Effective Field Theories and Quantum Chromodynamics on the Lattice
A. Ali Khan
Institut für Physik, Humboldt-Universität zu Berlin,
12489 Berlin, Germany

Abstract
We give a selection of results on spectrum and decay constants of light and heavy-light hadrons. Effective fields theories relevant for their lattice calculation, namely non-relativistic QCD (NRQCD) for heavy quarks on the lattice and Chiral Perturbation Theory for light quarks, are briefly discussed.

1 INTRODUCTION
The Standard Model of the strong and electroweak interactions is based on a $SU(3) \times SU(2) \times U(1)$ gauge symmetry with three generations of quarks and leptons as fermionic matter fields and a scalar field, the Higgs, which is responsible for the masses of the weak $SU(2)$ gauge bosons and of the fermions. For a recent review about the status of the Standard Model and new physics see e.g. [1].

The $SU(3)$ ‘sector’ of the model is Quantum Chromodynamics (QCD), a gauge theory of the strong interaction. With relativistic Dirac quarks, the model can be described classically by the Lagrangian

$$\mathcal{L}_{\text{QCD}} = \bar{q} (i \gamma^\mu D_\mu - m) q - \frac{1}{4} F^c_{\mu\nu} F^c_{\mu\nu}.$$  \hspace{1cm} (1)

The $q$ fields are 4-component Dirac spinors, and the $D_\mu$ are covariant derivatives, e.g., $D_\mu = \partial_\mu - ig_s A_\mu^c t^c$ with $[D_\mu, D_\nu] = -ig_s F^c_{\mu\nu} t^c$, where $F^c_{\mu\nu}$ are the field strength tensors, $g_s$ is the coupling constant, and the $t^c$ are generators of $SU(3)$ in the fundamental representation. A consequence of the self-interactions among the gluon fields $A_\mu^c$ is asymptotic freedom, i.e., the interactions between particles become weak at short distances and can be described with perturbation theory in the strong coupling $\alpha_s = g_s^2/(4\pi)$. At larger distance, the forces become strong, and non-perturbative methods are necessary to understand how hadron masses arise and whether it is possible to explain the hadron spectrum from first principles within the theory of strong interactions.

Although rather successful, the Standard Model by itself does not seem completely satisfactory. On the experimental side, recent discoveries such as neutrino mixings, new results from accelerator experiments [3,4] and indications for ‘dark energy’ in the cosmos indicate a need for an extension of the model. The Higgs particle has not yet been found; recent reviews of the status of Higgs searches are [5]. Further there are theoretical motivations to search for physics beyond the Standard Model (for a discussion see e.g. [2]). The Standard Model contains a considerably large set of coupling constants and masses as input parameters. It does not explain the values of typical energy scales such as the masses of the weak gauge bosons.

A strategy in the research is to simultaneously measure as many physical quantities as possible, test the results for self-consistency within the Standard Model and search for indications of new physics. Among the most interesting search grounds are the elements of
the Cabibbo-Kobayashi-Maskawa (CKM) matrix which parameterizes the flavor changing weak currents and provides a mechanism for CP violation within the Standard Model. Those CKM matrix elements which are relevant to reactions of heavy, for example $b$ and $c$, quarks are at present studied intensively in experiment and theory. We introduce the CKM matrix with an emphasis on $B$ meson decays in the framework of the weak effective theory in Section 1.1. The status of the CKM matrix is reviewed in [6]. For a review about recent results on quark masses see Ref. [7].

Description of the long-range interactions of QCD requires non-perturbative techniques. Using a four-dimensional lattice description of space and time it is possible to calculate matrix elements numerically on a computer within a path integral formalism. A brief introduction to the lattice formalism is given in Section 2; for detailed recent reviews see [8].

Ideally, the lattice extent $L$ should be much larger than the extent of the Compton wavelength of the particles that are supposed to be described, and the inverse lattice spacing $a$ should be much larger than the masses and momenta in the theory in order to avoid cutoff effects. The lightest hadrons, the pions, have a mass of around 140 MeV, whereas the $B$ meson has a mass of 5.28 GeV and contains a heavy quark with a mass of 5 GeV. The problem is how lattice simulations can accommodate this large range of scales.

| Ideally | $L^{-1}$ $\ll$ masses and energy splittings $\ll$ $a^{-1}$ |
|---------|----------------------------------------------------------|
| In Reality | $L = 2 - 3$ fm
$\left(L^{-1} = 0.07 - 0.1$ GeV$\right)$
finite size effects | $a^{-1} = 2 - 4$ GeV
cutoff effects |

To calculate properties of hadrons with $b$ quarks on the lattice, one can for example simulate at lighter quark masses where discretization errors are under better control, and use extrapolations in the heavy quark mass. Fortunately the energy level splittings of $b$ hadrons are much smaller than their masses: of the order of $\Lambda_{QCD} = 200 - 500$ MeV or smaller, where $\Lambda_{QCD}$ is the energy scale where QCD becomes non-perturbative. The dynamics of the heavy quarks can be accounted for as small corrections proportional to powers of the inverse heavy quark mass. This is the basis for effective field theories developed for heavy quarks: Heavy Quark Effective Theory (HQET) [9] and non-relativistic QCD (NRQCD) [11,12] (for reviews see [10]), which can be used to simulate heavy quarks directly on the lattice while avoiding large discretization errors due to the large mass. We discuss NRQCD in Section 3.

Practical simulations with light quarks are computationally expensive and sensitive to the finite lattice volume. Therefore one often uses quark masses much heavier than $u$ and $d$ quark masses and extrapolates the results to the physical values of the quark masses. A formalism for this can be derived using chiral perturbation theory ($\chi PT$), an expansion around the chiral (zero quark mass) limit describing low-energy degrees of freedom of QCD such as pions and nucleons. This is introduced in Section 3.2.

Lattice results for light and heavy-light hadron masses and heavy-light current matrix elements are discussed in Section 4.
1.1 Heavy quark decays

Study of weak decays of quarks is of interest for determinations of elements of the CKM matrix which parameterizes the mixings of quark generations in the Standard Model:

\[ \mathcal{L} = - \frac{g_2}{\sqrt{2}} (\bar{u}_L c_L \bar{t}_L) \gamma^\mu (V_{q_1 q_2}) \left( \begin{array}{c} u_L \\ s_L \\ c_L \\ t_L \\ b_L \end{array} \right) W^\dagger_{\mu} + \text{h.c.}, \] (2)

where \( u_L, d_L, c_L, s_L, t_L, b_L \) are left-handed quark spinors, \( W_\mu \) a charged weak gauge boson and \( g_2 \) the weak gauge coupling. \( V \) is a unitary matrix. There are indications that some of its elements have a non-trivial complex phase giving rise to CP violation. The CKM matrix elements with presently the largest uncertainties are the ones relevant to decays or mixings of the \( b \) quark: \( V_{cb}, V_{ub} \) and \( V_{td} \). \(|V_{ub}|^2 \) describes for example the leptonic meson decay \( B^+ \to l^+ \nu_l \), where \( l \) is a lepton (\( e, \mu \) or \( \tau \)) and \( \nu_l \) the corresponding neutrino and semileptonic decays into a light meson \( (B \to \pi, \rho, \omega) \) and a lepton-neutrino pair, and \(|V_{cb}|^2 \) determines the semileptonic decay \( B \to D \) and a lepton-neutrino pair. \(|V_{td}|^2 \) is proportional to the oscillation frequency between the mass eigenstates of the \( B^0 - \bar{B}^0 \) mixing, which is described by the left and middle diagrams in Figure 1 in the electroweak theory. Processes at energy scales much less than the \( W \) boson mass can be calculated within the weak effective theory where interactions mediated by the \( W \) or \( Z \) particles can be described by point interactions. \( B^0 - \bar{B}^0 \) is described by the third diagram in Figure 1.

To relate the weak processes between quarks with exclusive reaction rates of mesons, one uses form factors which get contributions from long-distance QCD interactions, and therefore have to be calculated nonperturbatively. This can be done from first principles using the lattice. In the effective theory the \( B \) meson decay is described by a matrix element of the heavy-light axial vector current

\[ \langle 0 | A_\mu(x) | B(p) \rangle = i f_B p_\mu e^{-i p x}, \] (3)

where \( f_B \) is the \( B \) decay constant. The branching ratio for the decay \( B^+ \to l^+ \nu_l \) is

\[ BR(B^+ \to l^+ \nu_l) = \frac{G_F^2 m_B m_l^2}{8 \pi} \left( 1 - \frac{m_l^2}{m_B^2} \right)^2 f_B^2 |V_{ub}|^2 \tau_B, \] (4)

where \( G_F = g_2^2/(8 M_W^2) \) is the Fermi constant and \( \tau_B \) the \( B \) lifetime. If \( f_B \) is known, \(|V_{ub}|^2 \) can in principle be determined experimentally from this decay. There exist only
experimental upper bounds on $f_B$, $f_{Bs}$ and $f_D$, but there are results on $f_{Ds}$:

$$f_{Ds} = 280(17)(25)(35) \text{ MeV} \ [13] \text{ and }$$

$$= 285(19)(40) \text{ MeV} \ [14].$$

In the weak effective theory, the form factor for the $B^0 - \bar{B}^0$ mixing matrix element can be parameterized as $f_B^2 B_B$, where the “bag parameter” $B_B$ quantifies to what extent the matrix element is described by $B$-to-vacuum currents:

$$B_B = \frac{3}{2} \frac{\langle B^0 |(\bar{b}_L \gamma_\mu d_L)(\bar{b}_L \gamma_\mu d_L)|B^0\rangle}{\langle B^0 |b_\gamma \mu_5 d|0\rangle \langle 0 |b_\gamma \gamma_5 d|B^0\rangle}. \quad (5)$$

