A NOTE ON ATIYAH’S Γ-INDEX THEOREM IN HEISENBERG CALCULUS

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Abstract. In this note, we prove an index theorem on Galois coverings for Heisenberg elliptic differential operators, but not elliptic, which is analogous to Atiyah’s Γ-index theorem. This note also contains an example of Heisenberg differential operators with a non-trivial Γ-index.

Introduction

M. F. Atiyah [1] introduced the notion of the Γ-index \( \text{index}_\Gamma(D) \) for a lifted elliptic differential operator \( \tilde{D} \) on a Galois \( \Gamma \)-covering over a closed manifold and proved that the Γ-index \( \text{index}_\Gamma(D) \) of a lifted operator \( \tilde{D} \) equals the Fredholm index \( \text{index}(D) \) of the elliptic differential operator \( D \) on the base manifold. On the other hand, Atiyah [1] also investigated properties of a Γ-trace \( \text{tr}_\Gamma \) at the same time. The Γ-trace is a trace of the Γ-trace operators, so it induces a homomorphism \( \text{tr}_\Gamma \), from \( K_0 \)-group of the Γ-compact operators to the real numbers. Out of a lifted elliptic differential operator \( \tilde{D} \), we can define the Γ-index class \( \text{Ind}_\Gamma(D) \) by using the Connes-Skandalis idempotent [3, II.9.α (p.131)] and send it by the induced homomorphism \( \text{tr}_\Gamma \), then the image \( \text{tr}_\Gamma(\text{Ind}_\Gamma(D)) \) equals the Γ-index \( \text{index}_\Gamma(D) \) and thus the Fredholm index \( \text{index}(D) \) of the elliptic differential operator \( D \) on the base manifold.

On the other hand, there is another pseudo-differential calculus on Heisenberg manifolds which is called the Heisenberg calculus; see, for instance [5]. Roughly speaking, Heisenberg calculus is “weighted” calculus and the product of the “Heisenberg principal symbols” are defined by convolution product. When the Heisenberg principal symbol of \( P \) is invertible, we call \( P \) a Heisenberg elliptic operator. Note that any Heisenberg elliptic operator is not elliptic. For a Heisenberg elliptic operator \( P \), we can construct a parametrix by using its inverse, so \( P \) is a Fredholm operator if the base manifold is closed. Thus the Fredholm index of \( P \) on a closed manifold is well defined, but a solution of an index problem of \( P \) does not obtained in general. However, index problems for Heisenberg elliptic operators on contact manifolds or foliated manifolds are solved by E. van Erp and P. F. Baum; see [2], [6], [7], [9].

In this note, we study that we can define the Γ-index \( \text{index}_\Gamma(P) \) and the Γ-index class \( \text{Ind}_\Gamma(P) \) for a lifted Heisenberg elliptic differential operator \( \tilde{P} \) by using a parametrix. Once these ingredients are defined, the proof of the matching of three ingredients \( \text{index}_\Gamma(P) \), \( \text{tr}_\Gamma(\text{Ind}_\Gamma(P)) \) and \( \text{index}(P) \), is straightforward; see

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subsection [2.1] We also investigate an example of Heisenberg differential operators on a contact manifold with non-trivial \( \Gamma \)-index by using the index formula in [2]; see subsection [2.2].

1. Short review of Atiyah’s \( \Gamma \)-index theorem

In this section, we recall Atiyah’s \( \Gamma \)-index theorem in ordinary pseudodifferential calculus. The main reference of this section is Atiyah’s paper [1]. Let \( \tilde{M} \to M \) be a Galois covering with a deck transformation group \( \Gamma \) over a closed manifold \( M \) with a smooth measure \( \mu \) and \( D : C^\infty(E) \to C^\infty(F) \) an elliptic differential operator on Hermitian vector bundles \( E, F \to M \). We lift these ingredients on \( \tilde{M} \) and denote by \( \tilde{D} : C^\infty(\tilde{E}) \to C^\infty(\tilde{F}) \) and \( \tilde{\mu} \). Let \( \text{Ker}_{L^2}(\tilde{D}) \) (resp. \( \text{Ker}_{L^2}(\tilde{D}^*) \)) be the \( L^2 \)-solutions of \( \tilde{D}u = 0 \) (resp. \( \tilde{D}^*u = 0 \)) and denote by \( \Pi_0 \) (resp. \( \Pi_1 \)) the orthogonal projection on a closed subspace \( \text{Ker}_{L^2}(\tilde{D}) \) (resp. \( \text{Ker}_{L^2}(\tilde{D}^*) \)) of the \( L^2 \)-sections.

