On the ratios of Barnes’ multiple gamma functions to the $p$-adic analogues

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Abstract

Let $F$ be a totally real field. For each ideal class $c$ of $F$ and each real embedding $ι$ of $F$, Hiroyuki Yoshida defined an invariant $X(c, ι)$ as a finite sum of log of Barnes’ multiple gamma functions with some correction terms. Then the derivative value of the partial zeta function $ζ'(0, c) = \sum _{ι} X(c, ι)$, where $ι$ runs over all real embeddings of $F$. Yoshida studied the relation between $exp(X(c, ι))$’s, Stark units, and Shimura’s period symbol. Yoshida and the author also defined and studied the $p$-adic analogue $X_p(c, ι)$: In particular, we discussed the relation between the ratios $[exp(X(c, ι)) : exp_p(X_p(c, ι))]$ and Gross-Stark units. In a previous paper, the author proved the algebraicity of some products of $exp(X(c, ι))$’s. In this paper, we prove its $p$-adic analogue. Then, by using these algebraicity properties, we discuss the relation between the ratios $[exp(X(c, ι)) : exp_p(X_p(c, ι))]$ and Stark units.

1 Introduction

Let $F$ be a totally real field, $𝓞_F$ the ring of integers of $F$, $ℐ$ an integral ideal of $F$, $I_ℐ$ the group of all fractional ideals of $F$ relatively prime to $ℐ$. We consider the narrow ideal class group modulo $ℐ$ defined as

$$C_ℐ := I_ℐ/\{(α) \in I_ℐ \mid α \in F^{×}, \ α \equiv 1 \ mod^* ℐ, \ α \gg 0\}.$$ 

Here $α \gg 0$ means that $α$ is totally positive. We denote the class in $C_ℐ$ of a fractional ideal $a$ by $[a]$. Shintani [Shin2] gave an explicit formula for the derivative value $ζ'(0, c)$ of the partial zeta function $ζ(s, c) := \sum _{𝓞_F \ni a \in C} Na^{-s}$ with $c \in C_ℐ$. The main term is a finite sum of log of Barnes’ multiple gamma functions $Γ(z, v)$ (Definition 2) of the following form:

$$ζ'(0, c) = \sum _{ι \in Hom(F, ℜ)} \left( \sum _{j \in J} \sum _{x \in R(c, v_j)} \log(Γ(ι(x^tv_j), ι(v_j))) \right) + \text{correction terms.} \quad (1)$$

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Here $v_j$ are suitable vectors whose entries are totally positive integers of $F$, $R(c,v_j)$ are finite sets of vectors whose entries are positive rational numbers (for details, see Theorem 1, Definition 6). We denote by $x^t v_j$ the inner product, by Hom($F, \mathbb{R}$) the set of all real embeddings of $F$. Yoshida [Yo, Chap. III, (3.6)–(3.9)] discovered an appropriate decomposition $\sum_{\iota \in \text{Hom}(F, \mathbb{R})} W(c, \iota) + V(c, \iota)$ of the “correction terms” and defined the class invariant $X(c, \iota) := \sum_{j \in J} \sum_{x \in R(c, v_j)} \log(\Gamma(x^t v_j, \iota(v_j))) + W(c, \iota) + V(c, \iota)$.

It follows that we have

$$\zeta'(0, c) = \sum_{\iota \in \text{Hom}(F, \mathbb{R})} X(c, \iota).$$

Since Shintani’s expression [1] is not unique, Yoshida’s invariant has an ambiguity: $\exp(X(c, \iota))$ is well-defined up to multiplication by a rational power of a unit of $\iota(F)$. Yoshida conjectured that each value of Shimura’s period symbol $p_K$ (Shimura, Theorem 32.5) can be expressed as a product of rational powers of $\exp(X(c, \iota))$’s [Yo, Chap. III, Conjecture 3.9]. Yoshida also studied the relation between Stark units and $\exp(X(c, \iota))$’s. On the other hand, Yoshida and the author defined and studied the $p$-adic analogue $X^p(c, \iota)$ in [KY1, KY2]. In particular we formulated a refinement of a $p$-adic analogue of the Stark conjecture by Gross. Let us explain more precisely: Let $K/F$ be an abelian extension of number fields, $S$ a finite set of places of $F$. We assume that

- $S$ contains all infinite places of $F$ and all ramified places in $K/F$.
- $S$ contains a distinguished place $v$ which splits completely in $K/F$.
- $\#S > 2$. (This assumption is for simplicity. $\#S > 1$ is essential.)

We fix a place $w$ of $K$ dividing $v$. Then the rank 1 abelian Stark conjecture, which is a version of Stark’s conjectures in [St], implies the following statement.

**Conjecture 1.** There exists a $v$-unit $u \in K$, which is called a Stark unit, satisfying

$$\log \|u^\tau\|_w = -W_K \zeta'_S(0, \tau) \quad (\tau \in \text{Gal}(K/F)).$$

Here $W_K$ is the number of roots of unity in $K$. We define the partial zeta function associated with $S, \tau$ by

$$\zeta_S(s, \tau) := \sum_{(\frac{K/F}{a}) = \tau, (a, S) = 1} Na^{-s},$$

where $a$ runs over all integral ideals of $F$, relatively prime to each finite place in $S$, whose images under the Artin symbol $(\frac{K/F}{a})$ equal $\tau$. If $w$ is a finite place, we put $\|x\|_w := Nw^{-\text{ord}_w(x)}$. Otherwise we take an embedding $t_w : K \hookrightarrow \mathbb{C}$ associated with $w$ and
put \( \|x\|_w := |\iota_w(x)| \) or \( |\iota_w(x)|^2 \) for a real or complex place \( w \) respectively. Note that when \( v \) is a real place, assuming the Stark conjecture, we can write a Stark unit explicitly as

\[
\exp(2\zeta'_S(0, \tau)) \in \iota_w(\mathcal{O}_K^\times).
\] (2)

Here we see that \( W_K = 2 \) since \( K \) has a real place \( w \).

Next we consider the case when \( v \) is a finite place lying above a rational prime \( p \). We additionally assume that

- \( F \) is a totally real field, \( K \) is a CM-field.
- \( S \) contains all places of \( F \) lying above \( p \).

Throughout this paper, we regard each number field as a subfield of \( \overline{\mathbb{Q}} \) and fix two embeddings \( \overline{\mathbb{Q}} \to \mathbb{C}, \overline{\mathbb{Q}} \to \mathbb{C}_p \). Here we denote by \( \mathbb{C}_p \) the \( p \)-adic completion of the algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \). For a number field \( L \), we denote by \( p_L := p_{L, \text{id}} \) the prime ideal corresponding to the \( p \)-adic topology on \( L \) induced by \( \text{id}: L \to \overline{\mathbb{Q}} \to \mathbb{C}_p \). We assume that

- \( \nu = \nu_F \), \( \nu = \nu_K \).

Put \( R := S - \{ \nu_F \} \). We see that \( \zeta'_S(s, \tau) = (1 - N_{p_F^{-s}}) \zeta_R(s, \tau) \) since \( p_F \) splits completely in \( K \). It follows that

\[
\zeta'_R(0, \tau) = \zeta_R(0, \tau) \log N_{p_F}.
\]

Hence, by \( \zeta_R(0, \tau) \in \mathbb{Q} \), there exist a positive integer \( W \) and a \( p_F \)-unit \( u' \in K \) which satisfy

\[
\log \|u'^\tau\|_w = -W\zeta'_R(0, \tau) \quad (\tau \in \text{Gal}(K/F)).
\] (3)

The Stark conjecture with \( v = p_F \) states that we can take \( W = W_K \). Gross conjectured the following property of \( u' \) as a \( p \)-adic number.

**Conjecture 2** ([Gr2, Conjecture 3.13]). Let \( u' \in K \) be a \( p_F \)-unit satisfying (3). Then we have

\[
\log_p N_{K_{p_K}/\mathbb{Q}_p}(u'^\tau) = -W\zeta'_{p,S}(0, \tau) \quad (\tau \in \text{Gal}(K/F)).
\]

Here \( \log_p \) denotes Iwasawa’s \( p \)-adic log function and \( \zeta'_{p,S}(s, \tau) \) denotes the \( p \)-adic interpolation function for \( \zeta_S(s, \tau) \). Dasgupta-Darmon-Pollack ([DDP] proved that Conjecture 2 holds true under certain conditions. In [KY1], we formulated a conjecture which expresses \( \log_p u'^\tau \) (without the norm \( N_{K_{p_K}/\mathbb{Q}_p} \)) by using not only \( X_p(c, \iota) \) but also \( X(c, \iota) \). More precisely, we see that the ratio \( [\exp(X(c, \iota)) : \exp_p(X_p(c, \iota))] \) is well-defined up to roots of unity (Corollary 1-(i)), and our conjecture (Conjecture 3) states that \( u' \) can be expressed as a product of \( \exp_p(X(c, \iota)) \)'s, up to \( \ker \log_p \) (Proposition 6-(ii)). One of the main results in this paper states that we can express a Stark unit as another product of \( \exp(X(c, \iota)) \)'s uniformly (Remark 6).
To clarify the meaning of the main results in this paper, we briefly describe the results in [Ka2]. Let $F = \mathbb{Q}$, $v$ the unique real place of $\mathbb{Q}$. Then the Stark conjecture implies the following “reciprocity law” on the sine function (in other words, on cyclotomic units):

$$\sin\left(\frac{\alpha}{m}\pi\right) = \pm \sin\left(\tau\left(\frac{\alpha}{m}\right)\pi\right) \quad \left(\frac{\alpha}{m} \in \mathbb{Q} \cap (0, 1), \ \tau \in \text{Aut}(\mathbb{C})\right). \tag{4}$$

Here we define $\tau\left(\frac{\alpha}{m}\right) \in \mathbb{Q} \cap (0, 1)$ by $\tau(\zeta_m) = e^{\pi i \frac{\alpha}{m}}$ with $\zeta_m := e^{2\pi i m}$. In [Ka2] we proved a reciprocity law on a period-ring valued beta function, which is a refinement of (4) in the following sense: For simplicity, we assume that $p \mid m, p \neq 2$ here. We use the following notation.

$$[a : b][c : d] := [ac : bd],$$

$$[a : b] = [c : d] \text{ means } \frac{a}{c} = \frac{b}{d},$$

$$[a : b] \equiv [c : d] \text{ mod } \mu_\infty \text{ means } \frac{ad}{cb} \in \mu_\infty,$$

where $\mu_\infty$ denotes the group of all roots of unity.

