Dynamical Primal-Dual Accelerated Method with Applications to Network Optimization

Xianlin Zeng, Jinlong Lei, and Jie Chen

Abstract

This paper develops a continuous-time primal-dual accelerated method with an increasing damping coefficient for a class of convex optimization problems with affine equality constraints. This paper analyzes critical values for parameters in the proposed method and prove that the rate of convergence in terms of the duality gap function is $O(\frac{1}{t^2})$ by choosing suitable parameters. As far as we know, this is the first continuous-time primal-dual accelerated method that can obtain the optimal rate. Then this work applies the proposed method to two network optimization problems, a distributed optimization problem with consensus constraints and a distributed extended monotropic optimization problem, and obtains two variant distributed algorithms. Finally, numerical simulations are given to demonstrate the efficacy of the proposed method.

Index Terms

Nesterov’s accelerated method, primal-dual method, network optimization, continuous-time algorithm.

I. INTRODUCTION

Many large-scale problems in control and optimization (including optimal consensus of multi-agents [1], network flow [2], energy dispatch of power grids [3], etc.) can be formulated as distributed optimization problems that studied in [4], [5]. In distributed optimization, information

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is allocated to multiple agents with no central agent, and certain equality constraints such as consensus constraints [6], resource allocation constraints [3], and monotropic constraints [2] should be satisfied. Distributed optimization problems are often solved by first-order methods, whose implementation is simpler than that of higher-order algorithms. The discrete-time distributed algorithms such as distributed subgradient methods [4], distributed Nesterov’s gradient methods [7], distributed gradient tracking methods [8], and distributed primal-dual methods [9], [10] have been extensively studied, of which the convergence analysis is specific to that designed algorithm while not within a unified framework. Recently, the study of the continuous-time distributed optimization algorithms have also drawn much attention from researchers, see e.g., [6], [11], [12].

It is well-known that the fastest rate of convergence of first-order methods for convex optimization in the worst case is $O(\frac{1}{t^2})$, see [13]. Most existing approaches with rate $O(\frac{1}{t^2})$ mainly focus on primal algorithms, which, however, can not be directly applied to the primal-dual framework. Because the primal-dual framework is a powerful technique for constrained optimization with equality/inequality constraints, which primal-based methods can not deal with, accelerating the rate of convergence of primal-dual methods is of great importance. The primary objective of this paper is to propose a dynamical primal-dual accelerated method for convex network optimization with convergence rate $O(\frac{1}{t^2})$. The previous works demonstrated that primal-dual methods for distributed optimization can guarantee asymptotic convergence (see [11]) or a rate of convergence $O(\frac{1}{t^2})$ (see [2]). Hence, for modern large-scale network optimization problems with equality constraints, the convergence rate of existing distributed primal-dual methods is slower than that of algorithms for centralized convex optimization without equality constraints. Thus, it is important to design a primal-dual accelerated method for convex network optimization problems.

A. Related Work

1) Distributed optimization algorithms: Due to the rapid growth in the scale and complexity of the network optimization problems, distributed first-order gradient-based primal-dual methods have been gaining more attention because they are easy for distributed implementation. For distributed strongly convex optimization problems, distributed solvers with linear or exponential rates have been proposed in [3], [14]. Furthermore, by using the notion of metric subregularity, some existing distributed algorithms (see [15]) have been proved to have linear convergence
rates if non-strongly convex cost functions satisfy some properties. If cost functions are arbitrary convex functions, distributed primal-dual algorithms are only proved to have asymptotic convergence (see [9], [11], [16]–[19]) or convergence rate $O(\frac{1}{t})$ (see [2]). Recently, a distributed algorithm that achieves convergence rate $O(\frac{1}{\sqrt{t}})$ has been proposed in [7], and an accelerated distributed algorithm using a gradient estimation scheme with rate $O(\frac{1}{\sqrt{t}})$ has been proposed in [20]. However, both [7] and [20] consider consensus constraints, cannot achieve the optimal rate $O(\frac{1}{t})$, and are not primal-dual algorithms.

2) Centralized Nesterov’s accelerated gradient methods: The Nesterov accelerated method, using a vanishing damping coefficient, was developed in [21] and proved to have a rate of convergence $O(\frac{1}{t^2})$. As a first-order algorithm, the Nesterov’s accelerated method and its different variants (see [13]) are proved to be optimal in some sense (see [22]) and have been widely studied in different settings [23], [24]. Recently, growing attention has been dedicated to the design and analysis of continuous-time Nesterov’s accelerated methods (see [25]–[28]). On one hand, ordinary differential equations (ODEs) often exhibit similar convergence properties to their discrete-time counterparts and thus can serve as a tool for algorithm design and analysis. On the other hand, continuous-time algorithms may allow for a better understanding of intuitive and ideas in the design. [25] showed that the continuous-time counterpart of the Nesterov’s accelerated method is a second-order ODE, and proved that the suboptimality gap is $O(\frac{1}{t^2})$ for parameter $\alpha \geq 3$ in the ODE; [29] further proved that the generated trajectory converges to a minimizer of $\phi$ as $t \to \infty$ for $\alpha > 3$; while [27] proved that the rate of convergence of the ODE proposed in [25] is $O(t^{-2+\frac{1}{\alpha}})$ for $0 < \alpha \leq 3$.

To develop new insights into accelerated algorithms and obtain new algorithms, there are ongoing recent research on discretization of accelerated algorithms. [30] studied a mixed forward/backward Euler scheme for continuous-time algorithms for constrained optimization and proved an analogous $O(\frac{1}{t^2})$ rate when the step size is small enough. [31] proposed alternative ODEs of Nesterov’s accelerated methods and their discretizations, which are called high-resolution ODEs because they use Hessians of cost functions to distinguish between Nesterov’s accelerated gradient method for strongly convex functions and Polyak’s heavy-ball method. [32] considered three discretization schemes: an explicit Euler scheme, an implicit Euler scheme, and a symplectic scheme, and applied the symplectic scheme to a high-resolution ODE proposed by [31] for minimizing smooth strongly convex functions. [33] showed that the Nesterov’s method arises as a straightforward discretization of a modified ODE. [34] further derived an ODE model
of the accelerated triple momentum algorithm for strongly convex optimization and investigated the convergence behavior of the ODE model. However, these works only focus on unconstrained optimization problems, while can not be applied to the widely used primal-dual framework.

