The integrals of motion for the elliptic deformation of the Virasoro and $W_N$ algebra

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Abstract

We review the free field realization of the deformed Virasoro algebra $Vir_{q,t}$ and the deformed $W$ algebra $W_{q,t}(\hat{gl}_N)$. We explicitly construct two classes of infinitely many commutative operators $I_m, G_m$, $(m \in \mathbb{N})$, in terms of these algebras. They can be regarded as the elliptic deformation of the local and nonlocal integrals of motion for the conformal field theory [1,2,3,4,5]. This review is based on the works [15,16,17].

Key words: Exactly Solved Model, Virasoro algebra, W-algebra, Quantum group, Conformal field theory, Elliptic quantum group, Deformed $W$-algebra

1 Introduction

The Korteweg-de Vries (KdV) equation occupies a central place in the modern theory of completely integrable systems. Because of its integrability, the KdV equation has infinitely many conservation laws. The Hamiltonian aspects of the KdV theory connected it to the conformal field theory. The quantization of the second Poisson bracket $\{,\}_{P.B.}$ of the KdV gives rise to the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c_{CFT}m(m^2-1)\delta_{m+n,0}$. The quantum field theory of the KdV theory becomes the conformal field theory associated with the Virasoro algebra [1,2,3]. V.Bazhanov, S.Lukyanov, Al.Zamolodchikov [4] constructed quantum field theoretical analogue of the commuting transfer matrix $T(z)$ acting the highest weight module of the Virasoro algebra. The commuting transfer matrix $T(z)$ is constructed as the trace of an image of the universal $R$-matrix associated with the quantum affine symmetry $U_q(\widehat{sl}_2)$.
Hence the commutativity \([T(z), T(w)] = 0\) is a direct consequence of the Yang-Baxter relation. We call the coefficients of the asymptotic expansion of \(\log T(z), (z \to \infty)\), the local integrals of motion for the Virasoro algebra. They recover the conservation laws of the KdV in the classical limit \(c_{CFT} \to \infty\). We call the coefficients of the Taylor expansion of \(T(z)\) the nonlocal integrals of motion for the Virasoro algebra. See also the generalization to the \(W_N\) algebra [4,5].

In this paper we construct the elliptic deformation of the integrals of motion for the conformal field theory [1,4,5]. In this paper we construct two classes of infinitely many commutative operators \(I_m, \mathcal{G}_m, (m \in \mathbb{N})\), associated with the deformed Virasoro algebra and the deformed \(W\)-algebra \(W_{q,t}(\hat{gl}_N)\). Because it is not so easy to calculate the trace of the image of the universal \(R\)-matrix of the elliptic quantum group, we prefer the completely different method of the construction for the integrals of motion in the elliptic deformation of the conformal field theory. Instead of considering the transfer matrix \(T(z)\), we directly give the explicit formulae of the integrals of motion \(I_n\) and \(G_n\) for the deformed \(W\)-algebra \(W_{q,t}(\hat{gl}_N)\). The commutativity of the integrals of motion are not understood as direct consequence of the Yang-Baxter relation. They are understood as consequence of the commutative family of the Feigin-Odesskii algebra [14].

The organization of this paper is as follows. In section 2 we give reviews on the deformed Virasoro algebra and the deformed \(W\)-algebra \(W_{q,t}(\hat{gl}_N)\). In section 3 we give explicit formulae of the integrals of motion for the deformed Virasoro algebra and the deformed \(W\)-algebra, and state the main theorem.

