The Bergman-Shilov boundary for subfamilies of $q$-plurisubharmonic functions

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Abstract

We introduce a notion of the Bergman-Shilov (or Shilov) boundary for some subclases of upper semi-continuous functions on a compact Hausdorff space. It is by definition the smallest closed subset of the given space on which all functions of that subclass attain their maximum. For certain subclasses with simple structure one can show the existence and uniqueness of the Shilov boundary. Then we provide its relation to the set of peak points and establish Bishop-type theorems. As an application we obtain a generalization of Bychkov’s theorem which gives a geometric characterization of the Shilov boundary for $q$-plurisubharmonic functions on convex bounded domains. In the case of bounded pseudoconvex domains with smooth boundary we also show that some parts of the Shilov boundary for $q$-plurisubharmonic functions are foliated by $q$-dimensional complex submanifolds.

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Introduction

In his article [Byc81] from 1981, S.N. Bychkov gave a geometric characterization of the Shilov boundary for bounded convex domains in $\mathbb{C}^n$. The aim of our paper is to generalize his result to the Shilov boundary with respect to $q$-plurisubharmonic and $q$-holomorphic functions.

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functions on bounded convex domains. These classes of functions were already studied by different authors, e.g., R. Basener in [Bas76], R.L. Hunt and J.J. Murray in [HM78] or Z. Slodkowski in [Slo86], [Slo84]. It was H.J. Bremermann in [Bre59] who observed that there is a characterization of a Bergman-Shilov boundary (or, for short, Shilov boundary) based on plurisubharmonic functions without showing its existence. This gap was filled by, e.g., J. Siciak in [Sic62]. Given a compact Hausdorff space $\mathcal{K}$ and a subclass $\mathcal{A}$ of upper semi-continuous functions on $\mathcal{K}$, the Shilov boundary for $\mathcal{A}$ is the smallest closed subset of $\mathcal{K}$ on which all functions from $\mathcal{A}$ attain their maximum. Existence and uniqueness for such a subset is guaranteed if $\mathcal{A}$ has some simple structure, e.g., if $\mathcal{A}$ forms a cone and sublevel sets of finitely many functions from $\mathcal{A}$ generate the topology of $\mathcal{K}$ (see Theorem 1' in [Sic62]). For $q$-plurisubharmonic functions the condition on $\mathcal{A}$ to be a cone is too strong, since $q$-plurisubharmonicity is not stable under addition. It turns out that the mentioned above condition can be relaxed so that the existence of the Shilov boundary for a wide class of upper semi-continuous functions can be guaranteed. This will be the main part of the first chapter.

In the second chapter we define the closure of a subclass of upper semi-continuous functions to be the collection of all limits of decreasing sequences of functions from that subclass. In our context it plays a role similar to the uniform closure of a subset of continuous functions on a compact Hausdorff space: The Shilov boundary for the subclass and its closure coincide.

The third chapter brings the Shilov boundary into connection to peak points. E. Bishop proved in [Bis59] that, if the compact Hausdorff space is assumed to be metrizable, then the closure of the set of peak points and the Shilov boundary for uniform subalgebras of continuous functions coincide. This is also true for any Banach subalgebra of continuous functions due to the results of H.G. Dales [Dal71] (see also [Hon88]). Note that using upper semi-continuous functions similar identities were obtained in [Sic62] and [Wit83]. We apply these results to unions of uniform algebras and establish additional Bishop-type theorems.

In the fourth chapter we introduce the notions of $q$-plurisubharmonic and $q$-holomorphic functions and give a list of their properties. For the proofs and further results on these classes of functions we refer also to [Slo86], [Die06] and [PZ13].

In chapter five the results from the first three chapters are applied to subclasses of $q$-plurisubharmonic functions.

In the sixth chapter Bychkov’s theorem is generalized as follows: a boundary point of a convex bounded domain does not lie in the Shilov boundary for $q$-plurisubharmonic or $q$-holomorphic functions if and only if it is contained in an open part of a complex plane of dimension at least $q + 1$ which is fully contained in the boundary of the given convex set.

It seems still to be an open question whether the Hausdorff dimension of Shilov boundary for holomorphic functions on compact sets in $\mathbb{C}^N$ is greater or equal to $N$. E. Bishop gave in [Bis59] a positive answer to this question in the special case $N = 2$. In this context, we consider the Hausdorff dimension of the Shilov boundary for $q$-plurisubharmonic functions on a convex bounded domain in chapter seven.

In chapter eight we show that the Shilov boundary for $q$-plurisubharmonic and the Shilov boundary for $C^2$-smooth $q$-plurisubharmonic functions defined near a compact set coincide due to approximation techniques of Bungart [Bun90], Slodkowski [Slo84] and Demailly [Dem12]. As an application we prove that if the given domain $D$ is bounded and smoothly bounded, then the Shilov boundary for $q$-plurisubharmonic functions defined near $\overline{D}$ is exactly the closure of the set of all strictly $q$-pseudoconvex points of the boundary of $D$. Using a rank condition on the Levi form of a defining function of $D$ which was established by M. Freeman in [Fre74], we obtain a foliation of parts of the Shilov boundary for $q$-plurisubharmonic functions on $\overline{D}$ by complex $q$-dimensional submanifolds.
1 Shilov boundary for upper semi-continuous functions

In this chapter we will define the Bergman-Shilov boundary for subclasses of upper semi-continuous functions and show its existence and uniqueness in certain cases. For the sake of abbreviation, we will simply talk about the Shilov boundary instead of the Bergman-Shilov boundary. Anyway, we have to point out that the concept of a distinguished boundary of certain domains in $\mathbb{C}^2$ was already introduced by S.Bergman in [Ber31].

At first, we recall some basic definitions and facts about upper semi-continuous functions on a compact Hausdorff space $K$.

**Definition 1.1** A function $f : K \to [-\infty, \infty)$ is called upper semi-continuous on $K$ if the sub-level set $\{ x \in K : f(x) < c \}$ is open in $K$ for every $c \in \mathbb{R}$. We denote then by $\mathcal{USC}(K)$ the set of all upper semi-continuous functions on $K$ and by $\mathcal{C}(K) = \mathcal{C}(K, \mathbb{C})$ the set of all complex-valued continuous functions on $K$.

We will outline an important example for an upper semi-continuous function.

**Example 1.2** Let $S$ be a closed subset of $K$. Then the characteristic function $\chi_S$ of $S$ (in $K$) given by

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \in K \setminus S \end{cases}$$

is upper semi-continuous on $K$.

The following statement is a well known fact.

**Lemma 1.3** Every function $f \in \mathcal{USC}(K)$ attains its maximum on $K$, i.e., there exists a point $x_0$ in $K$ such that

$$\max \{ f(x) : x \in K \} := f(x_0) = \sup \{ f(x) : x \in K \}.$$

From now on, $\mathcal{A}$ is always a subset of $\mathcal{USC}(K)$. Our main object of study is the Shilov boundary for $\mathcal{A}$.

**Definition 1.4** For a given function $f \in \mathcal{USC}(K)$ we set

$$S(f) := \{ x \in K : f(x) = \max_K f \}.$$

A subset $S$ of $K$ is called a boundary for $\mathcal{A}$ or $\mathcal{A}$-boundary if $S \cap S(f) \neq \emptyset$ for every $f \in \mathcal{A}$. We denote by $b_\mathcal{A}$ the set of all closed boundaries for $\mathcal{A}$. The set $\tilde{S}_\mathcal{A} := \bigcap_{S \in b_\mathcal{A}} S$ is called the Shilov boundary for $\mathcal{A}$.

We give first some simple examples.

**Example 1.5** (1) Let $f_1 = \chi_{[0,1]}$ and $f_2 = \chi_{[1,2]}$ considered as upper semi-continuous functions on the interval $K = [0,2]$. For $\mathcal{A} = \{ f_1, f_2 \}$ we have that $\{0,2\}, \{1\} \in b_\mathcal{A}$, $S(f_1) \cap S(f_2) = \{1\}$ and that $\tilde{S}_\mathcal{A}$ is empty.

(2) For $f_1 = \chi_{[0]}$ and $f_2 = \chi_{[1]}$ considered as functions on $K = [0,1]$ we take $\mathcal{A} = \{ f_1, f_2 \}$ and observe that $\{0,1\} \in b_\mathcal{A}$, $S(f_1) \cap S(f_2) = \emptyset$ and $\tilde{S}_\mathcal{A} = \{0,1\}$.

(3) Consider the functions $f_1 = \chi_{[-1,1]}$ and $f_2 = \chi_{[0]}$ defined on $[-1,1]$ and set $\mathcal{A} = \{ f_1, f_2 \}$. Then $\{-1,0\}, \{0,1\} \in b_\mathcal{A}$, so $\tilde{S}_\mathcal{A} = \{0\}$. But $\tilde{S}_\mathcal{A}$ can not be an $\mathcal{A}$-boundary because the function $f_1$ attains its maximum outside of zero.

We have the following properties of Shilov boundaries.
Proposition 1.6

(1) The set \( \hat{S}_A \) is closed and possibly empty, whereas \( b_A \) is never empty.

(2) \( S(f) \) is a closed non-empty subset of \( K \).

(3) If the set \( T := \bigcap_{f \in A} S(f) \) consists of more than two elements, then \( \hat{S}_A \) is empty.

(4) If the set \( T \) from above consists of one single element \( x_0 \in K \) and \( \hat{S}_A \neq \emptyset \), then \( \hat{S}_A = \{x_0\} \).

(5) The set \( S := \bigcup_{f \in A} S(f) \) is an \( A \)-boundary.

(6) If \( A_1 \subset A_2 \subset USC(K) \), then we have the following inclusions,

\[
b_{A_2} \subset b_{A_1} \quad \text{and} \quad \hat{S}_{A_1} \subset \hat{S}_{A_2}.
\]

(7) Let \( A = \bigcup_{j \in J} A_j \), where \( A_j \) are subsets of \( USC(K) \). If \( \hat{S}_{A_j} \) are \( A_j \)-boundaries, then \( \hat{S}_A \) is an \( A \)-boundary and

\[
\hat{S}_A = \bigcup_{j \in J} \hat{S}_{A_j}.
\]

Proof.

(1) The set \( \hat{S}_A \) is closed as intersection of closed sets. Example 1.5 (1) shows that \( \hat{S}_A \) might be empty. The set \( b_A \) contains at least the ambient space \( K \).

(2) Since \( f \in USC(K) \), the set \( \{x \in K : f(x) < \max_K f\} \) is open in \( K \), so the set \( S(f) = K \setminus \{x \in K : f(x) < \max_K f\} \) is a closed subset of \( K \). It is non-empty due to Lemma 1.3.

(3) Pick two distinct elements \( x_0, x_1 \) from \( T \). By definition \( \{x_0\} \) and \( \{x_1\} \) are \( A \)-boundaries and, thus, \( \hat{S}_A \subset \{x_0\} \cap \{x_1\} = \emptyset \).

(4) In this case \( \{x_0\} \in b_A \). Thus, \( \emptyset \neq \hat{S}_A \subset \{x_0\} \) which yields \( \hat{S}_A = \{x_0\} \).

(5) The set \( S \) is an \( A \)-boundary because \( S \cap S(f) = S(f) \neq \emptyset \) for every \( f \in A \).

(6) This fact follows directly from definition.

(7) The previous points (1) and (7) imply the inclusion \( S := \bigcup_{j \in J} \hat{S}_{A_j} \subset \hat{S}_A \). By assumption, the set \( S \) and, therefore, the set \( \hat{S}_A \) are non-empty.

Since an arbitrary function \( f \in A \) is contained in \( A_j \) for some \( j \in J \) and by the assumption that \( \hat{S}_{A_j} \) is an \( A_j \)-boundary, we obtain that

\[
\emptyset \neq S(f) \cap \hat{S}_{A_j} \subset S(f) \cap S \subset S(f) \cap \hat{S}_A.
\]

This means that \( S \) is an \( A \)-boundary and, thus, \( \hat{S}_A \subset S \). By the previous discussions above, we have that \( S = \hat{S}_A \) is an \( A \)-boundary.

We can easily bring our concept of the Shilov boundary into relation with the classical Shilov boundary for uniform subalgebras of \( \mathcal{C}(K) \).

Remark 1.7 Let \( B \) be a subset of \( \mathcal{C}(K) \). The classical Shilov boundary for \( B \) is the smallest closed subset \( S \) of \( K \) fulfilling \( \max_S |f| = \max_K |f| \) for every \( f \in B \). Clearly, it corresponds to the Shilov boundary for the class \( \log |B| := \{\log |f| : f \in B\} \). It then makes sense to simply write \( b_B \) and \( \hat{S}_B \) instead of \( b_{\log |B|} \) and \( \hat{S}_{\log |B|} \). It is clear that for the uniform closure \( \overline{B} \) of \( B \) in \( \mathcal{C}(K) \) we have that \( \hat{S}_B = \hat{S}_{\overline{B}} \).
Now we recall the classical result of Shilov.

**Theorem (Shilov)** Let $K$ be a compact Hausdorff space and $\mathcal{B}$ a Banach subalgebra of $\mathcal{C}(K)$. Then $\mathcal{S}_\mathcal{B}$ is non-empty and, moreover, it is a boundary for $\log |\mathcal{B}|$.

In this theorem the Banach algebra structure of $\mathcal{B}$ is heavily involved. We will extract the essential properties from that structure in order to establish similar results for Shilov boundaries for subclasses of upper semi-continuous functions.

**Definition 1.8** Let $\mathcal{A}$ be a subset of $\mathcal{USC}(K)$.

1. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two subfamilies of $\mathcal{USC}(K)$, then

   \[ \mathcal{A}_1 + \mathcal{A}_2 := \{ f + g : f \in \mathcal{A}_1, g \in \mathcal{A}_2 \}. \]

2. The family $\mathcal{A}$ is a **scalar cone** if $nf + b$ lies in $\mathcal{A}$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $f \in \mathcal{A}$ and $b \in \mathbb{R}$. Here we use the convention $-\infty \cdot 0 = 0$.

3. The set $\mathcal{A}$ is a **cone** if $af + bg$ is contained in $\mathcal{A}$ for every $a, b \in [0, +\infty)$ and $f, g \in \mathcal{A}$.

4. An open set $V$ in $K$ is an **$\mathcal{A}$-polyhedron** if there exist finitely many functions $f_1, \ldots, f_n$ in $\mathcal{A}$ and real numbers $C_1, \ldots, C_n$ such that

   \[ V = V(f_1, \ldots, f_n) = \{ x \in K : f_1(x) < C_1, \ldots, f_n(x) < C_n \}. \]

5. The set $\mathcal{A}$ **generates the topology of $K$** if for every point $x \in K$ and every neighborhood $U$ of $x$ in $K$ there is an $\mathcal{A}$-polyhedron $V$ such that $x \in V \subset U$.

Now we are able to show that the Shilov boundary for $\mathcal{A}$ is a non-empty boundary for $\mathcal{A}$ if $\mathcal{A}$ possesses some simple structure. The following two statements are based on standard arguments used in the case of Banach subalgebras of continuous functions (see e.g. [AW98], Theorem 9.1). First, we need the following lemma.

**Lemma 1.9** Let $\mathcal{A}$ be a scalar cone. Assume that there exist an $\mathcal{A}$-boundary $S \in b_\mathcal{A}$ and an $\mathcal{A}$-polyhedron $V = V(f_1, \ldots, f_n)$ such that $S \cap V = \emptyset$ and $\mathcal{A} + \{ f_j \} \subset \mathcal{A}$ for $j = 1, \ldots, n$. Given another $\mathcal{A}$-boundary $E \in b_\mathcal{A}$, it follows that $E \setminus V \in b_\mathcal{A}$.

