ON THE DOMAIN OF FOUR-DIMENSIONAL FORWARD DIFFERENCE MATRIX IN SOME DOUBLE SEQUENCE SPACES

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ABSTRACT. In this paper, we introduce some new double sequence spaces $\mathcal{M}_u(\Delta)$ and $\mathcal{C}_\vartheta(\Delta)$, where $\vartheta \in \{b, bp, r, r0\}$ as the domains of the four-dimensional forward difference matrix in the double sequence spaces $\mathcal{M}_u$ and $\mathcal{C}_\vartheta$, respectively. Then we investigate some topological and algebraic properties. Moreover, we determine the $\alpha$, $\beta(\vartheta)$, and $\gamma$-duals of the new spaces $\mathcal{M}_u(\Delta)$ and $\mathcal{C}_\vartheta(\Delta)$. Finally, we characterize four-dimensional matrix classes $(\lambda(\Delta), \mu)$ and $(\mu, \lambda(\Delta))$, where $\lambda = \{\mathcal{M}_u, \mathcal{C}_\vartheta\}$ and $\mu = \{\mathcal{M}_u, \mathcal{C}_\vartheta\}$.

1. INTRODUCTION

By $\Omega := \{x = (x_{mn}) : x_{mn} \in \mathbb{C}, \forall m, n \in \mathbb{N}\}$, we denote the set of all complex valued double sequences; $\Omega$ is a vector space with coordinatewise addition and scalar multiplication and any vector subspace of $\Omega$ is called a double sequence space. A double sequence $x = (x_{mn})$ is called convergent in Pringsheim's sense to a limit point $L$, if for every $\epsilon > 0$ there exists a natural number $n_0 = n_0(\epsilon)$ and $L \in \mathbb{C}$ such that $|x_{mn} - L| < \epsilon$ for all $m, n > n_0$, where $\mathbb{C}$ denotes the complex field; this is denoted by $L = \lim_{m,n \to \infty} x_{mn}$. The space of all double sequences that are convergent in the Pringsheim sense is denoted by $\mathcal{C}_p$, which is a linear space with coordinatewise addition and scalar multiplication. Mōricz [1] proved that the double sequence space $\mathcal{C}_p$ is a complete seminormed space with the seminorm

$$
\|x\|_\infty = \lim_{N \to \infty} \sup_{m,n \geq N} |x_{mn}|.
$$

The space of all null double sequences in Pringsheim’s sense is denoted by $\mathcal{C}_{p0}$.

A double sequence $x = (x_{mn})$ of complex numbers is called bounded if $\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$, and the space of all bounded double sequences is denoted by $\mathcal{M}_u$, that is,

$$
\mathcal{M}_u := \{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty\};
$$

it is a Banach space with the norm $\| \cdot \|_\infty$.

Unlike as in the case of single sequences there are double sequences which are convergent in Pringsheim’s sense but unbounded. That is, the set $\mathcal{C}_p \setminus \mathcal{M}_u$ is not empty. Boos [2] defined the sequence $x = (x_{mn})$ by

$$
x_{mn} = \begin{cases} n & , m = 0, n \in \mathbb{N} \\
0 & , m \geq 1, n \in \mathbb{N},
\end{cases}
$$

which is obviously in $\mathcal{C}_p$, i.e., $p - \lim_{m,n \to \infty} x_{mn} = 0$, but not in the set $\mathcal{M}_u$, i.e., $\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| = \infty$. Thus, $x \in \mathcal{C}_p \setminus \mathcal{M}_u$.

We also consider the set $\mathcal{C}_{bp}$ of double sequences which are both convergent in Pringsheim’s sense and bounded, that is,

$$
\mathcal{C}_{bp} := \mathcal{C}_p \cap \mathcal{M}_u = \left\{x = (x_{mn}) \in \mathcal{C}_p : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\}.
$$

The set $\mathcal{C}_{bp}$ is a Banach space with the norm

$$
\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty.
$$

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Hardy [8] called a sequence in the space $C_p$ regularly convergent if it is a convergent single sequence with respect to each index. We denote the set of such double sequences by $C_r$, that is,

$$C_r := \{ x = (x_{mn}) \in C_p : \forall m \in \mathbb{N} (x_{mn})_m \in c, \quad \text{and} \quad \forall n \in \mathbb{N} (x_{mn})_n \in c \},$$

where $c$ denotes the set of all convergent single sequences of complex numbers. Regular convergence requires the boundedness of double sequences; this is the main difference between regular convergence and the convergence in Pringsheim’s sense. We also use the notations $C_{bp0} = M_u \cap C_{p0}$ and $C_{r0} = C_r \cap C_{p0}$.

Throughout the text, unless otherwise stated we mean by the summation $\sum_{k,l} x_{kl}$ without limits run from 0 to $\infty$ is $\sum_{k,l=0}^{\infty} x_{kl}$.

The space $L_q$ of all absolutely $q$–summable double sequences was introduced by Başar and Sever [2] as follows

$$L_q := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^q < \infty \right\}, \quad (1 \leq q < \infty)$$

which is a Banach space with the norm $\| \cdot \|_q$ defined by

$$\|x\|_q = \left( \sum_{k,l} |x_{kl}|^q \right)^{1/q}.$$

Moreover, Zeltser [3] introduced the space $L_u$ which is the special case of the space $L_q$ for $q = 1$.

The double sequence spaces $BS$, $CS_\vartheta$, where $\vartheta \in \{ p, bp, r \}$, and $BV$ were introduced by Altay and Başar [6]. The set $BS$ of all double series whose sequences of partial sums are bounded is defined by

$$BS = \left\{ x = (x_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} |s_{mn}| < \infty \right\}$$

where the sequence $s_{mn} = \sum_{k,l=0}^{m,n} x_{kl}$ is the $(m,n)–th$ partial sum of the series. The series space $BS$ is a Banach space with norm defined as

$$(1.1) \quad \|x\|_{BS} = \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=0}^{m,n} x_{kl} \right|,$$

which is linearly isomorphic to the sequence space $M_u$. The set $CS_\vartheta$ of all series whose sequences of partial sums are $\vartheta$–convergent in Pringsheim’s sense is defined by

$$CS_\vartheta = \{ x = (x_{kl}) \in \Omega : (s_{mn}) \in C_\vartheta \}$$

where $\vartheta \in \{ p, bp, r \}$. The space $CS_p$ is a complete seminormed space with the seminorm defined by

$$\|x\|_\infty = \lim_{n \to \infty} \left( \sup_{k,l \geq n} \left| \sum_{i,j=0}^{k,l} x_{ij} \right| \right),$$

which is isomorphic to the sequence space $C_p$. Moreover, the sets $CS_{bp}$ and $CS_r$ are also Banach spaces with the norm (1.1) and the inclusion $CS_r \subset CS_{bp}$ holds. The set $BV$ of all double sequences of bounded variation is defined by Altay and Başar [8] as follows

$$BV = \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}| < \infty \right\}.$$

The space $BV$ is Banach space with the norm defined by

$$\|x\|_{BV} = \sum_{k,l} |x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}|,$$

which is linearly isomorphic to the space $L_u$ of absolutely convergent double series. Moreover, the inclusions $BV \subset C_\vartheta$ and $BV \subset M_u$ strictly hold.
Let $E$ be any double sequence space. Then,

$$dE := \left\{ x = (x_{kl}) \in \Omega : \left\{ \frac{1}{kl}x_{kl} \right\}_{k,l \in \mathbb{N}} \in E \right\},$$

$$E := \left\{ x = (x_{kl}) \in \Omega : \{klix_{kl}\}_{k,l \in \mathbb{N}} \in E \right\},$$

$$E^{\beta(\vartheta)} := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{CS}_\vartheta, \text{ for every } x = (x_{kl}) \in E \right\},$$

$$E^\alpha := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{L}u, \text{ for every } x = (x_{kl}) \in E \right\},$$

$$E^\gamma := \left\{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl}\} \in \mathcal{BS}, \text{ for every } x = (x_{kl}) \in E \right\}.$$

Therefore, let $E_1$ and $E_2$ are arbitrary double sequences with $E_2 \subset E_1$ then the inclusions $E_1^\alpha \subset E_2^\alpha$, $E_1^\gamma \subset E_2^\gamma$ and $E_1^{\beta(\vartheta)} \subset E_2^{\beta(\vartheta)}$ hold. But the inclusion $E_1^\gamma \subset E_1^{\beta(\vartheta)}$ does not hold, since $\mathcal{C}_p \setminus \mathcal{M}_u$ is not empty.

