Remarks on $\eta$-Einstein unit tangent bundles

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Abstract

We study the geometric properties of the base manifold for the unit tangent bundle satisfying the $\eta$-Einstein condition with the standard contact metric structure. One of the main theorems is that the unit tangent bundle of 4-dimensional Einstein manifold, equipped with the canonical contact metric structure, is $\eta$-Einstein manifold if and only if base manifold is the space of constant sectional curvature 1 or 2.

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1 Introduction

We consider the $\eta$-Einstein condition, which is suitable for contact metric manifold in general, that is, the Ricci tensor is of the form $\rho(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$ with $\alpha$ and $\beta$ being smooth functions. In [4], Boeckx and Vanhecke determined the unit tangent bundles which are Einstein with respect to the canonical contact metric structure. In the present paper, we shall extend their result to the $\eta$-Einstein case. The scalar curvature of an $\eta$-Einstein contact metric manifold is not necessarily constant in general, however, for some special $\eta$-Einstein contact metric manifolds, we may expect the scalar curvature to be constant. The main theorems are the following:

**Theorem 1.** Let $M$ be an $n(\geq 2)$-dimensional Riemannian manifold and $T_1M$ be the unit tangent bundle of $M$ equipped with the canonical contact metric structure. If $T_1M$ is an $\eta$-Einstein manifold, then $\alpha$ and $\beta$ are both constant valued ones on $T_1M$.

Let $\tau$ be the scalar curvature of $M$, $\rho$ be the Ricci curvature tensor of $M$, $R$ be the Riemann curvature tensor of $M$ and $\bar{\tau}$ be the scalar curvature of $T_1M$. Then we have the following theorems.
Theorem 2 Let \((T_1M, \eta, \bar{g}, \phi, \xi)\) be an \(\eta\)-Einstein manifold. Then \(\tau, |\rho|^2, |R|^2, \) and \(\bar{\tau}\) are all constant.

Theorem 3 Let \(M\) be a 4-dimensional Einstein manifold and \((T_1M, \eta, \bar{g}, \phi, \xi)\) be the unit tangent bundle of \(M\) equipped with the canonical contact metric structure. If \(T_1M\) is an \(\eta\)-Einstein manifold if and only if \((M, g)\) is the space of constant sectional curvature 1 or 2.

Question 1 Can we extend the above Theorem 3 to higher dimensional cases?

From our arguments in the present paper, the following question will naturally arise:

Question 2 Does there exist \(n(\geq 4)\) dimensional Riemannian manifold which is not a space of constant sectional curvature 1 or \(n-2\), whose unit tangent bundle is \(\eta\)-Einstein?

In the last section, we consider \(\eta\)-Einstein unit tangent bundles of some special base Riemannian manifolds.

## 2 Unit tangent bundle with contact metric structure

First, we give some preliminaries on a contact metric manifold. We refer to [2] for more details. A differentiable \((2n - 1)\)-dimensional manifold \(\bar{M}\) is said to be a contact manifold if it admits a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^{n-1} \neq 0\) everywhere on \(\bar{M}\), where the exponent denotes the \((n-1)\)-th exterior power. We call such \(\eta\) a contact form of \(\bar{M}\). It is well known that given a contact form \(\eta\), there exists a unique vector field \(\xi\), which is called the characteristic vector field, satisfying \(\eta(\xi) = 1\) and \(d\eta(\xi, \bar{X}) = 0\) for any vector field \(\bar{X}\) on \(\bar{M}\). A Riemannian metric \(\bar{g}\) is an associated metric to a contact form \(\eta\) if there exists a \((1, 1)\)-tensor field \(\phi\) satisfying

\[
\eta(\bar{X}) = \bar{g}(\bar{X}, \xi), \quad d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y}), \quad \phi^2 \bar{X} = -\bar{X} + \eta(\bar{X})\xi \tag{2.1}
\]

where \(\bar{X}\) and \(\bar{Y}\) are vector fields on \(\bar{M}\). From (2.1) it follows that

\[
\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{g}(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).
\]

