The $\alpha$-normal labeling method for computing the $p$-spectral radii of uniform hypergraphs

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Abstract

Let $G$ be an $r$-uniform hypergraph of order $n$. For each $p \geq 1$, the $p$-spectral radius $\lambda^{(p)}(G)$ is defined as

$$\lambda^{(p)}(G) := \max_{\sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} = 1} |x_1|^p + |x_2|^p + \cdots + |x_n|^p.$$ 

The $p$-spectral radius was introduced by Keevash-Lenz-Mubayi, and subsequently studied by Nikiforov in 2014. The most extensively studied case is when $p = r$, and $\lambda^{(r)}(G)$ is called the spectral radius of $G$. The $\alpha$-normal labeling method, which was introduced by Lu and Man in 2014, is effective method for computing the spectral radii of uniform hypergraphs. It labels each corner of an edge by a positive number so that the sum of the corner labels at any vertex is 1 while the product of all corner labels at any edge is $\alpha$. Since then, this method has been used by many researchers in studying $\lambda^{(r)}(G)$. In this paper, we extend Lu and Man’s $\alpha$-normal labeling method to the $p$-spectral radii of uniform hypergraphs for $p \neq r$; and find some applications.

Keywords: Uniform hypergraph; $p$-spectral radius; $\alpha$-normal labeling; weighted incidence matrix.

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1. Introduction

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ the $n$-dimensional real space. Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and a real number $p \geq 1$, we denote $||\mathbf{x}||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. We also denote $\mathbb{S}_p^{n-1}$ ($\mathbb{S}_{p,+}^{n-1}$, $\mathbb{S}_{p,+}^{n-1}$) the set of all (nonnegative, positive) real vectors $\mathbf{x} \in \mathbb{R}^n$ with $||\mathbf{x}||_p = 1$.

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Let $G$ be an $r$-uniform hypergraph of order $n$, the polynomial form of $G$ is a multilinear function $P_G(x) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $x \in \mathbb{R}^n$ as

$$P_G(x) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$ 

For any real number $p \geq 1$, the $p$-spectral radius of $G$ is defined as

$$\lambda^{(p)}(G) := \max_{||x||_p = 1} P_G(x).$$

If $x \in S^{n-1}_p$ is a vector such that $\lambda^{(p)}(G) = P_G(x)$, then $x$ is called an eigenvector to $\lambda^{(p)}(G)$. Note that $P_G(x)$ can always reach its maximum at some nonnegative vectors. By Lagrange’s method, we have the eigenequation for $\lambda^{(p)}(G)$ and $x \in S^{n-1}_{p,+}$ as follows:

$$\sum_{\{i, i_2, \ldots, i_r\} \in E(G)} x_{i_2} \cdots x_{i_r} = \lambda^{(p)}(G) x_i^{p-1} \text{ for } x_i \neq 0. \quad (1.1)$$

The $p$-spectral radius has been introduced by Keevash, Lenz and Mubayi [7] and subsequently studied by Nikiforov [11, 12, 5] and Chang et al. [3]. Note that the $p$-spectral radius $\lambda^{(p)}(G)$ shows remarkable connections with some hypergraph invariants. For instance, $\lambda^{(1)}(G)/r$ is equal to the Lagrangian of $G$, which has been investigated by Talbot [15], $\lambda^{(r)}(G)$ is the usual spectral radius introduced by Cooper and Dutle [4], and $\lambda^{(\infty)}(G)/r$ is the number of edges of $G$ (see [11]). It should be announced that we modified the definition of $p$-spectral radius by removing a constant factor $(r-1)!$ from [7], so that the $p$-spectral radius is the same as the one in [4] when $p = r$. This is not essential and does not affect the results at all.

Recall that a weighted incidence matrix $B = (B(v, e))$ of a hypergraph $G$ is a $|V| \times |E|$ matrix such that for any vertex $v$ and any edge $e$, the entry $B(v, e) > 0$ if $v \in e$ and $B(v, e) = 0$ if $v \notin e$. In [10], Lu and Man introduced the $\alpha$-normal labeling method for computing the spectral radii of uniform hypergraphs as follows:

**Theorem 1.1** ([10]) Let $G$ be a connected $r$-uniform hypergraph. Then the spectral radius of $G$ is $\rho(G)$ if and only if there is a weighted incidence matrix $B$ satisfying:

1. $\sum_{v \in V} B(v, e) = 1$, for any $v \in V(G)$;

2. $\prod_{v \in v} B(v, e) = \alpha = \rho(G)^{-r}$, for any $e \in E(G)$;

3. $\prod_{i=1}^\ell \frac{B(v_{i-1}, e_{i-1})}{B(v_i, e_i)} = 1$, for any cycle $v_0 e_1 v_1 e_2 \cdots v_{\ell-1} e_\ell(v_\ell = v_0)$.

The weighted incidence matrix $B(v, e)$ can be viewed as a labeling on the corners of edges. This $\alpha$-normal labeling method has been proved by many researches [6, 9, 8, 1, 13, 17, 18, 16] to be a simple and effective method in the study of spectral radii of uniform hypergraphs. In this paper, we extend Lu and Man’s method to the $p$-spectral radii of uniform hypergraphs for $p \neq r$. The $\alpha$-normal labeling method (for $p \neq r$) is very
different from the CSRH algorithm developed by Chang-Ding-Qi-Yan [3] to compute the $p$-spectrum radii of uniform hypergraphs numerically. Although our method can also be used to compute the $p$-spectral radius of a hypergraph $G$ when $G$ is highly symmetric or hypertree-like, this is not our main purpose. The goal of this paper is to provide a powerful tool to analyze the properties of the $p$-spectral radii of hypergraphs in the same way for the special case $p = r$. We illustrate this by giving several interesting applications. We discover two new monotonic functions characterizing the growth rate of $\lambda^{(p)}(G)$ with respective to the maximum degree and the minimum degrees (Theorem 3.2). We also prove two convex results (Theorem 3.4 and Theorem 3.5) of the $p$-spectral radius. We obtain a tight upper bound using degrees (Theorem 3.1). We determine the $p$-spectral radius of $G_1 \ast G_2$ (Theorem 3.6) and $G_1 \times G_2$ (Theorem 3.7). We study the $p$-spectral radius of the extension of a hypergraph (Theorem 3.8).

The paper is organized in the following way: in Section 2, we develop the $\alpha$-normal labeling method for $p > r$. In Section 3, we present many applications. The $\alpha$-normal labeling method for $p < r$ is handled the last section.

### 2. The $\alpha$-normal labeling method for $p > r$

In this section, we will establish a relation between $\lambda^{(p)}(G)$ and its weighted incidence matrix of a uniform hypergraph $G$. For concepts on hypergraphs we refer the reader to [2]. Before continuing, we need the following Perron–Frobenius theorem for uniform hypergraphs.

**Theorem 2.1** ([11]) Let $G$ be an $r$-uniform hypergraph with no isolated vertices. If $p > r$, then there exists a unique positive eigenvector to $\lambda^{(p)}(G)$.

Given an $r$-uniform hypergraph $G$, for each edge $e \in E(G)$, we put a weight $w(e) > 0$ on $e$. In the following, we always assume that $p > r$.

**Definition 2.1** An $r$-uniform hypergraph $G$ is called $\alpha$-normal if there exist a weighted incidence matrix $B$ and weights $\{w(e)\}$ satisfying

1. $\sum_{e \in E(G)} w(e) = 1$;

2. $\sum_{v, e \in e} B(v, e) = 1$, for any $v \in V(G)$;

3. $w(e)^{p-r} \cdot \prod_{v \in e} B(v, e) = \alpha$, for any $e \in E(G)$.

Moreover, the weighted incidence matrix $B$ and weights $\{w(e)\}$ are called consistent if for any $v \in V(G)$ and $v \in e_i, i = 1, 2, \ldots, d$,

$$\frac{w(e_1)}{B(v, e_1)} = \frac{w(e_2)}{B(v, e_2)} = \cdots = \frac{w(e_d)}{B(v, e_d)}.$$
Lemma 2.1 Let $G$ be an $r$-uniform hypergraph with no isolated vertices. Then the $p$-spectral radius of $G$ is $\lambda^{(p)}(G)$ if and only if $G$ is consistently $\alpha$-normal with $\alpha = r^{p-r}/(\lambda^{(p)}(G))^p$.

