Multiparticle singlet states cannot be maximally entangled for the bipartitions

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One way to explore multiparticle entanglement is to ask for maximal entanglement with respect to different bipartitions, leading to the notion of absolutely maximally entangled states or perfect tensors. A different path uses unitary invariance and symmetries, resulting in the concept of multiparticle singlet states. We show that these two concepts are incompatible in the sense that the space of pure multiparticle singlet states does not contain any state for which all partitions of two particles versus the rest are maximally entangled. This puts restrictions on the construction of quantum codes and contributes to discussions in the context of the AdS/CFT correspondence and quantum gravity.

I. INTRODUCTION

The notion of multiparticle entanglement is relevant for different fields in physics, such as condensed matter physics or quantum information processing. A fundamental problem, however, lies in the exponential scaling of the dimension of the underlying Hilbert space, rendering an exhaustive classification difficult. So, in order to gain insight into multiparticle entanglement phenomena as well as to identify potentially interesting quantum states, different concepts based on symmetries [1–5], graphical representations [6–8], matrix product approximations [9, 10] or entanglement quantification [11–16] can be used.

Indeed, it is a natural question to ask for states with maximal entanglement. A pure two-particle quantum state is maximally entangled if the reduced state for one particle is maximally mixed [17]. One can extend this definition to the multiparticle case by considering bipartitions of the particles into two groups and asking whether the global state is maximally entangled for the bipartitions. A multiparticle state is then absolutely maximally entangled (AME) if it is maximally entangled for all bipartitions. A weaker requirement is that all bipartitions of $k$ particles versus the rest shall be maximally entangled, these states are called $k$-uniform.

As AME states and $k$-uniform states are central for quantum error correction, they are under intensive research [18–20]. Indeed, some recently solved questions concerning the sheer existence of AME states have been highlighted as central problems in quantum information theory [21, 22]. In addition to their use in quantum information processing, AME states, under the name of perfect tensors, have been used to construct toy models for the AdS/CFT correspondence [23, 24] and to study the entanglement entropy in conformal field theories [25].

A different concept to explore the phenomena of multiparticle entanglement is the use of symmetries. Here, the study of unitary invariance has some tradition [1, 2, 26, 27]. Consider the two-qubit singlet state

\[ |\psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \]

This is not only a maximally entangled state (as the reduced density matrices are maximally mixed), it also has the property that it is form-invariant under simultaneous application of local unitaries, that is, $U \otimes U |\psi^-\rangle = e^{i\theta} |\psi^-\rangle$. Historically, this property was the key to understand the difference between quantum entanglement and the violation of Bell inequalities [1].

For more particles, one finds more states with this kind of unitary symmetry, and such states are called general singlet states. For three three-level systems, there is the totally antisymmetric state \[ |\psi_3\rangle = (|012\rangle + |201\rangle + |120\rangle - |210\rangle - |102\rangle - |021\rangle) / \sqrt{6}, \]

and for four qubits, the unitarily invariant states form a two-dimensional subspace [28–31], spanned by a two-copy extension of the singlet state (1) and the four-qubit singlet state [32, 33]

\[ |\psi_4\rangle = \frac{1}{\sqrt{3}} (|0011\rangle + |1100\rangle - \frac{1}{2} (|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle)). \]

This two-dimensional subspace can be used to encode a quantum bit such that it is immune against collective decoherence [32–34]. In addition, multiparticle singlet states have turned out to be useful for various quantum information tasks, such as secret sharing and liar detection, see Ref. [26] for an overview. Finally, states with unitary symmetry have attracted interest from the perspective of quantum gravity [35–37, 39].

It is a natural question to ask whether these two research lines are connected, in the sense that one can find $k$-uniform states or even AME states which belong the the invariant subspace of singlet states. This question was first studied in Ref. [38] where it has been shown that for four particles and arbitrary dimensions no AME singlet state can be found. Using methods from quantum gravity it was shown recently in Ref. [39] that for six qubits the singlet states cannot be AME. Combined with known results on the non-existence of AME states, this shows that multi-qubit singlet states cannot be AME (see also [40]).

In this paper we prove that the space of pure multiparticle singlet states of any particle number and any dimension does not contain states which are two-
There are only two functions that are compatible with the definition of the permutation-phase function $f$, namely either $f(\pi) = 1, \forall \pi \in S_d$ or $f(\pi) = \text{sgn}(\pi)$, the signum of the permutation.

**Proof.** Since $\xi$ is a group homomorphism, $f$ is a group homomorphism, too. So we only need to consider the images of the generators of $S_d$ under $f$ in order to characterize $f$.

