_{0i} + t_i b_{1i}, 1\sigma^2),
\]

\[
M_3 : f(y_i | \theta_3, \phi_3) = \mathcal{N}(1\beta_0 + t_i b_{1i}, \Sigma_{AR}).
\]  

(15)

Both competing models \(M_2\) and \(M_3\) are somewhat simpler than \(M_1\). Model \(M_2\) exchanges the autoregressive term for an individual random effect in the intercept \(b_{0i}\), with \((b_{0i} \mid \sigma_0) \sim \mathcal{N}(0, \sigma_0^2)\), and model \(M_3\) with only a random slope effect but with an autoregressive element in the measurement errors.

We complete the Bayesian models by eliciting a prior distribution for the subsequent parameters and hyperparameters. We assume prior independence among them.
and select a uniform distribution $U(0, 10)$ for the standard deviation parameters $\sigma, \sigma_0, \sigma_1$ and $\sigma_w$, and a $U(-1, 1)$ for the autoregressive parameter $\rho$. The normal distribution $N(0, 10^2)$ is chosen for the common intercept $\beta_0$.

Table 2: Mean (standard deviation) of the ten replicates of the approximate log evidence for models in (14) and (15) computed by means of the power posterior method.

| $M_1$       | $M_2$       | $M_3$       |
|-------------|-------------|-------------|
| $-1123.54 (0.48)$ | $-1158.45 (1.17)$ | $-1131.04 (1.13)$ |

Table 2 shows, for each model, the mean and the standard deviation of the 10 replicates of the approximate logarithmic evidence. Model $M_1$, which is the true sampling model, presents the highest evidence ($-1123.54$), followed by $M_3$ ($-1131.04$). The lowest marginal likelihood value ($-1158.45$) is for model $M_2$. Note that the variability associated with the computation of the model predictive probability is higher than that of the previous simulation study due to the complexity of these models.

5 Sardine landings in the Mediterranean Sea

Small pelagic fish species are key elements of the Mediterranean pelagic ecosystem (Albo-Puigserver et al., 2015). Fluctuations in populations of these species can provide serious ecological and socio-economic consequences (Pennino et al., 2020).

Catches in the Mediterranean Sea are dominated by small pelagics representing nearly 49% of the harvest (Ramírez et al., 2021). Among them, the European sardine ($Sardina pilchardus$, Walbaum, 1792) is one of the most commercial species which has also shown the highest over-exploitation rates in the 20 years (Coll et al., 2008). Mediterranean fisheries are highly diverse and geographically varied, not only because of the existence of different marine environments, but also because of different socio-economic situations, and fisheries status (Pennino et al., 2017).

Landing data (tonnes) of the European sardine caught by Mediterranean countries were extracted from Sea Around Us (Zeller and Pauly, 2016) from 1970 to 2014, directly to the online databases (www.seaaroundus.org). Countries participating in the study were Albania, Algeria, Bosnia and Herzegovina (B&H), Croatia, France, Greece, Italy, Montenegro, Morocco, Slovenia, Spain and Turkey. Data from countries recognised as sovereign after 1970 by the international community (B&H, Croatia, Montenegro, and Slovenia) were imputed based on information from Exclusive Economic Zones (Zeller and Pauly, 2016). This online database is derived mainly from FAO global fisheries catch statistics, complemented by the statistics of various international and national agencies, and reconstructed datasets. It is important to note that the data we are working with come from official fisheries and probably many more sardines are actually caught than those our data reflect (i.e., due illegal, unreported and unregulated fishing). Finally, a logarithmic transformation was applied to the landing data set in order to approach the normality assumption. Figure 3 shows the temporal pattern of the logarithm of the tonnes of sardines caught per country.
Figure 3: Annual log tons of European sardine (Sardina pilchardus) caught per country, from 1970 to 2014.

The amount of fish caught at the beginning of the study is very variable among the different countries. B&H is initially well below the level of the other countries. The temporal evolution of this quantity in most of the countries seems to be rather stable, although a slightly increasing trend can be seen in most countries, especially in B&H and Turkey. Slovenia’s behaviour in recent years has been different from that of the others, with a decreasing trend in the number of catches in the recent years. The annual changes that we observe in the same country with respect to the previous years are generally not very large, so it would seem reasonable to consider an autoregressive component when modelling the evolution of the sardine landings.

