Entropy from scaling symmetry breaking

NEYMAR CAVALCANTE\textsuperscript{1(a)}, SAULO DILES\textsuperscript{2(b)}, KUMAR S. GUPTA\textsuperscript{3(c)} and AMILCAR R. DE QUEIROZ\textsuperscript{1,4(d)}

\textsuperscript{1} Instituto de Física, Universidade de Brasília - Caixa Postal 04455, 70919-970, Brasília, DF, Brazil
\textsuperscript{2} Instituto de Física - Universidade Federal do Rio de Janeiro - Caixa Postal 68528, 21945-970 Rio de Janeiro, RJ, Brazil
\textsuperscript{3} Theory Division, Saha Institute of Nuclear Physics - 1/AF Bidhannagar, Kolkata 700064, India
\textsuperscript{4} Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza - 50009 Zaragoza, Spain

received 21 December 2014; accepted in final form 21 May 2015
published online 10 June 2015

PACS 89.70.Cf – Entropy and other measures of information
PACS 11.10.-z – Field theory

Abstract – The scaling symmetry in conformal quantum mechanics (CQM) can be broken due to the boundary conditions that follow from the requirement of a unitary time evolution of the Hamiltonian. We show that the scaling symmetry of CQM can be restored by introducing a suitable mixed state, which is associated with a nonvanishing von Neumann entropy. We give an explicit formula for the entropy arising from the mixed state in CQM. Our work provides a direct link between the restoration of a broken symmetry and the von Neumann entropy.

Copyright © EPLA, 2015

Introduction. – The representations of \(so(2,1)\) Lie algebra describe the physical states of a system governed by conformal quantum mechanics (CQM) \cite{1}. In such systems, the Hamiltonian \(H\) together with the generators of dilatation \(G\) and the special conformal transformation \(K\) span the spectrum generating algebra. The \(so(2,1)\) Lie algebra appears in the description of a large class of physical systems, including molecules \cite{2–4}, black holes \cite{5–7}, graphene \cite{8–10} and various types of rational Calogero models \cite{11–13}. It also appears in the study of instabilities of the Coulomb phase in QCD and confinement \cite{14–16}. It is also associated with renormalization group \cite{17} and dimensional transmutation \cite{18–20} in quantum mechanics. It is a remarkable fact that the essential physical features of such a large class of apparently unrelated physical systems are characterized by the representations of \(so(2,1)\) Lie algebra.

The Hamiltonian in CQM is an unbounded operator on a Hilbert space, which requires a specification of its domain \cite{21}. In such systems, the Hamiltonian \(H\) together with the generators of dilatation \(G\) and the special conformal transformation \(K\) span the spectrum generating algebra. The \(so(2,1)\) Lie algebra appears in the description of a large class of physical systems, including molecules \cite{2–4}, black holes \cite{5–7}, graphene \cite{8–10} and various types of rational Calogero models \cite{11–13}. It also appears in the study of instabilities of the Coulomb phase in QCD and confinement \cite{14–16}. It is also associated with renormalization group \cite{17} and dimensional transmutation \cite{18–20} in quantum mechanics. It is a remarkable fact that the essential physical features of such a large class of apparently unrelated physical systems are characterized by the representations of \(so(2,1)\) Lie algebra.

The Hamiltonian in CQM is an unbounded operator on a Hilbert space, which requires a specification of its domain \cite{21}. The domain, or equivalently the boundary conditions, are obtained by demanding a unitary time evolution, which is generated by a self-adjoint Hamiltonian. The boundary conditions leading to a unitary time evolution in CQM may not be unique. For certain values of

\textsuperscript{(a)}E-mail: neymarnepomuceno@gmail.com
\textsuperscript{(b)}E-mail: smdiles@gmail.com
\textsuperscript{(c)}E-mail: kumars.gupta@saha.ac.in
\textsuperscript{(d)}E-mail: amilcarg@unb.br
mixed state. We also obtain an entropy formula similar to that of Cardy [26] for the case of conformal systems with Virasoro algebra with a central charge [5–7]. Cardy formula uses Virasoro algebra with a central charge and modular invariance [26]. We rather use the idea of mixed states to derive the entropy formula. One advantage of our formulation is that it simply uses the so(2, 1) Lie algebra structure. This makes our approach amenable to a wider class of applications independent of the Virasoro algebra.

