MAGNETIC PARAMETERS INVERSION METHOD
WITH FULL TENSOR GRADIENT DATA

YANFEI WANG∗

Key Laboratory of Petroleum Resources Research
Institute of Geology and Geophysics, Chinese Academy of Sciences
Beijing 100029, China
University of the Chinese Academy of Sciences
Beijing 100049, China
Institutions of Earth Science, Chinese Academy of Sciences
Beijing 100029, China

DMITRY LUKYANENKO AND ANATOLY YAGOLA

Department of Mathematics, Faculty of Physics
Lomonosov Moscow State University
Moscow 119991, Russia

(Communicated by Peijun Li)

ABSTRACT. Retrieval of magnetization parameters using magnetic tensor gradient measurements receives attention in recent years. Determination of subsurface properties from the observed potential field measurements is referred to as inversion. Little regularizing inversion results using full tensor magnetic gradient modeling so far has been reported in the literature. Traditional magnetic inversion is based on the total magnetic intensity (TMI) data and solving the corresponding mathematical physical model. In recent years, with the development of the advanced technology, acquisition of the full tensor gradient magnetic data becomes available. In this paper, we study invert the magnetic parameters using the full tensor magnetic gradient data. A sparse Tikhonov regularization model is established. In solving the minimization model, the conjugate gradient method is addressed. Numerical and field data experiments are performed to show feasibility of our algorithm.

1. Introduction. In geophysical prospecting, data measured at, above, or below the ground are obtained during field survey (a forward problem), and extraction of the physical properties of the Earth from the data is a mathematical problem (an inverse problem) which is essential for processing and interpretation. Meanwhile, the use of magnetics for geophysical exploration is widely studied.

Traditional magnetic data are total magnetic intensity (TMI). With the development of a high temperature superconducting quantum interference devices (SQUIDs) operating in liquid nitrogen, a novel rotating magnetic gradiometer system has been designed. In China, we have designed a low temperature SQUIDs system, and we perform successively the field work in 2016. This system allows to measure components of the gradient tensor. Gradient measurements also provide

2010 Mathematics Subject Classification. Primary: 86A22; Secondary: 45B05, 45Q05, 65R32, 65F22.

Key words and phrases. Full tensor magnetic gradient, magnetic parameters reconstruction, regularization.

∗ Corresponding author: Yanfei Wang.
valuable additional information, compared to conventional total-field measurements, when the field is undersampled. Many discussions are given on the advantages of magnetic gradient tensor surveys as compared to the conventional total magnetic intensity (TMI) surveys [13, 14, 3, 12, 4].

Inversion of physical parameters, such as the magnetic susceptibility and the magnetization, are main scientific problems using magnetic field data [9, 10]. A lot of research works have been done so far. Wang and Hansen [18] reformulated the gravimetric-magnetic model in wavenumber domain into coordinates invariance form, and extended the original magnetic inversion method CompuDept into three-dimensional case, which allowed a large amount of airborne magnetic data being involved in inversion; Li and Oldenburg [5] recovered 3D susceptibility models by incorporating a priori information into the model objective function using one or more appropriate weighting functions; Pignatelli et al. [8] considered using dipole source to approximate the discrete gridded model of the anomaly, encompassed the depth weighting function into the discrete potential field function and employed the L–M method to solve the corresponding linear equation to get the solution with depth resolution.

According to the potential theory, magnetic inverse problems are ill-posed [17]. The main problems are the nonuniqueness and instability of the solution. Therefore it is crucial to choose proper norm of the model to restrict the solution space of the model. This is meaningful in reducing ill-posedness and enhancing numerical stability. The norm of the model should be chosen according to the a priori information of the model. Due to the fact that the magnetic data lack the resolution in depth, pure norm constraint on the model is not sufficient to reflect the medium layers. To overcome this problem, there are two ways. One is based on the Tarantola’s statistical theory [15], assuming that the data and the model are both uncertain and obey the Gaussian distribution, and constructing the fitting function using the maximum a posteriori likelihood function; another is based on Tikhonov regularization theory [16]. It can be proved these two forms are equivalent under proper conditions. Retrieval of magnetization parameters using magnetic tensor gradient measurements receives attention in recent years. The direct determination of subsurface properties (e.g., position, orientation, magnetic susceptibility) from the observed potential field measurements is referred to as inversion.

