Existence, uniqueness and stability of semi-linear rough partial differential equations

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Berlin, December 8, 2018
Rough PDE via Feynman-Kac integrals
  Rough PDEs: motivation from stoch. filtering
  Rough linear PDE
  Rough semilinear PDE

Rough PDE via Davies’ method
  Motivation: Burgers equation
  Variational approach
  Rough differential equations according to Davie
  A priori estimates
  Stability
Motivation: Stochastic filtering theory

Estimating partially observed diffusion process

\[(S) \quad dX_t = b(X_t) \, dt + \sigma(X_t) \, dB_t \quad \in \quad \mathbb{R}^d\]

\[(O) \quad dW_t = \gamma(X_t) \, dt + d\tilde{B}_t \quad \in \quad \mathbb{R}^e\]

for (ind.) Brownian motions \(B, \tilde{B}\)

cond. expectation of (S) given \(\mathcal{W}_{s:t} = \sigma(\mathcal{W}_u : u \in [s, t])\)

\[
E^{t,x} \left[ g(X_T) \mid W_{t,T} \right] = \frac{E^{t,x}_0 \left[ g(X_T) \exp \left( \int_t^T \gamma(X_s) \, dW_s - \frac{1}{2} \int_t^T \| \gamma(X_s) \|^2 \, ds \right) \right]}{E^{t,x}_0 \left[ \exp \left( \int_t^T \gamma(X_s) \, dW_s - \frac{1}{2} \int_t^T \| \gamma(X_s) \|^2 \, ds \right) \right]}
\]

where \(E^{t,x}_0\) denotes expectation w.r.t. \(B\) only (!)

Rem

▶ crucial pb: robustness w.r.t. data \(W_{0:t}\), (in part. \(\int_t^T \gamma(X_s) \, dW_s\))

▶ our Ansatz consider the rough path topology
Rough path integrals - main idea

- recall Young's ineq
  \[ \left| \int_s^t Y_r - Y_s \, dX_r \right| \leq C \beta, \alpha |t - s|^{\alpha + \beta} \| Y \|_\beta \| X \|_\alpha, \]
  \[ \alpha + \beta > 1 \]

- but for \( Y_t = F(X_t), \quad F \in C^2 \)

\[
\int_s^t F(X_r) \, dX_r = F(X_s)(X_t - X_s) + \int_s^t F(X_r) - F(X_s) \, dX_r
\]

\[ = F(X_s)X_{st} + DF(X_s) \int_s^t X_{sr} \, dX_r + R_{st} \]

with \( \| R \|_{3\alpha} \leq C \| D^2 F \|_\infty \| X \|_\alpha^2 \), the integral becomes continuous w.r.t. \((X_t, X_{st})\) in \( \| X \|_\alpha + \| X \|_{2\alpha} \), sufficient for \( \alpha > \frac{1}{3} \), since then

\[
\int_s^t F(X_r) \, dX_r := \lim_{|P| \to 0} \sum_{[u,v] \in P} F(X_u)X_{uv} + DF(X_u)X_{uv} \quad \text{well-defined}
\]

- admissible integrands: \( Y \in C^\alpha \) s.th. \( Y_{st} = Y_{st}'X_{st} + R_{st} \) with \( Y' \in C^\alpha \)
  and \( R \in C^{2\alpha} \)
Back to Stochastic Filtering

\[ E^{t,x}_t [g(X_T) \mid \mathcal{W}_{t,T}] = \frac{E^{t,x}_0 \left[ g(X_T) \exp \left( \int_t^T \gamma(X_s) \, dW_s - \frac{1}{2} \int_t^T \| \gamma(X_s) \|^2 \, ds \right) \right]}{E^{t,x}_0 \left[ \exp \left( \int_t^T \gamma(X_s) \, dW_s - \frac{1}{2} \int_t^T \| \gamma(X_s) \|^2 \, ds \right) \right]} = \frac{u(t, x)}{u_0(t, x)} \]

where

\[ u(t, x) = g(x) + \int_t^T Lu(r, x) \, dr + \int_t^T \gamma_k u(r, x) \, dW^k_r \]

