Deformations of functions and F-manifolds

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Abstract

We study deformations of functions on isolated singularities. Give a unified proof of the equality of Milnor and Tjurina numbers for functions on isolated complete intersections singularities and space curves. We define the structure of $F$-manifold on the base space of the miniversal deformation in both of the above cases. As a corollary, we proved a conjecture of V. Goryunov stating that the critical values of the miniversal unfolding of a function on a space curve are generically local coordinates on the base space of the deformation.

1 Introduction

The theory of Frobenius manifolds plays a central role in Mirror Symmetry, after the construction by Givental and Barannikov ([2]) of an isomorphism between the small quantum cohomology of $\mathbb{C}^n$ and the base space of the miniversal deformation of the linear function $f = x_1 + \cdots + x_{n+1}$ on the divisor $D = \{x_1 \cdots x_{n+1} = 1\}$. There are now a number of conjectures stating similar isomorphisms between quantum cohomology rings of algebraic varieties and unfoldings of functions on affine varieties. In this article we propose a Singularity Theory framework in which at least one of the ingredients making up the definition of Frobenius manifolds, namely the multiplication, can be naturally defined. This structure is known as $F$-manifold ([10], [11]).

A seemingly inescapable feature of this construction is that the multiplication is not defined on the whole tangent sheaf of the base space but only on a certain subsheaf, that of logarithmic vector fields to the discriminant. Contrary to those Frobenius manifolds constructed from unfoldings of isolated hypersurface singularities, our construction does contain some promising candidates to mirrors of algebraic varieties.

The main result of this article can be stated as follows:

**Theorem 1.1.** Let $f : (X,x) \to (\mathbb{C},0)$ be a function-germ with an isolated singularity on a isolated complete intersection or a space curve. Then the sheaf $\Theta(-\log \Delta)$ of logarithmic vector fields of the discriminant of its miniversal deformation is in a natural manner an (logarithmic) $F$-manifold. Moreover, each stratum of the logarithmic stratification of the base space inherits this structure.

The content of the article is as follows. First we provide a construction of the miniversal deformation of a function on a singular variety. We define a morphism closely related to the Kodaira-Spencer map that will be used to define the multiplication. Secondly we state a condition that ensures the equality of the dimension of the miniversal base space (Tjurina number) and the number of critical points of an unfolding of $f$ in the smooth fiber of the deformation. We then show that the condition holds for functions on isolated complete intersection singularities and (reduced) space curves. This provides a unified treatment to the $\mu = \tau$-type results of V. Goryunov ([7]) in the case of functions on isolated complete intersections singularities and of D. van Straten and the second author in the case of functions on space curves ([15]). Our methods are closer to those of [15]. To finish, we prove that the multiplication satisfies an integrability condition, making it into a logarithmic $F$-manifold.

Before going into the technical details we would like to work out a relatively simple example in which a full Frobenius structure can be constructed, namely that of function $f = x^p + y^q$ on the ordinary double point $X : xy = 0 \hookrightarrow \mathbb{C}^2$. This case is closely related to the construction of Frobenius manifolds on Hurwitz spaces by B. Dubrovin ([6], [17]), although as we are also collapsing the curve a new structure on the discriminant is made apparent. The aim of this example is firstly to guide

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the reader through the rest of the article and secondly to show how indeed our construction contains some interesting examples in Mirror Symmetry. It appears to be known among specialists that the resulting Frobenius manifold is the mirror of the orbifold $\mathbb{CP}(p,q)$.

### Functions on the double point

Let us consider a function germ $f = x^p + y^q$ on the $A_1$-singularity $X$: $xy = 0$. The result of the calculation that we are going to carry out can be resumed in the following theorem. We remark that certain aspects of the proof, particularly the multiplication, will only be evident after applying the results given in the main body of this article. The divisor $\Delta$ denotes the discriminant of the miniversal deformation of $f$.

**Theorem 1.2.** The sheaf $\Theta(-\log \Delta)$ is naturally endowed with a multiplication $\ast$ and a flat bilinear pairing $\langle \cdot, \cdot \rangle$ satisfying

$$\langle u \ast v, w \rangle = \langle u, v \ast w \rangle$$

for any $u, v, w \in \Theta(-\log \Delta)$.

