Functional Inequalities and Bounds for the Generalized Marcum Function of the Second Kind

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Abstract. In this paper, we consider the generalized Marcum function of the second kind as an analogous function of the so-called generalized Marcum $Q$-function. We provide the log-convexity (log-concavity) property for its unit complement and improve some of our previous results on it. One of the transformed functions of the generalized Marcum function of the second kind is discussed in details in this paper. This form of the generalized Marcum function of the second kind supplies various important inequalities. We also discuss the Turán type inequality for the generalized Marcum $Q$-function. Additionally, we provide the bounds for the generalized Marcum function of the second kind as well as for its symmetric difference.

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1. Introduction

The generalized Marcum function of the second kind of order $\nu > 0$ is defined by [4]

$$R_{\nu}(a, b) = \frac{e_{a, \nu}}{a^{\nu - 1}} \int_b^\infty t^\nu e^{-\frac{t^2 + a^2}{2}} K_{\nu - 1}(at) dt,$$  (1.1)
where, $a > 0, b \geq 0$, $K_\nu$ stands for the modified Bessel function of the second kind and

$$c_{a,\nu} = \frac{2}{\Gamma(\nu)\Gamma(1 - \nu, \frac{a^2}{2})}.$$ 

This function is an analogous function of the so-called generalized Marcum $Q$-function, which has applications in radar communication, see for example [13,14]. In these articles [13,14] the authors have used the following equation

$$\tilde{Q}_\nu(a, b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^{\nu-1/2} e^{-\frac{t^2 + a^2}{2}} I_{\nu-1}(\sqrt{at})dt$$

as a definition of the generalized Marcum $Q$-function, while many others have used the following definition

$$Q_\nu(a, b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^{\nu-1/2} e^{-\frac{t^2 + a^2}{2}} I_{\nu-1}(at)dt.$$ 

The generalized Marcum $Q$-function $Q_\nu(a, b)$ and its particular case $Q_1(a, b)$, the so-called Marcum function, are important functions of the electrical engineering literature, and their analytical properties, approximations and computations have been investigated by many researchers (mostly engineers) in the last decades, see for example [21] and the references therein. It turned out that the generalized Marcum $Q$-function possesses some quite interesting properties and its investigation is rather interesting from the mathematical point of view too. From the above two definitions we can conclude that

$$Q_\nu(\sqrt{a}, \sqrt{b}) = \tilde{Q}_\nu(a, b)$$

and analogously to $Q_\nu(\sqrt{a}, \sqrt{b})$, we can derive

$$R_\nu(\sqrt{a}, \sqrt{b}) = \frac{c_{\sqrt{a},\nu}}{2a^{\nu-1/2}} \int_b^\infty t^{\nu-1/2} e^{-\frac{t^2 + a^2}{2}} K_{\nu-1}(\sqrt{at})dt,$$

by replacing $a$ and $b$ by $\sqrt{a}$ and $\sqrt{b}$ respectively, in (1.1). In view of [18, eq. 10.27.3]

$$K_{-\nu}(x) = K_\nu(x)$$

and [18, eq. 8.6.6]

$$\Gamma(\mu, z) = 2 \frac{z^{\frac{\mu}{2}} e^{-z}}{\Gamma(1 - \mu)} \int_0^\infty e^{-t} t^{-\frac{\mu}{2}} K_\mu(2\sqrt{zt})dt$$

as $\mu < 1$, we get

$$R_\nu(\sqrt{a}, \sqrt{b}) = \frac{\int_b^\infty t^{\nu-1/2} e^{-\frac{t^2}{2}} K_{\nu-1}(\sqrt{at})dt}{\int_0^\infty u^{\nu-1/2} e^{-\frac{u^2}{2}} K_{\nu-1}(\sqrt{au})du} = \frac{A_\nu(a, b)}{A_\nu(a, 0)} \quad (1.2)$$

where

$$A_\nu(a, b) = \int_b^\infty t^{\nu-1/2} e^{-\frac{t^2}{2}} K_{\nu-1}(\sqrt{at})dt.$$
Both forms (1.1) and (1.2) have been used in [4], which contains a detailed study of the generalized Marcum function of the second kind. In particular, the study includes monotonicity, convexity, recurrence relation, closed form expression, and tight bounds for the generalized Marcum function of the second kind. In [5], extremely tight bounds of the generalized Marcum function of the second kind were obtained. It is interesting to note that the generalized Marcum function of the second kind is the survival function of the truncated distribution of a special case of the modified Bessel distribution of the second kind considered by Nadarajah [16], see [4] for more details. Therefore, in view of [4, Eq. (1.4)] the corresponding cumulative distribution function (the unit complement to the generalized Marcum function of the second kind) is given by

\[
S_\nu(a, b) = 1 - R_\nu(a, b) = \frac{c_{a, \nu}}{a^\nu - 1} \int_0^b t^\nu e^{-\frac{t^2 + a^2}{2}} K_{\nu-1}(at) dt. \tag{1.3}
\]

The above function is similar to the unit complement to the generalized Marcum Q-function, which has been discussed in [21]. Moreover, the authors of [21] have found some kind of analytical properties for this function. We also have the transformed form \( R_\nu(\sqrt{a}, \sqrt{b}) \) of the generalized Marcum function of the second kind in (1.2), which has a similar form as the normalized function \( f_a(x) \) [2], defined by

\[
f_a(x) = \frac{\Gamma(a, x)}{\Gamma(a, 0)} \quad \text{and} \quad \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt.
\]

Alzer and Baricz [2] have derived various inequalities for the normalized function \( f_a(x) \). Motivated by [2,21], in this paper, our aim is to present various inequalities involving the transformed form of the generalized Marcum function of the second kind with some analytical properties for \( S_\nu(a, b) \). We have also found some Turán type inequalities for the generalized Marcum function of the second kind and the generalized Marcum Q-function by a unified approach. However, Baricz and Sun have already developed the Turán type inequality for the generalized Marcum Q-function in [20], that is later on directly concluded by the log-concavity property of the function \( \nu \mapsto Q_\nu(a, b) \) on \([1, \infty)\) for all \( a, b \geq 0 \), [21]. Additionally, we provide some bounds for the generalized Marcum function of the second kind and its symmetric difference, which is defined by

\[
\Delta R_\nu(a, b) = R_\nu(a, b) - R_\nu(b, a)
\]

analogously to the symmetric difference of the generalized Marcum Q-function discussed in [6,11]. All the main results that include the inequalities and analytical properties with some other results are provided in the next section, while Sect. 3 contains the proofs of the main results.
2. Main Results

This section is divided into five subsections: in the first subsection, we present inequalities for the transformed form (1.2) of the generalized Marcum function of the second kind. The second subsection contains several properties of the generalized Marcum function of the second kind (1.1) and its unit complement (1.3). This subsection also deals with the Turán type inequality for $R_{\nu}(a, b)$, while the third subsection provides the Turán type inequality for the generalized Marcum $Q$-function. The fourth subsection is dedicated to some new bounds for the generalized Marcum function of the second kind, especially, to a uniform upper and lower bound. In addition, we also provide some lower and upper bounds for the symmetric difference of the generalized Marcum function of the second kind in the last subsection.

2.1. Inequalities for the Generalized Marcum Function of the Second Kind

In this subsection, we discuss the transformed generalized Marcum function of the second kind which is defined in Eq. (1.2). This transformed function is very important as we can see in [4], and it is used to obtain the monotonicity of the generalized Marcum function of the second kind with respect to the parameters $a$ and $\nu$. Now, we use this transformed function $R_{\nu}(\sqrt{a}, \sqrt{b})$ to derive various inequalities. We use the notation $R_{\nu}(a, b)$ for this transformed function, i.e., $R_{\nu}(a, b) = R_{\nu}(\sqrt{a}, \sqrt{b})$, for the sake of convenience, however it is used only in Theorem 1 and Lemma 2. The following theorem includes various inequality for $R_{\nu}(a, b)$ which are motivated by the results of [2]. Before stating the theorem, we describe here a few notations: The power sum and power mean of order $t$ are denoted by

$$S_t(x_1, x_2, \ldots, x_n) = (x_1^t + x_2^t + \cdots + x_n^t)^{1/t}$$

and

$$M_t(x_1, x_2, \ldots, x_n) = \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{1/t}.$$

The other notations are

$$M_{-\infty}(x_1, x_2, \ldots, x_n) = \min(x_1, \ldots, x_n),$$

$$M_{\infty}(x_1, x_2, \ldots, x_n) = \max(x_1, \ldots, x_n).$$

and

$$M_0(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_n)^{1/n}.$$

**Theorem 1.** Let $a > 0, \nu > 0$ and $x_1, \ldots, x_n, n \geq 2$, be positive real numbers.

a. If $p \geq 1$ then the inequality

$$R_{\nu}(a, x_1) \cdots R_{\nu}(a, x_n) \leq R_{\nu}(a, S_p(x_1, \ldots, x_n)) \quad (2.1)$$

holds. Its converse is also true if $\nu \geq 1$. 
b. The inequality
\[ R_{\nu}(a, M_r(x_1, \ldots, x_n)) \leq \frac{R_{\nu}(a, x_1) + \cdots + R_{\nu}(a, x_n)}{n} \leq R_{\nu}(a, M_s(x_1, \ldots, x_n)) \]
holds if and only if \( r \geq 1 \) and \( s = -\infty \).

c. The inequality
\[ R_{\nu}(a, M_r(x_1, \ldots, x_n)) \leq (R_{\nu}(a, x_1) \cdots R_{\nu}(a, x_n))^{1/n} \]
holds if and only if \( r \geq 1 \).

d. The inequality
\[ R_{\nu}(a, M_s(x_1, \ldots, x_n)) \leq \frac{n}{R_{\nu}(a, x_1) + \cdots + \frac{1}{R_{\nu}(a, x_n)}} \]
holds if and only if \( s = \infty \).

e. The inequality
\[ 0 \leq R_{\nu}(a, x_1^{1/a}) + \cdots + R_{\nu}(a, x_n^{1/a}) - R_{\nu}(a, (x_1 + \cdots + x_n)^{1/a}) \leq n - 1 \]
holds for all \( a > 1 \) and bounds are the sharpest.

f. Let \( p \neq 0 \) and \( q \neq 0 \) be real numbers. The inequality
\[ \left(R_{\nu}(a, (x + y)^p)\right)^q \leq R_{\nu}(a, x^p)^q + (R_{\nu}(a, y^p))^q \]
holds for all positive real numbers \( x \) and \( y \), i.e., the function \( x \mapsto R_{\nu}(a, x^p)^q \) is strictly subadditive on \((0, \infty)\) if and only if \( pq > 0 \).

g. The inequality
\[ R_{\nu}(a, x) + R_{\nu}(a, y) \leq 1 + R_{\nu}(a, z) \]
holds for all positive real numbers \( x, y \) and \( z \).

h. The function \( x \mapsto R_{\nu}(a, x) \) is completely monotonic on \((0, \infty)\) for all \( a > 0 \) and \( \nu > 0 \).