(and \( u_0 \) the same backward eq with terminal condition \( g(x) \equiv 1 \))

with

\[ Lu := \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T (x) D^2 u \right] + \langle b(x), Du \rangle \]

formally differentiating \( W^k \) yields the (backward rough) PDE

\[
\begin{cases}
- \partial_t u(t, x) = Lu(t, x) + \gamma_k(x) u(t, x) \dot{W}^k_t \\
u(T, x) = g(x), t \in [0, T], x \in \mathbb{R}^d.
\end{cases}
\]
Rough linear PDE

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u(t, x) = Lu(t, x) + \Gamma_k u(t, x) \dot{W}_t^k \\
u(0, x) = g(x), \ t \in [0, T], \ x \in \mathbb{R}^d.
\end{array} \right.
\end{aligned}
\]  
(1)

where \( W \) is a (geometric) \( \alpha \)-Hölder rough path, \( \alpha \in (1/3, 1/2] \) and

\[
Lu := \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T (x) D^2 u \right] + \langle b(x), Du \rangle + c(x)u(x)
\]

\[
\Gamma_k u := \langle \beta_k(x), Du \rangle + \gamma_k(x) u
\]

**Ansatz** in [Friz, Diehl, S. 2017]: (rough path) Feynman-Kac representation

\[
u^W(t, x) = \mathbb{E}^x \left[ g(X_t) \exp \left( \int_0^t c(X_r) \, dr + \int_0^t \gamma(X_r) \dot{W}_r \, dr \right) \right]
\]  
(2)

with

\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt + \beta(X_t) \dot{W}_t \, dt
\]

where \( B \) is a Brownian motion.
Rough linear PDE, ctd.

\[
\begin{align*}
\begin{cases}
\partial_t u(t,x) &= Lu(t,x) + \Gamma_k u(t,x) \dot{W}_t^k \\
\quad u(0,x) &= g(x), t \in [0,T], x \in \mathbb{R}^d.
\end{cases}
\end{align*}
\]  

(1)

where \( W \) is a (geometric) \( \alpha \)-Hölder rough path, \( \alpha \in (1/3, 1/2] \) and

\[
Lu := \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T(x) D^2 u \right] + \langle b(x), Du \rangle + c(x) u(x)
\]

\[
\Gamma_k u := \langle \beta_k(x), Du \rangle + \gamma_k(x) u
\]

**Ansatz** in [Friz, Diehl, S. 2017]: understand how

\[
u^W(t,x) = \mathbb{E}^x \left[ g(X_t) \exp \left( \int_0^t c(X_r) \, dr + \int_0^t \gamma(X_r) \, \dot{W}_r \, dr \right) \right]
\]  

(2)

depends on \( W = (W, \dot{W}) \) in the **rough path metric**, reduces to understand stability of \( W \mapsto X \), where

\[
dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt + \beta(X_t) \, \dot{W}_t \, dt
\]
Rough linear PDE: weak solutions

**Theorem**

[Friz, Nilssen, S. 2018]

Assume $\sigma_{i,k}, \beta_j \in C^3_b(\mathbb{R}^d)$, $b_j, \gamma_j, c \in C^1_b(\mathbb{R}^d)$. Given $g \in L^p(\mathbb{R}^d)$, the Feynman-Kac integral (2) yields an analytically weak solution $u$ of (1) satisfying

$$\sup_{t \in [0, T]} \| u(t) \|_{L^p} \leq C \| g \|_{L^p}$$

where $C$ depends on $\mathbf{W}$ only through the rough path metric.

**Rem** In the simple case $b = c = \gamma = 0$

- $C = E \left[ \sup_{x \in \mathbb{R}} | \det D\Phi_{0,t}^{-1}(x) | \right]$
- $\Phi$ being the flow generated by $dX_t = \sigma(X_t) dB_t + \beta(X_t) dW_t$
Theorem
[Friz, Nilssen, S. 2018]
Assume $\sigma_{i,k}, \beta_j, \gamma_j \in C^6_b(\mathbb{R}^d)$, $b_j, c \in C^4_b(\mathbb{R}^d)$. Given $g \in W^{3,p}(\mathbb{R}^d)$, the Feynman-Kac integral (2) yields a weak solution $u \in W^{3,p}$ of (1) satisfying

$$\sup_{t \in [0,T]} \|u(t)\|_{W^{3,p}} \leq C \|g\|_{W^{3,p}}$$

where $C$ depends on $W$ only through the rough path metric.

