Improved Friedrichs inequality for a subhomogeneous embedding

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Abstract

For a smooth bounded domain $\Omega$ and $p \geq q > 2$, we establish quantified versions of the classical Friedrichs inequality $\|\nabla u\|_p^p - \lambda_1 \|u\|_q^q \geq 0$, $u \in W^{1,p}_0(\Omega)$, where $\lambda_1$ is a generalized least frequency. We apply one of the obtained quantifications to show that the resonant equation $-\Delta_p u = \lambda_1 \|u\|^{p-q} |u|^{q-2} u + f$ coupled with zero Dirichlet boundary conditions possesses a weak solution provided $f$ is orthogonal to the minimizer of $\lambda_1$.

Keywords: $p$-Laplacian; sublinear; improved Friedrichs inequality; improved Poincaré inequality; Steklov inequality.

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1. Introduction

Let $p \geq q > 1$ and let $\Omega \subset \mathbb{R}^N$ be a domain of finite (Lebesgue) measure, $N \geq 1$. The embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is well known to be compact and the best constant of this embedding can be characterized by means of the generalized least frequency

$$\lambda_1 = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p \, dx}{\left( \frac{\int_{\Omega} |u|^q \, dx}{{\Omega}} \right)^{\frac{q}{p}}} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}. \quad (1.1)$$

Let us recall several classical facts about $\lambda_1$ and its minimizers. We have $\lambda_1 > 0$ and $\lambda_1$ is attained by a function $\varphi_1 \in W^{1,p}_0(\Omega)$. In view of the subhomogeneity assumption $p \geq q$, the minimizer $\varphi_1$
is known to be *unique* modulo scaling, see, e.g., [22, 23, 30]. The regularity results [11, 31, 36] imply that \( \varphi_1 \) is bounded and belongs to \( C^{1,\alpha}_{\text{loc}}(\Omega) \) with some \( \beta \in (0, 1) \). If, in addition, \( \Omega \) is of class \( C^{1,\alpha} \), \( \alpha \in (0, 1) \), then \( \varphi_1 \in C^{1,2}(\Omega) \), see [26]. By the strong maximum principle (see, e.g., [32]), \( \varphi_1 \) cannot attain zero values in \( \Omega \). Accordingly, we assume, without loss of generality, that \( \varphi_1 > 0 \) in \( \Omega \). We refer to [5, 18] for a comprehensive overview of the properties of \( \lambda_1, \varphi_1 \), and other critical values and points of the Rayleigh-type quotient in (1.1).

For functions from \( W^{1,p}_0(\Omega) \), the definition (1.1) leads to the standard *Friedrichs inequality*\(^1\)

\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \left( \int_{\Omega} |u|^q \, dx \right)^{\frac{p}{q}} \geq 0,
\]  
(1.2)

where the equality takes place if and only if \( u \) is either a minimizer of \( \lambda_1 \) or trivial, i.e., \( u \in \mathbb{R}\varphi_1 \). We are interested in obtaining a quantification of the inequality (1.2). In the homogeneous linear case \( p = q = 2 \), in which (1.2) is often referred to as the *Poincaré inequality*, one easily sees that (1.2) can be refined via the spectral decomposition as

\[
\int_{\Omega} |\nabla u|^2 \, dx - \lambda_1 \int_{\Omega} u^2 \, dx \geq \frac{\lambda_2 - \lambda_1}{\lambda_2} \int_{\Omega} |\nabla u^\perp|^2 \, dx,
\]  
(1.3)

where \( \lambda_2 \) is the second eigenvalue of the Dirichlet Laplacian in \( \Omega \) and \( u^\perp \) is the orthogonal projection in \( L^2(\Omega) \) on the orthogonal complement of the eigenspace \( \mathbb{R}\varphi_1 \). In the homogeneous nonlinear case \( p = q > 2 \), an improvement of the Poincaré inequality was obtained by Fleckinger-Pellé & Takáč in [16]. Under certain regularity assumptions on \( \Omega \), which will be discussed below, the authors proved that

\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx \geq C \left( \|\tilde{u}\|^{p-2} \int_{\Omega} |\nabla \varphi_1|^{p-2} |\nabla \tilde{u}^\perp|^2 \, dx + \int_{\Omega} |\nabla \tilde{u}^\perp|^p \, dx \right),
\]  
(1.4)

where \( \tilde{u} \in \mathbb{R} \) and \( \tilde{u}^\perp \in W^{1,p}_0(\Omega) \) are defined from the \( L^2(\Omega) \)-decomposition \( u = \tilde{u}\varphi_1 + \tilde{u}^\perp \) by

\[
\tilde{u} = \frac{\int_{\Omega} \varphi_1 u \, dx}{\int_{\Omega} \varphi_1^2 \, dx} \quad \text{and} \quad \int_{\Omega} \varphi_1 \tilde{u}^\perp \, dx = 0.
\]  
(1.5)

This result helps to provide fine estimates on function sequences and corresponding energy levels of various nonlinear problems. In particular, (1.4) found its importance in the development of the Fredholm alternative for the \( p \)-Laplacian at \( \lambda_1 \), see the overviews [12, 34, 35], as well as in the investigation of other related equations, see, e.g., [3, 4, 9, 21]. We refer to [1] and [13] for versions of the improved Poincaré inequality (1.4) in the entire space case and an exterior domain case, respectively, and to [14] for the consideration of the case \( p \in (1, 2) \). In the case \( p = 2 \) and \( q > 2 \), a quantification of a Friedrichs-type inequality for the function space \( W^{1,2}(\mathbb{S}^N) \) has been found in [17].

The aim of the present work is to obtain qualitative improvements of the nonlinear Friedrichs inequality (1.2) for \( p \geq q \geq 2 \) in a form similar to (1.3) and (1.4). For that purpose, we employ a different decomposition than (1.5) which seems to be more suitable to work with nonhomogeneous problems. Note that since \( \varphi_1 \) is a critical point of the Rayleigh quotient in (1.1), it is a weak solution of the corresponding Lane-Emden problem:

\[
\int_{\Omega} |\nabla \varphi_1|^{p-2} \langle \nabla \varphi_1, \nabla v \rangle \, dx - \lambda_1 \left( \int_{\Omega} \varphi_1^p \, dx \right)^{\frac{p-1}{p}} \int_{\Omega} \varphi_1^{p-1} v \, dx = 0 \quad \text{for all } v \in W^{1,p}_0(\Omega).
\]  
(1.6)

Hereinafter, by \( \langle \cdot, \cdot \rangle \) we denote the standard scalar product in \( \mathbb{R}^N \). In view of the form of (1.6), we decompose an arbitrary function \( u \in W^{1,p}_0(\Omega) \) as

\[
u = u^\parallel \varphi_1 + u^\perp,
\]  
(1.7)

\(^1\)The name for the inequality might not be optimal from the historical perspective, but it is frequently used in the contemporary literature. We refer to [24] and [29, Chapter II] for a comprehensive historical overview.
where \( u^\parallel \in \mathbb{R} \) and \( u^\perp \in W_0^{1,p}(\Omega) \) are defined by
\[
\|u^\parallel\|_\Omega = \frac{\int_\Omega \varphi_\parallel^{-1} u \, dx}{\int_\Omega \varphi_1^{-1} \, dx} \quad \text{and} \quad \int_\Omega \varphi_\parallel^{-1} u^\perp \, dx = 0.
\] (1.8)

We also introduce the following mild regularity assumption on \( \Omega \):

(A) In the case \( N \geq 2 \), \( \Omega \) is a bounded domain of class \( C^{1,\alpha} \) for some \( \alpha \in (0, 1) \). In the case \( N = 1 \), \( \Omega \) is a bounded open interval.

Let us state our first main result on the improved Friedrichs inequality.

**Theorem 1.1.** Let \( p \geq q \geq 2 \) with \( p > 2 \), and (A) be satisfied. Then there exists \( C = C(p, q, \Omega) > 0 \) such that
\[
\int_\Omega |\nabla u|^p \, dx - \lambda_1 \left( \int_\Omega |u|^q \, dx \right)^\frac{p}{q} \geq C \left( |u^\parallel|^{p-2} \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla u^\perp|^2 \, dx + \int_\Omega |\nabla u^\perp|^p \, dx \right)
\] (1.9)
for any \( u \in W_0^{1,p}(\Omega) \).

In the norm notation, the inequality (1.9) can be written as
\[
\|\nabla u\|_p^p - \lambda_1 \|u\|_\Omega^p \geq C \left( |u^\parallel|^{p-2} \|u^\perp\|_{\varphi_1}^2 + \|\nabla u^\perp\|_p^p \right),
\] (1.10)
where \( \|\cdot\|_r \) denotes the usual norm in \( L^r(\Omega) \), \( r \in [1, \infty) \), i.e., \( \|u\|_r^r = \int_\Omega |u|^r \, dx \), and the norm \( \|\cdot\|_{\varphi_1} \) is introduced in (2.7) below. Lemma 2.1 (i) can be used to estimate \( \|u^\parallel\|_{\varphi_1} \) from below by \( \|u^\parallel\|_\kappa \) for some \( \kappa > 2 \), which might be convenient in applications.

We remark that, in view of the decomposition (1.8), the right-hand side of (1.9) is 0-homogeneous with respect to \( \varphi_1 \) and \( p \)-homogeneous with respect to \( u \). In particular, the constant \( C \) does not depend on the normalization of \( \varphi_1 \).

In the case \( p = q = 2 \), the improved Friedrichs inequality (1.9) coincides with (1.3) up to a nonquantified constant \( C \). In the case \( p = q > 2 \), (1.9) has the same form as the improved Poincaré inequality (1.4) from [16], but we emphasize that (1.4) is formulated in terms of the decomposition (1.5) which slightly differs from (1.8), see Section 1.1 for a further generalization. We also note that [16] asks for stronger regularity assumptions on \( \Omega \) than (A), see [16, (H1)] and (H2), pp. 956-957. While the assumption [16, (H1)] is essentially equivalent to (A) (see a discussion in the proof of Lemma 2.1 (i) below), the assumption [16, (H2)] is less constructive and was shown to be valid only when \( N = 1 \) or when \( \partial \Omega \) is connected (in the case \( N \geq 2 \)). At the same time, it was conjectured in [33, Section 2.1] that [16, (H2)] is always satisfied provided [16, (H1)] holds. Recent embedding results from [6] (which are based on [10]) yield an affirmative answer to this conjecture, see Remark 2.2 below. For convenience, we update the statement of [16, Theorem 3.2] accordingly.

**Theorem 1.2** ([16]). Let \( p > 2 \) and (A) be satisfied. Then there exists \( C = C(p, \Omega) > 0 \) such that (1.4) holds for any \( u \in W_0^{1,p}(\Omega) \).

This result allows to weaken assumptions on \( \partial \Omega \) imposed, e.g., in [3, Theorem 2.6], [4, Theorem 1.11], and in other related results, see a discussion in [3, Remark 5] and [4, Remark 1.13].

Our proof of Theorem 1.1 is based on the strategy developed in [16]. Namely, we divide the consideration into two cases, by considering the following two function cones. For a fixed \( \gamma \in (0, \infty) \), we define a cone “around” \( \mathbb{R} \varphi_1 \):
\[
C_\gamma = \{ u \in W_0^{1,p}(\Omega) : \|\nabla u^\perp\|_p \leq \gamma \|u^\parallel\| \},
\] (1.11)
and the complementary cone
\[
C'_\gamma = \{ u \in W_0^{1,p}(\Omega) : \|\nabla u^\perp\|_p \geq \gamma \|u^\parallel\| \}.
\] (1.12)

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The proof of the inequality (1.9) in the complementary cone \( C'_\gamma \), given in Section 3.2, is rather straightforward thanks to the fact that functions from \( C'_\gamma \) are “far” from \( \mathbb{R}\varphi_1 \). The proof in the cone \( C_\gamma \) is more subtle since we are “close” to \( \mathbb{R}\varphi_1 \), and we deal with this case in Section 3.1. It requires working with the linearization of the \( p \)-Laplacian, and we provide corresponding results in Section 2.

### 1.1. Generalization

Let us discuss a generalization of Theorems 1.1 and 1.2, as well as of the inequality (1.3), suggested by the fact that the right-hand sides of the improved inequalities (1.4) and (1.9) are structurally similar. It is not hard to observe that the decompositions given by (1.5) and (1.8) provide particular examples of the direct sum decomposition of \( W^{1,p}_0(\Omega) \):

\[
W^{1,p}_0(\Omega) = \mathbb{R}\varphi_1 \oplus \text{Ker}(l),
\]

where \( l : W^{1,p}_0(\Omega) \to \mathbb{R} \) is a bounded linear functional satisfying \( l[\varphi_1] \neq 0 \), and \( \text{Ker}(l) \) is its kernel. More precisely, in the case of (1.5), we have \( l[u] = \int_\Omega \varphi_1 u \, dx \), and in the case of (1.8), we have \( l[u] = \int_\Omega \varphi_1^{-1} u \, dx \). Clearly, there are many similar functionals, and a particular choice may depend on an application.

Our second main theorem generalizes Theorems 1.1 and 1.2 for the decomposition (1.13).

