A FREE BOUNDARY PROBLEM FOR A PREY-PREDATOR MODEL WITH DEGENERATE DIFFUSION AND PREDATOR-STAGE STRUCTURE

SIYU LIU
School of Mathematics, Harbin Institute of Technology
Harbin 150001, China

HAOMIN HUANG
Department of Mathematics, Southern University of Science and Technology
Shenzhen 518000, China
and
School of Mathematics and Statistics, Wuhan University
Wuhan 430000, China

MINGXIN WANG*
School of Mathematics, Harbin Institute of Technology
Harbin 150001, China

Abstract. In this paper we consider a free boundary problem for a prey-predator model with degenerate diffusion and predator-stage structure. In our model, the individuals of a new or invasive predatory species are classified as belonging to either the immature or mature case. Firstly, the global existence, uniqueness, regularity of the solution are derived. And then when vanishing happens, we get uniform estimates and the long time behavior of the solution. At last, a sharp criterion governing spreading and vanishing for the free boundary problem is studied by the upper and lower solution method.

1. Introduction. The model. The dynamical relationship between a predator and a prey has long been among the dominant topics in mathematical ecology due to its universal existence and importance. In the real world, most species go through several stages during their lifetime, such as immature and mature stage. The vital rates (rates of survival, development, and reproduction) almost always depend on the development stage, among many other factors. Hence, it is significant and practical to introduce the stage structure into models. In recent years, prey-predator models with stage structure have attracted much attention, see [2]–[8] and the references therein.

In [6], Du, Pang and Wang proposed a diffusive prey-predator model with stage structure for the predator,
The boundary condition is the homogeneous Neumann boundary condition, which indicates that this system is self-contained with zero population flux across the boundary.

Their work inspires us in two sides. On one hand, if these predators are birds and the prey species are worms, then it is well known that the diffusion of birds are relatively faster than the worm, while that of the worm is so slow that we can omit its diffusion. Thus, the differential equations of (1) can be written as

\[
\begin{cases}
  u_t - D_1 \Delta u = au - u^2 - \varepsilon uv - uw, & x \in \Omega, \ t > 0, \\
v_t - D_2 \Delta v = kw - v, & x \in \Omega, \ t > 0, \\
w_t - D_3 \Delta w = bv - mw, & x \in \Omega, \ t > 0, \\
\partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial \Omega, \ t > 0, \\
w(x, 0) \geq 0, \ v(x, 0) \geq 0, \ w(x, 0) \geq 0, & x \in \Omega.
\end{cases}
\]

where \( u, v, w \) are respectively the population densities of prey, immature predator and mature predator, and \( a, \varepsilon, k, b, m \) and diffusion coefficients \( D_1, D_2, D_3 \) are all positive constants. The boundary condition is the homogeneous Neumann boundary condition, which indicates that this system is self-contained with zero population flux across the boundary.

On the other hand, in order to study the nature of spreading of multiple species, which give us a better understanding in the ecological complexity, we introduce a free boundary condition in this model. In this paper, we assume that the prey is a native species and occupies the whole space at the initial state. And the predatory species is a new or invasive species and exists initially in a bounded domain. In general, the predator will have a tendency to spread across the boundaries to expand their habitats, i.e., the initial habitats of the predator will evolve into the expanding regions as time increases. Because immature predators live depending on mature predators, they will share the same habitat. Based on the deduction of free boundary conditions given in [3], the free boundary conditions are \( h'(t) = -\mu w_x(t, h(t)) - \beta v_x(t, h(t)) \) and \( g'(t) = -\mu w_x(t, g(t)) - \beta v_x(t, g(t)) \), where \( \mu \) and \( \beta \) are positive constants. However, the migration of the immature depends on that of the mature completely. Hence the influence of the immature on the expanding boundary should be small, and we can consider that \( \beta = 0 \). It is worth to mention that methods and results of this paper are all effective when \( \beta > 0 \).

On account of the above discussions, the model we are concerned here is the following free boundary problem

\[
\begin{cases}
  u_t = au - u^2 - \varepsilon uv - uw, & t > 0, \ x \in \mathbb{R}, \\
v_t - d_1 v_{xx} = kw - v, & t > 0, \ g(t) < x < h(t), \\
w_t - d_2 w_{xx} = bv - mw, & t > 0, \ g(t) < x < h(t), \\
g'(t) = -\mu w_x(t, g(t)), & t \geq 0, \\
h'(t) = -\mu w_x(t, h(t)), & t \geq 0, \\
v(t, x) = w(t, x) = 0, & t \geq 0, \ x \notin (g(t), h(t)), \\
w(0, x) = u_0(x), & x \in \mathbb{R}, \\
v(0, x) = v_0(x), \ w(0, x) = w_0(x), & -h_0 \leq x \leq h_0, \\
-g(0) = h(0) = h_0,
\end{cases}
\]

where \( a, \varepsilon, k, b, m \) are as above and \( d_1, d_2, h_0, \mu \) are given positive constants. The initial functions \( u_0 \) and \( v_0 \) satisfy
• $u_0 \in C^1_0(\mathbb{R}) \cap W^{2,p}_{loc}(\mathbb{R}), u_0 > 0$ in $\mathbb{R}$, $v_0 w_0 \in W^2_p((-h_0, h_0)), v_0(\pm h_0) = w_0(\pm h_0) = 0$, and $u_0, v_0, w_0 > 0$ in $(-h_0, h_0),$

where $p > 3$, $C^1_0(\mathbb{R}) = \{ f \in C^1(\mathbb{R}) : f, f' \in L^\infty(\mathbb{R}) \}.$

Ecologically, this problem (2) describes the spreading of a predatory species with immature population density $v(t, x)$ and mature population density $w(t, x)$, which exists initially in $(-h_0, h_0)$ and disperses through random diffusion over $(g(t), h(t))$. The moving boundaries $g(t)$ and $h(t)$ can be considered as invading fronts.

The well-known Stefan type free boundary conditions arise in many other applications. For instance, it was used to describe wound healing [4] and melting of ice in contact with water [15]. As far as population models, Wang and Zhao [25, 30] consider such conditions for a predator-prey system in one dimension and higher dimension with radially symmetric solution respectively. Recently, Wang and Zhang [21] studied a Leslie-Gower prey-predator model with double free boundaries in one dimension. For other prey-predator models, please refer to [5]–[32] (the invasive species exists initially in a ball and invades into the asymptotic behavior of the solution to the corresponding initial problem, a spreading-vanishing dichotomy and a sharp criterion governing spreading and vanishing for the free boundary problem are derived.

There are also many theoretical developments for diffusive two-species competition models and mutualistic models with free boundaries. For competition models, please refer to [5]–[32] (the invasive species exists initially in a ball and invades into the new environment, while the resident species distributes in the whole space), and [10, 26] (two competition species are assumed to spread along the same free boundary), and [9]–[27] (with different free boundaries). For mutualistic models, please refer to [11, 28].

Main results. In this paper, we first obtain the global existence, uniqueness, regularity and estimates of the solution of (2). And then, dynamics of (2) are investigated. If $\lim_{t \to \infty} [h(t) - g(t)] < \infty$, we call vanishing happens. In this case, we have the following long time behavior:

$$\lim_{t \to \infty} \| v(t, \cdot) \|_{C^1([g(t), h(t)])} = 0, \quad \lim_{t \to \infty} \| w(t, \cdot) \|_{C^1([g(t), h(t)])} = 0,$$

$$\lim_{t \to \infty} u(t, x) = a \quad \text{uniformly on any compact subset of} \; \mathbb{R}.$$  

On the other hand, if $\lim_{t \to \infty} [h(t) - g(t)] = \infty$, we call spreading happens.

At last, we discuss conditions for spreading or vanishing. When $m \geq abk$, vanishing always happens; in the case $m < abk$, we obtain a criterion as follows: if the size of initial area $[-h_0, h_0]$ is larger than a critical number, namely $2h_0 \geq \Lambda$, where

$$\Lambda = \pi \left( \frac{2d_1d_2}{\sqrt{(d_2 + md_1)^2 + 4d_1d_2(abk - m) - (d_2 + md_1)}} \right)^{1/2},$$

then spreading always happens regardless of the moving parameter $\mu$ and initial population density $(u_0, v_0, w_0)$. On the other hand, if $2h_0 < \Lambda$, then whether spreading or vanishing occurs is determined by the initial population density $(u_0, v_0, w_0)$ and moving parameter $\mu$.  

$\bullet$
Then (2) can be transformed into
\[ C \Pi \] depending on
\[ \text{Theorem 2.1.} \]
For any given \( g \) solution \((2)\) in vanishing case. In Section 4, we provide the criterion for spreading and vanishing by the use of the upper and lower solutions method. In the last section we give a brief discussion.

2. Existence, uniqueness, regularities and estimates of global solution of \((2)\). We first state the local existence, uniqueness, regularity and estimates of solution \((u,v,w)\) of \((2)\). Before stating our results, we give some notations. Set \( g^* = -\mu w_0'(-h_0), \) and
\[
\Pi := \{ \| u_0 \|_{\infty}, \| v_0, w_0 \|_{W^2([-h_0,h_0])}, \alpha, k, b, m, h_0, g^*, h^* \}.
\]

**Theorem 2.1.** For any given \( \alpha \in (0,1) \) and \( p > 3/(1 - \alpha) \), there exists \( T > 0 \) depending on \( \Pi \) such that the problem \((2)\) admits a unique solution \((u,v,w,g,h)\) in \( C^1([0,T]; L^\infty(\mathbb{R})) \times [W^{1,2}_p(D_T) \cap C^{2+1/3,1+\alpha}(\partial D_T)] \times [C^{1+2/3}([0,T])] \), where
\[
D_T = \{ 0 < t \leq T, g(t) < x < h(t) \}.
\]

Moreover, \( u > 0 \) in \([0,T] \times \mathbb{R}, v, w > 0 \) in \( D_T, g'(t) < 0, h'(t) > 0 \) in \([0,T], \) and
\[
\| v, w \|_{W^{1,2}(D_T)} + \| v, w \|_{C^{2+1/3,1+\alpha}(\partial D_T)} + \| g, h \|_{C^{1+2/3}([0,T])} \leq C, \quad (3)
\]
where \( C \) depends on \( T \) and \( \Pi \).

**Proof.** This theorem can be proved similarly as [13, Theorem 2.1] (unpublished paper). For the convenience of readers, we present the details of the proof here. Generally speaking, the proof is divided into four steps. And positive constants \( C_j \) only depend on \( \Pi \), which we no longer emphasize this at every step.

