A unified differential equation solver approach for separable convex optimization: splitting, acceleration and nonergodic rate *

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Abstract
This paper provides a self-contained ordinary differential equation solver approach for separable convex optimization problems. A novel primal-dual dynamical system with built-in time rescaling factors is introduced, and the exponential decay of a tailored Lyapunov function is established. Then several time discretizations of the continuous model are considered and analyzed via a unified discrete Lyapunov function. Moreover, two families of accelerated proximal alternating direction methods of multipliers are obtained, and nonergodic optimal mixed-type convergence rates shall be proved for the primal objective residual, the feasibility violation and the Lagrangian gap. Finally, numerical experiments are provided to validate the practical performances.

Keywords: Separable convex optimization, linear constraint, dynamical system, exponential decay, primal-dual method, acceleration, splitting, linearization, nonergodic rate

1 Introduction
Consider the separable convex optimization problem:
\[
\min_{x \in \mathcal{X}, y \in \mathcal{Y}} F(x, y) := f(x) + g(y) \quad \text{s.t.} \quad Ax + By = b, \tag{1}
\]
where \(\mathcal{X} \subset \mathbb{R}^m\) and \(\mathcal{Y} \subset \mathbb{R}^n\) are two closed convex sets, \(A \in \mathbb{R}^{r \times m}\) and \(B \in \mathbb{R}^{r \times n}\) are linear operators, \(b \in \mathbb{R}^r\) is a given vector, and \(f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\) and \(g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) are two properly closed convex functions. We are mainly interested in first-order primal-dual methods for (1) based on the Lagrange function
\[
\mathcal{L}(x, y, \lambda) := F(x, y) + \delta_{\mathcal{X} \times \mathcal{Y}}(x, y) + \langle \lambda, Ax + By - b \rangle, \tag{2}
\]
where \((x, y, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^r\) and \(\delta_{\mathcal{X} \times \mathcal{Y}}\) denotes the indicator function of \(\mathcal{X} \times \mathcal{Y}\). Throughout, assume \((x^*, y^*, \lambda^*) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^r\) is a saddle-point of \(\mathcal{L}\), which means
\[
\mathcal{L}(x^*, y^*, \lambda^*) \leq \mathcal{L}(x, y^*, \lambda^*) \leq \mathcal{L}(x^*, y, \lambda^*) \quad \forall (x, y, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^r.
\]

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Then \((x^*, y^*)\) is a solution to (1) and for simplicity we set \(F^* = F(x^*, y^*).\)

In this work, we propose new accelerated primal-dual splitting methods for the separable optimization problem (1) via a unified differential equation solver approach. To be more specific, we shall first introduce a novel continuous dynamical system

\[
\begin{align*}
    x' &= v - x, \\
    \gamma v' &\in \mu_f(x - v) - \partial_x \mathcal{L}(x, y, \lambda), \\
    \theta \lambda' &= \nabla_\lambda \mathcal{L}(v, w, \lambda), \\
    \beta w' &\in \mu_g(y - w) - \partial_y \mathcal{L}(x, y, \lambda), \\
    y' &= w - y,
\end{align*}
\]

where \(\partial_x \mathcal{L}\) means the subdifferential (cf. (14)) with respect to \(x = x\) or \(y\), and \(\mu_f, \mu_g \geq 0\) correspond to the strong convexity parameters of \(f\) and \(g\). Moreover, \(\gamma, \beta\) and \(\theta\) are three time scaling factors and governed by \(\gamma' = \mu_f - \gamma, \beta' = \mu_g - \beta\) and \(\theta' = -\theta\), respectively. We equip (3) with a tailored Lyapunov function

\[
E(t) = \mathcal{L}(x, y, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda) + \frac{\gamma}{2} \|v - x^*\|^2 + \frac{\beta}{2} \|w - y^*\|^2 + \frac{\theta}{2} \|\lambda - \lambda^*\|^2,
\]

and establish the exponential decay \(E(t) \leq E(0)e^{-t}\) under the assumption that both \(f\) and \(g\) have Lipschitzian gradients. Note that our previous accelerated primal-dual flow model in [56, Section 2] can also be applied to the separable case (1) but it treats \((x, y)\) as an entire variable and only involves a single time scaling parameter for \(F\). However, the current one (3) adopts different scaling factors \(\gamma\) and \(\beta\) respectively for \(f\) and \(g\). This not only allows us to handle the partially strongly convex case \(\mu_f + \mu_g > 0\) but also paves the way for designing new primal-dual splitting algorithms.

Indeed, based on proper numerical discretizations of the continuous model (3), we propose several families of accelerated primal-dual splitting methods for the original optimization problem (1). Using a discrete analogue of (4), we establish the corresponding nonergodic, mixed-type and optimal convergence rates for the quantity

\[
\mathcal{L}(x_k, y_k, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda_k) + |F(x_k, y_k) - F^*| + \|Ax_k + By_k - b\|.
\]

Here, we note that (i) “nonergodic” means the estimate is proved for the last iterate \((x_k, y_k, \lambda_k)\) instead of its historic average (cf. (10)); (ii) “mixed-type” says the decay rate provides explicit dependence on \(\|A\|, \|B\|, \mu_f, \mu_g\) and the Lipschitz constant of \(\nabla f\) (and/or \(\nabla g\)) (cf. (9)); (iii) by “optimal” we mean the iteration complexities achieve the lower bounds of first-order primal-dual methods for problem (1); see [50, 66, 86, 90].

1.1 Outline

The rest of this paper is organized as follows. In the introduction part, we shall complete the literature review of existing methods for (1). Then in Section 2, we introduce our continuous model and establish the exponential decay of the Lyapunov function (2) under the smooth setting. After that, we propose two classes of methods in Sections 3 and 4, respectively, and prove nonergodic mixed-type convergence rates via a unified discrete Lyapunov function. Finally, we provide several numerical experiments in Section 5 and give some concluding remarks and discussions in Section 6.

1.2 Brief review of the one block case

Let us start with one-block setting:

\[
\min_{x \in \mathcal{X}} f(x) \quad \text{s.t.} \quad Ax = b.
\]

The augmented Lagrangian method (ALM) reads as [68]

\[
x_{k+1} = \arg\min_{x \in \mathcal{X}} \left\{ \mathcal{L}(x, \lambda_k) + \sigma \|Ax - b\|^2 \right\}, \quad \lambda_{k+1} = \lambda_k + \sigma (Ax_{k+1} - b),
\]

where \(\mathcal{L}(x, \lambda) = f(x) + g(x) + \langle \lambda, Ax - b \rangle\).
where \( \sigma > 0 \) denotes the penalty parameter and \( \mathcal{L}(x, \lambda) \) is defined by (2) without \( y, g \) and \( B \). Combining Nesterov’s extrapolation technique [62, 82], ALM can be further accelerated, and faster rate \( O(1/k^2) \) for the dual objective residual has been proved in [43, 45, 47, 48, 74]. The accelerated linearized ALM in [87] and some quadratic penalty methods [49, 81] can achieve the nonergodic rates \( O(1/k) \) and \( O(1/k^2) \) respectively for \( \mu_f = 0 \) and \( \mu_f > 0 \), in terms of the primal objective residual \( |f(x_k) - f(x^*)| \) and the feasibility violation \( \|Ax_k - b\| \). Moreover, primal-dual methods in [17, 58, 63, 88] possess optimal mixed-type convergence rates.

1.3 State-of-the-art methods for two-block case

When applied to (1), the classical ALM (7) has to minimize the augmented Lagrangian

\[
\mathcal{L}_\sigma(x, y, \lambda) := \mathcal{L}(x, y, \lambda) + \frac{\sigma}{2} \|Ax + By - b\|^2,
\]

which is not separable for any \( \sigma > 0 \). Hence, the original ALM (7) is further relaxed as the alternating direction method of multipliers (ADMM) [29]

\[
\begin{align*}
    x_{k+1} &= \argmin_{x \in \mathcal{X}} \mathcal{L}_\sigma(x, y_k, \lambda_k), \\
    y_{k+1} &= \argmin_{y \in \mathcal{Y}} \mathcal{L}_\sigma(x_{k+1}, y, \lambda_k), \\
    \lambda_{k+1} &= \lambda_k + \sigma(Ax_{k+1} + By_{k+1} - b),
\end{align*}
\]

which minimizes \( \mathcal{L}_\sigma(\cdot, \cdot, \lambda) \) with respect to \( x \) and \( y \) successively (like the Gauss-Seidel iteration).

So far, there are vast variants of ADMM, with proximal preconditioning [26, 44], symmetrization [39, 52], over-relaxation [22, 23] and parallelization [16, 33]. As showed in [22, 28], the Douglas–Rachford splitting [21] and the Peaceman–Rachford splitting [67] lead to equivalent forms of ADMM for solving the dual problem of (1). The primal–dual hybrid gradient framework [9, 10, 24, 42, 93] for bilinear saddle-point problems provides linearized versions of ADMM. Besides, accelerated ADMM with extrapolation can be found in [30, 31, 65].

The convergence rates \( O(1/k) \) and \( O(1/k^2) \) of ADMM and its variants have been proved in [20, 31, 44, 60, 71, 75, 87]. In [65], Ouyang et al. proposed an accelerated linearized ADMM and established the mixed-type convergence rate

\[
O\left(\frac{L_f}{k^2} + \frac{\|A\|}{k}\right),
\]

where \( L_f \) denotes the Lipschitz constant of \( \nabla f \). Although this yields the final \( O(1/k) \) rate, it does make sense because the dependence on \( L_f \) and \( \|A\| \) is optimal [66]. However, we mention that most existing works provide only ergodic convergence rates. In other words, the error is not measured at the last iterate \( X_k = (x_k, y_k, \lambda_k) \) but its average \( \bar{X}_k = (\bar{x}_k, \bar{y}_k, \lambda_k) \) (cf. [50, Definition 1]):

\[
\bar{X}_k = \frac{1}{k} \sum_{i=1}^{k} a_i X_i, \quad \text{with} \quad \sum_{i=1}^{k} a_i = 1, \quad a_i > 0.
\]

As mentioned in [50, 80], this might violate some key properties such as sparsity and low-rankness. To achieve nonergodic rates for the primal objective residual \( |F(x_k, y_k) - F^*| \) and the feasibility violation \( \|Ax_k + By_k - b\| \), Li and Lin [50] and Tran-Dinh et al. [76, 80, 81, 78] proposed new accelerated ADMM. It should be noticed that, in each iteration, the methods of Tran-Dinh et al. require one more proximal calculation than ADMM, and the final rates become ergodic if the extra proximal step is replaced by averaging.

Recently, we were aware of the works of Sabach and Teboulle [69] and Zhang et al. [91]. Both two proposed accelerated ADMM, and their ingredients are the so-called primal algorithmic map and the prediction-correction framework [44], which are different from our differential equation solver approach. They also established nonergodic rates \( O(1/k) \) and \( O(1/k^2) \) respectively for convex and (partially) strongly convex objectives, but have not derived delicate mixed-type estimates.
1.4 Dynamical system approach

As we can see, continuous dynamical system approaches [2, 3, 15, 12, 13, 46, 53, 55, 59, 72, 73, 83, 84, 85] for the unconstrained convex optimization have been extended to linearly constrained problems. Zeng et al. [89] generalized the continuous-time model of Nesterov accelerated gradient method [62] derived by Su et al. [73] to the one block case (6), and established the decay rate $O(1/t^2)$ via a new Lyapunov function. Further extensions with Bregman divergence and perturbation are given in [36, 92]. Based on time discretizations of proper continuous models, Luo [56, 57, 58], He et al. [37, 38] and Boţ et al. [7] proposed the corresponding accelerated ALM with nonergodic rate $O(1/k^2)$, and Chen and Wei [14] obtained linear convergence without strong convexity assumption.

Continuous dynamical systems for the separable problem (1) can be found in [1, 5, 6, 27, 35], and for general saddle-point systems, we refer to [18, 54, 70]. However, it is rare to see new primal-dual splitting algorithms with provable nonergodic convergence rates based on dynamical models.

2 Continuous Dynamical Systems

2.1 Preliminaries

Let $(\cdot, \cdot)$ and $\|\cdot\|$ be the usual inner product and the Euclidean norm, respectively. For any properly closed convex function $f$ on $\mathcal{X}$, we say $f \in S^0_\mu(\mathcal{X})$ with $\mu \geq 0$ if

$$f(x_1) - f(x_2) - \langle p, x_1 - x_2 \rangle \geq \frac{\mu}{2} \|x_1 - x_2\|^2 \quad \forall (x_1, x_2) \in \mathcal{X} \times \mathcal{X},$$

(11)

where $p \in \partial f(x_2)$ with $\partial f(x_2)$ denoting the subdifferential of $f$ at $x_2 \in \mathcal{X}$. We write $f \in S^{1,1}_{\mu,L}(\mathcal{X})$ if $f \in S^0_\mu(\mathcal{X})$ has $L$-Lipschitz continuous gradient:

$$f(x_1) - f(x_2) - \langle \nabla f(x_2), x_1 - x_2 \rangle \leq \frac{L}{2} \|x_1 - x_2\|^2 \quad \forall (x_1, x_2) \in \mathcal{X} \times \mathcal{X}.$$  

(12)

The function classes $S^0_\mu(\mathcal{Y})$ and $S^{1,1}_{\mu,L}(\mathcal{Y})$ are defined analogously. When $\mathcal{X}(\mathcal{Y})$ becomes the entire space $\mathbb{R}^m(\mathbb{R}^n)$, it shall be omitted for simplicity.