There exists a conformal Euler vector field $E$, that is:

$$\text{Lie}_E(\ast) = \ast \text{ and } \text{Lie}_E(\langle \cdot, \cdot \rangle) = \langle \cdot, \cdot \rangle$$

**Proof.** Let us begin by constructing the multiplication. The miniversal deformation of $f$ is given by the function $F = c + \sum_{i=1}^{p-1} a_i x^i + x^p + \sum_{j=1}^{q-1} b_j y^j + y^q$ on the fibration $\pi(x, y, a, b, c) = (xy, a, b, c)$, where $a = (a_p, \ldots, a_1)$ and $b = (b_q, \ldots, b_1)$ (see Cor. 2.3 and the paragraph below). Let $\Delta : \epsilon = 0$ be the (smooth) discriminant of $\pi$. The multiplication is defined by the following lifting process: lift a vector field $u \in \Theta(-\log \Delta)$ to $\tilde{u}$ such that $tr(\tilde{u}) = u \circ \pi$. Differentiating $F$ with respect to $\tilde{u}$ we obtain an element in the ring of germs $O_{X,0} = \mathbb{C}[x, y, a, b, c]$. We denote by $t' F(u)$ its class in the quotient $O_{X,0}/(H)$, where $H$ is the Jacobian determinant

$$\frac{\partial(F, \pi_1 = xy)}{\partial(x, y)} = \sum_{i=1}^{p-1} i a_i x^i + px^p - \sum_{j=1}^{q-1} j b_j y^j - qy^q$$

It will clear from later constructions (although it can be checked directly) that the map $t' F$ so constructed is an isomorphism of free modules over the base of rank $p + q$. We use it to pull back the algebra structure on $O_{X,0}/(H)$ so defining a multiplication $\ast$ in $\Theta(-\log \Delta)$.

To define the multiplication, we consider the relative dualising form $\alpha = dx \wedge dy / \partial \pi_1$ and use it to identify $O_{X,0}$ with $\omega_{X/B,0}$. Hence we have $dF = H \alpha$ and consider the Grothendiek residue pairing on $\omega_{X/B,0}/O_{X,0}(dF)$. We use $t' F$ to define a multiplicatively invariant non-degenerate bilinear pairing on $\Theta(-\log \Delta)$. For $u, v \in \Theta(-\log \Delta)$, it is explicitly given by

$$\langle u, v \rangle = \int_{\partial X_b} \frac{t' F(u) t' F(v)}{H} \alpha$$

being $\partial X_b$ the boundary of an appropriate representative of the fiber $\pi^{-1}(b)$. For $b \in B \setminus \Delta$, the fiber $X_b$ is a smooth rational curve with two points deleted, say $\infty_1$ and $\infty_2$, corresponding to $x = 0$ and $y = 0$. Hence the pairing can be expressed as

$$\langle u, v \rangle = -\operatorname{Res}_{\infty_1} \frac{t' F(u) t' F(v)}{H} \alpha + \operatorname{Res}_{\infty_2} \frac{t' F(u) t' F(v)}{H} \alpha$$

(1)

If we take the free basis of $\Theta(-\log \Delta)$ given by $e_\frac{\partial}{\partial \pi_1}$ and the rest of coordinate vector fields, the decomposition (1) allows us to express the matrix of $\langle \cdot, \cdot \rangle$ as a sum, each summand corresponding to the residues at each point. A direct calculation, necessary for what follows, shows that the matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & 4^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_{\infty_1} & 0 \\ 4^{-1} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 4^{-1} \\ 0 & M_{\infty_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4^{-1} & 0 & 0 & 0 \end{pmatrix}$$

(2)

where $M_{\infty_1} = \begin{pmatrix} 2b_2 & 3b_3 & 4b_4 & \ldots & (q-1)b_{q-1} & q^{-1} \\ 3b_3 & 4b_4 & \ldots & \ldots & \ldots & \ldots \\ 4b_4 & \ldots & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$ and analogously for $M_{\infty_2}$.

To show that the pairing is indeed flat, we compute flat coordinates. Let $b = (\epsilon_0, a_0, b_0) \in B \setminus \Delta$. As $F$ has a pole of order $p$ at $\infty_2$, we can find a local coordinate $u$ at $\infty_2$ such that $F = u^{-p}$. On
the other hand, the function $xu$ is holomorphic and not vanishing at $\infty_2$. Fixing a branch of log we can expand it as a power series:

$$\log xu = t_0 + t_1u + \cdots + t_{p-1}u^{p-1} + O(u^p)$$

Arguing as above, we find a coordinate $v$ such that $F = v^{-8}$ around $\infty_2$ and a series

$$\log yv = s_0 + s_1v + \cdots + s_{q-1}v^{q-1} + O(v^q)$$

Write $t = (t_1, \ldots, t_{p-1})$ and $s = (s_1, \ldots, s_{q-1})$. The interested reader will check, by series expansion of $x = u^{-1}\exp(\sum_{i \geq 0} t_i u^i)$ and analogously for $y$, the following claim:

*The functions $(t', s') = (\log \epsilon, t, s, c)$ form a coordinate system. The functions $t$, resp. $s$, only depend on $a$, resp. $b$. We can now show that $(\cdot)$ has a constant matrix in these coordinates. Let us take for example $\frac{\partial}{\partial t_i}$. We have

$$\frac{1}{x} \frac{\partial x}{\partial t_i} = u^i, \quad \frac{1}{y} \frac{\partial y}{\partial t_j} = -u^j \quad \frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial t} = \left( x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \right) u^i = Hu^i$$

(3)

As the functions $t$ only depend on $a$, according to (2) we only need to look at the residues at $\infty_2$. Hence

$$\frac{\partial}{\partial t^i} = \frac{\partial}{\partial t^j} \frac{(Hu^i)(Hu^j)}{H} = \frac{\partial}{\partial t^j} u^{i+j} H \alpha = \frac{\partial}{\partial t^j} u^{i+j} H \alpha$$

(4)

A similar calculation, together with the orthogonality relations between $a$ and $b$, and hence between $t$ and $s$, deduced from (2) proves that $(\cdot)$ is flat.

To finish, we prove the last claim. The Euler vector field corresponds to the class of $F$ in $O_{X,x}$. It is given by

$$E = \left( \frac{1}{p} + \frac{1}{q} \right) \epsilon \frac{\partial}{\partial c} + \sum_{i=1}^{p-1} \frac{p-i}{p} \alpha_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{q-1} \frac{q-i}{q} \beta_i \frac{\partial}{\partial b_i} + \epsilon \frac{\partial}{\partial c}$$

(5)

Giving weights $1/p + 1/q$ to the variable $\epsilon$, $p - i/p$ to $a_i$, $q - i/q$ to $b_i$ and 1 to $c$, we have that a polynomial $h(\epsilon, a, b, c)$ is quasi-homogeneous of degree $d$ if and only if $\text{Lie}_E(h) = d \cdot h$. From (2) we see that the entry in the position $ij$ of $\text{M}_{\infty_2}^{-1}$ (resp. $\text{M}_{\infty_2}^{-1}$) is (if not constant) quasi-homogeneous of degree $(i + j - p)/p$ (resp. $(i + j - q)/q$). This proves the claim.

2 Versal deformations of functions on isolated singularities

Given a reduced analytic variety $(X,x)$ and a germ $f \in \mathfrak{m}_{X,x}$, we will say that $f$ has an isolated singularity if there exists a representative $f: U \to S$ onto the complex line $S$ such that $U \setminus \{x\}$ is smooth and $f$ is submersive at any point of $U \setminus \{x\}$. The deformation problem with which we will be concerned is referred to as deformations of $X$ over $S$, that is, we will consider diagrams

$$\begin{array}{ccc}
(X,x) & \xrightarrow{i} & (X,x) \\
\downarrow & & \downarrow \\
(S,0) & \xleftarrow{f} & (B,0)
\end{array}$$

The notions of induced, pull-back or isomorphism of diagrams are defined in the customary fashion through maps on the base spaces, keeping the complex line $(S,0)$ fixed. This deformation theory is sometimes denoted by $\text{Def}(X/S)$ and from a purely algebraic point of view, it corresponds to the study of the deformations of $O_{X,x}$ as $O_{S,0}$ algebra.

As in any deformation theory, we have the powerful theory of the cotangent cohomology modules at our disposal. Given any holomorphic map $h: A \to B$ between analytic spaces, and a $O_A$-module
Most of the usefulness of the cotangent modules, as for any cohomology theory, resides in the short exact sequences derived from short exact sequences of modules, but also from homomorphism of the base rings (a neat review of the properties we will use can be found in [3]). Back to our function from short exact sequences of modules, it is the Zariski-Jacobi long exact sequence associated to the ring homomorphism \( C \to O_{X,x} \to O_{X,x} \). It begins

\[
0 \to T^0_{/B,x} \to T^1_{X,x} \to T^0_{B,0}(O_{X,x}) \to T^1_{X,B,x} \to T^1_{X,x} \to \ldots
\]  

(6)

The composite of \( \Theta_{B,0} \to \Theta(\pi)_x \) with the connecting homomorphism of (6) is the Kodaira-Spencer map of the deformation. Its kernel is the submodule of liftable vector fields and we will denote it by \( \mathcal{L}_{\pi,0} \). In many interesting cases it coincides with those vector fields tangent to the discriminant of \( \pi \).

