A formula of total probability with interference term and the Hilbert space representation of the contextual Kolmogorovian model
Andrei Khrennikov
International Center for Mathematical Modeling in Physics and Cognitive Sciences,
University of Växjö, S-35195, Sweden

Abstract

We compare the classical Kolmogorov and quantum probability models. We show that the gap between these models is not so huge as it was commonly believed. The main structures of quantum theory (interference of probabilities, Born’s rule, complex probabilistic amplitudes, Hilbert state space, representation of observables by operators) are present in a latent form in the Kolmogorov model. In particular, we obtain “interference of probabilities” without to appeal to the Hilbert space formalism. We interpret “interference of probabilities” as a perturbation (by a cos-term) of the conventional formula of total probability. Our classical derivation of quantum probabilistic formalism can stimulate applications of quantum methods outside of microworld: in psychology, biology, economy,...

Key words: formula of total probability, contextual Kolmogorov model, quantum representation, interference of probabilities, Born’s rule
MSC: 46N30, 60A99

1 Introduction

There is a rather common opinion that the quantum model of probability theory (i.e., the calculus on probabilities based on the complex Hilbert space) differs essentially from the classical (measure-theoretic) Kolmogorov model [1], [2]; see, e.g., [3]–[5] for details and discussions. Among distinguishing features of quantum probability there are typically mentioned:

a) The use of complex amplitudes of probabilities, \( \psi(x) \), (wave functions);
b) Born’s rule for probabilities. Probability of the event $B_x$ – to find a particle at the point $x$ – is given by

$$P_\psi(B_x) = |\psi(x)|^2. \quad (1)$$

c) Interference of probabilities. We present this phenomenon by coupling it to the formula of total probability. We consider the simplest partition of the sample space $\mathcal{A} = \{A_1, A_2\}$. Here we have, see, for example, [2]:

$$P(B|C) = \sum P(A_j|C)P(B|A_jC). \quad (2)$$

However, in the quantum probabilistic formalism there was derived a different formula:

$$P(B|C) = \sum P(A_j|C)P(B|A_j) + \sqrt{P(A_1|C)P(B|A_1)P(A_2|C)P(B|A_2)} \cdot 2 \cos \theta(B|A, C) \quad (3)$$

where $\theta(B|A, C)$ is an angle (“phase”) depending on the event $B$, partition $\mathcal{A}$ and the condition $C$ under which the event $B$ occurs. The presence of a new trigonometric term is interpreted as *interference of probabilities*, see, e.g., [6]. In [6] it was emphasized that the presence of interference of probabilities in quantum formalism is an exhibition of violation of fundamental laws of classical probability.

d). Representation of physical observables by noncommutative operators in the complex Hilbert space. (We recall that in the Kolmogorov model there are used random variables – measurable functions on the sample space).

The aim of this paper is to show that in fact the gap between quantum model (Dirac-von Neumann [7], [8]) and classical model (Kolmogorov [1]) is not as large as it is commonly believed.\footnote{We do not claim that all problems are solved. In this paper we do not consider composite systems. Therefore we do not even discuss such things as Bell's inequality and quantum nonlocality, see [3], [4] for details.} All mentioned distinguished features of quantum probability, a)–d), are present in a latent form in the classical Kolmogorov model.

The crucial point is that all probabilities should be considered as *contextual probabilities*. Here a context $C$ is any complex of conditions, physical, biological, economic, financial. Therefore it is meaningless to speak about
an abstract probability $P$ which has no relation to a concrete context. Any probability should be related to some fixed context $C$.\footnote{Of course, there is nothing new for probabilists. For example, A. N. Kolmogorov pointed out to the role of complexes of experimental conditions in defining probability in his famous book \cite{Kolmogorov1933} and especially in \cite{Kolmogorov1956}. Similar views are presented in the books of Gnedenko \cite{Gnedenko1949} and Renye \cite{Reny1976}. We can also say that von Mises' frequency probability \cite{vonMises1928} is contextual: a collective is defined by a complex of experimental conditions.}

Our main contribution is the contextual probabilistic analysis of the formula of total probability \cite{totalprobability} and derivation of the “quantum formula of total probability” \cite{quantumtotalprobability} (which is typically referred to as “interference of probabilities”). Starting with this formula (derived in the classical measure-theoretic framework with the Kolmogorov probability space: $\mathcal{P} = (\Omega, \mathcal{F}, P)$) we reproduce other distinguished features of the quantum probabilistic formalism.

The starting point of our analysis is the contextual interpretation of conditional probabilities. Typically conditional probability $P(A|C)$ is interpreted as the probability of occurrence of the event $A$ under the condition that the event $C$ occurred. This interpretation can be called the event conditioning. But we would not like to consider conditioning by occurrence of an event. In general it is impossible to identify, e.g., a collection of equipment in a laboratory with an event. We consider conditioning by a complex of, e.g., physical conditions $C$. So our conditioning is conditioning by context and not event.

An important consequence of this new interpretation of conditional probabilities $P(A|C)$ in the Kolmogorov model is that we are not able to apply Boolean algebra to sets $C$ representing contexts – complexes of e.g. physical conditions. For two events, say $C_1$ and $C_2$, it is always possible to consider the event corresponding to their simultaneous occurrence. By the Boolean algebra it is realized as $C = C_1 C_2$. This is a very natural operation on the algebra of events. But for two contexts it is not always possible to define their simultaneous realization. Therefore if such contexts are represented by sets $C_1$ and $C_2$ belonging the $\sigma$-algebra $\mathcal{F}$ of the Kolmogorov space, then by considering the set $C = C_1 C_2$ we cannot be sure that it would represent a physically meaningful context.

Thus we cannot consider the whole $\sigma$-algebra $\mathcal{F}$ of the Kolmogorov space as a set-representation of contexts. Depending on a problem under consideration conditional probabilities $P(A|C)$ can be considered only for contexts $C$ belonging some special collection $\mathcal{C} \subset \mathcal{F}$. (An event $A$ is still represented by an arbitrary element of the $\mathcal{F}$).
We shall show that such a “cutoff” of the Kolmogorov $\sigma$-algebra $\mathcal{F}$ can induce quantum probabilistic formalism. In such an approach quantum formalism arises as a special representation of the contextual Kolmogorov model: $\mathcal{P}_{\text{cont}} = (\Omega, \mathcal{F}|\mathcal{C}, \mathbf{P})$ for a special choice of the collection of contexts $\mathcal{C}$.

