CHARACTERS FOR PROJECTIVE MODULES IN THE BGG CATEGORY $\mathcal{O}$ FOR THE ORTHOSYMPELECTIC LIE SUPERALGEBRA $\mathfrak{osp}(3|4)$

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Abstract. We determine the Verma multiplicities of standard filtrations of projective modules for integral atypical blocks in the BGG category $\mathcal{O}$ for the orthosymplectic Lie superalgebras $\mathfrak{osp}(3|4)$ by way of translation functors. We then explicitly determine the composition factor multiplicities of Verma modules using BGG reciprocity.

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1. Introduction

1.1. A central problem in representation theory is understanding the representations of a given algebraic object, like a semisimple Lie algebra, and in particular, determining the irreducible constituents. A class of representations in which this problem is accessible is the BGG category $\mathcal{O}$ of modules of semisimple Lie algebras. This category exhibits rich and deep theory and a broad survey of results can be found in [Hum08]. A generalization of semisimple Lie algebras are basic Lie superalgebras, which exhibit many of the same phenomena (for reference, see [CW12; Mus12]). The BGG category $\mathcal{O}$ can analogously be defined for basic Lie superalgebras, and many of the results from the semisimple case extend. Among the most conceptual objects in this category are the Verma modules. In this paper, we determine Verma multiplicities of standard filtrations of projective modules of integral atypical highest weight in the BGG category $\mathcal{O}$ for the basic Lie superalgebra $\mathfrak{osp}(3|4)$. We then use BGG reciprocity to determine the composition factors in Verma modules.

1.2. Atypicality of weights is a phenomenon present in Lie superalgebras that has no analogue for semisimple Lie algebras. It allows an integral block in $\mathcal{O}$ whose degree of atypicality is greater than 0 to have infinitely many simple modules. The principal block in $\mathcal{O}$ for $\mathfrak{osp}(2m + 1|2n)$, which contains the trivial module, always has nonzero degree of atypicality when $m, n \geq 1$.

Atypicality arises due to the presence of isotropic roots (i.e. roots of length zero) in the root system, which expand the notion of linkage beyond the orbit of the Weyl group. For
\( \mathfrak{osp}(2m + 1|2n) \), the degree of atypicality is an integer in the range 0 to \( \min(m, n) \), inclusive. In the integral case, any typical (i.e. degree of atypicality 0) block can be reduced to the semisimple Lie algebra case via an equivalence of categories (cf. [Gor02b]). Therefore, the new cases arise primarily when the degree of atypicality is nonzero.

1.3. The problem of determining the irreducible representations that appear in a Jordan-Hölder series of a Verma module of a semisimple Lie algebra has a detailed history. For a dominant integral weight, Kazhdan and Lusztig conjectured that these multiplicities could be determined in terms of certain recursively defined polynomials generated from the Weyl group of the semisimple Lie algebra (cf. [LK79]), and this can be extended to an arbitrary integral weight by Jantzen’s translation functors (cf. [Jan79]). The Kazhdan-Lusztig conjecture was proven via geometric methods in the 1980s by Beilinson & Bernstein ([BB81]) and Brylinski & Kashiwara ([BK81]).

Generalizing to the basic Lie superalgebra case has been difficult because the Weyl group no longer solely dictates linkage, but some progress has been made. (cf. [Bru03; BLW16; CLW11; CLW15]). An entirely different approach (and therefore solving the problem for certain semisimple Lie algebras in a novel way) was done for \( \mathfrak{osp}(l|2n) \) by way of quantum symmetric pairs by Bao and Wang (cf. [Bao17; BW18]).

Nonetheless, these methods do not readily offer concrete multiplicities. By way of translation functors, we explicitly compute standard filtration formulae for projectives. This method is used to solve a similar problem for \( \mathfrak{gl}(3|1) \) and \( \mathfrak{gl}(2|2) \) in [Kan19], for \( G(3) \) in [CW18], and \( D(2|1; \zeta) \) in [CW19].

1.4. In this work, we use the tool of translation functors to determine the characters of projective modules in the BGG category \( \mathcal{O} \) for the orthosymplectic Lie superalgebras \( \mathfrak{osp}(3|4) \). Specifically, we explicitly determine the Verma multiplicities of standard filtration of projective modules in atypical blocks in \( \mathcal{O} \). There are infinitely many inequivalent atypical blocks. Then, BGG reciprocity allows us to convert these formulae to formulae for composition multiplicities, which we also explicitly state.

1.5. Our general approach of using translation functors is as follows. Given some projective cover \( P_\lambda \) for which we wish to deduce Verma multiplicities, we find some \( P_\mu \) with known Verma multiplicities and some finite-dimensional representation \( N \) such that the Verma module \( M_\lambda \) appears in a standard filtration of \( P_\mu \otimes N \). If \( \lambda \) is the lowest weight appearing among all the weights linked to \( \lambda \) appearing in the Verma flag, then \( P_\lambda \) is a direct summand for the projection of \( P_\mu \otimes N \) on to the block corresponding to \( \lambda \). In most cases, it is the only direct summand.

A particularly useful set of criteria for determining whether a summand is direct and for verifying indecomposability is stated in Proposition 2.7. These criteria follow from similar criteria on tilting modules (cf. [CW18]) derived from the Super Jantzen sum formula (cf. [Mus12]). Verifying indecomposability is a non-trivial step, as it is not evident whether or not translation functors yield an indecomposable projective. See §2.8 and §2.9 for explicit details and justification.

Our approach shows that in the cases we consider, standard filtrations always have Verma modules with multiplicity 1 or 2. By BGG reciprocity, these formulae determine the composition factors for Verma modules in \( \mathcal{O} \).
1.6. In §2, we recall basic structure theorems for \( \mathfrak{osp}(2m+1|2n) \), fix a Cartan subalgebra, a root system, a fundamental system, and define linkage. Also, we recall the BGG category \( \mathcal{O} \), review relevant results in the super case, and offer conditions when Verma modules appear in the standard filtration of projective modules.

The section §3 contains our original results. We find standard filtration multiplicities for projective modules of atypical integral highest weight when \( g = \mathfrak{osp}(3|4) \). These results are justified using the general facts in §2 and the strategy of translation functors. We then compute the composition multiplicities of Verma modules of atypical integral highest weight for \( \mathfrak{osp}(3|4) \).

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2. Preliminaries

We shall recall elementary properties about the structure of the Lie superalgebra \( \mathfrak{osp}(2m+1|2n) \) and introduce some basic notations.

2.1. Basic definitions. Suppose \( V = \mathbb{C}^{k|l} = \mathbb{C}^k \oplus \mathbb{C}^l \). Let \( \{ \overline{1}, \overline{2}, \ldots, \overline{k} \} \) and \( \{ 1, 2, \ldots, l \} \) parametrize the standard bases for the even and odd subspaces of \( V \), \( \mathbb{C}^k \) and \( \mathbb{C}^l \), respectively. Denote

\[
I(k, l) = \{ \overline{1}, \overline{2}, \ldots, \overline{k}; 1, 2, \ldots, l \}
\]

where we impose the total order

\[
\overline{1} < \cdots < \overline{k} < 0 < 1 < \cdots < l.
\]

The Lie superalgebra \( \mathfrak{gl}(k|l) \) is the Lie superalgebra of \( k \times l \) matrices over \( \mathbb{C} \) with bracket to be defined. The basis \( I(k, l) \) for \( V \) induces a basis for \( \mathfrak{gl}(k|l) \) given by \( \{ E_{ij} : i, j \in I(k, l) \} \), where \( E_{ij} \) is the elementary matrix with a 0 in every entry except for a 1 in the \( i \)-th row and \( j \)-th column \( (i, j \in I(m, n)) \). The even subalgebra \( \mathfrak{gl}(k|l)_0 \) of \( \mathfrak{gl}(k|l) \) has a basis \( \{ E_{ij} : i, j < 0, i, j > 0, i, j \in I(k, l) \} \) and the odd subspace \( \mathfrak{gl}(k|l)_\overline{0} \) has a basis \( \{ E_{ij} : i < 0 < j, j < 0 < i, i, j \in I(k, l) \} \). An element that is either purely even or purely odd is said to be homogeneous, and its parity (denoted \( \vert \cdot \vert \)) is 0 or 1, respectively. The Lie superbracket is defined on homogeneous elements \( x, y \in \mathfrak{gl}(k|l) \)

\[
[x, y] = xy - (-1)^{|x||y|}yx
\]

and extended by bilinearity. Define the supertranspose \( x^{st} \) of an element \( x \in \mathfrak{gl}(k|l) \) in \( (k|l) \)-block form \( x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) by \( x^{st} = \begin{pmatrix} a^t & c^t \\ -b^t & d^t \end{pmatrix} \), where \( t \) denotes the regular transpose. Then, we define the Lie superalgebra \( \mathfrak{osp}(2m+1|2n) \) by stabilizing a non-degenerate even supersymmetric bilinear form as follows:
Furthermore, with this choice of \( h \) we have a triangular decomposition \( \mathfrak{osp}(2m + 1|2n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) and root system \( \Phi = \Phi_\mathfrak{h} \cup \Phi_\mathfrak{T} \) where

\[
\Phi_\mathfrak{T} = \{ \pm 2\delta_j; \ \pm \delta_j \pm \delta_k; \ \pm \epsilon_i; \ \pm \epsilon_i \pm \epsilon_l \}
\]

\[
\Phi_\mathfrak{h} = \{ \pm \delta_j; \ \pm \epsilon_i \pm \epsilon_l \}
\]

is the even and odd root decomposition, where \( 1 \leq j \leq n, 1 \leq j < k \leq n, 1 \leq i \leq m, 1 \leq i < l \leq m \) and signs are taken independently. Call a root \( \alpha \in \Phi \) isotropic if \( (\alpha, \alpha) = 0 \). Let
The weights $\lambda$ and $\mu$ are said to be linked if $\lambda \sim \mu$. It can be shown that linkage is an equivalence relation.

Given a fundamental root system $\Pi$, we can establish the Bruhat order on $\mathfrak{h}^*$ as follows. Let $\lambda, \mu \in \mathfrak{h}^*$. We say $\lambda \geq \mu$ if $\lambda \sim \mu$ and $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Pi$ (i.e. the nonnegative sum of simple roots).
We introduce notation for both convenience and to make the degree of atypicality clear. If $\lambda = \sum_{j=1}^{m} q_j \delta_j + \sum_{i=1}^{n} r_i \epsilon_i \in \mathfrak{h}^*$, use the notation $\lambda = (q_1, q_2, \ldots, q_m \mid r_1, r_2, \ldots, r_n)$. Furthermore, the action of the Weyl group $W$ is clear. We can permute with signs everything to the left of the bar and to the right of the bar, but no coefficient may cross the bar.

The degree of atypicality of the weight $(q_1, q_2, \ldots, q_m \mid r_1, r_2, \ldots, r_n) - \rho$ is read by counting the number of pairs $(q_i, r_j)$ such that $|q_i| \neq |r_j|$, with the important stipulation no $q_i$ or $r_j$ be reused. The corresponding set of mutually orthogonal roots are $\delta_i - \epsilon_j$ if $q_i = -r_j$, and $\delta_i + \epsilon_j$ if $q_i = r_j$ for each pair $(i, j)$. The degree of the atypicality is also given by the size of the multiset $\{|q_i|\}_{i=1}^{m} \cap \{|r_j|\}_{j=1}^{n}$. In particular, if none of the $|q_i|$ coincide with the $|r_j|$, the weight is typical.

2.3. The Lie superalgebra $\mathfrak{osp}(3|4)$. Since the Lie superalgebra $\mathfrak{osp}(3|4)$ is of primary interest, we explicitly restate some of the previous facts for this Lie superalgebra. The even subalgebra is $\mathfrak{h}_e = \mathfrak{osp}(4) \oplus \mathfrak{so}(3)$. We write $\epsilon$ to abbreviate $\epsilon_1$. The positive system is given by $\Phi^+ = \Phi^+_0 \cup \Phi^+_1 = \{2\delta_1, 2\delta_2, \delta_1 \pm \delta_2, \epsilon\} \cup \{\delta_1, \delta_2, \delta_1 \pm \epsilon, \delta_2 \pm \epsilon\}$. The integral weight lattice in $\mathfrak{h}^*$ is given by $X = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\epsilon$. The Weyl vector is given by $\rho = \frac{1}{2}\delta_1 - \frac{1}{2}\delta_2 + \frac{1}{2}\epsilon$. Notice that any vector in $X + \rho$ is half-integer and therefore not orthogonal to the non-isotropic odd roots $\delta_1$ and $\delta_2$.

