Some remarks about non-minimally coupled scalar field models

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Abstract
Several results related to flat Friedmann–Lemaître–Robertson–Walker models in the conformal (Einstein) frame of scalar–tensor gravity theories are extended. Scalar fields with arbitrary (positive) potentials and arbitrary coupling functions are considered. Mild assumptions under such functions (differentiable class, number of singular points, asymptotes, etc) are introduced in a straightforward manner in order to characterize the asymptotic structure on a phase space. We pay special attention to the possible scaling solutions. Numerical evidence confirming our results is presented.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Recent astrophysical observations suggest that the universe is currently experiencing an accelerated expansion [1–21]. To explain this feature of the universe one choice is to introduce the concept of dark energy (DE; see [22–24] and references therein), which could be a cosmological constant, a quintessence field [25–31], a phantom field [32–37], or the quintom field [38–49], among other examples. Another choice is to consider higher order gravity (HOG) theories, for example the $F(R)$-models (see [50–58] and references therein). Other modified gravitational scenarios are the extended non-linear massive gravity scenario [59–62], the teleparallel DE (TDE) model [63–65] and some generalizations of TDE [66, 67]. All of these scenarios have very interesting cosmological features.
Another interesting effective scalar field model, which is related to inhomogeneous cosmologies, is the so-called ‘morphon’ field which arises as an effective scalar field model for averaged cosmologies [68, 69]. In this case the scalar field is not interpreted as a source of the Einstein equations, but as a mean field description of averaged inhomogeneities. In this case the ‘back-reaction’ effects (due to averaged expansion and shear fluctuations, the averaged 3-Ricci curvature, averaged pressure gradients and frame fluctuation terms) are formally equivalent to the dynamics of a homogeneous, minimally coupled scalar field. This relation was completely addressed in [68].

Although our results are more widely applicable if the scope is widened to effective scalar fields such as the morphon field, in this paper the DE contribution is modeled as a conventional self-interacting quintessence scalar field, $\phi$, with potential $V(\phi)$ [25–31].

In the inflationary universe scenarios, the potential $V(\phi)$ must satisfy some requirements which are necessary for the early-time acceleration of the expansion [70–74]. Exponential (de Sitter) expansion arises, for example, as a result of considering a constant potential $V(\phi) = V_0$, whereas the power-law inflationary solutions arise when considering an exponential potential $V(\phi) = V_0 \exp(-\lambda \phi)$ [75, 76]. In [77], a minimally coupled scalar field evolving in the quadratic potential $V(\phi) = \frac{1}{2} m^2 \phi^2$ has been investigated in the flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric. Therein a global picture of the solution space by means of a regular global dynamical system defined in extended compact space was provided. This system is suitable for obtaining global piecewise approximations for the late-time attractor solution due to the large range of convergence for center manifold expansions and the associated approximants as compared with the slow-roll approximation and associated slow-roll approximants [77].

Several gravitational theories consider multiple scalar fields with exponential potential, e.g. assisted inflation scenarios [78–83], the quintom DE paradigm [39, 40, 42, 43, 45] and others. The potentials have been considered as positive and negative exponentials [84], single and double exponentials [85–106], etc. Multiple scalar fields are discussed in [107–116, 118].

Some very interesting cases are the non-minimally coupled scalar fields that appear in the context of string theory [119] or in the context of scalar–tensor theories (STT) [120–129]. Coupled quintessence models were investigated by means of phase space studies in, for example, [94–96, 98, 100, 102, 130–132]. A specific non-minimally coupled subclass of Horndeski STT arising from the decoupling limit of massive gravity by covariantization was studied in [133, 134]. In [135], a flat FLRW scalar field with potential of types $V(\phi) = \phi^n$ and $V(\phi) = \phi^n + \phi^m$ was investigated, conformally coupled to the Ricci scalar, $R$, through the function $-\xi B(\phi) R$, where $\xi$ is the coupling constant and $B(\phi) = \phi^N$. The authors worked in the Jordan frame in the absence of matter. There, a global picture of the phase space by means of compact variables was presented. Some exact solutions for some choices of the slopes of the potential and the coupling function were discussed. In [136], they investigated a scalar field non-minimally coupled to the Ricci scalar evolving in Higgs-like (quadratic) potentials plus a negative cosmological constant. The double exponential potential and exponential coupling function were discussed in [91] as well as in [92, 93, 95, 98], under the ansatz $\phi = \lambda H$.

In [137], a detailed dynamical analysis of the Kantowski–Sachs and locally rotationally symmetric (LRS) Bianchi I and LRS Bianchi III models was performed by means of the method of f-devisers presented in [138] and applied in [139] to scalar field cosmologies in the framework of a generalized Chaplygin gas. This method, which allows us to perform the whole analysis for a wide range of potentials, is a modification of the method first introduced in [140]. The original method was used for investigating flat FLRW scalar field cosmologies.
and was generalized to several cosmological contexts in [145–148]. A drawback of the method of $f$-devisers is that it cannot be applied to some specific inflationary potentials such as the logarithmic $V(\phi) \propto \phi^n \ln^n(\phi)$ [149] and the generalized exponential one $V(\phi) \propto \phi^n \exp(-\lambda \phi^m)$ [150] since the resulting $f$-functions are not single valued, therefore one should apply asymptotic techniques in order to extract the dominant branch at large $\phi$-values as in [149, 150].

The idea of obtaining general results for scalar field cosmologies by only providing general features of the potentials and coupling functions is not new. Preceding works for a large variety of non-negative potentials are [151–158]. In [158], many of the results obtained in [156] have been extended by considering arbitrary potentials. In [155], was shown that for a large class of flat FLRW cosmologies with scalar fields with arbitrary potential, the past attractor corresponds to exactly integrable cosmologies with a massless scalar field. A list of integrable models with a minimally coupled scalar field, including double exponential potentials, in the absence of matter, is presented in [159]. Integrable non-minimally coupled scalar field models in the Jordan frame were investigated in [160]. In [161], flat FLRW cosmologies based on STTs were investigated. The new asymptotic expansions for the cosmological solutions near the initial space–time singularity obtained there contain as particular cases those studied in [155]. The proof of the local initial singularity theorem presented in [161] has been improved in [162]. Additionally, in [162] several results corresponding the late-time dynamics for the case of a scalar field non-minimally coupled to dark matter (DM) were presented. In this paper we extent these results by adding a radiation fluid. This leads to a more realistic cosmological model in the framework of the so-called complete cosmological dynamics [163], that is, a viable cosmological model should describe a radiation dominated era (RDE) before entering a matter dominated era (MDE), which should be succeeded by the current late-time accelerated expansion [49, 163, 164]. The transition from one era to the next can be understood, in the language of dynamical systems, in terms of the so-called heteroclinic sequences [144, 165–167].

Models arising in the conformal frame of $F(R)$ theories were considered in, e.g. [162, 168–171]. In [168] flat and negatively curved FLRW models with a perfect fluid matter source and a non-minimally coupled scalar field $\phi$ with potential $V(\phi)$ (related to the $F(R)$ function) were investigated. There they proved that for potentials that eventually become non-negative as $\phi \to \pm \infty$ and with a finite number of critical points, the non-negative local minima and the horizontal asymptotes approached from above by the potential are asymptotically stable. For a non-degenerated minimum with zero critical value and $\gamma > 1$, there is a transfer of energy from the fluid to the scalar field, which eventually dominates the expansion in a generic way. In [170], a mathematical procedure was developed for investigating the dynamics when $|\phi| \to +\infty$ in non-minimally coupled scalar fields models interacting with dark matter in the presence of radiation. The modified gravity models $F(R) = R + \alpha R^2$ (quadratic gravity) and $F(R) = R^n$ in the STT frame were studied there. For quadratic gravity, the equilibrium point corresponding to the de Sitter solution is locally asymptotically unstable (a saddle point).

The aim of this paper is to extend several results in [91, 155, 156, 161, 168, 169, 171] for the general case of arbitrary potentials and arbitrary couplings. They are described the early- and late-time dynamics of the models and we pay special attention to the possible scaling solutions. We follow the method first introduced in [155] and extended in [170] for the analysis of the limit $\phi \to +\infty$. We examine the example of a double exponential potential and our new results complement those in [91]. Additionally, we revisit the example of a power-
law coupling function and an Albrecht–Skordis potential, first introduced in [161] and then extended in section 4.4 of [170].

2. Basic framework

The action for a general class of STT, written in the so-called Einstein frame (EF), is given by [172]:

\[ \int d^4x \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} V_\mu \phi V_\nu \phi - V(\phi) + \chi(\phi)^{-2} \mathcal{L} \left( \mu, V_\mu, \chi(\phi)^{-1} g_{\alpha\beta} \right) \right\}. \tag{1} \]

We use a system of units in which \( \hbar = c = \pi G = 1 \). In this equation \( R \) is the curvature scalar, \( \phi \) is the scalar field, \( V_\mu \) is the covariant derivative, \( V(\phi) \) is the quintessence self-interaction potential, \( \chi(\phi)^{-2} \) is the coupling function, \( \mathcal{L} \) is the matter Lagrangian and \( \mu \) is a collective name for the matter degrees of freedom.

The matter energy–momentum tensor is defined by

\[ T_{\alpha\beta} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\alpha\beta}} \left\{ \sqrt{|g|} \chi^{-2} \mathcal{L} \left( \mu, V_\mu, \chi^{-1} g_{\alpha\beta} \right) \right\}. \tag{2} \]

Let us define

\[ Q_\beta \equiv V^a T_{a\beta} = -\frac{1}{2} \frac{1}{\chi(\phi)} \frac{d \chi(\phi)}{d \phi} V_\beta \phi, \quad T = T^a_a. \]

Since there is an exchange of energy between the scalar and the background fluids, the energy is not separately conserved for each component. Instead, the continuity equation for each fluid reads [185]:

\[ \rho_n + 3H (\rho_n + p_n) = Q, \tag{3} \]
\[ \rho_{DE} + 3H (\rho_{DE} + p_{DE}) = -Q, \tag{4} \]

where the dot accounts for the derivative with respect to the cosmic time and \( Q \) is the interaction term. Now, defining the total energy density and the total pressure as \( \rho_T = \rho_n + \rho_{DE} \) and \( p_T = p_n + p_{DE} \), respectively, then the total energy density is indeed conserved in the sense \( \rho_T + 3H (\rho_T + p_T) = 0 \). To specify the general form of the interaction term we can look at a STT of gravity (1) and the interaction term \( Q \) in (3) and (4), can be written in the following form:

\[ Q = -\frac{1}{2} (4 - 3\gamma) \rho_n H \left[ \frac{d \ln \chi}{d \alpha} \right], \tag{5} \]

where we have assumed that the coupling \( \chi \) can be written as a function of the scale factor through \( \chi(\phi(\alpha)) \). Comparing this with other interaction terms from the references, one can obtain the functional form of the coupling function \( \chi \) for each case. In the [186], for instance, \( Q = 3H c^2 (\rho_{DE} + p_n) = c^2 H \theta_n (r + 1)/r \), where \( c^2 \) denotes the transfer strength and \( r \equiv \Omega_m/\Omega_{DE} \). If one compares this expression with (5) one obtains the following coupling function:

\[ \chi = \text{(some function of } \rho_n, p_n, \theta_n). \]

\(^3\) For a discussion of the regularity of the conformal transformation, or the equivalence issue of the two frames, see for example [173–184] and references therein.
\( \chi(a) = \chi_0 \exp \left[ -\frac{6}{4 - 3\gamma} \int \frac{da}{a} \left( \frac{r + 1}{r} \right) c^2 \right]. \) (6)

where \( \chi_0 \) is an arbitrary integration constant. If \( c^2 = c_0^2 = \text{const.} \) and \( r = r_0 = \text{const.} \), then \( \chi = \chi_0 a^{-\frac{\sqrt{2} \gamma}{1 - \sqrt{2} \gamma}}. \) It is well known that a suitable coupling can produce scaling solutions, although the way to fix the coupling is not unequivocally determined. In reference [130, 187], for instance, the coupling is introduced by manually. In [188–190], the form of the interaction term is fixed by the requirement that the ratio of the energy densities of DM and quintessence has a stable fixed point during the evolution which solves the coincidence; in [188] a suitable interaction between the quintessence field and DM leads to a transition from the domination matter era to an accelerated expansion epoch for the model proposed in [188–190]. In [191], the coupling function is chosen as a Fourier expansion around some minimum of the (dilaton) scalar field.

