On the interplay between the loop-tree duality and helicity amplitudes

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ABSTRACT: The spinor-helicity formalism has proven to be very efficient in the calculation of scattering amplitudes in quantum field theory. In view of the developments to compute loop amplitudes based on the loop-tree duality (LTD) theorem, we exploit its features in illustrative one-loop processes. Since LTD is aimed at the cancellation at integrand level of ultraviolet (UV) and infrared (IR) singularities, by adding suitable UV counterterms and cancelling IR singularities through momentum mappings, or unsubtractions, with the real contributions. we explore their combination in a fully-automated code to render the expressions integrable in four space-time dimensions. In this paper, we focus our discussion on the local UV renormalisation. To this end, we consider processes that are IR finite and present a fully numerical implementation.
1 Introduction

In order to unveil the composition of matter and its interactions, it is necessary to analyse highly-precise experimental data obtained from colliders using accurate predictions. However, the established theoretical models, i.e. the Standard Model, involves dealing with very complicated mathematical equations, whose exact solutions are unknown in many physically relevant processes. Thus, most of the computations performed nowadays rely on the perturbative approach, which naturally leads to the appearance of Feynman amplitudes and integrals.

With the purpose of achieving a higher accuracy in the predictions, it is mandatory to explore higher perturbative orders and compute multi-loop amplitudes with high multiplicity. To tackle these calculations, several methods have been developed in the last years. On one hand, there has been an enormous progress in the algebraic handling of scattering amplitudes in gauge theories due to the use of alternative kinematic variables as the ones provided by the spinor-helicity formalism [1]. Also, there was an important improvement due to the study of the mathematical properties of the scattering amplitudes. For instance in the colour sector [2–4]. These techniques lead to a much more efficient treatment of the amplitudes, exploiting several symmetries to simplify the underlying expressions. On the other hand, there were also great advances in the calculation of Feynman integrals, both analytically and numerically. In particular, pointing towards a more efficient numerical implementation, we have been developing a novel strategy for computing Feynman integrals based on the loop-tree duality (LTD) theorem [5–8]. This theorem allows to decompose any loop amplitude (or integral) as the sum of tree-level like objects integrated over a proper phase-space region. From the physical point of view, loop particles are converted into real-radiation ones. From the mathematical side,
the integration domain is transformed from a Minkowski to an Euclidean space. In fact, the numerical evaluation of multi-loop integrals through LTD is, w.r.t. the approaches that pass by Feynman parametrisation or Mellin-Barnes transformations, more efficient due to the reduced number of integrations to be performed. Indeed, our methods motivated alternative studies of LTD [9, 10].

In this paper, we study how the LTD formalism can also be applied to the calculation of loop helicity amplitudes. This is done because LTD works at the level of denominators and the structure of the numerator does not generate any additional difficulty. To this end, we start considering illustrative examples in which the simplicity of the latter is displayed. In order to generate helicity amplitudes, we make use of the spinor-helicity formalism, where we write definite states for the external wave functions. On top of it, to have a very compact expression for the integrand and, hence, amplitude, we also use the momentum twistors’ variables [11]. These variables, due to their properties, allow us to express any kinematic process in terms of the minimal set of variables. In other words, for a process with \( n \) external massless particles we have \( 3n - 10 \) invariants to deal with [12]. Likewise, the extension to massive particles is straightforward.

Besides the clearness LTD offers us to compute any loop amplitude, in this paper, we also want to stress on the local UV renormalisation. Hence, for the sake of the simplicity, we consider processes that are IR safe but might have a local UV behaviour. For the latter, it is known that UV finite integrals might be locally divergent in the high-energy region. Therefore, a careful treatment has to be performed. For instance, at one-loop level, we refer the reader to Refs. [13–15] (and references therein). Whereas at two-loop level, and in general, we have extended this method in Ref. [16]. We remark that the idea of performing a local UV renormalisation is to obtain well-defined integrands in four space-time dimensions that allow a straightforward numerical evaluation of the integrals.

The paper is organised as follows. In Sec. 2, we recall the basis of the LTD formalism, stressing on the formulae applied in this work. We briefly present a discussion regarding the treatment of amplitudes with multiple powers of the propagators. In Sec. 3, we provide a description of the generation of kinematical variables by using the spinor-helicity formalism. In Sec. 3.1, we focus on the parametrisation of the loop three-momentum to integrate the dual contributions when external momenta are complex. The introduction of local UV renormalisation counter-terms is reviewed in Sec. 4. The main part of this manuscript is presented in Sec. 5, where we show numerical results for some explicit examples at one-loop level. This allows us to demonstrate the feasibility of the LTD-based numerical strategy with realistic scattering processes, as well as its efficiency. Conclusions and future research directions are analysed in Sec. 6.