2 Elliptic deformation of the Virasoro algebra and the \(W_N\)-algebra

In this section we review the elliptic deformation of the Virasoro algebra and the \(W_N\)-algebra. We fix three parameters \(x, r, s\) such that \(0 < x < 1\), \(\text{Re}(r) > 0\) and \(\text{Re}(s) > 0\). Let us set \(r^* = r - 1\). We set the parameters \(\tau\) by \(x = \exp (-\pi \sqrt{-1/r}\tau)\). We relate two variables \(z\) and \(u\) by \(z = x^{2u}\). The symbol \([u]_r\) stands for the Jacobi theta function

\[
[u]_r = x^{r^2-u} \frac{\Theta_{x^{2r}}(z)}{(x^{2r}; x^{2r})^\infty}, \quad \Theta_q(z) = (z; q)_\infty(q/z; q)_\infty(q; q)_\infty,
\]

where \((z; q)_\infty = \prod_{j=0}^\infty (1 - q^j z)\). The elliptic theta function satisfies the quasi-periodicities,

\[
[u + r]_r = -[u]_r, \quad [u + r \tau]_r = -e^{-\pi \sqrt{-1/r} - 2\pi \sqrt{-1/u}} [u]_r.
\]
The symbol \([a]\) stands for \(q\)-integer \([a] = \frac{x^a - x^{-a}}{x - x^{-1}}\).

### 2.1 Bosons

For \(N = 2, 3, 4, \cdots\), we introduce the bosons \(\beta_m^j, (m \in \mathbb{Z} \neq 0; 1 \leq j \leq N)\), which satisfy the commutation relation,

\[
[\beta_n^i, \beta_m^j] = n \frac{(r-1)n}{rn} \delta_{n+m,0} \times \left\{ \begin{array}{cl} \frac{(s-1)n}{sn} & (1 \leq i = j \leq N) \\ -\frac{n}{sn} x^{sn} \text{sgn}(i-j) & (1 \leq i \neq j \leq N) \end{array} \right.
\]  

(3)

For \(N = 2, 3, 4, \cdots\), we introduce the zero-mode operators \(P_\lambda\) and \(Q_\lambda\). Let \(\varepsilon_j, (1 \leq j \leq N)\) be an orthonormal basis in \(\mathbb{R}^N\) relative to the standard inner product \((\cdot | \cdot)\). Let us set \(\bar{\varepsilon}_i = \varepsilon_i - \varepsilon, \varepsilon = \frac{1}{N} \sum_{j=1}^{N} \varepsilon_j\). Let us set \(\alpha_j = \bar{\varepsilon}_j - \bar{\varepsilon}_{j+1}\). Let \(P_\lambda, Q_\lambda\) be the zero mode operators defined by the commutation relation

\[
[i P_\lambda, Q_\mu] = (\lambda|\mu), \quad (\lambda, \mu \in \sum_{j=1}^{N} \mathbb{Z} \bar{\varepsilon}_j).
\]  

(4)

The action of the Dynkin-diagram automorphism \(\eta\) on the bosons is given by

\[
\eta(\beta_m^1) = x^{-\frac{2}{r-1} m} \beta_m^2, \cdots, \eta(\beta_m^{N-1}) = x^{-\frac{2}{r-1} m} \beta_m^N, \eta(\beta_m^N) = x^{\frac{2}{r-1} (N-1) m} \beta_m^1.
\]  

(5)

The action of the Dynkin-diagram automorphism \(\eta\) on the zero-mode operator is given by

\[
\eta(P_\lambda) = P_{\eta(\lambda)}, \quad \eta(Q_\lambda) = Q_{\eta(\lambda)}, \quad \eta(\bar{\varepsilon}_j) = \bar{\varepsilon}_{j+1}, \quad (1 \leq j \leq N),
\]  

(6)

where we understand \(\bar{\varepsilon}_1 = \bar{\varepsilon}_{N+1}\). Let us introduce the Fock space \(\mathcal{F}_{l,k}, (l, k \in \sum_{j=1}^{N} \mathbb{Z} \bar{\varepsilon}_j)\), of the bosons, generated by \(\beta_m^j, (m > 0)\) over the vacuum vector \(|l, k\rangle, (l, k \in \sum_{j=1}^{N} \mathbb{Z} \bar{\varepsilon}_j)\),

\[
\beta_m^j |l, k\rangle = 0, \quad (m > 0; j = 1, 2, \cdots, N),
\]  