It is well known that the HOG theories [50–58] derived from the action

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} F(R) + \mathcal{L}(\mu, V_\mu, g_{\alpha\beta}) \right\}, \] (7)

and the STT with action

\[ \tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} - (V(\phi))^2 - V(\phi) + e^{-\sqrt{2\gamma} \phi} \mathcal{L}(\mu, V_\mu, e^{-\sqrt{2\gamma} \phi} g_{\alpha\beta}) \right\}, \] (8)

are conformally equivalent under the transformation,

\[ \tilde{g}_{\mu\nu} = F'(R) g_{\mu\nu}, \] (9a)

\[ \phi = \sqrt{\frac{3}{2}} \ln F'(R), \] (9b)

\[ V(R(\phi)) = \frac{1}{2(F'(R))^2} (RF'(R) - F(R)), \] (9c)

where it is assumed that (9b) can be solved for \( R \) to obtain a function \( R(\phi) \), in order to obtain the potential (9c) as an explicit function of \( \phi \). It is easy to note that the model arising from the action (8) can be obtained from (1) with the choice \( \chi(\phi) = e^{\sqrt{2\gamma} \phi} \). Thus, the results in [161] and in [168] can be obtained as particular cases by investigating a general class of models containing both STTs and \( F(R) \) gravity.

Using the above approach, we obtain that the quadratic gravity model \( F(R) = R + \alpha R^2 \) is conformally equivalent to a non-minimally coupled scalar field with the potential \( V(\phi) = \frac{1}{8\alpha} \left( 1 - e^{-\sqrt{2\gamma} \phi} \right)^2 \). \( \alpha > 0 \). This potential has only one local minimum at \( \phi = 0 \), and the asymptote \( V_\phi = \frac{1}{8\alpha} \), which is approached from below by the potential as \( \phi \to +\infty \). The zero minimum at \( \phi = 0 \) of this potential is stable, but it cannot provide the mechanism for the late-time acceleration since \( V \) and \( H \) asymptotically approach zero [156, 157, 192]. On the other hand, concerning the upper asymptote of \( V \) as \( \phi \to +\infty \), it follows that the potential has an exponential order of zero (see definition 2) as \( \phi \to +\infty \). Thus, using proposition 6 of [170], it follows that the de Sitter configuration \( \phi \to +\infty \), \( V(\phi) = V_\phi, H(\phi) = \sqrt{\frac{V_\phi}{3}} \) is unstable to perturbations along the \( \phi \)-axis in the neighborhood of ‘infinity’, and thus it cannot represent the late-time solution.

In the reference [192], conditions on the function \( F(R) \) were imposed with corresponding potential (9c) using the restrictions on \( V(\phi) \) obtained in [151–154]. There, the connection
between $F(R)$-gravity and non-minimally coupled scalar fields was exploited from the mathematical viewpoint and was proved with mathematically rigorous results. In [193], a generic class of $F(R)$ models for the Kantowski–Sachs metric using dynamical systems tools was investigated.

2.1. The field equations

In this section we investigate the action (1) for the flat FLRW metric:

$$\text{d}s^2 = -\text{d}r^2 + a(t)^2 \left( \text{d}r^2 + r^2 \left( \text{d}\theta^2 + \sin^2\theta \text{d}\varphi^2 \right) \right),$$

where $a(t)$ is the scale factor. The Hubble expansion scalar is $H = \dot{a}/a$, where the dot means the derivative with respect to time. We assume that the energy–momentum tensor (2) is $T^\mu_\nu = \text{diag}(-\rho_m, p_m, p_m, p_m)$, where $\rho_m$ and $p_m$ are, respectively, the isotropic energy density and the isotropic pressure of barotropic matter with the equation of state $p_m = (\gamma - 1)\rho_m$. We include a radiation source with energy density $\rho_r$ since we want to investigate the possible scaling solutions in the radiation regime. We neglect ordinary (uncoupled) barotropic matter.

The cosmological equations with the above ‘ingredients’ are

$$\dot{H} = -\frac{1}{2} \left( \gamma \rho_m + \frac{4}{3} \rho_r + \phi^2 \right),$$

$$\dot{\rho}_m = -3H\rho_m - \frac{1}{2} (4 - 3\gamma)\rho_m \frac{d\ln\chi(\phi)}{d\phi},$$

$$\dot{\rho}_r = -4H\rho_r,$$

$$\dot{\phi} = -3H\phi - \frac{dV(\phi)}{d\phi} + \frac{1}{2} (4 - 3\gamma)\rho_m \frac{d\ln\chi(\phi)}{d\phi},$$

$$3H^2 = \frac{1}{2} \phi^2 + V(\phi) + \rho_m + \rho_r.$$

We assume the general hypothesis $V(\phi) \in C^3$, $V(\phi) > 0$, $\chi(\phi) \in C^3$ and $\chi(\phi) > 0$ and thus the dynamical system (11) is of class $C^2$. In order to derive our results we will consider further assumptions which will be clearly stated when necessary. We assume $\rho_m, \rho_r \geq 0$ and $0 < \gamma < 2$, $\gamma \neq \frac{4}{3}$. The latter assumption excludes the possibility that the background matter behaves as radiation, which in rigor is automatically decoupled from the scalar field (since the energy–momentum tensor for radiation is traceless).

3. Dynamical systems analysis

In the following, we study the late-time behavior of solutions of (11), which are expanding at some initial time, i.e., $H(0) > 0$. The state vector of the system is $\{\phi, \dot{\phi}, \rho_m, \rho_r, H\}$. Defining $y := \dot{\phi}$, we rewrite the autonomous system as

$$\dot{H} = -\frac{1}{2} \left( \gamma \rho_m + \frac{4}{3} \rho_r + y^2 \right),$$

$$\dot{\rho}_m = -3\gamma H\rho_m - \frac{1}{2} (4 - 3\gamma)\rho_m \frac{d\ln\chi(\phi)}{d\phi}. $$
\[ \dot{\rho}_m = -4H\rho_m, \quad (12c) \]
\[ \dot{\rho}_r = -3Hy - \frac{dV(\phi)}{d\phi} + \frac{1}{2}(4-3\gamma)\rho_m \frac{d\ln\chi(\phi)}{d\phi}, \quad (12d) \]
\[ \phi = y, \quad (12e) \]
subject to the constraint
\[ 3H^2 = \frac{1}{2}y^2 + V(\phi) + \rho_m + \rho_r. \quad (13) \]

**Remark 1.** Using the standard arguments of ordinary differential equation (ODE) theory, it follows from equations (12b) and (12c) that the signs of \( \rho_m \) and \( \rho_r \), respectively, are invariant. This means that if \( \rho_m > 0 \) and \( \rho_r > 0 \) for some initial time \( t_0 \), then \( \rho_m(t) > 0 \) and \( \rho_r(t) > 0 \) throughout the solution. From (12a) and (13) and only if additional conditions are assumed, for example \( V(\phi) \geq 0 \) and \( V(\phi_{*}) = 0 \) for some \( \phi_{*} \), it follows that the sign of \( H \) is invariant.

From (12c) and (12a) it follows that \( \rho_r \) and \( H \) decrease. Also, defining \( \epsilon = \frac{1}{2}y^2 + V(\phi) \), it follows from (12b), (12d) and (12c) that
\[ \dot{\epsilon} + \rho_m + \dot{\rho}_r = -3H\left(y^2 + \gamma\rho_m + \frac{4}{3}\beta\right). \quad (14) \]
Thus, the total energy density contained in the dark sector is decreasing.

The system (12) defines a dynamical system in the phase space
\[ \Omega = \left\{ (H, \rho_m, \rho_r, y, \phi) \in \mathbb{R}^5 \mid 3H^2 = \frac{1}{2}y^2 + V(\phi) + \rho_m + \rho_r \right\}. \quad (15) \]

Let us assume in the first place that the potential function has a local minimum \( V(0) = 0 \). This implies that the point \((0, 0, 0, 0, 0)\) is a singular point of (12) which implies that an initially expanding universe \( (H > 0) \) should expand forever. Indeed, the set \( \left\{ (H, \rho_m, \rho_r, y, \phi) \in \Omega \mid H = 0 \right\} \) is invariant under the flow of (12). In addition, the sign of \( H \) is invariant. Otherwise, if the sign of \( H \) changes, a trajectory with \( H(0) > 0 \) can pass through \((0, 0, 0, 0, 0)\), violating the existence an uniqueness theorem for ODEs.

Proposition 2 of [156] can be generalized in this context as follows.

**Proposition 1.** Suppose that \( V \geq 0 \) and \( V(\phi) = 0 \iff \phi = 0 \). Let \( A \) be such that \( V \) bounded in \( A \) implies \( V(\phi) \) is bounded in \( A \). If there exists a constant \( K, K \neq 0 \) such that
\[ \chi'(\phi)/\chi(\phi) \leq 2K/(2-\gamma)(4-3\gamma). \]
Then,
\[ \lim_{t \to +\infty} (\rho_m, \rho_r, y) = (0, 0, 0). \]

**Proof.** Consider the trajectory passing through an arbitrary point \((H, \rho_m, \rho_r, y, \phi) \in \Omega \) with \( H > 0 \) at \( t = t_0 \). Since \( H \) is positive and decreasing we have that \( \lim_{t \to +\infty} H(t) \) exists and it is a non-negative number \( \eta \); in addition, \( H(t) \leq H(t_0) \) for all \( t \geq t_0 \). Then, from (15) it follows that each term \( \rho_m, \rho_r, 1/2y^2 \) and \( V(\phi) \) is bounded by \( 3H(t_0)^2 \) for all \( t \geq t_0 \).
Let us define \( A = \{ \phi: V(\phi) \leq 3 H(t_0)^2 \} \). Then, the trajectory is such that \( \phi \) remains in the interior of \( A \) and additionally \( V'(\phi) \) is bounded for \( \phi \in A \).

From equation (12a) it follows that

\[
- \int_{t_0}^{t} \left( \frac{1}{2} \gamma^2 + \frac{2}{3} \rho_n + \frac{4}{3} \beta \right) dt = H(t) - H(t_0).
\]

Taking the limit \( t \to +\infty \), we obtain

\[
\frac{1}{2} \int_{t_0}^{+\infty} \left( \gamma^2 + \gamma \rho_n + \frac{4}{3} \beta \right) dt = H(t_0) - \eta \Rightarrow \int_{t_0}^{+\infty} \left( \gamma^2 + \gamma \rho_n + \frac{4}{3} \beta \right) dt < +\infty. \tag{16}
\]

Taking the time derivative of \( f(t) = \gamma^2 + \gamma \rho_n + \frac{4}{3} \beta \) and making use of the hypothesis for \( \chi(\phi) \) we obtain

\[
\frac{d}{dt} \left( \gamma^2 + \gamma \rho_n + \frac{4}{3} \beta \right) \leq -2 V'(\phi) + K \rho_n - \frac{16}{3} \beta H.
\]

As we have seen, \( \gamma, \rho_n, \rho \), and \( H \) are bounded for \( t \geq t_0 \), and by the hypothesis for \( V(\phi) \), \( V'(\phi) \) is bounded. From these facts it follows that the time derivative of \( f \) is bounded. Since \( f \) is a non-negative function, the convergence of \( \int_{t_0}^{+\infty} f(t) dt \) implies \( \lim_{t \to +\infty} f(t) = 0 \). Hence, we have that

\[
\lim_{t \to +\infty} (\rho_n, \rho, \gamma) = (0, 0, 0).
\]

The hypotheses in 1 concerning the scalar field self-interacting potential are not very restrictive [156]. The hypothesis for \( \chi(\phi) \) is satisfied by a large class of coupling functions as well, including the exponential ones.

