Quantum Computing Using an Open System and Projected Subspace

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Abstract

Using the subdynamical kinetic equation for an open quantum system, a formulation is presented for performing decoherence-free (DF) quantum computing in Rigged Liouville Space (RLS). Three types of interactions were considered, and in each case, stationary and evolutionary states were evaluated for DF behavior in both the total space and the projected subspace. Projected subspaces were found using the subdynamics kinetic equation. It was shown that although the total space may be decoherent, the subspace can be DF. In the projected subspace, the evolution of the density operator may be time asymmetric. Hence, a formulation for performing quantum computing in RLS or rigged Hilbert space (RHS) was proposed, and a quantum Controlled-Not Logical gate with corresponding operations in RLS (RHS) was constructed. A generalized quantum Turing machine in RHS was also discussed.

Key Words: Quantum Computing, Subdynamics, Rigged Liouville Space, Decoherence, Open System

PACS: 05.30. − d + 85.30 + 82.20.Db + 84.35. + i
I. INTRODUCTION

Although an ideal quantum computing system is an isolated quantum system whose evolution is described by a unitary operator with time reversal symmetry, in practice, it is often not practical to isolate a real quantum computing system from its environment. The interaction between a real system and its environment introduces decoherence which destroys the superpositions of qubits that enable quantum logical operations to be validated\[1,2\]. Several authors have formulated theories for decoherence-free (DF) subspaces in which quantum computing can be performed. Theory is developed starting from a master equation (such as the Lindblad equation) for an open system within the powerful semigroup approach. This shows that such DF subspaces do indeed exist, allowing logical qubits to be encoded and not to decohere\[3,4\] under the Born-Markov approximation and with restrictions on the type of decoherence (i.e., symmetric or collective decoherence). This motivated us to identify appropriately designed open quantum computing systems useful for canceling the effects of intrinsic decoherence. Such systems would thus behave as ideal quantum computing systems in the appropriate subspaces without introducing approximations or restrictions on the type of decoherence. In such open systems, self-adjoint operators and unitary evolution groups are not intrinsically necessary to govern or operate quantum computation. Quantum computation can then be performed in a more general functional space which permits irreversibility, such as Rigged Hilbert Space (RHS) or Rigged Liouville Space (RLS), rather than just Hilbert or Liouville Space. These are kinds of triplet space structure, such as the dense subspace and its topological dual space rigged to Hilbert space, $\Phi \subset \mathcal{H} \subset \Phi^\times$, (RHS) or dense space and its topological dual space rigged to Liouville space, $\Phi \otimes \Phi \subset \mathcal{H} \otimes \mathcal{H} \subset \Phi^\times \otimes \Phi^\times$, (RLS)\[5,6\]. These generalized functional spaces were introduced in physics in order to provide insight into singular entities like Dirac’s delta functions and formulations of
quantum mechanics\textsuperscript{[7–8]}. Since then, they have been intensively developed for describing irreversible process, non-integrable systems and chaotic dynamical systems. The latter includes RHS of the Hardy class used to formulate the generalized eigenvectors and complex spectral decomposition of the Liouvillian for the Friedrichs model\textsuperscript{[9–11]} and for the general spectral decomposition of chaotic maps\textsuperscript{[6]}. Such an approach can also be used to formulate open quantum systems in a suitable functional space, revealing non-unitary evolution semi-group irreversibility. It may also be suitable for describing quantum logical operations in open quantum systems subject to decoherence.

Based on the above concepts, we propose a type of projected subspace for open quantum computing systems, to perform DF quantum logical operations for three different types of interactions, and investigate quantum logical operations in such subspaces based on the kinetic equation introduced from subdynamics\textsuperscript{[6,12]}. Compared with recent publications on DF subspaces, our approach is general and not restricted to the Born-Markov approximation. In section II, a subdynamic formulation and kinetic equation for an open system is derived using a simpler approach than previously published references\textsuperscript{[6,12]}. In sections III, IV and V, three types of coherence / DF conditions for the projected subspaces or total spaces are analyzed under respective interactions. The role of eigenvalues and eigenvectors of the Hamiltonian in the decoherent / DF processes of the systems are discussed. In sections VI and VII, a formulation for quantum logical operations in the projected space is investigated, and a formulation of quantum computation in RLS (RHS) and a generalized Turing machine are proposed.

**II. SUBDYNAMICAL KINETIC EQUATION**

In general, consider an open quantum computing system $S$, which may consist of real spins, pseudospins, or number states of a quantized field (such as photons), coupled with an infinite or finite environment $R$, and where the Hamiltonian operator for the total system is given by
\[ H(t) = H_0(t) + \lambda H_1(t), \]

and \( H_0(t) \equiv H_S(t) + H_R, \) \( H_S(t), \) \( H_R \) and \( H_I(t) \) correspond to the Hamiltonian of \( S, \) \( R \) and the interaction between \( S \) and \( R. \) \( \lambda \) is a coupling constant for this interaction. The corresponding Liouville equation for the total density operator \( \rho(t) \) is given by

\[ i \frac{d}{dt} \rho(t) = [H_0(t) + \lambda H_1(t), \rho(t)] = (L_0(t) + \lambda L_1(t)) \rho(t). \]

Choosing the time-independent eigenprojectors of \( L_0(t) \) as \( P_{\nu} \) and \( Q_{\nu} \) with \( Q_{\nu} + P_{\nu} = 1, \) respectively, then the eigenprojectors \( \Pi_\nu(t) \) for the total Liouvillian \( L(t) \) can be written in terms of the Heisenberg equation as \( i \frac{d}{dt} \Pi_\nu(t) = [L(t), \Pi_\nu(t)]. \) \( \Pi_\nu(t) \) satisfy the usual properties of projection operators, such as \( \sum_\nu \Pi_\nu(t) = 1, \) \( \Pi_\nu^2(t) = 1, \) \( \Pi_\nu(t) \Pi_\mu(t) = \delta_{\nu\mu}, \) and is analytic with respect to \( \lambda: \lim_{\lambda \to 0} \Pi_\nu(t) \to P_{\nu}. \) From the definition of the eigenprojectors \( \Pi_\nu(t) \) we have \( Q_\nu L(t) (Q_{\nu} + P_{\nu}) \Pi_\nu(t) = Z_{\nu\alpha} Q_\nu \Pi_\nu(t) \) \((Z_{\nu\alpha} \) is a corresponding eigenvalue), giving

\[ Q_\nu \Pi_\nu(t) = C_\nu(t) \Pi_\nu(t), \quad (1) \]

where the creation operator is defined by \( C_\nu(t) \equiv \sum_\alpha C_\nu(Z_{\nu\alpha}, t) P_{\nu} P_{\nu} \) with \( C_\nu(z, t) \equiv \frac{1}{z - Q_\nu L(t)} Q_\nu L(t) P_{\nu}, \) for \( z \in \) complex plane \( C. \) In the same way, the destruction operator \( D_\nu(t) \) is defined by

\[ \Pi_\nu(t) Q_\nu = \Pi_\nu(t) D_\nu(t). \quad (2) \]