Due to the difficulty of acquiring field measured magnetic gradient tensor data, most of the results in literature are based on the total magnetic intensity (TMI) data. In methodology, previous studies mainly focus on statistical regularization and transform-based filtering method to recover physical parameters. Little optimizing and regularizing inversion results using gradient tensor modeling so far has been reported in the literature. In this paper, we will report our recent results using our device. In inversion realization, we study magnetic parameters (dipole source) inversion with regularization using magnetic gradient tensor data in this paper. Our airborne magnetic field survey using the low temperature SQUID system is performed in an area consists of paramagnetic and ferromagnetic material. Our new contributions to literature in this paper are: (1) a three-dimensional Tikhonov regularization model is established for the magnetic parameters (dipole source) inversion using magnetic gradient tensor (MGT) data; (2) this is the first time to report inversion results using our new data with our device; and other than TMI data, full tensor gradient data are compared to draw the conclusion that better inversion results can be obtained with full tensor gradient data.
2. Method. In this Section we describe the mathematical modeling of the considered problem involving not only classical approach connected with using total magnetic intensity data but also with tensor gradient data.

2.1. Mathematical modeling. The equation describing magnetic field $B_{\text{field dipole}}$ of dipole sources $m$ is defined as

$$B_{\text{field dipole}} = \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot r)r - m}{r^3} \right),$$

where $m = m_x i + m_y j + m_z k$, $r = (x - x_s)i + (y - y_s)j + (z - z_s)k$, $r = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}$ is a distance between point $(x_s, y_s, z_s)$, which corresponds to allocation of the triaxial sensor that measures magnetic field $B_{\text{field dipole}}$, and point $(x, y, z)$ of dipole source $m$, $\mu_0$ is a permeability in vacuum.

Transforming $B_{\text{field dipole}}$ into following form

$$B_{\text{field dipole}} = B_x \text{dipole}i + B_y \text{dipole}j + B_z \text{dipole}k =$$

$$= \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot r)(x - x_s)}{r^3} - \frac{m_x}{r^3} \right)i + \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot r)(y - y_s)}{r^3} - \frac{m_y}{r^3} \right)j +$$

$$+ \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot r)(z - z_s)}{r^3} - \frac{m_z}{r^3} \right)k$$

and redefining the variables as $i = x, y, z$ and $p = (p_x, p_y, p_z) \equiv (x_s, y_s, z_s)$, we have following representation for components of vector $B_{\text{field dipole}}$:

$$B_i \text{dipole} = \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot r)(i - p_i)}{r^3} - m_i \right).$$

Taking derivative of $B_i \text{dipole}$ with respect to spatial variable $i = x, y, z$ and $j = x, y, z \neq i$, we have the diagonal elements and non-diagonal elements of tensor matrix $B_{\text{tensor}}$:

$$B_{ii} = \frac{\mu_0}{4\pi} \left( \frac{6m_i(i - p_i)}{r^5} + \frac{3(m \cdot r)}{r^3} - \frac{15(m \cdot r)(i - p_i)(i - p_i)}{r^7} \right),$$

$$B_{ij} = \frac{\mu_0}{4\pi} \left( \frac{3m_j(j - p_j)}{r^5} + \frac{3m_i(i - p_i)}{r^5} - \frac{15(m \cdot r)(i - p_i)(j - p_j)}{r^7} \right).$$

Note, that we define full tensor magnetic gradient $B_{\text{tensor}}$, which unlike to magnetic induction $B_{\text{field dipole}}$ (that has only 3 components) has 9 components and can be written in the following matrix form:

$$B_{\text{tensor}} \equiv [B_{ij}] = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial z} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial z} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{yx} & B_{yy} & B_{yz} \\ B_{zx} & B_{zy} & B_{zz} \end{bmatrix},$$

where $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$, $\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y}$, $\frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$, and $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$. So, actually, we have only 5 different components of the tensor matrix.

