Some Results on $\delta$-Semiperfect Rings and $\delta$-Supplemented Modules

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Abstract. In [9], the author extends the definition of lifting and supplemented modules to $\delta$-lifting and $\delta$-supplemented by replacing “small submodule” with “$\delta$-small submodule” introduced by Zhou in [13]. The aim of this paper is to show new properties of $\delta$-lifting and $\delta$-supplemented modules. Especially, we show that any finite direct sum of $\delta$-hollow modules is $\delta$-supplemented. On the other hand, the notion of amply $\delta$-supplemented modules is studied as a generalization of amply supplemented modules and several properties of these modules are given. We also prove that a module $M$ is Artinian if and only if $M$ is amply $\delta$-supplemented and satisfies Descending Chain Condition (DCC) on $\delta$-supplemented modules and on $\delta$-small submodules. Finally, we obtain the following result: a ring $R$ is right Artinian if and only if $R$ is a $\delta$-semiperfect ring which satisfies DCC on $\delta$-small right ideals of $R$.

1. Introduction

Throughout this paper, we will assume that $R$ is an associative ring with unity and all modules are unital right $R$-modules.

We recall some basic notions related to our topic. A submodule $N$ of a module $M$ is called small in $M$, written $N \ll M$, if, whenever $M = L + N$ for any...
submodule $L$ of $M$, we have $L = M$. A module $M$ is called *lifting* if, for every submodule $N$ of $M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll M$ ([11]). In [13], the author defined the notion of $\delta$-small submodules as follows. A submodule $N$ of a module $M$ is called a *$\delta$-small submodule*, written as $N \ll \delta M$, if, whenever $M = N + X$ with $M/X$ is singular, we have $M = X$. Following Koşan [9], a module $M$ is called *$\delta$-lifting* if, for every submodule $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is $\delta$-small in $M$. It is obvious that every lifting module is $\delta$-lifting and every singular $\delta$-lifting module is lifting.

**Lemma 1.1.** ([9, Lemma 2.9]) Assume that $N$ and $L$ are two submodules of the module $M$. Then the following conditions are equivalent:

1. $M = N + L$ and $N \cap L$ is $\delta$-small in $L$.

2. $M = N + L$ and for any proper submodule $K$ of $L$ with $L/K$ singular, $M \neq N + K$.

A submodule $L$ of $M$ is called a *$\delta$-supplement* of $N$ in $M$ if $N$ and $L$ satisfy the conditions in Lemma 1.1. A module $M$ is called *$\delta$-supplemented* if every submodule of $M$ has a $\delta$-supplement in $M$ (see [9]). It is clear that every supplemented module is $\delta$-supplemented and every singular $\delta$-supplemented module is supplemented.

In Section 2, we give some properties of $\delta$-supplements. We prove that any factor module of a $\delta$-supplemented module is $\delta$-supplemented and that any finite sum of $\delta$-supplemented modules is $\delta$-supplemented.

In Section 3, we give some results of decompositions and direct sums of $\delta$-lifting modules. In particular, the main result in the third section shows that if $M = M_1 \oplus M_2$ is a direct sum of $\delta$-lifting modules $M_1$ and $M_2$ such that $M_i$ is $M_j$-projective ($i=1,2$), then $M$ is a $\delta$-lifting module.

In Section 4, we study the notion of amply $\delta$-supplemented modules as a generalization of amply supplemented modules. Recall that a submodule $N$ of $M$ has *amply supplements in $M$* if, for every $L \subset M$ with $N + L = M$, there is a supplement $L'$ of $N$ with $L' \subset L$. Recall also that a module $M$ is called *amply supplemented* if all submodules have amply supplements in $M$. We call a module $M$ *amply $\delta$-supplemented* if for any submodules $N$ and $K$ of $M$ with $M = N + K$, $K$ contains a $\delta$-supplement of $N$ in $M$. It is clear that every amply supplemented module is amply $\delta$-supplemented and every singular amply $\delta$-supplemented module is amply supplemented. It is proved in this section that if $M$ is an amply $\delta$-supplemented module such that every $\delta$-supplement submodule of $M$ is a direct summand, then $M$ is $\delta$-lifting. Recall that a ring $R$ is $\delta$-semiperfect if the module $R_R$ is $\delta$-supplemented (see [9, Theorem 3.3]). We also characterize $\delta$-semiperfect rings in terms of amply $\delta$-supplemented modules.

2. *Some Properties of $\delta$-Supplemented Submodules*

We start with some general results on $\delta$-small submodules which are taken from
[13, Lemmas 1.2 and 1.3].

