An unique alternative non-negative gravitational energy tensor to the Bel-Robinson tensor in the quasilocal small sphere limit

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Abstract

The Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ gives a positive definite gravitational energy in the quasilocal small sphere limit approximation. However, there is an alternative tensor $V_{\alpha\beta\mu\nu}$ that was proposed recently that offers the same positivity as $B_{\alpha\beta\mu\nu}$ does. We have found that $V_{\alpha\beta\mu\nu}$ is the unique alternative tensor with $B_{\alpha\beta\mu\nu}$ which implies that these two tensors are a basis for expressions that have the desirable non-negative gravitational energy in the small sphere limit. In other words, the ‘energy-momentum’ density according to $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are on equal footing at the same limit.

1 Introduction

The Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ has many nice properties. It is completely symmetric, completely trace free and completely divergence free. It is usually regarded as being related to gravitational energy. In particular, the gravitational energy-momentum density in the small sphere vacuum limit is generally expected to be proportional to the Bel-Robinson tensor. This expectation is related to the requirement of energy positivity [1].

It should be emphasized that, generally speaking, a positive energy proof for a quasilocal expression is not easy. Here we consider specifically pseudotensor expressions. In fact, the quasilocal methods are not fundamentally different from pseudotensor methods [2]. The gravitational energy expression in a small region limit can be investigated through the pseudotensors. Normally, the expansion of a pseudotensor expression up to second order can be represented by certain tensors $B_{\alpha\beta\mu\nu}$, $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ [3][4]. In other words, quasilocal expression is a fancy name for a pseudotensor. Even though a pseudotensor is not a tensorial object, this does not imply that it is useless. The second order expansion expression provides guidance as to whether the gravitational energy expression is positive or not. More precisely, a negative quasilocal gravitational energy expression on a small scale definitely guarantees that it can be negative on the large scale. Conversely, a positive energy expression in the small region implies that there may be a chance to obtain positivity in a large region.

However, it is natural to question whether studying this kind of quasilocal formulation has physical significance. It is well known that the gravitational energy density cannot be detected at a point because of the equivalence principle (see section 20.4 in [1]). Fortunately, the quasilocal idea has physical meaning, i.e., the gravitational
energy density is well defined at the quasilocal level theoretically [5, 6, 7]. Practically, pseudotensors can be used to calculate the tidal heating [8] (e.g., Jupiter and Io) as well as using the quasilocal formalism [9]. Moreover, from the last decade, there are many researchers who believe that finding a good quasilocal expression (especially one that is locally positive) is meaningful and worthwhile, see e.g., [1, 10] and the many references contained therein.

In the past, the Bel-Robinson tensor has been considered to be the only tensor which contributes positivity in the small sphere limit. However, we recently found another tensor $V_{\alpha\beta\mu\nu}$ (defined in (29) below), which is also quadratic in the curvature, and which enjoys the same positivity properties as $B_{\alpha\beta\mu\nu}$. More precisely, the associated 4 momentum in a small sphere is a Lorentz-covariant future pointing non-spacelike vector (see 4.2.2 of [1]). For short we call this property causality. Furthermore, we found that $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are a basis for expressions which have the desirable non-negative gravitational energy in the small sphere vacuum limit. As we found, $V_{\alpha\beta\mu\nu}$ fulfills the weak energy condition which gives positive energy in the small sphere limit, this is then sufficient to argue that the ‘energy-momentum’ density according to $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are on an equal footing in the same limit. We will prove in section 3 that $V_{\alpha\beta\mu\nu}$ is the unique alternative tensor that has the desirable positive energy property in the small region limit.

In this work, we examine some properties of $V_{\alpha\beta\mu\nu}$ and some other quadratic in curvature tensors, $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$, which have shown up in the expansion of energy in the small sphere limit. For instance, it has already been shown that $V_{\alpha\beta\mu\nu}$ does not have the dominant energy condition [3]. We also found another tensor $V'_{\alpha\beta\mu\nu}$, which is not restricted to the pseudotensor conservation of energy-momentum requirement, but does satisfy the weak energy condition requirement.

Here we are concerned with finding a suitable form for a pseudotensor (equivalently, a quasilocal Hamiltonian expression, see [11]) in the small region limit. For the zeroth order term, the pseudotensor gives the mass density as the equivalence principle demands. Moreover, we also need to consider the ADM mass at the spatial infinity. Combining these two constraints [3] can confine a suitable pseudotensor expression. Finally, the non-vanishing second order terms contribute the gravitational energy-momentum in a small region limit; these terms are quadratic in the curvature tensor.

## 2 Quadratic curvature tensors

There are three basic tensors that commonly occur in the gravitational pseudotensor expression [11, 12]

\[
B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma}R_{\beta\sigma}^{\lambda\nu} + R_{\alpha\lambda\nu\sigma}R_{\beta\sigma}^{\lambda\mu} - \frac{1}{8} g_{\alpha\beta}g_{\mu\nu} R^2,
\]

\[
S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma}R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}^{\lambda\sigma} + \frac{1}{4} g_{\alpha\beta}g_{\mu\nu} R^2,
\]

\[
K_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma}R_{\mu\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma}R_{\beta\mu}^{\lambda\sigma} - \frac{3}{8} g_{\alpha\beta}g_{\mu\nu} R^2.
\]
where $R^2 = R_{\rho\tau\xi\kappa}R^{\rho\tau\xi\kappa}$. Some properties of $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ \cite{3} that are easily verified using the above definitions and the well known vacuum identity \cite{13}

$$R_{\alpha\lambda\sigma\tau}R_{\beta}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}R^2$$

are

$$S_{\alpha\beta\mu\nu} \equiv S_{(\alpha\beta)(\mu\nu)} \equiv S_{(\mu\nu)(\alpha\beta)}, \quad S_{\alpha\beta\mu} \equiv \frac{3}{2}g_{\alpha\beta}R^2, \quad S_{\alpha\beta\mu}^\nu \equiv 0, \quad (4)$$

$$K_{\alpha\beta\mu\nu} \equiv K_{(\alpha\beta)(\mu\nu)} \equiv K_{(\mu\nu)(\alpha\beta)}, \quad K_{\alpha\beta\mu} \equiv -\frac{3}{2}g_{\alpha\beta}R^2, \quad K_{\alpha\beta\mu}^\nu \equiv 0. \quad (5)$$

