Representations of Regular Trees and Invariants of AR-Components for Generalized Kronecker Quivers

Daniel Bissinger

Abstract. We investigate the generalized Kronecker algebra $K_r = k\Gamma_r$ with $r \geq 3$ arrows. Given a regular component $C$ of the Auslander-Reiten quiver of $K_r$, we show that the quasi-rank $\text{rk}(C) \in \mathbb{Z}_{\leq 1}$ can be described almost exactly as the distance $W(C) \in \mathbb{N}_0$ between two non-intersecting cones in $C$, given by modules with the equal images and the equal kernels property; more precisely, we show that the two numbers are linked by the inequality

$$-W(C) \leq \text{rk}(C) \leq -W(C) + 3.$$ 

Utilizing covering theory, we construct for each $n \in \mathbb{N}_0$ a bijection $\varphi_n$ between the field $k$ and \{C | C regular component, W(C) = n\}. As a consequence, we get new results about the number of regular components of a fixed quasi-rank.

Introduction

Let $k$ be an algebraically closed field of arbitrary characteristic. The finite-dimensional algebra $K_r$ is defined as the path algebra of the quiver $\Gamma_r$ with vertices $1, 2, r \in \mathbb{N}$ arrows $\gamma_1, \ldots, \gamma_r : 1 \to 2$ and called the generalized Kronecker algebra. We denote by $\text{mod}K_r$ the category of finite-dimensional left-modules of $K_r$.

It is well known that for $r \geq 3$ the hereditary algebra $K_r$ is of wild representation type [11, 1.3, 1.5], every regular component in the Auslander-Reiten quiver of $K_r$ is of type $\mathbb{Z}A_\infty$ [16] and there is a bijection between the regular components and the ground field $k$ [2, XVIII 1.8]. Therefore, the problem of completely understanding the category $\text{mod}K_r$ or all regular components is considered hopeless and it is desirable to find invariants which give more specific information about the regular components.

One important invariant (for any wild hereditary algebra), introduced in [9], is the quasi-rank $\text{rk}(C) \in \mathbb{Z}$ of a regular component $C$. For a quasi-simple module $X$ in $C$, $\text{rk}(C)$ is defined as

$$\text{rk}(C) := \min \{l \in \mathbb{Z} \mid \forall m \geq l : \text{rad}(X, \tau^m X) \neq 0\},$$

where $\text{rad}(X,Y)$ is the space of non-invertible homomorphisms from $X$ to $Y$. Another interesting invariant $W(C) \in \mathbb{N}_0$ was recently introduced in [22]. Motivated by the representation theory of group algebras of $p$-elementary abelian groups of characteristic $p > 0$, the author defines the category EKP of modules with the equal kernels property and the category EIP of modules with the equal images property in the framework of $K_r$. She shows the existence of uniquely determined quasi-simple modules $M_C$ and $W_C$ in $C$ such that the cone ($M_C \to$) of all successors of $M_C$ satisfies ($M_C \to$) = EKP $\cap C$ and the cone ($\to W_C$) of all predecessors of $W_C$ satisfies ($\to W_C$) = EIP $\cap C$. The width $W(C)$ of $C$ is defined as the unique number $W(C) \in \mathbb{N}_0$ such that $\tau^{W(C)+1}M_C = W_C$, i.e. the distance between the two cones. Utilizing homological descriptions of EKP and EIP from [22] involving a family of elementary modules, we show that the two invariants $\text{rk}(C)$ and $W(C)$ are linked by the inequality

$$-W(C) \leq \text{rk}(C) \leq -W(C) + 3.$$
Motivated by this connection, we construct for each $n \in \mathbb{N}$ a regular component $C$ with $W(C) = n$. In order to do so, we consider representations over the universal covering $C_r$ of $\Gamma_r$.

We define classes $\text{Inj}, \text{Surj}$ of representations over $C_r$ such that $M \in \text{rep}(C_r)$ is in $\text{Inj}$ (resp. $\text{Surj}$) if and only if for each arrow $\delta: x \to y$ of $C_r$ the linear map $M(\delta): M_x \to M_y$ is injective (resp. surjective).

Let $\pi_\lambda: \text{rep}(C_r) \to \text{rep}(\Gamma_r)$ be the push-down functor [8, 2.7] and $M \in \text{rep}(C_r)$ be indecomposable. We prove that $M$ is in $\text{Inj}$ (resp. $\text{Surj}$) if and only if $\pi_\lambda(M)$ is in EKP (resp. EIP). Since a component $D$ of the Auslander-Reiten quiver of $C_r$ which is taken to a regular component $C := \pi_\lambda(D)$ is also of type $ZA_{xy}$, we can lift the definition of $W(C)$ to $D$. We define $W_C(D) \in \mathbb{N}_0$ as the distance between the cones $\text{Surj} \cap D$ and $\text{Inj} \cap D$ and show that $W_C(D) = W(C)$.

For $X \in D$, we denote by $q\ell(X)$ its quasi-length. If $X$ has certain properties, we show how to construct a short exact sequence $\delta_X = 0 \to Y \to E \to X \to 0$ with indecomposable middle term $E$ in a component $\mathcal{E}$ such that

\[(*) \quad W_C(\mathcal{E}) = W_C(D) + q\ell(X) - 1.\]

The construction of $\delta_X$ relies on the fact that $C_r$ is an infinite $r$-regular tree with bipartite orientation. Using $(*)$, we construct for each $n \in \mathbb{N}$ a component $D_n$ with $W_C(D_n) = n$.

In conjunction with a natural action of $\text{GL}_r(k)$ on $\text{rep}(\Gamma_r)$, we arrive at our main theorem:

**Theorem.** Let $n \in \mathbb{N}_0$. There is a bijection $k \to \{C \mid C$ regular component of $\Gamma_r, W(C) = n\}$.

As an immediate consequence we get the following statements, which are generalizations of results by Kerner and Lukas [13, 3.1], [13, 5.2] for the Kronecker algebra.

**Corollary.** Let $r \geq 3$, then for each $n \in \mathbb{N}$ there are exactly $|k|$ regular components with quasi-rank in $\{-n, -n+1, -n+2, -n+3\}$.

**Corollary.** Assume that $k$ is uncountable and $q \in \mathbb{N}$. The set of components of quasi-rank \( \leq -q \) is uncountable.

## 1. Preliminaries and Motivation

### 1.1. Notations and basic results.

Throughout this article let $k$ be an algebraically closed field of arbitrary characteristic. For a quiver $Q = (Q_0, Q_1, s, t)$, $x \in Q_0$ let

\[x^+ := \{y \in Q_0 \mid \exists \alpha: x \to y\} \quad \text{and} \quad x^- := \{y \in Q_0 \mid \exists \alpha: y \to x\}.\]

Moreover let $n(x) := x^+ \cup x^-$. If $\alpha: x \to y$, then by definition $s(\alpha) = x$ and $t(\alpha) = y$. We say that $\alpha$ starts in $s(\alpha)$ and ends in $t(\alpha)$.

**Definition.** A quiver $Q$ is called

(a) locally finite if $n(x)$ is finite for each $x \in Q_0$,

(b) of bounded length if for each $x \in Q_0$ there is $N_x \in \mathbb{N}$ such that each path in $Q$ which starts or ends in $x$ is of length $\leq N_x$,

(c) locally bounded if $Q$ is locally finite and of bounded length.

From now on we assume that $Q$ is locally bounded. Note that this implies that $Q$ is acyclic. Moreover, we assume that $Q$ is connected. A finite dimensional representation $M = ((M_x)_{x \in Q_0}, (M(\alpha))_{\alpha \in Q_1})$ over $Q$ consists of vector spaces $M_x$ and linear maps $M(\alpha): M_x \to M_y$ such that $\dim_k M := \sum_{x \in Q_0} \dim_k M_x$ is finite. A morphism $f: M \to N$ between representations is a collection of linear maps $(f_x)_{x \in Q_0}$ such for each arrow $\alpha: x \to y$ there is a commutative diagram
The category of finite dimensional representations over $Q$ is denoted by $\text{rep}(Q)$. The path category $k(Q)$ has $Q_0$ as set of objects and $\text{Hom}_{k(Q)}(x,y)$ is the vector space with basis given by the paths from $x$ to $y$. The trivial arrow in $x$ is denoted by $\epsilon_x$. Since $Q$ is locally bounded, the category $k(Q)$ is locally bounded in the sense of [7, 2.1]. A finite-dimensional module over a locally bounded category $A$ is a functor $F: A \to \text{mod} k$ such that $\sum_{x \in A} \dim_k F(x)$ is finite. The category $\text{mod} A$ has Auslander-Reiten sequences (see [7, 2.2]). Since a finite dimensional module over $k(Q)$ is the same as a representation of $Q$, the category $\text{rep}(Q)$ has Auslander-Reiten sequences.

If moreover $Q_0$ is a finite set, we denote with $kQ$ the path algebra of $Q$ with idempotents $e_x, x \in Q_0$. In this case $kQ$ is a finite dimensional, associative, basic and connected $k$-algebra. We denote by mod $kQ$ the class of finite-dimensional $kQ$ left modules. Given $M \in \text{mod} kQ$ we let $M_x := e_x M$. The categories mod $kQ$ and $\text{rep}(Q)$ are equivalent (see for example [1, III 1.6]). We will therefore switch freely between representations of $Q$ and modules of $kQ$, if one of the approaches seems more convenient for us. We assume that the reader is familiar with Auslander-Reiten theory and basic results on wild hereditary algebras. For a well written survey on the subjects we refer to [1], [10] and [11]. Recall the definition of the dimension function

$$\dim: \text{mod} kQ \to \mathbb{Z}^{Q_0}, M \mapsto (\dim_k M_x)_{x \in Q_0}.$$ 

If $0 \to A \to B \to C \to 0$ is an exact sequence, then $\dim A + \dim C = \dim B$. A finite quiver $Q$ defines a (non-symmetric) bilinear form $\langle -,- \rangle: \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$, given by $\langle (x_i), (y_j) \rangle := \sum_{i \in Q_0} x_i y_j - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}$, which coincides with the Euler-Ringel form [17] on the Grothendieck group $K_0(kQ) \cong \mathbb{Z}^{Q_0}$, i.e. for $M, N \in \text{mod} kQ$ we have

$$\langle \dim M, \dim N \rangle = \dim_k \text{Hom}(M,N) - \dim_k \text{Ext}(M,N).$$

1.2. The Kronecker algebra and $ZA_\infty$ components. We always assume that $r \geq 3$. Denote by $\Gamma_r$ the $r$-Kronecker quiver, which is given by two vertices 1, 2 and arrows $\gamma_1, \ldots, \gamma_r: 1 \to 2$.

![Figure 1](image)  
Figure 1. The Kronecker quiver $\Gamma_r$.

We set $K_r := k\Gamma_r$ and $P_1 := K_re_2, P_2 := K_re_1$. The modules $P_1$ and $P_2$ are the indecomposable projective modules of $\text{mod} K_r$, $\dim_k \text{Hom}(P_1, P_2) = r$ and $\dim_k \text{Hom}(P_2, P_1) = 0$. As Figure 1 suggests, we write $\dim M = (\dim_k M_1, \dim_k M_2)$. For example $\dim P_1 = (0,1)$ and $\dim P_2 = (1, r)$.

Figure 2 shows the notation we use for the components $\mathcal{P}, \mathcal{I}$ in the Auslander-Reiten quiver of $K_r$ which contain the indecomposable projective modules $P_1, P_2$ and indecomposable injective modules $I_1, I_2$. The set of all other components is denoted by $\mathcal{R}$.

