Every compact group arises as the outer automorphism group of a II$_1$ factor

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Abstract

We show that any compact group can be realized as the outer automorphism group of a factor of type II$_1$. This has been proved in the abelian case by Ioana, Peterson and Popa [6] applying Popa’s deformation/rigidity techniques to amalgamated free product von Neumann algebras. Our methods are a generalization of theirs.

Introduction

The outer automorphism group Out($M$) of a II$_1$ factor $M$ provides, in principle, a useful invariant to distinguish between families of II$_1$ factors. But this group Out($M$) is extremely hard to compute. Breakthrough rigidity results in the theory of II$_1$ factors were obtained recently by Popa (see [10, 11, 12]) and are based on Popa’s deformation/rigidity technique. Techniques and results of Popa were applied in several papers (see also [15] for an overview) and two papers included as an application complete computations of outer automorphism groups of certain II$_1$ factors.

• In [6], it is shown that there exists, for every compact abelian group $K$, a type II$_1$ factor $M$ with Out($M$) $\cong K$. The result in [6] is an existence theorem and answered in particular the long standing open problem on the existence of II$_1$ factors without outer automorphisms.

• In [14], type II$_1$ factors $M$ with Out($M$) any discrete group of finite presentation are explicitly constructed. This in particular gave the first explicit examples of II$_1$ factors without outer automorphisms.

In our paper, the methods of [6] are applied to prove the existence of II$_1$ factors $M$ such that Out($M$) is any, possibly non-abelian, compact group. In fact, given a minimal action of a compact group $G$ on the hyperfinite II$_1$ factor $R$, we prove the existence of an action of $\Gamma = \text{SL}(3, \mathbb{Z})$ on the fixed point algebra $R^G$ such that the II$_1$ factor $M$ given as the amalgamated free product

$$M = \left( R^G \rtimes \Gamma \right) \ast_{R^G} R$$

satisfies Out($M$) $\cong G$.

The first rigidity results for II$_1$ factors are due to Connes in [1], where it is shown in particular that Out($N$) is a countable group whenever $N = \mathcal{L}(\Gamma)$ is the group von Neumann algebra of an ICC property (T) group $\Gamma$. So, for concrete ICC property (T) groups $\Gamma$, the group Out($\mathcal{L}(\Gamma)$) is in principle computable, although we do not know of any explicit computation.

Type II$_1$ factors admit a more general type of symmetry, under the form of finite index bimodules. The finite index $M$-$M$-bimodules (modulo isomorphism) form a fusion algebra that we denote as $\text{FAlg}(M)$. Such a fusion algebra is a set equipped with an additive (direct sum) and a multiplicative

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Another generalization of \([6]\) was given by the second author in \([16]\), where the existence of II_1 factors \(M\) with trivial fusion algebra, was shown. In our paper the fusion algebra FA_{\text{Alg}}(R) of the hyperfinite II_1 factor plays an important role: we make use of the fact that two countable fusion subalgebras of FA_{\text{Alg}}(R) become free after conjugating one of them by a well chosen automorphism of \(R\) (see \([16]\)).

1 Preliminaries

We denote by \((M, \tau)\) a von Neumann algebra \(M\) equipped with a faithful normal tracial state \(\tau\). We denote \(M^n := M_n(\mathbb{C}) \otimes M\) for all \(n \in \mathbb{N}\).

Let \((M, \tau)\) be a tracial von Neumann algebra and \(N \subset M\) a von Neumann subalgebra. The *-algebra of elements quasi-normalizing \(N\) is defined as

\[
Q_{\text{N}} M(N) := \{a \in M \mid \exists a_1, \ldots, a_n, b_1, \ldots, b_m \in M \text{ such that } Na \subset \sum_{i=1}^{n} a_i N \text{ and } aN \subset \sum_{i=1}^{m} Nb_i\}
\]

The inclusion \(N \subset M\) is called quasi-regular if \(Q_{\text{N}} M(N)^{\prime\prime} = M\).

If \(N\) is a von Neumann subalgebra of a von Neumann algebra \(M\), we denote by \(\text{Aut}(N \subset M)\) the group of automorphisms of \(M\) leaving \(N\) globally invariant.

1.1 Amalgamated free products

We make use of amalgamated free product factors and recall some basic facts and notations (see \([9]\) and \([17]\) for more details). Let \((M_0, \tau_0)\) and \((M_1, \tau_1)\) be tracial von Neumann algebras with a common von Neumann subalgebra \(N\) such that \(\tau_0|_N = \tau_1|_N\). We denote by \(E_i\) the unique \(\tau\)-preserving conditional expectation of \(M_i\) onto \(N\). The amalgamated free product \(M_0 \star_N M_1\) is, up to \(E\)-preserving isomorphism, the unique pair \((M, E)\) satisfying the following two conditions.

- The von Neumann algebra \(M\) is generated by embeddings of \(M_0\) and \(M_1\) that are identical on \(N\), and is equipped with a conditional expectation \(E : M \to N\).

- The subalgebras \(M_0\) and \(M_1\) are free with amalgamation over \(N\) with respect to \(E\). This means that \(E(x_1 \cdots x_n) = 0\) whenever \(x_j \in M_{i_j}\) such that \(E_{i_j}(x_j) = 0\) and \(i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n\).

