ON THE CONVERGENCE TO THE CONTINUUM OF FINITE RANGE LATTICE COVARIANCES

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Abstract: In J Stat Phys. 115, 415-449 (2004) Brydges, Guadagni and Mitter proved the existence of multiscale expansions of a class of lattice Green’s functions as sums of positive definite finite range functions (called fluctuation covariances). The lattice Green’s functions in the class considered are integral kernels of inverses of second order positive self-adjoint elliptic operators with constant coefficients and fractional powers thereof. The fluctuation covariances satisfy uniform bounds and the sequence converges in appropriate norms to a smooth, positive definite, finite range continuum function. In this note we prove that the convergence is actually exponentially fast.

1. Introduction

In [BGM], Brydges, Guadagni and Mitter proved the existence of multiscale expansions of a class of lattice Green’s functions as sums of positive definite finite range functions (fluctuation covariances). The lattice Green’s functions that were considered are integral kernels of inverses of self-adjoint lattice elliptic operators. The construction in [BGM] was given 1) for the resolvent operator \((a - \Delta)^{-1}\) with \(a \geq 0\) on \(\mathbb{Z}^d\) where \(\Delta\) is the standard lattice Laplacian, the resolvent parameter \(a \geq 0\), and \(d \geq 3\) and 2) for the Lévy Green’s function \((-\Delta)^{-\alpha/2}\) on \(\mathbb{Z}^d\) with \(d \geq 3\), and \(0 < \alpha < 2\). This has been extended in [BT] to Green’s functions of more general self-adjoint elliptic operators. The summands, called fluctuation covariances, after rescaling live on finer and finer lattices, have uniformly bounded support (finite range property) and it was proved in [BGM] that their Fourier transforms satisfy bounds independent of lattice spacing and have strong decay properties. It was also proved that the sequence of rescaled fluctuation covariances converge in appropriate norms to a smooth positive definite finite range continuum function. In the
present note (which is a sequel to the above paper and should be read as such) we prove that the convergence is exponentially fast. This is of some importance for renormalization group applications. An example is furnished in the work of Mitter and Scoppola, [MS]. The exponential convergence is stated in Theorem 1.1, page 931 of [MS] and then used in the construction of the stable manifold in section 6 of that paper. The present work furnishes the promised proof of that result. Another application is in the forthcoming work of R. Bauerschmidt (in preparation) where, amongst other things, exponential convergence is extended to the mass derivative of the finite range expansion. Applications of finite range multiscale expansions in rigorous renormalization group analysis include the work of Brydges and Slade on weakly self-avoiding simple random walks in \( \mathbb{Z}^d, d \geq 4 \) [BS], a new proof of the thermodynamic limit for the dipole gas by Dimock [D], the critical line in the Kosterlitz-Thouless transition by P. Falco [F] and forthcoming work by Stefan Adams et al. on gradient models.

2. Summary of earlier results and main theorem

In this section we will first summarize the results in [BGM] to the extent we will need them in order to be able to state our basic convergence estimates (Theorem 2.1, Corollaries 2.2 and 2.3). The proof of Theorem 2.1 will be given in section 3. Throughout this paper we will use the notations, definitions and results given in [BGM]. Let \( L = 2^p \) be a dyadic integer. \( L \) must be chosen sufficiently large depending on the dimension \( d \) and on the rate of decay given by the parameter \( k \) that appears in all our estimates starting with (1.13).

It is assumed to be large. We define \( \varepsilon_n = L^{-n}, n \geq 0 \). For \( n = 0, 1, 2, \ldots \) we have a sequence of lattices \( (\varepsilon_n \mathbb{Z})^d \subset \mathbb{R}^d \) which are nested, \( (\varepsilon_n \mathbb{Z})^d \subset (\varepsilon_{n+1} \mathbb{Z})^d \). We assume \( d \geq 3 \).

Let \( \Delta_{\varepsilon_n} \) be the lattice Laplacian acting on functions on \( (\varepsilon_n \mathbb{Z})^d \). For \( a \geq 0 \) the resolvent \( G_{\varepsilon_n}^a = (-\Delta_{\varepsilon_n} + a)^{-1} \) has the Fourier transform

\[
G_{\varepsilon_n}^a(x-y) = \int_{B_{\varepsilon_n}} \frac{d^d p}{(2\pi)^d} e^{i p \cdot (x-y)} (a - \hat{\Delta}_{\varepsilon_n}(p))^{-1}
\]

where \( B_{\varepsilon_n} = [\frac{-\pi}{\varepsilon_n}, \frac{\pi}{\varepsilon_n}]^d \) and

\[
\hat{\Delta}_{\varepsilon_n}(p) = 2\varepsilon_n^{-2} \sum_{\mu=1}^{d} (\cos(\varepsilon_n p_\mu) - 1).
\]

Let \( U_c(R_m) = (-\frac{R_m}{2}, \frac{R_m}{2})^d \subset (\mathbb{R})^d \) denote a continuum cube of edge length \( R_m = L^{-(m-1)} \). Here \( m = 0, 1, 2, \ldots, n \). Then \( U_{\varepsilon_n}(R_m) = U_c(R_m) \cap (\varepsilon_n \mathbb{Z})^d \) defines a cube in the lattice \( (\varepsilon_n \mathbb{Z})^d \). The boundary \( \partial U_{\varepsilon_n}(R_m) \) is defined to consist of lattice points not in \( U_{\varepsilon_n}(R_m) \) which have a nearest neighbour in \( U_{\varepsilon_n}(R_m) \).
Remark: The choice \( L = 2^p \), in particular that \( L \) is even, implies that the boundary of the continuum cube passes through lattice points. Therefore the boundary \( \partial U_{\varepsilon_n}(R_m) \) of the lattice cube is contained in the boundary \( \partial U_c(R_m) \) of the continuum cube. This is used in the proof of Lemma 6.5 of [BGM]. If one prefers, for example \( L = 3^p \), then replacing \( R/2 \) by \( R/3 \) in the definition of \( U_c(R_m) \) retains this boundary property.