Note that unlike $B_{\alpha\beta\mu\nu}$, both $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ are neither totally symmetric nor totally trace free. Obviously, (4) and (5) already indicate that $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ do not have the completely trace free property. For the non-completely symmetric property, one can verify this by using Petrov type D \cite{14}. In particular, we found a case where

$$S_{0011} \neq S_{0101}, \quad K_{0011} \neq K_{0101}, \quad (6)$$

where we have evaluated them using the electric part $E_{ab}$ and magnetic part $H_{ab}$, defined in terms of the Weyl tensor \cite{15} as follows:

$$E_{ab} := C_{a060}, \quad H_{ab} := *C_{a060}. \quad (7)$$

In order to appreciate the nice properties of the Bel-Robinson tensor, we compare some components of $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$. For the analog of the electromagnetic stress tensor energy density,

$$B_{0000} = E_{ab}E^{ab} + H_{ab}H^{ab}, \quad S_{0000} = 2(E_{ab}E^{ab} - H_{ab}H^{ab}), \quad K_{0000} = -E_{ab}E^{ab} + 3H_{ab}H^{ab}. \quad (8)$$

Likewise for the momentum density (i.e., the Poynting vector)

$$B_{000i} = 2\epsilon_{ijk}E^{jd}H_{d}^{k}, \quad S_{000i} = 0, \quad K_{000i} = 2\epsilon_{ijk}E^{jd}H_{d}^{k}. \quad (9)$$

Finally, the stress,

$$B_{00ij} = \delta_{ij}(E_{ab}E^{ab} + H_{ab}H^{ab}) - 2(E_{id}E_{j}^{d} + H_{id}H_{j}^{d}), \quad (10)$$

$$S_{00ij} = 2\left[\delta_{ij}(-E_{ab}E^{ab} + H_{ab}H^{ab}) + 4(-E_{id}E_{j}^{d} + H_{id}H_{j}^{d})\right], \quad (11)$$

$$K_{00ij} = \delta_{ij}(E_{ab}E^{ab} - 3H_{ab}H^{ab}) - 4E_{id}E_{j}^{d}. \quad (12)$$

From the above comparison, it is clear that the Bel-Robinson tensor indeed has the best analog with the electromagnetic stress tensor $T_{\mu\nu}$. In detail, in Minkowski coordinates $(\tau, x, y, z)$ the components of the electrodynamic stress tensor are

$$T^{\tau\tau} = \frac{1}{2}(E_{a}E^{a} + B_{a}B^{a}), \quad (13)$$

$$T^{0i} = \delta^{ij}\epsilon_{jab}E^{a}B^{b} = (\vec{E} \times \vec{B})^{i}, \quad (14)$$

$$T^{ij} = \frac{1}{2}\left[\delta^{ij}(E_{a}E^{a} + B_{a}B^{a}) - 2\left(E^{i}E^{j} + B^{i}B^{j}\right)\right]. \quad (15)$$
where $\vec{E}$ and $\vec{B}$ refer to the electric and magnetic field density.

Using a Taylor series expansion, the metric tensor can be written as

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + \partial_\mu g_{\alpha\beta}(0) x^\mu + \frac{1}{2} \partial_\mu \partial_\nu g_{\alpha\beta}(0) x^\mu x^\nu + \ldots$$  \hspace{1cm} (18)$$

At the origin in Riemann normal coordinates (RNC)

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \quad \partial_\mu g_{\alpha\beta}(0) = 0, \quad -3\partial_\mu g_{\alpha\beta}(0) = g_{\mu\nu} R_{\alpha\beta} + R_{\alpha\mu\nu},$$  \hspace{1cm} (19)$$

For the quadratic curvature tensors, there are 4 independent basis expressions, we may use

$$\tilde{B}_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\nu\sigma} R_{\beta\lambda\nu} + R_{\alpha\nu\lambda\sigma} R_{\beta\lambda\mu} = B_{\alpha\beta\mu\nu} + \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2,$$  \hspace{1cm} (21)$$

$$\tilde{S}_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\nu} R_{\beta\lambda\nu\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\lambda\mu} = S_{\alpha\beta\mu\nu} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R^2,$$  \hspace{1cm} (22)$$

$$\tilde{K}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\nu} R_{\lambda\mu} + R_{\alpha\lambda\beta\sigma} R_{\lambda\mu} = K_{\alpha\beta\mu\nu} + \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} R^2,$$  \hspace{1cm} (23)$$

$$\tilde{T}_{\alpha\beta\mu\nu} := -\frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2.$$  \hspace{1cm} (24)$$

