Asymptotic Stability of Stationary Solutions of a Free Boundary Problem Modeling the Growth of Tumors with Fluid Tissues

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Abstract

This paper aims at proving asymptotic stability of the radial stationary solution of a free boundary problem modeling the growth of nonnecrotic tumors with fluid-like tissues. In a previous paper we considered the case where the nutrient concentration $\sigma$ satisfies the stationary diffusion equation $\Delta \sigma = f(\sigma)$, and proved that there exists a threshold value $\gamma_\ast > 0$ for the surface tension coefficient $\gamma$, such that the radial stationary solution is asymptotically stable in case $\gamma > \gamma_\ast$, while unstable in case $\gamma < \gamma_\ast$. In this paper we extend this result to the case where $\sigma$ satisfies the non-stationary diffusion equation $\varepsilon \partial_t \sigma = \Delta \sigma - f(\sigma)$. We prove that for the same threshold value $\gamma_\ast$ as above, for every $\gamma > \gamma_\ast$ there is a corresponding constant $\varepsilon_0(\gamma) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(\gamma)$ the radial stationary solution is asymptotically stable with respect to small enough non-radial perturbations, while for $0 < \gamma < \gamma_\ast$ and $\varepsilon$ sufficiently small it is unstable under non-radial perturbations.

AMS subject classification: 35R35, 35B35, 76D27.

Key words and phrases: Free boundary problem; tumor growth; Stokes equations; radial stationary solution; asymptotic stability.

1 Introduction

This paper is concerned with the following free boundary problem modelling the growth of tumors with fluid-like tissues:

$$\varepsilon \partial_t \sigma = \Delta \sigma - f(\sigma) \quad \text{in } \Omega(t), \ t > 0,$$

$$\nabla \cdot \textbf{v} = g(\sigma) \quad \text{in } \Omega(t), \ t \geq 0,$$

$$-\nu \Delta \textbf{v} + \nabla p - \frac{\nu}{3} \nabla (\nabla \cdot \textbf{v}) = 0 \quad \text{in } \Omega(t), \ t \geq 0,$$

$$\sigma = \bar{\sigma} \quad \text{on } \partial \Omega(t), \ t \geq 0,$$

$$\mathbf{T}(\textbf{v}, p)\mathbf{n} = -\gamma k \mathbf{n} \quad \text{on } \partial \Omega(t), \ t \geq 0,$$

$$V_n = \textbf{v} \cdot \mathbf{n} \quad \text{on } \partial \Omega(t), \ t \geq 0,$$
where \( \sigma = \sigma(t, x) \), \( v = v(t, x) \) \((= (v_1(t, x), v_2(t, x), v_3(t, x)) \) and \( p = p(t, x) \) are unknown functions representing the concentration of nutrient, the velocity of fluid and the internal pressure, respectively, \( f \) and \( g \) are given functions representing the nutrient consumption rate and tumor cell proliferation rate, respectively, and \( \Omega(t) \) is an a priori unknown bounded domain in \( \mathbb{R}^3 \) representing the region occupied by the tumor at time \( t \). Besides, \( \varepsilon, \nu, \bar{\sigma} \) and \( \gamma \) are positive constants, among which \( \varepsilon \) is the ratio between typical, \( \nu \) is the viscosity coefficient of the tumor tissue, \( \gamma \) is the surface tension coefficient of the tumor surface, and \( \bar{\sigma} \) is the concentration of nutrient in tumor’s host tissues, \( \kappa \), \( V_n \) and \( n \) denote the mean curvature, the normal velocity and the unit outward normal, respectively, of the tumor surface \( \partial \Omega(t) \), and \( T(v, p) \) denotes the stress tensor, i.e.,

\[
T(v, p) = \nu [\nabla \otimes v + (\nabla \otimes v)^T] - (p + \frac{2\nu}{3} \nabla \cdot v) I,
\]

where \( I \) denotes the unit tensor. We note that the sign of the mean curvature \( \kappa \) is defined such that it is nonnegative for convex hyper-surfaces. Without loss of generality, later on we assume that \( \nu = 1 \) and \( \bar{\sigma} = 1 \). Note that the general situation can be easily reduced into this special situation by rescaling. As in [14], throughout this paper we assume that \( f \) and \( g \) are generic smooth functions satisfying the following assumptions:

(A1) \( f \in C^\infty[0, \infty), f'(\sigma) > 0 \) for \( \sigma \geq 0 \) and \( f(0) = 0 \).

(A2) \( g \in C^\infty[0, \infty), g'(\sigma) > 0 \) for \( \sigma \geq 0 \) and \( g(\bar{\sigma}) = 0 \) for some \( \bar{\sigma} > 0 \).

(A3) \( \bar{\sigma} < \sigma \).

The above problem is a simplified form of the tumor models proposed by Franks et al in literatures [4–7], which mimic the early stages of the growth of ductal carcinoma in breast, and was first studied by Friedman in [8]. Local well-posedness of this problem in Hölder spaces has been established by Friedman [8] in a more general setting. Moreover, in [8] it is also proved that, for the special case \( f(\sigma) = \lambda \sigma \) and \( g(\sigma) = \mu(\sigma - \bar{\sigma}) \), the problem (1.1)–(1.10) has a unique radially symmetric stationary solution \((\sigma_s, v_s, p_s, \Omega_s)\). In [11] Friedman and Hu proved that there exists a threshold value \((\mu/\gamma)_*\) such that in the case \( \mu/\gamma < (\mu/\gamma)_* \) this radial stationary solution is linearly asymptotically stable, i.e. the trivial solution of the linearization at \((\sigma_s, v_s, p_s, \Omega_s)\) of the original problem is asymptotically stable, and in the case \( \mu/\gamma > (\mu/\gamma)_* \) this stationary solution is unstable. However, whether or not in the case \( \mu/\gamma < (\mu/\gamma)_* \) this stationary solution is asymptotically stable, namely, whether or not \((\sigma_s, v_s, p_s, \Omega_s)\) is asymptotically stable under arbitrary sufficiently small non-radial perturbations, which is the Problem 3 of [8] (see also the Open Problem (i) in Section 2 of [9]), was not answered by these mentioned literatures.

In a previous work (see [14]) we studied the above problem for the model simplified from (1.1)–(1.10) by taking \( \varepsilon = 0 \), and proved that there exists a threshold value \( \gamma_* > 0 \) for the surface...
tension coefficient $\gamma$, such that in the case $\gamma > \gamma_s$ the radial stationary solution is asymptotically stable with respect to small enough non-radial perturbations, while in case $\gamma < \gamma_s$ this stationary solution is unstable under non-radial perturbations. The aim of the present work is to extend this result to the case where $\varepsilon$ is non-vanishing but small. We shall prove that for the same threshold value $\gamma_s$ as above, for every $\gamma > \gamma_s$ there is a corresponding constant $\varepsilon_0(\gamma) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(\gamma)$ the radial stationary solution is asymptotically stable with respect to small enough non-radial perturbations, while for $0 < \gamma < \gamma_s$ and $\varepsilon$ sufficiently small it is unstable under non-radial perturbations. To give a precise statement of our main result, let us first introduce some notations.

As in [14], we denote by $(\sigma_0, v_0, p_0, \Omega_0)$ the unique radial stationary solution of (1.1)–(1.8), i.e., $\Omega_0 = \{x \in \mathbb{R}^3 : |x| < R_0\}$ and

$$\sigma_0 = \sigma_0(r), \quad v_0 = v_0(r) \frac{x}{r}, \quad p_0 = p_0(r) \quad \text{for} \ x \in \Omega_0,$$

and for any $x_0 \in \mathbb{R}^3$ we denote by $(\sigma_{[x_0]}, v_{[x_0]}, p_{[x_0]}, \Omega_{[x_0]})$ the stationary solution of (1.1)–(1.8) obtained by the coordinate translation $x \to x + x_0$ of the stationary solution $(\sigma_0, v_0, p_0, \Omega_0)$. Given $\rho \in C^1(\partial \Omega_s)$ with $\|\rho\|_{C^1(\partial \Omega_s)}$ sufficiently small, we denote by $\Omega_\rho$ the domain enclosed by the hypersurface $r = R_0 + \rho(\xi)$, where $\xi \in \partial \Omega_s$. Since we shall only be concerned with small perturbations of the stationary solution $(\sigma_0, v_0, p_0, \Omega_0)$, there exist functions $\rho(t) (= \rho(\xi, t))$ and $\rho_0 (= \rho_0(\xi))$ on $\partial \Omega_s$ such that $\Omega(t) = \Omega_\rho(t)$ and $\Omega_0 = \Omega_{\rho_0}$. Using these notations, the initial condition (1.10) can be rewritten as follows:

$$\rho(\xi, 0) = \rho_0(\xi) \quad \text{for} \ \xi \in \partial \Omega_s.$$  \hspace{1cm} (1.12)

The solution $(\sigma, v, p, \Omega)$ of the problem (1.1)–(1.9) will be correspondingly rewritten as $(\sigma, v, p, \rho)$, and the radially symmetric stationary solution $(\sigma_0, v_0, p_0, \Omega_0)$ will be re-denoted as $(\sigma_0, v_0, p_0, 0)$.

The main result of this paper is the following theorem:

**Theorem 1.1** Assume that Assumptions (A1)–(A3) hold. For given $m \in \mathbb{N}$, $m \geq 3$, and $0 < \theta < 1$, we have the following assertion: There exists a positive threshold value $\gamma_s$ such that for any $\gamma > \gamma_s$, the radially symmetric stationary solution $(\sigma_0, v_0, p_0, 0)$ is asymptotically stable for small $\varepsilon$ in the following sense: There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists a corresponding constant $\varepsilon > 0$, such that for any $\rho_0 \in C^{m+\theta}(\partial \Omega_s)$ satisfying $\|\rho_0\|_{C^{m-1+\theta}(\partial \Omega_s)} < \varepsilon$, the problem (1.1)–(1.9) has a unique solution $(\sigma, v, p, \rho)$ for all $t \geq 0$, and there are positive constants $\omega, K$ independent of the initial data and a point $x_0 \in \mathbb{R}^3$ uniquely determined by the initial data, such that the following holds for all $t \geq 1$:

$$\|\sigma(\cdot, t) - \sigma_{[x_0]}\|_{C^{m+\theta}(\Omega(t))} + \|v(\cdot, t) - v_{[x_0]}\|_{C^{m-1+\theta}(\Omega(t))} + \|p(\cdot, t) - p_{[x_0]}\|_{C^{m-2+\theta}(\partial \Omega(t))} \leq Ke^{-\omega t}. \hspace{1cm} (1.13)$$

For $\gamma < \gamma_s$ and $\varepsilon$ sufficiently small, the stationary solution $(\sigma_0, v_0, p_0, 0)$ is unstable. \hfill $\Box$

As in [14] we shall use a functional approach to prove this result, namely, we shall first reduce the problem (1.1)–(1.10) into a differential equation in a Banach space, and next use the geometric theory for differential equations in Banach spaces to study the asymptotic behavior of the reduced equation. However, unlike the case $\varepsilon = 0$ in which the reduced equation is a
scalar (first-order) nonlinear parabolic pseudo-differential equation on the compact manifold $S^2$ which does not have a boundary, in the present case $\varepsilon \neq 0$ the reduced equation is a system of equations, one of which has a similar feature as the equation in the case $\varepsilon = 0$, while the other of which is defined on the domain $\Omega_s$ complemented with a Dirichlet boundary condition. This determines that in the present case we are forced to deal with a number of new difficulties. The first difficulty lies in computation of the spectrum of the linearized operator, because we now encounter a matrix operator which is not of the diagonal form. To overcome this difficulty we shall use a technique developed in [2] to show that the linearized operator is similar to a small perturbation of a matrix operator possessing a triangular structure. The second difficulty is caused by the Dirichlet boundary condition, which determines that we cannot find a suitable continuous interpolation space as our working space to make the center manifold analysis. More precisely, as in the case $\varepsilon = 0$, 0 is an eigenvalue of the linearized operator, so that the standard linearized stability principle does not apply. In the case $\varepsilon = 0$, this difficulty is overcome with the aid of the center manifold analysis technique developed in [3]. Since this technique requires that the working space must be a continuous interpolation space, it fails to apply to the present case $\varepsilon \neq 0$. To overcome this difficulty we shall use the idea of Lie group action developed in [2] and apply Theorem 2.1 of [2] to solve this problem.

The structure of the rest part is as follows. In Section 2 we first convert the problem into an equivalent initial-boundary value problem on a fixed domain by using Hanzawa transformation, and next we further reduce it into a differential equation in a Banach space. In Section 3 we study the linearization of (1.1)–(1.8) at the radial stationary solution, and compute the spectrum of the linearized operator. In the last section we give the proof of Theorem 1.2.

