BOUNDING G-THEORY WITH FIBRED CONTROL

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ABSTRACT. We use filtered modules over a Noetherian ring and fibred bounded control on homomorphisms to construct a new kind of controlled algebra with applications in geometric topology. The resulting theory can be thought of as a “pushout” of bounded K-theory with fibred control and bounded G-theory constructed and used by the authors. Bounded G-theory was geared toward constructing a G-theoretic version of assembly maps and proving the Novikov injectivity conjecture for them. The G-theory with fibred control is needed in the study of surjectivity of the assembly map. The relation between the K- and G-theories is the classical one: K-theory is meaningful, however G-theory is easier to compute, and the relationship is expressed via a Cartan map. This map turns out to be an equivalence under very mild constraints in terms of metric geometry such as finite decomposition complexity. The fibred theory is certainly more complicated than the absolute theory. This paper contains the non-equivariant theory including fibred controlled excision theorems known to be crucial for computations.

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1. Introduction

The purpose of this paper is to use filtered modules over a Noetherian ring with a fibred bounded control on homomorphisms to construct a bounded $G$-theory with fibred control. This theory can be thought of as a “pushout” of the bounded $K$-theory with fibred control constructed by the authors in [8] and the controlled $G$-theory constructed in [6]. Here is a summary of this situation:

\[
\begin{array}{ccc}
K(X,R) & \longrightarrow & K_X(Y) \\
\downarrow & & \downarrow \\
G(X,R) & \longrightarrow & G_X(Y)
\end{array}
\]

**Figure 1.** Bounded $G$-theory with fibred control as a “pushout”.

Throughout this paper, metric spaces such as $X$ and $Y$ that appear in the square will be proper metric spaces in the sense that every closed bounded subspace is compact. The space or spectrum in the upper left corner represents the indispensable in modern geometric topology bounded $K$-theory of Pedersen and Weibel [14, 15], reviewed here in the beginning of section 2.1. This theory is defined for any ring of coefficients $R$. It is built out of free $R$-modules with generating sets parametrized over the metric space $X$. This allows to impose geometric control conditions on the homomorphisms $f: F \rightarrow G$. The bounded control condition postulates that there is number $b \geq 0$ so that the image of every basis element in $F$ associated to some point $x$ in $X$ is spanned by basis elements in $G$ that are referenced by points within $b$ from $x$. We will review precise definitions shortly.

The spectrum in the upper right corner $K_X(Y)$ is a generalization of this theory to the situation when the modules are parametrized by the product of two metric spaces $X$ and $Y$, and the control imposed on the homomorphisms is relaxed: it is essentially the bounded control across $X$ but the bound is allowed to change in the complementary direction $Y$ as one varies the $X$-coordinate. This theory becomes useful when one considers “bundle phenomena”. For example, the space $X$ can be the universal cover of the tangent bundle of a manifold embedded in a Euclidean space or even its discrete model such as the fundamental group with a word metric. The space $Y$ can be the universal cover of the normal bundle to the embedding with a variety of useful metrics. This situation comes up the authors’ work in geometric topology. The fibred $K$-theory is still defined for any coefficient ring $R$.

To describe the bottom row in the square and for the rest of the paper, we restrict to Noetherian rings $R$.

In place of parametrizations used to control homomorphisms between free modules, one can use filtrations of arbitrary $R$-modules by subsets of the metric space $X$ and impose control conditions in terms of the filtrations. This was done in [6] for a single space $X$. The result was the bounded $G$-theory spectrum $G(X, R)$. The definition involved promoting the setting from the additive structure for free modules in the definition of bounded $K$-theory to a specific non-split Quillen exact structure on a category of filtered $R$ modules with morphisms satisfying control conditions.
and the admissible morphisms satisfying further “bi-control” conditions. Regardless of the significant change in techniques, literally every theorem about bounded $K$-theory has an exact (accidental pun) counterpart in $G$-theory.

Now it is clear what the “pushout” $G_X(Y)$ is supposed to mean. We want to look at the $K$-theory of a category built out of filtered modules over the product $X \times Y$ where the morphisms have the fibred control condition of the type described for fibred $K$-theory. This time we are interested in very specific excision results designed to deconstruct only the “fiber” direction. In our applications of this material we want to perform what we call here relative excision in the normal bundle direction. This is greatly facilitated by the hybrid conditions imposed on the objects themselves. We include several remarks in the paper regarding the options and why our choices seem to be optimal. Long story short, we have resolved in this paper the problems that may be much harder to solve, if solvable at all, for the straightforward combination of the theories in the corners of the diagram. We resolve them for a carefully crafted theory that has all the desired properties and yet specializes to precisely $G(X, R)$ when localized near the subspace $X \times 0$ in $X \times Y$.

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2. Elements of bounded $G$-theory

Bounded $G$-theory defined in [6] is a variant of bounded $K$-theory of Pedersen and Weibel made applicable to more general, non-split exact structures. It was designed by the authors for a different purpose than the one in this paper. The old focus was on the equivariant theory in addition to very basic excision that were sufficient for introducing an assembly map in $G$-theory for all finitely generated groups and proving the injectivity Novikov type theorem for a large class of groups. We will review and augment some material from [6] in the form best fit for the fibred theory.

2.1. Basic definitions. We start with a brief recollection of the bounded $K$-theory setup. The coefficients $R$ for this theory can be an arbitrary associative ring. The bounded category $C(X, R)$ is the additive category of geometric $R$-modules whose objects are functions $F: X \to \text{Free}_{fg}(R)$ which are locally finite assignments of free finitely generated $R$-modules $F_x$ to points $x$ of $X$. The local finiteness condition requires precisely that for any bounded subset $S \subset X$ the restriction of $F$ to $S$ has finitely many nonzero values. Let $d$ be the distance function in $X$. The morphisms in $C(M, R)$ are the $R$-linear homomorphisms

$$\phi: \bigoplus_{x \in X} F_x \to \bigoplus_{y \in X} G_y$$

with the property that the components $F_x \to G_y$ are zero for $d(x, y) > b$ for some fixed real number $b = b(\phi) \geq 0$. The associated $K$-theory spectrum is denoted by $K(X, R)$ and is called the bounded $K$-theory of $X$.

2.1.1. Remark. We would like to remind the reader that the original paper of Pedersen and Weibel [14] was already written in greater generality. If $\mathcal{A}$ is any additive category, it can be used as coefficients in this construction in place of finitely generated free $R$-modules using the same formulas as above. The outcome is the bounded category $C(X, \mathcal{A})$ which is again an additive category with the evident notion of
split exact sequences. It is now possible to iterate this construction: when there are two metric spaces $X$ and $Y$, Pedersen and Weibel built the additive category $C(X, C(Y, R))$. The objects of this category can be identified with the objects of $C(X \times Y, R)$ where the product is given a reasonable metric such as the max metric. The morphisms are, however, very different. They are $R$-homomorphisms which are still controlled over $X$ in the standard fashion but the non-zero components $F_x \to G_y$ are now allowed to vary in range with the $X$-coordinates of $x$ and $y$. In contrast, morphisms in $C(X \times Y, R)$ require one single number to work as a bound for all of these components.

Pedersen and Weibel were more interested in the product situation and, in fact, repaired the non-uniform boundedness properties of morphisms in $C(X, C(Y, R))$ by filtering the morphism sets. With that fix the category becomes isomorphic to $C(X \times Y, R)$. We, instead, embraced the flexibility of this construction in [S] with the idea of exploiting the additional deformations in $K(X, C(Y, R))$, the $K$-theory with fibred control, that the construction allows. This is the spectrum that shows up in Figure 4 as $K_X(Y)$.

2.1.2. Notation. For a subset $S \subset X$ and a real number $r \geq 0$, $S[r]$ will stand for the metric $r$-enlargement $\{x \in X \mid d(x, S) \leq r\}$. In this notation, the metric ball of radius $r$ centered at $x$ is $\{x\}[r]$ or simply $x[r]$.

A variation of the basic construction of bounded $K$-theory is based on the following observation. For every object $F$ and a subset $S$ there is a free $R$-module $F(S) = \bigoplus_{m \in S} F_m$. In this context we say an element $x \in F$ is supported on a subset $S$ if $x \in F(S)$. Now the restriction from arbitrary $R$-linear homomorphisms to the bounded ones can be described entirely in terms of these subobjects: $\phi$ is controlled as above precisely when there is a number $b \geq 0$ so that $\phi F(S) \subset F(S[b])$ for all choices of $S$.

In the rest of this section and the rest of the paper, we will restrict to the case of a Noetherian ring $R$.

Let $\mathcal{P}(X)$ denote the power set of $X$ partially ordered by inclusion and viewed as a category. If $F$ is a left $R$-module, let $\mathcal{I}(F)$ denote the family of all $R$-submodules of $F$ partially ordered by inclusion.

2.1.3. Definition. An $X$-filtered $R$-module is a module $F$ together with a functor $\mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of $X$ to the family of $R$-submodules of $F$, both ordered by inclusion, such that the value on $X$ is $F$. It will be most convenient to think of $F$ as the functor above and use notation $F(S)$ for the value of the functor on $S$. We will call $F$ reduced if $F(\emptyset) = 0$.

An $R$-homomorphism $f : F \to G$ of $X$-filtered modules is boundedly controlled if there is a fixed number $b \geq 0$ such that the image $f(F(S))$ is a submodule of $G(S[b])$ for all subsets $S$ of $X$.

The objects of the category $\mathbb{U}(X, R)$ are the reduced $X$-filtered $R$-modules, and the morphisms are the boundedly controlled homomorphisms.

The category $\mathbb{U}(X, R)$ we constructed is clearly an additive category, but the more interesting structure for developing its $K$-theory is a certain Quillen exact structure. For a good modern exposition of exact categories we refer to Keller [11]; there is also a leisurely review of relevant basic theory in [R section 2].

Let us recall some standard terms. If a category has kernels and cokernels for all morphisms, it is called preabelian. If, in addition, the canonical map $\text{coim}(f) \to
im(f) for each morphism f is monic and epic but not necessarily invertible, we will say the category is semi-abelian, cf. [16] pages 167-168 and [18]. It is abelian if it is also balanced in the sense that the canonical map is an isomorphism. Recall also that a category is called cocomplete if it contains colimits of arbitrary small diagrams, cf. Mac Lane [13] chapter V].

2.1.4. Remark. If \( X \) is unbounded, \( U(X, R) \) is not a balanced category and therefore not an abelian category. For an explicit description of a boundedly controlled morphism in \( U(\mathbb{Z}, R) \) which is an isomorphism of left \( R \)-modules but whose inverse is not boundedly controlled, we refer to [14] Example 1.5.

It turns out that the kernels and cokernels in \( U(X, R) \) can be characterized using an additional property a boundedly controlled morphism may or may not have.

2.1.5. Definition. A morphism \( f: F \to G \) in \( U(X, R) \) is called boundedly bicontrolled if there exists a number \( b \geq 0 \) such that in addition to inclusions of submodules
\[
f(F(S)) \subset G(S[b]),
\]
there are inclusions
\[
f(F) \cap G(S) \subset fF(S[b])
\]
for all subsets \( S \subset X \). In this case we will say that \( f \) has filtration degree \( b \) and write \( \text{fil}(f) \leq b \).

2.1.6. Definition. We define the admissible monomorphisms in \( U(X, R) \) be the boundedly bicontrolled homomorphisms \( m: F_1 \to F_2 \) such that the map \( F_1(X) \to F_2(X) \) is a monomorphism. We define the admissible epimorphisms be the boundedly bicontrolled homomorphisms \( e: F_1 \to F_2 \) such that \( F_1(X) \to F_2(X) \) is an epimorphism.

Let the class \( \mathcal{E} \) of exact sequences consist of the sequences
\[
F^\cdot: \quad F'' \xrightarrow{i} F \xrightarrow{j} F'',
\]
where \( i \) is an admissible monomorphism, \( j \) is an admissible epimorphism, and \( \text{im}(i) = \ker(j) \).

There are numerous examples of semi-abelian categories that are classical or have appeared recently in analysis and algebra that are listed in section 4 of [2] or in [16]. For us \( U(X, R) \) is of major interest.

2.1.7. Theorem. \( U(X, R) \) is a cocomplete semi-abelian category. The class of exact sequences \( \mathcal{E} \) gives an exact structure on \( U(X, R) \).

Proof. This fact is contained in Proposition 2.6 and Theorem 2.13 of [6].

We want to recall the explicit construction of kernels and cokernels \( U(X, R) \) for future reference. For a boundedly controlled morphism \( f: F \to G \), the kernel of \( f \) in \( \text{Mod}(R) \) has the \( X \)-filtration \( K \) where \( K(S) = \ker(f) \cap F(S) \). This gives a kernel of \( f \) in \( U(X, R) \). Similarly, let \( I \) be the \( X \)-filtration of the image of \( f \) in \( \text{Mod}(R) \) by \( I(S) = \text{im}(f) \cap G(S) \). Let \( H(X) \) be the cokernel of \( f \) in \( \text{Mod}(R) \), which is the quotient \( G(X)/fF(X) \). Then there is a filtration \( H \) of \( H(X) \) defined by \( H(S) = \text{im}(G(S)/I(S) \to H(X)) \) where the maps between quotients are induced from the structure maps of \( G \). The resulting epimorphism \( \pi: G(X) \to H(X) \) gives a boundedly bicontrolled morphism of filtration 0. Lemma 2.5 of [6] verifies the universal properties of the cokernel \( H \).
Now suppose $A$ is an arbitrary cocomplete semi-abelian category. All of the definitions and proofs presented so far can be interpreted verbatim to describe a bounded category $U(X,A)$ if instead of $R$-modules one uses objects from $A$. We should point out that the exposition in [3] is constructed in lesser generality with $A$ assumed to be abelian. It is important for the fibred version to relax abelian to semi-abelian. However, [3] can still serve as a good reference because throughout that paper only semi-abelian properties of $A$ are used.

In particular we have this conclusion.

2.1.8. **Theorem.** For any cocomplete semi-abelian category $A$, $U(X,A)$ is a co-complete semi-abelian category.

Of course, the category of $R$-modules $\text{Mod}(R)$ is a cocomplete abelian category and so can serve as a basic example of cocomplete semi-abelian coefficients $A$. In this case $U(X,A)$ is precisely $U(X,R)$.

2.2. **Properties of filtered objects.** In this section we will start imposing several conditions on objects in the bounded category $U(X,A)$. These conditions are agnostic to the nature of $A$, and so we can use the simpler notation $U(X)$ for this category. These conditions and results about them will be used in the contexts of both module categories and abstract semi-abelian categories as coefficients.

2.2.1. **Definition.** Let $F$ be an $X$-filtered $R$-module.

- $F$ is called lean or $D$-lean if there is a number $D \geq 0$ such that $F(S) \subset \sum_{x \in S} F(x[D])$

  for every subset $S$ of $X$.

- $F$ is called split or $D'$-split if there is a number $D' \geq 0$ such that we have $F(S) \subset F(T[D']) + F(U[D'])$

  whenever a subset $S$ of $X$ is written as a union $T \cup U$.

- $F$ is called insular or $d$-insular if there is a number $d \geq 0$ such that $F(S) \cap F(U) \subset F(S[d] \cap U[d])$

  for every pair of subsets $S, U$ of $X$.