Proof. (\(\Rightarrow\)) By Theorem 2.1, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in S_{p, +}^{n-1}$ be an eigenvector to $\lambda^{(p)}(G)$. Define a weighted incidence matrix $B$ and \{w(e)\} as follows:

$$B(v, e) = \begin{cases} \frac{\prod_{u \in e} x_u}{\lambda^{(p)}(G)x_v^p} & \text{if } v \in e, \\ 0, & \text{otherwise,} \end{cases}$$

$$w(e) = \frac{r\prod_{u \in e} x_u}{\lambda^{(p)}(G)}.$$

For any $v \in V(G)$, using the eigenequation (1.1) gives

$$\sum_{e \ni v \in e} B(v, e) = \frac{\sum_{e \ni v \in e} \prod_{u \in e} x_u}{\lambda^{(p)}(G)x_v^p} = 1.$$

Also, we see that

$$\sum_{e \in E(G)} w(e) = \frac{r}{\lambda^{(p)}(G)} \sum_{e \in E(G)} \prod_{u \in e} x_u = \frac{\lambda^{(p)}(G)}{\lambda^{(p)}(G)} = 1.$$

Therefore items (1) and (2) of Definition 2.1 are verified. For item (3), we check that

$$w(e)^{p-r} \cdot \prod_{v \in e} B(v, e) = \left(\frac{r}{\lambda^{(p)}(G)} \prod_{u \in e} x_u\right)^{p-r} \cdot \prod_{v \in e} \frac{\prod_{u \in e} x_u}{\lambda^{(p)}(G)x_v^p} = \frac{r^{p-r}}{(\lambda^{(p)}(G))^p} = \alpha.$$

To show that $B$ is consistent, for any $v \in V(G)$ and $v \in e_i$, $i = 1, 2, \ldots, d$, we have

$$\frac{w(e_1)}{B(v, e_1)} = \frac{w(e_2)}{B(v, e_2)} = \cdots = \frac{w(e_d)}{B(v, e_d)} = r x_v^p.$$

(\(\Leftarrow\)) Assume that $G$ is consistently $\alpha$-normal with weighted incidence matrix $B$ and \{w(e)\}. For any nonnegative vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in S_{p, +}^{n-1}$, by Hölder’s inequality and AM–GM inequality, we have

$$P_G(\mathbf{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}$$

$$\leq \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} w(e)^{(p-r)/p} \right)^{(p-r)/p} \left( \sum_{e \in E(G)} (B(v, e))^{1/r} x_v^{p/r} \right)^{r/p}.$$
\[
\begin{align*}
&= \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} \prod_{v \in e} \left( B(v, e) \right)^{1/r} x_v^{p/r} \right)^{r/p} \\
&\leq \frac{r^{1-r/p}}{\alpha^{1/p}} \left( \sum_{e \in E(G)} \sum_{v \in e} B(v, e) x_v^p \right)^{r/p} \\
&= \frac{r^{1-r/p}}{\alpha^{1/p}} \cdot ||x||_p^r = \frac{r^{1-r/p}}{\alpha^{1/p}}.
\end{align*}
\]

This inequality implies \( \lambda^{(p)}(G) \leq r^{1-r/p} \alpha^{-1/p} \).

The equality holds if \( G \) is \( \alpha \)-normal and there is a nonzero solution \( x \) to the following equations:

\[
B(i_1, e)x_{i_1}^p = B(i_2, e)x_{i_2}^p = \cdots = B(i_r, e)x_{i_r}^p = \frac{w(e)}{r},
\]

for any \( e = \{i_1, i_2, \ldots, i_r\} \in E(G) \). Define

\[
x_v^* = \left( \frac{w(e)}{rB(v, e)} \right)^{1/p}, \quad v \in e.
\]

The consistent condition guarantees that \( x_v^* \) is independent of the choice of the edge \( e \).

It is easy to check that \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) is a solution of (2.1). Equation (2.2) also implies that

\[
rB(v, e)(x_v^*)^p = w(e), \quad v \in e.
\]

Hence, the \( \ell^p \)-norm of \( x^* \) is

\[
\sum_{v \in V(G)} (x_v^*)^p = \sum_{e \in E(G)} \sum_{u \in e} B(u, e)(x_u^*)^p \\
= \sum_{e \in E(G)} rB(u, e)(x_u^*)^p \\
= \sum_{e \in E(G)} w(e) = 1,
\]

which follows that \( \lambda^{(p)}(G) = r^{1-r/p} \alpha^{-1/p} \). \( \square \)

**Example 2.1** Consider the following grid hypergraph \( G_1 \), which is a 4-uniform hypergraph with 25 vertices and 16 edges generated by subdividing a square. Let

\[
w_1 = \frac{1}{4(1 + 4^{1/(p-2)})^2}, \quad w_2 = \frac{4^{1/(p-2)}}{4(1 + 4^{1/(p-2)})^2}, \quad w_3 = \frac{4^{2/(p-2)}}{4(1 + 4^{1/(p-2)})^2}.
\]

For each vertex \( v \in V(G_1) \) and edge \( e \in E(G_1) \) with \( v \in e \), we put a weight \( w(e) \) at the center of \( e \), and label the value \( B(v, e) \) at each corner of the edge \( e \) as follows:
It can be checked that the grid hypergraph $G_1$ is consistently $\alpha$-normal with

$$\alpha = \frac{1}{4^{p-4}(1 + 4^{1/(p-2)})^2(p-2)}.$$

Therefore, the $p$-spectral radius of $G_1$ is

$$\lambda^{(p)}(G_1) = (16)^{1-4/p}(1 + 4^{1/(p-2)})^2(p-2)/p.$$  

In [11], Nikiforov proved that the $p$-spectral radius is a Lipschitz function in $p$. Taking $p \to 4^+$, we obtain that the spectral radius of $G_1$ is $\rho(G_1) = 3$.

The consistent condition in Definition 2.1 shows that

$$B(v, e) = \frac{w(e)}{\sum_{f: v \in f} w(f)}$$

for any $v \in e$. Therefore we immediately have the following statement: Let $G$ be an $r$-uniform hypergraph, then the $p$-spectral radius of $G$ is $\lambda^{(p)}(G)$ if and only if there exist weights $\{w(e)\}$ such that $\sum_{e \in E(G)} w(e) = 1$ and for each $e \in E(G)$,

$$\frac{w(e)^p}{\prod_{v \in e} \sum_{f: v \in f} w(f)} = \alpha,$$

with $\alpha = r^{p-r}/(\lambda^{(p)}(G))^p$. In some cases, the above conclusion is convenient to calculate $\lambda^{(p)}(G)$.

**Example 2.2** Consider the following 3-uniform hypergraph $G_2$ with 8 vertices and 4 edges.
Putting a weight $w_i$ on each edge $e_i$, $i = 1, 2, 3, 4$, the consistent condition shows that $w_1 = w_4$, and $w_2 = w_3$. We can obtain $\lambda(p)(G_2)$ by solving the following system of equations:

$$\begin{cases} 
\frac{w_1^p}{2w_1^2(2w_1 + 2w_2)} = \alpha \\
\frac{w_2^p}{2w_2^2(2w_1 + 2w_2)} = \alpha \\
2w_1 + 2w_2 = 1.
\end{cases}$$

We get $w_1 = \frac{2^{1/(p-2)}}{2(1+2^{1/(p-2)})}$, $w_2 = \frac{1}{2(1+2^{1/(p-2)})}$, and $\alpha = 2^{2-p}(1 + 2^{1/(p-2)})^{2-p}$. Thus, the $p$-spectral radius of $G_2$ is

$$\lambda(p)(G_2) = 3^{1-3/p} \cdot 2^{1-2/p} \cdot (1 + 2^{1/(p-2)})^{1-2/p}.$$ 

In particular, taking $p \to 3^+$, we get $\rho(G_2) = 3\sqrt{6}$.

