G-CONVERGENCE AND HOMOGENIZATION OF VISCOELASTIC FLOWS.

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ABSTRACT. The paper is devoted to homogenization of two-phase incompressible viscoelastic flows with disordered microstructure. We study two cases. In the first case, both phases are modeled as Kelvin-Voight viscoelastic materials. In the second case, one phase is a Kelvin-Voight material, and the other is a viscous Newtonian fluid. The microscale system contains the conservation of mass and balance of momentum equations. The inertial terms in the momentum equation incorporate the actual interface advected by the flow. In the constitutive equations, a frozen interface is employed. The interface geometry is arbitrary: we do not assume periodicity, statistical homogeneity or scale separation. The problem is homogenized using G-convergence and oscillating test functions. Since the microscale system is not parabolic, previously known constructions of the test functions do not work here. The test functions developed in the paper are non-local in time and satisfy divergence-free constraint exactly. The latter feature enables us to avoid working with pressure directly. We show that the effective medium is a single phase viscoelastic material that is not necessarily of Kelvin-Voight type. The effective constitutive equation contains a long memory viscoelastic term, as well as instantaneous elastic and viscous terms.

1. INTRODUCTION

Formulation of constitutive equations of multiphase materials under flow is a fundamental problem of continuum mechanics. Using mathematical homogenization theory [3], [15], [25] to solve this problem is intuitively appealing, but not easy. The reasons for this can be summarized as follows: it is difficult to homogenize evolution equations, non-linear equations, and equations involving general geometric distribution of the constituents. In particular, addressing the last difficulty is the necessary first step in developing a complete homogenization theory for moving interface problems.

Consider a composite material with two constituents which we call phases. During flow, the interface between the phases is advected by the flow velocity. Therefore, the interface motion is coupled to the flow dynamics. A priori, one cannot expect that a geometry that is random homogeneous at the initial time remains random homogeneous at future times. Scale separation also cannot be expected to hold for all times in the interval of interest. Therefore, homogenization techniques that require a specific type of geometry, e.g. two-scale convergence ([11], [21]) and ergodic theorems, cannot be used in general. This leaves G-convergence [20], [23], [27], [28], [31], [32] and, whenever variational formulation is available, Γ-convergence ([5], [8]). The problem studied in this paper is not variational, so G-convergence is chosen as the main technical tool.

G-convergence is a general notion of functional analysis and operator theory. In fact, G-convergence of an operator sequence \( A_\varepsilon \) can be identified with the Painleve-Kuratowski set convergence [16], [24] of the corresponding operator graphs \( \Gamma_\varepsilon \) (see e.g. the definition of G-convergence in [23]). Set convergence is sequentially compact provided the topology of \( \Gamma_\varepsilon \) has a countable base. Therefore, existence of an abstract G-limit operator \( A \) is easily obtained for \( A_\varepsilon \) that may be non-linear, non-local and multi-valued. Once existence of \( A \) is established, one needs to describe the structure of \( A \), which is a problem of characterization. To solve this problem we use the method of oscillating test functions [9], [10], [20], [23]. Our point of view is somewhat different from the standard one. To explain this further, consider a sequence of problems

\[
A_\varepsilon u_\varepsilon = f,
\]

where \( A_\varepsilon : X \rightarrow X^* \) are linear operators, \( X \) is a separable Banach space and \( X^* \) is its topological dual. Suppose that \( A_\varepsilon \) G-converges to an operator \( A \). This means that \( u_\varepsilon \rightarrow \overline{u} \) weakly in \( X \) and \( A\overline{u} = f \) for each \( f \in X^* \). Instead of choosing a sequence \( u_\varepsilon \) of oscillating test functions, we prescribe a sequence of corrector

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1 not necessarily random homogeneous
operators $N^\varepsilon : X \to X$ and then define the test functions by
\[ w^\varepsilon = N^\varepsilon w, \]
where $w$ is an arbitrary fixed test function from a dense subset of $X$.

It turns out that the characterization problem can be solved if $N^\varepsilon$ satisfy

\begin{align}
& (1.1) \quad N^\varepsilon w \rightharpoonup w \text{ weakly in } X, \\
& (1.2) \quad (A^\varepsilon)^\top N^\varepsilon w \rightharpoonup \overline{A}^\top w \text{ strongly in } X^*,
\end{align}

possibly along a subsequence. Here, $(A^\varepsilon)^\top$ is the formal adjoint of $A^\varepsilon$. The operator $\overline{A}^\top$ must be described as explicitly as possible.

Indeed, if (1.1) holds then
\[ \langle A^\varepsilon u^\varepsilon, N^\varepsilon w \rangle = \langle f, N^\varepsilon w \rangle \to \langle f, w \rangle = \langle A\overline{u}, w \rangle, \]
and if (1.2) is true, then
\[ \langle A^\varepsilon u^\varepsilon, N^\varepsilon w \rangle = \langle u^\varepsilon, (A^\varepsilon)^\top N^\varepsilon w \rangle \to \langle \overline{u}, \overline{A}^\top w \rangle = \langle A\overline{u}, w \rangle, \]

Since $w$ is any function in a dense subset of $X$, (1.1), (1.3) imply $A\overline{u} = \overline{A}\overline{u}$.

In practice, finding a corrector operator is a difficult task, and at present there are no general recipes for doing it. Known methods of $G$-limit characterization are specific to a certain class of problems: elliptic equations in divergence form [20], elliptic and parabolic problems with coercivity [31, 32], linear elasticity [22], operator equations with monotone [7], and pseudo-monotone [23] operators.

In most of these cases, the $G$-limit problem has the same general structure as the $\varepsilon$-problems. For example, mixing two linear elastic materials produces an effective material that is also linear elastic. However, in mechanics of multi-phase flows, the phases may have different physical properties and the general nature of the effective equations is often unknown. Even in the relatively simple case of dense suspensions of rigid particles in a Stokes fluid, effective equations are subject to debate ([4, 14, 26]). Therefore, the first goal of homogenization would be to determine the general structure of the effective equations: one-phase or two-phase, simple or not, type of memory dependence, presence of additional state variables etc. $G$-convergence is particularly well suited for answering questions of this kind.

In this paper, we use $G$-convergence to study homogenization of two-phase incompressible viscoelastic flows. The phases are modeled as Kelvin-Voight materials: the elastic stresses satisfy Hook’s law written in the spatial (Eulerian) formulation, and the viscous stresses obey Newton’s law. Despite the fact that constitutive equations for the phases look similar, this model can describe the mixture of two materials, one of which is fluid-like (small elasticity), and the other is solid-like (small viscosity). We also study the fluid-structure interaction problem, where one phase is a Kelvin-Voight material, and the other is a viscous Newtonian fluid. At the initial moment of time the two phases are finely mixed, and both phases occupy connected domains. The latter assumption excludes particle flows, since particles may collide, and that requires more complicated interface conditions. The density of each phase is constant, but the initial density of the mixture is highly oscillatory. This means that the sequence of initial densities does not converge strongly. No further assumptions are made about the initial interface geometry. In particular, we do not assume periodicity, statistical homogeneity or scale separation.

The system of equations contains the mass balance and the momentum balance equations. The mass balance is needed because the initial density of the mixture is not constant, and the interface geometry changes in time. We also make the following choice regarding modeling of the interface. The mass balance equation and the convective terms in the momentum equation incorporate a moving interface advected by the flow velocity. The constitutive equations for the stress employ a frozen interface, that is the interface that existed at the initial time. This choice makes sense both physically and mathematically.

From the point of view of physics, freezing the interface in constitutive equations is the only option compatible with the Hook’s law. In linear elasticity, deformations are assumed to be small, and thus spatial and referential descriptions are identified. In the referential description, the interface is always fixed. The reason for this is simple: a material particle that belongs initially to phase one remains in that phase at future times. Therefore, in the framework of linear elasticity, the interface must be fixed in both descriptions.
Otherwise elastic forces may lead to non-physical energy dissipation. More details illustrating this point are given in Appendix A.

This may seem like a reason to use fixed interface in all terms and work with the formally linearized equations of motion. This is not satisfactory for viscoelastic composites, because convective effects may be significant even when the interface is nearly stationary. Indeed, when one phase is fluid-like, it may flow at large Reynolds number in the cavities of a matrix made from the second, solid-like phase. In that case the deformations inside the solid-like phase are small, as are the deformations of the interface. However, the deformations in the fluid-like phase may be large, and convective effects inside that phase cannot be neglected. Formal calculations, presented in Appendix A, show that presence of convective terms is incompatible with the frozen interface. It is also impossible to meet the natural interface conditions unless convective terms are present in both phases, and not just in a fluid-like phase.

To summarize, the microscale model in the paper partially accounts for the shape variation of the interface and satisfies all of the above natural requirements. In addition, these equations provide a good benchmark model for studying more complicated problems with moving interface such as flows of two immiscible fluids with surface tension, flows of nonlinear viscoelastic materials, fluid-particle flows etc.

Mathematical study of the above microscale problem makes sense because little is known about $G$-limits of non-parabolic systems of fluid mechanics. Since the sequences of initial data do not converge strongly, the known compactness results for weak solutions such as Theorem 2.4. in [18] and Theorem 5.1 in [19] cannot be expected to hold. In a sense, the problems of the type studied here are more difficult than existence questions. Because of the lack of compactness, the structure of $G$-limits for generic continuum mechanics systems may be different from the structure of $\varepsilon$-problems, but this is not known. We show that in the present case, a new long memory term appears in the effective constitutive equations. Such results were previously obtained for linear equations with fixed interface. Formally this was done for locally periodic geometry in [25] and [6], and rigorously for periodic geometry in [12] using two-scale convergence. In [13], oscillating test functions were used for non-periodic scale separated geometries (some details of the proofs in that paper were omitted). In [12] and [13], the oscillating test functions did not satisfy divergence-free constraint. Using arbitrary test functions in incompressible problems makes it necessary to work with pressure directly. This is a serious obstacle, since good estimates on the pressure are not available even for density dependent Navier Stokes equations [18]. In this paper, we incorporate physically meaningful convective terms, allow for interfacial motions, and propose a construction of divergence-free oscillating test functions. The construction of the corrector operators involves certain auxiliary functions satisfying auxiliary problems. The right hand sides in auxiliary problems are chosen to satisfy condition (1.1) on the corrector operators $N^\varepsilon$. The treatment here is inspired by the "condition N" in the papers [31] [32] of Zhikov, Kozlov and Oleinik (see also the book [22] for an application to linear elasticity). It seems that many ideas in these works can be extended to non-parabolic evolution problems, including problems with moving interface.

The main result of the paper are Theorem 3.1 in Sect. 3, and Theorem 7.1 in Sect. 7. There we show that the effective system of equations describes a single-phase incompressible viscoelastic material. The effective system contains equations of mass and momentum balance. The effective constitutive equations contain a linearly elastic term, a linearly viscous term and a viscoelastic term that models a long term memory dependence of the effective material.

The paper is organized as follows. Section 1 is introductory. Microscale problem formulation and properties of finite energy weak solutions are given in Section 2. An outline of the existence proof for each fixed $\varepsilon > 0$ is presented in Appendix B. With some minor changes due to the presence of the elastic stress, we follow closely the existence proof for incompressible density-dependent Navier-Stokes equations in [18], Sect. 2.3, 2.4. The global weak solutions of Leray type obtained in this way satisfy the energy inequality. Section 3-6 contain a detailed study of the the case of two Kevin-Voigt materials. In Section 3, we formulate the main theorem and provide an outline of the proof. Section 4 is devoted to constructing the corrector operators and oscillating test functions. Section 5 deals with passage to the limit in the inertial terms of the momentum balance equation. In Section 6 we obtain the effective constitutive equations and combine all the results to finish the proof of the main theorem. Finally, in Section 7 we indicate the changes necessary to treat the fluid-structure interaction case, and state the main theorem for this case.
2. Micro-scale problem.

2.1. Equations of balance and constitutive equations.

2.1.1. Choice of a model. We consider two-phase materials in which at least one of the phases resist shearing, and the material stress tensor can be written as a sum of an elastic (conservative) and dissipative stresses. To avoid mathematical difficulties in dealing with nonlinear elasticity we limit our investigation to flows for which the Hook’s law of linear elasticity is an appropriate model of the elastic stress. We further suppose that deformation of the interface is small. In this case, the equations of motion are often formally linearized, assuming that the density is constant, and spatial and referential description are identified. As a consequence, the interface (always fixed in the referential description) is fixed in the spatial description. This is unsatisfactory when the physical properties of the phases are different, e.g. one phase is solid-like and the other one fluid-like. In this case, large deformations of the fluid-like phase may occur even when the deformations in the solid-like phase are small. Consequently, the contribution of the inertial terms to the overall momentum balance cannot be neglected. In addition, a correct model should respect physically realistic interface conditions (continuity of velocity and equality of tractions) as well as physically correct (formal) energy balance. These requirements make it necessary to consider inertia in both phases, with the corresponding terms that include a moving interface and densities governed by the mass conservation equations. By contrast, Hook’s law for the elastic stress may lead to the non-physical dissipation of the elastic energy unless the interface in the constitutive equations is frozen. In Appendix A we present some (formal) calculations illustrating the above points.

In this paper, we work with the micro-scale equations that employ a moving interface in the inertial terms and a frozen interface in the constitutive equations for the stress. This model can be viewed as a transition between a completely linear model with constant densities and static interface, and a fully nonlinear nonlinear that involves nonlinear constitutive equations for the elastic stress.

2.1.2. Micro-scale equations. Let $ρ^ε, v^ε$ denote, respectively, the density and velocity of the composite. We also define

(2.1) $u^ε(t, x) = \int_0^t v^ε(\tau, x)d\tau.$

**Interface evolution equation.** Evolution of $V^ε(t), W^ε(t)$ can be described by the interface evolution equation. Let $θ^ε(t, x)$ denote the characteristic function of $V^ε(t)$. In the referential description, $V^ε$ is fixed. Therefore, the material derivative of $θ^ε$ is zero. In the spatial description we have

(2.2) $∂tθ^ε + v^ε \cdot \nabla θ^ε = 0$ in $U.$

Equation (2.2) is supplemented with the initial condition

(2.3) $θ^ε(0, x) = θ^ε_0(x)$ in $U.$

**Mass conservation equation.** Next, we state the mass conservation equation. The composite density satisfies

(2.4) $∂tρ^ε + \text{div}(ρ^ε v^ε) = 0,$ in $U$

with the initial condition

(2.5) $ρ^ε(0, x) = ρ_1 θ^ε_0(x) + ρ_2 (1 − θ^ε_0(x)), \ x ∈ U,$

where $ρ_1, ρ_2$ are the densities of the respective phases. These densities are assumed to be constant and bounded below by a positive constant.