2.2.2. **Proposition.** The properties of being lean, split, and insular are preserved under isomorphisms in $U(X)$. Also, a $D$-lean filtered module is $D$-split.

**Proof.** If $f : F_1 \rightarrow F_2$ is an isomorphism with $\text{fil}(f) \leq b$, and $F_1$ is $D$-lean, $D'$-split, and $d$-insular, then $F_2$ is $(D + b)$-lean, $(D' + b)$-split, and $(d + 2b)$-insular. For the other statement, we have $F(T \cup U) \subset \sum_{x \in T} F(x[D]) + \sum_{x \in U} F(x[D]) \subset F(T[D]) + F(U[D])$

since in general $\sum_{x \in S} F(x[D]) \subset F(S[D])$. □

A collection of objects in an exact category is said to be **closed under extensions** if the middle term of an exact sequence belongs to the collection in case both of the extreme terms belong to the collection.
2.2.3. **Lemma.** (1) Lean objects are closed under extensions.
(2) Insular objects are closed under extensions.
(3) Split objects are closed under extensions.

**Proof.** For an exact sequence $E' \xrightarrow{f} E \xrightarrow{g} E''$ in $\text{U}(X)$, let $b \geq 0$ be a common filtration degree for $f$ and $g$. The first two statements follow from parts (1) and (2) of [2], Proposition 2.18. It is shown there that if $E'$ and $E''$ are $D$-lean then $E$ is $(4b+D)$-lean. Also, if both $E'$ and $E''$ are $d$-insular then $E$ is $(4b+2d)$-insular.

To prove (3), suppose both $E'$ and $E''$ are $D'$-split. We have

$$gE(T \cup U) \subset E''(T[b] \cup U[b]),$$

because in general $(T \cup U)[b] \subset T[b] \cup U[b]$. So

$$g(T \cup U) \subset E''(T[b+D']) + E''(U[b+D']) \subset gE(T[2b+D']) + gE(U[2b+D']).$$

If $z \in E(T \cup U)$ then we can write $g(z) = g(z_1) + g(z_2)$ where $z_1 \in E(T[2b+D'])$ and $z_2 \in E(U[2b+D'])$. Since $z - z_1 - z_2$ is an element of $\ker(g) \cap E(T[2b+D'] \cup U[2b+D'])$, we have an element

$$k \in E'(T[3b+D'] \cup U[3b+D']) \subset gE(T[3b + 2D']) + E'(U[3b + 2D'])$$

such that

$$z = f(k) + z_1 + z_2 \in E(T[4b+2D']) + E(U[4b+2D']).$$

So $E$ is $(4b+2D')$-split. \hfill \qed

2.2.4. **Lemma.** Let $E' \xrightarrow{f} E \xrightarrow{g} E''$ be an exact sequence in $\text{U}(X)$.

(1) If the object $E$ is lean then $E''$ is lean.
(2) If $E$ is split then $E''$ is split.
(3) If $E$ is insular then $E'$ is insular.
(4) If $E$ is insular and $E'$ is lean then $E''$ is insular.
(5) If $E$ is insular and $E'$ is split then $E''$ is insular.
(6) If $E$ is split and $E''$ is insular then $E'$ is split.

**Proof.** Let $b \geq 0$ be a common filtration degree for $f$ and $g$. If $E$ is $D$-lean, $D'$-split, or $d$-insular, it is easy to show that $E''$ is $(D + 2b)$-lean or $(D' + 2b)$-split and $E'$ is $(d + 2b)$-insular respectively, which verifies (1), (2), and (3).

Statement (4) follows from the proof of part (3c) of [2], Proposition 2.18. It is shown there that if $E'$ is $D$-lean and $E$ is $d$-insular then $E''$ is $(4b + D + d)$-insular. The same proof actually shows statement (5). The only equation in that proof that uses $D$-leanness of $E'$ is only used to get a consequence that is in fact immediate from the assumption that $E'$ is $D$-split.

(6) Suppose $E$ is $D'$-split and $E''$ is $d$-insular. Given $z \in E'(T \cup U)$, we have $f(z) \in E(T[b] \cup U[b])$. Now $f(z) \in E(T[b+D']) + E(U[b+D'])$, so we can write
accordingly \( f(z) = y_1 + y_2 \). Now \( f(z) \in \ker(g) \), because \( g(y_1) + g(y_2) = 0 \). Since \( E'' \) is \( d \)-insular,
\[
g(y_1) = -g(y_2) \in E''(T[2b + D' + d] \cap U[2b + D' + d]),
\]
so we are able to find
\[
y \in E(T[3b + D' + d] \cap U[3b + D' + d])
\]
such that \( g(y) = g(y_1) = -g(y_2) \), because generally \( (S \cap P)[b] \subset S[b] \cap P[b] \). Thus
\[
f(z) = y_1 + y_2 = (y_1 - y) + (y_2 + y)
\]
and
\[
y_1 - y \in E(T[3b + D' + d]), \quad y_2 + y \in E(U[3b + D' + d]).
\]
Let \( z_1 = f^{-1}(y_1 - y) \) and \( z_2 = f^{-1}(y_2 + y) \), and we have \( z = z_1 + z_2 \) such that
\[
z_1 \in E'(T[4b + D' + d]), \quad z_2 \in E'(U[4b + D' + d]),
\]
so \( E' \) is \((4b + D' + d)\)-split. \( \square \)

**2.2.5. Corollary.** Let \( E' \xrightarrow{f} E \xrightarrow{g} E'' \) be an exact sequence in \( U(X) \). If \( E \) is split and insular then \( E'' \) is insular if and only if \( E' \) is split.

**Proof.** This fact is the combination of parts (5) and (6) of the Lemma. \( \square \)

**2.2.6. Remark.** The last Corollary is in contrast with the absence of the analogous general fact if one substitutes the lean property for the split property. However, the analogue is true in the presence of certain geometric assumptions on the metric space. For example, suppose \( X \) has finite asymptotic dimension. Then from the main theorem of [5], we have the following counterpart to part (6) of the Lemma: if \( E \) is lean and \( E'' \) is insular then \( E' \) is lean. This fact is not needed in this paper. Here, the excision properties of the theory rely only on the properties of the cokernels. For the applications in [7], properties of the kernels become crucial in dealing with coherence issues, and the geometric conditions need to be imposed.

**2.2.7. Definition.** We define \( L(X) \) as the full subcategory of \( U(X) \) on objects that are lean and insular with the induced exact structure. Similarly, \( S(X) \) is the full subcategory of \( U(X) \) on objects that are split and insular.

Exact structures in \( L(X) \) and \( S(X) \) can be induced from \( U(X) \). A full subcategory \( H \) of an exact category \( C \) is said to be **closed under extensions** or **thick** in \( C \) if

1. \( H \) contains the zero object, and
2. for any exact sequence \( C' \rightarrow C \rightarrow C'' \) in \( C \), if \( C' \) and \( C'' \) are isomorphic to objects from \( H \) then so is \( C \).

It is known (cf. [2] Lemma 10.20) that a subcategory closed under extensions in \( C \) inherits the exact structure from \( C \).

**2.2.8. Theorem.** \( L(X) \) and \( S(X) \) are closed under extensions in \( U(X) \). Therefore, \( L(X) \) and \( S(X) \) are exact subcategories of \( U(X) \), so we have a sequence of exact inclusions

\[
L(X) \longrightarrow S(X) \longrightarrow U(X).
\]

**Proof.** The first fact follows from parts (1) and (2) of Lemma [2.2.8] the second from (2) and (3). \( \square \)
2.3. Local finiteness property. Finally, there is an additional property that will consider only in module categories.

2.3.1. Definition. An $X$-filtered $R$-module $F$ is \textit{locally finitely generated} if $F(S)$ is a finitely generated $R$-module for every bounded subset $S \subset X$.

The category $\mathbf{BL}(X,R)$ is the full subcategory of $\mathbf{L}(X,R)$ on the locally finitely generated objects. Similarly, the companion category $\mathbf{BS}(X,R)$ is the full subcategory of $\mathbf{S}(X,R)$ on the locally finitely generated objects.

2.3.2. Theorem. The category $\mathbf{BL}(X,R)$ is closed under extensions in $\mathbf{L}(X,R)$. Similarly, the category $\mathbf{BS}(X,R)$ is closed under extensions in $\mathbf{S}(X,R)$.

Proof. If $f: F \rightarrow G$ is an isomorphism with $\text{fil}(f) \leq b$ and $G$ is locally finitely generated, then $F(U)$ are finitely generated submodules of $G(U[b])$ for all bounded $U$, since $R$ is a Noetherian ring. Suppose

$$\xymatrix{ F' \ar[r]^f & F \ar[r]^g & F'' \ar[r] & F'}$$

is an exact sequence and let $b \geq 0$ be a common filtration degree for both $f$ and $g$. Assume that $F'$ and $F''$ are locally finitely generated. For every bounded subset $U \subset X$ the restriction $g: F(U) \rightarrow gF(U)$ is an epimorphism onto a submodule of the finitely generated $R$-module $F''(U[b])$. The kernel of $g|F(U)$ is a submodule of $F'(U[b])$, which is also finitely generated. So the extension $F(U)$ is finitely generated. $\square$

2.3.3. Corollary. $\mathbf{BL}(X,R)$ and $\mathbf{BS}(X,R)$ are exact categories. The additive category $\mathcal{C}(X,R)$ of geometric $R$-modules with the split exact structure is an exact subcategory of $\mathbf{BL}(X,R)$, so there is a sequence of exact inclusions

$$\xymatrix{ \mathcal{C}(X,R) \ar[r] & \mathbf{BL}(X,R) \ar[r] & \mathbf{BS}(X,R) \ar[r] & \mathbf{U}(X,R) }. \quad \square$$

2.3.4. Remark. We want to briefly explain the roles played by the two conditions, lean and split, that distinguish the two categories $\mathbf{BL}(X,R)$ and $\mathbf{BS}(X,R)$. $\mathbf{BL}(X,R)$ was used exclusively in [6], where it was proven to have good excision properties. There is a separate important issue of homological coherence that still requires the lean condition for its resolution, cf. [5, 7]. The setting with the split condition in $\mathbf{BS}(X,R)$ is much more streamlined for the excision arguments but has insufficient coherence properties. In the next section, we will pursue the goal of combining the two different conditions in the “base” and “fibre” in order to achieve required coherence in the base and “fibrewise” excision properties. The hybrid lean/split condition will provide a considerable advantage because the fibred setting is more complicated than the absolute case.

Recall that a morphism $e: F \rightarrow F$ is an idempotent if $e^2 = e$. Categories in which every idempotent is the projection onto a direct summand of $F$ are called \textit{idempotent complete}.

2.3.5. Proposition. $\mathbf{BL}(X,R)$ and $\mathbf{BS}(X,R)$ are idempotent complete.

Proof. First note that a regular preabelian category is idempotent complete. The proof is exactly the same as for an abelian category: if $e$ is an idempotent then its kernel is split by $1 - e$. Since the restriction of an idempotent $e$ to the image of $e$ is the identity, every idempotent here is boundedly bicontrolled of filtration 0. It follows easily that the splitting of $e$ in $\text{Mod}(R)$ is in fact a splitting in $\mathbf{BL}(X,R)$ or $\mathbf{BS}(X,R)$.

$\square$
Finally, we need to address (the lack of) inheritance in filtered modules. It is immediate that a submodule of an insular filtered module is also insular with respect to the standard filtration induced on the submodule. However, a submodule of a lean filtered module is not necessarily lean.

2.3.6. Definition. An $X$-filtered object $F$ is called strict if there exists an order preserving function $\ell: \mathcal{P}(X) \to [0, +\infty)$ such that for every $S \subset X$ the submodule $F(S)$ is $\ell_S$-lean and $\ell_S$-insular with respect to the standard $X$-filtration $F(S)(T) = F(S) \cap F(T)$.

It is important to note that this property is not preserved under isomorphisms, so the subcategory of strict objects is not essentially full in $\text{BL}(X, R)$.

The bounded category $\text{B}(X, R)$ was defined in [6] as the full subcategory of $\text{BL}(X, R)$ on objects isomorphic to strict objects. Now this category is closed under exact extensions in $U(X, R)$ according to [6] Theorem 2.22 and so is an exact category.

A consequence of strictness, or more generally being isomorphic to a strict object, is the following feature. Given a filtered module $F$ in $\text{B}(X, R)$, a lean grading of $F$ is a functor $\tilde{F}: \mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of $X$ to the submodules of $F$ such that

1. each $\tilde{F}(S)$ is an object of $\text{BL}(X, R)$ when given the standard filtration,
2. there is a number $K \geq 0$ such that

$$F(S) \subset \tilde{F}(S) \subset F(S[K])$$

for all subsets $S$ of $X$.

Clearly, each $\tilde{F}(S)$ is an object of $\text{B}(X, R)$. Also an actual strict object has a lean grading by $\tilde{F}(S) = F(S)$ with $K = 0$.

We note for the interested reader that the theory in [6], including the excision theorems, could be alternately developed for modules with lean gradings in place of $\text{B}(X, R)$. We do not require such theory in this paper. Instead, we develop a similar but more relaxed notion of gradings in $\text{BS}(X, R)$.

2.4. Graded objects and their closure properties.

2.4.1. Definition. Given a filtered module $F$ in $\text{BS}(X, R)$, a grading of $F$ is a functor $F: \mathcal{P}(X) \to \mathcal{I}(F)$ such that

1. each $F(S)$ is an object of $\text{BS}(X, R)$ when given the standard filtration,
2. there is a number $K \geq 0$ so that

$$F(S) \subset F(S) \subset F(S[K])$$

for all subsets $S$ of $X$.

We will say that a filtered module $F$ is graded if it is possible to equip it with a grading, but there is no specific choice of grading that is specified.

2.4.2. Proposition. The graded objects are closed under isomorphisms.

Proof. If $f: F \to F'$ is an isomorphism and $F$ has a grading $F$, a grading for $F'$ is given by $F'(C) = fF(C[K + b])$, where $b$ is a filtration bound for $f$. \hfill $\square$

2.4.3. Definition. We define $\text{G}(X, R)$ as the full subcategory of $\text{BS}(X, R)$ on the locally finitely generated graded filtered modules.
2.4.4. Proposition. \( G(X, R) \) is closed under extensions in \( BS(X, R) \). Therefore \( G(X, R) \) is an exact subcategory of \( BS(X, R) \).

Proof. Given an exact sequence \( F \xrightarrow{j} G \xrightarrow{q} H \) in \( BS(X, R) \), let \( b \geq 0 \) be a common filtration degree for both \( f \) and \( g \) as boundedly bicontrolled maps, and assume that \( F \) and \( H \) are graded modules in \( G(X, R) \) with the associated functors \( F \) and \( H \).

