Research Article

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Anomalous pseudo-parabolic Kirchhoff-type dynamical model

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Abstract: In this paper, we study an anomalous pseudo-parabolic Kirchhoff-type dynamical model aiming to reveal the control problem of the initial data on the dynamical behavior of the solution in dynamic control system. Firstly, the local existence of solution is obtained by employing the Contraction Mapping Principle. Then, we get the global existence of solution, long time behavior of global solution and blowup solution for \( J(u_0) \leq d \), respectively. In particular, the lower and upper bound estimates of the blowup time are given for \( J(u_0) < d \). Finally, we discuss the blowup of solution in finite time and also estimate an upper bound of the blowup time for high initial energy.

Keywords: Pseudo-parabolic Kirchhoff-type equation; Global existence; Asymptotic behavior; Blowup

MSC: 35B40, 35R11, 35K55

1 Introduction and main result

The paper is devoted to the study of an anomalous pseudo-parabolic Kirchhoff-type dynamical model as follows

\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
\frac{\partial u}{\partial t} + M([u]_s^2)(-\Delta)^s u + (-\Delta)^s u_t = |u|^{q-2} u, & \text{in } \Omega \times \mathbb{R}^+, \\
u(x, 0) = u_0(x), & \text{in } \Omega, \\
u(x, t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+_0,
\end{array}
\right.
\end{aligned}
\]

(1.1)

where \( s \in (0, 1), N > 2s, \Omega \subset \mathbb{R}^N \) is a bounded domain with Lipschitz boundary \( \partial \Omega \). The Kirchhoff function \( M(t) = \frac{t^\lambda}{\lambda} \) for \( t \in \mathbb{R}^+_0 \), here \( 1 \leq \lambda < \frac{N}{N-2s} \), and \( q \) satisfy \( 2\lambda < q \leq 2s^*_f \), where \( s^*_f \) is the fractional critical exponent given by

\[
2s^*_f := \frac{2N}{N - 2s}.
\]

And \([u]_s\) is the Gagliardo seminorm of \( u \) defined by

\[
[u]_s := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

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\((-\Delta)^s\) is the fractional Laplacian which, up to a normalization constant, is defined for any \(x \in \mathbb{R}^N\)
\[
(-\Delta)^s \varphi(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy,
\]
for any \(\varphi \in C^\infty_0(\mathbb{R}^N)\), where \(B_\varepsilon(x)\) denotes the ball in \(\mathbb{R}^N\) with radius \(\varepsilon > 0\) centered at \(x \in \mathbb{R}^N\). We can refer to [5, 23–25, 35–37] for more details on nonlocal operators and nonlocal Sobolev spaces.

Problem (1.1) is a class of nonlocal fractional diffusion problem, which is related to the anomalous diffusion theory. A usual model for anomalous diffusion is the linear evolution equation involving the fractional Laplacian
\[
\partial_t u + (-\Delta)^s u = 0,
\]
which derives asymptotically from basic random walks models, see [2, 22, 39] and references therein. We denote by \(u(x, t)\) the probability of finding the particle at the point \(x\) at time \(t\). Through a series of calculations, we can obtain \(\partial_t u(x, t) = -c_n,s(-\Delta)^s u(x, t)\) for some \(c_n,s > 0\), which shows that, for small time and space steps, the above probabilistic process approaches a fractional heat equation. Another nonlinear anomalous diffusion equation is the fractional porous medium equation \(\partial_t u + (-\Delta)^s (u^m) = 0\) with \(0 < s < 1\) and \(m > 0\), which was first proposed by De Pablo et al. in [32]. Many important results on these equations have been obtained, see an overview in [41] and references therein.

To the best of our knowledge, fractional Laplacian operator and related equations have a growing wide utilization in many important fields, as explained by Caffarelli in [3] and Vázquez in [40]. In particular, the steady state of problem (1.1) without strong damping term, first proposed by Fiscella and Valdinoci in [12] by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, is a fractional version of the so-called stationary Kirchhoff model. Subsequently, the existence of weak solutions which solves the above stationary problem was obtained. Later, Fu and Pucci in [9] proved the existence of global solutions with exponential decay and showed the blow-up in finite time of solutions to the space-fractional diffusion equation
\[
u_t + (-\Delta)^s u = |u|^{p-1} u, \quad \text{in } \Omega \times \mathbb{R}^+,
\]
where \(\Omega \subset \mathbb{R}^N\) is a bounded domain, \(p\) satisfies \(1 < p \leq 2^*_s - 1 = (N + 2s)/(N - 2s)\) and \(N > 2s\).

In recent years, much interest has grown on Kirchhoff-type problems, see for example [12, 28]. In these papers, to obtain the existence of weak solutions, the authors always assume that the Kirchhoff function \(M : \mathbb{R}_0^+ \to \mathbb{R}^+\) is a continuous and nondecreasing function and satisfies the following condition:
\[
\text{there exists } m_0 > 0 \text{ such that } M(t) \geq m_0 \text{ for all } t \in \mathbb{R}_0^+.
\]
A typical example is \(M(t) = m_0 + bt^m\) with \(m_0 > 0\), \(b \geq 0\) for all \(t \in \mathbb{R}_0^+\). Hence, we can divide the problem into degenerate and non-degenerate cases according to \(M(0) = 0\) and \(M(0) > 0\) respectively. For the non-degenerate Kirchhoff-type problem, we can refer to [10, 30]. It is worthwhile pointing out that the degenerate case is rather interesting and is treated in well-known papers in Kirchhoff theory, see for example [4]. From a physical point of view, the fact that \(M(0) = 0\) means that the base tension of the string is zero. For some recent results in the degenerate case, see for instance [1, 26, 29, 43, 45]. In this regard, Pan et al. in [31] studied for the first time the degenerate Kirchhoff-type diffusion problem involving fractional \(p\)-Laplacian of following equation
\[
u_t + [u^{(\lambda-1)p}(-\Delta)^s u] = |u|^{q-2} u, \quad \text{in } \Omega \times \mathbb{R}^+,
\]
where \(\Omega \subset \mathbb{R}^N\), \(p < q < Np/(N - sp)\) with \(1 < p < N/s\) and \(1 \leq \lambda < N/(N - sp)\). They obtained the existence of a global solution by combining the Galerkin method with potential well theory to discover the control mechanisms of the initial data on the dynamical behavior of the solution. Yang et al. in [50] studied the same problem with \(p \lambda < q < Np/(N - sp)\), they obtained the blow up of solutions by applying the concave method. Moreover, the authors estimated an upper bound of blow-up time in the sub-critical initial energy case \(f(u_0) < d\) and arbitrary positive initial energy case \(f(u_0) > 0\). Later, by implementing the same theory as
shown in [31], for the initial boundary value problem of (1.3) with $\lambda = 1$ and the more general nonlinearity $f(u)$ instead of $|u|^{q-2}u$, authors in [19] studied the existence and nonexistence of global weak solutions in the cases of $f(u_0) < d$, $f(u_0) = d$ and $f(u_0) > d$, respectively. Moreover, the authors estimated an upper bound of blow-up time for low and high initial energies. Xiang et al. [44] considered the initial boundary value problem of the following Kirchhoff type equation

$$u_t + M(|u|^{q-2}u)\Delta u = |u|^{p-2}u, \quad \text{in } \Omega \times \mathbb{R}^+,$$

where $\Omega \subset \mathbb{R}^N$, $M : [0, +\infty) \to [0, +\infty)$ is a continuous function and there exist two constants $m_0$ and $\lambda > 1$ such that $M(\sigma) \geq m_0\sigma^{\lambda-1}$, $\sigma \in (0, +\infty)$. For $1 < \lambda < 2N/(N - 2s)$, the local existence of nonnegative solutions of (1.4) is obtained by applying the Galerkin method. Moreover, the blowup conditions for nonnegative weak solution were obtained for $f(u_0) < d$. Then for the initial boundary value problem (1.4), Ding and Zhou [6] proved the global existence and finite time blowup of solution when $f(u_0) \leq d$ for the case $M(\sigma) = m_0\sigma^{\lambda-1}$. Moreover, they showed that the nonnegative weak solution exists globally for $\lambda \geq 2N/(N - 2s)$. Also Ding and Zhou in [7] gave the global existence and finite time blowup results with $f(u_0) > d$.

If $s \uparrow 1, M \equiv 1$, then the equation in (1.1) reduces to the following equation

$$u_t - \Delta u_t - \Delta u = u^p, \quad \text{in } \Omega \times \mathbb{R}^+.$$  

In [48], Xu and Su used the family of the potential wells to prove the nonexistence of solutions with initial energy $f(u_0) \leq d$, and obtained finite time blowup with high initial energy $f(u_0) > d$ by comparison principle. Later, Xu et al. [49] discussed the same problem, they established a new finite time blowup theorem for problem (1.5) and estimated the upper bound of blowup time for $f(u_0) > 0$. Previously, Liu and Zhao in [21] considered the initial-boundary value problem $u_t - \Delta u = f(u)$ with initial data $f(u_0) < d$ for $I(u_0) < 0$ and $I(u_0) \geq 0$, and initial data $f(u_0) = d$ for $I(u_0) > 0$. Xu in [46] studied the same problem with critical initial data $f(u_0) = d$, $I(u_0) < 0$. A powerful technique for treating the above problem is the so-called potential well method, which was established by Payne and Sattinger in [27]. Since then, the potential well method has been widely used to study the well-posedness of solution for evolution equations, such as [18, 46, 47]. Gazzola and Weth in [13] studied the initial-boundary value problem of $u_t - \Delta u = |u|^{p-1}u$, they proved finite time blow-up of solutions with high initial energy $f(u_0) > d$ by the comparison principle and variational methods. Recently, the threshold results of global existence and finite time blowup for several types of pseudo-parabolic equations were established in [33, 42].

It is worthy pointing out that the Kirchhoff-type parabolic problem

$$u_t - M\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u = |u|^{q-1}u, \quad \text{in } \Omega \times \mathbb{R}^+$$

was studied by Han et al. [14] where the global existence and finite time blowup of solutions were proved in the sub-critical, critical and super-critical cases. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain, $M(\tau) = a + b\tau$ with $a, b > 0$. In [15], by some differential inequalities, the authors investigated the upper and lower bounds of blowup time for the weak solution to (1.6).

Motivated by the above works, the main objective of this paper is to consider a more complicated case of the problem (1.4) studied in [6, 7, 44] by taking damping term $(-\Delta)^s u_t$ in the fractional setting. More precisely, we focus on the local and global well-posedness of degenerate Kirchhoff’s model of parabolic type (1.1), by using potential well theory and concave function method.

The outline of this paper is as follows. In Section 2, we recall some necessary definitions and properties of the fractional Sobolev spaces and introduce the family of potential wells. In Section 3, we prove the existence of local solutions for problem (1.1). In Section 4, we prove the global existence, the finite time blow-up, the asymptotic behavior for problem (1.1) and give the lifespan estimates of the blowup solution with $f(u_0) < d$. In Section 5, we parallelly extend some conclusions for the sub-critical initial energy case to the critical initial energy. In Section 6, by constructing a new unstable set and using some differential inequality techniques, we also study the finite time blowup solution for problem (1.1) and give the upper bound estimate of the blowup time at arbitrary positive initial energy level.
2 Preliminaries

2.1 Functional spaces

In this section, we first recall some necessary definitions and properties of the fractional Sobolev spaces, see also [5, 11] for further details.

Throughout the paper, $s \in (0, 1)$, $N > 2s$ and $2 \lambda < q < 2\lambda'$. We denote $Q = \mathbb{R}^{2N}\setminus \mathbb{S}$, where $\mathbb{S} = \mathbb{C}(\Omega) \times \mathbb{C}(\Omega) \subset \mathbb{R}^{2N}$, and $\mathbb{C}(\Omega) = \mathbb{R}^{N} \setminus \Omega$. $W$ is a linear space of Lebesgue measurable functions from $\mathbb{R}^{N}$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $u$ in $W$ belongs to $L^{2}(\Omega)$ and

$$\int_{Q} |u(x) - u(y)|^{2} |x - y|^{N+2s} dxdy < \infty.$$ 

The space $W$ is equipped with the norm

$$\|u\|_{W} := \left( \|u\|_{L^{2}(\Omega)}^{2} + \int_{Q} |u(x) - u(y)|^{2} |x - y|^{N+2s} dxdy \right)^{\frac{1}{2}},$$

It is easy to get that $\| \cdot \|_{W}$ is a norm on $W$. We shall work in the closed linear subspace

$$W_{0} := \{ u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^{N}\setminus \Omega \}. \quad (2.1)$$

By [35], we can get an equivalent norm on $W_{0}$ defined as

$$\|v\|_{W_{0}} := \left( \int_{Q} \frac{|v(x) - v(y)|^{2}}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

We put

$$(u, v)_{X_{0}} := (u, v) + (u, v)_{W_{0}}, \quad \|u\|_{X_{0}}^{2} := \|u\|_{2}^{2} + \|u\|_{W_{0}}^{2},$$

where $\| \cdot \|_{X_{0}}$ is an equivalent norm over $W_{0}$.