The oscillation frequency of the mass eigenstates is proportional to the mass difference and related to the form factors by

$$\Delta M_d \propto |V_{tb}^* V_{td}|^2 f_B^2 B_B. \quad (6)$$

2 QCD ON THE LATTICE

2.1 Gauge fields

Matter fields, e.g. quarks, sit on the lattice sites which are separated by a spacing $a$. The gauge fields on the lattice are represented by fields $U_{x,\mu} = \exp[ig_s a \int_{x+\mu} dz A_c(\mu)(z) t^c] = \exp[-ig_s a A_c(\mu)(x) t^c]$, parallel transporters between neighboring lattice sites. Line integrals of gauge fields over closed paths are called Wilson loops. The smallest $(1 \times 1)$ Wilson loop on the lattice is a product of gauge links over the the nearest neighbors, called plaquette,

$$P_{\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu} U_{x,\nu}.$$  

By expanding around $a \simeq 0$, one finds a tree-level relation between the plaquette and the continuum field strengths. Thereby one obtains the Wilson (or ‘plaquette’) lattice action as a discretization of the continuum gauge field action of Eq. (1),

$$S_g = -\frac{\beta}{N_c} \sum_\Box \text{Re Tr} \Box, \quad \beta = \frac{2N_c}{g_s^2}, \quad N_c: \text{ number of colors}, \quad (7)$$

which has lattice spacing errors at $O(a^2)$. To further reduce discretization effects, actions can be improved. For gauge field actions, this consists of adding larger Wilson loops ($1 \times 2, \ldots$). Improvement can be done by removing discretization effects order by order in $a$ (originally suggested by Symanzik for scalar field theory [15], and developed into an
improvement program for on-shell quantities in QCD by Ref. [16]), or with renormalization
group methods to obtain renormalization group improved (RG) [17] or perfect [18] actions.

At typical values of $\beta$ in lattice simulations, there are large corrections due to gauge
field loops on the lattice which shift the expectation value of $\bar{U}$ substantially with respect
to the free field value, one. The perturbative corrections can be reduced with ‘mean-
field’ (or ‘tadpole’) improvement [12]: the gauge links $U$ are divided by their expectation
value which can be calculated in perturbation theory or determined nonperturbatively in
simulations.

2.2 Lattice fermions

Discretization of the Euclidean continuum Dirac action by substituting the covariant
derivatives by covariant symmetric lattice differences gives the ‘naive’ lattice fermion
action
\[
S_F = a^4 \left( \frac{1}{2a} \sum_{x,\mu} \bar{q}_x \gamma_\mu \left[ U_{x,\mu} q_{x+\mu} - U_{x-\mu,\mu} q_{x-\mu} \right] + \sum_x m \bar{q}_x q_x \right), \tag{8}
\]
which is chirally symmetric if $m \to 0$ and has $O(a^2)$ errors. However, the naive discretiza-
tion leads to a flavor multiplication, the so-called ‘doublers’. At $m = 0$, the free fermion
propagator has a pole at $k_\mu = 0$ as in the continuum but also poles at $k_\mu = \pi/a$. There
are 16 species of fermions, which occur in pairs of opposite chirality.

Wilson’s solution to the doubling problem is to add a term of the form $a^4 \sum_x \bar{q}_x \Delta q_x$ to
the action, where $\Delta$ is a covariant second derivative. The doublers obtain masses which
remain finite in lattice units: $ma \neq 0$ if $a \to 0$. Chiral symmetry receives corrections
at $a \neq 0$, and $O(am), O(ap)$ discretization errors occur. $O(a)$ errors can be removed
from the action with the clover term proportional to $a^4 \sum_x \bar{q}_x \sigma_{\mu\nu} G^{\mu\nu} q_x$, where $G^{\mu\nu}$ is
a discretized version of the field strength tensor using four neighboring plaquettes [19].
The coefficient of the clover term can be calculated in perturbation theory (a common
choice is at tree-level using tadpole-improvement). Most recent calculations use a non-
perturbative determination of the clover coefficient [20] and are $O(a)$ improved to all
orders in perturbation theory.

Staggered fermions are obtained from a spin-diagonalization of naive fermions:
\[
q_x = \gamma_x \chi_x, \quad \bar{q}_x = \bar{\chi}_x \gamma_x^\dagger,
\]
with $\gamma_x = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, where $x = (x_1, x_2, x_3, x_4)$. The $\chi$ fields are one-component. In the
massless case, the theory has a $U(1) \times U(1)$ chiral symmetry at finite $a$ and a $U(4) \times U(4)$
chiral symmetry in the continuum limit. Discretization errors are $O(a^2)$. Improvement is
possible by adding higher dimensional operators.

Lattice formalisms for doubler-free, chiral fermions are given in [21–23].

2.3 Extracting physical quantities

Green functions can be calculated by evaluating the path integral over the lattice degrees
of freedom numerically. For example, a two-point function of a field $O(x)$ which can be
composed of more elementary fields $\{\phi_i(x)\}$ is given by
\[
\langle O^\dagger(x)O(0) \rangle = \frac{1}{Z} \int \mathcal{D}\phi O^\dagger(x)O(0)e^{-S[\phi]}, \tag{10}
\]
where $D\phi$ denotes integration over all dynamical fields (gauge, fermion, etc...) in the theory. To determine for example $f_B$ from the lattice, it is necessary to calculate the renormalization factors to match the unrenormalized lattice matrix element of the axial vector current to the corresponding matrix element in continuum QCD.

Ideally, these calculations are done at various values of the lattice spacings, and the continuum estimate is obtained by extrapolating as a function of $a$ to $a \to 0$. In practice, some lattice calculations are performed only at one or two values of $a$, in which case a continuum limit cannot be taken, and the discretization effects have to be included into the estimate of systematic errors. With NRQCD calculations, higher dimensional operators are included as discussed in Section 3.1, and an $a \to 0$ extrapolation cannot be done out of principle. Calculation at several values of $a$ then serves to determine the systematic error from keeping the lattice spacing finite.

In full QCD, the path integral includes gauge and fermionic fields $\int DUD\bar{q}Dq$. To decrease computational expenses, many calculations are done in the quenched approximation, i.e. the vacuum polarization due to quark loops is neglected.

To use lattice results in phenomenology, it is necessary to estimate systematical errors as accurately as possible. The most important sources are:

- Finite lattice spacing
- Finiteness of lattice volume
- Quenching (unphysical number of dynamical quarks)

3 EFFECTIVE THEORIES AND THE LATTICE

3.1 NRQCD

Non-relativistic QCD (NRQCD) is an effective theory formulated for heavy quarks assuming that their dynamics is non-relativistic, with correction terms which can be added within a systematic expansion. For quarkonia the higher order interactions are arranged in a $v^2$ expansion, where $v$ is the heavy quark velocity (see e.g. [12]). In heavy-light systems it is an expansion in $v$ or $1/M$, where $M$ is the heavy quark mass. At infinite mass, the heavy quark is just a source of the color electromagnetic field, whereas at finite $M$, there is a recoil of the heavy quark due to the interaction with soft gluons with typical momenta of $O(\Lambda_{QCD})$. One can argue that the heavy quark momentum $P_Q$ and light quark momentum $p_q$ are equal due to momentum conservation within the rest frame of the meson:

$$Mv \simeq P_Q = p_q \sim O(\Lambda_{QCD}).$$

(11)

Therefore $v \sim \Lambda_{QCD}/M \sim 0.1$ in B mesons and should be a reasonable expansion parameter to specify corrections to the $M \to \infty$ (static) limit. Contributions at $O(1/M)$ are the kinetic and the spin-colormagnetic energy of the heavy quark; at $O(1/M^2)$ a heavy quark spin-orbit interaction and a Darwin term are added. The $O(1/M^2)$ Lagrangian for the heavy quark is given by

$$\mathcal{L} = \psi^\dagger (D_t + H) \psi,$$

(12)
with heavy quark Pauli spinor $\psi$ and the Hamiltonian

$$\begin{align*}
H &= -\frac{\vec{D}^2}{2M} - \frac{g_s c_4}{M}\vec{\sigma} \cdot \vec{B} \\
&\quad + \frac{ig_s c_2}{8M^2}(\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D}) - \frac{g_s c_3}{8M^2}\vec{\sigma} \cdot (\vec{E} \times \vec{D}) - \frac{c_1(\vec{D}^2)^2}{8M^3}.
\end{align*}$$

(13)

The last term is the first relativistic correction to the kinetic energy of the heavy quark, which is usually included in calculations at $O(1/M^2)$. The coefficients of the various terms can be found with matching calculations to full QCD in the continuum. In the lattice calculations described in Section 4, they are set to their tree-level value one, using mean field improved gauge links.

