GENERALIZED WENTZELL BOUNDARY CONDITIONS FOR SECOND ORDER OPERATORS WITH INTERIOR DEGENERACY

Genni Fragnelli
Department of Mathematics, University of Bari Aldo Moro
Via E.Orabona 4, 70125 Bari, Italy

Gisèle Ruiz Goldstein and Jerome Goldstein
Department of Mathematical Sciences, University of Memphis
373 Dunn Hall, Memphis, TN 38152-3240, USA

Rosa Maria Mininni and Silvia Romanelli
Department of Mathematics, University of Bari Aldo Moro
Via E.Orabona 4, 70125 Bari, Italy

Abstract. We consider operators in divergence form, \( A_1 u = (a u')' \), and in nondivergence form, \( A_2 u = a u'' \), provided that the coefficient \( a \) vanishes in an interior point of the space domain. Characterizing the domain of the operators, we prove that, under suitable assumptions, the operators \( A_1 \) and \( A_2 \), equipped with general Wentzell boundary conditions, are nonpositive and selfadjoint on spaces of \( L^2 \) type.

1. Introduction. It is well known that degenerate parabolic equations are widely used as mathematical models in the applied sciences to describe the evolution in time of a given system. For this reason, in recent years an increasing interest has been devoted to the study of second order differential degenerate operators in divergence or in nondivergence form. A wide non exhaustive description of them can be found in [9, Introduction] and in the references therein. In particular, operators of the type \( A_1 u := (a u')' \), \( A_2 u := a u'' \) with suitable domains involving different boundary conditions, arise in a natural way in several contexts as aeronautics (Crocco equation), physics (boundary layer models), genetics (Wright-Fisher and Fleming-Viot models), mathematical finance (Black-Merton-Scholes models).

Here the novelty is that we deal with existence and regularity of solutions of Cauchy problems associated with parabolic equations having coefficients which degenerate in the interior of the spatial domain and satisfy general Wentzell boundary conditions in spaces of \( L^2 \)-type.

To our best knowledge, [16] is the first paper treating the existence of a solution for the Cauchy problem associated to a parabolic equation which degenerates in the interior of the spatial domain in the space \( L^2(0,1) \), while in [9] both the degenerate operators \( A_1 \) and \( A_2 \) in the space \( L^2(0,1) \), with or without weight, were

2010 Mathematics Subject Classification. Primary: 47D06; Secondary: 35K65, 47B25, 47N20.
Key words and phrases. Second order operators in divergence and nondivergence form, interior degeneracy, generalized Wentzell boundary conditions.
examined. In particular, the authors proved that both the operators are nonpositive and selfadjoint, hence they generate cosine families and, as a consequence, analytic semigroups, provided that the coefficient $a$ vanishes in an interior point of the spatial domain and Dirichlet boundary conditions hold.

These results are complemented in [12] and in [13], where other aspects of the associated parabolic problem, such as Carleman estimates and null controllability, are considered. We refer to the recent paper [4] for the analogous results under Neumann boundary conditions.

For this kind of equations, in addition to well posedness, controllability and Carleman estimates, other features have been studied in several papers, obtaining substantial progresses. Among these papers, we cite [3], [5], [10] and [11], where the authors obtain results concerning inverse problems, stability, optimal existence and regularity theory also in higher dimension.

On the other hand, the importance of the study of the operators $A_1$ and $A_2$ equipped with general Wentzell boundary conditions (GWBC) for shortness) in spaces of $L^p$-type recently received a lot of attention, after the new directions opened in [8]. Indeed, it is worth to mention that, in the case of heat equations, (GWBC) allow to take into account the action of heat sources on the boundary (see [14]).

Additional motivation for the study of evolution equations with (GWBC) comes from their possible interpretation as evolution equations with dynamical boundary conditions. For a general view on the role of Wentzell boundary conditions we refer to [7], while for a physical interpretation of them see [14]. In this framework, up to now, operators with interior degeneracy were never considered. Thus we intend to fill this gap according to the ideas of [9] by using Hilbert spaces depending on the boundary conditions introduced in [8]. Indeed in the main theorems (Sections 3-5) we will prove that, under suitable assumptions, the operators $A_1$ and $A_2$ equipped with (GWBC) are nonpositive and selfadjoint in suitable spaces of $L^2$-type. It is worth noting that in the present work we deal with real function spaces, but the assertions can be easily extended to the complex case.

2. Basic assumptions and preliminary results. In the following we will introduce the notions of weak and strong degeneracy for a real-valued function $a$ defined on the interval $[0, 1]$.

Accordingly, we will define suitable weighted spaces and prove some formulas of Green type. These results will play a key role for the study of the operators $A_1$ and $A_2$ considered in Sections 3-5.

**Definition 2.1.** A function $a \in C[0,1]$ is said to be weakly degenerate if there exists $x_0 \in (0, 1)$ such that $a(x_0) = 0$, $a(x) > 0$ in $[0,1] \setminus \{x_0\}$, and $\frac{1}{a} \in L^1(0,1)$.

**Example 2.2.** We can take $a(x) = |x-x_0|^\alpha$, $0 < \alpha < 1$, as an example of a weakly degenerate function.

For any weakly degenerate $a \in C[0,1]$, let us introduce the following weighted spaces:

\[ H^1_a(0,1) := \{ u \in L^2(0,1) \mid u \text{ absolutely continuous in } [0,1], \sqrt{au'} \in L^2(0,1) \}, \]

\[ H^2_a(0,1) := \{ u \in H^1_a(0,1) \mid au' \in H^1(0,1) \}. \]
We can take Example 2.6. If \( H \) coincide algebraically and their norms are equivalent. Hence Lemma 2.4. Proposition 2.3.

(i):

(ii):

\[
L^2_\alpha(0,1) := \left\{ u \in L^2(0,1) \mid \int_0^1 \frac{u^2}{a} \, dx < \infty \right\}, \quad H^1_\alpha(0,1) := L^2_\alpha(0,1) \cap H^1(0,1), \quad H^2_\alpha(0,1) := \left\{ u \in H^1_\alpha(0,1) \mid u' \in H^1(0,1) \right\},
\]

endowed with the respective norms defined by

\[
\|u\|_{L^2_\alpha(0,1)}^2 := \|u\|_{L^2(0,1)}^2 + \|\sqrt{a}u'\|^2_{L^2(0,1)}, \quad \text{for all } u \in H^1_\alpha(0,1),
\]

\[
\|u\|_{H^1_\alpha(0,1)} := \left\| \int_0^1 \frac{u^2}{a} \, dx \right\|, \quad \text{for all } u \in L^2_\alpha(0,1),
\]

\[
\|u\|_{H^2_\alpha(0,1)} := \left\| \int_0^1 \frac{u^2}{a} \, dx \right\| + \|u'\|_{L^2(0,1)}^2, \quad \text{for all } u \in H^1_\alpha(0,1),
\]

\[
\|u\|_{H^2_\alpha(0,1)} := \left\| \int_0^1 \frac{u^2}{a} \, dx \right\| + \|u'\|_{L^2(0,1)}^2 + \|u''\|_{L^2(0,1)}^2, \quad \text{for all } u \in H^2_\alpha(0,1).
\]

Similar arguments as in [9, Corollary 3.1] lead to the following result.

**Proposition 2.3.** If \( a \) is weakly degenerate, then the spaces \( H^1_\alpha(0,1) \) and \( H^1(0,1) \) coincide algebraically and their norms are equivalent. Hence \( C^\infty[0,1] \) is dense in \( H^1_\alpha(0,1) \).

The following Green formulae are analogous to those proved in [9].

**Lemma 2.4.** If \( a \) is weakly degenerate, then

(i): for all \((u,v) \in H^2_\alpha(0,1) \times H^1_\alpha(0,1)\):

\[
\int_0^1 (au')'v \, dx = [auv]_{x=0}^1 - \int_0^1 au'v' \, dx. \tag{2.1}
\]

(ii): for all \((u,v) \in H^2_\alpha(0,1) \times H^1_\alpha(0,1)\):

\[
\int_0^1 uu''v \, dx = [uvv]_{x=0}^1 - \int_0^1 u''v' \, dx.
\]

The proof of Lemma 2.4 is given in the Appendix.

Now let us introduce another notion of interior degeneracy.

**Definition 2.5.** A function \( a \in W^{1,\infty}(0,1) \) is called strongly degenerate if there exists \( x_0 \in (0,1) \) such that \( a(x_0) = 0, a(x) > 0 \) in \([0,1] \setminus \{x_0\}\), and \( \frac{1}{a} \notin L^1(0,1) \).