**Definition 2.2** An $r$-uniform hypergraph $G$ is called $\alpha$-subnormal if there exist a weighted incidence matrix $B$ and weights $\{w(e)\}$ satisfying

1. $\sum_{e \in E(G)} w(e) \leq 1$;
2. $\sum_{e \in E(G)} B(v, e) \leq 1$, for any $v \in V(G)$;
3. $w(e)^{p-r} \cdot \prod_{v \in e} B(v, e) \geq \alpha$, for any $e \in E(G)$.

Moreover, $G$ is called strictly $\alpha$-subnormal if it is $\alpha$-subnormal but not $\alpha$-normal.

**Lemma 2.2** Let $G$ be an $r$-uniform hypergraph. If $G$ is $\alpha$-subnormal, then the $p$-spectral radius of $G$ satisfies

$$\lambda(p)(G) \leq \frac{r^{1-r/p}}{\alpha^{1/p}}.$$ 

**Proof.** For any nonnegative vector $x = (x_1, x_2, \ldots, x_n)^T \in S_{p,+}^{n-1}$, by Hölder’s inequality and AM–GM inequality, we have

$$r \sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} \leq r^{1/p} \frac{\alpha^{1/p}}{\alpha^{1/p}} \sum_{e \in E(G)} \left( w(e)^{p-r}/p \cdot \prod_{v \in e} (B(v, e))^{1/p} x_v \right)$$
\[
\leq \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} w(e) \right)^{(p-r)/p} \left( \sum_{e \in E(G)} \prod_{v \in e} (B(v, e))^{1/r} x_v^{p/r} \right)^{r/p}
\]

which yields \( \lambda^{(p)}(G) \leq r^{1-r/p} \alpha^{-1/p} \). When \( G \) is strictly \( \alpha \)-subnormal, this inequality is strict, and therefore \( \lambda^{(p)}(G) < r^{1-r/p} \alpha^{-1/p} \). \( \square \)

**Definition 2.3** An \( r \)-uniform hypergraph \( G \) is called \( \alpha \)-supernormal if there exist a weighted incidence matrix \( B \) and weights \( \{w(e)\} \) satisfying

1. \( \sum_{e \in E(G)} w(e) \geq 1; \)
2. \( \sum_{e : v \in e} B(v, e) \geq 1, \) for any \( v \in V(G); \)
3. \( w(e)^{p-r} \cdot \prod_{v \in e} B(v, e) \leq \alpha, \) for any \( e \in E(G). \)

Moreover, \( G \) is called **strictly \( \alpha \)-supernormal** if it is \( \alpha \)-supernormal but not \( \alpha \)-normal.

**Lemma 2.3** Let \( G \) be an \( r \)-uniform hypergraph. If \( G \) is consistently \( \alpha \)-supernormal, then the \( p \)-spectral radius of \( G \) satisfies

\[
\lambda^{(p)}(G) \geq \frac{r^{1-r/p}}{\alpha^{1/p}}.
\]

**Proof.** The consistent condition implies that there exists a vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in S_{p,++}^{n-1} \) satisfying (2.1). Therefore

\[
\sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} \geq \frac{r}{\alpha^{1/p}} \sum_{e \in E(G)} \left( w(e)^{(p-r)/p} \cdot \prod_{v \in e} (B(v, e))^{1/p} x_v \right)
\]

\[
= \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} w(e)^{(p-r)/p} \cdot \prod_{v \in e} (B(v, e))^{1/r} x_v^{p/r} \right)^{r/p}
\]

\[
\geq \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} \prod_{v \in e} (B(v, e))^{1/r} x_v^{p/r} \right)^{r/p}
\]
\[
\alpha_{1/p} \left( \sum_{e \in E(G)} \sum_{v \in e} B(v, e) x_e^p \right)^{r/p} \\
\geq \frac{r - r/p}{\alpha_{1/p}},
\]
which yields \( \lambda^{(p)}(G) \geq r^{1 - r/p}/\alpha_{-1/p} \). When \( G \) is strictly \( \alpha \)-supernormal, this inequality is strict, and therefore \( \lambda^{(p)}(G) > r^{1 - r/p}/\alpha_{-1/p} \). \( \square \)

3. Applications for \( p > r \)

In this section, we give some applications of \( \alpha \)-normal labeling method for the range \( p > r \). Let \( G \) be an \( r \)-uniform hypergraph, and \( G_i \) be the connected components of \( G \), \( i \in [k] \). If \( 1 \leq p \leq r \), Nikiforov [11] proved that

\[
\lambda^{(p)}(G) = \max_{1 \leq i \leq k} \{ \lambda^{(p)}(G_i) \},
\]
while the statement is different for \( p > r \). Here we use Lemma 2.1 to give a new proof for the case \( p > r \).

**Proposition 3.1** ([11]) Let \( p > r \geq 2 \), and let \( G_1, G_2, \ldots, G_k \) be the components of an \( r \)-uniform hypergraph \( G \). If \( G \) has no isolated vertices, then

\[
\lambda^{(p)}(G) = \left( \sum_{i=1}^{k} \left( \lambda^{(p)}(G_i) \right)^{p/(p-r)} \right)^{(p-r)/p}.
\]

**Proof.** For any \( i \in [k] \), let \( G_i \) be consistently \( \alpha_i \)-normal with weighted incidence matrix \( B_i \) and \( \{ w_i(e) \} \), where \( \alpha_i = r^{p-r}/(\lambda^{(p)}(G_i))^p \). That is

\[
\begin{align*}
\sum_{e \in E(G_i)} w_i(e) &= 1, \\
\sum_{e \in E(G_i), v \in e} B_i(v, e) &= 1, \text{ for any } v \in V(G_i), \\
w_i(e)^{p-r} \prod_{v \in e} B_i(v, e) &= \alpha_i, \text{ for any } e \in E(G_i).
\end{align*}
\]

For convenience, we denote

\[
C := \sum_{i=1}^{k} \frac{1}{\alpha_i^{1/(p-r)}} = \frac{1}{r} \left( \sum_{i=1}^{k} \left( \lambda^{(p)}(G_i) \right)^{p/(p-r)} \right).
\]

Now we construct a weighted incidence matrix \( B \) and \( \{ w(e) \} \) for \( G \) as follows:

\[
B(v, e) = \begin{cases} 
B_i(v, e), & \text{if } v \in V(G_i), e \in E(G_i), \\
0, & \text{otherwise},
\end{cases}
\]

\[
w(e) = \frac{w_i(e)}{C \alpha_i^{1/(p-r)}}, \text{ if } e \in E(G_i).
\]
For any $v \in V(G)$, assume that $v \in V(G_i)$, then

$$\sum_{e \in E(G) : v \in e} B(v, e) = \sum_{e \in E(G_i) : v \in e} B_i(v, e) = 1.$$ 

For each edge $e \in E(G)$, assume that $e \in E(G_j)$, then

$$w(e)^{p-r} \prod_{v \in e} B(v, e) = \frac{w_j(e)^{p-r} \prod_{v \in e} B_j(v, e)}{C^{p-r} \alpha_j} = \frac{1}{C^{p-r}}.$$ 

Also we have

$$\sum_{e \in E(G)} w(e) = \frac{1}{C} \sum_{i=1}^{k} \frac{\sum_{e \in E(G_i)} w_i(e)}{\alpha_i^{1/(p-r)}} = \frac{1}{C} \sum_{i=1}^{k} \frac{1}{\alpha_i^{1/(p-r)}} = 1.$$ 

Clearly, $\{B(v, e)\}$ and $\{w(e)\}$ are consistent labeling of $G$. Therefore $G$ is consistently $\alpha$-normal with $\alpha = C^{-(p-r)}$. By Lemma 2.1 we obtain

$$\lambda^{(p)}(G) = \left(\frac{r^{p-r}}{\alpha}\right)^{1/p} = (rC)^{(p-r)/p}$$

$$= \left(\sum_{i=1}^{k} \left(\lambda^{(p)}(G_i)\right)^{p/(p-r)}\right)^{(p-r)/p},$$

completing the proof. $\square$

**Theorem 3.1** Let $G$ be an $r$-uniform hypergraph, and $d_v$ be the degree of vertex $v$. If $p > r$, then

$$\lambda^{(p)}(G) \leq \left(\frac{r}{\alpha} \right)^{1/p} \left(\sum_{e \in E(G) : v \in e} d_v^{1/(p-r)}\right)^{(p-r)/p}.$$ 