Ringel has proven [16, 2.3] that every component in $\mathcal{R}$ is of type $ZA_\infty$. A module in such a component is called regular and the class of all regular indecomposable modules is denoted.
by \( \text{ind} \mathcal{R} \). An irreducible morphism in a component of type \( \mathbb{Z}A_\infty \) (for any algebra) is injective if the corresponding arrow is uprising and surjective otherwise. A representation \( M \) in a \( \mathbb{Z}A_\infty \) component is called quasi-simple if the AR sequence terminating in \( M \) has an indecomposable middle term. These modules are the indecomposable modules in the bottom layer of the component. If \( M \) is quasi-simple in a component \( C \) of type \( \mathbb{Z}A_\infty \), then there is an infinite chain (ray) of irreducible monomorphisms (resp. epimorphisms)

\[
M = M[1] \to M[2] \to M[3] \to \cdots \to M[l] \to \cdots
\]

and for each indecomposable module \( X \) in \( C \) there are unique quasi-simple modules \( N, M \) and \( l \in \mathbb{N} \) with \((l)M = X = N[l]\). The number \( q_l(X) := l \) is called the quasi-length of \( X \).

The indecomposable modules in \( \mathcal{P} \) are called preprojective modules and the modules in \( \mathcal{I} \) are called preinjective modules. Moreover we call an arbitrary module preprojective (resp. preinjective, regular) if all its indecomposable direct summands are preprojective (resp. preinjective, regular). We have \( P \in \mathcal{P} \) (\( I \) in \( \mathcal{I} \)) if and only if there is \( l \in \mathbb{N}_0 \) with \( \tau^l P = P \) (\( \tau^{-l} I = I \)) for \( i \in \{1, 2\} \). Let \( \text{mod}_{I} K_{r} \) be the subcategory of all modules without non-zero projective direct summands and \( \text{mod}_{I} K_{r} \) the subcategory of all modules without non-zero injective summands. Since \( K_{r} \) is a hereditary algebra, the Auslander-Reiten translation \( \tau : \text{mod} K_{r} \to \text{mod} K_{r} \) induces an equivalence from \( \text{mod}_{I} K_{r} \) to \( \text{mod}_{I} K_{r} \), that will be often used later on without further notice.

1.3. The connection between \( \text{rk}(C) \) and \( W(C) \). Let us start by recalling the definitions of \( \text{rk}(C) \) and \( W(C) \).

**Definition.** Let \( A = kQ \) be a wild hereditary algebra and \( C \) be a regular component with a quasi-simple module \( X \) in \( C \), then

\[
\text{rk}(C) := \min \{ l \in \mathbb{Z} \mid \forall m \geq l : \text{rad}(X, \tau^m X) \neq 0 \}.
\]

It was shown in [9], that \( \bar{t}(A) := \max \{ \text{rk}(C) \mid C \text{ regular component} \} \) is finite and \( \bar{t}(K_{r}) = 1 \). In other words, there are lots of morphisms in \( \tau \)-direction. Also in \( \tau^{-1} \)-direction one finds a lot of morphisms since \( \bar{t}(A) := \inf \{ \text{rk}(C) \mid C \text{ regular component} \} = -\infty \) (see [13, 3.1]).

Another invariant that can be attached to a regular component \( C \) of the Kronecker algebra is defined in [22] and denoted by \( W(C) \). In order to define \( W(C) \), we first recall the definitions of \( \text{EKP} \) and \( \text{EIP} \). For \( \alpha \in k^r \setminus \{0\} \) and \( M \in \text{rep}(\Gamma_r) \) we consider the \( k \)-linear map \( M^\alpha := \sum_{i=1}^{r} \alpha_i M(\gamma_i) : M_1 \to M_2 \) and let \( X_\alpha \in \text{rep}(\Gamma) \) be the cokernel of the embedding \( (0, \iota_2) : P_1 \to P_2 \) where \( \iota_2 : k \to k^r, x \mapsto x\alpha \).

**Definition.** [22, 2.1] We define the classes of representations with the equal kernels property and with the equal images property as

1. \( \text{EKP} := \{ M \in \text{rep}(\Gamma_r) \mid \forall \alpha \in k^r : M^\alpha \text{ is injective} \} \) and
2. \( \text{EIP} := \{ M \in \text{rep}(\Gamma_r) \mid \forall \alpha \in k^r : M^\alpha \text{ is surjective} \} \).

Given a regular component \( C \), there exist uniquely determined quasi-simple representations \( M_C \) and \( W_C \) in \( C \) such that \( (M_C \to) = \text{EKP} \cap C \) and \( (\to W_C) = \text{EIP} \cap C \) (see [22, 3.3]). Now \( W(C) \) is defined as the unique integer with \( \tau^{W(C) + 1} M_C = W_C \). Since \( \text{EKP} \cap \text{EIP} = \{ \} \) it follows \( W(C) \geq 0 \).
**Proposition 1.3.1.** Let $C$ be a regular component, then $-\mathcal{W}(C) \leq \text{rk}(C) \leq -\mathcal{W}(C) + 3$.

**Proof.** On the one hand let $M := \tau^{-1}W_C$, then there is $\alpha \in k^r \setminus \{0\}$ with $0 \neq \text{Ext}(X_\alpha, M)$ [22, 2.5]. The Auslander-Reiten formula [11, 2.3] yields $0 \neq \text{Hom}(\tau^{-1}M, X_\alpha)$. On the other hand let $N := \tau^N_C$, then there is $\beta \in k^r \setminus \{0\}$ with $0 \neq \text{Hom}(X_\beta, N) \cong \text{Hom}(\tau X_\beta, \tau N)$ [22, 2.5]. By the Euler-Ringel form we have

$$0 > 2 - r = 1 + (r - 1)^2 - r(r - 1) = \langle \text{dim} X_\beta, \text{dim} X_\alpha \rangle = \text{dim}_k \text{Hom}(X_\beta, X_\alpha) - \text{dim}_k \text{Ext}(X_\beta, X_\alpha).$$

Hence $0 \neq \text{dim}_k \text{Ext}(X_\beta, X_\alpha) = \text{dim}_k \text{Hom}(\tau^{-1}X_\alpha, X_\beta) = \text{dim}_k \text{Hom}(X_\alpha, \tau X_\beta)$. Since $X_\alpha$ and $\tau X_\beta$ are elementary [3, 2.1.4], we get a non-zero morphism by [3, 2.1.1]

$$\tau^{-1}M \to X_\alpha \to \tau X_\beta \to \tau N,$$

and

$$0 \neq \text{Hom}(M, \tau^2 N) = \text{Hom}(\tau^{-1}W_C, \tau^2 \tau^{-\mathcal{W}(C)}W_C) = \text{Hom}(W_C, \tau^{-\mathcal{W}(C)+3}W_C).$$

Hence $\text{rad}(W_C, \tau^{-\mathcal{W}(C) + 3}W_C) \neq 0$, since [22, 4.10] together with $W_C$ is not a brick. By [9, 1.7] it follows that $\text{rk}(C) \leq -\mathcal{W}(C) + 3$. The second inequality follows from the proof of [22, 3.1.3].

In [22], the inequality $-\mathcal{W}(C) \leq \text{rk}(C)$ in conjunction with $\mathcal{L}(K_r) = -\infty$ was used to prove that $\sup\{\mathcal{W}(D) \mid D \in \mathcal{R}\} = \infty$. We choose a different approach and study the number $\mathcal{W}(C)$ to draw conclusions for $\text{rk}(C)$.

**Remark.** There are regular components with $\text{rk}(C_i) = 1$ and $\mathcal{W}(C_i) = i$ for $0 \leq i \leq 2$ [22, 3.3.1] and a component $D$ with $\text{rk}(D) = 0$ and $\mathcal{W}(D) = 0$ [3, 3.3.3].

### 2. Covering Theory

2.1. **General Theory.** We follow [19] and [20] and consider the universal cover $C_r$ of the quiver $\Gamma_r$. The underlying graph of $C_r$ is an $r$-regular tree and $C_r$ has bipartite orientation. That means each vertex $x \in (C_r)_0$ is a sink or a source and $|n(x)| = r$. In the following we recall the construction of $C_r$.

For a quiver $Q = (Q_0, Q_1, s, t)$ with arrow set $Q_1$ we write $(Q_1)^{-1} := \{\alpha^{-1} \mid \alpha \in Q_1\}$ for the formal inverses of $Q_1$. Moreover we extend the functions $s$ and $t$ to $(Q_1)^{-1}$ by defining $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$. A walk $w$ in $Q_1$ is a formal sequence $w = \alpha_n \cdots \alpha_1$ with $\alpha_i \in Q_1$, $\varepsilon \in \{1, -1\}$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $i < n$, where $\alpha^1 := \alpha$ for all $\alpha \in Q_1$. We set $t(w) := t(\alpha_n^\varepsilon)$ and $s(w) := s(\alpha_1^\varepsilon)$.

Let $\sim$ be the equivalence relation on the set of walks $W$ of $\Gamma_r$ generated by

$$\gamma_i^{-1} \gamma_i \sim \varepsilon_1 \text{ and } \gamma_i \gamma_i^{-1} \sim \varepsilon_2.$$

Let $^{-1}: W \to W$ be the involution on $W$ given by $(\alpha_n^\varepsilon \cdots \alpha_1^\varepsilon)^{-1} = \alpha_1^{-\varepsilon_1} \cdots \alpha_n^{-\varepsilon_n}$. Now consider the fundamental group $\pi(\Gamma_r)$ of $\Gamma_r$ in the point 1, i.e. the elements of $\pi(\Gamma_r)$ are the equivalence classes of unoriented paths starting and ending in 1, with multiplication given by concatenation of paths, inverse elements $[w]^{-1} := [w^{-1}]$ and identity element $[\varepsilon_1]$. Note that $\pi(\Gamma_r)$ is a free group in the $r - 1$ generators $\{[\gamma_j^{-1} \gamma_i] \mid 2 \leq j \leq r\}$ and in particular torsionfree.

The quiver $C_r$ is given by the following data:

(a) $(C_r)_0$ is the set of equivalence classes of paths starting in 1.

(b) There is an arrow from $[w]$ to $[w']$ whenever $w' \sim \gamma_i w$ for some $i \in \{1, \ldots, r\}$.

Let $\pi: C_r \to \Gamma_r$ be the quiver morphism given by $[w] \mapsto t(w)$ and $([w] \to [\gamma_i w]) \mapsto \gamma_i$. The morphisms $\pi$ is a $G$-Galois cover for $G = \pi(\Gamma_r)$, where the action of $G$ on $C_r$ is given by concatenation of paths: if $g = [w] \in \pi(\Gamma_r)$ and $[v], [u] \in (C_r)_0$ with arrow $[u] \to [\gamma_i u]$ then

$$g.[v] = [vw^{-1}] \text{ and } g.([u] \to [\gamma_i u]) = ([uw^{-1}] \to [\gamma_i uw^{-1}]).$$
We define $C^+_r := \pi^{-1}(\{1\})$, $C^-_r := \pi^{-1}(\{2\})$ and get an induced action on $\text{rep}(C_r)$ by shifting the support of representations via $G = \pi(\Gamma_r)$: Given $M \in \text{rep}(C_r)$ and $g \in G$ we define $M^g := (((M^g)_x)_{x \in (C_r)_0}, (M^g(\alpha))_{\alpha \in (C_r)_1})$, where

$$(M^g)_x := M_{g.x} \text{ and } M^g(\alpha) := M(g.\alpha).$$

By identifying the orbit quiver $C_r/G$ with $\Gamma_r$ we define the push-down functor $\pi_\lambda : \text{rep}(C_r) \to \text{rep}(\Gamma_r)$ on the objects via $\pi_\lambda(M) := (\pi_\lambda(M)_1, \pi_\lambda(M)_2; (\pi_\lambda(M)(\gamma_i))_{1 \leq i \leq r})$, where

$$\pi_\lambda(M)_i := \bigoplus_{\pi(\gamma)=i} M_g \text{ and }$$

$$\pi_\lambda(M)(\gamma_i) := \bigoplus_{\pi(\beta)=\gamma_i} M(\beta) : \pi_\lambda(M)_1 \to \pi_\lambda(M)_2 \text{ for } 1 \leq i \leq r.$$ 

If $f = (f_x)_{x \in (C_r)_0} : M \to N$ is a morphism in $\text{rep}(C_r)$ then $\pi_\lambda(f) = (g_{\pi(x)})_{x \in (C_r)_0} = (g_1, g_2)$ with

$$g_i := \bigoplus_{\pi(\gamma)=i} f_g : \pi_\lambda(M)_i \to \pi_\lambda(N)_i.$$

By [7, 3.2] $\pi_\lambda$ is an exact functor.