**Theorem 1.3.** Let \( p \geq q \geq 2 \). Let \( (A) \) be satisfied provided \( p > 2 \). Let \( l : W^{1,p}_0(\Omega) \to \mathbb{R} \) be a bounded linear functional such that \( l[\varphi_1] = 1 \), and let \( P : W^{1,p}_0(\Omega) \to \text{Ker}(l) \) be the corresponding projection operator:

\[
Pu = u - l[u]\varphi_1.
\]

Then there exists \( C = C(p,q,\Omega,l) > 0 \) such that

\[
\|\nabla u\|_p^p - \lambda_1 \|u\|_q^q \geq C \left( \|l[u]\|^{p-2}_p \|Pu\|_{\varphi_1}^2 + \|\nabla Pu\|_p^p \right)
\]

for any \( u \in W^{1,p}_0(\Omega) \).

The proof of Theorem 1.3 is placed in Section 4. It is worth noting that the dependence of the right-hand side of (1.14) on the exponent \( q \) is, in general, hidden only in the constant \( C \). The normalization \( l[\varphi_1] = 1 \) is imposed solely for brevity.

Let us also observe that in contrast to Theorems 1.1 and 1.2 the formulation of Theorem 1.3 includes the homogeneous linear case \( p = q = 2 \). In this case, (1.14) reads as

\[
\int_\Omega |\nabla u|^2 \, dx - \lambda_1 \int_\Omega u^2 \, dx \geq C \int_\Omega |\nabla Pu|^2 \, dx,
\]

and this generalization of the improved Poincaré inequality (1.3) does not seem straightforward to us since it covers the case of nonorthogonal decompositions of \( W^{1,2}_0(\Omega) \).

### 1.2. Application to the nonlinear Fredholm alternative

In order to justify the utility of the improved Friedrichs inequality (1.9), we consider the following model example of the boundary value problem at resonance:

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda_1 \|u\|_q^{p-q} |u|^{q-2} u + f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( p > q \geq 2 \) and \( f \in (D_{\varphi_1})^* \setminus \{0\} \). Here, \((D_{\varphi_1})^* \) is the dual of \( D_{\varphi_1} \). As a practical case, we see from Lemma 2.1 (i) that \( L^\infty(\Omega) \subseteq (D_{\varphi_1})^* \), where the inclusion is understood in the sense that for every \( f \in L^\infty(\Omega) \) there exists \( \bar{f} \in (D_{\varphi_1})^* \) such that \( \bar{f}[\xi] = \int_\Omega f \xi \, dx \) for all \( \xi \in D_{\varphi_1} \).
In the case \( p = q = 2 \), the classical Fredholm alternative asserts that (1.16) possesses a (nonunique) solution if and only if \( f[\varphi_1] = 0 \). In the nonlinear case \( p = q \neq 2 \), the situation is drastically different. In particular, the existence might occur for some \( f \) with \( f[\varphi_1] \neq 0 \). We refer to [12, 34, 35] for an overview. Note also that the problem (1.16) with \( f \equiv 0 \) is a nonlinear (but homogeneous) eigenvalue problem (1.6) corresponding to \( \lambda_1 \), see [5, 18].

The improved Friedrichs inequality (1.9) allows to give a simple proof of the following existence result which can be regarded as a particular case of the generalized Fredholm alternative.

**Theorem 1.4.** Let \( p > q \geq 2 \) and (A) be satisfied. Let \( f \in (D_{\varphi_1})^* \setminus \{0\} \) be such that \( f[\varphi_1] = 0 \). Then (1.16) possesses a weak solution.

The proof of Theorem 1.4 is placed in Section 5.

### 1.3. Alternative improvements of the Friedrichs inequality

The inequalities (1.4), (1.9), and (1.14) are not the only possible refinements of the Friedrichs (Poincaré, when \( p = q \)) inequality (1.2). The following quantification was kindly indicated to us by L. Brasco and it is a consequence of the enhanced hidden convexity established in [7] (see [7, Eq. (2.10)]) and also [25, Eq. (2), p. 178]). We place the proof in Section 6.

**Theorem 1.5.** Let \( p \geq q > 1 \) and \( \Omega \) be a domain of finite measure. Then there exists \( C = C(p,q,\Omega) > 0 \) such that

\[
\|\nabla u\|_p^p - \lambda_1 \|u\|_q^p \geq C \|u\|_q^p \max_{t \in [0,1]} \left[ t(1-t) \int_\Omega R_p \left( \frac{\|\varphi_1\|_q}{\|u\|_q}, \varphi_1; t \right) \, dx \right]
\]

(1.17)

for any \( u \in W^{1,p}_0(\Omega) \), where

\[
R_p(v,w;t) = \begin{cases} 
\frac{|wv - v^p|}{(1-t)v^p + tw^p}, & \text{if } p \geq 2, \\
\frac{(\|w\|_q^2 + \|v\|_q^2)^{t-2}}{(1-t)v^p + tw^p} \|wv - v^p\|^2, & \text{if } 1 < p < 2.
\end{cases}
\]

Moreover, in the homogeneous case \( p = q \), the following simpler inequality holds:

\[
\int_\Omega |\nabla u|^p \, dx - \lambda_1 \int_\Omega |u|^p \, dx \geq C \int_\Omega \|\varphi_1\|_{\Omega}^p, \varphi_1; 1 \, dx.
\]

(1.19)

In particular, if \( p = q \geq 2 \), then

\[
\int_\Omega |\nabla u|^p \, dx - \lambda_1 \int_\Omega |u|^p \, dx \geq C \int_\Omega \left( \frac{u}{\varphi_1} \right)^p \varphi_1^p \, dx.
\]

(1.20)

We note that (1.19) is not a direct corollary of (1.17). It can be observed that the right-hand sides of the inequalities (1.17) and (1.19) are 0-homogeneous with respect to \( \varphi_1 \) and \( p \)-homogeneous with respect to \( u \). Moreover, they measure the distance between \( \varphi_1 \) and \( u \), although in a different way than in the inequalities (1.4), (1.9), and (1.14). Compare also (1.3), (1.15), and (1.20) (with \( p = 2 \)). Applications of (1.17) and (1.19) are yet to be found.

By sacrificing the optimality of \( \lambda_1 \) in (1.2), another improvements of the Poincaré and Friedrichs inequalities can be found as refinements of the Hardy inequality; for instance,

\[
\int_\Omega |\nabla u|^p \, dx - \tilde{\lambda} \left( \int_\Omega |u|^q \, dx \right)^\frac{p}{q} \geq \frac{(N - p)p}{p^p} \int_\Omega \frac{|u|^p}{|x|^p} \, dx,
\]

(1.21)

see [8, Theorem 4.1 and Extension 4.3] for \( N \geq 3, p = 2, \) and \( 1 < q < 2^* \), and [20, Theorem 1] for \( 1 < p < N \) and \( 1 < q \leq p \). We also refer to [2] and references therein for a discussion about similar
inequalities with the right-hand side containing a term of the form \( \int_{\Omega} \frac{|u|^{p}}{\text{dist}(x,\Omega)^{p}} \, dx \). As was noted, the value \( \lambda \) on the left-hand side of (1.21) and related inequalities is compelled to satisfy \( \lambda > \lambda_1 \), which can be seen by substituting a minimizer of \( \lambda \) to get the equality \( \lambda \) scaling) minimizer of \( W \). We know from (1.21) that \( \lambda \) is a unique (modulo scaling) minimizer of \( \lambda_1 \), the equality \( \lambda[u] = 0 \) holds if and only if \( u = t\varphi_1 \), \( t \in \mathbb{R} \).

Throughout this section, we always assume that \( p \geq q \geq 2 \) with \( p > 2 \), and that \( \Omega \) is a domain of finite measure, unless otherwise explicitly stated.

2. Preliminaries

Let us introduce a functional \( J \in C^2(W_0^{1,p}(\Omega), \mathbb{R}) \) associated with the Friedrichs inequality (1.2):

\[
J[u] = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda_1}{p} \left( \int_{\Omega} |u|^q \, dx \right)^\frac{p}{q}.
\]

We know from (1.2) that \( \lambda[u] \geq 0 \) for all \( u \in W_0^{1,p}(\Omega) \). Moreover, since \( \varphi_1 \) is a unique (modulo scaling) minimizer of \( \lambda_1 \), the equality \( J[u] = 0 \) holds if and only if \( u = t\varphi_1 \), \( t \in \mathbb{R} \).

The Gateaux derivative of \( J[u] \) in a direction \( v \in W_0^{1,p}(\Omega) \) is given by

\[
DJ[u](v) = \int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla v) \, dx - \lambda_1 \left( \int_{\Omega} |u|^q \, dx \right)^\frac{p}{q} \int_{\Omega} |u|^{q-2} uv \, dx.
\]

In particular, by (1.6), we get \( DJ[\varphi_1](v) = 0 \).

Let us now take any \( v \in W_0^{1,p}(\Omega) \) and consider the function \( f(s) = J[\varphi_1 + sv] \), \( s \in \mathbb{R} \). Noting that this function is of class \( C^2(\mathbb{R}) \), we can apply the Taylor formula with the reminder in the integral form,

\[
f(1) = f(0) + f'(0) + \int_0^1 f''(s)(1-s) \, ds,
\]

to get

\[
J[\varphi_1 + v] = \int_0^1 D^2J[\varphi_1 + sv](v, v) (1-s) \, ds, \tag{2.1}
\]

thanks to \( f(0) = J[\varphi_1] = 0 \) and \( f'(0) = DJ[\varphi_1](v) = 0 \). The right-hand side of (2.1) suggests to consider the following quadratic form for any \( u, v \in W_0^{1,p}(\Omega) \):

\[
Q_u(v, v) = \int_0^1 D^2J[\varphi_1 + su](v, v) (1-s) \, ds
\]

\[
g = \int_0^1 \left( \int_{\Omega} A(\nabla \varphi_1 + su) \nabla v \, dx \right) (1-s) \, ds
\]

\[- \lambda_1(q-1) \int_0^1 \left( \int_{\Omega} |\varphi_1 + su|^q \, dx \right)^\frac{p}{q} \left( \int_{\Omega} |\varphi_1 + su|^{q-2} v^2 \, dx \right) (1-s) \, ds
\]

\[- \lambda_1(p-q) \int_0^1 \left( \int_{\Omega} |\varphi_1 + su|^q \, dx \right)^\frac{p}{q} \left( \int_{\Omega} |\varphi_1 + su|^{q-2}(\varphi_1 + su) v \, dx \right)^2 (1-s) \, ds.
\]
Here, $A$ is a symmetric $N \times N$-matrix defined as
\[
A(a) = |a|^{p-2} \left( I + (p-2)\frac{a \otimes a}{|a|^2} \right), \quad a \in \mathbb{R}^N \setminus \{0\},
\]
where $a \otimes a$ is a matrix defined as $a \otimes a = (a_i a_j)_{i,j=1}^N$, and we set $A(0)$ to be a zero matrix. The matrix $A$ corresponds to the linearization of the $p$-Laplacian and it is not hard to see that
\[
|a|^{p-2}|b|^2 \leq (A(a)b, b) \leq (p-1)|a|^{p-2}|b|^2
\]
for any $a, b \in \mathbb{R}^N$, see, e.g., [33, Section 3].

Since we might have $p < 2q$, the following remark is necessary. If $\int_\Omega |\varphi_1 + s_0 u|^q \, dx = 0$ for some $u \in W_0^{1,p}(\Omega)$ and $s_0 \in (0, 1)$, then $u = -s_0^{-1} \varphi_1$ a.e. in $\Omega$. In this case, we have
\[
\int_0^1 \left( \int_\Omega |\varphi_1 + s u|^q \, dx \right)^{\frac{2-2q}{q}} \left( \int_\Omega |\varphi_1 + s u|^{q-2}(\varphi_1 + s u) v \, dx \right)^2 \left( 1 - s \right) ds
\]
\[
= \left( \int_\Omega |\varphi_1|^q \, dx \right)^{\frac{2-2q}{q}} \left( \int_\Omega |\varphi_1|^{q-1} v \, dx \right)^2 \int_0^1 \left| 1 - s s_0^{-1} |p-2| (1-s) ds, \right.
\]
where the integral over $s$ in (2.3) is finite. That is, the quadratic form $Q_u(v,v)$ is well-defined for any $u, v \in W_0^{1,p}(\Omega)$, regardless the relation between $p$ and $2q$.