Applying the contextual approach to the formula of total probability (2), we see that using of probabilities of the type $\mathbf{P}(B|A_jC)$, i.e., conditioning by “intersection of contexts”, in general is meaningless. And we see that in the “quantum formula of total probability” (3) such probabilities were really excluded from consideration. Probabilities $\mathbf{P}(B|AC)$ are not defined in the physical framework. Therefore in (3), instead of $\mathbf{P}(B|AC)$, there were considered “experimental conditional probabilities” $\mathbf{P}(B|A_j)$. But in general we have the inequality:

$$\mathbf{P}(B|C) \neq \sum \mathbf{P}(A_j|C)\mathbf{P}(B|A_j)$$

(4)

that can be also interpreted as the equality:

$$\mathbf{P}(B|C) = \sum \mathbf{P}(A_j|C)\mathbf{P}(B|A_j) + \delta(B|A, C),$$

(5)

where a perturbation term $\delta(B|A, C)$ is defined as the difference of the left-hand and right-hand sides of (4). In this way, for a special system of contexts $\mathcal{C}^{\text{tr}}$, see section 2, we obtain the “quantum formula of total probability” (3): and with the aid of this formula we construct a representation of the collection of contexts $\mathcal{C}^{\text{tr}}$ in the unit sphere of the complex Hilbert space. This is the crucial step to reproduce a)–d) in the classical, but contextual probabilistic framework.

What are main purposes of such a construction? On one hand, we are able to demystify quantum probability and connect it in a rather simple way with the classical Kolmogorov model. On the other hand, by reproducing quantum

---

4Finally, we remark that our construction – the contextual Kolmogorov model – is very close to Renye’s model [11]. Renye also introduced a special collection of sets, say $\mathcal{C}_{\text{REN}}$, representing conditions. But collections of contexts $\mathcal{C}$ of our contextual Kolmogorov model do not satisfy conditions of Renye’s model. This gives us the possibility to reproduce quantum probabilistic formalism that was impossible to do in Renye’s model. The latter model is more general from the measure-theoretic viewpoint. In principle, we could explore this generality even in our contextual approach. But we shall not do this in the present paper. We want to show that even the Kolmogorov model contains (in a latent form) main quantum probabilistic structures. We emphasize again that typically the presence of such structures was considered as an exhibition of non-Kolmogorovness.
probabilistic calculus, in particular, “interference of probabilities”, in the measure-theoretic framework we see that there are no reasons to restrict applications of this calculus to description of processes in the microworld. By using contextual approach we can construct the quantum representation for statistical models in any domain of science, for example, biology, psychology, economics.\textsuperscript{5} We remark that the first derivation of the “quantum formula of total probability” \textsuperscript{3} without to appeal to the Hilbert space was done in papers [19], [20] in the \textit{von Mises frequency framework}; see also [5] for using of the law of large numbers for this purpose.\textsuperscript{6}

2 Interference formula of total probability

We consider the conventional formula of total probability \textsuperscript{2} in a special case. Let \(a\) and \(b\) be dichotomous random variables, \(a = a_1, a_2\) and \(b = b_1, b_2\). We have

\[
P(b = b_i | C) = \sum_n P(a = a_n | C)P(b = b_i | a = a_n, C).
\]

If a measurement of the variable \(a\) disturbs essentially the context \(C\), then we would not be able to create the context corresponding to nondisturbing measurement of \(a\) under the complex of experimental conditions \(C\). Therefore we should modify this formula and exclude probabilities \(P(b = b_i | a = a_n, C)\).

The following notion is well known in measurement theory of quantum mechanics, see [3], [4]. Let us denote by \(A_j\) the selection-context with respect to the value \(a = a_j\) of the random variable \(a\) (for example, in quantum mechanics there are considered momentum-selections: there are selected all particles with a fixed value of momentum). These contexts \((j = 1, 2\) in our case) are represented in the measure-theoretic approach by sets \(A_j = \ldots\)

\textsuperscript{5}Why can such a representation be fruitful? In our approach the quantum representation is a \textit{projection of the classical probability model}. This is an essential simplification of the classical probabilistic description. Such a simplified description can be useful for models in that the detailed classical probabilistic description is extremely complicated, for example, for applications to cognitive sciences and psychology and cognitive sciences see [13]–[15], in game theory [16], in financial mathematics [17], in classical theory of disordered systems [18].

\textsuperscript{6}Recent years there were also a few attempts to use non-Kolmogorovian, but measure-theoretic models to reproduce some predictions of quantum mechanics, see, e.g., [4] and [22].
\{\omega \in \Omega : a(\omega) = a_j\}. We also introduce the selection-contexts for the \(b\)-variable. They are represented by sets \(B_i = \{\omega \in \Omega : b(\omega) = b_i\}\). We consider partitions \(A = \{A_1, A_2\}\) and \(B = \{B_1, B_2\}\) of the sample space \(\Omega\).

A set \(C\) belonging to \(\mathcal{F}\) is said to be a nondegenerate context with respect to the partition \(A\) if \(P(A_nC) \neq 0\) for all \(n\). We denote the set of all \(A\)-nondegenerate contexts by the symbol \(\mathcal{C}_{A, nd}\). The partitions \(A\) and \(B\) are said to be incompatible if \(P(B_nA_k) \neq 0\) for all \(n\) and \(k\). Thus \(B\) and \(A\) are incompatible iff every \(B_n\) is a nondegenerate context with respect to \(A\) and vice versa. Random variables \(a\) and \(b\) inducing incompatible partitions \(A\) and \(B\) are said to be incompatible. (We remark that we defined incompatibility in purely measure-theoretic framework.)

Everywhere below \(a\) and \(b\) are incompatible random variables. Let \(B \in \mathcal{C}_{A, nd}\). We define a coefficient of interference of random variables \(a\) and \(b\) by:

\[
\lambda(B|A, C) = \frac{\delta(B|A, C)}{2\sqrt{P(A_1|C)P(B|A_1)P(A_2|C)P(B|A_2)}}
\]

where \(\delta(B|A, C) = P(B|C) - \sum_{j=1}^{2}P(B|A_j)P(A_j|C)\). We shall see that the “perturbed formula of total probability” (5) has interesting consequences if the perturbation \(\delta\) be represented in the form:

\[
\delta(B|A, C) = \lambda(B|A, C)\sqrt{P(A_1|C)P(B|A_1)P(A_2|C)P(B|A_2)}
\]

We set \(\mathcal{C}_{tr} = \{C \in \mathcal{C}_{A, nd} : |\lambda(B_j|A, C)| \leq 1\}\) We call elements of \(\mathcal{C}_{tr}\) trigonometric contexts. We consider the contextual Kolmogorov model with this collection of contexts:

\[
\mathcal{P}_{cont, tr} = (\Omega, \mathcal{F}|\mathcal{C}_{tr}, P)
\]