A Riemannian manifold \(\bar{M}\) equipped with structure tensors \((\eta, \bar{g}, \phi, \xi)\) satisfying (2.1) is said to be a contact metric manifold. We assume that a contact metric manifold
$\hat{M} = (\bar{M}, \eta, \bar{g}, \phi, \xi)$ is always oriented by the $(2n-1)$-form $\eta \wedge (d\eta)^{n-1}$. We denote by $dV$ the volume form of $\hat{M}$ with respect to the metric $\bar{g}$. Then we may easily observe that $dV = C \eta \wedge (d\eta)^{n-1}$, where $C = \frac{1}{(n-1)!}$. We now review some elementary facts in a contact metric manifold. First, for the characteristic vector field $\xi$, $L_\xi \eta = 0$ follows from $\eta(\xi) = 1$, $d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi \bar{Y})$ and $d\eta(\xi, \bar{X}) = 0$. Here $L$ denotes Lie derivation.

Next, since $d \circ L_\xi = L_\xi \circ d$, by taking account of $L_\xi \eta = 0$, we have

$$L_\xi dV = C L_\xi (\eta \wedge (d\eta)^{n-1})$$
$$= C (L_\xi \eta) \wedge (d\eta)^{n-1}$$
$$+ C \eta \wedge (L_\xi d\eta) \wedge d\eta \wedge \cdots \wedge d\eta$$
$$+ \cdots + C \eta \wedge d\eta \wedge \cdots \wedge (L_\xi d\eta)$$
$$= C \eta \wedge d(L_\xi \eta) \wedge d\eta \wedge \cdots \wedge d\eta$$
$$+ \cdots + C \eta \wedge d\eta \wedge \cdots \wedge d(L_\xi \eta)$$
$$= 0. \tag{2.2}$$

Since $L_\xi dV = (\text{div} \xi) dV$, by the definition of the divergence $\text{div} \xi$ with respect to $dV$ and by (2.2), we have

$$\text{div} \xi = 0 \quad \text{(i.e.,} \quad \nabla_i \xi^i = 0). \tag{2.3}$$

Since $\nabla_\bar{X} \xi$ is orthogonal to $\xi$, we have immediately

$$\nabla_\bar{X} \eta \xi = 0 \tag{2.4}$$

for any vector field $\bar{X}$ on $\hat{M}$.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\nabla$ the associated Levi-Civita connection. Its Riemann curvature tensor $R$ is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for all vector fields $X, Y$ and $Z$ on $M$. The tangent bundle of $(M, g)$ is denoted by $TM$ and consists of pairs $(p, u)$, where $p$ is a point in $M$ and $u$ a tangent vector to $M$ at $P$. The mapping $\pi : TM \to M$, $\pi(p, u) = p$ is the natural projection from $TM$ onto $M$. For a vector field $X$ on $M$, its vertical lift $X^v$ on $TM$ is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where $\omega$ is a 1-form on $M$. For a Levi-Civita connection $\nabla$ on $M$, the horizontal lift $X^h$ of $X$ is defined by $X^h\omega = \nabla_X \omega$. The tangent bundle $TM$ can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called Sasaki metric, depending only on the Riemannian metric $g$ on $M$. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields $X$ and $Y$ on $M$. Also, $TM$ admits an almost complex structure tensor $J$ defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $g$ is Hermitian metric for the
almost complex structure \( J \). We note that \( J \) is integrable if and only if \((M, g)\) is locally flat \([7]\).

The unit tangent bundle \( \pi : T_1 M \to M \) is a hypersurface of \( TM \) given by \( g_p(u, u) = 1 \). Note that \( \pi = \pi \circ i \), where \( i \) is the immersion. A unit normal vector \( N = u^v \) to \( T_1 M \) is given by the vertical lift of \( u \) for \((p, u)\). The horizontal lift of a vector is tangent to \( T_1 M \), but the vertical lift of vector is not tangent to \( T_1 M \) in general. So, we define the tangential lift of \( X \) to \((p, u) \in T_1 M \) by

\[
X^{t}_{(p,u)} = (X - g(X, u)u)^v. 
\]

Clearly, the tangent space \( T_{(p,u)} T_1 M \) is spanned by vectors of the form \( X^h \) and \( X^t \), where \( X \in T_p M \).

