Signless Laplacian spectral radius and fractional matchings of graphs

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Abstract

A fractional matching of a graph $G$ is a function $f$ giving each edge a number in $[0,1]$ so that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to $v$. The fractional matching number of $G$, written $\alpha'(G)$, is the maximum of $\sum_{e \in E(G)} f(e)$ over all fractional matchings $f$. In this paper, we propose the relations between the fractional matching number and the signless Laplacian spectral radius of a graph. As applications, we also give sufficient spectral conditions for existence of a fractional perfect matching in a graph in terms of the signless Laplacian spectral radius of the graph and its complement.

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Key words: Signless Laplacian spectral radius; Fractional matching; Fractional perfect matching

1 Introduction

Throughout this paper, all graphs are simple and undirected. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, and let $|V(G)|$ be the order of $G$ and $|E(G)|$ be the size of $G$. We denote by $N(v)$ the neighbor set of a vertex $v$, and denote by $|N(v)|$ the degree of $v$. Let $\delta$ and $\overline{d}$ be minimum degree and average degree of $G$, respectively. The complete product $G_1 \vartriangle G_2$ of graphs $G_1$ and $G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of $G_1$ to every vertex of $G_2$. For a vertex subset $S \subseteq V(G)$, the induced subgraph $G[S]$ is the subgraph of $G$ whose vertex set is $S$ and whose edge set consists of all edges of $G$ which have both ends in $S$. Let $G^c$ be the complement of $G$.

The adjacency matrix of $G$ is defined to be the matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. Its eigenvalues can be arranged in non-increasing order as

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The degree matrix of $G$ is denoted by $D(G) = \text{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian or the $Q$-matrix of $G$. Note that $Q(G)$ is nonnegative, symmetric and positive semidefinite, so its eigenvalues are real and can be arranged in non-increasing order as follows:

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0,$$

where $q_1(G)$ is the signless Laplacian spectral radius of graph $G$.

A matching in a graph is a set of edges no two of which are adjacent. The matching number $\alpha'(G)$ is the size of a largest matching. A fractional matching of a graph $G$ is a function $f$ giving each edge a number in $[0, 1]$ so that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to $v$. If $f(e) \in \{0, 1\}$, for every edge $e$, then $f$ is just a matching. The fractional matching number of $G$, written $\alpha'_f(G)$, is the maximum of $\sum_{e \in E(G)} f(e)$ over all fractional matchings $f$. It was shown in [15] that $\alpha'_f(G) \leq n/2$.

A fractional perfect matching is a fractional matching $f$ with $\sum_{e \in E(G)} f(e) = n/2$, that is, $\alpha'_f(G) = n/2$. If a fractional perfect matching takes only the values 0 and 1, it is a perfect matching.

Let $i(G)$ stand for the numbers of isolated vertices vertices of a graph $G$. In [15], the following results on fractional matchings and fractional perfect matchings were proved.

- For any graph $G$, $\alpha'_f(G) \geq \alpha'(G)$.
- If $G$ is bipartite, then $\alpha'_f(G) = \alpha'(G)$.
- For any graph $G$, $2\alpha'_f(G)$ is an integer.
- (The fractional analogue of Tutte’s 1-Factor Theorem) A graph $G$ has a fractional perfect matching if and only if

$$i(G - S) \leq |S|$$

for every vertex subset $S \subseteq V(G)$.
- (The fractional analogue of Berge-Tutte Formula) For any graph $G$,

$$\alpha'_f(G) = \frac{1}{2}(n - \max\{i(G - S) - |S|\}),$$

where the maximum is taken over all $S \subseteq V(G)$.

In recent years, the problem of finding tight sufficient spectral conditions for a graph possessing certain properties has received much attention. Especially, the study on the relations between the eigenvalues and the matching number was initiated by Brouwer and Haemers [2], for regular graphs, they obtained a sufficient condition on $\lambda_3$ for existence of a perfect matching.
Subsequently, Cioabă et al. [5, 6, 7] refined and generalized the above result to obtain a best upper bound on $\lambda_3$ to guarantee the existence of a perfect matching. Further, O and Cioabă [14] determined the relations between the eigenvalues of a $t$-edge-connected $k$-regular graph and its matching number when $t \leq k - 2$. Very recently, Feng and his coauthors [8, 12] presented sufficient spectral conditions of a connected graph to be $\beta$-deficient, where the deficiency of a graph is the number of vertices unmatched under a maximum matching in $G$.