A \( \Gamma \)-invariant bounded operator \( T \) on the \( L^2 \)-sections \( L^2(\tilde{E}) \) of \( \tilde{E} \) is of \( \Gamma \)-trace class if \( \phi T^* \psi \in L^2(\tilde{E}) \) is of trace class for any compactly supported smooth functions \( \phi, \psi \) on \( \tilde{M} \). Denote by \( \mathcal{L}_1^\Gamma \) the set of \( \Gamma \)-trace class operators and \( \text{tr}_\Gamma(T) \) the \( \Gamma \)-trace of a \( \Gamma \)-trace class operator \( T \) defined by

\[
\text{tr}_\Gamma(T) = \text{Tr}(\phi T^* \psi) \in \mathbb{C}.
\]

Here, the right hand side is the trace of a trace class operator \( \phi T^* \psi \) and this quantity does not depend on the choice of functions \( \phi, \psi \). By using ellipticity of \( \tilde{D} \), operators \( \phi \Pi_0 \psi \) and \( \phi \Pi_1 \psi \) are smoothing operators on compact sets. Thus \( \Pi_0 \) and \( \Pi_1 \) are of \( \Gamma \)-trace class and thus one obtains the \( \Gamma \)-index of \( \tilde{D} \):

\[
\text{index}_\Gamma(\tilde{D}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) \in \mathbb{R}.
\]

In the context of the \( \Gamma \)-index theorem, the most important class of \( \Gamma \)-trace class operators is the lifts of almost local smoothing operators on \( M \). Let \( S \) be an almost local smoothing operator with a smooth kernel \( k_S \) and \( \tilde{S} \) a lift of \( S \). Then \( \tilde{S} \) is of \( \Gamma \)-trace class and its \( \Gamma \)-trace is calculated by the following:

\[
(*) \quad \text{tr}_\Gamma(\tilde{S}) = \int_M \text{tr}(k_S(x, x))d\mu = \text{Tr}(S).
\]

Denote by \( \mathcal{K}_\Gamma \) the \( C^* \)-closure of \( \mathcal{L}_1^\Gamma \) and \( K_0(\mathcal{K}_\Gamma) \) the analytic \( K_0 \)-group. Then \( \text{tr}_\Gamma \) induces a homomorphism of abelian groups by substitution:

\[
(\text{tr}_\Gamma)_* : K_0(\mathcal{K}_\Gamma) \to \mathbb{R}.
\]

On the other hand, since \( D \) is an elliptic differential operator, there exist an almost local parametrix \( Q \) and almost local smoothing operators \( S_0, S_1 \) such that one has \( QD = 1 - S_0 \) and \( DQ = 1 - S_1 \). Denote by \( \tilde{Q}, \tilde{S}_0 \) and \( \tilde{S}_1 \) lifts of these operators and then one has same relations \( \tilde{Q}\tilde{D} = 1 - \tilde{S}_0 \) and \( \tilde{D}\tilde{Q} = 1 - \tilde{S}_1 \). Set

\[
e_\tilde{D} = \begin{bmatrix} S_0^2 & \tilde{S}_0(1 + S_0)\tilde{Q} \\ \tilde{S}_1\tilde{D} & 1 - \tilde{S}_1^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

By \( \tilde{Q}\tilde{S}_1 = \tilde{S}_0\tilde{Q} \) and \( \tilde{S}_1\tilde{D} = \tilde{D}\tilde{S}_0 \), one has \( e_{\tilde{D}}^2 = e_{\tilde{Q}} \), that is, \( e_{\tilde{D}} \) is an idempotent. Note that this idempotent \( e_{\tilde{D}} \) is called the Connes-Skandalis idempotent; see, for
instance \( [3] \Pi.9.\alpha \) (p.131)]. Moreover, a difference \( e_{\tilde{\mathcal{D}}} - e_1 \) is of \( \Gamma \)-trace class. Hence we can define a \( \Gamma \)-index class

\[
\text{Ind}_\Gamma(\tilde{D}) = [e_{\tilde{\mathcal{D}}} - e_1] \in K_0(\mathcal{K}_\Gamma).
\]

