Generalized Fourier–Feynman transforms and generalized convolution products on Wiener space II

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Received: 14 August 2019 / Accepted: 17 September 2019 / Published online: 1 January 2020
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Abstract
The purpose of this article is to present the second type fundamental relationship between the generalized Fourier–Feynman transform and the generalized convolution product on Wiener space. The relationships in this article are also natural extensions (to the case on an infinite dimensional Banach space) of the structure which exists between the Fourier transform and the convolution of functions on Euclidean spaces.

Keywords Wiener space · Gaussian process · Generalized Fourier–Feynman transform · Generalized convolution product

Mathematics Subject Classification 46G12 · 28C20 · 60G15 · 60J65

1 Introduction
Given a positive real $T > 0$, let $C_0[0, T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on $[0, T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let $m$ denote Wiener measure. Then, as is well-known, $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space.

In [5–7,13] Huffman, Park, Skoug and Storvick established fundamental relationships between the analytic Fourier–Feynman transform (FFT) and the convolution product (CP) for functionals $F$ and $G$ on $C_0[0, T]$, as follows:

Communicated by Constantin Niculescu.
\[ T_q^{(p)}((F \ast G)_q)(y) = T_q^{(p)}(F) \left( \frac{y}{\sqrt{2}} \right) T_q^{(p)}(G) \left( \frac{y}{\sqrt{2}} \right) \]  \hspace{1cm} (1.1) \]

and

\[ (T_q^{(p)}(F) \ast T_q^{(p)}(G))_{-q}(y) = T_q^{(p)} \left( F \left( \frac{y}{\sqrt{2}} \right) G \left( \frac{y}{\sqrt{2}} \right) \right)(y) \]  \hspace{1cm} (1.2) \]

for scale-almost every \( y \in C_0[0, T] \), where \( T_q^{(p)}(F) \) and \((F \ast G)_q\) denote the \( L_p\) analytic FFT and the CP of functionals \( F \) and \( G \) on \( C_0[0, T] \). For an elementary introduction of the FFT and the corresponding CP, see [14].

For \( f \in L_2(\mathbb{R}) \), let the Fourier transform of \( f \) be given by

\[ \mathcal{F}(f)(u) = \int_{\mathbb{R}} e^{iuv}f(v)dm^n_L(v) \]

and for \( f, g \in L_2(\mathbb{R}) \), let the convolution of \( f \) and \( g \) be given by

\[ (f \ast g)(u) = \int_{\mathbb{R}} f(u-v)g(v)dm^n_L(v) \]

where \( dm^n_L(v) \) denotes the normalized Lebesgue measure \((2\pi)^{-1/2}dv\) on \( \mathbb{R} \). As commented in [3], the Fourier transform \( \mathcal{F} \) acts like a homomorphism with convolution \( \ast \) and ordinary multiplication on \( L_2(\mathbb{R}) \) as follows: for \( f, g \in L_2(\mathbb{R}) \)

\[ \mathcal{F}(f \ast g) = \mathcal{F}(f)\mathcal{F}(g). \]  \hspace{1cm} (1.3) \]

But the Fourier transform \( \mathcal{F} \) and the convolution \( \ast \) have a dual property such as

\[ \mathcal{F}(f) \ast \mathcal{F}(g) = \mathcal{F}(fg). \]  \hspace{1cm} (1.4) \]

Equations (1.1) and (1.2) above are natural extensions (to the case on an infinite dimensional Banach space) of Eqs. (1.3) and (1.4), respectively.

In [2,8], the authors extended the relationships (1.1) and (1.2) to the cases between the generalized FFT (GFFT) and the generalized CP (GCP) of functionals on \( C_0[0, T] \). The definition of the ordinary FFT and the corresponding CP are based on the Wiener integral, see [5–7]. While the definition of the GFFT and the GCP studied in [2,8] are based on the generalized Wiener integral [4,12]. The generalized Wiener integral (associated with Gaussian process) was defined by \( \int_{C_0[0,T]} F(Z_h(x, \cdot))dm(x) \) where \( Z_h \) is the Gaussian process on \( C_0[0, T] \times [0, T] \) given by \( Z_h(x, t) = \int_t^0 h(s)\tilde{d}x(s) \), and where \( h \) is a nonzero function in \( L_2[0, T] \) and \( \int_0^t h(s)\tilde{d}x(s) \) denotes the Paley–Wiener–Zygmund stochastic integral [9–11].

On the other hand, in [3], the authors defined a more general CP (see, Definition 2.3 below) and developed the relationship, such as (1.1), between their GFFT and the GCP (see, Theorem 3.4 below). Equation (3.3) in Theorem 3.4 is useful in that it permits one to calculate the GFFT of the GCP of functionals on \( C_0[0, T] \) without actually calculating the GCP.
In this paper we work with the second relationship, such as Eq. (1.2), between the GFFT and the GCP of functionals on \( C_0[0, T] \). Our new results corresponds to Eq. (1.4) rather than Eq. (1.3). It turns out, as noted in Remark 3.7 below, that our second relationship between the GFFT and the CP also permits one to calculate the GCP of the GFFT of functionals on \( C_0[0, T] \) without actually calculating the GCP.

2 Preliminaries

In order to present our relationship between the GFFT and the GCP, we follow the exposition of [3].

A subset \( B \) of \( C_0[0, T] \) is said to be scale-invariant measurable provided \( \rho B \in \mathcal{M} \) for all \( \rho > 0 \), and a scale-invariant measurable set \( N \) is said to be scale-invariant null provided \( m(\rho N) = 0 \) for all \( \rho > 0 \). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional \( F \) is said to be scale-invariant measurable provided \( F \) is defined on a scale-invariant measurable set and \( F(\rho \cdot) \) is Wiener-measurable for every \( \rho > 0 \). If two functionals \( F \) and \( G \) are equal s-a.e., we write \( F \approx G \).

Let \( \mathbb{C} \), \( \mathbb{C}_+ \) and \( \tilde{\mathbb{C}}_+ \) denote the set of complex numbers, complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively. For each \( \lambda \in \mathbb{C} \), \( \lambda^{1/2} \) denotes the principal square root of \( \lambda \); i.e., \( \lambda^{1/2} \) is always chosen to have positive real part, so that \( \lambda^{-1/2} = (\lambda^{-1})^{1/2} \) is in \( \mathbb{C}_+ \) for all \( \lambda \in \mathbb{C}_+ \).

Let \( h \) be a function in \( L_2[0, T] \backslash\{0\} \) and let \( F \) be a \( \mathbb{C} \)-valued scale-invariant measurable functional on \( C_0[0, T] \) such that

\[
\int_{C_0[0,T]} F(\lambda^{-1/2} Z_h(x, \cdot)) dm(x) = J(h; \lambda)
\]

exists as a finite number for all \( \lambda > 0 \). If there exists a function \( J^*(h; \lambda) \) analytic on \( \mathbb{C}_+ \) such that \( J^*(h; \lambda) = J(h; \lambda) \) for all \( \lambda > 0 \), then \( J^*(h; \lambda) \) is defined to be the generalized analytic Wiener integral (associated with the Gaussian process \( Z_h \)) of \( F \) over \( C_0[0, T] \) with parameter \( \lambda \), and for \( \lambda \in \mathbb{C}_+ \) we write

\[
\int_{C_0[0,T]}^{\text{anw}_\lambda} F(Z_h(x, \cdot)) dm(x) = J^*(h; \lambda).
\]

Let \( q \neq 0 \) be a real number and let \( F \) be a functional such that

\[
\int_{C_0[0,T]}^{\text{anw}_\lambda} F(Z_h(x, \cdot)) dm(x)
\]

exists for all \( \lambda \in \mathbb{C}_+ \). If the following limit exists, we call it the generalized analytic Feynman integral of \( F \) with parameter \( q \) and we write

\[
\int_{C_0[0,T]}^{\text{anf}_q} F(Z_h(x, \cdot)) dm(x) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\text{anw}_\lambda} F(Z_h(x, \cdot)) dm(x).
\]
Next (see [2,3,8]) we state the definition of the GFFT.