Let $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ be an infinite four-dimensional matrix and $E_1, E_2 \in \Omega$. We write

$$y_{mn} = A_{mn}(x) = \vartheta - \sum_{k,l} a_{mnk}x_{kl} \text{ for each } m, n \in \mathbb{N}.$$ (1.2)

We say that $A$ defines a matrix transformation from $E_1$ to $E_2$ if

$$A(x) = (A_{mn}(x))_{m,n \in E_2} \text{ for all } x \in E_1.$$ (1.3)

The $\vartheta$-summability domain $E_A^{(\vartheta)}$ of a four-dimensional infinite matrix $A$ in a double sequence space $E$ is defined by

$$E_A^{(\vartheta)} := \left\{ x = (x_{kl}) \in \Omega : Ax = \left( \vartheta - \sum_{k,l} a_{mnk}x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } E \right\},$$

which is a sequence space. The above notation (1.3) says that $A = (a_{mnkl})_{m,n,k,l \in \mathbb{N}}$ maps the space $E_1$ into the space $E_2$ if $E_1 \subset (E_2)_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices that map the space $E_1$ into the space $E_2$ by $(E_1 : E_2)$. Thus, $A \in (E_1 : E_2)$ if and only if the double series on the right side of (1.3) $\vartheta$–converges for each $m, n \in \mathbb{N}$, i.e, $A_{mn} \in (E_1)^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and we have $Ax \in E_2$ for all $x \in E_1$.

Adams [7] defined that the four-dimensional infinite matrix $A = (a_{mnkl})$ is a triangular matrix if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both. We also say by Wilansky [8, Theorem 4.4.2, p. 66] that a triangular matrix $A = (a_{mnkl})$ is called a triangle if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$. One can be observed easily that if $A$ is triangle, then $E_A^{(\vartheta)}$ and $E$ are linearly isomorphic.

Wilansky [8] Theorem 4.4.2, p. 66] defined that if $E$ is a sequence space, then the continuous dual $E_A^*$ of the space $E_A$ is given by

$$E_A^* = \{ f : f = g \circ A, g \in E^* \}.$$ 

Zeltser [9] stated the notations of the double sequences $e^{kl} = (e_{mn}^{kl})$, $e^1$, $e_k$ and $e$ by

$$e_{mn}^{kl} = \begin{cases} 1 & , \quad (k,l) = (m,n); \\ 0 & , \quad \text{otherwise}. \end{cases}$$
\[ e^1 = \sum_k e^{kl}; \text{ the double sequence that all terms of } l\text{-th column are one and other terms are zero,} \]
\[ e_k = \sum_l e^{kl}; \text{ the double sequence that all terms of } k\text{-th row are one and other terms are zero,} \]
\[ e = \sum_{kl} e^{kl}; \text{ the double sequence that all terms are one} \]

for all \( k, l, m, n \in \mathbb{N} \).

The four-dimensional forward difference matrix \( \Delta = (\delta_{mnkl}) \) is defined by
\[ \delta_{mnkl} := \begin{cases} (-1)^{m+n-k-l}, & m \leq k \leq m + 1, \ n \leq l \leq n + 1, \\ 0, & \text{otherwise} \end{cases} \]
for all \( m, n, k, l \in \mathbb{N} \). The \( \Delta \)-transform of a double sequence \( x = (x_{mn}) \) is given by
\[ y_{mn} := \{\Delta x\}_{mn} = x_{mn} - x_{m+1,n} - x_{m,n+1} + x_{m+1,n+1} \]
for all \( m, n \in \mathbb{N} \). We shall briefly discuss \( \Delta^{-1} \) which is the inverse of four-dimensional forward difference matrix \( \Delta \), where \( (\Delta^{-1}\Delta)(x_{kl}) = x_{kl} \). Let \( \Delta^{-1}y_{kl} = x_{kl} \). Then we can show that \( x_{kl} \) is a finite summation of the original double sequence \( y_{kl} \).

\[ (1.4) \quad \Delta(\Delta^{-1}y_{kl}) = \Delta x_{kl} = x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}. \]

If we write the equation (1.4) for \( y_{00}, y_{01}, y_{10}, \ldots, y_{kl} \)

\[
\begin{align*}
\Delta(\Delta^{-1}y_{00}) &= \Delta x_{00} = x_{00} - x_{10} - x_{01} + x_{11} \\
\Delta(\Delta^{-1}y_{01}) &= \Delta x_{01} = x_{01} - x_{11} - x_{02} + x_{12} \\
\Delta(\Delta^{-1}y_{10}) &= \Delta x_{10} = x_{10} - x_{20} - x_{11} + x_{21} \\
\Delta(\Delta^{-1}y_{11}) &= \Delta x_{11} = x_{11} - x_{21} - x_{12} + x_{22} \\
&\vdots \\
\Delta(\Delta^{-1}y_{kl}) &= \Delta x_{kl} = x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}.
\end{align*}
\]

Then we add the left hand sides up to \( y_{00} + y_{01} + y_{10} + \ldots + y_{kl} \)

\[
\sum_{i,j=0}^{k,l} y_{i,j} = x_{k+1,l+1} + x_{00} - x_{k+1,0} - x_{0,l+1}
\]

for all \( k, l \in \mathbb{N} \). To be able to have \( x_{kl} \) instead of having \( x_{k+1,l+1} \) we must write it as

\[ (1.5) \quad x_{kl} = \sum_{i,j=0}^{k-1,l-1} y_{i,j} - x_{00} + x_{k,0} + x_{0,l} \]

for all \( k, l \in \mathbb{N} \). With this result we can introduce the role of inverse four-dimensional forward difference operator \( \Delta^{-1} \) on the double sequence \( y_{kl} \), where \( x_{kl} = \Delta^{-1}y_{kl} \), as the \((k - 1, l - 1)^{th}\)–partial sum of the double sequence \( y_{kl} \) plus arbitrary constants on the first row and the first column of the double sequence \( x = (x_{kl}) \).
2. New double sequence spaces

In this section, we introduce new double sequence spaces \( \mathcal{M}_u(\Delta) \), \( \mathcal{C}_\vartheta(\Delta) \), where \( \vartheta \in \{bp, rp, r0\} \), as the matrix domains of the four-dimensional matrix of the forward differences in the sequence spaces \( \mathcal{M}_u \) and \( \mathcal{C}_\vartheta \) as follow:

\[
\mathcal{M}_u(\Delta) := \left\{ x = (x_{kl}) \in \Omega : \sup_{k,l \in \mathbb{N}} |y_{kl}| < \infty \right\},
\]

\[
\mathcal{C}_\vartheta(\Delta) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni \vartheta - \lim_{k,l \to \infty} |y_{kl} - L| = 0 \right\},
\]

\[
\mathcal{C}_{\vartheta0}(\Delta) := \left\{ x = (x_{kl}) \in \Omega : \vartheta - \lim_{k,l \to \infty} |y_{kl}| = 0 \right\},
\]

where \( y_{kl} = \Delta x_{kl} = (x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}) \) for all \( k, l \in \mathbb{N} \).