These four tensors are manifestly symmetric in the last two indices which means $\tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{\alpha\beta(\mu\nu)}$. Then it automatically imply $\tilde{M}_{\alpha\beta(\mu\nu)} = \tilde{M}_{(\alpha\beta)(\mu\nu)}$. Moreover, it also naturally turns out that $\tilde{M}_{(\alpha\beta)(\mu\nu)} = \tilde{M}_{(\mu\nu)(\alpha\beta)}$. Explicitly, they fulfill the symmetry $\tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{\alpha\beta(\mu\nu)} = \tilde{M}_{(\alpha\beta)(\mu\nu)}$. Although there exists some other tensors different from $\tilde{B}_{\alpha\beta\mu\nu}$, $\tilde{S}_{\alpha\beta\mu\nu}$, $\tilde{K}_{\alpha\beta\mu\nu}$ and $\tilde{T}_{\alpha\beta\mu\nu}$, they are just linear combinations of these four. For instance

$$\tilde{T}_{\alpha\mu\beta\nu} + \tilde{T}_{\alpha\nu\beta\mu} \equiv \tilde{B}_{\alpha\beta\mu\nu} + \frac{1}{2} \tilde{S}_{\alpha\beta\mu\nu} - \tilde{K}_{\alpha\beta\mu\nu} + 2 \tilde{T}_{\alpha\beta\mu\nu}.$$  \hspace{1cm} (25)$$

The above identity can be obtained by making use of the completely symmetric property of the Bel-Robinson tensor. Using (25), we can rewrite the Bel-Robinson tensor in a different representation [16]:

$$B_{\alpha\beta\mu\nu} \equiv -\frac{1}{2} S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} + \frac{5}{8} g_{\alpha\beta} g_{\mu\nu} R^2 - \frac{1}{8} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) R^2.$$  \hspace{1cm} (26)$$

This equation will be used in section 3.

### 3 An unique alternative non-negative gravitational energy tensor in small sphere limit

#### 3.1 Proof the unique alternative non-negative energy tensor $V_{\alpha\beta\mu\nu}$

Using RNC Taylor series expansion around any point (i.e., at any preselected point we consider a small coordinate sphere in RNC, see e.g., [1, 7, 17, 18]), consider all the
possible combinations of the small region energy-momentum density in vacuum. In the neighbourhood of any preselected point in RNC, the pseudotensor then has the form \[ 19 \]
\[ 2 \kappa t_{\alpha}^{\beta} = 2G_{\alpha}^{\beta} + \left( a_1 \tilde{B}_{\alpha}^{\beta \mu \nu} + a_2 \tilde{S}_{\alpha}^{\beta \mu \nu} + a_3 \tilde{K}_{\alpha}^{\beta \mu \nu} + a_4 \tilde{T}_{\alpha}^{\beta \mu \nu} \right) x^\mu x^\nu + \mathcal{O}(\text{Ricci}, x) + \mathcal{O}(x^3), \]
(27)
where \( \kappa = 8\pi G/c^4 \) (here we take units such that \( c = 1 \) for simplicity) and \( a_1 \) to \( a_4 \) are real numbers. Here \( G_{\alpha \beta} \) is the Einstein tensor, but we will consider the vacuum case, so \( G_{\alpha \beta} = \kappa T_{\alpha \beta} = 0 \).
(28)
Then the first order linear in Ricci terms \( \mathcal{O}(\text{Ricci}, x) \) vanish. The lowest order non-vanishing term is of second order, compared to this in the small sphere limit we can ignore the third order terms \( \mathcal{O}(x^3) \). From now on, the second order term will be kept but the others are dropped. The essential purpose of the present paper is to prove that \( \tilde{B}_{\alpha \beta \mu \nu}, \tilde{S}_{\alpha \beta \mu \nu}, \tilde{K}_{\alpha \beta \mu \nu}, \tilde{T}_{\alpha \beta \mu \nu} \) are a basis for positive gravitational energy in the small sphere limit. There are two physical conditions which can constrain the unlimited combinations between \( \tilde{B}_{\alpha \beta \mu \nu}, \tilde{S}_{\alpha \beta \mu \nu}, \tilde{K}_{\alpha \beta \mu \nu}, \tilde{T}_{\alpha \beta \mu \nu} \).

(i) First condition: energy-momentum conservation. Consider (27) as follows
\[ 0 = \partial_\beta t_{\alpha}^{\beta} = \left( a_1 \tilde{B}_{\alpha}^{\beta \mu \nu} + a_2 \tilde{S}_{\alpha}^{\beta \mu \nu} + a_3 \tilde{K}_{\alpha}^{\beta \mu \nu} + a_4 \tilde{T}_{\alpha}^{\beta \mu \nu} \right) (\delta^\mu_\beta x^\nu + x^\mu \delta^\nu_\beta) \]
\[ = 2 \left( a_1 \tilde{B}_{\alpha}^{\beta \mu \beta} + a_2 \tilde{S}_{\alpha}^{\beta \mu \beta} + a_3 \tilde{K}_{\alpha}^{\beta \mu \beta} + a_4 \tilde{T}_{\alpha}^{\beta \mu \beta} \right) x^\mu \]
\[ = \frac{1}{4} (a_1 - 2a_2 + 3a_3 - a_4) g_{\alpha \beta} x^\beta R^2. \]
(30)
Therefore, the constraint for the conservation of the energy-momentum density is
\[ a_4 = a_1 - 2a_2 + 3a_3. \]
(31)
Although there are an infinite number of combinations which can fulfill the above constraint, it has removed one degree of freedom. As each single tensor of \( \tilde{B}_{\alpha \beta \mu \nu}, \tilde{S}_{\alpha \beta \mu \nu}, \tilde{K}_{\alpha \beta \mu \nu}, \tilde{T}_{\alpha \beta \mu \nu} \) cannot satisfy the conservation requirement, but a linear combination of them can. One can simplify the situation by eliminating \( \tilde{T}_{\alpha \beta \mu \nu} \) which is absorbed by \( \tilde{B}_{\alpha \beta \mu \nu}, \tilde{S}_{\alpha \beta \mu \nu}, \tilde{K}_{\alpha \beta \mu \nu} \). Then there are only 3 basis tensors left. Thus one can rewrite (27) as
\[ 2 \kappa t_{\alpha}^{\beta} = \left[ a_1 (\tilde{B}_{\alpha}^{\beta \mu \nu} + \tilde{T}_{\alpha}^{\beta \mu \nu}) + a_2 (\tilde{S}_{\alpha}^{\beta \mu \nu} - 2\tilde{T}_{\alpha}^{\beta \mu \nu}) + a_3 (\tilde{K}_{\alpha}^{\beta \mu \nu} + 3\tilde{T}_{\alpha}^{\beta \mu \nu}) \right] x^\mu x^\nu \]
\[ = \left( a_1 B_{\alpha}^{\beta \mu \nu} + a_2 S_{\alpha}^{\beta \mu \nu} + a_3 K_{\alpha}^{\beta \mu \nu} \right) x^\mu x^\nu. \]
(32)
Paying attention to [11, 20], when we consider all the possible expressions for the pseudotensors which including the flat metric, there does appear a linear combination of these three tensors. We defined \( K_{\alpha \mu \nu} \) just for convenience and without
any physical reason. In the beginning, it seems an interesting and even a mysterious combination that why only $B_{\alpha\beta\mu\nu}, S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ always showed up in the expression. Nothing more, nothing less. Now, we discovered that it is not an accident but becomes a necessary requirement because only $B_{\alpha\beta\mu\nu}, S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ can satisfy the condition of the energy-momentum density conservation.