Clearly, if $f \in S^0_{\mu_f}(\mathcal{X})$ and $g \in S^0_{\mu_g}(\mathcal{Y})$ with $\mu_f, \mu_g \geq 0$, then for all $(x_1, y_1, \lambda)$ and $(x_2, y_2, \lambda) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^r$, we have

$$\frac{\mu_f}{2} \|x_1 - x_2\|^2 + \frac{\mu_g}{2} \|y_1 - y_2\|^2 \leq \mathcal{L}(x_1, y_1, \lambda) - \mathcal{L}(x_2, y_2, \lambda) - \langle p, x_1 - x_2 \rangle - \langle q, y_1 - y_2 \rangle,$$

(13)

where $p \in \partial_x \mathcal{L}(x_2, y_2, \lambda)$ and $q \in \partial_y \mathcal{L}(x_2, y_2, \lambda)$. Above and in what follows, we set

$$\partial_x \mathcal{L}(x, y, \lambda) := \partial f(x) + A^T \lambda + N_X(x), \quad \partial_y \mathcal{L}(x, y, \lambda) := \partial g(y) + B^T \lambda + N_Y(y),$$

(14)

with $N_X(x)$ and $N_Y(y)$ being the norm cone of $\mathcal{X}$ and $\mathcal{Y}$ at $x$ and $y$, respectively.

We also introduce the notation $M \lesssim N$, which means $M \leq CN$ with some generic bounded constant $C > 0$ that is independent of $A, B, \mu_f, \mu_g, \gamma_0$ and $\beta_0$ (the initial conditions to (16)) but can be different in each occurrence.

2.2 Continuous-time model and exponential decay

Motivated by the accelerated primal-dual flow [56], we consider the following differential inclusion

$$\begin{cases}
0 \in \gamma x'' + (\gamma + \mu_f) x' + \partial_x \mathcal{L}(x, y, \lambda), \\
0 = \theta x' - \nabla x \mathcal{L}(x + x', y + y', \lambda), \\
0 \in \beta y'' + (\beta + \mu_g) y' + \partial_y \mathcal{L}(x, y, \lambda),
\end{cases}$$

(15)
where the parameters \((\theta, \gamma, \beta)\) are governed by
\[
\theta' = -\theta, \quad \gamma' = \mu_f - \gamma, \quad \beta' = \mu_g - \beta, \tag{16}
\]
with positive initial conditions: \(\theta(0) = 1, \gamma(0) = \gamma_0 > 0\) and \(\beta(0) = \beta_0 > 0\). Here, the new model (15) utilizes the separable structure of (1) and adopts different rescaling factors for \(x\) and \(y\), respectively.

It is not hard to obtain the exact solution of (16):
\[
\theta(t) = e^{-t}, \quad \gamma(t) = \gamma_0 e^{-t} + \mu_f (1 - e^{-t}), \quad \beta(t) = \beta_0 e^{-t} + \mu_g (1 - e^{-t}).
\]

Besides, as introduced in (3), an alternative presentation of (15) reads as
\[
\begin{align*}
x' &= x - v, \tag{17a} \\
\gamma v' &= \mu_f (x - v) - \partial_y \mathcal{L}(x, y, \lambda), \tag{17b} \\
\theta \lambda' &= \nabla \lambda \mathcal{L}(v, w, \lambda), \tag{17c} \\
\beta w' &= \mu_g (x - v) - \partial_y \mathcal{L}(x, y, \lambda), \tag{17d} \\
y' &= w - y. \tag{17e}
\end{align*}
\]

This seems a little bit complicated but for algorithm designing and convergence analysis, it is more convenient for us to start form (17) and treat \((\theta, \gamma, \beta)\) as unknowns that solve (16).

Let \(\Theta = (\theta, \gamma, \beta)\) and \(X = (x, y, v, w, \lambda)\) and define a Lyapunov function
\[
\mathcal{E}(\Theta, X) := \mathcal{L}(x, y, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda) + \frac{\theta}{2} \| \lambda - \lambda^* \|^2 + \frac{\beta}{2} \| w - y^* \|^2 + \frac{\gamma}{2} \| v - x^* \|^2, \tag{18}
\]
where \((x^*, y^*, \lambda^*)\) is a saddle-point of (2). We aim to establish the decay rate of \(\mathcal{E}\), by taking derivative with respect to \(t\). However, we have to mention that (i) solution existence of (17) (or (15)) in proper sense has not been given and (ii) smoothness property of the solution is also unknown. We set those aspects aside as they are beyond the scope of this work. For more discussions, we refer to Section 6.1.

To show the usefulness of our model, we prove the exponential decay under the smooth assumption: \(f \in \mathcal{S}^{1,1}_{\mu_f, L_f}\) and \(g \in \mathcal{S}^{1,1}_{\mu_g, L_g}\). In this setting, the differential inclusion (17) becomes a standard first-order dynamical system, with subgradients being Lipschitzian gradients, and it is not hard to conclude the well-posedness of a classical C\(^1\) solution by standard theory of ordinary differential equations.

**Theorem 2.1.** Assume \(f \in \mathcal{S}^{1,1}_{\mu_f, L_f}\) and \(g \in \mathcal{S}^{1,1}_{\mu_g, L_g}\) with \(\mu_f, \mu_g \geq 0\). Let \(\Theta = (\theta, \gamma, \beta)\) solve (16) and \(X = (x, y, v, w, \lambda)\) be the unique C\(^1\) solution to (17), then it holds that
\[
\frac{d}{dt} \mathcal{E}(\Theta, X) \leq -\mathcal{E}(\Theta, X) - \frac{\mu_f}{2} \| x' \|^2 - \frac{\mu_g}{2} \| y' \|^2, \tag{19}
\]
which yields the exponential decay
\[
2e^t \mathcal{E}(\Theta(t), X(t)) + \int_0^t e^s \left( \mu_f \| x'(s) \|^2 + \mu_g \| y'(s) \|^2 \right) ds \leq 2 \mathcal{E}(\Theta(0), X(0)), \tag{20}
\]
for all \(0 \leq t < \infty\).

**Proof.** As (20) can be obtained directly from (19), it is sufficient to establish the latter. Let us start from the identity \(\frac{d}{dt} \mathcal{E}(\Theta, X) = \langle \nabla_{\Theta} \mathcal{E}, \Theta' \rangle + \langle \nabla_X \mathcal{E}, X' \rangle\). By (16) and (18), it is trivial that
\[
\langle \nabla_{\Theta} \mathcal{E}, \Theta' \rangle = -\frac{\theta}{2} \| \lambda - \lambda^* \|^2 + \frac{\mu_f - \gamma}{2} \| v - x^* \|^2 + \frac{\mu_g - \beta}{2} \| w - y^* \|^2,
\]
and according to (17), a direct computation gives
\[
\langle \nabla_X \mathcal{E}, X' \rangle = \langle \lambda - \lambda^*, \nabla_\lambda \mathcal{L}(v, w, \lambda) \rangle + \langle v - x, \nabla_x \mathcal{L}(x, y, \lambda^*) \rangle + \langle w - y, \nabla_y \mathcal{L}(x, y, \lambda^*) \rangle + \langle v - x^*, \mu_f (x - v) - \nabla_x \mathcal{L}(x, y, \lambda) \rangle + \langle w - y^*, \mu_g (y - w) - \nabla_y \mathcal{L}(x, y, \lambda) \rangle.
\]
Shifting \( \lambda \) to \( \lambda^* \) yields
\[
- \langle v - x^*, \nabla_x \mathcal{L}(x, y, \lambda) \rangle - \langle w - y^*, \nabla_y \mathcal{L}(x, y, \lambda) \rangle
\]
\[
= - \langle v - x^*, \nabla_x \mathcal{L}(x, y, \lambda^*) \rangle - \langle w - y^*, \nabla_y \mathcal{L}(x, y, \lambda^*) \rangle - \langle \lambda - \lambda^*, Av + Bw - b \rangle,
\]
where we have used the optimality condition \( Ax^* + By^* = b \). It follows from (13) that
\[
\langle \nabla_X \mathcal{E}, X' \rangle = \langle x^* - x, \nabla_x \mathcal{L}(x, y, \lambda^*) \rangle + \langle y^* - y, \nabla_y \mathcal{L}(x, y, \lambda^*) \rangle + \mu_f \langle x - v, v - x^* \rangle + \mu_g \langle y - w, w - y^* \rangle
\]
\[
\leq \mathcal{L}(x^*, y^*, \lambda) - \mathcal{L}(x, y, \lambda^*) - \frac{\mu_f}{2} \|x - x^*\|^2 - \frac{\mu_g}{2} \|y - y^*\|^2
\]
\[
+ \mu_f \langle x - v, v - x^* \rangle + \mu_g \langle y - w, w - y^* \rangle.
\]
(21)

In view of the trivial but useful identity of vectors
\[
2 \langle u - z, z - a \rangle = \|u - a\|^2 - \|z - a\|^2 - \|u - z\|^2 \quad \forall u, z, a,
\]
we rearrange the last two cross terms in (21) and put everything together to get
\[
\frac{d}{dt} \mathcal{E}(\Theta, X) \leq -\mathcal{E}(\Theta, X) - \frac{\mu_f}{2} \|x - v\|^2 - \frac{\mu_g}{2} \|y - w\|^2.
\]

Observing that \( x - v = x' \) and \( y - w = y' \), we obtain (19) and complete the proof.

\[ \blacksquare \]

**Remark 2.1.** Thanks to the three scaling parameters introduced in (16), the exponential decay of the Lyapunov function (18) holds uniformly for \( \mu_f, \mu_g \geq 0 \). In discrete level, it allows us to treat convex and (partially) strongly convex cases in a unified manner and obtain automatically changing parameters by implicit discretization of (16), which is the key for our delicate mixed-type estimates.

**Remark 2.2.** From (20) we have the exponential decay rate of the Lagrangian duality gap:
\[
\mathcal{L}(x(t), y(t), \lambda^*) - \mathcal{L}(x^*, y^*, \lambda(t)) = O(e^{-t}).
\]

Invoking the proofs of [56, Lemma 2.1] and [57, Corollary 2.1], we can further establish
\[
\|Ax(t) + By(t) - b\| + |F(x(t), y(t)) - F^*| = O(e^{-t}).
\]

Moreover, by (13) and the above estimates, we conclude that
\[
\mu_f \|x(t) - x^*\|^2 + \mu_g \|y(t) - y^*\|^2 = O(e^{-t}),
\]
which means strong convergence \( x(t) \to x^* \) (or \( y(t) \to y^* \)) follows if \( \mu_f > 0 \) (or \( \mu_g > 0 \)).

### 3 The First Family of Methods

We now turn to the numerical aspect of our continuous model (17). In view of (14), the ways to discretize \((x, \lambda)\) in (17b) and \((y, \lambda)\) in (17d) are crucial, and \(\lambda\) plays an important role of decoupling \(x\) and \(y\). In this work, we always use the same discretization for \(\lambda\) in (17b) and (17d), and for the case of different choices, we refer to the discussion in Section 6.3.

In this section, we impose the following assumption:

**Assumption 1.** \( f \in \mathcal{S}_{\mu_f}^0(X) \) with \( \mu_f \geq 0 \) and \( g \in \mathcal{S}_{\mu_g}^0(Y) \) with \( \mu_g \geq 0 \).
For this nonsmooth setting, we adopt implicit discretizations \((x_{k+1}, \lambda_{k+1})\) and \((y_{k+1}, \bar{y}_{k+1})\) for (17b) and (17d), where \(\lambda_{k+1}\) is to be determined. That is, given the initial guess \((x_0, v_0, y_0, w_0, \lambda_0)\), consider an implicit discretization for (17):

\[
\begin{align*}
\frac{x_{k+1} - x_k}{\alpha_k} &= v_{k+1} - x_{k+1}, \quad \text{(23a)} \\
v_{k+1} - v_k &= \gamma_k \left( x_{k+1} - v_{k+1} \right) \quad \in \partial f(x_{k+1}, y_{k+1}, \lambda_{k+1}), \quad \text{(23b)} \\
\theta_k \frac{\lambda_{k+1} - \lambda_k}{\alpha_k} &= \nabla \mathcal{L}(v_{k+1}, w_{k+1}, \lambda_{k+1}), \quad \text{(23c)} \\
\beta_k \frac{w_{k+1} - w_k}{\alpha_k} &= \mu_g (y_{k+1} - w_{k+1}) - \partial g(x_{k+1}, y_{k+1}, \bar{\lambda}_{k+1}), \quad \text{(23d)} \\
y_{k+1} - y_k &= w_{k+1} - y_{k+1}, \quad \text{(23e)}
\end{align*}
\]

where \(\alpha_k > 0\) is the step size and the parameter system (16) is discretized implicitly by

\[
\frac{\theta_{k+1} - \theta_k}{\alpha_k} = -\gamma_{k+1} - \frac{\gamma_k}{\alpha_k} = \mu_f - \gamma_{k+1}, \quad \frac{\beta_{k+1} - \beta_k}{\alpha_k} = \mu_g - \beta_{k+1}, \quad \text{(24)}
\]

with initial conditions: \(\theta_0 = 1, \gamma_0 > 0\) and \(\beta_0 > 0\).