**Theorem 2.1.** The spaces \( \mathcal{M}_u(\Delta) \) and \( \mathcal{C}_\vartheta(\Delta) \), where \( \vartheta \in \{bp, bp0, r, r0\} \) are Banach spaces with the norm

\[
\|x\|_{\mathcal{M}_u(\Delta)} := |x_{k,0} + x_{0,l} - x_{00}| + \|\Delta x\|_{\mathcal{M}_u} := |x_{k,0} + x_{0,l} - x_{00}| + \sup_{k,l \in \mathbb{N}} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}|.
\]

**Proof.** The linearity of those spaces is clear. Suppose that \( x^i = (x^i_{kl}) \) is a Cauchy sequence in the space \( \mathcal{M}_u(\Delta) \) for all \( k, l \in \mathbb{N} \). Then

\[
\|x^i - x^j\|_{\mathcal{M}_u(\Delta)} = |(x^i_{k,0} - x^j_{k,0}) + (x^i_{0,l} - x^j_{0,l}) - (x^i_{00} - x^j_{00})|
\]

\[+ \sup_{k,l \in \mathbb{N}} |\Delta(x^i_{kl} - x^j_{kl})| \to 0 \]

as \( i, j \to \infty \). Thus, we obtain \( |x^i_{kl} - x^j_{kl}| \to 0 \) for \( i, j \to \infty \) and for every \( k, l \in \mathbb{N} \). Hence \( x^i = (x^i_{kl}) \) is a Cauchy sequence in \( \mathbb{C} \) for each \( k, l \in \mathbb{N} \). Since \( \mathbb{C} \) is complete, then it converges to a sequence \( x = (x_{kl}) \), i.e., we have

\[
\lim_{i \to \infty} x^i_{kl} = x_{kl}
\]

for each \( k, l \in \mathbb{N} \). Therefore, for every \( \epsilon > 0 \), there exits a natural number \( N(\epsilon) \), such that for all \( i, j \geq N(\epsilon) \), and for all \( k, l \in \mathbb{N} \) we have

\[
|x^i_{k,0} - x^j_{k,0}| < \frac{\epsilon}{4}, |x^i_{0,l} - x^j_{0,l}| < \frac{\epsilon}{4}, |x^i_{0,0} - x^j_{0,0}| < \frac{\epsilon}{4}, |\Delta(x^i_{kl} - x^j_{kl})| < \frac{\epsilon}{4}.
\]

Moreover,

\[
\lim_{j \to \infty} |x^i_{k,0} - x^j_{k,0}| = |x^i_{k,0} - x_{k,0}| < \frac{\epsilon}{4},
\]

\[
\lim_{j \to \infty} |x^i_{0,l} - x^j_{0,l}| = |x^i_{0,l} - x_{0,l}| < \frac{\epsilon}{4},
\]

\[
\lim_{j \to \infty} |x^i_{0,0} - x^j_{0,0}| = |x^i_{0,0} - x_{0,0}| < \frac{\epsilon}{4},
\]

\[
\lim_{j \to \infty} |\Delta(x^i_{kl} - x^j_{kl})| = |\Delta(x^i_{kl} - x_{kl})| < \frac{\epsilon}{4}
\]

for all \( i \geq N(\epsilon) \). Hence, we obtain that

\[
\|x^i - x\|_{\mathcal{M}_u(\Delta)} = |(x^i_{k,0} - x_{k,0}) + (x^i_{0,l} - x_{0,l}) - (x^i_{00} - x_{00})|
\]

\[+ \sup_{k,l \in \mathbb{N}} |\Delta(x^i_{kl} - x_{kl})| \leq |x^i_{k,0} - x_{k,0}| + |x^i_{0,l} - x_{0,l}| + |x^i_{0,0} - x_{0,0}|
\]

\[+ \sup_{k,l \in \mathbb{N}} |\Delta(x^i_{kl} - x_{kl})| < \epsilon.
\]
Now we must show that \( x \in \mathcal{M}_u(\Delta) \).

\[
\sup_{k,l \in \mathbb{N}} |\Delta x_{kl}| = \sup_{k,l \in \mathbb{N}} |x_{kl} - x_{k,l+1} + x_{k+1,l+1} - x_{k+1,l}| = \sup_{k,l \in \mathbb{N}} |x_{kl} - x_{k,l+1} + x_{k+1,l} + x_{k+1,l+1} - x_{k+1,l+1} - x_{k,l+1}| + |x_{k+1,l+1} - x_{k+1,l+1} + x_{k+1,l+1}| \leq \sup_{k,l \in \mathbb{N}} \left| \Delta x_{kl} \right| + \sup_{k,l \in \mathbb{N}} \left| \Delta x_{kl} - \Delta x_{kl} \right| < \infty
\]

Hence \( x = (x_{kl}) \in \mathcal{M}_u(\Delta) \). This completes the proof. \( \square \)

Let \( \vartheta = \{bp, bp0, r, r0\} \). We define the operator \( P \) form \( \lambda(\Delta) \) into itself, where \( \lambda \in \{\mathcal{M}_u, \mathcal{C}_0\} \) as

\[
P : \lambda(\Delta) \rightarrow \lambda(\Delta)
\]

\[
x \rightarrow P_x = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & x_{11} & x_{12} & x_{13} \\
0 & x_{21} & x_{22} & x_{23} \\
0 & x_{31} & x_{32} & x_{33} \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

for all \( x = (x_{kl}) \in \lambda(\Delta) \). Clearly \( P \) is a linear and bounded operator on \( \lambda(\Delta) \).

Now we show that the four-dimensional forward difference operator \( \Delta \) is a linear homeomorphism.

(2.2) \[
\Delta : P(\lambda(\Delta)) \rightarrow \lambda
\]

\[
x \rightarrow \Delta x = y = (x_{kl} - x_{k,l+1} - x_{k,l+1} - x_{k+1,l+1})
\]

where the set \( P(\lambda(\Delta)) \) is defined by

\[
P(\lambda(\Delta)) := \{ x = (x_{kl}) \in \mathbb{C} : x \in \lambda(\Delta) \text{ and } x_{00} = x_{0,0} = x_{0,l} = 0, \forall k, l \in \mathbb{N} \} \subset \lambda(\Delta)
\]

and

\[
\| x \|_{P(\lambda(\Delta))} = \| \Delta x \|_{\lambda}.
\]

Therefore, the spaces \( P(\lambda(\Delta)) \) and \( \lambda \) are equivalent as topological spaces, and the \( \Delta \) and \( \Delta^{-1} \) are norm preserving and \( \| \Delta \| = \| \Delta^{-1} \| = 1 \). We prove the following Lemma \ref{lem2.2} for the case \( \lambda = \mathcal{C}_{r0} \) by using the results in \cite{1} Theorem 5., Remark 3., P.132]. Since the proofs of the other cases are similar to that of following Lemma \ref{lem2.2} we left them as an exercise to the reader.

**Lemma 2.2.** A linear functional \( f_\Delta \) on \( P(\mathcal{C}_{r0}(\Delta)) \) is continuous if and only if there exists a double sequence \( a_{kl} \), \( k, l \geq 1 \in \mathcal{L}_u \) such that

(2.3) \[
f_\Delta(x) = \sum_{k,l=1}^{\infty} a_{kl}(\Delta x)_{kl}
\]

for all \( x \in P(\mathcal{C}_{r0}(\Delta)) \).