**Incompressibility:**

(2.6) $\text{div} v^ε = 0.$

**Momentum balance for the composite:**

(2.7) $∂t v^ε + \text{div}(ρ^ε v^ε \otimes v^ε) − \text{div}(T^ε_1 − P^ε_1 I + T^ε_2 − P^ε_2 I) = 0.$
Here \( T^s - P^s I \) is the stress tensor in the phase \( s \), \( P^s \) is the pressure, and \( I \) denotes the unit tensor. The initial conditions for (2.7) are
\[
\nu^s(0, x) = \nu_0(x),
\]
where \( \nu_0 \) does not depend on \( \varepsilon \). In addition, (2.11) implies that \( \nu^s(0, x) = 0 \).

On the boundary \( \partial U \), the condition
\[
\nu^s(t, x) = 0
\]
is imposed.

**Constitutive equations.** As explained above, static interface seems to be a natural choice that is relatively easy to handle (unlike combining referential formulation for the elastic stress with the spatial formulation for the viscous stress), and compatible with the spatial form of the Hook’s law for the elastic part of the stress. We therefore define
\[
T^s = T^s_1 + T^s_2 - P^s I, \quad \text{where} \quad P^s = P^s_1 + P^s_2, \quad T^s_1(t, x) = \theta^s_0(x) \left( A^1 e(u^s) + B^1 e(v^s) \right), \quad T^s_2(t, x) = \left( 1 - \theta^s_0(x) \right) \left( A^2 e(u^s) + B^2 e(v^s) \right),
\]
and \( A^s, B^s, s = 1, 2 \) are constant material tensors. In (2.10), \( e = \frac{1}{2} (\nabla + \nabla^T) \) is the symmetric part of the gradient. We assume that both phases are isotropic. In that case
\[
A^s_{ijkl} = \mu^s \delta_{ik} \delta_{jl}, \quad B^s_{ijkl} = \nu^s \delta_{ik} \delta_{jl}, \quad s = 1, 2,
\]
where \( \mu^s \) are the elastic moduli and \( \nu^s \) are the viscosities of the phases. All these constants are supposed to be positive.

We also assume that the tensors \( A^s, B^s \) satisfy
\[
\alpha_1 \xi \cdot \xi \leq A^s \xi \cdot \xi \leq \alpha_2 \xi \cdot \xi, \quad \beta_1 \xi \cdot \xi \leq B^s \xi \cdot \xi \leq \beta_2 \xi \cdot \xi,
\]
for each \( \xi \in \mathbb{R}^{3 \times 3} \), with \( \alpha > 0, \beta > 0 \) independent of \( \varepsilon \).

**Remark.** Together, (2.10), (2.4), (2.7) form a closed system, and thus the interface evolution equation (2.2) can be dropped. This fact is important for compressible flows for which the mass conservation equation is stable with respect to weak convergence, while equation (2.2) is not. For incompressible flows considered in this paper, the interface evolution equation has the same structure \( \partial_t + \text{div}(\nu^s \cdot \cdot \cdot) \) as the mass conservation. Moreover, if the initial densities \( \rho_1, \rho_2 \) are constant, then the densities of the phases remain constant during the motion. In this case, the interface evolution equation and the mass conservation equation are essentially equivalent.

**Interface conditions.** There are two interface conditions: the first is continuity of \( \nu \) across the interface (which is the actual moving interface governed by (2.2)), and the second is the equality of tractions \( (T^s - P^s I) \nu_s \) on the frozen interface. Here \( \nu_s \) denotes the exterior (to the phase \( s \)) unit normal to the frozen interface.

### 2.1.3. Weak formulation of the micro-scale problem.

In this section we provide the weak formulation of the problem to be homogenized. It consists of the mass conservation and momentum balance equations.

**Mass conservation.**
\[
\int_U \rho^s(0, x) \phi(0, x) dx - \int_{I_T} \int_U \rho^s \partial_t \phi dx dt - \int_{I_T} \int_U \rho^s \nu^s \cdot \nabla \phi dx dt = 0,
\]
where \( \rho^s(0, x) \) is given by (2.5), and \( I_T = [0, T) \).

Equation (2.13) is supposed to hold for each smooth test function \( \phi \), equal to zero on \( \partial U \) and vanishing for \( t \geq T \).

**Momentum balance.**
\[
- \int_U \rho^s(0, x) \nu_0 \cdot \psi dx - \int_{I_T} \int_U \rho^s \nu^s \cdot \partial_t \psi dx dt - \int_{I_T} \int_U \rho^s \nu^s \otimes \nu^s \cdot \nabla \psi dx dt \\
+ \int_{I_T} \int_U (T^s_1 + T^s_2) \cdot \psi dx dt = 0.
\]
Equation (2.14) holds for all smooth test-functions \( \psi \), such that \( \text{div} \, \psi = 0 \), \( \psi \) equal to zero on \( \partial U \), \( \psi(t, x) = 0 \) when \( t \geq T \). The dependence of \( T^e \) on \( \psi^e \) and \( \theta_0^e \) is given by (2.10).

Remark. It is important to note that, because of the condition \( \psi^e = 0 \) on \( \partial U \), the identity (2.14) also holds for test functions \( \psi \) with the condition \( \psi = 0 \) replaced by a less restrictive \( \psi \cdot \nu = 0 \) on \( \partial U \), \( \nu \) is the exterior unit normal to \( \partial U \). This fact will be used in Section 4 to construct oscillating test functions.

2.2. Finite energy weak solutions and bounds. We suppose that the initial conditions satisfy

\[
0 < C_1 \leq \rho^e(0, x) \leq C_2, \\
\v_0 \in H_0^1(U)
\]

with \( C_1, C_2 \) independent of \( \varepsilon \). The system (2.13), (2.14) closely resembles the system of density-dependent Navier-Stokes equations with density-dependent viscosity studied in [18], ch. 2. The only difference is the presence of the strain-dependent terms of the type \( A\varepsilon(u^e) \) in the constitutive equations. When the viscosity does not vanish, as is the case here, the existence and overall properties of the weak solutions are determined by the viscosity, and not elasticity, of the medium. The proof of existence for each fixed \( \varepsilon > 0 \) is outlined in Appendix B. It yields, for each \( \varepsilon \in \{ \varepsilon_k \}_{k=1}^\infty \), existence of the finite energy weak solutions of (2.13), (2.14) with the same properties as in [18], theorem 2.1, namely

\[
\rho^e \in L^\infty(I_T \times U) \cap C(I_T, L^p(U)), \text{ for all } 1 \leq p < \infty, \\
\v^e \in L^2(I_T, H_0^1(U)),
\]

satisfying the energy inequalities

\[
\frac{1}{2} \int_U \rho^e |v^e|^2 \, dx(t) + \int_U \left[ A^1 \theta_0^e(x) + A^2 (1 - \theta_0^e(x)) \right] |e(u^e)|^2 \, dx(t) \\
+ \int_{I_T} \int_U \left[ B^1 \theta_0^e(x) + B^2 (1 - \theta_0^e(x)) \right] |e(v^e)|^2 \, dx \, dt \leq \\
\frac{1}{2} \int_U \rho^e |v^e|^2 \, dx(0) + \int_U \left[ A^1 \theta_0^e(x) + A^2 (1 - \theta_0^e(x)) \right] |e(u^e)|^2 \, dx(0).
\]

Let us list the implications of the above estimates. First, renormalizing solutions of the mass conservation equations, as in [18], sect 2.3, we obtain

\[
\| \rho^e \|_{L^\infty(I_T \times U)} \leq C_2,
\]

Next we note that (2.11) implies \( u^e(0, x) = 0 \), and by (2.15), (2.16), the other initial conditions are bounded independent of \( \varepsilon \). This implies that the left hand side of (2.10) is bounded independent of \( \varepsilon \), so that

\[
\| T^e \|_{L^2(I_T \times U)} \leq C,
\]

with \( C \) independent of \( \varepsilon \). Combining (2.21) with the first Korn inequality for functions with zero trace on the boundary (see, e.g. [22], th. 2.1), and then with Poincaré inequality, we deduce

\[
\| \v^e \|_{L^2(I_T, H_0^1(U))} \leq C,
\]

\[
\| \u^e \|_{L^\infty(I_T, H_0^1(U))} \leq C,
\]

with \( C \) independent of \( \varepsilon \). Then it follows that

\[
\| \rho^e |v^e|^2 \|_{L^\infty(I_T, L^1(U))} \leq C,
\]

which, together with (2.20) yields

\[
\| \rho^e \v^e \|_{L^2(I_T, L^2(U))} \leq C,
\]

with \( C \) independent of \( \varepsilon \).
Theorem 3.1. The limits \( \bar{\rho}, \bar{\mathbf{u}}, \bar{T} \) satisfy
\[
\text{div} \, \mathbf{v} = 0,
\]
and the integral identities
\[
\int_U \bar{\rho}_0 \phi(0, x) dx - \int_{I_T} \int_U \bar{\rho} \partial_t \phi d\mathbf{x} dt - \int_{I_T} \int_U \bar{\mathbf{v}} \cdot \nabla \phi d\mathbf{x} dt = 0,
\]
\[
- \int_U \bar{\rho}_0 \mathbf{v}_0 \cdot \psi d\mathbf{x} - \int_{I_T} \int_U \bar{\mathbf{v}} \cdot \partial_t \psi d\mathbf{x} dt - \int_{I_T} \int_U \bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \cdot \nabla \psi d\mathbf{x} dt + \int_{I_T} \int_U \mathbf{T} \cdot e(\psi) d\mathbf{x} dt = 0.
\]
for all smooth test functions \( \phi, \psi \), such that \( \text{div} \, \psi = 0 \), and \( \phi, \psi \) are equal to zero on \( \partial U \) and vanish for \( t \geq T \).

Moreover, there exist the effective tensors \( \mathbf{A} \in L^2(U), \mathbf{B} \in L^2(U) \) and \( \mathbf{C} \in L^2(I_T \times U) \) such that the effective deviatoric stress \( \mathbf{T} \) satisfies
\[
\mathbf{T} = \mathbf{A} e(\mathbf{v}) + \mathbf{B} e(\mathbf{v}) + \int_0^t \mathbf{C}(t - \tau) e(\mathbf{v})(\tau) d\tau.
\]

Remark. Equations (7.11)–(7.14) essentially mean that the effective equations for \( \bar{\rho}, \bar{\mathbf{v}}, \bar{\mathbf{u}}, \bar{T} \) are
\[
\partial_t \bar{\rho} + \text{div}(\bar{\rho} \bar{\mathbf{v}}) = 0,
\]
\[
\text{div} \, \bar{\mathbf{v}} = 0,
\]
with initial and boundary conditions
\[
\bar{\mathbf{v}}(0, x) = \mathbf{v}_0, \quad \bar{\rho}(0, x) = \bar{\rho}_0, \quad \bar{\mathbf{v}} = 0 \text{ on } \partial U,
\]
and \( \bar{T} \) given by (7.14).

The result follows from a number of propositions and theorems. Here, the outline of the proof is presented for the reader’s convenience.

1. Construct the corrector operators \( N^\varepsilon \mathbf{w} = \mathbf{w}^\varepsilon \).
   \( N^\varepsilon \) are defined by (4.1)–(4.3). The proposed construction is non-local in \( t \) and satisfies the divergence-free constraint.

2. Auxiliary problems. Equation (4.1) includes two types of auxiliary functions: \( \mathbf{n}^{pq, \varepsilon} \) and \( \mathbf{m}^{pq, \varepsilon} \). An expression of the time derivative of \( \mathbf{w}^\varepsilon \) additionally contains the final value \( \mathbf{m}^{pq, \varepsilon}(T) \). These three families of functions satisfy auxiliary problems. The right hand sides in the auxiliary problems are chosen so
that the sequences of solutions converge to zero weakly in the appropriate Sobolev type spaces. The choice of the right hand sides involves abstract G-limit operators corresponding to each of the auxiliary problems. An analysis of auxiliary problems is presented in Sections 4.1, 4.2. Propositions 4.5, 4.8 are used repeatedly in the remainder of the proof.

3. **Convergence of \( w^\varepsilon \).** Using estimates from step 2, show that
\[
w^\varepsilon \to w, \quad \partial_t w^\varepsilon \to \partial_t w \text{ weakly in } L^2(I_T, H^1_0(U)).
\]
This is done in Proposition 5.2. This convergence corresponds to the condition (1.1) on \( N^\varepsilon \).

4. **Convergence of the inertial terms.** Proposition 5.2 combined with the Lemma 5.3 (Lemma 5.1 from [18]) implies that
\[
\int_U \rho^\varepsilon(0,x)w_0 \cdot w^\varepsilon + \int_0^T \int_U \rho^\varepsilon v^\varepsilon \cdot w^\varepsilon + \int_0^T \int_U \rho^\varepsilon v^\varepsilon \otimes v^\varepsilon \cdot \nabla w^\varepsilon
\]
converges to
\[
\int_U \mathcal{P}(0,x)v_0 \cdot w + \int_0^T \int_U \mathcal{P} \cdot w_t + \int_0^T \int_U \mathcal{P} \otimes \mathcal{P} \cdot \nabla w.
\]
This is done in Proposition 5.4.

5. **Compensated compactness of the stress.** Convergence of the inertial terms implies
\[
\int_0^T \int_U T^\varepsilon \cdot e(w^\varepsilon) \to \int_0^T \int_U T \cdot e(w).
\]
This is shown in Proposition 6.2.

6. **Effective stress.** Characterization of \( \mathbf{T} \) is obtained in Theorem 6.1. This step corresponds to establishing condition (1.2) on \( N^\varepsilon \). The main idea is to write
\[
\int_0^T \int_U T^\varepsilon \cdot e(w^\varepsilon) = -\langle u^\varepsilon, \text{div} \left( A^\varepsilon - B^\varepsilon \partial_t \right) w^\varepsilon \rangle.
\]
The expression for \( \langle u^\varepsilon, \text{div} \left( A^\varepsilon - B^\varepsilon \partial_t \right) w^\varepsilon \rangle \) contains a number of terms depending on \( n^{pq,\varepsilon}, m^{pq,\varepsilon}, m_p^{\varepsilon} \) and the corresponding pressures. In some of these terms we can pass to the limit using Lemma 5.1 from [18], since \( w^\varepsilon \) has Sobolev regularity. In other terms this is not possible, but these terms vanish by design of the auxiliary problems. The effective tensors \( \overline{\mathbf{A}}, \overline{\mathbf{B}} \) and \( \overline{\mathbf{C}} \) are obtained as weak limits of the three fluxes that appear in the auxiliary problems for, respectively, \( n^{pq,\varepsilon}, m^{pq,\varepsilon} \) and \( m^{\varepsilon} \).