**Step 1. Straighten the free boundary.** Let
\[
x = \frac{(h(t) - g(t))y + h(t) + g(t)}{2},
\]
\[
r(t, y) = u \left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right),
\]
\[
z(t, y) = v \left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right),
\]
\[
q(t, y) = w \left( t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2} \right).
\]

Then \((2)\) can be transformed into
\[
\begin{align*}
\begin{cases}
u_t = au - u^2 - \varepsilon uv - uw, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
z_t - d_1 \rho^2(t)z_{yy} - \zeta(t, y)z_y = krq - z, & t > 0, \quad |y| < 1, \\
q_t - d_2 \rho^2(t)q_{yy} - \zeta(t, y)q_y = bz - mq, & t > 0, \quad |y| < 1, \\
z(t, \pm 1) = q(t, \pm 1) = 0, & t \geq 0, \\
z(0, y) = v(0, h_0y) := z_0(y), \quad q(0, y) = w(0, h_0y) := q_0(y), & |y| \leq 1,
\end{cases}
\end{align*}
\]

where \( d_1, d_2, \rho(t), k, b, m, h_0, q_0 \) are positive constants.
and
\[
\begin{align*}
 g'(t) &= -\mu \rho(t) q_y(t, -1), \quad h'(t) = -\mu \rho(t) q_y(t, 1), \quad t > 0, \\
-g(0) &= h(0) = h_0,
\end{align*}
\]
where
\[
\rho(t) = \frac{2}{h(t) - g(t)}, \quad \zeta(t, y) = \frac{h'(t) + g'(t)}{h(t) - g(t)} + \frac{h'(t) - g'(t)}{h(t) - g(t)} y.
\]
Let
\[
T_1 = \min \left\{ 1, \frac{h_0}{2(2 + |g^*| + |h^*|)} \right\}, \quad g^* = -\mu w_0(-h_0), \quad h^* = -\mu w_0(h_0).
\]
For $0 < T \leq T_1$, denote $I_T = [0, T] \times [-1, 1]$ and
\[
D_{T,1}^1 = \{ z \in C(I_T) : z(0, y) = z_0(y), \ z(t, \pm 1) = 0, \ ||z - z_0||_{C(I_T)} \leq 1 \},
\]
\[
D_{T,2}^2 = \{ q \in C(I_T) : q(0, y) = q_0(y), \ q(t, \pm 1) = 0, \ ||q - q_0||_{C(I_T)} \leq 1 \},
\]
\[
D_{T,3}^3 = \{ g \in C^1([0, T]) : g(0) = -h_0, \ g'(0) = g^*, \ ||g' - g^*||_{C([0, T])} \leq 1 \},
\]
\[
D_{T,4}^4 = \{ h \in C^1([0, T]) : h(0) = h_0, \ h'(0) = h^*, \ ||h' - h^*||_{C([0, T])} \leq 1 \}.
\]
Clearly, $D_T = D_{T,1}^1 \times D_{T,2}^2 \times D_{T,3}^3 \times D_{T,4}^4$ is a bounded and closed convex set of $[C(I_T)]^2 \times [C^1([0, T])]^2$.

When $(g, h) \in D_{T,1}^1 \times D_{T,4}^4$. Due to the choice of $T$, we can extend $g, h$ to new functions, denoted by themselves, such that $(g, h) \in \Omega_{T,1}^1 \times \Omega_{T,4}^4$, where
\[
\Omega_{T,1}^1 = \{ g \in C^1([0, T_1]) : g(0) = -h_0, \ g'(0) = g^*, \ ||g' - g^*||_{C([0, T_1])} \leq 2 \},
\]
\[
\Omega_{T,4}^4 = \{ h \in C^1([0, T_1]) : h(0) = h_0, \ h'(0) = h^*, \ ||h' - h^*||_{C([0, T_1])} \leq 2 \}.
\]
Therefore, when $(g, h) \in D_{T}^1 \times D_{T}^4$ we have $(g, h) \in \Omega_{T,1}^1 \times \Omega_{T,4}^4$ and
\[
||g(t) + h_0|| + ||h(t) - h_0|| \leq T_1(||g'||_{C([0, T])} + ||h'||_{C([0, T])}) \leq h_0/2, \quad \forall \ t \in [0, T_1],
\]
which implies
\[
h_0 \leq h(t) - g(t) \leq 3h_0, \quad \forall \ t \in [0, T_1].
\]
Thus, functions $\rho(t)$ and $\zeta(t, y)$ are well defined on $[0, T_1]$.

**Step 2. Existence of the solution to** $(4)$-$(6)$. For any given $(z, q, g, h) \in D_T$, we first set $z = q = 0$ in $[0, T] \times ((-\infty, 1] \cup [1, \infty))$ and then extend $z$ and $q$ to $[T, T_1] \times \mathbb{R}$ by setting $z(t, y) = z(T, y)$ and $q(t, y) = q(T, y)$ for all $T < t \leq T_1$. Next, by taking
\[
v(t, x) = z \left( t, \frac{2x - g(t) - h(t)}{h(t) - g(t)} \right), \quad w(t, x) = q \left( t, \frac{2x - g(t) - h(t)}{h(t) - g(t)} \right)
\]
in $(4)$, we get a Cauchy problem. It is easy to show that this problem has a unique solution $u \in C^1([0, T_1]; L^\infty(\mathbb{R}))$. Moreover, $u > 0$ in $(0, T_1) \times \mathbb{R}$ by the maximum principle. Thus $r \in L^\infty((0, T_1) \times \mathbb{R})$ and $r > 0$ in $(0, T_1) \times \mathbb{R}$.

Then for such $r(t, y)$, $z(t, y)$ and $q(t, y)$, consider the following initial-boundary value problem
\[
\begin{aligned}
\dot{z} - d_1 \rho^2(t) \dot{z}_{yy} - \zeta(t, y) \dot{z}_y + \ddot{z} = k r q, & \quad t > 0, \quad |y| < 1, \\
\dot{q} - d_2 \rho^2(t) \dot{q}_{yy} - \zeta(t, y) \dot{q}_y + m \ddot{q} = b z, & \quad t > 0, \quad |y| < 1, \\
\ddot{z}(t, \pm 1) = q(t, \pm 1) = 0, & \quad t \geq 0, \\
\dot{z}(0, y) = z_0(y), \quad \dot{q}(0, y) = q_0(y), & \quad |y| \leq 1.
\end{aligned}
\]
Taking advantage of the $L^p$ theory with $p > 3/(1 - \alpha)$ and the Sobolev embedding theorem we have that the problem (7) admits a unique solution $(\tilde{z}, \tilde{q}) \in [W^{1,2}_p(I_{T_1}) \cap C^{(1+\alpha)/2,1+\alpha}(I_{T_1})]^2$, and
\[
\|\tilde{z}, \tilde{q}\|_{W^{1,2}_p(I_{T_1})} + \|\tilde{z}, \tilde{q}\|_{C^{(1+\alpha)/2,1+\alpha}(I_{T_1})} \leq C_1.
\]

Let
\[
\tilde{g}(t) = -h_0 - \mu \int_0^t \rho(\tau) \tilde{q}_b(\tau, -1) d\tau, \quad \tilde{h}(t) = h_0 - \mu \int_0^t \rho(\tau) \tilde{q}_b(\tau, 1) d\tau.
\]
Then $\tilde{g}(0) = -h_0, \tilde{h}(0) = h_0, \tilde{g}'(0) = g^*, \tilde{h}'(0) = h^*$, and $\tilde{g}', \tilde{h}' \in C^{\alpha/2}([0, T_1])$,
\[
\|\tilde{g}', \tilde{h}'\|_{C^{\alpha/2}([0, T_1])} \leq C_2.
\]

Subsequently, $\tilde{g}, \tilde{h} \in C^{1+\alpha/2}([0, T_1])$.

For $0 < T \leq T_1$, it is clear that
\[
\|\tilde{z}, \tilde{q}\|_{W^{1,2}_p(I_T)} + \|\tilde{z}, \tilde{q}\|_{C^{(1+\alpha)/2,1+\alpha}(I_T)} \leq C_1, \quad \|\tilde{g}', \tilde{h}'\|_{C^{\alpha/2}((0, T))} \leq C_2. \tag{8}
\]

Moreover, we have $\tilde{z}, \tilde{q} > 0$ for $(t, x) \in [0, T] \times (-1, 1)$. Then the Hopf boundary lemma gives $\tilde{q}_b(t, -1) > 0$ and $\tilde{q}_b(t, 1) < 0$ for $t \in [0, T]$, and hence $-\tilde{g}'(t), \tilde{h}'(t) > 0$ for $t \in [0, T]$.

Define a mapping $F : D_T \to [C(I_T)]^2 \times [C^1([0, T])]^2$ by
\[
F(z, q, g, h) = (\tilde{z}, \tilde{q}, \tilde{g}, \tilde{h}).
\]

Obviously, $F$ is continuous in $D_T$, and $(z, q, g, h) \in D_T$ is a fixed point of $F$ if and only if $(u, v, z, w, q, g, h)$ solves (4)-(6). According to (8), we know that $F$ is compact. Since $\tilde{z}(0, y) = z_0(y), \tilde{q}(0, y) = q_0(y), \tilde{g}'(0) = g^*$ and $\tilde{h}'(0) = h^*$, it yields
\[
\|\tilde{z} - z_0\|_{L^\infty(I_T)} \leq \|\tilde{z}\|_{C^{1+\alpha/2}(I_T)} T^{1+\alpha/2} \leq C_1 T^{1+\alpha/2},
\]
\[
\|\tilde{q} - q_0\|_{L^\infty(I_T)} \leq \|\tilde{q}\|_{C^{1+\alpha/2}(I_T)} T^{1+\alpha/2} \leq C_1 T^{1+\alpha/2},
\]
\[
\|\tilde{g}' - g^*\|_{L^\infty([0, T])} \leq \|\tilde{g}'\|_{C^{\alpha/2}([0, T])} T^{\alpha/2} \leq C_2 T^{\alpha/2},
\]
\[
\|\tilde{h}' - h^*\|_{L^\infty([0, T])} \leq \|\tilde{h}'\|_{C^{\alpha/2}([0, T])} T^{\alpha/2} \leq C_2 T^{\alpha/2}.
\]

Therefore, $F$ maps $D_T$ into itself if
\[
T \leq \min \left\{ T_1, C_1^{-2/(1+\alpha)}, C_2^{-2/\alpha} \right\}.
\]

Consequently, $F$ has at least one fixed point $(z, q, g, h) \in D_T$ by the Schauder fixed point theorem.

And so (4)-(6) have at least one solution $(u(t, x), v(t, y), z(t, x), w(t, y), q(t, y), g(t), h(t))$ defined in $[0, T]$.

