On R-matrix-valued Lax pairs for Calogero-Moser models

A. Grekov\textsuperscript{1} \hspace{1cm} A. Zotov\textsuperscript{2}

Abstract

The article is devoted to the study of $R$-matrix-valued Lax pairs for $N$-body (elliptic) Calogero-Moser models. Their matrix elements are given by quantum $\text{GL}_N$ $R$-matrices of Baxter-Belavin type. For $\tilde{N} = 1$ the widely known Krichever’s Lax pair with spectral parameter is reproduced. First, we construct the $R$-matrix-valued Lax pairs for Calogero-Moser models associated with classical root systems. For this purpose we study generalizations of the D’Hoker-Phong Lax pairs. It appeared that in the $R$-matrix-valued case the Lax pairs exist in special cases only. The number of quantum spaces (on which $R$-matrices act) and their dimension depend on the values of coupling constants. Some of the obtained classical Lax pairs admit straightforward extension to the quantum case. In the end we describe a relationship of the $R$-matrix-valued Lax pairs to Hitchin systems defined on $\text{SL}_{\tilde{N}N}$ bundles with nontrivial characteristic classes over elliptic curve. We show that the classical analogue of the anisotropic spin exchange operator entering the $R$-matrix-valued Lax equations is reproduced in these models.

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\textsuperscript{1}Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina St., Moscow 119991, Russia; e-mail: grekovandrew@mail.ru

\textsuperscript{2}Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina St., Moscow 119991, Russia; e-mail: zotov@mi.ras.ru
1 Introduction

In this paper we consider the Calogero-Moser models \[11\] and their generalizations of different types. The Hamiltonian of the elliptic classical sl\(_N\) model

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i>j}^{N} \wp(q_i - q_j)
\]  
(1.1)

together with the canonical Poisson brackets

\[
\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0.
\]  
(1.2)

provides equations of motion for \(N\)-particle dynamics:

\[
\dot{q}_i = p_i, \quad \ddot{q}_i = \nu^2 \sum_{k:k\neq i}^{N} \wp(q_{ik}),
\]  
(1.3)

All variables and the coupling constant \(\nu\) are assumed to be complex numbers. Equations (1.3) can be written in the Lax form. The Krichever’s Lax pair with spectra \(l\) parameter \[30\] reads as follows

\[
L(z) = \sum_{i,j=1}^{N} E_{ij} L_{ij}(z), \quad L_{ij}(z) = \delta_{ij} p_i + \nu(1 - \delta_{ij})\phi(z, q_{ij}), \quad q_{ij} = q_i - q_j,
\]  
(1.4)

\[
M_{ij}(z) = \nu d_i \delta_{ij} + \nu(1 - \delta_{ij})f(z, q_{ij}), \quad d_i = \sum_{k:k\neq i}^{N} E_2(q_{ik}) = -\sum_{k:k\neq i}^{N} f(0, q_{ik}),
\]  
(1.5)

i.e. the Lax equations

\[
\dot{L}(z) \equiv \{H, L(z)\} = [L(z), M(z)]
\]  
(1.6)

are equivalent to (1.3) identically in \(z\). The definitions and properties of elliptic functions entering (1.1)-(1.5) are given in the Appendix. The proof is based on the identity (A.14) written as

\[
\phi(z, q_{ab})f(z, q_{bc}) - f(z, q_{ab})\phi(z, q_{bc}) = \phi(z, q_{ac})(f(0, q_{bc}) - f(0, q_{ab})).
\]  
(1.7)

and

\[
\phi(z, q_{ab})f(z, q_{ba}) - f(z, q_{ab})\phi(z, q_{ba}) = \wp'(q_{ab}).
\]  
(1.8)

These are particular cases of the genus one Fay identity (A.13)

\[
\phi(z, q_{ab})\phi(w, q_{bc}) = \phi(w, q_{ac})\phi(z - w, q_{ab}) + \phi(w - z, q_{ac})\phi(z, q_{bc}).
\]  
(1.9)

The model (1.1)-(1.3) is included into a wide class of Calogero-Moser models associated with root systems \[38\]. The corresponding Lax pairs with spectral parameter were found in \[15\] \[9\]. In particular, for the BC\(_N\) root system described by the Hamiltonian

\[
H = \frac{1}{2} \sum_{a=1}^{N} p_a^2 - \nu^2 \sum_{a<b}^{N} (\wp(q_a - q_b) + \wp(q_a + q_b)) - \mu^2 \sum_{a=1}^{N} \wp(2q_a) - g^2 \sum_{a=1}^{N} \wp(q_a)
\]  
(1.10)

\(\{E_{ij} \in \text{Mat}(N), \ i, j = 1...N\} - \text{is the standard basis in Mat}(N): (E_{ij})_{kl} = \delta_{ik}\delta_{jl}.\)
there exists the Lax pair with spectral parameter of size \((2N + 1) \times (2N + 1)\) if (as in 38)

\[
g(g^2 - 2\nu^2 + \nu \mu) = 0. \tag{1.11}
\]

Let us remark that the Lax pairs of size \(3N \times 3N\) \([26]\) or \(2N \times 2N\) \([16]\) corresponding to the general case (all constants are arbitrary) are not considered in this paper.

The Lax pair \((1.4)-(1.5)\) of the \(sl_N\) model \((1.1)-(1.3)\) has the following generalization \([32]\)

\[
\mathcal{L}(z) = \sum_{i,j=1}^{N} E_{ij} \otimes \mathcal{L}_{ij}(z), \quad \mathcal{L}_{ij}(z) = 1^{\otimes N} \delta_{ij} p_i + \nu (1 - \delta_{ij}) R_{ij}^z(q_{ij})
\]

\[
\mathcal{M}_{ij}(z) = \nu d_i \delta_{ij} + \nu (1 - \delta_{ij}) R_{ij}^z(q_{ij}) + \nu \delta_{ij} \mathcal{F}^0, \quad d_i = - \sum_{k,k \neq i}^{N} F_{ik}^0(q_{ik}) \tag{1.13}
\]

where \(F_{ij}^z(q) = \partial_q R_{ij}^z(q), \ F_{ij}^0(q) = F_{ij}^z(q)|_{z=0} = F_{ij}^0(-q) \tag{A.21}\) and

\[
\mathcal{F}^0 = \sum_{k,m=1}^{N} F_{km}^0(q_{km}) = \frac{1}{2} \sum_{k,m=1}^{N} F_{km}^0(q_{km}). \tag{1.14}
\]

It has block-matrix structure\(^4\). The blocks are enumerated by \(i, j = 1...N\) as matrix elements in \((1.4)\). Each block of \(\mathcal{L}(z)\) is some \(GL(\tilde{N})\)-valued \(R\)-matrix in fundamental representation, acting on the \(N\)-th tensor power of \(\tilde{N}\)-dimensional vector space \(\mathcal{H} = (C^\tilde{N})^{\otimes N}\). So the size of
each block is \(\dim \mathcal{H} \times \dim \mathcal{H}\), and \(\dim \mathcal{H} = \tilde{N}^N\), i.e. \(\mathcal{L}(z) \in Mat_{\tilde{N}} \otimes Mat_{\tilde{N}}^{\otimes N}\). We will refer to \(Mat_{\tilde{N}}\) component as auxiliary space, and to \(Mat_{\tilde{N}}^{\otimes N} \cong \mathcal{H}^{\otimes 2}\) as “quantum” space since \(\mathcal{H}\) is the Hilbert space of \(GL(\tilde{N})\) spin chain (in fundamental representation) on \(N\) sites.

An \(R\)-matrix \(R_{ij}\) acts trivially in all tensor components except \(i, j\). It is normalized in a way that for \(\tilde{N} = 1\) it is reduced to the Kronecker function \(\phi(z, q_{ij})\) \([A.6]\) \([48]\). For instance, in one of the simplest examples \(R_{ij}\) is the Yang’s \(R\)-matrix \([50]\):

\[
R_{12}^\eta(q) = \frac{1}{q} + \frac{\tilde{N} P_{12}}{q}, \tag{1.15}
\]

where \(P_{12}\) is the permutation operator \([A.5]\). In general (and as a default) case \(R_{ij}\) is the Baxter-Belavin \([4, 5]\) elliptic \(R\)-matrix \([A.18]\). The properties of this \(R\)-matrix are very similar to those of the function \(\phi(z, q)\). The key equation for \(R_{ij}\) (which is needed for existence of the \(R\)-matrix-valued Lax pair) is the associative Yang-Baxter equation \([19]\)

\[
R_{ab}^z R_{bc}^w = R_{ac}^w R_{bc}^z + R_{bc}^{w,z} R_{ac}^z, \quad R_{ab}^z = R_{ab}^z(q_a - q_b). \tag{1.16}
\]

It is a matrix generalization of the Fay identity \((1.9)\), and it is fulfilled by the Baxter-Belavin \(R\)-matrix \([40]\). The degeneration of \((1.16)\) similar to \((1.7)\) is of the form:

\[
R_{ab}^z F_{bc}^z - F_{ab}^z R_{bc}^z = F_{bc}^0 R_{ac}^z - R_{ac}^z F_{ab}^0. \tag{1.17}
\]

\(^4\)Equations of motion following from \((1.12)-(1.13)\) contain the coupling constant \(\tilde{N} \nu\) instead of \(\nu\) in \((1.1), \ (1.3)\).

\(^5\)The operator-valued Lax pairs with a similar structure are known \([27, 23, 24, 6, 28]\). We discuss it below.
It underlies the Lax equations for the Lax pair (1.12)-(1.13). The last term $F^0$ in (1.13) is not needed in (1.5) since for $\tilde{N} = 1$ it is proportional to the identity $N \times N$ matrix. But it is important for $\tilde{N} > 1$ since it changes the order of $R$ and $F^0$ in the r.h.s. of (1.17). Namely,

$$[R_{ac}, F^0] + \sum_{b \neq a,c} R_{ab}^z F_{bc}^z - F_{ab}^z R_{bc}^z = \sum_{b \neq c} R_{ac}^z F_{bc}^0 - \sum_{b \neq a} F_{ab}^0 R_{ac}^z, \quad \forall a \neq c. \quad (1.18)$$

This identity provides cancellation of non-diagonal blocks in the Lax equations. See [40, 41, 33, 34, 52] for different properties and applications of $R$-matrices of the described type. Here we need two more important properties. These are the unitarity

$$R_{12}^z(q_{12}) R_{21}^z(q_{21}) = 1 \otimes 1 \tilde{N}^2 (\wp(\tilde{N}z) - \wp(q_{12})) \quad (1.19)$$

and the skew-symmetry

$$R_{ab}^z(q) = -R_{ba}^{-z}(-q). \quad (1.20)$$

On the one hand these properties are needed for the Lax equations since they lead to

$$F^0_{ab}(q) = F^0_{ba}(-q) \quad (1.21)$$

and to the analogue of (1.8) (obtained by differentiating the identity (1.19))

$$R_{ab}^z F_{ba}^z - F_{ab}^z R_{ba}^z = \tilde{N}^2 \wp'(q_{ab}), \quad (1.22)$$

which provides equations of motion in each diagonal block in the Lax equations.