On the fractional matching number, O [13] studied the connections between the fractional matching number and the spectral radius of a connected graph with given minimum degree. Xue et al. [16] considered the relations between the fractional matching number and the Laplacian spectral radius of a graph. Along this line, in this paper, we use the technique of proof in [13] and propose the relations between the fractional matching number and the signless Laplacian spectral radius of a graph. As applications, we also give sufficient spectral conditions for existence of a fractional perfect matching in a graph in terms of the signless Laplacian spectral radius of the graph and its complement.

2 Relations between $\alpha'_s(G)$ and $q_1(G)$ of graphs involving minimum degree

Before proving the main results, we list two useful lemmas. The first one is a famous result due to Dirac.

**Lemma 2.1** ([1]) Let $G$ be a simple graph of minimum degree $\delta$.

1. If $\delta \geq n/2$ and $n \geq 3$, then $G$ is hamiltonian.
2. If $G$ is 2-connected and $\delta \leq n/2$, then $G$ contains a cycle of length at least $2\delta$.

Given two non-increasing real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to interlace the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is tight if exists an integer $k \in [0, m]$ such that $\theta_i = \eta_i$ for $1 \leq i \leq k$ and $\theta_{n-m+i} = \eta_i$ for $k + 1 \leq i \leq m$.

Consider an $n \times n$ matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix $R$ of the matrix $M$ is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of $M$. The partition is equitable if each block $M_{i,j}$ of $M$ has constant row (and column) sum.
Lemma 2.2 ([3, 11]) Let $M$ be a real symmetric matrix. Then the eigenvalues of every quotient matrix of $M$ interlace the ones of $M$. Furthermore, if the interlacing is tight, then the partition is equitable.

From the Perron-Frobenius Theorem of non-negative matrices, we have the following lemma.

Lemma 2.3 ([9]) If $H$ is a subgraph of a connected graph $G$, then $q_1(G) \geq q_1(H)$.

Lemma 2.4 ([4]) Let $G$ be a graph on $n$ vertices with minimum, average and maximum vertex degrees $\delta, \overline{\delta}$ and $\Delta$. Then

$$2\delta \leq 2\overline{\delta} \leq q_1(G) \leq 2\Delta.$$

The equalities hold if and only if $G$ is regular.

Let $k$ be a positive integer. Let $B(\delta, k)$ be the set of connected bipartite graphs with minimum degree $\delta$ and the bipartitions $X$ and $Y$ such that:

(i) every vertex in $X$ has degree $\delta$,
(ii) $|X| = |Y| + k$, and
(iii) the degrees of vertices in $Y$ are equal.

Clearly, $|Y| \geq \delta$, and the complete bipartite graph $K_{\delta+k, \delta} \in B(\delta, k)$.

The following lemma can be found in [13]. To make our paper self-contained, here we provide a distinct proof again.

Lemma 2.5 ([13]) If $H \in B(\delta, k)$, then $\alpha'_H \leq \frac{|V(H)|-k}{2}$.

Proof. Note that $H$ is a bipartite graph, then $\alpha'_H = \alpha'(H) = |Y| = \frac{|V(H)|-k}{2}$. \hfill $\boxtimes$

Next we are going to determine the signless Laplacian spectral radius of $H$ in $B(\delta, k)$. The above notion of equitable partition of a vertex set in a graph is used. Consider a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ of the vertex set of a graph $G$ into $t$ non-empty subsets. For $1 \leq i, j \leq t$, let $b_{i,j}$ denote the average number of neighbours in $V_j$ of the vertices in $V_i$. The quotient matrix of this partition is the $t \times t$ matrix whose $(i,j)$-th entry equals $b_{i,j}$. This partition is equitable if for each $1 \leq i, j \leq t$, any vertex $v \in V_i$ has exactly $b_{i,j}$ neighbours in $V_j$. In this case, the eigenvalues of the quotient matrix are eigenvalues of $G$ and the spectral radius of the quotient matrix equals the spectral radius of $G$ (see [3, 10] for more details).