Rem In this case $x \mapsto \Phi_{0,t}(x)$ is $C^3_b(\mathbb{R})$. 
Rough semilinear PDE

Extensions to the semilinear case

\[
\begin{aligned}
\partial_t u(t, x) &= Lu(t, x) + F(u)(t, x) + \Gamma_k u(t, x) \dot{W}_k^t \\
u(0, x) &= g(x), \quad t \in [0, T], \quad x \in \mathbb{R}^d.
\end{aligned}
\]

(3)

using Duhamels principle

\[
u(t) = P_{0,t}^W g + \int_0^t P_{s,t}^W F(u(s)) \, ds \quad \text{(e.g.) in } L^2
\]

where \( P_{s,t}^W \) is the \( L^2 \)-propagator of the linear Pb

requires a priori energy estimates of the type

- \( \|u(t)\|_{L^2} \leq C \|g\|_{L^2} \)
- \( \int_0^t \|u(r)\|_{W^{1,2}}^2 \, dr \leq C \|g\|_{L^2}^2 \)
Linear rough PDE: energy estimates

**Ansatz** Consider the rough heat equation

\[
\begin{aligned}
\partial_t u(t, x) &= \Delta u(t, x) + \Gamma_k u(t, x) \dot{W}_t^k \\
u(0, x) &= g(x), t \in [0, T], x \in \mathbb{R}^d.
\end{aligned}
\]

for smooth \( W \)

usual integration by parts leads to

\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(r)\|_{L^2}^2 \, dr = \|u(0)\|_{L^2}^2 + 2 \int_0^t (u(r)^2, \Gamma_k^* 1) \, dW_r^k
\]

major problem: the elementary control of the 'rough integral'

\[
\int_0^t (u(r)^2, \Gamma_k^* 1) dW_r^k \, dr \leq \|\Gamma_k^* 1\|_{L^\infty} \int_0^t \|u(r)\|_{L^2}^2 \, d|W_r^k|
\]

is not continuous in the rough path metric
Introducing the backward rough pde

\[ \partial_t \varphi^W(t, x) + \Delta \varphi^W(t, x) + \Gamma^*_k \nabla \varphi^W(t, x) \dot{W}^k_t + \Gamma^*_k 1(x) \varphi^W(t, x) \dot{W}^k_t = 0 \]

with nonnegative terminal condition, e.g. \( \varphi^W(T, x) \equiv 1 \) leads to

\[
\int u(t)^2 \varphi^W(t) \, dx + 2 \int_0^t \int |\nabla u(r)|^2 \varphi^W(r) \, dx \, dr = \int u(0)^2 \varphi^W(0) \, dx \quad t \in [0, T]
\]

yields a priori energy estimate

as long as \( W \mapsto \varphi^W(t) \) is cont. in the rough path metric
(e.g. via Feynman-Kac representation)
Rough semilinear PDEs - main result

Theorem

[Friz, Nilssen, S. 2018]

Assume $\sigma_{i,k}, \beta_j, \gamma_j \in C^6_b(\mathbb{R}^d)$, $b_j, c \in C^4_b(\mathbb{R}^d)$ as well as

\(\lambda |\xi|^2 \leq |\sigma^T \xi|^2\) for some $\lambda > 0$,

\(\|F(u) - F(v)\|_{L^2} \leq C\|u - v\|_{W^{1,2}(\mathbb{R}^d)}\).