**Proof.** Since $\mathcal{A}$ is a scalar cone and $\mathcal{A} + \{ f_j \} \subset \mathcal{A}$ for $j = 1, \ldots, n$, the constant function 0 and, thus, $f_1, \ldots, f_n$ lie in $\mathcal{A}$. Hence, we can assume that $V$ is of the form $V = \{ x \in K : f_1(x) < 0, \ldots, f_n(x) < 0 \}$.

Notice first that $E \setminus V$ is non-empty. Otherwise, $E \subset V$, so $\max_{E \setminus V} f_j < 0$ for every $j = 1, \ldots, n$. Since $S$ does not meet $V$, there has to be an index $j_0 \in \{1, \ldots, n\}$ such that $\max_S f_{j_0} \geq 0$. We obtain the contradiction $0 \leq \max_S f_{j_0} = \max_E f_{j_0} < 0$.

Suppose that the statement of the lemma is false, i.e., there are a point $y \in K$ and a function $f \in \mathcal{A}$ such that $\max_{E \setminus V} f < \max_K f = f(y)$. Since $\mathcal{A}$ is a scalar cone and $S \in b_\mathcal{A}$, we can assume that $f(y) = 0$ and $y \in S$. Consider for $m \in \mathbb{N}$ the functions $g_j := mf + f_j \in \mathcal{A}$, $j=1, \ldots, n$. If $m$ is large enough, then $\max_{E \setminus V} g_j < 0$ for each $j=1, \ldots, n$. Since $\max_K f = 0$, it follows from the definition of $V$ that for every $j=1, \ldots, n$ we have that $g_j(x) < 0$ for every $x \in V$. Hence, $\max_K g_j = \max_{E \setminus V} g_j < 0$ for every $j=1, \ldots, n$.

We conclude that $y \in V$. If not, there is an index $j_1 \in \{1, \ldots, n\}$ with $f_{j_1}(y) \geq 0$ and, thus, $g_{j_1}(y) \geq 0$, which is impossible. Thus, $y \in V \cap S = \emptyset$, a contradiction. \[\Box\]
Theorem 1.10 If $A$ contains a subset $A_0$ which generates the topology of $K$ such that $A + A_0 \subset A$, then the Shilov $A$-boundary is an $A$-boundary; i.e., $\hat{S}_A \in b_A$.

Proof. At first, assume that $A$ is a scalar cone. If $\hat{S}_A = K$, then there is nothing to show. So we can assume that $\hat{S}_A \neq K$. We first treat the case $\hat{S}_A \neq \emptyset$. Suppose $\hat{S}_A \notin b_A$, then there is a function $f \in A$ such that $\max_{\hat{S}_A} f < \max_K f$. Since $f$ is upper semi-continuous on $K$, there is a neighborhood $U$ of $\hat{S}_A$ such that $f(x) < \max_K f$ for every $x \in U$. Then, since $A_0$ generates the topology of $K$, we conclude that for every $y \in L := K \setminus U$ there are an $A_0$-polyhedron $V_y$ and an $A$-boundary $S_y \in b_A$ such that $y \in V_y$ and $V_y \cap S_y = \emptyset$. The family $\{V_y\}_{y \in L}$ covers $L$. Hence, by the compactness of $L$, there are finitely many points $y_1, \ldots, y_\ell \in L$ such that the subfamily $\{V_{y_j}\}_{j=1, \ldots, \ell}$ covers $L$. Since $A + A_0 \subset A$, we can apply iteratively the previous Lemma 1.9 in order to obtain that

$$E := (((K \setminus V_{y_1}) \setminus V_{y_2}) \setminus \ldots \setminus V_{y_\ell}) = K \setminus \bigcup_{j=1}^\ell V_{y_j} \in b_A.$$ 

Notice that, by the construction, the set $\hat{S}_A$ lies in $E$ and, hence, $E$ is non-empty. Moreover, $E \subset U$ and, thus, $\max_E f < \max_K f$. But this contradicts to the fact that $E \in b_A$. Hence, $\hat{S}_A \in b_A$.

In the case $\hat{S}_A = \emptyset$, we pick an arbitrary point $p \in K$ and a neighborhood $U$ of $p$ in $K$ which is an $A_0$-polyhedron of the form $U = \{x \in K : f_1(x) < 0, \ldots, f_k(x) < 0\}$ such that $U \neq K$. Observe that for every $y \in K \setminus U$ there exists an $A$-boundary $S_y$ with $y \notin S_y$, since otherwise $y \in \hat{S}_A$. Then we can choose an $A_0$-polyhedron $V_y$ such that $y \in V_y$ and $S_y \cap V_y$ is empty. By the same argument as above we can construct an $A$-boundary $E$ such that $p \in E \subset U$. But since $U \neq K$, there exists a point $x_0 \in K \setminus U$ and an index $k_0 \in \{1, \ldots, k\}$ such that $f_{k_0}(x_0) \geq 0$. This leads to the contradiction

$$0 \leq f_{k_0}(x_0) \leq \max_k f_{k_0} = \max_E f_{k_0} < 0.$$ 

Thus, $\hat{S}_A$ can not be empty.

If $A$ is not necessarily a scalar cone, consider the set

$$\tilde{A} := \{nf + c : n \in N_0, \ f \in A, \ c \in \mathbb{R}\}.$$ 

Since $A$ lies in $\tilde{A}$, we have that $b_{\tilde{A}} \subset b_A$ and $\hat{S}_{\tilde{A}} \subset \hat{S}_A$. Pick an arbitrary $A$-boundary $S$ and a function $nf + c \in \tilde{A}$, where $f \in A$, $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Since $f$ and $nf + c$ attain their maximum at the same points, we have that

$$S \cap S(nf + c) = S \cap S(f) \neq \emptyset.$$ 

But this means that $S$ is also an $\tilde{A}$-boundary, so $b_{\tilde{A}} = b_A$ and $\hat{S}_{\tilde{A}} = \hat{S}_A$.

Now observe that the family $A_0 := \{nf + c : n \in N_0, \ f \in A_0, \ c \in \mathbb{R}\}$ generates the topology of $K$, since it contains $A_0$. Moreover, we have that $\tilde{A} + A_0 \subset \tilde{A}$ and that $\tilde{A}$ is a scalar cone. Thus, by the previous discussions, we conclude that $\hat{S}_A = \hat{S}_{\tilde{A}} \in b_{\tilde{A}} = b_A$. This finishes the proof. \qed

2 Closure of a subfamily of upper semi-continuous functions

As in the previous section, $K$ will always be a compact Hausdorff space and $A$ a subfamily of upper semi-continuous functions on $K$. 

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The limit of a decreasing sequence of upper semi-continuous functions is again upper semi-continuous. This simple fact will allow us to introduce the notion of the closure of \( \mathcal{A} \) and, hence, will give the possibility to compare the initial class with an approximating subclass.

**Definition 2.1** The closure \( \overline{\mathcal{A}}^\uparrow \) of \( \mathcal{A} \) is the set of pointwise limits of all decreasing sequences of functions in \( \mathcal{A} \). The set \( \mathcal{A} \) is closed if \( \overline{\mathcal{A}}^\uparrow = \mathcal{A} \).

**Remark 2.2**

1. If \( \mathcal{A} \) is a (scalar) cone, then its closure \( \overline{\mathcal{A}}^\uparrow \) is also a (scalar) cone.

2. The family \( \mathcal{A} \) generates the topology of \( K \) if and only if \( \overline{\mathcal{A}}^\uparrow \) generates it. Indeed, one inclusion is trivial, since \( \mathcal{A} \) is always contained in \( \overline{\mathcal{A}}^\uparrow \). So let \( \overline{\mathcal{A}}^\uparrow \) generate the topology of \( K \).

   Given a point \( p \in K \) and an open neighborhood \( U \) of \( p \) in \( K \) there exists an \( \overline{\mathcal{A}}^\uparrow \)-polyhedron \( V = \{ x \in K : f_1(x) < c_1, \ldots, f_k(x) < c_k \} \) such that \( p \in V \subset U \). Since \( f_1, \ldots, f_k \in \overline{\mathcal{A}}^\uparrow \), for every \( j = 1, \ldots, k \) there exists a sequence \( (f_{j,n_j})_{n_j \in \mathbb{N}} \) of functions \( f_{j,n_j} \in \mathcal{A} \) which decreases to \( f_j \) as \( n_j \) tends to \( \infty \). For large enough \( n_0 \) we have that \( f_j(p) \leq f_{j,n_0}(p) < c_j \) for every \( j = 1, \ldots, k \).

   Then
   \[
   p \in V_0 = \{ x \in K : f_{1,n_0}(x) < c_1, \ldots, f_{k,n_0}(x) < c_k \} \subset V \subset U
   \]
   and \( V_0 \) is an \( \mathcal{A} \)-polyhedron. Thus, \( \mathcal{A} \) generates the topology of \( K \).

   The notion of closure introduced above has not the same meaning as the notion 'closure' in the topological sense, since in general it will not lead to a closed subclass of upper semi-continuous functions. It becomes then an interesting question whether there is a better definition of the closure of \( \mathcal{A} \) which yields a closed set in our sense. Nevertheless, we will see later on that the notion introduced above is sufficient for our purposes.

**Example 2.3**

1. Consider the following upper semi-continuous functions on the compactification \( K = [0, +\infty] \) of the interval \([0, +\infty)\). For an integer \( n \in \mathbb{N} \) we set
   \[
   f_n := \chi_{[1-\frac{1}{n+1}, 1]} \quad \text{and} \quad g_{n,k} := 1/k \cdot \chi_{[1-\frac{1}{n+1}, 1]} + f_n.
   \]
   The functions \( f_n \) decrease to \( f_0 := \chi_{[1, 1]} \). Now if \( \mathcal{A} \) is the set \( \{ g_{k,n} : k, n \in \mathbb{N} \} \), then \( \overline{\mathcal{A}}^\uparrow = \mathcal{A} \cup \{ f_n : n \in \mathbb{N} \} \) and \( \overline{\mathcal{A}} = \mathcal{A} \cup \{ f_0 \} \), but it is easy to see that \( f_0 \) can not be the limit of a decreasing sequence of functions from \( \mathcal{A} \).

2. One can think that after closing \( \mathcal{A} \) finitely many times we obtain a closed set. But this turns out to be wrong. Define for \( k \in \mathbb{N} \) iteratively the \( k \)-th closure \( \overline{\mathcal{A}}^\uparrow \) of \( \mathcal{A} \) by \( (\overline{\mathcal{A}}^\uparrow)^{(k-1)} \).

   Given \( k \in \mathbb{N} \) and \( n_0, \ldots, n_k \in \mathbb{N} \) consider the following upper semi-continuous function
   \[
   h_{n_0, \ldots, n_k}(x) := \sum_{j=0}^{k-1} g_{n_j, n_{j+1}}(x - j),
   \]
   where \( x \in [0, +\infty] \) and \( g_{n_j, n_{j+1}} \) are the functions from the previous example. We set \( \mathcal{A} := \{ h_{n_0, \ldots, n_k} : k \in \mathbb{N}, n_0, \ldots, n_k \in \mathbb{N} \} \). Then we conclude that \( \overline{\mathcal{A}}^\uparrow \) contains the function \( \chi_{[1, \ldots, k]} \), but not \( \chi_{[1, \ldots, k+1]} \).

3. Even if we take the union of all \( \ell \)-th closures it will not lead to a closed set. Consider now for given integers \( k \in \mathbb{N} \) and \( n_0, \ldots, n_k \in \mathbb{N} \) the functions
   \[
   G_k := \chi_{\{\infty\}} + \sum_{j=k+1}^{\infty} (1 + 1/j) \chi_{\{j\}} \quad \text{and} \quad H_{n_0, \ldots, n_k} := h_{n_0, \ldots, n_k} + G_k,
   \]
   Then for large enough \( n_0 \) we have that \( f_j(p) \leq f_{j,n_0}(p) < c_j \) for every \( j = 1, \ldots, k \).

   Then
   \[
   p \in V_0 = \{ x \in K : f_{1,n_0}(x) < c_1, \ldots, f_{k,n_0}(x) < c_k \} \subset V \subset U
   \]
   and \( V_0 \) is an \( \mathcal{A} \)-polyhedron. Thus, \( \mathcal{A} \) generates the topology of \( K \).

   The notion of closure introduced above has not the same meaning as the notion 'closure' in the topological sense, since in general it will not lead to a closed subclass of upper semi-continuous functions. It becomes then an interesting question whether there is a better definition of the closure of \( \mathcal{A} \) which yields a closed set in our sense. Nevertheless, we will see later on that the notion introduced above is sufficient for our purposes.
where \( h_{n_0, \ldots, n_k} \) are the functions from the example above. Now consider the family \( \mathcal{A} := \{ H_{n_0, \ldots, n_k} : k \in \mathbb{N}, n_0, \ldots, n_k \in \mathbb{N} \} \). Then by the same argument as before we can derive that \( \bigcup_{\ell \in \mathbb{N}} \overline{\mathcal{A}^\ell} \) contains \( \chi_{\{1, \ldots, k\}} + g_k \) for every \( k \in \mathbb{N} \), but it does not contain \( \chi_{\{1, 2, \ldots, \infty\}} \).

Anyway, the functions \( \chi_{\{1, \ldots, k\}} + g_k \) decrease to \( \chi_{\{1, 2, \ldots, \infty\}} \) as \( k \) tends to \( \infty \). Hence,

\[
\chi_{\{1, 2, \ldots, \infty\}} \in \left( \bigcup_{\ell \in \mathbb{N}} \overline{\mathcal{A}^\ell} \right),
\]

but \( \chi_{\{1, 2, \ldots, \infty\}} \notin \overline{\mathcal{A}^\ell} \) for every \( \ell \in \mathbb{N} \).

We have seen by the previous examples that each iterate closure of \( \mathcal{A} \) might lead to a larger set. Nevertheless, this additional functions will not contribute to the Shilov boundary in the following sense.

**Lemma 2.4** Let \( f \) be upper semi-continuous on \( K \) and \( (f_n)_{n \in \mathbb{N}} \) a sequence of upper semi-continuous functions decreasing to \( f \). Assume that \( f \) is bounded above by a function \( g \) which is lower semi-continuous on \( K \), i.e., \( f < g \) on \( K \). Then there is an index \( n_0 \) such that \( f_n < g \) on \( K \) for every \( n \geq n_0 \).

**Proof.** Take a point \( x \in K \). Then there is an index \( n_x \in \mathbb{N} \) such that \( f(x) \leq f_{n_x}(x) < g(x) \). Since \( f_{n_x} - g \) is upper semi-continuous on \( K \), we can find an open neighborhood \( U_x \) of \( x \) in \( K \) such that \( f_{n_x}(y) < g(y) \) for every \( y \in U_x \). By compactness of \( K \) we can cover \( K \) by finitely many open sets \( U_{x_1}, \ldots, U_{x_\ell} \) from the covering \( \{U_x\}_{x \in K} \). We set \( n_0 := \max\{n_{x_j} : j = 1, \ldots, \ell\} \). Since \( (f_n)_{n \in \mathbb{N}} \) is decreasing, we obtain that \( f_n \leq f_{n_0} < g \) on \( K \) for every \( n \geq n_0 \). \( \square \)

**Lemma 2.5** Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of upper semi-continuous functions on \( K \) decreasing to \( f \). Then \( \lim_{n \to \infty} \max_K f_n = \max_K f \).

