Some strong forms of connectedness in topological spaces

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Abstract

In this paper, we study new separations of sets called half separated, half $\alpha$-separated, half semi-separated, half pre-separated, half $\beta$-separated sets and corresponding to these notions introduced half connected, half $\alpha$-connected, half semi-connected, half pre-connected, half $\beta$-connected topological spaces, respectively. These are stronger forms of connectedness, $\alpha$-connectedness, semi-connectedness, pre-connectedness, $\beta$-connectedness respectively. The properties of these notions follow the same pattern.

Keywords: Half connectedness, half $\alpha$-connectedness, half semi-connectedness, half pre-connectedness, half $\beta$-connectedness, topological space.

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1. Introduction

The concepts of semi-connectedness [23], pre-connectedness [24] and $\beta$-connectedness [1, 13, 31] in topological spaces are based on the notions of semi-open set [15], pre-open set [11], $\beta$-open set [10], and $b$-open sets [2], respectively. The classes of $\beta$-connected, semi-connected, pre-connected topological spaces are subclasses of the class of connected topological spaces. Recently, Noiri and Modak [22] introduced half $b$-connectedness in topological spaces. Tyagi et al. studied several forms of connectedness in topological spaces using the ideal notions (see [4, 5]) and also in generalized topological spaces (see [30, 6, 29]). In this paper we introduce the notions of half connectedness, half $\alpha$-connectedness, half semi-connectedness, half pre-connectedness and half $\beta$-connectedness in topological space. These properties are strong forms of connectedness, $\alpha$-connectedness, semi-connectedness, pre-connectedness and $\beta$-connectedness, respectively. The properties of these forms follow similar pattern.

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This paper is organized as follows: Section 2 contains preliminaries. Section 3 contains the definitions of half connectedness, half α-connectedness, half semi-connectedness, half pre-connectedness and half β-connectedness. In Section 4, we discuss the properties of these various strong forms of connectedness. The interaction of these properties with various types of mappings for examples continuous, α-continuous, semi-continuous, precontinuous and β-continuous etc. is investigated in section 5.

2. Preliminaries

(X, τ) or X be a topological space or space. We will denote by Cl(A) and Int(A) the closure of A and the interior of A, respectively. A subset A is said to be preopen [11] (resp., semi-open [15], α-open [20], β-open [10], regular open [28], b-open [2]) if A ⊆ Int(Cl(A)) (resp., A ⊆ Cl(Int(A)), A ⊆ Int(Cl(Int(A))), A ⊆ Cl(Cl(Int(A))), A = Int(Cl(A)), A ⊆ Cl(Cl(A)) ∪ Int(Cl(A))). The complement of a preopen (resp., semi-open, α-open, β-open, regular open, b-open) set is said to be preclosed (resp., semiclosed, α-closed, β-closed, regular closed, b-closed). The family of all pre-open (resp., semi-open, α-open, β-open, regular open, b-open) subsets of a topological space (X, τ) is denoted by PO(X) (resp., SO(X), αO(X), βO(X), RO(X), BO(X)). The family of all pre-closed (resp., semi-closed, α-closed, β-closed, regular closed, b-closed) subsets of a topological space (X, τ) is denoted by PC(X) (resp., SC(X), αC(X), βC(X), RC(X), BC(X)). The preclosure (resp., semi-closure, α-closure, β-closure, b-closure) of a set A ⊆ X is the intersection of pre-closed (resp., semi-closed, α-closed, β-closed, b-closed) sets containing A and are denoted by pCl(A) (resp., sCl(A), αCl(A), βCl(A), bCl(A)).

Definition 2.1. A pre-open (resp., α, b-open, β-open) subset A of a topological space X is said to be Pß-open [27] (resp., αß-open [3], ßß-open [26]) if for each x ∈ A there exists a β-closed set F such that x ∈ F ⊆ A. That is, the pre-open (resp., α-open, b-open, β-open) set is expressed as a union of β-closed sets.

Definition 2.2. Two subsets A and B in a space X are said to be half b-separated [22] (resp., Cl-Cl-separated [19]) if and only if A ∩ bCl(B) = φ or bCl(A) ∩ B = φ (resp., Cl(A) ∩ Cl(B) = φ).

Definition 2.3. A subset A of a space X is said to be half b-connected (resp. Cl-Cl-connected [19]) if A is not the union of two nonempty half b-separated (resp. Cl-Cl-separated) sets in X.

