Sphaleron transitions in a realistic heat bath

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August 1993

Abstract

We measure the diffusion rate of Chern-Simons number in the (1+1)-dimensional Abelian Higgs model interacting with a realistic heat bath for temperatures between 1/13 and 1/3 times the sphaleron energy. It is found that the measured rate is close to that predicted by one-loop calculation at the lower end of the temperature range considered but falls at least an order of magnitude short of one-loop estimate at the upper end of that range. We show numerically that the sphaleron approximation breaks down as soon as the gauge-invariant two-point function yields correlation length close to the sphaleron size.
Anomalous electroweak baryon-number violation may have played an important role in setting the baryon number of the Universe to its present value \cite{1, 2}. At temperatures above the gauge-boson mass scale electroweak baryon-number nonconservation is dominated by hopping over the finite-energy barriers separating topologically distinct vacua of the bosonic sector. Determination of the corresponding transition rate is a challenging nonperturbative problem, even in the range of validity of the classical approximation. At the lower temperature end of that range the barrier crossings are likely to occur in the vicinity of the saddle point (known as a sphaleron) of the energy functional. The rate can then be estimated using a field-theoretic extension of transition-state theory (TST) \cite{17, 18}. At higher temperatures this analytic tool is no longer available, and direct measurement of the rate in real-time numerical simulations of a lattice gauge-Higgs system is the only remaining possibility.

Analytical saddle-point estimates of the rate in the Standard Model are complicated by the fact that the corresponding sphaleron field configuration is not known exactly. At the same time, numerical real-time simulations of that system in its low-temperature regime carry an enormous computational cost and are yet to be performed \cite{1, 3}. In this situation lower-dimensional models become a very useful test ground on which activation-theory predictions can be confronted by numerical experiments \cite{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}. For this reason the (1+1)-dimensional Abelian Higgs model (AHM) studied numerically in this work has attracted much attention recently \cite{9, 12}.

Determination of the transition rate should include its proper averaging over the canonical ensemble in the phase space of a system in question. One way to achieve that would be to generate the canonical ensemble of initial configurations, subject each of these configurations to the Hamiltonian evolution, and average the transition rate over the initial states of the system. Such procedure, while being perfectly valid, is very costly computationally. To date, a single or a small number of initial configurations have been used in Hamiltonian simulations of AHM \cite{9, 12}. In addition, preparing initial configurations in case of a gauge theory presents a technical difficulty: if a standard importance-sampling method is used, resulting configurations will in general violate Gauss’ law. A special cooling procedure is required to eliminate static charge \cite{9}. It is not clear that such cooling does not cause the sample to deviate from the intended canonical ensemble.

Another way to obtain the canonical ensemble average of the rate is to replace Hamiltonian evolution by evolution in a heat bath. The simplest form of the latter is implemented using phenomenological Langevin equations of motion. While Langevin approach guarantees thermalization of the system, it does so at the expense of introducing an artificial viscosity parameter, thereby altering bulk dynamical properties of a field-theoretic system. Numerical studies performed on different models show that transition rates indeed strongly depend on viscosity \cite{7, 8}. Recent analytical work \cite{16} has also shown the impact of heat-bath properties on quantities of transition-rate type. In case of AHM, there also is a technical difficulty with the conventional Langevin approach: in order to maintain gauge invariance, one is forced to use polar coordinates for the Higgs field; whenever the latter vanishes, the equations of motion are singular \cite{4, 12}.

Recently we have proposed and tested a new method in which a field-theoretic system
interacts with a heat bath at its boundaries [3]. The heat bath is constructed so as to imitate an infinite extension of the system beyond the boundaries. Technically this means that the fields in the bulk of the system evolve according to the Hamiltonian equations of motion, while boundary fields are subject to Langevin evolution with a non-Markovian friction kernel and colored noise. Our construction approximates a natural situation in which open systems are immersed in a similar environment. In this way, dynamical evolution and canonical ensemble averaging occur at the same time while bulk dynamical properties of the system are intact. Moreover, the new procedure does not suffer from the technical difficulties of the two old ones. In this work we apply the realistic heat bath (RHB) method to the study of sphaleron transitions in AHM.