**Lemma 2.1.** Let $M$ be an $R$-module.

1. If $N \leq M$ and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule $Y$ with $Y \subseteq N$.

2. If $K \leq M$ and $f : M \to N$ is a homomorphism, then $f(K) \leq N$. In particular, if $K \leq M \leq N$, then $K \leq N$.

3. Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq M_1 \oplus M_2$ if and only if $K_1 \leq M_1$ and $K_2 \leq M_2$.

**Lemma 2.2.** Let $A$, $B$ and $C$ be submodules of an $R$-module $M$. If $M = A + B$, $B \leq C$, and $C/B \leq M/B$, then $(A \cap C)/(A \cap B) \leq M/(A \cap B)$.

**Proof.** Let $X$ be a submodule of $M$ such that $A \cap B \leq X$, $M/X$ is singular and $M/(A \cap B) = (A \cap C)/(A \cap B) + X/(A \cap B)$. Then $M = (A \cap C) + X = C + (A \cap X)$ by [1, Lemma 1.24]. Therefore $M/B = C/B + ((A \cap X) + B)/B$. Note that

\[
\frac{(M/B)/[(A \cap X) + B]/B}{(A \cap X) + B}/B = \frac{((A \cap C) + B + (A \cap X))}{B + (A \cap X)} = \frac{(A \cap C)/(A \cap C \cap (B + (A \cap X)))}{(A \cap C) + X} = \frac{(A \cap C) + X}{X} = M/X.
\]

Since $M/X$ is singular, it follows that $M = (A \cap X) + B$. Now, since $M = A + X$ we get $M = X + (A \cap B) = X$. Hence, $(A \cap C)/(A \cap B) \leq M/(A \cap B)$. \qed

**Proposition 2.3.** Let $M$ be an $R$-module.

1. Suppose that $K$ and $L$ are submodules of $M$ such that $K \leq L$.

   a. If $L$ is a $\delta$-supplement in $M$, then $L/K$ is a $\delta$-supplement in $M/K$.

   b. If $L$ has a $\delta$-supplement $H$ in $M$, then $(H + K)/K$ is a $\delta$-supplement of $L/K$ in $M/K$.

2. Let $B \leq C \leq M$ be submodules of $M$. If $C/B$ is a $\delta$-supplement in $M/B$ and $B$ is a $\delta$-supplement in $M$, then $C$ is a $\delta$-supplement submodule of $M$.

3. Assume that $M = M_1 \oplus M_2$. If $A$ is a $\delta$-supplement of $A'$ in $M_1$ and $B$ is a $\delta$-supplement of $B'$ in $M_2$, then $A \oplus B$ is a $\delta$-supplement of $A' \oplus B'$ in $M$.

**Proof.** (1)(a) Let $N$ be a submodule of $M$ such that $L + N = M$ and $L \cap N \leq L$. Therefore $L/K + (N + K)/K = M/K$ and $(L \cap (N + K))/K = [(L \cap N) + K]/K \leq M/K$ by Lemma 2.1(2).

(b) This can be proved by following the same method as in (a).

(2) Assume that $C/B$ is a $\delta$-supplement of $X/B$ in $M/B$ and $B$ is a $\delta$-supplement of $Y$ in $M$. Then $M/B = (C/B) + (X/B)$ and $(C/B) \cap (X/B) \leq C/B$. Moreover,
By hypothesis, we have
\[ C = C \cap (B + Y) = B + (C \cap Y). \]
Since \((C \cap X)/B \ll_D C/B\), it follows from Lemma 2.2 that \((C \cap X)/Y)/(B \cap Y) \ll_D C/(B \cap Y)\).

As \(B \cap Y \ll_D C\) we have \((C \cap X \cap Y) \ll_D C\). Since \(X = X \cap (B + Y) = B + (X \cap Y)\), we see that \(M = C + X = C + (X \cap Y)\). Therefore \(C\) is a \(\delta\)-supplement of \(X \cap Y\) in \(M\).

(3) By assumption, we have \(M_1 = A + A'\) and \(A \cap A' \ll_D A\). Moreover, \(M_2 = B + B'\) and \(B \cap B' \ll_D B\). Then \(M = (A \oplus B) + (A' \oplus B')\). By Lemma 2.1(3), \((A \cap A') \oplus (B \cap B') \ll_D A \oplus B\). Since \((A \oplus B) \cap (A' \oplus B') = (A \cap A') \oplus (B \cap B')\), it follows that \(A \oplus B\) is a \(\delta\)-supplement of \(A' \oplus B'\) in \(M\).