**Theorem 2.1.1.** [8, 3.6], [20, 6.2, 6.3] The following statements hold.

(a) $\pi_\lambda$ sends indecomposable representations in $\text{rep}(C_r)$ to indecomposable representations in $\text{rep}(\Gamma_r)$.

(b) If $M \in \text{rep}(C_r)$ is indecomposable then $\pi_\lambda(M) \cong \pi_\lambda(N)$ if and only if $M^g \cong N$ for some $g \in G$.

(c) $\pi_\lambda$ sends AR sequences to AR sequences and $\pi_\lambda$ commutes with the Auslander-Reiten translates, i.e. $\tau \circ \pi_\lambda = \pi_\lambda \circ \tau_{C_r}$.

(d) If $M \in \text{rep}(C_r)$ is indecomposable in a component $D$ with $\pi_\lambda(M)$ in a component $C$, then $\pi_\lambda$ induces a covering $D \to C$ of translation quivers.

**Definition.** Let $M \in \text{rep}(C_r)$ be indecomposable. $M$ is called regular if $\pi_\lambda(M)$ is regular. A component $D$ of $\text{rep}(C_r)$ is called regular if it contains a regular representation $M$. In this case we denote by $\pi_\lambda(D)$ the component containing $\pi_\lambda(M)$. Moreover we let $\mathcal{R}(C_r)$ be the set of all regular components of the Auslander-Reiten quiver of $\text{rep}(C_r)$.

![Illustration of $C_r$ for $r = 4$.](image-url)
Corollary 2.1.2. Let \( \mathcal{D} \) be regular component, then the covering \( \mathcal{D} \to \pi(\mathcal{D}) \) is an isomorphism of translation quivers. In particular, \( \mathcal{D} \) is of type \( \mathcal{Z}\mathcal{A}_\infty \). Moreover a component \( \mathcal{E} \) is regular if and only if \( \mathcal{E} \) is of type \( \mathcal{Z}\mathcal{A}_\infty \).

Proof. By [7, 1.7] \( \pi(\mathcal{D}) \cong \mathcal{Z}\mathcal{A}_\infty \) is a simply connected translation quiver. By [7, 1.6], [15, 1.7] the quiver morphism \( \mathcal{D} \to \pi(\mathcal{D}) \) is an isomorphism. If \( \pi(\mathcal{E}) \in \{\mathcal{I}, \mathcal{P}\} \) then there exists a vertex \( x \) with \( r \geq 3 \) successors (see Figure 2), since a covering is surjective on arrows. Hence \( \mathcal{E} \) is not of type \( \mathcal{Z}\mathcal{A}_\infty \). \( \square \)

2.2. Duality. Recall [22, 2.2] that the duality \( D: \text{rep}(\Gamma) \to \text{rep}(\Gamma^o) \) is defined by setting \( (DM)_x := (M_{\psi(x)})^* \) and \( (DM)(\gamma) := (M(\gamma))^* \), where \( \psi: \{1, 2\} \to \{1, 2\} \) is the involution with \( \psi(1) = 2 \).

We define an involution \( \varphi_0: (C_r)_0 \to (C_r)_0 \) via \( \varphi_0([w]) = [w \gamma_1] \), where \( \gamma_1 := \epsilon_2, \gamma_2 := \epsilon_1 \) and \( \alpha_{n}^{2} \cdot \alpha_{1}^{\epsilon_1} := \alpha_{n}^{\epsilon_2} \cdot \alpha_{1}^{\epsilon_1} \). This induces a quiver anti-morphism \( \varphi: C_r \to C_r \) in the following way.

If \( [w] \to [\gamma_i w] \) is an arrow of \( C_r \), then by definition there is a unique arrow \( \varphi([w] \to [\gamma_i w]) \) starting in \( \varphi_0([\gamma_i w]) = [\gamma_i \gamma_1 w] \) and ending in \( \varphi_0([w]) = [w \gamma_1] \), since \( \gamma_i \gamma_1 w = \gamma_1 \gamma_i w \). Note that \( \varphi(C_r^+) = C_r^+, \varphi(C_r^-) = C_r^- \) and \( \pi(\varphi(\alpha)) = \pi(\alpha) \).

We define a duality \( D_{C_r}: \text{rep}(C_r) \to \text{rep}(C_r) \) by setting \( D_{C_r}M := ((D_{C_r}M)(x \in (C_r)_0), (D_{C_r}M(\alpha))_{\alpha \in Q_1}) \) where \( (D_{C_r}M)_x := (M_{\varphi(x)})^* \) and \( D_{C_r}M(\alpha) := (M(\varphi(\alpha))^* \). By construction we have \( \pi(\alpha D_{C_r} = D \circ \pi(\alpha) \).

3. Lifting EKP and EIP to \( \text{rep}(C_r) \)

In the following we give a characterization of the equal images and equal kernels property for indecomposable representations of the form \( \pi(\alpha) \). Let \( \gamma_i: (C_r)_1 \to \{1, \ldots, r\} \) be the unique map with \( \gamma_i = \pi(\beta) \) for all \( \beta \in (C_r)_1 \). Note that if \( x \in (C_r)^+ \) then the restriction of \( \gamma_i \) to \( \{\alpha \in (C_r)_1 | s(\alpha) = x\} \) (resp. \( \{\alpha \in (C_r)_1 | t(\alpha) = x\} \)) is a bijective map to \( \{1, \ldots, r\} \).

Definition. Let \( X \neq \emptyset \subseteq \{C_r\}_0 \) be a set of vertices and \( T \subseteq C_r \) be a tree.

(a) The unique minimal tree containing \( X \) is denoted by \( T\langle X \rangle \).

(b) A vertex \( x \in T \) is called a leaf of \( T \), if \( |n(x) \cap T| \leq 1 \).

Definition. Let \( x \in (C_r)_0 \) and \( M \) be a representation of \( C_r \).

(a) The set \( \text{supp}(M) := \{y \in (C_r)_0 | M_y \neq 0\} \) is called the support of \( M \).

(b) For \( V \subseteq \text{supp}(M) \) we let \( M_V \) be the induced representation with \( \text{supp}(M_V) = V \).

(c) The vertex \( x \) is a leaf of \( M \) if \( x \) is a leaf of \( T\langle M \rangle := T\langle \text{supp}(M) \rangle \).

(d) \( M \) is called balanced provided that \( M \) is indecomposable and \( M \) has leaves in \( C_r^+ \) and \( C_r^- \).

Observe that if \( M \) is indecomposable we have \( T\langle M \rangle = \text{supp}(M) \).

Definition. We define

\( \text{Inj} := \{M \in \text{rep}(C_r) | \forall \delta \in (C_r)_1 : M(\delta) \text{ is injective}\} \)

\( \text{Surj} := \{M \in \text{rep}(C_r) | \forall \delta \in (C_r)_1 : M(\delta) \text{ is surjective}\} \).

Theorem 3.1. Let \( M \in \text{rep}(C_r) \) be an indecomposable representation. The following statements are equivalent:

(a) \( N := \pi(\alpha) \) has the equal kernels property.

(b) \( N(\gamma_i) \) is injective for all \( i \in \{1, \ldots, r\} \).

(c) \( M \in \text{Inj} \).

Proof. (a) \( \Rightarrow \) (b): Clear from the definition of EKP.

(b) \( \Rightarrow \) (c): Let \( \alpha: x \to y \) and \( m_x \in \ker M(\alpha) \). Denote by \( \iota_x: M_x \to \bigoplus_{z \in C_r^+} M_z \) the natural
embedding and let \( \iota_y : M_y \to \bigoplus_{z \in C_r} M_z \). We conclude

\[
0 = \iota_y \circ M(\alpha)(m_x) = \left[ \bigoplus_{\pi(\beta) = \pi(\alpha)} M(\beta) \right] \circ \iota_x(m_x) = (N(\gamma_\alpha) \circ \iota_x)(m_x).
\]

Hence \( \iota_x(m_x) \in \ker N(\gamma_\alpha) = \{0\} \) and \( m_x = 0 \).

\( (c) \Rightarrow (a) \): Let \( \alpha \in k^r \setminus \{0\}, f := \sum_{i=1} C_r \alpha_i N(\gamma_i) : N_1 \to N_2 \) and \( m \in \ker f \). We assume w.l.o.g. that \( \alpha_1 \neq 0 \). Write \( m = (m_z)_{z \in C_r^+} \). We have to show that \( m_z = 0 \) for all \( z \in C_r^+ \).

Let \( S := \{ z \in (C_r)_0 | m_z \neq 0 \} \subseteq C_r^+ \) and suppose that \( S \neq \emptyset \). Let \( T(S) \) be the minimal tree that contains \( S \). Since \( T(S)_0 \subseteq \text{supp}(M) \) every leaf belongs to \( C_r^+ \cap S \), by the minimality of \( T(S) \). Fix a leaf \( x \) in \( T(S) \) and let \( \gamma : x \to y \) be the unique arrow with \( \pi(\gamma) = \gamma_1 \). Then

\[
0 = (f(m))_y = \sum_{\beta \in (C_r)_1, t(\beta) = y} \alpha_\pi M(\beta)(m_{s(\beta)}) = \alpha_1 M(\gamma)(m_x) + \sum_{\gamma \neq \beta \in (C_r)_1, t(\beta) = y} \alpha_\pi M(\beta)(m_{s(\beta)}).
\]

By the injectivity of \( M(\gamma) \) there is \( \delta : z \to y \in (C_r)_1 \setminus \{\gamma\} \) with \( t(\delta) = y \) and \( 0 \neq \alpha_\pi M(\delta)(m_z) \). It follows \( m_z \neq 0 \) and \( z, x \in S \). Since \( C_r \) is a tree we get \( y \in T(S)_0 \).

Since \( \delta \neq \pi = 1 \), we assume without loss of generality that \( \delta = 2 \), so that \( \alpha_2 \neq 0 \). Let \( \eta : x \to a \) be the unique arrow with \( \eta = 2 \). Then

\[
0 = (f(m))_a = \sum_{\beta \in (C_r)_1, t(\beta) = a} \alpha_\pi M(\beta)(m_{s(\beta)}) = \alpha_2 M(\eta)(m_x) + \sum_{\eta \neq \beta \in (C_r)_1, t(\beta) = a} \alpha_\pi M(\beta)(m_{s(\beta)}).
\]

Hence there is \( \zeta : b \to a \in (C_r)_1 \setminus \{\eta\} \) with \( 0 \neq \alpha_\pi M(\zeta)(m_b) \) and \( a, b \) are in \( T(S)_0 \). We have shown that \( a \) and \( y \) are in \( T(S)_0 \). This is a contradiction since \( x \) is a leaf. \( \square \)

**Corollary 3.2.** Let \( M \in \text{rep}(C_r) \) be an indecomposable representation. The following statements are equivalent.