Under the same hypothesis of proposition 1, we can generalize proposition 3 in [156].

**Proposition 2.** Suppose that \( V'(\phi) > 0 \) for \( \phi > 0 \) and \( V'(\phi) < 0 \) for \( \phi < 0 \). Then, under the same hypotheses as in proposition 1, \( \lim_{t \to +\infty} \phi(t) \) exists and is equal to \(+\infty, 0\) or \(-\infty\).

**Proof.** Using the same argument as in proposition 1, \( \exists \lim_{t \to +\infty} H(t) = \eta \). If \( \eta = 0 \), then by the restriction (15) we obtain \( \lim_{t \to +\infty} V(\phi(t)) = 0 \). Since \( V \) is continuous and \( V(\phi) = 0 \iff \phi = 0 \), this implies that \( \lim_{t \to +\infty} \phi(t) = 0 \).

Suppose that \( \eta > 0 \). From (15) we obtain that \( \lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \). Therefore, there exists \( t' \) such that \( V(\phi) > 3\eta^2/2 \) for all \( t > t' \). From this fact it follows that \( \phi \) cannot be zero for some \( t > t' \) because \( \phi = 0 \iff V(\phi) = 0 \). Then, the sign of \( \phi \) is invariant for all \( t > t' \).

Suppose that \( \phi \) is positive for all \( t > t' \). Since \( V \) is an increasing function of \( \phi \) in \((0, +\infty)\), we have that \( \lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \leq \lim_{t \to +\infty} V(\phi) \). By the continuity and monotony of \( V \) it is obvious that the equality holds if and only if, \( \lim_{t \to +\infty} \phi(t) = +\infty \).

If \( \lim_{t \to +\infty} V(\phi(t)) < \lim_{t \to +\infty} V(\phi) \), then there exists \( \phi \geq 0 \) such that

\[
\lim_{t \to +\infty} V(\phi(t)) = V(\phi).
\]

Since \( V \) is continuous and strictly increasing we have that

\[
\lim_{t \to +\infty} \phi = \phi.
\]
By proposition 1, \( \lim_{t \to +\infty} (\rho_m(t), \rho_r(t), y(t)) = (0, 0, 0) \). In addition, \( H \) and \( \chi'(\phi)/\chi(\phi) \) are bounded. Therefore, taking the limit as \( t \to +\infty \) in (12) we find that
\[
\lim_{t \to +\infty} \frac{d}{dt} y = -V'(\phi) < 0.
\]
Hence, there exist \( t^* > t' \) such that \( \frac{d}{dt} y < -V'(\phi)/2 \) for all \( t \geq t^* \). This implies
\[
y(t) - y(t') = \int_{t'}^{t} \left( \frac{d}{dt'} y \right) dt < -\frac{V'(\phi)}{2}(t - t'),
\]
that is, \( y(t) \) takes negative values with an arbitrary large modulus as \( t \) increases, which is not possible since \( \lim_{t \to +\infty} (\phi(t)) = 0 \).

Hence, there exist \( t^* > t' \) such that \( \lim_{t \to +\infty} \phi(t) < -\phi(t') \). This implies \( \lim_{t \to +\infty} \phi(t) = -\infty \). Similarly, when \( \phi(t) > 0 \), we have \( \lim_{t \to +\infty} \phi(t) = +\infty \).

From this we conclude that, if initially \( \lim_{t \to +\infty} y(t) = 0 \), then \( \lim_{t \to +\infty} H(t) = 0 \). Indeed, we have that \( \lim_{t \to +\infty} \phi(t) \) is equal to \( +\infty, 0 \) or \( -\infty \). If \( \lim_{t \to +\infty} \phi(t) = +\infty \), then from the restriction (15) it follows that
\[
3\eta^2 = \lim_{t \to +\infty} V'(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) > 3 H(t_0)^2.
\]
This is impossible since \( H(t) \) is a decreasing function and \( H(t_0) \geq \eta \). In the same way, \( \lim_{t \to +\infty} \phi(t) = -\infty \) leads to a contradiction. Then, \( \lim_{t \to +\infty} \phi(t) = 0 \) and this implies \( \lim_{t \to +\infty} V'(\phi(t)) = 0 \), and again by (15), \( \lim_{t \to +\infty} H(t) = 0 \).

Thus, we have proved that if the potential has a local minimum at zero, if the derivative of the potential is bounded in the same set where the potential itself is, and provided that the derivative of the logarithm of the coupling function is bounded by above, then the energy densities of the DM and radiation and the kinetic energy density of the DE tend to zero as the time goes forward. Hence, the Universe would expand forever in a de Sitter phase. Also we have proved, in a similar way to proposition 3 in [156], that under the additional assumption of \( V'(\phi) \) being strictly decreasing (increasing) if \( \phi < 0 \) (\( \phi > 0 \)), the scalar field can be either zero or divergent into the future (the former case holds if the Hubble scalar vanishes asymptotically).

In order to complement the previous ideas, let us consider a non-negative potential without necessarily a local minimum at \( (0, 0) \), let us find conditions for the stability of de Sitter solutions and let us characterize the asymptotic properties of the scalar field at late times.

**Proposition 3.** Suppose that there exists a non-zero constant \( K \), such that \( \chi'(\phi)/\chi(\phi) \leq 2K/(2-\gamma)(4-3\gamma) \). Let \( V \) be a potential function with the properties:

(i) \( V \geq 0 \) and \( \lim_{\phi \to +\infty} V'(\phi) = +\infty \).

(ii) \( V' \) is continuous and \( V'(\phi) < 0 \).

(iii) If \( A \subset \mathbb{R} \) is such that \( V \) is bounded in \( A \), then, \( V'(\phi) \) is bounded in \( A \).

Then, \( \lim_{t \to +\infty} (\rho_m, \rho_r, y) = (0, 0, 0) \) and \( \lim_{t \to +\infty} \phi = +\infty \).

**Proof.** From (12b) and (12c), it follows that the sets \( \rho_m > 0 \) and \( \rho_r > 0 \) are invariant under the flow of (12) with restriction (15); in addition, \( \rho_m \) and \( \rho_r \) are different from zero if \( \rho_m(t_0) \) and \( \rho_r(t_0) \) are at the initial time. From this fact we have that \( H \) is never zero (and thus does not
have changes of sign) since by \((15)\), \(\rho_3(t) \geq 0\) for all \(t > t_0\), then, \(H\) is always non-negative if it is initially so. In addition, from \((12a)\) it follows that \(H\) is decreasing, then 

\[
\lim_{t \to +\infty} H(t) = \eta \geq 0
\]

and

\[
\frac{1}{2} \int_{t_0}^{+\infty} \left( y^2 + \gamma_\rho \rho + \frac{4}{3} \rho \right) dt = H(t_0) - \eta < +\infty.
\]

As in proposition 1, the total time derivative of \(y^2 + \gamma_\rho \rho + \frac{4}{3} \rho\) is bounded. It can be proved that \(\lim_{t \to +\infty} \rho_3 = +\infty\) in the same way as proved in 2.

From \((15)\) we have that \(\lim_{t \to +\infty} V(\phi) = 3\eta^2\). Since \(V\) is strictly decreasing with respect to \(\phi\); then \(V(\phi) > \lim_{t \to +\infty} V(\phi)\) for all \(\phi\), therefore \(\lim_{t \to +\infty} V(\phi(t)) \geq \lim_{t \to +\infty} V(\phi)\). We will consider two cases:

(i) If \(\lim_{t \to +\infty} V(\phi(t)) = \lim_{t \to +\infty} V(\phi)\), by the continuity of \(V\) it is obvious that \(\lim_{t \to +\infty} V = +\infty\).

(ii) If \(\lim_{t \to +\infty} V(\phi(t)) > \lim_{t \to +\infty} V(\phi)\), then there exists a unique \(\phi\) such that 

\[
\lim_{t \to +\infty} V(\phi(t)) = V(\phi).
\]

Since \(V\) is continuous and strictly decreasing, it follows that

\[
\lim_{t \to +\infty} \phi = \bar{\phi}.
\]

From \((12d)\) it follows that

\[
\lim_{t \to +\infty} \frac{d}{dt} y = -V'(\bar{\phi}) > 0,
\]

therefore, there exist \(t'\) such that \(\frac{d}{dt} y > -V'(\bar{\phi})/2\) for all \(t \geq t'\). From this fact we conclude that

\[
y(t) - y(t') > -\frac{V'(\bar{\phi})}{2}(t - t'),
\]

which is impossible since \(\lim_{t \to +\infty} y(t) = 0\). Finally \(\lim_{t \to +\infty} \phi = +\infty\). \(\square\)

If, additionally, the potential is such that \(\lim_{\phi \to +\infty} V(\phi) = 0\), then we conclude that \(H \to 0\) as \(t \to +\infty\).

The previous results are extensions of remark 1 and propositions 4, 5 and 6 discussed in [162] when the radiation is included in the cosmic budget.

For completeness let us show our proposition 3 in [170], which is an extension of proposition 1 of [168] for flat FLRW models since we have included radiation. This proposition gives a characterization of the future attractor of the system \((12)\) under some mild assumptions for the potential.

First, let us formalize the notion of the degenerate local minimum introduced in [168].

**Definition 1.** The function \(V(\phi)\) is said to have a degenerate local minimum at \(\phi_*\) if

\[
V'(\phi_*) = V''(\phi_*) = \ldots = V^{(2n-1)}(\phi_*) = 0,
\]

vanish at \(\phi_*\), and \(V^{(2n)}(\phi_*) > 0\), for some integer \(n\).

Then, we have the following proposition.
Proposition 4 \hspace{1mm} (Proposition 3 in [170]). Suppose that $V(\phi) \in C^2(\mathbb{R})$ satisfies the following conditions\footnote{The empty set is bounded and finite, and it is not excluded from the hypotheses (i) and (ii).}.

(i) The set $\{ \phi: V(\phi) < 0 \}$ is bounded.

(ii) The set of singular points of $V(\phi)$ is finite.

Let $\phi_\ast$ be a strict local minimum (possibly degenerate) for $V(\phi)$, with $V(\phi_\ast) \geq 0$. Then $p_\ast := \left( \phi_\ast, y_\ast = 0, \rho_{m\ast} = 0, \beta_\ast = 0, H = \sqrt{\frac{V(\phi_\ast)}{3}} \right)$ is an asymptotically stable singular point for the flow of (12).

Proof. The demonstration of the proof proceeds in an analogous way to the proof of proposition 1 in [168]. The main difference is that we have considered radiation but in a flat FLRW geometry ($k = 0$). In this case the function

$$W(\phi, y, \rho_m, H) \equiv H^2 - \frac{1}{3} \left( \frac{1}{2} y^2 + V(\phi) + \rho_m \right) = \frac{1}{3} \rho_\ast^2$$

(17)

evolves as

$$W = -4 HW$$

(18)

which decays faster to zero than the function $W(\phi, y, \rho_m, H) = -ka^{-2}$, $k = -1$, 0 defined in [168] as $a \to +\infty$. (The complete proof is offered in [170].) \hfill \square

4. Dynamical analysis for $\phi \to +\infty$

In this section we will investigate the flow as $\phi \to +\infty$ following the nomenclature and formalism introduced in [155] (see also [58] and [170]). Analogous results hold as $\phi \to -\infty$.

Definition 2 \hspace{1mm} (A function well behaved at infinity [155]). Let $V: \mathbb{R} \to \mathbb{R}$ be a $C^2$ non-negative function. Let there exist some $\phi_0 > 0$ for which $V(\phi) > 0$ for all $\phi > \phi_0$ and some number $N$ such that the function $W_V: [\phi_0, +\infty) \to \mathbb{R}$,

$$W_V(\phi) = \frac{V'(\phi)}{V(\phi)} - N$$

satisfies

$$\lim_{\phi \to +\infty} W_V(\phi) = 0.$$  \hspace{1mm} (19)

Then we say that $V$ is well behaved at infinity (WBI) of exponential order $N$.