Recently there have been empirical studies connecting symmetry breaking to entropy [27]. It is plausible that our work could be related to such experiments.

The paper is organized as follows. In the second section, we present the CQM of a massless particle on a plane. We then discuss the appearance of a nontrivial family of boundary conditions that renders the Hamiltonian as an appropriate self-adjoint operator. In the third section, we construct the mixed state that restores the scaling symmetry. This computation consists basically in the appropriate definition of the measure in the space of boundary conditions over which one averages pure states to derive the entropy formula. One advantage of our formulation is that it simply uses the so(2, 1) Lie algebra structure. This makes our approach amenable to a wider class of applications independent of the Virasoro algebra.

As mentioned before, a wide variety of physical systems admits such an algebraic structure. We here consider a prototype physical system that captures the essential physics of CQM.

Our system is described by a particle moving on a plane and interacting with a point defect [22]. Such defect consists of a flux tube perpendicular to the plane or any physical object which produces an inverse-square interaction. The Hamiltonian is given by

\[ H = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\alpha}{r^2} - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}, \]  

where \((r, \varphi)\) denote the polar coordinates on the plane and the dimensionless parameter \(\alpha\) captures the strength of the inverse-square potential. We redefine the variables, so that this Hamiltonian is effectively written as

\[ H = -\frac{\partial^2}{\partial r^2} - \frac{1 - 4\alpha}{4r^2} - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}. \]

We have basically transformed the initial measure \(r^2 dr\) for the radial part to just \(dr\). Henceforth we will consider only the latter.

We next consider that the eigenvalue equation is \(H\psi = E\psi\) with the separation of variables \(\psi(r, \varphi) = f(r)\chi(\varphi)\). The effective angular Hamiltonian

\[ H_\varphi = -\frac{\partial^2}{\partial \varphi^2} \] is self-adjoint for the domains [28]

\[ D_\theta = \{ L^2(S^1, d\varphi) : \chi(2\pi) = e^{i\theta}\chi(0), \chi'(2\pi) = e^{i\theta}\chi'(0) \}, \]  

parametrized by \(\theta \in [0, 2\pi]\). The spectrum of this angular operator is \(\lambda_n = (n + \frac{\theta}{2\pi})^2, n \in \mathbb{Z}\). This provides a one-parameter family of inequivalent quantization of the angular operator in (6).

As discussed in [24, 25], the self-adjoint extension parameter \(\theta\) breaks the parity \(P\) and time reversal \(T\) symmetries, except when \(\theta = 0, \pi\). It was also proposed in [24] that an appropriate impure (or mixed) state can be used to restore the \(P\) and \(T\) symmetries and the associated RG procedure was discussed in [25]. Here we shall consider the radial dynamics in detail and see that the boundary conditions can break the continuous scaling symmetry of the system.

The eigenvalue problem for the family of radial Hamiltonian parameterized by \(\delta\) is

\[ H_r f(r) \equiv \left[ -\frac{\partial^2}{\partial r^2} + \frac{\delta}{r^2} \right] f(r) = Ef(r), \]

where

\[ \delta = -\frac{1}{4} + \lambda_n + \alpha. \]

The radial operator \(H_r\) is symmetric in the domain \(D(H_r) \equiv \{ f(0) = f'(0) = 0, f, f' \text{ are absolutely continuous}, f \in L^2(R_+, dr) \}\). According to the value of the parameter \(\delta\), it is well known [21] that

1) \(\delta \geq \frac{3}{4}\): in this case \(H_r\) is essentially self-adjoint in \(D(H_r)\).
2) \(-\frac{1}{4} \leq \delta < \frac{3}{4}\). In this case \(H_r\) is not self-adjoint in \(D(H_r)\) but admits a one-parameter family of self-adjoint extensions. The self-adjoint extensions are characterized by a real parameter \(\gamma \in [0, 1]\).
3) \(\delta < -\frac{4}{3}\). In this case, the ground-state energy is unbounded from below and hence the system is unphysical. RG techniques can be used to address this case [17].