A measure on the lattice is just a weighted sum of point masses at lattice points, but it facilitates comparison with the continuum to write sums as integrals over such measures. \( P^a_{\partial U_{\varepsilon_n}(R_m)}(x, du) \) denotes the Poisson kernel measure on \( \partial U_{\varepsilon_n}(R_m) \). By definition this is such that if \( f \) is a function on \( \partial U_{\varepsilon_n}(R_m) \) then

\[
h_{\varepsilon_n,m}(x) = P^a_{\partial U_{\varepsilon_n}(R_m)}(x, f) = \int_{\partial U_{\varepsilon_n}(R_m)} P^a_{\partial U_{\varepsilon_n}(R_m)}(x, du)f(u)
\]

solves the Dirichlet problem

\[
(-\Delta_{\varepsilon_n} + 1)h_{\varepsilon_n,m}(x) = 0 : \quad x \in U_{\varepsilon_n}(R_m)
\]  

\[
h_{\varepsilon_n,m}(x) = f(x) : \quad x \in \partial U_{\varepsilon_n}(R_m).
\]

The Poisson kernel measure exists and a probabilistic representation was given (and exploited) in [BGM]. For \( a = 0 \), the Poisson kernel measure is a probability measure, otherwise \( (a > 0) \) it is a defective measure (total mass is less than 1). In [BGM] an averaging map \( f \to A^a_{\varepsilon_n,m}(R_m)f \) was introduced for functions \( f \) defined on \((\varepsilon_n\mathbb{Z})^d\). This uses the Poisson kernel measure above. In the next paragraph we recall the definition of the averaging operation and then the fluctuation measures which enter in the finite range multiscale expansion of Green’s functions established in [BGM].

Let \( g \) be a non-negative, rotationally invariant, \( C^\infty \) function on \( \mathbb{R}^d \) of finite range \( \frac{L}{4} \). In other words \( g(x) \) vanishes for \( |x| \geq \frac{L}{4} \). \( g \) is chosen to be normalized so that \( \int_{\mathbb{R}^d} dx \, g(x) = 1 \). Define the sequence of functions \( g_n \) by \( g_n(x) = L^{nd}g(L^nx) \) for \( n = 0, 1, 2,... \). Then the functions \( g_n \) have mass 1 and finite range \( \frac{L}{4}L^{-(n-1)} \). Restrict \( g \) to the lattice \((\varepsilon_n\mathbb{Z})^d\) and let \( c_{\varepsilon_n} \) be the positive constant so that \( c_{\varepsilon_n} \int_{(\varepsilon_n\mathbb{Z})^d} dx \, g(x) = 1 \). Here the integration is with respect to the “Lebesgue” measure on the lattice \((\varepsilon_n\mathbb{Z})^d\) i.e. the counting measure times \( \varepsilon_n^d \). The constants \( c_{\varepsilon_n} \) converge to 1 as \( n \to \infty \). We have \( \int_{(\varepsilon_n\mathbb{Z})^d} dx \, c_{\varepsilon_n-m}g_m(x) = \int_{(\varepsilon_{n-m}\mathbb{Z})^d} dx \, c_{\varepsilon_n-m}g(x) = 1 \). Let \( f \) be a function on \((\varepsilon_n\mathbb{Z})^d\). For \( m = 0, 1, ..., n \) we define a sequence of (averaging) maps \( f \to A^a_{\varepsilon_n,m}(R_m)f \) and their kernels \( A^a_{\varepsilon_n,m}(R_m)(x,u) \) by

\[
(A^a_{\varepsilon_n,m}(R_m)f)(x) = \int_{(\varepsilon_n\mathbb{Z})^d} dz \, c_{\varepsilon_n-m}g_m(z-x)P^a_{\partial U_{\varepsilon_n}(R_m)}(x-z, f(z+\cdot))
\]

\[
= \int_{(\varepsilon_n\mathbb{Z})^d} du \, A^a_{\varepsilon_n,m}(R_m)(x,u)f(u)
\]  

where (see [BGM], page 423-424) \( A^a_{\varepsilon_n,m}(R_m)(x,u) du \) is a family of translation invariant (defective) probability measures on \((\varepsilon_n\mathbb{Z})^d\). The support property of \( g_m \) makes sure that
the Poisson kernel entering above is never evaluated on \( x \) near the boundary point \( u \) where derivatives become large.

