Mathematical inequalities for some weighted means, II

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Abstract. We give a refined Young inequality which generalizes the inequality by Zou–Jiang. We also show the upper bound for the logarithmic mean by the use of the weighted geometric mean and the weighted arithmetic mean. Furthermore, we show some inequalities among the weighted means.

Keywords: Weighted logarithmic means, nested means, Heinz mean, Young inequality, relative operator entropy and operator inequality

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1 Introduction

The logarithmic mean is defined by
\[ L(a, b) := \frac{a - b}{\log a - \log b} = \int_0^1 a^t b^{1-t} \, dt, \quad a \neq b \] (1)
for two positive numbers \( a \) and \( b \). (We usually define \( L(a, b) = a \), if \( a = b \).) It is known the inequality:
\[ L(a, b) \leq \frac{2}{3} G(a, b) + \frac{1}{3} A(a, b), \] (2)
which is called the classical Pólya inequality in \[11\], where \( A(a, b) := \frac{1}{2}(a + b) \), \( G(a, b) := \sqrt{ab} \).
It is also known the inequality \[7\]:
\[ G(a, b) \leq L(a, b) \leq \left( \frac{a^{1/3} + b^{1/3}}{2} \right)^3. \] (3)

We have the following relation \[4\, \text{Lemma 1.1]}:
\[ L(a, b) \leq \left( \frac{a^{1/3} + b^{1/3}}{2} \right)^3 \leq \frac{2}{3} G(a, b) + \frac{1}{3} A(a, b), \] (4)
which is a refinement of the Pólya inequality.

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Recently, the weighted logarithmic mean is introduced in [8] as
\[ L_v(a, b) := \frac{1}{\log a - \log b} \left( \frac{1-v}{v} (a - a^{1-v}b^v) + \frac{v}{1-v} (a^{1-v}b^v - b) \right) \] (5)
and studied in [5, 9]. In [5, Corollary 2.2] and [9, Theorem 2.2], the following inequality is shown.
\[ L_v(a, b) \leq \frac{1}{2} G_v(a, b) + \frac{1}{2} A_v(a, b), \] (6)
where \( A_v(a, b) := (1-v) a + vb \) is the weighted arithmetic mean and \( G_v(a, b) := a^{1-v} b^v \) is the weighted geometric mean. However, the following inequality does not hold in general.
\[ L_v(a, b) \leq \frac{2}{3} G_v(a, b) + \frac{1}{3} A_v(a, b), \] (7)
since we have counter-examples. See [8] for example.

We have the following relation for \( a, b > 0 \) and \( 0 \leq v \leq 1 \),
\[ H_v(a, b) \leq G_v(a, b) \leq L_v(a, b) \leq A_v(a, b), \] (8)
where \( H_v(a, b) := \{ (1-v)a^{-1} + vb^{-1} \}^{-1} \) is the weighted harmonic mean. The inequality \( G_v(a, b) \leq A_v(a, b) \) is often called the Young inequality. Many refinements and reverses for this inequality have been studied. See [2, Chapter 2] for example. In the paper [3], one of authors studied some inequalities on the weighted means, especially the weighted logarithmic mean. In this paper, we give the further results on the weighted mean and obtain some new inequalities for them.

2 Main results

We firstly give a new refinement of the Young inequality which is a generalization for the known result by using the weighted parameter \( v \in [0, 1] \).

**Theorem 2.1.** For \( a, b > 0 \) and \( 0 \leq v \leq 1 \), we have
\[ G_v(a, b) \leq \left\{ 1 + \frac{\mu^2}{2} (\log a - \log b)^2 \right\} G_v(a, b) \leq A_v(a, b), \] (9)
where \( \mu := \min\{1-v, v\} \).

**Proof.** The first inequality of (9) is trivial. To prove the second inequality of (9), we assume \( 0 \leq v \leq \frac{1}{2} \) and set
\[ f_v(t) := (1-v)t + v - t^{1-v} - \frac{v^2}{2} (\log t)^2 t^{1-v}, \quad t > 0. \]

Then we have
\[ \frac{df_v(t)}{dt} = \frac{g_v(t)}{2t^v}, \quad g_v(t) := 2(1-v)(t^v - 1) - 2v^2 \log t - v^2 (1-v) (\log t)^2. \]

We also have
\[ \frac{dg_v(t)}{dt} = \frac{2v}{t} h_v(t), \quad h_v(t) := (1-v)(t^v - \log t^v) - v. \]
Since \( \frac{dh_v(t)}{dt} = \frac{v(1-v)(t^r-1)}{t} \), we have \( h_v(t) \geq h_v(1) = 1 - 2v \geq 0 \) which implies \( \frac{dg_v(t)}{dt} \geq 0 \). Thus we have \( g_v(t) \geq g_v(1) = 0 \) for \( t \geq 1 \), and \( g_v(t) \leq g_v(1) = 0 \) for \( 0 < t \leq 1 \), so that we have \( \frac{dg_v(t)}{dt} \geq 0 \) for \( t \geq 1 \), and \( \frac{dg_v(t)}{dt} \leq 0 \) for \( 0 < t \leq 1 \). Therefore we have \( f_v(t) \geq f_v(1) = 0 \). For the case \( 1/2 \leq v \leq 1 \) can be proven similarly. Finally, putting \( t := a/b \) and then multiplying \( b > 0 \) to both sides, we obtain the desired result.