**Example 2.6.** We can take \( a(x) = |x-x_0|^\alpha, \alpha \geq 1 \), as an example of a strongly degenerate function.

For any strongly degenerate \( a \in W^{1,\infty}(0,1) \), let us introduce the corresponding weighted spaces

\[
H^1(0,1) := \left\{ u \in L^2(0,1) \mid u \text{ locally absolutely continuous in } [0, x_0) \cup (x_0, 1], \right. \quad \sqrt{a}u' \in L^2(0,1),
\]

\[
H^2(0,1) := \left\{ u \in H^1(0,1) \mid au' \in H^1(0,1), \right\}
\]

and consider the spaces \( L^2_\alpha(0,1), H^1_\alpha(0,1) \) and \( H^2_\alpha(0,1) \) introduced in the weakly degenerate case.
Since in this situation a function \( u \in H^2_a(0,1) \) is locally absolutely continuous in \([0,1]\setminus \{x_0\}\) and not necessarily absolutely continuous in \([0,1]\) as for the weakly degenerate case, the equality (2.1) is not true a priori. Now let us provide some useful results.

**Proposition 2.7.** Let \( a \) be strongly degenerate and define

\[
X := \{ u \in L^2(0,1) \mid u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \sqrt{au'} \in L^2(0,1), au \in H^1(0,1) \text{ and } (au)(x_0) = 0 \},
\]

\[
Z := \{ u \in X \mid au' \in H^1(0,1), (au')(x_0) = 0 \}.
\]

Then

\[
(i) \quad H^1_a(0,1) = X
\]

\[
(ii) \quad H^2_a(0,1) = Z.
\]

**Proof.** (i) Of course \( X \subseteq H^1_a(0,1) \). Conversely, let us take \( u \in H^1_a(0,1) \). Proceeding as in [12], we prove that \( au \) is continuous at \( x_0 \), in particular \( (au)(x_0) = 0 \) that is, \( u \in X \). To this aim, observe that, by the assumption on \( u \), \( au' \in L^2(0,1) \) and, since \( a \in W^{1,\infty}(0,1) \), then \( (au)' = a' u + au' \in L^2(0,1) \). Therefore for \( 0 \leq x < x_0 \), one has

\[
(au)(x) = \int_0^x (au')(t) \, dt + (au)(0)
\]

(observe that \( (au)(0) \) exists since \( au \) is continuous away from \( x_0 \)). This implies that there exists

\[
\lim_{x \to x_0^-} (au)(x) = \int_0^{x_0} (au')(t) \, dt + (au)(0) = L \in \mathbb{R}.
\]

If \( L \neq 0 \), then there exists \( C > 0 \) such that

\[
|(au)(x)| \geq C
\]

for all \( x \) in a left neighborhood of \( x_0, x \neq x_0 \). Thus setting \( C_1 := \frac{C^2}{\max_{[0,1]} a(x)} > 0 \), it follows that

\[
|u^2(x)| \geq \frac{C^2}{a^2(x)} \geq \frac{C_1}{a(x)},
\]

for all \( x \) in a left neighborhood of \( x_0, x \neq x_0 \). But, since \( a \) is strongly degenerate, \( \frac{1}{a} \notin L^1(0,1) \) thus \( u \notin L^2(0,1) \). Hence \( L = 0 \). Analogously, one can prove that

\[
\lim_{x \to x_0^-} (au)(x) = 0
\]

and thus \( au \) is continuous at \( x_0 \) and \( (au)(x_0) = 0 \). Therefore, it easily follows that \( (au)' \) is the distributional derivative of \( au \), and so \( au \in H^1(0,1) \), i.e. \( u \in X \).

(ii) The inclusion \( Z \subseteq H^2_a(0,1) \) is obvious. Conversely, if \( u \in H^2_a(0,1) \), then we only need to show that \( (au')(x_0) = 0 \). Similar arguments as in the proof of (i) imply that

\[
\lim_{x \to x_0} (au')(x) = L \in \mathbb{R}.
\]

If \( L \neq 0 \), then there exists \( C > 0 \) such that

\[
|(au')(x)| \geq C
\]

for all \( x \) in a left neighborhood of \( x_0, x \neq x_0 \). It follows that

\[
|\sqrt{au'}(x)| \geq \frac{C}{\sqrt{a(x)}},
\]

\[
\lim_{x \to x_0} \sqrt{au'}(x) = L \in \mathbb{R}.
\]
for all $x$ in a left neighborhood of $x_0$, $x \neq x_0$. Hence $|(a(u'))^2(x)| \geq \frac{c^2}{a(x)}$ for all $x$ in a left neighborhood of $x_0$, $x \neq x_0$. But $a$ is strongly degenerate, so $\frac{1}{a} \not\in L^1(0,1)$ and thus $\sqrt{a}u' \notin L^2(0,1)$. Hence $L = 0$. Analogously, one can prove that $\lim_{x \to x_0^+} (au')(x) = 0$ and thus $au'$ can be extended by continuity at $x_0$ setting $(au')(x_0) = 0$.

We point out that Proposition 2.7 is based on the following

**Lemma 2.8.** (see [9, Lemma 2.5]) If $a$ is strongly degenerate, then for all $u \in Z$ we have that

$$|(au)(x)| \leq \|(au)'\|_{L^2(0,1)} \sqrt{|x - x_0|},$$

and

$$|(au')(x)| \leq \|(au)''\|_{L^2(0,1)} \sqrt{|x - x_0|},$$

(2.2)

for all $x \in [0,1]$.

As for the weakly degenerate case and using the previous characterization, we can prove the following Green’s formulae. (See Appendix.)

**Lemma 2.9.** If $a$ is strongly degenerate, then

(i): for all $(u,v) \in H^2_\alpha(0,1) \times H^1_\alpha(0,1)$ one has

$$\int_0^1 (au')'vdx = [au']_{x=0}^1 - \int_0^1 au'v'dx.$$

(ii): for all $(u,v) \in H^2_\alpha(0,1) \times H^1_\alpha(0,1):

$$\int_0^1 u''vdx = [u']_{x=0}^1 - \int_0^1 u'v'dx.$$

Similar arguments as in [9, Propositions 3.6, 3.8] allow to characterize the spaces $H^1_\frac{1}{2}(0,1)$ and $H^2_\frac{1}{2}(0,1)$.

Let us make the following additional assumption on $a$.

**Hypothesis 2.1** There exists a positive constant $K$ such that

$$\frac{1}{a(x)} \leq \frac{K}{|x - x_0|^2}, \quad \text{for all } x \in [0,1] \setminus \{x_0\}.$$

(e.g. $a(x) = |x - x_0|^{-\alpha}$, $1 \leq \alpha \leq 2$.)

**Proposition 2.10.** If $a$ is strongly degenerate and satisfies Hypothesis 2.1 then

(i):

$$H^1_\frac{1}{2}(0,1) = \{u \in H^1_\frac{1}{2}(0,1) \mid u(x_0) = 0\},$$

and the norms $\|u\|_{H^1_\frac{1}{2}(0,1)}$ and $\left(\int_0^1 (u')^2dx\right)^{1/2}$ are equivalent.

(ii):

$$H^2_\frac{1}{2}(0,1) = \{u \in H^1_\frac{1}{2}(0,1) \mid au'' \in L^2_\frac{1}{2}(0,1), au' \in H^1(0,1), u(x_0) = (au')(x_0) = 0\}.$$
3. Operators in divergence form with (GWBC): The weakly degenerate case. Let us fix $\beta_j, \gamma_j \in \mathbb{R}$, $j = 0, 1$, such that $\beta_j > 0$ and $\gamma_j \geq 0$, $j = 0, 1$. Consider a weakly or strongly degenerate function $a$, define the operator in divergence form $A_1 u = (au')'$ equipped with

$$A_1 u(j) + (-1)^{j+1} \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1,$$

and the space

$$X_\mu := L^2([0, 1], d\mu)$$

associated with (GWBC) (see [8]). Here

$$d\mu := dx \mid_{(0, 1)} \otimes \frac{adS}{\beta} \mid_{(0, 1)},$$

d$x$ denotes the Lebesgue measure on $(0, 1)$, $\beta = (\beta_0, \beta_1)$, and $\frac{adS}{\beta}$ denotes the natural (Dirac) measure $dS$ on $\{0, 1\}$ with weight $\frac{a}{\beta}$. More precisely, $X_\mu$ is a Hilbert space with respect to the inner product given by

$$(f, g)_{X_\mu} = \int_0^1 f(x)g(x)\, dx + \frac{a(1)f(1)g(1)}{\beta_1} + \frac{a(0)f(0)g(0)}{\beta_0},$$

where $f, g \in X_\mu$ are written as $(f \chi_{(0, 1)}), (f(0), f(1)))$, $(g \chi_{(0, 1)}), (g(0), g(1)))$, and $\chi_{(0, 1)}$ is the characteristic function of the interval $(0, 1)$.

