Abstract

We propose and study a scheme combining the finite element method and machine learning techniques for the numerical approximations of coupled nonlinear forward-backward stochastic partial differential equations (FBSPDEs) with homogeneous Dirichlet boundary conditions. Precisely, we generalize the pioneering work of Dunst and Prohl [SIAM J. Sci. Comp., 38(2017), 2725–2755] by considering general nonlinear and nonlocal FBSPDEs with more inclusive coupling; self-contained proofs are provided and different numerical techniques for the resulting finite dimensional equations are adopted. For such FBSPDEs, we first prove the existence and uniqueness of the strong solution as well as of the weak solution. Then the finite element method in the spatial domain leads to approximations of FBSPDEs by finite-dimensional forward-backward stochastic differential equations (FBSDEs) which are numerically computed by using some deep learning-based schemes. The convergence analysis is addressed for the spatial discretization of FBSPDEs, and the numerical examples, including both decoupled and coupled cases, indicate that our methods are quite efficient.

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1 Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space on which is defined a \(k\)-dimensional Wiener process \(W = \{W_t : t \in [0, \infty)\}\) such that \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration generated by \(W\) and augmented by all the \(\mathbb{P}\text{-null sets in} \mathcal{F}\). We denote by \(\mathcal{P}\) the \(\sigma\)-algebra of the predictable sets on \(\Omega \times [0, T]\) associated with \(\{\mathcal{F}_t\}_{t \geq 0}\).

In this paper, we consider the following coupled forward and backward stochastic partial
differential equations with homogeneous Dirichlet boundary conditions:

\[
\begin{align*}
\begin{cases}
\frac{d\rho(t,x)}{dt} & = \left( \Delta \rho(t,x) + F(t,x,\rho(t),\nabla \rho(t),u(t),\nabla u(t),\psi(t)) \right) \quad \text{for } t \in [0,T] \times D; \\
- \sum_{i=1}^{k} f^i(t,x,\rho(t),u(t)) \, dW^i_t, & \quad (t,x) \in [0,T] \times D; \\
\rho(0,x) & = \rho_0(x), \quad x \in D; \\
\rho(t,x) \bigg|_{\partial D} & = 0, \quad t \in [0,T],
\end{cases}
\end{align*}
\]

(1.1)

and

\[
\begin{align*}
\begin{cases}
- \frac{du(t,x)}{dt} & = \left( \Delta u(t,x) + G(t,x,\rho(t),\nabla \rho(t),u(t),\nabla u(t),\psi(t)) \right) \quad \text{for } t \in [0,T] \times D; \\
- \sum_{i=1}^{k} \psi^i(t,x) \, dW^i_t, & \quad (t,x) \in [0,T] \times D; \\
u(T,x) & = g(x,\rho(T)), \quad x \in D; \\
u(t,x) \bigg|_{\partial D} & = 0, \quad t \in [0,T].
\end{cases}
\end{align*}
\]

(1.2)

Here and throughout this paper, \( D \subset \mathbb{R}^d \) is a bounded domain with \( C^2 \) boundary \( \partial D \) and \( T \in (0,\infty) \) a finite deterministic time.

The forward-backward stochastic partial differential equation (FBSPDE) comprising of two equations like (1.1) and (1.2) arises naturally in many applications of probability theory and stochastic processes, for instance in the nonlinear filtering and stochastic control theory for processes with incomplete information, as the (usually coupled) system of the Duncan-Mortensen-Zakai filtration equation (or controlled SPDE) and its adjoint equation (for instance, see \([2, 10, 13, 19]\)); along this line, the study of FBSPDEs can date back to about forty years ago (see \([2]\)). On the other side, in the mean-field game theory certain classes of FBSPDEs (1.1)-(1.2) are raised as the mean-field game system with common noise; the former is a forward stochastic Kolmogorov equation describing the evolution of the conditional distributions of the states of the players given the common noise, while the latter is the stochastic Hamilton-Jacobi-Bellman equation characterizing the value function of the optimization problem when the flow of conditional distributions is given; more details are referred to \([3, 4, 5]\) for instance.

For the decoupled case when \( F \) is independent of \((u, \nabla u, \psi)\) or \((G,g)\) is independent of \( \rho \), the equations (1.1) and (1.2) may be solved separately and they have been extensively studied in the literature; see \([1, 6, 7, 9, 22, 24, 25, 27, 30, 32]\) among many others. However, there are few results on the wellposedness of coupled FBSPDEs, let alone numerical approximations. Indeed, a class of fully coupled FBSPDEs on the whole space was studied in \([33]\) where the FBSPDEs are viewed as natural extensions of (finite dimensional) forward-backward stochastic differential equations (FBSDEs) and the existence and uniqueness of weak solution (in the PDE/SPDE theory) is proved in the spirit of approaches for FBSDEs, while in \([4]\) the wellposedness in Hölder spaces is addressed for a class of FBSPDEs with linear coefficient \( f \) and periodic boundary conditions under certain strong assumptions; meanwhile, numerical methods for a special class of coupled linear FBSPDEs with

\[
F(t,x,\rho(t),\nabla \rho(t),u(t),\nabla u(t),\psi(t)) = u(t), \quad f^i(t,x,\rho(t),u(t)) = -\nu^i(t)\rho(t), \quad i = 1, \ldots, k,
\]
\begin{align*}
G(t, x, \rho(t), \nabla \rho(t), u(t), \nabla u(t), \psi(t)) &= \sum_{i=1}^{k} \nu^i(t)\psi^i(t) + h(t), \quad G(x, \rho(T)) = \Psi(x),
\end{align*}

may be found in the pioneering work of Dunst and Prohl [10] where the convergence analysis is established with finite element method for spatial discretization and the least square Monte Carlo simulation mixed with Picard type iterations or stochastic gradient method for the approximations of the resulting (finite-dimensional) FBSDEs.

In this work, we consider coupled FBSPDEs like (1.1)-(1.2) with homogeneous Dirichlet boundary conditions where coefficients may be \textit{nonlinear} and \textit{nonlocal}. The existence and uniqueness of strong solution of coupled FBSPDEs is derived under Lipschitz conditions. For numerical simulations, the coupled FBSPDE is discretized in spatial domain with finite element method, which results in finite dimensional coupled FBSDEs in temporal domain. We address the wellposedness of such FBSDEs as well as the convergence rate for the spatial discretization. Finally, the resulting FBSDEs are numerically computed with some existing deep learning-based schemes and we present two numerical examples which include both decoupled and coupled cases showing the efficiency of our methods. The approaches mix the existing probability theory and stochastic analysis, (S)PDE theory, and the numerical analysis in both deterministic and stochastic settings.

To overcome the so-called curse of dimensionality, several deep learning-based algorithms have been proposed and studied for numerical computations of partial differential equations (PDEs); see [11, 15, 16, 20, 21] among many others. As these deep learning schemes are based on the equivalence representation relationship between deterministic PDEs and associated Markovian FBSDEs, such numerical methods for PDEs and FBSDEs are one and the same. This paper extends the applications of these numerical methods (or their modifications) to FBSPDEs like (1.1)-(1.2) that may be coupled, nonlinear, and/or nonlocal; nevertheless, the high-dimensionality is not due to the spatial domain of (1.1)-(1.2) but from the resulting finite-dimensional FBSDEs after the spatial discretization of FBSPDE (1.1)-(1.2) with finite element methods, and the deep learning schemes are used to numerically compute solutions of these approximating (finite-dimensional) FBSDEs. On the other hand, many of such FBSPDEs arise from the non-Markovian type stochastic controls/games, with the associated representation systems (FBSDEs) being non-Markovian and even of McKean–Vlasov type (see [3, 4, 5, 27, 29, 31]), and this incurs the inapplicability of the existing deep learning-based methods that are only working under Markovian framework. Because of this, we adopt in this work the strategy: first discretize the FBSPDE (1.1)-(1.2) in spatial domain and then numerically compute the resulting finite-dimensional FBSDEs with the existing deep learning methods.

The rest of this paper is organized as follows. In section 2, we give the notation and assumptions as well as a brief introduction on the finite element methods and the deep neural networks. Section 3 is devoted to the wellposedness of coupled FBSPDEs for both weak and strong solutions. Then the rate of convergence for semi-discrete approximations is proved in Section 4 where the wellposedness of the finite dimensional approximating FBSDEs is also addressed. In Section 5, we introduce and discuss three different deep learning-based methods for the numerical approximations of Markovian FBSDEs. Finally, two numerical examples are presented in Section 6 and the proof of Lemma 4.1 is given in the appendix.
2 Preliminary

2.1 Notations and assumptions

Denote by $| \cdot |$ the norm in Euclidean spaces. For each $l \in \mathbb{N}^+$ and domain $D \subset \mathbb{R}^d$, denote by $C^\infty_c(D; \mathbb{R}^l)$ the space of infinitely differentiable functions $f : D \to \mathbb{R}^l$ with compact supports in $D$. We write $C^\infty_c := C^\infty_c(\mathbb{R}^l) = C^\infty_c(D; \mathbb{R}^l)$ when there is no confusion on the dimension. The Lebesgue measure in $\mathbb{R}^d$ will be denoted by $dx$. Also when there is no confusion on the dimension we write $L^2 := L^2(\mathbb{R}^l) = L^2(D; \mathbb{R}^l)$ for the usual Lebesgue integrable space with scalar product and norm defined

$$
\langle \phi, \psi \rangle = \frac{1}{l} \int_{D} \phi^j(x) \psi^j(x) dx, \quad \| \phi \| = (\langle \phi, \phi \rangle)^{1/2}, \ \forall \phi, \psi \in L^2(D; \mathbb{R}^l).
$$

In addition, for each $(n, p) \in \mathbb{R} \times [1, \infty]$ we define the $n$-th order Sobolev space $(H^{n,p}(\mathbb{R}^l), \| \cdot \|_{n,p})$ as usual; for simplicity, we may write $H^{n,p}(\mathbb{R}^l)$ as $H^{n,p}$ when there is no ambiguity about the dimension. Denote by $H^{1,2}_0$ the space of $f \in H^{1,2}$ with vanishing traces on $\partial D$, i.e., $H^{1,2}_0 = \{ f \in H^{1,2} : f \cdot 1_{\partial D} = 0 \}$. Write $H^{2,2}_0 = H^{2,2} \cap H^{1,2}_0$.

Let $V$ be a Banach space equipped with norm $\| \cdot \|_V$. For $p \in [1, \infty)$, $S^p(V)$ is the set of all the $V$-valued, $(\mathcal{F}_t)$-adapted and continuous processes $\{X_t\}_{t \in [0, T]}$ such that

$$
\|X\|_{S^p(V)} := \left( \sup_{t \in [0, T]} \|X(t)\|_V \right)^{1/p} < \infty.
$$

Denote by $L^p(V)$ the space of all the $V$-valued, $(\mathcal{F}_t)$-adapted processes $\{X_t\}_{t \in [0, T]}$ such that

$$
\|X\|_{L^p(V)} := \left( \mathbb{E} \left[ \int_0^T \|X(t)\|_V^p dt \right] \right)^{1/p} < \infty.
$$

Obviously, $(S^p(V), \| \cdot \|_{S^p(V)})$ and $(L^p(V), \| \cdot \|_{L^p(V)})$ are Banach spaces. By convention, we treat elements of spaces like $S^2(H^{n,2})$ and $L^2(H^{n,2})$ as functions rather than distributions or classes of equivalent functions, and if a function of such class admits a version with better properties, we always denote this version by itself. For example, if $u \in L^2(H^{n,2})$ and $u$ admits a version lying in $S^2(H^{n,2})$, we always adopt the modification $u \in L^2(H^{n,2}) \cap S^2(H^{n,2})$.

For the FBSPDE (1.1)-(1.2), following are the assumptions we use throughout this paper.

**Assumption 2.1.**

(a) For each $(\rho, \tilde{\rho}, u, \tilde{u}, \psi) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^k)$, the function

$$
F(\omega, t, x, \rho, \tilde{\rho}, u, \tilde{u}, \psi) : \Omega \times [0, T] \times D \to \mathbb{R}
$$

is $\mathcal{F} \otimes B(\Omega)$-measurable. There exist positive constants $L^\rho_1, L^\rho_2$ such that for all $(\rho_1, \tilde{\rho}_1, u_1, \tilde{u}_1, \psi_1), (\rho_2, \tilde{\rho}_2, u_2, \tilde{u}_2, \psi_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^k)$ and $(\omega, t) \in \Omega \times [0, T]$,

$$
\| F(\omega, t, \rho_1, \tilde{\rho}_1, u_1, \tilde{u}_1, \psi_1) - F(\omega, t, \rho_2, \tilde{\rho}_2, u_2, \tilde{u}_2, \psi_2) \|
\leq L_1^\rho \left( \| \rho_1 - \rho_2 \| + \| \tilde{\rho}_1 - \tilde{\rho}_2 \| \right) + L_2^\rho \left( \| u_2 - u_1 \| + \| \tilde{u}_2 - \tilde{u}_1 \| + \| \psi_2 - \psi_1 \| \right).
$$
(b) For each \((\rho, u) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})\), the function
\[
f(\omega, t, x, \rho, u) : \Omega \times [0, T] \times D \to \mathbb{R}^k
\]
is \(\mathcal{P} \otimes \mathcal{B}(D)\)-measurable. There exists positive constants \(L^f_1, L^f_2\) and \(\bar{L}^f\) such that for all \(\rho_1, \rho_2, u_1, u_2 \in L^2(\mathbb{R})\) and \(\rho, u \in H^{1,2}_0\) and \((\omega, t) \in \Omega \times [0, T]\),
\[
\|f(\omega, t, \rho_1, u_1) - f(\omega, t, \rho_2, u_2)\| \leq L^f_1 \|\rho_1 - \rho_2\| + L^f_2 \|u_1 - u_2\|
\]
\[
\|f(\omega, t, \rho, u)\|_{1,2} \leq \bar{L}^f(1 + \|\rho\|_{1,2} + \|u\|_{1,2}).
\]

(c) \(\rho_0 \in L^2(\Omega; \mathcal{F}_0, H^{1,2}_0)\) and \(F^0_t \in L^2(\mathcal{L}^2), f^0_t \in L^2(H^{1,2}_0)\) where
\[
F^0_t = F(\omega, t, 0, 0, 0, 0, 0), f^0_t = f(\omega, t, 0, 0) ; \ (\omega, t) \in \Omega \times [0, T].
\]

**Assumption 2.2.**

(a) For each \((\rho, \bar{\rho}, u, \bar{u}, \psi) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^k)\), the function
\[
G(\omega, t, x, \rho, \bar{\rho}, u, \bar{u}, \psi) : \Omega \times [0, T] \times D \to \mathbb{R}
\]
is \(\mathcal{P} \otimes \mathcal{B}(D)\)-measurable. There exist positive constants \(L^G_1, L^G_2\) such that for all \((\rho_1, \bar{\rho}_1, u_1, \bar{u}_1, \psi_1), (\rho_2, \bar{\rho}_2, u_2, \bar{u}_2, \psi_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^k)\) and \((\omega, t) \in \Omega \times [0, T]\),
\[
\|G(\omega, t, \rho_1, \bar{\rho}_1, u_1, \bar{u}_1, \psi_1) - G(\omega, t, \rho_2, \bar{\rho}_2, u_2, \bar{u}_2, \psi_2)\|
\]
\[
\leq L^G_1 \left(\|\rho_1 - \rho_2\| + \|\bar{\rho}_1 - \bar{\rho}_2\|\right) + L^G_2 \left(\|u_2 - u_1\| + \|\bar{u}_2 - \bar{u}_1\| + \|\psi_2 - \psi_1\|\right).
\]

(b) For each \(\rho \in L^2(\mathbb{R})\), the function
\[
g(\omega, x, \rho) : \Omega \times D \to \mathbb{R}
\]
is \(\mathcal{F}_T \otimes \mathcal{B}(D)\)-measurable. There exist positive constants \(L^g\) and \(\bar{L}^g\) such that for all \(\rho_1, \rho_2 \in L^2, \rho_3 \in L^2(\Omega; \mathcal{F}_T, H^{1,2}_0)\) and \(\omega \in \Omega\),
\[
\|g(\omega, \rho_1) - g(\omega, \rho_2)\| \leq L^g \|\rho_1 - \rho_2\|
\]
and
\[
\|g(\omega, \rho_3)\|_{1,2} \leq \bar{L}^g \left(1 + \|\rho_3\|_{1,2}\right).
\]

(c) \(g^0 \in L^2(\Omega; \mathcal{F}_T, H^{1,2}_0)\) and \(G^0_t \in L^2(\mathcal{L}^2)\) where
\[
g^0 = g(\omega, 0), G^0_t = G(\omega, t, 0, 0, 0, 0, 0) ; \ (\omega, t) \in \Omega \times [0, T].
\]

**Remark 2.1.** The above assumptions regarding Lipschitz continuity and linear growth are more or less standard. However, it is worth noting that the dependence of coefficients \(F, G, g\) on the unknown random fields may be nonlocal; for instance, the assumption on \(g\) covers some classes of functions of the following form:
\[
g(x, \rho(T)) = \int_D h(x, y)\rho(T, y)\, dy, \ \text{for} \ x \in D, \ \text{given function} \ h : D \times D \to \mathbb{R}.
\]
Such nonlocal dependence is substantially demanding in the mean-field game systems with common noise (see [3, 4, 5] for instance). In the existing literature, this nonlocal dependence is not taken into account in [10, 33]. In the theory of FBSPDEs for solutions in Hölder spaces in [4], the nonlocal dependence is demanded and allowed; nevertheless, the FBSPDEs therein are equipped with linear coefficients \( \mathbf{f}^j(t, x, \rho(t), u(t)) = \nu^j \sum_{i=1}^{d} \nabla_i \rho(t, x) \) for some constants \( \nu^j \), \( j = 1, \ldots, k \), and the associated stochastic integral can be and is actually disappeared in [4] by using the Itô-Kunita-Wentzell formula.

### 2.2 Finite Element Approximations

Let \( T_h = \{ K \} \) be a triangulation of the convex polyhedral domain \( D \subset \mathbb{R}^d \) into regular simplicial elements \( K \) with \( h := \max \{ \text{diam}(K) : K \in T_h \} \). The intersection of two different elements is either empty, or a vertex, or an entire edge of both elements. \( T_h \) is locally quasi-uniform, i.e., each element contains a ball of radius \( c_1 h \) and is contained in a circle of radius \( c_2 h \), where the constants \( c_1 > 0 \) and \( c_2 < \infty \) do not depend on \( K \) or \( h \). For each element \( K \in T_h \), let \( P^q(K) \) be the set of all polynomials of degree less than or equal to \( q \).

Now we define the finite dimensional space \( V_h^0 \subset H_0^{1, 2} \) consisting of the continuous piecewise linear functions on \( T_h \) by

\[
V_h^0 = \{ \phi : \phi|_K \in P^1(K) \forall K \in T_h, \phi \text{ is continuous on } D \text{ and } \phi = 0 \text{ on } \partial D \}.
\]

Let \( \{ N_1, \ldots, N_L \} \) be an enumeration of internal nodes of \( T_h \) and the space \( V_h^0 \) is spanned by the set of nodal basis functions \( \{ \phi_h^1, \ldots, \phi_h^L \} \). By \( \Pi_h : L^2 \to V_h^0 \), we denote the \( L^2 \)-projection of a given function \( \xi \) onto finite dimensional space, i.e., \( \langle \Pi_h \xi - \xi, \phi_h \rangle = 0 \) for all \( \phi_h \in V_h^0 \). The Ritz projection \( R_h : H_0^{1, 2} \to V_h^0 \) is defined by \( \langle \nabla [R_h \xi - \xi], \nabla \phi_h \rangle = 0 \) for all \( \phi_h \in V_h^0 \). Discrete Laplace operator \( \Pi_h \Delta \equiv \Delta_h : V_h^0 \to V_h^0 \) is given by \( -\langle \Delta_h \xi_h, \phi_h \rangle = \langle \nabla \xi_h, \nabla \phi_h \rangle \), \( \forall \phi_h, \xi_h \in V_h^0 \).