5.1 Modelling of sardine fisheries in the Mediterranean Sea

Let $y_{it}$ be the logarithm of the total tonnage of sardines caught in country $i$ ($i = 1, \ldots, 12$) during year $t$ ($t = 0, \ldots, 44$). Calendar time is the natural time scale of the study and $t = 0$ corresponds to 1970, the first year of the study.

We consider three longitudinal models, $M_1$, $M_2$, and $M_3$, to assess the dynamics of the official sardine fishery carried out by country $i$ in the Mediterranean Sea from 1970 to 2014. They were defined as follows

$$M_1 : f(y_{i|\theta_1, \phi_1}) = \mathcal{N}(\mathbf{1} \beta_0 + \mathbf{1} b_{0i} + t b_{1i}, \mathbf{1} \sigma^2),$$

$$M_2 : f(y_{i|\theta_2, \phi_2}) = \mathcal{N}(\mathbf{1} \beta_0 + t b_{1i}, \Sigma_{AR}),$$

$$M_3 : f(y_{i|\theta_3, \phi_3}) = \mathcal{N}(\mathbf{1} \beta_0 + t b_{1i} + w_{i}, \mathbf{1} \sigma^2),$$

(16)

where in all models $\mathbf{1}$ is now a vector of ones of dimension 45, $t = (0, 1, \ldots, 44)'$, $\beta_0$ is a common intercept and $\sigma^2$ a common variance. Individual random effects $b_{0i}$ and
random slopes $b_{1i}$ are mutually independent and conditionally normally distributed as 
$$(b_{0i} \mid \sigma_0^2) \sim \mathcal{N}(0, \sigma_0^2)$$ and $$(b_{1i} \mid \sigma_1^2) \sim \mathcal{N}(0, \sigma_1^2)$$. The elements of the variance-
covariance matrix in model $M_2$ are as in (3), and the autoregressive element $w_i$ of
model $M_2$ is defined as in (5).

We complete the Bayesian models in (16) by eliciting a prior distribution for the
subsequent parameters and hyperparameters of each model. In all of them we assume
prior independence and a vague prior scenario. We consider uniform distributions for
the standard deviation parameters $\pi(\sigma) = \pi(\sigma_0) = \pi(\sigma_1) = \pi(\sigma_w) = U(0, 5)$ and
$\pi(\rho) = U(-1, 1)$ for the autoregressive parameter in models $M_2$ and $M_3$. A normal
distribution is considered for the common intercept parameter $\pi(\beta_0) = N(0, 5^2)$.

The estimation of the log-evidence for all three models in (16) was derived through
the power posterior according to the same strategy followed in the previous section
by means of the Gibbs sampling (see Appendix 7). We considered two hundred
temperatures for models $M_1$ and $M_2$, 80,000 iterations, and a burn-in of 20,000
for each temperature value $\tau_r$. However, due to the complexity of the model $M_3$,
we increased the number of temperature values to 500, and obtain the discretisation
$0 = \tau_0 < \tau_1 < \cdots < \tau_{499} = 1$, where $\tau_r = (r/499)^2$ and $r = 0, \ldots, 499$, as
well as to increase the number of iterations to 200,000 and the burn-in to 50,000. Table 3 shows the mean and the standard deviation of ten replicates of the approximate
log-evidence of the three models.

Table 3: Mean (standard deviation) of the approximate log evidence for each model in
(16) by means of the power posterior method.

| Model | Mean (standard deviation) |
|-------|---------------------------|
| $M_1$ | $-515.57 (0.29)$          |
| $M_2$ | $-191.38 (0.14)$          |
| $M_3$ | $-193.36 (0.79)$          |

The evidence value of the model $M_1$ is clearly lower than the other two models
$M_2$ and $M_3$, this indicates that the autoregressive term is relevant in modelling sardine
fishing in the Mediterranean Sea. The last two have similar log-evidence values but
if one of them should be selected, this would be $M_2$, albeit by a very small margin.
Note that the approximate Bayes factor of model $M_2$ to $M_3$ is 7.25, which gives
evidence in favour of $M_2$, but not very strong. An alternative possibility that could be
interesting, but beyond our scope in this study, would be to deal with models $M_2$ and
$M_3$ through model averaging procedures (Hoeting et al. 1999).

5.2 Longitudinal modeling of European sardine landings

We will then carry out a Bayesian statistical analysis of the sardine landings by means
of model $M_2$ and discuss some of the relevant outputs derived from the subsequent
posterior distribution.