To define a grading for \( G \), consider a subset \( S \) and suppose \( H(S[b]) \) is \( D \)-split and \( d \)-insular. The induced epimorphism \( g: G(S[2b]) \cap g^{-1}H(S[b]) \to H(S[b]) \) extends to another epimorphism

\[
g' : fF(S[3b]) + G(S[2b]) \cap g^{-1}H(S[b]) \to H(S[b])
\]

with \( \ker(g') = F(S[3b]) \). Without loss of generality, suppose \( F(S[3b]) \) is \( D \)-split and \( d \)-insular. We define

\[
G(S) = fF(S[3b]) + G(S[2b]) \cap g^{-1}H(S[b]).
\]

From parts (2) and (3) of Lemma 2.2.4, the module \( G(S) \) with the standard filtration is \((4b + 2d)\)-split and \((4b + 2d)\)-insular. Since \( G(S) \subset g^{-1}H(S[b]) \), we have \( G(S) \subset G(S) \). On the other hand, if the grading \( F \) has characteristic number \( K \geq 0 \) then \( G(S) \subset G(S[4b + K]) \). The last fact together with Theorem 2.3.2 shows that \( G(S) \) is finitely generated.

The relations between the categories in this section can be summarized as a commutative diagram of fully exact inclusions

\[
\begin{align*}
& BL(X, R) \quad BS(X, R) \\
\downarrow & \downarrow & \downarrow \\
C(X, R) & = BS(X, R) \\
\downarrow & \downarrow & \downarrow \\
B(X, R) & \quad G(X, R)
\end{align*}
\]

The advantage of working with the category \( G(X, R) \) is that one can readily localize to geometrically defined subobjects.

2.4.5. Lemma. Suppose \( G \) is a graded \( X \)-filtered module with a grading \( G \). Let \( F \) be a submodule which is split with respect to the standard filtration. Then \( F(S) = F \cap G(S) \) is a grading of \( F \).

We will call the grading of a split submodule \( F \) obtained in Lemma 2.4.5 the standard grading of the submodule.

Proof. Of course, \( F(S) = F \cap G(S) \subset F \cap G(S) = F(S) \). On the other hand, there is \( d \geq 0 \) such that \( G(S) \subset G(S[d]) \), so \( F(S) \subset F \cap G(S[d]) = F(S[d]) \).

Consider the inclusion of modules \( i: F \to G \), and take the quotient \( q: G \to H \). Both \( F \) and \( G \) are split and insular, so \( H \) is split and insular by parts (2) and (4) of Lemma 2.2.4 with respect to the quotient filtration. We define \( H(S) \) as the partial image \( qG(S) \) and give \( H(S) \) the standard filtration in \( H \). Then \( H(S) \) is split as the image of a split \( G(S) \) and insular since \( H \) is insular. Now the kernel of the epimorphism \( q: G(S) \to H(S) \) is \( F \cap G(S) \), is split by part (6) of Lemma 2.2.4. Since \( F \) is insular, \( F(S) \) is also insular. This shows that \( F(S) \) gives a grading for \( F \).

This can be promoted to the following result.
2.4.6. **Proposition.** Given a boundedly bicontrolled epimorphism \( g : G \to H \) in \( \mathbf{BS}(X, R) \), suppose \( F \) is a submodule of \( G \) which is the kernel of \( g \) in \( \mathbf{Mod}(R) \). It is given the standard filtration. If \( G \) is graded and \( F \) is split then both \( H \) and \( F \) are graded.

**Proof.** The grading for \( H \) is given by \( H(S) = gG(S)[b] \), where \( b \) is a chosen bicontrol bound for \( g \). Each \( H(S) \) is split and insular as in the proof of Lemma 2.4.3. The inclusions \( H(S) \subset gG(S)[b] \subset gG(S)[b] = H(S) \) and \( gG(S)[b] \subset gG(S[b + K]) \subset H(S[2b + K]) \) show that \( H \) is a grading. The same argument as in Lemma 2.4.9 shows that \( F(S) = F \cap G(S[b]) \) gives a grading for \( F \).

Applying this fact, we are able to characterize admissible monomorphisms in \( \mathbf{G}(X, R) \) as follows.

2.4.7. **Proposition.** The inclusion of a subobject \( i : F \to G \) in \( \mathbf{G}(X, R) \) is an admissible monomorphism if and only if \( F \) is split.

**Proof.** The cokernel \( H \) of \( i \) in \( \mathbf{U}(X, R) \) has the filtration described in the proof of Theorem 2.1.7. From parts (2) and (5) of Lemma 2.2.4, \( H \) is split and insular. In fact, \( H \) is a cokernel of \( i \) in \( \mathbf{BS}(X, R) \). From Proposition 2.4.7 \( H \) is graded, so it is also a cokernel of \( i \) in \( \mathbf{G}(X, R) \).

We will use the following convention. When \( d \leq 0 \), the notation \( S[d] \) will stand for the subset \( S \setminus (X \setminus S)[−d] \).

2.4.8. **Corollary.** Given an object \( F \) in \( \mathbf{G}(X, R) \) and a subset \( S \) of \( X \), there is a number \( K \geq 0 \) and an admissible subobject \( i : F_S \to F \) in \( \mathbf{G}(X, R) \) with the property that \( F_S \subset F(S[K]) \). Moreover, the cokernel \( q: F \to H \) has the property that \( H(X) = H((X \setminus S)[2D']) \), where \( D' \) is a splitting constant for \( F \).

**Proof.** The object \( F \) has a grading \( \mathcal{F} \) with a characteristic constant \( K \). Property (1) in Definition 2.4.1 guarantees that \( \mathcal{F}(S) \) is a split object for any subset \( S \). For the first statement, choose \( F_S = \mathcal{F}(S) \) with the grading defined in Lemma 2.4.5 and apply Proposition 2.4.7. The second statement is shown as follows. By part (2) of Lemma 2.2.4, since \( \text{fil}(q) = 0 \), if \( F \) is \( D' \)-split then \( H \) is \( D' \)-split. Let \( T = S[−D'] \), then \( T[D'] \subset S \), so

\[
H(T[D']) = qF(T[D']) \subset qF(S) \subset qF_S = 0.
\]

Using the decomposition \( X = T \cup (X \setminus T) \) we can write

\[
H(X) = H(T[D']) + H((X \setminus T)[D']) = H((X \setminus T)[D']) = H((X \setminus S)[2D']).
\]

The last three results can be summarized as follows.

2.4.9. **Theorem.** Given a graded object \( F \) in \( \mathbf{G}(X, R) \) and a subset \( S \) of \( X \), we assume that \( F \) is \( D' \)-split and \( d \)-insular and is graded by \( \mathcal{F} \). The submodules \( \mathcal{F}(S) \) have the following properties:

1. each \( \mathcal{F}(S) \) is graded by \( \mathcal{F}(T) = \mathcal{F}(S) \cap \mathcal{F}(T) \);
2. \( \mathcal{F}(S) \subset \mathcal{F}(S) \subset F(S[K]) \) for some fixed number \( K \geq 0 \);
3. suppose \( q: F \to H \) is the cokernel of the inclusion \( i : \mathcal{F}(S) \to F \), then \( H \) is supported on \( (X \setminus S)[2D'] \);
4. \( H(S[−2D′−2d]) = 0 \).
Proof. Properties (1), (2), (3) are consequences of the last four results. (4) follows from the fact that a $d$-insular filtered module is $2d$-separated, in the sense that for any pair of subsets $S$ and $T$ such that $S[2d] \cap T = \emptyset$ we have $S[d] \cap T[d] = \emptyset$ so $F(S) \cap F(T) = 0$. Now $H(S[-2d'] - 2d) \cap H((X \setminus S)[2D']) = 0$, but $H((X \setminus S)[2D']) = H(X)$, thus $H(S[-2D' - 2d]) = 0$. \hfill \Box

2.4.10. Remark. Functoriality properties in controlled theories are well-understood. As expected, bounded $G$-theory is covariantly functorial from the category of proper metric spaces and uniformly expansive maps to exact categories when $R$ is fixed and is similarly a covariant functor from Noetherian rings to exact categories when $X$ is fixed. These details become important in the construction of the equivariant theory and in specific applications. We avoid questions of functoriality in this paper as we concentrate on computational tools such as excision.

3. Fibred bounded $G$-theory

3.1. Introduction of fibred control in $G$-theory. Suppose $X$ and $Y$ are two proper metric spaces and $R$ is a Noetherian ring. The product metric $d((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}$. There is certainly the semi-abelian category $U(X \times Y, R)$, the exact category $L(X \times Y, R)$ and, further, the bounded category $BL(X \times Y, R)$. We wish to construct a larger fibred bounded category $B_X(Y)$. The result will involve a mix of features from $BL(X, A)$ and $B_S(Y, R)$ and contain $BL(X \times Y, R)$ as an exact subcategory.

One definition can be made by simply imitating $K$-theory with fibred control as described in Remark 2.4.1. It is obtained as an iterated construction from the end of section 2.1 on the level of unrestricted category as $U(X, U(Y, R))$. From Theorem 2.1.3 $U(X, U(Y, R))$ is a complete semi-abelian category with complete semi-abelian coefficients $A = U(Y, R)$. This concept has a transparent definition but is not well-suited for a fibred theory. In particular, while it does contain $C(X \times Y, R)$, it does not contain the fibred bounded category $C(X, C(Y, R))$ as a subcategory. Now we proceed to develop a different, more explicit set-up in terms of a category $U_X(Y)$ that naturally contains $U(X \times Y, R)$ and $C(X, C(Y, R))$ as exact subcategories.

3.1.1. Definition. Given an $R$-module $F$, an $(X, Y)$-filtration of $F$ is a functor $\phi_F: \mathcal{P}(X \times Y) \to \mathcal{I}(F)$ from the power set of the product metric space to the partially ordered family of $R$-submodules of $F(X \times Y)$. Whenever $F$ is given a filtration, and there is no ambiguity, we will denote the values $\phi_F(U)$ by $F(U)$. We assume that $F$ is reduced in the sense that the value on the empty subset is 0.

The associated $X$-filtered $R$-module $F_X$ is given by $$F_X(S) = F(S \times Y).$$ Similarly, for each subset $S \subset X$, one has the $Y$-filtered $R$-module $F^S$ given by $$F^S(T) = F(S \times T).$$ In particular, $F_X(T) = F(X \times T)$.

We will use the following notation generalizing enlargements in a metric space.

3.1.2. Notation. Given a subset $U$ of $X \times Y$ and a function $k: X \to [0, +\infty)$, let $$U[k] = \{(x, y) \in X \times Y \mid \text{there is } (x, y') \in U \text{ with } d(y, y') \leq k(x)\}.$$
If in addition we are given a number $K \geq 0$ then
\[ U[K, k] = \{(x, y) \in X \times Y \mid \text{there is } (x', y) \in U[k] \text{ with } d(x, x') \leq K \}. \]
So $U[k] = U[0, k]$. Notice that if $U$ is a single point $(x, y)$ then
\[ U[K, k] = x[K] \times y[k(x)] = (x, y)[K, 0] \times (x, y)[0, k(x)]. \]
More generally, one can equivalently write
\[ U[K, k] = \bigcup_{(x, y) \in U} x[K] \times y[k(x)]. \]
If $U$ is a product set $S \times T$, it will be convenient to use the notation $(S, T)[K, k]$ in place of $(S \times T)[K, k]$. More generally, because the roles of the factors are very different when working with $(X, Y)$-filtrations, we will use the notation $(X, Y)$ for the product metric space so that the order of the factors is unambiguous. Similarly, we will use the notation $(S, T)$ for the product subset $S \times T$ in $(X, Y)$.

3.1.3. **Definition.** We will refer to the pair $(K, k)$ in the notation $U[K, k]$ as the **enlargement data**.

It is clear that when $Y = \text{pt}$, $U[K, k] = U[K]$ for any function $k$ under the identification $X \times Y = X$.

3.1.4. **Notation.** Let $x_0$ be a chosen fixed point in $X$. Given a monotone function $h : [0, +\infty) \to [0, +\infty)$, there is a function $h_{x_0} : X \to [0, +\infty)$ defined by
\[ h_{x_0}(x) = h(d_X(x_0, x)). \]

3.1.5. **Definition.** Given two $(X, Y)$-filtered modules $F$ and $G$, an $R$-homomorphism $f : F(X \times Y) \to G(X \times Y)$ is **boundedly controlled** if there are a number $b \geq 0$ and a monotone function $\theta : [0, +\infty) \to [0, +\infty)$ such that
\[ (\dagger) \quad fF(U) \subset G(U[b, \theta_{x_0}]) \]
for all subsets $U \subset X \times Y$ and some choice of $x_0 \in X$. It is clear that the condition is independent of the choice of $x_0$.

The **unrestricted fibred bounded category** $U_X(Y)$ has $(X, Y)$-filtered modules as objects and the boundedly controlled homomorphisms as morphisms.

3.1.6. **Theorem.** $U_X(Y)$ is a cocomplete semi-abelian category.

First we require a very useful fact.

A morphism $f : F \to G$ in $U_X(Y)$ is **boundedly bicontrolled** if there is filtration data $b \leq 0$ and $\theta : [0, +\infty) \to [0, +\infty)$ as in Definition 3.1.5 and in addition to $(\dagger)$ one also has the containments $fF \cap G(U) \subset fF(U[b, \theta_{x_0}])$. In this case, we will use the notation $\text{fil}(f) \leq (b, \theta)$.

3.1.7. **Lemma.** Let $f_1 : F \to G$, $f_2 : G \to H$ be in $U_X(Y)$ and $f_3 = f_2f_1$.

- (1) If $f_1$, $f_2$ are boundedly bicontrolled morphisms and either $f_1 : F(X \times Y) \to G(X \times Y)$ is an epi or $f_2 : G(X \times Y) \to H(X \times Y)$ is a monic, then $f_3$ is also boundedly bicontrolled.

- (2) If $f_1$, $f_3$ are boundedly bicontrolled and $f_3$ is epic then $f_2$ is also boundedly bicontrolled; if $f_3$ is only boundedly controlled then $f_2$ is also boundedly controlled.
(3) If \( f_2, f_3 \) are boundedly bicontrolled and \( f_2 \) is monic then \( f_1 \) is also boundedly bicontrolled; if \( f_3 \) is only boundedly controlled then \( f_1 \) is also boundedly controlled.

**Proof.** Suppose \( \text{fil}(f_i) \leq (b, \theta) \) and \( \text{fil}(f_j) \leq (b', \theta') \) for \( \{i, j\} \subset \{1, 2, 3\} \), then in fact \( \text{fil}(f_{6-i-j}) \leq (b + b', \theta + \theta') \) in each of the three cases. For example, there are factorizations

\[
\begin{align*}
  f_2 G(U) &\subset f_2 f_1 F(U[b, \theta x_0]) = f_3 f_1 F(U[b, \theta x_0]) \subset H(U[b + b', \theta x_0 + \theta' x_0]) \\
  f_2 G(X) \cap H(U) &\subset f_3 f_1 F(U[b', \theta' x_0]) = f_2 f_1 F(U[b', \theta' x_0]) \subset f_2 G(U[b + b', \theta x_0 + \theta' x_0])
\end{align*}
\]

which verify part 2 with \( i = 1, j = 3 \).