(a) \( N := \pi_\lambda(M) \) has the equal images property.

(b) \( N(\gamma_i) \) is surjective for all \( i \in \{1, \ldots, r\} \).

(c) \( M \in \text{Surj} \).

**Proof.** \( (c) \Rightarrow (a) \): Let \( M(\alpha) \) be surjective for each \( \alpha \in (C_r)_1 \), then \( (D_r, M)(\alpha) \) is injective for each \( \alpha \in (C_r)_1 \). By \( [3.1] \), the representation \( \pi_\lambda(D_r, M) \cong D\pi_\lambda(M) \) has the equal kernels property. Therefore \( \pi_\lambda(M) \) has the equal images property, since \( D(\text{EKP}) = \text{EIP} \). \( \square \)

**Corollary 3.3.** Let \( \mathcal{D} \in \mathcal{R}(C_r) \) and \( \mathcal{C} := \pi_\lambda(\mathcal{D}) \). Then there exist uniquely determined quasi-simple representations \( I_D \) and \( S_D \) in \( \mathcal{D} \) such that

(a) \( \pi_\lambda(I_D) = M_C \) and \( \pi_\lambda(S_D) = W_C \).

(b) \( \text{Surj} \cap \mathcal{D} = (\to S_D) \) and \( \text{Inj} \cap \mathcal{D} = (I_D \to) \).

(c) The unique integer \( W_C(D) \) with \( \tau_{C_r}^{W_C(D)+1}(I_D) = S_D \) is given by \( W_C(D) = W_C(\mathcal{C}) = N_0 \).

**Corollary 3.4.** Assume that \( M \in \text{rep}(C_r) \) is balanced, then \( M \) is regular.

**Proof.** If an indecomposable representation \( X \in \text{rep}(C_r) \) is in \( \text{Inj} \) (respectively \( \text{Surj} \)), then all leaves of \( X \) are in \( C_r^+ \) (resp. \( C_r^- \)). Hence \( \pi_\lambda(M) \notin \text{EIP} \cup \text{EKP} \) by \([3.1] \). By \([22, 2.7] \) the representations of the components \( \mathcal{P} \) and \( \mathcal{I} \) are contained in \( \text{EKP} \cup \text{EIP} \). \( \square \)

From now on we write \( x_0 := [e_1] \in (C_r)_0 \) for the vertex in \( (C_r)_0 \) given by the trivial walk starting in the vertex 1.

**Definition.** For \( i \in \{1, \ldots, r\} \), we let \( \beta_i \in (C_r)_1 \) be the unique arrow with \( s(\beta_i) = x_0 \) and \( \pi(\beta_i) = \gamma_i \). Moreover let \( z_i := t(\beta_i) \). We define an indecomposable representation \( X^i \) in \( \text{rep}(C_r) \) via:

\[
(X^i)_y := \begin{cases} k, & y \in \{x_0\} \cup x_0^+ \setminus \{z_i\} \\ 0, & \text{else} \end{cases}
\]
Lemma 3.6. Let $X^i(\beta_j) := \text{id}_k$ for all $j \neq i$. By definition we have $\dim \pi(X^i) = (1, r - 1)$ and $\pi(X^i) \cong X_e$. In view of [22, 2.5], [8, 3.6(c)] 2 and 3.1 we conclude the following.

**Corollary 3.5.** Let $M \in \text{rep}(C_r)$ be an indecomposable representation. The following statements are equivalent:

(a) $N := \pi(M)$ has the equal kernels property.
(b) $\text{Hom}(\pi(X^i), N) = 0$ for all $i \in \{1, \ldots, r\}$.
(c) $\text{Hom}(X^i)^g, M) = 0$ for all $i \in \{1, \ldots, r\}$ and all $g \in G$.

The following Lemma will be needed later on.

**Lemma 3.6.** Let $n \in \mathbb{N}$, $M$ be regular indecomposable, $i \in \{1, \ldots, r\}$ and $g \in G = \pi(\Gamma_r)$.

(a) For $n \geq 2$, the linear map $(\tau^n_{C_r}(X^i))^g(\alpha)$ is surjective for all $\alpha \in (C_r)_0$.
(b) If $n \geq 1$ and $f = (f_x)_{x \in (C_r)_0} : (\tau^n_{C_r}(X^i))^g \rightarrow M$ is a non-zero morphism, then each $f_x$ is injective.
(c) If $n \geq 2$, $x \in \text{supp}((\tau^n_{C_r}(X^i))^g) \cap C_r^{-}$ and $0 \neq f : (\tau^n_{C_r}(X^i))^g \rightarrow M$, then $\text{supp}(M) \cap n(x) = \text{supp}(M) \cap x^- = x^-$. This means $|\text{supp}(M) \cap x^-| = r$.

**Proof.** (a) For $n \geq 2$, we have $\pi(X^i) = \pi(\tau^n_{C_r}(X^i)) = \tau^n X_e \in EIP$ (see [22, 3.3]).
(b) For $n \geq 1$, it is known that each proper factor of $\tau^n X_e$ is preinjective, see for example [3, 2.1.4]. Let $0 \neq f \in \text{Hom}((\tau^n_{C_r}(X^i))^g, M)$, then $0 \neq \pi(X)f = (g_l)_{1 \leq l \leq 2} : \tau^n X_e \rightarrow \pi(M)$ is injective, where

$$g_i = \bigoplus_{\pi(y)=i} f_y : \bigoplus_{\pi(y)=i} ((\tau^n_{C_r}(X^i))^g)_y \rightarrow \bigoplus_{\pi(y)=i} M_y.$$ 

So each $f_x$ is injective, since $\pi^{-1}(\{1, 2\}) = (C_r)_0$.
(c) Let $x \in \text{supp}((\tau^n_{C_r}(X^i))^g) \cap C_r^{-}$ and $z \in n(x)$. Since $x$ is a sink, there is $\alpha : z \rightarrow x$ and by (a), $(\tau^n_{C_r}(X^i))(\alpha)$ is surjective. Hence $((\tau^n_{C_r}(X^i))^g)_{z} \neq 0$. By (b), we get $M_z \neq 0$ and $z \in \text{supp}(M) \cap x^-$. 

4. Considerations in the universal covering

Let us recall what we have shown so far. Given a component $D$ in $\mathcal{R}(C_r)$, the natural number $W_C(D)$ is the distance between the two non-empty, non-intersecting cones $\text{Inj} \cap D$ and $\text{Surj} \cap D$. Moreover we know that $W(\pi(X)) = W_C(D)$.

Let now $X \in D$ be indecomposable. Then there exists an integer $l \in \mathbb{Z}$ such that $\tau^l_{C_r} X \in \text{Surj}$, since $\text{Surj} \cap D$ is non-empty. We also find $n \geq 1$ with $\tau^n_{C_r} X \notin \text{Surj}$. Since $\text{Surj} \cap D$ is closed under $\tau_C$, we conclude $-n \leq l$. Therefore the following minima exist.

**Definition.** Let $X \in \text{rep}(C_r)$ be a regular indecomposable representation. We define

$$d^{\ast}(X) := \min\{l \in \mathbb{Z} \mid \tau^{\ast l} X \in \text{Inj}\} \text{ and } d^{+}(X) := \min\{l \in \mathbb{Z} \mid \tau^l X \in \text{Surj}\}.$$ 

Note that $|d^{\ast}(X)| \in \mathbb{N}_0$ is the distance of $X$ to the border of the cone $\text{Inj} \cap D$.

**Lemma 4.1.** Let $X \in \text{rep}(C_r)$ be indecomposable in a regular component $D$. Then

$$W_C(D) = d^{+}(X) + d^{\ast}(X) - \text{ql}(X).$$

**Proof.** Since the equality is obvious for $X$ quasi-simple we assume $l := \text{ql}(X) > 1$. Let $Z$ be the unique quasi-simple representation with $X = Z[l]$. By induction we get $W_C(D) = d^{+}(Z[l-1]) + d^{\ast}(Z[l-1]) - (l-1)$. Now observe that $d^{+}(Z[l-1]) + 1 = d^{+}(Z[l])$ and $d^{\ast}(Z[l-1]) = d^{\ast}(Z[l])$. Hence $W_C(D) = d^{+}(Z[l]) - 1 + d^{\ast}(Z[l]) - (l-1) = d^{+}(X) + d^{\ast}(X) - \text{ql}(X)$. 

We show in the following how to modify a regular representation $X \rightsquigarrow X'$ such that the obtained representation $X'$ is regular, $d^{+}(X) = d^{+}(X')$, $d^{\ast}(X) = d^{\ast}(X')$ and $\text{ql}(X') = 1$. 

4.1. Indecomposable representations arising from extensions. In this section, we show how to construct non-split exact sequences with indecomposable middle term in \( \text{rep}(C_r) \).

**Definition.** Let \( M, N \in \text{rep}(C_r) \) be indecomposable. The pair \((N, M)\) is called leaf-connected if there is \( \alpha \colon x \to y \in (C_r)_1 \) s.t.

(a) \( x \) is a leaf of \( M \), \( y \) is a leaf of \( N \) and

(b) \( \text{supp}(M) \cap \text{supp}(N) = \emptyset \).

**Remark.** Note that the assumption \( M \) and \( N \) being indecomposable together with properties (a) and (b) already implies the uniqueness of \( \alpha \). If \((N, M)\) is leaf-connected, \( \alpha \) is called the connecting arrow and \((M, N)\) is not leaf-connected.

**Definition.** Let \((N, M)\) be leaf-connected with connecting arrow \( \alpha : x \to y \) and \( f : M_x \to N_y \) a non-zero linear map. We define a representation \( N \circ f M \in \text{rep}(C_r) \) by setting

\[
\text{supp}(N \circ f M) := \text{supp}(N) \cup \text{supp}(M)
\]

with \((N \circ f M)_{\text{supp}(X)} = X \) for \( X \in \{M, N\} \) and \((N \circ f M)(\alpha) := f : M_x \to N_y\). Moreover we denote by \( \iota_f : N \to N \circ f M \) and \( \pi_f : N \circ f M \to M \) the natural morphisms of quiver representations. The \( k \)-linear map along the connecting arrow is called a connecting map for \((N, M)\).

**Remark.** Note that \( N \circ f M \) is just an extension of \( M \) by \( N \) with corresponding exact sequence \( \delta_f : 0 \to N \xrightarrow{i} N \circ f M \xrightarrow{\pi_f} M \to 0 \). The next lemma shows that \( M \circ f N \) is indecomposable. In particular, \( \delta_f \) does not split. One can also show that the map

\[
\Phi : \text{Hom}_k(M_x, N_y) \to \text{Ext}(M, N), f \mapsto [\delta_f],
\]

is an isomorphism of vector spaces, where \([\delta_0]\) is the neutral element in the abelian group \( \text{Ext}(M, N) \) with respect to the Baer sum.

**Lemma 4.1.1.** Let \((N, M)\) be leaf-connected.

(a) The representation \( N \circ f M \) is indecomposable.

(b) If \( M \) and \( N \) are regular, then \( N \circ f M \) is regular.

**Proof.** (a) Let \( U_1, U_2 \in \text{rep}(C_r) \) with \( N \circ f M = U_1 \oplus U_2 \). Hence we get

\[
M = (N \circ f M)_{\text{supp}(M)} = (U_1)_{\text{supp}(M)} \oplus (U_2)_{\text{supp}(M)}.
\]

Since \( M \in \text{rep}(C_r) \) is indecomposable, there is a unique \( i \in \{1, 2\} \) with \((U_i)_{\text{supp}(M)} = M\). We assume \( i = 1 \). By the same token there is a unique \( j \) with \((U_j)_{\text{supp}(N)} = N\). Since \((M \circ f N)(\alpha) = f \neq 0\), it follows that \( j = i = 1 \). Hence \( U_2 = (0) \) and \( N \circ f M \) is indecomposable.