Rearrange (23) in the usual primal-dual formulation:

\[
\begin{align*}
v_{k+1} &= x_{k+1} + (x_{k+1} - x_k)/\alpha_k, \quad \text{(25a)} \\
x_{k+1} &= \arg \min_{x \in \mathcal{X}} \left\{ \mathcal{L}(x, y_{k+1}, \bar{\lambda}_{k+1}) + \frac{\eta_{f,k}}{2\alpha_k} \|x - \bar{x}_k\|^2 \right\}, \quad \text{(25b)} \\
\lambda_{k+1} &= \lambda_k + \alpha_k / \theta_k (A v_{k+1} + B w_{k+1} - b), \quad \text{(25c)} \\
y_{k+1} &= \arg \min_{y \in \mathcal{Y}} \left\{ \mathcal{L}(x_{k+1}, y, \bar{\lambda}_{k+1}) + \frac{\eta_{g,k}}{2\alpha_k} \|y - \bar{y}_k\|^2 \right\}, \quad \text{(25d)} \\
w_{k+1} &= y_{k+1} + (y_{k+1} - y_k)/\alpha_k, \quad \text{(25e)}
\end{align*}
\]

where \(\eta_{f,k} := (\alpha_k + 1) \gamma_k + \mu_f \alpha_k\), \(\eta_{g,k} := (\alpha_k + 1) \beta_k + \mu_g \alpha_k\) and

\[
\bar{x}_k := x_k + \frac{\alpha_k \gamma_k}{\eta_{f,k}} (v_k - x_k), \quad \bar{y}_k := y_k + \frac{\alpha_k \beta_k}{\eta_{g,k}} (w_k - y_k). \quad \text{(26)}
\]

Note that (25) is an informal expression since the term \(\bar{\lambda}_{k+1}\) has not been determined yet. It brings hidden augmented terms for (25b) and (25d) with possible linearization and decoupling, and different choices lead to our first family of methods. Specifically, we shall adopt two semi-implicit candidates (39) and (52) and the explicit one (54); see Sections 3.2, 3.3 and 3.4 for more details.

Below, we give a one-iteration analysis for the implicit scheme (23). Then the nonergodic mixed-type convergence rates of our first family of methods can be obtained.

### 3.1 A single-step analysis

For the sequence \(\{(x_k, v_k, y_k, w_k, \lambda_k)\}_{k=0}^\infty\) generated by (23) and the parameter sequence \(\{(\theta_k, \gamma_k, \beta_k)\}_{k=0}^\infty\) defined by (24), we introduce a discrete Lyapunov function

\[
\mathcal{E}_k := \mathcal{L}(x_k, y_k, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda_k) + \frac{\gamma_k}{2} \|v_k - x^*\|^2 + \frac{\beta_k}{2} \|w_k - y^*\|^2 + \frac{\theta_k}{2} \|\lambda_k - \lambda^*\|^2, \quad \text{(27)}
\]

which is the discrete analogue of (18) and will be used for all the forthcoming methods.
Lemma 3.1. Let $k \in \mathbb{N}$ be fixed. For the implicit scheme (23) with Assumption 1 and the step size $\alpha_k > 0$, we have that

$$E_{k+1} - E_k \leq -\alpha_k E_{k+1} + \frac{\theta_k}{2} \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2 - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 - \frac{\beta_k}{2} \|w_{k+1} - w_k\|^2. \quad (28)$$

Proof. Let us calculate the difference $E_{k+1} - E_k = I_1 + I_2 + I_3 + I_4$, where

$$I_1 := \mathcal{L}(x_{k+1}, y_{k+1}, \lambda^*) - \mathcal{L}(x_k, y_k, \lambda^*),$$

$$I_2 := \frac{\theta_k+1}{2} \|\lambda_{k+1} - \lambda^*\|^2 - \frac{\theta_k}{2} \|\lambda_k - \lambda^*\|^2,$$

$$I_3 := \frac{\gamma_k+1}{2} \|v_{k+1} - x^*\|^2 - \frac{\gamma_k}{2} \|v_k - x^*\|^2,$$

$$I_4 := \frac{\beta_k+1}{2} \|w_{k+1} - y^*\|^2 - \frac{\beta_k}{2} \|w_k - y^*\|^2. \quad (29)$$

In what follows, we aim to estimate the above four terms one by one.

In view of (25b) and (25d), it is clear that $(x_{k+1}, y_{k+1}) \in \mathcal{X} \times \mathcal{Y}$. By (23b) and (23d), we have

$$\begin{align*}
p_{k+1} &:= \mu_f(x_{k+1} - v_{k+1}) - \gamma_k \frac{v_{k+1} - v_k}{\alpha_k} \in \partial_x \mathcal{L}(x_{k+1}, y_{k+1}, \tilde{\lambda}_{k+1}), \\
q_{k+1} &:= \mu_g(y_{k+1} - w_{k+1}) - \beta_k \frac{w_{k+1} - w_k}{\alpha_k} \in \partial_y \mathcal{L}(x_{k+1}, y_{k+1}, \tilde{\lambda}_{k+1}). \quad (30)
\end{align*}$$

Thanks to the inequality (13), it follows that

$$I_1 = \mathcal{L}(x_{k+1}, y_{k+1}, \tilde{\lambda}_{k+1}) - \mathcal{L}(x_k, y_k, \tilde{\lambda}_{k+1})$$

$$+ \langle \lambda^* - \tilde{\lambda}_{k+1}, A(x_{k+1} - x_k) + B(y_{k+1} - y_k) \rangle$$

$$\leq \langle p_{k+1}, x_{k+1} - x_k \rangle + \langle q_{k+1}, y_{k+1} - y_k \rangle + \langle \lambda^* - \tilde{\lambda}_{k+1}, A(x_{k+1} - x_k) + B(y_{k+1} - y_k) \rangle. \quad (32)$$

By the equation of the sequence $\{\theta_k\}_{k=0}^\infty$ in (24), there holds

$$I_2 = \frac{\theta_k+1 - \theta_k}{2} \|\lambda_{k+1} - \lambda^*\|^2 + \frac{\theta_k}{2} \left(\|\lambda_{k+1} - \lambda^*\|^2 - \|\lambda_k - \lambda^*\|^2\right)$$

$$= -\frac{\alpha_k \theta_k+1}{2} \|\lambda_{k+1} - \lambda^*\|^2 + \theta_k \langle \lambda_{k+1} - \lambda_k, \lambda_{k+1} - \lambda^* \rangle - \frac{\theta_k}{2} \|\lambda_{k+1} - \lambda_k\|^2.$$  

To match the term $\tilde{\lambda}_{k+1}$ in (32), we use (23c) to rewrite the last two terms

$$\begin{align*}
\theta_k &\langle \lambda_{k+1} - \lambda_k, \lambda_{k+1} - \lambda^* \rangle - \frac{\theta_k}{2} \|\lambda_{k+1} - \lambda_k\|^2 \\
&= \theta_k \langle \lambda_{k+1} - \lambda_k, \lambda_{k+1} - \tilde{\lambda}_{k+1} + \tilde{\lambda}_{k+1} - \lambda^* \rangle - \frac{\theta_k}{2} \|\lambda_{k+1} - \lambda_k\|^2 \\
&= \alpha_k \langle Av_{k+1} + Bw_{k+1} - b, \tilde{\lambda}_{k+1} - \lambda^* \rangle + \frac{\theta_k}{2} \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2 - \frac{\theta_k}{2} \|\lambda_k - \tilde{\lambda}_{k+1}\|^2.
\end{align*}$$

This implies the estimate

$$I_2 \leq \alpha_k \langle Av_{k+1} + Bw_{k+1} - b, \tilde{\lambda}_{k+1} - \lambda^* \rangle + \frac{\theta_k}{2} \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2 - \frac{\alpha_k \theta_k+1}{2} \|\lambda_{k+1} - \lambda^*\|^2. \quad (33)$$

Similarly, using the equation of $\{\gamma_k\}_{k=0}^\infty$ in (24), we have

$$I_3 = \frac{\gamma_k+1 - \gamma_k}{2} \|v_{k+1} - x^*\|^2 + \frac{\gamma_k}{2} \left(\|v_{k+1} - x^*\|^2 - \|v_k - x^*\|^2\right)$$

$$= \frac{\alpha_k (\mu_f - \gamma_k+1)}{2} \|v_{k+1} - x^*\|^2 - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 + \gamma_k \langle v_{k+1} - v_k, v_{k+1} - x^* \rangle. \quad (34)$$
In view of (30), we rewrite the last cross term by that

$$\gamma_k \langle v_{k+1} - v_k, v_{k+1} - x^* \rangle = \mu_f \alpha_k \langle x_{k+1} - v_{k+1}, v_{k+1} - x^* \rangle - \alpha_k \langle p_{k+1}, v_{k+1} - x^* \rangle.$$  

Using (22) and (25a) and summarizing the above decompositions yield that

$$\mathbb{I}_3 = -\frac{\alpha_k \gamma_{k+1}}{2} \|v_{k+1} - x^*\|^2 - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 - \frac{\mu_f \alpha_k}{2} \|x_{k+1} - v_{k+1}\|^2$$

$$+ \frac{\mu_f \alpha_k}{2} \|x_{k+1} - x^*\|^2 - \alpha_k \langle p_{k+1}, x_{k+1} - x^* \rangle - \langle p_{k+1}, x_{k+1} - x_k \rangle. \quad (35)$$

Analogously, by (24), (31) and (25e), we have

$$\mathbb{I}_4 = -\frac{\alpha_k \beta_{k+1}}{2} \|w_{k+1} - y^*\|^2 - \frac{\beta_k}{2} \|w_{k+1} - w_k\|^2 - \frac{\mu_g \alpha_k}{2} \|y_{k+1} - w_{k+1}\|^2$$

$$+ \frac{\mu_g \alpha_k}{2} \|y_{k+1} - y^*\|^2 - \alpha_k \langle q_{k+1}, y_{k+1} - y^* \rangle - \langle q_{k+1}, y_{k+1} - y_k \rangle. \quad (36)$$

Now, collecting (33), (35), (36) and (13), we arrive at the upper bound

$$\mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 \leq -\alpha_k \tilde{\mathcal{E}}_{k+1} + \frac{\theta_k}{2} \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2 - \frac{\beta_k}{2} \|w_{k+1} - w_k\|^2$$

$$- \langle p_{k+1}, x_{k+1} - x_k \rangle - \langle q_{k+1}, y_{k+1} - y_k \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2$$

$$- \langle \lambda^* - \tilde{\lambda}_{k+1}, A(x_{k+1} - x_k) + B(y_{k+1} - y_k) \rangle.$$

Plugging (32) into the above estimate gives

$$\tilde{\mathcal{E}}_{k+1} - \mathcal{E}_k \leq -\alpha_k \tilde{\mathcal{E}}_{k+1} + \frac{\theta_k}{2} \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2 - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 - \frac{\beta_k}{2} \|w_{k+1} - w_k\|^2.$$  

This establishes (28) and finishes the proof of this lemma.  

### 3.2 A semi-implicit choice

According to the single step estimate (28), it is evident that the implicit choice $\tilde{\lambda}_{k+1} = \lambda_{k+1}$ indicates the contraction

$$\tilde{\mathcal{E}}_{k+1} - \mathcal{E}_k \leq -\alpha_k \tilde{\mathcal{E}}_{k+1}, \quad (37)$$

which holds for any $\alpha_k > 0$, even for $\mu_f = \mu_g = 0$. This together with the fact (cf.(24))

$$\theta_{k+1} = \frac{\theta_k}{1 + \alpha_k} \quad \Rightarrow \quad \theta_k = \prod_{i=0}^{k-1} \frac{1}{1 + \alpha_i} \quad (38)$$

implies that $\mathcal{E}_k \leq \theta_k \mathcal{E}_0$ and the linear rate $\theta_k \leq (1 + \alpha_{\text{min}})^{-k}$ follows immediately if $\alpha_k \geq \alpha_{\text{min}} > 0$. But this does not lead to a splitting method since by (25c), $\lambda_{k+1}$ depends on $v_{k+1}$ and $w_{k+1}$. In other words, $x_{k+1}$ and $y_{k+1}$ are coupled with each other; see Section 6.2 for more discussions.

Hence, let us consider other semi-implicit choices that decouple $x_{k+1}$ and $y_{k+1}$. Recall again the estimate (28), which says if we want to maintain the contraction property (37), then the positive gain $\|\lambda_{k+1} - \tilde{\lambda}_{k+1}\|^2$ shall be controlled by additional two negative square norm terms $-\|v_{k+1} - v_k\|^2$ and $-\|w_{k+1} - w_k\|^2$. To do this, the relation

$$\|\lambda_{k+1} - \tilde{\lambda}_{k+1}\| = O(\|w_{k+1} - w_k\|) \quad \text{or} \quad \|\lambda_{k+1} - \tilde{\lambda}_{k+1}\| = O(\|v_{k+1} - v_k\|)$$

is important to be satisfied. In view of (25c), we are suggested to consider

$$\tilde{\lambda}_{k+1} = \lambda_k + \alpha_k / \theta_k (Av_{k+1} + Bw_k - b), \quad (39)$$
which gives the desired identity
\[ \lambda_{k+1} - \bar{\lambda}_{k+1} = \alpha_k / \theta_k B(w_{k+1} - w_k). \] (40)

Then by (28), the contraction (37) follows directly, provided that
\[ \frac{\theta_k}{2} \left\| \frac{1}{\lambda_{k+1}} - \bar{\lambda}_{k+1} \right\|^2 = \frac{\alpha_k^2}{2\theta_k} \left\| B(w_{k+1} - w_k) \right\|^2 \leq \frac{\beta_k}{2} \left\| w_{k+1} - w_k \right\|^2, \] (41)

which can be easily promised if \( \alpha_k^2 \| B \|^2 \leq \theta_k \beta_k \).