- Let $\Gamma_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be a generalization [Ka2, Lemma 4.2] of Morita’s $p$-adic gamma function. When $F = \mathbb{Q}, \gamma = (m), c = [(a)] \in C_m (a \in \mathbb{N}, (a, m) = 1),$ we see that

$$[\exp(X([a]), \text{id})] : \exp_p(X_p([a]), \text{id})] \equiv [\Gamma\left(\frac{a}{m}\right)/\sqrt{2\pi} : \Gamma_p\left(\frac{a}{m}\right)] \mod \mu_\infty.$$

Moreover we may regard this ratio as a refinement of a cyclotomic unit:

$$[\Gamma\left(\frac{a}{m}\right)/\sqrt{2\pi} : \Gamma_p\left(\frac{a}{m}\right)][\Gamma\left(\frac{m-a}{m}\right)/\sqrt{2\pi} : \Gamma_p\left(\frac{m-a}{m}\right)] \equiv \left[1 : 2\sin\left(\frac{\alpha}{m}\pi\right)\right] \mod \mu_\infty.$$

- We consider the $m$th Fermat curve $F_m : x^m + y^m = 1$ and its differential forms $\eta_{a,b,m} := x^{a-1}y^{b-m}dx$ $(0 < a,b < m, a+b \neq m)$. Although the period integral $\int_\gamma \eta_{a,b,m} \text{ depends on the choice of a closed path } \gamma \subset F_m(\mathbb{C}),$ the ratio $[\int_\gamma \eta_{a,b,m} : \int_\gamma \eta_{\frac{a}{m} : \frac{b}{m}}]$ is constant. Here $\int_\gamma \eta_{a,b,m} \in B_{\text{DR}}$ is the $p$-adic period defined by the $p$-adic Hodge theory, $B_{\text{DR}}$ is Fontaine’s $p$-adic period ring. Moreover we have

$$[\int_\gamma \eta_{a,b,m} : \int_\gamma \eta_{\frac{a}{m} : \frac{b}{m}}][\int_\gamma \eta_{m-a,m-b,m} : \int_\gamma \eta_{\frac{m-a}{m} : \frac{m-b}{m}}] = [2\pi i : (2\pi i)_p],$$

where $(2\pi i)_p \in B_{\text{DR}}$ is the $p$-adic counterpart of $2\pi i$.

- Rohrlich’s formula in [Gr] implies that $\int_\gamma \eta_{\frac{a}{m} : \frac{b}{m}} \in \mathbb{Q}$. It follows that

$$\mathfrak{B}(a,m) := \frac{\Gamma\left(\frac{a}{m}\right)\Gamma\left(\frac{b}{m}\right)}{\Gamma\left(\frac{m-a}{m}\right)\Gamma\left(\frac{m-b}{m}\right)} \int_\gamma \eta_{\frac{a}{m} : \frac{b}{m}} \in B_{\text{DR}} \quad (p \nmid ab(a+b))$$

are well-defined up to $\mu_\infty$. 

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• Coleman’s formula in [Co] implies that [Ka2, Theorem 7.2-(ii)]

\[
\Phi_\tau(\mathfrak{B}(\frac{a}{m}, \frac{b}{m})) \equiv p^{\deg \tau} \mathfrak{B}(\tau(\frac{a}{m}), \tau(\frac{b}{m})) \mod \mu_\infty \quad (\tau \in W_p).
\]

Here we denote by \(W_p \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) the Weil group, by \(\Phi_\tau\) the absolute Frobenius automorphism associated with \(\tau \in W_p\), acting on a subring of \(B_{dR}\).

• We can express \(2 \sin(\frac{a}{m} \pi)\) as a product of rational powers of \(\mathfrak{B}(\frac{a}{m}, \frac{b}{m})'\)’s up to \(\mu_\infty\). Hence the reciprocity law (5) implies (4), up to \(\mu_\infty\).

Summarizing the above, when \(F = \mathbb{Q}\) we gave an alternative proof of a part of the Stark conjecture by studying the ratios \([\Gamma\text{-function : } p\text{-adic } \Gamma\text{-function }],[\text{ periods of Fermat curves : } p\text{-adic periods of Fermat curves }].\) The ratio \([\exp(X(c, \iota)) : \exp_p(X_p(c, \iota))]\) is a generalization of the former one. The main result in this paper, which expresses Stark units and Gross-Stark units in terms of \(\exp(X_p(c, \iota))'\)’s, is a significant step toward the generalization of the above results in [Ka2].

**Remark 1.**

(i) For a generalization of the ratios \([\text{ periods of Fermat curves : } p\text{-adic periods of Fermat curves }],[\text{ CM-periods : } p\text{-adic CM periods }]\) under a certain condition.

(ii) Yoshida’s conjecture [Yoshida, Chap. III, Conjecture 3.9] is a generalization of Rohrlich’s formula. There is a slight generalization [Ka3, Conjecture 5.5].

(iii) Yoshida and the author formulated conjectures in [KY1, KY2] which are generalizations of Coleman’ formula.

**Remark 2.** Dasgupta formulated a conjecture [Da, Conjecture 3.21] which is also a refinement of Conjecture [3]. A modified version of \(u'\) of (3) is expressed in terms of a \(p\)-adic measure associated with zeta values. This conjecture is a further refinement of Conjecture [3] by \(\ker \log_p\). More precisely, we can show the following: Let \(u_T(b, D)\) be as in [Da, Conjecture 3.21], \(Y_p(\tau)\) for \(\tau \in \text{Gal}(H/F)\) as in Proposition 6-(ii), and \((\frac{H/F}{\eta})\) the Artin symbol. For simplicity we take \(T := \{\eta\}\) with \(\eta\) a prime ideal of \(F\). Then we can show that (a preprint is in preparation)

\[
\log_p(u_\eta(b, D)) = -Y_p(\frac{(H/F)}{\eta}) + N\eta Y_p(\frac{(H/F)}{\eta \eta^{-1}}).
\]

The outline of this paper is structured as follows. In §2, we introduce some basic properties of Barnes’ multiple zeta functions, multiple gamma functions, and their \(p\)-adic analogues. In §3, we recall a Proposition by Shintani, which is needed in order to relate the derivative values of the classical (resp. \(p\)-adic) partial zeta functions to the classical (resp. \(p\)-adic) multiple gamma functions in Theorem [2] (resp. Theorem [3]) in §4. We note that the classical multiple gamma functions and the \(p\)-adic analogues have not yet been mixed at this point. In §4, we recall the definition and some properties of Yoshida’s class invariant \(X(c, \iota)\). We also provide their \(p\)-adic analogues by quite similar arguments. There are two applications of these properties: First, in §5, we prove the algebraicity of some products of \(\exp_p(X_p(c, \iota))'\)’s, which is the \(p\)-adic analogue of the main
results in [Ka3]. Then we can clarify the relation between Stark units and the ratios \([\exp(X(c, \iota)) : \exp_p(X_p(c, \iota))].\) Next, in §6, we prove that a conjecture [KY1 Conjecture 5.10] on the ratios \([\exp(X(c, \iota)) : \exp_p(X_p(c, \iota))]\) holds true, and also clarify the relation between Gross-Stark units and the same ratios.

## 2 Multiple zeta and gamma functions

In this section, we review some basic properties of Barnes’ multiple zeta and gamma functions, and their \(p\)-adic analogues. For omitted proofs and details, we refer to [Yo, Chap. I, §1], [Ka1]. We denote by \(\mathbb{R}_+\) the set of all positive real numbers, by \(\mathbb{Z}_{\geq 0}\) the set of all non-negative integers.

### Definition 1. (i) Let \(z \in \mathbb{R}_+, \, \mathbf{v} \in \mathbb{R}^r_+.\) Then Barnes’ multiple zeta function is defined as

\[
\zeta(s, \mathbf{v}, z) := \sum_{m \in \mathbb{Z}_{\geq 0}} (z + m^t \mathbf{v})^{-s} \quad (\text{Re}(s) > r),
\]

where \(m^t \mathbf{v}\) denotes the inner product.

(ii) Let \(k\) be a field of characteristic \(0.\) (We may assume that \(k = \mathbb{C}\) or \(\mathbb{C}_p.\) Let \(\mathbf{v} = (v_1, \ldots, v_r) \in (k^\times)^r, \, \mathbf{x} = (x_1, \ldots, x_r) \in k^r.\) Then we define “formal multiple zeta values” for \(m \in \mathbb{Z}_{\geq 0}\) as

\[
\zeta\text{fml}(−m, \mathbf{v}, \mathbf{x}^t \mathbf{v}) := (-1)^r m! \sum_{|\mathbf{l}| = m+r} \prod_{i=1}^{r} \frac{B_{l_i}(x_i)v_i^{l_i-1}}{l_i!} \in k.
\]

Here we denote the \(l_i\)th Bernoulli polynomial by \(B_{l_i}(x_i)\) and \(\mathbf{l}\) in the sum runs over all \(\mathbf{l} = (l_1, \ldots, l_r) \in \mathbb{Z}_{\geq 0}^r\) satisfying \(|\mathbf{l}| := l_1 + \cdots + l_r = m + r.\)

### Proposition 1. The series in the definition of \(\zeta(s, \mathbf{v}, z)\) converges for \(\text{Re}(s) > r,\) has a meromorphic continuation to \(\mathbb{C},\) which is analytic at \(s = 0, -1, -2, \ldots.\) Moreover when \(\mathbf{v} \in \mathbb{R}^r_+, \, \mathbf{x}^t \mathbf{v} \in \mathbb{R}_+, \, m \in \mathbb{Z}_{\geq 0}\) we have

\[
\zeta(−m, \mathbf{v}, \mathbf{x}^t \mathbf{v}) = \zeta\text{fml}(−m, \mathbf{v}, \mathbf{x}^t \mathbf{v}).
\]

### Definition 2. Let \(z \in \mathbb{R}_+, \, \mathbf{v} \in \mathbb{R}^r_+.\) Then we define Barnes’ multiple gamma function by

\[
\Gamma(z, \mathbf{v}) := \exp \left( \frac{\partial}{\partial s} \zeta(s, \mathbf{v}, z)|_{s=0} \right).
\]

Note that this definition is modified from the original one \(\Gamma(z, \mathbf{v}) := \exp \left( \frac{\partial}{\partial s} \zeta(s, \mathbf{v}, z)|_{s=0} \right)\) with a correction term \(\rho(z).\)

We fix embeddings \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \, \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p\) throughout this paper. In particular \(p^r \in \mathbb{C}_p\) is well-defined for each \(r \in \mathbb{Q}.\)
Definition 3. (i) We put

\[ P := \{ p^r \in \mathbb{C}_p \mid r \in \mathbb{Q} \}, \]
\[ 1 + M := \{ z \in \mathbb{C}_p \mid |z - 1|_p < 1 \}, \]
\[ \mu_p := \{ z \in \mathbb{C}_p \mid z^n = 1 \text{ for some } n \in \mathbb{N} \text{ with } p \nmid n \} \]

and consider the following decomposition

\[ \mathbb{C}_p^\times \cong P \times \mu_p \times 1 + M, \]
\[ z \mapsto (p^{\text{ord}_p(z)}, \theta_p(z), \langle z \rangle). \]

More precisely, let

\[ \text{ord}_p : \mathbb{C}_p^\times \to \mathbb{Q}, \quad \theta_p : \mathbb{C}_p^\times \to \mu_p \]

be unique group homomorphisms satisfying

\[ |p^{-\text{ord}_p(z)}\theta_p(z)^{-1}z - 1|_p < 1. \]

Then we put

\[ \langle z \rangle := p^{-\text{ord}_p(z)}\theta_p(z)^{-1}z. \]

(ii) For \( z \in 1 + M, s \in \mathbb{Z}_p \), we put

\[ z^s := \sum_{k=1}^{\infty} \binom{s}{k} (z - 1)^k, \]

where \( \binom{s}{k} \) denotes the binomial coefficient.