Thus, for the whole object, for volume $V$ of which we want to restore the magnetic moment density $M$ ($M = M_x i + M_y j + M_z k$), we have the following 3D Fredholm
integral equations of the 1st kind:
\[B_{\text{field dipole}} = \frac{\mu_0}{4\pi} \iiint_V \left( \frac{3(M \cdot r)}{r^5} - \frac{M}{r^3} \right) \, dv,\]
\[B_{ij} = \frac{\mu_0}{4\pi} \iiint_V \left( \frac{6m_i(i-p_i)}{r^5} + \frac{3(M \cdot r)}{r^5} - \frac{15(M \cdot r)(i-p_i)(i-p_i)}{r^7} \right) \, dv,\]
\[B_{ij} = \frac{\mu_0}{4\pi} \iiint_V \left( \frac{3m_j(i-p_i)}{r^5} + \frac{3m_i(i-p_i)}{r^5} - \frac{15(M \cdot r)(i-p_i)(i-p_j)}{r^7} \right) \, dv,\]

which can be rewritten as the following system of two 3D Fredholm integral equations of the 1st kind:
\[
\begin{align*}
B_{\text{field dipole}}(x_s, y_s, z_s) &= \frac{\mu_0}{4\pi} \iiint_V K_{\text{TMI}}(x-x_s, y-y_s, z-z_s)M(x, y, z) \, dv, \\
B_{\text{tensor dipole}}(x_s, y_s, z_s) &= \frac{\mu_0}{4\pi} \iiint_V K_{\text{MGT}}(x-x_s, y-y_s, z-z_s)M(x, y, z) \, dv,
\end{align*}
\]

where \(B_{\text{field dipole}} = [B_x \ B_y \ B_z]^T\) and \(B_{\text{tensor dipole}} = [B_{xx} \ B_{xy} \ B_{xz} \ B_{yz} \ B_{zz}]^T\). Kernels \(K_{\text{TMI}}\) and \(K_{\text{MGT}}\) of these integral equations can be written as
\[
K_{\text{TMI}}(x-x_s, y-y_s, z-z_s) = \frac{3}{r^5} \begin{bmatrix}
3(x-x_s)^2 - r^2 & 3(x-x_s)(y-y_s) & 3(x-x_s)(z-z_s) \\
3(y-y_s)(x-x_s) & 3(y-y_s)^2 - r^2 & 3(y-y_s)(z-z_s) \\
3(z-z_s)(x-x_s) & 3(z-z_s)(y-y_s) & 3(z-z_s)^2 - r^2
\end{bmatrix},
\]
and
\[
K_{\text{MGT}}(x-x_s, y-y_s, z-z_s) = \frac{3}{r^7} \begin{bmatrix}
(x-x_s)[3r^2 - 5(x-x_s)^2] & (y-y_s)[r^2 - 5(x-x_s)^2] & (z-z_s)[r^2 - 5(x-x_s)^2] \\
(y-y_s)[r^2 - 5(x-x_s)^2] & (x-x_s)[3r^2 - 5(y-y_s)^2] & 5(x-x_s)(y-y_s)(z-z_s) \\
(z-z_s)[r^2 - 5(x-x_s)^2] & 5(x-x_s)(y-y_s)(z-z_s) & (x-x_s)[3r^2 - 5(z-z_s)^2] \\
-5(x-x_s)(y-y_s)(z-z_s) & (z-z_s)[r^2 - 5(y-y_s)^2] & (y-y_s)[r^2 - 5(z-z_s)^2] \\
-5(x-x_s)(y-y_s)(z-z_s) & (z-z_s)[r^2 - 5(y-y_s)^2] & (y-y_s)[r^2 - 5(z-z_s)^2] \\
(x-x_s)[r^2 - 5(z-z_s)^2] & (y-y_s)[r^2 - 5(z-z_s)^2] & (z-z_s)[3r^2 - 5(z-z_s)^2]
\end{bmatrix}.
\]

