A Certified Reduced Basis Approach for Parametrized Linear-Quadratic Optimal Control Problems with Control Constraints

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Abstract In this paper, we employ the reduced basis method for the efficient and reliable solution of parametrized optimal control problems governed by elliptic partial differential equations. We consider the standard linear-quadratic problem setting with distributed control and two-sided control constraints, which play an important role in many industrial and economical applications. For this problem class, we propose two different reduced basis approximations and associated error estimation procedures. In our first approach, we directly consider the resulting optimality system, introduce suitable reduced basis approximations for the state, adjoint, control, and Lagrange multipliers, and use a projection approach to bound the error in the reduced optimal control. For our second approach, we first reformulate the optimal control problem using two slack variables, we then develop a reduced basis approximation for both slack problems by suitably restricting the solution space, and derive error bounds for the slack based optimal control. We discuss benefits and drawbacks of both approaches and substantiate the comparison by presenting numerical results for a model problem.
1 Introduction

Optimal control problems governed by partial differential equations (PDEs) appear in a wide range of applications in science and engineering, such as heat phenomena, crystal growth, and fluid flow (see, e.g., [7, 3]). Their solution using classical discretization techniques such as finite elements (FE) or finite volumes can be computationally expensive and time-consuming. Often, additional parameters enter the problem, e.g., material or geometry parameters in a design exercise.

Previous work on reduced order methods for optimal control problems considered distributed but unconstrained controls or constrained but scalar controls. Elliptic optimal control problems with distributed control have been considered recently by Negri et al. [10]. The proposed error bound is based on the Banach-Nečas-Babuška (BNB) theory applied to the first order optimality system. The approach thus provides a combined bound for the error in the state, adjoint, and control variable, but it is only applicable to problems without control constraints. Since the bound requires the very costly computation of a lower bound to the inf-sup constant, Negri et al. [9] compute error estimates using a heuristic interpolant surrogate of that constant.

Based on the ideas in Tröltzsch and Volkwein [12], Kärcher et al. [5] proposed rigorous and online-efficient control error bounds for reduced basis (RB) approximations of scalar elliptic optimal control problems. These ideas are extended and improved in [6] to distributed control problems.

In a recent paper [1], we employed the RB method as a surrogate model for the solution of distributed and one-sided constrained optimal control problems governed by parametrized elliptic partial differential equations. In this paper we extend this work to two-sided control constraints. After stating the problem in Section 2 we present the following contributions:

- In Section 3 we extend previous work on reduced basis methods for variational inequalities in [2, 1] to the optimal control setting with two-sided control constraints. While we can derive an offline-online decomposable RB optimality system, we are only able to derive a partially offline-online decomposable control error bound that depends on the FE dimension of the control.

- In Section 4 we build on the recent work in [13, 1] and propose an RB slack approach for optimal control. We introduce two slack formulations for the optimal control problem, which we obtain by shifting the optimal control by each constraint. We are thus able to derive an offline-online decomposable RB optimality systems and control error bound. The evaluation of this bound is independent of the FE dimension of the problem, but requires the solution of three RB systems.

In Section 5 we propose a greedy sampling procedure to construct the RB spaces and in Section 6 we assess the properties of our methods by presenting numerical results for a Graetz flow problem.
2 General Problem Statement and Finite Element Discretization

In this section we introduce the parametrized linear-quadratic optimal control problem with elliptic PDE constraint and a constrained distributed control. We introduce a finite element (FE) discretization for the continuous problem and recall the first-order necessary (and in the convex setting sufficient) optimality conditions.

2.1 Preliminaries

Let $Y_e$ with $H^1_e(\Omega) \subset Y_e \subset H^1(\Omega)$ be a Hilbert space over the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$, with boundary $\Gamma$.

The subscript "e" denotes the "exact" infinite-dimensional continuous problem setting.

Furthermore, let $Y_e$ be a prescribed Hilbert space $U_e = L^2(\Omega)$, together with its inner product $(\cdot, \cdot)_Y$, induced norm $\| \cdot \|_Y = \sqrt{(\cdot, \cdot)_Y}$, and associated dual space $U_e'$. Furthermore, let $\mathcal{P} \subset \mathbb{R}^p$ be a prescribed $P$-dimensional compact parameter set in which the $P$-tuple (input) parameter $\mu = (\mu_1, \ldots, \mu_P)$ resides.

We directly consider a FE "truth" approximation for the exact infinite-dimensional optimal control problem. To this end, we define two conforming FE spaces $Y \subset Y_e$ and $U \subset U_e$ and denote their dimensions by $\mathcal{N}_Y = \dim(Y)$ and $\mathcal{N}_U = \dim(U)$. We assume that the truth spaces $Y$ and $U$ are sufficiently rich such that the FE solutions guarantee a desired accuracy over $\mathcal{P}$.

We next introduce the $\mu$-dependent bilinear form $a(\cdot, \cdot; \mu) : Y \times Y \to \mathbb{R}$, and shall assume that $a(\cdot, \cdot; \mu)$ is (i) continuous for all $\mu \in \mathcal{P}$ with continuity constant $\gamma_a(\mu) \leq \infty$ and (ii) coercive for all $\mu \in \mathcal{P}$ with coercivity constant $\gamma_a(\mu) > 0$. Furthermore, we introduce the $\mu$-dependent continuous linear functional $f(\cdot; \mu) : Y \to \mathbb{R}$ and the bilinear form $b(\cdot, \cdot; \mu) : U \times Y \to \mathbb{R}$ with continuity constant $\gamma_b(\mu) < \infty$.

In anticipation of the optimal control problem, we introduce the parametrized control constraints $u_a(\mu), u_b(\mu) \in U$ and a desired state $y_d \in D$. Here, $D \subset L^2(\Omega_d)$ is a suitable FE space for the observation subdomain $\Omega_d \subset \Omega$. Furthermore, we note that the semi-norm $| \cdot |_D$ for $y \in L^2(\Omega)$ is induced by the $L^2(\Omega)$-norm.

The involved bilinear and linear forms as well as the control constraint are assumed to depend affinely on the parameter. For example we require for all $w, v \in Y$ and all parameters $\mu \in \mathcal{P}$ that $a(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta^a_q(\mu) d^a(w, v)$ and $u_a(x; \mu) = \sum_{q=1}^{Q_a} \Theta^a_{ua}(\mu) u^a_d(x)$, for some (preferably) small integers $Q_a$ and $Q_{ua}$. Here, the coefficient functions $\Theta^a_q(\cdot) : \mathcal{P} \to \mathbb{R}$ are continuous and depend on $\mu$; whereas, the continuous bilinear and linear forms, e.g., $d^a(\cdot, \cdot)$ and $u^a_d \in U$ do not depend on $\mu$. Although we choose $y_d(x)$ to be parameter-independent, our approach directly extends to an affinely parameter-dependent $y_d(x; \mu)$ (see Kärcher et al. [6]).