We remark that in general the system of sets \(\mathcal{C}_{tr}\) is not an algebra: \(C_1, C_2 \in \mathcal{C}_{tr}\) does not imply that \(C = C_1C_2 \in \mathcal{C}_{tr}\). Our main result can be formulated in the form of the following theorem (which will be proved in a few steps):

**Theorem 2.1.** The “quantum formula of total probability” (3) can be derived in the Kolmogorov probability framework. On the basis of this formula we can construct a map from the set of trigonometric contexts \(\mathcal{C}_{tr}\) into the unit sphere \(S\) of the complex Hilbert space \(\mathcal{H}\) (space of complex amplitudes). Such a map is determined by a pair \(a, b\) of incompatible random variables.
(reference variables) that are represented by noncommutative operators \( \hat{a}, \hat{b} \). Unitarity of the matrix \( V^{b/a} \) of transition from the basis \( \{ e^a_i \} \) to the basic \( \{ e^b_i \} \) (these bases correspond to random variables \( a \) and \( b \)) is equivalent to Born’s rule for both reference variables. This construction can be realized only for a double stochastic matrix of transition probabilities.

First by using the relation (7) we see that the “perturbed formula of total probability” \( (3) \) can be written as:

\[
P(B|C) = \sum P(A_j|C)P(B|A_j) + 2\lambda(B|A, C) \sqrt{P(A_1|C)P(B|A_1)}P(A_2|C)P(B|A_2)
\]

1. Suppose that the interference coefficients \( |\lambda(B|A, C)| \leq 1 \) for every \( B \in \mathcal{B} \). We introduce new statistical parameters \( \theta(B|A, C) \in [0, 2\pi] \) and represent the coefficients in the trigonometric form: \( \lambda(B|A, C) = \cos \theta(B|A, C) \). Parameters \( \theta(B|A, C) \) are said to be relative phases of an event \( B \) with respect to the partition \( A \) in the context \( C \). In this case the “perturbed formula of total probability” given in the form \( (3) \) coincides with the “quantum formula of total probability” \( (9) \).

2. Suppose that \( |\lambda(B|A, C)| \geq 1 \) for every \( B \in \mathcal{B} \). We set \( \theta(B|A, C) \in (-\infty, +\infty) \) and represent the coefficients in the hyperbolic form: \( \lambda(B|A, C) = \pm \cosh \theta(B|A, C) \). In this case \( (9) \) has the form of “hyperbolic interference of probabilities”

\[
P(B|C) = \sum P(A_j|C)P(B|A_j) \pm 2 \cosh \theta(B|A, C) \sqrt{P(A_1|C)P(B|A_1)}P(A_2|C)P(B|A_2)
\]

In this paper we shall concentrate our considerations on the first case.\(^7\)

\(^7\)This is nothing other than the famous formula of interference of probabilities. Typically this formula is derived by using the Hilbert space (unitary) transformation corresponding to the transition from one orthonormal basis to another and Born’s probability postulate. The orthonormal basis under quantum consideration consist of eigenvectors of operators (noncommutative) corresponding to quantum physical observables \( a \) and \( b \).

\(^8\)We just mention that in the second case we can obtain a representation of the contextual Kolmogorov model \( \mathcal{P}_{cont, hyp} = (\Omega, \mathcal{F}_{hyp}|\mathcal{P}) \), where \( \mathcal{C}_{hyp} = \{ C \in \mathcal{C}_{A, nd} : |\lambda(B_j|A, C)| \geq 1 \} \), in so called hyperbolic Hilbert space: a Hilbert module over the two dimensional Clifford algebra (i.e., the commutative algebra with basis \( e_1 = 1 \) and \( e_2 = j \), where \( j^2 = +1 \), see [21] for details). Therefore it is impossible to represent the whole Kolmogorov \( \sigma \)-algebra \( \mathcal{F} \) in the complex Hilbert space. Moreover, \( \mathcal{C}_{A, nd} \cup \mathcal{C}_{hyp} \) is a proper sub-system of \( \mathcal{F} \). For example, there exist mixed hyper-trigonometric contexts: one \( \lambda \leq 1 \) and another \( \lambda \geq 1 \). There also exist degenerate contexts \( C \) for that interference coefficients are not defined at all.
Everywhere below $B = B_x$, $x = b_1, b_2$, and we shall often use the symbols
\[ \lambda(b = x|a, C) \] instead of \[ \lambda(B_x|A, C) \].

### 3 Extraction of complex probability amplitudes and Born’s rule from the Kolmogorov model

We recall that we study the case of incompatible dichotomous random variables $a = a_1, a_2, b = b_1, b_2$. This pair of variables will be fixed. We call such variables **reference variables**.

For each fixed pair $a, b$ of reference variables we construct a representation of the contextual Kolmogorov model $\mathcal{P}_{\text{cont,tr}} = (\Omega, \mathcal{F}[C^\text{tr}, \mathbf{P}])$ in the complex Hilbert space. We set $Y = \{a_1, a_2\}, X = \{b_1, b_2\}$ ("spectra" of random variables $a$ and $b$).

Let $C \in C^\text{tr}$.

\begin{align}
 p_C^a(y) &= \mathbf{P}(a = y|C), \quad p_C^b(x) = \mathbf{P}(b = x|C), \quad p(x|y) = \mathbf{P}(b = x|a = y),
 x \in X, y \in Y.
\end{align}

The formula (3) can be written as
\begin{align}
 p_C^b(x) &= \sum_{y \in Y} p_C^a(y)p(x|y) + 2 \cos \theta_C(x) \sqrt{\Pi_{y \in Y} p_C^a(y)p(x|y)}, \quad (11)
\end{align}
where $\theta_C(x) = \theta(b = x|a, C) = \pm \arccos \lambda(b = x|a, C), x \in X$. Here $\delta(b = x|a, C) = p_C^b(x) - \sum_{y \in Y} p_C^a(y)p(x|y)$ and $\lambda(b = x|a, C) = \frac{\delta(b = x|a, C)}{2\sqrt{\Pi_{y \in Y} p_C^a(y)p(x|y)}}$.