**Step 3. Existence and estimates of the solution** $(u, v, w, g, h)$ to (2). It is easy to see that $(u, v, w, g, h)$ solves (2), and satisfies
\[
u \in C^1([0, T]; L^\infty(R)), \quad \nu > 0 \text{ in } [0, T] \times R,
\]
\[
v, w \in W^{1,2}_p(D_T) \cap C^{1+\alpha,1+\alpha}(D_T), \quad v, w > 0 \text{ in } D_T,
\]
\[
g, h \in C^{1+\alpha/2}(0, T), \quad -g', h' > 0 \text{ in } (0, T].
\]
By estimates (8), we have \( \|v, w\|_{C^{(1+\alpha)/2, 1+\alpha}(\bar{D}_T)} \leq C_3 \) for some constant \( C_3 \). Thus
\[
\|z, q, \zeta_y, q_y\|_{L^\infty(\bar{D}_T)} \leq C_1, \quad \|v, w, v_x, w_x\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_3, \tag{9}
\]
where we have tacitly assumed \( v(t,x) = w(t,x) = 0 \) when \( x \notin (g(t), h(t)) \).

Recalling \( u_0 \in C^1_0(\mathbb{R}) \) and \( \|v, w\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_3 \), and using the continuous dependence of the solution on parameters, we can show that \( u(t, \cdot) \in C^1_0(\mathbb{R}) \). Moreover, by (4), one can easily derives that there exists a constant \( C_4 \) such that
\[
\|u, u_x\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_4. \tag{10}
\]

**Step 4. Uniqueness of the solution** \((u, v, w, g, h)\) to (2). For simplify notations, we will often use the notation \( X \leq Y \) whenever there exists some constant \( C \) so that \( X \leq CY \); as before, \( C \) can depend on \( \Pi \) but not on \( T \) when \( 0 < T \leq 1 \).

Let \((u_i, v_i, w_i, g_i, h_i), i = 1, 2\), be two solutions of (2), which are defined for \( t \in [0,T] \) with \( 0 < T \ll 1 \), and denote
\[
\begin{align*}
  r_i(t, y) &= u_i \left( t, \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2} \right), \quad 0 \leq t \leq T, \quad |y| \leq 1, \\
  z_i(t, y) &= v_i \left( t, \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2} \right), \quad 0 \leq t \leq T, \quad |y| \leq 1, \\
  q_i(t, y) &= w_i \left( t, \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2} \right), \quad 0 \leq t \leq T, \quad |y| \leq 1.
\end{align*}
\]

By the continuity we may think of (reducing \( T \) if necessary)
\[
- h_0 - 1 \leq g_i(t) \leq - h_0, \quad h_0 \leq h_i(t) \leq h_0 + 1 \quad \text{in } [0,T],
\]
\[
\|u_i - u_0, v_i - v_0, w_i - w_0\|_{L^\infty([0,T] \times \mathbb{R})} \leq 1.
\]

By (8), (9), and (10), we have
\[
\begin{align*}
\begin{cases}
  \|z_i, q_i\|_{W^{1,2}(\bar{D}_T)} \leq C_1, \\
  \|z_{iy}, q_{iy}\|_{L^\infty(\bar{D}_T)} \leq C_1, \\
  \|v_{ix}, w_{ix}\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_3, \\
  \|u_i, u_{ix}\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_4.
\end{cases}
\end{align*} \tag{11}
\]

Clearly, \((z_i, q_i, g_i, h_i)\) solves (5) and (6) with \( r = r_i \). Set \( U = u_1 - u_2, R = r_1 - r_2, V = v_1 - v_2, Z = z_1 - z_2, W = w_1 - w_2, Q = q_1 - q_2, G = g_1 - g_2 \) and \( H = h_1 - h_2 \). Then we have
\[
\begin{align*}
\begin{cases}
  U_t - aU + (u_1 + u_2)U + \varepsilon v_2U + w_2U = -u_1W - \varepsilon u_1V, & 0 < t \leq T, \quad x \in \mathbb{R}, \\
  U(0, x) = 0, & x \in \mathbb{R}, \\
  Z_t - \zeta_1(t,y)Z_y - d_1\rho_1^2(t)Z_{yy} + Z - kr_1Q - kq_2R &= [\zeta_1(t,y) - \zeta_2(t,y)]Z_{2y} + d_1[\rho_1^2(t) - \rho_2^2(t)]Z_{2yy}, & 0 < t \leq T, \quad |y| < 1, \\
  Q_t - \zeta_1(t,y)Q_y - d_2\rho_2^2(t)Q_{yy} + mQ - \rho Z &= [\zeta_1(t,y) - \zeta_2(t,y)]Q_{2y} + d_2[\rho_1^2(t) - \rho_2^2(t)]Q_{2yy}, & 0 < t \leq T, \quad |y| < 1, \\
  Z(0, y) = Q(0, y) = 0, & |y| \leq 1, \\
  Z(t, \pm 1) = Q(t, \pm 1) = 0, & t \geq 0.
\end{cases}
\end{align*} \tag{12}
\]
Indeed, for any given \((v)\) way. Note that \(g\)

\[
\begin{aligned}
G'(t) &= -\mu\rho_1(t)Q_y(t, -1) - \mu(\rho_1(t) - \rho_2(t))q_2(y(t, -1)), \quad 0 < t \leq T, \\
H'(t) &= -\mu\rho_1(t)Q_y(t, 1) - \mu(\rho_1(t) - \rho_2(t))q_2(y(t, 1)), \quad 0 < t \leq T, \\
G(0) &= H(0) = 0,
\end{aligned}
\]

where

\[
\rho_i(t) = \frac{2}{h_i(t) - g_i(t)}, \quad \zeta_i(t, y) = \frac{h'_i(t) - g'_i(t)}{h_i(t) - g_i(t)}y + \frac{h'_i(t) + g'_i(t)}{h_i(t) - g_i(t)}, \quad i = 1, 2.
\]

By using (11), it follows from (12) and (13) that

\[
\begin{aligned}
\|U\|_{L^\infty([0,T] \times \mathbb{R})} &\lesssim T\|V, W\|_{L^\infty([0,T] \times \mathbb{R})}, \\
\|Z\|_{W^{\alpha,2}_p(I_T)} &\lesssim \|Q\|_{L^\infty(I_T)} + \|R\|_{L^\infty(I_T)} + \|G, H\|_{C^1([0,T])}, \\
\|Q\|_{W^{\alpha,2}_p(I_T)} &\lesssim \|Z\|_{L^\infty(I_T)} + \|G, H\|_{C^1([0,T])}.
\end{aligned}
\]

Next we will show

\[
\|V, W\|_{L^\infty([0,T] \times \mathbb{R})} \lesssim \|G, H\|_{C^1([0,T])} + \|Z, Q\|_{L^\infty(I_T)}.
\]

Indeed, for any given \((t, x) \in [0, T] \times \mathbb{R},\) without loss of generality, we only consider the case \(g_1(t) < g_2(t) < h_1(t) < h_2(t).\) Other cases can be obtained in a similar way. Note that \(v_1(t, x) = v_2(t, x) = w_1(t, x) = w_2(t, x) = 0\) when \(x \leq g_1(t)\) or \(x \geq h_2(t).\) It is enough to consider the following three cases:

1. \(x \in [g_2(t), h_1(t)].\) Then \(|x| \leq h_1(t) + |g_2(t)|.\) Let \(y_1(t, x) = \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)} ,\)
   \(i = 1, 2.\) By note (11), the direct calculation yields
   \[
   |v_1(t, x) - v_2(t, x)| = \left|z_1(t, y_1(t, x)) - z_2(t, y_2(t, x))\right|
   \leq \|z_1\|_{L^\infty([0,T] \times \mathbb{R})}|y_1(t, x) - y_2(t, x)| + |z_1(t, y_2(t, x)) - z_2(t, y_2(t, x))|
   \lesssim \|G, H\|_{C^1([0,T])} + \|Z\|_{L^\infty(I_T)}.
   \]

   Similarly, \(|w_1(t, x) - w_2(t, x)| \lesssim \|G, H\|_{C^1([0,T])} + \|Q\|_{L^\infty(I_T)}.
   
2. \(x \in (g_1(t), g_2(t)).\) In such case, \(v_2(t, x) = w_2(t, x) = 0.\) Using (11), we have
   \[
   |v_1(t, x) - v_2(t, x)| = |v_1(t, x) - v_1(t, g_1(t))| + |v_1(t, g_1(t)) - v_1(t, x)|
   \leq \|v_1\|_{L^\infty([0,T] \times \mathbb{R})}|x - g_1(t)|
   \lesssim \|G\|_{C^1([0,T])}.
   \]

   Similarly, \(|w_1(t, x) - w_2(t, x)| \lesssim \|H\|_{C^1([0,T])}.
   
3. \(x \in (h_1(t), h_2(t)).\) Similar to Case 2, we have
   \[
   |v_1(t, x) - v_2(t, x)| \lesssim \|H\|_{C^1([0,T])}, \quad |w_1(t, x) - w_2(t, x)| \lesssim \|G\|_{C^1([0,T])}.
   \]

Combing above discussion, one can get (18). Substituting (18) into (15), one has

\[
\|U\|_{L^\infty([0,T] \times \mathbb{R})} \lesssim T(\|G, H\|_{C^1([0,T])} + \|Z, Q\|_{L^\infty(I_T)}).
\]

