Isoperimetric Inequalities in Normed Planes

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Abstract. The classical isoperimetric inequality can be extended to a general normed plane [2]. In the Euclidean plane, the defect in the isoperimetric inequality can be calculated in terms of the signed areas of some singular sets. In this paper we consider normed planes with smooth by parts unit balls and the corresponding class of admissible curves. For such an admissible curve, the singular sets are defined as projections in the subspaces of symmetric and constant width admissible curves. In this context, we obtain some improved isoperimetric inequalities whose equality hold for symmetric or constant width curves.

Mathematics Subject Classification (2010). 52A10, 52A40.

Keywords. Minkowski geometry; Curves of constant width; Wigner caustic; Isoperimetrix.

1. Introduction

The isoperimetric inequality in the plane is an old problem: It states that

$$\frac{L^2}{4\pi} \geq A(\gamma),$$

(1.1)

for any simple convex curve $\gamma$, with equality holding only for circles. The isoperimetric inequality has been extended to an arbitrary normed plane by Busemann. In this case, the curve of a fixed area that minimizes length is not the unit circle, but the dual unit circle, also called isoperimetrix ([2, 6, 7, 10, 11]). On the other hand, the isoperimetric defect in the Euclidean plane can be calculated in terms of the signed areas of 2 sets, the Wigner caustic and the constant width measure set of the curve $\gamma$ ([12, 13]).

The authors wants to thank CNPq and CAPES (financial code 001) for financial support during the preparation of this manuscript. This manuscript arises from the Ph.D. dissertation of the first author under the supervision of the second author ([9]).

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Normed planes with smooth strictly convex unit balls were considered in [1]. In such a normed plane, denote by $\mathcal{C}$ the space of smooth Lagrangian curves of degree 1 and consider the subspaces $\mathcal{C}_0^{sym}$ of symmetric curves and $\mathcal{C}_0^{cw}$ of constant width curves, both with zero dual length. Then an inner product can be defined in $\mathcal{C}$ such that $\mathcal{C}_0^{sym}$ and $\mathcal{C}_0^{cw}$ become orthogonal and there exist orthonormal basis of cycloids for both subspaces, where a cycloid here means an eigenvector of the double evolute operator. In a normed plane with polygonal unit ball, by considering the class $\mathcal{C}$ of polygons with sides parallel to the unit ball, one can describe a similar discrete construction ([5]).

In this paper we put together all these ingredients. We consider normed planes whose unit ball is smooth by parts, which include normed planes with smooth strictly convex and polygonal unit balls as particular cases. For such normed planes, we define a class $\mathcal{C}$ of admissible curves and the subspaces of symmetric and constant width admissible curves, which are orthogonal with respect to the inner product (with signal) defined by the mixed area. For $\gamma \in \mathcal{C}$, we give adequate definitions for the Wigner caustic and the constant width measure set of $\gamma$ and show that they are exactly the orthogonal projections in the above subspaces. Then we prove a formula for the isoperimetric defect based on the signed areas of these sets. Our approach, besides providing another proof of the isoperimetric inequality of [2], also gives rise to some improved isometric inequalities whose equalities hold only for symmetric or constant width curves.

The paper is organized as follows: In section 2 we define the admissible curves associated with the unitary ball of a normed plane. In section 3 we discuss mixed areas. In section 4 we decompose the space of admissible curves with zero dual length in 2 orthogonal subspaces and define the Wigner caustic and constant width measure set as the projections of a curve in these subspaces. In section 5 we prove the improved isoperimetric inequalities, which are the main results of the paper.

2. Unit Ball and Admissible Curves

2.1. Unit ball and its dual

Consider a normed plane with unit ball $U$. We shall assume that the boundary of $U$, $u = \partial U$, is smooth by parts. More precisely, we shall assume that $u = \bigcup_{1 \leq i \leq 2n} u_i$, where $u_i$ are smooth arcs with $u_{i+n} = -u_i$. We shall also assume that each $u_i$ is either a smooth strictly convex arc or a straight segment.