(7)

\[
P_\alpha |l, k\rangle = \left( \alpha \left| l, \sqrt{\frac{r}{r-1}} - k, \sqrt{\frac{r-1}{r}} \right) |l, k\rangle, \right.
\]  

(8)

\[
|l, k\rangle = \exp \left( \sqrt{\frac{r}{r-1}} Q_l - \sqrt{\frac{r-1}{r}} Q_k \right) |0, 0\rangle.
\]  

(9)
2.2 Deformed W-algebra

In this section, we review the deformed Virasoro algebra and the deformed W algebra $W_{q,t}(\hat{gl}_N)$, following [7891011151617].

**Definition 1** For $N = 2, 3, 4, \ldots$, the deformed W-algebra $W_{q,t}(\hat{gl}_N)$ is generated by the generators $T^j_m$, $(1 \leq j \leq N, m \in \mathbb{Z})$, with the defining relations (10) of the series $T_j(z) = \sum_{m \in \mathbb{Z}} T^j_m z^{-m}$.

\[
f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) = c \sum_{k=1}^{\infty} \frac{\Delta(x^{2k+1})}{x^k} \left( \delta \left( x^{j-i+2k}z_2 \right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^k z_2) - \delta \left( x^{j-i-2k}z_2 \right) f_{i-k,j+k}(x^{j+i})T_{i-k}(x^k z_1)T_{j+k}(x^{-k}z_2) \right),
\]

where we used the delta-function $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. Here we have set the constant $c$ and the structure functions $\Delta(z)$ and $f_{i,j}(z)$, $(1 \leq i, j \leq N)$ by

\[
c = \frac{(1 - x^{2r})(1 - x^{-2(r-1)})}{(1 - x^2)}, \quad \Delta(z) = \frac{(1 - x^{2r-1}z)(1 - x^{1-2r}z)}{(1 - xz)(1 - x^{-1}z)},
\]

\[
f_{i,j}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{(1 - x^{2rm})(1 - x^{-2(r-1)m})(1 - x^{2m\min(i,j)})(1 - x^{-2m\max(i,j)}))}{m(1 - x^{2m})(1 - x^{-2sm})} (x^{[i-j]}z)^m \right).
\]

**Example** Upon the specialization $N = s = 2$, we have the deformed Virasoro algebra $Vir_{q,t}$. Upon this specialization the generators $T^j_1$ can be regarded as $T^j_1 = 1$. The generators $T^j_0 = T^j$ satisfy the following defining relation.

\[
\sum_{l=0}^{\infty} f_l(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) = c(c^2 - x^{-2}) \delta_{n+m,0},
\]

where the structure constant $f_l$ is given by $\sum_{l=0}^{\infty} f_l z^l = f_{1,1}(z)$. In the CFT limit ($x \to 1$), we get the Virasoro algebra with the central charge $c_{CFT} = 1 - \frac{6}{r(r-1)}$.

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12} c_{CFT} m(m^2 - 1) \delta_{m+n,0}.
\]

**Proposition 2** For $N = 2$ the deformed W-algebra $W_{q,t}(\hat{gl}_2)$ is realized by the bosons $|3\rangle$, $|4\rangle$ on the Fock space.
\[ T_1(z) = \Lambda_1(z) + \Lambda_2(z), \quad T_2(z) =: \Lambda_1(x^{-1}z)\Lambda_2(xz) : , \tag{15} \]

where we have set
\[
\Lambda_1(z) = x^{-\sqrt{r(r-1)}P_{a_1}} : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} - x^{-rm}) \beta_{m}^1 z^{-m} \right) : , \tag{16} 
\]
\[
\Lambda_2(z) = x^{-\sqrt{r(r-1)}P_{a_2}} : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} - x^{-rm}) \beta_{m}^2 z^{-m} \right) : . \tag{17} 
\]

**Proposition 3**  
For \( N = 3, 4, \cdots \) the deformed \( W \)-algebra \( W_{q,t}(gl_N) \) is realized by the bosons (3), (4) on the Fock space.

