Non-commutative M-branes from Open Pure Spinor Supermembrane

Sota Hanazawa* and Makoto Sakaguchi†

* Graduate School of Science and Engineering, Ibaraki University, Mito 310-8512, Japan
† Department of Physics, Ibaraki University, Mito 310-8512, Japan

Abstract

Open supermembrane with a constant three-form flux in the pure spinor formalism is examined. The BRST symmetry of the open supermembrane action leads to non-commutative (NC) M-branes. In addition to the NC M5-brane with a self-dual two-form flux, we find a NC M9-brane with an electric flux and a NC M9-brane with a magnetic flux. The former reduces in the critical electric flux limit to an M2-brane on the M9-brane, while the latter reduces in the strong magnetic flux limit to infinitely many Kaluza-Klein monopoles dissolved into the M9-brane. These NC M-branes are shown to preserve a half of 32 supersymmetries.
1 Introduction

Supermembrane theory in eleven-dimensions [1] is expected to be closely related to a formulation of M-theory [2], despite the fact that its microscopic degrees of freedom have not been completely understood yet. Fortunately at the semiclassical level, M-theory is an eleven-dimensional supergravity theory interacting with 1/2 BPS objects: M2-branes, M5-branes, M9-branes, M-waves and Kaluza-Klein (KK) monopoles. It is known that an open supermembrane can end on Dirichlet $p$-branes with $p = 1, 5$ and $9$ [3][4]. The $p = 5$ case corresponds to the M5-brane and the $p = 9$ case to the M9-brane which is the boundary of the eleven-dimensional spacetime in the Hořava-Witten formulation of the heterotic string theory [5]. In [6], the M-wave [7] of eleven-dimensional supergravity are identified with D0-branes of IIA supergravity and their dual D6-branes are shown to be the KK monopole [8] in eleven-dimensions.

Dirichlet branes of a $\kappa$-symmetric open supermembrane are investigated from the $\kappa$-symmetry argument [3]. Furthermore, non-commutative (NC) M-branes are discussed from $\kappa$-symmetry of a $\kappa$-symmetric open supermembrane with a constant three-form flux and it is...
found that the self-duality of the two-form gauge field on the M5-brane world-volume follows from \( \kappa \)-symmetry of the open supermembrane [9]. In addition, intersecting NC M-branes are discussed in [10, 11].

In this paper we examine an open supermembrane with a constant three-form flux in the pure spinor formalism [12]. In the pure spinor formalism, the \( \kappa \)-symmetry is replaced with the BRST symmetry. One of advantages of our approach is to consider BRST symmetry which is expected to survive quantum corrections. It implies that our analysis in this paper may give a quantum consistency check for the \( \kappa \)-symmetry arguments. In addition, we will derive a NC M9-brane with an electric flux and a NC M9-brane with a magnetic flux. The former reduces in the critical electric flux limit to an M2-brane on the M9-brane. The latter reduces in the strong magnetic flux limit to infinitely many KK monopoles dissolved into the M9-brane, and is identified with a bound state of an M9-brane and KK monopoles.

We will examine the supersymmetry variation of an open supermembrane in the pure spinor formalism, and find that the boundary condition for the BRST symmetry solves those for supersymmetry. This shows that the NC M-branes derived in this paper preserve a half of 32 supersymmetries and should be 1/2 BPS objects.

This paper is organized as follows. In the next section, we introduce an open supermembrane action with constant fluxes in the pure spinor formalism, and give the BRST transformation law and the supersymmetry transformation law under which the action is invariant. We derive surface terms of the BRST transformation and deduce boundary conditions to eliminate them in section 3. In section 4, as solutions of the boundary conditions, we obtain NC M-branes; a NC M5-brane with self-dual three-form fluxes, a NC M9-brane with an electric flux and a NC M9-brane with a magnetic flux. In addition, the NC M-branes are shown to be half supersymmetric, namely 1/2 BPS. The last section is devoted to summary and discussions. In the appendix A, we give a derivation of NC M5-branes, and we describe a derivation of surface terms for the supersymmetry transformation of the action in the appendix B.
2 Pure Spinor Supermembrane with constant three-form fluxes

Before introducing the pure spinor supermembrane action, we introduce the \( \kappa \)-symmetric supermembrane action in \( d = 11 \). It is composed of two parts [1]

\[
S = \int_{\Sigma} d^3 \tau (\mathcal{L}_0 + \mathcal{L}_{WZ}) ,
\]

\[
\mathcal{L}_0 = P_\mu \Pi_{\alpha}^\mu + e^0 (P_\mu P_\mu + \det(\Pi_{IJ}^\mu \Pi_{JI}^\mu)) + e^I P_\mu \Pi_{IJ}^\mu ,
\]

\[
\mathcal{L}_{WZ} = \epsilon^{ijk} \left[ \frac{1}{6} \mathcal{H}_{\mu \nu \rho} \partial_i x^\mu \partial_j x^\nu \partial_k x^\rho - \frac{i}{4} \left\{ \bar{\partial} \Gamma^i_\mu \partial_j \theta \partial_j x^\nu \partial_k x^\rho \right. \right.
\]

\[
+ \frac{i}{2} \bar{\partial} \Gamma^i_\mu \partial_i \theta \bar{\partial} \Gamma^\rho_\mu \partial_j \theta \partial_k x^\nu - \frac{1}{12} \bar{\partial} \Gamma^i_\mu \partial_i \theta \bar{\partial} \Gamma^\rho_\mu \partial_j \theta \partial_k x^\nu \left. \theta \partial_k \theta \right] \right) ,
\]

where \( x^\mu (\mu = 0, 1, \ldots, 9, 10) \) are flat spacetime coordinates, and \( \tau^i (i = 0, 1, 2, \text{ and } I = 1, 2 \) are space indices) are coordinates on the world-volume \( \Sigma \). We have introduced \( \Pi_{ij}^\mu \equiv \partial_i x^\mu + \frac{i}{2} \bar{\partial} \Gamma^\mu_\nu \partial_\nu \theta \). The \( \Gamma^\mu \) denote 32 × 32 gamma matrices and \( \theta^\alpha \) is a 32-component Majorana spinor in \( d = 11 \). We define \( \bar{\theta} = \theta \Gamma^1 C \) with the charge conjugation matrix \( C \) so that \((C \Gamma^\nu \cdots \Gamma_\alpha)^{\alpha \beta}\) is symmetric under the exchange \( \alpha \leftrightarrow \beta \) if \( n = 1, 2 \) mod 4. The \( e^0 \) and \( e^I \) are Lagrange multipliers for reparametrization constraints. By eliminating \( P_\mu \) by its equation of motion, \( \mathcal{L}_0 \) reduces to the Nambu-Goto Lagrangian \( \sqrt{-\det \Pi_{ij}^\mu \Pi_{ji}^\mu} \) [13].

We have introduced a constant three-form flux \( \mathcal{H} = C - db \) in \( \mathcal{L}_{WZ} \) where \( C \) and \( b \) denote the three-form gauge potential and the two-form gauge field on the boundary brane, respectively. It should be noted that the action (2.1) is \( \kappa \)-symmetric and supersymmetric even in the presence of \( \mathcal{H} \).