**Definition 2.1** Let \( h \) be a function in \( L_2[0, T] \backslash \{0\} \). For \( \lambda \in \mathbb{C}_+ \) and \( y \in C_0[0, T] \), let

\[
T_{\lambda,h}(F)(y) = \int_{C_0[0,T]} F(y + Z_{\lambda,h}(x, \cdot))dm(x).
\]

For \( p \in (1, 2) \) we define the \( L_p \) analytic GFFT (associated with the Gaussian process \( Z_{\lambda} \)), \( T_{q,\lambda}^{(p)}(F) \) of \( F \), by the formula,

\[
T_{q,\lambda}^{(p)}(F)(y) = \text{l.i.m.} \frac{T_{\lambda,h}(F)(y)}{\lambda \to -iq \lambda \in \mathbb{C}_+}
\]

if it exists; i.e., for each \( \rho > 0 \),

\[
\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_{\lambda,h}(F)(\rho y) - T_{q,\lambda}^{(p)}(F)(\rho y) \right|^{p'} dm(y) = 0
\]

where \( 1/p + 1/p' = 1 \). We define the \( L_1 \) analytic GFFT, \( T_{q,\lambda}^{(1)}(F) \) of \( F \), by the formula

\[
T_{q,\lambda}^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,h}(F)(y)
\]

for s-a.e. \( y \in C_0[0, T] \) whenever this limit exists.

We note that for \( p \in [1, 2] \), \( T_{q,\lambda}^{(p)}(F) \) is defined only s-a.e. We also note that if \( T_{q,\lambda}^{(p)}(F) \) exists and if \( F \approx G \), then \( T_{q,\lambda}^{(p)}(G) \) exists and \( T_{q,\lambda}^{(p)}(G) \approx T_{q,\lambda}^{(p)}(F) \). One can see that for each \( h \in L_2[0, T] \), \( T_{q,\lambda}^{(p)}(F) \approx T_{q,-\lambda}^{(p)}(F) \) since

\[
\int_{C_0[0,T]} F(x)dm(x) = \int_{C_0[0,T]} F(-x)dm(x).
\]

**Remark 2.2** Note that if \( h \equiv 1 \) on \([0, T]\), then the generalized analytic Feynman integral and the \( L_p \) analytic GFFT, \( T_{q,\lambda}^{(1)}(F) \), agree with the previous definitions of the analytic Feynman integral and the analytic FFT, \( T_{q}^{(p)}(F) \), respectively [5–7,13] because \( Z_1(x, \cdot) = x \) for all \( x \in C_0[0, T] \).

Next (see [3]) we give the definition of our GCP.

**Definition 2.3** Let \( F \) and \( G \) be scale-invariant measurable functionals on \( C_0[0, T] \). For \( \lambda \in \mathbb{C}_+ \) and \( h_1, h_2 \in L_2[0, T] \backslash \{0\} \), we define their GCP with respect to \( \{Z_{h_1}, Z_{h_2}\} \) (if it exists) by
\[(F \ast G)_{\lambda}^{(h_1,h_2)}(y) = \begin{cases} \int_{C_0[0,T]}^{\text{anw}_\lambda} F\left(\frac{y+Z_{h_1}(x,\cdot)}{\sqrt{2}}\right) G\left(\frac{y-Z_{h_2}(x,\cdot)}{\sqrt{2}}\right) dm(x), & \lambda \in \mathbb{C}_+ \\ \int_{C_0[0,T]}^{\text{anf}_q} F\left(\frac{y+Z_{h_1}(x,\cdot)}{\sqrt{2}}\right) G\left(\frac{y-Z_{h_2}(x,\cdot)}{\sqrt{2}}\right) dm(x), & \lambda = -iq, \ q \in \mathbb{R}, \ q \neq 0. \end{cases} \tag{2.1}\]

When \(\lambda = -iq\), we denote \((F \ast G)_{\lambda}^{(h_1,h_2)}\) by \((F \ast G)_{q}^{(h_1,h_2)}\).

**Remark 2.4** (i) Given a function \(h\) in \(L_2[0, T]\) \(\setminus\{0\}\) and letting \(h_1 = h_2 = h\), Eq. (2.1) yields the convolution product studied in [2, 8]:

\[(F \ast G)_{q}^{(h,h)}(y) \equiv (F \ast G)_{q,h}(y) = \int_{C_0[0,T]}^{\text{anf}_q} F\left(\frac{y+Z_h(x,\cdot)}{\sqrt{2}}\right) G\left(\frac{y-Z_h(x,\cdot)}{\sqrt{2}}\right) dm(x). \]

(ii) Choosing \(h_1 = h_2 = 1\), Eq. (2.1) yields the convolution product studied in [5–7, 13]:

\[(F \ast G)_{q}^{(1,1)}(y) \equiv (F \ast G)_{q}(y) = \int_{C_0[0,T]}^{\text{anf}_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) dm(x). \]

In order to establish our assertion we define the following conventions. Let \(h_1\) and \(h_2\) be nonzero functions in \(L_2[0, T]\). Then there exists a function \(s \in L_2[0, T]\) such that

\[s^2(t) = h_1^2(t) + h_2^2(t) \tag{2.2}\]

for \(m_L\text{-a.e. } t \in [0, T]\), where \(m_L\) denotes Lebesgue measure on \([0, T]\). Note that the function ‘s’ satisfying (2.2) is not unique. We will use the symbol \(s(h_1, h_2)\) for the functions ‘s’s’ that satisfy (2.2) above. Given nonzero functions \(h_1\) and \(h_2\) in \(L_2[0, T]\), infinitely many functions, \(s(h_1, h_2)\), exist in \(L_2[0, T]\). Thus \(s(h_1, h_2)\) can be considered as an equivalence class of the equivalence relation \(\sim\) on \(L_2[0, T]\) given by

\[s_1 \sim s_2 \iff s_1^2 = s_2^2 \text{ } m_L\text{-a.e.}. \]

But we observe that for every function \(s\) in the equivalence class \(s(h_1, h_2)\), the Gaussian random variable \(Z_s(x, T)\) has the normal distribution \(N(0, ||h_1||_2^2 + ||h_2||_2^2)\).

Inductively, given a sequence \(\mathcal{H} = \{h_1, \ldots, h_n\}\) of nonzero functions in \(L_2[0, T]\), let \(s(\mathcal{H}) \equiv s(h_1, h_2, \ldots, h_n)\) be the equivalence class of the functions \(s\) which satisfy the relation

\[s^2(t) = h_1^2(t) + \cdots + h_n^2(t) \]

for \(m_L\text{-a.e. } t \in [0, T]\). Throughout the rest of this paper, for convenience, we will regard \(s(\mathcal{H})\) as a function in \(L_2[0, T]\). We note that if the functions \(h_1, \ldots, h_n\) are
in $L_\infty[0, T]$, then we can take $s(\mathcal{H})$ to be in $L_\infty[0, T]$. By an induction argument it follows that
\[ s(s(h_1, h_2, \ldots, h_{k-1}), h_k) = s(h_1, h_2, \ldots, h_k) \]
for all $k \in \{2, \ldots, n\}$.