1 The subscript “e” denotes the “exact” infinite-dimensional continuous problem setting.

2 The framework of this work directly extends to Neumann boundary controls $U_e = L^2(\partial\Omega)$ or finite dimensional controls $U_e = \mathbb{R}^m$. Also distributed controls on a subdomain $\Omega_d \subset \Omega$ or Neumann boundary controls on a boundary segment $\Gamma_d \subset \partial\Omega$ are possible.
For the development of a posteriori error bounds also require additional ingredients. We assume that we are given a positive lower bound $\alpha_u^{LB}(\mu) : \mathcal{D} \to \mathbb{R}_+$ for the coercivity constant $\alpha_u(\mu)$ of $a(\cdot, \cdot ; \mu)$ such that $\alpha_u^{LB}(\mu) \leq \alpha_u(\mu) \forall \mu \in \mathcal{D}$. Furthermore, we assume that we have upper bounds available for the constant $C_D^{UB} \geq C_D = \sup_{u \in \mathcal{Y} \setminus \{0\}} \frac{\|u\|_D}{\|u\|_U} \geq 0 \forall \mu \in \mathcal{D}$, and the continuity constant of the bilinear form $b(\cdot, \cdot ; \mu) : \mathcal{H}_b^{LB}(u) \geq \gamma_b(\mu) \forall \mu \in \mathcal{D}$. In our setting, it is possible to compute these constants (or their bounds) efficiently using an offline-online procedure (see [11, 6]).

### 2.2 Abstract Formulation of Linear-Quadratic Optimal Control Problems and the First-Order Optimality Conditions

We consider the following the FE optimal control problem in a weak form with $u_a(\mu) < u_b(\mu)$

$$\min_{\hat{y}, \hat{u}} J(\hat{y}, \hat{u}) = \frac{1}{2} \|\hat{y} - y_a\|^2_D + \frac{\lambda}{2} \|\hat{u}\|^2_U, \quad \lambda > 0$$

(P)

s.t. $(\hat{y}, \hat{u}) \in Y \times U$ solves

$$a(\hat{y}, \phi; \mu) = b(\hat{u}, \phi; \mu) + f(\phi; \mu) \quad \forall \phi \in Y,$$

$$(u_a(\mu), \rho)_U \leq (\hat{u}, \rho)_U \leq (u_b(\mu), \rho)_U \quad \forall \rho \in U^+,$$

where $U^+ := \{ \rho \in U : \rho \geq 0 \text{ almost everywhere} \}$ and we dropped the $\mu$-dependence of the state and control $(\hat{y}, \hat{u})$ for the sake of readability. We note that the last line of (P) is equivalent to $\hat{u}$ being in the convex admissible set

$$U_{ad} = \{ \psi \in U : (u_a(\mu), \psi)_U \leq (\psi, \rho)_U \leq (u_b(\mu), \psi)_U \forall \rho \in U^+ \}.$$  (1)

In the following we call problem (P) the “primal” problem, for which the existence and uniqueness of the solution is standard (see, e.g., [3]). The derivation of the necessary and sufficient first-order optimality system is straightforward: Given $\mu \in \mathcal{D}$, the optimal solution $(y, p, u, \sigma, \sigma_b) \in Y \times Y \times U \times U \times U$ satisfies

$$a(y, \phi; \mu) = b(u, \phi; \mu) + f(\phi; \mu) \quad \forall \phi \in Y, \quad (2a)$$

$$a(\phi, p; \mu) = (\phi - y, \phi)_D \quad \forall \phi \in Y, \quad (2b)$$

$$(\lambda u, \psi)_U - b(\psi, p; \mu) = (\sigma, \psi)_U - (\sigma_b, \psi)_U \quad \forall \psi \in U, \quad (2c)$$

$$(u_a(\mu) - u, \rho)_U \leq 0 \quad \forall \rho \in U^+,$$

$$(u_a(\mu) - u, \sigma)_U = 0, \quad \sigma \geq 0, \quad (2d)$$

$$(u_b(\mu) - u, \rho)_U \geq 0 \quad \forall \rho \in U^+,$$

$$(u_b(\mu) - u, \sigma_b)_U = 0, \quad \sigma_b \geq 0. \quad (2e)$$

Note that we follow a first-discretize-then-optimize approach here, for a more detailed discussion see [3, Section 3.2.4]).

In the following we comment on the FE-setting in this paper. We assume that the state variable is discretized by $P_1$, i.e., continuous and piecewise linear, and the control variable by $P_0$, i.e., piecewise constant finite elements. Next, we introduce
two bases for the FE spaces \( Y \) and \( U \), such that
\[
Y = \text{span}\{ \phi_i^y, i = 1, \ldots, N_Y \} \quad \text{and} \quad U = \text{span}\{ \phi_i^u, i = 1, \ldots, N_U \},
\]
where \( \phi_i^y \geq 0, i = 1, \ldots, N_Y \), and \( \phi_i^u \geq 0, i = 1, \ldots, N_U \), are the usual hat and bar basis functions. Using these basis functions we can express the functions \( y, u, \sigma, \sigma_b \in U \) as, e.g., \( y = \sum_{i=1}^{N_Y} y_i \phi_i^y \). The corresponding FE coefficient vectors are given by, e.g., \( y = (y_1, \ldots, y_{N_Y})^T \in \mathbb{R}^{N_Y} \). Note that by definition of \( U^+ \) and since \( \phi_i^u \geq 0 \), the condition \( \rho \in U^+ \) in (2d) and (2e) translates into the condition \( \rho \geq 0 \) for the corresponding coefficient vector. Further, we also introduce the control mass matrix \( M_U \) with entries \( (M_U)_{ij} = (\phi_i^u, \phi_j^u)_U \), which is for a \( P_0 \) control discretization a positive diagonal matrix. Hence the point-wise and the 'weak' (averaged) constraint formulations are equivalent \( u(x) \geq u_a(x; \mu) \iff (u, \rho)_U \geq (u_a(\mu), \rho)_U \forall \rho \in U^+ \).
However, this is in general not true for other control discretizations, e.g., \( P_1 \).

Based on the truth FE primal problem (P) we derive an RB primal problem \( \text{(P}_N\text{)} \) and a rigorous \textit{a posteriori} error bound for the error between the truth and RB control approximation in Theorem 1.