Following are some standard results about the stability of \( L^2 \)-projection onto finite element spaces and the associated approximation error estimates; refer to [12] for instance.

**Theorem 2.1.** In the following assertions, the constant \( C_e \) only depends on the domain and the regularity constants of the mesh but does not depend on \( h \):

(a) The \( L^2 \)-projection is stable on \( L^2 \), i.e., for all \( \xi \in L^2 \) we have \( \| \Pi_h \xi \| \leq \| \xi \| \). For locally quasi-uniform mesh, the \( L^2 \)-projection is also stable on \( H_0^{1, 2} \), i.e., for all \( \xi \in H_0^{1, 2} \) we have \( \| \Pi_h \xi \|_{1, 2} \leq C_e \| \xi \|_{1, 2} \).

(b) The \( L^2 \)-projection of a function into finite element space \( V_h^0 \) is the best approximation in \( V_h^0 \), i.e., \( \| \xi - \Pi_h \xi \| \leq \| \xi - \phi_h \| \) for all \( \phi_h \in V_h^0 \).

(c) The error estimates for \( L^2 \)-projection are given as follow:

\[
\| \xi - \Pi_h \xi \| \leq C_e h \| \xi \|_{1, 2}, \quad \forall \xi \in H_0^{1, 2};
\]

\[
\| \xi - \Pi_h \xi \| \leq C_e h^2 \| \xi \|_{2, 2}, \quad \forall \xi \in H_0^{2, 2};
\]

\[
\| \xi - \Pi_h \xi \|_{1, 2} \leq C_e h \| \xi \|_{2, 2}, \quad \forall \xi \in H_0^{2, 2}.
\]
2.3 Deep Neural Networks

Deep learning provides a very powerful framework for high dimensional function approximation. In what follows, we shall introduce the architecture of deep neural networks and associated universal approximation results.

Consider a deep neural network with input dimension \( d_i \), output dimension \( d_o \), number of layers \( N+1 \in \mathbb{N} \setminus \{1, 2\} \), and number of neurons \( m_n, n = 0, \cdots, N \) on each layer. Here, \( m_0 = d_i, m_N = d_o \) and for simplicity we choose an identical number of neurons for all hidden layers, that is, \( m_n = m, n = 1, \cdots, N - 1 \). Then a feed-forward neural network may be thought of as a function from \( \mathbb{R}^{d_i} \) to \( \mathbb{R}^{d_o} \) defined by compositions of simple functions as

\[
x \in \mathbb{R}^{d_i} \mapsto A_N \circ \cdots \circ A_2 \circ A_1(x) \in \mathbb{R}^{d_o},
\]

where \( f_1 \circ f_2(x) = f_1(f_2(x)) \). Here, \( A_1 : \mathbb{R}^{d_i} \mapsto \mathbb{R}^{m}, A_N : \mathbb{R}^{m} \mapsto \mathbb{R}^{d_o}, \) and \( A_n : \mathbb{R}^{m} \mapsto \mathbb{R}^{m}, n = 2, \cdots, N - 1 \) are affine transformations that take place inside a whole layer and defined by

\[
A_n(x) = W_n x + \beta_n.
\]

Here matrix \( W_n \) and vector \( \beta_n \) are called weight and bias respectively for the \( n \)th layer of the network. For the last layer we choose the identity function as activation function, and the activation function \( \alpha \) here applied component-wise on the outputs of \( A_n \).

We denote by \( \theta = (W_n, \beta_n)_{n=1}^N \) the parameters of neural network. Given \( d_i, d_o, N \) and \( m \), the total number of parameters in a network is \( N_{\theta} = \sum_{n=0}^{N-1} (m_n + 1)m_{n+1} = (d_0 + 1)m + (m + 1)m(N - 2) + (m + 1)d_1 \) and thus \( \theta \in \mathbb{R}^{N_{\theta}} \). Let \( \Theta \) be the set of all possible values of \( \theta \) and if there are no constraint on parameters then \( \Theta = \mathbb{R}^{N_{\theta}} \). By \( \mathcal{X}^\mathcal{N}(\vdash \theta) \) we denote the neural network function defined in \( (2.1) \) and the set of all such neural networks \( \mathcal{X}^\mathcal{N}(\vdash \theta), \theta \in \Theta \) within a fixed structure determined by \( d_i, d_o, N, m \) and \( \alpha \) is denoted by \( \mathcal{N}^\mathcal{N}^\alpha_{d_i,d_o,N,m}(\Theta) \).

Deep neural networks are very efficient for approximations of functions even in high-dimensional spaces. The following fundamental result is from [17, 18]:

**Theorem 2.2** (Universal Approximation Therorem). If \( \alpha \) is continuous and non-constant, it holds that:

(i) The set \( \cup_{m \in \mathbb{N}} \mathcal{N}^\mathcal{N}^\alpha_{d_i,d_o,N,m}(\Theta)(\mathbb{R}^{N_{\theta}}) \) is dense in \( L^2(\nu) \) for any finite measure \( \nu \) on \( \mathbb{R}^{d_i} \).

(ii) If we further have \( \alpha \in C^k \), then \( \cup_{m \in \mathbb{N}} \mathcal{N}^\mathcal{N}^\alpha_{d_o,d_i+2,m}(\mathbb{R}^{2m}) \) approximate any function and its derivatives up to order \( k \), arbitrary well on any compact set of \( \mathbb{R}^{d_i} \).

3 Wellposedness of FBSPDEs

The wellposedness of FBSPDE \([1.1]-[1.2]\) will be addressed in this section, before which we first introduce the definitions of weak and strong solutions.

**Definition 3.1.** Given \( (u, \psi) \in \left( \mathcal{L}^2(H_0^{1,2}) \cap \mathcal{S}^2(L^2) \right) \times \mathcal{L}^2(L^2) \), the random function \( \rho \in \mathcal{L}^2(H_0^{1,2}) \cap \mathcal{S}^2(L^2) \) is said to be a weak solution to FSPDE \([1.1]\) if for each \( \phi \in C_c^{\infty} \), the equality

\[
\langle \rho(t), \phi \rangle = \langle \rho_0, \phi \rangle + \int_0^t \left( -\langle \nabla \rho(t), \nabla \phi \rangle + \langle F(t, x, \rho, \nabla \rho, u, \nabla u, \psi), \phi \rangle \right) dt - \int_0^t \sum_{i=1}^k \langle f^i(t, \rho(t), u(t)), \phi \rangle dW^i(t)
\]

(3.1)
holds for all \( t \in [0, T] \) with probability 1. If we further have \( \rho \in L^2(H_0^{2,2}) \cap S^2(H_0^{1,2}) \) for given \((u, \psi) \in \left(L^2(H_0^{2,2}) \cap S^2(H_0^{1,2})\right) \times L^2(H_0^{1,2})\), then the solution \( \rho \) is called a strong solution.

**Definition 3.2.** Given \( \rho \in L^2(H_0^{1,2}) \cap S^2(L^2) \), the pair \((u, \psi) \in \left(L^2(H_0^{1,2}) \cap S^2(L^2)\right) \times L^2(L^2)\) is called a weak solution to BSPDE (1.2) if for each \( \phi \in C_c^\infty \), the equality

\[
\langle u(t), \phi \rangle = \langle g, \phi \rangle + \int_t^T \left( -\langle \nabla u(t), \nabla \phi \rangle + \langle G(t, x, \rho, \nabla \rho, u, \nabla u, \psi), \phi \rangle \right) dt
\]

holds for all \( t \in [0, T] \) with probability 1. If we further have \((u, \psi) \in \left(L^2(H_0^{2,2}) \cap S^2(H_0^{1,2})\right) \times L^2(H_0^{1,2})\) for given \( \rho \in L^2(H_0^{2,2}) \cap S^2(H_0^{1,2})\), the solution pair \((u, \psi)\) is called a strong solution.

**Definition 3.3.** The tuple \((\rho, u, \psi) \in \left(L^2(H_0^{1,2}) \cap S^2(L^2)\right) \times \left(L^2(H_0^{1,2}) \cap S^2(L^2)\right) \times L^2(L^2)\) is called a weak solution to FBSPDE (1.1)-(1.2) if for each \( \phi \in C_c^\infty \), the equalities

\[
\langle \rho(t), \phi \rangle = \langle \rho_0, \phi \rangle + \int_0^t \left( -\langle \nabla \rho(t), \nabla \phi \rangle + \langle F(t, x, \rho, \nabla \rho, u, \nabla u, \psi), \phi \rangle \right) dt
\]

and

\[
\langle u(t), \phi \rangle = \langle g, \phi \rangle + \int_t^T \left( -\langle \nabla u(t), \nabla \phi \rangle + \langle G(t, x, \rho, \nabla \rho, u, \nabla u, \psi), \phi \rangle \right) dt
\]

hold for all \( t \in [0, T] \) with probability 1. If we further have \((\rho, u, \psi) \in \left(L^2(H_0^{2,2}) \cap S^2(H_0^{1,2})\right) \times \left(L^2(H_0^{2,2}) \cap S^2(H_0^{1,2})\right) \times L^2(H_0^{1,2})\), the solution tuple \((\rho, u, \psi)\) is called a strong solution.

**Theorem 3.1.** Let assumptions \( 2.1, 2.2 \) hold. Then there exists \( \bar{C} = \bar{C}(L^1_1, L^F_1, L^G_2) \) such that if

\[
\bar{C} e^{\bar{C} T} \cdot \max \{|L^G_1|^2 T^2 + |L^G_2|^2 T + |L^F_1|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T + |L^F_2|^2 T\} < 1,
\]

FBSPDE (1.1)-(1.2) admits a unique weak solution \((\rho, u, \psi)\), with

\[
E\left[ \sup_{t \in [0, T]} \|\rho(t)\|^2 + \sup_{t \in [0, T]} \|u(t)\|^2 \right] + E\int_0^T \left( \|\nabla \rho(s)\|^2 + \|\nabla u(s)\|^2 + \|\psi(s)\|^2 \right) ds
\]

\[
\leq C E\left[ \|\rho_0\|^2 + \|g\|^2 + \int_0^T \|F_s^0\|^2 ds + \int_0^T \|f_s^0\|^2 ds + \int_0^T \|G_s^0\|^2 ds \right],
\]
Proof of Theorem 3.1.\]

Moreover, this weak solution is a strong one satisfying

\[
E \left[ \sup_{t \in [0,T]} \| \nabla \rho(t) \|^2 + \sup_{t \in [0,T]} \| \nabla u(t) \|^2 \right] + E \int_0^T \left( \| \Delta \rho(s) \|^2 + \| \Delta u(s) \|^2 + \| \psi(s) \|^2_{1,2} \right) ds \\
\leq C E \left[ 1 + \| \rho_0 \|^2_{1,2} + \| g_0 \|^2 + \int_0^T \| F_0^0 \|^2 ds + \int_0^T \| f_0^0 \|^2 ds + \int_0^T \| G_0^0 \|^2 ds \right],
\]

with constant \( C = C(L_1^f, L_2^f, L_1^\rho, L_2^\rho, L_1^F, L_2^F, L_1^G, L_2^G, T) \).

**Remark 3.1.** Recalling that in Assumptions 2.1, 2.2, the Lipschitz constant \((L_2^f, L_2^\rho)\) characterizes the dependence of the forward equation \((1.1)\) (resp. backward equation \((1.2)\)) on the solution of backward equation \((1.2)\) (resp. forward equation \((1.1)\)), we may see that when either \((L_2^f, L_2^\rho)\) or \((L_2^f, L_2^\rho)\) takes sufficiently small value, the extent of coupling can be thought of to be weak and condition \((4,5)\) guarantees the wellposedness of FBSPDE \((1.1)-(1.2)\). Such an assertion/observation does not exist in [33], because therein, the Lipschitz constants on solutions of forward and backward equations are not separated out as in Assumptions 2.1, 2.2 and the function spaces for the solutions \(\rho\) and \(u\) and the associated computations are different from ours. To discuss the numerical approximations, we focus Theorem 3.1 on the wellposedness of both weak and strong solution, while in [33], only the weak solution is concerned but for a more general class of FBSPDEs without any numerical discussions. Some numerical methods and wellposeness of solutions in Sobolev spaces for coupled FBSPDEs may be found in [10], but the FBSPDEs therein are restricted to linear ones with coefficients of the following form:

\[
F(t, x, \rho(t), \nabla \rho(t), u(t), \nabla u(t), \psi(t)) = u(t), \quad f^i(t, x, \rho(t), u(t)) = -\nu^i(t)\rho(t), \quad i = 1, \ldots, k,
\]

\[
G(t, x, \rho(t), \nabla \rho(t), u(t), \nabla u(t), \psi(t)) = \sum_{i=1}^k \nu^i(t)\psi^i(t) + h(t), \quad G(x, \rho(T)) = \Psi(x).
\]

We would note that neither of the papers [33, 10] incorporate the nonlocal dependence as stated in Remark 2.1.

In addition, the wellposedness in Hölder spaces is addressed in [11] for a class of FBSPDEs with linear coefficient \(f\) and periodic boundary conditions under certain strong assumptions; no numerical approximation is discussed and the readers may refer to Remark 2.1 for more comparisons.

The proof below will be divided into two steps; the first time reader may skip the detailed proof to enjoy the numerical analysis in the next sections.

**Proof of Theorem 3.1.** For each \(u_1', u_2' \in L^2(H_0^{1,2}) \cap S^2(L^2)\) and \(\psi_1', \psi_2' \in L^2(L^2)\), the standard SPDE theory (see [28] for instance) indicates that there are unique solutions, denoted by \(\rho_1\) and \(\rho_2\) respectively, to the following SPDEs

\[
\begin{aligned}
\frac{d\rho_1(t, x)}{dt} &= \left( \Delta \rho_1(t, x) + F(t, x, \rho_1(t), \nabla \rho_1(t), u_1'(t), \nabla u_1'(t), \psi_1'(t)) \right) dt \\
&\quad - \sum_{i=1}^k f^i(t, x, \rho_1(t), u_1'(t)) dW^i_t, \quad (t, x) \in [0, T] \times D; \\
\rho_1(0, x) &= \rho_0(x), \quad x \in D; \\
\rho_1(t, x) &= 0, \quad t \in [0, T],
\end{aligned}
\]  

\[(3.6)\]
In view of the Lipschitz continuity in Assumption 2.1, we have two parts.

(3.6) gives forward equation (1.1). Applying Itô formula for square norm (see \cite{23}, Theorem 3.1) for instance

\[\|\rho_0\|^2 = \|\rho_0\|^2 - 2 \int_0^t \langle \nabla \rho_1(s), \nabla \rho_1(s) \rangle ds - 2 \sum_{i=1}^k \int_0^t \langle f^i(s, \rho_1(s), u_1'(s)), \rho_1(s) \rangle dW^i_t \]

\[+ \sum_{i=1}^k \int_0^t \| f^i(s, \rho_1(s), u_1'(s)) \|^2 ds + 2 \int_0^t \langle F(s, \rho_1(s), \nabla \rho_1(s), u_1'(s), \nabla u_1'(s), \psi_1'(s)), \rho_1(s) \rangle ds. \]

In view of the Lipschitz continuity in Assumption 2.1, we have

\[2 \int_0^t \langle F(s, \rho_1(s), \nabla \rho_1(s), u_1'(s), \nabla u_1'(s), \psi_1'(s)), \rho_1(s) \rangle ds\]
\[ \begin{align*}
& \leq 2 \int_0^t \| F(s, \rho_1(s), \nabla \rho_1(s), u_1'(s), \nabla u_1'(s), \psi_1'(s)) \| \cdot \| \rho_1(s) \| ds \\
& \leq 2 \int_0^t \left( \| F_0(s) \| + L_1 F \| \rho_1(s) \| + \| \nabla \rho_1(s) \| \right) + L_2 F \left( \| u_1'(s) \| + \| \nabla u_1'(s) \| + \| \psi_1'(s) \| \right) \cdot \| \rho_1(s) \| ds \\
& \leq 2 \int_0^t \| F_0(s) \| \cdot \| \rho_1(s) \| ds + 2L_1 F \int_0^t \| \rho_1(s) \|^2 ds + 2L_1 F \int_0^t \| \nabla \rho_1(s) \| \cdot \| \rho_1(s) \| ds \\
& \quad + 2L_2 F \int_0^t \| u_1'(s) \| \cdot \| \rho_1(s) \| ds + 2L_2 F \int_0^t \| \nabla u_1'(s) \| \cdot \| \rho_1(s) \| ds + 2L_2 F \int_0^t \| \psi_1'(s) \| \cdot \| \rho_1(s) \| ds.
\end{align*} \]

Notice that
\[ 2L_2 F \int_0^t \| u_1'(s) \| \cdot \| \rho_1(s) \| ds \leq 2 \sup_{s \in [0, t]} \| \rho_1(s) \| L_2 F \int_0^t \| u_1'(s) \| ds \]
\[ \leq \varepsilon \sup_{s \in [0, t]} \| \rho_1(s) \|^2 + \frac{1}{\varepsilon} L_2 F^2 \left( \int_0^t \| u_1'(s) \| ds \right)^2 \]
\[ \leq \varepsilon \sup_{s \in [0, t]} \| \rho_1(s) \|^2 + \frac{1}{\varepsilon} L_2 F^2 \tau t \int_0^t \| u_1'(s) \|^2 ds. \]

Using similar computations as above and taking supremum over \( t \in [0, \tau] \) for \( \tau \in [0, T] \), we may arrive at
\[ \begin{align*}
& \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 + 2 \int_0^\tau \| \nabla \rho_1(s) \|^2 ds \\
& \leq \| \rho_0 \|^2 + \int_0^\tau \| f(s, \rho_1(s), u_1'(s)) \|^2 ds + 2L_1 F \int_0^\tau \| \rho_1(s) \|^2 ds \\
& \quad + 2 \int_0^\tau \| F_0(s) \| \cdot \| \rho_1(s) \| ds + 2L_1 F \int_0^\tau \| \nabla \rho_1(s) \| \cdot \| \rho_1(s) \| ds + \frac{1}{\varepsilon_1} |L_2|^2 \tau \int_0^\tau \| u_1'(s) \|^2 ds \\
& \quad + \frac{1}{\varepsilon_2} |L_2|^2 \tau \int_0^\tau \| \nabla u_1'(s) \|^2 ds + \frac{1}{\varepsilon_3} |L_2|^2 \tau \int_0^\tau \sum_{i=1}^k \| \psi_1^{i}(s) \|^2 ds + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 \\
& \quad + 2 \sup_{t \in [0, \tau]} \left| \int_0^t \sum_{i=1}^k \langle f^i(s, \rho_1(s), u_1'(s)), \rho_1(s) \rangle dW^i_s \right|.
\end{align*} \]

A straightforward application of Young’s inequality further gives
\[ \begin{align*}
(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3) \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 + (2 - \varepsilon_4) \int_0^\tau \| \nabla \rho_1(s) \|^2 ds \\
& \leq \| \rho_0 \|^2 + \varepsilon_5 \int_0^\tau \| F_0(s) \|^2 ds + 3 \int_0^\tau \| f_0(s) \|^2 ds + \frac{1}{\varepsilon_1} |L_2|^2 \tau \int_0^\tau \| u_1'(s) \|^2 ds \\
& \quad + \frac{1}{\varepsilon_2} |L_2|^2 \tau \int_0^\tau \| \nabla u_1'(s) \|^2 ds + \frac{1}{\varepsilon_3} |L_2|^2 \tau \int_0^\tau \sum_{i=1}^k \| \psi_1^{i}(s) \|^2 ds + 3|L_2|^2 \int_0^\tau \| \psi_1'(s) \|^2 ds \\
& \quad + 2 \sup_{t \in [0, \tau]} \left| \int_0^t \sum_{i=1}^k \langle f^i(s, \rho_1(s), u_1'(s)), \rho_1(s) \rangle dW^i_s \right| + \left( 2L_1 F + \frac{|L_2|^2}{\varepsilon_4} + \frac{1}{\varepsilon_5} + 3|L_1|^2 \right) \int_0^\tau \| \rho_1(s) \|^2 ds.
\end{align*} \]

Letting \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1/6, \varepsilon_4 = 7/4, \varepsilon_5 = 1 \), it follows that
\[ \begin{align*}
& \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 + \frac{1}{2} \int_0^\tau \| \nabla \rho_1(s) \|^2 ds
\end{align*} \]
\[ 11 \]
\[ \leq 2 \left( \| \rho_0 \|^2 + \int_0^T \| F^0_s \|^2 ds + 3 \sum_{i=1}^k \int_0^T \| f^i_s \|^2 ds \right) + 6 | L^f_{2} |^2 \int_0^T \| u'_1(s) \|^2 ds \\
+ 12 | L^f_{2} |^2 \tau \left( \int_0^T \| u'_1(s) \|^2 ds + \int_0^T \| \nabla u'_1(s) \|^2 ds + \int_0^T \| \psi'_1(s) \|^2 ds \right) \\
+ C_1 \int_0^T \| \rho_1(s) \|^2 ds + 4 \sup_{t \in [0, \tau]} \left| \int_0^T \sum_{i=1}^k \langle f^i(s, \rho_1(s), u'_1(s), \rho_1(s))dW^*_s \rangle \right|, \quad (3.10) \]

where constant \( C_1 = C_1(L^f_{1}, L^F_{1}) \).