The amalgamated free product \(M_0 \star_N M_1\) has a dense *-subalgebra given by:

\[
N \oplus \bigoplus_{n \geq 1} \left( \bigoplus_{i_1 \neq i_2, \ldots, i_{n-1} \neq i_n} M_{i_1}^\circ \cdots M_{i_n}^\circ \right)
\]

where \(M_{i_k}^\circ := M_{i_k} \ominus N\). The von Neumann algebra \(M_0 \star_N M_1\) has a trace, defined by \(\tau := \tau_0 \circ E = \tau_1 \circ E\).
1.2 Popa’s intertwining-by-bimodules technique

In this paper, we use Popa’s intertwining-by-bimodules technique (see \([10, 11, 12]\)) that we briefly recall now. Let \((B, \tau)\) be a tracial von Neumann algebra and \(H_B\) a right Hilbert \(B\)-module. There exists a projection \(p \in \text{B}(\ell^2(\mathbb{N})) \otimes B\) such that \(H_B \cong p(\ell^2(\mathbb{N}) \otimes L^2(B, \tau))_B\) and this projection is uniquely defined up to equivalence of projections in \(\text{B}(\ell^2(\mathbb{N})) \otimes B\). We denote \(\dim(H_B) := (\text{Tr} \otimes \tau)(p)\). Observe that the number \(\dim(H_B)\) depends on the choice of tracial state \(\tau\) in the non-factorial case.

Suppose now that \(A\) and \(B\) are two possibly non-unital von Neumann subalgebras of a tracial von Neumann algebra \((N, \tau)\). We say that \(A\) embeds into \(B\) inside \(N\), and write \(A \preceq_N B\), if there exists a non-zero \(A-B\)-subbimodule \(H\) of \(1_A L^2(N) 1_B\) such that \(\dim(H) < +\infty\). The relation \(A \preceq_N B\) is independent of the choice of tracial state \(\tau\) and is equivalent with the existence of \(n \in \mathbb{N}, v \in M_{1,n}(\mathbb{C}) \otimes 1_A N 1_B\) a non-zero partial isometry and \(\psi : A \to M_n(\mathbb{C}) \otimes B\) a possibly non-unital \(*\)-homomorphism satisfying \(av = v\psi(a)\) for all \(a \in A\). For details, we refer to Section 2 in \([10]\) and Appendix C in \([15]\).

1.3 Fusion algebras

We first recall the abstract notion of a fusion algebra and give below the basic example of the fusion algebra \(\text{FAlg}(M)\) of finite index bimodules over the \(\Pi_1\) factor \(M\).

**Definition 1.1.** A fusion algebra \(\mathcal{A}\) is a free \(\mathbb{N}\)-module \(\mathbb{N}[\mathcal{G}]\) equipped with the following additional structure:

- an associative and distributive product operation, and a multiplicative unit element \(e \in \mathcal{G}\),
- an additive, anti-multiplicative, involutive map \(x \mapsto \overline{x}\), called conjugation,

satisfying Frobenius reciprocity: defining the numbers \(m(x, y; z) \in \mathbb{N}\) for \(x, y, z \in \mathcal{G}\) through the formula

\[
xy = \sum_z m(x, y; z)z
\]

one has \(m(x, y; z) = m(\overline{x}, z; y)\) for all \(x, y, z \in \mathcal{G}\).

The base \(\mathcal{G}\) of the fusion algebra \(\mathcal{A}\) is canonically determined: these are exactly the non-zero elements of \(\mathcal{A}\) that cannot be expressed as the sum of two non-zero elements. The elements of \(\mathcal{G}\) are called the irreducible elements of the fusion algebra \(\mathcal{A}\) and we sometimes write \(\mathcal{G} = \text{Irred}\mathcal{A}\). Notice that conjugation preserves irreducibility. The intrinsic group \(\text{grp}(\mathcal{A})\) of the fusion algebra \(\mathcal{A}\) consists of the irreducible elements \(x \in \mathcal{A}\) such that \(x\overline{x} = e\). Equivalently, \(x \in \mathcal{A}\) belongs to the intrinsic group if and only if \(x\overline{x}\) is irreducible. It is easy to check that the intrinsic group of a fusion algebra is indeed a group. If \(x, y \in \mathcal{A}\), we sometimes say that \(x\) is included in \(y\), if there exists a \(z\) such that \(y = x + z\).

A dimension function on a fusion algebra \(\mathcal{A}\) is an additive, multiplicative, unital map \(d : \mathcal{A} \to \mathbb{R}^+\) satisfying \(d(\overline{x}) = d(x)\) for all \(x \in \mathcal{A}\). Suppose \(d\) is a dimension function on the fusion algebra \(\mathcal{A}\). Whenever \(x \in \mathcal{A}\) is non-zero, \(e\) is included in \(x\overline{x}\) and so \(d(x) \geq 1\). It then follows that \(x \in \mathcal{A}\) belongs to the intrinsic group of \(\mathcal{A}\) if and only if \(d(x) = 1\). Moreover, if \(x \in \mathcal{A}\) is non-zero and not in the intrinsic group, the same reasoning yields \(d(x) \geq \sqrt{2}\).

Two examples of fusion algebras with a dimension function arise as follows.
Let $\Gamma$ be a group and define $\mathcal{A} = \mathbb{N}[\Gamma]$. Define $d$ such that $d(s) = 1$ for all $s \in \Gamma$.

Let $G$ be a compact group and define the fusion algebra $\text{Rep}(G)$ as the set of equivalence classes of finite dimensional unitary representations of $G$. The operations on $\text{Rep}(G)$ are of course given by direct sum and tensor product of representations, while the dimension function $d$ is given by the ordinary Hilbert space dimension of the representation space.

