HÖLDER CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR EXTENDED HARPER’S MODEL WITH LIOUVILLE FREQUENCY

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Abstract. In this paper, we study the non-self-dual extended Harper’s model with a Liouville frequency. Based on the work of [20], we show that the integrated density of states (IDS for short) of the model is $\frac{1}{2}$-Hölder continuous. As an application, we also obtain the Carleson homogeneity of the spectrum.

1. Introduction and main results

Let us consider the extended Harper’s model (EHM for short), which is given by

$$H_{\lambda, \alpha, x} u_n = c(x + n\alpha)u_{n+1} + \overline{c}(x + (n - 1)\alpha)u_{n-1} + 2\cos 2\pi(x + n\alpha)u_n,$$

where $u = \{u_n\} \in \ell^2(\mathbb{Z})$ and

$$c(x) = c_\lambda(x) = \lambda_1 e^{-2\pi i(x+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(x+\frac{\alpha}{2})},$$

$$\overline{c}(x) = \overline{c}_\lambda(x) = \lambda_1 e^{2\pi i(x+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{-2\pi i(x+\frac{\alpha}{2})}.$$

We call $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$ the coupling, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the frequency and $x \in \mathbb{R}$ the phase. The EHM was originally proposed by Thouless [22] and if $\lambda_1 = \lambda_3 = 0$ it reduces to the famous almost Mathieu operator (AMO for short). Physically, the EHM describes the influence of a transversal magnetic field of flux $\alpha$ on a single tight-binding electron in a 2-dimensional crystal layer (see [4, 22]).

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For irrational frequency $\alpha$, the spectrum does not depend on $x$ and we denote it by $\Sigma_{\lambda,\alpha}$. Actually, the properties of $\Sigma_{\lambda,\alpha}$ rely heavily on $\lambda,\alpha$. In general, we split the coupling region into three parts (see Figure 1):

\begin{align*}
I &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 : 0 < \max\{\lambda_1 + \lambda_3, \lambda_2\} < 1 \}, \\
II &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 : 0 < \max\{\lambda_1 + \lambda_3, 1\} < \lambda_2 \}, \\
III &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 : 0 < \max\{\lambda_2, 1\} < \lambda_1 + \lambda_3 \}.
\end{align*}

According to the duality map $\sigma : (\lambda_1, \lambda_2, \lambda_3) = \lambda \rightarrow \overline{\lambda} = (\frac{\lambda_2}{\lambda_3}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$, region I and region II are dual to each other and region III is the self-dual regime. Note that region I is the regime of positive Lyapunov exponent. When considering $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we call $\alpha$ a Liouville frequency if $\beta(\alpha) > 0$, where

\begin{equation}
\beta(\alpha) = \limsup_{k \to \infty} -\ln \frac{\|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}
\end{equation}

and $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{k \in \mathbb{Z}} |x - k|$. On the contrary, $\alpha$ is called a Diophantine frequency for $\beta(\alpha) = 0$.

In the present paper, we focus on the regularity of the integrated density of states $N_{\lambda,\alpha}(\cdot)$ (see subsection 2.2 for details) and homogeneity of the spectrum in the sense of Carleson for EHM. The first main result of this paper is the following theorem.

**Theorem 1.1.** Suppose $0 < \beta(\alpha) < \infty$ and $\lambda \in \Pi$. Then there is an absolute constant $C > 0$ such that if $L_\lambda > C \beta(\alpha)$, we have for $E_1, E_2 \in \mathbb{R}$,

\begin{equation}
|N_{\lambda,\alpha}(E_1) - N_{\lambda,\alpha}(E_2)| \leq C_* |E_1 - E_2|^{\frac{1}{2}},
\end{equation}

where $C_* > 0$ is a constant depending on $\alpha, \lambda$ and

\begin{equation}
L_\lambda = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{\max\{\lambda_1 + \lambda_3, 1\} + \sqrt{\max\{\lambda_1 + \lambda_3, 1\}^2 - 4\lambda_1 \lambda_3}}.
\end{equation}
Consequently, we also have

**Theorem 1.2.** Assume the conditions of Theorem 1.1 hold. Then for any \( \epsilon > 0 \), there exists \( \sigma_* = \sigma_*(\lambda, \alpha, \epsilon) > 0 \) such that for all \( E \in \Sigma_{\lambda, \alpha} \) and \( \sigma \in (0, \sigma_*) \), we have

\[
\text{Leb}((E - \sigma, E + \sigma) \cap \Sigma_{\lambda, \alpha}) \geq (1 - \epsilon)\sigma,
\]

where \( \text{Leb}(\cdot) \) is the Lebesgue measure.

Let us recall some history about the regularity of the IDS for one-frequency quasi-periodic operators first. On one hand, we consider in the regime of positive Lyapunov exponent. In [11], Goldstein-Schlag proved Hölder continuity of the IDS for quasi-periodic Schrödinger operator with large analytic potential and Diophantine frequency. Later, Bourgain [5] showed the IDS for almost Mathieu operator is \((\frac{1}{2} - \epsilon)\)-Hölder continuous (for any small \( \epsilon > 0 \)) if the coupling \( \lambda_2 \) is small and the frequency is Diophantine. Recently, Tao-Voda [21] dealt with quasi-periodic Jacobi operators and obtained especially that the IDS for EHM is \((\frac{1}{2} - \epsilon)\)-Hölder continuous if the Lyapunov exponent is positive and the frequency is strong Diophantine. On the other hand, in the subcritical regime Amor [13] proved that the IDS for Schrödinger operator with Diophantine frequency and small (in perturbative sense) analytic potential is \( \frac{1}{2} \)-Hölder continuous. After that, Avila-Jitomirskaya [2] got \( \frac{1}{2} \)-Hölder continuity of the IDS for almost Mathieu operator for \( \lambda_2 \neq \pm 1 \) and Diophantine frequency and they also obtained \( \frac{1}{2} \)-Hölder continuity of the IDS for Schrödinger operator with small (in non-perturbative sense) analytic potential and Diophantine frequency. Note that all above mentioned results are in Diophantine frequency case and You-Zhang [23] extended Goldstein-Schlag’s results to weak Liouville frequency case. In [19], Liu-Yuan improved Avila-Jitomirskaya’s results to Liouville frequency case. In a recent work by Cai-Chavaudret-You-Zhou [6], they proved \( \frac{1}{2} \)-Hölder continuity of the IDS for Schrödinger operator with small (perturbative) finitely differentiable potential and Diophantine frequency.