In the pure spinor formalism, \( \kappa \)-symmetry is replaced with BRST symmetry. The supermembrane action in the pure spinor formalism [12] is given as

\[
S_{\text{pure}} = \int_{\Sigma} d^3 \tau \left[ \mathcal{P}_\mu \Pi_{\alpha}^\mu + \mathcal{L}_{WZ} + d_\alpha \partial_\theta \theta^\alpha + w_\alpha \partial_\theta \lambda^\alpha + (d \Gamma_\mu \partial_i \theta) \Pi_{IJ}^\mu \epsilon^{IJ} 
\]

\[
- \frac{1}{2} (\mathcal{P}_\mu \mathcal{P}_\mu + \det(\Pi_{IJ}^\mu \Pi_{JI}^\mu)) + e^I (\mathcal{P}_\mu \Pi_{IJ}^\mu + d \partial_i \theta + w \partial_i \lambda)
\]

\[
+ (w \Gamma_\mu \partial_i \lambda) \Pi_{IJ}^\mu \epsilon^{IJ} - i \epsilon^{IJ} (w \Gamma_\mu \partial_i \theta) (\bar{\lambda} \Gamma^i_\mu \partial_j \theta) + i \epsilon^{IJ} (w \partial_i \theta)(\bar{\lambda} \partial_j \theta) \right] ,
\]

where \( \mathcal{P}_\mu \) denotes \( \mathcal{P}_\mu \equiv P_\mu - \frac{1}{2} B_{MN} \partial_\lambda Z^M \partial_\lambda Z^N \epsilon^{IJ} \) with \( Z^M = (x^\mu, \theta^\alpha) \). The \( B_{MNP} \) is defined by \( \mathcal{L}_{WZ} \equiv \frac{1}{6} \epsilon^{ijk} B_{MNP} \partial_i Z^M \partial_j Z^N \partial_k Z^P \) where \( \mathcal{L}_{WZ} \) is given in (2.3). Define the momentum
conjugate to $\theta^\alpha$ by $p_\alpha \equiv \partial \lambda^\alpha / \partial \theta^\alpha$, and then we introduce $d_\alpha$ as 32 fermionic constraints

$$d_\alpha = p_\alpha - i \frac{1}{2} \tilde{P}^\mu (CT_\mu \theta)_\alpha - \frac{1}{2} e^{IJ} B_{M\alpha} \partial_I Z^\alpha \partial_J Z^N . \tag{2.5}$$

The Grassmann-even spinor fields $(\lambda^\alpha, w_\alpha)$ are pure spinor ghosts. The Lagrange multiplier $e^0$ has been set to $-1/2$.

The supermembrane action (2.4) is invariant under the supersymmetry transformations

$$\delta \theta^\alpha = \epsilon^\alpha , \; \delta x^\mu = i \tilde{\theta} \Gamma^\mu \epsilon , \; \delta e^0 = \delta \epsilon^I = 0 , \; \delta \lambda^\alpha = \delta x w_\alpha = 0 , \; \delta \tilde{P}_\mu = \delta \epsilon d_\alpha = 0 , \tag{2.6}$$

where $\tilde{P}_\mu$ and $d_\alpha$ are defined to be invariant under supersymmetry transformations.

The BRST operator is defined by $Q \equiv \lambda^\alpha d_\alpha$ which acts on a field $f$ by $Q f = i \{ Q, f \}$. By using (anti-)commutation relations $\{ p_\alpha, \theta^\beta \} = -i \delta^\beta_\alpha$, $[ P_\mu, x^\kappa ] = i \delta^\mu_\kappa$ and $[ \lambda^\alpha, w_\beta ] = -i \delta^\beta_\alpha$, we may derive

$$Q \theta^\alpha = \lambda^\alpha , \; Q x^\mu = i \tilde{\theta} \Gamma^\mu \epsilon , \; Q d_\alpha = -i \tilde{P}^\mu_0 (CT_\mu \lambda)_\alpha + \frac{i}{2} e^{IJ} \tilde{P}^\mu_I \Pi^\nu_J (CT_\mu \lambda)_\alpha , \; Q \lambda^\alpha = 0 , \; Q w_\alpha = d_\alpha , \; Q \tilde{P}_\mu = -i \tilde{\lambda} \Gamma^\mu \partial_\mu \theta \Pi^\nu_\nu \epsilon^{IJ} . \tag{2.7}$$

For the nilpotency of the BRST operator, we impose the pure spinor constraint $\tilde{\lambda} \Gamma^\mu \lambda = 0$, and its secondary constraints $\{ \tilde{\lambda} \Gamma^\mu \lambda \} \Pi^\mu_I = 0$ and $\tilde{\lambda} \partial_\mu \lambda = 0$. The equation of motion for $P_\mu$ determines $\tilde{P}^\mu = \Pi^\mu_0 + e^I \Pi^\mu_I \equiv \tilde{\Pi}^\mu_0$. The last equation in (2.7) follows from the equation of motion $\nabla \theta \equiv \partial_\theta \theta + e^I \partial_\theta \theta - 2 e^0 - \Gamma^\mu_\mu \partial_\theta \theta \Pi^\nu_\nu \epsilon^{KL} = 0$ with $e^0 = -1/2$. We assume that $Q e^I = -i e^I \tilde{\lambda} \partial_\mu \theta$, which is suggested by the $\kappa$-transformation of $e^I$ [12].

The BRST invariance of $S_{\text{pure}}$ can be shown [12] by following the method used in [16]. First, we note that

$$S_{\text{pure}} - \tilde{S} = \int d^3 \tau Q [w \nabla \theta] , \tag{2.8}$$

where $\tilde{S}$ denotes $S$ in (2.1) with the replacements $P_\mu \rightarrow \tilde{P}_\mu$ and $e^0 \rightarrow -1/2$. It is convenient for us to introduce a Grassmann-odd parameter $\epsilon$ to the BRST transformations: $\delta f = \epsilon Q f$, namely

$$\delta \theta^\alpha = \epsilon \lambda^\alpha , \; \delta x^\mu = \frac{i}{2} \tilde{\epsilon} \tilde{\lambda} \Gamma^\mu \epsilon , \; \delta e^I = -i \epsilon^{IJ} \tilde{\epsilon} \tilde{\lambda} \partial_\mu \theta , \; \delta \tilde{P}_\mu = -i \tilde{\epsilon} \tilde{\lambda} \Gamma^\mu \partial_\mu \theta \Pi^\nu_\nu \epsilon^{IJ} . \tag{2.9}$$

As in [12], one may show that

$$\delta \tilde{S} = i \int d^3 \tau \tilde{\epsilon} \tilde{\lambda} \left( \Gamma^\mu \tilde{P}_\mu - \frac{1}{2} \Pi^\nu_\nu \epsilon^{IJ} \Pi^\mu_\nu \right) \nabla \theta , \tag{2.10}$$

---

$^\dagger$The derivative with a superscript $r$ denotes the right derivative.

$^\ddagger$The double spinor formalism [14] sheds some light on a derivation of the BRST operator for the pure spinor supermembrane.