For the terms involving stochastic integrals, we use BDG inequality and obtain

\[ 4E \left[ \sup_{t \in [0, \tau]} \left| \int_0^T \sum_{i=1}^k \langle f^i(s, \rho_1(s), u'_1(s), \rho_1(s))dW^*_s \rangle \right| \right] \]
\[ \leq \frac{\tilde{C}}{3} \left[ \left( \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 \int_0^T \left( \| f^0_s \|^2 + L^f_{1} \| \rho_1(s) \|^2 + L^f_{2} \| u'_1(s) \|^2 \right) ds \right) \right]^\frac{1}{2} \]
\[ \leq \frac{\tilde{C}}{3} \left[ \left( \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 \int_0^T \left( \| f^0_s \|^2 + L^f_{1} \| \rho_1(s) \|^2 + L^f_{2} \| u'_1(s) \|^2 \right) ds \right) \right]^\frac{1}{2} \]
\[ \leq \frac{\tilde{C}^2 | L^f_{1} |^2}{\varepsilon_6} E \int_0^T \| \rho_1(s) \|^2 ds + \frac{\tilde{C}^2 | L^f_{2} |^2}{\varepsilon_6} E \int_0^T \| u'_1(s) \|^2 ds \]
\[ + \frac{\tilde{C}^2}{\varepsilon_6} \int_0^T \| f^0_s \|^2 ds. \]

Taking expectations on both sides of (3.10) and using above deductions with \( \varepsilon_6 = 1/2 \), we have

\[ E \left[ \sup_{t \in [0, \tau]} \| \rho_1(t) \|^2 \right] + E \int_0^T \| \nabla \rho_1(s) \|^2 ds \]
\[ \leq 4E \left[ \| \rho_0 \|^2 + \int_0^T \| F^0_s \|^2 ds + (\tilde{C}^2 + 3) \int_0^T \| f^0_s \|^2 ds \right] + C_1 E \int_0^T \| \rho_1(s) \|^2 ds \\
+ 24 | L^f_{2} |^2 \tau E \left[ \int_0^T \| u'_1(s) \|^2 ds + \int_0^T \| \nabla u'_1(s) \|^2 ds + \int_0^T \| \psi'_1(s) \|^2 ds \right] \\
+ 2(\tilde{C}^2 + 3) | L^f_{2} |^2 \tau E \left[ \sup_{t \in [0, \tau]} \| u'_1(t) \|^2 \right], \]

which by Gronwall’s inequality implies

\[ E \left[ \sup_{t \in [0, T]} \| \rho_1(t) \|^2 \right] + E \int_0^T \| \nabla \rho_1(s) \|^2 ds \]

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\[
\leq \bar{C}_1 \mathbb{E} \left[ \|\rho_0\|^2 + \int_0^T \|F^0_s\|^2 ds + \int_0^T \|f^0_s\|^2 ds \right] + \bar{C}_1 |L^f_2|^2 T \mathbb{E} \left[ \sup_{t \in [0,T]} \|u'_1(t)\|^2 \right]
\]
\[
+ \bar{C}_1 |L^f_2|^2 T E \left[ \int_0^T \left( \|u'_1(s)\|^2 + \|\nabla u'_1(s)\|^2 + \|\psi'_1(s)\|^2 \right) ds \right],
\]
with the constant \( \bar{C}_1 = 24 \left( \bar{C}^2 + 3 \right) e^{\bar{C}_1 T} \).

Put \( \tilde{\rho} = \rho_1 - \rho_2, \tilde{u}' = u'_1 - u'_2, \) and \( \tilde{\psi}' = \psi'_1 - \psi'_2. \) Through analogous applications of Itô formula to \( \tilde{\rho} \) together with similar computations, we may arrive at
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|\tilde{\rho}(t)\|^2 \right] + E \int_0^T \|
abla \tilde{\rho}(s)\|^2 ds
\]
\[
\leq \bar{C}_1 |L^f_2|^2 T E \int_0^T \left( \|\tilde{u}'(s)\|^2 + \|\nabla \tilde{u}'(s)\|^2 + \|\tilde{\psi}'(s)\|^2 \right) ds + \bar{C}_1 |L^f_2|^2 T E \left[ \sup_{t \in [0,T]} \|\tilde{u}'(t)\|^2 \right]
\]
\[
\leq \bar{C}_1 |L^f_2|^2 T E \int_0^T \left( \|\nabla \tilde{u}'(s)\|^2 + \|\tilde{\psi}'(s)\|^2 \right) ds + \bar{C}_1 T \left( |L^f_2|^2 T + |L^f_2|^2 \right) E \left[ \sup_{t \in [0,T]} \|\tilde{u}'(t)\|^2 \right]
\]
\[
\leq \bar{C}_1 T \cdot \max \{|L^f_2|^2, |L^f_2|^2 T + |L^f_2|^2 \} E \left[ \sup_{t \in [0,T]} \|\tilde{u}'(t)\|^2 + \int_0^T \left( \|\nabla \tilde{u}'(s)\|^2 + \|\tilde{\psi}'(s)\|^2 \right) ds \right].
\]

(3.11)

**Step 2.** Then we conduct the computations for the backward equation (1.12) and derive the wellposedness of FBSPDE (1.11)-(1.12). Applying Itô formula to (3.8) yields that
\[
\|u_1(t)\|^2
\]
\[
= \|g(\rho_1(T))\|^2 - 2 \int_T^t \langle \nabla u_1(s), \nabla u_1(s) \rangle ds - 2 \sum_{i=1}^k \int_T^t \langle \psi_i'(s), u_1(s) \rangle dW^i_s - \int_T^t \|\psi_1(s)\|^2 ds
\]
\[
+ 2 \int_T^t \langle G(s, \rho_1(s), \nabla \rho_1(s), u_1(s), \nabla u_1(s), \psi_1(s)), u_1(s) \rangle ds,
\]
which by Assumption [2.2](c) implies
\[
\|u_1(t)\|^2 + 2 \int_T^t \|\nabla u_1(s)\|^2 ds + \int_T^t \|\psi_1(s)\|^2 ds
\]
\[
\leq 2\|g\|^2 + 2|L^g|^2\|\rho_1(T)\|^2 + 2 \int_T^t \langle G(s, \rho_1(s), \nabla \rho_1(s), u_1(s), \nabla u_1(s), \psi_1(s)), u_1(s) \rangle ds
\]
\[
- 2 \sum_{i=1}^k \int_T^t \langle \psi_i'(s), u_1(s) \rangle dW^i_s. \]  
(3.12)

Further by Assumption [2.2](a) and (c), we have
\[
2 \int_T^t \langle G(s, \rho_1(s), \nabla \rho_1(s), u_1(s), \nabla u_1(s), \psi_1(s)), u_1(s) \rangle ds
\]
\[
\leq 2 \int_T^t \left[ \|G^0\| + L^G_1\|\rho_1(s)\| + \|\nabla \rho_1(s)\| + L^G_2\|u_1(s)\| + \|\nabla u_1(s)\| + \|\psi_1(s)\| \right] \|u_1(s)\| ds
\]
\[
\leq 2 \int_T^t \left( \|G^0\| \cdot \|u_1(s)\| + L^G_2\|u_1(s)\| \cdot \|u_1(s)\| + \|\nabla u_1(s)\| \cdot \|u_1(s)\| + \|\psi_1(s)\| \cdot \|u_1(s)\| \right) ds
\]
\[ + 2L_1^G \int_t^T \left( \| \rho_1(s) \| \cdot \| u_1(s) \| + \| \nabla \rho_1(s) \| \cdot \| u_1(s) \| \right) \, ds \]
\[ \leq \int_t^T \left( \varepsilon_1 \| G_0^s \|^2 + \varepsilon_2 \| \nabla u_1(s) \|^2 + \varepsilon_3 \| \psi_1(s) \|^2 \right) \, ds \]
\[ + \left( \frac{1}{\varepsilon_1} + \frac{|L_1^G|^2}{\varepsilon_2} + \frac{|L_2^G|^2}{\varepsilon_3} + 2L_2^G \right) \int_t^T \| u_1(s) \|^2 \, ds \]
\[ + 2L_1^G \sup_{s \in [t,T]} \| u_1(s) \| \int_t^T \left( \| \rho_1(s) \| + \| \nabla \rho_1(s) \| \right) \, ds \]
\[ \leq \int_t^T \left( \varepsilon_1 \| G_0^s \|^2 + \varepsilon_2 \| \nabla u_1(s) \|^2 + \varepsilon_3 \| \psi_1(s) \|^2 \right) \, ds \]
\[ + \left( \frac{1}{\varepsilon_1} + \frac{|L_1^G|^2}{\varepsilon_2} + \frac{|L_2^G|^2}{\varepsilon_3} + 2L_2^G \right) \int_t^T \| u_1(s) \|^2 \, ds + \varepsilon_4 \sup_{s \in [t,T]} \| u_1(s) \|^2 \]
\[ + |L_1^G|^2 (T-t) \frac{2}{\varepsilon_4} \int_t^T \left( \| \rho_1(s) \|^2 + \| \nabla \rho_1(s) \|^2 \right) \, ds - 2 \sum_{i=1}^k \int_t^T \langle \psi_1^i(s), u_1(s) \rangle \, dW_s^i,\]

Taking \( \varepsilon_1 = 1, \varepsilon_2 = 3/2, \) and \( \varepsilon_3 = 1/2 \) and combining the above computations with (3.12) yield that
\[ \| u_1(t) \|^2 + \frac{1}{2} \int_t^T \| \nabla u_1(s) \|^2 \, ds + \frac{1}{2} \int_t^T \| \psi_1(s) \|^2 \, ds \]
\[ \leq 2 \| g^0 \|^2 + 2 |L^g|^2 \| \rho_1(T) \|^2 + \int_t^T \| G_0^s \|^2 \, ds + C_2 \int_t^T \| u_1(s) \|^2 \, ds + \varepsilon_4 \sup_{s \in [t,T]} \| u_1(s) \|^2 \]
\[ + (T-t) |L_1^G|^2 \frac{2}{\varepsilon_4} \int_t^T \left( \| \rho_1(s) \|^2 + \| \nabla \rho_1(s) \|^2 \right) \, ds - 2 \sum_{i=1}^k \int_t^T \langle \psi_1^i(s), u_1(s) \rangle \, dW_s^i, \]

where the constant \( C_2 = C_2(L_2^G); \) in particular, taking expectations gives
\[ \mathbb{E} \int_t^T \| \psi_1(s) \|^2 \, ds \]
\[ \leq 4 \mathbb{E} \left[ \| g^0 \|^2 + |L^g|^2 \| \rho_1(T) \|^2 \right] + 2 \mathbb{E} \int_t^T \| G_0^s \|^2 \, ds + 2 \varepsilon_4 \mathbb{E} \left[ \sup_{s \in [t,T]} \| u_1(s) \|^2 \right] \]
\[ + 4(T-t) |L_1^G|^2 \frac{1}{\varepsilon_4} \mathbb{E} \int_t^T \left( \| \rho_1(s) \|^2 + \| \nabla \rho_1(s) \|^2 \right) \, ds + C_2 \mathbb{E} \int_t^T \| u_1(s) \|^2 \, ds. \]

On the other hand, taking supremum over \( t \in [\tau,T] \) for \( \tau \in [0,T] \) in (3.12), we have
\[ \sup_{t \in [\tau,T]} \| u_1(t) \|^2 + 2 \int_{\tau}^T \| \nabla u_1(s) \|^2 \, ds + \int_{\tau}^T \| \psi_1(s) \|^2 \, ds \]
\[ \leq 2 \| g^0 \|^2 + 2 |L^g|^2 \| \rho_1(T) \|^2 + 2 \sup_{t \in [\tau,T]} \left| \int_t^T \sum_{i=1}^k \langle \psi_1^i(s), u_1(s) \rangle \, dW_s^i \right| \]
\[ + 2 \int_{\tau}^T \langle G(s, \rho_1(s), \nabla \rho_1(s), u_1(s), \nabla u_1(s), \psi_1(s)), u_1(s) \rangle \, ds. \]

Rewriting (3.13) as
\[ 2 \int_{\tau}^T \langle G(s, \rho_1(s), \nabla \rho_1(s), u_1(s), \nabla u_1(s), \psi_1(s)), u_1(s) \rangle \, ds \]
Again, we use BDG inequality to deal with the stochastic integrals and obtain
\[
\begin{align*}
&\leq \int_{\tau}^{T} \left( \varepsilon_1 \|G_s^0\|^2 + \varepsilon_2 \|\nabla u_1(s)\|^2 + \varepsilon_3 \|\psi_1(s)\|^2 \right) ds + \varepsilon_5 \sup_{t \in [\tau, T]} \|u_1(t)\|^2 \\
&\quad + \left( \frac{1}{\varepsilon_1} + \frac{|L_2 G|^2}{\varepsilon_2} + \frac{|L_2 G|^2}{\varepsilon_3} + 2L_2 G \right) \int_{\tau}^{T} \|u_1(s)\|^2 ds \\
&\quad + 2|L_1 G|^2 (T - \tau) \frac{1}{\varepsilon_5} \int_{\tau}^{T} \left( \|\rho_1(s)\|^2 + \|\nabla \rho_1(s)\|^2 \right) ds,
\end{align*}
\]
and choosing \( \varepsilon_1 = 1, \varepsilon_2 = 7/4, \varepsilon_3 = 3/4, \) and \( \varepsilon_5 = 1/2, \) we have
\[
\sup_{t \in [\tau, T]} \|u_1(t)\|^2 + \frac{1}{2} \int_{\tau}^{T} \|\nabla u_1(s)\|^2 ds + \frac{1}{2} \int_{\tau}^{T} \|\psi_1(s)\|^2 ds \\
\leq 4 \left( \|g^0\|^2 + |L^g|^2 \|\rho_1(T)\|^2 \right) + 8|L_1 G|^2 (T - \tau) \int_{\tau}^{T} \left( \|\rho_1(s)\|^2 + \|\nabla \rho_1(s)\|^2 \right) ds \\
+ 4 \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| + C_2 \int_{\tau}^{T} \|u_1(s)\|^2 ds + 2 \int_{\tau}^{T} \|G_s^0\|^2 ds,
\]
with \( C_2 = C_2(L_2 G), \) which by taking expectations on both sides implies
\[
\begin{align*}
&\mathbb{E} \left[ \sup_{t \in [\tau, T]} \|u_1(t)\|^2 \right] + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \|\nabla u_1(s)\|^2 ds + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \|\psi_1(s)\|^2 ds \\
&\leq 4 \mathbb{E} \left[ \|g^0\|^2 + |L^g|^2 \|\rho_1(T)\|^2 \right] + 2 \mathbb{E} \left[ \int_{\tau}^{T} \|G_s^0\|^2 ds \right] + 4 \mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \right] \\
&\quad + 8|L_1 G|^2 (T - \tau) \mathbb{E} \left[ \int_{\tau}^{T} \left( \|\rho_1(s)\|^2 + \|\nabla \rho_1(s)\|^2 \right) ds \right] + C_2 \mathbb{E} \int_{\tau}^{T} \|u_1(s)\|^2 ds. \tag{3.15}
\end{align*}
\]
Again, we use BDG inequality to deal with the stochastic integrals and obtain
\[
\begin{align*}
&4 \mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \right] \\
&= 4 \mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i - \int_{\tau}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \right] \\
&\leq 4 \mathbb{E} \left[ \int_{\tau}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right] + \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \\
&\leq 4 \mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \right] + \sup_{t \in [\tau, T]} \left| \int_{t}^{T} \sum_{i=1}^{k} \langle \psi_1^i(s), u_1(s) \rangle dW_s^i \right| \\
&\leq \tilde{C} \mathbb{E} \left[ \int_{\tau}^{T} \sum_{i=1}^{k} \left| \langle \psi_1^i(s), u_1(s) \rangle \right|^2 ds \right]^{1/2} \\
&\leq \tilde{C} \mathbb{E} \left[ \sup_{t \in [\tau, T]} \|u_1(t)\|^2 \int_{t}^{T} \sum_{i=1}^{k} \|\psi_1^i(s)\|^2 ds \right]^{1/2} \\
&\leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \in [\tau, T]} \|u_1(t)\|^2 \right] + \tilde{C}^2 \mathbb{E} \int_{\tau}^{T} \|\psi_1(s)\|^2 ds,
\end{align*}
\]
which together with (3.14) and (3.15) implies

\[
\frac{3}{4} \mathbb{E} \left[ \sup_{t \in [\tau, T]} \| u(t) \|^2 \right] + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \| \nabla u(s) \|^2 ds + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \| \psi_1(s) \|^2 ds \\
\leq \hat{C}_1 \mathbb{E} \left[ \| g^0 \|^2 + |L^0|^2 \| \rho_1(T) \|^2 \right] + \hat{C}_1 \mathbb{E} \left[ \int_{\tau}^{T} \| G_0^s \|^2 ds \right] + \hat{C}_2 \mathbb{E} \int_{\tau}^{T} \| u_1(s) \|^2 ds \\
+ \hat{C}_1 |L^G_1|^2 (T - \tau) \mathbb{E} \left[ \int_{\tau}^{T} \left( \| \rho_1(s) \|^2 + \| \nabla \rho_1(s) \|^2 \right) ds \right] + 2\hat{C}_2 \varepsilon_4 \mathbb{E} \left[ \sup_{t \in [\tau, T]} \| u_1(t) \|^2 \right],
\]