We now define the standard contact metric structure of the unit tangent bundle \( T_1 M \) of a Riemannian manifold \((M, g)\). The metric \( g' \) on \( T_1 M \) is induced from the Sasaki metric \( \tilde{g} \) on \( TM \). Using the almost complex structure \( J \) on \( TM \), we define a unit vector field \( \xi' \), a 1-form \( \eta' \) and a \((1,1)\)-tensor field \( \phi' \) on \( T_1 M \) by

\[
\xi' = -JN, \quad \phi' = J - \eta' \otimes N.
\]

Since \( g'(X, \phi'Y) = 2d\eta'(X, Y) \), \((\eta', g', \phi', \xi')\) is not a contact metric structure. If we rescale by

\[
\xi = 2\xi', \quad \eta = \frac{1}{2} \eta', \quad \phi = \phi', \quad \tilde{g} = \frac{1}{4} g',
\]

we get the standard contact metric structure \((\eta, \tilde{g}, \phi, \xi)\). These tensors are given by

\[
\begin{align*}
\xi &= 2u^h, \\
\phi X^t &= -X^h + \frac{1}{2} g(X, u) \xi, \quad \phi X^h = X^t, \\
\eta(X^t) &= 0, \quad \eta(X^h) = \frac{1}{2} g(X, u), \\
\tilde{g}(X^t, Y^t) &= \frac{1}{4} (g(X, Y) - g(X, u) g(Y, u)), \\
\tilde{g}(X^t, Y^h) &= 0, \\
\tilde{g}(X^h, Y^h) &= \frac{1}{4} g(X, Y),
\end{align*}
\]  

(2.5)

where \( X \) and \( Y \) are vector fields on \( M \). From now on, we consider \( T_1 M = (T_1 M, \eta, \tilde{g}, \phi, \xi) \) with the standard contact metric structure.
The Levi Civita connection $\bar{\nabla}$ of $T_1 M$ is described by

$$
\begin{align*}
\bar{\nabla}_X Y^t &= -g(Y, u)X^t, \\
\bar{\nabla}_X Y^h &= \frac{1}{2} (R(u, X)Y)^h, \\
\bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2} (R(u, X)^h, X, Y)^t, \\
\bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2} (R(X, Y)^h, u)
\end{align*}
$$

(2.6)

for all vector fields $X$ and $Y$ on $M$.

Also the Riemann curvature tensor $\bar{R}$ of $T_1 M$ is given by

$$
\begin{align*}
\bar{R}(X^t, Y^t) Z^t &= -(g(X, Z) - g(X, u)g(Z, u)Y^t + (g(Y, Z) - g(Y, u)g(Z, u))X^t, \\
\bar{R}(X^t, Y^t) Z^h &= \{R(X - g(X, u)Y - g(Y, u)Z)^h \\
\quad + \frac{1}{4} \{[R(u, X), R(u, Y)]Z\}^h, \\
\bar{R}(X^h, Y^t) Z^t &= -\frac{1}{2} \{R(Y - g(Y, u)Z - (g(Z, u))X)^h \\
\quad - \frac{1}{2} \{R(u, Y)R(u, Z)(X)^h, \\
\bar{R}(X^h, Y^t) Z^h &= \frac{1}{2} \{R(X, Z)(Y - g(Y, u))\}^t t - \frac{1}{4} \{R(X, R(u, Y)Z)u\}^t t \\
\quad + \frac{1}{2} \{(\nabla_X R)(u, Y)Z\}^h, \\
\bar{R}(X^h, Y^h) Z^t &= \{R(X, Y)(Z - g(Z, u))\}^t t \\
\quad + \frac{1}{4} \{R(Y, R(u, Z)X)u - R(X, R(u, Z)Y))^t \\
\quad + \frac{1}{2} \{(\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X\}^h, \\
\bar{R}(X^h, Y^h) Z^h &= (R(X, Y)^h + \frac{1}{2} \{R(u, R(X, Y)u)Z\}^h \\
\quad - \frac{1}{4} \{R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y\}^h \\
\quad + \frac{1}{2} \{(\nabla_Z R)(X, Y)u\}^t
\end{align*}
$$

(2.7)

for all vector fields $X$, $Y$ and $Z$ on $M$.