Notice that, since \( \Pi_\nu \) is the projector, therefore \( Q_\nu \neq C_\nu(t) \neq D_\nu(t) \) in Eqs.\( (1) \) and \( (2). \) Here \( C_\nu(z, t) \) is the collision operator\(^{13,14}\) used in non-equilibrium statistical mechanics. The operator \( C_\nu(t) \) creates the \( Q_\nu\)-part of \( \Pi_\nu(t) \) from \( P_{\nu} \) and the operator \( D_\nu(t) \) provides the opposite operation of the creation operator since \( C_\nu(t) = Q_\nu C_\nu(t) P_{\nu} \) and \( D_\nu(t) = P_{\nu} D_\nu(t) Q_\nu. \) This allows the construction of kinetic equations for each \( P_{\nu}\)-component of \( \Pi_\nu(t) \) by
\[
\frac{i\partial}{\partial t} (P_\nu \Pi_\nu (t) \rho (t)) = iP_\nu \left[ \frac{\partial}{\partial t} \Pi_\nu (t) \right] \rho (t) + iP_\nu \Pi_\nu (t) \frac{\partial}{\partial t} \rho (t)
\]
(3)
\[
= P_\nu [L (t), \Pi_\nu (t)] \rho (t) + P_\nu \Pi_\nu (t) L (t) \rho (t)
= P_\nu L (t) (P_\nu + Q_\nu (t)) \Pi_\nu (t) \rho (t)
= P_\nu L (t) (P_\nu + C_\nu (t)) \Pi_\nu (t) \rho (t)
= \Theta_\nu (t) (P_\nu \Pi_\nu (t) \rho (t)) ,
\]
where \( \Theta_\nu (t) \equiv P_\nu L (t) P_\nu + P_\nu L (t) C_\nu (t) P_\nu \). A kinetic equation for all of \( P_\nu \) is given
\[
\frac{i}{\partial t} \sum_\nu (P_\nu \Pi_\nu (t) \rho (t)) = \sum_\nu \Theta_\nu (t) \sum_\mu (P_\mu \Pi_\mu (t) \rho (t))
= \Theta (t) \rho^{\text{proj}} (t) ,
\]
(4)
where we define the total intermediate operator as \( \Theta (t) \equiv \sum_\nu \Theta_\nu (t) \) and the total projector as \( \rho^{\text{proj}} (t) \equiv \sum_\nu \rho^{\text{proj}}_\nu (t) \equiv \sum_\nu P_\nu \Pi_\nu (t) \rho (t) \). We refer to Eq.(4) as the subdynamical kinetic equation, and the operator \( \Theta (t) \) as the generator of the subdynamical kinetic equation. The second-order approximation for each \( \Theta_\nu (t) \) corresponds to the Boltzmann, Pauli, and Fokker-Planck equations of kinetic theory and Brownian motion\[6,12\]. The creation and destruction operators can be obtained from the basic operator equations within the subdynamics formulation\[6,12\],
\[
i \frac{\partial}{\partial t} C_\nu (t) + [L_0, C_\nu (t)] = \lambda (C_\nu - Q_\nu) L_1 (P_\nu + C_\nu) ,
\]
(5)
\[
i \frac{\partial}{\partial t} D_\nu (t) + [L_0, D_\nu (t)] = \lambda (P_\nu + D_\nu) L_1 (Q_\nu - D_\nu) .
\]
(6)
The construction of a kinetic equation for the whole system can be accomplished by solving the equations for the \( C_\nu (t) \), or \( D_\nu (t) \), with retarded or advanced integrations; for example,
\[
C_\nu (t) = -i \int_t^{\infty} d\tau \hat{T} e^{-i \int_\tau^t L_0 (\tau') d\tau'} \lambda (C_\nu (\tau) - Q_\nu) L_1 (\tau) (P_\nu + C_\nu (\tau)) \hat{T} e^{i \int_\tau^t L_0 (\tau') d\tau'},
\]
(7)
where \( \hat{T} \) is the Dyson time-ordered operator. A similar approach can be used for the destruction operator. Neglecting higher terms \( 0 (C_\nu^2 (t)) \), \( 0 (D_\nu^2 (t)) \), Eqs.(5) and (6) become
\[ i \frac{\partial}{\partial t} C_\nu (t) + [L_0, C_\nu (t)] = -\lambda Q_\nu L_1 (P_\nu + C_\nu) , \]  
\[ i \frac{\partial}{\partial t} D_\nu (t) + [L_0, D_\nu (t)] = -\lambda (P_\nu + D_\nu) L_1 Q_\nu. \]

The exact solutions for Eqs.(8) and (9) are:

\[ C_\nu (t) = -i\lambda e^{-i \int_{\pm \infty}^t Q_\nu L (\tau') Q_\nu d\tau'} \left[ \int_{\pm \infty}^t e^{i \int_{\pm \infty}^\tau Q_\nu L (\tau') Q_\nu d\tau'} Q_\nu L (\tau) P_\nu e^{i P_\nu L_0 P_\nu t} d\tau \right] e^{i P_\nu L_0 P_\nu t}, \]

(10)

and

\[ D_\nu (t) = -i\lambda e^{i P_\nu L_0 P_\nu t} \left[ \int_{\pm \infty}^t e^{-i P_\nu L_0 P_\nu t} Q_\nu L (\tau) P_\nu e^{i \int_{\pm \infty}^\tau Q_\nu L (\tau') Q_\nu d\tau'} d\tau \right] e^{-i \int_{\pm \infty}^t Q_\nu L (\tau') Q_\nu d\tau'}. \]

(11)

The general evolution formulae for the density operator can be deduced from Eq.(3). Indeed, since

\[ P_\nu \Pi_\nu (t) \rho (t) = (P_\nu + D_\nu (t) C_\nu (t))^{-1} (P_\nu + D_\nu (t)) \rho (t) \]
\[ = (P_\nu + D_\nu (t) C_\nu (t))^{-1} (P_\nu + D_\nu (t)) \sum_{\nu'} (P_{\nu'} + C_{\nu'} (t)) \rho_{\nu'} (t) \]
\[ = \delta_{\nu \nu'} (P_\nu + D_\nu (t) C_\nu (t))^{-1} (P_{\nu'} + D_{\nu'} (t) C_{\nu'} (t)) \rho_{\nu'}^{proj} (t) \]
\[ = \rho_{\nu}^{proj} (t), \]

and

\[ \rho (t) \equiv \sum_{\nu} (P_\nu + C_\nu (t)) \rho_{\nu}^{proj} (t) \]
\[ = \Omega \rho_{\nu}^{proj} (t), \]

(13)

where \( \Omega \) is the similarity operator defined as

\[ \Omega \equiv \sum_{\nu} (P_\nu + C_\nu (t)). \]

(14)

Eq.(3) can be understood as

\[ i \frac{\partial}{\partial t} \rho_{\nu}^{proj} (t) = i \frac{\partial}{\partial t} P_\nu \Pi_\nu (t) \rho (t) \]
\[ = \Theta_\nu (t) P_\nu \Pi_\nu (t) \rho (t) \]
\[ = \Theta_\nu (t) \rho_{\nu}^{proj} (t) \]
\[ = (P_\nu L P_\nu + P_\nu L_0 C_\nu (t) P_\nu) \rho_{\nu}^{proj} (t), \]

where

\[ \Theta_\nu (t) = \sum_{\nu'} (P_\nu + C_\nu (t)) \rho_{\nu'}^{proj} (t) \]
\[ = (P_\nu L P_\nu + P_\nu L_0 C_\nu (t) P_\nu) \rho_{\nu}^{proj} (t), \]
which gives the formal solution for $\rho_{\nu}^{\text{proj}} (t)$,

$$
\rho_{\nu}^{\text{proj}} (t) = \hat{T} e^{-i \int_{t_0}^{t} dt' \Theta_n (t')} \rho_{\nu}^{\text{proj}} (t_0) \\
= \hat{T} e^{-i \int_{t_0}^{t} dt' \Theta_n (t')} (P_{\nu} + D_{\nu} (t_0) C_{\nu} (t_0))^{-1} (P_{\nu} + D_{\nu} (t_0)) \rho_{\nu}^{\text{proj}} (t_0) .
$$

Eq. (16), for each $P_{\nu}$-component of $\Pi_{\nu} (t)$ is related to the dynamics of $\rho_{\nu}^{\text{proj}} (t)$ and includes the master equation as the second-order approximation of $\Theta_{\nu} (t)$ with respect to weak coupling$^{[12]}$. The Liouville equation for the reduced projected density of the quantum computing subsystem $S$ can be obtained by taking the part trace $Tr_R$ of Eq. (16). It can be shown that the Liouvillian $L (t)$ and the total intermediate operator $\Theta (t)$ hold a similarity relation (i.e., in the time-dependent case, $L (t) = \left(i \frac{\partial \Theta (t)}{\partial t} + \Omega (t) \Theta (t) \right) \Omega^{-1} (t)$). In the time independent case, $L = \Omega \Theta \Omega^{-1}$, which can replace the Hamiltonian and relevant projections$^{[15]}$. Therefore, the eigenvectors of $\Theta$ in the time-independent case, can be transformed to $L$ with the same structure of eigenvalues for $\Theta$ and $L$. Thus Eq. (15) or (4) is the general kinetic equation with significant physical meaning, and can exactly describe the dynamics of the open quantum computing system $S$ without invoking any approximations. Hence, it provides the starting point for the discussions below.