In the case in which \(-\frac{1}{4} \leq \delta < \frac{3}{4}\), the boundary condition arising from the self-adjoint extension breaks the scaling symmetry for generic values of the self-adjoint extension parameter \(\gamma\) [11]. This happens since the domain in which \(H_r\) is self-adjoint is not kept invariant by
the generator of dilatations. To see this, consider the action of the dilatation operator $G = \frac{1}{2}(r \frac{d}{dr} + \frac{d}{dr})$ on a generic element $\phi_+(r) = \phi_+(r) + e^{iv} \phi_-(r) \in D_\gamma(H_r)$, where $\phi_+(r) = \sqrt{r} H^{(1)}(r e^{-i \frac{\pi}{4}})$ and $\phi_-(r) = \sqrt{r} H^{(1)}(r e^{i \frac{\pi}{4}})$, with $H^{(1)}$ being Hankel functions [30] and $v^2 = \gamma + \frac{1}{4}$. In the limit $r \to 0$, one obtains [11]

$$G\phi(r) \to \frac{1}{\sin v \pi} \left[ (1 + \nu) \frac{r^{\nu + \frac{1}{2}} (e^{-i \frac{\pi}{4}} - e^{i(\gamma + \frac{\pi}{4})})}{2^\nu \Gamma(1 + \nu)} + (1 - \nu) \frac{r^{-\nu + \frac{1}{2}} (e^{i(\gamma + \frac{\pi}{4})} - e^{-i \frac{\pi}{4}})}{2^{-\nu} \Gamma(1 - \nu)} \right].$$

Now, $G\phi(r) \in D_\gamma(H_r)$ only if $G\phi(r) \sim C\phi(r)$, where $C$ is a constant. This is not possible except for special values of the self-adjoint extension parameter $\gamma [11]$, so that $G\phi(r)$ in general does not belong to $D_\gamma(H_r)$. This shows that, when $-1/4 \leq \delta < 3/4$, the scaling symmetry is broken at the quantum level for generic values of the self-adjoint extension parameter $\gamma$.

**Mixed states for CQM.** From now on, we only consider the case $-1/4 \leq \delta < 3/4$, when $H_r$ admits a one-parameter family of self-adjoint extensions. In this case, the scaling symmetry is broken due to quantum effects for generic values of the self-adjoint extension parameter [11]. In particular, when $\alpha = 0$, the zero mode $n = 0$, when $\theta = 0$, and $n = 0, 1$ when $\theta > 0$, the dilatation operator $G$ changes the boundary conditions of the radial Hamiltonian $H_r$. Therefore, it does not make sense to implement the commutator of $G$ and $H_r$ on physical states.

We here use a mechanism to restore broken symmetries by mixed states as suggested by Balachandran and Queiroz in [24], see also [31–33]. Our strategy is to construct an appropriate mixed state on which the commutators between $H_r(\gamma)$ and $G$ can be implemented unambiguously. Such an appropriate mixed state consists of the average over all possible boundary conditions obtained from the action of the operator $G$.