Theorem 1 \hspace{1mm} (Theorem 2 [155]). Let $V$ be a WBI function of exponential order $N$, then, for all $\lambda > N$,

$$\lim_{\phi \to +\infty} e^{-\lambda \phi} V(\phi) = 0.$$
Definition 3. Let there be some coordinate transformation \( \varphi = f(\phi) \) mapping a neighborhood of infinity to a neighborhood of the origin. If \( g \) is a function of \( \phi \), \( \varphi \) is the function of \( \varphi \) whose domain is the range of \( f \) plus the origin, which takes the values:

\[
\varphi(\varphi) = \begin{cases} 
g\left(f^{-1}(\varphi)\right), & \varphi > 0 \\
\lim_{\phi \to +\infty} g(\phi), & \varphi = 0 \end{cases}
\]

Definition 4 (Class \( k \) WBI functions [155]). A \( C^k \) function \( V \) is class \( k \) WBI if it is WBI and if there exist \( \phi_0 > 0 \) and a coordinate transformation \( \varphi = f(\phi) \) which maps the interval \([\phi_0, +\infty)\) onto \((0, \epsilon)\), where \( \epsilon = f(\phi_0) \) and \( \lim_{\phi \to +\infty} f = 0 \), with the following additional properties.

(i) \( f \) is \( C^{k+1} \) and strictly decreasing.

(ii) The functions \( \frac{dV}{d\varphi} \) and \( \frac{d}{d\varphi} \) are \( C^k \) on the closed interval \([0, \epsilon]\).

(iii) \( \frac{d}{d\varphi} \left(\frac{dV}{d\varphi}\right)(0) = \frac{d}{d\varphi} \left(\frac{dV}{d\varphi}\right)(0) = 0 \).

We designate the set of all class \( k \) WBI functions \( \mathcal{E}_k^k \).

By assuming that \( V, \chi \in \mathcal{E}_k^k \), with exponential orders \( N \) and \( M \) respectively, we can define a dynamical system well suited to investigate the dynamics near the initial singularity. We will investigate the singular points therein, particularly those representing scaling solutions and those associated with the initial singularity [170].

Let us define the new Hubble-normalized dimensionless variables [170]:

\[
\sigma_1 = \phi, \quad \sigma_2 = \frac{\phi}{\sqrt[6]{H}}, \quad \sigma_3 = \frac{\sqrt{\mu_0}}{\sqrt[3]{3H}}, \quad \sigma_4 = \frac{\sqrt{V}}{\sqrt[3]{3H}}, \quad \sigma_5 = \frac{\sqrt{\mu}}{\sqrt[3]{3H}}
\]

and the time coordinate

\[
d\tau = 3H dt.
\]

Using these coordinates, the equations (12) can be recast as an autonomous system satisfying an inequality arising from the Friedmann equation (11e). This system is given by [170]:

\[
\begin{align*}
\sigma_1' &= \sqrt{\frac{2}{3}} \sigma_2 \\
\sigma_2' &= \sigma_2^3 + \frac{1}{6} \left(3\gamma \sigma_2^2 + 4\sigma_2^2 - 6\right) \sigma_2 - \frac{\sigma_2^2}{2\sqrt{6}} \frac{d\ln V}{d\sigma_1} + \frac{(4 - 3\gamma)\sigma_2 \sigma_3 d\ln \chi}{2\sqrt{6} \sigma_1}, \\
\sigma_3' &= \frac{1}{6} \sigma_3 \left(6\sigma_2^2 + 3\gamma \left(\sigma_2^2 - 1\right) + 4\sigma_2^2\right) - \frac{(4 - 3\gamma)\sigma_2 \sigma_3 d\ln \chi}{2\sqrt{6} \sigma_1}, \\
\sigma_4' &= \frac{1}{6} \sigma_4 \left(6\sigma_2^2 + 3\gamma \sigma_2^2 + 4\sigma_2^2\right) + \sqrt{\frac{6}{6}} \sigma_2 \sigma_4 \frac{d\ln V}{d\sigma_1}, \\
\sigma_5' &= \frac{1}{6} \sigma_5 \left(6\sigma_2^2 + 3\gamma \sigma_2^2 + 4\sigma_2^2 - 4\right).
\end{align*}
\]
The system (22) defines a flow in the phase space
\[ \Sigma := \left\{ \sigma \in \mathbb{R}^5; \sum_{j=2}^{5} \sigma_j^2 = 1, \sigma_j \geq 0, j = 3, 4, 5 \right\}. \]

(23)

Let \( \Sigma_\varepsilon = \{ (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in \Sigma; \sigma_1 > \varepsilon^{-1} \} \) where \( \varepsilon \) is any positive constant which is chosen sufficiently small so as to avoid any points where \( V \) or \( \chi = 0 \), thereby ensuring that \( W_V(\varphi) \) and \( W_\chi(\varphi) \) are well defined\(^5\).

Let the projection map be defined as
\[ \pi_1; \Sigma_\varepsilon \rightarrow \Omega_\varepsilon \]
\[ (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \rightarrow (\sigma_1, \sigma_2, \sigma_4, \sigma_5) \]

(24)

where
\[ \Omega_\varepsilon := \left\{ \sigma \in \mathbb{R}^4; \sigma_1 > \varepsilon^{-1}, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_j \geq 0, j = 4, 5 \right\}. \]

(25)

In \( \Omega_\varepsilon \) let the coordinate transformation be defined as \( (\sigma_1, \sigma_2, \sigma_4, \sigma_5) \xrightarrow{\varphi'(\sigma)} (\sigma, \sigma_2, \sigma_4, \sigma_5) \) where \( f(\sigma) \) tends to zero as \( \sigma_1 \) tends to \( +\infty \) and has been chosen so that the conditions (i)–(iii) of definition 4 are satisfied with \( k = 2 \).

The flow of (22) defined on \( \Sigma \) is topologically equivalent (under \( f \circ \pi_1 \)) to the flow of the four-dimensional dynamical system [170]:

\[ \varphi' = \sqrt{\frac{2}{3}} \gamma \sigma_2, \]

(26a)

\[ \sigma'_i = \sigma_i + \left( \frac{2\sigma_i^2}{3} - 1 \right) \sigma_2 - \frac{(W_V + N)\sigma_i^2}{\sqrt{6}} + \left( \frac{(W_M + M)(4 - 3\gamma)}{2\sqrt{6}} + \frac{\sigma_2^2}{2} \right)(1 - \sigma_2^2 - \sigma_4^2 - \sigma_5^2) \]

(26b)

\[ \sigma'_4 = \frac{1}{6} \sigma_4 \left( \sqrt{6} (W_V + N) \sigma_3 + 3(2 - \gamma) \sigma_2^2 + 3\gamma \left( 1 - \sigma_2^2 \right) + (4 - 3\gamma) \sigma_4^2 \right), \]

(26c)

\[ \sigma'_5 = \frac{1}{6} \sigma_5 \left( 3(2 - \gamma) \sigma_2^2 - 3\gamma \sigma_4^2 - (4 - 3\gamma) \left( 1 - \sigma_4^2 \right) \right), \]

(26d)

defined in the phase space
\[ \Omega_\varepsilon = \left\{ (\varphi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4; 0 \leq \varphi \leq f(e^{-1}), \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0 \right\}. \]

(27)

It can easily be proved that (27) defines a manifold with a boundary of dimension four. Its boundary, \( d\Omega_\varepsilon \), is the union of the sets \( \{ p \in \Omega_\varepsilon; \varphi = 0 \}, \{ p \in \Omega_\varepsilon; \varphi = f(e^{-1}) \}, \{ p \in \Omega_\varepsilon; \sigma_4 = 0 \}, \{ p \in \Omega_\varepsilon; \sigma_5 = 0 \} \) with the unitary three-sphere. The coordinates, existence conditions and stability of the singular points of the flow of (26) in the phase space (27) are presented in appendix A. Finally, the physical description of the solutions and connection with observables is discussed in appendix B. The results discussed in both appendices were first published in [170].

\(^5\) See 3 for the definition of functions with a bar.

\(^6\) For notational simplicity we will denote the image of \( \Omega_\varepsilon \) under \( f \) by the same symbol.
5. Examples

In this section we discuss the example of a double exponential potential, presented in [91], applying the procedure of [155, 170]. Our new results complement those in [91]. Next we revisit the example of a power-law coupling function and an Albrecht–Skordis potential, first introduced in [161], and then extended in section 4.4 of [170].

5.1. Double exponential potential and exponential coupling function

First let us discuss the case presented in [91]. In this example the potential is a double exponential:

\[ \psi_{\alpha \beta} = \psi_1 e^{-\alpha \psi} + \psi_2 e^{-\beta \psi}, \quad 0 < \alpha < \beta \quad \text{and} \quad \chi = \chi_0 \exp \left[ \frac{2h}{4 - 3\gamma} \right], \]

where \( \lambda \) is a constant. Then, for \( \gamma < \frac{4}{3} \), choose \( K \geq \frac{(2 - \gamma)\lambda}{2} \) and for \( \frac{4}{3} < \gamma \leq 2 \) choose \( K \leq \frac{(2 - \gamma)\lambda}{2} \). Thus, in our results the hypothesis for \( \chi \) is easily fulfilled. Since \( V > V_0 \), we have \( \phi' < \psi_0 \), \( \forall \phi \). Furthermore, under the hypothesis \( 0 < \alpha < \beta \), we have the strong restriction \( -\beta V(\phi) < \psi(\phi) < -\alpha V(\phi), \forall \phi \). Thus, the hypotheses of our proposition 3 are satisfied and we find proposition 1 of [91].

Additionally, we have that under the hypothesis \( 0 < \alpha < \beta \), \( V(\phi) = \psi_1 e^{-\alpha \phi} + \psi_2 e^{-\beta \phi} \) is well behaved at \( \phi \to +\infty \) with exponential order \( N = -\alpha \). On the other hand, the coupling function is well behaved with exponential order \( M = \frac{\lambda}{4 - 3\gamma} \). Under the transformation \( \phi = \psi^{-1} \) we have that \( V, \chi \in \mathbb{C}^4 \), and

\[
\mathcal{W}_\chi(\phi) = 0, \quad (28)
\]

\[
\begin{cases}
V_2(\alpha - \beta) e^\psi, & \phi > 0 \\
V_1 e^\psi + V_2 e^\psi, & \phi = 0
\end{cases}
\]

\[
\mathcal{J}(\phi) = -\phi^2, \quad (29)
\]

In this example, the evolution equations for \( \phi, \sigma_2, \sigma_4 \) and \( \sigma_5 \) are given by the equations (26) with \( M = \frac{\lambda}{4 - 3\gamma} \), \( N = -\alpha \) and \( \mathcal{W}_\psi(\phi) \), \( \mathcal{W}_\chi(\psi) \) and \( \mathcal{J} \), given by (28), (29) and (30) respectively. The state space is defined by

\[
\Omega_\phi = \left\{ (\psi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4: 0 \leq \psi \leq \epsilon, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0 \right\}.
\]

Now, since the function \( \mathcal{W}_\psi \) defined by (29) is a transcendental function for \( \phi = 0 \), we can introduce the new variable

\[
\nu = -\frac{V_2(\alpha - \beta) e^\psi}{V_1 e^\psi + V_2 e^\psi} \geq 0
\]

\[
(31)
\]

which is better for numerical integrations. It can be proved that under the hypothesis \( 0 < \alpha < \beta, \nu \to 0^+ \) as \( \phi \to 0^+ \).

