Integrability of auto-Bäcklund transformations and solutions of a torqued ABS equation

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Abstract

An auto-Bäcklund transformation for the quad equation $Q_1\delta$ is considered as a discrete equation, called $H_2^a$, which is a so called torqued version of $H_2$. The equations $H_2^a$ and $Q_1\delta$ compose a consistent cube, from which an auto-Bäcklund transformation and a Lax pair for $H_2^a$ are obtained. More generally it is shown that auto-Bäcklund transformations admit auto-Bäcklund transformations. Using the auto-Bäcklund transformation for $H_2^a$ we derive a seed solution and a one-soliton solution. From this solution it is seen that $H_2^a$ is a semi-autonomous lattice equation, as the spacing parameter $q$ depends on $m$ but it disappears from the plane wave factor.

Keywords: auto-Bäcklund transformation, consistency, Lax pair, soliton solution, torqued ABS equation, semi-autonomous

1. Introduction

The subtle concept of integrability touches on global existence and regularity of solutions, exact solvability, as well as compatibility and consistency (see [1]). In the past two decades, the study of discrete integrable systems has achieved a truly significant development, which mainly relies on the effective use of the property of multidimensional consistency (MDC). In the two-dimensional case, MDC means the equation is consistent around the cube (CAC) and this implies it can be embedded consistently into lattices of dimension 3 and higher [2–4]. In 2003, Adler, Bobenko and Suris (ABS) classified scalar quadrilateral equations that are CAC (with extra restrictions: affine linear, D4 symmetry and tetrahedron property) [5]. The complete list contains 9 equations.

In this paper, our discussion will focus on two of them, namely

$$Q_1\delta(u, \hat{u}, \tilde{u}, \check{u}; p, q) = p(u - \hat{u})(u - \tilde{u}) - q(u - \hat{u})(u - \check{u}) + \delta pq(p - q) = 0,$$

and

$$H_2^a(u, \hat{u}, \tilde{u}, \check{u}; p, q) = (u - \hat{u})(\tilde{u} - \check{u}) + (q - p)(u + \hat{u} + \tilde{u} + \check{u}) + q^2 - p^2 = 0.$$ (1.2)

Here $u = u(n, m)$ is a function on $\mathbb{Z}^2$, $p$ and $q$ are spacing parameters in the $n$ and $m$ direction respectively, $\delta$ is an arbitrary constant which we set equal to 1 in the sequel, and conventionally, tilde and hat denote shifts, i.e.

$$u = u(n, m), \quad \hat{u} = u(n + 1, m), \quad \tilde{u} = u(n, m + 1), \quad \check{u} = u(n + 1, m + 1).$$ (1.3)

H2 is a new equation due to the ABS classification, while $Q_1\delta$ extends the well known cross-ratio equation, or lattice Schwarzian Korteweg–de Vries equation $Q_1\delta$. Note that spacing parameters $p$ and $q$ can depend on $n$ and $m$ respectively, which leads to nonautonomous equations.

For a quadrilateral equation that is CAC the equation itself defines its own (natural) auto-Bäcklund transformation (auto-BT), see [5]. For example, the system

$$Q_1\delta(u, \hat{u}, \tilde{u}, \check{u}; p, q; r) = 0, \quad Q_1\delta(u, \hat{u}, \tilde{u}, \check{u}; q, r) = 0,$$

where $r$ acts as a wave number, composes an auto-BT between $Q_1\delta(u, \hat{u}, \tilde{u}, \check{u}; p, q) = 0$ and $Q_1\delta(\tilde{u}, \check{u}, \hat{u}, \tilde{u}; P, q) = 0$. 

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Such a property has been employed in solving CAC equations, see e.g.
[6–10].

Some CAC equations allow auto-BTs of other forms. For example, in [11] it was shown that the coupled system

\[
\begin{align*}
A: & \quad (u - \tilde{u})(\tilde{\alpha} - \alpha) - p(u + \tilde{u} + \alpha + \tilde{\alpha} + p + 2r) = 0, \\
B: & \quad (u - \tilde{u})(\tilde{\alpha} - \alpha) - q(u + \tilde{u} + \alpha + \tilde{\alpha} + q + 2r) = 0
\end{align*}
\]

provides an auto-BT between

\[
\begin{align*}
Q: & \quad Q_1(u, \tilde{u}, \alpha, \tilde{\alpha}; p, q) = 0 \\
\bar{Q}: & \quad \bar{Q}_1(\alpha, \tilde{\alpha}, \alpha, \tilde{\alpha}; p, q) = 0
\end{align*}
\]