\[
T_j(z) = \sum_{1 \leq s_1 < s_2 < \cdots < s_j \leq N} : \Lambda_{s_1}(x^{-j+1}z)\Lambda_{s_2}(x^{-j+3}z) \cdots \Lambda_{s_j}(x^{j-1}z) : , \tag{18} 
\]

where we have set
\[
\Lambda_j(z) = x^{-\sqrt{r(r-1)}P_{j}} : \exp \left( \sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \beta_{m}^j z^{-m} \right) : (1 \leq j \leq N). \tag{19} 
\]

Here the symbol : \( * : \) stands for usual normal ordering of bosons, i.e. \( \beta_{m}^j \) with \( m > 0 \) should be moved to the right.

### 2.3 Screening current

In this section we review the screening currents for the deformed Virasoro algebra and the deformed \( W \) algebra, following [7, 8, 9, 10, 11, 12, 13, 15, 16, 17].

**Definition 4**  
For \( N = 2 \) we introduce the operator \( F_j(z), (j = 1, 2) \), called the screening current for the deformed \( W \)-algebra \( W_{q,t}(gl_2) \). We define

\[
F_1(z) = e^{-i\sqrt{P_{a_1}}z} x^{-\sqrt{P_{a_1}}P_{a_1}} + \hat{\phi} : \exp \left( \sum_{m \neq 0} \frac{1}{m} (\beta_{m}^1 - \beta_{m}^2) z^{-m} \right) : , \tag{20} 
\]
\[
F_2(z) = e^{-i\sqrt{P_{a_2}}z} x^{-\sqrt{P_{a_2}}P_{a_2}} + \hat{\phi} : \exp \left( \sum_{m \neq 0} \frac{1}{m} (-x^{sm} \beta_{m}^1 + x^{-sm} \beta_{m}^2) z^{-m} \right) : . \tag{21} 
\]
Definition 5  For $N = 3, 4, \cdots$ we introduce the operator $F_j(z), (1 \leq j \leq N)$, called the screening current for the deformed $W$-algebra $W_{q,t}(gl_N)$. Let us set

$$F_j(z) = e^{-i\sqrt{r^\tau}Q_{n_j}(x^{\frac{2N}{N}} - 1)z} - \sqrt{r^\tau}P_{n_j} + \frac{i}{r^\tau}$$

$$\times : \exp \left( \sum_{m \neq 0} \frac{1}{m} (\beta^j_m - \beta^{j+1}_m)(x^{\frac{2N}{N}}z)^{-m} \right) : , \ (1 \leq j \leq N - 1), \quad (22)$$

$$F_N(z) = e^{-i\sqrt{r^\tau}Q_{n_N}(x^{2s-N}z)} - \sqrt{r^\tau}P_{n_N} + \frac{i}{r^\tau}z^2 \sqrt{r^\tau}P_{n_N} + \frac{i}{r^\tau}z$$

$$\times : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{-2sm} \beta^N_m - \beta^1_m z)^{-m} \right) : . \quad (23)$$

Proposition 6  For $N = 2$ the screening currents $F_1(z), F_2(z)$ satisfy the following commutation relations.

$$[u_1 - u_2]_r[u_1 - u_2 + 1]_rF_j(z_1)F_j(z_2)$$

$$= [u_1 - u_2]_r[u_2 - u_1 + 1]_rF_j(z_2)F_j(z_1), \quad (j = 1, 2), \quad (24)$$

$$\left[ u_1 - u_2 + \frac{s}{2} - 1 \right]_r \left[ u_1 - u_2 - \frac{s}{2} \right]_r F_2(z_1)F_1(z_2)$$

$$= \left[ u_2 - u_1 + \frac{s}{2} - 1 \right]_r \left[ u_2 - u_1 - \frac{s}{2} \right]_r F_1(z_2)F_2(z_1). \quad (25)$$

