Controlling a Markov Decision Process with an Abrupt Change in the Transition Kernel

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Abstract—We consider the control of a Markov decision process (MDP) that undergoes an abrupt change in its transition kernel (mode). We formulate the problem of minimizing regret under control switching based on mode change detection, compared to a mode-observing controller, as an optimal stopping problem. Using a sequence of approximations, we reduce it to a quickest change detection (QCD) problem with Markovian data, for which we characterize a state-dependent threshold-type optimal change detection policy. Numerical experiments illustrate various properties of our control-switching policy.

I. INTRODUCTION

Control engineering has perfected the art of controller synthesis to optimize system costs, given a static environment, often modeled as an MDP. Practical engineering contexts are often non-stationary, i.e., the underlying dynamics of the environment encoded in the transition kernel of the MDP changes at some point in time. A controller that is agnostic to that change can perform poorly against the changed environment. If the change point is known a priori, or is revealed when it happens, optimizing the control design becomes simple — design controllers for the two “modes” of the system and then switch between the controllers, when the mode change happens. We tackle the question of mode change detection, followed by switching between mode-specific controllers. Specifically, we consider the case where the underlying transition kernel of an MDP changes at a random point, and the change must be detected by observing the state dynamics.

The mode of the system for our problem functions essentially as a hidden state. As a result, controlling a non-stationary MDP can be formulated as a partially-observed MDP (POMDP). Even for small, finite state spaces, solving a POMDP can be computationally challenging. See [1] for details. The change-detection based control switching paradigm provides a computationally cheaper alternative.

Detection of a change in the statistics of a stochastic process has a long history in statistics and optimal control literature, with roots in [2]. Quickest change detection (QCD) theory seeks a causal control policy that an observer of the stochastic process can use to detect a change. In designing such a detector, one must balance between two competing erroneous declarations. A hasty detector might declare the change too early, leading to a false alarm. With controller switching against non-stationary MDPs, an early change will incur extra cost of using a possibly sub-optimal controller over the period when the controller has changed, but the transition kernel has not. A lethargic detector who declares the change too late, pays the delay penalty of continuing to use the wrong controller, even after the transition kernel has changed. Thus, an optimal change detection mechanism balances between the possibilities of false alarm and delay, optimizing the costs incurred in these situations. In this paper, we formalize the controllers under the changing environment in Section II and define the regret of change detection-based controller switching in Section III.

When the pre- and post-change data is independent and identically distributed (i.i.d.), the optimal Bayesian change detector with geometric prior distribution assumed over the change point, was derived in [3]. Alternate formulations without such prior distributions have been derived in [4]–[6]. See [7], [8] for detailed surveys on this history. In this paper, we assume a geometric prior on the change point. In Section IV, we simplify the regret expression for controller switching via approximations that rely on fast mixing of the Markov chains under the mode-specific controllers. Then in Section V, we formulate the question of optimal controller switching for minimizing the approximate regret as a QCD problem with Markovian data, and characterize an optimal change detector. Our main result (Theorem 3) proves that the optimal switching policy is threshold-type, where the thresholds depend on the observed Markov state of the system. While our goal in this paper is (approximate) regret minimization with switching between mode-specific controllers for a single change in the transition kernel of an MDP, Theorem 3 applies more broadly to general QCD problems with Markovian observations. We remark that while our results are presented more broadly to general QCD problems with Markovian observations.

In Section VI, we present numerical experiments to demonstrate the performance of the controller switching policy that we design on example non-stationary MDPs. Our results demonstrate that the change-detection based controller switching performs very similarly to that using mode-observation under a variety of parameter choices.

Perhaps the closest in spirit to our work is [9], where the authors provide several interesting insights into the use of QCD-based heuristics in controller switching for non-stationary MDPs. Compared to [9], however, our primary goal is theoretical analysis to establish the optimality of the state-dependent threshold-type Bayesian change detector for approximate regret minimization, and to provide conditions under which such approximations are expected to be accurate. We conclude the paper in Section VII. Proofs of results are included in the appendix.
II. FORMULATING THE QUICKEST TRANSITION KERNEL CHANGE DETECTION PROBLEM

Consider a controlled, non-stationary Markovian dynamical system evolving over a finite state space $\mathcal{X}$ in discrete time ($t \in \mathbb{Z}_+$) under the action of a controller choosing actions from a finite action space $\mathcal{U}$. The system is non-stationary in the sense that the state transition kernel remains fixed up until a random time $\tau$, when the system changes its mode. When the mode change happens, the state evolution, i.e., the transition kernel, changes. The pre and post-change kernels are denoted as $P_1(\cdot|x, u)$ and $P_2(\cdot|x, u)$, respectively, for $x \in \mathcal{X}, u \in \mathcal{U}$, and the non-stationary state transition kernel is then given by

$$P_t(x_{t+1}|x_t, u_t) = \begin{cases} P_1(x_{t+1}|x_t, u_t), & \text{if } t < \Gamma, \\ P_2(x_{t+1}|x_t, u_t), & \text{otherwise}. \end{cases} \quad (1)$$

In general, the setting described thus far could additionally include mode-dependent stage-cost functions. For simplicity, assume that stage costs are not mode dependent and given by the time-invariant function $c : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$. Using the notation $Y^0_t$ to denote the sequence $(Y_0, \ldots, Y_t)$ for an arbitrary variable $Y$, we define $b_t := (x^0_t, u^{t-1}_0)$ as the state-action history available prior to taking an action at time $t$.

The objective is to choose a policy $\pi$ to minimize the long-term cost, i.e., for all $x \in \mathcal{X}$,

$$J^*(x) = \min_{\pi \in \Pi} J^\infty(x; \pi)$$

$$= \min_{\pi \in \Pi} \mathbb{E}_{x_t \sim P_t} \left[ \sum_{t=0}^{\infty} \gamma^t c(x_t, U_t)|x_0 = x \right]. \quad (2)$$

As the change point $\tau$ is not observable, a Markovian policy may not, generally speaking, be optimal in (2). Considering the mode at time $t$, denoted $\Theta_t \in \{1, 2\}$, together with $X_t$, as part of an augmented state process $S_t := (X_t, \Theta_t)$, (2) can be formulated as a partially observed Markov decision process (POMDP). Direct solutions to POMDPs are computationally intensive. We propose and analyze an alternative approach—employ change detection and switch between two mode-specific controllers, i.e., optimal stationary policies for each of the system modes. We denote the mode-specific policies as $\pi_i : \mathcal{X} \to \mathcal{U}$ for $i = 1, 2$, denoting that in the fixed mode settings we may restrict to deterministic, Markovian policies without loss of optimality [10]. In particular, $\pi_i \in \Pi$ for $i = 1, 2$, and

$$J^\infty(x; \pi_i, P_i) = J^*(x; P_i) := \min_{\pi \in \Pi} J^\infty(x; \pi, P_i)$$

$$= \min_{\pi \in \Pi} \mathbb{E}_{x_t \sim P_t} \left[ \sum_{t=0}^{\infty} \gamma^t c(x_t, \pi(x_t))|x_0 = x \right]. \quad (3)$$