$^\S$For further discussion on this issue see [12][15].
and that
\[ \delta \int d^3 \tau Q[w \nabla \theta] = -i \int d^3 \tau \varepsilon \left( \Gamma_\mu \hat{\Pi}_0^\mu - \frac{1}{2} \Gamma_{\mu \nu} \epsilon^{J J_I^\mu \Pi_I^\nu} \right) \nabla \theta. \] (2.11)
Gathering these together, we may conclude that \( S_{\text{pure}} \) is BRST invariant
\[ \delta S_{\text{pure}} = \delta \tilde{S} + \delta \int d^3 \tau Q[w \nabla \theta] = 0. \] (2.12)

## 3 Open Supermembrane and BRST surface terms

For an open supermembrane in the pure spinor formalism, the BRST transformation of the action leaves surface terms. We will show that the boundary condition to eliminate them leads to a classification of Dirichlet-branes of an open supermembrane. Furthermore we note that an open supermembrane with constant three-form fluxes may attach to non-commutative M-branes.

In this section, we will derive surface terms of the BRST transformation of the open pure spinor supermembrane action. First of all, we consider boundary conditions for bosonic variables. The bosonic part of the pure spinor supermembrane action (2.4) is the same as the bosonic part of the action (2.1) with \( e^0 = -1/2 \). In studying the bosonic part, we will restore \( e^0 \) as a Lagrange multiplier, and we consider the bosonic part of (2.1) which is classically equivalent to the Nambu-Goto action [13]. Varying it with respect to \( x^\mu \), we obtain the surface term
\[ \delta S_{\text{bos}}^{\text{pure}} \bigg|_{\partial \Sigma} = \int_{\partial \Sigma} \left( \partial_i^j x_\mu + \frac{1}{2} \epsilon^{ijk} \mathcal{H}_{\mu \rho \sigma} \partial_\rho x^\nu \partial_\sigma x^\rho \right) \delta x^\mu n_i dS, \] (3.1)
where \( n_i \) is the unit vector normal to \( \partial \Sigma \).

We will turn on a constant \( \mathcal{H} \) along the world-volume of the Dirichlet \( p \)-brane of the open supermembrane. In order to eliminate the surface term, we must impose either of the following boundary conditions [17]: the Neumann boundary condition \( \partial_n x^\mu + \mathcal{H}^\mu_{\nu \rho} \partial_\nu x^\rho \partial_\tau x^\rho = 0 \) for Neumann directions \( x^{\mu_a} (a = 0, 1, \cdots, p) \), or the Dirichlet boundary condition \( \partial_t x^{\mu} = \partial_\tau x^{\mu} = 0 \) for Dirichlet directions \( x^{\mu_a} (a = p + 1, \cdots, 10) \). We have defined \( \partial_n \equiv n^i \partial_i \) and \( \partial_t \equiv t^i \partial_i \) with \( t \) and \( \tau \) being vectors tangent to \( \partial \Sigma \). The Neumann boundary condition above reduces to the ordinary Neumann boundary condition when \( \mathcal{H} = 0 \), while it mixes Neumann and Dirichlet boundary conditions for \( \mathcal{H} \neq 0 \).

### 3.1 BRST surface terms

Now we shall consider the surface terms of the BRST transformation. For the BRST symmetry to be unbroken in the presence of the boundary, these surface terms must be eliminated by appropriate boundary conditions on the fermionic variables.
We find that the surface terms of the BRST transformation of $S_{\text{pure}}$ come from the WZ term and take the form
\[
\delta S_{\text{pure}}| = \int_{\partial \Sigma} (\mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)}) dS, \tag{3.2}
\]
where we have introduced $\xi$ by $\xi \equiv \varepsilon \lambda$, and $\mathcal{L}^{(n)}$ denotes the $n$-th order terms in $\xi$ as well as $\theta$. It is worth noting that if we set $\xi = \delta \kappa \theta$, the above surface term coincides with that of the $\kappa$-symmetry transformation of the $\kappa$-symmetric open supermembrane action examined in [3, 9, 10, 11]. For the present paper to be self-contained, we will derive conditions on $\theta$ and $\xi$ for the surface terms to be deleted.

First we will show that $\mathcal{L}^{(6)}$ in (3.5) vanishes due to the Fierz identity
\[
(CT_{\mu \nu})_{(\alpha \beta)}(CT^\nu)_{\gamma \delta} = 0. \tag{3.6}
\]
One finds that
\[
\mathcal{L}^{(6)} = -\frac{i}{24} \varepsilon^{ijk} \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \bar{\theta} \Gamma^\mu \partial_k \theta \cdot \bar{\theta} \Gamma^\nu \xi n_i
\]
\[
= -\frac{i}{24} \varepsilon^{ijk} \bar{\theta} \Gamma_{\mu \nu} \partial_k \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta \cdot \bar{\theta} \Gamma^\nu \xi n_i
\]
\[
= 0. \tag{3.7}
\]
In the first equality the Fierz identity $\bar{\theta} \Gamma_{\mu \nu} \xi \cdot \bar{\theta} \Gamma^\nu \partial_j \theta - \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \bar{\theta} \Gamma^\nu \xi = 0$ has been used, and in the second equality we have used the Fierz identity $\bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \bar{\theta} \Gamma^\mu \partial_k \theta - \bar{\theta} \Gamma_{\mu \nu} \partial_k \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta = 0$. The third equality follows from the anti-symmetry of three indices of $\varepsilon^{ijk}$.

Next we consider $\mathcal{L}^{(2)}$ in (3.3). The bosonic boundary condition implies that $\mathcal{L}^{(2)} = -\frac{i}{2} (\mathcal{H}_{\bar{\mu} \bar{\nu} \bar{\rho}} \bar{\theta} \Gamma^\rho \xi + \bar{\theta} \Gamma_{\bar{\mu} \bar{\nu}} \xi) \partial_\tau x^\alpha \partial_\tau x^\beta$, and then we require that
\[
\mathcal{H}_{\bar{\mu} \bar{\nu} \bar{\rho}} \bar{\theta} \Gamma^\rho \xi + \bar{\theta} \Gamma_{\bar{\mu} \bar{\nu}} \xi = 0. \tag{3.8}
\]
We demand that the boundary condition on $\xi$ is the same as that on $\theta$. This is because the BRST transformation (2.9) relates them each other as $\delta \theta = \xi$. It implies that the BRST symmetry is preserved even in the presence of the boundary. Before solving the boundary condition (3.8), we consider $\mathcal{L}^{(4)}$ in (3.4). The bosonic boundary condition reduces it to
\[
\mathcal{L}^{(4)} = \frac{1}{4} (\bar{\theta} \Gamma_{\mu \nu} \xi \cdot \bar{\theta} \Gamma^\mu \partial_\theta \partial_\theta \cdot \bar{\theta} \Gamma^\nu \xi) \partial_\tau x^\mu - (t \leftrightarrow \tau)
\]
\[
+ \frac{1}{4} (\bar{\theta} \Gamma_{\mu \nu} \xi \cdot \bar{\theta} \Gamma^\mu \partial_\theta \partial_\theta \cdot \bar{\theta} \Gamma^\nu \xi) \partial_\tau x^\mu - (t \leftrightarrow \tau). \tag{3.9}
\]
The relation (3.8) makes the first line of the form \[ \frac{1}{2} \mathcal{H}_{\mu\nu\rho}(\bar{\theta} \Gamma^\rho \xi \cdot \bar{\theta} \Gamma^\rho \partial_\rho \theta + \bar{\partial} \Gamma^\rho \partial_\rho \theta \cdot \bar{\partial} \Gamma^\rho \xi) \partial_\tau x^\rho - (t \leftrightarrow \tau), \]
which vanishes due to the anti-symmetricity of the three indices of \( \mathcal{H}_{\mu\nu\rho} \). As a result, for \( \mathcal{L}^{(4)} = 0 \) we require
\[ \bar{\partial} \Gamma^\rho \xi = 0 \quad \text{or} \quad \bar{\partial} \Gamma^\rho \xi = 0. \] (3.10)

Summarizing, the surface terms should disappear if (3.8) and either of two equations in (3.10) are satisfied.