Hence $X_\mu$ is equipped with the norm defined by

$$\|f\|_{X_\mu}^2 = \int_0^1 |f(x)|^2 \, dx + \frac{a(1)|f(1)|^2}{\beta_1} + \frac{a(0)|f(0)|^2}{\beta_0},$$

for any $f \in X_\mu$, provided that $f$ is written as $(f \chi_{(0, 1)}, (f(0), f(1)))$.

Now we define the weighted spaces

$$\tilde{H}_a^1(0, 1) := \{u \in X_\mu \mid u \text{ absolutely continuous in } [0, 1], \sqrt{a}u' \in X_\mu\}$$

and

$$\tilde{H}_a^2(0, 1) := \{u \in \tilde{H}_a^1(0, 1) \mid au' \in H^1(0, 1)\},$$

equipped with the norms defined, respectively, by

$$\|u\|_{\tilde{H}_a^1(0, 1)} := \|u\|_{X_\mu}^2 + \|\sqrt{a}u'\|_{X_\mu}^2,$$

for all $u \in \tilde{H}_a^1(0, 1)$

and

$$\|u\|_{\tilde{H}_a^2(0, 1)} := \|u\|_{\tilde{H}_a^1(0, 1)}^2 + \|(au')'\|_{X_\mu}^2,$$

for all $u \in \tilde{H}_a^2(0, 1)$.

**Remark 3.1.** Let us observe that $\tilde{H}_a^1(0, 1) \subset H_a^1(0, 1)$ and $\tilde{H}_a^2(0, 1) \subset H_a^2(0, 1)$. It follows that for any $(u, v) \in \tilde{H}_a^2(0, 1) \times \tilde{H}_a^1(0, 1)$ the Green formula in Lemma 2.4(i) holds. Moreover $C^\infty[0, 1] \subset \tilde{H}_a^2(0, 1)$.

Further, let us define the domain of $A_1$ to be the following subspace of $X_\mu$:

$$D(A_1) = \{u \in \tilde{H}_a^2(0, 1) \mid A_1 u(j) + (-1)^{j+1} \beta_j u'(j) + \gamma_j u(j) = 0, \ j = 0, 1\}.$$

Now we are ready for the main results of this Section.

**Theorem 3.2.** If $a$ is weakly degenerate, then the operator $A_1$ with domain $D(A_1)$ is nonpositive and selfadjoint on $X_\mu$. 
Proof. Observe that \( D(A_1) \) is dense in \( X_\mu \). Indeed, if we introduce the space

\[
W_{1,a}(0,1) = \{u \in \hat{H}_a^2(0,1) | \supp \{u\} \text{ compact}, \supp \{u\} \subset [0,1] \setminus \{x_0\}, \quad A_j u(j) + (-1)^{j+1} \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0,1, \}
\]

endowed with the norm

\[
\|u\|_{W_{1,a}(0,1)}^2 := \|u\|_{\hat{H}_a^2(0,1)}^2,
\]

it is clear that \( W_{1,a}(0,1) \subset D(A_1) \subset X_\mu \).

Moreover, \( W_{1,a}(0,1) \) is dense in \( X_\mu \) by using the following argument: take \( u \in X_\mu \) and define \( u_n := \xi_n u, \quad n \geq \max \{ \frac{a}{x_0}, \frac{1-a}{1-x_0} \} \), where

\[
\xi_n = \begin{cases} 
0, & \text{in } [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}], \\
1, & \text{in } [0, x_0 - \frac{2}{n}] \cup [x_0 + \frac{2}{n}, 1], \\
n(x_0 - x) - 1, & \text{in } (x_0 - \frac{2}{n}, x_0 - \frac{1}{n}), \\
n(x - x_0) - 1, & \text{in } (x_0 + \frac{1}{n}, x_0 + \frac{2}{n}).
\end{cases}
\] (3.1)

It follows that \( u_n \to u \) in \( X_\mu \) as \( n \to \infty \). Hence \( D(A_1) \) is dense in \( X_\mu \). In order to show that \( A_1 \) is nonpositive and selfadjoint it suffices to prove that \( A_1 \) is symmetric, nonpositive and \( (I - A_1)(D(A_1)) = X_\mu \) (see e.g. [1, Theorem B.14] or [15]).

\( A_1 \) is symmetric.

By using Lemma 2.4(i) and (GWBC), for any \( u, v \in D(A_1), \) one has

\[
\langle A_1 u, v \rangle_{X_\mu} = \int_0^1 v (au')' dx + \frac{(au)'(0) a(0) v(0)}{\beta_0} + \frac{(au)'(1) a(1) v(1)}{\beta_1} = \int_0^1 au' v' dx + (\beta_0 u'(0) - \gamma_0 u(0)) \frac{a(0) v(0)}{\beta_0} - (\beta_1 u'(1) + \gamma_1 u(1)) \frac{a(1) v(1)}{\beta_1}
\]

\[
= -\int_0^1 au' v' dx - \frac{\gamma_0}{\beta_0} u(0) a(0) v(0) - \frac{\gamma_1}{\beta_1} u(1) a(1) v(1) = \langle u, A_1 v \rangle_{X_\mu}.
\]

\( A_1 \) is nonpositive.

For any \( u \in D(A_1), \) according to the previous calculations, one has

\[
\langle A_1 u, u \rangle_{X_\mu} = -\int_0^1 a |u'|^2 dx - \frac{\gamma_0}{\beta_0} a(0) |u(0)|^2 - \frac{\gamma_1}{\beta_1} a(1) |u(1)|^2 \leq 0.
\]

\( I - A_1 \) is surjective.

Observe that \( \hat{H}_a^1(0,1) \) is a Hilbert space with respect to the inner product

\[
(u, v)_{\hat{H}_a^1(0,1)} = \int_0^1 (uv + au' v') dx + \frac{a(0) u(0) v(0)}{\beta_0} + \frac{a(1) u(1) v(1)}{\beta_1},
\] (3.2)

for any \( u, v \in \hat{H}_a^1(0,1) \). Moreover, we have that

\[
\hat{H}_a^1(0,1) \hookrightarrow X_\mu \hookrightarrow (\hat{H}_a^1(0,1))^*,
\]
where \((\hat{H}_a^1(0,1))^*\) is the dual space of \(\hat{H}_a^1(0,1)\) with respect to \(X_\mu\). Let \(f \in X_\mu\) and define \(F : \hat{H}_a^1(0,1) \to \mathbb{R}\) such that

\[
F(v) = \int_0^1 f v \, dx + \frac{a(0)f(0)v(0)}{\beta_0} + \frac{a(1)f(1)v(1)}{\beta_1}.
\]

From \(\hat{H}_a^1(0,1) \to X_\mu\), it follows that \(F \in (\hat{H}_a^1(0,1))^*\). As a consequence, by Riesz’s Theorem, there exists a unique \(u \in \hat{H}_a^1(0,1)\) such that for any \(v \in \hat{H}_a^1(0,1)\)

\[
(u, v)_{\hat{H}_a^1(0,1)} = F(v) = \int_0^1 f v \, dx + \frac{a(0)f(0)v(0)}{\beta_0} + \frac{a(1)f(1)v(1)}{\beta_1}.
\]

From (3.2), the above equality means that

\[
\int_0^1 au'v' \, dx = \int_0^1 (f - u)v \, dx + \frac{a(0)(f - u)(0)v(0)}{\beta_0} + \frac{a(1)(f - u)(1)v(1)}{\beta_1}, \quad (3.3)
\]

for all \(v \in \hat{H}_a^1(0,1)\).