**Proof of Theorem 3.1.7** The additive properties are inherited from \( \text{Mod}(R) \), so the biproduct is given by the filtration-wise operation \((F \oplus G)(U) = F(U) \oplus G(U)\) in \( \text{Mod}(R) \). For any boundedly controlled morphism \( f: F \to G \), the kernel of \( f \) in \( \text{Mod}(R) \) has the standard \((X, Y)-filtration\) \( K \) where \( K(S) = \ker(f) \cap F(S) \) which gives the kernel of \( f \) in \( \text{U}_X(Y) \). The canonical monic \( \kappa: K \to F \) has filtration data \((0, 0)\) and is therefore boundedly bicontrolled. It follows from part 3 of Lemma 3.1.7 that \( K \) has the universal properties of the kernel in \( \text{U}_X(Y) \).

Similarly, let \( I \) be the standard \((X, Y)-filtration\) of the image of \( f \) in \( \text{Mod}(R) \) by \( I(U) = \text{im}(f) \cap G(U) \). Then there is a presheaf \( C \) over \((X, Y)\) with \( C(U) = G(U) / I(U) \) for \( U \subset (X, Y) \). Of course \( C(X \times Y) \) is the cokernel of \( f \) in \( \text{Mod}(R) \). Consider an \((X, Y)-filtered\) object \( \overline{C} \) associated to \( C \) given by \( \overline{C}(U) = \text{im}(C(U, X \times Y)) \) and \( \pi: G(X \times Y) \to C(X \times Y) \) gives a boundedly bicontrolled morphism \( \pi: G \to \overline{C} \) of filtration \((0, 0)\) since 

\[
\text{im}(\pi G(U, X \times Y)) = \text{im}(C(U, X \times Y)) = \overline{C}(U).
\]

This in conjunction with part 2 of Lemma 3.1.7 also verifies the universal cokernel properties of \( \overline{C} \) and \( \pi \) in \( \mathbf{U}_X(Y) \).

We mention one useful perspective on \( \mathbf{U}_X(Y) \).

**3.1.8. Proposition.** Suppose \( f: F \to G \) in \( \mathbf{U}_X(Y) \) is boundedly controlled with control data \((b, \theta)\). Then

1. \( f \) is bounded by \( b \) when viewed as a morphism \( F_X \to G_X \) in \( \mathbf{U}(X, R) \), and
2. for each bounded subset \( S \subset X \), the restriction \( f\vert_S: F_X(S) \to G_X(S[b]) \) is bounded when viewed as a morphism \( F^S \to G^{S[b]} \) of \( Y\)-filtered modules in \( \mathbf{U}(Y) \).

**Proof.** If \( f: F \to G \) is \((b, \theta)-controlled\) then for any subset \( S \subset X \) we have \( f F_X(S) \subset G((S, Y)[b, \theta x_0]) \subset G(S[b], Y) = G_X(S[b]). \) So \( f: F_X \to G_X \) is bounded by \( b \). Now for a given subset \( S \subset X \), let us define \( \theta_S = \sup_{x \in S} \theta x_0(x) \). Then \( f F_X(S)(T) = f F(S, T) \subset G(S[b], T[\theta_S]) = G_X(S[b])(T[\theta_S]) \) verifying that \( f\vert_S: F^S \to G^{S[b]} \) is bounded by \( \theta_S \).

**3.2. Properties of fibred objects.**

**3.2.1. Definition.** An \((X, Y)-filtered\) module \( F \) is called

- **lean** or \((D, \Delta)-lean** if there is a number \( D \geq 0 \) and a monotone function \( \Delta: [0, +\infty) \to [0, +\infty) \) so that

\[
F(U) \subset \sum_{(x, y) \in U} F(x \vert D \times y(\Delta x_0(x))) = \sum_{(x, y) \in U} F((x, y)\vert D, \Delta x_0)
\]
for any subset $U$ of $X \times Y$,

- split or $(D', \Delta')$-split if there is a number $D' \geq 0$ and a monotone function $\Delta': [0, +\infty) \to [0, +\infty)$ so that
  \[
  F(U_1 \cup U_2) \subset F(U_1[D', \Delta'_x]) + F(U_2[D', \Delta'_x])
  \]
  for each pair of subsets $U_1$ and $U_2$ of $X \times Y$,

- lean/split or $(D, \Delta')$-lean/split if there is a number $D \geq 0$ and a monotone function $\Delta': [0, +\infty) \to [0, +\infty)$ so that
  - the $X$-filtered module $F_X$ is $D$-lean, while
  - the $(X, Y)$-filtered module $F$ is $(D, \Delta')$-split,

- insular or $(d, \delta)$-insular if there is a number $d \geq 0$ and a monotone function $\delta: [0, +\infty) \to [0, +\infty)$ so that
  \[
  F(U_1) \cap F(U_2) \subset F(U_1[d, \delta_x] \cap U_2[d, \delta_x])
  \]
  for each pair of subsets $U_1$ and $U_2$ of $X \times Y$.

### 3.2.2. Proposition

Suppose $F$ is an $(X, Y)$-filtered $R$-module.

1. If $F$ is $(D, \Delta)$-lean then the corresponding $X$-filtered module $F_X$ defined by assigning $F_X(S) = F(S \times Y)$ is $D$-lean.

2. Similarly, if $F$ is $(d, \delta)$-insular then $F_X$ is $d$-insular.

3. If $F$ is $(D, \Delta)$-lean then it is $(D, \Delta)$-split and, further, $(D, \Delta)$-lean/split.

4. An $(X, Y)$-filtered module $F$ which is lean/split and insular can be thought of as an object $F_X$ of $\mathbf{L}(X, R)$.

**Proof.** (1) Since $(x, y)[D, \Delta_x] \subset x[D] \times Y$, we have
\[
F_X(S) \subset \sum_{x \in S} \sum_{y \in Y} F((x, y)[D, \Delta_x]) \subset \sum_{x \in S} F_X(x[D]).
\]
(2) $F(\sum_{y \in Y} F_X(T)) \subset F(S[d] \times Y \cap \sum_{y \in Y} T[d] \times Y) = F_X(S[d] \cap \sum_{y \in Y} T[d])$.

3. The split property follows directly from definitions, and so the lean/split property follows in view of part (1).

4. (4) follows from (2). \hfill \Box

### 3.2.3. Definition

There are two subcategories nested in $\mathbf{U}_X(Y)$:

- $\mathbf{LS}_X(Y)$ is the full subcategory of $\mathbf{U}_X(Y)$ on objects $F$ that are lean/split and insular,
- $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{LS}_X(Y)$ on objects $F$ such that $F(U)$ is a finitely generated submodule whenever $U \subset X \times Y$ is bounded. Equivalently, the subcategory $\mathbf{B}_X(Y)$ is full on objects $F$ such that all $Y$-filtered modules $F_S$ associated to bounded subsets $S \subset X$ are locally finitely generated.

Clearly, $\mathbf{B}_X(Y)$ is a generalization of the bounded category $\mathbf{B}(X, R)$: if $Y = \text{pt}$ then $\mathbf{B}_X(Y)$ is precisely $\mathbf{B}(X, R)$. On the other hand, if $X = \text{pt}$ then $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{BS}(Y, R)$ on locally finitely generated objects.

We proceed to define appropriate exact structures in these categories.

### 3.2.4. Definition

Let the *admissible monomorphisms* in $\mathbf{U}_X(Y)$ be the boundedly bicontrolled homomorphisms $m: F_1 \to F_2$ such that the module homomorphism $F_1(X \times Y) \to F_2(X \times Y)$ is a monomorphism. Let the *admissible epimorphisms* be the boundedly bicontrolled homomorphisms $e: F_1 \to F_2$ such that $F_1(X \times
Those other words, the exact structure \( \mathcal{E} \) is isomorphic to the \( \mathcal{E} \) possessing filtration-wise restrictions. Therefore, the class of admissible epimorphisms is closed under base change by \( j \).

**3.2.5. Theorem.** \( \mathcal{U}_X(Y) \) is a Quillen exact category.

**Proof.** We will verify the axioms for exact structures due to Quillen with some simplifications due to B. Keller \[11, 12\], cf. section 2 of [1].

It follows from Lemma 3.1.7 that the collections of admissible monomorphisms and admissible epimorphisms are closed under composition and that any short exact sequence isomorphic to some sequence in \( \mathcal{E} \).

Now suppose we are given an exact sequence \( F' \xrightarrow{i} F \xrightarrow{j} F'' \) in \( \mathcal{E} \) and a morphism \( f: A \rightarrow F'' \) in \( \mathcal{U}_X(Y) \). Let \((b_j, \theta_j)\) be some injection data for \( j \) as a boundedly controlled epi and let \((bf, \theta_f)\) be some control data for \( f \) as a boundedly controlled map. There is a base change diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{i} & F \\
\downarrow & & \downarrow \quad f' \\
F' & \xrightarrow{j} & F''
\end{array}
\]

where \( m: F \times_f A \rightarrow F \oplus A \) is the kernel of the epi \( j \circ pr_1 - f \circ pr_2: F \oplus A \rightarrow F'' \) and \( f' = pr_1 \circ m, j' = pr_2 \circ m \). The \((X,Y)\)-filtration on \( F \times_f A \) is the standard filtration as a subobject of the product \( F \times A \). The induced map \( j' \) has the same kernel as \( j \) and is bounded by 0. In fact, \( fA(U) \subset E''(U|bf, \theta_{f,x_0}) \), so \( fA(S) \subset jE(U|bf + bj, \theta_{f,x_0} + \theta_{j,x_0}) \), and

\[
\text{im}(j') \cap A(U) \subset j'(E \times_f A)(U|bf + bj, \theta_{f,x_0} + \theta_{j,x_0})).
\]

This shows that \( j' \) is boundedly bicontrolled with filtration data \((bf + bj, \theta_{f,x_0} + \theta_{j,x_0})\).

Therefore, the class of admissible epimorphisms is closed under base change by arbitrary morphisms in \( \mathcal{U}_X(Y) \). Cobase changes by admissible monomorphisms are similar. \( \square \)

**3.2.6. Proposition.** The admissible monomorphisms are precisely the morphisms isomorphic in \( \mathcal{U}_X(Y) \) to the filtration-wise monomorphisms and the admissible epimorphisms are those morphisms isomorphic to the filtration-wise epimorphisms. In other words, the exact structure \( \mathcal{E} \) in \( \mathcal{U}_X(Y) \) consists of sequences isomorphic to those

\[
E' : \quad E' \xrightarrow{i} E \xrightarrow{j} E''
\]

which possess filtration-wise restrictions

\[
E'(U) : \quad E'(U) \xrightarrow{i} E(U) \xrightarrow{j} E''(U)
\]

for all subsets \( U \subset (X,Y) \), and each \( E'(U) \) is an exact sequence of \( R \)-modules.

**Proof.** Each of the sequences \( E' \) in the statement is an exact sequence in \( \mathcal{U}_X(Y) \) because the restriction \( i: E'(U) \rightarrow E(U) \) is monic and \( j: E(U) \rightarrow E''(U) \) is epic, therefore \( i: E' \rightarrow E \) and \( j: E \rightarrow E'' \) are both bicontrolled of filtration \((0,0)\).
Suppose $F'$ is a sequence isomorphic to such $F''$, so there is a commutative diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{f} & F \\
\cong & & \cong \\
E' & \xrightarrow{i} & E \\
\end{array}
\]

Then $f$ and $g$ are compositions of two isomorphisms (which are clearly boundedly bicontrolled) which are either preceded by a boundedly bicontrolled monic or followed by a boundedly bicontrolled epi. By part (1) of Lemma 3.1.4 both $f$ and $g$ are boundedly bicontrolled.

Now suppose $F'$ is an exact sequence in $\mathcal{E}$. Let $K = \ker(g)$ and $C = \operatorname{coker}(f)$, then we obtain a commutative diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{f} & F \\
\cong & & \cong \\
K & \xrightarrow{i} & F \\
\end{array}
\]

where the vertical maps are the canonical isomorphisms. By the construction of kernels and cokernels in the proof of Theorem 3.1.6, there are exact sequences $K(U) \xrightarrow{f} F(U) \xrightarrow{g} C(U)$ for all subsets $U \subset (X, Y)$.

3.2.7. Proposition. In the exact category $U_X(Y)$,

1. the lean/split objects are closed under extensions,
2. the insular objects are closed under extensions.

\[\text{Suppose } E' \xrightarrow{i} E \xrightarrow{b} E'' \text{ is an exact sequence in } U_X(Y).\]

3. If the object $E$ is lean/split then $E''$ is lean/split.
4. If $E$ is insular then $E'$ is insular.
5. Suppose $E$ is insular then $E''$ is insular if $E'$ is lean/split.

Proof. All parts are proved by adapting the proofs of Lemmas 2.2.3 and 2.2.4. To illustrate, suppose that in the exact sequence, as given in the statement, $(b, \theta)$ is common filtration data for $F$ and $G$ and both $E'$ and $E''$ are $(D, \Delta')$-lean/split. For the first statement of part (2), notice that $E_X$ is $(4b+D)$-lean by part (1) of Lemma 2.2.3, so we need to verify that split objects are closed under extensions. Consider two subsets $U_1$ and $U_2$ of $X \times Y$. Then

\[
gE(U) \subset E''((U_1 \cup U_2)[b, \theta_0]) = E''(U_1[b, \theta_0] \cup U_2[b, \theta_0]) \subset E''(U_1 [b + D, \theta_0 + \Delta'_x]) + E''(U_2 [b + D, \theta_0 + \Delta'_y]),
\]

Therefore

\[
E(U) \subset E(U_1 [2b + D, 2\theta_0 + \Delta'_x]) + E(U_2 [2b + D, 2\theta_0 + \Delta'_y]) + fE(U_1 [3b + 2D, 3\theta_0 + 2\Delta'_x]) + fE(U_2 [3b + 2D, 3\theta_0 + 2\Delta'_y]) \\
\subset E(U_1 [4b + 2D, 4\theta_0 + 2\Delta'_x]) + E(U_2 [4b + 2D, 4\theta_0 + 2\Delta'_y]),
\]

showing that $E$ is $(4b + 2D, 4\theta + 2\Delta')$-lean/split. \qed
3.2.8. **Theorem.** \(\text{LS}_X(Y)\) is closed under extensions in \(U_X(Y)\). In turn, \(B_X(Y)\) is closed under extensions in \(\text{LS}_X(Y)\). Therefore, \(B_X(Y)\) is an exact category, and the inclusion \(e \colon C_X(Y) \rightarrow B_X(Y)\) is an exact embedding.

**Proof.** The first statement follows from parts (2) and (3) of Proposition 3.2.7. Suppose \(f : F \rightarrow G\) is an isomorphism with \(\text{fil}(f) \leq (b, \theta)\) and \(G\) is locally finitely generated, then \(F(U)\) is a finitely generated submodule of \(G(U[b, \theta])\) for any bounded subset \(U \subset X \times Y\) since \(R\) is Noetherian.