We are interested in the fusion algebra $\text{FAlg}(M)$ of a II$_1$ factor $M$. First of all, an $M$-$M$-bimodule $MH_M$ is said to be of finite Jones index if $\dim(MH) < \infty$ and $\dim(H_M) < \infty$. We define $\text{FAlg}(M)$ as the set of finite index $M$-$M$-bimodules modulo unitary equivalence.

Whenever $\psi : M \to pM^n p$ is a finite index inclusion in the sense of Jones [7], for some non-zero projection $p \in M^n$, we define the $M$-$M$-bimodule $H(\psi)$ on the Hilbert space $(M_{1,n} \otimes L^2(M))p$ with left and right module actions given by

$$a \cdot \xi := a \xi \quad \text{and} \quad \xi \cdot a = \xi \psi(a).$$

Every finite index $M$-$M$-bimodule is unitarily equivalent with some $H(\psi)$. Moreover, given finite index inclusions $\psi : M \to pM^n p$ and $\eta : M \to qM^m q$, we have $H(\psi) \cong H(\eta)$ if and only if there exists a unitary $u \in p(M_{n,m} \otimes M)q$ satisfying $\psi(a) = u \eta(a) u^*$ for all $a \in M$.

Addition in $\text{FAlg}(M)$ is given by the obvious direct sum of bimodules, while multiplication in $\text{FAlg}(M)$ is given by the Connes tensor product of $M$-$M$-bimodules that we recall now (see also V. Appendix B in [2]). Let $H$ be an $M$-$M$-bimodule. Define $\mathcal{H}$ as the dense subspace of $H$ consisting of bounded vectors:

$$\mathcal{H} := \{\xi \in H \mid \exists c > 0, \forall a \in M, \|\xi a\|_2 \leq c\|a\|_2\}.$$ 

For all $\xi \in \mathcal{H}$ and $a \in M$ we define $L_\xi(a) = \xi a$. By definition this map extends to a bounded operator $L_\xi : L^2(M) \to H$. We set:

$$\langle \xi, \eta \rangle_M := L_\xi^* L_\eta \in M, \forall \xi, \eta \in \mathcal{H}.$$ 

It is easy to check that this formula defines an $M$-valued scalar product on $\mathcal{H}$. Then the Connes tensor product of the $M$-$M$-bimodules $MH_M$ and $MK_M$ is defined as the separation and completion of the algebraic tensor product $\mathcal{H} \otimes_{alg} K$ for the scalar product

$$\langle a \otimes \xi, b \otimes \eta \rangle := \langle \xi, \langle a, b \rangle_M \eta \rangle.$$ 

The Connes tensor product is denoted by $H \otimes_M K$ and it is an $M$-$M$-bimodule:

$$a \cdot (b \otimes \xi) = ab \otimes \xi \quad \text{and} \quad (b \otimes \xi) \cdot a = b \otimes (\xi a).$$

Note that $H(\psi) \otimes_M H(\eta) = H((\text{id} \otimes \psi) \eta)$.

If $MH_M \in \text{FAlg}(M)$, the conjugate bimodule $M\overline{H}_M$ lives on the conjugate Hilbert space $\overline{H} = H^*$ with bimodule action given by

$$a \cdot \overline{\xi} = \overline{a \xi^*} \quad \text{and} \quad \overline{\xi} \cdot a = \overline{a^\ast \xi}.$$ 

The elements of the intrinsic group $\text{grp}(M)$ of $\text{FAlg}(M)$ are exactly the bimodules $H(\pi)$, where $\pi : M \to pM^n p$ is a $^*$-isomorphism. Denote by $\mathcal{F}(M)$ the fundamental group of $M$. We then get a short exact sequence $e \to \text{Out}(M) \to \text{grp}(M) \to \mathcal{F}(M) \to e$, mapping $\sigma \in \text{Aut}(M)$ to $H(\sigma) \in \text{grp}(M)$ and mapping $H(\pi) \in \text{grp}(M)$ to $\text{Tr}(p)$. 

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The fusion algebra \( \text{FA}_\text{Alg}(M) \) has a natural dimension function: the dimension of \( H(\psi) \) is defined as the square root of the minimal index of \( \psi(M) \subset pM^np \). Since for an irreducible subfactor the minimal index equals the usual Jones index, the dimension function \( d \) is given by

\[
d_{(M\text{H}_M)} = \sqrt{\dim(H_M) \dim(M_H)},
\]
whenever \( M_{\text{H}_M} \) is an irreducible \( M-M \)-bimodule. We refer to [8, 4] for details.

1.4 Minimal actions of compact groups and fusion algebras

A continuous action \( G \curvearrowright M \) of a compact group \( G \) on the \( \text{II}_1 \) factor \( M \) is said to be minimal if the map \( G \to \text{Aut}(M) \) is injective and if \( M \cap (M^G)' = \mathbb{C}1 \). Here, \( M^G \) denotes the von Neumann algebra of \( G \)-fixed points in \( M \).

Given such a minimal action \( G \curvearrowright M \), set \( N := M^G \). We get a canonical, dimension preserving, embedding \( \text{Rep}(G) \hookrightarrow \text{FA}_\text{Alg}(N) \) of fusion algebras, defined as follows. Let \( \pi : G \to U(n) \) be an irreducible unitary representation of \( G \). We choose a unitary \( V_\pi \in M_n(\mathbb{C}) \otimes M \) satisfying

\[
(id \otimes \sigma_g)(V_\pi) = V_\pi(\pi(g) \otimes 1)
\]
for all \( g \in G \). We then define the finite index inclusion

\[
\psi_\pi : N \to M_n(\mathbb{C}) \otimes N : \psi_\pi(a) = V_\pi(1 \otimes a)V_\pi^*. 
\]

It is easily checked that the \( N-N \)-bimodule \( H(\psi_\pi) \) is irreducible and, up to unitary equivalence, independent of the choice of \( V_\pi \). The map \( \pi \mapsto H(\psi_\pi) \) extends to an embedding \( \text{Rep}(G) \hookrightarrow \text{FA}_\text{Alg}(N) \).