**Lemma 2.1** ([35, Lemma 6]). Let $K : \mathbb{R}^{N}\setminus \{0\} \rightarrow (0, \infty)$ satisfy $K(x) = |x|^{-\lambda(N+2s)}$. Then there exists a positive constant $C_{0} = C_{0}(N, 2, s)$ such that for any $v \in W_{0}$ and $v \in [1, 2]$, 

$$\|v\|_{L^{\lambda}(\Omega)}^{2} \leq C_{0} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{2}}{|x - y|^{N+2s}} dxdy \leq \tilde{C} \int_{Q} \frac{|v(x) - v(y)|^{2}}{|x - y|^{N+2s}} dxdy.$$

**Lemma 2.2** (Gronwall inequality). Assume $y(t) \in L^{1}[0, T]$, and there exist constants $a$ and $b$ such that

$$y(t) \leq a + b \int_{0}^{T} y(z) dz, \quad 0 \leq t \leq T,$$

then

$$y(t) \leq ae^{bt}, \quad 0 \leq t \leq T. \quad (2.3)$$

**Definition 2.1** (Weak solution). A function $u \in L^{\infty}(0, \infty; W_{0})$ is said to be a weak solution of problem (1.1), if $u_{t} \in L^{2}(0, \infty; X_{0})$ and $u_{0} \in W_{0}$ for a.e. $t > 0$,

$$\int_{\Omega} \partial_{t} uv dx + \langle u, v \rangle_{W_{0}} + \langle u_{t}, v \rangle_{W_{0}} = \int_{\Omega} |u|^{{q-2}} u \phi dx, \quad (2.4)$$
Moreover, there holds
\[ (u_t, \phi)_{W_0} := \iint_Q \frac{(u_t(x) - u_t(y)) \cdot (v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy \]
and
\[ (u, v)_{W_0} := M(\|u\|_{W_0}^2) \iint_Q \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy \]
for any \( v \in W_0 \).

Then we define the potential energy functional of problem (1.1) as follows
\[ J(u) := \frac{1}{2A} \|u\|_{W_0}^2 - \frac{1}{q} \|u\|_q^q, \tag{2.5} \]
the Nehari functional
\[ I(u) := \|u\|_{W_0}^2 - \|u\|_q^q. \tag{2.6} \]

Moreover, there holds
\[ \int_0^t \|u_t\|_{X_0}^2 \, dt + f(u) \leq J(u_0). \tag{2.7} \]

Next we introduce the Nehari manifold
\[ N := \{ u \in X_0 \mid I(u) = 0, \|u\|_{X_0} \neq 0 \}. \]
Furthermore, we set
\[ N_+ := \{ u \in X_0 \mid I(u) > 0 \}, \]
\[ N_- := \{ u \in X_0 \mid I(u) < 0 \}. \]

The potential well depth is defined as
\[ d := \inf_{u \in N} J(u). \tag{2.8} \]

Further we give some sets as follows
\[ W_p := \{ u \in X_0 \mid J(u) < d, I(u) > 0 \} \cup \{ 0 \}, \]
\[ V_p := \{ u \in X_0 \mid J(u) < d, I(u) < 0 \}. \]

### 2.2 Family of potential wells

In this section, we shall introduce a family of potential wells \( W_\delta \) and \( V_\delta \), and give a series of their properties for problem (1.1). Firstly, let the definitions of functionals \( J(u) \), \( I(u) \) and the potential well \( W_p \) with its depth \( d \) given above hold. Next, we give some properties of above sets and functionals as follows.

**Lemma 2.3.** Let \( u \in X_0 \) and \( \|u\|_{X_0} \neq 0 \). Then
(i) \( \lim_{\theta \to 0} J(\theta u) = 0 \), \( \lim_{\theta \to \infty} J(\theta u) = -\infty \).
(ii) On the interval \( 0 < \theta < \infty \), there exists a unique \( \theta^* = \theta^*(u) \), such that
\[ \frac{d}{d\theta} J(\theta u) \bigg|_{\theta = \theta^*} = 0. \]
(iii) \( J(\theta u) \) is increasing for \( 0 < \theta < \theta^* \), decreasing for \( \theta^* \leq \theta < \infty \) and take the maximum at \( \theta = \theta^* \).
(iv) \( I(\theta u) > 0 \) for \( 0 < \theta < \theta^* \), \( I(\theta u) < 0 \) for \( \theta^* < \theta < \infty \) and \( I(\theta^* u) = 0 \).
Proof.

(i) By (2.5), we know
\[ J(\theta u) = \frac{\theta^{2\lambda}}{2\lambda} \| u \|_{W_0}^{2\lambda} - \frac{\theta^{q}}{q} \| u \|_q^q, \]
which gives
\[ \lim_{\theta \to 0} J(\theta u) = 0 \]
and
\[ \lim_{\theta \to \infty} J(\theta u) = -\infty \]
by the fact that \( q > 2\lambda \geq 2 \).

(ii) An easy calculation shows that
\[ \frac{d}{d\theta} J(\theta u) = \theta^{2\lambda-1} \| u \|_{W_0}^{2\lambda} - \theta^{q-1} \| u \|_q^q, \]
let \( \frac{d}{d\theta} J(\theta u) = 0 \), we get
\[ \theta^* = \left( \frac{\| u \|_{W_0}^{2\lambda}}{\| u \|_q^q} \right)^{\frac{1}{2\lambda - 1}}, \]
which leads to the conclusion.

(iii) By a direct calculation (2.9) gives \( \frac{d}{d\theta} J(\theta u) > 0 \) for \( 0 < \theta < \theta^* \), \( \frac{d}{d\theta} J(\theta u) < 0 \) for \( \theta^* < \theta < \infty \). Hence, the conclusion of (iii) holds.

(iv) Since
\[ I(\theta u) = \theta^{2\lambda} \| u \|_{W_0}^{2\lambda} - \theta^{q} \| u \|_q^q = \theta \frac{d}{d\theta} J(\theta u), \]
then the conclusion follows immediately.

Now, for \( \delta > 0 \), we define
\[ I_\delta(u) := \delta \| u \|_{W_0}^{2\lambda} - \| u \|_q^q, \]
\[ N_\delta := \{ u \in X_0 \mid I_\delta(u) = 0, \| u \|_{X_0} \neq 0 \} \]
and
\[ d(\delta) := \inf_{u \in N_\delta} J(u). \]

Further we set
\[ W_\delta := \{ u \in X_0 \mid I_\delta(u) > 0, J(u) < d(\delta) \} \cup \{0\}, \]
\[ V_\delta := \{ u \in X_0 \mid I_\delta(u) < 0, J(u) < d(\delta) \} \]
and
\[ r(\delta) := \left( \frac{\delta}{C^*} \right)^{\frac{1}{2\lambda - 1}}, \]
where \( C^* \) is the embedding constant for \( W_0 \to L_q(\Omega) \).

Lemma 2.4. Let \( u \in X_0 \) and \( 0 < \delta < \frac{q}{2\lambda} \), then we have

(i) \( I_\delta(u) \geq 0 \), provided \( 0 < \| u \|_{W_0} \leq r(\delta) \). Particularly, \( I(u) \geq 0 \) when \( 0 < \| u \|_{W_0} \leq r(1) \).

(ii) \( \| u \|_{W_0} > r(\delta) \), provided \( I_\delta(u) < 0 \). Particularly, \( \| u \|_{W_0} > r(1) \) when \( I(u) < 0 \).

(iii) \( \| u \|_{W_0} \geq r(\delta) \) or \( \| u \|_{W_0} = 0 \), provided \( I_\delta(u) = 0 \). Particularly, \( \| u \|_{W_0} \geq r(1) \) or \( \| u \|_{W_0} = 0 \) when \( I(u) = 0 \).

(iv) \( J(u) > 0 \) for \( 0 < \delta < \frac{q}{2\lambda} \), provided \( I_\delta(u) = 0 \) and \( \| u \|_{W_0} \neq 0 \).
Proof.

(i) From \(0 < \|u\|_{W_0} \leq r(\delta)\), it follows that
\[
\|u\|_q^q \leq C_\delta^q \|u\|_{W_0}^q = C_\delta^q \|u\|_{W_0}^{2\lambda} \|u\|_{W_0}^{q-2\lambda} \leq \delta \|u\|_{W_0}^{2\lambda},
\]
that is \(I_\delta(u) \geq 0\).

(ii) It is easy to see \(\|u\|_{W_0} \neq 0\) by \(I_\delta(u) < 0\). Thus from
\[
\delta \|u\|_{W_0}^{2\lambda} < \|u\|_q^q \leq C_\delta^q \|u\|_{W_0}^q = C_\delta^q \|u\|_{W_0}^{2\lambda} \|u\|_{W_0}^{q-2\lambda},
\]
it implies \(\|u\|_{W_0} > r(\delta)\).

(iii) When \(\|u\|_{W_0} = 0\), we can obviously get \(I_\delta(u) = 0\). When \(\|u\|_{W_0} \neq 0\) and \(I_\delta(u) = 0\), from
\[
\delta \|u\|_{W_0}^{2\lambda} = \|u\|_q^q \leq C_\delta^q \|u\|_{W_0}^{2\lambda} \|u\|_{W_0}^{q-2\lambda},
\]
it follows that \(\|u\|_{W_0} \geq r(\delta)\).

(iv) By combining Lemma 2.4 (iii) and \(I_\delta(u) = 0\), there holds
\[
J(u) = \left( \frac{1}{2\lambda} - \frac{\delta}{q} \right) \|u\|_{W_0}^{2\lambda} + \frac{\delta}{q} \|u\|_{W_0}^{2\lambda} - \frac{1}{q} \|u\|_q^q
= \left( \frac{1}{2\lambda} - \frac{\delta}{q} \right) \|u\|_{W_0}^{2\lambda} + \frac{1}{q} I_\delta(u)
= \left( \frac{1}{2\lambda} - \frac{\delta}{q} \right) \|u\|_{W_0}^{2\lambda}.
\]

Obviously, (iv) follows.

Lemma 2.5. Let \(\delta > 0\), then the properties of \(d(\delta)\) can be summarized as follows:

(i) \(d(\delta) = a(\delta)r^{2\lambda}(\delta)\) for \(0 < \delta < \frac{q}{2\lambda}\) and \(a(\delta) = \frac{1}{2\lambda} - \frac{\delta}{q}\).

(ii) \(\lim_{\delta \to 0} d(\delta) = 0\), \(\lim_{\delta \to \infty} d(\delta) = 0\) and \(d(\delta) < 0\) for \(\delta > \frac{q}{2\lambda}\).

(iii) \(d(\delta)\) is increasing for \(0 < \delta \leq 1\), decreasing for \(1 < \delta \leq \frac{q}{2\lambda}\) and takes the maximum at \(\delta = 1\).

Proof.

(i) When \(u \in N_\delta\), then from lemma 2.4 (ii) it gives \(\|u\|_{x_0} \geq r(\delta)\). Further by (2.8) and
\[
J(u) = \left( \frac{1}{2\lambda} - \frac{\delta}{q} \right) \|u\|_{W_0}^{2\lambda} + \frac{1}{q} I_\delta(u)
= \left( \frac{1}{2\lambda} - \frac{\delta}{q} \right) \|u\|_{W_0}^{2\lambda} + \frac{1}{q} I_\delta(u)
= a(\delta)\|u\|_{W_0}^{2\lambda} = a(\delta)r^{2\lambda}(\delta),
\]
it follows that \(d(\delta) = a(\delta)r^{2\lambda}(\delta)\).

(ii) By the conclusion of (i), obviously, (ii) holds.

(iii) For any \(u \in N_\delta\), we shall prove that for all \(1 < \delta'' < \delta' < q/(2\lambda)\) or \(0 < \delta' < \delta'' < 1\), there exist \(\epsilon(\delta', \delta'') > 0\) and \(\nu \in N_\delta\) satisfying \(J(u) - J(\nu) > \epsilon(\delta', \delta'')\). Actually, we can define \(\theta(\delta)\) by \(u \in N_\delta\), then \(I_\delta(\theta(\delta)u) = 0\) and \(\theta(\delta'') = 1\). Taking \(z(\theta) = J(\theta u)\), we get
\[
\frac{d}{d\theta} z(\theta) = \frac{1}{2\lambda} \left( (1 - \delta)||u||_{W_0}^{2\lambda} + I_\delta(\theta u) \right) = \theta^{2\lambda-1}(1 - \delta)||u||_{W_0}^{2\lambda}.
\]

Let \(\nu = \theta(\delta')u\), then \(\nu \in N_\delta\).

For \(0 < \delta' < \delta'' < 1\), we have
\[
J(u) - J(\nu) = z(1) - z(\theta(\delta'))
\]
\[\begin{aligned}
&= \int_{\theta(\delta')}^1 \frac{d}{d\theta} (z(\theta)) d\theta \\
&= \int_{\theta(\delta')}^1 \left( \theta^2 \lambda - 1 \right) \|u\|_{W_0^1}^{2\lambda} - \theta^{q-1} \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( \frac{1}{\delta} \|\theta u\|_{W_0^1}^{2\lambda} - \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta) \|\theta u\|_{W_0^1}^{2\lambda} + \delta \|\theta u\|_{W_0^1}^{2\lambda} - \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta) \|\theta u\|_{W_0^1}^{2\lambda} \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta^{2\lambda-1}) \|u\|_{W_0^1}^{2\lambda} \right) d\theta \\
&> \int_{\theta(\delta')}^1 (1 - \delta^{2\lambda-1}) \frac{(\delta^{2\lambda-1} - 1)}{2\lambda} \theta d\theta \\
&= (1 - \delta^{2\lambda-1}) \left( \frac{1}{2\lambda} \theta \right)
\end{aligned}\]

For \(1 < \delta'' < \delta' < \frac{2\lambda}{q+1}\), we have

\[
J(u) - J(v) = z(1) - z(\theta(\delta'))
\]

\[\begin{aligned}
&= \int_{\theta(\delta')}^1 \frac{d}{d\theta} (z(\theta)) d\theta \\
&= \int_{\theta(\delta')}^1 \left( \theta^2 \lambda - 1 \right) \|u\|_{W_0^1}^{2\lambda} - \theta^{q-1} \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( \frac{1}{\delta} \|\theta u\|_{W_0^1}^{2\lambda} - \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta) \|\theta u\|_{W_0^1}^{2\lambda} + \delta \|\theta u\|_{W_0^1}^{2\lambda} - \|u\|_q^q \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta) \|\theta u\|_{W_0^1}^{2\lambda} \right) d\theta \\
&= \int_{\theta(\delta')}^1 \left( (1 - \delta^{2\lambda-1}) \|u\|_{W_0^1}^{2\lambda} \right) d\theta \\
&> \int_{\theta(\delta')}^1 (1 - \delta^{2\lambda-1}) \frac{(\delta^{2\lambda-1} - 1)}{2\lambda} \theta d\theta \\
&= (1 - \delta^{2\lambda-1}) \left( \frac{1}{2\lambda} \theta \right)
\end{aligned}\]
Therefore, the conclusion of (iii) is proved.