Discretizing the Lagrangian (12) one can simulate $b$ quarks directly on the lattice, since there are no $O((aM)^n)$ discretization errors. Errors $O(a^2\vec{p}^2)$ and $O(aMv^2/2)$ arising from discretization of the spatial and temporal derivatives in the NRQCD Lagrangian can be corrected for by adding further terms to the Hamiltonian (13). If matrix elements of operators are to be calculated in this formalism, the $1/M$ corrections to the operators have to be taken into account as well. Simulation of the $1/M$ corrections within HQET on the lattice using nonperturbative renormalization is discussed in [24].

Other methods to avoid large discretization errors used in the calculations discussed here are to simulate heavy quarks around the charm and extrapolate to the $b$, or to use a non-relativistic interpretation of a Wilson or clover heavy quark action called FNAL [25] in this article. Refs. [25] and [26] formulate on-shell improvement programs for heavy quarks by adding further operators to the Wilson or clover action. For recent reviews see also [27, 28].

### 3.2 Chiral perturbation theory

The chiral symmetry of massless QCD can be understood as spontaneously broken. As a consequence one would expect massless Goldstone bosons in the spectrum. One can identify the physical pions with the Goldstone bosons if small quark mass terms are included in the QCD Lagrangian resulting in a small explicit breaking of chiral symmetry and a finite but small pion mass $m_\pi$. Typical momenta of low-energy interactions of pions will be $O(m_\pi)$. Chiral perturbation theory can be formulated as an effective theory expanding in powers of the pion mass and external momenta $p = O(m_\pi)$, the so-called $p$ expansion [29]. It is used to describe physical interactions at low energy scales, for example pion-nucleon scattering. It gives predictions for the expansion of hadron masses around the zero quark mass limit, which can be used in the analysis of lattice calculations to extrapolate lattice hadron masses simulated at larger quark mass to the physical light $(u,d)$ quark mass. The leading dependence at $O(p^2)$ is expected to be linear in the quark mass $m_q$ where $m_q \propto m_\pi^2$, with corrections at $O(p^3)$ etc. To facilitate inclusion of interactions with baryons within the $p$ expansion, Heavy Baryon Chiral Perturbation Theory ($HB\chi PT$) was developed [30], which works in the limit of infinite baryon mass.

The quark masses in present simulations are such that pions are not much lighter than the lattice nucleons, and a formulation with relativistic nucleons seems more appropriate [31,32].

For the pions one can use a representation based on their nature as Goldstone bosons of the broken $SU(2) \times SU(2)$ chiral symmetry of massless QCD with two flavors of $u$ and
\( d \) quarks with
\[
\mathcal{L}_\pi = \frac{1}{4} f_\pi^2 \text{Tr} \left[ \partial_\mu U^\dagger \partial^\mu U + \chi^\dagger U + \chi U^\dagger \right].
\]
(14)

\( U = \exp(\frac{1}{f_\pi} \vec{\pi} \pi) \) transforms according to the \((2, 2)\) representation of \( SU(2) \times SU(2) \). \( \chi = 2B_0M, M \) is the quark mass matrix and \( B_0 \) is proportional to the chiral condensate. The \( \vec{\pi} \) are the pion fields, and the \( \tau_i \) are Pauli matrices. \( f_\pi \) is the pion decay constant.

The nucleon Lagrangian at lowest order \((O(p^1))\) is
\[
\mathcal{L}^{(1)} = \overline{\Psi} (i \gamma_\mu D^\mu - m_0) \Psi + \frac{1}{2} g_A \overline{\Psi} \gamma_5 u^\mu \Psi,
\]
(15)

with \( u^2 = U, D_\mu = \partial_\mu + \frac{i}{2}[u^\dagger, \partial_\mu u] \) and \( u_\mu = iu^\dagger \partial_\mu u^\dagger \). \( \Psi \) is the Dirac spinor of the nucleon, \( m_0 \) the nucleon mass in the chiral limit, and \( g_A \) the nucleon axial vector coupling in the chiral limit. The \( O(p^2) \) Lagrangian is given by
\[
\mathcal{L}^{(2)} = c_1 \text{Tr}(\chi^+) \overline{\Psi} \Psi + \frac{c_2}{4m_0^2} \text{Tr}(u_\mu u_\nu) (\overline{\Psi} D^\mu D^\nu \Psi + h.c.)
\]
\[
+ \frac{c_3}{2} \text{Tr}(u_\mu u^\mu) \overline{\Psi} \Psi - \frac{c_4}{4} \overline{\Psi} \gamma_5 \gamma_\nu [u_\mu, u_\nu] \Psi,
\]
(16)

with \( \chi^+ = u^\dagger \chi u^\dagger + u \chi^\dagger u \). \( \text{Tr}() \) refers to the trace over the flavor indices.

Chiral extrapolation of nucleon masses from the lattice and calculation of the effect of the lattice size on nucleon masses are discussed in Section 4.3.

4 LATTICE RESULTS

4.1 Setting the scale

Masses and decay constants coming out of a simulation are at first dimensionless numbers in units of the lattice spacing. The value of the lattice spacing is determined by calculating a suitable quantity \( aM \) on the lattice and adjusting the corresponding dimensionful quantity \( M \) to its physical value. If the calculation is free of systematic errors such as lattice spacing, finite volume and quenching effects, using any quantity should give the same result. In practical calculations all of those errors can occur. Then, ‘suitable’ means that systematic errors of the quantity used to set the scale and the quantity that is supposed to be calculated cancel as well as possible. For example, the lattice scale can be determined using the static quark potential, which has small discretization errors \( (O(a^2)) \) or higher. A typical length scale is \( r_0 \) related to the interquark force [34] with \( r_0^2 \frac{\partial V}{\partial r} \bigg|_{r=r_0} = 1.65 \), which can be calculated on the lattice with high precision [35]. The physical values corresponding to potential models are around \( r_0 = 0.49 - 0.5 \) fm. Unless noted otherwise we use \( r_0 = 0.5 \) fm. For the string tension \( \sigma \), usually experimental values of \( \sqrt{\sigma} = 427 \) or 440 MeV are assumed. Other quantities frequently used to set the scale are the \( \rho \) meson mass \( m_\rho \), the nucleon mass, the decay constants \( f_\pi \) and \( f_K \), and charmonium and bottomonium level splittings.
In Fig. 2 we show examples for the discrepancy between lattice spacings from different physical quantities in the quenched approximation. With two flavors ($N_f = 2$), the agreement is improved: using Wilson gauge fields and two flavors of $O(a)$ improved clover sea quarks, Ref. [36] quotes an agreement of scales from $m_\rho$, $f_K$ and $r_0 = 0.5$ fm. However, in the two flavor calculations of [37, 38] (Wilson gauge fields, staggered sea and clover valence quarks) and of [39, 41] (RG gauge fields and tadpole-improved clover sea and valence quarks) at $a \approx 0.5$ GeV$^{-1}$, a $\sim 20\%$ discrepancy between lattice spacings from $\chi_b - \Upsilon$ mass splittings and $m_\rho$ remains.

With two flavors of light and one flavor of strange dynamical quarks, using a 1-loop Symanzik $O(a^2)$ improved gauge action and a tree-level tadpole $O(a^2)$ improved staggered sea quark action Ref. [40] finds an agreement of a variety of physical quantities with experiment (see Fig. 3).

Figure 2: Discrepancy of lattice spacings from various physical quantities on quenched lattices. Results are from [35, 37, 39, 41, 42, 44, 45]. Average over spin orientations is denoted by an overbar.

Figure 3: Comparison of lattice with experimental results from [40], using zero (left) and $N_f = 2$ light + 1 strange flavors (right). $a$ from the $\Upsilon' - \Upsilon$ mass splitting.
Figure 4: Unquenched light baryon spectrum from the lattice. Left and middle: results from [41]. Right figure: from [42]. Open symbols denote quenched, filled symbols unquenched results. $K$ input: strange quark mass set by fixing the $K$ meson mass to the physical value, $\phi$ input: strange quark mass set by fixing the $\phi$ meson mass.

4.2 The light baryon spectrum from the lattice

In quenched calculations it was found that the features of the experimental light hadron spectrum are described well by the lattice [46, 47]. It is of interest to study whether unquenching improves the agreement. In Fig. 4 we plot the baryon spectrum from the recent unquenched simulations of [41,42]. Discrepancies with experiment of $\sim 2\sigma$ remain. A reason may be uncertainty in the chiral extrapolation. Ref. [42] assigns additional systematic errors of up to 25 MeV from the chiral extrapolation uncertainty and the determination of $r_0$. In Table 1 we give results for light baryon mass splittings corresponding to the results shown in Fig. 4. For the splittings, the agreement with experiment is at the $1-2\sigma$ level. A recent calculation [43] with $N_f = 2_{\text{light}} + 1_{\text{strange}}$ finds a $\Delta - N$ splitting which

| scale | $\Lambda - N$ [MeV] | $\Delta - N$ [MeV] | $\Sigma^* - \Sigma$ [MeV] | $\Xi^* - \Xi$ [MeV] |
|-------|-----------------|-----------------|-----------------|-----------------|
| $N_f = 0$ |
| [41]  | $m_\rho$ | 151(34)(25) | 346(41) | 252(33)(10) | 247(26)(6) |
| [42]  | $m_\rho$ | 116(23)(26) | 314(37) | 252(30)(10) | 250(23)(11) |
| $N_f = 2$ |
| [41]  | $m_\rho$ | 124(49)(0) | 358(68) | 288(60)(0) | 279(56)(6) |
| [42]  | $m_\rho$ | 128(26)(16) | 313(31) | 248(27)(0) | 246(23)(0) |
| $N_f = 2_{\text{light}} + 1_{\text{strange}}$ |
| [43]  | $\chi_b - T$ | 293(54) | | | |
| Model calculations |
| [71]  | 155 | 270 | 180(15) | 200(15) |
| Experiment |
| | 177 | 294 | 191 | 215 |

Table 1: Light baryon mass splittings. The first error is statistical, the second is the difference from fixing the strange quark mass using the $K$ or $\phi$ meson where applicable. The quantity used to fix the lattice scale is indicated.
agrees well with experiment.