If we now consider an extension \( F \) of \( f \) to the total space \( (X, x) \), we can write \( \varphi = (\pi, F) \). The second sequence is also a Zariski-Jacobi sequence, this time corresponding to \( O_{B,0} \to O_{S \times B,0} \to O_{X,x} \):

\[
0 \to T^0_{X/S \times B,x} \to T^0_{X/B,x} \to T^0_{S \times B,0}(O_{X,x}) \to T^1_{X/S \times B,x} \to T^1_{X/B,x} \to \ldots
\]  

(7)

As before, we will be specially interested in a kernel, this time that of the map \( T^0_{X/S \times B,x} \to T^1_{X/B,x} \).

We will denote it by \( M_{\varphi,x} \). In fact, this module is readily described in more familiar terms using the exactness of (7). If \( tF : O_{X,x} \to \Theta(F)_x \) denotes the tangent map of \( F \), we have

\[
M_{\varphi,x} = \frac{\Theta(F)_x}{tF(\Theta(\pi/B)_x)}
\]  

(8)

After all these clarifications, we can state the main lemma of this section. The proof is so straightforward that it can be safely left to reader. It neatly separates the problem of finding a versal deformation of a function on a singular germ into firstly, versally deforming \((X, x)\) and secondly, versally unfolding \( f \).

**Lemma 2.1.** There is a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & \mathcal{L}_{\pi,0} & \to & \Theta_{B,0} & \to & T^1_{X/B,x} \\
\downarrow{tF} & & \downarrow{\ldots} & & \downarrow{i} & & \downarrow{1} \\
0 & \to & M_{\varphi,x} & \to & T^1_{X/S \times B,x} & \to & T^1_{X/B,x} & \to 0
\end{array}
\]

where \( tF \) is defined as follows: for \( u \in \mathcal{L}_{\pi,0} \), let \( \tilde{u} \in \Theta_{X,x} \) be a lift of \( u \). Then \( tF(v) \) is the class of \( tF(\tilde{u}) \) in \( M_{\varphi,x} \).

**Remark 2.2.** The vertical arrow in the middle is the Kodaira-Spencer map of the the map \( \varphi \) understood as a deformation of \( f : (X, x) \to (S, 0) \). It is the composite of \( \Theta_{B,0} \to \Theta(\varphi)_x \) with the connecting homomorphism of Zariski-Jacobi sequence derived from \( O_{S,0} \to O_{S \times B,0} \to O_{X,x} \).

We deduce the following criterion for versality:

**Corollary 2.3.** A deformation \( \varphi = (F, \pi) \) of \( f : (X, x) \to (S, 0) \) is versal if and only if \( \pi \) is versal as a deformation of \( (X, x) \) and \( tF \) is surjective.

Versal deformations can be now easily constructed from a versal deformations \( \pi \) of \((X, x)\). We take \( f_1, \ldots, f_l \) generators of the vector space \( \coker tF/\coker(tF + \varphi(f)) \) and consider the function \( F = f + a_1f_1 + \cdots + a_lf_l \), adding new parameters \( a_1, \ldots, a_l \). Requiring that \( \pi \) be miniversal and \( f_1, \ldots, f_l \) be a basis we will obtain a miniversal deformation. We will later see examples where this can be explicitly carried out.
3 Milnor and Tjurina numbers

An important feature of unfoldings of isolated singularities on smooth spaces is the conservation of the Milnor number. This invariant can be defined, among other ways, as the length of the Jacobian $\mathcal{O}_{C^n+1,0}((\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})).$ It is therefore both the number of non-degenerate critical points of a generic unfolding and the minimal number of parameters needed to versally unfold $f$. From this latter point of view, it could also be called the Tjurina number of the deformation problem defined by right equivalence of functions.

In our situation, even if the singularity $(X, x)$ is smoothable and we can speak of non-degenerate critical points of an unfolding, we might have a different number of those in non-isomorphic Milnor fibres. An example of this phenomenon is provided by the linear section $f = x_0 + x_1 + x_2 + x_3 + x_4$ on the germ $(X, 0)$ of the cone over the rational normal curve of degree 4 ([16]). On the other hand, we do have a well defined Tjurina number as the dimension of the vector space of first order infinitesimal deformations, namely, the length $\tau(X/S)$ of $T_{X,S,x}^1$. The next theorem tell us of conditions in which the Tjurina number indeed coincides with the number of non-degenerate critical points in every generic deformation.