4 Non-commutative M-branes

In this section, we will fix the fermionic boundary conditions which solve (3.8) and (3.10). We shall impose the same boundary condition on \( \theta \) and \( \xi \)
\[ \theta = M \theta, \quad \xi = M \xi, \] (4.1)
where \( M \) is the gluing matrix. This is because the BRST transformation relates them each other as \( \delta \theta = \xi \). In addition to the NC M5-brane obtained in [9], two kinds of NC M9-branes will be presented below.

4.1 Non-commutative M5-brane

Here we present the boundary condition for a NC M5-brane which solve (3.8) and (3.10). Because the derivation of the boundary condition is similar to those given in [9], we put it in the appendix A.

A gluing matrix and fluxes for a NC M5-brane\(^\|\) are \( M = e^{\varphi \Gamma^{345}} \Gamma^{01\ldots5} \), and \( \mathcal{H}_{012} = \sin \varphi \) and \( \mathcal{H}^{345} = \tan \varphi \). As explained in the appendix A, the fluxes satisfy
\[ \frac{1}{(\mathcal{H}_{012})^2} - \frac{1}{(\mathcal{H}^{345})^2} = 1, \] (4.2)
which is nothing but the self-duality condition [18] for the two-form gauge field on the M5-brane world-volume [19]. We note that the self-duality condition for the two-form gauge fields has been derived from the BRST symmetry of an open supermembrane.

For \( \varphi = 0 \), it represents a commutative M5-brane as \( M = \Gamma^{01\ldots5} \) and \( \mathcal{H}_{012} = \mathcal{H}^{345} = 0 \). On the other hand for \( \varphi \to \pi/2 \), the gluing matrix reduces to \( M \to \Gamma^{012} \), while fluxes are \( \mathcal{H}_{012} \to 1 \) and \( \mathcal{H}^{345} \to \infty \). It seems that this may describe an M2-brane with a critical flux \( \mathcal{H}_{012} = 1 \). However this limit is nothing but the OM limit [20], so that this M2-brane should
\(^\|\) The gluing matrix \( M = e^{\varphi \Gamma^{012}} \Gamma^{01\ldots5} \) with fluxes \( \mathcal{H}_{345} = \sinh \varphi \) and \( \mathcal{H}^{012} = \tanh \varphi \) gives a different parametrization of the NC M5-brane above.
be one of infinitely many M2-branes dissolved into the M5-brane**. This is consistent with the fact that there must be the M5-brane for the charge conservation [22]. Consequently the NC M5-brane should be regarded as a bound state of an M5-brane and M2-branes.

## 4.2 Non-commutative M9-branes

We will consider two types of non-commutative M9-branes. We may choose \{0, 1, \cdots , 9\} as the M9-brane world-volume directions without loss of generality. The Dirichlet direction is the 10-th direction which is denoted as ♮ below to avoid confusion.

### 4.2.1 Non-commutative M9-brane with an electric flux

First we consider the following gluing matrix

\[
M = h_0 \Gamma^{01\cdots 9} + h_1 \Gamma^{34\cdots 9} , \tag{4.3}
\]
which reduces to the gluing matrix for an M9-brane when \( h_1 = 0 \). We shall turn on \( \mathcal{H}_{012} \), and examine

\[
\mathcal{H}_{012} \tilde{\theta} \Gamma^2 \xi + \tilde{\theta} \Gamma_{01} \xi = 0 . \tag{4.4}
\]

Since

\[
\tilde{\theta} = \tilde{\theta} M' , \quad M' \equiv -h_0 \Gamma^{01\cdots 9} + h_1 \Gamma^{34\cdots 9} , \tag{4.5}
\]
we derive

\[
\tilde{\theta} \Gamma^2 \xi = \frac{1}{2} \tilde{\theta} (M' \Gamma^2 + \Gamma^2 M) \xi = -h_0 \tilde{\theta} \Gamma^{01\cdots 9} \Gamma^2 \xi = h_0 \tilde{\theta} \Gamma^{013\cdots 9} \xi , \tag{4.6}
\]
\[
\tilde{\theta} \Gamma_{01} \xi = \frac{1}{2} \tilde{\theta} (M' \Gamma_{01} + \Gamma_{01} M) \xi = h_1 \tilde{\theta} \Gamma^{34\cdots 9} \Gamma_{01} \xi = -h_1 \tilde{\theta} \Gamma^{013\cdots 9} \xi . \tag{4.7}
\]

This shows that (4.4) is satisfied when \( \mathcal{H}_{012} h_0 - h_1 = 0 \), i.e. \( \mathcal{H}_{012} = h_1 / h_0 \). The equation (4.4) with the replacement (012) \( \rightarrow \) (120) and that with the replacement (012) \( \rightarrow \) (201) are treated similarly, and satisfied when \( \mathcal{H}_{012} = h_1 / h_0 \). It is obvious that the latter equation in (3.10) is satisfied

\[
\tilde{\theta} \Gamma^2 M = -M' \Gamma^2 . \tag{4.8}
\]

This shows that (4.4) is satisfied when \( \mathcal{H}_{012} h_0 - h_1 = 0 \), i.e. \( \mathcal{H}_{012} = h_1 / h_0 \). The equation (4.4) with the replacement (012) \( \rightarrow \) (120) and that with the replacement (012) \( \rightarrow \) (201) are treated similarly, and satisfied when \( \mathcal{H}_{012} = h_1 / h_0 \). It is obvious that the latter equation in (3.10) is satisfied

\[
\tilde{\theta} \Gamma^2 \xi = \frac{1}{2} \tilde{\theta} (M' \Gamma^2 + \Gamma^2 M) \xi = 0 , \tag{4.8}
\]

since \( \Gamma^2 M = -M' \Gamma^2 \). As a result, the gluing matrix (4.3) with \( \mathcal{H}_{012} = h_1 / h_0 \) eliminates the BRST surface terms. For \( M^2 = h_0^2 - h_1^2 \equiv 1 \), we choose \( h_0 = \cosh \varphi \) and \( h_1 = \sinh \varphi \). In this parametrization, \( M \) and \( \mathcal{H}_{012} \) are expressed as

\[
M = e^{\varphi \Gamma^{01\cdots 9}} , \quad \mathcal{H}_{012} = \tanh \varphi . \tag{4.9}
\]

**See [21] for the relation to the BLG model.
When $\varphi = 0$, it represents a commutative M9-brane characterized by $M = \Gamma^{01\cdots9}$ and $H_{012} = 0$. On the other hand, when $\varphi \to \infty$, the boundary condition $\theta = M\theta$ reduces to $0 = (\Gamma^{01\cdots9} + \Gamma^{3\cdots9})\theta = \Gamma^{01\cdots9}(1 - \Gamma^{012})\theta$, i.e. $\theta = \Gamma^{012}\theta$. As $H_{012} \to 1$, it may represent an M2-brane with a critical flux $H_{012} = 1$. For the charge conservation, there must be another brane other than the M2-brane. We may expect that there should be an M9-brane behind the M2-brane with a critical flux. To confirm this expectation we need to know the M9-brane effective action which describes the coupling to the M2-brane. Further study is needed to clarify this point.