**Proof.** The limit \( a := \lim_{n \to \infty} \max_K f_n \) exists because \( (\max_K f_n)_{n \in \mathbb{N}} \) is a decreasing sequence bounded below by \( \max_K f \). Assume that \( a > \max_K f \). By the previous Lemma [2.4] we can find a large enough integer \( n_0 \) such that \( a > f_{n_0}(y) \) for every \( y \in K \), which is a contradiction to the definition of \( a \). \( \square \)

**Corollary 2.6** The set of all \( \mathcal{A} \)-boundaries coincides with the set of all \( \overline{\mathcal{A}}^\ell \)-boundaries, i.e.,

\[
b_A = b_{\overline{\mathcal{A}}^\ell} \quad \text{and} \quad \check{S}_A = \check{S}_{\overline{\mathcal{A}}^\ell}.
\]

### 3 Minimal boundary and peak points

In this section we discuss the relation between the Shilov boundary and peak points based on the main result of Bishop in [Bis59]. As before, let \( \mathcal{A} \) always be a subfamily of upper semi-continuous functions on a compact Hausdorff space \( K \).

**Definition 3.1**

1. We denote by \( B_\mathcal{A} \) the set of all (possibly non-closed) boundaries for \( \mathcal{A} \) (recall Definition [1.4]).
(2) If there exists a subset \( m_A \) in \( B_A \) such that \( m_A \) is contained in every boundary for \( A \), then this set will be called the minimal boundary for \( A \).

(3) A point \( x \in K \) is called peak point for \( A \) if there is a function \( f \in A \) such that \( S(f) = \{x\} \). We say that \( f \) peaks at \( x \). We denote by \( P_A \) the set of all peak points for \( A \).

The sets \( m_A, P_A \) and \( \tilde{S}_A \) are possibly empty. If \( m_A \) is non-empty, it is not necessarily closed, while \( \tilde{S}_A \) is by definition always a closed subset of \( K \). The following examples show that the sets \( m_A, \tilde{S}_A \) and \( P_A \) may differ or might be empty.

**Example 3.2** (1) We enumerate the subset \( L = [0,1] \cap \mathbb{Q} \) of \( K = [0,1] \) by a sequence \((x_n)_{n \in \mathbb{N}}\). For the subclass \( A = \{\chi_{\{x_n\}} : n \in \mathbb{N}\} \) of upper semi-continuous functions on \( K \), we have that \( P_A = m_A = L \subset \tilde{S}_A = [0,1] \).

(2) There exists a separating Banach algebra of continuous functions on a compact set with no minimal boundary.

There exists a Banach algebra of continuous functions on a compact set such that the minimal boundary is an open non-empty set. For both examples we refer to [Bis59].

(3) By Example [3.3] (1) we can see that there is a subclass \( A \) of \( \mathcal{USC}(K) \) such that \( \tilde{S}_A, P_A \) and \( m_A \) are all empty.

**Remark 3.3** Given \( f \in \mathcal{USC}(K) \) define \( A' := A \cup \{f\} \). The existence of \( m_A \) does not imply the existence of \( m_A \) in general. To see this consider \( A = \{\chi_{\{0\}}\} \) and \( f = \chi_{[-1,1]} \) on the interval \([-1,1]\). Even though \( m_A = \{0\}, m_A' \) does not exist. On the other hand it is easy to verify that, if we choose \( f \) which peaks at some point \( x \in K \), then \( m_A' = m_A \cup \{x\} \).

We give some properties and relations between the above defined sets.

**Proposition 3.4**

(1) The set \( P_A \) lies in every \( A \)-boundary \( S \) from \( B_A \). If \( P_A \) is itself an \( A \)-boundary, then it is the minimal boundary \( m_A \).

(2) The inclusions \( P_A \subset m_A \subset \tilde{S}_A \) hold whenever \( m_A \) exists.

(3) \( \tilde{S}_A = \overline{m_A} \) if \( m_A \) exists.

(4) Let \( A_1 \subset A_2 \subset \mathcal{USC}(K) \). Then we have the following inclusions,

\[
B_{A_2} \subset B_{A_1} \quad \text{and} \quad P_{A_1} \subset P_{A_2}.
\]

If \( m_{A_1} \) and \( m_{A_2} \) exist, then \( m_{A_1} \subset m_{A_2} \).

(5) Let \( A = \bigcup_{j \in J} A_j \), where \( A_j \) are subsets of \( \mathcal{USC}(K) \). Then \( P_A = \bigcup_{j \in J} P_{A_j} \). If \( m_{A_j} \) exists for every \( j \in J \), then \( m_A \) exists and \( m_A = \bigcup_{j \in J} m_{A_j} \).

**Proof.**

(1) Let \( x \in P_A \) and \( f \in A \) such that \( f \) peaks at \( x \). Given an \( A \)-boundary \( S \), it is clear that \( S \cap S(f) = \{x\} \). In particular, the point \( x \) lies in \( S \). This yields the inclusion \( P_A \subset S \).

Now if \( P_A \) lies in \( B_A \), then by the previous discussion and by the definition of the minimal boundary for \( A \), we have that \( P_A = m_A \).

(2) Since \( m_A \in B_A \) and by the previous property (1), we obtain that \( P_A \subset m_A \). Since \( m_A \) is the smallest \( A \)-boundary, \( m_A \subset S \) for every \( S \in b_A \). Hence, \( m_A \subset \tilde{S}_A \).
(3) Since $\tilde{S}_A$ is closed and $m_A \subset \tilde{S}_A$ by the previous point (2), $\overline{m}_A$ is a subset of $\tilde{S}_A$. On the other hand, $m_A$ is contained in $B_A$, and therefore $\overline{m}_A$ is a closed $A$-boundary. By definition this means that $\tilde{S}_A \subset \overline{m}_A$. Hence, $\overline{m}_A = \tilde{S}_A$.

(4) This inclusions follow immediately from the definitions of the corresponding sets.

(5) The identity $P_A = \bigcup_{j \in J} P_{A_j}$ is obvious. We show that $m := \bigcup_{j \in J} m_{A_j}$ is a minimal $A$-boundary. Pick an arbitrary function $f \in A$. Then $f \in A_j$ for some index $j \in J$. By assumption $m_{A_j}$ is a minimal boundary for $A_j$. Thus, we obtain that $\emptyset \neq S(f) \cap m_{A_j} \subset S(f) \cap m$.

But this implies that $m \in B_A$. Now let $S$ be an arbitrary $A$-boundary. By point (4) we have that $S \in B_A \subset B_{A_j}$ for every $j \in J$. Then $m_{A_j} \subset S$ for all $j \in J$ and, thus, $m \subset S$. This shows the minimality of $m$, so $m_A = m$. □

In what follows, we present some Bishop-type theorems for subclasses of upper semi-continuous functions on a metrizable compact space $K$.

**Definition 3.5**

(1) A topological space $K$ is metrizable if it has a metric which induces the given topology. In this case its topology admits a countable base.

(2) A subset $B$ of $C(K)$ is separating or separating points of $K$ if for every $x, y \in K$ there exists a function $f \in B$ such that $f(x) \neq f(y)$.

(3) Given a subclass $A$ of $USC(K)$, it is strictly separating or strictly separating points of $K$ if for every $x, y \in K$ there exist functions $f_1, f_2 \in A$ such that $f_1(x) > f_1(y)$ and $f_2(x) < f_2(y)$.

We recall Bishop’s theorem. Further generalizations can be found in [Sic62], [Dal71] and [Hon88].

**Definition 3.6** For $B$ being a subset of $C(K)$ we use the same simplification of notations as in Remark 1.7 above. Namely, we write $B_B$, $m_B$ and $P_B$ instead of $B_{\log|B|}$, $m_{\log|B|}$ and $P_{\log|B|}$, respectively.

**Theorem (Bishop, [Bis59])** Let $K$ be a compact metrizable Hausdorff space and $B$ a separating uniform subalgebra of $C(K)$. Then the minimal boundary of $B$ exists and is exactly the set of all peak points for $B$.

**Corollary 3.7** Suppose $B$ is a union of separating uniform subalgebras $(B_j)_{j \in J}$ of $C(K)$, where $K$ is a metrizable compact Hausdorff space. Then $m_B$ exists and

$$m_B = P_B \quad \text{and} \quad \tilde{S}_B = \overline{m}_B.$$

**Proof.** By Bishop’s theorem $m_{B_j}$ exists and coincides with $P_{A_j}$ for every $j \in J$. By Proposition 3.4 (5), we obtain that $m_B$ is the minimal boundary for $B$ and

$$P_B = \bigcup_{j \in J} P_{B_j} = \bigcup_{j \in J} m_{B_j} = m_B.$$

The identity $\tilde{S}_B = \overline{P}_B$ follows now from Proposition 3.4 (3). □
For closed cones of upper semi-continuous functions, i.e., for cones \( \mathcal{A} \) having the property \( \mathcal{A}^k = \mathcal{A} \), we are able to obtain another Bishop type theorem. The proof is nearly the same as in Theorem 1 in [Bis59]. A similar result for subfamilies of non-negative continuous functions was already obtained by Siciak in [Sic02] (see Theorem 3).

**Theorem 3.8** Let \( K \) be a metrizable compact Hausdorff space. Let \( \mathcal{A} \) be a closed cone in \( \text{USC}(K) \) containing real constants and strictly separating points of \( K \). Then \( m_\mathcal{A} \) exists and coincides with the set of all peak points for \( \mathcal{A} \); i.e.,

\[
m_\mathcal{A} = P_\mathcal{A} \quad \text{and} \quad \hat{S}_\mathcal{A} = \overline{P}_\mathcal{A}.
\]

**Proof.** In view of Proposition 3.4 (1) and (2), we only need to show that \( P_\mathcal{A} \) is a non-empty \( \mathcal{A} \)-boundary or, equivalently, \( P_\mathcal{A} \cap S(f) \neq \emptyset \) for every \( f \in \mathcal{A} \).

Fix a function \( f \in \mathcal{A} \). Denote by \( \Gamma \) the set of all peak sets for \( \mathcal{A} \); i.e.,

\[
\Gamma := \{ \gamma \subset K : \exists f_\gamma \in \mathcal{A} \text{ such that } S(f_\gamma) = \gamma \}.
\]

Let \( \mathcal{S} \) be the set of subsets \( \tilde{\Gamma} \) of \( \Gamma \) which contain \( S(f) \) and which have the finite intersection property (fip), i.e., for every finite family \( \{ \tilde{\gamma}_i \}_{i \in I} \) of elements in \( \tilde{\Gamma} \) its intersection \( \bigcap_{i \in I} \tilde{\gamma}_i \) is non-empty. Let \( (\tilde{\Gamma}_j)_{j \in J} \) be a totally ordered set in \( \mathcal{S} \). We infer that \( U := \bigcup_{j \in J} \tilde{\Gamma}_j \) is an upper bound for elements in \( (\tilde{\Gamma}_j)_{j \in J} \) and that it is contained in \( \mathcal{S} \). Indeed, it is obvious that \( U \) bounds all elements of \( (\tilde{\Gamma}_j)_{j \in J} \) and that \( S(f) \) lies in \( U \). Let \( \{ \gamma_i \}_{i \in I} \) be a finite family in \( U \). Since \( (\tilde{\Gamma}_j)_{j \in J} \) is totally ordered, there is an index \( j_0 \in J \) such that \( \gamma_i \in \tilde{\Gamma}_{j_0} \) for every \( i \in I \). But \( \tilde{\Gamma}_{j_0} \) has the (fip). Therefore, \( \bigcap_{i \in I} \tilde{\gamma}_i \) is non-empty. This implies that \( U \) has the (fip) and, thus, it is contained in \( \mathcal{S} \). By Zorn’s Lemma \( \mathcal{S} \) has a maximal element \( \Gamma_0 \). It contains \( S(f) \), has the (fip) and no larger subset of \( \Gamma \) has the (fip).

Since all the sets in \( \Gamma_0 \) are closed (see Proposition 3.4 (2)), the set \( K \) is compact and \( \Gamma_0 \) has the (fip), it follows that the set \( D := \bigcap_{\gamma \in \Gamma_0} \gamma \) is closed and non-empty. The set \( K \setminus D \) is covered by open sets \( K \setminus \gamma \), where \( \gamma \in \Gamma_0 \). Since \( K \) and, therefore, also \( K \setminus D \) are metrizable, we can choose a countable sequence \( (\gamma_n)_{n \in \mathbb{N}} \) of sets \( \gamma_n \in \Gamma_0 \) such that \( D = \bigcap_{n \in \mathbb{N}} \gamma_n \).

Let \( x_0 \) be a point in \( D \). Then the functions \( f_n := f_\gamma - f_\gamma(x_0) \) fulfill \( f_n \in \mathcal{A} \), \( S(f_n) = S(f_\gamma) = \gamma_n \), \( f_n \leq 0 \) on \( K \) and \( f_n(x_0) = 0 \).

Define as the function \( g := \sum_{n=1}^{\infty} f_n \). Since \( g \) is a limit of a decreasing sequence of functions \( \sum_{n=1}^{k} f_n \) in \( \mathcal{A} \), and since \( \mathcal{A} \) is closed, we deduce that \( g(x_0) = 0 \), \( g \leq 0 \) on \( K \) and that \( g \) is contained in \( \mathcal{A} \). In addition, if \( x \in K \setminus \gamma_n \), then \( f_n(x) < 0 \). This implies that \( g \) can not attain a maximum on \( K \) in \( x \) because \( g(x) < 0 \) and \( g(x_0) = 0 \). Therefore, \( g \) can only attain maximal values on \( K \) in \( \gamma_n \), i.e., \( S(g) \subset \gamma_n \). Since this inclusion holds for all \( n \in \mathbb{N} \), we have that \( S(g) \) is a subset of \( D \).

We claim that \( S(g) \) contains only a single point, namely \( x_0 \). If this is true, \( x_0 \) is a peak point for \( \mathcal{A} \). Then by the construction of \( g \) and the definition of \( D \) we have that \( x_0 \in S(g) \subset D \subset S(f) \). But this means that \( x_0 \in P_\mathcal{A} \cap S(f) \). In particular, the intersection \( P_\mathcal{A} \cap S(f) \) is non-empty. Since \( f \) is an arbitrary function in \( \mathcal{A} \), it follows from the definition that \( P_\mathcal{A} \) is an \( \mathcal{A} \)-boundary which implies that \( m_\mathcal{A} \subset P_\mathcal{A} \).

It remains to prove the claim above that \( S(g) \) consists only of a single point. Assume that this is false so that \( S(g) \) contains more than a single point. Since \( \mathcal{A} \) separates the points of \( K \), there is a function \( h \) in \( \mathcal{A} \) and a point \( x_1 \in S(g) \) such that \( h(x_1) = \max_{S(g)} h = 0 \) and \( h \) is not constant on \( S(g) \). Then the set \( E := \{ x \in S(g) : h(x) = 0 \} \) is a proper closed subset of \( S(g) \) containing \( x_1 \).