By using the elementary formula: $D = A + B + 2\sqrt{AB}\cos \theta = |\sqrt{A} + e^{i\theta}\sqrt{B}|^2$, for $A, B > 0, \theta \in [0, 2\pi]$, we can represent the probability $p_C^b(x)$ as the square of the complex amplitude (Born’s rule):
\begin{align}
 p_C^b(x) &= |\varphi_C(x)|^2. \quad (12)
\end{align}

We set
\begin{align}
 \varphi(x) &\equiv \varphi_C(x) = \sqrt{p_C^a(a_1)p(x|a_1) + e^{i\theta_C(x)}\sqrt{p_C^a(a_2)p(x|a_2)}}. \quad (13)
\end{align}

It is important to underline that since for each $x \in X$ phases $\theta_C(x)$ can be chosen in two ways (by choosing signs + or -) a representation of contexts by complex amplitudes is not uniquely defined.\(^9\)

---

\(^9\)To fix a representation of the contextual Kolmogorov space $\mathcal{P}_{\text{cont,tr}}$ we should fix phases. We shall see that to obtain a “good representation” we should choose phases in a special way.
We denote the space of functions: \( \varphi : X \to \mathbb{C} \), where \( \mathbb{C} \) is the field of complex numbers, by the symbol \( E = \Phi(X, \mathbb{C}) \). Since \( X = \{b_1, b_2\} \), the \( E \) is the two dimensional complex linear space. Dirac’s \( \delta \)-functions \( \{\delta(b_1 - x), \delta(b_2 - x)\} \) form the canonical basis in this space. We shall see (Proposition 5.1) that under natural assumption on the matrices of transition probabilities \( \varphi_{B_j}(x) = \delta(b_j - x) \). For each \( \varphi \in E \) we have \( \varphi(x) = \varphi(b_1)\delta(b_1 - x) + \varphi(b_2)\delta(b_2 - x) \). By using the representation (13) we construct the map

\[ J^{b|a} : C^\text{tr} \to \Phi(X, \mathbb{C}) \]  

(14)

The \( J^{b|a} \) maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation (12) of probability as the square of the absolute value of the complex \((b|a)\)-amplitude is nothing other than the famous **Born rule**. The complex amplitude \( \varphi_C(x) \) can be called a wave function of the complex of physical conditions (context \( C \)) or a pure state. We set \( b_x^b(\cdot) = \delta(x-\cdot) \). The Born’s rule for complex amplitudes (12) can be rewritten in the following form:

\[ p_C^b(x) = \left| (\varphi_C, e_x^b) \right|^2, \]

where the scalar product in the space \( E = \Phi(X, \mathbb{C}) \) is defined by the standard formula: \((\varphi, \psi) = \sum_{x \in X} \varphi(x)\overline{\psi}(x) \). The system of functions \( \{e_x^b\}_{x \in X} \) is an orthonormal basis in the Hilbert space \( H = (E, (\cdot, \cdot)) \) Let \( X \subset \mathbb{R} \), where \( \mathbb{R} \) is the field of real numbers. By using the Hilbert space representation (15) of the Born’s rule we obtain the Hilbert space representation of the expectation of the (Kolmogorovian) random variable \( b \):

\[ E(b|C) = \sum_{x \in X} x p_C^b(x) = \sum_{x \in X} x |\varphi_C(x)|^2 = \sum_{x \in X} x (\varphi_C, e_x^b)(\overline{\varphi_C}, e_x^b) = (\hat{b}\varphi_C, \varphi_C), \]

(16)

where the (self-adjoint) operator \( \hat{b} : E \to E \) is determined by its eigenvectors: \( \hat{b} e_x^b = x e_x^b, x \in X \). This is the multiplication operator in the space of complex functions \( \Phi(X, \mathbb{C}) : \hat{b}\varphi(x) = x \varphi(x) \). By (16) the conditional expectation of the Kolmogorovian random variable \( b \) is represented with the aid of the self-adjoint operator \( \hat{b} \). Therefore it is natural to represent this random variable (in the Hilbert space model) by the operator \( \hat{b} \). We shall use the following notations:

\[ u_j^a = \sqrt{p_C^b(a_j)}, u_j^b = \sqrt{p_C^b(b_j)}, p_{ij} = p(b_j|a_i), u_{ij} = \sqrt{p_{ij}}, \theta_j = \theta_C(b_j). \]

(17)

We remark that the coefficients \( u_j^a, u_j^b \) depend on a context \( C \); so \( u_j^a = u_j^a(C), u_j^b = u_j^b(C) \). We also consider the matrix of transition probabilities
\(P^{b|a} = (p_{ij})\). It is always a stochastic matrix.\(^{10}\) We have \(\varphi_{C} = v_{1}^{b}e_{1}^{b} + v_{2}^{b}e_{2}^{b}\), where \(v_{j}^{b} = u_{1j}^{a}u_{1j} + u_{2j}^{a}u_{2j}e^{i\theta_{j}}\). Hence

\[
p_{C}(b_{j}) = |v_{j}^{b}|^{2} = |u_{1j}^{a}u_{1j} + u_{2j}^{a}u_{2j}e^{i\theta_{j}}|^{2}.
\] (18)

This is the interference representation of probabilities that is used, e.g., in quantum formalism. We recall that we obtained (18) starting with the interference formula of total probability, (11).

We would like to have Born’s rule not only for the \(b\)-variable, but also for the \(a\)-variable. As we shall see, we cannot be lucky in the general case. Starting from two arbitrary incompatible (Kolmogorovian) random variables \(a\) and \(b\) we obtained a complex linear space representation of the probabilistic model which is essentially more general than the standard quantum representation. In our (more general) linear representation the “conjugate variable” \(a\) need not be represented by a symmetric operator (matrix) in the Hilbert space \(H\) generated by the \(b\). We recall that in QM both reference variables (the position and the momentum) are represented in the same Hilbert space.

For any context \(C_{0}\), we can represent the corresponding wave function \(\varphi = \varphi_{C_{0}}\) in the form:

\[
\varphi = u_{1}^{a}e_{1}^{a} + u_{2}^{a}e_{2}^{a},
\] (19)

where

\[
e_{1}^{a} = (u_{11}, u_{12}), \quad e_{2}^{a} = (e^{i\theta_{1}}u_{21}, e^{i\theta_{2}}u_{22})
\] (20)

Here \(\{e_{i}^{a}\}\) is a system of vectors in \(E\) corresponding to the \(a\)-observable. We suppose that vectors \(\{e_{i}^{a}\}\) are linearly independent, so \(\{e_{i}^{a}\}\) is a basis in \(E\). We have: \(e_{1}^{a} = v_{11}e_{1}^{b} + v_{12}e_{2}^{b}\), \(e_{2}^{a} = v_{21}e_{1}^{b} + v_{22}e_{2}^{b}\). Here \(V = (v_{ij})\) is the matrix corresponding to the transformation of complex amplitudes: \(v_{11} = u_{11}, v_{21} = u_{21}\) and \(v_{12} = e^{i\theta_{1}}u_{21}, v_{22} = e^{i\theta_{2}}u_{22}\). We would like to find a class of matrices \(V\) such that Born’s rule (in the Hilbert space form), see [15], holds true also in the \(a\)-basis:

\[
p_{C}(a_{j}) = |(\varphi, e_{j}^{a})|^{2}.
\] (21)

By (19) we have the Born’s rule (21) iff \(\{e_{i}^{a}\}\) was an orthonormal basis, i.e., the \(V\) was a unitary matrix.