The posterior distribution for the selected model $M_2$ was evaluated through Gibbs
sampling, running a chain of 250,000 iterations after a burn-in of 50,000, and thinning
the chain at every 250th iteration to reduce its autocorrelation. A summary of the
posterior outputs is shown in Table 4.

Table 4: Summary of the approximate posterior distribution of the parameters and hyperparameters in model $M_2$.

|   | mean | sd  | $q_{0.025}$ | $q_{0.975}$ |
|---|------|-----|-------------|-------------|
| $\beta_0$ | 7.92 | 0.61 | 6.48        | 9.05        |
| $\sigma_0$ | 3.02 | 0.95 | 0.68        | 4.67        |
| $\sigma_1$ | 0.03 | 0.02 | 0.00        | 0.06        |
| $\sigma$   | 0.31 | 0.01 | 0.29        | 0.34        |
| $\rho$     | 0.97 | 0.03 | 0.91        | 1           |

At the beginning of the study, $t = 0$, we notice a substantial common intercept as well as a large heterogeneity among the different countries. The country-specific variability related to the slope is not very large although its magnitude may be relevant due to the magnitude of the $t$ values. The variability associated with the model is around 0.31 and it is worth noting the high precision of this estimate. Finally, we observe very high and positive values of the correlation coefficient $\rho$ (around 0.97), which would indicate a fairly stable temporal dynamics of the fisheries.

![Figure 4: Random intercepts vs. random slopes. Approximate posterior mean for the random intercepts and the slope effects by country according to $M_2$.](image)

The posterior distribution of the random effects associated with each country provides us with useful information on the individual country patterns. Figure 4 shows the posterior mean of the random effect associated with the intercept and the slope of the countries in the study. It can be seen that a group of countries, including Algeria,
Croatia and Morocco, had high landing values at the beginning of the time series than the overall average and maintained this superiority over time. However, Greece and Spain and, to a greater extent, Italy and France also started with above-mean levels of fishing but their growth seems to have slowed down over time. B&H, Turkey, and Slovenia are countries with quite different dynamics from the rest. B&H started with a level of fishing well below the common mean, but increased the landings during the time-series (possibly as a consequence of the end of the Balkan wars) and now it is above the rest of the countries. The same trend can be observed in Turkey, although in the beginning it was in the mean of the rest of the countries. In contrast, Slovenia’s growth has slowed down over the years from an initial state in the mean of the rest of countries.

Finally, we can use the MCMC sample to investigate the autocorrelation function which depends on the autoregressive parameter $\rho$. Figure 5 shows its approximate posterior mean and 95% credible interval. As expected, the correlation decreases with the increasing of the time lag. However, the uncertainty involved in the posterior distribution is higher as the lag between the two response variables increases.

![Figure 5: Posterior mean (line) and 95% credible interval (shaded area) of the autocorrelation function according to the posterior distribution of the autoregressive parameter $\rho$ in $\mathcal{M}_2$.](image)

The analysis we have presented on sardine landing in the Mediterranean is a small illustration of the potential of Bayesian longitudinal models to analyse the general and individual behaviour of the different elements, in this case countries, of the study. A more detailed study of the problem can be found in Calvo et al. (2020).
6 Conclusions

The power posterior distribution, the key concept in the power posterior methodology, allows the computation of marginal likelihoods by extending the Gibbs sampling process quite naturally, i.e., by doing Gibbs sampling in each of the power posterior distributions, which are as simple to derive as the posterior distribution. This makes the implementation of this method ideal for Bayesian longitudinal models with different types of random effects and different levels of complexity. Some variations in the power posterior algorithm can be implemented using importance sampling to avoid sampling from the posterior distribution (Xie et al., 2011) or from distributions close to the prior (Fan et al., 2011). These improvements may slightly reduce the computational cost of the method. In addition, small changes in trapezoidal rule for estimating the evidence on the on the logarithmic scale can be applied in order to reduce the bias of the approximation (Friel et al., 2014).

In the two studies based on simulated data that we have examined in the paper, the correct model (i.e., that generated the data) is always selected among competing models with different sources of random variation, demonstrating the efficiency of the power posterior method. Moreover, following this methodology, the conclusion in the analysis of the European Sardine landings in the Mediterranean Sea is that the autoregressive term is relevant in its modelling. Actually, the model that includes two sources of random variation (random effects and autoregressive errors) is that with the highest marginal likelihood value.