In particular, if we denote by \(C^\infty_c(0,1)\) the space of all \(C^\infty(0,1)\) functions with compact support in \((0,1)\), from \(C^\infty_c(0,1) \subset \hat{H}_a^1(0,1)\), it follows that (3.3) holds for all \(v \in C^\infty_c(0,1)\). This implies

\[
\int_0^1 au'v' \, dx = \int_0^1 (f - u)v \, dx, \quad v \in C^\infty_c(0,1).
\]

Thus the weak derivative \((au')'\) in this context exists and \((au')' \in L^2(0,1)\). Hence \(u \in D(A_1)\) and

\[
u - A_1\ u = f.
\]

Thanks to Theorem 3.2, one has that the problem

\[
\begin{align*}
&\begin{cases}
\ u_\tau - A_1 u = h(t,x), \quad (t,x) \in Q_T := (0,T) \times (0,1),
\end{cases} \\
&(\text{GWBC}),
\end{align*}
\]

\[
u(0,x) = u_0(x), \quad x \in (0,1),
\]

(3.4)

is wellposed in the sense of Theorem 3.4 below. But first we recall the following definition.

**Definition 3.3.** Assume that \(u_0 \in X_\mu\) and \(h \in L^2(0,T; X_\mu)\). A function \(u\) is said to be a weak solution of (3.4) if

\[
u \in C([0,T]\cap L^2(0,T; \hat{H}_a^1(0,1))
\]

and satisfies

\[
\begin{align*}
\int_0^1 u(T,x) \varphi(T,x) \, dx - \int_0^1 u_0(x) \varphi(0,x) \, dx & - \int_0^T \int_0^1 \varphi_t(t,x) u(t,x) \, dx \, dt \\
& + \frac{a(1)u(T,1)\varphi(T,1)}{\beta_1} - \frac{a(1)u_0(1)\varphi(0,1)}{\beta_1} - \frac{a(1)}{\beta_1} \int_0^T u(t,1) \varphi_t(t,1) \, dt \\
& + \frac{a(0)u(T,0)\varphi(T,0)}{\beta_0} - \frac{a(0)u_0(0)\varphi(0,0)}{\beta_0} - \frac{a(0)}{\beta_0} \int_0^T u(t,0) \varphi_t(t,0) \, dt =
\end{align*}
\]
Assume that Theorem 3.4. where $u$ and $\phi$ for all $K$ where

$$-\int_0^T \int_0^1 a u_x \varphi_x dx dt - \frac{\gamma_1}{\beta_1} \int_0^T a(1) u(t, 1) \varphi(t, 1) dt - \frac{\gamma_0}{\beta_0} \int_0^T a(0) u(t, 0) \varphi(t, 0) dt$$

$$+ \int_0^T \int_0^1 h(t, x) \varphi(t, x) dx dt + \int_0^T \frac{a(1)}{\beta_1} h(t, 1) \varphi(t, 1) dt + \int_0^T \frac{a(0)}{\beta_0} h(t, 0) \varphi(t, 0) dt,$$

for all $\varphi \in H^1(0, T; X_\mu) \cap L^2(0, T; \hat{H}_1^1(0, 1))$.

**Theorem 3.4.** Assume that $a$ is weakly degenerate. Then for all $h \in L^2(0, T; X_\mu)$ and $u_0 \in X_\mu$, there exists a unique weak solution $u$ of (3.4) such that

$$\sup_{t \in [0, T]} \frac{\|u(t)\|_{X_\mu}^2}{\|X_\mu\|} \leq C \left[ \|u_0\|^2_{X_\mu} + \|h\|^2_{L^2(Q_T)} \right],$$

where $C > 0$ is a suitable constant.

Moreover, if $u_0 \in D(A_1)$, then $u \in C([0, T]; D(A_1)) \cap C^1([0, T]; X_\mu)$ and the following inequality holds

$$\sup_{t \in [0, T]} \frac{\|u(t)\|_{X_\mu}^2}{\|X_\mu\|} + \int_0^T \frac{\|u(t)\|_{X_\mu}^2}{\|X_\mu\|} dt \leq K \left[ \frac{\|u_x(\cdot, 0)\|_{L^2(0, T)}^2}{\|X_\mu\|} + \|u_0\|^2_{X_\mu} + \frac{1}{2} \|h(t)\|_{X_\mu}^2 \right],$$

where $K$ is a positive constant depending on $T, a, \beta$.

**Proof.** The assertion concerning the assumption $u_0 \in X_\mu$ and the regularity of the solution $u$ when $u_0 \in D(A_1)$ is a consequence of the results in [2] and of [6, Lemma 4.1.5 and Proposition 4.3.9]. We only need to prove (3.5). Now let us fix $u_0 \in D(A_1)$ and consider that the corresponding weak solution $u$ is in $C([0, T]; D(A_1)) \cap C^1([0, T]; X_\mu)$. In the differential equation of (3.4) take the inner product in $X_\mu$ of each term by $u(t)$, for any $t \in (0, T)$. The result is

$$\frac{1}{2} \frac{d}{dt} \left( \frac{\|u(t)\|_{X_\mu}^2}{\|X_\mu\|} + \frac{\|u_x(t, \cdot)\|_{X_\mu}^2}{\|X_\mu\|} \right) - \frac{\|a u_x(t, 1)\|_{X_\mu}^2}{\beta_1} - \frac{\|a(0) u_x(t, 0)\|_{X_\mu}^2}{\beta_0}$$

$$+ \frac{\gamma_0}{\beta_0} a(0) u^2(t, 0) + \frac{\gamma_1}{\beta_1} a(1) u^2(t, 1) \leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{X_\mu}^2 + \frac{1}{2} \|h(t)\|_{X_\mu}^2,$$

and the regularity of $u(t)$ implies that also $u_x(\cdot, 0)$ and $u_x(\cdot, 1)$ are in $L^2(0, T)$. Hence we deduce that

$$\frac{d}{dt} \|u(t)\|_{X_\mu}^2 \leq \frac{d}{dt} \|u(t)\|_{X_\mu}^2 + 2 \|u_x(t, \cdot)\|_{X_\mu}^2$$

$$\leq 2 \left[ \frac{\|a(1) u_x(t, 1)\|_{X_\mu}^2}{\beta_1} + \frac{\|a(0) u_x(t, 0)\|_{X_\mu}^2}{\beta_0} \right] + \|u(t)\|_{X_\mu}^2 + \|h(t)\|_{X_\mu}^2.$$  \hspace{1cm} (3.6)

By Gronwall’s Lemma, for any $t \in [0, T]$ one has

$$\|u(t)\|_{X_\mu}^2 \leq e^T \left[ \left( \int_0^T u_x^2(t, 1) \right) \cdot \frac{2 a^2(1)}{\beta_1} + \left( \int_0^T u_x^2(t, 0) \right) \cdot \frac{2 a^2(0)}{\beta_0} \right] + \|u_0\|_{X_\mu}^2 + \|h(t)\|_{X_\mu}^2.$$
Thus, there exists a positive constant $\hat{C}$ such that
\[
\sup_{t \in [0,T]} \| u(t) \|_{X_\mu} \leq \hat{C} \left[ \| u_x(\cdot, 1) \|_{L^2(0,T)}^2 + \| u_x(\cdot, 0) \|_{L^2(0,T)}^2 + \| u_0 \|_{X_\mu}^2 + \| h \|_{L^2(Q_T)}^2 \right].
\] (3.7)

Observe that, by integrating the second inequality of (3.6) over $(0, T)$ and using (3.7), we have
\[
\int_0^T \| \sqrt{a} u_x(t) \|_{X_\mu}^2 \, dt \leq \hat{C} \left[ \| u_x(\cdot, 1) \|_{L^2(0,T)}^2 + \| u_x(\cdot, 0) \|_{L^2(0,T)}^2 + \| u_0 \|_{X_\mu}^2 + \| h \|_{L^2(Q_T)}^2 \right],
\]
for a suitable positive constant $\hat{C}$. We can conclude that there exists a positive constant $K$ such that
\[
\sup_{t \in [0,T]} \| u(t) \|_{X_\mu} + \int_0^T \| u(t) \|_{H^1_x(0,1)}^2 \, dt \leq K \left[ \| u_x(\cdot, 0) \|_{L^2(0,T)}^2 + \| u_0 \|_{X_\mu}^2 + \| h \|_{L^2(Q_T)}^2 \right],
\] (3.8)
where $K$ depends on $T, a, \beta$. Then the assertion follows.

