We present results from a lattice Monte Carlo study of the Nambu – Jona-Lasinio model in 3+1 dimensions with a baryon chemical potential $\mu \neq 0$. As $\mu$ is increased there is a transition from a chirally-broken phase to relativistic quark matter, in which baryon number symmetry appears spontaneously broken by a diquark condensate at the Fermi surface, implying a superfluid ground state. Finite volume corrections to this relativistic BCS scenario, however, are anomalously large.

The proposal that at high baryon density a condensation of diquark pairs in degenerate quark matter takes place, leading to color superconducting ground states, has generated a profusion of theoretical and phenomenological interest $^1$. Most approaches assume the existence of a Fermi surface and then consider its instability with respect to diquark condensation arising from an attractive $qq$ interaction. The resulting excitation energy gap $\Delta$ can be calculated self-consistently in close analogy to the BCS gap in electronic superconductors. Model calculations yield gaps as large as 50 - 100MeV $^2$.

A first-principles calculation of the ground state of nuclear or quark matter using the methods of lattice QCD, however, is hampered by the “sign problem”, i.e. $\exp(-S/\hbar)$ is no longer positive definite, and cannot be used as a probability measure in importance sampling the Euclidean path integral, once baryon chemical potential $\mu \neq 0$. It is still possible, however, to study simpler models of strong interactions with $\mu \neq 0$ using lattice methods $^3$. The Nambu – Jona-Lasino (NJL) model, which closely resembles the original BCS model, is particularly interesting in the current context, because it is the only simulable model which exhibits a Fermi

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surface for sufficiently large $\mu$. The Lagrangian density is

$$L = \bar{\psi}(\not{\partial} + m_0 + \mu \gamma_0)\psi - \frac{g^2}{2} \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2 \right].$$

(1)

Here $\psi$ and $\bar{\psi}$ are isospinor “quark” fields carrying baryon charge; note that both $\bar{q}q$ and $qq$ interactions are attractive in the isoscalar channel.

For coupling strength $g^2$ sufficiently large the vacuum ground state has chiral SU(2)$_L \otimes$SU(2)$_R$ symmetry spontaneously broken to SU(2)$_{\text{isospin}}$ by generation of a chiral condensate and resulting “constituent” quark mass $\Sigma \approx g^2 \langle \bar{\psi}\psi \rangle$. At zero temperature this state is stable as $\mu$ is increased until at $\mu_c \sim \Sigma$ chiral symmetry is restored, at which point the baryon density $n_B = \langle \bar{\psi}\gamma_0\psi \rangle$ rises from zero. For $\mu > \mu_c$ the ground state is thus relativistic quark matter with Fermi momentum $k_F \sim \mu$. A natural question is whether $qq$ attraction in this phase leads to a diquark condensate spontaneously breaking the U(1)$_B$ symmetry of (1) leading (since in this case the broken symmetry is global rather than local) to superfluidity.

The NJL model in 2+1 dimensions has been studied extensively using Monte Carlo simulations. While there is enhanced $qq$ pairing in the scalar isoscalar channel for $\mu > \mu_c$, simulations with an explicit U(1)$_B$-violating source term $jqq$ found no evidence for diquark condensation in the $j \to 0$ limit. Instead, the condensate scales approximately as $\langle qq(j) \rangle \propto j^{\frac{1}{2}}$, the exponent $\delta$ increasing with $\mu$. Moreover, the quasiparticle dispersion relation in the spin-$\frac{1}{2}$ channel is consistent with a vanishing BCS gap $\Delta = 0$. The interpretation is that long-wavelength fluctuations of the complex phase of the $qq$ wavefunction destroy the condensate, but that long-range phase coherence remains (ie. correlations are dominated by “spin-waves” and remain algebraic functions of spatial separation) just as in the low-$T$ phase of the 2$d$ XY model. Superfluidity in two dimensions is realised in Kosterlitz-Thouless mode; namely, persistent flow patterns can only be disrupted by creating a vortex/anti-vortex excitation and then translating one member of the pair around the universe, which in the low-$T$ phase entails an energy cost increasing logarithmically with system size.

This rather exotic scenario, namely gapless thin film superfluidity, is inherently non-perturbative and is not exposed by self-consistent approaches. It thus justifies the application of numerical simulations, and begs the question of whether similar surprises will emerge from a study of the phenomenologically-motivated case of NJL$_{3+1}$. In 3+1$d$ the model is non-renormalisable, and hence its physical predictions are sensitive to the details of the ultra-violet cutoff. We have matched our lattice model to
low-energy QCD following the procedure of Klevansky, working to leading order in an expansion in $1/N_c$, where $N_c$ is the number of quark “colors”, and calculating $\Sigma$, $m_\pi$ and $f_\pi$ in terms of bare parameters $g^2$ and $m_0$. Since we use staggered lattice fermions $\chi, \bar{\chi}$, we have $N_c = 4$, in which case reasonable low-energy phenomenology emerges for the choice $a^2/g^2 = 0.565$, $m_0a = 0.002$, with inverse lattice spacing $a^{-1} \simeq 1\text{GeV}$. Figure 1 shows

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Chiral condensate $\langle \bar{\chi} \chi \rangle$, baryon density, and diquark condensate as functions of $\mu$, showing both large-$N_c$ solution (solid curve) and simulation results (points).}
\end{figure}

results for $\langle \bar{\chi} \chi \rangle$ and $n_B$ and confirms the transition between vacuum and quark matter at $\mu_c \sim \Sigma \sim g^2/\pi \langle \bar{\chi} \chi \rangle$. The discrepancy between line and points is ascribed to $O(1/N_c)$ corrections and amounts to a 30% effect. In contrast to 2+1d where a sharp first-order transition is observed, the transition appears continuous in the chiral limit $m_0 \to 0$, although this may be sensitive to the cutoff.