2 Loop-tree duality in a nutshell

The loop-tree duality (LTD) theorem [5, 6] is a useful tool to rewrite any loop integral in terms of tree-level-like objects defined in an Euclidean space. It relies on a suitable application of Cauchy’s residue theorem on the energy component of the loop momenta, which translates into cutting (i.e. putting on-shell) internal lines. For illustrative purposes, let us consider a generic one-loop scalar integral for an \( N \)-particle process, where the external momenta are labeled as \( p_i \) with \( i \in \{1, \ldots, N\} \). Then, we use the notation \( q_i = \ell + k_i \) and \( k_i = p_1 + \ldots + p_i \) to express the internal momenta; the momentum conservation condition reads \( k_N = 0 \).

In this way, the scalar integral is given by

\[
L^{(1)}(p_1, \ldots, p_N) = \int \prod_{i=1}^{N} G_F(q_i) = \int \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0},
\]

(2.1)

where \( G_F(q_i) \) stands for the customary Feynman propagator and

\[
\int = -4\pi^{d-4} \int \frac{d^d\ell}{(2\pi)^d},
\]

(2.2)
for the usual one-loop integration measure, with $\mu$ an arbitrary energy scale to restore the proper units after the extension to a $d$-dimensional space-time. The application of the LTD theorem leads to

$$L^{(1)}(p_1, \ldots, p_N) = -\sum_{i=1}^{N} I_i,$$

$$I_i = \int_{\ell} \delta(q_i) \prod_{j \neq i, j=1}^{N} \frac{1}{q_j^2 - m_j^2 - i0 \eta \cdot (q_j - q_i)},$$

with $m_j$ the mass associated to the internal line with momenta $q_i$, $\eta$ is an arbitrary future-like vector (i.e. $\eta^2 \geq 0$) and

$$\delta(q_i) = 2\pi i \delta(q_i^2 - m_i^2) \theta(q_i, 0),$$

forces the cut line to fulfil the physical (i.e. positive energy mode) on-shell condition. It is important to notice that the usual Feynman prescription is converted into the so-called dual prescription, which accounts for the information contained in the multiple cuts defined within the Feynman Tree theorem (FTT) [17]. The associated dual propagators are denoted as

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0 \eta \cdot k_{ji}},$$

where $q_j$ is the momenta flowing through the line, $q_i$ corresponds to the one that is set on-shell and $k_{ji} = q_j - q_i$.

As we can appreciate from Eq. (2.6), the dual prescription depends linearly on the momenta carried by the cut line and the propagating particle. Also, the loop measure is transformed into a phase-space one by inserting the factor $\delta(q_i)$, that forces the momenta $q_i$ to represent a physical on-shell state with positive energy.

### 2.1 Dealing with multiple poles

When dealing with multi-loop diagrams or local UV counter-terms, it is possible to have contributions with multiple powers of the propagators. Using integration-by-parts identities (IBPs) [18–20], they can be reduced to linear combinations of other integrals containing only single powers of the propagators, as done in [7]. However, this modifies the local behaviour of the integrands and might spoil the point-by-point cancellation of IR singularities present in the real-emission contribution. Thus, we will stick to the local approach and avoid using IBPs.

The extension of the LTD formulae to the multi-pole case is achieved by using the Cauchy’s residue theorem [7, 14]. Let us consider a generic one-loop integral with only a propagator raised to the power $m \geq 1$,

$$L^{(2)}_{m} (p_1, \ldots, p_N) = \int_{\ell} f(q_i, \{q_j\}; \{p_r\}) = \int_{\ell} G^{m}_{F}(q_i) \left( \prod_{j \neq i} G_{F}(q_j) \right) \mathcal{N}(q_i, q_j; \{p_r\}).$$

According to the LTD theorem, we need to extend the energy variable and choose an integration countour that encloses the point in the lower-part of the complex plane. By doing so, we get

$$L^{(2)}_{m} (p_1, \ldots, p_N) = 2\pi i \sum_{i} \int d^{d-1} \tilde{f} \text{Res}_{\text{Im}q_i, 0 < 0} \left\{ f(q_i, \{q_j\}; \{p_r\}); q_i^{(+)}; q_{i, 0}^{(+)m} \right\},$$

with

$$\text{Res} \left\{ f(q_i, \{q_j\}; \{p_r\}); q_i^{(+)}; q_{i, 0}^{(+)m} \right\} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dq_{i, 0}^{(+)m}} \left( \frac{\mathcal{N}(q_i, q_j; \{p_r\})}{q_{i, 0}^{(+)m}} \left( \prod_{j \neq i} G_{F}(q_j) \right) \right).$$
and \( q_{i,0}^{(+)} = \sqrt{\|\vec{q}_i\|^2 + m_i^2 - i0} \) the positive-energy solution of the corresponding on-shell condition. To explicitly compute the derivatives, it is useful to apply the identities