**Example 2.5** Let $h_1(t) = t^4$, $h_2(t) = \sqrt{2}t^3$, $h_3(t) = \sqrt{3}t^2$, $h_4(t) = \sqrt{2}t$, $h_5(t) = 1$, and $s(t) = t^4 + t^2 + 1$ for $t \in [0, T]$. Then $\mathcal{H} = \{h_1, h_2, h_3, h_4, h_5\}$ is a sequence of functions in $L_\infty[0, T]$ and it follows that
\[ s^2(t) = h_1^2(t) + h_2^2(t) + h_3^2(t) + h_4^2(t) + h_5^2(t). \]
Thus we can write $s = s(h_1, h_2, h_3, h_4, h_5)$. Furthermore, one can see that
\[ (-1)^m s = s((-1)^{n_1}h_1, (-1)^{n_2}h_2, (-1)^{n_3}h_3, (-1)^{n_4}h_4, (-1)^{n_5}h_5) \]
with $m, n_1, n_2, n_3, n_4, n_5 \in \{1, 2\}$. On the other hand, it also follows that
\[ s(h_1, h_2, h_3, h_4, h_5)(t) = s(g_1, g_2, g_3)(t) \]
for each $t \in [0, T]$, where $g_1(t) = -t^4 - 1$, $g_2(t) = \sqrt{2}t\sqrt{t^4 + 1}$, and $g_3(t) = t^2$ for $t \in [0, T]$.

**Example 2.6** Let $h_1(t) = t^4 + t^2$, $h_2(t) = t^4 - t^2$, $h_3(t) = \sqrt{2}t^3$, and $s(t) = \sqrt{2}(t^5 + t^4)$ for $t \in [0, T]$. Then, by the convention for $s$, it follows that
\[ s(t) = s(h_1, h_2)(t) = s(\sqrt{2}h_2, \sqrt{2}h_3)(t). \]

**Example 2.7** Using the well-known formulas for trigonometric and hyperbolic functions, it follows that
\[ \sec \left( \frac{\pi}{4}t \right) = s(\tan (\frac{\pi}{4} \cdot))(t) \]
\[ = s (\sin, \cos, \tan (\frac{\pi}{4} \cdot))(t) \]
\[ = s (\sin (\frac{\pi}{4} \cdot), \cos (\frac{\pi}{4} \cdot), \tan (\frac{\pi}{4} \cdot))(t), \]
\[ \cosh t = s(1, \sinh)(t) = s(-1, \sinh)(t) = s(\sin, \cos, \sinh)(t), \]
and
\[ -\coth \left( t + \frac{1}{2} \right) = s(1, \csch (\cdot + \frac{1}{2}))(t) = s(-\sin, \cos, -\csch (\cdot + \frac{1}{2}))(t) \]
for each $t \in [0, T]$.
3 The relationship between the GFFT and the GCP

The Banach algebra \( S(L_2[0, T]) \) consists of functionals on \( C_0[0, T] \) expressible in the form

\[
F(x) = \int_{L_2[0, T]} \exp[i \langle u, x \rangle] df(u)
\]  

(3.1)

for s-a.e. \( x \in C_0[0, T] \), where the associated measure \( f \) is an element of \( \mathcal{M}(L_2[0, T]) \), the space of \( \mathbb{C} \)-valued countably additive (and hence finite) Borel measures on \( L_2[0, T] \), and the pair \( \langle u, x \rangle \) denotes the Paley–Wiener–Zygmund stochastic integral \( Z_u(x, T) \equiv \int_0^T u(s) \tilde{a}x(t) \). For more details, see [1,4,8,13].

We first present two known results for the GFFT and the GCP of functionals in the Banach algebra \( S(L_2[0, T]) \).

**Theorem 3.1** [8] Let \( h \) be a nonzero function in \( L_\infty[0, T] \), and let \( F \in S(L_2[0, T]) \) be given by Eq. (3.1). Then, for all \( p \in [1, 2] \), the \( L^p \) analytic GFFT, \( T_{q,h}^{(p)}(F) \) of \( F \) exists for all nonzero real numbers \( q \), belongs to \( S(L_2[0, T]) \), and is given by the formula

\[
T_{q,h}^{(p)}(F)(y) = \int_{L_2[0, T]} \exp[i \langle u, y \rangle] df_t^h(u)
\]

for s-a.e. \( y \in C_0[0, T] \), where \( f_t^h \) is the complex measure in \( \mathcal{M}(L_2[0, T]) \) given by

\[
f_t^h(B) = \int_B \exp\left\{-\frac{i}{2q} \|uh\|_2^2\right\} df(u)
\]

for \( B \in \mathcal{B}(L_2[0, T]) \).

**Theorem 3.2** [3] Let \( k_1 \) and \( k_2 \) be nonzero functions in \( L_\infty[0, T] \) and let \( F \) and \( G \) be elements of \( S(L_2[0, T]) \) with corresponding finite Borel measures \( f \) and \( g \) in \( \mathcal{M}(L_2[0, T]) \). Then, the GCP \( (F \ast G)^{(k_1, k_2)}_q \) exists for all nonzero real \( q \), belongs to \( S(L_2[0, T]) \), and is given by the formula

\[
(F \ast G)^{(k_1, k_2)}_q(y) = \int_{L_2[0, T]} \exp[i \langle w, y \rangle] d\varphi^{k_1,k_2}_c(w)
\]

for s-a.e. \( y \in C_0[0, T] \), where

\[
\varphi^{k_1,k_2}_c = \varphi^{k_1,k_2} \circ \phi^{-1},
\]

\( \varphi^{k_1,k_2} \) is the complex measure in \( \mathcal{M}(L_2[0, T]) \) given by

\[
\varphi^{k_1,k_2}(B) = \int_B \exp\left\{-\frac{i}{4q} \|uk_1 - vk_2\|_2^2\right\} df(u)dg(v)
\]
for $B \in \mathcal{B}(L^2_{2}[0, T])$, and $\phi : L^2_{2}[0, T] \to L_2[0, T]$ is the continuous function given by $\phi(u, v) = (u + v)/\sqrt{2}$.

The following corollary and theorem will be very useful to prove our main theorem (namely, Theorem 3.6) which we establish the relationship between the GFFT and the GCP such as Eq. (1.2). The following corollary is a simple consequence of Theorem 3.1.

**Corollary 3.3** Let $h$ and $F$ be as in Theorem 3.1. Then, for all $p \in [1, 2]$, and all nonzero real $q$,

$$T_{-q, h}^{(p)}(T_{q, h}^{(p)}(F)) \approx F. \quad (3.2)$$

As such, the GFFT, $T_{q, h}^{(p)}$, has the inverse transform $\{T_{q, h}^{(p)}\}^{-1} = T_{-q, h}^{(p)}$.

The following theorem is due to Chang, Chung and Choi [3].