(iii) We denote the Iwasawa \( p \)-adic logarithmic function by \( \log_p \), which is defined as

\[ \log_p z := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\langle z \rangle - 1)^k}{k} \quad (z \in \mathbb{C}_p^\times). \]

(iv) Let \( r \in \mathbb{N} \). For a function \( f : \mathbb{Z}_p^r \to \mathbb{C}_p \), we put

\[ I(f) := \lim_{l_1 \to \infty} \ldots \lim_{l_r \to \infty} \frac{1}{p^{l_1 + \ldots + l_r}} \sum_{n_1=0}^{p^{l_1-1}} \ldots \sum_{n_r=0}^{p^{l_r-1}} f(n_1, \ldots, n_r) \]

whenever the limit exists.

(v) Let \( z \in \mathbb{C}_p^\times, \mathbf{v} = (v_1, \ldots, v_r) \in (\mathbb{C}_p^\times)^r \). For simplicity we assume that

\[ \text{ord}_p(z) < \text{ord}_p(v_1), \ldots, \text{ord}_p(v_r). \] (6)

Then we put

\[ f_{z,v,s} : \mathbb{Z}_p^r \to \mathbb{C}_p, \quad \mathbf{x} \mapsto \frac{(z + x^tv)^r}{v_1 \cdots v_r} (z + x^tv)^{-s} \]

and define the \( p \)-adic multiple zeta function by

\[ \zeta_p(s, \mathbf{v}, z) := \frac{(-1)^r I(f_{z,v,s})}{(r-s)(r-1-s) \cdots (1-s)}. \]
Proposition 2.  (i) ζ_p(s, v, z) is analytic at s = 0, and continuous on s ∈ Z_p − \{1, 2, ..., r\}.

(ii) ζ_p(s, v, z) is continuous for v, z.

(iii) Let z ∈ Q, v = (v_1, ..., v_r) ∈ Q^r. We assume that

\[ z, v_1, ..., v_r ∈ R_+ \text{ (via the embedding } Q ↦ C), \]

\[ \text{ord}_p(z) < \text{ord}_p(v_1), ..., \text{ord}_p(v_r) \text{ (via the embedding } Q ↦ C_p). \]

Then ζ_p(s, v, z) satisfies the following p-adic interpolation property:

\[ ζ_p(s, v, z) = p^{-\text{ord}_p(z)} θ_p(z)^{-m} ζ(-m, v, z) \quad (m ∈ Z_{≥0}). \]

Proof. (i), (iii) follow from [Ka1, Lemma 5.3-1, Theorem 5.1] respectively. Let z, z' ∈ C × p, v, v' ∈ (C × p)^r satisfy (6). When z' → z, we may assume that ord_p(z) = ord_p(z'), θ_p(z) = θ_p(z'). Then we see that for m ∈ Z_{≥0}

\[ \sup_{x ∈ Z_p} |f_{z,v,-m}(x) - f_{z',v',-m}(x)|_p → 0 (z' → z, v' → v). \]

Hence by [Ka1, (5.3)] we have

\[ |ζ_p(-m, v, z) - ζ_p(-m, v', z')|_p → 0 (z' → z, v' → v). \]

Since non-positive integers are dense in Z_p, the assertion (ii) is clear. □

Definition 4. Let z ∈ C^× p, v ∈ (C^× p)^r satisfy (6). Then we define the p-adic log multiple gamma function by

\[ LΓ_p(z, v) := \frac{∂}{∂s} ζ_p(s, v, z)|_{s=0}. \]

Proposition 3.  (i) Let z ∈ R_+, v ∈ R_+^r, α ∈ R_+. Then we have

\[ ζ(s, αv, αz) = α^{-s} ζ(s, v, z), \]

\[ Γ(αz, αv) = Γ(z, v)α^{-ζ(0, v, z)}. \]

(ii) Let z ∈ C^× p, v ∈ (C^× p)^r satisfy (6). Let α ∈ C^× p. Then we have

\[ ζ_p(s, αv, az) = (α)^{-s} ζ_p(s, v, z), \]

\[ LΓ_p(αz, αv) = LΓ_p(z, v) - ζ_p(0, v, z) log_α. \]

Proof. Follows immediately from Definitions 1(i), 2-4. □
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Definition 5. Let $F$ be a totally real field of degree $n$.

(i) Let $v_1, \ldots, v_r \in \mathbb{R}^n$ be linearly independent vectors. We define an $(r$-dimensional open simplicial) cone with basis $v_1, \ldots, v_r$ as

$$C(v_1, \ldots, v_r) := \{ t_1v_1 + \cdots + t_rv_r \mid t_i \in \mathbb{R}_+ \} \subset \mathbb{R}^n.$$ 

(ii) We identify $F \otimes \mathbb{R} = \mathbb{R}^n$, $k \sum_{i=1}^r \alpha_i \otimes \beta_i \mapsto \left( \sum_{i=1}^r \iota(\alpha_i)\beta_i \right)_{\iota \in \text{Hom}(F, \mathbb{R})}$. In particular, we regard $F$ as a subset of $F \otimes \mathbb{R} = \mathbb{R}^n$. We say that a cone $C(v_1, \ldots, v_r)$ is a cone of $F$ if $v_1, \ldots, v_r \in \mathcal{O}_F$.

(iii) Considering $F \subset F \otimes \mathbb{R} = \mathbb{R}^n$, we put

$$F \otimes \mathbb{R}_+ := \mathbb{R}^n_+,$$

$$F_+ := F \cap (F \otimes \mathbb{R}_+) = \{ z \in F \mid \iota \in \text{Hom}(F, \mathbb{R}) \Rightarrow \iota(z) > 0 \},$$

$$\mathcal{O}_{F,+} := \mathcal{O}_F \cap F_+,$$

$$E_{F,+} := \mathcal{O}_F^\times \cap F_+.$$ 

Theorem 1 ([Shin1, Proposition 4]). We consider the natural action $E_{F,+} \curvearrowright F \otimes \mathbb{R}_+$, $u(\alpha \otimes \beta) := (u\alpha) \otimes \beta$. Then there exists a fundamental domain $D$ of $F \otimes \mathbb{R}_+/E_{F,+}$ which can be written as a finite disjoint union of cones of $F$. Namely there exist $v_{ij} \in \mathcal{O}_{F,+}$ $(j \in J, 1 \leq i \leq r(j), |J| < \infty, r(j) \in \mathbb{N})$ satisfying

$$F \otimes \mathbb{R}_+ = \bigcoprod_{u \in E_{F,+}} uD, \quad D = \bigcoprod_{j \in J} C(v_{j1}, \ldots, v_{jr(j)}).$$

We call such a $D$ a Shintani domain.

4 The class invariants $X(c, \iota)$, $X_p(c, \iota)$

In this section, we summarize the definitions and some properties of $X(c, \iota)$ and $X_p(c, \iota)$. The class invariant $X(c)$ was introduced and studied by Yoshida [Yo] ($X(c, \iota)$ in this paper is almost equal to $X(\iota(c))$ in [Yo]). Yoshida and the author studied the $p$-adic analogue $X_p(c)$ in [KY1, KY2]. Throughout this section let $F$ be a totally real field of degree $n$, $\mathfrak{f}$ an integral ideal of $F$, $c \in C_\mathfrak{f}$, $\iota \in \text{Hom}(F, \mathbb{R})$, and $D = \bigcoprod_{j \in J} C(v_j)$ a disjoint union of cones with $v_j \in \mathcal{O}_F^{\mathfrak{f}(j)}$. Unless otherwise specified, we do not assume that $D$ is a Shintani domain in the sense of Theorem 1. Let $\pi : C_\mathfrak{f} \rightarrow C(1)$ be the natural projection. We take an integral ideal $\mathfrak{a}_c$ satisfying $\mathfrak{a}_c \mathfrak{f} \in \pi(c)$ for each $c \in C_\mathfrak{f}$.
4.1 The case of a totally positive $D$

In this subsection, we assume that $D = \bigsqcup_{j \in J} C(v_j)$ ($v_j \in \mathcal{O}_F^{r(j)}$) is totally positive, that is

$$D \subset F \otimes \mathbb{R}^+,$$

or equivalently, $v_j \in \mathcal{O}_{F,+}^{r(j)}$.

We introduce Yoshida’s invariants $G, W, V, X$ in [Yo] Chap. III, (3.6)–(3.9), (3.28)–(3.31)], which is slightly modified in [KY1] (4.3), [Ka3] §2. The equalities in Definitions 7, 9 follow from [Yo, Chap. II, Lemma 3.2].