Definition 2.4. A space X is said to be Pß-connected [27] (resp. αß-connected [3], ßß-connected [26]) if X cannot be expressed as the union of two disjoint nonempty Pß-open (resp., αß-open, ßß-open, ßß-open) subsets of X.

3. Half (semi, pre, α, β) connected sets

Definition 3.1. Two non-empty subsets A and B in a space X are said to be separated (resp., semi-separated [23], pre-separated [24], α-separated [13], β-separated [13]) if A ∩ Cl(B) = φ = Cl(A) ∩ B (resp., A ∩ sCl(B) = φ = sCl(A) ∩ B, A ∩ pCl(B) = φ = pCl(A) ∩ B, A ∩ αCl(B) = φ = αCl(A) ∩ B, A ∩ βCl(B) = φ = βCl(A) ∩ B).

Definition 3.2. Two non-empty subsets A and B in a space X are said to be half separated (resp., half semi-separated, half pre-separated, half α-separated, half β-separated) if A ∩ Cl(B) = φ or Cl(A) ∩ B = φ (resp., A ∩ sCl(B) = φ or sCl(A) ∩ B = φ, A ∩ pCl(B) = φ or pCl(A) ∩ B = φ, A ∩ αCl(B) = φ or αCl(A) ∩ B = φ, A ∩ βCl(B) = φ or βCl(A) ∩ B = φ).

From the above definitions, we have the following implications. However, converse of these are not always true as shown in the following examples.

Example 3.3. In the real line R, sets A = (−∞, 0) and B = (0, ∞) are separated but not Cl − Cl-separated.

Example 3.4. Let X = {a, b, c} with a topology τ = {φ, {a}, {b}, {a, b}, X}. SO(X) = {φ, {a}, {b}, {a, b}, {a, c}, {b, c}, X} and αO(X) = {φ, {a}, {b}, {a, b}, X}. Then A = {a} and B = {b, c} are semi-separated but not α-separated.
Example 3.5. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. \(BO(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}\) and \(SO(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, X\}\). Here $\{b\}$ and $\{c\}$ are b-separated but not semi-separated.

Example 3.6. In Example 3.5, the sets $\{a, b\}$ and $\{c, d\}$ are $\beta$-separated but not b-separated.

Example 3.7. In any space with more then one point with indiscrete topology, any two non-empty disjoint sets are b-separated but not $\alpha$-separated.

Example 3.8. Suppose that $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Here $\{a\}$ and $\{b, c\}$ are b-separated but not pre-separated.

Example 3.9. In $\mathbb{R}$ with usual topology, the sets $A = [0, 1]$ and $B = (1, 2)$ are half separated but not separated.

Example 3.10. In $\mathbb{R}$ with usual topology on $\mathbb{R}$ the sets $A = (0, 3]$ and $B = (3, 5)$ are half $\alpha$-separated but not $\alpha$-separated, half semi separated but not semi-separated, half b-separated but not b-separated, half $\beta$-separated but not $\beta$-separated, half pre-separated but not pre-separated, half separated but not $Cl-Cl$-separated.

Example 3.11. In Example 3.4, the sets $A = \{a\}$ and $B = \{b, c\}$ are half semi-separated but not half $\alpha$-separated.

Example 3.12. In Example 3.5, the sets $A = \{b\}$ and $B = \{c\}$ are half b-separated but not half semi-separated.

Theorem 3.13. Let $A$ and $B$ be non-empty sets in a space $X$. The following statements hold:

(i) If $A$ and $B$ are half separated (half semi-separated, half pre-separated, half $\alpha$-separated, half $\beta$-separated) and $A_1 \subseteq A$ and $B_1 \subseteq B$, then $A_1$ and $B_1$ are so, respectively.

(ii) If $A \cap B = \emptyset$ and one of $A$ and $B$ is closed (semi-closed, pre-closed, $\alpha$-closed, $\beta$-closed) or open (semi-open, pre-open, $\alpha$-open, $\beta$-open), then $A$ and $B$ are half separated (resp., half semi-separated, half pre-separated, half $\alpha$-separated, half $\beta$-separated).