By the definition of a map \((\text{tr}_\Gamma)_* \) and Atiyah’s paper, one has the following:

**Theorem 1.1** (Atiyah’s \( \Gamma \)-index theorem \([11 \text{ Theorem 3.8}] \)). In the above settings, we have the following equality:

\[
\text{index}_\Gamma(\tilde{D}) = (\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{D})) = \text{index}(D) \in \mathbb{Z}.
\]

As described in subsection 2.1 Atiyah’s proof of matching of these ingredients in the above equality does not essentially use ellipticity. Note that ellipticity of \( D \) and \( \tilde{D} \) is only used in the definition of these ingredients.

2. **Atiyah’s \( \Gamma \)-index theorem in Heisenberg calculus**

Let \((M, H)\) be a closed Heisenberg manifold, that is, \( M \) is a closed manifold and \( H \subset TM \) is a hyperplane bundle. Let \( P : C^\infty(E) \to C^\infty(F) \) be a Heisenberg elliptic differential operator on Hermitian vector bundles \( E, F \to (M, H) \), that is, the Heisenberg principal symbol \( \sigma_H(P) \) of \( P \) is an invertible element. In this section, we prove the \( \Gamma \)-index theorem for \( P \), which is analogous to Atiyah’s \( \Gamma \)-index theorem. Note that \( P \) is not an elliptic operator in the sense of ordinary pseudo-differential calculus.

2.1. **Statement and proof.** By \([5 \text{ Proposition 3.3.1}] \), there exist parametrix \( Q \) and smoothing operators \( S_0, S_1 \) such that one has \( PQ = 1 - S_0 \) and \( PQ = 1 - S_1 \). Thus \( P \) is a Fredholm operator and one has the Fredholm index \( \text{index}(P) \in \mathbb{Z} \) of \( P \) by compactness of \( M \). Moreover, since a integral kernel of \( Q \) is smooth off the diagonal, we can choose \( Q \) as an almost local operator and then \( S_0 \) and \( S_1 \) are also almost local operators.

Let \( \tilde{M} \to M \) be a Galois covering with a deck transformation group \( \Gamma \) over a closed manifold \( M \) with a smooth measure \( \mu \). We lift all structures on \( M \) to \( \tilde{M} \). Then \((\tilde{M}, \tilde{H})\) is a Heisenberg manifold, \( \tilde{P} : C^\infty(\tilde{E}) \to C^\infty(\tilde{F}) \) is a Heisenberg elliptic differential operator and one has \( \tilde{Q}\tilde{P} = 1 - \tilde{S}_0 \) and \( \tilde{P}\tilde{Q} = 1 - \tilde{S}_1 \).

Since \( P \) is a differential operator (in particular, \( P \) is local), there exists a constant \( C = C(P, \phi) > 0 \) such that we have an inequality

\[
\|\tilde{P}(\phi f)\|_{L^2} \leq C(\|\chi\tilde{P}f\|_{L^2} + \|\chi f\|_{L^2})
\]

for any \( f \in C^\infty(\tilde{E}) \); see \([5 \text{ Proposition 3.3.2}] \). Here, \( \phi, \chi \in C^\infty_c(\tilde{M}) \) are compactly supported smooth functions and one assumes \( \chi = 1 \) on the support of \( \phi \). Thus by using Atiyah’s technique of the proof of \([11 \text{ Proposition 3.1}] \), we have the following:

**Lemma 2.1.** The minimal domain of \( \tilde{P} \) equals the maximal domain of \( \tilde{P} \).

By Lemma 2.1 \( \tilde{P} \) has the unique closed extension denoted by the same letter \( \tilde{P} \) and thus the closure of the formal adjoint of \( \tilde{P} \) (the formal adjoint is also Heisenberg elliptic) equals the Hilbert space adjoint \( \tilde{P}^* \).