Given $g \in L^2(\mathbb{R}^d)$, there exists a unique weak solution $u \in C([0, T]; L^2) \cap L^2([0, T]; W^{1,2})$ of (3).
Burgers equation

Navier Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

$$\text{div} \ u = 0, \quad u(0) = u_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d).$$

have no meaning when $d = 1$

[Burgers, 1974] introduced the 1d toy-equation

$$\partial_t u + u \partial_x u = \nu \partial^2_x u, \quad (t, x) \in [0, T] \times \mathbb{R}$$

neglecting the pressure in the Navier-Stokes equations.
Stochastic Burgers equation

Adding randomness to the equation to incorporate

- fluctuations in classical continuum limits
- incorporate highly oscillating terms on small time-scales as statistical model for turbulence

first approach: additive, space-time white noise, $\xi$

$$\partial_t u = \nu \partial_x^2 u - u \partial_x u + \xi,$$

- [Bertini, Cancrini, Jona-Lasinio 1994]
- [DaPrato, Debussche, Temam 1994] (1st step towards regularity structures)
Stochastic Burgers equation with transport noise

Lagrangian viewpoint

\[ \phi_t(x) := \text{position of a fluid particle at time } t \text{ starting in } x \]

assume the following decomposition

\[ \dot{\phi}_t(x) = u(t, \phi_t(x)) - \beta_j(\phi_t(x)) \dot{Z}_t^j, \quad \phi_0(x) = x, \]

with \((Z^j)_j\) modelling a highly oscillating part

associated velocity field \(u\) is generated by

\[
\partial_t u = \nu \partial_x^2 u - u \partial_x u + \beta_j \partial_x u \dot{Z}_t^j \tag{4}
\]

Ansatz [Hocquet, Nilssen, S. 2018] prove stability of

\[(Z, \mathbb{Z}) \mapsto u \quad \text{w.r.t. rough path metric}\]
Variational approach

consider the weak formulation

\[ \partial_t \int u_t \phi \, dx = - \int \nabla u_t \cdot \nabla \phi \, dx - \int u_t \text{div}(\beta_j \phi) \, dx \dot{Z}_t^j \]

for all \( \phi \in W^{1,2}(\mathbb{R}) \) w.r.t. the Gelfand triple

\[ W^{1,2}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset W^{-1,2}(\mathbb{R}) \]

testing \( u \) against itself yields the energy estimate

\[ \|u_t\|_{L^2}^2 + 2 \int_0^t \|\nabla u_r\|_{L^2}^2 \, dr = \|u_0\|_{L^2}^2 - \int_0^t (u_r^2, \text{div}(\beta_j)) \, dZ_r^j \leq \|u_0\|_{L^2}^2 + \|\text{div}(\beta_j)\|_{L^\infty} \int_0^t \|u_r\|_{L^2}^2 \, d|Z^j|_r \]

because of \( \int u_t \partial_x u_t u_t \, dx = -\frac{1}{3} \int \partial_x (u_t^3) \, dx = 0 \)

Rem

- again, no easy a priori (energy) estimates
- but also no easy perturbative approach
Davies’ method for solving rough ode’s

**Main idea** consider the rough ode

\[
dy_t = g(y_t) dX_t,
\]

a Taylor expansion yields

\[
y_{st} := y_t - y_s = g(y_s) X_{st} + \nabla g(y_s) g(y_s) X_{st} + y_{st},
\]

**Definition** \( y \) is called a solution if \( y^\triangledown \) is more regular in time in the sense that

\[
|y_{st}^\triangledown| \lesssim |t - s|^{1+}
\]
Davies’ method for solving rough pde’s

consider pure transport eqn (and a smooth path \( Z \))

\[
\partial_t f = (\beta_j \cdot \nabla) f \dot{Z}_t^j, \quad f|_{t=0} \in L^2(\mathbb{R}^d)
\]

a similar Taylor expansion now yields

\[
f_{st} = (\beta_j \cdot \nabla) f_s Z_{st}^j + (\beta_j \cdot \nabla)(\beta_k \cdot \nabla) f_s Z_{st}^{k,j} + f_{st}^h
\]

where

\[
f_{st}^h(x) = \int_s^t \int_{r_1}^{r_2} \int_{r_1}^{r_2} (\beta_j \cdot \nabla)(\beta_k \cdot \nabla)(\beta_i \cdot \nabla) f_{r_3}(x) dZ_{r_3}^i dZ_{r_2}^k dZ_{r_1}^j
\]