In recent years, accelerated primal-dual methods have received lots of attention due to its application in constrained optimization and minimax optimization (or saddle point problems). Focusing on duality gaps of the Lagrangian function or minimax optimization, which often converge to 0 with rate $O\left(\frac{1}{k}\right)$, various accelerated primal-dual algorithms have been investigated. In [35], a primal-dual algorithm with a rate of convergence $O\left(\frac{1}{k^2}\right)$ has been proposed for saddle point problems, whose primal or dual cost function is uniformly convex. In [35], since each iteration involves solving an optimization problem, cost functions need to be “simple”. [36] proposed an accelerated design for primal-dual block coordinate method that has a rate of convergence $O\left(\frac{1}{k^2}\right)$ for strongly convex functions. [37] proposed a linearized augmented Lagrangian method with full acceleration $O\left(\frac{1}{k^2}\right)$ for strongly convex functions with adaptive parameters.

Different from previous literature, this work is the first attempt to propose a dynamical primal-dual method for constrained convex optimization problems with a rate of convergence $O\left(\frac{1}{t^2}\right)$.

B. Contributions

This paper has made the following contributions.

1) Considering convex optimization problems with equality constraints, this paper proposes a primal-dual Nesterov’s accelerated method, which has a convergence rate $O\left(\frac{1}{t^2}\right)$. The novel part in the design is the use of the derivative information in the Lagrangian saddle point dynamics to obtain the accelerated convergence of the proposed first-order primal-dual method. To our best knowledge, this is the first continuous-time primal-dual accelerated method owning a convergence rate $O\left(\frac{1}{t^2}\right)$.

2) This paper further analyzes convergence properties of the proposed primal-dual accelerated method for different choices of parameters, and figures out the best choice of parameters for the optimal convergence rate $O\left(\frac{1}{t^2}\right)$. To be specific, $\alpha > 3$ and $\beta = \frac{1}{2}$ are shown in Section III.

3) This paper applies the proposed primal-dual accelerated method to two classes of widely studied network optimization problems: distributed optimization with consensus constraints...
and distributed extended monotropic optimization. This leads to two distributed primal-dual Nesterov’s accelerated algorithms with convergence guarantees. The numerical experiments show faster convergence performances than that of existing results on distributed optimization problems [2], [5].

C. Organization

This paper is organized as follows. Section II provides necessary mathematical preliminaries and formulates a convex optimization problem with linear equality constraints. Section III proposes a continuous-time primal-dual Nesterov’s accelerated method and gives its convergence properties. Based on the proposed method, Section IV further designs two primal-dual Nesterov’s accelerated algorithms for two widely studied network optimization problems and presents the simulation results. Section V concludes this paper.

II. MATHEMATICAL PRELIMINARY AND PROBLEM SETUP

In this section, we introduce some mathematical notations and give the problem statement.

A. Notation

The symbol \( \mathbb{R} \) denotes the set of real numbers; \( \mathbb{R}^n \) denotes the set of \( n \)-dimensional real column vectors; \( \mathbb{R}^{n \times m} \) denotes the set of \( n \)-by-\( m \) real matrices; \( I_n \) denotes the \( n \times n \) identity matrix; \( (\cdot)^\top \) denotes transpose. We write \( \text{rank}(A) \) for the rank of the matrix \( A \), \( \text{range}(A) \) for the range of the matrix \( A \), \( \ker(A) \) for the kernel of the matrix \( A \), \( 1_n \) for the \( n \times 1 \) ones vector, \( 0_n \) for the \( n \times 1 \) zeros vector, and \( A \otimes B \) for the Kronecker product of matrices \( A \) and \( B \). Furthermore, \( \| \cdot \| \) denotes the Euclidean norm; \( A > 0 \) (\( A \geq 0 \)) denotes that matrix \( A \in \mathbb{R}^{n \times n} \) is positive definite (positive semi-definite); Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous function. \( f(t) = O\left(\frac{1}{t^n}\right) \) indicates that there exist constants \( C > 0 \) and \( t_0 \geq 0 \) such that \( f(t) \leq \frac{C}{t^n} \) for all \( t \geq t_0 \).

A set \( \Omega \) is convex if \( \lambda z_1 + (1 - \lambda)z_2 \in \Omega \) for all \( z_1, z_2 \in \Omega \) and \( \lambda \in [0, 1] \). A function \( f : \Omega \to \mathbb{R} \) is convex (strictly convex) if \( f(\lambda z_1 + (1 - \lambda)z_2) \leq (\prec \lambda f(z_1) + (1 - \lambda)f(z_2) \) for any \( z_1, z_2 \in \Omega \), \( z_1 \neq z_2 \), and \( \lambda \in (0, 1) \). If function \( f : \Omega \to \mathbb{R} \) is (strictly) convex, it is well-known that \( (z_1 - z_2)^\top (\nabla f(z_1) - \nabla f(z_2)) \geq (\succ 0 \) for any \( z_1, z_2 \in \Omega \) and \( z_1 \neq z_2 \). Given a differentiable function \( f(x, y) \), \( \nabla_x f(x, y) \) denotes the partial gradient of function \( f(x, y) \) with respect to \( x \).

An undirected graph \( G \) is denoted by \( G(V, E, A) \), where \( V = \{1, \ldots, n\} \) is a set of nodes, \( E \subset V \times V \) is a set of edges, and \( A = [a_{i,j}] \in \mathbb{R}^{n \times n} \) is an adjacency matrix such that \( a_{i,j} = a_{j,i} > 0 \)
if \((j, i) \in E\) and \(a_{i,j} = 0\) otherwise. The Laplacian matrix is \(L_n = D - A\), where \(D \in \mathbb{R}^{n \times n}\) is diagonal with \(D_{i,i} = \sum_{j=1}^{n} a_{i,j}, i \in \{1, \ldots, n\}\). If the graph \(G\) is undirected and connected, then \(L_n = L_n^T \geq 0\), \(\text{rank}(L_n) = n - 1\) and \(\ker(L_n) = \{k1_n : k \in \mathbb{R}\}\).

### B. Problem Formulation

Consider a convex optimization problem with an affine equality constraint given by

\[
\min_{x \in \mathbb{R}^q} \phi(x), \quad \text{s.t.} \ A x - b = 0_m, \tag{1}
\]

where \(A \in \mathbb{R}^{m \times q}\), \(b \in \mathbb{R}^m\), and \(\phi : \mathbb{R}^q \to \mathbb{R}\) is a convex and twice differentiable cost function.