**Remark 2.2.** It is remarkable that Theorem 2.1 recovers the following inequality [12]:

\[
\left\{ 1 + \frac{1}{8} (\log a - \log b)^2 \right\} G(a, b) \leq A(a, b)
\]

when \( v = 1/2 \). In addition, the following reverse of the second inequality in [9]:

\[
A_v(a, b) \leq \left\{ 1 + \frac{\lambda^2}{2} (\log a - \log b)^2 \right\} G_v(a, b), \quad \lambda := \max\{1 - v, v\}
\]

does not hold in general, because of [10].

Considering \( r \)-logarithmic function which is defined by \( \ln_r x := \frac{x^r - 1}{r} \) for \( x > 0 \) and \( r \neq 0 \), we find the following corollary of Theorem 2.1. We note that \( \lim_{r \to 0} \ln_r x = \log x \).

**Corollary 2.3.** Let \( 0 \leq v \leq 1 \), \( r \neq 0 \) and \( a, b > 0 \). If we have the conditions (i) \( r > 0 \) and \( 0 < t \leq 1 \), or (ii) \( r < 0 \) and \( t \geq 1 \), then we have the following inequalities.

\[
G_v(a, b) \leq \left\{ 1 + \frac{\mu^2}{2} (\ln_r \frac{a}{b})^2 \right\} G_v(a, b) \leq A_v(a, b),
\]

where \( \mu := \min\{1 - v, v\} \).

**Proof.** The first inequality of (11) is trivial. We consider the function

\[
f_{v,r}(t) := (1-v)t + v - t^{1-v} - \frac{\mu^2}{2} (\ln_r t)^2 t^{1-v}, \quad (t > 0, \quad r \neq 0, \quad 0 \leq v \leq 1).
\]

Then we have

\[
\frac{df_{v,r}(t)}{dr} = \frac{\mu^2 t^{1-v}(1-t^{r})}{r^3} (1 - t^r + t^r \log t^r).
\]

Putting \( x := 1/t^r \) in the fundamental inequality \( \log x \leq x - 1 \) for \( x > 0 \), we have \( 1 - t^r + t^r \log t^r \geq 0 \) for \( t > 0 \) and \( r \in \mathbb{R} \). Thus we have \( \frac{df_{v,r}(t)}{dr} \geq 0 \) for \( r > 0 \) and \( 0 < t \leq 1 \), which implies \( f_{v,r}(t) \geq f_{v,0}(t) \geq 0 \). The last inequality is thanks to the second inequality of (9) with the fact \( \lim_{r \to 0} \ln_r x = \log x \). We also have \( \frac{df_{v,r}(t)}{dr} \leq 0 \) for \( r < 0 \) and \( t \geq 1 \), which implies \( f_{v,r}(t) \geq f_{v,0}(t) \geq 0 \), similarly. Therefore, if we have the conditions (i) \( r > 0 \) and \( 0 < t \leq 1 \), or (ii) \( r < 0 \) and \( t \geq 1 \), then we have the following inequalities:

\[
(1-v)t + v - t^{1-v} - \frac{\mu^2}{2} (\ln_r t)^2 t^{1-v} \geq 0.
\]

Putting \( t := a/b \) in the above and multiplying \( b > 0 \) to both sides, we get the second inequality of (11).
It is well known that
\[ G(a, b) \leq H_{\nu}(a, b) \leq A(a, b), \]  
where \( H_{\nu}(a, b) := A(G_{\nu}(a, b), G_{1-\nu}(a, b)) = \frac{a^{1-b}b^v + a^v b^{1-v}}{2} \) is the Heinz mean. Then we have the following inequalities from Theorem 2.1.

**Corollary 2.4.** For \( a, b > 0 \) and \( 0 \leq v \leq 1 \), we have
\[ G(a, b) \leq H_{v}(a, b) \leq \left\{ 1 + \frac{\mu^2}{2} (\log a - \log b)^2 \right\} H_{\nu}(a, b) \leq A(a, b). \]

**Proof.** Replacing \( v \) by \( 1-v \) in (9) and adding it to (9) and then dividing 2, we get the result. \( \square \)

The inequality
\[ L(a, b) \leq \frac{1}{2} A_{v}(a, b) + \frac{1}{2} G_{v}(a, b), \quad a, b > 0, \quad 0 \leq v \leq 1 \]  
does not hold in general, since we have counter-examples such as
\[ \frac{1}{2} A_{1/4}(1/2, 1) + \frac{1}{2} G_{1/4}(1/2, 1) - L(1/2, 1) \approx -0.223091 \]  
and
\[ \frac{1}{2} A_{3/4}(2, 1) + \frac{1}{2} G_{3/4}(2, 1) - L(2, 1) \approx -0.446183. \]

However, we have the following inequality for the weighted mean.