Next, to calculate the Ricci tensor $\bar{\rho}$ of $T_1 M$ at the point $(p, u) \in T_1 M$, let $e_1, \cdots, e_n = u$ be an orthonormal basis of $T_p M$. Then $2e_1^t, \cdots, 2e_{n-1}^t, 2e_1^h, \cdots, 2e_n^h = \xi,$
is an orthonormal basis for $T(p,u)T_1M$ and $\bar{\rho}$ is given by

$$\bar{\rho}(X^t, Y^t) = (n - 2)(g(X, Y) - g(X, u)g(Y, u)) + \frac{1}{4} \sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i),$$

$$\bar{\rho}(X^t, Y^h) = \frac{1}{2}((\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y)),$$

$$\bar{\rho}(X^h, Y^h) = \rho(X, Y) - \frac{1}{2} \sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y),$$

(2.8)

where $\rho$ denotes the Ricci curvature tensor of $M$. From this, the scalar curvature $\bar{\tau}$ is given by

$$\bar{\tau} = \tau + (n - 1)(n - 2) - \frac{1}{4} \sum_{i,j=1}^{n} g(R(u, e_i)e_j, R(u, e_i)e_j),$$

(2.9)

where $\tau$ is the scalar curvature of $M$.

3 Unit tangent bundles with $\eta$-Einstein structure

We shall introduce the definition of $\eta$-Einstein manifold.

**Definition 1** If the Ricci tensor $\bar{\rho}$ of a contact metric manifold $(\bar{M}, \eta, \bar{g}, \phi, \xi)$ is of the form

$$\bar{\rho}(\bar{X}, \bar{Y}) = \alpha \bar{g}(\bar{X}, \bar{Y}) + \beta \eta(\bar{X})\eta(\bar{Y})$$

for smooth functions $\alpha$ and $\beta$, then $\bar{M}$ is called an $\eta$-Einstein manifold.

Now, let $M = (M, g)$ be a Riemannian manifold and $(T_1M, \eta, \bar{g}, \phi, \xi)$ be the unit tangent bundle of $(M, g)$ equipped with the canonical contact metric structure $(\eta, \bar{g}, \phi, \xi)$ defined as in section 2. Take the $\phi$-basis $\{\bar{e}_i, \bar{e}_{i^*} = \phi \bar{e}_i, \xi = \bar{e}_s\}$ on $T_1M$. Then the Ricci tensor $\bar{\rho}$ with respect to the $\phi$-basis should be

$$\bar{\rho} = \begin{pmatrix}
\bar{\rho}(\bar{e}_i, \bar{e}_j) & \bar{\rho}(\bar{e}_i, \bar{e}_{j^*}) & \bar{\rho}(\bar{e}_i, \bar{e}_s) \\
\bar{\rho}(\bar{e}_{i^*}, \bar{e}_j) & \bar{\rho}(\bar{e}_{i^*}, \bar{e}_{j^*}) & \bar{\rho}(\bar{e}_{i^*}, \bar{e}_s) \\
\bar{\rho}(\bar{e}_s, \bar{e}_j) & \bar{\rho}(\bar{e}_s, \bar{e}_{j^*}) & \bar{\rho}(\bar{e}_s, \bar{e}_s)
\end{pmatrix}. $$

(3.1)

In particular, if $T_1M$ is $\eta$-Einstein, by the definition, the Ricci tensor $\bar{\rho}$ is given by

$$\bar{\rho} = \begin{pmatrix}
\alpha & 0 & \cdots & 0 & 0 \\
0 & \alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha & 0 \\
0 & 0 & \cdots & 0 & \alpha + \beta
\end{pmatrix} $$

(3.2)
for some two smooth functions $\alpha$ and $\beta$ on $T_1M$. From (2.8), we have the following theorem.

**Theorem 4** Let $M$ be an $n$-dimensional Riemannian manifold. Then $T_1M$ is $\eta$-Einstein if and only if

$$\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i) = (\alpha - 4n + 8)(g(X, Y) - g(X, u)g(Y, u)),$$

$$g(R(u, e_i)X, R(u, e_i)Y) = 2\rho(X, Y) - \frac{1}{2} \alpha g(X, Y) - \frac{1}{2} \beta g(X, u)g(Y, u).$$

**Proof of Theorem 4** Let $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ be the unit tangent bundle equipped with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ and assume that $T_1M$ is an $\eta$-Einstein manifold. Then, by the definition, the Ricci tensor $\bar{\rho}$ of $T_1M$ takes the following form:

$$\bar{\rho} = \alpha \bar{g} + \beta \eta \otimes \eta$$

for some smooth functions $\alpha$ and $\beta$ on $T_1M$.