For \( \tau > 0 \), introduce the proximal operator of \( g \) by that
\[ \text{prox}_{\tau g}(z) := \arg\min_{y \in \mathcal{Y}} \left\{ g(y) + \frac{1}{2\tau} \| y - z \|^2 \right\} \quad \forall z \in \mathbb{R}^n. \] (42)

With the choice (39), we reformulate (25) as the following iteration

\[
\begin{align*}
\bar{\lambda}_k &= \lambda_k - \theta_k^{-1}(Ax_k + By_k - b) + \alpha_k / \theta_k B(w_k - y_k), \\
x_{k+1} &= \arg\min_{x \in \mathcal{X}} \left\{ \mathcal{L}_k(x, y_k, \bar{\lambda}_k) + \frac{\eta f_k}{2\theta_k} \| x - \bar{x}_k \|^2 \right\}, \quad \sigma_k = 1 / \theta_{k+1}, \\
v_{k+1} &= x_{k+1} + (x_{k+1} - x_k) / \alpha_k, \\
\hat{\lambda}_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_{k+1} + Bw_k - b), \quad \tau_k = \frac{\alpha_k}{\eta b_k}, \\
y_{k+1} &= \text{prox}_{\tau_k}^g(\bar{y}_k - \tau_k B^T \hat{\lambda}_{k+1}), \\
w_{k+1} &= y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
\lambda_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_{k+1} + Bw_{k+1} - b),
\end{align*}
\] (43)

where \((\bar{x}_k, \bar{y}_k)\) are defined by (26) and \((\eta f_k, \eta b_k)\) are the same as that in (25). As \( \bar{\lambda}_{k+1} \) depends only on \( v_{k+1} \) and \( w_k \), we see that (i) \( x_{k+1} \) and \( y_{k+1} \) are weakly coupled with each other, in the sense that they can be updated sequentially; (ii) the augmented term is used for computing \( x_{k+1} \) but it has been linearized for updating \( y_{k+1} \), which involves only the proximal calculation of \( g \).

For simplicity, in the rest of this paper, we set
\[ \gamma_0 = \mu_f \text{ for } \mu_f > 0, \quad \text{and } \beta_0 = \mu_g \text{ for } \mu_g > 0. \] (44)

Then by (24), both \( \{\gamma_k\}_{k=0}^\infty \) and \( \{\beta_k\}_{k=0}^\infty \) are decreasing, and it is clear that \( \mu_f \leq \gamma_k \leq \gamma_0 \) and \( \mu_g \leq \beta_k \leq \beta_0 \) for all \( k \in \mathbb{N} \). Moreover, we claim that
\[ \beta_k \geq \theta_k \beta_0, \quad \gamma_k \geq \theta_k \gamma_0. \] (45)

**Theorem 3.1.** If \( \bar{\lambda}_{k+1} \) is chosen from (39), then (25) reduces to (43). Under Assumption 1, the initial setting (44) and the condition \( \alpha_k^2 \| B \|^2 = \theta_k \beta_k \), it holds that \( \mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1} \). Moreover, we have \( \{(x_k, y_k)\}_{k=1}^\infty \subset \mathcal{X} \times \mathcal{Y} \) and
\[
\left\{ \begin{array}{l}
\| Ax_k + By_k - \theta_k R_0, \\
\mathcal{L}(x_k, y_k, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda_k) \leq \theta_k \mathcal{E}_0, \\
| F(x_k, y_k) - F^* | \leq \theta_k (\mathcal{E}_0 + \| \lambda^* \| R_0). 
\end{array} \right. \] (46)

Above, \( R_0 := \sqrt{2\mathcal{E}_0} + \| \lambda_0 - \lambda^* \| + \| A x_0 + B y_0 - b \| \) and
\[ \theta_k \leq \min \left\{ \frac{Q}{Q + \sqrt{\beta_0 k}}, \frac{4Q^2}{(2Q + \sqrt{\mu_g k})^2} \right\}, \] (47)

where \( Q = \| B \| + \sqrt{\beta_0} \).
Proof. Based on the above discussions, to get the contraction \( \mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1} \), we only need to verify (41) under the condition \( \alpha_k^2 \| B \|^2 = \beta_k \theta_k \), which is trivial. By (38), this gives \( \mathcal{E}_k \leq \theta_k \mathcal{E}_0 \) and also implies that
\[
\mathcal{L}(x_k, y_k, \lambda^*) - \mathcal{L}(x^*, y^*, \lambda_k) \leq \theta_k \mathcal{E}_0.
\]
Following the proof of [55, Theorem 3.1], we can establish
\[
\| Ax_k + By_k - b \| \leq \theta_k \mathcal{R}_0, \quad \text{and} \quad \| F(x_k, y_k) - F^* \| \leq \theta_k (\mathcal{E}_0 + \| \lambda^* \| \mathcal{R}_0),
\]
which yields (46).

It remains to verify the decay estimate (47). Since \( \alpha_k^2 \| B \|^2 = \theta_k \beta_k \), we obtain
\[
\alpha_k = \frac{\sqrt{\beta_k \theta_k}}{\| B \|} \leq \frac{\sqrt{\beta_0}}{\| B \|} \implies \frac{\theta_{k+1}}{\theta_k} = \frac{1}{1 + \alpha_k} \geq \frac{\| B \|}{\| B \| + \sqrt{\beta_0}}.
\]
By (24) and (45), it holds that
\[
\theta_{k+1} - \theta_k = -\alpha_k \theta_{k+1} = -\frac{\sqrt{\beta_k \theta_k}}{\| B \|} \leq -\frac{\sqrt{\beta_0}}{\| B \|} \theta_{k+1},
\]
and using Lemma C.1 implies
\[
\theta_k \leq \frac{\| B \| + \sqrt{\beta_0}}{\| B \| + \sqrt{\beta_0} + \sqrt{\beta_0} k}.
\]

On the other hand, since \( \beta_k \geq \mu_g \), (48) becomes
\[
\theta_{k+1} - \theta_k \leq -\frac{\sqrt{\mu_g \theta_{k+1}}}{\| B \|} \quad \text{by Lemma C.1} \implies \theta_k \leq \frac{4(\| B \| + \sqrt{\beta_0})^2}{(2(\| B \| + \sqrt{\beta_0}) + \sqrt{\mu_g} k)^2}.
\]

Note that this estimate and the previous one (49) hold true simultaneously. This yields (47) and concludes the proof of this theorem.

In Theorem 3.1, we have established the same convergence rate for the objective residual and the feasibility violation. For the special case: \( B = -I, b = 0 \), the separable problem (1) is equivalent to the unconstrained composite optimization
\[
\min_{x \in \mathcal{X}} P(x) := f(x) + g(Ax),
\]
where \( P(x) = F(x, Ax) \) and the minimal value is \( P^* = F^* \). As a corollary of Theorem 3.1, we can derive the convergence rate with respect to the composite objective in (50).

**Corollary 3.1.** Assume \( B = -I, b = 0 \) and let \( \{ x_k \}_{k=1} \subset \mathcal{X} \) be generated by (43) under the assumptions of Theorem 3.1. If \( g \) is \( M_g \)-Lipschitz continuous, then
\[
0 \leq P(x_k) - P^* \leq \theta_k (\mathcal{E}_0 + (\| \lambda^* \| + M_g) \mathcal{R}_0),
\]
where \( \theta_k \) satisfies the decay estimate (47).

**Proof.** It follows that
\[
P(x_k) - P^* = F(x_k, Ax_k) - F^* = g(Ax_k) - g(y_k) + F(x_k, y_k) - F^*
\leq |g(Ax_k) - g(y_k)| + |F(x_k, y_k) - F^*|
\leq M_g \| Ax_k - y_k \| + |F(x_k, y_k) - F^*|
\leq \theta_k (\mathcal{E}_0 + (\| \lambda^* \| + M_g) \mathcal{R}_0).
\]
In the last step, we used (46). This concludes the proof.

**Remark 3.1.** Observing the proof of Corollary 3.1, the estimate (51) depends solely on \( \| Ax_k - y_k \| \) and \( |F(x_k, y_k) - F^*| \). Hence, we claim that it holds true for all the rest methods with the corresponding decay rate of \( \theta_k \).
3.3 Another semi-implicit choice

As the roles of \((x, f, A)\) and \((y, g, B)\) are symmetric in (25), the previous choice (39) is also equivalent to

\[
\tilde{\lambda}_{k+1} = \lambda_k + \alpha_k / \theta_k (Av_k + Bw_{k+1} - b),
\]

which leads to

\[
\begin{align*}
\tilde{\lambda}_k &= \lambda_k - \theta_k^{-1} (Ax_k + By_k - b) + \alpha_k / \theta_k A(v_k - x_k), \\
y_{k+1} &= \arg\min_{y \in \mathcal{Y}} \left\{ \mathcal{L}_{\sigma_k}(x_k, y, \tilde{\lambda}_k) + \frac{\eta_{\theta_k}}{2\alpha_k} \| y - \tilde{y}_k \|^2 \right\}, \quad \sigma_k = 1 / \theta_{k+1}, \\
w_{k+1} &= y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
\tilde{\lambda}_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_k + Bw_{k+1} - b), \\
x_{k+1} &= \text{prox}_{\lambda_k/\theta_k}^X (\tilde{x}_k - s_k A^\top \tilde{\lambda}_{k+1}), \quad s_k = \alpha_k^2 / \eta_{f,k}, \\
v_{k+1} &= x_{k+1} + (x_{k+1} - x_k) / \alpha_k, \\
\lambda_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_k + Bw_{k+1} - b),
\end{align*}
\]

where \((\tilde{x}_k, \tilde{y}_k, \eta_{f,k}, \eta_{g,k})\) are the same as in (43) and the proximal operator \(\text{prox}_{\lambda_k/\theta_k}^X\) of \(f\) can be defined similarly as (42).

Below, we state the convergence rate of (53) but omit the detailed proof, which is almost identical to that of Theorem 3.1.

**Theorem 3.2.** Applying the choice (52) to (25) gives (53). In addition, under Assumption 1, the initial setting (44) and the condition \(\alpha_k^2 \| A \|^2 = \gamma_k \theta_k\), we have \(\{(x_k, y_k)\}_{k=1}^\infty \subset \mathcal{X} \times \mathcal{Y}\), and the estimate (46) holds true with

\[
\theta_k \leq \min \left\{ \frac{Q}{Q + \sqrt{\gamma_k}}, \frac{4Q^2}{(2Q + \sqrt{\mu_f k})^2} \right\},
\]

where \(Q = \| A \| + \sqrt{\gamma_0}\).

3.4 The explicit choice

Now, let us consider the explicit one:

\[
\tilde{\lambda}_{k+1} = \lambda_k + \alpha_k / \theta_k (Av_k + Bw_k - b),
\]

which yields the following method

\[
\begin{align*}
\tilde{\lambda}_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_k + Bw_k - b), \\
x_{k+1} &= \text{prox}_{\lambda_k/\theta_k}^X (\tilde{x}_k - s_k A^\top \tilde{\lambda}_{k+1}), \quad s_k = \alpha_k^2 / \eta_{f,k}, \\
v_{k+1} &= x_{k+1} + (x_{k+1} - x_k) / \alpha_k, \\
y_{k+1} &= \text{prox}_{\lambda_k/\theta_k}^Y (\tilde{y}_k - \tau_k B^\top \tilde{\lambda}_{k+1}), \quad \tau_k = \alpha_k^2 / \eta_{g,k}, \\
w_{k+1} &= y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
\lambda_{k+1} &= \lambda_k + \alpha_k / \theta_k (Av_k + Bw_{k+1} - b),
\end{align*}
\]

where \((\tilde{x}_k, \tilde{y}_k, \eta_{f,k}, \eta_{g,k})\) are the same as in (43). Note that (55) is a parallel linearized proximal ADMM since the two proximal steps in (55b) and (55d) are independent.

Recall that \(M \leq N\) means \(M \leq CN\) with some generic bounded constant \(C > 0\) that is independent of \(A, B, \mu_f, \mu_g, \gamma_0\) and \(\beta_0\) but can be different in each occurrence.
Theorem 3.3. Applying the explicit choice (54) to (25) leads to (55). Under Assumption 1, the initial setting (44) and the condition

$$2\alpha_k^2 (\beta_k \|A\|^2 + \gamma_k \|B\|^2) = \gamma_k \beta_k \theta_k,$$

we have \(\{(x_k, y_k)\}^\infty_{k=1} \subset X \times Y\) and \(\mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1}\). Moreover, if \(\gamma_0 \beta_0 \leq 2\beta_0 \|A\|^2 + 2\gamma_0 \|B\|^2\), then the estimate (46) holds true with

$$\theta_k \lesssim \min \left\{ \frac{\|A\|}{\sqrt{\gamma_0 k}}, \frac{\|A\|^2}{\mu_f k^2} \right\} + \min \left\{ \frac{\|B\|}{\sqrt{\beta_0 k}}, \frac{\|B\|^2}{\mu_g k^2} \right\}. \quad (57)$$

Proof. By (54) and (55f), we have

$$\lambda_{k+1} - \tilde{\lambda}_{k+1} = \alpha_k/\theta_k A (v_{k+1} - v_k) + \alpha_k/\theta_k B (w_{k+1} - w_k). \quad (58)$$

Taking this into the one-iteration estimate (28) gives

$$\mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1} + \frac{2\alpha_k^2 \|A\|^2 - \gamma_k \theta_k}{2\theta_k} \|v_{k+1} - v_k\|^2 + \frac{2\alpha_k^2 \|B\|^2 - \beta_k \theta_k}{2\theta_k} \|w_{k+1} - w_k\|^2,$$

and invoking the relation (56), we obtain the contraction \(\mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1}\). By using the proof of Theorem 3.1, the estimate (46) can still be verified.