The symmetric group $S_d$ is generated by the transpositions $\theta_k = (k, k + 1)$ of two neighboring elements with $k \in \{0, \ldots, d - 2\}$, so $f$ is determined by $f(\theta_k)$. Further, the transpositions fulfill the following relations

$$\theta_k^2 = \text{id} \quad (7)$$
$$\theta_k \theta_m = \theta_m \theta_k, \text{ for } |k - m| \geq 2 \quad (8)$$
$$\theta_k \theta_m \theta_k = \theta_m \theta_k \theta_m, \text{ for } |k - m| = 1. \quad (9)$$

Since $f$ is a homomorphism, Eq. (7) gives $f(\theta_k) = \pm 1$ and Eq. (6) gives

$$f(\theta_k) = f(\theta_m) \text{ for } |k - m| = 1. \quad (10)$$

Hence, we only have to distinguish two cases: Either $f(\theta_k) = 1 \forall k$ which implies that $f \equiv 1$ or (2) $f(\theta_k) = -1 \forall k$, which means that $f(\pi) = \text{sgn}(\pi)$. □

**Lemma 1.** There are only two functions that are compatible with the definition of the permutation-phase function $f$, namely either $f(\pi) = 1, \forall \pi \in S_d$ or $f(\pi) = \text{sgn}(\pi)$, the signum of the permutation.

**Lemma 2.** Let $|\psi\rangle = \sum_i t_i |i\rangle \in (\mathbb{C}^d)^{\otimes n}$ be a pure singlet state and $\pi \in S_d$ a permutation. Then it holds that

$$t_{\pi(i)} = f(\pi) t_i \quad (12)$$

where $f$ is the permutation-phase function of $|\psi\rangle$.

**Proof.** Taking the identity $V(\sigma)^{\otimes n} |\psi\rangle = f(\sigma) |\psi\rangle$ and comparing the coefficients in front of the basis vector $|i\rangle$ results in

$$t_{\sigma^{-1}(i)} = f(\sigma) t_i. \quad (13)$$

Choosing $\sigma = \pi^{-1}$ and using the fact that $f(\sigma) = f(\sigma^{-1})$ (which follows from $f(\sigma) = \pm 1$) proves the claim. □
B. The phase function for diagonal unitaries

We now characterize the phase function further by considering the restriction of \( \zeta \) to the diagonal unitaries. The key result is that for a multiparticle singlet state expanded in the computational basis many coefficients have to vanish. We start with a technical lemma.

**Lemma 3.** Let \( |\psi\rangle = \sum_{i} t_{i} |i\rangle \) be a pure \( n \)-particle singlet state. Then, for any \( k \in \{0, \ldots, d - 1\} \), there exists a number \( N_{k} \in \mathbb{N} \) such that for any non-zero coefficient \( t_{j} \neq 0 \), the multi-index \( l \) contains the value \( k \) exactly \( N_{k} \) times.

In other words, the non-vanishing terms of a singlet state in the computational basis are tensor products \( |t_{1}, t_{2}, t_{3}, \ldots, t_{n}\rangle \) of \( N_{0} \) times the single-particle state \( |0\rangle \), \( N_{1} \) times the single-particle state \( |1\rangle \), etc. This naturally implies \( \sum_{k=0}^{d-1} N_{k} = n \).

**Proof.** We consider Eq. (3) for diagonal unitary matrices. Diagonal unitary matrices are generated by the matrices

\[
U_{k} = 1 + (e^{i\phi_{k}} - 1) |k\rangle \langle k|,
\]

where all diagonal entries, except the \( k \)-th one, are 1, while the \( k \)-th diagonal entry is \( e^{i\phi_{k}} \). Such a diagonal matrix acts on a single-particle computational basis vector \( |a\rangle \) as

\[
U_{k} |a\rangle = \begin{cases} 
  e^{i\phi_{k}} |a\rangle, & \text{if } a = k, \\
  |a\rangle, & \text{else}.
\end{cases}
\]

We now determine the phase function \( \zeta(U_{k}) \) from Eq. (3). This yields

\[
U_{k}^{\otimes n} |\psi\rangle = \sum_{l} t_{l} U_{k}^{i_{1}} |i_{1}\rangle \otimes \cdots \otimes U_{k}^{i_{n}} |i_{n}\rangle = \sum_{l} e^{i\phi_{l}} K_{l} t_{l} |l\rangle,
\]

where \( K_{l} \) is the number of times that \( k \) appears in the multi-index \( l \). This equation holds for arbitrary \( \phi_{k} \); moreover, \( \zeta(U_{k}) \) affects all coefficients \( t_{j} \neq 0 \) in the same way. Hence, all indices \( l \), for which \( t_{l} \neq 0 \) must contain \( k \) the same number of times.