**Proof.** First we show that \( \Delta : P(\mathcal{C}_{r0}(\Delta)) \rightarrow \mathcal{C}_{r0}, \Delta x_{kl} = x_{kl} - x_{k,l+1} + x_{k,l+1} + x_{k+1,l+1} \) with \( x_{00} = x_{k,0} = x_{0,l} = 0 \) for each \( k, l \in \mathbb{N} \) is an isometric linear isomorphism, that is, we prove that \( \Delta \) is a bijection between \( P(\mathcal{C}_{r0}(\Delta)) \) and \( \mathcal{C}_{r0} \) by \( \Delta x_{kl} = x_{kl} - x_{k,l+1} + x_{k,l+1} + x_{k+1,l+1} \) with \( x_{00} = x_{k,0} = x_{0,l} = 0 \) for each \( k, l \in \mathbb{N} \). Linearity is clear. Moreover, \( x = 0 \) whenever \( \Delta x = 0 \), and hence \( \Delta \) is injective. Now suppose that \( y = (y_{kl}) \in \mathcal{C}_{r0} \), we define the sequence \( x = (x_{kl}) \) by \( x_{kl} = \sum_{i,j=0}^{k-1,l-1} y_{ij} \) with \( x_{00} = x_{k,0} = x_{0,l} = 0 \).
for each \( k, l \in \mathbb{N} \). Then we have,

\[
\| x \|_{P(\mathcal{C}_{r_0}(\Delta))} = \sup_{k,l \in \mathbb{N}} |\Delta x_{kl}| \\
= \sup_{k,l \in \mathbb{N}} \left| \Delta \left( \sum_{i,j=0}^{k-1,l-1} y_{ij} \right) \right| \\
= \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=0}^{k-1,l-1} y_{ij} - \sum_{i,j=0}^{k-1} y_{ij} - \sum_{i,j=0}^{l-1} y_{ij} + \sum_{i,j=0}^{k,l} y_{ij} \right| \\
= \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=0}^{k-1,l-1} y_{ij} - \sum_{j=0}^{l-1} y_{kj} + \sum_{i,j=0}^{k,l} y_{ij} \right| \\
\leq \sup_{k,l \in \mathbb{N}} |y_{kl}| = \| y \|_\infty < \infty.
\]

It shows that \( x \in P(\mathcal{C}_{r_0}(\Delta)) \) and consequently \( \Delta \) is surjective and norm preserving. It completes the first part of the proof.

Now suppose that \( f_\Delta \) is a linear functional on \( P(\mathcal{C}_{r_0}(\Delta)) \). If \( f_\Delta \) is continuous, then \( f_\Delta \circ \Delta^{-1} \) is a continuous linear functional on \( \mathcal{C}_{r_0} \). Then by \([1]\) Remark 3, there exists a double sequence \( a = (a_{kl})_{k,l \geq 1} \in \mathcal{L}_u \) such that

\[
f_\Delta \circ \Delta^{-1}(y) = \sum_{k,l=0}^\infty a_{kl}y_{kl}
\]

for all \( y \in \mathcal{C}_{r_0} \). It gives

\[
f_\Delta(x) = (f_\Delta \circ \Delta^{-1})(\Delta x) = \sum_{k,l=0}^\infty a_{kl}(\Delta x)_{kl}
\]

for all \( x \in P(\mathcal{C}_{r_0}(\Delta)) \). Conversely, if \( f_\Delta(x) = \sum_{k,l=1}^\infty a_{kl}(\Delta x)_{kl} \) for all \( x \in P(\mathcal{C}_{r_0}(\Delta)) \) and for some \( a = (a_{kl}) \in \mathcal{L}_u \), then

\[
|f_\Delta(x)| = \left| \sum_{k,l=0}^\infty a_{kl}(\Delta x)_{kl} \right| \leq \sum_{k,l=1}^\infty |a_{kl}| |(\Delta x)_{kl}| \\
\leq \| x \|_{P(\mathcal{C}_{r_0}(\Delta))} \sum_{k,l=0}^\infty |a_{kl}| \\
= \| x \|_{P(\mathcal{C}_{r_0}(\Delta))} \| a \|_{\mathcal{L}_u}.
\]

Therefore, \( \| f_\Delta \| \leq \| a \|_{\mathcal{L}_u} \) and then we see that \( f_\Delta \) is a bounded(continuous) linear functional on \( P(\mathcal{C}_{r_0}(\Delta)) \). This completes the proof.

\[\square\]

**Definition 2.3.** Let \( X \) and \( Y \) be Banach spaces, and \( B(X,Y) \) be the space of bounded linear operators from \( X \) into \( Y \). An operator \( T \in B(X,Y) \) is called an isometry if \( \|Tx\| = \|x\| \) for all \( x \in X \).

Now we denote the continuous duals of \( P(\lambda(\Delta)) \) and \( \lambda \) by \( [P(\lambda(\Delta))]^* \) and \( \lambda^* \), respectively. We may now show that the operator

\[
T : [P(\lambda(\Delta))]^* \to \lambda^* \\
f_\Delta \to f = f_\Delta \circ (\Delta^{-1})
\]
is a linear isometry. Hence, \( [P(\mathcal{M}_u(\Delta))]^* \cong \mathcal{M}^*_u \), by [1] Remark 3., we have \( [P(\lambda(\Delta))]^* \cong \lambda^* \cong \mathcal{L}_u \), where \( \lambda \in \{C_r, C_{r0}\} \), by [1] Theorem 8., we have \( [P(\mu(\Delta))]^* \cong \mu^* \cong \ell_1(e^*_\infty) \), where \( \mu \in \{C_{bp}, C_{bp0}\} \), and the sets \( \ell_1 \) and \( \ell_\infty \) represent absolutely summable and bounded single sequence spaces, respectively.

Now we prove the following Theorem only for the case \( \lambda = C_{r0} \).

**Theorem 2.4.** The continuous dual \( [P(C_{r0}(\Delta))]^* \) is isometrically isomorphic to \( C_{r0}^* \cong \mathcal{L}_u \).

**Proof.** Let us define an operator

\[
T : [P(C_{r0}(\Delta))]^* \to C_{r0}^* \cong \mathcal{L}_u
\]

with \( T(f_\Delta) = (f_\Delta(e^{kl}))_{k,l \geq 1} \),

\[
T(f_\Delta(x)) = T((f_\Delta \circ \Delta^{-1})(\Delta x)) = \sum_{k,l=1}^\infty a_{kl}T((\Delta x)_{kl})
\]

where \( a = (a_{kl}) \in \mathcal{L}_u \). Therefore, \( T \) is a surjective linear map by Lemma 2.2. Moreover, since \( T(f_\Delta(e^{kl})) = 0 = (0,0,0,...) \) implies \( f_\Delta = 0 \), where \((x_{kl}) = e^{kl}\) is Schauder basis for \( C_{r0} \) by the definition of double Schauder basis [10], Definition 4.2., p. 14, \( T \) is injective. Let \( f_\Delta \in [P(C_{r0}(\Delta))]^* \) and \( x \in P(C_{r0}(\Delta)) \). Then we have

\[
|f_\Delta(x)| = \left| f_\Delta \left( \sum_{k,l=1}^\infty (\Delta x)_{kl}e^{kl} \right) \right| = \left| \sum_{k,l=1}^\infty (\Delta x)_{kl}f_\Delta(e^{kl}) \right| \leq \sum_{k,l=1}^\infty \left| f_\Delta(e^{kl}) \right| |(\Delta x)_{kl}|
\]

\[
\leq \sup_{k,l \in \mathbb{N}} \left| |(\Delta x)_{kl}| \sum_{k,l=1}^\infty \left| f_\Delta(e^{kl}) \right| \right|
\]

\[
\leq \| x \|_{P(C_{r0}(\Delta))} \| T(f_\Delta) \|_{\mathcal{L}_u}.
\]

Then we obtain

\[
(2.4) \quad \| f_\Delta \|_\infty \leq \| T(f_\Delta) \|_{\mathcal{L}_u}.
\]

Furthermore, since \( |f_\Delta(e^{kl})| \leq \| f_\Delta \|_\infty |e^{kl}|_{P(C_{r0}(\Delta))} = \| f_\Delta \|_\infty \), then we have

\[
(2.5) \quad \| T(f_\Delta) \|_{\mathcal{L}_u} = \sup_{k,l \in \mathbb{N}} |f_\Delta(e^{kl})| \leq \| f_\Delta \|_\infty.
\]

We obtain by (2.4) and (2.5) that \( \| T(f_\Delta) \|_{\mathcal{L}_u} = \| f_\Delta \|_\infty \). This completes the proof. \( \square \)

### 3. Dual Spaces of the New Double Sequence Spaces

In this section, we determine the \( \alpha-\), \( \beta(\vartheta)-\) and \( \gamma-\) duals of our new double sequence spaces. First, we begin with some lemmas to determine the \( \alpha-\), \( \beta(\vartheta)-\) and \( \gamma-\) duals of the spaces \( \mathcal{M}_u(\Delta) \), \( \mathcal{C}_0(\Delta) \), where \( \vartheta \in \{bp, r\} \).