Also note that the coefficients of \( V_\pi \) quasi-normalize \( N \) and so, the inclusion \( N \subset M \) is quasi-regular.

1.5 Freeness and free products of fusion algebras

**Definition 1.2.** Let \( \mathcal{A} \) be a fusion algebra and \( \mathcal{A}_i \subset \mathcal{A} \) fusion subalgebras for \( i = 1, 2 \). We say that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are free inside \( \mathcal{A} \) if every alternating product of irreducibles in \( \mathcal{A}_i \setminus \{e\} \), remains irreducible and different from \( \{e\} \).

Given fusion algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), there is up to isomorphism a unique fusion algebra \( \mathcal{A} \) generated by copies of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) that are free. We call this unique \( \mathcal{A} \) the free product of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) and denote it by \( \mathcal{A}_1 \ast \mathcal{A}_2 \). Of course, the free product can be constructed in a concrete way as follows: given \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), set \( \mathcal{G}_i = \text{Irred}(_i \mathcal{A}) \). Define \( \mathcal{G} \) as the set of words with letters alternatingly from \( \mathcal{G}_1 \setminus \{e\} \) and \( \mathcal{G}_2 \setminus \{e\} \). Denote the empty word as \( e \). Then, \( \mathcal{A}_1 \ast \mathcal{A}_2 = \mathbb{N}[\mathcal{G}] \). The product on \( \mathbb{N}[\mathcal{G}] \) is the unique associative and distributive operation satisfying the following two conditions:

- The embeddings \( \mathcal{A}_i \hookrightarrow \mathbb{N}[\mathcal{G}] \) are multiplicative.
- If the last letter of the alternating word \( x \in \mathcal{G} \) and the first letter of the alternating word \( y \in \mathcal{G} \) belong to different fusion algebras \( \mathcal{A}_i \), the product of \( x \) and \( y \) is again irreducible and given by concatenation of \( x \) and \( y \).
Denote by $R$ the hyperfinite II$_1$ factor. It is a crucial ingredient of this paper that FA$(R)$ is huge, in the sense that FA$(R)$ contains many free fusion subalgebras. More precisely, it was shown in Theorem 5.1 of [16] that countable fusion subalgebras of FA$(R)$ can be made free by conjugating one of them with an automorphism of $R$ (see Theorem 1.3 below). Note that the same result has first been proven for countable subgroups of Out$(R)$ in [6]. In both cases, the key ingredients come from [13].

Let $M$ be a II$_1$ factor and $MK_M \in$ FA$(M)$. Whenever $\alpha \in$ Aut$(M)$, we define the conjugation of $K$ by $\alpha$ as the bimodule $K^\alpha := H(\alpha^{-1}) \otimes_M K \otimes_M H(\alpha)$. Of course, $K^\alpha$ has $K$ as its underlying Hilbert space with new left and right module action given by $\xi \cdot_{\text{new}} a = \xi \cdot_{\text{old}} \alpha(a)$ and $a \cdot_{\text{new}} \xi = \alpha(a) \cdot_{\text{old}} \xi$.

**Theorem 1.3** (Thm. 5.1 in [16]). Let $R$ be the hyperfinite II$_1$ factor and $A_0, A_1$ two countable fusion subalgebras of FA$(R)$. Then, 

$$\{\alpha \in \text{Aut}(R) \mid A_0^\alpha \text{ and } A_1 \text{ are free}\}$$

is a $G_\delta$-dense subset of Aut$(R)$.

1.6 Property (T) and relative property (T) for II$_1$ factors

Property (T) for finite von Neumann algebras was defined by Connes and Jones in [3]: a II$_1$ factor $(N, \tau)$ has property (T) if and only if there exists $\epsilon > 0$ and a finite subset $F \subset N$ such that every $N$-$N$-bimodule $H$ that has a unit vector $\xi$ satisfying $\|x\xi - \xi x\| \leq \epsilon, \forall x \in F$, actually has a non-zero $N$-central vector $\xi_0$, meaning that $x\xi_0 = \xi_0 x, \forall x \in N$.

Note that an ICC group $\Gamma$ has property (T) if and only if the II$_1$ factor $L(\Gamma)$ has property (T) in the sense of Connes and Jones.

Relative property (T) for inclusions $B \subset (N, \tau)$ of finite von Neumann algebras was defined by Popa in [12]. We do not really use relative property (T) in this paper, just the trivial observation that $B \subset N$ has the relative property (T) if $B$ has itself property (T).

2 Statement of the main result

The main result that we prove is that any compact group $G$ can be realized as the outer automorphism group of a type II$_1$ factor, see Theorem 2.3. A more precise theorem can be formulated as follows.

Note that any character $\omega \in \text{Char}(\Gamma)$ defines an automorphism $\theta_\omega$ of any crossed product $N \rtimes \Gamma$ acting identically on $N$ and multiplying by $\omega$ on $\Gamma$.

