Definability and continuity of the SU-rank in unidimensional supersimple theories

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Abstract

We prove, in particular, that in a supersimple unidimensional theory the $SU$-rank is continuous and the $D$-rank is definable.

1 Introduction

Analysis of "Forking sets", i.e. invariant sets built by applying first-order operations on the predicates $R(a,b)$ if "$\phi(x,a)$ forks over $b$", has powerful implications on the structure of definable sets in simple theories, e.g. Lstp=stp for low theories, elimination of hyperimaginaries in supersimple theories, supersimplicity of countable hypersimple unidimensional theories and of a large class (non s-essentially 1-based theories) of uncountable hypersimple unidimensional theories and more (a theory is hypersimple if it is simple and eliminates hyperimaginaries).

It is well known that a unidimensional stable theory is superstable. Stable unidimensional theories share some nice definability properties: the $SU$-rank is continuous and Shelah’s $D$-rank is definable. In this paper, we generalize and strengthen these properties to simple theories. We prove this by an argument that relies on this kind of analysis. We will assume basic knowledge of simple theories as in [K1],[KP],[HKP]. A good text book on simple theories that covers much more is [W]. The notations are standard, and throughout the paper we work in a highly saturated, highly strongly-homogeneous model $\mathcal{C}$ of a complete first-order theory $T$ in a language $L$. 
2 Preliminaries

In this section $T$ is assumed to be simple. We quote several known facts that we will apply.

2.1 Lowness

We will say that the formula $\phi(x, y) \in L$ is low in $x$ if there exists $k < \omega$ such that for every $\emptyset$-indiscernible sequence $(b_i | i < \omega)$, the set $\{\phi(x, b_i) | i < \omega\}$ is inconsistent iff every subset of it of size $k$ is inconsistent. Note that every stable formula $\phi(x, y)$ is low in both $x$ and $y$.

Remark 2.1 Note that if $\phi(x, y) \in L$ is low in $x$ then the relation $F_\phi$ defined by $F_\phi(b, A) \iff \phi(x, b)$ forks over $A$ is type-definable.

2.2 Almost internality and analyzability

Let $P$ be an $A$-invariant family of partial types. We say that $p \in S(A)$ is (almost-) $P$-internal if there exists a realization $a$ of $p$ and there exists $b$ with $a \downarrow_A b$ such that for some tuple $c$ of realizations of partial types in $P$ over $Ab$ we have $a \in \text{dcl}(b, c)$ (respectively, $a \in \text{acl}(b, c)$). We say that $p$ is analyzable in $P$ if there exists a sequence $I = \langle a_i | i \leq \alpha \rangle$ in $C^{eq}$ such that $a_\alpha \models p$ and $tp(a_i/\{a_j | j < i\} \cup A)$ is almost $P$-internal for every $i \leq \alpha$.

First, the following fact is straightforward.

Fact 2.2 1) Assume $tp(a_i/A)$ are (almost) $P$-internal for $i < \alpha$. Then $tp(\langle a_i | i < \alpha \rangle/A)$ is (almost) $P$-internal. Thus, if $tp(a_i/A)$ are analyzable in $P$ for $i < \alpha$. Then $tp(\langle a_i | i < \alpha \rangle/A)$ is analyzable in $P$.
2) If $tp(a/A)$ almost $P$-internal, so is $tp(a/B)$ for any set $B \supseteq A$.

The following Fact [S0, Theorem 2.2] will useful (a quite similar result has been proved independently in [W, Proposition 3.4.9]).

Fact 2.3 Let $T$ be simple. Let $p(x) \in S(A)$ be a hyperimaginairy amaligamtion base and let $U$ be an $A$-invariant set of hyperimaginaries. Suppose $p$ is almost $U$-internal. Then for every Morley sequence $\bar{a}$ of length $\omega$ of $p$ there is a type-definable one-to-bounded relation $S(x, \bar{y}, \bar{a})$ (i.e. for every $\bar{y}$ there
are boundedly many $x$-s for which $S(x, \bar{y}, \bar{a})$ holds) which covers $p$ by $\mathcal{U}$. If $p$ and $\mathcal{U}$ are real then $S$ can be chosen to be definable.

