On radical square zero rings

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Abstract. Let $\Lambda$ be a connected left artinian ring with radical square zero and with $n$ simple modules. If $\Lambda$ is not self-injective, then we show that any module $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$ is projective. We also determine the structure of the artin algebras with radical square zero and $n$ simple modules which have a non-projective module $M$ such that $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$.

Xiao-Wu Chen [C] has recently shown: given a connected artin algebra $\Lambda$ with radical square zero then either $\Lambda$ is self-injective or else any CM module is projective. Here we extend this result by showing: If $\Lambda$ is a connected artin algebra with radical square zero and $n$ simple modules then either $\Lambda$ is self-injective or else any module $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$ is projective. Actually, we will not need the assumption on $\Lambda$ to be an artin algebra; it is sufficient to assume that $\Lambda$ is a left artinian ring. And we show that for artin algebras the bound $n + 1$ is optimal by determining the structure of those artin algebras with radical square zero and $n$ simple modules which have a non-projective module $M$ such that $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$.

From now on, let $\Lambda$ be a left artinian ring with radical square zero, this means that $\Lambda$ has an ideal $I$ with $I^2 = 0$ (the radical) such that

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Λ/I is semisimple artinian. We also assume that Λ is connected (the only central idempotents are 0 and 1). The modules to be considered are usually finitely generated left Λ-modules. Let n be the number of (isomorphism classes of) simple modules.

Given a module M, we denote by PM a projective cover, by QM an injective envelope of M. Also, we denote by ΩM a syzygy module for M, this is the kernel of a projective cover PM → M. Since Λ is a ring with radical square zero, all the syzygy modules are semisimple. Inductively, we define Ω0M = M, and Ωi+1M = Ω(ΩiM) for i ≥ 0.

Lemma 1. If M is a non-projective module with Exti(M, Λ) = 0 for 1 ≤ i ≤ d + 1 (and d ≥ 1), then there exists a simple non-projective module S with Exti(S, Λ) = 0 for 1 ≤ i ≤ d.

Proof. The proof is obvious: We have Exti(M, Λ) ≃ Exti−1(ΩM, Λ), for all i ≥ 2. Since M is not projective, ΩM ≠ 0. Now ΩM is semisimple. If all simple direct summands of ΩM are projective, then also ΩM is projective, but then the condition Ext1(M, Λ) = 0 implies that Ext1(M, ΩM) = 0 in contrast to the existence of the exact sequence 0 → ΩM → PM → M → 0. Thus, let S be a non-projective simple direct summand of ΩM.

Lemma 2. If S is a non-projective simple module with Ext1(S, Λ) = 0, then PS is injective and ΩS is simple and not projective.

Proof. First, we show that PS has length 2. Otherwise, ΩS is of length at least 2, thus there is a proper decomposition ΩS = U ⊕ U′ and then there is a canonical exact sequence

$$0 \rightarrow PS \rightarrow PS/U \oplus PS/U' \rightarrow S \rightarrow 0,$$

which of course does not split. But since Ext1(S, Λ) = 0, we have Ext1(S, P) = 0, for any projective module P. Thus, we obtain a contradiction.

This shows also that ΩS is simple. Of course, ΩS cannot be projective, again according to the assumption that Ext1(S, P) = 0, for any projective module P.

Now let us consider the injective envelope Q of ΩS. It contains PS as a submodule (since PS has ΩS as socle). Assume that Q is of length at least 3. Take a submodule I of Q of length 2 which is different from PS
and let $V = PS + I$, this is a submodule of $Q$ of length 3. Thus, there are the following inclusion maps $u_1, u_2, v_1, v_2$:

$$
\begin{array}{c}
\Omega S \xrightarrow{u_1} PS \\
v_1 \downarrow \quad \downarrow u_2 \\
I \xrightarrow{v_2} V
\end{array}
$$

The projective cover $p : PI \to I$ has as restriction a surjective map $p' : \text{rad} PI \to \Omega S$. But $\text{rad} PI$ is semisimple, thus $p'$ is a split epimorphism, thus we obtain a map $w : \Omega S \to PI$ such that $pw = v_1$. We consider the exact sequence induced from the sequence $0 \to \Omega S \to PI \to S$ by the map $w$:

$$
\begin{array}{c}
0 \xrightarrow{} \Omega S \xrightarrow{u_1} PS \xrightarrow{c_1} S \xrightarrow{} 0 \\
w \downarrow \\
0 \xrightarrow{} PI \xrightarrow{u'_1} N \xrightarrow{e'_1} S \xrightarrow{} 0
\end{array}
$$