2 Reduction of the problem

In this section we reduce the problem (1.1)–(1.10) into a differential equation in a Banach space. For simplicity of the notation, later on we always assume that $R_s = 1$. Note that this assumption is reasonable because the case $R_s \neq 1$ can be easily reduced into this case after a rescaling. It follows that

$$\Omega_s = \mathbb{B}^3 = \{x \in \mathbb{R}^3 : |x| < 1\} \quad \text{and} \quad \partial\Omega_s = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}.$$ 

Let $m$ and $\theta$ be as in Theorem 1.1. We introduce an operator $\Pi \in L(C^{m+\theta}(S^2), C^{m+\theta}(\mathbb{B}^3))$ as follows: Given $\rho \in C^{m+\theta}(S^2)$, we define $u = \Pi(\rho) \in C^{m+\theta}(\mathbb{B}^3)$ to be the unique solution of the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{B}^3, \quad u = \rho \quad \text{on} \quad S^2. \quad (2.1)$$ 

It is clear that $\Pi \in L(C^{m+\theta}(S^2), C^{m+\theta}(\mathbb{B}^3))$, and $\Pi$ is a right inverse of the trace operator, i.e., we have $\text{tr}_{S^2}(\Pi(u)) = u$ for any $u \in C^{m+\theta}(S^2)$. Let $E \in L(C^{m+\theta}(\mathbb{B}^3), BUC^{m+\theta}(\mathbb{R}^3))$ be an extension operator, i.e., $E$ has the property that $E(u)(x) = u(x)$ for any $u \in C^{m+\theta}(\mathbb{B}^3)$ and $x \in \mathbb{B}^3$. Here $BUC^{m+\theta}(\mathbb{R}^3)$ denotes the space of all $C^m$ functions $u$ on $\mathbb{R}^3$ such that $u$ itself and all its partial derivatives of order $\leq m$ are bounded and uniformly $\theta$-th order Hölder continuous in $\mathbb{R}^3$. We denote $\Pi_1 = E \circ \Pi$. Then clearly $\Pi_1 \in L(C^{m+\theta}(S^2), BUC^{m+\theta}(\mathbb{R}^3))$, so that there
exists a constant $C_0 > 0$ such that
\[
\|\Pi_1(\rho)\|_{BUC^{m+\theta}(\mathbb{R}^3)} \leq C_0\|\rho\|_{C^{m+\theta}(\mathbb{S}^2)} \quad \text{for } \rho \in C^{m+\theta}(\mathbb{S}^2).
\] (2.2)

Take a constant $0 < \delta < \min\{1/6, 1/(3C_0)\}$ and fix it, where $C_0$ is the constant in (2.2). We choose a cut-off function $\chi \in C^\infty(0, \infty)$ such that
\[
0 \leq \chi \leq 1, \quad \chi(\tau) = \begin{cases} 1, & \text{for } |\tau| \leq \delta, \\ 0, & \text{for } |\tau| \geq 3\delta, \end{cases} \quad \text{and} \quad |\chi'(\tau)| \leq \frac{2}{3\delta}.
\] (2.3)

We denote
\[
O^{m+\theta}_\delta(\mathbb{S}^2) = \{\rho \in C^{m+\theta}(\mathbb{S}^2) : \|\rho\|_{C^{m+\theta}(\mathbb{S}^2)} < \delta\}.
\]

Given $\rho \in O^{m+\theta}_\delta(\mathbb{S}^2)$, we define the Hanzawa transformation $\Phi_\rho : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:
\[
\Phi_\rho(x) = x + \chi(r - 1)\Pi_1(\rho)(x)\frac{x}{r} \quad \text{for } x \in \mathbb{R}^3.
\] (2.4)

Using (2.2) and (2.3) we can easily verify that
\[
\Phi_\rho \in \text{Diff}^{m+\theta}(\mathbb{R}^3, \mathbb{R}^3) \quad \text{and} \quad \Phi_\rho(x) = x \quad \text{if} \quad \text{dist}(x, \mathbb{S}^2) > 3\delta.
\]

We define $\phi_\rho = \Phi_\rho|_{\mathbb{S}^2}$ and $\Gamma_\rho = \text{Im}(\phi_\rho)$, and denote by $\Omega_\rho$ the domain enclosed by $\Gamma_\rho$. Clearly,
\[
\phi_\rho(\omega) = [1 + \rho(\omega)]\omega \quad \text{for } \omega \in \mathbb{S}^2.
\]

Thus, in the polar coordinates $(r, \omega)$ of $\mathbb{R}^3$, where $r = |x|$ and $\omega = x/|x|$, the hyper-surface $\Gamma_\rho$ has the following equation: $r = 1 + \rho(\omega)$.

Next, given $\rho \in C([0, T], O^{m+\theta}_\delta(\mathbb{S}^2))$, for each $t \in [0, T]$ we define $\Gamma_\rho(t) = \Gamma_\rho(t)$ and $\Omega_\rho(t) = \Omega_\rho(t)$. Since our purpose is to study asymptotical stability of the radially symmetric stationary solution, later on we always assume the initial domain $\Omega_0$ lies in a small neighborhood of $\Omega_s$. More precisely, we assume $\Gamma_0 := \partial\Omega_0 = \text{Im}(\phi_{\rho_0})$ for some $\rho_0 \in O^{m+\theta}_\delta(\mathbb{S}^2)$.

Let $\rho$ be as above, and let $\Phi^i_\rho$ be the $i$-th component of $\Phi_\rho$, $i = 1, 2, 3$. We denote
\[
[D\Phi^i_\rho]_{ij} := \partial_i \Phi^j_\rho = \frac{\partial \Phi^j_\rho}{\partial x_i}, \quad a_{ij}^\rho(x) = [D\Phi_\rho(x)]_{ij}^{-1} \quad (i, j = 1, 2, 3),
\]
\[
G_\rho(x) = \det(D\Phi_\rho(x)) \quad \text{for } x \in \mathbb{R}^3,
\]
\[
H_\rho(\omega) = |\phi_\rho|^2 \sqrt{1 + |\nabla_\omega \phi_\rho|^2} \quad \text{for } \omega \in \mathbb{S}^2,
\]
where $\nabla_\omega$ represents the orthogonal projection of the gradient $\nabla_{x}$ onto the tangent space $T_\omega(\mathbb{S}^2)^1$.

Here and hereafter, for a matrix $A$ we use the notation $A_{ij}$ to denote the element of

\[
1)\text{In the coordinate } \omega = \omega(\vartheta, \varphi) = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \quad (0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2\pi) \quad \text{of the sphere we have}
\]
\[
\nabla_\omega f(\omega) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, -\sin \vartheta)\partial_\vartheta f(\omega(\vartheta, \varphi)) + \frac{1}{\sin \vartheta}(-\sin \varphi, \cos \varphi, 0)\partial_\varphi f(\omega(\vartheta, \varphi)).
\]

Note also that $\nabla_{x} f = \frac{\partial f}{\partial x} + \frac{1}{r} \nabla_\omega f$. 

A in the \((i, j)\)-th position. From (2.3) we see that \([\rho \to \Phi_{\rho}] \in C^\infty(O_\delta^{m+\theta}(S^2), \text{Diff}^{m+\theta}(\mathbb{R}^3, \mathbb{R}^3))\). Thus we have

\[
\begin{align*}
[\rho &\to a_{ij}^p] \in C^\infty(O_\delta^{m+\theta}(S^2), C^{m-1+\theta}(\mathbb{R}^3)), \quad i, j = 1, 2, 3, \\
[\rho &\to G_{\rho}] \in C^\infty(O_\delta^{m+\theta}(S^2), C^{m-1+\theta}(\mathbb{R}^3)), \\
[\rho &\to H_{\rho}] \in C^\infty(O_\delta^{m+\theta}(S^2), C^{m-1+\theta}(S^2)).
\end{align*}
\]

(2.5)

We now introduce four partial differential operator \(A(\rho), \vec{B}(\rho), \vec{B}(\rho)\cdot\) and \(\vec{B}(\rho)\otimes\) on \(\mathbb{R}^3\) as follows:

\[
A(\rho)u(x) = a_{ij}^p(x)\partial_j (a_{ik}^p(x)\partial_k u(x)) \quad \text{for scalar function } u,
\]

\[
\vec{B}(\rho)u(x) = \left( a_{ij}^p(x)\partial_j u(x), a_{ij}^p(x)\partial_j u(x), a_{ij}^p(x)\partial_j u(x) \right) \quad \text{for scalar function } u,
\]

\[
\vec{B}(\rho) \cdot v(x) = a_{ij}^p(x)\partial_j v_i(x) \quad \text{for vector function } v = (v_1, v_2, v_3),
\]

\[
\vec{B}(\rho) \otimes v(x) = (a_{ik}^p(x)\partial_k v_j(x)) \quad \text{for vector function } v = (v_1, v_2, v_3).
\]

Here and hereafter we use the convention that repeated indices represent summations with respect to these indices, and \(\partial_j = \partial/\partial x_j, \ j = 1, 2, 3\). These definitions can be respectively briefly rewritten as follows:

\[
A(\rho)u = (\Delta (u \circ \Phi_{\rho}^{-1})) \circ \Phi_{\rho}, \quad \vec{B}(\rho)u = (\nabla (u \circ \Phi_{\rho}^{-1})) \circ \Phi_{\rho},
\]

\[
\vec{B}(\rho) \cdot v = (\nabla \cdot (v \circ \Phi_{\rho}^{-1})) \circ \Phi_{\rho}, \quad \vec{B}(\rho) \otimes v = (\nabla \otimes (v \circ \Phi_{\rho}^{-1})) \circ \Phi_{\rho}.
\]

By (2.5) we have

\[
\begin{align*}
[\rho &\to A(\rho)] \in C^\infty(O_\delta^{m+\theta}(S^2), L(C^{m+\theta}(\mathbb{R}^3), C^{m-2+\theta}(\mathbb{R}^3))), \\
[\rho &\to \vec{B}(\rho)] \in C^\infty(O_\delta^{m+\theta}(S^2), L(C^{m+\theta}(\mathbb{R}^3), (C^{m-1+\theta}(\mathbb{R}^3))^3)), \\
[\rho &\to \vec{B}(\rho)\cdot] \in C^\infty(O_\delta^{m+\theta}(S^2), L((C^{m+\theta}(\mathbb{R}^3))^3, C^{m-1+\theta}(\mathbb{R}^3))), \\
[\rho &\to \vec{B}(\rho)\otimes] \in C^\infty(O_\delta^{m+\theta}(S^2), L((C^{m+\theta}(\mathbb{R}^3))^3, (C^{m-1+\theta}(\mathbb{R}^3))^3))^3).
\end{align*}
\]

(2.6)

Next we introduce the boundary operator \(\vec{D}(\rho)\): \((C^{m-1+\theta}(\mathbb{R}^3))^3 \rightarrow C^{m-1+\theta}(S^2)\)

\[
\vec{D}(\rho)v = \text{tr}_{S^2}(v) \cdot [\omega - \frac{1}{1+\rho} \nabla \omega],
\]

and the bilinear operator \(C(\rho): C^{m+\theta}(\mathbb{R}^3) \times (C^{m-1+\theta}(\mathbb{R}^3))^3 \rightarrow C^{m-1+\theta}(\mathbb{R}^3)\)

\[
C(\rho)[u, v] = \chi(r - 1)\Pi_1(\vec{D}(\rho)v)\vec{B}(\rho)u \cdot e_r.
\]

Here and hereafter we use the notation \(e_r\) to denote the vector function on \(\mathbb{R}^3\setminus\{0\}\) defined by \(e_r(x) = \omega(x) = x/r\). Note that since \(\chi(r - 1) = 0\) for \(0 \leq r \leq 1 - 3\delta\), we see that \(C(\rho)[u, v]\) is a well-defined function on \(\mathbb{R}^3\). Again, by (2.5) we have

\[
\begin{align*}
[\rho &\to \vec{D}] \in C^\infty(O_\delta^{m+\theta}(S^2), L((C^{m-1+\theta}(\mathbb{R}^3))^3, C^{m-1+\theta}(S^2))), \\
[\rho &\to C(\rho)] \in C^\infty(O_\delta^{m+\theta}(S^2), BL(C^{m+\theta}(\mathbb{R}^3) \times (C^{m-1+\theta}(\mathbb{R}^3))^3, C^{m-1+\theta}(\mathbb{R}^3))).
\end{align*}
\]

(2.7)

Here the notation \(BL(\cdot \times \cdot, \cdot)\) denotes the Banach space of bounded bilinear mappings with respect to the indicated Banach spaces.
Let \( n \) and \( \kappa \) be respectively the unit outward normal and the mean curvature of \( \Gamma_\rho \) (see (1.5)). We denote
\[
\tilde{n}_\rho(x) = n(\phi_\rho(x)), \quad \tilde{\kappa}_\rho(x) = \kappa(\phi_\rho(x)) \quad \text{for} \ x \in \mathbb{S}^2.
\]
A direct computation shows that
\[
\tilde{n}_\rho(x) = \frac{x \cdot [(D\Phi_\rho(x))^{-1}]^T}{|x \cdot [(D\Phi_\rho(x))^{-1}]|} = \frac{a_{ij}^\rho(x) x_j e_i}{|a_{ij}^\rho(x) x_j e_i|},
\]
where
\[
e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1),
\]
and
\[
\tilde{\kappa}_\rho(x) = \frac{1}{2} a_{ij}^\rho(x) \partial_j \tilde{n}_\rho^i(x).
\]
By (2.5) we have
\[
\begin{cases}
\{ [\rho \to \tilde{\kappa}_\rho] \in C^\infty(O_\delta^{m-\theta}(\mathbb{S}^2), C^{m-2+\theta}(\mathbb{S}^2)), \\
\{ [\rho \to \tilde{n}_\rho] \in C^\infty(O_\delta^{m-\theta}(\mathbb{S}^2), (C^{m-1+\theta}(\mathbb{S}^2))^3).
\end{cases}
\] (2.8)

As in \([8]\) we introduce the following vector functions:
\[
w_1(x) = (0, x_3, -x_2), \quad w_2(x) = (-x_3, 0, x_1), \quad w_3(x) = (x_2, -x_1, 0).
\]
Then clearly \( v \times x = (v \cdot w_1, v \cdot w_2, v \cdot w_3) \).