**Theorem 2.1.** Let $M_1$ be the hyperfinite II$_1$ factor and $G$ a compact group acting on $M_1$. Denote $N = M_1^G$, the von Neumann algebra of $G$-fixed points in $M_1$. Let $\Gamma$ be an ICC group acting on $N$. Denote $M_0 := N \rtimes \Gamma$. Assume that

1. the action $G \ract M_1$ is minimal,
2. the action $\Gamma \ract N$ is outer and $M_0$ has the property (T),
3. the natural images of $\text{Rep} G \hookrightarrow \text{FA}(N)$ and $\text{Aut}(N \subset M_0) \xrightarrow{\text{restr}} \text{Out}(N) \subset \text{FA}(N)$ inside the fusion algebra FA$(N)$, are free in the sense of Definition 1.2.
Then, the homomorphism
\[
\text{Char}(\Gamma) \times G \to \text{Aut}(M_0 \ast M_1) : (\omega, g) \mapsto \theta_\omega \ast \sigma_g
\]
induces an isomorphism \(\text{Char}(\Gamma) \times G \cong \text{Out}(M_0 \ast M_1)\).

Combining Theorems 1.3 and 2.1, we shall prove the following.

**Corollary 2.2.** Let \(G\) be a compact, second countable group and \(G \curvearrowright R\) a minimal action on the hyperfinite II\(_1\) factor \(R\). Then there exists an outer action of \(SL(3, \mathbb{Z})\) on the fixed point algebra \(R^G\), such that for \(M\) given as the amalgamated free product \(M = (R^G \times \Gamma) \ast R\), the natural homomorphism
\[
G \to \text{Aut}(M) : g \mapsto \text{id} * \sigma_g
\]
induces an isomorphism \(G \cong \text{Out}(M)\).

Of course, every compact, second countable group \(G\) admits a minimal action on the hyperfinite II\(_1\) factor. A possible construction goes as follows: take an amenable ICC group \(\Lambda\) and define the Bernoulli action crossed product \(R = L^\infty(\prod \Lambda(G, \text{Haar})) \ast \Lambda\) with \(G \curvearrowright R\) acting by diagonal left translation on \(\prod \Lambda(G, \text{Haar})\) and trivially on \(\Lambda\).

So, we immediately get the following result.

**Theorem 2.3.** Let \(G\) be a compact, second countable group. There exists a type II\(_1\) factor \(M\) with \(\text{Out}(M) \cong G\).

### 3 A Kurosh automorphism theorem for fusion algebras

An important ingredient in the analysis of all automorphisms of an amalgamated free product \(M = M_0 \ast_N M_1\) as in Theorem 2.1 above, is a generalization of the Kurosh automorphism theorem to automorphisms of free products of fusion algebras. We do not prove a general result, but a rather easy theorem sufficient for our purposes.

Recall that a group is said to be freely indecomposable if it cannot be written as a non-trivial free product.

**Theorem 3.1.** Let \(\Gamma\) be a countable group non isomorphic to \(\mathbb{Z}\) and freely indecomposable. Let \(A\) be an abelian fusion algebra with a dimension function \(d\) and non-isomorphic to the group \(\mathbb{Z}\).

Every dimension preserving automorphism \(\alpha\) of \(N[\Gamma] \ast A\) is of the form \((\text{Ad} u) \circ (\alpha_0 \ast \alpha_1)\) for some \(u \in \Gamma \ast \text{grp}(A)\), \(\alpha_0 \in \text{Aut}(\Gamma)\) and \(\alpha_1\) a dimension preserving automorphism of \(A\).

**Proof.** Let \(\alpha\) be a dimension preserving automorphism of \(N[\Gamma] \ast A\).

We denote \(\Lambda = \text{grp}(A)\), the intrinsic group of \(A\), and \(\Delta = \Gamma \ast \Lambda\), which is as well the intrinsic group of \(N[\Gamma] \ast A\). We also write \(G = \text{Irred} A\), which means that \(A = N[G]\). We may of course assume that \(G \neq \Lambda\), because the group case of our theorem is covered by the classical Kurosh theorem. Finally, we set \(G^0 = G \setminus \{e\}\) and \(\Gamma^0 = \Gamma \setminus \{e\}\).

If \(u \in \Delta\), we write \(u^{-1}\) instead of \(u^{-1}\).

**Claim.** There exists \(x \in G \setminus \Lambda\) and \(u \in \Delta\) such that \(\alpha(x) \in u(G \setminus \Lambda)u^{-1}\).

**Proof of the claim.** Define \(\lambda = \inf\{d(x) \mid x \in G \setminus \Lambda\} \geq \sqrt{2}\). Take \(x \in G \setminus \Lambda\) with \(d(x) < \sqrt{2}\lambda\). Write \(\alpha(x)\) as an alternating word in \(G^0\) and \(\Gamma^0\). Suppose that in this expression of \(\alpha(x)\), there
appears twice a letter from \( G \setminus \Lambda \). Then the dimension of these two letters is greater or equal than
\( \lambda \geq \sqrt{2} \), making \( d(x) = d(\alpha(x)) \geq \sqrt{2}\lambda \) ; a contradiction. So, we have shown that \( \alpha(x) = uyv^{-1} \)
with \( y \in G \setminus \Lambda \) and \( u, v \in \Delta \). We may assume that \( u, v \) are either equal to \( e \), either end with a letter
from \( \Gamma^0 \). Expressing the commutation of \( \alpha(x) \) and \( \alpha(\overline{x}) \), we find that \( uy\overline{y}u^{-1} = v\overline{y}vy^{-1} \). Since
\( y \not\in \Lambda \), we find that \( y \overline{y} \neq e \) and so \( u = v \), proving the claim.