If \(F' \xrightarrow{f} F \xrightarrow{g} F''\) is an exact sequence in \(\text{LS}_X(Y)\), \(F'\) and \(F''\) are locally finitely generated, and \((b, \theta)\) is common filtration data for \(f\) and \(g\), then \(gF(U)\) is a finitely generated submodule of \(F''(U[b, \theta])\) for any bounded subset \(U\). The kernel of the restriction of \(g\) to \(F(U)\) is a finitely generated submodule of \(F'(U[b, \theta])\), so the extension \(F(U)\) is finitely generated. \(\square\)

3.2.9. **Remark.** (1) There is certainly an exact embedding \(\iota \colon B(X \times Y, R) \rightarrow B_X(Y)\) which is given by the identity on objects. Because of the relaxation of the control conditions on homomorphisms, the morphism sets in the image of \(\iota\) are in general properly smaller than in \(B_X(Y)\). However \(\iota\) is also proper on objects. For example, the lean objects in \(\text{BL}(X \times Y, R)\) are generated by the submodules \(f(S \times T)\) where the diameters of \(S\) and \(T\) are uniformly bounded from above. This is different from the weaker condition in \(B_X(Y)\).

(2) While there is no functor between the categories \(U_X(Y)\) and \(U(X, U(Y))\), there is a “forgetful” function associating to objects of \(U_X(Y)\) some objects of \(U(X, U(Y))\). It is defined by \(\phi(F)(S) = F(S, Y)\) with the \(Y\)-filtration given by \(F(S, Y)(T) = F(S, T)\).

This relationship can be made much more fruitful if the nature of the objects in \(U(X, U(Y))\) is shifted to be functors from the category of bounded subsets of \(X\) to subobjects of \(U(Y)\) leading to a category that can be thought of as \(B(X, B(Y, R))\). It turns out that on this level there is a well-defined exact functor \(B_X(Y) \rightarrow B(X, B(Y, R))\). We don’t require this type of theory but it has been developed and applied in [1, 3].

3.3. **Fibrewise restriction.** We begin to prepare for the development of fibred localization exact sequences and fibred bounded excision theorems. The model for localization and fibration theorems in controlled \(G\)-theory [6, sections 3 and 4] can be implemented here as well.

There are two complementary ways to introduce support in \(B_X(Y)\).

(1) Let \(B_{<Z}(Y)\) be the full subcategory of \(B_X(Y)\) on objects \(F\) supported near \(Z\) when viewed as objects \(F_X\) in \(\text{BL}(X, R)\). In other words, \(F\) is an object of \(B_{<Z}(Y)\) if \(F_X \subset F_X(Z[d]) = F(Z[d] \times Y)\) for some number \(d \geq 0\).

(2) Let \(B_X(Y)_{<C}\) be the full subcategory of \(B_X(Y)\) on objects \(F\) such that

\[
F(X, Y) \subset F((X, C)[r, \rho_{x_0}])
\]

for some number \(r \geq 0\) and an order preserving function \(\rho : [0, +\infty) \rightarrow [0, +\infty)\).

The first version of support is a straightforward generalization of support for geometric modules that was exploited in [6]. In this paper we are more interested in the latter, fibrewise version (2) of support.
3.3.1. **Proposition.** Suppose \( F \) is a \((D, \Delta)\)-lean/split object of \( \mathcal{B}_X(Y) \). The following are equivalent statements.

1. \( F \) is an object of \( \mathcal{B}_X(Y)_{<C} \).
2. There is a number \( k \geq 0 \) and an order preserving function \( \lambda : \mathcal{B}(X) \to [0, +\infty) \) such that
   \[
   F^S \subseteq F^{S[k]}(C[\lambda(S)])
   \]
   for all bounded subsets \( S \subseteq X \).
3. There is a number \( k \geq 0 \) and a monotone function \( \Lambda : [0, +\infty) \to [0, +\infty) \)
   such that
   \[
   F^x[D] \subseteq F^{x[D+k]}(C[\Lambda_{x_0}(x)])
   \]
   for all \( x \in X \).

**Proof.** (2) \( \iff \) (3): If \( F \) satisfies (2) then \( F^x[D] \subseteq F^{x[D+k]}(C[\lambda(x[D])]) \). It suffices to define \( \Lambda \) such that \( \lambda(x[D]) \leq \Lambda_{x_0}(x) = \Lambda(d(x_0, x)) \). Since \( x[D] \leq x_0[d(x_0, x) + D] \) and \( \lambda \) is order preserving, one can take \( \Lambda(r) = \lambda(x_0[r + D]) \).

In the opposite direction, given a bounded subset \( S \subseteq X \),
\[
F^S \subseteq \sum_{x \in S} F^x[D] \subseteq \sum_{x \in S} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \subseteq F^{x[D+k]}(C[\lambda(S)])
\]
when \( \lambda(S) = \sup\{\Lambda_{x_0}(x) \mid x \in S\} \).

(1) \( \iff \) (3): If \( F \) is in \( \mathcal{B}_X(Y)_{<C} \) then \( F^x[D] \subseteq F((X, C)[r, \rho]) \), so
\[
F^x[D] \subseteq F^x[D] \cap F((X, C)[r, \rho]).
\]

If \( F \) is \((d, \delta)\)-insular then
\[
F^x[D] \subseteq F((x, C)[D + r + d, \rho + \delta]) \subseteq F^{x[D+d+r]}(C[\Lambda_{x_0}(x)])
\]
for \( \Lambda(a) = \sup\{(\delta + \rho)(z) \mid d(x_0, y) \leq a + D + d + r\} \).

In the opposite direction, we have
\[
F \subseteq \sum_{x \in X} F^x[D] \subseteq \sum_{x \in X} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \subseteq F((X, C)[D + k, \Lambda_{x_0}])
\]
for an object \( F \) of \( \mathcal{B}_X(Y) \) satisfying (3). \( \Box \)

3.3.2. **Definition.** A **Serre subcategory** of an exact category is a full subcategory which is closed under exact extensions and closed under passage to admissible subobjects and admissible quotients.

Note that this property is relative to the choice of exact structure.

3.3.3. **Proposition.** \( \mathcal{B}_X(Y)_{<C} \) is a Serre subcategory of \( \mathcal{B}_X(Y) \).

**Proof.** First we show closure under exact extensions. Let \( F \xrightarrow{f} G \xrightarrow{g} H \) be an exact sequence in \( \mathcal{B}_X(Y) \), let \((b, \theta)\) be common set of filtration data for \( f \) and \( g \), and assume all objects be \((D, \Delta)\)-lean/split. We assume that \( F \) and \( H \) are objects of \( \mathcal{B}_X(Y)_{<C} \), so there is a number \( r \geq 0 \) and a monotone function \( \rho : [0, +\infty) \to [0, +\infty) \) such that at the same time \( F(X, Y) = F((X, C)[r, \rho_{x_0}]) \) and \( H(X, Y) = H((X, C)[r, \rho_{x_0}]) \) for some choice of a base point \( x_0 \) in \( X \). Therefore
\[
fF(X, Y) = fF((X, C)[r, \rho_{x_0}]) \subseteq G((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]).
\]
In particular, the image $I = \operatorname{im}(f)$ with the standard filtration $I^S(T) = I \cap G^S(T)$ is an object of $\mathcal{B}_X(Y)_{<C}$. Now

$$H(X,Y) = gG(X,Y) \cap H((X,C)[r,\rho_{x_0}])) \subset gG((X,C)[r + b,\rho_{x_0} + \theta_{x_0}])).$$

Let $L = G((X,C)[r + b,\rho_{x_0} + \theta_{x_0}])$ viewed as a subobject of $G$ with the standard filtration. Since $G = I + L$ for any submodule $L$ with $g(L) = H$, we have

$$G(X,Y) = G((X,C)[r + b,\rho_{x_0} + \theta_{x_0}])).$$

so $G$ is an object of $\mathcal{B}_X(Y)_{<C}$.

Suppose $f: F \rightarrow G$ is an admissible monomorphism in $\mathcal{B}_X(Y)$, which is a boundedly bicontrolled monic with $\operatorname{fil}(f) \leq (b,\theta)$, $F$ is $(D',\Delta')$-lean/split, $G$ is $(D,\Delta)$-lean/split for some $D \geq D' + b$, and $G$ is $(d,\delta)$-insular.

If $G$ is an object of $\mathcal{B}_X(Y)_{<C}$, according to Proposition 3.3.1,

$$G^S \subset G^{S[k]}(C[\lambda(S)])$$

for some number $k \geq 0$, an order preserving function $\lambda: \mathcal{B}(X) \rightarrow [0,\infty)$, and all bounded subsets $\mathcal{S} \subset X$. Then

$$fF^{x[D']} \subset G^{x[D' + b]} \subset G^{x[D' + b + k]}(C[\lambda(x[D' + b])]) \subset G^{x[D + k]}(C[\lambda(x[D])]),$$

using the fact that $\lambda$ is order preserving. Since

$$G^{x[D + k]}(Y - C[\lambda(x[D + k])] + \Delta(x[D]) + \theta(x[D']) + 2\delta(x[D + k])) = 0,$$

we have

$$F^{x[D']} (Y - C[\lambda(x[D + k])] + \Delta(x[D]) + 2\theta(x[D']) + 2\delta(x[D + k])) = 0.$$

Therefore

$$F^{x[D']} \subset F^{x[D']} (C[\lambda(x[D])] + \Delta(x[D]) + \Delta'(x[D']) + 2\theta(x[D']) + 2\delta(x[D + k]),$$

so $F$, which is generated by $F^{x[D']}$, is also an object of $\mathcal{B}_X(Y)_{<C}$.

On the other hand, let $g: G \rightarrow H$ be an admissible quotient with $\operatorname{fil}(g) \leq (b,\theta)$ and suppose $G$ is an object of $\mathcal{B}_X(Y)_{<C}$ so that there is a number $r \geq 0$ and a monotone function $\rho: [0,\infty) \rightarrow [0,\infty)$ such that

$$G(X,Y) = G((X,C)[r,\rho_{x_0}])).$$

This implies that

$$H(X,Y) = gG(X,Y) \subset H((X,C)[r + b,\rho_{x_0} + \theta_{x_0}])).$$

so $H$ is also in $\mathcal{B}_X(Y)_{<C}$. \hfill \Box

3.4. Fibrewise gradings. The gradings from Definition 2.4.1 can be generalized to gradings of objects from $\mathcal{B}_X(Y)$.

3.4.1. Definition. Given an object $F$ of $\mathcal{B}_X(Y)$, a grading of $F$ is a covariant functor $F: \mathcal{P}(X,Y) \rightarrow \mathcal{I}(F)$ with the following properties:

1. if $F(C)$ is given the standard filtration, it is an object of $\mathcal{B}_X(Y)$,
2. there is an enlargement data $(K,k)$ such that
$$F(C) \subset F(C) \subset F(C[K,k_{x_0}]),$$

for all subsets $C$ of $(X,Y)$.

3.4.2. Remark. If $C = (X,S)$ then $F(C)$ is an object of $\mathcal{B}_X(Y)_{<S}$. 
We are concerned with localizations to a specific type of subspaces of \((X,Y)\). This makes the following partial gradings sufficient and easier to work with.

3.4.3. **Definition.** Let \(\mathcal{M}^{\geq 0}\) be the set of all monotone functions \(\delta: [0, +\infty) \to [0, +\infty)\). Let \(\mathcal{P}_X(Y)\) be the subcategory of \(\mathcal{P}(X,Y)\) consisting of all subsets of the form \((X,C)[D,\delta_{x_0}]\) for some choices of a subset \(C \subset Y\), a number \(D \geq 0\), and a function \(\delta \in \mathcal{M}^{\geq 0}\).

Given an object \(F\) of \(B_X(Y)\), a \(Y\)-grading of \(F\) is a functor \(\mathcal{F}: \mathcal{P}_X(Y) \to \mathcal{T}(F)\) with the following properties:

1. the submodule \(\mathcal{F}((X,C)[D,\delta_{x_0}])\) with the standard filtration is an object of \(B_X(Y)\),
2. there is an enlargement data \((K,k)\) such that \(\mathcal{F}((X,C)[D,\delta_{x_0}]) \subset \mathcal{F}((X,C)[D,\delta_{x_0}]) \subset \mathcal{F}((X,C)[D + K,\delta_{x_0} + k_{x_0}])\),

for all subsets in \(\mathcal{P}_X(Y)\).

Since \(U[D + K,\delta_{x_0} + k_{x_0}] = U[D,\delta_{x_0}][K,k_{x_0}]\) for general subsets \(U\), the third, largest submodule is independent of the choice of \(D, \delta_{x_0}\).

We say that an object \(F\) of \(B_X(Y)\) is \(Y\)-graded if there exists a \(Y\)-grading of \(F\), but the grading itself is not specified, and define \(G_X(Y)\) as the full subcategory of \(B_X(Y)\) on \(Y\)-graded filtered modules.

3.4.4. **Proposition.** The \(Y\)-graded objects in \(B_X(Y)\) are closed under isomorphisms. The subcategory \(G_X(Y)\) is closed under extensions in \(B_X(Y)\). Therefore, \(G_X(Y)\) is an exact subcategory of \(B_X(Y)\).

**Proof.** Suppose \(f: F \to F'\) is an isomorphism between a \(Y\)-graded module \(F\) and \(F'\) in \(B_X(Y)\). If \(\text{fil}(f) \leq (b,\theta)\) and \(\mathcal{F}\) is a \(Y\)-grading for \(F\) then \(\mathcal{F}((X,C)[D,\delta_{x_0}] = f\mathcal{F}((X,C)[D + B,\delta_{x_0} + k_{x_0} + \theta_{x_0}])\).

Let \(f: \xrightarrow{\mathcal{F}} G \xrightarrow{g} H\) be an exact sequence in \(B_X(Y)\). Suppose \(\text{fil}(f) \leq (b,\theta)\) and \(\text{fil}(g) \leq (b,\theta)\) for the same set of bicontrol bound data and suppose that \(F'\) and \(H\) are graded modules in \(G_X(Y)\) with the associated functors \(\mathcal{F}\) and \(\mathcal{H}\).

The assignment \(G(U) = f\mathcal{F}(U[3b,3\theta]) + G(S[2b,2\theta]) \cap g^{-1}\mathcal{H}(S[b,\theta])\)
gives a grading of \(G\) as an object of \(G_X(Y)\). \(\square\)

As with the category \(G(X,R)\), the advantage of working with \(G_X(Y)\) as opposed to \(B_X(Y)\) is that we are able to localize to the grading subobjects associated to subsets from the family \(\mathcal{P}_X(Y)\).

3.4.5. **Lemma.** Let \(F\) be a submodule of a \(Y\)-filtered module \(G\) in \(G_X(Y)\) which is lean/split with respect to the standard filtration. Then \(\mathcal{F}(U) = F \cap \mathcal{G}(U)\) is a \(Y\)-grading of \(F\).

We will call this induced \(Y\)-grading of \(F\) the \textit{standard \(Y\)-grading} of the submodule.

**Proof.** The proof is reduced to checking that \(\mathcal{F}(U)\) is an object of \(B_X(Y)\) for each subset \(U \in \mathcal{P}_X(Y)\). Suppose \(i: F \to G\) is the inclusion and \(q: G \to H\) is the quotient of \(i\). Since \(F\) is insular by part (4) of Proposition 3.2.7, both \(F\) and \(G\) are
lean/split and insular. Thus $H$ is lean/split and insular by parts (3) and (5) of 3.2.7. Let $\mathcal{H}(U) = g\mathcal{G}(U)$ with the standard filtration in $H$. Then $\mathcal{H}(U)$ is lean/split by part (3) and insular as a submodule of insular $H$. The kernel $\mathcal{F}(U)$ of the filtration $(0, 0)$ map $q: \mathcal{G}(U) \to \mathcal{H}(U)$ is lean/split by part (5) of 3.2.7 and is insular as a submodule of insular $F$. Locally finite generation of $\mathcal{F}(U)$ follows from the same property of $\mathcal{G}(U)$.