On the other hand, any \( L^2 \)-solutions of \( \tilde{P}u = 0 \) and \( \tilde{P}^*u = 0 \) are smooth by existence of a parametrix. Thus the orthogonal projection \( \Pi_0 \) (resp. \( \Pi_1 \)) onto a closed subspace \( \text{Ker}_{L^2}(\tilde{P}) \) (resp. \( \text{Ker}_{L^2}(\tilde{P}^*) \)) of the \( L^2 \)-sections is of \( \Gamma \)-trace class.
since operators $\phi \Pi_0 \psi$ and $\phi \Pi_1 \psi$ are smoothing operators on compact sets. Thus one obtains the well-defined $\Gamma$-index of $\tilde{P}$.

**Definition 2.2.** The $\Gamma$-index of $\tilde{P}$ is defined to be

$$\text{index}_\Gamma(\tilde{P}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) \in \mathbb{R}.$$ 

By using operators $\tilde{P}, \tilde{Q}, \tilde{S}_0$ and $\tilde{S}_1$, we define

$$e_\tilde{P} = \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(1 + \tilde{S}_0)\tilde{Q} \\ \tilde{S}_1 \tilde{P} & 1 - \tilde{S}_1^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Since a difference $e_\tilde{P} - e_1$ is of $\Gamma$-trace class, one can define a $\Gamma$-index class of $\tilde{P}$.

**Definition 2.3.** We define $\Gamma$-index class $\text{Ind}_\Gamma(\tilde{P})$ of $\tilde{P}$ by

$$\text{Ind}_\Gamma(\tilde{P}) = [e_\tilde{P}] - [e_1] \in K_0(K_\Gamma).$$ 

By using a $\Gamma$-trace, we have the $\Gamma$-index theorem in Heisenberg calculus.

**Theorem 2.4.** Let $P$ be a Heisenberg elliptic differential operator on a closed Heisenberg manifold $(M, H)$ and $\tilde{P}$ its lift as previously. Then one has

$$\text{index}_\Gamma(\tilde{P}) = (\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P})) = \text{index}(P) \in \mathbb{Z}.$$ 

**Proof.** First, note that equalities

$$1 - S_0^2 = 1 - (1 - QP)^2 = (2Q - PQ)P \quad \text{and} \quad 1 - S_1^2 = 1 - (1 - PQ)^2 = P(2Q - PQ),$$

and note that operators $2Q - PQ$, $S_0^2$ and $S_1^2$ are almost local operators. Thus by Atiyah’s technique in [1, Section 5], one has

$$\text{index}_\Gamma(\tilde{P}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2).$$

Next, the definition of a map $(\text{tr}_\Gamma)_*$, one has

$$(\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P})) = \text{tr}_\Gamma \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(1 + \tilde{S}_0)\tilde{Q} \\ \tilde{S}_1 \tilde{P} & -\tilde{S}_1^2 \end{bmatrix} = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2).$$

Since operators $\tilde{S}_0^2$ and $\tilde{S}_1^2$ are lifts of almost local smoothing operators, one has

$$\text{index}(P) = \text{Tr}(S_0^2) - \text{Tr}(S_1^2) = \text{tr}_\Gamma(\tilde{S}_0^2) - \text{tr}_\Gamma(\tilde{S}_1^2)$$
by using (11) in Section 11. This proves the equality in the theorem. □

**Remark 2.5.** As pointed out in [3, Section 4], the results in [5, Section 3] hold verbatim for arbitrary codimension. That is, we do not need to assume that a distribution $H$ is of codimension 1.
2.2. Example. Index problems for Heisenberg elliptic operators on arbitrary closed Heisenberg manifolds are not solved yet. However, van Erp [6,7] and Baum and van Erp [2] solved the index problem on contact manifolds, which are good examples of Heisenberg manifolds. In this subsection, we investigate an example of Heisenberg elliptic differential operators with non-trivial Γ-index on a Galois cover ring over a Heisenberg manifold. Erp [2] solved the index problem on contact manifolds, which are good examples of Heisenberg manifolds.

Let $T^2 = S^1 \times S^1 = \{(e^{ix}, e^{iy})\}$ be a 2-torus and set

$$e(x, y) = \begin{bmatrix} f(x) & g(x) + h(x)e^{iy} & 1 - f(x) \\ g(x) + h(x)e^{-iy} & f(x) & 1 - f(x) \end{bmatrix}.$$ 

Here, let $f$ be a $[0, 1]$-valued $2\pi\mathbb{Z}$-periodic function on $\mathbb{R}$ such that $f(0) = 1$ and $f(\pi) = 0$ and set $g(x) = \chi_{[0, \pi]}(x)\sqrt{f(x) - f(x)^2}$ and $h(x) = \chi_{[\pi, 2\pi]}(x)\sqrt{f(x) - f(x)^2}$; see [4, Section I, 2]. Moreover, we assume $f, g$ and $h$ are smooth functions. Then $e$ defines an $M_2(\mathbb{C})$-valued smooth function on $T^2$.