Consider the sets \( T_n := \{ x \in \mathcal{A} : h(x) \geq 1/n \} \), where \( n \in \mathbb{N} \). These sets are closed and disjoint from \( S(g) \). This means that \( g < 0 \) on \( T_n \) for every \( n \in \mathbb{N} \). Hence, for each \( n \in \mathbb{N} \) we can choose a large enough constant \( c_n \in \mathbb{N} \) such that \( \max_{T_n} \{ h + c_n g \} < 0 \). Recall that \( g \) is non-negative on \( K \).
Define the function \( \varphi \) by \( \varphi := h + \sum_{n=1}^{\infty} c_n g \). Since \( \varphi \) is the limit of a decreasing sequence of functions \( h + \sum_{n=1}^{k} c_n g \in \mathcal{A} \), it also lies in \( \mathcal{A} \). For a point \( x \in K \setminus \bigcup_{n} T_n \) we have that \( h(x) \leq 0 \) and \( g(x) \leq 0 \) and therefore \( \varphi(x) \leq 0 \). In addition, on \( T_j \) it holds that

\[
\varphi = h + \sum_{n=1}^{\infty} c_n g \leq \max_{T_j} \{ h + cj g \} + \sum_{n=1, n \neq j}^{\infty} c_j g < 0.
\]

This implies that \( x_1 \in S(\varphi) \) because \( \varphi(x_1) = 0 \) and \( \varphi \leq 0 \) on \( K \). Moreover, we have that \( S(\varphi) \in \Gamma \) and \( x_1 \in S(g) \cap S(\varphi) \). We assert that \( S(\varphi) \in \Gamma_0 \). To show this, we set \( \Gamma_1 := \Gamma_0 \cup \{ S(\varphi) \} \). Since \( S(g) \subset D \), we have that

\[
\emptyset \neq S(g) \cap S(\varphi) \subset S(\varphi) \cap D \subset S(\varphi) \cap \gamma
\]

for every \( \gamma \in \Gamma_0 \). Together with the (fip) of \( \Gamma_0 \) it follows that \( \Gamma_1 \in \mathfrak{S} \). Since \( \Gamma_0 \subset \Gamma_1 \) and by the maximality of \( \Gamma_0 \), we conclude that \( \Gamma_0 = \Gamma_1 \) and, thus, \( S(\varphi) \in \Gamma_0 \).

Therefore, \( S(g) \subset D \subset S(\varphi) \), since \( D = \bigcap_{\gamma \in \Gamma_0} \gamma \). Recall that \( S(g) \setminus E \neq \emptyset \). Now if \( x \in S(g) \setminus E \), we have that \( g(x) = 0 \) but \( h(x) < 0 \) and thus \( \varphi(x) < 0 \). Hence, \( S(g) \setminus E \) and \( S(\varphi) \) are disjoint, which is a contradiction to \( S(g) \subset S(\varphi) \). Finally, we have shown that \( m_A \) is contained in \( P_A \).

Since the families of functions we will define later do not form cones, we need another peak point theorem.

**Definition 3.9** Let \( \mathcal{A} \) be a subclass of upper semi-continuous functions on \( K \) and let \( \Theta \) be a subset of non-negative continuous functions on \( K \) with the following property: for each \( x \in K \) and each closed subset \( S \) of \( K \) with \( x \not\in S \) there exists a function \( \vartheta \in \Theta \) such that \( S(\vartheta) = \{ x \} \) and \( \vartheta \) vanishes on \( S \). We say that a function \( f \in \mathcal{A} \) is a strictly-\( \mathcal{A} \)-function with respect to \( \Theta \) if for every \( \vartheta \in \Theta \) there is a number \( \varepsilon_0 > 0 \) such that \( f + \varepsilon \vartheta \in \mathcal{A} \) for every \( \varepsilon \in (\varepsilon_0, \varepsilon_0) \). The subfamily of \( \mathcal{A} \) consisting of all strictly-\( \mathcal{A} \)-functions with respect to \( \Theta \) is denoted by \( \mathcal{A}[\Theta] \).

**Theorem 3.10** Let \( \mathcal{A} \) be a subclass of upper semi-continuous functions on \( K \). Suppose that there exist a subclass \( \Theta \) as in the previous definition and a positive function \( \omega \in \mathcal{A}[\Theta] \) such that \( \mathcal{A} + \{ \varepsilon \omega \} \in \mathcal{A}[\Theta] \) for every positive number \( \varepsilon > 0 \). Then \( \hat{S}_A = \overline{P_A} \in b_A \).

**Proof.** First, observe that \( P_{\mathcal{A}[\Theta]} \) is non-empty. Indeed, the function \( \omega \) attains its maximum on \( K \), say at a point \( x_0 \in K \). Pick a function \( \vartheta \in \Theta \) with \( S(\vartheta) = \{ x_0 \} \). Then there is a positive number \( \delta > 0 \) such that \( \omega + \delta \vartheta \) lies in \( \mathcal{A} \). But then \( 2\omega + \delta \vartheta \) is in \( \mathcal{A}[\Theta] \) by the assumption made on \( \omega \). Moreover, \( S(2\omega + \delta \vartheta) = \{ x_0 \} \) and, thus, \( x_0 \in P_{\mathcal{A}[\Theta]} \).

The set \( P_{\mathcal{A}[\Theta]} \) is a subset of \( \hat{S}_{\mathcal{A}[\Theta]} \). We show that \( S := \overline{P_{\mathcal{A}[\Theta]}} \) is a boundary for the class \( \mathcal{A}[\Theta] \). If not, there exists a function \( f \in \mathcal{A}[\Theta] \) such that \( \max_K f > \max_S f \). For a small enough number \( \varepsilon_0 > 0 \) we have that \( \max_K g > \max_S g \), where \( g := f + \varepsilon_0 \omega \). Then there exists a point \( x_1 \in K \setminus S \) such that \( g(x_1) = \max_K g \). Let \( \theta \) be a function from \( \Theta \) such that \( S(\theta) = \{ x_1 \} \) and \( \theta \) vanishes on \( S \). In particular, \( \theta(x_1) > 0 \). Then for a small enough number \( \varepsilon_1 > 0 \) the function \( f + \varepsilon_1 \theta \) is in \( \mathcal{A} \). Hence, the function \( h := g + \varepsilon_1 \theta = f + \varepsilon_1 \theta + \varepsilon_0 \omega \) lies in \( \mathcal{A}[\Theta] \) and fulfills \( S(h) = \{ x_1 \} \). Thus, \( x_1 \in P_{\mathcal{A}[\Theta]} \subset S \). But this contradicts to

\[
\max_S h = \max_S g < \max_K g = g(x_1) < h(x_1) \leq \max_S h.
\]

Therefore, \( \hat{S}_{\mathcal{A}[\Theta]} \) is contained in \( S \). Altogether, we have that \( \overline{P_{\mathcal{A}[\Theta]}} = \hat{S}_{\mathcal{A}[\Theta]} \).

Let \( f \) be an arbitrary function from \( \mathcal{A} \). Then the sequence \( (f_n)_{n \in \mathbb{N}} \) of functions \( f_n := f + (1/n) \omega \) in \( \mathcal{A}[\Theta] \) decreases to \( f \). This implies that \( \mathcal{A} \) lies in the closure of \( \mathcal{A}[\Theta] \). Since
$\mathcal{A}[\Theta]$ lies in $\mathcal{A}$ and in view of Corollary 2.6 we have that $b_\mathcal{A} = b_{\mathcal{A}[\Theta]}$ and $\tilde{\mathcal{S}}_{\mathcal{A}[\Theta]} = \tilde{\mathcal{S}}_\mathcal{A}$. Finally, the proof is done due to the following inclusions,

$$\tilde{\mathcal{S}}_\mathcal{A} = \tilde{\mathcal{S}}_{\mathcal{A}[\Theta]} = \overline{\mathcal{P}_{\mathcal{A}[\Theta]}} \subset \overline{\mathcal{P}_\mathcal{A}} \subset \tilde{\mathcal{S}}_\mathcal{A}.$$

\[\square\]

4 \hspace{1em} \textit{q-plurisubharmonic and q-holomorphic functions}

We give a short overview of what is known about $q$-plurisubharmonic and $q$-holomorphic functions.

\textbf{Definition 4.1} Let $U$ be an open set in $\mathbb{C}^N$ and $u : U \to [-\infty, \infty)$ be an upper semi-continuous function on $U$, i.e., $\{z \in U : u(z) < c\}$ is open for every $c \in \mathbb{R}$. Let $q \in \{0, 1, \ldots, N-1\}$.

1. The function $u$ is called \textit{subplurisubharmonic} on $U$ if for every ball $B \Subset U$ and every function $h$ which is pluriharmonic in a neighborhood of the closure of $B$ and fulfills $u \leq h$ on $\partial B$ one has that $u \leq h$ on $B$.

2. The function $u$ is called \textit{$q$-plurisubharmonic} in $U$ if it is subplurisubharmonic in $U \cap \pi$ for every $(q+1)$-dimensional complex affine plane $\pi \subset \mathbb{C}^N$.

3. By $\mathcal{P}_{\mathcal{SH}}(U)$ we denote the set of all $q$-plurisubharmonic functions on $U$. If $q$ is an integer with $q \geq N$, we simply define $\mathcal{P}_{\mathcal{SH}}(U) := \mathcal{USC}(U)$.

4. Given a compact set $K$ in $\mathbb{C}^N$ the set $\mathcal{P}_{\mathcal{SH}}(K)$ denotes the set of all functions $v \in \mathcal{USC}(K)$ which have an $q$-plurisubharmonic extension into an open neighborhood of $K$, i.e., there exists an open neighborhood $U$ of $K$ and a function $v \in \mathcal{P}_{\mathcal{SH}}(U)$ such that $u|K = v$.

5. We set $\mathcal{AP}_{\mathcal{SH}}(K) := \mathcal{P}_{\mathcal{SH}}(\text{int}K) \cap \mathcal{C}(K)$.

We give an overview of the basic properties of $q$-plurisubharmonic functions although we might not use them explicitly in the following sections. We refer to [Dic00], [HM78], [Slo86], [Slo84] and [PZ13] for details and further properties.

\textbf{Proposition 4.2} Let $U$ be an open set in $\mathbb{C}^N$.

1. The 0-plurisubharmonic functions are the classical plurisubharmonic functions.

2. $\mathcal{P}_{\mathcal{SH}}(U) \subset \mathcal{P}_{\mathcal{SH}}(U) \subset \ldots \subset \mathcal{P}_{\mathcal{SH}}(U) \subset \mathcal{USC}(U)$

3. Given $c \geq 0$ and two functions $u \in \mathcal{P}_{\mathcal{SH}}(U)$ and $v \in \mathcal{P}_{\mathcal{SH}}(U)$,

$$cu \in \mathcal{P}_{\mathcal{SH}}(U), \hspace{1em} \max\{u, v\} \in \mathcal{P}_{\mathcal{SH}}(U), \hspace{1em} u + v \in \mathcal{P}_{\mathcal{SH}}(U), \hspace{1em} \min\{u, v\} \in \mathcal{P}_{\mathcal{SH}}(U).$$

4. The $C^2$-smooth function $u$ lies in $\mathcal{P}_{\mathcal{SH}}(U)$ if and only if its complex Hessian $(u_{x_k \bar{z}_\ell})_{k, \ell=1, \ldots, N}$ has at least $N-q$ non-negative eigenvalues at each point in $U$.

5. Let $u_j$ be $q_j$-plurisubharmonic functions, $j = 1, \ldots, k$. If $\chi : \mathbb{R}^k \to \mathbb{R}$ is a $C^2$-smooth convex function which is non-decreasing in each variable, then the composition $\chi(u_1, \ldots, u_k)$ is a $\hat{q}$-plurisubharmonic function with $\hat{q} = q_1 + \ldots + q_k$. 

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(6) If $\psi \in \mathcal{PSH}_q(U)$, then $\psi \circ h \in \mathcal{PSH}_q(W)$ for every holomorphic mapping $h : W \to U$, where $W$ is an open set in $\mathbb{C}^n$.

(7) If $(u_n)_{n \in \mathbb{N}}$ is a decreasing sequence of functions in $\mathcal{PSH}_q(U)$, then the limit $\lim_{n \to \infty} u_n$ lies in $\mathcal{PSH}_q(U)$.

(8) Let $\{u_j\}_{j \in J}$ be a locally bounded family of functions in $\mathcal{PSH}_q(U)$. Then $u^* = \left(\sup_{j} u_j\right)^*$ lies in $\mathcal{PSH}_r(U)$. Here, $u^*$ means the upper semi-continuous regularization of $u$, i.e., $u^*(z) := \limsup_{\zeta \to z, \zeta \in U} u(\zeta)$ for $z \in U$.

(9) Let $V$ be an open subset of $U$. Let $v \in \mathcal{PSH}_q(V)$ and $u \in \mathcal{PSH}_q(U)$ such that $\limsup_{\zeta \to z, \zeta \in V} v(\zeta) \leq u(z)$ for every $z \in U \cap bV$. Then we have that
$$\varphi = \begin{cases} u & \text{on } U \setminus V \\ \max\{u, v\} & \text{on } V \end{cases} \in \mathcal{PSH}_q(U).$$

(10) Let $q < N$ and $U \subseteq \mathbb{C}^N$. Then every function $u \in \mathcal{PSH}_q(U) \cap \mathcal{USC}(\overline{U})$ satisfies the maximum principle, i.e.,
$$\max_M u = \max_{bU} u.$$

Proof. The properties (1) and (2) follow directly from the definition. Regarding (3), it is not hard to verify that $cu \in \mathcal{PSH}_q(U)$ and $\max\{u, v\} \in \mathcal{PSH}_{\max\{q, r\}}(U)$. The proofs of the properties $u + v \in \mathcal{PSH}_{q+r}(U)$ and $\min\{u, v\} \in \mathcal{PSH}_{q+r+1}(U)$ can be found in [Slo84]. For the proofs of (4), (9) and (10) we refer to [HM78]. The proofs of (5) and (6) can be found in, e.g., [PZ13]. The properties (7) and (8) are easy to verify. □

A generalization of holomorphic functions is given by the so-called $q$-holomorphic functions which were already studied by, e.g., Basener in [Bas76] and [Bas78] and Hunt and Murray in [HM78].

**Definition 4.3** Let $U$ be an open set in $\mathbb{C}^N$.

(1) Given an integer $q \geq 0$, the set of $q$-holomorphic functions on $U$ is defined by
$$\mathcal{O}_q(U) := \{ f \in \mathcal{C}^2(U) : \bar{\partial} f \wedge (\bar{\partial} \partial f)^q = 0 \}.$$

(2) Let $K$ be a compact set in $\mathbb{C}^N$. The set $\mathcal{O}_q(K)$ denotes the set of all continuous functions $f$ on $K$ which have a $q$-holomorphic extension into an open neighborhood of $K$, i.e., there exist an open neighborhood $U$ of $K$ and a function $F \in \mathcal{O}_q(U)$ such that $F|K = f$.

(3) We define $A_q(K) := \mathcal{O}_q(\text{int}K) \cap \mathcal{C}(K)$.

The next proposition is a collection of properties of $q$-holomorphic functions.

**Proposition 4.4** Let again $U$ be an open set in $\mathbb{C}^N$.

(1) The $0$-holomorphic functions are the usual holomorphic functions.

(2) If $q \geq N$, then $\mathcal{O}_q(U) = \mathcal{C}^2(U)$.

(3) $\mathcal{O}(U) \subset \mathcal{O}_1(U) \subset \ldots \subset \mathcal{O}_{N-1}(U) \subset \mathcal{O}_N(U) = \mathcal{C}^2(U)$.
(4) A function $f \in \mathcal{C}^2(U)$ lies in $\mathcal{O}_q(U)$ if and only if
\[
\begin{pmatrix}
  f_{\bar{z}_1} & \cdots & f_{\bar{z}_N} \\
  f_{z_1 \bar{z}_1} & \cdots & f_{z_1 \bar{z}_N} \\
  \vdots & \ddots & \vdots \\
  f_{z_N \bar{z}_1} & \cdots & f_{z_N \bar{z}_N}
\end{pmatrix}
\]

\[
\mathrm{rank} \leq q \quad \text{on } U.
\]

(5) If $f \in \mathcal{O}_q(U)$, $g \in \mathcal{O}_r(U)$, $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, then
\[
f^m, \lambda f \in \mathcal{O}_q(U) \quad \text{and} \quad fg, f + g \in \mathcal{O}_{q+r}(U).
\]

(6) If $W$ is an open set in $\mathbb{C}^n$, $f \in \mathcal{O}_q(W)$ and $h : U \to W$ a holomorphic mapping, then $f \circ h \in \mathcal{O}_q(U)$.