7. **Mass conservation.** Mass conservation equation is weakly stable. This is a well known fact (see [18]).

### 4. Corrector operators and oscillating test functions

We look for corrector operators of the form
\[
N^\varepsilon w \equiv w^\varepsilon(t,x) = w(t,x) + n^{pq,\varepsilon}(x)\epsilon(w)_{pq} + \int_t^T m^{pq,\varepsilon}(t - \tau + T, x)\epsilon(w_t)_{pq}(\tau, x)d\tau + \nabla \phi^\varepsilon,
\]
(Summation over \( p, q \in \{1, 2, 3\} \) is assumed). Here, \( w \in C^\infty_0(I_T \times U) \), \( \text{div} \ w = 0 \) is an arbitrary test function, \( n^{pq,\varepsilon} \in H^1_0(U) \), \( m^{pq,\varepsilon} \in L^2(I_T, H^1_0(U)) \) are to be specified. So far we require that
\[
\text{div} \ n^{pq,\varepsilon} = 0, \text{ div} \ m^{pq,\varepsilon} = 0.
\]
More conditions will be imposed below.

The function \( \phi^\varepsilon \in L^2(I_T, H^1(U)) \) satisfies
\[
\Delta \phi^\varepsilon = -n^{pq,\varepsilon} \cdot \nabla \epsilon(w)_{pq} - \int_t^T m^{pq,\varepsilon}(t - \tau + T, x) \cdot \nabla \epsilon(w_t)_{pq}(\tau, x)d\tau,
\]
\[
\nabla \phi^\varepsilon \cdot \nu = 0 \text{ on } \partial U.
\]

The choice of the first three terms in (4.1) is motivated by similar expressions used in periodic [12] and scale-separated [13] homogenization. The last term are need to enforce divergence-free constraint. This is necessary in order to avoid dealing with pressure in (2.14) which is not \( L^1_{\text{loc}}(U) \) in general.
Note also that \( \nabla \phi^\varepsilon \cdot \nu \) is zero on \( \partial U \) for almost all \( t \). This makes \( w^\varepsilon \) correctly defined test functions for \((2.14)\) (see the Remark following that equation). Moreover,

**Lemma 4.1.** The function \( w^\varepsilon \) defined by \((4.1)\)–\((4.3)\) satisfies

\[
\text{div} \ w^\varepsilon = 0.
\]

**Proof.** Taking divergence of \((4.1)\) and using \((4.2)\) we find

\[
\text{div} \left( w(t, x) + n^{pq, \varepsilon} \psi_{pq}(\tau) + \int_0^\tau m^{pq, \varepsilon}(t - \tau, x) e(\psi_{pq}(\tau, x)) d\tau \right) = n^{pq, \varepsilon} \cdot \nabla e(w)_{pq} + \int_0^\tau m^{pq, \varepsilon}(t - \tau, x) \cdot \nabla e(\psi_{pq}(\tau, x)) d\tau,
\]

and the claim follows from the condition \((4.3)\).

\[\Box\]

**4.1. Auxiliary problem for \( m^{pq, \varepsilon} \).** In this subsection, \( p, q \) are fixed, so we drop them to simplify notations, and write \( m^\varepsilon \) instead of \( m^{pq, \varepsilon} \) and so on. We look for \( m^\varepsilon \) that solve the auxiliary problem

\[
- \text{div} \ (A^\varepsilon e(m^\varepsilon) - B^\varepsilon e(m^\varepsilon_T)) - \nabla P^\varepsilon = f, \quad \text{div} \ m^\varepsilon = 0,
\]

satisfying the condition

\[
m^\varepsilon(T, x) = m^\varepsilon_T.
\]

The objective of this section is to show that the right hand side \( f \) can be chosen so that \( m^\varepsilon \) converges weakly to zero in an appropriate space.

Let \( \psi(t) \in C^\infty(I_T) \) satisfy \( \psi(0) = 0 \) and \( \psi(T) = 1 \). We use this function to reduce \((4.4), (4.5)\) to a problem with a different right hand side and zero condition at \( t = T \). Writing

\[
m^\varepsilon = \hat{m}^\varepsilon + \psi m^\varepsilon_T
\]

we deduce that \( \hat{m}^\varepsilon \) solves

\[
- \text{div} \ (A^\varepsilon e(\hat{m}^\varepsilon) - B^\varepsilon e(\hat{m}^\varepsilon_T)) - \nabla P^\varepsilon = f + \text{div} \ (A^\varepsilon e(\psi m^\varepsilon_T) - B^\varepsilon e(\psi m^\varepsilon_T)), \quad \text{div} \ \hat{m}^\varepsilon = 0,
\]

with the initial condition

\[
\hat{m}(T, x) = 0.
\]

Define the spaces

\[
\mathcal{V} = \{ v \in L^2(I_T, H^1_0(U)), \ \text{div} \ v = 0 \},
\]

\[
\mathcal{W} = \{ v \in \mathcal{V} : v_t \in \mathcal{V} \},
\]

\[
\mathcal{W}_T = \{ v \in \mathcal{W} : v(T) = 0 \}.
\]

The space \( \mathcal{V} \) is equipped with a norm

\[
\| v \|_{\mathcal{V}} = \left( \int_0^T \int_U e(v) \cdot e(v) dx dt \right)^{1/2}.
\]

This norm is induced by the norm

\[
\| v \|_{H_0^1(U)} = \left( \int_U e(v) \cdot e(v) dx \right)^{1/2}.
\]

By Korn inequality \((22)\), \((4.12)\) is a norm equivalent to the standard one. Also, \( \mathcal{W}_T \) is dense in \( \mathcal{V} \). This can be proved in the same way as e.g. Thm. 2.1 in \([17]\).

A weak solution \( \hat{m}^\varepsilon \) of \((4.13), (4.14)\) is an element of \( \mathcal{W}_T \) satisfying

\[
\int_0^T \int_U (A^\varepsilon e(\hat{m}^\varepsilon) - B^\varepsilon e(\hat{m}^\varepsilon_T)) \cdot e(\phi) dx dt = (f - g^\varepsilon, \phi)_{\mathcal{V}, \mathcal{V}}.
\]
for all \( \phi \in \mathcal{V} \). Here,
\[
g^* = -\text{div} (\mathcal{A}^\varepsilon e(\psi \mathbf{m}_T^\varepsilon)) - \mathcal{B}^\varepsilon e(\psi_0, \mathbf{m}_0^\varepsilon))\]

Equation (4.13) can be stated as
\[
\mathcal{G}^* \mathbf{m}^\varepsilon = \mathbf{f} - g^* ,
\]
with the operator \( \mathcal{G}^* : \mathcal{W}_T \to \mathcal{V}^* \). We consider it as an unbounded operator on \( \mathcal{V} \) with the domain \( \mathcal{W}_T \). The corresponding bilinear form is defined as
\[
\langle \mathcal{G}^* \mathbf{u}, \mathbf{v} \rangle = \int_0^T \int_U (\mathcal{A}^\varepsilon e(\mathbf{u}) - \mathcal{B}^\varepsilon e(\mathbf{u}_t)) \cdot e(\mathbf{v}) \, dx \, dt
\]
for each \( \mathbf{u} \in \mathcal{W}_T, \mathbf{v} \in \mathcal{V} \). Finally, we note that the adjoint operator \( \mathcal{G}^{*\prime} \) with the domain
\[\mathcal{W}_0 = \{ \mathbf{v} \in \mathcal{W} : \mathbf{v}(0) = 0 \}\]
is defined by
\[
\langle \mathcal{G}^{*\prime} \mathbf{u}, \mathbf{v} \rangle = \int_0^T \int_U (\mathcal{A}^\varepsilon e(\mathbf{u}) + \mathcal{B}^\varepsilon e(\mathbf{u}_t)) \cdot e(\mathbf{v}) \, dx \, dt
\]

Proposition 4.2. (i) \( \mathcal{G}^* \) is strongly coercive:
\[
\langle \mathcal{G}^* \mathbf{u}, \mathbf{u} \rangle \geq \alpha_1 \| \mathbf{u} \|_{\mathcal{V}}^2
\]
where \( \alpha_1 \) is a constant from (2.12) (and thus independent of \( \varepsilon \)); (ii) \( \mathcal{G}^* \) has a bounded inverse satisfying
\[
\| (\mathcal{G}^*)^{-1} \mathbf{f} \|_{\mathcal{V}} \leq \frac{1}{\alpha_1} \| \mathbf{f} \|_{\mathcal{V}^*}
\]
for each \( \mathbf{f} \in \mathcal{V}^* \).

Proof. If \( \mathbf{u} \) is sufficiently smooth then after integrating by parts in (4.11) we would have
\[
\langle \mathcal{G}^* \mathbf{u}, \mathbf{u} \rangle = \int_0^T \int_U \mathcal{A}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \, dt + \frac{1}{2} \left( \int_U \mathcal{B}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \right)(0),
\]
where we took into account that \( \mathbf{u}(T) = 0 \). However, for an arbitrary \( \mathbf{u} \in \mathcal{W}_T \) the second term in the right hand side may not be well defined. To bypass this difficulty, observe that for almost all \( t \in I_T \),
\[
\frac{1}{2} \left( \int_U \mathcal{B}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \right)(t)
\]
is finite. For such \( t \) we have
\[
\int_t^T \int_U \mathcal{A}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) - \mathcal{B}^\varepsilon e(\mathbf{u}_t) \cdot e(\mathbf{u}) \, dx \, dt
\]
\[
= \int_t^T \int_U \mathcal{A}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \, dt + \frac{1}{2} \left( \int_U \mathcal{B}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \right)(t)
\]
\[
\geq \int_t^T \int_U \mathcal{A}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \, dt \geq \alpha_1 \int_t^T \int_U e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \, dt
\]
The last inequality follows from (2.12). Using absolute continuity in \( t \) of the first and last terms in the above inequality, we can pass to the limit \( t \to 0^+ \) and obtain
\[
\langle \mathcal{G}^* \mathbf{u}, \mathbf{u} \rangle = \lim_{t \to 0^+} \int_t^T \int_U \mathcal{A}^\varepsilon e(\mathbf{u}) \cdot e(\mathbf{u}) - \mathcal{B}^\varepsilon e(\mathbf{u}_t) \cdot e(\mathbf{u}) \, dx \, dt
\]
\[
\geq \lim_{t \to 0^+} \alpha_1 \int_t^T \int_U e(\mathbf{u}) \cdot e(\mathbf{u}) \, dx \, dt = \alpha_1 \| \mathbf{u} \|_{\mathcal{V}}^2
\]
which proves (i).

(ii) follows from (i). This is known (see, e.g., [32], Lemma 1). We only sketch the proof for completeness. Since \( \mathcal{G}^* \) is closed, passing to the limit in (4.17) we obtain that the image of \( \mathcal{G}^* \) is closed in \( \mathcal{V}^* \). If this image
does not contain all \( V^* \), then, because of density of \( W_T \) in \( V \), there is \( g \in V \) such that \( \langle G^* u, g \rangle = 0 \) for all \( u \in W_T \). This yields \( g \in W_0 \) (domain of \( G^{*,*} \)) and \( G^{*,*} g = 0 \). Next we observe that \( G^{*,*} \) satisfies (4.17) which yields \( g = 0 \) and gives a contradiction. Thus \( G^* \) is onto. The estimate (4.18) follows from (4.17).

**Remark.** (ii) implies existence of the pressure \( P_3^* \in L^2(I_T, L^2(U)) \). This follows using standard arguments from [29] combined with the inclusion \( A^\varepsilon e(m^{pq}) + B^\varepsilon e(m^{pq}) \in L^2(I_T, L^2(U)) \). Moreover, \( P_3^* \) is bounded in \( L^2(I_T, L^2(U)) \) independent of \( \varepsilon \). Therefore, extracting a subsequence if necessary, we can assume that \( P_3^* \to \mathcal{T}_3 \) weakly in \( L^2(I_T, L^2(U)) \).

**Definition 4.3.** We say that the sequence \( G^* \) \( G \)-converges to an operator \( G : D(G) \subset W_T \to V^* \), if for each \( f \in V^* \) the sequence \( u^\varepsilon = (G^*)^{-1} f \) converges to some \( u \in D(G) \) weakly in \( W_T \). In this case we define \( Gu = f \).

**Proposition 4.4.** The sequence \( G^* \) contains a \( G \)-convergent subsequence. The limiting operator \( G \) has the following properties:

(i) 
\[
\langle Gu, u \rangle \geq \alpha_1 \| u \|_V^2
\]

for each \( u \in D(G) \);

(ii) \( D(G) = W_T \).

**Proof.** Let us write \( G^* = A^\varepsilon - B^\varepsilon \partial_t \) where \( A^\varepsilon, B^\varepsilon : V \to V^* \) are operators induced by the bilinear forms

\[
a^\varepsilon(u, v) \equiv \int_0^T \int_U A^\varepsilon e(u) \cdot e(v) \, dx \, dt,
\]

\[
b^\varepsilon(u, v) \equiv \int_0^T \int_U B^\varepsilon e(u) \cdot e(v) \, dx \, dt,
\]

respectively. Ellipticity assumptions (2.12) imply that \( A^\varepsilon, B^\varepsilon \) are coercive and bounded with coercivity constants and bounds independent of \( \varepsilon \). In particular coercivity of \( B^\varepsilon \) implies that there exists a bounded inverse \( (B^\varepsilon)^{-1} \) defined on \( V^* \), satisfying \( \| (B^\varepsilon)^{-1} \| \leq \frac{1}{\beta_1} \), where \( \beta_1 \) is the lower bound from (2.12). Therefore, if \( (A^\varepsilon - B^\varepsilon \partial_t) u^\varepsilon = f \) then \( \partial_t u^\varepsilon = (B^\varepsilon)^{-1} (A^\varepsilon u - f) \). This implies

\[
\| \partial_t u^\varepsilon \|_V \leq \frac{\alpha_2}{\beta_1} \| (G^*)^{-1} f \|_V + \| f \|_{V^*} \leq \left( \frac{\alpha_2}{\alpha_1 \beta_1} + 1 \right) \| f \|_{V^*}.
\]

Thus, if \( u^\varepsilon \rightharpoonup u \) weakly in \( W_T \) then

\[
\| u \|_{W} \leq C(\alpha_1, \alpha_2, \beta_1) \| f \|_{V^*}.
\]

Since \( V^* \) is separable, we can use diagonal procedure to find a subsequence, non relabeled, such that \( u^\varepsilon = (G^*)^{-1} f \) converges weakly in \( W_T \) to \( u = G^{-1} f \) for all \( f \) in a dense subset of \( V^* \). Inequality (4.20) implies that convergence also holds for all \( f \in V^* \), and the operator \( G^{-1} \) is bounded.