**Corollary 2.4.** Every factor module of a \(\delta\)-supplemented module is \(\delta\)-supplemented.

**Lemma 2.5.** Let \(M_1\) and \(M_2\) be submodules of \(M\) such that \(M_1\) is \(\delta\)-supplemented and \(M_1 + M_2\) has a \(\delta\)-supplement in \(M\). Then \(M_2\) has a \(\delta\)-supplement in \(M\).

**Proof.** By assumption, there exists a submodule \(N\) of \(M\) such that \(M_1 + M_2 + N = M\) and \((M_1 + M_2) \cap N \ll_D N\). Moreover, since \(M_1\) is \(\delta\)-supplemented, \((M_2 + N) \cap M_1\) has a \(\delta\)-supplement in \(M_1\). Then there exists \(L \leq M_1\) such that \(M_1 = (M_2 + N) \cap M_1 + L\) and \((M_2 + N) \cap L \ll_D L\). Then we have \(M = M_1 + M_2 + N = (M_2 + N) \cap M_1 + L\). Moreover, we have \(M_2 \cap (L + N) \leq [(M_2 + L) \cap N] + [(M_2 + N) \cap L] \leq [(M_2 + M_1) \cap N] + [(M_2 + N) \cap L]\). Now, it follows that \(M_2 \cap (L + N) \ll_D (L + N)\). Hence, \(L + N\) is a \(\delta\)-supplement of \(M_2\) in \(M\).

**Proposition 2.6.** Any finite sum of \(\delta\)-supplemented modules is \(\delta\)-supplemented.

**Proof.** We prove it for two modules; the finite case can be proved similarly. Let \(M_1\) and \(M_2\) be two submodules of a module \(M\) such that \(M = M_1 + M_2\) and \(M_1\) and \(M_2\) are \(\delta\)-supplemented. It is easily seen that for every submodule \(N\) of \(M\), \(M_1 + (M_2 + N)\) has a \(\delta\)-supplement in \(M\). Hence by Lemma 2.5, \(M_2 + N\) has a \(\delta\)-supplement in \(M\). Applying Lemma 2.5 again we conclude that \(N\) has a \(\delta\)-supplement in \(M\).

**Corollary 2.7.** Let \(M\) be a \(\delta\)-supplemented module. Then every finitely \(M\)-generated module is \(\delta\)-supplemented.

**Proof.** By Corollary 2.4 and Proposition 2.6.

Recall that an \(R\)-module \(M\) is said to be **hollow** (respectively **\(\delta\)**-hollow) if every proper submodule of \(M\) is small (respectively \(\delta\)**-small) in \(M\). It is clear that every hollow module is \(\delta\)**-hollow. In [3], the author called a module \(M\) **\(\delta\)**-local if, \(\delta(M) \ll_D M\) and \(\delta(M)\) is a maximal submodule of \(M\). Moreover, the author also shows in [3] that a local module needs not to be \(\delta\)**-local in general.

**Proposition 2.8.** Let \(M\) be a \(\delta\)**-hollow module. Then \(\delta(M) = M\) or \(M\) is a local and a \(\delta\)**-local module.

**Proof.** Suppose that \(\delta(M) \neq M\). Then \(\delta(M) \ll_D M\) and \(\text{Rad}(M) \neq M\) since \(\text{Rad}(M) \leq \delta(M)\) (see [13, Lemma 1.5(1)]). Let \(N\) be a maximal submodule of \(M\).

By hypothesis, we have \(N \ll_D M\). Therefore \(N \leq \delta(M)\). It follows that \(\delta(M) = N\). Then \(\text{Rad}(M) = \delta(M)\) is the only maximal submodule of \(M\). Consequently, \(M\) is a
δ-local module. On the other hand, it is easy to see that δ(M) is small in M. This implies that M is a local module.

The proof of the following two results are clear.

**Proposition 2.9.** Let K be a δ-hollow submodule of the module M. Then K is a δ-supplement of each proper submodule L of M such that K + L = M.

**Proposition 2.10.** Every δ-hollow module is δ-supplemented.

**Corollary 2.11.** Any finite sum of δ-hollow modules is δ-supplemented.

**Proof.** It can be obtained by using Propositions 2.6 and 2.10.

Recall that a module M is called cofinitely δ-supplemented if every submodule N of M such that M/N is finitely generated has a δ-supplement in M.

Also recall that a module is called coatomic if every proper submodule is contained in a maximal one.

**Proposition 2.12.** Let M be a coatomic module. Then the following are equivalent:

(i) M is cofinitely δ-supplemented.