2.2. Monotonicity Patterns of the Generalized Marcum Function of the Second Kind

From [4] we recall that the function \( b \mapsto R_{\nu}(a, b) \) is logarithmically concave on \((0, \infty)\) for each \( \nu > 3/2 \) and \( a > 0 \), and thus for all \( a > 0 \), \( \nu > 3/2 \), \( b_1, b_2 \geq 0 \) and \( b_1 \neq b_2 \), it satisfies the following inequalities:
\[ R_{\nu}(a, b_1 + b_2) < R_{\nu}(a, b_1)R_{\nu}(a, b_2) < R_{\nu}^2(a, \frac{b_1 + b_2}{2}). \]

The left-hand side inequality is nothing but the new is better than used inequality, which has several applications in the economic theory, while the other part of the inequality is the inequality directly obtained by log-concave property of the function \( b \mapsto R_{\nu}(a, b) \). In part c of [4, Theorem 1] we also showed that the transformed generalized Marcum function of the second kind stated
in (1.2) is log-convex with respect to $b$ on the interval $(0, \infty)$ when $a > 0$ and $\nu \geq 1$. This property implies the inequality

$$R_\nu^2 \left( \sqrt{a}, \sqrt{\frac{b_1 + b_2}{2}} \right) \leq R_\nu(\sqrt{a}, \sqrt{b_1}) R_\nu(\sqrt{a}, \sqrt{b_2}) \leq R_\nu(\sqrt{a}, \sqrt{b_1 + b_2}),$$

where $b_1 \neq b_2$. The right-hand side inequality is reversed to the new is better than used inequality, that sometimes called the new is worst then used (nws) inequality. In this subsection, we discuss similar kind of properties for the unit complement to the generalized Marcum function of the second kind which is defined in (1.3). In particular, we prove that the function $b \mapsto S_\nu(a,b)$ is log-concave on $(0, \infty)$ for all $a > 0$ and $\nu > 3/2$. Unfortunately, the unit complement to $R_\nu(\sqrt{a}, \sqrt{b})$, i.e., $S_\nu(\sqrt{a}, \sqrt{b})$ do not have the log-convex property with respect to $b$, since part b of [3, Theorem 1] says the cumulative distribution function (cdf) $F : [a,b] \subseteq \mathbb{R} \mapsto [0,1]$ is log-convex if the corresponding probability density function (pdf) $f : [a,b] \subseteq \mathbb{R} \mapsto \mathbb{R}$ is log-convex with $f(a) = 0$, and the fact

$$\lim_{t \to 0} t^{\nu-1} K_{\nu-1}(\sqrt{at}) = \frac{a^{\nu-1} \Gamma(\nu - 1)}{2^{2-\nu}} \neq 0.$$

Our first theorem answers the question about logarithmic and geometric convexity/concavity of the unit complement to the generalized Marcum function of the second kind.

**Theorem 2.** The following assertions are true:

a. The function $b \mapsto S_\nu(a,b)$ is log-concave on $(0, \infty)$ for each $\nu > 3/2$ and $a > 0$. Furthermore, the following inequality holds for all $a > 0$, $\nu > 3/2$, $b_1, b_2 \geq 0$ and $b_1 \neq b_2$,

$$S_\nu(a, b_1 + b_2) < S_\nu(a, b_1) S_\nu(a, b_2) < S_\nu^2 \left( a, \frac{b_1 + b_2}{2} \right).$$

(2.8)

b. The function $b \mapsto S_\nu(a, \sqrt{b})$ is log-concave on $(0, \infty)$ for each $\nu > 0$ and $a > 0$.

c. The function $b \mapsto S_\nu(a, b)$ is geometrically concave on $(0, \infty)$ for all $\nu > 0$ and $a > 0$.

d. The function $\nu \mapsto S_\nu(a, b)/c_{a, \nu}$ is log-convex on $(0, \infty)$ for every $b > 0$ and $a > 0$.

e. The function $a \mapsto S_\nu(\sqrt{a}, \sqrt{b})/c_{\sqrt{a}, \nu}$ is log-convex on $(0, \infty)$ for every $b > 0$ and $\nu > 1/2$.

f. The function $\nu \mapsto R_\nu(a, b)/c_{a, \nu}$ is log-convex on $(0, \infty)$ for each $b \geq 0$ and $a > 0$.

g. The function $a \mapsto R_\nu(\sqrt{a}, \sqrt{b})/c_{\sqrt{a}, \nu}$ is log-convex on $(0, \infty)$ for each $b \geq 0$ and $\nu > 1/2$. 
Part f of Theorem 2 gives the following inequality

$$R^2_{\sqrt{\frac{1}{2}+\nu^2}}(a, b) \leq \left( \frac{c_{\nu_1, \nu_2}^2}{c_{a, \nu_1}c_{a, \nu_2}} \right) R_{\nu_1}(a, b)R_{\nu_2}(a, b), \quad (2.9)$$

We also have the inequality

$$R^2_{\nu}\left(\sqrt{\frac{a_1 + a_2}{2}}, \sqrt{b}\right) \leq \left( \frac{c^2_{\sqrt{\frac{a_1 + a_2}{2}}, \nu}}{c_{\sqrt{a_1}, \nu}c_{\sqrt{a_2}, \nu}} \right) R_{\nu}(\sqrt{a_1}, \sqrt{b})R_{\nu}(\sqrt{a_2}, \sqrt{b}). \quad (2.10)$$

due to the property of the function $a \mapsto R_{\nu}(a, b)/c_{a, \nu}$ provided in part g of Theorem 2. In view of (1.3), the monotone property of the function $S_{\nu}(a, b)$ can be obtained form the monotonicity of $R_{\nu}(a, b)$, [4, Theorem 1]. Thus the function $S_{\nu}(a, b)$ is monotone increasing with respect to the parameter $a$ and $b$ but decreasing with respect to the parameter $\nu$.

**Remark 1.** Part d and e of Theorem 2 provide similar inequalities for $S_{\nu}(a, b)/c_{a, \nu}$ and $S_{\nu}(\sqrt{a}, \sqrt{b})/c_{\sqrt{a}, \nu}$ such as we have (2.9) and (2.10) respectively for $R_{\nu}(a, b)$. We also get the following inequalities which are based on log-concavity of the function $b \mapsto S_{\nu}(\sqrt{a}, \sqrt{b})$ and geometric-concavity property of the function $b \mapsto S_{\nu}(a, b)$,

$$S_{\nu}(a, \sqrt{b_1 + b_2}) < S_{\nu}(a, \sqrt{b_1})S_{\nu}(a, \sqrt{b_2}) < S^2_{\nu}\left(a, \sqrt{\frac{b_1 + b_2}{2}}\right) \quad (2.11)$$

and

$$S^2_{\nu}(a, \sqrt{b_1b_2}) \geq S_{\nu}(a, b_1)S_{\nu}(a, b_2) \quad (2.12)$$

for all $\nu > 0$ and $a > 0$ respectively. The right hand side of the inequality (2.8) is weaker than the inequality (2.12) since the function $b \mapsto S_{\nu}(a, b)$ is increasing on $(0, \infty)$ and $(b_1 + b_2)/2 \geq \sqrt{b_1b_2}$ is true for all $b_1, b_2 \geq 0$ and $\nu > 0$, while the log-concavity property gives stronger inequality than the geometric-concavity property for $R_{\nu}(a, b)$. It is worth mentioning here that the inequality (2.8) is valid only for $\nu > 3/2$ but by using the part c, we extend the range of $\nu$ from $\nu > 3/2$ to $\nu > 0$ for the inequality (2.8) since an increasing geometrically concave function is a log-concave function too.

**Remark 2.** Part c of [4, Theorem 1] says that the function $b \mapsto R_{\nu}(a, \sqrt{b})$ is log-convex on $(0, \infty)$ for all $a > 0$ and $\nu \geq 1$, and this can be improved as follows. Assume that $0 < \nu < 1$. Consider the function $f_{\nu}(t) = \frac{c_{\nu_1, \nu_2}^{\nu_1 - 1}}{2a^{\nu_1 - 1}\nu_1} t^{-\nu_1 + 1} e^{-\frac{t\nu_2}{\nu_1} - K_{\nu_1 - 1}}$. By using the same technique as we used in part e, the function $t \mapsto f_{\nu}(t)$ is log-convex for all $1 - \nu > -1/2$, i.e., $\nu < 3/2$. Hence by [3, Theorem 1], the function $b \mapsto R_{\nu}(\sqrt{a}, \sqrt{b})$ is log-convex on $(0, \infty)$ for all $a > 0$ and $\nu \leq 3/2$. Combining this with part c of [4, Theorem 1] we get the following statement: the function $b \mapsto R_{\nu}(a, \sqrt{b})$ is log-convex on $(0, \infty)$ for all $a > 0$ and $\nu \geq 0$. 


Next, our aim is to find some Turán-type inequality for \( R_\nu(a, b) \). One way of getting such type of inequality is to find the log-concavity/convexity property for the functions with respect to their parameters. Our numerical results suggest the following conjectures:

**Conjecture 1.** The function \( \nu \mapsto R_\nu(a, b) \) is log-concave on \((0, \infty)\) for all \( a > 0 \) and \( b \geq 0 \).

**Conjecture 2.** The function \( a \mapsto R_\nu(\sqrt{a}, \sqrt{b}) \) is log-convex on \((0, \infty)\) for all \( \nu > 0 \) and \( b \geq 0 \).

**Theorem 3.** Let \( R'_\nu(a, b) \) denote the differentiation of \( R_\nu(a, b) \) with respect to the parameter \( b \). Then the following assertions are true.

a. The function \( \nu \mapsto R'_\nu(a, b)/R_\nu(a, b) \) is increasing on \((0, \infty)\) for \( a > 0 \) and \( b \geq 0 \).

b. The function \( a \mapsto c_{a,\nu+1}R_\nu(a, b)/c_{a,\nu}R_{\nu+1}(a, b) \) is decreasing on \((0, \infty)\) for \( \nu > 0 \) and \( b > 0 \).

c. The function \( a \mapsto R'_\nu(a, b)/R_\nu(a, b) \) is decreasing on \((0, \infty)\) for \( \nu > 0 \) and \( b > 0 \).

d. The function \( a \mapsto c_{a,\nu+1}R_\nu(a, b)/c_{a,\nu}R_{\nu+1}(a, b) \) is increasing on \((0, \infty)\) for \( \nu > 0 \) and \( b > 0 \).