Next, consider a sequence \( u^\varepsilon \) such that \( G^* u^\varepsilon = f \). Then, by the preceding, \( u^\varepsilon \) converges weakly to \( u \), and \( Gu = f \) by definition of the \( G \)-limit. Since \( \langle G^* u^\varepsilon, u^\varepsilon \rangle = \langle f, u^\varepsilon \rangle = \langle Gu, u^\varepsilon \rangle \), we can pass to the limit and obtain

\[
\lim_{\varepsilon \to 0} \langle G^* u^\varepsilon, u^\varepsilon \rangle = \langle Gu, u \rangle.
\]

This, together with lower semicontinuity of the norm with respect to weak convergence, allows passage to the limit in (4.17) which yields (4.19).

To prove (ii), observe first that by (4.20), \( D(G) \subset W_T \). To prove equality, we first show that \( G^{-1} \) is injective: \( G^{-1} f = 0 \) implies \( f = 0 \). Arguing by contradictions, suppose that there is \( g \in V^*, g \neq 0 \) such that \( G^{-1} g = 0 \). Consider the sequence \( u^\varepsilon_g = (G^*)^{-1} g \). By definition of \( G \), \( u^\varepsilon_g \) converges to zero weakly in \( W_T \). Then by (4.17),

\[
\langle g, u^\varepsilon_g \rangle = \langle G^* u^\varepsilon_g, u^\varepsilon_g \rangle \geq \alpha_2 \| u^\varepsilon_g \|_V^2.
\]
passing to the limit $\varepsilon \to 0$ we obtain that $\|u^\ast_{g,\varepsilon}\|_V \to 0$ and thus $u^\ast_{g,\varepsilon}$ converges to zero strongly in $V$. Next, we use $u^\ast_{g,\varepsilon}$ as the test function in the weak formulation of $G^\varepsilon u^\ast_{g,\varepsilon} = g$. Integrating by parts (which can be justified as in the proof of Proposition 4.12 and using coercivity of $B^\varepsilon$ we obtain

$$\langle g, u^\ast_{g,\varepsilon}\rangle = \|G^\varepsilon u^\ast_{g,\varepsilon}\|_2 \geq \beta_1 \|u^\ast_{g,\varepsilon}\|_V,$$

and thus $u^\ast_{g,\varepsilon}$ converges to zero strongly in $V$. Next, using uniform boundedness of $A^\varepsilon, B^\varepsilon$, we write

$$\| g \|_V = \| A^\varepsilon u^\ast_{g,\varepsilon} - B^\varepsilon u^\ast_{g,\varepsilon}\|_V \leq \alpha_2 \| u^\ast_{g,\varepsilon}\|_V + \beta_2 \| u^\ast_{g,\varepsilon}\|_V$$

where $\alpha_2, \beta_2$ are constants from (2.12). Passing to the limit in the above we deduce $g = 0$, which contradicts assumption $g \neq 0$. Thus $G^{-1}$ is injective.

Next we show that $D(G)$ (equivalently, the range of $G^{-1}$) is dense in $V$. If this were false, there would be $h \neq 0, h \in V^\ast$ such that $\langle h, G^{-1}f \rangle = 0$ for all $f \in V^\ast$. Let $u_h = G^{-1}h$. Choosing $f = h$, we obtain using (4.19):

$$0 = \langle h, G^{-1}h \rangle = \langle Gu_h, u_h \rangle \geq \alpha_1 \| u_h \|_V.$$

Therefore, $u_h = 0$. Then $h = 0$ by injectivity of $G^{-1}$. This contradicts the assumption $h \neq 0$.

Finally, observe that the norm of $W_T$ induces a scalar product

$$(u, v)_{W_T} = \int_0^T \int_U (e(u_t) \cdot e(v_t) + e(u) \cdot e(v)) \, dx \, dt.$$

In search of a contradiction, suppose that $D(G)$ is a proper subset of $W_T$. Then there is $\overline{u} \neq 0, \overline{u} \in W_T$ such that $(\overline{u}, G^{-1}f)_{W_T} = 0$ for all $f \in V^\ast$. Expression $(\overline{u}, v)_{W_T}$ defines a bounded linear functional $l(\overline{u})$ on $D(G)$ which by Hahn-Banach theorem can be extended to a bounded linear functional $L(\overline{u})$ on $V$ and this extension has same norm as $l(\overline{u})$. Therefore, $(\overline{u}, G^{-1}f)_{W_T} = 0$ implies $(L(\overline{u}), G^{-1}f) = 0$ for all $f \in V^\ast$. Density of the range of $G^{-1}$ implies $L(\overline{u}) = 0$. But then $(\overline{u}, \overline{u})_{W_T} = 0$ which contradicts the assumption $\overline{u} \neq 0$. Thus (ii) is proved.

**Proposition 4.5.** There exists $f \in V^\ast$ such that the sequence of solutions $m^\varepsilon$ of (4.4), with this choice of the right hand side, contains a subsequence (not relabeled) satisfying

(i) $m^\varepsilon \to 0$ weakly in $W_T$,

(ii) $m^\varepsilon \to 0$ strongly in $L^2(I_T, L^2(U))$.

**Proof.** By Proposition 4.3 proved below in Sect. 4.2 we can assume that $m^\varepsilon_T$ converges to zero weakly in $H^1_0(U)$. Then $\psi m^\varepsilon_T \to 0$ and $\partial_t \psi m^\varepsilon_T \to 0$ weakly in $V$. Since $m^\varepsilon = m^\varepsilon + \psi m^\varepsilon_T$, (i) will be proved if we show that there is choice of $f$ such that $m^\varepsilon \to 0$ weakly in $W_T$.

Consider (4.14). In view of (2.12) and uniform bounds on $m^\varepsilon_T$, the sequence $g^\varepsilon$ is bounded in $V^\ast$. Therefore, the sequence $(G^\varepsilon)^{-1} g^\varepsilon$ is bounded in $W_T$, and we can extract a subsequence that converges weakly to some $q \in W_T$. By Proposition 4.3 (ii), $q \in D(G)$. Therefore, we can choose

$$f = Gq.$$

Then

$$\dot{m}^\varepsilon = (G^\varepsilon)^{-1} (f - g^\varepsilon) = (G^\varepsilon)^{-1} Gq - (G^\varepsilon)^{-1} g^\varepsilon.$$

By Proposition 4.3 $(G^\varepsilon)^{-1} Gq \to q$ weakly in $W_T$, up to extraction of a subsequence. Hence, $\dot{m}^\varepsilon \to 0$.

Thus (i) is proved.

To prove (ii), observe first that $m^\varepsilon_T$ converges to zero strongly in $L^2(U)$ and thus $\psi m^\varepsilon_T \to 0$ and $\partial_t \psi m^\varepsilon_T \to 0$ strongly in $L^2(I_T, L^2(U))$. Therefore, to prove strong convergence of $m^\varepsilon$ it is enough to prove strong convergence of $\dot{m}^\varepsilon$.

Next, note that $\dot{m}^\varepsilon_T$ is bounded in $V$ independent of $\varepsilon$. Now strong convergence of $\dot{m}^\varepsilon$ is deduced from (i) and J. L. Lions’ compactness theorem (see e.g. [29], thm. 2.1, ch.III).
4.2. Auxiliary problems for \( n^{pq,\varepsilon}, m^{pq,\varepsilon} \). In this section, \( p, q \) are once again fixed, so we drop them to simplify notations, and write \( n^{\varepsilon} \) instead of \( n^{pq,\varepsilon} \) and so on.

We seek \( n^{\varepsilon}, m^{\varepsilon}_T \) satisfying, respectively, the auxiliary problems

\[
\begin{align*}
(4.22) \quad - \text{div} \left( A^\varepsilon (I^{pq} + e(n^{\varepsilon})) \right) - \nabla P^\varepsilon_1 &= f_1, \\
(4.23) \quad - \text{div} \left( B^\varepsilon (I^{pq} + e(n^{\varepsilon})) - e(m^{\varepsilon}_2) \right) - \nabla P^\varepsilon_2 &= f_2,
\end{align*}
\]

with suitably chosen \( f_1, f_2 \). First, we find \( n^{\varepsilon} \) from (4.22). Then this \( n^{\varepsilon} \) should be plugged into (4.23), and then \( m^{\varepsilon}_T \) can be found. The goal, as before, to choose \( f_1, f_2 \) so that \( n^{\varepsilon}, m^{\varepsilon}_T \) would have subsequences that converge weakly to zero. Let

\[ V = \{ v \in H^1_0 : \text{div} \ v = 0 \}, \]

equipped with the norm (4.12). Given \( f_1 \in V^*, \ n^{\varepsilon} \in V \) is a weak solution of (4.22) provided

\[
\int_{\Omega} A^\varepsilon (I^{pq} + e(n^{\varepsilon})) \cdot e(w)dx = (f_1, w)_{V,V^*},
\]

for all \( w \in V \). Weak solutions of (4.23) are defined similarly. This identity can be written as an operator equation

\[
(4.25) \quad A^\varepsilon n^{\varepsilon} = f_1 - g^{\varepsilon}_1,
\]

where

\[
\begin{align*}
g^{\varepsilon}_1 &= - \text{div} \left( A^\varepsilon I^{pq} \right) \in V^*, \\
A^\varepsilon : V \rightarrow V^* \text{ is the operator induced by the bilinear form}
\end{align*}
\]

and

\[
\begin{align*}
a^\varepsilon(u,v) &= \int_{\Omega} A^\varepsilon e(u) \cdot e(v)dx.
\end{align*}
\]

Similarly, (4.23) can be written as

\[
(4.26) \quad B^\varepsilon m^{\varepsilon}_T = f_2 - g^{\varepsilon}_2,
\]

where the operator \( B^\varepsilon \) is induced by the form

\[
\begin{align*}
\epsilon^\varepsilon(u,v) &= \int_{\Omega} B^\varepsilon e(u) \cdot e(v)dx,
\end{align*}
\]

and

\[
\begin{align*}
g^{\varepsilon}_2 &= - \text{div} \left( B^\varepsilon (I^{pq} + e(n^{\varepsilon})) \right).
\end{align*}
\]

By (2.12), operators \( A^\varepsilon, B^\varepsilon \) satisfy

\[
\begin{align*}
(4.27) \quad \langle A^\varepsilon u, u \rangle_{V,V^*} \geq \alpha_1 \parallel u \parallel^2_V, \\
(4.28) \quad \parallel A^\varepsilon u \parallel_{V^*} \leq \alpha_2 \parallel u \parallel_V, \\
(4.29) \quad \langle B^\varepsilon u, u \rangle_{V,V^*} \geq \beta_1 \parallel u \parallel^2_V, \\
(4.30) \quad \parallel B^\varepsilon u \parallel_{V^*} \leq \beta_2 \parallel u \parallel_V.
\end{align*}
\]

Lax-Milgram lemma implies existence of unique solutions of (4.25), (4.26). These solutions satisfy

\[
\begin{align*}
n^{\varepsilon} = (A^\varepsilon)^{-1} (f_1 - g^{\varepsilon}_1), \quad \parallel n^{\varepsilon} \parallel_{V^*} \leq \frac{1}{\alpha_1} \parallel f_1 - g^{\varepsilon}_1 \parallel_{V^*}, \\\nm^{\varepsilon}_T = (B^\varepsilon)^{-1} (f_2 - g^{\varepsilon}_2), \quad \parallel m^{\varepsilon}_T \parallel_{V^*} \leq \frac{1}{\beta_1} \parallel f_2 - g^{\varepsilon}_2 \parallel_{V^*}.
\end{align*}
\]

Remark. Existence of the pressures \( P^\varepsilon_1 \in L^2(U), P^\varepsilon_2 \in L^2(U) \) follows using standard arguments from [29]. Moreover, these pressures are bounded in \( L^2(U) \) independent of \( \varepsilon \). Therefore, extracting a subsequence if necessary, we can assume that \( P^\varepsilon_j \rightarrow P^j, j = 1, 2 \) weakly in \( L^2(U) \).

Definition 4.6. The sequence of operators \( A^\varepsilon : V \rightarrow V^* \) \( G \)-converges to an operator \( A \) if \( (A^\varepsilon)^{-1} f \) converges to some \( u \in V \) weakly in \( V \), for each \( f \in V^* \). We also define \( f = Au \).

Proposition 4.7. The sequences \( A^\varepsilon, B^\varepsilon \) contain \( G \)-convergent subsequences.

Proof. This is known [20], thm. 2.
Proposition 4.8. There exists \( f_1 \in V^* \) (respectively \( f_2 \)) such that, up to extraction of a subsequence, \( n^\varepsilon \) (respectively \( m^\varepsilon \)) converge to zero weakly in \( V \).

Proof. Consider (4.23). Since \( g_2^\varepsilon \) is bounded in \( V^* \), and \( (A^\varepsilon)^{-1} \) is bounded independent of \( \varepsilon \), the sequence \( (A^\varepsilon)^{-1} g_1^\varepsilon \) is bounded in \( V \). Thus we can extract a subsequence that converges weakly in \( V \) to some \( \overline{u}_1 \in V \). Choose
\[
(4.31) \quad f_1 = A\overline{u}_1.
\]
Then
\[
n^\varepsilon = (A^\varepsilon)^{-1} A\overline{u}_1 - (A^\varepsilon)^{-1} g_1^\varepsilon.
\]
By definition of \( A \), the first term in the right hand side converges to \( \overline{u}_1 \) weakly in \( V \), and so does the second. Hence \( n^\varepsilon \to 0 \) weakly in \( V \). For (4.26) the procedure is the same. Up to extraction of a subsequence, \( (B^\varepsilon)^{-1} g_2^\varepsilon \to \overline{u}_2 \) weakly in \( V \), and we choose
\[
(4.32) \quad f_2 = B\overline{u}_2.
\]
\[\square\]

5. Inertial terms in the momentum balance equation

In the remainder of the paper, we assume that the oscillating test functions \( w^\varepsilon \) are defined as follows.

