THE HYPERMULTIPLE GAMMA FUNCTIONS OF BM-TYPE

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ABSTRACT. In this paper, we introduce the hypermultiple gamma functions of BM-type and prove the asymptotic expansion of these functions.

1. INTRODUCTION

In Barnes [1], he introduced the multiple gamma function
\[
\Gamma_r(w; \omega) = \exp \left( \partial_s \zeta_r(s, w; \omega) \bigg|_{s=0} \right),
\]
where \( \zeta_r \) means the Barnes multiple zeta function
\[
\zeta_r(s, w; \omega) = \sum_{n \geq 0} (n \cdot \omega + w)^{-s},
\]
and \( \omega = (\omega_1, \ldots, \omega_r), n = (n_1, \ldots, n_r) \in \mathbb{Z}^r, n \cdot \omega = n_1 \omega_1 + \cdots + n_r \omega_r \) and \( n \geq 0 \) means \( n_i \geq 0 \) for all \( i \). He also proved the asymptotic expansion of this function:

**Theorem 1.1.** As \( w \to \infty \), we have asymptotically
\[
\log \Gamma_r(w + a; \omega) = \sum_{N=0}^{r} a_{r-N}(a; \omega) \frac{(w)^N}{N!} (H_N - \log w) + a_{r,k+1}(a; \omega) \frac{1}{w} + O \left( \frac{1}{w^2} \right),
\]
where \( \text{Re}(a) > 0, |\omega| = \omega_1 \cdots \omega_r, H_N \) means \( N \)-th harmonic number
\[
H_N = \sum_{i=1}^{N} \frac{t^{-1}}{i}.
\]
and \( a_{r,N} \) means the \( N \)-th multiple Bernoulli polynomial
\[
\prod_{i=1}^{r} (1 - e^{-\omega_i t})^{-1} = \sum_{N=-r}^{\infty} a_{r,N}(w; \omega)t^N.
\]
Katayama-Ohtsuki [4] gave a simple proof of this behavior based on the integral representation of \( \log \Gamma_r \);
\[
\log \Gamma_r(w; \omega) = \int_{I(\lambda, \infty)} f_\omega(t) e^{-wt} t^{-1} \left( \frac{1}{2\pi i} \log t + \left( \frac{\gamma}{2\pi i} - \frac{1}{2} \right) \right) dt,
\]
where \( f_\omega(t) = \prod_{i=1}^{r} (1 - e^{-\omega_i t})^{-1}, 0 < \lambda < \min_{1 \leq i \leq r} \frac{2\pi}{|\omega_i|} \) and \( I(\lambda, \infty) \) is the path consisting of the infinite line from \( \infty \) to \( \lambda \), the circle of radius \( \lambda \) around 0 in the positive sense and the infinite line from \( \lambda \) to \( \infty \).

While, Kurokawa-Ochiai [6] introduced the multiple gamma functions of BM (Barnes-Milnor) type:
\[
\Gamma_{r,k}(w; \omega) = \exp \left( \partial_s \zeta_r(s, w; \omega) \bigg|_{s=-k} \right).
\]
for non-negative integers $k$. They also generalized Kinkelin’s formulas:

**Theorem 1.2.** For $k \geq 1$, we have

$$
\int_0^w \left( \log \Gamma_{r,k-1}(t; \omega) - \frac{1}{k} \zeta_r(1 - k, t; \omega) \right) \, dt = \frac{1}{k} \log \frac{\Gamma_{r,k}(w; \omega)}{\Gamma_{r,k}(0; \omega)}.
$$

This theorem can be written as

$$
\frac{\partial}{\partial w} \log \Gamma_{r,k}(w; \omega) = k \log \Gamma_{r,k} - \zeta_r(1 - k, w; \omega).
$$

The term $\zeta_r(1 - k, w; \omega)$ can be regarded as a “gap” of hierarchy of $\Gamma_{r,k}$. To correct this gap, the “balanced” multiple gamma functions $P_{r,k}$ were defined in the author’s preprint [5]:

$$
P_{r,k}(w; \omega) = \frac{(-1)^k}{k!} \log \Gamma_{r,k}(w; \omega) + H_k a_{r,k}(w; \omega)
$$

for $k \geq 0$. Then hierarchy of $P_{r,k}$ is simpler than that of $\Gamma_{r,k}$:

$$
\frac{\partial}{\partial w} P_{r,k}(w; \omega) = -P_{r,k-1}(w; \omega).
$$

This fact shows that $P_{r,k}$ can be defined for any $k \in \mathbb{Z}$. Moreover, we can investigate the asymptotic behavior of $P_{r,k}$ in the same way as we prove that of $\Gamma_r$:

**Theorem 1.3.** As $w \to \infty$, we have asymptotically

$$
P_{r+l,k}(w; (\omega, \alpha)) = \sum_{N=-l}^{r+k} a_{l,N}(a; \alpha) P_{r-k-N}(w; \omega) + \frac{a_{l,r+k+1}(a; \alpha)}{|\omega|^{x_l}} \frac{1}{w} + O \left( \frac{1}{w^2} \right),
$$

where $\alpha = (\alpha_1, \cdots, \alpha_l)$ with $\text{Re}(\alpha_j) > 0$ ($j = 1, \cdots, l$).