### 2.3 The extension property

We recall some natural extensions of notions from [BPV]. By a pair $(M, P^M)$ of $T$ we mean an $L_P = L \cup \{P\}$-structure, where $M$ is a model of $T$ and $P$ is a new predicate symbol whose interpretation is an elementary submodel of $M$. For the rest of this subsection, by a $|T|$-small type we mean a complete hyperimaginary type in $\leq |T|$ variables over a hyperimaginary of length $\leq |T|$ (i.e. a sequence of length $\leq |T|$ modulo a $\emptyset$-type-definable equivalence relation).

**Definition 2.4** Let $\mathcal{P}_0, \mathcal{P}_1$ be $\emptyset$-invariant families of $|T|$-small types.

1) We say that a pair $(M, P^M)$ satisfies the extension property for $\mathcal{P}_0$ if for every $L$-type $p \in S(A)$, where $A$ is a hyperimaginary with $A \in dcl(M)$ and $p \in \mathcal{P}_0$, there is $a \in p^M$ such that $a \downarrow A \subseteq P^M$.

2) Let

$$T_{Ext,\mathcal{P}_0} = \bigcap \{Th_{L_P}(M, P^M) | \text{the pair } (M, P^M) \text{ satisfies the extension property w.r.t. } \mathcal{P}_0 \}.$$ 

3) We say that $\mathcal{P}_0$ dominates $\mathcal{P}_1$ w.r.t. the extension property if $(M, P^M)$ satisfies the extension property for $\mathcal{P}_1$ for every $|T|^+$-saturated pair $(M, P^M) \models T_{Ext,\mathcal{P}_0}$. In this case we write $\mathcal{P}_0 \succeq_{Ext} \mathcal{P}_1$.

4) We say that the extension property is first-order for $\mathcal{P}_0$ if $\mathcal{P}_0 \succeq_{Ext} \mathcal{P}_0$ (i.e. every $|T|^+$-saturated model of $T_{Ext,\mathcal{P}_0}$ satisfies the extension property for $\mathcal{P}_0$).

We say that the extension property is first-order if the extension property is first-order for the family of all $|T|$-small types (equivalently, for the family of all real types over sets of size $\leq |T|$).

**Fact 2.5** [BPV, Proposition 4.5] The extension property is first-order in $T$ iff for every formulas $\phi(x, y), \psi(y, z) \in L$ the relation $Q_{\phi, \psi}$ defined by:

$$Q_{\phi, \psi}(a) \iff \phi(x, b) \text{ doesn’t fork over } a \text{ for every } b \models \psi(y, a)$$

is type-definable (here $a$ can be an infinite tuple from $C$ whose sorts are fixed).
Fact 2.6 [S1, Lemma 3.7] Let $P_0$ be an $\emptyset$-invariant family of $|T|$-small types. Assume $P_0$ is extension-closed and that the extension property is first-order for $P_0$. Let $P^*$ be the maximal class of $|T|$-small types such that $P_0 \supseteq_{est} P^*$. Then $P^* \supseteq An(P_0)$, where $An(P_0)$ denotes the class of all $|T|$-small types analyzable in $P_0$ by hyperimaginaries.

We will need the following consequence.

Corollary 2.7 Assume $\theta = \theta(x) \in L$, $SU(\theta) = 1$ and $tp(a)$ is analyzable in $\theta$ for every $a \in C$. Then the extension property is first-order in $T$.

Proof: First, note the following slightly stronger version of Hrushovski's Lemma [H, Lemma 4.3]:

Claim 2.8 Let $\theta(x) \in L$, $SU(\theta(x)) = 1$ and let $\chi(x,y) \in L$ be such that $\chi(x,y) \vdash \theta(x)$. Then there exists $N < \omega$ such that for all $a \in C$, $\chi(C,a)$ is finite iff its cardinality is smaller than $N$.

Now, by Fact 2.6 it will sufficient to show that the extension property is first-order for the family of complete types over $\emptyset$ that extends $\theta$ (for short we will say that the extension property is first-order for $\theta$.) To see this, for any $\chi(x,\bar{y}) \in L$ with $\chi(x,\bar{y}) \vdash \theta(x)$, consider the following $L_P$ formula (by Claim 2.8 it is a formula):

$$S_\chi(x) = \forall \bar{y} [(P(\bar{y}) \land \exists^{<\infty} x \chi(x,\bar{y})) \rightarrow \neg \chi(x,\bar{y})].$$

For every finite set $\Delta$ of formulas of the form $\chi(x,\bar{y})$ such that $\chi(x,\bar{y}) \vdash \theta(x)$ ($x$ is the fixed variable of $\theta$, $\bar{y}$ any tuple of variables), and consistent $\phi(x) \vdash \theta(x)$, let

$$\Psi_{\Delta,\phi} = \exists x [\phi(x) \land \bigwedge_{\chi \in \Delta} S_\chi(x)].$$

The following two claims shows that the extension property is first-order for $\theta$:

Claim 2.9 For any pair $\hat{M} = (M,P^M)$ of $T$ that satisfies the extension property for $\theta$, we have $\hat{M} \models \Psi_{\Delta,\phi}$ for any finite set $\Delta$ (as above) and any $\phi(x) \vdash \theta(x)$.