Definition 5.1. Let \( w \in C^\infty(I_T \times U) \), \( \text{div} w = 0 \) be arbitrary, and define \( w^\varepsilon \) by (4.11). In (4.11), choose \( n^{pq,\varepsilon}, m^{pq,\varepsilon} \) that solve, respectively (4.22), (4.23) with the right hand sides chosen according to (4.31), (4.32). Also, let \( m^{pq,\varepsilon} \) satisfy (4.4) with the right hand side chosen according to (4.21).

Proposition 5.2. The sequence \( w^\varepsilon \) defined as above satisfies
\[
(5.1) \quad w^\varepsilon \to w \text{ in } L^2(I_T, L^2(U)), \quad w^\varepsilon_t \to w_t \text{ in } L^2(I_T, L^2(U)).
\]
Also, \( w^\varepsilon \in L^\infty(I_T, H^1_0(U)) \), and
\[
(5.2) \quad \|w^\varepsilon\|_{L^\infty(I_T, H^1_0(U))} \leq T^{1/2} \|w^\varepsilon_t\|_{L^2(I_T, H^1_0(U))} \leq CT^{1/2}
\]
with \( C \) independent of \( \varepsilon \).

Proof. First we need a formula for the time derivative of \( w^\varepsilon \). After taking time derivative of (4.11), integrating by parts in the time convolution (which involves putting time differentiation on \( w_t \) instead of \( m^{pq,\varepsilon} \)) and using \( w(T) = 0 \) we obtain
\[
(5.3) \quad w^\varepsilon_t = w_t + n^{pq,\varepsilon}(x)e(w_t)_{pq} + \int_T^t m^{pq,\varepsilon}(t - \tau + T)e(w_t)_{pq}(\tau)d\tau + \nabla \phi^\varepsilon.
\]
To prove strong convergence of \( w^\varepsilon_t \) we need to prove that all terms in the right hand side of (5.3) converge to zero strongly in \( L^2(I_T, L^2(U)) \).

Step 1. Show that
\[
n^{pq,\varepsilon}e(w)_{pq} + \int_T^t m^{pq,\varepsilon}(t - \tau + T)e(w_t)_{pq}(\tau)d\tau \to 0,
\]
and
\[
n^{pq,\varepsilon}(x)e(w_t)_{pq} + \int_T^t m^{pq,\varepsilon}(t - \tau + T)e(w_t)_{pq}(\tau)d\tau \to 0
\]
strongly in \( L^2(I_T \times U) \).

By Propositions 4.8, \( n^{pq,\varepsilon} \to 0 \) weakly in \( H^1_0(U) \) and thus strongly in \( L^2(U) \). By Proposition 4.8, \( m^{pq,\varepsilon} \to 0 \) strongly in \( L^2(I_T, L^2(U)) \), and we conclude.

Step 2. Show that \( \nabla \phi^\varepsilon \to 0 \) and \( \partial_i \nabla \phi^\varepsilon \to 0 \) strongly in \( L^2(I_T \times U) \).

First we estimate \( e(\nabla \phi^\varepsilon) \). Note that \( e(\nabla \phi^\varepsilon)_{ij} = \partial_i \partial_j \phi^\varepsilon \) and write \( \partial_i \partial_j \phi^\varepsilon = \partial_i \partial_j E*(\Delta \phi^\varepsilon) + \partial_i \partial_j K*(\Delta \phi^\varepsilon) \), where \( E = \frac{1}{|x|} \) is a fundamental solution of the Laplacian, and \( K \) is a harmonic function, which depends
only on $U$. To estimate $\partial_i \partial_j E \ast (\Delta \phi^e)$ we use Calderón-Zygmund inequality (see, e.g. [11], Theorem 9.9) with $p = 2$ and obtain

$$\| \partial_i \partial_j E \ast (\Delta \phi^e) \|_{L^2(I_T \times U)} \leq \| \Delta \phi^e \|_{L^2(I_T \times U)}$$

Since $\partial_i \partial_j K$ is a smooth function, there exists a constant $C(U)$ depending only on $U$ such that

$$\| \partial_i \partial_j K \ast (\Delta \phi^e) \|_{L^2(I_T \times U)} \leq C(U) \| \Delta \phi^e \|_{L^2(I_T \times U)}$$

Thus

$$\| \partial_i \partial_j \phi^e \|_{L^2(I_T \times U)} \leq (1 + C(U)) \| \Delta \phi^e \|_{L^2(I_T \times U)}.$$  \hspace{1cm} (5.4)

Combining (5.4) with (4.3) we find

$$\int_{I_T} \int_{U} |e(\nabla \phi^e)|^2 dx dt \leq C(w, U) \sum_{p,q=1}^3 \left( \| n^{pq,e} \|_{L^2(U)}^2 + \| m^{pq,e} \|_{L^2(I_T \times U)}^2 \right)$$

so $e(\nabla \phi^e)$ converges to zero strongly in $L^2(I_T \times U)$. This also implies that the components of the Hessian of $\phi^e$ converge to zero strongly in $L^2(I_T \times U)$.

Finally, the standard a priori estimate for the Neumann problem $\Delta \phi^e = f^e, f \in L^2(U)$, satisfying $\nabla \phi^e \cdot \nu = 0$ on the boundary, yields

$$\int_{I_T} \nabla \phi^e \cdot \nabla \phi^e \, dx \leq \| f^e \|_{L^2(U)} \| \phi^e \|_{L^2(U)} \leq C \| f^e \|_{L^2(U)} \left( \int_{I_T} \nabla \phi^e \cdot \nabla \phi^e \, dx \right)^{1/2}.$$  \hspace{1cm} (5.6)

Here, $C$ is the constant in Poincaré inequality. Poincaré inequality applies after we impose the condition $\int_{\partial U} \phi^e dS = 0$, standard for Neumann problem. Since $f^e$, given by the right hand side of (4.3), converges to zero strongly in $L^2(I_T, L^2(U))$, (5.6) implies that $\nabla \phi^e$ converges to zero strongly in $L^2(I_T, L^2(U))$.

Differentiating (4.3) in $t$ and integrating by parts as in (5.3) we find

$$\Delta \phi^e_t = -n^{pq,e} \cdot \nabla e(w_t)_{pq} - \int_{I_T} m^{pq,e}(t - \tau + T) \cdot \nabla e(w_t)_{pq}(\tau) d\tau.$$  \hspace{1cm} (5.7)

Therefore, arguing as above we have

$$\int_{I_T} \int_{U} |e(\nabla \phi^e_t)|^2 \, dx dt \leq C(w, U) \sum_{p,q=1}^3 \left( \| n^{pq,e} \|_{L^2(U)}^2 + \| m^{pq,e} \|_{L^2(I_T \times U)}^2 \right),$$

which yields $e(\nabla \phi^e_t) \to 0$, and then $\nabla \phi^e_t \to$ strongly in $L^2(I_T \times U)$.

**Step 4.** Prove (5.2).

Since $w^r(t) = -\int_t^T w^r_\tau(\tau) d\tau$, we obtain for almost all $t \in I_T$

$$\| w^r \|_{H^1_0(U)}^2(t) \leq \int_0^T \| w^r_\tau \|_{H^1_0(U)}^2(\tau) d\tau \leq T^{1/2} \left( \int_0^T \| w^r_\tau \|_{H^1_0(U)}^2(\tau) d\tau \right)^{1/2},$$

and (5.2) follows.

Next we will need the following lemma ([11], lemma 5.1).

**Lemma 5.3.** Let $g^n, h^n$ converge weakly to $g, h$, respectively in $L^{p_1}(0, T; L^{p_2}(\Omega)), L^{q_1}(0, T; L^{q_2}(\Omega))$, where $1 \leq p_1, p_2 \leq \infty$, $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1$.

We assume in addition that $\partial_t g^n$ is bounded in $L^1(0, T; W^{-m,1}(\Omega))$ for some $m \geq 0$ independent of $n$ and

$$\| h^n - h^n(\cdot + \xi, t) \|_{L^{q_1}(0, T; L^{q_2}(\Omega))} \to 0$$

as $|\xi| \to 0$, uniformly in $n$.

Then $g^n h^n$ converges to $gh$ in the sense of distributions on $\Omega \times (0, T)$.

This lemma can be used to obtain the effective mass conservation equation and to pass to the limit in the inertial terms in the momentum equation (2.14).
Proposition 5.4. Let \( w^\varepsilon \) be functions from Definition 5.1. Then

\[
(5.9) \quad \lim_{\varepsilon \to 0} \int_U \rho^\varepsilon(0, x) v_0 \cdot w^\varepsilon \, dx = \int_U \overline{\rho}(0, x) v_0 \cdot w \, dx,
\]

\[
(5.10) \quad \lim_{\varepsilon \to 0} \int_{I_T \times U} \rho^\varepsilon(0, x) v_0 \cdot \partial_t w^\varepsilon \, dx = \int_{I_T \times U} \overline{\rho} \overline{v} \cdot \partial_t w \, dx dt,
\]

and

\[
(5.11) \quad \lim_{\varepsilon \to 0} \int_{I_T \times U} \rho^\varepsilon \nabla v^\varepsilon \cdot \nabla w^\varepsilon \, dx dt = \int_{I_T \times U} \overline{\rho} \overline{\nabla} \cdot \nabla w \, dx dt.
\]

Proof. By Lemma 5.8 \( \rho^\varepsilon \rightarrow \overline{\rho} \overline{v} \) in the sense of distributions, and thus also weakly in \( L^2(I_T \times U) \). By \( (5.1) \), \( w^\varepsilon, \partial_t w^\varepsilon \) converge to \( w, \partial_t w \) strongly in \( L^2(I_T \times U) \). This permits passage to the limit in the products and yields \( (5.9) \).

Since \( w^\varepsilon - w \in C(I_T, L^2(U)) \),

\[
\| w^\varepsilon - w \|_{L^2(U)}(0) \leq \int_0^T \| w^\varepsilon_t - w_t \|_{L^2(U)}(\tau) d\tau \leq T^{1/2} \left( \int_0^T \| w^\varepsilon_t - w_t \|_{L^2(U)}(\tau) d\tau \right)^{1/2}.
\]

Noting that \( w^\varepsilon_t \rightarrow w_t \), strongly in \( L^2(I_T \times U) \), we obtain that \( (w^\varepsilon - w)(0) \rightarrow 0 \) strongly in \( L^2(U) \). Since \( \rho^\varepsilon(0, x) \) converges weakly-\( * \) to \( \overline{\rho}(0, x) \), strong convergence of \( w^\varepsilon(0, x) \) permits passage to the limit in the product \( \rho^\varepsilon(0, x)v_0(0, x) \cdot w^\varepsilon(0, x) \) and yields \( (5.9) \).

Next, fix \( j \in \{1, 2, 3\} \), pick a function \( \eta \in C_0^\infty(I_T \times U) \), and insert the test function \( (w^\varepsilon - w) \eta \) into the weak formulation of the mass balance equation. This yields

\[
(5.12) \quad \int_U \rho^\varepsilon(0, x)(w^\varepsilon - w)\eta(0, x) dx - \int_{I_T \times U} \rho^\varepsilon \partial_t ((w^\varepsilon - w)\eta) dx dt - \int_I \int_U \rho^\varepsilon \cdot \nabla ((w^\varepsilon - w)\eta) dx dt = 0.
\]

Strong convergence of \( (w^\varepsilon - w)(0) \) to zero implies

\[
(5.13) \quad \lim_{\varepsilon \to 0} \int_U \rho^\varepsilon(0, x)(w^\varepsilon - w)\eta(0, x) dx = 0.
\]

Next, note that

\[
(5.14) \quad \lim_{\varepsilon \to 0} \int_{I_T \times U} \rho^\varepsilon \partial_t ((w^\varepsilon - w)\eta) dx dt = 0
\]

because \( (w^\varepsilon_t - w_t) \rightarrow 0 \) strongly in \( L^2(I_T \times U) \), and \( \rho^\varepsilon \) is bounded in \( L^\infty(I_T \times U) \) independent of \( \varepsilon \). Now from \( (5.12), (5.13) \) and \( (5.14) \) we deduce

\[
(5.15) \quad \lim_{\varepsilon \to 0} \int_{I_T \times U} \rho^\varepsilon \cdot \nabla ((w^\varepsilon - w)\eta) dx dt = 0.
\]

Since \( \rho^\varepsilon \) is bounded in \( L^2(I_T \times U) \), \( w^\varepsilon \rightarrow w \) strongly in \( L^2(I_T \times U) \), \( (5.15) \) implies

\[
(5.16) \quad \lim_{\varepsilon \to 0} \int_{I_T \times U} \eta \rho^\varepsilon \cdot \nabla (w^\varepsilon - w) dx dt = 0.
\]

Since \( \eta \in C_0^\infty(I_T \times U) \) is an arbitrary test function, \( \rho^\varepsilon \nabla (w^\varepsilon - w) \) is bounded in \( L^2(I_T, L^{5/4}(U)) \) independent of \( \varepsilon \). This follows from Sobolev imbedding for \( \nabla \) and H"older inequality. Application of H"older inequality yields

\[
\left( \int_U \left| \rho^\varepsilon \partial_k w^\varepsilon \right|^s \, dx \right)^{1/s} \leq \| \rho^\varepsilon \|_{L^\infty(U)}^{s} \left( \int_U \left| v_0^\varepsilon \right|^s \right)^{1/s} \left( \int_U \left| \partial_k w^\varepsilon \right|^q \right)^{1/q} \left( \int_U \left| \partial_k w^\varepsilon \right|^q \right)^{1/q} (t).
\]
Here $s, q \geq 1$ and $\frac{1}{s} + \frac{1}{q} = 1$. Hence,

\[
\int_{I_T} \left( \int_{U} \left| \rho^x v^x_k \partial_k w^s_j \right|^2 \, dx \right) \frac{d\tau}{dt} \leq \int_{I_T} \left( \int_{U} \left| \partial_k w^s_j \right|^q \right) \frac{d\tau}{dt} \left( \int_{U} \left| v^x_k \right|^q \, dx \right) \frac{d\tau}{dt} dt
\]

\[
\leq \left\| \rho^x \right\|_{L^5(I_T \times U)} \left\| \partial_k w^s_j \right\|_{L^5(L^{s'})} \left( \int_{I_T} \left( \int_{U} \left| v^x_k \right|^q \, dx \right) \frac{d\tau}{dt} \right) dt
\]