### 4.3 Nucleon mass: chiral extrapolation and finite size effects

![Graph showing nucleon masses](image)

**Figure 5:** Chiral extrapolation of nucleon masses on large lattices in two-flavor QCD using non-relativistic (left: from Ref. [48]) and relativistic (right: from Ref. [50]) $\chi PT$ at $O(p^4)$. The dot-dashed line on the left shows the non-relativistic $O(p^3)$ result.

$HB\chi PT$ predicts at $O(p^3)$ a correction $\sim m_\pi^2$ to the quadratic dependence on $m_\pi$, but with a coefficient which is very different from the value found from fits to the lattice data. In Ref. [48], a good description of lattice data up to pion masses $\sim 600$ MeV could be achieved using the non-relativistic formalism at $O(p^4)$ (see Fig. 5 on the left). With relativistic $\chi PT$ at $O(p^4)$ [49], the agreement with the lattice data is also good up to rather large pion masses, as shown in Fig. 5 on the right [50].

Having ensured that relativistic $\chi PT$ at $O(p^4)$ indeed describes the nucleon mass on very large lattices, it should be possible to calculate the finite size effects on lattices which are not too small within this formalism. Calculating the difference of the nucleon self-energy in a spatially finite and infinite volume within $\chi PT$ at $O(p^4)$ [50], assuming an infinite temporal extent of the lattice, one finds a good agreement with the finite size behavior of the lattice results. An example for a pion mass around 550 MeV is given in Fig. 6.

The non-relativistic formalism at $O(p^3)$ predicts finite size effects which are clearly smaller than the finite size effects of the lattice data [51].

### 4.4 The spectrum of hadrons with a $b$ quark

A heavy quark with infinite mass can be regarded as a color source which is static in the rest frame of the hadron and whose spin is not relevant to the interactions. Corrections due to the finiteness of the heavy quark mass can be included in a $1/M$ expansion. Within the Heavy Quark Effective Theory (HQET), the mass of a heavy-light hadron $H$ can be
thought of as consisting of the following contributions:

\[ M_H = M_Q + \bar{\Lambda} - \frac{1}{2M_Q} \left[ \frac{\langle H | \overline{Q} \vec{B}^2 Q | H \rangle}{2M_H} + \frac{\langle H | \overline{Q} \vec{\sigma} \cdot \vec{B} Q | H \rangle}{2M_H} \right] + O(1/M_Q^2), \tag{17} \]

where \( Q \) is the heavy quark spinor, \( M_Q \) the heavy quark mass, \( \bar{\Lambda} \) the binding energy of the meson for \( M_Q \to \infty \), and the other two terms the expectation values of the heavy quark kinetic energy and the spin-colormagnetic interaction energy respectively. We give a brief summary of lattice results on the hyperfine splittings \( B^* - B \) and \( B_s^* - B_s \) in Section 4.4.1. Results on \( P \) wave states and baryons are given in Tables 2, 3 and 4. Masses averaged over spin-orientations (spin-averaged) are denoted by an overbar. The first error on the individual lattice results includes statistical errors and uncertainties fixing the masses to the physical values, the second, where applicable, is a chiral extrapolation uncertainty. To calculate weighted averages, we include an estimate of systematic errors from the actions. Most of the calculations use NRQCD, except for [52] who uses heavy clover quarks and [53] who simulates \( B_s \) mesons in the static approximation and interpolates between the static and experimental \( D_s \) mesons.

Discretization errors with non-perturbatively \( O(a) \) improved clover light quarks (finer lattice of [44] and [55]) are \( O(a^2 \Lambda_{QCD}^2) \), whereas the tadpole-improved light clover action has \( O(a^2 \Lambda_{QCD}^2) \) and \( O(\alpha_s a \Lambda_{QCD}) \) errors (coarser lattice of [44] and [56,57]). Refs. [58,59] use \( O(a^2) \) tree-level tadpole-improved clover light actions respectively. Ref. [60] uses staggered light quarks. The scale has been set with \( m_{\rho} \) except in Ref. [57] sets the scale with \( \sqrt{\sigma} = 427 \) MeV. Ref. [53] and [54] uses \( r_0 \) with physical values of 0.5 and 0.525 fm respectively, and [60] uses \( r_0 = 0.5 \) fm and quarkonia at and around the charm [61]. All other calculations from the set discussed here use \( m_{\rho} \). The NRQCD action has errors \( O(\alpha_s \Lambda_{QCD}/M) \) from corrections to the spin-magnetic coefficient. Errors on spin splittings are treated as being dominated by an error on the spin-magnetic coefficient of \( O(\alpha_s) \sim 20 - 30\% \).
The systematical error of each result is divided into a part common to all calculations, which is taken to be of the order of the error of the calculation with the smallest uncertainty, and and a rest which is treated as independent. The error on the average is rescaled by \( r = \sqrt{\frac{\chi^2}{(N - 1)}} \), where \( N \) is the number of results, if \( r > 1 \). The second error on the averages comes from the variation due to the 10\% ambiguity between using \( a \) from \( m_\rho \) and \( a \) from \( r_0 = 0.5 \) fm in the quenched case, and asymmetric chiral extrapolations where applicable. The \( \chi_b - \Upsilon \) mass difference is not included in the estimate of the scale variation since it gives values for spin-independent mass splittings which are much higher than experiment. For example, Ref. [38] quotes \( B_s - B \) and the \( \Lambda_b - B \) splittings of 118 and 670 MeV if the scale is set with the \( \chi_b - \Upsilon \) splitting instead with \( m_\rho \). The experimental values are 90 and 345 MeV respectively [62].

If a collaboration quotes results from several lattice spacings, they are plotted starting from the coarsest lattice on the left. Asymmetric errors are added linearly in the plots.

For the error estimates we use nominal values of \( \Lambda_{QCD} = 400 \) MeV, \( M = 5 \) GeV and \( \alpha_s = \alpha_V(1/a) \), where \( \alpha_V(q^\ast) \) is defined in the potential scheme described in [63] at the scale \( q^\ast \). Since the lattice results have rather varying central values we do not calculate the error in percent of the individual lattice splittings but of the experimental splittings or nominal estimates thereof.

### 4.4.1 \( B^\ast - B \) splitting

Results from quenched lattice NRQCD calculations of the \( B^\ast - B \) and \( B_s^\ast - B_s \) splittings fixing the scale with \( m_\rho \) (e.g. [44, 56, 57, 59, 64]), and \( r_0 = 0.5 \) fm [60] are found to be around 25 - 35 MeV, compared to experimental values of 45.8(4) MeV and 47.0(2.6) MeV [62], respectively. Using relativistic \( O(a) \) improved heavy quarks, Ref. [45] obtains a splitting around 10 - 20 MeV with an error of 10 MeV. A quenched calculation using the FNAL action [65] setting \( a \) with the charmonium 1\( P - 1\S \) splitting quotes a \( B_s^\ast - B_s \) splitting of around 40(10) MeV. A preliminary calculation comparing results with zero and two flavors of 3\( \times \) the strange quark mass [57] on lattices with \( a \sim 0.2 \) fm finds an unquenched value of around 33 MeV with only an insignificant increase compared to the quenched result from coarse lattices. A recent NRQCD calculation with 2 + 1 dynamical flavors [66] finds a \( B_s^\ast - B_s \) splitting of 42.5(3.7) MeV.

### 4.4.2 Orbitally excited \( B \) mesons

For heavy-light mesons it seems appropriate to use a hydrogen-like picture for the coupling of angular momenta of the quarks. In the infinite mass limit, there are two \( P \) wave energy levels with light quark angular momentum \( j_l = 1/2 \) and 3/2. At finite \( M \) there is an additional hyperfine structure due to the coupling of the heavy quark spin. This results in one level with angular momentum zero (\( B_0^\ast \)), two with angular momentum one (\( B_1^\ast \) and \( B_1 \)), and one with angular momentum two (\( B_2^\ast \)). Here we discuss the \( B_0^\ast \) and \( B_2^\ast \) level splittings.

Lattice results are presented in Table\textsuperscript{2} In the error estimation, we use guessed values where no experimental value is available: 400 MeV for \( B_0^\ast - B \) and \( B_{s0}^\ast - B_s \) and 500 MeV for \( B_2^\ast - B \) and \( B_{s2}^\ast - B_s \).