On the other hand, together with (1.19) and (1.20) the associative Yang-Baxter equation leads to the quantum Yang-Baxter equation

$$R_{ab}^\eta R_{ac}^\eta R_{bc}^\eta = R_{bc}^\eta R_{ac}^\eta R_{ab}^\eta. \quad \text{In this respect we deal with the quantum } R\text{-matrices satisfying (1.16), (1.19), (1.20), and the Planck constant of } R\text{-matrix plays the role of the spectral parameter for the Lax pair (1.12)-(1.13). In trigonometric case the } R\text{-matrices satisfying the requirements include the standard GL}(\tilde{N}) \text{ XXZ } R\text{-matrix [31] and its deformation [13, 1] (GL}(\tilde{N}) \text{ extension of the 7-vertex } R\text{-matrix). In the rational case the set of the } R\text{-matrices includes the Yang’s one (1.15) and its deformations [13, 46] (GL}(\tilde{N}) \text{ extension of the 11-vertex } R\text{-matrix).}

The aim of the paper is to clarify the origin of the $R$-matrix-valued Lax pairs and examine some known constructions, which work for the ordinary Lax pairs.

First, we study extensions of (1.12)-(1.13) to other root systems. More precisely, we propose $R$-matrix-valued extensions of the D’Hoker-Phong Lax pairs for (untwisted) Calogero-Moser models associated with classical root systems and $BC_N$ (1.10). The auxiliary space in these cases is given by $\text{Mat}_{2N}$ or $\text{Mat}_{2N+1}$ because such root systems are obtained from $\mathfrak{sl}_{2N}$ or $\mathfrak{sl}_{2N+1}$ cases by discrete reduction. There are two natural possibilities for arranging tensor components of the quantum spaces. The first one is to keep $2N + 1$ (or $2N$) components of the quantum spaces in the reduced root system. The second – is to leave only $N$ (or $N + 1$) components. We study both cases.

Next, we proceed to quantum Calogero-Moser models [11, 39, 12]. To some extent they are described by quantum analogue of the Lax equations (1.6) [47, 10]:

$$[\hat{H}, \hat{L}(z)] = \hbar [\hat{L}(z), M(z)], \quad (1.23)$$

where $\hat{H}$ is the quantum Hamiltonian (it is scalar in the auxiliary space), $\hat{L}(z)$ is the quantum Lax matrix and $\hbar$ is the Planck constant. The operators $\hat{H}$ and $\hat{L}(z)$ are obtained from the
classical (1.1) and (1.4) by replacing momenta \( p_i \) with \( \hbar \partial_{q_i} \), and the coupling constant in the Hamiltonian acquires the quantum correction. We verify if the obtained \( R \)-matrix-valued Lax pairs are generalized to quantum case in a similar way. It appears that (besides the \( sl_N \) case) only models associated with SO type root systems are generalized. As a result we show:

**Proposition 1.1** The D’Hoker-Phong Lax pairs for (untwisted) classical Calogero-Moser models associated with classical root systems and \( BC_N \) admit \( R \)-matrix-valued extensions with additional constraints:

- for the coupling constants in \( C_N \) and \( BC_N \) cases;
- for the size of \( R \)-matrix (\( \tilde{N} = 2 \)) in \( B_N \) and \( D_N \) cases.

The latter cases are directly generalized to quantum Lax equations, while the \( C_N \) and \( BC_N \) cases are not. The \( A_N \) Lax pair is generalized to the quantum case straightforwardly without any restrictions.

The Calogero-Moser models [11, 39] possess also spin generalizations [20]. Its Lax description is known at classical [8] and quantum levels [27, 23, 24, 6, 28]. It is important to note that for the quantum Calogero-Moser models with spin the quantum Lax pairs have the same operator-valued (tensor) structure as in (1.12)-(1.13). The term analogues to \( F_0 \) (1.14) is treated as a part of the quantum Hamiltonian, describing interaction of spins. We explain (in Section 3) how the \( R \)-matrix-valued Lax pairs generalize (and reproduce) the previously known results.

Finally, we discuss an origin of the \( R \)-matrix-valued Lax pairs (for \( sl_N \) case (1.12)-(1.13) with \( GL_{\tilde{N}} \) \( R \)-matrices) by relating them to Hitchin systems on \( SL(N\tilde{N}) \)-bundles over elliptic curve. Originally systems of this type were derived by A. Polychronakos from matrix models [43] and later were described as Hitchin systems with nontrivial characteristic classes [51, 35]. It is also known as the model of interacting tops since it is Hamiltonian (or equations of motion) are treated as interaction of \( N \) \( SL(\tilde{N}) \)-valued elliptic tops.

The relation between the \( R \)-matrix-valued Lax pairs and the interacting tops comes from rewriting the Lax equation for (1.12)-(1.13) in the form

\[
\{H, L\} + [\nu F^0, L(z)] = [L(z), \tilde{M}(z)],
\]

where in contrast to (1.13) \( \tilde{M} \) does not include the \( F^0 \) term (1.14). In this respect the \( R \)-matrix-valued Lax pair is "half-quantum": the spin variables are quantized in the fundamental representation, while the positions and momenta of particles remain classical. The \( F^0 \) term in this treatment is the (anisotropic) spin exchange operator. We will show that the classical analogue for such spin exchange operator appear in the above mentioned Hitchin systems. Alternatively, the result is formulated as follows.

**Proposition 1.2** The quantum Hamiltonian \( \hat{H}^{\text{tops}} \) of the model of \( N \) interacting \( SL(\tilde{N}) \) elliptic tops (with spin variables being quantized in the fundamental representation) coincides with the sum of the quantum Calogero-Moser Hamiltonian (1.23) and \( F^0 \)-term (1.14)

\[
\hat{H}^{\text{tops}} = \hat{H}^{\text{CM}} + \hbar \nu F^0 + 1^\otimes N_{\text{const}}
\]

up to a constant proportional to identity matrix in \( \text{End}(\mathcal{H}) \) and redefinition of the coupling constants.

We prove this Proposition in the end of Section 4.

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6Some more details are given in the Conclusion.
2 Classical root systems

In [15] the following Lax pair was found for the model (1.10):

\[
L = \begin{pmatrix}
P + A & B_1 & C_1 \\ 
B_2 & -P + A^T & C_2 \\ 
C_1^T & C_2^T & 0
\end{pmatrix}
\quad M = \begin{pmatrix}
A' + d & B_1' & C_1' \\ 
B_2' & A'^T + d & C_2' \\ 
C_1'^T & C_2'^T & d_0
\end{pmatrix}
\]

(2.1)

where \( A, B, P, D \in \text{Mat}_N \), \( C_1, C_2 \) are columns of length \( N \) and

\[
P_{ab} = \delta_{ab} p_a, \quad A_{ab} = \nu(1 - \delta_{ab}) \phi(z, q_a - q_b),
\]

\[
(B_1)_{ab} = \nu(1 - \delta_{ab}) \phi(z, q_a + q_b) + \mu \delta_{ab} \phi(z, 2q_a),
\]

\[
(B_2)_{ab} = \nu(1 - \delta_{ab}) \phi(z, -q_a - q_b) + \mu \delta_{ab} \phi(z, -2q_a),
\]

\[
(C_1)_a = g \phi(z, q_a), \quad (C_2)_a = g \phi(z, -q_a),
\]

\[
d_{ab} = \delta_{ab} d_a, \quad d_a = \frac{g^2}{\nu} \phi(q_a) + \mu \phi(2q_a) + \nu \sum_{b \neq a} (\phi(q_a - q_b) + \phi(q_a + q_b)),
\]

\[
d_0 = 2\nu \sum_c \phi(q_c).
\]

(2.2)

(2.3)

The superscript \( T \) stands for transposition, and the prime means the derivative with respect to the second argument of functions \( \phi(z, q) \), i.e. \( \phi(z, q) \) are replaced by \( f(z, q) \) as in (1.5).

The Lax pair (2.1), (2.2), (2.3) satisfies the Lax equations (1.6) with the Hamiltonian (1.10) if the additional constraint (1.11) for the coupling constants \( \nu, \mu \) and \( g \) holds true. In the following particular cases the classical root systems arise:

- \( B_N \) (so\(_{2N+1}\)): \( \mu = 0, \ g^2 = 2\nu^2 \);
- \( C_N \) (sp\(_{2N}\)): \( g = 0 \);
- \( D_N \) (so\(_{2N}\)): \( \mu = 0, \ g = 0 \).

In order to deal with \( 2N \times 2N \) matrices in \( C_N \) and \( D_N \) cases one may subtract \( d_0 1_{2N+1} \) from the \( M \)-matrix.

Our purpose is to generalize (2.1), (2.2), (2.3) to \( R \)-matrix-valued Lax pair of (1.12), (1.13) type. The problem is that \( R_{ij}^k(q) \) carries the indices, which enumerate tensor components in quantum space. Hence we should define a number of these components and arrange them. There are two natural possibilities to do it:

- the first one is to keep the number of components to be equal to the Lax matrix size, i.e. to \( 2N + 1 \) (or, to \( 2N \) if \( g = 0 \)).

\[\text{7There is also a kind of intermediate case in the so-called universal Lax pairs [28]. The number of quantum spaces (sites) is equal to the Lax matrix size, and the } R \text{-matrix is proportional to the permutation operator defined by Weyl group elements of the corresponding root system. For example, in } D_N \text{ case the permutation of } i\text{-th and } j\text{-th sites (} 1 \leq i, j \leq N) \text{ includes also permutation of } i + N\text{-th and } j + N\text{-th sites. Elliptic } R \text{-matrix exists for } GL \text{ case only. For this reason we do not know generalization for these operators to (elliptic) } R \text{-matrix satisfying the necessary conditions.} \]
• the second possibility is to leave only half of this set (coming from \(s_{l2N+1}\)) likewise it is performed for spin chains with boundaries \([15][17]\). Put it differently, the number of "spin sites" in quantum space is equal to the rank of the root systems.

The first possibility leads to \(\mu = \nu\), and we are left with \(BC_N\) and \(C_N\) cases. They are described by straightforward reductions from \(s_{l2N+1}\) and \(s_{l2N}\) respectively\(^8\), so in this case \(g = \nu\) as well. At the same time the case \(g = -\nu\) for \(BC_N\) root system also works. See the explanation below (2.19).

The second possibility implies \(\mu = 0\) by the following reason. The diagonal elements in the blocks \(B_1\) and \(B_2\) (standing behind \(E_a,N,a\) and \(E_{N+a,a}\), \(a = 1,...,N\)) should have elements with \(R\)-matrices restricted to a single quantum space, i.e. \(R^a_{aa}(\pm 2q_a)\). But such terms can not be involved into ansatz based on \((1.16)\), which is identity in \(\text{Mat}_{N}^\otimes 3\). Thus we are left with \(B_N\) and \(D_N\) cases. Moreover, we will see that only \(\tilde{N} = 2\) is possible and \(g = \pm \sqrt{2}\nu\).