\[
G_F^{-1}(q_i) = (q_{i,0} - q_{i,0}^{(+)})(q_{i,0} + q_{i,0}^{(+)}), \quad q_i^{(+)} = (q_{i,0}^{(+)}, \vec{q}_i),
\]

where we introduce the on-shell vector associated to the cut line \( i \). A careful discussion about the computation of the residue is presented in Refs. [5, 7]. It is important to take care of the dual prescription for the contributions associated to the original amplitude, since it may contain thresholds in the low energy region. In that case, the propagators associated to uncut lines must be promoted to dual propagators. On the contrary, when applying the LTD formalism to the UV counter-terms, we can neglect the prescriptions and straightforwardly use the Cauchy’s formula for computing the residue.

### 3 Generation of the kinematics

In order to provide helicity amplitudes, we take advantage of the momentum twistor parametrisation proposed in Ref. [11], where the standard spinor products, \( ⟨\bullet\bullet⟩, [\bullet\bullet] \), are replaced by a minimal set of independent variables, \( z \). The number of variables in the latter depends on the kinematic process. In particular, any \( n \)-point massless amplitude can be expressed in terms of \( 3n - 10 \) independent variables. Hence, the extension to amplitudes with massive particles is straightforward. In Appendix A, we briefly recall the main features of these variables.

Within the LTD approach, the evaluation of integrals is performed in the momentum space instead of \( d \)-dimensional objects into a four-dimensional space by making use of the following properties\(^1\):

\[
q_{i,[d]} \cdot P_{j,[4]} = q_{i,[4]} \cdot P_{j,[4]}, \quad q_{i,[d]} \cdot ε_{j,[4]} = q_{i,[4]} \cdot ε_{j,[4]},
\]

where we contracted the loop momentum with external momenta or polarization vectors. Therefore, we only need to keep track of squared loop momenta, \( q_{i,[d]} \cdot q_{j,[d]} \). It turns out that due to the cuts performed within the LTD formalism, we can easily remove this dependence and work with objects in four dimensions. Let us also remark that, within this approach, we do not need to include extra-dimensional products, i.e. \( q_{i,[d-4]} \cdot q_{j,[d-4]} \).

Then, working in four space-time dimensions, we can parametrise the loop momenta in terms of a four-dimensional basis, i.e. \( E = \{ e_i \} \). Therefore, to reduce as much as possible the number of scalar products to be evaluated, we choose \( E = \{ p_1, p_2, ε_{12}, ε_{21} \} \). With this choice, the loop momenta is expressed as

\[
q_i^α = x_{i,1} p_1^α + x_{i,2} p_2^α + x_{i,3} ε_{12}^α + x_{i,4} ε_{21}^α,
\]

where \( p_1 \) and \( p_2 \) are the massless momenta built from the parametrisation obtained from the momentum twistors of an \( n \)-point kinematics and \( ε_{ij}^α = \frac{1}{2} (i|γ^α |j) \). We remark that for the elements of the basis, we explicitly work with the components of the four-vectors. Hence, with this decomposition, all the scalar products involving external momenta or polarization vectors contracted with the loop momenta can always

\(^1\)We closely follow the convention of Ref. [24].
be expressed in terms of scalar products among the elements of the basis and the loop momenta, i.e. \( q_i \cdot e_j \). The aim of this refinement is twofold. Firstly, to reduce the number of scalar products required for the computation. In second place, to cancel redundant expressions that appear at integrand level. This prevents some non-contributing terms that pop up in intermediate steps of the computation, before performing an explicit evaluation.

### 3.1 Parametrisation of the loop momentum

As discussed in Sec. 2, once LTD is applied to any loop integral or virtual amplitude, the integration over the loop energy component is removed and the remaining one is performed over an Euclidean space. Thus, the loop three-momentum needs to be properly parametrised to improve the computational efficiency. We remark that we are considering a complex-valued parametrisation of the external momenta. Explicitly, the second component of the three-momentum is purely imaginary; this is due to the method applied to build their representation starting from scalar invariants\(^2\). It is worth noticing, however, that the scalar products among themselves do not contain any phase (i.e. they are purely real), as expected in any physical kinematic configuration. Hence, to overcome any possible issue when using a real parametrisation of the loop three-momentum, we express it in cylindrical coordinates,

\[
q_i = (\xi_i \cos \phi_i, \rho_i, \xi_i \sin \phi_i),
\]

for the \( i \)-th cut. Then, the resulting integral is given by

\[
I_i = \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \xi_i \, d\xi_i \, d\phi_i \, d\rho_i \, I_i(\xi_i, \phi_i, \rho_i),
\]