(7) If $f \in \mathcal{O}_q(U)$ and $h$ is a complex valued holomorphic function defined in the neighborhood of the image of $f$, then $h \circ f \in \mathcal{O}_q(U)$.

(8) If $q < N$, then every function $f \in \mathcal{O}_q(U)$ admits the local maximum modulus principle, i.e., for every relatively compact set $D \Subset U$ we have that
\[
\max_{\overline{D}} |f| = \max_{\partial D} |f|.
\]

(9) If $f \in \mathcal{O}_q(U)$, then $\text{Ref}$ and $\log |f|$ lie in $\mathcal{PSH}_q(U)$.

Proof. The statements (1), (2) and (3) follow from definition. The proofs of (4), (5), (6), (7) and (8) can be found in [Bas76]. (9) has been proven in [HM78].

We give some examples of $q$-holomorphic functions, which can be also found in [Bas76].

**Example 4.5** (1) Every pluriharmonic or anti-holomorphic function is 1-holomorphic.

(2) If $V$ is a complex submanifold of $U$, then every restriction of a function $f \in \mathcal{O}_q(U)$ to $V$ lies in $\mathcal{O}_q(V)$ because the inclusion mapping $i : V \hookrightarrow U$ is holomorphic.

(3) If there are local coordinates $z_1, \ldots, z_N$ such that a given $C^2$-smooth function depends holomorphically in $N-q$ variables $z_1, \ldots, z_{N-q}$, then $g$ is $q$-holomorphic.

(4) Let $h = (h_1, \ldots, h_q)$ be a holomorphic mapping from $U$ into $\mathbb{C}^q$ and $V := \{z \in U : h(z) = 0\}$. Then
\[
\chi_{m,V} = \frac{1}{1 + m(h_1^2 + \ldots + h_q^2)}
\]
lies in $\mathcal{O}_q(U)$ for every $m \in \mathbb{N}$ due to the previous property (3). The sequence $(\chi_{m,V})_{m \in \mathbb{N}}$ decreases to the characteristic function $\chi_V$ of $V$ in $U$.

(5) Let $L$ be an affine complex plane in $\mathbb{C}^N$ of codimension $q \in \{0, \ldots, N-1\}$ and let $U$ be an open set in $\mathbb{C}^N$. Consider a function $h \in \mathcal{C}^2(U)$ which is holomorphic on $U \cap L'$ for every parallel copy $L'$ of the plane $L$. Then by property (3) above the function $h$ lies in $\mathcal{O}_q(U)$.

The last example leads to another subfamily of $q$-holomorphic functions which will serve later for the characterization of the Shilov boundary of bounded convex domains.

**Definition 4.6** Let $U$ be an open set and $K$ be a compact set in $\mathbb{C}^N$. Let $L$ be an affine complex plane of codimension $q \in \{0, \ldots, N-1\}$. 
(1) We denote by $\mathcal{O}(L, U)$ the set of functions described in the last example, part (5).

(2) The class $\mathcal{O}_q^*(U)$ is the union of all sets $\mathcal{O}(L, U)$, where $L$ varies among all affine complex planes in $\mathbb{C}^N$ of codimension $q$.

(3) The set $\mathcal{O}(L, K)$ will mean the set of all continuous functions $f$ on $K$ such that there exist a neighborhood $U$ of $K$ and a function $F \in \mathcal{O}(L, U)$ with $F|K = f$. The class $\mathcal{O}_q^*(K)$ is then the union of all sets $\mathcal{O}(L, K)$, where $L$ again varies among all affine complex planes of codimension $q$.

(4) We set $A(L, K) := \mathcal{O}(L, \text{int}K) \cap \mathcal{C}(K)$ and $A_q^*(K) := \mathcal{O}_q^*(\text{int}K) \cap \mathcal{C}(K)$.

We have the following properties for this new class of functions.

**Proposition 4.7** Let $U, K, L$ and $q$ be as in the previous Definition 4.6. Then we have the following properties.

1. $\mathcal{O}_q^*(U) \subset \mathcal{O}_q(U)$
2. $\mathcal{O}_q^{N}(U) = \mathcal{O}_q(U) \subset \mathcal{O}_q^* \subset \mathcal{O}_q(U) \subset \mathcal{C}^2(U)$
3. The family $A(L, K)$ and the uniform closure of $\mathcal{O}(L, K)$ are uniform subalgebras of $\mathcal{C}(K)$.

**Proof.** Property (1) follows from Example 4.6 (5). The inclusions in point (2) follow directly from the definition. The statement in point (3) is easy to verify, since the uniform limit of a sequence of holomorphic functions remains holomorphic. \(\square\)

5 Shilov boundary for $q$-plurisubharmonic functions

We first prove the existence of the Shilov boundary for the subclasses of $q$-plurisubharmonic functions defined in the previous section.

**Proposition 5.1** Let $U \subset \mathbb{C}^N$ be open, $K \subset \mathbb{C}^N$ be compact and $L$ be a complex plane in $\mathbb{C}^N$ of codimension $q \in \{0, \ldots, N - 1\}$. Then we have the following properties.

1. Recall that $\log |B| = \{\log |f| : f \in B\}$ for a subfamily $B$ of complex-valued continuous functions. Then

   \[
   \log |\mathcal{O}_0(U)| \subset \log |\mathcal{O}(L, U)| \subset \log |\mathcal{O}_q^*(U)| \subset \log |\mathcal{O}_q(U)| \subset \mathcal{P}\mathcal{S}\mathcal{H}_q(U).
   \]

2. For the respecting Shilov boundaries we have that

   \[
   \hat{\mathcal{S}}_{\mathcal{O}_0(K)} \subset \hat{\mathcal{S}}_{\mathcal{O}_q(L, K)} \subset \hat{\mathcal{S}}_{\mathcal{O}_q^*(K)} \subset \hat{\mathcal{S}}_{\mathcal{O}_q(K)} \subset \hat{\mathcal{S}}_{\mathcal{P}\mathcal{S}\mathcal{H}_q(K)} \subset bK
   \]

   and

   \[
   \hat{\mathcal{S}}_{\mathcal{A}_0(K)} \subset \hat{\mathcal{S}}_{\mathcal{A}_q(L, K)} \subset \hat{\mathcal{S}}_{\mathcal{A}_q^*(K)} \subset \hat{\mathcal{S}}_{\mathcal{A}_q(K)} \subset \hat{\mathcal{S}}_{\mathcal{A}\mathcal{P}\mathcal{S}\mathcal{H}_q^0(K)} \subset bK.
   \]

3. Let $B \in \{\mathcal{O}_0(K), \mathcal{O}(L, K), \mathcal{O}_q^*(K), \mathcal{O}_q(K), \mathcal{P}\mathcal{S}\mathcal{H}_q(K)\}$. Then

   \[
   \overline{\mathcal{T}}_B = \hat{\mathcal{S}}_{bB} \subset bB \quad \text{and} \quad \overline{\mathcal{T}}_{\mathcal{P}\mathcal{S}\mathcal{H}_q(K)} = \hat{\mathcal{S}}_{b\mathcal{P}\mathcal{S}\mathcal{H}_q(K)} \subset b\mathcal{P}\mathcal{S}\mathcal{H}_q(K)
   \]

4. Let $B \in \{\mathcal{A}_0(K), \mathcal{A}(L, K), \mathcal{A}_q^*(K), \mathcal{A}_q(K), \mathcal{A}\mathcal{P}\mathcal{S}\mathcal{H}_q^0(K)\}$. Then

   \[
   \overline{\mathcal{T}}_B = \hat{\mathcal{S}}_b \subset bB.
   \]
Proof. (1) This follows from Example 4.5 (5) and Proposition 4.4 (9).

(2) This is a consequence of part (1), Proposition 4.6 (6), Remark 4.7, Proposition 4.2 (10) and Corollary 4.3 together with the following fact: If \( f \) is a function from \( A_q(K) \), then the functions \( \psi_n := \max\{|f|, -n\}, \ n \in \mathbb{N} \), define a sequence of functions from \( \mathcal{APSH}_q(K) \) decreasing to \( |f| \). Thus, \( \log |A_q(K)| \) lies in \( \mathcal{APSH}_q(K) \). Thus, \( \hat{\mathcal{S}}_{A_q(K)} \) is contained in \( \hat{\mathcal{S}}_{\mathcal{APSH}_q(K)} \).

(3) Let \( A \in \{ \log |B|, \mathcal{PSH}_q(K) \} \). It is obvious that \( A_0 := \log |O_0(K)| \) generates the topology of \( K \). By Proposition 4.3 (5), (9) and Proposition 4.2 (3) we deduce that the set \( A \) is a scalar cone such that \( \log |O_0(K)| + A \subset A \). Then it follows from Theorem 1.10 that \( \hat{S_A} \) is an \( \mathcal{A} \)-boundary, so \( \hat{S_A} \in b_A \).

If \( B = O_0(K) \) or \( B = O(L, K) \), we can directly apply Bishop’s Theorem to the Banach subalgebras \( \mathcal{B} \) in order to obtain \( \mathcal{P}_{\mathcal{B}} = \hat{\mathcal{S}}_{\mathcal{B}} \).

Let \( \mathbb{L}_q \) be the set of all complex planes of codimension \( q \) in \( \mathbb{C}^N \). By Proposition 4.6 (6), Bishop’s Theorem and Proposition 3.4 (4) we conclude that

\[
\hat{S}_{O_q(K)} = \bigcup_{L' \in \mathbb{L}_q} \hat{S}_{O_q(L', K)} = \bigcup_{L' \in \mathbb{L}_q} \mathcal{P}_{O_q(L', K)} \subset \mathcal{P}_{\bigcup_{L' \in \mathbb{L}_q} O_q(L', K)} \subset \mathcal{P}_{\mathcal{O}_q(K)} \subset \hat{S}_{O_q(K)}.
\]

Given a function \( f \in O_q(K) \) let \( B_f \) be the uniform algebra in \( \mathcal{C}(K) \) generated by \( f \) and \( O_0(K) \). It follows from Proposition 4.3 (5) that this is really an algebra. We set \( \mathcal{M} := \bigcup_{f \in O_q(K)} B_f \). Then we have the inclusions \( O_q(K) \subset \mathcal{M} \subset \mathcal{O}_q(K) \) and, thus, \( \hat{S}_{\mathcal{M}} = \hat{S}_{O_q(K)} = \hat{S}_{\mathcal{O}_q(K)} \). Now by Proposition 3.4 we obtain that

\[
\hat{S}_{O_q(K)} = \hat{S}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}} \subset \mathcal{P}_{\mathcal{O}_q(K)} \subset \hat{S}_{\mathcal{O}_q(K)} = \hat{S}_{O_q(K)}.
\]

Hence, for \( B \in \{ \mathcal{O}_q(K), \mathcal{O}_q(K) \} \) we have that \( \mathcal{P}_{\mathcal{B}} = \hat{S}_{\mathcal{B}} \).

(4) By setting again \( A_0 := \log |O_0(K)| \) and by the same reasons as in the first part of the previous point (3), we can deduce that the Shilov boundary for the class \( \mathcal{B} \) exists.

Since the families \( A_q(K) \) and \( A(L, K) \) are uniform subalgebras of \( \mathcal{C}(K) \) we can apply Bishop’s thereom in order to obtain the corresponding peak point property.

The family \( A^p_q(K) \) is the union of uniform algebras of the form \( A(L', K) \), where \( L' \) varies among all complex planes \( L' \) of codimension \( q \). For given \( f \in A_q(K) \) the family \( A_f \) denotes the uniform closure of the algebra generated by \( f \) and \( A_0(K) \). The family \( A_q(K) \) is then exactly the union of all such families \( A_f \), where \( f \in A_q(K) \). Then by the same arguments as in the middle part of the previous point, we obtain peak point properties for the families \( A^p(K) \) and \( A_q(K) \).

The last peak point property for the class \( \mathcal{APSH}_q(K) \) is again due to Theorem 3.10 by using \( \omega(z) := 1 + |z|^2, \ z \in \mathbb{C}^N. \)

For certain subfamilies of \( q \)-plurisubharmonic or \( q \)-holomorphic functions we are already able to classify the Shilov boundary.

Remark 5.2 (1) Let \( K \) be a compact set in \( \mathbb{C}^N \). Then the Shilov boundary for the family

\[ A := \mathcal{PSH}_q(\text{int} K) \cap \mathcal{USC}(K) \]
of all \(g\)-plurisubharmonic functions on the interior of \(K\) which are upper semi-continuous up to the boundary \(K\) is exactly the whole boundary of \(K\). Indeed, by the maximum principle (Proposition 1.2 [10]), the Shilov boundary for \(\mathcal{A}\) is contained in the boundary of \(K\). On the other hand, pick a point \(x\) in the boundary of \(K\). Then the characteristic function \(\chi_x\) of the set \(\{x\}\) in \(K\) lies in \(\mathcal{A}\). Moreover, it peaks at \(x\). Hence, we have that the whole boundary of \(K\) is the set \(P_A\). Since this set lies in the Shilov boundary for \(\mathcal{A}\), we conclude that \(\hat{S}_A = bK\).

(2) The following function \(f\) from Example 5 in [Bas76] is \((N - 1)\)-holomorphic on \(\mathbb{C}^N \setminus \{0\}\) and has an isolated non-removable singularity at the origin,

\[
f(z) = \frac{\bar{z}_1 + \ldots + \bar{z}_N}{|z_1|^2 + \ldots + |z_N|^2}.
\]

Let \(p\) be a boundary point of a compact set \(K\) in \(\mathbb{C}^N\) and let \((p_n)_{n \in \mathbb{N}}\) be a sequence of points \(p_n \notin K\) which converges to \(p\) outside \(K\). For \(n \in \mathbb{N}\) consider the function \(f_n(z) := f(z - p_n)\), which is \((N - 1)\)-holomorphic on \(\mathbb{C}^N \setminus \{p_n\}\). Now if \(n\) tends to \(+\infty\), the absolute values \(|f_n(p)|\) tend to \(+\infty\). Hence, for every small enough neighborhood \(U\) of \(p\) there is an index \(n \in \mathbb{N}\) such that \(U\) contains \(p_n\) and \(|f_n|\) attains its maximum on \(K\) only inside the set \(U \cap K\). By the definition of the Shilov boundary, the set \(U\) intersects \(\hat{S}_{O_{N-1}(K)}\). Since \(U\) was an arbitrary small neighborhood of \(p\), the point \(p\) itself is contained in \(\hat{S}_{O_{N-1}(K)}\). Therefore, the whole boundary \(bK\) of \(K\) is contained \(\hat{S}_{O_{N-1}(K)}\). Now the local maximum modulus principle in Proposition 4.3 [8] yields

\[
\hat{S}_{O_{N-1}(K)} = bK.
\]

In the next statement we compare the Shilov boundary for subclasses of \(g\)-holomorphic functions defined on subspaces of different dimensions. Some ideas of its proof are similar to the arguments given in the proof of Theorem 3 in [Bas78].

**Proposition 5.3** Let \(K\) be a compact set in \(\mathbb{C}^N\) which admits a Stein neighborhood basis. Given a complex plane \(L\) of codimension \(q \in \{0, \ldots, N - 1\}\) it holds that

\[
\hat{S}_{O(K \cap L)} = \hat{S}_{O(L, K)} \cap L.
\]

**Proof.**

Observe that \(K \cap L\) is non-empty if and only if \(\hat{S}_{O(L, K)} \cap L\) is non-empty. Indeed, assume that \(K \cap L\) is non-empty, but \(L\) does not intersect \(S_0 := \hat{S}_{O(L, K)}\). For \(n \in \mathbb{N}\) let \(\chi_n := \chi_{n, L}\) be the functions from the part (4) of Example 4.3. It is obvious that \(\chi_n \in O(L, K)\), since it is constant on each plane of codimension \(q\) parallel to \(L\). Recall that \(\chi_n\) decreases to the characteristic function of \(L\). Then for large enough integer \(n \in \mathbb{N}\), we can arrange that

\[
\max_K \chi_n \geq \max_{K \cap L} \chi_n > \max_{S_0} \chi_n,
\]

which is a contradiction to the definition of the Shilov boundary for the class \(O(L, K)\). The other direction is obvious, because \(\hat{S}_{O(L, K)}\) is a non-empty subset of \(K\) due to Proposition 5.1 (3).