(b) By construction \( N \) is a subrepresentation of \( M \circ f N \) and \( M \) a factor representation. Since \( \pi_\lambda(N \circ f M) \) is indecomposable with regular factor representation \( \pi_\lambda(M) \), it is regular itself or preprojective. By the same token \( \pi_\lambda(N \circ f M) \) is regular or preinjective. \( \square \)
Proposition 4.1.5. Hence we find that $(N^g, M)$ is leaf-connected with connecting arrow $x \to g.y$.

Proof. Since $x, y$ are leaves, we find at most one arrow $\delta_M : x \to x_1$ and at most one arrow $\delta_N : y_1 \to y$ with $x_1 \in \text{supp}(M)$ and $y_1 \in \text{supp}(N)$. Since $r \geq 3$ we find an arrow $\alpha : x \to z$ with $\pi(\delta_M) \neq \pi(\alpha) \neq \pi(\delta_N)$. In particular $\alpha \neq \delta_M$ and $x_1 \neq z$. Now let $g \in \pi(\Gamma_r)$ be the unique element with $g.z = y$. Then $z \in n(x)$ is a leaf of $N^g$. By construction, we have $\pi(g.\alpha) = \pi(\alpha) \neq \pi(\delta_N)$ and conclude $x \notin \text{supp}(N^g)$. It follows that $\text{supp}(M) \cap \text{supp}(N^g) = \emptyset$. Hence $(N^g, M)$ is leaf-connected with connecting arrow $\alpha$. □

Definition. Let $n \geq 2$ and $M_1, \ldots, M_n \in \text{rep}(C_r)$ be indecomposable. The tuple $(M_n, \ldots, M_1)$ is called leaf-connected, provided that $(M_{i+1}, M_i)$ is leaf-connected for all $1 \leq i < n$. A tuple $(f_{n-1}, \ldots, f_1)$ of $k$-linear maps is called a connecting map for $(M_n, \ldots, M_1)$ if $f_i$ is a connecting map for $(M_{i+1}, M_i)$ for all $1 \leq i < n$.

Remark. Let $(M, L)$ and $(L, N)$ be leaf-connected with connecting maps $f$ and $g$, then $\text{supp}(M) \cap \text{supp}(N) = \emptyset$, since $C_r$ is a tree. Hence $\text{supp}(M * f L) \cap \text{supp}(N) = \emptyset$ and $(M * f L, N)$ is connected. Therefore the next definition is well-defined.

Definition. Let $n \geq 2$, $(M_n, \ldots, M_1)$ be leaf-connected with connecting map $(f_{n-1}, \ldots, f_1)$. Then we define inductively $M_n * f_{n-1} M_{n-1} * f_{n-2} \cdots * f_1 M_1 := (M_n * f_{n-1} M_{n-1} * \cdots * f_{i+1} M_{i+1}) * f_i M_i$ for all $2 \leq i < n$.

From now on, we assume that $(M_n, \ldots, M_1)$ is leaf-connected with connecting map $(f_{n-1}, \ldots, f_1)$. For $1 \leq i \leq n$, we define $*_{\geq i} M_j := M_n * f_{n-1} M_{n-1} * \cdots * f_i M_i$ and $*_{\leq i} M_j := M_i * f_{i-1} M_{i-1} * \cdots * f_1 M_1$. Moreover, we set $*_{\geq n} M_j = M_n$ and $*_{\leq 1} M_j = M_1$.

Lemma 4.1.3. Let $n \geq 2$, $(M_n, \ldots, M_1)$ be leaf-connected and $1 \leq i < n$.

(a) The representation $*_{\geq i} M_j$ is indecomposable.

(b) There is a short exact sequence $0 \to *_{\geq i+1} M_j \to *_{\geq i} M_j \to M_i \to 0$.

(c) If $M_n, M_1$ are regular, then $*_{\geq i} M_j$ is regular.

Proof. For (c), just note that $M_i$ is balanced for $2 \leq i \leq n - 1$, hence $M_i$ is regular. □

Lemma 4.1.4. Let $X, Y \in \text{rep}(C_r)$ be regular indecomposable. Then the following statements are equivalent.

(a) There is $g \in G$ such that $\text{Hom}(X^g, Y) \neq 0$.

(b) There is $h \in G$ such that $\text{Hom}(\tau_{C_r} X^h, \tau_{C_r} Y)$.

(c) There is $l \in G$ such that $\text{Hom}(\tau_{C_r}^{-1} X^l, \tau_{C_r}^{-1} Y)$.

Proof. We only show (a) ⇒ (b). Let $g \in G$, such that $0 \neq \text{Hom}(X^g, Y)$. Then $0 \neq \text{Hom}(\pi_{\lambda}(X), \pi_{\lambda}(Y)) \cong \text{Hom}(\tau \circ \pi_{\lambda}(X), \tau \circ \pi_{\lambda}(Y)) \cong \text{Hom}(\pi_{\lambda}(\tau_{C_r} X), \pi_{\lambda}(\tau_{C_r} Y))$. Hence we find $h \in G$ such that $0 \neq \text{Hom}(\tau_{C_r} X^h, \tau_{C_r} Y)$. □

Proposition 4.1.5. Let $n \geq 2$, $(M_n, \ldots, M_1)$ be leaf-connected and $M_1, M_n$ regular, then

(a) $\max\{d^-(*_{\geq i} M_j) \mid 1 \leq i \leq n\} \leq d^-(*_{\geq i+1} M_j) \leq \max\{d^-(M_i) \mid 1 \leq i \leq n\}$.

(b) $\max\{d^+(*_{\leq i} M_j) \mid 1 \leq i \leq n\} \leq d^+(*_{\leq i+1} M_j) \leq \max\{d^+(M_i) \mid 1 \leq i \leq n\}$.

Proof. Note that we have a filtration of $*_{\geq 1} M_j$ by regular subrepresentations

$0 \subset M_n \subset M_n * M_{n-1} \subset M_n * M_{n-1} * M_{n-2} \subset \cdots \subset *_{\geq 2} M_j \subset *_{\geq 1} M_j$,

with $*_{\geq 1} M_j / *_{\geq i+1} M_j \cong M_i$ regular for all $1 \leq i \leq n$, where $*_{\geq n+1} M_j := 0$.

(a) Let $i \in \{1, \ldots, n\}$ and $Z := *_{\geq i} M_i$. Consider the short exact sequence

$0 \to Z \to *_{\geq i} M_j \to (*_{\geq i} M_j)/Z \to 0$. 
Now \( l \in \mathbb{Z} \) such that \( \tau_{C_l}^{-1}Z \notin \text{Inj} \). Then there exists \( i \in \{1, \ldots, r\} \) and \( g \in G \) such that \( \text{Hom}((X^i)^g, \tau_{C_l}^{-1}Z) \neq 0 \). Hence we find \( h \in G \) with \( \text{Hom}((\tau_{C_l}^i X^i)^h, Z) \) for some \( h \in G \). Left-exactness of \( \text{Hom}((\tau_{C_l}^i X^i)^h, -) \) ensures that \( 0 \neq \text{Hom}((\tau_{C_l}^i X^i)^h, \ast_{j \geq 1} M_j) \) and therefore we find \( g \in G \) with \( 0 \neq \text{Hom}(X^i_1, \tau_{C_l}^{-1}(\ast_{j \geq 1} M_j)) \). Hence \( \tau_{C_l}^{-1}(\ast_{j \geq 1} M_j) \notin \text{Inj} \) and therefore \( d^-(X) \leq d^-(\ast_{j \geq 1} M_j) \).

If \( \text{Hom}((X^i)^g, \tau_{C_l}^{-1}(\ast_{j \geq 1} M_j)) \neq 0 \) for some \( g \in G \) and \( i \in \{1, \ldots, r\} \), then we get find \( h \in G \) with \( 0 \neq \text{Hom}(\tau_{C_l}^i (X^i)^h, \ast_{j \geq 1} M_j) \). By [11, 1.9] we find \( 1 \leq p \leq n \) with \( 0 \neq \text{Hom}(\tau_{C_l}^i (X^i)^h, M_p) \). Hence there is \( g \in G \) with \( 0 \neq \text{Hom}((X^i)^g, \tau_{C_l}^{-1} M_p) \). Hence \( d^-(\ast_{j \geq 1} M_j) \leq d^-(M_p) \).

(b) Note for \( i \geq 2 \) that \( \ast_{j \geq 1} M_j / \ast_{j \geq i} M_j \cong \ast_{j < i} M_j \). Hence \( D_{C_l}(\ast_{j \geq 1} M_j) \) has a filtration

\[
0 \subset D_{C_l} M_1 \subset D_{C_l} M_1 \ast D_{C_l} M_2 \subset \cdots \subset \ast_{j \leq n-1} D_{C_l} M_j \subset D_{C_l}(\ast_{j \geq 1} M_j)
\]

Now apply (a) and note that \( d^+(X) = d^- (D_{C_l} X) \) for each regular indecomposable representation \( X \).

4.2. Small representations and trees.

**Definition.** A balanced representation \( N \) is called small if \( 1 \leq d^-(N), d^+(N) \leq 2 \).

Note that \( N \) being balanced always implies \( d^-(N) \geq 1 \) and \( d^+(N) \geq 1 \).

**Definition.** Denote with \( \overline{C}_r \) the underlying graph of \( C_r \). Then \( (\overline{C}_r, 0, d) \) obtains the structure of a metric space, where \( d(x, y) \in \mathbb{N}_0 \) denotes the length of the unique path in \( \overline{C}_r \) connecting vertices \( x \) and \( y \).

**Definition.** Let \( T \subseteq C_r \) be a finite subtree. \( T \) is called small if

(a) \( T \) has leaves in \( C_r^+ \) and \( C_r^- \),

(b) for all \( x \in T_0 \), we have \( |T_0 \cap n(x)| \leq 3 \),

(c) if \( |T_0 \cap n(x)| = 3 = |T_0 \cap n(y)| \) then \( x = y \) or \( d(x, y) \geq 3 \).

**Examples.** Let \( l \in \mathbb{N} \) and \( n \in 2\mathbb{N} \) with \( n \geq 4l \). We denote with \( A_{4l,n} \subseteq C_r \) a (small) subtree of the following form:

\[
\begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
| & | & | & | & | & | & | & | & |
\end{array}
\]

\[
t_0 \quad t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_{4l-3} \quad t_{4l-2} \quad t_{4l-1} \quad t_{4l}
\]

\[
a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad \cdots \quad a_{4l-3} \quad a_{4l-2} \quad a_{4l-1} \quad a_{4l}
\]

**Lemma 4.2.1.** Assume \( L \) is an indecomposable representation with small tree \( T(L) \). Then \( L \) is small.

**Proof.** Since \( T(L) \) is small with \( T(L)_0 = \text{supp}(L) \), \( L \) is balanced and therefore regular.

Let \( l \in \mathbb{Z} \) be such that \( \tau_{C_l}^{-1}L \notin \text{Inj} \). By [3.5] and [4.1.4] we find \( i \in \{1, \ldots, r\} \) and \( g \in G \) with \( 0 \neq \text{Hom}((\tau_{C_l}^i X^i)^g, L) \). Fix \( h : (\tau_{C_l}^i X^i)^g \rightarrow L \) non-zero.

