Dispersion properties of uniform trapezoidal optical waveguides

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Abstract. A method for calculation of the propagation constant of guided modes in the optical waveguides with trapezoidal cross-sections is proposed. The method enables to calculate a propagation constant correction factor that arises due to perturbations of the cross-section of a reference rectangular waveguide. The correction factor is analytically calculated within the framework of the coupled-mode theory. A refined form of the coupling coefficient is obtained owing to the application of a concept of effective excitation sources and accurate account of boundary conditions. An impact of the waveguide cross-section shape on its dispersion properties is analysed. The necessity to take into account the proposed form of the coupling coefficient is demonstrated by comparison of the simulation results with other methods of dispersion calculation.

1. Introduction

A thin-film dielectric optical waveguide is the key element of the integrated microwave photonics (IMWP) circuitry [1, 2] from which ring resonators, bend waveguide sections and Bragg gratings can be built. Specific nature of the planar technology used to produce the optical waveguides often results in deviation of their cross-section from an intended rectangular shape [3]. Such a deviation needs to be taken into account as it affects the dispersion of waveguide modes. Hence, it is necessary that already existing theories for calculation of mode dispersion be augmented.

Besides numerical modesolvers there exist several analytical techniques for guided modes propagation constant calculation in optical waveguides with arbitrary cross-sections. They include various forms of point-matching method [4–6], a perturbation theory method [7], a finite difference method [8], and an equivalent circuit method [9] to name a few. It is worth noting that methods [4–6] are rather cumbersome and numerically unstable [10] which may lead to an imprecise estimation of the propagation constant.

In this paper, we turn our attention to a calculation method based on the coupled-mode theory combined with the concept of effective excitation sources [11]. This method allows for the analytical description of dispersion properties of the waveguides with arbitrary behaviour of their cross-section.

2. Dispersion characteristics of a rectangular dielectric waveguide

In this section we analyse properties of the guided modes in the reference lossless dielectric waveguide with a rectangular cross-section (figure 1). The waveguide core has width 2a,
height $2b$, and dielectric permittivity $\varepsilon_1$. The permittivities of cladding regions outside of
the waveguide core are $\varepsilon_0 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 \equiv \varepsilon_2$. In this section we follow the method proposed
in [12] which is a modification of the Marcatili’s method of approximate mode analysis.

The guided mode inside wavenumber $k_1$ can be expressed via its components the following way:

$$k_1^2 = \omega^2 \varepsilon_1 \mu_0 = k_{1x}^2 + k_{1y}^2 + \beta^2,$$

where $\mu_0$ stands for the vacuum permeability, and $\beta$ is the sought longitudinal propagation
constant. Within the framework of the method the transversal distributions of electric and
magnetic fields in the waveguide core (regions 1, $-a < x < a$ and $-b < y < b$) are represented
as the following products:

$$\vec{E}_{1z} = E_1^{m} \sin(\varepsilon k_{1x} \cos(y k_{1y})$$

$$\vec{H}_{1z} = H_{1}^{m} \cos(\varepsilon k_{1x} \sin(y k_{1y})),$$

where $E_1^{m}$ and $H_1^{m}$ are field amplitudes. The following derivation concerns two fundamental
guided modes, namely $E_{1x}$ (the upper line in (2)) and $E_{1y}$ (the lower line in (2)). The derivation
for the other modes is made in a similar manner. Four combinations of trigonometric functions
are distinguished in the boundary value problem, each describing a set of guided eigenmodes,
which together form an infinite set of guided modes.

Outside the waveguide core, in the regions 3 ($x \geq a, 0 < y < b$) and 0 ($y \geq b, 0 < x < a$), the fields decay exponentially. Transversal electric field distributions are represented the following way:

$$\vec{E}_{3x} = \vec{E}_{1x}(a, y) \exp[-(x - a) k_{3x}], \quad \vec{E}_{0y} = \vec{E}_{1y}(x, b) \exp[-(y - b) k_{0y}].$$

Expressions for the magnetic field distribution have a similar form. In (3) $k_{3x}$ and $k_{0y}$ are the transverse components of the guided mode outside wavenumber $k_0 = \omega^2 \varepsilon_0 \mu_0$. Their respective expressions are:

$$k_{0y} = \sqrt{\omega^2 (\varepsilon_1 - \varepsilon_0) \mu_0 - k_{1y}^2},$$

$$k_{3x} = \sqrt{\omega^2 (\varepsilon_1 - \varepsilon_3) \mu_0 - k_{1x}^2}.$$

In four corner regions ($x \geq a$ and $y \geq b$, etc) the field values are considered to be infinitesimally
small. This assumption restrains the applicability of the method to well-confined guided modes.