### 3 Reduced Basis Method for the Primal Problem

#### 3.1 Reduced Basis Approximation

To begin, we define the RB spaces \( Y_N \subset Y, U_N, \Sigma_N, \Sigma_{b,N} \subset U \) as well as the convex cones \( \Sigma_N^+ \subset U^+, \Sigma_{b,N}^+ \subset U^+ \) as follows: given \( N \) parameter samples \( \mu_1, \ldots, \mu_N \), we set
\[
Y_N = \text{span}\{ \xi_1^y, \ldots, \xi_{N_N}^y \} = \text{span}\{ y(\mu_1), \rho(\mu_1), \ldots, y(\mu_N), \rho(\mu_N) \}, \tag{3a}
\]
\[
U_N = \text{span}\{ \xi_1^u, \ldots, \xi_{N_N}^u \} = \text{span}\{ u(\mu_1), \sigma(\mu_1), \sigma_b(\mu_1), \ldots, u(\mu_N), \sigma(\mu_N), \sigma_b(\mu_N) \}, \tag{3b}
\]
\[
\Sigma_N = \text{span}\{ \xi_1^\sigma, \ldots, \xi_{N_N}^\sigma \} = \text{span}\{ \sigma(\mu_1), \ldots, \sigma(\mu_N) \}, \tag{3c}
\]
\[
\Sigma_{b,N} = \text{span}\{ \xi_1^{\sigma_b}, \ldots, \xi_{N_{N_b}}^{\sigma_b} \} = \text{span}\{ \sigma_b(\mu_1), \ldots, \sigma_b(\mu_N) \}, \tag{3d}
\]
\[
\Sigma_N^+ = \text{span}_{+}\{ \xi_1^\sigma, \ldots, \xi_{N_N}^\sigma \} \quad \text{and} \quad \Sigma_{b,N}^+ = \text{span}_{+}\{ \xi_1^{\sigma_b}, \ldots, \xi_{N_{N_b}}^{\sigma_b} \}, \tag{3e}
\]
where we assume that the basis functions, \( \xi_1^*, \ldots, \xi_{N_N}^* \), are linearly independent. Note that we employ integrated spaces for the state and adjoint as well as for the control (see Remarks 1 and 2). For the spaces \( Y_N \) and \( U_N \) we additionally assume that the basis functions are orthogonal, i.e., \( (\xi_i^y, \xi_j^y)_Y = \delta_i j \) and \( (\xi_i^u, \xi_j^u)_U = \delta_i j \), where \( \delta_i j \) is the Kronecker delta. This orthogonality is favorable to keep the condition of the RB algebraic linear systems small [11]. More importantly, we do not orthogonalize the basis \( \xi_1^\sigma, \xi_1^{\sigma_b} \) of the cones \( \Sigma_N^+, \Sigma_{b,N}^+ \subset U^+ \), because we rely on the non-negativity of the basis functions. We describe the greedy sampling approach to construct the RB spaces in Section 5. Next we derive the RB problem. Given the RB spaces in (3) we
derive the RB primal problem
\[
\min_{\tilde{y}_N, \tilde{u}_N} J(\tilde{y}_N, \tilde{u}_N) = \frac{1}{2} \|y_d - \tilde{y}_N\|^2_U + \frac{\lambda}{2} \|\tilde{u}_N\|^2_U
\] (P)
\[
\text{s.t. } (\tilde{y}_N, \tilde{u}_N) \in Y_N \times U_N \text{ solves } a(\tilde{y}_N, \phi; \mu) = b(\tilde{u}_N, \phi; \mu) + f(\phi; \mu) \forall \phi \in Y_N,
\]
\[
(u_a(\mu), \rho)_U \leq (\tilde{u}_N, \rho)_U \forall \rho \in \Sigma^+_N, (u_b(\mu), \rho)_U \geq (\tilde{u}_N, \rho)_U \forall \rho \in \Sigma^+_b N.
\]
The last two lines of (P) define the admissible set for \(u_N: U_{ad,N} = \{ \psi \in U_N; (u_a(\mu), \rho)_U \leq (\psi, \rho)_U \forall \rho \in \Sigma^+_N, (u_b(\mu), \rho)_U \geq (\psi, \rho)_U \forall \rho \in \Sigma^+_b N \}\), which is in general not a subset of \(U_{ad} \) in (1). Analogously to the primal problem (P) we obtain the RB optimality system: Given \(\mu \in \mathcal{U} \), the optimal solution \((y_N, p_N, u_N, \sigma_N, \sigma_b N) \in Y_N \times Y_N \times U_N \times \Sigma_N \times \Sigma_{b,N} \) satisfies
\[
a(y_N, \phi; \mu) = b(u_N, \phi; \mu) + f(\phi; \mu) \quad \forall \phi \in Y_N, \quad (4a)
\]
\[
a(\phi, p_N; \mu) = (y_d - y_N, \phi)_D \quad \forall \phi \in Y_N, \quad (4b)
\]
\[
(\lambda u_a, \psi)_U - b(\psi, p_N; \mu) = (\sigma_N, \psi)_U - (\sigma_{b,N}, \psi)_U \quad \forall \psi \in U_N, \quad (4c)
\]
\[
(u_a(\mu) - u_N, \rho)_U \leq 0 \forall \rho \in \Sigma^+_N, (u_b(\mu) - u_N, \sigma_N)U = 0, \quad (4d)
\]
\[
(u_a(\mu) - u_N, \rho)_U \geq 0 \forall \rho \in \Sigma^+_b N, (u_b(\mu) - u_N, \sigma_{b,N})U = 0, \quad (4e)
\]

Remark 1 (Existence, uniqueness, integrated space \(Y_N \)). Since (P) is a linear-quadratic optimal control problem over the closed convex admissible set \(U_{ad,N} \), the existence and uniqueness of the RB optimal control \(u_N \) follows from standard arguments (see, e.g., [3, Theorem 1.43]). Also note that we use a single “integrated” reduced basis trial and test space \(Y_N \) for the state and adjoint equations as one ingredient to ensure stability of the system (4), see e.g. Kärcher [4].

Remark 2 (Stability, integrated space \(U_N \)). For the stability of the RB solutions we need to show that the RB inf-sup constants
\[
\beta_N := \inf_{\psi_N \in \Sigma_N, \psi_N \in U_N} \sup_{\psi_N \in \Sigma_N, \psi_N \in U_N} \frac{(|\psi|, \psi_N)_U}{\|\psi_N\|_U}, \quad \beta_{b,N} := \inf_{\psi_N \in \Sigma_N, \psi_N \in U_N} \sup_{\psi_N \in \Sigma_N, \psi_N \in U_N} \frac{(|\psi|, \psi_N)_U}{\|\psi_N\|_U}
\]
are bounded away from zero. We guarantee that \(\beta_N, \beta_{b,N} \geq \beta > 0\) by enriching the RB control space with suitable supremizers [8]. Here, these supremizers are just the multiplier snapshots \(\sigma(\mu^n), \sigma_{b}(\mu^n), 1 \leq n \leq N \); we thus have \(\beta_N = \beta_{b,N} = \beta = 1 \).

### 3.2 Primal Error Bound

We next propose an \textit{a posteriori} error bound for the optimal control. The bound is based on [1], which uses an RB approach for variational inequalities of the first kind [2], and (ii) an RB approach for optimal control problems with a PDE constraint [6]. Before stating the main result, we define the following approximation errors (omitting \(\mu\)-dependencies) of the RB primal system (4)
\[ e_y = y - y_N, \quad e_p = p - p_N, \quad e_u = u - u_N, \quad e_{\sigma} = \sigma - \sigma_N, \quad e_{\sigma_0} = \sigma_0 - \sigma_{0,N} \]

as well as the residuals in the next definition.