**Observation 1.** If \( x \in \Delta \), \( y \in \Gamma^0 \) and \( xy = yx \), then \( x \in \Lambda \). This follows by analyzing reduced
words in \( \Gamma^0 \) and \( \Gamma^0 \).

Because of the claim and replacing \( \alpha \) by \( (\Ad u^{-1}) \circ \alpha \), we may from now on assume the existence
of \( x, y \in G \setminus \Lambda \) with \( \alpha(x) = y \). Whenever \( a \in \Lambda \), \( \alpha(a) \) belongs to \( \Delta \) and commutes with \( y \).

**Observation 1.** above implies that \( \alpha(\Lambda) \subset \Lambda \). Similarly, \( \alpha^{-1}(\Lambda) \subset \Lambda \) so that \( \alpha(\Lambda) = \Lambda \). It follows
that the restriction of \( \alpha \) to \( \Delta \) defines an automorphism of \( \Gamma \) that globally preserves \( \Lambda \). The
classical Kurosh theorem implies that \( \alpha(\Gamma) = \Gamma \).

**Observation 2.** If \( z \in \Gamma^0 \) and \( \alpha(z) \in \Delta \Gamma^0 \Delta \), then actually \( \alpha(z) \in \Gamma^0 \). Indeed, write \( \alpha(z) = urv^{-1} \)
for \( r \in \Gamma^0 \) and \( u, v \in \Delta \) either equal to \( e \) or with their last letter in \( \Gamma^0 \). Writing out that \( \alpha(z) = urv^{-1} \) and \( y = \alpha(x) \) commute, it follows that \( u = v = e \). A similar observation holds for \( \alpha^{-1} \).

It remains to prove that \( \alpha(G) = G \). Assume the contrary and define
\[ \delta = \inf \{ d(z) \mid z \in G, \ (\alpha(z) \not\in G \text{ or } \alpha^{-1}(z) \not\in G \} \].

Take \( z \in \Gamma^0 \) with \( d(z) < \sqrt{2} \delta \) such that \( \alpha(z) \not\in \Gamma^0 \) or \( \alpha^{-1}(z) \not\in \Gamma^0 \). Assume that we are in the case
\( \alpha(z) \not\in \Gamma^0 \). By construction \( \alpha(r), \alpha^{-1}(r) \in \Gamma^0 \) for every \( r \in \Gamma^0 \) with \( d(r) < \delta \). Write \( \alpha(z) \) as an alternating
word in \( \Gamma^0 \) and \( \Gamma^0 \). By observation 2, the expression for \( \alpha(z) \) contains at least twice a letter from \( G \setminus \Lambda \). Hence every letter in the expression for \( \alpha(z) \) has dimension strictly smaller than \( \delta \). Applying \( \alpha^{-1} \) and using the fact that \( \alpha^{-1}(\Gamma^0) = \Gamma^0 \), we have written \( z \in \Gamma^0 \) as an alternating
word in \( \Gamma^0 \) and \( \Gamma^0 \) with more than 2 letters; a contradiction.

\[ \square \]

### 4 Proof of the main results

Before proving Theorem 2.1 and Corollary 2.2 we state the following lemma. It is a consequence
of Lemma 8.4 in [3] (see also Props. 3.3 and 3.5 in [10]).

**Lemma 4.1.** Let \( \Gamma_0, \Gamma_1 \) be ICC groups acting outerly on the \( II_1 \) factors \( A_0 \) and \( A_1 \) respectively. Set
\( M := A_0 \times \Gamma_0 \) and suppose that \( \alpha : A_0 \times \Gamma_0 \to A_1 \times \Gamma_1 \) is an isomorphism such that \( \alpha(A_0) \preceq_M A_1 \)
and \( A_1 \preceq_M \alpha(A_0) \).

Then, there exists a unitary \( u \in \mathcal{U}(M) \) such that \( u\alpha(A_0)u^* = A_1 \).

A first step in the proof of Theorem 2.1 is the following lemma. The crucial ingredients of its proof
are Theorems 1.2.1 and 4.3 in [3] (see also Thms. 4.6 and 5.6 in [5] for alternative proofs).

**Lemma 4.2.** Suppose that the assumptions of Theorem 2.1 are fulfilled. Set \( M = M_0 \ast_N M_1 \). For
every \( \alpha \in \text{Aut}(M) \), there exists \( u \in \mathcal{U}(M) \) such that \( (\Ad u) \circ \alpha \in \text{Aut}(N \subset M) \).

Note that in fact assumption 3 in Theorem 2.1 will not be used in the proof of this lemma.
Proof. By 4.3 in [6] and because $M_0$ has property (T), there exists $i \in \{0, 1\}$ such that $\alpha(M_0) \prec_M M_i$. Since $M_1$ is hyperfinite, it follows that $i = 0$. So, we can take a projection $p \in M_0'$, a non-zero partial isometry $v \in M_1, n(C) \otimes M$ and a unital $^*$-homomorphism $\psi : M_0 \to pM_0^*p$ satisfying $\alpha(a)v = v\psi(a)$ for all $a \in M_0$. Since $\psi(M_0)$ has property (T), we know that $\psi(M_0) \not\prec M_0 N^n$. By 1.2.1 in [6], it follows that $\psi(M_0)^{\prime} \cap pM_0^*p \subset pM_0^*p$. In particular, $v^*v \in pM_0^*p$. So, we may assume that $p = v^*v$. Since also $\alpha(M_0)^{\prime} \cap M = M_0^0 \cap M = C1$, we have $vv^* = 1$. Factoriality of $M_0$ now allows to assume that $v \in U(M)$ and $v^*\alpha(M_0)v \subset M_0$.