with \( \hat{C}_1 = \hat{C}_2^2 + 8 \) and \( \hat{C}_2 = \hat{C}_2 C_2 + C_2 \). Taking \( \varepsilon_4 = \frac{1}{8c^2} \) gives

\[
\mathbb{E} \left[ \sup_{t \in [\tau, T]} \| u_1(t) \|^2 \right] + \mathbb{E} \int_{\tau}^{T} \| \nabla u_1(s) \|^2 ds + \mathbb{E} \int_{\tau}^{T} \| \psi_1(s) \|^2 ds \\
\leq 2\hat{C}_1 \mathbb{E} \left[ \| g^0 \|^2 + |L^0|^2 \| \rho_1(T) \|^2 \right] + 2\hat{C}_1 \mathbb{E} \int_{\tau}^{T} \| f_0^s \|^2 ds + 2\hat{C}_2 \mathbb{E} \int_{\tau}^{T} \| u_1(s) \|^2 ds \\
+ 2\hat{C}_1 |L^G_1|^2 (T - \tau) \mathbb{E} \left[ \int_{\tau}^{T} \left( \| \rho_1(s) \|^2 + \| \nabla \rho_1(s) \|^2 \right) ds \right],
\]

which by Gronwall’s inequality indicates that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \hat{u}(t) \|^2 \right] + \mathbb{E} \int_{0}^{T} \left( \| \nabla \hat{u}(s) \|^2 + \| \hat{\psi}(s) \|^2 \right) ds \\
\leq \hat{C}_2 |L^0|^2 \mathbb{E} \left[ \| \hat{\rho}(T) \|^2 \right] + \hat{C}_2 |L^G_1|^2 T \mathbb{E} \left[ \int_{0}^{T} \left( \| \hat{\rho}(s) \|^2 + \| \nabla \hat{\rho}(s) \|^2 \right) ds \right] \\
\leq \hat{C}_2 |L^0|^2 \mathbb{E} \left[ \sup_{s \in [0, T]} \| \hat{\rho}(s) \|^2 \right] + \hat{C}_2 |L^G_1|^2 T \mathbb{E} \left[ \cdot \sup_{s \in [0, T]} \| \hat{\rho}(s) \|^2 + \int_{0}^{T} \| \nabla \hat{\rho}(s) \|^2 ds \right] \\
\leq \hat{C}_2 \cdot \max \left\{ |L^G_1|^2 T^2 + |L^0|^2, |L^G_1|^2 T \right\} \mathbb{E} \left[ \sup_{s \in [0, T]} \| \hat{\rho}(s) \|^2 + \int_{0}^{T} \| \nabla \hat{\rho}(s) \|^2 ds \right].
\]

(3.16)

Substituting (3.11) into (3.16) we finally have

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \hat{u}(t) \|^2 \right] + \mathbb{E} \int_{0}^{T} \left( \| \nabla \hat{u}(s) \|^2 + \| \hat{\psi}(s) \|^2 \right) ds \\
\leq \hat{C} \mathbb{E} \left[ \sup_{s \in [0, T]} \| \hat{u}'(s) \|^2 + \int_{0}^{T} \left( \| \nabla \hat{u}'(s) \|^2 + \| \hat{\psi}'(s) \|^2 \right) ds \right],
\]

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where \( \hat{C} = \hat{C}e^{CT} \cdot \max \{|L_1^G|^2T^2 + |L_2^G|^2T \} \cdot T \cdot \max \{|L_2^F|^2, |L_2^F|^2T + |L_2^F|^2 \} \) with \( \hat{C} \) depending on \( L_1^G, L_2^F \) and \( L_2^G \). Hence we can conclude that as long as

\[
\hat{C} = \hat{C}e^{CT} \cdot \max \{|L_1^G|^2T^2 + |L_2^G|^2T \} \cdot T \cdot \max \{|L_2^F|^2, |L_2^F|^2T + |L_2^F|^2 \} < 1,
\]

the mapping \((\rho'_1, \psi'_1) \mapsto (u_1, \psi_1)\) is a contraction and FBSPDE (1.1)-(1.2) admits a unique weak solution \((\rho, u, \psi)\) which satisfies

\[
E \left[ \sup_{t \in [0,T]} \|\rho(t)\|^2 + \sup_{t \in [0,T]} \|u(t)\|^2 \right] + E \int_0^T (\|\nabla \rho(s)\|^2 + \|\nabla u(s)\|^2 + \|\psi(s)\|^2) \, ds \\
\leq CE \left[ \|\rho_0\|^2 + \|g_0\|^2 + \int_0^T \|F_s^0\|^2 \, ds + \int_0^T \|L_s^0\|^2 \, ds + \int_0^T \|G_s^0\|^2 \, ds \right], \tag{3.17}
\]

with \( C = C(L_1^f, L_2^f, L_2^g, L_1^f, L_2^f, L_1^g, L_2^g, T) \).

Now, as \((\rho, u, \psi)\) is the unique weak solution to FBSPDE (1.1)-(1.2), Assumptions 2.1-2.2 allow us to further check that \( F(\rho, \nabla \rho, u, \nabla u, \psi) \in L^2(L^2) \) and \( f(\rho) \in L^2(H^{1,2}) \). Then by [8, Theorem 2.5], \( \rho \) is the strong solution of FSPDE (1.1); similarly, [9, Theorem 3.1] implies that \((u, \psi)\) is the strong solution of BSPDE (1.2). Therefore, FBSPDE (1.1)-(1.2) admits a unique strong solution \((\rho, u, \psi)\) and as a straightforward consequence of (3.17) and the estimates in [8, Theorem 2.5] and [9, Theorem 3.1], it holds that

\[
E \left[ \sup_{t \in [0,T]} \|\nabla \rho(t)\|^2 + \sup_{t \in [0,T]} \|\nabla u(t)\|^2 \right] + E \int_0^T (\|\Delta \rho(s)\|^2 + \|\Delta u(s)\|^2 + \|\psi(s)\|^2) \, ds \\
\leq CE \left[ \|\rho_0\|^2 + \|\nabla \rho_0\|^2 + \|g_0\|^2 + \int_0^T \|F_s^0\|^2 \, ds + \int_0^T \|L_s^0\|^2 \, ds + \int_0^T \|G_s^0\|^2 \, ds \right],
\]

with \( C = C(L_1^f, L_2^f, L_2^g, L_1^f, L_2^f, L_1^g, L_2^g, T) \).

\[\square\]

4 Rate of Convergence for Semi-Discrete Approximation

Let \((\rho, u, \psi)\) the solution to FBSPDE (1.1)-(1.2). With the finite element method, we approximate it by the triple \((\rho_h, u_h, \psi_h)\) satisfying the following FBSDH:

\[
\begin{cases}
\frac{d\rho_h(t)}{dt} = \left( \Delta_h \rho_h(t) + \Pi_h F(t, x, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t)) \right) \, dt \\
- \sum_{i=1}^k \Pi_h f^i(t, \rho_h(t), u_h(t)) \, dW^i(t), \quad t \in (0, T]; \\
\rho_h(0) = \rho_0,
\end{cases}
\tag{4.1}
\]

and

\[
\begin{cases}
-\frac{du_h(t)}{dt} = \left( \Delta_h u_h(t) + \Pi_h G(t, x, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t)) \right) \, dt \\
- \sum_{i=1}^k \psi_h^i(t) \, dW^i(t), \quad t \in (0, T]; \\
u_h(T) = \Pi_h g(\rho_h(T)).
\end{cases}
\tag{4.2}
\]

\[\text{As the equations (4.1) and (4.2) are essentially finite-dimensional, we call it an FBSDE here.}\]
In what follows, we set
\[ \varrho := \rho - \rho_h, \quad \mathcal{U} := u - u_h, \quad \zeta := \psi - \psi_h. \]

First comes the wellposedness of FBSDE (4.1)-(4.2).

**Lemma 4.1.** Let assumptions [2.1] hold. Then for each fixed \( h > 0 \) there exists \( \bar{C} = C(L^f, L^g, L^G) \) such that if
\[
\bar{C} e^{CT} \cdot \max \{ |L^G_1|^2 T^2 + |L^g|^2, |L^g|^2 T \} \cdot T \cdot \max \{ |L^F_2|^2, |L^F_1|^2 T + |L^f|^2 \} < 1, \quad (4.3)
\]
FBSDE (4.1)-(4.2) admits a unique solution \((\rho_h, u_h, \psi_h) \in \left( L^2(V^0_h) \cap S^2(V^0_h) \right) \times \left( L^2(V^0_h) \cap S^2(V^0_h) \right) \times L^2(V^0_h), \)

\begin{align*}
(1) \quad & \mathbb{E} \left[ \sup_{t \in [0,T]} \| \rho_h(t) \|^2 + \sup_{t \in [0,T]} \| u_h(t) \|^2 \right] + \mathbb{E} \left[ \int_0^T \left( \| \nabla \rho_h(s) \|^2 + \| \nabla u_h(s) \|^2 + \| \psi_h(s) \|^2 \right) ds \right] \\
& \leq C_1 \mathbb{E} \left[ \| \rho_0 \|^2 + \| g^0 \|^2 + \int_0^T \left( \| f^0_s \|^2 + \| F^0_s \|^2 + \| G^0_s \|^2 \right) ds \right], \\
(2) \quad & \mathbb{E} \left[ \sup_{t \in [0,T]} \| \nabla \rho_h(t) \|^2 + \sup_{t \in [0,T]} \| \nabla u_h(t) \|^2 \right] \\
& + \mathbb{E} \left[ \int_0^T \left( \| \Delta_h \rho_h(s) \|^2 + \| \Delta_h u_h(s) \|^2 + \| \psi_h(s) \|^2 \right) ds \right] \\
& \leq C_2 \mathbb{E} \left[ 1 + \| \rho_0 \|^2 + \| \rho_0 \|^2 + \| g^0 \|^2 + \int_0^T \left( \| f^0_s \|^2 + \| F^0_s \|^2 + \| G^0_s \|^2 \right) ds \right],
\end{align*}

where the constants \( C_1 = C_1(L^f_1, L^f_2, L^g, L^g_1, L^F_1, L^F_2, L^G, L^G_1, L^G_2, T, C_e) \) and \( C_2 = C_2(L^f_1, L^f_2, L^g, L^g_1, L^g_2, L^F_1, L^F_2, L^G, L^G_1, L^G_2, T, C_e) \) are independent of \( h \).

As the computations involved in the proof of Lemma 4.1 are more or less standard, its proof is postponed to the appendix. In what follows, we denote
\[
Q_0 := \mathbb{E} \left[ 1 + \| \rho_0 \|_{1,2}^2 + \| g^0 \|_{1}^2 + \int_0^T \left( \| f^0_s \|^2 + \| F^0_s \|^2 + \| G^0_s \|^2 \right) ds \right].
\]

Next, we shall prove the error estimate for \( \rho \) in terms of \((\mathcal{U}, \zeta)\). Recalling the \( L^2 \)-projection \( \Pi_h \) and the associated error estimates in Theorem 2.1, we have for \( \xi \in H^{1,2}_0 \):
\[
\| \xi \|^2 = \| \Pi_h \xi \|^2 + \| (id - \Pi_h) \xi \|^2 \leq \| \Pi_h \xi \|^2 + C^2 e h^2 \| \xi \|_{1,2}^2, \quad \text{and} \quad \| \Pi_h \xi \|^2 \leq \| \xi \|^2. \quad (4.4)
\]
If we assume further \( \xi \in H^{2,2}_0 \), applying the error estimates in Theorem 2.1 and the classical Elliptic equation theory (see [14] Lemma 9.17 for instance) gives that
\[
\| [id - \Pi_h] \xi \|_{1,2} \leq C e h \| \xi \|_{2,2} \leq Ch \| \Delta \xi \|, \quad (4.5)
\]
where the constant \( C \) only depends on the domain \( D \) and dimension \( d \), being independent of \( h \).
Lemma 4.2. Given \((u, \psi) \in (L^2(H^0_0)^2) \cap S^2(H^0_0)) \times L^2(H^1_0), \) let \(\rho\) be the strong solution of FSPDE \([1.1]\) and given \((u_0, \psi_0) \in (L^2(V_0^0) \cap S^2(V_0^0)) \times L^2(V_0^0), \) \(\rho_h\) be the solution of FSDE \([4.1]\). Then it holds that for all \(\tau \in [0, T],\)

\[
\begin{align*}
E & \left[ \sup_{t \in [0, \tau]} \| \varphi(t) \| ^2 \right] + E \int_0^\tau \| \nabla \varphi(s) \| ^2 ds \\
\leq CQ_0 h^2 + C_1 \tau \cdot \max\{|L_2^F|, |L_2^F| \tau + |L_2^F| \}
\end{align*}
\]

where constants \(C = C(L_1^F, L_2^F, \bar{L}_1^F, L_2^F, L_2^G, L_2^G, T, d, D), \) \(C_1 = C_1 e^{C_1 \tau}, \) and \(C_1 = C_1 (L_1^F, L_2^F)\) are independent of \(h.\)

Proof. Subtracting \([4.1]\) from \([1.1]\) leads to for \(0 \leq t \leq T\)

\[
\begin{cases}
\frac{d}{dt} \Pi_h \varphi(t) = \Delta_h \mathcal{R}_h \varphi(t) dt - \sum_{i=1}^k \Pi_h \left[ f^i(t, \rho_h(t), u_h(t)) - f^i(t, \rho(t), u(t)) \right] dW^i_t \\
+ \Pi_h \left[ F(t, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t)) - F(t, \rho(t), \nabla \rho(t), u(t), \nabla u(t), \psi(t)) \right] dt,
\end{cases}
\]

\(\Pi_h \varphi(0) = 0.\)

(4.6)

Fix \(t \in [0, T].\) Applying Itô’s formula to equation \([4.6]\) for \(\Pi_h \varphi\) yields that \(\mathbb{P}\)-a.s.

\[
\begin{align*}
\| \Pi_h \varphi(t) \|^2 & - \int_0^t \left\| \Pi_h \left[ f(s, \rho_h(s), u_h(s)) - f(s, \rho(s), u(s)) \right] \right\|^2 ds \\
- 2 \int_0^t \left\langle \Pi_h \left[ F(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)) \right. \right.
\left. - F(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \right], \Pi_h \varphi(s) \right\rangle ds \\
- 2 \sum_{i=1}^k \int_0^t \left\langle \Pi_h \left[ f^i(s, \rho_h(s), u_h(s)) - f^i(s, \rho(s), u(s)) \right], \Pi_h \varphi(s) \right\rangle dW^i_s \\
= 2 \int_0^t (\Delta_h \mathcal{R}_h \varphi(s), \Pi_h \varphi(s)) ds \\
= -2 \int_0^t (\nabla \mathcal{R}_h \varphi(s), \nabla \Pi_h \varphi(s)) ds \\
= -2 \int_0^t (\nabla \varphi(s), \nabla \Pi_h \varphi(s)) ds,
\end{align*}
\]

and this gives

\[
\begin{align*}
\| \Pi_h \varphi(t) \|^2 & + 2 \int_0^t \| \nabla \varphi(s) \|^2 ds \\
= 2 \int_0^t (\nabla \varphi(s), \nabla [id - \Pi_h] \varphi(s)) ds + \sum_{i=1}^k \int_0^t \left\| \Pi_h \left[ f^i(s, \rho_h(s), u_h(s)) - f^i(s, \rho(s), u(s)) \right] \right\|^2 ds \\
+ 2 \int_0^t \left\langle \Pi_h \left[ F(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)) \right. \right.
\left. - F(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \right] \right\rangle dt,
\end{align*}
\]

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Using the stability of $L^2$-projection and the Lipschitz property from Assumption 2.1, we have

\[
\begin{align*}
- F(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \bigg| \Pi_h \varrho(s) \bigg) ds \\
- 2 \sum_{i=1}^{k} \int_0^t \left\langle \Pi_h \left[ f_i(s, \rho_h(s), u_h(s)) - f_i(s, \rho(s), u(s)) \right], \Pi_h \varrho(s) \right\rangle dW^i_s.
\end{align*}
\]

(4.7)

Taking supremum over $t \in [0, \tau]$ for $\tau \in [0, T]$ in (4.7), and using deductions as above, we may arrive at

\[
(1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \sup_{t \in [0, \tau]} \| \Pi_h \varrho(t) \|^2 + (2 - \varepsilon_5) \int_0^\tau \| \nabla \varrho(s) \|^2 ds
\]

\[
\leq \frac{1}{\varepsilon_5} \int_0^\tau \| \nabla [id - \Pi_h] \varrho(s) \|^2 ds + \varepsilon_1 \int_0^\tau \| \nabla \varrho(s) \|^2 ds + \left( \frac{|L^2_1|^2}{\varepsilon_1} + 2L^2_2 + 2|L^2_1|^2 \right) \int_0^\tau \| \varrho(s) \|^2 ds
\]

\[
+ |L^2_2|^2 \tau \int_0^\tau \left( \frac{1}{\varepsilon_2} \| \mathcal{U}(s) \|^2 + \frac{1}{\varepsilon_3} \| \nabla \mathcal{U}(s) \|^2 + \frac{1}{\varepsilon_4} \| \nabla \mathcal{U}(s) \|^2 \right) ds + 2|L^2_2|^2 \int_0^\tau \| \mathcal{U}(s) \|^2 ds
\]

\[
+ 2 \sup_{t \in [0, \tau]} \sum_{i=1}^{k} \left| \int_0^t \left\langle \Pi_h \left[ f_i(s, \rho_h(s), u_h(s)) - f_i(s, \rho(s), u(s)) \right], \Pi_h \varrho(s) \right\rangle dW^i_s \right|.
\]

Take $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1}{6}, \varepsilon_5 = 1$, and $\varepsilon_1 = \frac{3}{4}$. It follows that

\[
\sup_{t \in [0, \tau]} \| \Pi_h \varrho(t) \|^2 + \frac{1}{2} \int_0^\tau \| \nabla \varrho(s) \|^2 ds
\]

\[
\leq 2 \int_0^\tau \| \nabla [id - \Pi_h] \varrho(s) \|^2 ds + 12|L^2_2|^2 \tau \int_0^\tau \left( \| \mathcal{U}(s) \|^2 + \| \nabla \mathcal{U}(s) \|^2 + \| \nabla \mathcal{U}(s) \|^2 \right) ds
\]

\[
+ 4|L^2_2|^2 \tau \int_0^\tau \| \mathcal{U}(s) \|^2 ds + C_1 \int_0^\tau \| \varrho(s) \|^2 ds
\]

\[
+ 4 \sup_{t \in [0, \tau]} \sum_{i=1}^{k} \left| \int_0^t \left\langle \Pi_h \left[ f_i(s, \rho_h(s), u_h(s)) - f_i(s, \rho(s), u(s)) \right], \Pi_h \varrho(s) \right\rangle dW^i_s \right|.
\]
with the constant $C'_1 = C'_1(L_1^f, L_1^F)$.