(ii) Every maximal submodule of M has a δ-supplement.

(iii) \[ M = \sum_{i \in I} H_i \] where each \( H_i \) is either simple or δ-local.

**Proof.** The equivalences clearly hold if M is semisimple. So, assume that M is not semisimple.

(i)⇒(ii) Clear.

(ii)⇒(iii) Let K be the sum of all δ-supplement submodules of maximal submodules L of M with \( \text{Soc}(M) \leq L \). By [3, Lemma 3.4], K is a sum of δ-local submodules of M. Suppose that \( M \neq \text{Soc}(M) + K \). Then there is a maximal submodule N of M such that \( \text{Soc}(M) + K \leq N \). By hypothesis, N has a δ-supplement H in M. Thus \( H \leq K \leq N \) and \( N = M \), a contradiction. It follows that \( M = \text{Soc}(M) + K \), and the proof is complete.

(iii)⇒(i) It follows from [1, Proposition 2.5] and [3, Lemma 3.3].

**Corollary 2.13.** If M is a coatomic δ-supplemented module, then \( M = \sum_{i \in I} H_i \) where each \( H_i \) is either simple or δ-local.

**Proof.** This is clear by Proposition 2.12.

3. Decompositions and Direct Sums of δ-Lifting Modules

Following [13], a projective module P is called a projective δ-cover of a module M if there exists an epimorphism \( f : P \rightarrow M \) with \( \text{Ker}(f) \ll_\delta P \), and a ring R is called δ-semiperfect if every simple R-module has a projective δ-cover. In [9], it is proved that a ring R is δ-semiperfect if and only if the R-module \( R_R \) is δ-supplemented. The following example shows that a δ-lifting module need not be lifting.
Example 3.1. Let $F$ be a field, $I = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$ and $R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots)| n \in \mathbb{N}, x_i \in \mathbb{M}_2(F), x \in I\}$. By [13, Example 4.3], the ring $R$ is $\delta$-semiperfect and $\delta(R) = \{(x_1, x_2, \ldots, x_n, x, x, \ldots)| n \in \mathbb{N}, x_i \in \mathbb{M}_2(F), x \in J\}$ where $J = \left( \begin{array}{cc} 0 & F \\ 0 & 0 \end{array} \right)$. Therefore the module $R_R$ is $\delta$-lifting by [13, Lemma 2.4 and Theorem 3.6]. On the other hand, by [13, Example 4.3], $R$ is not semiregular. Hence $R_R$ is not supplemented. Thus $R_R$ is not lifting.

Lemma 3.2. (See [9, Lemma 2.3])

(1) The following conditions are equivalent for a module $M$:
   (a) $M$ is $\delta$-lifting.
   (b) For every $N \leq M$, there exists a decomposition $N = A \oplus B$ such that $A$ is a direct summand of $M$ and $B \ll \delta M$.

(2) If $M$ is $\delta$-lifting, then any direct summand of $M$ is $\delta$-lifting.

Proposition 3.3. Let $M$ be an indecomposable module. Then $M$ is $\delta$-lifting if and only if $M$ is $\delta$-hollow.

Proof. Let $M$ be a $\delta$-lifting indecomposable module. Let $N$ be a proper submodule of $M$. Since $M$ is $\delta$-lifting, we have a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is $\delta$-small in $B$ for some submodules $A$ and $B$ of $M$. Since $M$ is indecomposable and $N \neq M$, we have $A = 0$, and so $M = B$. Therefore $N \ll \delta M$. Hence, $M$ is $\delta$-hollow. The converse is clear.

Proposition 3.4.

(1) If $M$ is a $\delta$-lifting module, then $M/\delta(M)$ is a semisimple module.

(2) If $M$ is a $\delta$-lifting module, then any submodule $N$ of $M$ with $N \cap \delta(M) = 0$ is semisimple.

(3) If the module $R_R$ is $\delta$-lifting, then $M/\delta(M)$ is a semisimple module for every $R$-module $M$.

Proof. (1) See [9, Lemma 2.12].

(2) Since $N \cong (N \oplus \delta(M))/\delta(M) \leq M/\delta(M)$ is semisimple by (1), then $N$ is semisimple.

(3) Let $M$ be an $R$-module. By hypothesis and (1), $R/\delta(R)$ is a semisimple ring. But, on the other hand $M\delta(R) = \delta(M)$ by [13, Theorem 1.8]. Thus, $M/\delta(M)$ is a semisimple module.