(ii) Second condition: non-negative gravitational energy in the small sphere limit. The purpose of the pseudotensor is for determining the gravitational energy-momentum, the associated energy-momentum can be calculated as

$$2\kappa P_\mu = \int_{t=0}^t t^\rho \xi \xi x^t \frac{d\Sigma_\rho}{\Sigma_\rho} = t^0 \mu l m \int_{t=0}^t x^l x^m d^3 x$$

$$= t^0 \mu l m \frac{\delta^l_m}{3} \int r^2 d^3 x = t^0 \mu l \frac{4 \pi r^5}{15}, \quad (33)$$

where $l, m = 1, 2, 3$. Using this calculation method, the energy-momentum in the small sphere limit for (32) becomes

$$P_\mu = (-E, \vec{P}) = -\frac{r^5}{60G} \left( a_1 B_{\mu l}^l + a_2 S_{\mu l}^l + a_3 K_{\mu l}^l \right). \quad (34)$$

The ‘energy-momentum’ values associated with $B_{\alpha\beta\mu\nu}, S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ are proportional to

$$B_{\mu l}^l = (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon_{cab} E^{ad} H^{b}_d), \quad (35)$$

$$S_{\mu l}^l = -10(E_{ab} E^{ab} - H_{ab} H^{ab}, 0), \quad (36)$$

$$K_{\mu l}^l = B_{\mu l}^l - S_{\mu l}^l. \quad (37)$$

Looking back at (32), we are interested in the positive gravitational energy within a small sphere limit, the Bel-Robinson tensor already satisfies this condition. Precisely

$$B_{00}^l = E_{ab} E^{ab} + H_{ab} H^{ab} \geq 0. \quad (38)$$

The rest of the job is to find the coefficients $a_2$ and $a_3$. Using (37), rewrite (34) as

$$P_\mu = -\frac{r^5}{60G} \left[ (a_1 + a_3) B_{\mu l}^l + (a_2 - a_3) S_{\mu l}^l \right]. \quad (39)$$

Equation (36) shows that $S_{\mu l}^l$ cannot ensure positivity, since we should allow for any magnitude of $|E_{ab}|$ and $|H_{ab}|$. In other words, for $S_{\alpha\beta\mu\nu}$ the sign of the ‘energy’ density is uncertain. Therefore the only possibility for (39) to guarantee positivity is when $a_1 + a_3 \geq 10 |a_2 - a_3|$. However, if we consider that the momentum should be future pointing and non-spacelike (i.e., inside the light cone such that $-P_0 \geq |\vec{P}|$), the unique requirement for (39) to assure non-negative is when $a_2 = a_3$. In other words, causality. Moreover, using (29) and (37), we obtained

$$V_{\mu l}^l = S_{\mu l}^l + K_{\mu l}^l = B_{\mu l}^l. \quad (40)$$

Consequently (34) becomes

$$P_\mu = -\frac{r^5}{60G} \left( a_1 B_{\mu l}^l + a_2 V_{\mu l}^l \right) = -\frac{r^5}{60G} (a_1 + a_2) B_{\mu l}^l. \quad (41)$$

Hence the proof is completed. Indeed $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are a basis for expressions which have non-negative gravitational ‘energy’ density in vacuum.
3.2 Physical meaning of the completely traceless property of $B_{\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$

For the quasilocal small sphere region, there are four fundamental quadratic Weyl curvature tensors. We know that $\tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}(\alpha\beta)(\mu\nu) = M_{(\mu\nu)(\alpha\beta)}$ for all $\tilde{M}_{\alpha\beta\mu\nu} \in \{B_{\alpha\beta\mu\nu}, \tilde{S}_{\alpha\beta\mu\nu}, \tilde{K}_{\alpha\beta\mu\nu}, \tilde{T}_{\alpha\beta\mu\nu}\}$. In order to check the completely trace free property of the linear combinations of $B_{\alpha\beta\mu\nu}, \tilde{S}_{\alpha\beta\mu\nu}, \tilde{K}_{\alpha\beta\mu\nu}$ and $\tilde{T}_{\alpha\beta\mu\nu}$. We only need to consider two cases.