In the following, we combine this result with the previous results on local permutations of the basis vectors. Intuitively, one may expect that the numbers \( N_{k} \) in Lemma 3 can not depend on \( k \), as one can always change the index \( k \) by local permutations, without affecting the singlet state too much. This will indeed turn out to be the case.

Before formulating this in a precise manner, note that we can describe the action of a permutation \( \omega \in S_{n} \) of the particles on a multi-index as

\[
\omega(l) = (l_{\omega(1)}, \ldots, l_{\omega(n)}).
\]

In this language, if \( |\psi\rangle = \sum_{i} t_{i} |i\rangle \) is a pure singlet state where \( j, l \) are two multi-indices such that \( t_{j} \neq 0 \) and \( t_{l} \neq 0 \). Then there exists a permutation \( \omega \in S_{n} \) of the particles such that \( j = \omega(l) \). This follows directly from Lemma 3. More generally, we have:

**Lemma 4.** Let \( |\psi\rangle = \sum_{i} t_{i} |i\rangle \) be a pure singlet state. Then there exists a number \( K \in \mathbb{N} \), such that for any non-zero coefficient \( t_{j} \neq 0 \), the multi-index \( l \) contains each value \( k \in \{0, \ldots, d - 1\} \) exactly \( K \) times.

**Proof.** Consider two values \( k, m \in \{0, \ldots, d-1\} \) and let \( N_{k} \) and \( N_{m} \) be defined as in Lemma 3. So, \( N_{k} \) denotes how many times \( k \) is contained in the multi-index \( l \) of any coefficient \( t_{l} \neq 0 \). Now consider the local permutation \( \pi = (k, m) \in S_{d} \), which just replaces \( k \) with \( m \) and vice versa. According to Lemma 2, we have

\[
t_{\pi(l)} = f((k, m)) t_{l}.
\]

Therefore, \( t_{l} \neq 0 \) implies \( t_{\pi(l)} \neq 0 \) due to Lemma 1. However, \( \pi(l) \) contains \( k \) exactly \( N_{m} \) times and \( m \) exactly \( N_{k} \) times. Thus, we must have \( N_{m} = N_{k} \).

It follows that for any pure singlet state \( |\psi\rangle \in (C^{d})^{\otimes n} \), \( n \) is always an integer multiple of \( d \), that is \( n = Kd \) for some \( K \in \mathbb{N} \).

In view of the previous results, the structure of the four-qubit singlet state in Eq. (2) becomes clearer now. For this state, we have \( K = 2 \), so it must be a superposition of tensor products with two \( |0\rangle \) and two \( |1\rangle \) factors. The phase function of two possible permutations is just the identity. Note that Lemma 4 also implies that pure singlet states cannot exist for an odd number of qubits.

III. MAIN RESULTS

With the previous insights, we have characterized pure singlet states for our needs and we can now consider the second property, absolutely maximally entangled states and \( k \)-uniform states.

We start with introducing some notation on truncated multi-indices. Let \( A \subset \{1, \ldots, n\} \) be a set of particles and \( B = A^{c} = \{1, \ldots, n\} \setminus A \) be its complement. For a multi-index \( i = (i_{a})_{a \in \{1, \ldots, n\}} \) we define the truncated multi-indices

\[
i_{A} = (i_{a})_{a \in A} \quad \text{and} \quad i_{B} = (i_{a})_{a \in B}.
\]

Using this notation we express the marginal state of the subsystems contained in \( A \) as

\[
\rho_{A} = \text{Tr}_{B} (|\psi\rangle \langle \psi|) = \sum_{i_{A}, j_{A}} e_{A}(i_{A}; j_{A}) |i_{A}\rangle \langle j_{A}|,
\]

with the coefficients of the marginal density matrix

\[
e_{A}(i_{A}; j_{A}) = \sum_{i_{B}} t_{i_{A}, i_{B}} t_{j_{A}, i_{B}}^{*}.
\]
Now we can define AME states and k-uniform states in a very explicit manner that is useful for our purpose. We call a state \( |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} \) a k-uniform state, if for any set of parties \( A \) with \( |A| \leq k \) the corresponding marginal is a maximally mixed state, that is

\[
\rho_A(i_A; j_A) = \begin{cases} \frac{1}{d}, & \text{if } i_A = j_A \\ 0, & \text{else}. \end{cases} \tag{22}
\]

A quantum state is absolutely maximally entangled (AME) if it is k-uniform with \( k = \lceil n/2 \rceil \).