**Theorem 3.4** Let $k_1, k_2, F, G$ be as in Theorem 3.2, and let $h$ be a nonzero function in $L_{\infty}[0, T]$. Assume that $h^2 = k_1 k_2$ $m_{L}$-a.e. on $[0, T]$. Then, for all $p \in [1, 2]$ and all nonzero real $q$,

$$T_{q, h}^{(p)}(((F \ast G)_{q}^{(k_1, k_2)})(y) = T_{q, s(h, k_1)/\sqrt{2}}^{(p)}(F) \left(\frac{y}{\sqrt{2}}\right) T_{q, s(h, k_2)/\sqrt{2}}^{(p)}(G) \left(\frac{y}{\sqrt{2}}\right) \quad (3.3)$$

for $s$-a.e. $y \in C_0[0, T]$, where $s(h, k_j)$’s, $j \in \{1, 2\}$, are the functions which satisfy the relation (2.2), respectively.

**Remark 3.5** In Eq. (3.3), choosing $h = k_1 = k_2 \equiv 1$ yields Eq. (1.1) above. Also, letting $h = k_1 = k_2$ yields the results studied in [2,8]. As mentioned above, Eq. (3.3) is a more general extension of Eq. (1.3) to the case on an infinite dimensional Banach space.

We are now ready to establish our main theorem in this paper.

**Theorem 3.6** Let $k_1, k_2, F, G, h$ be as in Theorem 3.4. Then, for all $p \in [1, 2]$ and all nonzero real $q$,

$$\left(T_{q, s(h, k_1)/\sqrt{2}}^{(p)}(F) \ast T_{q, s(h, k_2)/\sqrt{2}}^{(p)}(G)\right)_{-q}^{(k_1, k_2)}(y) = T_{q, h}^{(p)}\left(F \left(\frac{y}{\sqrt{2}}\right) \ast G \left(\frac{y}{\sqrt{2}}\right)\right)(y) \quad (3.4)$$

for $s$-a.e. $y \in C_0[0, T]$, where $s(h, k_j)$’s, $j \in \{1, 2\}$, are the functions which satisfy the relation (2.2), respectively.
Proof Applying (3.2), (3.3) with $F$, $G$, and $q$ replaced with $T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F)$, $T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G)$, and $-q$, respectively, and (3.2) again, it follows that for s-a.e. $y \in C_0[0, T]$,

$$
\left( T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F) \ast T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G) \right)_{-q}^{(k_1,k_2)}(y) 
= T_{q,h}^{(p)} \left( T_{-q,h}^{(p)} \left( \left( T_{-q,s(h,k_1)/\sqrt{2}}^{(p)}(F) \ast T_{-q,s(h,k_2)/\sqrt{2}}^{(p)}(G) \right)_{-q}^{(k_1,k_2)} \right) \right)(y) 
= T_{q,h}^{(p)} \left( T_{-q,s(h,k_1)/\sqrt{2}}^{(p)} \left( T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F) \right) \left( \frac{\cdot}{\sqrt{2}} \right) \right) \times \left( T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G) \left( \frac{\cdot}{\sqrt{2}} \right) \right)(y) 
= T_{q,h}^{(p)} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right)(y)
$$
as desired. \qed

Remark 3.7 (i) Equation (3.3) shows that the GFFT of the GCP of two functionals is the ordinary product of their transforms. On the other hand, Eq. (3.4) above shows that the GCP of GFFTs of functionals is the GFFT of product of the functionals. These equations are useful in that they permit one to calculate $T_{q,h}^{(p)} ((F \ast G)_q^{(k_1,k_2)})$ and $T_{-q,s(h,k_1)/\sqrt{2}}^{(p)}(F) \ast T_{-q,s(h,k_2)/\sqrt{2}}^{(p)}(G)_{-q}^{(k_1,k_2)}$ without actually calculating the GCPs involved them, respectively. In practice, Eq. (3.4) tells us that to calculate $T_{q,h}^{(p)} (F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right))$ is easier to calculate than are $T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F)$, $T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(G)$, and $(T_{q,s(h,k_1)/\sqrt{2}}^{(p)}(F) \ast T_{q,s(h,k_2)/\sqrt{2}}^{(p)}(G))_{-q}^{(k_1,k_2)}$.

(ii) Equation (3.4) is a more general extension of Eq. (1.4) to the case on an infinite dimensional Banach space.

Corollary 3.8 (Theorem 3.1 in [13]) Let $F$ and $G$ be as in Theorem 3.2. Then, for all $p \in [1, 2]$ and all real $q \in \mathbb{R}\setminus\{0\}$,

$$
\left( T_{q}^{(p)}(F) \ast T_{q}^{(p)}(G) \right)_{-q}(y) = T_{q,h}^{(p)} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right)(y)
$$

for s-a.e. $y \in C_0[0, T]$, where $T_{q}^{(p)}(F)$ denotes the ordinary analytic FFT of $F$ and $(F \ast G)_q$ denotes the CP of $F$ and $G$ (see Remarks 2.2 and 2.4).

Proof In Eq. (3.4), simply choose $h = k_1 = k_2 \equiv 1$. \qed

Corollary 3.9 (Theorem 3.2 in [2]) Let $F$, $G$, and $h$ be as in Theorem 3.4. Then, for all $p \in [1, 2]$ and all real $q \in \mathbb{R}\setminus\{0\}$,

$$
\left( T_{q,h}^{(p)}(F) \ast T_{q,h}^{(p)}(G) \right)_{-q}(y) = T_{q,h}^{(p)} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right)(y)
$$
for s.a.e. \( y \in C_0[0, T] \), where \((F \ast G)_q \equiv (F \ast G)^{(h, h)}_q\) denotes the GCP of F and G studied in [2,8] (see Remark 2.4).

**Proof** In Eq. (3.4), simply choose \( h = k_1 = k_2 \).

\[ \square \]

### 4 Examples

The assertion in Theorem 3.6 above can be applied to many Gaussian processes \( Z_h \) with \( h \in L_\infty[0, T] \). In view of the assumption in Theorems 3.4 and 3.6, we have to check that there exist solutions \( \{h, k_1, k_2, s_1, s_2\} \) of the system

\[
\begin{align*}
(i) \quad & h^2 = k_1 k_2, \\
(ii) \quad & s_1 = s(h, k_1) \text{ m}_L\text{-a.e. on } [0, T], \\
(iii) \quad & s_2 = s(h, k_2) \text{ m}_L\text{-a.e. on } [0, T],
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
(i) \quad & h^2 = k_1 k_2, \\
(ii) \quad & s_1^2 = h^2 + k_1^2 \text{ m}_L\text{-a.e. on } [0, T], \\
(iii) \quad & s_2^2 = h^2 + k_2^2 \text{ m}_L\text{-a.e. on } [0, T].
\end{align*}
\]

Throughout this section, we will present some examples for the solution sets of the system (4.1). To do this we consider the Wiener space \( C_0[0, 1] \) and the Hilbert space \( L_2[0, 1] \) for simplicity.