Limiting our search to policies that switch between $\pi_1$ and $\pi_2$ at a single time $t = \tau$, the more general policy design problem in (2) reduces to an optimal stopping problem, where stopping corresponds to switching from $\pi_1$ to $\pi_2$. In this context, a natural performance baseline policy is one which observes the mode directly and switches controllers at time $t = \Gamma$. Intuitively, if an algorithm can accurately detect the change point, then a switching policy employing such an algorithm should well approximate the performance achieved when mode observations are available. Our goal is to utilize tools from the QCD literature to develop a change detection algorithm for the described non-stationary MDP setting, and characterize the regret of the resulting change-detection based controller with respect to the mode-observing baseline.

We remark that our regret analysis will not exactly compare a CD-based algorithm with the “optimal” control. To see why, note that $\pi_1$ is designed to minimize the long-run discounted cost, assuming that the system remains in mode 1 for the entire infinite horizon. In other words, the design of $\pi_1$ is oblivious to the possibility of a mode change, and hence, is not strictly optimal in our non-stationary setting, even when $\Theta_\tau = 1$. The choice of operating with $\pi_2$, however, is optimal post-change, as we preclude the possibility of the mode switching back from $\Theta_\tau = 2$ to $\Theta_\tau = 1$. On the other hand, optimal policies arising from the POMDP formulation previously mentioned take into account the potential mode change, and continuously adjust actions in accordance with a belief regarding the system mode. Tackling such complications is relegated to future endeavors.

III. DEFINING THE REGRET

Let $X = \{X_t\}_{t \geq 0}$ and $X' = \{X'_t\}_{t \geq 0}$ denote the sequence of state observations generated by the change detection (CD) based and mode observing controllers, under the same random change point $\tau$. Note that the state trajectories of the CD and mode observing controllers will always agree prior to the change point, or the time at which the CD controller switches to $\pi_2$, whichever is earlier.

We seek to identify a causal switching rule $\tau$ that defines an extended integer-valued random variable, adapted to the filtration $\{\sigma(X_1, \ldots, X_t)\}$, generated by the stochastic process. We will often use $\tau$ to denote the random stopping time chosen by the underlying decision rule. Note that we need not adapt $\tau$ to $U = \{U_t\}_{t \geq 0}$, as the observed states are mapped deterministically to actions by both $\pi_1$ and $\pi_2$.

Let $\Theta_t \in \{1, 2\}$ denote the mode of the system at time $t$. Further, let $D_t$ denote the switching decision at time $t$, i.e., $D_t = 1$ for $t < \tau$, and $D_t = 2$ otherwise. Then, the regret minimization problem can be written as

$$\min_{\tau \in \mathcal{S}} \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_t(X_t, D_t, \Theta_t) \right] =: \mathcal{R}(\tau), \quad (4)$$

where $\mathcal{S}$ is the set of all causal switching policies satisfying $P(\tau < \infty) = 1$, and

$$r_t(X_t, D_t, \Theta_t) = \begin{cases} 0, & \text{if } D_t = 1, \Theta_t = 1, \\ c(X_t, \pi_1(X_t)) - c(X'_t, \pi_2(X'_t)), & \text{if } D_t = 1, \Theta_t = 2, \\ c(X_t, \pi_2(X_t)) - c(X'_t, \pi_1(X'_t)), & \text{if } D_t = 2, \Theta_t = 1, \\ c(X_t, \pi_2(X_t)) - c(X'_t, \pi_2(X'_t)), & \text{if } D_t = 2, \Theta_t = 2. \end{cases} \quad (5)$$

In order to further analyze the expected regret in (4), we need to introduce additional notation. $J^m_{ij}(Q)$ denotes the expected $m$ step horizon cost, when the initial state is distributed according to $Q$ and policy $\pi_i$, is used under system mode $j$, for $i, j = 1, 2$. If the initial state $X$ is given, we write $J^m_{ij}(\delta_X)$, where $\delta_X$ is the measure concentrated at $X$. 3402
Let \( P_{m}^{ij}(\delta_X) \) be the \( m \) step probability distribution, given that the system started in state \( X \), and policy \( \pi_i \) was used in system mode \( j \). Additionally, we denote the stationary state distribution arising from the use of policy \( \pi_i \) in mode \( j \), for \( i, j = 1, 2 \) as \( \Delta_{ij} \), and the associated Markov chain as \( M_{ij} \).

With this notation in hand, we can give expected regret terms corresponding to the false alarm and delay cases, conditioned on the history of state observations until time \( t \).

For switching policies, \( D_t = 2 \) implies that \( D_{t+k} = 2 \) for all \( k > 0 \). The decision to switch ends at the first time when \( D_t = 2 \). Thus, we can fold the remaining infinite horizon cost-to-go into the cost of switching, and the optimal switching policy for (4) can be obtained from dynamic programming. Specifically, consider the Bellman equation:

\[
V_i(X^0_t) = \min \left\{ \left( r^{FA}_t + r_t^D \right) \mathbb{I}_{d_t=2} + \mathbb{E} \left[ \mathbb{I}_{1 \leq \Gamma < \infty} \left( c(x_t, \pi_1(x_t)) - c(x'_t, \pi_2(x'_t)) \right) \right] + \gamma V_{i+1}(X^t_{x'_t}, d_t=1) \right\},
\]

where \( r^{FA}_t \) and \( r_t^D \) give the expected regret (cost) to go due to false alarm and delay when stopping at time \( t \), respectively, conditioned on \( X^0_t \). The following proposition gives expressions for \( r^{FA}_t \) and \( r_t^D \). See Appendix VIII-A for a detailed derivation. Since \( U_t = \pi_1(x_t) \) for \( t < \tau \), we do not explicitly include conditioning on \( \{U^0_t\} \) in (6). We likewise suppress inclusion of \( \{U^0_t\} \) in the subsequent development in expressions where the use of \( \pi_1 \) or \( \pi_2 \) is clear from context.