Applying the same reasoning to $\alpha^{-1}$, we also get a unitary $w \in U(M)$ satisfying $w^*M_0w \subset \alpha(M_0)$. It follows that $(wv)^*M_0(wv) \subset M_0$. Another application of 1.2.1 in [6] implies that $wv \in M_0$. But then all the inclusions $(wv)^*M_0(wv) \subset v^*\alpha(M_0)v \subset M_0$ are equalities. So, after a unitary conjugacy, we may assume that $\alpha(M_0) = M_0$.

Quasi-regularity of $N \subset M$ combined with 1.2.1 in [6], implies that $\alpha(N) \prec_M N$. Similarly $N \prec_M \alpha(N)$. The lemma then follows by applying Lemma 1.1. 

Proof of Theorem 2.1. We still denote $M = M_0 \ast_N M_1$. Let $\alpha \in \text{Aut}(M)$. We prove below that after a unitary conjugacy of $\alpha$, one has $\alpha(a) = a$ for all $a \in N$ and $\alpha(M_i) = M_i$ for $i \in \{0, 1\}$. This then implies that $\alpha|_{M_0} = \theta$, for some $\theta \in \text{Char}(\Gamma)$ and $\alpha|_{M_1} = \sigma$ for some $\sigma \in G$. Hence, it implies the surjectivity of the homomorphism $\text{Char}(\Gamma) \times G \to \text{Out}(M)$. The injectivity of this homomorphism follows from the irreducibility $N' \cap M = C1$ that we prove now as the consequence of more general needed considerations.

Let $I$ be a complete set of inequivalent irreducible unitary representations of $G$. For every $\pi \in I$, choose a unitary $V_\pi \in B(H_\pi) \otimes M_1$ satisfying $(id \otimes \sigma_g)(V_\pi) = V_\pi(\pi(g) \otimes 1)$. Define $K_0(\pi) \subset M_1$ as the linear span of

$$(\xi^* \otimes a)V_\pi(\eta \otimes 1) , \xi, \eta \in H_\pi , \ a \in N .$$

It follows that the closure $K(\pi)$ of $K_0(\pi)$ is a finite index $N$-$N$-submodule of $L^2(M_1)$. Denote by

$$\Psi : \text{Rep} G \hookrightarrow \text{FAlg}(N)$$

the embedding defined in Subsection 1.4. It follows that $K(\pi) \cong (\text{dim} \pi) \cdot \Psi(\pi)$. Moreover, we have the following orthogonal decomposition of $L^2(M_1)$.

$$L^2(M_1) = \bigoplus_{\pi \in I} K(\pi) .$$

In the same way, we define for every $s \in \Gamma$, the subspace $H_0(s) \subset M_0$ given by $H_0(s) = Nu_s$, with closure $H(s) \subset L^2(M_0)$.

Whenever $w = s_0 \pi_1 s_1 \cdots s_{n-1} \pi_n s_n$ is an alternating word in $\Gamma \setminus \{e\}$ and $I \setminus \{e\}$, we define the $N$-$N$-submodule $H(w) \subset L^2(M)$ as the closure of $H_0(s_0)K_0(\pi_1) \cdots K_0(\pi_n)H_0(s_n)$. One then obtains the orthogonal decomposition of $L^2(M)$ given by

$$L^2(M) = \bigoplus_{w \text{ alternating word}} H(w) .$$

Moreover, as $N$-$N$-bimodules, we get the unitary equivalences

$$H(w) \cong H(s_0) \otimes_N K(\pi_1) \otimes_N \cdots \otimes_N K(\pi_n) \otimes_N H(s_n)$$

$$\cong \text{ a multiple of } H(s_0) \otimes_N \Psi(\pi_1) \otimes_N \cdots \otimes_N \Psi(\pi_n) \otimes_N H(s_n) .$$

(2)
Denote by $\text{FAlg}(N \subset M)$ the fusion subalgebra of $\text{FAlg}(N)$ generated by the finite index $N$-$N$-subbimodules of $L^2(M)$. Using assumption 3 in Theorem 2.1 and (2) above, $\Psi$ extends to an isomorphism $\Psi : N[\Gamma] \ast \text{Rep}(G) \rightarrow \text{FAlg}(N \subset M)$ such that $H(w)$ is a multiple of $\Psi(w)$ for every alternating word $w$. We see in particular that $L^2(M)$ contains only once the trivial $N$-$N$-bimodule (for $w$ the empty alternating word and $H(w) = L^2(N)$). This means that $N' \cap M = \mathbb{C}1$ as was needed above.