The dual ball $U^*$ is defined as

$$U^* = \{ f \in (\mathbb{R}^2)^* | ||f|| \leq 1 \},$$
where $||f|| = \sup\{|f(x)| | x \in U\}$. Each functional $f \in (\mathbb{R}^2)^*$ can be represented by a vector $v \in \mathbb{R}^2$ by the relation

$$f(x) = [x, v], \ x \in \mathbb{R}^2,$$

(2.1)

where $[a, b]$ denotes the determinant of the $2 \times 2$ matrix whose columns are $a, b \in \mathbb{R}^2$. We shall represent $U^*$ by $V$ under this identification.

Given $v \in \mathbb{R}^2$, let $L_v$ denote the support line of $U$ parallel to $v$, and assume $L_v$ touches $U$ in $u \in \partial U$. Then $v \in V$ if and only if $[u, v] = 1$. Thus if $u$ belongs to a smooth arc, the corresponding $v$ also belongs to a smooth arc. If $u$ belongs to a straight segment, then $v$ is constant. Finally if $u$ is a vertex, then $v$ describes a segment. An example of a smooth by parts unit ball $U$ and its dual $V$ can be seen in Figure 1.

![Figure 1](image1.png)

**Figure 1.** An example of a smooth by parts unit ball and its dual.

### 2.2. Parameterization of the unit circle

Consider a parameterization $u(t)$ of $u$ such that $u'(t) \neq 0$, for any $t \in [t_i, t_{i+1}]$, where $u(t_i)$ are the vertices of $u$. There are 2 types of intervals $[t_i, t_{i+1}]$: The smooth intervals, where $u$ and $v$ are strictly convex, and the linear intervals, where $u$ is a straight segment and $v$ is constant. At smooth intervals we shall assume also that $[u'(t), u''(t)] \neq 0$. For any interval $t_i \leq t \leq t_{i+1}$, we can write

$$v(t) = \frac{u'(t)}{[u(t), u'(t)]}.$$  

(2.2)

In fact, the functional associated with $v(t)$ under the identification (2.1) is zero in the direction $u'(t)$ and is one at $u(t)$. Thus it belongs to the dual unit circle at direction $u'(t)$. Since the straight lines are not included in Equation (2.2), this formula parameterizes only a part of the dual circle, but in fact this part is all that we need in the paper.
Example 1. Consider the unit ball shown in Figure 1. Then a parameterization for the unit circle is given by

\[ u(t) = \begin{cases} 
(1 - t, t), & 0 \leq t \leq 1, \\
(cos(\frac{\pi}{2} t), sin(\frac{\pi}{2} t)), & 1 \leq t \leq 2, \\
(t - 3, 2 - t), & 2 \leq t \leq 3, \\
(cos(\frac{\pi}{2} t), sin(\frac{\pi}{2} t)), & 3 \leq t \leq 4,
\end{cases} \]

The corresponding points in the dual circle are given by

\[ v(t) = \begin{cases} 
(-1, 1), & 0 \leq t \leq 1, \\
(- sin(\frac{\pi}{2} t), cos(\frac{\pi}{2} t)), & 1 \leq t \leq 2, \\
(1, -1), & 2 \leq t \leq 3, \\
(- sin(\frac{\pi}{2} t), cos(\frac{\pi}{2} t)), & 3 \leq t \leq 4.
\end{cases} \]

Note that this parameterization does not include the straight lines of the dual ball.

2.3. Admissible curves

Denote by \( C = C(U) \) the class of smooth by parts closed curves \( \gamma \) parameterized by \( 0 \leq t \leq 2T \), \( \gamma(0) = \gamma(2T) \) such that, for \( t_i \leq t \leq t_{i+1} \),

\[ \gamma'(t) = r(t)u'(t), \tag{2.3} \]

for some scalar function \( r(t) \). We remark that when \( u \) is smooth, the class \( C \) includes all the convex smooth curves ([4]), and when \( u \) is polygonal, the class \( C \) consists of all polygons with sides parallel to those of \( u \) ([5]).