Lemma 2.6 If $H \in B(\delta, k)$, then $q_1(H) = \frac{2|V(H)|\delta}{|V(H)|-k}$.

Proof. Let $Q(H)$ be the signless Laplacian matrix of the graph $H$. The quotient matrix $R$ of $Q(H)$ on the partitions $X$ and $Y$ is

$$R = \begin{pmatrix} \delta & \delta \\ \delta |X| & \delta |X| \end{pmatrix}.$$ 

By a simple calculation, the largest eigenvalue of $R$ is $\lambda_1(R) = \delta(1 + \frac{|X|}{|Y|}) = \frac{2|V(H)|\delta}{|V(H)|-k}$. Note that the partition is equitable. By the above fact mentioned in the paragraph before Lemma 2.6, we have

$$q_1(H) = \lambda_1(R) = \frac{2|V(H)|\delta}{|V(H)|-k}.$$ \hfill $\boxtimes$
Theorem 2.7 Let $G$ be a connected graph of order $n \geq 3$ and minimum degree $\delta$, and let $k$ be a real number belonging to $[0,n]$. If $q_1(G) < \frac{2n\delta}{n-k}$, then $\alpha'_s(G) > \frac{n-k}{2}$.

Proof. We distinguish the following two cases to prove.

Case 1. $\delta > \frac{n-k}{2}$.

If $\delta \geq \frac{n}{2}$, by Lemma 2.1 (1), then $G$ contains $C_n$. Construct a fractional matching $f$ of $G$: $f(e) = \frac{1}{2}$ for any $e \in E(C_n)$, otherwise $f(e) = 0$. Then we have

$$\alpha'_s(G) \geq \sum_{e \in E(G)} f(e) = \frac{n}{2} > \frac{n-k}{2}.$$  

Otherwise $\frac{n}{2} > \delta > \frac{n-k}{2}$, if $G$ is 2-connected, by Lemma 2.1 (2), then $G$ contains a cycle of length at least $2\delta$. Taking $\frac{1}{2}$ for each edge of the cycle and 0 for other edges of $G$, it is easy to see that

$$\alpha'_s(G) \geq \delta > \frac{n-k}{2}.$$  

Otherwise $G$ contains a cut vertex $u$. We can assume that $V_1$ and $V_2$ are two of connected components of $G - u$. Let $v_1$ and $v_2$ be two of neighbors of $u$, where $v_1 \in V_1$ and $v_2 \in V_2$. Note that the minimum degree of the vertices in $V_1$ or $V_2$ is $\delta - 1$. Hence induced subgraphs $G[V_1]$ and $G[V_2]$ contain paths $v_1'P_1v_1$ and $v_2P_2v_2'$ of length $\delta - 1$, respectively. So we can find a path $v_1'P_1v_1uv_2P_2v_2'$ of length $2\delta$ in $G$. Taking $\frac{1}{2}$ for each edge of the path and 0 for others of $G$, then we have

$$\alpha'_s(G) \geq \delta > \frac{n-k}{2}.$$  

Case 2. $\delta \leq \frac{n-k}{2}$.

On the contrary, suppose that $\alpha'_s(G) \leq \frac{n-k}{2}$. By Equation (1), there exists a vertex subset $S \subseteq V(G)$ satisfying $i(G - S) - |S| \geq |k|$. Let $T$ be the set of isolated vertices in $G - S$. Note that the neighbors of each vertex of $T$ only belong to $S$, then there exist at least $\delta$ vertices in $S$. Let $|T| = t$ and $|S| = s$, then $t = i(G - S) \geq s + |k|$. Consider the bipartite subgraph $H$ with the partitions $S$ and $T$. Let $a$ be the number of edges having one end-vertex in $S$ and the other in $T$. Then $a \geq t\delta$. For the partitions $S$ and $T$ in $H$, the quotient matrix of $Q(H)$ is

$$R = \left( \begin{array}{cc} \frac{s}{s} & \frac{k}{s} \\ \frac{k}{s} & \frac{s}{s} \end{array} \right).$$  