**Definition 1 (Residuals).** The residuals of the state equation, the adjoint equation w.r.t. (2a)–(2c) are defined for all \( \mu \in \mathcal{D} \) by

\[ r_y(\phi; \mu) = b(y_N, \phi; \mu) + f(\phi; \mu) - a(y, \phi; \mu) \quad \forall \phi \in Y, \quad (5a) \]
\[ r_p(\phi; \mu) = (y_N - y_N, \phi)_D - a(\phi, p_N; \mu) \quad \forall \phi \in Y, \quad (5b) \]
\[ r_u(\psi; \mu) = -\lambda (u_N, \psi) + b(\psi, p_N; \mu) + (\sigma_N, \psi)_U - (\sigma_{b,N}, \psi)_U \quad \forall \psi \in U. \quad (5c) \]

**Theorem 1 (Primal error bound).** Let \( u \) and \( u_N \) be the optimal controls of the FE primal problem \((P)\) and of the RB primal problem \((P_N)\), respectively. Then the error in the optimal control satisfies for any given parameter \( \mu \in \mathcal{D} \)

\[ \|e_u\|_U \leq \Delta_N^p(\mu), \]

where \( \Delta_N^p(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)} \) with nonnegative coefficients

\[ c_1(\mu) = \frac{1}{2\lambda} \left( \|r_u\|_U + \frac{\gamma_{LUB}^U}{\alpha_a^L} \|r_p\|_Y + \lambda (\delta_1 + \delta_{1b}) \right), \quad (6a) \]
\[ c_2(\mu) = \frac{1}{\lambda} \left[ \frac{2}{\alpha_a^L} \|r_y\|_Y \|r_p\|_Y + \frac{1}{4} \left( \frac{C_{LUB}^U}{\alpha_a^L} \left( \|r_y\|_Y + \gamma_{LUB}^U (\delta_1 + \delta_{1b}) \right) \right)^2 \right. \]
\[ \left. + \left( \|r_u\|_U + \frac{\gamma_{LUB}^U}{\alpha_a^L} \|r_p\|_Y + \sqrt{2(\sigma_N, \sigma_{b,N})_U} (\delta_1 + \delta_{1b}) + \delta_2 + \delta_{2b} \right) \right], \quad (6b) \]

and \( \delta_1 = \|[u - u_N]_+\|_U, \quad \delta_2 = (|u - u_N|_+, \sigma_N)_U, \quad \delta_{1b} = \|[u_N - u_b]_+\|_U, \quad \delta_{2b} = (|u_N - u_b|_+, \sigma_{b,N})_U. \)

Here \([.]_+ = \max(\cdot, 0)\) denotes the positive part (a.e.). Note that we sometimes use \( r_\cdot \) instead of \( r_\cdot(\cdot; \mu) \) and omit the \( \mu \)-dependencies on the r.h.s. of (6) for a better readability.

**Proof.** This proof follows the proof of the primal error bound from [1]. Since the FE optimal solution \((y, p, u, \sigma, \sigma_0)\) satisfies the optimality conditions (2), we obtain the following error-residual equations:

\[ a(e_y, \phi; \mu) - b(e_u, \phi; \mu) = r_y(\phi; \mu) \quad \forall \phi \in Y, \quad (7a) \]
\[ a(\phi, e_p; \mu) + (e_y, \phi)_D = r_p(\phi; \mu) \quad \forall \phi \in Y, \quad (7b) \]
\[ \lambda(e_u, \psi)_U - b(\psi, e_p; \mu) - (e_{\sigma}, \psi)_U + (e_{\sigma_0}, \psi)_U = r_u(\psi; \mu) \quad \forall \psi \in U. \quad (7c) \]

From (7a) with \( \phi = e_y \), (7b) with \( \varphi = e_p \), and \( \alpha_{a}^{LUB}(\mu) \leq \alpha_a(\mu) \) we infer that

\[ \|e_y\|_Y \leq \frac{1}{\alpha_{a}^{LUB}} \left( \|r_y\|_Y + \gamma_{LUB}^U \|e_u\|_U \right), \quad \|e_p\|_Y \leq \frac{1}{\alpha_{a}^{LUB}} \left( \|r_p\|_Y + C_{LUB}^{U} \|e_y\|_D \right). \quad (8) \]
Choosing $\phi = e_p$, $\varphi = e_y$, $\psi = e_u$ in (7), adding (7b) and (7c), and subtracting (7a) results in

$$
\lambda ||e_u||_U^2 + |e_y||D| \leq ||r_y||_U^2 ||e_p||_U^2 + ||r_p||_U^2 ||e_y||_U + ||r_u||_U^2 ||e_u||_U + (e_\sigma - e_{\sigma_0}, e_u)_U. \tag{9}
$$

Next, we bound $(e_\sigma, e_u)_U$ and $-(e_{\sigma_0}, e_u)_U$. We first consider $(e_\sigma, e_u)_U$ and note that

$$(e_\sigma, e_u)_U = (\sigma - \sigma_N, u - u_N)_U = (\sigma, u - u_N)_U + (\sigma_N, u_N - u)_U$

$$
= (\sigma, u - u_a(\mu))_U + (\sigma_N, u_N - u_a(\mu))_U + (\sigma_N, u_N - u_a(\mu))_U + (\sigma_N, u_N - u_a(\mu))_U,$$

where, except for the second term, all terms are nonpositive, see (2d) and (4d).

Hence

$$(e_\sigma, e_u)_U \leq (\sigma, u_a(\mu) - u_N)_U \leq (\sigma, [u_a(\mu)]^+) U\tag{10}
= (\sigma - \sigma_N, u_a(\mu) - u_N)_U + (\sigma_N, [u_a(\mu) - u_N]) U \leq ||e_\sigma||_U |\delta_1 + \delta_2|.
$$

Analogously, we bound $-(e_{\sigma_0}, e_u)_U \leq ||e_{\sigma_0}||_U |\delta_1| + |\delta_2|$. Most significantly, it remains to bound the terms $||e_\sigma||_U$ and $||e_{\sigma_0}||_U$, which we achieve in two steps: First, we relate $||e_\sigma||_U$ and $||e_{\sigma_0}||_U$ with $||e_\sigma - e_{\sigma_0}||_U$ by

$$||e_\sigma||_U^2 + \|e_{\sigma_0}\|_U^2 = ||e_\sigma - e_{\sigma_0}||_U^2 + 2 (\sigma, \sigma - \sigma_N, \sigma_N)_U + (\sigma, \sigma_N, \sigma_N)_U - (\sigma, \sigma_N, \sigma_N)_U$$

If we employ $(\sigma, \sigma_N)_U = 0$, $(\sigma, \sigma_N)_U \geq 0$, and $(\sigma, \sigma_N)_U \geq 0$, we obtain

$$||e_\sigma||_U \cdot ||e_{\sigma_0}||_U \leq ||e_\sigma - e_{\sigma_0}||_U + \sqrt{2(\sigma, \sigma_N)}.$$

Second, we focus on the optimality residual (7c), use the inf-sup stability of $(\cdot, \cdot)_U$ and (8) to derive $||e_\sigma - e_{\sigma_0}||_U \leq ||r_u||_U + \lambda ||e_u||_U + \frac{\lambda}{\alpha^U} \|r_p\|_U ||e_y||_U + \gamma^U \|e_{\sigma_0}||_D$.