We consider the notation $H_r(\gamma)$, with $0 \leq \gamma \leq 1$. For a fixed $D_\gamma$, we take a fixed general state $|\gamma\rangle$. The mixed state that restores the scaling symmetry is of the form

$$\omega = \int d\mu(\gamma) \lambda_\gamma \hat{\omega}_\gamma, \tag{12}$$

where $d\mu(\gamma)$ is a $U(1)$-invariant Haar measure, with $U(1)$ being isomorphic to the space of all self-adjoint extensions we are considering, $\hat{\omega}_\gamma$ is a pure state ($\hat{\omega}_\gamma^2 = \hat{\omega}_\gamma$) in the domain $D_{\gamma}$ and $\lambda_\gamma$ is a weight satisfying

$$\lambda_\gamma \geq 0, \quad \int d\mu(\gamma) \lambda_\gamma = 1. \tag{13}$$

The von Neumann entropy of $\omega$ is given by

$$S(\omega) = -\int d\mu(\gamma) \lambda_\gamma \log \lambda_\gamma. \tag{14}$$

In the previous section we argued that the requirement of covariance of $\omega$ under the action by $G$ forces the decomposition (12) to be uniform, that is, $\lambda_\gamma = 1/\mathcal{N}$, for any $\gamma$, where $\mathcal{N}$ is an appropriate normalization factor. The proposed anomaly-restoring mixed state is therefore (10). The normalization factor is associated with the volume of a $G$-orbit in the $U(1)$-family of domains. But there are ambiguities in the definition of the entropy to be dealt with. The uniform decomposition of the mixed state $\omega$ has associated entropy

$$S(\omega) = \log \mathcal{N}. \tag{15}$$

Our problem now is to find $\mathcal{N}$.

Naively, $\mathcal{N}$ would be the volume of the $U(1)$-family of self-adjoint domains of the Hamiltonian. This would give 1 if $U(1)$ is modelled after an interval with length 1. However, the uniformity of the weights in the mixed states brings in with an ambiguity in the evaluation of $\mathcal{N}$ (a counting ambiguity). This ambiguity is resolved if one
uses an appropriate Gibbs-like factor associated with the volume of the symmetry group generating this ambiguity.

We start by considering a finite interval as a model for the $U(1)$-family of boundary conditions, that is, $0 \leq \gamma \leq 1$. We proceed by considering a uniform discretization of this interval. Each discrete point of the interval is labelled by $j$ and the inter-site distance is denoted by $a$. Other regularizations of the interval leads to the same final conclusion. We keep this uniform regularization. The regularized mixed state becomes

$$\omega_N = \frac{1}{N^N} \sum_{j=1}^{N} \omega_j,$$

where $N$ is the number of discrete points uniformly filling the interval. The discrete points $j$ can be reshuffled without affecting the counting. This implies an ambiguity in the counting associated with the symmetric group $S_N$. We then fix a choice of ordering of $j$, that is, we fix a particular $S_N$ gauge orbit. In other words, we account for the distinct manners of ordering the points $j$. The result is that $N^N$ is proportional to the number of conjugacy classes of the group $S_N$. This is similar to accounting for the volume of the orbit space of a gauge group and the fact that this volume is equal to the number of conjugacy classes of $S_N$. Recall that the number of conjugacy classes of $S_N$ is equal to the number of inequivalent irreducible representations of $S_N$. Hence, the normalization factor is $N^N = (aN)|\mathcal{C}_{S_N}|$, where $|\mathcal{C}_{S_N}|$ is the number of conjugacy classes of the symmetric group $S_N$. It is a well-known result [36] that $|\mathcal{C}_{S_N}| = p(N)$, where $p(N)$ is the number of partitions of the integer $N$. In other words, each inequivalent irreducible representation of $S_N$ is labelled by a partition of $N$.