Here, $N$ is the pushout of the two maps $u_1$ and $w$. Since we know that $u_2u_1 = v_2v_1 = v_2pw$, there is a map $f : N \to V$ such that $fu'_1 = v_2p$ and $fw' = u_2$. Since the map $\begin{bmatrix} v_2p & u_2 \end{bmatrix} : PI \oplus PS \to V$ is surjective, also $f$ is surjective.

But recall that we assume that $\text{Ext}^1(S, \Lambda) = 0$, thus $\text{Ext}^1(S, PI) = 0$. This means that the lower exact sequence splits and therefore the socle of $N = PI \oplus S$ is a maximal submodule of $N$ (since $I$ is a local module, also $PI$ is a local module). Now $f$ maps the socle of $N$ into the socle of $V$, thus it maps a maximal submodule of $N$ into a simple submodule of $V$. This implies that the image of $f$ has length at most 2, thus $f$ cannot be surjective. This contradiction shows that $Q$ has to be of length 2, thus $Q = PS$ and therefore $PS$ is injective.

Lemma 3. If $S$ is a non-projective simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_i S$ with $0 \leq i \leq d$ are simple and not projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$.

Proof. The proof is again is obvious, we use induction. If $d \geq 2$, we know by induction that the modules $S_i$ with $0 \leq i \leq d - 1$ are simple and not projective, and that the modules $P(S_i)$ are injective for $0 \leq i < d - 1$. 

\hfill $\square$
But $\text{Ext}^1(\Omega_{d-1}S, \Lambda) \simeq \text{Ext}^d(S, \Lambda) = 0$, thus Lemma 2 asserts that also $S_d$ is simple and not projective and that $P(S_{d-1})$ is injective. □

**Lemma 4.** Let $S_0, S_1, \ldots, S_b$ be simple modules with $S_i = \Omega_i(S_0)$ for $1 \leq i \leq b$. Assume that there is an integer $0 \leq a < b$ such that the modules $S_i$ with $a \leq i < b$ are pairwise non-isomorphic, whereas $S_b$ is isomorphic to $S_a$. In addition, we assume that the modules $P(S_i)$ for $a \leq i < b$ are injective. Then $S_a, \ldots, S_{b-1}$ is the list of all the simple modules and $\Lambda$ is self-injective.

*Proof.* Let $S$ be the subcategory of all modules with composition factors of the form $S_i$, where $a \leq i < b$. We claim that this subcategory is closed under projective covers and injective envelopes. Indeed, the projective cover of $S_i$ for $a \leq i < b$ has the composition factors $S_i$ and $S_{i+1}$ (and $S_b = S_a$), thus it is in $S$. Similarly, the injective envelope for $S_i$ with $a < i < b$ is $Q(S_i) = P(S_{i-1})$, thus it has the composition factors $S_{i-1}$ and $S_i$, and $Q(S_a) = Q(S_b) = P(S_{b-1})$ has the composition factors $S_{b-1}$ and $S_a$. Since we assume that $\Lambda$ is connected, we know that the only non-trivial subcategory closed under composition factors, extensions, projective covers and injective envelopes is the module category itself. This shows that $S_a, \ldots, S_{b-1}$ are all the simple modules. Since the projective cover of any simple module is injective, $\Lambda$ is self-injective. □

**Theorem 1.** Let $\Lambda$ be a connected left artinian ring with radical square zero. Assume that $\Lambda$ is not self-injective. If $S$ is a non-projective simple module such that $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_i S$ with $0 \leq i \leq d$ are pairwise non-isomorphic simple and non-projective modules and the modules $P(S_i)$ are injective for $0 \leq i < d$.

*Proof.* According to Lemma 3, the modules $S_i$ (with $0 \leq i \leq d$) are simple and non-projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$. If at least two of the modules $S_0, \ldots, S_d$ are isomorphic, then Lemma 4 asserts that $\Lambda$ is self-injective, but this we have excluded. □

**Theorem 2.** Let $\Lambda$ be a connected left artinian ring with radical square zero and with $n$ simple modules. The following conditions are equivalent:

(i) $\Lambda$ is self-injective, but not a simple ring.
(ii) There exists a non-projective module $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n + 1$.