To investigate the diquark sector, we introduce a source term $j_{\pm}(q^{tr}q \pm \bar{q}\bar{q}^{tr})$ into the quark operator used for measurements (since the simulation dynamics are performed with $j_{\pm} = 0$, this is a “partially quenched” approximation) and define the following:

$$\langle qq_{\pm} \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial j_{\pm}}; \quad \chi_{\pm} = \frac{\partial \langle qq_{\pm} \rangle}{\partial j_{\pm}}; \quad R(j_+) = \left| \frac{\chi_+}{\chi_-} \right|_{j_- = 0}. \tag{2}$$
The nature of the ground state is then determined by

\[ \lim_{j+ \to 0} \langle qq+ \rangle = 0, \quad \lim_{j+ \to 0} R = 1 \quad \text{if } U(1)_B \text{ manifest} \]

\[ \lim_{j+ \to 0} \langle qq+ \rangle \neq 0, \quad \lim_{j+ \to 0} R = 0 \quad \text{if } U(1)_B \text{ broken} \]

(3)

where the vanishing of \( R \) in the broken phase follows from the Goldstone nature of the \( qq- \) boson in this case.

Figure 2. Susceptibility ratio \( R \) (left) and diquark condensate \( \langle qq+ \rangle \) (right) as functions of \( j_+ \) for various \( \mu \).

Figure 2 shows results for \( R \) and \( \langle qq+ \rangle \) from data taken on \( 12^4, 16^4 \) and \( 20^4 \) lattices and then linearly extrapolated to the limit \( L^{-1}_t = T \to 0 \). For \( \mu = 0 \) the data appear to extrapolate smoothly to \( R = 1, \langle qq+ \rangle = 0 \) as \( j_+ \to 0 \). For \( \mu a = 1.0 \) by contrast, which lies in the region of high baryon density and restored chiral symmetry, then if data with \( j_+ a \leq 0.2 \) are ignored the extrapolated values are \( R \simeq 0, \langle qq+ \rangle a^3 \simeq 0.28 \), suggesting superfluidity. For intermediate \( \mu \) the situation is not so clear cut, but the results of a quadratic extrapolation of \( \langle qq+ \rangle \) to \( j_+ = 0 \) are shown in Figure 1. For \( 0.4 \lesssim \mu a \lesssim 0.9 \) the condensate scales as \( \mu^2 \), which is expected if diquark pairs within a distance \( \Delta \) of a Fermi surface of radius \( \mu \) participate in the condensate leading to \( \langle qq+ \rangle \sim \Delta \mu^2 \). The value of \( \langle qq+ \rangle \) at \( \mu a = 1.0 \), together with an estimated \( \Sigma \sim 350 \text{MeV} \), is thus plausibly consistent with the gap values reported in [2]. The positive curvature contrasts with the behaviour of the superfluid condensate in Two Color QCD in which \( qq \) pairs are tightly bound and form a conventional Bose-Einstein condensate.

Since our interpretation of broken \( U(1)_B \) symmetry at high baryon density relies on discarding some data, the above conclusions are necessarily provisional. As spontaneous symmetry breaking does not occur in a finite
volume, it is natural to attribute the depletion of $\langle qq^+ \rangle$ at small $j^+$ to the effects of a finite $L_s$. Note however that the threshold $ja \sim 0.3$ required for behaviour characteristic of the thermodynamic limit seems anomalously large compared with that required to observe chiral symmetry breaking ($m_0 a \sim 0.002$) on the same volume, or indeed for thin film superfluidity ($ja \sim 0.1$) in NJL$_{2+1}$.

We suspect that finite volume effects in systems exhibiting a Fermi surface are unconventional, and indeed data on systems with varying $L_s, L_t$ suggest the approach to the thermodynamic limit at small $j$ is non-monotonic. Solution of the gap equation on finite systems shows that $\Delta(L_s)$ oscillates before approaching its infinite volume limit for $L_s \gtrsim 8$fm, which translates to a $40^3$ spatial lattice for our model.

In future work we plan to study the quasiparticle spectrum which in principle permits a direct estimate of the superfluid gap $\Delta$. It might also prove interesting to choose different bare parameters, since a small change in the cutoff might actually change the chiral symmetry restoring transition to first order, making it easier to distinguish the properties of high- and low-$\mu$ phases. Finally, if our conclusion that NJL$_{3+1}$ is superfluid at high density persists, then it will be interesting to examine the stability of the condensate as either temperature or $\mu_{\text{isospin}}$ (which has the effect of separating $u$ and $d$ Fermi surfaces and hence inhibiting pairing in the isoscalar channel) is raised from zero.

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