Case (i). Consider the trace on the first and third indices:

$$a_1 \tilde{B}^\alpha_{\mu\alpha\nu} + a_2 \tilde{S}^\alpha_{\mu\alpha\nu} + a_3 \tilde{K}^\alpha_{\mu\alpha\nu} + a_4 \tilde{T}^\alpha_{\mu\alpha\nu} = \frac{1}{8} (a_1 - 2a_2 + 3a_3 - a_4) g_{\mu\nu} R^2. \quad (42)$$

Case (ii). Consider the trace on the first pair:

$$a_1 \tilde{B}^\alpha_{\alpha\mu\nu} + a_2 \tilde{S}^\alpha_{\alpha\mu\nu} + a_3 \tilde{K}^\alpha_{\alpha\mu\nu} + a_4 \tilde{T}^\alpha_{\alpha\mu\nu} = \frac{1}{2} (a_1 + a_2 - a_4) g_{\mu\nu} R^2. \quad (43)$$

For completely traceless, (42) and (43) have to vanish at the same time

$$a_1 - 2a_2 + 3a_3 - a_4 = 0, \quad (44)$$

$$a_1 + a_2 - a_4 = 0. \quad (45)$$

Notice that (44) is the same constraint for the energy-momentum conservation expressed in (31). This means that one of the mathematical trace free condition turns out to be one of the physical criteria. Solving the above two equations, we recovered the same requirement for the gravitational energy-momentum (i.e., casuality) which was indicated in (39), explicitly $a_2 = a_3$. Moreover, using this totally traceless property at the quasilocal small sphere limit shown in (44) and (45), we recovered the same result: that there are two tensors that generate the basis mentioned in section 3.1. The proof is follows

$$a_1 \tilde{B}_{\alpha\beta\mu\nu} + a_2 \tilde{S}_{\alpha\beta\mu\nu} + a_3 \tilde{K}_{\alpha\beta\mu\nu} + a_4 \tilde{T}_{\alpha\beta\mu\nu} = a_1 B_{\alpha\beta\mu\nu} + a_2 V_{\alpha\beta\mu\nu}, \quad (46)$$

where

$$V_{\alpha\beta\mu\nu} = \tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}. \quad (47)$$

The completely traceless property turns out to be a remarkable result because it is not just a mathematical property, but it reveals some physical meaning and conditions. Namely, the conservation of the energy-momentum and casuality.

In other words, we have discovered necessary and sufficient conditions. From considering the quasilocal small sphere limit, we find the completely traceless for these two fourth rank tensors $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ guarantees the fulfillment of the energy-momentum conservation and casuality. Conversely, if any fourth rank tensor satisfies the conservation of energy-momentum and casuality, then it must be a linear combination of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$. 
3.3 Counting the independent components of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$

As $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are different but share the same gravitational ‘energy-momentum’ density, then one may interested to know how many non-vanishing independent components $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ have. Using (26) and (29), here we write the alternative relationship between $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$

$$V_{\alpha\beta\mu\nu} := B_{\alpha\beta\mu\nu} + W_{\alpha\beta\mu\nu}, \quad (48)$$

where we have defined

$$W_{\alpha\beta\mu\nu} := \frac{3}{2}S_{\alpha\beta\mu\nu} - \frac{5}{8}g_{\alpha\beta}g_{\mu\nu}R^2 + \frac{1}{8}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu})R^2. \quad (49)$$

Basically, these three tensors ($B$, $V$, $W$) are fourth rank, in principle they could have 256 components. However, after considering the symmetry properties, they only have a relatively small amount of independent components. It may be important to do this simple counting because it reduces the workload (i.e., computer algebra) when one calculates all the components of these three tensors.

First of all, we count the number of components of $B_{\alpha\beta\mu\nu}$. In principle, as $B_{\alpha\beta\mu\nu}$ is completely symmetric, by explicit examination it reduces to 35 components. There is a formula that directly gives this number. A $k$th rank totally symmetric tensor in $n$ dimensional space has $C_n^{n+k-1}$ components. For our case

$$C_4^{4+4-1} = 35. \quad (50)$$

Since $B_{\alpha\beta\mu\nu}$ is completely trace free, there are 10 constraints which can be replaced by the other components. Finally, we only left 25 independent components for $B_{\alpha\beta\mu\nu}$ (see [14]).

Secondly, we count $V_{\alpha\beta\mu\nu}$. Tensor $V_{\alpha\beta\mu\nu}$ does not have the completely symmetric property, but fulfills some certain symmetries $V_{\alpha\beta\mu\nu} = V_{(\alpha\beta)(\mu\nu)} = V_{(\mu\nu)(\alpha\beta)}$. In principle, this reduces $V_{\alpha\beta\mu\nu}$ to 55 components. However, when we consider the totally trace free property of $V_{\alpha\beta\mu\nu}$ (but not completely symmetric), then there are two extra constraints need to be taken into account

$$V^\alpha_{\alpha\mu\nu} = 0, \quad V^\alpha_{\mu\alpha\nu} = 0. \quad (51)$$

Hence we have $55 - 10 - 10 = 35$ independent components for $V_{\alpha\beta\mu\nu}$.

Finally, for the completeness, we count $W_{\alpha\beta\mu\nu}$. Note that $W_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are similar. In detail

$$W_{\alpha\beta\mu\nu} = W_{(\alpha\beta)(\mu\nu)} = W_{(\mu\nu)(\alpha\beta)}, \quad W^\alpha_{\alpha\mu\nu} = 0, \quad W^\alpha_{\mu\alpha\nu} = 0. \quad (52)$$

In principle, there should be at most 35 components. However, one must consider the extra constraint

$$W_{\alpha\beta\mu\nu} + W_{\alpha\mu\nu\beta} + W_{\alpha\nu\beta\mu} = 0. \quad (53)$$

Finally we have $35 - 25 = 10$ independent components for $W_{\alpha\beta\mu\nu}$. 