Let us prove the mixed-type estimate (57). Since \(\beta_k \leq \beta_0, \gamma_k \leq \gamma_0\) and \(\theta_k \leq 1\), by (56), we have

$$\alpha_k \leq \frac{\sqrt{\gamma_0 \beta_0}}{\sqrt{2\beta_0 \|A\|^2 + 2\gamma_0 \|B\|^2}} := \sigma \leq 1,$$

and it follows that \(\theta_{k+1}/\theta_k = 1/(1 + \alpha_k) \geq 1/2\). Analogously to (48), one has

$$\theta_k \lesssim \theta_k \theta_{k+1} \leq -\sigma \theta_k \theta_{k+1} \quad \text{by Lemma C.1} \implies \theta_k \lesssim \frac{\|A\|}{\sqrt{\gamma_0 k}} + \frac{\|B\|}{\sqrt{\beta_0 k}}. \quad (59)$$

On the other hand, as \(\beta_k \geq \mu_g\), we obtain

$$\theta_k \lesssim \frac{\sqrt{\gamma_0 \mu_g \theta_k \theta_{k+1}}}{2\mu_g \|A\|^2 + 2\gamma_0 \|B\|^2} \quad \text{by Lemma C.2} \implies \theta_k \lesssim \frac{\|A\|}{\sqrt{\gamma_0 k}} + \frac{\|B\|^2}{\mu_g k^2}.$$

Consequently, we have

$$\theta_k \lesssim \frac{\|A\|}{\sqrt{\gamma_0 k}} + \min \left\{ \frac{\|B\|}{\sqrt{\beta_0 k}}, \frac{\|B\|^2}{\mu_g k^2} \right\}. \quad (59)$$

Moreover, since \(\gamma_k \geq \mu_f\), repeating the above discussions and using Lemmas C.1 and C.2, we conclude that

$$\theta_k \lesssim \frac{\|A\|^2}{\mu_f k^2} + \min \left\{ \frac{\|B\|}{\sqrt{\gamma_0 k}}, \frac{\|B\|^2}{\mu_g k^2} \right\}. \quad (59)$$

Therefore, combining this with (59) leads to (57) and completes the proof. ■

Remark 3.2. Our method (55) is close to the parallel type ADMM, such as the predictor corrector proximal multipliers (PCPM) method [16], the proximal-center based decomposition method (PCBDM) [61] and the decomposition algorithms in [79]. However, it is rare to see mixed-type estimates like (57), especially for partially strongly convex case \(\mu_f + \mu_g > 0\).
4 The Second Family of Methods

We then focus on the second class of primal-dual splitting methods that apply semi-implicit and implicit discretizations to \( x \) and \( y \), separately, and also consider different discretizations for \( \lambda \) as before.

To do this, let us start from the following scheme

\[
\begin{align*}
\frac{x_{k+1} - x_k}{\alpha_k} &= u_k - x_{k+1}, \\
\gamma_k \frac{v_{k+1} - v_k}{\alpha_k} &\in \mu_f(x_{k+1} - v_{k+1}) - \partial_x \mathcal{L}(x_{k+1}, y_{k+1}, \bar{\lambda}_{k+1}), \\
\theta_k \frac{\lambda_{k+1} - \bar{\lambda}_k}{\alpha_k} &= \nabla_{\bar{\lambda}} \mathcal{L}(v_{k+1}, w_{k+1}, \lambda_{k+1}), \\
\beta_k \frac{w_{k+1} - w_k}{\alpha_k} &\in \mu_y(y_{k+1} - w_{k+1}) - \partial_y \mathcal{L}(x_{k+1}, y_{k+1}, \bar{\lambda}_{k+1}), \\
y_{k+1} - y_k &= w_{k+1} - y_{k+1},
\end{align*}
\]

where \( \bar{\lambda}_{k+1} \) is to be determined and the parameter system (16) is still discretized by (24). Note that \( x_{k+1} \) is calculated easily from (60a), and to update \( v_{k+1} \) via (60b), one has to compute the subgradient \( p_{k+1} \in \partial f(x_{k+1}) \). However, as a convex combination of \( x_k \) and \( v_k \), \( x_{k+1} \) might be outside the constraint set \( \mathcal{X} \) since (60b) cannot promise \( \{v_k\}_{k=1}^\infty \subset \mathcal{X} \).

To avoid this, we apply implicit discretization to \( \partial f \). Or more generally, we consider the composite case \( f = f_1 + f_2 \) with \( f_1 \in S^1_{\mu_f, L_f}(\mathcal{X}) \) and \( f_2 \in S_0^0(\mathcal{X}) \). Therefore, in this section, we impose the following assumption.

**Assumption 2.** \( g \in S^0_{\mu_g}(\mathcal{X}) \) with \( \mu_g \geq 0 \) and \( f = f_1 + f_2 \) where \( f_2 \in S_0^0(\mathcal{X}) \) and \( f_1 \in S^1_{\mu_f, L_f}(\mathcal{X}) \) with \( 0 \leq \mu_f \leq L_f < \infty \).

To utilize the separable structure of \( f \), we adopt the operator splitting technique and to promise the contraction of the Lyapunov function \( \mathcal{E}_k \), we borrow the correction idea from [59, Section 7.3] and propose the following modified scheme

\[
\begin{align*}
\frac{u_k - x_k}{\alpha_k} &= v_k - u_k, \\
\gamma_k \frac{v_{k+1} - v_k}{\alpha_k} &\in \mu_f(u_k - v_{k+1}) - \mathcal{G}_x(u_k, v_{k+1}, \bar{\lambda}_{k+1}), \\
\theta_k \frac{\lambda_{k+1} - \bar{\lambda}_k}{\alpha_k} &= \nabla_{\bar{\lambda}} \mathcal{L}(v_{k+1}, w_{k+1}, \lambda_{k+1}), \\
\beta_k \frac{w_{k+1} - w_k}{\alpha_k} &\in \mu_y(y_{k+1} - w_{k+1}) - \partial_y \mathcal{L}(x_{k+1}, y_{k+1}, \bar{\lambda}_{k+1}), \\
y_{k+1} - y_k &= w_{k+1} - y_{k+1},
\end{align*}
\]

where \( \mathcal{G}_x(u_k, v_{k+1}, \bar{\lambda}_{k+1}) = \nabla f_1(u_k) + \partial f_2(v_{k+1}) + A^T \bar{\lambda}_{k+1} + \mathcal{N}_X(v_{k+1}) \). Above, we replaced \( x_{k+1} \) in (60a) and (60b) by \( u_k \) and updated it by (61c), which is an extra correction step. Similarly with (25), we have
an informal primal-dual formulation:

\[
\begin{align*}
    u_k &= (x_k + \alpha_k v_k) / (1 + \alpha_k), \\
    v_{k+1} &= \text{argmin}_{v \in \mathcal{X}} \left\{ f_2(v) + \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v \rangle + \frac{\eta_{f,k}}{2\alpha_k} \| v - \tilde{v}_k \|^2 \right\}, \\
    x_{k+1} &= (x_k + \alpha_k v_{k+1}) / (1 + \alpha_k), \\
    y_{k+1} &= \text{argmin}_{y \in \mathcal{Y}} \left\{ g(y) + \langle B y, \lambda_{k+1} \rangle + \frac{\eta_{g,k}}{2\alpha_k} \| y - \tilde{y}_k \|^2 \right\}, \\
    w_{k+1} &= y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
    \lambda_{k+1} &= \lambda_k + \alpha_k / \theta_k (A v_{k+1} + B w_{k+1} - b),
\end{align*}
\]

where \((\tilde{y}_k, \eta_{g,k})\) are the same as that in (25) and

\[
    \tilde{v}_k = \frac{1}{\eta_{f,k}} (\gamma_k v_k + \mu_f \alpha_k u_k) \quad \text{with} \quad \eta_{f,k} := \gamma_k + \mu_f \alpha_k.
\]

To compute \(\nabla f_1(u_k)\), the step (62b) needs \(u_k \in \mathcal{X}\). In view of (62a), this is true if \((x_k, v_k) \in \mathcal{X} \times \mathcal{X}\). Thanks to the correction (62c), we conclude that \(\{(x_k, u_k, v_k)\}_{k=0}^{\infty} \subset \mathcal{X} \times \mathcal{X}\) as long as \((x_0, v_0) \in \mathcal{X} \times \mathcal{X}\).

Similarly with the previous section, different choices of \(\lambda_{k+1}\) (cf. (39), (52) and (54)) result in our second family of methods. One thing that we shall emphasis is, the first class of methods in Section 3 require no correction step since both \(x\) and \(y\) are discretized implicitly. However, all the methods in this section consider semi-implicit discretization for \(x\) and thus need proper correction (cf. (61c)) to promise the contraction property of the discrete Lyapunov function (27). By symmetry, the second class of methods can be easily rewritten and applied to the case \(F(x, y) = f(x) + (g_1(y) + g_2(y))\). For simplicity, we omit the detailed presentations.

### 4.1 The one-iteration estimate

Analogously to Lemma 3.1, we establish the one-iteration analysis in Lemma 4.1, which helps us prove the nonergodic rates of the second family of methods.

**Lemma 4.1.** Let \(k\) be fixed. For the scheme (61) with Assumption 2 and \((x_k, v_k) \in \mathcal{X} \times \mathcal{X}\), we have \((u_k, x_{k+1}, v_{k+1}) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}\) and

\[
\mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1} + \frac{L_f \alpha_k^2 \theta_{k+1} - \gamma_k \theta_k}{2\theta_k} \| v_{k+1} - v_k \|^2 + \frac{\theta_k}{2} \| \lambda_{k+1} - \lambda_{k+1} \|^2 - \frac{\beta_k}{2} \| w_{k+1} - w_k \|^2.
\]

**Proof.** As before, we calculate the difference \(\mathcal{E}_{k+1} - \mathcal{E}_k = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4\) with \(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\) and \(\mathcal{I}_4\) being defined in (29).

Expand the first term \(\mathcal{I}_1\) as follows

\[
\mathcal{I}_1 = f(x_{k+1}) - f(x_k) + g(y_{k+1}) - g(y_k) + \langle \lambda^*, A(x_{k+1} - x_k) + B(y_{k+1} - y_k) \rangle,
\]

and then duplicate the estimate (33):

\[
\mathcal{I}_2 \leq \alpha_k \left( A v_{k+1} + B w_{k+1} - b, \lambda_{k+1} - \lambda^* \right) + \frac{\theta_k}{2} \| \lambda_{k+1} - \lambda_{k+1} \|^2 - \frac{\alpha_k \theta_{k+1}}{2} \| \lambda_{k+1} - \lambda^* \|^2.
\]

In addition, we claim that the relation (36) holds true here:

\[
\mathcal{I}_4 = -\frac{\alpha_k \beta_{k+1}}{2} \| w_{k+1} - y^* \|^2 - \frac{\beta_k}{2} \| w_{k+1} - w_k \|^2 - \frac{\mu_g \alpha_k}{2} \| y_{k+1} - w_{k+1} \|^2
\]

\[
+ \frac{\mu_g \alpha_k}{2} \| y_{k+1} - y^* \|^2 - \alpha_k \langle q_{k+1}, y_{k+1} - y^* \rangle - \langle q_{k+1}, y_{k+1} - y_k \rangle.
\]
where \( q_{k+1} \in \partial_y \mathcal{L}(x_{k+1}, y_{k+1}, \lambda_{k+1}) \) has been defined by (31). By (13) and the fact \( y_{k+1} \in \mathcal{Y} \) (cf. (62d)), we obtain that
\[
\frac{\mu_k \alpha_k}{2} \left\| y_{k+1} - y^* \right\|^2 - \alpha_k \langle q_{k+1}, y_{k+1} - y^* \rangle - \langle q_{k+1}, y_{k+1} - y_k \rangle \\
\leq \alpha_k \left[ g(y^*) - g(y_{k+1}) + \langle \lambda_{k+1}, B(y^* - y_{k+1}) \rangle \right] + g(y_k) - g(y_{k+1}) + \langle \lambda_{k+1}, B(y_k - y_{k+1}) \rangle.
\]
Dropping the negative square term \(- \| y_{k+1} - w_{k+1} \|^2 \) and shifting \( \lambda_{k+1} \) to \( \lambda^* \), we get
\[
I_4 \leq \alpha_k \left[ g(y^*) - g(y_{k+1}) + \langle \lambda^*, B(y^* - y_{k+1}) \rangle \right] - \frac{\alpha_k \beta_{k+1}}{2} \left\| w_{k+1} - y^* \right\|^2 \\
- \frac{\beta_k}{2} \| w_{k+1} - w_k \|^2 + \alpha_k \langle \lambda_{k+1} - \lambda^*, B(y^* - y_{k+1}) \rangle \\
+ g(y_k) - g(y_{k+1}) + \langle \lambda_{k+1}, B(y_k - y_{k+1}) \rangle.
\]
The estimate for \( I_3 \) starts from (34) but is more subtle. We list the desired result below:
\[
I_3 \leq \alpha_k \left[ f(x^*) - f(x_{k+1}) + \langle \lambda^*, A(x^* - x_{k+1}) \rangle \right] - \frac{\alpha_k \gamma_{k+1}}{2} \left\| v_{k+1} - x^* \right\|^2 \\
+ f_1(x_k) - f_1(x_{k+1}) - \alpha_k f_2(v_{k+1}) - f_2(x_{k+1}) - \frac{\gamma_k}{2} \| v_{k+1} - v_k \|^2 \\
- \alpha_k \| u_{k+1} - u_k \|^2 - \| v_{k+1} - v_k \|^2 - \beta_k \left\| w_{k+1} - w_k \right\|^2.
\]