Rem requires 3 derivatives of \( f \)
Davies’ method for solving rough pde’s, ctd.

for the variational approach therefore need

\[ W^{3,2}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset W^{-3,2}(\mathbb{R}^d) \]

define the unbounded rough drivers

\[ A_{st}^1 \phi := (\beta_j \cdot \nabla)\phi Z^j_{st} \quad A_{st}^2 \phi := (\beta_j \cdot \nabla)(\beta_k \cdot \nabla)\phi Z^{k,j}_{st} \]

satisfying

\[ |A_{st}^1|_{L(W^n, W^{n-1})} \lesssim |t - s|^{\alpha} \quad |A_{st}^2|_{L(W^n, W^{n-2})} \lesssim |t - s|^{2\alpha} \]

and Chen’s relation \[ A_{st}^2 - A_{s\theta}^2 - A_{\theta t}^2 = A_{\theta t}^1 A_{s\theta}^1 \]

Definition \( f : [0, T] \rightarrow L^2(\mathbb{R}^d) \) is called a solution if

\[ f_{st} = A_{st}^1 f_s + A_{st}^2 f_s + f_{st}^b \]

with

\[ |f_{st}^b(\phi)| \lesssim |t - s|^{1+} |\phi|_{W^3}. \]
A priori estimates

consider the equation perturbed by a drift

\[ f_{st} = \mu_{st} + A_{st}^1 f_s + A_{st}^2 f_s + f_{st}^h \]

where \( \mu : [0, T] \to \mathcal{W}^{-2,2}(\mathbb{R}^d) \) is Lipschitz, e.g. \( \mu = \Delta f \)

Proposition

[Deya, Gubinelli, Hofmanova, Tindel 2016]

Suppose we have a solution to the above equation. Then

\[ |f_{st}^h(\phi)| \lesssim (\mu_{Lip} \|A\|_\alpha + |f|_{L_{T,x}^\infty,2} \|A\|_{2,\alpha}^2) \|\phi\|_{\mathcal{W}^3} |t - s|^{1+}. \]
Stability

to obtain stability of

\[ \nu_{st} = \int_s^t \Delta \nu_r \, dr + A_{st}^1 \nu_s + A_{st}^2 \nu_s + \nu_{st}^\bullet \]

in the rough path topology we now need 3 derivatives

using a tensorization argument and involved approximation by smooth solutions we obtain

\[ \nu_{st}^2(\phi) = -2 \int_s^t |\nabla \nu_r|^2(\phi) + \nabla \nu_r(\nu_r \nabla \phi) \, dr \]

\[ + \nu_s^2(A_{st}^1, \phi) + \nu_s^2(A_{st}^2, \phi) + \nu_{st}^2,\bullet(\phi) \]

with the a priori estimate

\[ |\nu_{st}^2,\bullet(\phi)| \lesssim A \left( \|\nabla \nu\|^2_{L^2([s,t] \times \mathbb{R}^d)} + \|\nu\|^2_{L^\infty([s,t];L^2(D))} \right) \|\phi\|_{W^{3,\infty}} |t-s|^{1+} \]
Stability, ctd.

to obtain stability of

$$v_{st} = \int_s^t \Delta v_r \, dr + A_{st}^1 v_s + A_{st}^2 v_s + v_{st}^\#$$

in the rough path topology we now need 3 derivatives

letting $\phi = 1$ we get

$$(\|v\|_{L^2}^2)_{st} + 2 \int_s^t \|\nabla v_r\|_{L^2}^2 \, dr = v_{s}^2 (A_{st}^1 , 1) + v_{s}^2 (A_{st}^2 , 1) + v_{st}^2 (1)$$

$$\lesssim \|A\|_\alpha |t - s|^{\alpha} (\|\nabla v\|_{L^2([s,t] \times \mathbb{R}^d)}^2 + \|v\|_{L^\infty([s,t]; L^2(\mathbb{R}^d))}^2)$$