The scalar \( r(t) = r_\gamma(t) \) is called the curvature radius of \( \gamma \) at \( \gamma(t) \) ([1],[8]). Up to sign, the curvature radius is independent of the choice of the parameterizations of \( \gamma \) and \( u \). At an interval where \( u \) is a straight segment, Equation (2.3) implies that \( \gamma \) is also a straight segment and \( r(t) \) is the rate between the lengths of the \( \gamma \) and \( u \) segments. At a smooth interval, we have that

\[ [\gamma'(t), \gamma''(t)] = r(t)^2[u', u''](t), \]

which implies that \( \gamma \) has no inflection points. It is clear from Equation (2.3) that, up to a translation, we can recover \( \gamma \) from \( r \). It is easy to see from the curvature radius whether or not the curve is convex, as next lemma shows:

**Lemma 2.1.** The curve \( \gamma \in C \) is convex if and only if \( r(\gamma) \) does not change sign.

**Proof.** Denote \( \gamma'(t^+) = \lim_{s \to t^+} \gamma'(t) \) and \( \gamma'(t^-) = \lim_{s \to t^-} \gamma'(s) \). If \( r \) changes sign at a point \( t_0 \) of a smooth interval, then \( \gamma'(t_0^+) \) and \( \gamma'(t_0^-) \) are pointing in opposite directions and so the curve is not convex at this point. Similarly, if \( r \) changes sign at a vertex \( t_0 \), then \( \gamma'(t_0^+) \) and \( \gamma'(t_0^-) \) make an angle bigger than \( \pi \), and again the curve is not convex at this point.

Conversely, if \( r \) does not change sign, the tangents at vertices are making angles strictly smaller than \( \pi \) and so the curve is locally convex at the vertices. Since the curve has no inflection points, it is also locally convex at the smooth arcs. Finally, since the index of \( \gamma \) is \( \pm 1 \), the curve \( \gamma \) is necessarily convex. \( \square \)
Corollary 2.2. Given $\gamma \in C$, there exists a constant $K > 0$ such that $\gamma + Ku$ is convex.

Proof. The curvature radius of $\gamma + Ku$ is $r(t) + K$. If we choose $K \geq -\min\{r(t)\}$, we obtain a convex curve. \qed

2.4. Dual length

Given $\gamma \in C$, from Equations (2.2) and (2.3), we can write

$$\gamma'(t) = r(t)[u, u'](t)v(t),$$

for any $t$ in a smooth or linear interval. The (signed) dual length of $\gamma \in C$ is defined by

$$L_*(\gamma) = \int_0^{2T} r(t)[u, u'](t)dt. \quad (2.4)$$

When $\gamma$ is convex, $L_*(\gamma)$ coincides with the dual length of $\gamma$.

Example 2. Let

$$\gamma(t) = \begin{cases} 
(2 - t, 1 + t), & 0 \leq t \leq 1, \\
\frac{1}{\sqrt{15 \cos^2(\frac{\pi}{2} t) + 1}}(16 \cos(\frac{\pi}{2} t), \sin(\frac{\pi}{2} t)) + (1, 1), & 1 \leq t \leq 2, \\
(-11 + 4t, 9 - 4t), & 2 \leq t \leq 3, \\
\frac{1}{\sqrt{15 \sin^2(\frac{\pi}{2} t) + 1}}(\cos(\frac{\pi}{2} t), 16 \sin(\frac{\pi}{2} t)) + (1, 1), & 3 \leq t \leq 4,
\end{cases}$$

(see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{curve.png}
\caption{A curve in $C$ (Example 2).}
\end{figure}

The curvature radius is given by

$$r(t) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
\frac{16}{\sqrt{(15 \cos^2(\frac{\pi}{2} t) + 1)^3}}, & 1 \leq t \leq 2, \\
4, & 2 \leq t \leq 3, \\
\frac{16}{\sqrt{(15 \sin^2(\frac{\pi}{2} t) + 1)^3}}, & 3 \leq t \leq 4.
\end{cases}$$

We can calculate the dual length of $\gamma$ by equations (2.4) to obtain

$$L_*(\gamma) \approx 13.58.$$
3. Mixed areas

3.1. Basic properties

Consider 2 curves $\gamma_1, \gamma_2 \in C$. Then the mixed area is defined by

$$A(\gamma_1, \gamma_2) = \frac{1}{2} \int_0^{2T} [\gamma_1, \gamma_2]'(t) dt.$$ 

The signed area of $\gamma \in C$ is defined as $A(\gamma) = A(\gamma, \gamma)$. If $\gamma$ is convex, $A(\gamma)$ is the area of the region bounded by $\gamma$.