Claim 2.10 Assume $\hat{M} = (M,P^M)$ is a pair of $T$ that is $|T|^+$-saturated and satisfies $\Psi_{\Delta,\phi}$ for any finite set $\Delta$ (as above) and any $\phi(x) \vdash \theta(x)$. Then $\hat{M} = (M,P^M)$ satisfies the extension property for $\theta$. 

□
3 Type-definability of the D-rank and more

Recall the following lemma [S1, Lemma 8.4].

**Fact 3.1** Assume the extension property is first-order in $T$. Let $\psi(x, z_1, ..., z_m)$ be a Stone-open relation over $\emptyset$ and let $\phi_j(x, y_j) \in L$ for $j = 0, .., m$. Let $U$ be the following invariant set. For all $d_1 \in C$, $U(d_1)$ \iff

$$\exists a \exists d_2...d_m[\psi(a, d_1, ...d_m) \land \bigwedge_{j=0}^{m} (\phi_j(a, y_j) \text{ forks over } d_1...d_j)].$$

Then $U$ is a $\tau^f$-open set over $\emptyset$. If each $\phi_j(x, y_j)$ is assumed to be low in $y_j$ and $\psi$ is assumed to be definable, then $U$ is a basic $\tau^f_\infty$-open set.

The following variation of the above fact will be useful to us.

**Lemma 3.2** Assume the extension property is first-order in $T$. Let $\psi(x, z_1, ..., z_m)$ be a Stone-open relation over $\emptyset$ and let $\phi_j(x, y_j) \in L$ for $j = 1, .., m$. Let $U$ be the following invariant set. For all $d_1 \in C$, $U(d_1)$ \iff

$$\exists a \exists d_2...d_m[\psi(a, d_1, ...d_m) \land \bigwedge_{j=1}^{m} (\phi_j(a, y_j) \text{ forks over } d_1...d_j)].$$

Then $U$ is a Stone-open set over $\emptyset$. Moreover, if we assume in addition that each $\phi_j(x, y_j)$ is low in $y_j$ and $\psi(x, z_1, ..., z_m)$ is definable, then $U$ is a definable set over $\emptyset$.

**Remark 3.3** The point in Lemma 3.2 is that the index $j$ starts from 1 instead of 0 in Fact 3.1.

**Proof of Lemma 3.2** First, the proof of the following claim is identical to the proof of [S1, Subclaim 8.5].

**Claim 3.4** Let $\Gamma'$ be defined by $\Gamma'(d_1)$ \iff

$$\bigwedge_{\eta = \{\eta_j\}_{j<m} \in L} \forall d_2...d_m[[(\bigwedge_{j=1}^{m} \eta_j(d_1...d_m, y_j) \text{ forks over } d_1...d_j) \rightarrow \forall a \Lambda_\eta(a, d_1, ..., d_m)].$$

where $\Lambda_\eta$ is defined by

$$\Lambda_\eta(a, d_1, ...d_m) \iff \psi(a, d_1, ...d_m) \rightarrow \bigvee_{j=1}^{m} [\phi_j(a, y_j) \land \neg \eta_j(d_1...d_m, y_j) \text{ dinfo } d_1...d_m].$$

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where $\eta_m$ denotes a contradiction. Then $\Gamma' = \Gamma$ (Note that when $m = 1$ the above empty conjunction from $j = 1$ to $m - 1$ is interpreted as "True").