(iii) If one of $A$ and $B$ is closed (semi-closed, pre-closed, $\alpha$-closed, $\beta$-closed) or open (semi-open, pre-open, $\alpha$-open, $\beta$-open) and if $H = A \cap (X - B)$ and $G = B \cap (X - A)$, then $H$ and $G$ are half separated (resp., half semi-separated, half pre-separated, half $\alpha$-separated, half $\beta$-separated).
Proof. (i) Obvious.
(ii) In case $A$ is open, by $A \cap B = \emptyset$, $A \cap \text{Cl}(B) = \emptyset$. In case $A$ is closed, $\text{Cl}(A) \cap B = A \cap B = \emptyset$. Therefore, $A$ and $B$ are half separated.
(iii) (1) Let $A$ be closed. Then $\text{Cl}(H) \cap G \subset \text{Cl}(A) \cap (X - A) = A \cap (X - A) = \emptyset$.
(2) Let $A$ be open. Then $H \cap \text{Cl}(G) \subset A \cap \text{Cl}(X - A) = A \cap (X - A) = \emptyset$.
The proof is similar in other cases.

**Theorem 3.14.** The subsets $A$ and $B$ of a space $X$ are half separated (resp., half semi-separated, half pre-separated, half $\alpha$-separated, half $\beta$-separated) if and only if there exists $U$ in $\tau(X)$ (resp., $S\alpha(X)$, $P\alpha(X)$, $\alpha O(X)$, $\beta O(X)$) such that $A \subset U$ and $B \cap U = \emptyset$ or there exists $V$ in $\tau(X)$ (resp., $S\alpha(X)$, $P\alpha(X)$, $\alpha O(X)$, $\beta O(X)$) such that $B \subset V$ and $A \cap V = \emptyset$.

Proof. Let $A$ and $B$ be half separated sets. Then $A \cap \text{Cl}(B) = \emptyset$ or $\text{Cl}(A) \cap B = \emptyset$. Suppose that $A \cap \text{Cl}(B) = \emptyset$. Set $U = X - \text{Cl}(B)$. Then $U \in \tau(X)$, $A \subset U$ and $B \cap U = \emptyset$. Conversely, suppose that there exists $U \in \tau(X)$ such that $A \subset U$ and $B \cap U = \emptyset$. Then $\text{Cl}(B) \cap U = \emptyset$ and hence $A \cap \text{Cl}(B) = \emptyset$. Thus, $A$ and $B$ are half separated.

**Definition 3.15.** A subset $S$ of a space $X$ is said to be connected (resp., semi-connected [23], pre-connected [24], $\alpha$-connected [13], $\beta$-connected [13]) if there are no two separated subsets $A$ and $B$ (resp., semi-separated, pre-separated, $\alpha$-separated, $\beta$-separated) such that $S = A \cup B$.

**Definition 3.16.** A subset $A$ of a space $X$ is said to be half connected (resp., half semi-connected, half pre-connected, half $\alpha$-connected, half $\beta$-connected) if $A$ is not the union of two non-empty half separated (resp., half semi-separated, half pre-separated, half $\alpha$-separated, half $\beta$-separated) sets in $X$.

**Theorem 3.17.** [9] A space $(X, \tau)$ is b-connected if and only if it is $\beta$-connected.

**Corollary 3.18.** A space $(X, \tau)$ is half b-connected if and only if it is half $\beta$-connected.

**Theorem 3.19.** For a topological space following statements hold:
1. [27] $X$ is $P\beta$-connected if and only $X$ is pre-connected;
2. [3] $X$ is $\alpha\beta$-connected if and only $X$ is $\alpha$-connected if and only $X$ is connected;
3. [26] $X$ is $b\beta$-connected if and only $X$ is $b$-connected if and only $X$ is $\beta$-connected if and only if $X$ is $\beta\beta$-connected.

From definitions, Theorem 3.17 and Theorem 3.19, we have following theorem.

**Theorem 3.20.** For a topological space following statements hold:
1. $X$ is half pre-connected then $X$ is $P\beta$-connected;
2. $X$ is half connected or half connected then $X$ is $\alpha\beta$-connected;
3. $X$ is half $b$-connected or half $\beta$-connected then $X$ is $b\beta$-connected, $b$-connected, $\beta$-connected and $\beta\beta$-connected.