Earlier real-time simulations of AHM [8, 4] found that at low temperatures (about 0.1 of the sphaleron energy) the temperature dependence of the rate qualitatively agrees with that predicted by the 1-loop (TST) calculation of Ref. [6]. The temperature range of our simulation is wider and includes somewhat higher temperatures, up to about 1/3 the sphaleron energy, for which the TST result may no longer be reliable. This allows us to estimate the temperature at which TST loses validity. As will be demonstrated in the following, this is the temperature at which the scalar field correlation length falls below the linear size of the sphaleron. It is also interesting to determine the sign and magnitude of the rate deviation from the TST prediction. If the rate we measure falls considerably short of the latter, it might be indicative of the entropic rate suppression which reflects the difficulty of creating a coherent configuration in high-temperature plasma. Our results, presented in the following, do indeed show dramatic slowdown of the rate growth.

Our starting point is the (1+1)-dimensional lattice AHM Lagrangian which in suitably chosen units reads [4]

\[
L = \frac{a}{2} \sum_j \left[ \frac{1}{\xi} (\dot{A}_{j+1,j} - \frac{A_{j+1}^0 - A_j^0}{a})^2 + |(\partial_0 - iA_j^0)\phi_j|^2 
- a^{-2}|\phi_{j+1} - \exp(iA_{j,j+1})\phi_j|^2 - \frac{1}{2} (|\phi_j|^2 - 1)^2 \right].
\] (1)

Here \( j \) labels sites of a chain whose lattice spacing is \( a \). The temporal component of a vector potential, \( A_j^0 \), and a complex scalar field \( \phi_j \) reside on sites of the chain, whereas the spatial component of the vector potential, \( A_{j,j+1}^1 \), resides on links. Imposing the \( A^0 = 0 \) condition one obtains a Hamiltonian (we shall drop the Lorentz index of \( A^1 \) from now on)

\[
H = \frac{a}{2} \sum_j \left[ \left( \frac{\xi E_{j,j+1}}{a} \right)^2 + \frac{|\pi_j|^2 + |\phi_{j+1} - \exp(iA_{j,j+1})\phi_j|^2 + \frac{1}{2} (|\phi_j|^2 - 1)^2}{a} \right],
\] (2)

where \( \pi_j \) and \( E_{j,j+1} \) are canonically conjugate momenta of \( \phi_j \) and \( A_{j,j+1} \), respectively. The Hamiltonian equations of motion

\[
\dot{A}_{j,j+1} = \frac{\xi}{a} E_{j,j+1},
\]
\[ E_{j,j+1} = i\phi_j \left( \exp(iaA_{j,j+1})\phi^*_j + \phi^*_j \right) + \text{h.c.}, \]
\[ \dot{\phi}_j = \frac{1}{a} \pi^\rho_j, \]
\[ \dot{\pi}_j = \frac{1}{a} \left( \exp(iaA_{j,j+1})\phi^*_j + \exp(-iaA_{j-1,j})\phi^*_j - 2\phi^*_j \right) \]
\[ -a\phi^*_j \left( |\phi_j|^2 - 1 \right) \]

are supplemented by the Gauss’ law constraint
\[ \frac{1}{a} (E_{j,j+1} - E_{j-1,j}) = \text{Im} \left( \pi_j \phi^*_j \right). \]

The same dynamics can be described in terms of gauge-invariant variables. To this end, the Higgs field is rewritten in polar coordinates: \( \phi_j = \rho_j \exp(i\alpha_j) \). Defining \( b_{j,j+1} = \alpha_{j+1} - \alpha_j - aA_{j,j+1} \), \( \epsilon_{j,j+1} = \frac{1}{a} E_{j,j+1} \) and introducing canonical momentum \( \pi^\rho_j \) for \( \rho_j \) we see that (3) together with (4) is equivalent to
\[ \dot{\epsilon}_{j,j+1} = \frac{1}{a} \rho_j \rho_{j+1} \sin b_{j,j+1}, \]
\[ \dot{b}_{j,j+1} = \frac{1}{a} \left( \frac{\epsilon_{j+1,j+2} - \epsilon_{j,j+1}}{\rho^2_{j+1}} - \frac{\epsilon_{j,j+1} - \epsilon_{j-1,j}}{\rho^2_j} \right) - a\epsilon_{j,j+1}, \]
\[ \dot{\rho}_j = \frac{1}{a} \pi^\rho_j, \]
\[ \dot{\pi}^\rho_j = \frac{(\epsilon_{j,j+1} - \epsilon_{j-1,j})}{a\rho^3_j} + \frac{1}{a} \left( \rho_{j+1} \cos b_{j,j+1} + \rho_{j-1} \cos b_{j-1,j} - 2\rho_j \right) \]
\[ -a\rho_j \left( \rho^2_j - 1 \right) \].

