The Relativistic Particle and its d-brane Cousins

Vesselin G. Gueorguiev

Department of Physics and Astronomy,
Louisiana State University, Baton Rouge, LA 70803
email: vesselin@phys.lsu.edu

Abstract

We study properties of classical reparametrization-invariant matter systems, mainly the relativistic particle and its $d$-brane generalization. The corresponding matter Lagrangian naturally contains background interaction fields, such as a 1-form field, analogous to the electromagnetic vector potential, and a metric tensor. In order to make the theory free of background fields and prepare for quantum theory of fields, we discuss the field Lagrangians consistent with the gauge symmetries presented in the equations of motion for the matter.

Keywords: matter Lagrangian, homogeneous singular Lagrangian, extended objects, d-branes, interaction fields.

1 Introduction

Probing and understanding physical reality goes through a classical interface that shapes our thoughts in classical causality chains. Therefore, understanding of the essential mathematical constructions in classical mechanics and classical field theory is important, even though quantum mechanics and quantum field theory are regarded as more fundamental than their classical counterparts. Two approaches, the Hamiltonian and the Lagrangian, are very useful in physics [1, 2, 3, 4, 7]. In general, there is a transformation that relates these two approaches. For a reparametrization-invariant theory, however, there are problems in changing to the Hamiltonian approach [2, 3, 4, 5, 6].

Fiber bundles provide the mathematical framework for classical mechanics, field theory, and even quantum mechanics if viewed as a classical field theory. Parallel transport, covariant differentiation, and gauge symmetry are very important structures [8] associated with fiber bundles. When asking what structures are important to physics we should also ask why one fiber bundle should be more “physical” than another, why the “physical” base manifold seems to be a four-dimensional Lorentzian manifold [9, 10, 11], and how one should construct an action integral for a given fiber bundle [1, 7, 12, 13, 14].

Starting with the tangent or cotangent bundle seems natural because these bundles are related to the notion of a classical point-like matter. Since we accrue and test our knowledge via experiments that involve classical apparatus, the physically accessible fields should be generated by matter and should couple with matter as well. Therefore, the matter Lagrangian should contain the fields, not their derivatives, with which classical matter interacts [15].

We study the properties of reparametrization-invariant matter systems, mainly the relativistic particle [5, 8, 12, 16] and its extended object ($d$-brane) generalization. We try to find the answer to the question: “What is the Lagrangian for matter?” The corresponding matter Lagrangian naturally contains background interaction fields, such as a 1-form field, analogous to the electromagnetic vector potential, and a metric tensor. We discuss the guiding principles for construction of field Lagrangians. Due to the limited space available, we will not discuss the “non-relativistic” limit, Klein-Gordon equation, relativistic mass-shell equation, and Dirac equation here, these topics are covered in ref. [23].

In section two we consider an example of reparametrization-invariant action, the Lagrangian for a relativistic particle. In the third section we argue in favor of first order homogeneous Lagrangians. In section four we consider a possible generalization to the $D$-dimensional extended objects ($d$-branes). Section five contains a review of the field Lagrangians relevant for the interaction fields. Our conclusions and discussions are in section six.
2 The Matter Lagrangian for the Relativistic Particle

From everyday experience we know that localized particles move with a finite 3D speed. In an extended 4D configuration space-time, when time is added as a coordinate \((x^0 = c t)\), particles move with a constant 4-velocity. The 4-velocity is constant because of its definition \(v^\mu = dx^\mu / d\tau\) that uses the invariance of the proper time \((\tau)\) defined via the metric tensor \((g_{\mu\nu})\) \(d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu\). In this case, the action for a massive relativistic particle has a nice geometrical meaning: it is the distance along the trajectory [8]:

\[
S_1 = \int d\tau L_1(x, v) = \int d\tau \sqrt{g_{\mu\nu} v^\mu v^\nu}, \quad \sqrt{g_{\mu\nu} v^\mu v^\nu} \to 1 \Rightarrow S_1 = \int d\tau. \quad (1)
\]

However, for massless particles, such as photons, the length of the 4-velocity is zero \((g_{\mu\nu} v^\mu v^\nu = 0)\). Thus one has to use a different Lagrangian to avoid problems due to division by zero. The appropriate ‘good’ action is [8]:

\[
S_2 = \int L_2(x, v) d\tau = \int g_{\mu\nu} v^\mu v^\nu d\tau. \quad (2)
\]