The following implications are immediate:
4. Properties of Half (semi, pre, \(\alpha\), \(\beta\))-Connected sets

**Theorem 4.1.** A space \(X\) is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) if and only if it cannot be expressed as the disjoint union of non-empty open (resp., semi-open, pre-open, \(\alpha\)-open, \(\beta\)-open) set and a non-empty closed (resp., semi-closed, pre-closed, \(\alpha\)-closed, \(\beta\)-closed) set.

*Proof.* Let \(X\) be a half connected space. Suppose that \(X = U \cup F\), where \(U \cap F = \emptyset\), \(U\) is a non-empty open set and \(F\) is a non-empty closed set in \(X\). Since \(F\) is a closed set in \(X\), \(U \cap \text{Cl}(F) = \emptyset\) and so \(U\) and \(F\) are half separated. Therefore, \(X\) is not half connected space, a contradiction. Conversely, suppose that \(X\) is not a half connected space. Then there exist non-empty half separated sets \(A\) and \(B\) such that \(X = A \cup B\). Let \(A \cap \text{Cl}(B) = \emptyset\). Set \(U = X - \text{Cl}(B)\) and \(F = \text{Cl}(B)\). Then \(U \cup F = X\) and \(U \cap F = \emptyset\). Also \(U\) is a non-empty open set and \(F\) is a non-empty closed set.

**Theorem 4.2.** Let \(X\) be a space. If \(A\) is a half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) subset of \(X\) and \(H\) and \(G\) are half separated (resp., half semi-separated, half pre-separated, half \(\alpha\)-separated, half \(\beta\)-separated) subsets of \(X\) with \(A \subset H \cup G\), then either \(A \subset H\) or \(A \subset G\).

*Proof.* Let \(A\) be a half connected set and \(A \subset H \cup G\). Since \(H\) and \(G\) are half separated, \(G \cap \text{Cl}(H) = \emptyset\) or \(\text{Cl}(G) \cap H = \emptyset\). Let \(G \cap \text{Cl}(H) = \emptyset\). Since \(A = (A \cap H) \cup (A \cap G)\), then \((A \cap G) \cup \text{Cl}(A \cap H) \subset G \cap \text{Cl}(H) = \emptyset\). If \(A \cap H\) and \(A \cap G\) are non-empty, then \(A\) is not half connected, a contradiction. Thus, either \(A \cap H = \emptyset\) or \(A \cap G = \emptyset\). This implies that \(A \subset H\) or \(A \subset G\).

**Theorem 4.3.** If \(A\) and \(B\) are half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) sets of a space \(X\) and \(A\) and \(B\) are not half separated (resp., half semi-separated, half pre-separated, half \(\alpha\)-separated, half \(\beta\)-separated), then \(A \cup B\) is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected).

*Proof.* Let \(A\) and \(B\) be half connected sets in \(X\). Suppose \(A \cup B\) is not half connected. Then there exist two non-empty half separated sets \(G\) and \(H\) such that \(A \cup B = G \cup H\). Since \(G\) and \(H\) are half separated, \(G \cap \text{Cl}(H) = \emptyset\) or \(\text{Cl}(G) \cap H = \emptyset\). Suppose that \(G \cap \text{Cl}(H) = \emptyset\), since \(A\) and \(B\) are half connected by Theorem 4.2, either \(A \subset G\) or \(A \subset H\). Similarly \(B \subset G\) or \(B \subset H\). If \(A \subset G\) and \(B \subset G\), then \(H\) is empty, a contradiction. If \(B \subset G\) and \(A \subset H\), then \(\text{Cl}(A) \cap B \subset \text{Cl}(H)\cap G = \emptyset\). Thus, \(A\) and \(B\) are half separated, a contradiction. Thus, \(A \cup B\) is half connected.
Theorem 4.4. If \(|M_i : i \in I|\) is a non-empty family of half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) sets of a space \(X\) and \(\cap_{i\in I}M_i \neq \emptyset\), then \(\cup_{i\in I}M_i \neq \emptyset\) is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected).

Proof. Suppose that \(\cup_{i\in I}M_i\) is not half connected. Then \(\cup_{i\in I}M_i = H \cup G\), where \(H\) and \(G\) are non-empty half separated sets in \(X\). Since \(\cap_{i\in I}M_i \neq \emptyset\), we have a point \(x \in \cap_{i\in I}M_i\). Since \(x \in \cup_{i\in I}M_i\), either \(x \in H\) or \(x \in G\). Suppose that \(x \in H\). By Theorem 4.2, \(M_i \subset H\) for all \(i\). Thus, \(\cup_{i\in I}M_i \subset H\). Then \(G\) is empty. Therefore, \(\cup_{i\in I}M_i\) is half connected.