where \( I_i \) is the integrand after plugging the explicit parametrisation of the loop three-momentum (3.3). We note that carefully integrating \( I_i \) over \( \rho_i \) brings a lot of cancellations, in particular, when considering kinematical configurations below threshold. This is because the imaginary part introduced by the prescriptions must cancel in these configurations, and the \( \rho_i \) variable captures all the imaginary contributions due to the explicit functional form of the parametrisation external momenta. Of course, we are excluding from this claim the presence of imaginary terms introduced by the numerators (for instance, originated by the polarisation vectors). Hence, to account for the simplifications that occur in the \( \rho_i \)-integration, we re-write Eq. (3.4) as

\[
I_i = \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \xi_i \, d\xi_i \, d\phi_i \, d\rho_i \left[ I_i(\xi_i, \phi_i, \rho_i) + I_i(\xi_i, \phi_i, -\rho_i) \right],
\]

which turns out to be equivalent to consider the real-part of the integrand in the previously mentioned conditions.

Furthermore, we notice that the \((\xi_i, \rho_i)\)-plane can be compactified by changing variables and using polar coordinates. Explicitly, we define

\[
(\xi_i, \rho_i) \rightarrow \frac{x_i}{1 - x_i} \left( \cos \theta_i, \sin \theta_i \right),
\]

with \( 0 \leq x_i < 1 \) and \( 0 \leq \theta_i < \pi/2 \). In the last part, we restricted the angular integration to the first quadrant because both \( \xi_i \) and \( \rho_i \) are positive.

### 4 Local UV renormalisation

Since we are aiming for a complete numerical implementation, it is necessary to build integrand-level counterterms, in order to cancel the local singular behaviour for very high energies of the amplitudes under

\(^2\)For more details about the construction of the momentum parametrisation, see Appendix A.
consideration. In the following, we recall how it is possible to generate these counter-terms very easily from
the original amplitudes [16, 21].

For a given loop momentum \( \ell_j \), we consider the integrand-level replacement

\[
S_{j,\text{UV}} : \{ \ell_j^2 \mid \ell_j \cdot k_i \} \rightarrow \{ \lambda^2 q_{j,\text{UV}}^2 + (1 - \lambda^2) \mu_{\text{UV}}^2 \mid \lambda q_{j,\text{UV}} \cdot k_i \}, \tag{4.1}
\]

where \( \mu_{\text{UV}} \) is an arbitrary scale that can be identified with the renormalisation scale, and \( q_{j,\text{UV}} = \ell_j + k_{j,\text{UV}} \), with \( k_{j,\text{UV}} \) arbitrary, that we will set to 0 for simplicity. By applying \( S_{j,\text{UV}} \) to an unintegrated and uncut one-loop amplitude \( A^{(1)} \) with loop momentum \( \ell_j \), and then expanding in \( \lambda \) around infinity up to logarithmic degree (this operation will be represented by the operator \( L_\lambda \) in the following), we directly obtain an integrand-level expression that cancels the local UV singularities \(^3\) exhibited by \( A^{(1)} \). It is important to note, though, that this counter-term may generate a finite part after integration, which must be fixed through a scheme fixing parameter \( d_{j,\text{UV}} \). Therefore, the counter-term reads,

\[
A^{(1)}_{j,\text{UV}} = L_\lambda \left( A^{(1)} \bigg|_{S_{j,\text{UV}}} \right) - d_{j,\text{UV}} \mu_{\text{UV}}^2 \int_{\ell_j} (G_F(q_{j,\text{UV}}))^3, \tag{4.2}
\]

where the integral multiplying \( d_{j,\text{UV}} \) integrates to the same finite quantity in both 4 and \( d \) dimensions. The quantity \( A^{(1)}_{j,\text{UV}} \) properly cancels the UV behaviour of \( A^{(1)} \) while giving the required finite part (which is 0 for instance in the \( \text{MS} \) scheme).

For a two-loop amplitude \( A^{(2)} \), the general local renormalisation procedure has been extended in Ref. [16]. In the two-loop case, it is necessary to consider three UV regimes involving the two internal momenta, \( \ell_1 \) and \( \ell_2 \). For instance, we can consider the regimes

\[
\begin{align*}
|\ell_1| & \rightarrow \infty, & |\ell_1| \text{ fixed}, & |\ell_2| \rightarrow \infty, & |\ell_2| \rightarrow \infty.
\end{align*}
\tag{4.3}
\]