**Proposition 1.** The following assertions hold:

\[
r^{FA}_t = \mathbb{E} \left[ \mathbb{I}_{t \geq \Gamma} \left( J_{2|1}^{\Gamma-t}(\delta_{X^t_i}) - J_{1|1}^{\Gamma-t}(\delta_{X^t_i}) \right) \right] + \gamma \mathbb{E} \left[ \mathbb{I}_{t \geq \Gamma} \left( J_{2|2}^\infty(\delta_{X^t_i}) - J_{1|2}^\infty(\delta_{X^t_i}) \right) \right] |X^0_0|,
\]

\[
r^D_t = \mathbb{E} \left[ \mathbb{I}_{t \geq \Gamma} \left( J_{2|1}^\infty(\delta_{X^t_i}) - J_{1|1}^\infty(\delta_{X^t_i}) \right) \right] |X^0_t|.
\]

Expressions (7) and (8) provide further insight into the sources of expected regret when stopping at time \( t \). In the false alarm scenario, the CD based controller will erroneously use \( \pi_2 \) in mode 1 for \( \Gamma - t \) time steps. The regret incurred during this period consists of a transient component as \( M_{2|1} \) and \( M_{1|1} \) mix to \( \Delta_{2|1} \) and \( \Delta_{1|1} \), as well as a steady state component due to the difference between \( \Delta_{2|1} \) and \( \Delta_{1|1} \). Similar transient and steady state components will generally arise when either the mode changes, or the CD based controller declares a change.

Continuing in (7), there is a discrepancy between the distributions of the states \( X_T \) and \( X'_T \) due to the difference in control policies used by the CD and mode observing controllers for \( \Gamma - t \) steps. Similarly, when switching in the delayed detection scenario, there is a discrepancy in the distribution of states \( X_t \) and \( X'_t \) and thus the expected long term costs over the remaining horizon in mode 2.

In terms of selecting a stopping rule, the form of (7) and (8) introduce substantial difficulties. Application of dynamic programming techniques to (6) requires explicit evaluation of (7) and (8), needing computation of finite and infinite horizon expected discounted costs, which often do not yield closed form expressions, requiring numerical estimation. As \( \Gamma \) is unobserved, these discounted costs must be computed over ranges of potential values for \( \Gamma \). Further, as (8) includes terms involving \( X_{\Gamma} \), and the continuation option in (6) involves an expectation over \( X_{\Gamma} \) assuming the change point occurred at some \( \Gamma < t \), the entire history of state observations must be retained. In light of these difficulties, we now develop approximations to \( r^{FA}_t \) and \( r^D_t \), which in turn yield a more tractable approximation of the expected regret in (4).

**IV. APPROXIMATING THE TOTAL REGRET \( R(\tau) \)**

A primary component of the difficulty in direct use of (7) and (8) lies in the state dependency of the discounted costs involved. This state dependency follows from the mixing properties of Markov chains associated with combinations of system modes and mode specific policies. Our problem simplifies significantly under the assumption that these Markov chains are fast mixing, i.e., the impact of an initial state or state distribution on expected future states is minimal. We have the following result on convergence, i.e., mixing of a Markov chain to its stationary distribution starting from an arbitrary distribution on \( X \). Let \( P_X \) denote the set of all probability measures on \( X \).

**Theorem 1** ([11] Theorem 4.9). Let \( M \) be an irreducible and aperiodic Markov chain on \( X \), with \( t \)-step probabilities \( P^t \) and stationary distribution \( \Delta \). Then there exists \( \beta \in (0, 1) \) and \( B > 0 \) such that

\[
\sup_{\mu \in P_X} \| \mu P^t - \Delta \|_{TV} \leq B \beta^t.
\]

When there is a change in policy, mode or both, the overall system switches from one Markov chain into another, with initial state distribution affected by the preceding trajectory. Theorem 1 allows us to bound the difference in cost to go between an arbitrary initial state distribution, and the stationary distribution of the Markov chain entered. Let \( c_i := [c(x_{(1)}, \pi_i(x_{(1)})), \ldots, c(x_{(|X|)}, \pi_i(x_{(|X|)}))]^T \in \mathbb{R}^{|X|} \) denote the cost vector when using policy \( i = 1, 2 \), where \( \{x_{(k)}\}_{k=1}^{|X|} \) is an ordering of the states compatible with \( \Delta_{ij} \) for \( i, j = 1, 2 \). Let \( E^k(\mu, \Delta_{ij}) \) denote the difference between the \( k \)-step cost-to-go functions, given that the system enters \( MC(\Delta_{ij}) \) with initial state distributed according to \( \mu \), i.e.,

\[
E^k(\mu, \Delta_{ij}) := |J^k_{i|j}(\mu)| - |J^k_{i|j}(\Delta_{ij})| = |J^k_{i|j}(\mu)| - \frac{1 - \gamma^k}{1 - \gamma} c_i^\top \Delta_{ij},
\]

where \( c_i \) is defined in (10). Assuming fast mixing, i.e., \( \beta \approx 0 \), we neglect error terms of the form in (11) and use the approximation

\[
J^k_{i|j}(\mu) \approx J^k_{i|j}(\Delta_{ij}) = \frac{1 - \gamma^k}{1 - \gamma} c_i^\top \Delta_{ij}.
\]
for $i, j = 1, 2$ and $k \leq \infty$. An upper bound on the resulting error is then given in the following result that is proven in Appendix VII-B.

**Lemma 2.** Let $\beta_{ij} \in (0, 1)$ and $B_{ij} > 0$ be the mixing coefficients of $M_{ij}$ per Theorem 1. If $\mu \in \mathcal{P}_x$, then

$$E^k(\mu, \Delta_{ij}) \leq 2\|c_i\|\infty B_{ij} \frac{1 - (\gamma \beta_{ij})^k}{1 - \gamma \beta_{ij}}. \quad (13)$$

Under the assumption of fast mixing, if we switch policies at time $t$, the regret-to-go under false alarm and delay situations can be approximated using Proposition 1 as follows.

$$r_t^{FA} \approx E \left[ \textbf{1}_{t < \Gamma} \left( \frac{1 - \gamma^{t-\Gamma}}{1 - \gamma} (c^2_1 \Delta_{2|1} - c^1_1 \Delta_{1|1}) + \frac{\gamma^{t-\Gamma}}{1 - \gamma} c^2_2 (\Delta_{2|2} - \Delta_{2|2}) \right) | X_0^t \right] \quad (14)$$

$$= (c^2_1 \Delta_{2|1} - c^1_1 \Delta_{1|1}) E \left[ \textbf{1}_{t < \Gamma} \frac{1 - \gamma^{t-\Gamma}}{1 - \gamma} | X_0^t \right], \quad (15)$$