If we take into account that \(V \subset P = \{(x, y, z) : L_x \leq x \leq R_x, L_y \leq y \leq R_y, L_z \leq z \leq R_z\}\) and the system of sensor planes is restricted by rectangular parallelepiped \(Q = \{(x_s, y_s, z_s) \equiv (s, t, r) : L_s \leq s \leq R_s, L_t \leq t \leq R_t, L_r \leq r \leq R_r\}\), we can rewrite the system (1) in the following operator form
\[
AM = \frac{\mu_0}{4\pi} \iiint_{L_x \times L_y \times L_z} K(s, t, r, x, y, z)M(x, y, z) \, dx \, dy \, dz = B(s, t, r),
\]
where \(B(s, t, r)\) and \(M(x, y, z)\) are vector–functions: \(B = [B_x \ B_y \ B_z \ B_{xx} \ B_{xy} \ B_{xz} \ B_{yz} \ B_{zz}]^T\) and \(M = [M_x \ M_y \ M_z]^T\), kernel \(K(s, t, r, x, y, z)\) is a matrix-function: \(K = [K_{\text{TMI}} \ K_{\text{MGT}}]^T\). (\(K = K_{\text{TMI}}\) in the case of total magnetic intensity model without using full tensor magnetic gradient data and \(K = K_{\text{MGT}}\) in the opposed case).

In this paper we consider the model included the magnetic gradient tensor data only (thus \(K = K_{\text{MGT}}\).
2.2. Tikhonov regularization. Then we will assume that $M \in W^2_2(P)$, $B \in L_2(Q)$, and operator $A$ with kernel $K$ is continuous and unique. Norms of the right-hand side of equation (2) and the solution are introduces as follows:

$$
\|B\|_{L_2} = \sqrt{\|B_x\|^2_{L_2} + \|B_y\|^2_{L_2} + \|B_z\|^2_{L_2} + \|B_{xy}\|^2_{L_2} + \|B_{xz}\|^2_{L_2} + \|B_{yz}\|^2_{L_2} + \|B_{zz}\|^2_{L_2}},
$$

$$
\|M\|_{W^2_2} = \sqrt{\|M_x\|^2_{W^2_2} + \|M_y\|^2_{W^2_2} + \|M_z\|^2_{W^2_2}}.
$$

Suppose that instead of accurately known $\bar{B}$ and $A$ their approximate values $B_\delta$ and $A_\delta$ are known, such that $\|B_\delta - \bar{B}\|_{L_2} \leq \delta$, $\|A - A_\delta\|_{W^2_2} \leq h$. So, the inverse problem is ill-posed and it is necessary to build a regularizing algorithm for its solving. We use the algorithm based on minimization of the Tikhonov functional [16]

$$
F^\alpha[M] = \|A_\delta M - B_\delta\|^2_{L_2} + \alpha\|M\|^2_{W^2_2}.
$$

For any $\alpha > 0$ an unique extremal of the Tikhonov functional $M_\alpha^\eta$, $\eta = \{\delta, h\}$, which implements minimum of $F^\alpha[M]$, exists. To select the regularization parameter the generalized discrepancy principle can be used [2]. When we choose the parameter $\alpha = \alpha(\eta)$ accordingly to the generalized discrepancy principle

$$
\rho(\alpha) = \|A_\delta M_\eta^\alpha - B_\delta\|^2_{L_2} - \left(\delta + h\|M_\eta^\alpha\|_{W^2_2}\right)^2 = 0
$$

$M_\alpha^\eta$ tends to exact solution as $\eta \to 0$ in $W^2_2$. The minimal element of the Tikhonov functional for fixed $\alpha > 0$ can be found by the application of the conjugate gradient method.

2.3. Numerical aspects of the algorithm. For numerical minimization of functional (3) we used algorithms which were described in details at works [6, 7], including some recommendations of its effective parallelization.

After discretization an approximate solution $M$, which realizes the minimum of functional (3), can be found as a solution of the system

$$
(A_h^T A_h + \alpha R^T R) M = A_h^T B_\delta,
$$

where $R$ — finite-difference approximation of the operator $R$: $\|M\|_{W^2_2} = \|RM\|_{L_2}$, dimensions of matrix $A$: $(N_A \times N)$, matrix $R$: $(N_R \times N)$, vector $M$: $(N \times 1)$.