The outcome of the above argument is

$$S(\omega_N) = \log N^N = \log(aN)p(N).$$

We now renormalize this counting formula (17) by taking $N \to \infty$ in a continuous manner, that is, simultaneously sending the inter-site distance $a$ to zero while keeping the unit length of the interval fixed. Therefore, $aN \to 1$. At the same time, this large $N$ limit leads to the notorious asymptotic formula by Hardy-Ramanujan [37] or Rademacher [38],

$$p(N) \sim \frac{1}{4\sqrt{3}N} e^{\pi \sqrt{\frac{2}{3}N}}.$$  \hspace{1cm} (18)

The entropy is therefore

$$S(\omega_N \to \infty) \sim \pi \sqrt{\frac{2N}{3}} - \log(4\sqrt{3}).$$  \hspace{1cm} (19)

In order to compensate for the divergence in $N \to \infty$, we properly normalize the entropy by $\sqrt{N}$, so that

$$S(\omega_N \to \infty) = \frac{S(\omega_N)}{\sqrt{N}} = \pi \sqrt{\frac{2}{3}}.$$  \hspace{1cm} (20)

This expression is similar to Cardy’s formula for CFT with $c = 1$ [26]. We will discuss more on the relation of our derivation and the derivation of Cardy’s formula in the next section.

Similar arguments work for other regularizations. The final result depends on the uniform continuous $S_\infty$ sum of pure states (10).

In [39,40], the origin of the ambiguity for the computation of an entropy of certain mixed states was extensively analysed from an algebraic perspective. It was shown there that degeneracy leads to an ambiguity of the irreducible representation of the algebra of observable. We were inspired by the above works to properly count states taking into account the mentioned ambiguity.

**Many particles.** The generalization of the previous counting argument to more particles is straightforward. We provide here the detailed argument.

We first consider independent distinguishable particles. Independence means that the particles do not interact among themselves. Obviously, each distinct particle independently obeys conformal mechanics.

Suppose there are $c$ of such independent distinguishable particles. Then, in the previous counting we have to replace $N$ to $cN$. This means that each particle carries a $U(1)$-family of self-adjoint extensions. They are independent so that we end up with $c$ of such families. Independence here leads to the fixing of the form of the self-adjoint family. In principle, for $c$ particles, we could have an $U(c)$-family, suggesting the existence of boundary conditions that transmute one particle into another. This more general case of $U(c)$ is also associated with a quantum topological change [41,42]. The new formulae become

$$S(\omega_N^c) = \log N^c = \log(cN)p(cN)),$$  \hspace{1cm} (21)

$$p(cN) \sim \frac{1}{4\sqrt{3}cN} e^{\pi \sqrt{\frac{2}{3}cN}}.$$  \hspace{1cm} (22)

$$S(\omega_N^c \to \infty) \sim \pi \sqrt{\frac{2cN}{3}} - \log(4\sqrt{3}).$$  \hspace{1cm} (23)

We then quotient out this last expression by $\sqrt{N}$ and take the limit $N \to \infty$, so that

$$S(\omega_N^c) = \frac{S(\omega_N^c \to \infty)}{\sqrt{N}} = \pi \sqrt{\frac{2c}{3}}.$$  \hspace{1cm} (24)

This formula resembles Cardy’s formula for general $c$.

If the particles are indistinguishable, then we need to take extra care due to the statistics of such particles. Once the statistic group, which is associated with the fundamental group of the underlying configuration space of the system, is taken into account, the procedure of counting is the same as stated above.

**Conclusion.** – The continuous scaling symmetry in CQM can be broken due to quantization. This happens when the boundary conditions, or equivalently the domain of the Hamiltonian is not preserved by the
action of the dilatation operator [22]. This is a purely quantum-mechanical symmetry breaking, analogous to what happens for the 2D delta-function potential [18]. We have shown here that a suitable mixed state can be used to restore the broken scaling symmetry. Such a mixed state is associated with a nonvanishing von Neumann entropy. It thus appears that the symmetry which is broken due to the choice of boundary conditions in quantum mechanics may be restored at the expense of a mixed state with nonvanishing entropy. Our analysis provides a connection between the von Neumann entropy and the restoration of scaling symmetry in CQM.

The mixed state $\omega$ in eq. (10) can be interpreted as a sum over classes of topology. Indeed, in [41] it is shown that distinct values of the self-adjoint extension parameter corresponds to classes of topology. Now, it is an interesting perspective to sharpen this statement in our model. In particular, one can ask if there is a correspondence between the entropy we have computed and the number of classes of topology. We postpone the analysis of this interesting problem to another work.