Proposition 7  For $N = 2$ the commutation relations between $\Lambda_j(z)$ and $F_j(z)$ are given by

$$[\Lambda_1(z_1), F_1(z_2)] = (x^{-r^*} - x^{r^*}) \delta(x^{r^*}z_1/z_2)\mathcal{A}(x^{r^*}z_2), \quad (26)$$

$$[\Lambda_2(z_1), F_1(z_2)] = (x^{r^*} - x^{-r^*}) \delta(x^{-r^*}z_1/z_2)\mathcal{A}(x^{r^*}z_2), \quad (27)$$

$$[\Lambda_1(z_1), F_2(z_2)] = (x^{r^*} - x^{-r^*}) \delta(x^{-r^*}z_1/z_2)\eta(\mathcal{A}(x^{r^*}z_2)), \quad (28)$$

$$[\Lambda_2(z_1), F_2(z_2)] = (x^{-r^*} - x^{r^*}) \delta(x^{r^*}z_1/z_2)\eta(\mathcal{A}(x^{r^*}z_2)), \quad (29)$$

where we have set

$$\mathcal{A}(z) = e^{i\sqrt{r^\tau}Q_{n_1}z} - \sqrt{r^\tau}P_{n_1} + \frac{i}{r^\tau}$$

$$\times : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} \beta^1_m - x^{-rm} \beta^2_m)z^{-m} \right) : . \quad (30)$$

Proposition 8  For $N = 3, 4, 5, \cdots$, the screening currents $F_j(z)$ satisfy the following commutation relations.
\[
\left[u_1 - u_2 - \frac{s}{N}\right] F_j(z_1) F_{j+1}(z_2) = \left[u_2 - u_1 + \frac{s}{N} - 1\right] F_{j+1}(z_2) F_j(z_1), \quad (31)
\]

\[
[u_1 - u_2, [u_1 - u_2 + 1, F_j(z_1) F_j(z_2)] = [u_2 - u_1, [u_2 - u_1 + 1, F_j(z_2) F_j(z_1)]], \quad (32)
\]

for \(1 \leq j \leq N\). We understand \(F_{N+1}(z) = F_1(z)\). We have

\[
F_i(z_1) F_j(z_2) = F_j(z_2) F_i(z_1), \quad \text{otherwise.} \quad (33)
\]

**Proposition 9**  
For \(N = 3, 4, 5, \ldots\) the commutation relations between \(\Lambda_j(z)\) and \(F_j(z)\) are given by

\[
[\Lambda_j(z_1), F_j(z_2)] = (x^{-r^*} - x^{-r^*}) \delta(x^{-\frac{2j}{N}} z_1/z_2) \Lambda_j(x^{\frac{2j}{N}} z_1), \quad (1 \leq j \leq N - 1), \quad (34)
\]

\[
[\Lambda_{j+1}(z_1), F_j(z_2)] = (x^{-r^*} - x^{-r^*}) \delta(x^{-\frac{2j}{N} - r} z_1/z_2) \Lambda_j(x^{\frac{2j}{N} + r} z_2), \quad (1 \leq j \leq N - 1), \quad (35)
\]

\[
[\Lambda_N(z_1), F_N(z_2)] = (x^{-r^*} - x^{-r^*}) \delta(x^{-2} z_1/z_2) \Lambda_N(x^{-r} z_2), \quad (36)
\]

\[
[\Lambda_1(z_1), F_N(z_2)] = (x^{-r^*} - x^{-r^*}) \delta(x^{-r} z_1/z_2) \Lambda_N(x^{-r} z_2), \quad (37)
\]

where we have set

\[
\mathcal{A}_j(z) = e^{i\sqrt{\frac{2}{r}} Q_{o,j} x^{-\sqrt{\frac{2}{r}} (P_j + P_{j+1})} (x^{-j} z)} \sqrt{\frac{2}{r}} P_{o,j} + \frac{x^*}{r},
\]

\[
\times : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} \beta_m^j - x^{-rm} \beta_{m+1}^j) z^{-m} \right) ; \quad (1 \leq j \leq N - 1), \quad (38)
\]