3.4.6. Proposition. Suppose $g: G \to H$ is a boundedly bicontrolled epimorphism in $B_X(Y)$ and suppose $F$ is the kernel of $g$ in $\textbf{Mod}(R)$. If $G$ is $Y$-graded and $F$ is lean/split with respect to the standard $Y$-filtration then both $H$ and $F$ are $Y$-graded.

Proof. The $Y$-grading for $H$ is given by $\mathcal{H}(U) = g\mathcal{G}(U[b, \theta])$, where $(b, \theta)$ is a chosen set of filtration data for $g$. The argument for Lemma 3.4.6 applies directly to show that $H$ is indeed a grading of $H$, and the assignment $\mathcal{F}(U) = F \cap \mathcal{G}(U[b, \theta])$ gives a $Y$-grading of $F$.

This allows to characterize admissible monomorphisms in $G_X(Y)$.

3.4.7. Proposition. The inclusion of a subobject $i: F \to G$ in $G_X(Y)$ is an admissible monomorphism if and only if $F$ is lean/split.

Proof. Let $H$ be a cokernel of $i$ in $U_X(Y)$. As verified in Lemma 3.4.6, $H$ is lean/split and insular and is, in fact, a cokernel of $i$ in $B_X(Y)$. From Proposition 3.4.6, $H$ is graded, so it is also a cokernel of $i$ in $G(X, R)$.

In the case $K \leq 0$ and $k$ a non-positive function, we will use notation $U[K, k]$ for the subset $U \setminus ((X, Y) \setminus U)[−K, −k]$ of $(X, Y)$. The following is a direct analogue of Corollary 2.4.8 with exactly the same proof.

3.4.8. Corollary. Given an object $F$ in $G_X(Y)$ and a subset $U$ from the family $P_X(Y)$, there is a set of enlargement data $(K, k)$ and an admissible subobject $i: F_U \to F$ in $G_X(Y)$ with the property that $F_U \subset F(U[K, k])$. If $G$ is $(D, \Delta')$-lean/split then the quotient $q: F \to H$ of the inclusion has the property that $H((X, Y) \setminus U)[2D, 2\Delta']$.

Now we can summarize the preceding results.

3.4.9. Theorem. Given a graded object $F$ in $G_X(Y)$ and a subset $U$ from the family $P_X(Y)$, we assume that $F$ is $(D, \Delta')$-split and $(d, \delta)$-insular and is graded by $F$. The submodule $F(U)$ has the following properties:

1. $F(U)$ is graded by $F_U(T) = F(U) \cap F(T)$,
2. $F(U) \subset F(U) \subset F(U[K, k])$ for some fixed enlargement data $(K, k)$,
3. if $q: F \to H$ is the quotient of the inclusion $i: F(U) \to F$ and $F$ is $(D, \Delta')$-lean/split, then $H$ is supported on $(X \setminus U)[2D, 2\Delta']$,
4. $H(U[−2D − 2d, −2\Delta' − 2\delta]) = 0$.

Proof. Property (1) follows from Lemma 3.4.6. Properties (2) and (3) follow from Corollary 3.4.8. Property (1) follows from the fact that a $d$-insular filtered module is $(2d, 2\delta)$-separated, in the sense that for any pair of subsets $U$ and $V$ of $(X, Y)$ such that $U[2d, 2\delta] \cap V = \emptyset$ we have $U[d, \delta] \cap V[d, \delta] = \emptyset$ so $F(U) \cap F(V) = 0$. Now

\[ H(U[−2D − 2d, −2\Delta' − 2\delta]) \cap H(((X, Y) \setminus U)[2D, 2\Delta']) = 0, \]

but $H(((X, Y) \setminus U)[2D, 2\Delta']) = H(X, Y)$, thus $H(U[−2D − 2d, −2\Delta' − 2\delta]) = 0$. □
3.5. Localization fibration sequence. We will use the localization theorem of Schlichting [17] for Serre subcategories of exact categories. These techniques require the Serre subcategory to satisfy some additional assumptions that we verify next.

3.5.1. Definition. A class of morphisms \( \Sigma \) in an additive category \( A \) admits a calculus of right fractions if

1. the identity of each object is in \( \Sigma \),
2. \( \Sigma \) is closed under composition,
3. each diagram \( F \xrightarrow{f} G \xleftarrow{s} G' \) with \( s \in \Sigma \) can be completed to a commutative square

\[
\begin{array}{ccc}
F' & \xrightarrow{f'} & G' \\
\downarrow{t} & & \downarrow{s} \\
F & \xrightarrow{f} & G
\end{array}
\]

with \( t \in \Sigma \), and
4. if \( f \) is a morphism in \( A \) and \( s \in \Sigma \) such that \( sf = 0 \) then there exists \( t \in \Sigma \) such that \( ft = 0 \).

In this case there is a construction of the localization \( A[\Sigma^{-1}] \) which has the same objects as \( A \). The morphism sets \( \text{Hom}(F,G) \) in \( A[\Sigma^{-1}] \) consist of equivalence classes of diagrams

\[
(s,f): \quad F \xleftarrow{s} F' \xrightarrow{f} G
\]

with the equivalence relation generated by \((s_1,f_1) \sim (s_2,f_2)\) if there is a map \( h: F'_1 \rightarrow F'_2 \) so that \( f_1 = f_2h \) and \( s_1 = s_2h \). Let \((s,f)\) denote the equivalence class of \((s,f)\). The composition of morphisms in \( A[\Sigma^{-1}] \) is defined by \((s,f) \circ (t|g) = (st'|gf')\) where \( f' \) and \( t' \) fit in the commutative square

\[
\begin{array}{ccc}
F'' & \xrightarrow{f'} & G' \\
\downarrow{t'} & & \downarrow{t} \\
F & \xrightarrow{f} & G
\end{array}
\]

from axiom 3.

3.5.2. Proposition. The localization \( A[\Sigma^{-1}] \) is a category. The morphisms of the form \((\text{id} | s)\) where \( s \in \Sigma \) are isomorphisms in \( A[\Sigma^{-1}] \). The rule \( P_\Sigma(f) = (\text{id} | f) \) gives a functor \( P_\Sigma: A \rightarrow A[\Sigma^{-1}] \) which is universal among the functors making the morphisms \( \Sigma \) invertible.

Proof. The proofs of these facts can be found in Chapter I of [10]. The inverse of \((\text{id} | s)\) is \((s|\text{id})\). \(\square\)

From Proposition 3.5.1, given a subset \( C \) of \( Y \), the category \( B_X(Y)_{<C} \) is a Serre subcategory of \( B_X(Y) \).

3.5.3. Proposition. The restriction to \( Y \)-gradings in \( B_X(Y)_{<C} \) gives a full exact subcategory \( G_X(Y)_{<C} \) which is a Serre subcategory of \( G_X(Y) \).

Proof. One only needs to observe that if the modules at the ends of the exact sequence in the proof of Proposition 3.4.3 have \( Y \)-gradings in \( B_X(Y)_{<C} \) then the displayed grading of \( G \) shows that \( G \) is an object of \( G_X(Y)_{<C} \). \(\square\)
The following shorthand notation is convenient when the choice of \( C \) is clear.

**3.5.4. Notation.** The category \( \mathbf{G} \) is the exact subcategory of \( Y \)-graded objects in \( \mathbf{B}_X(Y) \). When the choice of the subset \( C \subset Y \) is understood, we will use notation \( \mathbf{C} \) for the Serre subcategory \( \mathbf{G}_X(Y)_{\subset C} \) of \( \mathbf{G} \).

**3.5.5. Definition.** Define the class of weak equivalences \( \Sigma(C) \) in \( \mathbf{G} \) to consist of all finite compositions of admissible monomorphisms with kernels in \( \mathbf{C} \) and admissible epimorphisms with kernels in \( \mathbf{C} \).

We need the class \( \Sigma(C) \) to admits calculus of right fractions. This follows from \cite{17} Lemma 1.13 as soon as we prove the following fact.

A Serre subcategory \( \mathbf{C} \) of an exact category \( \mathbf{G} \) is right filtering if each morphism \( f: F_1 \to F_2 \) in \( \mathbf{G} \), where \( F_2 \) is an object of \( \mathbf{C} \), factors through an admissible epimorphism \( e: F_1 \to \overline{F}_2 \), where \( \overline{F}_2 \) is in \( \mathbf{C} \).

**3.5.6. Lemma.** The subcategory \( \mathbf{C} = \mathbf{G}_X(Y)_{\subset C} \) of \( \mathbf{G} = \mathbf{G}_X(Y) \) is right filtering.

**Proof.** For a morphism \( f: F_1 \to F_2 \) in \( \mathbf{G} \) with \( F_2 \) in \( \mathbf{C} \), we assume that both \( F_1 \) and \( F_2 \) are \((D,\Delta')\)-lean/split and \((d,\delta)\)-insular. Suppose \( f \) is bounded by \((b,\theta)\) and let \( r \geq 0 \) and \( \rho: [0,\infty) \to [0,\infty) \) be a monotone function such that \( F_2(X,Y) \subset F_2((X,C)[r,\rho_{x_0}]) \).

Now for any characteristic set of data \((K,k)\) for the grading \( \mathcal{F}_1 \) and any subset \( R \) we have

\[
\{f\mathcal{F}(R) \subset f\mathcal{F}_1(R[K,k_{x_0}]) \subset F_2(R[K+b,k_{x_0}+\theta_{x_0}])\}.
\]

By part (3) of Theorem 3.4.3 \( F_2(R[K+b,k_{x_0}+\theta_{x_0}]) \cap F_2((X,C)[r,\rho_{x_0}]) = 0 \) for any \( R \) such that

\[
R[K+b+2D+2d,k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}] \cap (X,C)[r,\rho_{x_0}] = \emptyset.
\]

If we choose

\[
R = (X,Y) \setminus (X,C)[K+b+2D+2d+r,k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]
\]

and define \( E = \mathcal{F}_1(R) \), then \( fE = 0 \). Let \( \mathcal{T}_2 \) be the cokernel of the inclusion \( E \to F_1 \).

Then \( \mathcal{T}_2 \) is lean/split and insular and has a grading given by \( \mathcal{T}_2(S) = q\mathcal{F}_1(S[b,\theta_{x_0}]) \).

Since

\[
\mathcal{T}_2(X,Y) \subset \mathcal{T}_2((X,C)[K+b+2D+2d+r,k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]),
\]

the quotient \( \mathcal{T}_2 \) is in \( \mathbf{C} \), and \( f \) factors as \( F_1 \to \mathcal{T}_2 \to F_2 \) in the right square in the map of exact sequences

\[
\begin{array}{ccc}
E & \longrightarrow & F_1 \\
\downarrow & & \downarrow f' \\
K & \longrightarrow & \mathcal{T}_2
\end{array}
\]

as required. \( \square \)

**3.5.7. Definition.** The category \( \mathbf{G}/\mathbf{C} \) is the localization \( \mathbf{G}[\Sigma(C)^{-1}] \).

It is clear that the quotient \( \mathbf{G}/\mathbf{C} \) is an additive category, and \( P_{\Sigma(C)} \) is an additive functor. In fact, we have the following.

**3.5.8. Theorem.** The short sequences in \( \mathbf{G}/\mathbf{C} \) which are isomorphic to images of exact sequences from \( \mathbf{G} \) form a Quillen exact structure.
Proof. This will be a consequence from [17] Proposition 1.16. Since \( C \) is right filtering by Lemma 3.5.6 it remains to check that \( C \) right s-filtering in \( G \) in the following sense. A subcategory \( C \) of an exact category \( G \) is right \( s \)-filtering if given an admissible monomorphism \( f: F_1 \to F_2 \) with \( F_1 \) in \( C \), there exist \( E \) in \( C \) and an admissible epimorphism \( e: F_2 \to E \) such that the composition \( ef \) is an admissible monomorphism.

Suppose that \( F_1 \) and \( F_2 \) have the same properties as in the proof of Lemma 3.5.6 and \( \text{fil}(f) \leq (b, \theta) \). Since \( F_1 \) is in \( C \), there are \( r \geq 0 \) and a monotone function \( \rho: [0, +\infty) \to [0, +\infty) \) such that \( F_1(X, Y) \subseteq F_1((X, C)|r, \rho_{x_0}|) \). Then let \( F'_2 = F_2(T) \) where

\[
T = (X, Y) \setminus (X, C)[K + b + 2D + 2d + r, k_{x_0} + \theta_{x_0} + 2\Delta'_{x_0} + 2\delta_{x_0} + \rho_{x_0}].
\]

Define \( E \) as the cokernel of the inclusion \( F'_2 \to F_2 \) and let \( e: F_2 \to E \) be the quotient map. The composition \( ef \) is an admissible monomorphism with \( \text{fil}(ef) = \text{fil}(f) \leq (b, \theta) \).

3.5.9. Notation. If \( C \) is a subset of \( Y \) as before, \( G_X(Y, C) \) will stand for the exact category \( G/C \) and \( G_X(Y, C) \) for its Quillen \( K \)-theory.

The main tool in proving controlled excision theorems will be the following localization sequence.

3.5.10. Theorem (Theorem 2.1 of Schlichting [17]). Let \( Z \) be an idempotent complete right \( s \)-filtering subcategory of an exact category \( E \) which is full and closed under exact extensions. Then the sequence of exact categories \( Z \to E \to E/Z \) induces a homotopy fibration of Quillen \( K \)-theory spectra

\[
K(Z) \to K(E) \to K(E/Z).
\]

3.5.11. Corollary. There is a homotopy fibration

\[
G_X(Y)_{<C} \longrightarrow G_X(Y) \longrightarrow G_X(Y, C).
\]

There is a more intrinsic statement of the same fact.

3.5.12. Theorem (Localization). There is a homotopy fibration

\[
G_X(C) \longrightarrow G_X(Y) \longrightarrow G_X(Y, C).
\]

Theorem 3.5.12 is a consequence of Corollary 3.5.11 as soon as we show that \( G_X(C) \) and \( G_X(Y)_{<C} \) are weakly equivalent.

Recall that the essential full image of a functor \( F: C \to D \) is the full subcategory of \( D \) whose objects are those \( D \) that are isomorphic to \( F(C) \) for some \( C \) from \( C \).

3.5.13. Lemma. Given a pair of proper metric spaces \( C \subset Y \), there is a fully faithful embedding \( \epsilon: G_X(C) \to G_X(Y) \). The Serre subcategory \( G_X(Y)_{<C} \) is the essential full image of \( G_X(C) \) in \( G_X(Y) \). Therefore, the inclusion \( C \subset Y \) induces a weak equivalence

\[
G_X(C) \to G_X(Y)_{<C}.
\]

Proof. Suppose \( F \) is an object of \( G_X(C) \). The embedding \( \epsilon \) is given by \( \epsilon(F)(U) = F((X, C) \cap U), \epsilon(F)(S) = F((X, C) \cap U) \). It is clear that \( \epsilon(F) \) is in \( G_X(Y)_{<C} \).