Therefore

\[
\left\| \rho^x v^x_k \partial_k w^s_j \right\|_{L^5(I_T \times U) \cap L^5(U)} \leq \left\| \rho^x \right\|_{L^5(I_T \times U)} \left\| \partial_k w^s_j \right\|_{L^5(L^{s'})} \left\| v^x_k \right\|_{L^5(I_T, L^{s}(U))}
\]

We need to choose $s, q$ so that (i) the right hand side of (5.17) is finite; and (ii) $v^x \in L^2(I_T, L^s(U))$, where $\frac{1}{s} + \frac{1}{q} = 1$. By Sobolev imbedding, $s' \leq 6$, and therefore

\[
s \geq 6.
\]

By (5.2), $\left\| \partial_k w^s_j \right\|_{L^5(L^{s'}(U))}$ is finite if

\[
1 \leq s q' \leq 2 \iff 1 - \frac{1}{q} \leq s \leq 2 - \frac{2}{q},
\]

(sq' < 2 are allowed because $U$ is bounded). Also, by Sobolev imbedding

\[
1 \leq s q \leq 6 \iff 1 - \frac{1}{q} \leq s \leq \frac{6}{q}.
\]

The solution set of inequalities (5.18)–(5.20) is a non-empty, convex quadrilateral in the $1/q - s$-plane. For example, we can choose $s = \frac{6}{5}$ and any $q$ satisfying $\frac{1}{s} \leq \frac{1}{q} \leq \frac{2}{s}$. If, for example, $1/q = \frac{2}{5}$, then $sq = \frac{12}{5}$ and (5.17) becomes

\[
\left\| \rho^x v^x \cdot \nabla w^x \right\|_{L^5(I_T, L^{1/5}(U))} \leq C \left\| \rho^x \right\|_{L^5(I_T \times U)} \left\| \nabla w^x \right\|_{L^5(I_T, L^{1/5}(U))} \left\| w^x \right\|_{L^5(I_T, L^{3/5}(U))}.
\]

Since the right hand side of (5.17) is bounded independent of $\varepsilon$, the claim is proved.

Together with (5.16), this yields $\rho^x v^x \cdot \nabla (w^x - w) \to 0$ weakly in $L^2(I_T, L^{6/5}(U))$. Therefore, the weak limit of $\rho^x v^x \cdot \nabla w^x$ is the same as the weak limit of $\rho^x v^x \cdot \nabla w$ in $L^2(I_T, L^{6/5}(U))$. By Lemma 5.3

\[
\rho^x v^x \cdot \nabla w \to \bar{\rho} \bar{v} \cdot \nabla \bar{w}
\]

weakly in $L^2(I_T, L^{6/5}(U))$. The bound on $\partial_i [\rho^x v^x \cdot \nabla w^x]$ in a negative Sobolev space follow from the corresponding bounds on $\rho^x v^x$ and the fact that $w^x$ is bounded in $L^2(I_T, H^1_{x}(U))$.

Now application of Lemma 5.3 with $g^x = \rho^x v^x \cdot \nabla w^x$, $h^x = v^x$ yields

\[
\lim_{\varepsilon \to 0} \int_{I_T \times U} \rho^x v^x \otimes v^x \cdot \nabla w^x \eta \, dx dt = \int_{I_T \times U} \overline{\rho} \overline{v} \otimes \overline{v} \cdot \nabla \eta \, dx dt.
\]

For each $\eta \in C_0^\infty(I_T \times U)$. From Sobolev imbedding and bounds on $\nabla w^x$, in the same way as (5.17) was analyzed, choosing $1/q = 2/5, s = 6/5$, we obtain

\[
\left\| \rho^x v^x \otimes v^x \cdot \nabla w^x \right\|_{L^5(I_T \times U)} \leq C \left\| \rho^x \right\|_{L^5(I_T \times U)} \left\| \nabla w^x \right\|_{L^5(I_T \times L^2(U))} \left\| \rho^x \cdot v^x \right\|_{L^5(I_T, L^3(U))}
\]

with $C$ independent of $\varepsilon$. Note that the right hand side is bounded independent of $\varepsilon$. Therefore, $\rho^x v^x \otimes v^x \cdot \nabla w^x$ converges to $\overline{\rho} \overline{v} \otimes \overline{v} \cdot \nabla \bar{w}$ weakly in $L^2(I_T, L^{5}(U))$. This implies convergence of integrals of $\rho^x v^x \otimes v^x \cdot \nabla w^x$ over subsets of $I_T \times U$, and in particular (5.11).
6. Effective deviatoric stress. Proof of the main theorem

**Theorem 6.1.** There exist a subsequence, not relabeled, and effective material tensors \( \overline{A} \in L^2(U) \), \( \overline{B} \in L^2(I_T \times U) \) and \( \overline{C} \in L^2(I_T \times U) \) such that for each \( w \in C^0_c(I_T \times U) \) with \( \text{div} \ w = 0, w(T,x) = 0 \),

\[
\int_{I_T} \int_U \mathbf{T} \cdot e(w) \, dx \, dt = \lim_{\varepsilon \to 0} \int_{I_T} \int_U \mathbf{T}^\varepsilon \cdot e(w^\varepsilon) \, dx \, dt
\]

\[
= \int_{I_T} \int_U \left( \overline{A} \varepsilon(w) + \overline{B} e(\mathbf{v}) + \int_0^1 \overline{C}(t - \tau, \cdot) e(\mathbf{v})(\tau, \cdot) \right) \cdot e(w) \, dx \, d\tau
\]

as \( \varepsilon \to 0 \) along this subsequence.

**Proof.** The theorem follows from several propositions. First, we prove that convergence of inertial terms implies compensated compactness of stress.

**Proposition 6.2.** Let \( w^\varepsilon \) be test functions from Definition 5.1. Then

\[
\lim_{\varepsilon \to 0} \int_{I_T} \int_U \mathbf{T}^\varepsilon \cdot e(w^\varepsilon) \, dx \, dt = \int_{I_T} \int_U \mathbf{T} \cdot e(w) \, dx \, dt.
\]

**Proof of the proposition.** First, use \( w \) as a test function in (2.14) and pass to the limit \( \varepsilon \to 0 \). Repeated application of Lemma 5.3 in the inertial terms yields

\[
- \int_U \overline{p}(0, x) \nabla w \, dx - \int_{I_T \times U} \overline{v} \cdot \partial_t w \, dx \, dt
\]

\[
- \int_{I_T \times U} \overline{p} \nabla \cdot \nabla w \, dx \, dt + \int_{I_T \times U} \mathbf{T} \cdot e(w) \, dx \, dt = 0
\]

Then insert \( w^\varepsilon \) into (2.14) and pass to the limit \( \varepsilon \to 0 \). By Proposition 6.4, the integrals corresponding to the inertial terms will converge to the corresponding integrals of the limiting functions \( \overline{\rho}, \overline{v} \). This yields

\[
- \int_U \overline{p}(0, x) \nabla w \, dx - \int_{I_T \times U} \overline{v} \cdot \partial_t w \, dx \, dt
\]

\[
- \int_{I_T \times U} \overline{p} \nabla \cdot \nabla w \, dx \, dt + \lim_{\varepsilon \to 0} \int_{I_T \times U} \mathbf{T}^\varepsilon \cdot e(w^\varepsilon) \, dx \, dt = 0
\]

Comparison of (6.1) and (6.2) finishes the proof.

Next, using symmetry of \( A^\varepsilon, B^\varepsilon \) we have

\[
\int_{I_T \times U} \mathbf{T}^\varepsilon \cdot e(w^\varepsilon) \, dx \, dt = \int_{I_T \times U} e(w^\varepsilon) \cdot [A^\varepsilon e(w^\varepsilon) - B^\varepsilon e(w^\varepsilon)] \, dx \, dt.
\]

Differentiation in (4.1) yields

\[
A^\varepsilon e(w^\varepsilon) - B^\varepsilon e(w^\varepsilon) = \mathcal{F}_1^{pq,\varepsilon}(e(w)_{pq}) + \mathcal{F}_2^{pq,\varepsilon}(e(w)_{pq}) + \int_t^T \mathcal{F}_3^{pq,\varepsilon}(t - \tau + T) e(w_{pq})(\tau) \, d\tau
\]

\[
= A^\varepsilon(g_1^\varepsilon + g_2^\varepsilon) - B^\varepsilon(\partial_t g_1^\varepsilon + \partial_t g_2^\varepsilon)
\]

\[
+ A^\varepsilon(\nabla \phi^\varepsilon) - B^\varepsilon e(\nabla \phi^\varepsilon),
\]

where

\[
\mathcal{F}_1^{pq,\varepsilon}(x) = A^\varepsilon(I^{pq} + e(n^{pq,\varepsilon})), \quad I^{pq} = \frac{1}{2}(e_p \otimes e_q + e_q \otimes e_p),
\]

\[
\mathcal{F}_2^{pq,\varepsilon}(x) = B^\varepsilon(I^{pq} + e(n^{pq,\varepsilon}) - e(m^{pq,\varepsilon})),
\]

\[
\mathcal{F}_3^{pq,\varepsilon}(t, x) = A^\varepsilon e(m_{pq,\varepsilon}^{pq}) - B^\varepsilon e(m_{pq,\varepsilon}^{pq}),
\]

(6.4)

\[
g_1^\varepsilon = \frac{1}{2} \left( n^{pq,\varepsilon} \otimes \nabla e(w)_{pq} + (n^{pq,\varepsilon} \otimes \nabla e(w)_{pq})^T \right),
\]

(6.5)

\[
g_2^\varepsilon = \frac{1}{2} \int_t^T \left( m^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w)_{pq}(\tau) + (m^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w)_{pq}(\tau))^T \right) \, d\tau.
\]
Next we show that the only terms in (6.3) that contribute to the effective stress are the terms containing $F_{pq}^{\varepsilon}$. 

**Proposition 6.3.** Let $w^\varepsilon$ be as is Definition 5.1. Then

\[
\lim_{\varepsilon \to 0} \int_I \int_U e(u^\varepsilon) \cdot [A^\varepsilon (g_1^\varepsilon + g_2^\varepsilon) - B^\varepsilon (\partial_t g_1^\varepsilon + \partial_x g_2^\varepsilon)] \, dx \, dt = 0,
\]

\[
\lim_{\varepsilon \to 0} \int_I \int_U e(u^\varepsilon) \cdot [A^\varepsilon e(\nabla \phi^\varepsilon) - B^\varepsilon e(\nabla \phi^\varepsilon)] \, dx \, dt = 0.
\]

**Proof of the proposition.** Since $e(u^\varepsilon)$ converges to $e(\bar{u})$ weakly in $L^2(I_T \times U)$, it is enough to show that all terms in brackets in (6.6), (6.7) converge to zero strongly in $L^2(I_T \times U)$.

**Step 1. Prove (6.6).**

By Proposition 4.8 $m^{pq, \varepsilon}$ converges to zero strongly in $L^2(I_T \times U)$. Therefore, $g_1^\varepsilon, \partial_t g_1^\varepsilon$ in (6.3) converge to zero strongly in $L^2(I_T \times U)$. By Proposition 4.9 $m^{pq, \varepsilon}$ converge to zero strongly in $L^2(I_T \times U)$. Hence, $g_2^\varepsilon$ in (7.3) converges to zero strongly in $L^2(I_T \times U)$. Next, differentiate $g_2^\varepsilon$ in $t$ and integrate by parts in the time convolution exactly as in the proof of Proposition 6.2. Then

\[
\partial_t g_2^\varepsilon = \frac{1}{2} \int_t^T \left( m^{pq, \varepsilon}(t - \tau + T) \otimes \nabla e(w(t)) + (m^{pq, \varepsilon}(t - \tau + T) \otimes \nabla e(w(t))) \right) \, d\tau.
\]

which converges to zero strongly in $L^2(I_T \times U)$. Next, since $A^\varepsilon, B^\varepsilon$ are bounded pointwise independent of $\varepsilon$, we deduce that

\[
A^\varepsilon (g_1^\varepsilon + g_2^\varepsilon) - B^\varepsilon (\partial_t g_1^\varepsilon + \partial_x g_2^\varepsilon)
\]

converges to zero strongly in $L^2(I_T \times U)$.

**Step 2. Prove (6.7).**

From (5.5) we have $e(\nabla \phi^\varepsilon) \to 0$, and by (5.8), $e(\nabla \phi^\varepsilon) \to 0$ strongly $L^2(I_T \times U)$. Hence, $A^\varepsilon e(\nabla \phi^\varepsilon), B^\varepsilon e(\nabla \phi^\varepsilon)$ also converge to zero strongly in $L^2(I_T \times U)$.