**Remark 3.** If Conjecture 1 is true then in view of the inequality (2.9) we get

\[
0 \leq R_{\nu+1}^2(a, b) - R_{\nu}(a, b)R_{\nu+2}(a, b) \leq \left(1 - \frac{c_{a,\nu}c_{a,\nu+2}}{c_{a,\nu+1}^2}\right)R_{\nu+1}^2(a, b). \tag{2.13}
\]

Consider the function

\[
\phi_\nu(a, b) := \frac{R_{\nu+1}^2(a, b) - R_{\nu}(a, b)R_{\nu+2}(a, b)}{R_{\nu+1}^2(a, b)}.
\]

Taking into consideration the asymptotic formula (2.23) we have

\[
\lim_{b \to \infty} \phi_\nu(a, b) = 1 - \frac{c_{a,\nu}c_{a,\nu+2}}{c_{a,\nu+1}^2}
\]

and it is easy to calculate

\[
\lim_{b \to 0} \phi_\nu(a, b) = 0.
\]

Thus, the inequality (2.13) with the above discussion gives the following tight bounds for \( \phi_\nu(a, b) \)

\[
\lim_{b \to 0} \phi_\nu(a, b) \leq \phi_\nu(a, b) \leq \lim_{b \to \infty} \phi_\nu(a, b).
\]

Our numerical results also suggest the following conjecture.

**Conjecture 3.** The function

\[
b \mapsto \phi_\nu(a, b) = \frac{R_{\nu+1}^2(a, b) - R_{\nu}(a, b)R_{\nu+2}(a, b)}{R_{\nu+1}^2(a, b)}
\]
is monotonic increasing on the interval \((0, \infty)\) for all \(a > 0\) and \(\nu > 0\). Conjecture 3 is also supported by Remark 3 and if this conjecture is true then the inequality (2.13) is the sharpest one and thus cannot be further improved.

2.3. Turán Type Inequality of the Marcum Q-Function

It is well-known that the generalized Marcum Q-function has the log-concavity property with respect to all the parameters. This is proved in the papers [20, 21]. With the help of log-concavity property of \(Q_{\nu}(a, b)\) we can have the Turán type inequality for the generalized Marcum Q-function. In this subsection, our aim is to find the Turán type inequality for the generalized Marcum Q-function with another method. The method is same as we used in the previous subsection to obtain the Turán type inequality for the generalized Marcum function of the second kind. Now, we recall first some important formulae related to the generalized Marcum Q-function.

The generalized Marcum Q-function is defined by [20]

\[
Q_{\nu}(a, b) = \begin{cases} 
\frac{1}{a^{\nu-1}} \int_b^{\infty} t^{\nu} e^{-\frac{a^2+t^2}{2}} I_{\nu-1}(at) dt; & a > 0 \\
\frac{1}{2^{\nu-1} \Gamma(\nu)} \int_b^{\infty} t^{2\nu-1} e^{-\frac{t^2}{2}} dt; & a = 0,
\end{cases}
\]  

(2.14)

where \(a \geq 0\), \(b, \nu > 0\) and \(I_{\nu}\) stands for the modified Bessel function of the first kind. Integrating by parts on (2.14) yields

\[
Q_{\nu}(a, b) + Q_{\nu-1}(a, b) = \left( \frac{b}{a} \right)^{\nu-1} e^{-\frac{b^2+a^2}{2}} I_{\nu-1}(ab).
\]  

(2.15)

Now, we give the main theorem of this subsection.

**Theorem 4.** Let \(Q'_{\nu}(a, b)\) denote the differentiation of \(Q_{\nu}(a, b)\) with respect to the parameter \(b\). Then the following assertions are true.

- **a.** The function \(\nu \mapsto Q'_{\nu}(a, b)/Q_{\nu}(a, b)\) is increasing on \((1, \infty)\) for \(a > 0\) and \(b > 0\).
- **b.** The function \(a \mapsto Q'_{\nu}(a, b)/Q_{\nu}(a, b)\) is increasing on \((0, \infty)\) for \(\nu > 1/2\) and \(b > 0\).

From Theorem 4 we conclude the following corollary.

**Corollary 1.** The function \(\nu \mapsto Q_{\nu-1}(a, b)/Q_{\nu}(a, b)\) is an increasing function on \((1, \infty)\) for all \(a > 0\) and \(b > 0\), while \(a \mapsto Q_{\nu-1}(a, b)/Q_{\nu}(a, b)\) is an increasing function on \((0, \infty)\) for all \(\nu > 1\) and \(b > 0\). Moreover, the Turán type inequality

\[
Q_{\nu}(a, b)Q_{\nu+2}(a, b) \leq Q_{\nu+1}^2(a, b)
\]

holds for all \(a > 0, b > 0\) and \(\nu > 0\).

**Proof.** Re-write the recurrence relation (2.15) as

\[
1 - \frac{Q_{\nu-1}(a, b)}{Q_{\nu}(a, b)} = - \frac{Q'_{\nu}(a, b)}{bQ_{\nu}(a, b)} \quad \text{for} \quad \nu > 1.
\]
This implies that the functions $Q_{\nu-1}(a,b)/Q_{\nu}(a,b)$ and $Q'_{\nu}(a,b)/Q_{\nu}(a,b)$ have the same monotonic property with respect to $\nu$ as well as $a$. Hence the result follows. Furthermore, the monotonicity of the function $\nu \mapsto Q_{\nu}(a,b)/Q_{\nu+1}(a,b)$ gives the desired inequality. □

2.4. Some New Bounds for the Generalized Marcum Function of the Second Kind

In [4,5], we have found various upper and lower bounds for generalized Marcum function of the second kind. These bounds were obtained by using the monotonicity of functions which involve the modified Bessel function of the second kind. In [4, Theorem 8], we found a uniform upper bound for the generalized Marcum function of the second kind, which is given by

$$R_{\nu}(a,b) \leq \frac{\Gamma(\nu,b^2)}{\Gamma(\nu)} = Q_{\nu}(0,b).$$

Motivated by this result our main focus in this subsection is to obtain a uniform upper and lower bound for the generalized Marcum function of the second kind. But first we provide some new bounds for the Marcum and generalized Marcum function of the second kind.

In view of [7, Corollary 3.4]

$$\sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} < K_0(x) < \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}},$$

we conclude the following bounds for the Marcum function of the second kind

$$\sqrt{\frac{\pi}{2}} c_{a,1} \int_b^{\infty} \frac{t}{\sqrt{at + \frac{1}{2}}} e^{-\frac{(a+t)^2}{2}} dt < R_1(a,b) < \sqrt{\frac{\pi}{2}} c_{a,1} \int_b^{\infty} \sqrt{t} e^{-\frac{(a+t)^2}{2}} dt. \quad (2.17)$$

In view of part g of [4, Theorem 1] the function $\nu \mapsto R_{\nu}(a,b)$ is increasing on $(0, \infty)$ for all $a > 0$ and $b \geq 0$ and a well-known result is that $a \mapsto Q_{\nu}(a,b)$ is increasing on $(0, \infty)$, see [21, Theorem 1]. Thus, the inequalities (2.16) and (2.17) give the following

$$\sqrt{\frac{\pi}{2}} c_{a,1} \int_b^{\infty} \frac{t}{\sqrt{at + \frac{1}{2}}} e^{-\frac{(a+t)^2}{2}} dt < R_1(a,b) < R_{\nu}(a,b) < Q_{\nu}(0,b) < Q_{\nu}(a,b). \quad (2.18)$$

Next, by using the monotonicity of the function $t \mapsto e^{-\frac{t^2}{2}}$ and the differential formula [18, 10.29.4]

$$\frac{\partial}{\partial x}(x^\nu K_{\nu}(x)) = -x^\nu K_{\nu-1}(x) \quad (2.19)$$

in Eq. (1.1), we get the following upper bound

$$R_{\nu}(a,b) \leq c_{a,\nu} \left( \frac{b}{a} \right)^\nu e^{-\frac{a^2+b^2}{2}} K_{\nu}(ab). \quad (2.20)$$
Finally, we derive a uniform upper and lower bound for the generalized Marcum function of the second kind. For this, we again use part $g$ of [4, Theorem 1] in the recurrence relation [18, Eq. (2.7)]

$$R_\nu(a, b) + \frac{c_{a,\nu}}{c_{a,\nu-1}} R_{\nu-1}(a, b) = c_{a,\nu} \left( \frac{b}{a} \right)^{\nu-1} K_{\nu-1}(ab) e^{-\frac{a^2+b^2}{2}}, \quad \text{where} \quad \nu > 1.$$  

(2.21)

Then we have

$$R_\nu(a, b) \left( \frac{1}{c_{a,\nu-1}} + \frac{1}{c_{a,\nu}} \right) \geq \left( \frac{b}{a} \right)^{\nu-1} K_{\nu-1}(ab) e^{-\frac{a^2+b^2}{2}}.$$  

By using

$$\frac{1}{c_{a,\nu}} + \frac{1}{c_{a,\nu+1}} = \frac{\Gamma\nu}{2} e^{-\frac{a^2}{2}} \left( \frac{a^2}{2} \right)^{-\nu},$$

we have

$$R_\nu(a, b) \geq 2e^{-\frac{a^2}{2}} \left( \frac{ab}{2} \right)^{\nu-1} \frac{K_{\nu-1}(ab)}{\Gamma(\nu - 1)}.$$  

Similarly, we can find that

$$R_\nu(a, b) \leq 2e^{-\frac{b^2}{2}} \left( \frac{ab}{2} \right)^{\nu} \frac{K_{\nu}(ab)}{\Gamma(\nu)}.$$  

Hence we have the following uniform upper and lower bound

$$2e^{-\frac{b^2}{2}} \left( \frac{ab}{2} \right)^{\nu-1} \frac{K_{\nu-1}(ab)}{\Gamma(\nu - 1)} \leq R_\nu(a, b) \leq 2e^{-\frac{a^2}{2}} \left( \frac{ab}{2} \right)^{\nu} \frac{K_{\nu}(ab)}{\Gamma(\nu)}.$$  

(2.22)

for all $\nu > 1$, while the right hand side is valid for $\nu > 0$. In view asymptotic formulas [18, 10.25.3]

$$K_{\nu}(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$$  

(2.23)

and [18, 10.30.2]

$$K_{\nu}(x) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{x}{2} \right)^{-\nu} \quad \text{where} \quad \nu > 0,$$  

(2.24)

we can conclude that the bounds obtained in (2.22) are tight when $b \to \infty$ as well as $b \to 0$.