Let \( T \) be a given positive number and consider a function \( \rho : [0, T] \to O_\delta^{m+\theta}(\mathbb{S}^2) \). We assume that \( \rho \in C([0, T], O_\delta^{m+\theta}(\mathbb{S}^2)) \). Given such \( \rho \), we denote
\[
\Gamma_\rho(t) = \Gamma_{\rho(t)}, \quad \Omega_\rho(t) = \Omega_{\rho(t)} \quad (0 \leq t \leq T).
\]

Finally, for \( \sigma, v \) and \( p \) as in (1.1)–(1.9), we denote
\[
\tilde{\sigma} = \sigma \circ \Phi_\rho, \quad \tilde{v} = v \circ \Phi_\rho, \quad \tilde{p} = p \circ \Phi_\rho.
\]
We also denote \( \tilde{w}_j^\rho = w_j \circ \Phi_\rho, \ j = 1, 2, 3. \)

Using these notations, we claim that the Hanzawa transformation transforms the equations (1.1)–(1.10) into the following equations, respectively:
\[
\varepsilon \partial_t \tilde{\sigma} - A(\rho) \tilde{\sigma} - \varepsilon C(\rho)[\tilde{\sigma}, \tilde{v}] = -f(\tilde{\sigma}) \quad \text{in} \ \mathbb{B}^3 \times \mathbb{R}_+,
\]
\[
\tilde{G}(\rho) \cdot \tilde{v} = g(\tilde{\sigma}) \quad \text{in} \ \mathbb{B}^3 \times \mathbb{R}_+,
\]
\[
-A(\rho) \tilde{v} + \tilde{G}(\rho) \tilde{p} - \frac{1}{3} \tilde{G}(\rho) (\tilde{G}(\rho) \cdot \tilde{v}) = 0 \quad \text{in} \ \mathbb{B}^3 \times \mathbb{R}_+,
\]
\[
\tilde{\sigma} = 1 \quad \text{on} \ \mathbb{S}^2 \times \mathbb{R}_+,
\]
\[
\tilde{T}_\rho(\tilde{v}, \tilde{p}) \tilde{n}_\rho = -\gamma \tilde{\kappa}_\rho \tilde{n}_\rho \quad \text{on} \ \mathbb{S}^2 \times \mathbb{R}_+,
\]
\[
\partial_t \rho = \tilde{D}(\rho) \tilde{v} \quad \text{on} \ \mathbb{S}^2 \times \mathbb{R}_+,
\]
\[ \int_{|x|<1} \bar{v}(x)G_\rho(x)dx = 0, \quad t > 0, \quad (2.15) \]

\[ \int_{|x|<1} \bar{v}(x) \cdot \bar{w}_j^\rho(x)G_\rho(x)dx = 0, \quad j = 1, 2, 3, \quad t > 0. \quad (2.16) \]

\[ \bar{\sigma}(x, 0) = \bar{\sigma}_0(x) \quad \text{for } x \in \mathbb{R}^3, \quad (2.17) \]

\[ \rho(\omega, 0) = \rho_0(\omega) \quad \text{for } \omega \in \mathbb{S}^2. \quad (2.18) \]

Here \( \bar{T}_\rho(\tilde{v}, \tilde{p}) = [\bar{\mathcal{B}}(\rho) \otimes \tilde{v} + (\bar{\mathcal{B}}(\rho) \otimes \tilde{v})^T] - [\tilde{p} + (2/3)\bar{\mathcal{B}}(\rho) \cdot \tilde{v}]I. \)

Indeed, it is immediate to see that under the Hanzawa transformation, the equations (1.2)–(1.5) and (1.7)–(1.9) are respectively transformed into the equations (2.10)–(2.13) and (2.15)–(2.18), which is a rewritten form of (1.10). In what follows we prove that (1.1) and (1.6) are transformed into the equations (2.10)–(2.13) and (2.15)–(2.18).

Let \( \psi_\rho(x, t) = r - 1 - \rho(\omega, t), \) where \( r = |x| \) and \( \omega = x/|x| \). Then \( x \in \Gamma_\rho(t) \) if and only if \( \psi_\rho(x, t) = 0 \). It follows that the normal velocity of \( \Gamma_\rho(t) \) is as follows (see [3]):

\[ V_n(x, t) = \frac{\partial_1 \rho(\omega, t)}{|\nabla_x \psi_\rho(x, t)|} \quad \text{for } x \in \Gamma_\rho(t), \quad t > 0. \]

Moreover, \( n(x, t) = \nabla_x \psi_\rho(x, t)/|\nabla_x \psi_\rho(x, t)| \). Hence (1.6) can be rewritten as follows:

\[ \partial_1 \rho(\omega, t) = \nabla_x \psi_\rho(x, t) \quad \text{for } x \in \Gamma_\rho(t), \quad t > 0, \]

where \( \omega = x/|x| \). Since \( \nabla_x \psi_\rho = \frac{\partial \psi_\rho}{\partial r} \omega + \frac{1}{r} \nabla_\omega \psi_\rho \), we see that after the Hanzawa transformation, this equation has the following form:

\[ \partial_1 \rho(\omega, t) = \bar{\nabla}(\omega, t) \cdot \left[ \omega - \frac{\nabla_\omega \rho(\omega, t) - r}{1 + \rho(\omega, t)} \right] \quad \text{for } \omega \in \mathbb{S}^2, \quad t > 0. \]

Recalling the definition of the operator \( \bar{\mathcal{D}}(\rho) \), we see that the equation (2.14) follows. Next, by differentiating the relation \( \bar{\sigma} = \sigma \circ \Phi_\rho \) in \( t \) and using the equations (1.1), (2.4) and (2.14) we see that

\[ \partial_t \bar{\sigma} = \partial_t \sigma \circ \Phi_\rho + \partial_t \Phi_\rho \cdot (\nabla_x \sigma \circ \Phi_\rho) \]

\[ = \frac{1}{\varepsilon} \Delta \sigma \circ \Phi_\rho - \frac{1}{\varepsilon} f(\sigma \circ \Phi_\rho) + \chi (r - 1) \Pi_1 (\partial_1 \rho_\varepsilon \cdot (\nabla_x \sigma \circ \Phi_\rho) \]

\[ = \frac{1}{\varepsilon} A(\rho) \bar{\sigma} - \frac{1}{\varepsilon} f(\bar{\sigma}) + \chi (r - 1) \Pi_1 (\bar{\mathcal{D}}(\rho) \tilde{v}) e_r \cdot \mathcal{B}(\rho) \bar{\sigma} \]

\[ = \frac{1}{\varepsilon} A(\rho) \bar{\sigma} - \frac{1}{\varepsilon} f(\bar{\sigma}) + C(\rho)[\bar{\sigma}, \bar{v}]. \]

Hence (2.9) follows.

The above deduction yields the following lemma:

**Lemma 2.1** If \((\sigma, v, p, \rho)\) is a solution of the problem (1.1)–(1.10), then by letting \( \bar{\sigma} = \sigma \circ \Phi_\rho, \bar{v} = v \circ \Phi_\rho \) and \( \bar{p} = p \circ \Phi_\rho \), we have that \((\bar{\sigma}, \bar{v}, \bar{p}, \rho)\) is a solution of the problem (2.9)–(2.18). Conversely, If \((\bar{\sigma}, \bar{v}, \bar{p}, \rho)\) is a solution of the problem (2.9)–(2.18), then by letting \( \sigma = \bar{\sigma} \circ \Phi_\rho^{-1}, v = \tilde{v} \circ \Phi_\rho^{-1} \) and \( p = \tilde{p} \circ \Phi_\rho^{-1} \), we have that \((\sigma, v, p, \rho)\) is a solution of the problem (1.1)–(1.10).
Proof: The above deduction shows that if $(\sigma, v, p, \rho)$ is a solution of (1.1)–(1.10), then $(\tilde{\sigma}, \tilde{\nu}, \tilde{p}, \rho)$ satisfies (2.9)–(2.18). The converse can be similarly verified. □

We now proceed to reduce the problem (2.9)–(2.18) into evolution equations only in $\tilde{\sigma}$ and $\rho$. The idea is to solve equations (2.10), (2.11), (2.13), (2.15) and (2.16) to get $\tilde{\nu}$ and $\tilde{p}$ as functionals of $\tilde{\sigma}$ and $\rho$, and next substitute $\tilde{\nu}$ obtained in this way into equations (2.9) and (2.14). Thus, for given $\rho \in O_\delta^{m+\theta}(S^2)$ we consider the following boundary value problem:

\begin{align}
\tilde{\mathcal{B}}(\rho) \cdot \tilde{v} &= \varphi & \text{in } \mathbb{B}^3, \\
-\mathcal{A}(\rho)\tilde{v} + \tilde{\mathcal{B}}(\rho)\tilde{p} &= g & \text{in } \mathbb{B}^3, \\
\tilde{T}_p(\tilde{v}, \tilde{p})n_\rho &= h & \text{on } S^2, \\
\int_{|x|<1} \tilde{v}(x)G_\rho(x)dx &= 0, \\
\int_{|x|<1} \tilde{v}(x) \cdot \mathbf{w}_j^\rho(x)G_\rho(x)dx &= 0, & j = 1, 2, 3,
\end{align}

where $\varphi \in C^{m-k+\theta}(\mathbb{B}^3)$, $g \in (C^{m-k-2+\theta}(\mathbb{B}^3))^3$ and $h \in (C^{m-k+1+\theta}(S^2))^3$ for some $0 \leq k \leq m-2$.

**Lemma 2.2** Let $\delta$ be sufficiently small and let $\rho \in O_\delta^{m+\theta}(S^2)$ be given. A necessary and sufficient condition for (2.19)–(2.23) to have a solution is that $\varphi$, $g$ and $h$ satisfy the following relations:

\begin{align}
\int_{|x|<1} (g(x) - \frac{1}{3} \tilde{\mathcal{B}}(\rho)\varphi(x)) \cdot \mathbf{w}_j^\rho(x)G_\rho(x)dx + \int_{|x|=1} h(x) \cdot \mathbf{e}_j h_\rho(x) dS_x &= 0, & j = 1, 2, 3, \\
\int_{|x|<1} (g(x) - \frac{1}{3} \tilde{\mathcal{B}}(\rho)\varphi(x)) \cdot \mathbf{e}_j G_\rho(x)dx + \int_{|x|=1} h(x) \cdot \mathbf{e}_j H_\rho(x) dS_x &= 0, & j = 1, 2, 3.
\end{align}

If this condition is satisfied, then (2.19)–(2.23) has a unique solution $(\tilde{\nu}, \tilde{p}) \in (C^{m-k+\theta}(\mathbb{B}^3))^3 \times C^{m-k+1+\theta}(\mathbb{B}^3)$). Moreover, we have $\tilde{v} = \tilde{p}(\varphi) + Q(\rho)g + R(\rho)h$, where

\begin{align}
\tilde{p} &\in \bigcap_{k=0}^{m-2} C^\infty(O_\delta^{m+\theta}(S^2), L(C^{m-k+1+\theta}(\mathbb{B}^3))^3), (C^{m-k+\theta}(\mathbb{B}^3))^3), \\
Q &\in \bigcap_{k=0}^{m-2} C^\infty(O_\delta^{m+\theta}(S^2), L((C^{m-k-2+\theta}(\mathbb{B}^3))^3), (C^{m-k+\theta}(\mathbb{B}^3))^3), \\
R &\in \bigcap_{k=0}^{m-2} C^\infty(O_\delta^{m+\theta}(S^2), L((C^{m-k+1+\theta}(\mathbb{B}^3))^3), (C^{m-k+\theta}(\mathbb{B}^3))^3)).
\end{align}

**Proof:** See Lemmas 2.3 and 2.4 of [14]. □

Now, let $\tilde{\sigma} \in C^{m+\theta}(\mathbb{B}^3)$, $\rho \in O_\delta^{m+\theta}(S^2)$ and we consider the system of equations (2.10), (2.11), (2.13), (2.15) and (2.16). These equations can be rewritten in the form of (2.19)–(2.23), with

\begin{align}
\varphi = g(\tilde{\sigma}), & \quad g = \frac{1}{3} \tilde{\mathcal{B}}(\rho)g(\tilde{\sigma}), & \quad h = -\gamma \tilde{\kappa_\rho} \tilde{n}_\rho.
\end{align}
As was shown in [14], the relations (2.24) and (2.25) are satisfied by these functions. Besides, it is obvious that \( \varphi \in C^{m+\theta}(\mathbb{R}^3) \subseteq C^{m-2+\theta}(\mathbb{R}^3) \) and \( g \in (C^{m-1+\theta}(\mathbb{R}^3))^3 \subseteq (C^{m-3+\theta}(\mathbb{R}^3))^3 \). Furthermore, by (2.8) we see that \( h \in (C^{m-2+\theta}(\mathbb{S}^2))^3 \). Hence, by Lemma 2.2 (with \( k = 1 \)) we infer that these equations have a unique solution \((\widetilde{\nu}, \widetilde{p}) \in (C^{m-1+\theta}(\mathbb{R}^3))^3 \times C^{m-2+\theta}(\mathbb{R}^3)\), and

\[
\widetilde{\nu} = \widetilde{V}(\sigma, \rho) = \widetilde{\mathcal{F}}(\rho)g(\sigma) + \frac{1}{3} \mathcal{Q}(\rho)\widetilde{B}(\rho)g(\sigma) - \gamma \mathcal{R}(\rho)(\mathcal{K}(\rho)\widetilde{N}(\rho)).
\]  

(2.28)

where \( \mathcal{K}(\rho) = \tilde{K}_\rho \) and \( \widetilde{N}(\rho) = \tilde{n}_\rho \). We note that

\[
\mathcal{K} \in C^\infty(O_{\delta}^{m+\theta}(\mathbb{S}^2), C^{m-2+\theta}(\mathbb{S}^2)), \quad \widetilde{N} \in C^\infty(O_{\delta}^{m+\theta}(\mathbb{S}^2), (C^{m-1+\theta}(\mathbb{S}^2))^3).
\]  

(2.29)

Substituting the above expression of \( \widetilde{\nu} \) into (2.9) and (2.14), and introducing operators \( \mathcal{F} : C^{m+\theta}(\mathbb{R}^3) \times O_{\delta}^{m+\theta}(\mathbb{S}^2) \to C^{m+\theta}(\mathbb{S}^2) \) and \( \mathcal{G} : C^{m+\theta}(\mathbb{R}^3) \times O_{\delta}^{m+\theta}(\mathbb{S}^2) \to C^{m-1+\theta}(\mathbb{S}^2) \) respectively by

\[
\mathcal{F}(\sigma, \rho) = \frac{1}{\varepsilon} A(\rho)\sigma - \chi(|x| - 1)\Pi_1(\widetilde{B}(\rho)\widetilde{V}(\sigma, \rho))\widetilde{B}(\rho)\sigma \cdot e_r - \frac{1}{\varepsilon} f(\sigma),
\]  

(2.30)

\[
\mathcal{G}(\sigma, \rho) = \widetilde{\mathcal{D}}(\rho)\widetilde{V}(\sigma, \rho) = \text{tr}_{S^2}[\widetilde{V}(\sigma, \rho)] \cdot \left[\omega - \frac{1}{1+\rho} \nabla \omega \rho\right],
\]  

(2.31)

(for \( \sigma \in C^{m+\theta}(\mathbb{R}^3) \) and \( \rho \in O_{\delta}^{m+\theta}(\mathbb{S}^2) \)), where as before \( \omega \) represents the variable in \( \mathbb{S}^2 \) and \( \omega(x) = x/|x| \) for \( x \in \mathbb{R}^3 \setminus \{0\} \), we see that the problem (2.9)–(2.18) is reduced to the following problem:

\[
\begin{align*}
\partial_t \sigma &= \mathcal{F}(\sigma, \rho) \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+,
\partial_t \rho &= \mathcal{G}(\sigma, \rho) \quad \text{on} \quad \mathbb{S}^2 \times \mathbb{R}_+,
\sigma &= 1 \quad \text{on} \quad \mathbb{S}^2 \times \mathbb{R}_+,
\sigma|_{t=0} &= \sigma_0 \quad \text{in} \quad \mathbb{R}^3,
\rho|_{t=0} &= \rho_0 \quad \text{on} \quad \mathbb{S}^2.
\end{align*}
\]  

(2.32)

We summarize:

**Lemma 2.3** Let \((\sigma, \tilde{\nu}, \tilde{p}, \rho)\) be a solution of the problem (2.9)–(2.18). Then \((\sigma, \rho)\) is a solution of the problem (2.32). Conversely, if \((\sigma, \rho)\) is a solution of (2.32), then by letting \((\tilde{\nu}, \tilde{p})\) be the unique solution of the problem (2.19)–(2.23) in which \( \varphi, g \) and \( h \) are given by (2.27), we have that \((\sigma, \tilde{\nu}, \tilde{p}, \rho)\) is a solution of (2.9)–(2.18).

