CHARACTER DEGREE SUMS IN FINITE NONSOLVABLE GROUPS

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Abstract. Let \( N \) be a minimal normal nonabelian subgroup of a finite group \( G \). We will show that there exists a nontrivial irreducible character of \( N \) of degree at least 5 which is extendible to \( G \). This result will be used to settle two open questions raised by Berkovich and Mann, and Berkovich and Zhmud.

1. Introduction and notations

All groups are finite. Let \( G \) be a group. Denote by \( \text{Irr}(G) \) the set of all complex irreducible characters of \( G \). Let \( N \) be a normal subgroup of \( G \). Let \( \theta \in \text{Irr}(N) \) be an irreducible character of \( N \). We say that \( \theta \) is extendible to \( G \) if there exists \( \chi \in \text{Irr}(G) \) such that the restriction of \( \chi \) to \( N \) is \( \theta \), that is \( \chi_N = \theta \). There are many papers devoted to finding sufficient conditions for \( \theta \) to be extendible to \( G \) (see Gallagher [7], Gagola [6] and [8, Chapter 8 and 11]). In this paper, we are interested in the existence problem, that is, assume \( N \) is a normal subgroup of \( G \), is there any non-trivial irreducible character of \( N \) that extends to \( G \)? We are mostly concerned with nonsolvable groups. Suppose that \( N \) is a minimal normal nonabelian subgroup of a group \( G \). In [3, Lemma 5], it is shown that there exists a nontrivial irreducible character \( \theta \) of \( N \) which is extendible to \( G \). In Theorem 1.1 below, we will show that \( \theta \) can be chosen with \( \theta(1) \geq 5 \). Using this result, we answer two open problems raised by Berkovich and Mann, and Berkovich and Zhmud.

Theorem 1.1. Suppose that \( N \) is a minimal normal nonabelian subgroup of a group \( G \). Then there exists an irreducible character \( \theta \) of \( N \) such that \( \theta \) is extendible to \( G \) with \( \theta(1) \geq 5 \).

Let \( T(G) \) be the sum of degrees of complex irreducible characters of \( G \), i.e \( T(G) = \sum_{\chi \in \text{Irr}(G)} \chi(1) \). Let \( k(G) \) be the number of conjugacy classes of \( G \) and let \( b(G) \) be the largest irreducible character degree of \( G \). Let \( N \) be a normal subgroup of \( G \). Denote by \( \text{Irr}(G,N) \) the set of all complex irreducible characters \( \chi \) of \( G \) such that \( N \not\leq \ker \chi \) and by \( T(G,N) \) the corresponding sum of degrees of all characters in \( \text{Irr}(G,N) \). It is obvious that \( \text{Irr}(G) = \text{Irr}(G/N) \cup \text{Irr}(G,N) \) and \( T(G) = T(G/N) + T(G,N) \). In [11, Theorem 8], Y. Berkovich and A. Mann showed that if \( G \) is nonsolvable then \( T(G) > 2|G : G'| \) and they asked whether or not \( T(G) > 2T(G/N) \), where \( N \) is a nonsolvable normal subgroup of \( G \). Here is our first result.

Theorem 1.2. Let \( N \) be a nonsolvable normal subgroup of a group \( G \). Then \( T(G) \geq 6T(G/N) \).

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This settles Question 4 in [1] or Problem 138 in [2]. By Schwarz inequality, it is easy to see that $T(G)^2 \leq |G|k(G)$. Hence $T(G)$ can be used to estimate $k(G)$. Moreover, as $T(G) \leq k(G)b(G)$, we can also get a lower bound for $b(G)$ in terms of $T(G)$ and $k(G)$. The reason that we are interested in the character degree sums comes from a question of Jan Saxl ([9, Problem 9.56]) which asked for the classification of groups in which the square of every irreducible character is multiplicity free. In fact if we could prove that $T(G/F(G)) < b(G/F(G))^2$, provided $G$ is nonsolvable, where $F(G)$ is the Fitting subgroup of $G$, then such a group in Jan Saxl's question is solvable. This will limit the possibilities for such groups. The proof for this fact is quite straightforward. Let $G$ be a minimal counter-example to the assertion that the square of every irreducible character of $G$ is multiplicity free but $G$ is not solvable. We first observe that if $N \trianglelefteq G$ and $\chi \in \text{Irr}(G/N)$ then $\chi \in \text{Irr}(G)$ and every irreducible constituent of $\chi^2$ in $G$ is also an irreducible character of $G/N$ so that $\chi^2$ is multiplicity free in $G/N$ since it is multiplicity free in $G$. Thus $G/N$ satisfies Saxl’s condition. Secondly, for any $\chi \in \text{Irr}(G)$, as $\chi^2$ is multiplicity free, it follows that $\chi(1)^2 = \chi(1) \leq T(G)$. Now combining these two observations for the quotient group $G/F(G)$, we obtain $b(G/F(G))^2 \leq T(G/F(G))$, where $b(G/F(G))$ is the largest character degree of $G/F(G)$. As $G$ is nonsolvable and the Fitting subgroup $F(G)$ is solvable, $G/F(G)$ is nonsolvable. Then the inequality mentioned above would provide a contradiction.