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7 Appendix. Complete posterior conditional distribution associated of the power posteriors

In this appendix the posterior conditional densities needed to compute the power posteriors via the Gibbs sampling are listed. The conditional densities of the three most general models ($M_1$, $M_2$, $M_3$ of the second simulation study) are fully described. Variability associated with the normal distributions are expressed in terms of the variance.

7.1 Study 2: an unbalanced longitudinal data with serial correlation

7.1.1 Model $M_1$

- $\pi(\beta_0 | y_{obs}, b, w, \sigma, \tau) = N\left( \frac{10^2 \tau \sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - (b_0 + t_{ij} + w_{ij}))}{\sigma^2 + 10^2 \tau \sum_{i=1}^{N} n_i}, \frac{10^2 \sigma^2}{\sigma^2 + 10^2 \tau \sum_{i=1}^{N} n_i} \right)$,
- $\pi(b_{1i} | y_{obs}, \beta_0, w_i, \sigma, \sigma_1, \tau) = N\left( \frac{\sigma_1^2 \tau \sum_{j=1}^{n_i} (y_{ij} - (\beta_0 + w_{ij}))}{\sigma^2 + \sigma_1^2 \tau \sum_{j=1}^{n_i} t_{ij}^2} \right)$,
- $\pi(\omega_i | y_{obs}, \beta_0, w_{-i}, \rho, \sigma, \sigma_w, \tau) = N\left( \frac{\sigma_w^2 \tau (y_{ij} - (\beta_0 + b_1 t_{ij})) + \sigma^2 \rho (w_{i,j-1} + w_{i,j+1})}{\sigma^2(1 + \rho^2) + \sigma_w^2 \tau}, \frac{\sigma_w^2}{\sigma^2 + \sigma_w^2 \tau} \right)$,
- $\pi(w_{ij} | y, \beta_0, w_{-ij}, \rho, \sigma, \sigma_w, \tau) = \frac{1}{\sigma} \exp \left( -\frac{\tau \sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - (\beta_0 + b_1 t_{ij} + w_{ij}))^2}{2\sigma^2} \right) U(0, 10)$,
- $\pi(\sigma | y_{obs}, b) \propto \frac{1}{\sigma} \exp \left( -\frac{\sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - (\beta_0 + b_1 t_{ij} + w_{ij}))^2}{2\sigma^2} \right) U(0, 10)$,
- $\pi(\sigma_w | y_{obs}, \rho, w) \propto \frac{(1 - \rho^2)^{N/2}}{\sigma_{w}^N} \exp \left( -\frac{(1 - \rho^2) \sum_{i=1}^{N} w_{i1}^2}{2\sigma_{w}^2} \right) \exp \left( -\frac{\sum_{i=1}^{N} \sum_{j=1}^{n_i} (w_{ij} - \rho w_{i,j-1})^2}{2\sigma_{w}^2} \right) U(0, 10)$,
- $\pi(\rho | y_{obs}, \sigma_w, w) \propto \frac{(1 - \rho^2)^{N/2}}{\sigma_{w}^N} \exp \left( -\frac{(1 - \rho^2) \sum_{i=1}^{N} w_{i1}^2}{2\sigma_{w}^2} \right) \exp \left( -\frac{\sum_{i=1}^{N} \sum_{j=1}^{n_i} (w_{ij} - \rho w_{i,j-1})^2}{2\sigma_{w}^2} \right) U(-1, 1)$.
7.1.2 Model $M_2$

- $\pi(\beta_0 | y_{obs}, b, \sigma, \tau) = \mathcal{N} \left( \frac{10^2 \sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - (b_0 + b_1 t_{ij}))}{\sigma^2 + 10^2 \tau \sum_{i=1}^{N} n_i}, \frac{10^2 \sigma^2}{\sigma^2 + 10^2 \tau \sum_{i=1}^{N} n_i} \right)$,

- $\pi(b_0 | y_{obs}, \beta_0, b_1, \sigma, \sigma_0, \tau) = \mathcal{N} \left( \frac{\sigma^2 \sum_{i=1}^{N} t_{ij} (y_{ij} - (\beta_0 + b_1 t_{ij}))}{\sigma^2 + \sigma_0^2 \tau n_i}, \frac{\sigma_0^2 \sigma^2}{\sigma^2 + \sigma_0^2 \tau n_i} \right)$,