We can also consider the 3-dimensional problem of the quantization of an electron in the field of a magnetic monopole, the Dirac-Fierz problem as described by Hurst in [43]. Equation (5) of [43] can be appropriately written in a form analogous to our equation (8), where $\delta$ would be now a function of the eigenvalues of $J^2$ and $\mu = eg$. Our analysis can then be applied to this case. Furthermore, the meaning of the entropy and its consequences to this case are certainly an interesting problem.

It is known that the near-horizon conformal structure of certain black holes can be described by a CQM, which is associated with a Virasoro algebra [5–7]. Using the Cardy formula, the central charge of the Virasoro algebra can be related to the Bekenstein-Hawking formula [44]. Here we have argued that the restoration of the symmetry in CQM also leads to an entropy formula. It is thus possible that these two apparently different mechanisms of entropy generation are related, which is under investigation.

***

We thank A. P. BALACHANDRAN, Bruno Carneiro da Cunha, Manuel Asorey, José García Esteve, Fernando Falceto, Filiberto Ares for discussions. NC thanks Prof. Mirjam Cvetic and acknowledges the kind hospitality at the study group of the Department of Physics and Astronomy at the University of Pennsylvania. KG thanks Prof. Álvaro Ferraz and acknowledges the kind hospitality at IIP-UFRN, Natal, Brazil, where part of this work was done. ARQ acknowledges the kind hospitality at Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza. NC is supported by the Programa Ciência sem Fronteiras under CAPES process No. 99999.003034/2014-03. SD is supported by CNPq. ARQ is supported by CAPES process No. BEX 8713/13-8.

