Extreme Value Condition of Stabilizing Lead/Lag Region

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Abstract. This paper considers the problem of stabilizing linear time-invariant plants by a lead/lag compensator. The extreme value condition of the stabilizing lead/lag region is given. The coordinates of the local extreme value point is obtained. At the local extreme value point, the closed-loop system contains a pair of conjugate imaginary poles and a pole at the origin. If the zero gain is fixed, the stability region in the plane with respect to the proportional gain and the pole gain is determined by plotting the stability boundary locus. The effectiveness of the method presented is illustrated a numerical example.

Introduction

The lead/lag compensator is widely used in industrial processes. It has simple structure and gives a satisfactory performance. The lead/lag compensator can be tuned both in theoretical and practical ways [5]. To tune a lead/lag compensator in theoretical way, its basis is to find the stabilizing region in the parameters space. Since the structure of the lead/lag compensator is similar to the PID controller, the methods using to determine the stabilizing PID region of linear time-invariant plants are introduced. A linear programming method based on the generalized version of the Hermite-Biehler theorem is adopted to obtain all stabilizing PID controllers [1, 2]. This method is extended to determine the stabilizing lead/lag region of linear time-invariant plants [4]. However, the computation time of this method will increase exponentially as the order of the plant varies. Another way to determined the stabilizing PID region is given in [6, 7]. The admissible range of the proportional gain in stabilizing PID region is derived [6]. In [7], the extreme value condition of stabilizing PID region is given. The entire stability region is obtained by plotting the stability boundary locus. In this paper, the method using in [7] is extended to determine the stabilizing lead/lag region. The content of this paper is arranged as follows. Firstly, the extreme value condition of stabilizing lead/lag region is given; Secondly, the coordinates of the local extreme value point are derived; finally, a numerical example is used to illustrate the proposed method.

System Model

The structure of the lead/lag control system is shown in Fig. 1.

Figure 1. The structure of the lead/lag control system.

The lead/lag compensator has the form

$$G_c(s) = \frac{ks + a}{s + b}$$

(1)

Where $K_p$, $a$, and $b$ are proportional gain, zero gain and pole gain respectively.
The transfer function of the plant is

\[ G_p(s) = \frac{N(s)}{D(s)} \]  

(2)

where \( D(s) \) and \( N(s) \) are polynomials, also \( N(0) \neq 0 \) holds.

The transfer function of the closed-loop system can be described by

\[ G(s) = \frac{(ks + a)N(s)}{(s + b)D(s) + (ks + a)N(s)} \]  

(3)

The characteristic equation of the closed-loop system is

\[ C(s) = (s + b)D(s) + (ks + a)N(s) = 0 \]  

(4)

If \( \omega > 0 \), we define

\[ \frac{D(j\omega)}{N(j\omega)} = A(\omega) + jB(\omega) \]  

(5)

where \( A(\omega) \) and \( B(\omega) \) are polynomials.

Using this definition and substituting \( s = j\omega(\omega > 0) \) into (4), we obtain

\[ a = \omega B(\omega) - bA(\omega) \]  

(6)

\[ k = -A(\omega) - b \frac{B(\omega)}{\omega} \]  

(7)

Substituting \( s = 0 \) into (4) yields

\[ \frac{a}{b} = -\frac{D(0)}{N(0)} \]  

(8)

Assume that the plant can be stabilized by a lead/lag compensator. If the zero gain is fixed, the curves described by (6) to (8) are called the stability boundary locus in the \((a,k)\) plane [3]. For each region bounded by the stability locus, a testing point is needed to verify the closed-loop stability. Since the number of the regions is finite, the stability region can be obtained in this way.

**Extreme Value Condition of Stabilizing lead/lag Region**

Theorem 1: If the zero gain reaches the extreme value, then \( A(\omega) = 0 \).

Proof: From (6), we can get

\[ \frac{\partial a}{\partial b} = -A(\omega) \]  

(9)

Also we get

\[ \frac{\partial a}{\partial k} = \frac{\partial b}{\partial k} = \frac{\omega A(\omega)}{B(\omega)} \]  

(10)

Based on the extreme value condition of multi-variable function, if the zero gain reaches the extreme value, then

\[ A(\omega) = 0 \]  

(11)
By disproof method, assume that \( B(\omega) = 0 \), then we get

\[ a = bk \]  \hspace{1cm} (13)

Substituting (13) into (8) yields

\[ k = -\frac{D(0)}{N(0)} \]  \hspace{1cm} (14)

The equation (14) shows that the proportional gain is a constant at the local extreme value point. So there exists a contradiction.