### 2.1 \(C_N\) Case

In this case \(g = 0\) and \(\mu = \nu\), the auxiliary space is \(\text{Mat}_{2N}\), the quantum space is \(\text{Mat}^\otimes _N^{2N}\), and \(\tilde{N}\) is arbitrary. The Lax pair:

\[
\mathcal{L} = \begin{pmatrix} P + A_1 & B_1 \\ B_2 & -P + A_2 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} A'_1 + D_1 + F & B'_1 \\ B'_2 & A'_2 + D_2 + F \end{pmatrix}
\]

where

\[
P = \sum_a p_a E_{aa} \otimes 1_N^\otimes 2N, \quad A_1 = \nu \sum_{a,b} E_{ab} \otimes R_{ab}^z(q_a - q_b), \quad A_2 = \nu \sum_{a,b} E_{ab} \otimes R_{a+N,b+N}^z(-q_a + q_b),
\]

\[
B_1 = \nu \sum_{a,b} E_{ab} \otimes R_{a,b+N}^z(q_a + q_b), \quad B_2 = \nu \sum_{a,b} E_{ab} \otimes R_{a+N,b}^z(-q_a - q_b),
\]

\[
A'_1 = \nu \sum_{a,b} E_{ab} \otimes F_{ab}^z(q_a - q_b), \quad A'_2 = \nu \sum_{a,b} E_{ab} \otimes F_{a+N,b+N}^z(-q_a + q_b),
\]

\[
B'_1 = \nu \sum_{a,b} E_{ab} \otimes F_{a,b+N}^z(q_a + q_b), \quad B'_2 = \nu \sum_{a,b} E_{ab} \otimes F_{a+N,b}^z(-q_a - q_b),
\]

\[
D_1 = \nu \sum_a E_{aa} \otimes d_a, \quad d_a = - \sum_{c,c' \neq a} F_{ac}^0(q_a - q_c) - \sum_c F_{a,c+N}^0(q_a + q_c),
\]

\[
D_2 = \nu \sum_a E_{aa} \otimes d_{a+N}, \quad d_{a+N} = - \sum_c F_{a+N,c}^0(q_a + q_c) - \sum_{c,c' \neq a} F_{a+N,c+N}^0(q_a - q_c)
\]

and \(F = \nu 1_N \otimes F^0\) with

\[
F^0 = \frac{1}{2} \sum_{c \neq d} (F_{cd}^0(q_c - q_d) + F_{c+N,d}^0(q_c - q_d)) + \frac{1}{2} \sum_{c,d} (F_{c,d+N}^0(q_c + q_d) + F_{c+N,d}^0(q_c + q_d)).
\]

\(^8\)For example, the reduction from \(s_{l2N}\) with positions of particles \(u_i, i = 1...2N\) to \(C_N\) is achieved by identifying \(u_i = q_i\) and \(u_{i+N} = -q_i, i = 1...N\) (and the same for momenta).
2.2 BC\(_N\) case

Here \(\pm g = \mu = \nu\), the auxiliary space is Mat\(_{2N+1}\), the quantum space is Mat\(_{\tilde{N}}\bigotimes(2N+1)\), and \(\tilde{N}\) is arbitrary.

\[
\mathcal{L} = \begin{pmatrix}
P + A_1 & B_1 & C_1 \\
B_2 & -P + A_2 & C_2 \\
C_2^T & C_1^T & 0
\end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix}
A_1' + D_1 + \mathcal{F} & B_1' & C_1' \\
B_2' & A_2' + D_2 + \mathcal{F} & C_2' \\
C_2'^T & C_1'^T & D_3 + \mathcal{F}
\end{pmatrix}
\]

(2.8)

where the blocks \(A, B\) and \(P\) are the same as in \(C_N\) case and

\[
(C_1)_a = \pm \nu R_{a,2N+1}^z(q_a), \quad (C_2)_a = \pm \nu R_{a+N,2N+1}^z(-q_a),
\]

\[
(C_1^T)_a = \pm \nu R_{2N+1,a+N}^z(q_a), \quad (C_2^T)_a = \pm \nu R_{2N+1,a+N}^z(-q_a),
\]

\[
(C_1')_a = \pm \nu F_{a,2N+1}^z(q_a), \quad (C_2')_a = \pm \nu F_{a+N,2N+1}^z(-q_a),
\]

\[
D_3 = \nu d_{2N+1}, \quad d_{2N+1} = -\sum_c F_{c,2N+1}^0(q_c) - \sum_c F_{c+N,2N+1}^0(q_c).
\]

The blocks \(D_1 = \nu \sum_a E_{aa} \otimes d_a, D_2 = \nu \sum_a E_{aa} \otimes d_{a+N}\) and \(\mathcal{F} = \nu 1_N \otimes \mathcal{F}^0\) are given by:

\[
d_a = -\sum_{c; c \neq a} F_{ac}^0(q_a - q_c) - \sum_c F_{a,c+N}^0(q_a + q_c) - F_{a,2N+1}^0(q_a),
\]

\[
d_{a+N} = -\sum_c F_{a+N,c}^0(q_a + q_c) - \sum_{c; c \neq a} F_{a+N,c+N}^0(q_a - q_c) - F_{a+N,2N+1}^0(q_a)
\]

(2.10)

and

\[
\mathcal{F}^0 = \frac{1}{2} \sum_{c \neq d} (F_{cd}^0(q_c - q_d) + F_{c+N,d+N}^0(q_c - q_d)) +
\]

\[
+ \frac{1}{2} \sum_{c,d} (F_{c,d+N}^0(q_c + q_d) + F_{c+N,d}(q_c + q_d)) + \sum_c F_{c,2N+1}^0(q_c) + \sum_c F_{c+N,2N+1}^0(q_c).
\]

(2.11)

For shortness we can also write \((2.10)-(2.11)\) as

\[
d_{a,BC(N)}^B = d_{a,Sp(2N)}^0 - F_{a,2N+1}^0(q_a), \quad d_{a+N,BC(N)}^B = d_{a+N,Sp(2N)}^0 - F_{a+N,2N+1}^0(q_a),
\]

\[
\mathcal{F}_{BC(N)}^0 = \mathcal{F}_{Sp(2N)}^0 - d_{2N+1}.
\]

(2.12)

Let us also comment on the necessity of \(\pm g = \mu = \nu\). Consider, for example, the block \(\{13\}\) of the Lax equation:

\[
[\mathcal{L}, \mathcal{M}]^{13} = PC'_1 + AC'_1 + B_1C'_2 + C_1D_3 + [C_1, \mathcal{F}] - A'C_1 - D_1C_1 - B'_1C_2.
\]

(2.13)

The term \(PC'_1\) gives the equation of motion. Next,

\[
(AC'_1 - A'C_1)_a = \nu g \sum_b R_{ab}^z(q_a - q_b) F_{b,2N+1}^z(q_b) - F_{ab}^z(q_a - q_b) R_{b,2N+1}^z(q_b) =
\]

\[
= \nu g \sum_b F_{ab}^0(q_a - q_b) R_{b,2N+1}^z(q_b) - R_{b,2N+1}^z(q_b) F_{ab}^0(q_a - q_b).
\]

(2.14)
The rest of the terms (after applying the unitarity condition) yield:

\[ [\mathcal{L}, \mathcal{M}]_a^{13} = \text{eq. of motion} + \]

\[ + \nu g \sum_{b, c \neq a} [F_{b2N+1}^0(q_b) R_{a,2N+1}^z(q_a) - F_{a,2N+1}^z(q_a) F_{a,2N+1}^0(q_a - q_b)] + \]

\[ + \nu g \sum_{b, c \neq a} [F_{b2N+1}^0(q_b) R_{a,2N+1}^z(q_a) - R_{a,2N+1}^z(q_a) F_{a,b+N}^0(q_a + q_b)] + \]

\[ + \mu g (F_{a,2N+1}^0(q_a) R_{a,2N+1}^z(q_a) - R_{a,2N+1}^z(q_a) F_{a,b+N}^0(2q_a)) + \]

\[ + \nu g R_{a,2N+1}^z(q_a) d_{2N+1} - g D_{1a} R_{a,2N+1}^z(q_a) + \nu g [R_{a,2N+1}^z(q_a), \mathcal{F}_a^0]. \]

Here \( D_{1a} \) and \( \mathcal{F}_a^0 \) have the following form:

\[ D_{1a} = -\frac{g^2}{\nu} F_{a,2N+1}^0 - \mu F_{a,a+N}^0 - \nu \sum_{b, c \neq a} F_{a,b+N}^0(q_a - q_b) + F_{a,b+N}^0(q_a + q_b), \quad (2.16) \]

\[ \mathcal{F}_a^0 = \frac{1}{2} \sum_{c \neq d} \left( F_{c,d}^0(q_c - q_d) + F_{c,d+1}^0(q_c - q_d) \right) + \]

\[ + \frac{1}{2} \sum_{c \neq d} \left( F_{c,d+N}^0(q_c + q_d) + F_{c,d+1}^0(q_c + q_d) \right) + \frac{\mu}{\nu} \sum_c \left( F_{c,c+N}^0(2q_c) + F_{c,c+1}^0(2q_c) \right) + \]

\[ + \sum_c F_{c,2N+1}^0(q_c) + \sum_c F_{c+1,2N+1}^0(q_c). \]

After rearranging the summands we obtain:

\[ \nu g \left[ R_{a,2N+1}^z(q_a), \mathcal{F}_a^0 - \sum_{b, c \neq a} (F_{a,b+N}^0(q_a - q_b) + F_{a,b+N}^0(q_a + q_b)) - \right. \]

\[ - \frac{\mu}{\nu} F_{a,a+N}^0(2q_a) - \sum_c F_{c,2N+1}^0(q_c) \right] + \]

\[ + \left( \frac{g^3}{\nu} F_{a,2N+1}^0(q_a) - \nu g F_{a,2N+1}^0(q_a) + \mu g F_{a,a+N,2N+1}^0(q_a) - \nu g F_{a,a+N,2N+1}^0(q_a) \right) \times \]

\[ \times R_{a,2N+1}^z(q_a). \]

The commutator vanishes because all terms, which act non-trivially in \( a \) and \( 2N + 1 \) quantum spaces are subtracted from \( \mathcal{F}_a^0 \), and we are left with the following expression:

\[ \left( \frac{g^3}{\nu} - \nu g \right) F_{a,2N+1}^0(q_a) + (\mu g - \nu g) F_{a,a+N,2N+1}^0(q_a) \]

\[ \times R_{a,2N+1}^z(q_a). \]

(multiplied by \( R_{a,2N+1}^z(q_a) \)), which must be equal to zero for validity of the Lax equations. In the scalar case \((\tilde{N} = 1)\) \( F_{b,2N+1}^0 \) is proportional to \( F_{b+N,2N+1}^0 \), and this requirement leads only to the condition \( [\mathcal{L}, \mathcal{M}]_a^{13} = 0 \). But in the case \( \tilde{N} \neq 1 \) it totally fixes the constants (up to a sign). Calculations for blocks \{23\}, \{31\} and \{32\} are absolutely analogous. And all other blocks in \([\mathcal{L}, \mathcal{M}]\) depend only on square of \( g \), consequently are not sensitive to its sign. The natural way to solve such kind of problem is to reduce the number of quantum spaces. This is what we do in the next paragraph.
2.3 $D_N$ case with $N$ quantum spaces

$R$-matrix-valued $SO_{2N}$ Lax pair with $N$ quantum spaces has the same block structure as the one for $Sp(2N)$ with $2N$ quantum spaces \([2.4]\):

$$
\mathcal{L} = \begin{pmatrix} P + A_1 & B_1 \\ B_2 & -P + A_2 \end{pmatrix} \quad \mathcal{M} = \begin{pmatrix} A'_1 + D_1 + F & B'_1 \\ B'_2 & A'_2 + D_2 + F \end{pmatrix}
$$