It follows from the Karush-Kuhn-Tucker (KKT) optimality condition (see [38, Theorem 3.34]) that \(x^* \in \mathbb{R}^q\) is a solution to the problem (1) if and only if there exists \(\lambda^* \in \mathbb{R}^m\) such that

\[
0_q = \nabla \phi(x^*) + A^\top \lambda^*, \tag{2a}
\]
\[
0_m = A x^* - b. \tag{2b}
\]

Define the augmented Lagrangian function \(L : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}\) as

\[
L(x, \lambda) = \phi(x) + \lambda^\top (A x - b) + \frac{1}{2} \|A x - b\|^2. \tag{3}
\]

It is well-known that \((x^*, \lambda^*) \in \mathbb{R}^q \times \mathbb{R}^m\) satisfies (2) if and only if \((x^*, \lambda^*) \in \mathbb{R}^q \times \mathbb{R}^m\) is a saddle point of (3), that is,

\[
L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda), \quad \forall (x, \lambda) \in \mathbb{R}^q \times \mathbb{R}^m.
\]

Given a continuous-time algorithm, the rate of convergence of the algorithm is said to be \(O(\frac{1}{t^p})\) if the duality gap satisfies \(L(x(t), \lambda^*) - L(x^*, \lambda^*) = O(\frac{1}{t^p})\), where \(p > 0\). This paper aims to design a primal-dual accelerated method that has a convergence rate faster than \(O(\frac{1}{t})\), the rate of first-order primal-dual methods for convex optimization.

Furthermore, the formulation (1) captures two important scenarios of network optimization problems.

**Scenario 1: Distributed Optimization with Consensus Constraints.** Consider a network of \(n\) agents interacting over a graph \(G\). The distributed agents cooperate to solve the following problem

\[
\min_{x \in \mathbb{R}^{nq}} \ f(x), \quad f(x) = \sum_{i=1}^{n} f_i(x_i), \tag{4a}
\]
\[
\text{s.t.} \ x_i = x_j, \quad i,j \in \{1, \ldots, n\}, \tag{4b}
\]
where agent \( i \) only knows its local cost function \( f_i : \mathbb{R}^q \to \mathbb{R} \) and the shared information of its neighbors through local communications.

Problem (4) is a widely investigated model that has many applications such as optimal consensus of agents [1] and distributed machine learning [39].

Scenario 2: Distributed Extended Monotropic Optimization. Given a network \( G \) composed of \( n \) agents, the distributed extended monotropic optimization problem is

\[
\begin{aligned}
\min_{y \in \mathbb{R}^q} h(y), \quad h(y) &= \sum_{i=1}^n h_i(y_i), \\
\text{s.t.} \quad W y &= \sum_{i=1}^n W_i y_i = \sum_{i=1}^n d_i = d_0, \quad i \in \{1, \ldots, n\},
\end{aligned}
\]  

(5a)

where \( y_i \in \mathbb{R}^{q_i}, \ y \triangleq [y_1^\top, \ldots, y_n^\top]^\top \in \mathbb{R}^q \) with \( q \triangleq \sum_{i=1}^n q_i, \ W_i \in \mathbb{R}^{m \times q_i}, \ d_0, d_i \in \mathbb{R}^m, \) and \( W = [W_1, \ldots, W_n] \in \mathbb{R}^{m \times q} \). In this problem, agent \( i \) has its state \( y_i \in \mathbb{R}^{q_i} \), objective function \( h_i(y_i) \), constraint matrix \( W_i \in \mathbb{R}^{m \times q_i} \), information from neighboring agents, and a vector \( d_i \in \mathbb{R}^m \) such that \( \sum_{i=1}^n d_i = d_0 \).

Problem (5) covers many network optimization problems such as resource allocation problems and network flow problems [2], [3], [9]. Different from problem (4), \( y_i \)'s appear in the same constraint (5b).

To ensure the wellposedness of problems (4) and (5), the following assumption is needed.

**Assumption 2.1:**

1) Graph \( G \) is connected and undirected.
2) There exists at least one finite solution to problems (4) and (5).
3) Function \( f_i(\cdot) \) in problem (4) \( (h_i(\cdot) \) in problem (5)) is convex and twice continuously differentiable.

### III. PRIMAL-DUAL NESTEROV’S ACCELERATED METHOD

In this section, we propose a dynamical primal-dual Nesterov’s accelerated method for problem (1). We then investigate convergence properties of the proposed method and analyze critical values of algorithm parameters.

**A. Algorithm Design**

We propose a primal-dual accelerated method as follows:

\[
\ddot{x}(t) = -\alpha t \dot{x}(t) - \nabla \phi(x(t)) - A^\top(\lambda(t) + \beta t \dot{\lambda}(t)) - A^\top(A x(t) - b),
\]  

(6a)
\[ \ddot{\lambda}(t) = -\frac{\alpha}{t} \dot{\lambda}(t) + A(x(t) + \beta t \dot{x}(t)) - b, \]

(6b)

where \( t_0 > 0, \alpha > 3, \beta = \frac{1}{2}, x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0, \lambda(t_0) = \lambda_0, \) and \( \dot{\lambda}(t_0) = \dot{\lambda}_0. \) In the remaining of this paper, we omit \((t)\) in the algorithm and analysis without causing confusions. For example, we use \( x \) and \( \dot{x} \) to denote \((x(t), \dot{x}(t))\). Then with the definition of \( L(x, \lambda) \) in (3), the algorithm (6) can be written as

\[
\begin{align*}
\dot{x} &= -\frac{\alpha}{t} \dot{x} - \nabla_x L(x, \lambda + \beta t \dot{\lambda}), \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \\
\dot{\lambda} &= -\frac{\alpha}{t} \dot{\lambda} + \nabla_\lambda L(x + \beta t \dot{x}, \lambda), \quad \lambda(t_0) = \lambda_0, \quad \dot{\lambda}(t_0) = \dot{\lambda}_0.
\end{align*}
\]

Remark 3.1: Since \( \phi(\cdot) \) is twice differentiable, \( \nabla \phi(\cdot) \) is locally Lipschitz continuous. It follows from [40] Theorem 2.38, pp. 96 that algorithm (6) has a unique trajectory. The initial time \( t_0 > 0 \) avoids the singularity of the damping coefficient \( \frac{\alpha}{t} \) at zero. Although algorithm (6) uses \( t \dot{x}(t) \) and \( t \dot{\lambda}(t) \) in the righthand side, Section III-B will show that \( t \dot{x}(t) \) and \( t \dot{\lambda}(t) \) are bounded for all \( t \geq t_0 \). Hence, algorithm (6) is well defined with a bounded right-hand side.