The weights in $\mathfrak{h}^*$ of $\mathfrak{osp}(3|4)$ are of the form $(a, b \mid c)$ in our notation. We are mainly interested in modules of integral highest weight $\lambda \in X$, frequently written as $\lambda = (a, b \mid c) - \rho$ where $a, b, c \in \mathbb{Z} + \frac{1}{2}$.

The Weyl group is $W = W_{\mathfrak{sp}(4)} \times W_{\mathfrak{so}(3)} \cong (\mathbb{Z}_2^2 \rtimes S_2) \times \mathbb{Z}_2$ is the product of dihedral groups. $W_{\mathfrak{sp}(4)}$ acts on a weight $\lambda = (a, b \mid c)$ by signed permutations of $a$ and $b$ and $W_{\mathfrak{so}(3)}$ acts by sign changes of $c$. A weight $\lambda = (a, b \mid c) - \rho$ is atypical (of degree one) if and only if $c \in \{\pm a, \pm b\}$.

Denote by $r$ the reflection associated with $\delta_1 - \delta_2$, by $s$ the reflection associated with $2\delta_2$, and by $t$ the reflection associated with $\epsilon$. Then, the respective actions on $\mathfrak{h}^*$ are given by permuting $\delta_1$ and $\delta_2$, negating $\delta_2$, and negating $\epsilon$. As a Coxeter group, the Weyl group has a presentation $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^4, (rt)^2, (st)^2 \rangle$. The first two reflections $r$ and $s$ generate $W_{\mathfrak{sp}(4)}$, and $t$ generates $W_{\mathfrak{so}(3)}$. We impose the Bruhat order on $W$, writing $w' \leq w$ if a reduced word for $w'$ appears in some reduced word for $w$ for $w', w \in W$. By the BGG theorem, this order is compatible with the partial order above on $\mathfrak{h}^*$ in the sense that if $\lambda - \rho$ is typical, dot-regular, and antidominant, then $w' \leq w$ if and only if $w'\lambda \leq w\lambda$ (cf. [Hum08]). $W_{\mathfrak{sp}(4)}$ is dihedral and therefore the restricted Bruhat order is determined by comparing the lengths of elements. The Bruhat graph of $W_{\mathfrak{sp}(4)}$ is given below:

![Bruhat Graph](image)

Combined with the fact that $t$ is central, this makes clear the Bruhat order on $W$. 

\[ r \rightarrow rsr \]

\[ s \rightarrow srsr \]
2.4. The BGG category $\mathcal{O}$. From now on, let $\mathfrak{g} = \mathfrak{osp}(3|4) = \mathfrak{g}_\mathfrak{tr} \oplus \mathfrak{g}_\mathfrak{tr}$ with the standard associated bilinear form, root system, and triangular decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that the BGG category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-modules $M$ subject to the following three conditions:

1. $M$ is finitely generated.
2. $M$ is $\mathfrak{h}$-semisimple: $M = \bigoplus_{\lambda \in \mathcal{X}} M^\lambda$, where $M^\lambda = \{v \in M \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$ is a nonzero weight space.
3. $M$ is locally $\mathfrak{n}^+$-finite: $U(\mathfrak{n}^+) \cdot v$ is finite dimensional for all $v \in M$.

Observe that the abelian quotient algebra $\mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$. Thus, any $\lambda \in \mathfrak{h}^*$ naturally defines a one-dimensional $\mathfrak{b}$-module with trivial $\mathfrak{n}^+$-action, which we denote as $\mathbb{C}_\lambda$. Specifically, if $v \in \mathbb{C}_\lambda$, then $h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}$. Now, define

\[(2.15) \quad M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda - \rho},\]

where $\rho$ is the Weyl vector. This is naturally a left $U(\mathfrak{g})$-module. This is called a Verma module of highest weight $\lambda - \rho$.

We let $L_\lambda$ denote the unique simple quotient of $M_\lambda$ of highest weight $\lambda - \rho$, and use the notation $[M_\mu : L_\lambda]$ to denote the multiplicity of $L_\lambda$ in a composition series of $M_\mu$. Such a series exists for all $M$ in $\mathcal{O}$.

In the notation introduced in \[2.2\] if $\lambda = (a, b \mid c)$, write $M_{a,b|c}$ to denote $M_\lambda$ and $L_{a,b|c}$ to denote $L_\lambda$.

2.5. Blocks in $\mathcal{O}$. The integral blocks in $\mathcal{O}$ can be divided into typical and atypical blocks. By definition, any simple module in a typical block has typical highest weight. The typical blocks in $\mathcal{O}$ are described by Gorelik (see section 8.5.1 in [Gor02a] and theorem 1.3.1 in [Gor02b]). Because any $\rho$-shifted integral weight is strongly typical in the sense of [Gor02b], we get

**Proposition 2.1** (Gorelik). Any typical block in $\mathcal{O}$ is equivalent to a block in the BGG category $\mathcal{O}$ of $\mathfrak{g}_\mathfrak{tr}$-modules of integral weights.

For $\mathfrak{osp}(3|4)$, the central characters associated to two different weights are the same if and only they are linked (cf. [CW12]). Therefore, blocks in $\mathcal{O}$ are indexed by linkage classes. In particular, each $a \in \mathbb{Z}_{\geq 0} + 1/2$ specifies a different block $B_a$, with a corresponding linkage class representative given by $(a, b \mid b) - \rho$ with $b \in \mathbb{Z} + 1/2$. All integral atypical blocks are given this way. In particular, the principal block $B_{1/2}$ contains the trivial module.

2.6. Key results in $\mathcal{O}$. The primary means by which the goals of this paper are achieved are by using translation functors. We restate the necessary results to justify our steps. This collection of results is justified in [Hum08, Chap. 1-3] for the BGG category $\mathcal{O}$ for semisimple Lie algebras; similar arguments extend them to the BGG category $\mathcal{O}$ of $\mathfrak{osp}(3|4)$-modules.

**Theorem 2.2.** Let $N$ be a finite dimensional $U(\mathfrak{g})$-module. For any $\lambda \in \mathfrak{h}^*$, the tensor module $T := M_\lambda \otimes N$ has a finite filtration with quotients isomorphic to Verma modules of the form $M_{\lambda + \mu}$, where $\mu$ ranges over the weights of $N$, each occurring $\dim N^a$ times in the filtration.

A module $N \in \mathcal{O}$ has a standard filtration or a Verma flag if there is a sequence of submodules $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = N$ such that each $N_i/N_{i-1}$ $1 \leq i \leq k$ is
isomorphic to a Verma module. The number of times the Verma module $M_\lambda$ appears in a standard filtration of $N$ is denoted by $(N : M_\lambda)$.

It can be shown that the length and the Verma multiplicities in a standard filtration are independent of choice of a standard filtration. Therefore, the following informal notation to indicate a standard filtration of a module is useful. If $M_\lambda$, $\lambda \in h^*$, $1 \leq i \leq k$ are the Verma modules appearing with multiplicity $c_i \in \mathbb{Z}_{>0}$ in a standard filtration of a module $N$, we write:

$$N = c_1 M_{\lambda_1} + c_2 M_{\lambda_2} + \cdots + c_k M_{\lambda_k}$$  \hspace{1cm} (2.16)

Similarly, if $L_{\mu_i}$, $\mu_i \in h^*$, $1 \leq i \leq k$ are the irreducibles appearing with multiplicity $d_i \in \mathbb{Z}_{>0}$ in a composition series of a module $N$, we write

$$N = d_1 L_{\mu_1} + d_2 L_{\mu_2} + \cdots + d_k L_{\mu_k}$$  \hspace{1cm} (2.17)

We let $P_\lambda$ denote the (unique) projective cover for $L_\lambda$ for all $\lambda \in h^*$, that is the indecomposable projective such that $P_\lambda \rightarrow L_\lambda \rightarrow 0$. We recall the following facts about projectives.

1. All projectives have a standard filtration.
2. The category $O$ has enough projectives.
3. If $P = Q \oplus R$ with $P, Q, R \in O$, $P$ is projective if and only if $Q$ and $R$ are projective.
4. If $P \in O$ is projective and indecomposable, then $P \cong P_\lambda$ for some $\lambda \in h^*$.
5. The Verma modules $M_\mu$ which appear in a standard filtration of $P_\lambda$ satisfy $\mu \geq \lambda$ in the Bruhat ordering, and $M_\lambda$ appears with multiplicity 1.

These facts yield the following lemma.

**Lemma 2.3.** If $\lambda - \rho$ is the lowest weight in a standard filtration of a projective object $P$, then $P_\lambda$ is a direct summand of $P$.

The following proposition, which follows from Theorem 2.2, is a critical part of our translation functor arguments.

**Proposition 2.4.** If a projective $P$ has a standard filtration given by $P_\lambda = \sum_{\nu} M_\nu$, the $\nu$ not necessarily distinct, then for any finite-dimensional representation $N$ with weights $\mu$, the standard filtration for $P \otimes N$ is given by $\sum_{\nu} \sum_{\mu} M_{\nu + \mu}$, where $\mu$ appears in the sum with multiplicity given by $\dim N^\mu$.

Knowing the Verma flag structure of typical projectives will be key in determining those of atypical projectives. We have the following lemmas.

**Lemma 2.5.** If $\lambda \in X + \rho$ is such that $\lambda - \rho$ is typical and dot-regular, then the Verma modules that appear in a standard filtration of $P_\lambda$ are of the form $M_{w\lambda}$, where $w \in W$ such that $w\lambda \geq \lambda$, and each Verma module appears with multiplicity 1.

**Proof.** By Proposition 2.1 we have an equivalence of categories to the Lie algebra $\mathfrak{g}_\Sigma = \mathfrak{sp}(4) \oplus \mathfrak{so}(3)$. Since the Weyl group $W$ is the product of dihedral groups, it is well known in this case that the Kazhdan-Lusztig polynomials are all 1 (and for our particular case it can be directly verified by computation). The result follows by the Kazhdan-Lusztig conjecture. \hfill \Box

The lemma also extends to typical and dot-singular weights.
Lemma 2.6. Let $\lambda \in X + \rho$ be such that $\lambda - \rho$ is a typical anti-dominant dot-singular weight. Let $W^{\lambda}$ be a minimal set of left-coset representatives of $W/W_{\lambda}$, where $W_{\lambda} = \{ w \in W \mid w \lambda = \lambda \}$. Then, if $\sigma \in W^{\lambda}$,

\[ P_{\sigma \lambda} = \sum_{\tau \geq \sigma, \tau \in W^{\lambda}} M_{\tau \lambda}. \tag{2.18} \]

Proof. The proof is analogous to that of Lemma 3.5 in [CW18]. Since $\lambda = a \delta_1 + b \delta_2 + c \varepsilon$ with $a, b, c \in \mathbb{Z} + \frac{1}{2}$ is singular and in particular $c \neq 0$, changing the sign of $c$ does not stabilize $\lambda$. Hence, the action of $W_{\mathfrak{so}(3)}$ is always regular. Therefore, $\{ e \} \neq W_{\lambda} \subseteq W_{\mathfrak{sp}(4)}$. The central character corresponding to the integral weight $\lambda - \rho$ is strongly typical in the sense of Gorenik (cf. [Gor02a]) and by Proposition 2.1 we have an equivalence of categories between the block containing the irreducible module $L_{\lambda}$ and a singular integral block of $\mathfrak{sp}(4) \oplus \mathfrak{so}(3)$-modules.

Since the action of $W_{\mathfrak{so}(3)}$ is regular, it suffices to check the analog of (2.18) for a singular integral block of $W_{\mathfrak{sp}(4)}$ modules. Since the corresponding Weyl group is dihedral and the Kazhdan-Lusztig polynomials are 1, the lemma follows by Theorem 3.11.4 in [BGS96]. □

Lastly, we recall BGG reciprocity.

\[ (P_{\lambda} : M_{\mu}) = [M_{\mu} : L_{\lambda}], \quad \lambda, \mu \in \mathfrak{h}^*. \tag{2.19} \]

2.7. Some representations of $\mathfrak{osp}(3|4)$. The strategy of using translation functors involves choosing appropriate representations to tensor with projective modules to produce new modules.