III. DECOHERENCE AND DF INTERACTIONS IN TOTAL SPACE FOR A DIAGONAL INTERACTION

Using Eq. (15) or (4), it is apparent that the second term on the right tends to zero corresponding to a complete DF subspace for the bipartite system $S + R$. This suggests that one should try to identify bipartite systems which, under “diagonal interaction” with the environment, still permit the second term to be zero and remain coherent. Here the term “diagonal interaction” is taken to mean that the eigenfunctions of the interaction part of the Hamiltonian are still the original product states of the unperturbed part of the Hamiltonian$^{[16]}$. Indeed, if the Hamiltonian of an open system has a diagonal interaction form, then the interaction term of Eq. (15) or (4), should equal zero and a DF condition exist for the stationary states of the system.
To illustrate this point, consider a two-level atom interacting with a single cavity mode. Although this model does not describe decoherence from a Bosonic bath (since it only deals with one bosonic degree of freedom) it is sufficient to show the underlying principles. In the case of an off-resonant interaction, the effective Hamiltonian can be expressed as[^16,^17]

\[ H = \omega_0 \sigma^z + \omega a^+ a + g a^+ a |2\rangle \langle 2|, \]  

where \( \sigma^z \equiv |2\rangle \langle 2| - |1\rangle \langle 1| \), \( \omega_0 \) and \( \omega \) are angular frequencies, \( g \) is the coupling number, and \( a^+, a \) are creation and annihilation operators for the cavity mode, respectively.

Without exercising the diagonal interaction, \( g a^+ a |2\rangle \langle 2| \), the spectral decomposition of the total Hamiltonian for this system can be expressed as

\[ H_0 = \sum_n \sum_{j=1}^{2} (-1)^j \omega_0 + n\omega \left| j (1) \otimes n (2) \right\rangle \left\langle n (2) \otimes j (1) \right| \]  

with

\[ |\psi\rangle = \sum_n \sum_{j=1}^{2} \langle n (2) \otimes j (1)| \psi \rangle |j (1) \otimes n (2)\rangle, \text{ for any } |\psi\rangle \text{ of this system.} \]  

(19)

Exercising the diagonal interaction, \( g a^+ a |2\rangle \langle 2| \), the spectral decomposition of the total Hamiltonian is

\[ H = \sum_n \sum_{j=1}^{2} \left( (-1)^j \omega_0 + n\omega + \frac{1+(-1)^j}{2}ng \right) \left| j (1) \otimes n (2) \right\rangle \left\langle n (2) \otimes j (1) \right| . \]  

(20)

It is therefore apparent that the eigenvectors of this Hamiltonian \( H \) are still the original eigenvectors of \( H_0 \) although the eigenvalues change to \( (-1)^j \omega_0 + n\omega + \frac{1+(-1)^j}{2}ng \) from \( (-1)^j \omega_0 + n\omega \). \( H \) is diagonal with respect to the basis \( \{ |j (1) \rangle \otimes |n (2) \rangle ; \langle n (2) | \otimes \langle j (1) | \} \), and therefore the off-diagonal matrix elements of the corresponding Liouvillian are equal to zero,

\[ P_{\nu} L_1 (t) Q_{\nu} = 0, \text{ for } \nu = (j, j'; n, n'), \]  

(21)

Eq. (21) yields

\[ P_{\nu} L_1 (t) Q_{\nu} C_{\nu} (t) P_{\nu} = 0, \]  

(22)
where the eigenprojectors defined in Liouville space are

\[ P_\nu \equiv \left| \nu \right\rangle \langle \nu \right| = \left| j (i) \otimes n (k) \right\rangle \left\langle j' (i) \otimes n' (k) \right| \left| j (i) \otimes n (k) \right\rangle \langle j (i) \otimes n (k) \rangle . \]  

(23)

Thus Eq. (19) remains invariant with or without the diagonal interaction; that is, there is no decoherence introduced by the diagonal interaction for the stationary states. But since the eigenvalues have been changed by the interaction, the evolution of the density operators are still subject to a type of “decoherence” (unitary error) introduced by this change. That is, before the interaction the evolution is

\[ |\psi (t)\rangle_b = \sum_{jn} e^{-i(-1)^j_2 \omega_0 + n \omega} t \langle j (i) \otimes n (k) | \psi (t) \rangle | j (i) \otimes n (k) \rangle , \]  

(24)

while after interaction, the evolution is

\[ |\psi (t)\rangle_a = \sum_{jn} e^{-i(-1)^j_2 \omega_0 + n \omega + \frac{1+i(-1)^j}{2} n g} t \langle j (i) \otimes n (k) | \psi (t) \rangle | j (i) \otimes n (k) \rangle , \]  

(25)

and generally we have

\[ |\psi (t)\rangle_b \neq |\psi (t)\rangle_a . \]  

(26)

In other words, Eqs. (19) and (21) reveal that the eigenvectors of the total Hamiltonian $H$ with or without the diagonal interaction, have the same tensor products, and are all diagonal with respect to the set of these tensor products, showing that the diagonal interaction does not introduce any decoherence in the stationary states. Eqs. (24) and (25) show that the eigenvalues of the total Hamiltonian are changed by the interaction. In this case, the interaction introduces a phase shift which may introduce a type of “decoherence” (unitary error) in the evolution of the states, although the fidelity\cite{18} $Tr \left( \sqrt{\rho (0) \rho (t)} \right)$, remains 1 for both cases (i.e., with or without the interaction).

**IV. DF SUBSPACE FOR TRIANGULAR INTERACTION**

The above discussion for a diagonal interaction, refers to a rather ideal case. Another less ideal DF situation is now discussed - it is the “triangular interaction”. This is defined
as when the off-diagonal matrix elements (of the interaction part of the total Hamiltonian) are up or lower triangular. For example, if the interaction part of the Hamiltonian can be expressed in a triangular form using the previous model

\[ H(1, 2) = g \left( a^+ a \sigma^z + a^- \sigma^- \right) , \]  

then the subdynamical kinetic equation (15) gives

\[
i \frac{\partial}{\partial t} \rho_\nu(t) = \left[ P_\nu L_0 P_\nu \rho_\nu(t) + P_\nu L_R P_\nu + P_\nu L_1 Q_\nu(t) P_\nu \right] \rho_\nu(t) = P_\nu (L_S + L_R) P_\nu \rho_\nu(t) ,
\]

where from the triangular property of the matrix elements we have

\[
\langle i \otimes k | H(1, 2) | j \otimes n \rangle = \begin{cases} 
(-1)^j g n, & i = j, k = n \\
g\sqrt{n-1}, & k = n - 1, i = j - 1 \\
0, & \text{otherwise}
\end{cases} .
\]

This causes the second term in Eq.(15) to be zero, since the \( C_\nu \) is a function of \( Q_\nu L_1 P_\nu \) from subdynamics theory\[6,7\]:

\[ P_\nu L_1 Q_\nu C_\nu P_\nu = 0. \]  