But now the corresponding blocks have the form:

$$
(A_1)_{ab} = \nu(1 - \delta_{ab}) R_{ab}^z(q_a - q_b), \quad (A_2)_{ab} = \nu(1 - \delta_{ab}) R_{ab}^z(-q_a + q_b),
$$

$$
(B_1)_{ab} = \nu(1 - \delta_{ab}) R_{ab}^z(q_a + q_b), \quad (B_2)_{ab} = \nu(1 - \delta_{ab}) R_{ab}^z(-q_a - q_b),
$$

$$
(A'_1)_{ab} = \nu(1 - \delta_{ab}) F_{ab}^z(q_a - q_b), \quad (A'_2)_{ab} = \nu(1 - \delta_{ab}) F_{ab}^z(-q_a + q_b),
$$

$$
(B'_1)_{ab} = \nu(1 - \delta_{ab}) F_{ab}^z(q_a + q_b), \quad (B'_2)_{ab} = \nu(1 - \delta_{ab}) F_{ab}^z(-q_a - q_b),
$$

\begin{equation}
(D_1)_a = (D_2)_a = D_a = -\nu \sum_{c, c' \neq a} \left( F_{ac}^0(q_a - q_c) + F_{ac}^0(q_a + q_c) \right),
\end{equation}

$$
\mathcal{F}_{ab} = \nu \delta_{ab} F^0,
$$

\begin{equation}
F^0 = \frac{1}{2} \sum_{c \neq d} \left( F_{cd}^0(q_c - q_d) + F_{cd}^0(q_c + q_d) \right).
\end{equation}

Notice that blocks $B_1$ and $B_2$ are off-diagonal here. Representing the Lax pair schematically as $\mathcal{L} = \mathcal{P} + R$, $\mathcal{M} = D + \mathcal{F} + F$, where $\mathcal{P}$ consists of momenta part, $R$ includes $A_1, A_2, B_1, B_2$ and similarly for $\mathcal{M}$, we can rewrite the r.h.s. of the Lax equations in the form:

$$
[\mathcal{L}, \mathcal{M}] = [\mathcal{P} + R, D + \mathcal{F} + F] = [\mathcal{P}, F] + [R, D] + [R, \mathcal{F}] + [R, F].
$$

The calculations are performed for $N \times N$ ($\otimes \text{Mat}_{N}^{\otimes N}$) blocks \{11\}, \{12\}, \{21\}, \{22\} separately.

**Proof for Block \{11\}**

The first summand in \(2.23\):

$$
[P, F] = \sum_{c, a \neq b} \nu p_c [E_{cc} \otimes 1, E_{ab} \otimes F_{ab}^z(q_a - q_b)] = \sum_{a \neq b} \nu(q_a - q_b) E_{ab} \otimes F_{ab}^z(q_a - q_b).
$$

The second summand in \(2.23\):

$$
[R, D] = \sum_{a \neq b} \nu^2 E_{ab} \otimes \left( \sum_{c, c' \neq a} F_{ac}^0(q_a - q_c) R_{ab}^z(q_a - q_b) + F_{ac}^0(q_a + q_c) R_{ab}^z(q_a - q_b) - R_{ab}^z(q_a - q_b) F_{bc}^0(q_b - q_c) + R_{ab}^z(q_a - q_b) F_{bc}^0(q_b + q_c) \right) =
$$

\begin{equation}
= \sum_{a \neq b} \nu^2 E_{ab} \otimes \sum_{c, c' \neq a, b} \left( F_{ac}^0(q_a - q_c) R_{ab}^z(q_a - q_b) + F_{ac}^0(q_a + q_c) R_{ab}^z(q_a - q_b) - R_{ab}^z(q_a - q_b) F_{bc}^0(q_b - q_c) - R_{ab}^z(q_a - q_b) F_{bc}^0(q_b + q_c) + F_{ab}^0(q_a - q_b) R_{ab}^z(q_a - q_b) + F_{ab}^0(q_a + q_b) R_{ab}^z(q_a - q_b) - R_{ab}^z(q_a - q_b) F_{ab}^0(q_a + q_b) - R_{ab}^z(q_a - q_b) F_{ab}^0(q_a - q_b) \right).
\end{equation}
The last summand in (2.23): 

\[
[R, F](\text{off-diagonal}) = \\
= \sum_{a \neq b} \nu^2 E_{ab} \otimes \sum_{c, c \neq a, b} \left( R_{ac}^z(q_a - q_c) F_{cb}^z(q_c - q_b) - F_{ac}^z(q_a - q_c) R_{cb}^z(q_c - q_b) + \\
+ R_{ac}^z(q_a + q_c) F_{cb}^z(-q_c - q_b) - F_{ac}^z(q_a + q_c) R_{cb}^z(-q_c - q_b) \right) \\
= \sum_{a \neq b} \nu^2 E_{ab} \otimes \sum_{c, c \neq a, b} \left( F_{cb}^0(q_c - q_b) R_{ac}^z(q_a - q_b) - R_{ab}^z(q_a - q_b) F_{ac}^0(q_a - q_c) + \\
+ F_{cb}^0(-q_c - q_b) R_{ab}^z(q_a - q_b) - R_{ab}^z(q_a - q_b) F_{ac}^0(q_a + q_c) \right). \\
\]

Hence 

\[
\hat{L} = [P, F] + [R, F](\text{diagonal}) \\
(2.28)
\] provides equations of motion (with \(\nu\) replaced by \(\tilde{\nu}\)). Let us verify that the sum of the remaining terms in the r.h.s. of the Lax equations is equal to zero. Indeed,

\[
[R, F](\text{off-diagonal}) + [R, F] + [R, D] = \\
= \sum_{a \neq b} \nu^2 E_{ab} \otimes \left[ R_{ab}^z(q_a - q_b), F_{ac}^0(q_a - q_c) + F_{bc}^0(q_b - q_c) \right] \\
- \sum_{c, c \neq a, b} \left( F_{ac}^0(q_a - q_c) + F_{bc}^0(q_b - q_c) \right) \\
\]

This equals zero, since there are no terms in the right part of the commutator acting non-trivially in \(a\)-th and \(b\)-th quantum spaces. All such terms are cancelled by \(F^0\).

**Proof for block \{12\}**

The r.h.s. of the Lax equations:

\[
[L, M]_{a,b+N} = [P + R, D + F + F]_{a,b+N} = \\
= [P, F]_{a,b+N} + [R, D]_{a,b+N} + [R, F]_{a,b+N} + [R, F]_{a,b+N}. \\
(2.30)
\]

As in the case of block \{12\} the term \([P, F]_{a,b+N}\) is cancelled by the off-diagonal part in the block \{12\} of the l.h.s. of the Lax equation. But \{12\} block of the Lax operator has no diagonal part in this new case. So the sum of the remaining terms in (2.30) should vanish:

\[
[R, D]_{a,b+N} + [R, F]_{a,b+N} + [R, F]_{a,b+N} = 0 \\
(2.31)
\]
Let us verify it. First, consider the last term in (2.31):

\[ [R, F]_{a,b+N} \text{(off-diagonal)} = \]

\[
= \nu^2 \sum_{c \neq a,b} \left( R^{z}_{ab}(q_a - q_c)F^{z}_{ca}(q_c + q_b) - F^{z}_{ac}(q_a - q_c)R^{z}_{ca}(q_c + q_a) + R^{z}_{ac}(q_a + q_c)F^{z}_{cb}(q_b - q_c) - F^{z}_{ac}(q_a + q_c)R^{z}_{cb}(-q_c + q_b) \right)
\]

\[ \xlongequal{(1.17)} (2.32) \]

For the diagonal part there is no appropriate identity, and we leave it as it is:

\[ [R, F]_{a,a+N} \text{(diagonal)} = \]

\[
= \nu^2 \sum_{c \neq a} \left( R^{z}_{ac}(q_a - q_c)F^{z}_{ca}(q_c + q_a) - F^{z}_{ac}(q_a - q_c)R^{z}_{ca}(q_c + q_a) + R^{z}_{ac}(q_a + q_c)F^{z}_{ab}(q_b - q_c) - F^{z}_{ac}(q_a + q_c)R^{z}_{ab}(-q_c + q_a) \right)
\]

The first term in (2.31) is again simplified through (1.17):

\[ [R, D]_{a,b+N} = (R^{z}_{ab}(q_a - q_b)D_b - D_a R^{z}_{ab}(q_a - q_b)) = \]

\[
= \nu^2 \sum_{c \neq a,b} \left( F^{0}_{ac}(q_a - q_c)R^{z}_{ab}(q_a + q_b) + F^{0}_{ac}(q_a + q_c)R^{z}_{ab}(q_a + q_b) - R^{z}_{ab}(q_a + q_c)F^{0}_{bc}(q_b - q_c) - F^{z}_{ab}(q_a + q_b)F^{0}_{bc}(q_b - q_c) + F^{0}_{ab}(q_b - q_a)F^{0}_{ba}(q_b - q_a) \right)
\]

\[ \xlongequal{(2.34)} (2.33) \]

Gathering all the off-diagonal terms in (2.31) we get the commutator

\[ [R, F]_{a,b+N} \text{(off-diagonal)} + [R, D]_{a,b+N} + [R, F]_{a,b+N} = \]

\[
= \nu^2 \left[ R^{z}_{ab}(q_a + q_b), F^{0}_{ab}(q_a - q_b) - F^{0}_{ab}(q_a + q_b) - \right.
\]

\[
\left. - \sum_{c \neq a,b} \left( F^{0}_{ac}(q_a - q_c) + F^{0}_{ac}(q_a + q_c) + F^{0}_{bc}(q_b - q_c) + F^{0}_{bc}(q_b + q_c) \right) \right],
\]

which equals zero by the same reason as for block \{11\} (2.29). Therefore, we are left with the diagonal term

\[ [L, M]_{a,a+N} = [R, F]_{a,a+N}, \]

(2.36)

and it vanishes if

\[ R^{z}_{ab}(u)F^{z}_{ba}(v) - F^{z}_{ab}(v)R^{z}_{ba}(u) = 0. \]

(2.37)
The latter is of course not true in general case, but is true in some particular cases. For example, it is obviously holds true for the Yang’s case \((1,15)\). What is more important for us is that \((2.37)\) holds true in \(N = 2\) case with the Baxter’s \(R\)-matrix since it is of the form
\[
R_{i2}(u) = \sum_{\alpha=0}^{3} \varphi_{\alpha}^{(u)}(u) \sigma_{\alpha} \otimes \sigma_{\alpha}.
\]
It is easy to check that for such \(R\)-matrices
\[
[R_{ab}, R_{ab}(v)] = 0 \quad (2.38)
\]
due to the properties of the Pauli matrices. By differentiating \((2.38)\) with respect to \(v\) (and using the additional symmetry \(R_{ab}^{z}(u) = R_{ba}^{z}(u)\)) one finds \((2.37)\). With this property we have
\[
R_{ac}^{z}(q_{a} - q_{c})F_{ca}(q_{c} + q_{a}) - F_{ac}^{z}(q_{a} - q_{c})R_{ca}^{z}(q_{c} + q_{a}) +
+ R_{ac}^{z}(q_{a} + q_{c})F_{ca}^{z}(-q_{c} + q_{a}) - F_{ac}^{z}(q_{a} + q_{c})R_{ca}^{z}(-q_{c} + q_{a}) = 0,
\]
and this is the expression, standing under the sum in \([R, F]_{a,a+N}\).

The proofs for blocks \(\{21\}\) and \(\{22\}\) are performed in a similar way.