**Example 4.1** (Polynomials) The set \( \mathcal{P} = \{h, k_1, k_2, s_1, s_2\} \) of functions in \( L_\infty[0, 1] \) with

\[
\begin{align*}
h(t) &= 2t(t^2 - 1) \\
k_1(t) &= (t^2 - 1)^2, \\
k_2(t) &= 4t^2, \\
s_1(t) &= (t^2 - 1)(t^2 + 1), \\
s_2(t) &= 2t(t^2 + 1)
\end{align*}
\]

is a solution set of the system (4.1). Thus

\[ s(h, k_1)(t) \equiv s_1(t) = (t^2 - 1)(t^2 + 1), \]

and

\[ s(h, k_2)(t) \equiv s_2(t) = 2t(t^2 + 1) \]

for all \( t \in [0, 1] \). In this case, Eq. (3.4) with the functions in \( \mathcal{P} \) holds for any functionals in \( F \) and \( G \) in \( S(L_2[0, 1]) \).
\textbf{Example 4.2} (Trigonometric functions I) The set $\mathcal{T}_1 = \{h, k_1, k_2, s_1, s_2\}$ of functions in $L_\infty[0, 1]$ with
\[
\begin{align*}
  h(t) &= \sin 2t = 2 \sin t \cos t, \\
  k_1(t) &= 2 \sin^2 t, \\
  k_2(t) &= 2 \cos^2 t, \\
  s_1(t) &= 2 \sin t, \\
  s_2(t) &= 2 \cos t
\end{align*}
\]
is a solution set of the system (4.1). Thus
\[
s(h, k_1)(t) \equiv s_1(t) = s(2 \sin \cos, 2 \sin^2)(t) = 2 \sin t,
\]
and
\[
s(h, k_2)(t) \equiv s_2(t) = s(2 \sin \cos, 2 \cos^2)(t) = 2 \cos t
\]
for all $t \in [0, 1]$. Also, using Eq. (3.4), it follows that for all $p \in [1, 2]$, all nonzero real $q$, and all functionals $F$ and $G$ in $\mathcal{S}(L_2[0, 1])$,
\[
\left( T_{q, \sqrt{2} \sin}^{(p)} (F) \ast T_{q, \sqrt{2} \cos}^{(p)} (G) \right)_{-q}^{(2 \sin^2, 2 \cos^2)} (y) = T_{q, 2 \sin}^{(p)} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right) (y)
\]
for s-a.e. $y \in C_0[0, 1]$.

\textbf{Example 4.3} (Trigonometric functions II) The set $\mathcal{T}_2 = \{h, k_1, k_2, s_1, s_2\}$ of functions in $L_\infty[0, 1]$ with
\[
\begin{align*}
  h(t) &= \sqrt{2} \sin t, \\
  k_1(t) &= \sqrt{2} \sin t \tan t, \\
  k_2(t) &= \sqrt{2} \cos t, \\
  s_1(t) &= \sqrt{2} \tan t, \\
  s_2(t) &= \sqrt{2}
\end{align*}
\]
is a solution set of the system (4.1). Thus
\[
s(h, k_1)(t) \equiv s_1(t) = s(\sqrt{2} \sin, \sqrt{2} \sin \tan)(t) = \sqrt{2} \tan t,
\]
and
\[
s(h, k_2)(t) \equiv s_2(t) = s(\sqrt{2} \sin, \sqrt{2} \cos)(t) = \sqrt{2} \quad (\text{constant function})
\]
for all $t \in [0, 1]$. 
Example 4.4 (Hyperbolic functions) The hyperbolic functions are defined in terms of the exponential functions $e^x$ and $e^{-x}$. The set $\mathcal{H} = \{ h, k_1, k_2, s_1, s_2 \}$ of functions in $L_\infty[0, 1]$ with

$$
\begin{align*}
  h(t) &= 1, \\
  k_1(t) &= \sinh (t + \frac{1}{2}), \\
  k_2(t) &= \text{csch}(t + \frac{1}{2}), \\
  s_1(t) &= \cosh (t + \frac{1}{2}), \\
  s_2(t) &= \coth (t + \frac{1}{2})
\end{align*}
$$

is a solution set of the system (4.1). Thus

$$
\begin{align*}
  s(h, k_1)(t) &= s_1(t) = s \left( 1, \sinh \left( \cdot + \frac{1}{2} \right) \right)(t) = \cosh \left( t + \frac{1}{2} \right), \\
  s(h, k_2)(t) &= s_2(t) = s \left( 1, \text{csch} \left( \cdot + \frac{1}{2} \right) \right)(t) = \coth \left( t + \frac{1}{2} \right)
\end{align*}
$$

for all $t \in [0, 1]$.

5 Iterated GFFTs and GCPs

In this section, we present general relationships between the iterated GFFT and the GCP for functionals in $\mathcal{S}(L_2[0, T])$ which are developments of (3.4). To do this we quote a result from [3].

Theorem 5.1 Let $F \in \mathcal{S}(L_2[0, T])$ be given by Eq. (3.1), and let $\mathcal{H} = \{ h_1, \ldots, h_n \}$ be a finite sequence of nonzero functions in $L_\infty[0, T]$. Then, for all $p \in [1, 2]$ and all nonzero real $q$, the iterated $L_p$ analytic GFFT,

$$
T_{q, h_n}^{(p)} \left( T_{q, h_{n-1}}^{(p)} \left( \cdots \left( T_{q, h_2}^{(p)} \left( T_{q, h_1}^{(p)} (F) \right) \right) \right) \cdots \right)
$$

of $F$ exists, belongs to $\mathcal{S}(L_2[0, T])$, and is given by the formula

$$
T_{q, h_n}^{(p)} \left( T_{q, h_{n-1}}^{(p)} \left( \cdots \left( T_{q, h_2}^{(p)} \left( T_{q, h_1}^{(p)} (F) \right) \right) \right) \cdots \right)(y) = \int_{L_2[0, T]} \exp \{ i \langle u, y \rangle \} df_{t_1^{h_1}, \ldots, t_n^{h_n}}(u)
$$

for $s$-a.e. $y \in C_0[0, T]$, where $f_{t_1^{h_1}, \ldots, t_n^{h_n}}$ is the complex measure in $\mathcal{M}(L_2[0, T])$ given by

$$
f_{t_1^{h_1}, \ldots, t_n^{h_n}}(B) = \int_B \exp \left\{ - \frac{i}{2q} \sum_{j=1}^n \| u h_j \|_2^2 \right\} df(u)
$$
for \( B \in \mathcal{B}(L_2[0, T]) \). Moreover it follows that

\[
T_{q,h_n}^{(p)}(T_{q,h_{n-1}}^{(p)}(\cdots (T_{q,h_2}^{(p)}(T_{q,h_1}^{(p)}(F))\cdots ))) (y) = T_{q,s(H)}^{(p)}(F)(y) \quad (5.1)
\]

for \( s \)-a.e. \( y \in C_0[0, T] \), where \( s(H) \equiv s(h_1, \ldots, h_n) \) is a function in \( L_\infty[0, T] \) satisfying the relation

\[
s(H)^2(t) = h_1^2(t) + \cdots + h_n^2(t) \quad (5.2)
\]

for \( m_L \)-a.e. \( t \in [0, T] \).

We next establish two types of extensions of Theorem 3.6 above.