The counter-terms relative to the singular behaviour of the first two regimes can be generated using the replacement in Eq. (4.1). To build the local counter-term needed to cancel the third regime, we need the additional replacement

\[
S_{\text{UV}^2} : \{ \ell_j^2 \mid \ell_j \cdot \ell_k \mid \ell_j \cdot k_i \} \rightarrow \{ \lambda^2 q_{j,\text{UV}}^2 + (1 - \lambda^2) \mu_{\text{UV}}^2 \mid \lambda^2 q_{j,\text{UV}} \cdot q_k,\text{UV} + (1 - \lambda^2) \mu_{\text{UV}}^2/2 \mid \lambda q_{j,\text{UV}} \cdot k_i \}, \tag{4.4}
\]

to build the counter-term

\[
A_{\text{UV}^2} = L_\lambda \left( A - \sum_{j=1,2} A_{j,\text{UV}} \right) \bigg|_{S_{\text{UV}^2}} - d_{\text{UV}^2} \mu_{\text{UV}}^4 \int_{\ell_1} \int_{\ell_2} (G_F(q_{1,\text{UV}}))^3 (G_F(q_{12,\text{UV}}))^3, \tag{4.5}
\]

where once again, the term proportional to the scheme-fixing coefficient \( d_{\text{UV}^2} \) integrates to a finite quantity.

The one- and two-loop versions of this algorithm were explicitly implemented in a MATHEMATICA code [16]. It is fully process-independent and can be directly applied to any scattering amplitude, producing the appropriate local counter-term to regularise the divergent behaviour in the high-energy region.

5 Applications at one-loop

In this section, we give some explicit examples in which the techniques described in Secs. 3 and 4 are applied. We focus on some processes that contain two to four kinematic invariants, and we consider the

\(^3\)This procedure is equivalent to expanding around the UV propagator \((G_F(q_{\text{UV}}))^{-1} = q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0 \) and then keeping only the divergent terms [13–15].
non-vanishing helicity configurations. We summarise the description of our examples in Table 1. In the following, the kinematic invariants are implicitly given in GeV$^2$.

Since we are aiming at a calculation performed purely in four space-time dimensions, we restrict the analysis presented in this article to processes that are simultaneously IR and UV finite. Although the processes under consideration exhibit these features, they might still possess a local UV-divergent behaviour that prevents to perform the calculation directly in four space-time dimensions, without introducing any additional regularisation. This is because, in the most general case, the associated integrands turn out to be non-integrable functions in the high-energy limit (or UV limit).

Eventually, in the context of dimensional regularisation (DREG), setting $d = 4$ from the beginning of the calculation can generate wrong results. This situation was exhaustively discussed in Ref. [21] for the computation of the decay width of $H \rightarrow \gamma\gamma$ at leading order. Therefore, we need to build local UV counter-terms that take care of the singularities that appear at integrand level in the UV limit. In other words, we need to locally renormalise our amplitude, as explained in Sec. 4, to render the expressions integrable.

| Process   | Kinematic scales | Helicity configuration |
|-----------|-----------------|------------------------|
| $H \rightarrow \gamma\gamma$ | $s_{12}, m_f^2$ | ++                     |
| $\gamma\gamma \rightarrow \gamma\gamma$ | $s, t, m_f^2$ | + + + +                |
| $H \rightarrow ggg$ | $s_{12}, s_{13}, s_{23}, m_f^2$ | + + +                  |

Table 1. Processes considered at one-loop level with their kinematic scales. We indicate the non-vanishing helicity configurations.

The calculation of the amplitude $H \rightarrow \gamma\gamma$ performed in Ref. [21], through the form factor decomposition, exploited several analytical properties in order to simplify the results. In particular, due to gauge invariance, it was possible to remove vanishing terms at integrand level. On the contrary, in the present calculation, we directly generate the proper UV counter-term to render the amplitude integrable in four space-time dimensions, without taking into account any kind of analytical property to achieve further simplifications. The numerical integration performed by LTD was compared with the analytic expression of the amplitude. For the latter, we rely on two MATHEMATICA packages, the integral reduction provided FEYNCALC [25–27] and the analytic expressions for the one-loop scalar integrals collected in PACKAGE-X [28]. Our results are shown in Fig. 1, where we plot the value of the amplitude as a function of the fermion internal mass, $m_f^2$, for different values of $s_{12}$. An excellent agreement is found, as expected from our previous studies of this process [16, 21].

For the processes including more kinematic scales, namely $\gamma\gamma \rightarrow \gamma\gamma$ and $H \rightarrow ggg$, we do not rely on FEYNCALC because it becomes inefficient when the rank of the loop momentum in the numerator starts increasing. Therefore, instead of decomposing the integrals, we work at the integrand level by reducing the amplitudes to scalar one-loop integrals. In order to do so, we follow the Ossola-Papadopoulos-Pittau (OPP) method [29]. For the evaluation of the scalar one-loop integrals we keep using PACKAGE-X. Our results are shown in Figs. 2 and 3, where we plot the amplitudes as a function of the fermion internal mass $m_f^2$. For $\gamma\gamma \rightarrow \gamma\gamma$, we fixed $s = -5$ and considered $t = \{-8, -10, -12\}$. In the case of $H \rightarrow ggg$, $s_{12} = -1/3$ and $s_{23} = -1/7$ remained fixed while we varied $s_{13} \in [8, 12]$. The agreement is very good for both processes, in all the kinematical and helicity configurations that we explored. Small numerical instabilities arise for $m_f^2 > 180$ in $H \rightarrow gggg$, although they can be fixed by slightly increasing the numerical precision of the integration.