**Lemma 2.6.** For \( u \in X_0 \) and \( 0 < \delta_1 < 1 < \delta_2 < \frac{q}{2} \), if \( 0 < J(u) < d \) and \( d(\delta_1) = d(\delta_2) = J(u) \). Then \( I_\delta(u) \) keeps the invariance of sign for \( \delta \in (\delta_1, \delta_2) \).

**Proof.** \( J(u) > 0 \) implies \( \|u\|_{W_0} \neq 0 \). If the sign of \( I_\delta(u) \) is changeable for \( \delta_1 < \delta < \delta_2 \), by the continuity of \( I_\delta(u) \) in \( \delta \), then there exists a \( \delta \in (\delta_1, \delta_2) \) such that \( I_\delta(u) = 0 \). Therefore, by (iii) of Lemma 2.5 and (2.10), it follows that \( J(u) > d(\delta) \), which obviously contradicts \( d(\delta_1) = d(\delta_2) = J(u) < d(\delta) \).

### 3 Existence and uniqueness of local solution

Inspired by [38], in which Taniguchi considered the existence of a local solution to a Kirchhoff-type wave equation with damping. In this section, we shall prove the local well-posedness of solution to the Kirchhoff-type pseudo-parabolic equation of the form (1.1).

For a given \( T > 0 \), we consider the space \( \mathcal{H} = C([0, T], X_0) \) endowed with the norm

\[
\|u\|_{\mathcal{H}} := \max_{\tau \in [0, T]} \|u\|_{X_0}.
\]

In the following, the existence and uniqueness of solution for the linear problem corresponding to (1.1) is proved.

**Lemma 3.1.** For every \( T > 0 \), every \( u \in \mathcal{H} \) and \( u_0 \in X_0 \), there exists a unique \( v \) satisfying

\[
v \in C([0, T], X_0) \cap C^1([0, T], L^2(\Omega)), \quad v_t \in L^2([0, T], X_0),
\]

which solves the linear problem

\[
\begin{align*}
&v_t + M(\|u\|^2_{W_0})|(-\Delta)^{\frac{q}{2}}v + (-\Delta)^{\frac{q}{2}}v_t| = |u|^{q-2}u, \quad (x, t) \in \Omega \times \mathbb{R}^+ \\
&v(x, 0) = u_0, \quad x \in \Omega, \\
v(x, t) = 0, \quad (x, t) \in (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+,
\end{align*}
\]

**Proof.** The assertion follows from an application of the Galerkin method. By [36], for every \( h \geq 1 \) let \( W_h = \text{Span}\{\omega_1, \cdots, \omega_h\} \), where \( \{\omega_j\} \) is the orthogonal complete system of eigenfunctions of \((-\Delta)^{\frac{q}{2}}\) in \( W_0 \) such that \( \|\omega_j\|_{W_0} = 1 \) and \( \|\omega_j\|_2 = 1 \) for all \( j \). Then, \( \{\omega_j\} \) is orthogonal and complete in \( L^2(\Omega) \) and \( W_0 \); denote by \( \{\lambda_j\} \) the related eigenvalues repeated according to their multiplicity. Let

\[
u_h^0 = \sum_{j=1}^{h} \left( \int_{\Omega} u_0 \omega_j \right) \omega_j,
\]

so that \( u_h^0 \in W_h, u_h^0 \to u_0 \) in \( W_0 \) as \( h \to \infty \). For all \( h \geq 1 \) we seek \( h \) functions \( \gamma_1^h, \cdots, \gamma_h^h \in C^1[0, T] \) such that

\[
v_h(t) = \sum_{j=1}^{h} \gamma_j^h(t) \omega_j,
\]

solving the problem

\[
\begin{align*}
&\int_{\Omega} \left( \dot{v}_h(t) + M(\|u\|^2_{W_0})(-\Delta)^{\frac{q}{2}}v_h + (-\Delta)^{\frac{q}{2}}\dot{v}_h - |u|^{q-2}u \right) \eta \, dx = 0, \\
v_h(0) = u_h^0,
\end{align*}
\]

\[
(3.4)
\]
for every $\eta \in W_h$ and $t \geq 0$. For $j = 1, \cdots, h$, taking $\eta = \omega_j$ in (3.4) yields the following Kirchhoff fractional Laplacian problem for a linear ordinary differential equation with unknown $\gamma_j^h$

$$\begin{align*}
\gamma_j^h(t) + M(\|u\|_{W_0}^2)\gamma_j^h(t) + \lambda_j \gamma_j^h(t) &= \psi_j(t), \\
\gamma_j^h(0) &= \int_\Omega u_0 \omega_j,
\end{align*}$$

(3.5)

where $\psi_j(t) = \int_\Omega |u|^{q-2} u \omega_j \, dx \in C[0, T]$. For all $j$, the above Kirchhoff fractional Laplacian problem yields a unique global solution $\gamma_j^h \in C^1[0, T]$. In turn, this gives a unique $\nu_h$ defined by (3.3) and satisfying (3.4). In particular, (3.3) implies that $\nu_h(t) \in W_0$ for every $t \in [0, T]$, so that fractional Sobolev inequality entails

$$\|\nu_h(t)\|_q \leq c_1 \|\nu_h(t)\|_{W_0},$$

(3.6)

Next, the proof is divided into the following two cases.

**Case 1:** $M(\|u\|_{W_0}^2) \geq m_0 > 0$ for any $u \in W_0$, where $m_0$ is a constant.

Taking $\eta = \nu_h(t)$ into (3.4) and integrating over $[0, t] \subset [0, T]$, we obtain

$$\begin{align*}
\frac{1}{2} M(\|u\|_{W_0}^2)\|\nu_h(t)\|_{W_0}^2 + \int_0^t \|\dot{\nu}_h(t)\|_{X_0}^2 d\tau &= \int_0^t |u|^{q-2} u \dot{\nu}_h(\tau) d\tau + \frac{1}{2} M(\|u\|_{W_0}^2)\|u_0\|_{W_0}^2 + \int_0^t (u, u_t)_{W_0} M(\|u\|_{W_0}^2)\|\nu_h(t)\|_{W_0}^2 d\tau,
\end{align*}$$

(3.7)

for every $h \geq 1$. Since $u \in \mathcal{H}$, $\|u\|_{X_0}$ is bounded. We estimate the last term in the right-hand side thanks to Hölder and Young inequalities

$$\begin{align*}
\int_0^t \int_\Omega |u|^{q-2} u \dot{\nu}_h(\tau) d\tau &\leq \int_0^t \|u\|^{q-1}_{\mathcal{H}} \|\dot{\nu}_h(\tau)\|_{X_0} \frac{\|u\|_{\mathcal{H}}}{\|u\|_{\mathcal{H}}} d\tau, \\
&\leq \int_0^t c\|u\|^{q-1}_{W_0} \|\dot{\nu}_h\|_{W_0} d\tau, \\
&\leq cT + \int_0^t \frac{1}{2} \|\dot{\nu}_h(\tau)\|_{X_0}^2 d\tau,
\end{align*}$$

(3.8)

where $c > 0$ and represent different constants between different lines.

By combining (3.7), (3.8) and Hölder inequality, we obtain

$$\begin{align*}
\frac{1}{2} M(\|u\|_{W_0}^2)\|\nu_h(t)\|_{W_0}^2 + \int_0^t \frac{1}{2} \|\dot{\nu}_h(\tau)\|_{X_0}^2 d\tau &\leq cT + M(\|u_0\|_{W_0}^2)\|u_0\|_{W_0}^2 + \int_0^t (u, u_t)_{W_0} M(\|u\|_{W_0}^2)\|\nu_h\|_{W_0}^2 d\tau, \\
&\leq cT + M(\|u_0\|_{W_0}^2)\|u_0\|_{W_0}^2 + \int_0^t \|u\|_{W_0} \|u_t\|_{W_0} M(\|u\|_{W_0}^2)\|\nu_h\|_{W_0}^2 d\tau, \\
&\leq cT + M(\|u_0\|_{W_0}^2)\|u_0\|_{W_0}^2 + \int_0^t \|u\|_{W_0} \|u_t\|_{W_0} M(\|u\|_{W_0}^2)\|\nu_h\|_{W_0}^2 d\tau, \\
&\leq L + \int_0^t \|u\|_{W_0} \|u_t\|_{W_0} M(\|u\|_{W_0}^2)\|\nu_h\|_{W_0}^2 d\tau,
\end{align*}$$

(3.9)
for every $h \geq 1$, where $L := cT + M(\|u_0\|_{W_0}^2)\|u_0^h\|_{W_0}^2$ is a constant. Further by $u \in \mathcal{H}$, we can deduce that there exists a $L_1$ such that $\|u\|_{W_0}\|u_t\|_{W_0}M(\|u\|_{W_0}^2) \leq L_1$. Hence from $M(\|u\|_{W_0}^2) \geq m_0 > 0$ and (3.9) we derive

$$\frac{m_0}{2} \|v_h(t)\|_{W_0}^2 \leq L + L_1 \int_0^t \|v_h\|_{W_0}^2 d\tau.$$ 

By Gronwall’s inequality, we obtain

$$\int_0^t \|v_h\|_{W_0}^2 d\tau \leq \frac{L}{L_1} (e^{\frac{m_0}{2} t} - 1),$$

then (3.9) yields that

$$\frac{1}{2} M(\|u\|_{W_0}^2)\|v_h(t)\|_{W_0}^2 + \int_0^t \|\tilde{v}_h(\tau)\|_{X_0}^2 d\tau \leq L + L(e^{\frac{m_0}{2} T} - 1) := C_T,$$

where $C_T > 0$ is independent of $h$. By this uniform estimate, the embedding $W_0 \hookrightarrow L^2(\Omega)$ and using (3.4), we have

$$\{v_h\} \text{ is bounded in } L^\infty([0, T], W_0); \\
\{\tilde{v}_h\} \text{ is bounded in } L^2([0, T], X_0).$$

Case 2: There is at least a $\tilde{u} \in W_0$ such that $M(\|\tilde{u}\|_{W_0}^2) = 0$.

Taking $\eta = v_h(t)$ into (3.4) and integrating over $[0, t] \subset [0, T]$, we obtain

$$2 \int_0^t M(\|\tilde{u}\|_{W_0}^2)\|v_h\|_{W_0}^2 d\tau + \|v_h\|_{X_0}^2 = \|u_0^h\|_{X_0}^2 + 2 \int_0^t |\tilde{u}|^{q-2}\tilde{u}v_h d\tau. \quad (3.11)$$

Since $\tilde{u} \in \mathcal{H}$, $\|\tilde{u}\|_{X_0}$ is bounded. We estimate the last term in the right-hand side thanks to Hölder and Young inequalities

$$2 \int_0^t \int_\Omega |\tilde{u}|^{q-2}\tilde{u}v_h(\tau) d\Omega d\tau \leq c\int_0^t \|\tilde{u}\|_{W_0}^{q-1}\|v_h\|_{W_0} d\tau \leq cT + \int_0^t \|v_h(\tau)\|_{X_0}^2 d\tau. \quad (3.12)$$

Substituting (3.12) into (3.11), we have

$$2 \int_0^t M(\|\tilde{u}\|_{W_0}^2)\|v_h\|_{W_0}^2 + \|v_h\|_{X_0}^2 \leq \|u_0^h\|_{X_0}^2 + cT + \int_0^t \|v_h(\tau)\|_{X_0}^2 d\tau$$

$$= A_1 + \int_0^t \|v_h(\tau)\|_{X_0}^2 d\tau, \quad (3.13)$$

where $A_1 := \|u_0^h\|_{X_0}^2 + cT$ is a constant.

By using Gronwall’s inequality again, we have

$$\int_0^t \|v_h(\tau)\|_{X_0}^2 d\tau \leq A_1(e^t - 1),$$
then
\[ \|v_h\|_{X_0}^2 \leq A_1 + A_1(e^T - 1) := A_T. \] (3.14)

So
\[ \{v_h\} \text{ is bounded in } L^\infty([0,T],X_0). \]

Therefore, up to a subsequence, we may pass to the limit in (3.4) and obtain a weak solution \( v \) of (3.2) with the above regularity. Then by \( v \in L^\infty([0,T],X_0) \) we deduce that \( v \in L^4([0,T],X_0) \), which together with \( \dot{v} \in L^2([0,T],X_0) \) gives that \( v \in W^{1,2}([0,T],X_0) \), here \( W^{1,2} \) denotes the Sobolev space consisting all functions \( v \in L^2([0,T],X_0) \) such that \( \dot{v} \in L^2([0,T],X_0) \). Further by Theorem 2 in [8, Chapter 5] we can derive that \( v \in C([0,T],X_0) \). Naturally, it follows that \( v \in C([0,T],X_0) \) and \( v \in C([0,T],L^2(\Omega)) \). Finally, from (3.2) we have \( v \in C^1([0,T],L^2(\Omega)) \). The existence of \( v \) solving (3.2) and satisfying (3.1) is so proved.