To show that \( G_X(Y)_{<C} \) is the essential full image, for an object \( G \) of \( G_X(Y)_{<C} \) assume that \( C \subset G((X, C)[r, \rho_{x_0}]) \) for some number \( r \geq 0 \) and a monotone function \( \rho: [0, +\infty) \to [0, +\infty) \). Choose any set function \( \tau: (X, C)[r, \rho_{x_0}] \to X \times C \) with the properties

\[
\text{ef}
\]
Then the $Y$-filtered module $E$ associated to $G$ given by $E(S) = G(\tau^{-1}(S))$ with the grading $E(U') = G(\tau^{-1}(U'))$ is an object of $G_X(C)$. Indeed, if $G(\tau^{-1}(U'))$ is $(D, \Delta')$-lean/split and $(d, \delta)$-insular then $E(U')$ is $(D + r, \Delta' + \rho)$-lean/split and $(d + r, \delta + \rho)$-insular. The identity map is an isomorphism in $G_X(Y)$ with $\text{fil}(\text{id}) \leq (2r, 2\rho + 2r).$ \hfill $\Box$

4. Fibrewise excision theorems

4.1. Waldhausen categories and $K$-theory. Our main reference for Waldhausen $K$-theory terminology and notation is Thomason [19].

A Waldhausen category $D$ with weak equivalences $w(D)$ is often denoted by $wD$ as a reminder of the choice. A functor between Waldhausen categories is exact if it preserves the chosen zero objects, cofibrations, weak equivalences, and cobase changes. Let $D$ be a small Waldhausen category with respect to two categories of weak equivalences $v(D) \subset w(D)$ with a cylinder functor $T$ both for $vD$ and for $wD$ satisfying the cylinder axiom for $wD$. Suppose also that $w(D)$ satisfies the extension and saturation axioms. Define $vD^w$ to be the full subcategory of $vD$ whose objects are $F$ such that $0 \to F \in w(D)$. Then $vD^w$ is a small Waldhausen category with cofibrations $\text{co}(D^w) = \text{co}(D) \cap D^w$ and weak equivalences $v(D^w) = v(D) \cap D^w$. The cylinder functor $T$ for $vD$ induces a cylinder functor for $vD^w$. If $T$ satisfies the cylinder axiom then the induced functor does so too.

4.1.1. Theorem (Approximation Theorem). Let $E: D_1 \to D_2$ be an exact functor between two small saturated Waldhausen categories. It induces a map of $K$-theory spectra

$$K(E): K(D_1) \to K(D_2).$$

Assume that $D_1$ has a cylinder functor satisfying the cylinder axiom. If $E$ satisfies two conditions:

(1) a morphism $f \in D_1$ is in $w(D_1)$ if and only if $E(f) \in D_2$ is in $w(D_2),$

(2) for any object $D_1 \in D_1$ and any morphism $g: E(D_1) \to D_2$ in $D_2$, there is an object $D'_1 \in D_1$, a morphism $f: D_1 \to D'_1$ in $D_1$, and a weak equivalence $g': E(D'_1) \to D_2$ in $w(D_2)$ such that $g = g'E(f),$

then $K(E)$ is a homotopy equivalence.

Proof. This is Theorem 1.6.7 of [20]. The presence of the cylinder functor with the cylinder axiom allows to make condition (2) weaker than that of Waldhausen, see point 1.9.1 in [19]. \hfill $\Box$

4.1.2. Definition. In any additive category, a sequence of morphisms

$$E^\cdot: 0 \to E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E^n \to 0$$

is called a (bounded) chain complex if the compositions $d_{i+1}d_i$ are the zero maps for all $i = 1, \ldots, n - 1$. A chain map $f: E^\cdot \to E^\cdot$ is a collection of morphisms $f^i: E^i \to E^i$ such that $f^i d_i = d_i f^i$. A chain map $f$ is null-homotopic if there are morphisms $s_i: f^{i+1} \to E^i$ such that $f = ds + sd$. Two chain maps $f, g: E^\cdot \to E^\cdot$ are chain homotopic if $f - g$ is null-homotopic. Now $f$ is a chain homotopy equivalence
if there is a chain map \( h: E^i \to F^i \) such that the compositions \( fh \) and \( hf \) are chain homotopic to the respective identity maps.

The Waldhausen structures on categories of bounded chain complexes are based on homotopy equivalence as a weakening of the notion of isomorphism of chain complexes.

A sequence of maps in an exact category is called *acyclic* if it is assembled out of short exact sequences in the sense that each map factors as the composition of the cokernel of the preceding map and the kernel of the succeeding map.

It is known that the class of acyclic complexes in an exact category is closed under isomorphisms in the homotopy category if and only if the category is idempotent complete, which is also equivalent to the property that each contractible chain complex is acyclic, cf. [12, sec. 11].

Given an exact category \( E \), there is a standard choice for the Waldhausen structure on the category \( E' \) of bounded chain complexes in \( E \) where the degree-wise admissible monomorphisms are the cofibrations and the chain maps whose mapping cones are homotopy equivalent to acyclic complexes are the weak equivalences \( v(E') \).

The following fact is well-known, cf. point 1.1.2 in [19].

4.1.3. **Proposition.** The category \( vE' \) is a Waldhausen category satisfying the extension and saturation axioms and has cylinder functor satisfying the cylinder axiom.

4.1.4. **Example.** There are two choices for the Waldhausen structure on the category of bounded chain complexes \( G' = G(X, R)' \). One is the standard choice \( vG' \) as above. Given a subset \( C \subset Y \), another choice for the weak equivalences \( w(G') \) is the chain maps whose mapping cones are homotopy equivalent to acyclic complexes in the quotient \( G/C \).

4.1.5. **Corollary.** The categories \( vG' \) and \( wG' \) are Waldhausen categories satisfying the extension and saturation axioms and have cylinder functors satisfying the cylinder axiom.

**Proof.** All axioms and constructions, including the cylinder functor, for \( wG' \) are inherited from \( vG' \). \( \square \)

The \( K \)-theory functor from the category of small Waldhausen categories \( D \) and exact functors to the category of connective spectra is defined in terms of \( S \)-construction as in Waldhausen [20]. It extends to simplicial categories \( D \) with cofibrations and weak equivalences and inductively delivers the connective spectrum \( n \mapsto |wS^{(n)} D| \). We obtain the functor assigning to \( D \) the connective \( \Omega \)-spectrum

\[
K(D) = \Omega^\infty |wS^{(\infty)} D| = \operatorname{colim}_{n \geq 1} \Omega^n |wS^{(n)} D|
\]

representing the Waldhausen algebraic \( K \)-theory of \( D \). For example, if \( D \) is the additive category of free finitely generated \( R \)-modules with the canonical Waldhausen structure, then the stable homotopy groups of \( K(D) \) are the usual \( K \)-groups of the ring \( R \). In fact, there is a general identification of the two theories. Recall that for any exact category \( E \), the category \( E' \) of bounded chain complexes has the Waldhausen structure \( vE' \) as in Example 4.1.3.
4.1.6. **Theorem.** The Quillen $K$-theory of an exact category $\mathbf{E}$ is equivalent to the Waldhausen $K$-theory of $w\mathbf{E}'$.

**Proof.** The proof is based on repeated applications of the Additivity Theorem, cf. Thomason’s Theorem 1.11.7 from [19]. Thomason’s proof of his Theorem 1.11.7 can be repeated verbatim here. It is in fact simpler in this case since his condition 1.11.3.1 is not required. □

4.2. **Controlled excision theorems.** These are the major computational tools in controlled $K$-theory. We develop excision results $G_X(Y)$ with respect to specific coverings of the variable $Y$ in this section.

Suppose $Y_1$ and $Y_2$ are subsets of a proper metric space $Y$, and $Y = Y_1 \cup Y_2$. We use the notation $G = G_X(Y)$, $G_i = G_X(Y)_{<Y_i}$ for $i = 1$ or 2, and $G_{12}$ for the intersection $G_1 \cap G_2$. Theorem 3.5.10 can be applied to two inclusions of categories, $G_{12} \rightarrow G_1$ and $G_2 \rightarrow G$. The resulting homotopy fibrations are the rows in a commutative diagram of $K$-theory spectra

\[
\begin{array}{cccc}
K(G_{12}) & \longrightarrow & K(G_1) & \longrightarrow & K(G_1/G_{12}) \\
\downarrow & & \downarrow & & \downarrow K(I) \\
K(G_2) & \longrightarrow & K(G) & \longrightarrow & K(G/G_2)
\end{array}
\]

The vertical maps are induced by exact inclusions, including $K(I)$ induced by the exact functor $I: G_1/G_{12} \rightarrow G/G_2$ which is itself induced from the exact inclusion $G_1 \rightarrow G$.

4.2.1. **Remark.** This is precisely the commutative diagram from Cardenas-Pedersen [4, section 8] transported from bounded $K$-theory to fibred $G$-theory. Cardenas and Pedersen use Karoubi quotients and the Karoubi fibrations in order to generate their diagram. One of the crucial points in [4] is that the functor $I$ between the Karoubi quotients is an isomorphism of categories. In fibred $G$-theory the situation is more complicated: $I$ is not necessarily full and, therefore, not an isomorphism of categories. We will use the Approximation Theorem to prove that $K(I)$ is nevertheless an equivalence of spectra.

4.2.2. **Proposition.** $K(wG') \simeq K(G/C)$.

**Proof.** This follows from Lemma 2.3 in [17] as part of the proof of Theorem 3.5.10 where $K(wG')$ from Waldhausen’s Fibration Theorem is identified with the Quillen $K$-theory spectrum $K(G/C)$. □

4.2.3. **Lemma.** If $f': F' \rightarrow G'$ is a degreewise admissible monomorphism with cokernel in $\mathcal{C}$ then $f'$ is a weak equivalence in $wG'$.

**Proof.** The mapping cone $Cf'$ is quasi-isomorphic to the cokernel of $f'$, by Lemma 11.6 of [12], which is zero in $G/C$. □

The exact inclusion $I$ induces the exact functor $wG'_1 \rightarrow wG'$.

4.2.4. **Lemma.** The map $K(wG'_1) \rightarrow K(wG')$ is a weak equivalence.

**Proof.** Applying the Approximation Theorem, condition (1) is clear, so we need to check condition (2). Consider

\[
F': 0 \rightarrow F^1 \xrightarrow{\phi_1} F^2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} F^n \rightarrow 0
\]
in $G_1$ and a chain map $g: F^i \to G^i$ for some complex

$$G^i: \quad \cdots \to G^1 \xrightarrow{\psi_1} G^2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} G^n \longrightarrow 0$$

in $G$. Suppose all $F^i$ and $G^i$ are $(D, \Delta')$-lean/split and $(d, \delta)$-insular. Also assume that there is a fixed number $r \geq 0$ and a monotone function $\rho: [0, +\infty) \to [0, +\infty)$ such that $F^i(X,Y) \subset F^i((X,C)[r, \rho x])$ holds for all $0 \leq i \leq n$. If the pair $(b, \theta)$ serves as bounded control data for all $\phi_i$, $\psi_i$, and $g_i$, we define the submodule

$$F^{\psi_i} = G^i((X, Y_1)[r + 3ib, \rho x + 3i\theta x + kx])$$

and define $\xi_i: F^{\psi_i} \to F^{\psi_{i+1}}$ to be the restrictions of $\psi_i$ to $F^{\psi_i}$. This gives a chain subcomplex $(F^{\psi_i}, \xi_i)$ of $(G^i, G^i)$ in $G$ with the inclusion $i: F^{\psi_i} \to G^i$. Notice that we have the induced chain map $\overline{f}: F^i \to F^i$ in $G_1$ so that $g = iI(\overline{f})$.

Once we establish that $C^i = \text{coker}(i)$ is in $G_2$, $K(I)$ is a weak equivalence by Lemma 4.2.5.

Since

$$F^{\psi_i} \subset G^i((X, Y_1)[r + 3ib + K, \rho x + 3i\theta x + kx]),$$

each $C^i$ is supported on

$$(X, Y) \setminus Y_1)[2D + 2d - r - 3ib - K, 2\Delta'_{\rho x} + 2\delta x - \rho x - 3i\theta x - kx]\subset (X, Y_2)[2D + 2d, 2\Delta'_{\rho x} + 2\delta x],$$

cf. Lemma 5.5.6. So the complex $C^i$ is indeed in $G_2$. \hfill $\square$

The excision theorems are best stated in terms of non-connecitive deloopings of the $K$-theory spectra. Following Pedersen and Weibel we can use the same kind of diagram to first deloop $K$-theory and then reuse it to prove the excision theorem.

Let $\mathbb{R}, \mathbb{R}^{\geq 0}$, and $\mathbb{R}^{\leq 0}$ denote the metric spaces of the reals, the nonnegative reals, and the nonpositive reals with the restriction of the usual metric on the real line $\mathbb{R}$. Then there is the following instance of commutative diagram (2)

$$
\begin{array}{cccc}
G_X(Y) & \longrightarrow & G_X(Y \times \mathbb{R}^{\geq 0}) & \longrightarrow & K(G_1/G_{12}) \\
\downarrow & & \downarrow & & \downarrow \\
G_X(Y \times \mathbb{R}^{\geq 0}) & \longrightarrow & G_X(Y \times \mathbb{R}) & \longrightarrow & K(G/G_2)
\end{array}
$$

We already know that $K(I)$ is an equivalence.

4.2.5. Lemma. The spectra $G_X(Y \times \mathbb{R}^{\geq 0})$ and $G_X(Y \times \mathbb{R}^{\leq 0})$ are contractible.

Proof. This follows from the fact that these controlled categories are flasque, that is, the evident shift functor $T$ in the positive (respectively negative) direction along $\mathbb{R}^{\geq 0}$ (respectively $\mathbb{R}^{\leq 0}$) interpreted in the obvious way is an exact endofunctor, and there is a natural equivalence $1 \oplus T \cong T$. Contractibility follows from the Additivity Theorem, cf. Pedersen–Weibel [14]. \hfill $\square$

In view of Lemma 4.2.4 we obtain a map $G_X(Y) \to \Omega G_X(Y \times \mathbb{R})$ which inducess isomorphisms of $K$-groups in positive dimensions. Weak equivalences

$$\Omega^k G_X(Y \times \mathbb{R}^k) \longrightarrow \Omega^{k+1} G_X(Y \times \mathbb{R}^{k+1})$$

are obtained by iterating this construction for $k \geq 2$. 
4.2.6. **Definition.** The *nonconnective fibred bounded* $G$-theory over the pair $(X, Y)$ is the spectrum

$$G_X^{-\infty}(Y) \overset{\text{def}}{=} \text{hocolim}_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k).$$

Since $\text{BL}(X, R)$ can be identified with $\text{B}_X(\text{pt})$, this definition also gives a nonconnective delooping of the $G$-theory of $X$:

$$G^{-\infty}(X, R) = \text{hocolim}_{k>0} \Omega^k G_X(R^k).$$

The subcategory $G_X(Y \times \mathbb{R}^k)_{\leq C \times \mathbb{R}^k}$ is evidently a Serre subcategory of $G_X(Y \times \mathbb{R}^k)$ for any choice of the subset $C \subset Y$.