Let us stress that in the processes we consider within the LTD approach, we do not perform any integral
Figure 1. $H \to \gamma \gamma$ at one-loop as a function of the internal mass $m_f^2$. We plot the predictions for $s \in \{8, 10, 12\}$. The solid blue lines correspond to the analytical results, while the red points are computed through the LTD-based numerical approach.

Figure 2. One-loop contributions to the process $\gamma \gamma \to \gamma \gamma$, as a function of the internal mass $m_f^2$. We consider all the possible helicity configurations for a fixed ordering of the external legs: $+++$, $-++$, $-++$ and $-++$. In each case, we fix $s = -5$ and plot the predictions for $t \in \{-8, -10, -12\}$. The solid blue lines correspond to the analytical results, while the red points were computed through the LTD-based numerical approach.
or integrand reduction. We directly evaluate them with the proper inclusion of the UV local counter-terms, as explained in Sec. 4. This approach indeed straightforwardly allows the evaluation of the amplitude in four space-time dimensions. Regarding the evaluation of the required integrals, we use the built-in Mathematica function NIntegrate on a desktop machine with an Intel i7 (3.4GHz) processor with 8 cores and 16 GB of RAM. The time for phase-space point was $O(30')$.

6 Conclusions and future directions

In this article, we have explored the features of a numerical implementation based on the loop-tree duality theorem and its interplay with the momentum twistor parametrisation. We applied the dual decomposition to change the integration domain of one-loop amplitudes, from a Minkowski to an Euclidean space-time. And, simultaneously, the application of the helicity formalism lead to very compact expressions. Thus, the resulting implementation turns out to be both analytically and numerically efficient.

The computational framework that we developed was successfully applied to some benchmark processes at one-loop. In particular. Since the results were available through other codes or explicit analytical expressions, we were able to perform a comparison and we found a complete agreement with our predictions. Moreover, we have shown in a previous publication [16] the feasibility of the local renormalisation procedure to deal with scattering two-loop amplitudes.

This work constitutes an important step towards the automation of a LTD-based framework to compute physical observables in a fully numerical approach. By canceling the singularities directly at integrand level, we prevent them to manifest when performing the numerical evaluations and we have to deal only with integrable expressions. In this way, the algebraic handling of the virtual amplitudes produces a four-dimensional representation of IR-safe finite observables. Moreover, inside our framework, we include an algorithm to achieve a local renormalisation, i.e. a point-by-point cancellation of UV singularities. We defer for future research the demonstration of a fully consistent local renormalisation method to tackle multi-loop scattering amplitudes.

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A Momentum twistor parametrisation

For the sake of simplicity, we remark the main features of the twistor variables. Nevertheless, for an exhaustive study of them, we refer the reader to Ref. [12].

A.1 Little group scaling

Let us remark the little group scaling, a group of transformations that leaves the momentum of an on-shell particle invariant. Hence, the spinors $|i\rangle$ and $|i\rangle$ can be re-scaled according to

$$|i\rangle \rightarrow t|i\rangle, \quad |i\rangle \rightarrow t^{-1}|i\rangle.$$  \hspace{1cm} (A.1)

This transformation turns out to be very interesting at the amplitude level. This is because amplitudes with massless particles can always be written in terms of spinorial products. Then, we have that:

- scalars do not scale,
- fermions with spin $1/2$ scale as $t^{-2h}$ for $h = \pm \frac{1}{2},$
- polarisation vectors with spin $1$ scale as $t^{-2h}$ for $h = \pm 1.$

This implies that an $n$-point amplitude, after one of the massless particles is re-scaled according to Eq. (A.1), can be expressed as

$$A_n ([1], [1], h_1, \ldots, \{t_i | i\rangle, t_i^{-1} | i\rangle, h_i, \ldots\}) = t_i^{-2h_i} \times A_n ([1], [1], h_1, \ldots, \{t_i | i\rangle, t_i^{-1} | i\rangle, h_i, \ldots\}),$$  \hspace{1cm} (A.2)

with $h_i$ the helicity of the particle $i.$

A.2 Momentum twistor variables

The momentum conservation rule implies that the vectors representing the different momenta close into a contour, which can be defined by the edges or by the cusps. The former is the usual representation, $p_1 + p_2 + \ldots + p_n = 0,$ whereas, the latter correspond to locate a point $y_{ij}^\alpha$ in a dual space. In fact, these points can be expressed in terms of momentum vectors

$$p_i^\alpha = (y_i - y_{i+1})^\alpha.$$  \hspace{1cm} (A.3)

