The Cheshire Cat Bag Model: Color Anomaly and $\eta'$ Properties

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Abstract

We show that color can leak from a QCD bag if we allow for pseudoscalar isoscalar singlet ($\eta'$) coupling at the surface. To enforce total confinement of color an additional boundary term is suggested. New relations between the $\eta'$ mass and decay constant and the QCD gluon condensates are derived and compared with the empirical parameters.

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1. INTRODUCTION

The bag model was originally introduced in order to have color confinement by hand \[1\]: the bag exterior acts as a perfect color dielectric, while the color degrees of freedom are confined to the bag interior. The latter has been achieved by forbidding color to flow from the bag interior to the exterior. This means that, at the surface of the bag, the gluons are subject to the following confining boundary equations

\[ \hat{n} \cdot \vec{E}^a = 0, \quad \hat{n} \times \vec{B}^a = 0, \]  

(1)

where \( \hat{n} \) is the outward normal to the bag surface. The color electric fields \( E^a_\mu \) have to point along the surface of the bag (\( a \) is the color index), while the color magnetic fields \( B^a_\mu \) have to be orthogonal to the surface. For the quarks, the confining boundary equation states that the flux of the isoscalar current through the bag surface is forbidden: \( n_\mu \bar{q} \gamma^\mu q = 0 \), where \( n_\mu = (0, \hat{n}) \) is the space-like 4-vector generalization of the normal vector at the bag surface. In comparison to this traditional picture of the bag model, the bag from the “Cheshire cat” point of view serves just as a formal separation of space-time in a bag region where the underlying QCD still acts, an exterior region where QCD is “integrated out” (practically meaning “replaced”) by an effective action formulated in hadronic degrees of freedom, and a boundary where both regimes are matched so generally that it doesn’t matter where the bag walls are located \[2, 3, 4\]. In other words, the bag itself has no physical significance, only its formalism is used. Therefore the name “Cheshire cat bag” as inspired by the vanishing of the Cheshire cat in the Lewis Caroll’s tale Alice in Wonderland \[5\]. Of course it should be clear that in 3+1 dimensions the Cheshire cat bag can only work approximately in practice, since we know from e.g. large \( N_c \) arguments that an effective action trying to replace QCD must eventually involve infinitely many local terms. In the Cheshire cat formulation of the bag model it is obvious that the confining conditions for gluons and quarks will be not satisfied in general, since the bag wall can be moved at will.

2. TOY MODEL IN 1+1 DIMENSIONS

As mentioned, we can at most expect an approximate formulation of the Cheshire cat bag model in the physical 3+1 space-time, since we have only an approximate bosonization of QCD at our disposal, as inspired by large \( N_c \) arguments. In the 1 + 1 dimensional toy world, however, fermion and boson formulations are exactly equivalent. Because of this, we can expect that the 1 + 1 dimensional analogon of the 3 + 1 Cheshire cat bag will work exactly \[3, 4\]. For that purpose, let us start out with the simplest case: massless fermions in the bag interior, massless bosons in the exterior, and the usual hybrid bag model coupling \[3\] between the fermions and bosons at the boundary in order to guarantee chiral invariance. The bag lagrangian reads therefore

\[ \mathcal{L}_{\text{bag}} = \left( i \bar{\psi} \gamma^\mu \partial_\mu \psi \right) \Theta_{\text{in}} - \frac{1}{2} \bar{\psi} e^{i \gamma_5 \phi / f} \psi \Delta_S + \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \Theta_{\text{out}} \]  

(2)

where \( \psi \) is the fermion field, \( \phi \) the (pseudo-)scalar boson field, \( \Theta_{\text{in/out}} \) are step-functions with support on the bag interior/exterior and \( \Delta_S \) is the surface delta-function. We know
that massless fermions can be exactly bosonized into massless bosons – both theories
are equivalent. Thus we can ask the question what will happen to a e.g. right-moving
(right-handed) fermion wave packet prepared in the bag interior, when it hits the bag
wall \cite{4}. The underlying bosonization arguments would tell that the fermion wave packet
should pass the bag wall in an undisturbed fashion. On the other hand, the linear quark
boundary condition

\[ -i\gamma_a \gamma^\mu \psi(t, x_{bw}) = e^{i\gamma_5 \phi(x_{bw})/f} \psi(t, x_{bw}) \]  