Note that since the extension property is first-order in $T$, the relation $\Lambda_0^0$ defined by $\Lambda_0^0(d_1, \ldots, d_m) \equiv \forall a \Lambda_\eta(a, d_1, \ldots, d_m)$ is type-definable by Fact 2.5. Thus we conclude that if $\Gamma$ is the complement of $U$ then for all $d_1$, $\Gamma(d_1)$ iff

$$\bigwedge_{\eta = \{\eta_j\}_{j < m} \in L} \forall d_2 \ldots d_m [\neg \Lambda_0^0(d_1, \ldots, d_m) \rightarrow \bigvee_{j=1}^{m-1} \eta_j(d_1 \ldots d_m, y_j) \text{ dnf } d_1 \ldots d_j].$$

Thus, we conclude that if $m = 1$, $\Gamma$ is type-definable (as the disjunction in the last formula from $j = 1$ to $m - 1$ is interpreted as "False") and if $m > 1$, we are done by the induction hypothesis. The "Moreover" claim follows easily by Remark 2.1 and compactness.

**Corollary 3.5** Let $T$ be any simple theory in which that extension property is first-order. Let $\phi(x, y) \in L$ and let $k < \omega$. Then the set

$$D_{\phi}^{\leq k} \equiv \{a | D(\phi(x, a)) \leq k\}$$

is a type-definable set (over $\emptyset$).

**Proof:** Straightforward by 3.2

### 4 Continuity of the SU-rank

In this section we work in $\mathcal{C}$ unless stated otherwise and all tuples $a, b, c, \ldots$ are assumed to be finite.

**Notation 4.1** For a formula $\theta \in L$ and finite tuple of sorts $s$, let

$$An^s(\theta) = \{a \in C^s | \text{tp}(a) \text{ is analyzable in } \theta\}.$$ 

For finite tuples sorts $s_0, s_1$, let $An^{s_0 \times s_1}(\theta) = An^{s_0}(\theta) \times An^{s_1}(\theta)$.

**Fact 4.2** For every formula $\theta \in L$, and tuple of sorts $s$, $An^s(\theta)$ is a Stone-open set.
Lemma 4.3 Let $T$ be a simple theory in which the extension property is first-order. Assume $\theta(x) \in L$ and $SU(\theta) = 1$. Let $r < \omega$ and let $s_0, s_1$ be finite tuple of sorts. Then the set
\[ SU_{\leq r, s_0, s_1}^{\text{int}}(\theta) \equiv \{ (a, a') \in C_s \times C_s \mid SU(a/a') \leq r \text{ and } tp(a/a') \text{ is almost internal in } \theta \} \]
is a Stone-open set (over $\emptyset$).

Proof: Assume $(a, a') \in SU_{\leq r, s_0, s_1}^{\text{int}}(\theta)$, i.e. $tp(a/a')$ is almost $\theta$-internal and $SU(a/a') = k \leq r$. By Fact 2.3 there exists $b \supseteq a'$ such that $b \setminus a'$ is a tuple of realizations of $tp(a)$, and there exists a formula $\chi(x, y, b)$ such that $\forall \bar{y} \exists^x \chi(x, \bar{y}, b)$ and such that for all $\bar{a} \models Lstp(a/a')$, there is a tuple $\bar{c} = c_1, ..., c_n$ of realizations of $\theta$ such that $\models \chi(\bar{a}, \bar{c}, b)$. Let $s$ be the sequence of sorts of $b$. Let $\lambda = (2^{|T|})^+$. Let $U$ be the subset of $C_s \times C_s$ defined by
\[ U(\hat{a}, \hat{a'}) \iff \exists \{ b_i \}_{i < \lambda}, b_i \in C_s, b_i \supseteq \hat{a} \quad [\forall \bar{a} \models tp(\hat{a}/a') \exists \bar{c} \subseteq \theta^a(\chi(\hat{a}, \bar{c}, b_i))]. \]

Now, for $l \leq n$, let $\delta_{l,n}$ be defined in the following way: $\delta_{l,n}(\bar{d}, e)$ iff $\bar{d} = d_1, ..., d_n$, where each $d_i$ has the sort of a single realization of $\theta$ and for some distinct $1 \leq i_1, i_2, ..., i_l \leq n$, $N_{j=1}^l d_{i_j} \not\subseteq acl(e, d_{i_1}...d_{i_{j-1}})$. Note that if $\bar{d} = d_1, ..., d_n$ and each $d_i$ realizes $\theta$, then for all $e$, $\delta_{l,n}(\bar{d}, e)$ iff $\text{dim}(\bar{d}/e) \geq l$ iff $SU(\bar{d}/e) \geq l$.