Next, we employ the inequalities (8) and (11) in (9) to obtain

$$
\lambda ||e_u||_U^2 + |e_y||D| \leq ||e_u||_U \left(||r_u||_U + \frac{\gamma^U}{\alpha^U} ||r_p||_U ||e_y||_U + \lambda (|\delta_1| + |\delta_2|) \right)
+ \frac{2}{\alpha^U} ||r_y||_U ||e_p||_U ||e_y||_U + \gamma^U \left(||r_p||_U + \gamma^U ||e_{\sigma_0}||_D \right) \left(|\delta_1| + |\delta_2| \right)
+ \left(||r_u||_U + \frac{\gamma^U}{\alpha^U} ||r_p||_U ||e_y||_U + \frac{\gamma^U}{\alpha^U} ||e_{\sigma_0}||_D \right) (|\delta_1| + |\delta_2|) + |\delta_2| + |\delta_2|
$$

It thus follows from applying Young’s inequality to the $|e_y|_D$-terms in (12) that

$$||e_u||_U^2 - 2c_1(\mu) ||e_u||_U - c_2(\mu) \leq 0,$$

where $c_1(\mu)$ and $c_2(\mu)$ are given in (6). Solving the last inequality for the larger root yields $||e_u||_U \leq c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)} = \Delta^p_0(\mu)$. 
We note that most of the ingredients of the primal error bound $\Delta_{pr}^{\mu}(\mu)$ introduced in Theorem 1 are standard, i.e., the dual norms of state, adjoint, and control residuals, as well as coercivity and continuity constants or rather their lower and upper bounds [11, 6]. The only non-standard terms are $\delta_1, \delta_2, \delta_{1b}$ and $\delta_{2b}$, which measure the constraint-violation of the RB optimal control $u_N$. As a result, the online computational cost to evaluate $\delta$ — and hence the error bound $\Delta_{pr}^{\mu}(\mu)$ — depends on the FE control dimension $N_U$, requiring $O((N_U + N_\sigma + N_{\sigma b})N_U)$ operations.

4 Slack Problem and the Primal-Slack Error Bound

In this section we introduce a reformulation of the original primal problem by means of a slack variable. We extend the ideas presented for the one-sided control-constrained problem in [1] to the two-sided control-constrained problem. First, we reformulate the original optimization problem (P) by replacing the control variable with a slack variable that depends on one of the two constraints $u_a(\mu)$ or $u_b(\mu)$. Second, we use snapshots of the slack variable to construct an associated convex cone, leading to strictly feasible approximations w.r.t. either the lower or upper constraint. Third, we derive two RB slack problems by restricting the RB-slack coefficients to a convex cone. And finally, we propose an a posteriori $N$-independent error bound for RB slack approximation w.r.t. either the lower or upper constraint in Theorem 2.

4.1 FE and RB Slack Problem

We consider the FE optimization problem (P) and introduce the slack variable $s \in U^+$ given by $s = u - u_a(\mu)$ together with the corresponding FE coefficient vector $\bar{s} = u - u_a(\mu)$, where we state the slack variable w.r.t. $u_a(\mu)$. Here, we again omit the explicit dependence of $u$ and $s$ on the parameter $\mu$. We note that, by construction, the feasibility of $u$ w.r.t. $u_a(\mu)$ is equivalent to $M_U \bar{s} \geq 0$, which in turn is equivalent to $\bar{s} \geq 0$, if we are using $P_0$ elements.

If we substitute $u$ by $s + u_a(\mu)$ in (P), we obtain the “slack” optimization problem

$$\min_{\bar{s}, \hat{s}} J_s(\hat{s}, \bar{s}) = \frac{1}{2} |\hat{s} - y_d|_D^2 + \frac{\lambda}{2} \|\bar{s} + u_a(\mu)\|_U^2$$

\text{s.t.} \quad (\bar{s}, \hat{s}) \in Y \times U^+ \quad \text{solves} \quad a(\hat{s}, \varphi; \mu) = b(\bar{s} + u_a(\mu), \varphi; \mu) + f(\varphi; \mu) \forall \varphi \in Y,

\quad (u_a(\mu), \rho)_U \geq (\bar{s} + u_a(\mu), \rho)_U \quad \forall \rho \in U^+.

Analogously, we define a slack variable $s_b = u_b(\mu) - u_b^b$ w.r.t. $u_b(\mu)$ and recast (P) w.r.t. $s_b$. We do not state this minimization problem explicitly since it is analogous to (S).

In the following we derive two RB slack problems w.r.t. $u_a(\mu)$ and $u_b(\mu)$. We start with the former and reuse the RB space $Y_N$, introduced in Section 3.1, for
the state and adjoint variables. Furthermore, for the RB approximation of the slack variable we simply introduce an RB slack space $S_N$ and a convex cone by shifting the control snapshots of (P) with the control constraint $u_a(\mu)$

$$S_N = \text{span}\{\xi_1^s, \ldots, \xi_N^s\} = \text{span}\{u(\mu^1) - u_a(\mu^1), \ldots, u(\mu^N) - u_a(\mu^N)\},$$

$$S_N^+ = \text{span}_+\{\xi_1^s, \ldots, \xi_N^s\} \subset U^+.$$ (13a)

$$S_N^+ = \text{span}_+\{\xi_1^s, \ldots, \xi_N^s\} \subset U^+.$$ (13b)

We assume that the snapshots $\xi_1^s, \ldots, \xi_N^s$ are linearly independent and not orthogonalized. Further, we need to consider a Lagrange multiplier $\sigma_a^b \in U^+$ for the constraint $u_b(\mu)$ by incorporating the RB space $\Sigma_{b,N} \subset U$, as well as the convex cone $\Sigma_{b,N}^+$ from (3d) and (3e). Overall, we notice that the positive cone $S_N^+$ satisfies $S_N^+ \subset U^+$. If we now require the RB slack approximation $s_N$ of the slack variable to lie in $S_N^+$, it follows that $s_N \geq 0$. This results in a feasible w.r.t. $u_a(\mu)$ (not necessarily w.r.t. $u_b(\mu)$) control approximation $u^* = s_N + u_a(\mu)$. Note that $s_N \in S_N^+$ is equivalent to nonnegative RB coefficients $\sigma_a^b$.

Overall, employing the RB spaces in (S) results in the RB slack problem

$$\min J_N(\hat{y}^s_N, \hat{s}_N) = \frac{1}{2} \|\hat{y}_N - y_d\|_D^2 + \frac{1}{2} \|\hat{s}_N + u_a(\mu)\|_U^2$$

s.t. $(\hat{y}^s_N, \hat{s}_N) \in Y_N \times S_N^+$ solves $a(\hat{y}^s_N, \phi; \mu) = b(\hat{s}_N + u_a(\mu), \phi; \mu) + f(\phi; \mu) \forall \phi \in Y_N$, $(u_b(\mu), \rho)_U \geq (\hat{s}_N + u_a(\mu), \rho)_U \forall \rho \in \Sigma_{b,N}^+.$

As in the RB primal problem (P_N), the existence and uniqueness of the RB optimal control follows from the same arguments as in the Remark 1. Next, we derive the optimality conditions for $s_N$; however, we here follow the ‘first-discretize-then-optimize’ approach that will eventually lead to a feasible —w.r.t. $u_a(\mu)$— approximation of the control. We perform two steps.