Recalling from (2.1) that $J[\varphi_1 + v] = Q(v,v)$, we deduce from (1.2) that
\[
Q(v,v) \geq 0 \quad \text{for any} \quad v \in W_0^{1,p}(\Omega), \quad \text{and} \quad Q_{\varphi_1}(\varphi_1, \varphi_1) = 0. \tag{2.4}
\]

Due to the homogeneity, we have $Q_{tv}(v,v) = |t|^{-2} Q_{tv}(tv,tv) \geq 0$ for all $t \in \mathbb{R} \setminus \{0\}$, which yields, by the continuity,
\[
Q_0(v,v) \geq 0 \quad \text{for any} \quad v \in W_0^{1,p}(\Omega), \quad \text{and} \quad Q_0(\varphi_1, \varphi_1) = 0, \tag{2.5}
\]
where
\[
Q_0(v,v) = \frac{1}{2} \int_\Omega \langle A(\nabla \varphi_1) \nabla v, \nabla v \rangle \, dx - \frac{\lambda_1(q-1)}{2} \left( \int_\Omega \varphi_1^q \, dx \right)^{\frac{2}{q-2}} \int_\Omega \varphi_1^{-2} v^2 \, dx
\]
\[
- \frac{\lambda_1(p-q)}{2} \left( \int_\Omega \varphi_1^q \, dx \right)^{\frac{2}{q-2}} \left( \int_\Omega \varphi_1^{-1} v \, dx \right)^2. \tag{2.6}
\]

### 2.2. Weighted space and embeddings

Let us now discuss a natural domain for the quadratic form $Q_0$. We see from (2.2) that
\[
\|v\|_{\varphi_1}^2 \leq \int_\Omega \langle A(\nabla \varphi_1) \nabla v, \nabla v \rangle \, dx \leq (p-1)\|v\|_{\varphi_1}^2
\]
for any $v \in W_0^{1,p}(\Omega)$, where
\[
\|v\|_{\varphi_1} := \left( \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla v|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.7}
\]

The seminorm (2.7) is actually a norm in $W_0^{1,p}(\Omega)$, which follows, e.g., from the inequalities (A.2) below, or from the fact that the critical set $\{ x \in \Omega : |\nabla \varphi_1(x)| = 0 \}$ has zero measure (see [27] or [6, 10]). Following [33, Section 2.1], we denote by $D_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. Clearly, $D_{\varphi_1}$ is a Hilbert space with the scalar product induced by (2.7).
Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, it is also dense in $D_{\varphi_1}$. Taking into account that the embedding $D_{\varphi_1} \hookrightarrow L^p(\Omega)$ is continuous (which follows from Lemma 2.1 (i) below), we conclude that $D_{\varphi_1}$ coincides with the spaces $X_0^{1,2}(\Omega; |\nabla \varphi_1|^{p-2})$ (in the notation of [6]) and $H_0^{1,2}(\Omega)$ with $p = |\nabla \varphi_1|^{p-2}$ (in the notation of [10]), which are defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_2 + \| \cdot \|_{\varphi_1}$.

In the following lemma, we collect several embedding results for the space $D_{\varphi_1}$ which are essentially based on [6, 10, 19, 33]. We denote $r^* = \frac{N r}{N - r}$ if $r < N$ and $r^* = \infty$ if $r \geq N$.

**Lemma 2.1.** Let $p > 2$, $1 < q < p^*$, and $(A)$ be satisfied. Then the following assertions hold:

(i) $D_{\varphi_1} \hookrightarrow L^\kappa(\Omega)$ compactly for some $\kappa \in (2, 2^*)$;

(ii) $D_{\varphi_1} \hookrightarrow W_0^{1,\theta}(\Omega)$ continuously for some $\theta > 1$;

(iii) $W_0^{1,p}(\Omega) \hookrightarrow D_{\varphi_1}$ continuously.

In fact, in the proofs of our main theorems, we use the compactness of the embedding $D_{\varphi_1} \hookrightarrow L^\kappa(\Omega)$ only in the case $\kappa = 2$. However, we prove the more general statement, as it is interesting on its own. We place the proof of Lemma 2.1 in Appendix A.

**Remark 2.2.** The continuous embedding $D_{\varphi_1} \hookrightarrow W_0^{1,\theta}(\Omega)$ guarantees that no function $u \in D_{\varphi_1}$ can be of the form $u = \varphi \chi_S$ a.e. in $\Omega$, where $S \subset \Omega$ is a (Lebesgue) measurable set satisfying $0 < |S| < |\Omega|$ and $\chi_S$ is the characteristic function of $S$, as it follows, e.g., from [15, Theorem 2, p. 164]. In particular, this shows that the assumption [33, (H2)] (or, equivalently, [16, (H2)]) is always satisfied whenever $(A)$ holds, as conjectured in [33, Section 2.1].

### 2.3. Characterization of $\lambda_1$ via the quadratic form

It can be shown exactly as in the proof of [6, Proposition 3.5] that any sequence $\{v_n\} \subset C_0^\infty(\Omega)$ converging in $D_{\varphi_1}$ to some $v \in D_{\varphi_1}$ satisfies

$$
\lim_{n \to \infty} \int_\Omega \langle A(\nabla \varphi_1)\nabla v_n, \nabla v_n \rangle \, dx = \int_\Omega \langle A(\nabla \varphi_1)\nabla v, \nabla v \rangle \, dx.
$$

Therefore, recalling that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ is dense in $D_{\varphi_1}$, and using the continuity of the embedding $D_{\varphi_1} \hookrightarrow L^2(\Omega)$ which follows from Lemma 2.1 (i) (under the assumption $(A)$), we deduce from (2.5) that

$$
Q_0(v, v) \geq 0 \quad \text{for any } v \in D_{\varphi_1}.
$$

Define the following critical value:

$$
\mu_1 = \inf_{v \in D_{\varphi_1} \setminus \{0\}} \left\{ \frac{\int_\Omega \langle A(\nabla \varphi_1)\nabla v, \nabla v \rangle \, dx}{(q - 1) \left( \int_\Omega \varphi_1^q \, dx \right)^{\frac{p-q}{q}}} \right\}.
$$

The inequality (2.9) and the definition (2.6) of $Q_0(v, v)$ immediately yield $\mu_1 = \lambda_1$, which is therefore an alternative characterization of $\lambda_1$. We state this result explicitly.

**Lemma 2.3.** Let $p \geq q \geq 2$ with $p > 2$, and $(A)$ be satisfied. Then $\mu_1 = \lambda_1$.

One of the most essential parts in our analysis is the following nondegeneracy result showing that the functions $u = t \varphi_1$ (with $t \in \mathbb{R}$) are the only elements of $D_{\varphi_1}$ satisfying $Q_0(u, u) = 0$. In the case $p = q > 2$, this result is given by [16, Lemma 5.2] (or, equivalently, [33, Proposition 4.4]) in combination with Remark 2.2, and by [6, Proposition 3.5].
Lemma 2.4. Let $p \geq q \geq 2$ with $p > 2$, and (A) be satisfied. Then the set of minimizers of $\mu_1$ is exhausted by $t \varphi_1$ with $t \neq 0$.

Proof. Thanks to the references provided above, it is sufficient to consider only the case $p > q \geq 2$. We argue in a way similar to that from [6, Proposition 3.5] and only the last part of our proof is different (in fact, simpler, thanks to $p > q$). First, in view of (2.8), $\mu_1$ can be equivalently characterized through the minimization over $C_0^\infty(\Omega)$ instead of $\mathcal{D}_{\varphi_1}$, i.e.,

$$
\mu_1 = \inf_{v \in C_0^\infty(\Omega) \setminus \{0\}} \left\{ \frac{\int_\Omega \langle A(\nabla \varphi_1) \nabla v, \nabla v \rangle \, dx}{(q - 1) \left( \int_\Omega \varphi_1^q \, dx \right) \frac{p - q}{q} \int_\Omega \varphi_1^{q-2} v^2 \, dx + (p - q) \left( \int_\Omega \varphi_1^q \, dx \right) \frac{p - q}{q} \int_\Omega \varphi_1^{q-1} v \, dx} \right\}.
$$

Suppose, by contradiction, that there exists a minimizer $v \in \mathcal{D}_{\varphi_1}$ of $\mu_1$ such that $v \notin \mathbb{R} \varphi_1$. From the properties of quadratic forms, it is not hard to observe that any linear combination of $\varphi_1$ and $v$ also minimizes $\mu_1$. Alternatively, one could directly calculate $Q_0(\varphi_1 + \delta v, \varphi_1 + \delta v)$ for $\delta \in \mathbb{R}$ by noting that the chain of equalities

$$
\int_\Omega \langle A(\nabla \varphi_1) \nabla \varphi_1, \nabla w \rangle \, dx = \int_\Omega \langle A(\nabla \varphi_1) \nabla w, \nabla \varphi_1 \rangle \, dx
$$

holds for any $w \in W_0^{1,p}(\Omega)$, to deduce that $Q_0(\varphi_1 + \delta v, \varphi_1 + \delta v) = 0$. As a consequence, if $\int_\Omega \varphi_1^{-1} v \, dx \neq 0$, then there exists $\delta_0 \neq 0$ such that $\int_\Omega \varphi_1^{-1}(v + \delta_0 v) \, dx = 0$. Therefore, we can assume, without loss of generality, that the minimizer $v$ satisfies $Q_0(v, v) = 0$ and $\int_\Omega \varphi_1^{-1} v \, dx = 0$.

Let $\{v_n\} \subset C_0^\infty(\Omega)$ be a sequence converging to $v$ in $\mathcal{D}_{\varphi_1}$. Using $v_n^2/\varphi_1$ as test functions in (2.11), we apply the Picone inequality from [6, Lemma A.1] to obtain

$$
(p - 1) \lambda_1 \left( \int_\Omega \varphi_1^q \, dx \right) \frac{p - q}{q} \int_\Omega \varphi_1^{q-2} v_n^2 \, dx = \int_\Omega \langle A(\nabla \varphi_1) \nabla \varphi_1, \nabla \left( \frac{v_n^2}{\varphi_1} \right) \rangle \, dx,
$$

(2.12)

Let us now pass to the limit in (2.12) as $n \to \infty$. The passage to limit on the left-hand side (i.e., the weak term in (2.12)) can be performed thanks to Lemma 2.1 (i), while the right-hand side of (2.12) converges thanks to (2.8). Thus, since $v$ satisfies $Q_0(v, v) = 0$ and $\int_\Omega \varphi_1^{-1} v \, dx = 0$, we conclude from (2.6) and (2.12) that

$$
\lambda_1 (p - 1) \left( \int_\Omega \varphi_1^q \, dx \right) \frac{p - q}{q} \int_\Omega \varphi_1^{q-2} v^2 \, dx \leq \int_\Omega \langle A(\nabla \varphi_1) \nabla v, \nabla v \rangle \, dx
$$

(2.13)

$$
\leq \lambda_1 (q - 1) \left( \int_\Omega \varphi_1^q \, dx \right) \frac{p - q}{q} \int_\Omega \varphi_1^{q-2} v^2 \, dx.
$$

Since $\varphi_1 > 0$ in $\Omega$, we have $\int_\Omega \varphi_1^{q-2} v^2 \, dx > 0$, and hence we arrive at a contradiction to the assumption $p > q$.

3. Proof of Theorem 1.1

As we discussed in Section 1, the proof of Theorem 1.1 splits into two cases: when $u \in C_\gamma$ (for a sufficiently small $\gamma > 0$) and when $u \in C'_\gamma$ (for any $\gamma > 0$), where the cones $C_\gamma$ and $C'_\gamma$ are defined in (1.11) and (1.12), respectively. The analysis in the cone $C_\gamma$ is more subtle than that in $C'_\gamma$ since functions from $C_\gamma$ are “close” to the subspace $\mathbb{R} \varphi_1$, and we start the proof with this case.

For convenience, throughout the proof, we denote by $C > 0$ a universal constant whose value may vary from inequality to inequality, but its exact value is irrelevant for our purposes.
3.1. Proof in $C_\gamma$

For brevity, we introduce the following notation:

$$P_1(t,v) = \int_0^1 \left( \int_\Omega \langle A(\nabla \varphi_1 + st \nabla v), \nabla v \rangle \, dx \right) (1-s) \, ds,$$

$$P_0(t,v) = (q-1) \int_0^1 \left( \int_\Omega |\varphi_1 + stv|^q \, dx \right)^\frac{q}{q-2} \left( \int_\Omega |\varphi_1 + stv|^{q-2}v^2 \, dx \right) (1-s) \, ds$$

$$+ (p-q) \int_0^1 \left( \int_\Omega |\varphi_1 + stv|^q \, dx \right)^\frac{q}{q-2} \left( \int_\Omega |\varphi_1 + stv|^{q-2}(\varphi_1 + stv)v \, dx \right)^2 (1-s) \, ds.$$  

Accordingly, in the case $t = 0$, we have

$$P_1(0,v) = \frac{1}{2} \int_\Omega \langle A(\nabla \varphi_1), \nabla v \rangle \, dx,$$

$$P_0(0,v) = \frac{q-1}{2} \left( \int_\Omega \varphi_1^q \, dx \right)^\frac{q}{q-2} \left( \int_\Omega \varphi_1^{q-2}v^2 \, dx \right) + \frac{p-q}{2} \left( \int_\Omega \varphi_1^q \, dx \right)^\frac{q}{q-2} \left( \int_\Omega \varphi_1^{q-2}v \, dx \right)^2.$$  

In view of the inequalities in (2.4) and (2.5), the quadratic form $Q_{tv}(v,v)$ satisfies

$$Q_{tv}(v,v) = P_1(t,v) - \lambda_1 P_0(t,v) \geq 0 \quad \text{for all } t \in \mathbb{R}, \ v \in W^{1,p}_0(\Omega).$$  

(3.4)

Observe that $P_0(t,v) > 0$ for any $t \in \mathbb{R}$ and $v \in W^{1,p}_0(\Omega) \setminus \{0\}$, cf. (2.3). We will also need the following inequality given by [16, Lemma A.2] (see also [16, (5.3)]): there exists $C > 0$ such that

$$C \left( \int_\Omega |\varphi_1|^{p-2} |\nabla v|^2 \, dx + |t|^{p-2} \int_\Omega |v|^p \, dx \right) \leq P_1(t,v) \quad \text{for all } t \in \mathbb{R}, v \in W^{1,p}_0(\Omega).$$

(3.5)

For any $\gamma > 0$, we consider the minimization problem

$$\widetilde{\Lambda}_\gamma = \inf \left\{ \frac{P_1(1,v)}{P_0(1,v)} : v \in W^{1,p}_0(\Omega) \setminus \{0\}, \ ||\nabla v||_p \leq \gamma, \ \int_\Omega \varphi_1^{q-1}v \, dx = 0 \right\}.$$  

(3.6)

It is readily seen from the inequality in (3.4) that $\lambda_1 \leq \widetilde{\Lambda}_\gamma$. Moreover, $\widetilde{\Lambda}_\gamma \leq \Lambda_{\gamma_2}$ provided $\gamma_2 \leq \gamma_1$, thanks to the “monotonicity” of the constraint $||\nabla v||_p \leq \gamma$ with respect to $\gamma$. The main property of $\Lambda_{\gamma_0}$ needed to prove Theorem 1.1 in $C_\gamma$ consists in the fact that $\lambda_1 < \Lambda_\gamma$ for any sufficiently small $\gamma > 0$. More precisely, we have the following separation result.