Eqs.(28) and (29) show that the density operator \( \rho_\nu(t) \) is DF under the triangular interaction in the subspaces. That is, with or without loading the triangular interaction, the spectral decomposition for the intermediate operator \( \Theta \) is invariant,

\[
\Theta = \sum_\nu P_\nu L_0 P_\nu = \sum_\nu \sum_{n \neq n'} \sum_{j \neq j'} (-1)^j n g |jn, j'n'\rangle \langle j'n', j, n| ,
\]

so that an arbitrary expanded projected density operator and its evolution remains invariant with or without the interaction,

\[
|\rho^{proj}\rangle = \sum_\nu \sum_{n \neq n'} \sum_{j \neq j'} (-1)^j n g |jn, j'n'\rangle \langle j'n', j, n| \rho^{proj}_\nu |jn, j'n'\rangle ,
\]

and

\[
|\rho^{proj}(t)\rangle = \sum_\nu \sum_{n \neq n'} \sum_{j \neq j'} e^{-i(-1)^j n g t} |jn, j'n'\rangle \langle j'n', j, n| \rho^{proj}_\nu(t) \rangle |jn, j'n'\rangle .
\]
This shows that no decoherence is introduced by the triangular interaction. However, this does not mean that the total density operator \( \rho(t) \) is DF in the total space. Indeed, taking into account Eq.(14) and using the similarity operator \( \Omega \) on Eq.(15), gives

\[
i \frac{\partial}{\partial t} \Omega \rho_\nu(t) = i \frac{\partial}{\partial t} \rho(t) = \Omega \Theta_\nu \Omega^{-1} \rho_\nu(t) = \Omega \Theta_\nu \Omega^{-1} \rho(t),
\]

and yields

\[
L = \sum_\nu \Omega \Theta_\nu \Omega^{-1} \sum_\nu (\nu | L_0 | \nu) (P_\nu + C_\nu) (\nu | P_\nu + D_\nu)
\]

\[
= \sum_\nu \sum_{n \neq n'} \sum_{j=1}^2 (-1)^j g (n - n') \left( |\nu\rangle + \sum_{\nu' < \nu} (\nu' | C_\nu | \nu) |\nu\rangle \right) \left( |\nu\rangle + \sum_{\nu' < \nu} (\nu' | D_\nu | \nu) (\nu) \right),
\]

where \((\nu' | C_\nu | \nu)\) can be obtained from the recurrence formula (7) or (10),

\[
(\nu' | C_\nu | \nu) \quad (35)
\]

\[
= \frac{1}{l_\nu - l_{\nu'}} g (\nu' | L_1 (P_\nu + C_\nu) | \nu)
\]

\[
= \frac{1}{((-1)^j n - (-1)^j n')} \left( (\nu' | \nu) \right) (\nu | L_1 (P_\nu + C_\nu) | \nu)
\]

\[
= \frac{1}{((-1)^j n - (-1)^j n')} \left( (\nu' | \nu) \right) (\nu | L_1 (P_\nu + C_\nu) | \nu)
\]

\[
= \frac{1}{(\nu' | \nu) \left( \nu | L_1 (P_\nu + C_\nu) | \nu \right)}
\]

\[
= \frac{1}{(\nu' | \nu) \left( \nu | L_1 (P_\nu + C_\nu) | \nu \right)}
\]

\[
= \frac{1}{(\nu' | \nu) \left( \nu | L_1 (P_\nu + C_\nu) | \nu \right)}
\]

Hence the eigenvectors of the total Hamiltonian can be formally written as

\[
|\varphi_\nu\rangle = |\nu\rangle + \sum_{\nu' < \nu} (\nu' | C_\nu (t) | \nu) |\nu'\rangle,
\]

and

\[
(\varphi_\nu) = (\nu | + \sum_{\nu' < \nu} (\nu' | D_\nu (t) | \nu') (\nu').
\]

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This shows that the eigenvectors of the total Hamiltonian are a combination of the eigenvectors of the intermediate operator $\Theta$, and the total system is still subject to decoherence. Therefore, it can be concluded that the system, under a triangular interaction in each projected $P_{\nu}$ subspace, is DF for both stationary and evolutionary states, but suffers from decoherence in the total space.

V. DF AND DECOHERENT SUBSPACES FOR A GENERAL INTERACTION

Since the triangular interaction does not introduce decoherence in the corresponding subdynamical projected subspaces, this suggests that there should exist a more general class of DF subspaces. In fact, as mentioned in the introduction, the existence of such DF subspaces has been shown by several approaches, such as projection onto a symmetric subspace of multiple copies of a quantum computing system\cite{4}. In this treatment, we focus on providing an argument based on a subdynamics approach. Generally, it is assumed that the interaction parts of the Hamiltonian for the previous model are non-diagonal or non-triangular in form; moreover, for practical use (i.e., to construct a quantum logical gate), a model is considered consisting of two atoms with two-levels simultaneously interacting with a single optical mode\cite{19,20} and influenced by an environment consisting of a set of infinite Harmonic oscillators. The Hamiltonian is given by

$$ H = H_0 + \lambda H_I $$

with

$$ H_0 = \sum_{j=1}^{2} \left[ \omega_j \sigma_j^z + g (a^+ \sigma_j^- + a \sigma_j^+) \right] + \omega a^+ a + \sum_k \omega_k b_k^+ b_k, $$

$$ H_I = \sum_k \sum_{j=1}^{2} g_k \left( b_k^+ + b_k \right) \left( \sigma_j^- + \sigma_j^+ \right), $$

where $b_k^+$, $b_k$ are creation and annihilation operators of the oscillator. The eigenvalues of the equation for $H_0$ are given by first writing the matrix expression of $H_0$ in
Hilbert space \( \{|\pm (1), \pm (2), n, \{n_k\}\}\) and then diagonalizing it. Using the matrix elements \( \langle \pm (1), \pm (2), n', \{n'_k\}| H_0 |\pm (1), \pm (2), n, \{n_k\}\rangle \), the matrix expression for \( H_0 \) in the subspace \( \{\varphi_{n,k}^+, \varphi_{n+1,k}^+, \varphi_{n+1,k}^-\}\),

\[
\varphi_{n,k}^+ = |+1), +(2), n, \{n_k\}\rangle, \\
\varphi_{n+1,k}^+ = |-(1), +(2), n + 1, \{n_k\}\rangle, \\
\varphi_{n+1,k}^- = |+1), -(2), n + 1, \{n_k\}\rangle,
\]

is given by

\[
H_0 = \begin{pmatrix}
\frac{\omega_1 + \omega_2}{2} + n\omega + \sum_k n_k\omega_k & g\sqrt{n + 1} & g\sqrt{n + 1} \\
g\sqrt{n + 1} & (n + 1)\omega + \sum_k n_k\omega_k & 0 \\
g\sqrt{n + 1} & 0 & (n + 1)\omega + \sum_k n_k\omega_k
\end{pmatrix}
\]

Following the eigenvalue equation, \( H_0 |f\rangle = \varepsilon |f\rangle \), gives a characteristic equation,

\[
\begin{vmatrix}
a & \gamma & \gamma \\
\gamma & b & 0 \\
\gamma & 0 & b
\end{vmatrix} = 0,
\]

its solutions give the eigenvalues as:

\[
\varepsilon_{n,k}^{++} = b, \\
\varepsilon_{n+1,k}^{++} = \frac{1}{2}b + \frac{1}{2}a + \frac{1}{2}\sqrt{(b^2 - 2ab + a^2 + 8\gamma^2)}, \\
\varepsilon_{n+1,k}^{+-} = \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}\sqrt{(b^2 - 2ab + a^2 + 8\gamma^2)},
\]

and the corresponding eigenvectors are

\[
|f_{n,k}^{++}\rangle = -|\varphi_{n+1,k}^-\rangle + |\varphi_{n+1,k}^+\rangle, \\
|f_{n+1,k}^{++}\rangle = -\frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}\sqrt{(b^2 - 2ab + a^2 + 8\gamma^2)}\gamma|\varphi_{n,k}^+\rangle + |\varphi_{n+1,k}^+\rangle + |\varphi_{n+1,k}^-\rangle, \\
|f_{n+1,k}^{+-}\rangle = -\frac{1}{2}b - \frac{1}{2}a - \frac{1}{2}\sqrt{(b^2 - 2ab + a^2 + 8\gamma^2)}\gamma|\varphi_{n,k}^+\rangle + |\varphi_{n+1,k}^+\rangle + |\varphi_{n+1,k}^-\rangle.
\]
where we denote $a \equiv \frac{\omega_1 + \omega_2}{2} + \omega + \sum_k n_k \omega_k - \varepsilon$, $b \equiv (n + 1) \omega + \sum_k n_k \omega_k - \varepsilon$ and $\gamma \equiv \sqrt{g + 1}$.