The detailed proof can be found in Appendix A. Consequently, combining these estimates from \( I_1 \) to \( I_4 \) gives
\[
\mathcal{E}_{k+1} - \mathcal{E}_k \leq - \alpha_k \mathcal{E}_{k+1} + (1 + \alpha_k) f_2(x_{k+1}) - f_2(x_k) - \alpha_k f_2(v_{k+1}) \\
+ (1 + \alpha_k) \left( f_1(x_{k+1}) - f_1(u_k) \right) - \alpha_k \langle \nabla f_1(u_k), v_{k+1} - u_k \rangle \\
+ \frac{\theta_k}{2} \left\| \lambda_{k+1} - \lambda_k \right\|^2 - \frac{\gamma_k}{2} \left\| v_{k+1} - v_k \right\|^2 - \frac{\beta_k}{2} \left\| w_{k+1} - w_k \right\|^2.
\]
Notice that by (62c), \( x_{k+1} \) is a convex combination of \( x_k \) and \( v_{k+1} \) and
\[
(1 + \alpha_k) f_2(x_{k+1}) - f_2(x_k) - \alpha_k f_2(v_{k+1}) \leq 0.
\]
By (12) and Assumption 2, it follows immediately that
\[
f_1(x_{k+1}) - f_1(u_k) \leq \langle \nabla f_1(u_k), x_{k+1} - u_k \rangle + \frac{L_f}{2} \left\| x_{k+1} - u_k \right\|^2.
\]
Besides, by (62a) and (62c) we have
\[
x_{k+1} - u_k = \alpha_k (v_{k+1} - v_k) / (1 + \alpha_k),
\]
which implies
\[
(1 + \alpha_k) \left( f_1(x_{k+1}) - f_1(u_k) \right) - \alpha_k \langle \nabla f_1(u_k), v_{k+1} - v_k \rangle \leq \frac{L_f \alpha_k^2}{2 + 2 \alpha_k} \left\| v_{k+1} - v_k \right\|^2.
\]
Therefore, plugging (66) and (68) into (65) gives
\[
\mathcal{E}_{k+1} - \mathcal{E}_k \leq - \alpha_k \mathcal{E}_{k+1} + \frac{L_f \alpha_k^2}{2 + 2 \alpha_k} \left\| v_{k+1} - v_k \right\|^2 + \frac{\theta_k}{2} \left\| \lambda_{k+1} - \lambda_k \right\|^2 - \frac{\beta_k}{2} \left\| w_{k+1} - w_k \right\|^2.
\]
In view of the relation \( \theta_k = \theta_{k+1} (1 + \alpha_k) \), we obtain (63) and finish the proof. ■
4.2 The semi-implicit choice (39)

By Lemma 4.1, if \( L_f \alpha_k^2 \leq \gamma_k(1 + \alpha_k) \), then \( \tilde{\lambda}_{k+1} = \lambda_{k+1} \) leads to (37). However, this does not give a splitting algorithm. Thus, as before, we consider other semi-implicit and explicit choices.

Different from the first class of methods in which (39) and (52) are equivalent, the scheme (62) loses this symmetric property. In this part, we consider the first one (39), which gives

\[
\begin{aligned}
\{ u_k \} &= \{ \sum_{k=0}^{\infty} \left( \frac{x_k + \alpha_k v_k}{1 + \alpha_k} \right) \} / \{ \lambda_k \}, \\
\{ d_k \} &= \{ \nabla f_1(u_k) + A^T \lambda_k \}, \\
\{ v_{k+1} \} &= \arg\min_{v \in X} \left\{ \frac{1}{2} f_2(v) + (d_k, v) + \frac{\alpha_k}{2 \theta_k} \| Av + Bw_k - b \|^2 + \frac{\theta_f}{2 \alpha_k} \| v - \tilde{v}_k \|^2 \right\}, \\
\{ x_{k+1} \} &= \{ \frac{x_k + \alpha_k v_{k+1}}{1 + \alpha_k} \}, \\
\{ \tilde{\lambda}_{k+1} \} &= \{ \tilde{\lambda}_k + \alpha_k / \theta_k (Av_{k+1} + Bw_k - b) \}, \\
\{ y_{k+1} \} &= \text{prox}_{\tau_k} \left( \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{\gamma_k \| B \|^2}{\theta_k} \right) \right. \\
\{ w_{k+1} \} &= \{ y_{k+1} + (y_{k+1} - y_k) / \alpha_k \}, \\
\{ \tilde{\lambda}_{k+1} \} &= \{ \tilde{\lambda}_k + \alpha_k / \theta_k (Av_{k+1} + Bw_k - b) \},
\end{aligned}
\]

(69)

where \( \{ u_k \}, \{ v_{k+1} \}, \{ y_{k+1} \}, \{ w_{k+1} \} \) are the same as that in (62) and \( \{ x_0, v_0 \} \in X \times Y \). According to Lemma 4.1, we have \( \{ (x_k, y_k) \}_{k=1}^{\infty} \subset X \times X \) and the following result should appear natural.

**Theorem 4.1.** If \( \tilde{\lambda}_{k+1} \) is chosen from (39), then (62) reduces to (69). Besides, under the initial setting (44), Assumption 2 and the condition

\[
(L_f \beta_k \theta_k + \gamma_k \| B \|^2) \alpha_k^2 = \gamma_k \beta_k \theta_k,
\]

(70)

we have \( \{ (x_k, y_k) \}_{k=1}^{\infty} \subset X \times Y \) and

\[
\| Ax_k + B y_k - b \| \leq \theta_k \mathcal{R}_0, \quad | F(x_k, y_k) - F^* | \leq \theta_k (\mathcal{E}_0 + \| \lambda^* \| \mathcal{R}_0 ).
\]

(71)

Above, \( \mathcal{R}_0 \) is defined in Theorem 3.1 and \( \theta_k \) satisfies

\[
\theta_k \leq \min \left\{ \frac{\| B \|}{\sqrt{\beta_0 \theta_k}}, \frac{\| B \|^2}{\mu_g k^2} \right\} + \min \left\{ \frac{L_f}{\gamma_0 k^2}, \exp \left( - \frac{k}{4 \sqrt{L_f}} \right) \right\},
\]

(72)

provided that \( \gamma_0 \beta_0 \leq L_f \beta_0 + \gamma_0 \| B \|^2 \).

**Proof.** In view of Lemma 4.1 and the relation (40), it follows that

\[
\mathcal{E}_{k+1} - \mathcal{E}_k \leq - \alpha_k \mathcal{E}_k + \frac{L_f \alpha_k^2 \theta_k - \eta_k \theta_k}{2 \theta_k} \| v_{k+1} - v_k \|^2 + \frac{\alpha_k^2 \| B \|^2 - \beta_k \theta_k \| w_{k+1} - w_k \|^2}{2 \theta_k}.
\]

Thanks to the condition (70) and the fact \( \theta_k+1 \leq \theta_k \), the above two square terms can be dropped. This promises \( \mathcal{E}_k \leq \theta_k \mathcal{E}_0 \) and thus implies (71), by repeating the proof of (46). Then using Lemmas C.1, C.2 and C.3, the proof of the mixed-type estimate (72) is in line with that of (57).
and a similar argument as before, we conclude the following.

\[ \begin{align*}
& \begin{cases}
    u_k = (x_k + \alpha_k v_k) / (1 + \alpha_k), \\
    \hat{\lambda}_k = \lambda_k - \theta_k^{-1} (A x_k + B y_k - b) + \alpha_k / \theta_k A (v_k - x_k), \\
    y_{k+1} = \arg\min_{y \in \mathcal{Y}} \left\{ L_{\sigma_k} (x_k, y, \hat{\lambda}_k) + \frac{\eta_{g,k}}{2 \alpha_k} \| y - \bar{y}_k \|^2 \right\}, \quad \sigma_k = 1 / \theta_{k+1}, \\
    w_{k+1} = y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
    \tilde{\lambda}_{k+1} = \lambda_k + \alpha_k / \theta_k (A v_k + B w_{k+1} - b), \\
    v_{k+1} = \text{prox}^{\mathcal{X}}_{s_k f_2} \left[ \bar{v}_k - s_k (\nabla f_1 (u_k) + A^T \tilde{\lambda}_{k+1}) \right], \quad s_k = \alpha_k / \tilde{\eta}_{f,k}, \\
    x_{k+1} = (x_k + \alpha_k v_{k+1}) / (1 + \alpha_k), \\
    \tilde{\lambda}_{k+1} = \lambda_k + \alpha_k / \theta_k (A v_{k+1} + B w_{k+1} - b),
\end{cases} \\
\end{align*} \tag{73} \]

where \((\bar{v}_k, \bar{y}_k, \tilde{\eta}_{f,k}, \eta_{g,k})\) are the same as that in (62) and \((x_0, v_0) \in \mathcal{X} \times \mathcal{X} \). By (52) and the last equation of (73), we have

\[ \lambda_{k+1} - \tilde{\lambda}_{k+1} = \alpha_k / \theta_k A (v_{k+1} - v_k). \]

Plugging this into Lemma 4.1, one finds that

\[ \mathcal{E}_{k+1} - \mathcal{E}_k \leq -\alpha_k \mathcal{E}_{k+1} + \frac{1}{2 \theta_k} \left( (L_f \theta_{k+1} + \| A \|^2) \alpha_k^2 - \gamma_k \theta_k \right) \| v_{k+1} - v_k \|^2. \]

Thus under the condition (74), the contraction follows easily. As the mixed-type estimate (75) of \( \theta_k \) can be proved by using Lemma C.2 and a similar argument as before, we conclude the following.

**Theorem 4.2.** If \( \tilde{\lambda}_{k+1} \) is chosen from (52), then (62) becomes (73). Under the initial setting (44), Assumption 2 and the condition

\[ (L_f \theta_k + \| A \|^2) \alpha_k^2 = \gamma_k \theta_k, \tag{74} \]

we have \( \{(x_k, y_k)\}_{k=1}^\infty \subset \mathcal{X} \times \mathcal{Y} \) and \( \mathcal{E}_k \leq \theta_k \mathcal{E}_0 \). Moreover, if \( \gamma_0 \leq L_f + \| A \|^2 \), then the estimate (71) holds true with

\[ \theta_k \lesssim \min \left\{ \frac{\| A \|}{\sqrt{\gamma_0 k}}, \frac{L_f}{\gamma_0 k^2}, \frac{\| A \|^2}{\mu_f k^2} + \exp \left( - \frac{k}{4} \sqrt{\mu_f L_f} \right) \right\}. \tag{75} \]

\[ \begin{align*}
& \begin{cases}
    u_k = (x_k + \alpha_k v_{k+1}) / (1 + \alpha_k), \\
    \tilde{\lambda}_{k+1} = \lambda_{k+1} + \alpha_k / \theta_k (A v_k + B w_{k+1} - b), \\
    v_{k+1} = \text{prox}^{\mathcal{X}}_{s_k f_2} \left[ \bar{v}_k - s_k (\nabla f_1 (u_k) + A^T \tilde{\lambda}_{k+1}) \right], \quad s_k = \alpha_k / \tilde{\eta}_{f,k}, \\
    x_{k+1} = (x_k + \alpha_k v_{k+1}) / (1 + \alpha_k), \\
    \tau_k = \alpha_k / \eta_{g,k}, \\
    y_{k+1} = \text{prox}^{\mathcal{Y}}_{\tau_k g} (\bar{y}_k - \tau_k B^T \tilde{\lambda}_{k+1}), \\
    w_{k+1} = y_{k+1} + (y_{k+1} - y_k) / \alpha_k, \\
    \lambda_{k+1} = \lambda_k + \alpha_k / \theta_k (A v_{k+1} + B w_{k+1} - b),
\end{cases} \\
\end{align*} \tag{76} \]

where \((\bar{v}_k, \bar{y}_k, \tilde{\eta}_{f,k}, \eta_{g,k})\) are the same as that in (62) and \((x_0, v_0) \in \mathcal{X} \times \mathcal{X} \).
By (54) and the last equation of (76), we see that (58) still holds true and invoking Lemma 4.1, we obtain
\[ \mathcal{E}_{k+1} - \mathcal{E}_k \leq - \alpha_k \mathcal{E}_{k+1} + \frac{2 \alpha^2}{2 \theta_k} \|B\|^2 - \beta_k \theta_k \|w_{k+1} - w_k\|^2 + \frac{1}{2 \theta_k} ((L_f \theta_{k+1} + 2 \|A\|^2) \alpha^2 - \gamma_k \theta_k) \|v_{k+1} - v_k\|^2. \]

Hence, it is not hard to conclude the following result from this. By using Lemmas C.2 and C.3, the proof of the mixed-type estimate (77) is a little bit tedious but similar with the spirit of (75).

**Theorem 4.3.** Applying (54) to (62) leads to (76). In addition, under the initial setting (44), Assumption 2 and the condition
\[ (L_f \beta_k \theta_k + 2 \beta_k \|A\|^2 + 2 \gamma_k \|B\|^2) \alpha^2 = \gamma_k \beta_k \theta_k, \]
we have \( \{(x_k, y_k)\}_{k=1}^\infty \subset X \times Y \) and \( \mathcal{E}_k \leq \theta_k \mathcal{E}_0 \). If \( \gamma_0 \beta_0 \leq L_f \beta_0 + 2 \beta_0 \|A\|^2 + 2 \gamma_0 \|B\|^2 \), then the estimate (71) holds true with
\[ \theta_k \leq \min \left\{ \frac{\|B\|}{\sqrt{\alpha_k} k}, \frac{\|B\|^2}{\mu_k k^2} \right\} + \min \left\{ \frac{\|A\|}{\sqrt{\gamma_0} k}, \frac{L_f}{\gamma_0 k^2} \right\} \|A\|^2 + \exp \left( - \frac{k}{4} \sqrt{\frac{\|B\|^2}{L_f}} \right) \right\}. \quad (77) \]

**Remark 4.1.** Based on the discretization (61), one can further apply the operator splitting technique to \( \partial_y L(x, y, \lambda) \) and replace (61c) and (61f) by that
\[
\begin{align*}
\frac{z_k - y_k}{\alpha_k} &= w_k - z_k, \\
\frac{\beta_k}{\alpha_k} w_{k+1} - w_k &\in \mu_g (z_k - w_k) + G_g (z_k, w_{k+1}, \bar{\lambda}_{k+1},), \\
\frac{y_{k+1} - y_k}{\alpha_k} &= w_{k+1} - y_{k+1},
\end{align*}
\]
where \( G_x (u_k, v_{k+1}, \bar{\lambda}_{k+1}) \) is the same as that in (61) and
\[ G_g (z_k, w_{k+1}, \bar{\lambda}_{k+1}) = \nabla g_1 (z_k) + \partial g_2 (w_{k+1}) + B^\top \bar{\lambda}_{k+1} + N_y (w_{k+1}). \]
This yields the third family of methods by considering different choices of \( \bar{\lambda}_{k+1} \) (cf. (39), (52) and (54)).