Experimental knowledge of the \( P \) state level structure is still sparse. The Particle Data Book lists two candidates. There is the \( B_s^\ast(5732) \) resonance with a width of \( \sim 130 \) MeV, which is believed to come from several narrow and broad \( P \) wave states, and the \( B_{sJ}^\ast(5850) \) signal with a width of \( \sim 50 \) MeV, which can be interpreted as stemming from
excited $B_s$ states. A comparison of the lattice results with experiment and with model calculations is given in Fig. 7.

The sign of the $B_s^* - B_s^0$ mass difference is disputed among potential model calculations (e.g. [67–70]). Individual lattice calculations [44, 58] find a splitting around zero and are within errors compatible with a small negative splitting, but the lattice average for $B_s^* - B_s^0$ is positive.

4.4.3 $b$ baryons

Baryons with one $b$ quark can be thought of as two light quarks coupling to form a spin zero or spin one diquark. The state with a spin zero diquark is the $\Lambda_b$. If the diquark has spin one, the heavy quark can couple to a spin 1/2 state, the $\Sigma_b$, and a spin 3/2 state, the $\Sigma_b^*$. If the light quarks in the $\Sigma_b$ and the $\Sigma_b^*$ are substituted by strange quarks one obtains the $\Omega_b$ and the $\Omega_b^*$.

In Table 4 we summarize results for the spin-independent splittings $\Lambda_b - \overline{B}$ and $\Sigma_b - \Lambda_b$ with $M(\Sigma_b) = (2M(\Sigma_b) + 4M(\Sigma_b^*)) / 6$ and $M(\overline{B}) = (3M(B^*) + M(B)) / 4$, and the $\Sigma_b^* - \Sigma_b$ hyperfine splitting. The values quoted for [55] are obtained by interpolating the heavy-
strange meson mass to the $B_s$ mass and interpolating the baryon splittings to the thus obtained $b$ quark mass.

We compare lattice results with calculations within a constituent quark model [71] and a Skyrme model [72] where the baryon is described as a bound state between a soliton and a heavy quark.

The experimental values of the $\Lambda_b - \bar{B}$ and the $\Lambda_c - \bar{D}$ splittings are very close: 311(9) and 310(2) MeV respectively. The only $1/M$ correction to these splittings comes from the heavy quark kinetic energy. Its contribution to the $\Lambda_b$ mass appears to be very close to the contribution to the $\bar{B}$ mass.

The quenched lattice average differs from the experimental value by less than 2σ. Preliminary results with two flavors of dynamical quarks around the strange quark mass [38, 57] are even higher. Ref. [57] finds an increase of $\sim 15\%$ if $N_f$ is changed from zero to two dynamical quarks of around 3 times the strange quark mass. Calculation at smaller quark masses would clarify whether the difference is related to a chiral extrapolation uncertainty.

In Fig. 8 we show results for the spin-averaged $\Sigma_b - \Lambda_b$ splitting. Ref. [72] gives a relation between the heavy-light and light baryon splittings, $\Delta M (\Sigma_Q - \Lambda_Q) / \Delta M (\Delta - N) = 2/3$, for $Q = c, b$. For $Q = c$, the equality holds well experimentally. Lattice results for the ratio with $Q = b$ vary between 0.5 and 1.

In Fig. 9 we also give results for the $\Sigma_b^* - \Sigma_b$ hyperfine splitting. The expectation from HQET is that the hyperfine splitting is generated by the spin-chromomagnetic interaction (see Eq. (17)) and should be proportional to $1/M_Q$. Rescaling the experimental value of $\Delta M (\Sigma_c^* - \Sigma_c) \approx 67$ MeV by the ratio of $b$ and $c$ quark masses, one expects $\Delta M (\Sigma_b^* - \Sigma_b) \sim 20$ MeV, and the lattice results are compatible with this expectation. The error on the $\Sigma_b^* - \Sigma_b$ and $\Omega_b^* - \Omega_b$ splittings is estimated to be $O(\alpha_s) \times 20$ MeV.

Figure 8: $\Lambda_b - \bar{B}$ splitting from quenched (left) and $N_f = 2$ (middle) lattices (Refs. [38, 52, 55–57, 59]). The lattice average shown on the right is quenched. CP-Q and CP-2 denote quenched and unquenched results from [57], respectively. Model results are from Refs. [71, 72].
Figure 9: Above: $\Sigma_b - \Lambda_b$ splitting from the lattice (left) and model calculations (right). The dashed line shows the experimental value for the $\Sigma_c - \Lambda_c$ splitting. Below: $\Sigma_b^* - \Sigma_b$ splitting. Lattice data from Refs. [52,55,56,59], model results from Refs. [71,72].

4.5 $f_B$

In Table 5 we summarize lattice results for $f_B$ and $f_{Bs}$ since 1998. The first error given in the Table is statistical, and the second is the systematical error given by the authors added in quadrature.

First we address unquenching effects on $f_B$. They depend on how the scale is set. In Table 6 we compare ratios of decay constants from quenched and two-flavor simulations with the same gauge field and valence quark actions and find an increase of $\sim 10\%$ if $a$ is set with $f_\pi$, $10 - 20\%$ if $a$ is set with $m_\rho$ and no increase with $\Upsilon$ and (for $f_{Ds}$) with $r_0$.

We calculate weighted averages for quenched, $N_f = 2$ and $N_f = 2 + 1$ results of $f_B$ and $f_{Bs}$. Since the methods for error estimation can vary considerably between different collaborations, even if similar lattice actions and parameters are used, we make new assignments motivated by the error analysis of the authors themselves. We assign common systematic errors to the calculations with RG gauge fields using NRQCD [73] and using the FNAL heavy quark action as non-relativistic effective field theory without taking the continuum limit [74], and to the calculations using Wilson gauge fields and NRQCD (quenched are [64,75,77]). For quenched configurations with clover light quarks, we assign systematic errors of 20 and 22 MeV respectively for $f_B$ and $f_{Bs}$. Ref. [60] uses NRQCD with an $O(a^2)$ tadpole improved gauge action and staggered light valence quarks, and we use their own systematic error assignment.

The quenched calculations of Refs. [45,78–80] use Wilson gauge and clover quarks at $a = 0.35 - 0.37$ GeV$^{-1}$ ($\beta = 6.2$) simulated at the charm quark mass. Ref. [79]
uses tree-level tadpole-improved clover quarks without including $O(\alpha_s \times a)$ terms in the renormalization. Ref. [78] uses a non-perturbatively $O(a)$ improved clover quark action and a partly non-perturbative current renormalization. Refs. [45,80] use nonperturbative $O(a)$ improvement except for a perturbative value for the $O(\alpha_s a m_q)$ quark mass correction to the renormalization constant. Although different degrees of improvement are used, and the scaling behaviour is found to be different, the results for $f_D$, agree at $\beta = 6.2$. We therefore assign a common systematic error to these results. According to the estimate of the discretization error given in [79] (8%) and of a $1/M$ extrapolation error of $\sim 9\%$ given in [45] we use 23 MeV for $f_B$ and 26 MeV for $f_{B_s}$.

Refs. [76,81,82] use heavy quarks in the FNAL formalism and extrapolate their results to $a \rightarrow 0$. Refs. [83,84] use a step scaling method with the Schrödinger functional and clover heavy quarks. Part of the renormalization factors is calculated nonperturbatively. Their results are continuum extrapolated. Ref. [85] uses an interpolation between static and clover charm quarks which are non-perturbatively improved using the Schrödinger functional and continuum extrapolated.

The second error on the quenched results includes the ambiguity between scales from $m_{\rho}$ and $\Upsilon$ level splittings by varying the result by $+30\%$ if the scale is taken from $m_{\rho}$ or $\sqrt{\sigma} = 427$ MeV, $(-3 + 27)\%$ if the scale is set with $f_{\rho}$, and $(-12 + 18)\%$ if the scale is set with $r_0 = 0.5$ fm. The lattice spacings calculated in [79,80] from $K$ physics are close to the results using $f_{\pi}$ at the same $\beta$ values, and the scales from $f_{K}$ determined in [81] are close to the ones using $m_{\rho}$ from the same actions and $\beta$ values. We also quote the upper bound from the variation between scales from $m_{\rho}$ and $r_0 = 0.5$ fm (second error given in square brackets) which is slightly lower than but within errors compatible with the unquenched central value with $N_f = 2 + 1$. The uncertainty in fixing the strange quark mass is also included. Where it is not given by the authors, we include a (+7) MeV error.

For two-flavor QCD, we estimate the systematic errors from non-relativistic methods [36,38,73,74] to be 21 MeV for $f_B$ and 25 MeV for $f_{B_s}$. For the systematic error of the continuum extrapolated results with FNAL heavy quarks [82,88] we use the estimate of Ref. [82] including errors due to continuum extrapolation, perturbation theory, $1/M$ extrapolation and, where applicable, the spin-magnetic coefficient. Ref. [36] makes an estimate of the light quark mass dependence of $f_B$ using 1-loop $\chi$PT and quotes an uncertainty $(9_b)$ MeV on $f_B$ from the chiral extrapolation. We assign the same error also to the $N_f = 2$ results of Refs. [38,73,74,88]. For the values of Ref. [82] we use their own estimate of the chiral extrapolation error. The uncertainty in the chiral extrapolations, a $\sim 50$ MeV increase in the decay constants if the $\chi_b - \Upsilon$ splitting instead of $m_{\rho}$ is used to set the scale quoted by [38] and [73], and the variation from determining the strange quark mass by setting the $K$ or $\phi$ meson mass to the physical value gives the second error on the $N_f = 2$ averages. The variation between using $m_{\rho}$ and $f_{\pi}$ to set the scale and the chiral extrapolation uncertainty determined by [82], and the variation in the strange quark mass give the second error in square brackets.