For numerical solving of system (5) we use the conjugate gradient in the following form.

Let $M^{(s)}$ — minimizing sequence, $p^{(s)}$, $q^{(s)}$ — auxiliary vectors, $p^{(0)} = 0$, $M^{(1)}$ — any arbitrary point. Then formulæ of the conjugate gradient method for searching of solution $M^{(N)}$ of system (5) can be rewritten as follow:

$$
r^{(s)} = A_h^T (A_h M^{(s)} - B_\delta) + \alpha R^T (RM^{(s)}), \quad \text{if} \quad s = 1,
$$

$$
r^{(s-1)} - q^{(s-1)} / (r^{(s-1)}, q^{(s-1)}), \quad \text{if} \quad s \geq 2,
$$

$$
p^{(s)} = p^{(s-1)} + \frac{r^{(s)}}{(r^{(s)}, r^{(s)})},
$$

$$
q^{(s)} = A_h^T (A_h p^{(s)}) + \alpha R^T (R p^{(s)}),
$$

$$
M^{(s+1)} = M^{(s)} - \frac{p^{(s)}}{(p^{(s)}, q^{(s)})}. $$
It should be noted that in numerical experiments we can put $\alpha = 0$, $M^{(1)} = 0$ and use the iteration number $s$ as the regularization parameter (in this case the stopping criterion is $\|A h M^{(s+1)} - B\|_2^2 \leq (\delta + \bar{h} \|M_0\| W_2^2)^2$) [1].

3. Results of calculations.

3.1. Theoretical example. At first, we wanted to compare different models (TMI, TMI+ MGT and MGT models). In order to do this we performed the simulation of the experimental data with noise level $\delta \sim 4\%$ (both TMI and MGT) which are close enough to the real field experiments. For testing calculation we used the integral domain $[x = -5000–5000 ; y = -5000–5000 ; z = -105–95]$ (that equal to the integral plane for $z = -100$) with grids $(Nx,Ny,Nz) = (80,80,1)$, and the observation section $[x = -400–4000 ; y = -4000–4000 ; z = 2000]$ with grids $(Ns,Nt,Nr) = (350,20,1)$ (corresponds to the simulated field data). The normalized magnitude of the model magnetic moment density $M$ is represented on Figure 1(a). The results of calculations which used TMI, TMI+MGT and MGT models are represented of Figure 1(b,c,d). The root-mean-square error in the components of the reconstructed vector $M$ is 0.12263 for TMI-model, 0.12262 — for TMI+MGT-model and 0.12527 for MGT-model. This means that all of the models produce “equal” results, but Figure 1(c) shows that MGT-model produce more detailed solution concerned to the small details.

The main conclusion from testing calculations is that 1) MGT-model is able to produce the better reconstruction for the magnitude of the small details of the solution, 2) the MGT-model should be used alone without combining TMI- and MGT-data.

3.2. Field data applications. Then we performed some testing calculations with real field data (there are only the tensor data with the first 5 tensor components of the MGT-data, other 3 components of the dipole field TMI-data are absent).

Our data was measured using the low-temperature SQUID. With a superconducting loop, SQUIDs can detect minute changes of flux. In our device, only changes perpendicular to the loop are detected, hence they are vector sensors. Using the new device, we can obtain abundant information of the magnetism with low noise, which may yield high resolution inversion results and give us sufficient quantitative analysis. Our airborne magnetic field survey using the low temperature SQUID system is performed in an area consists of paramagnetic and ferromagnetic material, which is a typical test area in North China. With the new type of data, we perform numerical inversion using our proposed regularization method.