The equations of motion in this form involve only two pairs of real canonical variables, namely, \( \rho_j, \pi^\rho_j \) and \( \epsilon_{j,j+1}, b_{j,j+1} \). It is easy to see that (5) follow from the Hamiltonian
\[ H' = \frac{a}{2} \sum_j \left( \xi \epsilon^2_{j,j+1} + \left( \frac{\epsilon_{j,j+1} - \epsilon_{j-1,j}}{a\rho_j} \right)^2 + \left( \frac{\pi^\rho_j}{a} \right)^2 + \frac{2}{a^2} \left( \rho^2_j - \rho_j \rho_{j+1} \cos b_{j,j+1} \right) \right) \]
\[ + \frac{a}{4} \sum_j \left( \rho^2_j - 1 \right)^2 \]

obtained from polar-coordinate form of (2) by substituting (4).

In the following we shall use both presented forms of the equations of motion. On one hand, the Cartesian form (3) allows better numerical handling of sphaleron-like field configurations in which the Higgs field is close to zero at one or more sites. For this reason we use it for real-time evolution in the bulk of an open gauge-Higgs system. On the other hand, the gauge-invariant form (5) involves less degrees of freedom and lends itself easier to linearization. We therefore use it for the heat bath construction.

As a heat bath we take AHM linearized in the vicinity of one of its gauge-equivalent vacua, which, as is well known, is a system of two free fields: the radial Higgs field \( \varphi \)
and the gauge field $\varepsilon$ whose masses are $\sqrt{2}$ and $\sqrt{\xi}$, respectively. Those are coupled at the boundary site of the AHM ((2) or (3)) each to its interacting counterpart, \textit{i.e.} $\rho$ to $\rho$, and $\varepsilon$ to $\varepsilon$. Suppose for definiteness that the left boundary of the interacting system separating it from the linear heat bath is at $j = 0$ site of the chain. The field equations at the boundary are then modified compared to (5), namely, the second and the fourth equation of (5) are replaced by

\begin{align*}
\dot{b}_{-1,0} &= \frac{1}{a}(\varepsilon_{0,1} - 2*\varepsilon_{-1,0}) - a\xi\varepsilon_{-1,0} + D(\sqrt{\xi}, [\varepsilon_{-1,0}]) + F(\sqrt{\xi}, t), \\
\dot{\pi}_0 &= \frac{1}{a}(\rho_1 \cos b_{0,1} - 2\rho_0 + 1) - 2a(\rho_0 - 1) + D(\sqrt{2}, [\rho_0 - 1]) + F(\sqrt{2}, t).
\end{align*}

In going from (5) to (7) we linearized in the vicinity of $\rho_0 = 1$, $b_{-1,0} = 0$ and, following Ref. [5], introduced two terms describing interaction with the linear heat bath at the boundary. The $D(m, [\sigma])$ term represents the reaction to the motion of a boundary field $\sigma$ from the heat bath. The mass of the corresponding heat-bath field is $m$. Explicitly,

\begin{equation}
D(m, [\sigma]) = \int_{-\infty}^{t} \sigma(t')\chi_m(t - t')dt',
\end{equation}

where the Fourier image of the causal response function $\chi_m(t)$ is

\begin{equation}
\tilde{\chi}_m(\omega) = 2i\text{sign}(\omega)\sqrt{(\omega^2 - m^2)(1 + a^2(m^2 - \omega^2)/4)}
\end{equation}

for frequencies $m < |\omega| < \sqrt{m^2 + 4/a^2}$ and vanishes outside this range. The $D(m, [\sigma])$ term thus describes dissipation processes: the choice of $\chi_m(t)$ ensures that the waves traveling across the boundary into the heat bath are completely absorbed. In order for the system to reach thermal equilibrium with the heat bath, the $F(m, t)$ term describing thermal fluctuations of a heat-bath field at the boundary should be included. According to fluctuation-dissipation theorem, $F(m, t)$ is a Gaussian random variable whose time autocorrelation is related to $\chi_m(t)$:

\begin{equation}
\langle F(m, t)F(m, t + \tau) \rangle = \theta \int_{0}^{\tau} \chi_m(t')dt',
\end{equation}

where $\theta$ is the temperature. We solve numerically the system of equations (3) together with (7) and its right-boundary analog. Numerical implementation of the boundary heat bath is explained in detail in Ref. [4]. As has already been mentioned, the Cartesian form of equations of motion in the bulk is used due to its superior numerical properties. On the other hand, we use polar coordinates to evolve boundary fields. In order to be able to transform from polar to Cartesian coordinates at the boundary, we need to keep track of the angular variable $\alpha$ of the boundary Higgs field. This is done with the help of Gauss’ law which for the linearized system gives

\begin{equation}
\dot{\alpha}_j = \frac{1}{a}(E_{j,j+1} - E_{j-1,j}).
\end{equation}
We use second-order Runge-Kutta algorithm for numerical solution of (3,7,11). A number of criteria were applied in evaluating the algorithm performance and in determining the value of the time step. In particular, we tested the absorption properties of the simulated heat bath by dropping the noise term from the equations of motion and cooling an initially hot system. The resulting cooling curve was then compared to that of the same system in a large real zero-temperature heat bath. In a similar way we studied thermalization of an initially cold system in the heat bath. The temperature was measured by averaging the kinetic energy of the radial Higgs field over the system and over the time history. In all our simulations the temperature of a thermalized system was found to be within 3% of the assigned value.

One special numerical issue to be dealt with in a real-time simulation of a gauge theory is the accuracy of the Gauss’ constraint. The equations we solve are consistent with the Gauss’ law. However, numerical errors give rise to a small spurious static charge density which should be kept in check in order to have no impact on the quantities we measure, in particular, the sphaleron transition rate. Since the rate is exponentially sensitive to the sphaleron energy, we computed the perturbative correction to the latter in presence of a small static charge distribution \( q(x) \), \( \Delta E_{sph}([q]) \). The corresponding expression is derived in the Appendix. We then used \( \beta \Delta E_{sph}([q]) \) averaged over the sphaleron positions as a criterion of Gauss’ law violation (here and in the following \( \beta \) denotes inverse temperature). In all our simulations the amount of the spurious charge was far too small to have any measurable impact on the sphaleron transition rate.

Our attention in this work is focused on the temperature dependence of the rate. For this reason we performed all our simulations at a fixed value of the gauge coupling \( \xi = 10 \), with the exception of preliminary study at \( \xi = 0.5 \). Most of our measurements were done for a chain of length \( L = 100 \) and lattice spacing \( a = 0.5 \). Since our results are to be compared to the analytical prediction for AHM in the continuum, we checked their dependence on both lattice cutoffs by performing additional simulations at \( a = 0.25 \) and at \( L = 50 \). The Runge-Kutta time step was 0.01 for \( a = 0.5 \) and 0.004 for \( a = 0.25 \). For each set of \( a, L \), and the temperature the simulation time was \( 10^5 \) time units. To add confidence to our measurements we also performed microcanonical simulations at a number of parameter values. The agreement with the canonical results was good except for the \( \beta = 5, L = 100, a = 0.5 \) case for which the canonical simulation gives somewhat higher value of the rate.

Following Ref. [4], we extracted the sphaleron transition rate from \( \Delta_{CS}(t) \), the time-averaged squared deviation of the Chern-Simons variable \( N_{CS}(x) \equiv (2\pi)^{-1} \int A(x)dx \) for a lag \( t \). For lags shorter than the average time between consecutive sphaleron transitions \( \Delta_{CS}(t) \) is determined by fluctuations of \( N_{CS} \) in the vicinity of one of its vacuum values. At lags much longer than the lifetime of a vacuum many uncorrelated sphaleron transitions would have occurred, each changing the value of \( N_{CS} \) by an integer, and a random-walk behavior sets in:

\[
\Delta_{CS}(t) = \Gamma L t,
\]

where \( \Gamma \) is the sphaleron transition rate per unit length. Figure[4] illustrates the described lag dependence of \( \Delta_{CS}(t) \).
Once $\Delta_{CS}(t)$ is known, it can be (at large enough values of $t$) fitted to a straight line through the origin to yield $\Gamma$. Since the values of $\Delta_{CS}(t)$ at different $t$ are strongly correlated, we found that, while the quality of fit remains high as more and more $\Delta_{CS}(t)$ data points are included, the error on $\Gamma$ is not reduced significantly by fitting with more degrees of freedom. We therefore simplified the procedure and extracted $\Gamma$ from a single value of $\Delta_{CS}(t)$ at $t = 1000$. This choice of a lag is suitable since, on one hand, it is much shorter than our total simulation time ($10^5$), while on the other hand it is at least several times longer than the average time between consecutive sphaleron transitions in the temperature range considered. We also verified that $\Gamma$ remains constant in a wide range of lags including the chosen one.