Uniqueness follows arguing for contradiction: if \( v \) and \( \omega \) are two solutions of (3.2) which share the same initial data, by subtracting the equations and testing with \( v_t - \omega_t \), instead of (3.7) we can get
\[ \frac{1}{2} M(\|u\|_{W_0}^2) \|v(t) - \omega(t)\|_{W_0}^2 + \int_0^t \|v_t(\tau) - \omega_t(\tau)\|_{X_0}^2 d\tau = 0, \]
which immediately yields \( \omega \equiv v \). The proof of the lemma is now complete. \( \square \)

Next, we establish local existence and uniqueness of (1.1).

**Theorem 3.1.** Let \( u_0 \in X_0 \), then there exist \( T > 0 \) and a unique local solution of (1.1) over \([0,T]\).

**Proof.** Let \( R^2 := \frac{2}{m_0} M(\|u_0\|_{W_0}^2) \|u_0\|_{W_0}^2 \) and for any \( T > 0 \) we consider
\[ M_T := \{ u \in \mathcal{H} : u(0) = u_0 \text{ and } \|u\|_{\mathcal{H}}^2 \leq R^2 \}. \]

By lemma 3.1, for any \( u \in M_T \), we may define \( v := \Phi(u) \), being \( v \) the unique solution to problem (3.2). We claim that, for a suitable \( T > 0 \) and given \( u \in M_T \), \( \Phi \) is a contractive map satisfying \( \Phi(M_T) \subseteq M_T \). Moreover, the corresponding solution \( v = \Phi(u) \) satisfies for all \( t \in (0,T] \) the energy identity
\[ \int_0^t \frac{1}{2} M(\|u\|_{W_0}^2) \|v\|_{W_0}^2 + \int_0^t \|v_t(\tau)\|_{X_0}^2 d\tau = \frac{1}{2} M(\|u_0\|_{W_0}^2) \|u_0\|_{W_0}^2 + \int_0^t (u, u_t)_{W_0} M(\|u\|_{W_0}^2) \|v\|_{W_0}^2 d\tau + \int_0^t \int_\Omega |u(\tau)|^{q/2} u(\tau)v_t(\tau) dx d\tau. \] (3.15)

Next, we still divide the proof to two cases corresponding to Lemma 3.1.

Case 1: \( M(\|u\|_{W_0}^2) \geq m_0 > 0 \) for any \( u \in W_0 \), where \( m_0 \) is a constant.

For the last term on the right-hand side of (3.15), we argue in the same spirit (although slightly differently) as for (3.8) and we get
\[ \int_0^t \int_\Omega |u(\tau)|^{q/2} u(\tau)v_t(\tau) dx d\tau \leq c \int_0^t \|u(\tau)\|_{W_0}^{q-1} \|v_t(\tau)\|_{W_0}^2 d\tau \]
\[ \leq c \int_0^t \|u(\tau)\|_{W_0}^{q-1} \|v_t(\tau)\|_{W_0}^2 d\tau \]
\[ \leq c T R^{2(q-1)} + \int_0^t \|v_t\|_{X_0}^2 d\tau \]
\[ \leq c T R^{2(q-1)} + \int_0^t \|v_t\|_{X_0}^2 d\tau \]
\[ \leq c T R^{2(q-1)} + \int_0^t \|v_t\|_{X_0}^2 d\tau \] (3.16)
for all \( t \in (0, T) \). Combining (3.15) with (3.16) we can get

\[
\frac{1}{2} M(\|u\|_{W_0}^2)\|v\|_{\bar{W}_0}^2 \\
\leq c T R^{2(q-1)} + \frac{1}{2} M(\|u_0\|_{\bar{W}_0}^2)\|u_0\|_{\bar{W}_0}^2 + \int_0^t (u, u_t)_{W_0} M'(\|u\|_{\bar{W}_0}^2)\|v\|_{\bar{W}_0}^2 d\tau
\]

(3.17)

where \( L_2 := c T R^{2(q-1)} + \frac{1}{2} M(\|u_0\|_{\bar{W}_0}^2)\|u_0\|_{\bar{W}_0}^2 \) is a constant. By using Gronwall’s inequality, it gives

\[
\|v\|_{\bar{W}_0}^2 \leq \frac{2L_2}{m_0} e^{\frac{2L_1}{m_0} T},
\]

so

\[
\int_0^t \|v\|_{\bar{W}_0}^2 d\tau \leq \frac{L_2}{2L_1} (e^{\frac{L_1}{m_0} T} - 1).
\]

Then taking the maximum over \([0, T]\) gives

\[
m_0\|v\|_{\bar{W}_0}^2 \leq M(\|u\|_{\bar{W}_0}^2)\|v\|_{\bar{W}_0}^2 \leq c T R^{2(q-1)} + M(\|u_0\|_{\bar{W}_0}^2)\|u_0\|_{\bar{W}_0}^2 + L_2 (e^{\frac{L_1}{m_0} T} - 1).
\]

(3.18)

Choosing \( T \) sufficiently small, we get \( \|v\|_{\bar{W}_0}^2 \leq R^2 \).

Case 2: There is at least a \( \tilde{u} \in \mathcal{X}_0 \) such that \( M(\|\tilde{u}\|_{\bar{W}_0}^2) = 0 \).

In this regard, let \( R^2 = \|u_0\|_{\mathcal{X}_0}^2 \) and for any \( T > 0 \) consider

\[ M_T = \{ u \in \mathcal{X} : u(0) = u_0 \text{ and } \|u\|_{\mathcal{X}_T}^2 \leq R^2 \}. \]

Similar to (3.11) in lemma 3.1, the corresponding solution \( v = \Phi(u) \) satisfies for all \( t \in (0, T) \) the energy identity

\[
2 \int_0^t M(\|\tilde{u}\|_{\bar{W}_0}^2)\|v\|_{\bar{W}_0}^2 d\tau + \|v\|_{\bar{W}_0}^2 = \|u_0\|_{\mathcal{X}_0}^2 + 2 \int_0^t |\tilde{u}|^{q-2} \tilde{u} v d\tau,
\]

(3.19)

by

\[
2 \int_0^t \int_\Omega |\tilde{u}|^{q-2} \tilde{u} v \tau dx d\tau \leq c T R^{2(q-1)} + \int_0^t \|v(\tau)\|_{\mathcal{X}_0}^2 d\tau
\]

(3.20)

then

\[
\int_0^t M(\|\tilde{u}\|_{\bar{W}_0}^2)\|v\|_{\bar{W}_0}^2 + \|v\|_{\mathcal{X}_0}^2 d\tau \leq \|u_0\|_{\mathcal{X}_0}^2 + c T R^{2(q-1)} + \int_0^t \|v(\tau)\|_{\mathcal{X}_0}^2 d\tau
\]

(3.21)

\[
= A_2 + \int_0^t \|v(\tau)\|_{\mathcal{X}_0}^2 d\tau,
\]

where \( A_2 := \|u_0\|_{\mathcal{X}_0}^2 + c T R^{2(q-1)} \) is a constant. By Gronwall’s inequality

\[
\int_0^t \|v(\tau)\|_{\mathcal{X}_0}^2 d\tau \leq A_2 (e^t - 1).
\]
So
\[ \|v\|_{X_t}^2 \leq \|u_0\|_{X_0}^2 + cTR^{2(q-1)} + A_2(e^T - 1). \] (3.22)
Choosing \( T \) sufficiently small, we get \( \|v\|_{X_T}^2 \leq R^2 \).

Combining Case 1 and Case 2, we show that \( \Phi(M_T) \subseteq M_T \). Next we prove \( \Phi \) is a contraction. Now take \( \omega_1 \) and \( \omega_2 \) in \( M_T \), subtracting the two equations (3.2) for \( v_1 = \Phi(\omega_1) \) and \( v_2 = \Phi(\omega_2) \), and setting \( v = v_1 - v_2 \) we obtain for all \( \eta \in W_0 \) and a.e. \( t \in [0, T] \)
\[ (v_t, \eta) + M((u_t^2)((-\Delta)^{q-2}v_t, \eta) + ((-\Delta)^{q-2}v_t, \eta) \]
\[ = \int_{\Omega} \left( (\omega_1(t)^{q-2} - (\omega_2(t)^{q-2} \omega_2(t) \right) \eta \]
\[ = \int_{\Omega} \xi(t) (\omega_1(t) - \omega_2(t)) \eta, \] (3.23)
where \( \xi = \xi(x, t) \geq 0 \) is given by Lagrange Theorem so that \( \xi(t) \leq (q-1)(|\omega_1(t)| + |\omega_2(t)|)^{q-2} \). Therefore, by taking \( \eta = v_1 \) in (3.23) and arguing as above, we obtain
\[ \|\Phi(\omega_1) - \Phi(\omega_2)\|_{M_T}^2 = \|v\|_{X_T}^2 \leq \xi \|\omega_1 - \omega_2\|_{X_T}^2, \]
for some \( 0 < \xi < 1 \) provided \( T \) is sufficiently small. This proves the claim. By the Contraction Mapping Principle, there exists a unique weak solution to problem (1.1) defined on \([0, T]\). The main statement of Theorem 3.1 is thus proved.

4 Sub-critical initial energy \( J(u_0) < d \)

In this section, we prove the invariance of some sets under the flow of problem (1.1).

**Definition 4.1** (Maximal existence time). Let \( u(t) \) be a weak solution of problem (1.1). We define the maximal existence time \( T_{\max} \) of \( u(t) \) as follows:
(i) If \( u(t) \) exists for \( 0 \leq t < \infty \), then \( T_{\max} = \infty \).
(ii) If there exists a \( t_0 \in (0, \infty) \) such that \( u(t) \) exists for \( 0 \leq t < t_0 \), but doesn’t exist at \( t = t_0 \), then \( T_{\max} = t_0 \).

**Lemma 4.1** (Invariant sets when \( J(u_0) < d \)). Assume that \( u_0 \in X_0, 0 < e < d \), \( \delta_1 < \delta_2 \) are the two roots of equation \( d(\delta) = e \) for \( 0 < \delta_1 < 1 < \delta_2 < \frac{q}{q-1} \), \( T_{\max} \) is the maximal existence time of \( u(t) \). Then
(i) All weak solutions \( u \) of problem (1.1) with \( J(u_0) = e \) belong to \( W_\delta \) for \( \delta_1 < \delta < \delta_2, 0 \leq t < T_{\max} \), provided \( I(u_0) > 0 \).
(ii) All weak solutions \( u \) of problem (1.1) with \( J(u_0) = e \) belong to \( V_\delta \) for \( \delta_1 < \delta < \delta_2, 0 \leq t < T_{\max} \), provided \( I(u_0) < 0 \).

**Proof.**
(i) Let \( u(t) \) be any weak solution of problem (1.1) with \( J(u_0) = e, I(u_0) > 0 \) or \( \|u_0\|_{X_0} = 0 \). \( T_{\max} \) is the existence time of \( u(t) \). If \( \|u_0\|_{X_0} = 0 \), then \( u_0(x) \in W_\delta \). If \( I(u_0) > 0 \) then from Lemma 2.6, it follows \( I_\delta(u_0) > 0 \) and \( J(u_0) < d(\delta) \). Then \( u_0(x) \in W_\delta \) for \( \delta_1 < \delta < \delta_2 \) and \( 0 < t < T_{\max} \). Arguing by contradiction, by the continuity of \( I(u(t)) \) in \( t \), we suppose that there exists a first time \( t_1 \in (0, T_{\max}) \) and \( \delta_0 \in (\delta_1, \delta_2) \) such that \( u(t_1) \in \partial W_{\delta_0}, \)
\[ \text{i.e., } I_{\delta_0}(u(t_1)) = 0, \|u(x)\|_{X_0} \neq 0 \text{ or } J(u(t_1)) = d(\delta_0). \]
From
\[ \int_0^t \|u(t)\|_{X_0}^2 dt \geq J(u_0) - d(\delta), \delta_1 < \delta < \delta_2, 0 \leq t < T_{\max}, \] (4.1)
we can see that \( J(u(t_0)) \neq d(\delta_0) \). If \( I_{\delta_0}(u(t_0)) = 0, \|u(x)\|_{X_0} \neq 0 \), then by the definition of \( d(\delta) \) we have \( J(u(t_0)) \geq d(\delta_0), \) which contradicts (4.1).
Let \( u(t) \) be any weak solution of problem (1.1) with \( J(u_0) = e, I(u_0) < 0 \). From \( J(u_0) = e, I(u_0) < 0 \) and Lemma 2.6, it follows \( I_\delta(u_0) < 0 \) and \( J(u_0) < d(\delta) \). Then \( u_0(x) \in V_\delta \) for \( \delta_1 < \delta < \delta_2 \). We prove \( u(t) \in V_\delta \) for \( \delta_1 < \delta < \delta_2 \) and \( 0 < t < T_{max} \). Arguing by contradiction, by the continuity of \( I(u(t)) \) with respect to \( t \), we suppose that there exists a first time \( t_1 \in (0, T_{max}) \) and \( \delta_0 \in (\delta_1, \delta_2) \) satisfying \( u(t_1) \in \partial V_{\delta_0} \), i.e., \( I_{\delta_0}(u(t_1)) = 0 \) or \( J(u(t_1)) = d(\delta_0) \). However, from (4.1) we can deduce that \( J(u(t_1)) \neq d(\delta_0) \). Besides for \( I_{\delta_0}(u(t_1)) = 0 \), we have \( I_{\delta_0}(u(t)) < 0 \) for \( 0 < t < t_1 \). Then by (ii) of Lemma 2.4, it follows that \( \|u(t_1)\|_{X_0} > r(\delta_0) \). Obviously, \( J(u(t_1)) \neq d(\delta_0) \) and \( \|u(t_1)\|_{X_0} > r(\delta_0) \), this contradicts (4.1).

\[
\square
\]

**Remark 4.1.** If in Lemma 4.1 the assumption \( J(u_0) = e \) is replaced by \( 0 < J(u_0) \leq e \), then the conclusion of Lemma 4.1 also holds.