Remark 3.2: The use of derivative information \( \beta t \dot{x}(t) \) and \( \beta t \dot{\lambda}(t) \) in the update (6) can be interpreted from the following perspectives. From the control perspective, it may be viewed as a “derivative feedback” design and plays a role as damping terms. From the optimization perspective, \( \beta t \dot{x}(t) \) and \( \beta t \dot{\lambda}(t) \) point to the future moving direction of \( x(t) \) and \( \lambda(t) \). Thus, algorithm (6) uses the estimated “future” position \( x(t) + \beta t \dot{x}(t) \) and \( \lambda(t) + \beta t \dot{\lambda}(t) \).

Remark 3.3: Without loss of generality, the order for the rate of convergence of algorithm (6) remains unchanged if the right-hand side of (6) is multiplied by any positive constant gain. However, one should be very careful choosing time-varying gains because an infinitely large increasing gain may produce unbounded variable derivatives and make algorithms impractical.

B. Convergence Analysis for \( \alpha > 3 \) and \( \beta = \frac{1}{2} \)

In this subsection, we investigate the convergence results of the algorithm (6) when \( \alpha > 3 \) and \( \beta = \frac{1}{2} \). To be specific, the following theorem shows that algorithm (6) has convergence rate \( O(\frac{1}{t^2}) \).

Theorem 3.1: Suppose the problem (1) has a nonempty solution set \( S \). Let \((x(t), \lambda(t))\) be a trajectory generated by the algorithm (6) with \( \alpha > 3 \) and \( \beta = \frac{1}{2} \). Then we have the following.

(i) Trajectories of \((x(t), \lambda(t), t \dot{x}(t), t \dot{\lambda}(t))\) and \((\dot{x}(t), \dot{\lambda}(t))\) are bounded for \( t \geq t_0 \).
(ii) The trajectory \((x(t), \lambda(t), \dot{x}(t), \dot{\lambda}(t))\) satisfies the convergence properties \(L(x(t), \lambda^*) - L(x^*, \lambda^*) = O\left(\frac{1}{t^2}\right), \|Ax(t) - b\|^2 = O\left(\frac{1}{t}\right), \|\dot{x}(t)\| = O\left(\frac{1}{t}\right), \text{and } \|\dot{\lambda}(t)\| = O\left(\frac{1}{t}\right)\).

(iii) The trajectory of \(x(t)\) converges to the solution set \(S\) as \(t \to \infty\).

Proof: Let \((x^*, \lambda^*) \in \mathbb{R}^q \times \mathbb{R}^m\) satisfy (2). Define function
\[
V(t, x, \lambda, \dot{x}, \dot{\lambda}) = V_1(t, x) + V_2(t, x, \dot{x}) + V_3(t, \lambda, \dot{\lambda})
\]
such that
\[
\begin{align*}
V_1 &= t^2[L(x, \lambda^*) - L(x^*, \lambda^*)], \\
V_2 &= 2\|x + \beta t \dot{x} - x^*\|^2 + 2(\alpha \beta - \beta - 1)\|x - x^*\|^2, \\
V_3 &= 2\|\lambda + \beta t \dot{\lambda} - \lambda^*\|^2 + 2(\alpha \beta - \beta - 1)\|\lambda - \lambda^*\|^2,
\end{align*}
\]
where \(L(\cdot, \cdot)\) is defined in (3). By the property of saddle points of \(L(\cdot, \cdot)\), \(L(x, \lambda^*) \geq L(x^*, \lambda^*)\) for all \(x \in \mathbb{R}^q\). Hence, function \(V\) is positive definite with respect to \((x, \lambda, t\dot{x}, t\dot{\lambda})\) for all \(t \geq t_0\).

(i) The derivatives of \(V_i\)'s \(i = 1, 2, 3\), along the trajectory of algorithm (6) satisfy that
\[
\begin{align*}
\dot{V}_1 &= 2t[\phi(x) - \phi(x^*) + \lambda^T A(x - x^*) + \frac{1}{2}\|Ax - b\|^2] \\
&\quad + t^2[\nabla \phi(x) + A^T \lambda^* + A^T (Ax - b)]^T \dot{x}, \\
\dot{V}_2 &= 4(x + \beta t \dot{x} - x^*)^T((1 + \beta) \dot{x} + \beta t \ddot{x}) + 4(\alpha \beta - \beta - 1)(x - x^*)^T \dot{x} \\
&\quad - 4\beta t(x - x^*)^T(\nabla \phi(x) + A^T \lambda) - 4\beta t\|Ax - b\|^2 - 4\beta^2 t^2(x - x^*)^T A^T \dot{\lambda} \\
&\quad + 4\beta(1 + \beta - \alpha \beta)\|\dot{x}\|^2 + 4\beta^2 t^2 \dot{x}^T \nabla \phi(x) + A^T \lambda) - 4\beta^3 t^3 \dot{x}^T A^T \dot{\lambda} \\
&\quad - 4\beta^2 t^2 \dot{x}^T A^T (Ax - b), \\
\dot{V}_3 &= 4(\lambda + \beta t \dot{\lambda} - \lambda^*)^T((1 + \beta) \dot{\lambda} + \beta t \ddot{\lambda}) + 4(\alpha \beta - \beta - 1)(\lambda - \lambda^*)^T \dot{\lambda} \\
&\quad - 4\beta t(\lambda - \lambda^*)^T (Ax - b) + 4\beta^2 t^2(\lambda - \lambda^*)^T A \dot{x} + 4\beta(1 + \beta - \alpha \beta) t\|\dot{\lambda}\|^2 \\
&\quad + 4\beta^2 t^2 \dot{\lambda}^T (Ax - b) + 4\beta^3 t^3 \dot{x}^T A^T \dot{\lambda}.
\end{align*}
\]

Plug (2) in (12) and (13), and rearrange the terms. We have
\[
\begin{align*}
\dot{V}_2 + \dot{V}_3 &= -4\beta t((x - x^*)^T(\nabla \phi(x) + A^T \lambda^*) + \frac{1}{2}\|Ax - b\|^2) \\
&\quad - 4\beta^2 t^2 \dot{x}^T(\nabla \phi(x) + A^T \lambda^* + A^T (Ax - b)) + N
\end{align*}
\]
with \(N \triangleq 4\beta(1 + \beta - \alpha \beta) t\|\dot{x}\|^2 + 4\beta(1 + \beta - \alpha \beta) t\|\dot{\lambda}\|^2 - 2\beta t\|Ax - b\|^2\).
Plugging $\beta = \frac{1}{2}$ in (14), it follows from (11) and (14) that

$$V = 2t[\phi(x) - \phi(x^*) - (x - x^*)^T \nabla \phi(x)] + N,$$

where $N = (3 - \alpha)t\|\dot{x}\|^2 + (3 - \alpha)t\|\dot{\lambda}\|^2 - t\|Ax - b\|^2$.