Transversal field components can be expressed via the longitudinal ones from Maxwell
equations. Imposition of the continuity boundary conditions on the transverse field components
along $x = a$ and $y = b$ waveguide walls produces a system of equations for $k_{1x}$ and $k_{1y}$:

$$\begin{cases}
k_{1x} k_{3x} \cot(ak_{1x}) - k_{1y} k_{2y} \tan(bk_{1y}) \pm k_{2t}^2 = 0, \\
k_{1x} k_{3x} \tan(ak_{1x}) - k_{1y} k_{2y} \cot(bk_{1y}) \mp \varepsilon_r k_{2t}^2 = 0,
\end{cases}$$

where $\varepsilon_r = \varepsilon_1/\varepsilon_2$, and $k_{2t}$ is a transversal outside wavenumber:

$$k_{2t}^2 = k_0^2 (\varepsilon_1 - \varepsilon_2) - k_{1x}^2 - k_{1y}^2.$$
3. Introduction of effective sources

3.1. Modal expansions of fields

In this section a coupled-mode theory formalism is introduced that allows us to describe the waveguide shape perturbation in terms of effective excitation sources. First, we write down the electric and magnetic fields as eigenmode expansions [13]:

\[ E = \sum_n A_n \hat{E}_n e^{-i\beta_n z}, \quad H = \sum_n A_n \hat{H}_n e^{-i\beta_n z}, \quad (7) \]

where \( A_n(\omega) \) are mode excitation amplitudes, \( \hat{E}_n \) and \( \hat{H}_n \) are the fields of waveguide eigenmodes derived in the previous section. The eigenmodes form a Hilbert space spanned on basis functions \{ \hat{E}_n, \hat{H}_n \}. These basis functions satisfy a normalisation condition:

\[ \int_C [\hat{E}_m^* \times \hat{H}_n + \hat{E}_n^* \times \hat{H}_m^*] \cdot e_z \, dC = \delta_{mn} N_{mn}, \quad (8) \]

where \( N_{mm} \equiv N_m \) is the \( m \)th mode norm, which is related to power flow density of the mode. Integration in (8) is performed over all of the regions where power is being carried by guided modes. In expansions (7) radiation modes are not explicitly emphasized. However, they can be taken into account by considering the summation signs in generalized sense, including integration over a continuous argument.

The calculation of dispersion properties of a trapezoidal waveguide is based on the introduction of the correction factor for the propagation constant of the reference rectangular waveguide subjected to cross-section shape perturbation. This perturbation gives rise to effective excitation sources inside the waveguide. Note that excitation can result either from the presence of real electromagnetic field sources or waveguide medium parameter modifications. Any excitation type is described by means of the excitation sources that enter Maxwell equations.

Second, following the coupled-mode formalism, inside the excitation regions amplitudes \( A_n \) obtain longitudinal dependence, and expansions (7) are supplemented with longitudinal fields. Hence, inside the excitation regions expansions of the electric and magnetic fields assume the form:

\[ E = \sum_n A_n(z) \hat{E}_n e^{-i\beta_n z} + E_b, \quad H = \sum_n A_n(z) \hat{H}_n e^{-i\beta_n z} + H_b, \quad (9) \]

where \( E_b \) and \( H_b \) are orthogonal complementary fields. They represent an orthogonal complement to the eigenmode Hilbert space.

3.2. Effective excitation sources

The perturbation of the medium modifies field distributions and gives rise to excess inductions in the form of the constitutive equations as follows:

\[ \Delta D = \Delta \varepsilon E, \quad \Delta B = \Delta \mu H, \quad (10) \]
where
\[ \Delta \varepsilon = \varepsilon_p - \varepsilon, \quad \Delta \mu = \mu_p - \mu, \tag{11} \]
where \( \varepsilon_p \) and \( \mu_p \) are the permittivity and permeability of a perturbed medium, while \( \varepsilon \) and \( \mu \) are the permittivity and permeability of a non-perturbed medium. At optical frequencies \( \mu_p = \mu = 1 \), which gives \( \Delta \mu = 0 \). Hence, in our case the excess induction \( \Delta B = 0 \).