Using the coordinate transformation (31), we obtain the dynamical system

\[
\nu' = \sqrt{\frac{2}{3}} \sigma_2^2 (\alpha - \beta + \nu), \quad (32a)
\]
\[ \sigma_1' = \frac{\alpha^2}{\sqrt{6}} + \frac{1}{6} \sigma_1^2 \left( -3\varphi \left( \sigma_2^2 + \sigma_4^2 + \sigma_5^2 - 1 \right) + 6\sigma_2^2 + 4\sigma_5^2 - 6 \right) \]
\[ - \lambda \left( \sigma_2^2 + \sigma_4^2 + \sigma_5^2 - 1 \right), \quad (32b) \]
\[ \sigma_4' = -\frac{1}{6} \sigma_4 \left( \sqrt{6} \alpha \sigma_2 + 3(\gamma - 2)\sigma_2^2 + 3\varphi \left( \sigma_4^2 + \sigma_5^2 - 1 \right) - 4\sigma_5^2 \right) - \frac{\sigma_4 v}{\sqrt{6}}, \quad (32c) \]
\[ \sigma_5' = \frac{1}{6} \sigma_5 \left( -3\varphi \left( \sigma_2^2 + \sigma_4^2 + \sigma_5^2 - 1 \right) + 6\sigma_2^2 + 4\sigma_5^2 - 4 \right), \quad (32d) \]
defined in the phase space
\[
\Psi_c = \left\{ \left( v, \sigma_2, \sigma_4, \sigma_5 \right) \in \mathbb{R}^4 : 0 \leq v \leq \frac{V_2(\beta - \alpha)e^{\alpha \varphi}}{V_1 e^{2\beta \varphi} + V_2 e^{2\alpha \varphi}}, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0 \right\}.
\]
In table 1, the location, existence conditions and stability of the singular points of the system \((32)\) are summarized. The stability is analyzed for the flow restricted to the invariant set \(v = 0\), i.e. we are not taking into account perturbations along the \(v\)-axis.

Let us discuss some physical properties of the cosmological solutions associated with the singular points displayed in table 1.

- \(P_{1,2}\) represent kinetic-dominated cosmological solutions. They behave as stiff-like matter.
  The associated cosmological solution satisfies
  \[ H = \frac{1}{3t - c_1}, \quad a = \sqrt{3t - c_1}c_2, \quad \phi = c_3 \pm \frac{2}{\sqrt{3}} \ln \left( 3t - c_1 \right), \]
  where \(c_j, j = 1, 2, 3\) are integration constants. These solutions are associated with the local past attractors of the system for an open set of values of the parameters \(\alpha\) and \(\lambda\).
- \(P_3\) represents a matter–kinetic scaling solution. The associated asymptotic cosmological solutions satisfy the rates
  \[ H = \frac{8 - 4\varphi}{4(\gamma - 2)c_1 + t \left( \lambda^2 - 6(\gamma - 2)\right)^{\frac{4(\gamma - 2)}{2\beta - 6(\gamma - 2)\varphi}}}, \]
  \[ a = c_2 \left( 4(\gamma - 2)c_1 + t \left( \lambda^2 - 6(\gamma - 2)\right)^{\frac{4(\gamma - 2)}{2\beta - 6(\gamma - 2)\varphi}} \right)^{\frac{4(\gamma - 2)}{2\beta - 6(\gamma - 2)\varphi}} + c_3, \]
  \[ \beta_m = \frac{48(\gamma - 2)^2 - 8\lambda^2}{\left( t \left( 6(\gamma - 2)c_1 - 4(\gamma - 2)c_1 \right)^2 + c_3, \right.} \]
  \[ \phi = \ln \left[ \left( 4(\gamma - 2)c_1 + t \left( \lambda^2 - 6(\gamma - 2)\right)^{\frac{41}{2\beta - 6(\gamma - 2)\varphi}} \right)^{\frac{4(\gamma - 2)}{2\beta - 6(\gamma - 2)\varphi}} + c_4, \right. \]
where \(c_j, j = 1, 2, 3, 4\) are integration constants. These solutions are stable or saddle depending on the parameters \(\alpha\) and \(\gamma\).
- Point \(R_3\) represents a radiation dominated solution and asymptotically the cosmological solutions satisfy the rates
Table 1. Location of the singular points of the flow of (32) defined in the invariant set \( p \in \mathcal{I}_p : \varphi = 0 \) for \( M = \frac{1}{4 + \beta} \) and \( N = -\alpha \). We use the definitions \( \Gamma^{-}(\alpha, \gamma) = \alpha \leq \sqrt{\alpha^2 + 6\gamma^2 - 12\gamma} \) and \( Y(\gamma) = \sqrt[2]{(4 - 3\gamma)(2 - \gamma)} \).

| Label | \((\sigma_2, \sigma_3, \sigma_3)\) | Existence | Stability\(^a\) |
|-------|-------------------------------|-----------|----------------|
| \(P_1\) | \((-1, 0, 0)\) | always unstable | if \( \alpha > -\sqrt{6} \), \( \lambda > -\sqrt{6} (2 - \gamma) \) \(\lambda < \sqrt{6}(2 - \gamma)^2\) and saddle otherwise. |
| \(P_2\) | \((1, 0, 0)\) | always unstable | if \( \alpha < \sqrt{6} \), \( \lambda < \sqrt{6} (2 - \gamma) \) saddle otherwise. |
| \(P_3\) | \((\frac{\lambda}{\sqrt{6}(2 - \gamma)}, 0, 0)\) | \(\lambda^2 \leq 6(2 - \gamma)^2\) | stable for \(0 < \gamma \leq \frac{2}{3}\), and \(\sqrt{6}(2 - \gamma)^2 < \alpha \leq \frac{\sqrt{8(2 - \gamma)}}{\sqrt{4 - 3\gamma}}\), and \(\Gamma^{-}(\alpha, \gamma) < \lambda < \Gamma^{+}(\alpha, \gamma)\) or \(0 < \gamma \leq \frac{2}{3}\), and \(\alpha > \frac{\sqrt{8(2 - \gamma)}}{\sqrt{4 - 3\gamma}}\), and \(\Gamma^{-}(\alpha, \gamma) < \lambda < Y(\gamma)\) or \(\frac{2}{3} < \gamma < \frac{4}{3}\), and \(\alpha > \frac{\sqrt{8(2 - \gamma)}}{\sqrt{4 - 3\gamma}}\), and \(\Gamma^{-}(\alpha, \gamma) < \lambda < Y(\gamma)\) saddle otherwise. |
| \(R_1\) | \((0, 0, 1)\) | always unstable | saddle for \(0 < \alpha < 2\), \(\lambda > \frac{2\alpha^2 - 6\gamma}{\alpha}\) saddle otherwise. |
| \(P_4\) | \((\frac{\alpha}{\sqrt{8\gamma}}, \sqrt{1 - \frac{\alpha^2}{8\gamma}}, 0)\) | always | \(\alpha^2 \leq 6\) stable for \(0 < \alpha < 2\), saddle otherwise. |
Table 1. (Continued)

| Label | \((\sigma_2, \sigma_3, \sigma_4)\) | Existence | Stability\(^a\) |
|-------|---------------------------------|-----------|-----------------|
| \(R_{5,6}\) | \(\left(\frac{\sqrt{\beta_1}}{2a-2}, \frac{\sqrt{2\alpha^{2} - 6\gamma - 2(2\gamma + \lambda)}^{2}}{\beta_1-2a}, 0\right)\) | \(\frac{a(\lambda - 2a) + 6\gamma}{(\lambda - 2a)^3} \leq 0\) | numerical inspection |
| \(R_2\) | \(\left(\frac{\gamma}{\sqrt{4 - 3\gamma}}\right)\), 0, \(\sqrt{6\gamma^2 + 20\gamma + \lambda^2 - 16}\) | \(\gamma < \frac{4}{3}, \lambda^2 > \gamma (\gamma)^2\) | stable for |
| \(\begin{cases} \gamma < \frac{4}{3}, \\
\alpha > \frac{8(2 - \gamma)}{\sqrt{4 - 3\gamma}}, \\
\gamma (\gamma) < \lambda < \frac{1}{2} \alpha (4 - 3\gamma) \end{cases}\) | saddle otherwise |
| \(R_3\) | \(\left(\frac{\sqrt{\alpha}}{a}, \frac{2}{\sqrt{3a}}, \frac{\sqrt{\alpha^2 - a}}{a}\right)\) | \(\alpha \geq 2\) | stable if \(\alpha > 2, \lambda > \frac{1}{2} \alpha (4 - 3\gamma)\), saddle otherwise |

\(^a\) The stability is analyzed for the flow restricted to the invariant set \(v = 0\).
It is always a saddle point.

- $P_4$ represents power-law scalar field dominated inflationary cosmological solutions. As $\phi \to +\infty$ the potential behaves as $V \sim V_1 \exp \left[-a\phi^2 \right]$. Thus it is easy to obtain the asymptotic exact solution:

$$H = \frac{1}{2l - c_1}, \quad a = c_2 \sqrt{2l - c_1}, \quad \beta = c_3 = \frac{6}{c_1 - 2l}.$$ 

- $P_{5,6}$ represent matter-kinetic-potential scaling solutions. In the limit $\phi \to +\infty$ we have the asymptotic expansions:

$$H = \frac{2a^2 - \lambda}{2a1 + c_1\lambda + 3a\gamma l}, \quad a = c_2 \left(c_1(\lambda - 2a) + 3a\gamma\right) \frac{2a - \lambda}{3a\gamma},$$

$$\beta = \frac{6}{(c_1(\lambda - 2a) + 3a\gamma\gamma)^2} + c_3, \quad \phi = \ln \left[\left(c_1(\lambda - 2a) + 3a\gamma\gamma\right)^2\right] + c_4.$$ 

- $R_3$ represents a radiation-matter-scaler field scaling solution. In the limit $\phi \to +\infty$ we have the asymptotic expansions:
Figure 2. Projection in the plane $(\sigma_2, \sigma_4)$ of the flow of (32) restricted to the invariant set $\{\sigma_4 = 0\} \subset \mathcal{P}$ for the $V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi}$ and the coupling function $\chi = \chi_0 \exp\left[\frac{i \phi}{4 - \gamma}\right]$. We select the following values for the parameters: $V_1 = 1, V_2 = 2, \alpha = 1.5, \beta = 2, \lambda = -5$ and $\gamma = 1$. The stability of $P_6$ is illustrated in the figure. $P_1$ is a source and $P_2$ and $P_4$ are saddles. $P_3$ does not exist.

Figure 3. Some orbits in the invariant set $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$ for the flow of (32) restricted to the invariant set $\{v = 0\} \subset \mathcal{P}$ for the potential $V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi}$ and the coupling function $\chi = \chi_0 \exp\left[\frac{i \phi}{4 - \gamma}\right]$. We select the following values for the parameters: $V_1 = 1, V_2 = 2, \alpha = 2.4, \beta = 3, \lambda = 1.1$ and $\gamma = 1$. The stability of $P_6$ is illustrated in the figure. $P_1$ and $P_2$ are sources whereas $P_3, P_4, R_1$ and $R_3$ are saddles. $R_2$ does not exist.
\[ H = \frac{1}{2t - c_1}, \quad a = c_2 \sqrt{2t - c_1}, \quad \rho_m = \frac{16 - 12\gamma}{\lambda^2 (c_1 - 2t)^2} + c_3, \]
\[ \rho_s = c_4 - \frac{6(-6\gamma^2 + 20\gamma + \lambda^2 - 16)}{\lambda^2 (c_1 - 2t)}, \quad \phi = \ln \left( \lambda (2t - c_1) \right)^{\frac{(4 - 3\gamma)}{\lambda}} + c_5. \]

- \( R_3 \) represents radiation–kinetic-potential scaling solutions. The associated cosmological solutions satisfy the asymptotic expansions:

\[ H = \frac{1}{2t - c_1}, \quad a = c_2 \sqrt{2t - c_1}, \quad \rho_s = c_3 - \frac{6(\sigma^2 - 4)}{a^2 (c_1 - 2t)}, \]
\[ \phi = \ln \left( 2at - a\sigma c_1 \right)^2 + c_4 \quad \text{as} \quad \phi \to +\infty. \]

In order to illustrate our analytical results we proceed to some numerical experimentation.

In figures 1 and 2 projections of the flow of (32) are presented, restricted to the invariant set \( \{ \sigma_5 = 0 \} \subset \Psi \). On the planes \( (\sigma_2, \nu) \) and \( (\sigma_2, \sigma_4) \), respectively, for the potential.
\[ V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi} \] and the coupling function \( \chi = \chi_0 \exp\left[ \frac{J \phi}{3 - 3\gamma} \right] \). We select the following values for the parameters: \( V_1 = 1, V_2 = 2, \alpha = 1.5, \beta = 2, \lambda = -5 \) and \( \gamma = 1 \). These numerical simulations confirm the stability of \( P_6 \) for this choice of parameters.