Because $\phi(\cdot)$ is convex, it is clear that $\phi(x) - \phi(x^*) - (x - x^*)^T \nabla \phi(x) \leq 0$. It follows from (15) that

$$\dot{V} \leq (3 - \alpha)t\|\dot{x}\|^2 + (3 - \alpha)t\|\dot{\lambda}\|^2 - t\|Ax - b\|^2 \leq 0,$$

which means that $V(t, x, \lambda, \dot{x}, \dot{\lambda})$ is non-increasing on $[t_0, \infty)$. Hence,

$$V_2(t, x, \dot{x}) \leq V(t, x, \lambda, \dot{x}, \dot{\lambda}) \leq V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0) \triangleq V_0.$$

Recall the definition of $V_2(\cdot)$. It follows that $\|x + \beta t\dot{x} - x^*\| \leq \sqrt{\frac{V_0}{2}}$ and $\|x - x^*\| \leq \sqrt{\frac{V_0}{2(\alpha\beta - \beta - 1)}}$ for all $t \geq t_0$. Clearly, $x(t)$ is bounded for all $t \geq t_0$. In addition, it follows from the triangle inequality that $\|\beta t\dot{x}\| \leq \|x + \beta t\dot{x} - x^*\| + \|x^* - x\| \leq \sqrt{\frac{V_0}{2}} + \sqrt{\frac{V_0}{2(\alpha\beta - \beta - 1)}}$ for all $t \geq t_0$. Hence, $\|t\dot{x}\|$ is bounded for all $t \geq t_0$. Similarly, one can prove that $\|\lambda\|$ and $\|t\dot{\lambda}\|$ are bounded for all $t \geq t_0$. To sum up, the trajectory of $(x(t), \lambda(t), t\dot{x}(t), t\dot{\lambda}(t))$ is bounded for $t \geq t_0$. By (6), $(\ddot{x}(t), \ddot{\lambda}(t))$ is bounded for $t \geq t_0$.

(ii) Since $\dot{V}(t, x, \lambda, \ddot{x}, \ddot{\lambda}) \leq 0$, then $V(t, x(t), \lambda(t), \ddot{x}(t), \ddot{\lambda}(t)) \leq V(t_0, x_0, \lambda_0, \ddot{x}_0, \ddot{\lambda}_0) \triangleq V_0$. It follows from the definition of $V(\cdot)$ that

$$t^2(L(x(t), \lambda^*) - L(x^*, \lambda^*)) = V_1(t, x) \leq V(t, x(t), \lambda(t), \ddot{x}(t), \ddot{\lambda}(t)) \leq V_0.$$}

Hence, $L(x(t), \lambda^*) - L(x^*, \lambda^*) = O\left(\frac{1}{t^2}\right)$. Due to the convexity of $\phi(\cdot)$ and (2a),

$$\phi(x) - \phi(x^*) + \lambda^T A(x - x^*) \geq (x - x^*)^T (\nabla \phi(x^*) + A^T \lambda^*) = 0.$$

Then it follows from (3) and $L(x(t), \lambda^*) - L(x^*, \lambda^*) = O\left(\frac{1}{t^2}\right)$ that $\frac{1}{2}\|Ax(t) - b\|^2 \leq L(x(t), \lambda^*) - L(x^*, \lambda^*)$ and $\|Ax(t) - b\|^2 = O\left(\frac{1}{t^2}\right)$. In addition, since we have proved the boundedness of $t\dot{x}(t)$ and $t\dot{\lambda}(t)$, it is clear that $\|\dot{x}(t)\| = O\left(\frac{1}{t}\right)$ and $\|\dot{\lambda}(t)\| = O\left(\frac{1}{t}\right)$.

(iii) By part i), $(x(t), \lambda(t), t\dot{x}(t), t\dot{\lambda}(t))$ is bounded in a compact set for $t \geq t_0$. Because $\|Ax(t) - b\|^2 \to 0$ as $t \to \infty$, $x(t)$ converges to the set of feasible points of problem (1). Now, suppose, ad absurdum, that $x(t)$ does not converge to $S$ as $t \to \infty$. Recall that $x(t)$ stays in a compact set for all $t \geq t_0$. There exists an unbounded positive sequence $\{t_k\} \subset [t_0, \infty)$ such that $\lim_{k \to \infty} x(t_k)$ exists and $\text{dist}(\lim_{k \to \infty} x(t_k), S) = \lim_{k \to \infty} \text{dist}(x(t_k), S) \geq \epsilon > 0$. Since $\|Ax(t) - b\|^2 \to 0$ and $L(x(t), \lambda^*) - L(x^*, \lambda^*) \to 0$ as $t \to \infty$,

$$\lim_{k \to \infty} \phi(x(t_k)) - \phi(x^*) = \phi(\lim_{k \to \infty} x(t_k)) - \phi(x^*)$$
\[ L(\lim_{k \to \infty} x(t_k), \lambda^*) - L(x^*, \lambda^*) = 0, \]

which contradicts \( \lim_{k \to \infty} \text{dist}(x(t_k), S) \geq \epsilon > 0 \). Thus, \( x(t) \) converges to the solution set \( S \) as \( t \to \infty \).

**Remark 3.4:** Main challenges of proving Theorem 3.1 are twofold. Firstly, one needs to find a suitable Lyapunov function to prove the convergence properties of algorithm (6) when \( \alpha > 3 \). In the proof of Theorem 3.1, the augmented Lagrangian function and quadratic functions are elegantly combined to overcome this challenge. Secondly, one needs to find suitable choices for \( \beta \) when \( \alpha > 3 \). In the proof of Theorem 3.1, \( \beta = \frac{1}{2} \) is used to prove (15). In fact, by comparing parameters in (11) and (14), one might find that \( \beta = \frac{1}{2} \) is the only choice to obtain the convergence proof.

One should note that the \( O(\frac{1}{t^2}) \) convergence rate is not tight for strongly convex optimization.

Suppose \( \phi \) is a strongly convex or quadratic convex function. The primal-dual dynamics has a linear rate, which is faster than \( O(\frac{1}{t^2}) \).

**C. Analysis of** \( 0 < \alpha \leq 3 \) **and** \( \beta = \frac{3}{2\alpha} \)

We proceed to discuss the case with algorithm parameters \( 0 < \alpha \leq 3 \) and \( \beta = \frac{3}{2\alpha} \), and show in the following theorem that the best rate of convergence is \( O(t^{-\frac{2\alpha}{\alpha}}) \).

**Theorem 3.2:** Denote \( S \) as the set of solutions to problem (1) and assume \( S \neq \emptyset \). Let \((x(t), \lambda(t))\) be a trajectory of algorithm (6), where \( 0 < \alpha \leq 3 \) and \( \beta = \frac{3}{2\alpha} \).