**Proposition 3.1.** Let $p \geq q \geq 2$ with $p > 2$, and (A) be satisfied. Then there exists $\gamma_0 > 0$ such that $\lambda_1 < \Lambda_{\gamma_0}$.

**Proof.** Suppose, by contradiction, that there exists a decreasing sequence $\{\gamma_n\} \subset (0, \infty)$ such that $\gamma_n \rightarrow 0$ and $\lambda_1 = \Lambda_{\gamma_n}$. Let $\{v_n\} \subset W^{1,p}_0(\Omega) \setminus \{0\}$ be a sequence which satisfies $||\nabla v_n||_p \rightarrow 0$, $\int_\Omega \varphi_1^{q-1}v_n \, dx = 0$, and

$$\frac{P_1(1,v_n)}{P_0(1,v_n)} \rightarrow \lambda_1 \quad \text{as } n \rightarrow \infty.$$  

(3.7)

The existence of such a sequence follows from the diagonal argument performed over minimizing sequences of $\Lambda_{\gamma_n}$ along $n$.

We pass to the normalized sequence $\{w_n\}$ consisting of functions $w_n = v_n/t_n$, where $t_n \in \mathbb{R} \setminus \{0\}$ is chosen to satisfy $P_0(t_n, w_n) = 1$. Such $t_n$ always exists since $0 < P_0(1,v_n) = t_n^2 v_n^2_0 P_0(t_n, w_n)$. More precisely, $t_n^2 = P_0(1,v_n)$. Noting that $P_1(1,v_n) = t_n^2 P_1(t_n, w_n)$, we have $P_1(t_n, w_n) \rightarrow \lambda_1$, and the “orthogonality” condition $\int_\Omega \varphi_1^{q-1}w_n \, dx = 0$ is satisfied.
Let us observe that \( t_n \to 0 \). Indeed, thanks to the convergence \( \|\nabla v_n\|_p \to 0 \), we apply the triangle inequality and the Hölder inequality to estimate \( P_0(1, v_n) \) from above in terms of \( \|\nabla v_n\|_p \) and hence to deduce that \( P_0(1, v_n) \to 0 \), which yields \( t_n \to 0 \).

Since \( P_1(t_n, w_n) \to \lambda_1 \), the inequality (3.5) gives the boundedness of \( \{w_n\} \) in \( D_{\varphi_1} \) and \( \{\frac{\nabla w_n}{\|\nabla w_n\|_p}\} \) in \( W_0^{1,p}(\Omega) \). Consequently, there exists \( w_0 \in D_{\varphi_1} \) such that the following convergences take place along a common subsequence of indices (see Lemma 2.1 and the Rellich-Kondrachov theorem):

(i) \( w_n \to w_0 \) weakly in \( D_{\varphi_1} \) and \( W_0^{1,q}(\Omega) \) for some \( q > 1 \), and strongly in \( L^2(\Omega) \).

(ii) \( \frac{\nabla w_n}{\|\nabla w_n\|_p} \to 0 \) weakly in \( W_0^{1,p}(\Omega) \) and strongly in \( L^r(\Omega) \), \( r \in (1, p^*) \). Here, the limit is zero since \( t_n \to 0 \) implies \( \frac{\nabla w_n}{\|\nabla w_n\|_p} \to 0 \) strongly in \( L^2(\Omega) \) by (i).

(iii) \( \frac{\nabla w_n}{\|\nabla w_n\|_p} \to 0 \) strongly in \( L^q(\Omega) \). Indeed, applying the Hölder inequality and using (i) and (ii), we get

\[
\|\frac{\nabla w_n}{\|\nabla w_n\|_p}\|_q^q = \int_\Omega \frac{\nabla w_n}{\|\nabla w_n\|_p} |\nabla w_n|^q dx \leq \left( \int_\Omega \frac{\nabla w_n}{\|\nabla w_n\|_p} |\nabla w_n|^p dx \right)^{\frac{q}{p}} \left( \int_\Omega \frac{\nabla w_n}{\|\nabla w_n\|_p} |\nabla w_n|^2 dx \right)^{\frac{q-2}{p}}
\]

\[
= \|\frac{\nabla w_n}{\|\nabla w_n\|_p}\|_p^{\frac{q}{p}} \|\nabla w_n\|_2^{\frac{q-2}{p}} \to 0.
\]

(iv) \( t_n w_n \to 0 \) strongly in \( W_0^{1,p}(\Omega) \) and \( L^r(\Omega) \), \( r \in (1, p^*) \), and a.e. in \( \Omega \), since \( v_n = t_n w_n \) and \( \|\nabla v_n\|_p \to 0 \).

Moreover, we have \( \int_\Omega \varphi_1^{q-1} w_0 dx = 0 \), thanks to (i).

Our aim now is to prove the following two facts:

\[
\lim_{n \to \infty} P_0(t_n, w_n) = P_0(0, w_0) = 1 \tag{3.8}
\]

and

\[
\liminf_{n \to \infty} P_1(t_n, w_n) \geq P_1(0, w_0). \tag{3.9}
\]

We start with the convergence (3.8) and recall that \( P_0(t_n, w_n) = 1 \). Since \( \int_\Omega \varphi_1^{q-1} w_0 dx = 0 \), we see from (3.3) that

\[
P_0(0, w_0) = \frac{q-1}{2} \left( \int_\Omega \varphi_1 dx \right)^{\frac{q-1}{q}} \int_\Omega \varphi_1^{q-2} w_0^2 dx.
\]

At the same time, in view of the \( L^q(\Omega) \)-convergence given in (iv), the triangle inequality yields

\[
\left( \int_\Omega |\varphi_1 + s t_n w_n|^q dx \right)^{\frac{1}{q}} \to \left( \int_\Omega \varphi_1 dx \right)^{\frac{1}{q}} > 0 \tag{3.10}
\]

uniformly with respect to \( s \in [0,1] \). Therefore, in order to establish (3.8), it is sufficient to prove that

\[
\int_\Omega |\varphi_1 + s t_n w_n|^{q-2} w_n^2 dx \to \int_\Omega \varphi_1^{q-2} w_0^2 dx \tag{3.11}
\]

and

\[
\int_\Omega |\varphi_1 + s t_n w_n|^{q-2} (\varphi_1 + s t_n w_n) w_n dx \to 0, \tag{3.12}
\]

both convergences being uniform with respect to \( s \in [0,1] \).

First, we justify (3.11). If \( q = 2 \), then the convergence is given by (i). So, assume that \( q > 2 \). We have

\[
\left| \int_\Omega |\varphi_1 + s t_n w_n|^{q-2} w_n^2 dx - \int_\Omega \varphi_1^{q-2} w_0^2 dx \right|
\]
3.8

3.15

3.11

3.10

3.16

3.1

3.14

and the fact that

(iv)

3.14

3.12

3.11

3.8

3.9

3.12

3.2

and recalling that

Ω

{[90x-88]and thus strongly in $L^\infty(\Omega \setminus E_\varepsilon)$. Consequently, $|\varphi_1 + st_n w_n|^{q-2} \lor 1^{q-2}$ strongly in $L^\infty(\Omega \setminus E_\varepsilon)$ and, clearly, uniformly with respect to $s \in [0, 1]$. Therefore, decomposing

\[
\int_{\Omega} \left| \varphi_1 + st_n w_n \right|^{q-2} - \varphi_1^{q-2} \left| w_n^2 \right| dx \leq \int_{\Omega \setminus E_\varepsilon} \left| \varphi_1 + st_n w_n \right|^{q-2} - \varphi_1^{q-2} \left| w_n^2 \right| dx + \int_{E_\varepsilon} \left| \varphi_1 + st_n w_n \right|^{q-2} - \varphi_1^{q-2} \left| w_n^2 \right| dx
\]

(3.14)

and recalling that \{w_n\} is bounded in $L^2(\Omega)$ by (i), we deduce that the first integral on the right-hand side of (3.14) converges to zero uniformly with respect to $s \in [0, 1]$. As for the second integral on the right-hand side of (3.14), we estimate it roughly as follows:

\[
\int_{E_\varepsilon} |\varphi_1 + st_n w_n|^{q-2} - \varphi_1^{q-2} |w_n|^2 dx \leq C \int_{E_\varepsilon} \varphi_1^{q-2} |w_n|^2 dx + C s \int_{E_\varepsilon} t_n^{q-2} |w_n|^2 dx. \quad (3.15)
\]

Since $\varphi_1$ is bounded in $\Omega$ and \{w_n\} converges in $L^2(\Omega)$ by (i), we get

\[
\int_{E_\varepsilon} \varphi_1^{q-2} |w_n|^2 dx = \int_{E_\varepsilon} \varphi_1^{q-2} |w_0|^2 dx + o(1),
\]

and $\int_{E_\varepsilon} \varphi_1^{q-2} |w_0|^2 dx$ converges to zero as $\varepsilon \to 0$ by the absolute continuity of the Lebesgue integral.

The second integral on the right-hand side of (3.15) converges to zero as $n \to \infty$ thanks to (iii). Summarizing, we obtain the desired convergence (3.11) by successively passing to the limit as $n \to \infty$ and then as $\varepsilon \to 0$.

Let us now discuss the convergence (3.12). Decomposing

\[
\int_{\Omega} |\varphi_1 + st_n w_n|^{q-2} (\varphi_1 + st_n w_n) w_n dx = \int_{\Omega} |\varphi_1 + st_n w_n|^{q-2} \varphi_1 w_n dx + st_n \int_{\Omega} |\varphi_1 + st_n w_n|^{q-2} w_n^2 dx,
\]

(3.16)

we see that the second integral on the right-hand side of (3.16) converges to zero in view of (3.11) since $t_n \to 0$. The convergence to zero of the first integral on the right-hand side of (3.16) can be established in much the same way as above, by appealing to Egorov’s theorem and the absolute continuity of the Lebesgue integral. We omit details. Combining the convergences (3.10), (3.11), and (3.12), we finish the proof of the convergence (3.8). Also, we observe that (3.8) implies $w_0 \neq 0$ a.e. in $\Omega$.

Let us turn to the justification of the weak lower semicontinuity type inequality (3.9):

\[
\liminf_{n \to \infty} \mathcal{P}_1(t_n, w_n) \geq \mathcal{P}_1(0, w_0). \quad (3.9)
\]

Expanding, for convenience, $\mathcal{P}_1(t_n, w_n)$ and $\mathcal{P}_1(0, w_0)$ given by (3.1) and (3.2), respectively, we have

\[
\mathcal{P}_1(t_n, w_n) = \int_0^1 \left[ \int_{\Omega} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 dx + (p - 2) \int_{\Omega} |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} (\nabla \varphi_1 + st_n \nabla w_n, \nabla w_n)^2 dx \right] (1 - s) ds
\]

and

\[
\mathcal{P}_1(0, w_0) = \frac{1}{2} \int_{\Omega} |\nabla \varphi_1|^{p-2} |\nabla w_0|^2 dx + \frac{p-2}{2} \int_{\Omega} |\nabla \varphi_1|^{p-4} (\nabla \varphi_1, \nabla w_0)^2 dx.
\]

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Thanks to Fatou’s lemma and the superadditivity of the limit inferior, the validity of (3.9) will be implied by the separate validity of the following two inequalities for any fixed $s \in [0, 1]$: \[ \liminf_{n \to \infty} \int_{\Omega} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 \, dx \geq \int_{\Omega} |\nabla \varphi_1|^{p-2} |\nabla w_0|^2 \, dx \] (3.17) and \[ \liminf_{n \to \infty} \int_{\Omega} |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} \left( |\nabla \varphi_1 + st_n \nabla w_n|^2 - |\nabla w_n|^2 \right) \, dx \geq \int_{\Omega} |\nabla \varphi_1|^{p-4} \left( |\nabla \varphi_1|^2 - |\nabla w_0|^2 \right) \, dx. \] (3.18)

Prior to the proof of (3.17) and (3.18), let us observe that the weak convergence $w_n \rightharpoonup w_0$ in $W^{1,\theta}_0(\Omega)$ stated in (i) implies the weak convergence $\nabla w_n \rightharpoonup \nabla w_0$ in $L^\theta(\Omega; \mathbb{R}^N)$, since the operator $T : W^{1,\theta}_0(\Omega) \to L^\theta(\Omega; \mathbb{R}^N)$ defined as $T(u) = \nabla u$ is linear and bounded. As a consequence, $\nabla w_n - \nabla w_0 \rightharpoonup 0$ weakly in $L^\theta(\Omega \setminus E; \mathbb{R}^N)$ for any measurable set $E \subset \Omega$.