We continue by proving the inclusion \(\hat{S}_{O(K \cap L)} \subset \hat{S}_{O(L, K)} \cap L\). Let again be \(S_0 := \hat{S}_{O(L, K)}\). We have to show that \(\max_{K \cap L} |f| = \max_{S_0 \cap L} |f|\) for every function \(f \in O(K \cap L)\). Pick an arbitrary function \(f \in O(K \cap L)\). Then \(f \in O(U \cap L)\) for some open neighborhood \(U\) of \(K\). Since \(K\) has a Stein neighborhood basis, we can assume that \(U\) is pseudoconvex. Let \(F\) be a holomorphic extension of \(f\) to the whole of \(U\). Then \(F_n := F \cdot \chi_{n, L} \in O(L, K)\) for every \(n \in \mathbb{N}\). Furthermore, we have that

\[
\max_{K \cap L} |f| = \lim_{n \to \infty} \max_K |F_n| = \lim_{n \to \infty} \max_{S_0} |F_n| = \max_{S_0 \cap L} |f|.
\]
By the definition it means that \( \tilde{S}_{O(K \cap L)} \) is contained in \( S_0 \cap L \). For the other inclusion, take a point \( p \in L \cap P_{O(L,K)} \). Then there is a peak function \( f \) in \( \overline{O}(L,K) \) such that 
\[
\{ z \in K : |f(z)| = \max_{K} |f| = \{ p \} .
\]
It is easy to see that \( g = f|L \) lies in \( \overline{O}(K \cap L) \) and that 
\[
\{ z \in K \cap L : |g(z)| = \max_{K \cap L} |g| = \{ p \} ,
\]
because \( p \in K \cap L \). Thus, \( p \) is also a peak point for \( \overline{O}(K \cap L) \). We obtain that
\[
L \cap P_{O(L,K)} \subset P_{O(K \cap L)} \subset \tilde{S}_{O(K \cap L)} = \tilde{S}_{O(K \cap L)}.
\]
Together with Proposition 5.1 (3) we conclude that
\[
L \cap \tilde{S}_{O(L,K)} = L \cap \overline{P_{O(L,K)}} \subset \tilde{S}_{O(K \cap L)}.
\]
\[\square\]

6 Generalization of Bychkov’s theorem

In [Byc81], S.N. Bychkov gave a characterization of the Shilov boundary for bounded convex domains \( D \subset \mathbb{C}^N \). Our goal in this section is to generalize this theorem to Shilov boundaries for subclasses of \( q \)-plurisubharmonic and \( q \)-holomorphic functions (see Theorem 6.19 below).

First, we introduce one more subclass of continuous functions which is usually used when working with the classical Shilov boundary for holomorphic functions. Namely, given a compact set \( K \subset \mathbb{C}^N \), the set \( A_0(K) := \mathcal{O}_0(\text{int}(K)) \cap \mathcal{C}(K) \) forms a uniform subalgebra of \( \mathcal{C}(K) \).

We recall the main result of Bychkov’s article [Byc81].

**Theorem 6.1 (Bychkov, 1981)** Let \( D \) be a bounded convex open set in \( \mathbb{C}^2 \). A boundary point \( p \in bD \) is not in \( \tilde{S}_{A_0(D)} \) if and only if there is a neighborhood \( U \) of \( p \) in \( bD \) such that \( U \) consists only of complex points (see Definition 6.7).

**Remark 6.2** If \( D \subset \mathbb{C}^N \) is a bounded convex domain, it is easy to verify that \( D \) has a Stein neighborhood basis and that
\[
\tilde{S}_{A_0(D)} = \tilde{S}_{O_1(D)}, \quad \tilde{S}_{A_0(D)} = \tilde{S}_{O_1(D)} \quad \text{and} \quad \tilde{S}_{A_0(D)} = \tilde{S}_{P_{SH_0(D)}(D)} \subset \mathbb{C}^N.
\]

We recall some definitions from convexity theory given in Bychkov’s article [Byc81]. We also mainly use his notations.

**Definition 6.3**

1. A set \( K \subset \mathbb{R}^m \) is called convex if for every two points \( x_1, x_2 \) contained in \( K \) the segment \( \{ (1-t)x_1 + tx_2 : 0 \leq t \leq 1 \} \) also lies in \( K \). The dimension of the smallest (real) plane containing \( K \) is the dimension of \( K \).

2. Let \( K \) be a convex body, i.e., a compact convex set with non-empty interior, and let \( p \) a boundary point of \( K \). Every hyperplane \( H \) in \( \mathbb{R}^m \) splits the space \( \mathbb{R}^m \) into two halfspaces \( H^+ \) and \( H^- \). The hyperplane \( H \) is then said to be supporting for \( K \) at \( p \) if \( H \) contains \( p \) and \( K \) lies in one of the closed halfspaces \( H \cup H^+ \) or \( H \cup H^- \).

3. A subset of the boundary \( bK \) of \( K \) which results from an intersection of \( K \) with supporting hyperplanes is called a face of \( K \). A face is again a lower dimensional convex set. The empty set and \( K \) itself are also considered to be faces. A face of a face of \( K \) does not need to be a face of \( K \). The arbitrary intersection of faces of \( K \) is again a face of \( K \).
Remark 6.4 Given a convex body $K$, there exists a unique minimal face $F_1 = F_{\min}(p, K)$ of $F_0 := K$ in the boundary of $K$ containing the point $p$. It can be defined as the intersection of $K$ and all supporting hyperplanes for $K$ at $p$. Then there are two options for $p$: either it is an inner point of the convex body $F_1$ or it lies on the boundary of $F_1$. In the second case, the point $p$ might again lie either in the interior of the minimal face $F_2 = F_{\min}(p, F_1)$ of $F_1$ or in the boundary of $F_2$. Inductively, we obtain a finite sequence $(F_j)_{j=0,\ldots,j(p)}$ of convex bodies $F_j$ in $K$ of dimension $m_j$ such that $F_{j+1} = F_{\min}(p, F_j) \supset F_j$ for each $j \in \{0, \ldots, j(p)-1\}$ and such that either, if $m_{j(p)} > 0$, the point $p$ is an interior point of $F_{j(p)}$, or, if $m_{j(p)} = 0$, the minimal face $F_{j(p)}$ consists only of the point $p$.

Definition 6.5 The convex body $F_p(K) := F_{j(p)}$ obtained in Remark 6.4 above will be called the face essentially containing $p$. It is contained in a plane $E_p(K)$ of dimension $m_{j(p)}$ which admits $E_{j(p)} \cap K = F_{j(p)}$.

Example 6.6 Let $\Delta$ be the unit disc in $\mathbb{R}^2$ and consider the set $K = \bigcup \{[-1,1] \times [-1,0]\}$. It is a convex body in $\mathbb{R}^2$. The plane $\pi_1 = \{1\} \times \mathbb{R}$ is the only supporting hyperplane of $K$ at $p = 1$ in $\mathbb{R}^2$. Thus, the minimal face of $K$ containing $p$ is the segment $F_1 = \pi_1 \cap K = \{1\} \times [-1,0]$. The point $p$ lies in the boundary of $F_1$ in $\{1\} \times \mathbb{R}$. Then the set $\pi_2 := \{1\}$ is the only supporting hyperplane of $F_1$ at $p$ in $\{1\} \times \mathbb{R}$. Hence, the minimal face of $F_1$ having $p$ inside is the set $F_2 = \pi_2 \cap K = \{p\}$. Therefore, the face essentially containing $p$ is the set $F_2 = \{1\}$.

In the following, let $D$ be always a bounded convex domain in $\mathbb{C}^N$.

Definition 6.7 Let $p \in bD$ and let $E^C_p(D)$ be the largest complex plane inside $E_p(D)$ passing through $p$. We define $\nu(p)$ to be the complex dimension of $E^C_p(D)$. If $\nu(p) = 0$, then $E_p(D)$ is totally real and we say that the point $p$ is real.

The set $\Pi_{p}(D)$ will denote the set of all complex planes $\pi$ in $\mathbb{C}^N$ such that there exists a domain $G \subset \mathbb{C}^N$ with $p \in G \cap \pi \subset bD$. If $\Pi_{p}(D)$ is not empty, then $p$ is called complex.

We restate Lemma 2.5 in [Bycz84] and its important corollary.

Lemma 6.8 If $I \subset bD$ is an open segment containing $p \in bD$, then $I \subset F_{p}(D)$.

Corollary 6.9 A boundary point $p \in bD$ is either real or complex.

From this we can derive further consequences.

Corollary 6.10 If $p \in bD$ is complex, then $E^C_{p}(D) \in \Pi_{p}(D)$.

Proof. Since $p$ is complex, it can not be real due to the previous Corollary 6.9. Thus, $E^C_p(D)$ is not empty and the face essentially containing $p$ can not be a single point. The point $p$ is then an inner point of the convex body $F_{p}(D)$ in $E_p(D)$. Hence, there is an open ball $B'$ with center $p$ inside $F_{p}(D)$, and we can find an open ball $B$ in $\mathbb{C}^N$ with center $p$ such that $B \cap E_p(D) = B'$. Then we obtain that

$$E^C_p(D) \cap B \subset E_p(D) \cap B = B' \subset F_p(D) \subset bD.$$ 

It follows now from the definition of $\Pi_{p}(D)$ that $E^C_p(D)$ lies in $\Pi_{p}(D)$.

Corollary 6.11 If $\pi \in \Pi_{p}(D)$, then $\pi$ lies in $E^C_p(D)$. 

\[ \square \]
Proof. Let $G$ be an open neighborhood of $p$ in $\mathbb{C}^N$ such that $p \in G \cap \pi \subset bD$. It follows from Lemma 6.13 that $p \in G \cap \pi$ lies in $F_p(\partial \Omega)$. Since $G \cap \pi$ is open in $\pi$, we have that $\pi$ is contained in $E_p(\partial \Omega)$. Since $\pi$ is a complex plane containing $p$ and $E_p(\partial \Omega)$ is the largest complex plane inside $E_p(\partial \Omega)$, we conclude that $\pi$ lies in $E^C_p(\partial \Omega)$. □

We specify complex points in the following way.

Definition 6.12 Given $q \in \{1, \ldots, N-1\}$, a complex point $p$ is called $q$-complex if $\nu(p) \geq q$.

The previous classifications can be reduced to the following simple observation.

Remark 6.13 A boundary point $p$ in $bD$ is $q$-complex if and only if there is a domain $G$ in $\mathbb{C}^N$ and a complex plane of dimension at least $q$ such that $p \in G \cap \pi \subset bD$.

The next lemma asserts that a complex point $p$ is a lower dimensional real point when intersecting the convex body with a complex plane containing $p$ transversal to $E^C_p(\partial \Omega)$.

Lemma 6.14 Let $p$ be a complex point in $bD$. Let $\pi$ be a complex affine plane of codimension $\nu(p)$ such that $E^C_p(\partial \Omega) \cap \pi = \{p\}$. Then $p$ is a real boundary point of $\partial \Omega \cap \pi$.

Proof. If $\nu(p) = N-1$, the statement is obviously true, since every boundary point of $\partial \Omega \cap \pi$ is real.

Suppose that $\nu(p) \leq N-2$ and that the statement is false. Then by Corollary 6.9 the point $p$ is a complex boundary point of $\partial \Omega \cap \pi$. By Corollary 6.10 there exist a domain $G \subset \mathbb{C}^N$ and a complex line $L$ in $\pi$ such that $p \in G \cap L \subset bD \cap \pi \subset bD$. Hence, $L \in \Pi_p(\partial \Omega)$. By Corollary 6.11 the line $L$ lies in $E^C_p(\partial \Omega)$. But since $E^C_p(\partial \Omega) \cap \pi = \{p\}$ and $L \subset \pi$, it follows that $L = \{p\}$, which is absurd. □

We generalize now Proposition 2.6 in [Byc81] which states that a real boundary point always lies in the Shilov boundary for the class $A_0(\partial \Omega)$.

Proposition 6.15 If $p \in bD$, then $p \in \hat{S}_{O_{\nu(p)}(\partial \Omega)}$.

Proof. By Corollary 6.9 $p$ is either real or complex.

If $p$ is real, then by Proposition 2.6 in [Byc81] and Remark 6.2 we have that

$$p \in \hat{S}_{A_0(\partial \Omega)} = \hat{S}_{O_0(\partial \Omega)} = \hat{S}_{O_0^+(\partial \Omega)}.$$ 

Recall that it follows from definition that $O_0(\partial \Omega) = O_0^+(\partial \Omega)$.

If $p$ is complex, then there are a domain $G$ and a $\nu(p)$-dimensional complex plane $\pi$ such that $p \in G \cap \pi \subset bD$. Let $L$ be a complex affine plane of codimension $\nu(p)$ such that $\pi \cap L = \{p\}$. Then, by Lemma 6.14 the point $p$ is a real boundary point of the convex body $\partial \Omega \cap L$. By Proposition 2.6 in [Byc81] and by Propositions 5.3 and 5.1 (2) we obtain that

$$p \in \hat{S}_{A_0(\partial \Omega \cap L)} = \hat{S}_{O_0(\partial \Omega \cap L)} \subset \hat{S}_{O(\partial \Omega \cap L)} \subset \hat{S}_{O_{\nu(p)}(\partial \Omega)}.$$ 

□

As a first consequence, we obtain a characterization of the Shilov boundary for the family of $(N-1)$-plurisubharmonic functions. Compare also Remark 6.2 (2).
**Corollary 6.16**  The Shilov boundaries for the classes $\mathcal{O}_{N-1}(\overline{D})$, $\mathcal{O}_{N-1}(\overline{D})$ and $\mathcal{P}_{\mathcal{S}_{H,N-1}(\overline{D})}$ coincide with the topological boundary of $D$; i.e.,

$$\hat{S}_{\mathcal{O}_{N-1}(\overline{D})} = \hat{S}_{\mathcal{O}_{N-1}(\overline{D})} = \hat{S}_{\mathcal{P}_{\mathcal{S}_{H,N-1}(\overline{D})}} = bD.$$ 

**Proof.** If $p \in bD$, then $p$ is real or complex and $0 \leq \nu(p) \leq N - 1$. Thus, the previous proposition and Propositions 5.1 (2) and 4.2 (10) imply that

$$p \in \hat{S}_{\mathcal{O}_{\nu(p)}(\overline{D})} \subset \hat{S}_{\mathcal{O}_{N-1}(\overline{D})} \subset \hat{S}_{\mathcal{O}_{N-1}(\overline{D})} \subset \hat{S}_{\mathcal{P}_{\mathcal{S}_{H,N-1}(\overline{D})}} \subset bD.$$

\[\square\]

We will need the following lemma.

**Lemma 6.17**  Let $p \in bD$ and $q \in \{0, \ldots, N - 2\}$. If there exists an at least $(q+1)$-dimensional complex analytic set in $bD$ containing $p$, then $p$ is not a peak point for the class $P_{\mathcal{S}_{H,q}(\overline{D})}$. In particular, no $(q+1)$-complex point can be contained in $P_{\mathcal{S}_{H,q}(\overline{D})}$.