These dual variables satisfy momentum conservation after imposing a periodicity relation, namely $y_{n+1} = y_1.$ For the sake of the simplicity, we take into account the ordering of the external particles. Hence, we define

$$y_{ij}^\alpha = (y_i - y_j)^\alpha = (p_i + p_{i+1} + \ldots + p_{j-1})^\alpha.$$  \hspace{1cm} (A.4)

Furthermore, because all the particles are massless (i.e $p_i^\alpha = 0$), we write the Dirac equation in terms of holomorphic spinors,

$$p_i | i\rangle = \left(\hat{y}_i - \hat{y}_{i+1}\right) | i\rangle = 0,$$  \hspace{1cm} (A.5)
and we define a new variable $|\mu_i\rangle$, according to

$$|\mu_i\rangle = y_i^+|\bar{i}\rangle = y_{i+1}^+|\bar{i}\rangle.$$  \hfill (A.6)

With these two independent variables, $|i\rangle$ and $|\mu_i\rangle$, we build a new four-component spinor variable $Z_i$, usually called momentum twistor. Nevertheless, the anti-holomorphic spinors $|i\rangle$, can be written as

$$|i\rangle = \frac{\langle i+1\rangle [\mu_{i-1}] + \langle i \rangle [\mu_{i+1}] + \langle i-1 \rangle [\mu_{i}]}{\langle i-1 \rangle (i+1)} ,$$  \hfill (A.7)

due to the Gordon identity. Given $n$ momentum twistors, denoted $(Z_1, Z_2, \ldots, Z_n)$, they must fulfill Poincaré and $U(1)$ symmetries, besides satisfying momentum conservation and on-shellness. These symmetries allow us to express any $n$-point massless amplitude in terms for $3n-10$ variables, which is the minimal quantity required$^4$.

Since we are interested in parametrising the external momenta in terms of the minimal set of variables, we follow the representation used in Ref. [12]. In particular, for a four-point kinematics, we have

$$Z = \begin{pmatrix} |1\rangle \vert 2\rangle \vert 3\rangle \vert 4\rangle \vert \mu_1\rangle \vert \mu_2\rangle \vert \mu_3\rangle \vert \mu_4\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{z_1} & 0 & \frac{1}{z_2} + \frac{1}{z_3} \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} ,$$  \hfill (A.8)

where we can relate $z_1$ and $z_2$ to the kinematic invariants according to

$$z_1 = s_{12} , \quad z_2 = \frac{s_{14}}{s_{12}} .$$  \hfill (A.9)

Likewise, we obtain a particular generalisation for $n \geq 5$

$$Z = \begin{pmatrix} 1 & 0 & f_1 & f_2 & f_3 & \cdots & f_{n-3} & f_{n-2} \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & z_{n-1} & z_n & \cdots & z_{2n-6} & 1 \\ 0 & 0 & 1 & 1 & z_{2n-5} & \cdots & z_{3n-11} & 1 - \frac{z_{2n-10}}{z_{n-1}} \end{pmatrix} ,$$  \hfill (A.10)

with

$$f_i = \sum_{k=1}^{i} \frac{1}{\prod_{k=1}^{i} z_i} ,$$  \hfill (A.11)

and

$$z_i = \begin{cases} s_{12} & i = 1 \\ \frac{-(i+1)(i+2)}{(i+1)(i+2)} & i = 2, \ldots, n-2 \\ \frac{s_{12}}{s_{12}} & i = n-1 \\ \frac{i-n+4}{i(i-n+5)} & i = n, \ldots, 2n-6 \\ \frac{\sum_{j=2}^{i-n+5} \frac{(i-n+5)(i-1)}{i(i-n+5)}}{\sum_{j=2}^{i-n+4} \frac{(i-1)(i-n+5)}{i(i-n+5)}} & i = n+1, \ldots, 3n-11 \\ \frac{s_{12}}{s_{12}} & i = 3n-10 \end{cases} .$$  \hfill (A.12)

We remark that with this configuration of external momenta, we drop the physical phase of the amplitude, namely, the information that accounts for parity invariance. However, it can be straightforwardly restored by using the prefactor

$$\left( \frac{\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle} \right)^{-h_1} \prod_{i=2}^{n} \left( \frac{(1i)^2}{\langle 13 \rangle} \right)^{-h_i} ,$$  \hfill (A.13)

where $h_i$ are the helicities of the external massless momenta.

$^4$For an extensive review of the derivation of the momentum twistors, we refer the reader to Refs. [12].
B  External momenta

In this appendix we give the external momenta in terms of the kinematic scales shown in Sec. 5.