There are also many works on the Carleson homogeneity of the spectrum for quasi-periodic operators. In continuous quasi-periodic Schrödinger operator with Diophantine frequency case, Damanik and Goldstein [7] set up Carleson homogeneity of the spectrum for small analytic potential. Later, in the regime of positive Lyapunov exponent Goldsein-Damanik-Schlag-Voda [10] proved Carleson homogeneity of the spectrum for quasi-periodic Schrödinger operator with Diophantine frequency. In [12], Goldstein-Schlag-Voda got the Carleson homogeneity of the spectrum for Diophantine multi-frequency quasi-periodic Schrödinger operator. Recently, in the subcritical regime Leguil [17] obtained Carleson homogeneity of the spectrum for quasi-periodic Schrödinger operator with Diophantine frequency. Actually, we remark that all these results are attached to Diophantine frequency and Liu-Shi in [18] extended Leguil’s results to Liouville frequency case. In [5], Fillman-Lukic established Carleson homogeneity of the spectrum for limit-periodic Schrödinger operator.

The present paper is organized as follows. In section 2, we give some basic concepts and notations. In section 3, we will prove \( \frac{1}{2} \)-Hölder continuity of the IDS by
establishing some quantitative almost reducibility results. The proof of Theorem 1.2 is included in Appendix B.

2. Some basic concepts and notations

2.1. Cocycle, transfer matrix and Lyapunov exponent. Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{B}) \) be the set of all analytic mappings from \( \mathbb{R}/\mathbb{Z} \) to some Banach space \( (\mathcal{B}, \| \cdot \|) \). By a cocycle, we mean a pair \( (\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{R})) \) and we can regard it as a dynamical system on \( (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \) with \( (\alpha, A) : (x, v) \mapsto (x + \alpha, A(x)v), \ (x, v) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \).

For \( k > 0 \), we define the \( k \)-step transfer matrix as

\[
A_k(x) = \prod_{l=k}^{1} A(x + (l - 1)\alpha),
\]

and the Lyapunov exponent for \( (\alpha, A) \) as

\[
\mathcal{L}(\alpha, A) = \lim_{k \to +\infty} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_k(x) \| \, dx = \inf_{k > 0} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_k(x) \| \, dx.
\]

2.2. Spectral measures and the IDS. Let \( H \) be a bounded self-adjoint operator on \( \ell^2(\mathbb{Z}) \). Then \( (H - z)^{-1} \) is analytic in \( \mathbb{C} \setminus \Sigma(H) \), where \( \Sigma(H) \) is the spectrum of \( H \), and we have for \( f \in \ell^2(\mathbb{Z}) \),

\[
\Im \langle (H - z)^{-1} f, f \rangle = \Im z \cdot \| (H - z)^{-1} f \|_{\ell^2(\mathbb{Z})}^2,
\]

where \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( \ell^2(\mathbb{Z}) \). Thus \( \phi_f(z) = \langle (H - z)^{-1} f, f \rangle \) is an analytic function in the upper half plane with \( \Im \phi_f \geq 0 \) (\( \phi_f \) is the so-called Herglotz function). Therefore, one has a representation

\[
\phi_f(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu_f(x),
\]

where \( \mu_f \) is the spectral measure associated to vector \( f \). Alternatively, for any Borel set \( \Omega \subseteq \mathbb{R} \),

\[
\mu_f(\Omega) = \langle \mathbb{E}(\Omega)f, f \rangle,
\]

where \( \mathbb{E} \) is the corresponding spectral projection of \( H \).

Denote by \( \mu_{\lambda, \alpha, x}^f \) the spectral measure of the operator \( H_{\lambda, \alpha, x} \) and vector \( f \) as above with \( \| f \|_{\ell^2(\mathbb{Z})} = 1 \). The IDS \( \mathcal{N}_{\lambda, \alpha} : \mathbb{R} \to [0, 1] \) is obtained by averaging the spectral measure \( \mu_{\lambda, \alpha, x}^f \) with respect to \( x \), i.e.,

\[
\mathcal{N}_{\lambda, \alpha}(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu_{\lambda, \alpha, x}^f(-\infty, E) \, dx.
\]

It is a continuous, non-decreasing surjective function and the definition is independent of the choice of \( f \).
2.3. Gap labelling and IDS. Each connected component of \([E_{\min}, E_{\max}] \setminus \Sigma_{\lambda, \alpha}\) is called a spectral gap, where \(E_{\min} = \min\{E : E \in \Sigma_{\lambda, \alpha}\}\) and \(E_{\max} = \max\{E : E \in \Sigma_{\lambda, \alpha}\}\). By the well-known gap labelling theorem [8, 10], for every spectral gap \(G\) there exists unique nonzero integer \(m\) such that \(N_{\lambda, \alpha}|_G = m\alpha \mod \mathbb{Z}\) and

\[
(E^-_m, E^+_m) = \{E_{\min} \leq E \leq E_{\max} : N_{\lambda, \alpha}(E) = m\alpha \mod \mathbb{Z}\}.
\]

2.4. Extended Harper’s cocycle. Recalling (1.1), for \(c(x) \neq 0\) the equation

\[
H_{\lambda, \alpha, \theta}u = Eu
\]

is equivalent to

\[
\begin{pmatrix}
  u_{k+1} \\
  u_k
\end{pmatrix} = A_{\lambda, E}(x + k\alpha) \begin{pmatrix}
  u_k \\
  u_{k-1}
\end{pmatrix},
\]

where \(A_{\lambda, E}(x) = \frac{1}{\sqrt{|c(x)|}} \begin{pmatrix}
  E - 2\cos 2\pi x & -c(x - \alpha) \\
  c(x) & -E
\end{pmatrix} \cdot Q_{\lambda}(x + \alpha)\).

Since in general, \(A_{\lambda, E}(x) \notin \text{SL}(2, \mathbb{R})\), we need make a few modifications and consider the “renormalized” \(\text{SL}(2, \mathbb{R})\)-cocycle

\[
\mathcal{A}_{\lambda, E}(x) = \frac{1}{\sqrt{|c(x)|}} \begin{pmatrix}
  E - 2\cos 2\pi x & -|c(x - \alpha)| \\
  |c(x)| & 0
\end{pmatrix} \cdot Q_{\lambda}(x + \alpha)\).
\]

where \(|c(x)| = \sqrt{c(x)\overline{c(x)}}\) and \(Q_{\lambda}, Q_{\lambda}^{-1}\) are analytic in \(\{x \in \mathbb{C}/\mathbb{Z} : |\Im x| \leq \frac{C_{\lambda}}{T}\}\) if \(L_{\lambda} \geq 5\beta(\alpha)\) (see Lemma A.1 of the Appendix for details). We call \((\alpha, \mathcal{A}_{\lambda, E})\) the extended Harper’s cocycle and denote by \(L(\alpha) = \mathcal{L}(\alpha, \mathcal{A}_{\lambda, E})\) its Lyapunov exponent. Actually, there is a direct definition of the Lyapunov exponent \(L(\alpha, A_{\lambda, E})\) for \((\alpha, A_{\lambda, E})\) (see [15] for details) and \(L_{\lambda}(E) = L(\alpha, A_{\lambda, E})\) (ignoring the dependence on \(\alpha\)).