**Theorem 5.2** Let \( k_1, k_2, F, \) and \( G \) be as in Theorem 3.2, and let \( H = \{h_1, \ldots, h_n\} \) be a finite sequence of nonzero functions in \( L_\infty[0, T] \). Assume that

\[
s(H)^2 \equiv s(h_1, \ldots, h_n)^2 = k_1k_2
\]

for \( m_L \)-a.e. on \([0, T]\), where \( s(H) \) is the function in \( L_\infty[0, T] \) satisfying (5.2) above. Then, for all \( p \in [1, 2] \) and all nonzero real \( q \),

\[
\left( T_{q,k_1/\sqrt{2}}^{(p)}(T_{q,h_n/\sqrt{2}}^{(p)}(\cdots (T_{q,h_2/\sqrt{2}}^{(p)}(T_{q,h_1/\sqrt{2}}^{(p)}(F))\cdots ))) \right) * T_{q,k_2/\sqrt{2}}^{(p)}(T_{q,h_n/\sqrt{2}}^{(p)}(\cdots (T_{q,h_2/\sqrt{2}}^{(p)}(T_{q,h_1/\sqrt{2}}^{(p)}(G))\cdots )))^{(k_1,k_2)}_q (y) = T_{q,s(H,k_1)\sqrt{2}}^{(p)}(F) * T_{q,s(H,k_2)\sqrt{2}}^{(p)}(G)^{(k_1,k_2)}_q (y) = T_{q,s(H)}^{(p)}\left( F\left( \frac{\cdot}{\sqrt{2}} \right) G\left( \frac{\cdot}{\sqrt{2}} \right) \right) (y) \quad (5.3)
\]

for \( s \)-a.e. \( y \in C_0[0, T] \), where \( s(H, k_1) \) and \( s(H, k_2) \) are functions in \( L_\infty[0, T] \) satisfying the relations

\[
s(H, k_1)^2 \equiv s(h_1, \ldots, h_n, k_1)^2 = h_1^2 + \cdots + h_n^2 + k_1^2
\]

and

\[
s(H, k_2)^2 \equiv s(h_1, \ldots, h_n, k_2)^2 = h_1^2 + \cdots + h_n^2 + k_2^2
\]

for \( m_L \)-a.e. on \([0, T]\), respectively.

**Proof** Applying (5.1), the first equality of (5.3) follows immediately. Next using (3.4) with \( h \) replaced with \( s(H) \), the second equality of (5.3) also follows. \( \square \)

In view of Eqs. (5.1) and (3.4), we also obtain the following assertion.
Theorem 5.3 Let $F$ and $G$ be as in Theorem 3.2. Given a nonzero function $h$ in $L_\infty[0, T]$ and finite sequences $K_1 = \{k_{11}, k_{12}, \ldots, k_{1n}\}$ and $K_2 = \{k_{21}, k_{22}, \ldots, k_{2m}\}$ of nonzero functions in $L_\infty[0, T]$, assume that

$$h^2 = s(K_1)s(K_2)$$

for $m_L$-a.e. on $[0, T]$. Then, for all $p \in [1, 2]$ and all nonzero real $q$,

$$\begin{align*}
(T^{(p)}_{q,h/\sqrt{2}}(T^{(p)}_{q,k_{1n}/\sqrt{2}}(\cdots (T^{(p)}_{q,k_{12}/\sqrt{2}}(T^{(p)}_{q,k_{11}/\sqrt{2}}(F)) \cdots)))
& \ast T^{(p)}_{q,h/\sqrt{2}}(T^{(p)}_{q,k_{2m}/\sqrt{2}}(\cdots (T^{(p)}_{q,k_{22}/\sqrt{2}}(T^{(p)}_{q,k_{21}/\sqrt{2}}(G)) \cdots)))
\right)_{-q}^{(s(K_1), s(K_2))} \\
& = (T^{(p)}_{q,h/\sqrt{2}}(T^{(p)}_{q,s(h,K_1)/\sqrt{2}}(F)) \ast T^{(p)}_{q,h/\sqrt{2}}(T^{(p)}_{q,s(h,K_2)/\sqrt{2}}(G)))_{-q}^{(s(K_1), s(K_2))} \\
& = (T^{(p)}_{q,h}(F(\frac{\cdot}{\sqrt{2}})G(\frac{\cdot}{\sqrt{2}})))_{-q}^{(s(K_1), s(K_2))}
\end{align*}$$

(5.4)

for $s$-a.e. $y \in C_0[0, T])$, where $s(h, s(K_1))$, and $s(h, s(K_2))$ are functions in $L_\infty[0, T]$ satisfying the relations

$$s(h, s(K_1))^2 = h^2 + s(K_1)^2 = h^2 + k_{11}^2 + \cdots + k_{1n}^2,$$

and

$$s(h, s(K_2))^2 = h^2 + s(K_2)^2 = h^2 + k_{21}^2 + \cdots + k_{2m}^2$$

for $m_L$-a.e. on $[0, T]$, respectively.

Remark 5.4 Note that given the functions $\{s(\mathcal{H}), k_1, k_2, s(\mathcal{H}, k_1), s(\mathcal{H}, k_2)\}$ in Theorem 5.2, the set $\mathcal{F} = \{h, k_1, k_2, s_1, s_2\}$ of functions in $L_\infty[0, T]$ with

$$\begin{align*}
h(t) &= s(\mathcal{H})(t), \\
s_1(t) &= s(\mathcal{H}, k_1)(t), \\
s_2(t) &= s(\mathcal{H}, k_2)(t)
\end{align*}$$

is a solution set of the system (4.1). Also, given the functions

$$\{h, s(K_1), s(K_2), s(h, s(K_1)), s(h, s(K_2))\}$$
in Theorem 5.3, the set $\mathcal{F} = \{h, k_1, k_2, s_1, s_2\}$ of functions in $L_\infty[0, T]$ with

$$
\begin{align*}
    k_1(t) &= s(K_1)(t), \\
    k_2(t) &= s(K_2)(t), \\
    s_1(t) &= s(h, s(K_1))(t), \\
    s_2(t) &= s(h, s(K_2))(t)
\end{align*}
$$

is a solution set of the system (4.1).

In the following two examples, we also consider the Wiener space $C_0[0, 1]$ and the Hilbert space $L_\infty[0, 1]$ for simplicity.

**Example 5.5** Let $h_1 = \sin \frac{\pi}{4}(t + \frac{1}{2})$, $h_2 = \cos \frac{\pi}{4}(t + \frac{1}{2})$, $h_3 = \tan \frac{\pi}{4}(t + \frac{1}{2})$, $k_1(t) = \tan \frac{\pi}{4}(t + \frac{1}{2})$, and $k_2(t) = \sec \frac{\pi}{4}(t + \frac{1}{2}) \csc \frac{\pi}{4}(t + \frac{1}{2})$ on $[0, 1]$. Then $\{h_1, h_2, h_3, k_1, k_2\}$ is a set of functions in $L_\infty[0, 1]$, and given the set $\mathcal{H} = \{h_1, h_2, h_3\}$, it follows that

$$
s(\mathcal{H})(t) = s(h_1, h_2, h_3)^2(t) = s(\sin \frac{\pi}{4}(\cdot + \frac{1}{2}), \cos \frac{\pi}{4}(\cdot + \frac{1}{2}), \tan \frac{\pi}{4}(\cdot + \frac{1}{2}))^2(t) = \sec^2 \frac{\pi}{4}(t + \frac{1}{2}) = k_1(t)k_2(t),
$$

and

$$
s(\mathcal{H}, k_1)^2(t) = s(h_1, h_2, h_3, k_1)^2(t) = \sec^2 \frac{\pi}{4}(t + \frac{1}{2}) + \tan^2 \frac{\pi}{4}(t + \frac{1}{2}) = s(\mathcal{H}, k_1)^2(t),
$$

for all $t \in [0, 1]$. From this we see that the set $\mathcal{F}_1 = \{h, k_1, k_2, s_1, s_2\}$ of functions in $L_\infty[0, 1]$ with

$$
\begin{align*}
    h(t) &= s(h_1, h_2, h_3)(t) = \sec \frac{\pi}{4}(t + \frac{1}{2}), \\
    k_1(t) &= \tan \frac{\pi}{4}(t + \frac{1}{2}), \\
    k_2(t) &= \sec \frac{\pi}{4}(t + \frac{1}{2}) \csc \frac{\pi}{4}(t + \frac{1}{2}), \\
    s_1(t) &= s(\mathcal{H}, k_1)(t), \\
    s_2(t) &= s(\mathcal{H}, k_2)(t)
\end{align*}
$$

is a solution set of the system (4.1), and Eq. (5.3) holds with the sequence $\mathcal{H} = \{h_1, h_2, h_3\}$ and the functions $k_1$ and $k_2$.