From these eigenvectors, the eigenprojectors $P_{nm,kj}^{\pm \mp}$ of the corresponding unperturbed Liouvillian $L_0$ are defined as

$$P_{nm,kj}^{\pm \mp} = \left| f_{n,k}^\pm \right\rangle \left\langle f_{m,j}^{\mp} \right| \left( f_{m,j}^{\mp} \right\rangle \left\langle f_{n,k}^\pm \right|$$

with

$$Q_{nm,kj}^{\pm \mp} = 1 - P_{nm,kj}^{\pm \mp}.$$ (47)

Without the interaction, the spectral decomposition for $L_0$ is given by

$$L_0 = \sum_{\pm,\mp} \sum_{nm,kj} \left( \varepsilon_{n,k}^\pm - \varepsilon_{m,j}^{\mp} \right) \left| \phi_{nm,jk}^{\pm \mp} \right\rangle \left\langle \phi_{nm,jk}^{\pm \mp} \right|$$

$$= \sum_{\pm,\mp} \sum_{nm,kj} \left( \varepsilon_{n,k}^\pm - \varepsilon_{m,j}^{\mp} \right) P_{nm,kj}^{\pm \mp}.$$ (49)

The corresponding spectral decomposition of the intermediate operator $\Theta$, in terms of subdynamics theory, is the same as that for $L_0$, i.e.,

$$\Theta = \sum_{\pm,\mp} \sum_{nm,kj} P_{nm,kj}^{\pm \mp} L_0 P_{nm,kj}^{\pm \mp} = \sum_{\pm,\mp} \sum_{nm,kj} \left( \varepsilon_{n,k}^\pm - \varepsilon_{m,j}^{\mp} \right) \left| \phi_{nm,jk}^{\pm \mp} \right\rangle \left\langle \phi_{nm,jk}^{\pm \mp} \right|.$$ (50)

Now considering interactions occurring to this system, the Liouvillian becomes

$$L = L_0 + \lambda L_1$$

with

$$L_1 = \sum_{nm,jk \neq n'm'j'k'} \left( \phi_{nm,jk}^{\pm \mp} \left| L_1 \right| \phi_{n'm'j'k'}^{\pm \mp} \right) \left( \phi_{n'm'j'k'}^{\pm \mp} \right\rangle \left\langle \phi_{nm,jk}^{\pm \mp} \right|.$$ (52)

Then, in terms of the subdynamical equation (3), the operator $\Theta$ can be expressed as

$$\Theta = \sum_{\pm,\mp} \sum_{nm,kj} \left( P_{nm,kj}^{\pm \mp} L_0 P_{nm,kj}^{\pm \mp} + P_{nm,kj}^{\pm \mp} L_1 Q_{nm,kj}^{\pm \mp} C_{nm,kj}^{\pm \mp} P_{nm,kj}^{\pm \mp} \right),$$ (53)

and the corresponding spectral decomposition is
This again means that the eigenvectors of $\Theta$, $\phi_{nm,jk}^{\pm \mp}$, with or without coupling are the same, while the corresponding eigenvalues change to $\{(\varepsilon_{n,k}^{\pm} - \varepsilon_{m,j}^{\pm}) + (\phi_{nm,jk}^{\pm \mp} | L_1 C_{nm,jk}^{\pm \mp} | \phi_{nm,jk}^{\pm \mp})\}$ (with the interaction) from $\{(\varepsilon_{n,k}^{\pm} - \varepsilon_{m,j}^{\pm})\}$ (without the interaction). Using the set of the eigenvectors $\{\phi_{nm,jk}^{\pm \mp}\}$ of the free Liouvillian $L_0$ as a basis for expanding an arbitrary state of the system $\rho_{proj}$, in the projected subspace space, the stationary projected density operator remains invariant with or without the interaction,

$$
|\rho_{proj}\rangle \equiv \sum_{\pm} \sum_{nm,kj} P_{nm,kj}^{\pm \mp} \Pi_{nm,kj}^{\pm \mp} \rho = \sum_{\pm} \sum_{nm,kj} \left( \phi_{nm,jk}^{\pm \mp} | P_{nm,kj}^{\pm \mp} \Pi_{nm,kj}^{\pm \mp} \rho \right) \phi_{nm,jk}^{\pm \mp},
$$

while the evolution of the projected density operator changes, upon loading the interaction, to

$$
|\rho_{proj}^\prime(t)\rangle \equiv \sum_{\pm} \sum_{nm,kj} P_{nm,kj}^{\pm \mp} \Pi_{nm,kj}^{\pm \mp} \rho(t) = \sum_{\pm} \sum_{nm,kj} e^{-i\{(\varepsilon_{n,k}^{\pm} - \varepsilon_{m,j}^{\pm}) + (\phi_{nm,jk}^{\pm \mp} | L_1 C_{nm,jk}^{\pm \mp} | \phi_{nm,jk}^{\pm \mp})\}t} \left( \phi_{nm,jk}^{\pm \mp} | P_{nm,kj}^{\pm \mp} \Pi_{nm,kj}^{\pm \mp} \rho(0) \right) \phi_{nm,jk}^{\pm \mp}.
$$

This is similar to the case for the diagonal interaction, but it takes place in the projected subspace, showing that no decoherence occurs to the stationary states, but the evolutionary states are subject to a phase shift introduced by the change in the eigenvalues in the subdynamical projected subspace. This phase shift induces a type of “decoherence” (like a unitary error) to the evolutionary states. Even so, the fidelity of the mixed states in the subspace can be shown to equal 1,
\[ F(t) = Tr \sqrt{\rho_{\text{proj}}(0) \rho_{\text{proj}}(t)} \]

\[ = Tr \sqrt{\rho_{\text{proj}}(0) \hat{T} e^{-i \int \Theta(t') dt'} \rho_{\text{proj}}(0) \hat{T} e^{i \int \Theta(t') dt'}} \]

\[ = Tr \left\{ \sum_{\pm} \sum_{nm,kj} (\rho_{\text{proj}})_{nm,kj}^{\pm} P_{nm,kj}^{\pm} \left( \hat{T} e^{-i \int \Theta(t') dt'} \sum_{\pm} \sum_{nm,kj} (\rho_{\text{proj}})_{nm,kj}^{\pm} P_{nm,kj}^{\pm} e^{i \int \Theta(t') dt'} \right) \right\} \]

\[ = Tr \left\{ \sum_{\pm} \sum_{nm,kj} \left[ (\rho_{\text{proj}})_{nm,kj}^{\pm} \right]^2 e^{-i \left[ (\epsilon_{n,k} - \epsilon_{m,j}) + (\phi_{nm,jk}^\pm L_{1} C_{nm,jk}^\pm |\phi_{nm,jk}^\pm|) \right] t} \right\} \]

\[ = Tr \left( \sqrt{\rho_{\text{proj}}(0) \rho_{\text{proj}}(t)} \right) = 1, \]

where \((\rho_{\text{proj}})_{nm,kj}^{\pm}\) is the matrix element of the density operator \(\rho_{\text{proj}}(t_0)\) with respect to \(|\phi_{nm,jk}^\pm\rangle\), \(|\phi_{nm,jk}^\pm\rangle\), and \(\rho_{\text{proj}}(t_0)\) can be expanded by the set of the eigenprojectors \(P_{nm,kj}^{\pm}\).

Since the eigenprojectors remain invariant, the initial density operator remains the same with or without the interaction under the assumption of decoupling of initial states. Eq. (57) shows that the constructed projected subspace is DF! Whatever the subspace, the total density operator \(\rho\) is decoherent in the total space. This suggests a universal procedure for constructing a DF subdynamical projected subspace by choosing a set of one-dimensional eigenprojectors \(P_\nu\) of the unperturbed Liouville operator \(L_0(t)\) through the kinetic equation (15) or (8). In this constructed subspace, the governing equation is the subdynamical kinetic equation and the interaction part of the total Liouvillian cannot change the diagonal property of the intermediate operator, if each of \(P_\nu\) in the complete set is chosen as a one-dimensional projector by construction (note if \(P_\nu\) is over one-dimensional the intermediate operator generally is not diagonal).