We assume that \( l \geq 2 \). By [3.6] \( h \) is a monomorphism and \( \text{supp}((\tau_{C_l}^i X^i)^g) \subseteq \text{supp}(L) \). Let \( s \) be a sink of \( (\tau_{C_l}^i X^i)^g \). By [3.6] we have \( |\text{supp}(L) \cap n(s)| = r \). Since \( T(L) \) is small we get \( 3 \leq r = |\text{supp}(L) \cap n(s)| \leq 3 \). Hence

\[
(*) \quad r = 3 = |n(s) \cap \text{supp}(L)|.
\]

Now let \( t_1, t_2 \in \text{supp}((\tau_{C_l}^i X^i)^g) \) be sinks. Then (*) yields

\[
|n(t_1) \cap \text{supp}(L)| = 3 = |n(t_2) \cap \text{supp}(L)|.
\]

Since \( T(L) \) is small we get \( t_1 = t_2 \) or \( d(t_1, t_2) \geq 3 \). Since \( C_r \) has bipartite orientation and \( \text{supp}((\tau_{C_l}^i X^i)^g) \) is connected, it follows \( t_1 = t_2 \). Hence \( \text{supp}((\tau_{C_l}^i X^i)^g) \) contains exactly one sink \( s \). Write \( n(s) = \{a, b, c\} \). Since \( l \geq 2 \), [3.6] \( (a) \) implies \( \text{supp}((\tau_{C_l}^i X^i)^g) = \{s, a, b, c\} \). Hence \( Z := (\tau_{C_l}^i X^i)^g \) can be considered as a representation of the Dynkin diagram \( D_4 \) with unique sink \( s \) such that all linear maps are surjective. It follows that \( Z_x = 1 \) for all \( x \in \{s, a, b, c\} \). Hence
\[ \dim \pi_\lambda(Z) = (3, 1) \] and \( \pi_\lambda(Z) \) is indecomposable. But the only indecomposable representation \( I \in \text{rep}(\Gamma_3) \) with dimension vector \((3, 1)\) is injective. This is a contradiction since \( Z \) is regular. Therefore \( l \leq 1 \) and \( d^-(L) \leq 2 \). For the other inequality note that \( T(L) \) is small if and only if \( T(D_C, L) \) is small and \( d^+(L) = d^-(D_C, L) \leq 2 \). \( \square \)

5. The main theorem

**Lemma 5.1.** Let \( M \in \text{rep}(\Gamma_r) \) be an indecomposable representation with dimension vector \( \dim M = (a + 1, a) \), \( a \geq 1 \). Then \( M \) is a regular and quasi-simple representation.

**Proof.** That \( M \) is regular follows immediately from \([5, 2.1]\). By \([5, 3.4]\) it suffices to show that \( A_t \) is not a common divisor of \( a + 1 \) and \( a \) for all \( t \geq 2 \), where \( \dim \tilde{P}_i = (A_{i-1}, A_i) \) is the dimension vector of the preprojective indecomposable representation \( \tilde{P}_i \) (see Figure \( 2 \)). But this is trivial since \( \gcd(a + 1, a) = 1 \). \( \square \)

**Theorem 5.2.** Let \( M \in \text{rep}(C_r) \) be balanced in the regular component \( D \) and \( 2 \leq d^+(M), d^-(M) \). There is \( n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \) there is a regular component \( D_n \) with \( W_C(D_n) = W_C(D) + ql(M) - 1 \). Moreover \( D_n \) contains a balanced quasi-simple representation \( F_n \) with \( \dim \pi_\lambda(F_n) = (n, n + 1) \) or \( \dim \pi_\lambda(F_n) = (n, n + 1) \) and \( D_i \neq D_j \) for \( i \neq j \geq n_0 \).

**Proof.** Write \( \dim \pi_\lambda(M) = (a, b) \). After dualising \( M \) we can assume \( a \leq b \). Set \( l := 2(b - a) + 1 \geq 1 \) and \( p_0 := 4l \). Now let \( p \geq p_0 \) with \( p \in 2\mathbb{N} \). Consider an indecomposable and thin representation \( L \) (i.e. \( \dim L \leq 1 \) for all \( x \in (C_r)_0 \) such that \( (L, M) \) is leaf-connected and \( T(L) = T(\text{supp}(L)) \) is of type \( A_{l,p} \)). Let \( g \in G \) be such that \( (M^g, L) \) is leaf-connected. We conclude for \( n := 2b + \frac{1}{2}p \in \mathbb{N} \), \( n_0 := 2b + \frac{1}{2}p_0 \), and \( F_n := M^g * L * M \) that

\[
\dim \pi_\lambda(F_n) = \dim \pi_\lambda(M^g * L * M) = 2 \dim \pi_\lambda(M) + \dim \pi_\lambda(L) = 2(a, b) + \left( \frac{1}{2}p + l, \frac{1}{2}p \right)
\]

\[
= (2a + 2b - 2a + 1, 2b) + \frac{1}{2}(p, p) = (n, n).
\]

By \([4.3]\) \( F_n \) is a regular indecomposable representation and by \([5.1]\) \( \pi_\lambda(F_n) \) is quasi-simple. Therefore \( F_n \) is quasi-simple in a regular component \( D_n \). We conclude with \([4.4.1]\)

\[ d^-(M) = d^-(M^g) \leq d^-(F_n) \leq \max\{d^-(M), d^-(L)\} = \max\{d^-(M), 2\} = d^-(M), \]

i.e. \( d^-(M) = d^-(F_n) \). By the same token we have \( d^+(M) = d^+(F_n) \) and conclude

\[ W_C(D_n) = d^+(F_n) + d^-(F_n) - ql(F_n) = d^+(M) + d^-(M) - 1 \]

\[= (d^+(M) + d^-(M) - ql(M)) + ql(M) - 1 = W_C(D) + ql(M) - 1. \]

It follows immediately from the construction that the regular components are pairwise distinct, since \( F_i, F_j \) are non-isomorphic and satisfy \( ql(F_i) = ql(F_j) \) and \( d^-(F_i) = d^-(F_j) \) for \( i \neq j \geq n_0 \). \( \square \)

**Corollary 5.3.** Let \( M \in \mathcal{D} \) be balanced and \( 2 \leq d^+(M), d^-(M) \). Then there exists a balanced and quasi-simple representation \( F \) in a regular component \( \mathcal{E} \) such that \( W_C(\mathcal{E}) = W_C(\mathcal{D}) + ql(M) - 1 \). Moreover there is a leaf \( x \in C_r^+ \) with \( \dim_k F_x = 1 = \dim_k F_y \) for the unique element \( y \in x^+ \cap \text{supp}(F) \).

**Proof.** Fix \( n \geq n_0 \) in the proof of the theorem and set \( F := F_n = M^g * L * M \). The last claim follows since \( L \) is a thin representation of type \( A_{l,p} \) which has \( l + 1 \geq 2 \) leaves in \( C_r^+ \) (see Figure \( 5 \)). \( \square \)
Figure 5. Illustration of the proof with supp(L) of type $\mathbb{A}_{3,12}$

6. Applications

6.1. Regular components for every width. The aim of this section is to construct for each $n \in \mathbb{N}$ a regular component $D$ with $\mathcal{W}_C(D) = n$. Although each indecomposable representation has a leaf, it is in general not true that a regular representation has leaves in $C^+_\tau$ and $C^-_\tau$. For example if $M$ is indecomposable in Inj, then each leaf of $M$ is a sink by [3.1]. The next results shows that self-dual representations have leaves in $C^+_\tau$ and $C^-_\tau$.

Lemma 6.1.1. Let $M \in \text{rep}(C_\tau)$ be indecomposable such that $D\pi_\lambda(M) \cong \pi_\lambda(M)$, then $M$ is balanced.

Proof. Since $\text{supp}(M)$ is finite there exists a leaf $x$ of $M$. Without loss of generality we assume that $x \in C^+_\tau$. We get $\pi_\lambda(M) \cong D\pi_\lambda(M) \cong \pi_\lambda(D_C(M))$. Therefore we find $h \in G$ such that $M \cong (D_C(M))^h$. Since $h^{-1}.\varphi(x) \in C^-_\tau$ (see [2.2]) is a leaf of $(D_C(M))^h$ the claim follows. □

We denote with $\sigma C_\tau$ the quiver obtained by changing the orientation of all arrows in $C_\tau$. Note that $\sigma^2 C_\tau = C_\tau$ and $\sigma C_\tau \cong C_\tau$. We denote by $\Phi^+$ the composition of the Bernstein-Gelfand-Ponomarev reflection functors [1] VII 5.5 for all the sources of $C_\tau$. $\Phi^+$ is a well-defined functor $\Phi^+ : \text{rep}(C_\tau) \to \text{rep}(\sigma C_\tau)$ (see [19] 2.3)). By the same token we have a functor $\Phi^- : \text{rep}(\sigma C_\tau) \to \text{rep}(C_\tau)$ given by the composition of the reflection functors for all the sources of $\sigma C_\tau$. Then $\mathcal{F} := \Phi^- \circ \Phi^+ : \text{rep}(C_\tau) \to \text{rep}(C_\tau)$ satisfies $\mathcal{F}(M) \cong \tau^{-1}_C M$ for $M \in \text{rep}(C_\tau)$ indecomposable and non-injective [19] 2.3], [1] VII 5.8. Therefore statements (a) and (b) of the next Lemma follow immediately from the definition of the reflection functors.

Lemma 6.1.2. Let $M \in \text{rep}(C_\tau)$ indecomposable and not injective.

(a) For each $x \in C^+_\tau$ we have $\dim_k (\tau^{-1}_C M)_x = (\sum_{y \in x^+} \dim_k M_y) - \dim_k M_x$.

(b) For each $y \in C^-_\tau$ we have $\dim_k (\tau^{-1}_C M)_y = (\sum_{x \in y^+} \dim_k (\tau^{-1}_C M)_x) - \dim_k M_y$.

(c) Let $0 \to A \to B \to C \to 0$ be an almost split sequence with $B$ indecomposable. If $a \in C^-_\tau$ is a leaf of $A$, then $B$ has a leaf in $C^-_\tau$.

(d) Let $0 \to A \to B \to C \to 0$ be an almost split sequence with $B$ indecomposable. If $a \in C^+_\tau$ is a leaf of $A$ and $b \in a^+$ satisfies $\dim_k A_b = \dim_k A_a$, then $a$ is a leaf of $B$.

Proof. (c) Consider a path $a \leftarrow b \to c$ such that $b, c$ are not in $\text{supp}(A)$ as illustrated in Figure 6. Since $b$ is in $C^+_\tau$ we get with (a) that

$$\dim_k C_b = \dim_k (\tau^{-1}_C A)_b = (\sum_{y \in b^+} \dim_k A_y) - \dim_k A_b = \dim_k A_a - \dim_k A_b = \dim_k A_a \neq 0.$$ 

Now let $d \in n(c) \setminus \{b\}$. Then $\dim_k C_d = (\sum_{y \in d^+} \dim_k A_y) - \dim_k A_d = 0$. Hence we get that $\dim_k C_c = \dim_k (\tau^{-1}_C A)_c \overset{(b)}{=} (\sum_{x \in c^+} \dim_k (\tau^{-1}_C A)_x) - \dim_k A_c = \dim_k C_b - \dim_k A_c = \dim_k C_b \neq 0$. Hence $c \in \text{supp}(C)$ is a leaf of $C$ and since $\text{supp}(B) = \text{supp}(A) \cup \text{supp}(C)$ we conclude that $c \in C^-_\tau$ is a leaf of $B$. 