Imposing the following condition:

**Assumption 3.** \( f = f_1 + f_2 \) where \( f_2 \in S^0_\mu (X) \) and \( f_1 \in S^1_{\mu_f, L_f} (X) \) with \( 0 \leq \mu_f \leq L_f < \infty \), and \( g = g_1 + g_2 \) where \( g_2 \in S^0_\mu (Y) \) and \( g_1 \in S^1_{\mu_g, L_g} (Y) \) with \( 0 \leq \mu_g \leq L_g < \infty \).

Analogously to Lemma 4.1, the one step analysis reads as follows
\[ \mathcal{E}_{k+1} - \mathcal{E}_k \leq - \alpha_k \mathcal{E}_{k+1} + \frac{L_f \alpha^2 \theta_{k+1} - \gamma_k \theta_k}{2 \theta_k} \|v_{k+1} - v_k\|^2 \]
\[ + \frac{\theta_k}{2} \|\lambda_{k+1} - \bar{\lambda}_{k+1}\|^2 + \frac{L_f \alpha^2 \theta_{k+1} - \beta_k \theta_k}{2 \theta_k} \|w_{k+1} - w_k\|^2. \]
Then, nonergodic optimal mixed-type convergence rates can be established as well. For simplicity, we omit the detailed presentations of these methods and their proofs as well.
5 Numerical Experiments

In this part, we investigate the practical performances of our methods on the least absolute deviation (LAD) regression and the support vector machine (SVM), both of which admit the following form

$$\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} F(x, y) := f(x) + g(y) \quad \text{s.t.} \ Ax - y = 0. \quad (78)$$

For all cases in the sequel, $f$ and $g$ are nonsmooth but have explicit proximal calculations. Hence, we focus only on the semi-implicit scheme (53) (denoted by Semi-APD) and report the detailed comparisons with related algorithms:

- the standard linearized ADMM (LADMM) [71, Algorithm 2],
- the accelerated linearized ADMM (ALADMM) [87, Algorithm 2],
- the fast alternating minimization algorithm (Fast-AMA) [31, Algorithm 9],
- the accelerated LADMM with nonergodic rate (ALADMM-NE) [50, Algorithm 1],
- the new primal-dual (New-PD) algorithm [81, Scheme (39)],
- the Chambolle–Pock (CP) method [9].

All these methods (including our Semi-APD) linearize the augmented term and thus share the same proximal operations of $f$ and $g$ and the matrix-vector multiplications of $A$ and $A^\top$. We mention that ALADMM-NE is designed only for convex problems and the convergence rate is $O(1/k)$. Both New-PD and Fast-AMA require strong convexity and possess the fast rate $O(1/k^2)$. Our Semi-APD, ALADMM and CP enjoy the rates $O(1/k)$ and $O(1/k^2)$ respectively for convex and partially strongly convex objectives, but the latter two use ergodic sequences.

To measure the convergence behavior, we look at three relative errors:

- the objective residual: $|F(x_k, y_k) - F^*|/|F(x_0, y_0)|$,
- the violation of feasibility: $\|Ax_k - y_k\| / \|Ax_0 - y_0\|$,
- the composite objective residual: $(P(x_k) - P^*)/|P(x_0)|$,

where the composite objective is $P(x) = f(x) + g(Ax)$, and the minimal value $F^* = P^*$ is approximated by running LADMM with enough iterations. Moreover, for the LAD regression problem, we also illustrate the capability of each algorithm for maintaining the sparsity.

5.1 LAD regression

Consider the LAD regression problem

$$\min_{x \in \mathbb{R}^n} P(x) := f(x) + \|Ax - b\|_1, \quad (79)$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given data with $m \ll n$, and $f$ is a regularization function. Clearly, problem (79) is equivalent to (78) with $g(y) = \|y - b\|_1$. Here, we choose two types of regularizer:

- **Case 1**: $f(x) = \lambda \|x\|_1$,
- **Case 2**: $f(x) = \lambda \|x\|_1 + \mu f/2 \|x\|^2$, 

20
where the regularization parameter is $\lambda = 2$ and the strong convexity constant is $\mu_f = 0.1$. Similarly with [81], we generate the matrix $A$ from the standard normal distribution and set $b = Ax^\# + e$, where $x^\#$ is a sparse vector and $e$ is a Gaussian noise with variance $\sigma^2 = 0.01$.

Figure 1: Numerical results of the LAD regression problem (79) under Case 1. The problem size is $(m, n) = (400, 4000)$ and the sparse vector $x^\#$ has 10% nonzero elements.

Numerical outputs of Case 1 and Case 2 are displayed respectively in Figs. 1 and 2. In Case 1, our Semi-APD performs the best for the objective residual $|F(x_k, y_k) - F^*|$ (top left) and the composite objective residual $|P(x_k) - P^*|$ (bottom left). For the violation of feasibility $\|Ax_k - y_k\|$ (top right), however, Semi-APD is inferior to ALADMM-NE but still better than others. As the theoretical rates of ALADMM and CP are in ergodic sense, we also plot the errors in terms of the averaged sequences. It can be seen that ergodic convergence is much slower than that in nonergodic sense.

In the bottom right part of Fig. 1, we also report the sparsity of all iterative sequences. As we can see, except ALADMM-NE, all the methods maintain nice sparsity. More precisely, the standard LADMM provides a very sparse solution, and the sequences of the rest methods are dense in the beginning but become more sparse as the iteration step grows up. Besides, ergodic sequences perform not well because the average operation breaks the sparsity.

For Case 2, the objective $f$ is strongly convex. From Fig. 2, we observe that Semi-APD has fast convergence for the composite objective residual but is not competitive with New-PD for the objective residual and the violation of feasibility. However, New-PD requires three proximal calculations in each iteration and provides poor sparsity. As a contrast, our Semi-APD generates almost the same sparsity as ALADMM, Fast-AMA and LADMM. Again, ergodic sequences are inferior to those in nonergodic sense, for both convergence rate and sparsity.
5.2 Support vector machine

Given a matrix $W \in \mathbb{R}^{m \times n}$, the bias vector $b \in \mathbb{R}^m$ and the classify vector $c \in \mathbb{R}^m$, consider the SVM problem

$$\min_{x \in \mathbb{R}^n} F(x) := g(x) + \frac{1}{m} \sum_{j=1}^{m} \ell(c_j, w_j^\top x - b_j),$$

(80)

where $w_j$ is the $j$-th column of $W$, $\ell(a, b) := \max(0, 1 - ab)$ is the Hinge loss function and $g : \mathbb{R}^n \to \mathbb{R}^+$ is a regularization function. We follow [87] to generate the problem data and consider

- Binary linear SVM : $g = \rho \| \cdot \|_1$ with $\rho = 0.2$,
- Elastic net regularized SVM : $g = \rho_1/2 \| \cdot \|_2^2 + \rho_2 \| \cdot \|_1$, with $\rho_1 = 0.05$ and $\rho_2 = 0.5$.

Numerical outputs of two SVM problems are plotted in Figs. 3 and 4. For both two cases, our Semi-APD outperforms others on the objective residual $|F(x_k, y_k) - F^*|$ and the composite objective residual $|P(x_k) - P^*|$. In Fig. 3, it provides the smallest violation of feasibility which is comparable with that of ALADMM. While in Fig. 4, ALADMM is superior than our Semi-APD for the violation of feasibility. In addition, except the objective residual in Fig. 3, the ergodic sequences of CP and ALADMM provide slow convergence.
Figure 3: Numerical results of the binary linear SVM with \((m, n) = (100, 500)\).

Figure 4: Numerical results of the elastic net regularized SVM with \((m, n) = (100, 500)\).
6 Conclusions and Discussions

In this work, we present a self-contained differential equation solver approach for separable convex optimization problems. A novel dynamical system is introduced, and proper time discretizations lead to two families of primal-dual methods with acceleration, linearization and splitting. Besides, nonergodic optimal mixed-type convergence rates are established by a unified Lyapunov function.

We also conduct some numerical experiments to validate the practical performances of the proposed method, regrading the objective residual, the feasibility violation and the capability for sparsity recovering. Although it does not always outperform existing algorithms on the convergence behavior, it maintains desired sparsity and does never work significantly worse.

Below, we summarize some discussions and perspectives.

6.1 Well-posedness of the nonsmooth case

To study the differential inclusion (15), the Moreau–Yosida approximation is an effective tool for solution existence (cf. [55]). In [1], Attouch et al. established the existence of a global $C^1$ solution to a temporally rescaled inertial augmented Lagrangian system, which is a second order inclusion system for the convex separable problem (1). However, as mentioned in [1, Section 4], well-posedness under general nonsmooth setting deserves further study.

6.2 The implicit discretization of $\lambda$

Among the proposed algorithms in this work, we excluded the implicit choice $\tilde{\lambda}_{k+1} = \lambda_{k+1}$, which makes $x_{k+1}$ and $y_{k+1}$ coupled with each other. Let us take the scheme (25) as an example, which can be formulated by that

\[
\begin{align*}
    x_{k+1} &= \text{prox}_{\bar{X}_f} \left( \bar{x}_k - s_k A^T \lambda_k \right), \quad s_k = \frac{\alpha_k^2}{\eta_{f,k}}, \\
    y_{k+1} &= \text{prox}_{\bar{Y}_g} \left( \bar{y}_k - \tau_k B^T \lambda_k \right), \quad \tau_k = \frac{\alpha_k^2}{\eta_{g,k}}, \\
    \lambda_{k+1} &= \bar{\lambda}_k + \theta_{k+1}^{-1} (A x_{k+1} + B y_{k+1} - b), \quad \bar{\lambda}_{k} = \lambda_k - \theta_k^{-1} (A x_k + B y_k - b).
\end{align*}
\]

Eliminating $x_{k+1}$ and $y_{k+1}$ gives a nonlinear equation in terms of $\lambda_{k+1}$:

\[
\theta_{k+1} \lambda_{k+1} - A \text{prox}_{\bar{X}_f} \left( \bar{x}_k - s_k A^T \lambda_{k+1} \right) - B \text{prox}_{\bar{Y}_g} \left( \bar{y}_k - \tau_k B^T \lambda_{k+1} \right) = \theta_{k+1} \bar{\lambda}_k - b. \tag{81}
\]

In addition, applying $\bar{\lambda}_{k+1} = \lambda_{k+1}$ to (61), we obtain the corresponding nonlinear equation that enjoys a similar structure with (81). Following the spirit of [51, 56, 57, 64], one can call the semi-smooth Newton iteration [25] to solve (81) efficiently, provided that the problem itself has nice properties such as sparsity and semismoothness.

6.3 Successive choice of $\lambda$

As announced in Section 3, we restricted ourselves to the same choice $\lambda = \bar{\lambda}_{k+1}$ for both $\partial_x \mathcal{L}(x, y, \lambda)$ and $\partial_y \mathcal{L}(x, y, \lambda)$. Thus, unlike the original ADMM (8) and existing accelerated ADMM [65, 77, 87], our methods do not involve simultaneously the augmented terms of $x$ and $y$. This can be recovered if we adopt different choices for $\lambda$. For example, one can apply (39) and $\bar{\lambda}_{k+1} = \lambda_{k+1}$ to (25b) and (25d), respectively. However, this successive way brings more cross terms, and the one-iteration analysis (cf. Lemmas 3.1 and 4.1) deserves further study.

24
6.4 The multi-block case

Consider the multi-block case:

\[ F(x) = \sum_{i=1}^{M} f_i(x_i), \quad \sum_{i=1}^{M} A_i x_i = b, \quad x = (x_1, x_2, \ldots, x_M), \quad M \geq 3. \] \hfill (82)

It has been showed in [11] that the direct extension of ADMM is not necessarily convergent unless each \( f_i \) is strongly convex (cf. [32]). For general convex case, some variants have been proposed with provable convergence [19, 33, 40] and the sublinear rate \( O(1/k) \) [4, 41, 34].

We claim that the continuous model (15) and Theorem 2.1 can be extended to the multi-block case (82). As for the discrete level, parallel type methods (cf. (55) and (76)) are more likely to be generalized to this case but more efforts are needed to study the rest Gauss-Seidel type algorithms.