The problem (2.30) can be rewritten as an initial value problem of a differential equation in a Banach space. For this purpose we denote

\[
\mathbb{X} = C^{m-2+\theta}(\mathbb{R}^3) \times C^{m-1+\theta}(\mathbb{S}^2), \quad \mathbb{X}_0 = (C^{m+\theta}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)) \times C^{m+\theta}(\mathbb{S}^2),
\]

\[
\mathbb{O}_\delta = (C^{m+\theta}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)) \times O^{m+\theta}(\mathbb{S}^2),
\]

where \( C_0(\mathbb{R}^3) = \{ u \in C(\mathbb{R}^3) : u|_{\mathbb{S}^2} = 0 \} \), and define a bounded nonlinear operator \( \mathcal{F} \) in \( \mathbb{X} \) with domain \( \mathbb{O}_\delta \) (i.e., \( \mathcal{F} : \mathbb{O}_\delta \to \mathbb{X} \)) as follows:

\[
\mathcal{F}(U) = (\mathcal{F}(u + 1, \rho), \mathcal{G}(u + 1, \rho)) \quad \text{for} \quad U = (u, \rho) \in \mathbb{O}_\delta.
\]  

(2.33)
Then (2.32) can be rewritten as an initial value problem of a differential equation in $X$:

$$
\begin{cases}
\frac{dU}{dt} = F(U) & \text{for } t > 0, \\
U|_{t=0} = U_0,
\end{cases}
$$

(3.34)

where $U_0 = (\bar{\sigma} - 1, \rho_0)$. The relation between solutions of (2.32) and (3.34) is that $U = (\bar{\sigma} - 1, \rho)$.

We note that $X_0$ is not dense in $X$.

## 3 Linearization of $F(U)$

Let $(\sigma, v_s, p_s, R_s)$ be the radially symmetric stationary solution of the problem (1.1)–(1.10) (recall that $R_s = 1$) and denote $U_s = (\sigma - 1, 0)$. Then $U_s$ is a stationary solution of the differential equation in (2.31), so that $F(U_s) = 0$.

From (2.6), (2.7) and (2.29) it can be easily seen that $F \in C^\infty(\Omega, X)$, where $\Omega$ is regarded as an open subset of $X_0$. It follows that the Fréchet derivative $DF \in C^\infty(\Omega, L(X_0, X))$. In this section we first derive a useful expression of $DF(U_s)$, and next use it to prove that $DF(U_s)$ is an infinitesimal generator of an analytic semigroup in $X$ with domain $X_0$.

By (2.33) we see that for $V = (v, \eta) \in X_0$, we have

$$
DF(U_s)V = (D_{\bar{\sigma}}F(\sigma_s, 0)v + D_{\rho}F(\sigma_s, 0)\eta, D_{\bar{\sigma}}G(\sigma_s, 0)v + D_{\rho}G(\sigma_s, 0)\eta),
$$

(3.1)

where $D_{\bar{\sigma}}F$ and $D_{\rho}F$ represent Fréchet derivatives of $F(\sigma, \rho)$ in $\bar{\sigma}$ and $\rho$, respectively, and similarly for $D_{\bar{\sigma}}G$ and $D_{\rho}G$. In what follows we deduce expressions of these Fréchet derivatives.

We first note that, clearly,

$$
A(0)u = \Delta u, \quad \tilde{B}(0)u = \nabla u, \quad \tilde{B}(0) \cdot v = \nabla \cdot v, \quad \tilde{B}(0) \otimes v = \nabla \otimes v,
$$

(3.2)

$$
\tilde{D}(0)v = \text{tr}_{S^2}(v) \cdot n_0, \quad K(0) = \tilde{K}_{\rho=0} = 1, \quad \tilde{N}(0) = n_{\rho=0} = n_0,
$$

(3.3)

$$
G_{\rho}(x)|_{\rho=0} = 1, \quad H_{\rho}(x)|_{\rho=0} = 1, \quad \tilde{w}^j(x)|_{\rho=0} = w_j(x).
$$

(3.4)

In (3.3) $n_0$ denotes the unit outward normal of the unit sphere $S^2$, and this notation will be used throughout the remaining part of this paper. We also denote

$$
M(\eta) = \chi(r - 1)\Pi_1(\eta), \quad U(\eta) = \lim_{\epsilon \to 0} \frac{G_{\epsilon\eta} - 1}{\epsilon}, \quad W_j(\eta) = \lim_{\epsilon \to 0} \frac{\tilde{w}^j \eta - w_j}{\epsilon}.
$$

They are evidently linear operators in $\eta$.

**Lemma 3.1** We have

$$
[A'(0)\eta]_{\sigma} = -[\Delta - f'(\sigma_s(r))][\sigma_s'(r)M(\eta)] \quad \text{in } B^3,
$$

(3.5)

$$
[\tilde{B}(0)\eta] \cdot v_s + [\tilde{D}(0)\eta] p_s - \frac{1}{3}[\tilde{B}(0)\eta] g(\sigma_s) = \frac{1}{3} \nabla[g'(\sigma_s)\sigma_s'(r)M(\eta)]
$$

$$
- \nabla[p_s'(r)M(\eta)] + \Delta [v_s'(r)M(\eta) e_r] \quad \text{in } B^3,
$$

(3.6)

$$
- [A'(0)\eta] v_s + [\tilde{B}(0)\eta] p_s - \frac{1}{3}[\tilde{B}(0)\eta] g(\sigma_s) = \frac{1}{3} \nabla[g'(\sigma_s)\sigma_s'(r)M(\eta)]
$$

$$
- \nabla[p_s'(r)M(\eta)] + \Delta [v_s'(r)M(\eta) e_r] \quad \text{in } B^3.
$$

(3.7)
\[
\left[ [\tilde{B}'(0) \eta] \otimes v_s + \left( [\tilde{B}'(0) \eta] \otimes v_s \right)^T \right] n_0 = - \left[ \nabla \otimes [v_s'(r) M(\eta) e_r] + \left( \nabla \otimes [v_s'(r) M(\eta) e_r] \right)^T \right] n_0 \\
+ \left[ p_s'(r) + \frac{2}{3} g'(\sigma_s) \sigma_s'(r) - 4g(1) \right] \eta n_0 \quad \text{on } S^2, \quad (3.8)
\]

\[
[\tilde{B}'(0) \eta] v = -\text{tr}_{S^2}(v) \cdot \nabla_\omega \eta \quad \text{in } \mathbb{B}^3, \quad (3.9)
\]

\[
\int_{|x|<1} \left[ v_s'(r) M(\eta) e_r + U(\eta) v_s \right] dx = 0, \quad (3.10)
\]

\[
\int_{|x|<1} v_s \cdot W_j(\eta) dx = 0, \quad j = 1, 2, 3. \quad (3.11)
\]

**Proof:** (3.5) follows from (5.8) of [2]. To prove (3.6) we denote \( \sigma_{s,\epsilon \eta} = \sigma_s \circ \Phi_{\epsilon \eta} \) and \( v_{s,\epsilon \eta} = v_s \circ \Phi_{\epsilon \eta} \). By making Hanzawa transformation to the equation \( \nabla \cdot v_s = g(\sigma_s) \) we have

\[
\tilde{B}(\epsilon \eta) \cdot v_{s,\epsilon \eta} = g(\sigma_{s,\epsilon \eta}). \quad (3.12)
\]

Since \( v_{s,\epsilon \eta}|_{\epsilon = 0} = v_s \) and \( \sigma_{s,\epsilon \eta}|_{\epsilon = 0} = \sigma_s \), we get

\[
[\tilde{B}(\epsilon \eta) - \tilde{B}(0)] \cdot v_{s,\epsilon \eta} + \tilde{B}(0) \cdot [v_{s,\epsilon \eta} - v_s] = g(\sigma_{s,\epsilon \eta}) - g(\sigma_s).
\]

Dividing both sides with \( \epsilon \), then letting \( \epsilon \to 0 \) and using the relations

\[
\lim_{\epsilon \to 0} \frac{\sigma_{s,\epsilon \eta} - \sigma_s}{\epsilon} = \sigma_s'(r) M(\eta), \quad \lim_{\epsilon \to 0} \frac{v_{s,\epsilon \eta} - v_s}{\epsilon} = v_s'(r) M(\eta) e_r, \quad (3.13)
\]

we see that (3.6) follows. To prove (3.7) we denote \( p_{s,\epsilon \eta} = p_s \circ \Phi_{\epsilon \eta} \). By making Hanzawa transformation to the equation \( -\Delta v_s + \nabla p_s - \frac{1}{3} \nabla (g(\sigma_s)) = 0 \) we get

\[
- A(\epsilon \eta) v_{s,\epsilon \eta} + \tilde{B}(\epsilon \eta) p_{s,\epsilon \eta} - \frac{1}{3} \tilde{B}(\epsilon \eta)(g(\sigma_{s,\epsilon \eta})) = 0. \quad (3.14)
\]

Since \( p_{s,\epsilon \eta}|_{\epsilon = 0} = p_s \) and

\[
\lim_{\epsilon \to 0} \frac{p_{s,\epsilon \eta} - p_s}{\epsilon} = p_s'(r) M(\eta), \quad (3.15)
\]

by a similar argument as before we obtain (3.7).

Next we prove (3.8). We denote \( \tilde{e}''_{\epsilon \eta} = e_r \circ \Phi_{\epsilon \eta} \). By making Hanzawa transformation to the equation \( T(v_s, p_s)|_{S^2} n_0 = -\gamma n_0 \) and noticing that \( e_r|_{S^2} = n_0 \), we get

\[
T_{\epsilon \eta}(v_{s,\epsilon \eta}, p_{s,\epsilon \eta}) \tilde{e}''_{\epsilon \eta} + \gamma \tilde{e}''_{\epsilon \eta} = 0 \quad \text{on } \Phi_{\epsilon \eta}^{-1}(S^2). \quad (3.16)
\]

By (3.13) and (3.15) we have \( v_{s,\epsilon \eta} = v_s + \epsilon v_s'(r) M(\eta) e_r + o(\epsilon) \) and \( p_{s,\epsilon \eta} = p_s + \epsilon p_s'(r) M(\eta) + o(\epsilon) \). Thus

\[
T_{\epsilon \eta}(v_{s,\epsilon \eta}, p_{s,\epsilon \eta}) = \left[ \tilde{B}(\epsilon \eta) \otimes v_{s,\epsilon \eta} + \left( \tilde{B}(\epsilon \eta) \otimes v_{s,\epsilon \eta} \right)^T \right] - \left[ p_{s,\epsilon \eta} + \frac{2}{3} \tilde{B}(\epsilon \eta) \cdot v_{s,\epsilon \eta} \right] I
\]

\[
= \left[ \tilde{B}(0) \otimes v_s + \left( \tilde{B}(0) \otimes v_s \right)^T \right] - \left[ p_s + \frac{2}{3} \tilde{B}(0) \cdot v_s \right] I
\]

\[
+ \epsilon \left[ \tilde{B}'(0) \eta \otimes v_s + \left( \tilde{B}'(0) \eta \otimes v_s \right)^T \right] - \frac{2}{3} \tilde{B}'(0) \eta \cdot v_s I
\]

\[
+ \tilde{B}(0) \otimes [v_s'(r) M(\eta) e_r] + \left( \tilde{B}(0) \otimes [v_s'(r) M(\eta) e_r] \right)^T
\]

\[
- p_s(r) M(\eta) I - \frac{2}{3} \tilde{B}(0) \cdot [v_s'(r) M(\eta) \omega] I \right) + o(\epsilon).
\]
Noticing that
\[
\left[ \tilde{E}(0) \otimes v_s + (\tilde{E}(0) \otimes v_s)^T \right] - \left[ p_s + \frac{2}{3} \tilde{E}(0) \cdot v_s \right] I = T(v_s, p_s),
\]
and denoting by \(L(\eta)\) the expression in the braces, we see that the above result can be briefly rewritten as follows:
\[
\tilde{T}_{\eta}(v_s, p_s, \epsilon) = T(v_s, p_s) + \epsilon L(\eta) + o(\epsilon).
\]
Since for \(x \in \Phi^{-1}(\mathbb{S}^2)\) we have
\[
\Phi_{\eta}(x) = x + \epsilon \chi(r - 1) \Pi_1(\eta)(x)x/r = [r + \epsilon \chi(r - 1) \Pi_1(\eta)(x)]\omega(x),
\]
where \(\omega(x) = x/r\), and \(\Phi_{\eta}(x) \in \mathbb{S}^2\), we see that \(\Phi_{\eta}(x) = \omega(x)\) for all \(x \in \Phi^{-1}(\mathbb{S}^2)\). This implies that \(\tilde{e}_{\eta}(x) = e_r(\omega(x)) = n_0(\omega(x))\) for all \(x \in \Phi^{-1}(\mathbb{S}^2)\). Hence, from (3.16) and (3.17) we get
\[
0 = \left[ T_{\eta}(v_s, p_s, \epsilon) e_{\eta} + \gamma \tilde{e}_{\eta} \right]_{\Phi^{-1}_{\eta}(\mathbb{S}^2)} = \left[ T(v_s, p_s) n_0 + \gamma n_0 - \epsilon \eta \frac{\partial}{\partial r} \left( T(v_s, p_s) n_0 \right) \right]_{\mathbb{S}^2} + \epsilon L(\eta) + o(\epsilon).
\]
Points on \(\Phi^{-1}_{\eta}(\mathbb{S}^2)\) and \(\mathbb{S}^2\) such that the last equality holds are related by the relation \(\omega = \omega(x)\) for \(x \in \Phi^{-1}_{\eta}(\mathbb{S}^2)\) and \(\omega \in \mathbb{S}^2\), and in getting the last equality we used the following relations:
\[
\left[ T(v_s, p_s) n_0 + \gamma n_0 - \epsilon \eta \frac{\partial}{\partial r} \left( T(v_s, p_s) n_0 \right) \right]_{\mathbb{S}^2} + o(\epsilon),
\]
\[
L(\eta)_{\Phi^{-1}_{\eta}(\mathbb{S}^2)} = L(\eta)_{\mathbb{S}^2} + O(\epsilon).
\]
The proof of the first relation uses a similar argument as that used in (4.29) of \([10]\), and the second relation is immediate. Since \(\left| T(v_s, p_s) n_0 + \gamma n_0 \right|_{\mathbb{S}^2} = 0\), \(M(\eta)|_{\mathbb{S}^2} = \eta\), and by the result in Appendix A of \([14]\) we have
\[
\frac{\partial}{\partial r} \left( T(v_s, p_s) n_0 \right)_{\mathbb{S}^2} = 2v_s'\prime(r) - p_s'(r) - \frac{2}{3} g'(\sigma_s) \sigma_s'(r)|_{r=1} n_0 = -4g(1)n_0,
\]
by dividing (3.18) with \(\epsilon\), then letting \(\epsilon \to 0\) and using (3.6), we see that (3.8) follows.