\( \text{(i)} \) The trajectory of \((t^{\frac{\alpha}{4}}x(t), t^{\frac{\alpha}{4}}\lambda(t), t^{\frac{\alpha}{4}}\dot{x}(t), t^{\frac{\alpha}{4}}\dot{\lambda}(t))\) is bounded for \( t \geq t_0 \) and \( 0 < \alpha < 3 \).

\( \text{(ii)} \) The trajectory \((x(t), \lambda(t), \dot{x}(t), \dot{\lambda}(t))\) satisfies the convergence properties \( L(x(t), \lambda^*) - L(x^*, \lambda^*) = O(t^{-\frac{2\alpha}{\alpha}}) \) and \( \|Ax(t) - b\|^2 = O(t^{-\frac{2\alpha}{\alpha}}) \). If, in addition, \( 0 < \alpha < 3 \), then \( \|\dot{x}(t)\| = O(t^{-\frac{\alpha}{2}}) \) and \( \|\dot{\lambda}(t)\| = O(t^{-\frac{\alpha}{2}}) \).

\( \text{(iii)} \) If, in addition, the trajectory of \( x(t) \) is bounded, then \( x(t) \) converges to the solution set \( S \) as \( t \to \infty \).

**Proof:** To show that the rate of convergence is \( O(t^{-\frac{2\alpha}{\alpha}}) \), we define scalar \( p \in (0, 1] \) and functions \( \theta : [t_0, \infty) \to \mathbb{R}_+ \) and \( \eta : [t_0, \infty) \to \mathbb{R}_+ \). Define the function

\[ V(t, x, \lambda, \dot{x}, \dot{\lambda}) = V_1(t, x) + V_2(t, x, \dot{x}) + V_3(t, \lambda, \dot{\lambda}) \]

such that

\[ V_1 = t^{2p}[L(x, \lambda^*) - L(x^*, \lambda^*)], \]
\[ V_2 = 0.5\|\theta(t)(x - x^*) + tp^*\hat{x}\|^2 + \eta(t)/2\|x - x^*\|^2, \]
\[ V_3 = 0.5\|\theta(t)(\lambda - \lambda^*) + t^p\hat{\lambda}\|^2 + \eta(t)/2\|\lambda - \lambda^*\|^2, \]

where \(L(\cdot, \cdot)\) is defined in (3) and \((x^*, \lambda^*)\) satisfies (2).

(i) It follows from (2) and (6) that derivatives of \(V_i\)'s along the trajectory of algorithm (6) satisfy

\[
\dot{V}_1 = 2pt^{2p-1}[\phi(x) - \phi(x^*) + \lambda^\top A(x - x^*) + \frac{1}{2}\|Ax - b\|^2] \\
+ t^{2p}[\nabla \phi(x) + A^\top \lambda^* + A^\top (Ax - b)]^\top \hat{x},
\]

\[
\dot{V}_2 = (\theta(t)(x - x^*) + t^p\hat{x})^\top \left(\dot{\theta}(t)(x - x^*) + (\theta(t) + pt^{p-1})\hat{x} + t^p\hat{x}\right) \\
+ \dot{\eta}(t)/2\|x - x^*\|^2 + \eta(t)(x - x^*)^\top \hat{x} \\
= (\theta(t)(x - x^*) + t^p\hat{x})^\top \left(\dot{\theta}(t)(x - x^*) + (\theta(t) + (p - \alpha)t^{p-1})\hat{x} \\
+ t^p(-\nabla \phi(x) - A^\top (\lambda^* + \beta t\hat{\lambda}) - A^\top (Ax - b))\right) \\
+ \dot{\eta}(t)/2\|x - x^*\|^2 + \eta(t)(x - x^*)^\top \hat{x} \\
= \left(\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2}\right)(x - x^*)^2 + \left(\theta^2(t) + (p - \alpha)\theta(t)t^{p-1} + t^p\dot{\theta}(t) + \eta(t)\right)(x - x^*)^\top \hat{x} \\
- \theta(t)t^p(x - x^*)^\top (\nabla \phi(x) - \nabla \phi(x^*)) - \theta(t)t^p(x - x^*)^\top A^\top (\lambda - \lambda^*) - \theta(t)t^p\|Ax - b\|^2 \\
- \theta(t)\beta t^{p+1}(x - x^*)^\top A^\top \dot{\lambda} + t^p(\theta(t) + (p - \alpha)t^{p-1})\hat{x}^2 - t^{2p}(\nabla \phi(x) - \nabla \phi(x^*))^\top \hat{x} \\
- t^{2p}\dot{\lambda}^\top A^\top (\lambda - \lambda^* - \beta t^{2p+1}\hat{x}^\top A^\top \dot{\lambda} - t^{2p}\dot{x}^\top A^\top (Ax - b));
\]

\[
\dot{V}_3 = (\theta(t)(\lambda - \lambda^*) + t^p\dot{\lambda})^\top \left(\dot{\theta}(t)(\lambda - \lambda^*) + (\theta(t) + pt^{p-1})\dot{\lambda} + t^p\dot{\lambda}\right) \\
+ \dot{\eta}(t)/2\|\lambda - \lambda^*\|^2 + \eta(t)(\lambda - \lambda^*)^\top \dot{\lambda} \\
= (\theta(t)(\lambda - \lambda^*) + t^p\dot{\lambda})^\top \left(\dot{\theta}(t)(\lambda - \lambda^*) + (\theta(t) + (p - \alpha)t^{p-1})\dot{\lambda} + t^p(A(x + \beta t\hat{x}) - b)\right) \\
+ \dot{\eta}(t)/2\|\lambda - \lambda^*\|^2 + \eta(t)(\lambda - \lambda^*)^\top \dot{\lambda} \\
= \left(\theta(t)\dot{\theta}(t) + \frac{\dot{\eta}(t)}{2}\right)(\lambda - \lambda^*)^2 + \left(\theta^2(t) + (p - \alpha)\theta(t)t^{p-1} + t^p\dot{\theta}(t) + \eta(t)\right)(\lambda - \lambda^*)^\top \dot{\lambda} \\
+ \theta(t)t^p(\lambda - \lambda^*)^\top A(x - x^*) + \theta(t)\beta t^{p+1}(\lambda - \lambda^*)^\top A\hat{x} \\
+ t^p(\theta(t) + (p - \alpha)t^{p-1})\dot{\lambda}^2 + t^{2p}\dot{\lambda}^\top A(x - x^*) + \beta t^{2p+1}\dot{\lambda}^\top A\hat{x}.\]
Summing (18)-(20) and rearranging terms, it follows from (2) and (18)-(20) that \( \dot{V}(t, x, \lambda, \dot{x}, \dot{\lambda}) = \sum_{i=1}^{5} M_i \), where