The Thouless formula relates the Lyapunov exponent to the integrated density of states,

\[
L_{\lambda}(E) = -\int_{\mathbb{R}/\mathbb{Z}} \ln |c_{\alpha}(x)| \, dx + \int_{\mathbb{R}} \ln |E' - E| \, dN_{\lambda, \alpha}(E').
\]

2.5. Aubry duality. The map \(\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \rightarrow \overline{\lambda} = (\frac{\lambda_1}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_3}{\lambda_2})\) induces the duality between region I and region II, and we call \(H_{\overline{\lambda}, \alpha, x}\) the Aubry duality of \(H_{\lambda, \alpha, x}\). We have \(\Sigma_{\lambda, \alpha} = \lambda_2 \Sigma_{\overline{\lambda}, \alpha}\) for \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\).

Aubry duality expresses an algebraic relation between the families of operators \(\{H_{\overline{\lambda}, \alpha, x}\}_{\alpha \in \mathbb{R}}\) and \(\{H_{\lambda, \alpha, x}\}_{\alpha \in \mathbb{R}}\) by Bloch waves, i.e., if \(u : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}\) is an \(L^2\) function whose Fourier coefficients \(\hat{u}\) satisfy \(H_{\overline{\lambda}, \alpha, \theta} \hat{u} = \frac{E}{\lambda_2} \hat{u}\), then there exist \(\theta \in \mathbb{R}\), such that

\[
U(x) = \begin{pmatrix}
  e^{2\pi i \theta} u(x) \\
  u(x - \alpha)
\end{pmatrix}
\]

satisfies

\[
A_{\lambda, E}(x) \cdot U(x) = e^{2\pi i \theta} U(x + \alpha).
\]

\(^1\overline{c(x)}\) is the complex conjugate of \(c(x)\) for \(x \in \mathbb{R}/\mathbb{Z}\) and its analytic extension for \(x \notin \mathbb{R}\).
2.6. Some notations. We briefly comment on the constants and norms in the following proofs. Let $C(\alpha)$ be a large constant depending on $\alpha$ and $C_\ast$ (resp. $c_\ast$) be a large (resp. small) constant depending on $\lambda$ and $\alpha$. Define the strip $\Delta_s = \{ z \in \mathbb{C}/\mathbb{Z} : |3z| < s \}$ and let $\|v\|_s = \sup_{z \in \Delta_s} |v(z)|$, where $v$ is a mapping from $\Delta_s$ to some Banach space ($B, \| \cdot \|$). In this paper, $B$ may be $\mathbb{C}$, $\mathbb{C}^2$ or $\text{SL}(2, \mathbb{C})$.

3. $\frac{1}{2}$-Hölder continuity of the IDS

In this section we will prove the $\frac{1}{2}$-Hölder continuity of the IDS for EHM. To this end, one needs to establish quantitative (almost) reducibility results for the extended Harper’s cocycle. Let us begin with some useful definitions and lemmata.

**Definition 3.1.** Fix $\theta \in \mathbb{R}, \epsilon_0 > 0$. We call $n \in \mathbb{Z}$ an $\epsilon_0$-resonance of $\theta$ if

$$
\min_{|k| \leq |n|} \|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \|2\theta - n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\epsilon_0|n|}.
$$

Given $\theta \in \mathbb{R}$, we order all the $\epsilon_0$-resonances of $\theta$ as $0 < |n_1| \leq |n_2| < \cdots$. We say $\theta$ is $\epsilon_0$-resonant if the set of all $\epsilon_0$-resonances of $\theta$ is infinite and $\epsilon_0$-non-resonant for otherwise. Supposing $\{0, n_1, \cdots, n_j\}$ is the set of all $\epsilon_0$-resonances of $\theta$, we let $n_{j+1} = \infty$.

**Lemma 3.2** (Theorem 3.3 of [2]). Let $E \in \Sigma_{\lambda, \alpha}$. Then there exist $\theta = \theta(E) \in \mathbb{R}$ and solution $u$ of $\mathcal{H}_{\lambda, \alpha} u = \frac{E}{\lambda} u$ with $u_0 = 1, |u_k| \leq 1$.

Throughout this section we fix $E, \theta = \theta(E)$ and $u$ which are given by Lemma 3.2. In the following, we let $C_2, C_1 \ (C_2 \gg C_1)$ be large absolute constants which are bigger than any positive absolute constant $C$. Moreover, we assume $\lambda \in \Pi$ and

$$
h = \frac{L_{\lambda}}{200\pi}, \quad L_{\lambda} > C_2\beta(\alpha).
$$

From Theorem 3.3 in [20], we have

\begin{equation}
|u_k| \leq C_\ast e^{-2\pi h |k|}, \text{ for } 3|n_j| < |k| < \frac{|n_{j+1}|}{3},
\end{equation}

where $\{n_j\}$ is the set of all $C_1\beta(\alpha)$-resonances of $\theta = \theta(E)$.

**Lemma 3.3** (Lemma 6.6 of [20]). We have

\begin{equation}
\sup_{0 \leq k \leq \frac{E}{2\pi h}} \|A_k\|_{h} \leq C_\ast e^{C\beta(\alpha)n},
\end{equation}

where $A_k(x)$ denotes the $k$-step transfer matrix of $(\alpha, A_{\lambda, E})$ and $C > 0$ is some absolute constant.

**Lemma 3.4** (Theorem 2.6 of [1]). Given $\eta > 0$, we let $U : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^2$ be analytic in $\Delta_\eta$ and satisfy $\delta_1 \leq \|U(x)\| \leq \delta_2^{-1}$ for $\forall x \in \Delta_\eta$. Then there exists $B(x) : \mathbb{C}/\mathbb{Z} \to \text{SL}(2, \mathbb{C})$ being analytic in $\Delta_\eta$ with first column $U(x)$ and $\|B\|_{\eta} \leq C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2))$, where $C > 0$ is some absolute constant.
For simplicity, we write \( n = \lfloor n_j \rfloor < \infty \) and \( N = \lfloor n_j+1 \rfloor \) in the following.