Since $e$ is an idempotent of rank 1, $e$ defines an complex line bundle $E$ on $T^2$. As well known, the first Chern class $c_1(E)$ of $E$ is given by a 2-form

$$\frac{-1}{2\pi} \text{tr}(e(de)^2) = \frac{-1}{\pi}(hh' + 2f'h^2 - 2fhh')dx \wedge dy.$$ 

Thus, by using an equality $h^2 = f^2$ on $[\pi, 2\pi]$, we can calculate the first Chern number of $E$:

$$\int_{T^2} c_1(E) = -\int_{\pi}^{2\pi} f' dx = -1.$$ 

Let $T^3 = T^2 \times S^1 = \{(e^{ix}, e^{iy}, e^{iz})\}$ be a 3-torus and $q : T^3 \to T^2$ the projection onto $T^2$ of the first component. Set $\theta_k = \cos(kz)dx - \sin(kz)dy$ for a positive integer $k$, $H_k = \ker(\theta_k)$, $f_l(x, y, z) = e^{itz} + 1$ for a integer $l$ and $F = q^*E$. Then $(T^3, H_k)$ is a contact manifold and $H_k$ is a flat vector bundle. Denote by $T_k$ the Reeb vector field for $\theta_k$ and $\Delta^F_{T_k} = -\nabla^E_{X_k} \nabla^E_{X_k} - \nabla^E_{Y_k} \nabla^E_{Y_k}$ the sum of squares on $F$, where $\{X_k, Y_k\}$ is a local frame of $H_k$. Set

$$P_{k,l} = \Delta^F_{T_k} + if_l \nabla^F_{T_k}.$$ 

Since the values of $f_l - n$ contained in $\mathbb{C}^\times$ for any odd integer $n$, an operator $P_{k,l} : C^\infty(F) \to C^\infty(F)$ is a Heisenberg elliptic differential operator of Heisenberg order 2. By the index formula for $P_{k,l}$ in [2] Example 6.5.3], one has

$$\text{index}(P_{k,l}) = \int_{T^3} \frac{-1}{2\pi i} e^{-itz} de^{itz} \wedge (\text{tr}(F) + 1) = \frac{-1}{2\pi i} \int_{S^1} e^{-itz} de^{itz} \int_{T^2} c_1(E) = l.$$ 

Note that a contact structure $H_k$ is a lift of $H_1$ by a $k$-fold cover $p_k : T^3 \to T^3$:

$$\begin{cases} e^{ix}, e^{iy}, e^{iz} \\ e^{ix}, e^{iy}, e^{iktz} \end{cases} \to \begin{cases} e^{ix}, e^{iy}, e^{iktz} \end{cases}.$$ 

Since the lift $\widetilde{P}_{k,l}$ of a subLaplacian $P_{k,l}$ by $p_k$ equals $P_{k,kl}$, we have the $\Gamma(= \mathbb{Z}/k\mathbb{Z})$-index of $\widetilde{P}_{k,l}$:

$$\text{index}_{\Gamma}(\widetilde{P}_{k,l}) = \frac{1}{k} \text{index}(\widetilde{P}_{k,l}) = \frac{1}{k} \text{index}(P_{k,kl}) = l = \text{index}(P_{k,l}).$$ 

Next, we consider a general Galois covering of $T^3$. Let $X \to T^3$ be a Galois covering with a deck transformation group $\Gamma$, which is a quotient of $\pi_1(T^3) = \mathbb{Z}^3$, for example, $X = \mathbb{R}^3$ and $\Gamma = \mathbb{Z}^3$ the universal covering. By Theorem 2.4, we have a non-trivial $\Gamma$-index as follows:

$$\text{index}_{\Gamma}(\widetilde{P}_{k,l}) = \text{index}(P_{k,l}) = l.$$
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