**Theorem 2.5.** For \( a, b > 0 \) and \( 0 \leq v \leq 1 \), we have
\[ L(a, b) \leq \frac{1}{2} A_{v}(a, b) + \frac{1}{2} G_{1-v}(a, b). \]  

**Proof.** To prove (16), it is sufficient to prove \( f_v(t) \geq 0 \) for \( t > 0 \) and \( 0 \leq v \leq 1 \), where
\[ f_v(t) := t^v + (1-v)t + v - \frac{2(t-1)}{\log t}, \quad t > 0. \]

Then we have \( \frac{d f_v(t)}{dv} = 1-t + t^v \log t \) and \( \frac{d^2 f_v(t)}{dv^2} = t^v (\log t)^2 \geq 0 \). Then \( f_v(t) \) takes a minimum value at \( v = v_{min} \), that is,
\[ \frac{f(v_{min})}{dv} = 0 \Leftrightarrow v_{min} = \frac{t-1}{\log t} \Leftrightarrow v_{min} = \log_t \left( \frac{t-1}{\log t} \right). \]

Therefore we have
\[ f_v(t) \geq f_{v_{min}}(t) = t - \frac{(t-1)}{\log t} \log \left( \frac{t-1}{\log t} \right) - \frac{(t-1)}{\log t}. \]

We here prove \( f_{v_{min}}(t) \geq 0 \) for \( t \geq 1 \). Then we calculate
\[ \frac{df_{v_{min}}}{dt} = \frac{g(t)}{t \left( \log t \right)^2}, \quad g(t) := (t \log t + t - 1) \log \left( \frac{t-1}{\log t} \right) + t (\log t)^2 - 2t \log t + 2(t-1). \]

Then we calculate
\[ \frac{dg(t)}{dt} = \frac{(t \log t + 1 - t) (t \log t + t - 1)}{t(t-1) \log t} + (\log t)^2 + (\log t + 2) \left( \frac{t-1}{\log t} \right) \geq 0, \quad t \geq 1, \]
since \( t \log t + 1 - t \geq 0 \) for \( t > 0 \). Thus we have \( g(t) \geq g(1) = 0 \), since \( \lim_{t \to 1} \frac{t - 1}{\log t} = 0 \). Since we have \( \frac{df_{v_{\text{min}}}(t)}{dt} \geq 0 \), we have \( f_{v_{\text{min}}}(t) \geq f_{v_{\text{min}}}(1) = 0 \). Therefore we have

\[
t^v + (1 - v)t + v - \frac{2(t - 1)}{\log t} \geq 0, \quad t \geq 1.
\]

Putting \( t := 1/s \) with \( 0 < s \leq 1 \) in (17), replacing \( v \) by \( 1 - v \), and multiplying \( s > 0 \) to both sides, we have

\[
s^v + (1 - v)s + v - \frac{2(s - 1)}{\log s} \geq 0, \quad 0 < s \leq 1.
\]

Therefore we have \( f_v(t) \geq 0 \) for all \( t > 0 \).

**Remark 2.6.** Replacing \( v \) by \( 1 - v \) in (16) with same procedure in the Corollary 2.4, we get the following inequality.

\[
L(a, b) \leq \frac{1}{2} A(a, b) + \frac{1}{2} H z_v(a, b).
\]

This inequality is also proven by the use of (6) with \( v = 1/2 \) and (12) as

\[
L(a, b) \leq \frac{1}{2} A(a, b) + \frac{1}{2} G(a, b) \leq \frac{1}{2} A(a, b) + \frac{1}{2} H z_v(a, b).
\]

We study the properties on the function which is a representing function of the weighted logarithmic mean.

\[
L_v(t) := \frac{1}{\log t} \left( \frac{1 - v}{v} (t - t^{1-v}) + \frac{v}{1 - v} (t^{1-v} - 1) \right).
\]

We easily see \( bL_v(a/b) = L_v(a, b) \), \( L_{1/2}(t) = \frac{t - 1}{\log t} \), \( \lim_{v \to 0} L_v(t) = t \) and \( \lim_{v \to 1} L_v(t) = 1 \). We also see \( \lim_{t \to 1} L_v(t) = 1 \) and \( \lim_{t \to 0} L_v(t) = 0 \). We have the following property.

**Proposition 2.7.** The function \( L_v(t) \) given in (19) is increasing with respect to \( v \) when \( 0 < t \leq 1 \) and decreasing with respect to \( v \) when \( t \geq 1 \).

**Proof.** We calculate

\[
\frac{dL_v(t)}{dv} = \frac{f_v(t)}{v^2(1 - v)^2 t^{v-1} \log t}, \quad f_v(t) := -(1-v)^2 t^v - v^2 t^{v-1} + (2v^2 - 2v + 1) + v(2v^2 - 3v + 1) \log t.
\]

We also calculate

\[
\frac{df_v(t)}{dt} = \frac{v(1 - v)}{t} g_v(t), \quad g_v(t) := -(1 - v) t^v + vt^{v-1} + 1 - 2v.
\]