(iii) There exists a non-projective simple module $S$ with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n$.

Proof. First, assume that $\Lambda$ is self-injective, but not simple. Since $\Lambda$ is not semisimple, there is a non-projective module $M$. Since $\Lambda$ is self-injective, $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$. This shows the implication (i) $\implies$ (ii). The implication (ii) $\implies$ (iii) follows from Lemma 1. Finally, for the implication (iii) $\implies$ (i) we use Theorem 1. Namely, if $\Lambda$ is not self-injective, then Theorem 1 asserts that the simple modules $S_i = \Omega_i S$ with $0 \leq i \leq n$ are pairwise non-isomorphic. However, these are $n + 1$ simple modules, and we assume that the number of isomorphism classes of simple modules is $n$. This completes the proof of Theorem 2.

Note that the implication (ii) $\implies$ (i) in Theorem 2 asserts in particular that either $\Lambda$ is self-injective or else that any CM module is projective, as shown by Chen [C]. Let us recall that a module $M$ is said to be a CM module provided $\text{Ext}^i(M, \Lambda) = 0$ and $\text{Ext}^i(\text{Tr} M, \Lambda) = 0$, for all $i \geq 1$ (here $\text{Tr}$ denotes the transpose of the module); these modules are also called Gorenstein-projective modules, or totally reflexive modules, or modules of G-dimension equal to 0. Note that in general there do exist modules $M$ with $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$ which are not CM modules, see [JS].

We also draw the attention to the generalized Nakayama conjecture formulated by Auslander-Reiten [AR]. It asserts that for any artin algebra $\Lambda$ and any simple $\Lambda$-module $S$ there should exist an integer $i \geq 0$ such that $\text{Ext}^i(S, \Lambda) \neq 0$. It is known that this conjecture holds true for algebras with radical square zero. The implication (iii) $\implies$ (i) of Theorem 2 provides an effective bound: If $n$ is the number of simple $\Lambda$-modules, and $S$ is simple, then $\text{Ext}^i(S, \Lambda) \neq 0$ for some $0 \leq i \leq n$. Namely, in case $S$ is projective or $\Lambda$ is self-injective, then $\text{Ext}^0(S, \Lambda) \neq 0$. Now assume that $S$ is simple and not projective and that $\Lambda$ is not self-injective. Then there must exist some integer $1 \leq i \leq n$ with $\text{Ext}^i(S, \Lambda) \neq 0$, since otherwise the condition (iii) would be satisfied and therefore condition (i).

Theorem 1 may be interpreted as a statement concerning the Ext-quiver of $\Lambda$. Recall that the Ext-quiver $\Gamma(R)$ of a left artinian ring $R$ has as vertices the (isomorphism classes of the) simple $R$-modules, and if $S, T$
are simple $R$-modules, there is an arrow $T \to S$ provided $\text{Ext}^1(T, S) \neq 0$, thus provided that there exists an indecomposable $R$-module $M$ of length 2 with socle $S$ and top $T$. We may add to the arrow $\alpha : T \to S$ the number $l(\alpha) = ab$, where $a$ is the length of soc $PT$ and $b$ is the length of $QS/$soc (note that $b$ may be infinite). The properties of $\Gamma(R)$ which are relevant for this note are the following: the vertex $S$ is a sink if and only if $S$ is projective; the vertex $S$ is a source if and only if $S$ is injective; finally, if $R$ is a radical square zero ring and $S, T$ are simple $R$-modules then $PT = QS$ if and only if there is an arrow $\alpha : T \to S$ with $l(\alpha) = 1$ and this is the only arrow starting at $T$ and the only arrow ending in $S$.

Theorem 1 assert the following: Let $\Lambda$ be a connected left artinian ring with radical square zero. Assume that $\Lambda$ is not self-injective. Let $S$ be a non-projective simple module such that $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, and let $S_i = \Omega_i S$ with $0 \leq i \leq d$. Then the local structure of $\Gamma(\Lambda)$ is as follows:

such that there is at least one arrow starting in $S_d$ (but maybe no arrow ending in $S_0$). To be precise: the picture is supposed to show all the arrows starting or ending in the vertices $S_0, \ldots, S_d$ (and to assert that the vertices $S_0, \ldots, S_d$ are pairwise different).