The simplest module we use is the seven-dimensional natural representation $V = \mathbb{C}^{3|4}$ of $\mathfrak{osp}(3|4)$. We also use the adjoint representation (also denoted $\mathfrak{g}$) of $\mathfrak{osp}(3|4)$. Finally, we use the second exterior power $\Lambda^2 V$ of the natural representation (call it the wedge-squared of the natural). In general, the $k$-th exterior power of a vector superspace $W = W_\uparrow \oplus W_\uparrow$ is defined as:

\[ \Lambda^k(W) := \bigoplus_{i+j=k} (\Lambda^i(W_\uparrow) \otimes S^j(W_\uparrow)) \tag{2.20} \]

where $\Lambda^i$ and $S^j$ acting on vector spaces are the $i$-th exterior power and $j$-th symmetric power in the traditional sense, respectively.

In the case of $\mathfrak{osp}(3|4)$, the natural representation has dimension 7, the wedge-squared of the natural has dimension 24, and the adjoint has dimension 25.

2.8. Conditions for nonzero Verma flag multiplicities in projective modules. We have the following proposition, which uses BGG reciprocity to reformulate the conditions for tilting modules in [CW12, Proposition 2.2] as conditions for projective modules.

Proposition 2.7. Suppose that $\lambda \in X, \alpha_i \in \Phi_0^+, 1 \leq i \leq k$, and $\beta, \gamma \in \Phi_1^+$. Let $w = s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1} \in W$.

1. Suppose that $\langle \lambda, \alpha_i^\vee \rangle < 0$. Then $(P_{\lambda} : M_{s_{\alpha_1} \lambda}) > 0$.
2. Suppose that $\langle s_{\alpha_{i-1}} \cdots s_{\alpha_1} \lambda, \alpha_i^\vee \rangle < 0$ for all $i = 1, 2, \ldots, k$. Then $(P_{\lambda} : M_{w \lambda}) > 0$.
3. Suppose that $(\lambda, \beta) = 0$. Then $(P_{\lambda} : M_{\lambda+\beta}) > 0$.
4. Suppose that $(\lambda, \beta) = 0$ and $\langle s_{\alpha_{i-1}} \cdots s_{\alpha_1} (\lambda + \beta), \alpha_i^\vee \rangle < 0$ for all $i = 1, 2, \ldots, k$. Then $(P_{\lambda} : M_{w (\lambda+\beta)}) > 0$. 
(5) Suppose that \((\lambda, \beta) = (\lambda + \beta, \gamma) = 0\) and \(\text{ht}(\beta) < \text{ht}(\gamma)\). Then \((P_\lambda : M_{\lambda+\beta+\gamma}) > 0\).

(6) Suppose that \((\lambda, \beta) = (\lambda + \beta, \gamma) = 0\), \(\text{ht}(\beta) < \text{ht}(\gamma)\), and \((s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda + \beta + \gamma), \alpha_i^\vee) < 0\) for all \(i \in 1, 2, \ldots, k\). Then \((P_\lambda : M_{\lambda+\beta+\gamma}) > 0\).

Proof. The proposition is originally derived using the Super Jantzen sum formula (cf. Gor02a, Mus12), giving conditions for composition factors. BGG Reciprocity (2.19) immediately translates the conditions from those on tilting modules to those on projective modules. □

We now rederive a well-known but useful corollary, which goes back to a fundamental lemma of Penkov and Serganova.

Corollary 2.8. Suppose \(\lambda - \rho \in \mathfrak{h}^*\) is atypical. Then \(P_\lambda\) must have a Verma flag of length greater than 1.

Proof. \(M_\lambda\) appears in the standard filtration. Furthermore, because \(\lambda - \rho\) is atypical, there exists \(\beta\) such that \(\beta \in \Phi_+^\vee\) and \((\lambda, \beta) = 0\). Therefore, apply Proposition 2.7(4) to see that \(M_{\lambda+\beta}\) also appears in the standard filtration. □

2.9. Strategy. Given an atypical \(\lambda - \rho \in \mathfrak{h}^*\), we seek to deduce the standard filtration formula of \(P_\lambda\). To do so, we choose a \(\mu \in \mathfrak{h}^*\) such that we know a standard filtration for \(P_\mu\). This is often accomplished by letting \(\mu := \lambda - \nu\), where \(\nu\) is a weight (often the lowest) in some finite-dimensional representation \(W\) such that \(\mu - \rho\) is typical; Lemma 2.3 and Lemma 2.6 tell us the structure of \(P_\mu\). Proposition 2.4 can be used to deduce the Verma modules which appear in a standard filtration of the projective \(P_\mu \otimes W\), which must include \(M_\lambda\). Our next step is to project \(P_\mu \otimes W\) onto the block corresponding to the linkage class of \(\lambda - \rho\). We denote the resulting projection as \(\text{pr}_\lambda(P_\mu \otimes W)\). By Lemma 2.3 if \(M_\lambda\) has the lowest weight of all the Verma modules in the standard filtration of the projection, \(P_\lambda\) must appear in that projection as a direct summand. The projection itself is done by collecting all Verma modules in the standard filtration whose weights are linked to \(\lambda - \rho\).

In this projection, we apply Proposition 2.7 to see which Verma modules appear in the standard filtration of \(P_\lambda\). These necessarily appear in the projection because \(P_\lambda\) is a direct summand. Then, we generally try to argue that there is no other direct summand (i.e. \(P_\lambda\) is the projection). This is often done by showing that no other indecomposable projective can appear in the projection, since there are not enough terms. In certain special cases, this method fails, and we get two possible standard filtrations of \(P_\lambda\). To determine which one is correct, we generally show that one of them is not a projective.

For convenience, we introduce the following notation which we use extensively in the presentation of our results and proofs to save space and improve clarity. Let \(\lambda \in X + \rho\) be such that \(\lambda - \rho\) is anti-dominant. Let \(W^\lambda\) be a minimal set of left-coset representatives of \(W/W_\lambda\), where \(W_\lambda = \{w \in W \mid w\lambda = \lambda\}\). Then, if \(\sigma \in W^\lambda\), we denote

\[
\sum_{\tau \geq \sigma, \tau \in W^\lambda} M_{\tau\lambda}
\]

by

\[
\sum_{\sigma} M_{\sigma\lambda}.
\]

For example, we may write

\[
M_{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{3}{2}, \frac{1}{2}} + M_{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}}.
\]
as

\[ \sum M_{\frac{1}{2}, -\frac{3}{2} | \frac{1}{2}}. \]

### 3. Character Formulae for \( \mathfrak{osp}(3|4) \)

In this section, we determine Verma multiplicities for standard filtration formulae for projective covers of simple modules of \( \mathfrak{osp}(3|4) \) with integral, atypical highest weight.

#### 3.1. Results

Let \( \mathfrak{g} = \mathfrak{osp}(3|4) \) have the standard choices of Cartan subalgebra, bilinear form, root system, positive, and fundamental system as described in \( \S 2 \). Recall the notation described in \( \S 2.2 \) to describe a weight in \( \mathfrak{h}^* \). We have the following Theorems 3.1 to 3.4 that describe standard filtrations of projectives in these blocks.

**Theorem 3.1.** Let \( \lambda - \rho = (a, b | c) - \rho \) be an atypical weight with \( a, b, c \in \frac{1}{2} + \mathbb{Z}, a, b > 0, \) and \( c \in \{ \pm a, \pm b \} \). The projective modules \( P_\lambda \) of highest weight \( \lambda - \rho \) have the following Verma flag formulae.

1. Suppose that \( a > b > 0 \).
   1.1. When \( c = a \), we have
   \[
   P_{a,b|a} = M_{a,b|a} + M_{a+1,b|a+1}.
   \]
   1.2. When \( c = -a \), we have
   \[
   P_{a,b|-a} = M_{a,b|-a} + M_{a,b|a} + M_{a+1,b|-a-1} + M_{a+1,b|a+1}.
   \]
   1.3. When \( c = b \), we have
   \[
   P_{a,b|b} = M_{a,b|b} + M_{a,b+1|b+1}
   \]
   for \( b < a - 1 \), and
   \[
   P_{a,a-1|a-1} = M_{a,a-1|a-1} + M_{a,a|a} + M_{a+1,a|a+1}.
   \]
   1.4. When \( c = -b \), we have
   \[
   P_{a,b|-b} = M_{a,b|-b} + M_{a,b|b} + M_{a,b+1|-b-1} + M_{a,b+1|b+1}
   \]
   for \( b < a - 1 \), and
   \[
   P_{a,a-1|-a+1} = M_{a,a-1|-a+1} + M_{a,a-1|a-1} + M_{a,a|-a} + M_{a,a|a}
   \]
   \[
   + M_{a+1,a|-a-1} + M_{a+1,a|a+1}.
   \]
2. Suppose that \( b > a > 0 \).
   2.1. When \( c = a \), we have
   \[
   P_{a,b|a} = M_{a,b|a} + M_{b,a|a} + M_{a+1,b|a+1} + M_{b,a+1|a+1}
   \]
   for \( b > a + 1 \), and
   \[
   P_{a,a+1|a} = M_{a,a+1|a} + M_{a+1,a|a} + M_{a+1,a+1|a+1}.
   \]
   2.2. When \( c = -a \), we have
   \[
   P_{a,b|-a} = M_{a,b|-a} + M_{a,b|a} + M_{b,a|-a} + M_{b,a|a}
   \]
   \[
   + M_{a+1,b|-a-1} + M_{a+1,b|a+1} + M_{b,a+1|-a-1} + M_{b,a+1|a+1}
   \]
for $b > a + 1$, and
\[
P_{a,a+1|a} = M_{a,a+1|a} + M_{a,a+1|a} + M_{a+1,a|a} + M_{a+1,a+1|a+1}.
\]

(2.3) When $c = b$, we have
\[
P_{a,b|b} = M_{a,b|b} + M_{a,b+1|b} + M_{b+1,a|b+1}.
\]

(2.4) When $c = -b$, we have
\[
P_{a,b|-b} = M_{a,b|-b} + M_{a,b|b} + M_{b,a|-b} + M_{b,b|b} + M_{a+1,b|b+1} + M_{b+1,a|-b+1} + M_{b+1,a|b+1}
\]
\[
= \sum M_{a,b|-b} + \sum M_{a,b|-b-1}.
\]

(3) Suppose that $a = b > 0$.

(3.1) When $c = a$, we have
\[
P_{a,a|a} = M_{a,a|a} + M_{a,a+1|a+1} + M_{a+1,a|a+1}.
\]

(3.2) When $c = -a$, we have
\[
P_{a,a|-a} = M_{a,a|-a} + M_{a,a|a} + M_{a,a+1|-a-1} + M_{a,a+1|a+1}
\]
\[
+ M_{a+1,a|-a-1} + M_{a+1,a|a+1}.
\]

Theorem 3.2. Let $\lambda - \rho = (a, b | c) - \rho$ be an atypical weight with $a, b, c \in \frac{1}{2} + \mathbb{Z}$, $a > 0 > b$, and $c \in \{\pm a, \pm b\}$. The projective modules $P_{\lambda}$ of highest weight $\lambda - \rho$ have the following Verma flag formulae.

(1) Suppose that $a > -b > 0$

(1.1) When $c = a$,
\[
P_{a,b|a} = M_{a,b|a} + M_{a,-b|a} + M_{a+1,b|a+1} + M_{a+1,-b|a+1}.
\]

(1.2) When $c = -a$,
\[
P_{a,b|-a} = M_{a,b|-a} + M_{a,b|a} + M_{a,-b|-a} + M_{a,-b|a}
\]
\[
+ M_{a+1,b|-a-1} + M_{a+1,b|a+1} + M_{a+1,-b|-a-1} + M_{a+1,-b|a+1}.
\]

(1.3) When $c = -b$,
\[
P_{a,b|-b} = M_{a,b|-b} + M_{a,-b|-b} + M_{a,b+1|-b+1} + M_{a,-b-1|-b-1}
\]

for $b < -\frac{1}{2}$, and
\[
P_{a,-\frac{1}{2}|-\frac{1}{2}} = M_{a,-\frac{1}{2}|-\frac{1}{2}} + M_{a,-\frac{1}{2}|-\frac{1}{2}} + M_{a,-\frac{1}{2}|-\frac{1}{2}} + M_{a,-\frac{1}{2}|-\frac{1}{2}}
\]

for $a > \frac{3}{2}$, and
\[
P_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}} = M_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}}
\]
\[
+ M_{\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}}.
\]
(1.4) When $c = b$,

$$P_{a,b|b} = M_{a,b|b} + M_{a,b|b} + M_{a,-b|b} + M_{a,-b|b} + M_{a,b+1|b+1} + M_{a,b-1|b+1} + M_{a,-b-1|b+1} + M_{a,-b-1|b+1}$$

for $b < -\frac{1}{2}$, and

$$P_{a,-\frac{1}{2}|\frac{1}{2}} = M_{a,-\frac{1}{2}|\frac{1}{2}} + M_{a,-\frac{1}{2}|\frac{1}{2}} + M_{a,\frac{1}{2}|\frac{1}{2}} + M_{a,\frac{1}{2}|\frac{1}{2}} = \sum M_{a,-\frac{1}{2}|\frac{1}{2}}.$$

(2) Suppose that $-b > a > 0$.