### 4.2.2 Non-commutative M9-brane with a magnetic flux

Next, we shall consider the following gluing matrix

$$ M = h_0\Gamma^{01\cdots9} + h_1\Gamma^{01\cdots6}, \quad (4.10) $$

and turn on $H_{789}$. Since

$$ \bar{\theta} = \bar{\theta}M', \quad M' \equiv -h_0\Gamma^{01\cdots9} + h_1\Gamma^{01\cdots6}, \quad (4.11) $$

we derive

$$ \bar{\theta}\Gamma^9\xi = \frac{1}{2} \bar{\theta}(M'\Gamma^9 + \Gamma^9 M)\xi = -h_0\bar{\theta}\Gamma^{01\cdots9}\Gamma^9\xi = -h_0\bar{\theta}\Gamma^{01\cdots8}\xi, \quad (4.12) $$

$$ \bar{\theta}\Gamma_{78}\xi = \frac{1}{2} \bar{\theta}(M'\Gamma_{78} + \Gamma_{78} M)\xi = h_1\bar{\theta}\Gamma^{01\cdots6}\Gamma_{78}\xi = h_1\bar{\theta}\Gamma^{01\cdots8}\xi. \quad (4.13) $$

This shows that

$$ H_{789}\bar{\theta}\Gamma^9\xi + \bar{\theta}\Gamma_{78}\xi = 0 \quad (4.14) $$

is satisfied when $-H_{789}h_0 + h_1 = 0$, i.e. $H_{789} = h_1/h_0$. The equation (4.14) with the replacement $(789) \rightarrow (897)$ and that with the replacement $(789) \rightarrow (978)$ are shown to be treated similarly and are satisfied when $H_{789} = h_1/h_0$. It is obvious that the latter equation in (3.10) is satisfied since $\Gamma^2 M = -M'\Gamma^2$. As a result, the gluing matrix (4.10) with $H_{789} = h_1/h_0$ eliminates the BRST surface terms. For $M^2 = h_0^2 + h_1^2 \equiv 1$, we choose $h_0 = \cos\varphi$ and $h_1 = \sin\varphi$. In this parametrization, $M$ and $H_{789}$ are expressed as

$$ M = e^{\varphi\Gamma^{789}\Gamma^{01\cdots9}}, \quad H_{789} = \tan\varphi. \quad (4.15) $$

When $\varphi = 0$, it represents a commutative M9-brane characterized by $M = \Gamma^{01\cdots9}$ and $H_{789} = 0$. On the other hand, when $\varphi = \pi/2$, the gluing matrix reduces to $M = \Gamma^{01\cdots6}$ and the flux diverges $H_{789} \to \infty$. It seems that this describes a 6-brane, but there is no
seven-form gauge potential in eleven-dimensions. In ten-dimensions, however, we have a RR seven-form gauge potential $C_7$ which is dual to a RR one-form gauge potential $C_1$. The gauge potential $C_7$ couples to a D6-brane which is characterized by a harmonic function $H$ on the space $\mathbb{E}^3$ transverse to the world-volume $\mathbb{E}^{1,6}$. An eleven-dimensional lift of the D6-brane is known as a KK monopole which is magnetically charged with respect to $C_1$ and takes the form [6]

$$\begin{align*}
\text{ds}^2 &= \text{ds}^2(\mathbb{E}^{1,6}) + H(y)\text{d}y^i\text{d}y_i + H(y)^{-1}(\text{d}z + \text{d}y^iC_i(y))^2, \\
F_{ij} &\equiv \partial_i C_j(y) - \partial_j C_i(y) = \epsilon_{ijk}\partial_k H(y),
\end{align*}$$

(4.16)

where $y^i (i = 1, 2, 3)$ are coordinates on $\mathbb{E}^3$. The direct dimensional reduction with respect to $z$ leads to the D6-brane solution. We will interpret the boundary characterized by $M = \Gamma^{01 \cdots 6}$ as the KK monopole. In the present context, the KK monopole extends along $\mathbb{E}^{1,6}$ spanned by $\{0, 1, \cdots , 6\}$ and $H$ is a harmonic function on $\mathbb{E}^3$ spanned by $\{7, 8, 9\}$. As $H_{789} \to \infty$, there should be infinitely many KK monopoles so that $H(y)$ has infinitely many poles on $\mathbb{E}^3$. These KK monopoles have dissolved inside an M9-brane. Consequently the NC M9-brane should be regarded as a bound state of an M9-brane and KK monopoles.

### 4.3 Supersymmetry of non-commutative M-branes

In this subsection, we shall show that the NC M-branes derived in the sections 4.1 and 4.2 are half supersymmetric. For this purpose we will examine the surface terms of the supersymmetry transformations and show that they are deleted by the boundary conditions examined in the previous subsections.

To see that the action (2.4) is invariant under the supersymmetry transformations (2.6), we need to perform partial integration. For an open supermembrane we are considering, the partial integration leaves a surface term which has to be deleted by an appropriate boundary condition for some amount of supersymmetry to be preserved. We find that the surface terms come from the Wess-Zumino term (2.3) and take the form

$$\delta_{\epsilon}S_{\text{pure}} = \int_{\partial \Sigma} \left( \mathcal{L}_{\text{SUSY}}^{(2)} + \mathcal{L}_{\text{SUSY}}^{(4)} + \mathcal{L}_{\text{SUSY}}^{(6)} \right) \text{d}S,$$

(4.17)

where

$$\begin{align*}
\mathcal{L}_{\text{SUSY}}^{(2)} &= \frac{i}{4} \epsilon^{ijk} (\mathcal{H}_{\mu\nu\rho} \bar{\theta} \Gamma^\rho \epsilon + \bar{\theta} \Gamma_{\mu\nu} \epsilon) \partial_j x^\mu \partial_k x^\nu n_i, \\
\mathcal{L}_{\text{SUSY}}^{(4)} &= \frac{1}{24} \epsilon^{ijk} (\epsilon \Gamma_{\mu\nu} \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \partial_\theta + \bar{\theta} \Gamma_{\mu\nu} \partial_\theta \cdot \epsilon \Gamma^\mu \theta) \partial_k x^\nu n_i, \\
\mathcal{L}_{\text{SUSY}}^{(6)} &= \frac{i}{144} \epsilon^{ijk} (\epsilon \Gamma_{\mu\nu} \theta \cdot \bar{\theta} \Gamma^\nu \partial_j \partial_\theta + \bar{\theta} \Gamma_{\mu\nu} \partial_\theta \cdot \epsilon \Gamma^\nu \theta) \bar{\theta} \Gamma^\nu \partial_k \theta n_i.
\end{align*}$$

(4.18-4.20)
A derivation of them was given in the appendix B. Now, we shall compare these surface terms with the BRST surface terms in (3.3),(3.4) and (3.5). We found that

$$L^{(2)}_{\text{SUSY}} = L^{(2)}|_{\xi = -\epsilon}, \quad L^{(4)}_{\text{SUSY}} = L^{(4)}|_{\xi = -\epsilon/3}, \quad L^{(6)}_{\text{SUSY}} = L^{(6)}|_{\xi = -\epsilon/3}. \quad (4.21)$$

It implies that the boundary conditions which eliminate the BRST surface terms also eliminate the surface terms of supersymmetry\footnote{For the \(\kappa\)-symmetric open supermembrane, see [3], where it was shown that the boundary conditions to eliminate the \(\kappa\)-symmetry surface term will preserve a half of the supersymmetries.}. As a result, we may conclude that the NC M-branes obtained in the sections 4.1 and 4.2 would preserve a half of 32 supersymmetries so that they are 1/2 BPS objects.