**Proof.** This follows immediately from the local maximum principle for $q$-plurisubharmonic functions on analytic sets (see Corollary 5.3 in [Slo86]). \[\square\]

We are now able to generalize Bychkov’s theorem.

**Definition 6.18**  For $q \in \{1, \ldots, N - 1\}$ denote by $\Gamma_{q}(\overline{D})$ the set of all boundary points of $D$ which have a neighborhood $U$ in $bD$ such that $U$ consists only of $q$-complex points.

**Theorem 6.19**  Let $q \in \{0, \ldots, N - 2\}$. Then

$$\hat{S}_{\mathcal{O}_{q}(\overline{D})} = \hat{S}_{\mathcal{O}_{q}(\overline{D})} = \hat{S}_{\mathcal{P}_{\mathcal{S}_{H,q}(\overline{D})}} = bD \setminus \Gamma_{q+1}(\overline{D}).$$

**Proof.** If $p \in bD \setminus \hat{S}_{\mathcal{O}_{q}(\overline{D})}$, then there is a neighborhood $U$ of $p$ in $bD$ such that $U \cap \hat{S}_{\mathcal{O}_{q}(\overline{D})} = \emptyset$. Thus, if $w \in U$, then $\nu(w) \geq q + 1$ due to Proposition 6.15. This means that $U$ consists only of $(q+1)$-complex points. Hence, $p \in \Gamma_{q+1}(\overline{D})$. We conclude that

$$bD \setminus \Gamma_{q+1}(\overline{D}) \subset \hat{S}_{\mathcal{O}_{q}(\overline{D})}.$$ 

On the other hand, if there is a neighborhood $U$ of $p$ in $bD$ such that $U$ contains only $(q+1)$-complex points, then, by Lemma 6.17, we obtain that $U \cap P_{\mathcal{S}_{H,q}(\overline{D})} = \emptyset$. This implies that $p \notin P_{\mathcal{S}_{H,q}(\overline{D})}$. Since, by Proposition 5.1 (3), the latter set coincides with $\hat{S}_{\mathcal{P}_{\mathcal{S}_{H,q}(\overline{D})}}$, we obtain the other inclusion

$$\hat{S}_{\mathcal{P}_{\mathcal{S}_{H,q}(\overline{D})}} \subset bD \setminus \Gamma_{q+1}(\overline{D}).$$

In view of Proposition 5.1 (2) this completes the proof. \[\square\]

Now we give an interesting observation following from the previous Theorem.

**Remark 6.20**  Given an integer $q \in \{1, \ldots, N - 1\}$ let $\Gamma_{q}^{A}(\overline{D})$ be the set of all boundary points $p$ of $D$ such that there exists a neighborhood $U$ of $p$ in $bD$ so that for each point $z \in U$ there is a complex analytic set in $U$ of dimension at least $q$ containing $z$. Then

$$\Gamma_{q}^{A}(\overline{D}) = \Gamma_{q}(\overline{D}).$$
Indeed, the inclusion $\Gamma_q(\mathcal{D}) \subset \Gamma_q^q(\mathcal{D})$ follows directly from the definition of these two sets and the definition of $q$-complex points.

Now let $p \in \Gamma_q^q(\mathcal{D})$. Then Lemma 6.17 and Proposition 5.1 (3) imply that $p \notin \hat{S}_{PSH_{q-1}}(\mathcal{D})$. Thus, by Theorem 6.19 we have that $p$ is contained in $\Gamma_q(\mathcal{D})$. This shows the other inclusion.

In the end of this section, we check for an analytic structure of the Shilov boundary of $q$-plurisubharmonic functions on convex sets.

**Theorem 6.21** Let $q \in \{1, \ldots, N - 1\}$ and assume that $\{z \in bD : \nu(z) \geq q + 1\}$ is open. If it is non-empty, then the following open part

$$\mathcal{F}_q(\mathcal{D}) := \text{int}_{bD} \left( \hat{S}_{PSH_q}(\mathcal{D}) \setminus \hat{S}_{PSH_{q-1}}(\mathcal{D}) \right)$$

of the Shilov boundary for $PSH_q(\mathcal{D})$ in $bD$ locally admits a complex foliation by complex $q$-dimensional planes in the following sense: for every point $p \in \mathcal{F}_q(\mathcal{D})$ there exists a neighborhood $U$ of $p$ in $bD$ such that for each $z \in U$ there is a domain $G_z$ in $\mathbb{C}^N$ and a unique complex $q$-dimensional plane $\pi_z$ with $z \in \pi_z \cap G_z \subset U$. In the special case $q = N - 1$, these complex (hyper-)planes are aligned parallelly.

**Proof.** We set $\Gamma_N := \emptyset$. By Theorem 6.19 and by Corollary 6.16 we have that $\mathcal{F}_q(\mathcal{D}) = \Gamma_q(\mathcal{D}) \setminus \Gamma_{q+1}(\mathcal{D})$. If the set $\{z \in bD : \nu(z) \geq q + 1\}$ is open, then it coincides with $\Gamma_{q+1}(\mathcal{D})$. Thus,

$$\mathcal{F}_q(\mathcal{D}) = \Gamma_q(\mathcal{D}) \setminus \{z \in bD : \nu(z) \geq q + 1\}.$$  

Now if $p$ is an arbitrary point from $\mathcal{F}_q(\mathcal{D})$, then there is a neighborhood $W$ of $p$ in $\mathcal{F}_q(\mathcal{D})$ such that $\nu(z) = q$ for every point $z \in W$. Hence, the open set $\mathcal{F}_q(\mathcal{D})$ consists only of exactly $q$-complex points. Then Corollaries 6.10 and 6.11 imply existence and uniqueness of an open part of a complex $q$-dimensional plane $\pi_z = E_z^q(\mathcal{D})$ containing $z$ and lying in $U$.

For the special case of $q = N - 1$ the set $\mathcal{F}_{N-1}(\mathcal{D})$ is a convex hypersurface foliated by complex hyperplanes. By a result of Beloshapka and Bychkov in [BB86], they have to be aligned parallelly. (See also Example 6.24 and the remark before this example.)

At the end of this section, we give an example for a convex domain $D$ in $\mathbb{C}^3$ such that the part $\mathcal{F}_D$ does not admit a foliation in the sense of the previous theorem if the assumption on the openness of $\{z \in bD : \nu(z) \geq 2\}$ is dropped.

**Example 6.22** Consider the domain $G$ in $\mathbb{C} \times \mathbb{R}$ given by

$$G = \{(x, y, u) \in \mathbb{C} \times \mathbb{R} : x^2 + (1 - y^2)u^2 < (1 - y^2), |y| < 1\}.$$  

It is easy to compute that the function $h(y, u) := \sqrt{(1 - y^2)(1 - u^2)}$ is concave on $[-1, 1]^2$. Since $G$ is the intersection of the sublevel set $\{x < h(y, u)\}$ of the concave function $h$ and the superlevel set $\{x > -h(y, u)\}$ of the convex function $-h$ over $[-1, 1]^2$, it is convex in $\mathbb{C} \times \mathbb{R}$.

The boundary of $G$ contains the flat parts $\{\pm i\} \times (-1, 1)$ and $\{0\} \times [-1, 1] \times \{\pm 1\}$ whereas the rest of the boundary consists of strictly convex points. By putting $D := G \times (-1, 1)^3$ we obtain a convex domain $D$ in $\mathbb{C}^3$ such that

$$\{z \in bD : \nu(z) \geq 2\} = \{\pm i\} \times (-1, 1)^4.$$  

and $\Gamma_1(\mathcal{D})$ is the whole boundary of $D$. In particular, $\Gamma_2(\mathcal{D})$ is empty. Thus, the boundary points $z$ in $bD$ with $\nu(z) \geq 2$ lie in $\Gamma_1(\mathcal{D})$, but there is no unique foliation by complex one-dimensional planes near these points.
7 Hausdorff dimension of the Shilov boundary

In this section we prove some estimates on the Hausdorff dimension of the Shilov boundary for $q$-plurisubharmonic functions on convex bodies.

**Definition 7.1** Let $(X,d)$ be a metric space.

1. For a subset $U$ of $X$ denote by $\text{diam}(U)$ the diameter of $U$, i.e.,
   \[
   \text{diam}(U) := \sup\{d(x,y) : x, y \in U\}.
   \]

2. Given a subset $E$ of $X$ and positive numbers $s$ and $\varepsilon$ we set
   \[
   H_s^\varepsilon(E) := \inf\left\{\sum_{i=1}^\infty \text{diam}(U_i)^s : E \subset \bigcup_{i=1}^\infty U_i, \text{diam}(U_i) < \varepsilon \forall i \in \mathbb{N}\right\}.
   \]
   The $s$-dimensional Hausdorff measure is then defined by
   \[
   H^s(E) := \lim_{\varepsilon \to 0} H_s^\varepsilon(E).
   \]

3. For every subset $E$ of $X$ there is a number $s_0 \in [0, +\infty[$ such that
   \[
   H^s(E) = \infty \text{ for } s \in (0, s_0) \quad \text{and} \quad H^s(E) = 0 \text{ for } s \in (s_0, \infty).
   \]
   The number $\dim_H E := s_0$ is called the Hausdorff (or metric) dimension of $E$.

The next statement can be found in, e.g., [Fal03], Corollary 7.12.

**Proposition 7.2** Let $I$ be a $m$-dimensional cube in $\mathbb{R}^m$, $J$ be a $n$-dimensional cube in $\mathbb{R}^n$ and $F$ be a subset of $I \times J$. For a given point $x \in I$ consider the slice $F_x := F \cap (\{x\} \times J)$. If $\dim_H F_x \geq \alpha$ for every $x \in I$, then $\dim_H F \geq \alpha + m$.

It was shown in [Byc81] that the Hausdorff dimension of the Shilov boundary of a convex body in $\mathbb{C}^2$ is not less than 2. We partially generalize this result.

**Theorem 7.3** Let $D$ be a convex bounded domain in $\mathbb{C}^N$ and $q \in \{0, \ldots, N-2\}$. Suppose that there are a constant $\alpha \geq 0$ and a complex $q$-codimensional plane $\pi_0$ intersecting $D$ such that
   \[
   \dim_H \mathring{S}_{O_q(D\cap\pi)} \geq \alpha
   \]
   for every complex $q$-codimensional plane $\pi$ which lies nearby $\pi_0$ and which is parallel to $\pi_0$. Then
   \[
   \dim_H \mathring{S}_{O_q(D)} \geq \alpha + 2q.
   \]
   In particular, $\dim_H \mathring{S}_{O_{N-2}(D)} \geq 2N - 2$.

**Proof.** Denote by $\Pi$ the set of the complex planes mentioned in the assumptions of this theorem. Then, by Proposition 5.3 and Proposition 5.1 we have that
   \[
   \bigcup_{\pi \in \Pi} \mathring{S}_{O_q(D\cap\pi)} \subset \bigcup_{\pi \in \Pi} \mathring{S}_{O_q(D\cap\pi)} \subset \mathring{S}_{O_q(D)}.
   \]
   It follows then from Proposition 7.2 that $\dim_H \mathring{S}_{O_q(D)} \geq \alpha + 2q$.

Let now $q = N-2$. It was shown in [Byc81], Theorem 3.1, that $\dim_H \mathring{S}_{O_q(D\cap\pi)} \geq \alpha = 2$ for every complex two dimensional affine plane $\pi$ such that $\pi \cap D \neq \emptyset$. Hence, we conclude that
   \[
   \dim_H \mathring{S}_{O_{N-2}(D)} \geq 2 + 2(N - 2) = 2N - 2.
   \]
To show that the Hausdorff dimension of $\hat{S}_{A_0(D)}$ is not less than two if $D \subset \mathbb{C}^2$ is a convex domain, Bychkov used that $\Gamma_1(D)$ admits a local foliation by complex lines which are aligned parallelly to each other. More general, if a convex hypersurface (i.e., an open part of the boundary of a convex body) is foliated by complex hyperplanes, then, by a result of Beloshapka and Bychkov in [BB86], they are always parallel to each other. Especially, this holds for the open set $\Gamma_{N-1}(D)$, provided it is not empty. But, in general, it fails for lower dimensional complex foliations as the following example from [NTT12] shows.

**Example 7.4** Consider the function $\varrho(z) = (\text{Re}z_2)^2 - (\text{Re}z_1)(\text{Re}z_3)$ for $z \in \mathbb{C}^3$. Then the set
\[
D := \{z \in \mathbb{C}^3 : \text{Re}(z_1) > 0, \varrho(z) < 0\}
\]
is convex and an open part of its boundary is foliated by a real 3-dimensional parameter family of open parts of non-parallel complex lines of the form
\[
\{(a^2\zeta + ib, a\zeta + ic, \zeta) \in \mathbb{C}, \quad a, b, c \in \mathbb{R}\}.
\]

### 8 Shilov boundary for smooth $q$-plurisubharmonic functions

In this section, we give a characterization of the Shilov boundary for $C^2$-smooth $q$-plurisubharmonic functions defined near the closure of a compact set.

**Definition 8.1** Let $K$ be a compact set in $\mathbb{C}^N$.

1. We denote by $\mathcal{PSH}_{q}^2(K)$ the set of all functions which are $C^2$-smooth and $q$-plurisubharmonic in some neighborhood of $K$.

2. The set $\mathcal{PSH}_{q}^c(K)$ is the set of all functions which are continuous on some neighborhood of $K$ and locally the maximum of finitely many $C^2$-smooth $q$-plurisubharmonic functions.

3. The set $\mathcal{PSH}_{q}^0(K)$ is formed by all functions which are continuous and $q$-plurisubharmonic in some neighborhood of $K$.

**Remark 8.2** Since $A_0 := \mathcal{PSH}_{q}^0(K)$ generates the topology of $K$ and fulfills $A_0 + A \subset A$, where $A \in \{\mathcal{PSH}_{q}^0(K), \mathcal{PSH}_{q}^c(K), \mathcal{PSH}_{q}^2(K)\}$, we can apply Theorem 1.10 in order to obtain that $\hat{S}_A$ is a non-empty $A$-boundary. If we put $\omega(z) := 1 + |z|^2$, $z \in \mathbb{C}^N$, then by Theorem 3.10 we get the peak property $\overline{P}_A = \hat{S}_A$ for the subfamilies of $q$-plurisubharmonic functions defined above.

In the following, we present a useful regularization technique derived from [Dem12], Lemma (5.18) in chapter 5.

**Definition 8.3** Let $\theta$ be a non-negative $C^\infty$-smooth function on $\mathbb{R}$ with compact support in the unit interval $(-1,1)$ such that $\int_{\mathbb{R}} \theta(s)ds = 1$ and $\theta(-t) = \theta(t)$ for all $t \in \mathbb{R}$. Given positive numbers $\varepsilon_1, \ldots, \varepsilon_\ell \in (0,\infty)$ and $t_1, \ldots, t_\ell \in \mathbb{R}$, we define the regularized maximum by
\[
\max_{(\varepsilon_1, \ldots, \varepsilon_\ell)}(t_1, \ldots, t_\ell) := \int_{\mathbb{R}^\ell} \max\{t_1 + \varepsilon_1 s_1, \ldots, t_\ell + \varepsilon_\ell s_\ell\} \theta(s_1) \cdots \theta(s_\ell) ds_1 \cdots ds_\ell.
\]
For a single positive number $\varepsilon > 0$ we set $\max_{\varepsilon} := \max_{(\varepsilon, \ldots, \varepsilon)}$.
The regularized maximum has the following properties.