**Lemma 3.1.** $A(\gamma_1, \gamma_2)$ is a symmetric bilinear map in $C$. Moreover,

$$L_*(\gamma) = 2A(u, \gamma).$$  \hspace{1cm} (3.1)

**Proof.** Since $[\gamma_1, \gamma_2]' = [\gamma'_1, \gamma_2] + [\gamma_1, \gamma'_2]$ we obtain

$$\int_0^{2T} [\gamma_1, \gamma_2]'(t) dt = \int_0^{2T} [\gamma_2, \gamma'_1](t) dt.$$ 

For the second assertion observe that

$$L_*(\gamma) = \int_0^{2T} r(t)[u, u'](t) dt = \int_0^{2T} [u, \gamma'](t) dt = 2A(u, \gamma),$$

thus proving the lemma. \hfill \Box

3.2. Minkowski inequality

The Minkowski inequality says that for convex curves $\gamma \in C$,

$$A(\gamma_1, \gamma_2)^2 \geq A(\gamma_1)A(\gamma_2),$$

with equality if and only if $\gamma_1$ is a multiple of $\gamma_2$.

**Lemma 3.2.** For any $\gamma \in C$,

$$L_*(\gamma)^2 \geq 4A(\gamma)A(u),$$

with equality if and only if $\gamma$ is a multiple $u$.

**Proof.** Since $\gamma \in C$, by Corollary 2.2 there exists a constant $K > 0$ such that $\gamma + Ku$ are convex. Then, by Minkowski inequality,

$$A(\gamma + Ku, u)^2 \geq A(\gamma + Ku)A(u).$$

Since $A(\gamma + Ku) = A(\gamma) + K^2 A(u) + 2KA(\gamma, u)$ and $A(\gamma + Ku, u) = A(\gamma, u) + KA(u)$, we conclude that

$$A(\gamma, u)^2 \geq A(\gamma)A(u),$$

thus proving the lemma. \hfill \Box
3.3. Curves of constant width

We say that $\gamma \in \mathcal{C}$ is of constant width if

$$[\gamma, v](t + T) + [\gamma, v](t) = c,$$

for some constant $c$.

**Lemma 3.3.** For constant width curves, we have that

$$L^*(\gamma) = 2cA(U)$$

and so necessarily $c = w_\gamma$.

**Proof.** Since

$$2A(\gamma, u) = \int_0^{2T} [\gamma, u'](t) dt = \int_0^T [u, u'] ([\gamma, v](t) + [\gamma(t + T), v(t + T)]) dt$$

we conclude that $L^*(\gamma) = 2cA(U)$. \qed

**Corollary 3.4.** A curve $\gamma \in \mathcal{C}$ with zero dual length is of constant width if and only if

$$[\gamma, v](t + T) + [\gamma, v](t) = 0. \tag{3.2}$$

4. Wigner Caustic and Constant Width Measure Set

In this section we consider the mixed area as a signed inner product in $\mathcal{C}$.

4.1. An orthogonal decomposition of $\mathcal{C}$

Denote by $\mathcal{U}$ the 1-dimensional subspace of $\mathcal{C}$ consisting of the constant multiples of the unit ball, by $\mathcal{C}_{0}^{\text{sym}} \subset \mathcal{C}$ the subspace consisting of symmetric curves with respect to the origin with zero dual length and by $\mathcal{C}_{0}^{\text{cw}} \subset \mathcal{C}$ the subspace consisting of constant width curves, also with zero dual length. Then Equation [3.1] implies that both $\mathcal{C}_{0}^{\text{sym}}$ and $\mathcal{C}_{0}^{\text{cw}}$ are orthogonal to $\mathcal{U}$.