The results of this procedure are included in Table 5 as average1. Refs. [74] and [38] quote statistical errors on their quenched and $N_f = 2$ results on $f_{B_s}$ respectively which are about half the statistical errors of other calculations with a similar ensemble size. We therefore also calculate the average with their statistical errors and the error of Ref. [84] enlarged by a factor of two. The result, which is very close to average1, is quoted as average2 in Table 5. Except for the uncertainties due to fixing the scale and reaching the physical quark masses, the errors on these averages are only few percent. The double ratio of $B$ and $D$ decay constants $f_B \sqrt{M_B}/(f_B \sqrt{M_B} \times f_D \sqrt{M_D}/(f_{D_s} \sqrt{M_{D_s}}))$ should be independent of the exact form of the chiral extrapolation.
up to $1/M_Q$ corrections, as is supported by the results of a two-flavor calculation using FNAL heavy quarks [88]. As argued by Ref. [87] using $\chi PT$, the chiral extrapolation uncertainty of the ratio $f_{B_s}\sqrt{M_{B_s}}/(f_{B}\sqrt{M_B}) \times f_{\pi}/f_K$ should also be small.

Employing unquenched staggered ($N_f = 2+1$) MILC gauge field ensembles at $a \simeq 0.13$ fm, the NRQCD estimate of [86] is:

$$f_{B_s} = 260(7)(28),$$

where systematical errors are added in quadrature. Calculations of the decay constants with the staggered $N_f = 2+1$ configurations using NRQCD [89] and FNAL [90] heavy quarks are in further progress. Within the statistical and systematical errors quoted in Table 5, the results with $N_f = 0, 2$ and $2 + 1$ agree among each other.

We relate this to the experimental value for $f_{D_s}$ using unquenched lattice results for the ratio $f_{B_s}/f_{D_s}$ from two-flavor calculations which work directly at the $b$ and $c$ quark masses without using extrapolations.

Taking the experimental value $f_{D_s} = 283(45)$ MeV (Eq. (5)), and the range of values for the ratio $f_{B_s}/f_{D_s}$ from Table 6 one obtains $f_{B_s} = 230 - 260$ MeV.

Other recent review articles [92–96] quote lattice estimates for $f_B$ and $f_{B_s}$ which are within errors in agreement with the averages quoted in Table 5.

In Table 5 we compare the lattice results with recent sum rule [97–100] and potential model calculations [101], and we find that they are within errors in agreement.

## 5 Conclusions

Applications of non-relativistic QCD and chiral perturbation theory in lattice calculations are presented. The status of lattice results on the light and heavy-light hadron spectrum and the decay constants $f_B$ and $f_{B_s}$ is summarized, and weighted lattice averages for $b$ hadron mass splittings and decay constants are calculated. The agreement of the hadron spectrum with experiment is a major success of lattice QCD in general, and of non-relativistic methods for heavy quarks in particular, and supports the reliability of lattice predictions of hadronic matrix elements. The lattice has become instrumental in QCD calculations. Work on further understanding and reduction of lattice errors is in progress and will enable very precise checks.

## Acknowledgements

I thank D. Ebert, R.N. Faustov, V.O. Galkin, A. Schäfer and G. Schierholz for discussions and comments on the manuscript, and C. Bernard, V.M. Braun, J. Koponen, L. Lellouch and C. Michael for discussions. I thank D. Toussaint for the numbers for the unquenched light baryons from MILC ( [43]).

I am grateful for a personal fellowship of the Deutsche Forschungsgemeinschaft. I would like to thank the group “Theory of Elementary Particles/Phenomenology-Lattice Gauge Theories” of the Humboldt University Berlin and the NIC/DESY Zeuthen for their kind hospitality.

The numerical computations relevant to my publications were performed at the computer centers EPCC (Edinburgh), LANL (Los Alamos), LRZ (München), NIC (Jülich) and NIC (Zeuthen), NCSA (Urbana-Champaign), RCCP (Tsukuba) and SCRI (Tallahassee). I thank all institutions for their support.
References

[1] Wilczek, F. 2003, Summary talk at ICHEP2002, 24-31 July 2002, Amsterdam, Nucl. Phys. Proc. Suppl. 117, 410.

[2] Wilczek F. 1999, Invited theoretical summary talk at 18th International Conference on Neutrino Physics and Astrophysics (NEUTRINO 98), Takayama, Japan, 4-9 Jun 1998, Nucl. Phys. (Proc. Suppl.) 77, 511.

[3] Ligeti Z. 2004, Plenary talk at ICHEP2004, 16-22 August, 2004, Beijing, China, to appear in the proceedings, hep-ph/0408267.

[4] Höcker A. 2004, talk at ICHEP2004, 16-22 August, 2004, Beijing, China, to appear in the proceedings, hep-ph/0410081.

[5] Djouadi A. 2005, hep-ph/0503172; Djouadi A. 2005, hep-ph/0503173.

[6] Ali, A. 2003, Lectures at the International Meeting on Fundamental Physics, Soto de Cangas (Asturias), Spain, February 23 - 28, 2003, to appear in the proceedings, hep-ph/0312303; Ligeti Z. 2004, Plenary talk at ICHEP2004, 16-22 August, 2004, Beijing, China, to appear in the proceedings, hep-ph/0408267.

[7] Gupta, R. 2003, Summary talk at the 2nd Workshop CKM Unitarity Triangle, 5-9 April 2003, IPPP Durham, UK, hep-ph/0311033.

[8] Sharpe S. 1995, Lectures given at Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 94), Boulder, CO, 29 May - 24 Jun 1994, in: CP Violation and the limits of the Standard Model, J.F. Donoghue World Scientific, Singapore; Davies, C.T.H. 2001, Lectures at 55th Scottish Universities Summer School, St Andrews, August 2001, in: Heavy Flavour Physics, C.T.H. Davies and S.M. Playfer (Eds.), Scottish Graduate Textbook Series, Institute of Physics 2002; Kronfeld, A.S. 2002, in: At the Frontiers of Physics: Handbook of QCD, Vol. 4, M. Shifman (Ed.), World Scientific, Singapore, hep-lat/0205021; C. McNeile, 2003, submitted to Int. Rev. Nucl. Phys, Vol.9, Hadronic Physics from Lattice QCD, A.M. Green (Ed.), hep-lat/0307027; DeGrand T. 2004, Int. J. Mod. Phys. A 19, 1337.

[9] Isgur N. and Wise M.B. 1992, Invited talk at Hadron 91, College Park, MD, Aug 12-16, 1991, Adv. Ser. Direct. High Energy Phys. 10, 549.

[10] Isgur, N. and Wise, M.B. 1992, in: Heavy Flavors, A.J. Buras and M. Lindner (Eds.), World Scientific, Singapore, 234; Neubert M. 1994, Phys. Rep. 245, 259; Grinstein, B. 1995, hep-ph/9508227; Manohar A. and Wise M.B. 2000, Heavy Quark Physics, Cambridge University Press, Cambridge.

[11] Thacker, B.A., and Lepage, G.P. 1991, Phys. Rev. D 43, 196.