Let $\text{Char}G \subset \mathcal{I}$ be the subset of $\mathcal{I}$ consisting of one-dimensional unitary representations of $G$. Then, $\text{Char}G$ is as well the intrinsic group of the fusion algebra $\text{Rep}(G)$. Whenever $\pi \in \text{Char}G$, we have $V_\pi \in \mathcal{U}(M_1)$ and $V_\pi$ normalizes $N$. Whenever $w \in \Gamma \ast \text{Char}G$, write $w = s_0 \pi_1 s_1 \cdots s_{n-1} \pi_n s_n$ as an alternating word in $\Gamma \setminus \{e\} \cup \text{Char}G \setminus \{e\}$ and define the unitary $U(w) := u_{s_0} V_{\pi_1} \cdots V_{\pi_n} u_{s_n}$ normalizing $N$. We are now ready to complete the proof of the theorem. So, let $\alpha \in \text{Aut}(M)$. We have to prove that after a unitary conjugacy of $\alpha$, one has $\alpha(a) = a$ for all $a \in N$ and $\alpha(M_i) = M_i$ for $i \in \{0, 1\}$. By Lemma 3.2 we may assume that $\alpha(N) = N$. But then, the conjugation map $K \mapsto K^\alpha$ defines an automorphism of the fusion subalgebra $\text{FAlg}(N \subset M)$ of $\text{FAlg}(N)$. Define the automorphism $\eta$ of $N[\Gamma] \ast \text{Rep}(G)$ such that $\Psi(\eta(w)) = (\Psi(w))^\alpha$ in $\text{FAlg}(N)$ for all $w \in N[\Gamma] \ast \text{Rep}(G)$. By Theorem 3.1, we find an element $v$ in $\Gamma \ast \text{Char}(G)$ such that $(\text{Ad} v) \circ \eta$ globally preserves $\Gamma$ and $\text{Rep}(G)$.

Replacing $\alpha$ by $(\text{Ad} U(v)) \circ \alpha$, we may assume that $\eta$ preserves globally $\Gamma$ and $\text{Rep}(G)$. The equality $\alpha(N) = N$ remains true. The restrictions of $\eta$ yield an automorphism of the group $\Gamma$ and a permutation of $\mathcal{I}$ respecting the fusion rules. Moreover, we have $K(\pi) = K(\eta(\pi))$ for every $\pi \in \mathcal{I}$ and $H(s) = H(\eta(s))$ for every $s \in \Gamma$. Choose $s \in \Gamma$. Note that $H(s)$ is isomorphic as an $N$-$N$-bimodule with the closure of $\alpha^{-1}(Nu_s)$ in $N L^2(M)_N$. Since the $N$-$N$-bimodule $H(\eta(s))$ appears with multiplicity 1 in the decomposition (11) of $L^2(M)$, we conclude that the closure of $\alpha^{-1}(Nu_s)$ inside $L^2(M)$ equals $H(\eta(s))$ for all $s \in \Gamma$. It follows that $\alpha(M_0) = M_0$. A similar reasoning shows that $\alpha(M_1) = M_1$.

Since $\alpha \in \text{Aut}(N \subset M_0)$, assumption 3 in Theorem 2.1 implies that the $N$-$N$-bimodule $H(\alpha|_N)$ is free with respect to $\Psi(\text{Rep}(G))$ inside $\text{FAlg}(N)$. But the formula $K(\pi) = K(\eta(\pi))$ means that $H(\alpha|_N)$ normalizes $\Psi(\text{Rep}(G))$. Both statements can only be true at the same time if $H(\alpha|_N)$ is the trivial $N$-$N$-module. So, $\alpha|_N$ is an inner automorphism of $N$ and we are done.

**Proof of Corollary 2.2.** It suffices to give an example of an outer action of $\Gamma = \text{SL}(3, \mathbb{Z})$ on the hyperfinite $\Pi_1$ factor $N$ such that $N \rtimes \Gamma$ has property (T). Indeed, starting from a minimal action of $G$ on the hyperfinite $\Pi_1$ factor $M_1 = R$, set $N = M_1^G$ and take an outer action of $\Gamma$ on $N$ such that $M_0 := N \rtimes \Gamma$ has property (T). By Theorem 4.4 in [12], $\text{Aut}(N \subset N \rtimes \Gamma) \cap \text{AdU}(N)$ is a countable group. By Theorem 1.3 we can take an automorphism $\alpha \in \text{Aut}(R)$ and replace $\Gamma \subset \text{Aut}(N)$ by $\alpha \Gamma \alpha^{-1}$ in such a way that all conditions of Theorem 2.1 are fulfilled. Since $\text{Char} \Gamma = \{e\}$, the corollary then follows from Theorem 2.1.

Take $\Gamma_1 := \text{SL}(3, \mathbb{Z}) \ltimes (\mathbb{Z}^3 \oplus \mathbb{Z}^3)$ with the action of $\text{SL}(3, \mathbb{Z})$ on $\mathbb{Z}^3 \oplus \mathbb{Z}^3$ given

$$A \cdot (x, y) := (Ax, (A^{-1})^t y).$$

Note that $\Gamma_1$ is a property (T) group. Take $k \in \mathbb{R} \setminus 2\pi \mathbb{Q}$ and define the non degenerate 2-cocycle $\Omega \in Z^2(\mathbb{Z}^3 \oplus \mathbb{Z}^3, S^1)$ by the formula

$$\Omega((x, y); (x', y')) := e^{ik((x, y)-(x', y'))}.$$
where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{Z}^3$. The 2-cocycle $\Omega$ is $\text{SL}(3, \mathbb{Z})$-invariant and hence extends to a 2-cocycle $\tilde{\Omega} \in Z^2(\Gamma_1, S^1)$ by the formula

$$\tilde{\Omega}(((x, y), A); ((x', y'), B)) := \Omega((x, y); A \cdot (x', y')) .$$

The twisted group von Neumann algebra $L_{\tilde{\Omega}}(\Gamma_1)$ still has property (T) and can be regarded as well as $L_{\Omega}(\mathbb{Z}^3 \oplus \mathbb{Z}^3) \rtimes \text{SL}(3, \mathbb{Z})$. Since $L_{\Omega}(\mathbb{Z}^3 \oplus \mathbb{Z}^3)$ is the hyperfinite $\text{II}_1$ factor, we are done. \qed

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