**Proof.** For convenience, denote

$$C := \sum_{e \in E(G) : v \in e} d_v^{1/(p-r)}.$$ 

Define a weighted incident matrix $B$ and $\{w(e)\}$ for $G$ as follows:

$$B(v, e) = \begin{cases} 1/d_v, & \text{if } v \in e, \\ 0, & \text{otherwise,} \end{cases}$$

$$w(e) = \frac{1}{C} \prod_{v \in e} d_v^{1/(p-r)}.$$ 

It can be checked that $G$ is $\alpha$-subnormal with $\alpha = C^{-(p-r)}$. By Lemma 2.2 we have

$$\lambda^{(p)}(G) \leq \left(\frac{1}{\alpha} \right)^{1/p} = \left(\frac{r}{\alpha} \right)^{1/r} \left(\sum_{e \in E(G) : v \in e} d_v^{1/(p-r)}\right)^{(p-r)/p}.$$ 

The proof is completed. $\square$
From Theorem 3.1, we immediately obtain the following result.

**Corollary 3.1** Let $G$ be an $r$-uniform hypergraph with $m$ edges. If $p > r$, then

$$\lambda^{(p)}(G) \leq (rm)^{1-r/p} \cdot \max_{e \in E(G)} \prod_{v \in e} d_v^{1/p}.$$ 

In the following, we present some properties of the function $\lambda^{(p)}(G)$ for a fixed $r$-uniform hypergraph $G$.

**Lemma 3.1** Suppose that $G$ is consistently $\alpha$-normal with weights $\{w(e)\}$. Let $\delta$ and $\Delta$ be the minimum degree and maximum degree of $G$, respectively. If $p > r$, then

$$(\alpha \delta^r)^{1/(p-r)} \leq w(e) \leq (\alpha \Delta^r)^{1/(p-r)}.$$ 

**Proof.** Without loss of generality, assume $w(e_1) = \min\{w(e) : e \in E(G)\}$ and $w(e_2) = \max\{w(e) : e \in E(G)\}$. According to (2.3), we have

$$w(e_1)^p = \alpha \prod_{v \in e_1} \sum_{f : v \in f} w(f) \geq \alpha (\delta \cdot w(e_1))^r,$$

which yields $w(e_1) \geq (\alpha \delta^r)^{1/(p-r)}$. Similarly, we have

$$w(e_2)^p = \alpha \prod_{v \in e_2} \sum_{f : v \in f} w(f) \leq \alpha (\Delta \cdot w(e_2))^r,$$

which follows that $w(e_2) \leq (\alpha \Delta^r)^{1/(p-r)}$. The proof is completed. 

**Theorem 3.2** Suppose that $G$ is an $r$-uniform hypergraph and $p > r$. Let

$$f_G(p) := \left(\frac{\lambda^{(p)}(G)}{\Delta}\right)^{p/(p-r)}, \quad g_G(p) := \left(\frac{\lambda^{(p)}(G)}{\delta}\right)^{p/(p-r)}.$$

Then $f_G(p)$ is non-decreasing in $p$ while $g_G(p)$ is non-increasing in $p$.

**Proof.** Let $G$ be consistently $\alpha$-normal with weighted incidence matrix $B$ and weights $\{w(e)\}$ for $\lambda^{(p)}(G)$, i.e.,

$$\sum_{e \in E(G)} w(e) = 1,$$

$$\sum_{v \in e} B(v, e) = 1, \text{ for any } v \in V(G),$$

$$w(e)^{p-r} \prod_{v \in e} B(v, e) = \alpha, \text{ for any } e \in E(G).$$

Let $r < p < p'$. We now define a weighted incidence matrix $B'$ and $\{w'(e)\}$ for $\lambda^{(p')} (G)$ as follows:

$$B'(v, e) = B(v, e), \quad w'(e) = w(e).$$

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Therefore, we obtain
\[ \sum_{e \in E(G)} w'(e) = \sum_{e \in E(G)} w(e) = 1 \]
and for any \( v \in V(G) \),
\[ \sum_{e \in E(G) : v \in e} B'(v, e) = \sum_{e \in E(G) : v \in e} B(v, e) = 1. \]
Using Lemma 3.1 gives
\[
w'(e)^{p'-r} \prod_{v \in e} B'(v, e) = w(e)^{p'-r} \alpha \leq (\alpha \Delta^r)^{(p'-p)/(p-r)} \alpha = \alpha^{(p'-r)/(p-r)} \Delta^r \Big( \frac{\alpha}{\lambda(G)} \Big)^{(p'-p)/(p-r)} \] \[ \sum_{e \in E(G)} B'(v, e) = 1. \]
Hence, \( G \) is consistently \( \alpha' \)-supernormal for \( \lambda(p')(G) \) with
\[ \alpha' = r^{p'-r} \left( \frac{\Delta^r \lambda(p')}{\lambda(p)(G)^{p(p'-r)}} \right)^{1/(p-r)}. \]
By Lemma 2.3 we have
\[ \lambda(p')(G) \geq \left( \frac{1}{\alpha'} \right)^{1/p'} = \left( \frac{\Delta^r \lambda(p')(G)^{p(p'-r)}}{\lambda(p)(G)^{p(p'-r)}} \right)^{1/(p'(p-r))}, \]
which implies that
\[ \left( \frac{\lambda(p')(G)}{\Delta} \right)^{p'/(p'-r)} \geq \left( \frac{\lambda(p)(G)}{\Delta} \right)^{p/(p-r)}. \]
Similarly, we can obtain
\[ \left( \frac{\lambda(p')(G)}{\delta} \right)^{p'/(p'-r)} \leq \left( \frac{\lambda(p)(G)}{\delta} \right)^{p/(p-r)}. \]
The proof is completed. \( \square \)

Assume that \( G \) has \( m \) edges. In [11], Nikiforov proved that the function \( (\lambda(p)(G)/(rm))^p \) is non-increasing in \( p \). Here we give a new proof for \( p > r \) using \( \alpha \)-normal labeling method.

**Theorem 3.3 ([11])** Let \( G \) be an \( r \)-uniform hypergraph with \( m \) edges. Then the function \( (\lambda(p)(G)/(rm))^p \) is non-increasing in \( p \).

**Proof.** Let \( G \) be consistently \( \alpha \)-normal with weighted incidence matrix \( B \) and weights \( \{ w(e) \} \) for \( \lambda(p)(G) \), where \( \alpha = r^{p-r}/(\lambda(p)(G))^p \). Let \( r < p < p' \). We define a weighted incidence matrix \( B' \) and \( \{ w'(e) \} \) for \( \lambda(p')(G) \) as follows:
\[ B'(v, e) = B(v, e), \quad w'(e) = \left( \frac{w(e)^{p-r}}{m^{p'-p}} \right)^{1/(p'-r)}. \]
It is obvious that
\[
\sum_{e \in E(G): v \in e} B'(v, e) = \sum_{e \in E(G): v \in e} B(v, e) = 1.
\]

By Hölder’s inequality, we see
\[
\sum_{e \in E(G)} w'(e) = \frac{1}{m^{(p'-p)/(p'-r)}} \sum_{e \in E(G)} w(e)^{(p-r)/(p'-r)} \leq 1.
\]

For each edge \( e \in E(G) \), we have
\[
w'(e)^{p'-r} \prod_{v \in e} B'(v, e) = \frac{w(e)^{p-r}}{m^{p'-p}} \prod_{v \in e} B(v, e) = \frac{\alpha}{m^{p'-p}}.
\]

Therefore, \( G \) is \( \alpha' \)-subnormal for \( \lambda^{(p')}(G) \), where \( \alpha' = \alpha m^{p-p'} \). Using Lemma 2.2 gives
\[
(\lambda^{(p')}(G))^{p'} \leq \frac{r^{p'-r}}{\alpha'} = (rm)^{p'-p} (\lambda^{(p)}(G))^p,
\]
the result follows. \( \square \)

**Theorem 3.4** For any r-uniform hypergraph \( G \), the function \( h_G(p) := p \log \lambda^{(p)}(G) \) is concave upward on \((r, \infty)\).