3.2.1. Real Field Example 1. For testing calculation we used the integral domain $[x = -1000–1000 ; y = -1000–1000 ; z = -505–495]$ (that equal to the integral plane for $z = -500$) with grids $(Nx,Ny,Nz) = (40,40,1)$, and the observation section $[x = -215–215 ; y = -250–250 ; z = 0–0]$ with grids $(Ns,Nt,Nr) = (43,50,1)$ (corresponds to the field data). These parameters allow to preliminary allocate the area of the magnetic sources. The result of calculations for magnitude of the vector without regularization is represented on Figure 2. Then, based on the received information, we specify the dimensions of the integral domain: $[x = -500,–500 ; y = -500–500 ; z = -505–495]$ (with grids $(Nx,Ny,Nz) = (40,40,1)$, and the same observation section with same grids). Then we perform the calculations with regularization for the error-level of input field data $\delta \sim 5\%$. 
Figure 1. Results of testing calculations: a) model solution (the normalized value of the magnitude of the magnetic moment density $M$), b) retrieved solution for the TMI-model, c) retrieved solution for the MGT-model, d) retrieved solution for the TMI+MGT-model. The MGT-model produces the better reconstruction for the magnitude of the small details of the model solution. The use of the combined TMI+MGT-data does not give any advantages in reconstruction quality comparing with the using of TMI-data only.

3.2.2. Real Field Example 2. For testing calculation we used the integral domain $[x = -1000–1000 ; y = -1000–1000 ; z = -205–195]$ (that equal to the integral plane for $z = -200$) with grids $(Nx,Ny,Nz) = (40,40,1)$, and the observation section $[x = -215–215 ; y = -250–250 ; z = 0,0]$ with grids $(Ns,Nt,Nr) = (43,50,1)$ (corresponds to the field data). These parameters allow to preliminary allocate the area of the magnetic sources. The result of calculations for magnitude of the vector without regularization is represented on Figure 3 Then, based on the received information, we specify the dimensions of the integral domain: $[x = -500–500 ; y = -500–500 ; z = -205–195]$ (with grids $(Nx,Ny,Nz) = (40,40,1)$, and the same observation section with same grids). Then we perform the calculations with regularization for the error-level of input field data $\delta \sim 5\%$. 

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Figure 2. Results of calculation for Real Field Example 1. The main figure shows the result of preliminary allocation of the magnetic sources using quite large area as integral domain for calculations. The mini-figure shows more accurate results of second calculations for the integral domain with specified dimensions.

Figure 3. Results of calculation for Real Field Example 2. The main figure shows the result of preliminary allocation of the magnetic sources using quite large area as integral domain for calculations. The mini-figure shows more accurate results of second calculations for the integral domain with specified dimensions.
Magnetic inversion with full tensor data

For calculations in both examples we used 128 processors of the shared research facilities of HPC computing resources “Lomonosov–1” at Lomonosov Moscow State University [11]. Time of calculations $\sim 5$ minutes.

4. Conclusion. Using full tensor magnetic gradient data to retrieve the interested medium parameters is an important inverse problem in geophysics. Usually it is hard to obtain the practical full tensor magnetic gradient data. In this paper, we report our recent results using data measured by the low temperature SQUIDs system designed by SIMIT, Chinese Academy of Sciences. We establish a three-dimensional Tikhonov regularization model and use the conjugate gradient method to solve the large scale inverse problem. We first perform synthetic tests using the proposed regularization method to invert the physical parameters, then practical data are performed. Our inversion results reveal that the full tensor magnetic gradient data can distinguish fine structures of anomalies underground very well. Therefore, using full tensor magnetic gradient data for geophysical prospecting is a useful technique in the future.

Acknowledgments. We are grateful to the referees for their constructive comments, valuable suggestions for our paper. The research is supported by National Natural Science Foundation of China (grant No. 91630202), the NSFC-RFBR (grant No. 41611530693), the RFBR-NSFC (grant No. 17-51-53002) and National Key R & D Program of the Ministry of Science and Technology of China with the Project “Integration Platform Construction for Joint Inversion and Interpretation of Integrated Geophysics” (grant No. 2018YFC0603500). The research is carried out using the equipment of the shared research facilities of HPC computing resources at Lomonosov Moscow State University [11].

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Received April 2018; revised January 2019.

E-mail address: yfwang@mail.iggcas.ac.cn
E-mail address: lukyanenko@physics.msu.ru
E-mail address: yagola@physics.msu.ru