Since we have

\[
\frac{dg_v(t)}{dt} = -v(1 - v)(t + 1) t^{v-2} \leq 0,
\]

we have \( g_v(t) \geq g_v(1) = 0 \) for \( 0 < t \leq 1 \) and \( g_v(t) \leq g_v(1) = 0 \) for \( t \geq 1 \). That is, we have \( \frac{df_v(t)}{dt} \geq 0 \) for \( 0 < t \leq 1 \) and \( \frac{df_v(t)}{dt} \leq 0 \) for \( t \geq 1 \). Thus we have \( f_v(t) \leq f_v(1) = 0 \). Therefore we have \( \frac{dL_v(t)}{dv} \geq 0 \) for \( 0 < t \leq 1 \) and \( \frac{dL_v(t)}{dv} \leq 0 \) for \( t \geq 1 \).
Since \( \frac{d}{dv} ((1 - v)t + v) = 1 - t \), \( \frac{d}{dv} t^{1-v} = -t^{1-v} \log t \) and \( \frac{d}{dv} \left( \frac{t}{(1 - v) + vt} \right) = \frac{t(1 - t)}{(1 - v) + vt} \), we easily see that the representing functions for the weighted arithmetic mean, the weighted geometric mean and the weighted harmonic mean have similar properties. Proposition 2.7 will be applied to the proof of Proposition 2.8.

We further study the inequalities among means. In the sequel, we consider the bounds of the nested means for the weighted means. From the simple calculations and numerical computations, we see

\[
A(A_v(a, b), A_{1-v}(a, b)) = A(a, b), \quad G(G_v(a, b), G_{1-v}(a, b)) = G(a, b),
\]

and

\[
H(H_v(a, b), H_{1-v}(a, b)) = H(a, b), \quad L(L_v(a, b), L_{1-v}(a, b)) \neq L(a, b).
\]

Because we have

\[
L(L_{1/4}(10, 1), L_{3/4}(10, 1)) - L(10, 1) \simeq 0.0173327 \tag{20}
\]

as an example.

As given in (13), the distance between the geometric mean and the arithmetic mean is not so tight. Therefore the proof for the following relations are not so difficult.

**Proposition 2.8.** For \( 0 \leq v \leq 1 \) and \( a, b > 0 \), we have

\[
G(a, b) \leq G(A_v(a, b), A_{1-v}(a, b)) \leq A(a, b) \tag{21}
\]

and

\[
G(a, b) \leq A(G_v(a, b), G_{1-v}(a, b)) \leq A(a, b). \tag{22}
\]

**Proof.** The inequalities (21) can be proven by

\[
A(a, b) = A(A_v(a, b), A_{1-v}(a, b)) \geq G(A_v(a, b), A_{1-v}(a, b)) \geq G(G_v(a, b), G_{1-v}(a, b)) = G(a, b).
\]

The inequalities (21) are just same to the inequalities given in (12). \( \square \)

From Proposition 2.8 we have the following.

**Proposition 2.9.** For \( 0 \leq v \leq 1 \) and \( a, b > 0 \), we have

\[
H(a, b) \leq G(H_v(a, b), H_{1-v}(a, b)) \leq G(a, b) \tag{23}
\]

and

\[
H(a, b) \leq H(G_v(a, b), G_{1-v}(a, b)) \leq G(a, b). \tag{24}
\]

**Proof.** Replacing \( a \) and \( b \) by \( 1/a \) and \( 1/b \) in Proposition 2.8 respectively, and taking inverse we have (23) and (24), since \( A(1/a, 1/b)^{-1} = H(a, b) \), \( G(1/a, 1/b)^{-1} = G(a, b) \),

\[
G(A_v(1/a, 1/b), A_{1-v}(1/a, 1/b))^{-1} = G(H_v(a, b), H_{1-v}(a, b))
\]

and

\[
A(G_v(1/a, 1/b), G_{1-v}(1/a, 1/b))^{-1} = H(G_v(a, b), G_{1-v}(a, b)). \quad \square
\]

We have the following relation on the arithmetic mean and the logarithmic mean. Their proofs are not so easy, since the distance between the logarithmic mean and the arithmetic mean is relatively tight.
Theorem 2.10. For \(0 \leq v \leq 1\) and \(a, b > 0\), we have
\[
L(a, b) \leq A(L_v(a, b), L_{1-v}(a, b)) \leq A(a, b)
\] (25)
and
\[
L(a, b) \leq L(A_v(a, b), A_{1-v}(a, b)) \leq A(a, b).
\] (26)

Proof. Since \(L_v(a, b) \leq A_v(a, b)\) and \(L_{1-v}(a, b) \leq A_{1-v}(a, b)\), we have the second inequality of (25). As for the first inequality of (25), it is sufficient to prove \(f_v(t) \geq 0\) for \(t > 0\) and \(0 \leq v \leq 1\), where \(f_v(t) := \frac{1}{2}L_v(t) + \frac{1}{2}L_{1-v}(t) - L_{1/2}(t)\) and \(L_v(t)\) is given in (19). Since \(\lim_{v \to 0} L_v(t) = t\) and \(\lim_{v \to 1} L_v(t) = 1\), for the special cases such as \(v = 0\) or \(v = 1\), \(f_v(t) \geq 0\) is equivalent to \(\frac{t+1}{2} \geq L(t)\) which is known. So we assume \(0 < v < 1\).