[12] Lepage, G.P. et al. 1992, Phys. Rev. D 46, 4052.
[13] Chadha, M. et al. 1998, CLEO Collaboration, Phys. Rev. D 58, 032002.
[14] Heister, A. et al. (2002), ALEPH Collaboration, Phys. Lett. B 528, 1.
[15] Symanzik, K. 1980, in: Recent Developments in Gauge Theories, G. t’Hooft et al. (Eds.), Plenum, New York, 313;
Symanzik, K. 1982, in: Mathematical Problems in Theoretical Physics, R. Schrader et al. (Ed.), Springer, New York;
Symanzik K., 1983, Nucl Phys. B226, 187, 205.
[16] Lüscher M. and Weisz P. 1985, Commun. Math. Phys. 97, 59; Erratum ibid. 98, 433;
Lüscher M. and Weisz P. 1985, Phys. Lett. B 158, 250.
[17] Iwasaki Y. 1983, Univ. of Tsukuba Report No. UTHEP-118;
Iwasaki Y. 1985, Nucl. Phys. B258, 141;
[18] Hasenfratz P. and Niedermayer F. 1994, Nucl. Phys. B414, 785.
[19] Sheikholeslami B. and Wohler R. 1985, Nucl. Phys. B259, 572.
[20] Lüscher M. et al. 1997, ALPHA Collaboration, Nucl. Phys. B491, 323;
Jansen K. et al. 1998, ALPHA Collaboration, Nucl. Phys. B530, 185 and 2002, erratum ibid. B643, 517.
[21] Kaplan D. 1992, Phys. Lett. B 288, 342; Shamir Y. 1993, Nucl. Phys. B406, 90;
Fuhrman V. and Shamir Y. 1995, Nucl. Phys. B439, 54.
[22] Neuberger H. 1998, Phys. Rev. D 57, 5417.
[23] Hasenfratz P. 1998, Nucl. Phys. (Proc. Suppl.) 63, 53;
Hasenfratz P., Laliena V. and Niedermayer F. 1998, Phys. Lett. B 427, 125.
[24] Heitger J. and Sommer R. 2004, JHEP 0402, 022.
[25] El-Khadra A.X., Kronfeld A.S. and Mackenzie P.B. 1997, Phys. Rev. D 55, 3933;
Oktay M. et al. 2004, Nucl. Phys. (Proc. Suppl.) 129, 349.
[26] Aoki S., Kuramashi Y. and Tominaga S. 2002, Nucl. Phys. (Proc. Suppl.) 106, 349;
Aoki S., Kuramashi Y. and Tominaga S. 2003, Prog. Theor. Phys. 109, 383.
[27] Bernard C. 2001, plenary talk at Lattice 2000, Nucl. Phys. (Proc. Suppl.) 94, 159 (2001).
[28] Kronfeld A.S. 2004, Nucl. Phys. (Proc. Suppl.) 129, 46.
[29] Gasser J. and Leutwyler H. 1984, Ann. Phys. 158, 142;
Gasser J. and Leutwyler H. 1985, Nucl. Phys. B250, 465;
Weinberg S. 1996, The Quantum Theory of Fields, Vol. II, Cambridge University Press, Cambridge.
[30] Jenkins E. and Manohar A.V. 1991, Phys. Lett. B 255, 558;
Bernard V., Kaiser N. and Meißner U.-G. 1995, Int. J. Mod. Phys. E 4, 193.
[31] Weinberg S. 1979, Physica 96A, 327; Tang H.-B. 1996, hep-ph/9607436; Gasser J., Sainio M.E. and Svarc A. 1988, Nucl. Phys. B307, 779.

[32] Becher T. and Leutwyler H. 1999, Eur. Phys. J. C9, 643.

[33] Gasser J. and Leutwyler H. 1987, Phys. Lett. B 188, 477; Neuberger H. 1988, Phys. Rev. Lett. 60, 889; Nucl. Phys. B300, 180.

[34] Sommer R. 1994, Nucl. Phys. B411, 839.

[35] Guagnelli M., Sommer R. and Wittig H. 1998, ALPHA Collaboration, Nucl. Phys. B535, 389.

[36] Aoki S. et al. 2003, JLQCD Collaboration, Phys. Rev. Lett. 91, 212001.

[37] Davies C.T.H. et al. 1997, Phys. Rev. D 56, 2755.

[38] Collins S. et al. 1999, SGO Collaboration, Phys. Rev. D 60, 074504.

[39] Manke T. et al. 2000, Phys. Rev. D 62, 114508.

[40] Gottlieb S. 2004, plenary talk at Lattice 2003, Nucl. Phys. (Proc. Suppl.) 129, 17.

[41] Ali Khan A. et al. 2003, CP-PACS Collaboration, Phys. Rev. D 65, 054505 (2002) and erratum ibid. 67, 05901 (2003).

[42] Aoki S. et al. 2003, JLQCD Collaboration, Phys. Rev. D 68, 054502.

[43] Aubin C. et al. 2004, Phys. Rev. D 70, 094505.

[44] Hein J. et al. 2000, Phys. Rev. D 62, 074503.

[45] Bowler K.C. et al. 2001, UKQCD Collaboration, Nuclear Physics B 619, 507.

[46] Butler F. et al. 1994, Nucl. Phys. B430, 179.

[47] Aoki S. et al. 2003, CP-PACS Collaboration, Phys. Rev. D 67, 034503.

[48] Bernard V., Hemmert T.R. and Meißner U.-G. 2004, Nucl. Phys. A732, 149.

[49] Procura M., Hemmert T.R. and Weise W., 2004, Phys. Rev. D 69, 034505.

[50] Ali Khan A. et al. 2004, QCDSF collaboration and UKQCD collaboration, Nucl. Phys. B689, 175.

[51] Ali Khan A. et al. 2003, Nucl. Phys. B (Proc. Suppl.) 119, 419; Ali Khan A. et al. 2004, in: Color Confinement and Hadrons in Quantum Chromodynamics, Proceedings of Confinement 2003, Tokyo, 21–24 July 2003, eds. H. Suganuma et al., World Scientific, Singapore, 383.

[52] Bowler K.C. et al. 1996, UKQCD Collaboration, Phys. Rev. D 54, 3619.

[53] Green A.M. et al. 2004, UKQCD Collaboration, Phys. Rev. D 69, 094505.
Burch T., Gattringer C. and Schäfer A. 2005, Nucl. Phys. (Proc.Suppl.) 140, 347.

Aoki S. et al. 2003, JLQCD Collaboration, Phys. Rev. D 69, 094512.

Ali Khan A. et al. 2000, GLOK Collaboration, Phys. Rev. D 62, 054505.

Ali Khan A. et al. 2000, CP-PACS Collaboration, Nucl. Phys. B (Proc. Suppl.) 83, 265.

Lewis R. and Woloshyn R.M. 2000, Phys. Rev. D 62, 114507.

Mathur N., Lewis R. and Woloshyn R.M. 2002, Phys. Rev. D 66, 014502.

Wingate M. et al. 2003, Phys. Rev. D 67, 054505.

Alford M.G., Klassen, T.R. and Lepage G.P. 1998, Phys. Rev. D 58, 034503.

Particle Data Group, Hagiwara K. et al. 2002, Phys. Rev. D 66, 010001.

Lepage G.P. and Mackenzie P.B. 1993, Phys. Rev. D 48, 2250.

Ishikawa K.-I. et al. 2000, JLQCD Collaboration, Phys. Rev. D 61, 074501.

Mackenzie P., Ryan S. and Simone J. 1998, Nucl. Phys. B (Proc. Suppl.) 63, 305.

Wingate M. et al. 2003, Nucl. Phys. Proc. Suppl. 119, 604.

Godfrey S. and Isgur N. 1985, Phys. Rev. D 32, 189; Godfrey S. and Kokoski R. ibid. 1991, 43, 1679.

Isgur N. 1998, Phys. Rev. D 57, 4041.

Ebert D., Faustov R.N. and Galkin V.O. 1998, Phys. Rev. D 57, 5663 and erratum ibid. 59, 019902.

Di Pierro M. and Eichten E. 2001, Phys. Rev. D 64, 114004.

Capstick S. and Isgur N. 1986, Phys. Rev D 34, 2809.

Jenkins E. and Manohar A.V. 1992, Phys. Lett. B 294, 273.

Ali Khan A. et al. 2001, CP-PACS Collaboration, Phys. Rev. D 64, 054504.

Ali Khan A. et al. 2001, CP-PACS Collaboration, Phys. Rev. D 64, 034505.

Collins S. et al. 2001, Phys. Rev. D 63, 034505.

Aoki S. et al. 1998, Phys. Rev. Lett. 80, 5711.

Ali Khan A. et al. 1998, GLOK Collaboration, Phys. Lett. B427, 132.

Abada A. et al. 2000, Nucl. Phys. B (Proc. Suppl.) 83, 268.

Lellouch L. and Lin C.-J.D. 2001, UKQCD Collaboration, Phys. Rev. D 64, 094501.
Becirevic D. et al. 2001, Nucl. Phys. B618, 241.

El-Khadra A.X. et al. 1998, Fermilab Collaboration, Phys. Rev. D 58, 014506.

Bernard C. et al. 2002, Phys. Rev. D 66, 094501.

Guagnelli M. et al. 2002, Phys. Lett. B546, 237.

de Divitiis G.M. et al. 2003, Nucl. Phys. B672, 372.

Della Morte M. et al. 2004, ALPHA Collaboration, Phys. Lett. B581, 93;
Jüttner A. and Rolf J. 2003, ALPHA Collaboration, Phys. Lett. B560 (2003) 59;
Rolf J. and Sint S. 2002, ALPHA Collaboration, JHEP 12, 007.

Wingate M. et al. 2004, Phys. Rev. Lett. 92, 162001.

Becirevic D. et al. 2003, Phys. Lett. B 563, 150.

Onogi T. et al. 2004, Nucl. Phys. B (Proc. Suppl.) 129 373.

Gray A. et al. 2005, Nucl. Phys. (Proc. Suppl.) 140, 446.

Bernard C. et al. 2005, Nucl. Phys. (Proc. Suppl.) 140, 449.

Maynard C.M. 2002, UKQCD Collaboration, Nucl. Phys. B (Proc. Suppl.) 106, 388 (2002).

Ryan S.M. 2002, plenary talk at Lattice 2001, Nucl. Phys. (Proc. Suppl.) 106, 86.

Lellouch L. 2003, plenary talk at ICHEP 2002, Nucl. Phys. (Proc. Suppl.) 117, 127.

Becirevic D. 2003, invited talk at the Workshop on the CKM Unitarity Triangle, IPPP Durham, 5-9 April 2003, hep-ph/0310072.