First, we use the RB-representations of $\hat{y}^s_N, s_N$ with their RB-coefficient vectors $\hat{y}^s_N, \hat{s}_N$ to discretize (S_N). Since the algebraic RB slack problem is simple to derive, we only state the main crucial condition that $\hat{s}_N \geq 0$. Next, we derive the first-order optimality conditions. We introduce a discrete Lagrange multiplier $\hat{\omega}_N \in \mathbb{R}^{N_s}$, $\hat{\omega}_N \geq 0$, ensuring the non-negativeness of $\hat{s}_N$ and derive the following necessary (and here sufficient) first-order optimality system: Given $\mu \in \mathcal{D}$, the optimal RB slack solution coefficients $(\hat{y}^s_N, \hat{s}_N, \hat{\omega}_N, \hat{\omega}_N) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_s} \times \mathbb{R}^{N_s} \times \mathbb{R}^{N_b} \times \mathbb{R}^{N_b}$ satisfy (omitting all $\mu$-dependencies)

$$A_N^T \hat{Y}_N \hat{s}_N = F_N + B_N^s \hat{Y}_N + B_a^s,$$ (14a)

$$\lambda U_N^s + \lambda U_a^s - (B_N^s)^T \hat{p}_N^s = \hat{\omega}_N - U_N^{\sigma_a^b} \hat{\sigma}_b^s,$$ (14b)

$$\lambda U_{b,N}^s \hat{s}_N + \lambda U_{a,N}^s - (B_{b,N}^s)^T \hat{p}_{b,N}^s = \hat{\omega}_N - U_N^{\sigma_b^a} \hat{\sigma}_a^s.$$ (14c)

$$\hat{s}_N = 0, \hat{\omega}_N = 0, \hat{\omega}_N \geq 0.$$ (14d)

$$(U_{b,N}^{\sigma_a^b} - U_{a,N}^{\sigma_b^a} \hat{Y}_N^s, \hat{\sigma}_b^s) = 0, (U_{b,N}^{\sigma_b^a} - U_{a,N}^{\sigma_a^b} \hat{Y}_N^s, \hat{\sigma}_a^s) \geq 0,$$ (14e)

where the reduced basis matrices and vectors are given by
Theorem 2 (Primal-slack error bound).

\[ \| \mu \|_{r_{\text{residual}}} \]

4.2 Primal-Slack Error Bound

We state the main result in the following theorem.

**Theorem 2 (Primal-sslack error bound).** Let \( u, s_N, \) and \( s_{b, N} \) be the optimal solutions of the FE primal problem \((P)\) and the RB slack problems \((S_N)\) and its equivalent w.r.t. \( u_b(\mu) \), respectively. Then the error in the optimal control satisfies for all parameters \( \mu \in \mathcal{S} \)

\[
\| e_1^s \|^2_{U} \leq \Delta_{N}^{\text{pr-sl}}(\mu),
\]

where \( \Delta_{N}^{\text{pr-sl}}(\mu) := c_1^s(\mu) + \sqrt{c_2^s(\mu)} \) with nonnegative coefficients

\[
c_1^s(\mu) = \frac{1}{2\lambda} \left( \| s_{pr}^u \|_{U}^2 + \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y}^2 + \| u^b \|^2_{U} \right),
\]

\[
c_2^s(\mu) = \frac{1}{\lambda} \left[ \frac{2}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} \| s_{pr}^p \|_{Y} + \frac{1}{3} \left( \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} + \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| u^b \|^2_{U} \right)^2 + (\sigma_N, s_N)_U \right] \frac{2}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} + \frac{1}{3} \left( \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} + \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| u^b \|^2_{U} \right)^2 + (\sigma_{b, N}, s_{b, N})_U \right] \frac{2}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} + \frac{1}{3} \left( \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| s_{pr}^u \|_{Y} + \frac{\gamma^B_{UB}}{\alpha^L_{UB}} \| u^b \|^2_{U} \right)^2 + (\sigma_{b, N}, s_{b, N})_U \right]
\]

**Proof.** Let the FE primal solution \((y, p, u, \sigma, \sigma_b)\) satisfy the optimality conditions (2). We follow the proof of Theorem 1, and derive analogously to (9) the inequality.
\[ \lambda \| e_n^u \|^2_U + \| e_n^s \|^2_U \leq \| r^p \|^2 \| Y \| + \| r^s \|^2 \| Y \| + \| e_n^p \|^2 \| Y \| + \| e_n^s \|^2 \| U \| + (e_\alpha - e_\alpha, e_n^\nu U). \] (16)

We first focus on \((e_\sigma, e_n^\nu U)\) and exploit the feasibility of \(u^i\) w.r.t. \(u_0(\mu)\). Again we have \((e_\sigma, e_n^\nu U) = -(\sigma, u_0(\mu) - u) - (\sigma, s_N) - (\sigma_N, u_0(\mu) - u) + (\sigma_N, s_N) U\), where the first three terms are nonpositive and hence \((e_\sigma, e_n^\nu U) \leq (\sigma_N, s_N) U\). In order to bound \(- (e_\sigma, e_n^\nu U)\), we need to solve the second RB slack problem for \(u^b\) and derive

\[- (e_\sigma, e_n^\nu U) = (e_\sigma, u^i - u^b + u^b - u) \leq \| e_\sigma \|_U \| u^i - u^b \|_U + (\sigma_{b,N}, s_{b,N}) U.\]

We restate that \(\| e_\sigma \|_U \leq \| e_\sigma - e_\sigma \|_U + \sqrt{2(\sigma_N, \sigma_{b,N}) U}\) and \(\| e_\sigma - e_\sigma \|_U\) is bounded by \(\| e_\sigma - e_\sigma \|_U \leq \| e_\sigma \|_U + \| e_\sigma \|_U + \frac{\sqrt{2(\sigma_N, \sigma_{b,N}) U}}{\sqrt{2e_0}} \| u^i - u^b \|_U\). Using the bounds for \((e_\sigma, e_n^\nu U) - (e_\sigma, e_n^\nu U)\) and the inequalities (8) in (16) obtain

\[
\lambda \| e_n^\nu U \|^2 + \| e_n^s \|^2 \leq \| r^p \|_U \left( \| r^p \|_U \left( \| r^p \|_U \left( \| e_n^\nu U \|^2 + \| e_n^s \|^2 \right) \right) + (\sigma_N, s_N) U \right) + \frac{2}{\sqrt{2e_0}} \left( \right) + \left( \right) + \left( \right) + \left( \right) + \left( \right) + \left( \right) + \left( \right) + \left( \right) = \Delta_{pr-sl}^N(\mu).
\]

It thus follows from employing Young’s inequality to the \(|e_0|_D\)-term that \(\| e_n^\nu U \|^2 - 2c_1(\mu) \| e_n^\nu U \|^2 - c_2(\mu) \leq 0\), where \(c_1(\mu)\) and \(c_2(\mu)\) are given in (15). Solving the last inequality for the larger root yields \(\| e_n^\nu U \| \leq c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)} = \Delta_{pr-sl}^N(\mu)\).