### 2.4 \(B_{N}\) case with \(N + 1\) quantum spaces

Here \(R\)-matrix-valued Lax pair with \(N + 1\) quantum spaces has the same block structure as the one for BC(\(N\)) with \(2N + 1\) quantum spaces \((2.8)\)
\[
\mathcal{L} = \begin{pmatrix}
P + A_{1} & B_{1} & C_{1} \\
B_{2} & -P + A_{2} & C_{2} \\
C_{2}^{T} & C_{1}^{T} & 0
\end{pmatrix} \quad \mathcal{M} = \begin{pmatrix}
A'_{1} + D_{1} + F & B'_{1} & C'_{1} \\
B'_{2} & A'_{2} + D_{2} + F & C'_{2} \\
C'_{2}^{T} & C'_{1}^{T} & D_{3} + F
\end{pmatrix} \quad (2.40)
\]
but now the corresponding blocks have the following form. The blocks \(A, B\) and \(P\) are the same as in the previous case \((2.21)\), and the rest are:
\[
(C^{1}_{1})_{a} = \pm \sqrt{2\nu} R_{a,N+1}^{z}(q_{a}), \quad (C^{2}_{1})_{a} = \pm \sqrt{2\nu} R_{a,N+1}^{z}(-q_{a}),
\]
\[
(C^{1}_{1})_{a} = \pm \sqrt{2\nu} R_{N+1,a}^{z}(q_{a}), \quad (C^{2}_{1})_{a} = \pm \sqrt{2\nu} R_{N+1,a}^{z}(-q_{a}),
\]
\[
(C^{1}_{1})_{a} = \pm \sqrt{2\nu} F_{a,N+1}^{z}(q_{a}), \quad (C^{2}_{1})_{a} = \pm \sqrt{2\nu} F_{a,N+1}^{z}(-q_{a}),
\]
\[
(C^{1}_{1})_{a} = \pm \sqrt{2\nu} F_{N+1,a}^{z}(q_{a}), \quad (C^{2}_{1})_{a} = \pm \sqrt{2\nu} F_{N+1,a}^{z}(-q_{a}),
\]
\[
D_{3} = -2\nu \sum_{c} F_{c,N+1}^{0}(q_{c}).
\]
The \(D\) and \(F\) terms are given by:
\[
(D^{1})_{a} = (D^{2})_{a} = -\nu \sum_{c: c \neq a} \left( F_{ac}^{0}(q_{a} - q_{c}) + F_{ac}^{0}(q_{a} + q_{c}) \right) - 2\nu F_{a,N+1}^{0}(q_{a}), \quad (2.42)
\]
and
\[
F_{ab} = \nu \delta_{ab} F_{0}^{0}, \quad F_{0}^{0} = \frac{1}{2} \sum_{c \neq d} \left( F_{cd}(q_{c} - q_{d}) + F_{cd}(q_{c} + q_{d}) \right) + 2 \sum_{c} F_{c,N+1}^{0}(q_{c}). \quad (2.43)
\]
Or, equivalently

\[(D_1)^{BC(N)}_a = (D_2)^{BC(N)}_a = D_a^{SO(2N)} + 2\nu F_{a,N+1}^0(q_a), \quad (2.44)\]

\[\nu F_{BC(N)}^0 = \nu F_{SO(2N)}^0 - D_3.\]

As in the previous case the calculations are performed separately for \(N \times N\) \((\mathbb{S}^N)\) blocks \{11\}, \{12\}, \{21\}, \{22\}; for \(N \times 1\) \((\mathbb{S}^N)\) blocks \{13\}, \{23\}; for \(1 \times N\) \((\mathbb{S}^N)\) blocks \{31\}, \{32\} and for \(1 \times 1\) \((\mathbb{S}^N)\) block \{33\}. Evaluations for the blocks \{11\}, \{12\}, \{21\}, \{22\} are similar. In particular, we again come to condition (2.37). The calculation for the block \{33\} is as follows:

\[\{L, M\}_{2N+1,2N+1} = (C_2^T C_1^T - C_2^T C_1 + C_1^T C_2 - C_1^T C_2) = \]

\[= 2\nu^2 \sum_a \left( R_{\tilde{N}+1,a}^z (-q_a) F_{\tilde{N}+1,a}^z(q_a) - F_{\tilde{N}+1,a}^z(-q_a) R_{\tilde{N}+1,a}^z(q_a) + R_{\tilde{N}+1,a}^z(q_a) F_{\tilde{N}+1,a}^z(-q_a) - F_{\tilde{N}+1,a}^z(q_a) R_{\tilde{N}+1,a}^z(-q_a) \right) \]

\[= 2\nu^2 \sum_a (\tilde{N}^2 \varphi'(-q_a) + \tilde{N}^2 \varphi'(q_a)) = 0. \quad (2.45)\]

In the end of the Section let us remark that the Hamiltonian (1.10) is a particular case of the most general model [26]

\[H = \frac{1}{2} \sum_{a=1}^{N} p_a^2 - \nu^2 \sum_{a<b}^{N} (\varphi(q_a - q_b) + \varphi(q_a + q_b)) - \sum_{\gamma=0}^{3} \sum_{a=1}^{N} \nu_a^2 \varphi(q_a + \omega_{\gamma}), \quad (2.46)\]

which has \(3N \times 3N\) Lax representation (and \(2N \times 2N\) Lax representation [16]). In the last term \(\omega_{\gamma}\) are the half-periods of the elliptic curve with moduli \(\tau\): \(\{0, 1/2, \tau/2, 1/2 + \tau/2\}\), and all five constants in (2.46) are arbitrary. Unfortunately, we do not know how to extend \(3N \times 3N\) (or \(2N \times 2N\)) Lax pair for (2.46) to the \(R\)-matrix-valued case. At the same time the \(BC_1\) case (one degree of freedom and four arbitrary constants) with \(N = 2\) was suggested in [33].

3 Quantum Lax pairs and spin models

Spinless case. The classical Lax pairs for Calogero-Moser models can be quantized through the quantum Lax equation [47, 10]

\[\{\hat{H}, \hat{L}(z)\} = h \{\hat{L}(z), M(z)\}. \quad (3.1)\]

Before discussing the \(R\)-matrix-valued Lax pairs consider the ordinary case \(\tilde{N} = 1\). As mentioned before, in this case the Lax pair (1.12)-(1.13) differs from (1.4)-(1.5) by the scalar (in the auxiliary space) part \(F^0\) (1.14) of \(M\)-matrix. For \(N = 1\)

\[F^0_{\tilde{N}=1} = -\nu \sum_{i>j} \varphi(q_{ij}) + \text{const}, \quad \text{const} = \frac{N^2 - N}{3} \frac{\varphi''(0)}{\varphi'(0)}. \quad (3.2)\]
This term is cancelled out from the classical Lax equations. But it becomes important in quantum case because it is treated as a part of the quantum Hamiltonian in (3.1). With this term the coupling constant in

\[ \hat{H} = \frac{1}{2} \sum_{i=1}^{N} \hat{p}_{i}^2 - \nu(\nu + \hbar) \sum_{i>j} \varphi(q_i - q_j) , \quad \hat{p}_i = \hbar \partial_{q_i} \]  

(3.3)

acquires the quantum correction. Another reason for treating (3.2) as a part of the Hamiltonian is as follows. After subtracting \( F^0 \) from \( M \) we are left (in \( \tilde{N} = 1 \) case) with the original \( M \)-matrix (1.5). In rational and trigonometric cases (A.8) this \( M \)-matrix satisfies the so-called sum up to zero condition:

\[ \sum_{j : j \neq i} M_{ij} = \sum_{j : j \neq i} M_{ji} = 0 \quad \text{for all } i . \]  

(3.4)

It can be proved [47] that with this condition the total sums

\[ \hat{I}_k = \text{ts}(\hat{L}^k) = \sum_{i,j} (\hat{L}^k)_{ij} , \quad k \in \mathbb{Z}_+ \]  

(3.5)

are the quantum integrals of motion, i.e. \([\hat{H}, \hat{I}_k] = 0 \) (while in the classical case the integrals of motion can be defined as \( \text{tr}(L^k) \)). The second Hamiltonian \( \hat{I}_2 \) yields \( \hat{H} \) (3.3).

In the elliptic case the sum up to zero condition (3.4) is not fulfilled. At the same time the quantum Lax equation holds true.

**Spin case.** Similar construction works for the spin Calogero-Moser model [20]. The operator valued Lax pair [27, 23, 6, 28] is of the form

\[ \hat{L}^{\text{spin}}(z) = \sum_{i,j=1}^{N} E_{ij} \otimes \hat{L}^{\text{spin}}_{ij}(z) , \quad \hat{L}^{\text{spin}}_{ij}(z) = \delta_{ij} \hat{p}_i + \nu(1 - \delta_{ij}) \phi(z,q_{ij}) P_{ij} , \]  

(3.6)

\[ \hat{M}^{\text{spin}}_{ij}(z) = \nu d_i \delta_{ij} + \nu(1 - \delta_{ij}) f(z,q_{ij}) P_{ij} , \quad d_i = \sum_{k : k \neq i}^{N} E_2(q_{ik}) P_{ik} = - \sum_{k : k \neq i}^{N} f(0,q_{ik}) P_{ik} , \]  

(3.7)

where \( P_{ij} \in \text{Mat}_{\tilde{N}} \otimes \tilde{N} \) is the permutation operator (of \( i \)-th and \( j \)-th tensor components) in quantum space \( \mathcal{H} = (\mathbb{C}^N)^{\otimes \tilde{N}} \), or the spin exchange operator. The tensor structure of (3.6)-(3.7) is the same as for the \( R \)-matrix-valued Lax pair (1.12)-(1.13). The term \( F^0 \) (1.13) in this case

\[ \mathcal{F}^0 = -\nu \sum_{i>j} E_2(q_i - q_j) P_{ij} \]  

(3.8)

is treated as a part of the quantum Hamiltonian

\[ \hat{H}^{\text{spin}} = \frac{1}{2} \sum_{i=1}^{N} \hat{p}_{i}^2 - \sum_{i>j} \nu(\nu + \hbar) P_{ij} E_2(q_i - q_j) \]  

(3.9)

likewise it was performed in the spinless case (3.3). It describes the spin exchange interaction. Again, in the rational and trigonometric cases the sum up to zero condition is fulfilled, and
ts(\(L^k\)) provides the higher Hamiltonians including the one (3.9) for \(k = 2\). Another recipe for constructing the higher Hamiltonians comes from the underlying Yangian structure [6].

From the point of view of \(R\)-matrix-valued case (1.12)-(1.13) the Lax pair \(\hat{L}^{\text{spin}}(z)\) and \(\hat{M}^{\text{spin}}(z) + 1_N \otimes F^0\) corresponds to the special case when \(R^z_{ij}(q_{ij})\) is replaced by \(\phi(z, q_{ij}) P_{ij}\). The associating Yang-Baxter equation (1.16) holds true in this case since the matrix-valued expressions \(P_{ab}P_{bc} = P_{ac}P_{ab} = P_{bc}P_{ac}\) are cancelled out, and we are left with the scalar identity (1.9).