(2.1) When $c = a$, we have

$$P_{a,b|a} = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}.$$

(2.2) When $c = -a$, we have

$$P_{a,b|-a} = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}.$$

(2.3) When $c = -b$, we have

$$P_{a,b|-b} = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}.$$

(2.4) When $c = b$, we have

$$P_{a,b|b} = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.$$

(3) Suppose that $a = -b > 0$.

(3.1) When $c = a$, we have

$$P_{a,-a|a} = M_{a,-a|a} + M_{a,a|a} + M_{a,-a+1|a-1} + M_{a,a-1|a-1} + M_{a+1,-a|a+1} + M_{a+1,a|a+1}$$

for $a > \frac{1}{2}$, and

$$P_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} = M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2},\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2},\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2},\frac{1}{2}|\frac{1}{2}} + 2M_{\frac{1}{2},\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2},\frac{1}{2}|\frac{1}{2}}.$$

(3.2) When $c = -a$, we have

$$P_{a,-a|-a} = \sum M_{a,-a|-a} + \sum M_{a,-a+1|-a+1} + \sum M_{a+1,-a|-a-1}$$

for $a > \frac{1}{2}$, and

$$P_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} = \sum M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}}.$$

**Theorem 3.3.** Let $\lambda - \rho = (a, b | c) - \rho$ be an atypical weight with $a, b, c \in \frac{1}{2} + \mathbb{Z}$, $b > 0 > a$, and $c \in \{\pm a, \pm b\}$. The projective modules $P_\lambda$ of highest weight $\lambda - \rho$ have the following Verma flag formulae.

(1) Suppose that $a < -b < 0$.

(1.1) When $c = -a$, we have

$$P_{a,b|-a} = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}.$$
(1.2) When $c = a$, we have
$$P_{a,b|a} = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}.$$  

(1.3) When $c = b$, we have
$$P_{a,b|b} = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.$$  

(1.4) When $c = -b$, we have
$$P_{a,b|-b} = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}.$$  

(2) Suppose that $-b < a < 0$.

(2.1) When $c = -a$, we have
$$P_{a,b|-a} = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}$$
for $a < -\frac{1}{2}$, and
$$P_{-\frac{1}{2},b|\frac{1}{2}} = \sum M_{-\frac{1}{2},b|\frac{1}{2}} + M_{\frac{1}{2},b|-\frac{1}{2}} + M_{\frac{1}{2},b|\frac{1}{2}} + M_{\frac{3}{2},b|\frac{3}{2}}$$
for $b > \frac{3}{2}$, and
$$P_{-\frac{1}{2},\frac{3}{2}} = \sum M_{-\frac{1}{2},\frac{3}{2}} + M_{\frac{1}{2},\frac{3}{2}|\frac{1}{2}} + M_{\frac{3}{2},\frac{3}{2}|\frac{3}{2}} + M_{\frac{5}{2},\frac{3}{2}|\frac{3}{2}}.$$  

(2.2) When $c = a$, we have
$$P_{a,b|a} = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}$$
for $a < -\frac{1}{2}$, and
$$P_{-\frac{1}{2},b|\frac{1}{2}} = \sum M_{-\frac{1}{2},b|\frac{1}{2}}.$$  

(2.3) When $c = b$, we have
$$P_{a,b|b} = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.$$  

(2.4) When $c = -b$, we have
$$P_{a,b|-b} = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}.$$  

(3) Suppose that $a = -b < 0$.

(3.1) When $c = -a$, we have
$$P_{a,-a|-a} = M_{a,-a|-a} + M_{a,-a|-a} + 2M_{a,-a|-a}$$
$$+ M_{a,-a+1|-a+1} + M_{a,-a+1|-a+1} + M_{a+1,a|-a+1} + M_{a+1,a|-a+1} + M_{a+1,-a|-a+1}$$
$$+ M_{a+1,a|-a-1} + M_{a+1,-a|-a-1} + M_{a,a+1|-a-1} + M_{a,a+1|-a-1}$$
$$= \sum M_{a,-a|-a} + M_{a,-a|-a} + \sum M_{a,-a+1|-a+1} + \sum M_{a+1,-a|-a-1}.$$
for \( a < -\frac{1}{2} \), and
\[
P_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} = M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}
+ M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}
= \sum M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + \sum M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}.
\]

(3.2) When \( c = a \), we have
\[
P_{a, a|a} = \sum M_{a, a|a} + M_{-a, a|a} + M_{-a, a|a}
+ \sum M_{a, a+1|a-1} + \sum M_{a+1, a|a+1}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} = \sum M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} + \sum M_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}.
\]

**Theorem 3.4.** Let \( \lambda - \rho = (a, b \mid c) - \rho \) be an atypical weight with \( a, b, c \in \frac{1}{2} + \mathbb{Z} \), \( a, b < 0 \), and \( c \in \{ \pm a, \pm b \} \). The projective modules \( P_{\lambda} \) of highest weight \( \lambda - \rho \) have the following Verma flag formulae.

(1) Suppose that \( a < b < 0 \).

(1.1) When \( c = -a \), we have
\[
P_{a, b|-a} = \sum M_{a, b|-a} + \sum M_{a+1, b|-a-1}.
\]

(1.2) When \( c = a \), we have
\[
P_{a, b|a} = \sum M_{a, b|a} + \sum M_{a+1, b|a+1}.
\]

(1.3) When \( c = -b \), we have
\[
P_{a, b|-b} = \sum M_{a, b|-b} + \sum M_{a, b+1|-b-1}
\]
for \( b < -\frac{1}{2} \), and
\[
P_{a, -\frac{1}{2}, -\frac{1}{2}} = \sum M_{a, -\frac{1}{2}, -\frac{1}{2}} + \sum M_{-a, -\frac{1}{2}, -\frac{1}{2}}
+ M_{a, -\frac{1}{2}, -\frac{1}{2}} + M_{-a, -\frac{1}{2}, -\frac{1}{2}} + M_{a, -\frac{1}{2}, -\frac{1}{2}} + M_{-a, -\frac{1}{2}, -\frac{1}{2}}
+ M_{a, -\frac{1}{2}, -\frac{1}{2}} + M_{-a, -\frac{1}{2}, -\frac{1}{2}} + \sum M_{a, -\frac{1}{2}, -\frac{1}{2}}.
\]

(1.4) When \( c = b \), we have
\[
P_{a, b|b} = \sum M_{a, b|b} + \sum M_{a, b+1|b+1}
\]
for \( b < -\frac{1}{2} \), and
\[
P_{a, -\frac{1}{2}, -\frac{1}{2}} = \sum M_{a, -\frac{1}{2}, -\frac{1}{2}}.
\]

(2) Suppose that \( b < a < 0 \).
(2.1) When \( c = -a \), we have
\[
P_{a,b|-a} = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},b|-\frac{1}{2}} = \sum M_{-\frac{1}{2},b|-\frac{1}{2}} + M_{-\frac{1}{2},-\frac{1}{2}} + M_{-\frac{1}{2},-1} + M_{-\frac{1}{2},-\frac{1}{2}}
\]
\[
+ \sum M_{-\frac{1}{2},b|-\frac{1}{2}}.
\]

(2.2) When \( c = a \), we have
\[
P_{a,b|a} = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},b|-\frac{1}{2}} = \sum M_{-\frac{1}{2},b|-\frac{1}{2}}.
\]

(2.3) When \( c = -b \), we have
\[
P_{a,b|-b} = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}
\]
for \( b < a - 1 \), and
\[
P_{a,a-1|-a+1} = \sum M_{a,a-1|-a+1} + \sum M_{a+1,a|-a-1}
\]
\[
+ \sum M_{a,a|-a} + \sum M_{-a,a|-a}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} = \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + M_{-\frac{1}{2},-\frac{1}{2}} + M_{-\frac{1}{2},\frac{1}{2}}
\]
\[
+ \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}}.
\]

(2.4) When \( c = b \), we have
\[
P_{a,b|b} = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}
\]
for \( b < a - 1 \), and
\[
P_{a,a-1|a+1} = \sum M_{a,a-1|a+1} + \sum M_{a+1,a|a+1}
\]
\[
+ \sum M_{a,a|a} + \sum M_{-a,a|a}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} = \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}}.
\]

(3) Suppose that \( a = b < 0 \).
(3.1) When \( c = -a \), we have
\[
P_{a,a|-a} = \sum M_{a,a|-a} + \sum M_{a,a+1|-a-1}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} = \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} + \sum M_{-\frac{1}{2},-\frac{3}{2}}.
\]

(3.2) When \( c = a \), we have
\[
P_{a,a|a} = \sum M_{a,a|a} + \sum M_{a,a+1|a+1}
\]
for \( a < -\frac{1}{2} \), and
\[
P_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}} = \sum M_{-\frac{1}{2},-\frac{1}{2}|-\frac{1}{2}}.
\]

3.2. Proof. In this subsection, we prove Theorems 3.1 through 3.4. We use the method of translation functors by effecting certain finite-dimensional representations. These representations, which are all irreducible, highest-weight, and self-dual (cf. [CW12]), and their weights are given below. All weights, except the zero weight, appear with multiplicity 1. The zero weight is stated with its total multiplicity (i.e. \( 3 \cdot 0 \) means the zero-weight space is three-dimensional).

| Representation | Weights | Dimension | Highest Weight |
|----------------|---------|-----------|----------------|
| \( V \)        | \( \pm\{\delta_1, \delta_2, \epsilon\} \cup \{0\} \) | 7           | \( \delta_1 \) |
| \( \wedge^2 V \) | \( \pm\{\delta_1 \pm \delta_2, \delta_1 \pm \epsilon, \delta_1, \delta_2 \pm \epsilon, \delta_2, 2\epsilon, \epsilon\} \cup \{4 \cdot 0\} \) | 24          | \( \delta_1 + \delta_2 \) |
| \( \mathfrak{g} \) | \( \pm\{2\delta_1, \delta_1 \pm \delta_2, \delta_1 \pm \epsilon, \delta_1, 2\delta_2, \delta_2 \pm \epsilon, \delta_2, \epsilon\} \cup \{3 \cdot 0\} \) | 25          | \( 2\delta_1 \) |

In particular, we have as \( \text{osp}(3|4) \)-modules, \( V \cong L_{3/2, -1/2|1/2} = L_{\delta_1 + \rho} \), \( \wedge^2 V \cong L_{3/2, 1/2|1/2} = L_{\delta_1 + \delta_2 + \rho} \), and \( \mathfrak{g} \cong L_{5/2, -1/2|1/2} = L_{2\delta_1 + \rho} \).

We now offer justification for the formulae above, separated into cases that have different formulae, based on the strategy in [2.9] Our proof will be more explicit in the earlier cases and cases which require more sophisticated techniques; those which lack much explanation follow the strategy almost directly and list only the choices of \( P_{\mu} \) and representation for translation functor.

Proof of Theorem 3.1. Let \( \lambda - \rho = (a,b \mid c) - \rho \) be an atypical weight with \( a, b, c \in \frac{1}{2} + \mathbb{Z} \) and \( a, b > 0 \).

(1) Suppose that \( a > b > 0 \).