Consider first the case \( m = 0 \) and recall that \( R_0 = L \). We define a fluctuation covariance

\[
\Gamma^a_{\varepsilon_n}(x - y) = G^a_{\varepsilon_n}(x - y) - (A^a_{\varepsilon_n,0}(R_0)G^a_{\varepsilon_n}A^a_{\varepsilon_n,0}(R_0)^*)(x - y). \tag{2.7}
\]

\( \Gamma^a_{\varepsilon_n} \) is a positive definite function of finite range \( L \) and \( \Gamma^a_{\varepsilon_n}(p) \) is continuous in \( p \) including at \( p = 0 \), (Lemma 3.1, [BGM]). For \( n \geq 1 \) define

\[
\Gamma^a_n = A^a_n \Gamma^a_{\varepsilon_n} A^a_n^* \tag{2.8}
\]

where

\[
A^a_n = \prod_{m=1}^{n} A^a_{\varepsilon_n,m}(R_m) \tag{2.9}
\]

and the product above is given by a multiple convolution. For \( n = 0 \) we set \( A^a_0 = 1 \). \( \Gamma^a_n \) is a positive definite function with finite range \( 6L \), (Lemma 3.2, [BGM]). Let \( G^a =: G^a_{\varepsilon_0} \) be the unit lattice resolvent.

\[
G^a(x - y) = \sum_{n \geq 0} L^{-n(d-2)} \Gamma^a_n \left( \frac{x - y}{L^n} \right) \tag{2.10}
\]

where \( a_n = L^{2n}a \).

**Remark 1:** The factor 6 in the range \( 6L \) of \( \Gamma^a_n \) is an artifact. By scaling down the edge length \( R_m = L^{-(m-1)} \) of the cube \( U_{\varepsilon_n}(R_m) \) to \( R_m = \frac{1}{16} L^{-(m-1)} \) and the range of \( g \) from \( L/4 \) to \( L/64 \) we get \( \Gamma^a_n \) to have finite range (less than) \( L/2 \).

Let \( G = (-\Delta)^{-\frac{\alpha}{2}}, 0 < \alpha < 2 \), be the Green’s function of a Lévy walk in \( \mathbb{Z}^d \). \( G \) has the integral representation \( G = \text{const} \int_0^\infty da \, a^{-\alpha/2} G^a \). Integrating (2.10) with the measure \( da \, a^{-\alpha/2} \) we get the finite-range multiscale expansion for \( G \)

\[
G(x - y) = \sum_{n \geq 0} L^{-n[\varphi]} \Gamma_n \left( \frac{x - y}{L^n} \right) \tag{2.11}
\]

where \( [\varphi] = \frac{d-\alpha}{2} \) and

\[
\Gamma_n = \int_0^\infty da \, a^{-\alpha/2} \Gamma^a_n. \tag{2.12}
\]

These formulae make sense by virtue of the following bound provided in Theorem 5.5, page 434, [BGM]:

Let \( B_{\varepsilon_n} = [\frac{\pi}{\varepsilon_n}, \frac{\pi}{\varepsilon_n}]^d \), the first Brillouin zone of the dual lattice. Then for all \( n \geq 0 \) and all \( k \geq 0 \), there is a constant \( c_{k,L} \) independent of \( n \) such that for \( p \in B_{\varepsilon_n} \),
The continuum Poisson kernel measure and the continuum averaging operation $f \rightarrow \hat{A}_{c,m}^a(R_m)f$ is now defined as in (2.6) using the continuum Poisson kernel measure and the $z$-integration is in $\mathbb{R}^d$. The continuum fluctuation covariances $\hat{\Gamma}_c^a$, and $\hat{\Gamma}_{c,n}^a$ are defined by the continuum analogues of (2.7)-(2.9). Using Fourier transforms we have

$$\hat{\Gamma}_c^a(p) = \hat{G}_c^a(p) - |\hat{A}_{c,0}^a(R_0)(p)|^2\hat{G}_c^a(p)$$

$$\hat{\Gamma}_{c,n}^a(p) = |\hat{A}_{c,n}^a(p)|^2\hat{\Gamma}_c^a(p) = \prod_{m=1}^n |\hat{A}_{c,m}^a(R_m)(p)|^2\hat{\Gamma}_c^a(p).$$

The continuum analogue of the elliptic estimates ( Appendix A, [BGM]) used in the proof of Theorem 5.5 of [BGM] imply that the bounds (2.13),(2.14) continue to hold in $\mathbb{R}^d$ for
\( n \geq 1 \). Thus we have that for all \( n \geq 1 \) and all \( k \geq 0 \) there is a constant \( c_{k,L} \) independent of \( n \) such that

\[
|\hat{\Gamma}_{c,n}(p)| \leq c_{k,L}(1 + a)^{-1}(1 + p^2)^{-2k} \tag{2.19}
\]

and its improvement

\[
|\hat{\Gamma}_{c,n}(p)| \leq c_{k,L}e^{-ca \frac{1}{2}}(1 + p^2)^{-2k} \tag{2.20}
\]

The following statements are proved in Section 6 of [BGM], (see Theorem 6.1 and its proof).

1. **Continuum covariances:** The uniformly bounded sequence \( \{\hat{\Gamma}_{c,n}(p)\}_{n \geq 1} \), (see above), is Cauchy so that pointwise in \( p \),

\[
\hat{\Gamma}_{c,n}(p) \to \hat{\Gamma}_{c,*}(p) \tag{2.21}
\]

and \( \hat{\Gamma}_{c,*}(p) \) satisfies the bounds (2.19), (2.20). \( \hat{\Gamma}_{c,*}(x) \) is in \( H_k(\mathbb{R}^d) \) for all \( k \geq 0 \). Therefore by Sobolev embedding \( \hat{\Gamma}_{c,*}(x) \) is a smooth function.