B.1  $H \rightarrow \gamma \gamma$

We focus on the process

$$H (-p_3) \rightarrow \gamma (p_1) + \gamma (p_2) , \quad (B.1)$$

with the kinematics,

$$p_1^\mu = \frac{1}{2} \{ -1, 1, t, -1 \} ,$$
$$p_2^\mu = \frac{s t_2}{2} \{ 0, -1, t, 0 \} ,$$
$$\epsilon_+^\mu (p_1) = \frac{1}{\sqrt{2}} \{ 1, -1, t, -1 \} ,$$
$$\epsilon_-^\mu (p_1) = \frac{1}{\sqrt{2}} \{ -1, 0, 0, -1 \} ,$$
$$\epsilon_+^\mu (p_2) = \frac{s t_2}{\sqrt{2}} \{ 1, 0, 0, 1 \} ,$$
$$\epsilon_-^\mu (p_2) = \frac{1}{\sqrt{2} s t_2} \{ -1, 1, -t, 1 \} . \quad (B.2)$$

B.2  $\gamma \gamma \rightarrow \gamma \gamma$

We consider the light-by-light scattering,

$$\gamma (-p_3) \gamma (-p_4) \rightarrow \gamma (p_3) \gamma (p_4) , \quad (B.3)$$

with the kinematics,

$$p_1^\mu = \frac{1}{2} \{ -1, 1, t, -1 \} ,$$
$$p_2^\mu = \frac{s}{2} \{ 0, -1, t, 0 \} ,$$
$$p_3^\mu = \frac{1}{2} \{ s t + 1, s + t, i(t - s), 1 - st \} ,$$
$$\epsilon_+^\mu (p_1) = \frac{1}{\sqrt{2}} \{ 1, -1, t, -1 \} ,$$
$$\epsilon_-^\mu (p_1) = \frac{1}{\sqrt{2}} \{ -1, 0, 0, -1 \} ,$$
$$\epsilon_+^\mu (p_2) = \frac{s}{\sqrt{2}} \{ 1, 0, 0, 1 \} ,$$
$$\epsilon_-^\mu (p_2) = \frac{1}{\sqrt{2} s} \{ -1, 1, -t, -s \} ,$$
$$\epsilon_+^\mu (p_3) = \frac{1}{\sqrt{2} (s + 1)} \{ s - 1, - (s - 1), i(s + 1), - (s + 1) \} ,$$
$$\epsilon_-^\mu (p_3) = \frac{s t}{\sqrt{2}} \{ 0, 1, t, 0 \} ,$$
$$\epsilon_+^\mu (p_4) = \frac{1}{\sqrt{2} s t_2} \{ - s t + t + 1, (s - 1) t - 1, - i(s t + t + 1), s t + t + 1 \} . \quad (B.4)$$
We consider the Higgs decay into thee gluons,

\[ H \rightarrow g (p_2) g (p_3) g (p_4) , \]  

with,

\[
p_1^\mu = \frac{1}{2} \left\{ \frac{s_{12} + s_{13}}{s_{23}}, 1, \frac{s_{12} + s_{13}}{s_{23}} \right\} ,
\]

\[
p_2^\mu = \frac{s_{12}}{2} \left\{ 0, -1, \epsilon, 0 \right\} ,
\]

\[
p_3^\mu = \frac{1}{2} \left\{ s_{23} + 1, \frac{s_{23}}{s_{12}}, \frac{s_{23}}{s_{12}} - s_{12}, 1 - s_{23} \right\} ,
\]

\[
\epsilon_+^p (p_1) = \frac{1}{\sqrt{2}} \left\{ 1, -1, \epsilon, -1 \right\} ,
\]

\[
\epsilon_-^p (p_1) = \frac{1}{\sqrt{2}} \left\{ -1, 0, 0, -1 \right\} ,
\]

\[
\epsilon_+^p (p_2) = \frac{s_{12}}{\sqrt{2}} \left\{ 1, 0, 0, 1 \right\} ,
\]

\[
\epsilon_-^p (p_2) = \frac{1}{\sqrt{2} s_{12}} \left\{ -1, 1, -\epsilon, 1 \right\} ,
\]

\[
\epsilon_+^p (p_3) = \frac{1}{\sqrt{2}} \left\{ -s_{12}, -s_{23}, -18 s_{23}, -s_{12} \right\} ,
\]

\[
\epsilon_-^p (p_3) = \frac{1}{\sqrt{2} s_{12} s_{13} s_{23}} \left\{ - (s_{12} s_{23} + s_{12} + s_{13}), - (s_{12} s_{13} + s_{23}), t(s_{12} s_{13} + s_{23}), - (s_{12} + s_{13} - s_{12} s_{23}) \right\} .
\]

\[ \text{(B.5)} \]

\[ \text{(B.6)} \]

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