Notice that the Euler-Lagrange equations obtained from \(S_1\) and \(S_2\) are equivalent, and both are equivalent to the geodesic equation as well:

\[
\frac{d}{d\tau} \vec{v} = D_{\bar{g}} \vec{v} = v^\beta \nabla_\beta \vec{v} = 0, \quad v^\beta \left( \frac{\partial v^\alpha}{\partial x^\beta} - \Gamma^\alpha_{\beta\gamma} v^\gamma \right) = 0. \quad (3)
\]

In general relativity the Levi-Civita connection \(\nabla_\beta\), with Christoffel symbols \(\Gamma^\alpha_{\beta\gamma}\), preserves the length of the vectors \((\nabla g(\vec{v}, \vec{v}) = 0)\) [8]. Therefore, these equivalences are not surprising because the Lagrangians in \((1)\) and \((2)\) are functions of the preserved length \(g(\vec{v}, \vec{v}) = v^2\). However, the parallel transport for a general connection \(\nabla_\beta\) may not preserve the length of the vectors.

The equivalence between \(S_1\) and \(S_2\) is very robust. Since \(L_2\) is a homogeneous function of order 2 with respect to \(\vec{v}\), the corresponding Hamiltonian function \((h = v^\beta L / \partial v - L)\) is exactly equal to \(L\) \((h(x, v) = L(x, v))\). Thus \(L_2\) is conserved, and so is the length of \(\vec{v}\). Any homogeneous Lagrangian of order \(n \neq 1\) is conserved because \(h = (n - 1)L\). When \(dL/d\tau = 0\), then one can show that the Euler-Lagrange equations for \(L\) and \(L' = f(L)\) are equivalent under certain restrictions on \(f\). This is an interesting type of equivalence that applies to homogeneous Lagrangians. It is different from the usual equivalence \(L \to L' = L + d\Lambda/d\tau\) or the more general equivalence discussed in ref. [17]. Any solution of the Euler-Lagrange equation for \(L' = L^\alpha\) would conserve \(L = L_1\) since \(h = (\alpha - 1)L^\alpha\). All these solutions are solutions of the Euler-Lagrange equation for \(L\) as well; thus \(L^\alpha < L\). In general, conservation of \(L_1\) is not guaranteed since \(L_1 \to L_1 + d\Lambda/d\tau\) is also a homogeneous Lagrangian of order one equivalent to \(L_1\). This suggests that there may be a choice of \(\Lambda\), a “gauge fixing”, so that \(L_1 + d\Lambda/d\tau\) is conserved even if \(L_1\) is not. The above discussion applies to a general homogeneous Lagrangian.

3 Homogeneous Lagrangians of First Order

Suppose we don’t know anything about classical physics, which is mainly concerned with trajectories of point particles in some space \(M\), but we are told we can derive it from a variational principle if we use the right action integral \(S = \int L d\tau\). By following the above example we wonder: ‘should the smallest ‘distance’ be the guiding principle?’ when constructing \(L\). If yes, “How should it be defined for other field theories?” It seems that a reparametrization-invariant theory can provide us with a metric-like structure [5], and thus a possible link between field models and geometric models [13].

In the example of the relativistic particle, the Lagrangian and the trajectory parameterization have a geometrical meaning. In general, however, parameterization of a trajectory is quite arbitrary for any observer. Leaving aside recent hints for quantum space-time from loop gravity and other theories, we ask: “Should there be any preferred trajectory parameterization in a smooth 4D space-time?” and “Aren’t we free to choose the standard of distance (time, using natural units \(c = 1\))?” If

\[\text{If there is a smallest time interval that sets a space-time scale, then this would imply a discrete space-time structure since there may not be any events in the smallest time interval. The Planck scale is often considered to be such an essential scale } [5].\]
so, then we really have a smooth continuous manifold and our theory should not depend on the choice of parameterization.