Define \( I_2 = \left[ -\left\lfloor \frac{N}{7} \right\rfloor, \left\lfloor \frac{N}{7} \right\rfloor \right] \) and

\[
U_{I_2}^*(x) = \left( e^{2\pi i \theta} \sum_{k \in I_2} u_k e^{2\pi i kx} \right) + \left( \sum_{k \in I_2} u_k e^{2\pi i k(x-\alpha)} \right),
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \in \mathbb{R} \). Suppose \( U_{I_2}^*(x) = Q_{\lambda}(x) \cdot U_{I_2}(x) \).

Recalling \((2.3)\) and \((3.1)\), we have

\[
(3.3) \quad \mathcal{A}_{\lambda,E}(x)U_{I_2}^*(x) = e^{2\pi i \theta} U_{I_2}^*(x + \alpha) + G_*(x),
\]

where

\[
(3.4) \quad \|G_*\|_{L^\infty} \leq C_* e^{-\frac{h}{10} N}.
\]

We have the following useful estimate.

**Lemma 3.5** (Lemma 6.6 in \([20]\)). We have for \( n > n(\lambda, \alpha) \),

\[
(3.5) \quad \inf_{x \in \Delta_{\frac{h}{3}}} \|U_{I_2}^*(x)\| \geq e^{-C\beta(\alpha)n},
\]

where \( C > 0 \) is some absolute constant.

We now turn to the upper bound. From \((3.1)\) and the definition of \( u \) in Lemma 3.2, one has

\[
(3.6) \quad \|U_{I_2}^*(x)\|_{C_{\beta}(\alpha)} \leq C_* \sum_{|k| \leq 3n} |u_k| e^{2\pi C_{1}\beta(\alpha)|k|} + C_* \sum_{3n < |k| \leq \frac{N}{7}} |u_k| e^{2\pi C_{1}\beta(\alpha)|k|} \leq C_* e^{CC_{1}\beta(\alpha)n}.
\]

The purpose of the following is to construct quantitative almost reducibility (in \( SL(2, \mathbb{C}) \)) results. Suppose now \( B(x) \) is as in Lemma 3.4 with \( U(x) = U_{I_2}^*(x) \) and \( \eta = C_{1}\beta(\alpha) \). Then from \((3.5)\) \((3.6)\) and Lemma 3.4, we obtain

\[
(3.7) \quad \|B\|_{C_{1}\beta(\alpha)} \|B^{-1}\|_{C_{1}\beta(\alpha)} \leq C_* e^{C_{1}\beta(\alpha)n}.
\]

More precisely, by letting \( B(x) = (U_{I_2}^*(x), V(x)) \) and recalling \((3.3)\), we have

\[
\mathcal{A}_{\lambda,E}(x)B(x) = \left[ e^{2\pi i \theta} U_{I_2}^*(x + \alpha) + G_*(x), \mathcal{A}_{\lambda,E}(x)V(x) \right] = B(x + \alpha) \left[ e^{2\pi i \theta} 0 \right] + \left[ G_*(x), \mathcal{A}_{\lambda,E}(x)V(x) - e^{-2\pi i \theta} V(x + \alpha) \right].
\]

In other words,

\[
(3.8) \quad B^{-1}(x + \alpha)\mathcal{A}_{\lambda,E}(x)B(x) = \left[ e^{2\pi i \theta} 0 \right] + \left[ \beta_1(x) b(x) \beta_2(x) \beta_3(x) \right].
\]

From \((3.4)\) and \((3.7)\), we get

\[
(3.9) \quad \|\beta_1\|_{C_{1}\beta(\alpha)}, \|\beta_2\|_{C_{1}\beta(\alpha)} \leq C_* e^{-\frac{h}{20} N},
\]
and
\[(3.10)\]
\[\|b\|_{C_1\beta(\alpha)} \leq C_4 e^{CC_1\beta(\alpha)n}.\]
By taking determinant on \((3.8)\) and noting \(\bar{A}, E, B \in \text{SL}(2, \mathbb{C})\), one has
\[(3.11)\]
\[\|\beta\|_{C_1\beta(\alpha)} \leq \|b\|_{C_1\beta(\alpha)} \|\beta\|_{C_1\beta(\alpha)} + \|\beta\|_{C_1\beta(\alpha)} \leq C_4 e^{-\frac{h}{m}N}.\]

Actually, one can obtain the following refinement.

**Theorem 3.6.** Under the previous assumptions, there exists \(\Phi(x) : \mathbb{C}/\mathbb{Z} \to \text{SL}(2, \mathbb{C})\)
being analytic in \(\Delta \leq \frac{1}{2} C_1\beta(\alpha)\) with \(\|\Phi\|_{\frac{1}{2} C_1\beta(\alpha)} \leq C_4 e^{CC_1\beta(\alpha)n}\) such that
\[(3.12)\]
\[\Phi^{-1}(x + \alpha) \bar{A}_E(x) \Phi(x) = \begin{bmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{bmatrix} + \begin{bmatrix} \beta_1'(x) & b'(x) \\ \beta_2'(x) & \beta_3'(x) \end{bmatrix}\]
with
\[(3.13)\]
\[\|\beta_1'\|_{\frac{1}{2} C_1\beta(\alpha)}, \|\beta_2'\|_{\frac{1}{2} C_1\beta(\alpha)}, \|\beta_3'\|_{\frac{1}{2} C_1\beta(\alpha)} \leq C_4 e^{-\frac{h}{m}N},\]
and
\[(3.14)\]
\[\|b'\|_{\frac{1}{2} C_1\beta(\alpha)} \leq C_4 e^{-\frac{1}{2} C_1\beta(\alpha)n}.\]

**Proof.** We assume \(n > n(\lambda, \alpha)\), otherwise this theorem is trivial. Recalling \((3.8)\),
we can write \(b(x) = b'(x) + b''(x)\), where \(b'(x) = \sum_{|k| < C_1n, k \neq n j} \tilde{b}_k e^{2\pi i kx}\),
\(b''(x) = \tilde{b}_n e^{2\pi i n j x}\), and \(b''(x) = \sum_{|k| > C_1n} \tilde{b}_k e^{2\pi i kx}\). Then by \((3.10)\),
\[(3.15)\]
\[\|b''\|_{\frac{1}{2} C_1\beta(\alpha)} \leq \sum_{|k| > C_1n} \|b\|_{C_1\beta(\alpha)} e^{-\pi C_1\beta(\alpha)|k|} \leq C_4 e^{-2 C_1\beta(\alpha)n}.\]