(3)

in the Weyl representation for the $\gamma$-matrices implies that in the usual static bag approx-
imation the right moving (right handed) quark wave packet $\psi_R$ is – modulo a constant
phase shift – completely reflected into a left moving (left handed) one $\psi_L$ at the bag
surface:

\[ i\psi_L(t + x) = e^{i\phi/f} \psi_R(t + x - 2x_{bw}). \]  

(4)

This paradox can be resolved, if the boundary condition is generalized to involve time-
dependent $\phi$-fields, $\psi(t, x_{bw})$. In this case the $\phi$-field can be Taylor-expanded around the
collision time $t_0$ of the center of the wave packet to linear order such that the reflected
particle gets an extra time-dependent (an via translation also space-dependent) phase
factor $\exp(i(t + x)\dot{\phi}/f)$. This means that the wave packet is not only reflected at the
bag wall, but also acquires an extra momentum $\Delta p = \dot{\phi}/f$ and an extra energy kick
$\Delta E = -\dot{\phi}/f$ via the time-dependent boundary. At the same time the temporal shift in $\phi$
causes a boson excitation to be sent to the right. Using phase space arguments it can be
shown that when exactly one fermion state is “drowned” in the Dirac sea ($\Delta Q = 1$), the
boson excitation carries exactly one winding number, $\Delta \phi = 2\pi f$:

\[ \Delta Q = \frac{1}{\hbar} \Delta p \Delta x = \frac{\dot{\phi}}{2\pi f} c \Delta t \equiv \dot{Q} \Delta t, \]  

(5)

with Planck’s constant $\hbar = 2\pi$ and the velocity of light $c = 1$. Thus the paradox is solved by
“drowning” the reflected fermion wave packet in the Dirac sea and by simultaneously
exciting a soliton boson configuration which carries now (via bosonization) the informa-
tion that originally a right-moving fermion wave packet was created. From eq.(5) one reads off the following fermion number anomaly, $\dot{Q} = \dot{\phi}/(2\pi f)$ valid at the bag wall. Let
us now extend the model by coupling a U(1) gauge field $A_\mu(x, t)$ to the fermions in the
bag interior, $e \bar{\psi} \gamma^\mu A_\mu \psi$. This so-called Schwinger model can still be exactly bosonized into
a – now – massive boson field in case the boson mass in the term $\frac{1}{2} m^2 \phi^2$ is fine-tuned to
$m = e/\sqrt{\pi}$. Is the usual hybrid bag coupling (see eq.(3)) still sufficient to guarantee a
Cheshire cat bag situation? The answer is no: Since the fermions carry not only fermion
number charge, but now also electric charge, the fermion number anomaly (5) induces an
anomaly in the electric charge $e \int d^4 x' \dot{\phi}/(2\pi f)$. This non-conserved electric charge must
be compensated by a counter term which can only act at the bag surface, since the boson field itself is charge-neutral and since the fermion number anomaly (and thus the charge
anomaly, too) acts at the boundary: $\mathcal{L}_{CT} = e A_\mu \dot{\phi}/(2\pi f) \Delta S$. Covariantly, the counter
lagrangian reads

\[ \mathcal{L}_{CT} = -\frac{e}{2\pi} \epsilon^\mu_{\nu\rho} n_\nu A_\mu \frac{\dot{\phi}}{f} \Delta S, \]  

(6)
where $\epsilon^{\mu\nu}$ is the anti-symmetric tensor in 1+1 dimensions. Note that the counter term is not gauge invariant in order to cancel the non-gauge invariance resulting from the charge anomaly of the fermions. As a consequence of the new counter term the electric field $E$ satisfies the following generalized boundary equation

$$E = -\frac{e^\phi}{2\pi f}.$$  \hspace{1cm} (7)

at the bag surface instead of the 1+1 dimensional analog of the M.I.T. boundary equations (1), $E = 0$.

3. GENERALIZATION TO 3+1 DIMENSIONS

We will here only present a heuristic argument for the color anomaly. An exact argument based on the multiple reflection method [8] and completely analogous to the fractional baryon number calculation of Goldstone and Jaffe [9] can be found in ref.[7].