**Proposition 3.1.** Let $\varphi = (\pi, \phi): (X, 0) \rightarrow (S \times B, 0)$ be a 1-parameter deformation of $f$. Assume that the following extendability condition is satisfied:

\[ (*) \quad \text{any vector field tangent to the fibres of } f \text{ can be extended to a vector field tangent to the fibres of } \varphi, \]

then both $T_{X/S \times B, x}^1$ and $M_{\varphi,x}$ are free $\mathcal{O}_{B,0}$-modules. Moreover, if $T_{X,x}^2 = 0$ and the generic fibre of $\pi$ is smooth, their ranks coincide.

**Proof.** Let $y$ be a parameter in $(B, 0)$. The exact sequence

\[ 0 \rightarrow \mathcal{O}_{X,x} \rightarrow_{\varphi} \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \rightarrow 0 \]

induces a long exact sequence:

\[ 0 \rightarrow \Theta_{X/S \times B, x} \rightarrow_{\varphi} \Theta_{X/S \times B, x} \rightarrow \Theta_{X/S, x} \rightarrow T_{X/S \times B, x}^1 \rightarrow T_{X/S, x}^1 \rightarrow \ldots \]

The condition $(\ast)$ implies that map $\Theta_{X/S \times B, x} \rightarrow \Theta_{X/S, x}$ is surjective and hence $T_{X/S \times B, x}^1 \rightarrow T_{X/S, x}^1$ injective. Therefore $T_{X/S \times B, x}^1$ and $M_{\varphi,x}$ are flat over $\mathbb{C}(y)$ and hence free.

For the second statement, we first show that the condition $T_{X,x}^2 = 0$ also implies $T_{X,x}^2 = 0$. Associated to $C \rightarrow \mathcal{O}_{S,0} \rightarrow \mathcal{O}_{X,x}$ we have a long exact sequence:

\[ \ldots \rightarrow \Theta_{X/S, x} \rightarrow \Theta_{X,S,x} \rightarrow \Theta_{X/S,x} \rightarrow T_{X/S,x}^1 \rightarrow T_{X/S,x}^1 \rightarrow \ldots \]

As $(S, 0)$ is smooth, $T_{X,S,x}^i = 0$ for $i \geq 1$, so that $T_{X/S,x}^i = T_{X,x}^i$ for $i \geq 2$. Finally if the generic fibre of $g$ is a smooth, then $T_{X/S \times B, x}^i$ is annihilated by a power of the maximal ideal $m_{B,0}$, and hence it is Artinian. The exact sequence then contains the short exact sequence:

\[ 0 \rightarrow T_{X/S \times B, x}^1 \rightarrow T_{X,S,x}^1 \rightarrow \mathcal{O}_{X,x} \rightarrow T_{X/S,x}^{t+1} \rightarrow \ldots \]

It follows that $\text{rk} T_{X/S \times B, x}^i = \dim_{\mathbb{C}} T_{X/S,x}^i$. To see that this is also the rank of $M_{\varphi,x}$ we write one more exact sequence:

\[ 0 \rightarrow M_{\varphi,x} \rightarrow T_{X/S \times B, x}^1 \rightarrow T_{X/B, x}^1 \rightarrow 0 \]

and notice that $T_{X/B, x}^1$ is supported at 0.

From now on, we restrict ourselves to situations in which all the conditions of the above theorem are satisfied, namely, functions on smoothable and unobstructed singularities for which the condition $(\ast)$ holds for any 1-parameter deformation. We will now show that this family of functions includes some interesting examples. First, note that it follows from the above proposition that not only $\tau(X/S)$ coincides with the number of Morse critical points in the generic deformation, but also that for the miniversal deformation of $f$, the map

\[ t' F: \mathcal{L}_{x,0} \rightarrow M_{\varphi,x} \]

extends to an isomorphism of free sheaves. In particular, the sheaf of liftable vector fields is necessarily free. We will now have a close look at two situations for which we can prove the extendability.
condition: the case described in the previous remark and that of isolated complete intersection singularities. But let us first introduce a piece of notation. The module $M_{\varphi,x}$ is not independent of the given deformation $\varphi$. Even its length is not a well-defined invariant of the function $f$. To avoid such a dependence, we consider the miniversal deformation of $(X,x)$ alone, say $\pi: (X,0) \to (B,0)$ and take any extension $F$ to the total space. We define

$$M_f = \frac{M_{\varphi,x}}{m_{B,0}M_{\varphi,x}}$$

Note that this module is well-defined as any two extensions of $f$ differ by an element of the maximal ideal $m_{B,0}$.