In [9, Example 2.4], it is proved that if $R = \mathbb{Z}_8$, then the $R$-module $M = R \oplus (2R/4R)$ is not $\delta$-lifting, although the $R$-modules $R_R$ and $(2R/4R)_R$ are $\delta$-lifting. The following result deals with a special case of a direct sum of two $\delta$-lifting modules.
The following theorem may be seen in the literature but we want to give it here for the readers.

**Theorem 3.5.** Let $M = M_1 \oplus M_2$. If $M_1$ and $M_2$ are $\delta$-lifting modules such that $M_i$ is $M_j$-projective $(i=1,2)$, then $M$ is a $\delta$-lifting module.

**Proof.** Let $N \leq M$. Since $M_1$ is $\delta$-lifting, there exist submodules $K$ and $K'$ of $M_1$ such that $M_1 = K \oplus L$, $K \leq M_1 \cap (N + M_2)$ and $L \cap (N + M_2) \ll_{\delta} M_1$. Therefore $M_1 = K \oplus L \oplus M_2 = N + (L \oplus M_2)$. On the other hand, since $M_2$ is $\delta$-lifting, there exist submodules $X$ and $Y$ of $M_2$ such that $M_2 = X \oplus Y$, $X \leq M_2 \cap (N + L)$ and $Y \cap (N + L) \ll_{\delta} M_2$. Hence $M = (X \oplus K) \oplus (L \oplus Y)$ and $M = N + (L \oplus Y)$. Since $M_1$ is $M_2$-projective, $X \oplus K$ is $(L \oplus Y)$-projective by [11, Propositions 4.32 and 4.33]. Then there exists a submodule $N_1$ of $N$ such that $M = N_1 \oplus (L \oplus Y)$. Then we have $N \cap (L \oplus Y) \leq Y \cap (N + L) + L \cap (N + Y)$. But $Y \cap (N + L) \ll_{\delta} M_2$ and $L \cap (N + M_2) \ll_{\delta} M_1$. Thus, $N \cap (L \oplus Y) \ll_{\delta} M$ by Lemma 2.1. \hfill \qedsymbol

**Corollary 3.6.** Let $M = M_1 \oplus M_2$. If $M_1$ and $M_2$ are $\delta$-lifting projective modules, then $M$ is $\delta$-lifting.

**Proof.** This is clear by Theorem 3.5. \hfill \qedsymbol

**Corollary 3.7.** Let $R$ be a $\delta$-semiperfect ring. Then every free module of finite rank is $\delta$-lifting.

**Proof.** This is clear by Corollary 2.6. \hfill \qedsymbol

**Theorem 3.8.** (i) If $M$ is a $\delta$-lifting module, then $M$ has a decomposition $M = M_1 \oplus M_2$ such that $M_1$ is semisimple, $M_2$ is $\delta$-lifting and $\delta(M_2)$ is an essential submodule of $M_2$.

(ii) If $M$ is a $\delta$-lifting module, then $M$ has a decomposition $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ are $\delta$-lifting modules, $\delta(M_1) = M_1$ and $\delta(M_2) \ll_{\delta} M_2$.

**Proof.** (i) This is by [9, Proposition 2.13] and Lemma 3.2(2).

(ii) Since $M$ is $\delta$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq \delta(M)$ and $\delta(M) \cap M_2 \ll_{\delta} M_2$. Now, $\delta(M) = \delta(M_1) \oplus \delta(M_2)$ implies that $\delta(M) \cap M_2 = \delta(M_2) \oplus (M_2 \cap \delta(M_1)) = \delta(M_2) \ll_{\delta} M_2$. On the other hand, $\delta(M) \cap M_1 = \delta(M_1) \oplus (M_1 \cap \delta(M_2)) = \delta(M_1)$. Moreover, $M_1$ and $M_2$ are $\delta$-lifting by Lemma 3.2(2). \hfill \qedsymbol

**Proposition 3.9.** If $M$ is a $\delta$-lifting module such that $\delta(M)$ has a supplement in $M$, then we have a decomposition $M = M_1 \oplus M_2$ such that $M_1$ is a lifting module and $M_2$ is $\delta$-lifting with $\delta(M_2) = M_2$.