By a simple calculation, the largest eigenvalue of $R$ is $\lambda_1(R) = \frac{1}{2} + \frac{s}{t}$. According to Lemmas 2.2 and 2.3, we have

$$q_1(G) \geq q_1(H) \geq \lambda_1(R) \geq \delta \left( \frac{t}{s} + 1 \right) \geq \delta \left( \frac{k}{s} + 2 \right) \geq \delta \left( \frac{2\left| k \right|}{n - \left| k \right|} + 2 \right) = \frac{2n\delta}{n - \left| k \right|},$$

satisfying $a \geq t\delta, t \geq s + \left| k \right|$ and $n \geq t + s \geq 2s + \left| k \right|$. This yields a contradiction. \hfill \Box

From Lemmas 2.5 and 2.6, we can see that there exists graphs $H$ with minimum degree $\delta$ such that $\alpha'_s(H) = \frac{n-k}{2}$ and $q_1(H) = \frac{2n\delta}{n-k}$. In the sense that Theorem 2.7 is best possible. As an application, we obtain the following relationship between $\alpha'_s(G)$ and $q_1(G)$ of graphs involving minimum degree $\delta$.  

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Theorem 2.8 Let $G$ be a connected graph of order $n \geq 3$ with minimum degree $\delta$. Then

$$\alpha'_*(G) \geq \frac{n\delta}{q_1(G)},$$

with the equality holding if and only if $G \in \mathcal{B}(\delta, k)$ and $k = n - \frac{2n\delta}{q_1(G)}$ is an integer.

Proof. For simplicity, we denote $q_1(G)$ and $\alpha'_*(G)$ by $q_1$ and $\alpha'_*$. Theorem 2.7 tells us that if $q_1 < \frac{2n\delta}{n-k}$, then $\alpha'_* > \frac{n-k}{2}$. Note that $\frac{1}{n-k}$ is an increasing function of $x$ on $[0, n)$, $\frac{2n\delta}{n-k}$ decreases towards $q_1$ as $k$ decreases towards $k^*$, where $k^* = n - \frac{2n\delta}{q_1}$. By Theorem 2.7, $\alpha'_* > \frac{n-k}{2}$ for each value of $k \in (k^*, n)$. Letting $k$ tend to $k^*$ and finally equal to $k^*$, we obtain $\alpha'_* \geq \frac{2-nk^*}{2} = \frac{n\delta}{q_1}$, as desired.

Next we consider the equality. If $G \in \mathcal{B}(\delta, k)$ and $k = n - \frac{2n\delta}{q_1(G)}$ is an integer. By Lemma 2.5, $\alpha'_*(G) = \frac{n-k}{2} = \frac{n\delta}{q_1}$. For the necessity, suppose that $\alpha'_*(G) = \frac{n\delta}{q_1}$. This needs equality in each step of the above proof. Hence we have $k = k^* = n - \frac{2n\delta}{q_1}$, that is to say, $q_1 = \frac{2n\delta}{n-k}$. Hence the each step in (2) of the proof of Theorem 2.7 must be an equality. Since $\lfloor k \rfloor = k, k = n - \frac{2n\delta}{q_1(G)}$ must be an integer. Furthermore, since $a = t\delta, t = k + s$ and $n = 2s + k$ in (2) of Theorem 2.7, and $s \geq \delta$, then $G \in \mathcal{B}(\delta, k)$.

3 Relations between fractional perfect matchings and $q_1(G)$ of graphs involving minimum degree

Using Theorem 2.7, we obtain the following sufficient condition for existence of a fractional perfect matching in a graph $G$ in terms of the signless Laplacian spectral radius of $G$.

Theorem 3.1 Let $G$ be a connected graph of order $n \geq 3$ with minimum degree $\delta$. If $q_1(G) < \frac{2n\delta}{n-k}$, then $G$ has a fractional perfect matching.

Proof. By Theorem 2.7, taking $k = 1$, then we have $\alpha'_*(G) > \frac{n\delta}{2}$. Recalling that $2\alpha'_*(G)$ is an integer, then $\alpha'_*(G) = \frac{n\delta}{2}$. Thus $G$ has a fractional perfect matching. □

Further, we consider the sufficient condition for existence of a fractional perfect matching in a graph $G$ in terms of the signless Laplacian spectral radius of its complement.