Denote by $T_1(G)$ the sum of degrees of nonlinear irreducible characters of $G$. Let $\text{Irr}_2(G) = \{\chi \in \text{Irr}(G) \mid \chi(1) > 2\}$ and let $T_2(G)$ be the sum of degrees of characters in $\text{Irr}_2(G)$. Observe that if $G$ does not have any irreducible characters of degree 2 then $T_1(G) = T_2(G)$, for example, this is the case if $G$ is a nonabelian simple group. The following result is a generalization of [1] Theorem 8.

**Theorem 1.3.** If $G$ is nonsolvable then $T_2(G) \geq 5|G : G'|$.

It is well known that a group $G$ is abelian if and only if $T(G) = k(G)$. The following theorem shows that the structure of $G$ is very restricted when $T(G)$ is small in terms of $k(G)$.

**Theorem 1.4.** If $T(G) \leq 2k(G)$ then $G$ is solvable.

This gives a positive answer to Problem 24 in [2]. We note that this property does not characterize the solvability of groups. In fact, let $G \cong 3^2 : 2S_4$, which is a maximal parabolic subgroup of $\text{PSL}(3,3)$. We have $T(G) = 50, k(G) = 11, T(G) > 4k(G)$ and $G$ is solvable. Now if $G \cong A_5$, then $T(G) = 16, k(G) = 5, T(G) < 4k(G)$ and $G$ is nonsolvable. We conjecture that a group $G$ is solvable provided that $T(G) \leq 3k(G)$.

2. Preliminaries

**Lemma 2.1.** Let $T$ be a non-abelian simple group. Then there exists a nontrivial irreducible character $\varphi$ of $T$ that extends to $\text{Aut}(T)$ with $\varphi(1) \geq 5$.

This is essentially Lemma 4.2 in [10] or [3, Theorems 2, 3, 4]. However, the fact that $\varphi(1) \geq 5$ is not explicitly stated there so that we will give a proof for completeness.

**Proof.** According to the Theorem of Classification of Finite Simple Groups, every nonabelian simple group is isomorphic to the alternating group of degree $n \geq 7$, a
sporadic group or a finite group of Lie type. We will consider the Tits group $^2F_4(2)$ as a sporadic rather than a finite group of Lie type. For alternating group $A_n, n \geq 7$, the irreducible character $\varphi$ corresponding to the partition $(n - 1, 1)$ extends to $S_n$ and $\varphi(1) = n - 1 \geq 6$. If $T$ is a sporadic group, Tits group or $A_5$, by inspecting [4], we can see that there exists an irreducible character $\varphi$ of $T$ that extends to $\text{Aut}(T)$ with $\varphi(1) \geq 5$. Finally assume $T$ is a finite group of Lie type defined over a field of size $q = p^l$, where $p$ is prime. Choose $\varphi$ to be the Steinberg character of $T$ of degree $|T|_p$, the order of the $p$-Sylow subgroup of $T$. Then $\varphi$ is extendible to $\text{Aut}(T)$ (see [5]). Moreover, we can easily check that $|T|_p > 5$ provided that $T \not\cong L_2(4) \cong L_2(5) \cong A_5$. Thus $\varphi(1) \geq 5$. This completes the proof. □