**Lemma 3.1.** We have \( \sup_{k,l \in \mathbb{N}} |\Delta x_{kl}| < \infty \) if and only if

(i) \( \sup_{k,l \in \mathbb{N}} k|x_{kl}| < \infty \),

(ii) \( \sup_{k,l \in \mathbb{N}} k|\Delta (\frac{1}{k}x_{kl})| < \infty \).

**Proof.** Suppose that there exists a positive real number \( M \) such that

\[
\sup_{k,l \in \mathbb{N}} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| \leq M.
\]

Then

\[
|x_{kl}| = |x_{k,0} + x_{0,l} - x_{00} + x_{kl}| = \left| \sum_{i,j=0}^{k-1,l-1} \Delta x_{ij} \right| \leq \sum_{i,j=0}^{k-1,l-1} |\Delta x_{ij}| \leq M(kl).
\]
It is clearly seen that (i) is necessary. Moreover, by considering the condition (i) there exists positive real numbers \(N_1, N_2, N_3\) such that

\[
\begin{align*}
(3.1) \quad & \sup_{k,l \in \mathbb{N}} \frac{1}{(k+1)l} |x_{k+1,l}| \leq N_1, \\
(3.2) \quad & \sup_{k,l \in \mathbb{N}} \frac{1}{k(l+1)} |x_{k,l+1}| \leq N_2, \\
(3.3) \quad & \sup_{k,l \in \mathbb{N}} \frac{1}{(k+1)(l+1)} |x_{k+1,l+1}| \leq N_3.
\end{align*}
\]

Then we have

\[
kl \left| \Delta \left( \frac{1}{kl} x_{kl} \right) \right| = kl \left| \frac{1}{kl} x_{kl} - \frac{1}{(k+1)l} x_{k+1,l} - \frac{1}{k(l+1)} x_{k,l+1} \right|
\]

\[
\leq kl \left( \left| \frac{1}{kl} \Delta x_{kl} \right| + \left| \frac{1}{kl} (k+1) x_{k+1,l} \right| + \left| \frac{1}{k(l+1)} x_{k,l+1} \right| \right)
\]

\[
\leq kl \left( \left| \frac{1}{kl} \Delta x_{kl} \right| + \left| \frac{1}{kl} (k+1) x_{k+1,l} \right| + \left| \frac{1}{k(l+1)} x_{k,l+1} \right| \right)
\]

\[
\leq \frac{kl}{k+1} x_{k+1,l+1}
\]

where \(M' = M + N_1 + N_2 + N_3\). So it gives the necessity of (ii).

Now let us suppose that the conditions (i) and (ii) hold. By only considering the following inequality

\[
kl \left| \Delta \left( \frac{1}{kl} x_{kl} \right) \right| = kl \left| \frac{1}{kl} x_{kl} - \frac{1}{(k+1)l} x_{k+1,l} - \frac{1}{k(l+1)} x_{k,l+1} \right|
\]

\[
\leq kl \left( \left| \frac{1}{kl} \Delta x_{kl} \right| + \left| \frac{1}{kl} (k+1) x_{k+1,l} \right| + \left| \frac{1}{k(l+1)} x_{k,l+1} \right| \right)
\]

\[
\leq kl \left( \left| \frac{1}{kl} \Delta x_{kl} \right| + \left| \frac{1}{kl} (k+1) x_{k+1,l} \right| + \left| \frac{1}{k(l+1)} x_{k,l+1} \right| \right)
\]

we can see the necessity of \(\sup_{k,l \in \mathbb{N}} \left| \Delta x_{kl} \right| < \infty\).

\[\square\]

**Lemma 3.2.** Let \(\Delta x_{kl} = y_{kl}\). If

\[
\sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} y_{kl} \right| < \infty
\]

then

\[
\sup_{m,n \in \mathbb{N}} \left( (m+1)(n+1) \left| \sum_{k,l=1}^{\infty} \frac{y_{m+k-1,n+l-1}}{(m+k)(n+l)} \right| \right) < \infty
\]
Proof. Let us consider Abel’s double partial summation on the \( (s, t) \)-th - partial sum of the series \( \sum_{k, l=1}^{\infty} \frac{y_{m+k+1, n+l-1}}{(m+k)(n+l)} \) as in the following equation.

\[
(3.4) \quad \sum_{k, l=1}^{s, t} \frac{y_{m+k-1, n+l-1}}{(m+k)(n+l)} = \sum_{k, l=1}^{s-1, t-1} \left( \sum_{i, j=1}^{k, t} y_{m+i-1, n+j-1} \right) \Delta_{11}^{k, t} \left( \frac{1}{(m+k)(n+l)} \right) \\
+ \sum_{k=1}^{s-1} \left( \sum_{i, j=1}^{k, t} y_{m+i-1, n+j-1} \right) \Delta_{10}^{k, t} \left( \frac{1}{(m+k)(n+l)} \right) \\
+ \sum_{l=1}^{t-1} \left( \sum_{i, j=1}^{s, l} y_{m+i-1, n+j-1} \right) \Delta_{01}^{l, t} \left( \frac{1}{(m+s)(n+l)} \right) \\
+ \sum_{i, j=1}^{s, t} y_{m+i-1, n+j-1} \left( \frac{1}{(m+s)(n+t)} \right)
\]

where for the double sequence \( a_{k, l} = \frac{1}{(m+k)(n+l)} \)

\[
\Delta_{10}^{k, t} a_{k, l} = a_{k, l} - a_{k+1, l} \\
\Delta_{10}^{k, l} a_{k, l} = a_{k, l} - a_{k, l+1} \\
\Delta_{11}^{k, l} a_{k, l} = \Delta_{10}^{k, l} (\Delta_{10}^{k, l} a_{k, l}) = \Delta_{10}^{k, l} (\Delta_{10}^{k, l} a_{k, l}) = a_{k, l} - a_{k+1, l} - a_{k, l+1} + a_{k+1, l+1}.
\]

Since there exists a positive real number \( M \) such that

\[
(3.5) \quad \sup_{m, n \in \mathbb{N}} \sum_{k, l=1}^{m, n} y_{k, l} \leq M,
\]

the equation (3.4) is written as

\[
\sum_{k, l=1}^{s, t} \frac{y_{m+k-1, n+l-1}}{(m+k)(n+l)} \leq M \left[ \sum_{k, l=1}^{s-1, t-1} \left( \frac{1}{(m+k)(n+l)} - \frac{1}{(m+k+1)(n+l+1)} \right) \\
+ \sum_{k=1}^{s-1} \left( \frac{1}{(m+k)(n+l+1)} + \frac{1}{(m+k+1)(n+l+1)} \right) \\
+ \sum_{l=1}^{t-1} \left( \frac{1}{(m+l+1)} - \frac{1}{(m+s)(n+l+1)} \right) \\
+ \frac{1}{(m+s)(n+l)} \right] \\
= \frac{M}{(m+1)(n+1)}.
\]

Therefore by passing to \( \vartheta \)-limit as \( s, t \to \infty \), where \( \vartheta = \{bp, r\} \), and taking supremum over \( m, n \in \mathbb{N} \), then the condition

\[
\sup_{m, n \in \mathbb{N}} \left( \frac{m+1}{(m+1)(n+1)} \sum_{k, l=1}^{\infty} \frac{y_{m+k-1, n+l-1}}{(m+k)(n+l)} \right) < \infty
\]

is immediate. \( \square \)
Lemma 3.3. Let $\vartheta \in \{bp, r\}$. If the series $\sum_{k,l=1}^{\infty} \Delta x_{kl}$ is $\vartheta-$convergent, then

$$\vartheta - \lim_{m,n \to \infty} \left( (m+1)(n+1) \left| \sum_{k,l=1}^{\infty} \frac{y_{m+k-1,n+l-1}}{(m+k)(n+l)} \right| \right) = 0$$