2. **Lattice covariances:** Pick any integer \( l \geq 1 \) and let \( p \in B_{\varepsilon_l} \). For \( n \geq l \) the sequence \( \{\hat{\Gamma}_n(p)\}_{n \geq 1} \), (see (2.13), (2.14)) converges to the continuum limit function above:

\[
\hat{\Gamma}_n(p) \to \hat{\Gamma}_{c,*}(p). \tag{2.22}
\]

The next theorem, which is our main result, shows that the convergence is exponentially fast. It is stated in terms of a Sobolev space \( L_k^1((\varepsilon_l\mathbb{Z})^d) \) which is discussed below the theorem.

**Theorem 2.1:** Pick any integer \( l \geq d \). Restrict \( \Gamma_{c,*}^a \) to \((\varepsilon_l\mathbb{Z})^d\). Then for all \( n \geq l \) and all \( k \geq 0 \) there is a constant \( c_{k,L} \) independent of \( n \) such that

\[
\|\Gamma_n^a - \Gamma_{c,*}^a\|_{L_k^1((\varepsilon_l\mathbb{Z})^d)} \leq c_{k,L}L^{-\frac{d}{2}}e^{-ca \frac{1}{2}}. \tag{2.23}
\]

**Remark:** Let \( \Omega \subset \mathbb{R}^d \) be an open set. Let \( C_0^\infty(\Omega) \) be the space of \( C^\infty \) functions of compact support in \( \Omega \). Then \( L_k^1(\Omega) \) is the Banach space (also known as \( W_0^{1,k}(\Omega) \)) obtained by completing \( C_0^\infty(\Omega) \) in the norm

\[
\|f\|_{L_k^1(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^1(\Omega)}. \tag{2.24}
\]

Let now \( \Omega \) be a bounded open cube. Then, as is well known, repeated application of the Poincaré inequality gives the equivalent norm (see e.g. [A])

\[
\|f\|_{L_k^1(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^1(\Omega)} \tag{2.25}
\]
The same definitions are adapted to the lattice with integrals and derivatives being replaced by sums and finite differences (forward lattice derivatives). Just as in the continuum the equivalent norm is proved by repeated applications of the lattice Poincaré inequality (proved in Lemma B2 of Appendix B of [BGM]). The $L^1_k$ Sobolev spaces of index larger than $d$ embed into spaces of continuous functions (see e.g. [A]) and the same proof works in the continuum and the lattice. This can be seen in the proof of Lemma B.1 of Appendix B of [BGM] in which the first equation together with the argument in the last four lines of the proof implies, for $k > d + j$, that

$$
\| f \|_{C^j_0(\Omega)} \leq C_{\Omega,j,k} \| f \|_{L^1_k(\Omega)}
$$

(2.26)

where the lattice norm denoted by $C^j_0(\Omega)$ is defined as the supremum over the lattice derivatives of orders up to $j$ of functions of compact support in $\Omega$. We can use the spaces obtained by completing smooth functions of compact support because $\Gamma_a(x), \Gamma_{c,*}(x)$ are of finite range $6L$, i.e. they vanish for $|x| \geq 6L$. The norm in (2.23) can therefore be taken in the finite cube $\Omega_{\varepsilon_n} = U_{\varepsilon_n}(6L)$.

Let $\partial_{\varepsilon_n}^{\alpha} = \prod_{j=1}^d \partial_{\varepsilon_n,e_j}^{\alpha_j}$, $\alpha = (\alpha_1, ..., \alpha_d)$, $\alpha_j$ non-negative integers, denote a multiple $\varepsilon_n$-lattice partial derivative. Here $\partial_{\varepsilon_n,e_j}^{\alpha_j}$ is the forward $\varepsilon_n$-lattice derivative in direction $e_j$. The $e_1, .., e_d$ are unit vectors specifying the orientation of $\mathbb{R}^d$ and all embedded lattices $(\varepsilon_n \mathbb{Z})^d$. Let $\partial_{c}^{\alpha}$ be a multiple continuum partial derivative. Then (2.23) implies by Sobolev embedding of high degree lattice $L^1_k$ spaces:

**Corollary 2.2**: For all $|\alpha| \geq 0$, and for all $n \geq l \geq d$

$$
\| \partial_{\varepsilon_n}^{\alpha} \Gamma_n^a - \partial_{c}^{\alpha} \Gamma_{c,*}^a \|_{L^\infty((\varepsilon_l \mathbb{Z})^d)} \leq c_{k,L} L^{-\frac{n}{2}} e^{-ca^\frac{1}{2}}.
$$

(2.27)

**Proof**: We have

$$
\| \partial_{\varepsilon_n}^{\alpha} \Gamma_n^a - \partial_{c}^{\alpha} \Gamma_{c,*}^a \|_{L^\infty((\varepsilon_l \mathbb{Z})^d)}
\leq \| \partial_{\varepsilon_n}^{\alpha} \Gamma_n^a - \partial_{\varepsilon_n}^{\alpha} \Gamma_{c,*}^a \|_{L^\infty((\varepsilon_l \mathbb{Z})^d)} + \| \partial_{\varepsilon_n}^{\alpha} \Gamma_{c,*}^a - \partial_{c}^{\alpha} \Gamma_{c,*}^a \|_{L^\infty(\mathbb{R}^d)}
$$