For a while, we adopt the traditional convention for the notations in the classical tensor analysis. In the local coordinate neighborhood, from (3.6), we get

$$\bar{\rho}_{ij} = \alpha \bar{g}_{ij} + \beta \eta_{i} \eta_{j}.$$

Operating $\bar{\nabla}^i = \bar{g}^{ia} \bar{\nabla}_a$ on both sides of (3.7), we get

$$\bar{\nabla}^i \bar{\rho}_{ij} = (\bar{\nabla}^i \alpha) \bar{g}_{ij} + (\bar{\nabla}^i \beta) \eta_{i} \eta_{j} + \beta (\bar{\nabla}^i \eta_{k}) \eta_{j} + \beta (\bar{\nabla}^i \eta_{j}) \eta_{k}$$

$$= \bar{\nabla}_{j} \alpha + (\bar{\nabla}_{i} \beta) \xi_{j} + \beta (\text{div} \xi) \eta_{j} + \beta \xi^i \bar{\nabla}^i \eta_{j}.$$ (3.8)

Transvecting $\xi^j$ with (3.8), we have

$$\xi^j \bar{\nabla}^i \bar{\rho}_{ij} = \xi \alpha + \xi \beta + \beta (\text{div} \xi) \eta_{j} + \beta (\bar{\nabla} \xi \eta) \xi.$$ (3.9)

Here, taking account of the second Bianchi identity, we get

$$\bar{\nabla}^i \bar{\rho}_{ij} = \frac{1}{2} \bar{\nabla}_{j} \bar{\tau}$$

and hence the left-hand side of (3.9) implies $\frac{1}{2} \xi \bar{\tau}$. Thus, from (2.3), (2.4) and (3.9) we have

$$\xi \bar{\tau} = 2 \xi \alpha + 2 \xi \beta.$$ (3.10)
On one hand, by (3.7), we get
\[ \bar{\tau} = (2n - 1)\alpha + \beta. \]

Thus, we have also
\[ \xi \bar{\tau} = (2n - 1)\xi \alpha + \xi \beta. \quad (3.11) \]

Then from (3.10) and (3.11), we have
\[ (2n - 3)\xi \alpha - \xi \beta = 0. \quad (3.12) \]

Next, let \( \bar{X} = (X^j) \) be a vector field on \( T_1 M \) with \( \eta(\bar{X}) = 0 \). Transvecting \( X^j \) with (3.8), we have also
\[ X^j \bar{\nabla}^i \bar{\rho}_{ij} = \bar{X} \alpha + \beta (\bar{\nabla} \xi \eta)(\bar{X}) \]
and hence
\[ \frac{1}{2} \bar{X} \bar{\tau} = \bar{X} \alpha + \beta (\bar{\nabla} \xi \eta)(\bar{X}). \quad (3.13) \]

Here, we get
\[ (\bar{\nabla} \xi \eta)(\bar{X}) = -\eta(\bar{\nabla} \xi \bar{X}) \]
\[ = -\eta(\bar{\nabla} \bar{X} \xi + [\xi, \bar{X}]) \]
\[ = -\eta([\xi, \bar{X}]). \quad (3.14) \]

On one hand, we get
\[ -\eta([\xi, \bar{X}]) = \xi(\eta(\bar{X})) - \bar{X}(\eta(\xi)) - \eta([\xi, \bar{X}]) \]
\[ = d\eta(\xi, \bar{X}) \]
\[ = \bar{g}(\xi, \phi \bar{X}) \]
\[ = 0. \quad (3.15) \]

Thus from (3.13) \( \sim \) (3.15), we have
\[ \bar{X} \bar{\tau} = 2 \bar{X} \alpha \quad (3.16) \]

for vector field \( \bar{X} \) with \( \eta(\bar{X}) = 0 \). Since \( \bar{\tau} = (2n - 1)\alpha + \beta \) holds on \( T_1 M \), we have also
\[ \bar{X} \bar{\tau} = (2n - 1)\bar{X} \alpha + \bar{X} \beta. \quad (3.17) \]

Thus, by (3.16) and (3.17), we have
\[ (2n - 3)\bar{X} \alpha + \bar{X} \beta = 0 \quad (3.18) \]

for vector field \( \bar{X} \) with \( \eta(\bar{X}) = 0 \).
From now on, we state some fundamental properties of the \( \eta \)-Einstein contact metric structure \((\eta, \bar{g}, \phi, \xi)\) on \( T_1M \), by making use of the facts in the above. First of all, by \( (3.4) \), we see that the scalar curvature \( \tau \) of the base manifold \((M, g)\) (dim\(M \geq 2\)) is constant.