Since we study the two-dimensional case (i.e., dichotomous random variables), \(V \equiv V^{b|a}\) is unitary iff the matrix of transition probabilities \(P^{b|a}\) is

\(^{10}\)So \(p_{i1} + p_{i2} = 1, i = 1, 2.\)
double stochastic and $e^{i\theta_1} = -e^{i\theta_2}$ or

$$\theta_{C_0}(b_1) - \theta_{C_0}(b_2) = \pi \mod 2\pi \quad (22)$$

We recall that a matrix is double stochastic if it is stochastic, i.e., $p_{j1} + p_{j2} = 1$, and, moreover, $p_{1j} + p_{2j} = 1, j = 1, 2$. Any matrix of transition probabilities is stochastic, but in general it is not double stochastic. We remark that the constraints (22) on phases and the double stochasticity constraint are not independent:

**Lemma 3.1.** Let the matrix of transition probabilities $P^{b|a}$ be double stochastic. Then:

$$\cos \theta_C(b_2) = -\cos \theta_C(b_1) \quad (23)$$

for any context $C \in C^{tr}$.

By Lemma 3.1 we have two different possibilities to choose phases:

$$\theta_{C_0}(b_1) + \theta_{C_0}(b_2) = \pi \text{ or } \theta_{C_0}(b_1) - \theta_{C_0}(b_2) = \pi \mod 2\pi$$

By (22) to obtain the Born’s rule for the $a$-variable we should choose phases $\theta_{C_0}(b_i), i = 1, 2$, in such a way that

$$\theta_{C_0}(b_2) = \theta_{C_0}(b_1) + \pi. \quad (24)$$

If $\theta_{C_0}(b_1) \in [0, \pi]$ then $\theta_{C_0}(b_2) \in [\pi, 2\pi]$ and vice versa. Lemma 3.1 is very important since by it (in the case when reference observables are chosen in such way that the matrix of transition probabilities is double stochastic) we can always choose $\theta_{C_0}(b_j), j = 1, 2$, to satisfy (21).

The delicate feature of the presented construction of the $a$-representation is that the basis $e^a_j$ depends on the context $C_0 : e^a_j = e^a_j(C_0)$. And the Born’s rule, in fact, has the form: $p^a_{C_0}(a_j) = |\langle \varphi_{C_0}, e^a_j(C_0) \rangle|^2$. We would like to use (as in the conventional quantum formalism) one fixed $a$-basis for all contexts $C \in C^{tr}$. We may try to use for all contexts $C \in C^{tr}$ the basis $e^a_j \equiv e^a_j(C_0)$ corresponding to one fixed context $C_0$. We shall see that this is really the fruitful strategy.

**Lemma 3.2** Let the matrix of transition probabilities $P^{b|a}$ be double stochastic and let for any context $C \in C^{tr}$ phases $\theta_C(b_j)$ be chosen as

$$\theta_C(b_2) = \theta_C(b_1) + \pi \mod 2\pi. \quad (25)$$
Then for any context $C \in \mathcal{C}^{tr}$ we have the Born’s rule for the basis $e_j^a \equiv e_j^a(C_0)$ constructed for a fixed context $C_0$:

$$p_C^a(a_j) = |\langle \varphi_C, e_j^a \rangle|^2$$  \hspace{1cm} (26)

**Proof.** Let $C_0$ be some fixed context. We take the basic $\{e_j^a(C_0)\}$ (and the matrix $V(C_0)$) corresponding to this context. For any $C \in \mathcal{C}^{tr}$, we would like to represent the wave function $\phi_C$ as $\phi_C = v_1^a(C)e_1^a(C_0) + v_2^a(C)e_2^a(C_0)$, where $|v_j^a(C)|^2 = p_C^a(a_j)$. It is clear that, for any $C \in \mathcal{C}^{tr}$, we can represent the wave function as

$$\phi_C(b_1) = u_1^a(C)v_{11}(C_0) + e^{i[\theta_C(b_1) - \theta_C_0(b_1)]}u_2^a(C)v_{12}(C_0)$$

$$\phi_C(b_2) = u_1^a(C)v_{21}(C_0) + e^{i[\theta_C(b_2) - \theta_C_0(b_2)]}u_2^a(C)v_{22}(C_0)$$

Thus we should have: $\theta_C(b_1) - \theta_C_0(b_1) = \theta_C(b_2) - \theta_C_0(b_2) \mod 2\pi$. for any pair of contexts $C_0$ and $C_1$. By using the relations (25) between phases $\theta_C(b_1), \theta_C(b_2)$ and $\theta_C_0(b_1), \theta_C_0(b_2)$ we obtain: $\theta_C(b_2) - \theta_C_0(b_2) = (\theta_C(b_1) + \pi - \theta_C_0(b_1) - \pi) = \theta_C(b_1) - \theta_C_0(b_1) \mod 2\pi$.

The constraint (25) essentially restricted the class of complex amplitudes which can be used to represent a context $C \in \mathcal{C}^{tr}$. Any $C$ can be represented only by two amplitudes $\varphi(x)$ and $\bar{\varphi}(x)$ corresponding to the two possible choices of $\theta_C(b_1)$ (in $[0, \pi]$ or $(\pi, 2\pi]$).

By Lemma 3.2 we obtain the following part of the Theorem 2.1: We can construct the Hilbert space representation of the contextual Kolmogorov model $\mathcal{P}_{cont, tr}$ such that the Born’s rule holds true for both reference variables iff the matrix of transition probabilities $P^{[\theta]}_b$ is double stochastic.

If $P^{[\theta]}_b$ is double stochastic we have a quantum-like representation not only for the conditional expectation of the variable $b$, see (16), but also for the variable $a$:

$$E(a|C) = \sum_{y \in Y} yp_C^a(y) = \sum_{y \in Y} y|\langle \varphi_C, e_y^a \rangle|^2 = \langle \hat{a}\varphi_C, \varphi_C \rangle$$  \hspace{1cm} (27)

where the self-adjoint operator (symmetric matrix) $\hat{a} : E \rightarrow E$ is determined by its eigenvectors: $\hat{a}e_j^a = a_j e_j^a$. By (27) it is natural to represent the random variable $a$ by the operator $\hat{a}$.