(1.1) When \( \lambda = (a,b \mid a) \),
\[
\text{pr}_\lambda (P_{a+1,b \mid a} \otimes V) = M_{a,b \mid a} + M_{a+1,b \mid a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to \( P_\lambda \).
(1.2) When $\lambda = (a, b| - a)$,
\[
\text{pr}_\lambda \left( P_{a+1,b|-a} \otimes V \right) = M_{a,b|-a} + M_{a,b|a} + M_{a+1,b|-a-1}M_{a+1,b|a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(1.3) When $\lambda = (a, b|b)$:
(i) If $b < a - 1$,
\[
\text{pr}_\lambda \left( P_{a,b+1|b} \otimes V \right) = M_{a,b|b} + M_{a,b+1|b+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.
(ii) If $b = a - 1$,
\[
\text{pr}_\lambda \left( P_{a+1,a-1|a-1} \otimes V \right) = M_{a,a-1|a-1} + M_{a,a|a} + M_{a+1,a|a+1}
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(1.4) When $\lambda = (a, b| - b)$:
(i) If $b < a - 1$,
\[
\text{pr}_\lambda \left( P_{a,b+1|-b} \otimes V \right) = M_{a,b|-b} + M_{a,b|b} + M_{a,b+1|-b-1} + M_{a,b+1|b+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.
(ii) If $b = a - 1$,
\[
\text{pr}_\lambda \left( P_{a+1,a-1|-a+1} \otimes V \right) = M_{a,a-1|-a+1} + M_{a,a-1|a-1} + M_{a,a|a} + M_{a+1,a|a+1}
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(2) Suppose that $b > a > 0$.

(2.1) When $\lambda = (a, b|a)$:
(i) If $b > a + 1$,
\[
\text{pr}_\lambda \left( P_{a+1,b|a} \otimes V \right) = M_{a,b|a} + M_{b,a|a} + M_{a+1,b|a+1} + M_{b,a+1|a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.
(ii) If $b = a + 1$,
\[
\text{pr}_\lambda \left( P_{a+1,a+1|a} \otimes V \right) = M_{a,a+1|a} + M_{a+1,a|a} + M_{a+1,a+1|a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(2.2) When $\lambda = (a, b| - a)$:
(i) If $b > a + 1$,
\[
\text{pr}_\lambda \left( P_{a+1,b|-a} \otimes V \right) = M_{a,b|-a} + M_{a,b|a} + M_{b,a|-a} + M_{b,a|a}
\]
\[+ M_{a+1,b|-a-1} + M_{a+1,b|a+1} + M_{b,a+1|-a-1} + M_{b,a+1|a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.
(ii) If $b = a + 1$,
\[
\text{pr}_\lambda \left( P_{a+1,a+1|-a} \otimes V \right) = M_{a,a+1|-a} + M_{a,a+1|a} + M_{a+1,a|-a} + M_{a+1,a|a}
\]
\[+ M_{a+1,a+1|-a-1} + M_{a+1,a+1|a+1}.
\]
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. 
Proof of Theorem 3.2. Let \( \lambda = (a, b|b) \),

\[
\text{pr}_\lambda \left( P_{a, b+1|b} \otimes V \right) = M_{a, b|b} + M_{b, a|b} + M_{a, b+1|b+1} + M_{b+1, a|b+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

(2.4) When \( \lambda = (a, b|b) \),

\[
\text{pr}_\lambda \left( P_{a, b+1|b} \otimes V \right) = M_{a, b|b} + M_{b, a|b} + M_{a, b+1|b+1} + M_{b+1, a|b+1} = \sum M_{a, b|b} + \sum M_{a, b+1|b+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

(3) Suppose that \( a = b > 0 \).
(3.1) When \( \lambda = (a, a|a) \),

\[
\text{pr}_\lambda \left( P_{a, a|a+1} \otimes V \right) = M_{a, a|a} + M_{a, a+1|a+1} + M_{a+1, a|a+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

(3.2) When \( \lambda = (a, a|a) \),

\[
\text{pr}_\lambda \left( P_{a, a|a+1} \otimes V \right) = M_{a, a|a} + M_{a, a+1|a+1} + M_{a+1, a|a+1} + M_{a+1, a|a+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

\( \square \)

Proof of Theorem 3.2. Let \( \lambda - \rho = (a, b|c) - \rho \) be an atypical weight with \( a, b, c \in \frac{1}{2} + \mathbb{Z} \) and \( a > 0 > b \).

(1) Suppose that \( a > -b > 0 \).
(1.1) When \( \lambda = (a, b|a) \),

\[
\text{pr}_\lambda \left( P_{a+1, b|a} \otimes V \right) = M_{a, b|a} + M_{a, b|b} + M_{a+1, b|a+1} + M_{a+1, b|a+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

(1.2) When \( \lambda = (a, b|a) \),

\[
\text{pr}_\lambda \left( P_{a+1, b|a} \otimes V \right) = M_{a, b|a} + M_{a, b|b} + M_{a, b|a+1} + M_{a+1, b|a+1} + M_{a+1, b|a+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).

(1.3) When \( \lambda = (a, b|b) \):
(i) If \( b < -\frac{1}{2} \),

\[
\text{pr}_\lambda \left( P_{a, b+1|b} \otimes V \right) = M_{a, b|b} + M_{a, b+1|b} + M_{a, b+1|b+1} + M_{a+1, b+1|b+1}. 
\]

By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).
(ii) If \( b = -\frac{1}{2} \) and \( a > \frac{3}{2} \),

\[
\text{pr}_\lambda \left( P_{a, b+1|b} \otimes V \right) = M_{a, b|b} + M_{a, b+1|b} + M_{a, b+1|b} + M_{a, b+1|b}. 
\]

By Lemma 2.3 \( P_\lambda \) must appear in the projection as a direct summand, and Proposition 2.7 ensures that the first three terms appear in \( P_\lambda \). However,
(2) Suppose that 

\[ \lambda (2.2) \text{when} \]

By Lemma 2.3 and Proposition 2.7, the projection is equal to 

\[ P \] 

(1.4) When \( \lambda = (a, b) \): 

(i) If \( b < -\frac{1}{2} \) and \( a = \frac{3}{2} \), 

\[ \text{pr}_\lambda \left( P_{a, \frac{3}{2}} | V \right) = M_{a, b} + M_{a, b - 1} + M_{a, b - 2} + M_{a, b - 3} \]

\[ + M_{a, b - 4} + 3 M_{a, \frac{3}{2}, \frac{3}{2}}. \]

By Lemma 2.3 and Proposition 2.7 the projection is equal to \( P \). 

(ii) If \( b = -\frac{1}{2} \) and \( a > \frac{3}{2} \), 

\[ \text{pr}_\lambda \left( P_{a, -\frac{1}{2}} \otimes V \right) = 2 M_{a, -\frac{1}{2}} + 2 M_{a, -\frac{1}{2} - \frac{1}{2}} + 3 M_{a, \frac{3}{2}, \frac{3}{2}} \]

By Lemma 2.3 \( P \) must appear twice in the projection as a direct summand, and By Proposition 2.7 one copy of each of the first four terms must be in \( P \). Now, one copy of the fourth term and the last term remain. However, since only one copy of these two terms remains, they cannot appear in \( P \). 

Thus, we get that 

\[ \text{pr}_\lambda \left( P_{a, -\frac{1}{2}} \otimes V \right) = 2 P_{a, -\frac{1}{2}} + P_{a, \frac{3}{2}, \frac{3}{2}} \]

and 

\[ P_{a, -\frac{1}{2}} = M_{a, -\frac{1}{2}} + M_{a, -\frac{1}{2} - \frac{1}{2}} + M_{a, \frac{3}{2}, \frac{3}{2}}. \]

(iii) If \( b = -\frac{1}{2} \) and \( a = \frac{3}{2} \), we get a formula consistent with the previous case by applying the same method.

(2) Suppose that \( -b > a > 0 \).

(2.1) When \( \lambda = (a, b) \), 

\[ \text{pr}_\lambda \left( P_{b+1, a} \otimes V \right) = \sum M_{a, b} + \sum M_{b+1, a+1}. \]

By Lemma 2.3 and Proposition 2.7 the projection is equal to \( P \). Note that when \( b = -a - 1 \), \( \sum M_{b+1, a+1} \) has two instead of four terms.

(2.2) When \( \lambda = (a, b - a) \), 

\[ \text{pr}_\lambda \left( P_{b+1, a} \otimes V \right) = \sum M_{b, a} + \sum M_{b+1, a+1}. \]

By Lemma 2.3 and Proposition 2.7 the projection is equal to \( P \). Note that when \( b = -a - 1 \), \( \sum M_{b+1, a+1} \) has four instead of eight terms.
(2.3) When $\lambda = (a, b) - b$, $\Pr_{\lambda} \left(P_{a, b+1|-b} \otimes V\right) = \sum M_{a, b|-b} + \sum M_{a, b+1|-b-1}$.

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_{\lambda}$. Note that when $b = -a - 1$, $\sum M_{a, b+1|-b-1}$ has two instead of four terms.

(2.4) When $\lambda = (a, b | b)$, $\Pr_{\lambda} \left(P_{a, b+1|b} \otimes V\right) = \sum M_{a, b|b} + \sum M_{a, b+1|b+1}$.

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_{\lambda}$. Note that when $b = -a - 1$, $\sum M_{a, b+1|b+1}$ has four instead of eight terms.

(3) Suppose that $a = -b > 0$.

(3.1) When $\lambda = (a, -a | a)$:

(i) If $a > \frac{1}{2}$,

$$\Pr_{\lambda} \left(P_{a+1,-a|a} \otimes V\right) = M_{a,-a|a} + M_{a,-a+1|a-1} + M_{a,a-1|a-1} + M_{a+1,-a+1|a+1}.$$  

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_{\lambda}$.

(ii) If $a = \frac{1}{2}$,

$$\Pr_{\lambda} \left(P_{a+1,-a|a} \otimes V\right) = M_{a,-a|a} + M_{a,-a+1|a-1} + M_{a,a-1|a-1} + M_{a+1,-a+1|a+1}.$$  

By Lemma 2.3, $P_{\lambda}$ must appear in the projection. By Proposition 2.7, the first six term and one copy of $M_{a,-a|a}$ must appear in $P_{\lambda}$. However, we run into some trouble here, since the two remaining terms, $M_{a,-a+1|a-1}$, $M_{a,a-1|a-1}$ could actually form the projective $P_{\frac{1}{2}, \frac{1}{2}}$, which means we have to devise some different method to show they are also included in $P_{\lambda}$.

We have two possible standard filtrations of $P_{\lambda}$. Call the shorter one, which do not include the two unexplained terms, $Q$, and call $P_{\frac{1}{2}, \frac{1}{2}} = M_{\frac{1}{2}, \frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}} + R$. We shall show that $P_{\lambda}$ has the longer standard filtration, which we shall denote by abuse of notation $Q + R$, by proving that $Q$ is not a projective. We calculate the projections $\Pr_{\mu} (Q \otimes g)$ and $\Pr_{\mu} (R \otimes g)$.

| Projective | Terms | $\Pr_{\mu} (- \otimes g)$ |
|------------|-------|---------------------------|
| $Q$        | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{1}{2}}$, $M_{\frac{1}{2}, -\frac{1}{2}}$, $M_{\frac{1}{2}, -\frac{1}{2}}$, $M_{\frac{1}{2}, -\frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
| $R$        | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
|            | $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$, $M_{\frac{1}{2}, \frac{1}{2}}$ |
If $Q$ were a projective, then $\text{pr}_\mu(Q \otimes g)$ is again a projective and by Lemma 2.3 must split into indecomposable projectives. We see that the lowest weight appearing is $\left(\frac{1}{2}, -\frac{5}{2} | \frac{1}{2} \right)$, so $P_{\frac{1}{2}, -\frac{5}{2} | \frac{1}{2}}$ must appear, and its terms are colored red. Next, we must have $P_{\frac{1}{2}, -\frac{5}{2} | -\frac{1}{2}}$ appear, whose terms are colored blue. Then, as $\left(\frac{5}{2}, -\frac{1}{2} | \frac{1}{2} \right)$ is the next lowest weight, $P_{\frac{5}{2}, -\frac{1}{2} | \frac{1}{2}}$ must appear (colored violet). However, we see that there are not enough terms left in $\text{pr}_\mu(Q \otimes g)$. Thus, $Q$ is not a projective and we must have $P_\lambda = Q + R$. It turns out that

$$\text{pr}_\mu((Q + R) \otimes g) = P_{\frac{1}{2}, -\frac{3}{4} | \frac{3}{4}} + P_{\frac{3}{2}, -\frac{3}{4} | \frac{1}{4}} + P_{\frac{5}{2}, -\frac{3}{4} | \frac{1}{2}}.$$  

(3.2) When $\lambda = (a, -a) - a$:

(i) If $a > \frac{1}{2}$,

$$\text{pr}_\lambda \left( P_{a+1, -a} \otimes V \right) = \sum M_{a, -a} + \sum M_{a, a+1} + \sum M_{a+1, -a}.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(ii) If $a = \frac{1}{2}$,

$$\text{pr}_\lambda \left( P_{\frac{1}{2}, -\frac{1}{4} | \frac{1}{4}} \otimes V \right) = \sum M_{\frac{1}{2}, -\frac{1}{4} | \frac{1}{4}} + \sum M_{\frac{3}{2}, -\frac{1}{4} | \frac{1}{4}}.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. 