**Lemma 8.4**

1. The function \((t_1, \ldots, t_\ell) \mapsto \max_{t_1, \ldots, t_\ell}(t_1, \ldots, t_\ell)\) is a \(C^\infty\)-smooth convex function on \(\mathbb{R}^\ell\) which is non-decreasing in every variable \(t_1, \ldots, t_\ell\).
2. It holds that \(\max\{t_1, \ldots, t_\ell\} \leq \max_{t_1, \ldots, t_\ell}(t_1, \ldots, t_\ell) \leq \max\{t_1 + \epsilon_1, \ldots, t_\ell + \epsilon_\ell\}\).
3. If \(t_j + \epsilon_j < \max_{i \neq j}\{t_i - \epsilon_i\}\), then we have that
   \[
   \max_{t_1, \ldots, t_\ell}(t_1, \ldots, t_\ell) = \max_{t_1, \ldots, t_\ell}(t_1 - \epsilon_1, \ldots, t_j - \epsilon_j, t_{j+1}, \ldots, t_\ell).
   \]

We can apply the regularized maximum to \(q\)-plurisubharmonic functions.

**Lemma 8.5** Let \(\psi_1, \ldots, \psi_k\) be finitely many \(C^2\)-smooth functions on an open set \(U\) in \(\mathbb{C}^N\) such that for each \(j \in \{1, \ldots, k\}\) the function \(\psi_j\) is \(q_j\)-plurisubharmonic on \(U\). Then for every tuple of positive numbers \((\epsilon_1, \ldots, \epsilon_k)\) the regularized maximum \(\max_{(\epsilon_1, \ldots, \epsilon_k)}(\psi_1, \ldots, \psi_k)\) is \(C^2\)-smooth and \(q\)-plurisubharmonic on \(U\), where \(q = q_1 + \ldots + q_k\).

**Proof.** This is a consequence of Lemma 8.4 and Proposition 2.11 in [PZ 13].

The regularized maximum allows to compare the Shilov boundaries of the families of smooth and non-smooth \(q\)-plurisubharmonic functions introduced in Definition 8.1.

**Proposition 8.6** Given a compact set \(K\) in \(\mathbb{C}^N\) we have that

\[
P_{\mathcal{PSH}^q_0(K)} = P_{\mathcal{PSH}^0_q(K)}
\]

and

\[
\hat{S}_{\mathcal{PSH}^q_0(K)} = \hat{S}_{\mathcal{PSH}^0_q(K)} = \hat{S}_{\mathcal{PSH}_q(K)}.
\]

**Proof.** Since \(\mathcal{PSH}^q_0(K) \subset \mathcal{PSH}^0_q(K) \subset \mathcal{PSH}^0_q(K) \subset \mathcal{PSH}_q(K)\), we derive for the set of peak points of these classes that

\[
P_{\mathcal{PSH}^q_0(K)} \subset P_{\mathcal{PSH}^0_q(K)} \subset P_{\mathcal{PSH}^0_q(K)} \subset P_{\mathcal{PSH}_q(K)}
\]

(1)

By the peak point property \(\overline{P}_A = \hat{S}_A\) for these families (see Remark 8.2) it follows that

\[
\hat{S}_{\mathcal{PSH}^q_0(K)} \subset \hat{S}_{\mathcal{PSH}^0_q(K)} \subset \hat{S}_{\mathcal{PSH}^0_q(K)} \subset \hat{S}_{\mathcal{PSH}_q(K)}.
\]

Assume now that there is a function \(\psi \in \mathcal{PSH}^c_q(K)\) such that \(\psi\) peaks at some point \(p \in bK\). Then there are a neighborhood \(U\) of \(p\) and finitely many \(C^2\)-smooth functions \(\psi_1, \ldots, \psi_k\) on \(U\) such that \(\psi = \max_{j=1,\ldots,k}\psi_j\) on \(U\). By picking a slightly smaller neighborhood of \(p\), we can arrange that the functions \(\psi_j\), \(j = 1, \ldots, k\), are defined in a neighborhood of \(\overline{U}\). Let \(j_0\) be an index from \(\{1, \ldots, k\}\) such that \(\psi(p) = \psi_{j_0}(p)\). Since \(\psi\) peaks at \(p\), we have that

\[
\psi_{j_0}(p) = \psi(p) > \psi(z) \geq \psi_{j_0}(z)
\]

for every \(z \in (U \cap K) \setminus \{p\}\). Hence, \(\psi_{j_0}\) peaks at \(p\) in \(K \cap U\). Since \(\psi_{j_0}\) is continuous on \(\overline{U}\), we can choose a suitable constant \(c \in \mathbb{R}\) such that

\[
\psi_{j_0}(p) > c \geq \max_{bU \cap K} \psi_{j_0}.
\]
By Lemma 8.3, the function \( \varphi := \max_{z} \left\{ \psi_{p_{0}}, c \right\} \) is \( C^2 \)-smooth and \( q \)- plurisubharmonic in a neighborhood of \( U \cap K \). If we choose \( \varepsilon > 0 \) small enough, then due to Lemma 8.3 (3) we can derive that the function \( \varphi \) peaks at \( p \) in \( K \) and fulfills \( \varphi = c \) on \( bU \cap K \). In view of the previous property, we can extend \( \varphi \) by the constant \( c \) into a neighborhood of \( K \) in order to obtain a function from \( \mathcal{PSH}^{0}_q(K) \) which peaks at \( p \). Since \( p \) was an arbitrary peak point for the class \( \mathcal{PSH}^{0}_q(K) \), together with the inclusions in (1) above, we conclude that

\[
P_{\mathcal{PSH}^{0}_q(K)} = P_{\mathcal{PSH}^{0}_q(K)}.
\]

By the peak point property for the \( q \)- plurisubharmonic functions from Definition 8.1 we obtain that

\[
\tilde{\mathcal{PSH}}^{0}_q(K) = \tilde{\mathcal{PSH}}^{0}_q(K).
\]

Now Bungart’s approximation theorem (see Corollary 5.4 in [Bun90] and Slodkowski’s approximation theorem (see Theorem 2.9 in [Slo84]) imply that

\[
\mathcal{PSH}^{0}_q(K) \subset \overline{\mathcal{PSH}^{0}_q(K)}^1 \quad \text{and} \quad \mathcal{PSH}_q(K) \subset \overline{\mathcal{PSH}^{0}_q(K)}^1.
\]

Therefore, Corollary 2.6 yields

\[
\tilde{\mathcal{PSH}}^{0}_q(K) \subset \tilde{\mathcal{PSH}}^{0}_q(K) = \tilde{\mathcal{PSH}}^{0}_q(K) \subset \tilde{\mathcal{PSH}}^{0}_q(K).
\]

Hence, we obtain the remaining identities \( \tilde{\mathcal{PSH}}^{0}_q(K) = \tilde{\mathcal{PSH}}^{0}_q(K) = \tilde{\mathcal{PSH}}^{0}_q(K) \). \( \square \)

**Remark 8.7** From the proof of the previous result we can derive the following local peak point property: Let \( p \) be a boundary point of a compact set \( K \) in \( \mathbb{C} \). If \( p \) is a local peak point for \( C^2 \)- smooth \( q \)- plurisubharmonic functions, i.e., there is a neighborhood \( U \) of \( p \) and a \( C^2 \)-smooth \( q \)- plurisubharmonic function \( \psi \) on \( U \) such that \( \psi(p) > \psi(z) \) for every \( z \in (U \cap K) \setminus \{p\} \), then \( p \) is a (global) peak point for \( \mathcal{PSH}^{0}_q(K) \).

We recall the definition of a strictly \( q \)- pseudoconvex boundary point of a smoothly bounded domain.

**Definition 8.8** Let \( D \) be an open set in \( \mathbb{C}^N \) with \( C^2 \)- smooth boundary, \( p \) be a boundary point of \( D \) and \( q \in \{0, \ldots, N-1\} \). If there are a neighborhood \( U \) of \( p \) and a \( C^2 \)-smooth strictly \( q \)- plurisubharmonic function \( \varphi \) on \( U \) such that \( d\varphi(p) \neq 0 \) and \( U \cap \partial D = \{ z \in U : \varphi(z) < 0 \} \), then \( D \) is said to be strictly \( q \)- pseudoconvex at \( p \). The set of all points \( p \in bD \) such that \( D \) is strictly \( q \)- pseudoconvex at \( p \) is denoted by \( S_q(D) \).

Now we give a characterization of the Shilov boundary for \( q \)- plurisubharmonic functions on bounded domains with \( C^2 \)- smooth boundary.

**Theorem 8.9** Let \( q \in \{0, \ldots, N-1\} \) and let \( D \) be a bounded domain in \( \mathbb{C}^N \) with \( C^2 \)- smooth boundary. Then

\[
\tilde{\mathcal{PSH}}^{0}_q(D) = \tilde{\mathcal{PSH}}^{0}_q(D) = \overline{S_q(D)}.
\]

**Proof.** It follows from Theorem 5.6 in [HM78] that

\[
P_{\mathcal{PSH}^{0}_q(D) \cap C^0(D)} \subset \overline{S_q(D)} \quad \text{and} \quad S_q(D) \subset P_{\mathcal{PSH}^{0}_q(D) \cap C^0(D)}.
\]

It follows from the definition that \( \mathcal{PSH}^{0}_q(D) \subset \mathcal{PSH}^{0}_q(D) \cap C^0(D) \) and, therefore,

\[
P_{\mathcal{PSH}^{0}_q(D)} \subset \overline{S_q(D)}.
\]

(2)

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Hence, by the previous Proposition 8.6 and the peak point property for $C^2$-smooth $q$-plurisubharmonic functions (see Remark 8.2) we obtain that
\[ \tilde{S}_{PSH^2_q(D)} = \tilde{S}_{PSH^2_q(D)} = \overline{P_{PSH^2_q(D)}} \subset \overline{S_q(D)}. \]  
(3)

On the other hand, let $p \in S_q(D)$. Then there is a neighborhood $U$ of $p$ and a $C^2$-smooth strictly $q$-plurisubharmonic function $\rho$ on $U$ such that $\rho$ vanishes on $bD \cap U$ and $\rho(z) < 0$ if $z \in U \cap D$. Since $\rho$ is strictly $q$-plurisubharmonic, there is a positive constant $\varepsilon > 0$ such that $\varphi(z) := \rho(z) - \varepsilon|z - p|^2$ is $C^2$-smooth and $q$-plurisubharmonic on $U$. Moreover, $\varphi(p) = 0$ and $\varphi(z) < 0$ for every $z \in (U \cap D) \setminus \{p\}$. In view of Remark 8.7, the point $p$ is also a peak point for the family $PSH^2_q(D)$. Since $p$ is an arbitrary point from $S_q(D)$, it follows that $S_q(D)$ lies in $P_{PSH^2_q(D)}$. In view of inclusion (3) above and the peak point property for $C^2$-smooth $q$-plurisubharmonic functions, we obtain that
\[ \tilde{S}_{PSH^2_q(D)} = \overline{P_{PSH^2_q(D)}} = \overline{S_q(D)}. \]

The statement follows now from the inclusions (3) above. \qed

For the special case $q = N - 1$ we obtain the following improvement of Remark 5.2 and Corollary 6.18 in the case of smoothly bounded domains.

**Theorem 8.10** If $D$ is a bounded domain in $\mathbb{C}^N$ with $C^2$-smooth boundary, then we have that
\[ P_{PSH^3_{N-1}(\overline{D})} = \tilde{S}_{PSH^3_{N-1}(\overline{D})} = bD. \]

**Proof.** Since $D$ is bounded and has a $C^2$-smooth boundary, it is easy to construct a global defining function $\rho$ for $D$, i.e., a $C^2$-smooth functions in a neighborhood $U$ of $\overline{D}$ such that $D = \{z \in U : \rho(z) < 0\}$ and $d\rho \neq 0$ on $bD$. Then for a large enough constant $c > 0$, the function $\psi := e^{\rho} - 1$ is strictly $(N - 1)$-plurisubharmonic and $C^2$-smooth in a neighborhood $V \subset U$ of $bD$. By shrinking $V$, we can assume that $\psi$ is defined in a neighborhood of $\overline{V}$ in $U$. For an appropriate choice of positive constants $\delta > 0$ and $b > 0$ we have that $\delta|z|^2 - b < \psi(z)$ for every $z \in bD$ and that $\psi(z) < \delta|z|^2 - b$ for every $z \in bV \cap D$. For a positive number $\eta > 0$ we put $\tilde{\psi}(z) := \max\{\psi(z), \delta|z|^2 - b\}$. Then, by Lemma 8.3 (3), we can choose $\eta > 0$ so small that $\tilde{\psi}(z) = \psi(z)$ for every $z$ in some neighborhood of $bD$ in $V$ and such that $\tilde{\psi}(z) = \delta|z|^2 - b$ for every $z$ in some neighborhood of $bV \cap D$. Then we can extend $\tilde{\psi}(z)$ by $\delta|z|^2 - b$ into $D \setminus V$. We denote this extension again by $\tilde{\psi}$. Observe that $\tilde{\psi}$ is now strictly $(N - 1)$-plurisubharmonic in some neighborhood of $\overline{D}$. Therefore, for every boundary point $p$ of $D$ there is a positive constant $\varepsilon = \varepsilon(p)$ such that $\tilde{\psi}(z) - \varepsilon|z - p|^2$ is $(N - 1)$-plurisubharmonic and $C^2$-smooth in some neighborhood of $\overline{D}$. Moreover, it peaks at $p$. Hence, we derive that
\[ bD \subset P_{PSH^3_{N-1}(\overline{D})} \subset \tilde{S}_{PSH^3_{N-1}(\overline{D})} \subset bD. \]

We also mention here the following result obtained by Basener in [Bas78] (see Theorem 5).

**Theorem 8.11** Let $q \in \{0, \ldots, N - 1\}$. Then $\tilde{S}_{A_q(D)}$ is contained in $\overline{S_q(D)}$.

**Remark 8.12** The lack of appropriate gluing techniques for $q$-holomorphic functions does not permit to obtain a converse results, i.e., it remains an open question whether the inclusion $\tilde{S}_{A_q(D)} \supset S_q(D)$ is also true.
As in the convex case (see Theorem 6.21) we can find a complex foliation in some parts of the Shilov boundary for \( q \)-plurisubharmonic functions on smoothly bounded domains. For further results on complex foliations of real submanifolds we refer to [Fre74].

**Theorem 8.13** Let \( q \) be an integer from \( \{1, \ldots, N - 1\} \) and let \( D \) be a bounded pseudo-convex domain in \( \mathbb{C}^N \) with \( C^2 \)-smooth boundary. Then the open part
\[
\mathcal{F}_q(D) := \text{int}_{bD} \left( \overline{S_{PSH_q(D)}} \setminus \overline{S_{PSH_{q-1}(D)}} \right)
\]
of the Shilov boundary for \( PSH_q(D) \) in \( bD \) locally admits a foliation by complex \( q \)-dimensional submanifolds, provided it is not empty.

**Proof.** By Theorem 8.9 we have that
\[
\mathcal{F}_q(D) = \text{int}_{bD} \left( S_q(D) \setminus S_{q-1}(D) \right) = S_q(D) \setminus S_{q-1}(D). \tag{4}
\]
Given a defining function \( q \) of \( D \), by the definition of the set \( S_q(D) \), by pseudoconvexity of \( D \) and by the identities (4) above, for each point \( p \in \mathcal{F}_q(D) \) the complex Hessian \( L \) of \( q \) at \( p \) has exactly \( N - q - 1 \) positive and \( q \) zero eigenvalues on the holomorphic tangent space \( H_p bD \) to \( bD \) at \( p \). Then, by Theorem 1.1 in [Fre74], the set \( \mathcal{F}_q(D) \) locally admits a foliation by complex \( q \)-dimensional submanifolds. \( \square \)

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