Note that for multiparticle singlet states the single-particle marginals are always maximally mixed, so they are automatically 1-uniform. This follows from the fact that the single-particle reduced state \( \rho \) is also unitarily invariant, that is \( U \rho U^\dagger = \rho \), and for a single particle this implies that \( \rho_1 = \mathbf{1}/d \). But can singlet states be two-uniform?

We are ready to prove that this is never the case. To introduce the argument in a simplified setting, consider a putative six-qubit singlet state, which should be two-uniform. A six-qubit singlet state has terms consisting of three “0” and three “1” in the computational basis \( (K = 3 \text{ in Lemma 4}) \). For a two uniform state, we should have \( \rho_{1,2} = \mathbf{1}/4 \), fixing the diagonal elements in any basis. So, for the diagonal element \( \rho_{1,2}(0,0; 0,0) \) in the computational basis it should hold that

\[
\rho_{1,2}(0,0; 0,0) = |t_{0,0,0,1,1,1}|^2 + |t_{0,0,1,0,1,1}|^2 \\
+ |t_{0,0,1,1,0,1}|^2 + |t_{0,0,1,1,1,0}|^2 \leq \frac{1}{4}. \tag{23}
\]

If we consider also \( \rho_{1,2}(1,1; 1,1) \) and sum over all qubit pairs \( a, \beta \) we obtain after a short calculation the condition

\[
\sum_{a, \beta} |t_{a, \beta}(0,0; 0,0) + t_{a, \beta}(1,1; 1,1)|^2 = 2 \left( \frac{3}{2} \right) \sum_{i} |t_i|^2 \geq \frac{1}{4} \times 2 \times \left( \frac{6}{2} \right) = \frac{15}{2}. \tag{24}
\]

where the prefactor \( 2 \left( \frac{3}{2} \right) = 6 \) can be understood as coming from the fact that any \( |t_i|^2 \) contributes to \( 2 \left( \frac{3}{2} \right) \) terms of the form \( t_{a, \beta}(\ell, \ell, \ell, \ell) \). But then, this condition is in direct contradiction to the normalization condition \( \sum_{i} |t_i|^2 = 1 \), so a six-qubit singlet state cannot be two-uniform.

This reasoning can directly be generalized to arbitrary dimensions and numbers of particles. Indeed, in the general case with \( K \geq 2 \), we have

\[
\sum_{a, \beta} \sum_{\ell} \rho_{a, \beta}(\ell, \ell, \ell, \ell) = d \left( \frac{K}{2} \right) \sum_{i} |t_i|^2 \geq \frac{1}{d} \times d \times \left( \frac{n}{2} \right)^K, \tag{25}
\]

where, as before, \( K = n/d \). Again, it can easily be seen that this is in contradiction to the normalization \( \sum_{i} |t_i|^2 = 1 \). For the case \( K = 1 \), however, one has \( t_{a, \beta}(0,0; 0,0) = 0 \) for any \( a, \beta \), consequently the marginals cannot be maximally mixed. In summary, we have proved the following:

**Theorem 5.** Let \( |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} \) be a multiparticle singlet state. Then, not all two-particle reduced density matrices can be maximally mixed. In other words, singlet states cannot be two-uniform.

For \( n \geq 4 \) AME states need to be two-uniform, while for \( n \leq 3 \) AME states need to be one-uniform only. This directly implies:

**Corollary 6.** Let \( |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} \) be an AME singlet state. Then either \( n = d = 2 \) or \( n = d = 3 \). For these two cases AME singlet states indeed exist as explained in the introduction. For all other cases of \( n \) and \( d \) there are no AME singlet states.

**IV. CONCLUSION**

We have shown that the space of unitarily invariant pure quantum states does not contain any two-uniform state. Especially, an invariant state cannot be absolutely maximally entangled. This generalizes previous results and closes some debates from the literature [38–40].

There are several ways in which our work can be generalized. First, it would be interesting to weaken the condition of unitary invariance, by considering a subgroup of the unitary group \( U(d) \). This may help to understand the set of decoherence processes, under which code words of a given quantum code can be made robust. Second, the characterisation of entangled subspaces has attracted recent interest [41–43]. In this terminology, we characterized the entangled subspace of multiparticle singlets. It would be very interesting whether similar results can also be obtained for other constructions of entangled subspaces.

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