Theorem 4.5. Let \(X\) be a space, \(|A_\alpha : \alpha \in \Delta|\) be a family of half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) sets and \(A\) is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected). If \(A \cap A_\alpha \neq \emptyset\) for every \(\alpha \in \Delta\), then \(A \cup (\cup_{\alpha \in \Delta}A_\alpha)\) is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected).

Proof. Since \(A \cap A_\alpha \neq \emptyset\) for each \(\alpha \in \Delta\), by Theorem 3.4, \(A \cup A_\alpha\) is half connected for each \(\alpha \in \Delta\). Moreover, \(A \cup (\cup_{\alpha \in \Delta}A_\alpha) = (\cup(A \cup A_\alpha))\) and \(\emptyset \neq A \cap \cup_{\alpha \in \Delta}(A \cup A_\alpha)\). Thus, by Theorem 4.4, \(A \cup (\cup_{\alpha \in \Delta}A_\alpha)\) is half connected.

Definition 4.6. A space \(X\) is said to be \(T_0\) (resp., \(sT_0\) [16], \(pT_0\) [14], \(aT_0\) [8], \(\beta T_0\)) if for each pair of distinct points in \(X\), there exists an open (resp., semi-open, pre-open, \(\alpha\)-open, \(\beta\)-open) set containing one of them but not the other.

Theorem 4.7. Let \(X\) be a \(T_0\) (resp., \(sT_0\), \(pT_0\), \(aT_0\), \(\beta T_0\)) topological space where \(|X| \geq 2\). Then \(X\) is not half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected).

Proof. Let \(x, y\) be distinct points of \(X\). Then there exists an open set \(U\) such that \(x \in U\) and \(y \notin U\) or \(y \in U\) and \(x \notin U\). Let \(x \in U\) and \(y \notin U\). Then \(y \in X - U\), and \(X - U\) is closed. Then \(X = U \cup (X - U)\). By Theorem 4.1, \(X\) is not half connected.

5. Strong form of connectedness and mappings

Definition 5.1. A function \(f : X \to Y\) is said to be:

(i) continuous (resp., semi-continuous [15], pre-continuous [11], \(\alpha\)-continuous [25], \(\beta\)-continuous [10]) if the inverse image under \(f\) of each open set in \(Y\) is open (resp., semi-open, pre-open, \(\alpha\)-open, \(\beta\)-open) in \(X\).

(ii) closed (resp., semi-closed [21], pre-closed [18], \(\alpha\)-closed [12], \(\beta\)-closed) if the image under \(f\) of each closed set in \(X\) is closed (resp., semi-closed, pre-closed, \(\alpha\)-closed, \(\beta\)-closed) in \(Y\).

(iii) irresolute [7] (resp., pre-irresolute [25], \(\alpha\)-irresolute [17], \(\beta\)-irresolute) if for each point \(x \in X\) and each open (resp., semi-open, pre-open, \(\alpha\)-open, \(\beta\)-open) set \(V\) of \(Y\) containing \(f(x)\), there exist a open (resp., semi-open, pre-open, \(\alpha\)-open, \(\beta\)-open) set \(U\) of \(X\) containing \(x\) such that \(f(U) \subset V\).

Theorem 5.2. The continuous (resp., irresolute, pre-irresolute, \(\alpha\)-irresolute, \(\beta\)-irresolute) image of a half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected) space is half connected (resp., half semi-connected, half pre-connected, half \(\alpha\)-connected, half \(\beta\)-connected).