We are interested in approximating the regret for discounted cost settings, where the discount factor is close to unity, as we seek to achieve good performance in both the pre- and post-change regimes. Smaller discount factors effectively penalize false alarms more heavily than delayed detection as the penalty on future costs are discounted more heavily. Thus, with $\gamma = 1 - \epsilon$ with $\epsilon \ll 1$, we have

$$\lim_{\epsilon \to 0} \frac{1 - \gamma^m}{1 - \gamma} = \lim_{\epsilon \to 0} \frac{1 - (1 - \epsilon)^m}{1 - \epsilon} = m \quad (16)$$

for any $m > 0$, using which, we simplify $r_t^{FA}$ as

$$r_t^{FA} \approx (c^2_1 \Delta_{2|1} - c^1_1 \Delta_{1|1}) E \left[ \textbf{1}_{t < \Gamma} (\Gamma - t) | X_0^t \right], \quad (17)$$

implying

$$E[r_t^{FA}] \approx (c^2_1 \Delta_{2|1} - c^1_1 \Delta_{1|1}) E[(\Gamma - t) +]. \quad (18)$$

Again with fast mixing, the single-step component of the continuation cost in (6) can be approximated as

$$E \left[ \textbf{1}_{\Gamma < t} (c(X_t, \pi_1 (X_t)) - c(X_t', \pi_2 (X_t'))) | X_0^t \right] \approx E \left[ \textbf{1}_{\Gamma < t} (c(X_t, \pi_1 (X_t)) - c^1_1 \Delta_{2|2}) | X_0^t \right], \quad (19)$$

since, given the event $\{ \Gamma < t \}$, $X_t'$ will be distributed according to approximately $\Delta_{2|2}$. Approximating $\gamma \approx 1$, we may now write the following approximation of the Bellman equation in (6),

$$\hat{V}(X_0^t) = \min_{d_t} \left\{ (c^2_1 \Delta_{2|1} - c^1_1 \Delta_{1|1}) E[(\Gamma - t) + | X_0^t] \textbf{1}_{d_t = 2} \right.$$}

$$\left. + E \left[ \textbf{1}_{\Gamma < t} (c(X_t, \pi_1 (X_t)) - c^1_1 \Delta_{2|2}) \right] + \hat{V}(X_{0}^{t+1}) | X_0^t, d_t \right\}. \quad (20)$$

The optimal value functions $\hat{V}$ satisfy the Bellman equation of the following optimal control problem that optimizes over all causal control-switching policies $S$

$$\min_{r \in S} E \left[ \sum_{t=1}^{\tau} c(X_t, \pi_1 (X_t)) - c^1_1 \Delta_{2|2} (\tau - \Delta) + c^2_2 \Delta_{2|2} (\tau - \Gamma) \right]. \quad (21)$$

Reapplying the fast mixing approximation, we have

$$\min_{r \in S} E \left[ \sum_{t=1}^{\tau} c(X_t, \pi_1 (X_t)) \right] \approx c^1_1 \Delta_{1|2} (\tau - \Gamma) \quad (22)$$

for all $\tau \in S$. Combining (21) and (22), we arrive at the approximate regret minimization problem, given by

$$\min_{r \in S} E \left[ \sum_{t=1}^{\tau} c(X_t, \pi_1 (X_t)) \right] \approx c^1_1 \Delta_{1|2} (\tau - \Gamma) \quad (23)$$

In view of Lemma 2, $\tilde{R}(\tau)$ will be a good approximation to $R(\tau)$, when the constants $\beta_{ij}$ for $i, j = 1, 2$ are small. Roughly speaking, $\beta_{ij}$ is small when the spectral gap of $M_{ij}$ is large, i.e., $\beta_{ij}$ depends largely on the second leading eigenvalue of $M_{ij}$ (see [11] Chapter 12). Additionally, a switching policy optimizing (23) will prioritize minimization of false alarm and delay based regret based upon the difference in average stage costs in the involved Markov chains. For instance, when $c^2_2 \Delta_{2|2} < c^1_2 \Delta_{1|1} \ll c^1_2 \Delta_{1|2} - c^1_2 \Delta_{2|2}$, the controller may exhibit longer average false alarm periods. However, if the expected unit step loss in using $\pi_2$ in mode 1 is not too large, then assuming fast mixing, the aggregate regret due to such errors should likewise be small.

In what follows, we seek to identify stopping rules that exactly solve (23). Assume a geometric prior on the change point $\Gamma$ with $P\{\Gamma = t\} = \rho (1 - \rho)^{t-1}$ for $t \geq 1$. Then, the memoryless property of geometric distributions yields $E[\Gamma - t \gamma] = P\{\Gamma > t\}/\rho$. Assuming $c^1_2 \Delta_{1|2} - c^1_2 \Delta_{2|2} > 0$ and $c^2_2 \Delta_{2|1} - c^1_2 \Delta_{1|1} > 0$ we further get

$$\tilde{R}(\tau) \propto E[(\Gamma - t) +] + \lambda \rho \{\Gamma > t\}, \quad (24)$$

where

$$\lambda \rho = (c^2_1 \Delta_{2|1} - c^1_2 \Delta_{1|1})/(c^1_1 \Delta_{1|2} - c^1_1 \Delta_{2|2}). \quad (25)$$

The above argument requires both the numerator and the denominator in (25) to be non-negative. Notice that when $X_0 \sim \mu_0$, we have

$$J_{1|1}^{\infty}(\mu_0) = \frac{1}{1 - \gamma} c^1_2 \Delta_{1|1} + E^{\infty}(\mu_0, \Delta_{1|1})$$

$$\leq \frac{1}{1 - \gamma} c^2_2 \Delta_{2|2} + E^{\infty}(\mu_0, \Delta_{2|2}), \quad (26)$$

owing to the optimality of $\pi_1$ for the MDP with in mode 1. In essence, this implies that

$$c^2_2 \Delta_{2|1} - c^1_1 \Delta_{1|1} \geq (1 - \gamma) \left[ E^{\infty}(\mu_0, \Delta_{1|1}) - E^{\infty}(\mu_0, \Delta_{2|1}) \right].$$

As long as $M_{2|1}$ and $M_{1|1}$ are fast-mixing, we expect the errors due to mixing to be small, and the left-hand side to be positive. A similar argument applies to (25)’s denominator.
V. OPTIMAL QCD POLICY FOR APPROXIMATE REGRET MINIMIZATION