In addition, let $\Theta$ be the subset of $C_s \times C_s$ defined by
\[ \Theta(\hat{a}, \hat{a'}) \iff \forall b' \in C_s[b' \supseteq \hat{a'} \rightarrow \bigwedge_{l=k+1}^n (\Phi_l(\hat{a}, b') \rightarrow \Psi_{l-k}(\hat{a}, b'))] \]
where, $\Phi_l$, for $l < \omega$, is the subset of $C_s \times C_s$ defined by
\[ \Phi_l(\hat{a}, b') \iff \exists \bar{c} = c_1, c_2, ..., c_n \in \theta^a [\chi(\hat{a}, \bar{c}, b') \wedge \delta_{l,n}(\bar{c}, b')], \]
and $\Psi_l$, for $l < \omega$, is the subset of $C_s \times C_s$ defined by
\[ \Psi_l(\hat{a}, b') \iff \exists \bar{c} = c_1, c_2, ..., c_n \in \theta^a [\chi(\hat{a}, \bar{c}, b') \wedge \delta_{l,n}(\bar{c}, \hat{a}b')]. \]

Subclaim 4.4 $(a, a') \in U \cap \Theta$.

Proof: Since the number of extensions of $tp(a/a')$ to $bdd(a')$ is small, we get the required $b_i$-s in the definition of $U$ for ensuring that $U(a, a')$ holds. Now,
to show that \((a, a') \in \Theta\), assume \(b' \in C^s\) and \(b' \supseteq a'\) and \(\Phi_l(a, b')\) holds for some \(k + 1 \leq l \leq n\). Then, there is \(\bar{c} = c_1, c_2, \ldots c_n \in \Theta^c\) such that \(|\models \chi(a, \bar{c}, b')\) and \(SU(\bar{c}/b') \geq l\). Thus \(SU(\bar{c}/b') = SU(\bar{c}a/b') \leq SU(a/b') \oplus SU(\bar{c}/ab')\).

Since we know that \(SU(a/b') \leq k\) (as \(SU(a/a') = k\) and \(b' \supseteq a'\)), we conclude that \(SU(\bar{c}/ab') \geq l - k\).

**Subclaim 4.5** \(U \cap \Theta \subseteq SU_{\leq r, s_0, s_1}^{Aint}(\theta)\).

**Proof:** Assume \((\hat{a}, \hat{a}') \in U \cap \Theta\). Since \((\hat{a}, \hat{a}') \in U\), by extension there exists \(b^* \in C^s\) such that \(b^* \supseteq \hat{a}'\) and \(b^* \downarrow \hat{a}\) and such that for some \(c = c_1, c_2, \ldots c_n \in \Theta^c\) we have \(|\models \chi(\hat{a}, \bar{c}, b^*)\). We may assume that \(SU(\bar{c}*/b^*) \geq SU(\bar{c}'/b^*)\) for all tuples \(\bar{c}' = c_1', c_2', \ldots c_n' \in \Theta^c\) such that \(|\models \chi(\hat{a}, \bar{c}', b^*)\). Let \(l = SU(\bar{c}*/b^*)\). Then \(|\models \Phi_l(\hat{a}, b^*)\). By \(|\models \Theta(\hat{a}, \hat{a}')\), we conclude that \(|\models \Psi_{l-k}(\hat{a}, b^*)\).

So, there exists a tuple \(\bar{c}^* = c_1^*, c_2^*, \ldots c_n^* \in \Theta^c\) such that \(|\models \chi(\bar{a}, \bar{c}^*, b^*)\) and \(SU(\bar{c}^*/\bar{a}b^*) \geq l - k\). Now, as \(SU(\bar{a}c^*/b^*) < \omega\),

\[SU(\bar{c}^*/b^*) = SU(\bar{a}c^*/b^*) = SU(\bar{a}/b^*) + SU(\bar{c}^*/\bar{a}b^*).\]

By maximality of \(l\), \(SU(\bar{c}^*/b^*) \leq l\); so it follows that \(SU(\bar{a}/b^*) \leq k\). Since \(b^* \downarrow \hat{a}\) and \(b^* \supseteq \hat{a}'\), we conclude that \(SU(\bar{a}/\bar{a}') \leq k\).