**Proof.** For any \( r < p_1 < p < p_2 \), write \( p = \mu p_1 + (1 - \mu)p_2 \), where \( \mu = (p_2 - p)/(p_2 - p_1) \).

According to Lemma 2.1, let \( G \) be consistently \( \alpha_i \)-normal with weighted incident matrix \( B_i \) and \( \{w_i(e)\} \) for \( \lambda^{(p_i)}(G) \), where \( \alpha_i = r^{p_i-r}/(\lambda^{(p_i)}(G))^{p_i}, i = 1, 2 \). That is

\[
\begin{cases}
\sum_{e \in E(G)} w_i(e) = 1, \\
\sum_{e \in E(G)} B_i(v, e) = 1, \text{ for any } v \in V(G), \\
w_i(e)^{p_i-r} \prod_{v \in e} B_i(v, e) = \alpha_i, \text{ for any } e \in E(G).
\end{cases}
\]

Let \( \xi = \frac{(p_1-r)(p_2-p)}{(p-r)(p_2-p_1)} \in (0, 1) \). We define a weighted incidence matrix \( B \) and \( \{w(e)\} \) for \( \lambda^{(p)}(G) \) as follows:

\[
B(v, e) = \mu B_1(v, e) + (1 - \mu) B_2(v, e), \\
w(e) = \xi w_1(e) + (1 - \xi) w_2(e).
\]

In the following we shall prove \( \{B(v, e)\} \) and \( \{w(e)\} \) are \( \alpha \)-subnormal labeling for \( \alpha = \alpha_1^\mu \alpha_2^{1-\mu} \) (and for \( \lambda^{(p)}(G) \)). For any vertex \( v \in V(G) \),

\[
\sum_{e \in E(G): v \in e} B(v, e) = \mu \sum_{e \in E(G): v \in e} B_1(v, e) + (1 - \mu) \sum_{e \in E(G): v \in e} B_2(v, e)
= \mu + (1 - \mu) = 1.
\]
We also have
\[
\sum_{e \in E(G)} w(e) = \xi \sum_{e \in E(G)} w_1(e) + (1 - \xi) \sum_{e \in E(G)} w_2(e) = \xi + (1 - \xi) = 1.
\]

For any edge \( e \in E(G) \), we have
\[
w(e)^{p-r} \prod_{v \in e} B(v, e) = [\xi w_1(e) + (1 - \xi)w_2(e)]^{p-r} \prod_{v \in e} (\mu B_1(v, e) + (1 - \mu)B_2(v, e))
\]
\[
\geq w_1(e)^{(p-r)}w_2(e)^{(1-\xi)(p-r)} \prod_{v \in e} (B_1(v, e))^\mu(B_2(v, e))^{1-\mu}
\]
\[
= w_1(e)^{(p_1-r)}w_2(e)^{(1-\mu)(p_2-r)} \prod_{v \in e} (B_1(v, e))^\mu(B_2(v, e))^{1-\mu}
\]
\[
= \left( w_1(e)^{p_1-r} \prod_{v \in e} B_1(v, e) \right)^\mu \left( w_2(e)^{p_2-r} \prod_{v \in e} B_2(v, e) \right)^{1-\mu}
\]
\[
= \alpha_1^{\mu} \alpha_2^{1-\mu}.
\]

By Lemma 2.2, we have
\[
p \log \lambda^{(p)}(G) \leq p \log(r^{1-r/p} \alpha^{-1/p})
\]
\[
= (p - r) \log r - \mu \log \alpha_1 - (1 - \mu) \log \alpha_2
\]
\[
= \mu([p_1 - r] \log r - \log \alpha_1] + (1 - \mu)[(p_2 - r) \log r - \log \alpha_2]
\]
\[
= \mu p_1 \log \lambda^{(p_1)}(G) + (1 - \mu)p_2 \log \lambda^{(p_2)}(G).
\]

Thus the function \( h_G(p) = p \log \lambda^{(p)}(G) \) is concave upward on \((r, \infty)\). \( \square \)

**Corollary 3.2** The function \( \lambda^{(p)}(G) \) is differentiable on \((r, \infty)\) except countable many \( p \)'s. Moreover, the left-hand derivative and the right-hand derivative always exist for any \( p > r \).

**Remark 3.1** In [11], Nikiforov provides an example showing that \( \lambda^{(p)}(G) \) may not be differentiable at \( p = 2, 3, \ldots, r \). He asked whether \( \lambda^{(p)}(G) \) is continuously differentiable on \((r, \infty)\). This corollary give a weak solution toward this problem.

**Theorem 3.5** For any \( r \)-uniform hypergraph \( G \) and \( p > r \), the function \( \log \lambda^{(p)}(G) \) is concave upward in \( 1/p \).

**Proof.** For any \( p_i > r, i = 1, 2, \) write
\[
\frac{1}{p} = \frac{\mu}{p_1} + \frac{1-\mu}{p_2}, \text{ where } \mu = \frac{p_1(p_2 - p)}{p(p_2 - p_1)}.
\]

Let \( G \) be consistently \( \alpha_i \)-normal with weighted incident matrix \( B_i \) and \( \{w_i(e)\} \) for \( \lambda^{(p_i)}(G) \), where \( \alpha_i = r^{p_i-r}/(\lambda^{(p_i)}(G))^{p_i}, i = 1, 2. \) Therefore
\[
\begin{cases}
\sum_{e \in E(G)} w_i(e) = 1, \\
\sum_{e \in E(G)} B_i(v, e) = 1, \text{ for any } v \in V(G), \\
w_i(e)^{1-r/p_i} \prod_{v \in e} (B_i(v, e))^{1/p_i} = \alpha_i^{1/p_i}, \text{ for any } e \in E(G).
\end{cases}
\]
Furthermore, let
\[ \eta = \frac{p}{p_1} \mu, \quad \xi = \frac{p(p_1 - r)}{p_1(p - r)} \mu. \]

We define a weighted incidence matrix \( B \) and \( \{w(e)\} \) for \( \lambda^{(p)}(G) \) as follows:
\[
B(v, e) = \eta B_1(v, e) + (1 - \eta) B_2(v, e), \\
\sum_{e \in \mathbb{E}}(B_2(v, e) - (1 - \eta)B_2(v, e)) = 1.
\]

Thus the function \( \log \lambda^{(p)} \) is concave upward in \( 1/p \).

By Lemma 2.2, we have
\[
\log \lambda^{(p)}(G) \leq \left( 1 - \frac{r}{p} \right) \log r - \frac{1}{p} \log \alpha \\
= \mu \log \lambda^{(p_1)}(G) + (1 - \mu) \log \lambda^{(p_2)}(G).
\]

Thus the function \( \log \lambda^{(p)}(G) \) is concave upward in \( 1/p \).

Now we consider the applications of the \( \alpha \)-normal labeling method in the study of the products of hypergraphs. Let \( G_i \) be \( r_i \)-uniform hypergraph, \( i = 1, 2, \) with \( V(G_1) \cap V(G_2) = \emptyset \). Define an \( (r_1 + r_2) \)-uniform hypergraph \( G_1 * G_2 \) by
\[
V(G_1 * G_2) = V(G_1) \cup V(G_2), \quad E(G_1 * G_2) = \{e \cup f \mid e \in E(G_1), f \in E(G_2)\}.
\]

In [11], Nikiforov investigated the \( p \)-spectral radius of \( G_1 * tK_1 \) and \( G_1 * K^i_t \) for \( p \geq 1 \), where \( K^i_t \) is a \( t \)-uniform hypergraph with only one edge. In the following, we give a generalized result for large \( p \).