Let us introduce the quivers $\Delta(n, t)$, where $n, t$ are positive integers. The quiver $\Delta(n, t)$ has $n$ vertices and also $n$ arrows, namely the vertices labeled $0, 1, \ldots, n - 1$, and arrows $i \to i + 1$ for $0 \leq i \leq n - 1$ (modulo $n$) (thus, we deal with an oriented cycle); in addition, let $l(\alpha) = t$ for the arrow $\alpha : n - 1 \to 0$ and let $l(\beta) = 1$ for the remaining arrows $\beta$: 
Note that the Ext-quiver of a connected self-injective left artinian ring with radical square zero and \( n \) vertices is just \( \Delta(n, 1) \). Our further interest lies in the cases \( t > 1 \).

**Theorem 3.** Let \( \Lambda \) be a connected left artinian ring with radical square zero and with \( n \) simple modules.

(a) If there exists a non-projective simple modules \( S \) with \( \text{Ext}^i(S, \Lambda) = 0 \) for \( 1 \leq i \leq n - 1 \), or if there exists a non-projective module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \), then \( \Gamma(\Lambda) \) is of the form \( \Delta(n, t) \) with \( t > 1 \).

(b) Conversely, if \( \Gamma(\Lambda) = \Delta(n, t) \) and \( t > 1 \), then there exists a unique simple module \( S \) with \( \text{Ext}^i(S, \Lambda) = 0 \) for \( 1 \leq i \leq n - 1 \), namely the module \( S = S(0) \) (and it satisfies \( \text{Ext}^n(S, \Lambda) \neq 0 \)).

(c) If \( \Gamma(\Lambda) = \Delta(n, t) \) and \( t > 1 \), and if we assume in addition that \( \Lambda \) is an artin algebra, then there exists a unique indecomposable module \( M \) with \( \text{Ext}^i(M, \Lambda) = 0 \) for \( 1 \leq i \leq n \), namely \( M = \text{Tr} D(S(0)) \) (and it satisfies \( \text{Ext}^{n+1}(M, \Lambda) \neq 0 \)).

Here, for \( \Lambda \) an artin algebra, \( D \) denotes the \( k \)-duality, where \( k \) is the center of \( \Lambda \) (thus \( D = \text{Hom}_k(-, E) \), where \( E \) is a minimal injective cogenerator in the category of \( k \)-modules); thus \( D \text{Tr} \) is the Auslander-Reiten translation and \( \text{Tr} D \) the reverse.

**Proof of Theorem 3.** Part (a) is a direct consequence of Theorem 1, using the interpretation in terms of the Ext-quiver as outlined above. Note that we must have \( t > 1 \), since otherwise \( \Lambda \) would be self-injective.

(b) We assume that \( \Gamma(\Lambda) = \Delta(n, t) \) with \( t > 1 \). For \( 0 \leq i < n \), let \( S(i) \) be the simple module corresponding to the vertex \( i \), let \( P(i) \) be its projective cover, \( I(i) \) its injective envelope. We see from the quiver that all the projective modules \( P(i) \) with \( 0 \leq i \leq n - 2 \) are injective, thus \( \text{Ext}^j(-, \Lambda) = \text{Ext}^j(-, P(n-1)) \) for all \( j \geq 1 \). In addition, the quiver shows that \( \Omega S(i) = S(i+1) \) for \( 0 \leq i \leq n - 2 \). Finally, we have \( \Omega S(n-1) = S(0)^a \) for some positive integer \( a \) dividing \( t \) and the injective envelope of \( P(n-1) \) yields an exact sequence

\[
0 \to P(n-1) \to I(P(n-1)) \to S(n-1)^{t-1} \to 0 \quad (*)
\]

(namely, \( I(P(n-1)) = I(\text{soc } P(n-1)) = I(S(0)^a) = I(S(0))^a \) and \( I(S(0))/\text{soc} \) is the direct sum of \( b \) copies of \( S(n-1) \), where \( ab = t \); thus
the cokernel of the inclusion map $P(n-1) \rightarrow I(P(n-1))$ consists of $t-1$ copies of $S(n-1))$.