We then eliminate the term \(b'(x)\) by solving some homological equation. From the
definition of \(\beta(\alpha)\) in \((1.2)\), we have the small divisor estimate
\[(3.16)\]
\[\|\tilde{k}_\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq C(\alpha) e^{-\frac{2}{2} \beta(\alpha)|k|}, \text{ for } k \neq 0.\]
Together with the definition of \(\epsilon_0\)-resonance, one has for \(|k| \leq C_1n\) and \(k \neq n j,\)
\[(3.17)\]
\[\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|(n j - k)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - n j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq C_4 e^{-3 C_1\beta(\alpha)n}.\]
Let \(\tilde{w}_k = \tilde{b}_k \frac{e^{2\pi i \theta} \Phi}{e^{2\pi i \theta} - e^{-2\pi i \theta}}\) for \(|k| \leq C_1n\) and \(k \neq n j,\) and \(\tilde{w}_k = 0\) for \(|k| > C_1n\) or \(k = n j,\). Consequently, the function \(w(x) = \sum_{k \in \mathbb{Z}} \tilde{w}_k e^{2\pi i kx}\) will satisfy
\[\|w\|_{\frac{1}{2} C_1\beta(\alpha)} \leq C_4 e^{CC_1\beta(\alpha)n}\]
from \((3.10)\) and \((3.17)\). If we define
\[W(x) = \begin{bmatrix} 1 & w(x) \\ 0 & 1 \end{bmatrix},\]
then we must have
\[W^{-1}(x + \alpha) \begin{bmatrix} e^{2\pi i \theta} & b'(x) \\ 0 & e^{-2\pi i \theta} \end{bmatrix} W(x) = \begin{bmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{bmatrix},\]
and
\[(3.18)\]
\[\|W\|_{\frac{1}{2} C_1\beta(\alpha)} \leq C_4 e^{CC_1\beta(\alpha)n}.\]
We now set $\Phi(x) = B(x)W(x)$ and then $\|\Phi\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star e^{CC_1\beta(\alpha)n}$. By direct computation, we have

$$\Phi^{-1}(x + \alpha)\overline{A}_\lambda E(x)\Phi(x) = Z(x) + \Psi(x),$$

with

$$Z(x) = \begin{bmatrix} e^{2\pi i \theta} & b^r(x) \\ 0 & e^{-2\pi i \theta} \end{bmatrix}$$

and

$$\Psi(x) = \begin{bmatrix} \beta_1^r(x) & \beta_1(x) \\ \beta_2^r(x) & \beta_2(x) \\ \beta_3^r(x) & \beta_3(x) \end{bmatrix} = W^{-1}(x + \alpha) \begin{bmatrix} \beta_1(x) & b_1^r(x) \\ \beta_2(x) & b_2(x) \\ \beta_3(x) & b_3(x) \end{bmatrix} W(x).$$

Hence we can obtain (3.13) and

$$\|\Psi\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star e^{-C_1^2\beta(\alpha)n}$$

from (3.9), (3.11), (3.15) and (3.18).

Thus what remains is to estimate the term $b^r(x)$. For $s \in \mathbb{N}$, we set

$$Z_s(x) = \prod_{k=s-1}^0 Z(x + k\alpha) = \begin{bmatrix} e^{2\pi is\theta} & b^r_s(x) \\ 0 & e^{-2\pi is\theta} \end{bmatrix},$$

where

$$b^r_s(x) = \hat{b}_{n_j} e^{2\pi i((s-1)\theta + n_j x)} \sum_{k=0}^{s-1} e^{-2\pi ik(2\theta - n_j \alpha)}.$$ 

Therefore,

$$|b^r_s|_0 = \left| \hat{b}_{n_j} \frac{\sin \pi s(2\theta - n_j \alpha)}{\sin \pi (2\theta - n_j \alpha)} \right|$$

if $\sin \pi (2\theta - n_j \alpha) \neq 0$, and $|b^r_s|_0 = s|\hat{b}_{n_j}|$ otherwise. Noting that

$$2\|x\|_{\mathbb{R}/\mathbb{Z}} \leq \pi \|x\|_{\mathbb{R}/\mathbb{Z}} \leq \pi \|x\|_{\mathbb{R}/\mathbb{Z}},$$

we have for $0 \leq s \leq \frac{1}{2}\|2\theta - n_j \alpha\|_{\mathbb{R}/\mathbb{Z}}^{-1}$,

$$\frac{2s}{\pi} |\hat{b}_{n_j}| \leq |b^r_s|_0 \leq s|\hat{b}_{n_j}|.$$

Therefore, for $0 \leq s \leq \frac{1}{2}\|2\theta - n_j \alpha\|_{\mathbb{R}/\mathbb{Z}}^{-1}$,

$$\frac{2s}{\pi} |\hat{b}_{n_j}| \leq \|Z_s\|_0 \leq 1 + s|\hat{b}_{n_j}| \leq C_\star (1 + s) e^{CC_1\beta(\alpha)n}. $$

Because of

$$\Phi^{-1}(x + s\alpha)\overline{A}_s(x)\Phi(x) = Z_s(x) + \sum_{k=1}^{s} \sum_{s-1\geq j_1 > j_2 > \cdots > j_k \geq 0} \Psi(x + j_1\alpha) \cdots \Psi(x + j_k\alpha) \times Z_{s-1-j_1}(x + (j_1 + 1)\alpha)Z_{j_1-j_2-1}(x + (j_2 + 1)\alpha) \cdots Z_{j_k}(x)$$
and combining with (3.19) and (3.20), we have for $s \sim e^{\frac{1}{100}C_1^2(\alpha)n} < \frac{1}{2} \|2\theta - n_j \alpha\|_{B/Z}^{-1}$,

$$
\|A_s\|_0 \geq \|\Phi\|_0^{-2} \left( \|Z_s\|_0 - \sum_{k=1}^{s} \left( \frac{1}{k} \right) \|\Psi\|_0 \left( \max_{0 \leq j \leq s-1} \|Z_j\|_0 \right)^{1+k} \right) 
$$

$$
\geq \|\Phi\|_0^{-2} \left( \|Z_s\|_0 - C_s e^{\frac{1}{100}C_1^2(\alpha)n} \sum_{k=1}^{s} \left( \frac{1}{k} \right) 2^k e^{-\frac{1}{2}C_1^2(\alpha)n} \right) 
$$

$$
\geq \|\Phi\|_0^{-2} \left( \|Z_s\|_0 - C_s e^{\frac{1}{100}C_1^2(\alpha)n} \left( (1 + 2e^{-\frac{1}{2}C_1^2(\alpha)n})s - 1 \right) \right) 
$$

$$
\geq c_s e^{-CC_1^2(\alpha)n} \left( \|Z_s\|_0 - C_s e^{-\frac{1}{100}C_1^2(\alpha)n} \right).
$$