**Definition 6.** Let $v = (v_1, \ldots, v_r) \in \mathcal{O}_{F,+}^r$ be linearly independent. Then we put

$$R(c, v) := R(c, v, a_c) := \{ x \in ((0, 1] \cap \mathbb{Q})^r \mid \mathcal{O}_F \supset (x^t v) a_c f \in c \}.$$

**Definition 7.** We put

$$G(c, \iota, D, a_c) := \frac{d}{ds} \sum_{z \in (a_c f)^{-1} \cap D, (z)a_c f \in c} \iota(z)^{-s} = \sum_{j \in J} \sum_{x \in R(c, v_j)} \log \Gamma(\iota(x^t v_j), \iota(v_j)).$$

**Definition 8.** We define group homomorphisms $\log_\iota : I_F \to \mathbb{R}$ for $\iota \in \text{Hom}(F, \mathbb{R})$ as follows: For each prime ideal $p$ of $F$, we choose a generator $\pi_p \in \mathcal{O}_{F,+}$ of the principal ideal $p^{h_{F,+}}$, where $h_{F,+} = |C(1)|$ is the narrow class number. Then we put

$$\log_\iota p := \frac{1}{h_{F,+}} \log \iota(\pi_p).$$

We extend this linearly to $\log_\iota : I_F \to \mathbb{R}$.

We consider a principal ideal $(\alpha) = \alpha \mathcal{O}_F$ with $\alpha \in F^\times$. The difference between $\log \iota(\alpha)$ and $\log_\iota(\alpha)$ is as follows: By definition there exists a generator $\alpha' \in (\alpha)^{h_{F,+}}$ satisfying $\log_\iota(\alpha) = \frac{1}{h_{F,+}} \log \iota(\alpha')$. Then we see that

$$\log \iota(\alpha) - \log_\iota(\alpha) = \frac{1}{h_{F,+}} \log \iota(u_\alpha) \quad (u_\alpha := \frac{\alpha^{h_{F,+}}}{\alpha'} \in E_{F,+}). \quad (7)$$

**Definition 9.** We put

$$W(c, \iota, D, a_c) := -\left[ \sum_{z \in (a_c f)^{-1} \cap D, (z)a_c f \in c} \iota(z)^{-s} \right]_{s=0} \log_\iota a_c f$$

$$= -\left( \sum_{j \in J} \sum_{x \in R(c, v_j)} \zeta(0, \iota(v_j), \iota(x^t v_j)) \right) \log_\iota a_c f.$$

**Definition 10.** For $\iota, \iota' \in \text{Hom}(F, \mathbb{R})$ with $\iota \neq \iota'$, we put

$$v_{\iota, \iota'} := \left[ \frac{d}{ds} \sum_{z \in (a_c f)^{-1} \cap D, (z)a_c f \in c} (\iota(z)\iota'(z))^{-s} - \iota(z)^{-s} - \iota'(z)^{-s} \right]_{s=0}. $$
Then we define
\[ V(c, \iota, D, a_c) := \frac{2}{n} \sum_{\iota' \neq \iota} v_{\iota, \iota'} - \frac{2}{n^2} \sum_{\iota' \neq \iota''} v_{\iota, \iota', \iota''}. \]

Here \( \iota' \) runs over all \( \iota' \in \text{Hom}(F, \mathbb{R}) \) with \( \iota' \neq \iota \) in the first sum, \( \iota', \iota'' \) run over all \( \iota', \iota'' \in \text{Hom}(F, \mathbb{R}) \) with \( \iota' \neq \iota'' \) in the second sum.

**Definition 11.** We put
\[ X(c, \iota, D, a_c) := G(c, \iota, D, a_c) + W(c, \iota, D, a_c) + V(c, \iota, D, a_c). \]

If \( D \) is a Shintani domain in the sense of Theorem 7 and if we fix \( D, a_c \), then we put
\[ X(c, \iota) := X(c, \iota, D, a_c), \quad G(c, \iota) := G(c, \iota, D, a_c). \]

The following Theorem is essentially due to Yoshida: Yoshida transformed Shintani’s formula \([\text{Shin}2]\) into a similar form. Yoshida and the author modified the \( W \)-term slightly in \([\text{KY}1], [\text{Ka}3]\) into the present form.

**Theorem 2 ([Yo, Chap. III, (3.11)], [Ka3, Theorem 2.5]).** Let \( D \) be a Shintani domain. Then we have
\[ \zeta'(0, c) = \sum_{\iota \in \text{Hom}(F, \mathbb{R})} X(c, \iota). \]

The following Lemma also is essentially due to Yoshida, although \( \exp(X(\iota(c))) \) in \([Yo]\) was well-defined up to \( \iota(F_+)^{\mathbb{Q}} \).

**Lemma 1 ([Yo, Chap. III, §3.6, 3.7], [Ka3, Lemma 3.11]).** We consider the following operations on \( D, a_c. \)

(I) Let \( j_0 \in J \). We decompose \( C(v_{j_0}) = \bigsqcup_{l \in L} C(v_l) \) and replace
\[ D = \bigsqcup_{j \in J} C(v_j) \Rightarrow D = \left( \bigsqcup_{j \in J - \{j_0\}} C(v_j) \right) \bigsqcup \left( \bigsqcup_{l \in L} C(v_l) \right). \]

Note that a replacement of basis (\(|L| = 1\)) is included in this case.

(II) Let \( j_0 \in J, \epsilon \in E_{F_+}. \) We replace \( C(v_{j_0}) \) by \( C(\epsilon v_{j_0}) \), that is
\[ D = \bigsqcup_{j \in J} C(v_j) \Rightarrow D = \left( \bigsqcup_{j \in J - \{j_0\}} C(v_j) \right) \bigsqcup C(\epsilon v_{j_0}). \]

(III) Let \( \alpha \in a_c^{-1} \cap F_+ \). We replace
\[ a_c \Rightarrow \alpha a_c, \]
\[ C(v_j) \Rightarrow \alpha^{-1} C(v_j) = C(v_j') \]
simultaneously. Here we take \( v_j' \) satisfying \( \alpha^{-1} C(v_j) = C(v_j') \), \( v_j' \in \mathcal{O}^{(j)}_{F_+} \) for each \( j \in J \).
In order to simplify the following expressions, we put

\[ Z_j := \sum_{x \in R(c,v_j)} \zeta_{\text{fml}}(0, v_j, x^t v_j) \in F \quad (j \in J). \]

In particular

\[ \iota(Z_j) = \sum_{x \in R(c,v_j)} \zeta(0, \iota(v_j), \iota(x^t v_j)) \in \iota(F), \]

\[ \text{Tr}_{F/Q} Z_j = \sum_{x \in R(c,v_j)} \sum_{\iota \in \text{Hom}(F, \mathbb{R})} \zeta(0, \iota(v_j), \iota(x^t v_j)) \in \mathbb{Q} \]

have meanings.

(i) \( G(c, \iota, D, a_c) \) changes as follows by the operations (I), (II), (III).

(I) Stays constant.

(II) \( G(c, \iota, D, a_c) - \iota(Z_j) \log \iota(\epsilon) \).

(III) \( G(c, \iota, D, a_c) + \sum_{j \in J} \iota(Z_j) \log \iota(\alpha) \).

(ii) \( W(c, \iota, D, a_c) \) changes as follows by the operations (I), (II), (III).

(I) Stays constant.

(II) Stays constant.

(III) \( W(c, \iota, D, a_c) - \sum_{j \in J} \iota(Z_j) \log_\alpha(\alpha) \).

(iii) \( V(c, \iota, D, a_c) \) changes as follows by the operations (I), (II), (III).

(I) Stays constant.

(II) \( V(c, \iota, D, a_c) + \left( \iota(Z_j) - \frac{\text{Tr}_{F/Q}(Z_j)}{n} \right) \log \iota(\epsilon) \).

(III) \( V(c, \iota, D, a_c) + \sum_{j \in J} \left( \iota(Z_j) - \frac{\text{Tr}_{F/Q}(Z_j)}{n} \right) \left( \frac{1}{n} \log(N_{F/Q}(\alpha)) - \log \iota(\alpha) \right) \).

(iv) Additionally assume that \( D \) is a Shintani domain. Then \( X(c, \iota, D, a_c) \) changes as follows by the operations (I), (II), (III).

(I) Stays constant.

(II) \( X(c, \iota, D, a_c) - \frac{\text{Tr}_{F/Q}(Z_j)}{n} \log \iota(\epsilon) \).

(III) \( X(c, \iota, D, a_c) - \frac{\zeta(0, c)}{h_{F,+}} \log \iota(u_\alpha) \). Here we take \( u_\alpha \in E_{F,+} \) is as in (7).
In particular, \(\exp(X(c, \iota)) \mod \iota(E_{F,+})^Q\) does not depend on \(D, a_c\), where we put \(\iota(E_{F,+})^Q := \{\iota(u)^{\frac{1}{N}} \mid u \in E_{F,+}, N \in \mathbb{N}\}. More precisely, for Shintani domains \(D, D'\), integral ideals \(a_c, a'_c\) satisfying \(a_c \mathcal{f}, a'_c \mathcal{f} \in \pi(c)\), there exist \(N \in \mathbb{N}, u_{c,D,D',a_c,a'_c} \in E_{F,+}\) satisfying

\[
X(c, \iota, D', a'_c) - X(c, \iota, D, a_c) = \frac{1}{N} \log \iota(u_{c,D,D',a_c,a'_c}).
\]

**Proof.** For comparison to the \(p\)-adic case (Lemma 4), we give a brief sketch of the proof of (i)-(II). Let \(D' := (\coprod_{j \in J-\{j_0\}} C(\nu_j)) \coprod C(\epsilon \nu_{j_0})\). By definition we have

\[
G(c, \iota, D', a_c) - G(c, \iota, D, a_c) = \left[\frac{d}{ds} (\iota(s)^{-1} - 1) \right] \sum_{z \in (a_c \mathcal{f})^{-1} \cap C(\epsilon \nu_{j_0}), (z) a_c f \in c} \iota(z)^{-s} ds
\]

By noting that

\[
\{z \in (a_c \mathcal{f})^{-1} \cap C(\epsilon \nu_{j_0}) \mid (z) a_c f \in c\} = \epsilon \{z \in (a_c \mathcal{f})^{-1} \cap C(\epsilon \nu_{j_0}) \mid (z) a_c f \in c\}, \tag{8}
\]

we can rewrite

\[
G(c, \iota, D', a_c) - G(c, \iota, D, a_c) = -\iota(\mathcal{Z}_{j_0}) \log \iota(\epsilon)
\]

as desired. The other cases are similar. \(\square\)

### 4.2 The case of a non-totally positive \(D\)

In this subsection, we fix an embedding \(\iota_0 \in \text{Hom}(F, \mathbb{R})\). We put

\[
F \otimes \mathbb{R}_{(n-1)+} := \left\{ \sum_{i=1}^{k} \alpha_i \otimes \beta_i \in F \otimes \mathbb{R} \mid \text{Hom}(F, \mathbb{R}) \ni \iota \neq \iota_0 \Rightarrow \sum_{i=1}^{k} \iota(\alpha_i)\beta_i \in \mathbb{R}_+ \right\}.
\]

For each subset \(A \subset F\), we put

\[
A_{(n-1)+} := A \cap (F \otimes \mathbb{R}_{(n-1)+}) = \{a \in A \mid \text{Hom}(F, \mathbb{R}) \ni \iota \neq \iota_0 \Rightarrow \iota(a) \in \mathbb{R}_+\}.
\]

We introduce generalizations of Yoshida’s invariants in \([K3]\).