5 Summary and discussions

We examined boundary conditions for the BRST symmetry of the open supermembrane with a constant flux in the pure spinor formalism. The boundary conditions lead to a possible Dirichlet branes of an open supermembrane. It is found that the surface terms coincide with those for the \(\kappa\)-variation of the \(\kappa\)-symmetric open supermembrane examined in [3, 9, 10, 11] if we replace \(\xi\) for \(\delta_\kappa \theta\). So we have obtained the NC M5-brane derived there. In addition to the NC M5-brane, we found two types of NC M9-branes in this paper. One is the NC M9-brane with an electric flux characterized by (4.9). It reduces in the critical electric flux limit to an M2-brane on an M9-brane. Another is the NC M9-brane with a magnetic flux characterized by (4.15). It reduces in the strong flux limit to infinitely many KK monopoles dissolved into an M9-brane. It is argued that this NC M9-brane should be regarded as a bound state of an M9-brane and KK monopoles. Furthermore, we have examined the surface terms for the supersymmetry transformations of the open supermembrane action with a constant three-form flux in the pure spinor formalism. We found that the surface term \(L^{(n)}_{\text{SUSY}}\) in (4.17) for the supersymmetry variation is proportional to the surface term \(L^{(n)}\) in (3.2) for the BRST variation if we replace the supersymmetry parameter \(\epsilon\) for \(\xi\). Consequently, we have concluded that the NC M-branes obtained here should preserve a half of 32 supersymmetries and are 1/2 BPS objects.

In this paper, we have examined an open supermembrane in the pure spinor formalism, instead of the \(\kappa\)-symmetric open supermembrane. One of advantages of our approach is to consider the BRST symmetry which is expected to survive quantum corrections. We found that our results are consistent with the previous ones obtained from the \(\kappa\)-symmetry arguments. This implies that our results may give a quantum consistency check for the previous ones.
For the NC M9-brane with an electric flux characterized by (4.9), we assumed that there should be an M9-brane behind the M2-brane in the critical flux limit. To confirm this point we need to know the M9-brane effective action which describes the coupling to the M2-brane. It is interesting for us to clarify this point.

We obtained NC M-branes as solutions of boundary conditions for an open supermembrane. They are expected to be obtained also as classical solutions of M-brane world-volume equations of motion. Especially they should preserve a half of 32 supersymmetries from our analysis in the section 4.3. Solutions of the world-volume equations of motion will help us to examine properties of the NC M-branes obtained in this paper. It is also interesting to pursue supergravity solutions of NC M-branes.

It is known that in a general background, the classical BRST invariance of an open pure spinor superstring implies that the background fields satisfy full non-linear equations of motion for a supersymmetric Born-Infeld action [23]. This is the open string version of [24] in which the classical BRST invariance of a closed pure spinor superstring in a curved background is shown to imply that the background fields satisfy full non-linear equations of motion for the type-II supergravity. There are similar results for the classical $\kappa$-invariance of an open Green-Schwarz superstring [25] and a closed Green-Schwarz superstring [26], respectively. It is interesting to extract non-commutative M-brane equations of motion by requiring BRST invariance in the pure spinor supermembrane with background fields.

An obvious generalization of our analysis is to examine NC M-branes in curved backgrounds, such as the AdS$_4 \times$S$^7$ background. In [27], commutative M-branes are derived as Dirichlet branes of the $\kappa$-symmetric open supermembrane in AdS$_4 \times$S$^7$ and AdS$_7 \times$S$^4$. It is interesting to examine NC M-branes in AdS$_4 \times$S$^7$ by using open supermembrane with constant fluxes in the pure spinor formalism. As for D-branes in AdS$_5 \times$S$^5$, we examined them from the BRST symmetry of the pure spinor superstring in [28]. Furthermore recently world-sheet supersymmetries are examined in [29]. It is also interesting to examine NC D-branes by using the pure spinor superstring with two-form fluxes. We expect that this analysis may support the previous results obtained from the $\kappa$-symmetry arguments [30]. We hope to report these issues in the near future.

Acknowledgments

The authors would like to thank Takanori Fujiwara, Yoshifumi Hyakutake and Kentaroh Yoshida for useful comments. MS would like to appreciate the organizers of the conference “Progress in Quantum Field Theory and String Theory II” held at Osaka City University, March 27-31, 2017 for their kind hospitality. There he reported the main results in this paper.
A NC M5-brane

We will derive the NC M5-branes given in the section 4.1. Consider the gluing matrix of the form

\[
M = h_0 \Gamma^{\bar{\mu}_0 \cdots \bar{\mu}_5} + h_1 \Gamma^{\bar{\mu}_6 \cdots \bar{\mu}_2} .
\]  

(A.1)

We note that this reduces to the gluing matrix for an M5-brane when \( h_1 = 0 \). For \( M^2 = 1 \), we demand \(-s_0 h_0^2 - s_1 h_1^2 \equiv 1 \). We have introduced \( s_0 \) and \( s_1 \) such that \( s_0 = -1 \) when \( 0 \in \{ \bar{\mu}_0, \cdots, \bar{\mu}_5 \} \) and \( s_0 = +1 \) otherwise, while \( s_1 = -1 \) when \( 0 \in \{ \bar{\mu}_0, \cdots, \bar{\mu}_2 \} \) and \( s_1 = +1 \) otherwise. For reality of \( \theta \), either \( s_0 \) or \( s_1 \) must be \(-1 \). Noting that \( s_1 = 1 \) follows from \( s_0 = 1 \), we set \( s_0 = -1 \). Introducing fluxes \( \mathcal{H}_{\bar{\mu}_0 \cdots \bar{\mu}_2} \) and \( \mathcal{H}^{\bar{\mu}_3 \cdots \bar{\mu}_5} \) we obtain a non-trivial solution. Since

\[
\bar{\theta} = \theta^T M^T C = \bar{\theta} M' , \quad M' \equiv -h_0 \Gamma^{\bar{\mu}_0 \cdots \bar{\mu}_5} + h_1 \Gamma^{\bar{\mu}_6 \cdots \bar{\mu}_2} ,
\]  