(2.28)

By Sobolev embedding followed by Theorem 2.1 with $l = n$ and $k$ sufficiently large, the first term is bounded as required by the right hand side of (2.27) so we now consider the second term. We bound the $L^\infty$ norm by the $L^1$ norm of the Fourier transform. The derivatives $\partial_{\varepsilon_n}^{\alpha} - \partial_{c}^{\alpha}$ give rise to a factor

$$
\left| \prod_{j=1}^d \left( \varepsilon_n^{-1} (e^{i\varepsilon_n k \cdot e_j} - 1) \right)^{\alpha_j} - \prod_{j=1}^d (ik \cdot e_j)^{\alpha_j} \right| \leq \frac{|\alpha|}{2} \varepsilon_n |k|^{|\alpha|+1}
$$

in the $L^1$ norm of the Fourier transform. The desired result then follows from the continuum version of (2.14).
Applying the above to the sequence of Lévy fluctuation covariances \( \{ \Gamma_n \}_n \geq 0 \) we have

**Corollary 2.3:** Under the same conditions as above we have for all \( k \geq 0 \)

\[
\| \Gamma_n - \Gamma_{c,*} \|_{L_k^1((\varepsilon_i\mathbb{Z})^d)} \leq c_{k,L}L^{-\frac{n}{2}}. \tag{2.29}
\]

Moreover for all \( |\alpha| \geq 0 \)

\[
\| \partial_{\alpha} \Gamma_n - \partial_{\alpha} \Gamma_{c,*} \|_{L_\infty((\varepsilon_i\mathbb{Z})^d)} \leq c_{k,L}L^{-\frac{n}{2}}. \tag{2.30}
\]

### 3. Proof of Theorem 2.1

Theorem 2.1 follows by combining the following two lemmas whose proofs are given below. The first is about the convergence of continuum covariances and the second is about lattice covariances.

**Lemma 3.1:** For \( k \geq 0 \), there is a constant \( c_{k,L} \) such that for \( a \geq 0 \) and \( n \geq 2 \)

\[
| \hat{\Gamma}_{c,n}^a (p) - \hat{\Gamma}_{c,*}^a (p) | \leq c_{k,L} e^{-ca^\frac{1}{2}} (1 + p^2)^{-k}L^{-\frac{n}{2}}. \tag{3.1}
\]

**Lemma 3.2:** Let \( p \in B_{\varepsilon_n} \). Then for all \( n \geq d \) and all \( k \geq 0, a \geq 0 \) there is a constant \( c_{k,L} \) independent of \( n \) such that

\[
| \hat{\Gamma}_{n}^a (p) - \hat{\Gamma}_{c,n}^a (p) | \leq c_{k,L} e^{-ca^\frac{1}{2}} (1 + p^2)^{-k} (1 + (bp^2 + a)^{-1})L^{-n} \tag{3.2}
\]

where \( b \) is a positive constant independent of \( n \) and other parameters.

**Proof of Theorem 2.1:** Since Lemmas 2.1 and 2.2 hold for all \( k \geq 0 \) we can replace \( k \) by \( k + d + 1 \). From Lemma 3.1 and Lemma 3.2 we get

\[
| \hat{\Gamma}_{n}^a (p) - \hat{\Gamma}_{c,*}^a (p) | \leq c_{k,L} e^{-ca^\frac{1}{2}} (1 + p^2)^{-(k+d+1)} (1 + (bp^2 + a)^{-1})L^{-\frac{n}{2}} \tag{3.3}
\]

for all \( k \geq 0 \). By definition

\[
\| \Gamma_n^a - \Gamma_{c,*}^a \|_{L_{2k}^1((\varepsilon_i\mathbb{Z})^d)} = \| \Gamma_n^a - \Gamma_{c,*}^a \|_{L_{2k}^1(U_{\varepsilon_1}(6L))} = \sum_{|\alpha|=2k} \int_{U_{\varepsilon_1}(6L)} dx |D^\alpha (\Gamma_n^a (x) - \Gamma_{c,*}^a (x)) |.
\]

After introducing a Fourier transform we get

\[
\| \Gamma_n^a - \Gamma_{c,*}^a \|_{L_{2k}^1((\varepsilon_i\mathbb{Z})^d)} \leq \sum_{|\alpha|=2k} \int_{U_{\varepsilon_1}(6L)} dx \int_{B_{\varepsilon_1}} \frac{dp}{(2\pi)^d} |D^\alpha e^{ipx} | | \hat{\Gamma}^a (p) - \hat{\Gamma}_{c,*}^a (p) |.
\]