**Theorem 3.6** Suppose that \( G_i \) is an \( r_i \)-uniform hypergraph, \( i = 1, 2 \). If \( p > r_1 + r_2 \), then
\[
\lambda^{(p)}(G_1 * G_2) = \frac{(r_1 + r_2)^{1-(r_1+r_2)/p}}{r_1^{1-r_1/p}r_2^{1-r_2/p}} \lambda^{(p)}(G_1) \lambda^{(p)}(G_2).
\]
Proof. According to Lemma 2.1, let $G_i$ be consistently $\alpha_i$-normal with weighted incident matrix $B_i$ and $\{w_i(e)\}$ for $\lambda(p)(G_i)$, where $\alpha_i = r_i^{p-r_i}/(\lambda(p)(G_i))^p$, $i = 1, 2$. That is

$$\begin{cases}
    \sum_{e \in E(G_i)} w_i(e) = 1, \\
    \sum_{e \in E(G_i) : v \in e} B_i(v, e) = 1, \text{ for any } v \in V(G_i), \\
    w_i(e)^{p-r_i} \prod_{v \in e} B_i(v, e) = \alpha_i, \text{ for any } e \in E(G_i).
\end{cases}$$

For any $v \in V(G_1 \ast G_2)$ and $e \cup f \in E(G_1 \ast G_2)$, we define a weighted incidence matrix $B$ and $\{w(e \cup f)\}$ for $G_1 \ast G_2$ as follows:

$$B(v, e \cup f) = \begin{cases}
    B_1(v, e) \cdot w_2(f), & \text{if } v \in e, \\
    B_2(v, f) \cdot w_1(e), & \text{if } v \in f, \\
    0, & \text{otherwise},
\end{cases} \quad (3.1)$$

$$w(e \cup f) = w_1(e) \cdot w_2(f). \quad (3.2)$$

In the following we shall prove that $\{B(v, e \cup f)\}$ and $\{w(e \cup f)\}$ are consistent $\alpha$-normal labeling of $G_1 \ast G_2$ with $\alpha = \alpha_1 \alpha_2$.

(i). By (3.2) we have

$$\sum_{e \cup f \in E(G_1 \ast G_2)} w(e \cup f) = \left( \sum_{e \in E(G_1)} w_1(e) \right) \left( \sum_{f \in E(G_2)} w_2(f) \right) = 1.$$

(ii). For any $v \in V(G_1 \ast G_2)$, if $v \in V(G_1)$ we have

$$\sum_{e \cup f : v \in e \cup f} B(v, e \cup f) = \sum_{e \cup f : v \in e \cup f} B_1(v, e) w_2(f)$$

$$= \sum_{e \in E(G_1) : v \in e} \sum_{f \in E(G_2)} B_1(v, e) w_2(f)$$

$$= \left( \sum_{e \in E(G_1) : v \in e} B_1(v, e) \right) \left( \sum_{f \in E(G_2)} w_2(f) \right)$$

$$= 1.$$

If $v \in V(G_2)$, we can prove $\sum_{e \cup f : v \in e \cup f} B(v, e \cup f) = 1$ similarly.

(iii). For any edge $e \cup f \in E(G_1 \ast G_2)$, it follows from (3.1) and (3.2) that

$$w(e \cup f)^{p-(r_1+r_2)} \prod_{v \in e \cup f} B(v, e \cup f)$$

$$= (w_1(e)w_2(f))^{p-(r_1+r_2)} \prod_{v \in e} B(v, e \cup f) \prod_{u \in f} B(u, e \cup f)$$
Lemma 2.1, let 

\[ = w_1(e)^{p-r_1} \prod_{v \in e} B_1(v, e) \cdot w_2(f)^{p-r_2} \prod_{u \in f} B_2(u, f) \]

= \alpha_1 \alpha_2.

Finally, consider each vertex \( v \in V(G_1 \ast G_2) \) and \( v \in e \cup f \in E(G_1 \ast G_2) \). By (3.1) and (3.2) we see that

\[
\frac{w(e \cup f)}{B(v, e \cup f)} = \begin{cases} 
\frac{w_1(e)}{B_1(v, e)}, & \text{if } v \in V(G_1), \\
\frac{w_2(f)}{B_2(v, f)}, & \text{if } v \in V(G_2).
\end{cases}
\]

Hence, \( G_1 \ast G_2 \) is consistently \( \alpha \)-normal with \( \alpha = \alpha_1 \alpha_2 \). Using Lemma 2.1 gives

\[
\lambda^{(p)}(G_1 \ast G_2) = \frac{(r_1 + r_2)^{1-(r_1+r_2)/p}}{(\alpha_1 \alpha_2)^{1/p}}
\]

\[= \frac{(r_1 + r_2)^{1-(r_1+r_2)/p}}{r_1^{1-r_1/p}r_2^{1-r_2/p}} \lambda^{(p)}(G_1)\lambda^{(p)}(G_2). \]

The proof is completed. \( \square \)

Let \( G_1 \) and \( G_2 \) be two \( r \)-uniform hypergraphs. The direct product \( G_1 \times G_2 \) of \( G_1 \) and \( G_2 \) is defined as \( V(G_1 \times G_2) = V(G_1) \times V(G_2) \), and \( \{(i_1, j_1), \ldots, (i_r, j_r)\} \in E(G_1 \times G_2) \) if and only if \( \{i_1, \ldots, i_r\} \in E(G_1) \) and \( \{j_1, \ldots, j_r\} \in E(G_2) \). For an edge \( f = \{(i_1, j_1), \ldots, (i_r, j_r)\} \in E(G_1 \times G_2) \), we denote \( \pi_1(f) := \{i_1, \ldots, i_r\} \in E(G_1) \) and \( \pi_2(f) := \{j_1, \ldots, j_r\} \in E(G_2) \).

In what follows, we give an extension to a result of Shao [14].

**Theorem 3.7** Let \( G_1 \) and \( G_2 \) be two \( r \)-uniform hypergraphs. If \( p > r \), then

\[
\lambda^{(p)}(G_1 \times G_2) = (r-1)!\lambda^{(p)}(G_1)\lambda^{(p)}(G_2).
\]

**Proof.** By Lemma 2.1, let \( \{B_i(v, e)\} \) and \( \{w_i(e)\} \) be the consistent \( \alpha_i \)-normal labeling of \( G_i \) with \( \alpha_i = r_i^{p-r_i}/(\lambda^{(p)}(G_i))^p \), \( i = 1, 2 \).

Define a weighted incident matrix \( B \) and \( \{w(f)\} \) for \( G_1 \times G_2 \) as follows:

\[
B((u, v), f) = \begin{cases} 
B_1(u, \pi_1(f))B_2(v, \pi_2(f)), & \text{if } (u, v) \in f, \\
0, & \text{otherwise},
\end{cases}
\]

\[w(f) = \frac{w_1(\pi_1(f))w_2(\pi_2(f))}{r!}. \quad (3.4)
\]

In what follows, we shall prove that \( \{B((u, v), f)\} \) and \( \{w(f)\} \) are consistent \( \alpha \)-normal labeling of \( G_1 \times G_2 \) with \( \alpha = r^{r_1} \alpha_1 \alpha_2 / (r!)^p \).