(A.2)

\[
\bar{\theta} \Gamma_{\bar{\mu}_0 \bar{\mu}_1} \xi = -\frac{1}{2} \bar{\theta} (M' \Gamma_{\bar{\mu}_0 \bar{\mu}_1} + \Gamma_{\bar{\mu}_0 \bar{\mu}_1} M) \xi = -\frac{1}{2} \bar{\theta} \Gamma_{\bar{\mu}_0 \bar{\mu}_1} (M' + M) \xi = -h_0 \bar{\theta} \Gamma^{\bar{\mu}_2} \xi ,
\]  

(A.3)

\[
\bar{\theta} \Gamma_{\bar{\mu}_5} \xi = \frac{1}{2} \bar{\theta} (M' \Gamma_{\bar{\mu}_5} + \Gamma_{\bar{\mu}_5} M) \xi = \frac{1}{2} \bar{\theta} \Gamma_{\bar{\mu}_5} (-M' + M) \xi = -h_0 \bar{\theta} \Gamma^{\bar{\mu}_0 \cdots \bar{\mu}_4} \xi ,
\]  

(A.4)

\[
\bar{\theta} \Gamma^{\bar{\mu}_3 \bar{\mu}_4} \xi = -\frac{1}{2} \bar{\theta} (M' \Gamma^{\bar{\mu}_3 \bar{\mu}_4} + \Gamma^{\bar{\mu}_3 \bar{\mu}_4} M) \xi = -\frac{1}{2} \bar{\theta} \Gamma^{\bar{\mu}_3 \bar{\mu}_4} (M' + M) \xi = h_1 \bar{\theta} \Gamma^{\bar{\mu}_0 \cdots \bar{\mu}_4} \xi ,
\]  

(A.5)

\[
\bar{\theta} \Gamma_{\bar{\mu}_2 \bar{\mu}_3} \xi = -\frac{1}{2} \bar{\theta} (M' \Gamma_{\bar{\mu}_2 \bar{\mu}_3} + \Gamma_{\bar{\mu}_2 \bar{\mu}_3} M) \xi = -\frac{1}{2} \bar{\theta} \Gamma_{\bar{\mu}_2 \bar{\mu}_3} (-M + M) \xi = 0 ,
\]  

(A.6)

(3.8) implies that

\[
\mathcal{H}_{\bar{\mu}_0 \cdots \bar{\mu}_2} - h_1 = 0 , \quad -h_0 \mathcal{H}^{\bar{\mu}_3 \cdots \bar{\mu}_5} + h_1 = 0 .
\]  

(A.7)

Substituting them into \( h_0^2 - s_1 h_1^2 = 1 \), we obtain

\[
\frac{1}{(\mathcal{H}_{\bar{\mu}_0 \cdots \bar{\mu}_2})^2} - \frac{1}{(\mathcal{H}^{\bar{\mu}_3 \cdots \bar{\mu}_5})^2} = -s_1 .
\]  

(A.8)

This is nothing but the self-duality condition [18] for the two-form gauge field on the M5-brane world-volume [19]. It is straightforward to see that the latter condition in (3.10) is satisfied.

For \( s_1 = -1 \), we may take \( M = M = h_0 \Gamma^{012345} + h_1 \Gamma^{012} \) without loss of generality. Parametrizing \( h_0 \) and \( h_1 \) as \( h_0 = \cos \varphi \) and \( h_1 = \sin \varphi \), we can express it as \( M = e^{i \varphi} \Gamma^{012345} \Gamma^{01 \cdots 5} \). Fluxes are \( \mathcal{H}_{012} = \sin \varphi \) and \( \mathcal{H}^{345} = \tan \varphi \). On the other hand for \( s_1 = 1 \), we may take \( M \) as

SH would like to thank the Yukawa Institute for Theoretical Physics at Kyoto University for hospitality during the workshop YITP-W-17-08 "Strings and Fields 2017."
\[ M = h_0 \Gamma^{012345} + h_1 \Gamma^{345} \] without loss of generality. Parametrizing \( h_0 \) and \( h_1 \) as \( h_0 = \cosh \varphi \) and \( h_1 = \sinh \varphi \), we can express it as \( M = e^{\varphi \Gamma^{012}} \). Fluxes are \( \mathcal{H}_{345} = \sinh \varphi \) and \( \mathcal{H}^{012} = \tanh \varphi \).

B Supersymmetry surface term

We shall derive the supersymmetry surface terms (4.19) and (4.20) in section 4.3 below.

The surface term \( \mathcal{L}^{(4)}_{\text{susy}} \) given in (4.19) may be derived from terms contained in \( \delta_r \mathcal{L}_{\text{WZ}} \) which are the four-th order terms in fermions as follows

\[
\delta_r \mathcal{L}_{\text{WZ}}|_{\theta^*} = \frac{1}{8} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \partial_i \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta - \bar{\theta} \Gamma_{\mu \nu} \partial_i \theta \cdot \epsilon \Gamma^\mu \partial_j \theta \right) \partial_k x^\nu \tag{B.1}
\]

\[
= \frac{1}{8} \epsilon^{ijk} \left( \frac{1}{2} \epsilon \Gamma_{\mu \nu} \partial_i \theta \Gamma^\mu \partial_j \theta + \frac{1}{2} \partial_i \partial_j \partial_k \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \partial_k x^\nu \tag{B.2}
\]

\[
= - \frac{1}{2} \delta_r \mathcal{L}_{\text{WZ}}|_{\theta^*} + \partial_i \left[ \frac{1}{16} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \partial_i \theta \Gamma^\mu \partial_j \theta + \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \partial_k x^\nu \right] \tag{B.3}
\]

\[
= \partial_i \left[ \frac{1}{24} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \partial_i \theta \Gamma^\mu \partial_j \theta + \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \partial_k x^\nu \right]. \tag{B.4}
\]

In the second equality we have utilized the Fierz identity, and for the third equality partial integration has been performed for \( \partial_i \bar{\theta} \). Similarly, examining terms contained in \( \delta_r \mathcal{L}_{\text{WZ}} \) which are six-th order in fermions, we derive (4.20) as follows.

\[
\delta_r \mathcal{L}_{\text{WZ}}|_{\bar{\theta}^*} = \frac{i}{48} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \bar{\partial}_i \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta - \bar{\theta} \Gamma_{\mu \nu} \bar{\partial}_i \theta \cdot \epsilon \Gamma^\mu \partial_j \theta \right) \bar{\theta} \Gamma^\nu \partial_k \theta \tag{B.5}
\]

\[
= \frac{i}{48} \epsilon^{ijk} \left( \frac{1}{2} \epsilon \Gamma_{\mu \nu} \bar{\partial}_i \theta \Gamma^\mu \partial_j \theta + \frac{1}{2} \partial_i \partial_j \partial_k \theta \cdot \bar{\theta} \Gamma^\mu \partial_k \theta \right) \bar{\theta} \Gamma^\nu \partial_k \theta \tag{B.6}
\]

\[
= - \frac{1}{2} \delta_r \mathcal{L}_{\text{WZ}}|_{\bar{\theta}^*} + \partial_i \left[ \frac{1}{96} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \bar{\partial}_i \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta + \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \bar{\theta} \Gamma^\nu \partial_k \theta \right]
\]

\[ + \frac{i}{96} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \bar{\partial}_i \theta \Gamma^\mu \partial_j \theta + \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \partial_j \bar{\theta} \Gamma^\nu \partial_k \theta \tag{B.7}
\]

\[ = \partial_i \left[ \frac{1}{144} \epsilon^{ijk} \left( \epsilon \Gamma_{\mu \nu} \bar{\partial}_i \theta \cdot \bar{\theta} \Gamma^\mu \partial_j \theta + \bar{\theta} \Gamma_{\mu \nu} \partial_j \theta \cdot \epsilon \Gamma^\mu \partial_k \theta \right) \bar{\theta} \Gamma^\nu \partial_k \theta \right]. \tag{B.8}
\]

In the second equality we have utilized the Fierz identity, and for the third equality partial integration has been performed for \( \partial_i \bar{\theta} \). The last line in (B.7) is eliminated by the Fierz identity.