In the next example, the kernel functions of the Gaussian processes defining the transforms and convolutions involve trigonometric and hyperbolic (and hence exponential) functions.
**Example 5.6** Consider the function
\[ h(t) = 2\sqrt{\csc \frac{\pi}{4}}(t + \frac{1}{2}) \cosh \frac{\pi}{4}(t + \frac{1}{2}) \]
on [0, 1], and the finite sequences
\[ K_1 = \{ 2\tanh \frac{\pi}{4}(t + \frac{1}{2}), 2\text{sech} \frac{\pi}{4}(t + \frac{1}{2}), 2\cot \frac{\pi}{4}(t + \frac{1}{2}) \} \]
and
\[ K_2 = \{ \sqrt{2}\sin \frac{\pi}{4}(t + \frac{1}{2}), \sqrt{2}\cos \frac{\pi}{4}(t + \frac{1}{2}), \sqrt{2}\sinh \frac{\pi}{4}(t + \frac{1}{2}), \sqrt{2}\cosh \frac{\pi}{4}(t + \frac{1}{2}) \} \]
of functions in \( L_\infty[0, 1] \). Then using the relationships among hyperbolic functions and among trigonometric functions, one can see that
\[ s(K_1)(t) = 2\csc \frac{\pi}{4}(t + \frac{1}{2}) \text{ and } s(K_2)(t) = 2\cosh \frac{\pi}{4}(t + \frac{1}{2}) \]
on [0, 1]. From this we also see that the set \( F_1 = \{ h, k_1, k_2, s_1, s_2 \} \) of functions in \( L_\infty[0, 1] \) with
\[
\begin{align*}
  h(t) &= 2\sqrt{\csc \frac{\pi}{4}}(t + \frac{1}{2}) \cosh \frac{\pi}{4}(t + \frac{1}{2}), \\
  k_1(t) &= s(K_1)(t) = 2\csc \frac{\pi}{4}(t + \frac{1}{2}), \\
  k_2(t) &= s(K_2)(t) = 2\cosh \frac{\pi}{4}(t + \frac{1}{2}), \\
  s_1(t) &= s(h, s(K_1))(t), \\
  s_2(t) &= s(h, s(K_2))(t)
\end{align*}
\]
is a solution set of the system (4.1), and Eq. (5.4) holds with the function \( h \), and the sequences \( K_1 \) and \( K_2 \).

### 6 Further results

In this section, we derive a more general relationship between the iterated GFFT and the GCP for functionals in \( S(L_2[0, T]) \). To do this we also quote a result from [3].

**Theorem 6.1** Let \( F \) and \( \mathcal{H} = \{ h_1, \ldots, h_n \} \) be as in Theorem 5.1. Assume that \( q_1, q_2, \ldots, q_n \) are nonzero real numbers with \( \text{sgn}(q_1) = \cdots = \text{sgn}(q_n) \), where ‘\( \text{sgn} \)’ denotes the sign function. Then, for all \( p \in [1, 2] \),
\[
T^{(p)}_{g_n, h_n} \left( T^{(p)}_{q_{n-1}, h_{n-1}} \left( \cdots \left( T^{(p)}_{q_2, h_2} \left( T^{(p)}_{q_1, h_1} (F) \right) \right) \cdots \right) \right)(y)
\]
\[
= T^{(p)}_{\alpha_n, \tau_n^{(n)}} \left( T^{(p)}_{\alpha_n, \tau_n^{(n-1)}} \left( \cdots \left( T^{(p)}_{\alpha_n, \tau_n^{(2)}} \left( T^{(p)}_{\alpha_n, \tau_n^{(1)}} (F) \right) \right) \cdots \right) \right)(y)
\]
for s-a.e. \( y \in C_0[0, T] \), where \( \alpha_n \) is given by

\[
\alpha_n = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n}
\]

and \( \tau_n^{(j)} = \sqrt{\alpha_n/q_j} \) for each \( j \in \{1, \ldots, n\} \). Moreover it follows that

\[
T^{(p)}_{q_n,h_n} (T^{(p)}_{q_{n-1},h_{n-1}} \left( \cdots (T^{(p)}_{q_2,h_2} (T^{(p)}_{q_1,h_1} (F)) \cdots ) \right))(y) = T^{(p)}_{\alpha_n,s(\tau H)} (F)(y)
\]

for s-a.e. \( y \in C_0[0, T] \), where \( s(\tau H) \equiv s(\tau_1^{(1)} h_1, \ldots, \tau_n^{(n)} h_n) \) is a function in \( L_\infty[0, T] \) satisfying the relation

\[
s(\tau H)^2(t) = (\tau_1^{(1)} h_1)^2(t) + \cdots + (\tau_n^{(n)} h_n)^2(t)
\]

for \( m_L \)-a.e. \( t \in [0, T] \).

Next, by a careful examination we see that for all \( F \in S(L^2[0, T]) \) and any positive real \( \beta > 0 \),

\[
T^{(p)}_{\beta q,h} (F) \approx T^{(p)}_{q,h/\sqrt{\beta}} (F).
\]

Using (6.1) and (3.4), we have the following lemma.

**Lemma 6.2** Let \( k_1, k_2, F, G, \) and \( h \) be as in Theorem 3.4. Let \( q, q_1, \) and \( q_2 \) be nonzero real numbers with \( \text{sgn}(q) = \text{sgn}(q_1) = \text{sgn}(q_2) \). Then, for all \( p \in [1, 2] \),

\[
\left( T^{(p)}_{q_1,\sqrt{2}/(2q) s(h,k_1)} (F) * T^{(p)}_{q_2,\sqrt{2}/(2q) s(h,k_2)} (G) \right)^{(k_1,k_2)} (y) = T^{(p)}_{q,h} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right)(y)
\]

for s-a.e. \( y \in C_0[0, T] \).

Finally, in view of Theorem 6.1 and Lemma 6.2, we obtain the following assertion.