(d) Application of (a) yields (see Figure 7)
\[
\dim_k C_a = \dim_k (\tau_{C_a}^{-1} A)_a = \left( \sum_{z \in a^+} \dim_k A_z \right) - \dim_k A_a = \dim_k A_b - \dim_k A_a = 0.
\]

Now fix \( c \in a^+ \setminus \{ b \} \), then \( c \in C^- \). Let \( d \in c^- \setminus \{ a \} \) then \( \dim_k A_f = 0 \) for all \( f \in d^+ \cup \{ d \} \), since \( f \notin \text{supp}(A) \). Hence we get
\[
\dim_k C_d = \dim_k (\tau_{C_d}^{-1} A)_d = \left( \sum_{y \in d^+} \dim_k A_y \right) - \dim_k A_d = 0.
\]

We conclude
\[
\dim_k C_c = \dim_k (\tau_{C_c}^{-1} A)_c = \left( \sum_{z \in c^-} \dim_k (\tau_{C_c}^{-1} A)_z \right) - \dim_k A_c = \left( \sum_{z \in c^-} \dim_k C_z \right) - \dim_k C_c = 0 - 0 = 0.
\]

We have shown that \( (a^+ \setminus \{ b \}) \cap (\text{supp}(A) \cup \text{supp}(C)) = \emptyset \). Since \( \text{supp}(A) \cup \text{supp}(C) = \text{supp}(B) \) we get \( |\text{supp}(B) \cap n(a)| = |(\text{supp}(A) \cup \text{supp}(C)) \cap n(a)| \leq |\{ b \}| = 1 \). Since \( a \in \text{supp}(A) \subseteq \text{supp}(B) \) the vertex \( a \in C^- \) is a leaf of \( T(\text{supp}(B)) \). Since \( B \) is indecomposable we have \( T(B) = T(\text{supp}(B)) \).

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{Illustration of the setup for (d).}
\end{figure}
\]

Let \( M_1 \in \text{rep}(C_r) \) be regular with \( \dim_k M_1 = 2 \). We define inductively a sequence of indecomposable representations in the regular component \( \overline{D} \) of \( M_1 \). The representation \( M_1 \) is quasi-simple. Assume that \( M_n \) is already defined. If \( n \) is odd, then \( M_{n+1} \) is the unique indecomposable representation with irreducible epimorphism \( M_{n+1} \to M_n \); if \( n \) is even, then \( M_{n+1} \) is the unique indecomposable representation with irreducible monomorphism \( M_n \to M_{n+1} \). The component \( \overline{D} \) is shown in Figure 8. We have \( W_C(\overline{D}) = W(\pi_\lambda(\overline{D})) = 1 \) [22, Example 3.3].

**Theorem 6.1.3.** Let \( r \geq 3 \).

(a) For each \( m \geq 1 \) there is \( n_0 \geq 1 \) and a family \( (D_n)_{n \geq n_0} \) of regular components with \( W_C(D_n) = m \) and \( D_n \) contains a quasi-simple representation \( E_n \) with \( \dim \pi_\lambda(E_n) = (n + 1, n) \).

(b) \( N \subseteq \{ W_C(E) \mid E \in \mathcal{R}(C_r) \} \subseteq N_0 \).

(c) \( \{ W(C) \mid C \in \mathcal{R} \} = N_0 \).

**Proof.** Recall that \( W_C(\overline{D}) = 1 \) and \( W_C(\overline{E}) = 2 \) for the regular component \( \overline{E} \) containing \( X_1 \). Fix \( l \geq 1 \). Then we have \( D \pi_\lambda(M_{2l+1}) \cong \pi_\lambda(M_{2l+1}) \) by [22, Example 3.3]. By [6.1.1] \( M_{2l+1} \) is balanced. Moreover we have \( d^-(M_{2l+1}), d^+(M_{2l+1}) \geq 2 \). Hence [5.2] yields \( n_0 \in \mathbb{N} \) and an infinite family of components \( (D_n)_{n \geq n_0} \) of width \( W_C(D_n) = W_C(\overline{D}) + ql(M_{2l+1}) - 1 = 2l + 1 \).
By Corollary 5.3, we find a balanced and quasi-simple representation \( A \in D_n \) that satisfies the assumption of \([6,1.2](d)\). Consider the AR sequence \( 0 \to A \to B \to C \to 0 \). Then \( B \) is balanced by \([6.1.2](c),(d), \) \( 2 \leq d^+(B), d^-(B) \) and \( ql(B) = 2 \). By \( 5.2 \), we get an infinite family of components of width \( \mathcal{W}C(D_n) + ql(B) - 1 = (2l + 1) + 2 - 1 = 2l + 2 \). This proves (a) and (b). For (c) observe that there exist regular components \( C \in R \) with \( \mathcal{W}(C) = 0 \) \([22,3.3]\). \( \square \)

6.2. Counting regular components of fixed width. This section is motivated by the following result by Kerner and Lukas.

**Proposition.** \([10,5.2]\) Assume that \( k \) is uncountable and \( A \) is a wild hereditary algebra with \( n \geq 2 \) simple modules. Then the number of regular component of \( A \) with quasi-rank \(-1\) is uncountable. Moreover the set of components of quasi-rank \( \leq -1 \) for the Kronecker algebra is uncountable.

The proof of the second statement uses the first statement for the path algebra \( A \) of the wild quiver \( 1 \to 2 \rightrightarrows 3 \) with \( n(A) = 3 \) simple modules and the existence of a regular tilting module \( T_r \) in mod \( A \) that induces a bijection

\[
\varphi: \{ C \mid C \text{ regular component of } A \} \to R
\]

with \( \text{rk}(\varphi(C)) \leq \text{rk}(C) \) for all \( C \in \{ D \mid D \text{ regular component of } A \} \). To generalize the arguments from \( \leq -1 \) to \( \leq -p \) for \( p \in \mathbb{N} \) one would need the existence of bricks of arbitrary quasi-length. Unfortunately for each hereditary algebra there is a finite upper bound for the quasi-length of regular bricks given by the number of simple modules \(-1\), which is in our case \( n(A) - 1 = 2 \).

We show how to circumvent this obstacle by considering an action of the general linear group \( \text{GL}_r(k) \) on \( \text{rep}(\Gamma_r) \).

**Definition.** \([4,3.6]\) Denote with \( \text{GL}_r(k) \) the group of invertible \( r \times r \)-matrices with coefficients in \( k \) which acts on \( \bigoplus_{i=1}^r k \gamma_i \) via \( A \gamma_j = \sum_{i=1}^r a_{ij} \gamma_i \) for \( 1 \leq j \leq r, A \in \text{GL}_r(k) \). For \( A \in \text{GL}_r(k) \), let \( \varphi_A : \mathcal{K}_r \to \mathcal{K}_r \) the algebra homomorphism with \( \varphi_A(e_1) = e_1, \varphi_A(e_2) = e_2 \) and \( \varphi_A(\gamma_i) = A \gamma_i, 1 \leq i \leq r. \) For a \( \mathcal{K}_r \)-module \( M \) denote the pullback of \( M \) along \( \varphi_{A^{-1}} \) by \( A.M \). The module \( M \) is called \( \text{GL}_r(k) \)-stable if \( A.M \cong M \) for all \( g \in \text{GL}_r(k) \), in other words if \( \text{GL}_r = \text{GL}_r(k)_M := \{ A \in \text{GL}_r(k) \mid A.M \cong M \} \).

**Examples.** (a) The simple representations of \( \Gamma_r \) are \( \text{GL}_r(k) \)-stable and by \([6,2.2]\) every preinjective and every preprojective representation is \( \text{GL}_r(k) \)-stable.

(b) There are \( \text{GL}_r(k) \)-stable representations that are regular \([22,1.2]\). In this case all representations in the same component are also \( \text{GL}_r(k) \)-stable.

(c) Recall that the preinjective representation \( I_3 = \tau I_1 \) has dimension vector \( (3r - 1, r) \). Let \( M \) be in \( \text{rep}(C_r) \) with \( \pi_\lambda(M) \cong I_3 \). The support of \( M \) for \( r = 3 \) is shown in Figure 9. Let \( c \in T(M)_0 \) be the vertex \( \dim_k M_c = 2 \).
The underlying tree of $\text{supp}(M)$ is symmetric in the following sense. The quiver $T(M) \setminus \{c\}$ is not connected and consists of $r = 3$ isomorphic trees $T_1, T_2, T_3$. Moreover for $n \in \{1, 2\}$ the sum $\dim_k M_x$ for $x \in (T_i)_0$ with distance $d(c, x) = n$ is independent of $i \in \{1, 2, 3\}$. We will prove that this is not a coincidence. We show that every representation $M$ such that $\pi_{\lambda}(M)$ is $\text{GL}_r(k)$-stable, has a central point.

### 6.2.1. $S_r$-stability

Denote by $S_r$ the symmetric group on $\{1, \ldots, r\}$. Then each for each $\sigma \in S_r$ there is an induced bijection on $\{1, \ldots, r\} \to (\Gamma_r)_1$ given by $i \mapsto \gamma_{\sigma(i)}$ which extends in a natural way to the set of equivalence classes of walks in $\Gamma_r$. By abuse of notation we denote by $\sigma: C_r \to C_r$ the induced quiver automorphism. Let $\alpha: [w] \to [\gamma_i w]$ be an arrow in $C_r$, then by definition $\sigma(\alpha)$ is the unique arrow $\sigma(\alpha): \sigma([w]) \to [\gamma_{\sigma(i)} \sigma([w])]$. Note that $\pi(\alpha) = \gamma_i$ and $\pi(\sigma(\alpha)) = \gamma_{\sigma(i)}$.

Now let $M \in \text{rep}(C_r)$ be an indecomposable representation. We define $\sigma(M)$ to be the indecomposable representation with

$$\sigma(M)_x := M_{\sigma(x)} \text{ and } \sigma(M)(\alpha) := M(\sigma(\alpha)).$$

We say that $M$ is $S_r$-stable if for each $\sigma \in S_r$ there is $g_{\sigma}$ with $M \cong \sigma(M)^{g_{\sigma}}$. This definition is motivated by the following obvious result:

**Corollary 6.2.1.** Let $M \in \text{rep}(C_r)$ be an indecomposable representation. If $\pi_{\lambda}(M)$ is $\text{GL}_r(k)$-stable then $M$ is $S_r$-stable.

**Proof.** We let $I(\sigma)$ be the permutation matrix given by $\sigma$, i.e. $I(\sigma)_{ij} = 1$ if and only if $\sigma(i) = j$ and $I(\sigma)_{ij} = 0$ otherwise. Now we assume that $\pi_{\lambda}(M)$ is $\text{GL}_r(k)$-stable. Then we get for each $\sigma \in S_r$ that

$$\pi_{\lambda}(\sigma(M)) = I(\sigma) \pi_{\lambda}(M) \cong \pi_{\lambda}(M).$$

Hence we find $g_{\sigma} \in G$ such that

$$M \cong \sigma(M)^{g_{\sigma}}.$$

Note that $\sigma([\epsilon_1]) = [\epsilon_1]$ and since $G$ acts freely $C_r$, the element $g_{\sigma}$ is uniquely determined. In the following we study the quiver automorphisms $\sigma \circ g_{\sigma}: C_r \to C_r$. 

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**Figure 9.** Support of $M$ with $\pi_{\lambda}(M) \cong I_3$ for $r = 3$. 

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6.2.2. Automorphisms of trees. A group $G$ is said to have property FA \[21\] I.6.1 if every action of $G$ on a tree $T$ by graph automorphisms (which do not invert an edge) has a global fixed point $z \in T_0$, i.e. $gz = z$ for all $g \in G$. It is known \[21\] I.6.3.1 that all finitely generated torsion groups have the property FA. In particular, for every group action of a finite group acting on a quiver which underlying graph is a tree, there is a global fixed point.