Following the proof of [19, Theorem 1.1], we can show that

\[
\begin{aligned}
|Z|_{C^{\overline{\alpha}}(I_T)} &\lesssim \|Z\|_{W^{\alpha,2}_p(I_T)}, \quad \|Q, Q_y\|_{C^{\overline{\alpha}}(I_T)} \lesssim \|Q\|_{W^{\alpha,2}_p(I_T)}.
\end{aligned}
\]
where $\| \cdot \|^{1/2}_{C^{\alpha/2,\infty}}$ is the Holder semi-norm. Note $Z(0, y) = 0$, for any $(t, y) \in I_T$, 
\[
|Z(t, y)| = |Z(t, y) - Z(0, y)| \leq T^{\frac{\alpha}{2}} |Z|^{1/2}_{C^{\alpha/2,\infty}(I_T)} \lesssim T^{\frac{\alpha}{2}} \| Z \|_{W^{1,2}_p(I_T)},
\]
which implies 
\[
\| Z \|_{L^\infty(I_T)} \lesssim T^{\frac{\alpha}{2}} \| Z \|_{W^{1,2}_p(I_T)}.
\]
Similarly $\| Q \|_{L^\infty(I_T)} \lesssim T^{\alpha/2} \| Q \|_{W^{1,2}_p(I_T)}$. Then by (16) and (17), we have 
\[
\begin{align*}
\| Z \|_{L^\infty(I_T)} & \lesssim T^{\frac{\alpha}{2}} (\| Q \|_{L^\infty(I_T)} + \| R \|_{L^\infty([0,T] \times \mathbb{R})} + \| G, H \|_{C^1([0,T])}), \\
\| Q \|_{L^\infty(I_T)} & \lesssim T^{\frac{\alpha}{2}} (\| Z \|_{L^\infty(I_T)} + \| G, H \|_{C^1([0,T])}) \\
& \lesssim T^{\alpha} (\| Q \|_{L^\infty(I_T)} + \| R \|_{L^\infty([0,T] \times \mathbb{R})} + \| G, H \|_{C^1([0,T])}) \\
& \lesssim \| R \|_{L^\infty([0,T] \times \mathbb{R})} + \| G, H \|_{C^1([0,T])} \text{ for sufficiently small } T.
\end{align*}
\]
This combined with (19) and (21) derives 
\[
\| U \|_{L^\infty([0,T] \times \mathbb{R})} \lesssim T \| (G, H) \|_{C^1([0,T])} + \| R \|_{L^\infty(I_T)}.
\]
Denote $x_i = \frac{1}{2} \{ h_i(t) - g_i(t) \} y + h_i(t) + g_i(t)$. One can calculate that 
\[
|r_1(t, y) - r_2(t, y)| = |u_1(t, x_1(t, y)) - u_2(t, x_2(t, y))| \\
\leq |u_1|_{L^\infty([0,T] \times \mathbb{R})} |x_1(t, y) - x_2(t, y)| \\
+ |u_1(t, x_2(t, y)) - u_2(t, x_2(t, y))| \\
\lesssim \| G, H \|_{C^1([0,T])} + \| U \|_{L^\infty([0,T] \times \mathbb{R})},
\]
which implies 
\[
\| R \|_{L^\infty([0,T] \times \mathbb{R})} \lesssim \| G, H \|_{C^1([0,T])} + \| U \|_{L^\infty([0,T] \times \mathbb{R})}.
\]
Hence for sufficiently small $T$, 
\[
\begin{align*}
\| U \|_{L^\infty([0,T] \times \mathbb{R})} & \lesssim \| G, H \|_{C^1([0,T])}, \\
\end{align*}
\]
Besides, taking advantage of (14), (20), (22) and (23), it can be deduced that 
\[
\begin{align*}
|G', H'|^{1/2}_{C^{\alpha/2}([0,T])} & \lesssim \| Q \|_{W^{1,2}_p(I_T)} + \| G, H \|_{C^1([0,T])} \\
& \lesssim \| Q \|_{W^{1,2}_p(I_T)} + \| G, H \|_{C^1([0,T])} + \| U \|_{L^\infty([0,T] \times \mathbb{R})}.
\end{align*}
\]
Noticing that $G'(0) = G(0) = 0$ and $H'(0) = H(0) = 0$, we have 
\[
\begin{align*}
\| G, H \|_{C^1([0,T])} & \lesssim T^{\frac{\alpha}{2}} (T + 1) \| G', H' \|_{C^{\alpha/2}([0,T])} \\
& \lesssim T^{\frac{\alpha}{2}} (\| G, H \|_{C^1([0,T])} + \| U \|_{L^\infty([0,T] \times \mathbb{R})}) \\
& \lesssim T^{\frac{\alpha}{2}} \| U \|_{L^\infty([0,T] \times \mathbb{R})} \text{ for sufficiently small } T.
\end{align*}
\]
This combined with (24) allows us to derive 
\[
\| U \|_{L^\infty([0,T] \times \mathbb{R})} \lesssim T^{\frac{\alpha}{2}} \| U \|_{L^\infty([0,T] \times \mathbb{R})},
\]
which implies $u_1 = u_2$ for sufficiently small $T$. Hence $v_1 = v_2$, $w_1 = w_2$, $g_1 = g_2$, and $h_1 = h_2$, i.e., (2) has a unique local classical solution $(u, v, w, g, h)$. The proof is completed. \hfill \Box
Next, we will extend the local solution to a global one by making some suitable estimates. For the sake of the exposition, it is convenient to define a $2 \times 2$ matrix

$$
\mathcal{L}_u := \begin{pmatrix} -1 & ku \\ b & -m \end{pmatrix}
$$

for all $u \in \mathbb{R}$.

**Theorem 2.2.** The problem (2) admits a unique global solution $(u, v, w, g, h)$ and $(u, v, w, g, h) \in C^1(\mathbb{R}^+; L^\infty(\mathbb{R})) \times [W_p^{1,2}(D_\infty) \cap C^{\frac{1+\alpha}{1+\alpha}}(\overline{D_\infty})]^2 \times [C^{1+\frac{\alpha}{2}}(\mathbb{R}^+)]^2$.

**Proof.** Let $T_0$ be the maximal existence time of $(u, v, w, g, h)$.

Firstly, consider the unique solution $\hat{u}$ of

$$
\begin{cases}
\hat{u}_t = \hat{u}(a - \hat{u}), & t > 0, \\
\hat{u}(0) = \max_{x \in \mathbb{R}} \|u_0\|_\infty.
\end{cases}
$$

And from the comparison principle and the proof of Theorem 2.1 we have

$$
0 < u \leq \max\{a, \|u_0\|_\infty\} := M_1 \text{ in } [0, T_0) \times \mathbb{R},
$$

$v, w > 0$ in $[0, T_0) \times (g(t), h(t))$, $g' < 0$, $h' > 0$ in $[0, T_0)$, where $[0, T_0) \times (g(t), h(t)) = \{0 \leq t < T_0, g(t) < x < h(t)\}$. Then $(v, w)$ satisfies

$$
\begin{cases}
v_t - d_1 v_{xx} \leq kM_1 w - v, & t \in (0, T_0), \quad x \in (g(t), h(t)), \\
w_t - d_2 w_{xx} = bv - mw, & t \in (0, T_0), \quad x \in (g(t), h(t)), \\
v(t, x) = w(t, x) = 0, & t \in [0, T_0), \quad x = g(t), h(t).
\end{cases}
$$

Then consider the unique solution $(\hat{v}, \hat{w})$ of

$$
\begin{cases}
\hat{v}_t = kM_1 \hat{w} - \hat{v}, & t > 0, \\
\hat{w}_t = b\hat{v} - m\hat{w}, & t > 0, \\
\hat{v}(0) = \|v_0\|_\infty, \quad \hat{w}(0) = \|w_0\|_\infty.
\end{cases}
$$

It is obvious that there exists a constant $C > 0$ such that $\hat{v}, \hat{w} \leq Ce^{\lambda_+ t}$ for $t > 0$, where $\lambda_+$ is the larger eigenvalue of matrix $\mathcal{L}_{M_1}$. On the other hand, by comparison principle, we have $v \leq \hat{v}, w \leq \hat{w}$ for $t \in [0, T_0)$ and $x \in (g(t), h(t))$. Thus

$$
v, w \leq Ce^{\lambda_+ t} \text{ in } [0, T_0) \times (g(t), h(t)).
$$

By above estimates and (3), we can extend the unique local solution $(u, v, w, g, h)$ obtained in Theorem 2.1 to a global solution and

$$(u, v, w, g, h) \in C^1(\mathbb{R}^+; L^\infty(\mathbb{R})) \times [W_p^{1,2}(D_\infty) \cap C^{\frac{1+\alpha}{1+\alpha}}(\overline{D_\infty})]^2 \times [C^{1+\frac{\alpha}{2}}(\mathbb{R}^+)]^2,$$

see the proof of [21, Theorem 1.1] for the details. The proof is completed.

In order to obtain the long time behavior, we need some uniform estimates for the solution $(u, v, w, g, h)$. Inspired by [25, 14, 22, 1], we have the following theorem.

**Theorem 2.3.** Let $(u, v, w, g, h)$ be the solution of (2). If $h_\infty - g_\infty < \infty$, then there exists a constant $K > 0$, such that

$$
\|v(t, \cdot), w(t, \cdot)\|_{C^1([g(t), h(t)])} \leq K, \quad \forall t > 1; \quad \|g', h'\|_{C^{\frac{\alpha}{2}}([1, \infty))} \leq K. \quad (25)
$$
Proof. At first, similar as above, we have

\[
\begin{align*}
0 < u(t,x) & \leq \max\{a, \|u_0\|_{\infty}\} := M_1 \quad \text{for} \ t > 0, \ x \in \mathbb{R}, \\
v(t,x), \ w(t,x) & > 0 \quad \text{for} \ t > 0, \ x \in (g(t), h(t)), \\
v_x(t, h(t)), \ w_x(t, h(t)) & < 0, \ v_x(t, g(t)), \ w_x(t, g(t)) > 0 \quad \text{for} \ t > 0.
\end{align*}
\]

(26)

Next, we will use a bootstrap argument to give the uniform \(L^\infty\) estimates for \((v, w)\). The idea of this step comes from [14] and [1]. Introduce functions \(r(t,y), z(t, y), q(t, y)\) as above. Then \((r, z, q)\) satisfies

\[
\begin{align*}
\left\{ \begin{array}{ll}
rt - \zeta(t,y)ry &= ar - r^2 - \varepsilon rz - rq, & t > 0, \ |y| < 1, \\
z_t - d_1 \rho^2(t)z_{yy} - \zeta(t,y)z_y &= krq - z, & t > 0, \ |y| < 1, \\
q_t - d_2 \rho^2(t)q_{yy} - \zeta(t,y)q_y &= bz - mq, & t > 0, \ |y| < 1, \\
z(t, \pm 1) &= q(t, \pm 1) = 0, & t \geq 0, \\
z(0, y) &= z_0(y), \ q(0, y) = q_0(y), & |y| < 1,
\end{array} \right.
\end{align*}
\]

(27)

where \(\rho(t)\) and \(\zeta(t, y)\) are defined as above. Moreover, by (26) we have

\[
\begin{align*}
\left\{ \begin{array}{ll}
0 < r(t,y) & \leq M_1, \ z(t,y), \ q(t,y) \geq 0, & t > 0, \ -1 \leq y \leq 1, \\
z_y(t,1), \ q_y(t,1) & < 0, \ z_y(t,-1), \ q_y(t,-1) > 0, & t > 0.
\end{array} \right.
\end{align*}
\]

We will give the uniform \(L^1\) estimates for \((z, q)\) at first. Multiplying the first equation of (27) by \(k\), adding the second equation of (27), and integrating on the interval \((-1,1)\), we have

\[
\frac{d}{dt} \int_{-1}^{1} (kr + z)dy = - \int_{-1}^{1} (kr + z)dy + \int_{-1}^{1} kr(a + 1 - r)dy - k \int_{-1}^{1} rzdy + d_1 \rho^2 \int_{-1}^{1} \zeta_y(t,1) - \zeta_y(t,-1)dy + \int_{-1}^{1} \zeta(kr + z)dy.
\]

Set \(A(t) = \frac{h'(t)-g'(t)}{h(t)-g(t)}\). As \(h'(t) > 0, g'(t) < 0\), we have

\[
\zeta(kr + z) \bigg|_{-1}^{1} = k \frac{2h'(t)}{h(t) - g(t)} r(t,1) - k \frac{2g'(t)}{h(t) - g(t)} r(t,-1) \leq 2kA(t)M_1.
\]

Thus,

\[
\int_{-1}^{1} \zeta(kr + z)dy = \zeta(kr + z) \bigg|_{-1}^{1} - \int_{-1}^{1} \zeta_y(kr + z)dy \leq 2kA(t)M_1 - A(t) \int_{-1}^{1} (kr + z)dy.
\]