The above discussions are summarized below in tabular form:
where \( St \) and \( Et \) refer to stationary states and evolutionary states, and \( PE, D \) and \( DF \) refer to a phase error, decoherent and decoherence-free, respectively. The table shows that it is possible to construct a DF subdynamical projected subspace for an open quantum computing system with three types of interaction occurring with the environment. This allows quantum logical computations to be performed by using the projected density operator on the corresponding subspaces, which may be beyond Hilbert space. For example, these can include rigged Hilbert spaces. To clarify this point, a model of two two-level atoms is discussed in detail in the following subsection. A second-order calculation is presented, but the approximation does not restrict the validity of the result, since there is no problem in allowing high order calculations to be performed based on the same methodology.

VI. QUANTUM LOGICAL OPERATION IN SUBSPACES

The expected evolution of the projected density operator, in the projected subspaces, can be formally solved from Eq. (4),

\[
\rho_{\text{proj}}(t) = \sum_{\pm} \sum_{nmjk} e^{-i\Theta_{nm,kj}^T t} \left| P_{nm,kj} \Pi_{nm,kj}^T \rho(0) \right> \left< P_{nm,kj} \Pi_{nm,kj}^T \rho(0) \right|
\]

\[
= \sum_{\pm} \sum_{nmjk} e^{-iE_{nm,kj}^T t} \left( \phi_{nm,kj}^T \left| \rho_{nm,kj}^T (0) \right> \left< \rho_{nm,kj}^T (0) \right| \phi_{nm,kj} \right),
\]

where the difference in energy \( E_{nm,kj}^T \) is defined as \( E_{nm,kj}^T = \varepsilon_{n,k}^T - \varepsilon_{m,j}^T + \lambda \left( \phi_{nm,kj}^T \left| L_1 C_{nm,kj}^T \phi_{nm,kj} \right> \right), \) and the projected density operator is determined from \( \rho(t), \)

\[
P_{nm,kj} \Pi_{nm,kj} \rho(t) \equiv \rho_{nm,kj}^T (t)
\]

\[
= \left( P_{nm,kj} + D_{nm,kj} C_{nm,kj} \right)^{-1} \left( P_{nm,kj} + D_{nm,kj} \right) \rho(t).
\]

\[
\]
Here $D_{nm,kj}^{±±∓∓}$ is given by Eq. (11),

$$D_{nm,kj}^{±±∓∓} = -i\langle \frac{P_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} \rangle \left[ \int_{-\infty}^{t} e^{-iP_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} P_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} \right] t_{1} Q_{nm,kj}^{±±∓∓} e^{iQ_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} d\tau \right]\times e^{-iQ_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} t}$$

$$= -\lambda P_{nm,kj}^{±±∓∓} L_{1} Q_{nm,kj}^{±±∓∓} \frac{1}{Q_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} - (\varepsilon_{n,k} - \varepsilon_{m,j})}.$$  

Taking into account the Born approximation, the second order term $D_{nm,kj}^{[2]±±∓∓}$ is

$$D_{nm,kj}^{[2]±±∓∓} = \sum_{±± n' m'' j'' k''} -\lambda \left[ \left| \phi_{nm,kj}^{±±∓∓} \right| \left( L_{1} \phi_{n'' m'', j'' k''}^{±±∓∓} / (\varepsilon_{n', k'} - \varepsilon_{m', j'}) \right) \left( \phi_{nm,kj}^{±±∓∓} \right) - \phi_{n'' m'', j'' k''}^{±±∓∓} \left| \phi_{nm,kj}^{±±∓∓} \right| + 0 (\lambda^{2}) \right]$$

which gives the second order projected density operator as

$$\rho_{nm,kj}^{[2]±±∓∓} (t) = \left[ P_{nm,kj}^{±±∓∓} + D_{nm,kj}^{[2]±±∓∓} \right] \rho (t),$$

where $\left( \phi_{nm,kj}^{±±∓∓} \left| L_{1} \phi_{n'' m'', j'' k''}^{±±∓∓} \right| \right) = g_{j} (\sqrt{(j+1)} j + \sqrt{(j-1)} j) - g_{k} (\sqrt{(k+1)} k + \sqrt{(k-1)} k).$

On the other hand, $E_{nm,kj}^{±±∓∓}$ can be approximately found by calculating $C_{nm,kj}^{[1]±±∓∓}$ from Eq. (10). For example, the first order $C_{nm,kj}^{[1]±±∓∓}$ is

$$C_{nm,kj}^{[1]±±∓∓} = -\lambda \sum_{±± n' m'' j'' k''} \frac{1}{Q_{nm,kj}^{±±∓∓} L_{nm,kj}^{±±∓∓} - (\varepsilon_{n,k} - \varepsilon_{m,j})} Q_{nm,kj}^{±±∓∓} L_{1} P_{nm,kj}^{±±∓∓}$$

$$= -\lambda \sum_{±± n' m'' j'' k''} \left( \phi_{n'' m'', j'' k''}^{±±∓∓} \left| \phi_{nm,kj}^{±±∓∓} \right| \left| \phi_{nm,kj}^{±±∓∓} \right| \right) + 0 (\lambda),$$

which then gives the second order term $E_{nm,kj}^{[2]±±∓∓}$,

$$E_{nm,kj}^{[2]±±∓∓} = \left( \varepsilon_{n,k}^{±∓} - \varepsilon_{m,j}^{∓∓} \right) - \lambda^{2} \sum_{±± n' m'' j'' k''} \left( \phi_{n'' m'', j'' k''}^{±±∓∓} / (\varepsilon_{n', k'} - \varepsilon_{m', j'}) \right) \left( \phi_{nm,kj}^{±±∓∓} \right)$$

$$= \left( \varepsilon_{n,k}^{±∓} - \varepsilon_{m,j}^{∓∓} \right) - \lambda \sum_{±± n' m'' j'' k''} g_{j} (\sqrt{(j+1)} j + \sqrt{(j-1)} j) - g_{k} (\sqrt{(k+1)} k + \sqrt{(k-1)} k).$$
and hence giving the second order evolution of $\rho_{\text{proj}}^{[2]}(t)$,

$$|\rho_{\text{proj}}^{[2]}(t)\rangle = \sum_{\pm\mp} \sum_{nmjk} |\rho_{nm,kj}^{[2]\pm\mp\mp\mp}(0)\rangle \langle \rho_{nm,kj}^{\pm\mp\mp\mp} |,$$

(65)

Considering this evolution, a quantum XOR operator for the system may be constructed by applying a sequence of operations\cite{20,21} which is related to the swap operator,

$$U_{sw} |i, j\rangle = e^{-iLt_{sw}} |i, j\rangle = |j, i\rangle. \quad (66)$$

Suppose there is no coupling interaction, the ideal swap operator is then given by adjusting the coupling time $t_{sw}$

$$U_{sw} = \sum_{\pm\mp} \sum_{nmjk} e^{-iE_{0}^{n,k,mj}t_{sw}} \quad (67)$$

and loading the interaction, the nonideal action of the swap operator is given by the evolution operator (since the energy shift induced an error), i.e.,

$$U_{sw} = \sum_{\pm\mp} \sum_{nmjk} e^{-i(E_{0}^{n,k,mj}+\Delta E_{n,k,mj})(t_{sw}+\Delta t)} \langle \phi_{nm,jk}^{\pm\mp\mp\mp} | \phi_{nm,jk}^{\pm\mp\mp\mp} \rangle, \quad (68)$$

To cancel the decoherence, we have to have

$$e^{-i\epsilon_{n,k}^{\pm\mp} t_{sw}} = e^{-iE_{0}^{n,m,kj}(t_{sw}+\Delta t)}, \quad (69)$$

which gives

$$\Delta t_{sw} = \frac{t_{sw}}{E_{0}^{n,m,kj}} + 1, \quad (70)$$

where we have assumed that the distribution of energy is homogeneous, $\frac{E_{0}^{n,m,kj}}{\Delta E}$ = constant, to determine the universal time $\Delta t$. For example, if the second order nonideal swap operator is
demonstrates an important advantage of performing quantum logical operations in the projected subspace.

\[ \rho = \sum_{\pm} e^{i \sum_{n,k} g_j (\sqrt{(j+1)j} + \sqrt{(j-1)j}) - g_k (\sqrt{(k+1)k} + \sqrt{(k-1)k})} \left( t_{sw} + \triangle t_{sw} \right) \]

then the second order \( \triangle t_{sw}^{[2]} \) is

\[ \triangle t_{sw}^{[2]} = \left( \frac{\varepsilon_{n,k} - \varepsilon_{m,j}}{g_j (\sqrt{(j+1)j} + \sqrt{(j-1)j}) - g_k (\sqrt{(k+1)k} + \sqrt{(k-1)k})} \right) \]

This shows that: (1) the change of the eigenvalues after loading the coupling interaction, introduces a phase error in the swap operator, (2) the error can easily be cancelled by adjusting the coupling time \( (\triangle t_{sw}) \) based on \( t_{sw} \), since the eigenvectors are invariant. This demonstrates an important advantage of performing quantum logical operations in the projected subspace.