3.4 Physical application for $V_{\alpha\beta\mu\nu}$

The physical application of $V_{\alpha\beta\mu\nu}$ is similar to $B_{\alpha\beta\mu\nu}$ in the small region limit. For instance, the Einstein pseudotensor does not have a suitable positivity expressions as the sign of the energy density is uncertain. Recall the result for Einstein in vacuum \[^{[11]}\]

$$2\kappa E_{\alpha}^{\beta} = \frac{1}{18}(4B_{\alpha}^{\beta\mu\nu} - S_{\alpha}^{\beta\mu\nu})x^{\mu}x^{\nu}. \quad (54)$$

Referring to \[^{[13]}\], the corresponding gravitational energy is

$$P_{0} = -\frac{r^5}{60G}(4B_{00} - S_{00}) = -\frac{r^5}{30G}(7E_{ab}E^{ab} - 3H_{ab}H^{ab}). \quad (55)$$

However, the Papapetrou pseudotensor gives a value which is a linear combination of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ at the second order \[^{[3, 11]}\]. This indicates that there is a chance to obtain a positive energy expression in the large region. Recall the result for Papapetrou in vacuum:

$$2\kappa P_{0}^{\alpha\beta} = \frac{1}{9}(4B_{0\beta}^{\alpha\mu} - V_{\alpha\beta}^{\mu\nu})x^{\mu}x^{\nu}. \quad (56)$$

Similarly, the corresponding gravitational energy from \[^{[33]}\] is

$$P_{0} = -\frac{r^5}{60G}(4B_{00} - V_{00}) = -\frac{r^5}{20G}(E_{ab}E^{ab} + H_{ab}H^{ab}). \quad (57)$$

Before we go on to study any further, however, there comes a question whether $V_{\alpha\beta\mu\nu}$ and $B_{\alpha\beta\mu\nu}$ are totally equivalent? Although they have some components that are exactly the same (e.g., $V_{\mu000} \equiv B_{\mu000}$), we find that $V_{\alpha\beta\mu\nu}$ and $B_{\alpha\beta\mu\nu}$ are indeed different tensors \[^{[3]}\]. It is easy to clarify that these two are different, since they are defined by different fundamental quadratic curvatures, explicitly

$$B_{\alpha\beta\mu\nu} = \tilde{B}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\mu\nu}, \quad (58)$$

$$V_{\alpha\beta\mu\nu} = \tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\mu\nu}. \quad (59)$$

In particular, $V_{\alpha\beta\mu\nu}$ is completely trace free but not completely symmetric \[^{[3]}\]. The following lists some properties

$$V_{\alpha\beta\mu\nu} \equiv V_{(\alpha\beta)(\mu\nu)} \equiv V_{(\mu\nu)(\alpha\beta)}, \quad V_{\alpha\beta\mu}^{\nu} \equiv 0 \equiv V_{\alpha\mu\beta}^{\nu}, \quad (60)$$

$$V_{0000} \equiv V_{0000} \equiv V_{m}^{m} \equiv V_{m}^{ml} \equiv E_{ab}E^{ab} + H_{ab}H^{ab} \equiv B_{0000}, \quad (61)$$

$$V_{\mu000} \equiv V_{\mu000} \equiv V_{\mu000} \equiv (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{cab}E^{cd}H^{ab}) \equiv B_{\mu0l}^{l}. \quad (62)$$

It is known that $B_{\alpha\beta\mu\nu}$ has the dominant energy property \[^{[21, 22]}\]

$$B_{\alpha\beta\mu\nu} w_{1}^{\alpha} w_{2}^{\beta} w_{3}^{\mu} w_{4}^{\nu} \geq 0, \quad (63)$$

where $w_{1}, w_{2}, w_{3}, w_{4}$ are any future-pointing causal vectors. While $V_{\alpha\beta\mu\nu}$ only satisfies the weak energy condition and $W_{\alpha\beta\mu\nu}$ fulfills none of them. However, we found $W_{\alpha\beta\mu\nu}$ has some interesting properties. A simple computation using \[^{[2]}\] shows that

$$W_{\alpha\beta\mu\nu} u^{\alpha} u^{\beta} u^{\mu} u^{\nu} = 0, \quad W_{\alpha\beta\mu\nu} u^{\alpha} u^{\beta} u^{\mu} u^{\nu} = 0, \quad (64)$$
where \( t \) is a timelike unit normal vector and \( u \) can be timelike or null. Looking at \([18]\), \( V_{\alpha \beta \mu \nu} \) contains more information than \( B_{\alpha \beta \mu \nu} \), however it seems that \( B_{\alpha \beta \mu \nu} \) is the important part of \( V_{\alpha \beta \mu \nu} \) and \( W_{\alpha \beta \mu \nu} \) is a kind of gauge freedom (i.e., it has no important physical effect).

A physical reasonable energy-momentum tensor should fulfill certain energy conditions. In particular, the local energy density measured by the observer with a 4-velocity should be non-negative. This energy condition must be true for all timelike unit normal vectors \([23]\). In fact, we found \( V_{\alpha \beta \mu \nu} \) has the non-negative ‘energy’ property

\[
V_{\alpha \beta \mu \nu} t^\alpha t^\beta t^\mu t^\nu \equiv B_{\alpha \beta \mu \nu} t^\alpha t^\beta t^\mu t^\nu = E_{ab} \Pi_{ab} + H_{ab} H_{ab} \geq 0. \tag{65}
\]

Looking at this weak energy condition and from the reason of the continuity \([23]\), the above inequalities must still be true if the timelike vector \( t \) is replaced by a null vector \( v \). Indeed, we found

\[
V_{\alpha \beta \mu \nu} v^\alpha v^\beta v^\mu v^\nu \equiv B_{\alpha \beta \mu \nu} v^\alpha v^\beta v^\mu v^\nu \geq 0. \tag{66}
\]

Here we take a simple test, with 3 different cases, to evaluate the inequality of \((66)\) and get some idea what is the value can be. Without loss of generality, consider 3 simple cases from the unit normal null vector \( v_1 = k(1, 1, 0, 0) \), \( v_2 = k(1, 0, 1, 0) \), \( v_3 = (1, 0, 0, 1) \) and \( k \) is a constant. We found

\[
B_{\alpha \beta \mu \nu} v^\alpha_i v^\beta_i v^\mu_i v^\nu_i = k^4 (B_{0000} + 4B_{000i} + 6B_{00ii} + 4B_{0iii} + 4B_{iiii}), \tag{67}
\]

where \( i = 1, 2, 3 \). More precisely, we have calculated the results referring to the five different distinct Petrov types \([14]\) as shown in Table 1.