By Proposition 6.3

\[
\int_I \int_U \mathbf{T} \cdot e(w) \, dx \, dt = \lim_{\varepsilon \to 0} \int_I \int_U \mathbf{T} \cdot e(w^\varepsilon) \, dx \, dt = \lim_{\varepsilon \to 0} I(u^\varepsilon, w^\varepsilon),
\]

where

\[
I(u^\varepsilon, w^\varepsilon) = \int_I \int_U e(u^\varepsilon) \cdot \left( F_1^{pq, \varepsilon} w + F_2^{pq, \varepsilon} w(t) + \int_t^T F_3^{pq, \varepsilon}(t - \tau + T) e(w(t)) \, d\tau \right) \, dx \, dt.
\]

**Proposition 6.4.** There exist the effective tensors $\mathbf{A} \in L^2(U), \mathbf{B} \in L^2(U)$ and $\mathbf{C} \in L^2(I_T \times U)$ such that, up to extraction of a subsequence,

\[
\lim_{\varepsilon \to 0} I(u^\varepsilon, w^\varepsilon) = -\langle \mathbf{u}, \nabla \mathbf{F}^{pq, \varepsilon} \rangle + \langle \mathbf{F}^{pq, \varepsilon}_2 w(t) \rangle - \langle \mathbf{u}, \nabla \mathbf{F}^{pq, \varepsilon}_3 \rangle + \int_0^T (\mathbf{C}(t - \tau, \cdot) e(w(t)) \, d\tau) \cdot e(w) \, dx \, dt,
\]

**Proof of the proposition.** First, we note that $F_{j}^{pq, \varepsilon}, P_{j}^{pq, \varepsilon}, j = 1, 2$ are bounded in $L^2(U)$ independent of $\varepsilon$. Therefore, extracting weakly convergent subsequences, not relabeled, we have that $F_{j}^{pq, \varepsilon} \to F_{j}^{pq}, P_{j}^{pq, \varepsilon} \to P_{j}^{pq},$ and passing to the limit in (1.22) and (1.23) we obtain

\[
\text{div} F_{j}^{pq, \varepsilon} - \nabla P_{j}^{pq, \varepsilon} = f_j = \text{div} F_{j}^{pq} - \nabla P_{j}^{pq}.
\]

and thus

\[
\text{div} F_{j}^{pq, \varepsilon} = \nabla \left( P_{j}^{pq, \varepsilon} - P_{j}^{pq} \right) + \text{div} F_{j}^{pq}, \ j = 1, 2.
\]
Similarly, \( \mathcal{F}^{pq,\varepsilon}_3, P^{pq,\varepsilon} \) are bounded in \( L^2(I_T \times U) \) independent of \( \varepsilon \). Therefore, extracting weakly convergent subsequences as before, we obtain

\[
\text{div}\mathcal{F}^{pq,\varepsilon}_3 = \nabla \left( P^{pq,\varepsilon}_3 - \mathcal{T}^{pq}_3 \right) + \text{div}\mathcal{F}^{pq}_3.
\]

From (6.10), (6.11) we deduce

\[
\text{div}\left( \mathcal{F}^{pq,\varepsilon}_1 e(w)_{pq} \right) = \text{div}\left( \mathcal{T}^{pq}_1 e(w)_{pq} \right) + \nabla \left( \left( P^{pq,\varepsilon}_1 - \mathcal{T}^{pq}_1 \right) e(w)_{pq} \right)
\]

and

\[
\text{div}\left( \mathcal{F}^{pq,\varepsilon}_j e(w)_{pq} \right) = \text{div}\left( \mathcal{T}^{pq}_j e(w)_{pq} \right) + \nabla \left( \left( P^{pq,\varepsilon}_j - \mathcal{T}^{pq}_j \right) e(w)_{pq} \right).
\]

Next, since \( \text{div} u^\varepsilon = 0 \),

\[
\langle u^\varepsilon, \nabla \left( P^{pq,\varepsilon}_1 - \mathcal{T}^{pq}_1 \right) e(w)_{pq} \rangle = 0,
\]

\[
\langle u^\varepsilon, \nabla \left( P^{pq,\varepsilon}_2 - \mathcal{T}^{pq}_2 \right) e(w)_{pq} \rangle = 0,
\]

\[
\langle u^\varepsilon, \nabla \int_T \left( P^{pq,\varepsilon}_3 - \mathcal{T}^{pq}_3 \right) e(w)_{pq}(t) \, d\tau \rangle = 0.
\]

Integrating by parts and using (6.12), (6.13) and (6.14) we have

\[
I(u^\varepsilon, w^\varepsilon) = -\langle u^\varepsilon, \text{div}\left( \mathcal{T}^{pq}_1 e(w)_{pq} \right) + \text{div}\left( \mathcal{T}^{pq}_2 e(w)_{pq} \right) \rangle
\]

\[
-\langle u^\varepsilon, \text{div}\left( \int_T \mathcal{T}^{pq}_3 (t - \tau) e(w)_{pq}(\tau) \, d\tau \right) \rangle + \mathcal{R}^\varepsilon
\]

where

\[
\mathcal{R}^\varepsilon = -\langle u^\varepsilon, \left( \mathcal{F}^{pq,\varepsilon}_1 - \mathcal{T}^{pq}_1 \right) \cdot \nabla e(w)_{pq} \rangle - \langle \mathcal{F}^{pq,\varepsilon}_2 - \mathcal{T}^{pq}_2 \rangle \cdot \nabla e(w)_{pq} \rangle
\]

\[
-\langle u^\varepsilon, \int_T \left( \mathcal{F}^{pq,\varepsilon}_3 - \mathcal{T}^{pq}_3 \right) (t - \tau + T) \cdot \nabla e(w)_{pq}(\tau) \rangle \, d\tau \rangle
\]

\[
-\langle u^\varepsilon, \left( P^{pq,\varepsilon}_1 - \mathcal{T}^{pq}_1 \right) \cdot \nabla e(w)_{pq} \rangle - \langle u^\varepsilon, \left( P^{pq,\varepsilon}_2 - \mathcal{T}^{pq}_2 \right) \cdot \nabla e(w)_{pq} \rangle
\]

\[
-\langle u^\varepsilon, \int_T \left( P^{pq,\varepsilon}_3 - \mathcal{T}^{pq}_3 \right) (t - \tau) \cdot \nabla e(w)_{pq}(\tau) \rangle \, d\tau \rangle
\]

since \( u^\varepsilon \in L^2(I_T, H^1_0(U)) \), we can apply Lemma 5.3 which yields \( \lim_{\varepsilon \to 0} \mathcal{R}^\varepsilon = 0 \). Then we have (6.15), where the components of the effective tensors \( \underline{A}(x), \underline{B}(x), \underline{C}(t, x) \) are defined by

\[
\underline{A}_{pqij} = \mathcal{F}^{pq,\varepsilon}_{1,ij}, \quad \underline{B}_{pqij} = \mathcal{F}^{pq,\varepsilon}_{2,ij}, \quad \underline{C}_{pqij} = \mathcal{F}^{pq,\varepsilon}_{3,ij}.
\]

This completes the proof of the theorem (6.1).

**Proof of the main theorem.**

To obtain (7.12), we pass to the limit in (2.13) using Lemma 5.3. Next, insert \( w^\varepsilon = N^\varepsilon w \) into (2.14). The limit of the inertial terms is given in Proposition 5.4 and the limit of the term containing \( T^\varepsilon \cdot e(w^\varepsilon) \) is provided by Theorem 6.1. Together, these results yield (7.12). The divergence-free constraint (7.11) is obtained by straightforward passing to the limit in \( \text{div} u^\varepsilon = 0 \).
7. Fluid-structure interaction

Compared to the previous sections, the main difference now is lack of ellipticity in $A^\varepsilon$. In this section we assume $A^\varepsilon = A_1^\varepsilon \theta_0$. This means that phase one is a Kelvin-Voight viscoelastic material, and phase two is a Newtonian fluid. To deal with degeneration of $A^\varepsilon$ we modify (4.1) as follows.

\begin{equation}
\tag{7.1}
N^\varepsilon w \equiv w^\varepsilon = w + \int_t^T n^{pq,\varepsilon}(t - \tau + T) e(w)_{pq}(\tau) d\tau + \int_t^T m^{pq,\varepsilon}(t - \tau + T) e(w_t)_{pq}(\tau) d\tau + \nabla \phi^\varepsilon.
\end{equation}

Here, $n^{pq,\varepsilon}, m^{pq,\varepsilon}, \phi^\varepsilon$ are as in (4.1), (8.3), respectively. We note, however, that the auxiliary problems for $n^{pq,\varepsilon}, m^{pq,\varepsilon}$ will be different. Differentiating (7.1) we obtain

\begin{equation}
\tag{7.2}
A^\varepsilon e(w^\varepsilon) - B^\varepsilon e(w_t^\varepsilon) = \mathcal{F}_1^{pq,\varepsilon} e(w)_{pq} + (\mathcal{F}_2^{pq,\varepsilon})(w_t)_{pq} + \int_t^T \mathcal{F}_3^{pq,\varepsilon}(t - \tau + T)e(w)_{pq}(\tau) d\tau + \int_t^T \mathcal{F}_4^{pq,\varepsilon}(t - \tau + T)e(w_t)_{pq}(\tau) d\tau
\end{equation}

\begin{equation}
+ A^\varepsilon (g_1^\varepsilon + g_2^\varepsilon) - B^\varepsilon (\partial_t g_1^\varepsilon + \partial_t g_2^\varepsilon) + A^\varepsilon e(\nabla \phi^\varepsilon) - B^\varepsilon e(\nabla \phi_t^\varepsilon),
\end{equation}

where

\begin{align}
\mathcal{F}_1^{pq,\varepsilon} &= A^\varepsilon I^{pq} - B^\varepsilon e(n_{T}^{pq,\varepsilon}), \quad I^{pq} = \frac{1}{2} (e_p \otimes e_q + e_q \otimes e_p), \\
\mathcal{F}_2^{pq,\varepsilon} &= -B^\varepsilon (I^{pq} + e(m_{T}^{pq,\varepsilon})), \\
\mathcal{F}_3^{pq,\varepsilon}(x) &= A^\varepsilon e(n_{pq,\varepsilon}^\varepsilon) - B^\varepsilon e(n_{t}^{pq,\varepsilon}), \\
\mathcal{F}_4^{pq,\varepsilon}(t, x) &= A^\varepsilon e(m_{pq,\varepsilon}^\varepsilon) - B^\varepsilon e(m_{t}^{pq,\varepsilon}),
\end{align}

\begin{equation}
\tag{7.3}
g_1^\varepsilon = \frac{1}{2} \int_t^T \left( n^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w)_{pq}(\tau) + (n^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w)_{pq}(\tau))^{T} \right) d\tau.
\end{equation}

\begin{equation}
\tag{7.4}
g_2^\varepsilon = \frac{1}{2} \int_t^T \left( m^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w_t)_{pq}(\tau) + (m^{pq,\varepsilon}(t - \tau + T) \otimes \nabla e(w_t)_{pq}(\tau))^{T} \right) d\tau.
\end{equation}

In the statements of the following auxiliary problems we drop $p, q$ to simplify notations.

- **First auxiliary problem.** Find $n_T^\varepsilon \in H_0^1(U)$ satisfying

\begin{equation}
\tag{7.5}
\text{div} (A^\varepsilon I^{pq} - B^\varepsilon e(n_{T}^{pq,\varepsilon})) - \nabla P_1^\varepsilon = f_1, \quad \text{div} n_T^\varepsilon = 0,
\end{equation}

- **Second auxiliary problem.** Find $m_T^\varepsilon \in H_0^1(U)$ satisfying

\begin{equation}
\tag{7.6}
- \text{div} (B^\varepsilon (I^{pq} + e(m_{T}^{pq,\varepsilon}))) - \nabla P_2^\varepsilon = f_2, \quad \text{div} m_T^\varepsilon = 0,
\end{equation}

- **Third auxiliary problem.** Find $n^\varepsilon \in W$ satisfying

\begin{equation}
\tag{7.7}
- \text{div} ((A^\varepsilon - B^\varepsilon \partial_t) e(n^\varepsilon)) - \nabla P_3^\varepsilon = f_3, \quad \text{div} n^\varepsilon = 0, \quad n^\varepsilon(T) = n_T^\varepsilon.
\end{equation}

- **Fourth auxiliary problem.** Find $m^\varepsilon \in W$ satisfying

\begin{equation}
\tag{7.8}
- \text{div} ((A^\varepsilon - B^\varepsilon \partial_t) e(m^\varepsilon)) - \nabla P_4^\varepsilon = f_4, \quad \text{div} m^\varepsilon = 0, \quad m^\varepsilon(T) = m_T^\varepsilon.
\end{equation}

Since $B^\varepsilon$ is still elliptic, the problems (7.5), (7.6) can be analyzed exactly as problems in Section 4.2. All the results in that section apply without change. The problems (7.7), (7.8) were dealt with in Section 4.1. The most important condition is still ellipticity of $B^\varepsilon$, and the only change that is needed is in the proof of (i) in Proposition 4.2 where we used ellipticity of $A^\varepsilon$. Now, to prove (i) note that

\begin{equation}
\tag{7.9}
\int_t^T \int_U A^\varepsilon e(u) : e(u) dxd\tau + \frac{1}{2} \int_U B^\varepsilon e(u) : e(u) dx(t) = \int_t^T \langle f, u \rangle_{H^{-1}(U), H_0^1(U)}(\tau) d\tau
\end{equation}
Theorem 7.1. (7.11) \( \int_t^T \langle f, u \rangle_{H^{-1}(U), H_0^1(U)}(\tau) d\tau \leq \int_t^T \| f \|_{H^{-1}(U)}(\tau) \| u \|_{H_0^1(U)}(\tau) d\tau \)

\[
\leq \left( \int_t^T \| f \|_{H^{-1}(U)}(\tau) d\tau \right)^{1/2} \left( \int_t^T \| u \|_{H_0^1(U)}^2(\tau) d\tau \right)^{1/2} 
\leq \| f \|_{L^2(I_T, H^{-1}(U))} \| u \|_{L^2(I_T, H_0^1(U))}
\]

Since the first term in the left hand side of (7.9) is non-negative, from (7.9), (7.10) we deduce using ellipticity of \( B^2 \)

\[
\frac{1}{2} \beta_1 \sup_{t \in I_T} \int_U e(u) \cdot e(u) dx(t) \leq \| f \|_{L^2(I_T, H^{-1}(U))} \| u \|_{L^2(I_T, H_0^1(U))}.
\]

This yields (i) (with a different constant \( \frac{1}{2} \beta_1 T^{-1} \)) after observing that

\[
\| u \|_{L^2(I_T, H_0^1(U))}^2 = \int_0^T \int_U e(u) \cdot e(u) dx \leq T \sup_{t \in I_T} \int_U e(u) \cdot e(u) dx(t).
\]

All the arguments in Sections 5, 6 apply with minor changes due to the presence of four fluxes \( F_j \), \( j = 1, \ldots, 4 \), instead of three. The result can be summarized as a theorem.

**Theorem 7.1.** In the case if fluid-structure interaction, the limits \( \overline{p}, \overline{\nu}, \overline{\mu} \) satisfy

\[
(7.11) \quad \text{div} \overline{\nu} = 0,
\]

and the integral identities

\[
(7.12) \quad \int_U \overline{p}_0 \phi(0, x) dx - \int_{I_T} \int_U \overline{p} \partial_t \phi dx dt - \int_{I_T} \int_U \overline{\nu} \cdot \nabla \phi dx dt = 0,
\]

\[
(7.13) \quad \int_U \overline{p}_0 \psi dx - \int_{I_T} \int_U \overline{\nu} \cdot \partial_t \psi dx dt - \int_{I_T} \int_U \overline{\mu} \otimes \nabla \psi dx dt + \int_{I_T} \int_U \overline{T} \cdot e(\psi) dx dt = 0.
\]

for all smooth test functions \( \phi, \psi, \) such that\( \text{div} \psi = 0, \) and \( \phi, \psi \) are equal to zero on \( \partial U \) and vanish for \( t \geq T \).