Let us denote the unit sphere in the Hilbert space $E = \Phi(X, C)$ by the symbol $S$. The map $J^{[\theta]} : \mathcal{C}^{tr} \rightarrow S$ need not be a surjection (injection). In
general the set of (pure) states corresponding to a contextual Kolmogorov space

\[ S_{\text{ctr}} \equiv S_{\text{ctr}}^{b|a} = J^{b|a}(C^{\text{tr}}) \]

is just a proper subset of the sphere \( S \). The structure of the set of pure states \( S_{\text{ctr}} \) is determined by the Kolmogorov space and the reference variables \( a \) and \( b \).

4 Noncommutative operator-representation of Kolmogorovian random variables

Let the matrix of transition probabilities \( P^{b|a} \) be double stochastic. We consider in this section the case of real valued random variables. Here spectra of random variables \( b \) and \( a \) are subsets of \( \mathbb{R} \). We set \( q_1 = \sqrt{p_{11}} = \sqrt{p_{22}} \) and \( q_2 = \sqrt{p_{12}} = \sqrt{p_{21}} \). Thus the vectors of the \( a \)-basis, see (20), have the following form: \( e_1^a = (q_1, q_2) \), \( e_2^a = (e^{i\theta_1}q_2, e^{i\theta_2}q_1) \). Since \( \theta_2 = \theta_1 + \pi \), we get \( e_2^a = e^{i\theta_2}(-q_2, q_1) \). We now find matrices of operators \( \hat{a} \) and \( \hat{b} \) in the \( b \)-representation. The latter one is diagonal. For \( \hat{a} \) we have: \( \hat{a} = V \text{diag}(a_1, a_2)V^* \), where \( v_{11} = v_{22} = q_1, v_{21} = -v_{12} = q_2 \). Thus \( a_{11} = a_1q_1^2 + a_2q_2^2, a_{22} = a_1q_2^2 + a_2q_1^2, a_{12} = a_{21} = (a_1 - a_2)q_1q_2 \). Hence \( [\hat{b}, \hat{a}] = \hat{m} \), where \( m_{11} = m_{22} = 0 \) and \( m_{12} = -m_{21} = (a_1 - a_2)(b_2 - b_1)q_1q_2 \). Since \( a_1 \neq a_2, b_1 \neq b_2 \) and \( q_j \neq 0 \), we have \( \hat{m} \neq 0 \).

5 The role of simultaneous double stochasticity of \( P^{b|a} \) and \( P^{a|b} \)

Starting with the \( b \)-representation – complex amplitudes \( \phi_C(x) \) defined on the spectrum (range of values) of a random variable \( b \) – we constructed the \( a \)-representation. This construction is natural (i.e., it produces the Born’s probability rule) only when the \( P^{b|a} \) is double stochastic. We would like to have a symmetric model. So by starting with the \( a \)-representation – complex amplitudes \( \phi_C(y) \) defined on the spectrum (range of values) of a random variable \( a \) – we would like to construct the natural \( b \)-representation. Thus both matrices of transition probabilities \( P^{b|a} \) and \( P^{a|b} \) should be double stochastic.

**Theorem 5.1.** Let the matrix \( P^{b|a} \) be double stochastic. The contexts \( B_1, B_2 \) belong to \( C^{\text{tr}} \) iff the matrix \( P^{a|b} \) is double stochastic.
Lemma 5.1. Both matrices of transition probabilities \( P_{b|a} \) and \( P_{a|b} \) are double stochastic iff the transition probabilities are symmetric, i.e.,

\[
p(b_i|a_j) = p(a_j|b_i), \quad i, j = 1, 2.
\]

This is equivalent that random variables \( a \) and \( b \) have the uniform probability distribution: \( p^a(a_i) = p^b(b_i) = 1/2, \quad i = 1, 2. \)

This Lemma has important physical consequences. A natural (Bornian) Hilbert space representation of contexts can be constructed only on the basis of a pair of (incompatible) uniformly distributed random variables.

Lemma 5.2. Let both matrices \( P_{b|a} \) and \( P_{a|b} \) be double stochastic. Then

\[
\lambda(B_i|a, B_i) = 1.
\]

Proposition 5.1. Let both matrices of transition probabilities \( P_{b|a} \) and \( P_{a|b} \) be double stochastic. Then

\[
J_{b|a}(B_j)(x) = \delta(b_j - x), \quad x \in X, \quad \text{and} \quad J_{a|b}(A_j)(y) = \delta(a_j - y), \quad y \in Y.
\]

Thus in the case when both matrices of transition probabilities \( P_{a|b} \) and \( P_{b|a} \) are double stochastic (i.e., both reference variables \( a \) and \( b \) are uniformly distributed) the Born’s rule has the form: \( p^b_C(x) = |\langle \phi_C, \phi_{B_x} \rangle|^2. \)

6 Complex amplitudes of probabilities in the case of multivalued reference variables

The general case of random variables taking \( n \geq 2 \) different values can be (inductively) reduced to the case of dichotomous random variables. We consider two incompatible random variables taking \( n \) values: \( b = b_1, \ldots, b_n \) and \( a = a_1, \ldots, a_n. \) We start with some evident generalizations of results presented in section 2.

Lemma 6.1. Let \( B, C, D_1, D_2 \in \mathcal{F}, \) \( P(C) \neq 0 \) and \( D_1 \cap D_2 = \emptyset. \) Then

\[
P(B(D_1 \cup D_2)|C) = P(BD_1|C) + P(BD_2|C)
\]

Proposition 6.1. (The formula of total probability) Let conditions of Lemma 6.1 hold true and let \( P(D_jC) \neq 0. \) Then

\[
P(B(D_1 \cup D_2)|C) = P(B|D_1C)P(D_1|C) + P(B|D_2C)P(D_2|C)
\]
Proposition 6.2. (Contextual formula of total probability) Let conditions of Proposition 6.1 hold true and let \( P(BD_j) \neq 0, j = 1, 2 \). Then

\[
P(B(D_1 \cup D_2)|C) = P(B|D_1)P(D_1|C) + P(B|D_2)P(D_2|C) + 2\lambda(B|\{D_1, D_2\}, C)\sqrt{P(B|D_1)P(D_1|C)P(B|D_2)P(D_2|C)},
\]

where the "interference coefficient"

\[
\lambda(B|\{D_1, D_2\}, C) = \frac{\delta(B|\{D_1, D_2\}, C)}{2\sqrt{P(B|D_1)P(D_1|C)P(B|D_2)P(D_2|C)}}
\]

and \( \delta(B|\{D_1, D_2\}, C) = P(B(D_1 \cup D_2)|C) - \sum_{j=1}^{2} P(B|D_j)P(D_j|C)\)

\[
= \sum_{j=1}^{2} P(D_j|C)(P(B|D_j C) - P(B|D_j))
\]

We remark that if \( D = \{D_1, D_2\} \) is a partition of the sample space, then the formula [32] coincides with the interference formula of total probability, see section 2.