Also, it follows from (iv) that $t_n \nabla w_n \rightharpoonup \vec{0}$ strongly in $L^p(\Omega; \mathbb{R}^N)$ and hence a.e. in $\Omega$. Thus, by Egorov’s theorem, for any $\varepsilon > 0$ there exists a measurable subset $E_\varepsilon \subset \Omega$ such that $|E_\varepsilon| < \varepsilon$ and $t_n \nabla w_n \rightharpoonup 0$ uniformly in $\Omega \setminus E_\varepsilon$ and thus strongly in $L^\infty(\Omega \setminus E_\varepsilon; \mathbb{R}^N)$. This yields \[ \nabla \varphi_1 + st_n \nabla w_n \rightharpoonup \nabla \varphi_1 \quad \text{strongly in } L^\infty(\Omega \setminus E_\varepsilon; \mathbb{R}^N). \] (3.19)

Let us define \[ E_{\varepsilon,k} = E_\varepsilon \cup \{ x \in \Omega : |\nabla w_0| \geq k \}. \]

We see that $\nabla w_0$ is bounded in $\Omega \setminus E_{\varepsilon,k}$, and hence \[ |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_0| \quad \text{strongly in } L^\infty(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N), \] (3.20) since $\Omega \setminus E_{\varepsilon,k} \subset \Omega \setminus E_\varepsilon$. In particular, in view of the boundedness of $\Omega \setminus E_{\varepsilon,k}$, the Hölder inequality guarantees that the convergence (3.20) is also strong in $L^\theta(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N)$.

Let us now prove the inequality (3.17). Since the function $a \mapsto |a|^2$ (with $a \in \mathbb{R}^N$) is convex, we have \[ |a_n|^2 \geq |a|^2 + 2(a, a_n - a) \quad \text{for any } a_n, a \in \mathbb{R}^N. \] (3.21)

Substituting $a_n = \nabla w_n$ and $a = \nabla w_0$ into (3.21), we obtain \[ \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 \, dx \geq \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_0|^2 \, dx + 2 \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 \, dx - \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1|^{p-2} |\nabla w_0|^2 \, dx. \] (3.22)

The first integral on the right-hand side of (3.22) converges to $\int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1|^{p-2} |\nabla w_0|^2 \, dx$ thanks to (3.20). Recalling that the convergence (3.20) is strong also in $L^\theta(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N)$ and $\nabla w_n - \nabla w_0 \rightharpoonup 0$ weakly in $L^\theta(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N)$, we deduce that the second integral on the right-hand side of (3.22) tends to zero. Thus, we obtain the inequalities

\[ \liminf_{n \to \infty} \int_{\Omega} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 \, dx \geq \liminf_{n \to \infty} \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-2} |\nabla w_n|^2 \, dx \geq \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1|^{p-2} |\nabla w_0|^2 \, dx. \]

Passing successively to the limit as $k \to \infty$ and then as $\varepsilon \to 0$, we conclude that (3.17) is satisfied.

Let us now prove the inequality (3.18) using the same strategy as above. Define, for brevity, the vector-functions \[ \Phi_n = |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} \left( |\nabla \varphi_1 + st_n \nabla w_n| + 2 |\nabla w_0| \right) \left( |\nabla \varphi_1| + st_n |\nabla w_n| \right). \]
and
\[ \Phi = |\nabla \varphi_1|^{p-4} (\nabla \varphi_1, \nabla w_0) \nabla \varphi_1. \]
Consider the function \( f(x; a, b) = |x|^{p-4}(x, a)(x, b) \), which is \((p - 2)\)-homogeneous with respect to \( x \). It is not hard to see that \( f \) is continuous in \( \mathbb{R}^{3N} \), and hence \( f \) is uniformly continuous in any bounded subset of \( \mathbb{R}^{3N} \). Componentwise, we have
\[ (\Phi_n)_k = f(\nabla \varphi_1 + st_n \nabla w_n, \nabla w_0, e_k) \quad \text{and} \quad (\Phi)_k = f(\nabla \varphi_1; \nabla w_0, e_k), \quad k = 1, 2, \ldots, N, \]
where \( e_k \) is the unit \( k \)-th coordinate vector. Therefore, in view of (3.19) and the uniform continuity of \( f \), we deduce that \( (\Phi_n)_k \to (\Phi)_k \) strongly in \( L^\infty(\Omega \setminus E_{\varepsilon,k}) \) for any \( k = 1, 2, \ldots, N \), and hence \( \Phi_n \to \Phi \) strongly in \( L^\theta(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N) \). In particular, since \( \Omega \setminus E_{\varepsilon,k} \) is bounded, we also have \( (\Phi_n)_k \to (\Phi)_k \) strongly in \( L^\theta(\Omega \setminus E_{\varepsilon,k}; \mathbb{R}^N) \).

Observe that the function \( a \mapsto (x, a)^2 \) is convex for any \( x \in \mathbb{R}^N \), which yields the following inequality:
\[ \langle x, a_n \rangle^2 \geq (x, a)^2 + 2(x, a)(x, a_n - a) \quad \text{for any } x, a_n, a \in \mathbb{R}^N. \quad (3.23) \]
Substituting \( x = \nabla \varphi_1 + st_n \nabla w_n, a_n = \nabla w_n, \) and \( a = \nabla w_0 \) into (3.23), we obtain
\[
\begin{align*}
\int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} (\nabla \varphi_1 + st_n \nabla w_n, \nabla w_n)^2 & \ dx \\
\geq & \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} (\nabla \varphi_1 + st_n \nabla w_n, \nabla w_0)^2 & \ dx \\
& + 2 \int_{\Omega \setminus E_{\varepsilon,k}} |\nabla \varphi_1 + st_n \nabla w_n|^{p-4} (\nabla \varphi_1 + st_n \nabla w_n, \nabla w_0) (\nabla \varphi_1 + st_n \nabla w_n, \nabla w_n - \nabla w_0) & \ dx \\
= & \int_{\Omega \setminus E_{\varepsilon,k}} (\Phi_n, \nabla w_0) & \ dx + 2 \int_{\Omega \setminus E_{\varepsilon,k}} (\Phi_n, \nabla w_n - \nabla w_0) & \ dx.
\end{align*}
\]
From this point, we argue exactly as in the proof of (3.17) above (cf. (3.22)) and establish the inequality (3.18). We omit details. Combining now (3.17) and (3.18), we finish the proof of the inequality (3.9).

Finally, recalling that \( w_0 \in \mathcal{D}_{\varphi_1} \setminus \{0\} \) and \( \int_{\Omega} \varphi_1^{q-1} w_0 \ dx = 0 \), we use \( w_0 \) as an admissible function for the definition (2.10) of \( \mu_1 \) and, taking into account (3.8), (3.9), and the convergence (3.7), we arrive at
\[ \mu_1 \leq \frac{\mathcal{P}_1(0, w_0)}{\mathcal{P}_0(0, w_0)} \leq \liminf_{n \to \infty} \frac{\mathcal{P}_1(t_n, w_n)}{\mathcal{P}_0(t_n, w_n)} = \liminf_{n \to \infty} \frac{\mathcal{P}_1(1, v_n)}{\mathcal{P}_0(1, v_n)} = \lambda_1 = \mu_1, \]
where the last equality is given by Lemma 2.3. Consequently, \( w_0 \) is a minimizer of \( \mu_1 \). However, according to Lemma 2.4, \( w_0 \) has to coincide with \( \varphi_1 \) up to a nonzero constant factor, which is impossible since \( \int_{\Omega} \varphi_1^{q-1} w_0 \ dx = 0 \). This contradiction finishes the proof of the claimed inequality \( \lambda_1 < \tilde{\lambda}_0 \) for a sufficiently small \( \gamma_0 > 0 \). \( \square \)

**Remark 3.2.** In the case \( p = q > 2 \), the result of Proposition 3.1 is given by [16, Lemma 5.2]. However, while the convergences (3.8) and (3.9) look intuitively correct, their rigorous justification appears to be delicate, cf. [16, Proof of Lemma 5.2, p. 965]. Our proof of (3.9) relies on the continuous embedding \( \mathcal{D}_{\varphi_1} \hookrightarrow W_{0, \theta}^{1,q}(\Omega) \) provided by Lemma 2.1 (ii) (which is essentially due to [6, Corollary 2.8] and [19, Lemma 1.3, p. 238], see a discussion in the proof of Lemma 2.1 (ii) in Appendix A below) and inspired by a convexity argument from the proof of [6, Theorem 1.1]. It is interesting to know if (3.9) can be proved without this embedding result.

**Remark 3.3.** Our proof of the equalities (3.8) uses the assumption \( q \geq 2 \) in a principal way. However, we anticipate that it is a technical assumption and (3.8), as well as Theorems 1.1 and 1.3, in general, hold true for any \( q \in (1, p] \). Details are left for future investigation.
With the help of Proposition 3.1, we establish the result of Theorem 1.1 in the cone $C_{\gamma}$ for a sufficiently small $\gamma > 0$.

**Lemma 3.4.** Let $p \geq q \geq 2$ with $p > 2$, and (A) be satisfied. Let $\gamma_0 > 0$ be given by Proposition 3.1. Then there exists $C = C(\gamma_0, p, q, \Omega) > 0$ such that the improved Friedrichs inequality (1.9) is satisfied for any $u \in C_{\gamma_0}$.

**Proof.** Take any $u \in C_{\gamma_0}$ and recall that this assumption reads as $\|\nabla u^+\|_p \leq \gamma_0|u|$. If $u^+ \equiv 0$ a.e. in $\Omega$ or $u^\perp = 0$, then there is nothing to prove. Assume that $u^+ \neq 0$ a.e. in $\Omega$ and $u^\perp \neq 0$. Consider the normalized function $\omega = u/u^\perp$. We have $\omega = \varphi_1 + \omega^\perp$, where $\omega^\perp = u^+ / u^\perp$ satisfies $0 < \|\nabla \omega^\perp\|_p \leq \gamma_0$ and $\int_{\Omega} \varphi_1^{q-1} \omega^\perp dx = 0$. That is, $\omega^\perp$ is an admissible function for the definition (3.6) of $\Lambda_\gamma$. Using our notation, we write the following chain of equalities:

$$
\frac{1}{p} \int_{\Omega} |\nabla (\varphi_1 + \omega^\perp)|^p dx - \frac{\lambda_1}{p} \left( \int_{\Omega} |\varphi_1 + \omega^\perp|^q dx \right)^{\frac{p}{q}} = J[\varphi_1 + \omega^\perp] = Q_{\omega^\perp}(\omega^\perp, \omega^\perp) = P_1(1, \omega^\perp) - \lambda_1 P_0(1, \omega^\perp).
$$

Proposition 3.1 guarantees that

$$
P_1(1, \omega^\perp) - \lambda_1 P_0(1, \omega^\perp) \geq \left( 1 - \frac{\lambda_1}{\Lambda_{\gamma_0}} \right) P_1(1, \omega^\perp) \geq \frac{C}{p} \left( \int_{\Omega} |\nabla \varphi_1|^{p-2} |\nabla \omega^\perp|^2 dx + \int_{\Omega} |\nabla \omega^\perp|^p dx \right),
$$

where the last inequality is given by (3.5), $C > 0$ does not depend on $\omega^\perp$, and the factor $1/p$ is chosen for convenience. Multiplying these two displayed expressions by $p|u|^p$, we pass back to the function $u = u^\perp \varphi_1 + u^\perp \in C_{\gamma_0}$ and finally arrive at the claimed inequality:

$$
\int_{\Omega} |\nabla u|^p dx - \lambda_1 \left( \int_{\Omega} |u|^q dx \right)^{\frac{p}{q}} \geq C \left( |u|^p - 2 \int_{\Omega} |\nabla \varphi_1|^{p-2} |\nabla u^\perp|^2 dx + \int_{\Omega} |\nabla u^\perp|^p dx \right). \quad \square
$$

### 3.2. Proof in $C'_{\gamma}$

Let us fix any $\gamma > 0$ and define the following critical value:

$$
\Lambda_{\gamma} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{(\int_{\Omega} |u|^q dx)^{\frac{p}{q}}} : u \in C'_{\gamma} \setminus \{0\} \right\}. \quad (3.24)
$$

Clearly, we have $\|\nabla u^\perp\|_p > 0$ for any $u \in C'_{\gamma} \setminus \{0\}$, since the latter reads as $\|\nabla u^\perp\|_p \geq \gamma |u|$.

Therefore, $\Lambda_{\gamma}$ can be equivalently characterized as

$$
\Lambda_{\gamma} = \inf \left\{ \frac{\int_{\Omega} |t \nabla \varphi_1 + \nabla v|^p dx}{(\int_{\Omega} |t \varphi_1 + v|^q dx)^{\frac{p}{q}}} : (t, v) \in \mathbb{R} \times W_0^{1,p}(\Omega), |t| \leq \gamma^{-1}, \|\nabla v\|_p = 1, \int_{\Omega} \varphi_1^{q-1} v dx = 0 \right\}. \quad (3.25)
$$

It is evident from (3.24) that $\lambda_1 \leq \Lambda_{\gamma}$. The main property of $\Lambda_{\gamma}$ needed for the proof of Theorem 1.1 in $C'_{\gamma}$ is the fact that $\lambda_1 < \Lambda_{\gamma}$. We provide the following slightly more general result which is valid without assuming $p > 2$, $q \geq 2$, and the regularity (A) of $\Omega$.

**Lemma 3.5.** Let $p \geq q > 1$, $\gamma > 0$, and $\Omega$ be a domain of finite measure. Then $\lambda_1 < \Lambda_{\gamma}$.