4.2.7. **Definition.** We define

$$G_X^{-\infty}(Y, C) \overset{\text{def}}{=} \text{hocolim}_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k)_{\leq C \times \mathbb{R}^k}.$$

We also define

$$G_X^{-\infty}(Y, C_1, C_2) \overset{\text{def}}{=} \text{hocolim}_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k)_{\leq C_1 \times \mathbb{R}^k, C_2 \times \mathbb{R}^k}.$$

4.2.8. **Theorem** (Fibrewise Bounded Excision, Version One). Suppose $Y_1$ and $Y_2$ are subsets of a metric space $Y$, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra

$$
\begin{array}{ccc}
G_X^{-\infty}(Y, Y_1, Y_2) & \longrightarrow & G_X^{-\infty}(Y) \\
\downarrow & & \downarrow \\
G_X^{-\infty}(Y) & \longrightarrow & G_X^{-\infty}(Y)
\end{array}
$$

where the maps of spectra are induced from the exact inclusions. If $Y_1$ and $Y_2$ are mutually antithetic subsets of $Y$, there is a homotopy pushout

$$
\begin{array}{ccc}
G_X^{-\infty}(Y_1 \cap Y_2) & \longrightarrow & G_X^{-\infty}(Y_1) \\
\downarrow & & \downarrow \\
G_X^{-\infty}(Y_2) & \longrightarrow & G_X^{-\infty}(Y)
\end{array}
$$

**Proof.** Let us write $S^k G$ for $G_X(Y \times \mathbb{R}^k)$ whenever $G$ is the fibred bounded category for a pair $(X, Y)$. If $C$ represents a family of coarsely equivalent subsets in a coarse covering $\mathcal{U}$ of $Y$, consider the fibration

$$G_X(C) \longrightarrow G_X(Y) \longrightarrow K(G/C)$$

from Theorem 3.5.12. Notice that there is a map $K(G/C) \to \Omega K(S^k G/S^k C)$ which is an equivalence in positive dimensions by the Five Lemma. Defining

$$G_X^{-\infty}(Y, C) = K^{-\infty}(G/C) = \text{hocolim}_{k>0} \Omega^k K(S^k G/S^k C)$$

gives an induced fibration

$$G_X^{-\infty}(C) \longrightarrow G_X^{-\infty}(Y) \longrightarrow G_X^{-\infty}(Y, C).$$
The theorem follows from the commutative diagram

\[
\begin{array}{ccc}
G^\infty_X(Y)_{<Y_1, Y_2} & \longrightarrow & G^\infty_X(Y)_{<Y_1} & \longrightarrow & K^\infty(G_1/G_{12}) \\
\downarrow & & \downarrow & & \downarrow \\
G^\infty_X(Y)_{<Y_2} & \longrightarrow & G^\infty_X(Y) & \longrightarrow & K^\infty(G/G_2)
\end{array}
\]

and the fact that \(K^\infty(G_1/G_{12}) \rightarrow K^\infty(G/G_2)\) is a weak equivalence.

From Lemma 3.5.13 one has a weak equivalence \(G^\infty_X(Y)_{<C} \simeq G^\infty_X(C)\). If \(C_1\) and \(C_2\) are coarsely antithetic then the same construction shows that \(G_X(C_1 \cap C_2) \rightarrow G_X(Y)_{<C_1, C_2}\) is onto the essential full image, and so there is a weak equivalence \(G^\infty_X(Y)_{<C_1, C_2} \simeq G^\infty_X(C_1 \cap C_2)\). This allows to substitute the terms in the commutative diagram, giving the second statement.

\[\square\]

4.3. Fibred coarse coverings. To state the excision theorems properly in the coarse geometric setting, we develop the language of fibred coarse coverings.

Two subsets \(A, B\) of \((X, Y)\) are called coarsely equivalent if there is a set of enlargement data \((K, k)\) such that \(A \subset B[K, k_{x_0}]\) and \(B \subset A[K, k_{x_0}]\). We will use the notation \(A \parallel B\) for this equivalence relation.

A family of subsets \(A\) is called coarsely saturated if it is maximal with respect to this equivalence relation. Given a subset \(A\), we denote by \(S(A)\) the smallest boundedly saturated family containing \(A\).

A collection of subsets \(\mathcal{U} = \{U_i\}\) is a coarse covering of \((X, Y)\) if \((X, Y) = \bigcup S_i\) for some \(S_i \in S(U_i)\). Similarly, \(\mathcal{U} = \{A_i\}\) is a coarse covering by coarsely saturated families if for some (and therefore any) choice of subsets \(A_i \in A_i\), \(\{A_i\}\) is a coarse covering in the above sense.

We will say that a pair of subsets \(A, B\) of \((X, Y)\) are coarsely antithetic if for any two sets of enlargement data \((D_1, d_1)\) and \((D_2, d_2)\) there exist enlargement data \((D, d)\) such that

\[A[D_1, (d_1)_{x_0}] \cap B[D_2, (d_2)_{x_0}] \subset (A \cap B)[D, d_{x_0}].\]

We will write \(A \triangleleft B\) to indicate that \(A\) and \(B\) are coarsely antithetic.

Given two subsets \(A\) and \(B\), we define

\[S(A, B) = \{A' \cap B' \mid A' \in S(A), B' \in S(B), A' \triangleleft B'\}.\]

It is easy to see that \(S(A, B)\) is a coarsely saturated family.

4.3.1. Proposition. \(S(A, B)\) is a coarsely saturated family.

Proof. Suppose \(A_1, A'_1\) and \(A_2, A'_2\) are two coarsely antithetic pairs, and \(A_1 \subset A_2[D_12, (d_{12})_{x_0}], A'_1 \subset A'_2[D'_12, (d'_{12})_{x_0}]\) for some \(D_{12}, d_{12}, D'_{12}, d'_{12}\). Then

\[A_1 \cap A'_1 \subset A_2[D_{12}, (d_{12})_{x_0}] \cap A'_2[D'_{12}, (d'_{12})_{x_0}] \subset (A_2 \cap A'_2)[D, d_{x_0}]\]

for some \((D, d)\). \[\square\]

There is the straightforward generalization to the case of a finite number of subsets of \((X, Y)\). Similarly, we write \(\hat{\cdots} A_k\) if for arbitrary sets of data \((D_i, d_i)\) there is a set of enlargement data \((D, d)\) so that

\[A_1[D_1, (d_1)_{x_0}] \cap \cdots \cap A_k[D_k, (d_k)_{x_0}] \subset (A_1 \cap \cdots \cap A_k)[D, d_{x_0}]\]

and define

\[S(A_1, \ldots, A_k) = \{A'_1 \cap \cdots \cap A'_k \mid A'_i \in S(A_i), A_1 \triangleleft \cdots \triangleleft A_k\}.\]
Identifying any coarsely saturated family $\mathcal{A}$ with $\mathcal{S}(A)$ for $A \in \mathcal{A}$, one has the coarse saturated family $\mathcal{S}(A_1,\ldots,A_k)$. We will refer to $\mathcal{S}(A_1,\ldots,A_k)$ as the coarse intersection of $A_1,\ldots,A_k$. A coarse covering $\mathcal{U}$ is closed under coarse intersections if all coarse intersections $\mathcal{S}(A_1,\ldots,A_k)$ are nonempty and are contained in $\mathcal{U}$. If $\mathcal{U}$ is a given coarse covering, the smallest coarse covering that is closed under coarse intersections and contains $\mathcal{U}$ will be called the closure of $\mathcal{U}$ under coarse intersections.

All of the terms introduced above have absolute analogues obtained by simply restricting to the case $Y = pt$. So there are, in particular, finite coarse coverings of a single metric space.

4.3.2. Proposition. If $\mathcal{U}$ is a finite coarse antithetic covering of $Y$ then $(X,\mathcal{U})$ consisting of subsets $(X,U)$, $U \in \mathcal{U}$, is a coarse antithetic covering of $(X,Y)$. If $\mathcal{U}$ is closed under coarse intersections, $(X,\mathcal{U})$ is closed under coarse intersections.

Proof. Suppose $\mathcal{U} = \{A_i\}$ so that for $A_i \in \mathcal{A}$, \{A_i\} is a coarse covering of $Y$. Then $\{(X,A_i)\}$ is a covering of $(X,Y)$. Suppose $\mathcal{U}$ is coarsely antithetic, so given numbers $d_1, d_2$ there is a number $d$ so that $A_i[d_1] \cap A_j[d_2] \subset (A_i \cap A_j)[d]$. If $d_1, d_2$ are non-decreasing functions, these values give a non-decreasing function $d$. Now given enlargement data $(D_1,d_1)$ and $(D_2,d_2)$, we have

$$(X,A_i)[D_1, (d_1)_{x_0}] \cap (X,A_j)[D_2, (d_2)_{x_0}] \subset (X,A_i \cap A_j)[D,h],$$

where $D$ can be any non-negative number, and $h$ is the function $h(x) = d(d_X(x_0,x)+D_1 + D_2)$. So $(X,\mathcal{U})$ is a coarsely antithetic covering. A similar estimate gives the last statement. 

Suppose $\mathcal{U}$ is a finite coarse covering of $Y$ closed under coarse intersections. We can define the homotopy pushout

$$\mathcal{G}_X(Y;\mathcal{U}) = \hocolim_{U \in \mathcal{U}} G_X^{-\infty}(Y)_{<U}.$$

4.3.3. Theorem (Fibrewise Bounded Excision, Version Two). There is a weak equivalence

$$\mathcal{G}_X(Y;\mathcal{U}) \simeq G_X^{-\infty}(Y).$$

Proof. Apply Theorem 4.3.3 inductively to the sets in $\mathcal{U}$. 

4.4. Relative excision theorems. Fibred $G$-theory has a useful relative version, and there are generalizations of the excision theorems to relative statements.

4.4.1. Definition. Let $Y' \in \mathcal{A}$ for a coarse covering $\mathcal{U}$ of $Y$. Let $G = \mathcal{G}_X(Y;\mathcal{U})$ and $Y' = \mathcal{G}_X(Y'_{<\mathcal{U}}).$ The category $G_X(Y,Y')$ is the quotient category $G/Y'$.

It is now straightforward to define

$$G_X^{-\infty}(Y,Y') = \hocolim_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k),$$

$$G_X^{-\infty}(Y,Y')_{<C} = \hocolim_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k)_{<C \times \mathbb{R}^k},$$

and

$$G_X^{-\infty}(Y,Y')_{<C_1,C_2} = \hocolim_{k>0} \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k)_{<C_1 \times \mathbb{R}^k, C_2 \times \mathbb{R}^k}.$$ 

The theory developed in this section is spontaneously relativized to give the following excision theorem.
4.4.2. **Theorem** (Relative Fibrewise Excision, Version One). If $Y$ is the union of two subsets $U_1$ and $U_2$, there is a homotopy pushout diagram of spectra

$$
\begin{array}{ccc}
G_X^{-\infty}(Y,Y')_{<u_1,u_2} & \longrightarrow & G_X^{-\infty}(Y,Y')_{<u_1} \\
\downarrow & & \downarrow \\
G_X^{-\infty}(Y,Y')_{<u_2} & \longrightarrow & G_X^{-\infty}(Y,Y')
\end{array}
$$

where the maps of spectra are induced from the exact inclusions. In fact, if $Y$ is the union of two mutually antithetic subsets $U_1$ and $U_2$, and $Y'$ is antithetic to both $U_1$ and $U_2$, there is a homotopy pushout

$$
\begin{array}{ccc}
G_X^{-\infty}(U_1 \cap U_2, U_1 \cap U_2 \cap Y') & \longrightarrow & G_X^{-\infty}(U_1, U_1 \cap Y') \\
\downarrow & & \downarrow \\
G_X^{-\infty}(U_2, U_2 \cap Y') & \longrightarrow & G_X^{-\infty}(Y,Y')
\end{array}
$$

Finally, we want to state the relative excision theorem in the most familiar form.

4.4.3. **Proposition.** Given a subset $U$ of $Y'$, there is a weak equivalence

$$
G_X^{-\infty}(Y,Y') \simeq G_X^{-\infty}(Y-U,Y'-U).
$$

**Proof.** Consider the setup of Theorem 4.2.8 with $Y_1 = Y - U$ and $Y_2 = Y'$, then Lemma 4.2.4 shows that the map

$$
\frac{G_X(Y)_{<(Y-U)}}{G_X(Y)_{<(Y-U)} \cap G_X(Y)_{<Y'}} \longrightarrow \frac{G_X(Y)}{G_X(Y)_{<Y}},
$$

induces a weak equivalence on the level of $K$-theory.

Notice that, since $U$ is a subset of $Y'$, we have the interpretation

$$
G_X(Y)_{<(Y-U)} \cap G_X(Y)_{<Y'} = G_X(Y)_{<(Y'-U)}.
$$

Now the maps of quotients

$$
\frac{G_X(Y)}{G_X(Y')} \longrightarrow \frac{G_X(Y)}{G_X(Y)_{<Y}},
$$

and

$$
\frac{G_X(Y)_{<(Y-U)}}{G_X(Y)_{<(Y'-U)}} \longrightarrow \frac{G_X(Y-U)}{G_X(Y'-U)}
$$

induced by fully faithful embeddings also induce weak equivalences. Their composition gives the required equivalence.

The relative theorem can be restated using coarse coverings in terms of the homotopy pushout

$$
\mathcal{G}_X(Y,Y';U) = \text{hocolim}_{U \in \mathcal{U}} G_X^{-\infty}(Y,Y')_{<U}.
$$

4.4.4. **Theorem** (Relative Fibrewise Excision, Version Two). There is a weak equivalence

$$
\mathcal{G}_X(Y,Y';U) \simeq G_X^{-\infty}(Y,Y').
$$

**Proof.** Apply Theorem 4.4.2 inductively to the sets in $\mathcal{U}$. 

□
5. Conclusion

It is a familiar fact that $G$-theoretic approximations to the usually more meaningful $K$-theoretic invariants are easier to compute. This paper confirms the pattern in the controlled algebra setting. One approach to computing the $K$-theory leads one to consider $K$-theory with fibred control. It is in this setting that the tools of bounded $K$-theory become insufficient for computation. The paper [5, section 5.2] contains an explicit example of failure of the standard localization tools in bounded $K$-theory based on Karoubi filtrations. This paper, in contrast, uses a different technology of exact and Waldhausen categories. The Fibrewise Excision Theorems from section 4 suffice to resolve the example from [5] and perform computations in more general geometric settings. This material will appear in [9], while the relationship between the $K$-theory of group rings for finitely generated groups and a $G$-theoretic analogue based on controlled $G$-theory is studied in [5, 7]. For the purpose of stating the results we restrict to regular coefficient rings $R$ of finite global dimension. The conclusion is that the appropriate $G$-theory of the group ring is computable leveraging the results of this paper, while the Cartan comparison map from the $K$-theory is an equivalence for a remarkably large class of groups $\pi$ including all groups with finite $K(\pi, 1)$ and finite decomposition complexity.

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