**Definition 12.** For each \(\iota \in \text{Hom}(F, \mathbb{R})\), we take \(\nu_\iota \in \mathcal{O}_F\) satisfying

\[
\nu_\iota \equiv 1 \mod \mathfrak{f}, \ i(\mu_\iota) < 0, \ i'(\nu_\iota) > 0 \ (\iota \neq \iota' \in \text{Hom}(F, \mathbb{R}))
\]

and put

\[
c_\iota := [(\nu_\iota)] \in C_\mathfrak{f}.
\]
Remark 3. Let $H_i$ be the maximal ray class field modulo $\mathfrak{f}$ in the narrow sense, \text{Art}: C_i \to \text{Gal}(H_i/F)$ the Artin map. Then \text{Art}(c_i)$ is the complex conjugation at $i$ \cite[Chap. III, the first paragraph of §5.1]{Yao}. Hence the fixed subfield $H_i^{\text{Art}(c_i)}$ is the maximal subfield where the real place $i$ splits completely.

Definition 13. Let $v = (v_1, \ldots, v_r) \in \mathcal{O}_F^{(n-1)+}$ be linearly independent. Then we put

\[ R(c \cup cc_a, v) := R(c \cup cc_a, v, a_c) := \{ x \in ((0, 1] \cap \mathbb{Q})^r \mid \mathcal{O}_F \supset (x^tv)a_c \in c \cup cc_a \}. \]

Definition 14. Let $i \in \text{Hom}(F, \mathbb{R}), \neq i_0$. By replacing $c$ with $c \cup cc_a$ in Definition \[7\], we put

\[ G(c \cup cc_a, i, D, a_c) := \left[ \frac{d}{ds} \sum_{z \in (a_c)^{-1} \cap D, (z) a_c \in c \cup cc_a} i(z)^{-s} \right]_{s=0} = \sum_{j \in J} \sum_{x \in R(c \cup cc_a, v_j)} \log \Gamma(i(x^tv), i(v_j)). \]

Remark 4. When $C(v) \subset F \otimes \mathbb{R}^{(n-1)+}$, we consider the pair $c, cc_a$ for the following reason: Let $z \in (a_c)^{-1} \cap C(v)$ satisfy $(z) a_c \in c$. We write $v = (v_1, \ldots, v_r)$ and $z = x^tv$ with $x \in \mathbb{Q}^*_0$. If $C(v) \subset F \otimes \mathbb{R}^{(n-1)+}$, then we see that

\[ (z + v_i)a_c \in c, \]
\[ (z - v_i)a_c \in c (x_i > 1), \]

by noting that $(z \pm v_i)a_c = (1 \pm v_i/z)(z)a_c$. It follows that \( \sum_{z \in (a_c)^{-1} \cap C(v), (z) a_c \in c} i(z)^{-s} = \sum_{x \in R(c, v)} \zeta(s, i(v), i(x^tv)). \) If $C(v) \subset F \otimes \mathbb{R}^{(n-1)+}$, then it follows only that $(z \pm v_i)a_c \in c \cup cc_a$.

Definition 15. For each $i \in \text{Hom}(F, \mathbb{R})$ (including $i = i_0$), we put

\[ W(c \cup cc_a, i, D, a_c) := -\left( \sum_{j \in J} \sum_{x \in R(c \cup cc_a, v_j)} \zeta_{\text{finl}}(0, i(v_j), i(x^tv_j)) \right) \log_i a_c. \]

We note that $\zeta_{\text{finl}}(0, i(v_j), i(x^tv_j)) = \zeta(0, i(v_j), i(x^tv_j))$ if all entries of $i(v_j)$ are positive.

Definition 16. (i) Let $v = (v_1, \ldots, v_r) \in \mathcal{O}_F^r$ be linearly independent. Let $x = (x_1, \ldots, x_r) \in (\mathbb{Q} \cap (0, 1])^r$, $i \in \text{Hom}(F, \mathbb{R})$. We put

\[ V(v, x, i) := \frac{-1}{n} \sum \sum_{p=1}^r \zeta_{\text{finl}}(-1, \frac{i(v_p)}{i(v_{p'})}, \frac{i'(v_p)}{i'(v_{p'})}) \log \frac{i(v_p)}{i'(v_p)} \]
\[ + \frac{1}{2n^2} \sum \sum_{p=1}^r \zeta_{\text{finl}}(-1, \frac{i'(v_p)}{i'(v_{p'})}, \frac{i''(v_p)}{i''(v_{p'})}) \log \frac{i'(v_p)}{i''(v_p)}. \]

Here for an $r$-dimensional vector $(a_q) = (a_1, \ldots, a_r)$, we denote by $(a_q)_{q \neq p}$ the $(r-1)$-dimensional vector $(a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_r)$. 

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(ii) For each $\iota \in \text{Hom}(F, \mathbb{R})$ (including $\iota = \iota_0$), we put

$$V(c \cup cc_{i_0}, \iota, D, a_c) := \sum_{j \in J} \sum_{x \in R(c \cup cc_{i_0}, v_j)} V(v_j, x, \iota).$$

**Definition 17.** Let $\iota \in \text{Hom}(F, \mathbb{R})$, $\neq \iota_0$. We put

$$X(c \cup cc_{i_0}, \iota, D, a_c) := G(c \cup cc_{i_0}, \iota, D, a_c) + W(c \cup cc_{i_0}, \iota, D, a_c) + V(c \cup cc_{i_0}, \iota, D, a_c).$$

**Lemma 2 ([Ka3 Lemmas 3.6, 3.7, 3.10]).** Let $\iota \in \text{Hom}(F, \mathbb{R})$, $\neq \iota_0$. Then the same statements as in Lemma [4] hold, by replacing $c$ with $c \cup cc_{i_0}$. In particular we replace $Z_j$ with

$$Z_j = \sum_{x \in R(c \cup cc_{i_0}, v_j)} \zeta_{\text{fml}}(0, v_j, x^t v_j).$$

For (iv)-(III), we replace $\zeta(0, c)$ with

$$\zeta(0, c \cup cc_{i_0}) := \begin{cases} \zeta(0, c) & (c_{i_0} = [(1)] \text{ or } \mathcal{O}_{F, (n-1)+} = \emptyset), \\ \zeta(0, c) + \zeta(0, cc_{i_0}) & \text{(otherwise)}. \end{cases}$$

**Lemma 3 ([Ka3 Lemma 3.12]).** We assume that $D$ is a Shintani domain, and that $\iota \neq \iota_0$. Then the following assertions hold.

(i) We have

$$X(c \cup cc_{i_0}, \iota, D, (\nu_{i_0}) a_c) - X(c \cup cc_{i_0}, \iota, D, a_c) = \frac{\zeta(0, c)}{h_{F,+}} \log \iota(u_{\nu_{i_0}}).$$

Here, in the symbol $X(c \cup cc_{i_0}, \iota, D, (\nu_{i_0}) a_c)$, the roles of $c, c_{i_0}$ are exchanged. We take $u_{\nu_{i_0}} \in E_{F,+}$ as in [7].

(ii) If $c_{i_0} = [(1)]$ or if $\mathcal{O}_{F, (n-1)+} = \emptyset$, then we have

$$X(c \cup cc_{i_0}, \iota, D, a_c) = X(c, \iota, D, a_c).$$

Otherwise, we take an element $\epsilon \in \mathcal{O}_{F, (n-1)+}$. Then we have

$$X(c \cup cc_{i_0}, \iota, D, a_c) = X(c, \iota, D, a_c) + \frac{\zeta(0, cc_{i_0})}{h_{F,+}} \log \iota(u_{\nu_{i_0}} \epsilon^{-1}).$$

Here we take $u_{\nu_{i_0}} \epsilon^{-1} \in E_{F,+}$ as in [7].
4.3 \( p \)-adic analogues

We introduce the \( p \)-adic analogues of Yoshida’s invariants. The case when \( D \subset F \otimes \mathbb{R}_+ \)
was studied in \cite{KY1, KY2}. In the \( p \)-adic case, the Archimedean topology induced by
\( \iota \in \text{Hom}(F, \mathbb{R}) \) is not so important. Instead, we consider the prime ideal corresponding
to the \( p \)-adic topology.

**Definition 18.** Recall that we fixed embeddings \( \mathbb{Q} \hookrightarrow \mathbb{C}, \mathbb{Q} \hookrightarrow \mathbb{C}_p \). We identify
\( \text{Hom}(F, \mathbb{R}) = \text{Hom}(F, \mathbb{C}_p) \)
and define for \( \iota \in \text{Hom}(F, \mathbb{R}) \) (= \( \text{Hom}(F, \mathbb{C}_p) \))
\[ p_\iota := \{ z \in \mathcal{O}_F | |\iota(z)|_p < 1 \}. \]

In other words, \( \iota(p_\iota) \) corresponds to the \( p \)-adic topology on \( \iota(F) \subset \mathbb{C}_p \).

We fix \( \iota_0 \in \text{Hom}(F, \mathbb{R}) \) and define \( F \otimes \mathbb{R}_{(n-1)+} \subset F \otimes \mathbb{R} \) as in the previous subsection.
We note that \( \iota \) may be equal to \( \iota_0 \). Throughout this subsection, we assume that
\[ p_\iota \mid \mathfrak{f}. \]

Under this assumption \( (\iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \) with \( \mathbf{x} \in R(c, \mathbf{v}_j) \) satisfy \( \Box \). Namely, the \( p \)-adic interpolation function \( \zeta_p(s, \iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \) of \( \zeta(s, \iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \) is well-defined. Hence we
may consider the \( p \)-adic interpolation of \( \sum_{z \in (a_f)^{-1} \cap D, (z)_{a_f} \in c} \iota(z)^{-s} \) in Definition 7. In the
setting of Definition 14, \( \zeta_p(s, \iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \) for \( \mathbf{x} \in R(c \cup cc_{i_0}, \mathbf{v}_j) \) are also well-defined
whenever \( p_\iota \mid \mathfrak{f} \), although \( \zeta(s, \iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \) with \( \iota = \iota_0 \) may not be well-defined.