**Anisotropic spin case.** In [24] the anisotropic extension of the (trigonometric) spin Calogero-Moser-Sutherland model was suggested. It is based on the \(su(2)\) XXZ (trigonometric) classical \(r\)-matrix
\[
\hat{r}^{\text{XXZ}}_{12}(q) = (1 \otimes 1 + \sigma_3 \otimes \sigma_3) \cot(q) + (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \frac{1}{\sin(q)},
\]
(3.10)
which is used in the Lax pair in the same way as \(R^z_{ij}(q)\) in (1.12):
\[
\hat{L}^{\text{XXZ}}(z) = \sum_{i,j=1}^N E_{ij} \otimes \hat{L}^{\text{XXZ}}_{ij}(z), \quad \hat{L}^{\text{XXZ}}_{ij}(z) = \delta_{ij} p_i + \nu(1 - \delta_{ij}) r^{\text{XXZ}}_{ij}(q_{ij}).
\]
(3.11)
In this respect the Lax matrix (1.12) with the elliptic Baxter-Belavin \(R\)-matrix generalizes the case (3.10) in the same way as the elliptic Lax matrix (1.4) generalizes the one without spectral parameter for the classical (trigonometric) Sutherland model. Let us slightly modify (3.11) to have exactly the form (1.12). Consider (1.12)-(1.13) defined through the XXZ quantum \(R\)-matrix
\[
R^z_{12}(q) = \sum_{\alpha=0}^3 \varphi^z_{\alpha}(q) \sigma_{\alpha} \otimes \sigma_{\alpha}, \quad \varphi^z_{1}(q) = \varphi^z_{2}(q) = \frac{1}{\sin q},
\]
(3.12)
\[
\varphi^z_{0}(q) = (\cot z + \cot q + \frac{1}{\sin z}), \quad \varphi^z_{3}(q) = (\cot z + \cot q - \frac{1}{\sin z}).
\]
This results in the quantum Hamiltonian with the \(F^0\) term
\[
F^0 = \sum_{a<b} \left( \frac{1}{\sin^2 q_{ab}} (\sigma_a \otimes \sigma_0 + \sigma_3 \otimes \sigma_3) + \frac{\cos q_{ab}}{\sin^2 q_{ab}} (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \right),
\]
(3.13)
which was obtained in [24] from (3.11) via (3.5). It happened because the terms depending on \(z\) in (3.12), do not depend on \(q\). Hence, \(F^z_{12}(q) = F^0_{12}(q)\) and the condition (3.4) can be fulfilled.

For the elliptic \(R\)-matrix-valued Lax pairs with half amount of quantum spaces, found in the previous Section, the quantum Lax equations hold true in a similar way:

**Proposition 3.1** For the Lax pairs (1.12)-(1.13) for \(A_{N-1}\), (2.20) for \(D_N\) and (2.40) for \(B_N\) root systems one can define the quantum Lax pair by simple replacing momenta \(p_i\) with \(\hat{p}_i = \hbar \partial_q\). The quantum Lax equations hold true with the quantum Hamiltonian \(\hat{H}\) obtained from the classical \(H(p, q)\) as \(\hat{H}(\hat{p}, \hat{q})\).

The proof of this statement is direct. It is similar to the classical case and (3.6). The main idea, needed for the proof is that in the above mentioned cases
\[
[ P, D + F^0] = 0,
\]
(3.14)
where $D$ is a diagonal part of the $M$-operator. This happens because the only terms in $D$ depending on some $q_a$ are those, which simultaneously act non-trivially on the tensor component number $a$ in quantum space. We already know from the classical case that $D_a$ subtracts exactly these terms from $F^0$. For $C_N$ and $BC_N$ cases this statement is not true. For example, $D_{1a} - F^0$ contains the terms $F^0_{a+N,c+N}(-q_a + q_c)$, which do not commute with $\hat{p}_a$. 

From what has been said it seems also natural to remove the $F^0$ terms from the corresponding $M$-matrices and treat them as a part of the quantum Hamiltonians, which then take the form

$$\hat{H} = H(\hat{p}, q) + \hbar \nu F^0,$$

(3.15)

where the terms $F^0$ are given by (1.14) for $A_{N-1}$, (2.20) for $D_N$ and (2.43) for $B_N$ root systems.

Let us emphasize that $F^0$ term in the elliptic quantum Hamiltonian (3.15) does not follow from the Lax matrix. The construction (3.5) used in the rational and trigonometric cases does not work here since the sum up to zero condition (3.4) is not valid. Moreover, in the classical case the corresponding Lax matrices provide the Hamiltonians without $F^0$. However, we are going to argue that (3.15) is meaningful even in the classical case (see the next Section).

The quantum Lax matrices (3.6), are closely related to the Knizhnik-Zamolodchikov equations [29]. As was mentioned in [23, 24] the KZ connections (3.11) are obtained as

$$\nabla_i \psi = 0, \quad i = 1...N.$$

(3.17)

The interrelation between the quantum Calogero models and the KZ equations was observed by Matsuo and Cherednik [36, 12] (see also [17]). Their construction, in fact, underlies the construction of the operator valued Lax pairs.\footnote{It should be also mentioned that different types of spinless and spin Calogero-Moser models can be described in the framework of the Cherednik-Dunkl and/or exchange operator formalism [14, 42]. See e.g. [6, 7, 49, 18]. In our approach we do not use the particles exchange operators.}

The relation to KZ equations is modified in the elliptic case as follows. To the $R$-matrix-valued Lax pair (1.12)-(1.13) let us associate the $\mathfrak{gl}_N$ KZB equations on elliptic curve with $N$ punctures at $q_i$:

$$\nabla_i \psi = 0, \quad \nabla_i = \hbar \partial_i + \nu \sum_{j:j \neq i} r_{ij}(q_i - q_j),$$

$$\nabla_\tau \psi = 0, \quad \nabla_\tau = \hbar \partial_\tau + \nu \sum_{j:j \neq k} m_{jk}(q_j - q_k),$$

(3.18)

where $r_{ij}$ is the classical $r$-matrix, $m_{ij}$ is the next term in the classical limit (A.19), and $i = 1, ..., N$. The commutativity of the connections follows from the classical Yang-Baxter equation

$$[r_{ab}, r_{ac}] + [r_{ab}, r_{bc}] + [r_{ac}, r_{bc}] = 0$$

(3.19)

and the identities

$$[r_{ab}, m_{ac} + m_{bc}] + [r_{ac}, m_{ab} + m_{bc}] = 0,$$

(3.20)
which are obtained from the associative Yang-Baxter equation (1.16) and (1.19)-(1.20). It was shown in [32] that a solution of (3.18) satisfies also

\[-N\nu \hbar \partial_\tau \psi = \left( \frac{1}{2} \sum_{i=1}^{N} \hbar^2 \partial_{q_i}^2 - \nu^2 \tilde{N}^2 \sum_{i<j} \psi(q_i - q_j) + \hbar \nu \sum_{i<j} \partial_{r_{ij}}(q_i - q_j) + \text{const} \right) \psi \tag{3.21}\]

or, equivalently (A.21)

\[-N\nu \hbar \partial_\tau \psi = \left( H(p, q) + \hbar \nu F^0 + \text{const} \right) \psi \tag{3.22}\]

Therefore, in contrast to trigonometric case (3.16)-(3.17), the conformal block solving the KZ(B) equations in the elliptic case is not an eigenfunction of the quantum Hamiltonian, but a solution of quantum non-autonomous problem, or the non-stationary Shrödinger equation with the same quantum Hamiltonian.

In order to clarify the link between the KZ(B) equations and the $R$-matrix-valued Lax pairs, a precise relation is needed between the conformal blocks and the Baker-Akhiezer functions of the corresponding Lax pair. This issue remains open.

### 4 Spin exchange operators from Hitchin systems

The purpose of this Section is to explain the classical origin of the $F^0$ term (related to the elliptic $R$-matrix) in the Hamiltonian (3.15) and/or (1.24). We start with a brief description of the classical spin Calogero-Moser model, and then proceed to the model of $N$ interacting $\text{SL}_\tilde{N}$ tops, which provides the classical analogue of the Hamiltonian with the (elliptic) anisotropic spin exchange operator.

**Spin Calogero-Moser model.** In classical mechanics the $\text{sl}_N$ spin Calogero-Moser models are described as follows [8] (see also [3, 6]). The phase space is a direct product $\mathbb{C}^{2N} \times \mathcal{O}$: the particles degree of freedom $p_i, q_i, i = 1...N$ parameterize $\mathbb{C}^{2N}$ and the coadjoint orbit of $\text{GL}_N$ group $\mathcal{O}$ is parameterized by $S_{ij}, i, j = 1...N$, i.e. the classical spin variables are arranged into matrix $S = \sum_{ij} E_{ij} S_{ij} \in \text{Mat}_N$ with fixed eigenvalues. The Poisson structure is a direct sum of the canonical brackets (1.2) and the Lie-Poisson structure

\[
\{S_{ij}, S_{kl}\} = -S_{kl} \delta_{kj} + S_{kj} \delta_{il} . \tag{4.1}\]

The latter is realized by embedding $S$ into a (larger) space with canonical variables:

\[
S_{ij} = \sum_{a=1}^{\tilde{N}} \xi_i^a \eta_j^a , \quad \{\xi_i^a, \eta_j^b\} = \delta_{ij} \delta_{ab} . \tag{4.2}\]

The Lax pair

\[
L_{ij}^{\text{spin}}(z) = \delta_{ij}(p_i + S_{ii} E_1(z)) + (1 - \delta_{ij}) S_{ij} \phi(z, q_{ij}) , \tag{4.3}\]

\[
M_{ij}^{\text{spin}}(z) = (1 - \delta_{ij}) S_{ij} f(z, q_i - q_j) \tag{4.4}\]

and the Hamiltonian

\[
H^{\text{spin}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \sum_{i>j} S_{ij} S_{ji} E_2(q_i - q_j) . \tag{4.5}\]
satisfy the Lax equations (1.6) and define integrable system if the additional constraints hold:

\[ S_{ii} = \text{const}, \quad \text{for all } i. \]  

(4.6)

The are generated by the action of the Cartan subgroup of SL\(_N\).

The appearance of the spin exchange operator (3.8) comes from rewriting \( S_{ij}S_{ji} \) in the Hamiltonian in terms of \( N \) Mat\(_{\tilde{N}}\)-valued matrices:

\[ S_{ij}S_{ji} = \sum_{a,b=1}^{\tilde{N}} \xi^a_i \xi^b_{ij} \eta^b_j = \text{tr}(\tilde{B}B), \quad \tilde{B} = \sum_{a,b=1}^{\tilde{N}} \tilde{E}_{ab} \tilde{B}_{ab} \in \text{Mat}_{\tilde{N}}, \quad \tilde{B}_{ab} = \xi^a_i \eta^b_i. \]  

(4.7)

Due to (4.6) \( \text{tr} \tilde{B} = \text{const} \) for all \( i = 1, \ldots, N \). Hence, we come to \( N \) minimal coadjoint orbits of GL\(_{\tilde{N}}\). The quantization of \( \tilde{B}_{ab} \) in the fundamental representation of gl\(_{\tilde{N}}\) is given by \( \tilde{E}_{ab} \). This provides the permutation operator \( P_{ij} = \sum_{a,b=1}^{\tilde{N}} \tilde{E}_{ab} \tilde{E}_{ba} \in \text{Mat}_{\tilde{N}}, \quad \tilde{E}_{ab} = \xi^a_i \eta^b_i \).

Let us remark that the classical \( r \)-matrix structure for (4.3) is defined by dynamical \( r \)-matrix. Its quantization is performed in terms of quantum dynamical \( R \)-matrix \([3]\), while the approach using the quantum Lax pairs of (3.6)-(3.7) type is more like the Matsuo-Cherednik construction. It is dual to the direct one since it is based on gl\(_{\tilde{N}}\) KZ equations, and the positions of particles are the \( N \) punctures on the base spectral curve.