Thanks to \(z_y(t,1) < 0\) and \(z_y(t,-1) > 0\), it follows that

\[
\frac{d}{dt} \int_{-1}^{1} (kr + z)dy \leq -[1 + A(t)] \int_{-1}^{1} (kr + z)dy + 2kM_1[a + 1 + A(t)].
\]

Let \(W(t) = \int_{-1}^{1} (kr + z)dy\) and integrate the above inequality from 0 to \(t\) yielding

\[
W(t) \leq e^{-\int_{0}^{t} (A(s)+1)ds} \left( W(0) + \int_{0}^{t} [2kM_1 + 2kM_1(1 + A(\tau))]e^{\int_{0}^{\tau} (A(s)+1)ds} d\tau \right) \leq W(0) + 2(a + 1)kM_1 := C_1,
\]

which implies \(\int_{-1}^{1} z(t, y) \leq C_1\), i.e., \(\sup_{t \geq 0} \|z(t, \cdot)\|_{L^1((-1,1))} \leq C_1\).
Similarly, it can be derived from the third equation of (27) that
\[
\frac{d}{dt} \int_{-1}^{1} q \, dy = -[m + A(t)] \int_{-1}^{1} q \, dy + b \int_{-1}^{1} z \, dy + d_2 \rho^2(t)[q_y(t,1) - q_y(t,-1)]
\]
\[
\leq -[m + A(t)] \int_{-1}^{1} q \, dy + bC_1,
\]
which implies
\[
\int_{-1}^{1} q(t,y) \, dy \leq \int_{-1}^{1} q(0,y) \, dy + bC_1 := C_2, \quad \Rightarrow \sup_{t\geq 0} \|q(t,\cdot)\|_{L^1((-1,1))} \leq C_2.
\]

Next, we need the uniform \(L^2\) estimates for \(q\). Multiplying the second equation of (27) by \(z\) and the third equation by \(q\), summing them up, and integrating on \((-1,1)\), we have
\[
\frac{d}{dt} \left[ \frac{1}{2} \int_{-1}^{1} (z^2 + q^2) \, dy \right]
\]
\[
= -\rho^2(t) \left[ d_1 \int_{-1}^{1} |z|_y^2 \, dy + d_2 \int_{-1}^{1} |q_y|_y^2 \, dy \right] - \frac{A(t)}{2} \left[ \frac{1}{2} \int_{-1}^{1} (z^2 + q^2) \, dy \right]
\]
\[
+ \int_{-1}^{1} krq \, dy + \int_{-1}^{1} bq \, dy - \int_{-1}^{1} (z^2 + m q^2) \, dy
\]
\[
\leq -\rho^2(t) \left[ d_1 \int_{-1}^{1} |z|_y^2 \, dy + d_2 \int_{-1}^{1} |q_y|_y^2 \, dy \right] + \frac{kM_1 + b}{2} \int_{-1}^{1} (z^2 + q^2) \, dy,
\]
where the Hölder inequality is used. Recall the Gagliardo-Nirenberg interpolation inequality
\[
\|\vartheta\|_2 \leq C_0 \|\vartheta_y\|_{1/2}^{1/2} \|\vartheta\|_1^{1/2},
\]
where \(C_0\) is a positive constant and then with the help of Young’s inequality we obtain
\[
\|\vartheta\|_2^2 \leq \epsilon \|\vartheta_y\|_{1/2}^2 + C_0^3 \epsilon^{-2} \|\vartheta\|_1^2,
\]
where \(0 < \epsilon < \frac{1}{2}\). By replacing \(z\) with \(\vartheta\) and \(\vartheta_z\) in the place of \(\epsilon\) in (30) and then multiplying each side by \((kM_1 + b + \epsilon_z)/2\), we have
\[
- \left( \frac{kM_1 + b}{2} + \epsilon_z \right) \|z\|_2^2 + \left( \frac{kM_1 + b}{2} + \epsilon_z \right) C_0^3 \epsilon^{-2} \|z\|_1^2
\]
\[
\geq - \left( \frac{kM_1 + b}{2} + \epsilon_z \right) \epsilon_z \|z_y\|_2^2.
\]
Choosing \(\epsilon_z\) such that \(kM_1 + b + \epsilon_z \leq d_1 \rho^2(t)\), one can obtain from (29) that
\[
\frac{d}{dt} \|z\|_2^2 \leq -2\epsilon_z \|z\|_1^2 + (kM_1 + b + 2\epsilon_z) C_0^3 \epsilon^{-2} \|z\|_1^2.
\]
Repeating the above process in a similar way and choosing \(\epsilon_q\) such that \(kM_1 + b + \epsilon_q + \epsilon_q^2 \leq d_2 \rho^2(t)\), we can have
\[
\frac{d}{dt} \|q\|_2^2 \leq -2\epsilon_q \|q\|_1^2 + (kM_1 + b + 2\epsilon_q) C_0^3 \epsilon_q^{-2} \|q\|_1^2.
\]
After letting \( \epsilon_\ast = \min\{ \epsilon_x, \epsilon_q \} \), \( \epsilon^\ast = \max\{ \epsilon_x, \epsilon_q \} \) and summing (31) and (32) up, it is easy to derive that

\[
\frac{d}{dt}(\|z\|^2 + \|q\|^2) \leq -2\epsilon^\ast(\|z\|^2 + \|q\|^2) + (kM_1 + b + 2\epsilon^\ast) C_0^3(\epsilon^\ast)^{-\frac{3}{2}}(\|z\|^2 + \|q\|^2),
\]

which means

\[
\|z\|^2 + \|q\|^2 \leq \max\{C_0^3(C_1^2 + C_2^2)(kM_1 + b + 2\epsilon^\ast)(\epsilon^\ast)^{-\frac{3}{2}}(\epsilon^\ast)^{-1}, \|z_0\|^2 + \|q_0\|^2 \} := C_3.
\]

Hence \( \sup_{t \geq 0} \|q(t, \cdot)\|_{L^2([-1,1])} \leq C_3 \).

Now, we are ready to use a bootstrap argument to show the uniform \( L^\infty \) estimates for \( z \). Multiplying the second equation of (27) by \( z^{2\eta - 1} \) yields, after a partial integration,

\[
\frac{d}{dt} \int_{-1}^{1} z^{2\eta} \, dy + d_1 \rho^2(t) \frac{2\eta - 1}{\eta^2} \int_{-1}^{1} (z^{\eta})^2 \, dy + \frac{A(t)}{2\eta} \int_{-1}^{1} z^{2\eta} \, dy
\]

\[
= \int_{-1}^{1} (k\eta q z^{2\eta - 1} - z^{2\eta}) \, dy.
\]

After introducing of \( \xi = z^{\eta} \) as a new variable one gets

\[
\frac{d}{dt} \int_{-1}^{1} \xi^2 \, dy + d_1 \rho^2(t) \frac{2\eta - 1}{\eta^2} \int_{-1}^{1} \xi_\eta^2 \, dy + \frac{A(t)}{2\eta} \int_{-1}^{1} \xi^2 \, dy \leq kM_1 \int_{-1}^{1} q(1 + \xi^2) \, dy.
\]

Denote the left-hand side of (33) by \( L \). All norms are taken with respect to the spatial variable \( y \) whereas time occurs as a parameter.

\[
L \leq kM_1(\|q\|_1 + \|q\|_2(\|\xi\|^2_1))
\]

\[
\leq kM_1 \left[ \|q\|_1 + \|q\|_2 \left( \frac{C_0^2}{2} \left(\|\xi\|_1 + \|\xi\|_2^2 \right) \right)^\frac{3}{2} \right]
\]

\[
\leq kM_1 \left[ \|q\|_1 + \|q\|_2 C_0^2 \left(\|\xi\|_2^2 + \|\xi\|_1 \|\xi\|_2 \right) \right]
\]

\[
\leq kM_1 \left[ \|q\|_1 + \|q\|_2 C_0^2 \left(\|\xi\|_2^2 + \sigma \|\xi\|_2^2 + \sigma^{-1} \|\xi\|_2^2 \right) \right].
\]

In (35) the Gagliardo-Nirenberg inequality

\[
\|\xi\|_4 \leq \frac{C_4}{2} \left(\|\xi\|_1 + \|\xi\|_2^{\frac{3}{2}} \|\xi\|_2^{\frac{1}{2}} \right)
\]

in the version without boundary conditions is used and In Young’s inequality is used again in (36), hence \( \sigma > 0 \) is arbitrary.

By noting that \( h_\infty - g_\infty < \infty \), we have \( \rho_0 := \inf_{t \geq 0} \rho(t) > 0 \). Now setting

\[
\sigma = \frac{(2\eta - 1)d_1 \rho_0^2}{2\eta^2 kM_1 \|q\|_2 C_0^2}
\]

and using the Poincaré inequality

\[
C_5 \|\xi\|_2 \leq \|\xi\|_2^2
\]
(here $C_5$ is the first eigenvalue of operator $-\Delta$ on $(-1, 1)$ with homogeneous Dirichlet boundary condition), one arrives at

$$\frac{d}{dt} \|\xi\|^2 + C_5 d_1 \rho_0^2 (2\eta - 1) \eta^{-1} \|\xi\|^2 \leq 2\eta k M_1 (\|q\|_1 + C_4^2 \|q\|_2) \|\xi\|^2 + \|q\|_2 C_4^2 \sigma^{-1} \|\xi\|^2$$

$$= 2\eta k M_1 (\|q\|_1 + C_4^2 \|q\|_2) \|\xi\|^2 + \frac{8\eta^3 k^2 M_1 \|q\|^2 C_4^4}{(2\eta - 1) d_1 \rho_0^3} \|\xi\|^2.$$

Let $Z_0 := \|z(0, \cdot)\|_{\infty}$ and $Z_\eta := \max\{1, Z_0, \|z(t, \cdot)\|_{\eta}\}$. By integration one gets

$$Z_{2\eta}^2 \leq \max \left\{ 1, \frac{Z_0^2}{Z_{\eta}^2}, \left( 2\eta k M_1 (\|q\|_1 + C_4^2 \|q\|_2 Z_{2\eta}^2) \right)^{\frac{1}{2}} \right\} \left( \frac{4\eta^3 k^2 M_1^2 \|q\|^2 C_4^4}{(2\eta - 1) d_1 \rho_0^3} Z_{2\eta}^{\frac{1}{2}} \right) \leq \max \left\{ 1, \frac{Z_0^2}{Z_{\eta}^2}, (2C_6 Q\eta)^{\frac{1}{2}} Z_{\eta}^{\frac{1}{2}} \right\},$$

where

$$C_6 := \left( \frac{k M_1 (1 + C_4^2)}{2C_5 \rho_0^2 d_1 (2\eta - 1)} + \frac{\eta^2 k^2 M_1^2 C_4^4}{C_5 \rho_0^2 d_1^2 (2\eta - 1)^2} \right)^{\frac{1}{2}}, \quad Q := \max \{1, C_2, C_3\}.$$