Using the definition of lattice derivatives and multiderivatives we get the trivial inequality
\[ |D^\alpha e^{ipx}| \leq \prod_{j=1}^{d} |p_j|^{\alpha_j} \leq (p^2)^{\frac{\left|\alpha\right|}{2}} \]

which we use to majorize the inequality preceding it by

\[ \|\Gamma_n^a - \Gamma_{c,*}^a\|_{L_{2k}^1((\varepsilon,\mathbb{Z})^d)} \leq c_{k,L} \int_{B_{\varepsilon l}} \frac{dp}{(2\pi)^d} (p^2)^k |\hat{\Gamma}^a(p) - \hat{\Gamma}_{c,*}^a(p)| \]  \hspace{1cm} (3.4)

where the constant \(c_{k,L}\) depends on \(L\) through the volume of the cube \(U_{\varepsilon l}(6L)\) and on \(k\) because

\[ \sum_{|\alpha|=2k} 1 = c_k. \]

We majorize the right hand side of (3.4) using the bound (3.3). Note that \(d \geq 3\) so that integrability is assured uniformly in \(l\) for all \(a \geq 0\). We therefore get the bound

\[ \|\Gamma_n^a - \Gamma_{c,*}^a\|_{L_{2k}^1((\varepsilon,\mathbb{Z})^d)} \leq c_{k,L} L^{-\frac{k}{2}} e^{-ca^{1/2}} \]

which proves Theorem 2.1.

**Proof of Lemma 3.1:** We will divide the proof into two cases.

**Case 1.** Suppose \(|p| > L^{n/2}\) or \(a > L^n\).

Recall that \(\hat{\Gamma}_{c,n}^a(p)\) and its pointwise limit \(\hat{\Gamma}_{c,*}^a(p)\) satisfy the uniform bound (2.20). Therefore

\[ |\hat{\Gamma}_{c,n}^a(p) - \hat{\Gamma}_{c,*}^a(p)| \leq |\hat{\Gamma}_{c,n}^a(p)| + |\hat{\Gamma}_{c,*}^a(p)| \leq c_{k,L} e^{-ca^{1/2}} (1 + p^2)^{-2k}. \]  \hspace{1cm} (3.5)

Suppose \(|p| > L^{n/2}\). Then from the above

\[ |\hat{\Gamma}_{c,n}^a(p) - \hat{\Gamma}_{c,*}^a(p)| \leq c_{k,L} L^{-n} e^{-ca^{1/2}} (1 + p^2)^{-k} \]  \hspace{1cm} (3.6)

which gives the desired bound.

Suppose now that \(a > L^n\). Then for any \(O(1)\) constant \(c > 0\)

\[ e^{-ca^{1/2}} \leq e^{-\frac{c}{2}a^{1/2}} e^{-\frac{c}{2}L^{n/2}} \leq \frac{4}{c^2} L^{-n} e^{-\frac{c}{2}a^{1/2}}. \]

Inserting this bound in (3.5) we get with a new constant \(c_{k,L}\) and a new constant \(c\)

\[ |\hat{\Gamma}_{c,n}^a(p) - \hat{\Gamma}_{c,*}^a(p)| \leq c_{k,L} L^{-n} e^{-ca^{1/2}} (1 + p^2)^{-k} \]  \hspace{1cm} (3.7)

as desired.

**Case 2.** This is the converse of Case 1, namely \(|p| \leq L^{n/2}\) and \(a \leq L^n\).
From (2.21) and (2.18) we have

\[ \hat{\Gamma}^a_{c,\ast}(p) = \prod_{m=n+1}^{\infty} |\hat{A}^a_{c,m}(R_m)(p)|^2 \hat{\Gamma}^a_{c,n}(p). \]

Therefore

\[
|\hat{\Gamma}^a_{c,\ast}(p) - \hat{\Gamma}^a_{c,n}(p)| \leq |\hat{\Gamma}^a_{c,n}(p)| \left| \prod_{m=n+1}^{\infty} |\hat{A}^a_{c,m}(R_m)(p)|^2 - 1 \right|
\]

\[
\leq c_k L e^{-ca \frac{1}{2}} (1 + p^2)^{-2k} \prod_{m=n+1}^{\infty} |\hat{A}^a_{c,m}(R_m)(p)|^2 - 1
\]  

(3.8)

where we have used the bound (2.19). From the continuum version of the estimate (6.17) on page 442 of [BGM] we have

\[
|1 - \hat{A}_{c,m}(p)| \leq R_m |p| + aR_m^2
\]

where \( R_m = L^{-(m-1)} \). In the present Case 2 we have \( |p| \leq L^{n/2} \), \( a \leq L^n \), and in (3.8) \( m \geq n + 1 \) with \( n \geq 2 \). It is then easy to see that \( |1 - \hat{A}_{c,m}(p)| \leq 2L^{-\frac{(m-1)}{2}} \).

\[
(1 - 2L^{-\frac{(m-1)}{2}})^2 \leq |\hat{A}_{c,m}(p)|^2 \leq (1 + 2L^{-\frac{(m-1)}{2}})^2.
\]

(3.9)

1. From \( 1 + x \leq e^x \), for \( x \geq 0 \), and (3.9) we have \( |\hat{A}_{c,m}(p)|^2 \leq e^{4L^{-\frac{(m-1)}{2}}} \). Therefore

\[
\prod_{m=n+1}^{\infty} |\hat{A}^a_{c,m}(R_m)(p)|^2 \leq e^{4 \sum_{m=n+1}^{\infty} L^{-\frac{(m-1)}{2}}} \leq e^{O(1)L^{-\frac{n}{2}}}
\]

\[
\leq 1 + O(1)L^{-\frac{n}{2}}
\]

(3.10)