In figure 3 we show some orbits in the invariant set \( \sigma \phi + \theta = \frac{1}{2} \) for the potential \( V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi} \) and the coupling function \( \chi = \chi_0 \exp\left[ \frac{J \phi}{3 - 3\gamma} \right] \). For the choice of parameters \( V_1 = 1, V_2 = 2, \alpha = 2.4, \beta = 3, \lambda = 1.1 \) and \( \gamma = 1 \), the power-law solution \( P_6 \) is the attractor. \( P_1 \) and \( P_2 \) are sources whereas \( P_3, P_4, R_1 \) and \( R_2 \) are saddles. \( R_2 \) does not exist.

5.2. Coupling functions and potentials of exponential orders \( M = 0 \) and \( N = \mu \neq 0 \), respectively

As an example let us consider \( \chi, V \in \mathcal{E}^2 \) of exponential orders \( M = 0 \) and \( N = -\mu \), respectively [170]. This class of potentials contains the cases investigated in [85, 197] (there they did not considered coupling to matter, i.e. \( \chi(\phi) \equiv 1 \), in the second case, for flat FLRW cosmologies), the case investigated in [143] (for positive potentials and standard flat FLRW dynamics), the example examined in [161], etc.

In table 2 the location, existence conditions and stability of the singular points for the flow at the invariant set \( \phi = 0 \) are summarized.

Let us discuss some physical properties of the cosmological solutions associated with the singular points displayed in table 2 [170].

- \( R_{1,2} \) represent kinetic-dominated cosmological solutions. They behave as stiff-like matter and can be local past attractors of the systems for an open set of values of the parameter \( \mu \). The same rates for \( a, H \) and \( \phi \) presented in section (5.1) also apply here.
- \( P_3 \) represents matter-dominated cosmological solutions that satisfy

\[
H = \frac{2}{3t_\gamma - 2c_1}, \quad a = (3t_\gamma - 2c_1)^{\frac{3}{2}} c_2, \quad \rho_m = \frac{12}{(3t_\gamma - 2c_1)^2} + c_3.
\]

- \( R_1 \) represents a radiation-dominated cosmological solution satisfying

\[
H = \frac{1}{2t_\gamma - c_1}, \quad a = \sqrt{2t_\gamma - c_1} c_2, \quad \rho_r = \frac{3}{(2t_\gamma - c_1)^2} + c_3.
\]

- \( P_4 \) represents power-law scalar field dominated inflationary cosmological solutions. As \( \phi \to +\infty \) the potential behaves as \( V \sim V_0 \exp\left[ -\mu \phi \right] \). Thus it is easy to obtain the asymptotic exact solution:

\[
H = \frac{2}{\mu^2 - 2c_1}, \quad a = (\mu^2 - 2c_1)^{\frac{1}{2}} c_2, \quad \phi \sim \frac{1}{\mu} \ln \left[ \frac{V_0 \left( \mu_\gamma^2 - 2c_1 \right)^2}{2 \left( 6 - \mu_\gamma^2 \right)} \right].
\]

- \( P_{5,6} \) represent matter–kinetic-potential scaling solutions. As before, in the limit \( \phi \to +\infty \) we obtain the asymptotic expansions:
\[ \gamma \gamma \phi = -\gamma \gamma \sim -\gamma \gamma \]  
\[ \begin{bmatrix} \begin{array}{c} H \end{array} \\
\begin{array}{c} \mu \end{array} \end{bmatrix} \]  
\[ 2, 2, 1 \ln 2 \]  
\[ 0, 2, 1 \]  
\[ 0, 0, 2, 1 \]  
\[ R_3 \]  
\[ \chi \phi = (3t\gamma - 2c_1)^{\frac{2}{3\gamma} - c_2}, \quad \phi = -\frac{1}{\mu} \ln \left[ \frac{V_0\mu^2 (3t\gamma - 2c_1)^2}{18(2 - \gamma)\gamma} \right]. \]

\[ H = \frac{2}{3t\gamma - 2c_1}, \quad a = (3t\gamma - 2c_1)^{\frac{2}{3\gamma} - c_2}, \quad \phi = -\frac{1}{\mu} \ln \left[ \frac{V_0\mu^2 (3t\gamma - 2c_1)^2}{18(2 - \gamma)\gamma} \right]. \]

- \( R_3 \) represent radiation–kinetic-potential scaling solutions. As before, the following asymptotic expansions are deduced:

\[ H = \frac{1}{2t - c_1}, \quad a = \sqrt{2t - c_1} c_2, \quad \phi = -\frac{1}{\mu} \ln \left[ \frac{V_0\mu^2 (2t - c_1)^2}{4} \right]. \]

5.2.1. Power-law coupling and an Albrecht–Skordis potential. Now, let us revisit the example of a power-law coupling function and an Albrecht–Skordis potential, first introduced in [161] and then extended in section 4.4 of [170].

Let us consider the coupling function

\[ \chi(\phi) = \left( \frac{3\alpha}{8} \right)^{\frac{1}{\gamma}} \chi_0 \left( \phi - \phi_0 \right)^2, \quad \alpha > 0, \text{ const.}, \quad \phi_0 \geq 0. \] (33)

and the Albrecht–Skordis potential [198]:

\[ V(\phi) = e^{-\mu\phi} \left( A + (\phi - B)^2 \right). \] (34)

The coupling function (33) and the potential (34) are WBI functions of exponential orders \( M = 0 \) and \( N = -\mu \), respectively.
It is easy to prove that the power-law coupling and the Albrecht–Skordis potential are at least $\mathcal{E}^2_+$, under the admissible coordinate transformation [170]:

$$\varphi = \phi^{1/2} = f(\phi). \tag{35}$$

We fix here an error in formulas (B6)–(B9) in [161]. With the choice $\varphi = \phi^{-1}$, the resulting barred functions given by (B7)–(B9) there are not of the desired differentiable class.
Using the above coordinate transformation we find

\[ W_\sigma(\sigma) = \frac{2\sigma^2}{\alpha (1 - \sigma^2 \phi_0)} \] (36)

\[ W_\tau(\tau) = -\frac{2\tau^2 (B\tau^2 - 1)}{\lambda \tau^3 + (B\tau^2 - 1)^2} \] (37)

and

\[ \mathcal{F}(\phi) = -\frac{1}{2} \phi^3. \] (38)

In this example, the evolution equations for \( \phi, \sigma_2, \sigma_4 \) and \( \sigma_5 \) are given by the equations (26) with \( M = 0, N = -\mu \) and \( W_\sigma(\sigma), W_\tau(\tau) = 0 \) and \( \mathcal{F} \), given by (36), (37) and (38), respectively. The state space is defined by

\[ \Omega = \{ (\sigma_1, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4; 0 < \sigma_1 \leq \sqrt{\tau}, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0 \}. \]

Finally, let us discuss some numerical simulations.

In figure 4 some orbits in the invariant set \( \{ \sigma_1 = 0, \sigma_5 = 0 \} \subset \Omega \) are presented for the model with coupling function (33) potential (34). We select the following values for the parameters: \( \epsilon = 1.00, \mu = 2.00, A = 0.50, \alpha = 0.33, B = 0.5, \phi_0 = 0 \) and \( \gamma = 1 \). Observe that almost all the orbits are past asymptotic to \( P_1, P_2 \) is a saddle and the center manifold of \( P_4 \) attracts all the orbits in \( \{ \sigma_1 = 0 \} \). However, it is not an attractor in the invariant set \( \sigma_3 > 0, \sigma_5 = 0 \). In figure 5 some orbits in the invariant set \( \{ \phi = 0, \sigma_5 = 0 \} \subset \Omega \) are presented for the same values of the parameters as before. \( R_3 \) is a local past attractor, \( P_1 \) is the global past attractor, \( R_1, R_2 \) are saddles and \( P_5 \) is a local future attractor.

In figure 6 some orbits in the invariant set \( \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1 \) are displayed for the choice of \( \phi = 0 \) for the model with coupling function (33) potential (34). We select the following values for the parameters: \( \gamma = 1, \epsilon = 1.00, \mu = 2.10, A = 0.50, \alpha = 0.33, B = 0.5 \) and \( \phi_0 = 0 \). In this case \( P_5 \) is the local sink in this invariant set. \( R_3 \) exists and it is a saddle.

6. Conclusion

In this paper we have extended several results related to flat FLRW models in the conformal (Einstein) frame of scalar–tensor gravity theories. We have considered scalar fields with arbitrary (positive) potentials and arbitrary coupling functions. Then, we have introduced mild assumptions under such functions (differentiable class, number of singular points, asymptotes, etc.) in a straightforward manner in order to characterize the asymptotic structure on a phase space. We have also presented various examples of numerical evidence that confirm some of these results.

Our main results are the following.

(i) Proposition 1 states that for non-negative potentials with a local zero minimum at \( \phi = 0 \), such that its derivative is bounded in the same set where the potential is and provided the derivative of the logarithm of the coupling function has an upper bound, the energy densities of matter and radiation as well as the kinetic term tend to zero when time goes...
Thus, the Universe would expand forever in a de Sitter phase in the future. This result is an extension of proposition 2 in [156] in the non-minimal coupling context. It is also an extension of proposition 4 in [162] when the radiation is included in the cosmic budget.

(ii) Under the same hypotheses as in 1 and provided that \( V(\phi) \) is strictly decreasing (increasing) for negative (positive) values of the scalar field, then the scalar field diverges in the future or it equals zero (the last case holds only if the Hubble scalar vanishes towards the future). Proposition 2 is an extension of proposition 3 in [156] and of proposition 5 in [162] when the radiation is included in the background.

(iii) Assuming that the potential is non-negative (without necessarily a local minimum at \((0, 0)\)), such that for \( \phi \to +\infty \) it is unbounded, if its derivative is continuous and bounded on a set \( A \) where the potential is bounded, then the cosmological model evolves to a late-time de Sitter solution characterized by the divergence of the scalar field. Additionally, if the potential vanishes asymptotically, the Hubble scalar vanishes too (see proposition 3). Proposition 3 is an extension for \( \rho > 0 \) of proposition 6 discussed in [162].

(iv) We have formulated and proved proposition 4 (proposition 3 in [170]) generalizing analogous result in [168]. Our result states that if the potential \( V(\phi) \) is such that the (possibly empty) set where it is negative is bounded and the (possibly empty) set of singular points of \( V(\phi) \) is finite, then the singular point

\[
\mathbf{p}_s := \left\{ \phi_s, \gamma_s = 0, \rho_{\text{tot}} = 0, \rho_\phi = 0, H = \sqrt{\frac{V(\phi_s)}{3}} \right\},
\]

where \( \phi_s \) is a strict local minimum for \( V(\phi) \), corresponding to a de Sitter solution, is an asymptotically stable singular point for the flow.

(v) For the analysis of the system as \( \phi \to +\infty \), we have defined a suitable change of variables to bring a neighborhood of \( \phi = +\infty \) in a bounded set. In this regime we found: radiation-dominated cosmological solutions; power-law scalar field dominated inflationary cosmological solutions; matter–kinetic-potential scaling solutions and radiation–kinetic-potential scaling solutions. The rigorous mathematical apparatus was developed in section 4.

(vi) Using the above procedure we have investigated the behavior at the limit \( \phi \to +\infty \) for the following models:

(i) a double exponential potential \( V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi} \), \( 0 < \alpha < \beta \) and the coupling function \( \chi = \chi_0 \exp \left[ \frac{ib}{4 - 3\lambda} \right] \), where \( \lambda \) is a constant discussed in [91]; and

(ii) a general class of potentials containing the cases investigated in [85, 195] and in [198]. We have re-examined the toy model with power-law coupling and Albrecht–Skordis potential \( V(\phi) = e^{-ib(\phi - B)^2} \) [198] investigated in [161] in the presence of radiation.

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Appendix A. Singular points of the flow of (26) in the phase space (27).

The system (26) admits the following singular points (taken from [170]).