**Subclaim 4.6** \(U \cap \Theta\) is Stone-open.

**Proof:** It will be sufficient to prove that \(\Psi_l\) is Stone-open for every \(l \leq n\). Indeed, let us define for every distinct \(1 \leq i_1, i_2, \ldots, i_l \leq n\), a relation \(\Psi_{i_1, i_2, \ldots, i_l}\) by: for every \((\hat{a}, b') \in C^{s_0} \times C^{s_1}\), we have

\[\Psi_{i_1, i_2, \ldots, i_l}(\hat{a}, b') \text{ iff } \exists \bar{c} = c_1, c_2, \ldots c_n \in \Theta^c:: [\chi(\hat{a}, \bar{c}, b') \land \bigwedge_{j=1}^{l} (y = c_{i_j} \text{ forks over } \hat{a}b'c_{i_1} \ldots c_{i_{j-1}})].\]

Then, for every \((\hat{a}, b') \in C^{s_0} \times C^{s_1}\), we have \(\Psi_l(\hat{a}, b')\) iff there are distinct \(1 \leq i_1, i_2, \ldots, i_l \leq n\) such that \(\Psi_{i_1, i_2, \ldots, i_l}(\hat{a}, b')\). By Lemma 3.2, \(\Psi_{i_1, i_2, \ldots, i_l}\) is definable for every distinct \(1 \leq i_1, i_2, \ldots, i_l \leq n\). To see this, choose in Lemma 3.2 \(m = l\), \(d_1 = \hat{a}b'\), \(d_j\) remains the same for \(j = 2, \ldots, l\), \(a = c_1c_2 \ldots c_n\), \(\phi_j = (y = c_{i_j})\), and

\[\psi(a, d_1, \ldots, d_m) = \chi(\hat{a}, \bar{c}, b') \land \bigwedge_{j=2}^{l} (d_j = c_{i_{j-1}}) \land \bigwedge_{i=1}^{n} \theta(c_i).\]
Lemma 4.7 Let $T$ be any simple theory. Assume $\theta(x) \in L$ and $SU(\theta) = 1$. Let $r < \omega$ and let $s_0, s_1$ be sorts. Then the set

$$SU_{\leq r,s_0,s_1}^{An,Aint}(\theta) \equiv \{(a,a') \in An^{s_0,s_1}(\theta) \mid SU(a/a') \leq r \text{ and } tp(a/a') \text{ is almost internal in } \theta\}$$

is a Stone-open set (over $\emptyset$).

Proof: Let $(a,a') \in SU_{\leq r,s_0,s_1}^{An,Aint}(\theta)$, i.e. $tp(a/a')$ is almost $\theta$-internal, $tp(a')$ is analyzable in $\theta$ and $SU(a/a') \leq r$. Let $(a_1,a_2,...a_{n-1},a_n)$ be an $a$-analysis in $\theta$ with $a_{n-1} = a'$, $a_n = a$. Let $U(x_1, ..., x_n)$ be the Stone-open set over $\emptyset$ such that $(a'_1, ...a'_n) \models U$ iff $(a'_1, a'_2, ..., a'_n)$ is an $a$-analysis in $\theta$. Let $\phi(x_1, ..., x_n) \in L$ be such that $(a_1, a_2, ..., a_{n-1}, a_n) \models \phi$ and $\phi^c \subseteq U$. Let $V_0 = \emptyset$ and for $1 \leq i \leq n$ let $V_i$ be the projection of $\phi(x_1, ..., x_n)$ on the $i$-th coordinate. For $0 \leq i \leq n$ let $\hat{s}_i$ be the sort of $V_i$. By working in $C^{eq}$ we may assume that for all $0 \leq i < j \leq n$, $\hat{s}_i \neq \hat{s}_j$. Let $\bar{M} = (V_0, V_1, ..., V_n)$ be the structure whose universe is the disjoint union of the $V_i$-s, where the interpretation of $\hat{s}_i$ in $\bar{M}$ is $V_i$ and $\bar{M}$ is equipped with the induced structure from $C^{eq}$, that is, the $\emptyset$-definable subsets of $\bar{M}$ are precisely the $\emptyset$-definable sets of $C^{eq}$ that are subsets of the cartesian products of $V_i^{k_i}$ for some $i$-s and $k_i$-s. Clearly $\bar{M}$ is saturated.

In the following, if $p(x)$ is a partial type of $\bar{M}$, we will consider this type in $C^{eq}$ by replacing $p(x)$ with $p'(x) = p(x) \wedge \nu(x)$ where $\nu(x)$ is the formula that says ”$x$ belongs to $\bar{M}$”

Claim 4.8 1) Dividing of partial types of $\bar{M}$ is absolute between $\bar{M}$ and $C^{eq}$ and $\bar{T} = Th(\bar{M})$ is simple (thus forking is is absolute between $\bar{M}$ and $C^{eq}$).