2. It is easy to see from the lower bound in (3.9) that, for \( m \geq n + 1 \), and \( n \geq 2 \) we get

\[
|\hat{A}^a_{c,m}(R_m)(p)|^2 \geq e^{2 \log(1 - 2L^{-\frac{(m-1)}{2}})} = e^{-O(1)L^{-\frac{(m-1)}{2}}}
\]

whence

\[
\prod_{m=n+1}^{\infty} |\hat{A}^a_{c,m}(R_m)(p)|^2 \geq e^{-O(1) \sum_{m=n+1}^{\infty} L^{-\frac{(m-1)}{2}}} = e^{-O(1)L^{-\frac{n}{2}}(1 - L^{-1/2})^{-1}}
\]

\[
\geq e^{-O(1)L^{-\frac{n}{2}}}
\]

For \( x \geq 0 \) and sufficiently small we have \( e^{-x} \geq 1 - 2x \). Therefore we get from the previous inequality

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\[
\prod_{m=n+1}^{\infty} |\hat{A}_{c,m}^a(R_m)(p)|^2 \geq 1 - O(1)L^{-\frac{d}{2}}. \quad (3.11)
\]

From (3.10) and (3.11) we get

\[
\left| \prod_{m=n+1}^{\infty} |\hat{A}_{c,m}^a(R_m)(p)|^2 - 1 \right| \leq O(1)L^{-\frac{d}{2}}. \quad (3.12)
\]

Inserting the bound (3.12) in (3.8) gives

\[
|\hat{\Gamma}^a_{c,*}(p) - \hat{\Gamma}^a_{c,n}(p)| \leq c_{k,L}L^{-\frac{d}{2}}e^{-ca\frac{1}{2}}(1 + p^2)^{-2k}
\]

which completes the proof of Lemma 3.1. \qed

**Proof of Lemma 3.2:**

The proof of Lemma 3.2 reposes crucially on Lemma 6.7, [BGM, page 441], and Claim 2.3 to follow. According to Lemma 6.7 of [BGM], for \(0 \leq m \leq n\), there is a constant \(c_{L,m}\) independent of \(n\) such that

\[
|\hat{A}_{c,n,m}^a(R_m)(p) - \hat{A}_{c,m}^a(R_m)(p)| \leq c_{L,m}\varepsilon_n. \quad (3.13)
\]

It will be important to have a control on the \(m\)-dependence of the constant \(c_{L,m}\) in (3.13). This is provided by

**Claim 3.3:**

\[
c_{L,m} = O(1)L^{-\frac{(d-2)}{2}}mL^d. \quad (3.14)
\]

Sketch of proof: Claim 2.3 follows from an examination of proof of Lemma 6.7, [BGM]. This proof needs the Poisson kernel estimate (Proposition 5.2) and Lemma 6.5 both of which are proved in Appendix A of [BGM]. The Poisson kernel estimate gives a constant \(O(1)R_m^{d/2}\) where \(R_m = L^{-(m-1)}\). An additional \(L^m\) arises from a derivative on \(g_m\) (see the proof of Lemma 6.7). Therefore \(R_m^{d/2}L^m\) is the constant of Lemma 6.7 and gives the right hand side of (3.14). \qed

From (2.16) and (2.18)

\[
\hat{\Gamma}^a_n(p) - \hat{\Gamma}^a_{c,n}(p) = |\hat{A}_{c}^a(p)|^2(\hat{\Gamma}_{c}^a(p) - \hat{\Gamma}_{c}^a(p)) + (|\hat{A}_{c}^a(p)|^2 - |\hat{A}_{c,n}^a(p)|^2)\hat{\Gamma}_{c}^a(p)
\]

whence on using the bounds (2.15), (2.20) together with (see [BGM, page 435])

\[
|\hat{\Gamma}_{c}^a(p)| \leq c_L(1 + p^2)^{-1}
\]
and \(|A^2 - B^2| \leq |A - B||A + B|\), we get

\[
|\hat{\Gamma}_n^a(p) - \hat{\Gamma}_{c,n}^a(p)| \leq c_k L e^{-O(1)a^\frac{1}{2}} (1 + p^2)^{-k} \left( |\hat{\Gamma}_n^a(p) - \hat{\Gamma}_{c}^a(p)| + |\hat{A}_n^a(p) - \hat{A}_{c,n}^a(p)| \right). \tag{3.15}
\]

We will estimate the two terms within the big round brackets above.

1. From (2.9) and the continuum analogue of (2.9) we get

\[
\hat{A}_n^a(p) - \hat{A}_{c,n}^a(p) =
\]

\[
= \sum_{m=1}^n \prod_{i<m} \hat{A}_{\varepsilon,n,i}^a(R_i)(p) \left( \hat{A}_{\varepsilon,n,m}^a(R_m)(p) - \hat{A}_{c,n}^a(R_m)(p) \right) \prod_{j>m} \hat{A}_{c,j}^a(R_j)(p).
\]

We bound \(|\hat{A}_{\varepsilon,n,i}^a(R_i)(p)|\) and \(|\hat{A}_{c,j}^a(R_j)(p)|\) by 1 and the \(m\) factor by (3.13) and (3.14). We get

\[
|\hat{A}_n^a(p) - \hat{A}_{c,n}^a(p)| \leq \sum_{m=1}^n L^{-(n-\frac{d}{2})} O(1) L^{-\frac{(d-2)m}{2}}.
\]