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A Proof of the Estimate (64)

Recall the identity (34):

\[ I_3 = \frac{\alpha_k}{2} (\mu f - \gamma k+1) \left\| v_{k+1} - x^* \right\|^2 - \frac{\gamma_k}{2} \left\| v_{k+1} - v_k \right\|^2 + \gamma_k \langle v_{k+1} - v_k, v_{k+1} - x^* \rangle. \] \hfill (83)

By (61b), we have \( p_{k+1} \in \partial f_2(v_{k+1}) + N_X(v_{k+1}) \) where

\[ p_{k+1} := \mu f(u_k - v_{k+1}) - \gamma_k \frac{v_{k+1} - v_k}{\alpha_k} - \nabla f_1(u_k) - A^T \lambda_{k+1}. \]

Rewrite the last term in (83) by that

\[ \gamma_k \langle v_{k+1} - v_k, v_{k+1} - x^* \rangle = \mu f \alpha_k \langle u_k - v_{k+1}, v_{k+1} - x^* \rangle - \alpha_k \langle p_{k+1}, v_{k+1} - x^* \rangle - \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v_{k+1} - x^* \rangle. \]

Invoking (22), the first cross term is estimate as follows

\[ \mu f \alpha_k \langle u_k - v_{k+1}, v_{k+1} - x^* \rangle \leq \frac{\mu_f \alpha_k}{2} \left( \left\| u_k - x^* \right\|^2 - \left\| v_{k+1} - x^* \right\|^2 \right). \]

For the second term, we have

\[ - \alpha_k \langle p_{k+1}, v_{k+1} - x^* \rangle \leq - \alpha_k (f_2(v_{k+1}) - f_2(x^*)) = - \alpha_k (f_2(x_{k+1}) - f_2(x^*)) - \alpha_k (f_2(v_{k+1}) - f_2(x_{k+1})), \]

and summarizing the above results gives

\[ \begin{align*}
I_3 \leq & - \alpha_k (f_2(x_{k+1}) - f_2(x^*)) - \frac{\alpha_k \gamma k+1}{2} \left\| v_{k+1} - x^* \right\|^2 \\
& - \alpha_k (f_2(v_{k+1}) - f_2(x_{k+1})) - \frac{\gamma_k}{2} \left\| v_{k+1} - v_k \right\|^2 \\
& + \frac{\mu_f \alpha_k}{2} \left\| u_k - x^* \right\|^2 - \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v_{k+1} - x^* \rangle.
\end{align*} \] \hfill (84)
Let us focus on the last term in (84). By (61a), it follows that
\[- \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v_{k+1} - x^* \rangle \]
\[= - \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v_{k+1} - v_k \rangle - \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, u_k - x^* \rangle \]
\[- \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, u_k - x_k \rangle. \]

Using (11), the fact \((x_k, u_k) \in X \times X\) and Assumption 2, we obtain
\[- \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, u_k - x^* \rangle - \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, u_k - x_k \rangle \]
\[\leq \alpha_k \left( f_1(x^*) - f_1(u_k) + \langle \lambda_{k+1}, A(x^* - u_k) \rangle \right) - \frac{\mu \rho \alpha_k}{2} \| u_k - x^* \|^2 \]
\[+ f_1(x_k) - f_1(u_k) + \langle \lambda_{k+1}, A(x_k - u_k) \rangle. \]

We then shift \(u_k\) to \(x_{k+1}\) to get
\[\frac{\mu \rho \alpha_k}{2} \| u_k - x^* \|^2 - \alpha_k \langle \nabla f_1(u_k) + A^T \lambda_{k+1}, v_{k+1} - x^* \rangle \]
\[\leq \alpha_k \left( f_1(x^*) - f_1(x_{k+1}) + \langle \lambda_{k+1}, A(x^* - x_{k+1}) \rangle \right) \]
\[+ f_1(x_k) - f_1(x_{k+1}) + \langle \lambda_{k+1}, A(x_k - x_{k+1}) \rangle \]
\[+ (1 + \alpha_k) \left( f_1(x_{k+1}) - f_1(u_k) \right) - \alpha_k \langle \nabla f_1(u_k), v_{k+1} - v_k \rangle \]
\[+ (1 + \alpha_k) \langle \lambda_{k+1}, A(x_{k+1} - u_k) \rangle - \alpha_k \langle A^T \lambda_{k+1}, v_{k+1} - v_k \rangle. \]

Thanks to the relation (67), the last term vanishes. After switching \(\lambda_{k+1}\) to \(\lambda^*\), we plug the above estimates into (84) to obtain
\[I_3 \leq \alpha_k \left( f(x^*) - f(x_{k+1}) + \langle \lambda^*, A(x^* - x_{k+1}) \rangle \right) - \frac{\alpha_k \gamma k}{2} \| v_{k+1} - x^* \|^2 \]
\[+ f_1(x_k) - f_1(x_{k+1}) - \alpha_k (f_2(v_{k+1}) - f_2(x_{k+1})) - \frac{\gamma k}{2} \| v_{k+1} - v_k \|^2 \]
\[+ (1 + \alpha_k) \left( f_1(x_{k+1}) - f_1(u_k) \right) - \alpha_k \langle \nabla f_1(u_k), v_{k+1} - v_k \rangle \]
\[+ \alpha_k \langle \lambda_{k+1} - \lambda^*, A(x^* - x_{k+1}) \rangle + \langle \lambda_{k+1}, A(x_k - x_{k+1}) \rangle. \]

This establishes (64).

B An Auxiliary Differential Inequality

Denote by \(W^{1,\infty}(0, \infty)\) the usual Sobolev space [8] consisting of all real-valued functions, which, together with their generalized derivatives, belong to \(L^{\infty}(0, \infty)\). Assume \(y \in W^{1,\infty}(0, \infty)\) is positive and satisfies the differential inequality
\[y'(t) \leq -\frac{\nu}{\sqrt{Py'(t) + Qy(t) + R^2}}, \quad y(0) = 1, \quad (85)\]
where \(\nu, P, Q, R \geq 0\) are constants and \(\sigma \in L^1(0, \infty)\) is nonnegative. We shall establish sharp decay estimates of \(y(t)\) under two cases that are particularly interested in this paper, and the corresponding discrete versions will be presented later in Appendix C.

B.1 Case I

Let us first consider: \(\nu \geq 3/2, P = 0\) and \(Q > 0\). For this case, we cite the result from [58, Lemma 5.1].
Lemma B.1 ([58]). Let \( y \in W^{1,\infty}(0, \infty) \) be positive and satisfy (85) with \( \nu \geq 3/2 \), \( P = 0 \) and \( Q > 0 \). Then for all \( t > 0 \), we have

\[
y(t) \leq C_\nu \begin{cases} \left( \frac{\sqrt{Q}}{\Sigma(t)} \right)^{\frac{\nu^2}{2}} + \frac{R}{\Sigma(t)} & \text{if } \nu > 3/2, \\
\exp \left( -\frac{\Sigma(t)}{2\sqrt{Q}} \right) + \left( \frac{R}{\Sigma(t)} \right)^2 & \text{if } \nu = 3/2,
\end{cases}
\]

where \( \Sigma(t) := \int_0^t \sigma(s) \, ds \) and \( C_\nu > 0 \) depends only on \( \nu \).

B.2 Case II

We then move to another case: \( \nu = 2 \) and \( P > 0 \).

Lemma B.2. Let \( y \in W^{1,\infty}(0, \infty) \) be positive and satisfy (85) with \( \nu = 2 \) and \( P > 0 \). Then for all \( t > 0 \), we have

\[
y(t) \leq \exp \left( -\frac{\Sigma(t)}{2\sqrt{P}} \right) + \frac{36Q}{\Sigma(t)} + 6R, \tag{86}
\]

where \( \Sigma(t) = \int_0^t \sigma(s) \, ds \).

Proof. From (85) we obtain

\[
y'(t) \leq -\frac{\sigma(t)y^2(t)}{\sqrt{P}y(t) + \sqrt{Q}y(t) + R},
\]

which implies

\[
\left[ \sqrt{P}y^{-1}(t) + \sqrt{Q}y^{-3/2}(t) + Ry^{-2}(t) \right] y'(t) \leq -\sigma(t).
\]

Since \( y(0) = 1 \), integrating over \((0, t)\) gives

\[
\sqrt{P} \ln \frac{1}{y(t)} + 2\sqrt{Q}(y^{-1/2}(t) - 1) + R(y^{-1}(t) - 1) \geq \int_0^t \sigma(s) \, ds = \Sigma(t). \tag{87}
\]

Define \( G : (0, \infty) \rightarrow [0, \infty) \) as follows

\[
G(w) := \sqrt{P} \ln \frac{1}{w} + 2\sqrt{Q}(w^{-1/2} - 1) + R(w^{-1} - 1) \quad \forall w > 0.
\]

Besides, let

\[
Y_1(t) = \exp \left( \frac{\Sigma(t)}{2\sqrt{P}} \right), \quad Y_2(t) = \frac{Q}{\left( \sqrt{Q} + \frac{3}{6} \Sigma(t) \right)^2}, \quad \text{and} \quad Y_3(t) = \frac{R}{R + \frac{3}{6} \Sigma(t)}.
\]

One observes

\[
\sqrt{P} \ln \frac{1}{Y_1(t)} = 3\sqrt{Q}(Y_2^{-1/2}(t) - 1) = 3R(Y_3^{-1}(t) - 1) = \frac{1}{2} \Sigma(t),
\]

and it follows that

\[
G(Y(t)) \leq \sqrt{P} \ln \frac{1}{Y_1(t)} + 2\sqrt{Q}(Y_2^{-1/2}(t) - 1) + R(Y_3^{-1}(t) - 1) = \Sigma(t),
\]

where \( Y(t) = Y_1(t) + Y_2(t) + Y_3(t) \). As (87) implies \( G(y(t)) \geq \Sigma(t) \) and \( G(\cdot) \) is monotone decreasing, we conclude that

\[
y(t) \leq Y(t) \leq \exp \left( -\frac{\Sigma(t)}{2\sqrt{P}} \right) + \frac{36Q}{\Sigma(t)} + 6R, \quad \Sigma(t),
\]

which leads to (86) and completes the proof. \( \blacksquare \)
C Decay Estimates of Some Difference Equations

Lemma C.1. Let \( \{ \theta_k \}_{k=0}^{\infty} \) be a positive real sequence such that
\[
\theta_{k+1} - \theta_k \leq -\sigma \theta_k \theta_{k+1}, \quad \theta_0 = 1, \tag{88}
\]
where \( \sigma, \nu > 0 \). If \( \theta_{k+1}/\theta_k \geq \tau > 0 \) for all \( k \in \mathbb{N} \), then
\[
\theta_k \leq (1 + \sigma \tau k)^{-1/\nu} \quad \forall k \in \mathbb{N}. \tag{89}
\]
Proof. Define a piece-wise continuous linear function \( y \) on \([0, \infty) \) \rightarrow (0, \infty) by that
\[
y(t) := \theta_k(k + 1 - t) + \theta_{k+1}(t - k), \quad t \in [k, k+1) \quad \forall k \in \mathbb{N}. \tag{90}
\]
Clearly, \( y \in W^{1,\infty}(0, \infty) \) is decreasing and \( y(0) = 1 \). In addition, we have
\[
\theta_{k+1} \leq y(t) \leq \theta_k \quad \text{and} \quad \frac{\theta_{k+1}}{\theta_k} \geq \tau \quad \forall t \in [k, k+1]. \tag{91}
\]
According to (88), we obtain
\[
y'(t) \leq -\sigma \tau y^{1+\nu}(t) \implies y(t) \leq (1 + \sigma \tau t)^{-1/\nu}.
\]
Hence, (89) follows immediately from this estimate and the fact \( \theta_k = y(k) \). \hfill \Box

We then apply Lemmas B.1 and B.2 to obtain the optimal decay rates of two difference equations.

Lemma C.2. Let \( \{ \theta_k \}_{k=0}^{\infty} \) be a positive real sequence such that
\[
\theta_{k+1} - \theta_k \leq -\frac{\sigma \theta_k \theta_{k+1}}{\sqrt{Q \theta_k + R^2}}, \quad \theta_0 = 1, \tag{92}
\]
where \( \sigma, Q > 0, R \geq 0 \) and \( \nu \geq 1/2 \). If \( \theta_{k+1}/\theta_k \geq \tau > 0 \) for \( k \in \mathbb{N} \), then we have
\[
\theta_k \leq C_{\nu} \left\{ \begin{array}{ll}
\left( \frac{\sqrt{Q}}{\sigma \tau k} \right)^{2} + \left( \frac{R}{\sigma \tau k} \right)^{\frac{1}{\nu}}, & \text{if } \nu > 1/2, \\
\exp \left( -\frac{\sigma \tau k}{2\sqrt{Q}} \right) + \left( \frac{R}{\sigma \tau k} \right)^{2}, & \text{if } \nu = 1/2,
\end{array} \right. \tag{93}
\]
for \( k \geq 1 \), where \( C_{\nu} > 0 \) depends only on \( \nu \).
Proof. Again, we use the piece-wise continuous linear interpolation \( y(t) \) defined by (90). In view of (91) and (92), we find
\[
y'(t) \leq -\frac{\sigma \tau y^{1+\nu}(t)}{\sqrt{Q y(t) + R^2}}, \tag{94}
\]
and invoking Lemma B.1 proves (93). \hfill \Box

Lemma C.3. Let \( \{ \theta_k \}_{k=0}^{\infty} \) be a positive real sequence such that
\[
\theta_{k+1} - \theta_k \leq -\frac{\sigma \theta_k \theta_{k+1}}{\sqrt{P \theta_k^2 + Q \theta_k + R^2}}, \quad \theta_0 = 1, \tag{95}
\]
where \( \sigma, P > 0 \) and \( Q, R \geq 0 \). If \( \theta_{k+1}/\theta_k \geq \tau > 0 \) for all \( k \in \mathbb{N} \), then we have
\[
\theta_k \leq \exp \left( -\frac{\sigma \tau k}{2\sqrt{P}} \right) + \frac{36Q}{\sigma^2 \tau^2 k^2} + \frac{6R}{\sigma \tau k} \quad \forall k \geq 1. \tag{95}
\]
Proof. Similarly with (94), it is not hard to get
\[
y'(t) \leq -\frac{\sigma \tau y^2(t)}{\sqrt{P y^2(t) + Q y(t) + R^2}}.
\]
Applying Lemma B.2 gives (95) and completes the proof. \hfill \Box
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