### 5 Greedy Sampling Procedure

The reduced basis spaces for the two-sided control-constrained optimal control problem in sections 3.1, and 4.1 are constructed using the greedy sampling procedure outlined in Algorithm 1. Suppose \(X_{train} \subset D\) is a finite but suitably large parameter train sample, \(\mu^1 \in X_{train}\) is the initial parameter value, \(N_{max}\) the maximum number of greedy iterations, \(e_{\text{tol}, \text{min}} > 0\) is a prescribed desired error tolerance, and \(\Delta_{N}^*(\mu) / \| u_N \|_U, \mu \in \{pr, pr-sl\}\) is the primal or primal-slack error bound from (6) or (15) with \(u_N^0 \in \{u_0, u^i\}\).

We make two remarks: First, by using the bounds \(\Delta_{N}^*(\mu), \mu \in \{pr, pr-sl\}\) we only refer to the bounds derived for the primal error \(\| u - u_N \|_U\) and the slack error \(\| u - u_N \|_U\) w.r.t. to \(u_0(\mu)\). Therefore, using the primal-slack bound \(\Delta_{N}^{pr-sl}(\mu)\) in the greedy sampling procedure, we expect not only to construct an accurate RB space \(S_N\) for \(s_N\) but also an accurate RB space \(S_{b,N}\) for \(s_{b,N}\). Second, we comment again on two pathological cases: (1) if the one control constraint is fully active in each greedy step, i.e. we have, e.g., \(\mu(n) = u_n\), \(n = 1, \ldots, N\), we set \(S_N = \{\}\) and \(s_N = 0\) (analogously for \(u_b\) we set \(S_{b,N} = \{\}\) and \(s_{b,N} = 0\)); and (2) if the control constraint
Algorithm 1 Greedy Sampling Procedure

1: Choose $\mathcal{E}_{\text{init}} \subset \Omega$, $\mu^1 \in \mathcal{E}_{\text{train}}$ (arbitrary), $N_{\text{max}}$, and $\epsilon_{\text{tol, min}} > 0$
2: Set $N \leftarrow 1$, $Y_0 \leftarrow \{0\}$, $U_0 \leftarrow \{0\}$, $S_0 \leftarrow \{0\}$, $S_{b,0} \leftarrow \{0\}$, $\Sigma_0 \leftarrow \{0\}$, $\Sigma_{b,0} \leftarrow \{0\}$
3: Set $\Delta^*_N(\mu) \leftarrow -\infty$
4: while $\Delta^*_N(\mu)/\|u^*_N(\mu)\|_U > \epsilon_{\text{tol, min}}$ and $N \leq N_{\text{max}}$ do
   5: \hspace{1em} $Y_N \leftarrow Y_{N-1} \oplus \text{span}\{v(\mu^N), p(\mu^N)\}$
   6: \hspace{1em} $U_N \leftarrow U_{N-1} \oplus \text{span}\{u(\mu^N), \sigma(\mu^N), \sigma_b(\mu^N)\}$
   7: \hspace{1em} $S_N \leftarrow S_{N-1} \oplus \text{span}\{s(\mu^N)\}$
   8: \hspace{1em} $S_{b,N} \leftarrow S_{b,N-1} \oplus \text{span}\{s_b(\mu^N)\}$
   9: \hspace{1em} $\Sigma_N \leftarrow \Sigma_{N-1} \oplus \text{span}\{\sigma(\mu^N)\}$
 10: \hspace{1em} $\Sigma_{b,N} \leftarrow \Sigma_{b,N-1} \oplus \text{span}\{\sigma_b(\mu^N)\}$
11: \hspace{1em} $\mu^{N+1} \leftarrow \arg\max_{\mu \in \mathcal{E}_{\text{train}}} \Delta^*_N(\mu)/\|u^*_N(\mu)\|_U$
12: \hspace{1em} $N \leftarrow N + 1$
end while

is never active, i.e., for all snapshots $\sigma(\mu^n) = \sigma_b(\mu^n) = 0$, $n = 1, \ldots, N$, we set $\Sigma_N = \Sigma_N^+ = \Sigma_{b,N} = \Sigma_{b,N}^+ = \{\}$ and $\sigma_N = \sigma_{b,N} = 0$.

6 Numerical Results: Graetz Flow with Parametrized Geometry and Lower and Upper Control Constraints

We consider a Graetz flow problem, which describes heat convection and conduction in a duct. The main goal of this example is to demonstrate the different properties of the approximations and their error bounds. The problem is parametrized by a varying Péclet number $\mu_1 \in [5, 18]$ and a geometry parameter $\mu_2 \in [0.8, 1.2]$. Hence, the parameter domain is $\mathcal{P} = [5, 18] \times [0.8, 1.2]$. The parametrized geometry is given by $\Omega(\mu) = [0, 1.5 + \mu_2] \times [0, 1]$ and is subdivided into three subdomains $\Omega_1(\mu) = [0.2\mu_2, 0.8\mu_2] \times [0, 0.7]$, $\Omega_2(\mu) = [\mu_2 + 0.2, \mu_2 + 1.5] \times [0.3, 0.7]$, and $\Omega(\mu) = \Omega(\mu) \setminus \{\Omega_1(\mu) \cup \Omega_2(\mu)\}$. A sketch of the domain is shown in Figure 1. We impose boundary condition of homogeneous Neumann and non-homogeneous Dirichlet type: $y_n = 0$ on $\Gamma_N(\mu)$, and $y = 1$ on $\Gamma_D(\mu)$. Thus the trial space is given by $Y(\mu) = \{v \in H^1(\Omega(\mu)) : v|_{\Gamma_D(\mu)} = 1\}$. The amount of heat supply in the whole domain $\Omega(\mu)$ is regulated by the distributed control $u \in U(\mu) \subset U(\mu) = L^2(\Omega(\mu))$ and bounded by the lower and upper constraints $u_{\ell} = -0.5$ and $u_u = 1.25$. The observation domain is $\Omega_y(\mu) = \Omega_1(\mu) \cup \Omega_2(\mu)$ and the desired state is given by $y_d = 0.5$ on $\Omega_1(\mu)$ and $y_d = 2$ on $\Omega_2(\mu)$.