2) Almost internality for types in $\bar{M}$ is absolute between $C^{eq}$ and $\bar{M}$.

3) Every type of $\bar{M}$ is $a$-analyzable in $V_0 = \emptyset$ in $\bar{M}$ (and in $C^{eq}$). Clearly, $\emptyset$ is weakly minimal in $\bar{M}$ too, thus $\bar{T}$ is supersimple.

Proof: 1) Clearly dividing for partial types of $\bar{M}$ is absolute between $\bar{M}$ and $C^{eq}$. Thus, it is clear that every complete finitary type of $\bar{M}$ does not divide over a subset of size $\leq |T|$ of its domain, so $\bar{T}$ is simple. To prove 2), assume that $p \in S(A)$ and $q$ is a partial type over $A$ both of $\bar{M}$. By 1) if $p$ is almost internal in $q$ in the sense of $\bar{M}$ then the same holds in the sense of $C^{eq}$. Now, assume $p$ is almost internal in $q$ in the sense of $C^{eq}$, where $p$ and $q$ are types of $\bar{M}$ ($p$ is complete). By Fact 2.3 in $C^{eq}$ there exists $a \models p$ and a tuple $\bar{a}$ of realizations of $p$ that is independent from $a$ such that $a \in acl(q^c, \bar{a})$ (over
the corresponding parameters). Since \( a, \bar{a} \in \bar{M} \), using 1), the same is true in \( \bar{M} \). To prove 3) we need to prove that for every \( a, A \subseteq \bar{M} \), \( tp(a/A) \) is \( a \)-analyzable in \( V_0 \) in \( \bar{M} \). By Fact 2.2 we may assume that \( a \) is a singleton (rather than a tuple) and \( A = \emptyset \). If \( a \in V_0 \) we are clearly done. Otherwise, \( a \in V_i \) for some \( 1 \leq i \leq n \), so there are \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n \in C \) such that \( C^eq \models \phi(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots a_n) \). Then, in particular, in \( C^eq, (a_1, \ldots, a_{i-1}, a) \) is an \( a \)-analysis in \( \theta \). By 2), \( (a_1, \ldots, a_{i-1}, a) \) is an \( a \)-analysis in \( \theta \) in \( \bar{M} \). \( \square \)

**Claim 4.9** For every \( a, A \) in \( \bar{M} \) we have \( SU^M(a/A) = SU(a/A) \), where \( SU^M \) is the \( SU \)-rank in \( \bar{M} \) and \( SU \) is the usual \( SU \)-rank in \( C^eq \).

**Proof:** It will be sufficient to show that \( SU^M(a/A) \geq SU(a/A) \) for every \( a, A \subseteq \bar{M} \). We prove by induction on \( \alpha \) that for all \( a, A \), if \( SU(a/A) \geq \alpha \) then \( SU^M(a/A) \geq \alpha \). We may clearly assume \( \alpha = \beta + 1 \). We work in \( C^eq \).

Let \( B \supseteq A \) be such that \( a \not\subseteq B \) and \( SU(a/B) \geq \beta \). Let \( I \) be Morley sequence of \( Lstp(a/B) \) and let \( a' \models Lstp(a/B) \) be such that \( a' \upharpoonright bdd(B) \) (so, \( a' \in \bar{M} \)). Let \( e = Cb(a/bdd(B)) \). We conclude that \( a' \upharpoonright A I \) and \( SU(a'/AI) = SU(a'/Ae) = SU(a'/B) = SU(a/B) \) (as \( a' \upharpoonright Ae \) and \( e \in dcl(I) \)). So, \( SU(a'/AI) \geq \beta \). As \( AI \subseteq \bar{M} \), by Claim 4.8 1), we get that \( \bar{M} \models a' \upharpoonright A I \). Thus \( SU^M(a/A) = SU^M(a'/A) \geq \alpha = \beta + 1 \). \( \square \)

Now, let \( U_\bar{M} = (SU^\bar{M}_{\leq r, s_0, s_1}(\theta))^{\bar{M}} \) i.e. \( U_\bar{M} \) is the set defined in Lemma 4.3 in \( \bar{M} \). By Corollary 2.7 and Claim 4.8 the extension property is first-order in \( T = Th(\bar{M}) \). Thus, by Lemma 4.3 \( U_\bar{M} \) is a Stone-open set over \( \emptyset \) in \( \bar{M} \) and in particular in \( C^eq \). Since almost-internality is absolute between \( \bar{M} \) and \( C^eq \) by Claim 4.8 and \( SU^\bar{M} = SU \) by Claim 4.9 \( (a, a') \in U_\bar{M} \). By Claims 4.8 4.9 \( U_\bar{M} \subseteq SU^\bar{M}_{\leq r, s_0, s_1}(\theta) \). \( \square \)