In the construction of a Hilbert space representation of contexts for multivalued observables there will be used the following combination of formulas [30] and [32].

Lemma 6.2. Let conditions of Lemma 6.1 hold true and let \( P(BD_1) \), \( P(CD_1) \) and \( P(BD_2) \) be strictly positive. Then

\[
P(B(D_1 \cup D_2)|C) = P(B|D_1)P(D_1|C) + P(BD_2|C)
\]

\[+ 2\mu(B|\{D_1, D_2\}, C)\sqrt{P(B|D_1)P(D_1|C)P(BD_2|C)},\]

where \( \mu(B|\{D_1, D_2\}, C) = \frac{P(B(D_1 \cup D_2)|C) - P(B|D_1)P(D_1|C) - P(BD_2|C)}{2\sqrt{P(B|D_1)P(D_1|C)P(BD_2|C)}}\)

Suppose that coefficients \( \mu \) and \( \lambda \) are bounded by 1. Then we can represent them in the trigonometric form:

\[
\lambda(B|\{D_1, D_2\}, C) = \cos \theta(B|\{D_1, D_2\}, C)
\]

\[
\mu(B|\{D_1, D_2\}, C) = \cos \gamma(B|\{D_1, D_2\}, C)
\]
By inserting these cos-expressions in (32) and (33) we obtain trigonometric transformations of probabilities. We have (by Lemma 6.2):

\[ P(B_x|C) = P(B_x(A_1 \cup \ldots \cup A_n)|C) \]

= \[ P(B_x|A_1)P(A_1|C) + P(B_x|A_2 \cup \ldots \cup A_n)|C) \]

+2\mu(B_x\{A_1, A_2 \cup \ldots \cup A_n\}, C)\sqrt{P(B_x|A_1)P(A_1|C)P(B_x(A_2 \cup \ldots \cup A_n)|C)},

where \( \mu(B_x\{A_1, A_2, \ldots \cup A_n\}, C) \) is the phase of the complex amplitude:

\[ \varphi_C(x) \equiv \varphi_C^{(1)}(x) = \sqrt{P(B_x|A_1)P(A_1|C)} + e^{i\gamma_C^{(1)}(x)}\sqrt{P(B_x(A_2 \cup \ldots \cup A_n)|C)}, \]

where the phase \( \gamma_C^{(1)}(x) \equiv \gamma(B_x\{A_1, A_2 \cup \ldots \cup A_n\}, C) \). In the same way the probability in the second summand can be represented as:

\[ P(B_x(A_1 \cup \ldots \cup A_n)|C) = P(B_x|A_2)P(A_2|C) + P(B_x(A_3 \cup \ldots \cup A_n)|C) + \]

2\mu(B_x\{A_2, A_3 \cup \ldots \cup A_n\}, C)\sqrt{P(B_x|A_2)P(A_2|C)P(B_x(A_3 \cup \ldots \cup A_n)|C)},

where

\[ \mu(B_x\{A_2, A_3 \cup \ldots \cup A_n\}, C) \]

= \[ \frac{P(B_x(A_2 \cup \ldots \cup A_n)|C) - P(B_x|A_2)P(A_2|C) - P(B_x(A_3 \cup \ldots \cup A_n)|C)}{2\sqrt{P(B_x|A_2)P(A_2|C)P(B_x(A_3 \cup \ldots \cup A_n)|C)}}. \]

By supposing that these coefficients of statistical disturbance are bounded by 1 we represent the probability as the square of the absolute value of the complex amplitude:

\[ \varphi_C^{(2)}(x) = \sqrt{P(B_x|A_2)P(A_2|C)} + e^{i\gamma_C^{(2)}(x)}\sqrt{P(B_x(A_3 \cup \ldots \cup A_n)|C)}, \]

16
where \( \gamma_C^{(2)}(x) = \pm \arccos \mu(B_x|\{A_2, A_3, \ldots \cup A_n\}, C) \). On the \( j \)th step we represent \( P(B_x|A_j \cup \ldots \cup A_n)|C) \) as the square of the absolute value of the complex amplitude

\[
\varphi_C^{(j)}(x) = \sqrt{P(B_x|A_j)P(A_j|C)} + e^{i\gamma^{(j)}_C(x)} \sqrt{P(B_x|A_{j+1} \cup \ldots \cup A_n)|C)},
\]

where \( \gamma^{(j)}_C(x) \) is the phase of the coefficient

\[
\mu(B_x|\{A_j, A_{j+1} \cup \ldots \cup A_n\}, C) = \frac{P(B_x|A_j \cup \ldots \cup A_n)|C) - P(B_x|A_j)P(A_j|C) - P(B_x|A_{j+1} \cup \ldots \cup A_n)|C)}{2\sqrt{P(B_x|A_j)P(A_j|C)}P(B_x|A_{j+1} \cup \ldots \cup A_n)|C)}.
\]

It is supposed that at each step we obtain coefficients \( |\mu| \) bounded by 1. At the step \( j = n-1 \) we should represent the probability \( P(B_x|A_{n-1} \cup A_n)|C) \). Here we can already totally eliminate the \( C \)-contextuality for \( B_x \) :

\[
P(B_x|A_{n-1} \cup A_n)|C) = P(B_x|A_{n-1})P(A_{n-1}|C) + P(B_x|A_n)P(A_n|C)
\]

\[+2\lambda(B_x|\{A_{n-1}, A_n\}) \sqrt{P(B_x|A_{n-1})P(A_{n-1}|C)P(B_x|A_n)P(A_n|C)},\]

where the coefficient of statistical disturbance \( \lambda \) was defined by (33). And if \( |\lambda| \) is bounded by 1 then we can represent the probability as the square of the absolute value of the complex amplitude:

\[
\varphi_C^{(n-1)}(x) = \sqrt{P(B_x|A_{n-1})P(A_{n-1}|C)} + e^{i\theta_C(x)} \sqrt{P(B_x|A_n)P(A_n|C)},
\]

where \( \theta_C(x) = \pm \arccos \lambda(x|\{A_{n-1}, A_n\}, C) \).