\[
\mathcal{A}_N(z) = e^{i\sqrt{\frac{2}{r}} Q_{o,N} x^{-\sqrt{\frac{2}{r}} (P_{N} + P_1)} (x^{2s-N} z)} \sqrt{\frac{2}{r}} P_{o,N} + \frac{x^*}{r} z^{-\sqrt{\frac{2}{r}} P_1 + \frac{x^*}{r}},
\]

\[
\times : \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{(r-2)s} m \beta_m^N - x^{-rm} \beta_{m+1}^N) z^{-m} \right) ; \quad (39)
\]

### 3 Integrals of Motion

In this section we review the integrals of motion for the deformed Virasoro algebra and the deformed \(W\)-algebra, following \[15,16,17\].
3.1 Local integrals of motion $\mathcal{I}_n$

We define the operators $\mathcal{I}_n$, $(n = 1, 2, 3, \cdots)$, which we call the local integrals of motion for the $W_{q,t}(gl_N)$, $(N = 2, 3, 4, \cdots)$.

**Definition 10** For the regime $\text{Re}(s) > 2$ and $\text{Re}(r^*) < 0$, we define

\[
\mathcal{I}_n = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{z_j} T_1(z_1) T_1(z_2) T_1(z_3) \cdots T_1(z_n) \\
\times \prod_{1 \leq j < k \leq n} \frac{[u_k - u_j, s][u_k - u_j + r]}{[u_k - u_j + 1, s][u_k - u_j + r^*]}, \quad (n = 1, 2, 3, \cdots). \tag{40}
\]

Here the contour $C$ encircles $z_j = 0$ in such a way that $z_j = x^{-2+2sl}z_k, x^{-2r^*+2sl}z_k$, $(l = 0, 1, 2, \cdots)$ is inside and $z_j = x^{2-2sl}z_k, x^{2r^*-2sl}z_k$, $(l \in \mathbb{N})$ is outside for $1 \leq j < k \leq n$. We call $\mathcal{I}_n$ the local integrals of motion.

The definitions of the local integrals of motion $\mathcal{I}_n$ for generic $\text{Re}(s) > 0$ and $\text{Re}(r) > 0$ should be understood as analytic continuation.

3.2 Nonlocal integrals of motion $\mathcal{G}_n$

We define the operators $\mathcal{G}_m$, $(m = 1, 2, 3, \cdots)$, which we call the nonlocal integrals of motion for the $W_{q,t}(gl_N)$, $(N = 2, 3, 4, \cdots)$.

**Definition 11** For $N = 2$ and the regime $0 < \text{Re}(s) < 2$ and $\text{Re}(r) > 0$, we define

\[
\mathcal{G}_m = \int \cdots \int \prod_{t=1,2} \prod_{j=1}^{m} \frac{dz_j^{(t)}}{z_j^{(t)}} F_1(z_1^{(1)}) F_1(z_2^{(1)}) \cdots F_1(z_m^{(1)}) F_2(z_1^{(2)}) F_2(z_2^{(2)}) \cdots F_2(z_m^{(2)}) \\
\times \prod_{t=1,2} \prod_{1 \leq j < k \leq m} \left[ u_j^{(t)} - u_k^{(t)} \right]_r \left[ u_k^{(t)} - u_j^{(t)} - 1 \right]_r \\
\times \prod_{j,k=1}^{m} \left[ u_j^{(1)} - u_k^{(2)} + \frac{s}{2} \right]_r \left[ u_k^{(2)} - u_j^{(1)} + \frac{s}{2} - 1 \right]_r \\
\times \prod_{t=1,2} \sum_{j=1}^{m} \left[ u_j^{(t)} - u_j^{(t+1)} \right] - \sqrt{rr^*} P_{ct+1}, \quad (m = 1, 2, \cdots). \tag{41}
\]

Here the contour $I$ encircles $z_j^{(t)} = 0$, $(t = 1, 2; 1 \leq j \leq m)$ in such a way that
\[ |x^{s+2lr}z_k^{(2)}|, |x^{-s+2lr}z_k^{(2)}| < |z_j^{(1)}| < |x^{-s-2lr}z_k^{(2)}|, |x^{s-2-2lr}z_k^{(2)}|, \] (42)

for \(1 \leq j, k \leq m\) and \(l \in \mathbb{N}\). We call \(\mathcal{G}_m\) the nonlocal integrals of motion for the deformed \(W\)-algebra \(W_{q,t}(gl_N)\).