Note that different Petrov types corresponding to different values simply because they are evaluated from different cannonial frames. Likewise, it is not surprising that different frames associated with different values from the same superpotential such as the Freud superpotential \([12, 16, 18, 24]\) (i.e., using holonomic frames or orthonormal frames). The statement is correct according to \([23]\) for \( V_{\alpha \beta \mu \nu} \), which is based on the fact that \( B_{\alpha \beta \mu \nu} \) has the dominant energy property.

Following from \((63)\), the energy-momentum density for \( V_{\alpha \beta \mu \nu} \) in the small sphere limit is

\[
2 \kappa P_\mu = \frac{4\pi r^5}{15} (V^\alpha_{\mu \alpha} - V^0_{\mu 0}) = -\frac{4\pi r^5}{15} V_{0\mu 0}. \tag{68}
\]

| Type  | \( v^1 \)  | \( v^2 \)  | \( v^3 \)  |
|-------|---------|---------|---------|
| Type I | \( 4k^4 \left[ (E_{22} - E_{33})^2 + (H_{22} - H_{33})^2 \right] \) | \( 4k^4 \left[ (E_{11} - E_{33})^2 + (H_{11} - H_{33})^2 \right] \) | \( 4k^4 \left[ (E_{11} - E_{22})^2 + (H_{11} - H_{22})^2 \right] \) |
| Type D | \( 0 \)   | \( 9k^4 \left( E_{11}^2 + H_{11}^2 \right) \) | \( 9k^4 \left( E_{11}^2 + H_{11}^2 \right) \) |
| Type II| \( 64k^4 \left( E_{23}^2 + H_{23}^2 \right) \) | \( 4k^4 \left[ (E_{11} - E_{33})^2 + (H_{11} - H_{33})^2 \right] \) | \( 4k^4 \left[ (E_{11} - E_{22})^2 + (H_{11} - H_{22})^2 \right] \) |
| Type III| \( 0 \)   | \( 16k^4 \left( E_{12}^2 + H_{12}^2 \right) \) | \( 16k^4 \left( E_{12}^2 + H_{12}^2 \right) \) |
| Type N | \( 64k^4 \left( E_{22}^2 + H_{22}^2 \right) \) | \( 4k^4 \left( E_{22}^2 + H_{22}^2 \right) \) | \( 4k^4 \left( E_{22}^2 + H_{22}^2 \right) \) |

Table 1: Five different Petrov types
Or, more covariantly,

\[ 2\kappa P_\mu u^\mu = -\frac{4\pi r^5}{15} V_{\mu\alpha\beta\gamma} u^\mu t^\alpha t^\beta t^\gamma, \]  

(69)

where

\[ V_{\alpha\beta\mu\nu} t^\beta t^\mu t^\nu \equiv B_{\alpha\beta\mu\nu} t^\beta t^\mu t^\nu = (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon_{cab} E^{ad} H^{bd}), \]

(70)

and it should be recalled that

\[ E_{ab} E^{ab} + H_{ab} H^{ab} \geq |2\epsilon_{cab} E^{ad} H^{bd}|. \]

(71)

The physical meaning (non-spacelike energy-momentum) is here simpler and clearer than that of the dominant energy condition (63). Obviously, \( V_{\alpha\beta\mu\nu} \) can play the same role as \( B_{\alpha\beta\mu\nu} \), it ensures a causal 4 momentum in the small sphere limit. In other words, the ‘energy-momentum’ density according to \( B_{\alpha\beta\mu\nu} \) and \( V_{\alpha\beta\mu\nu} \) are on equal footing at the small sphere region limit. Complementary, the Bel-Robinson tensor is no longer the only tensor that have the unique preference for achieving the causal 4 momentum in the quasilocal small region, but \( V_{\alpha\beta\mu\nu} \) can play the same role and it becomes the unique alternative choice.

### 3.5 Positive energy for the general fourth rank tensor \( V'_{\alpha\beta\mu\nu} \)

From the technical point of view, if we are just interested in positive energy and relax the restriction on the pseudotensor constraint, which means the conservation of the energy-momentum, there are an infinite number of combinations that have the weak energy condition, not including \( B_{\alpha\beta\mu\nu} \). We define

\[ V'_{\alpha\beta\mu\nu} := \tilde{K}_{\alpha\beta\mu\nu} + s\tilde{S}_{\alpha\beta\mu\nu} + t_1\tilde{T}_{\alpha\beta\mu\nu} + t_2\tilde{T}_{\alpha\mu\beta\nu} + t_3\tilde{T}_{\alpha\nu\beta\mu}, \]

(72)

where \( s, t_1, t_2, t_3 \) are real numbers and \( t_1 + t_2 + t_3 = 1 \). Obviously, the energy-momentum contribution for \( \tilde{S}_{\alpha\beta\mu\nu} \) can be ignored according to (22). Explicitly

\[ \tilde{S}_{\alpha\beta\mu\nu} u^\alpha t^\beta t^\mu t^\nu \equiv 0 \equiv \tilde{S}_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu. \]

(73)

On the other hand,

\[ \tilde{T}_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \equiv \tilde{T}_{\alpha\mu\beta\nu} u^\alpha u^\beta u^\mu u^\nu \equiv \tilde{T}_{\alpha\nu\beta\mu} u^\alpha u^\beta u^\mu u^\nu. \]