2.5. Bounds for the Symmetric Difference of the Generalized Marcum Function of the Second Kind

In this subsection, we discuss the symmetric difference of generalized Marcum function of the second kind. This work is motivated by the results on the symmetric difference of the generalized Marcum $Q$-function. The symmetric difference of the generalized Marcum $Q$-function is defined as

$$\Delta Q_\nu(a, b) = Q_\nu(a, b) - Q_\nu(b, a),$$
and this function has been studied by various authors, see [6,11] and references therein, due to its use in digital communication. Since the function $Q_\nu(a,b)$ has opposite monotonicity property with respect to the parameter $a$ and $b$, the function $a \mapsto \Delta Q_\nu(a,b)$ is increasing on $(0,\infty)$, while the function $b \mapsto \Delta Q_\nu(a,b)$ is decreasing on $(0,\infty)$. In a similar manner, we define the symmetric difference of the generalized Marcum function of the second kind

$$\Delta R_\nu(a,b) = R_\nu(a,b) - R_\nu(b,a).$$

Unfortunately, the study of the symmetric difference of generalized Marcum function of the second kind is more complicated than the study of symmetric difference of Marcum $Q$-function because they do not have the same monotonic behavior throughout the interval $(0,\infty)$. All the same, it is easy to find bounds for the function $\Delta R_\nu(a,b)$, which gives a certain idea about its value. We use the bounds of the generalized Marcum function of the second kind obtained in [4,5] to produce bounds for the symmetric difference of the generalized Marcum function of the second kind. For convenience, we provide various existing upper and lower bounds for the generalized Marcum function of the second kind.

**Upper bounds:** [4, Eq. 2.17] If $\nu \geq 1/2$ and $a > b > 0$, then

$$R_\nu(a,b) \leq 1 - \frac{c_{a,\nu}}{(ab)^{\nu-1}} K_{\nu-1}(ab) e^{ab} \int_a^{b+a} (y-a)^{2\nu-1} e^{-\frac{y^2}{2}} dy. \quad (2.25)$$

[5, Eq. 2.25] If $\nu \geq 3/2$ and $a > b > 0$, then

$$R_\nu(a,b) \leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \int_a^{b+a} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy. \quad (2.26)$$

[4, Eq. 2.10] If $\nu \geq 1/2$ and $b \geq a > 0$, then

$$R_\nu(a,b) \leq \frac{c_{a,\nu}}{(ab)^{\nu-1}} K_{\nu-1}(ab) e^{ab} \int_{b+a}^{\infty} (y-a)^{2\nu-1} e^{-\frac{y^2}{2}} dy. \quad (2.27)$$

[5, Eq. 2.17] If $\nu \geq 3/2$ and $b \geq a > 0$, then

$$R_\nu(a,b) \leq \frac{c_{a,\nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \int_{b+a}^{\infty} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy. \quad (2.28)$$

**Lower bounds:** [4, Eq. 2.22] If $\nu \geq 1$ and $b \geq a > 0$, then

$$R_\nu(a,b) \geq c_{a,\nu} \left( \frac{b}{a} \right)^{\nu-1} b e^{ab} K_{\nu-1}(ab) \sqrt{\frac{\pi}{2}} \text{erfc} \left( \frac{b+a}{\sqrt{2}} \right). \quad (2.29)$$

[4, Eq. 2.11] If $\nu \geq 3/2$ and $b \geq a > 0$, then

$$R_\nu(a,b) \geq c_{a,\nu} \left( \frac{b}{a} \right)^{\nu-1} e^{ab} K_{\nu-1}(ab) \left[ e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \text{erfc} \left( \frac{b+a}{\sqrt{2}} \right) \right]. \quad (2.30)$$
If $\nu \geq 1$ and $a > b > 0$, then
\[
R_{\nu}(a, b) \geq 1 - c_{a, \nu} \left( \frac{b}{a} \right)^{\nu-1} be^{ab} K_{\nu-1}(ab) \sqrt{\frac{\pi}{2}} \left[ \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b + a}{\sqrt{2}} \right) \right].
\] (2.31)

If $\nu \geq 3/2$ and $a > b > 0$, then
\[
R_{\nu}(a, b) \geq 1 - c_{a, \nu} \left( \frac{b}{a} \right)^{\nu-1} e^{ab} K_{\nu-1}(ab) \left[ e^{-\frac{a^2}{2}} - e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \left( \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b + a}{\sqrt{2}} \right) \right) \right].
\] (2.32)

From the above mentioned bounds, we can find the bounds for the symmetric difference of $R_{\nu}(a, b)$. It is evident from the definition of symmetric difference that $\Delta R_{\nu}(a, b) = -\Delta R_{\nu}(b, a)$. Therefore, we only supply bounds for the case $a > b > 0$, while the other case $b > a > 0$ will follow by the above relation. It is clear that for $\nu \geq 1$, the upper bound for $\Delta R_{\nu}(a, b)$ is obtained by subtracting (2.29) from (2.25)
\[
\Delta R_{\nu}(a, b) \leq 1 - c_{a, \nu} \left( \frac{a}{b} \right)^{\nu-1} K_{\nu-1}(ab) e^{ab} \int_a^{b+a} (y - a)^{2\nu-1} e^{-\frac{y^2}{2}} dy - c_{b, \nu} \left( \frac{a}{b} \right)^{\nu-1} a e^{ab} K_{\nu-1}(ab) \sqrt{\frac{\pi}{2}} \text{erfc} \left( \frac{b + a}{\sqrt{2}} \right). \] (2.33)

Another upper bound is as follows
\[
\Delta R_{\nu}(a, b) \leq 1 - \frac{c_{a, \nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \int_a^{b+a} (y - a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy - c_{b, \nu} \left( \frac{a}{b} \right)^{\nu-1} e^{ab} K_{\nu-1}(ab) \left[ e^{-\frac{(b+a)^2}{2}} - b \sqrt{\frac{\pi}{2}} \text{erfc} \left( \frac{b + a}{\sqrt{2}} \right) \right],
\] (2.34)

which can be obtained by subtracting (2.30) from (2.26), but this bound is valid only for $\nu \geq 3/2$. However, it is much tighter than the bound obtained in (2.33). This conclusion can be drawn directly from the discussions provided in [4, Subsection 2.3] and [5, Subsection 3.1].

The lower bound for $\Delta R_{\nu}(a, b)$ when $\nu \geq 1$ is
\[
\Delta R_{\nu}(a, b) \geq 1 - c_{a, \nu} \left( \frac{b}{a} \right)^{\nu-1} be^{ab} K_{\nu-1}(ab) \sqrt{\frac{\pi}{2}} \left[ \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b + a}{\sqrt{2}} \right) \right] - \frac{c_{b, \nu}}{(ab)^{\nu-1}} K_{\nu-1}(ab) e^{ab} \int_{b+a}^{\infty} (y - b)^{2\nu-1} e^{-\frac{y^2}{2}} dy.
\] (2.35)
which is obtained by subtracting (2.27) from (2.31). From (2.32) and (2.28), we can find another lower bound

\[
\Delta R_\nu(a, b) \geq 1 - c_{a, \nu} \left( \frac{b}{a} \right)^{\nu - 1} e^{ab} K_{\nu - 1}(ab) \\
\left[ e^{-\frac{a^2}{2}} - e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \left( \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b+a}{\sqrt{2}} \right) \right) \right] \\
- \frac{c_{b, \nu}}{b^{\nu - 1}} \sqrt{a} K_{\nu - 1}(ab) e^{ab} \int_{b+a}^{\infty} (y-b)^{\nu - \frac{1}{2}} e^{-\frac{y^2}{2}} dy,
\]

(2.36)

which is valid for \( \nu \geq 3/2 \). Discussions provided in [4, Subsection 2.3] and [5, Subsection 3.1] imply that the bound given in (2.36) is much sharper than the bound obtained in (2.35).

3. Proof of Main Results

Here, first we give some important lemmas which will be useful to prove our main results.

**Lemma 1.** [15, pp. 22] If \( F \) is convex on \([0, \infty)\), then we have for \( x_1, \ldots, x_n > 0 \):

\[
F(x_1) + \cdots + F(x_n) \leq F(x_1 + \cdots + x_n) + (n-1)F(0).
\]

If \( F \) is concave on \([0, \infty)\), then the reversed inequality holds.

The following lemma is a useful tool to achieve the inequalities which are given in Theorem 1.

**Lemma 2.** The following are true:

a. The function \( b \mapsto R_\nu(a, b) \) is log-convex on \([0, \infty)\) for each \( a > 0 \) and \( \nu > 0 \).

b. The function \( b \mapsto R_\nu(a, b^{1/a}) \) is log-convex on \([0, \infty)\) for each \( a \geq 1 \) and \( \nu > 0 \).