Finally, (3.9) is immediate, and (3.10), (3.11) follow from the relations \(\int_{\mathbb{S}^2} v_s dx = 0\), \(\int_{\mathbb{S}^2} v_s \times x dx = 0\) and a similar argument as above, which we omit here. This completes the proof of Lemma 3.1. \(\square\)

**Lemma 3.2** For \(v \in C^{m+\theta}(\overline{\mathbb{S}^2})\) and \(\eta \in C^{m+\theta}(\mathbb{S}^2)\) we have
\[
D_\sigma \tilde{V}(\sigma_s, 0)v = \tilde{P}(0)|g'(\sigma_s)v| + \frac{1}{3} Q(0) \left\{ \nabla [g'(\sigma_s)v] \right\},
\]
\[
D_\eta \tilde{V}(\sigma_s, 0)\eta = v_s'(r) M(\eta) e_r - \tilde{P}(0) \left[ g'(\sigma_s) \sigma_s'(r) M(\eta) \right] - \frac{1}{3} Q(0) \left\{ \nabla [g'(\sigma_s) \sigma_s'(r) M(\eta)] \right\}
+ R(0) \left[ \gamma(\eta + \frac{1}{2} \Delta_\omega \eta) n_0 - 2g(1) \nabla_\omega \eta + 4g(1)\eta n_0 \right].
\]
Proof: From the definition of $\bar{V}(\bar{\sigma}, \rho)$ it is clear that

$$D_{\vec{g}}\bar{V}(\sigma_s, 0)v = \bar{P}(0)[g'(\sigma_s)v] + \frac{1}{3}Q(0)\bar{B}(0)[g'(\sigma_s)v].$$

Since $\bar{B}(0) = \nabla$, we see that (3.19) follows.

To compute $V \equiv D_{\rho}\bar{V}(\sigma_s, 0)\eta$ we denote $\bar{v} = \bar{V}(\sigma_s, \epsilon\eta)$, where $\eta \in C^{m+\theta}(\mathbb{S}^3)$ is given. By the definition of $\bar{V}(\bar{\sigma}, \rho)$ we see that there exists a function $\bar{p} \in C^{m-1+\theta}(\mathbb{B}^3)$ such that $(\bar{v}, \bar{p})$ is the unique solution of the problem

$$\bar{B}(\epsilon\eta) \cdot \bar{v} = g(\sigma_s) \quad \text{in } \mathbb{B}^3,$$

$$- A(\epsilon\eta)\bar{v} + \bar{B}(\epsilon\eta)\bar{p} = \frac{1}{3}\bar{B}(\epsilon\eta)(g(\sigma_s)) \quad \text{in } \mathbb{B}^3,$$

$$\bar{T}_{\epsilon\eta}(\bar{v}, \bar{p})\bar{n}_{\epsilon\eta} = -\gamma\bar{\kappa}_{\epsilon\eta}\bar{n}_{\epsilon\eta} \quad \text{on } \mathbb{S}^2,$$

$$\int_{|x|<1} \bar{v}(x)G_{\epsilon\eta}(x)dx = 0,$$

$$\int_{|x|<1} \bar{v}(x) \cdot \bar{w}_{\epsilon\eta}^j(x)G_{\epsilon\eta}(x)dx = 0, \quad j = 1, 2, 3.$$ (3.25)

We note that the above problem does have a unique solution. Indeed, from the proof of (2.34) of [14] we see that for any sufficiently small $\epsilon > 0$, the conditions (2.24) and (2.25) are satisfied by $\varphi = g(\sigma_s)$, $g = \frac{1}{3}\bar{B}(\epsilon\eta)(g(\sigma_s))$ and $h = -\gamma\bar{\kappa}_{\epsilon\eta}\bar{n}_{\epsilon\eta}$ with $\rho = \epsilon\eta$. Hence the desired assertion follows from Lemma 2.2.

Clearly, $\lim_{\epsilon \to 0} \bar{v} = v_s$, $\lim_{\epsilon \to 0} \bar{p} = p_s$ and $V = \lim_{\epsilon \to 0} \epsilon^{-1}(\bar{v} - v_s)$. Hence, by a similar argument as in the proof of (3.6) and (3.7) we get, from (3.21) and (3.22) respectively, that

$$\nabla \cdot V = -[\bar{B}(\epsilon\eta)] \cdot v_s \quad \text{in } \mathbb{B}^3,$$ (3.26)

and

$$- \Delta V + \nabla P = [A'(\epsilon\eta)]v_s - [\bar{B}(\epsilon\eta)]p_s + \frac{1}{3}[\bar{B}(\epsilon\eta)]g(\sigma_s) \quad \text{in } \mathbb{B}^3,$$ (3.27)

where $P = \lim_{\epsilon \to 0} \epsilon^{-1}(\bar{p} - p_s)$. Next, recalling that

$$\bar{n}_{\epsilon\eta} = n_0 - \epsilon \nabla_\omega \eta + o(\epsilon) \quad \text{and} \quad \bar{\kappa}_{\epsilon\eta} = 1 - \epsilon[\eta + \frac{1}{2}\Delta_\omega \eta] + o(\epsilon),$$

where as before $n$ and $\kappa$ are the unit outward normal and the mean curvature of $\Gamma_\rho(= \phi_\rho(\mathbb{S}^2))$, respectively, by a direct computation we easily obtain

$$\bar{n}_{\epsilon\eta} = n_0 - \epsilon \nabla_\omega \eta + o(\epsilon) \quad \text{and} \quad \bar{\kappa}_{\epsilon\eta} = 1 - \epsilon[\eta + \frac{1}{2}\Delta_\omega \eta] + o(\epsilon).$$

Thus similarly as in the proof of (3.8) we get from (3.23) that

$$[\nabla \otimes V + (\nabla \otimes V)^T]n_0 = -\left\{[\bar{B}(\epsilon\eta)] \otimes v_s + ([\bar{B}(\epsilon\eta)] \otimes v_s)^T\right\}n_0$$

$$+ \gamma\{\nabla_\omega \eta + [\eta + \frac{1}{2}\Delta_\omega \eta]n_0\} + Pn_0 + T(v_s, p_s)\nabla_\omega \eta \quad \text{on } \mathbb{S}^2.$$
A direct computation shows that (cf. (4.33) of [10])
\[ T(v_s, p_s) \big|_{S^2} \nabla_\omega \eta = -(\gamma + 2g(1))\nabla_\omega \eta. \]

Hence by using (1.11), (3.26) and the above result we obtain
\[ T(V, P) n_0 = [\nabla \otimes V + (\nabla \otimes V)^T] n_0 - [P + \frac{2}{3} \nabla \cdot V] n_0 \]
\[ = -\left\{ [\bar{B}'(0) \eta] \otimes v_s + ([\bar{B}'(0) \eta] \otimes v_s)^T \right\} n_0 + \frac{2}{3} \{ [\bar{B}'(0) \eta] \cdot v_s \} n_0 \]
\[ + \gamma [\eta + \frac{1}{2} \Delta_\omega \eta] n_0 - 2g(1)\nabla_\omega \eta \quad \text{on } S^2. \]

Finally, similarly as in the proof of (3.9) and (3.10) we get from (3.24) and (3.25) that
\[ \int_{|x| < 1} [D_\rho \bar{V}(\sigma_s, 0) \eta + U(\eta) v_s] \, dx = 0, \]
\[ \int_{|x| < 1} \left\{ [D_\rho \bar{V}(\sigma_s, 0) \eta] \cdot w_j + v_s \cdot W_j(\eta) \right\} \, dx = 0, \quad j = 1, 2, 3, \]
respectively. Now let \( V_1 = V - v'_s(r) M(\eta) n_0 \) and \( P_1 = P - p'_s(r) M(\eta) \). Then from (3.6)–(3.11) and (3.26)–(3.30) we easily obtain
\[ \nabla \cdot V_1 = -g'(\sigma_s) \sigma'_s(r) M(\eta) \quad \text{in } \mathbb{B}^3, \]
\[ -\Delta V_1 + \nabla P_1 = -\frac{1}{3} [g'(\sigma_s) \sigma'_s(r) M(\eta)] \quad \text{in } \mathbb{B}^3, \]
\[ T(V_1, P_1) n_0 = \gamma (\eta + \frac{1}{2} \Delta_\omega \eta) n_0 - 2g(1)\nabla_\omega \eta + 4g(1)\eta n_0 \quad \text{on } S^2, \]
\[ \int_{|x| < 1} V_1 \, dx = 0, \quad j = 1, 2, 3. \]

In getting (3.35) we also used the fact that \( n_0 \cdot w_j = 0 \) for \( j = 1, 2, 3 \). Using the relations
\[ \int_{|x| = 1} \nabla_\omega \eta \cdot w_j \, dS_\omega = -\int_{|x| = 1} \eta \nabla_\omega \cdot w_j \, dS_\omega = 0 \quad (j = 1, 2, 3) \]
we easily see that the relations (2.24) and (2.25) with \( \rho = 0 \) are satisfied by
\[ \varphi = -g'(\sigma_s) \sigma'_s(r) M(\eta), \quad g = -\frac{1}{3} \nabla [g'(\sigma_s) \sigma'_s(r) M(\eta)], \]
and
\[ h = \gamma (\eta + \frac{1}{2} \Delta_\omega \eta) n_0 - 2g(1)\nabla_\omega \eta + 4g(1)\eta n_0. \]

Hence by Lemma 2.2 we see that the problem (3.31)–(3.35) has a unique solution \( (V_1, P_1) \) and, in particular, \( V_1 \) is given by
\[ V_1 = -\bar{P}(0) [g'(\sigma_s) \sigma'_s(r) M(\eta)] - \frac{1}{3} \bar{Q}(0) \{ \nabla [g'(\sigma_s) \sigma'_s(r) M(\eta)] \}
+ R(0) [\gamma (\eta + \frac{1}{2} \Delta_\omega \eta) n_0 - 2g(1)\nabla_\omega \eta + 4g(1)\eta n_0], \]
from which (3.20) immediately follows. This completes the proof of Lemma 3.2. \qed

We are now ready to compute all the Fréchet derivatives appearing in the right-hand side of (3.1). First, by (2.31), (3.3) and (3.19) we have

$$\begin{align*}
D_\sigma G(\sigma_s,0)v &= \bar{\mathcal{D}}(0)[D_\sigma \bar{\mathcal{V}}(\sigma_s,0)v] = \text{tr}_{\mathbb{S}^2} \left\{ \bar{\mathcal{P}}(0)[g'(\sigma_s)v] + \frac{1}{3} Q(0)[\nabla (g'(\sigma_s)v)] \right\} \cdot n_0. \quad (3.37)
\end{align*}$$

Next, by (3.9) and the facts that $\bar{V}(\sigma_s,0) = v_s$, $\text{tr}_{\mathbb{S}^2}(v_s) = 0$ we have

$$\begin{align*}
[\bar{\mathcal{D}}'(0)\eta]\bar{\mathcal{V}}(\sigma_s,0) = -\text{tr}_{\mathbb{S}^2}[\bar{V}(\sigma_s,0)\cdot \nabla \omega \eta] = -\text{tr}_{\mathbb{S}^2}(v_s) \cdot \nabla \omega \eta = 0.
\end{align*}$$

Thus by (2.31), (3.3) and (3.20) we have

$$\begin{align*}
D_\rho G(\sigma_s,0)\eta &= [\bar{\mathcal{D}}'(0)\eta]\bar{\mathcal{V}}(\sigma_s,0) + \bar{\mathcal{D}}(0)[D_\rho \bar{\mathcal{V}}(\sigma_s,0)\eta] \\
&= \text{tr}_{\mathbb{S}^2}[D_\rho \bar{\mathcal{V}}(\sigma_s,0)\eta] \cdot n_0 \\
&= g(1) \eta - \text{tr}_{\mathbb{S}^2} \left\{ \bar{\mathcal{P}}(0) [g'(\sigma_s)\sigma'_s(r) M(\eta)] + \frac{1}{3} Q(0) \left\{ \nabla [g'(\sigma_s)\sigma'_s(r) M(\eta)] \right\} \right. \\
&- \left. R(0) \left\{ \gamma(\eta + \frac{1}{2} \Delta \omega \eta) n_0 - 2g(1) \nabla \omega \eta + 4g(1) \eta n_0 \right\} \right\} \cdot n_0. \quad (3.38)
\end{align*}$$