**Proof.** Suppose, by contradiction, that $\lambda_1 = \Lambda_{\gamma}$. Then there exists a sequence $\{(t_n, v_n)\}$ which satisfies the constraints in the definition (3.25) of $\Lambda_{\gamma}$ and such that

$$
\frac{\int_{\Omega} |t_n \nabla \varphi_1 + \nabla v_n|^p dx}{(\int_{\Omega} |t_n \varphi_1 + v_n|^q dx)^{\frac{p}{q}}} \to \lambda_1 \quad \text{as } n \to \infty. \quad (3.26)
$$

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In view of the boundedness of \( \{t_n\} \) and \( \{\|\nabla v_n\|_p\} \), we deduce the existence of \( t_0 \in [-\gamma^{-1}, \gamma^{-1}] \) and \( v_0 \in W^{1,p}_0(\Omega) \) such that \( t_n \to t_0 \), and \( v_n \to v_0 \) weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^q(\Omega) \), along a common subsequence of indices. Thus, \( t_n \varphi_1 + v_n \to t_0 \varphi_1 + v_0 \) weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^q(\Omega) \). Moreover, in view of the boundedness of \( \varphi_1 \) in \( \Omega \), we have \( \int_\Omega \varphi_1^{q-1} v_0 \, dx = 0 \). Let us show that \( t_0 \varphi_1 + v_0 \neq 0 \) a.e. in \( \Omega \). Suppose, by contradiction, that \( v_0 = -t_0 \varphi_1 \). If \( t_0 = 0 \), then we get a contradiction to the “orthogonality” \( \int_\Omega \varphi_1^{q-1} v_0 \, dx = 0 \) since \( \varphi_1 > 0 \) in \( \Omega \). If \( t_0 = 0 \), then we deduce from \( (3.26) \) and the strong convergence in \( L^q(\Omega) \) that \( \|\nabla v_n\|_p \to 0 \), which contradicts the assumption \( \|\nabla v_n\|_p = 1 \). Therefore, the function \( t_0 \varphi_1 + v_0 \) is nonzero.

Thanks to the weak lower semicontinuity of the norm in \( W^{1,p}_0(\Omega) \), we get

\[
0 < \int_\Omega |t_0 \nabla \varphi_1 + \nabla v_0|^p \, dx \leq \liminf_{n \to \infty} \int_\Omega |t_n \nabla \varphi_1 + \nabla v_n|^p \, dx,
\]

and hence the convergence \( (3.26) \) and the definition \( (1.1) \) of \( \lambda_1 \) yield \( t_n \varphi_1 + v_n \to t_0 \varphi_1 + v_0 \) strongly in \( W^{1,p}_0(\Omega) \). In particular, we deduce that \( t_0 \varphi_1 + v_0 \) is a minimizer of \( \lambda_1 \) and \( \|\nabla v_0\|_p = 1 \). Using the fact that the minimizer of \( \lambda_1 \) is unique modulo scaling, we obtain the existence of \( s \in \mathbb{R} \setminus \{0\} \) such that \( t_0 \varphi_1 + v_0 = s \varphi_1 \). Since \( v_0 \neq 0 \) a.e. in \( \Omega \), we get \( v_0 = (s - t_0) \varphi_1 \) and \( s \neq t_0 \). However, this again contradicts the “orthogonality” \( \int_\Omega \varphi_1^{q-1} v_0 \, dx = 0 \) since \( \varphi_1 > 0 \) in \( \Omega \).

**Remark 3.6.** It is evident from the proof that Lemma 3.5 remains valid without the assumption \( p \geq q \) provided that \( 1 < q < p^* \) and \( \varphi_1 \) is a unique (modulo scaling) minimizer of \( \lambda_1 \). In the case \( p = q > 1 \), Lemma 3.5 is given by [16, Lemma 5.1].

Using Lemma 3.5, we establish the result of Theorem 1.1 in the cone \( C'_\gamma \) for any \( \gamma > 0 \) and observe that it is valid for any \( q \in (1, p] \).

**Lemma 3.7.** Let \( p \geq q > 1 \) with \( p > 2 \), and (A) be satisfied. Then for any \( \gamma > 0 \) there exists \( C = C(\gamma, p, q, \Omega) > 0 \) such that the improved Friedrichs inequality \( (1.9) \) is satisfied for any \( u \in C'_\gamma \).

**Proof.** Let \( u \in C'_\gamma \). If \( u \equiv 0 \) a.e. in \( \Omega \), then there is nothing to prove. Assuming that \( u \in C'_\gamma \setminus \{0\} \), we see that \( u \) is an admissible function for the definition \( (3.24) \) of \( \Lambda_\gamma \), which yields

\[
\int_\Omega |\nabla u|^p \, dx - \lambda_1 \left( \int_\Omega |u|^q \, dx \right)^{\frac{p}{q}} \geq \left( 1 - \frac{\lambda_1}{\Lambda_\gamma} \right) \int_\Omega |\nabla u|^p \, dx.
\]

(3.27)

According to Lemma 3.5, we have \( \lambda_1 < \Lambda_\gamma \). We want to prove that the right-hand side of \( (3.27) \) can be estimated from below in the same form as the right-hand side of \( (1.9) \). If \( u \equiv 0 \), then this fact is evident. Assume that \( u \equiv 0 \neq 0 \). Dividing \( (3.27) \) by \( |u|^p \), we get

\[
\int_\Omega |\nabla(\varphi_1 + \omega^\perp)|^p \, dx - \lambda_1 \left( \int_\Omega |\varphi_1 + \omega^\perp|^q \, dx \right)^{\frac{p}{q}} \geq \left( 1 - \frac{\lambda_1}{\Lambda_\gamma} \right) \int_\Omega |\nabla(\varphi_1 + \omega^\perp)|^p \, dx,
\]

(3.28)

where \( \omega^\perp = u^\perp / |u| \) satisfies \( \|\nabla \omega^\perp\|_p \geq \gamma \) and \( \int_\Omega \varphi_1^{q-1} \omega^\perp \, dx = 0 \). First, we show the existence of \( C > 0 \) independent of \( \omega^\perp \) such that

\[
\int_\Omega |\nabla(\varphi_1 + \omega^\perp)|^p \, dx \geq C \int_\Omega |\nabla \omega^\perp|^p \, dx.
\]

(3.29)

Suppose, contrary to our claim, that there exists a sequence \( \{\omega^\perp_n\} \subset W^{1,p}_0(\Omega) \) satisfying \( \|\nabla \omega^\perp_n\|_p \geq \gamma \), \( \int_\Omega \varphi_1^{q-1} \omega^\perp_n \, dx = 0 \), and

\[
\int_\Omega |\nabla(\varphi_1 + \omega^\perp_n)|^p \, dx \leq \frac{1}{n} \int_\Omega |\nabla \omega^\perp_n|^p \, dx.
\]

(3.30)
It is not hard to see that \( \{ \| \nabla \omega_n^\perp \|_p \} \) is bounded, and hence there exists \( \omega_0^\perp \in W_0^{1,p}(\Omega) \) such that \( \omega_n^\perp \to \omega_0^\perp \) weakly in \( W_0^{1,p}(\Omega) \) and strongly in \( L^p(\Omega) \), up to a subsequence. In particular, we have

\[
\int_\Omega \nabla \omega_n^\perp \omega_0^\perp \, dx = 0.
\]

We deduce from (3.30) that

\[
0 \leq \int_\Omega |\nabla (\varphi_1 + \omega_n^\perp)|^p \, dx \leq \liminf_{n \to \infty} \int_\Omega |\nabla (\varphi_1 + \omega_n^\perp)|^p \, dx \leq \liminf_{n \to \infty} \frac{1}{n} \int_\Omega |\nabla \omega_n^\perp|^p \, dx = 0.
\]

Consequently, \( \int_\Omega |\nabla (\varphi_1 + \omega_n^\perp)|^p \, dx = 0 \), which leads to \( \omega_0^\perp = -\varphi_1 \). However, this is impossible since \( \int_\Omega \varphi_1^\perp \omega_0^\perp \, dx = 0 \). This proves the lower bound (3.29).

Second, we show the existence of \( C > 0 \) independent of \( \omega_\perp \) such that

\[
\int_\Omega |\nabla \omega_\perp|^p \, dx \geq C \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla \omega_\perp|^2 \, dx.
\]

(3.31)

We see that if \( \| \omega_\perp \|_{\varphi_1} \leq \gamma^{p/2} \), then (3.31) is satisfied with \( C = 1 \) since \( \| \nabla \omega_\perp \|_p \geq \gamma \) by the choice of \( u \). On the other hand, if \( \| \omega_\perp \|_{\varphi_1} \geq \gamma^{p/2} \), then the continuous embedding \( W_0^{1,p}(\Omega) \hookrightarrow \mathcal{D}_{\varphi_1} \) (see Lemma 2.1 (iii)) gives

\[
\int_\Omega |\nabla \omega_\perp|^p \, dx \geq C \left( \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla \omega_\perp|^2 \, dx \right)^{\frac{p}{2}} \geq C \gamma^{\frac{p-2}{2}} \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla \omega_\perp|^2 \, dx.
\]

Thus, the estimate (3.31) holds true.

Combining (3.28) with the lower bounds (3.29) and (3.31), and multiplying the resulting expression by \( |u|^p \), we pass back to the function \( u = u_\perp \varphi_1 + u_\perp \in C_0^\perp \) and deduce the desired inequality

\[
\int_\Omega |\nabla u|^p \, dx - \lambda_1 \left( \int_\Omega |u|^q \, dx \right)^{\frac{p}{q}} \geq C \left( |u|^p - \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla u_\perp|^2 \, dx + \int_\Omega |\nabla u_\perp|^p \, dx \right).
\]

The proof of Theorem 1.1 follows by combining Lemmas 3.4 and 3.7.

4. Generalization. Proof of Theorem 1.3

Let us denote the expression (without the constant \( C \)) on the right-hand side of the improved Friedrichs inequality (1.14) as \( M_l[u] \), i.e.,

\[
M_l[u] = \|l[u]\|_{\varphi_1}^p \|Pu\|^2_{\varphi_1} + \|\nabla Pu\|^p_p,
\]

where \( Pu = u - l[u] \varphi_1 \) is the projection to \( \text{Ker}(l) \), and \( l[\varphi_1] = 1 \).

Our goal is to prove the following equivalence statement.

**Proposition 4.1.** Let \( p \geq 2 \) and (A) be satisfied. Let \( l_1, l_2 : W_0^{1,p}(\Omega) \to \mathbb{R} \) be two distinct bounded linear functionals such that \( l_1[\varphi_1] = l_2[\varphi_1] = 1 \). Then there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 M_{l_1}[u] \leq M_{l_2}[u] \leq C_2 M_{l_1}[u]
\]

for any \( u \in W_0^{1,p}(\Omega) \).

Once Proposition 4.1 is established, Theorem 1.3 follows directly from Theorem 1.1 by taking one of the functionals, say, \( l_1 \), as \( l_1[u] = \int_\Omega \varphi_1^{l_1} u \, dx / \int_\Omega \varphi_1^{l_1} \, dx \). (In fact, in order to prove Theorem 1.1, it is sufficient to apply only one part of Proposition 4.1, i.e., that \( M_{l_2}[u] \leq C_2 M_{l_1}[u] \).

**Proof of Proposition 4.1.** We start with several auxiliary observations. Since \( l_1 \) and \( l_2 \) are distinct and not linearly dependent (in view of the normalization assumption \( l_1[\varphi_1] = l_2[\varphi_1] = 1 \)), their kernels do not coincide, and hence there exists \( v_1 \in W_0^{1,p}(\Omega) \) such that \( l_1[v_1] = 1 \) and \( l_2[v_1] = 0 \). Then we define \( v_2 = \varphi_1 - v_1 \), so that \( l_1[v_2] = 0 \) and \( l_2[v_2] = 1 \). Finally, we introduce \( \psi = v_1 - v_2 = \ldots \)
\[2v_1 - \varphi_1,\] and hence \(l_1[\psi] = 1\) and \(l_2[\psi] = -1.\) Notice that \(\psi \notin \mathbb{R}\varphi_1.\) With the help of \(\varphi_1\) and \(\psi,\) any \(u \in W^{1,p}_0(\Omega)\) can be decomposed as

\[u = \alpha \varphi_1 + \beta \psi + w,\]  
(4.2)

where

\[\alpha = \frac{l_1[u] + l_2[u]}{2}, \quad \beta = \frac{l_1[u] - l_2[u]}{2},\]

and \(w \in \text{Ker}(l_1) \cap \text{Ker}(l_2),\) i.e., \(l_1[w] = l_2[w] = 0.\) We have

\[l_1[u] = \alpha + \beta, \quad P_1 u = \beta(\psi - \varphi_1) + w,\]  
(4.3)

\[l_2[u] = \alpha - \beta, \quad P_2 u = \beta(\psi + \varphi_1) + w.\]  
(4.4)

Using (4.3) and (4.4), we rewrite \(M_{l_1}\) and \(M_{l_2}\) defined by (4.1) as

\[M_{l_1}[u] = |\alpha + \beta|^p - 2||\beta(\psi - \varphi_1) + w||_{\varphi_1}^2 + \|\beta \nabla(\psi - \varphi_1) + \nabla w||_p^p,\]

\[M_{l_2}[u] = |\alpha - \beta|^p - 2||\beta(\psi + \varphi_1) + w||_{\varphi_1}^2 + \|\beta \nabla(\psi + \varphi_1) + \nabla w||_p^p.\]