(74)

Once again, here \( u \) can be timelike or null. Using (59), rewrite (72)

\[ V'_{\alpha\beta\mu\nu} = V_{\alpha\beta\mu\nu} + (s - 1)\tilde{S}_{\alpha\beta\mu\nu} + (t_1 - 1)\tilde{T}_{\alpha\beta\mu\nu} + t_2\tilde{T}_{\alpha\mu\beta\nu} + t_3\tilde{T}_{\alpha\nu\beta\mu}. \]

(75)

Based on the weak energy condition and continuity property, we found

\[ V'_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \equiv V_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \equiv B_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \geq 0. \]

(76)

This illustrates that there does exist an infinite number of combinations which have positivity if we exclude the conservation of the energy-momentum requirement according to the pseudotensor restriction.
Furthermore, in order to obtain the dominant energy condition, the Bel-Robinson tensor is the unique tensor that has the suitable combination from the four fundamental quadratic curvature combinations, namely from $B_{\alpha\beta\mu\nu}$ to $W_{\alpha\beta\mu\nu}$. As a matter of fact, $B_{\alpha\beta\mu\nu}$ has more nice properties than the other quadratic curvature combinations generally (e.g., $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$). In particular, $B_{\alpha\beta\mu\nu}$ possesses the completely symmetric property. However, concerning the gravitational energy at the small sphere limit, we found that $V_{\alpha\beta\mu\nu}$ is the unique alternative choice to compare with $B_{\alpha\beta\mu\nu}$.

4 Conclusion

Using the four fundamental quadratic curvature tensors, we constructed all the possible combinations in the quasilocal small sphere region expression. We recovered that $B_{\alpha\beta\mu\nu}$ gives a definite positive gravitational energy (more previously a causal 4-momentum) in the small sphere limit approximation. However, we found an unique alternative, the recently proposed tensor $V_{\alpha\beta\mu\nu}$, which also contributes the same non-negative gravitational energy density at the same region limit. Based on the two physical conditions: energy-momentum conservation and casuality. We found that these two tensors can be classified as a basis for expressions which have the desirable non-negative gravitational energy in the small sphere region. In other words, $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are on equal footing in the small sphere limit. This means that if we obtain $B_{\alpha\beta\mu\nu}$ or $V_{\alpha\beta\mu\nu}$ from the gravitational expression at the small scale, either of them is good enough to search whether the expression is positive or not at the large scale. For example, the Papapetrou pseudotensor can be a good candidate to study the positivity energy expression, as it is proportional to the linear combination of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ at the second order evaluation.

We found that only a linear combination of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ satisfies the energy-momentum conservation and casuality physical conditions. Remarkably, the completely trace free property for both $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ demonstrates the same two physical requirements. It may be interesting and also from the practical reason to count the independent components for $B_{\alpha\beta\mu\nu}$, $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$. We found that there are 25 independent components for $B_{\alpha\beta\mu\nu}$, 35 components for $V_{\alpha\beta\mu\nu}$ and 10 for $W_{\alpha\beta\mu\nu}$. Moreover, we found that $W_{\alpha\beta\mu\nu}$, associated with $V_{\alpha\beta\mu\nu}$, behaves as a kind of gauge freedom. Furthermore, relaxing the restriction of the energy-momentum conservation requirement for the pseudotensor, $V'_{\alpha\beta\mu\nu}$ demonstrates that there are an infinite number of ways to obtain positivity, namely the weak energy condition. For the conserved expressions $B_{\alpha\beta\mu\nu}$ satisfies the dominant energy condition while $V_{\alpha\beta\mu\nu}$ does not, but does fulfill the weak energy condition.

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References

[1] Szabados L B 2009 Living Rev. Rel. 12 4
[2] Chang C C, Nester J M and Chen C M 1999 Phys. Rev. Lett 83 1897
[3] So L L 2009 Class. and Quantum Grav. 26 185004, So L L and Nester J M “Energy-momentum in small spheres: the classical pseudotensors” (in preparation)
[4] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[5] Hawking S W 1968 J. Math. Phys. 9 598
[6] Penrose R 1982 Proc. R. Soc. Lond. A 381 53
[7] Horowitz G T and Schmidt B G 1982 Proc. R. Soc. Lond. A 381 215
[8] Purdue P 1999 Phys. Rev. D 60 104054
[9] Booth I S and Creighton J D E 2000 Phys. Rev. D 62 067503
[10] Liu C C M and Yau S T 2003 Phys. Rev. Lett 90 231102
[11] So L L and Nester J M 2009 Class. and Quantum Grav. 26 085004
[12] So L L 2007 Int. J. Mod. Phys. D 16 875
[13] Yefremov A P 1975 Acta Phys. Pol. B6 667
[14] Gomez-Lobo A G P 2008 Class. Quantum. Grav. 25 015006
[15] Carmeli M “Classical Fields General relativity and Gauge Theory” (John Wiley & Sons 1982)
[16] Deser S, Franklin J S and Seminaea D 1999 Class. Quantum Grav. 16 2815
[17] Bergqvist G 1998 Class. Quantum Grav. 15 1535
[18] Garecki J 1973 Acta Phys. Pol. B4 347
[19] So L L 2008 Class. Quantum Grav. 25 175012
[20] So L L and Nester J M 2009 Phys. Rev. D 79 084028
[21] Penrose R and Rindler W 1984 Spinors and spacetime (Cambridge U.P., Cambridge) Vol. 1
[22] Senovilla J M M 2000 Class. Quantum Grav. 17 2799
[23] Stephani H, Kramer D, Maccallum M, Hoenselaers C and Herlt E 2003 “Exact Solutions of Einstein’s Field Equations. 2nd edition, Cambridge University Press
[24] So L L and Nester J M 2009 Chin. J. Phys. 47 10