By the conditions of Lemma 3.2, we may take \(L\) sufficiently large and \(d \geq 3\) so that the series is geometrically convergent and dominated by the first term. Therefore

\[
|\hat{A}_n^a(p) - \hat{A}_{c,n}^a(p)| \leq O(L) L^{-n}. \tag{3.16}
\]

2. We now estimate the first term within the big round brackets in (3.15). From (2.7) and (2.17) we get

\[
\hat{\Gamma}_n^a(p) - \hat{\Gamma}_c^a(p) = (\hat{G}_n^a(p) - \hat{G}_c^a(p)) \left( 1 - |\hat{A}_{c,0}^a(L)(p)|^2 \right) + \hat{G}_n^a(p) \left( |\hat{A}_{c,0}^a(L)(p)|^2 - |\hat{A}_{\varepsilon,n,0}^a(L)(p)|^2 \right).
\]

Using the bounds \(|\hat{A}_{c,0}^a(L)(p)| \leq 1\) and \(|\hat{A}_{\varepsilon,n,0}^a(L)(p)| \leq 1\) we get

\[
|\hat{\Gamma}_n^a(p) - \Gamma_c^a(p)| \leq 2 |\hat{G}_n^a(p) - \hat{G}_c^a(p)| + 2 |\hat{G}_n^a(p)| \left| \hat{A}_{c,0}^a(L)(p) - \hat{A}_{\varepsilon,n,0}^a(L)(p) \right|. \tag{3.17}
\]

We first bound the second term in (3.17). There exists a constant \(b\) independent of \(n\) such that (see equation (5.9) in [BGM], page 434, we have replaced the constant \(c\) by \(b\))

\[
0 \leq \hat{G}_n^a(p) \leq (a + bp^2)^{-1}. \tag{3.18}
\]

Furthermore from (3.13)

\[
|\hat{A}_{\varepsilon,n,0}^a(L)(p) - \hat{A}_{c,0}^a(L)(p)| \leq c_L \varepsilon_n. \tag{3.19}
\]
Therefore

\[ |\hat{G}^a_{\varepsilon_n}(p)| |\hat{A}^a_{c,0}(L)(p) - \hat{A}^a_{\varepsilon_n,0}(L)(p)| \leq c_L(a + bp^2)^{-1}\varepsilon_n. \quad (3.20) \]

Next we bound the first term on the right hand side of (3.17).

1. Consider first the case \(|p| \geq L^{n/2}\). Then from (3.18) and \(\hat{G}^a_c(p) = (a + p^2)^{-1}\) we get

\[ |\hat{G}^a_{\varepsilon_n}(p) - \hat{G}^a_c(p)| \leq |\hat{G}^a_{\varepsilon_n}(p)| + |\hat{G}^a_c(p)| \leq O(1)L^{-n}. \quad (3.21) \]

2. Next we consider the case \(|p| < L^{n/2}\). From the definition above of \(\hat{G}^a_c(p)\), and \(\hat{G}^a_{\varepsilon_n}(p)\) we get

\[
\hat{G}^a_{\varepsilon_n}(p) - \hat{G}^a_c(p) = \frac{p^2 + \hat{\Delta}_{\varepsilon_n}(p)}{(a + p^2)(a - \hat{\Delta}_{\varepsilon_n}(p))}.
\]

Now using the bound (3.18) we get

\[ |\hat{G}^a_{\varepsilon_n}(p) - \hat{G}^a_c(p)| \leq O(1)\frac{1}{(p^2)^2}|p^2 + \hat{\Delta}_{\varepsilon_n}(p)|. \]

From \(|p| < L^{n/2}\) and \(\varepsilon_n = L^{-n}\) we have \(\varepsilon_n|p| < L^{-n/2}\). Now expanding out \(\hat{\Delta}_{\varepsilon_n}(p)\) (see (2.2)) in an absolutely convergent series we easily get the estimate for \(* * * \) \(n \geq d\)

\[ |p^2 + \hat{\Delta}_{\varepsilon_n}(p)| \leq O(1)\varepsilon_n^2(p^2)^2. \]

Combining this with the earlier inequality we get

\[ |\hat{G}^a_{\varepsilon_n}(p) - \hat{G}^a_c(p)| \leq O(1)\varepsilon_n^2. \quad (3.22) \]

From (3.21) and (3.22) we get for all \(p \in B_{\varepsilon_n}\)

\[ |\hat{G}^a_{\varepsilon_n}(p) - \hat{G}^a_c(p)| \leq O(1)\varepsilon_n. \quad (3.23) \]

Inserting the bounds (3.20) and (3.23) in (3.17) we get

\[ |\hat{\Gamma}^a_{\varepsilon_n}(p) - \Gamma^a_c(p)| \leq c_L\varepsilon_n(1 + (a + bp^2)^{-1}). \quad (3.24) \]

From (3.15) and the bounds (3.16) and (3.24) we get

\[ |\hat{\Gamma}^a_n(p) - \hat{\Gamma}^a_{c,n}(p)| \leq c_kL^{-k} L^{-n}(1 + p^2)^{-k}(1 + (a + bp^2)^{-1}). \quad (3.25) \]
This proves Lemma 3.2. ■

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