### 4.1 Global existence and finite time blowup of solution

In this section, we prove a threshold result of global existence and nonexistence of solutions for problem (1.1) with the sub-critical initial energy \( J(u_0) < d \).

**Theorem 4.1** (Global existence when \( J(u_0) < d \)). Let \( u_0 \in X_0 \), \( J(u_0) < d \) and \( I(u_0) > 0 \). Then problem (1.1) admits a global weak solution \( u(t) \in L^\infty(0, \infty; X_0) \) with \( u_1(t) \in L^2(0, \infty; X_0) \) and \( u(t) \in W_p \) for \( 0 \leq t < \infty \).

**Proof.** Let \( \omega_j(x) \) be a system of base functions in \( X_0 \). Construct the approximate solutions of problem (1.1) as follows

\[
u_m(x, t) = \sum_{j=1}^{m} g_{jm}(t) \omega_j(x), \quad m = 1, 2, \cdots
\]

satisfying

\[
(u_{mt}, \omega_s) + (u_m, \omega_{ts})_{W_0} + (u_{mt}, \omega_{ts})_{W_0} = (|u_m|^{q-2} u_m, \omega_s), \quad s = 1, 2, \cdots, m
\]

(4.2)

\[
u_m(x, 0) = \sum_{j=1}^{m} a_{jm} \omega_j(x) \to u_0(x) \text{ in } X_0 \text{ as } m \to \infty.
\]

(4.3)

Multiplying (4.2) by \( g_{jm}'(t) \), summing for \( s \), and integrating over \( [0, t] \) in time, then

\[
\int_0^t \|u_{mt}\|^2_{X_0} d\tau + J(u_m) = J(u_m(0)), \quad 0 \leq t < \infty.
\]

By (4.3) we can get \( J(u_m(0)) \to J(u_0) \), then for sufficiently large \( m \), we have

\[
\int_0^t \|u_{mt}\|^2_{X_0} d\tau + J(u_m) < d, \quad 0 \leq t < \infty.
\]

(4.4)

Next, we prove \( u_m(x, t) \in W_p \) for sufficiently large \( m \) and \( 0 \leq t < \infty \). If it is false, then there exists \( t_0 \) such that \( u_m(x, t_0) \in \partial W_p \), then

\[
I(u_m(t_0)) = 0, \quad \|u_m(t_0)\|_{X_0} \neq 0 \text{ or } J(u_m(t_0)) = d.
\]

By (4.4), it implies that \( J(u_m(t_0)) = d < J(u_0(0)) \) is not true. On the other hand, If \( I(u_m(t_0)) = 0, \|u_m(t_0)\|_{X_0} \neq 0 \), according to the definition of \( d \), we have \( J(u_m(t_0)) \geq d \), which is also contradictory with (4.4). Hence \( u_m(x, t) \in W_p \) for all \( 0 \leq t < \infty \) and sufficiently large \( m \). Then by (4.4) and

\[
J(u_m) = \frac{1}{q} I(u_m) + \frac{q-2\lambda}{2q\lambda} \|u_m\|_{W_p}^2.
\]
we obtain
\[
\int_0^t \|u_{mr}\|^2_{X_0} \, d\tau + \frac{q-2\lambda}{2q\lambda} \|u_m\|^2_{W_0} < d, \quad 0 \leq t < \infty,
\] (4.5)
for sufficiently large \(m\), which yields
\[
\int_0^t \|u_{mr}\|^2_{X_0} \, d\tau < d, \quad 0 \leq t < \infty.
\]
Also, according to the embedding inequality \(\|u_m\|_2 \leq C\|u_m\|_{W_0}\), (4.5) implies
\[
\|u_m\|^2_{X_0} \leq (1 + C^2\lambda)\|u_m\|^2_{W_0} < \frac{2q\lambda d (1 + C^2\lambda)}{q - 2\lambda}, \quad 0 \leq t < \infty.
\]
So, we can get
\[
\|u_m\|^2 \leq C^2\|u_m\|^2_{W_0} \leq \frac{C^2}{\|u_m\|^2_{X_0}} < \frac{C^2}{\left(\frac{2q\lambda d (1 + C^2\lambda)}{q - 2\lambda}\right)^{\frac{2}{q-2}\lambda}},
\]
where \(C\) is the embedding constant from \(W_0 \rightarrow L^q(\Omega)\). Therefore, there exist a \(u\) and a subsequence \(u_m\), such that as \(m \rightarrow \infty\),
\[
\begin{align*}
&u_{mt} \rightharpoonup u_t \text{ weakly in } L^2(0, \infty; X_0), \\
&u_m \rightharpoonup u \text{ weakly star in } L^\infty(0, \infty; X_0), \\
&u_m \rightarrow u \text{ weakly in } L^\infty(0, \infty; L^q(\Omega)).
\end{align*}
\]
Thus in (4.2), for \(s \) fixed, letting \(m \rightarrow \infty\), then, we get
\[
(u_t, w_s) + \langle u, w_s \rangle_{W_0} + \langle u_t, w_s \rangle_{W_0} = \langle |u|^q u, w_s \rangle,
\]
for all \(s\). Further we have
\[
(u_t, v) + \langle u, v \rangle_{W_0} + \langle u_t, v \rangle_{W_0} = \langle |u|^q u, v \rangle, \quad \forall v \in X_0, \ t \in (0, \infty).
\]
Moreover, (4.3) give us \(u(x, 0) = u_0(x)\) in \(X_0\). That means \(u\) is a global weak solution of problem (1.1). \(\square\)

**Theorem 4.2** (Finite time blowup when \(J(u_0) < d\)). Suppose that \(u_0 \in X_0\) and \(u_0 \in V_p\), then any nontrivial solution to problem (1.1) blows up in finite time. In other words, there exists a finite time \(T\) such that
\[
\lim_{t \uparrow T} \int_0^t \|u\|^2_{X_0} \, d\tau = +\infty.
\] (4.6)

**Proof.** Arguing by contradiction, assume that the solution \(u(t)\) exists for all \(t \geq 0\). Let \(u(t)\) be any weak solution of problem (1.1) with \(J(u_0) < d, I(u_0) < 0\). For any \(t > 0\), we define
\[
H(t) := \int_0^t \|u\|^2_{X_0} \, d\tau + (T - t)\|u_0\|_{X_0}.
\] (4.7)
So we can get
\[
H'(t) = \|u\|^2_{X_0} - \|u_0\|^2_{X_0} = 2 \int_0^t \langle u, u_t \rangle_{X_0} \, d\tau,
\] (4.8)
and
\[
H''(t) = 2(u, u_t)_{X_0}.
\] (4.9)
Employing the Cauchy Schwartz inequality, we obtain
\[
\left( \int_0^t (u, u_r)_{X_0} \, dt \right)^2 \leq \int_0^t \|u\|_{X_0}^2 \, dt \int_0^t \|u_r\|_{X_0}^2 \, dt.
\] (4.10)

As a consequence, we read the differential inequality
\[
H(t)H''(t) - \frac{q}{2} (H'(t))^2 = H(t)H''(t) - 2q \left( \int_0^t (u, u_r)_{X_0} \, dt \right)^2 \\
\geq H(t)H''(t) - 2qH(t) \int_0^t \|u_r\|_{X_0}^2 \, dt \\
= H(t)\xi(t),
\] (4.11)

for almost every \( t \geq 0 \). Next we define
\[
\xi(t) := 2(u, u_r)_{X_0} - 2q \int_0^t \|u_r\|_{X_0}^2 \, dt.
\]

Setting \( \phi = u(t) \) in (2.4) and using (2.7), it follows that
\[
\xi(t) \geq -2qJ(u(t)) + \frac{q}{\lambda} \|u\|_{W_0}^2 - 2qJ(u_0) + 2qJ(u(t)) \geq \frac{q}{\lambda} \|u\|_{W_0}^2 - 2q\lambda.
\] (4.12)

Then we discuss the situation in two cases.

(i) If \( 0 < f(u_0) < d \), by (2.8) and Lemma 2.3 (ii) we have \( d \leq \frac{q\|u\|_{W_0}^2}{\lambda} \), where
\[
\theta = \left( \frac{\|u\|_{W_0}^2}{\|u\|_Y^2} \right)^{\frac{1}{\alpha}}.
\]
It follows Lemma 4.1 that \( I(u) < 0 \) for \( t > 0 \). This implies \( \theta < 1 \), then we can get
\[
d \leq \frac{q - 2\lambda}{2\lambda} \|u\|_{W_0}^2.
\] (4.13)

So
\[
\xi(t) > 0, \quad 0 \leq t \leq \infty,
\]
which implies
\[
H(t)H''(t) - \frac{q}{2} (H'(t))^2 > 0, \quad 0 \leq t \leq \infty.
\] (4.14)

Further by a simple computation, it gives
\[
(H^{-\alpha}(t))^\prime = \frac{-a}{H^{\alpha+2}}(t) \left( H(t)H''(t) - (\alpha + 1)(H'(t))^2 \right) \leq 0, \quad \alpha = \frac{q - 2}{2} > 0.
\] (4.15)

(ii) If \( f(u_0) \leq 0 \), by (4.12) and \( q > 2\lambda \), we get
\[
\xi(t) \geq \frac{q - 2\lambda}{\lambda} \|u\|_{W_0}^2 - 2qJ(u_0) > 0.
\]

Obviously, we can also derive (4.14) in this case. Moreover, by \( f(u_0) \leq 0 \) it implies that \( I(u) < 0 \) for all \( t \leq 0 \).
Then multiplying both sides of the first equation in (1.1) by $u$, it follows that $(u, u_t)_{X_0} = -I(u)$, which together with (4.9) and the fact $I(u) < 0$ gives $H'(t) > 0$ for all $t \geq 0$. Then by $H'(0) = 0$, it can be deduced that $H'(t) > 0$ for all $t > 0$. Note that

$$\left(H^{-a}(t)\right)' = \frac{-aH(t)}{H^{a+1}(t)} < 0$$

(4.16)

by $H'(t) > 0$ and $H(t) > 0$ for $t > 0$. From (4.15) and (4.16), it follows that there exists a finite time $T > 0$ such that

$$\lim_{t \to T} H^{-a}(t) = 0$$

and

$$\lim_{t \to T} H(t) = +\infty.$$ 

Obviously, this contradicts $T_{\text{max}} = \infty$. So we get

$$\lim_{t \to T} \int_0^t \|u\|_{X_0}^2 \, dt = +\infty.$$ 

Then the proof is completed. \hfill \qed 

### 4.2 Asymptotic behavior of solutions

Xu and Su in [48] studied the initial boundary value problem of semilinear pseudo-parabolic equation (1.5), obtained the asymptotic behavior of solutions with initial energy $J(u_0) < d$, which implies that the global solution to problem (1.5) decay exponentially. In this section, we shall consider the above decay behavior of the global solution in the fractional setting.

**Theorem 4.3** (Asymptotic behavior of solutions for $J(u_0) < d$). Let $u_0 \in X_0$, $J(u_0) < d$ and $I(u_0) > 0$. Then for the global weak solution $u$ of problem (1.1), when $\lambda = 1$, there exists a constant $\beta > 0$ such that

$$\|u\|_{X_0}^2 \leq \|u_0\|_{X_0}^2 e^{-\beta t}, \quad 0 \leq t < \infty,$$

(4.17)

when $\lambda > 1$, then

$$\|u\|_{X_0}^2 \leq \left(2(1-\delta_1)(\lambda-1)t + \|u_0\|_{X_0}^{2(\lambda-1)}\right)^{-\frac{1}{\lambda-1}}, \quad 0 \leq t < \infty,$$

(4.18)

where $\delta_1$ is same as that in Lemma 4.1.