This section is devoted to the derivation of the optimal change detector to minimize the approximate regret $R(\tau)$. Specifically, this regret under a causal control switching policy $\tau \in \mathcal{S}$ in (23) can be written as

$$R(\tau) \propto E \left[ \sum_{t=0}^{\tau} g_t(D_t, \Theta_t) \right],$$

where

$$g_t(D_t, \Theta_t) = \begin{cases} 1, & \text{if } D_t = 1, \Theta_t = 2, \\ \lambda, & \text{if } D_t = 2, \Theta_t = 1, \\ 0, & \text{otherwise}. \end{cases}$$

The optimal control problem in (27) is amenable to infinite horizon dynamic programming. For $t \geq 1$, define

$$p_t := P\{\Gamma \leq t-1 | X_t^0\} = P\{\Gamma < t | X_t^0\}$$

as the posterior probability at time $t$ that the change has taken place prior to time $t$, with $p_0 = 0$. This probability is a sufficient statistic in the sense that $(X_t, p_t)$ defines a Markovian state for dynamic programming calculations for this optimal control problem, following [12].

Let $\pi_{CD}$ denote the control policy that uses change detection to switch between $\pi_1$ and $\pi_2$. Specifically, we have

$$\pi_{CD}(X_t) = \begin{cases} \pi_1(X_t), & \text{if } t < \tau, \\ \pi_2(X_t), & \text{otherwise}. \end{cases}$$

Then, define

$$L(X_{t+1}, X_t) := \frac{P_2(X_{t+1} | X_t, \pi_{CD}(X_t))}{P_1(X_{t+1} | X_t, \pi_{CD}(X_t))}$$

Recall that $P_2$ encodes the transition kernel in mode $\theta = 1$. Thus, $L$ becomes the likelihood ratio of observing $X_{t+1}$ in mode 2 compared to that in mode 1. Further, define

$$\tilde{p}_t := p_t + \rho (1 - p_t).$$

Then, the evolution of $p_t$ via Bayes’ rule becomes

$$p_{t+1} = \Phi(X_{t+1}, X_t, p_t) := \frac{\tilde{p}_t L(X_{t+1}, X_t)}{\tilde{p}_t L(X_{t+1}, X_t) + (1 - \tilde{p}_t)}.$$  (33)

Per the dynamic programming principle, the optimal cost of the minimization of the right hand side of (27) is given by $E_{X_0 \sim \mu_0}[\hat{V}(0, X_0)]$, where the cost-to-go function $\hat{V}$ satisfies the Bellman equation,

$$\hat{V}(p_t, X_t) = \min_{d_t} \left\{ \lambda (1 - p_t) 1_{d_t = 2} \\
+ \left( p_t + E[\hat{V}(p_{t+1}, X_{t+1}) | p_t, X_t, d_t] \right) 1_{d_t = 1} \right\}$$  (34)

The expectation in the right hand side of the above equation is calculated as follows. First, fix $\pi_{CD}(X_t) = \pi_1(X_t)$. Then, the distribution of $X_{t+1}$ is either that dictated by $P_1$ or $P_2$, depending on whether the change point has happened before $t$, while the action is taken according to $\pi_1(X_t)$. For each candidate $X_{t+1}$, evaluate $L$ using (31) with $\pi_{CD}(X_t) = \pi_1(X_t)$ and $p_{t+1}$ via (33).

The following result, sketched in Appendix VIII-C, characterizes the nature of the optimal change detection policy.

The result needs additional notation. Let $C$ denote the set of nonnegative functions of $p \in [0, 1]$, dominated by $\lambda (1 - p)$. Define the right hand side of (34) as $BV$ for $B : C \rightarrow C$.

**Theorem 3.** There exists a unique solution to the fixed-point equation $BV = V$ in $C$, given by $V = \lim_{k \rightarrow \infty} B^k \hat{v}$, where $\psi(p, x) := \lambda (1 - p)$, for $p \in [0, 1]$ and $x \in \mathbb{X}$. Moreover, an optimal control-switching policy to minimize $R(\tau)$ is threshold-type, given by $\tau^* = \min\{t : p_t \geq p^\circ(X_t)\}$, where $p^\circ(X_t) \in [0, 1]$ is the unique solution of

$$\lambda (1 - p^\circ(X_t)) - p^\circ(X_t) = E[\hat{V}(\Phi(X_{t+1}, X_t, p^\circ(X_t)), X_{t+1}) | p^\circ(X_t), X_t, d_t = 1].$$

We prove the existence of value functions that satisfy the Bellman equation. In discounted-cost optimal control problems, such existence is typically argued via the Contraction Mapping Theorem. Our QCD problem for approximate regret is undiscounted; a classic contraction mapping-based argument does not work. Our proof structure shows the result over a finite horizon first, and then extends it to the infinite-horizon setting using an argument similar to that in [13].

We contextualize this result within prior art on QCD. Per [3], [8], Bayesian QCD with i.i.d. data under a geometric prior on the change point yields a single-threshold stopping rule. Unlike that setting, our data is Markovian. Theorem 3 reveals that a similar single-threshold policy is optimal with Markovian data too, but with the modification that the threshold is state-dependent. While we derive this stopping rule by viewing $R(\tau)$ as the approximate regret in switching control policies for MDPs under an environment with a changing transition kernel, the result applies more generally to Bayesian change detection with Markovian observations, and might be of independent interest to the QCD literature.

Leveraging Theorem 3, a solution to (34) can be obtained via successive application of $B$ on $\psi$, by discretizing the space of posterior probability $p_t$ to a set of points $\mathbb{D} \subset [0, 1]$ through value iteration. We follow the same procedure for our numerical experiments in Section VI. The cost of computing the optimal controllers for the mode-specific MDPs via value iteration is $O(|\mathbb{X}|^2 |\mathcal{A}|)$. The same for the value iteration to compute an approximately optimal switching policy becomes $O(|\mathbb{X}|^2 |\mathbb{D}|^2)$. One can alternately solve the hidden-mode MDP as a POMDP (equivalently, a belief-MDP) by discretizing the belief pertaining to the modes over $\mathbb{D}$, and then applying value iteration. Such a solution technique incurs a computational cost of $O(|\mathbb{X}|^2 |\mathbb{D}|^2 |\mathcal{A}|)$. Thus, for problems with large action spaces, our approach is more computationally efficient. A more precise comparison of QCD-based control of hidden-mode MDPs and direct POMDP solution is left to future work.