Thus, taking $1/(2\eta)$ power yields

$$Z_{2\eta} \leq \max \{1, Z_0, (2C_6 Q\eta)^{1/2} Z_{\eta}\}.$$

By induction, we have

$$Z_{2^\eta} \leq 2^{\delta_1} (C_6 Q\eta)^{\delta_2} Z_{\eta},$$

where

$$\delta_1 = \nu \frac{1}{2^{\eta}} + \frac{(\nu - 1)}{2^{\nu - 1} \eta} + \cdots + \frac{1}{2^\nu \eta} = \frac{2}{\eta} - \frac{1}{2^\nu - 1} \eta - \frac{1}{2^\nu \eta},$$

$$\delta_2 = \frac{1}{2^{\eta}} + \frac{1}{2^{\nu - 1} \eta} + \cdots + \frac{1}{2^\nu \eta} = \frac{2}{\eta} - \frac{1}{2^\nu \eta}.$$

By taking the limit $\nu \to \infty$, we get

$$\sup_{t \geq 0} \|z(t, \cdot)\|_{\infty} \leq \lim_{\nu \to \infty} Z_{2^{\nu + 1} \eta} \leq 2^{\frac{\delta_1}{2}} (C_6 Q\eta)^{\frac{\delta_2}{2}} Z_{\eta}, \quad \eta > 1/2.$$

So it is easy to derive that

$$\sup_{t \geq 0} \|z(t, \cdot)\|_{\infty} \leq 2^4 (C_7 Q)^2 Z_1 \leq 2^4 (C_7 Q)^2 \max \{1, Z_0, C_1\},$$

where

$$C_7 := \left( \frac{k M_1 (1 + C_4^2)}{2C_5 \rho_0^2 d_1} + \frac{\eta^2 k^2 M_1^2 C_4^4}{C_5 \rho_0^2 d_1^2} \right)^{\frac{1}{2}}.$$

Similarly, we can obtain that $\|q(t, \cdot)\|_{\infty}$ is bounded uniformly for $t > 0$. Thus there exists a constant $M_2$ such that

$$v(t, x), w(t, x) \leq M_2 \quad \text{for } t > 0, \ x \in (g(t), h(t)).$$

By using above uniform $L^\infty$ estimates, similar to the proof of [25, Lemma 2.1], we can obtain that

$$-2\mu M_2 M_3 \leq g'(t) < 0, \quad 0 < h'(t) \leq 2\mu M_2 M_3 \quad \text{for } t > 0,$$
where
\[
M_3 = \max \left\{ \frac{1}{h_0}, \sqrt{\frac{b}{2}}, \frac{\|w_0\|_{C^1([-h_0, h_0])}}{M_2} \right\}.
\]

Finally, following the proof of [25, Theorem 4.1] or [22, Theorem 2.2] step by step, we have (25) holds. The proof is completed. \( \square \)

3. The long time behavior in vanishing case. In this section, we will show the long time behavior of the solution to the problem (2) for vanishing case.

**Proposition 3.1.** [21, Proposition 2] Let \( d, C, \mu, \eta_0 \) be positive constants, \( w \in W^{1,2}_p((0, T) \times (0, \eta(t))) \) for some \( p > 1 \) and any \( T > 0 \), and \( \eta, \xi \in C([0, \infty) \times (0, \eta(t))) \), \( \eta \in C^1([0, \infty)) \). If \( (w, \eta) \) satisfies
\[
\begin{aligned}
w_t - dw_{xx} &\geq -Cw, \quad t > 0, \quad 0 < x < \eta(t), \\
w &\geq 0, \quad t > 0, \quad x = 0, \\
w = 0, \quad \eta'(t) &\geq -\mu w_x, \quad t > 0, \quad x = \eta, \\
w(0, x) = w_0(x) &\geq 0, \quad x \in (0, \eta_0), \\
\eta(0) &= \eta_0,
\end{aligned}
\]
and \( \lim_{t \to \infty} \eta(t) = \eta_\infty < \infty, \lim_{t \to \infty} \eta'(t) = 0, \)
\[
\|w(t, \cdot)\|_{C^1([0, \eta(t))]}) \leq M, \forall t > 1
\]
for some constant \( M > 0 \). Then
\[
\lim_{t \to \infty} \max_{0 \leq x \leq \eta(t)} w(t, x) = 0.
\]

We are ready to show the following long time behavior.

**Theorem 3.1.** Let \( (u, v, w, g, h) \) be any solution of (2). If \( h_\infty - g_\infty < \infty \), then
\[
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([\eta(t), \xi(t))]} = 0, \quad \lim_{t \to \infty} \|w(t, \cdot)\|_{C([\eta(t), \xi(t))]} = 0, \quad (37)
\]
\[
\lim_{t \to \infty} u(t, x) = a \quad \text{uniformly on any compact subset of } \mathbb{R}. \quad (38)
\]

**Proof.** Applying (25) and Proposition 3.1, we obtain \( \lim_{t \to \infty} \|w(t, \cdot)\|_{C([\eta(t), \xi(t))]} = 0 \) directly. Then from the second equation of (2), one can easily derives the first limit in (37).

Now, for any \( \sigma > 0 \), there exists \( T > 0 \) such that \( v, w \leq \sigma, \) for \( t > T, \ x \in \mathbb{R} \).
Hence, \( u \) satisfies
\[
u_t \geq u(a - \varepsilon \sigma - \sigma - u), \quad t > T, \quad x \in \mathbb{R}.
\]
For any \( L > 0 \), consider the unique solution \( \bar{u} \) of
\[
\begin{aligned}
u' &= \bar{u}(a - \varepsilon \sigma - \sigma - \bar{u}), \quad t > T, \\
u(T) &= \min_{[-L, L]} u(T, \cdot) > 0.
\end{aligned}
\]
It follows from the comparison principle that \( \bar{u}(t, x) \leq u(t, x) \) for \( t > T \) and \( x \in [-L, L] \). Hence
\[
\liminf_{t \to \infty} \min_{[-L, L]} u(t, \cdot) \geq \lim_{t \to \infty} u(t) = a - \varepsilon \sigma - \sigma.
\]
Since \( \sigma > 0 \) can be arbitrarily small, this implies that
\[
\liminf_{t \to \infty} \min_{x \in [-L, L]} u(t, x) \geq a. \quad (39)
\]
On the other hand, by noting that \( v, w \geq 0 \) for \( t > 0, \ x \in \mathbb{R} \), \( u \) satisfies
\[
u_t \leq u(a - u), \quad t > 0, \quad x \in \mathbb{R}.
\]

Similar as above, we can obtain that
\[
\limsup_{t \to \infty} \max_{x \in \mathbb{R}} u(t, x) \leq a.
\]

Combining this with (39), we have (38) holds. The proof is completed. \( \square \)

4. The criterion governing spreading and vanishing. This section is devoted to the criterion governing spreading and vanishing. The criterion will be discussed in two parts. If \( m \geq abk \), we will show that vanishing always happens. On the other hand, if \( m < abk \), some sufficient conditions for spreading or vanishing will be given.

It follows from the proof of Theorem 2.2 that \( g(t) \) is monotonic decreasing and \( h(t) \) is monotonic increasing. Therefore, there exist \( g_\infty \in [-\infty, -h_0) \) and \( h_\infty \in (h_0, \infty] \) such that \( \lim_{t \to \infty} g(t) = g_\infty \) and \( \lim_{t \to \infty} h(t) = h_\infty \).

**Theorem 4.1.** Let \((u, v, w, g, h)\) be any solution of (2). If \( m \geq abk \), then \( h_\infty - g_\infty < \infty \) and (37), (38) hold.

**Proof.** Let \( \tilde{u} \) be the unique positive solution of
\[
\tilde{u}' = a\tilde{u} - \tilde{u}^2, \quad t > 0; \quad \tilde{u}(0) = a + \|u_0\|_\infty.
\]

It is easy to derive that \( \tilde{u} \leq a + \|u_0\|_\infty e^{-at} \) for \( t > 0 \), and \( u \leq \tilde{u} \) for any \( t > 0 \) and \( x \in \mathbb{R} \) by comparison principle. Hence, \((v, w)\) satisfies
\[
\begin{cases}
v_t - d_1v_{xx} \leq kf(t)v - v, & t > 0, \quad x \in (g(t), h(t)), \\
w_t - d_2w_{xx} = bv - mw, & t > 0, \quad x \in (g(t), h(t)), \\
v(t, x) = 0, \quad w(t, x) = 0, & t \geq 0, \quad x = \{g(t), h(t)\},
\end{cases}
\]

where \( f(t) = a + \|u_0\|_\infty e^{-at} \). Let \((\tilde{v}, \tilde{w})\) be the unique solution of
\[
\begin{cases}
\tilde{v}' = kf(t)\tilde{v} - \tilde{v}, & t > 0, \\
\tilde{w}' = b\tilde{w} - m\tilde{w}, & t > 0, \\
\tilde{v}(0) = \|v_0\|_\infty, \quad \tilde{w}(0) = \|w_0\|_\infty.
\end{cases}
\]

By note that \( m \geq abk \), it is easy to see that there exists a constant \( C_1 > 0 \), such that
\[
\tilde{v}, \quad \tilde{w} \leq C_1 \quad \text{for} \quad t > 0.
\]

On the other hand, by the comparison principle, we have \( v \leq \tilde{v}, \ w \leq \tilde{w} \) for \( t > 0 \) and \( x \in (g(t), h(t)) \).