In this paper, we introduce a new generalization of the multiple gamma functions and call it the hypermultiple gamma functions of BM-type:

$$
m\Gamma_{r,k}(w; \omega) = \exp \left( \frac{\partial^m}{\partial s^m} \zeta_r(s, w; \omega) \bigg|_{s=-k} \right)
$$

This generalization includes both Kurokawa-Ochiai’s $\Gamma_{r,k}$ and Katayama’s hypermultiple gamma functions

$$
m\Gamma_r(w; \omega) = \exp \left( \frac{\partial^m}{\partial s^m} \zeta_r(s, w; \omega) \bigg|_{s=0} \right)
$$

introduced in Katayama [3]. We remark that $\log m\Gamma_r$ means the natural logarithm of $m\Gamma_r$, not $\log \Gamma_r / \log m$. The name ”hypermultiple” is derived from Katayama’s report [2].

Our second purpose is to construct the ”balanced” hypermultiple gamma functions of BM-type $mP_{r,k}$ and to show the asymptotic behavior of these functions. We shall give the definition of $mP_{r,k}$ in the section 2. Our main theorem is following:

**Theorem 1.4.** As $w \to \infty$, we have asymptotically

$$
mP_{r+l,k}(w; (\omega, \alpha)) = \sum_{N=-l}^{r+k} a_{l,N}(a; \alpha) mP_{r-k-N}(w; \omega) + O \left( \frac{(\log w)^{m-1}}{w} \right).
$$

2
2. Construction of $mP_{r,k}$

In this section, we construct the balanced hypermultiple gamma functions of BM-type $mP_{r,k}$. For $k \in \mathbb{Z}_{\geq 0}$, we define

$$e^{(s+k)x} / \Gamma(s)(e^{2\pi is} - 1) = \sum_{m=0}^{\infty} \frac{mQ_k(x)}{m!} (s+k)^m.$$ 

Then $mQ_k(x)$ is a polynomial of $x$ whose degree is $m$. From a well-known representation

$$\zeta_r(s, w; \omega) = \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{I(\lambda, \infty)} f_\omega(t) e^{-wt}s^{-1} \, dt,$$

we obtain a representation of the hypermultiple gamma functions of BM-type

$$\log m \Gamma_{r,k}(w; \omega) = \int_{I(\lambda, \infty)} f_\omega(t) e^{-wtk^{-1}} mQ_k(\log t) \, dt.$$

For $\mu \in \mathbb{Z}_{\geq 0}$, we define the multiple harmonic sum

$$H_k(\mu) = \sum_{0 < j_1 \leq \cdots \leq j_\mu \leq k} j_1^{-1} \cdots j_\mu^{-1}$$

and put

$$c^m_{\mu,k} = \frac{(-1)^k}{k!} \frac{m!}{(m-\mu)!} H_k(\mu).$$

We consider $H_0(\mu)$ as 0 and $H_k(0)$ as 1. Then we can define the balanced hypermultiple gamma functions of BM-type

$$mP_{r,k}(w; \omega) = \sum_{\mu=0}^{m} c^m_{m-\mu,k} \log m \Gamma_{r,k}(w; \omega).$$

**Proposition 2.1.** The functions $mP_{r,k}$ satisfy simplified Kinkelin’s formula. In other words, we have

$$\frac{\partial}{\partial w} mP_{r,k}(w; \omega) = -mP_{r,k-1}(w; \omega).$$

**Proof.** From (1), we have

$$\frac{\partial}{\partial w} \zeta_r(s, w; \omega) = -s \zeta_r(s+1, w; \omega).$$