**Definition.** For $x \in C_r^+$ and $1 \leq i \leq r$ denote by $T(x,i)$ the connected component of $C_r \setminus \{x\}$ containing $t(a_i)$, where $a_i : x \to t(a_i)$ is the unique arrow with $\pi(a_i) = \gamma_i$. Let $M$ in $\text{rep}(C_r)$ be indecomposable and $x \in \text{supp}(M)$, then we define $T(x,i,M) := T(M) \cap T(x,i)$.

Note that $\text{supp}(M) = \{x\} \cup \bigcup_{i=1}^r T(x,i,M)_0$.

**Proposition 6.2.2.** Let $M \in \text{rep}(C_r)$ be $S_r$-stable. Then there is $c \in \text{supp}(M)$ such that

(a) $\sigma \circ g_{\sigma}(c) = c$ for all $\sigma \in S_r$.

(b) For each $n \in \mathbb{N}$ the number $r$ divides $D(n,c) := \sum_{x \in \text{supp}(M),d(x,c) = n} M_x$.

**Proof.** Since $M \cong \sigma(M)^{\sigma}$, we have

\[ \text{supp}(M) = \text{supp}(\sigma(M)^{\sigma}) = \{g_{\sigma}^{-1}x \mid x \in \text{supp}(\sigma(M))\} = \{g_{\sigma}^{-1} \circ \sigma^{-1}(x) \mid x \in \text{supp}(M)\}. \]

We assume that $\dim_k M \neq 1$, otherwise there is nothing to show. Let $T \subseteq C_r$ be the finite subtree $T := T(M)$. Then $\sigma \circ g_{\sigma} : T \to T$ is a quiver automorphism of $T$. Since $T$ is finite, $\text{Aut}(T)$ is finite and there exists a vertex $c \in T_0$ with $\varphi(c) = c$ for all $\varphi \in \text{Aut}(T)$. We assume that $c \in C_r^+$. For $1 \leq i \leq r$ we let $\beta_i : c \to t(\beta_i)$ be the unique arrow with $\pi(\beta_i) = \gamma_i$ and set $T_i := T(c,i,M)$. Since $\dim_k M \neq 1$ and $M$ is indecomposable, we can assume w.l.o.g. that $c := t(\beta_1) \in \text{supp}(M)$. Since every automorphism of $C_r$ respects the metric (see [4.2]) we get

\[ 1 = d(c,e) = d(\sigma \circ g_{\sigma}(c), \sigma \circ g_{\sigma}(e)) = d(c, \sigma \circ g_{\sigma}(e)). \]

In particular, $\sigma \circ g_{\sigma}(\beta_1) \in \{\beta_1, \ldots, \beta_r\}$. Now fix $j \in \{1, \ldots, r\} \setminus \{1\}$ and $\sigma := (1,j) \in S_r$. Then we have $\pi(\sigma(\beta_1)) = \gamma_j$. Since $\pi \circ g = \pi$ for all $g \in G$, we get $\sigma \circ g_{\sigma}(\beta_1) = \beta_j$ and conclude $\sigma \circ g_{\sigma}(T_1) = T_j$. Hence $T_1, \ldots, T_r$ are non-empty isomorphic quivers. For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, r\}$ we define $d_{n,i,c} := \{x \in (T_i)_0 \mid d(x,c) = n\}$. Let now $x \in d_{n,1,c}$, then we have $\sigma \circ g_{\sigma}(x) \in d_{n,j,c}$ since $\sigma \circ g_{\sigma}(x) \in T_j$ and

\[ n = d(c,x) = d(\sigma \circ g_{\sigma}(c), \sigma \circ g_{\sigma}(x)) = d(c, \sigma \circ g_{\sigma}(x)). \]

Moreover we have $M_x = (\sigma(M)^{\sigma})_x = M_{\sigma \circ g_{\sigma}(x)}$. It follows $\sum_{y \in d_{n,1,c}} \dim_k M_y = \sum_{y \in d_{n,j,c}} \dim_k M_y$ and we conclude

\[ D(n,c) = \sum_{x \in \text{supp}(M),d(x,c) = n} \dim_k M_x = r \cdot \sum_{y \in d_{n,1,c}} \dim_k M_y. \]

\[ \square \]

**Corollary 6.2.3.** Assume that $M$ is $S_r$-stable. If $\dim \pi_\lambda(M) = (a,b)$ then $r$ divides $a$ or $b$.

**Proof.** Let $c \in \text{supp}(M)$ be as in [6.2.2]. Then we have

\[ \dim_k M = \dim_k M_c + \sum_{n \in 2\mathbb{N} - 1} D(n,c) + \sum_{n \in 2\mathbb{N}} D(n,c). \]

If $c \in C_r^+$, then $b = \sum_{n \in 2\mathbb{N} - 1} D(n,c)$ and $a = \sum_{n \in 2\mathbb{N} - 1} D(n,c)$ otherwise. Hence $r$ divides $b$ or $a$. \[ \square \]

As an application we get the following result for components of the Kronecker quiver $\Gamma_r$.

**Corollary 6.2.4.** Let $m \in \mathbb{N}$, then there exists a regular component $C$ with $W(C) = m$ and no representation in $C$ is $\text{GL}_r(k)$-stable.
Proof. Let \( m \geq 1 \). By \([5.2]\) there exists \( n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \) there is a regular component \( D_n \) with \( \mathcal{W}(D_n) = m \) and \( D_m \) contains a quasi-simple representation \( E_n = \pi_\lambda(F_n) \) with \( F_n \in \text{rep}(C) \) and \( \dim E_n = (n + 1, n) \). Since \( r \geq 3 \), we find \( l \geq n_0 \) (even infinitely many) such that \( r \) does not divide \( l \) and \( l + 1 \). Hence \( F_l \) is not \( S_r \)-stable and \( E_l \) not \( \text{GL}_r(k) \)-stable. Therefore no representation in \( D_l \) is \( \text{GL}_r(k) \)-stable by \([6.2.2]\).

6.2.3. The number of regular components in \( \text{rep}(\Gamma_r) \).

Definition. A locally closed set is an open subset of a closed set. A constructible set is a finite union of locally closed sets.

Lemma 6.2.5. Let \( M \in \text{rep}(\Gamma_r) \) with \( \text{GL}_r(k)_M \neq \text{GL}_r(k) \). There is an injection \( \iota : k \to \text{GL}_r(k)/\text{GL}_r(k)_M \).

Proof. By \([6.2.1]\) \( \text{GL}_r(k)_M \) is a closed subgroup of \( \text{GL}_r(k) \) and by \([14.5.5]\) an \( \text{GL}_r(k)/\text{GL}_r(k)_M \) is an algebraic variety. Hence we find an affine variety \( V \subseteq \text{GL}_r(k)/\text{GL}_r(k)_M \) with \( d := \dim V = \dim \text{GL}_r(k)/\text{GL}_r(k)_M \). Since \( \text{GL}_r(k) \) is irreducible we have \( \dim \text{GL}_r(k)/\text{GL}_r(k)_M = \dim \text{GL}_r(k) - \dim \text{GL}_r(k)_M \geq 1 \). Let \( k[t_1, \ldots, t_d] \to k[V] \) be a Noether-normalization and \( \varphi^* : V \to \mathbb{A}^d \) be the comorphism. Then \( \varphi^* \) is dominant. Hence there is a dominant morphism \( f : V \to \mathbb{A}^1 \). By Chevalley’s Theorem \( f(V) \) is constructible and hence finite or cofinite. Since \( f(V) \) is dense in \( \mathbb{A}^1 \), \( f(V) \) is not finite and therefore cofinite. That means \( |k \setminus C| \) is finite. Since \( k \) is infinite we have \( |f(V)| = |k| \). It follows \( |k| = |A| = |f(V)| \leq |V| \leq |\text{GL}_r(M)/\text{GL}_r(k)_M| \).

Theorem 6.2.6. Let \( m \in \mathbb{N} \). There is a bijection \( \{C \in \mathcal{R} \mid \mathcal{W}(C) = m\} \to k \).

Proof. It is well known that \( |\mathcal{R}| = |k| \), see \([2.2] \text{XVIII 1.8}\). In particular there is an injection \( \{C \in \mathcal{R} \mid \mathcal{W}(C) = m\} \to k \).

By \([6.2.4]\) there is a regular component \( C \) with \( \mathcal{W}(C) = m \in \mathbb{N} \) such that no representation in \( C \) is \( \text{GL}_r(k) \)-stable. For \( E \in C \) the map

\[
\text{GL}_r(k)/\text{GL}_r(k)_E \to \{Z \in \text{rep}(\Gamma_r) \mid \dim Z = \dim E\}; A \text{GL}_r(k)_E \to AE
\]

is well defined and injective. Since the number of representations in a regular component with given dimension vector \((a, b)\) is \( \leq 1 \) \([2.1 \text{ XIII 1.7}] \) and \( \text{GL}_r(k) \) acts via auto equivalences we get with \([6.2.5]\) an injection

\[
k \to \text{GL}_r(k)/\text{GL}_r(k)_E \to \{Z \in \text{rep}(\Gamma_r) \mid \dim Z = \dim E\} \to \{C \in \mathcal{R} \mid \mathcal{W}(C) = m\}.
\]

By the Schröder-Bernstein Theorem we get the desired bijection.

Remark. Note that we restrict ourselves to components of width \( \geq 1 \), since we dont know whether components in \( \text{rep}(\Gamma_r) \) of width \( 0 \) exist. Also the examples \([22.3.3]\) of components of width \( 0 \) in \( \text{rep}(\Gamma_r) \) are \( \text{GL}_r(k) \)-stable. For the case \( n = 0 \) we argue as follows.

Lemma 6.2.7. Let \( r \geq 3 \), then there exists a bijection \( k \to \{C \in \mathcal{R} \mid \mathcal{W}(C) = 0\} \).

Proof. The proof of \([3.3.3]\) yields an injective map \( \varphi : \text{ind} \mathcal{E}(X) \to \{C \in \mathcal{R} \mid \mathcal{W}(C) = 0\} \), where \( \text{ind} \mathcal{E}(X) \) are the indecomposable objects in a category \( \mathcal{E}(X) \) equivalent to the category of finite dimensional modules over the power-series ring \( k\langle X_1, \ldots, X_t \rangle \) in non-commuting variables \( X_1, \ldots, X_t \) and \( t \geq 2 \). Now let \( \lambda \in k \setminus \{0\} \) and consider the indecomposable module \( M_\lambda = k^2 \) given by \( X_1(a, b) = (ab, 0), X_2(a, b) = (b, 0) \) and \( X_i(a, b) = 0 \) for \( i > 2 \). Then \( M_\lambda \neq M_\mu \) for \( \lambda \neq \mu \) and we have an injection \( k \to \text{ind} \mathcal{E}(X) \). The claim follows as in \([6.2.6]\).

Corollary 6.2.8. Let \( r \geq 3 \), then for each \( n \in \mathbb{N} \) there are exactly \( |k| \) regular components such that \( \text{rk}(C) \in [-n, -n + 3] \).

Corollary 6.2.9. Assume that \( k \) is uncountable and \( p \in \mathbb{N} \). The set of components of quasi-rank \( \leq -p \) in \( \mathcal{R} \) is uncountable.
ACKNOWLEDGEMENT

The results of this article are part of my doctoral thesis, which I am currently writing at the University of Kiel. I would like to thank my advisor Rolf Farnsteiner for his continuous support and helpful comments. I also would like to thank the whole research team for the very pleasant working atmosphere and the encouragement throughout my studies. In particular, I thank Christian Drenkhahn for proofreading. Furthermore, I thank Claus Michael Ringel for fruitful discussions during my visits in Bielefeld.

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Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, 24098 Kiel, Germany
E-mail address: bissinger@math.uni-kiel.de