Using (4.5) we can have $\|\nabla [id - \Pi_h] \varrho(s)\| \leq C \|\varrho(s)\|_{1,2} \leq Ch \|\Delta \varrho(s)\|$, and thus

$$
\sup_{t \in [0, \tau]} \|\Pi_h \varrho(t)\|^2 + \frac{1}{2} \int_0^\tau \|\nabla \varrho(s)\|^2 ds
\leq Ch^2 \int_0^\tau \|\Delta \varrho(s)\|^2 ds + 12|L_1^F|^2 \tau \int_0^\tau \left(\|\mathcal{U}(s)\|^2 + \|\nabla \mathcal{U}(s)\|^2 + \|\zeta(s)\|^2\right) ds
+ 4 \sup_{t \in [0, \tau]} \sum_{i=1}^k \int_0^t \left\langle \Pi_h \left[ f^i(s, \rho_h(s), u_h(s)) - f^i(s, \rho(s), u(s)) \right], \Pi_h \varrho(s) \right\rangle dW^i(s)
+ 4|L_2^f|^2 \int_0^\tau \|\mathcal{U}(s)\|^2 ds + C'_1 \int_0^\tau \|\varrho(s)\|^2 ds,
$$

(4.8)

where $C = C(d, D)$.

We take expectations and use BDG inequality for the terms involving stochastic integrals to obtain

$$
4\mathbb{E}\left[ \sup_{t \in [0, \tau]} \int_0^t \sum_{i=1}^k \left(\Pi_h \left[ f^i(s, \rho_h(s), u_h(s)) - f^i(s, \rho(s), u(s)) \right], \Pi_h \varrho(s) \right) dW^i(s) \right]
\leq C\mathbb{E}\left[ \left( \int_0^\tau \sum_{i=1}^k \|\Pi_h \left[ f^i(s, \rho_h(s), u_h(s)) - f^i(s, \rho(s), u(s)) \right]\|^2 \cdot \|\Pi_h \varrho(s)\|^2 ds \right)^{\frac{1}{2}} \right]
\leq C\mathbb{E}\left[ \left( 2 \int_0^\tau \left( |L_1^f|^2 \|\varrho(s)\|^2 + |L_2^f|^2 \|\mathcal{U}(s)\|^2 \right) \cdot \|\Pi_h \varrho(s)\|^2 ds \right)^{\frac{1}{2}} \right]
\leq C\mathbb{E}\left[ \left( 2 \sup_{t \in [0, \tau]} \|\Pi_h \varrho(t)\|^2 \int_0^\tau \left( |L_1^f|^2 \|\varrho(s)\|^2 + |L_2^f|^2 \|\mathcal{U}(s)\|^2 \right) ds \right)^{\frac{1}{2}} \right]
\leq \frac{1}{2} \mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\Pi_h \varrho(t)\|^2 \right] + 4|CL_1^f|^2 \mathbb{E} \int_0^\tau \|\varrho(s)\|^2 ds + 4|CL_2^f|^2 \mathbb{E} \int_0^\tau \|\mathcal{U}(s)\|^2 ds,
$$

which yields from (4.8) that

$$
\mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\Pi_h \varrho(t)\|^2 \right] + \mathbb{E} \int_0^\tau \|\nabla \varrho(s)\|^2 ds
\leq Ch^2 \mathbb{E} \int_0^\tau \|\Delta \varrho(s)\|^2 ds + 24|L_2^F|^2 \tau \mathbb{E} \int_0^\tau \left( \|\mathcal{U}(s)\|^2 + \|\nabla \mathcal{U}(s)\|^2 + \|\zeta(s)\|^2 \right) ds + C_1 \mathbb{E} \int_0^\tau \|\varrho(s)\|^2 ds
+ 4(C^2 + 1)|L_2^F|^2 \tau \mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\mathcal{U}(t)\|^2 \right],
$$

with $C_1 = C_1(L_1^f, L_1^F)$.

By (4.4), there holds

$$
\mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\Pi_h \varrho(t)\|^2 \right] \geq \mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\varrho(t)\|^2 \right] - Ch^2 \mathbb{E}\left[ \sup_{t \in [0, \tau]} \|\varrho(t)\|^2_{1,2} \right].
$$

(4.9)
In view of Theorem 3.1 and Lemma 4.1, we have

\[
\mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \varrho(t) \|^2 \right] + \mathbb{E} \int_0^\tau \| \nabla \varrho(s) \|^2 ds \\
\leq CQ_0 h^2 + 24 |L_f^2|^2 \tau \mathbb{E} \int_0^\tau \left( |\mathcal{U}(s)|^2 + \| \nabla \mathcal{U}(s) \|^2 + |\zeta(s)|^2 \right) ds + C_1 \mathbb{E} \int_0^\tau |\varrho(s)|^2 ds \\
+ 4(\bar{C}^2 + 1) |L_f^2|^2 \tau \mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \mathcal{U}(t) \|^2 \right],
\]

where \( C = C(L_1, L_2, \tilde{L}, L^9, L_1^F, L_2^F, L_1^G, L_2^G, T, d, D) \). This together with Gronwall’s inequality further implies that for all \( \tau \in [0, T] \)

\[
\mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \varrho(t) \| \right] + \mathbb{E} \int_0^\tau \| \nabla \varrho(t) \|^2 dt \\
\leq CQ_0 h^2 + \bar{C}_1 |L_f^2|^2 \tau \mathbb{E} \int_0^\tau \left( |\mathcal{U}(s)|^2 + \| \nabla \mathcal{U}(s) \|^2 + |\zeta(s)|^2 \right) ds + \bar{C}_1 |L_f^2|^2 \tau \mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \mathcal{U}(t) \|^2 \right] \\
\leq CQ_0 h^2 + \bar{C}_1 |L_f^2|^2 \tau \mathbb{E} \int_0^\tau \left( \| \nabla \mathcal{U}(s) \|^2 + |\zeta(s)|^2 \right) ds + \bar{C}_1 |L_f^2|^2 \tau \mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \mathcal{U}(t) \|^2 \right] \\
\leq CQ_0 h^2 + \bar{C}_1 \tau \cdot \max\{ |L_f^2|^2, |L_f^2|^2 + |L_f^2|^2 \} \mathbb{E} \left[ \sup_{t \in [0, \tau]} \| \mathcal{U}(t) \|^2 + \int_0^\tau \left( \| \nabla \mathcal{U}(s) \|^2 + |\zeta(s)|^2 \right) ds \right],
\]

where \( \bar{C}_1 = 24(\bar{C}^2 + 1) \cdot e^{C_1 \tau} \).

Now, we are ready to give the convergence rate for the semi-discrete approximation.

**Theorem 4.3.** Let \((\rho, u, \psi) \in \left( \mathcal{L}^2(H_0^{2,2}) \cap S^2(H_0^{1,2}) \right) \times \left( \mathcal{L}^2(H_0^{2,2}) \cap S^2(H_0^{1,2}) \right) \times \mathcal{L}^2(H_0^{1,2})\) be the strong solution of FBSDE (1.1)-(1.2) and \((\rho_h, u_h, \psi_h) \in \left( \mathcal{L}^2(V_h^0) \cap S^2(V_h^0) \right) \times \left( \mathcal{L}^2(V_h^0) \cap S^2(V_h^0) \right) \times \mathcal{L}^2(V_h^0)\) be the solution of FBSDE (4.1)-(4.2). Then \( \varrho(t) := \rho(t) - \rho_h(t) \), \( \mathcal{U}(t) := u(t) - u_h(t) \) and \( \zeta(t) := \psi(t) - \psi_h(t) \) satisfy

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \varrho(t) \|^2 + \sup_{t \in [0, T]} \| \mathcal{U}(t) \|^2 + \int_0^T \left( \| \nabla \varrho(t) \|^2 + \| \nabla \mathcal{U}(t) \|^2 + \| \zeta(t) \|^2 \right) dt \right] \leq CQ_0 h^2,
\]

where the constant \( C = C(L_1, L_2, \tilde{L}, L^9, L_1^F, L_2^F, L_1^G, L_2^G, T, d, D) \).

**Proof.** Subtracting (4.2) from (1.2) leads to for \( 0 \leq t \leq T \)

\[
\begin{cases}
-d\Pi_h \mathcal{U}(t) = \left( \Delta_h \mathcal{R}_h \mathcal{U}(t) + \Pi_h \left[ G(t, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t)) \right. \right. \\
\left. \left. - G(t, \rho(t), \nabla \rho(t), u(t), \nabla u(t), \psi(t)) \right] \right) dt - \sum_{i=1}^k \Pi_h \zeta(t) dW^i(t),
\end{cases}
\]

(4.10)

Our computations will be divided into two parts.
Using Assumption 2.2, we have

\[
\| \Pi_h \mathcal{U}(t) \|^2 + 2 \sum_{i=1}^{k} \int_t^T (\Pi_h \zeta^i(s), \Pi_h \mathcal{U}(s)) \mathrm{d}W^i_s + \int_t^T \| \Pi_h \zeta(s) \|^2 \mathrm{d}s - \int_t^T \Pi_h [g(\rho(T)) - g(\rho_h(T))] \mathrm{d}s
\]

\[\leq 2 \int_t^T \left\langle \Pi_h \left[ G(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)) - G(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \right], \Pi_h \mathcal{U}(s) \right\rangle \mathrm{d}s\]

\[= 2 \int_t^T \langle \Delta_h R_h \mathcal{U}(s), \Pi_h \mathcal{U}(s) \rangle \mathrm{d}s\]

\[= -2 \int_t^T \langle \nabla R_h \mathcal{U}(s), \nabla \Pi_h \mathcal{U}(s) \rangle \mathrm{d}s\]

\[= -2 \int_t^T \langle \nabla \mathcal{U}(s), \nabla \Pi_h \mathcal{U}(s) \rangle \mathrm{d}s. \tag{4.11}\]

Then using Assumption 2.2, we have

\[
2 \int_t^T \left\langle \Pi_h \left[ G(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)) - G(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \right], \Pi_h \mathcal{U}(s) \right\rangle \mathrm{d}s
\]

\[\leq 2 \int_t^T \left\| \Pi_h \left[ G(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)) - G(s, \rho(s), \nabla \rho(s), u(s), \nabla u(s), \psi(s)) \right] \right\| \cdot \left\| \Pi_h \mathcal{U}(s) \right\| \mathrm{d}s\]

\[\leq 2 \int_t^T \left( L^G_1 \left( \| \varrho(s) \| + \| \nabla \varrho(s) \| \right) + L^G_2 \left( \| \mathcal{U}(s) \| + \| \nabla \mathcal{U}(s) \| + \| \zeta(s) \| \right) \right) \cdot \left\| \Pi_h \mathcal{U}(s) \right\| \mathrm{d}s\]

\[\leq 2 |L^G_2|^2 (T - t) \frac{1}{\varepsilon_1} \int_t^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) \mathrm{d}s + \int_t^T \left( \varepsilon_3 \| \nabla \mathcal{U}(s) \|^2 + \varepsilon_4 \| \zeta(s) \|^2 \right) \mathrm{d}s\]

\[+ \left( \frac{|L^G_2|^2}{\varepsilon_3} + \frac{|L^G_2|^2}{\varepsilon_4} + 2L^G_2 \right) \int_t^T \| \Pi_h \mathcal{U}(s) \|^2 \mathrm{d}s + \varepsilon_1 \sup_{s \in [t, T]} \| \Pi_h \mathcal{U}(s) \|^2, \tag{4.12}\]

Recalling (4.4), (4.5), and the estimates in Theorem 3.1 and Lemma 4.1, we may insert above computations into (4.11) to obtain

\[
\| \mathcal{U}(t) \|^2 + (1 - \varepsilon_4) \sum_{i=1}^{k} \int_t^T \| \zeta^i(s) \|^2 \mathrm{d}s + (2 - \varepsilon_3 - \varepsilon_5) \int_t^T \| \nabla \mathcal{U}(s) \|^2 \mathrm{d}s
\]

\[\leq CQ_0 h^2 + |L^g|^2 \| \varrho(T) \|^2 + 2|L^G_1|^2 (T - t) \frac{1}{\varepsilon_1} \int_t^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) \mathrm{d}s\]

\[+ C_2 \int_t^T \| \mathcal{U}(s) \|^2 \mathrm{d}s + \varepsilon_1 \sup_{s \in [t, T]} \| \mathcal{U}(s) \|^2 - 2 \sum_{i=1}^{k} \int_t^T \langle \Pi_h \zeta^i(s), \Pi_h \mathcal{U}(s) \rangle \mathrm{d}W^i_s,\]

with \( C = C(L^F_1 L^F_2 L^G L^g L^F L^F L^G L^G T d D) \) and \( C_2 = C_2(L^G_2) \). Taking \( \varepsilon_3 = \varepsilon_4 = \frac{1}{2} \).
and \( \varepsilon_5 = 1 \), we have

\[
\| U(t) \|^2 + \frac{1}{2} \sum_{i=1}^{k} \int_t^T \| \zeta(s) \|^2 ds + \frac{1}{2} \int_t^T \| \nabla U(s) \|^2 ds
\]

\[
\leq CQ_0 h^2 + 2|L_4^G|^2 (T - t) \frac{1}{\varepsilon_1} \int_t^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) ds + C_2 \int_t^T \| U(s) \|^2 ds
\]

\[
+ |L|^2 \| \varrho(T) \|^2 + \varepsilon_1 \sup_{s \in [t,T]} \| U(s) \|^2 - 2 \sum_{i=1}^{k} \int_t^T \langle \Pi_h \zeta^i(s), \Pi_h U(s) \rangle dW_s^i.
\]

Taking expectations on both sides implies particularly that

\[
\mathbb{E} \left[ \int_t^T \| \zeta(s) \|^2 ds \right]
\]

\[
\leq CQ_0 h^2 + 4|L_4^G|^2 (T - t) \frac{1}{\varepsilon_1} \mathbb{E} \left[ \int_t^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) ds \right]
\]

\[
+ C_2 \mathbb{E} \left[ \int_t^T \| U(s) \|^2 ds \right] + 2|L|^2 \mathbb{E} \left[ \| \varrho(T) \|^2 \right] + 2\varepsilon_1 \mathbb{E} \left[ \sup_{s \in [t,T]} \| U(s) \|^2 \right].
\]

(4.13)

**Step 2.** Taking Supremum over \( t \in [\tau,T] \) with \( \tau \in [0,T] \) on both sides of (4.11) and conducting computations as in (4.12), we have

\[
\mathbb{E} \left[ \sup_{t \in [\tau,T]} \| U(t) \|^2 \right] + (1 - \varepsilon_4) \mathbb{E} \left[ \int_\tau^T \| \zeta(s) \|^2 ds \right] + (2 - \varepsilon_3 - \varepsilon_5) \mathbb{E} \left[ \int_\tau^T \| \nabla U(s) \|^2 ds \right]
\]

\[
\leq \frac{1}{\varepsilon_5} \mathbb{E} \int_\tau^T \| \nabla \left[ i d - \Pi_h U(s) \right] \| ds + 2|L_4^G|^2 (T - \tau) \frac{1}{\varepsilon_1} \mathbb{E} \int_\tau^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) ds
\]

\[
+ C_2 \mathbb{E} \int_\tau^T \| U(s) \|^2 ds + \varepsilon_1 \mathbb{E} \left[ \sup_{s \in [\tau,T]} \| U(s) \|^2 \right] + |L|^2 \mathbb{E} \left[ \| \varrho(T) \|^2 \right]
\]

\[
+ 2\mathbb{E} \left[ \sup_{t \in [\tau,T]} \left| \int_t^T \sum_{i=1}^{k} \langle \Pi_h \zeta^i(s), \Pi_h U(s) \rangle dW_s^i \right| \right],
\]

Take \( \varepsilon_1 = 1/2, \varepsilon_3 = \varepsilon_4 = 3/4 \) and \( \varepsilon_5 = 1 \). It holds that

\[
\mathbb{E} \left[ \sup_{t \in [\tau,T]} \| U(t) \|^2 \right] + \frac{1}{2} \mathbb{E} \int_\tau^T \| \zeta(s) \|^2 ds + \frac{1}{2} \mathbb{E} \int_\tau^T \| \nabla U(s) \|^2 ds
\]

\[
\leq 2\mathbb{E} \int_\tau^T \| \nabla \left[ i d - \Pi_h U(s) \right] \| ds + 8|L_4^G|^2 (T - \tau) \mathbb{E} \int_\tau^T \left( \| \varrho(s) \|^2 + \| \nabla \varrho(s) \|^2 \right) ds
\]

\[
+ 2|L|^2 \mathbb{E} \left[ \| \varrho(T) \|^2 \right] + 2C_2 \mathbb{E} \int_\tau^T \| U(s) \|^2 ds + 4\mathbb{E} \left[ \sup_{t \in [\tau,T]} \left| \int_t^T \sum_{i=1}^{k} \langle \Pi_h \zeta^i(s), \Pi_h U(s) \rangle dW_s^i \right| \right].
\]

(4.14)

Again we use BDG inequality for the terms involving stochastic integrals to obtain

\[
4\mathbb{E} \left[ \sup_{t \in [\tau,T]} \left| \int_t^T \sum_{i=1}^{k} \langle \Pi_h \zeta^i(s), \Pi_h U(s) \rangle dW_s^i \right| \right] \leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \in [\tau,T]} \| U(t) \|^2 \right] + C^2 \mathbb{E} \int_\tau^T \| \zeta(s) \|^2 ds,
\]

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which together with (4.5), (4.13), (4.14), and estimates in Theorem 3.1 and Lemma 4.1 implies that

\[
\frac{3}{4} E\left[ \sup_{t \in [r,T]} \|\mathcal{U}(t)\|^2 \right] + \frac{1}{2} E \int_\tau^T \|\zeta(s)\|^2 ds + \frac{1}{2} E \int_\tau^T \|\nabla \mathcal{U}(s)\|^2 ds \\
\leq C Q_0 h^2 + \tilde{C}_1 |L_1^G|^2 (T - \tau) E \left[ \int_\tau^T \left( \|\varrho(s)\|^2 + \|\nabla \varrho(s)\|^2 \right) ds \right] + \tilde{C}_2 E \int_\tau^T \|\mathcal{U}(s)\|^2 ds \\
+ \tilde{C}_1 |L_1^G|^2 E \left[ \|\varrho(T)\|^2 \right] + 2 \tilde{C}_2 \bar{\varepsilon}_1 E \left[ \sup_{s \in [t,T]} \|\mathcal{U}(s)\|^2 \right],
\]

with constants \( C = C(L_1^f, L_1^f, L_1^g, L_1^g, L_2^F, L_2^G, L_2^G, L_2^G, T, d, D) \), \( \tilde{C}_1 = \frac{4 \tilde{C}_2}{\bar{\varepsilon}_1} + 8 \) and \( \tilde{C}_2 = \tilde{C}^2 C_2 + C_2 \). Taking \( \bar{\varepsilon}_1 = \frac{1}{8 \tilde{C}_2} \), we have

\[
E \left[ \sup_{t \in [r,T]} \|\mathcal{U}(t)\|^2 \right] + E \sum_{i=1}^k \int_\tau^T \|\zeta^i(s)\|^2 ds + E \int_\tau^T \|\nabla \mathcal{U}(s)\|^2 ds \\
\leq C Q_0 h^2 + 2 \tilde{C}_1 |L_1^G|^2 (T - \tau) E \left[ \int_\tau^T \left( \|\varrho(s)\|^2 + \|\nabla \varrho(s)\|^2 \right) ds \right] + 2 \tilde{C}_2 E \int_\tau^T \|\mathcal{U}(s)\|^2 ds \\
+ 2 \tilde{C}_1 |L_1^G|^2 E \left[ \|\varrho(T)\|^2 \right],
\]

which by Gronwall’s inequality yields that

\[
E \left[ \sup_{t \in [0,T]} \|\mathcal{U}(t)\|^2 \right] + E \int_0^T \|\zeta(s)\|^2 ds + E \int_0^T \|\nabla \mathcal{U}(s)\|^2 ds \\
\leq C Q_0 h^2 + \tilde{C}_2 |L_1^G|^2 E \left[ \sup_{t \in [0,T]} \|\varrho(t)\|^2 \right] + \tilde{C}_2 |L_1^G|^2 T E \left[ \sup_{t \in [0,T]} \|\varrho(t)\|^2 + \int_0^T \|\nabla \varrho(s)\|^2 ds \right] \\
\leq C Q_0 h^2 + \tilde{C}_2 \cdot \max \{ |L_1^G|^2 T^2 + |L_1^G|^2, |L_1^G|^2 T \} E \left[ \sup_{t \in [0,T]} \|\varrho(t)\|^2 + \int_0^T \|\nabla \varrho(s)\|^2 ds \right].
\]