**Proof.**

a. This follows from part c of [4, Theorem 1] and Remark 2.

b. Since \( R_\nu(a, b) = A_\nu(a, b)/A_\nu(a, 0) \), the property with respect to \( b \) of \( R_\nu(a, b) \) can be verified from the function \( A_\nu(a, b) \). By some easy calculation we have

\[
A_\nu(a, b^{1/a}) = \frac{1}{a} \int_{b}^{\infty} u^{\frac{1}{a} - 1} e^{-\frac{u^{1/a}}{2}} u^{\frac{\nu - 1}{2a}} K_{\nu - 1}(\sqrt{au^{1/a}}) du.
\]

It is clear that

\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\frac{1}{a} - 1} e^{-\frac{u^{1/a}}{2}} \right) \right) = - \left( \frac{1}{a} - 1 \right) \frac{1}{u^2} - \frac{1}{2a} \left( \frac{1}{a} - 1 \right) u^{\frac{1}{a} - 2} \geq 0
\]
for all \( a \geq 1 \). The differentiation formula (2.19) leads to
\[
\frac{\partial}{\partial x} \left( x^{\nu-1} K_{\nu-1} \left( \sqrt{ax^2} \right) \right) = - \frac{1}{2\sqrt{a}} x^{\frac{1}{2} - 1} x^{\nu-2} K_{\nu-2}(\sqrt{ax^2}).
\]
Now
\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\nu-1} K_{\nu-1} \left( \sqrt{au^2} \right) \right) \right) = - \frac{1}{2\sqrt{a}} u^{\frac{1}{2} - 1} \frac{K_{\nu-2}(\sqrt{au^2})}{K_{\nu-1}(\sqrt{au^2})}.
\]
(3.1)
Further differentiation gives
\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\nu-1} K_{\nu-1} \left( \sqrt{au^2} \right) \right) \right) = - \left( u^{\frac{1}{2} - 1} \frac{\partial}{\partial u} \left( \frac{K_{\nu-2}(\sqrt{au^2})}{\sqrt{au^2} K_{\nu-1}(\sqrt{au^2})} \right) \right) + \left( \frac{1}{a} - 1 \right) u^{\frac{1}{2} - 1} \frac{K_{\nu-2}(\sqrt{au^2})}{\sqrt{au^2} K_{\nu-1}(\sqrt{au^2})}.
\]
(3.2)
In view of the result [9, p. 583] for \( t > 0 \) and \( \nu \geq 0 \),
\[
\frac{K_{\nu-1}(\sqrt{t})}{\sqrt{t} K_{\nu}(\sqrt{t})} = \frac{4}{\pi^2} \int_0^\infty \frac{\gamma_\nu(z)}{t + z^2} dz, \quad \text{where} \quad \gamma_\nu(z) = \frac{z^{-1}}{J_\nu(z) + Y_\nu(z)},
\]
\( J_\nu \) and \( Y_\nu \) stand for the Bessel function of the first and second kind, respectively. Eq. (3.2) leads to
\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\nu-1} K_{\nu-1}(\sqrt{au^2}) \right) \right) > 0 \quad \text{for all} \quad a \geq 1 \quad \text{and} \quad \nu \geq 1.
\]
For the case when \( \nu < 1 \), Eq. (3.1) reduces to
\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\nu-1} K_{\nu-1}(\sqrt{au^2}) \right) \right) = - \frac{1}{2\sqrt{a}} u^{\frac{1}{2} - 1} \frac{K_{2-\nu}(\sqrt{au^2})}{K_{1-\nu}(\sqrt{au^2})}.
\]
In view of Lemma 3 the above equation gives
\[
\frac{\partial^2}{\partial u^2} \left( \log \left( u^{\nu-1} K_{\nu-1}(\sqrt{au^2}) \right) \right) \geq 0
\]
when \( a > 1 \) and \( \nu < \frac{3}{2} \). Hence the function \( u \mapsto \log(u^{\nu-1} K_{\nu-1}(\sqrt{au^2})) \) is log-convex on \((0, \infty)\) when \( a \geq 1 \) and \( \nu > 0 \). Since the log-convexity property remains invariant after integration, \( A_\nu(a, b^{1/a}) \) is log-convex for all \( a \geq 1 \) and \( \nu > 0 \).

**Proof of Theorem 1.** a. Part a of Lemmas 2 and 1 together show
\[
\mathcal{R}_\nu(a, x_1) \cdots \mathcal{R}_\nu(a, x_n) \leq \mathcal{R}_\nu(a, x_1 + \cdots + x_n).
\]
Since the functions \( t \mapsto S_t(x_1, \ldots, x_n) \) is decreasing on \((0, \infty)\) (see [8, p. 28]), the function \( t \mapsto R_\nu(a, S_t(x_1, \ldots, x_n)) \) is increasing on \((0, \infty)\). Thus, we have

\[ R_\nu(a, x_1 + \cdots + x_n) \leq R_\nu(a, S_p(x_1, \ldots, x_n)) \] for all \( p \geq 1 \).

Thus the result holds. Conversely, let the inequality (2.1) hold. If we put \( x_1 = x_2 = x \) and \( x_3 = \cdots = x_n = 0 \), then the inequality (2.1) reduces to

\[ (R_\nu(a, x))^2 \leq R_\nu(a, 2^{1/p} x). \] (3.3)

Consider

\[ \psi_\nu(a, x) = R_\nu(a, cx) - (R_\nu(a, x))^2 \text{ where } c > 1. \]

Then

\[ \psi_\nu(a, 0) = 0 \]

and

\[ A_\nu(a, 0) \psi_\nu'(a, x) = 2x^{\nu+1} R_\nu(a, x)e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax}) - c(cx)^{\nu+1} e^{-\frac{cx}{2}} K_{\nu-1}(\sqrt{cax}). \]

In view of (2.24), for \( \nu > 1 \) we have

\[ \lim_{x \to 0} A_\nu(a, 0) \psi_\nu'(a, x) = \frac{(2 - c)}{2} \Gamma(\nu - 1) \left( \frac{\sqrt{a}}{2} \right)^{\nu+1} \]

\[ \begin{cases} > 0, & \text{if } c < 2 \\ < 0, & \text{if } c > 2. \end{cases} \]

In view of the monotone decreasing property of the function \( x \mapsto K_\nu(x) \) on \((0, \infty)\) for all \( \nu > 0 \) [18, 10.37] and the asymptotic formula \( K_0(x) \sim -\log x \) [18, 10.32.3] we get the similar inequalities as the above one for \( \nu = 1 \). Suppose \( p < 1 \). Then \( c = 2^{\frac{1}{p}} > 2 \) and hence \( x \mapsto \psi_\nu(a, x) \) is decreasing near \( x = 0 \). Thus we get \( (R_\nu(a, x))^2 \geq R_\nu(a, 2^{1/p} x) \) near \( x = 0 \), a contradiction to (3.3). This completes the proof.

b. It is well-known that the power mean \( t \mapsto M_t(x_1, \ldots, x_n) \) is an increasing function on \((0, \infty)\), see [8, p. 26]. This implies that the function \( t \mapsto R_\nu(a, M_t(x_1, \ldots, x_n)) \) is decreasing on \((0, \infty)\). Thus we have

\[ R_\nu(a, M_r(x_1, \ldots, x_n)) \leq R_\nu(a, M_1(x_1, \ldots, x_n)) \]

for all \( r \geq 1 \). Now in view of above inequality, it is enough to prove the left hand inequality of (2.2) only for \( r = 1 \). Using part b of Lemma 2 and the fact that every log-convex function is also a convex function, we have

\[ R_\nu(a, M_1(x_1, \ldots, x_n)) = R_\nu \left( a, \left( \frac{x_1 + \cdots + x_n}{n} \right) \right) \]

\[ \leq \frac{R_\nu(a, x_1) + \cdots + R_\nu(a, x_n)}{n}. \]
Since the function $x \mapsto \mathcal{R}_\nu(a, x)$ is a decreasing function on $(0, \infty)$, for all $i = 1, 2, \ldots, n$
\[ \mathcal{R}_\nu(a, x_i) \leq \mathcal{R}_\nu(a, \mathcal{M}_s(x_1, \ldots, x_n)) \quad \text{when} \quad s = -\infty. \]

By adding these inequalities, we get the right-hand side of the inequality (2.2).

Assume that the left-hand side of (2.2) is true for any positive real numbers $x_1, x_2, \ldots, x_n$. Then for $x, y > 0$, we have
\[ 0 \leq \mathcal{R}_\nu(a, x) + (n - 1)\mathcal{R}_\nu(a, y) - n\mathcal{R}_\nu(a, \mathcal{M}_r(x, y, \ldots, y)) = V_{a, r}(x, y). \]

It is evident that
\[ V_{a, r}(y, y) = 0 \quad \text{and} \quad \left( \frac{\partial}{\partial x} V_{a, r}(x, y) \right)_{x=y} = 0. \]

This implies
\[
\left( \frac{\partial^2}{\partial x^2} V_{a, r}(x, y) \right)_{x=y} = \frac{e^{-\frac{y}{2}}}{A_\nu(a, 0)} \left( 1 - \frac{1}{n} \right) y^{\nu-3} K_{\nu-1}(\sqrt{ay}) \left( \frac{\sqrt{ay} K_{\nu-2}(\sqrt{ay})}{2 K_{\nu-1}(\sqrt{ay})} + \frac{y}{2} + r - 1 \right) > 0. \tag{3.4}
\]

In view of (2.24) and [18, 10.31.3]
\[ K_0(x) \sim -\log x, \]

we have
\[ \lim_{y \to 0} \left( \frac{\sqrt{ay} K_{\nu-2}(\sqrt{ay})}{2 K_{\nu-1}(\sqrt{ay})} + \frac{y}{2} \right) = 0 \quad \text{for all} \quad \nu \geq 1. \tag{3.5}\]

Now take into consideration a well-known result from real analysis (that is, if $a + \epsilon > 0$ for all $\epsilon > 0$, then $a \geq 0$), and then Eqs. (3.4) and (3.5) together give $r \geq 1$ for all $\nu > 2$. This leads us to $r \geq 1$. Now for the case $0 < \nu < 1$, we rewrite the Eq. (3.4)
\[
\left( \frac{\partial^2}{\partial x^2} V_{a, r}(x, y) \right)_{x=y} = \frac{e^{-\frac{y}{2}}}{A_\nu(a, 0)} \left( 1 - \frac{1}{n} \right) y^{\nu-3} K_{\nu-1}(\sqrt{ay}) \left( \frac{ay K_{\nu-2}(\sqrt{ay})}{2 K_{\nu-1}(\sqrt{ay})} + (\sqrt{ay})^2 + (\sqrt{ay})(r - 1) \right) > 0. \tag{3.6}
\]

By using a similar argument as we used above, we get $r \geq 1$ since
\[ \lim_{y \to 0} \left( \frac{ay K_{\nu-2}(\sqrt{ay})}{2 K_{\nu-1}(\sqrt{ay})} + ay y \right) = 0. \]

Suppose that there exists a real number $s$ such that the right-hand side of (2.2) is satisfied for all $x_1, \ldots, x_n > 0$. Consider $s \geq 0$. If $x_1 \to \infty$, then clearly right hand inequality of (2.2) does not hold. Consider $s < 0$. Let us take $x_1 = x, x_2 = \cdots = x_n = y$ and allowing $y$ tends to $\infty$, then for all $x > 0$
\[ \phi_\nu(a, x) := n\mathcal{R}_\nu(a, cx) - \mathcal{R}_\nu(a, x) \geq 0 \quad \text{where} \quad c = n^{-\frac{1}{\nu}}. \]
By using the fact that the function \( t \mapsto K_\nu(t) \) is decreasing on \((0, \infty)\), we have
\[
\phi'_\nu(a, x) \geq \frac{K_{\nu-1}(\sqrt{ax})}{A_\nu(a, 0)} x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax}) (1 - nc^{\frac{\nu+1}{2}} e^{\frac{x}{2}(1-c)}).
\]
As we know \( c = n^{-\frac{1}{2}} > 1 \), therefore for all \( x > x^* \) we have
\[
\phi'_\nu(a, x) > 0,
\]
where \( x^* \) is the unique root of the equation
\[
1 - nc^{\frac{\nu+1}{2}} e^{\frac{x}{2}(1-c)} = 0.
\]
This shows that \( \phi_\nu(a, x) \) is an increasing function on \((x^*, \infty)\) and hence
\[
\phi_\nu(a, x) < \lim_{x \to \infty} \phi_\nu(a, x) = 0,
\]
which is a contradiction.