VI. NUMERICAL EXPERIMENTS

We now illustrate properties of our controller switching through numerical examples. In our simulations, the finite horizon and the geometric prior were such that pre- and post-change Markov chains mix sufficiently.
A. On a Random MDP Environment

We simulated the dynamics of a non-stationary MDP with $|X| = 5$ and $|U| = 3$. Entries of $P_1$ were generated from Unif[0,1], and then rescaled to yield a transition kernel. $P_2$ was obtained by permuting $P_1$ with respect to actions, e.g., $P_1(x'|x, a = 0) = P_2(x'|x, a = 1)$, and so on. Costs for each state/action pair were sampled from Unif[0,1]. We used a geometric prior for a range of $\rho$'s, listed in Table I. With $\gamma = 0.999$, we used value iteration to compute $\pi_1$ and $\pi_2$, by uniformly discretizing the posterior probability values to 1000 points in $[0,1]$. The change detection-based controller was computed via repeated application of the Bellman operator $B$, defined via (34), on $\psi$. Additionally, we computed a POMDP-based controller by discretizing the scalar belief over the unobserved mode to 1000 points in $[0,1]$. We recorded the discounted costs over horizons of lengths $H = [2/\rho]$, averaged over 35000 runs, calculated with mode observation-based controller switching in $J_{MO}$, that based on change detection in $J_{CD}$, and the POMDP-based controller in $J_{POMDP}$. The differences between the mean costs are all significant ($p$-values < 0.005). $J_{CD}$ is within 0.7% of $J_{MO}$, and within 0.4% of $J_{POMDP}$ implying that change detection-based switching performs quite well compared to mode-observed switching, as well as the optimal approach assuming no mode observations.

With the right hand side of (25) held constant, $\lambda$–the relative weight of regret due to false alarm compared to that due to delay–varies inversely with $\rho$. Thus, as $\rho$ increases, $\lambda$ decreases, and the state dependent thresholds $p^o$ decrease as Figure 1 reveals. In essence, the algorithm becomes more keen to switch from $\pi_1$ to $\pi_2$, incurring possibly higher regret due to lower false alarm penalty. Figure 1 confirms that the probability of false alarm grows with $\rho$.

![Fig. 1: State dependant switching thresholds and probability of false alarm (PFA) for the random MDP.](image)

B. Inventory Control Problem

Next, we present results on a non-stationary environment, studied in [9]. The state $X_t$ represents the inventory level at time $t$ with maximum inventory level $N$. Let $U_t$ denote the additional inventory to be ordered, yielding total inventory $I_t := \min\{X_t + U_t, N\}$. Having selected $U_t$, random integral demand $W_t$ is realized, giving residual inventory $X_{t+1} = \max\{0, X_t + U_t - W_t\}$. With $v$, $h$ and $d$ denoting the costs per unit ordered, inventory held, and demand lost,

$$c(X_t, U_t) := vU_t + h[I_t - W_t]_+ + d[W_t - I_t]_+.$$ (35)

The demands are distributed according to $W_t \sim \text{Poisson}(\nu)$ for $t < \Gamma$, and $W_t \sim \text{Unif}[0, N]$ afterwards. Conditioned on the change point, the demands are independent. We assume a geometric prior on the change point with $\rho = 0.01$. We run tests with $N \in \{10, 15\}$ and $d \in \{100, 200, 300\}$. The remaining parameters are set to $v = 1$, $h = 5$, $\gamma = 0.999$, $\nu = 2$. For this example, define

$$c_{ij} := [E_j[c(0, \pi_i(0)), \ldots, E_j(c(N, \pi_i(N)))]'] \in \mathbb{R}^{N+1},$$

where $E_j$ stands for expectation with respect to the demand distribution in mode $j \in \{1, 2\}$. In this example, the cost vectors depend on the mode as well as the policy, giving

$$\lambda p = \frac{c_{21} - c_{11}}{c_{12} - c_{22}},$$ (36)

differing from (25) with mode-dependent costs. We compute the optimal controllers in each mode via value iteration, determine $\lambda$ via (36) and then approximate thresholds in Theorem 3 by discretizing $p^o$ values to 100 uniformly spaced points in $[0,1]$, and iterating $B$.

**TABLE I: Performance comparison for random MDP environment of the discounted aggregate costs with mode-observed, POMDP, and change detection-based controller switching**, averaged over 35000 runs over horizon length $H = 2/\rho$.

| $\rho$ | $J_{MO}$ | $J_{POMDP}$ | $J_{CD}$ |
|-------|---------|-------------|---------|
| 0.0100 | 67.57  | 67.81  | 68.01  |
| 0.0078 | 84.94  | 85.21  | 85.38  |
| 0.0060 | 105.78 | 106.17 | 106.51 |
| 0.0046 | 130.92 | 131.16 | 131.47 |
| 0.0036 | 159.95 | 160.31 | 160.57 |
| 0.0028 | 192.64 | 193.06 | 193.37 |

We recorded the discounted costs over horizons of length 1000, and averaged over 4000 episodes. Across all cost and inventory parameters tested, the difference between mean costs $J_{MO}$ and $J_{CD}$ is less than 9% of $J_{MO}$, and the difference between $J_{POMDP}$ and $J_{CD}$ is less than 5.2% of $J_{POMDP}$. Thus, the QCD-based switching performs well with respect to the optimal (POMDP-based) approach in the absence of mode observation. Again, the difference between mean costs are significant ($p$-values < 0.002). Thus, QCD-based switching performs well with respect to the mode observation-based controller. Note that in this simulation we do not make use of cost observations in the QCD procedure, although the cost distribution changes with the mode. While incorporation of cost observations may improve the change detection performance, we leave such an extension to future work.

**TABLE II: Mean performance in the inventory control problem for the mode observing, POMDP, and change detection-based switching controllers with $\gamma = 0.999$ and 4000 episodes.**

| $N$   | $d$   | $\lambda$ | $J_{MO}$ | $J_{POMDP}$ | $J_{CD}$ |
|-------|-------|-----------|---------|-------------|---------|
| 10    | 100   | 19.39     | 18044  | 18404  | 18721  |
| 10    | 200   | 8.06      | 18206  | 18814  | 19381  |
| 10    | 300   | 7.10      | 18399  | 19020  | 19424  |
| 15    | 100   | 15.49     | 26170  | 26932  | 27535  |
| 15    | 200   | 6.97      | 26401  | 27475  | 28375  |
| 15    | 300   | 5.33      | 26597  | 27975  | 28917  |
VII. CONCLUSIONS

In this work, we examined the question of detecting a change in the transition kernel of an environment modeled as a finite state/action MDP in discrete time. We used an array of approximations to formulate a QCD problem to minimize the approximate regret against a mode-observing controller, and demonstrated that a threshold-type switching controller is optimal, where the thresholds are state-dependent.