Now setting \( X = Y = e_j \) in \( (3.3) \) and \( (3.5) \) and taking sum for \( j = 1, \cdots, n \), we obtain

\[
\sum_{i,j=1}^{n} g(R(u,e_j)e_i,R(u,e_j)e_i) = (\alpha - 4n + 8)(n - 1), \quad (3.19)
\]

\[
\sum_{i,j=1}^{n} g(R(u,e_i)e_j,R(u,e_i)e_j) = 2\tau - \frac{1}{2} n\alpha - \frac{1}{2} \beta. \quad (3.20)
\]

From \( (3.19) \) and \( (3.20) \), we have

\[
(3n - 2)\alpha + \beta = 4\tau + 8(n - 1)(n - 2). \quad (3.21)
\]

Since \( \tau \) is constant and \( \xi = 2u^h \), we have

\[
(3n - 2)u^h\alpha + u^h\beta = 0. \quad (3.22)
\]

And \( (3.12) \) can be rewritten as follows :

\[
(2n - 3)u^h\alpha - u^h\beta = 0. \quad (3.23)
\]

From \( (3.22) \) and \( (3.23) \), we have

\[
u^h\alpha = 0 \text{ and } u^h\beta = 0. \quad (3.24)
\]

Operating \( X^t \) \((X \in T_pM) \in T_{(p,u)}T_1M \) on the both sides of \( (3.21) \), we have

\[
(3n - 2)X^t\alpha + X^t\beta = 0. \quad (3.25)
\]

Since \( X^t \) is orthogonal to \( \xi \) (i.e., \( \eta(X^t) = 0 \)), from \( (3.18) \), we have

\[
(2n - 3)X^t\alpha + X^t\beta = 0. \quad (3.26)
\]

Thus, from \( (3.25) \) and \( (3.26) \), we have

\[
X^t\alpha = 0 \text{ and } X^t\beta = 0 \text{ at } (p,u). \quad (3.27)
\]

Similarly, operating \( X^h \) \((X \in T_pM) \in T_{(p,u)}T_1M \) on the both sides of \( (3.21) \) for vector field \( X \) on \( M \) such that \( g(X, u) = 0 \), we have

\[
X^h\alpha = 0 \text{ and } X^h\beta = 0 \text{ at } (p,u). \quad (3.28)
\]
Summing up (3.24), (3.27) and (3.28), we see that the smooth functions $\alpha$ and $\beta$ are constants. □

By Theorem 1, we immediately obtain that

**Corollary 5** $T_1M$ with $\eta$-Einstein structure has constant scalar curvature $\bar{\tau}$.

**Proof of Theorem 2.** For $T_1M$ with constant scalar curvature it holds

$$\sum_{i,j=1}^{n} g(R(u,e_j)e_i, R(u,e_j)e_i) = \frac{|R|^2}{n}, \quad (3.29)$$

where $|R|^2 = \sum_{i,j,k=1}^{n} g(R(e_i,e_j)e_k, R(e_i,e_j)e_k)$. From (3.19), (3.20) and (3.29), we have

$$\alpha = \frac{|R|^2}{n(n-1)} + 4(n-2), \quad (3.30)$$

$$\beta = 4\tau - 4n(n-2) - \frac{3n-2}{n(n-1)}|R|^2. \quad (3.31)$$

Next, we integrate (3.5) with $X = Y = u$ over $S^{n-1}(1)$ in $T_pM$. Then using the formula in [6], we have

$$\frac{1}{n(n+2)}(|\rho|^2 + \frac{3}{2}|R|^2) = \frac{2}{n} - \frac{1}{2} \alpha - \frac{1}{2} \beta. \quad (3.32)$$

Eliminating $\alpha$ and $\beta$ from (3.30) $\sim$ (3.32), we obtain the equation:

$$2|\rho|^2 - 3(n+1)|R|^2 = -4(n-1)(n+2)\tau + 4n(n-1)(n-2)(n+2). \quad (3.33)$$

In proof of Theorem 1 we know that $\alpha$, $\beta$ and $\tau$ are constant. Since $\alpha$ is constant, from (3.30), we see that $|R|^2$ is constant. Thus, by (3.32), we see also that $|\rho|^2$ is constant. Therefore we have Theorem 2. □

4 Special cases

(I) 2-dimensional case

It is well-known that $R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)$ always holds. So, we have $|R|^2 = 4\kappa^2$, $|\rho|^2 = 2\kappa^2$, $\tau = 2\kappa$, where $\kappa$ is the Gaussian curvature. From (3.33), we see that $M$ has Gaussian curvature $\kappa = 0$ or $\kappa = 1$.