References

[1] E. Bergshoeff, E. Sezgin and P. K. Townsend, “Supermembranes and Eleven-Dimensional Supergravity,” Phys. Lett. B 189 (1987) 75.
E. Bergshoeff, E. Sezgin and P. K. Townsend, “Properties of the Eleven-Dimensional Super Membrane Theory,” Annals Phys. 185 (1988) 330.

[2] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,” Nucl. Phys. B 438 (1995) 109 [hep-th/9410167].

E. Witten, “String theory dynamics in various dimensions,” Nucl. Phys. B 443 (1995) 85 [hep-th/9503124].

[3] K. Ezawa, Y. Matsuo and K. Murakami, “Matrix regularization of open supermembrane: Towards M theory five-brane via open supermembrane,” Phys. Rev. D 57 (1998) 5118 [hep-th/9707200].

[4] B. de Wit, K. Peeters and J. C. Plefka, “Open and closed supermembranes with winding,” Nucl. Phys. Proc. Suppl. 68 (1998) 206 [hep-th/9710215].

[5] P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven-dimensions,” Nucl. Phys. B 460 (1996) 506 [hep-th/9510209].

P. Horava and E. Witten, “Eleven-dimensional supergravity on a manifold with boundary,” Nucl. Phys. B 475 (1996) 94 [hep-th/9603142].

[6] P. K. Townsend, “The eleven-dimensional supermembrane revisited,” Phys. Lett. B 350 (1995) 184 [hep-th/9501068].

[7] C. M. Hull, “Exact pp Wave Solutions of Eleven-dimensional Supergravity,” Phys. Lett. 139B (1984) 39.

[8] R. D. Sorkin, “Kaluza-Klein Monopole,” Phys. Rev. Lett. 51 (1983) 87.

D. J. Gross and M. J. Perry, “Magnetic Monopoles in Kaluza-Klein Theories,” Nucl. Phys. B 226 (1983) 29.

[9] M. Sakaguchi and K. Yoshida, “Noncommutative M-branes from covariant open supermembranes,” Phys. Lett. B 642 (2006) 400 [hep-th/0608099].

[10] M. Sakaguchi and K. Yoshida, “Intersecting Noncommutative M5-branes from Covariant Open Supermembrane,” Nucl. Phys. B 781 (2007) 85 [hep-th/0702062 [HEP-TH]].

[11] M. Sakaguchi and K. Yoshida, “A Covariant Approach to Noncommutative M5-branes,” Prog. Theor. Phys. Suppl. 171 (2007) 275 [hep-th/0702132 [HEP-TH]].

[12] N. Berkovits, “Towards covariant quantization of the supermembrane,” JHEP 0209 (2002) 051 [hep-th/0201151].
[13] E. Bergshoeff, E. Sezgin and Y. Tanii, “Hamiltonian Formulation of the Supermembrane,” Nucl. Phys. B 298 (1988) 187.

[14] Y. Aisaka and Y. Kazama, “Towards pure spinor type covariant description of supermembrane: An Approach from the double spinor formalism,” JHEP 0605 (2006) 041 [hep-th/0603004].

[15] P. Fre’ and P. A. Grassi, “Pure Spinors, Free Differential Algebras, and the Supermembrane,” Nucl. Phys. B 763 (2007) 1 [hep-th/0606171].

M. Babalic and N. Wyllard, “Towards relating the kappa-symmetric and pure-spinor versions of the supermembrane,” JHEP 0810 (2008) 059 [arXiv:0808.3691 [hep-th]].

[16] I. Oda and M. Tonin, “On the Berkovits covariant quantization of GS superstring,” Phys. Lett. B 520, 398 (2001) [hep-th/0109051].

[17] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, “A Noncommutative M theory five-brane,” Nucl. Phys. B 590 (2000) 173 [hep-th/0005026].

[18] P. S. Howe, E. Sezgin and P. C. West, “Covariant field equations of the M theory five-brane,” Phys. Lett. B 399 (1997) 49 [hep-th/9702008].

P. S. Howe, E. Sezgin and P. C. West, “The Six-dimensional selfdual tensor,” Phys. Lett. B 400 (1997) 255 [hep-th/9702111].

[19] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909 (1999) 032 [hep-th/9908142].

[20] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, “(OM) theory in diverse dimensions,” JHEP 0008 (2000) 008 [hep-th/0006062].

[21] P. M. Ho and Y. Matsuo, “M5 from M2,” JHEP 0806 (2008) 105 [arXiv:0804.3629 [hep-th]].

P. M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, “M5-brane in three-form flux and multiple M2-branes,” JHEP 0808 (2008) 014 [arXiv:0805.2898 [hep-th]].

[22] A. Strominger, “Open p-branes,” Phys. Lett. B 383 (1996) 44 [hep-th/9512059].

[23] N. Berkovits and V. Pershin, “Supersymmetric Born-Infeld from the pure spinor formalism of the open superstring,” JHEP 0301 (2003) 023 [hep-th/0205154].

[24] N. Berkovits and P. S. Howe, “Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring,” Nucl. Phys. B 635 (2002) 75 [hep-th/0112160].
[25] C. S. Chu, P. S. Howe and E. Sezgin, “Strings and D-branes with boundaries,” Phys. Lett. B 428 (1998) 59 [hep-th/9801202].

[26] M. Grisaru, P. S. Howe, L. Mezincescu, B. E. W. Nilsson and P. K. Townsend, “N = 2 superstrings in a supergravity background,” Phys. Lett. B 162 (1985) 116.

[27] M. Sakaguchi and K. Yoshida, “Dirichlet branes of the covariant open supermembrane in AdS$_4$×S$^7$ and AdS$_7$×S$^4$,” Nucl. Phys. B 681 (2004) 137 [hep-th/0310035].
M. Sakaguchi and K. Yoshida, “Open M-branes on AdS$_4$/7×S$^{7/4}$ revisited,” Nucl. Phys. B 714 (2005) 51 [hep-th/0405109].

[28] S. Hanazawa and M. Sakaguchi, “D-branes from Pure Spinor Superstring in AdS$_5$×S$^5$ Background,” Nucl. Phys. B 914 (2017) 234 [arXiv:1609.05457 [hep-th]].

[29] J. Park and H. Shin, “Notes on worldvolume supersymmetries for D-branes on AdS$_5$×S$^5$ background,” JHEP 1709 (2017) 022 [arXiv:1705.06887 [hep-th]].

[30] M. Sakaguchi and K. Yoshida, “Noncommutative D-brane from Covariant AdS Superstring,” Nucl. Phys. B 797 (2008) 179 [hep-th/0604039].