c. From part a of Lemma 2 and the fact every log-convex function is convex, we have
\[
R_\nu(a, M_1(x_1, \ldots, x_n)) \leq (R_\nu(a, x_1) \ldots R_\nu(a, x_n))^{1/n}.
\]
Since the power mean is an increasing function with respect to its order [8, p. 26],
\[
R_\nu(a, M_r(x, y, \ldots, y)) \leq (R_\nu(a, x_1) \ldots R_\nu(a, x_n))^{1/n} \quad \text{where} \quad r \geq 1.
\]
Conversely, suppose that the inequality (2.3) is true for all \( x_1, \ldots, x_n \). Then we have
\[
(A_\nu(a, M_r(x_1, \ldots, x_n)))^n \leq A_\nu(a, x_1) \ldots A_\nu(a, x_n).
\]
Put \( x_1 = x, x_2 = \cdots = x_n = y \), then the above inequality can be written as
\[
\left( \frac{A_\nu(a, M_r)}{(M_r)^{\frac{\nu-1}{2}} e^{-\frac{M_r}{2}} K_{\nu-1}(\sqrt{aM_r})} \right)^n \leq \frac{A_\nu(a, x)}{x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax})} \frac{x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax})}{((M_r)^{\frac{\nu-1}{2}} e^{-\frac{M_r}{2}} K_{\nu-1}(\sqrt{aM_r}))^n} (A_\nu(a, y))^{n-1} \quad (3.7)
\]
where, \( M_r \) denotes \( M_r(x, y, \ldots, y) \). In view of asymptotic formula (2.23) and \( r < 1 \) we have
\[
\lim_{x \to \infty} \left( \frac{A_\nu(a, x)}{x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax})} \right) = 2
\]
and
\[
\lim_{x \to \infty} \frac{x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}} K_{\nu-1}(\sqrt{ax})}{((M_r)^{\frac{\nu-1}{2}} e^{-\frac{M_r}{2}} K_{\nu-1}(\sqrt{aM_r}))^n} = 0.
\]
The above two limits with inequality (3.7) give a contradiction. Hence \( r \geq 1 \).
d. For \( i = 1, 2, \ldots, n \), we have
\[
\frac{1}{R_\nu(a, x_i)} \leq \max_{1 \leq i \leq n} \frac{1}{R_\nu(a, x_i)}.
\]
This gives us
\[
\frac{1}{n} \left( \frac{1}{R_\nu(a, x_1)} + \cdots + \frac{1}{R_\nu(a, x_n)} \right) \leq \max_{1 \leq i \leq n} \frac{1}{R_\nu(a, x_i)}.
\]
Now, as the function \( t \mapsto 1/R_\nu(a, t) \) is an increasing function on \((0, \infty)\), we get
\[
\max_{1 \leq i \leq n} \frac{1}{R_\nu(a, x_i)} = \frac{1}{R_\nu(a, \max(x_1, \ldots, x_n))} = \frac{1}{R_\nu(a, \mathcal{M}_\infty(x_1, \ldots, x_n))}.
\]
Hence the inequality (2.4) holds. Now, to complete the proof we need to show that \( s = \infty \). Suppose that there exists a real number \( s \) such that the inequality (2.4) holds true. From the geometric-harmonic mean inequality
\[
\frac{1}{n} \left( \frac{1}{R_\nu(a, x_1)} + \cdots + \frac{1}{R_\nu(a, x_n)} \right) \leq \left( \frac{R_\nu(a, x_1) \cdots R_\nu(a, x_n)}{n} \right)^{1/n},
\]
and the inequality (2.4) together imply
\[
R_\nu(a, \mathcal{M}_s(x_1, \ldots, x_n)) \leq \left( R_\nu(a, x_1) \cdots R_\nu(a, x_n) \right)^{1/n}.
\]
Therefore by part c we have \( s \geq 1 \). Now set \( x_1 = x, x_2 = \cdots = x_n = y \) and allowing \( y \) tends to 0, then the inequality (2.4) implies
\[
(n - 1)R_\nu(a, cx) + \frac{R_\nu(a, cx)}{R_\nu(a, x)} \leq n, \quad \text{where} \quad c = n^{-\frac{1}{s}}. \tag{3.8}
\]
Taking into consideration the asymptotic formula (2.23) of the modified Bessel function of the second kind, we have
\[
\lim_{x \to \infty} \frac{R_\nu(a, \alpha x)}{R_\nu(a, x)} = \begin{cases} \infty, & \text{if } 0 < \alpha < 1 \\ 0, & \text{if } \alpha > 1 \\ 1, & \text{if } \alpha = 1 \end{cases}. \tag{3.9}
\]
Since \( c = n^{-1/s} < 1 \), then (3.9) gives
\[
\lim_{x \to \infty} \frac{R_\nu(a, cx)}{R_\nu(a, x)} = \infty,
\]
which is a contradiction for (3.8). Therefore, \( s \) is not a real number. Hence \( s = \infty \).

e. Part b of Lemmas 2 and 1 imply the inequality (2.5). The bounds are sharpest since if we take all \( x_i \) tend to \( \infty \) then we get value zero and if we set \( x_1 = x, x_2 = \cdots = x_n = y \) and taking \( y \) tends to zero then we get the value \( n - 1 \).

f. It is clear that
\[
\frac{d}{dx} \left( (R_\nu(a, x^p))^q \right) = -pq \frac{(R_\nu(a, x^p))^{q-1}}{A_\nu(a, 0)} x^{p-1} \left( x^p \right)^{\nu+1} e^{-x^p} K_{\nu-1}(\sqrt{ax^p})).
\]
If $pq > 0$, then it is decreasing. Hence inequality (2.6) follows. Suppose that inequality (2.6) is valid for all $x, y \geq 0$. If $x = y$ then
\[
\left( \frac{\mathcal{R}_\nu(a, 2^p x^p)}{\mathcal{R}_\nu(a, x^p)} \right)^q < 2.
\] (3.10)

Now if $p > 0$, then Eq. (3.9) implies
\[
\lim_{x \to \infty} \frac{\mathcal{R}_\nu(a, 2^p x^p)}{\mathcal{R}_\nu(a, x^p)} = 0.
\]
The above result together with inequality (3.10) give $q > 0$. Again, if $p < 0$, then Eq. (3.9) implies
\[
\lim_{x \to \infty} \frac{\mathcal{R}_\nu(a, 2^p x^p)}{\mathcal{R}_\nu(a, x^p)} = \infty.
\]
The above result together with inequality (3.10) give $q < 0$. This completes the proof.

g. Consider a function
\[
\gamma_\nu(a, x, y) = 1 + \mathcal{R}_\nu(a, \sqrt{x^2 + y^2}) - \mathcal{R}_\nu(a, x) - \mathcal{R}_\nu(a, y)
\] and
\[
\delta_\nu(a, t) = x^{\nu - \frac{3}{2}} e^{-\frac{\nu}{2} K_{\nu - 1}(\sqrt{ax})}.
\]
Then
\[
\frac{\partial}{\partial x} (\gamma_\nu(a, x, y)) = \frac{1}{A_\nu(a, 0)} x (\delta_\nu(a, x) - \delta_\nu(a, \sqrt{x^2 + y^2})).
\] (3.11)

Since the function $x \mapsto \delta_\nu(x, y)$ is a decreasing function on $(0, \infty)$, the Eq. (3.11) imply
\[
\frac{\partial}{\partial x} (\gamma_\nu(a, x, y)) \geq 0.
\]
Therefore, the function $\gamma_\nu(x, y)$ is increasing and hence
\[
\gamma_\nu(x, y) \geq \gamma_\nu(0, y).
\]
This gives the required result.

h. A function $f : [0, \infty) \mapsto R$ is called completely monotonic, if $f$ is continuous on $[0, \infty)$ and satisfies
\[
(-1)^n f^{(n)}(x) \geq 0 \quad (x > 0, n = 0, 1, 2, \ldots).
\]
On differentiating $\mathcal{R}_\nu(a, x)$ with respect to $x$, we have
\[
A_\nu(a, 0) \mathcal{R}_\nu'(a, x) = -(x)^{\nu - \frac{1}{2}} e^{-\frac{\nu}{2} K_{\nu - 1}(\sqrt{ax})}.
\] (3.12)
In view of the differentiation formula (3.13), we have
\[
\frac{\partial^n}{\partial x^n} (x^{\nu - \frac{1}{2}} K_{\nu - 1}(\sqrt{ax})) = (-1)^n \left( \frac{\sqrt{a}}{2} \right)^n x^{\nu - \frac{n-1}{2}} K_{\nu - n - 1}(\sqrt{at}).
\]
The Eq. (3.12) with the above relation show that \(-\mathcal{R}_\nu'(a, x)\) is the product of two completely monotonic functions. Hence \(x \mapsto \mathcal{R}_\nu(a, x)\) is completely monotonic on \((0, \infty)\).

**Lemma 3.** [10, Lemma 2.4] For each fixed \(\beta > 0\) and each fixed \(\nu > -\beta/2\), the function \(K_{\nu+\beta}(x)/K_\nu(x)\) decreases to 1 as \(x\) increases from 0 to \(\infty\).

**Lemma 4.** [4, Lemma 3] For any \(0 < x < y\), the following assertions are valid:

a. The function \(x \mapsto K_\nu(x)/K_\nu(y)\) is increasing on \((0, \infty)\).

b. The function \(x \mapsto x^\beta K_{\nu+\beta}(x)/K_\nu(x)\) is increasing on \((0, \infty)\) for any \(\beta > 0\) and \(\nu \in \mathbb{R}\). Moreover, the function \(x \mapsto (x/y)^\nu K_\nu(x)/K_\nu(y)\) is decreasing on \(\mathbb{R}\).

Now, we continue with the proof of Theorem 2.