The reason to introduce this module is that if the conditions of proposition 3.1 are fulfilled, its dual is an isomorphism everywhere it is an isomorphism whenever the fibre is smooth. As the generic fibre is indeed smooth, the set $F$ and if $\cl$ On the other hand, the relative class map $\cl$ the given deformation $\varphi$, $\phi$, and take any extension $F$ such that we consider the miniversal deformation of $\varphi$. Therefore obtained by considering the miniversal deformation of the curve together with the unfolding $\varphi$ to the total space. We define

$$M_f = \frac{M_{\varphi,x}}{m_{B,0}M_{\varphi,x}}$$

Proposition 4.1. For a function $f$ on a space curve,

$$M_f = \frac{\Theta(f)_x}{tf(\omega_{X,x})}$$

Proof. A space curve is a Cohen-Macaulay variety of codimension 2 and as such is defined by the maximal minors $\Delta_i$ of a $(m \times m - 1)$-matrix $M$ with coefficients in $O_{C^3,x}$. There deformations are well understood ([19]), they are also defined by the maximal minors $\Delta_i$ of a perturbation $\tilde{M}$ of $M$.

An identical calculation to that of [15], but using the relative class map for the miniversal family instead of that of $(X,x)$, shows that its dual in $\Theta_{X,x}$ is generated by the vector fields

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial \Delta_i} & \frac{\partial}{\partial \Delta_j} & \frac{\partial}{\partial \Delta_k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{vmatrix}, \quad 1 \leq i < j \leq l$$

On the other hand, the relative class map $cl_{X/B,x}: \Omega_{X/B,x} \to \omega_{X/B,x}$ (or rather, a representative of it) is an isomorphism whenever the fibre is smooth. As the generic fibre is indeed smooth, the set where it fails to be bijective is of codimension at least 2. Hence its dual is an isomorphism everywhere and if $F$ is any extension of $f$ to $(X,x)$ we have

$$M_f = \Theta(f)_x/t\Omega_{X/B} + m_{B,0} \Theta(f)_x = \frac{\Theta(f)_x}{tf(\omega_{X,x})}$$

Example 4.2. We can use the above calculation to compute versal deformations of functions on space curves. For example, the union of the three coordinate axis in $(C^3,0)$ is defined by the $2 \times 2$-minors of $M = \begin{pmatrix} x & y \\ 0 & y \end{pmatrix}$. The miniversal deformation of a function $f = x^p + y^q + z^r$ is therefore obtained by considering the miniversal deformation of the curve together with the unfolding $F = x^p + \sum_{i=1}^{p-1} a_i x^{p-i} + y^q + \sum_{i=1}^{q-1} b_i y^{q-i} + z^r + \sum_{i=1}^{r-1} c_i z^{r-i} + d$. In [16], where simple functions on curves are classified, this singularity is referred to as $C_{p,q,r}$. 6
We now go on to study the case of functions on complete intersections. Let $f : (X,x) \to (S,0)$ be a germ with an isolated singularity on a $n$-dimensional complete intersection. Let $g_1, \ldots, g_k$ be elements defining the ideal of $(X,x)$ in $(\mathbb{C}^{n+k},x)$.

A submodule of $\Theta_{X,x}$, whose members are clearly tangent to all the fibres of $f$ is generated by the maximal minors of the matrix

$$
\begin{pmatrix}
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_{n+k}} \\
\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_N} \\
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n+k}} \\
\cdots & \cdots & \cdots \\
\frac{\partial g_k}{\partial x_1} & \cdots & \frac{\partial g_k}{\partial x_{n+k}}
\end{pmatrix}
$$

(11)

Lemma 4.3. The vector fields in (11) generate $\Theta_{X/S,x}$.

Proof. Let $\varphi = (f, g_1, \ldots, g_k)$. The module $\Theta_{X/S,x}$ is the kernel of

$$\Theta_{\mathbb{C}^{n+k},x} \otimes O_{X,x} \overset{\varphi \otimes 1}{\longrightarrow} \Theta_{\mathbb{C}^{1+k},B} \otimes O_{X,x}$$

(12)

As $f$ has an isolated singularity and $(X,x)$ is a Cohen-Macaulay, the depth of the ideal in $O_{X,x}$ generated by the maximal minors of (12) is $n$, i.e., the greatest possible. It follows that the Eagon-Northcott complex is exact (9) and the kernel is generated by the above vector fields.