Thus from Lemma 3.3 and (3.20), we have for $s \sim e^{\frac{1}{100}C_1^2(\alpha)n}$,

$$
|\hat{b}_{nj}| \leq C_s e^{-\frac{1}{100}C_1^2(\alpha)n},
$$

and hence

$$
\|b^l\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{-\frac{1}{100}C_1^2(\alpha)n}.
$$

The proof is finished. \qed

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. If the energy $E$ is in the resolvent set, then $\mathcal{L}_\lambda$ is clearly Lipschitz continuous. Thus it is suffice to consider the case $E \in \Sigma_{\lambda, \alpha}$. Given $\epsilon > 0$, we define $D = \begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix}$ where $d = \|\Phi\|_{\frac{1}{2}C_1(\alpha)} e^{\frac{1}{2}}$ and $\Phi$ is given by Theorem 3.6.

Let $\Phi'(x) = \Phi(x)D$. If $\epsilon \leq C_s e^{-C_1^2(\alpha)n}$, we have

$$
\|\Phi'\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{-\frac{1}{4}}.
$$

Set $B'(x) = \Phi'^{-1}(x + \alpha)A_{\lambda, E}(x)\Phi'(x)$, then

$$
B'(x) = \begin{bmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{bmatrix} + \begin{bmatrix} \beta'_1(x) & d^2 b'(x) \\ d^{-2} \beta'_2(x) & \beta'_3(x) \end{bmatrix}
$$

with

$$
\|\beta'_1\|_{\frac{1}{2}C_1(\alpha)} , \|\beta'_2\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{-\frac{1}{100}hN},
$$

$$
\|d^2 b'\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{-\frac{1}{100}C_1^2(\alpha)n} e^{\frac{1}{4}},
$$

and

$$
\|d^{-2} \beta'_2\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{-\frac{1}{100}hN} e^{-\frac{1}{2}}.
$$

If $\epsilon \geq C_s e^{-\frac{1}{100}hN}$, then

$$
\|d^{-2} \beta'_2\|_{\frac{1}{2}C_1(\alpha)} \leq C_s e^{\frac{1}{2}},
$$

and

$$
\|B'\|_{\frac{1}{2}C_1(\alpha)} \leq 1 + C_s e^{\frac{1}{2}}.
$$
Thus recalling (3.23), we obtain
\[
L(\alpha, B') \leq \ln \| B' \| \frac{1}{2} C_1 \beta(\alpha) \leq \ln \left( 1 + C_* \epsilon \frac{1}{2} \right) \leq C_* \epsilon \frac{1}{2}.
\]

Define
\[
I_j := \{ \epsilon \in \mathbb{R} : C_* e^{-\frac{1}{100} h |n_j + 1|} \leq \epsilon \leq C_* e^{-C_2 \beta(\alpha) |n_j|} \}.
\]
Then for any small \( \epsilon_0 > 0 \), there exists \( j_0 \in \mathbb{Z}^+ \) such that \( [0, \epsilon_0] \subset \bigcup_{j \geq j_0} I_j \). Let \( \epsilon = |E - E'| \in [0, \epsilon_0] \) with \( E' \in \mathbb{C} \). Then by (3.21) and (3.22), one has
\[
L(\alpha, B') = L(\alpha, \Phi^{-1}(x + \alpha) \overline{A}_{\lambda, E'}(x) \Phi(x))
\]
\[
\leq \ln \| B' + \Phi^{-1}(x + \alpha) (\overline{A}_{\lambda, E'}(x) - \overline{A}_{\lambda, E}(x)) \Phi'(x) \| \frac{1}{2} C_1 \beta(\alpha)
\]
\[
\leq \ln \left( 1 + C_* \epsilon \frac{1}{2} \right) \leq C_* \epsilon \frac{1}{2}.
\]

Hence,
\[
|L(\alpha, B') - L(\alpha, E)| \leq C_* |E' - E| \frac{1}{2}.
\]

From the Thouless formula (2.2), we have
\[
\left| L(\alpha, \overline{A}_{\lambda, E + i \epsilon}) - L(\alpha, E) \right| = \frac{1}{2} \int \ln \left( 1 + \frac{\epsilon^2}{(E - E')^2} \right) dN_{\lambda, \alpha}(E')
\]
\[
\geq \frac{1}{2} \ln 2 \left( N_{\lambda, \alpha}(E + \epsilon) - N_{\lambda, \alpha}(E - \epsilon) \right).
\]

Thus recalling (3.23), we obtain
\[
N_{\lambda, \alpha}(E + \epsilon) - N_{\lambda, \alpha}(E - \epsilon) \leq C_* \epsilon \frac{1}{2},
\]
which means precisely that \( N_{\lambda, \alpha} \) is \( \frac{1}{2} \)-Hölder continuous. This finishes the proof of Theorem 1.1. \( \square \)

APPENDIX A.

Lemma A.1. Let \( 0 < \beta(\alpha) < \infty \) and \( \alpha \in \Pi \). If \( L_{\lambda} \geq 5 \beta(\alpha) \), then there are analytic mapping \( Q_\lambda \) from \( \Delta_{\frac{2\pi}{\alpha}} \) to \( M_2(\mathbb{C}) \) and its inverse \( Q_\lambda^{-1} \) which is analytic in the same region, such that for all \( x \in \Delta_{\frac{2\pi}{\alpha}} \),
\[
Q_\lambda^{-1}(x + \alpha) A_{\lambda, E}(x) Q_\lambda(x) = \overline{A}_{\lambda, E}(x),
\]
where \( M_2(\mathbb{C}) \) denotes the space of all \( 2 \times 2 \) complex matrices.