If we look at the Euler-Lagrange equations:
\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) = \frac{\partial L}{\partial \alpha},
\]
we see that any homogeneous Lagrangian of order \( n \) \( (L(x, \alpha \dot{\alpha}) = \alpha^n L(x, \dot{\alpha}) \) provides a reparametrization invariance \( (\tau \rightarrow \tau/\alpha, \dot{\alpha} \rightarrow \alpha \dot{\alpha}) \). Next, note that the action \( S \) involves an integration that is a natural structure for orientable manifolds \((M)\) with an \( n \)-form of the volume. Since a trajectory is a one-dimensional object, then what we are looking at is an embedding \( \phi : \mathbb{R}^1 \rightarrow M \). This means that we push forward the tangential space \( \phi_* : T(\mathbb{R}^1) = \mathbb{R}^1 \rightarrow T(M) \), and pull back the cotangent space \( \phi^* : T^*(\mathbb{R}^1) = \mathbb{R}^1 \leftarrow T^*(M) \). Thus a 1-form \( \omega \) on \( M \) that leaves in \( T^*(M) \) \( (\omega = A_\mu (x) dx^\mu) \) will be pulled back on \( \mathbb{R}^1 \) \( (\phi^*(\omega)) \) and there it is proportional to the volume form on \( \mathbb{R}^1 \) \( (\phi^*(\omega) = A_\mu (x) (dx^\mu / d\tau) d\tau \sim d\tau) \), allowing us to integrate \( \int \phi^*(\omega) : \int Ld\tau = \int A_\mu (x) v^\mu d\tau. \)

Therefore, by selecting a 1-form \( \omega = A_\mu (x) dx^\mu \) on \( M \) and using \( L = A_\mu (x) v^\mu \) we are actually solving for the embedding \( \phi : \mathbb{R}^1 \rightarrow M \) using a chart on \( M \) with coordinates \( x : M \rightarrow \mathbb{R}^n \). The Lagrangian obtained this way is homogeneous of first order in \( v \) with a very simple dynamics. The corresponding Euler-Lagrange equation is \( F_{\mu\nu} \dot{v}^{\mu} = 0 \) where \( F \) is a 2-form \((F = da)\); in electrodynamics this is the Faraday’s tensor. If we relax the assumption that \( L \) is a pulled back 1-form and assume that it is just a homogeneous Lagrangian of order one, then we find a reparametrization-invariant theory that may have an interesting dynamics.

### 3.1 Pros and Cons About Homogeneous Lagrangians of First Order

Some of the good things about a theory with a first order homogeneous Lagrangian are:

1. Any Lagrangian \( L(x, \frac{dx}{d\tau}) \) gives rise to a reparametrization-invariant Lagrangian\(^2\) by enlarging the space to an extended space-time: \( L(x, \frac{dx}{d\tau}) \rightarrow L(x, \frac{dx}{d\tau}, \frac{dv}{d\tau}) \).\(^3\)
2. There is a reparametrization invariance of the action \( S = \int L(x, \frac{dx}{d\tau}) d\tau. \)
3. Parameterization-independent path-integral quantization since the action \( S \) is reparametrization invariant.
4. The reparametrization invariance may help in dealing with singularities\(^4\).
5. It is easily generalized to \( D \)-dimensional extended objects \((D\)-branes\), which is the subject of the next section. Most of the features listed are more or less self-evident.

The list of bad things about a theory with a first order homogeneous Lagrangian includes:

1. There are constraints among the Euler-Lagrange equations since \( \det \left( \frac{\partial^2 L}{\partial v \partial \dot{v}} \right) = 0 \).\(^5\)
2. It follows that the Legendre transformation \((T(M) \leftrightarrow T^*(M))\), which exchanges coordinates \((x, v) \leftrightarrow (x, p)\), is problematic\(^6\).
3. There is a problem with the canonical quantization approach since the Hamiltonian function is identically ZERO \((h \equiv 0)\).\(^7\)

Constraints among the equations of motion are not an insurmountable problem since there are procedures for quantizing such theories\(^8\). For example, instead of using \( h \equiv 0 \) one can use some of the constraint equations available, or a conserved quantity, as the Hamiltonian for the quantization procedure\(^9\). Changing coordinates \((x, v) \leftrightarrow (x, p)\) seems to be difficult, but it may be resolved in some special cases by using the assumption that a gauge \( \Lambda \) has been chosen so that \( L \rightarrow L + \frac{d\Lambda}{d\tau} = L' = const. \) We would not discuss the above-mentioned quantization troubles since they are outside of the scope of this paper. A different approach is under investigation, for more details see ref.\(^{10}\).

\(^2\)It is an open question whether there is an equivalence of the corresponding Euler-Lagrange equations.
3.2 Canonical Form of the First Order Homogeneous Lagrangians

By now, we hope that the reader is puzzled, as we are, about the answer to the following question: “What is the general mathematical expression for first order homogeneous functions?” Below we define what we mean by the *canonical form of the first order homogeneous Lagrangian* and why we prefer such a mathematical expression.