and

\[
\begin{align*}
A: & \quad Q_1(u, \tilde{u}, \alpha, \tilde{\alpha}; p, r) = 0, \\
\bar{A}: & \quad \bar{Q}_1(\alpha, \tilde{\alpha}, \alpha, \tilde{\alpha}; p, r) = 0, \\
B: & \quad B_2(u, \tilde{u}, \alpha, \tilde{\alpha}; r, q) = 0
\end{align*}
\]

provides an auto-BT to

\[
\begin{align*}
A: & \quad p(u - \tilde{u})(\tilde{\alpha} - \alpha) - r(u - \tilde{u})(\alpha - \tilde{\alpha}) \\
& \quad + pr(p - r) = 0, \\
B: & \quad (u - \tilde{u})(\tilde{\alpha} - \alpha) - r(u + \tilde{u} + \alpha + \tilde{\alpha} + r + 2q) = 0
\end{align*}
\]

In section 2, we establish a simple but quite general result, namely that if a system of equations \(A = B = 0\) comprises an auto-BT, then both equations \(A = 0\) and \(B = 0\) admit an auto-BT themselves. In particular, the equation \(H_2^a\) given by (1.6) is CAC, with \(H_2^a\) and \(Q_1\), providing its an auto-BT. We construct a Lax pair for \(H_2^a\), which is asymmetric. In section 3, we employ the auto-BT for \(H_2^a\) to derive a seed-solution and the corresponding one-soliton solution. In the seed-solution the spacing parameter \(q\) depends explicitly on \(m\), which makes \(H_2^a\) inherit semi-autonomous. Some conclusions are presented in section 4.

2. Auto-BTs for auto-BTs and a Lax pair for \(H_2^a\)

To have a consistent cube with \(H_2^a\) and \(Q_1\) on the side faces, providing an auto-BT for \(H_2^a\), we assign equations to six faces as follows:

\[
\begin{align*}
Q: & \quad H_2^a(u, \tilde{u}, \alpha, \tilde{\alpha}; p, q) = 0, \\
\bar{Q}: & \quad \bar{H}_2^a(\alpha, \tilde{\alpha}, \tilde{\alpha}, \alpha; p, q) = 0
\end{align*}
\]

Then, given initial values \(u, \tilde{u}, \alpha, \tilde{\alpha}\), by direct calculation, one can find that the value \(\tilde{\alpha}\) is uniquely determined. Thus, the cube in figure 1 with (2.1) is a consistent cube.

By means of such a consistency, the side equations \(A\) and \(B\), i.e.

\[
\begin{align*}
A: & \quad p(u - \tilde{u})(\tilde{\alpha} - \alpha) - r(u - \tilde{u})(\alpha - \tilde{\alpha}) \\
& \quad + pr(p - r) = 0, \\
B: & \quad (u - \tilde{u})(\tilde{\alpha} - \alpha) - r(u + \tilde{u} + \alpha + \tilde{\alpha} + r + 2q) = 0
\end{align*}
\]

was identified as a torqued version of the equation \(H_2\). The superscript \(^a\) refers to the additive transformation of the spacing parameter. In [11], equation (1.6) appeared as part of an auto-BT for \(Q_1\). The corresponding consistent cube is a special case of [15, equation (3.9)]. In [14], equation (1.6) was shown to be an integrable equation in its own right, with an asymmetric auto-BT given by \(A = H_2^a = 0\) and \(B = H_2 = 0\). Here we provide an alternative auto-BT for equation (1.6) to the one that was provided in [14].

In figure 1. Consistent cube with equations \(A\), \(B\) and \(Q\) on its faces.

In [16], section 2.1 where the same idea was used to reduce the number of triplets of equations to consider for the classification of consistent cubes.

**Lemma 2.1.** Let

\[
\begin{align*}
A(u, \tilde{u}, \alpha, \tilde{\alpha}; p, r) = 0, \\
B(u, \tilde{u}, \alpha, \tilde{\alpha}; q, r) = 0
\end{align*}
\]

be an auto-BT for

\[
Q(u, \tilde{u}, \alpha, \tilde{\alpha}; p, q) = 0.
\]
Then we have (i)
\[ Q(u, \bar{u}, \bar{u}, \bar{\alpha}; p, r) = 0, \quad B(u, \bar{u}, \bar{u}, \bar{\alpha}; r, q) = 0 \] (2.5)
is an auto-BT for
\[ A(u, \bar{u}, \bar{\alpha}; p, q) = 0; \] (2.6)and (ii)
\[ Q(u, \bar{u}, \bar{\alpha}; r, p) = 0, \quad A(u, \bar{u}, \bar{\alpha}; r, q) = 0 \] (2.7)is an auto-BT for
\[ B(u, \bar{u}, \bar{\alpha}; p, q) = 0. \] (2.8)