**Definition 19.** When \( D \subset F \otimes \mathbb{R}_{(n-1)+} \), we put
\[ G_p(c \cup cc_{i_0}, \iota, D, a_c) := \sum_{j \in J} \sum_{\mathbf{x} \in R(c \cup cc_{i_0}, \mathbf{v}_j)} L \Gamma_p(\iota(\mathbf{x}^t\mathbf{v}_j), \iota(\mathbf{v}_j)). \]

When \( D \subset F \otimes \mathbb{R}_+ \), we also define \( G_p(c, \iota, D, a_c) \) by replacing \( R(c \cup cc_{i_0}, \mathbf{v}_j) \) with \( R(c, \mathbf{v}_j) \).

**Definition 20.** Let \( h_{F,+}, \pi_\mathfrak{p} \) be as in Definition 8 and we put
\[ \log_{\iota, \mathfrak{p}} := \frac{1}{h_{F,+}} \log_{\mathfrak{p}} \iota(\pi_\mathfrak{p}) \]
for prime ideals \( \mathfrak{p} \) of \( F \) and for \( \iota \in \text{Hom}(F, \mathbb{R}) \). We extend this linearly to \( \log_{\iota, \mathfrak{p}} : I_F \to \mathbb{C}_p \).

When \( D \subset F \otimes \mathbb{R}_{(n-1)+} \), we put
\[ W_p(c \cup cc_{i_0}, \iota, D, a_c) := - \left( \sum_{j \in J} \sum_{\mathbf{x} \in R(c \cup cc_{i_0}, \mathbf{v}_j)} \zeta_{\text{finl}}(0, \iota(\mathbf{v}_j), \iota(\mathbf{x}^t\mathbf{v}_j)) \right) \log_{\iota, \mathfrak{p}} a_c \mathfrak{f}. \]

When \( D \subset F \otimes \mathbb{R}_+ \), we also define \( W_p(c, \iota, D, a_c) \) similarly.
Definition 21. We define $V_p(c, \mathfrak{c}, x, \iota, d, a_c)$ by replacing $\log | \cdot |$ in Definition 16 with $\log_p(\cdot \cdot \cdot)$. When $D \subset F \otimes \mathbb{R}_{(n-1)+}$, we put

$$V_p(c \cup cc_{i_0}, \iota, D, a_c) := \sum_{j \in J} \sum_{x \in R(c \cup cc_{i_0}, v_j)} V_p(v_j, x, \iota).$$

When $D \subset F \otimes \mathbb{R}_{+}$, we also define $V_p(c, \iota, D, a_c)$ similarly.

Definition 22. When $D \subset F \otimes \mathbb{R}_{(n-1)+}$, we put

$$X_p(c \cup cc_{i_0}, \iota, D, a_c) := G_p(c \cup cc_{i_0}, \iota, D, a_c) + W_p(c \cup cc_{i_0}, \iota, D, a_c) + V_p(c \cup cc_{i_0}, \iota, D, a_c).$$

When $D \subset F \otimes \mathbb{R}_+$, we also define $X_p(c, \iota, D, a_c)$ similarly. If $D$ is a Shintani domain, we put

$$X_p(c, \iota) := X_p(c, \iota, D, a_c), \quad G_p(c, \iota, D, a_c).$$

Proposition 4. The definitions of $G_p, W_p, V_p, X_p$ does not depend on the choice of the cone decomposition $D = \bigsqcup_{j \in J} C(v_j)$ when we fix $D$.

Proof. We use a similar argument to the proof of [Ka3, Lemma 3.6]. We can reduce the problem to a refinement of a cone: Let $D = C(v)$ with $\mathfrak{v} = (v_1, \ldots, v_r) \in \mathcal{O}_c$. We abbreviate $(\mathfrak{v})$ as $v$. It suffices to show that $G_p, W_p, V_p$ does not change under the following operations.

1. Change the order of the basis $v$.
2. Replace $v_1$ by $nv_1$ with $n \in \mathbb{N}$.
3. Decompose $C(v)$ into $C(v) \bigsqcup C(v^a) \bigsqcup C(v^\alpha)$ with

$$v^a := (v_1, v_1 + v_2, v_3, \ldots, v_r), \quad v^\alpha := (v_1 + v_2, v_2, v_3, \ldots, v_r), \quad v^\beta := (v_1 + v_2, v_3, \ldots, v_r).$$

The case of $V_p$ is completely the same as [Ka3, Lemma 3.6]. The remaining cases follows from that

$$\zeta_p(s, (v_1, \ldots, v_r), z)$$

does not depend on the order of the basis $v_1, \ldots, v_r$,

$$\zeta_p(s, (v_1, v_2, \ldots, v_r), z) = \sum_{k=0}^{n-1} \zeta_p(s, (nv_1, v_2, \ldots, v_r), z + kv_1),$$

$$\zeta_p(s, v, x^t v) = \begin{cases} \zeta_p(s, v^a, x^t v^a) + \zeta_p(s, v^\alpha, x^t v^\alpha) & (x_1 \neq x_2), \\ \zeta_p(s, v^\beta, x^t v^\beta) + \zeta_p(s, v^\beta, x^t v^\beta) + \zeta_p(s, v^\beta, x^t v^\beta) & (x_1 = x_2). \end{cases}$$

for $x = (x_1, \ldots, x_r) \in R(c, v)$ or $x \in R(c \cup cc_{i_0}, v)$. Here we put

$$x^a := \begin{cases} (x_1 - x_2 + 1, x_2, x_3, \ldots, x_r) & \text{if } x_1 < x_2, \\ (x_1, x_2, x_3, \ldots, x_r) & \text{if } x_1 > x_2, \\ (1, x, x_2, \ldots, x_r) & \text{if } x_1 = x_2 = x, \end{cases}$$

$$x^\alpha := \begin{cases} (x_1, x_2 - x_1, x_3, \ldots, x_r) & \text{if } x_1 < x_2, \\ (x_1, x_2 - x_1 + 1, x_3, \ldots, x_r) & \text{if } x_1 > x_2, \\ (x_1, x_3, \ldots, x_r) & \text{if } x_1 = x_2 = x, \end{cases}$$

$$x^\beta := (x, x_3, x_4, \ldots, x_r)$$

if $x_1 = x_2 = x$. 

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When all \( v_i \) are positive, these equations follow from the \( p \)-adic interpolation property, since the same equations obviously hold for \( \zeta(s, v, z) \). Furthermore \( \zeta_p(s, v, z) \) is continuous in the sense of Proposition 2-(ii), so we can generalize equations to all \( v \).

We proved the following \( p \)-adic analogue of Theorem 2 although we will not use it in this paper.

**Theorem 3** ([Ka1, Theorem 6.2], [KY1, Theorem 3.1]). Assume that \( p \) divides \( f \) for any prime ideal \( p \) lying above \( p \). If \( p = 2 \), we further assume that \( 2p \) divides \( f \) for any \( p \) lying above \( 2 \). When \( D \) is a Shintani domain, we have

\[
\zeta'_p(0, c) = \sum_{\iota \in \text{Hom}(F, \mathbb{R})} X_p(c, \iota).
\]

Here \( \zeta_p(s, c) \) is the \( p \)-adic interpolation function of \( \zeta(s, c) \).

**Lemma 4.** Let \( \iota \in \text{Hom}(F, \mathbb{R}) \) satisfy \( p, | f \). Then the same statements as in Lemmas 3, 3 hold, by replacing \( X, G, W, V, \log, \log_{\iota} \) with \( X_p, G_p, W_p, V_p, \log_{\iota p}, \log_{\iota p}, \) not excepting \( \iota = \iota_0 \). We also replace \( \zeta((0, \iota(v_j), \iota(x^jv_j)), \zeta_{\text{fund}}(0, \iota(v_j), \iota(x^jv_j))) \) when \( \iota(v_j) \) has a negative entry.

**Proof.** The same proof as those of [Yo, Chap. III, §3.6, 3.7], [Ka3, Lemmas 3.10, 3.11, 3.12] works, by using Propositions 3-(ii), 4. For example, consider the \( p \)-adic analogue of Lemma 1-(i)-(II): Let \( D = \bigcup_{j \in J} C(v_j), D' = (\bigcup_{j \notin \{j_0\}} C(v_j)) \bigcup C(\iota v_{j_0}) \). Then we have

\[
G_p(\iota, c, D', a_c) - G_p(\iota, c, D, a_c) = \sum_{x \in R(c, \iota \epsilon v_{j_0})} L\Gamma_p(\iota(x^jv_{j_0}), \iota(x^jv_{j_0 })) - \sum_{x \in R(c, \iota v_{j_0})} L\Gamma_p(\iota(x^jv_{j_0}), \iota(x^jv_{j_0})).
\]

We see that \( R(c, \iota v_{j_0}) = R(c, \iota v_{j_0}) \) by (8). Hence Proposition 3-(ii) states that

\[
G_p(\iota, c, D', a_c) - G_p(\iota, c, D, a_c) = -\iota (\epsilon v_{j_0}) \log_p \iota(\epsilon)
\]

as desired. The other cases can be proved similarly.

5  The case when a real place splits completely

Theorem 4 is one of the main results in this paper. The Archimedean part (9) was proved in [Ka3]. The \( p \)-adic part (10) is a new result, although it is proved by quite similar arguments. Let \( c_i = [(v_i)] \in C_f \) be as in Lemma 12.