Moreover, there exist the effective tensors \( \overline{A} \in L^2(U), \overline{B} \in L^2(U) \) and \( \overline{C} \in L^2(I_T \times U), \overline{D} \in L^2(I_T \times U) \) such that the effective deviatoric stress \( \overline{T} \) satisfies

\[
(7.14) \quad \overline{T} = \overline{A} e(\overline{u}) + \overline{B} e(\overline{v}) + \int_0^t \overline{C}(t - \tau) e(\overline{w})(\tau) d\tau + \int_0^t \overline{D}(t - \tau) e(\nu)(\tau) d\tau
\]

Remark. The effective tensors are obtained as weak \( L^2(I_T \times U) \) limits of the four fluxes:

\[
(7.15) \quad \overline{A}_{pqij} = \overline{F}_{pqij}^{1,i,j}, \quad \overline{B}_{pqij} = \overline{F}_{pqij}^{2,i,j}, \quad \overline{C}_{pqij} = \overline{F}_{pqij}^{3,i,j}, \quad \overline{D}_{pqij} = \overline{F}_{pqij}^{4,i,j}.
\]

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APPENDIX A. MOVING INTERFACE IN THE INERTIAL TERMS AND FROZEN INTERFACE IN THE
CONSTITUTIVE EQUATIONS

In this section we present formal calculations leading to the weak formulation of the momentum balance equation.

1. Inertial terms.

\begin{equation}
\int_{I_T} \int_{V^*} \partial_t (\rho^\varepsilon v^\varepsilon) \cdot \psi \, dx \, dt = \int_{I_T} \int_{U} \theta^\varepsilon \partial_t (\rho^\varepsilon v^\varepsilon) \cdot \psi \, dx \, dt = - \int_{I_T} \int_{U} \theta^\varepsilon (\rho^\varepsilon v^\varepsilon) \cdot \partial_t \psi \, dx \, dt - \int_{I_T} \int_{U} (\rho^\varepsilon v^\varepsilon \cdot \psi) \partial_t \theta^\varepsilon \, dx \, dt - \int_{I_T} \rho^1 \theta_0 v_0 \cdot \psi (0, x) \, dx \, dt.
\end{equation}

\begin{equation}
\int_{I_T} \int_{V^*} \partial_j (\rho^\varepsilon v^\varepsilon_i v^\varepsilon_j) \psi_i \, dx \, dt = \int_{I_T} \int_{U} \theta^\varepsilon \partial_j (\rho^\varepsilon v^\varepsilon_i v^\varepsilon_j) \psi_i \, dx \, dt = - \int_{I_T} \int_{U} \theta^\varepsilon \rho^\varepsilon v^\varepsilon_i v^\varepsilon_j \partial_j \psi_i \, dx \, dt - \int_{I_T} \int_{U} (\rho^\varepsilon v^\varepsilon \cdot \psi)(v^\varepsilon \cdot \nabla \theta^\varepsilon) \, dx \, dt.
\end{equation}

Combining (A.1) and (A.2) we obtain

\begin{equation}
\int_{I_T} \int_{V^*} [\partial_t (\rho^\varepsilon v^\varepsilon) + \text{div}(\rho^\varepsilon v \otimes v)] \cdot \psi \, dx \, dt = - \int_{I_T} \rho^1 \theta_0 v_0 \cdot \psi (0, x) \, dx \, dt - \int_{I_T} \int_{U} \theta^\varepsilon (\rho^\varepsilon v^\varepsilon) \cdot \partial_t \psi \, dx \, dt - \int_{I_T} \int_{U} (\rho^\varepsilon v^\varepsilon \cdot \psi) \partial_t \theta^\varepsilon \, dx \, dt - \int_{I_T} \int_{U} (\rho^\varepsilon v^\varepsilon \cdot \psi)(v^\varepsilon \cdot \nabla \theta^\varepsilon) \, dx \, dt.
\end{equation}

When \( \theta^\varepsilon \) satisfies the interface evolution equation, the last term in the right hand side is zero. If the interface were frozen, then this term would be present in the weak formulation of the momentum equation. Since \( \theta_0 \) is piecewise constant, the weak formulation would contain a non-physical term supported on the interface.

To avoid such non-physical terms, one needs to use \( \theta^\varepsilon \) in the inertial terms of the momentum equation.

2. Constitutive equation. Moving interface assumption combined with the Hook’s law would lead to a non-physical dissipation of the elastic energy. Indeed, let the elastic part of the stress be written as

\begin{equation}
\theta_0^\varepsilon (x) A^1 e(u^\varepsilon) + (1 - \theta_0^\varepsilon (x)) A^2 e(u^\varepsilon),
\end{equation}

The important condition here is that \( \theta_0 \) is independent of \( t \). The stiffness tensors \( A^1, A^2 \) of the phases are supposed to be constant. Formally multiplying (A.4) by \( v^\varepsilon \) and integrating by parts we obtain

\begin{equation}
\int_0^T \int_U (\theta_0 A^1 + (1 - \theta_0) A^2) e(u^\varepsilon) \cdot e(\partial_t u^\varepsilon) \, dx \, dt = \frac{1}{2} \int_0^T \int_U \partial_t \left[ (\theta_0 A^1 + (1 - \theta_0) A^2) e(u^\varepsilon) \cdot e(u^\varepsilon) \right] \, dx \, dt.
\end{equation}

This expresses the fact that the elastic energy changes by the amount of work done by elastic forces, with no dissipation. If one were to use \( \theta^\varepsilon (t, x) \) in (A.5), differentiation in time would not commute with multiplication by \( \theta^\varepsilon A^1 + (1 - \theta^\varepsilon) A^2 \), and (A.6) could not be obtained.

APPENDIX B. EXISTENCE OF WEAK SOLUTIONS. OUTLINE OF THE PROOF.

In this Section we outline the proof of existence of global weak solutions for the system (2.13), (2.14) for each fixed \( \varepsilon > 0 \). Since \( \varepsilon \) is fixed, we drop superscript to simplify the notation. We follow closely [18], Sect. 2.3, 2.4.

1. The initial conditions. The initial conditions satisfy (2.15), (2.16).
2. Formal a priori estimates \([2.20]–[2.25]\) are obtained as explained in Sect. 2.2. In particular, renormalization as in \([18]\), Sect. 2.3 is used to get \(|\{x \in U : \alpha \leq \rho(t,x) \leq \beta\}|\) for each \(0 \leq \alpha \leq \beta < \infty\), where \(|\cdot|\) denotes Lebesgue measure. This implies \([2.20]\).

3. Compactness results. Since we wish to approximate exact solutions, a compactness result is needed. Suppose that we have two sequences \(\rho^n, v^n\) satisfying the conditions \(0 \leq \rho^n \leq C, \text{div} \, v^n = 0\), a. e. on \(I_T \times U\), \(\|v^n\|_{L^2(I_T, H^1_0(U))} \leq C, v^n \rightarrow v\) weakly in \(L^2(I_T, H^1_0(U))\). Moreover, assume that

\[
\partial_t \rho^n + \text{div}(\rho^n v^n) = 0
\]

in \(D'(I_T \times U)\), \(\rho^n|v^n|^2\) is bounded in \(L^\infty(I_T, L^1(U))\), and we have

\[
|(\partial_t (\rho^n v^n), \phi)| \leq C \|\phi\|_{L^q(I_T, W^{m,q}(U))}
\]

for all \(\phi \in L^q(I_T, W^{m,q}(U))\) such that \(\text{div} \, \phi = 0\).

Finally suppose that the initial conditions for the density \(\rho_0^n\) satisfy

\[
\rho_0^n \rightarrow \rho_0
\]

in \(L^1(U)\).

Then, by the compactness Theorem 2.4 in \([18]\) (it applies without any change), we have

\[
\rho^n \rightarrow \rho \text{ in } C([0,T], L^p(U)), \text{ for all } 1 \leq p < \infty,
\]

\[
\sqrt{\rho^n} \nu^n \rightarrow \sqrt{\rho} \nu \text{ in } L^p(I_T, L^r(U)), \text{ for } 2 < p < \infty, 1 \leq r \leq \frac{6p}{3p-4},
\]

\[
v^n \rightarrow v_i \text{ in } L^\theta(I_T, L^{3\theta}(U)), \text{ for } 1 \leq \theta < 2 \text{ on the set } \{\rho > 0\},
\]

4. Construction of smooth approximate solutions. As in \([18]\), Sect. 2.4, construct solutions \((\rho, v)\) of the approximate system

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0 \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) - \text{div} \, (A_j e(u) + B_j e(v)) + \nabla P_d &= 0 \text{ in } D',
\end{align*}
\]

div \(v = 0\) in \(D'\),

where \(v_d, A_j, B_j\), are smooth regularizations of the respective quantities. The initial conditions are regularizations of the original ones. Then, using a fixed point argument as in Theorem 2.6, we can prove existence of smooth solutions to \([B.1], [B.2]\). Existence of a fixed point follows from the a priori estimates. The only issue that needs to be explained here is bootstrap regularity of the constructed solutions. The following proposition replaces Proposition 2.1 in \([18]\).

**Proposition B.1.** Consider the system

\[
\begin{align*}
\partial_t u_i + b \cdot \nabla u_i - a \Delta u_i - m \Delta u_i + \partial_i P &= 0, \\
\text{div} \, v &= 0 \text{ in } I_T \times U, i = 1, 2, 3, \text{ with the initial conditions}
\end{align*}
\]

\[
\begin{align*}
u(0,\cdot) &= v_0, \quad u(0,\cdot) = 0.
\end{align*}
\]

Suppose that \(c \in L^\infty(I_T \times U), \ a, m \in L^\infty(I_T, W^{1,\infty}(U)), \ b \in L^2(I_T, L^\infty(U)), \ c \geq k, \ a \geq k, \ m \geq k \text{ a.e. on } I_T \times U \text{ for some } k > 0; v_0 \in H^1_0(U). \) Also, assume that \(a, m\) are independent of \(t\).

Then the system \([B.3], [B.4]\) has a unique solution \((v, P)\) such that \(v \in L^2(I_T, H^2(U)) \cap C([0,T], H^1_0(U)), \partial_t v \in L^2(I_T \times U), \nabla P \in L^2(I_T \times U)\).

**Outline of the proof.** Compared to the proof of Proposition 2.1 in \([18]\), we have one new term \(m \Delta u_i\). Multiplying this term by \(v_i\) and integrating by parts we have

\[- \int_U m \Delta u_i \partial_i v_i = d_i \int_U \nabla u_i \cdot m \nabla v_i - \int_U m |\nabla v_i|^2 + \int_U \nabla u_i \cdot \nabla m \partial_i v_i\]

Multiplying \([B.3]\) by \(\partial_i v_i\) using the above identity, integrating by parts and summing over \(i\) we find

\[
\int_U c |\partial_i v|^2 + \frac{1}{2} d_i \int_U a |\nabla v|^2 + d_i \int_U \nabla u \cdot m \nabla v - \int_U m |\nabla v|^2 = \int_U f,
\]
where \( f = b \cdot \nabla v \partial_t v - \nabla u \cdot (\nabla m \odot \partial_t v) \). Integrating this identity with respect to \( t \) we obtain
\[
k \int_0^t \int_U |\partial_t v|^2 + \frac{k}{2} \int_U |\nabla v|^2(t) \leq \int_U m|\nabla u||\nabla v|(t) + \int_0^t \int_U m|\nabla v|^2 + \int_0^t \int_U f + \frac{k}{2} \int_U a|\nabla v|(0).
\]
Next we write
\[
\int_U m|\nabla u||\nabla v|(t) \leq m \|L^\infty(U)\left(\frac{1}{2} \nu \int_U |\nabla v|^2(t) + \frac{1}{2\nu} \int_U |\nabla u|^2(t)\right),
\]
where we choose \( \nu = k/2 \). The term containing \( f \) is handled similarly, putting \( \nu \) front of \( \int_0^t \int_U |\partial_t v|^2 \).
Combining the previous two inequalities with the standard a priori bounds on \( \nabla u, \nabla v \) we have
\[
k \int_0^t \int_U |\partial_t v|^2 + \frac{k}{4} \int_U |\nabla v|^2(t) \leq C_k,
\]
where \( C_k \) depends only on \( k \) and the data. This yields a priori estimates on \( v \) in \( L^\infty(I_T, H_0^1(U)) \) and on \( \partial_t v \) in \( L^2(I_T \times U) \).

Now we can write (B.3) as
\[
\Delta (av + mu) - \nabla P = h, \ \text{div} \ v = 0, \ \text{in} \ U,
\]
\( v \in H_0^1(U), u \in H_0^1(U) \) for almost all \( t \in I_T \). Also, \( h \) is bounded in \( L^2(I_t \times U) \) in terms of the data. Next, estimating pressure \( P \) exactly as in [18], Prop. 2.1, we conclude that \( av + mu \in L^2(I_T, H^2(U)) \). Since \( a, m \) are smooth and positive, this implies
\[
\partial_t u + \frac{m}{a} u = g,
\]
where \( g \in L^2(I_T, H^2(U)) \). Now, formally, \( u(t, \cdot) = \int_0^t e^{-\frac{\pi}{\nu}(t-\tau)} g(\tau, \cdot) d\tau \), and hence
\[
\left| \partial_{x_i x_j}^2 u(t) \right|^2 \leq \left( \int_0^t \left| \partial_{x_i x_j}^2 u(t) \right|^2 d\tau \right) \leq t \int_0^t \left| \partial_{x_i x_j}^2 g(\tau) \right|^2 d\tau \leq T \int_0^T \left| \partial_{x_i x_j}^2 g(\tau) \right|^2 d\tau.
\]
Integrating over \( U \) we deduce that \( u \) is bounded in \( L^\infty(I_T, H^2(U)) \) in terms of the data. Then using \( v = -\frac{m}{a} u + g \) we obtain that \( v \) is bounded in \( L^2(I_T, H^2(U)) \) in terms of the data. The formal calculations can be easily justified by an approximation argument.

5. Passage to the limit. This is done using compactness from step 3 exactly as in [18].

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