Proof. Let \(f : X \to Y\) be an continuous function and \(X\) be a half connected space. Suppose that \(f(X)\) is not half connected subset of \(Y\). Then there exist half separated sets \(P\) and \(Q\) in \(Y\) such that \(f(X) = P \cup Q\). Since \(P\) and \(Q\) are half separated, \(Cl(P) \cap Q = \emptyset\) or \(P \cap Cl(Q) = \emptyset\). Since \(f\) is continuous, \(Cl(f^{-1}(P)) \cap f^{-1}(Q) = f^{-1}(Cl(P) \cap f^{-1}(Q)) = f^{-1}(Cl(P) \cap Q) = \emptyset\) or \(f^{-1}(P) \cap Cl(f^{-1}(Q)) = f^{-1}(P) \cap Cl(Q) = \emptyset\). Since \(P \neq Q\), there exists a point \(p \in X\) such that \(f(p) \in P\) and hence \(f^{-1}(P) \neq \emptyset\). Similarly, \(f^{-1}(Q) \neq \emptyset\). Therefore, \(f^{-1}(P)\) and \(f^{-1}(Q)\) are half separated sets such that \(X = f^{-1}(P) \cup f^{-1}(Q)\). Therefore, \(X\) is not a half connected space, a contradiction.

Lemma 5.3. Let \(f : X \to Y\) be a continuous (resp., semi-continuous [15], pre-continuous [11], \(\alpha\)-continuous [25], \(\beta\)-continuous [10]). Then \(Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))\) (resp., \(sCl(f^{-1}(B)) \subseteq f^{-1}(sCl(B))\), \(pCl(f^{-1}(B)) \subseteq f^{-1}(pCl(B))\), \(\alpha Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha Cl(B))\), \(\beta Cl(f^{-1}(B)) \subseteq f^{-1}(\beta Cl(B))\)) for each \(B \subseteq Y\).
Theorem 5.4. If \( f : X \rightarrow Y \) is a continuous (resp., semi-continuous, pre-continuous, \( \alpha \)-continuous, \( \beta \)-continuous) and \( K \) is half connected (resp., half semi-connected, half pre-connected, half \( \beta \)-connected) set in \( X \), then \( f(K) \) is \( Cl - Cl \)-connected in \( Y \).

Proof. Suppose \( f(K) \) is not \( Cl - Cl \)-connected sets in \( Y \). There exist two non-empty \( Cl - Cl \)-separated sets \( P \) and \( Q \) of \( Y \) such that \( f(K) = P \cup Q \), set \( A = K \cap f^{-1}(P) \) and \( B = K \cap f^{-1}(Q) \). Since \( f(K) \cap P \neq \emptyset \), then \( K \cap f^{-1}(P) \neq \emptyset \) and so \( A \neq \emptyset \). Similarly \( B \neq \emptyset \), moreover \( A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = (K \cap f^{-1}(P) \cup f^{-1}(Q)) = K \cap (f^{-1}(P) \cup Q) = K \cap (f^{-1}(f(K))) = K \). Since \( f \) is continuous, by Lemma 5.3, \( A \cap Cl(B) \subset f^{-1}(P) \cap Cl(f^{-1}(Q)) \subset f^{-1}(Cl(P)) \cap f^{-1}(f(Cl(Q))) = f^{-1}(Cl(P) \cap Cl(Q)) = f^{-1}(f(K)) = K \). This is contrary to the fact that \( K \) is half connected.

Corollary 5.5. If \( f : X \rightarrow Y \) is bijective closed (resp., semi-closed, pre-closed, \( \alpha \)-closed, \( \beta \)-closed) function and \( K \) is half connected (resp., half semi-connected, half pre-connected, half \( \beta \)-connected), then \( f^{-1}(K) \) is \( Cl - Cl \)-connected in \( X \).

Proof. Let \( f : X \rightarrow Y \) be a closed bijection. then \( f^{-1} : Y \rightarrow X \) is a continuous bijection. since \( K \) is half connected in \( Y \) by Theorem 5.2, \( f^{-1}(K) \) is \( Cl - Cl \)-connected in \( X \).

The proof is similar in other cases.

Theorem 5.6. If \( E \) is half connected (resp., half semi-connected, half pre-connected, half \( \alpha \)-connected, half \( \beta \)-connected), then \( Cl(E) \) is half connected (resp., half semi-connected, half pre-connected, half \( \alpha \)-connected, half \( \beta \)-connected).

Proof. Suppose that \( Cl(E) \) is not half connected. Then there are two half separated sets \( A \) and \( B \) in \( X \) such that \( Cl(E) = A \cup B \). Since \( E = (A \cap E) \cup (B \cap E) \) and \( (Cl(A \cap E)) \cap B = \emptyset \), \( (Cl(A \cap E)) \cap (B \cap E) = \emptyset \). Therefore, \( E \) is not half connected.

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