First, note that any symmetric tensor of rank $n$ ($S_{\alpha_1\alpha_2...\alpha_n} = S_{[\alpha_1\alpha_2...\alpha_n]}$), where $[\alpha_1\alpha_2...\alpha_n]$ is an arbitrary permutation of the indexes, defines a homogeneous function of order $n$ ($S_n(\vec{v},...,\vec{v}) = S_{\alpha_1\alpha_2...\alpha_n} v^\alpha_1...v^\alpha_n$). The symmetric tensor of rank two is denoted by $g_{\alpha\beta}$. Using this notation, the canonical form of the first order homogeneous Lagrangian is defined as:

$$L(\vec{x}, \vec{v}) = \sum_{n=1}^{\infty} \sqrt{S_n(\vec{v},...,\vec{v})} = A_\alpha v^\alpha + \sqrt{g_{\alpha\beta} v^\alpha v^\beta} + ... \sqrt{S_n(\vec{v},...,\vec{v})}. \quad (5)$$

Whatever is the Lagrangian for matter, it should involve interaction fields that couple with the velocity $\vec{v}$ to a scalar. Thus we must have $L_{\text{matter}}(\vec{x}, \vec{v}; \text{Fields})$. When the matter action is combined with the action for the interaction fields ($S = \int LdV$), we obtain a full background independent theory. Then the corresponding Euler-Lagrange equations contain “dynamical derivatives” on the left hand side and sources on the right hand side:

$$\partial_\gamma \left( \frac{\delta L}{\delta (\partial_\gamma \Psi^\alpha)} \right) = \frac{\delta L}{\delta \Psi^\alpha} + \frac{\partial L_{\text{matter}}}{\partial \Psi^\alpha}. \quad (6)$$

The advantage of the canonical form of the first order homogeneous Lagrangian (5) is that each interaction field, which is associated with a symmetric tensor, has a unique matter source that is a monomial in the velocities:

$$\frac{\partial L}{\partial S_{\alpha_1\alpha_2...\alpha_n}} = \frac{1}{n} (S_n(\vec{v},...,\vec{v}))^{\frac{1-n}{2}} v^{\alpha_1}...v^{\alpha_n}. \quad (6)$$

There are many other ways one can write first-order homogeneous functions (6). For example, one can consider the following expression $L(\vec{x}, \vec{v}) = (h_{\alpha\beta} v^\alpha v^\beta) (g_{\alpha\beta} v^\alpha v^\beta)^{-1/2}$ where $h$ and $g$ are seemingly different symmetric tensors. However, each of these fields ($h$ and $g$) has the same source type ($\sim v^\alpha v^\beta$):

$$\frac{\partial L}{\partial h_{\alpha\beta}} = \frac{L(\vec{x}, \vec{v})}{h_{\gamma\rho} v^\gamma v^\rho} v^\alpha v^\beta, \quad \frac{\partial L}{\partial g_{\alpha\beta}} = \frac{L(\vec{x}, \vec{v})}{g_{\gamma\rho} v^\gamma v^\rho} v^\alpha v^\beta.$$ 

Theories with two metrics have been studied before [26, 27]. At this stage, however, we cannot find any good reason why the same source type should produce different fields. Therefore, we prefer the canonical form (3) for our discussion.

4 $D$-dimensional Extended Objects

In the previous sections, we have discussed the classical mechanics of a point-like particle as a problem concerned with the embedding $\phi : \mathbb{R}^1 \to M$. The map $\phi$ provides the trajectory (the word line) of the particle in the target space $M$. In this sense, we are dealing with a 0-brane that is a one dimensional object. Although time is kept in mind as an extra dimension, we do not insist on any special structure that time flow. We think of an extended object as a manifold $D$ with dimension, denoted also by $D$, $\dim D = D = d + 1$ where $d = 0, 1, 2, ...$. In this sense, we have to solve for $\phi : D \to M$ such that some action integral is minimized. From this point of view, we are dealing with mechanics of a $d$-brane. In other words, how is this $D$-dimensional extended object submerged in $M$, and what are the relevant interaction fields? By using the coordinate charts on $M (x : M \to \mathbb{R}^n)$, we can think of this as a field theory over the $D$-manifold with a local fiber $\mathbb{R}^n$. Thus the field $\phi$ is such that $\phi^\alpha = x \circ \phi : D \to M \to \mathbb{R}^n$. Following the point particle discussion, we consider the space of the $D$-forms over the manifold $M$, denoted by $\Lambda^D (M)$, that has dimension $(m_D) = \frac{m!}{D(m-D)!}$. An element $\Omega$ in $\Lambda^D (M)$ has the form $\Omega = \Omega_{\alpha_1...\alpha_m} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge ...dx^{\alpha_m}$. We use an arbitrary label $\Gamma$ to index
different $D$-forms over $M, \Gamma = 1, 2, \ldots, \binom{D}{m}$; thus $\Omega \to \Omega^\Gamma = \Omega^\Gamma_{\alpha_1 \ldots \alpha_m} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots dx^{\alpha_m}$. Next we introduce “generalized velocity vectors” with components $\omega^\Gamma$:

$$\omega^\Gamma = \frac{\Omega^\Gamma}{dz} = \Omega^\Gamma_{\alpha_1 \ldots \alpha_D} \frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_D})}{\partial (z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_D)}, \quad dz = dz^1 \wedge dz^2 \wedge \ldots \wedge dz^D.$$ 

In the above expression, $\frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_D})}{\partial (z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_D)}$ represents the Jacobian of the transformation from coordinates $\{x^{\alpha}\}$ over the manifold $M$ to coordinates $\{z^{\alpha}\}$ over the $d$-brane. The pull back of a $D$-form $\Omega^\Gamma$ must be proportional to the volume form over the $d$-brane:

$$\phi^* (\Omega^\Gamma) = \omega^\Gamma dz^1 \wedge dz^2 \wedge \ldots \wedge dz^D = \Omega^\Gamma_{\alpha_1 \ldots \alpha_D} \frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_D})}{\partial (z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_D)} dz^1 \wedge dz^2 \wedge \ldots \wedge dz^D.$$ 

Therefore, it is suitable for integration over the $D$-manifold. Thus the action for $\phi$ is

$$S [\phi] = \int_D L (\phi, \bar{\omega}) dz = \int_D \phi^* (\Omega) = \int_D A_\Gamma (\phi) \omega^\Gamma dz.$$

This is a homogeneous function in $\omega$ and has reparametrization (diffeomorphism) invariance with respect to the $D$-manifold. If we relax the linearity $L (\phi, \bar{\omega}) = \phi^* (\Omega) = A_\Gamma (\phi) \omega^\Gamma$ in $\bar{\omega}$, then the canonical expression for the homogenous Lagrangian is:

$$L (\phi, \bar{\omega}) = \sum_{n=1}^{\infty} \sqrt{S_n (\bar{\omega}, \ldots, \bar{\omega})} = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega_1^\Gamma \omega_2^\Gamma + \ldots \sqrt{S_m (\bar{\omega}, \ldots, \bar{\omega})}}. \quad (7)$$

At this point, there is strong analogy between a point particle and a $d$-brane. However, there is a difference in the number of components; $\bar{x}, \bar{v}$, and $\tilde{\phi} = \bar{x} \circ \phi$ have the same number of components, but the “generalized velocity vector” $\bar{\omega}$ has $\binom{\dim D}{\dim \tilde{\Gamma}}$ components which are Jacobians $\bar{\omega}_\alpha^\beta$.

Some familiar Lagrangians include:

- The Lagrangian for a 0-brane (relativistic point particle in an electromagnetic field, dim $D = 1$ and $\omega^\Gamma \to v^\alpha = \frac{dx^\alpha}{dt}$) is:

$$L (\tilde{\phi}, \bar{\omega}) = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega_1^\Gamma \omega_2^\Gamma \rightarrow L (\bar{x}, \bar{v}) = qA_\alpha v^\alpha + m \sqrt{g_{\alpha\beta} v^\alpha v^\beta}. \quad (8)$$

- The Lagrangian for a 1-brane (strings, dim $D = 2$) is:

$$L (x^\alpha, \partial x^\beta) = \sqrt{Y^\alpha^\beta Y^\alpha_\beta},$$

using the notation:

$$\omega^\Gamma \rightarrow Y^{\alpha\beta} = \frac{\partial (x^\alpha, x^\beta)}{\partial (\tau, \sigma)} = \det \left( \begin{array}{cc} \partial_x x^\alpha & \partial_x x^\beta \\ \partial_x x^\beta & \partial_x x^\beta \end{array} \right) = \partial_\tau x^\alpha \partial_\sigma x^\beta - \partial_\sigma x^\alpha \partial_\tau x^\beta.$$ 