(i) By (3.4) we deduce that

\[
\sum_{f \in E(G_1 \times G_2)} w(f) = \frac{1}{r!} \sum_{f \in E(G_1 \times G_2)} w_1(\pi_1(f))w_2(\pi_2(f))
\]
\[
= \sum_{e_1 \in E(G_1)} \sum_{e_2 \in E(G_2)} w_1(e_1)w_2(e_2)
\]
\[
= 1.
\]

(ii). For any \((u, v) \in V(G_1 \times G_2)\), we have
\[
\sum_{f: (u, v) \in f} B((u, v), f) = \frac{1}{(r - 1)!} \sum_{f: (u, v) \in f} B_1(u, \pi_1(f))B_2(v, \pi_2(f))
\]
\[
= \sum_{e_1: u \in e_1} \sum_{e_2: v \in e_2} B_1(u, e_1)B_2(v, e_2)
\]
\[
= 1.
\]

(iii). For each edge \(f \in E(G_1 \times G_2)\), we obtain that
\[
w(f)^{p-r} \prod_{(u, v) \in f} B((u, v), f)
\]
\[
= \frac{[w_1(\pi_1(f))w_2(\pi_2(f))]^{p-r}}{(r!)^{p-r}} \prod_{(u, v) \in f} \frac{B_1(u, \pi_1(f))B_2(v, \pi_2(f))}{(r - 1)!}
\]
\[
= \frac{r^r \alpha_1 \alpha_2}{(r!)^p}.
\]

Finally, consider each vertex \((u, v) \in V(G_1) \times V(G_2)\) and \((u, v) \in f\). It follows from (3.3) and (3.4) that
\[
\frac{w(f)}{B((u, v), f)} = \frac{w_1(\pi_1(f))}{rB_1(u, \pi_1(f))} \cdot \frac{w_2(\pi_2(f))}{B_2(v, \pi_2(f))}.
\]

Therefore \(G_1 \times G_2\) is consistently \(\alpha\)-normal with \(\alpha = \frac{r^r \alpha_1 \alpha_2}{(r!)^p}\). Hence
\[
\lambda^{(p)}(G_1 \times G_2) = \frac{r^{1-r/p}}{\alpha^{1/p}} = (r - 1)! \lambda^{(p)}(G_1)\lambda^{(p)}(G_2).
\]

The proof is completed. \(\square\)

For a given \(r\)-uniform hypergraph \(G\), we let \(G^{r+1}\) be an \((r + 1)\)-uniform hypergraph obtained by adding a new vertex \(v_e\) in each edge \(e\) of \(G\) such that all these new vertices are pairwise disjoint. Following [6], \(G^{r+1}\) is a generalized power of \(G\). If we do not require that all \(v_e\) to be distinct, we get an extension of \(G\). Denote \(\mathcal{E}(G)\) the set of all extensions of \(G\).

**Theorem 3.8** Let \(G\) be an \(r\)-uniform hypergraph with no isolated vertices, and \(H\) be an extension of \(G\). If \(p > r\), then
\[
\left(\left(\frac{r + 1}{r}\right)^{p-r}(\lambda^{(p)}(G))^p\right)^{1/(p+1)} \leq \lambda^{(p+1)}(H) \leq \left(\frac{(r + 1)^{p-r}}{r^{p+1-r}}\right)^{1/(p+1)} \lambda^{(p+1)}(G),
\]
with the left equality holds if and only if \(H \cong G^{r+1}\), and the right equality holds if and only if \(H \cong G \ast K_1\).
Lemma 2.1, let $B$. We now define a weighted incident matrix which yields $\lambda$. Proof. Let $H \in E(G)$. If $i = m$, then $H \cong G^{r+1}$. If $i \leq m - 1$, we claim that there is an extension $H' \in E_{i+1}(G)$ such that $\lambda^{(p+1)}(H') < \lambda^{(p+1)}(H)$. Choose a vertex $v_0 \in V(H) \setminus V(G)$ with degree greater than one, and an edge $e_0 \in E(H)$ such that $v_0 \in e_0$. Let $H'$ be the extension of $G$ with $V(H') = V(H) \cup \{v_0\}$ and $E(H') = (E(H) \setminus e_0) \cup ((e_0 \setminus \{v_0\}) \cup \{u_0\})$, where $u_0 \notin V(H)$ is a new vertex. Assume $x$ is the positive eigenvector with $||x||_{p+1} = 1$ corresponding to $\lambda^{(p+1)}(H')$, we define a vector $y$ for $H$ as follows:

$$y_v = \begin{cases} x_v, & v \neq v_0, \\ \sqrt[p+1]{x_{e_0} + x_{v_0}}, & v = v_0. \end{cases}$$

It follows that

$$\lambda^{(p+1)}(H) - \lambda^{(p+1)}(H') \geq r \sum_{e \in E(H) \setminus e} \prod_{v \in v} y_v - r \sum_{e \in E(H') \setminus v} \prod_{v \in v} x_v$$

$$= r(y_{v_0} - x_{v_0}) \prod_{v \in e_0 \setminus \{v_0\}} x_v + r(y_{v_0} - x_{v_0}) \sum_{e \in E(H) \setminus e_0} \prod_{v \in e_0 \setminus \{v_0\}} x_v$$

$$> 0,$$

which yields $\lambda^{(p+1)}(H) > \lambda^{(p+1)}(H')$. Therefore $\lambda^{(p+1)}(H) \geq \lambda^{(p+1)}(G^{r+1})$, with equality if and only if $H \cong G^{r+1}$. Now it suffices to show that

$$\lambda^{(p+1)}(G^{r+1}) = \left(\left(\frac{r + 1}{r}\right)^{p-r} \lambda^{(p)}(G)^p\right)^{1/(p+1)}.$$

By Lemma 2.1, let $G$ be consistently $\alpha$-normal with weighted incident matrix $B$ and $\{w(e)\}$, where $\alpha = r^{p-r} / (\lambda(p)(G))^p$. That is

$$\begin{cases} \sum_{e \in E(G)} w(e) = 1, \\ \sum_{e \in E(G) \setminus e} B(v, e) = 1, \text{ for any } v \in V(G), \\ w(e)^{p-r} \prod_{v \in e} B(v, e) = \alpha, \text{ for any } e \in E(G). \end{cases}$$

We now define a weighted incident matrix $B' = (B'(v, e \cup \{v_e\}))$ and $\{w'(e \cup \{v_e\})\}$ for $G^{r+1}$ as follows:

$$B'(v, e \cup \{v_e\}) = \begin{cases} B(v, e), & \text{if } v \in e, \\ 1, & \text{if } v = v_e, \\ 0, & \text{otherwise}. \end{cases}$$
\[ w'(e \cup \{v_e\}) = w(e). \]

It can be checked that
\[
\sum_{e \cup \{v_e\} \in E(G^{r+1})} w'(e \cup \{v_e\}) = 1, \quad \sum_{e \cup \{v_e\} \cdot v \in e \cup \{v_e\}} B'(v, e \cup \{v_e\}) = 1,
\]
and for each edge \( e \cup \{v_e\} \in E(G^{r+1}) \), we have
\[
(w'(e \cup \{v_e\}))^{(p+1)-(r+1)} \prod _{v \in e \cup \{v_e\}} B'(v, e \cup \{v_e\}) = \alpha.
\]

Clearly, the weighted incidence matrix \( B' \) and \( \{w'(e \cup \{v_e\})\} \) are consistent. Using Lemma 2.1 gives
\[
\frac{r^{p-r}}{(\lambda^p(G))^p} = \alpha = \frac{(r+1)^{p-r}}{(\lambda^{p+1}(G^{r+1}))^{p+1}},
\]
as desired.

For the right inequality, we can prove \( \lambda^{p+1}(H) \leq \lambda^{p+1}(G*K_1) \) similarly. According to Theorem 3.6, we have
\[
\lambda^{p+1}(G*K_1) = \left( \frac{(r+1)^{p-r}}{r^{p+1-r}} \right)^{1/(p+1)} \lambda^{p+1}(G),
\]
the result follows. \( \square \)

4. The \( \alpha \)-normal labeling method for \( 1 \leq p < r \)

In this section, we make a brief discussion on the \( \alpha \)-normal labeling method for \( 1 \leq p < r \). Due to the fact that the Perron–Frobenius Theorem fails for general hypergraph \( G \) when \( 1 \leq p < r \), the theory is less effective than the case \( p \geq r \). However, we can still define the \( \alpha \)-normal labeling method as Definition 2.1.

Unlike the case \( p > r \), neither the existence nor the uniqueness can be said for the \( \alpha \)-normal labeling for general \( r \)-uniform hypergraph \( G \). However, we still have the following result.

**Theorem 4.1** For \( 1 \leq p < r \), and any \( r \)-uniform hypergraph \( G \) with \( p \)-spectral radius \( \lambda^p(G) \), there exists an induced sub-hypergraph \( G[S] \) such that \( G[S] \) is consistently \( \alpha \)-normal with \( \alpha = r^{p-r}/(\lambda^p(G))^p \).