Summary of all our rate measurements is presented in Table 1, while for Figure 2 we selected the results that best reflect the important features of $\Gamma$ dependence on the inverse temperature $\beta$, as well as on $a$. Obviously, no measurable dependence on $L$ is observed. The absence of finite-size effects is to be expected of our heat-bath construction. Namely, at low temperatures the linearized heat bath closely imitates the infinite extension of the nonlinear system beyond the boundaries. At high temperatures the system becomes less and less correlated in space and time, and, as a result, the boundary effects lose importance. We also observe no dependence of the rate on $a$, except for the highest temperature ($\beta = 3$) considered. The virtual independence of $\Gamma$ of the lattice spacing at low temperatures has been found in earlier work [3, 4] and is confirmed by our results. As will be shown shortly, the difference in the rates between the $a = 0.5$ and $a = 0.25$
Figure 2: Temperature dependence of the transition rate $\Gamma$. The solid curves correspond to the TST prediction (13), with the value of $\xi$ indicated near each curve.

Cases at $\beta = 3$ is consistent with other properties of the model at this temperature. Our rate measurements are to be compared to the TST prediction \[ \Gamma = \frac{\omega_0}{2\pi} \left( \frac{6\beta E_{\text{sph}}}{2\pi} \right)^{\frac{1}{2}} \left( \frac{\Gamma(\alpha + s + 1)\Gamma(\alpha - s)}{\Gamma(\alpha + 1)\Gamma(\alpha)} \right)^{\frac{1}{2}} \exp\left(-\beta E_{\text{sph}}\right), \] (13)

where the sphaleron energy $E_{\text{sph}} = 2\sqrt{2}/3$, $\alpha^2 = s(s + 1) = 2\xi$, and $\omega_0 = s + 1$ is the negative squared eigenfrequency corresponding to the sphaleron instability.

As Figure 2 clearly shows, the values of $\Gamma$ at low temperature are close to those given by (13). While the approach of measured $\Gamma$ to the TST prediction is slow, the two nearly coincide at the lowest temperature considered, $\beta = 14$. This nice agreement shows once again that our heat bath construction works as intended. But the most notable feature of our results is their dramatic departure from the TST-predicted values starting at about $\beta = 5$. For $\xi = 10$ the discrepancy is a factor of 5 already at $\beta = 5$, and grows at $\beta = 3$ to a factor of 10 for $a = 0.5$ and a factor of 20 for $a = 0.25$. The situation is similar for $\xi = 0.5$. Moreover, in the latter two cases the rate practically does not grow between $\beta = 5$ and $\beta = 3$. Note that the deviations are much larger than our measurement error bars and are therefore statistically significant.

It is natural to ask why this strong lagging of the measured rate behind the TST one begins in the vicinity of $\beta = 5$. To this end recall the underlying assumption of (13):
Table 1: Summary of transition rate measurements. The inverse temperature $\beta$ is given as deduced from the average kinetic energy of the radial Higgs field.

Chern-Simons number diffusion is dominated by evolution of configurations resembling the vacuum into those resembling the zero-temperature sphaleron. It is clear that the $N_{CS}$ diffusion can only be described in these terms as long as the Higgs field is correlated on a length scale larger than the sphaleron size (2$\sqrt{2}$). The corresponding correlation length $\lambda$ can be found by measuring a gauge-invariant two-point function \[ C_{jl} = \phi_j^* \phi_l \exp \left(-ia \sum_{k=j}^{l-1} A_{k,k+1} \right) = \rho_j \rho_l \exp \left(-i \sum_{k=j}^{l-1} b_{k,k+1} \right). \] (14)