**The classical version of the \( F^0 \) term** (3.15) coming from the elliptic \( R \)-matrix (A.18) is as follows:

\[ F^0 = \frac{1}{2} \sum_{i \neq j}^{N} F^0_{ij}(q_{ij}) = \frac{1}{2} \sum_{i \neq j}^{N} \partial \eta r_{ij}(q_{ij}), \]  

(4.8)

where using (A.20) and (A.10) we have

\[ \partial \eta r_{ij}(q_{ij}) = -E_2(q_{ij}) T_0 T_0^\dagger + \sum_{\gamma \in \mathbb{Z}_{\tilde{N}} \times \mathbb{Z}_{\tilde{N}}, \gamma \neq 0} \varphi_{\gamma}(q_{ij}, \omega_{\gamma})(E_1(q_{ij} + \omega_{\gamma}) - E_1(q_{ij}) + 2\pi i \partial r \omega_{\gamma}) T_\gamma T_{-\gamma}. \]  

(4.9)

It follows from the previous discussion that the classical analogue of (4.8)-(4.9) is

\[ F^0_{\text{class}} = \frac{1}{2} \sum_{i \neq j}^{N} \left( -E_2(q_{ij}) \tilde{B}_0 \tilde{B}_0^\dagger + \sum_{\gamma \neq 0} \varphi_{\gamma}(q_{ij}, \omega_{\gamma})(E_1(q_{ij} + \omega_{\gamma}) - E_1(q_{ij}) + 2\pi i \partial r \omega_{\gamma}) \tilde{B}_\gamma \tilde{B}_{-\gamma} \right), \]  

(4.10)

where \( \tilde{B} \) are the \( N \) Mat\(_{\tilde{N}}\)-valued matrices (classical spin variables), and \( \tilde{B}_{\alpha} \) – their components in the basis (A.1).

---

\(^{10}\)The expressions for the \( M \)-matrix (4.4) is valid only before the Poisson reduction is performed, which may led to additional (Dirac) terms. For instance, if \( \tilde{N} = 1 \) in (4.2) (the minimal orbit) then this reduction kills all spin variables, and the model is reduced to the spinless Calogero-Moser model (1.4). As a result the diagonal terms in (1.5) appear.
Hitchin systems on $\text{SL}_{N\bar{N}}$-bundles. In the Hitchin approach \cite{25} to elliptic models \cite{37} the positions of particles are treated as coordinates on the moduli space of Higgs bundles over (punctured) elliptic curve. The $\text{GL}_n$-bundles are classified according to Atiyah \cite{2}. Dimensions of the moduli spaces are obtained as the greatest common divisors of the rank $n$ and degrees of underlying holomorphic vector bundles. By changing the degrees we describe different type models. This leads to integrable systems called interacting elliptic tops \cite{51}. Similar construction exists for arbitrary complex simple Lie group \cite{35}. The classification is based on characteristic classes, which are in one to one correspondence with the center of the structure group.

Here we consider the model related to the $\text{SL}(N\bar{N}, \mathbb{C})$-bundle and the characteristic class is $N$ (or $\exp(2\pi i \frac{N}{\bar{N}})$). It means that the model contains $N$ particles degrees of freedom, and the Lax matrix is $\text{Mat}_{N\bar{N}}$-valued:

$$L(z) = \sum_{i,j=1}^{N} E_{ij} \otimes L_{ij}(z), \quad L_{ij}(z) \in \text{Mat}_{\bar{N}},$$

where

$$L_{ij}(z) =$$

$$= \delta_{ij} \left( p_i + S_0^{ii} T_0 E_1(z) + \sum_{\alpha \neq 0} S_\alpha^{ii} T_\alpha \varphi_\alpha(z, \omega_\alpha) \right) + (1 - \delta_{ij}) \sum_\alpha S_\alpha^{ij} T_\alpha \varphi_\alpha(z, \omega_\alpha - \frac{q_{ij}}{N}),$$

where $T_\alpha$ is the basis $\{\mathbf{A, I}\}$ and $S_\alpha^{ij}$ – the components in this basis of $N^2$ matrices $S^{ij} \in \text{Mat}_{\bar{N}}$. The variables $S_\alpha^{ij}$ are dual to the basis $E_{ij} \otimes T_\alpha$ in $\text{Mat}_{N\bar{N}}$. Therefore, the Poisson brackets are defined in the same way as the commutation relations for the set $\{E_{ij} \otimes T_\alpha\}$, i.e.

$$\{S_\alpha^{ij}, S_\beta^{kl}\} = \delta_{il} \delta_{ad} S_\alpha^{kj} - \delta_{kj} \delta_{cb} S_\alpha^{il}, \quad S^{ij} = \sum_{a,b=1}^{\bar{N}} \tilde{E}_{ab} S_{ab}^{ij}. \quad (4.13)$$

Of course $S^{ij}$ can be decomposed in the standard basis in $\text{Mat}_{\bar{N}}$ as well. Then the brackets (4.13) acquire the form:

$$\{S_{ab}^{ij}, S_{cd}^{kl}\} = \delta_{il} \delta_{ad} S_{cb}^{kj} - \delta_{kj} \delta_{cb} S_{ad}^{il}, \quad S^{ij} = \sum_{a,b=1}^{\bar{N}} \tilde{E}_{ab} S_{ab}^{ij}. \quad (4.14)$$

The analogue for constraints (4.6) is again generated by the coadjoint action of the Cartan subgroup of $\text{GL}_N \subset \text{GL}_{N\bar{N}}$:

$$S_0^{ii} = \text{const for all } i = 1, \ldots, N. \quad (4.15)$$

The quadratic Hamiltonian is evaluated from $\text{tr} L^2(z)$:

$$H_{\text{tops}} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha \neq 0} S_{\alpha}^{ii} S_{-\alpha}^{ii} E_2(\omega_\alpha) - \frac{1}{2} \sum_{i \neq j}^{N} \sum_{\alpha} S_{\alpha}^{ij} S_{-\alpha}^{ji} E_2(\omega_\alpha - \frac{q_{ij}}{N}). \quad (4.16)$$

Consider the case of the minimal coadjoint orbit of $\text{GL}_{N\bar{N}}$. Then the matrix

$$S = \sum_{i,j=1}^{N} E_{ij} \otimes S^{ij}, \quad S^{ij} \in \text{Mat}_{\bar{N}}, \quad i, j = 1, \ldots, N \quad (4.17)$$
is of rank 1, that is
\[
S^{ij} = \sum_{a,b=1}^{\tilde{N}} \bar{E}_{ab} \xi^a_i \eta^b_j
\]  
(4.18)

with \(\xi^a\) and \(\eta^b\) are as in (4.2). This parametrization is in agreement with the Poisson brackets (4.14) or (4.13). From (4.7) we also have
\[
S^{ii} = \tilde{B}.
\]  
(4.19)

Due to (A.4) \(S^{ij}_\alpha = \text{tr}(S^{ij} T_{-\alpha})/\tilde{N}\). Therefore, using the \(\tilde{N}\)-dimensional columns \(\xi^i\) and the \(\tilde{N}\)-dimensional rows \(\eta^j\) we have
\[
S^{ij}_\alpha S^{-ij}_{\alpha} = \frac{\text{tr}(\eta^j T_{-\alpha} \xi^i)}{\tilde{N}^2} - \frac{\text{tr}(\eta^j T_{-\alpha} \xi^i \eta^j T_{\alpha} \xi^j)}{\tilde{N}^2} - \frac{\text{tr}(\tilde{B} T_{\alpha} \tilde{B} T_{-\alpha})}{\tilde{N}^2}.
\]  
(4.20)

Plugging this expression into (4.16) we get
\[
H^{\text{tops}} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha \neq 0} \eta^i B_{\alpha} B_{-\alpha} E_2(\omega_\alpha) - \frac{1}{2} \sum_{i \neq j}^{N} \sum_{\alpha} \frac{\text{tr}(\tilde{B} T_{\alpha} \tilde{B} T_{-\alpha})}{\tilde{N}^2} E_2(\omega_\alpha - \frac{q_{ij}}{\tilde{N}}).
\]  
(4.21)

The Hamiltonians of this type were originally found in [43] from the study of matrix models. In [51] the model (4.21) was called the interacting elliptic tops, since the second term is the sum of Hamiltonians for \(\text{SL}_{\tilde{N}}\) elliptic integrable tops, and the last term describes their interaction.

Let us write the last term in (4.21) more explicitly. For this purpose plug \(\tilde{B} = \sum_{\gamma} T_{\gamma} \tilde{B}_\gamma\) and \(\tilde{B} = \sum_{\mu} \tilde{T}_{\mu} \tilde{B}_\mu\) into (4.21):
\[
\sum_{i \neq j}^{N} \sum_{\alpha} \text{tr}(\tilde{B} T_{\alpha} \tilde{B} T_{-\alpha}) E_2(\omega_\alpha - \frac{q_{ij}}{\tilde{N}}) = \sum_{i \neq j}^{N} \sum_{\alpha,\mu,\gamma} \kappa_{\alpha,\mu}^2 T_{\gamma} B_{\gamma} B_{\mu} E_2(\omega_\alpha - \frac{q_{ij}}{\tilde{N}}),
\]  
(4.22)

\[\sum_{i \neq j}^{N} \sum_{\alpha,\mu} \kappa_{\alpha,\mu}^2 \tilde{T}_{\mu} \tilde{B}_\mu E_2(\omega_\alpha - \frac{q_{ij}}{\tilde{N}}) = \tilde{N} \sum_{i \neq j}^{N} \sum_{\alpha,\mu} \kappa_{\alpha,\mu}^2 \tilde{T}_{\mu} \tilde{B}_\mu E_2(\omega_\alpha + \frac{q_{ij}}{\tilde{N}}),
\]

where we used \(T_{\alpha} T_{\mu} T_{-\alpha} = \kappa_{\alpha,\mu}^2 T_{\mu}\) coming from (A.3), and changed the indices \(i, j\) in the last line. Finally, we sum up over the index \(\alpha\) by applying the Fourier transformation formulae (A.24), (A.25). This yields
\[
\frac{1}{2} \sum_{i \neq j}^{N} \sum_{\alpha} \text{tr}(\tilde{B} T_{\alpha} \tilde{B} T_{-\alpha}) E_2(\omega_\alpha - \frac{q_{ij}}{\tilde{N}}) = -\tilde{N}^3 \mathcal{F}^0_{\text{class}},
\]  
(4.23)

where \(\mathcal{F}^0_{\text{class}}\) was defined in (4.10). The answer for the Hamiltonian (4.21) or (4.16) is as follows
\[
H^{\text{tops}} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha \neq 0} i B_{\alpha} B_{-\alpha} E_2(\omega_\alpha) + \tilde{N} \mathcal{F}^0_{\text{class}}.
\]  
(4.24)
Hence, we showed that the classical analogue of the anisotropic elliptic spin exchange operator (see (3.15), (3.22), (4.8)) arises in the model (4.11)-(4.16) likewise the classical version of the isotropic spin exchange operator (4.7) comes from the spin Calogero model (1.3), (1.5).