Direct calculation gives
\[
\frac{d}{dt} \int_{g(t)}^{h(t)} \left( v + \frac{w}{b} \right) dx = \int_{g(t)}^{h(t)} \left( v_t + \frac{w_t}{b} \right) dx
\]
\[
\leq d_1 \int_{g(t)}^{h(t)} v_{xx} dx + \frac{d_2}{b} \int_{g(t)}^{h(t)} w_{xx} dx + k\|u_0\|_\infty e^{-at} \int_{g(t)}^{h(t)} w dx
\]
\[
\leq -\frac{d_2}{b\mu} h'(t) + \frac{d_2}{b\mu} g'(t) + k\|u_0\|_\infty C_1 e^{-at} (h(t) - g(t)).
\]
Denote 
\[ l(t) = h(t) - g(t), \quad \phi(t) = k\|u_0\|_\infty C_1 e^{-at}, \quad f(t) = \int_{g(t)}^{h(t)} \left( v(t', \cdot) + \frac{w(t', \cdot)}{b} \right) dx. \]

Then we have
\[ d_2\phi(t) \leq \frac{b\mu}{d_2} [-f'(t) + \phi(t)] l(t). \]

Integrating from 0 to \( t \) one has
\[
l(t) \leq l(0) + \frac{b\mu}{d_2} \left[ \int_0^t l(s)\phi(s)ds - f(t) + f(0) \right],
\]
\[
\leq \frac{b\mu}{d_2} \int_0^t l(s)\phi(s)ds + l(0) + \frac{b\mu}{d_2} f(0).
\]

By Gronwall's inequality, we have
\[
l(t) \leq \left[ l(0) + \frac{b\mu}{d_2} f(0) \right] \exp \left\{ \frac{b\mu}{d_2} \int_0^t \phi(s)ds \right\} < \infty.
\]

Hence \( h_\infty - g_\infty < \infty \). Then (37) and (38) hold by Theorem 3.1. The proof is completed. \( \square \)

Now, we study the case: \( m < abk \). At first, consider the following eigenvalue problem
\[
\begin{aligned}
&-d_1 \phi_{xx} - kav\psi + \phi + \lambda\phi = 0, \quad l_1 < x < l_2, \\
&-d_2 \psi_{xx} - b\phi + m\psi + \lambda\psi = 0, \quad l_1 < x < l_2, \\
&\phi(l_1) = \psi(l_1) = \phi(l_2) = \psi(l_2) = 0.
\end{aligned}
\]

(40)

It is well known that \( \lambda \) is the eigenvalue of (40) if and only if \( \lambda \) is the eigenvalue of the matrix \(-\gamma_i D + L_a\) for some \( i \geq 1 \), where \( D = \text{diag}(d_1, d_2) \) and \( \gamma_i \) is an eigenvalue of operator \(-\Delta\) in \((l_1, l_2)\) with homogeneous Dirichlet boundary condition. Clearly, \( \lambda \) and \( \gamma_i \) depend only on the difference \( l_2 - l_1 \), and we write \( \lambda = \lambda(l_2 - l_1), \gamma_i = \gamma_i(l_2 - l_1) \) sometimes.

For \( i = 1 \), one can easily derives that the characteristic polynomial of \(-\gamma_1 D + L_a\) has two distinct real roots. We denote the larger root by \( \lambda^+ := \lambda^+(l_2 - l_1) \) and the corresponding eigenvector by \((1, \theta)^T\). It is easy to see that \( \theta > 0 \).

Direct calculations show that \( \lambda^+ \) is decreasing in \( \gamma_1 \). Moreover, it is well known that \( \gamma_1(l_2 - l_1) \) is decreasing in \( l_2 - l_1 \) and
\[
\lim_{l_2 - l_1 \to 0} \gamma_1(l_2 - l_1) = +\infty, \quad \lim_{l_2 - l_1 \to \infty} \gamma_1(l_2 - l_1) = 0.
\]

By note that \( m < abk \), we have \( \lambda^+(l_2 - l_1) \) is increasing in \( l_2 - l_1 \) and
\[
\lim_{l_2 - l_1 \to 0} \lambda^+(l_2 - l_1) = -\infty, \quad \lim_{l_2 - l_1 \to \infty} \lambda^+(l_2 - l_1) := \lambda^*_\infty > 0.
\]

Hence there exists a unique \( \Lambda \) such that \( \lambda^+(\Lambda) = 0 \). Direct calculations give that
\[
\Lambda = \pi \left( \frac{2d_1d_2}{\sqrt{(d_2 + md_1)^2 + 4d_1d_2(abk - m)} - (d_2 + md_1)} \right)^{1/2}.
\]

**Theorem 4.2.** Let \((u, v, w, g, h)\) be any solution of (2). If \( m < abk \) and \( h_\infty - g_\infty < \infty \), then \( h_\infty - g_\infty \leq \Lambda \). Hence, \( h_0 \geq \Lambda/2 \) implies \( h_\infty - g_\infty = \infty \) due to \( g'(t) < 0 \) and \( h'(t) > 0 \) for \( t > 0 \).
Proof. We assume $h_\infty - g_\infty > \Lambda$ to get a contradiction. By Theorem 3.1, the condition $h_\infty - g_\infty < \infty$ implies that
\[
\lim_{t \to \infty} \|v(t, \cdot)||_{C([g(t), h(t))]} = 0, \quad \lim_{t \to \infty} \|w(t, \cdot)||_{C([g(t), h(t))]} = 0, \tag{41}
\]
\[
\lim_{t \to \infty} u(t, x) = a \quad \text{uniformly in any compact subset of} \ \mathbb{R}.
\]

For any given $0 < \delta \ll 1$, there exists $\tau \gg 1$ such that
\[
u(t, x) \geq a - \delta, \quad \forall \ t \geq \tau, \ x \in [g_\infty, h_\infty]; \quad h(\tau) - g(\tau) > \Lambda_\delta,
\]
where
\[
\Lambda_\delta = \pi \left( \frac{2d_1d_2}{\sqrt{(d_2 + md_1)^2 + 4d_1d_2(\delta)(b - m) - (d_2 + md_1)^2}} \right)^{1/2}.
\]
Set $l_1 = h(\tau)$ and $l_2 = g(\tau)$, then $(v, w)$ satisfies
\[
\begin{align*}
v_t - d_1v_{xx} &\geq k(a - \delta)w - v, \quad t > \tau, \ x \in (l_1, l_2), \\
w_t - d_2w_{xx} &\geq bw - mw, \quad t > \tau, \ x \in (l_1, l_2), \\
v(t, x) &> 0, \quad w(t, x) > 0, \quad t \geq \tau, \ x \in (l_1, l_2).
\end{align*}
\]
Define
\[
\tilde{v}(t, x) = \sigma e^{\lambda_+^* t} \sin \left( \frac{\pi(x - l_1)}{l_2 - l_1} \right), \quad \tilde{w}(t, x) = \sigma \theta e^{\lambda_+^* t} \sin \left( \frac{\pi(x - l_1)}{l_2 - l_1} \right),
\]
where $\sigma > 0$ will be chosen later, $\lambda_+^*$ is the larger eigenvalue of matrix $-\gamma_1 D + \mathcal{L}_{a - \delta}$, and $(1, \theta)^T$ is the corresponding eigenvector. One can easily derives that $\lambda_+^* > 0$. Since $\gamma_1 = \pi^2/(l_2 - l_1)^2$, it is easy to verify that $\tilde{v}(t, x)$ and $\tilde{w}(t, x)$ satisfy
\[
\begin{align*}
v_t - d_1v_{xx} &\geq k(a - \delta)w - \tilde{v}, \quad t > \tau, \ x \in (l_1, l_2), \\
w_t - d_2w_{xx} &\geq bw - mw, \quad t > \tau, \ x \in (l_1, l_2), \\
v(t, x) = \tilde{v}(t, x) = 0, \quad t \geq \tau, \ x = l_1, l_2.
\end{align*}
\]
And we can choose a sufficiently small $\sigma$, such that $\tilde{v}(\tau, x) \leq v(\tau, x)$ and $\tilde{w}(\tau, x) \leq w(\tau, x)$ for $x \in [l_1, l_2]$. By comparison principle we can see that $v \leq \tilde{v}$ and $w \leq \tilde{w}$ in $[\tau, \infty) \times [l_1, l_2]$, which is contradict to (41). Thus we have $h_\infty - g_\infty = \infty$. The proof is completed. \hfill \Box

When $h_0 \geq \Lambda/2$, we have spreading always occurs. Next we consider $h_0 < \Lambda/2$.

Lemma 4.1. Suppose $m < abk$ and $h_0 < \Lambda/2$. Then there exists $\mu_0 > 0$ depending on $u_0$, $v_0$, $w_0$ and $h_0$ such that $h_\infty - g_\infty < \infty$ when $\mu \leq \mu_0$.

Proof. Take $\bar{u}$ be the unique solution of
\[
\begin{align*}
\bar{u}'(t) &= \bar{u}(a - \bar{u}), \quad t > 0, \\
\bar{u}(0) &= a + \|u_0\|_\infty.
\end{align*}
\]
Then $\bar{u}(t) > a$ for all $t \geq 0$ and $\lim_{t \to \infty} \bar{u}(t) = a$. By comparison principle, we have $u(t, x) \leq \bar{u}(t)$ for all $t \geq 0$ and $x \in \mathbb{R}$.

Denote $\sigma := \frac{h_0}{a + \frac{A}{\tau}}$, constant $\lambda_+^*$ is the larger eigenvalue of matrix $\gamma_1(\sigma) D + \mathcal{L}_a$, where $\gamma_1(\sigma)$ is the first eigenvalue of operator $-\Delta$ on $(-\sigma, \sigma)$ with homogeneous
Dirichlet boundary condition, and \((1, \theta)^T\) is the corresponding eigenvector. Clearly, we have \(\lambda^+_\sigma < 0, \theta > 0, \gamma_1(\sigma) = \pi^2/(2\sigma)^2\) and
\[
\begin{align*}
-\gamma_1(\sigma)d_1 - 1 + \theta ka &= \lambda^+_\sigma, \\
\eta - \theta(\gamma_1(\sigma)d_2 - m) &= \theta \lambda^+_\sigma.
\end{align*}
\tag{42}
\]
Define
\[
\begin{align*}
f(t) &= M \exp \left\{ \int_0^t [\theta k(u(s) - a) + \lambda^+_\sigma] \, ds \right\}, \quad t \geq 0, \\
\eta(t) &= \left(h_0^2 + \theta \mu \pi \int_0^t f(s) \, ds \right)^{1/2}, \quad t \geq 0, \\
\vec{v}(t, x) &= f(t) \cos \left( \frac{\pi x}{2\eta(t)} \right), \quad t \geq 0, \quad -\eta(t) \leq x \leq \eta(t), \\
\vec{w}(t, x) &= \theta f(t) \cos \left( \frac{\pi x}{2\eta(t)} \right), \quad t \geq 0, \quad -\eta(t) \leq x \leq \eta(t),
\end{align*}
\]
where the constant \(M > 0\) will be chosen later. One can easily derives that \(\eta'(t) > 0\) for \(t \geq 0\) and
\[
\frac{f'(t)}{f(t)} = \theta k(u - a) + \lambda^+_\sigma, \quad \forall \ t \geq 0.
\tag{43}
\]
As \(\lim_{t \to \infty} \vec{u}(t) = a\), it follows that
\[
\lim_{t \to \infty} \theta k(u(t) - a) + \lambda^+_\sigma < 0.
\]
Hence \(\int_0^\infty f(t) \, dt < \infty\). Define
\[
\mu_0 := \frac{\sigma^2 - h_0^2}{\theta \pi \int_0^\infty f(t) \, dt}.
\]
Then for any \(0 < \mu \leq \mu_0\), we have \(\eta(t) < \sigma\) for \(t \geq 0\).