We have:

\[
\varphi_C^{(j)}(x) = \sqrt{P(B_x|A_j \cup \ldots \cup A_n)|C)} e^{i \alpha^{(j)}_C(x)}.
\]

where \( \alpha^{(j)}_C(x) = \arg \varphi_C^{(j)}(x) = \arccos \frac{M_j}{N_j} \), where \( M_j = \sqrt{P(B_x|A_j)P(A_j|C)} + \mu(B_x|\{A_j, A_{j+1} \cup \ldots \cup A_n\}, C) \sqrt{P(B_x|A_{j+1} \cup \ldots \cup A_n)|C)} \)

\( N_j = \sqrt{P(B_x|A_j \cup \ldots \cup A_n)|C)} \). Finally, we have:

\[
\alpha_C^{(n-1)}(x) = \arg \varphi_C^{(n-1)}(x)
\]

\[
= \arccos \sqrt{P(B_x|A_{n-1})P(A_{n-1}|C)} + \lambda(B_x|\{A_{n-1}, A_n\}, C) \sqrt{P(B_x|A_n)P(A_n|C)}
\]

\[
\sqrt{P(B_x|A_{n-1} \cup A_n)|C)}.
\]
Thus we have:

$$\varphi_C(x) = \sqrt{P(B_x|A_1)}P(A_1|C) + e^{i[\alpha_1^{(1)}(x) - \gamma_1^{(2)}(x)]} \varphi_1^{(2)}(x)$$

$$= \sqrt{P(B_x|A_1)}P(A_1|C) + e^{i\beta_1^{(2)}(x)} \sqrt{P(B_x|A_2)}P(A_2|C)$$

$$+ e^{i\beta_1^{(3)}(x)} \varphi_1^{(3)}(x),$$

where

$$\beta_1^{(2)}(x) = \gamma_1^{(1)}(x) - \alpha_1^{(2)}(x), \beta_1^{(3)}(x) = \beta_1^{(2)}(x) + \gamma_1^{(2)}(x) - \alpha_1^{(3)}(x).$$

Finally, we obtain:

$$\varphi_C(x) = \sum_{j=1}^{n} e^{i\beta_j^{(2)}(x)} \sqrt{P(B_x|A_j)}P(A_j|C)$$

with $\beta_1^{(1)}(x) = 0$ (this is just due to our special choice of a representation) and $\beta_1^{(n)}(x) = \beta_1^{(n-1)}(x) + \theta_C(x)$.

Thus by inductive splitting of multivalued variables into dichotomous variables we represented contextual probabilities by complex amplitudes $\varphi_C(x)$. Here the Born’s rule holds true.

By using the standard in this paper symbols $p(x|y) = P(B_x|A_y)$ and $p_C^h(x) = P(B_x|C), p_C^p(y) = P(A_y|C)$ we write

$$\varphi_C(x) = \sum_{y} e^{i\beta_C^{(y)}(x)} \sqrt{p_C^h(y)p(x|y)}.$$
\[ +2 \sum_{y_1 < y_2} \cos[\beta_C^{(y_2)}(x) - \beta_C^{(y_1)}(x)] \sqrt{p_C(x|y_1)p_C(x|y_2)p(y_1)p(y_2)}. \]

We can proceed in the same way as in the case of dichotomous random variables, see section 3.4.

I would like to thank A. V. Bulinskii, A. N. Shirayev, A. S. Holevo and V.M. Maximov for discussions on probabilistic foundations.

References

1. A. N. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrech, Springer Verlag, Berlin, 1933; reprinted : Foundations of the Probability Theory, Chelsea Publ. Comp., New York, 1956.

2. A. N. Shiryayev, Probability, Springer, New York-Berlin-Heidelberg, 1984.

3. A. S. Holevo, Statistical structure of quantum theory, Springer, New York-Berlin-Heidelberg, 2001.

4. A. Yu. Khrennikov, Non-Kolmogorovian theories of probability and quantum physics, Nauka, Fizmatlit, Moscow 2003 (in Russian).

5. A. V. Bulinskii and A. Yu. Khrennikov, Nonclassical total probability formula and quantum interference probabilities, Statistics and Probability Letters, 70 (2004), pp. 49-58.

6. R. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New-York, 1965.

7. P. A. M. Dirac, The Principles of Quantum Mechanics, Oxford Univ. Press, Oxford, 1930.

8. J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press, Princeton, N.J., 1955.

9. A. N. Kolmogorov, The Theory of Probability. In: A. D. Alexandrov, A. N. Kolmogorov, M. A. Lavrent’ev, eds. Mathematics, Its Content, Methods, and Meaning 2, M.I.T. Press, Boston, 1965, pp. 110-118.

10. B. V. Gnedenko, The theory of probability, Chelsea Publ. Com., New-York, 1962.

11. A. Renye, On a new axiomatics of probability theory, Acta Mat. Acad. Sc. Hung., 6 (1955), pp. 285-335.

12. R. Von Mises, The mathematical theory of probability and statistics, Academic, London, 1964.

13. E. Conte, O. Todarello, A. Federici, F. Vitiello, M. Lopane, A. Yu. Khrennikov, A preliminar evidence of quantum-like behaviour in measurements of mental states, Proc. Int. Conf. Quantum Theory: Reconsideration
of Foundations, A. Yu. Khrennikov, ed, Ser. Math. Modelling, 10, Växjö Univ. Press, Växjö, 2004, pp. 441-454.

14. A. Yu. Khrennikov, Quantum-like formalism for cognitive measurements, Biosystems, 70 (2003), pp. 211-233.

15. A. Yu. Khrennikov, Information dynamics in cognitive, psychological and anomalous phenomena, Kluwer, Dordreht, 2004.

16. A. A. Grib, A. Yu. Khrennikov, K. Starkov, Probability amplitude in quantum-like games, Proc. Int. Conf. Quantum Theory: Reconsideration of Foundations, A. Yu. Khrennikov, ed., Ser. Math. Modelling, 10, Växjö Univ. Press, Växjö, 2004, pp. 703-722.

17. O. A. Choustova, Bohmian mechanics for financial processes, J. Modern Optics, 51 (2004), pp. 1111-1112.

18. A. Yu. Khrennikov and S. V. Kozyrev, Noncommutative probability in classical disordered systems, Physica A, 326 (2003), pp. 456-463.

19. A. Yu. Khrennikov, Linear representations of probabilistic transformations induced by context transitions, J. Phys.A: Math. Gen., 34 (2001), pp. 9965-9981.

20. A. Yu. Khrennikov, Contextual viewpoint to quantum statistics, J. Math. Phys., 44 (2003), pp. 2471-2478.

21. A. Yu. Khrennikov, Interference of probabilities and number field structure of quantum models, Annalen der Physik, 12 (2003), pp. 575-585.

22. V. M. Maximov, Abstract models of Probability, Proc. Conf. Foundations of Probability and Physics, Quantum Probability and White Noise Analysis, 13, 201-218, WSP, Singapore, 2001, pp.257-273.