Let us assume that there is a non-zero color magnetic field $\vec{B}^a$ present in the neighborhood of the bag, given in some fixed gauge which will be the same as the gauge of the color charge in the color anomaly. Because of the usual M.I.T. boundary condition for the gluons (1), the color magnetic fields point perpendicular to the bag surface and the quarks close to the bag surface grouped in the corresponding Landau levels will do the same. There are two types of Landau levels, the lowest ones have a linear dispersion, while the rest have a parabolic dispersion. Whereas the latter are passive for the argument to be presented, the former behave exactly as the fermion levels in the 1+1 dimensional bag in case we couple the quarks at the bag surface to the pseudo-scalar isoscalar 3+1 dimensional $\eta'$ field: $-\frac{1}{2}\tilde{g}\exp(i\gamma_5\eta'/f)q\Delta S$. If the $\eta'$ field at the bag surface is time-dependent, the “reflected” quarks moving in the linear Landau levels get again a “phase kick” in energy and momentum and there will be an anomaly in the quark number. The latter will induce a color anomaly, since the quarks carry color charge. Its value can be found by transcribing the integrated fermion number anomaly, $e\phi/(2\pi f)$, from 1+1 dimensions to the 3+1 dimensional situation, $\tilde{g}(s)\eta'/(2\pi f)$ and multiplying it with the number of Landau states $N_F\tilde{g}(s)\cdot \vec{B}^a/(2\pi)$ per area along the bag wall. $N_F$ is the effective number of light flavors and $\tilde{g}(s)$ is the quark-gluon coupling constant ($\tilde{g}(s)^2 = \frac{1}{2}g(s)^2$ in terms of the gluon coupling constant which is defined for a given scale $s$ associated with the size of the bag surface). The result is:

$$\Delta Q^a = N_F \frac{\tilde{g}(s)^2}{8\pi^2} \oint_{\text{Area}} d\Sigma_\beta \vec{B}^a(\beta) \cdot \hat{n}_\beta \frac{\eta'(\beta)}{f} \hspace{1cm} (8)$$

with $d\Sigma_\beta$ denoting the area element at the surface point $\beta$. Thus contrary to the common belief the chiral boundary $\{i\gamma_\mu\gamma^\mu + \exp(i\gamma_5\eta'/f)\}q(\beta) = 0$ ensures the conservation of color charge only at the classical level, but not at the quantum level. In order to restore gauge invariance at the quantum level a non-gauge invariant counter term acting at the bag surface has to be added. Here is the final result of ref.[7] which is formulated in terms...
of the Chern-Simons current $K^n_5 = \epsilon^{\mu\nu\alpha\beta}(A^n_\alpha G^a_{\alpha\beta} - \frac{2}{3} f^{abc} A^n_a A^b_\alpha A^c_\beta)$ and is the non-abelian generalization corresponding to the above derived result:

$$L_{CT} = \frac{g(s)^2}{16\pi^2} \int_{\Sigma} d\Sigma_\beta K^\mu_5 n_\mu \frac{\eta'}{f}$$

(9)

As consequence of the new counter term the restricted M.I.T. boundary equations for the gluons ([1]) are now replaced by

$$\hat{n} \cdot \vec{E}_a = -\frac{N_F g^2(s)}{8\pi^2 f} \hat{n} \cdot \vec{B}_a \eta'$$

(10)

$$\hat{n} \times \vec{B}_a = \frac{N_F g^2(s)}{8\pi^2 f} \hat{n} \times \vec{E}_a \eta'.$$

(11)

Note that for $\eta' = 0$ the old M.I.T. boundary terms are recovered.

4. CHESHIRE CAT DERIVATION OF THE $\eta'$ MASS

In the following we will combine the new derived boundary terms ([10] and [11]) with the Cheshire cat principle in order to derive the $\eta'$ mass. For this purpose the bag is used as a test bag which replaces in a small spatial volume of variable size $V$ the profile of a static $\eta'$ solution which still acts in the remainder of space. This should be done in such a way that after tuning the parameters of the $\eta'$ field (here the mass) it doesn’t matter whether the bag is inserted or not. Integrating the (linearized) static equation of motion of the $\eta'$ field over the volume $V$, we get the following set of relations

$$\int_V m^2 \eta' = \int_V \nabla^2 \eta' = \int_{\partial V} d\Sigma \cdot \nabla \eta' = -\frac{1}{2f} \int_{\partial V} d\Sigma \cdot \vec{j}_5^{\text{bos}}$$

(12)

where in the last step the bosonic expression for the axial current has been inserted. Because of continuity in the axial current at the bag surface, the fluxes of the boson and quark axial current are the same and we have

$$\text{eq. (12)} = \frac{1}{2f} \int_{\partial V} d\Sigma \cdot \vec{j}_5^{\text{quark}} = -\frac{1}{2f} \int_V \text{Anomaly}$$