**Proof of Theorem 2.**

**a.** By part a of [4, Lemma 2], the function \(t \mapsto t^{\nu-1}K_{\nu-1}(at)\) is log-concave on \((0, \infty)\) for all \(a > 0\) and \(\nu > 3/2\). It is quite clear that the function \(t \mapsto t^{\nu}e^{-\frac{t^2}{\nu}}\) is a log-concave function on \((0, \infty)\). Thus the product of these two functions, i.e., the function \(t \mapsto t^{\nu}e^{-\frac{t^2}{\nu}}K_{\nu-1}(at)\) is also a log-concave function on \((0, \infty)\) for \(a > 0\) and \(\nu > 3/2\) as the product of two log-concave functions is again a log-concave function. Therefore, the probability density function

\[
t \mapsto \frac{c_{a, \nu}}{a^{\nu-1}} t^{\nu-1}e^{-\frac{t^2}{\nu}} K_{\nu-1}(at)
\]

is log-concave on \((0, \infty)\) for all \(\nu > 3/2\). Thus, in view of [3, Theorem 1] the cumulative distributive function of a log-concave probability density function is log-concave, the cumulative distributive function i.e., \(S_\nu(a, b)\) is log-concave on \((0, \infty)\) for all \(\nu > 3/2\) and \(a > 0\). Furthermore, the inequality (2.8) follows by Lemma 1 and the log-concave property of the function \(b \mapsto S_\nu(a, b)\).

**b.** On replacing \(a\) and \(b\) by \(\sqrt{a}\) and \(\sqrt{b}\) respectively, in the Eq. (1.3), we have

\[
S_\nu(\sqrt{a}, \sqrt{b}) = \frac{1}{\Gamma(\nu)} \frac{1}{\Gamma(1-\nu, \frac{\sqrt{a}}{2})} \int_0^b t^{\nu-1} e^{-\frac{t^2}{\nu}} K_{\nu-1}(\sqrt{at}) dt.
\]

Formula (2.19) gives

\[
\frac{\partial}{\partial t} (t^{\nu-1} K_{\nu-1}(\sqrt{at})) = -\frac{\sqrt{a}}{2} t^{\nu-2} K_{\nu-2}(\sqrt{at})
\]

and it is well known that the function \(t \mapsto e^{-\frac{t^2}{\nu}}\) is decreasing on \((0, \infty)\). Thus, the function

\[
t \mapsto t^{\nu-1} e^{-\frac{t^2}{\nu}} K_{\nu-1}(\sqrt{at})
\]

is decreasing on \((0, \infty)\) for all \(\nu > 0\) and \(a > 0\). Thus, we conclude that the function \(b \mapsto S_\nu(\sqrt{a}, \sqrt{b})\) is concave on \((0, \infty)\) and hence log-concave. By replacing \(a\) by \(a^2\) we get the required result.
c. It is well-known that a differentiable function $f$ on $(0, \infty)$ is geometrically concave if and only if $xf'(x)/f(x)$ is decreasing $(0, \infty)$. Now in view of [18, 10.29.2]
\[
\frac{xK'_\nu(x)}{K_\nu(x)} = \nu - \frac{xK_{\nu+1}(x)}{K_\nu(x)},
\]
and part b of [4, Lemma 3](see also, [10, Lemma 2.6]), the function $x \mapsto xK'_\nu(x)/K_\nu(x)$ is a decreasing function on $(0, \infty)$ for all $\nu \in \mathbb{R}$. By using this result and the recurrence relation [18, 10.29.2]
\[
xK'_\nu(x) = -\nu - xK_{\nu-1}(x),
\]
the function $x \mapsto -xK_{\nu-1}(x)/K_\nu(x)$ is a decreasing function on $(0, \infty)$ for all $\nu \in \mathbb{R}$. Thus, in view of (2.19) we can conclude that the function $x \mapsto x^\nu K_\nu(x)$ is geometrically concave on $(0, \infty)$ for all $\nu \in \mathbb{R}$. Consequently, the function $t \mapsto t^{\nu-1}K_{\nu-1}(at)$ is geometrically concave on $(0, \infty)$ for all $\nu > 0$ and $a > 0$. Hence the probability density function
\[
t \mapsto \frac{c_{a, \nu}}{a^{\nu-1}} t^{\nu-1} e^{-\frac{t^2+a^2}{2}} K_{\nu-1}(at)
\]
is geometrically concave on $(0, \infty)$ for all $\nu > 0$ and $a > 0$ as the function $t \mapsto te^{-\frac{t^2}{2}}$ is geometrically concave on $(0, \infty)$ and the product of two positive geometrically concave functions is again a geometrically concave function. Finally, the result holds by [3, Theorem 3] which says that the cumulative distributive function of a geometrically concave probability function is geometrically concave.

d. We use the following notations
\[
F_\nu(a, b) = S_\nu(a, b) = \frac{1}{c_{a, \nu}} \quad \text{and} \quad G_\nu(a, t) = \frac{1}{a^{\nu-1}} t^{\nu} e^{-\frac{t^2+a^2}{2}} K_{\nu-1}(at).
\]
As the function $\nu \mapsto K_\nu(x)$ is log-convex on $(0, \infty)$ (see [4, Theorem 3.3]), therefore the function $\nu \mapsto G_\nu(a, b)$ is log-convex on $(0, \infty)$ for all $b \geq 0$ and $a > 0$. Now, by using the Hölder-Rogers inequality for integrals for $\alpha \in [0, 1]$, we have
\[
F_{\alpha \nu_1 + (1-\alpha)\nu_2}(a, b) = \int_b^\infty G_{\alpha \nu_1 + (1-\alpha)\nu_2}(a, t)dt \\
\leq \int_b^\infty (G_{\nu_1}(a, t))^\alpha (G_{\nu_2}(a, t))^{1-\alpha}dt \\
\leq \left(\int_b^\infty G_{\nu_1}(a, t)dt\right)^\alpha \left(\int_b^\infty G_{\nu_2}(a, t)dt\right)^{1-\alpha} \\
= (F_{\nu_1}(a, b))^\alpha (F_{\nu_2}(a, b))^{1-\alpha}
\]
Hence the result follows.
e. From (1.3) we have
\[ S_\nu(\sqrt{\alpha}, \sqrt{\beta}) = \frac{1}{2a^{\nu-1}} \int_0^b t^{\nu-1} e^{-t/2} K_{\nu-1}(\sqrt{at}) dt. \]

In view of [10, Lemma 2.4] and the differential formula [18, 10.29.4]
\[ (x^{-\nu} K_\nu(x))' = -x^{-\nu} K_{\nu+1}(x), \]
the function \( a \mapsto a^{1-\nu} K_{\nu-1}(at) \) is log-convex on \((0, \infty)\) for all \( \nu > 1/2 \) as well as decreasing. Thus, the function
\[ a \mapsto a^{\frac{1-\nu}{2}} K_{\nu-1}(\sqrt{at}) \]
is log-convex on \((0, \infty)\) for all \( \nu > 1/2 \) since square root function is concave and composition of a square root function followed by a log-convex function is again a log-convex function. Consequently, the function
\[ a \mapsto \frac{1}{2a^{\nu-1}} t^{\nu-1} e^{-t/2} K_{\nu-1}(\sqrt{at}) \]
is log-convex on \((0, \infty)\) for all \( \nu > 1/2 \). Now, by using the H"{o}lder-Rogers inequality for integrals, we conclude that the function \( a \mapsto S_\nu(\sqrt{\alpha}, \sqrt{\beta})/c_{\sqrt{\alpha}, \nu} \) is log-convex on \((0, \infty)\).

f. This follows from part d.

g. This follows from part e. \(\square\)

Remark 4. Part b of [4, Lemma 2] states the following: The function \( x \mapsto x^\nu K_\nu(x) \) is log-convex on \((0, \infty)\) for \( \nu \leq 1/2 \). It is important to mention here that, in the proof of the above result the following wrong argument has been used: The function \( \nu \mapsto K_\nu(x) \) is an increasing function on \( \mathbb{R} \) for a fixed \( x > 0 \). But this is not correct, since \( K_\nu(x) = K_{-\nu}(x) \), that is the function \( \nu \mapsto K_\nu(x) \) is an even function. The correct result is that the function \( \nu \mapsto K_\nu(x) \) is an increasing function on \((0, \infty)\) for fixed \( x > 0 \).

We would like to take this opportunity to correct the proof of part b of [4, Lemma 2] stated above. We prove this in two parts namely, when \( \nu < 0 \) and \( 0 \leq \nu \leq 1/2 \). First we assume \( \nu < 0 \). Now let \( \mu = -\nu \), and then \( \mu > 0 \). Consider \( f_\mu(x) = x^{-\mu} K_\mu(x) = x^{-\mu} K_\mu(x) \) as \( K_{-\mu}(x) = K_\mu(x) \). Now,
\[ (\log(f_\mu(x)))' = (\log(x^{-\mu} K_\mu(x)))' = -\frac{x^{-\mu} K_{\mu+1}(x)}{x^{-\mu} K_\mu(x)} = -\frac{K_{\mu+1}(x)}{K_\mu(x)}. \tag{3.14} \]

In view of [10, Lemma 2.4] for each fixed \( \beta > 0 \) and each fixed \( \nu \) satisfying \( \nu > -\beta/2 \), the function \( K_{\nu+\beta}(x)/K_\nu(x) \) decreases to 1 as \( x \) increases from 0 to \( \infty \), and thus we have that the function \( x \mapsto K_{\mu+1}(x)/K_\mu(x) \) is decreasing on \((0, \infty)\) for all \( \mu > -1/2 \) and in particular for all \( \mu > 0 \). Thus, by (3.14), the function \( x \mapsto f_\mu(x) \) is log-convex on \((0, \infty)\) for all \( \mu > 0 \). Equivalently, the function \( x \mapsto x^\mu K_\nu(x) \) is log-convex on \((0, \infty)\) for all \( \nu < 0 \).
Now we assume $0 \leq \nu \leq 1/2$. Then

$$
(\log(x^\nu K_\nu(x)))' = \frac{x^\nu K_{\nu-1}(x)}{x^\nu K_\nu(x)} - \frac{K_{1-\nu}(x)}{K_\nu(x)}.
$$

(3.15)

If we choose $\beta = 1-2\nu$, then $\beta > 0$ and $\nu > -\beta/2$. Thus, in view of [10, Lemma 2.4], the function $x \mapsto K_{\nu+\beta}(x)/K_\nu(x)$ is decreasing on $(0, \infty)$. Consequently, $x \mapsto K_{1-\nu}(x)/K_\nu(x)$ is decreasing on $(0, \infty)$. Hence from (3.15), the function $x \mapsto x^\nu K_\nu(x)$ is log-convex on $(0, \infty)$ for all $0 \leq \nu \leq 1/2$. This completes the proof.