4. Operators in divergence form with (GWBC): The strongly degenerate case. Now let us assume that $a$ is strongly degenerate and consider the operator $(A_1, \hat{D}(A_1))$, where $A_1$ is defined as in the previous Section and $\hat{D}(A_1)$ is obtained by replacing, in the definition of $D(A_1)$, the space $H^2_{a}(0,1)$ by the following
\[
\hat{H}^2_{a,s}(0,1) := \{ u \in \hat{H}^1_{a,s}(0,1) \, | \, au' \in H^1(0,1) \}.
\]
Here
\[
\hat{H}^1_{a,s}(0,1) := \{ u \in X_\mu \, | \, u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \sqrt{a} u' \in X_\mu \}.
\]

In analogy with Proposition 2.7 one has the following

**Proposition 4.1.** Let $a$ be strongly degenerate and define
\[
\hat{X} := \{ u \in X_\mu \, | \, u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \sqrt{a} u' \in X_\mu, au \in H^1(0,1) \text{ and } (au)(x_0) = 0 \},
\]
\[
\hat{Z} := \{ u \in \hat{X} \, | \, au' \in H^1(0,1), (au')(x_0) = 0 \}.
\]
Then
\[
\hat{H}^1_{a,s}(0,1) = \hat{X} \quad \text{and} \quad \hat{H}^2_{a,s}(0,1) = \hat{Z}.
\]

**Remark 4.2.** Let us observe that $\hat{H}^1_{a,s}(0,1) \subset H^1_{a}(0,1)$ and $\hat{H}^2_{a,s}(0,1) \subset H^2_{a}(0,1)$. It follows that for any $(u,v) \in \hat{H}^2_{a,s}(0,1) \times \hat{H}^1_{a,s}(0,1)$ the Green formula in Lemma 2.4(i) holds. Moreover $C^\infty[0,1] \subset \hat{H}^2_{a,s}(0,1)$.

As for the weakly degenerate case, one has the next result.
Theorem 4.3. If \( a \) is strongly degenerate, then the operator \((A_1, \hat{D}(A_1))\) is nonpositive and selfadjoint on \( X_\mu \).

Proof. Let us introduce the space

\[
W_{1,as}(0,1) = \{ u \in \tilde{H}^2_{as}(0,1) \mid \text{supp} \{ u \} \text{ compact, supp} \{ u \} \subset [0,1] \setminus \{ x_0 \}, \quad A_1 u(j) + (-1)^{j+1} \beta_j u'(j) + \gamma_j u(j) = 0, \ j = 0, 1, \}
\]

and observe that

\[
W_{1,as}(0,1) \subset \hat{D}(A_1) \subset X_\mu.
\]

Then, analogous arguments as in Theorem 3.1 imply that \( W_{1,as}(0,1) \) is dense in \( X_\mu \) and, hence, \( \hat{D}(A_1) \) is dense in \( X_\mu \). Moreover, as a consequence of Lemma 2.3 (i) and (GWBC), by arguing as in Theorem 3.1, one can show that \( A_1 \) is symmetric and nonpositive. In order to prove that \( I - A_1 \) is surjective, observe that \( W_{1,as}(0,1) \) is a Hilbert space with respect to the inner product

\[
(u,v)_{W_{1,as}} = \int_0^1 (uv + au'v') \, dx + \frac{a(0)u(0)v(0)}{\beta_0} + \frac{a(1)u(1)v(1)}{\beta_1},
\]

for any \( u, v \in W_{1,as}(0,1) \). Notice that

\[
W_{1,as}(0,1) \hookrightarrow X_\mu \hookrightarrow (W_{1,as}(0,1))^*,
\]

where \((W_{1,as}(0,1))^*\) is the dual space of \( W_{1,as}(0,1) \) with respect to \( X_\mu \). Let \( f \in X_\mu \) and define \( F : W_{1,as}(0,1) \mapsto \mathbb{R} \) such that

\[
F(v) = \int_0^1 f v \, dx + \frac{a(0)f(0)v(0)}{\beta_0} + \frac{a(1)f(1)v(1)}{\beta_1}.
\]

From \( W_{1,as}(0,1) \hookrightarrow X_\mu \) it follows that \( F \in (W_{1,as}(0,1))^* \). Hence, by the Riesz’s Theorem, there exists a unique \( u \in W_{1,as}(0,1) \) such that for any \( v \in W_{1,as}(0,1) \) we have

\[
(u,v)_{W_{1,as}} = F(v) = \int_0^1 f v \, dx + \frac{a(0)f(0)v(0)}{\beta_0} + \frac{a(1)f(1)v(1)}{\beta_1}.
\]

The above equality means that

\[
\int_0^1 au'v' \, dx = \int_0^1 (f - u) v \, dx + \frac{a(0)(f - u)(0)v(0)}{\beta_0} + \frac{a(1)(f - u)(1)v(1)}{\beta_1} \tag{4.1}
\]

for all \( v \in W_{1,as}(0,1) \).

Let us denote by \( C_c^\infty((0,1) \setminus \{ x_0 \}) \) the space of all \( C^\infty(0,1) \) functions that vanish in a neighborhood of \( x_0 \), with compact support in \((0,1) \setminus \{ x_0 \}\). From \( C_c^\infty((0,1) \setminus \{ x_0 \}) \subset W_{1,as}(0,1) \), it follows that (4.1) holds for all \( v \in C_c^\infty((0,1) \setminus \{ x_0 \}) \). This implies

\[
\int_0^1 au'v' \, dx = \int_0^1 (f - u) v \, dx
\]

for all \( v \in C_c^\infty((0,1) \setminus \{ x_0 \}) \). Thus the weak derivative \((au)'\) in this context exists and \((au)' \in L^2(0,1) \). Hence, \( u \in \hat{D}(A_1) \) and \( u - A_1 u = f \).

Thus the analogous of Theorem 3.4 holds for (3.4) in the case \( a \) strongly degenerate. In addition we have a characterization of \( \hat{D}(A_1) \).
Proposition 4.4. Let
\[
D := \{u \in X_\mu \mid u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\},
\]
\[au \in H^1(0,1), au' \in H^1(0,1), (au)(x_0) = (au')(x_0) = 0
\]
and \(A_1u(j) + (-1)^{j+1}\beta_ju'(j) + \gamma_ju(j) = 0, j = 0, 1\).

Then
\[
\hat{D}(A_1) = D.
\]

Proof. \(D \subseteq \hat{D}(A_1)\): Let \(u \in D\). It is sufficient to prove that \(\sqrt{a}u' \in X_\mu\). Since \(au' \in H^1(0,1)\) and \(u\) is locally absolutely continuous in \([0,1] \setminus \{x_0\}\), the terms \([au'u](1)\) and \([au'u](x)\), for \(x \in (x_0, 1]\), are indeed well defined, so we have
\[
\int_0^{1}[(au')u(s)]ds = \int_0^{1}(au')^2(s)ds - \int_0^{1}(a(\nu')^2(s)ds.
\]
Thus
\[
(au')(x) = (au')(1) - \int_0^{1}(au')u(s)ds - \int_0^{1}(a(\nu')^2(s)ds.
\]
Since \(u \in D\), then \((au') \in L^1(0,1)\). Hence there exists
\[
\lim_{x \to x_0^+} (au')(x) = L \in [-\infty, +\infty),
\]
since no integrability is known about \((au')^2\) and such a limit could be \(-\infty\). If \(L \neq 0\), there exists \(C > 0\) such that
\[
|\[(au')u(s)]| \geq C
\]
for all \(x\) in a right neighborhood of \(x_0, x \neq x_0\). Thus by (2.2) there exists \(C_1 > 0\) such that
\[
|u(x)| \geq \frac{C}{\|(au')(x)\|} \geq \frac{C_1}{\sqrt{x - x_0}},
\]
for all \(x\) in a right neighborhood of \(x_0, x \neq x_0\). This implies that \(u \notin L^2(0,1)\) and thus \(u \notin X_\mu\). Hence \(L = 0\) and
\[
\int_0^{1}[(au')u(s)]ds = [au'u](1) - \int_0^{1}(au')^2(s)ds.
\]
If \(x \in [0, x_0)\), proceeding as before, it follows that
\[
\int_0^{x_0}[(au')u(s)]ds = [au'u](0) - \int_0^{x_0}(au')^2(s)ds.
\]
By (4.2) and (4.3), it follows that
\[
\int_0^{1}[(au')u(s)]ds = [au'u]_{x=0} - \int_0^{1}(au')^2(s)ds.
\]
Since \((au')'u \in L^1(0,1)\), then \(\sqrt{a}u' \in L^2(0,1)\). Thus \(\sqrt{a}u' \in X_\mu\) and hence, \(D \subseteq \hat{D}(A_1)\).