**Proof.** Assume that $M$ is $\delta$-lifting and let $A$ be a supplement of $\delta(M)$ in $M$. Then we have a decomposition $M = M_1 \oplus M_2$ such that $A = M_1 \oplus (M_2 \cap A)$ and $M_2 \cap A \ll_{\delta} M_2$. Let $N$ be a submodule of $M_1$. Since $M_1$ is a $\delta$-lifting module by Lemma 3.2(2), we have a decomposition $M_1 = X \oplus Y$ such that $N = X \oplus (Y \cap N)$ and $Y \cap N \ll_{\delta} Y$. Since $A \cap \delta(M) \ll A$ and $Y \cap N \leq \delta(M) \cap A$, we obtain that $Y \cap N \ll A$. Hence $Y \cap N \ll Y$ by [11, Lemma 4.2(2)]. Therefore $M_1$ is a lifting module. Moreover, we have $M = \delta(M_2) + \delta(M_1) + M_1 = \delta(M_2) \oplus M_1$. This gives
\[ \delta(M_2) = M_2. \]

4. Amply \( \delta \)-Supplemented Modules

In this section we study the notion of amply \( \delta \)-supplemented modules. Several properties of this type of modules are proved. Recall that a module \( M \) is amply \( \delta \)-supplemented if for any submodules \( N \) and \( K \) of \( M \) with \( M = N + K \), \( K \) contains a \( \delta \)-supplement of \( N \) in \( M \). It is clear that every amply \( \delta \)-supplemented module is \( \delta \)-supplemented.

**Lemma 4.1.** Let \( M \) be an \( R \)-module. If every submodule of \( M \) is \( \delta \)-supplemented, then \( M \) is amply \( \delta \)-supplemented.

**Proof.** Let \( A \) and \( B \) be submodules of \( M \) such that \( M = A + B \). Since \( A \) is \( \delta \)-supplemented and \( A \cap B \leq A \), there is a submodule \( C \leq A \) such that \( A \cap B + C = A \) and \( A \cap B \cap C \leq \delta C \). Therefore \( C + B = M \). Since \( C \cap B = C \cap B \cap A \leq \delta C \), \( C \) is a \( \delta \)-supplement of \( B \) in \( M \). It follows that \( M \) is amply \( \delta \)-supplemented. \( \square \)

**Proposition 4.2.** If \( M \) is an amply \( \delta \)-supplemented module such that every \( \delta \)-supplement submodule of \( M \) is a direct summand, then \( M \) is a \( \delta \)-lifting module.

**Proof.** Let \( N \) be a submodule of \( M \). By assumption, \( N \) has a \( \delta \)-supplement \( K \) and \( K \) has a \( \delta \)-supplement \( L \) such that \( L \leq N \) and \( L \) is a direct summand of \( M \). Then \( M = L \oplus T = L + K \) for some submodule \( T \) of \( M \). Note that \( N = L \oplus (N \cap T) = L + (N \cap K) \). Let \( \pi \) denote the canonical projection \( \pi : L \oplus T \to T \). Then \( \pi(N) = \pi(N \cap K) = N \cap T \). Since \( K \) is a \( \delta \)-supplement of \( N \), we have \( N \cap K \leq \delta K \). Hence \( \pi(N \cap K) = N \cap T \leq \delta T \) by Lemma 2.1(2). Consequently, \( M \) is a \( \delta \)-lifting module by Lemma 3.2. \( \square \)

**Proposition 4.3.** Any epimorphic image of an amply \( \delta \)-supplemented module is again amply \( \delta \)-supplemented.

**Proof.** Let \( M \) be an amply \( \delta \)-supplemented module and let \( f : M \to N \) be an epimorphism, where \( N \) is an \( R \)-module. Let \( N = A + B \). Then \( f^{-1}(N) = M = f^{-1}(A) + f^{-1}(B) \). Since \( M \) is an amply \( \delta \)-supplemented module, there is a submodule \( X \leq f^{-1}(B) \) such that \( M = f^{-1}(A) \oplus X \) and \( f^{-1}(A) \cap X \leq \delta X \). Hence \( N = f(M) = A + f(X) \) and \( A \cap f(X) = f(f^{-1}(A) \cap X) \leq \delta f(X) \) by Lemma 2.1(2). This implies that \( f(X) \) is a \( \delta \)-supplement of \( A \) in \( M \). Moreover, we have \( f(X) \leq B \). Therefore \( N \) is amply \( \delta \)-supplemented. \( \square \)

Recall that a module \( M \) is called \( \pi \)-projective if for every two submodules \( N \) and \( L \) of \( M \) with \( M = N + L \), there exists an endomorphism \( \alpha \) of \( M \) such that \( \alpha(M) \leq N \) and \( (1 - \alpha)(M) \leq L \). It is well known that \( \pi \)-projective supplemented modules are amply supplemented. Next we prove an analogue for this result.

**Theorem 4.4.** Let \( M \) be a \( \pi \)-projective module. If \( M \) is \( \delta \)-supplemented, then \( M \) is amply \( \delta \)-supplemented.