The proof of Proposition 1.2 comes from quantization of (4.24). First, notice that the second term turns into a constant term proportional to identity matrix (in Mat$^{\otimes N}_{\bar{N}}$) when quantized in the fundamental representation since $\hat{T}_\alpha \hat{T}_{-\alpha} = 1_{\bar{N}}$. It could be added to the original definition of $F^0$ (1.14) as the terms corresponding to $m = k$ under the sum. The potential of the Calogero-Moser model comes from the scalar part (the first term in (4.9)) of the $F^0$ term (4.8). The difference between $E_2$ and $\wp$-functions (A.8) is also included into the constant term proportional to identity matrix. In this way come to the statement (1.25). Similarly, in classical mechanics the Calogero-Moser potential comes from the first term of (4.10) since $B_0$ are equal to the same constant (4.15) for all $i$. ■

5 Conclusion

We considered $R$-matrix-valued Lax pairs for $N$-body Calogero-Moser models. The one for $A_{N-1}$ root system was previously known [32]. We proposed their extensions to other root systems. Namely, we studied generalizations of the D'Hoker-Phong Lax pairs [15] for the classical roots systems in the untwisted case. These Lax pairs are block-matrices of $2N \times 2N$ or $(2N+1) \times (2N+1)$ size, and each block is of the size $\bar{N}_r \times \bar{N}_r$, where $r$ is the number of quantum spaces (spin sites). Two possibilities were considered. The first one is to keep all $2N$ (or $2N+1$) quantum spaces in $R$-matrices. This leads to the Lax pairs for $C_N$ and $B_{CN}$ cases. The second possibility is to leave only half ($N$ or $N+1$) quantum spaces. It results in constructing $B_N$ and $D_N$ models with $GL_2$ ($\bar{N} = 2$) Baxter’s $R$-matrix. The summary of admissible values of the coupling constants and the number of quantum spaces in $R$-matrices is presented in the table below (horizontally are the numbers of quantum spaces).

|       | $N$          | $N+1$       | $2N$     | $2N+1$    |
|-------|--------------|-------------|----------|-----------|
| SO(2N)| $g = 0$, $\mu = 0$ | $\bar{N} = 2$ |          |           |
| SO(2N+1) | $g = \pm \sqrt{2} \nu$, $\mu = 0$ | $\bar{N} = 2$ |          |           |
| Sp(2N) | $g = 0$, $\mu = \nu$ | $\bar{N} = \text{any}$ |          |           |
| BC(N)  | $g = \pm \nu$, $\mu = \nu$ | $\bar{N} = \text{any}$ |          |           |

Number of spin quantum spaces and values of coupling constants.

Recall that the ordinary Lax pairs (2.1)-(2.3) were defined for the following values of the coupling constants:

- $SO_{2N}$: $\mu = 0$, $g = 0$;
- $SO_{2N+1}$: $\mu = 0$, $g^2 = 2\nu^2$;
In this respect our results are as follows: the $R$-matrix-valued ansatz generalizing (2.1)-(2.3) works with additional constraints. For SO cases the additional condition is $\tilde{N} = 2$, while for $C_N$ and $BC_N$ cases there is no restriction on $\tilde{N}$ but the constants should satisfy $\mu = \nu$ together with $g = 0$ or $g = \pm \nu$ for $C_N$ or $BC_N$ root systems respectively.

Then we proceed to the quantum Lax pairs. A short summary is that the classical $R$-matrix-valued Lax pairs are generalized to quantum Lax pairs only for SO cases from the above table.

The quantum Lax pairs are naturally related to the spin Calogero-Moser models. The corresponding spin exchange operators $\mathcal{F}^0$ appear as a scalar parts of the $R$-matrix-valued $M$-matrices. On the other hand the same operators can be derived from KZ or KZB equations. We demonstrate these relations for $\text{sl}_N$ $R$-matrix-valued Lax pair. The link between the operator-valued Lax pairs and KZ equations comes from the Matsuo-Cherednik duality. Its quasi-classical version provides the so-called quantum-classical duality between the quantum spin chains (Gaudin models) and the classical many-body systems of Ruijsenaars-Schneider (Calogero-Moser) type [22]. In this paper we deal with another example of quantum-classical relation. We treat the Lax equations for the classical Calogero-Moser model (1.12)-(1.14) with $R$-matrix-valued Lax pairs as half-quantum model (1.24), which quantum part is described by the spin exchange operator known previously as the "noncommutative spin interactions" [43]. The spin variables are quantized in the fundamental representation, while the particles degrees of freedom remain classical. We show that the classical counterpart of the elliptic anisotropic spin exchange operator comes from the Hitchin type system on $\text{SL}_{N\tilde{N}}$-bundle with nontrivial characteristic class over elliptic curve. See the Proposition 1.2.

It was shown in [44] that the spin exchange operator $\mathcal{F}^0$ (4.9) for $\tilde{N} = 2$ being reduced to the equilibrium position $q_j = j/N$ provides the Hamiltonian for anisotropic extension of the Inozemtsev elliptic long-range chain. In view of the relation of the $R$-matrix-valued Lax pairs and the Hitchin systems on $\text{SL}(N\tilde{N})$-bundles we expect that these type long-range integrable spin chains admit Lax representations of size $N\tilde{N} \times N\tilde{N}$ at both - classical and quantum levels. They are obtained from the one for interacting tops (1.12) by the substitution $p_j = 0$, $q_j = j/N$. Such Lax pair allows to calculate the higher Hamiltonians. These questions will be discussed in our next publication [21].

6 Appendix

In addition to the standard basis in $\text{Mat}_{\tilde{N}}$ we use the one [3]

$$T_a = T_{a_1 a_2} = \exp\left(\frac{\pi i}{\tilde{N}} a_1 a_2\right) Q^{a_1} \Lambda^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_{\tilde{N}} \times \mathbb{Z}_{\tilde{N}}$$  \hspace{1cm} (A.1)

constructed by means of the finite dimensional representation of Heisenberg group

$$Q_{kl} = \delta_{kl} \exp\left(\frac{2\pi i k}{\tilde{N}}\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \text{mod} \tilde{N}}, \quad Q^{\tilde{N}} = \Lambda^{\tilde{N}} = 1_{\tilde{N} \times \tilde{N}}.$$  \hspace{1cm} (A.2)
The following relations hold
\[ T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = \exp \left( \frac{\pi i}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1) \right), \quad (A.3) \]

\[ \text{tr}(T_\alpha T_\beta) = \tilde{N} \delta_{\alpha,-\beta}, \quad (A.4) \]
where \( \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \). The permutation operator takes the form
\[ P_{12} = \sum_{i,j=1} \tilde{E}_{ij} \otimes \tilde{E}_{ji} = \frac{1}{N} \sum_{\alpha \in \mathbb{Z} \times \mathbb{Z}} T_\alpha \otimes T_{-\alpha}, \quad (A.5) \]
where \( \tilde{E}_{ij} \) is the standard basis in Mat\( \tilde{N} \).

The Kronecker function is defined in the rational, trigonometric (hyperbolic) and elliptic case as follows:
\[ \phi(\eta, z) = \begin{cases} 
\frac{1}{\eta} + \frac{1}{z} & \text{rational case}, \\
\coth(\eta) + \coth(z) & \text{trigonometric case}, \\
\frac{\vartheta'(0)\vartheta(y+z)}{\vartheta(\eta)\vartheta(z)} & \text{elliptic case}.
\end{cases} \quad (A.6) \]
In the latter case the theta-function is the odd one
\[ \vartheta(z) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (z + \frac{1}{2})(k + \frac{1}{2}) \right). \quad (A.7) \]
Similarly, the first Eisenstein (odd) function and the Weierstrass (even) \( \wp \)-function:
\[ E_1(z) = \begin{cases} 
\frac{1}{z}, \\
\coth(z), \\
\vartheta'(z)/\vartheta(z),
\end{cases} \quad \varphi(z) = \begin{cases} 
\frac{1}{z^2}, \\
\frac{1}{\sinh^2(z)}, \\
-\partial_z E_1(z) + \frac{1}{3} \varphi'''(0).
\end{cases} \quad (A.8) \]
The derivative
\[ E_2(z) = -\partial_z E_1(z) \quad (A.9) \]
is the second Eisenstein function. The derivative of the Kronecker function:
\[ f(z, q) \equiv \partial_q \phi(z, q) = \phi(z, q)(E_1(z + q) - E_1(q)). \quad (A.10) \]
Due to the following behavior of \( \phi(z, q) \) near \( z = 0 \)
\[ \phi(z, q) = z^{-1} + E_1(q) + z (E_1^2(q) - \varphi(q))/2 + O(z^2). \quad (A.11) \]
we also have
\[ f(0, q) = -E_2(q). \quad (A.12) \]
The Fay trisecant identity:
\[ \phi(z, q)\phi(w, u) = \phi(z - w, q)\phi(w, q + u) + \phi(w - z, u)\phi(z, q + u). \quad (A.13) \]
For the Lax equations the following degenerations of (A.13) are needed
\[ \phi(z, x)f(z, y) - \phi(z, y)f(z, x) = \phi(z, x + y)(\varphi(x) - \varphi(y)), \quad (A.14) \]
\[ \phi(\eta, z) \phi(\eta, -z) = \varphi(\eta) - \varphi(z) = E_2(\eta) - E_2(z). \]  
\[ \text{Also} \]
\[ \phi(z, q) \phi(w, q) = \phi(z + w, q)(E_1(z) + E_1(w) + E_1(q) - E_1(z + w + q)) = \]
\[ = \phi(z + w, q)(E_1(z) + E_1(w)) - f(z + w, q). \]  

The set of \( \tilde{N}^2 \) functions
\[ \varphi_a(z) = \exp(2\pi i \frac{a_1 + a_2 \tau}{\tilde{N}}) \phi(z, \eta + \frac{a_1 \omega_1 + a_2 \omega_2}{\tilde{N}}), \quad a = (a_1, a_2) \in \mathbb{Z}_{\tilde{N}} \times \mathbb{Z}_{\tilde{N}} \]
is used in the definition of the Baxter-Belavin’s \([4, 5]\) elliptic R-matrix
\[ R_{12}^a(z) = \sum_\alpha T_\alpha \otimes T_{-\alpha} \varphi_\alpha(z, \omega_\alpha + \eta). \]  

The classical limit (behavior near \( \eta = 0 \))
\[ R_{12}^a(z) = \frac{1}{\eta} \otimes 1 + r_{12}(z) + \eta m_{12}(z) + O(\eta^2) \]
is similar to \([A.11]\). The classical r-matrix
\[ r_{12}(z) = 1 \otimes 1 E_1(z) + \sum_{\alpha \neq 0} T_\alpha \otimes T_{-\alpha} \varphi_\alpha(z, \omega_\alpha) \]
is skew-symmetric due to \([1.20]\) or \([1.19]\). From \([A.19]\) we conclude that
\[ F_{12}^0(q) = \partial_q R_{12}^0(q)|_{\eta=0} = \partial_q r_{12}(q) = F_{21}^0(-q). \]  

The finite Fourier transformation for the set of functions \([A.17]\) is as follows (see e.g. \([53]\)):
\[ \frac{1}{\tilde{N}} \sum_\alpha \kappa_{\alpha, \gamma} \varphi_\alpha(\tilde{N} \eta, \omega_\alpha + \frac{z}{\tilde{N}}) = \varphi_\gamma(z, \omega_\gamma + \eta), \quad \forall \gamma. \]  

It is generated by the arguments symmetry (similarly to \( \phi(z, q) = \phi(q, z) \))
\[ R_{12}^z(q) P_{12} = R_{12}^{z/\tilde{N}}(\tilde{N} z). \]  

In particular, \([A.22]\) leads to
\[ \sum_\alpha E_2(\omega_\alpha + \eta) = \tilde{N}^2 E_2(\tilde{N} \eta) \quad \text{or} \quad \sum_\alpha \varphi(\omega_\alpha + \eta) = \tilde{N}^2 \varphi(\tilde{N} \eta) \]
and for \( \gamma \neq 0 \)
\[ \sum_\alpha \kappa_{\alpha, \gamma}^2 E_2(\omega_\alpha + \eta) = -\tilde{N}^2 \varphi_\gamma(\tilde{N} \eta, \omega_\gamma)(E_1(\tilde{N} \eta + \omega_\gamma) - E_1(\tilde{N} \eta) + 2\pi i \partial_\tau \omega_\gamma). \]  

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