\(\hat{D}(A_1) \subseteq D\): Let \(u \in \hat{D}(A_1)\). By Remark 4.2 and Proposition 2.7 we know that \(au \in H^1(0,1)\) and \((au)(x_0) = 0\). Thus it is sufficient to prove that \((au')(x_0) = 0\). Toward this
end, as in [9], observe that, since \( au' \in H^1(0, 1) \), there exists \( L \in \mathbb{R} \) such that
\[
\lim_{x \to x_0} (au')(x) = (au')(x_0) = L.
\]
If \( L \neq 0 \), there exists \( C > 0 \) such that
\[
|(au')(x)| \geq C,
\]
for all \( x \) in a neighborhood of \( x_0 \), \( x \neq x_0 \). Thus
\[
|(a(u')^2)(x)| \geq \frac{C^2}{a(x)},
\]
for all \( x \) in a neighborhood of \( x_0 \), \( x \neq x_0 \). This implies that \( \sqrt{a}u' \notin L^2(0, 1) \) and thus \( \sqrt{a}u' \notin X_\mu \). Hence \( L = 0 \) and so \( (au')(x_0) = 0 \).

We point out the fact that the condition \( \frac{1}{a} \notin L^1(0, 1) \) is crucial to prove the previous characterization.

5. Operators in non divergence form with (GWBC). Let us fix \( \beta_1, \gamma_j \in \mathbb{R} \) such that \( \beta_j > 0 \), \( \gamma_j \geq 0 \), \( j = 0, 1 \). Consider a weakly or strongly degenerate function \( a \) and define the operator in nondivergence form \( A_2u = au'' \) equipped with the general Wentzell boundary conditions
\[
A_2u(j) + (a - 1)\beta u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1 \tag{GWBC}
\]
and the space
\[
Y_\mu := L^2_\frac{1}{2}([0, 1], d\mu),
\]
where the space \( L^2_\frac{1}{2}(0, 1) \) has been defined in Section 2, and
\[
d\mu := \frac{dx}{a} |_{\{0,1\}} \otimes \frac{dS}{\beta} |_{\{0,1\}}.
\]
As usual, \( dx \) denotes the Lebesgue measure on \( (0, 1) \), \( \beta = (\beta_0, \beta_1) \), and \( dS \) denotes the natural (Dirac) measure \( dS \) on \( \{0, 1\} \) with weight \( \frac{1}{\beta} \). Thus \( Y_\mu \) is a Hilbert space with the inner product given by
\[
(f, g)_{Y_\mu} = \int_0^1 \frac{f(x)g(x)}{a} \, dx + \frac{f(1)g(1)}{\beta_1} + \frac{f(0)g(0)}{\beta_0},
\]
where \( f, g \in Y_\mu \), are written as \( (f \chi_{(0,1)}, (f(0), f(1))), (g \chi_{(0,1)}, (g(0), g(1))) \).

Hence \( Y_\mu \) is equipped with the norm defined by
\[
\|f\|_{Y_\mu}^2 = \int_0^1 \frac{|f(x)|^2}{a} \, dx + \frac{|f(1)|^2}{\beta_1} + \frac{|f(0)|^2}{\beta_0},
\]
for any \( f \in Y_\mu \), which is written as \( (f \chi_{(0,1)}, (f(0), f(1))) \).

Let us now introduce the following spaces
\[
\tilde{H}^1_\frac{1}{2}(0, 1) := Y_\mu \cap H^1(0, 1),
\]
\[
\tilde{H}^2_\frac{1}{2}(0, 1) := \left\{ u \in \tilde{H}^1_\frac{1}{2}(0, 1) \mid u' \in H^1(0, 1) \right\},
\]
endowed, respectively, with the associated norms
\[
\|u\|_{\tilde{H}^1_\frac{1}{2}(0, 1)}^2 := \|u\|^2_{Y_\mu} + \|u'\|^2_{L^2(0, 1)}, \quad \text{for all } u \in \tilde{H}^1_\frac{1}{2}(0, 1),
\]
and
\[
\|u\|_{\tilde{H}^2_\frac{1}{2}(0, 1)}^2 := \|u\|^2_{\tilde{H}^1_\frac{1}{2}(0, 1)} + \|au''\|^2_{Y_\mu}, \quad \text{for all } u \in \tilde{H}^2_\frac{1}{2}(0, 1).
\]
Remark 5.1. Let us observe that \( \overline{H}^1_\pi(0,1) \subset H^1_\pi(0,1) \) and \( \overline{H}^2_\pi(0,1) \subset H^2_\pi(0,1) \).

It follows that for any \((u,v) \in \overline{H}^2_\pi(0,1) \times \overline{H}^1_\pi(0,1)\) the Green formula in Lemma 2.4 (ii) holds.

Let us define the domain of \(A_2\) as follows

\[
D(A_2) := \{u \in \overline{H}^2_\pi(0,1)| A_2u(j) + (-1)^{j+1}\beta_j u'(j) + \gamma_j u(j) = 0, \ j = 0,1\}.
\]

As a consequence of the results in Section 2, one has the next result.

Theorem 5.2. If \(a\) is weakly degenerate, then the operator \((A_2,D(A_2))\) is selfadjoint and nonpositive on \(Y_\mu\).

Proof. Since \(C^\infty_\mathbb{C}(0,1) \subset D(A_2)\), the domain \(D(A_2)\) is dense in \(Y_\mu\). In order to show that \(A_2\) is nonpositive and selfadjoint it suffices to prove that \(A_2\) is symmetric, nonpositive and \((I - A_2)(D(A_2)) = Y_\mu\) (see e.g. [1, Theorem B.14] or [15]).

\(A_2\) is symmetric.

By Lemma 2.4(ii), for any \(u,v \in D(A_2)\), one has

\[
\langle A_2u,v \rangle_{Y_\mu} = \int_0^1 \frac{a'u''v}{a} dx + \frac{a(0)u''(0)v(0)}{\beta_0} + \frac{a(1)u''(1)v(1)}{\beta_1} = \int_0^1 u'v' + (\beta_0 u'(0) - \gamma_0 u(0))v(0) + (\beta_1 u'(1) + \gamma_1 u(1))v(1) = \langle u,A_2v \rangle_{Y_\mu}.
\]

\(A_2\) is nonpositive.

For any \(u \in D(A_2)\), according to the previous calculations, one has

\[
\langle A_2u,u \rangle_{Y_\mu} = -\int_0^1 |u'|^2 dx - \frac{\gamma_0}{\beta_0}|u(0)|^2 - \frac{\gamma_1}{\beta_1}|u(1)|^2 \leq 0.
\]

\(I - A_2\) is surjective.

Similar arguments as in Theorem 3.2 can be applied.

Further, if \(a\) is strongly degenerate analogous results as in Proposition 2.10 hold, provided that one replaces the spaces \(H^1_\pi(0,1)\) and \(H^2_\pi(0,1)\) by the spaces \(\overline{H}^1_\pi(0,1)\) and \(\overline{H}^2_\pi(0,1)\), respectively. Thus one can deduce the following.

Theorem 5.3. If \(a\) is strongly degenerate and Hypothesis 2.1 is satisfied, then the operator \((A_2,D(A_2))\) is selfadjoint and nonpositive on \(Y_\mu\).

Proof. First, by using similar arguments as in [9, Proposition 3.8] one can show that

\[
D(A_2) := \{u \in \overline{H}^1_\pi(0,1) | \text{ au'' } \in L^2_\pi(0,1), \text{ au' } \in H^1(0,1), \ u(x_0) = (au')(x_0) = 0, \text{ and } A_2u(j) + (-1)^{j+1}\beta_j u'(j) + \gamma_j u(j) = 0, \ j = 0,1\}.
\]
Now let us introduce the subspace

\[ W_{1,\frac{1}{2}}(0,1) = \{ v \in H^2(0,1) \cap L^2_{\frac{1}{2}}(0,1) \mid \text{supp}\{v\} \text{ compact, supp}\{v\} \subset [0,1) \setminus \{x_0\}, \]

\[ A_2 v(j) + (-1)^{j+1} \beta_j v'(j) + \gamma_j v(j) = 0, \quad j = 0,1, \]

endowed with the norm

\[ \|v\|_{W_{1,\frac{1}{2}}(0,1)}^2 := \|v\|_{\tilde{H}_{\frac{1}{2}}^2(0,1)}^2. \]

It is evident that

\[ W_{1,\frac{1}{2}}(0,1) \subset D(A_2) \subset Y_\mu. \]

Moreover, \( W_{1,\frac{1}{2}}(0,1) \) is dense in \( Y_\mu \) by using the following argument: take \( v \in Y_\mu \) and define \( v_n := \xi_n v, \ n \geq \max \left\{ \frac{2}{1-x_0}, \frac{4}{1-x_0} \right\} \), where \( \xi_n \) is defined in (3.1). It is clear that \( v_n \to v \) in \( Y_\mu \) as \( n \to \infty \). Hence \( D(A_2) \) is dense in \( Y_\mu \).