**Proof.** Let \( M = N + K \). Then there exists \( \alpha \in \text{End}(M) \) such that \( \alpha(M) \leq N \) and
(1 − α)(M) ∈ K. Since M is δ-supplemented, there exists a submodule L ≤ M such that M = N + L and N ∩ L ≪δ L. Clearly, (1 − α)(L) ≤ K and M = N + (1 − α)(L). Since N ∩ L ≪δ L, then N ∩ (1 − α)(L) = (1 − α)(N ∩ L) ≪δ (1 − α)(L). So M is amply δ-supplemented.

Corollary 4.5. Let M₁, M₂, ..., Mₖ be submodules of a projective module M such that M = ⊕ᵢ=₁^k Mᵢ. The following are equivalent:

(i) M is amply δ-supplemented.

(ii) For every i (1 ≤ i ≤ k), Mᵢ is amply δ-supplemented.

Proof. (i)⇒(ii) By Proposition 4.3.

(ii)⇒(i) Since for every 1 ≤ i ≤ k, Mᵢ is amply δ-supplemented, it follows from Proposition 2.6 that M = ⊕ᵢ=₁^k Mᵢ is δ-supplemented. By Theorem 4.4, M is amply δ-supplemented.

Proposition 4.6. Let M be an amply δ-supplemented module. Assume that for every submodule K of M such that K = K₁ ∩ K₂ where K₁ and K₂ are δ-supplement submodules in M with M = K₁ + K₂, every homomorphism β : M → M/K can be lifted to a homomorphism γ : M → M. Then M is π-projective.

Proof. Let A and B be submodules of M with M = A + B. Since M is an amply δ-supplemented module, there exist two submodules B' ≤ B and A' ≤ A such that M = A + B' = A' + B', A ∩ B' ≪δ B' and A' ∩ B' ≪δ A'. Now, we consider the homomorphism β : M → M/(A' ∩ B') defined by β(a' + b') = b' + A' ∩ B', where a' ∈ A' and b' ∈ B'. By hypothesis, β can be lifted to a homomorphism α : M → M. Moreover, we have α(M) ≤ B' and (1 − α)(M) ≤ A'. Hence M is π-projective.

Let M be a module and B ≤ A ≤ M. If A/B ≪ M/B then B is called a coessential submodule of A in M. If A has no proper coessential submodule in M, then A is called coclosed in M (see [8]).

If A/B ≪δ M/B and A/B is singular, then B will be called a δ-coessential submodule of A. If A has no proper δ-coessential submodule in M, then A is called δ-coclosed in M (see [3]). Clearly, every δ-coclosed submodule is coclosed.

Note that every δ-supplement submodule of a module M is δ-coclosed by [3, Corollary 2.6].

Let K ≤ N ≤ M. The submodule K is said to be a δ-coclosure of N in M if K is a δ-coessential submodule of N and K is δ-coclosed in M.

Proposition 4.7. Let M be a δ-lifting module. Then every singular δ-coclosed submodule of M is a direct summand.

Proof. Let N be a singular δ-coclosed submodule of M. Since M is δ-lifting, there exist submodules M₁ and M₂ of M such that M = M₁ ⊕ M₂, M₁ ≤ N and
Every finitely generated projective $M$. Therefore $N = M_1 \oplus (N \cap M_2)$ and $N \cap M_2 \ll_\delta N$ by [3, Corollary 2.6]. It follows that $N = M_1$ since $N/M_1$ is singular. □

**Lemma 4.8.** Let $M$ be an amply $\delta$-supplemented module. Then every submodule $N$ of $M$ has a $\delta$-coclosure in $M$.

**Proof.** The proof is clear.

A module $M$ is called *weakly $\delta$-supplemented* if for every submodule $N \leq M$, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \ll_\delta M$. It is clear that every $\delta$-supplemented module is weakly $\delta$-supplemented. □

**Proposition 4.9.** A module $M$ is amply $\delta$-supplemented if and only if $M$ is weakly $\delta$-supplemented and every submodule of $M$ has a $\delta$-coclosure in $M$.