Thirdly, from (2.30) and a direct computation we have

$$\begin{align*}
D_\sigma F(\sigma_s,0)v &= \varepsilon^{-1} A(0)v - \varepsilon^{-1} f'(\sigma_s)v + \chi(\sigma_s) \Pi_1 \{ \bar{\mathcal{D}}(0)[D_\sigma \bar{\mathcal{V}}(\sigma_s,0)v] \} \bar{\mathcal{B}}(0) \sigma_s \cdot \mathbf{e}_r \\
&+ \chi(\sigma_s) \Pi_1 \{ \bar{\mathcal{D}}(0) \bar{\mathcal{V}}(\sigma_s,0) \} \bar{\mathcal{B}}(0)v \cdot \mathbf{e}_r \\
&= \varepsilon^{-1} [\Delta - f'(\sigma_s)]v + \chi(\sigma_s) \Pi_1 \{ \bar{\mathcal{D}}(0)[D_\sigma \bar{\mathcal{V}}(\sigma_s,0)v] \} \sigma'_s(r) \\
&+ \chi(\sigma_s) \Pi_1 \{ \text{tr}_{\mathbb{S}^2}(v_s) \cdot n_0 \} \nabla v \cdot \mathbf{e}_r \\
&= \varepsilon^{-1} [\Delta - f'(\sigma_s)]v + \chi(\sigma_s) \Pi_1 \{ \text{tr}_{\mathbb{S}^2}(\bar{\mathcal{P}}(0)) [g'(\sigma_s)v] \} \\
&+ \frac{1}{3} Q(0) \left\{ \nabla [g'(\sigma_s)\sigma'_s(r)] \right\} \cdot n_0. \quad (3.39)
\end{align*}$$

Finally, from (2.30), (3.5), (3.38) and a direct computation we have

$$\begin{align*}
D_\rho F(\sigma_s,0)\eta &= \varepsilon^{-1} [A'(0)\eta] \sigma_s + \chi(\sigma_s) \Pi_1 \{ D_\rho G(\sigma_s,0)\eta \} \bar{\mathcal{B}}(0) \sigma_s \cdot \mathbf{e}_r \\
&+ \chi(\sigma_s) \Pi_1 \{ \bar{\mathcal{D}}(0) \bar{\mathcal{V}}(\sigma_s,0) \} \bar{\mathcal{B}}(0) \sigma_s \\
&= \varepsilon^{-1} [A'(0)\eta] \sigma_s + \chi(\sigma_s) \Pi_1 \{ D_\rho G(\sigma_s,0)\eta \} \sigma'_s(r) \\
&\quad \text{(because $\bar{\mathcal{D}}(0) \bar{\mathcal{V}}(\sigma_s,0) = \text{tr}_{\mathbb{S}^2}(v_s) \cdot n_0 = 0$)} \\
&= -\varepsilon^{-1} [\Delta - f'(\sigma_s)][\chi(\sigma_s) \Pi_1(\eta)] + g(1) \sigma'_s(r) M(\eta) \\
&- \chi(\sigma_s) \sigma'_s(r) \Pi_1 \{ \text{tr}_{\mathbb{S}^2}(\bar{\mathcal{P}}(0)) [g'(\sigma_s)\sigma'_s(r) M(\eta)] \} \\
&+ \frac{1}{3} Q(0) \left\{ \nabla [g'(\sigma_s)\sigma'_s(r) M(\eta)] \right\} - R(0) \left\{ \gamma(\eta + \frac{1}{2} \Delta \omega \eta) n_0 \\
&- 2g(1) \nabla \omega \eta + 4g(1) \eta n_0 \right\} \cdot n_0. \quad (3.40)
\end{align*}$$

In conclusion, we have the following lemma.

**Lemma 3.3** The Fréchet derivative $DF(U_s)$ is given by (3.1), in which $D_\sigma F(\sigma_s,0)$, $D_\rho F(\sigma_s,0)$, $D_\sigma G(\sigma_s,0)$ and $D_\rho G(\sigma_s,0)$ are given by (3.39), (3.40), (3.37) and (3.38), respectively. \qed
As usual, for a linear operator $L$ from a product space $X_1 \times X_2$ to another product space $Y_1 \times Y_2$ having the expression

$$L(x_1, x_2) = (L_{11}x_1 + L_{12}x_2, L_{21}x_1 + L_{22}x_2) \quad \text{for} \quad (x_1, x_2) \in X_1 \times X_2,$$

we write it as $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$. Then we have

$$DF(U_s) = \begin{pmatrix} D_\theta F(\sigma_s, 0) & D_\rho F(\sigma_s, 0) \\ D_\theta G(\sigma_s, 0) & D_\rho G(\sigma_s, 0) \end{pmatrix}.$$ 

In the sequel we follow the idea of [2] to study the property of this operator and compute its spectrum.

Recall that $m \geq 3$, $m \in \mathbb{N}$ and $\theta \in (0, 1)$. Let $\mathcal{A}_0 : C^{m+\theta}(\mathbb{B}^3) \to C^{m-2+\theta}(\mathbb{B}^3)$ and $\mathcal{J} : C^{m+\theta}(\mathbb{B}^3) \to C^{m+\theta}(\mathbb{S}^2)$ be the following operators:

$$\mathcal{A}_0 v = [\Delta - f'(\sigma_s)]v \quad \text{for} \quad v \in C^{m+\theta}(\mathbb{B}^3), \quad (3.41)$$

$$\mathcal{J} v = \text{tr}_{\mathbb{S}^2} [\mathcal{F}(0)[g'/(\sigma_s)v] + \frac{1}{3} \mathcal{Q}(0)(\nabla(g'(\sigma_s)v))] \cdot n_0 \quad \text{for} \quad v \in C^{m+\theta}(\mathbb{B}^3). \quad (3.42)$$

Clearly, $\mathcal{A}_0 \in L(C^{m+\theta}(\mathbb{B}^3), C^{m-2+\theta}(\mathbb{B}^3))$, and

$$\mathcal{J} \in L(C^{m+\theta}(\mathbb{B}^3), C^{m-2+\theta}(\mathbb{S}^2)) \cap L(C^{m-2+\theta}(\mathbb{B}^3), C^{m-2+\theta}(\mathbb{S}^2)). \quad (3.43)$$

Next let $\Pi_0 : C^{m-2+\theta}(\mathbb{S}^2) \to C^{m-2+\theta}(\mathbb{B}^3)$ be the operator defined by $\Pi_0(\eta) = u$ for $\eta \in C^{m-2+\theta}(\mathbb{S}^2)$, where $u \in C^{m-2+\theta}(\mathbb{B}^3)$ is the unique solution of the following boundary value problem:

$$[\Delta - f'(\sigma_s)]u = 0 \quad \text{in} \quad \mathbb{B}^3, \quad u = \eta \quad \text{on} \quad \mathbb{S}^2.$$

It is clear that $\Pi_0 \in L(C^{m-2+\theta}(\mathbb{S}^2), C^{m-2+\theta}(\mathbb{B}^3)) \cap L(C^{m+\theta}(\mathbb{S}^2), C^{m+\theta}(\mathbb{B}^3))$ and $\mathcal{A}_0 \Pi_0 = 0$. We also let $\mathcal{B}_\gamma$ be the following operator from $C^{m+\theta}(\mathbb{S}^2)$ to $C^{m-1+\theta}(\mathbb{S}^2)$:

$$\mathcal{B}_\gamma \eta = g(1)\eta - \sigma'_s(1)\mathcal{J}\Pi_0(\eta) + \text{tr}_{\mathbb{S}^2} \left\{ \mathcal{R}(0) \left[ \gamma(\eta) \frac{1}{3} \Delta \omega \eta \right] n_0 \\
-2g(1)\nabla\omega \eta + 4g(1)n_0 \right\} \cdot n_0 \quad \text{for} \quad \eta \in C^{m+\theta}(\mathbb{S}^2). \quad (3.44)$$

It is evident that $\mathcal{B}_\gamma \in L(C^{m+\theta}(\mathbb{S}^2), C^{m-1+\theta}(\mathbb{S}^2))$. Clearly,

$$\begin{cases}
D_\theta F(\sigma_s, 0)v = \varepsilon^{-1} \mathcal{A}_0 v + \chi(r-1)\sigma'_s(r)\Pi_1 \mathcal{J} v, \\
D_\theta G(\sigma_s, 0)v = \mathcal{J} v,
\end{cases}$$

$$\begin{cases}
D_\rho F(\sigma_s, 0)\eta = \chi(r-1)\sigma'_s(r)\Pi_1(\eta) \left[ \mathcal{B}_\gamma \eta + \sigma'_s(1)\mathcal{J}\Pi_0(\eta) - \mathcal{J}(\chi(r-1)\sigma'_s(r)\Pi_1(\eta)) \right] \\
D_\rho G(\sigma_s, 0)\eta = \mathcal{B}_\gamma \eta + \sigma'_s(1)\mathcal{J}\Pi_0(\eta) - \mathcal{J}(\sigma'_s(r)r\Pi_1(\eta)).
\end{cases} \quad (3.45)$$

Finally, let $\mathcal{M} : X_0 \to X$ and $\mathcal{T} : X \to X$ be the following operators:

$$\mathcal{M} = \begin{pmatrix} \mathcal{J} & \mathcal{A}_0 \mathcal{J} \\ \mathcal{B}_\gamma \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} I & -\sigma'_s(1)\Pi_1 + \chi(r-1)\sigma'_s(r)\Pi_1 \\ 0 & I \end{pmatrix}. $$
It is easy to see that $\mathcal{M} \in L(\mathcal{X}_0, \mathcal{X})$ and $\mathcal{T} \in L(\mathcal{X})$. Moreover, since $\text{tr}_{\mathcal{G}}\{ -\sigma_s'(1)\Pi_0(\eta) + \chi(r-1)\sigma_s'(r)\Pi_1(\eta) \} = -\sigma_s'(1)\eta + \sigma_s'(1)\eta = 0$ for any $\eta \in C^{m+\theta}(\mathcal{S}^2)$, we see that $\mathcal{T}$ maps $\mathcal{X}_0$ into itself, i.e. $\mathcal{T} \in L(\mathcal{X}) \cap L(\mathcal{X}_0)$. Besides, it can be easily seen that

$$\mathcal{T}^{-1} = \begin{pmatrix} I & \sigma_s'(1)\Pi_0 - \sigma_s'(r)\chi(r-1)\Pi_1 \\ 0 & I \end{pmatrix}.$$ 

By a simple computation we have

**Lemma 3.4** \(D\mathcal{F}(U_s) = \mathcal{T}\mathcal{M}\mathcal{T}^{-1}. \) \(\square\)

Given a closed linear operator $L$ on a Banach space, we denote by $\sigma(L)$ and $\sigma_p(L)$ respectively the spectrum and the set of all eigenvalues of $L$. As an immediate consequence of Lemma 3.4 we have the following preliminary result:

**Lemma 3.5** (i) Regarded as an unbounded linear operator in $\mathcal{X}$ with domain $\mathcal{X}_0$, $D\mathcal{F}(U_s)$ is an infinitesimal generator of an analytic semigroup in $\mathcal{X}$.

(ii) If $\delta$ is sufficiently small then for any $U \in \mathcal{Q}_\delta$ and in a neighborhood of $U_s$ in $\mathcal{X}_0$, we have that $D\mathcal{F}(U)$, regarded as an unbounded linear operator in $\mathcal{X}$ with domain $\mathcal{X}_0$, is an infinitesimal generator of an analytic semigroup in $\mathcal{X}$.

(iii) Let $V \in \mathcal{X}_0$ and $\lambda \in \mathbb{C}$. Then $D\mathcal{F}(U_s)V = \lambda V$ if and only if $\mathcal{M}(\mathcal{T}^{-1}V) = \lambda \mathcal{T}^{-1}V$.

(iv) $\sigma(D\mathcal{F}(U_s)) = \sigma_p(D\mathcal{F}(U_s)) = \sigma_p(\mathcal{M}) = \sigma(\mathcal{M})$.

**Proof:** (i) By Lemma 3.4 it suffices to prove that the operator $\mathcal{M}$, regarded as an unbounded linear operator in $\mathcal{X}$ with domain $\mathcal{X}_0$, is an infinitesimal generator of an analytic semigroup in $\mathcal{X}$. We denote

$$\mathcal{M}_1 = \begin{pmatrix} \varepsilon^{-1}A_0 + \sigma_s'(1)\Pi_0\mathcal{J} & \sigma_s'(1)\Pi_0\mathcal{B}_\gamma \\ 0 & \mathcal{B}_\gamma \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ \mathcal{J} & 0 \end{pmatrix}.$$ 

Then $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. From (3.43) it can be easily seen that $\mathcal{M}_2 \in L(\mathcal{X})$. Thus by a standard result for perturbations of generators of analytic semigroups (see [12]), we only need to show that $\mathcal{M}_1$, regarded as an unbounded linear operator in $\mathcal{X}$ with domain $\mathcal{X}_0$, is an infinitesimal generator of an analytic semigroup in $\mathcal{X}$.

Clearly, the operator $\varepsilon^{-1}A_0 + \sigma_s'(1)\Pi_0\mathcal{J}$ is an infinitesimal generator of an analytic semigroup in $C^{m-2+\theta}(\mathbb{B}^3)$ (with domain $C^{m+\theta}(\mathbb{B}^3)$). Next, from (3.42), (3.44) and the definition of $\Pi_0$ it can be easily seen that the operator $\mathcal{B}_\gamma$ can be rewritten in the following form:

$$\mathcal{B}_\gamma \eta = \text{tr}_{\mathcal{G}}(\tilde{v}) \cdot \boldsymbol{n}_0 + g(1)\eta \quad \text{for} \quad \eta \in C^{m+\theta}(\mathbb{S}^2),$$

where $\tilde{v}$ is the second component of the solution $(\phi, \tilde{v}, \psi)$ of the following problem:

$$\Delta \phi = f'(\sigma_s)\phi \quad \text{in} \quad \mathbb{B}^3,$$

$$\nabla \cdot \tilde{v} = g'(\sigma_s)\phi \quad \text{in} \quad \mathbb{B}^3,$$

$$-\Delta \tilde{v} + \nabla \psi - \frac{1}{3} \nabla (\nabla \cdot \tilde{v}) = 0 \quad \text{in} \quad \mathbb{B}^3,$$
\[ \phi = -\sigma'_s(1)\eta \quad \text{on } S^2, \]
\[ \mathbf{T}(\vec{v}, \psi)\mathbf{n}_0 = -2g(1)\nabla_\omega \eta + \gamma(\eta + \frac{1}{2}\Delta_\omega \eta)\mathbf{n}_0 + 4g(1)\eta\mathbf{n}_0 \quad \text{on } S^2, \]
\[ \int_{|\mathbf{x}|<1} \vec{v} \, d\mathbf{x} = 0, \]
\[ \int_{|\mathbf{x}|<1} \vec{v} \times \mathbf{x} \, d\mathbf{x} = 0. \]

This shows that \( \mathcal{B}_\gamma \) is the same operator as that given by (3.9) of [14] with the same notation. Thus by Lemma 2.6 and Lemma 3.1 of [14] we have:

For any \( \mathbf{F} \text{ from Lemma 3.2 and Lemma 3.3 of [14]} \) we see that \( \mathcal{B}_\gamma \) is an infinitesimal generator of an analytic semigroup in \( C^{m-1+\theta}(S^2) \) (with domain \( C^{m+\theta}(S^2) \)). Besides, for any \( \eta \in C^{m+\theta}(S^2) \) we have

\[ ||\sigma'_s(1)\Pi_0\mathcal{B}_\gamma\eta||_{C^{m-2+\theta}(\overline{\mathbb{F}})} \leq C ||\mathcal{B}_\gamma\eta||_{C^{m-2+\theta}(S^2)} \leq C ||\mathcal{B}_\gamma\eta||_{C^{m-1+\theta}(S^2)}, \]

i.e., \( \sigma'_s(1)\Pi_0\mathcal{B}_\gamma \) is \( \mathcal{B}_\gamma \)-bounded in the notion of [14]. Hence, by Corollary 3.3 of [14] (see also Lemma 3.2 of [1]) we see that \( \mathcal{M}_1 \) is an infinitesimal generator of an analytic semigroup in \( \mathbb{X} \) (with domain \( \mathbb{X}_0 \)), as desired. This proves the assertion \((i)\).