Using estimate for \( \varrho(t) \) from Lemma 4.2 we have

\[
E \left[ \sup_{t \in [0,T]} \|\mathcal{U}(t)\|^2 \right] + E \int_0^T \|\zeta(s)\|^2 ds + E \int_0^T \|\nabla \mathcal{U}(s)\|^2 ds \\
\leq C Q_0 h^2 + \hat{C} E \left[ \sup_{t \in [0,T]} \|\mathcal{U}(t)\|^2 + \int_0^T \left( \|\nabla \mathcal{U}(s)\|^2 + \|\zeta(s)\|^2 \right) ds \right],
\]

where \( \hat{C} = \tilde{C} e^{CT} \cdot \max \{ |L_1^G|^2 T^2 + |L_1^G|^2, |L_1^G|^2 T \} \cdot \max \{ |L_1^G|^2, |L_1^G|^2 T + |L_1^G|^2 \} \) and \( C = C(L_1^f, L_2^f, L_1^f, L_1^g, L_1^g, L_1^g, L_1^g, L_1^g, L_1^g, L_1^g, L_1^g, T, d, D) \).
Since we already have from Theorem 3.1 that
\[ \hat{C} = C e^{CT} \cdot \max \{ |L_1|^2 T^2 + |L_2|^2 , |L_3|^2 \} \cdot T \cdot \max \{ |L_2|^2 , |L_2|^2 T + |L_2|^2 \} < 1, \]
we have
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \| \mathcal{U}(t) \| \right]^2 + \mathbb{E} \sum_{i=1}^k \int_0^T \| \zeta^i(s) \|^2 ds + \mathbb{E} \int_0^T \| \nabla \mathcal{U}(s) \|^2 ds \leq C Q_0 h^2, \]
where the constant \( C = C(L_1^f, L_2^f, \tilde{L}_1^f, L_2^g, L_1^f, L_2^g, L_2^g, L_2^g, T, d, D) \) is independent of \( h \). Combining this estimate with estimate of \( g(t) \) from Lemma 4.2 we have finally,
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \| g(t) \| + \sup_{t \in [0,T]} \| \mathcal{U}(t) \|^2 + \int_0^T \left( \| \nabla g(t) \|^2 + \| \nabla \mathcal{U}(t) \|^2 + \| \zeta(t) \|^2 \right) dt \right] \leq C Q_0 h^2. \]

5 Finite dimensional approximating FBSDEs and deep learning-based algorithms

5.1 Finite dimensional approximating FBSDEs

As the approximations of the solution to FBSPDE (1.1)-(1.2), the solution to FBSDE (4.1)-(4.2) is valued on the finite dimensional space \( V^0_h \) and has the following form:
\[ \rho_h(t, x) = \sum_{l=1}^L [\bar{\rho}_h(t) \phi^l_h(x), \quad u_h(t, x) = \sum_{l=1}^L [\bar{u}_h(t) \phi^l_h(x), \quad \text{and} \quad \psi^l_h(t, x) = \sum_{l=1}^L [\bar{\psi}_h(t) \phi^l_h(x). \]

Then for each \( \phi^l_h \in V^0_h, l = 1, 2, \ldots, L \) we have
\[ \langle \rho_h(t), \phi^l_h \rangle = \langle \rho_0, \phi^l_h \rangle - \int_0^t \sum_{k=1}^k \langle f^k(s, \rho_h(s), u_h(s), \phi^l_h) \rangle dW^i(s) \]
\[ + \int_0^t \left( -\langle \nabla \rho_h(s), \nabla \phi^l_h \rangle + \langle F(s, x, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s), \phi^l_h \rangle \right) ds. \]

It may be written as
\[ A \tilde{\rho}_h(t) = \tilde{\rho}_0 + \int_0^t \left( -B \tilde{\rho}_h(s) + \tilde{F}(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s); \phi_h) \right) dt \]
\[ - \int_0^t \sum_{k=1}^k \tilde{f}^k(s, \rho_h(s), u_h(s); \phi_h) dW^i(s), \]
or equivalently,
\[ \tilde{\rho}_h(t) = A^{-1} \tilde{\rho}_0 + A^{-1} \int_0^t \left( -B \tilde{\rho}_h(s) + \tilde{F}(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s); \phi_h) \right) dt \]
\[ - A^{-1} \int_0^t \sum_{k=1}^k \tilde{f}^k(s, \rho_h(s), u_h(s); \phi_h) dW^i(s), \]
where $\mathbf{A} = (a_{ml})_{1 \leq m, l \leq L}$ and $\mathbf{B} = (b_{ml})_{1 \leq m, l \leq L}$ with $a_{ml} = \langle \phi_h^l, \phi_h^m \rangle$ and $b_{ml} = (\nabla \phi_h^l, \nabla \phi_h^m)$. The function $\bar{F}(t, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t); \phi_h)$ is $\mathbb{R}^L$-valued with $l$-th entry $(F(t, \rho_h(t), \nabla \rho_h(t), u_h(t), \nabla u_h(t), \psi_h(t)), \phi_h)$. Here, by $\bar{\xi}_\phi$ we denote a vector with $l$-th entry $[\bar{\xi}_\phi] := (x, \phi_h^l)$.

On the other hand, for each $\phi_h^l \in V_h^0, l = 1, \ldots, L$, we have

\[
\langle u_h(t), \phi_h^l \rangle = \langle g(\rho_h(T)), \phi_h^l \rangle - \int_t^T \sum_{i=1}^k \langle \psi_h^i(s), \phi_h^l \rangle dW^i(s)
- \int_t^T \left( (\nabla \rho_h(s), \nabla \phi_h^l) - (G(s, x, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)), \phi_h^l) \right) ds,
\]

which may be written equivalently as

\[
\tilde{u}_h(t) = \mathbf{A}^{-1} \bar{g}(\rho_h(T); \phi_h) - \int_t^T \sum_{i=1}^k \psi_h^i(s) dW^i(s)
- \mathbf{A}^{-1} \int_t^T \left( \mathbf{B} \tilde{u}_h(s) - \bar{G}(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s); \phi_h) \right) ds.
\]

Finally, we have the following finite dimensional coupled FBSDE

\[
\begin{align*}
\bar{\rho}_h(t) &= \mathbf{A}^{-1} \bar{\rho}_\phi + \mathbf{A}^{-1} \int_t^T \left( - \mathbf{B} \bar{\rho}_h(s) + \bar{F}(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s); \phi_h) \right) ds \\
- \mathbf{A}^{-1} \int_t^T \sum_{i=1}^k \bar{f}_i(s, \rho_h(s), u_h(s); \phi_h) dW^i(s),
\end{align*}
\]

\[
\bar{u}_h(t) = \mathbf{A}^{-1} \bar{g}(\rho_h(T); \phi_h) - \int_t^T \sum_{i=1}^k \psi_h^i(s) dW^i(s)
- \mathbf{A}^{-1} \int_t^T \left( \mathbf{B} \bar{u}_h(s) - \bar{G}(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s); \phi_h) \right) ds,
\]

In the above coupled FBSDE, the $\mathbb{R}^L$-valued random functions $\bar{\rho}_h(t), \bar{u}_h(t),$ and $\psi_h^i(t), i = 1, \ldots, k,$ are unknown expansion coefficients in the expressions for $\rho_h, u_h,$ and $\psi_h^i,$ respectively. We shall use existing deep learning-based algorithms to solve the finite dimensional coupled FBSDE of the form (5.1).

### 5.2 Deep learning algorithms for FBSDEs

Let us consider the following general form of coupled FBSDE:

\[
\begin{align*}
X(t) &= X_0 + \int_0^t \mu(s, X(s), Y(s), Z(s)) ds - \sum_{i=1}^k \int_0^t \sigma^i(s, X(s), Y(s)) dW^i(s), \\
Y(t) &= g(X(T)) + \int_t^T b(s, X(s), Y(s), Z(s)) ds - \sum_{i=1}^k \int_t^T Z^i(s) dW^i(s),
\end{align*}
\]

(5.2)

with the unknown processes $X(t), Y(t),$ and $Z^i(t)$ being $\mathbb{R}^L$-valued, for $i = 1, 2, \ldots, k.$ When $\mu$ and $\sigma$ do not depend on $Y$ or $Z,$ FBSDE (5.2) is decoupled. Some algorithms based on deep learning techniques that are highly capable of solving such decoupled FBSDEs (5.2) when $L$
is large have just been proposed; see \cite{11, 15, 20} for instance. For coupled FBSDEs, an
algorithm is proposed in \cite{16} with a convergence analysis, and three more algorithms for fully
coupled FBSDEs are also introduced in \cite{21}. Under the Markovian framework (i.e., when
all the coefficients $\mu$, $\sigma$, and $g$ are deterministic), the solution $(Y(t), Z(t))$ of the BSDE in
\cite{5, 2}, can be expressed as a function of $X(t)$, the solution of forward SDE in \cite{5, 2}, that is,
$(Y(t), Z(t)) = (\mathcal{Y}(t, X(t)), Z(t, X(t))), t \in [0, T]$ for some deterministic functions $\mathcal{Y}$ and $\mathcal{Z}.
This well known result is a key-ingredient for the approximations of BSDEs in all these deep
learning-based algorithms for FBSDEs. We will address these deep learning-based algorithms for
finite dimensional FBSDEs as Deep BSDE. Deep BSDE methods use neural networks to approx-
imate unknown functions and reformulates the original problem into a stochastic optimization
problem. Here we shall present two existing Deep BSDE algorithms: one is for decoupled FB-
SDE and another for coupled FBSDE. We will introduce a third algorithm as a modified version
of the second one for coupled FBSDEs.

To discuss the numerical algorithms, we first consider a partition of the time interval $[0, T]$ into grid
$\pi : t_0 = 0 < t_1 < \cdots < t_J = T$ with $|\pi| = \max_{j=0, \cdots, J-1} \Delta t_j$, $\Delta t_j := t_{j+1} - t_j$. Now let
us consider the forward representation of the BSDE in \cite{5, 2}, which is written as
\begin{equation}
Y(t) = Y(0) - \int_0^t b(s, X(s), Y(s), Z(s))ds + \sum_{i=1}^k \int_0^t Z_i(s)dW_i(s), \quad 0 \leq t \leq T. \quad (5.3)
\end{equation}
Without any loss of generality, we take $k = 1$ in the following subsections.

5.2.1 Deep BSDE-1 for decoupled Markovian FBSDEs

This Deep BSDE algorithm proposed in \cite{20} is for decoupled FBSDEs. The forward process $X$
of \cite{5, 2} is numerically approximated by $X^\pi$ using Euler Scheme on time grid $\pi$. For example,
forward Euler scheme can be used which is defined as
\begin{equation}
X_{t_{j+1}}^\pi = X_{t_j}^\pi + \mu(t_j, X_{t_j}^\pi) \Delta t_j - \sigma(t_j, X_{t_j}^\pi) W_{t_j}, \quad j = 0, \cdots, J - 1, \quad X_{t_0}^\pi = X_0, \quad (5.4)
\end{equation}
with $\Delta W_{t_j} := W(t_{j+1}) - W(t_j)$.
Here $Y(t)$ and $Z(t)$ are treated as functions of $X(t)$, that is, $Y(t) = \mathcal{Y}(t, X(t))$ and $Z(t) =
\mathcal{Z}(t, X(t))$ for some deterministic functions $\mathcal{Y}$ and $\mathcal{Z}$. This Deep FBSDE algorithm is based on
backward dynamic programming, and the discrete approximations of the functions $\mathcal{Y}(t, \cdot)$ and
$\mathcal{Z}(t, \cdot)$ on time grid $\pi$ are performed backwardly in time. These functions are approximated by
deep neural networks.

The algorithm starts with an estimation $Y_{t_j}^\pi$ of $\mathcal{Y}(t_j, X_{t_j}^\pi)$ with $Y_{t_j}^\pi = g(X_{t_j}^\pi)$. Then at each
time step $t_j : j = J - 1, \cdots, 1, 0$, given an estimation $Y_{t_{j+1}}^\pi$ of $\mathcal{Y}(t_{j+1}, X_{t_{j+1}}^\pi)$, two independent
deep neural networks $\mathcal{Y}_j^N(\cdot; \theta_{1,j})$ and $\mathcal{Z}_j^N(\cdot; \theta_{2,j})$ approximate, respectively, $\mathcal{Y}(t_j, \cdot)$ and $\mathcal{Z}(t_j, \cdot)$
by minimizing quadratic loss function
\begin{equation}
\hat{w}_j(\theta_j) := \mathbb{E} \left[ Y_{t_{j+1}}^\pi - Y^F(t_{j+1} | X_{t_{j+1}}^\pi, \mathcal{Y}_j^N(X_{t_j}^\pi; \theta_{1,j}), \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j})) \right]^2, \quad (5.5)
\end{equation}
with respect to its parameters $\theta_j = (\theta_{1,j}, \theta_{2,j})$ using gradient based method, where
\begin{equation}
Y^F(t_{j+1} | X_{t_{j+1}}^\pi, \mathcal{Y}_j^N(X_{t_j}^\pi; \theta_{1,j}), \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j})) = \mathcal{Y}_j^N(X_{t_j}^\pi; \theta_{1,j}) - b(t_j, X_{t_j}^\pi, \mathcal{Y}_j^N(X_{t_j}^\pi; \theta_{1,j}), \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j})) \Delta t_j + \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j}) W_{t_j}, \quad (5.6)
\end{equation}
is computed from the forward representation (5.3) of the backward equation. If

$$\theta_j^* = (\theta_{1,j}^*, \theta_{2,j}^*) \in \arg \min_{\theta_j \in \Theta} \Sigma_j(\theta_j),$$

then $\mathcal{Y}_j^N(X_{t_j}^\pi, \theta_{1,j}^*)$ is the approximation of $Y_{t_j}^\pi = \mathcal{Y}(t_j, X_{t_j}^\pi)$ and $\mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j}^*)$ the approximation of $Z_{t_j}^\pi = \mathcal{Z}(t_j, X_{t_j}^\pi)$. Finally, the backward induction on time step leads us to $\mathcal{Y}_0^N(X_{t_0}^\pi; \theta_{1,0}^*)$, the approximation of $Y_{t_0}^\pi = \mathcal{Y}(t_0, X_{t_0}^\pi)$. We refer to [21] for the convergence analysis and various numerical examples.

5.2.2 Deep BSDE-2 for coupled Markovian FBSDEs

This algorithm for fully coupled FBSDE is proposed in [21, Algorithm-2]. Here $Y(t)$ and $Z(t)$ are also treated as functions of $X(t)$, that is, $Y(t) = \mathcal{Y}(t, X(t))$ and $Z(t) = \mathcal{Z}(t, X(t))$ for some deterministic functions $\mathcal{Y}$ and $\mathcal{Z}$. This method starts with estimations $\mathcal{Y}_0$ and $\mathcal{Z}_0$ of $\mathcal{Y}(t_0, X_{t_0}^\pi) = Y_{t_0}^\pi$ and $\mathcal{Z}(t_0, X_{t_0}^\pi) = Z_{t_0}^\pi$ respectively and then calculate $X_{t_1}^\pi$ and $Y_{t_1}^\pi$ by using Euler scheme as

$$X_{t_1}^\pi = X_{t_0}^\pi + \mu(t_0, X_{t_0}^\pi, \mathcal{Y}_0, \mathcal{Z}_0) \Delta t_0 - \sigma(t_0, X_{t_0}^\pi, Y_{t_0}^\pi) \Delta W_{t_0},$$

$$Y_{t_1}^\pi = \mathcal{Y}_0 - b(t_0, X_{t_0}^\pi, \mathcal{Y}_0, \mathcal{Z}_0) \Delta t_0 + \mathcal{Z}_0 \Delta W_{t_0}.$$

When using two neural networks $\mathcal{Y}_1^N(\cdot; \theta_{1,1})$ and $\mathcal{Z}_1^N(\cdot; \theta_{2,1})$ to approximate respectively $Y_{t_1}^\pi = \mathcal{Y}(t_1, X_{t_1}^\pi)$ and $Z_{t_1}^\pi = \mathcal{Z}(t_1, X_{t_1}^\pi)$, the associated local loss function is defined as

$$\Sigma_1 := \Delta t_0 \cdot \mathbb{E} \left| Y_{t_1}^\pi - \mathcal{Y}_1^N(X_{t_1}^\pi; \theta_{1,1}) \right|^2.$$

Then for $j = 1, 2, \ldots, J-2$, using Euler scheme for $X_{t_{j+1}}^\pi$ and $Y_{t_{j+1}}^\pi$ gives

$$X_{t_{j+1}}^\pi = X_{t_j}^\pi + \mu(t_j, X_{t_j}^\pi, \mathcal{Y}_j^N(X_{t_j}^\pi; \theta_{1,j}), \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j})) \Delta t_j - \sigma(t_j, X_{t_j}^\pi, Y_{t_j}^\pi) \Delta W_{t_j},$$

$$Y_{t_{j+1}}^\pi = Y_{t_j}^\pi - b(t_j, X_{t_j}^\pi, Y_{t_j}^\pi, \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j})) \Delta t_j + \mathcal{Z}_j^N(X_{t_j}^\pi; \theta_{2,j}) \Delta W_{t_j},$$

where the two neural networks $\mathcal{Y}_j^N(\cdot; \theta_{1,j+1})$ and $\mathcal{Z}_j^N(\cdot; \theta_{2,j+1})$ approximate respectively $Y_{t_{j+1}}^\pi = \mathcal{Y}(t_{j+1}, X_{t_{j+1}}^\pi)$ and $Z_{t_{j+1}}^\pi = \mathcal{Z}(t_{j+1}, X_{t_{j+1}}^\pi)$ with associated local loss function given by

$$\Sigma_{j+1} := \Delta t_j \cdot \mathbb{E} \left| Y_{t_{j+1}}^\pi - \mathcal{Y}_{j+1}^N(X_{t_{j+1}}^\pi; \theta_{1,j+1}) \right|^2.$$

Finally, using Euler scheme for $X_{t_J}^\pi$ and $Y_{t_J}^\pi$ gives

$$X_{t_J}^\pi = X_{t_{J-1}}^\pi + \mu(t_{J-1}, X_{t_{J-1}}^\pi, \mathcal{Y}_{J-1}^N(X_{t_{J-1}}^\pi; \theta_{1,J-1}), \mathcal{Z}_{J-1}^N(X_{t_{J-1}}^\pi; \theta_{2,J-1})) \Delta t_{J-1} - \sigma(t_{J-1}, X_{t_{J-1}}^\pi, Y_{t_{J-1}}^\pi) \Delta W_{t_{J-1}},$$

$$Y_{t_J}^\pi = Y_{t_{J-1}}^\pi - b(t_{J-1}, X_{t_{J-1}}^\pi, Y_{t_{J-1}}^\pi, \mathcal{Z}_{J-1}^N(X_{t_{J-1}}^\pi; \theta_{2,J-1})) \Delta t_{J-1} + \mathcal{Z}_J^N(X_{t_{J-1}}^\pi; \theta_{2,J-1}) \Delta W_{t_{J-1}},$$

and define local loss function

$$\Sigma_J := \mathbb{E} \left| Y_{t_J}^\pi - g(X_{t_J}^\pi) \right|^2.$$

Now the scheme is to optimize the global loss function

$$\widehat{\Sigma} = \Sigma_0, \theta_{1,1}, \theta_{2,1}, \ldots, \theta_{1,J}, \theta_{2,J} := \sum_{j=1}^J \Sigma_j,$$
over all $\theta = (\gamma_0, Z_0, \theta_{1,1}, \theta_{2,1}, \ldots, \theta_{1,J}, \theta_{2,J})$, and for some $\theta^* = (\gamma_0^*, Z_0^*, \theta_{1,1}^*, \theta_{2,1}^*, \ldots, \theta_{1,J}^*, \theta_{2,J}^*)$ if

$$\theta^* \in \arg \min_{\theta \in \Theta} \widehat{L}(\theta),$$

then $\gamma_0^*$ is the desired approximation of $Y_0^*$ by Deep BSDE-2. We refer to [21] for various numerical examples, while the reader may refer to [16] for an alternative algorithm for a class of coupled Markovian FBSDEs with both convergence analysis and numerical examples.