Proof. Since the partial sum of the series $\sum_{k,l=1}^{\infty} \Delta x_{kl}$ is $\vartheta-$convergent, where $\vartheta \in \{bp, r\}$, we have

$$\sum_{i,j=1}^{k,l} y_{m+i-1,n+j-1} = \sum_{i,j=m,n}^{m+k-1,n+l-1} y_{ij} = O(1).$$

Then by using the equality (3.4) we write

$$(m+1)(n+1) \sum_{k,l=1}^{\infty} \frac{y_{m+k-1,n+l-1}}{(m+k)(n+l)} = O(1).$$

If we let $\vartheta-$limit as $m, n \to \infty$, we reach the proof. \hfill \square

Corollary 3.4. Let $\vartheta \in \{bp, r\}$ and $a = (a_{kl})$ be any double sequence. Then

(i) If $\sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} kla_{kl} \right| < \infty$, then

$$\sup_{m,n \in \mathbb{N}} \left| mn \sum_{k,l=m+1,n+1}^{\infty} a_{kl} \right| < \infty$$

(ii) If $\sum_{k,l=1}^{\infty} kla_{kl}$ is $\vartheta-$convergent, then

$$\vartheta - \lim_{m,n \to \infty} \left( mn \sum_{k,l=m+1,n+1}^{\infty} a_{kl} \right) = 0$$

(iii) $\sum_{k,l=1}^{\infty} kla_{kl}$ is $\vartheta-$convergent if and only if

$$\sum_{k,l=1}^{\infty} R_{kl}$$

is $\vartheta-$convergent with $mnR_{mn} = O(1),$

where $R_{mn} = \sum_{k,l=m+1,n+1}^{\infty} a_{kl}$

Proof. The proof of (i) and (ii) can be easily seen by writing $kla_{kl}$ instead of $y_{kl}$ in Lemma 3.2, and writing $(k+1)(l+1)a_{k+1,l+1}$ instead of $y_{kl}$ in Lemma 3.3 respectively.

To prove the corollary (iii), the following $(s, t)^{th}$ partial sum can be written by using Abel’s double summation formula that

$$\sum_{k,l=1}^{s,t} kla_{kl} = \sum_{k,l=1}^{s-1,t-1} \left( \sum_{i,j=0}^{k,l} a_{ij} \right) \Delta_{11}^{kl} + \sum_{k=1}^{s-1} \left( \sum_{i,j=0}^{k} a_{ij} \right) \Delta_{10}^{kl} + \sum_{i,j=0}^{s,t} a_{ij}(st)$$

$$= \sum_{k,l=1}^{s,t} \left( \sum_{i,j=k,l}^{s,t} a_{ij} \right) + st \sum_{k,l=s+1,t+1}^{\infty} a_{kl}.$$
\[ D_1 := \int \mathcal{L}_u := \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l=1}^{\infty} kl |a_{kl}| < \infty \right\} \]

\[ D_2 := \int \mathcal{C}S_{\varnothing} := \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l=1}^{\infty} kla_{kl} \text{ is } \varnothing - \text{convergent} \right\} \]

\[ D_3 := \int \mathcal{B}S := \left\{ a = (a_{kl}) \in \Omega : \sum_{m,n,k,l=1}^{\infty} kl |a_{kl}| < \infty \right\} \]

\[ D_4 := \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l=1}^{\infty} \left| \sum_{i,j=k,l}^{\infty} a_{ij} \right| < \infty \right\} \]

**Theorem 3.5.** Let \( \lambda \in \{ \mathcal{M}_u, \mathcal{C}_b, \mathcal{C}_r \} \). Then \([P(\lambda(\Delta))]^\alpha \subset D_1\)

**Proof.** We need to prove the existence of the inclusion relations \( D_1 \subset [P(\lambda(\Delta))]^\alpha \) and \([P(\lambda(\Delta))]^\alpha \subset D_1\).

Suppose that \( a = (a_{kl}) \in D_1 \), i.e., \( \sum_{k,l=1}^{\infty} kl |a_{kl}| < \infty \). Then by using Lemma 3.1 we have

\[ \sum_{k,l=1}^{\infty} |a_{kl} x_{kl}| = \sum_{k,l=1}^{\infty} kl |a_{kl}| \left( \frac{|x_{kl}|}{kl} \right) < \infty \]

for all \( x = (x_{kl}) \in P(\lambda(\Delta)) \). This shows that \( a = (a_{kl}) \in [P(\lambda(\Delta))]^\alpha \). Hence, the inclusion \( D_1 \subset [P(\lambda(\Delta))]^\alpha \) holds.

Now suppose that \( a = (a_{kl}) \in [P(\lambda(\Delta))]^\alpha \), i.e., \( \sum_{k,l=1}^{\infty} a_{kl} x_{kl} < \infty \) for all \( x = (x_{kl}) \in P(\lambda(\Delta)) \). If we consider the double sequence \( x = (x_{kl}) \) as

\[
(3.6) \quad x_{kl} := \begin{cases} 0, & k = 0, l \geq 0 \\ 0, & l = 0, k \geq 0 \\ kl, & k \geq 1, l \geq 1 \end{cases}
\]

Then we have

\[ \sum_{k,l=1}^{\infty} |a_{kl} x_{kl}| = \sum_{k,l=1}^{\infty} kl |a_{kl}| < \infty \]

which says \( a = (a_{kl}) \in D_1 \). Hence, the inclusion \([P(\lambda(\Delta))]^\alpha \subset D_1\) holds. This concludes the proof. \( \square \)

**Theorem 3.6.** Let \( \lambda \in \{ \mathcal{M}_u, \mathcal{C}_b, \mathcal{C}_r \} \). Then \([P(\lambda(\Delta))]^{\beta(\varnothing)} \subset D_2 \cap D_4\).

**Proof.** We should show the validity of the inclusions \( D_2 \cap D_4 \subset [P(\lambda(\Delta))]^{\beta(\varnothing)} \) and \([P(\lambda(\Delta))]^{\beta(\varnothing)} \subset D_2 \cap D_4\).

Suppose that the double sequence \( a = (a_{kl}) \in D_2 \cap D_4 \) and the sequence \( x = (x_{kl}) \in P(\lambda(\Delta)) \) are defined with the relation (2.2) between the terms of the sequence \( x = (x_{kl}) \) and \( y = (y_{kl}) \) as

\[
(3.7) \quad x_{kl} = \sum_{i,j=1}^{k,l} y_{i-1,j-1},
\]

where \( y = (y_{kl}) \in \lambda \) which is defined as

\[
(3.8) \quad y_{kl} := \begin{cases} x_{11}, & k = 0, l = 0 \\ -x_{11} + x_{12}, & k = 0, l = 1 \\ -x_{11} + x_{21}, & k = 1, l = 0 \\ x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}, & k \geq 1, l \geq 1 \end{cases}
\]
Then, we have the following \((s, t)\)th–partial sum of the series \(\sum_{k,l} a_{kl} x_{kl}\) that

\[
\sum_{k,l=1}^{s,t} a_{kl} x_{kl} = \sum_{k,l=1}^{s,t} a_{kl} \left( \sum_{i,j=1}^{k,l} y_{i,j-1,j-1} \right) \\
= \sum_{k,l=1}^{s-t-1} \left( \sum_{i,j=k,l}^{s-t-1} a_{ij} \right) y_{kl} \\
= \sum_{k,l=1}^{s-t-1} \left( \sum_{i,j=k,l}^{\infty} a_{ij} \right) y_{kl} - \sum_{k,l=1}^{s-t-1} \left( \sum_{i,j=s,t}^{\infty} a_{ij} \right) y_{kl} \\
= \sum_{k,l=1}^{s-t-1} R_{kl} y_{kl} - R_{st} \sum_{k,l=1}^{s-t-1} y_{kl}.
\]

Now, by the Corollary 3.4(iii), we can say that the sequence \(\sum_{k,l=1}^{s,t} a_{kl} x_{kl}\) is \(\varphi\)–convergent for every \(x = (x_{kl}) \in P(\lambda(\Delta))\), since \(\sum_{k,l=1}^{s-t-1} R_{kl} y_{kl}\) is \(\varphi\)–convergent with \(x_{st} R_{st} \to 0\) as \(s, t \to \infty\). This yields that \(a = (a_{kl}) \in [P(\lambda(\Delta))]^{\beta(\varphi)}\) and the inclusion \(D_2 \cap D_4 \subset [P(\lambda(\Delta))]^{\beta(\varphi)}\) holds.