Since $t > 1$, the exact sequence (∗) shows that $\text{Ext}^1(S(n-1), P(n-1)) \neq 0$. It also implies that $\text{Ext}^1(S(i), P(n-1)) = 0$ for $0 \leq i \leq n-2$, and therefore that

$$\text{Ext}^i(S(0), P(n-1)) = \text{Ext}^1(\Omega_{i-1}S(0), P(n-1))$$

$$= \text{Ext}^1(S(i-1), P(n-1))$$

$$= 0$$

for $1 \leq i \leq n-1$.

Since $\Omega_{n-i-1}S(i) = S(n-1)$ for $0 \leq i \leq n-1$, we see that

$$\text{Ext}^{n-i}(S(i), P(n-1)) = \text{Ext}^1(\Omega_{n-i-1}S(i), P(n-1))$$

$$= \text{Ext}^1(S(n-1), P(n-1))$$

$$\neq 0$$

for $0 \leq i \leq n-1$. Thus, on the one hand, we have $\text{Ext}^n(S(0), \Lambda) \neq 0$, this concludes the proof that $S(0)$ has the required properties. On the other hand, we also see that $S = S(0)$ is the only simple module with $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n-1$. This completes the proof of (b).

(c) Assume now in addition that $\Lambda$ is an artin algebra. As usual, we denote the Auslander-Reiten translation $D\text{Tr}$ by $\tau$. Let $M$ be a non-projective indecomposable module with $\text{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n$. The shape of $\Gamma(\Lambda)$ shows that $\Omega M = S^c$ for some simple module $S$ (and we have $c \geq 1$), also it shows that no simple module is projective. Now $\text{Ext}^i(S, \Lambda) = 0$ for $1 \leq i < n$, thus according to (b) we must have $S = S(0)$. It follows that $PM$ has to be a direct sum of copies of $P(n-1)$, say of $d$ copies. Thus a minimal projective presentation of $M$ is of the form

$$P(0)^c \rightarrow P(n-1)^d \rightarrow M \rightarrow 0,$$

and therefore a minimal injective copresentation of $\tau M$ is of the form

$$0 \rightarrow \tau M \rightarrow I(0)^c \rightarrow I(n-1)^d.$$

In particular, $\text{soc} \tau M = S(0)^c$ and $(\tau M)/\text{soc}$ is a direct sum of copies of $S(n-1)$. 

Assume that $\tau M \neq S(0)$, thus it has at least one composition factor of the form $S(n-1)$ and therefore there exists a non-zero map $f : P(n-1) \to \tau M$. Since $\tau M$ is indecomposable and not injective, any map from an injective module to $\tau M$ maps into the socle of $\tau M$. But the image of $f$ is not contained in the socle of $\tau M$, therefore $f$ cannot be factored through an injective module. It follows that

$$\text{Ext}^1(M, P(n-1)) \simeq \text{DHom}(P(n-1), \tau M) \neq 0,$$

which contradicts the assumption that $\text{Ext}^1(M, \Lambda) = 0$. This shows that $\tau M = S(0)$ and therefore $M = \text{Tr} DS(0)$.

Of course, conversely we see that $M = \text{Tr} DS(0)$ satisfies $\text{Ext}^i(M, P(n-1)) = 0$ for $1 \leq i \leq n$, and $\text{Ext}^{n+1}(M, P(n-1)) \neq 0$. □

Remarks.

(1) The module $M = \text{Tr} DS(0)$ considered in (c) has length $t^2 + t - 1$, thus the number $t$ (and therefore $\Delta(n, t)$) is determined by $M$.

(2) If $\Lambda$ is an artin algebra with Ext-quiver $\Delta(n, t)$, the number $t$ has to be the square of an integer, say $t = m^2$. A typical example of such an artin algebra is the path algebra of the following quiver

with altogether $n + m - 1$ arrows, modulo the ideal generated by all paths of length 2. Of course, if $\Lambda$ is a finite-dimensional $k$-algebra with radical square zero and Ext-quiver $\Delta(n, m^2)$, and $k$ is an algebraically closed field, then $\Lambda$ is Morita-equivalent to such an algebra.

Also the following artin algebras with radical square zero and Ext-quiver $\Delta(1, m^2)$ may be of interest: the factor rings of the polynomial ring $\mathbb{Z}[T_1, \ldots, T_{m-1}]$ modulo the square of the ideal generated by some prime number $p$ and the variables $T_1, \ldots, T_{m-1}$.
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