**Proof.** If \( A = B = 0 \) is an auto-BT of \( Q = 0 \), then they compose a consistent cube as in figure 1. We prove the result by relabeling the fields at the vertices, see [13, lemma 2.1]. For (i) we interchange \( \bar{u} \leftrightarrow \bar{\alpha} \) and \( q \leftrightarrow r \), and for (ii) we perform the cyclic shifts \( \bar{u} \rightarrow \bar{\alpha} \rightarrow \bar{\alpha} \rightarrow \bar{u} \) and \( q \rightarrow p \rightarrow r \rightarrow q \). □

Applying (i) to the consistent cube with (1.4a) and (1.5) we obtain (2.1a). Applying (ii) yields the same, as Q1, has D4 symmetry.

3D consistency can be used to construct Lax pairs for quadrilateral equations (see [3, 5, 17]). To achieve a Lax pair for H2\(^{n} \), we rewrite (2.2a) as
\[ \bar{\alpha} = \frac{u(pu - r\bar{u}) + (p - r)(pr - u\bar{u})}{(p - r)u + r\bar{u} - p\bar{u}}, \] (2.9a)
\[ \bar{u} = -r + \bar{u} - \frac{2r(q + \bar{u} + u)}{r - u + \bar{\alpha}}. \] (2.9b)

Then, introducing \( \bar{\varphi} = G/F \) and \( \varphi = (G, F)^{T} \), from (2.9a) we have
\[ \tilde{\varphi} = L\varphi, \quad \tilde{\varphi} = M\varphi, \] (2.10)

where
\[ L = \gamma\begin{pmatrix} -ur - (p-r)\bar{u} & pu\bar{u} + (p-r)pr \\ -p & (p-r)u + r\bar{u} \end{pmatrix}, \]
\[ M = \gamma\begin{pmatrix} \bar{u} - r & -(r + \bar{u})(r - u) - 2r(q + u + \bar{u}) \\ 1 & r - u \end{pmatrix}, \] with \( \gamma = \frac{1}{\sqrt{r - (u - \bar{u})^{2}}}, \) \( \gamma' = \frac{1}{\sqrt{q - u + \bar{u}}}. \) The linear system (2.10) is compatible for solutions of (1.6) in the sense that H2\(^{n} \) is a divisor of \( (LM)^{2} - (ML)^{2} \), where the square can be taken either as matrix multiplication, or as component-wise multiplication.

### 3. Seed and one-soliton solution

In this section, we use the auto-BT (2.2a) to construct solutions for (1.6). First, we need to have a simple solution as a ‘seed’. To find such a solution, we take \( \bar{\alpha} = u \) in the BT (2.2a), i.e.
\[ (u - \bar{u})^{2} = p(p - r), \quad u + \bar{u} = -q - \frac{r}{2}. \] (3.1)

This so-called *fixed point approach* has proved to be effective in finding seed solutions [6, 8].

**Proposition 3.1.** Parametrizing
\[ p = \frac{\alpha}{a}, \quad \alpha = \frac{-ac}{a^{2} - 1}, \quad q = (\text{-}1)^{m}\beta - \frac{c}{2}, \] (3.2)

and setting the seed BT parameter equal to \( r = c \), the equations (3.1) allow the solution
\[ u_{0} = (\text{-}1)^{m}(\alpha n + \beta m + c_{0}), \] (3.3)

where \( c_{0} \) is a constant.

**Proof.** By direct calculation, with the given parameterizations the equations (3.1) read
\[ (u - \bar{u})^{2} = \alpha^{2}, \quad u + \bar{u} = (\text{-}1)^{m+1}\beta. \] □

It can be verified directly that (3.3) also provides a solution to (1.6). Next, we derive the one-soliton solution for (1.6), from the auto-BT (2.2a) with \( u = u_{0} \) as a seed solution.

**Proposition 3.2.** The equation (1.6), with lattice parameters (3.2) admits the one-soliton solution
\[ u_{1} = (\text{-}1)^{m}\left(\alpha n + \beta m + c_{0} + \frac{ck}{1 - k^{2}}\frac{1 - \rho_{n,m}}{1 + \rho_{n,m}}\right), \] (3.4)

where
\[ \rho_{n,m} = \rho_{0,0}\left(\frac{a + k}{a - k}\right)^{m} \prod_{i=0}^{m-1} \left(\frac{\text{-}1}{}\right)^{i} k^{1} + k \] (3.5)

with constant \( \rho_{0,0} \) is the plane wave factor.