**Theorem 4.** Let \( c \in C_f, \ i_0 \in \text{Hom}(F, \mathbb{R}) \). Then there exist \( u \in E_{F,+}, m \in \mathbb{N} \) satisfying

\[
\exp(\text{X}(c, \iota)) \exp(\text{X}(cc_{i0}, \iota)) = \iota(u)^{\frac{1}{m}} \quad (\iota \in \text{Hom}(F, \mathbb{R}), \ i \neq i_0),
\]

\[
\text{X}_p(c, \iota) + \text{X}_p(cc_{i0}, \iota) = \frac{1}{m} \log_p \iota(u) \quad (\iota \in \text{Hom}(F, \mathbb{R}), \ p, | f).
\]
Proof. [9] was proved in [Ka3, Proof of Theorem 3.1] by using Lemmas 1, 2, 3, and [Ka3, Lemma 3.4]. We can repeat the same argument for the $p$-adic analogue (10) by Lemma 4. Let $D, \nu, X_t, \epsilon_t \ (t \in T, |T| < \infty)$ be as in [Ka3, Lemma 3.4]. Then we have

- $D$ is a Shintani domain, $\nu \in F_{(n-1)+}, \epsilon_t \in E_{F,+}$.
- Each $X_t$ is a subset of $F \otimes \mathbb{R}_{(n-1)+}$ which can be expressed as a finite disjoint union of cones of $F$: $X_t = \bigsqcup_{j \in \mathcal{J}_t} C(v_j)$.
- We denoted by $\bigcup$ the multiset sum. Then we have

$$
(D \bigsqcup \nu D) \bigcup \left( \bigcup_{t \in T} \epsilon_t X_t \right) = \bigcup_{t \in T} X_t.
$$

Then, by Lemma 2(iv)-(II) or its $p$-adic analogue in Lemma 4, we have for $* = \emptyset$ or $p$

$$X_\ast(c \cup c_{c_0}, \nu, D, a_c) \cup X_\ast(c \cup c_{c_0}, \nu D, a_c) = X_\ast(c \cup c_{c_0}, \nu D, a_c) = \sum_{t \in T} \frac{\text{Tr}_{F/Q}(Z_t)}{n} \log_* \epsilon_t(t) ,
$$

where we put

$$Z_t := \sum_{j \in J_t} \sum_{x \in \mathbb{R}(c_{c_{c_0}}, v_j)} \zeta_{\text{ml}}(0,v_j, x^t v_j).
$$

First we assume that $c_{c_0} = [(1)]$ or $\mathcal{O}_{F,(n-1)+} = \emptyset$. Then we have

$$X_\ast(c \cup c_{c_0}, \nu D, a_c) = X_\ast(c, \nu D, a_c),
$$

$$X_\ast(c \cup c_{c_0}, \nu D, a_c) = X_\ast(c \cup c_{c_0}, \nu D, a_c) - \frac{\zeta(0,c_{c_0})}{h_{F,+}} \log_* \nu_{v_0}.
$$

by Lemma 3(ii), (i), (ii), Lemma 1(iv), and their $p$-adic analogues respectively. Summarizing the above, we can write

$$X_\ast(c, \nu D, a_c) + X_\ast(c_{c_0}, \nu D, a_{c_{c_0}}) = \sum_{t \in T} \frac{\text{Tr}_{F/Q}(Z_t)}{n} \log_* \epsilon_t(t).
$$
Next, we assume that \( c_{i_0} \neq [(1)] \) and \( \mathcal{O}_{F,(n-1)+}^{\infty} \neq \emptyset \). We take an element \( \epsilon \in \mathcal{O}_{F,(n-1)+}^{\infty} \). Then we have

\[
X_*(c \cup cc_{i_0}, \nu, D, a_c) = X_*(c, \nu, D, a_c) + X_*(cc_{i_0}, \nu, D, (\nu_{i_0})a_c) + \frac{\zeta(0,cc_{i_0})}{h_{F,+}^{\nu}} \log^* \epsilon(u_{\nu_{i_0}}^{\nu-1}) \]

by Lemma 3(ii), (i), (ii), Lemma 4(iv) (3 times), and their \( p \)-adic analogues respectively. Summarizing the above, we can write

\[
X_*(c, \nu, D, a_c) + X_*(cc_{i_0}, \nu, D, a_{cc_{i_0}}) \]

\[
= X_*(c, \nu, D, a_c) + X_*(cc_{i_0}, \nu, D, a_{cc_{i_0}}) + \frac{\zeta(0,cc_{i_0})}{h_{F,+}^{\nu}} \log^* \epsilon(u_{\nu_{i_0}}^{\nu-1}) \]

Thus the assertion holds in both cases.

**Corollary 1.** Let \( p, \iota \) satisfy \( p, \iota \mid \mathfrak{f} \). Let \( \exp_p: \mathbb{C}_p \to \mathbb{C}_p^\times \) be any group homomorphism which coincides with the usual power series \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) on a neighborhood of 0. We denote by \( \mu_{\infty} \) the group of all roots of unity.

(i) The ratio \( [\exp(X(c, \iota)) : \exp_p(X(c, \iota))] \) mod \( \mu_{\infty} \) depends only on \( c, \iota \). Strictly speaking, let \( D, D' \) be Shintani domains, and \( a_c, a'_c \) integral ideals satisfying \( a_c \mid \mathfrak{f}, a'_c \in \pi(c) \).

Then we have

\[
\frac{\exp(X(c, \iota, D, a_c))}{\exp(X(c, \iota, D', a'_c))} = \frac{\exp_p(X(c, \iota, D, a_c))}{\exp_p(X(c, \iota, D', a'_c))} \mod \mu_{\infty}.
\]

(ii) Whenever \( p, \iota \mid \mathfrak{f} \) (\( p \) may vary), we have

\[
\frac{\exp(X(c, \iota)) \exp(X(cc_{i_0}, \iota))}{\exp_p(X(c, \iota)) \exp_p(X(cc_{i_0}, \iota))} \equiv \exp(\zeta'(0, c)) \exp(\zeta'(0, cc_{i_0})) \mod \mu_{\infty}.
\]
Proof. We see that \( \exp_p \circ \log_p = \text{id} \) on a neighborhood of 1. Hence we have for a large enough \( N \)

\[
(\exp_p(\log_p(z)))^p^N = \exp_p \circ \log_p((z)^p^N) = (z)^p^N \ (z \in \mathbb{C}_p^\times).
\]

In particular, when \( \text{ord}_p(z) = 0 \) we have

\[
\exp_p(\log_p(z)) \equiv z \mod \mu_{\infty}.
\]

Hence (i) follows from Lemmas 1, 4. The assertion (ii) follows from Theorems 2, 4 since

\[
\prod_{\iota \in \text{Hom}(F, \mathbb{R})} \iota(u) = N_{F/\mathbb{Q}}(u) = 1 \text{ for } u \in E_{F,+}. \]

6 The case when a finite place splits completely

In the previous section (especially in Corollary 1-(ii)), \( \exp(X(c, \iota)) \) are the main terms
and \( \exp_p(X_p(c, \iota)) \) are the correction terms. Their roles are exchanged in this section.

6.1 A brief review of the results in [KY1]

Let \( p_\iota \) be the prime ideal of \( F \) corresponding to the \( p \)-adic topology of \( F \) induced by \( F \hookrightarrow \mathbb{C}_p \) as in Definition 18.

Definition 23. (i) We denote by \( H_f \) the maximal ray class field modulo \( f \) in the narrow sense, by \( \text{Art}: C_f \rightarrow \text{Gal}(H_f/F) \) the Artin map.

(ii) We denote the group of all characters of \( C_f \) by \( \hat{C}_f \). Let \( \chi \in \hat{C}_f \). For an integral ideal \( g \) we denote the associated character \( \chi_g \) by \( \chi \in \hat{C}_f \). Namely, \( \chi_g \) is the composite map

\[
C_{fg} \rightarrow C_f \rightarrow \mathbb{C}_f.
\]

(iii) Let \( K \) be an extension field of \( F \). For \( \iota \in \text{Hom}(F, \mathbb{R}) \), we take a lift \( \tilde{\iota}: K \rightarrow \mathbb{C}_p \) of \( \iota: F \rightarrow \mathbb{C}_p \) and put

\[
p_{K,\tilde{\iota}} := \{ z \in \mathcal{O}_K \mid |\tilde{\iota}(z)|_p < 1 \}.
\]

Moreover we take a generator \( \alpha_{K,\tilde{\iota}} \) of the principal ideal \( p_{K,\tilde{\iota}}^{h_K} \), where \( h_K \) is the class number of \( K \).

(iv) We denote by \( \overline{Q} \log \overline{Q}^\times \) (resp. \( \overline{Q} \log_p \overline{Q}^\times \)) the \( \overline{Q} \)-subspace of \( \mathbb{C} \) (resp. \( \mathbb{C}_p \)) generated by \( \log a \) (resp. \( \log_p a \)) with \( a \in \overline{Q}^\times \). For \( \log a \), we take any branch of \( \log \).

(v) We define a \( \overline{Q} \)-linear mapping \( [\_]_p \) by

\[
[\_]_p : \overline{Q} \log \overline{Q}^\times \rightarrow \overline{Q} \log_p \overline{Q}^\times, \quad \sum_{i=1}^k a_i \log b_i \mapsto \sum_{i=1}^k a_i \log_p b_i.
\]

This map is well-defined [KY1, Lemma 5.1] by a well-known theorem of A. Baker.
Remark 5. Let $H$ be an intermediate field of $H_f/F$, $\chi'$ a character of $\text{Gal}(H/F)$. Then there exists a character $\chi \in \mathcal{C}_f$ corresponding to $\chi'$ via the Artin map. We note that $\chi$ may not be primitive. The relation between the Artin $L$-function $L(s, \chi')$ and the Hecke $L$-function $L(s, \chi)$ can be written as

$$L(s, \chi) = L(s, \chi') \prod_{q|\mathfrak{f}, \ q \nmid \mathfrak{f}_{\chi'}} (1 - \chi'(\text{Frob}_q Nq^{-s})).$$

Here $q$ runs over all prime ideal dividing $\mathfrak{f}$, not dividing the conductor $\mathfrak{f}_{\chi'}$ of $\chi'$. We denote by $\text{Frob}_q \in \text{Gal}(H/F)$ the Frobenius automorphism.

Proposition 5 ([KY1, Lemma 5.5, Proposition 5.6]). Let $\chi \in \mathcal{C}_f$. We assume that $p_\iota \nmid \mathfrak{f}, \chi(p_\iota) = 1$.

Then we have for $k \in \mathbb{N}$

$$\sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) X(c, \iota) \in \mathbb{Q} \log \mathbb{Q}^\times.\]$$

Moreover the quantity

$$\sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) X_p(c, \iota) - \left[ \sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) X(c, \iota) \right]_p \in \mathbb{C}_p$$

does not depend on the choices of $D$ and $a_c$’s.

Proof. The case $k = 1$ follows from [KY1, Lemma 5.5, Proposition 5.6]. We can use mathematical induction by [KY1, Lemmas 5.3, 5.4].