In view of (42) and (43), direct computation yields
\[
\begin{align*}
\vec{v}_t - d_1 \vec{v}_{xx} + \vec{v} - k \vec{u} \vec{w} \\
= f' \cos y + \frac{f \eta' y}{\eta} \sin y + d_1 f \left( \frac{\pi}{2\eta} \right)^2 \cos y + f(1 - \theta k \vec{u}) \cos y \\
&\geq f \cos y \left[ \theta k(u - a) + \lambda^+_\sigma + d_1 \left( \frac{\pi}{2\eta} \right)^2 + 1 - \theta k \vec{u} \right] \\
&= f \cos y \left[ d_1 \left( \frac{\pi}{2\eta} \right)^2 - d_1 \left( \frac{\pi}{2\sigma} \right)^2 \right] > 0
\end{align*}
\]
for all \(t > 0\) and \(-\eta(t) < x < \eta(t)\), where \(y = (\pi x)/(2\eta)\). Similarly,
\[
\begin{align*}
\vec{w}_t - d_2 \vec{w}_{xx} - b \vec{v} + m \vec{w} \\
= \theta f' \cos y + \frac{\theta f \eta' y}{\eta} \sin y + \theta d_2 f \left( \frac{\pi}{2\eta} \right)^2 \cos y + f(\theta m - b) \cos y \\
&\geq f \cos y \left[ \theta^2 k(u - a) + \theta \lambda^+_\sigma + \theta d_2 \left( \frac{\pi}{2\eta} \right)^2 + \theta m - b \right]
\end{align*}
\]
Suppose have the following lemma. The details of proof will be omitted here.

Then for any \( \theta \) depending on \( h \) and \( u \), we can prove the following theorem.

Choose \( M \) large enough such that

\[
\begin{align*}
v_0(x) &\leq M \cos \left( \frac{\pi x}{2h_0} \right) \quad \text{and} \quad w_0(x) \leq \theta M \cos \left( \frac{\pi x}{2h_0} \right) \quad \text{in} \quad [-h_0, h_0].
\end{align*}
\]

Then for any \( 0 < \mu \leq \mu_0 \), \((\bar{v}, \bar{w})\) satisfies

\[
\begin{align*}
\bar{v}_t - d_1 \bar{v}_{xx} &\geq k\bar{w} - \bar{v}, & t > 0, \quad -\eta(t) < x < \eta(t), \\
\bar{w}_t - d_2 \bar{w}_{xx} &\geq b\bar{u} - m\bar{w}, & t > 0, \quad -\eta(t) < x < \eta(t), \\
\bar{v}(t, \pm\eta(t)) &\equiv \bar{w}(t, \pm\eta(t)) = 0, & t > 0, \\
\eta'(t) &\equiv \pm\bar{w}_x(t, \pm\eta(t)), & t > 0, \\
\eta(0) &\geq h_0, \quad \bar{v}(0, x) \geq v_0(x), \quad \bar{w}(0, x) \geq w_0(x), & |x| \leq h_0.
\end{align*}
\]

By comparison principle, we have \(-\eta(t) \leq g(t), \eta(t) \geq h(t)\) for \( t > 0 \). As a consequence,

\[
g_\infty \geq \lim_{t \to \infty} \eta(t) > -\sigma, \quad h_\infty \leq \lim_{t \to \infty} \eta(t) < \sigma,
\]

which means \( h_\infty - g_\infty < \infty \). The proof has been completed.

By using Theorem 4.2 and similar arguments in [24, Lemma 3.1, Lemma 3.2], we have the following lemma. The details of proof will be omitted here.

Lemma 4.2. Suppose \( m < abk \) and \( h_0 < \Lambda/2 \). Then there exists \( \mu^0 > 0 \) such that \( h_\infty - g_\infty = \infty \) when \( \mu > \mu^0 \).

Combining Lemma 4.1 and 4.2, following the arguments in [25, Theorem 5.2] we can prove the following theorem.

Theorem 4.3. Suppose \( m < abk \) and \( h_0 < \Lambda/2 \). Then there exist \( \mu^* \geq \mu_\ast > 0 \), depending on \( u_0, \eta_0, v_0, w_0 \) and \( h_0 \), such that \( h_\infty - g_\infty = \infty \) if \( \mu > \mu^* \) and \( h_\infty - g_\infty \leq \Lambda \) if \( \mu \leq \mu_\ast \) or \( \mu = \mu^* \).

5. Discussion. In [6], Du, Pang and Wang investigate the initial-boundary value problem (1). The section 3 in that paper shows that when \( m > abk \), the stationary point \((a, 0, 0)\) is globally attractive, while \( m < abk \), the constant positive steady state is linearly stable, and hence asymptotically stable.

Based on the deduction of the model in [2], we introduced a free boundary problem where Stefan type free boundaries and degenerate diffusion have been considered in this paper. The following results give an accurate description of the long time behavior:

1. When \( m > abk \), vanishing always happens, i.e. the predator will spread within a bounded area and dies out, and the prey will converge to its predator-free carrying capacity in the long run. This result is similar to the initial-boundary value problem model in [6]. Besides, when \( m = abk \), our results demonstrate that the solution will tend to \((a, 0, 0)\).
2. When \( m < abk \), there is a criterion, i.e. if the size of habitats is larger than the criterion \( \Lambda \) at any time, then spreading happens. On the contrary, predator may vanish eventually. Hence, if the initial occupying area \([-h_0, h_0]\) is beyond the critical size, then spreading always happens regardless of the moving parameter \( \mu \) and initial population densities. On the other hand, if the size of initial habitats is less than the critical size, then whether spreading or vanishing occurs will be determined by moving parameter \( \mu \) and the initial population densities.

There are still some technical difficulties for this model. At first, when \( h_\infty-g_\infty=\infty \), we cannot give an uniform \( L^\infty \) estimate for \((v, w)\). Hence, it is hard to establish a long time behavior of the solution for the spreading case. Without these results, it seems to be impossible to estimate the asymptotic spreading speed, even if we have some results of corresponding traveling wave solution. These problems still need research deeply.

Acknowledgments. The authors would like to thank the anonymous referees for their helpful comments and suggestions.

REFERENCES
[1] N. D. Alikakos, An application of the invariance principle to reaction-diffusion equations, *J. Differential Equations*, 33 (1979), 201–225.
[2] S. M. Baer, B. W. Kooi, Y. A. Kuznetsov and H. R. Thieme, Multiparametric bifurcation analysis of a basic two-stage population model, *SIAM J. Appl. Math.*, 66 (2006), 1339–1365.
[3] H. Bunting, Y. H. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, 7 (2012), 583–603.
[4] X. F. Chen and A. Friedman, A free boundary problem arising in a model of wound healing, *SIAM J. Math. Anal.*, 32 (2000), 778–800.
[5] Y. H. Du and Z. G. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. Ser. B*, 19 (2014), 3105–3132.
[6] Y. H. Du, P. Y. H. Pang and M. X. Wang, Qualitative analysis of a prey-predator model with stage structure for the predator, *SIAM J. Appl. Math.*, 69 (2008), 596–620.
[7] Y. H. Du, M. X. Wang and M. L. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, *J. Math. Pures Appl.*, 107 (2017), 253–287.
[8] S. A. Gourley and Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, *J. Math. Biol.*, 49 (2004), 188–200.
[9] J. S. Guo and C. H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, *Nonlinearity*, 28 (2015), 1–27.
[10] J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differential Equations*, 24 (2012), 873–895.
[11] M. Li and Z. G. Lin, The spreading fronts in a mutualistic model with advection, *Discrete Contin. Dyn. Syst. Ser. B*, 20 (2015), 2089–2105.
[12] S. Y. Liu, H. M. Huang and M. X. Wang, Asymptotic spreading of a diffusive competition model with different free boundaries, *J. Differential Equations*, 266 (2019), 4769–4799.
[13] S. Y. Liu, H. M. Huang and M. X. Wang, Spatial spreading of a logistic SI epidemic model with degenerate diffusion and double free boundaries, preprint.
[14] F. Rothe, Uniform bounds from bounded \( L^p \) functionals in reaction-diffusion equations, *J. Differential Equations*, 45 (1982), 207–233.
[15] L. I. Rubinstein, *The Stefan Problem*, American Mathematical Society, Providence, RI, 1971.
[16] J. Wang, The selection for dispersal: A diffusive competition model with a free boundary, *Z. Angew. Math. Phys.*, 66 (2015), 2143–2160.
[17] J. Wang and L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment, *J. Math. Anal. Appl.*, 423 (2015), 377–398.
[18] J. P. Wang and M. X. Wang, The diffusive Beddington-DeAngelis predator-prey model with nonlinear prey-taxis and free boundary, *Math. Methods Appl. Sci.*, 41 (2018), 6741–6762.
M. X. Wang, Existence and uniqueness of solutions of free boundary problems in heterogeneous environments, *Discrete Contin. Dyn. Syst. Ser. B.*, 24 (2019), 415–421.

M. X. Wang, On some free boundary problems of the prey-predator model, *J. Differential Equations*, 256 (2014), 3365–3394.

M. X. Wang and Q. Y. Zhang, Dynamics for the diffusive Leslie-Gower model with double free boundaries, *Discrete Contin. Dyn. Syst. Ser. A.*, 38 (2018), 2591–2607.

M. X. Wang and Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, *J. Differential Equations*, 264 (2018), 3527–3558.

M. X. Wang and Y. Zhang, Note on a two-species competition-diffusion model with two free boundaries, *Nonlinear Anal.*, 159 (2017), 458–467.

M. X. Wang and Y. Zhang, Two kinds of free boundary problems for the diffusive prey-predator model, *Nonlinear Anal.: Real World Appl.*, 24 (2015), 73–82.

M. X. Wang and J. F. Zhao, A free boundary problem for a predator-prey model with double free boundaries, *J. Dynam. Differential Equations*, 29 (2017), 957–979.

M. X. Wang and J. F. Zhao, Free boundary problems for a Lotka-Volterra competition system, *J. Dynam. Differential Equations*, 26 (2014), 655–672.

C. H. Wu, The minimal habitat size for spreading in a weak competition system with two free boundaries, *J. Differential Equations*, 259 (2015), 873–897.

Q. Y. Zhang and M. X. Wang, Dynamics for the diffusive mutualist model with advection and different free boundaries, *J. Math. Anal. Appl.*, 474 (2019), 1512–1535.

J. F. Zhao, C. M. Song and H. T. Zhang, A diffusive stage-structured model with a free boundary, *Bound. Value Probl.*, 138 (2018), 1–23.

J. F. Zhao and M. X. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, *Nonlinear Anal.: Real World Appl.*, 16 (2014), 250–263.

Y. G. Zhao and M. X. Wang, Free boundary problems for the diffusive competition system in higher dimension with sign-changing coefficients, *IMA J. Appl. Math.*, 81 (2016), 255–280.

L. Zhou, S. Zhang and Z. H. Liu, An evolitional free-boundary problem of a reaction-diffusion-advection system, *Proc. Royal Soc. Edinburgh Sect. A.*, 147 (2017), 615–648.

Received March 2019; revised June 2019.

E-mail address: 16B912019@hit.edu.cn
E-mail address: huanghm@sustech.edu.cn
E-mail address: mxwang@hit.edu.cn