(i) The singular point $P_1$ with coordinates $\sigma_1 = 0, \sigma_2 = -1, \sigma_3 = 0, \sigma_4 = 0$ always exists. The eigenvalues of the linearization around the singular point are $0, \frac{1}{7}, 1 - \frac{\gamma}{\sqrt{6}}, \frac{M(4 - 3\gamma)}{\sqrt{6}} - \gamma + 2$. Thus:

(a) $P_1$ has a one-dimensional center manifold tangent to the $\sigma_1$-axis provided $N \neq \sqrt{6}$ and $M \neq -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4}$ (otherwise the center manifold would be two- or three-dimensional).

(b) $P_1$ admits a three-dimensional unstable manifold and a one-dimensional center manifold for

$$N < \sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M > -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4};$$

or

$$N < \sqrt{6}, \quad \frac{4}{3} < \gamma < 2, \quad M < -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4}.$$ 

In this case the center manifold of $P_1$ acts as a local source for an open set of orbits in (27).

(c) $P_1$ admits a two-dimensional unstable manifold, a one-dimensional stable manifold and a one-dimensional center if

$$N > \sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M > -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4};$$

or

$$N > \sqrt{6}, \quad \frac{4}{3} < \gamma < 2, \quad M < -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4};$$

or

$$N < \sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M < -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4};$$

or

$$N < \sqrt{6}, \quad \frac{4}{3} < \gamma < 2, \quad M > -\frac{\sqrt{6}(\gamma - 2)}{3\gamma - 4}.$$ 

(d) $P_1$ admits a one-dimensional unstable manifold, a two-dimensional stable manifold and a one-dimensional center manifold for
(ii) The singular point \( P_2 \) with coordinates \( \varphi = 0, \sigma_2 = 1, \sigma_4 = 0, \sigma_5 = 0 \) always exists. The eigenvalues of the linearization around the singular point are

\[
0, \quad \frac{1}{3}, \quad 1 + \frac{N}{\sqrt{6}}, \quad -\gamma + \frac{M(3\gamma - 4)}{\sqrt{6}} + 2.
\]

As before, let us determine the conditions on the free parameters for the existence of center, unstable and stable manifolds for \( P_2 \).

(a) If \( N \neq -\sqrt{6} \) and \( M \neq \frac{\sqrt{6}(y - 2)}{3\gamma - 4} \) there exists a one-dimensional center manifold tangent to the \( \varphi \)-axis, otherwise the center manifold would be two- or three-dimensional.

(b) \( P_2 \) has a three-dimensional unstable manifold and a one-dimensional center manifold (tangent to the \( \varphi \)-axis) if

\[
N > -\sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M < \frac{\sqrt{6}(y - 2)}{3\gamma - 4};
\]

or

\[
N > \frac{4}{3}, \quad 0 < \gamma < 2, \quad M > \frac{\sqrt{6}(y - 2)}{3\gamma - 4}.
\]

In this case the center manifold of \( P_2 \) acts as a local source for an open set of orbits in (27).

(c) \( P_2 \) has a two-dimensional unstable manifold, a one-dimensional stable manifold and a one-dimensional center manifold if

\[
N < -\sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M < \frac{\sqrt{6}(y - 2)}{3\gamma - 4};
\]

or

\[
N < \frac{4}{3}, \quad 0 < \gamma < 2, \quad M > \frac{\sqrt{6}(y - 2)}{3\gamma - 4};
\]

or

\[
N > -\sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M > \frac{\sqrt{6}(y - 2)}{3\gamma - 4};
\]

or

\[
N > \frac{4}{3}, \quad 0 < \gamma < 2, \quad M < \frac{\sqrt{6}(y - 2)}{3\gamma - 4}.
\]

(d) \( P_2 \) has a one-dimensional unstable manifold, a two-dimensional stable manifold and a one-dimensional center manifold if
\[ N < - \sqrt{6}, \quad 0 < \gamma < \frac{4}{3}, \quad M > \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4}; \]

or

\[ N < - \sqrt{6}, \quad \frac{4}{3} < \gamma < 2, \quad M < \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4}. \]

(iii) The singular point \( P_3 \) with coordinates

\[ \varphi = 0, \quad \sigma_2 = \frac{M (3 \gamma - 4)}{\sqrt{6} (\gamma - 2)}, \quad \sigma_4 = 0, \quad \sigma_5 = 0 \]

exists for

(a) \( 0 < \gamma < \frac{4}{3}, \quad - \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4} \leq M \leq \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4}; \)

or

(b) \( \frac{4}{3} < \gamma < 2, \quad \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4} \leq M \leq - \frac{\sqrt{6} (\gamma - 2)}{3 \gamma - 4}. \)

The eigenvalues of the linearization are

\[ \lambda_1 = \frac{(3 \gamma - 4) ((3 \gamma - 4) M^2 - 2 \gamma + 4)}{12 (\gamma - 2)}, \]
\[ \lambda_2 = \frac{-M^2 (4 - 3 \gamma)^2 + 6 (\gamma - 2) \gamma + 2 MN (3 \gamma - 4)}{12 (\gamma - 2)}, \]
\[ \lambda_3 = \frac{6 (\gamma - 2)^2 - M^2 (4 - 3 \gamma)^2}{12 (\gamma - 2)}. \]

As before, let us determine the conditions on the free parameters for the existence of center, unstable and stable manifolds for \( P_3 \).

(a) For \( \gamma, N \) and \( M \) such that \( \lambda_1 \neq 0, \lambda_3 \neq 0 \) the center manifold is one-dimensional and tangent to the \( \varphi \)-axis. Otherwise the center manifold would be two- or three-dimensional (it is never four-dimensional).

(b) \( P_3 \) admits a one-dimensional center manifold and a three-dimensional stable manifold for

\[ 0 < \gamma < \frac{4}{3}, \quad - \frac{\sqrt{2} \sqrt{\gamma - 2}}{\sqrt{3 \gamma - 4}} < M < 0, \quad N > \frac{M^2 (4 - 3 \gamma)^2 - 6 (\gamma - 2) \gamma}{2 M (3 \gamma - 4)}; \]

or

\[ 0 < \gamma < \frac{4}{3}, \quad 0 < M < \frac{\sqrt{2} \sqrt{\gamma - 2}}{\sqrt{3 \gamma - 4}}, \quad N < \frac{M^2 (4 - 3 \gamma)^2 - 6 (\gamma - 2) \gamma}{2 M (3 \gamma - 4)}. \]

(c) The unstable manifold of \( P_3 \) is two-dimensional (thus its stable and center manifolds are both one-dimensional) in the cases
\[
0 < \gamma < \frac{4}{3}, \quad \frac{\sqrt{6} (\gamma - 2)}{3\gamma - 4} < M < -\frac{\sqrt{2}}{\sqrt{3\gamma - 4}}, \quad N < \frac{M^2 (4 - 3\gamma)^2 - 6(\gamma - 2)\gamma}{2 M (3\gamma - 4)},
\]

or
\[
0 < \gamma < \frac{4}{3}, \quad \frac{\sqrt{2}}{\sqrt{3\gamma - 4}} < M < \frac{\sqrt{6} (\gamma - 2)}{3\gamma - 4}, \quad N > \frac{M^2 (4 - 3\gamma)^2 - 6(\gamma - 2)\gamma}{2 M (3\gamma - 4)};
\]

or
\[
\frac{4}{3} < \gamma < 2, \quad \frac{\sqrt{6} (\gamma - 2)}{3\gamma - 4} < M < 0, \quad N > \frac{M^2 (4 - 3\gamma)^2 - 6(\gamma - 2)\gamma}{2 M (3\gamma - 4)};
\]

or
\[
\frac{4}{3} < \gamma < 2, \quad M = 0, \quad N \in \mathbb{R};
\]

or
\[
\frac{4}{3} < \gamma < 2, \quad 0 < M < -\frac{\sqrt{6} (\gamma - 2)}{3\gamma - 4}, \quad N < \frac{M^2 (4 - 3\gamma)^2 - 6(\gamma - 2)\gamma}{2 M (3\gamma - 4)}.
\]

Otherwise, \( P_3 \) has a one-dimensional unstable manifold. Thus, it is never a local source since its unstable manifold is of dimension less than three.

(iv) The singular point \( R_1 \) with coordinates \( \varphi = 0, \sigma_2 = 0, \sigma_4 = 0, \sigma_5 = 1 \) always exists. The eigenvalues of the linearization are \( 0, -\frac{2}{3}, -\frac{4}{3} - \gamma \). The center manifold is one-dimensional and tangent to the \( \varphi \)-axis. The unstable (stable) manifold is one-dimensional (two-dimensional) if \( \gamma > \frac{4}{3} \), otherwise it is two-dimensional (one-dimensional).

(v) The singular point \( R_2 \) with coordinates
\[
\sigma_2 = \frac{\sqrt{2}}{M}, \quad \sigma_4 = 0, \quad \sigma_5 = \frac{\sqrt{\frac{4 - 2\gamma}{M^2} + 3\gamma - 4}}{\sqrt{3\gamma - 4}}
\]
exists for
\[
0 < \gamma < \frac{4}{3}, \quad M^2 \geq \frac{2(\gamma - 2)}{3\gamma - 4}.
\]
The eigenvalues of the linearization are
\[
0, -\frac{M + \sqrt{3 M^2 (4\gamma - 5) - 8(\gamma - 2)}}{6 M}, \quad \frac{\sqrt{3 M^2 (4\gamma - 5) - 8(\gamma - 2) - M}}{6 M} - \frac{1}{3} \left( \frac{N}{M} + 2 \right).
\]
Let us determine the conditions on the free parameters for the existence of center, unstable and stable manifolds for \( R_2 \).

(a) \( R_2 \) has a three-dimensional stable manifold and a one-dimensional center manifold if
\[
0 < \gamma < \frac{5}{4}, \quad -2 \sqrt{\frac{2}{3}} \sqrt{4\gamma - 5} \leq M < -\sqrt{\frac{2}{3}} \sqrt{\gamma - 2}, \quad N > -2 M;
\]
or
(b) By reversing the sign of the last inequality, i.e. the inequality solved for $N$, in the previous six cases we obtain conditions for $R_2$ having a two-dimensional stable manifold, a one-dimensional unstable manifold and a one-dimensional center manifold.

(vi) The singular point $P_4$ with coordinates

$$\varphi = 0, \quad \sigma_2 = -\frac{N}{\sqrt{6}}, \quad \sigma_4 = \sqrt{1 - \frac{N^2}{6}}, \quad \sigma_5 = 0$$

exists whenever $N^2 < 6$. The eigenvalues of the linearization are

$$0, \quad \frac{1}{6}(N^2 - 6), \quad \frac{1}{6}(N^2 - 4), \quad \frac{1}{3}N(2M + N) - \frac{1}{2}(MN + 2)\gamma.$$

The conditions for the existence of stable, unstable and center manifolds are as follows.