- The Lagrangian for a $d$-brane has the Dirac-Nambu-Goto term (DN) $\Gamma$:

$$L (x^\alpha, \partial D x^\beta) = \sqrt{Y^\Gamma Y}_{\Gamma}. \quad (9)$$

Notice that most of the Lagrangians above, except for the relativistic particle, are restricted only to gravity-like interactions. In the case of the charged relativistic particle, the electromagnetic interaction is very important. Thus similar interaction should be introduced in the string theory and in the DNG model.
5 The Background Fields and Their Lagrangians

The uniqueness of the interaction fields and source types has been essential for the selection of the matter Lagrangian (2). The first two terms in the Lagrangian are easily identified as electromagnetic and gravitational interaction. The other terms are somewhat new. It is not yet clear if these new terms are real or not, so we will not engage them actively in the following discussion. At this stage, we have a theory with background fields since we don’t know the equations for the interaction fields. To complete the theory, we need to introduce actions for these interaction fields.

One way to write the action integrals for the interaction fields $S_n$ in (3) follows the case of the $d$-brane discussion. There, we have been solving for $\phi : D \to M$ by selecting a Lagrangian that is more than a pull back of a $d$-form over the manifold $M$. In a similar way, we may view $S_n$ as an $M$-brane field theory, where $S_n : M \to S_nM$ and $S_nM$ is the fiber of symmetric tensors of rank $n$ over $M$. This approach, however, cannot terminate itself since new interaction fields would be generated as in the case of $\phi : D \to M$.

Another way assumes that $A\gamma$ is a 1-form. Thus we may use the external algebra structure $\Lambda (T^* M)$ over $M$ to construct objects proportional to the volume form over $M$. For any $n$-form $(A)$ objects proportional to the volume form $\Omega_{Vol}$ can be constructed by using the external derivative $d$, multiplication $\wedge$, and Hodge dual * operations in $\Lambda (T^* M)$. For example, $A \wedge * A$ and $dA \wedge * dA$ are forms proportional to the volume form.

The next important ingredient comes from the symmetry in the matter equation. That is, if there is a transformation $A \to A'$ that leaves the matter equations unchanged, then there is no way to distinguish $A$ and $A'$. Thus the action for the field $A$ should obey the same symmetry (gauge symmetry).

For example, the matter equation for 4D electromagnetic interaction is $d\vec{v}/d\tau = F \cdot \vec{v}$ where $F$ is the 2-form obtained by differentiation of the 1-form $(A)$ ($F = dA$), and the gauge symmetry for $A$ is $A \to A' = A + df$. The reasonable terms for a 1-form field in the field Lagrangian $\mathcal{L}(A)$ are: $A \wedge * A, dA \wedge * dA$, and $dA \wedge dA$. The first term does not conform with the gauge symmetry $A \to A' = A + df$ and the second term $(dA \wedge dA)$ is a boundary term since $dA \wedge dA = d(A \wedge dA)$ that gives $\int_M d(A \wedge dA) = A \wedge dA$ at the boundary of $M$. This term is interesting in the quantum Hall effect. Therefore, we are left with a unique action for electromagnetism:

$$S[A] = \int_M dA \wedge * dA = \int_M F \wedge * F.$$  

For our next example, we look at the terms in the matter equation that involve gravity. There are two possible choices of matter equation. The first one is the geodesic equation $d\vec{v}/d\tau = \vec{v} \cdot \Gamma \cdot \vec{v}$ where $\Gamma$ is considered as a connection 1-form that transforms in the usual way $\Gamma \to \Gamma + \partial g$ under coordinate transformations $(g)$. This type of transformation, however, is not a ‘good’ symmetry since restricting $\Gamma \to \Gamma + \Sigma$ to transformations $\Sigma = \partial g$, such that $\vec{v} \cdot \Sigma \cdot \vec{v} = 0$, would mean to select a subset of coordinate systems, inertial systems, for which the action $S$ is well defined and satisfies $S[\Gamma] = S[\Gamma + \Sigma]$. Selecting a class of observables for the description of the system is not desirable, so we shall not follow this road.