Conversely, we have
\[
\lambda^p(G) = r^{1-r/p} \max \{ \alpha_i^{-1/p} \},
\]
where the maximum is taken over all \( \alpha_i \) such that there is a consistent \( \alpha_i \)-normal labeling on some induced sub-hypergraph of \( G \).
Proof. Assume that $P_G(x)$ reaches the maximum at $x^* = (x_1, x_2, \ldots, x_n)^T \in S^{n-1}_{p,+}$. Let $S = \{i: x_i > 0\}$. Consider the induced hypergraph $G[S]$. Observe that $P_{G[S]}(x)$ reaches the maximum at $x^*_S \in S^{n-1}_{p,+}$, and therefore $\lambda_p(G[S]) = \lambda_p(G)$.

Define a weighted incidence matrix $B$ and \{w(e)\} (on $G[S]$) as follows:

$$B(v, e) = \begin{cases} \prod_{u \in e} x_u \lambda_p(G[S]) x_v^p, & \text{if } v \in e \text{ and } v \in S, e \in E(G[S]), \\ 0, & \text{otherwise}, \end{cases}$$

$$w(e) = \frac{r \prod_{u \in e} x_u}{\lambda_p(G[S])}.$$ 

Since $x_v \neq 0$ for any $v \in S$, the above $B$ and \{w(e)\} are well-defined on $G[S]$.

For any $v \in S$, using the eigenequation (1.1) gives

$$\sum_{e: v \in e} B(v, e) = \sum_{e: v \in e} \prod_{u \in e} x_u \lambda_p(G[S]) x_v^p = 1.$$ 

Also, we see that

$$\sum_{e \in E(G[S])} w(e) = \frac{r}{\lambda_p(G[S])} \sum_{e \in E(G[S])} \prod_{u \in e} \lambda_p(G[S]) x_u = 1.$$ 

Therefore items (1) and (2) of Definition 2.1 are verified. For item (3), we check that

$$w(e)^{p-r} \prod_{v \in e} B(v, e) = \left( \frac{r}{\lambda_p(G[S])} \prod_{u \in e} x_u \right)^{p-r} \prod_{v \in e} \frac{x_u}{\lambda_p(G[S]) x_v^p} = \alpha.$$ 

To show that $B$ is consistent, for any $v \in S$ and $v \in e_i$, $i = 1, 2, \ldots, d$, we have

$$\frac{w(e_1)}{B(v, e_1)} = \frac{w(e_2)}{B(v, e_2)} = \cdots = \frac{w(e_d)}{B(v, e_d)} = r x_v^p.$$ 

Conversely, assume that for some $S_i \subset V, G[S_i]$ is consistently $\alpha_i$-normal with weighted incidence matrix $B$ and weights \{w(e)\}. Define a vector $x = (x_1, x_2, \ldots, x_n)^T \in S^{n-1}_{p,+}$ for $G$ as follows:

$$x_v = \begin{cases} \left( \frac{w(e)}{r B(v, e)} \right)^{1/p}, & \text{if } v \in e \in E(G[S_i]), \\ 0, & \text{otherwise}. \end{cases}$$ 

The consistent condition guarantees that $x_v$ (for $v \in S$) is independent of the choice of the edge $e$. Clearly, $\|x\|_p = 1$. Hence,

$$\lambda_p(G) \geq P_G(x) = r \sum_{e \in E(G[S_i])} \prod_{v \in e} x_v.$$ 

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\[ r^{1-r/p} \sum_{e \in E(G[S_i])} \frac{w(e)^{r/p}}{\prod_{v \in e} (B(v, e))^{1/p}} \]

\[ = r^{1-r/p} \sum_{e \in E(G[S_i])} \frac{w(e)}{w(e)^{1-r/p} \prod_{v \in e} (B(v, e))^{1/p}} \]

\[ = \frac{r^{1-r/p}}{\alpha_i^{1/p}} \sum_{e \in E(G[S_i])} w(e) \]

Combining with the first part of the theorem, we have

\[ \lambda^{(p)}(G) = r^{1-r/p} \max_i \{ \alpha_i^{-1/p} \}. \]

The proof is completed. \(\square\)

**Example 4.1** Consider the following graph \(G\) with 6 vertices and 7 edges.

\[ \begin{array}{c}
    \text{v1} \\
    \text{v2} \\
    \text{v3} \\
    \text{v4} \\
    \text{v5} \\
    \text{v6}
\end{array} \]

When \(p = 1\), \(G[S]\) has a consistent \(\alpha_1\)-normal labeling if and only if \(S\) forms a clique of size 2 or 3 in \(G\). In particular, both \(G[\{v_1, v_2, v_3\}]\) and \(G[\{v_4, v_5, v_6\}]\) has a consistent \(\frac{3}{4}\)-normal labeling while \(G[\{u, v\}]\) has a consistent 1-normal labeling for each edge \(uv\). We have

\[ \lambda^{(1)}(G) = 2^{-1} \cdot \max \left\{ \frac{4}{3}, 1 \right\} = \frac{2}{3}. \]

We can also define the \(\alpha\)-subnormal for \(p \in [1, r)\) similar to the case for \(p = r\) (see [10, Definition 4]).

**Definition 4.1** For \(1 \leq p < r\), a hypergraph \(G\) with \(m\) edges is called \(\alpha\)-subnormal for \(p\) if there exists a weighted incidence matrix \(B\) satisfying

1. \(\sum_{v \in e} B(v, e) \leq 1\), for any \(v \in V(G)\);

2. \(m^{r-p} \prod_{v \in e} B(v, e) \geq \alpha\), for any \(e \in E(G)\).

**Theorem 4.2** Let \(G\) be an \(r\)-uniform hypergraph with \(m\) edges. If \(G\) is \(\alpha\)-subnormal for \(p \in [1, r)\), then the \(p\)-spectral radius of \(G\) satisfies

\[ \lambda^{(p)}(G) \leq \frac{(r/m)^{1-r/p}}{\alpha^{1/p}}. \]
Proof. For any nonnegative vector \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{S}^{n-1}_{p,+} \), we have

\[
\begin{align*}
\sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} & \leq m^{r/p-1} \frac{r}{\alpha^{1/p}} \sum_{v \in V(G)} \prod_{e \in E(G)} (B(v, e))^{1/p} x_v \\
& \leq m^{r/p-1} \frac{r}{\alpha^{1/p}} \left( \sum_{e \in E(G)} \prod_{v \in e} (B(v, e))^{1/r} x_v^{p/r} \right)^{r/p} \\
& \leq m^{r/p-1} \frac{r^{1-r/p}}{\alpha^{1/p}} \left( \sum_{e \in E(G)} \sum_{v \in e} B(v, e) x_v^p \right)^{r/p} \\
& \leq \frac{(r/m)^{1-r/p}}{\alpha^{1/p}},
\end{align*}
\]

which yields \( \lambda^{(p)}(G) \leq (r/m)^{1-r/p} \alpha^{-1/p} \). \( \square \)

From Definition 4.1, a hypergraph \( G \) (with \( m \) edges) is \( \alpha \)-subnormal for \( p \in [1, r) \) if and only if \( G \) is \( \alpha' \)-subnormal for any \( p' \in [1, r) \) with \( \alpha' = \alpha m^{p-p'} \). We have the following corollary.

**Corollary 4.1** Let \( G \) be an \( r \)-uniform hypergraph with \( m \) edges. If \( G \) is \( \alpha \)-subnormal for \( 1 \leq p < r \), then for any \( p' \in [1, r) \) the \( p' \)-spectral radius of \( G \) satisfies

\[
\lambda^{(p')}(G) \leq \frac{r^{1-r/p'}}{\alpha^{1/p'} m^{(p-r)/p'}}.
\]

For \( 1 \leq p < r \), denote \( G^r(p) \) the set of \( r \)-uniform hypergraph \( G \) for which \( P_G(x) \) reaches the maximum at some point in \( x \in \mathbb{S}^{\left| V(G) \right|-1}_{p,+} \). Then for any \( r \)-uniform hypergraph \( G \), there exists a set \( S \subset V \) such that \( G[S] \in G^r(p) \). Finally, we conclude this section with a problem of Nikiforov [11, Problem 5.9], which are related to the topic of this section.

**Problem 4.1** Given \( 1 \leq p < r \), characterize all \( r \)-uniform hypergraphs in \( G^r(p) \).

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