(II) 3-dimensional case
It is well-known that the curvature tensor $R$ of 3-dimensional Riemannian manifold $(M, g)$ is of the following form.

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

$$= \left\{g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W)$$

$$- g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z)\right\}$$

$$+ \frac{\tau}{2}\left\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\right\}$$

(4.1)

for all vector fields $X, Y, Z, W$ on $M$. From (4.1), by direct calculation, we get

$$|R|^2 = 4|\rho|^2 - \tau^2.$$  (4.2)

By (3.33) and (4.2), we have

$$23|\rho|^2 - 6\tau^2 - 20\tau + 60 = 0,$$

and thus

$$23 \left| \rho - \frac{\tau}{3} g \right|^2 + \frac{5}{3}(\tau - 6)^2 = 0.$$  (4.3)

From (4.3), we have $\rho = \frac{\tau}{3} g$ and $\tau = 6$ and hence

$$\rho = 2g.$$  (4.4)

Thus by (4.1) and (4.4), we have

$$R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W)$$

and hence $(M, g)$ is a space of constant sectional curvature 1. The above result has been proved in [5]. We may note that our proof is much simpler than their proof.

(III) Conformally flat case

By the similar arguments in [5], we can also have the following.

**Theorem 6** Let $M = (M, g)$ be an $n$-dimensional conformally flat manifold $(n \geq 4)$. Then $(\tilde{T}_1 M, \eta, \tilde{g}, \phi, \xi)$ is $\eta$-Einstein if and only if $(M, g)$ is a space of constant sectional curvature 1 or $n - 2$.

(IV) Einstein case

Let $M = (M, g)$ be an $n$-dimensional Einstein manifold $(n \geq 3)$. Then we have

$$\left| R + \frac{\tau}{2n(n - 1)} g \otimes g \right|^2 = |R|^2 - \frac{2\tau^2}{n(n - 1)},$$  (4.5)
where \((h \boxtimes k)(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W)\) for any \((0,2)\)-tensors \(h\) and \(k\). By (3.33) and (4.5), we have

\[
\frac{2\tau^2}{n} - 3(n+1) \left\{ \left| R + \frac{\tau}{2n(n-1)} g \boxtimes g \right|^2 + \frac{2\tau^2}{n(n-1)} \right\} = -4(n-1)(n+2)\tau + 4n(n-1)(n-2)(n+2)
\]

and hence

\[
-3(n+1) \left| R + \frac{\tau}{2n(n-1)} g \boxtimes g \right|^2 = \frac{4(n+2)}{n(n-1)} \{ \tau^2 - n(n-1)^2\tau + n^2(n-1)^2(n-2) \} = \frac{4(n+2)}{n(n-1)}(\tau - n(n-1))(\tau - n(n-1)(n-2)).
\]

Then from (4.6), we have

\[
n(n-1) \leq \tau \leq n(n-1)(n-2), \quad n \geq 3.
\] (4.7)

By Theorem 4, we see that \((M,g)\) is super-Einstein by virtue of (3.3). Since the scalar curvature of \(T_1M\) is constant as shown by Theorem 2, this also follows from the result of Boeckx and Vanhecke (3), Proposition 3.6). Thus we have

**Theorem 7** Let \((M,g)\) be an \(n\)-dimensional Einstein manifold and \((T_1M, \eta, \bar{g}, \phi, \xi)\) be the unit tangent bundle of \(M\) equipped with the canonical contact metric structure. If \(T_1M\) is \(\eta\)-Einstein, then \(M\) is super-Einstein and the scalar curvature \(\tau\) satisfies the above inequality (4.7).

In the remainder of this section, we shall consider the case that \((M,g)\) is a 4-dimensional Einstein manifold.