**Proof.** First, Theorem 4.1 gives the existence of global weak solutions for problem (1.1). Now we need to prove (4.17) and (4.18). Let $u(t)$ be any global weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) > 0$. Then (2.4) holds for $0 \leq t < \infty$. Multiplying (2.4) by any $d(t) \in [0, \infty)$, we get

$$(u_t, d(t)v) + \langle u, d(t)v \rangle_{W_0} + (u_t, d(t)v)_{W_0} = (|u|^{q-2}u, d(t)v), \quad v \in X_0, \quad d(t) \in C[0, \infty)$$

and

$$(u_t, w) + \langle u, w \rangle_{W_0} + (u_t, w)_{W_0} = (|u|^{q-2}u, w).$$

(4.19)

Setting $w = u$, (4.19) leads to

$$\frac{1}{2} \frac{d}{dt} \|u\|_{X_0}^2 + I(u) = 0, \quad 0 \leq t < \infty.$$ 

(4.20)
From Lemma 4.1 along with $0 < J(u_0) < d$ and $I(u_0) > 0$, we get $u(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$ and $t \in [0, \infty)$. Hence, it follows that $I_\delta(u) \geq 0$ and $I_{\delta_1}(u) \geq 0$ for $\delta \in (\delta_1, \delta_2)$ and $t \in [0, \infty)$. Then from (4.20) and the definition of $d(\delta)$, we have

$$\frac{1}{2} \frac{d}{dt} \|u\|_{X_0}^2 + (1 - \delta_1)\|u\|_{X_0}^{2\lambda} + I_{\delta_1}(u) = 0, \quad 0 \leq t < \infty.$$  \hfill (4.21)

From (4.21) we also have

$$\frac{1}{2} \frac{d}{dt} \|u\|_{X_0}^2 + (1 - \delta_1)\|u\|_{X_0}^{2\lambda} \leq 0, \quad 0 \leq t < \infty.$$  \hfill (4.22)

When $\lambda = 1$, by the Gronwall inequality, we have

$$\|u\|_{X_0}^2 \leq \|u_0\|_{X_0}^2 e^{-2(1 - \delta_1)t}, \quad 0 \leq t < \infty.$$  

Therefore, there exists a constant $\beta = 2(1 - \delta_1) > 0$ such that

$$\|u\|_{X_0}^2 \leq \|u_0\|_{X_0}^2 e^{-\beta t}, \quad 0 \leq t < \infty.$$  

When $\lambda > 1$, from (4.22) we have

$$\frac{d}{dt} \|u\|_{X_0}^2 \leq -2(1 - \delta_1)\|u\|_{X_0}^{2\lambda}, \quad 0 \leq t < \infty.$$  

So we get

$$\|u\|_{X_0}^2 \leq \left(2(1 - \delta_1)(\lambda - 1)t + \|u_0\|_{X_0}^{2(\lambda - 1)}\right)^{-\frac{1}{\lambda-1}}, \quad 0 \leq t < \infty.$$  

The proof is completed. \hfill \Box

### 4.3 Lower bound estimate of the blowup time

Luo [20] considered the semilinear pseudo-parabolic equation (1.5), obtained a lower bound for blow-up time at low initial energy. Inspired by Luo’s work. In this section, by the similar argument, we derive the lower bound estimate for blowup time of solution to problem (1.1) with $J(u_0) < d$.

**Theorem 4.4** (Lower bound of the blowup time when $J(u_0) < d$). Suppose $q > 2\lambda, u_0 \in X_0, J(u_0) < d, I(u_0) < 0$, then the solution $u(x, t)$ of problem blows up in finite time $T$ in $X_0$-norm. Moreover,

$$T \geq \frac{\|u_0\|_{X_0}^{-q+2}}{(q - 2)C^q}.$$  

**Proof.** First, from Theorem 4.2, we know that the solution $u$ of problem (1.1) blows up in finite time $T$. Let

$$\varphi(t) := \|u\|_{X_0}^2,$$  \hfill (4.23)

multiplying $u$ on two sides of equation (1.1), we have

$$(u_t, u) + ((-\Delta)^s u_t, u) = -|u|^q_{L^q}(-\Delta)^s u, u) + |u|^{q-2} u, u),$$  \hfill (4.24)

then by direct computation and (4.24), it follows that

$$\varphi'(t) = -2\|u\|_{W_0^{2\lambda}} + 2\|u\|_{W_q^q}^q.$$  \hfill (4.25)

Then by (4.25) and the embedding inequality $\|u\|_q \leq \|u\|_{X_0}$, it implies

$$\varphi'(t) \leq 2C^q(t)(\varphi(t))^q.$$
So we see the following inequality
\[
\frac{\varphi'(t)}{(\varphi(t))^2} \leq 2C_i^4.
\] (4.26)

Integrating the inequality (4.26) from 0 to \(t\), we have
\[
(\varphi(0))^{-\frac{q+2}{2}} - (\varphi(t))^{-\frac{q+2}{2}} \leq (q - 2)C_i^4 t.
\] (4.27)

So letting \(t \to T\) in (4.27), we can conclude that
\[
T \geq \frac{\|u_0\|_{\mathcal{X}_0}}{(q - 2)C_i^4}.
\]
The proof is completed. \(\square\)

### 4.4 Upper bound estimate of the blowup time

Sun et al. [34] obtained the finite time blowup results for (1.5) provided that the initial energy satisfies \(f(u_0) < d(\infty)\), where \(d(\infty)\) is a nonnegative constant, and also derive the estimates of the lower bound and upper bound for the blowup time. Similarly, we turn to the upper bound estimate for blowup time at sub-critical initial energy case of problem (1.1).

**Theorem 4.5** (Upper bound of the blowup time when \(0 < f(u_0) < d\)). For all \(q > 2\lambda\), Assume that \(u_0 \in \mathcal{X}_0\), \(0 < f(u_0) < d\), \(I(u_0) < 0\), then the solution of problem (1.1) blows up in finite time. Furthermore, the maximum existence time \(T\) of \(u(t)\) satisfies
\[
T \leq \frac{4(q - 1)\|u_0\|_{\mathcal{X}_0}^2}{q(q - 2)^2}.
\]

**Proof.** Since \(I(u_0) < 0\) and \(0 < f(u_0) < d\), by Lemma 4.1 (ii) we can get \(I(u(t)) < 0\), further
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{\mathcal{X}_0}^2 = \|u\|_q^q - \|u\|_{W_0}^{2\lambda} = -I(u(t)) > 0, \quad t \in [0, T].
\] (4.28)

For \(\hat{T} \in (0, T)\), we define
\[
F(t) := \int_0^t \|u\|_{\mathcal{X}_0}^2 d\tau + (\hat{T} - t)\|u_0\|_{\mathcal{X}_0}^2 + \beta(t + \sigma)^2,
\] (4.29)
where \(\beta\) and \(\sigma\) will be given in the following proof. Then we can write
\[
F'(t) \equiv \|u\|_{\mathcal{X}_0}^2 - \|u_0\|_{\mathcal{X}_0}^2 + 2\beta(t + \sigma)
\] (4.30)
\[
= \int_0^t \frac{d}{d\tau} \|u\|_{\mathcal{X}_0}^2 d\tau + 2\beta(t + \sigma)
\] (4.31)
\[
= 2\int_0^t (u, \tau)_{\mathcal{X}_0} d\tau + 2\beta(t + \sigma)
\]
\[
> 0.
\]

So \(F(t) \geq F(0) > 0\) and \(F(t)\) is strictly increasing on \([0, \hat{T}]\). Furthermore, by (2.7) we can deduce
\[
F''(t) = 2(u, u_t)_{\mathcal{X}_0} + 2\beta
\]
\[
= 2f(u(t)) + 2\beta
\]
\[
= -2qf(u(t)) + \frac{q - 2\lambda}{\lambda} \|u\|_{W_0}^{2\lambda} + 2\beta
\] (4.31)
\[
= -2qf(u_0) + 2q \int_0^t \|u(\tau)\|_{\mathcal{X}_0}^2 d\tau + \frac{q - 2\lambda}{\lambda} \|u\|_{W_0}^{2\lambda} + 2\beta.
\]
By Cauchy-Schwartz inequality and Hölder’s inequality we can get
\[
\left( \int_0^t (u, u_t)_{X_0} \, dt + \beta (t + \sigma) \right)^2 \leq \left( \int_0^t \| u \|_{X_0}^2 \, dt + \beta (t + \sigma)^2 \right) \left( \int_0^t \| u_t \|_{X_0}^2 \, dt + \beta \right),
\]
(4.32)
Therefore, in view of (4.29)-(4.32) and (4.13), we derive
\[
F(t)F'(t) - \frac{q}{2} (F'(t))^2
= F(t)F''(t) - 2q \left( \int_0^t (u, u_t)_{X_0} \, dt + \beta (t + \sigma) \right)^2
\geq F(t)F''(t) - 2q \left( \int_0^t \| u \|_{X_0}^2 \, dt + \beta (t + \sigma)^2 \right) \left( \int_0^t \| u_t \|_{X_0}^2 \, dt + \beta \right)
\geq F(t)F''(t) - 2qF(t) \left( \int_0^t \| u_t \|_{X_0}^2 \, dt + \beta \right)
= F(t) \left( -2qJ(u_0) + \frac{q - 2\lambda}{\lambda} \| u \|_{W_0}^{2\lambda} + 2\beta - 2q\beta \right)
> 2qF(t) \left( d - J(u_0) - \frac{(q - 1)\beta}{q} \right)
\]
(4.33)
for any \( t \in [0, \bar{T}] \) and restricting \( \beta \) to satisfy
\[
0 < \beta \leq \frac{q}{q - 1} (d - J(u_0)).
\]
(4.34)
Let \( G(t) := F^{\frac{q}{q-2}}(t) \) for \( t \in [0, \bar{T}] \), then by \( F(t) > 0, F'(t) > 0, q > 2 \) and the above inequality, we get
\[
G'(t) = -\frac{q - 2}{2} F^{-\frac{q}{q-2}}(t)F'(t) < 0,
\]
(4.35)
\[
G''(t) = \frac{2 - q}{2} F^{-\frac{q}{q-2}}(t) \left( F(t)F''(t) - \frac{q}{2} (F'(t))^2 \right) < 0
\]
for all \( t \in [0, \bar{T}] \). Then it follows from \( G''(t) < 0 \) that
\[
G(\bar{T}) - G(0) = G'(0)\bar{T} < G'(0)\bar{T}, \quad \gamma \in (0, \bar{T}).
\]
(4.36)
By the definition of \( G(t), (4.29), (4.30) \) and (4.35), we obtain
\[
G(0) = F^{\frac{q}{q-2}}(0) > 0, \quad G(\bar{T}) = F^{\frac{q}{q-2}}(\bar{T}) > 0,
G'(0) = 2 - q F^{-\frac{q}{q-2}}(0)F'(0) = (2 - q)\beta \sigma F^{-\frac{q}{q-2}}(0) < 0.
\]
Combining (4.36) and the above inequalities, we can deduce
\[
\bar{T} \leq \frac{G(\bar{T}) - G(0)}{G'(0)} = \frac{G(0) - G(0)}{G'(0)} = \frac{F(0)}{(q - 2)\beta \sigma}.
\]
Then it follows that
\[
\bar{T} \leq \frac{T \| u_0 \|_{X_0}^2 + \beta \sigma^2}{(q - 2)\beta \sigma} = \frac{\sigma}{q - 2} + \frac{\| u_0 \|_{X_0}^2}{(q - 2)\beta \sigma} - \bar{T}.
\]
Hence, letting \( \bar{T} \to T \), we get
\[
T \leq \frac{\| u_0 \|_{X_0}^2}{(q - 2)\beta \sigma} T + \frac{\sigma}{q - 2},
\]
(4.37)
Fix any $\beta$ satisfying (4.34), let $\sigma$ be large enough such that
\[
\frac{||u_0||_{X_0}^2}{(q-2)^2} < \sigma < +\infty, \tag{4.38}
\]
then (4.37) leads to
\[
T \leq \frac{\sigma}{q-2} \left( 1 + \frac{||u_0||_{X_0}^2}{(2-q)\beta\sigma} \right)^{-1} = \frac{\beta\sigma^2}{(q-2)\beta - ||u_0||_{X_0}^2}. \tag{4.39}
\]
Minimizing the last term of (4.39) for $\sigma$ satisfying (4.38) one has
\[
T \leq \frac{2||u_0||_{X_0}^2}{(q-2)^2\beta}. \tag{4.40}
\]
Minimizing the the last term of (4.39) for $\beta$ satisfying (4.34) we finally get
\[
T \leq \frac{4(q-1)||u_0||_{X_0}^2}{q(q-2)^2(d-f(u_0))}. \tag{4.41}
\]
The proof is completed. \hfill \Box

By the way, inspired by [20], we can also derive an upper bound for blow-up time when $J(u_0) < 0$.

**Theorem 4.6** (Upper bound of the blowup time when $J(u_0) < 0$). If $q > 2\lambda \geq 2$, $u_0 \in X_0$, $J(u_0) < 0$, then the solution of problem (1.1) blows up at some finite time $T$ and
\[
T \leq \frac{||u_0||_{X_0}^2}{(2-q)qJ(u_0)}. \tag{4.42}
\]

*Proof.* If we replace the auxiliary functions $\varphi(t)$ and $\psi(t)$ used in the proof of [20, Theorem 3.1] with the following form
\[
\varphi(t) := ||u||_{X_0}^2, \quad t \in [0, T]
\]
and
\[
\psi(t) := -2qJ(u), \quad t \in [0, T],
\]
which corresponds to problem (1.1). Then by similar arguments, we can derive this conclusion. \hfill \Box

**Remark 4.2.** The above conclusion does not give the upper bound of the blowup time when $J(u_0) = 0$. In fact, if we consider the case $d(\beta) = 0$ in Lemma 2.5 and the case $e = 0$ in Lemma 4.1, we will find Theorem 4.5 is also valid to estimate the upper bound of blowup time when $J(u_0) = 0$. It’s just that we don’t consider the non-positive situation of $d(\delta)$, even if $d(\delta) = 0$ and $e = 0$ satisfy the characteristics of Lemma 2.5 and Lemma 4.1, respectively.

## 5 Critical initial energy $J(u_0) = d$

### 5.1 Global existence and finite time blowup of solution

In this section, we prove a threshold result of global existence and nonexistence of solutions for problem (1.1) with the critical initial energy $J(u_0) = d$.