**Theorem 6.3** Let \( k_1, k_2, F, G, \) and \( h \) be as in Theorem 3.4. Let \( \mathcal{H}_1 = \{h_1\}_{j=1}^n \) and \( \mathcal{H}_2 = \{h_2\}_{l=1}^m \) be finite sequences of nonzero functions in \( L_\infty[0, T] \). Given nonzero real numbers \( q, q_1, q_{11}, \ldots, q_{1n}, q_2, q_{21}, \ldots, q_{2m} \) with

\[
\text{sgn}(q) = \text{sgn}(q_1) = \text{sgn}(q_{11}) = \cdots = \text{sgn}(q_{1n}) = \text{sgn}(q_2) = \text{sgn}(q_{21}) = \cdots = \text{sgn}(q_{2m}),
\]

let

\[
\alpha_{1n} = \frac{1}{q_{11}} + \frac{1}{q_{12}} + \cdots + \frac{1}{q_{1n}},
\]


\[ \alpha_{2m} = \frac{1}{\frac{1}{q_2} + \frac{1}{q_2} + \cdots + \frac{1}{q_{2m}}} , \]

\[ \beta_{1n} = \frac{1}{\frac{1}{q_1} + \frac{1}{q_1} + \frac{1}{q_{12}} + \cdots + \frac{1}{q_{1n}}} , \]

and

\[ \beta_{2m} = \frac{1}{\frac{1}{q_2} + \frac{1}{q_2} + \cdots + \frac{1}{q_{2m}}} . \]

Furthermore, assume that

\[ h^2 = s(\tau_{1n} \mathcal{H}_1) s(\tau_{2m} \mathcal{H}_2) \]

for \( m \) \( L \)-a.e. on \([0, T]\), where \( s(\tau_{1n} \mathcal{H}_1) \) and \( s(\tau_{2m} \mathcal{H}_2) \) are functions in \( L_\infty[0, T] \) satisfying the relation

\[ s(\tau_{1n} \mathcal{H}_1)^2 = s(\tau_{1n} h_{11}, \ldots, \tau_{1n} h_{1n})^2 = (\tau_{1n} h_{11})^2 + \cdots + (\tau_{1n} h_{1n})^2 \]

and

\[ s(\tau_{2m} \mathcal{H}_2)^2 = s(\tau_{2m} h_{21}, \ldots, \tau_{2m} h_{2m})^2 = (\tau_{2m} h_{21})^2 + \cdots + (\tau_{2m} h_{2m})^2 , \]

respectively, and where \( \tau_{1n}^{(j)} = \frac{\sqrt{\alpha_{1n}/q_{1j}}}{\sqrt{m}} \) for each \( j \in \{1, \ldots, n\} \), and \( \tau_{2m}^{(l)} = \frac{\sqrt{\alpha_{2m}/q_{2l}}}{\sqrt{m}} \) for each \( l \in \{1, \ldots, m\} \). For notational convenience, let

\[ h'_1 = \sqrt{\frac{q_1}{2q}} h, \quad h'_j = \sqrt{\frac{\alpha_{1n}/q_j}{2q}} h_{1j} , \quad j = 1, \ldots, n , \]

and let

\[ h'_2 = \sqrt{\frac{q_2}{2q}} h, \quad h'_{2l} = \sqrt{\frac{\alpha_{2m}/q_{2l}}{2q}} h_{2l} , \quad l = 1, \ldots, m . \]

Then, for all \( p \in [1, 2] \),

\[
\begin{align*}
&\left( T^{(p)}_{q_1, h'_1} \left( T^{(p)}_{q_{1n}, h_{1n}'} \left( \cdots \left( T^{(p)}_{q_{11}, h_{11}'} (F) \right) \cdots \right) \right) \right)
	\ast T^{(p)}_{q_2, h'_2} \left( T^{(p)}_{q_{2m}, h_{2m}'} \left( \cdots \left( T^{(p)}_{q_{21}, h_{21}'} (G) \right) \cdots \right) \right)_{-q}^{(s(\tau_{1n} \mathcal{H}_1), s(\tau_{2m} \mathcal{H}_2))} \\
&= \left( T^{(p)}_{q_1, \sqrt{q_1/(2q)} h} \left( T^{(p)}_{q_{1n}, \sqrt{\alpha_{1n}/q_{1n}}(2q)} s(\tau_{1n} \mathcal{H}_1) (F) \right) \right)
	\ast T^{(p)}_{q_2, \sqrt{q_2/(2q)} h} \left( T^{(p)}_{q_{2m}, \sqrt{\alpha_{2m}/q_{2m}}(2q)} s(\tau_{2m} \mathcal{H}_2) (G) \right)_{-q}^{(s(\tau_{1n} \mathcal{H}_1), s(\tau_{2m} \mathcal{H}_2))} \\
&= \left( T^{(p)}_{\beta_{1n}, \sqrt{\beta_{1n}/(2q)} s(h, s(\tau_{1n} \mathcal{H}_1)) (F) \right)
\end{align*}
\]
\[ *T_{\beta_2 m, \sqrt{2} \beta_2 m/(2q)}^{(p)} (s(\tau_1 H_1), s(\tau_2 H_2)) (y) = T_{q, h}^{(p)} \left( F \left( \frac{.}{\sqrt{2}} \right) G \left( \frac{.}{\sqrt{2}} \right) \right) (y) \]

for s-a.e. \( y \in C_0[0, T] \).

Acknowledgements  The authors would like to express their gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper. The present research was supported by the research fund of Dankook University in 2019.

References

1. Cameron, R.H., Storvick, D.A.: Some Banach Algebras of Analytic Feynman Integrable Functionals. Analytic Functions, Kozubnik 1979 (Proc. Seventh Conf., Kozubnik, 1979), Lecture Notes in Math., vol. 798, pp. 18–67. Springer, Berlin (1980)
2. Chang, K.S., Cho, D.H., Kim, B.S., Song, T.S., Yoo, I.: Relationships involving generalized Fourier–Feynman transform, convolution and first variation. Integral Transforms Spec. Funct. 16, 391–405 (2005)
3. Chang, S.J., Chung, H.S., Choi, J.G.: Generalized Fourier–Feynman transforms and generalized convolution products on Wiener space. Indag. Math. 28, 566–579 (2017)
4. Chung, D.M., Park, C., Skoug, D.: Generalized Feynman integrals via conditional Feynman integrals. Mich. Math. J. 40, 377–391 (1993)
5. Huffman, T., Park, C., Skoug, D.: Analytic Fourier–Feynman transforms and convolution. Trans. Am. Math. Soc. 347, 661–673 (1995)
6. Huffman, T., Park, C., Skoug, D.: Convolutions and Fourier–Feynman transforms of functionals involving multiple integrals. Mich. Math. J. 43, 247–261 (1996)
7. Huffman, T., Park, C., Skoug, D.: Convolution and Fourier–Feynman transforms. Rocky Mt. J. Math. 27, 827–841 (1997)
8. Huffman, T., Park, C., Skoug, D.: Generalized transforms and convolutions. Int. J. Math. Math. Sci. 20, 19–32 (1997)
9. Paley, R.E.A.C., Wiener, N., Zygmund, A.: Notes on random functions. Math. Z. 37, 647–668 (1933)
10. Park, C.: A generalized Paley–Wiener–Zygmund integral and its applications. Proc. Am. Math. Soc. 23, 388–400 (1969)
11. Park, C., Skoug, D.: A note on Paley–Wiener–Zygmund stochastic integrals. Proc. Am. Math. Soc. 103, 591–601 (1988)
12. Park, C., Skoug, D.: A Kac–Feynman integral equation for conditional Wiener integrals. J. Integral Equations Appl. 3, 411–427 (1991)
13. Park, C., Skoug, D., Storvick, D.: Relationships among the first variation, the convolution product, and the Fourier–Feynman transform. Rocky Mt. J. Math. 28, 1447–1468 (1998)
14. Skoug, D., Storvick, D.: A survey of results involving transforms and convolutions in function space. Rocky Mt. J. Math. 34, 1147–1175 (2004)