Now, suppose that \(a = (a_{kl}) \in [P(\lambda(\Delta))]^{\beta(\varphi)}\). Then the series \(\sum_{k,l=1}^{\infty} a_{kl} x_{kl}\) is \(\varphi\)–convergent for every \(x = (x_{kl}) \in P(\lambda(\Delta))\). If we consider the sequence \(x = (x_{kl})\) defined in (3.4). Then, we can observe that

\[
\sum_{k,l=1}^{\infty} a_{kl} x_{kl} = \sum_{k,l=1}^{\infty} kla_{kl}
\]

and by the equality \(y = \Delta x\) we have the following series

\[
\sum_{k,l=1}^{s,t} kla_{kl} = \sum_{k,l=1}^{s-t-1} \left( \sum_{i,j=k,l}^{\infty} a_{ij} \right) - \sum_{k,l=1}^{s-t-1} \left( \sum_{i,j=s,t}^{\infty} a_{ij} \right) \\
= \sum_{k,l=1}^{s-t-1} R_{kl} - st R_{st}
\]

which is \(\varphi\)–convergent as \(s, t \to \infty\). Thus, \(a = (a_{kl}) \in D_2\). Moreover, by Corollary 3.4(ii) we can write that \(st R_{st} \to 0\) as \(s, t \to \infty\) for every \(y = (y_{kl}) \in \lambda\), and \(\sum_{k,l=1}^{\infty} R_{kl} < \infty\). Therefore, \(a = (a_{kl}) \in D_4\).

Hence the inclusion \([P(\lambda(\Delta))]^{\beta(\varphi)} \subset D_2 \cap D_4\) holds. This completes the proof. \(\square\)

**Theorem 3.7.** Let \(\lambda \in \{M_1, C_\vartheta\}\). Then \([P(\lambda(\Delta))]^{\gamma} = D_3 \cap D_4\), where \(\vartheta \in \{bp, r\}\).

**Proof.** The proof can be done with the similar path as above by considering Corollary 3.4(i). So, we omit the repetition. \(\square\)

4. **Matrix Transformations**

In this section we characterize the four-dimensional matrix mapping from the sequence space \(\lambda(\Delta)\) to \(\mu\) and vice-versa. Then we conclude the section with some significant results.

**Theorem 4.1.** The four-dimensional matrix \(A = (a_{mnkl}) \in (\lambda(\Delta) : \mu)\) if and only if

\[
A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}} \in (\lambda(\Delta))^{\beta(\varphi)} \text{ for all } m, n \in \mathbb{N},
\]

\[
A_{mn}(kl) = \sum_{k,l=1}^{\infty} kla_{mnkl} \in \mu,
\]

\[
B = (b_{mnkl}) \in (\lambda : \mu),
\]

for all \(m, n, k, l \in \mathbb{N}\).
where the four-dimensional matrix

\[(4.4)\]

\[B = (b_{mnkl}) = \sum_{i,j=kl}^{\infty} a_{mnij} \text{ for all } m,n,k,l \in \mathbb{N}.\]

**Proof.** Suppose that \(A = (a_{mnkl}) \in (\lambda(\Delta) : \mu)\). Then, \(A_{mn}(x)\) exists for every \(x = (x_{kl}) \in \lambda(\Delta)\) and is in \(\mu\) for all \(m,n \in \mathbb{N}\). If we define the sequence \(x = (x_{kl})\) by

\[(4.5)\]

\[x_{kl} := \begin{cases} 1 & , \text{ } k = l \\ 0 & \text{, otherwise} \end{cases}\]

for all \(k,l \in \mathbb{N}\), then the necessity of \((4.1)\) is clear. If we define the sequence \(x = (x_{kl})\) as \(x_{kl} = kl\) for all \(k,l \in \mathbb{N}\), then the necessity of \((4.2)\) is also clear by Theorem 3.6. Moreover, by Theorem 3.6 we have \(\sum_{k,l=1}^{\infty} |a_{mnkl}| < \infty\) for each \(m,n \in \mathbb{N}\).

Now suppose that \(x = (x_{kl}) \in P(\lambda(\Delta)) \subset \lambda(\Delta)\) let us consider the \((s,t)^{th}\) partial sum of the series \(\sum_{k,l=1}^{\infty} a_{mnkl}x_{kl}\) by considering the relation \(x_{kl} = \sum_{i,j=0}^{k-1,l-1} y_{ij}\) between terms of the sequences \(x = (x_{kl})\) and \(y = (y_{kl})\) as in the following

\[A_{mn}^{st}(x) = \sum_{k,l=1}^{s,t} a_{mnkl}x_{kl}\]

\[= \sum_{k,l=1}^{s,t} a_{mnkl} \left( \sum_{i,j=0}^{k-1,l-1} y_{ij} \right)\]

\[= \sum_{k,l=1}^{s,t} a_{mnkl} \left( \sum_{i,j=k,l}^{s-1,t-1} a_{mnij} \right) y_{kl}\]

\[= \sum_{k,l=1}^{s,t} b_{mnkl} y_{kl} - b_{mnts} \sum_{k,l=1}^{s-1,t-1} y_{kl}\]

where \(y \in \lambda\). We obtain by letting \(\vartheta\)-limit as \(s,t \to \infty\) and by considering the Corollary 3.3(iii) that \(A_{mn}(x) = \sum_{k,l=1}^{\infty} b_{mnkl} y_{kl}\), that is \(Ax = By\) for each \(m,n \in \mathbb{N}\). Therefore, \(A = (a_{mnkl}) \in (\lambda(\Delta) : \mu)\) implies that \(B = (b_{mnkl}) \in (\lambda : \mu)\).

Now suppose that the conditions \((4.1)-(4.3)\) hold. Let us take a sequence \(x = (x_{kl}) \in \lambda(\Delta)\) defined by

\[x_{kl} := \begin{cases} x_{k,1} & , \text{ } k \geq 1, l = 1 \\ x_{1,l} & , \text{ } k = l, l \geq 1 \\ \tilde{x}_{kl} & , \text{ } k > l, l > 1 \end{cases}\]

where \(\tilde{x} = (\tilde{x}_{kl}) \in P(\lambda(\Delta))\). Then, if we write again the above \((s,t)^{th}\) partial sum of the series \(\sum_{k,l=1}^{\infty} a_{mnkl}x_{kl}\), we have

\[A_{mn}^{st}(x) = \sum_{k,l=1}^{s,t} a_{mnkl}x_{kl}\]

\[= a_{mn11}x_{11} + \sum_{l=2}^{t} a_{mn1,l}x_{1,l} + \sum_{k=2}^{s} a_{mn,k,1}x_{k,1} + \sum_{k,l=2}^{s,t} a_{mnkl}\tilde{x}_{kl}\]

\[= a_{mn11}x_{11} + \sum_{k=2}^{s-1} b_{mnk1} y_{k,1} + \sum_{l=2}^{t} b_{mn1,l} y_{1,l} + \sum_{k,l=1}^{s-1,t-1} b_{mnkl} y_{kl} - b_{mnts} \sum_{k,l=1}^{s-1,t-1} y_{kl}\]
Therefore, we obtain by letting limit as \( s, t \to \infty \) that
\[
A_{mn}(x) = a_{mn1}x_{11} + \sum_{k=2}^{\infty} b_{mnk,1}y_{k,1} + \sum_{l=2}^{\infty} b_{mn,1,l}y_{1,l} + \sum_{k,l=1}^{\infty} b_{mnkl}y_{kl}.
\]
Thus, \( A_{mn}(x) \) exists for each \( x = (x_{kl}) \in \lambda(\Delta) \) and is in \( \mu \) since \( B \in (\lambda : \mu) \). This completes the proof. \( \square \)

We list some four-dimensional matrix classes from and into the sequence spaces \( \lambda, \mu = \{M_u, C_{bp}, C_r\} \) as in the following table, which have been characterized in some distinguished papers (see [14, Theorem 3.5], [15, Lemma 3.2], [16, Theorem 2.2], [17, Theorem 3.2]).