**Proof.** ($\Rightarrow$) This is clear by Lemma 4.8.

($\Leftarrow$) Suppose that $M$ is weakly $\delta$-supplemented and every submodule of $M$ has a $\delta$-coclosure in $M$. Let $A$ and $B$ be two submodules of $M$ such that $M = A + B$. Since $M$ is weakly $\delta$-supplemented, there exists a submodule $C$ of $M$ such that $(A \cap B) + C = M$ and $(A \cap B) \cap C \ll_\delta M$. Then $(A \cap B) + (C \cap B) = B$. Thus $A + (C \cap B) = M$ by [4, Lemma 1.24]. Let $N$ be a $\delta$-coclosure of $C \cap B$ in $M$. Then $(C \cap B)/N$ is singular, $N$ is $\delta$-coclosed in $M$ and $(C \cap B)/N \ll_\delta M/N$. On the other hand, we have $[(A + N)/N] + (C \cap B)/N = M/N$ and $M/(A + N) \cong (C \cap B)/[(C \cap B) \cap (A + N)]$. Hence $M/(A + N) \cong (C \cap B)/[N + (A \cap B) \cap C]$ is a factor module of $(C \cap B)/N$. So $M/(A + N)$ is singular. It follows that $M = A + N$. Since $A \cap N \leq (A \cap B) \cap C \ll_\delta M$, we get $A \cap N \ll_\delta N$ by [3, Corollary 2.6]. Consequently, $N$ is a $\delta$-supplement of $A$ in $M$ with $N \leq B$. Therefore $M$ is amply $\delta$-supplemented. □

The next result gives a characterization of $\delta$-semiperfect rings in terms of $\delta$-supplemented modules. It is taken from [13, Theorem 3.6] and [9, Theorem 3.3].

**Lemma 4.10.** The following are equivalent for a ring $R$:

1. $R$ is a $\delta$-semiperfect ring.
2. $R/\delta(R)$ is a semisimple ring and idempotents lift modulo $\delta(R)$.
3. There exists a complete orthogonal set of idempotents $e_1, \ldots, e_n$ such that, for each $i$, either $e_i R$ is a simple $R$-module or $e_i R$ has a unique essential maximal submodule.
4. Every finitely generated $R$-module is $\delta$-supplemented.
5. Every finitely generated projective $R$-module is $\delta$-lifting.
6. Every finitely generated projective $R$-module is $\delta$-supplemented.
7. $R_R$ is $\delta$-supplemented.
It is well-known that a ring $R$ is semiperfect if and only if $R_R$ is supplemented if and only if $R_R$ is amply supplemented.

**Corollary 4.11.** The following are equivalent for a ring $R$:

1. $R$ is $\delta$-semiperfect.
2. $R_R$ is amply $\delta$-supplemented.
3. Every finitely generated module is amply $\delta$-supplemented.

**Proof.** (1)$\iff$(2) is follows from Theorem 4.4 and Lemma 4.10.
(2)$\implies$(3) By Proposition 4.3 and Corollary 4.5.
(3)$\implies$(2) Clear. $\square$

**Theorem 4.12.** Let $M$ be an $R$-module. Then $M$ is Artinian if and only if $M$ is amply $\delta$-supplemented and satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules.

**Proof.** The necessity is clear. Conversely, assume that $M$ is amply $\delta$-supplemented module which satisfies DCC on $\delta$-supplement submodules and on $\delta$-small submodules. By [10, Proposition 2.6], $\delta(M)$ is an Artinian module. We next show that $M/\delta(M)$ is an Artinian module. Let $\delta(M) \leq N_1 \leq N_2 \leq \cdots$ be an ascending chain of submodules of $M$. Since $M$ is amply $\delta$-supplemented, there exists a descending chain of submodules of $M$ such that $K_i$ is a $\delta$-supplement of $N_i$ in $M$ for each $i \geq 1$. By hypothesis, there exists a positive integer $n$ such that $K_n = K_{n+j}$ for each $j \geq 1$. Since $K_i \cap N_i \leq \delta(M)$, we have $M/\delta(M) = N_i/\delta(M)(K_i + \delta(M))/\delta(M)$ for each $i \geq n$. It follows that $N_i = N_n$ for each $i \geq n$. Thus $M/\delta(M)$ is Noetherian, and hence finitely generated. Moreover, $M/\delta(M)$ is a semisimple module by [9, Lemma 2.12]. Then $M/\delta(M)$ is Artinian. Consequently, $M$ is Artinian. $\square$

**Proposition 4.13.** Let $M$ be a finitely generated $\delta$-supplemented module. Then $M$ is Artinian if and only if $M$ satisfies DCC on $\delta$-small submodules.

**Proof.** By [9, Lemma 2.12] and [10, Proposition 2.6]. $\square$

**Corollary 4.14.** $R$ is a right Artinian ring if and only if $R$ is a $\delta$-semiperfect ring which satisfies DCC on $\delta$-small right ideals of $R$.

**Proof.** By Corollary 4.11 and Proposition 4.13.

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