**Notation 4.10** For a formula \( \theta = \theta(x, c) \in L(C) \) and sort \( s \), let \( \tilde{A}n^s(\theta) = \{ a \in C^s | tp(a/c') \) is analyzable in \( \theta(x, c') \) for all \( \theta \)-conjugate \( \theta(x, c') \) of \( \theta(x, c) \} \). For sorts \( s_0, s_1 \), let \( \tilde{A}n^{s_0, s_1}(\theta) = \tilde{A}n^{s_0}(\theta) \times \tilde{A}n^{s_1}(\theta) \).
Theorem 4.11 Let $T$ be a simple theory and work in $C^{eq}$. Assume $\theta \in L(C)$, $SU(\theta) = 1$. Let $r < \omega$ and let $s_0, s_1$ be sorts. Then the set

$$SU_{\leq r, s_0, s_1}(\theta) = \{(a, b) \in \tilde{A}n_{s_0,s_1}(\theta) | SU(a/b) \leq r\}$$

is a Stone-open set (over $\emptyset$).

Proof: First, assume that $\theta = \theta(x) \in L$. Note that $(a, b) \in SU_{\leq r, s_0, s_1}(\theta)$ iff there are $a_0, a_1, \ldots, a_n = a \in dcl(ab)$ and $r_0, \ldots, r_n < \omega$ such that $r_0 + \ldots + r_n = r$, $tp(b)$ is analyzable in $\theta$ and $tp(a_i/a_{<i}b)$ is (almost) $\theta$-internal and $SU(a_i/a_{<i}b) \leq r_i$ for all $i \leq n$. By Lemma 4.7 we are done in case $\theta \in L$. Assume now $\theta = \theta(x, c^*) \in L(C)$. By working over $c^*$ and what we have just proved, there is Stone-open set $U(x, y, c^*)$ over $c^*$ such that for all $(a, b)$ we have $U(a, b, c^*)$ iff $tp(a/c^*)$ and $tp(b/c^*)$ are analyzable in $\theta(x, c^*)$ and $SU(a/bc^*) \leq r$. Let $p = tp(c^*)$ and let $V(x, y) = \forall c(c \models p \rightarrow U(x, y, c))$. Then $V(x, y)$ is Stone-open over $\emptyset$ and $V^c = SU_{\leq r, s_0, s_1}(\theta)$ (whenever $(a, b) \models V$, there exists $c \models p$ independent from $ab$ over $\emptyset$ such that $U(a, b, c)$; thus $SU(a/b) \leq r$.) □

5 Definability of the D-rank

Theorem 5.1 Let $T$ be a simple theory in which the extension property is first-order. Work in $C^{eq}$. Assume $\theta(x) \in L(C)$, $SU(\theta) = 1$. Let $\phi(x, y) \in L$ be such that

$$\forall a, b [\phi(a, b) \Rightarrow a \in \tilde{A}n^x(\theta), b \in \tilde{A}n^y(\theta)].$$

Then the $D$-rank is definable for $\phi(x, y)$, namely, for all $r < \omega$, the set $D^r_\phi = \{b | D(\phi(x, b)) = r\}$ is definable.

Proof: By Corollary 3.5, $D^r_\phi = \{b | D(\phi(x, b)) \leq r\}$ is type-definable. Now, $D^r_\phi = \{b | \forall a (\phi(a, b) \Rightarrow SU(a/b) \leq r)\}$. By Theorem 4.11 and our assumption, $D^r_\phi$ is Stone-open so we are done.

We say that the $SU$-rank is uniformly continuous if for any given sorts $s_0, s_1$, the set $\{(a, b) \in C^{s_0} \times C^{s_1} | SU(a/b) \leq r\}$ is Stone open for any $r$. We say that the $D$-rank is definable if the $D$-rank is definable for $\phi(x, y)$ for any $\phi(x, y)$. 

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Corollary 5.2 Let $T$ be a supersimple unidimensional theory (e.g. $T$ is a countable hypersimple theory). Then the $SU$-rank is uniformly continuous and the $D$-rank is definable. In particular, for every finitary type $p \in S(A)$ we have $D(p) = SU(p)$.

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