Table 1. The characterizations of the matrix classes \((\lambda; \mu)\), where \( \lambda, \mu \in \{M_u, C_{bp}, C_r\} \).

| From \( \lambda \) \( \downarrow \) \( \to \mu \) | \( M_u \) | \( C_{bp} \) | \( C_r \) |
|---|---|---|---|
| \( M_u \) | 1 | 2 | * |
| \( C_{bp} \) | 3 | 4 | 4 |
| \( C_r \) | * | 5 | 5 |

We list the necessary and sufficient conditions for each class in the following table. Note that * shows the unknown characterization of respective four-dimensional matrix class.
Corollary 4.2. Let the four-dimensional matrix $B = (b_{mnkl})$ is defined as in (4.4). Then the followings hold for four-dimensional infinite matrix $A = (a_{mnkl})$.

(i) $A \in (M_u(\Delta), M_u)$ if and only if the conditions in (4.1) and (4.2) hold, and 1 holds in Table 2 with $b_{mnkl}$ instead of $a_{mnkl}$.

(ii) $A \in (M_u(\Delta), C_{bp})$ if and only if the conditions in (4.1) and (4.2) hold, and 2 holds in Table 2 with $b_{mnkl}$ instead of $a_{mnkl}$.

(iii) $A \in (C_{bp}(\Delta), M_u)$ if and only if the conditions in (4.1) and (4.2) hold, and 3 holds in Table 2 with $b_{mnkl}$ instead of $a_{mnkl}$.

(iv) Let $\vartheta = \{bp, r\}$. $A \in (C_{bp}(\Delta), C_{\vartheta})$ if and only if the conditions in (4.1) and (4.2) hold, and 4 holds in Table 2 with $b_{mnkl}$ instead of $a_{mnkl}$.

(v) Let $\vartheta = \{bp, r\}$. $A \in (C_{\vartheta}(\Delta), C_{\vartheta})$ if and only if the conditions in (4.1) and (4.2) hold, and 5 holds in Table 2 with $b_{mnkl}$ instead of $a_{mnkl}$.

Theorem 4.3. The four-dimensional matrix $A = (a_{mnkl}) \in (\mu : \lambda(\Delta))$ if and only if

\begin{equation}
A_{mn} \in \mu^{\lambda(\vartheta)},
\end{equation}

\begin{equation}
F = (f_{mnkl}) \in (\mu : \lambda),
\end{equation}

where the four-dimensional matrix

\begin{equation}
F = (f_{mnkl}) = \Delta_{mn}a_{mnij} = a_{mnij} - a_{m+n+1,i} + a_{m+1,n+1,i}.
\end{equation}

Proof. Suppose that $A = (a_{mnkl}) \in (\mu : \lambda(\Delta))$. Then, $A_{mn}(x)$ exists for every $x = (x_{kl}) \in \mu$ and is in $\lambda(\Delta)$ for all $m, n \in \mathbb{N}$. Thus, the necessity of (4.17) is immediate. Since $A_{mn}(x) \in \lambda(\Delta)$, then $\Delta A \in \lambda$ for every $x = (x_{kl}) \in \mu$. Clearly $\Delta A$ is the matrix $F$. Hence, the necessity of the condition $F = (f_{mnkl}) \in (\lambda : \mu)$ can be clearly seen. The rest of the theorem can be followed by the similar path as in Theorem 4.1. We omit the details.

Corollary 4.4. Let the four-dimensional matrix $F = (f_{mnkl})$ is defined as in (4.19). Then the followings hold for four-dimensional infinite matrix $A = (a_{mnkl})$.

(i) $A \in (M_u, M_u(\Delta))$ if and only if the condition in (4.17) holds, and 1 holds in Table 2 with $f_{mnkl}$ instead of $a_{mnkl}$.

(ii) $A \in (M_u, C_{bp}(\Delta))$ if and only if the condition in (4.17) holds, and 2 holds in Table 2 with $f_{mnkl}$ instead of $a_{mnkl}$.

(iii) $A \in (C_{bp}, M_u(\Delta))$ if and only if the condition in (4.17) holds, and 3 holds in Table 2 with $f_{mnkl}$ instead of $a_{mnkl}$.

(iv) Let $\vartheta = \{bp, r\}$. $A \in (C_{bp}, C_{\vartheta}(\Delta))$ if and only if the condition in (4.17) holds, and 4 holds in Table 2 with $f_{mnkl}$ instead of $a_{mnkl}$.

(v) Let $\vartheta = \{bp, r\}$. $A \in (C_{\vartheta}, C_{\vartheta}(\Delta))$ if and only if the condition in (4.17) holds, and 5 holds in Table 2 with $f_{mnkl}$ instead of $a_{mnkl}$.

Table 2. The necessary and sufficient conditions for $A \in (\lambda; \mu)$, where $\lambda, \mu \in \{M_u, C_{bp}, C_{\vartheta}\}$.

| $1 \text{ iff}$ | $2 \text{ iff}$ | $3 \text{ iff}$ | $4 \text{ iff}$ | $5 \text{ iff}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| (4.6)           | (4.6)           | (4.6)           | (4.6)           | (4.6)           |
| (4.7)           | (4.7)           | (4.7)           | (4.7)           | (4.7)           |
| (4.18)          | (4.18)          | (4.18)          | (4.18)          | (4.18)          |
| (4.19)          | (4.19)          | (4.19)          | (4.19)          | (4.19)          |
| (4.10)          | (4.10)          | (4.10)          | (4.10)          | (4.10)          |
| (4.10)          | (4.10)          | (4.10)          | (4.10)          | (4.10)          |

5. CONCLUSION

The four-dimensional backward difference matrix domain on some double sequence spaces has been studied by Demiriz and Duyar [12]. Then Başar and Tuğ [13], and Tuğ [14, 15, 16, 17, 18, 19, 20, 21, 22, 23] studied the four-dimensional generalized backward difference matrix and its domain in some double sequence spaces. Moreover, Tuğ at al. [21, 25] studied the sequentially defined four-dimensional
backward difference matrix domain on some double sequence spaces, and the space $\mathcal{BV}_{\vartheta_0}$ of double sequences of bounded variations, respectively.

In this work we defined the new double sequence spaces $\mathcal{M}_\vartheta(\Delta), \mathcal{C}_\vartheta(\Delta)$, where $\vartheta \in \{bp, r\}$ derived by the domain of four-dimensional forward difference matrix $\Delta$. Then we investigated some topological properties, determined $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals and characterized some four-dimensional matrix classes related with these new double sequence spaces.

The paper contribute nonstandard results and new contributions to the theory of double sequences. As a natural continuation of this work, the four-dimensional forward difference matrix domain in the double sequence spaces $\mathcal{C}_p$ and $\mathcal{L}_q$, where $0 < q < \infty$ are still open problem. Moreover, the four-dimensional forward difference matrix domain in the spaces $\mathcal{C}_f$, $\mathcal{BS}$, $\mathcal{CS}$ and $\mathcal{BV}$ can be calculated. Furthermore, Hahn double sequence space can be defined and studied by using some significant results stated in this work.

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