### 5.2.3 Deep BSDE-3 for coupled Markovian FBSDEs

This method is just a modified version of Deep BSDE-2. This method starts with estimations $\gamma_0$ and $Z_0$ of $\gamma(t_0, X_{t_0}^\pi) = Y_{t_0}^\pi$ and $Z(t_0, X_{t_0}^\pi) = Z_{t_0}^\pi$ respectively and then calculate $X_{t_1}^\pi$ and $Y_{t_1}^\pi$ by using Euler scheme as

$$X_{t_1}^\pi = X_{t_0}^\pi + \mu(t_0, X_{t_0}^\pi, \gamma_0, Z_0) \Delta t_0 - \sigma(t_0, X_{t_0}^\pi, Y_{t_0}^\pi) \Delta W_{t_0},$$

$$Y_{t_1}^\pi = \gamma_0 - b(t_0, X_{t_0}^\pi, \gamma_0, Z_0) \Delta t_0 + Z_0 \Delta W_{t_0},$$

and use two neural networks $\gamma_1^N(\cdot; \theta_{1,1})$ and $Z_1^N(\cdot; \theta_{2,1})$ to approximate respectively $Y_{t_1}^\pi = \gamma(t_1, X_{t_1}^\pi)$ and $Z_{t_1}^\pi = Z(t_1, X_{t_1}^\pi)$ with associated local loss function

$$\mathcal{L}_1 := \mathbb{E} \left| Y_{t_1}^\pi - \gamma_1^N(X_{t_1}^\pi; \theta_{1,1}) \right|^2.$$  

Then for $j = 1, 2, \ldots, J - 2$, using Euler scheme to calculate $X_{t_{j+1}}^\pi$ and $Y_{t_{j+1}}^\pi$ gives

$$X_{t_{j+1}}^\pi = X_{t_j}^\pi + \mu(t_j, X_{t_j}^\pi, \gamma_j^N(X_{t_j}^\pi; \theta_{1,j}), Z_j^N(X_{t_j}^\pi; \theta_{2,j})) \Delta t_j - \sigma(t_j, X_{t_j}^\pi, Y_{t_j}^\pi) \Delta W_{t_j},$$

$$Y_{t_{j+1}}^\pi = \gamma_j^N(X_{t_j}^\pi; \theta_{1,j}) - b(t_j, X_{t_j}^\pi, \gamma_j^N(X_{t_j}^\pi; \theta_{1,j}), Z_j^N(X_{t_j}^\pi; \theta_{2,j})) \Delta t_j + Z_j^N(X_{t_j}^\pi; \theta_{2,j}) \Delta W_{t_j}.$$

Here, the difference from Deep BSDE-2 is lying in that the computation of $Y_{t_{j+1}}^\pi$ is based on the neural networks $\gamma_j^N(X_{t_j}^\pi; \theta_{1,j})$ and $Z_j^N(X_{t_j}^\pi; \theta_{2,j})$, rather than $Y_{t_j}^\pi$ and $Z_{t_j}^\pi$. Further, using two neural networks $\gamma_{j+1}^N(\cdot; \theta_{1,j+1})$ and $Z_{j+1}^N(\cdot; \theta_{2,j+1})$ to approximate respectively $Y_{t_{j+1}}^\pi = \gamma(t_{j+1}, X_{t_{j+1}}^\pi)$ and $Z_{t_{j+1}}^\pi = Z(t_{j+1}, X_{t_{j+1}}^\pi)$ with associated local loss function

$$\mathcal{L}_{j+1} := \mathbb{E} \left| Y_{t_{j+1}}^\pi - \gamma_{j+1}^N(X_{t_{j+1}}^\pi; \theta_{1,j+1}) \right|^2.$$  

Finally, using Euler scheme to calculate $X_{t_j}^\pi$ and $Y_{t_j}^\pi$ gives

$$X_{t_j}^\pi = X_{t_{j-1}}^\pi + \mu(t_{j-1}, X_{t_{j-1}}^\pi, \gamma_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{1,j-1}), Z_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{2,j-1})) \Delta t_{j-1} - \sigma(t_{j-1}, X_{t_{j-1}}^\pi, Y_{t_{j-1}}^\pi) \Delta W_{t_{j-1}},$$

$$Y_{t_j}^\pi = \gamma_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{1,j-1}) - b(t_{j-1}, X_{t_{j-1}}^\pi, \gamma_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{1,j-1}), Z_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{2,j-1})) \Delta t_{j-1} + Z_{j-1}^N(X_{t_{j-1}}^\pi; \theta_{2,j-1}) \Delta W_{t_{j-1}},$$

and define local loss function

$$\mathcal{L}_j := \mathbb{E} \left| Y_{t_j}^\pi - g(X_{t_j}^\pi) \right|^2.$$  

The scheme is to minimize the global loss function

$$\widehat{L}(\gamma_0, Z_0, \theta_{1,1}, \theta_{2,1}, \ldots, \theta_{1,J}, \theta_{2,J}) := \sum_{j=1}^{J} \mathcal{L}_j.$$
over all \( \theta = (\mathcal{Y}_0, Z_0, \theta_{1,1}, \theta_{2,1}, \ldots, \theta_{1,J}, \theta_{2,J}) \). If

\[
\theta^* = (\mathcal{Y}_0^*, Z_0^*, \theta_{1,1}^*, \theta_{2,1}^*, \ldots, \theta_{1,J}^*, \theta_{2,J}^*) \in \arg \min_{\theta \in \Theta} \mathcal{L}(\theta),
\]

then \( \mathcal{Y}_0^* \) is the desired approximation of \( Y_{0}^* \) by Deep BSDE-3.

In contrast with Deep BSDE-2, \( Y_{t_j}^* \) and \( Z_{t_j}^* \) in Deep BSDE-3 are replaced by \( \mathcal{Y}_j^{N}(X_{t_j}^{*}; \theta_{1,j}) \) and \( \mathcal{Z}_j^{N}(X_{t_j}^{*}; \theta_{2,j}) \) in Euler scheme to calculate \( Y_{t_{j+1}}^* \) for \( j = 1, 2, \ldots, J - 1 \).

6 Numerical examples

In this section, first, we will discuss the finite-dimensional framework for FBSPDEs on the domain \( D = (0, 1) \) with homogeneous Dirichlet boundary conditions. Then, we will solve two examples of FBSPDEs with the finite element method and deep learning schemes.

6.1 Framework for Homogeneous Dirichlet Boundary

Let \( D \subset \mathbb{R} \) with \( D = (0, 1) \) and \( \mathcal{T}_h : 0 = x_0 < x_1 < \cdots < x_L < x_{L+1} = 1 \) be a partition of the domain \( D \) into \( L+1 \) subintervals \( I_j = (x_{j-1}, x_j) \) with \( h_j = |I_j| = x_j - x_{j-1} = 1, 2, \ldots, L, L+1 \). Define \( h = \max\{h_j : j = 1, 2, \ldots, L, L+1 \} \). Then let \( \{\phi^1_h, \phi^2_h, \ldots, \phi^L_h\} \) be the set of nodal basis functions corresponding to the internal nodes \( \{x_1, x_2, \ldots, x_L\} \) which span the finite dimensional function space \( V_h^0 \). Nodal basis functions \( \phi^i_h, i = 1, 2, \ldots, L \) are hat functions and given by

\[
\phi^i_h(x) = \begin{cases} 
\frac{x-x_{i-1}}{h_i}, & \text{for } x_{i-1} \leq x \leq x_i \\
\frac{x_{i+1}-x}{h_{i+1}}, & \text{for } x_i \leq x \leq x_{i+1} \\
0, & \text{else.}
\end{cases}
\] (6.1)

Set \( \phi^{i-1}_h(x) = \frac{x-x_{i-1}}{h_i} \) and \( \phi^{i+1}_h(x) = \frac{x_{i+1}-x}{h_{i+1}} \).

The mass matrix \( A = [a_{ij}]_{i,j=1}^{L} \) is endowed with entries \( a_{ij} = \langle \phi^i_h, \phi^j_h \rangle \). Basic calculations imply that the symmetric Mass matrix \( A \) is given by

\[
A = \begin{pmatrix}
\frac{h_1}{3} + \frac{h_2}{3} & \frac{h_2}{3} & 0 & \cdots & 0 & 0 \\
\frac{h_2}{3} & \frac{h_3}{3} + \frac{h_2}{3} & \frac{h_2}{3} & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{h_{L-1}}{3} & \frac{h_{L-1}}{3} & \frac{h_{L}}{3} \\
0 & 0 & \cdots & \frac{h_{L}}{3} & \frac{h_{L}}{3} & \frac{h_{L+1}}{3}
\end{pmatrix}_{L \times L}.
\]

In a similar way, the stiffness matrix \( B = [b_{ij}]_{i,j=1}^{L} \) is equipped with entries \( b_{ij} = \langle \nabla \phi^i_h, \nabla \phi^j_h \rangle \). Straightforward computations yield the Stiffness matrix:

\[
B = \begin{pmatrix}
\frac{1}{h_1} + \frac{1}{h_2} & \frac{1}{h_2} & \cdots & 0 & 0 & 0 \\
-\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & \cdots & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{1}{h_{L-1}} + \frac{1}{h_{L}} & \frac{1}{h_{L}} & \frac{1}{h_{L+1}} \\
0 & 0 & \cdots & \frac{1}{h_{L}} & \frac{1}{h_{L}} + \frac{1}{h_{L+1}} & \frac{1}{h_{L+1}}
\end{pmatrix}_{L \times L}.
\]
In addition, the $l$--th component of the vector $\bar{g}(\rho_h(T); \phi_h)$ is defined as $\langle g(\rho_h(T)), \phi_h^l \rangle$. The involved integrals may be evaluated via conventional numerical approximations. Here, to evaluate the integrals one can use numerical integration.

### 6.2 Example 1

Consider the following decoupled FBSPDE with homogeneous Dirichlet boundary conditions:

\[
\begin{aligned}
  &\frac{d\rho(t, x)}{dt} = \delta \Delta \rho(t, x) + \sum_{i=1}^{k} \gamma_i^t \delta W_t^i - \rho(t, x) \sum_{i=1}^{k} \gamma_i^t \psi_i^t(t, x), \\
  &\rho(0, x) = \rho_0(x), \\
  &\rho(t, x)_{\partial D} = 0, \\
  &-du(t, x) = \left( \delta \Delta u(t, x) + \sum_{i=1}^{k} \gamma_i^t \psi_i^t(t, x) + f(t, x, \rho(t, x)) \right) dt - \sum_{i=1}^{k} \psi_i^t(t, x) dW_t^i \\
  &u(T, x) = g(\rho(T, x)), \\
  &u(t, x)_{\partial D} = 0.
\end{aligned}
\]

(6.2)

Here, $k = 1, g(x) = 1 - e^{-x}, \rho_0 > 0, \gamma_i \equiv \gamma$ is a constant and $f(t, x, \rho(t, x))$ is given by

\[
f(t, x, \rho(t, x)) = \delta \left( \mathbb{E} [\nabla \rho_0(x + \sqrt{2\delta} B_t)] e^{-\gamma W_t - \frac{1}{2} \gamma^2 t} \right)^2 e^{-\rho(t, x)} + \frac{1}{2} \gamma^2 \rho^2(t, x) e^{-\rho(t, x)} + \gamma^2 (t, x) e^{-\rho(t, x)} - 2\delta \mathbb{E} [\Delta \rho_0(x + \sqrt{2\delta} B_t)] e^{-\gamma W_t - \frac{1}{2} \gamma^2 t} e^{-\rho(t, x)}.
\]

The analytic solution of above FBSPDE gives

\[
\rho(t, x) = \mathbb{E} [\rho_0(x + \sqrt{2\delta} B_t)] e^{-\gamma W_t - \frac{1}{2} \gamma^2 t},
\]

and

\[
u(t, x) = 1 - e^{\rho(t, x)}.
\]

The approximating finite dimensional FBSDEs are of the following form:

\[
\begin{aligned}
  &d\bar{\rho}(t) = \mu(t, \bar{\rho}(t)) dt - \sigma(t, \bar{\rho}(t)) dW(t), \quad \bar{\rho}(0) = A^{-1} \rho_0, \\
  &-d\bar{u}(t) = b(t, \bar{\rho}(t), \bar{u}(t), \bar{u}(t)) dt - \bar{\psi}(t) dW(t), \quad \bar{u}(T) = A^{-1} \bar{g}(\rho(T); \phi_h),
\end{aligned}
\]

(6.3)

with

\[
\begin{aligned}
  &\mu(t, \bar{\rho}) := -\delta A^{-1} B \bar{\rho}, \\
  &\sigma(t, \bar{\rho}) := \gamma \bar{u}, \\
  &b(t, \bar{\rho}(t), \bar{u}(t), \bar{u}(t)) := -\delta A^{-1} B \bar{u} + \gamma \bar{\psi} + A^{-1} \bar{f}(t, \rho(t); \phi_h).
\end{aligned}
\]

We choose $T = 0.5, \delta = 0.20, \gamma = 1.0, \rho_0(x) = \sin(\pi x)$ and the solution of above finite dimensional FBSDE is approximated by using Deep BSDE-1 algorithm. We adopt uniform mesh with $L$ internal nodes and uniform time grid. We use fully connected neural network comprising 2 hidden layers with $L + 10$ neurons in each layer. Hyperbolic tangent is used as activation function.
Figure 1: Example 1 (δ = 0.2, deep BSDE-1)  

Table 1: Relative Error

| L   | Relative Error | Δt |
|-----|----------------|----|
| 5   | 0.004395       | .05|
| 15  | 0.009749       | .05|
| 20  | 0.000893       | .05|
| 25  | 0.001699       | .025|
| 35  | 0.001905       | .0167|
| 50  | 0.000294       | .001|

for hidden layers and Adam optimizer adopted for training. We set batch size 512 for training purpose. The forward process is numerically approximated at the time grid by backward Euler scheme:

\[ \tilde{\rho}_h(t_{j+1}) = (I + \delta A^{-1} B \Delta t)^{-1} (\tilde{\rho}_h(t_j) - \gamma \tilde{\rho}_h(t_j) \Delta W_t), \quad j = 0, 1, \ldots, J - 1, \quad \tilde{\rho}_h(0) = A^{-1} \tilde{\rho}_0. \]

With Deep BSDE-1, we simulate the approximate solution for \( L = 5, 15, 20 \) with \( \Delta t = 0.05 \), \( L = 25 \) with \( \Delta t = 0.025 \), \( L = 35 \) with \( \Delta t = 0.167 \) and \( L = 50 \) with \( \Delta t = 0.001 \). Figure 1 shows the mean value of \( u(0, x) \) from 10 simulations and Table 1 shows the relative errors. Letting \( \hat{u}(0, x) \) be the approximation of \( u(0, x) \), we investigate the relative error:

\[
R_E = \frac{\int_D \left| u(0, x) - \frac{1}{10} \sum_{i=1}^{10} \hat{u}^i(0, x) \right|^2 \, dx}{\int_D |u(0, x)|^2 \, dx}.
\]

In Example-1, we can see that Deep BSDE-1 improves the accuracy of the approximations as mesh sizes for both space and time domains decrease.