Corollary 4.4. For germ $f : (X,x) \to (S,0)$ with an isolated singularity on a complete intersection, $\tau(X/S)$ coincides with the number of non-degenerate critical points of a generic deformation. If $n$ denotes the dimension of $(X,x)$, then

$$M_f \simeq \frac{\omega_{X,x}}{df \wedge \Omega_{X,x}^{n-1}}$$

(13)

Proof. In view of lemma 4.3, it is clear that the extendability condition holds. But lemma 4.3 is also telling us which vector fields are tangent to all the fibers of a deformation of a complete intersection. Simply take $(X,x)$ to be the ambient space $(\mathbb{C}^{n+k},x)$ and change $(S,0)$ by $(\mathbb{C}^k,0)$. We see that $\Theta_{X/B,x}$ is generated by the maximal minors of (11) with the row involving deleted. The equality (13) is now evident by differentiating $f$ with respect to this set of generators of $\Theta_{X/B,x}$.

Remark 4.5. The equality between Tjurina and Milnor numbers for functions on complete intersections is proven with unrelated methods in 7.

Remark 4.6. Using 18 and a well-known result 20, we can interpret the rank of $M_f$, and hence $\tau(X/S)$, as the rank of certain vanishing homology, namely $H_n(X_b,Y_b)$ for Milnor fibers of $(X,x)$ and $f$.

5 Multiplication on the sheaf of liftable vector fields

Whenever the map $t'F$ of 10 extends to an isomorphism of sheaves, we can use it to define a multiplication on $L_e$ by pulling-back the algebra structure on $M_e$. If $(X,x)$ is also smoothable then this defines a multiplication on the tangent bundle of the complement $B \setminus \Delta$ of the discriminant of the fibration. We begin recalling the definition of $F$-manifold from 11 Ch. 1.

Definition 5.1. A complex manifold with an associative and commutative multiplication $\ast$ on the tangent bundle is called an $F$-manifold if:

1. (unity) there exists a global vector field $e$ such that $e \ast u = u$ for any $u \in \Theta_M$ and,
2. (integrability) $\text{Lie}_u(x) = u \ast \text{Lie}_u(x) + \text{Lie}_u(x) \ast v$ for any $u, v \in \Theta_M$.

An Euler vector field $E$ (of weight 1) for $M$ is defined by the condition

$$\text{Lie}_E(x) = \ast$$

The main consequence of this definition is the integrability of multiplicative subbundles of $TM$, namely, if in a neighbourhood $U$ of a point $p \in M$ we can decompose $TU$ as a sum of unitary subalgebras $A \oplus B$ such that $A \ast B$, then $A$ and $B$ are integrable.

By choosing good representatives in the sense of 14 for all the germs involved, we have:
Proposition 5.2. The map $t' F$ endows the sheaf of liftable vector fields $\mathcal{L}_\pi$ with the structure of commutative and associative $\mathcal{O}_B$-algebra * such that, for any $u, v \in \mathcal{L}_\pi$:

$$\operatorname{Lie}_{u * v}(*) = \operatorname{Lie}_u(*) * v + u * \operatorname{Lie}_v(*)$$  \hspace{1cm} (14)

The class of $F$ in $\pi_* M_\varphi$ corresponds to an Euler vector field of weight 1.

Proof. It is enough to show (14) off $\Delta$. Let $\mu = \operatorname{rk} \pi_* M_\varphi$. For a generic point $b \in B \setminus \Delta$, the function $F$ has $\mu$ quadratic singularities on the smooth fiber $\pi^{-1}(b)$. Hence $\pi_* M_\varphi$ decomposes into $\mu$ 1-dimensional unitary subalgebras. In a neighbourhood $U \subset S \setminus \Delta$ of such a point, the integrability condition is equivalent to the image $L$ of the map

$$\operatorname{supp} M_\varphi \ni x \mapsto d_x F \in T^*_{\pi(x)} B$$  \hspace{1cm} (15)

being a Lagrangian subvariety of $T^* B$ (see [11], Th. 3.2). If $\alpha$ denotes the canonical 1-form on $T^* B$ and $p: T^* B \to B$ the projection, it is easy to check that the diagram