Since \(\varphi_1\) and \(\psi\) are fixed, we employ the triangle inequality to estimate \(M_{l_1}\) from above as follows:

\[M_{l_1}[u] \leq C_1|\alpha + \beta|^p - 2(||\beta||_{\varphi_1} + ||w||_{\varphi_1})^2 + C_1(||\beta|| + \|\nabla w\|_p)^p,\]  
(4.5)

where \(C_1 > 0\) does not depend on \(u.\) A similar estimate holds true also for \(M_{l_2}.\) On the other hand, recalling that \(W^{1,p}_0(\Omega) \subseteq D_{\varphi_1},\) \(l_1[\psi + \varphi_1] = 2,\) and \(w \in \text{Ker}(l_1),\) we apply Lemma 4.2 (see below) with \(l = l_1\) to provide the following lower estimate for \(M_{l_2}:\)

\[M_{l_2}[u] \geq C_2|\alpha - \beta|^p - 2(||\beta||_{\varphi_1} + ||w||_{\varphi_1})^2 + C_2(||\beta|| + \|\nabla w\|_p)^p,\]  
(4.6)

where \(C_2 > 0\) is independent of \(u.\) A similar estimate can be derived also for \(M_{l_1}.\)

Assume now, contrary to the claim of the proposition, that \(M_{l_1}\) and \(M_{l_2}\) are not equivalent. That is, noting that \(M_{l_1}\) and \(M_{l_2}\) are \(p\)-homogeneous, we can assume, without loss of generality, that there exists a sequence \(\{u_n\} \subset W^{1,p}_0(\Omega)\) such that \(M_{l_1}[u_n] = 1\) and \(M_{l_2}[u_n] \to 0.\) Using the decomposition (4.2), we write

\[u_n = \alpha_n \varphi_1 + \beta_n \psi + w_n.\]

We obtain from (4.6) that \(\beta_n \to 0\) and \(w_n \to 0\) strongly in \(W^{1,p}_0(\Omega).\) If \(p = 2,\) then we get a contradiction to (4.5) and \(M_{l_1}[u_n] = 1.\) Hence, assume that \(p > 2.\) The convergence \(w_n \to 0\) strongly in \(W^{1,p}_0(\Omega)\) implies that \(w_n \to 0\) strongly in \(D_{\varphi_1},\) see Lemma 2.1 (iii). We deduce from (4.5) and \(M_{l_1}[u_n] = 1\) that

\[\lim_{n \to \infty} |\alpha_n + \beta_n|^p - 2(||\beta_n|| + ||w_n||_{\varphi_1})^2 \geq C_1^{-1},\]

which yields \(\alpha_n \to \infty.\) But then

\[\lim_{n \to \infty} |\alpha_n - \beta_n|^p - 2(||\beta_n|| + ||w_n||_{\varphi_1})^2 \geq \lim_{n \to \infty} \frac{|\alpha_n - \beta_n|^p - 2}{|\alpha_n + \beta_n|^p - 2} \lim_{n \to \infty} \frac{|\alpha_n + \beta_n|^p - 2(||\beta_n|| + ||w_n||_{\varphi_1})^2}{|\alpha_n + \beta_n|^p - 2} \geq C_1^{-1},\]

which contradicts (4.6) in view of the assumption \(M_{l_2}[u_n] \to 0.\)

Let us provide the following auxiliary result about an inverse triangle type inequality which is used in the proof of Proposition 4.1.

**Lemma 4.2.** Let \(X\) be a Banach space. Let \(l : X \to \mathbb{R}\) be a bounded linear functional. Let \(\omega \in X\) be such that \(l[\omega] \neq 0.\) Then there exists a constant \(C > 0\) such that for any \(u \in \mathbb{R}\omega\) and \(v \in \text{Ker}(l)\) the following estimate holds:

\[||u + v|| \geq C(||u|| + ||v||).\]  
(4.7)
Proof. Assume, without loss of generality, that \( l[\omega] = 1 \). Let us take any \( u \in \mathbb{R} \omega \) and \( v \in \text{Ker}(l) \), and define \( \phi = u + v \). Since \( X = \mathbb{R} \omega \oplus \text{Ker}(l) \), we have \( \phi = l[\phi] \omega + P \phi \) and hence, by the uniqueness of the decomposition, \( u = l[\phi] \omega \) and \( v = P \phi \), where \( P : X \to \text{Ker}(l) \) is a projection operator defined as \( P \phi = \phi - l[\phi] \omega \). Then
\[
\|u\| + \|v\| = \|l[\phi]\| \|\omega\| + \|P \phi\| \leq 2 \|l[\phi]\| \|\omega\| + \|\phi\| \leq (2 \|l\|, \|\omega\| + 1) \|\phi\|,
\]
where \( \|\cdot\|_* \) is the operator norm. Denoting \( C = (2 \|l\|, \|\omega\| + 1)^{-1} \), we obtain (4.7).

\[\Box\]

5. Application. Proof of Theorem 1.4

The \( C^1(W^{1,p}_0(\Omega); \mathbb{R}) \)-energy functional associated with the problem (1.16) is given by
\[
E[u] = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda_1}{p} \left( \int_{\Omega} |u|^q \, dx \right)^{\frac{p}{q}} - f[u].
\]
Let us show that under the assumptions of Theorem 1.4 the functional \( E \) is bounded from below and satisfies \( E[u] \geq \sigma(1) \) as \( \|\nabla u\|_p \to \infty \). This will imply that \( E \) possesses a global minimizer. Throughout the proof, we denote by \( C > 0 \) a universal constant.

We decompose any \( u \in W^{1,p}_0(\Omega) \) as in (1.7), i.e., \( u = u^\parallel \varphi_1 + u^\perp \), where \( u^\parallel \) and \( u^\perp \) are defined by (1.8). By Lemma 2.1 (i), we have \( \|\nabla u^\parallel\|_p \geq C\|u^\parallel\|_{\varphi_1} \) and
\[
f[u] = f[u^\parallel] \leq \|f\|_\varphi \|u^\parallel\|_{\varphi_1} \leq C\|f\|_\varphi \|\nabla u^\parallel\|_p,
\]
where \( \|\cdot\|_\varphi \) stands for the operator norm. Applying Theorem 1.1 (see (1.10)), we deduce that
\[
E[u] \geq C \left( \|u^\parallel\|_{\varphi_1}^2 + \|\nabla u^\parallel\|_p^2 \right) - C\|f\|_\varphi \|\nabla u^\parallel\|_p \tag{5.1}
\]
and
\[
E[u] \geq C\|u^\parallel\|_{\varphi_1}^2 + C\|u^\parallel\|_{\varphi_1}^2 - \|f\|_\varphi \|u^\parallel\|_{\varphi_1}. \tag{5.2}
\]
We see from (5.1) or (5.2) that \( E \) is bounded from below. If \( \|\nabla u\|_p \to \infty \), then we have \( \|u^\parallel\| \to \infty \) or \( \|\nabla u^\parallel\|_p \to \infty \). In the latter case, (5.1) yields \( E[u] \to +\infty \). If \( \|u^\parallel\| \to \infty \) and \( \{u^\parallel\} \) is separated from zero, then \( E[u] \to +\infty \) as well, as it follows from (5.2). Finally, if \( \|u^\parallel\| \to \infty \) and \( \|u^\parallel\|_{\varphi_1} \to 0 \), then we deduce from (5.2) that \( E[u] \geq -\|f\|_2 \|u^\parallel\|_{\varphi_1} = o(1) \). On the other hand, since \( p, q > 1 \) and \( f \) is nonzero, it is not hard to observe that \( \inf_{W^{1,p}_0(\Omega)} E < 0 \).

Thus, any minimizing sequence for \( E \) is bounded in \( W^{1,p}_0(\Omega) \) and hence, by a standard argument, it converges strongly in \( W^{1,p}_0(\Omega) \) to a global minimizer of \( E \) which is a (weak) solution of (1.16), up to a subsequence.

\[\Box\]

6. Alternative improvement. Proof of Theorem 1.5

First, let us take any \( w \in W^{1,p}_0(\Omega) \setminus \{0\} \) such that
\[
\int_{\Omega} |w|^q \, dx = \int_{\Omega} \varphi_1^q \, dx. \tag{6.1}
\]
By the definition (1.1) of \( \lambda_1 \), we have
\[
\int_{\Omega} |\nabla w|^p \, dx \geq \int_{\Omega} |\nabla \varphi_1|^p \, dx. \tag{6.2}
\]
Set
\[
\sigma_t = ((1 - t)|w|^p + t\varphi_1^p)^\frac{1}{p}, \quad t \in [0, 1].
\]
By the enhanced hidden convexity [7, Eq. (2.10)] (see also [25, Eq. (2), p. 178] for the case \( p = 2 \) and \( t = 1/2 \), we have

\[
\int_{\Omega} |\nabla \sigma|^p \, dx + Ct(1-t) \int_{\Omega} \mathcal{R}_p(|w|, \varphi_1; t) \, dx \leq (1-t) \int_{\Omega} |\nabla w|^p \, dx + t \int_{\Omega} |\nabla \varphi_1|^p \, dx \leq \int_{\Omega} |\nabla w|^p \, dx,
\]

where the last inequality follows from (6.2) and \( \mathcal{R}_p \) is given by (1.18). In particular, \( \sigma_t \in W_0^{1,p}(\Omega) \) for all \( t \in [0, 1] \). On the other hand, by the concavity of the map \( s \mapsto s^q/p \) and the equality (6.1), we get

\[
\int_{\Omega} \sigma_t^q \, dx \geq (1-t) \int_{\Omega} |w|^q \, dx + t \int_{\Omega} \varphi_1^q \, dx = \int_{\Omega} |w|^q \, dx.
\]

(6.4)

Raising both sides of (6.4) to the power \( p/q \), multiplying the result by \( \lambda_1 \) and subtracting from (6.3), we obtain

\[
\int_{\Omega} |\nabla w|^p \, dx - \lambda_1 \left( \int_{\Omega} |w|^q \, dx \right)^\frac{p}{q} \geq Ct(1-t) \int_{\Omega} \mathcal{R}_p(|w|, \varphi_1; t) \, dx + \int_{\Omega} |\nabla \sigma|^p \, dx - \lambda_1 \left( \int_{\Omega} \sigma_t^q \, dx \right)^\frac{p}{q}
\]

for any \( t \in [0, 1] \). Using (1.2), we estimate the difference of integrals containing \( \sigma_t \) by zero and hence derive

\[
\int_{\Omega} |\nabla w|^p \, dx - \lambda_1 \left( \int_{\Omega} |w|^q \, dx \right)^\frac{p}{q} \geq C \max_{t \in [0, 1]} \left[ t(1-t) \int_{\Omega} \mathcal{R}_p(|w|, \varphi_1; t) \, dx \right]
\]

(6.5)

for any \( w \in W_0^{1,p}(\Omega) \) satisfying the normalization (6.1).

Now we take any \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \). Consider \( w = su \), where \( s = \| \varphi_1 \|_q / \| u \|_q \). Such \( w \) satisfies (6.1) and hence (6.5). We get

\[
s^p \left( \int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \left( \int_{\Omega} |u|^q \, dx \right)^\frac{p}{q} \right) \geq C \max_{t \in [0, 1]} \left[ t(1-t) \int_{\Omega} \mathcal{R}_p(s|u|, \varphi_1; t) \, dx \right],
\]

that is,

\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \left( \int_{\Omega} |u|^q \, dx \right)^\frac{p}{q} \geq C \frac{\| u \|^p}{\| u \|_q} \max_{t \in [0, 1]} \left[ t(1-t) \int_{\Omega} \mathcal{R}_p \left( \| \varphi_1 \|_q, u, \varphi_1; t \right) \, dx \right],
\]

which is exactly the improved Friedrichs inequality (1.17).

Let us now justify the improved Poincaré inequality (1.19). Take any \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \) and set, as above,

\[
\tilde{\sigma}_t = (1-t)|u|^p + t\varphi_1^p, \quad t \in [0, 1].
\]

By the enhanced hidden convexity, we have

\[
\int_{\Omega} |\nabla \tilde{\sigma}_t|^p \, dx + Ct(1-t) \int_{\Omega} \mathcal{R}_p(|u|, \varphi_1; t) \, dx \leq (1-t) \int_{\Omega} |\nabla u|^p \, dx + t \int_{\Omega} |\nabla \varphi_1|^p \, dx,
\]

(6.6)

and

\[
\lambda_1 \int_{\Omega} \tilde{\sigma}_t^q \, dx = \lambda_1 (1-t) \int_{\Omega} |u|^p \, dx + \lambda_1 t \int_{\Omega} \varphi_1^q \, dx.
\]

(6.7)

Subtracting (6.7) from (6.6), recalling that \( \varphi_1 \) is the minimizer of \( \lambda_1 \), and using (1.2) to estimate the difference of integrals containing \( \tilde{\sigma}_t \) by zero, we get

\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx \geq Ct \int_{\Omega} \mathcal{R}_p(|u|, \varphi_1; t) \, dx
\]
for any \( t \in [0, 1] \). Tending now \( t \) to \( 1 \), we derive the desired inequality (1.19). Finally, if \( p \geq 2 \), then the definition (1.18) of \( R_p \) gives

\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx \geq C \int_{\Omega} \left| \nabla u - \frac{|u|}{\varphi_1} \nabla \varphi_1 \right|^p \, dx.
\]

Observing that

\[
\left| \nabla u - \frac{|u|}{\varphi_1} \nabla \varphi_1 \right| = \left| \nabla u - \frac{u}{\varphi_1} \nabla \varphi_1 \right| = \left| \nabla \left( \frac{|u|}{\varphi_1} \right) \right| \varphi_1 \quad \text{a.e. in } \Omega,
\]

we obtain the inequality (1.20).

\[ \square \]

A. Appendix. Proof of Lemma 2.1

We recall that \( p > 2 \), \( 1 < q < p^* \), and \((A)\) is satisfied. For convenience, throughout the proof, we sometimes denote the norms in \( W_0^{1,q}(\Omega) \) and \( L^r(\Omega) \) (for \( r \geq 1 \)) as \( \| \cdot \|_{W_0^{1,q}(\Omega)} \) and \( \| \cdot \|_{L^r(\Omega)} \), respectively, to reflect the dependence on \( \Omega \). Moreover, we denote by \( C > 0 \) a universal constant whose value may vary from inequality to inequality.