(13)

where the fact has been used that the axial current is anomalous, $\partial_{\mu} j_5^\mu = \text{Anomaly}$. The term $\int_V \vec{q}^5/(2f)$ which would normally have appeared on the right hand side of the last equation has been dropped, since a time-variation in the axial charge $\int_V q^5$ would be in contradiction to the static ansatz. Since the location and the size of the volume $V$ can be chosen at will, we finally have the local operator identity

$$m^2 \eta' = -\frac{1}{2f} \text{Anomaly}$$

(14)

involving the usual chiral anomaly. Up to this point the derivation is completely general, valid for $1+1$ (where $\eta'$ should be replaced by $\phi$ of course) as well as for $3+1$ dimensions. If we (a) insert for the chiral anomaly the $1+1$ dimensional expression $(e/\pi)E$, (b) relate via the new boundary term ([7]) $E$ locally to $E = -e\phi/(2\pi f)$ and (c) use the fact that in the
1+1 case the decay constant \( f \) is just a \( c \)-number, \( f = 1/\sqrt{4\pi} \) (this can again be derived via the Cheshire cat principle see refs. \([2, 3]\)), we end up with the relation \( m^2 \phi = (e^2/\pi)\phi \). Thus the Cheshire cat derivation in 1+1 dimensions gives the exact Schwinger model result \( m^2 = e^2/\pi \) without ever making explicit use of the underlying bosonization. In the 3+1 case the chiral anomaly has the form \( N_F(g^2/4\pi^2)\vec{E}^a \cdot \vec{B}^a \). Under the M.I.T. boundary terms \([\Box]\) the scalar product \( \vec{E}^a \cdot \vec{B}^a \) would normally be zero; under the new boundary terms \([\Box] \) and \([\Box] \), however, this product can be re-expressed by the gluon condensate and the \( \eta' \) field: \( N_F(g^2/16\pi^2)\langle G^2 \rangle \eta'/f \), see ref. \([10]\) for more details where also the final result for the \( \eta' \) mass in the 3+1 dimensional world is listed:

\[
f^2m^2 = N_F \frac{g(s)^2}{8\pi^2} \langle 0| \frac{g(\mu)^2}{8\pi^2} G(\mu)^2 |0 \rangle.
\]

(15)

Inserting in this relation (which is analogous to the Gell-Mann-Oakes-Renner relations for the octet Goldstone bosons \([\Box]\)) the value \( \langle 0| (\alpha(\mu)/\pi) G(\mu)^2 |0 \rangle \sim (330 \text{ MeV})^4 \) borrowed from the QCD sum rule calculations \([\Box]\) and for \( g(s)^2 \) values for reasonable bag scales (see ref. \([10]\)) one ends up with \( \eta' \) masses between 0.33 GeV and 1.6 GeV compared to the empirical value of 0.96 GeV. In this calculation the \( \eta' \) decay constant was in turn derived from the Cheshire cat principle and had the reasonable value \( f_{\eta'} \equiv \sqrt{2/N_F}f = 100 \text{ MeV} \).

5. CONCLUSION

There is necessarily a color anomaly in any hybrid bag model with a chiral coupling to a pseudoscalar isoscalar \( (\eta') \) field at the bag surface. Consequently, a gauge-dependent counter term is required in order to restore the overall gauge invariance. The counter term induces changes in the gluon boundary equations, see \([\Box] \) and \([\Box] \) which relate the M.I.T. forbidden fields at the bag surface to the M.I.T. allowed ones times the \( \eta' \) field. The old M.I.T. boundary equations \([\Box] \) are recovered for a vanishing \( \eta' \) field. As a consequence non-zero matrix-elements \( \langle 0| \vec{E}^a \cdot \vec{B}^a |\eta' \rangle \) can easily exist in the bag under the new boundary terms in contrast to usual bag calculations. Furthermore, even for a spherical bag the color electric field can now point radially. Finally, using the new boundary terms combined with the Cheshire cat principle the \( \eta' \) mass, and – as shown in ref. \([10]\) – decay constant \( f_{\eta'} \) and 4-point vertex can be derived in reasonable agreement with the empirical information in 3+1 dimensions where the Cheshire cat principle is expected to work approximately and in exact agreement with the bosonization results in the 1+1 dimensional case.

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