6.3 Example 2

Consider the following coupled and nonlocal FBSPDE:

\[
\begin{align*}
\frac{d\rho(t, x)}{dt} &= \left( \delta \Delta \rho(t, x) + f_1(t, x, \rho(t, x), u(t, x)) \right) \, dt - f_3(t, x) \, dW_t, \\
\rho(0, x) &= \rho_0(x) = \frac{\pi}{2} \sin(\pi x) + \frac{1}{2} \sin(2\pi x), \\
\rho(t, x) \bigg|_{\partial D} &= 0, \\
-\frac{d\rho(t, x)}{dt} &= \left( \delta \Delta \rho(t, x) + f_2(t, x, \rho(t, x), u(t, x)) \right) \, dt - \psi(t, x) \, dW_t, \\
u(T, x) &= g(\rho(T, x)) = \arctan(\rho(T, x)), \\
u(t, x) \bigg|_{\partial D} &= 0,
\end{align*}
\]

(6.4)
with
\[ f_1(t, x, \rho(t), u(t)) = \alpha \cdot \cos(u(t)) - \frac{\alpha}{\sqrt{1 + \rho(t)^2}} + \delta \pi^2 \rho(t) + \delta \cdot \frac{2 + \cos(W_t)}{2} \cdot \pi^2 \sin(2\pi x) \]
\[ - \frac{\cos(W_t)}{12} \cdot \sin(2\pi x), \]
\[ f_2(t, x, \rho(t), u(t)) = \frac{2\delta \rho(t)}{(1 + \rho(t)^2)^2} \cdot \frac{\pi^2}{2} \cos(\pi x) + \frac{2 + \cos(W_t)}{3} \cdot \pi \cos(2\pi x) \]
\[ + \frac{2\delta}{1 + \rho(t)^2} \cdot \left( \sin(\pi x) + \frac{2 + \cos(W_t)}{3} \cdot 2\pi^2 \sin(2\pi x) \right) \]
\[ + \alpha \cdot u(t) - \alpha \cdot \arctan(\rho(t)) + \frac{\rho(t)}{(1 + \rho(t)^2)^2} \cdot \frac{\sin^2(W_t) \cdot \sin^2(2\pi x)}{36} \]
\[ - \frac{1}{1 + \rho(t)^2} \left( \delta \pi^2 \rho(t) + \delta \cdot \frac{2 + \cos(W_t)}{2} \cdot \pi^2 \sin(2\pi x) - \frac{\cos(W_t)}{12} \cdot \sin(2\pi x) \right) \]
\[ + \gamma \cdot \left( \int_0^1 \sin(2\pi x) \rho(t) dx - \frac{2 + \cos(W_t)}{12} \right), \]
\[ f_3(t, x) = \frac{\sin(W_t)}{6} \cdot \sin(2\pi x). \]

The analytic solution gives
\[ \rho(t, x) = \frac{\pi}{2} \sin(\pi x) + \frac{2 + \cos(W_t)}{6} \sin(2\pi x), \]
\[ u(t, x) = \arctan(\rho(t, x)). \]

The approximating finite dimensional FBSDE is given by
\[
\begin{cases}
\mathrm{d} \tilde{\rho}_h(t) = \mu(t, \tilde{\rho}_h(t), \tilde{u}_h(t)) \mathrm{d}t - \sigma(t) \mathrm{d}W_t, & \tilde{\rho}_h(0) = A^{-1} \rho_0, \\
-\mathrm{d} \tilde{u}_h(t) = b(t, \tilde{\rho}_h(t), \tilde{u}_h(t)) \mathrm{d}t - \psi_h(t) \mathrm{d}W(t), & \tilde{u}_h(T) = A^{-1} \tilde{g}(\rho_h(T); \phi_h),
\end{cases}
\] (6.5)

with
\[
\mu(t, \tilde{\rho}_h(t), \tilde{u}_h(t)) = -\delta A^{-1} B \tilde{\rho}_h(t) + A^{-1} \tilde{f}_1(t, \rho_h(t), u_h(t); \phi_h),
\]
\[
b(t, \tilde{\rho}_h(t), \tilde{u}_h(t)) = -\delta A^{-1} B \tilde{u}_h(t) + A^{-1} \tilde{f}_2(t, \tilde{\rho}_h(t), \tilde{u}_h(t); \phi_h),
\]
\[
\sigma(t) = A^{-1} \tilde{f}_3(t; \phi_h).
\]

When \( \alpha = 0.2 = \gamma \) and \( \delta = .001 \), the resulting finite dimensional FBSDE is coupled and the solution is approximated via both Deep BSDE-2 and Deep BSDE-3 algorithms. In both schemes, we consider \( T = 0.5 \), the uniform time grid with \( \Delta t = 0.05 \) and the uniform mesh with \( L \) internal nodes. We use fully connected neural network comprising 2 hidden layers with \( L + 10 \) neurons in each layer. Hyperbolic tangent is used as activation function for hidden layers and Adam optimizer is used for training. We use batch size 512 for training purpose. For time discretization of forward process, the backward Euler scheme is employed, which is written as:
\[
\tilde{\rho}_h(t_{j+1}) = (I + \delta A^{-1} B \Delta t)^{-1} \left( \tilde{\rho}_h(t_j) + A^{-1} \tilde{f}_1(t_j, \rho_h(t_j), u_h(t_j); \phi_h) \Delta t - A^{-1} \tilde{f}_3(t_j; \phi_h) \Delta W_{t_j} \right),
\]
\[ j = 0, 1, \ldots, J - 1, \]
With Deep BSDE-2, we compute the approximate solution for $L = 5$ and $L = 15$. Figure 2 shows the mean values of $u(0, x)$ from 10 runs and Table 2 shows the relative errors. The results from Deep BSDE-2 are not stable for $L \geq 20$ and the associated numerical results are not presented here.

It is worth noting that the approximating performance of Deep BSDE-2 may be improved when $\Delta t$ is smaller, while herein we fix $\Delta t = 0.05$ to compare these two methods: Deep BSDE-2 and Deep BSDE-3. Indeed, with Deep BSDE-3, we simulate the approximate solution for $L = 5, 15, 30$ and $L = 50$. Figure 3 shows the mean value of $u(0, x)$ from 10 runs and Table 3 shows the relative errors. Clearly, Deep BSDE-3 provides stable solutions for even bigger values of $L$ compared to Deep BSDE-2 and the accuracy of Deep BSDE-3 is also higher than Deep BSDE-2.

A Appendix

A.1 Proof of Lemma 4.1

Proof of Lemma 4.1. The proof of the existence and uniqueness of solution to FBSDE (4.1)-(4.2) and Assertion (i) is the same as that of Theorem 3.1. We only need to prove Assertion (ii). Computations involved in this proof will be divided into two parts.

Step 1. This part is devoted to some estimates associated to the forward equation (4.1). Fix $t \in [0, T]$. Applying Itô’s formula to equation (4.1) for $\nabla \rho_t(s)$ yields $\mathbb{P}$-a.s.

$$
\|\nabla \rho_t(s)\|^2 = \|\nabla \rho_0\|^2 + 2 \int_0^t \langle \nabla \Delta h \rho_h(s), \nabla \rho_h(s) \rangle ds + \int_0^t \|\nabla \Pi_h f(s, \rho_h(s), u_h(s))\|^2 ds \\
+ 2 \int_0^t \langle \nabla \Pi_h F(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \psi_h(s)), \nabla \rho_h(s) \rangle ds
$$
Using Lipschitz property from Assumption 2.1, we have

\[
-2 \sum_{i=1}^{k} \int_0^t \langle \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i \\
= \| \nabla \rho_0 \|^2 - 2 \int_0^t \| \Delta_h \rho_h(s) \|^2 ds + \int_0^t \| \nabla \Pi_h f(s, \rho_h(s), u_h(s)) \|^2 ds \\
+ 2 \int_0^t \langle \Pi_h F(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \nabla u_h(s), \psi_h(s)), \nabla \rho_h(s) \rangle ds \\
- 2 \sum_{i=1}^{k} \int_0^t \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i.
\]

(A.1)
we have further

Taking $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \frac{1}{4}$, we conclude from (A.1) that

$$
\|\nabla \rho_h(t)\|^2 + \frac{1}{2} \int_0^t \|\Delta_h \rho_h(s)\|^2 ds \\
\leq \|\nabla \rho_0\|^2 + \int_0^t \|\nabla \Pi_h f(s, \rho_h(s), u_h(s))\|^2 ds - 2 \sum_{i=1}^k \int_0^t \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i \\
+ 4 \int_0^t \left( \|F_s^0\|^2 + \|L_1^F\|^2 \|\rho_h(s)\|^2 + \|L_2^F\|^2 \|u_h(s)\|^2 + \|L_3^F\|^2 \|\nabla \rho_h(s)\|^2 + |L_4^F|^2 \|\psi(s)\|^2 \right) ds \\
+ 4|L_1^F|^2 \int_0^t \|\nabla \rho_h(s)\|^2 ds.
$$

Noticing $\|\nabla \Pi_h f(s, \rho_h(s), u_h(s))\| \leq C_e \|f(s, \rho_h(s), u_h(s))\|_{1,2} \leq C_e \tilde{L} \sum_{1}^{3} \|\rho_h(s)\|^2 + \|u_h(s)\|^2$,
we have further

$$
\|\nabla \rho_h(t)\|^2 + \frac{1}{2} \int_0^t \|\Delta_h \rho_h(s)\|^2 ds \\
\leq \|\nabla \rho_0\|^2 + 3C_e \tilde{L}^2 t + 4 \int_0^t \|F_s^0\|^2 ds - 2 \sum_{i=1}^k \int_0^t \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i \\
+ 4 \int_0^t \left( |L_1^F|^2 \|\rho_h(s)\|^2 + |L_2^F|^2 \|u_h(s)\|^2 + |L_3^F|^2 \|\nabla \rho_h(s)\|^2 + |L_4^F|^2 \|\psi(s)\|^2 \right) ds \\
+ \left( 4|L_1^F|^2 + 3C_e \tilde{L}^2 \right) \int_0^t \|\rho_h(s)\|^4_{2,2} ds + 3C_e \tilde{L}^2 \|\|\psi(s)\|^2 \right) ds.
$$

Taking supremum over $t \in [0, \tau]$ for $\tau \in [0, T]$ and then expectations on both sides we have

$$
\mathbb{E} \left[ \sup_{t \in [0, \tau]} \|\nabla \rho_h(t)\|^2 \right] + \frac{1}{2} \mathbb{E} \int_0^\tau \|\Delta_h \rho_h(s)\|^2 ds \\
\leq \mathbb{E} \left[ \|\nabla \rho_0\|^2 \right] + 4 \mathbb{E} \int_0^\tau \|F_s^0\|^2 ds + 2 \mathbb{E} \left[ \sup_{t \in [0, \tau]} \left( \int_0^t \sum_{i=1}^k \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i \right) \right] \\
+ \left( 4|L_1^F|^2 + 3C_e \tilde{L}^2 \right) \mathbb{E} \int_0^\tau \|\rho_h(s)\|^4_{2,2} ds + 3C_e \tilde{L}^2 \|\|\psi(s)\|^2 \right) ds.
$$

(A.2)

For the terms involving stochastic integrals, we use BDG inequality to obtain

$$
2 \mathbb{E} \left[ \sup_{t \in [0, \tau]} \left( \int_0^t \sum_{i=1}^k \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle dW_s^i \right) \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^\tau \sum_{i=1}^k \left( \langle \nabla \Pi_h f^i(s, \rho_h(s), u_h(s)), \nabla \rho_h(s) \rangle \right)^2 ds \right)^{\frac{1}{2}} \right].
$$

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\begin{align*}
\leq \tilde{C} & \mathbb{E}\left[ \left( \int_0^T |C_e \tilde{L} f|^2 \left( 1 + \|\rho_h(s)\|_{1,2} + \|u_h(s)\|_{1,2} \right)^2 \cdot \|\nabla \rho_h(s)\|^2 ds \right) \right] \\
\leq \tilde{C} & \mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|\nabla \rho_h(t)\|^2 \cdot |C_e \tilde{L} f|^2 \int_0^T \left( 1 + \|\rho_h(s)\|_{1,2} + \|u_h(s)\|_{1,2} \right)^2 ds \right) \right] \\
\leq \mathbb{E} & \left[ \varepsilon_7 \sup_{t \in [0,T]} \|\nabla \rho_h(t)\|^2 + \frac{|\tilde{C} C_e \tilde{L} f|^2}{\varepsilon_7} \int_0^T \left( 1 + \|\rho_h(s)\|_{1,2} + \|u_h(s)\|_{1,2} \right)^2 ds \right] \\
\leq \varepsilon_7 & \mathbb{E}\left[ \sup_{t \in [0,T]} \|\nabla \rho_h(t)\|^2 \right] + \frac{3|\tilde{C} C_e \tilde{L} f|^2}{\varepsilon_7} \mathbb{E}\int_0^T \|\rho_h(s)\|^2_{1,2} ds + \frac{3|\tilde{C} C_e \tilde{L} f|^2}{\varepsilon_7} \mathbb{E}\int_0^T \|u_h(s)\|^2_{1,2} ds \\
\quad + \frac{3|\tilde{C} C_e \tilde{L} f|^2}{\varepsilon_7} \tau,
\end{align*}

with \( \varepsilon_7 = \frac{1}{2} \). This together with (A.2) implies that

\begin{align*}
\mathbb{E} & \left[ \sup_{t \in [0,T]} \|\nabla \rho_h(t)\|^2 \right] + \mathbb{E} \int_0^T \|\Delta_h \rho_h(s)\|^2 ds \\
\leq 2 \mathbb{E} & \left[ \|\nabla \rho_0\|^2 \right] + 8 \mathbb{E} \int_0^T \left|F^0\right|^2 ds + \left( 8 |\tilde{C}|^2 + 4 (3|\tilde{C}|^2 + 1) |C_e \tilde{L} f|^2 \right) \mathbb{E} \int_0^T \|\rho_h(s)\|^2_{1,2} ds \\
& \quad + 8 \mathbb{E} \int_0^T \left( |\tilde{L} f|^2 \|\rho_h(s)\|^2 + |\tilde{L} f|^2 \|u_h(s)\|^2 + |\tilde{L} f|^2 \|\nabla u_h(s)\|^2 + |\tilde{L} f|^2 \|\psi_h(s)\|^2 \right) ds \\
& \quad + 4 (3|\tilde{C}|^2 + 1) |C_e \tilde{L} f|^2 \tau + |C_e \tilde{L} f|^2 (3 + 12 |\tilde{C}|^2) \mathbb{E} \int_0^T \|u_h(s)\|^2_{1,2} ds.
\end{align*}

Now using Gronwall’s inequality and estimate from Assertion (i) we have

\begin{align*}
\mathbb{E} & \left[ \sup_{t \in [0,T]} \|\nabla \rho_h(t)\|^2 \right] + \mathbb{E} \int_0^T \|\Delta_h \rho_h(s)\|^2 ds \\
\leq C & \mathbb{E}\left[ 1 + \|\rho_0\|^2 + \|\nabla \rho_0\|^2 + \|\psi_0\|^2 \right] + C \mathbb{E} \int_0^T \left( \sum_{i=1}^k \|f^i_s\|^2 + \|F^0_s\|^2 + \|G^0_s\|^2 \right) ds,
\end{align*}

with \( C = C(L_1^f, L_2^f, \tilde{L} f, L_0^f, L_1^f, L_2^f, L_1^G, L_2^G, T, C_e) \).

**Step 2.** Then we conduct the computations for the backward equation (4.2). Fix \( t \in [0,T] \). Applying Itô’s formula to equation (4.2) for \( \nabla u_h(t) \) yields \( \mathbb{P} \)-a.s.

\begin{align*}
\|\nabla u_h(t)\|^2 & = \|\nabla \Pi_h g(\rho_h(T))\|^2 + 2 \int_t^T \langle \nabla \Delta_h u_h(s), \nabla u_h(s) \rangle ds - 2 \sum_{i=1}^k \int_t^T \langle \nabla \psi^i_h(s), \nabla u_h(s) \rangle dW^i_s \\
& \quad - \int_t^T \|\nabla \psi_h(s)\|^2 ds + 2 \int_t^T \langle \nabla \Pi_h G(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \psi_h(s)), \nabla u_h(s) \rangle ds \\
& = \|\nabla \Pi_h g(\rho_h(T))\|^2 - 2 \int_t^T \langle \Delta_h u_h(s), \Delta_h u_h(s) \rangle ds - 2 \sum_{i=1}^k \int_t^T \langle \nabla \psi^i_h(s), \nabla u_h(s) \rangle dW^i_s \\
& \quad - \int_t^T \|\nabla \psi_h(s)\|^2 ds + 2 \int_t^T \langle \nabla \Pi_h G(s, \rho_h(s), \nabla \rho_h(s), u_h(s), \psi_h(s)), \nabla u_h(s) \rangle ds.
\end{align*}

Using \( \|\nabla \Pi_h g(\rho_h(T))\| \leq C_e \|g(\rho_h(T))\|_{1,2} \leq C_e \tilde{L} g \left( 1 + \|\rho_h(T)\|_{1,2} \right) \), we have

\begin{align*}
\|\nabla u_h(t)\|^2 + 2 \int_t^T \|\Delta_h u_h(s)\|^2 ds + \int_t^T \|\nabla \psi_h(s)\|^2 ds
\end{align*}
Then taking expectations on both sides we have in particular

In view of the Lipschitz property from Assumption 2.2, we have

Taking \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \frac{1}{4} \) and combining the above result with (A.3) yield that

Then taking expectations on both sides we have in particular

\[
\mathbb{E} \int_0^T \| \nabla \psi_h(s) \|^2 ds \\
\leq 2 \mathbb{E} \left[ C_T \tilde{L}^2 \| \rho_h(T) \|^2_{2,2} + 2 \mathbb{E} \int_t^T \| G_h^0 \|^2 ds + 4 \mathbb{E} \int_t^T \| G_h^1 \|^2 ds + 4 \mathbb{E} \int_t^T \| G_h^2 \|^2 ds \right].
\]
On the other hand, taking supremum over \( t \in [\tau, T] \) for \( \tau \in [0, T] \) and taking expectations in (A.4), we have

\[
\mathbb{E} \left[ \sup_{t \in [\tau, T]} \| \nabla u_h(t) \|^2 \right] + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \| \Delta_h u_h(s) \|^2 ds + \mathbb{E} \int_{\tau}^{T} \| \nabla \psi_h(s) \|^2 ds \\
\leq 2|C_e\tilde{L}^g|^2 + 2|C_eL^g|^2\mathbb{E} \left[ \| \rho_h(T) \|^2 \right] + 4\mathbb{E} \int_{\tau}^{T} \| G^0_s \|^2 ds + 4|L^g_2|^2 \mathbb{E} \int_{\tau}^{T} \| \nabla u_h(s) \|^2 ds \\
+ 4\mathbb{E} \int_{\tau}^{T} \left( |L^g_1|^2 \| \rho_h(s) \|^2 + |L^g_1|^2 \| \nabla \rho_h(s) \|^2 + |L^g_2|^2 \| u_h(s) \|^2 + |L^g_2|^2 \| \psi_h(s) \|^2 \right) ds \\
+ 2\mathbb{E} \left[ \sup_{t \in [\tau, T]} \int_{\tau}^{T} \sum_{i=1}^{k} (\nabla \psi^i_h(s), \nabla u_h(s)) dW_s^i \right].
\] (A.6)

Now we use BDG inequality for the terms involving stochastic integrals and obtain

\[
2\mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_{\tau}^{T} \sum_{i=1}^{k} (\nabla \psi^i_h(s), \nabla u_h(s)) dW_s^i \right| \right] \leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [\tau, T]} \| \nabla u_h(t) \|^2 \right] + 2\tilde{C} \mathbb{E} \int_{\tau}^{T} \| \nabla \psi_1(s) \|^2 ds,
\]

which together with (A.6) implies that

\[
\frac{1}{2} \mathbb{E} \left[ \sup_{t \in [\tau, T]} \| \nabla u_h(t) \|^2 \right] + \frac{1}{2} \mathbb{E} \int_{\tau}^{T} \| \Delta_h u_h(s) \|^2 ds + \mathbb{E} \int_{\tau}^{T} \| \nabla \psi_h(s) \|^2 ds \\
\leq 2|C_e\tilde{L}^g|^2 + 4\mathbb{E} \int_{\tau}^{T} \| G^0_s \|^2 ds + 2\tilde{C} \mathbb{E} \int_{\tau}^{T} \sum_{i=1}^{k} \| \nabla \psi^i_h(s) \|^2 ds + 4|L^g_2|^2 \mathbb{E} \int_{\tau}^{T} \| \nabla u_h(s) \|^2 ds \\
+ 4\mathbb{E} \int_{\tau}^{T} \left( |L^g_1|^2 \| \rho_h(s) \|^2 + |L^g_1|^2 \| \nabla \rho_h(s) \|^2 + |L^g_2|^2 \| u_h(s) \|^2 + |L^g_2|^2 \| \psi_h(s) \|^2 \right) ds \\
+ 2|C_eL^g|^2 \mathbb{E} \left[ \| \rho_h(T) \|^2 \right].
\]

Then using estimates from Assertion (i) and from (A.5), and applying Gronwall’s inequality, we have

\[
\frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, T]} \| \nabla u_h(t) \|^2 \right] + \frac{1}{2} \mathbb{E} \int_{0}^{T} \| \Delta_h u_h(s) \|^2 ds + \mathbb{E} \int_{0}^{T} \| \nabla \psi_h(s) \|^2 ds \\
\leq C + C\mathbb{E} \left[ \| \rho_0 \|^2 \right] + \mathbb{E} \int_{0}^{T} \left( \| f^0 \|^2 + \| F^0_s \|^2 + \| G^0_s \|^2 \right) ds,
\]

with \( C = C(L^f_1, L^f_2, \tilde{L}^f, L^g, \tilde{L}^g, L^f_1, L^g_1, L^f_2, L^g_2, T, C_e) \). Combining this with the estimate from Step 1 finally gives Assertion (ii).

\[ \square \]

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