**Definition 12** For \(N = 3, 4, 5, \cdots\) and the regime \(0 < \text{Re}(s) < N\) and \(\text{Re}(r) > 0\), we define

\[
\mathcal{G}_m = \int \cdots \int \prod_{t=1}^N \prod_{j=1}^m \frac{dz_j^{(t)}}{z_j^{(t)}} F_1(z_1^{(1)}) F_1(z_2^{(1)}) \cdots F_1(z_m^{(1)}) \\
\times F_2(z_1^{(2)}) F_2(z_2^{(2)}) \cdots F_2(z_m^{(2)}) \cdots F_N(z_1^{(N)}) F_N(z_2^{(N)}) \cdots F_N(z_m^{(N)}) \\
\times \prod_{t=1}^{N-1} \prod_{j=1}^m \left[ u_j^{(t)} - u_k^{(t)} \right]_r \left[ u_k^{(t)} - u_j^{(t)} - 1 \right]_r \\
\times \prod_{t=1}^N \prod_{j,k=1}^m \left[ u_j^{(t)} - u_k^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{j,k=1}^m \left[ u_j^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r \\
\times \prod_{t=1}^N \left[ \sum_{j=1}^m (u_j^{(t)} - u_j^{(t+1)}) - \sqrt{r}r^s P_{t+1} \right]_r , \ (m = 1, 2, \cdots). \] (43)

Here the contour \(I\) encircles \(z_j^{(t)} = 0\), \((1 \leq t \leq N; 1 \leq j \leq m)\) in such a way that

\[
|x^{\frac{2s}{N}+2lr}z_k^{(t+1)}| < |z_j^{(t)}| < |x^{-\frac{2s}{N}-2lr}z_k^{(t+1)}|, \ (1 \leq t \leq N - 1), \] (44)
\[
|x^{\frac{2s}{N}+2lr}z_k^{(1)}| < |z_j^{(N)}| < |x^{-\frac{2s}{N}-2lr}z_k^{(1)}|, \] (45)

for \(1 \leq j, k \leq m\) and \(l \in \mathbb{N}\). We call \(\mathcal{G}_m\) the nonlocal integrals of motion for the deformed \(W\)-algebra \(W_{q,t}(gl_N)\).

The definitions of the nonlocal integrals of motion \(\mathcal{G}_n\) for generic \(\text{Re}(s) > 0\) and \(\text{Re}(r) > 0\) should be understood as analytic continuation.

### 3.3 Main results

In this section we state the main results.

**Theorem 13** For \(N = 2, 3, 4, \cdots\), the local integrals of motion \(\mathcal{I}_n\) and the nonlocal integrals of motion \(\mathcal{G}_m\) commute with each other.

\[ [\mathcal{I}_n, \mathcal{I}_m] = [\mathcal{I}_n, \mathcal{G}_m] = [\mathcal{G}_n, \mathcal{G}_m] = 0, \ (m, n = 1, 2, \cdots). \] (46)
These commutativities are understood as consequence of commuting family of the Feigin-Odesskii algebra [14].

**Theorem 14** For $N = 2, 3, 4, \cdots$, the local integrals of motion $I_n$ and the nonlocal integrals of motion $G_n$ are invariant under the action of the Dynkin-diagram automorphism $\eta$.

$$\eta(I_n) = I_n, \quad \eta(G_n) = G_n, \quad (n = 1, 2, \cdots).$$

(47)

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