Moreover, from Eqs.(10), (11) and (33) it is evident that eigenvalue \( E_{nm,kj}^{\pm \mp \mp \mp} \) may be complex (in fact, this model is a type of Friedrichs model, possessing a complex spectrum in the sense of extending Hilbert space\(^{[22-24]}\)). This means that the time evolution of \( \rho_{nm,kj}^{\pm \mp \mp \mp} \) in Eq.(58) may be asymmetric. Thus, a Rigged Liouville Space (RLS) formulation can be chosen\(^{[6,6]}\) to describe the subsystem, since a Hilbert space formulation provides an inadequate description. The evolution of the projected density operator and its adjoint operator are expressed as

\[ \sum_{\pm} \sum_{nmjk} | \rho_{nm,kj}^{\pm \mp \mp \mp} (t) \rangle = \sum_{\pm} \sum_{nmjk} e^{-i E_{nm,kj}^{\pm \mp \mp \mp} t} \left( \phi_{nm,kj}^{\pm \mp \mp \mp} (0) \right) \left( \phi_{nm,kj}^{\pm \mp \mp \mp} \right), \]

\[ \sum_{\pm} \sum_{nmjk} (\bar{\rho}_{nm,kj}^{\pm \mp \mp \mp} (t)) = \sum_{\pm} \sum_{nmjk} e^{i E_{nm,kj}^{\pm \mp \mp \mp} t} \left( \phi_{nm,kj}^{\pm \mp \mp \mp} (0) \right) \left( \phi_{nm,kj}^{\pm \mp \mp \mp} \right), \]

where \( \rho_{nm,kj}^{\pm \mp \mp \mp} (t) \) may exist in the test space \( \Phi_{SR} \otimes \Phi_{SR} \), which is a dense subspace of the Liouville Space, representing the physical states which can be prepared in an actual experiment. Its adjoint \( \bar{\rho}_{nm,kj}^{\pm \mp \mp \mp} (t) \) lies in the dual space \( ^{*} \Phi_{SR} \otimes \Phi_{SR}^* \), representing a procedure.
that associates with each state a number, while preserving the linear structure which results from the superposition principle. This is a RLS structure which facilitates describing irreversible processes like decoherence and dissipation due to interaction with the environment. This poses an interesting question as to what is new in this formulation for an open quantum system, if it performs quantum logical operations in a RLS (RHS). For convenience we assume that $|\rho_{nm,kj}^{\pm\mp\mp\mp}(t)\rangle$, $(\tilde{\rho}_{nm,kj}^{\pm\mp\mp\mp}(t))$ are biorthonormalized with respect to each other, and the evolution of $\rho_{nm,kj}^{\pm\mp\mp\mp}(t)$ (in Eq.(58)) in the projected subspace permits the system to perform quantum computing in a RLS (RHS).

Indeed, if we construct a general quantum logical operator $Q$ given by some function $f$ of the biorthonormal states $\{|\rho_{nm,kj}^{\pm\mp}(t)\rangle, (\tilde{\rho}_{nm,kj}^{\pm\mp}(t))\}$ in a RLS as

$$Qg\left(|\rho_{nm,kj}^{\pm\mp}(t)\rangle\right) = g'\left(|\rho_{nm,kj}^{\pm\mp}(t)\rangle\right),$$

(74)

$$h\left(|\tilde{\rho}_{nm,kj}^{\pm\mp}(t)\rangle\right) = h'\left(|\tilde{\rho}_{nm,kj}^{\pm\mp}(t)\rangle\right),$$

(75)

and

$$Q \equiv f\left(|\rho_{nm,kj}^{\pm\mp}(t)\rangle\right) (\tilde{\rho}_{nm,kj}^{\pm\mp}(t)).$$

(76)

Then the states or observables are $g\left(|\rho_{nm,kj}^{\pm\mp}(t)\rangle\right)$, $g'\left(|\rho_{nm,kj}^{\pm\mp}(t)\rangle\right) \in \Phi_S \otimes \Phi_S$, $h\left(|\tilde{\rho}_{nm,kj}^{\pm\mp}(t)\rangle\right)$, $h'\left(|\tilde{\rho}_{nm,kj}^{\pm\mp}(t)\rangle\right) \in \chi \Phi_S \otimes \Phi_S^\ast$, and the space is closed to the dual pair spaces $\Phi_S \otimes \Phi_S$ and $\chi \Phi_S \otimes \Phi_S^\ast$. The RLS structure is $\Phi_S \otimes \Phi_S \subset \mathcal{H} \otimes \mathcal{H} \subset \chi \Phi_S \otimes \Phi_S^\ast$. For example, a quantum universal gate, such as a two qubit based quantum Controlled-Not logical gate (XOR gate) in RLS, can generally be constructed from a combination of the projection operators as:

$$\mathcal{CN} = |00\rangle (\tilde{00}) + |01\rangle (\tilde{01}) + |10\rangle (\tilde{11}) + |11\rangle (\tilde{10}),$$

(77)

which induces the corresponding quantum logical operations,

$$\mathcal{CN} |00\rangle = |00\rangle, \mathcal{CN} |01\rangle = |01\rangle,$$

(78)

$$\mathcal{CN} |10\rangle = |11\rangle, \mathcal{CN} |11\rangle = |10\rangle,$$

$$(\tilde{00})\mathcal{CN} = (\tilde{00}) , (\tilde{01})\mathcal{CN} = (\tilde{01}) ,$$

$$(\tilde{10})\mathcal{CN} = (\tilde{11}) , (\tilde{11})\mathcal{CN} = (\tilde{10}).$$
and demonstrates that the quantum logical operations are closed in RHS. The property of 
closure for quantum logical operators in RLS (RHS) is important for practical applications. 
This shows that quantum logical operations in RLS (RHS) inherit the essential properties of 
those in Liouville (Hilbert) space, if the biorthornomal property of generalized eigenstates 
still hold.