There are a number of interesting directions for future research. First, we aim to consider the setting that allows switching among multiple modes, possibly multiple times. Second, we want to extend our results to continuous state/action MDPs via kernel methods, and consider minimax formulations that relax the geometric prior on the change point. Third, we want to analyze change detection-based reinforcement learning in non-stationary environments that do not assume knowledge of system model in various modes.

REFERENCES

[1] G. E. Monahan, “State of the art—a survey of partially observable markov decision processes: theory, models, and algorithms,” Management science, vol. 28, no. 1, pp. 1–16, 1982.
[2] W. A. Shewhart, “The application of statistics as an aid in maintaining quality of a manufactured product,” Journal of the American Statistical Association, vol. 20, no. 152, pp. 546–548, 1925.
[3] A. N. Shiryaev, Optimal stopping rules. Springer Science & Business Media, 2007, vol. 8.
[4] G. Lorden, “Procedures for reacting to a change in distribution,” The annals of mathematical statistics, pp. 1897–1908, 1971.
[5] Y. Ritov, “Decision theoretic optimality of the cumsum procedure,” The Annals of Statistics, pp. 1464–1469, 1990.
[6] M. Pollak, “Optimal detection of a change in distribution,” The Annals of Statistics, pp. 206–227, 1985.
[7] A. S. Polunchenko and A. G. Tartakovsky, “State-of-the-art in sequential change-point detection,” Methodology and computing in applied probability, under policy $j$, the discounted $T$-step cost to go is given by

$$J_{P_{ij}^T}(\bar{\mu}) = E_{X_0 \sim \mu} \left[ \sum_{t=0}^{T-1} \gamma^t c(X_t, \pi_i(X_t)) \right] = \sum_{t=0}^{T-1} \gamma^t P_{ij}^T(\bar{\mu}).$$

Therefore, we have

$$|J_{P_{ij}^T}(\bar{\mu}) - J_{P_{ij}^T}(\Delta_{ij})| \leq ||c|| \sum_{t=0}^{T-1} \gamma^t ||P_{ij}^T(\bar{\mu}) - \Delta_{ij}||_1$$

$$= 2||c|| \sum_{t=0}^{T-1} \gamma^t ||P_{ij}^T(\bar{\mu}) - \Delta_{ij}||_\infty \leq 2B_{ij} ||c|| \frac{1 - (1 - \gamma_{ij})^T}{1 - \gamma_{ij}}.$$

C. Proof of Theorem 3

Consider the minimization of $\hat{R}(\tau)$ over a finite horizon of length $T$. Precisely, define the optimal cost of the following $T$-period optimal control problem,

$$\tilde{\nu}_*^{T}(p, x) := \min_{\tau \in S^T} E \left[ \sum_{t=0}^{\tau} \delta_t(D_t, \Theta_t)p_0 = p, X_0 = x \right].$$

where $S^T := \{ \tau \in S : |\tau| \leq T - 1 \}$. Here, $\tilde{\nu}_*^{T}(p, x)$ describes the optimal cost, assuming a starting state $x$ with an initial belief $\theta_0 = 2$ for all $t$. The sequence $\{p_t\}$ with $p_0 = p$ evolves according to the recursion $\Phi$ defined in (33). In this problem, if a change has not been declared at any time $t < T$, then it is declared at time $T$. As $S^T \subset S^{T+k}$ for $k > 0$, $\tilde{\nu}_*^{T}$ is decreasing in $T$. The optimal costs satisfy

$$\tilde{v}_0^{T}(p, x) = \lambda(1 - p),$$

$$\tilde{v}_*^{T+k}(p, x) = \min\{1 - p, p + E[\tilde{v}_*^{T}(\Phi(X^+, x, p), X^+)]|p, x, \pi_1(x)\},$$

for $T \geq 0$. The relation in (37b) holds as the change may be declared at time $t = 0$, or at a later time $t > 0$. In the latter case, due to the memoryless property of the geometric change point prior, the optimal cost following the transition due to state-action pair $(x, \pi_1(x))$ is precisely $\tilde{v}_*^{T}(p^+, X^+)$. Define the optimal cost over the infinite horizon as

$$\tilde{\nu}_*(p, x) := \inf_{\tau \in S} E \left[ \sum_{t=0}^{\tau} \delta_t(D_t, \Theta_t)p_0 = p, X_0 = x \right].$$

3407
We now show that \( \bar{v}_\epsilon = \lim_{T \to \infty} \bar{v}^T_\epsilon \). The limit exists, owing to the decreasing and nonnegative nature of \( \bar{v}^T_\epsilon \). Since \( S_T \subseteq S \), we have \( \bar{v} \leq \lim_{T \to \infty} \bar{v}^T_\epsilon \). To show the reverse inequality, fix \((p, x)\) and \( \epsilon > 0 \). Then, there exists an almost surely finite stopping time \( \tau_x \in S \) such that

\[
\bar{v}_\epsilon(p, x) + \epsilon \geq \mathbb{E} \left[ \sum_{t=0}^{\tau_x} g_t(D_t, \Theta_t) | p_0 = p, X_0 = x \right].
\]  

(39)

We then have that

\[
\mathbb{E} \left[ \sum_{t=0}^{\tau_x} g_t(D_t, \Theta_t) | p_0 = p, X_0 = x \right] \geq \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=0}^{\tau_x} g_t(D_t, \Theta_t) | p_0 = p, X_0 = x \right],
\]

where (a) uses the monotone convergence theorem and \( P(\tau_x < \infty) = 1 \), while (b) uses the dominated convergence theorem. Since \( \epsilon > 0 \) is arbitrary, \( \lim_{T \to \infty} \bar{v}^T_\epsilon(p, x) \leq \bar{v}_\epsilon(p, x) \), which completes the proof of \( \bar{v}_\epsilon = \lim_{T \to \infty} \bar{v}^T_\epsilon \).

For the \( T \)-period problem, dynamic programming implies the existence of optimal cost-to-go functions \( \bar{V}^T_0 \), \( \ldots \), \( \bar{V}^T_T \) that satisfies \( \bar{V}^T_0(p, x) = \bar{V}^\infty_0(p, x) \), where

\[
\bar{V}^T_T(p, x) = \lambda(1 - p),
\]

(40a)

\[
\bar{V}^T_t(p, x) = \min \left\{ \lambda(1 - p), p + A^T_t(p, x) \right\}
\]

(40b)

for \( t = 0, \ldots, T - 1 \), where

\[
A^T_t(p, x) := \mathbb{E} \left[ \bar{V}^T_{t+1}(\Phi(X^+, x, p), X^+) | p, x, \pi_1(x) \right].
\]

(41)

Here, \( X^+ \) denotes the random next state from \( x \) under action \( \pi_1(x) \), given belief \( p \) on the change. With this notation, we prove the result through the following sequence of steps.