**Lemma 4.1.** The subspaces $\mathcal{C}_{0}^{\text{sym}}$ and $\mathcal{C}_{0}^{\text{cw}}$ are orthogonal.

**Proof.** If $\gamma_1 \in \mathcal{C}_{0}^{\text{sym}}$ and $\gamma_2 \in \mathcal{C}_{0}^{\text{cw}}$, we have that

$$\gamma_1(t + T) = -\gamma_1(t), \quad [\gamma_2(t + T), v(t + T)] + [\gamma_2(t), v(t)] = 0.$$

Thus

$$A(\gamma_1, \gamma_2) = \int_0^T [\gamma_2, \gamma_1'](t) dt + \int_0^T [\gamma_2, \gamma_1'](t + T) dt,$$

$$= \int_0^T r_1(t)[u, u'](t) ([\gamma_2(t), v(t)] - [\gamma_2(t + T), v(t)]) dt$$

$$= \int_0^T r_1(t)[u, u'](t) ([\gamma_2(t), v(t)] + [\gamma_2(t + T), v(t + T)]) dt,$$

which proves that $A(\gamma_1, \gamma_2) = 0$. \qed
4.2. Orthogonal projections

For $\gamma \in \mathcal{C}$, define the \textit{Wigner caustic} of $\gamma$ by

$$WC(\gamma)(t) = \frac{1}{2} (\gamma(t) + \gamma(t+T)),$$

and the \textit{constant width measure set} of $\gamma$ by

$$CWMS(\gamma)(t) = \frac{1}{2} (\gamma(t) - \gamma(t+T) - w_\gamma u(t)),$$

where

$$w_\gamma = \frac{L_\star(\gamma)}{A_U}$$

is the \textit{mean width} of $\gamma$. It is clear that $WC: \mathcal{C} \to \mathcal{C}$ and $CWMS: \mathcal{C} \to \mathcal{C}$ are linear maps and any $\gamma \in \mathcal{C}$ can be decomposed as the sum of its Wigner Caustic, its Constant Width Measure Set and a multiple of the unit ball. In fact,

$$\gamma = WC(\gamma) + CWMS(\gamma) + \frac{w_\gamma}{2} u.$$

We shall prove that $CWMS$ is the orthogonal projection on $\mathcal{C}_0^{sym}$, while $WC$ is the orthogonal projection on $\mathcal{C}_0^{cw}$.

**Proposition 4.2.** We have that:

1. The image of $WC$ is $\mathcal{C}_0^{cw}$, while the image of $CWMS$ is $\mathcal{C}_0^{sym}$.
2. The kernel of $WC$ is $\mathcal{C}_0^{sym} \oplus \mathcal{U}$, while the kernel of $CWMS$ is $\mathcal{C}_0^{cw} \oplus \mathcal{U}$

**Proof.** Observe first that

$$[WC(\gamma)(t), v(t)] + [WC(\gamma)(t+T), v(t+T)] = [WC(\gamma)(t), v(t)] - [WC(\gamma)(t), v(t)] = 0,$$

which implies that $WC(\gamma)$ has constant width 0, i.e., $WC(\gamma) \in \mathcal{C}_0^{cw}$. Moreover, $CWMS(\gamma)$ is symmetric and

$$L_\star CWMS(\gamma) = L_\star(\gamma) - \frac{1}{2} \int_0^{2T} w_\gamma [u, u'](t) dt = 0,$$

thus proving that $CWMS(\gamma) \in \mathcal{C}_0^{sym}$. For the second item, observe that

$$\gamma(t) - WC(\gamma)(t) = \frac{1}{2} (\gamma(t) - \gamma(t+T))$$

is symmetric, which means $(\gamma - WC(\gamma)) \in \mathcal{C}_0^{sym} \oplus \mathcal{U}$. On the other hand

$$\gamma(t) - CWMS(\gamma)(t) = WC(\gamma)(t) + \frac{1}{2} w_\gamma u(t)$$

has constant width, which means $(\gamma - CWMS(\gamma)) \in \mathcal{C}_0^{cw} \oplus \mathcal{U}$.  