The assertion \((ii)\) is an easy consequence of the assertion \((i)\) and the fact that \( \mathbb{F} \in C^\infty(\mathbb{Q}_\delta, \mathbb{X}) \), and the assertion \((iii)\) is immediate. Finally, since \( \mathbb{X}_0 \) is compactly embedding into \( \mathbb{X} \), the assertion \((iv)\) follows from \((i)\) and \((iii)\). The proof is complete. \( \square \)

Later on we always assume that \( \delta \) is sufficiently small such that the open set \( \mathbb{Q}_\delta \) satisfies the condition of Lemma 3.4 \((i)\).

As in [14], for every integer \( l \geq 0 \) we let \( Y_{lm}(\omega) \), \( m = -l, -l+1, \ldots, l-1, l \), be a normalized orthogonal basis (in \( L^2(S^2) \) sense) of the space of all spherical harmonics of degree \( l \). It is well-known that

\[ \Delta_\omega Y_{lm}(\omega) = -(l^2 + l)Y_{lm}(\omega). \]

For every such \( l \) we denote by \( F_l(r) \) the unique solution of the following ODE problem:

\[ \begin{cases} F''_l(r) + \frac{2}{r} F'_l(r) - \frac{l^2 + l}{r^2} F_l(r) = f'(\sigma_s(r))F_l(r) & \text{for } 0 < r < 1, \\ F'_l(0) = 0, \quad F_l(1) = -\sigma'_s(1). \end{cases} \]

Next we denote

\[ \alpha_0 = g(1) + \int_0^1 g'(\sigma_s(r))F_0(r)r^2 \, dr, \]

\[ \alpha_l(\gamma) = -\frac{l(l+2)(2l+1)}{4(2l^2 + 4l + 3)}(\gamma - \gamma_l), \quad \text{for } l \geq 2, \]

where

\[ \gamma_l = \frac{4(2l+3)(l+1)}{l(l+2)(2l+1)}\left[g(1) + \int_0^1 g'(\sigma_s(r))F_l(r)r^{l+2} \, dr\right], \quad l \geq 2. \]

From Lemma 3.2 and Lemma 3.3 of [14] we have:

Lemma 3.6 \((i)\) \( \mathcal{B}_\gamma \) is a Fourier multiplication operator having the following expression: For any \( \eta \in C^\infty(S^2) \) with Fourier expansion \( \eta(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\omega) \), we have

\[ \mathcal{B}_\gamma \eta(\omega) = \alpha_0 c_{00} Y_{00} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \alpha_l(\gamma) c_{lm} Y_{lm}(\omega). \]
(ii) The spectrum of $B_\gamma$ is given by
\[
\sigma(B_\gamma) = \{\alpha_0, 0\} \cup \{\alpha_l(\gamma) : l = 2, 3, 4, \ldots\}.
\]
Moreover, the multiplicity of the eigenvalue $0$ is 3. \qed

By Lemma 3.4 (ii) of [14] we know that $\gamma_l > 0$ for all $l \geq 2$, and $\lim_{l \to \infty} \gamma_l = 0$. Thus as in [14] we define
\[
\gamma_* = \max_{l \geq 2} \gamma_l.
\]
Clearly $0 < \gamma_* < \infty$, and for $\gamma > \gamma_*$ we have $\alpha_l(\gamma) < 0$ for all $l \geq 2$, while if $\gamma < \gamma_*$ then there exists $l \geq 2$ such that $\alpha_l(\gamma) > 0$. Since clearly $\lim_{l \to \infty} \alpha_l(\gamma) = -\infty$, the following notation makes sense:
\[
\alpha^*_\gamma = \max\{\alpha_0, \alpha_l(\gamma), l \geq 2\}.
\]
By Lemma 3.4 (i) of [14] we know that $\alpha_0 < 0$. Thus $\alpha^*_\gamma < 0$ for all $\gamma > \gamma_*$.  

In the following lemma $\varepsilon$ is the constant appearing in the equation (1.1), which also appears in the expressions of $DF(U_\delta)$ and $M$.

Lemma 3.7 We have the following assertions:

(i) $0$ is an eigenvalue of $M$ of multiplicity 3.

(ii) For any $\gamma > 0$ there exists a corresponding constant $\varepsilon'_0 > 0$ and a bounded continuous function $\mu_{l,\gamma}$ defined on $(0, \varepsilon'_0]$, such that for any $0 < \varepsilon \leq \varepsilon'_0$ we have
\[
\sigma(M) \supseteq \{\lambda_{l,\gamma}(\varepsilon) = \alpha_l(\gamma) + \varepsilon \mu_{l,\gamma}(\varepsilon), \ l = 2, 3, \ldots\}, \tag{3.50}
\]
and for each $l \geq 2$, the eigenvectors of $M$ corresponding to the eigenvalue $\lambda_{l,\gamma}(\varepsilon)$ have the expression $U_{lm} = \left(\begin{array}{c} \varepsilon a_{l,\gamma}(r, \varepsilon) \\ 1 \end{array}\right) Y_{lm}(\omega)$, where $a_{l,\gamma}(r, \varepsilon)$ is a smooth function in $r \in [r_0, 1]$ and is bounded continuous in $\varepsilon \in (0, \varepsilon'_0]$.

(iii) For any $\gamma > \gamma_*$ there exists a corresponding constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, we have
\[
\sup\{\text{Re} \lambda : \lambda \in \sigma(M) \setminus \{0\}\} \leq \frac{1}{2} \alpha^*_\gamma < 0. \tag{3.51}
\]

Proof: We assert that for a vector $U = (v, \eta) \in X_0$, $MU = 0$ if and only if $v = 0$ and $B_\gamma \eta = 0$. Indeed, it is easy to see that $MU = 0$ if and only if $A_0 v = 0$ and $B_\gamma \eta = 0$. Since $U \in X_0$ implies that $v \in C^{m+\theta}(\overline{B^3}) \cap C_0(\overline{B^3})$, we see that the boundary value of $v$ is zero. Hence, by the maximum principle we see that $A_0 v = 0$ implies that $v = 0$. This proves the desired assertion. By this assertion and the fact that $0$ is an eigenvalue of $B_\gamma$ of multiplicity 3, we immediately get the assertion (i). Next, by making slight changes of the proof of Lemma 6.4 of [2], we get the assertion (ii). Finally, the assertion (iii) follows from a quite similar proof as that of Lemma 6.5 of [2]. \qed
4 The proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. By Lemmas 2.1 and 2.3, we only need to prove that the stationary point $U_\delta$ of the equation (2.34) is asymptotically stable in case $\gamma > \gamma_*$ whereas unstable in case $\gamma < \gamma_*$. By Lemma 3.5 (ii) we see that the equation (2.34) is of the parabolic type in a small neighborhood of $U_\delta$. It is thus natural to use the geometric theory for parabolic differential equations in Banach spaces to investigate this equation. In doing so, however, we meet with a serious difficulty that, by Lemma 3.5 (iv) and Lemma 3.7 (i), 0 is an eigenvalue of $DF(U_\delta)$, so that the standard linearized asymptotic stability principle cannot be used to tackle the equation (2.34). We shall as in [2] appeal to the Lie group action possessed by this equation to overcome this difficulty.

For $\tau > 0$ we denote by $B_\tau^3$ the ball in $\mathbb{R}^3$ centered at the origin with radius $\tau$. Regarding $B_\tau^3$ as a neighborhood of the unit element 0 of the commutative Lie group $\mathbb{R}^3$, we see that $G = B_\tau^3$ is a local Lie group of dimension 3. Given $z \in \mathbb{R}^3$, we denote by $S_z$ the translation in $\mathbb{R}^3$ induced by $z$, i.e.,

$$S_z(x) = x + z \quad \text{for} \quad x \in \mathbb{R}^3.$$

Let $\rho \in C^1(S^2)$ such that $\|\rho\|_{C^1(S^2)}$ is sufficiently small, say, $\|\rho\|_{C^1(S^2)} < \delta$ for some small $\delta > 0$. For any $z \in B_\tau^3$, consider the image of the hypersurface $r = 1 + \rho(\omega)$ under the translation $S_z$, which is still a hypersurface. This hypersurface has the equation $r = 1 + \tilde{\rho}(\omega)$ with $\tilde{\rho} \in C^1(S^2)$, and $\tilde{\rho}$ is uniquely determined by $\rho$ and $z$. We denote

$$\tilde{\rho} = S_z^*(\rho).$$

By some similar arguments as in the proof of Lemmas 4.1 and 4.3 of [2] we can show that if $\rho \in O_{\delta}^{m+\theta}(S^2)$ then $S_z^*(\rho) \in C^{m+\theta}(S^2)$ and $S_z^* \in C(O_{\delta}^{m+\theta}(S^2), C^{m+\theta}(S^2)) \cap C^1(O_{\delta}^{m+\theta}(S^2), C^{m+1+\theta}(S^2))$. Next, for each $z \in B_\tau^3$ and $\rho \in C^1(S^2)$ such that $\|\rho\|_{C^1(S^2)}$ is sufficiently small, let $P_{z,\rho} : C(B^3) \to C(B^3)$ be the mapping

$$P_{z,\rho}(u)(x) = u(\Phi_\rho^{-1}(\Phi_{S_z^*(\rho)}(x) - z)) \quad \text{for} \quad u \in C(B^3).$$

This mapping is well-defined and, actually, we have $P_{z,\rho} \in L(C(B^3))$. Indeed, letting $\tilde{\rho} = S_z^*(\rho)$ and denoting by $\Omega_\rho$ and $\Omega_{\tilde{\rho}}$ the domains enclosed by the hypersurfaces $r = 1 + \rho(\omega)$ and $r = 1 + \tilde{\rho}(\omega)$, respectively, we see that $x \in B^3$ if and only if $\Phi_{S_z^*(\rho)}(x) = \Phi_{\tilde{\rho}}(x) \in \Omega_{\tilde{\rho}}$, which is equivalent to $\Phi_{S_z^*(\rho)}(x) - z \in \Omega_\rho$. But we know that $y \in \Omega_{\tilde{\rho}}$ if and only if $\Phi_{\tilde{\rho}}^{-1}(y) \in B^3$. Hence, we see that $x \in B^3$ if and only if $\Phi_{\tilde{\rho}}^{-1}(\Phi_{S_z^*(\rho)}(x) - z) \in B^3$. Moreover, it can be easily seen that for $\rho \in C^1(S^2)$ the mapping $x \to \Phi_{\tilde{\rho}}^{-1}(\Phi_{S_z^*(\rho)}(x) - z)$ is a $C^{m+\theta}$-diffeomorphism from $B^3$ onto itself. Hence the desired assertion follows. Furthermore, it is also immediate to see that if $\rho \in O_{\delta}^{m+\theta}(S^2)$ then the mapping $x \to \Phi_{\tilde{\rho}}^{-1}(\Phi_{S_z^*(\rho)}(x) - z)$ is a $C^{m+\theta}$-diffeomorphism from $B^3$ onto itself, so that $P_{z,\rho} \in L(C(B^3))$.

Now, for each $z \in G = B_\tau^3$ we define a mapping $S_z^* : \mathcal{O}_\delta \to \mathcal{X}_0$ (recall that $\mathcal{O}_\delta = (C^{m+\theta}(B^3) \cap C_0(B^3)) \times O_{\delta}^{m+\theta}(S^2)$ and $\mathcal{X}_0 = (C^{m+\theta}(B^3) \cap C_0(B^3)) \times C^{m+\theta}(S^2)$) as follows: For any $U = (u, \rho) \in \mathcal{O}_\delta$, let

$$S_z^*(u, \rho) = (P_{z,\rho}(u), S_z^*(\rho)).$$
It is obvious that if \( u \in C_0(\mathbb{B}^3) \) then also \( P_{z,\rho}(u) \in C_0(\mathbb{B}^3) \). Thus for \( \tau \) and \( \delta \) sufficiently small this mapping is well-defined and it does map \( \mathcal{O}_\delta \) into \( \mathcal{X}_0 \). We have

**Lemma 4.1** Let \( m \geq 3 \) and \( 0 < \theta < 1 \). Let

\[
\mathcal{O}_\delta' = (C^{m-2+\theta}(\mathbb{B}^3) \cap C_0(\mathbb{B}^3)) \times O_\delta^{m-1+\theta}(S^2) \subseteq \mathcal{X}.
\]

For sufficiently small \( \tau > 0 \) and \( \delta > 0 \) we have the following assertions:

(i) For any \( z \in \mathbb{B}_\tau^3 \) we have \( \mathcal{S}_z^* \in C(\mathcal{O}_\delta', \mathcal{X}) \cap C(\mathcal{O}_\delta, \mathcal{X}_0) \).