In general, the Euler-Lagrange equations assume a background observer who defines a coordinate system. For electromagnetism, this is tolerable since neutral particles are such privileged observers. In gravity, however, there is no such observer, and the action should be related. Such an equation is the equation of the geodesic deviation: $d^2\xi/d\tau^2 = R \cdot \xi$, where $R$ is a Lie $TM$ valued curvature 2-form $R = d\Gamma + [\Gamma, \Gamma]$. A general curvature 2-form is denoted by $F \to (F_{\alpha\beta})^i_j$. Here, $\alpha$ and $\beta$ are related to the tangential space of the base manifold $M$. The $i$ and $j$ are related to the fiber structure of the bundle where is given the connection that defines $(F_{\alpha\beta})^i_j$. Clearly, the Ricci tensor $R$ is a very special curvature because all of its indices are of $TM$ type. For that reason, it is possible to contract the fiber degree of freedom with the base manifold degree of freedom (indices) Thus an action linear in $R$ is possible. In general, one needs to consider a quadratic action, i.e. trace of $F \wedge * F (F_{\alpha\beta}^i \wedge * F_{\alpha\beta}^j)$.

Using the symmetries of the Ricci tensor $R (R_{\alpha\beta,\gamma\rho} = -R_{\beta\alpha,\gamma\rho} = -R_{\alpha\beta,\rho\gamma} = R_{\gamma\rho,\alpha\beta})$ we have two possible expressions that can be proportional to the volume form $\Omega$. The first expression is present in all dimensions and is denoted by $R^*$, which means that a Hodge dual operation has been applied to the second pair of indices $(R_{\alpha\beta,\gamma\rho})$. The $R^*$ action seems to be related to the Cartan-Einstein action for gravity $S[R] = \int R_{\alpha\beta} \wedge *(dx^\alpha \wedge dx^\beta)$.
The other expression is only possible in a four-dimensional space-time and involves full anti-symmetrization of \( R (R_{\alpha[\beta,\gamma]\rho}) \) denoted by \( R^\wedge \). We have not been able to identify the role of the \( R^\wedge \) yet. Such a term in the action could produce a stabilizing effect or restore the renormalizability of the four dimensional gravity. If this happens, then it could be one of the reasons why the spacetime seems to be four dimensional. However, a statistical argument \([11]\) based on geometric and differential structure of various brane and target spaces seems to be a better explanation for why we are living in a 4D space-time. For example, if we make a statistical path-integral-like estimate using the following generating function:

\[
Z[S] = \sum_{\text{dim } M = 1}^{\infty} \left( \sum_{\text{M-topologies}} \left( \sum_{\text{dim } D = 1}^{\text{dim } M} \left( \sum_{\text{D-topologies}} \left( \sum_{\text{other structures}} e^{-S_m[\phi : D \to M; S_n]} - S_f[S_n] \right) \right) \right) \right),
\]

then we may find that the expectation value for the averaged space dimension is 4 because of the infinitely many homeomorphic but not diffeomorphic 4D spaces. In the expression above \( S_m[\phi : D \to M; S_n] \) is the action for matter with \( S_n \)-type fields, and \( S_f[S_n] \) is the corresponding action for the interaction fields.

### 6 Conclusions and Discussions

In summary, we have discussed the structure of the matter Lagrangian (\( L \)) for extended objects. Imposing reparametrization invariance of the action \( S \) naturally leads to a first order homogeneous Lagrangian. In its canonical form, \( L \) contains electromagnetic and gravitational interactions, as well as interactions that are not clearly identified yet. If one extrapolates from the strengths of the two known interactions, then one may suggest that the next terms should be important, if present at all, at big cosmological scales such as galactic cluster dynamics. The choice of the canonical Lagrangian is based on the assumption of one-to-one correspondence between interaction fields and the type of their sources. If one can show that any homogeneous function can be written in the canonical form suggested, then this would be a significant step in our understanding of the fundamental interactions. Note that an equivalent expression can be considered as well: \( L = A_\alpha(\vec{x}, \vec{v})v^\alpha \). This expression is simpler, and is concerned with the structure of the homogeneous functions of order zero \( A_\alpha(\vec{x}, \vec{v}) \). If one is going to study the new interaction fields \( S_n, n > 2 \), then the guiding principles for writing field Lagrangians, as discussed in the examples of electromagnetism and gravity, may be useful. It would be interesting to apply the outlined constructions to general relativity by considering it as a 3-brane in a 10 dimensional target space \((g_{\alpha\beta} : M \to S_2M)\).

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