Overal, the parametrized optimal control problem is given by
\[
\min_{\tilde{y} \in Y(\mu), \tilde{u} \in U(\mu)} J(\tilde{y}, \tilde{u}; \mu) = \frac{1}{2} \|	ilde{y} - y_f\|^2 + \frac{\lambda}{2} \|\tilde{u}\|^2_{L^2(\Omega(\mu))}
\]

s.t. \[
\frac{1}{\mu_1} \int_{\Omega(\mu)} \nabla \tilde{y} \cdot \nabla \phi \, dx + \int_{\Omega(\mu)} \beta(x) \cdot \nabla \tilde{y} \, dx = \int_{\Omega(\mu)} \tilde{u} \phi \, dx \quad \forall \phi \in Y(\mu),
\]

\[
(u_0, \rho)_{U(\mu)} \leq (\tilde{u}, \tilde{\rho})_{U(\mu)} \leq (u_h, \rho)_{U(\mu)} \quad \forall \rho \in U(\mu)^+,
\]

for the given parabolic velocity field \( \beta(x) = (x_2(1 - x_2), 0)^T \). The regularization parameter \( \lambda \) is fixed to 0.01.

After recasting the problem to a reference domain \( \Omega = \Omega(\mu^{\text{ref}}) = [0, 2.5] \times [0, 1] \) for \( \mu^{\text{ref}} = (5, 1) \), and introducing suitable lifting functions that take into account the non-homogeneous Dirichlet boundary conditions, we can reformulate the problem in terms of the parameter-independent FE space \( Y \subset H^1_0(\Omega) \) and \( U \subset U_0 = L^2(\Omega) \) \cite{11}. We then obtain the affine representation of all involved quantities with \( Q_y = Q_f = 4 \), \( Q_b = Q_d = Q_u = Q_{\text{pr}} = 2 \), and \( Q_{\text{in}} = 1 \). The details of these calculations are very similar to the details presented by Rozza et al. \cite{11} and Kärcher \cite{4}, and are thus omitted. The inner product for the state space is given by \( (w, v)_Y = \frac{1}{\mu_1} \int_{\Omega} \nabla w \cdot \nabla v \, dx + \frac{1}{\mu} \int_{\Omega} \beta(x) \cdot \nabla v \, dx \) and we obtain a lower bound \( \alpha_{\text{pr}}^Y(\mu) \) for the coercivity constant by the so-called min-theta approach \cite{11}. Note that for the control space we obtain a parameter-dependent inner product \( \langle \cdot, \cdot \rangle_{U(\mu)} \) from the affine geometry parametrization. Hence the control error is measured in the parameter-dependent energy norm \( ||\cdot||_{U(\mu)} \). The derivations of the primal and primal-slack error bounds remain the same in this case and they bound the control error in the energy norm.

Although the introduction of a domain parametrization seems to add an entirely new \( \mu \)-dependence to the primal and the slack problems (P) and (S), the reductions and the error bound derivations can be analogously derived w.r.t. \( \langle \cdot, \cdot \rangle_{U(\mu)} \) instead of \( \langle \cdot, \cdot \rangle_{U(\mu^{\text{ref}})} \). Also the definition of the integrated space \( U_N \) in (3) remains, while in the inf-sup condition of Remark 2 we use \( \langle \cdot, \cdot \rangle_{U(\mu)} \) instead of \( \langle \cdot, \cdot \rangle_U \).

We choose a \( P_1 \) discretization for the state and adjoint, and a \( P_0 \) discretization for the control to obtain \( \dim(Y) = \mathcal{N}_Y \approx 11,000 \) and \( \dim(U) = \mathcal{N}_U \approx 22,000 \). The chosen discretization induces a discretization error of roughly 2%. In Figure 2 we present control snapshots and associated active sets for two different parameters displayed on the reference domain \( \Omega(\mu_2^{\text{ref}} = 1) \). We observe strongly varying control solutions and active sets.

We construct the RB spaces using the greedy procedure described in Algorithm 1 by employing an equidistant train sample \( \mathcal{S}_{\text{train}} \subset \mathcal{D} \) of size \( 30 \cdot 30 = 900 \) (log-scale in \( \mu_1 \) and lin-scale in \( \mu_2 \)) and stop the greedy enrichment after 30 steps. We also introduce a test sample with \( 10 \cdot 5 \) (log \times lin) equidistant parameter points in \([5.2, 17.5] \times [0.82, 1.17] \subset \mathcal{D} \).

In Figure 3 we present, as a function of \( N \), the resulting energy norm errors and bounds over \( \mathcal{S}_{\text{test}} \). Here, the errors and bounds are defined as follows: the primal-slack bound is the maximum of \( \Delta_{\text{pr}}^{\text{sl}}(\mu)/\|u(\mu)\|_{U(\mu)} \) over \( \mathcal{S}_{\text{test}} \), the primal bound is the maximum of \( \Delta_{\text{pr}}^Y(\mu)/\|u(\mu)\|_{U(\mu)} \) over \( \mathcal{S}_{\text{test}} \), and the \( u' \) and \( u_N \) errors are the maxima of \( \|u(\mu) - u'(\mu)\|_{U(\mu)}/\|u(\mu)\|_{U(\mu)} \) and \( \|u(\mu) - u_N(\mu)\|_{U(\mu)}/\|u(\mu)\|_{U(\mu)} \).
over $\Xi_{\text{test}}$, respectively. We observe that both errors and both bounds decay very similarly. Quantitatively, the error bounds are comparable throughout all $N$, since the dominating primal-slack terms $\|r_u\|_{U(\mu)}$ and $\lambda \|u^d - u^b\|_{U(\mu)}$ are comparable to the dominating primal terms $\|r_{u_h}\|_{U(\mu)}$ and $\lambda (\delta_1 + \delta_{1b})$, resulting in $\Delta_{pr}(\mu) \approx \Delta_{pr-sl}(\mu)$.

We briefly report the computational timings: the solution of the FE optimization problem takes $\approx 4$ seconds (for a discretization error of 2%). The RB primal problem, for $N = 25$, is solved in $\approx 0.066s$ and the RB slack problem is solved faster in $\approx 0.029s$, since $\dim(S_N) = 25$. We turn to the evaluation of error bounds: the primal bound takes $0.01s$, whereas the primal-slack bound, given $\sigma_N$, $\sigma_{b,N}$, takes $0.0065s$. From this we can conclude that for $N = 25$ the overall cost for one primal bound evaluation is roughly $0.076s = 0.066s + 0.01s$ and for the primal-slack bound evaluation is roughly $0.13 \approx 0.029 + 0.029 + 0.066 + 0.0065s$, since it relies on three RB solutions.

7 Conclusions

In this paper we extended the ideas from [1] to propose two certified reduced basis approaches for distributed elliptic optimal control problems with two-sided control constraints: a primal and a primal-slack approach. Albeit the reduction for the primal approach was straightforward, the primal-slack approach needed more consideration. We proposed for each constraint a corresponding RB slack problem with an additional Lagrange multiplier. The primal $a posteriori$ error bound from [1] could
be extended for the two-sided case by special properties of the Lagrange multipliers of the two-sided problem. The primal-slack error bound also relies on these properties and in addition uses three RB solutions to derive an $N$-independent error bound. Both the primal and slack RB approximation can be evaluated efficiently using the standard offline-online decomposition. However, on the one hand the primal error bound depends on the FE control dimension and on the other hand the primal-slack error bound relies on three reduced order optimization problems.

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