\[
M_1 = 2pt^{2p-1}[\phi(x) - \phi(x^*) + \lambda^T A(x - x^*) + \frac{1}{2} \|Ax - b\|^2] - \theta(t)t^p(x - x^*)^T(\nabla \phi(x) - \nabla \phi(x^*)) - \theta(t)t^p\|Ax - b\|^2,
\]

\[
M_2 = \left( \theta(t)\dot{\theta}(t) + \frac{\dot{\theta}(t)}{2} \right) (x - x^*)^2 + \left( \theta(t)\dot{\theta}(t) + \frac{\dot{\theta}(t)}{2} \right) (\lambda - \lambda^*)^2,
\]

\[
M_3 = \left( \theta^2(t) + (p - \alpha)\theta(t)t^{p-1} + t^p\dot{\theta}(t) + \eta(t) \right) (x - x^*)^T \dot{x}
\]

\[
+ \left( \theta^2(t) + (p - \alpha)\theta(t)t^{p-1} + t^p\dot{\theta}(t) + \eta(t) \right) (\lambda - \lambda^*)^T \dot{\lambda},
\]

\[
M_4 = t^p(\theta(t) + (p - \alpha)t^{p-1})\dot{x}^2 + t^p(\theta(t) + (p - \alpha)t^{p-1})\dot{\lambda}^2,
\]

\[
M_5 = (t^{2p} - \theta(t)\beta t^{p+1})(x - x^*)^T A^T \dot{\lambda} + (\theta(t)\beta t^{p+1} - t^{2p})\dot{x}^T A^T (\lambda - \lambda^*).
\]

Since \( \nabla \phi(x^*) = -A^T \lambda^* \) by (2a), \( M_1 \) can be rewritten as:

\[
M_1 = 2pt^{2p-1} \left[ \phi(x) - \phi(x^*) - \frac{\theta(t)}{2p} t^{1-p}(x - x^*)^T \nabla \phi(x) + (1 - \frac{\theta(t)}{2p} t^{1-p}) \lambda^T A(x - x^*) + \frac{1}{2} \frac{\theta(t)}{2p} t^{1-p} \|Ax - b\|^2 \right].
\]

By considering \( M_1-M_5 \), one can verify that \( \dot{V}(t, x, \lambda, \dot{x}, \dot{\lambda}) \leq 0 \) if

\[
\frac{\theta(t)}{2p} t^{1-p} = 1, \quad (21)
\]

\[
\theta(t)\dot{\theta}(t) + \frac{\dot{\theta}(t)}{2} \leq 0, \quad (22)
\]

\[
\theta^2(t) + (p - \alpha)\theta(t)t^{p-1} + t^p\dot{\theta}(t) + \eta(t) = 0, \quad (23)
\]

\[
\theta(t) + (p - \alpha)t^{p-1} \leq 0, \quad (24)
\]

\[
t^{2p} - \theta(t)\beta t^{p+1} = 0. \quad (25)
\]

Next, we seek feasible choices of functions \((\theta(\cdot), \eta(\cdot))\) and the parameter \(p\) that satisfy (21)-(25). Clearly, (21) implies that

\[
\theta(t) = 2pt^{p-1}. \quad (26)
\]

Plugging (26) in (23)-(25), we have

\[
\eta(t) = (-2 + \frac{\alpha + 1}{2p})\theta^2(t), \quad (27)
\]

\[
p \leq \frac{\alpha}{3}, \text{ and } \beta = \frac{1}{2p}. \quad (28)
\]
Plugging (26) and (27) in (22) gives that
\[-1 + \frac{\alpha + 1}{2p}p - 1 \leq 0.\]  
(29)

Since \(\eta(t) \geq 0\), (27) implies that \(p \leq \frac{1 + \alpha}{4}\). This combined with (28), (29), and \(p \in (0, 1]\) proves that \(0 < p \leq \min\{1, \frac{\alpha}{3}, \frac{1 + \alpha}{4}\} = \frac{\alpha}{3}\).

Choose \(p = \frac{\alpha}{3}\), \(\theta(t) = \frac{2\alpha}{3} t_{\frac{\alpha}{3}}^{-1}\), and \(\eta(t) = \frac{2(3 - \alpha)\alpha}{9} t_{\frac{\alpha}{3}}^{-2}\). Then \(M_1 - M_5\) can be simplified as
\[
\begin{align*}
M_1 &= \frac{2\alpha}{3} t_{\frac{\alpha}{3}}^{-1}[\phi(x) - \phi(x^*) - (x - x^*)^\top \nabla \phi(x)] \leq 0, \\
M_2 &= \frac{2}{27} \alpha (\alpha^2 - 9) t_{\frac{\alpha}{3}}^{-3} [(x - x^*)^2 + (\lambda - \lambda^*)^2] \leq 0,
\end{align*}
\]
\(M_3 = M_4 = M_5 = 0\), and hence
\[
\dot{V} \leq \frac{2\alpha}{3} t_{\frac{\alpha}{3}}^{-1}[\phi(x) - \phi(x^*) - (x - x^*)^\top \nabla \phi(x)] \\
+ \frac{2}{27} \alpha (\alpha^2 - 9) t_{\frac{\alpha}{3}}^{-3} [(x - x^*)^2 + (\lambda - \lambda^*)^2] \leq 0.
\]

By plugging \(p = \frac{\alpha}{3}\), \(\theta(t) = \frac{2\alpha}{3} t_{\frac{\alpha}{3}}^{-1}\), and \(\eta(t) = \frac{2(3 - \alpha)\alpha}{9} t_{\frac{\alpha}{3}}^{-2}\) into \(V_2(t, x, \dot{x})\), we have
\[
\begin{align*}
V_2 &= \frac{1}{2} \left\| \frac{2\alpha}{3} t_{\frac{\alpha}{3}}^{-1} (x - x^*) + t_{\frac{\alpha}{3}} \dot{x} \right\|^2 + \frac{(3 - \alpha)\alpha}{9} t_{\frac{\alpha}{3}}^{-2} \left\| x - x^* \right\|^2 \\
&= \left\| \frac{\sqrt{\alpha^2 + 3\alpha}}{3} t_{\frac{\alpha}{3}}^{-1} (x - x^*) + \sqrt{\frac{\alpha}{\alpha + 3}} t_{\frac{\alpha}{3}} \dot{x} \right\|^2 + \frac{3 - \alpha}{2(\alpha + 3)} t_{\frac{\alpha}{3}}^{-2} \left\| \dot{x} \right\|^2.
\end{align*}
\]