**Proof of Theorem 3.** On differentiating both sides of the Eq. (1.1) with respect to $b$, we have

$$
R'_\nu(a, b) = -\frac{c_{a, \nu}}{c_{a, \nu-1}} b^\nu e^{-\frac{a^2 + b^2}{2}} K_{\nu-1}(ab).
$$

In view of the above differentiation formula the recurrence relation (2.21) becomes

$$
R_\nu(a, b) + \frac{c_{a, \nu}}{c_{a, \nu-1}} R_{\nu-1}(a, b) = -\frac{1}{b} R'_\nu(a, b).
$$

(3.16)

**a.** It is evident that

$$
\frac{R'_\nu(a, b)}{R_\nu(a, b)} = -\frac{b^\nu e^{-\frac{b^2}{2}} K_{\nu-1}(ab)}{\int_b^\infty t^\nu e^{-\frac{t^2}{2}} K_{\nu-1}(at)dt}.
$$

Then

$$
\frac{\partial}{\partial \nu} \left( \frac{R'_\nu(a, b)}{R_\nu(a, b)} \right) = -\frac{1}{B_\nu(a, b)^2} \int_b^\infty e^{-\frac{t^2}{2}} tb \frac{\partial}{\partial \nu} \left( \frac{b^\nu K_{\nu-1}(ab)}{t^\nu K_{\nu-1}(at)} \right)
$$

$$
\times (t^\nu K_{\nu-1}(at))^2 dt,
$$

where

$$
B_\nu(a, b) = \int_b^\infty t^\nu e^{-\frac{t^2}{2}} K_{\nu-1}(at)dt.
$$

In view of part **b** of Lemma 4, the function $\nu \mapsto (b/t)^\nu K_\nu(b)/K_\nu(t)$ is decreasing on $(0, \infty)$ for all $0 < b < t$ and $\nu > 0$. Therefore

$$
\frac{\partial}{\partial \nu} \left( \frac{R'_\nu(a, b)}{R_\nu(a, b)} \right) \geq 0.
$$

Hence the function $\nu \mapsto R'_\nu(a, b)/R_\nu(a, b)$ is increasing on $(0, \infty)$ for $a > 0$ and $b > 0$.

**b.** From the recurrence relation (3.16), we have

$$
\frac{c_{a, \nu}}{c_{a, \nu-1}} \frac{R_{\nu-1}(a, b)}{R_\nu(a, b)} = -\frac{1}{b} \frac{R'_\nu(a, b)}{R_\nu(a, b)} - 1.
$$

In view of part **a**, the function $\nu \mapsto c_{a, \nu} R_{\nu-1}(a, b)/c_{a, \nu-1} R_\nu(a, b)$ is a decreasing function on $(1, \infty)$. Hence the result follows. Another way to prove this is to use part **f** of Theorem 2, which says that the function $\nu \mapsto R_\nu(a, b)/c_{a, \nu}$ is log-convex on $(0, \infty)$ for all $a > 0$ and $b \geq 0$. This gives the required result.
c. It is easy to see that
\[ \frac{\partial}{\partial a} \left( \frac{R'_\nu(a, b)}{R_\nu(a, b)} \right) = -\frac{1}{B_\nu(a, b)} \left( \nu e^{-\frac{\nu^2}{2}} \int_b^\infty t^\nu e^{-\frac{t^2}{2}} \frac{\partial}{\partial a} \left( \frac{K_{\nu-1}(ab)}{K_{\nu-1}(at)} \right) K^2_{\nu-1}(at) dt \right), \]
where \( B_\nu(a, b) \) is defined in part a. By [4, Equation 3.7] and Lemma 4, we get
\[ \frac{\partial}{\partial a} \left( \frac{K_{\nu-1}(ab)}{K_{\nu-1}(at)} \right) > 0. \]
Therefore the function \( a \mapsto R'_\nu(a, b)/R_\nu(a, b) \) is decreasing on \((0, \infty)\) for \( a > 0 \) and \( b > 0 \).

d. Similarly as in part b, the function \( a \mapsto c_{a, \nu} R_\nu(a, b) / c_{a, \nu} R_{\nu+1}(a, b) \) has opposite monotonicity property than the monotonicity property satisfied by the function \( a \mapsto R'_\nu(a, b)/R_\nu(a, b) \). Hence the result follows.

The next lemma is a useful tool to prove Theorem 4.

**Lemma 5.** Let \( 0 < x < y \). Then the following are true.

a. The function \( x \mapsto x^\beta I_{\nu+\beta}(x)/I_\nu(x) \) is increasing on \((0, \infty)\) when \( \nu > -\beta/2 \) and \( \beta > 0 \). Consequently, the function \( \nu \mapsto (x/y)^\nu I_{\nu}(x)/I_\nu(y) \) is decreasing on \((0, \infty)\).

b. The function \( t \mapsto I_{\nu}(tx)/I_{\nu}(ty) \) is decreasing on \((0, \infty)\) for all \( \nu > -1/2 \).

**Proof.** a. In view of the derivative formula [18, 10.29.4]
\[ \frac{\partial}{\partial x} \left( x^\nu I_\nu(x) \right) = x^\nu I_{\nu-1}(x) \]
we have
\[ \frac{\partial}{\partial x} \left( \frac{x^\beta I_{\nu+\beta}(x)}{I_\nu(x)} \right) = \frac{x^\beta (I_\nu(x) I_{\nu+\beta-1}(x) - I_{\nu+\beta}(x) I_{\nu-1}(x))}{(I_\nu(x))^2}. \] (3.17)
In view of the formula [18, 10.32.15]
\[ I_\mu(x) I_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} I_{\mu+\nu}(2x \cos \theta) \cos ((\mu - \nu)\theta) d\theta, \quad \text{where} \quad \nu + \mu > -1, \]
the Eq. (3.17) reduces to
\[ \frac{\partial}{\partial x} \left( \frac{x^\beta I_{\nu+\beta}(x)}{I_\nu(x)} \right) = \frac{x^\beta}{(I_\nu(x))^2} \frac{2}{\pi} \int_0^{\pi/2} I_{2\nu+\beta-1}(2x \cos \theta)(2 \sin (\beta \theta) \sin \theta) d\theta, \]
which is positive for \( x \in (0, \infty), \nu > -\beta/2 \) and \( \beta \in [0, 2] \). Hence the function \( x \mapsto x^\beta I_{\nu+\beta}(x)/I_\nu(x) \) is increasing on \((0, \infty)\) when \( \nu > -\beta/2 \) and \( \beta \in [0, 2] \).

It is worth to mention here that according to Lorch [12] the function \( x \mapsto I_{\nu+\beta}(x)/I_\nu(x) \) is increasing on \((0, \infty)\) when \( \nu > -1, \nu > -\beta/2 \) and \( \beta > 0 \). Thus, the function \( x \mapsto x^\beta I_{\nu+\beta}(x)/I_\nu(x) \) is increasing on \((0, \infty)\) when \( \nu > -\beta/2 \) and \( \beta > 0 \) as a product of two positive and increasing functions.

This implies for any \( 0 < x < y \) we have
\[ \frac{x^\beta I_{\nu+\beta}(x)}{I_\nu(x)} \leq \frac{y^\beta I_{\nu+\beta}(y)}{I_\nu(y)}. \]
This gives the following inequality
\[
\frac{x^\nu + \beta I_{\nu + \beta}(x)}{y^\nu + \beta I_{\nu + \beta}(y)} \leq \frac{x^\nu I_{\nu}(x)}{x^\nu I_{\nu}(x)}.
\]
Hence the function \( \nu \mapsto (x/y)^\nu I_{\nu}(x)/I_{\nu}(y) \) is decreasing on \((0, \infty)\).

b. In view of the recurrence relation \([18, 10.29.3]\)
\[
\frac{\partial}{\partial x} (I_{\nu}(x)) = \frac{\nu}{x} I_{\nu}(x) + \frac{I_{\nu+1}(x)}{I_{\nu}(x)}
\]
we have
\[
\frac{\partial}{\partial t} \left( \frac{I_{\nu}(tx)}{I_{\nu}(ty)} \right) = \frac{I_{\nu}(tx)}{t} \left( \frac{(tx)I_{\nu+1}(tx)}{I_{\nu}(tx)} - \frac{(ty)I_{\nu+1}(ty)}{I_{\nu}(ty)} \right).
\]
Using part a we get that the function \( t \mapsto I_{\nu}(tx)/I_{\nu}(ty) \) is decreasing on \((0, \infty)\).

\(\square\)

Proof of Theorem 4. It is clear that
\[
\frac{Q_{\nu}'(a, b)}{Q_{\nu}(a, b)} = -\frac{b^\nu e^{-\frac{t^2}{2}} I_{\nu-1}(ab)}{\int_b^\infty t^\nu e^{-\frac{t^2}{2}} I_{\nu-1}(at) dt}.
\]
a. It is easy to see the following equation
\[
\frac{\partial}{\partial \nu} \left( \frac{Q_{\nu}'(a, b)}{Q_{\nu}(a, b)} \right) = -\frac{1}{H_{\nu}(a, b)} \int_b^\infty t^\nu e^{-\frac{t^2}{2}} \frac{\partial}{\partial \nu} \left( \frac{b^\nu I_{\nu-1}(ab)}{t^\nu I_{\nu-1}(at)} \right) \times (t^\nu I_{\nu-1}(at))^2 dt,
\]
where
\[
H_{\nu}(a, b) = \int_b^\infty t^\nu e^{-\frac{t^2}{2}} I_{\nu-1}(at) dt.
\]
Now using part a of Lemma 5 we get
\[
\frac{\partial}{\partial \nu} \left( \frac{Q_{\nu}'(a, b)}{Q_{\nu}(a, b)} \right) \geq 0 \text{ for all } \nu > 1.
\]
This proves the result.

b. Similarly, we have
\[
\frac{\partial}{\partial a} \left( \frac{Q_{\nu}'(a, b)}{Q_{\nu}(a, b)} \right) = -\frac{1}{B_{\nu}(a, b)} \int_b^\infty (tb)^\nu e^{-\frac{t^2+\nu^2}{2}} \frac{\partial}{\partial a} \left( \frac{I_{\nu-1}(ab)}{I_{\nu-1}(at)} \right) (I_{\nu-1}(at))^2 dt.
\]
Using part b of Lemma 5 in above equation we have
\[
\frac{\partial}{\partial a} \left( \frac{Q_{\nu}'(a, b)}{Q_{\nu}(a, b)} \right) \geq 0.
\]
This completes the proof. \(\square\)
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