**Proof.** Let
\[ u_{1} = u_{0} + (\text{-}1)^{m}(\kappa + \nu), \] (3.6)

where \( \kappa = kr \). With (3.2) and parametrizing the first BT parameter by
\[ r = \frac{c}{1 - k^{2}}, \] (3.7)

then substitution of \( u = u_{0} \) and \( \bar{\alpha} = u_{1} \) into the auto-BT (2.2a) yields
\[ \bar{\nu} = \frac{\nu E_{+}}{\nu + E_{-}}, \quad \bar{\nu} = \frac{\nu F_{+}(m)}{\nu + F_{-}(m)}, \] (3.8)

where
\[ E_{\pm} = -r(a \pm k), \quad F_{\pm}(m) = r((\text{-}1)^{m} \mp k). \] (3.9)
The difference system (3.8) can be linearized using \( \nu = \frac{f}{g} \) and \( \Phi = (f, g)^T \), which leads to

\[
\Phi(n + 1, m) = M \Phi(n, m), \quad \Phi(n, m + 1) = N(m) \Phi(n, m),
\]

where

\[
M = \begin{pmatrix} E_+ & 0 \\ 1 & E_- \end{pmatrix}, \quad N(m) = \begin{pmatrix} F_+ & 0 \\ 1 & F_- \end{pmatrix}.
\]

(3.10)

By ‘integrating’ (3.10) we have

\[
\Phi(n, m) = \mathcal{M}(n) \Phi(0, m), \quad \Phi(n, m) = \mathcal{N}(m) \Phi(n, 0),
\]

where

\[
\mathcal{M}(n) = \begin{pmatrix} E_+^n & 0 \\ \frac{E_+ E_- - E_-^n}{2n} & E_-^n \end{pmatrix},
\]

\[
\mathcal{N}(m) = \begin{pmatrix} 0 & \prod_{i=0}^{m-1} F_+(i) \\ \frac{1 - (-1)^m}{2} \prod_{i=0}^{m-2} F_+(i) \prod_{i=0}^{m-1} F_-(i) & 0 \end{pmatrix}.
\]

Thus, we get a solution to (3.12):

\[
\Phi(n, m) = \mathcal{M}(n) \mathcal{N}(m) \Phi(0, 0),
\]

from which \( \nu = f/g \) is obtained as

\[
\nu = \frac{E_+^n \prod_{i=0}^{m-1} F_+(i) \cdot \nu_{0,0}}{E_-^n \prod_{i=0}^{m-1} F_-(i) + \frac{(E_+^n \prod_{i=0}^{m-1} F_+(i) - E_-^n \prod_{i=0}^{m-1} F_-(i)) \nu_{0,0}}{2n}}.
\]

(3.14)

where \( \nu_{0,0} = \frac{f_{0,0}}{g_{0,0}} \). Introducing the plane wave factor

\[
\rho_{m,n} = \rho_{0,0} \frac{E_+^n \prod_{i=0}^{m-1} F_+(i)}{E_-^n \prod_{i=0}^{m-1} F_-(i)}
\]

\[
= \rho_{0,0} \left( \frac{a + k}{a - k} \right)^n \prod_{i=0}^{m-1} (-1)^{i-k} \prod_{i=0}^{m-1} (-1)^{i+k}
\]

(3.15)

with constant \( \rho_{0,0} \), the above \( \nu \) is written as

\[
\nu = -\frac{2k \rho_{m,n}}{1 + \rho_{m,n}},
\]

(3.16)

where some constants are absorbed into \( \rho_{0,0} = \frac{-\rho_{m,n}}{2k + \rho_{m,n}} \). Substituting (3.16) into (3.6) yields the one-soliton solution (3.4), which solves (1.6) with (3.2) and (3.7). Note that in the plane wave factor (3.15) \( m, n \in \mathbb{Z} \), and when \( m \leq 0 \) the product \( \prod_{i=0}^{m-1} \) is considered as \( \prod_{i=0}^{0} \).

4. Conclusions

In this paper, we have shown that equations which constitute an auto-BT for a quad equation admit auto-BTs themselves. We have focussed on one such equation, the torqued H2 equation denoted H20 (1.6), which forms an auto-BT for Q1. This equation is not part of the ABS list of CAC quad equations, as it is not symmetric with respect to \( (n, p) \leftrightarrow (m, q) \). The integrability of this equation is guaranteed as it is part of a consistent cube, see [14]. The equations H20 and Q1 comprise an auto-BT from which a Lax pair was obtained. Using this auto-BT we have derived a seed solution and a one-soliton solution. The parameterization of these solutions show that H20 is a semi-autonomous equation. We hope to be able to construct higher order soliton solutions in a future paper.

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