Definition 24. Let $\chi \in \mathcal{C}_f$ satisfy $p_\iota \nmid \mathfrak{f}, \chi(p_\iota) = 1$. For $k \in \mathbb{N}$ we define

$$Y_p(\chi_{p_\iota \mathfrak{f}}, \iota) := \sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) X_p(c, \iota) - \left[ \sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) X(c, \iota) \right]_p.$$

It follows that

$$Y_p(\chi_{p_\iota \mathfrak{f}}, \iota) = \sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) G_p(c, \iota) - \left[ \sum_{c \in C_{p_\iota \mathfrak{f}}^k} \chi_{p_\iota \mathfrak{f}}(c) G(c, \iota) \right]_p$$

since the $W, V$-terms are in $\mathbb{Q} \log \mathbb{Q}^\times$.

In [KY1], under the assumption $\chi(p_\iota) = 1$, we formulated the following two conjectures on the exact value of $Y_p(\chi_{p_\iota}, \iota)$, which is a refinement of Conjecture 2. The latter one has now become a corollary of Theorem 4.
Conjecture 3 ([KY1 Conjecture A']). Let $K$ be a CM-field which is abelian over $F$ with $f_{K/F}$ the conductor. We assume that $p_i$ splits completely in $K/F$. (Hence $p_i \nmid f_{K/F}$.) Then for any odd character $\chi'$ of $\text{Gal}(K/F)$, we have

$$Y_p(\chi_{p_i}, \iota) = \frac{L(0, \chi')}{2h_K} \sum_{\sigma \in \text{Gal}(K/F)} \chi' (\sigma) \log \tilde{i} \left( \frac{\alpha_{K,i}^{\sigma}}{\alpha_{K,i}^{0}} \right).$$

Here $\rho$ denotes the unique complex conjugation on $K$. We take $\chi \in \hat{C}_{L/F}$ corresponding to $\chi'$.

Theorem 5 ([KY1 Conjecture 5.10]). Let $\chi \in \hat{C}_f$. We assume that $\chi$ is not totally odd and satisfies $\chi(p_i) = 1$. (Hence $p_i \nmid f_i$.) Then we have

$$Y_p(\chi_{p_i}, \iota) = 0.$$

Proof. Since $\chi$ is not totally odd, there exists $\iota_0 \in \text{Hom}(F, \mathbb{R})$ satisfying $\chi(c_{\iota_0}) = 1$. Then we can write

$$Y_p(\chi_{p_i}, \iota) = \frac{1}{2} \sum_{c \in C_{p_i}} \chi(c) (X_p(c, \iota) + X_p(cc_{\iota_0}, \iota) - [X(c, \iota) + X(cc_{\iota_0}, \iota)]_p).$$

When $\iota \neq \iota_0$, we have $X_p(c, \iota) + X_p(cc_{\iota_0}, \iota) - [X(c, \iota) + X(cc_{\iota_0}, \iota)]_p = 0$ by Theorem 4. Even when $\iota = \iota_0$, by Corollary 1 and $\chi(p_i) = 1$, we have

$$Y_p(\chi_{p_i}, \iota) = \frac{1}{2} \sum_{c \in C_{p_i}} \chi(c) \log \tilde{p}_i \left( \zeta'_0(c, \iota_0) \zeta'(0, cc_{\iota_0}) \right) = \left[ \frac{d}{ds} L(s, \chi)(1 - Np_i^{-s}) \right]_p.$$

Since $\chi$ is not totally odd, we see that $\text{ord}_{s=0} L(s, \chi) \geq 1$, so $\frac{d}{ds} L(s, \chi)(1 - Np_i^{-s}) = 0$. Hence the assertion is clear. \hfill $\Box$

We reformulate Conjecture 3 in the remaining of this subsection.

Lemma 5 ([KY1 Lemma 6.4]). Let $\chi \in \hat{C}_f$ satisfy $\chi(p_i) = 1$. Then for an integral ideal $g$ of $F$ we have

$$Y_p(\chi_{p_i}, \iota) = \left( \prod_{q \mid g} (1 - \chi_{p_i}(q)) \right) Y_p(\chi_{p_i}, \iota).$$

Here $q$ runs over all prime ideals dividing $g$. In particular for $k \in \mathbb{N}$ we have

$$Y_p(\chi_{p_i^k}, \iota) = Y_p(\chi_{p_i}, \iota).$$

Proposition 6. Assume that $p_i \nmid f$. Let $H$ be the fixed subfield of $H_f$ under $\langle \text{Art}(p_i) \rangle$. Conjecture 3 is equivalent to each statement below.

(i) Let $\chi \in \hat{C}_f$. We assume that $\chi$ is totally odd and satisfies $\chi(p_i) = 1$. We denote the character of $\text{Gal}(H/F)$ corresponding to $\chi$ by $\chi'$. Then we have

$$Y_p(\chi_{p_i}, \iota) = \frac{-L(0, \chi)}{h_H} \sum_{\sigma \in \text{Gal}(H/F)} \chi'(\sigma) \log \tilde{i} (\alpha_{H,i}^{\sigma}).$$
Let $\tau \in \text{Gal}(H/F)$. We put

$$Y_p(\tau) := \sum_{c \in \mathcal{C} | \text{Art}(c) = \tau} X_p(c, \iota) - \left[ \sum_{c \in \mathcal{C} | \text{Art}(c) = \tau} X(c, \iota) \right]_p.$$ 

Here $c$ runs over all ideal classes whose images under the composite map $\mathcal{C} | \text{fp} \to \mathcal{C} | \to \text{Gal}(H_f/F) \to \text{Gal}(H/F)$ equal $\tau$. Then we have

$$Y_p(\tau) = -\frac{1}{h_H} \sum_{c \in \mathcal{C}_{l}} \zeta(0, c^{-1}) \log p \left( \alpha_{H, \iota} \right).$$

**Proof.** First we note that $\chi \in \hat{\mathcal{C}}_{l}$ is totally odd if and only if the intermediate field of $H_f/F$ corresponding to $\text{Art}(\ker \chi)$ is a CM-field. Hence the equivalence $\text{Conjecture} 3 \Leftrightarrow \text{(i)}$ follows from Lemma 5. If $\chi$ is not totally odd, then we have $L(0, \chi) = 0$, so the equation in (i) follows from Theorem 5. Therefore the equivalence (i) $\Leftrightarrow$ (ii) follows from the orthogonality of characters.

**Remark 6.** Assume that $\mathfrak{p}_{l} \mid \mathfrak{l}$. Let $H$ be an intermediate field of $H_f/F$. First we assume that the real place $\iota$ splits completely in $H$. Then, by Remark 3 and Corollary 7-(ii), we see that the Stark conjecture (2) with $v = \iota$ implies

$$\left( \prod_{c \in \phi_H^{-1}(\tau)} \frac{\exp(X(c, \iota))}{\exp_p(X_p(c, \iota))} \right)^2 \in \mathcal{O}_H^\times (\tau \in \text{Gal}(H/F)),$$

where $\phi_H$ denotes the composite map

$$\mathcal{C}_{l} \to \text{Gal}(H_f/F) \to \text{Gal}(H/F).$$

Strictly speaking, we put $\prod \exp_p(X_p(c, \iota)) = \prod \exp(X_p(c, \iota)) \exp_p(X(c, \iota))$. Next we assume that $\mathfrak{p}_{l}$ splits completely in $H$. Then Proposition 6-(ii) states that Conjecture 3 implies

$$\left( \prod_{c \in \phi_H^{-1}(\tau)} \frac{\exp(X(c, \iota))}{\exp_p(X_p(c, \iota))} \right)^{h_H W} \in \mathcal{I}(\alpha) \ker \log_p (\tau \in \text{Gal}(H/F)).$$

Here $W$ is the least common multiple of the denominators of $\zeta(0, c^{-1})$'s, $\alpha$ is a $\mathfrak{p}_{l}$-unit of $H$ defined as

$$\alpha := \prod_{c \in \mathcal{C}_{l}} \alpha_{H, \iota}^{\text{Art}(c) W \zeta(0, c^{-1})},$$

and $\ker \log_p$ is generated by rational powers of $p$ and the roots of unity.
References

[Co] R. Coleman, On the Frobenius matrices of Fermat curves, p-adic analysis, *Lecture Notes in Math.* **1454** (1990), 173–193.

[Da] S. Dasgupta, Shintani zeta-functions and Gross-Stark units for totally real fields, *Duke Math. J.* **143** (2008), no. 2, 225–279.

[DDP] S. Dasgupta, H. Darmon, R. Pollack, Hilbert modular forms and the Gross-Stark conjecture, *Ann. of Math. (2)* **174** (2011), no. 1, 439–484.

[dS] E. de Shalit, On monomial relations between p-adic periods, *J. Reine Angew. Math.* **374** (1987), 193–207.

[Gr1] B. H. Gross, On the periods of abelian integrals and a formula of Chowla and Selberg (with an appendix by D. E. Rohrlich), *Inv. Math.* **45** (1978), 193–211.

[Gr2] B. H. Gross, p-adic L-series at s = 0, *J. Fac. Sci. Univ. Tokyo* **28** (1981), 979–994.

[KY1] T. Kashio, and H. Yoshida, On p-adic absolute CM-Periods, I, *Amer. J. Math.* **130** (2008), no. 6, 1629–1685.

[KY2] T. Kashio, and H. Yoshida, On p-adic absolute CM-Periods, II, *Publ. Res. Inst. Math. Sci.* **45** (2009), no. 1, 187–225.

[Ka1] T. Kashio, On a p-adic analogue of Shintani’s formula, *J. Math. Kyoto Univ.* **45** (2005), 99–128.

[Ka2] T. Kashio, Fermat curves and the reciprocity law on cyclotomic units (arXiv:1502.04397), to appear in *J. Reine Angew. Math.*, the title was changed to “Fermat curves and a refinement of the reciprocity law on cyclotomic units”.

[Ka3] T. Kashio, On the algebraicity of some products of special values of Barnes’ multiple gamma function (arXiv:1510.01141), to appear in *Amer. J. Math.*

[Shim] G. Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Math. Ser. **46**, Princeton University Press, 1998.

[Shin1] T. Shintani, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **23** (1976), no. 2, 393–417.

[Shin2] T. Shintani, On values at s = 1 of certain L functions of totally real algebraic number fields, *Algebraic Number Theory, Proc. International Symp., Kyoto, 1976*, Kinokuniya, Tokyo (1977), 201–212.

[St] H. M. Stark, L-functions at s = 1. IV. First derivatives at s = 0, *Adv. in Math.* **35** (1980), no. 3, 197–235.

[Yo] H. Yoshida, *Absolute CM-Periods*, Math. Surveys Monogr. **106**, Amer. Math. Soc., Providence, RI, 2003.