Proof. Let
\[
\epsilon_* = \min \left\{ \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_1}, \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3} \right\}.
\]
Then as \( \lambda_2 > \lambda_1 + \lambda_3 \), we have for any \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_* \),
\[
\lambda_2 - (\lambda_1 e^{2\pi \epsilon} + \lambda_3 e^{-2\pi \epsilon}) > 0,
\]
\[
\lambda_2 - (\lambda_1 e^{-2\pi \epsilon} + \lambda_3 e^{2\pi \epsilon}) > 0.
\]
Thus \( \Re(e^{x + i\epsilon}) > 0, \Re(e^{x - i\epsilon}) > 0 \) for any \( x \in \mathbb{R}/\mathbb{Z} \). We have showed that \( c(x), \overline{\tau}(x) \) have no zeros on \( \Delta_{\lambda, \alpha} \). Recalling (A.1) and (A.2) again, the rotation numbers of \( c(x), \overline{\tau}(x) \) on \( \Delta_{\lambda, \alpha} \) are identically vanishing. Consequently, there are single-valued analytic functions \( g_1(x) = \log |c(x)| + i \arg c(x) \) and \( g_2(x) = \log |\overline{\tau}(x)| + i \arg \overline{\tau}(x) \) on \( \Delta_{\lambda, \alpha} \) such that \( c(x) = e^{g_1(x)}, \overline{\tau}(x) = e^{g_2(x)} \).

Noting for \( x \in \mathbb{R}/\mathbb{Z} \),

\[
\Re(c(1 - \alpha - x)) = \Re(c(x)), \Im(c(1 - \alpha - x)) = -\Im(c(x)),
\]

then we have

\[
\int_{\mathbb{R}/\mathbb{Z}} \arg c(x) dx = \int_{\frac{1}{2} - \frac{\alpha}{2}}^{\frac{1}{2} + \frac{\alpha}{2}} \arg c(x) dx + \int_{\frac{1}{2} - \frac{\alpha}{2}}^{1 - \frac{1}{2}} \arg c(x) dx
\]

\[
= -\int_{\frac{1}{2} - \frac{\alpha}{2}}^{\frac{1}{2} + \frac{\alpha}{2}} \arg c(1 - \alpha - x) dx + \int_{\frac{1}{2} - \frac{\alpha}{2}}^{1 - \frac{1}{2}} \arg c(x) dx
\]

\[
= 0.
\]

Similarly, \( \int_{\mathbb{R}/\mathbb{Z}} \arg \overline{\tau}(x) dx = 0 \). Hence \( \overline{g_1} - g_2(0) = \int_{\mathbb{R}/\mathbb{Z}} (g_1(x) - g_2(x)) dx = 0 \) and the function \( f(x) = \sum k e^{2\pi i k x} \) will solve the equation

\[
2f(x + \alpha) - 2f(x) = g_1(x) - g_2(x),
\]

where \( \hat{f}_0 = 0 \) and \( \hat{f}_k = \frac{g_1 - g_2(k)}{2(e^{2\pi i k} - 1)}, k \neq 0 \). Because of the small divisor estimate (3.16) and \( \mathcal{L}_\lambda \geq 5\beta(\alpha) \), \( f(x) \) must be analytic on \( \Delta_{\lambda, \alpha} \). Thus \( c(x) = |c(x)| e^{f(x+\alpha) - f(x)}, \overline{\tau}(x) = |c(x)| e^{-f(x+\alpha) + f(x)} \) for all \( x \in \Delta_{\lambda, \alpha} \).

Let

\[
Q_\lambda(x) = e^{f(x)} \sqrt{|c(x) - \alpha|} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{|c(x) - \alpha|} \end{bmatrix}.
\]

Then the proof follows (the detail computations are similar to that of [14]).

\[\square\]

**Appendix B. Carleson Homogeneity: proof of Theorem 1.2**

In this appendix, we will complete the proof of Theorem 1.2 and this follows from the Hölder continuity of the IDS together with the exponential decay of the lengths of the spectral gaps. For the convenience of readers, we include the details in the following.

**Lemma B.1** (Theorem 1.1 of [20]). Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( 0 \leq \beta(\alpha) < \infty \) and \( E_m^-, E_m^+ \) be given by (2.1). Then there exists absolute constant \( C > 1 \) such that, if \( \lambda \in \Pi \) and \( \mathcal{L}_\lambda > C\beta(\alpha) \), one has for \( |m| \geq m_* \),

\[
E_m^+ - E_m^- \leq e^{-C^{-1}\mathcal{L}_\lambda|m|},
\]

where \( m_* \) is a positive constant only depending on \( \lambda, \alpha \) and \( \mathcal{L}_\lambda \) is given by (1.3).
Lemma B.2. Let $G_m = (E_m^-, E_m^+)$ for $m \in \mathbb{Z} \setminus \{0\}$ and $G_0 = (-\infty, E_{\text{min}})$. Then for $m' \neq m \in \mathbb{Z} \setminus \{0\}$ with $|m'| \geq |m|$, we have

\begin{equation}
\text{dist}(G_m, G_{m'}) = \inf_{x \in G_m, x' \in G_{m'}} |x - x'| \geq c_\star e^{-6\beta(\alpha)|m'|},
\end{equation}

and for $m \in \mathbb{Z} \setminus \{0\}$

\begin{equation}
\text{dist}(G_m, G_0) \geq c_\star e^{-6\beta(\alpha)|m|}.
\end{equation}

Proof. From the small divisor condition (3.16), one has

\begin{equation}
\|(m - m')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{C(\alpha)}{e^{-3\beta(\alpha)|m|}}
\end{equation}

for $|m'| \geq |m|$. Without loss of generality, we assume $E_m^+ \leq E_{m'}^-$. By Theorem 1.1, (2.1) and (B.3), we have

\begin{equation}
\text{dist}(G_m, G_{m'}) = |E_m^- - E_{m'}^-| \geq \left(\frac{1}{C_\star} \left| N_{\lambda, \alpha}(E_m^-) - N_{\lambda, \alpha}(E_{m'}^-) \right| \right)^2 \geq c_\star \|(m - m')\alpha\|_{\mathbb{R}/\mathbb{Z}}^2, \end{equation}

which completes the proof of (B.1). The proof of (B.2) is similar. □

Now we can give the proof of Theorem 1.2

Proof of Theorem 1.2. Assume $0 < \sigma \leq \sigma_\star(\lambda, \alpha, \epsilon)$. For $E \in \Sigma_{\lambda, \alpha}$ and $\sigma$, let

\[ \mathcal{R}(E, \sigma) = \{m \in \mathbb{Z} \setminus \{0\} : (E - \sigma, E + \sigma) \cap G_m \neq \emptyset\}. \]

Define $m_0 \in \mathbb{Z} \setminus \{0\}$ with $|m_0| = \min_{m \in \mathcal{R}(E, \sigma)} |m|$. For any $m \in \mathcal{R}(E, \sigma)$, one has

\[ \text{dist}(G_m, G_{m_0}) \leq 2\sigma. \]