(rough) Gronwall lemma $\Rightarrow$ energy estimate
Application to Rough Burgers

\[ \partial_t u = \partial_x^2 u - u \partial_x u + \beta_j \partial_x u \dot{Z}_t \]

formally yields the following eqn for \( u^2 \)

\[ \partial_t u^2 = 2u \partial_x^2 u - 2u^2 \partial_x u + \beta_j \partial_x u^2 \dot{Z}_t \]

the remainder in this case can be estimated by

\[
|u^2_{st}^{\#}(\phi)| \lesssim_A \left( \| \nabla u \|^2_{L^2([s,t];L^2)} + \| u \|^2_{L^\infty([s,t];L^2)} + \| u \|^3_{L^\infty([s,t];L^3)} \right) \cdot \| \phi \|_{W^3,\infty} |t - s|^{1+}
\]

Rem cubic in \( u \), so rough Gronwall lemma not applicable
Weighted measure space approach

similar to part 1 introduce the backward rough pde

\[ \partial_t \varphi^Z_t = -\partial_x^2 \varphi^Z_t + \partial_x (\beta_j \varphi^Z_t) \dot{Z}^j_t, \quad \varphi^Z_T \equiv 1 \]

leads to the weighted energy equality

\[ \partial_t \int u^2_t \varphi^Z_t \, dx + 2 \int (\partial_x u_t)^2 \varphi^Z_t \, dx = 2 \int u^2_t \partial_x u_t \varphi^Z_t \, dx \]

if we can show that \( 0 < m_* \leq \varphi^Z_t (x) \leq m^* < \infty \)

\[ \| u_T \|_{L^2}^2 + \int_0^T \| \partial_x u_t \|_{L^2}^2 \, dt \leq m_* m^* \| u_0 \|_{L^2}^2 + 2 \int_0^T (|u_t|^2, |\partial_x u_t|) \, dt \]

still cubic in \( u \) on RHS
Use non-linear Gronwall:

**Theorem (Bihari-LaSalle)**

*Suppose $q > 1$ and two positive functions $x, k : [0, T] \rightarrow \mathbb{R}_+$ satisfy*

\[ x_t \leq x_0 + \int_0^t k_s x_s^q ds. \]

*Then for all $t$ such that $(q - 1)x_0 \int_0^t k_s ds < 1$ we have the following estimate*

\[ x_t \leq \frac{x_0}{\left(1 - (q - 1)x_0 \int_0^t k_s ds\right)^{1/(q-1)}}. \]
Backward equation

main advantage of the backward equation

\[ \partial_t \varphi^Z_t = -\partial^2_x \varphi^Z_t + \partial_x (\beta_j \varphi^Z_t) \dot{Z}_t^j, \quad m_T = 1 \]

admits the Feynman-Kac representation

\[ \varphi^Z_t(x) = E \left[ \exp \left( \int_t^T (\partial_x \beta_j)(\psi_{t,s}(x))dZ_s^j \right) \right] \]

where

\[ d\psi_{t,s}(x) = \sqrt{2}dB_s + \beta_j(\psi_{t,s}(x))dZ_s^j, \quad \psi_{t,t}(x) = x \]

So in fact this method can be seen as a flow-transformation on the energy space
Main result

This gives

\[ \partial_t (u_t^2, m_t) + 2(|\partial_x u_t|^2, m_t) = 2(u_t^2, \partial_x(u_t m_t)). \]

Upper and lower bound on \( m \) can be checked directly from the Feynman-Kac representation.

Theorem

[Hocquet, Nilssen, S. 2018]

There exists a maximal time \( T_0 \in (0, T] \) such that existence and uniqueness of finite-energy solutions \( u \) to

\[ \partial_t u = \partial_x^2 u - u \partial_x u - \beta_j \partial_x u \dot{Z}^j \]

holds on \( (0, T_0) \times \mathbb{R} \)

The value of \( T_0 \) depends only upon the quantities \( \|u_0\|_{L^2}, \|\beta_j\|_{C_b^6}, \|Z\|_{\alpha}, \|Z\|_{2\alpha} \)