**Theorem 5.1** (Global existence when $J(u_0) = d$). Assume that $u_0 \in X_0$, $I(u_0) > 0$ and $J(u_0) = d$. Then the weak solution of problem (1.1) exists globally, satisfying $u(t) \in L^\infty(0, \infty; X_0)$ with $u(t) \in L^2(0, \infty; X_0)$ and $u(t) \in W_p = W_p \cup \partial W_p$ for $0 \leq t < \infty$. 
Proof. From \( J(u_0) = d \), it can be deduced that \( \| u_0 \|_{X_0} \neq 0 \). Pick a sequence \( \theta_m = 1 - \frac{1}{m} \), such that \( 0 < \theta_m < 1 \), Let \( u_{0m}(x) = \theta_m u_0(x) \), \( m = 2, 3, \cdots \). Consider the initial condition \( u(x, 0) = u_{0m}(x) \) and the corresponding problem (1.1). From \( I(u_0) \geq 0 \) and lemma 2.3 (ii), we have

\[
\theta_* = \left( \frac{\| u_0 \|_{X_0}^{2\lambda W_0}}{\| u_0 \|_{W_0}^q} \right)^{\frac{1}{q-\gamma}} \geq 1.
\]

Thus, it follows that \( I(u_{0m}) = I(\theta_m u_0) > 0 \) and \( I(u_{0m}) = J(\theta_m u_0) < J(u_0) = d \). Then by Theorem 4.1, it can be deduced that for each \( m \) problem (1.1) admits a global solution \( u_m \in L^\infty(0, \infty; X_0) \) with \( u_{mt} \in L^2(0, \infty; X_0) \) and \( u_m \in V_p \) for \( 0 \leq t < \infty \), satisfying

\[
(u_{mt}, v) + (u_m, v)_{W_0} + (u_{mt}, v)_{W_0} = (\| u_m \|^{q-2} u_m, v), \ \forall v \in X_0, \ t \in \mathbb{R}_0^+,
\]

\[
\int_0^t \| u_{mt} \|_{X_0}^2 \, dt + J(u_m) < d, \ 0 \leq t < \infty.
\]

Then we can get

\[
\int_0^t \| u_{mt} \|_{X_0}^2 \, dt + \frac{q-2\lambda}{2q\lambda} \| u_m \|_{W_0}^{2\lambda} < d, \ 0 \leq t < \infty.
\]

The rest of the proof is similar as that in theorem 4.1. \( \square \)

Theorem 5.2 (Finite time blowup when \( J(u_0) = d \)). Suppose that \( u_0 \in X_0, I(u_0) < 0 \) and \( J(u_0) = d \). Then any nontrivial solution to problem (1.1) must blowup in finite time \( T \) satisfying

\[
\lim_{t \to T} \int_0^t \| u \|_{X_0}^2 \, dt = +\infty.
\]

Proof. Similar to the proof of Theorem 4.2, first we assume that the critical initial energy solution exists globally. From \( I(u_0) < 0 \) and the continuity of \( I(u(t)) \) in \( t \), it can be seen that there exists a sufficiently small \( \tilde{t} > 0 \) such that \( I(u(t)) < 0 \) for \( t \in [0, \tilde{t}] \). Moreover, by the fact that \( (u, u_t)_{X_0} = -I(u(t)) > 0 \) for \( t \in [0, \tilde{t}] \), we have \( u_t \neq 0 \) for \( t \in [0, \tilde{t}] \). Hence, by (2.7) and the continuity of \( J(u(t)) \) in \( t \), it follows that \( J(u(t)) < d \) for \( t \in (0, \tilde{t}] \). Taking \( \hat{t} \in (0, \tilde{t}] \) as the new initial time, obviously there holds \( I(u(\hat{t})) < 0 \) and \( J(u(\hat{t})) < d \). The remainder of the proof is similar to that of Theorem 4.2. \( \square \)

5.2 Asymptotic behavior of solutions

In this section, we consider the asymptotic behavior of solutions for problem (1.1) with the critical initial condition \( J(u_0) = d \). By the similar way of the proof of Theorem 4.3, we can give Theorem 5.3.

Theorem 5.3 (Asymptotic behavior of solutions for \( J(u_0) = d \)). Let \( u_0 \in X_0, J(u_0) = d \) and \( I(u_0) > 0 \). Then for the global weak solution \( u \) of problem (1.1), when \( \lambda = 1 \), there exists constants \( E > 0, t_1 > 0 \) and \( \gamma > 0 \) such that

\[
\| u \|_{X_0}^2 \leq E e^{-\gamma t}, \ t_1 \leq t < \infty.
\]

When \( \lambda > 1 \), then

\[
\| u \|_{X_0}^2 \leq \left( 2(1 - \delta_1)(\lambda - 1)(t - t_1) + \| u(t_1) \|_{X_0}^{2(\lambda-1)} \right)^{-\frac{1}{\gamma-1}}, \ t_1 \leq t < \infty.
\]

Proof. First, Theorem 5.1 gives the existence of a global weak solution for problem (1.1). In addition, from Remark 4.1, Theorem 5.1 and (2.7), it follows that if \( u(t) \) is a global weak solution of problem (1.1) with \( J(u_0) = d, I(u_0) > 0 \), we claim that \( J(u) < d \) and \( I(u) \geq 0 \) for \( 0 \leq t < \infty \). Next, we consider the following two cases.
(i) Assume that \( I(u) > 0 \) for \( 0 \leq t < \infty \). Then from \((u_t, u)_{X_0} = -I(u) < 0 \) and \( \|u_t\|_{X_0} > 0 \), it follows that \( \int_0^t \|u_t\|^2_{X_0} \, d\tau \) is increasing on \( t \in [0, \infty) \). Picking any \( t_1 > 0 \) and setting

\[
d_1 = d - \int_0^{t_1} \|u_t\|^2_{X_0} \, d\tau, \tag{5.5}
\]

by noticing (2.7), we have \( 0 < I(u) \leq d < d \) and \( u(t) \in W_\delta \) for \( \delta \in (\delta_1, \delta_2) \) and \( t \in [t_1, \infty) \), where \( \delta_1 < \delta_2 \) solve the equation \( d(\delta) = d_1 \). Thus, \( I_\delta(u) \geq 0 \) for \( t \geq t_1 \), which together with (4.21), gives that

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{X_0} + (1 - \delta_1)\|u\|^2_{X_0} \leq 0, \quad t_1 \leq t < \infty.
\]

When \( \lambda = 1 \), making use of Gronwall’s inequality, we can get

\[
\|u\|^2_{X_0} \leq \|u(t_1)\|^2_{X_0} e^{-2(1-\delta_1)(t-t_1)} = \|u(t_1)\|^2_{X_0} e^{2(1-\delta_1)t_1} e^{-2(1-\delta_1)t}.
\]

When \( \lambda > 1 \), it follows that

\[
\|u\|^2_{X_0} \leq \left(2(1-\delta_1)(\lambda - 1)(t-t_1) + \|u(t_1)\|^2_{X_0} e^{-2\lambda(t_1)}\right)^\frac{1}{\lambda}, \quad t_1 \leq t < \infty.
\]

(ii) Let us suppose by contradiction that \( t_0 > 0 \) is the first time such that \( I(u(t_0)) = 0 \). By (2.8), we get

\[
J(u(t_0)) \geq d.
\]

Meanwhile, (2.7) gives

\[
J(u(t_0)) \leq d - \int_0^{t_0} \|u_t\|^2_{X_0} \, d\tau \leq d. \tag{5.6}
\]

Hence we deduce \( J(u(t_0)) = d \). Again from (5.6) we get \( \int_0^{t_0} \|u_t\|^2_{X_0} \, d\tau = 0 \), that is \( u(t) \equiv 0 \) for \( 0 \leq t \leq t_1 \), which contradicts \( I(u_0) > 0 \). Hence we have \( I(u) > 0 \) and \( J(u) < d \) for \( 0 < t < \infty \).

By the continuity of the functionals \( I(u) \) and \( J(u) \) in \( t \), we reset the initial data to a small enough \( t_1 > 0 \) such that \( 0 < J(u(t_1)) < d \) and \( I(u(t_1)) > 0 \). By (4.21) we get

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{X_0} + (1 - \delta_1)\|u\|^2_{X_0} \leq 0, \quad t_1 \leq t < \infty.
\]

Making use of Gronwall’s inequality, when \( \lambda = 1 \) we can get

\[
\|u\|^2_{X_0} \leq \|u(t_1)\|^2_{X_0} e^{-2(1-\delta_1)(t-t_1)} = \|u(t_1)\|^2_{X_0} e^{2(1-\delta_1)t_1} e^{-2(1-\delta_1)t},
\]

when \( \lambda > 1 \), the result is same as the case (i). Therefore, when \( \lambda = 1 \), there exist constants \( E > 0, t_1 > 0 \) and \( \gamma > 0 \) such that

\[
\|u\|^2_{X_0} \leq E e^{-\gamma t}, \quad t_1 \leq t < \infty.
\]

The proof is completed. \( \square \)

### 6 Blowup for arbitrary positive initial energy \( J(u_0) > 0 \)

In this section, we establish a finite time blowup theorem for the solution of problem (1.1) with arbitrary high initial energy. At the same time, we estimate the upper bound of the blowup time. Firstly, the invariance of the set \( \mathcal{N}_- \) is proved as follows.
**Lemma 6.1** (The invariance of \( N \) when \( f(u_0) > 0 \). Assume that \( 2\lambda < q < 2 \lambda^* \), \( u_0 \in X_0 \), \( f(u_0) > 0 \) and the initial condition

\[
\frac{2\lambda q(1 + C^2)^{\lambda}}{q - 2\lambda} f(u_0) \leq \|u_0\|_W^\lambda \tag{6.1}
\]

holds. Then \( u \in N \) for all \( t \in [0, T] \), where \( C \) denotes the embedding constant for \( W_0 \to L^2(\Omega) \), \( T \) is maximum existence time of \( u(t) \).

**Proof.** Let \( u(t) \) be any weak solution of problem (1.1). Multiplying (1.1) by \( u_t(t) \) and integrating on \( \Omega \), then we have

\[
\|u_t\|_{X_0}^2 = -\frac{1}{2\lambda} \frac{d}{dt} \|u\|_W^2 + \frac{1}{q} \frac{d}{dt} \|u\|_q^q.
\]

Further we could obtain

\[
\frac{d}{dt} f(u) = -\|u_t(t)\|_{X_0}^2 \leq 0. \tag{6.2}
\]

Multiplying (1.1) by \( u \) and integrate on \( \Omega \times (0, t) \), we have

\[
\frac{1}{2} \|u\|_{X_0}^2 - \frac{1}{2} \|u_0\|_{X_0}^2 + \int_0^t (\|u\|_W^2 - \|u\|_q^q) dt = 0,
\]

that is

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{X_0}^2 = -I(u). \tag{6.3}
\]

Note that

\[
f(u_0) = \frac{q - 2\lambda}{2\lambda q} \|u_0\|_W^2 + \frac{1}{q} I(u_0),
\]

which together with (6.1) indicates that \( I(u_0) < 0 \). Next, we prove \( u(t) \in N \) for all \( t \in [0, T] \). Arguing by contradiction, by the continuity of \( I(t) \) in \( t \), we assume that there exists a \( \tilde{t} \in (0, T) \) such that \( u(t) \in N \) for \( 0 \leq t < \tilde{t} \) and \( u(\tilde{t}) \in N \), then by (6.3) we have

\[
\frac{d}{dt} \|u(t)\|_{X_0}^2 = -2I(u) > 0, \quad t \in [0, \tilde{t}], \tag{6.4}
\]

which implies that

\[
\|u_0\|_{X_0}^2 < \|u(\tilde{t})\|_{X_0}^2.
\]

Then, we have

\[
\|u_0\|_{W_0}^{2\lambda} < \|u(\tilde{t})\|_{W_0}^{2\lambda}. \tag{6.5}
\]

From (6.2) it follows that

\[
f(u(t)) \leq f(u_0) \quad \text{for all} \quad t \in [0, \tilde{t}]. \tag{6.6}
\]

By the definition of \( f(u) \) and \( u(\tilde{t}) \in N \), we derive to

\[
f(u(\tilde{t})) = \frac{q - 2\lambda}{2\lambda q} \|u(\tilde{t})\|_{W_0}^{2\lambda},
\]

which together with (6.1) and (6.6), we can get

\[
\frac{q - 2\lambda}{2\lambda q(1 + C^2)^{\lambda}} \|u(\tilde{t})\|_{X_0}^2 \leq \frac{q - 2\lambda}{2\lambda q} \|u(\tilde{t})\|_{W_0}^{2\lambda} \leq f(u_0) \leq \frac{q - 2\lambda}{2\lambda q(1 + C^2)^{\lambda}} \|u_0\|_{X_0}^2,
\]

i.e., \( \|u(\tilde{t})\|_{X_0}^{2\lambda} < \|u_0\|_{X_0}^{2\lambda} \), which contradicts (6.5). \( \square \)
Lemma 6.2. ([16, 17]) Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality

$$
\psi''(t)\psi(t) - (1 + \theta)(\psi'(t))^2 \geq 0, \quad t > 0,
$$

where $\theta > 0$ is some constant. If $\psi(0) > 0$ and $\psi'(0) > 0$, then there exists $0 < t_1 \leq \frac{\psi(0)}{\theta \psi'(0)}$ such that $\psi(t)$ tends to $\infty$ as $t \to t_1$.

Now we show high energy blowup and estimate the upper bound of the blowup time of solutions for problem (1.1).

**Theorem 6.1** (Finite time blowup when $J(u_0) > 0$). Let $u(t)$ be a weak solution to problem (1.1) with $u_0 \in X_0$. Suppose that $J(u_0) > 0$ and (6.1) holds, then the solution $u(t)$ blows up in finite time. In addition there exists a $t_1$ as

$$
0 < t_1 \leq \frac{2\eta(0)}{(\alpha - 1)\eta'(0)},
$$

such that

$$
\lim_{t \to t_1} \int_0^t \|u\|^2_{X_0} \, d\tau = +\infty,
$$

where $\alpha, \eta(0)$ and $\eta'(0)$ will be determined in the later proof.