Theorem 3.2 Let $G$ be a connected graph of order $n \geq 3$ with minimum degree $\delta$, and $G^c$ be the complement of $G$. If $q_1(G^c) < 2\delta$, then $G$ has a fractional perfect matching.

Proof. Suppose, for the sake of contradiction, that $\alpha'_*(G) < n/2$. By Equation (1), there exist a vertex set $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Let $T$ be the set of isolated vertices in $G - S$. Since the neighbors of each vertex of $T$ only belong to $S$, there are at least $\delta$ vertices in $S$. Hence $|T| \geq |S| + 1 \geq \delta + 1$. Note that $G^c[T]$ is a clique. By Lemma 2.3, we have

$$q_1(G^c) \geq q_1(G^c[T]) = 2(|T| - 1) \geq 2\delta,$$

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a contradiction.

Consider the complete bipartite graph $K_{\delta+1,\delta}$. Clearly $q_1(K_{\delta+1,\delta}) = 2\delta$. However, $K_{\delta+1,\delta}$ has no fractional perfect matching since $\alpha'_c(K_{\delta+1,\delta}) = \alpha'(K_{\delta+1,\delta}) = \delta < \frac{2\delta+1}{2}$. In this sense that the result of Theorem 3.2 is best possible.

In fact, the sufficient condition in Theorem 3.2 is very useful and important. There exist some graphs which can be checked for existence of a fractional perfect matching only by applying Theorem 3.2, not Theorem 3.1.

**Example 3.3** Let $H$ be a graph obtained by joining $t$ edges between an isolated vertex and the complete graph $K_{2t}$, where $2 \leq t < 2t$. Note that $H$ contains $C_{2t+1}$, hence $H$ has a fractional perfect matching. However by Lemma 2.4,

$$q_1(H) \geq 2d(H) \geq 2 \cdot \frac{t + t \cdot (2t - 1) + t \cdot 2t}{2t + 1} = \frac{8t^2}{2t + 1} > 2t + 1 = \frac{2|V(H)|\delta(H)}{|V(H)| - 1}.$$

Clearly, we can not use the sufficient condition in Theorem 3.1 to verify. Note that $q_1(H^c) = q_1(K_{1,t}) = t + 1 < 2t = 2\delta(H)$, so the sufficient condition in Theorem 3.2 is available.

Indeed, if we exclude some graphs, the upper bound in Theorem 3.2 can be slightly raised. Let $H = (\delta + 1)K_1 \triangledown H_\delta$, where $H_\delta$ be any graph of order $\delta$. Clearly $q_1(H^c) = 2\delta$. Note that there exists a vertex subset $V(H_\delta)$ such that $\iota(H - V(H_\delta)) > |V(H_\delta)|$, hence $H$ has no fractional perfect matching.

**Theorem 3.4** Let $G$ be a connected graph on $n \geq 3$ vertices with minimum degree $\delta$, and $G^c$ be the complement of $G$. If $q_1(G^c) < 2\delta + 2$, then $G$ has a fractional perfect matching unless $G \cong (\delta + 1)K_1 \triangledown H_\delta$.

**Proof.** Suppose that $\alpha'_c(G) < n/2$. By Equation (1), there exist a vertex set $S \subseteq V(G)$ such that $\iota(G - S) - |S| > 0$. Let $T$ be the set of isolated vertices in $G - S$. Note that the neighbors of each vertex of $T$ only belong to $S$, then $|T| \geq |S| + 1 \geq \delta + 1$.

We claim that $V(G) = T \cup S$. Otherwise we can find a clique of order $\delta + 2$ in $G^c$, and $q_1(G^c) \geq 2\delta + 2$, a contradiction. If $|T| \geq \delta + 2$, then there is a clique of order $\delta + 2$ in $G^c$, and $q_1(G^c) \geq 2\delta + 2$, a contradiction. Hence $|T| = \delta + 1$, and thus $|S| = \delta$, that is to say, $G \cong (\delta + 1)K_1 \triangledown H_\delta$. \[\square\]

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