Thus we get

$$\frac{\partial}{\partial w} \log m \Gamma_{r,k}(w; \omega) = k \log m \Gamma_{r,k-1}(w; \omega) - m \log m \Gamma_{r,k-1}(w; \omega)$$

for $m \geq 1$ and

$$\frac{\partial}{\partial w} \log \Gamma_{r,k}(w; \omega) = k \log \Gamma_{r,k}(w; \omega)$$

for $m = 0$. Then we obtain

$$\frac{\partial}{\partial w} mP_{r,k}(w; \omega) = \sum_{\mu=0}^{m-1} (kc^m_{m-\mu,k} - (\mu+1)c^m_{m-\mu-1,k}) \log m \Gamma_{r,k-1}(w; \omega) + kc^m_{0,k} \log m \Gamma_{r,k-1}(w; \omega)$$

and

$$mP_{r,k-1}(w; \omega) = \sum_{\mu=0}^{m-1} c^m_{m-\mu,k-1} \log m \Gamma_{r,k-1}(w; \omega) + c^m_{0,k-1} \log m \Gamma_{r,k-1}(w; \omega).$$
Therefore we only have to show
\[ kc_m^{\mu,k} - (m - \mu + 1)c_m^{m-1,k} + c_{\mu,k-1}^m = 0 \]
for \( \mu \geq 1 \) and
\[ kc_0^m + c_0^{m-1} = 0. \]
These identity are easily derived from a relation
(2) \[ kH_k(\mu) - H_k(\mu - 1) - kH_{k-1}(\mu) = 0. \]

If we put
\[ S_{m,k}(x) = \sum_{\mu=0}^{m} mQ_k(x)c_m^{m-\mu,k}, \]
we can obviously write the definition of \( mP_{r,k} \) as
\[ \log mP_{r,k}(w; \omega) = \int_{I(\lambda, \infty)} f_\omega(t)e^{-\omega t}k^{-1}S_{m,k}(\log t)\,dt. \]

3. PROOF OF THE MAIN THEOREM

In this section, we prove some important fact about \( S_{m,k}(x) \) and our main theorem. The following proposition plays an important role in proof of the main theorem.

**Proposition 3.1.** The polynomial \( S_{m,k}(x) \) is independent on \( k \).

**Proof.** Let \( F_k(s) \) be the generating function of \( H_k(\mu) \) for fixed \( k \):
\[ F_k(s) = \sum_{\mu=0}^{\infty} H_k(\mu)s^\mu. \]
From an easy identity
(3) \[ \sum_{m=0}^{\infty} \frac{S_{m,k}(x)}{m!} s^m = \frac{(-1)^k}{k!} F_k(s) \frac{e^{sx}}{\Gamma(s-k)(e^{2\pi i s} - 1)}, \]
we only have to show that the right side of (3) is independent on \( k \). Since (2), it follows that
\[ F_k(s) = F_{k-1}(s) = \frac{s}{k} F_k(s). \]
This relation and an identity \( F_0(s) = 1 \) shows that
\[ F_k(s) = \frac{k!}{(1-s)_k}, \]
where \( (a)_k = a(a+1) \cdots (a+k-1) \) is the Pochhammer symbol. Hence we have
(4) \[ \frac{(-1)^k}{k!} F_k(s) \frac{e^{sx}}{\Gamma(s-k)(e^{2\pi i s} - 1)} = \frac{e^{sx}}{\Gamma(s)(e^{2\pi i s} - 1)}. \]
This is clearly independent on \( k \). \( \square \)

**Corollary 3.2.** For \( m \geq 0 \), we have \( S_{m,k}(x) = mQ_0(x) \).
Proof. The right side of (4) is equal to
\[ \sum_{m=0}^{\infty} \frac{m!}{m!} Q_0(x)^{m}. \]
\[ \qed \]

Proof of Theorem 1.4. From Theorem 3.1, we obtain
\[ m P_{r+l,k}(w + a; (\omega, \alpha)) = \int_{I(\lambda, \infty)} f_\omega(t) e^{-wt} \left( \sum_{N=r+k+1}^{\infty} a_{l,N}(a; \alpha)t^N \right) t^{-k-1} s_{m,k}(\log t) \ dt 
+ \sum_{N=-l}^{r+k} a_{l,N}(a; \alpha) m P_{r,-N}(w; \omega). \]
(5)

From an identity
\[ \int_{I(\lambda, \infty)} f_\omega(t)e^{-wt} \left( \sum_{N=r+k+1}^{\infty} a_{l,N}(a; \alpha)t^N \right) t^{-k-1}(\log t)^{\nu} \ dt 
= \sum_{D=0}^{\nu-1} \binom{\nu}{D}(2\pi i)^{\nu-D} \int_0^{\infty} f_\omega(t) e^{-wt} \left( \sum_{N=r+k+1}^{\infty} a_{l,N}(a; \alpha)t^N \right) t^{-k-1}(\log t)^{D} \ dt, \]
the first term of the right side of (5) is \( O((\log w)^{m-1}/w). \) \[ \qed \]

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