A rough estimate of $\lambda$ at low temperatures is obtained by averaging $C_{jl}$ over the thermal ensemble with $H'$ of (8) replaced by its linearized version. Performing Gaussian integration over the $b$ variables one finds $\lambda = 2\beta$. This is, in fact, an overestimate of $\lambda$, since thermal fluctuations of the radial Higgs field are not taken into account. Figure 2 shows the values of $\lambda$ obtained by fitting $\langle C_{jl} \rangle$ to $\text{const} \times \exp(-|j - l|/\lambda)$. As expected, the breakdown of the saddle-point approximation for the rate occurs as $\lambda$ becomes smaller than the sphaleron size. Note that this is true for both values of $\xi$ considered. At $\beta = 3$ $\lambda = 1.47$ for $\xi = 10$, only about 3 times larger than the lattice spacing $a = 0.5$. Hence the field strongly fluctuates at length scales comparable to the lattice spacing. It is therefore not surprising that we find the $a$ dependence of the rate at this temperature.

At this point it is unclear what causes the sharp slowdown of the rate growth at the high-temperature end of our measurement range. We cannot exclude a possibility that at temperatures in question crossing the $N_{CS} = \text{half} – \text{integer separatrix}$ in the configuration space of the model \[ \text{in close vicinity of the sphaleron saddle point is still strongly preferred energetically, but is already suppressed entropically. It is usually assumed} \] that at temperatures above the sphaleron energy the rate grows like a power of the temperature. It could be that what we observe at $\beta \leq 5$ is a crossover.
Figure 3: Temperature dependence of the correlation length deduced from the gauge-invariant two-point function (14). The sphaleron size is shown by the dashed line for comparison.

from the exponential to power-law behavior of the rate. It is not clear, however, that at such a crossover the rate should stop growing as it does for $\xi = 10$, $a = 0.25$ and for $\xi = 0.5$, $a = 0.5$. The only way to resolve this puzzling situation is by rate measurements at still higher temperatures, as well as smaller lattice spacings. We plan to do so in the future.

To summarize, we performed an accurate measurement of the sphaleron transition rate in AHM averaged over the canonical ensemble. The latter was obtained by immersing the system in a realistic heat bath. The ergodicity of real-time evolution was thus achieved without having to introduce an artificial viscosity parameter. Our rate measurements approach the corresponding TST estimate at low temperature. This is in agreement with Ref. [9] where similar measurements were performed microcanonically. The highest temperature studied in that work was $0.103 E_{sph}$, well within the range of applicability of TST. In going beyond that range, we found dramatic slowdown of the rate growth, with suppression factor as large as 20 relative to TST at the highest temperature considered. Our measurements show that the breakdown of TST occurs as soon as the correlation length deduced from $C_{jl}$ (14) becomes comparable to the sphaleron size. This suggests that a similar object might serve as a criterion for applicability of the sphaleron approximation in other theories, including the realistic 3+1-dimensional case.
We gratefully acknowledge enlightening discussions with A. I. Bochkarev, Ph. de Forcrand, E. G. Klepfish, A. Kovner, and especially A. Wipf. Numerical simulations for this work were performed on the Cray YMP/464 supercomputer at ETH.

Appendix

In this Appendix we give a perturbative estimate of the shift in the sphaleron energy in presence of a small static charge density $q$. For simplicity we use continuum, rather than lattice, formulation of AHM. We shall also assume that the system has an infinite length. Analogous to (2), the Hamiltonian depends on three pairs of canonical variables $A,E, \rho, \pi, \rho_0, \alpha, \pi_\alpha$. Static configurations are obtained by minimizing the energy with respect to all the variables on a subspace constrained by Gauss’ law $\pi_\alpha = E' - q$ (prime means derivative with respect to the spatial variable $x$). Vacuum configurations correspond to the absolute minimum of the energy on that subspace, while sphalerons minimize the energy among configurations with $\rho = 0$ at one point. Two of the variables, $A$ and $\alpha$, enter the Hamiltonian only in combination $b = \alpha' - A$, thereby effectively reducing the number of variables by one. It is convenient to define a new complex field

$$\Phi(x) = \rho(x) \exp \left( i \int_{-\infty}^{x} b(x') dx' \right),$$

(15)