Proof of Theorem 3.3. Let $\lambda - \rho = (a, b | c) - \rho$ be an atypical weight with $a, b, c \in \frac{1}{2} + Z$ and $b > 0 > a$.

(1) Suppose that $a < -b < 0$.

(1.1) When $\lambda = (a, b) - a$,

$$\text{pr}_\lambda \left( P_{a+1, b} - a \otimes V \right) = \sum M_{a, b} + \sum M_{a+1, b} - a.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Notice that when $b = -a - 1$, $\sum M_{a+1, b} - a - 1$ has three instead of six terms.

(1.2) When $\lambda = (a, b | a)$,

$$\text{pr}_\lambda \left( P_{a+1, b} | a \otimes V \right) = \sum M_{a, b} + \sum M_{a+1, b} - a + 1.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Notice that when $b = -a - 1$, $\sum M_{a+1, b} - a - 1$ has six instead of twelve terms.

(1.3) When $\lambda = (a, b | b)$,

$$\text{pr}_\lambda \left( P_{a, b+1} | b \otimes V \right) = \sum M_{a, b} + \sum M_{a, b+1} | b + 1.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Notice that when $b = -a - 1$, $\sum M_{a, b+1} | b + 1$ has three instead of six terms.

(1.4) When $\lambda = (a, b) - b$,

$$\text{pr}_\lambda \left( P_{a, b+1} - b \otimes V \right) = \sum M_{a, b} - b + \sum M_{a, b+1} - b - 1.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Notice that when $b = -a - 1$, $\sum M_{a, b+1} - b - 1$ has six instead of twelve terms.
(2) Suppose that $-b < a < 0$.

(2.1) When $\lambda = (a, b|a)$:

(i) If $a < -\frac{1}{2}$,

$$\text{pr}_\lambda \left( P_{a+1, b-a} \otimes V \right) = \sum M_{a, b-a} + \sum M_{a+1, b-a-1}.$$ 

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_\lambda$.

(ii) If $a = -\frac{1}{2}$ and $b > \frac{3}{2}$,

$$\text{pr}_\lambda \left( P_{\frac{1}{2}, b|\frac{1}{2}} \otimes V \right) = \sum M_{-\frac{1}{2}, b|\frac{1}{2}} + M_{\frac{1}{2}, b|\frac{1}{2}} + M_{b, \frac{1}{2}|\frac{1}{2}} + M_{b, 0|\frac{1}{2}}.$$ 

By Lemma 2.3 and Proposition 2.7, $\sum M_{-\frac{1}{2}, b|\frac{1}{2}}$ must belong to $P_\lambda$. The lowest remaining term is $M_{\frac{1}{2}, b|\frac{1}{2}}$, and since the remaining terms do not contain the standard filtration of $P_{\frac{1}{2}, b|\frac{1}{2}}$, it must also belong to $P_\lambda$. By the same argument, each of the remaining terms belongs to $P_\lambda$.

(iii) If $a = -\frac{1}{2}$ and $b = \frac{3}{2}$,

$$\text{pr}_\lambda \left( P_{\frac{1}{2}, 3|\frac{1}{2}} \otimes V \right) = \sum M_{-\frac{1}{2}, 3|\frac{1}{2}} + M_{\frac{1}{2}, 3|\frac{1}{2}} + M_{3, \frac{1}{2}|\frac{1}{2}} + M_{3, 0|\frac{1}{2}}.$$ 

By Lemma 2.3 and Proposition 2.7, $\sum M_{-\frac{1}{2}, 3|\frac{1}{2}}$ must belong to $P_\lambda$. The lowest remaining term is $M_{\frac{1}{2}, 3|\frac{1}{2}}$, and since the remaining terms do not contain the standard filtration of $P_{\frac{1}{2}, 3|\frac{1}{2}}$, it must also belong to $P_\lambda$. By the same argument, each of the remaining terms belongs to $P_\lambda$.

(2.2) When $\lambda = (a, b|a)$:

(i) If $a < -\frac{1}{2}$,

$$\text{pr}_\lambda \left( P_{a+1, b|a} \otimes V \right) = \sum M_{a, b|a} + \sum M_{a+1, b|a+1}.$$ 

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_\lambda$.

(ii) If $a = -\frac{1}{2}$ and $b > \frac{3}{2}$,

$$\text{pr}_\lambda \left( P_{-\frac{1}{2}, b|\frac{1}{2}} \otimes V \right) = 2 \sum M_{-\frac{1}{2}, b|\frac{1}{2}} + M_{\frac{1}{2}, b|\frac{1}{2}} + M_{b, \frac{1}{2}|\frac{1}{2}} + M_{b, 0|\frac{1}{2}}.$$ 

By Lemma 2.3, two copies of $P_\lambda$ must appear in the projection. By Proposition 2.7, $\sum M_{-\frac{1}{2}, b|\frac{1}{2}}$ belongs to $P_\lambda$. Now, since the remaining four terms each only appear with multiplicity one, they cannot belong to $P_\lambda$. They form $P_{\frac{1}{2}, b|\frac{1}{2}}$.

(iii) If $a = -\frac{1}{2}$ and $b = \frac{3}{2}$, we get the same result with the same projection as above, except that the projection has three (instead of four) remaining terms after subtracting two copies of $\sum M_{-\frac{1}{2}, b|\frac{1}{2}}$. The three terms still form $P_{\frac{1}{2}, b|\frac{1}{2}}$, which has three instead of four terms when $b = \frac{3}{2}$.
(2.3) When $\lambda = (a, b|b)$,
$$\text{pr}_\lambda (P_{a,b+1|b} \otimes V) = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.$$  
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(2.4) When $\lambda = (a, b|\neg b)$,
$$\text{pr}_\lambda (P_{a,b+1|\neg b} \otimes V) = \sum M_{a,b|\neg b} + \sum M_{a,b+1|\neg b+1}.$$  
By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(3) Suppose that $a = -b < 0$.

(3.1) When $\lambda = (a, -a|\neg a)$:

(i) If $a < -\frac{1}{2}$,
$$\text{pr}_\lambda (P_{a+1,-a|\neg a} \otimes V) = \sum M_{a,-a|\neg a} + M_{a,-a|\neg a} + \sum M_{a,-a+1|\neg a+1} + \sum M_{a+1,-a|\neg a+1}.$$  

By Lemma 2.3 and Proposition 2.7 one copy of each term must appear in $P_\lambda$. Now, there remains only the second copy of the term $M_{a,-a|\neg a}$, and as it clearly cannot form a projective, it must also belong to $P_\lambda$.

(ii) If $a = -\frac{1}{2}$,
$$\text{pr}_\lambda (P_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} \otimes V) = \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} + \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} + \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}}.$$  

By Lemma 2.3 and Proposition 2.7 every term except for $M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}}$ must appear in $P_\lambda$. As the one remaining term cannot form a projective, it must also belong to $P_\lambda$. Notice that unlike the previous term, each term here only appears with multiplicity one.

(3.2) When $\lambda = (a, -a|a)$:

(i) If $a < -\frac{1}{2}$,
$$\text{pr}_\lambda (P_{a+1,-a|a} \otimes V) = \sum M_{a,-a|a} + M_{a,-a|a} + \sum M_{a,-a+1|a+1} + \sum M_{a+1,-a|a+1}.$$  

By Lemma 2.3 and Proposition 2.7 one copy of each term must appear in $P_\lambda$. Now, there remain only the second copies of the terms $M_{a,-a|a}$ and $M_{a,-a|a}$, and as they cannot form a projective, they must also belong to $P_\lambda$.

(ii) If $a = -\frac{1}{2}$,
$$\text{pr}_\lambda (P_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} \otimes V) = \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} + \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}} + \sum M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}}.$$  

By Lemma 2.3 and Proposition 2.7 one copy of each term must appear in $P_\lambda$. Now, there remain only the second copies of the terms $M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}}$ and $M_{\frac{1}{2},\frac{1}{2}|\neg \frac{1}{2}}$, and as they cannot form a projective, they must also belong to $P_\lambda$.  

□
Proof of Theorem 3.4. Let $\lambda - \rho = (a, b | c) - \rho$ be an atypical weight with $a, b, c \in \frac{1}{2} + \mathbb{Z}$ and $a, b < 0$.

(1) Suppose that $a < b < 0$.

(1.1) When $\lambda = (a, b | -a)$,

$$ \text{pr}_\lambda(P_{a+1, b|-a} \otimes V) = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}. $$

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Note that when $b = a + 1$, $\sum M_{a+1,b|-a-1}$ has four instead of eight terms.

(1.2) When $\lambda = (a, b | a)$,

$$ \text{pr}_\lambda(P_{a+1,b|a} \otimes V) = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}. $$

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. Note that when $b = a + 1$, $\sum M_{a+1,b|a+1}$ has eight instead of sixteen terms.

(1.3) When $\lambda = (a, b | -b)$:

(i) If $b < -\frac{1}{2}$,

$$ \text{pr}_\lambda(P_{a,b+1|-b} \otimes V) = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}. $$

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(ii) If $b = -\frac{1}{2}$, we start with the case $\lambda = (-\frac{3}{2}, -\frac{1}{2}).$ We project $P_{-\frac{3}{2}, -\frac{1}{2}} \otimes V$ onto the $\lambda$ linkage block. Notice that in the table below, to save space, we use the $\pm$ sign to combine two terms into one. For example, $M_{\frac{3}{2}, \pm \frac{1}{2}}$ represents the two terms $M_{\frac{3}{2}, \frac{1}{2}}$ and $M_{\frac{3}{2}, -\frac{1}{2}}$.

| $P_{-\frac{3}{2}, -\frac{1}{2}}$ | $\text{pr}_\lambda(P_{-\frac{3}{2}, -\frac{1}{2}} \otimes V)$ |
|-----------------|-----------------|
| $M_{\frac{3}{2}, \frac{1}{2}}$ | $M_{\frac{3}{2}, \frac{1}{2}}$ |
| $M_{\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{3}{2}, -\frac{1}{2}}$ |
| $M_{\frac{1}{2}, \pm \frac{1}{2}}$ | $M_{\frac{1}{2}, \pm \frac{1}{2}}$ |
| $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{1}{2}}$ |
| $M_{\frac{1}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$ |
| $M_{\frac{1}{2}, \pm \frac{1}{2}}$ | $M_{\frac{1}{2}, \pm \frac{1}{2}}$ |
| $M_{\frac{1}{2}, \frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{1}{2}}$ |
| $M_{\frac{1}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, \frac{1}{2}}$ |

We start by finding the terms that must appear in $P_\lambda$. By Proposition 2.7 the terms colored red belong to $P_\lambda$. Now, the lowest remaining term is $M_{-\frac{3}{2}, \frac{1}{2}}$, and since $P_{-\frac{3}{2}, \frac{1}{2}}$ does not appear in the projection, it must belong to $P_\lambda$. By the same reasoning, the second copy of $M_{-\frac{3}{2}, \frac{1}{2}}$ must also appear in $P_\lambda$. Now, the next lowest term is $M_{\frac{1}{2}, \frac{3}{2}}$. However, the three terms in the standard filtration of $P_{\frac{1}{2}, \frac{3}{2}}$ all remain in the projection, colored blue. The two remaining unsorted terms must belong to $P_\lambda$ as no more projective can form among them. Again, we face two possible standard filtrations of $P_\lambda$. Denote the one containing all red and black terms $Q$, and denote the three blue terms $R$. We shall show that $Q$ is not a projective and thereby prove that $P_\lambda = Q + R$. 