$$\begin{array}{c}
\operatorname{supp} M_\varphi \\
p \circ \mathcal{O}_L
\end{array} \xrightarrow{\Theta_B} \begin{array}{c}
\operatorname{supp} M_\varphi \\
\Theta_B
\end{array}$$

is commutative. The homomorphism on the right hand side is given by evaluation, so that it can also be expressed as $\alpha(\bar{u})$ where $\bar{u}$ is a lift of $u \in \Theta_B$ to $\Theta_{T^* B}$. Hence $\alpha_L$ is the relative differential of $F$ when thought of as a map on $L$ via the identification (13). It follows that $\alpha_L$ is the exact and hence closed, so that $L$ is Lagrangian. The statement about the Euler vector is an easy calculation that we leave to the reader (see [11], Th. 3.3). $\square$

The above proposition establishes the structure of $F$-manifold at least off $\Delta$. In the case where $\mathcal{L}_\pi$ coincides with the sheaf of tangent vector fields to $\Delta$, denoted by $\Theta(-\log \Delta)$, we can in fact define the $F$-manifold structure on each of the stratum of the logarithmic stratification induced by $\Theta(-\log \Delta)$ (13). First we need a lemma:

Lemma 5.3. For any ideal sheaf $I \subset \mathcal{O}_B$, the kernel of the map

$$\mathcal{L}_\pi / I \mathcal{L}_\pi \longrightarrow \Theta_B / I \Theta_B$$

is identified by $t' F$ with an ideal of $\pi_* M_\varphi / I \pi_* M_\varphi$.

Proof. Choosing a Stein representative of $\pi$, we can interpret the diagram in lemma (24) as a morphism of free resolutions of $\pi_* T^1_{X / B}$ where the row at the bottom is actually a complex of $g_* \mathcal{O}_X, x^*$-modules. Hence

$$\ker (\mathcal{L}_\pi / I \mathcal{L}_\pi \longrightarrow \Theta_B / I \Theta_B) = \operatorname{Tor}_1^{\mathcal{O}_B}(\pi_* T^1_{X / B}, \mathcal{O}_B / I)$$

is mapped by $t'F$ to the kernel of

$$\pi_* M_\varphi / I \pi_* M_\varphi \longrightarrow \pi_* T^1_{X / S \times B} / I \pi_* T^1_{X / S \times B}$$

and therefore it is an ideal. $\square$

Theorem 5.4. If $\mathcal{L}_\pi = \Theta(-\log \Delta)$, then each stratum of the logarithmic stratification has the structure of $F$-manifold with an Euler vector field of weight 1.

Proof. Let $b \in B$ and let $S_b$ be the stratum in which $b$ lies. Let $V$ be a open neighbourhood of $b$ in which $S_b \cap V$ is an analytic subset of $V$ defined by the ideal $I_{S_b}$. The sheaf $\Theta_{S_b \cap V}$ can be identified with

$$\frac{\operatorname{im} (\mathcal{L}_\pi|_\pi \longrightarrow \Theta_B|_\pi)}{I_{S_b} \operatorname{im} (\mathcal{L}_\pi|_\pi \longrightarrow \Theta_B|_\pi)}$$

Let $K$ denote the sheaf $\operatorname{Tor}_1^{\mathcal{O}_B}(\pi_* T^1_{X / B}, \mathcal{O}_B / I_{S_b})$. The map $t'F$ descends to the above quotient and it yields an isomorphism of $\mathcal{O}_{S_b \cap V}$-modules

$$\Theta_{S_b \cap V} \xrightarrow{\simeq} \frac{\pi_* M_\varphi|_V}{I_{S_b} \pi_* M_\varphi|_V + t' F(K|_V)}$$

According to the previous lemma, the right hand side is a $\mathcal{O}_B$-algebra. The above isomorphism defines the multiplication on the tangent bundle of the stratum $S_b$. From proposition (12) it follows that it is an $F$-manifold with Euler vector field of weight 1 given by the class of $F$ in the corresponding algebra. $\square$
Remark 5.5. If the stratum $S_b$ is a massive $F$-manifold, that is, there exist coordinates $u_1, \ldots, u_l$ such that
\[ \frac{\partial}{\partial u_i} \star \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i} \]
for all $i, j$, then the critical values of $F$ are generically local coordinates on $S_b$. In particular, this always holds on the stratum $B - \Delta_g$. In the case of space curves, that the critical values of $F$ off the bifurcation diagram are local coordinates is shown for simple functions in [9] who also conjectured the analogous result for non-simple functions.

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