Similar arguments as in the proof of Theorem 5.1 show that \( A_2 \) is a symmetric and nonpositive operator. Let us show that \( I-A_2 \) is surjective, i.e. \( (I-A_2)^{-1} \) exists and \( (I-A_2)^{-1} \) is dense.

As in Section 4, let us denote by \( C_c^\infty((0,1)\setminus \{x_0\}) \) the space of \( C_c^\infty((0,1)) \) functions that vanish in a neighborhood of \( x_0 \), with compact support in \( (0,1) \setminus \{x_0\} \). Since \( C_c^\infty((0,1)\setminus \{x_0\}) \subset W_{1,\frac{1}{2}}(0,1) \), (5.2) holds for all \( v \in C_c^\infty((0,1)\setminus \{x_0\}) \), i.e.

\[ \int_0^1 u v' \, dx = \int_0^1 \frac{(f-u) v}{a} \, dx + \frac{(f-u)(1)v(1)}{\beta_1} + \frac{(f-u)(0)v(0)}{\beta_0}, \]

for all \( v \in W_{1,\frac{1}{2}}(0,1) \).

As in Section 4, let us denote by \( C_c^\infty((0,1)\setminus \{x_0\}) \) the space of \( C_c^\infty((0,1)) \) functions that vanish in a neighborhood of \( x_0 \), with compact support in \( (0,1) \setminus \{x_0\} \). Since \( C_c^\infty((0,1)\setminus \{x_0\}) \subset W_{1,\frac{1}{2}}(0,1) \), (5.2) holds for all \( v \in C_c^\infty((0,1)\setminus \{x_0\}) \), i.e.

\[ \int_0^1 u v' \, dx = \int_0^1 \frac{(f-u) v}{a} \, dx, \quad v \in C_c^\infty((0,1)\setminus \{x_0\}). \]

Thus the weak derivative \( u'' \) in this context exists and \( au'' \in L^2_{\frac{1}{2}}(0,1) \). Hence, \( u \in D(A_2) \), and, according to (5.3) and Lemma 2.9 (ii), we have that

\[ u - A_2 u = f. \]
As a consequence of Theorems 5.2 and 5.3, one has that \( A_2 \) is the infinitesimal generator of a strongly continuous semigroup on \( Y_{\mu} \). Hence, the problem

\[
\begin{align*}
& \begin{cases} 
  u_t - A_2 u = h(t, x), & (t, x) \in Q_T, \\
  (GWBC),
\end{cases} \\
  u(0, x) = u_0(x), & x \in (0, 1),
\end{align*}
\]

is well-posed in the sense of evolution operator theory. In particular, the following theorem holds.

**Theorem 5.4.** Assume that \( a \) is weakly degenerate (resp. strongly degenerate and the Hypothesis 2.1 is satisfied). Then for all \( h \in L^2(0, T; Y_{\mu}) \) and \( u_0 \in Y_{\mu} \), there exists a unique weak solution \( u \in C([0, T]; Y_{\mu}) \cap L^2(0, T; H_{\mu}^1(0, 1)) \) of (5.4) such that

\[
\sup_{t \in [0, T]} \| u(t) \|_{Y_{\mu}}^2 \leq C \left[ \| u_0 \|_{Y_{\mu}}^2 + \| h \|_{L^2(Q_T)}^2 \right],
\]

where the constant \( C \) in (5.5) depends on \( T, a, \beta \), but is independent of \( u_0 \) and \( h \). Moreover, if \( u_0 \in D(A_2) \), then \( u \in C([0, T]; D(A_2)) \cap C^1([0, T]; Y_{\mu}) \).

**Appendix.** In the following we will give the proofs of Lemmas 2.4 and 2.9.

**Proof of Lemma 2.4** (i). The proof is analogous to that of [9, Lemma 2.1], but we present it for the readers’ convenience. Let \((u, v) \in H_a^2(0, 1) \times H_a^1(0, 1)\). For any sufficiently small \( \delta > 0 \) one has

\[
\begin{align*}
\int_0^1 (au')'vdx &= \int_0^{x_0-\delta} (au')'vdx + \int_{x_0-\delta}^{x_0+\delta} (au')'vdx + \int_{x_0+\delta}^1 (au')'vdx \\
&= (au')(x_0 - \delta) - (au')(0) \\
&- \int_0^{x_0-\delta} au'v'dx + \int_{x_0-\delta}^{x_0+\delta} (au')'vdx \\
&+ (au')(1) - (au')(x_0 + \delta) - \int_{x_0+\delta}^1 au'v'dx \\
&= [au']_{x=x_0}^{x=x_0+\delta} + (au')(x_0 - \delta) - \int_0^{x_0-\delta} au'v'dx + \int_{x_0-\delta}^{x_0+\delta} (au')'vdx \\
&- (au')(x_0 + \delta) - \int_{x_0+\delta}^1 au'v'dx,
\end{align*}
\]

(A-1)

since \( au' \in H^1(0, 1) \). Now we prove that

\[
\lim_{\delta \to 0} \int_{x_0-\delta}^{x_0} au'v'dx = \int_0^{x_0} au'v'dx, \quad \lim_{\delta \to 0} \int_{x_0+\delta}^1 au'v'dx = \int_{x_0}^1 au'v'dx
\]

and

\[
\lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} (au')'v dx = 0. \quad (A-2)
\]

Toward this end, observe that

\[
\int_0^{x_0-\delta} au'v'dx = \int_{x_0}^{x_0-\delta} au'v'dx - \int_0^{x_0} au'v'dx
\]

(A-3)
and

\[ \int_{x_0 + \delta}^{1} au'v' dx = \int_{x_0}^{1} au'v' dx - \int_{x_0}^{x_0 + \delta} au'v' dx. \]  \hspace{1cm} (A-4)

Moreover, \((au')'v\) and \(au'v'\) \(\in L^1(0,1)\). Thus for any \(\epsilon > 0\), by the absolute continuity of the integral, there exists \(\delta := \delta(\epsilon) > 0\) such that

\[ \left| \int_{x_0 - \delta}^{x_0} au'v' dx \right| \leq \int_{x_0 - \delta}^{x_0} |au'v'| dx < \epsilon, \]

\[ \left| \int_{x_0 + \delta}^{x_0} (au')'v dx \right| \leq \int_{x_0 - \delta}^{x_0 + \delta} |(au')'v| dx < \epsilon, \]

\[ \left| \int_{x_0}^{x_0 + \delta} au'v' dx \right| \leq \int_{x_0}^{x_0 + \delta} |au'v'| dx < \epsilon. \]

Now take such a \(\delta\) in (A-1). Thus \(\epsilon\) being arbitrary,

\[ \lim_{\delta \to 0} \int_{x_0 - \delta}^{x_0} au'v' dx = \lim_{\delta \to 0} \int_{x_0}^{x_0 + \delta} (au')'v dx = \lim_{\delta \to 0} \int_{x_0}^{x_0 + \delta} au'v' dx = 0. \]

The previous equalities and (A-3), (A-4) imply

\[ \lim_{\delta \to 0} \int_{x_0 - \delta}^{1} au'v' dx = \int_{0}^{1} au'v' dx \]

\[ \lim_{\delta \to 0} \int_{x_0 + \delta}^{1} au'v' dx = \int_{x_0}^{1} au'v' dx. \]

(A-5)

In order to obtain the desired result it is sufficient to prove that

\[ \lim_{\delta \to 0} (au'(x_0 - \delta)) = \lim_{\delta \to 0} (au'(x_0 + \delta)). \]

Since \(au' \in H^1(0,1)\) and \(v \in H^1_a(0,1)\),

\[ \lim_{\delta \to 0} (au'(x_0 - \delta)) = (au')(x_0) = \lim_{\delta \to 0} (au')(x_0 + \delta). \]  \hspace{1cm} (A-6)