(a) We first show that \( \bar{V}^T_t(p, x) \) for \( t = 0, \ldots, T \) and \( A^T_t(p, x) \) for \( t = 0, \ldots, T - 1 \) are nonnegative, concave functions of \( p \in [0, 1] \) for every \( x \in \mathbb{X} \).

(b) Next, we utilize the concavity of \( \bar{V}^T \) and \( A^T \) to deduce the threshold structure of an optimal stopping rule for the \( T \)-period problem.

(c) Then, we establish that \( \lim_{T \to \infty} \bar{V}^T_t \) exists and that \( \bar{V} = \bar{v}_\epsilon \).

(d) If \( BG = \mathbb{G} \) for any \( G \in \mathbb{G} \), then we show that \( G = \bar{V} \).

(e) We conclude the proof by showing \( \bar{V} = \lim_{T \to \infty} B^k \psi \).

Proofs of steps (a)-(e) are in an arXiv version of the paper [14]. Only a proof sketch of each step is included here.

- **Step (a):** We prove the claim using backward induction. Notice that \( \bar{V}^T_t(p, x) \) is a nonnegative, affine function of \( p \) in \([0, 1]\). Then, using elementary calculations, we show that

\[
A^T_{t-1}(p, x) = \lambda(1 - p)(1 - p),
\]

(42)

which is nonnegative and affine in \( p \), completing the base case. Next, assume that \( \bar{V}^T_{t+1}(p, x) \) is nonnegative and concave in \( p \) for all \( x \in \mathbb{X} \). The concavity of \( \bar{V}^T_{t+1}(p, x) \) implies the existence of a collection of affine functions such that

\[
\bar{V}^T_{t+1}(p, x) = \inf_{z \in \mathbb{Z}(x)} \{ a_z + b_z \} \text{ for scalars } b_z, a_z \text{ for each } z \in \mathbb{Z}(x).
\]

Then, (41) gives

\[
A^T_t(p, x) = \inf_{z \in \mathbb{Z}(x)} \{ a_z(p + p(1 - p)) + b_z \},
\]

(43)

proving that \( A^T_t(p, x) \) is concave. Nonnegativity of \( \bar{V}^T_{t+1}(p, x) \) yields the same for \( A^T_t(p, x) \). Further, from (40b), \( \bar{V}^T_t(p, x) \) is the pointwise minimum of a pair of concave and nonnegative functions of \( p \), and therefore, is itself concave and nonnegative in \( p \in [0, 1] \). This completes step (a).

- **Step (b):** In the right hand side of (40b), \( \lambda(1 - p) \) decreases monotonically from \( \lambda \) to zero, and \( p + A^T_t(p, x) \) is nonnegative and concave in \( p \in [0, 1] \). We then show that \( A^T_t(1, x) = 0 \) for all \( x \in \mathbb{X} \), and \( t = 0, \ldots, T - 1 \), which in turn yields that \( p + A^T_t(p, x) \) can intersect \( \lambda(1 - p) \) at most once. This establishes the it is optimal to stop at time \( t \), when \( p \) at time \( t \) exceeds \( \psi_t(p) \) that satisfies

\[
\lambda(1 - p_t(x)) = \psi_t(x) + A^T_t(p_t(x), x).
\]

(44)

This establishes the threshold structure of an optimal switching policy for the \( T \)-period problem.

- **Step (c):** We show that \( \bar{V}^T_t(p, x) \) decreases with \( T \). Being bounded below by zero, it converges to some \( \bar{V}^\infty_t(p, x) \). The memoryless property of the geometric prior yields

\[
\bar{V}^T_{t+1}(p, x) = \bar{V}^T_t(p, x) \implies \bar{V}^\infty_t(p, x) = \bar{V}^\infty_t(p, x).
\]

(45)

Elementary calculation then gives \( \bar{B}V = \bar{V} \). We establish the optimality of \( \bar{V}(p, x) \) from

\[
\bar{v}_\epsilon(p, x) = \lim_{T \to \infty} \bar{V}^T_t(p, x) = \lim_{T \to \infty} \bar{V}^\infty_t(p, x) = \bar{V}(p, x).
\]

(46)

Arguing along the lines of step (b), we infer that an optimal switching rule for the infinite horizon is threshold-type, where the threshold \( p^*(x) \) satisfies

\[
\lambda(1 - p^*(x)) = \psi^*(x) + A(p^*(x), x).
\]

(47)

Finally, we show that the switching time \( \tau \) implied by the bounded function \( \bar{v}_\epsilon = \bar{V} \) is almost surely finite by proving that \( \mathbb{E}[\tau | p_0 = p, X_0 = x] \) is bounded above by \( \lambda \bar{v}_\epsilon(p, x) + \mathbb{E}[\Gamma | p_0 = p, X_0 = x] \), where \( \lambda := 1/\min\{1, \lambda \} \).

- **Step (d):** Let \( BG = \mathbb{G} \) for some \( G \in \mathbb{G} \) and associate with \( G \) a threshold-type stopping rule. Then, \( BG = \mathbb{G} \) gives

\[
\mathbb{E}[\tau_G | p_0 = p, X_0 = x] \geq \lambda \mathbb{E}[\Gamma | p_0 = p, X_0 = x] + \lambda P(\Gamma > \tau_G | p_0 = p, X_0 = x).
\]

(48)

We utilize the bounded property of \( G \in \mathbb{G} \) to establish that \( \tau_G \) is finite almost surely, implying that \( \tau_G \in S \). Then, optimality of \( V \) yields \( G(p, x) \geq V(p, x) \). Finally, we establish the reverse by taking \( T \to \infty \) on \( G(p, x) \leq V^\infty_0(p, x) \) shown by backward induction, completing step (d).

- **Step (e):** We utilize induction to show that

\[
0 \leq [B^{k+1} \psi](p, x) \leq [B^k \psi](p, x)
\]

(49)

for all \( p \in [0, 1] \), \( x \in \mathbb{X} \), and \( k \geq 0 \). Then, the monotonically decreasing but bounded sequences \([B^k \psi](p, x)\) admit a limit. The rest follows from the observation that this limit must be a fixed point of \( B \), for which \( V \) is the unique candidate.