Similarly, we can obtain the following theorems.

Theorem 2: If the pole gain reaches the extreme value, then \( A(\omega) = 0 \).

Theorem 3: If the proportional gain reaches the extreme value, then \( A(\omega) = 0 \).

Assume that \( \omega = \omega^* (\omega^* > 0) \) is the root of \( A(\omega) = 0 \). From (6), we get

\[ a = a^* = \omega^* B(\omega^*) \]  \hspace{1cm} (15)

From (7), we can get

\[ k = -b \frac{B(\omega^*)}{\omega^*} \]  \hspace{1cm} (16)

So we can obtain the following thereom.

Thereom 4: The extreme value condition of stabilizing lead/lag region is

\[
\begin{align*}
  A(\omega^*) &= 0 \\
  a^* &= \omega^* B(\omega^*) \\
  k &= -b \frac{B(\omega^*)}{\omega^*}
\end{align*}
\]

From thereom 4, it can be obtain that if the zero gain reaches the extreme value, the stability region in the \((a, k)\) plane will reduce to a straight line described by

\[ a = a^* = \omega^* B(\omega^*) \]  \hspace{1cm} (17)

\[ k = -b \frac{B(\omega^*)}{\omega^*} \]  \hspace{1cm} (18)

From (17), (18) and (8), we can get

\[
\begin{align*}
  a &= a^* = \omega^* B(\omega^*) \\
  b &= b^* = -\frac{N(0)}{D(0)} \omega^* B(\omega^*) \\
  k &= k^* = \frac{N(0)}{D(0)} B^2(\omega^*)
\end{align*}
\]  \hspace{1cm} (19)
So the point which coordinates satisfy (19) is the possible local extreme value point. Moreover, the equation (19) shows that at the local extreme value point, the closed-loop system contains a pair of conjugate imaginary poles and a pole at the origin.

**Numerical Examples**

The transfer function of the plant has the form

\[ G_p(s) = \frac{1}{(s+1)^4} \]  

(20)

It can be obtained from (20) that

\[ A(\omega) = \omega^4 - 6\omega^2 + 1 \]  

(21)

\[ B(\omega) = 4\omega - 4\omega^3 \]  

(22)

Based on theorem 4, we get

\[ \omega_1^* = 0.4142 \]  

(23)

\[ \omega_2^* = 2.4142 \]  

(24)

If \( \omega_1^* = 0.4142 \), then the equation of the corresponding straight line is

\[ a_1^* = 0.5685 \]  

(25)

\[ k = -3.3137b \]  

(26)

The possible extreme value point is

\[ \begin{cases} a_1^* = 0.5685 \\ b_1^* = -0.5685 \\ k_1^* = 1.8840 \end{cases} \]  

(27)

If \( b = -0.56 \), the stability region in the \((a, k)\) plane is shown in Fig. 2.

![Figure 2. The stability region when \( b = -0.56 \).](image)
Referring to Fig. 2, if \( b \to -0.5685 \), then the stability region will reduce to a point. So the point which coordinates satisfy (27) is the local extreme value point.

If \( \omega_2^* = 2.4142 \), then the equation of the corresponding straight line is

\[
a_i^* = -112.5685
\]

\[
k = 19.3137b
\]

The possible extreme value point is

\[
\begin{align*}
  a_i^* &= -112.5685 \\
  b_i^* &= 112.5685 \\
  k_i^* &= 2174.1
\end{align*}
\]  

(30)

If \( b = 112.5685 \), the stability region in the \((a, k)\) plane is shown in Fig. 3.

![Figure 3. The stability region when \( b = 112.5685 \).](image)

Referring to Fig. 3, if \( b \to 112.5685 \), then the stability region will not reduce to a point. So the point which coordinates satisfy (30) is not the local extreme value point.

The stabilizing lead/lag region is obtained by plotting the stability boundary locus which is shown in Fig. 4.

![Figure 4. The stabilizing lead/lag region of the example.](image)
If the parameters described by (27) are adopted, the closed-loop response is shown in Fig. 5.

![Figure 5. The closed-loop response at the extreme value point.](image)

**Summary**

Extreme value condition of stabilizing lead/lag region is given. The coordinates of the local extreme value point is obtained. Compared with the existing methods, the proposed method is simple and fast, also the stabilizing lead/lag region is determined directly by plotting the stability boundary locus. The further work is to extend this method to time delay systems.

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