By the simple calculations, we have
\[
f_v(t) = \frac{g_v(t)}{2v(1-v)t^{v-1} \log t}, \quad g_v(t) := (1 - 2v)^2 t^{v-1}(t - 1) + (1 - 2v)(t^{2v-1} - 1).
\]

Then we have
\[
dg_v(t) = v(1 - 2v)^2 t^{v-2} h_v(t), \quad h_v(t) := \frac{1-t^v}{v} - (1-t).
\]

Since \(\frac{t^v-1}{v} \leq t-1\) for \(t > 0\) and \(0 < v < 1\), we have \(h_v(t) \geq 0\) which implies \(dg_v(t) \geq 0\). Thus we have \(g_v(t) \geq 0\) for \(t \geq 1\) and \(g_v(t) \leq 0\) for \(0 < t \leq 1\). Therefore we have \(f_v(t) \geq 0\) for \(t > 0\) and \(0 \leq v \leq 1\).

The second inequality of (26) can be easily proven by
\[
L(A_v(a, b), A_{1-v}(a, b)) \leq A(A_v(a, b), A_{1-v}(a, b)) = A(a, b).
\]

As for the first inequality of (26), it is sufficient to prove the following inequality
\[
\frac{t-1}{\log t} \leq \frac{(1-2v)(t-1)}{\log \{vt + (1-v)\}}
\] (27)
for \(t > 0\) and \(0 \leq v \leq 1\). Since the equality holds when \(v = 0, 1/2, 1\), we assume \(0 < v < 1\) with \(v \neq 1/2\).

(i) For the case \(t \geq 1\) and \(0 < v < 1/2\), the inequality (27) is equivalent to
\[
\log \frac{(1-v)t+v}{vt+(1-v)} \leq (1-2v) \log t \iff (1-v)t+v \leq vt^{2-2v} + (1-v)t^{1-2v}.
\]

So we set the function \(k_v(t) := vt^{2-2v} + (1-v)t^{1-2v} - (1-v)t - v\). Then we have \(dk_v(t) = 2v(1-v)t^{1-2v} + (1-v)(1-2v)t^{-2v} - (1-v)\) and \(\frac{d^2k_v(t)}{dt^2} = 2v(1-v)(1-2v)t^{-2v-1}(t-1) \geq 0\). Thus we have \(\frac{dk_v(t)}{dt} \geq \frac{dk_v(1)}{dt} = 0\) which implies \(k_v(t) \geq k_v(1) = 0\). Therefore we have (27) for \(t \geq 1\) and \(0 < v < 1/2\). Replacing \(t := 1/s \geq 1\) in (27) and multiplying \(s > 0\) to both sides, we have
\[
\frac{s-1}{\log s} \leq \frac{(1-2v)(1-s)}{\log \left\{\frac{(1-v)+vs}{s}\right\} - \log \left\{\frac{v+(1-v)s}{s}\right\}}
\] (28)
for \(0 < s \leq 1\) and \(0 < v < 1/2\). Thus we have the inequality (27) for \(t > 0\) and \(0 < v < 1/2\).
(ii) For the case $t \geq 1$ and $1/2 < v < 1$, the inequality \(27\) is equivalent to

$$
\log \left( \frac{(1 - v)t + v}{vt + (1 - v)} \right) \geq (1 - 2v) \log t \Leftrightarrow (1 - v)t + v \geq vt^{2-2v} + (1 - v)t^{1-2v}.
$$

So we set the function \(l_v(t) := (1 - v)t + v - vt^{2-2v} - (1 - v)t^{1-2v} \). Then we have \(\frac{dl_v(t)}{dt} = (1 - v) - 2v(1-v)t^{1-2v} - (1-v)(1-2v)t^{-2v}\) and \(\frac{d^2l_v(t)}{dt^2} = -2v(1-v)(1-2v)t^{-2v-1}(t-1) \geq 0\).

Thus we have \(\frac{dl_v(t)}{dt} \geq \frac{dl_v(1)}{dt} = 0\) which implies \(l_v(t) \geq l_v(1) = 0\). Therefore we have \(27\) for \(t \geq 1\) and \(1/2 < v < 1\). By the similar way to the last part of (i), we have the inequality \(27\) for \(t > 0\) and \(1/2 < v < 1\).

From (i) and (ii), we have the inequality \(27\) for \(t > 0\) and \(0 < v < 1\). □

**Theorem 2.11.** For \(0 \leq v \leq 1\) and \(a, b > 0\), we have

$$
G(a, b) \leq L(G_v(a, b), G_{1-v}(a, b)) \leq L(a, b). \tag{29}
$$

and

$$
G(a, b) \leq G(L_v(a, b), L_{1-v}(a, b)) \leq L(a, b) \tag{30}
$$

**Proof.** The first inequality of \(29\) is easily proven by

$$
L(G_v(a, b), G_{1-v}(a, b)) \geq G(G_v(a, b), G_{1-v}(a, b)) = G(a, b).
$$

To prove the second inequality of \(29\), we set the function (since \(L(G_v(a, b), G_{1-v}(a, b)) = L(a, b)\) for \(v = 1/2\))

$$
f_v(t) := (1 - 2v)(t - 1) - t^{1-v} + t^v, \quad t \geq 1, \quad 0 \leq v < 1/2.
$$