(ii) For any \( z, w \in \mathbb{B}_\tau^3 \) we have

\[
\mathcal{S}_z^* \circ \mathcal{S}_w^* = \mathcal{S}_{z+w}^*, \quad \mathcal{S}_0^* = \text{id}, \quad \text{and} \quad (\mathcal{S}_z^*)^{-1} = \mathcal{S}_{-z}^*.
\]

(iii) The mapping \( \mathcal{S}^* : z \to \mathcal{S}_z^* \) from \( \mathbb{B}_\tau^3 \) to \( C(\mathcal{O}_\delta', \mathcal{X}) \) is an injection, and

\[
\mathcal{S}^* \in C^k(\mathbb{B}_\tau^3, C^l(\mathcal{O}_\delta, C^{m-k-l+\theta}(\mathbb{B}^3) \times C^{m-k-l+\theta}(S^2))), \quad k \geq 0, \quad l \geq 0, \quad k + l \leq m. \tag{4.1}
\]

Moreover, for fixed \( z \in \mathbb{B}_\tau^3 \) we have

\[
\mathcal{D}\mathcal{S}_z^* \in C(\mathcal{O}_\delta, L(C^{m-1+\theta}(\mathbb{B}^3) \times C^{m-1+\theta}(S^2))), \tag{4.2}
\]

i.e., for any \( U \in \mathcal{O}_\delta \), the operator \( \mathcal{D}\mathcal{S}_z^*(U) \) (which is, by (4.1), an bounded linear operator from \( C^{m+\theta}(\mathbb{B}^3) \times C^{m+\theta}(S^2) \) to \( C^{m-1+\theta}(\mathbb{B}^3) \times C^{m-1+\theta}(S^2) \)) can be extended into a bounded linear operator from \( C^{m-1+\theta}(\mathbb{B}^3) \times C^{m-1+\theta}(S^2) \) to itself, and the mapping \( U \to \mathcal{D}\mathcal{S}_z^*(U) \) from \( \mathcal{O}_\delta \) to \( L(C^{m-1+\theta}(\mathbb{B}^3) \times C^{m-1+\theta}(S^2)) \) is continuous.

(iv) Define \( p : \mathbb{B}_\tau^3 \times \mathcal{O}_\delta' \to \mathcal{X} \) by \( p(z, U) = \mathcal{S}_z^*(U) \) for \( (z, U) \in \mathbb{B}_\tau^3 \times \mathcal{O}_\delta' \). Then for any \( U \in \mathcal{O}_\delta \) we have \( p(\cdot, U) \in C^3(\mathbb{B}_\tau^3, \mathcal{X}) \), and rank \( D_z p(z, U) = 3 \) for every \( z \in \mathbb{B}_\tau^3 \) and \( U \in \mathcal{O}_\delta \). If furthermore \( U \in \mathcal{Z} = C^\infty(\mathbb{B}^3) \times C^\infty(S^2) \) then \( p(\cdot, U) \in C^\infty(\mathbb{B}_\tau^3, \mathcal{Z}) \).

**Proof.** This lemma follows from a similar argument as that in the establishment of Lemma 4.4 of [2]. Indeed, to get the assertions of this lemma we only need to replace the Sobolev and corresponding Besov spaces \( W^{m,q}(\mathbb{B}^3) \), \( B^{m-1/q}(S^2) \) in [2] with the H"older spaces \( C^{m+\theta}(\mathbb{B}^3) \) and \( C^{m+\theta}(S^2) \), and after a such replacement all the analysis presented in Section 4 of [2] still holds. In particular, the proof of (4.1) uses a similar argument as that of the second assertion in (i) of Lemma 4.3 of [2]. Since this analysis is lengthy but does not have new ingredient different from that in [2], we omit it here. \( \square \)

**Corollary 4.2** Let assumptions be as in Lemma 4.1. Then for sufficiently small \( \tau > 0 \), \( \delta > 0 \) and fixed \( z \in \mathbb{B}_\tau^3 \) we also have the following relation:

\[
\mathcal{D}\mathcal{S}_z^*(u, \rho) = (P_{z,\rho}(u), \mathcal{S}_z^*(\rho)), \quad \text{we see that}
\]

\[
\mathcal{D}\mathcal{S}_z^*(u, \rho)(v, \eta) = (D_u P_{z,\rho}(u)v + D_{\rho} P_{z,\rho}(u)\eta, D\mathcal{S}_z^*(\rho)\eta).
\]

From (4.2) we have \( [(u, \rho) \to \mathcal{D}\mathcal{S}_z^*(u, \rho)] \in C(\mathcal{O}_\delta, L(C^{m+\theta}(S^2))) \) and \( [(u, \rho) \to D_{\rho} P_{z,\rho}(u)] \in C(\mathcal{O}_\delta, L(C^{m-1+\theta}(S^2))) \subseteq C(\mathcal{O}_\delta, L(C^{m-1+\theta}(S^2), C^{m-2+\theta}(S^2))) \). Since \( m - 1 \geq m - 2 \), by (4.2) we
also have \([u, \rho] \to D_u P_{z, \rho}(u) \in C(\Omega_\delta, L(C^{m-2+\theta}(\mathbb{R}^1)))\). Combining these assertions together, we see that (4.3) follows. \(\square\)

In the sequel, for \(\rho = \rho(t), u = u(x, t)\) and \(U = (u(x, t), \rho(t))\), we denote by \(P_{z, \rho}(u)\) the function \(\tilde{u}(x, t) = u(\Phi^{-1}_{\rho(t)}(\Phi_{S_{z}(\rho(t))}(x) - z), t)\), by \(S_{z}(\rho)\) the function \(\tilde{\rho}(t) = S_{z}^{*}(\rho(t))\), and by \(S_{z}^{*}(U)\) the vector function \((P_{z, \rho}(u), S_{z}^{*}(\rho)) = (\tilde{u}(x, t), \tilde{\rho}(t))\).

**Lemma 4.3** If \(U = (u, \rho)\) is a solution of the equation \(dU/dt = F(U)\) such that \(\|\rho\|_{C^1(\mathbb{R}^2)}\) is sufficiently small, then for any \(z \in \mathbb{R}^3\) such that \(|z|\) is sufficiently small, \(S_{z}^{*}(U) = (P_{z, \rho}(u), S_{z}^{*}(\rho))\) is also a solution of this equation.

**Proof:** It is easy to see that if \((\sigma, \nu, p, \Omega)\) is a solution of the system of equations (1.1)–(1.8), then for any \(z \in \mathbb{R}^3\), we have that \((\tilde{\sigma}, \tilde{\nu}, \tilde{p}, \tilde{\Omega})\) defined by
\[
\tilde{\sigma}(x, t) = \sigma(x - z, t), \quad \tilde{\rho}(x, t) = p(x - z, t), \quad \tilde{\nu}(x, t) = \nu(x - z, t), \quad \tilde{\Omega}(t) = \Omega(t) + z,
\]
is also a solution of (1.1)–(1.8). From this fact we see immediately that if \(U = (u, \rho)\) is a solution of the equation
\[
\frac{dU}{dt} = F(U), \tag{4.4}
\]
then \(\tilde{U} = (\tilde{u}, \tilde{\rho})\), where
\[
\tilde{u}(x, t) = u(\Phi^{-1}_{\rho(t)}(\Phi_{S_{z}(\rho(t))}(x) - z), t), \quad \tilde{\rho}(t) = S_{z}^{*}(\rho(t)),
\]
is also a solution of this equation, which is the desired assertion. \(\square\)

**Lemma 4.4** If \(\tau\) and \(\delta\) are sufficiently small then for any \(z \in \mathbb{B}_\tau^3\) and \(U = (u, \rho) \in \Omega_\delta\) we have
\[
F(S_{z}^{*}(U)) = D S_{z}^{*}(U) F(U). \tag{4.5}
\]

**Proof:** By Lemma 3.5 (ii) we see that the equation (4.1) is of the parabolic type in \(\Omega_\delta\), provided \(\delta\) is sufficiently small. Hence, by a well-known result in the theory of differential equations of the parabolic type in Banach spaces (cf. Theorem 8.1.1 of \([12]\)) we see that given any \(U_0 = (u, \rho) \in \mathbb{X}_0\) there exists \(t_0 > 0\) such that the equation (4.1) has a unique solution \(U = U(t)\) for \(0 \leq t \leq t_0\), which belongs to \(C([0, t_0], \mathbb{X}) \cap C((0, t_0], \Omega_\delta) \cap L^\infty((0, t_0), \mathbb{X})\) and satisfies the initial condition \(U(0) = U_0\). Let \(\tilde{U}(t) = S_{z}^{*}(U(t))\) for \(0 \leq t \leq t_0\). By Lemma 4.2, \(\tilde{U}\) is also a solution of (4.1), satisfying the initial condition \(\tilde{U}(0) = S_{z}^{*}(U_0)\). The fact that \(\tilde{U}\) is the solution of (4.1) implies that
\[
\frac{d\tilde{U}(t)}{dt} = F(\tilde{U}(t)) \quad \text{for} \quad 0 < t \leq t_0.
\]
On the other hand, since \(\tilde{U}(t) = S_{z}^{*}(U(t))\), we have
\[
\frac{d\tilde{U}(t)}{dt} = D S_{z}^{*}(U(t)) \frac{dU(t)}{dt} = D S_{z}^{*}(U(t)) F(U(t)) \quad \text{for} \quad 0 < t \leq t_0.
\]
Thus
\[
F(\tilde{U}(t)) = D S_{z}^{*}(U(t)) F(U(t)) \quad \text{for} \quad 0 < t \leq t_0. \tag{4.6}
\]
If \( U(t) \) is a strict solution (in the sense of [12]) then \( U \in C([0, t_0], \mathbb{O}_\delta) \cap C^1([0, t_0], \mathbb{X}) \) and clearly \( \tilde{U}(t) \) is also a strict solution, so that by directly letting \( t \to 0^+ \) we get

\[
\mathcal{F}(S^*_t(U_0)) = DS^*_t(U_0)\mathcal{F}(U_0).
\] (4.7)

If \( U(t) \) is not a strict solution then we establish this relation in the following way: Let

\[
\tilde{\mathcal{X}}_0 = C^{-m/2+\theta}(\mathbb{R}^3) \times C^{m/2-\theta}(S^2), \quad \tilde{\mathcal{X}} = C^{-m/2+\theta}(\mathbb{R}^3) \times C^{m/2-\theta}(S^2),
\]

and let \( \tilde{\mathcal{O}}_\delta \) be a small neighborhood of the origin of \( \tilde{\mathcal{X}}_0 \). Then \( \mathcal{F} \in C(\tilde{\mathcal{O}}_\delta, \tilde{\mathcal{X}}) \) and \( DS^*_t \in C(\tilde{\mathcal{O}}_\delta, L(\tilde{\mathcal{X}})) \) (by (4.3)). Hence, since \( U \in C([0, t_0], \mathbb{X}) \subseteq C([0, t_0], \tilde{\mathcal{X}}_0) \), which also ensures that \( \tilde{U} \in C([0, t_0], \tilde{\mathcal{X}}_0) \), by letting \( t \to 0^+ \) we see that (4.7) still holds.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1:** We first assume that \( \gamma > \gamma_s \). Then by Lemma 3.5 (iv) and Lemma 3.7 (i) and (iii) we see that

\[
\sup\{\Re \lambda : \lambda \in \sigma(D\mathcal{F}(U_s))\backslash\{0\}\} \leq \frac{1}{2} \alpha^*_\gamma < 0,
\]

and 0 is an eigenvalue of \( D\mathcal{F}(U_s) \) of multiplicity 3. By Lemma 4.1 we see that the mapping \( p : \mathbb{B}_3^3 \times \mathbb{O}_\delta \to \mathcal{X} \) defined by \( p(z, U) = S^*_t(U) \) for \( (z, U) \in \mathbb{B}_3^3 \times \mathbb{O}_\delta \) is a Lie group action of the local Lie group \( G = \mathbb{B}_3^3 \) to the space \( \mathcal{X} \), and by Lemma 4.4 we see that the equation \( dU/dt = F(U) \) is quasi-invariant (in the sense of [2]) under this Lie group action. Since for every \( z \in \mathbb{B}_3^3 \) and \( U \in \mathbb{O}_\delta \) we have \( \text{rank} \, D_z p(z, U) = 3 = \text{the multiplicity of the eigenvalue 0 of } D\mathcal{F}(U_s) \), we see that all the conditions of Theorem 2.1 of [2] are satisfied by the equation \( dU/dt = F(U) \) and the Lie group action \( (G, p) \). Hence, by Theorem 2.1 of [2] we conclude that there exist two neighborhoods \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) of \( U_s \) in \( \mathcal{X}_0 \), \( \mathcal{O}_1 \) is larger than \( \mathcal{O}_2 \) (i.e. \( \mathcal{O}_2 \subseteq \mathcal{O}_1 \)), such that for each \( U_0 \in \mathcal{O}_1 \) the initial value problem (2.34) has a unique solution \( U = U(t) \) for all \( t \geq 0 \), which belongs to \( C([0, \infty), \mathcal{X}_0) \cap C((0, \infty), \mathcal{O}_1) \cap L^\infty((0, \infty), \mathcal{X}_0) \cap C^1((0, \infty), \mathbb{X}) \), and for any \( U_0 \in \mathcal{O}_2 \) there exists corresponding \( V_0 \in \mathcal{O}_1 \) and \( z \in G = \mathbb{B}_3^3 \) which are uniquely determined by \( U_0 \), such that

\[
U_0 = S^*_t(V_0), \quad \lim_{t \to \infty} U(t; V_0) = U_s, \quad \text{and} \quad \lim_{t \to \infty} U(t; U_0) = S^*_t(U_s),
\]

where \( U(t; V_0) \) refers to the solution of (2.34) with initial value \( V_0 \), and similarly for \( U(t; U_0) \). Moreover, the convergence in the above limit relations is exponentially fast. From this result and Lemmas 2.1 and 2.3 we immediately obtain (1.13).

Next we assume that \( \gamma < \gamma_s \). Then there exists \( l \geq 2 \) such that \( \alpha_l(\gamma) > 0 \). By Lemma 3.5 (iv) and Lemma 3.7 (ii) we see that for \( \varepsilon > 0 \) sufficiently small, \( D\mathcal{F}(U_0) \) has an eigenvalue \( \lambda_{l, \gamma}(\varepsilon) = \alpha_l(\gamma) + \varepsilon \mu_{l, \gamma}(\varepsilon) \) which is clearly positive. Hence, by a standard result in the geometric theory of parabolic differential equations in Banach spaces (cf. Theorem 9.1.3 of [12]) we conclude that \( U_s \) is unstable as a stationary point of the equation \( dU/dt = F(U) \). Using again Lemmas 2.1 and 2.3, we get the last assertion of Theorem 1.1. This completes the proof.
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