Let $\mu = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$. We project $Q \otimes V$ and $R \otimes V$ onto the $\mu$ linkage block. In the table we use our notation of $\sum M_\lambda$ to combine terms.

| Projective | Terms | $\text{pr}_\mu(- \otimes V)$ |
|------------|-------|-----------------------------|
| $Q$        | $\sum M_{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}}$ | $\sum M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}}$ | $\sum M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}}$ |
| $M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}}$ | $M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}}$ |
| $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ |
| $M_{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{1}{2}, -1}$ | $M_{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{1}{2}, -1}$ |
| $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ |
| $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{-\frac{3}{2}, -\frac{3}{2}, -1}$ |

| $R$        | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ |
| $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ |
| $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ | $M_{\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}}$ |

We find indecomposable projectives in this projection starting with the lowest term. First, $P_\mu$ appears, colored red. The next lowest is $M_{-\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}}$, so $P_{-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}}$ must appear, colored blue. Since $Q$ does not have enough terms for this, it cannot be a projective and we must have that $P_\lambda = Q + R$.

(iii) Here we calculate the projective $P_\nu$ where $\nu = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. If we keep isolating projectives from the projection above, we find that the next projective that must appear is $P_{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}$. By Proposition [2.7], the terms colored violet must appear. Of the five remaining terms, the two in the projection (colored brown) of $Q$ must also belong to $P_{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}$, as they cannot belong to another projective, while the three in the projection of $R$ could form the projective $T = P_{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}}$. Thus, we have two possible standard filtrations of $P_{-\frac{1}{2}, -\frac{1}{2}, 1}$, which we denote by $S$ and $S + T$. Now, we project $S \otimes V$ and $T \otimes V$ back onto the linkage block of $\lambda = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$. 


By Lemma 2.3, $P_\lambda$ must appear in the projection, and we color its terms red. We see that $S$ does not have enough terms and thus cannot be a projective. Thus, $P_\nu = S + T$, and

$$\text{pr}_\lambda((S + T) \otimes V) = P_{-\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}} + P_{\frac{3}{2}, \frac{1}{2}|\frac{1}{2}}.$$  

(iv) If $b = -\frac{1}{2}$ and $a < -\frac{3}{2}$, we project $P_{a, \frac{1}{2}|\frac{1}{2}} \otimes V$ onto the $\lambda$ linkage block. Similar to the previous case, we obtain two possible standard filtrations, denoted $Q(a)$ and $(Q + R)(a)$. Now, we consider the specific case of $a = -\frac{5}{2}$, and we project the corresponding $Q(-\frac{5}{2}) \otimes V$ and $R(-\frac{5}{2}) \otimes V$ onto the $(-\frac{3}{2}, -\frac{1}{2}|\frac{1}{2})$ block. It turns out that $Q(-\frac{5}{2})$ is not a projective and

$$\text{pr}_{-\frac{5}{2}, -\frac{1}{2}|\frac{1}{2}}((Q + R) \otimes V) = P_{-\frac{7}{2}, -\frac{3}{2}|\frac{1}{2}} + P_{-\frac{7}{2}, -\frac{1}{2}|\frac{1}{2}}.$$  

Thus, $P_{-\frac{7}{2}, -\frac{3}{2}|\frac{1}{2}} = (Q + R)(-\frac{5}{2})$. Then, we proceed by induction, projecting $Q(a - 1) \otimes V$ and $R(a - 1) \otimes V$ onto the $(a, \frac{1}{2}|\frac{1}{2})$ block. Since we find that the projection of $(Q + R)(a - 1) \otimes V$ is equal to $P_{a, \frac{1}{2}|\frac{1}{2}} = (Q + R)(a)$, $Q(a - 1)$ does not have enough terms and thus is not a projective. This way, we show that $P_{a, \frac{1}{2}|\frac{1}{2}} = (Q + R)(a)$ for all $a < -\frac{3}{2}$.

(1.4) When $\lambda = (a, b|b)$:

(i) If $b < -\frac{3}{2}$,

$$\text{pr}_\lambda(P_{a, b+1|b} \otimes V) = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.$$  

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_\lambda$.

(ii) We defer the case of $\lambda = (a, -\frac{1}{2} - \frac{1}{2})$ to the next part, since the method requires a projective we have not yet calculated.

(2) Suppose that $a = b < 0$. (Note that we prove part 3 of Theorem 3.4 before part 2)

(2.1) When $\lambda = (a, a| - a)$,

(i) If $a < -\frac{1}{2}$,

$$\text{pr}_\lambda\left(\bigwedge^2 V\right) = \sum M_{a,a|-a} + \sum M_{a,a+1|-a-1}.$$  

By Lemma 2.3 and Proposition 2.7, the projection is equal to $P_\lambda$.

(ii) The case $\lambda = (-\frac{1}{2}, -\frac{1}{2}|\frac{1}{2})$ was resolved in case 1.
When $\lambda = (a, a|a)$,

(i) If $a < -\frac{1}{2}$,

$$\text{pr}_{\lambda}\left(P_{a+1, a+1|a} \otimes \bigwedge^2 V\right) = \sum M_{a,a|a} + \sum M_{a,a+1|a+1}.$$  

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_{\lambda}$.

(ii) If $a = -\frac{1}{2}$,

$$\text{pr}_{\lambda}\left(P_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}} \otimes V\right) = 3\sum M_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}} + 2\left(\sum M_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}} + M_{\frac{1}{2}, \frac{1}{2}|-\frac{1}{2}} + \sum M_{-\frac{1}{2}, \frac{3}{2}|\frac{1}{2}}\right).$$

Since the lowest term is $M_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}}$ and it appears with multiplicity 3, by Lemma 2.3, $P_{\lambda}$ must appear three times in the projection. By Proposition 2.7, $\sum M_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}}$ belongs to $P_{\lambda}$. Since these are also the only terms with multiplicity at least 3,

$$P_{\lambda} = \sum M_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}}.$$  

It turns out that

$$\text{pr}_{\lambda}\left(P_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}} \otimes V\right) = 3P_{-\frac{1}{2}, -\frac{1}{2}|-\frac{1}{2}} + 2P_{-\frac{1}{2}, \frac{1}{2}|-\frac{1}{2}}.$$  

(iii) Here we calculate the projective $P_{\nu}$ where $\nu = (a, -\frac{1}{2}|-\frac{1}{2})$, which we deferred from case 1. First, we consider the specific case where $a = -\frac{3}{2}$. We have that

$$\text{pr}_{\nu}\left(P_{a+1, -\frac{1}{2}|-\frac{1}{2}} \otimes V\right) = \sum M_{a, -\frac{1}{2}|-\frac{1}{2}}.$$  

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_{-\frac{3}{2}, -\frac{1}{2}|-\frac{1}{2}}$. Now, we proceed by induction,

$$\text{pr}_{(a, -\frac{1}{2}|-\frac{1}{2})}\left(P_{a+1, -\frac{1}{2}|-\frac{1}{2}} \otimes V\right) = \sum M_{a, -\frac{1}{2}|-\frac{1}{2}},$$

and get By Proposition 2.7 that

$$P_{a, -\frac{1}{2}|-\frac{1}{2}} = \sum M_{a, -\frac{1}{2}|-\frac{1}{2}}$$

for all $a < -\frac{1}{2}$ (in fact, for $a = -\frac{1}{2}$ as well, as shown in the previous subcase).

(3) Suppose that $b < a < 0$.

(3.1) When $\lambda = (a, b|-a)$:

(i) If $a < -\frac{1}{2}$,

$$\text{pr}_{\lambda}\left(P_{a+1, b|-a} \otimes V\right) = \sum M_{a,b|-a} + \sum M_{a+1,b|-a-1}.$$  

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_{\lambda}$.  

(2.2) When $\lambda = (a, a|a)$,
(ii) If \(a = -\frac{1}{2}\), we start with the case \(\lambda = (-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2})\). We have that

\[
\text{pr}_{\lambda}\left(P_{\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} \otimes V\right) = \sum M_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{-\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{-\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{-\frac{3}{2}, -\frac{3}{2}|\frac{1}{2}}.
\]

By Proposition 2.7, all terms except for the four colored red must appear in \(P_{\lambda}\). Proceeding from the lowest remaining term, we can show that each of the remaining term must appear in \(P_{\lambda}\) since no other projective can form in the projection.

(iii) If \(a = -\frac{1}{2}\) and \(b < -\frac{3}{2}\),

\[
\text{pr}_{\lambda}\left(P_{\frac{1}{2}, b|\frac{1}{2}} \otimes V\right) = \sum M_{-\frac{1}{2}, b|\frac{1}{2}} + M_{-b, -\frac{1}{2}|\frac{1}{2}} + M_{-b, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, b|\frac{1}{2}} + M_{\frac{3}{2}, b|\frac{1}{2}} + M_{-b, -\frac{3}{2}|\frac{1}{2}} + M_{-b, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, b|\frac{1}{2}}.
\]

By a similar argument as above, we can show that all terms except for the four terms colored red, which can form the projective \(P_{-b, -\frac{3}{2}|\frac{1}{2}}\), must appear in \(P_{\lambda}\). We now have two possible standard filtrations for \(P_{\lambda}\). As usual, call them \(Q(b)\) and \((Q + R)(b)\). First we consider the case \(b = -\frac{5}{2}\).

We project \(Q(-\frac{5}{2}) \otimes V\) and \(R(-\frac{5}{2}) \otimes V\) back to the \((-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2})\) block.

| Projective | Terms | pr_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}}(- \otimes V) |
|------------|-------|--------------------------------------------------|
| \(Q(-\frac{5}{2})\) | \(\sum M_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{-\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}}\) | \(\sum M_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{-\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}} + M_{\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} + M_{\frac{3}{2}, -\frac{1}{2}|\frac{1}{2}}\) |
| \(M_{1, \pm\frac{5}{2}|\frac{1}{2}}\) | \(M_{\frac{1}{2}, \pm\frac{5}{2}|\frac{1}{2}}\) | \(M_{\frac{3}{2}, \pm\frac{5}{2}|\frac{1}{2}}\) |
| \(M_{\frac{1}{2}, \pm\frac{3}{2}|\frac{1}{2}}\) | \(M_{\frac{3}{2}, \pm\frac{3}{2}|\frac{1}{2}}\) |
| \(M_{\frac{3}{2}, \pm\frac{1}{2}|\frac{1}{2}}\) |

Since the lowest term appearing in the projection is \(M_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}}\), the projective \(P_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}}\) must appear in the projection, with its terms colored red. We see that \(Q\) again does not have enough terms and is therefore not a projective. Thus, \(P_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}} = (Q + R)(-\frac{5}{2})\). As we have the base case now, we may proceed by induction. By projecting \(Q(b-1) \otimes V\) and \(R(b-1) \otimes V\) onto the \((-\frac{1}{2}, b|\frac{1}{2})\) block, we see that \(Q(b-1)\) does not have enough terms and

\[
\text{pr}_{-\frac{1}{2}, b|\frac{1}{2}} ((Q + R)(b-1) \otimes V) = (Q + R)(b).
\]

Thus, for all \(b < -\frac{3}{2}\), \(P_{-\frac{1}{2}, b|\frac{1}{2}} = (Q + R)(b)\).

(3.2) When \(\lambda = (a, b|a)\):
(i) If $a < -\frac{1}{2}$,

$$\text{pr}_\lambda \left( P_{a+1,b|a} \otimes V \right) = \sum M_{a,b|a} + \sum M_{a+1,b|a+1}. $$

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$.

(ii) If $a = -\frac{1}{2}$, we start with the case $\lambda = \left( -\frac{1}{2}, -\frac{3}{2} - \frac{1}{2} \right)$. We project $P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} \otimes V$ onto the $\lambda$ block. We get that

$$\text{pr}_\lambda \left( P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} \otimes V \right) = 2 \sum M_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} + 2 \sum M_{\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}} + 2 \sum M_{\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}} + M_{\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}}.$$

First, by Lemma 2.3 and Proposition 2.7, two copies of $P_\lambda$ must appear in the projection and the terms in $\sum M_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}}$ belong to $P_\lambda$. Now, the lowest remaining term is $M_{\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}}$, and since only one copy of it remains, it cannot belong to $P_\lambda$, which means $P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}}$ must appear in the projection as a separate projective. Now, the only remaining terms are the two copies of $R(-\frac{3}{2}) = \sum M_{\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}}$, which could form $P_{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}$. Thus, we again face two possibilities for $P_\lambda$, namely, $Q(-\frac{3}{2}) = \sum M_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}}$ and $(Q + R)(-\frac{3}{2})$.