Since

$$
\frac{df_v(t)}{dt} = (1 - 2v) - (1 - v)t^{-v} + vt^{-v-1}, \quad \frac{d^2f_v(t)}{dt^2} = v(1-v)t^{-v-1}(1 - t^{2v-1}) \geq 0,
$$
we have \(\frac{df_v(t)}{dt} \geq \frac{df_v(1)}{dt} = 0\) which implies \(f_v(t) \geq f_v(1) = 0\). By the similar way, we can prove \(t^{1-v} - t^v - (1 - 2v)(t - 1) \geq 0\) for \(t \geq 1\) and \(1/2 < v \leq 1\). Thus we have

$$
\frac{t^{1-v} - t^v}{(1 - 2v) \log t} \leq \frac{t - 1}{\log t}, \quad t \geq 1, \quad 0 \leq v \leq 1. \tag{31}
$$

Putting \(t := 1/s \geq 1\), we obtain

$$
\frac{s^{1-v} - s^v}{(1 - 2v) \log s} \leq \frac{s - 1}{\log s}, \quad 0 < s \leq 1, \quad 0 \leq v \leq 1.
$$

Thus we have

$$
\frac{t^{1-v} - t^v}{(1 - 2v) \log t} \leq \frac{t - 1}{\log t}, \quad t > 0, \quad 0 \leq v \leq 1. \tag{32}
$$

Putting \(t := a/b\) in \(32\) and multiplying \(b > 0\) to both sides, we obtain the second inequality of \(29\).

The first inequality of \(30\) is easily proven by

$$
G(L_v(a, b), L_{1-v}(a, b)) \geq G(G_v(a, b), G_{1-v}(a, b)) = G(a, b).
$$

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Since \( G(L_v(a,b), L_{1-v}(a,b)) \leq L(a,b) \) is equivalent to \( L_v(t)L_{1-v}(t) \leq L_{1/2}(t)^2 \) because we put \( t := a/b > 0 \) and multiply \( b^2 > 0 \) to both sides. Since \( (\log t)^2 > 0 \) for all \( t > 0 \), we set the function

\[
g_v(t) := (t - 1)^2 - \left\{ \frac{1 - v}{v} \left( t - t^{1-v} \right) + \frac{v}{1 - v} \left( t^{1-v} - 1 \right) \right\} \left\{ \frac{v}{1 - v} \left( t^v - 1 \right) + \frac{1 - v}{v} \left( t^v - 1 \right) \right\}.
\]

Since \( g_v(t) = g_{1-v}(t) \) and \( g_v(t) \geq 0, \ (t \geq 1) \) implies \( g_v(s) \geq 0, \ (0 < s \leq 1) \) by putting \( t := 1/s \geq 1 \), we have only to prove \( g_v(t) \geq 0 \) for \( t \geq 1 \) and \( 0 \leq v \leq 1/2 \). Then we calculate

\[
\frac{dg_v(t)}{dt} = \frac{(1 - 2v) t^{-v-1}}{v^2(1 - v)^2} \left\{ (v - 1)^3 t + v^2 (2 - v) t^2 + v^5 t^{2v} - (1 + v)(1 - v) t^{2v+1} + 2(1 - 2v) t^{v+1} \right\},
\]

\[
\frac{d^2 g_v(t)}{dt^2} = \frac{(1 - 2v) t^{-v-2}}{v(1 - v)} h_v(t), \quad h_v(t) := (1 - v)^2 t + v(2 - v) t^2 - v^2 t^{2v} - (1 - v)(1 + v) t^{2v+1}.
\]

We further calculate

\[
\frac{dh_v(t)}{dt} = (1 - v)^2 + 2v(2 - v) t - 2v^3 t^{2v-1} - (1 - v)(1 + v)(1 + 2v) t^{2v},
\]

\[
\frac{d^2 h_v(t)}{dt^2} = 2v(2 - v) - 2v(1 - v)(1 + v)(1 + 2v) t^{2v-1} + 2v^3 (1 - 2v) t^{2v},
\]

\[
\frac{d^3 h_v(t)}{dt^3} = 2v(1 - v)(1 - 2v) t^{2v-3} ((v + 1)(2v + 1)t - 2v^2) \geq 0, \quad (t \geq 1, \ 0 \leq v \leq 1/2).\]

Thus we have \( \frac{d^2 h_v(t)}{dt^2} \geq \frac{d^2 h_v(1)}{dt^2} = v(1 - v)(1 - 2v) \geq 0 \) which implies \( \frac{dh_v(t)}{dt} \geq \frac{dh_v(1)}{dt} = 0 \).

So we have \( h_v(t) \geq h_v(1) = 0 \) which means \( \frac{d^2 g_v(t)}{dt^2} \geq 0 \) which implies \( \frac{dg_v(t)}{dt} \geq \frac{dg_v(1)}{dt} = 0 \).

Therefore we have \( g_v(t) \geq g_v(1) = 0 \). \( \square \)

In the end of this section, we state some operator inequalities for the essential scalar inequalities which were obtained above. To this end, we give a notation for a self-adjoint operator \( A \). If a self-adjoint operator \( A \) satisfies \( \langle Ax, x \rangle \geq 0 \) for all vectors \( x \neq 0 \), then \( A \) is called a positive operator, and we use the notation \( A \geq 0 \). If \( \langle Ax, x \rangle > 0 \) for all vectors \( x \neq 0 \), then \( A \) is called strict positive operator, and we use the notation \( A > 0 \). It is known that the scalar order is equivalent to the operator partial order by Kubo-Ando theory [6]. Therefore it is often important to obtain a new scalar inequality. To express the logarithmic mean, we use the integral form such as \( L(t, 1) = L_{1/2}(t) = \int_0^1 t^x dx \) and \( L_v(t, 1) = L_v(t) = \frac{v}{1-v} \int_0^{1-v} t^x dx + \frac{1-v}{v} \int_{1-v}^1 t^x dx \).