We first assume $(E - \sigma, E + \sigma) \cap G_0 = \emptyset$. Recalling (B.1), we have for any $m \in \mathcal{R}(E, \sigma)$ with $m \neq m_0$,

\[ 2\sigma \geq c_\star e^{-6\beta(\alpha)|m|}, \]

that is

\begin{equation}
|m| \geq -\frac{\ln(C_\star \sigma)}{6\beta(\alpha)}. \end{equation}
Then by Lemma B.1, we obtain
\[ \sum_{m \in \mathbb{R}(E,\sigma), m \neq m_0} \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_m \right) \]
\[ \leq \sum_{m \in \mathbb{R}(E,\sigma), m \neq m_0} (E_m^+ - E_m^-) \]
\[ \leq \sum_{|m| \geq -\ln(C\sigma)} C e^{-C^{-1}|m|} \]
\[ \leq \varepsilon_\sigma. \]

On the other hand, \( E \in \Sigma_{\lambda,\alpha} \) implies \( E \not\in G_{m_0} \). Thus we have
\[ \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_{m_0} \right) \leq \sigma. \]  (B.6)

In this case, (B.5) and (B.6) implies
\[ \text{Leb}\left( (E - \sigma, E + \sigma) \cap \Sigma_{\lambda,\alpha} \right) \]
\[ \geq 2\sigma - \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_{m_0} \right) \]
\[ - \sum_{m \in \mathbb{R}(E,\sigma), m \neq m_0} \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_m \right) \]
\[ \geq 2\sigma - \sigma - \varepsilon_\sigma \geq (1 - \varepsilon)\sigma. \]

In the case \( (E - \sigma, E + \sigma) \cap G_0 \neq \emptyset \), we have
\[ 0 < E_m^+ - E_{\min} \leq 2\sigma \]
for any \( m \in \mathbb{R}(E,\sigma) \). Thus, (B.4) also holds for any \( m \in \mathbb{R}(E,\sigma) \) by (B.2). From the proof of (B.5), we have
\[ \sum_{m \in \mathbb{R}(E,\sigma)} \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_m \right) \leq \varepsilon_\sigma. \]  (B.7)

Noticing that \( E \in \Sigma_{\lambda,\alpha} \) and \( E \not\in G_0 \), one has
\[ \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_0 \right) \leq \sigma. \]  (B.8)

By (B.7) and (B.8), we obtained
\[ \text{Leb}\left( (E - \sigma, E + \sigma) \cap \Sigma_{\lambda,\alpha} \right) \]
\[ \geq 2\sigma - \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_0 \right) \]
\[ - \sum_{m \in \mathbb{R}(E,\sigma)} \text{Leb}\left( (E - \sigma, E + \sigma) \cap G_m \right) \]
\[ \geq 2\sigma - \sigma - \varepsilon_\sigma \geq (1 - \varepsilon)\sigma. \]

Putting all the cases together, we complete the proof of Theorem 1.2. \( \square \)
References

[1] A. Avila. The absolutely continuous spectrum of the almost Mathieu operator. arXiv:0810.2965, 2008.
[2] A. Avila and S. Jitomirskaya. Almost localization and almost reducibility. J. Eur. Math. Soc., 12:93–131, 2010.
[3] A. Avila and S. Jitomirskaya. Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators. Comm. Math. Phys., 301(2):563–581, 2011.
[4] A. Avila, S. Jitomirskaya, and C. A. Marx. Spectral theory of extended Harper’s model and a question by Erdős and Szekeres. to appear in Invent. Math, 2017.
[5] J. Bourgain. Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime. Lett. Math. Phys., 51(2):83–118, 2000.
[6] A. Cai, C. Chavaudret, J. You, and Q. Zhou. Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles. arXiv:1706.08649, 2017.
[7] D. Damanik, M. Goldstein, and M. Lukic. The spectrum of a Schrödinger operator with small quasi-periodic potential is homogeneous. arXiv:1408.4335, 2014.
[8] F. Delyon and B. Souillard. The rotation number for finite difference operators and its properties. Comm. Math. Phys., 89(3):415–426, 1983.
[9] J. Fillman and M. Lukic. Spectral homogeneity of limit-periodic Schrödinger operators. J. Spectr. Theory, 7(2):387–406, 2017.
[10] M. Goldstein, D. Damanik, W. Schlag, and M. Voda. Homogeneity of the spectrum for quasi-periodic Schrödinger operators. arXiv:1505.04904, 2015.
[11] M. Goldstein and W. Schlag. Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. Ann. of Math, 154(1):155–203, 2001.
[12] M. Goldstein, W. Schlag, and M. Voda. On localization and the spectrum of multi-frequency quasi-periodic operators. arXiv:1610.00380, 2016.
[13] S. Hadj Amor. Hölder continuity of the rotation number for quasi-periodic co-cycles in SL(2, R). Comm. Math. Phys., 287(2):565–588, 2009.
[14] R. Han. Dry ten Martini problem for non self-dual extended Harper’s model. to appear in Trans. Amer. Math. Soc., 2017.
[15] S. Jitomirskaya and C. A. Marx. Analytic quasi-periodic cocycles with singularities and the Lyapunov exponent of extended Harper’s model. Comm. Math. Phys., 317(1):237–267, 2012.
[16] R. Johnson and J. Moser. The rotation number for almost periodic potentials. Comm. Math. Phys., 90(2):317–318, 1983.
[17] M. Leguil. Exponential decay of the size of spectral gaps for quasiperiodic Schrödinger operators. arXiv:1607.03422v3, 2016.
[18] W. Liu and Y. Shi. Upper bounds on the spectral gaps of quasi-periodic Schrödinger operators with Liouville frequencies. arXiv:1708.01760, 2017.
[19] W. Liu and X. Yuan. Hölder continuity of the spectral measures for one-dimensional Schrödinger operator in exponential regime. J. Math. Phys., 56(1):012701, 21, 2015.
[20] Y. Shi and X. Yuan. Exponential decay of the lengths of spectral gaps for extended Harper’s model with Liouville frequency. arXiv:1708.01763, 2017.
[21] K. Tao and M. Voda. Hölder continuity of the integrated density of states for quasi-periodic Jacobi operators. J. Spectr. Theory, 7(2):361–386, 2017.
[22] D. J. Thouless. Bandwidths for a quasiperiodic tight-binding model. Phys. Rev. B, 28(8):4272–4276, 1983.
[23] J. You and S. Zhang. Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycle with weak Liouville frequency. Ergodic Theory Dynam. Systems, 34(4):1395–1408, 2014.