One may still argue that the advantages of the above approach for quantum logical opera-
tions in RLS (RHS), for an open computing system $S$, are not evident. We would argue that 
the Liouville (Hilbert) space approach generally fails to describe open quantum computing 
systems which undergo irreversible processes. These include semigroup evolution and non-
self-adjoint Hamiltonians of subsystems, far from equilibrium. This is something that the 
RLS (RHS) formulation is well suited to handling. To construct quantum logical operations 
in projected subspaces, the influence of decoherence induced by changes in the eigenvalues, 
may easily be cancelled since the eigenvectors remain invariant. Moreover, in this space 
the evolution of the states are permitted to be time asymmetric, providing a framework for 
describing the irreversibility of practical open systems. This irreversibility does not change 
quantum reversible logical operations to quantum irreversible logical operation in the quan-
tum universal Controlled-Not logical gate. To appreciate this, one must distinguish between 
irreversibility of a quantum logical operation, introduced by the structure of logical gate 
and irreversibility of the process, induced by interactions with the environment. Reversible 
computation means reversible logical operations on the structure of the logical gate. In this 
sense, quantum computing in RLS (RHS) does not adversely impact on reversible quantum 
logical operations and permits computing any reversible function, although irreversible pro-
cesses exist. This is further clarified below in relation to construction of a quantum Turing 
machine in RHS.
VII. CONSTRUCTION OF A TURING MACHINE IN RHS

Consider, in general, an open system with $N$ components (atoms) each of which can have two states. We design a quantum Turing machine consisting of $N = n + 1$ pseudospins or levels $|\pm (j)\rangle$, $j = h, 1, 2, \cdots, n$, in a $2^{n+1}$ dimensional test space $\Phi$ and $\langle \pm (j) |$, $j = h, 1, 2, \cdots, n$, in the dual space of the test space $\Phi^\times$ with bi Orthomodality and isometry,

$$\langle \pm (j) | \mp (j') \rangle = \delta_{\pm \mp} \delta_{jj'},$$  

(79)

$$\langle U^+ \pm (j) | U (\pm (j)) \rangle = \langle \pm (j) | \pm (j) \rangle,$$  

(80)

where the Turing head is denoted as $h$, and the Turing tape pseudospins are denoted as $1, \cdots, n$, and $\pm$ represent two states or two observables. The quantum Turing machine can be considered as an open system in which the Turing tape is a special finite bath coupled to the Turing head. The state of the quantum Turing machine $|\psi\rangle$ lies in a $2^{n+1}$ dimensional dense subspace $\Phi$ of the Hilbert Space and is spanned by $\{|\pm (j)\rangle\}$, and its observables $\langle \tilde{\psi} |$ are spanned by $\{\langle \pm (j) |\}$ in the dual space $\Phi^\times$. The evolution of a pure state $|\psi_0\rangle \langle \tilde{\psi}_0|$ is given by

$$|\psi_j\rangle \langle \tilde{\psi}_j| = U (j) |\psi_0\rangle \langle \tilde{\psi}_0| \tilde{U}^+ (j),$$  

(81)

where the non-unitary evolution operator $U$ can be expanded by the non-self-adjoint generators defined by

$$\hat{\lambda}_x (j) = P_{01} (j) + P_{10} (j),$$  

(82)

$$\hat{\lambda}_y (j) = iP_{01} (j) - iP_{10} (j),$$

$$\hat{\lambda}_z (j) = P_{11} (j) - P_{00} (j),$$

with

$$P_{sk} (j) = |i (j)\rangle \langle k (j)|.$$  

(83)
Following ref. [25], we restrict ourselves to the Bloch vector $\vec{\lambda}$ of the Turing head $h$ as

$$\lambda^h_i = \langle \tilde{\psi}_h | \tilde{\lambda}_i (h) \otimes 1 (1) \otimes \cdots \otimes 1 (n) | \psi_h \rangle.$$  \hspace{1cm} (84)

Hence, for $2^n$ biorthonormal initial tape states $|\phi_0\rangle = |\pm (1)\rangle \otimes |\pm (2)\rangle \otimes \cdots \otimes |\pm (n)\rangle$, $\langle \tilde{\phi}_0 | = \langle \pm (n) | \otimes \cdots \otimes \langle \pm (2) | \otimes \langle \pm (1) |$ and arbitrary Turing head $h$ states $|\varphi_h\rangle$, $\langle \tilde{\varphi}_h |$, the states of Turing machine $|\psi_h\rangle$, $\langle \tilde{\psi}_h |$ can be written as the tensor product states

$$|\psi_h\rangle = |\varphi_h \otimes \phi_0\rangle,$$  \hspace{1cm} (85)

$$\langle \tilde{\psi}_h | = \langle \tilde{\varphi}_h \otimes \tilde{\phi}_0 |,$$

and hence the Bloch vector of the Turing head $\lambda^h_i$ is described by the above tensor product states as

$$\lambda^h_i (\phi_0, \tilde{\phi}_0) = \langle \tilde{\varphi}_h \otimes \tilde{\phi}_0 | \tilde{\lambda}_i (h) \otimes 1 (1) \otimes \cdots \otimes 1 (n) | \varphi_h \otimes \phi_0 \rangle$$  \hspace{1cm} (86)

performing a pure state trajectory on the Bloch circle,

$$\lambda^h_y (\phi_0, \tilde{\phi}_0)^2 + \lambda^h_z (\phi_0, \tilde{\phi}_0)^2 = 1.$$  \hspace{1cm} (87)

Any initial state of the Turing machine $|\psi_0\rangle$, $\langle \tilde{\psi}_0 |$ with the Turing head $h$ in the pure state $|\varphi_0 (h)\rangle$, $\langle \tilde{\varphi}_0 (h) |$ can thus be written

$$|\psi_0\rangle = \sum_{j=1}^{2^n} a_j | \varphi_0 (h) \otimes \phi_j^0 \rangle,$$  \hspace{1cm} (88)

$$\langle \tilde{\psi}_0 | = \sum_{j=1}^{2^n} b_j \langle \tilde{\varphi}_0 (h) \otimes \tilde{\phi}_j^0 |.$$

This yields a motion of the Bloch vector of Turing head $h$ induced by the change of the tape states, given by

$$\lambda^h_k (\psi_0, \tilde{\psi}_0) = \sum_{j=1}^{2^n} a_j b_j \lambda^h_k (\phi_j^0, \tilde{\phi}_j^0),$$  \hspace{1cm} (89)
where \(a_j, b_j\) are expansion coefficients. This shows that the motion of the Bloch vector of Turing head \(\lambda^h_{k} \left( \psi_0, \bar{\psi}_0 \right)\) constructed in RHS can be expressed as \(2^n - 1\) entangled Bloch vectors of the Turing head \(\lambda^h_{k} \left( \phi^j_0, \bar{\phi}^j_0 \right)\) induced by a pure Turing head state \(|\varphi_0 (h)\rangle, \langle \bar{\varphi}_0 (h)|\), and \(2^n\) initial tape states \(|\phi_0\rangle, \langle \bar{\phi}_0|\). This means that the motion of the Turing head \(h, \lambda^h_{k} \left( \psi_0, \bar{\psi}_0 \right)\), can perform many parallel primitive trajectories of the Bloch vector of the Turing head \(h, \lambda^h_{k} \left( \phi^j_0, \bar{\phi}^j_0 \right)\), identical to the situation of the reversible Turing head constructed in the Hilbert Space formulation\(^{[25]}\). This is because the generalized basis in RHS remains biorthonormal and isometric, corresponding to the orthonormal property, and invariance of the inner product for the basis in Hilbert Space. This enables most properties of the quantum Turing machine in Hilbert Space to be inherited in RHS, with the exception of the reversible property of an ideal process.

VIII. CONCLUSIONS

In conclusion, a subdynamic formulation for an open quantum system is presented using a simpler approach than discussed previously. Based on the subdynamical kinetic equation for an open quantum system, the properties of coherence and decoherence-free conditions for three types of interactions were analyzed. A condition for a quantum computing system to remain in coherent states in the projected subspaces is found. That is, the interactions between the system and its environment should be “diagonal” or “triangular”. Moreover, one can find universal DF projected subspaces by using the subdynamical kinetic equation without restrictions on the type of decoherence and without introducing any approximations, although the total space may be decoherent. An implied universal concept in our analysis is that the invariant eigenvectors remain a DF structure for any stationary states of the system and that changes of the eigenvalues only induce a phase shift. This type of phase shift which may introduce a type of phase error for the evolved states does not change the DF property of the subspace. The possible phase error can be cancelled, due to the invariance of the eigenvectors in the projected subspace under the assumption of a homogeneous distribution.
of the energy shifts. In this projected subspace, one can construct a quantum XOR operator and show that the evolution of the SWAP operator is time asymmetric. Hence, we propose a formulation for performing quantum computing in RLS (RHS) by constructing a general quantum Controlled-Not logical gate, with corresponding operations in RLS (RHS), and a generalized quantum Turing machine in RHS.

ACKNOWLEDGEMENTS

We gratefully acknowledge financial support from NSERC, MITACS, CIPI, MMO, CITO and China State Key Projects of Basic Research and Natural Science foundation (G1999064509, N0 79970121, 60072032).
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