**Proof of Theorem 3** We may choose an orthonormal basis \(\{e_i\}\) (known as the Singer-Thorpe basis) at each point \(p \in M\) such that

\[
\begin{align*}
R_{1212} &= R_{3434} = a, & R_{1313} &= R_{2424} = b, & R_{1414} &= R_{2323} = c, \\
R_{1234} &= d, & R_{1342} &= e, & R_{1423} &= f, \\
R_{ijkl} &= 0 \text{ whenever just three of the indices } i, j, k, l \text{ are distinct (10)}.
\end{align*}
\] (4.8)

Note that \(d + e + f = 0\) by the first Bianchi identity and

\[
a + b + c = -\frac{\tau}{4}.
\] (4.9)
Further, by the direct calculation, we have
\[ |R|^2 = 8(a^2 + b^2 + c^2 + d^2 + e^2 + f^2), \]
\[ |\rho|^2 = 4(a + b + c)^2. \]  
(4.10)

From Theorem 7, since \( M \) is super-Einstein, we have (8, 9)
\[ \pm d = a + \frac{\tau}{12}, \quad \pm e = b + \frac{\tau}{12}, \quad \pm f = c + \frac{\tau}{12}. \]  
(4.11)

From (3.3), taking account of (4.8), we have easily
\[ 2(a^2 + d^2) = \alpha - 8, \]  
(4.12)
\[ 2(b^2 + e^2) = \alpha - 8, \]  
(4.13)
\[ 2(c^2 + f^2) = \alpha - 8. \]  
(4.14)

Thus from (4.12) and (4.13), taking account of (4.11), we have
\[ (a - b)(a + b + \frac{\tau}{12}) = 0. \]  
(4.15)

Similarly, we have
\[ (b - c)(b + c + \frac{\tau}{12}) = 0, \]  
(4.16)
\[ (c - a)(c + a + \frac{\tau}{12}) = 0. \]  
(4.17)

We first suppose that \( a \neq b, \) \( b \neq c, \) \( c \neq a. \) Then by (4.15) \( \sim \) (4.17), we get
\[ a + b + \frac{\tau}{12} = 0, \quad b + c + \frac{\tau}{12} = 0, \quad c + a + \frac{\tau}{12} = 0. \]
Thus we have \( a = b = c = -\frac{\tau}{24}. \) However this is a contradiction.

Next, we suppose that \( a \neq b, \) \( b \neq c, \) \( c = a \) (i.e., \( a = c, \) \( a \neq b \)). Then by (4.15) \( \sim \) (4.17), we have
\[ a + b + \frac{\tau}{12} = 0, \quad b + c + \frac{\tau}{12} = 0. \]  
(4.18)

By (1.9) and the hypothesis \( a = c, \) we have
\[ 2a + b = -\frac{\tau}{4}. \]  
(4.19)

Thus by (4.18) and (4.19), we have
\[ a = c = -\frac{\tau}{6}, \quad b = \frac{\tau}{12}. \]  
(4.20)

Thus by (4.11) and (4.20), we have
\[ \pm d = -\frac{\tau}{12}, \quad \pm e = \frac{\tau}{6}, \quad \pm f = -\frac{\tau}{12}. \]  
(4.21)
Thus from (4.10), (4.20) and (4.21), we have

\[ |R|^2 = \frac{5}{6} \tau^2, \quad |\rho|^2 = \frac{\tau^2}{4}. \]  

(4.22)

Then by (3.33) and (4.22), we obtain

\[ \tau^2 - 6\tau + 48 = 0. \]  

(4.23)

However, this quadratic equation (4.23) does not admit a real solution. This is also a contradiction. By the similar way, we can also deduce a contradiction in the cases \( b = c \neq a \) and \( a = b \neq c \). Thus, it must follow that \( a = b = c \) and hence by (4.9) and (4.11), we have

\[ a = b = c = -\frac{\tau}{12}, \quad d = e = f = 0. \]

Therefore, by (4.8), \((M, g)\) is a space of constant sectional curvature \( \frac{\tau}{12} \). Then we have

\[ |R|^2 = \frac{\tau^2}{6}, \quad |\rho|^2 = \frac{\tau^2}{4}. \]

Thus, by (3.33), we have

\[ (\tau - 12)(\tau - 24) = 0. \]  

(4.24)

Therefore, we have Theorem 3.

\[ \square \]

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