(i) Let us show the existence of \( \kappa \in (2, 2^*) \) such that the embedding \( \mathcal{D}_{\varphi_1} \hookrightarrow L^\kappa(\Omega) \) is compact. We start with several remarks. In the case \( p = q > 2 \), the compactness of \( \mathcal{D}_{\varphi_1} \hookrightarrow L^2(\Omega) \) is proved in [33, Lemma 4.2]. Although [33, Lemma 4.2] requires \( \Omega \) to satisfy the assumption \((A)\) and the interior sphere condition when \( N \geq 2 \), the latter requirement is used only to guarantee that the first eigenfunction \( \varphi_1 \) obeys the Hopf maximum principle on \( \partial\Omega \) (see [33, (2.2)])], which is by now known to be true assuming \((A)\) alone, see [28]. The arguments of [33, Lemma 4.2] can be slightly amended to cover the general case \( p > 2 \) and \( 1 < q < p^* \). In the case \( N = 1 \) and when \( p = q > 2 \), the compactness of the embedding \( \mathcal{D}_{\varphi_1} \hookrightarrow L^\kappa(\Omega) \) for any \( \kappa > 1 \) follows from [19, Lemma 1.3, p. 238], and analogous arguments can be applied to cover any \( p > 2 \) and \( q > 1 \). Thus, hereinafter, we will be interested only in the case \( N \geq 2 \).

The continuity of the embedding \( \mathcal{D}_{\varphi_1} \hookrightarrow L^\kappa(\Omega) \) for some \( \kappa > 2 \) follows from [10, Theorem 3.1, (3.6)] in combination with [10, Theorem 2.3]. The results of [10] are formulated under an abstract smoothness assumption on \( \Omega \). Nevertheless, the assumption \((A)\) is sufficient for our aims, which is shown in [6, Theorem 2.7]. In the current form, [6, Theorem 2.7] requires \( q \geq p \), but the proof remains valid with no changes at least for a fixed \( q \in (1, p^*) \). Indeed, the assumption \( q \geq p \) is used in the proof of [6, Theorem 2.7] only in the derivation of an explicit and uniform \( L^\infty(\Omega) \)-bound for minimizers of \( \lambda_1 \) with respect to \( q \), see [6, Proposition 2.4]. Since we are interested in a fixed \( q \in (1, p^*) \), it is sufficient to substitute [6, Proposition 2.4] with the fact that any minimizer of \( \lambda_1 \) belongs to \( L^\infty(\Omega) \), see, e.g., [31, Theorem II].

Based on the continuity of the embedding \( \mathcal{D}_{\varphi_1} \hookrightarrow L^\kappa(\Omega) \), we show that \( \mathcal{D}_{\varphi_1} \hookrightarrow L^\kappa(\Omega) \) compactly for some \( \kappa \in (2, \sigma) \). Our proof is inspired by [33, Lemma 4.2], but we provide more details in subtle places. We assume, without loss of generality, that \( \sigma \in (2, 2^*) \).

Let \( \Omega_\delta \) be a strip of width \( \delta > 0 \) around the boundary \( \partial \Omega \):

\[ \Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \}. \]

Under the assumption \((A)\), we have \( \varphi_1 \in C^1(\overline{\Omega}) \) (see [26]) and it follows from [28] that \( \varphi_1 \) satisfies \( \partial \varphi_1 / \partial \nu < 0 \) on \( \partial \Omega \), where \( \nu \) is the outward unit normal vector to \( \partial \Omega \). Thus, for any sufficiently small \( \delta > 0 \) there exists \( C > 0 \) such that \( |\nabla \varphi_1| \geq C > 0 \) in \( \Omega_\delta \). As a consequence, we can find \( C > 0 \) such that

\[
C^{-1} \|v\|_{W_0^{1,\sigma}(\Omega_\delta)} \leq \|v\|_{\varphi_1} \leq C \|v\|_{W_0^{1,\kappa}(\Omega_\delta)} \quad \text{for any } v \in C_0^\infty(\Omega_\delta). \tag{A.1}
\]

Set \( \kappa = \sigma / 2 + 1 \) and observe that \( \kappa \in (2, \sigma) \). Taking any \( v \in C_0^\infty(\Omega) \), we use \( |v|^\kappa \) as a test function
for (1.6) and get

$$\lambda_1 \left( \int_{\Omega} \varphi^q \, dx \right) \leq \int_{\Omega} \varphi^{q-1} |v|^\kappa \, dx = \int_{\Omega} |\nabla \varphi|^p \langle \nabla \varphi, \nabla (|v|^\kappa) \rangle \, dx$$

$$\leq \kappa \int_{\Omega} |\nabla \varphi|^{\frac{p}{\kappa} - 1} |\nabla v| |\nabla |\nabla v|^{\kappa - 1} \, dx \leq \kappa \|
abla \varphi\|_{\varphi} \left( \int_{\Omega} |\nabla |\nabla v|^{p} |v|^\kappa \, dx \right)^{\frac{1}{p}}. \quad (A.2)$$

Fix a cut-off function $\xi \in C_0^\infty(\Omega)$ such that

$$\xi(x) = 0 \text{ if } x \in \Omega_\delta \text{ and } \xi(x) = 1 \text{ if } x \in \Omega \setminus \Omega_{2\delta},$$

and consider the decomposition $v = \xi v + (1 - \xi)v$. We use the continuity of the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^p(\Omega)$ and the $C^1(\Omega)$-regularity of $\varphi_1$ to get

$$\|(1 - \xi)v\|_{p^*}^2 = \int_{\Omega} |\nabla \varphi|^{p-2}((1 - \xi)\nabla v - \nabla \xi \cdot v)^2 \, dx \leq C \|
abla v\|_{p^*}^p + C \|v\|^2_{L^2(\Omega)} \leq C \|v\|_{p^*}^2,$$

where $C > 0$ does not depend on $v \in C_0^\infty(\Omega)$. Thus, by the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_{\varphi_1}$, we conclude that the multiplication by $(1 - \xi)$ is a bounded linear operator in $\mathcal{D}_{\varphi_1}$. Consequently, the multiplication by $\xi$ is also a bounded linear operator in $\mathcal{D}_{\varphi_1}$.

Thanks to the continuity of the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^p(\Omega)$, any sequence $\{v_n\} \subset \mathcal{D}_{\varphi_1}$ which converges weakly in $\mathcal{D}_{\varphi_1}$ must also converge weakly in $L^p(\Omega)$ and $L^\infty(\Omega)$ to the same limit. Assume, without loss of generality, that this limit is zero and that $\{v_n\} \subset C_0^\infty(\Omega)$. Since the multiplications by $\xi$ and $(1 - \xi)$ are bounded linear operators in $\mathcal{D}_{\varphi_1}$, both $\{\xi v_n\}$ and $\{(1 - \xi)v_n\}$ also converge to zero weakly in $\mathcal{D}_{\varphi_1}$, $L^p(\Omega)$, and $L^\infty(\Omega)$. In order to establish the desired compactness of the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow L^\infty(\Omega)$, it is sufficient to prove that both $\{\xi v_n\}$ and $\{(1 - \xi)v_n\}$ converge to zero strongly in $L^\infty(\Omega)$. Since supp $(1 - \xi)v_n \subset \Omega_{2\delta}$ and $2 < \kappa < \sigma < 2^*$, the convergence of $\{(1 - \xi)v_n\}$ follows from (A.1), as $\mathcal{D}_{\varphi_1}$ restricted to functions supported in $\Omega_{2\delta}$ coincides with $W^{1,2}(\Omega_{2\delta})$ which is compactly embedded in $L^\infty(\Omega_{2\delta})$.

Let us investigate the convergence of $\{\xi v_n\}$. Take any $\eta > 0$ and define

$$U_\eta = \left\{ x \in \Omega : |\nabla \varphi_1(x)| > \frac{\eta}{2} \right\} \quad \text{and} \quad U_\eta' = \left\{ x \in \Omega : |\nabla \varphi_1(x)| < \eta \right\}.$$

Thanks to the $C^1$-regularity of $\varphi_1$, both $U_\eta$ and $U_\eta'$ are open and $U_\eta \cup U_\eta' = \Omega$. Since the weakly convergent sequence $\{\xi v_n\} \subset C_0^\infty(\Omega \setminus \Omega_\delta)$ is bounded in $\mathcal{D}_{\varphi_1}$ and $\varphi_1 > 0$ in $\Omega \setminus \Omega_\delta$, we plug $\xi v_n$ into (A.2) and obtain

$$C \int_{\Omega} |\xi v_n|^\kappa \, dx \leq C \left( \int_{U_\eta} |\nabla \varphi|^p |\xi v_n|^\sigma \, dx \right)^{\frac{1}{p}} + C \left( \int_{U_\eta'} |\nabla \varphi|^p |\xi v_n|^\sigma \, dx \right)^{\frac{1}{p}}. \quad (A.3)$$

Fix any $\varepsilon > 0$. The boundedness of $\{\xi v_n\}$ in $\mathcal{D}_{\varphi_1}$ implies its boundedness in $L^p(\Omega)$, and hence there exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ we get

$$C \left( \int_{U_\eta} |\nabla \varphi|^p |\xi v_n|^\sigma \, dx \right)^{\frac{1}{p}} \leq C \eta^{\frac{p}{p^*}} \|\xi v_n\|_{L^p(\Omega)}^{\frac{p}{p^*}} \leq C \eta^{\frac{p}{p^*}} \|\xi v_n\|_{L^\infty(\Omega)}^{\frac{p}{p^*}} \leq \varepsilon. \quad (A.4)$$

for all $n$. Taking any $\eta \in (0, \eta_0)$, let us now prove the existence of $n_0 = n_0(\varepsilon) > 0$ such that the first term on the right-hand side of (A.3) is also less than $\varepsilon$ for any $n \geq n_0$. Denote by $w_n$ the restriction of $\xi v_n$ to $U_\eta$, i.e., $w_n = (\xi v_n)|_{U_\eta}$. We have

$$\|w_n\|_{W^{1,2}(U_\eta)} = \int_{U_\eta} |\nabla w_n|^2 \, dx + \int_{U_\eta} |w_n|^2 \, dx$$

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Thus, the restriction to \( U_\eta \) is a bounded linear operator from \( \mathcal{D}_{\varphi_1} \) to \( W^{1,2}(U_\eta) \). Since bounded linear operators preserve the weak convergence, \( \{ w_n \} \) converges weakly to zero in \( W^{1,2}(U_\eta) \).

By the standard regularity theory, we have \( \varphi_1 \in C^\infty(\Omega \setminus Z) \), where \( Z = \{ x \in \Omega : |\nabla \varphi_1| = 0 \} \). Thus, Sard’s theorem asserts that the set of critical levels of the function \( |\nabla \varphi_1| \) has zero measure. Then, the implicit function theorem guarantees that for the continuity of the embedding \( D_{\varphi_1} \rightarrow L^2(\Omega) \), the strong convergence of \( \{ w_n \} \) implies that \( \{ v_n \} \) converges to zero strongly in \( L^2(U_\eta) \). Therefore, we can find the desired \( n_0 = n_0(\varepsilon) \) such that

\[
C \left( \int_{U_\eta} |\nabla \varphi_1|^p |\xi v_n|^p \right)^\frac{1}{p} \leq C \| \nabla \varphi_1 \|_\infty^{\frac{p}{2}} \left( \int_{U_\eta} |v_n|^p \right)^{\frac{1}{p}} \leq \varepsilon
\]  

(A.5)

for all \( n \geq n_0 \). Combining (A.4) and (A.5), we conclude from (A.3) that for any \( \varepsilon > 0 \) there exists \( n_1 \geq n_0 > 0 \) such that \( \| \xi v_n \|_{L^2(\Omega)} \leq \varepsilon \) for any \( n \geq n_1 \). This completes the proof of the strong convergence of \( \{ v_n \} \) to zero in \( L^2(\Omega) \), which gives the compactness of the embedding \( D_{\varphi_1} \hookrightarrow L^2(\Omega) \), where \( \kappa = \sigma/2 + 1 \in (2, \sigma) \).

(ii) For the continuity of the embedding \( D_{\varphi_1} \hookrightarrow W^{1,\theta}_0(\Omega) \) for some \( \theta > 1 \) in the case \( N \geq 2 \), we refer to [6, Corollary 2.8] which remains valid for any fixed \( 1 < q < p^* \) with no changes in the proof, as discussed in the proof of (i) above. In the case \( N = 1 \) and when \( p = q > 2 \), the claim is given by [19, Lemma 1.3, p. 238] with an explicit value of \( \theta \), and the same argument applies for any \( p > 2 \) and \( q > 1 \).

(iii) The continuity of the embedding \( W^{1,p}_0(\Omega) \hookrightarrow D_{\varphi_1} \) follows trivially from the regularity \( \varphi_1 \in C^1(\Omega) \) and the Hölder inequality.

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