Since [32] is rewritten as

\[
\frac{1}{1 - 2v} \int_v^{1-v} t^x dx \leq \int_0^1 t^x dx,
\]

we have for \( A, B > 0 \)

\[
\frac{1}{1 - 2v} \int_v^{1-v} A_x^x B dx \leq \int_0^1 A_x^x B dx, \quad 0 \leq v \leq 1
\]

by putting \( t := A^{-1/2} B A^{-1/2} \) and multiplying \( A^{1/2} \) to both sides, where

\[
A_x^x B := A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^x A^{1/2}, \quad 0 \leq x \leq 1
\]

is the weighted operator geometric mean for \( A, B > 0 \). In the proof of Theorem 2.11 we proved \( L_v(t)L_{1-v}(t) \leq L_{1/2}(t)^2 \) which is also rewritten as

\[
\left( \frac{v}{1-v} \int_0^{1-v} t^x dx + \frac{1-v}{v} \int_{1-v}^1 t^x dx \right) \left( \frac{1-v}{v} \int_0^v t^x dx + \frac{v}{1-v} \int_{1-v}^1 t^x dx \right) \leq \left( \int_0^1 t^x dx \right)^2.
\]
Thus we similarly have for $A, B > 0$

$$(A \ell_v B) A^{-1} (A \ell_{1-v} B) \leq (A \ell B) A^{-1} (A \ell B), \quad 0 \leq v \leq 1$$

where the weighted operator logarithmic mean is defined by

$$A \ell_v B := \frac{v}{1-v} \int_0^{1-v} A^* x B dx + \frac{1-v}{v} \int_{1-v}^1 A^* x B dx$$

and the operator logarithmic mean is written by $A \ell B := A \ell_{1/2} B = \int_0^1 A^* x B dx$. Since we proved $L_{1/2}(t) \leq \frac{1}{2} L_v(t) + \frac{1}{2} L_{1-v}(t)$ for $t > 0$ and $0 \leq v \leq 1$, we have for $A, B > 0$

$$A \ell B \leq \frac{1}{2} A \ell_v B + \frac{1}{2} A \ell_{1-v} B, \quad 0 \leq v \leq 1.$$

From (2.5), we also have for $A, B > 0$

$$A \ell B \leq \frac{1}{2} A \nabla_v B + \frac{1}{2} A \nabla_{1-v} B, \quad 0 \leq v \leq 1,$$

where $A \nabla_v B := (1-v)A + vB$ is the weighted operator arithmetic mean. From [9], we have for $A, B > 0$

$$0 \leq K^* (A^*_v B) K \leq A \nabla_v B - A^*_v B, \quad 0 \leq v \leq 1,$$

where $K := \frac{\mu}{\sqrt{2}} A^{-1} S(A|B)$, $\mu := \{1-v, v\}$ and $S(A|B) := A^{1/2} \log \left( A^{-1/2} B A^{-1/2} \right) A^{1/2}$ is known as the operator relative entropy [1]. We see that the second inequality in (33) gives a generalization of [12, Theorem 4.1]. Furthermore, the inequalities (11) is equivalent to the inequalities:

$$0 \leq \frac{\mu^2}{2} (\ln_r t)^v (\ln_r t) \leq (1-v) + vt - t^v$$

under the conditions (i) $r > 0$ and $0 < t \leq 1$, or (ii) $r < 0$ and $t \geq 1$. Therefore we have the following proposition.

**Proposition 2.12.** Under the conditions (i) $r > 0$ and $0 < B \leq A$, or (ii) $r < 0$ and $0 < A \leq B$, we have the following operator inequalities:

$$0 \leq K_r^* (A^*_v B) K_r \leq A \nabla_v B - A^*_v B, \quad 0 \leq v \leq 1,$$

where $K_r := \frac{\mu}{\sqrt{2}} A^{-1} S_r(A|B)$, $\mu := \{1-v, v\}$ and $S_r(A|B) := A^{1/2} \ln_r \left( A^{-1/2} B A^{-1/2} \right) A^{1/2}$ is known as the Tsallis operator relative entropy [10].

The other obtained scalar inequalities give the corresponding operator inequalities. However, we omit them.

### 3 Concluding remarks

Related to (13), we have the following result which does not contradicts with (14) and (15).