**Corollary 4.3.** We have that $A(WC(\gamma)) \leq 0$ with equality if and only if $\gamma$ is symmetric and $A(CWMS(\gamma)) \leq 0$ with equality if and only if $\gamma$ is of constant width.
Proof. This corollary follows immediately from the fact that
\[ L^*_s(\text{WC}(\gamma)) = L^*_s(\text{CWMS}(\gamma)) = 0 \]
and Lemma 3.2.

Example 3. The WC and CWMS of the curve of Example 2 can be seen in Figure 3.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The WC and CWMS of the curve $\gamma$. Their areas are approximately $-1.33$ and $-0.48$, respectively.}
\end{figure}

5. Improved Isoperimetric Inequalities

5.1. An isoperimetric equality

Proposition 5.1. Let $\gamma \in C$ be a convex curve. Then
\[ \frac{L^2_2(\gamma)}{4A_U} = A_\gamma - 2A_{WC(\gamma)} - A_{CWMS(\gamma)}. \] (5.1)

Proof. Consider the orthogonal decomposition of $\gamma$ given by Equation (4.4). Taking into account that WC is $T$-periodic, we obtain
\[ A(\gamma) = 2A(\text{WC}(\gamma)) + A(\text{CWMS}(\gamma)) + \frac{w^2_\gamma}{4} A(U). \]
We conclude that
\[ A(\gamma) - 2A(\text{WC}(\gamma)) - A(\text{CWMS}(\gamma)) = \frac{L^2_2(\gamma)}{4A(U)}, \]
thus proving the proposition.

\[ \square \]
5.2. Some consequences

The next corollary gives us two new improved isoperimetric inequalities:

**Corollary 5.2.** Let $\gamma \in C$ be a convex curve.

1. The following improved isoperimetric inequality holds:
   \[
   \frac{L_2^*(\gamma)}{4A_U} \geq A_\gamma - A_{CWMS(\gamma)},
   \]  
   with equality if and only if $\gamma$ is symmetric.
2. The following improved isoperimetric inequality holds:
   \[
   \frac{L_2^*(\gamma)}{4A_U} \geq A_\gamma - 2A_{WC(\gamma)},
   \]  
   with equality if and only if $\gamma$ is of constant width.

Next corollary recovers the isoperimetric inequality of Busemann in a more restricted case. In fact, it holds for any convex curve $\gamma$ and any normed plane (see [2]).

**Corollary 5.3.** Let $\gamma \in C$ be a convex curve. Then
   \[
   \frac{L_2^*(\gamma)}{4A_U} \geq A_\gamma,
   \]  
   and equality holds if and only if $\gamma$ is a multiple of the unit ball.

5.3. Lhuilier’s inequality

Consider a convex polygon $K$ and let $K_1$ be the polygon which is circumscribed about the unit circle and whose sides are respectively parallel to the sides of $K$. Let $L_* (K)$ denotes the dual length of $K$, $A(K)$ the area enclosed by $K$ and $A(K_1)$ the area enclosed by $K_1$. Then Lhuilier’s theorem ([3]) states that
   \[
   \frac{L_2^*(K)}{4A(K_1)} \geq A(K),
   \]
   with equality if and only if $K$ is a constant multiple of $K_1$.

We shall now proof a weaker version of Lhuilier’s inequality, namely
   \[
   \frac{L_2^*(K)}{4A(K_1^0)} \geq A(K),
   \]  
   where $K_1^0 = K_1 \cap (-K_1)$ is the symmetrization of $K_1$, with equality if and only if $K$ is a constant multiple of $K_1^0$. This inequality coincides with Lhuilier’s inequality if $K$ has parallel opposite sides. To prove the general case of Lhuilier’s inequality, we need to develop the results of this paper in the more general context of a non-symmetric unit ball.

To prove (5.5), consider the normed plane with unit ball $K_1^0$. Observe that the dual length $L_* (K)$ with respect to original normed plane is the same as the dual length of $K$ with respect to the normed plane with unit ball $K_1^0$. 

Then by inequality (5.4) applied to $K \in \mathcal{C}(K_1^0)$, i.e., in the normed plane whose unit ball is $K_1^0$, we obtain Inequality (5.5).

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