Thus by (A-1), (A-2), (A-5) and (A-6), it follows that

\[ \int_{0}^{1} (au')'v dx = [au'v]_{x=0}^{x=1} - \int_{0}^{1} au'v' dx. \]

\[ \square \]

**Proof of Lemma 2.4 (ii).** It is trivial, since \((u,v) \in H^2_\alpha(0,1) \times H^1_\beta(0,1)\). \[ \square \]

**Proof of Lemma 2.9 (i).** Let \((u,v) \in H^2_\alpha(0,1) \times H^1_\beta(0,1)\). As for the weak case, one can prove that, for any \(\delta > 0\),

\[ \int_{0}^{1} (au')'v dx = [au'v]_{x=0}^{x=1} + (au')(x_0 - \delta) - \int_{0}^{x_0 - \delta} au'v' dx \]

\[ + \int_{x_0 + \delta}^{x_0} (au')'v dx - (au')(x_0 + \delta) - \int_{x_0 + \delta}^{1} au'v' dx. \]

Moreover

\[ \lim_{\delta \to 0} \int_{0}^{x_0 - \delta} au'v' dx = \int_{0}^{x_0} au'v' dx, \quad \lim_{\delta \to 0} \int_{x_0 + \delta}^{1} au'v' dx = \int_{x_0}^{1} au'v' dx \]
and
\[ \lim_{\delta \to 0} \int_{x_0 - \delta}^{x_0 + \delta} (au')'vdx = 0. \]

In order to obtain the desired result it is sufficient to prove that
\[ \lim_{\delta \to 0} (au')(x_0 - \delta) = \lim_{\delta \to 0} (au')(x_0 + \delta). \] (A-7)

First of all, observe that, since \( au' \in H^1(0,1) \) and \( v \) is locally absolutely continuous on \([0, x_0) \cup (x_0, 1]\), the terms \((au')(0)\) and \((au')(1)\) are indeed well defined. Moreover,
\[ (au')(x_0 - \delta) = (au')(0) + \int_{0}^{x_0 - \delta} ((au')'(s))ds + \int_{0}^{x_0 - \delta} (au')(s)ds \]
and
\[ (au')(x_0 + \delta) = (au')(1) - \int_{x_0 + \delta}^{1} ((au')'(s))ds - \int_{x_0 + \delta}^{1} (au')(s)ds. \]

Since \((au')', v \in L^2(0,1)\) and \(\sqrt{au'}, \sqrt{au'} \in L^2(0,1)\), by Hölder’s inequality, \((au')'v \in L^1(0,1)\) and \(au'v \in L^1(0,1)\). Thus there exist \(L_1, L_2 \in \mathbb{R}\) such that
\[ \lim_{\delta \to 0} (au')(x_0 - \delta) = (au')(0) + \lim_{\delta \to 0} \int_{0}^{x_0 - \delta} ((au')'(s))ds + \lim_{\delta \to 0} \int_{0}^{x_0 - \delta} (au')(s)ds = L_1 \]
and
\[ \lim_{\delta \to 0} (au')(x_0 + \delta) = (au')(1) - \lim_{\delta \to 0} \int_{x_0 + \delta}^{1} ((au')'(s))ds - \lim_{\delta \to 0} \int_{x_0 + \delta}^{1} (au')(s)ds = L_2. \]

If \(L_1 \neq 0\), then there exists \(C > 0\) such that
\[ |(au')(x)| \geq C \]
for all \(x\) in a left neighborhood of \(x_0, x \neq x_0\). Thus by (2.2),
\[ |v(x)| \geq \frac{C}{|(au')(x)|} \geq \frac{C_1}{\sqrt{x_0 - x}} \]
for all \(x\) in a left neighborhood of \(x_0, x \neq x_0\), and for a suitable positive constant \(C_1\). This implies that \(v \notin L^2(0,1)\). Hence \(L_1 = 0\). Analogously, one can prove that \(L_2 = 0\). Thus (A-7) holds. In particular,
\[ \lim_{\delta \to 0} (au')(x_0 - \delta) = \lim_{\delta \to 0} (au')(x_0 + \delta) = 0 \]
and the desired result follows. \(\square\)

**Proof of Lemma 2.9 (ii).** It is trivial, since \((u, v) \in H^1_\pi (0,1) \times H^1_\pi (0,1)\). \(\square\)
Acknowledgments. Genni Fragnelli, Rosa Maria Mininni and Silvia Romanelli are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

The research of Genni Fragnelli is partially supported by the research project “Sistemi con operatori irregolari” of the GNAMPA-INdAM.

REFERENCES

[1] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, \textit{Vector-Valued Laplace Transforms and Cauchy Problems}, Monographs in Mathematics, \textbf{96}, Birkhäuser Verlag, Basel, 2001.

[2] J. M. Ball, Strongly continuous semigroups, weak solutions and the variation of constant formula, \textit{Proc. Amer. Math. Soc.}, \textbf{63} (1977), 370–373.

[3] G. I. Boutaayamou, G. Fragnelli and L. Maniar, Lipschitz stability for linear parabolic systems with interior degeneracy, \textit{Electron. J. Differential Equations}, \textbf{2014} (2014), 1–26.

[4] G. I. Boutaayamou, G. Fragnelli and L. Maniar, Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions, \textit{J. Anal. Math.}, in press. \texttt{arXiv:1509.00863}.

[5] G. I. Boutaayamou, G. Fragnelli and L. Maniar, Inverse problems for parabolic equations with interior degeneracy and Neumann boundary conditions, \textit{J. Inverse Ill-Posed Probl}, \textbf{15} (2015), 27–51.

[6] T. Cazenave and A. Haraux, \textit{An Introduction to Semilinear Evolution Equations}, Oxford Lecture Series in Mathematics and its Applications, \textbf{13} (1998), Oxford University Press.

[7] G. M. Coclite, A. Favini, C. G. Gal, G. R. Goldstein, J. A. Goldstein, E. Obrecht and S. Romanelli, The role of Wentzell boundary conditions in linear and nonlinear analysis, in \textit{Advances in nonlinear analysis: Theory, methods and applications}, Math. Probl. Eng. Aerosp. Sci. \textbf{3}, Camb. Sci. Publ., Cambridge, (2009), 277–289.

[8] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, \textit{The heat equation with generalized Wentzell boundary conditions}, \textit{J. Evol. Equ.}, \textbf{2} (2002), 1–19.

[9] G. Fragnelli, G. R. Goldstein, J. A. Goldstein and S. Romanelli, Generators with interior degeneracy on spaces of $L^2$ type, \textit{Electron. J. Differential Equations}, \textbf{2012} (2012), 1–30.

[10] G. Fragnelli, G. Marinoschi, R. M. Mininni and S. Romanelli, A control approach for an identification problem associated to a strongly degenerate parabolic system with interior degeneracy, in: \textit{New Prospects in direct, inverse and control problems for evolution equations (eds. A. Favini, G. Fragnelli and R.M. Mininni)}, Springer INdAM Ser. \textbf{10}, Springer, Cham, (2014), 121–139.

[11] G. Fragnelli, G. Marinoschi, R. M. Mininni and S. Romanelli, Identification of a diffusion coefficient in strongly degenerate parabolic equations with interior degeneracy, \textit{J. Evol. Equ.}, \textbf{15} (2015), 27–51.

[12] G. Fragnelli and D. Mugnai, Carleman estimates and observability inequalities for parabolic equations with interior degeneracy, \textit{Adv. Nonlinear Anal.}, \textbf{2} (2013), 339–378.

[13] G. Fragnelli and D. Mugnai, Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations, \textit{Mem. Amer. Math. Soc.}, in press, \textbf{242} (2016), \texttt{arXiv:1508.04014}.

[14] G. R. Goldstein, Derivation and physical interpretation of general Wentzell boundary conditions, \textit{Adv. Differential Equations}, \textbf{11} (2006), 457–480.

[15] J. A. Goldstein, \textit{Semigroups of Linear Operators and Applications}, Oxford Univ. Press, Oxford, New York, 1985.

[16] A. Stahel, Degenerate semilinear parabolic equations, \textit{Differential Integral Equations}, \textbf{5} (1992), 683–691.

Received April 2015; revised September 2015.

\textit{E-mail address: genni.fragnelli@uniba.it}
\textit{E-mail address: ggoldste@memphis.edu}
\textit{E-mail address: jgoldste@memphis.edu}
\textit{E-mail address: rosamaria.mininni@uniba.it}
\textit{E-mail address: silvia.romanelli@uniba.it}