After eliminating $\pi_\alpha$ with the help of Gauss’ law one obtains

$$H = \frac{1}{2} \int dx \left[ \xi E^2 + \pi_\rho^2 + \frac{(E' - q)^2}{|\Phi|^2} + |\Phi'|^2 + \frac{1}{2}(|\Phi|^2 - 1)^2 \right],$$

(16)

Extremization of energy leads, apart from the trivial condition $\pi_\rho = 0$, to a system of coupled equations for $\Phi$ and $E$:

$$\xi E - \left[ \frac{E' - q}{|\Phi|^2} \right]' = 0, \Phi'' + \frac{(E' - q)^2 \Phi}{|\Phi|^4} - \Phi(|\Phi|^2 - 1) = 0.$$  

(17)

These equations are simplified if we introduce an electrostatic potential for the electric field: $E = Y'$. If we require that $q$ vanishes for $|x|$ above certain value, $E$ and $|\Phi|$ must respectively approach 0 and 1 as $x \to \pm \infty$. The electrostatic potential $Y$ will then approach a constant value. If that constant is chosen to be 0, (17) takes form

$$Y'' - \xi |\Phi|^2 Y = q; \Phi'' + \xi^2 Y^2 \Phi - \Phi(|\Phi|^2 - 1) = 0.$$ 

(18)

We are interested in perturbative corrections to the lowest order in $q$ to the vacuum $Y = 0, \Phi = 1$ and to the sphaleron configuration $Y = 0, \Phi = \tanh(x/\sqrt{2})$. We will show in the following that in both cases the correction to $Y$ is first order in $q$. It is then clear from the second equation of (18) that the correction to $\Phi$ is at best second order. Inspecting the Hamiltonian (16) and bearing in mind that the unperturbed solutions extremize $H$ for $q = 0$ we conclude that the correction to the energy is second order in
$q$ and comes solely from the first-order correction to $Y$. Our task therefore reduces to solving the first equation of (18) in the unperturbed $\Phi$ background. Writing $Y(x)$ as $\int G(x, y)q(y)dy$ we obtain an equation for the Green’s function $G(x, y)$:

$$\left(\partial_x^2 - \xi |\Phi(x)|^2\right) G(x, y) = \delta(x - y).$$

(19)

It is easy to verify that

$$G_v(x, y) = -\frac{1}{2\sqrt{\xi}} \exp \left(-\sqrt{\xi}|x - y|\right)$$

(20)

for the vacuum ($\Phi(x) = 1$). For the sphaleron configuration ($\Phi(x) = \tanh(x/\sqrt{2})$) the solution of (19) is also straightforward but somewhat cumbersome. The result can be expressed in terms of hypergeometric functions:

$$G_s(x, y) = -C \cosh^{-\sqrt{2\xi}}(x/\sqrt{2}) \cosh^{-\sqrt{2\xi}}(y/\sqrt{2})$$

$$\times F\left(\alpha_+, \alpha_-; \gamma; \frac{\exp \left(x/\sqrt{2}\right)}{2 \cosh \left(x/\sqrt{2}\right)}\right)$$

$$\times F\left(\alpha_+, \alpha_-; \gamma; \frac{\exp \left(-y/\sqrt{2}\right)}{2 \cosh \left(y/\sqrt{2}\right)}\right)$$

(21)

if $y \geq x$, with $x$ and $y$ interchanged otherwise. Here

$$\alpha_\pm = \frac{1 + \sqrt{8\xi} \pm \sqrt{1 + 8\xi}}{2}; \gamma = \frac{1 + \alpha_+ + \alpha_-}{2};$$

$$C = \frac{\Gamma(\alpha_+)\Gamma(\alpha_-)}{4^{1+\sqrt{2\xi}} \Gamma(\gamma) \Gamma(\sqrt{2\xi})}.$$  (22)

Substituting the correction to electric field into (16) we find the energy shift of a state due to the static charge $q$:

$$\Delta H = -\frac{\xi}{2} \int dx dy G(x, y)q(x)q(y).$$

(23)

This result is hardly surprising: $G(x, y)$ is nothing but the Coulomb potential at $x$ due to a unit charge at $y$. Therefore to the lowest order in $q$ the energy shift is equal to the Coulomb energy of external charge distribution. The correction to the sphaleron energy barrier then follows immediately:

$$\Delta E_{sph} = -\frac{\xi}{2} \int dx dy \left(G_s(x, y) - G_v(x, y)\right) q(x)q(y).$$

(24)

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