**Proof.** Arguing by contradiction, we assume the existence time of solution $T = +\infty$. Integrating of (6.2) with respect to $t$, we have

$$
J(u) + \int_0^t \|u_\tau\|_{X_0}^2 \, d\tau = J(u_0).
$$

From (6.3) we have

$$
\frac{d}{dt}\|u\|_{X_0}^2 = -2I(u)
$$

$$
= -2\left(\|u\|_{W_0}^{2\lambda} - \|u\|_{q}^{q}\right)
$$

$$
= -4\lambda \left(\frac{1}{2\lambda}\|u\|_{W_0}^{2\lambda} - \frac{1}{q}\|u\|_{q}^{q}\right) + \left(2 - \frac{4\lambda}{q}\right)\|u\|_{q}^{q}
$$

$$
= -4\lambda J(u) + 2\frac{q - 4\lambda}{q}\|u\|_{q}^{q}.
$$

In the rest of the proof, we consider the following two cases.

(i) $J(u) \geq 0$, for all $t > 0$. From (6.1), we choose $\alpha$ satisfying

$$
1 < \alpha < \frac{(q - 2\lambda)\|u_0\|_{X_0}^{2\lambda}}{2\lambda q(1 + C^2)J(u_0)}.
$$

Substituting (6.7) into (6.8), as $J(u) \geq 0$ in this case we get

$$
\frac{d}{dt}\|u\|_{X_0}^2 = 4\lambda(\alpha - 1)J(u) - 4\lambda J(u) + \frac{(q - 2\lambda)}{q}\|u\|_{q}^{q}
$$

$$
\geq -4\lambda J(u_0) + 4\lambda J(u) + \int_0^t \|u_\tau\|_{X_0}^2 \, d\tau + \frac{2(q - 2\lambda)}{q}\|u\|_{q}^{q}.
$$

From Lemma 6.1 it follows that $J(u) < 0$, i.e.,

$$
\|u\|_{W_0}^{2\lambda} < \|u\|_{q}^{q}.
$$
Therefore, applying the basic inequality $s \leq s^a + 1$ for any $s \geq 0$ and $a \geq 1$, we can obtain

\[
\frac{d}{dt} \|u\|_{X_0}^2 > -4\lambda a f(u_0) + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr + \frac{2(q - 2\lambda)}{q} \|u\|_q^q
\]

\[
> -4\lambda a f(u_0) + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr + \frac{2(q - 2\lambda)}{q} \|u\|_W^2
\]

\[
> -4\lambda a f(u_0) + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr + \frac{2(q - 2\lambda)}{q (\|u\|_W^2 - 1)}
\]

\[
z - 4\lambda a f(u_0) + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr + \frac{2(q - 2\lambda)}{q (1 + C^2)} \|u\|_{X_0}^q - \frac{2(q - 2\lambda)}{q}
\]

where $C$ is the best embedding constant of inequality $\|u\|_2 \leq C \|u\|_W$. Then

\[
\frac{d}{dt} \|u\|_{X_0}^2 - \frac{2(q - 2\lambda)}{q (1 + C^2)} \|u\|_{X_0}^2 > -4\lambda a f(u_0) - \frac{2(q - 2\lambda)}{q}
\]

which yields

\[
\|u\|_{X_0}^2 > \|u_0\|_{X_0}^2 \left( \frac{2(q - 2\lambda)}{q (1 + C^2)} + \frac{q (1 + C^2)}{q - 2\lambda} \right) \left( 2\lambda a f(u_0) + \frac{q - 2\lambda}{q} \right) \left( 1 - e^{\frac{2(q - 2\lambda)}{q (1 + C^2)}} \right).
\]

Next, we define

\[
y(t) := \int_0^t \|u(\tau)\|_{X_0}^2 \, d\tau.
\]

Since the solution $u$ is global, thus the function $y(t)$ is bounded for all $t \geq 0$. Then we have

\[
y(t) = \|u(t)\|_{X_0}^2
\]

and

\[
y'(t) = \frac{d}{dt} \|u\|_{X_0}^2.
\]

Substituting (6.13) into (6.11), we get

\[
y''(t) > \frac{2(q - 2\lambda)}{q (1 + C^2)} \|u_0\|_{X_0}^2 - 4\lambda a f(u_0) - \frac{2(q - 2\lambda)}{q} \left( e^{\frac{2(q - 2\lambda)}{q (1 + C^2)}} + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr \right)
\]

\[
> 2\lambda a \varepsilon \|u_0\|_{X_0}^2 + 4\lambda a \int_0^t \|u_r\|_{X_0}^2 \, dr := A(t).
\]

By (6.9), we can take $\varepsilon > 0$ small enough such that

\[
\varepsilon < \frac{1}{2\lambda a \|u_0\|_{X_0}^2} \left( \frac{2(q - 2\lambda)}{q (1 + C^2)} \|u_0\|_{X_0}^2 - 4\lambda a f(u_0) - \frac{2(q - 2\lambda)}{q} \right).
\]

Then we pick $c > 0$ large enough such that

\[
c > \frac{1}{4} e^{-2} \|u_0\|_{X_0}^2.
\]

We now define the auxiliary function

\[
\eta(t) := y''(t) + e^{-1} \|u_0\|_{X_0}^2 y(t) + c.
\]
Hence
\[
\eta'(t) = (2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0}) y'(t),
\]
\[
(6.17)
\]
\[
\eta''(t) = (2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0}) y''(t) + 2(y'(t))^2.
\]
\[
(6.18)
\]
Setting \( \rho := 4c - \varepsilon^{-2}u_0 \|u_0\|_{X_0}^2 \), by (6.16) we know \( \rho > 0 \). Now, from (6.17) we can write
\[
(\eta'(t))^2 = (2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0})^2 (y'(t))^2
\]
\[
= (4y^2(t) + 4\varepsilon^{-1}u_0 \|u_0\|_{X_0} y(t) + \varepsilon^{-2}u_0 \|u_0\|_{X_0}^2) (y'(t))^2
\]
\[
= (4y^2(t) + 4\varepsilon^{-1}u_0 \|u_0\|_{X_0} y(t) + 4c - \rho) (y'(t))^2
\]
\[
= (4\varphi(t) - \rho)(y'(t))^2.
\]
\[
(6.19)
\]
The above equality yields
\[
4\eta(t)(y'(t))^2 = (\eta'(t))^2 + \rho(y'(t))^2.
\]
\[
(6.20)
\]
By integrating the following identity from 0 to \( t \), it gives
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{X_0}^2 = (u, u_t) + (u, u_t)_{W_0},
\]
\[
(6.21)
\]
i.e.,
\[
\frac{1}{2} \left( \|u(t)\|_{X_0}^2 - \|u_0\|_{X_0}^2 \right) = \int_0^t (u, u_t)_{X_0} d\tau.
\]
Hence
\[
\|u(t)\|_{X_0}^2 = \|u_0\|_{X_0}^2 + 2 \int_0^t (u, u_t)_{X_0} d\tau.
\]
This equality along with the Hölder and Young’s inequality gives
\[
(y'(t))^2 = \|u(t)\|_{X_0}^4 = \left( \|u_0\|_{X_0}^2 + 2 \int_0^t (u, u_t)_{X_0} d\tau \right)^2
\]
\[
(6.22)
\]
\[
\leq \left( \|u_0\|_{X_0}^2 + 2 \left( \int_0^t \|u\|_{X_0}^2 d\tau \right) \right)^{\frac{1}{2}} \left( \int_0^t \|u_t\|_{X_0}^2 d\tau \right)^{\frac{1}{2}}
\]
\[
\leq \|u_0\|_{X_0}^2 + 4y(t) \int_0^t \|u_t\|_{X_0}^2 d\tau + 2\varepsilon\|u_0\|_{X_0}^2 y(t) + 2\varepsilon^{-1}\|u_0\|_{X_0}^2 \int_0^t \|u_t\|_{X_0}^2 d\tau
\]
\[
= : B(t).
\]
From (6.18) and (6.20), we can get
\[
2\eta(t)\eta''(t) = 2 \left( 2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0} \right) y''(t) + 2(y'(t))^2 \eta(t)
\]
\[
= 2 \left( 2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0} \right) y''(t) \eta(t) + 4(y'(t))^2 \eta(t)
\]
\[
= 2 \left( 2y(t) + \varepsilon^{-1}u_0 \|u_0\|_{X_0} \right) y''(t) \eta(t) + (\eta'(t))^2 + \rho(y'(t))^2.
\]
\[
(6.23)
\]
By (6.15) and the fact that $e^{\frac{2\alpha - 2\varnothing}{2(\alpha - 1)}} > 1$ and $\eta(t) > 0$, we obtain

$$2\eta(t)\eta''(t) - (1 + \alpha)(\eta'(t))^2$$
$$> 2\eta(t)\left(2y(t) + \varepsilon^{-1}\|u_0\|_{\mathcal{X}_0}\right)\left(4\lambda\int_0^t \|u_\tau\|_{\mathcal{X}_0}^2 d\tau + 2\lambda\varepsilon\|u_0\|_{\mathcal{X}_0}^2\right) - 4\alpha\eta(t)B(t)$$
$$> 4\lambda\alpha\eta(t)\left(2y(t) + \varepsilon^{-1}\|u_0\|_{\mathcal{X}_0}\right)\left(2\int_0^t \|u_\tau\|_{\mathcal{X}_0}^2 d\tau + \varepsilon\|u_0\|_{\mathcal{X}_0}^2\right) - 4\alpha\eta(t)B(t)$$
$$= 4\lambda\alpha B(t)\eta(t) - 4\alpha B(t)\eta(t)$$
$$> 0,$$

i.e.,

$$\eta(t)\eta''(t) - \frac{1 + \alpha}{2}(\eta'(t))^2 > 0, \quad t \in [0, T],$$

which implies that

$$(\eta^{-\varepsilon}(t))^\alpha = \frac{\varepsilon}{(\eta(t))^{\varepsilon+2}}((\varepsilon + 1)(\eta'(t))^2 - \eta''(t)\eta(t)) < 0, \quad \varepsilon = \frac{\alpha - 1}{2} > 0.$$

Since $\eta(0) = c > \frac{1}{2}\varepsilon^{-2}\|u_0\|_{\mathcal{X}_0}^2 > 0$, $\eta'(0) = \varepsilon^{-1}\|u_0\|_{\mathcal{X}_0}^2 > 0$, by Lemma 6.2, it follows that there exists a

$$0 < t_\ast \leq \frac{2\eta(0)}{(\alpha - 1)\eta'(0)},$$

such that

$$\lim_{t \to t_\ast} \eta^{-\varepsilon}(t) = 0,$$

and

$$\lim_{t \to t_\ast} \eta(t) = +\infty.$$

As $\eta(t)$ is a continuous function with respect to $t$, we can conclude that $y(t)$ tends to $\infty$ at some $t_\ast$, which contradicts $T = +\infty$.

(ii) There exist some $i$ such that $f(u(i)) < 0$.

Since $f(u_0) > 0$, by the continuity of $f(u(t))$ in $t$, we can assume that there exists a first time $t_0 > 0$ such that $f(u(t_0)) = 0$ and $f(u(t)) < 0$ for some $i > t_0$. We take $u(t)$ as a new initial datum, then from Lemma 6.1, we have $u(t) \in \mathcal{N}_+$ for $t > t_0$. Then similar to the proof of Theorem 4.2, we can prove the finite time blowup of the solution.

Combining the above two cases, we conclude that $u(t)$ blows up in finite time. \hfill \Box

7 Conclusions and future works

Inspired by [35], it is natural to consider the following more general problem

$$\begin{cases}
  u_t + M(|u|^2)\mathcal{L}_K u + (-\Delta)^{\alpha} u_t = |u|^{q-2}u, & \text{in } \Omega \times \mathbb{R}^+, \\
  u(x, 0) = u_0(x), & \text{in } \Omega, \\
  u(x, t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}_0^+,
\end{cases} \tag{7.1}$$

the operator $\mathcal{L}_K$ is given by

$$\mathcal{L}_K \varphi(x) = \int_{\mathbb{R}^n} (2\varphi(x) - \varphi(x + y) + \varphi(x - y))K(y)dy \tag{7.2}$$
for every $x \in \mathbb{R}^n$, where the kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ satisfies the following assumption
\[
\begin{aligned}
\text{(H)} \quad \{m(x)K \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}; \\
\text{there exists } K_0 > 0, \quad \text{such that } K(x) \geq K_0 |x|^{-(N+2s)} \text{ for a.e. } x \in \mathbb{R}^n \setminus \{0\}.
\end{aligned}
\]

A typical example for $K$ is the singular kernel $K(x) = |x|^{-(N+2s)}$. In this case, up to some normalization constant, $\mathcal{L}_K \varphi(x) = (-\Delta)^s \varphi(x)$. Using the arguments similar to Sects. 4-6 of this paper, we get the existence and finite time blow-up of solutions, as well as the asymptotic behavior for problem (7.1). However, the global existence for super-critical initial energy, i.e., $J(u_0) > d$ can't be obtained because of the absence of the comparison principle. Thus, in order to prove the global well-posedness for problem (1.1) and (7.1) in the super-critical initial energy case, some new methods and strategies should be found, which will be the object of future work. At the same time, this work is helpful to analyze the observability and measurability of the control model in control system.

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