**Proposition 3.1.** For $a, b > 0$ and $0 \leq v \leq 1$, we have the following inequalities.

(i) For $0 \leq v \leq 1/2$ and $a \geq b$, we have

$$L(a, b) \leq \frac{1}{2} A_v(a, b) + \frac{1}{2} G_v(a, b).$$

(34)
(ii) $\frac{1}{2} \leq v \leq 1$ and $a \leq b$, we also have

$$L(a, b) \leq \frac{1}{2} A_v(a, b) + \frac{1}{2} G_v(a, b). \quad (35)$$

**Proof.** (i) For the case $a/b =: t \geq 1$, from Proposition 2.7 for $0 \leq v \leq \frac{1}{2}$ we have $L_{1/2}(t, 1) \leq L_v(t, 1)$ which implies $L(a, b) = L_{1/2}(a, b) \leq L_v(a, b)$. Thus we have $L(a, b) \leq L_v(a, b) \leq \frac{1}{2} A_v(a, b) + \frac{1}{2} G_v(a, b)$ for $0 \leq v \leq \frac{1}{2}$ from (6).

(ii) For the case $a/b =: t \leq 1$, from Proposition 2.7 we have similarly $L(a, b) = L_{1/2}(a, b) \leq L_v(a, b)$ for $1/2 \leq v \leq 1$. Thus we have $L(a, b) \leq L_v(a, b) \leq \frac{1}{2} A_v(a, b) + \frac{1}{2} G_v(a, b)$ for $1/2 \leq v \leq 1$ from (6).

It is quite natural to consider the maximum (optimal) value $p$ such that

$$L_v(a, b) \leq (1 - p) A_v(a, b) + p G_v(a, b), \quad (36)$$

By the numerical computation shows that

$$(1 - p) A_v(a, b) + p G_v(a, b) - L_v(a, b) \simeq -1.39948 \times 10^{-8}$$

when $a := 10^{-10}$, $b := 1$, $v := 1 - 10^{-10}$ and $p = \frac{13}{25}$. This means the inequality (36) does not hold when $p = \frac{13}{25} > \frac{1}{2}$.

We close this paper with the following conjecture. From (20) with several numerical computations indicate that the inequality seems to be true

$$L(a, b) \leq L(L_v(a, b), L_{1-v}(a, b)), \quad a, b > 0, \quad 0 \leq v \leq 1. \quad (37)$$

However we have not proven this inequality due to its complicated computations, and we also have not found any counter-examples. If the conjectured inequality (37) will be shown, then it will give a tight inequality for the first inequalities in (25) and (26).

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Appendix: Proof of (6) by elementary calculations

As we noted, (6) has been proven already by the use of the Hermite-Hadamard inequality for convex function. From the point of self-sufficiency, we give a direct and an elementary proof for (6).

Proof of (6): To prove (6), we firstly prove \( f_v(t) \geq 0 \) for \( t \geq 1 \) and \( 0 \leq v \leq 1 \), where

\[
f_v(t) := v(1-v)t^{1-v} \log t + v(1-v) \{(1-v)t + v\} \log t - 2 \{(1-v)^2(t-t^{1-v}) + v^2(t^{1-v} - 1)\}.
\]

Then we have \( f'_v(t) = \frac{(1-v)}{tv} g_v(t) \), where

\[
g_v(t) := (3v - 2 - v^2)t^v + v^2t^{v-1} + (2 - 3v) + v(1-v)(1 + t^v) \log t.
\]

Then we also have \( g'_v(t) = \frac{v(1-v)}{t} h_v(t) \), where

\[
h_v(t) := 1 - vt^{v-1} - (1-v)t^v + vt^v \log t.
\]

Since we have \( h'_v(t) = vt^{v-2} \{(1-v) + vt + vt \log t\} \geq 0 \) for \( t \geq 1 \) and \( 0 \leq v \leq 1 \), \( h_v(t) \geq h_v(1) = 0 \). Thus we have \( g'_v(t) \geq 0 \) which implies \( g_v(t) \geq g_v(1) = 0 \). Therefore we have \( f'_v(t) \geq 0 \) so that we have \( f_v(t) \geq f_v(1) = 0 \).

We secondly prove \( k_v(t) \geq 0 \) for \( 0 < t \leq 1 \) and \( 0 \leq v \leq 1 \), where

\[
k_v(t) := 2 \{(1-v)^2(t-t^{1-v}) + v^2(t^{1-v} - 1)\} - v(1-v)t^{1-v} \log t - v(1-v) \{(1-v)t - v\} \log t.
\]

Then we have \( k'_v(t) = \frac{(1-v)}{tv} l_v(t) \), where

\[
l_v(t) := (2 - 3v + v^2)t^v - v^2t^{v-1} + (3v - 2) - v(1-v)(1 + t^v) \log t.
\]
Then we also have \( l'_v(t) = \frac{v(1 - v)}{t} (m_v(t) - vt^v \log t) \), where

\[ m_v(t) := (1 - v)t^v + vt^{v-1} - 1. \]

Since \( m'_v(t) = v(1 - v)t^{v-2}(t - 1) \leq 0 \) for \( 0 < t \leq 1 \) and \( 0 \leq v \leq 1 \), we have \( m_v(t) \geq m_v(1) = 0 \). Thus we have \( l'_v(t) \geq 0 \) for \( 0 < t \leq 1 \) and \( 0 \leq v \leq 1 \) so that we have \( l_v(t) \leq l_v(1) = 0 \) which implies \( k'_v(t) \leq 0 \). Therefore for \( 0 < t \leq 1 \) and \( 0 \leq v \leq 1 \), we have \( k_v(t) \geq k_v(1) = 0 \). \( \square \)