(a) The center manifold is one-dimensional and the stable manifold is three-dimensional provided $N = 0$, $M \in \mathbb{R}$, $\gamma \neq \frac{4}{3}$; or

$$0 < \gamma < \frac{4}{3}, \quad -2 < N < 0, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};$$

or

$$0 < \gamma < \frac{4}{3}, \quad 0 < N < 2, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};$$

or

$$\frac{4}{3} < \gamma < 2, \quad -2 < N < 0, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};$$

or
\[
\frac{4}{3} < \gamma < 2, \quad 0 < N < 2, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}.
\]

(b) The stable manifold is two-dimensional, the unstable manifold is one-dimensional and the center manifold is one-dimensional provided

\[
0 < \gamma < \frac{4}{3}, \quad -\sqrt{6} < N < -2, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
0 < \gamma < \frac{4}{3}, \quad -2 < N < 0, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
0 < \gamma < \frac{4}{3}, \quad 0 < N < 2, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
0 < \gamma < \frac{4}{3}, \quad 2 < N < \sqrt{6}, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
\frac{4}{3} < \gamma < 2, \quad -\sqrt{6} < N < -2, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
\frac{4}{3} < \gamma < 2, \quad -2 < N < 0, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
\frac{4}{3} < \gamma < 2, \quad 0 < N < 2, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or

\[
\frac{4}{3} < \gamma < 2, \quad 2 < N < \sqrt{6}, \quad M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}.
\]

(c) The stable manifold is one-dimensional, the unstable manifold is two-dimensional and the center manifold is one-dimensional provided

\[
0 < \gamma < \frac{4}{3}, \quad -\sqrt{6} < N < -2, \quad M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)};
\]

or
\[
0 < \gamma < \frac{4}{3}, \quad 2 < N < \sqrt{6}, \quad M > \frac{2\left(N^2 - 3\gamma\right)}{N(3\gamma - 4)}; \\
\text{or} \\
\frac{4}{3} < \gamma < 2, \quad -\sqrt{6} < N < -2, \quad M > \frac{2\left(N^2 - 3\gamma\right)}{N(3\gamma - 4)}; \\
\text{or} \\
\frac{4}{3} < \gamma < 2, \quad 2 < N < \sqrt{6}, \quad M < \frac{2\left(N^2 - 3\gamma\right)}{N(3\gamma - 4)}.
\]

(vii) The singular point \( R_3 \) with coordinates

\[
\varphi = 0, \quad \sigma_2 = -\frac{2\sqrt{3}}{N}, \quad \sigma_4 = \frac{2}{\sqrt{3}|N|}, \quad \sigma_5 = \frac{\sqrt{N^2 - 4}}{|N|}
\]

exists for \( N^2 \geq 4 \). The eigenvalues of the linearization are

\[
0, \quad \frac{1}{6}\left(-\frac{64N^2 - 15N^4}{N^2} - 1\right), \quad \frac{1}{6}\left(\frac{64N^2 - 15N^4}{N^2} - 1\right) - \frac{2M + N(3\gamma - 4)}{3N}.
\]

The conditions for the existence of stable, unstable and center manifolds are as follows.

(a) The stable manifold is three-dimensional and the center manifold is one-dimensional provided

\[
0 < \gamma < \frac{4}{3}, \quad N < -\frac{8}{\sqrt{15}}, \quad M > -\frac{N}{2}; \\
\text{or} \\
0 < \gamma < \frac{4}{3}, \quad -\frac{8}{\sqrt{15}} \leq N < -2, \quad M > -\frac{N}{2}; \\
\text{or} \\
0 < \gamma < \frac{4}{3}, \quad 2 < N \leq \frac{8}{\sqrt{15}}, \quad M < -\frac{N}{2}; \\
\text{or} \\
0 < \gamma < \frac{4}{3}, \quad N > \frac{8}{\sqrt{15}}, \quad M < -\frac{N}{2}; \\
\text{or} \\
\frac{4}{3} < \gamma < 2, \quad 2 < N \leq \frac{8}{\sqrt{15}}, \quad M > -\frac{N}{2}; \\
\text{or}
\]

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(b) By reversing the sign of the last inequality, i.e. the inequality solved for $M$, in the previous eight cases we obtain the conditions for $R^3$ having a two-dimensional stable manifold, a one-dimensional unstable manifold and a one-dimensional center manifold.

(viii) The singular point $P_3$ with coordinates

$$\varphi = 0, \quad \sigma_2 = \frac{\sqrt{6\gamma}}{M(3\gamma - 4) - 2N},$$

$$\sigma_4 = \frac{\sqrt{M^2(4 - 3\gamma)^2 + MN(8 - 6\gamma) - 6(\gamma - 2)\gamma}}{2N + M(4 - 3\gamma)}, \quad \sigma_5 = 0$$

exists for

$$2(2M + N) > 3M\gamma, \quad M(3\gamma - 4)(M(3\gamma - 4) - 2N) \geq 6(\gamma - 2)\gamma,$$

and

$$\frac{3(MN + 2)\gamma - 2N(2M + N)}{(2N + M(4 - 3\gamma))^2} \leq 0.$$

The eigenvalues of the linearization are

$$0, \quad \frac{12M + 6N - 3(M + N)\gamma + \sqrt{3}f(\gamma, M, N)}{6(M(3\gamma - 4) - 2N)},$$

$$\frac{3N(\gamma - 2) + 3M(3\gamma - 4) + \sqrt{3}f(\gamma, M, N)}{6(2N + M(4 - 3\gamma))}, \quad (2M + N)(3\gamma - 4),$$

where

$$f(\gamma, M, N) = 2M^3N(3\gamma - 4)^3 + 2MN\left(4N^2 - 6\gamma^2 + 3\gamma - 6\right)(3\gamma - 4)$$

$$-M^2\left(8N^2 - 12\gamma - 3\right)(4 - 3\gamma)^2 + 3(\gamma - 2)\left(N^2(9\gamma - 2) - 24\gamma^2\right).$$

The stability conditions of $P_3$ are too complicated to display them here. Thus we must rely on numerical experimentation. We can obtain, however, some analytic results. For instance, there exists at least a one-dimensional center manifold. The unstable manifold is always of dimension lower than three. Thus the singular point is never a local source. If all the eigenvalues, apart from the zero one, have negative reals parts, then the center manifold of $P_3$ acts as a local sink. This means that the orbits in the stable manifold approach the center manifold of $P_3$ when time goes forward.

(ix) The singular point $P_6$ with coordinates

$$\varphi = 0, \quad \sigma_2 = \frac{\sqrt{6\gamma}}{M(3\gamma - 4) - 2N},$$

$$\sigma_4 = \frac{\sqrt{M^2(4 - 3\gamma)^2 + MN(8 - 6\gamma) - 6(\gamma - 2)\gamma}}{2N + M(4 - 3\gamma)}, \quad \sigma_5 = 0$$

exists for
\[ M(3\gamma - 4)(M(3\gamma - 4) - 2N) \geq 6(\gamma - 2)\gamma, \quad 2(2M + N) < 3M\gamma, \]

and

\[ \frac{3(MN + 2)\gamma - 2N(2M + N)}{(2N + M(4 - 3\gamma))^2} \leq 0. \]

The eigenvalues of the linearization are the same as displayed in the previous point. However, the stability conditions are rather different (since the existence conditions are different to those of \( P_5 \)). As before, the stability conditions are too complicated to display them here, but similar conclusions concerning the center and unstable manifolds to those for \( P_5 \) are obtained. For obtain further information about its stability we must to resort to numerical experimentation.

**Appendix B. A physical description of the solutions and connection with observables**

Let us now present the formalism of obtaining the physical description of a singular point and also connect with the basic observables relevant for a physical discussion (taken from \[170\]).

First, we obtain first-order expansions for \( H, a, \phi \), and \( \rho_m \) and \( \rho_r \) in terms of \( t \) around a singular point (considering the equations (11a), the definition of the scale factor \( a \) in terms of the Hubble factor \( H \), the definition of \( \sigma_2 \) and the matter conservation equations (11b) and (11c)) given, respectively, by

\[
2H(t) = H(t)^2 \left( 3(\gamma - 2)\sigma_2^{*2} + 3\gamma \left( \sigma_4^{*2} + \sigma_5^{*2} - 1 \right) - 4\sigma_5^{*2} \right),
\]

\[
\dot{a}(t) = a(t)H(t),
\]

\[
\dot{\phi}(t) = \sqrt{6}\sigma_2^{*} H(t),
\]

\[
\rho_m(t) = -\frac{3}{2}H(t)^3 \left( \sqrt{6}M(3\gamma - 4)\sigma_2^{*} - 6\gamma \right) \left( \sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1 \right),
\]

\[
\dot{\rho}_r(t) = -12\sigma_5^{*2} H(t)^3,
\]

where the star superscript denotes the evaluation at a specific singular point. The equation

\[
\ddot{\phi}(t) = \frac{3}{2}H(t)^2 \left( M(3\gamma - 4) \left( \sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1 \right) - 2N\sigma_4^{*2} + \sqrt{6}\sigma_2^{*} \right),
\]

derived from the equation of motion for the scalar field (11d) should be used as a consistency test for the above procedure. Solving the differential equations (B.1) and substituting the resulting expressions in (B.2) results in

\[
-6M(3\gamma - 4) \left( \sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1 \right) + 12N\sigma_4^{*2}
\]

\[
+ 2\sqrt{6}\sigma_2^{*} \left( 3\gamma \left( \sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1 \right) - 6\sigma_2^{*2} - 4\sigma_5^{*2} + 6 \right) = 0.
\]

This integrability condition should be (at least asymptotically) fulfilled.

Instead of applying this procedure to a generic singular point here, we direct the reader to section 5 for some worked examples where this procedure has been applied. However, we will discuss some cosmological observables.
Table B1. The observable cosmological quantities and physical behavior of the solutions at the singular points of the cosmological system. We use the notations 

\[ M_1(\gamma) = \frac{\delta(\gamma^2 - 8 + 8y)}{4 - 3y}, \quad M_2(\gamma) = \frac{\delta(\gamma^2 - 8 + 8y)}{4 - 3y}. \] (Taken from [170].)

| Cr.P. | $q$ | $w_{\text{eff}}$ | Solution/description |
|-------|-----|-----------------|----------------------|
| $P_1$ | 2   | 1               | decelerating         |
| $P_2$ | 2   | 1              | decelerating         |
| $P_3$ | $\frac{-M_1^2(4 - 3y^2 + 2(3y - 8) + 8)}{4(y - 2)}$ | $\frac{-M_1^2(4 - 3y^2)}{6(y - 2)} + \gamma - 1$ | accelerating for $0 < \gamma < \frac{2}{3}$, $-M_1(\gamma) < M < M_1(\gamma)$ |
| $P_4$ | $\frac{1}{2}(N^2 - 2)$ | $\frac{1}{2}(N^2 - 3)$ | accelerating for $-\sqrt{2} < N < \sqrt{2}$ |
| $P_5$ | $\frac{3(M + N)y - 2(2M + N)}{2N + M(4 - 3y)}$ | $\frac{M(4 - 3y) - 2N(y - 1)}{M(3y - 4) + 2N}$ | power-law-inflationary accelerating for $\frac{3(M + N)y - 2(2M + N)}{2N + M(4 - 3y)} < 0$ |
| $P_6$ | $\frac{3(M + N)y - 2(2M + N)}{2N + M(4 - 3y)}$ | $\frac{M(4 - 3y) - 2N(y - 1)}{M(3y - 4) + 2N}$ | matter–kinetic–potential scaling accelerating for $\frac{3(M + N)y - 2(2M + N)}{2N + M(4 - 3y)} < 0$ |
| $R_1$ | 1   | $\frac{1}{3}$ | decelerating, radiation-dominated |
| $R_2$ | 1   | $\frac{1}{3}$ | decelerating |
| $R_3$ | 1   | $\frac{1}{3}$ | radiation–kinetic–potential scaling |

We can calculate the deceleration parameter $q$ defined as usual as [165]

\[ q = -\frac{\ddot{a} \dot{a}}{a^2}. \] (B.4)

Additionally, we can calculate the effective (total) equation-of-state parameter of the universe $w_{\text{eff}}$, defined conventionally as

\[ w_{\text{eff}} \equiv \frac{p_{\text{tot}}}{\rho_{\text{tot}}}, \] (B.5)

where $p_{\text{tot}}$ and $\rho_{\text{tot}}$ are, respectively, the total isotropic pressure and the total energy density. Therefore, in terms of the auxiliary variables we have

\[ q = -\frac{3}{2}(\gamma - 2)\sigma_2^2 - \frac{3y\sigma_1^2}{2} + \frac{1}{2}(4 - 3y)\sigma_2^2 + \frac{1}{2}(3y - 2) \] (B.6)

\[ w_{\text{eff}} = (2 - \gamma)\sigma_2^2 - \gamma\sigma_1^2 + \frac{1}{3}(4 - 3y)\sigma_2^2 + \gamma - 1. \] (B.7)

First of all, for each singular point described in the last section we calculate the effective (total) equation-of-state parameter of the universe $w_{\text{eff}}$ using (B.7) and the deceleration parameter $q$ using (B.6). The results are presented in table B1. Furthermore, as usual, for an expanding universe $q < 0$ corresponds to accelerating expansion and $q > 0$ to decelerating expansion.
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