REFERENCES

[1] De Alfaro V., Fubini S. and Furlan G., Nuovo Cimento A, 34 (1976) 569.
[2] Camblong H. E., Epele L. N., Fanchiotti H. and García Canal C. A., Phys. Rev. Lett., 87 (2001) 220402.
[3] Camblong H. E. and Ordóñez C. R., Phys. Rev. D, 68 (2003) 125013.
[4] Giri P. R., Gupta Kumar S., Melianac S. and Samsarov A., Phys. Lett. A, 371 (2008) 2967, hep-th/0703121.
[5] Birmingham D., Gupta Kumar S. and Sen S., Phys. Lett. B, 505 (2001) 191, hep-th/0102051.
[6] Gupta Kumar S. and Sen Siddhartha, Phys. Lett. B, 526 (2002) 121, hep-th/0112041.
[7] Chakrabarti S. K., Gupta K. S. and Sen S., Int. J. Mod. Phys. A, 23 (2008) 2547, arXiv:0708.1667.
[8] Gupta Kumar S. and Sen Siddhartha, Phys. Rev. B, 78 (2008) 205429, arXiv:0808.2864 [hep-th].
[9] Gupta Kumar S., Samsarov A. and Sen Siddhartha, Eur. Phys. J. B, 73 (2010) 389, arXiv:0903.0272 [cond-mat.mes-hall].
[10] Chakraborty Baishali, Gupta Kumar S. and Sen Siddhartha, J. Phys. A: Math. Theor., 46 (2013) 055303, arXiv:1207.5705 [hep-th].
[11] Basu-Mallick B., Ghosh Pijush K. and Gupta Kumar S., Nucl. Phys. B, 659 (2003) 437, arXiv:hep-th/0207040.
[12] Basu-Mallick B., Ghosh Pijush K. and Gupta Kumar S., Phys. Lett. A, 311 (2003) 87, arXiv:hep-th/0208132.
[13] Melianac S., Samsarov A., Basu-Mallick B. and Gupta Kumar S., Eur. Phys. J. C, 49 (2007) 875, arXiv:hep-th/0609111.
[14] Asorey M. and Santagata A., PoS(Confinement X), (2012) 057.
[15] Asorey M. and Santagata A., AIP Conf. Proc., 1606 (2014) 407.
[16] Asorey M. and Santagata A., PoS(QCD-TNT-III), (2013) 004.
[17] Gupta K. S. and Rajeev S. G., Phys. Rev. D, 48 (1993) 5940, hep-th/9305052.
[18] Jackiw R., Delta-function Potentials in 2 and 3-Dimensional Quantum Mechanics, in MAB Beg Memorial Volume (World Scientific, Singapore).
[19] Camblong H. E., Epele L. N., Fanchiotti H. and García Canal C. A., Ann. Phys., 287 (2001) 14.
[20] Camblong H. E., Epele L. N., Fanchiotti H. and Canal C. A. G., Ann. Phys., 287 (2001) 57, hep-th/0003267.
[21] Reed M. and Simon B., Methods of Modern Mathematical Physics: Fourier Analysis, Self-adjointness, Vol. 2 (Academic Press) 1975.
[22] Esteve J. G., Phys. Rev. D, 66 (2002) 125013.
[23] Esteve J. G., Phys. Rev. D, 34 (1986) 674.
[24] Balachandran A. P. and Queiroz A. R., Phys. Rev. D, 85 (2012) 025017, arXiv:1108.3898.
[25] Gupta K. S. and Queiroz A., Anomalies and renormalization of impure states in quantum theories, arXiv:1306.5570.
[26] Cardy J. L., Nucl. Phys. B, 270 (1986) 186.
[27] Roldan E., Martinez I. A., Parrondo J. M. and Petrov D., Nat. Phys., 10 (2014) 457.
[28] Asorey M., Esteve J. G. and Pacheco A. F., Phys. Rev. D, 27 (1983) 1852.
[29] Ibarra A., Lledó F. and Pérez-Pardo J. M., On Self-Adjoint Extensions and Symmetries in Quantum Mechanics, to be published in Ann. Henri Poincaré (2014) DOI:10.1007/s00023-014-0379-4.
[30] Abromowitz M. and Stegun I. A., Handbook of Mathematical Functions (Dover Publications, New York) 1970.
[31] Balachandran A. P. and de Queiroz A. R., JHEP, 11 (2011) 126, arXiv:1109.5290.
[32] Balachandran A., Govindarajan T. and de Queiroz A. R., Eur. Phys. J. Plus, 127 (2012) 118, arXiv:1204.6609.
[33] de Queiroz A. R., Srivastava R. and Vaidya S., Renormalization of noncommutative quantum field theories, arXiv:1207.2358.
[34] Bargmann V., Ann. Math., 48 (1947) 568.
[35] Balachandran A., Govindarajan T., de Queiroz A. R. and Reyes-Lega A., Phys. Rev. A, 88 (2013) 022301, arXiv:1301.1300.
[36] Fulton W. and Harris J., Representation Theory: A First Course, Vol. 129 (Springer) 1991.
[37] Hardy G. H. and Ramanujan S., Proc. Lond. Math. Soc., 2 (1918) 75.
[38] Rademacher H., Topics in Analytic Number Theory (Springer) 1973.
[39] Balachandran A., de Queiroz A. and Vaidya S., Eur. Phys. J. Plus, 128 (2013) 112, arXiv:1212.1239.
[40] Balachandran A., de Queiroz A. R. and Vaidya S., Phys. Rev. D, 88 (2013) 025001, arXiv:1302.4924.
[41] Balachandran A. P., Bimonte G., Marmo G. and Simoni A., Nucl. Phys. B, 446 (1995) 299.
[42] Asorey M., Balachandran A. P., Marmo G., Silva I. P., de Queiroz A. R., Teotonio-Sobrinho P. and Vaidya S., Quantum physics and fluctuating topologies: Survey, arXiv:1211.6882.
[43] Hurst C. A., Ann. Phys., 50 (1968) 51.
[44] Carlip S., Phys. Rev. Lett., 82 (1999) 2828.