Now, by projecting $Q(-\frac{3}{2}) \otimes V$ and $R(-\frac{3}{2}) \otimes V$ onto the $(-\frac{1}{2}, -\frac{5}{2} - \frac{1}{2})$ block, we see that $P_{-\frac{1}{2}, -\frac{5}{2} - \frac{1}{2}}$, also has two possible standard filtrations $Q(-\frac{3}{2})$ and $(Q + R)(-\frac{3}{2})$, defined similarly. In addition, $P_{-\frac{1}{2}, -\frac{5}{2} - \frac{1}{2}} = Q(-\frac{3}{2})$ if and only if $P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} = Q(-\frac{3}{2})$, as otherwise when we project the shorter projective onto the block of the longer, there would not be enough terms. The same argument carries as we induct on $b$. Thus, it remains to find the correct filtration for any specific value of $b$.

We examine $\mu = \left( -\frac{1}{2}, -\frac{3}{2} - \frac{1}{2} \right)$ and show that $P_\mu = Q(-\frac{3}{2})$. Consider the projection $\text{pr}_\mu \left( P_{-\frac{1}{2}, -\frac{3}{2} - \frac{3}{2}} \otimes \Lambda^2 V \right)$, which has 180 terms. By applying Lemma 2.3, we find that $P_{-\frac{1}{2}, -\frac{3}{2} - \frac{3}{2}}$ and four copies of $P_{-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}}$ must appear in the projection. Now, 60 terms remain, and the lowest term is $M_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}}$, which appear 4 times, which means that $P_\mu$ must appear four times. However, $(Q + R)(-\frac{3}{2})$ has 16 terms and thus does not fit. Thus,

$$P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} = Q(b) = \sum M_{-\frac{1}{2}, b|-\frac{1}{2}}$$

for all $b < -\frac{1}{2}$. It turns out that

$$\text{pr}_\mu \left( P_{-\frac{1}{2}, -\frac{3}{2} - \frac{3}{2}} \otimes \Lambda^2 V \right) = P_{-\frac{1}{2}, -\frac{3}{2} - \frac{1}{2}} + 4P_{-\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}} + 4P_{\frac{3}{2}, -\frac{3}{2} - \frac{1}{2}} + P_{\frac{3}{2}, -\frac{3}{2} - \frac{3}{2}}.$$

(3.3) When $\lambda = (a, b| - b)$:

(i) If $b < a - 1$,

$$\text{pr}_\lambda \left( P_{a,b+1|-b} \otimes V \right) = \sum M_{a,b|-b} + \sum M_{a,b+1|-b-1}.$$ 

By Lemma 2.3 and Proposition 2.7 the projection is equal to $P_\lambda$. 

(ii) If \( b = a - 1 \) and \( a < -\frac{1}{2} \),
\[
\text{pr}_\lambda \left( P_{a+1,a-1|a+1} \otimes V \right) = \sum M_{a,a-1|a+1} + \sum M_{a+1,a|a-1} + \sum M_{a,a} + \sum M_{-a,a|a}.
\]
By Proposition 2.7, all terms except for \( \sum M_{-a,a|a} \) must belong to \( P_\lambda \). As no projective can form among the remaining two terms, the projection is equal to \( P_\lambda \).

(iii) If \( b = a - 1 \) and \( a = -\frac{1}{2} \),
\[
\text{pr}_\lambda \left( P_{\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} \otimes V \right) = \sum M_{\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}}.
\]
By Proposition 2.7, all terms except for \( \sum M_{-\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} \) and \( \sum M_{-\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} \) must belong to \( P_\lambda \). As no projective can form among the remaining six terms, the projection is equal to \( P_\lambda \).

(3.4) When \( \lambda = (a, b|b) \):
(i) If \( b < a - 1 \),
\[
\text{pr}_\lambda \left( P_{a,b+1|b} \otimes V \right) = \sum M_{a,b|b} + \sum M_{a,b+1|b+1}.
\]
By Lemma 2.3 and Proposition 2.7, the projection is equal to \( P_\lambda \).
(ii) If \( b = a - 1 \) and \( a < -\frac{1}{2} \),
\[
\text{pr}_\lambda \left( P_{a+1,a-1|a-1} \otimes V \right) = \sum M_{a,a-1|a-1} + \sum M_{a+1,a|a+1} + \sum M_{a,a} + \sum M_{-a,a|a}.
\]
By Proposition 2.7, all terms except for \( \sum M_{-a,a|a} \) must belong to \( P_\lambda \). As no projective can form among the remaining four terms, the projection is equal to \( P_\lambda \).
(iii) If \( b = a - 1 \) and \( a = -\frac{1}{2} \),
\[
\text{pr}_\lambda \left( P_{-\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} \otimes V \right) = \sum M_{-\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} + \sum M_{-\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} + \sum M_{\frac{1}{2},-\frac{1}{2}|\frac{1}{2}}.
\]
By Proposition 2.7, all terms except for \( \sum M_{-\frac{1}{2},-\frac{3}{2}|\frac{1}{2}} \) and \( \sum M_{-\frac{1}{2},-\frac{1}{2}|\frac{1}{2}} \) must belong to \( P_\lambda \). As no projective can form among the remaining four terms, the projection is equal to \( P_\lambda \).  

\[\square\]

4. JORDAN-H"OLDER FORMULAE FOR \( \text{osp}(3|4) \)

By BGG reciprocity, we can convert the standard filtration multiplicities for projective modules into Jordan-H"older multiplicities of irreducible modules for Verma modules.
Let $\lambda \in X + \rho$ such that $\lambda - \rho$ is atypical, integral, and antidominant. Let $W^\lambda$ be a minimal set of left-coset representatives of $W/W_\lambda$. Then, by applying the BGG reciprocity to Proposition 2.7, we immediately get that the composition series of $M_{\sigma\lambda}$ ($\sigma \in W^\lambda$) must include
\[
\sum_{\tau \leq \sigma, \tau \in W^\lambda} (L_{\tau\lambda} + L_{\tau\lambda-\alpha} + L_{\tau\lambda-\alpha-\beta}),
\]
where each term in the sum appears with multiplicity one only if it is linked to $\lambda$, and $\alpha, \beta \in \Phi^+$ and $\text{ht}(\alpha) > \text{ht}(\beta)$. For convenience, we denote this summation by
\[
\sum L_{\sigma\lambda}.
\]

**Theorem 4.1.** Let $\lambda - \rho = (a, b | c) - \rho$ be an atypical weight with $a, b, c \in \frac{1}{2} + \mathbb{Z}$ and $c \in \{\pm a, \pm b\}$. The Verma modules $M_\lambda$ of highest weight $\lambda - \rho$ have Jordan-Hölder formulae
\[
M_\lambda = \sum L_\lambda
\]
except in the following cases.

(i) Suppose that $\lambda - \rho = (a', b'| \frac{3}{2}) - \rho$ is atypical, and at least one of $a', b'$ is positive. Since at least one of $|a'|, |b'|$ is equal to $\frac{3}{2}$, suppose that $\{|a'|, |b'|\} = \{a, \frac{3}{2}\}$. Then, unless specified otherwise in cases below, $M_\lambda$ has the following composition series:
\[
M_\lambda = \sum L_\lambda + \sum_{*} L_{a, -\frac{1}{2}| \frac{1}{2}},
\]
where $\sum_{*} L_{a, -\frac{1}{2}| \frac{1}{2}}$ denotes the the sum of the terms in the set
\[
\{L_{-a, -\frac{1}{2}| \frac{1}{2}}, L_{a, -\frac{1}{2}| \frac{1}{2}}, L_{-\frac{1}{2}, -a| \frac{1}{2}}, L_{-\frac{1}{2}, a| \frac{1}{2}}\}
\]
that are lower than $L_\lambda$. The following subcases are exceptions to this case, which contain some additional terms than those given above.

(ii) When $\lambda = \left(\frac{3}{2}, -\frac{1}{2} \mid \frac{3}{2}\right)$. We have
\[
M_\lambda = \sum L_\lambda + L_{-\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}} + L_{-\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}},
\]
where we use red to emphasize terms with multiplicity two (its first copy appears in $\sum L_\lambda$).

(iii) When $\lambda = \left(\frac{3}{2}, \frac{1}{2} \mid \frac{3}{2}\right)$. We have
\[
M_\lambda = \sum L_\lambda + L_{-\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}} + L_{-\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}} + L_{\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}} + L_{\frac{1}{2}, -\frac{1}{2}| \frac{1}{2}}.
\]

(iv) When $\lambda = \left(\frac{3}{2}, \frac{3}{2} \mid \frac{3}{2}\right)$. We have
\[
M_\lambda = \sum L_\lambda + \sum_{*} L_{\frac{3}{2}, -\frac{1}{2}| \frac{1}{2}}
+ L_{-\frac{1}{2}, \frac{1}{2}| \frac{3}{2}} + L_{-\frac{1}{2}, \frac{1}{2}| \frac{3}{2}} + L_{-\frac{3}{2}, -\frac{1}{2}| \frac{3}{2}} + L_{-\frac{3}{2}, -\frac{1}{2}| \frac{3}{2}} + L_{-\frac{3}{2}, -\frac{1}{2}| \frac{3}{2}}.
(2) Suppose that \( \lambda - \rho = (a', b'|\frac{1}{2}) - \rho \) is atypical, and at least one of \( a' \), \( b' \) is greater than \( \frac{1}{2} \).

(i) When \( \lambda = (-\frac{1}{2}, b|\frac{1}{2}) \) or \( \lambda = (\frac{1}{2}, b|\frac{1}{2}) \) with \( b > \frac{1}{2} \). We have

\[
M_\lambda = \sum L_\lambda + L_{-b, -\frac{1}{2}|\frac{1}{2}}.
\]

(ii) Suppose that \( \lambda = (a, -\frac{1}{2}|\frac{1}{2}) \) or \( \lambda = (a, \frac{1}{2}|\frac{1}{2}) \) with \( a > \frac{1}{2} \). We have

\[
M_\lambda = \sum L_\lambda + L_{-\frac{1}{2}, -a|\frac{1}{2}} + L_{-a, -\frac{1}{2}|\frac{1}{2}}.
\]

(3) Suppose that \( \lambda - \rho = (a, b, c) - \rho \) is atypical, and \( a = |b| = |c| \).

(i) When \( \lambda = (a, -a, a) \), we have

\[
M_\lambda = \sum L_\lambda + L_{-a, -a-1|a-1}.
\]

(ii) When \( \lambda = (a, -a|a) \) and \( a \neq \frac{3}{2} \), we have

\[
M_\lambda = \sum L_\lambda + L_{-a, -a-1|a-1} + L_{-a, -a-1|a+1}.
\]

(iii) The case \( \lambda = (\frac{3}{2}, -\frac{3}{2}|\frac{3}{2}) \) is given above.

(iv) When \( \lambda = (a, a|a-a) \), we have

\[
M_\lambda = \sum L_\lambda + L_{-a,a|a-a} + L_{-a,-a-1|a-1}.
\]

(v) When \( \lambda = (a, -a|a) \) and \( a > \frac{3}{2} \), we have

\[
M_\lambda = \sum L_\lambda + L_{-a,a|a} + L_{-a,a-1|a-1} + L_{-a,-a-1|a+1}.
\]

(vi) The case \( \lambda = (\frac{3}{2}, \frac{3}{2}|\frac{3}{2}) \) is given above.

(vii) When \( \lambda = (\frac{1}{2}, \frac{1}{2}|\frac{1}{2}) \), we have

\[
M_\lambda = \sum L_\lambda + L_{-\frac{1}{2}, \frac{1}{2}|\frac{1}{2}} + L_{-\frac{1}{2}, \frac{1}{2}|\frac{1}{2}} + L_{-\frac{1}{2}, \frac{1}{2}|\frac{1}{2}} + L_{-\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}}.
\]

(4) When \( \lambda = (\frac{5}{2}, \frac{3}{2}|\frac{3}{2}) \), we have

\[
M_\lambda = \sum L_\lambda + L_{\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}}.
\]

(5) When \( \lambda = (\frac{5}{2}, \frac{3}{2}|\frac{3}{2}) \), we have

\[
M_\lambda = \sum L_\lambda + L_{\frac{3}{2}, -\frac{3}{2}|\frac{1}{2}}.
\]

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