It worth noting that the excess inductions employed here are similar to the notion of polarization perturbation introduced by Yariv in [16]. The excess inductions produce excess electric and magnetic displacement currents:
\[ J_e^b = i\omega \Delta D, \quad J_m^b = i\omega \Delta B = 0, \tag{12} \]
where \( b \) subscript emphasizes the bulk nature of the currents. We introduce the effective excitation sources in the form of the excitation currents to the Maxwell’s equations as follows:
\[ \nabla \times E = -i\omega B, \quad \nabla \times H = i\omega D + J_e^b, \tag{13} \]

By substitution of (9) in (13) the effective bulk electric current can be expressed in terms of the electrical orthogonal complementary field:
\[ E_b = -\frac{1}{i\omega \varepsilon} J_{k_z}^e, \quad H_b = 0. \tag{14} \]

3.3. Application of boundary conditions

Next, we demonstrate that besides bulk currents, the effective excitation sources are to include surface currents as well. Since the fields inside the excitation region (7) differ from the undisturbed fields (9) by the \( E_b \), we apply the boundary conditions on the contour \( L \) that encloses the cross-section of excitation region \( S \) (figure 2):
\[ n^+ \times E^+ + n^- \times E^- = -J_m^s, \quad n^+ \times H^+ + n^- \times H^- = J_e^s, \tag{15} \]
where \( s \) subscript underlines the surface nature of the currents. By substituting eigenmode decompositions (7) and (9) into the boundary conditions we obtain an expression for the effective surface currents in the following form:
\[ J_e^s = -n \times E_b \bigg|_L = -\frac{e_z \times n}{i\omega \varepsilon} J_{k_z}^e \bigg|_L, \quad J_m^s = n \times H_b \bigg|_L = 0, \tag{16} \]
where \( e_z \) is a longitudinal unit vector, and \( n \) is an outward normal to \( S \).

4. Correction factor for the propagation constant

To derive of the set of coupled mode equations for the perturbed waveguide it is necessary to obtain the expansions of effective excitation sources in terms of the eigenmodes \( \{ \hat{E}_n, \hat{H}_n \} \) of the unperturbed system. This is achieved by plugging in eigenmode expansions (7) into the reciprocity principle in a conjugate form which gives rise to an excitation equation for \( m \)th waveguide mode. The said derivation is rather bulky and can be referenced in a general form in the work already cited [17]. From there, we find the expression that describes excitation of \( m \)th waveguide mode by the set of all eigenmodes:
\[ da_m/dz = -i\beta_m a_m(z) + \sum_n c_{mn} a_n(z), \tag{17} \]
where \( a_n(z) = A_n(z) \exp(-i\beta_n z) \), and \( c_{mn} \) is the intermodal coupling coefficient that consists of two terms, each produced by the corresponding excitation current:
\[ c_{mn} = c_{mn}^{bulk} + c_{mn}^{surf}. \tag{18} \]
Note that the coupling coefficient obtained in our work differs from conventionally used one [18] by the surface constituent. Its occurrence in intermodal coupling is due to the introduction of effective excitation sources and their description in terms of orthogonal complementary fields. The expressions for the constituents of the coupling coefficient have the following form:

\[
c_{mn} = -\frac{i\omega}{N_m} \int_S (\Delta \vec{\tau}_c \cdot \hat{\mathbf{E}}_m) \cdot \hat{\mathbf{E}}_m^* dS - \frac{1}{N_m} \int_L (\Delta \vec{\tau}_c \cdot \hat{\mathbf{E}}_n) \cdot \hat{\mathbf{H}}_m^* dL,
\]

where \(N_m\) is a modal norm, \(\Delta \vec{\tau}_c\) and \(\Delta \vec{\tau}_c\) are static surface coupling tensors that describe geometrical perturbations of a waveguide. Their occurrence is not necessarily related to tensorial properties of a medium, but considerably clarifies the formulae. In the case of a trapezoidal cross-section the coupling tensors assume the following form:

\[
\Delta \vec{\tau}_c = \Delta \vec{\tau}_c \left[ \hat{\mathbf{1}} - \mathbf{e}_t \mathbf{e}_z \frac{\varepsilon_1}{\varepsilon_2} \right], \quad \Delta \vec{\tau}_c = \varepsilon_2 \mathbf{e}_z - \varepsilon_1 \frac{\mathbf{e}_z}{\varepsilon_2} \right|_L,
\]

where \(\mathbf{1}\) is a unity matrix, \(\mathbf{e}_t\) is a unit vector tangent to the contour \(L\), \(\mathbf{e}_z\) is a dyad.

To define integration limits in (19) the lateral walls of the trapezoidal cross-section are to be described by means of functions \(s_L(y)\) and \(s_R(y)\) (figure 2). In such a case the integrals (19) assume the following form:

\[
\begin{align*}
\varepsilon_{nm}^{\text{bulk}} &= -\frac{i\omega}{N_m} \int_a^{-a+s_L(y)} \int_{-b}^b \hat{\Psi} dxdy + \frac{i\omega}{N_m} \int_a^{a-s_R(y)} \int_{-b}^b \hat{\Psi} dxdy, \\
\varepsilon_{nm}^{\text{surf}} &= -\frac{1}{N_m} \int_L \hat{\Xi} dL + \frac{1}{N_m} \int_R \hat{\Xi} dR,
\end{align*}
\]

where

\[
\hat{\Psi} = (\varepsilon_1 - \varepsilon_2) (\hat{\mathbf{E}}_{mt}^* \cdot \hat{\mathbf{E}}_{nt} + \varepsilon_1 \varepsilon_2 \hat{\mathbf{E}}_{mz}^* \hat{\mathbf{E}}_{nz}), \quad \hat{\Xi} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} \left( \hat{\mathbf{H}}_m^* \cdot \mathbf{e}_t \right) \hat{\mathbf{E}}_{nz}.
\]

In the last expression the notation of \(\hat{\mathbf{E}}_{nt}\) is introduced from the expression \(\hat{\mathbf{E}}_n = \hat{\mathbf{E}}_{nt} + \mathbf{e}_z \hat{\mathbf{E}}_{nz}\).

The correction factor for \(m\)th guided mode can be obtained by calculating the integrals (21) for the case of self-coupling, i.e. for \(m = n\). The propagation constant of a guided mode of a non-rectangular waveguide \(\beta_{nm}^\prime\) is related to the propagation constant of the reference waveguide \(\beta_m\) by means of the self-coupling coefficient:

\[
\beta_{nm}^\prime = \beta_m + ic_{nm}.
\]

This last expression can be derived from (17) by taking element \(c_{nm}a_m\) out of the summation and neglecting the modes that differ considerably in their \(\beta_m\) value.

5. Simulation results

We compare the propagation constant values calculated using our method with the MPB modesolver [19] as well as with the results of point-matching method. On figure 3 the good correspondence is seen between normalized propagation constant values \(\beta\) in the range of normalized frequency \(V\) from 0.6 to 1.6. The lower frequency limit is imposed by the applicability of the method of approximate mode analysis, while the upper limit is imposed by the onset of the higher-order guided modes that need to be taken in consideration in (17). On figure 4 the propagation constant deviation on 1550 nm wavelength (\(\Delta \beta_{1550} = \frac{\beta_{1550} - \beta_{1550}^\text{MPB}}{\beta_{1550}^\text{MPB}} \times 100\%\)) is plotted for different waveguide cross-sections. In can be seen that account of the surface coupling term enables to more precisely predict propagation constant values. This can be justified by referencing to (22) where the surface term integrand includes the magnetic field which the common coupled-mode formulation does not take into account.
6. Conclusion
The method for calculation of the propagation constant of the dielectric optical waveguides with trapezoidal cross-sections is put forward. The coupled-mode theory together with the concept of effective excitation sources are employed. It worth noting that the method permits to treat waveguide structures with the perturbations of different origin, e.g. the periodical modulation of the waveguide cross-section or periodical modulation of the dielectric permittivity of the waveguide material, waveguide bends, the presence of external real excitation currents of different nature, etc.

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