Lemma 2.2. (Gallagher [3 Corollary 6.17]). Let $N \leq G$ and let $\chi \in \text{Irr}(G)$ be such that $\chi_N = \vartheta \in \text{Irr}(N)$. Then the characters $\beta \chi$ for $\beta \in \text{Irr}(G/N)$ are irreducible, distinct for distinct $\beta$ and are all of the irreducible constituents of $\vartheta^G$.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. Since $N$ is a minimal normal nonabelian subgroup of $G$, there exists a nonabelian simple group $T$ such that $N = T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T, i = 1, \ldots, k$. Let $\varphi$ be an irreducible character of $T$ obtained from Lemma 2.1 and let $\theta = \varphi \times \varphi \times \cdots \times \varphi$. By [3 Lemma 5], $\theta \in \text{Irr}(N)$ and it is extendible to $G$. As $\varphi(1) \geq 5$, we have $\theta(1) = \varphi(1)^k \geq 5$. The proof is now complete. □

Proof of Theorem 1.2. We argue by induction on the order of $G$. Assume first that $N$ is a minimal normal subgroup of $G$. By Theorem 1.1, there exists an irreducible character $\varphi$ of $N$ which extends to an irreducible character $\chi$ of $G$ with $\chi(1) \geq 5$. Now by Lemma 2.2 there is an injective map from $\text{Irr}(G/N)$ to $\text{Irr}(G, N)$ which maps $\beta \in \text{Irr}(G/N)$ to $\beta \chi \in \text{Irr}(G, N)$, so that $\chi(1)T(G/N) \leq T(G, N)$. Therefore

$$T(G) = T(G/N) + T(G, N) \geq (1 + \chi(1))T(G/N) \geq 6T(G/N).$$

Let $K$ be a minimal normal subgroup of $G$ which is contained in $N$. Then $K$ is a proper subgroup of $N$. If $K$ is solvable, then $N/K$ is nonsolvable, and by inductive hypothesis, we have

$$T(G) \geq T(G/K) \geq 6T((G/K)/(N/K)) = 6T(G/N).$$

If $K$ is nonsolvable, then we can apply the result proved in the first paragraph to deduce that $T(G) \geq 6T(G/K)$. As $K \leq N$, we have $T(G/K) \geq T(G/N)$ so that $T(G) \geq 6T(G/K) \geq 6T(G/N)$. The proof is now complete. □

Proof of Theorem 1.3. Let $N$ be the last term of the derived series of $G$ and let $K$ be maximal among the normal subgroups of $G$ that are contained in $N$. Since $N = N' \leq G'$, it suffices to prove the result for $G/K$ so that we can assume that $K = 1$ and hence $N$ is a minimal normal nonabelian subgroup of $G$. By Theorem 1.1 there exists an irreducible character $\chi \in \text{Irr}(G)$, with $\chi(1) \geq 5$, and $\chi_N = \varphi \in \text{Irr}(N)$. Let $\psi = \chi_{G'}$. As $\psi_N = \varphi \in \text{Irr}(N)$, it follows that $\psi \in \text{Irr}(G')$ and hence $\chi_{G'} = \psi \in \text{Irr}(G')$ with $\chi \in \text{Irr}(G)$ and $\chi(1) \geq 5$. Now by Lemma 2.2 there is an injective map from $\text{Irr}(G/G')$ to $\text{Irr}_{2}(G)$ which maps $\beta \in \text{Irr}(G/G')$ to $\beta \chi \in \text{Irr}_{2}(G)$, so that $\chi(1)|G : G'| \leq T_2(G)$. Thus $T_2(G) \geq 5|G : G'|$. This finishes
Proof of Theorem 1.4. By way of contradiction, assume that $G$ is nonsolvable. Let $a$ be the number of linear characters of $G$, let $b$ be the number of irreducible characters of $G$ of degree 2 and finally let $c$ be the number of irreducible characters of degree greater than 2. We have

\begin{align}
(1) & \quad k(G) = a + b + c \\
(2) & \quad T(G) = a + 2b + T_2(G) \\
(3) & \quad T(G) \geq a + 2b + 3c
\end{align}

Since $T(G) \leq 2k(G)$, it follows from (1) and (3) that

\[ a + 2b + 3c \leq 2a + 2b + 2c \]

and hence

\[ c \leq a \]

Since $T(G) \leq 2k(G)$, it follows from (1) and (2) that

\[ a + 2b + T_2(G) \leq 2a + 2b + 2c \]

and so

\[ T_2(G) \leq a + 2c \]

Combining with (4), we obtain

\[ T_2(G) \leq 3a = 3|G : G'|. \]

However, this contradicts Theorem 1.3. Thus $G$ must be solvable. This completes the proof.

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