Wittig H. 2004, invited talk at the International Europhysics Conference on High Energy Physics EPS-HEP2003, 17-23 July 2003, Aachen, Germany, Eur. Phys. J. C33, S890.

Hashimoto S. 2004, plenary talk at ICHEP 2004, August 16-22, 2004, Beijing, China, hep-ph/0411126.

Narison S. 2001, Phys. Lett. B 520, 115.

Penin A. and Steinhauser M. 2002, Phys. Rev. D 65 054006.

Jamin M. and Lange B.O. 2002, Phys. Rev. D 65, 056005.

Braun V.M. 1999, talk at Heavy Flavors 8, Southampton, U.K., 25–29 July 1999, hep-ph/9911206; Braun V.M. 2004, private communication.

Ebert D., Faustov R.N. and Galkin V.O. 2002, Mod. Phys. Lett. A 17, 803.
| Ref. | $\Delta M(B)$[MeV] | $\Delta M(B_s)$[MeV] |
|------|-------------------|------------------|
| $B_0^* - B$ |
| Lattice |  |
| [44], $a \sim 1.1$GeV$^{-1}$ | 400(30)(9) | 295(45)(11) |
| [44], $a \sim 2.6$GeV$^{-1}$ | 475(19)$^{(20)}$ | 451$^{(15)}_{(16)}$ |
| 58 | 374(37) | 357(27)$^{(9)}$ |
| 60, 1l | 442(56) |
| 60, 1l | 471(25) |
| 60, 1l | 315(105) |
| 60, 1l | 425(60) |
| 60, Asq | 285(78) |
| 60, Asq | 523(94) |
| 60, Asq | 403(56) |
| 53 | 386(31) |
| 54 | 408(67) | 419(37) |
| average | 402(41)$^{(33)}$ | 382(23)$^{(27)}_{(33)}$ |
| Model calculations |  |
| 67 | 450 | 440 |
| 69 | 453 | 466 |
| 70 | 427 | 431 |
| $B_2^* - B$ |
| Lattice |  |
| [44], $a \sim 1.1$GeV$^{-1}$ | 402(78)$^{(20)}_{(6)}$ |
| [44], $a \sim 2.6$GeV$^{-1}$ | 493$^{(20)}_{(22)}$ | 474(62)$^{(6)}_{(16)}$ |
| 58 | 526(45) | 493(26) |
| 57 | 426(17) |
| 53 | 534(52) |
| 54 | 440(77) | 455(41) |
| average | 498(49)$^{(22)}_{(28)}$ | 487(31)$^{(4)}_{(8)}$ |
| Model calculations |  |
| 67 | 450 | 490 |
| 69 | 453 | 469 |
| 70 | 435 | 447 |
| Preliminary experimental |  |
| 62 | 419(8) | 483(15) |

Table 2: $B$ orbital excitations. Only statistical errors and, where quoted by the authors, errors due to chiral extrapolation and fixing the $b$ quark mass are shown. For the $B_2^*$ of [44] at $a \sim 1.1$GeV$^{-1}$ we quote the result from the lattice operator they use for the $\overline{D}_0 - \overline{S}$ splitting. The other lattice operator corresponding to the $B_2^*$ in their calculation gives a $\sim 45$ MeV higher result.
Table 3: $P$ state fine structure of $B$ mesons. Only statistical errors and errors due to chiral extrapolation and fixing the $b$ quark mass are shown.
| Ref. | $\Lambda_b - B$[MeV] | $\Sigma_b - \Lambda_b$[MeV] |
|------|-----------------|-----------------|
| **Lattice** | | |
| 52 | 338(52) | 186(61) |
| 56 | 370(67) | 221(71) |
| 59, $a_s = 1.1$GeV$^{-1}$ | 361(103) | 156(39) |
| 59, $a_s = 0.9$GeV$^{-1}$ | 367(108) | 191(37) |
| 55 | 389(44) | 122(65) |
| 57, quenched | 361(22) | |
| 38, $N_f = 2$ | 545(40)(22) | |
| 57, $N_f = 2$ | 417(19) | |
| quenched average | 372(33)(33) | 174(28)(15) |
| **Models** | | |
| 72 | 312 | 196 |
| 71 | 217 |
| 71, c quark | 212 |
| **Experiment** | | |
| 62 | 311(10) | |

| Ref. | $\Sigma_b - \Omega_b$[MeV] | $\Omega_b - \Omega_b$[MeV] |
|------|-----------------|-----------------|
| **Lattice** | | |
| 56 | 19(7) | 18(4) |
| 59, $a_s = 1.1$GeV$^{-1}$ | 22(12) | 18(3) |
| 59, $a_s = 0.9$GeV$^{-1}$ | 24(13) | 20(9) |
| 55 | 10(12) | 7(4) |
| average | 18(8)(4) | 13(7)(3) |
| **Models** | | |
| 71 | 10 | |
| 72 | 8 | |

Table 4: $b$ baryons. Only statistical errors and, where applicable, systematical errors due to scale setting except for the quenched scale ambiguity and fitting are shown.
| Ref. | scale | $f_B$ [MeV] | $f_{B_s}$ [MeV] |
|------|--------|-------------|-----------------|
|      |        |             |                 |
| Lattice | $N_f = 0$ |             |                 |
| 76   | $m_\rho$ | 173(4)(13)  | 199(3)(14)     |
| 81   | $f_K$   | 164(11)(8)  | 185(8)(9)      |
| 77   | $m_\rho$ | 147(11)(16) | 175(8)(16)     |
| 64   | $\sqrt{\sigma} = 427$MeV | 170(5)(15)  | 191(4)(17)     |
| 78   | $\frac{M_K^*}{M_K}$, $M_K^*$ | 173(13)(24) | 196(11)(22)    |
| 45   | $f_\pi$ | 195(6)(23)  | 220(6)(28)     |
| 79   | $f_K$   | 177(17)(22) | 204(12)(24)    |
| 75   | $m_\rho$ | 187(4)(15)  | 187(4)(15)     |
| 74   | $m_\rho$ | 188(3)(26)  | 220(2)(31)     |
| 73   | $m_\rho$ | 191(4)(27)  | 220(4)(31)     |
| 80   | $\frac{M_K^*}{M_K}$, $M_K^*$ | 174(22)(8)  | 204(15)(8)     |
| 82   | $f_\pi$ | 173(6)(16)  | 199(5)(17)     |
| 83   | $r_0$   | 170(11)(23) | 192(9)(25)     |
| 84   | $r_0$   | 192(6)(4)   | 192(6)(4)      |
| 85   | $r_0$   | 205(12)     | 205(12)        |
| 60   | $r_0$   | 225(9)(34)  | 225(9)(34)     |
| average1 |        | 175(7)(48[21]) | 198(5)(46[9]) |
| average2 |        | 201(6)(51[13]) |             |
| $N_f = 2$ |             |             |                 |
| 38   | $m_\rho$ | 186(5)(25)  | 215(3)(20)     |
| 74   | $m_\rho$ | 208(10)(29) | 250(10)(35)    |
| 73   | $m_\rho$ | 204(8)(29)  | 242(9)(34)     |
| 82   | $f_\pi$ | 190(7)(23)  | 217(6)(36)     |
| 36   | $m_\rho$ | 191(10)(10) | 215(9)(13)     |
| 88   | $r_0 = 0.49$fm | 181(7)(23) |             |
| average1 |        | 190(10)(55[6]) | 226(15)(53[7]) |
| average2 |        | 226(15)(55[8]) |             |
| $N_f = 2 + 1$ |             |             |                 |
| 86   | $\Upsilon(2S - 1S)$ | 260(7)(28) |             |
| $N_f = 2 + 1$ |             |             |                 |
| Sum rules |             |             |                 |
| 97   |        | 203(23)     | 236(30)        |
| 98   |        | 206(20)     |                 |
| 99   |        | 210(19)     | 244(21)        |
| 100  |        | 180 - 190(30) |             |
| Potential models |             |             |                 |
| 101  |        | 178(15)     | 196(20)        |

Table 5: $f_B$ and $f_{B_s}$ from the lattice. Statistical errors and systematical errors given by the authors are included, adding the systematical errors in quadrature. The method to set the scale is indicated in the second column. The first error on the averages is due to the statistical and systematical errors of the individual results, while the second error is from chiral extrapolation uncertainties and scale ambiguities as explained in the text.
| Ref. | Scale | $f_B^{N_f=2}/f_B^{N_f=0}$ | Ref. | $f_B/f_D$ |
|------|-------|-----------------|------|---------|
| [82] | $f_\pi$ | 1.10(6) | [74]($N_f = 0$) | 0.88(1) |
| 82   | $m_\rho$ | 1.19(6) | [82]($N_f = 0$) | 0.891(12)(40)34 |
| 74   | $m_\rho$ | 1.11(6) | 85 ($N_f = 0$) | 0.81(6) |
| 73   | $m_\rho$ | 1.07(5) | 74 ($N_f = 2$) | 0.94(6) |
| [73] | $\Upsilon(P-S)$ | 0.97(5) | [82]($N_f = 2$) | 0.922(13)(68)55 |
| 91   | $f_D$ | 0.98(4) | |

Table 6: Ratios of decay constants. The first error is statistical, the second the systematical errors given by the authors added in quadrature, where applicable.