- $\pi(b_1 | y_{obs}, \beta_0, b_0, \sigma, \sigma_1, \tau) = \mathcal{N} \left( \frac{\sigma^2 \sum_{i=1}^{N} n_i (y_{ij} - (\beta_0 + b_0 t_{ij}))}{\sigma^2 + \sigma_1^2 \tau \sum_{i=1}^{N} t_{ij}^2}, \frac{\sigma_1^2 \sigma^2}{\sigma^2 + \sigma_1^2 \tau \sum_{i=1}^{N} t_{ij}^2} \right)$,

- $\pi(\sigma | y_{obs}, \beta_0, b, \nu, \tau) \propto \frac{1}{\sigma\sum_{i=1}^{N} n_i} \exp \left( - \frac{\tau \sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - (\beta_0 + b_0 t_{ij}))^2}{2\sigma^2} \right) \mathcal{U}(0, 10)$,

- $\pi(\sigma_0 | y_{obs}, b_0) \propto \frac{1}{\sigma_0} \exp \left( - \frac{\sum_{i=1}^{N} b_{0i}^2}{2\sigma_0^2} \right) \mathcal{U}(0, 10)$,

- $\pi(\sigma_1 | y_{obs}, b_1) \propto \frac{1}{\sigma_1} \exp \left( - \frac{\sum_{i=1}^{N} b_{1i}^2}{2\sigma_1^2} \right) \mathcal{U}(0, 10)$.

7.1.3 Model $M_3$

- $\pi(\beta_0 | y_{obs}, \rho, b_1, \sigma, \tau) = \mathcal{N} \left( \frac{10^2 \sum_{i=1}^{N} \sum_{j=1}^{n_i} (y_{ij} - b_1 t_{ij}) - (1 - \rho) \sum_{i=1}^{N} \sum_{j=2}^{n_i} (\rho y_{ij-1} - y_{ij}) + (1 - \rho) \beta_0}{\sigma^2 + 10^2 \tau (1 - \rho)^2 N + (1 - \rho)^2 \sum_{i=1}^{N} (n_i - 1)} \right)$,

- $\pi(b_1 | y_{obs}, \beta_0, \sigma, \sigma_1, \tau) = \mathcal{N} \left( \frac{\sigma^2 \sum_{i=1}^{N} t_{ij} (y_{ij} - (\beta_0 + (1 - \rho) \beta_{0i} - (1 - \rho) \rho y_{ij-1} + \rho t_{ij-1} - t_{ij}))}{\sigma^2 + \sigma_1^2 \tau ((1 - \rho)^2 t_{1i}^2 + \sum_{j=2}^{n_i} (t_{ij} - \rho t_{ij-1})^2)}, \frac{\sigma_1^2 \sigma^2}{\sigma^2 + \sigma_1^2 \tau ((1 - \rho)^2 t_{1i}^2 + \sum_{j=2}^{n_i} (t_{ij} - \rho t_{ij-1})^2)} \right)$,

- $f(\sigma | y_{obs}, \beta_0, \rho, b, \tau) \propto \frac{1}{\sigma \sum_{i=1}^{N} n_i} \exp \left( - \frac{\tau (1 - \rho)^2 \sum_{i=1}^{N} (y_{ij} - (\beta_0 + b_1 t_{ij}))^2}{2\sigma^2} \right) \exp \left( - \frac{\tau \sum_{i=1}^{N} \sum_{j=2}^{n_i} (y_{ij} - (1 - \rho) \beta_0 + b_1 (t_{ij} - \rho t_{ij-1} + \rho y_{ij-1}))^2}{2\sigma^2} \right) \mathcal{U}(0, 10)$,

- $\pi(\sigma_1 | y_{obs}, b_1) \propto \frac{1}{\sigma_1} \exp \left( - \frac{\sum_{i=1}^{N} b_{1i}^2}{2\sigma_1^2} \right) \mathcal{U}(0, 10)$,

- $\pi(\rho | y_{obs}, \beta_0, b_1, \tau) \propto \frac{(1 - \rho)^2 N}{\sigma \sum_{i=1}^{N} n_i} \exp \left( - \frac{\tau (1 - \rho)^2 \sum_{i=1}^{N} (y_{ij} - (\beta_0 + b_1 t_{ij}))^2}{2\sigma^2} \right) \exp \left( - \frac{\tau \sum_{i=1}^{N} \sum_{j=2}^{n_i} (y_{ij} - ((1 - \rho) \beta_0 + b_1 (t_{ij} - \rho t_{ij-1} + \rho y_{ij-1}))^2}{2\sigma^2} \right) \mathcal{U}(-1, 1)$.