Since \(V_1(t, x) \geq 0\) and function \(V_2\) is positive definite with respect to \((t_{\frac{\alpha}{3}}^{-1} x, t_{\frac{\alpha}{3}}^{-1} \lambda, t_{\frac{\alpha}{3}} \dot{x}, t_{\frac{\alpha}{3}} \dot{\lambda})\) for all \(t \geq t_0\) and \(0 < \alpha < 3\). There exists a class \(K\) function \(\kappa(\cdot)\) that \(V(t, x, \lambda, \dot{x}, \dot{\lambda}) \geq \kappa(t_{\frac{\alpha}{3}}^{-1} x, t_{\frac{\alpha}{3}}^{-1} \lambda, t_{\frac{\alpha}{3}} \dot{x}, t_{\frac{\alpha}{3}} \dot{\lambda})\). It follows from \(\dot{V} \leq 0\) that the trajectory of \((t_{\frac{\alpha}{3}}^{-1} x(t), t_{\frac{\alpha}{3}}^{-1} \lambda(t), t_{\frac{\alpha}{3}} \dot{x}(t), t_{\frac{\alpha}{3}} \dot{\lambda}(t))\) is bounded for \(t \geq t_0\) and \(0 < \alpha < 3\).

(ii) Note that \(V_2(t, x(t), \dot{x}(t)) \leq V(t, x(t), \lambda(t), \dot{x}(t), \dot{\lambda}(t)) \leq V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\) for \(t \geq t_0\).

It follows that for \(t \geq t_0\), \(\|\dot{x}(t)\|^2 \leq \frac{2(\alpha + 3)}{3 - \alpha} \frac{1}{t_{\frac{\alpha}{3}}} V_2(t, x(t), \dot{x}(t)) \leq \frac{2(\alpha + 3)}{3 - \alpha} \frac{1}{t_{\frac{\alpha}{3}}} V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\). Similarly, for \(t \geq t_0\), \(\|\dot{\lambda}(t)\|^2 \leq \frac{2(\alpha + 3)}{3 - \alpha} \frac{1}{t_{\frac{\alpha}{3}}} V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\). We have \(\|\dot{x}(t)\| = O(t^{-\frac{\alpha}{3}})\) and \(\|\dot{\lambda}(t)\| = O(t^{-\frac{\alpha}{3}})\).

As a result, \(V(t, x(t), \lambda(t), \dot{x}(t), \dot{\lambda}(t)) \leq V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\) for all \(t \geq t_0\). Because \(L(x(t), \lambda^*) - L(x^*, \lambda^*) = t^{-\frac{2\alpha}{3}} V_1(t, x(t))\) and \(V_1(t, x(t)) \leq V(t, x(t), \lambda(t), \dot{x}(t), \dot{\lambda}(t)) \leq V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\) for \(t \geq t_0\), we have \(L(x(t), \lambda^*) - L(x^*, \lambda^*) \leq t^{-\frac{2\alpha}{3}} V(t_0, x_0, \lambda_0, \dot{x}_0, \dot{\lambda}_0)\) and \(L(x(t), \lambda^*) - L(x^*, \lambda^*) = O(t^{-\frac{2\alpha}{3}})\). From the definition of \(L(\cdot, \cdot)\) in (3), it is clear that \(\frac{1}{2} \| A x(t) - b \|^2 \leq L(x(t), \lambda^*) - L(x^*, \lambda^*)\) and \(\| A x(t) - b \|^2 = O(t^{-\frac{2\alpha}{3}})\).
(iii) Suppose the trajectory of \( x(t) \) is bounded. Because \( \|Ax(t) - b\|^2 \to 0 \) as \( t \to \infty \), \( x(t) \) converges to the set of feasible points of the problem (1). Then \( \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O(t^{-\frac{\alpha}{2}}) \) implies that \( x(t) \) converges to the solution set \( S \).

Remark 3.5: In Theorem 3.2, we show that the rate of convergence of \( \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O(t^{-\frac{\alpha}{2}}) \) if \( 0 < \alpha \leq 3 \). However, the boundedness of \( (x(t), \lambda(t), t\dot{x}(t), t\dot{\lambda}(t)) \) is not guaranteed. In addition, combining the results of Theorem 3.1, \( \alpha > 3 \) and \( \beta = \frac{1}{2} \) are the optimal choice for parameters because they make algorithm (6) converge with rate \( O(\frac{1}{t^\alpha}) \).

Remark 3.6: Different from the proof of Theorem 3.1, the construction of Lyapunov function \( V \) in the proof of Theorem 3.2 is more challenging because some time-varying gains \( \theta(\cdot) \) and \( \eta(\cdot) \) in the Lyapunov function need to be found. By analyzing necessary conditions (21)-(25) for \( \dot{V} \leq 0 \), we find time-varying gains \( \theta(\cdot) \) and \( \eta(\cdot) \). If there is no dual variable in algorithm (6), the result is consistent with the result in [27] focusing the primal-based accelerated algorithm with \( \alpha \leq 3 \).

The key of proving Theorems 3.1 and 3.2 is finding appropriate Lyapunov functions. The design of Lyapunov functions is partially inspired by the results for primal-based accelerated algorithms (see [25], [27]). However, we have extended the algorithm design and the analysis to primal-dual cases, which are a more general formulation. The duality gap function \( \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) \) in the proof is often used to analyze primal-dual accelerated methods to show convergence orders (see [35], [36], [41]). The obtained convergence rates, i.e., \( O(\frac{1}{t^\alpha}) \) for the case \( \alpha > 3 \) and \( O(t^{-\frac{\alpha}{2}}) \) for the case \( 0 < \alpha < 3 \), are consistent to that of primal-based accelerated algorithms for unconstrained convex optimization problems [25], [27].

D. Discussion on accelerated method

In sections III-B and III-C, we have analyzed the primal-dual accelerated algorithm for different choices of \( \alpha \) and \( \beta \). The properties are summarized in Table I, which shows that 3 is the critical value for \( \alpha \). Based on the analysis of lower bounds for convergence rates derived from constructed Lyapunov functions, we show that convergence rates of \( \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*), \|Ax(t) - b\|^2, \|\dot{x}(t)\|, \) and \( \|\dot{\lambda}(t)\| \) for the case \( \alpha > 3 \) are faster than those for the case \( 0 < \alpha < 3 \). In addition, if \( \alpha > 3 \), the right hand side of (6) is bounded for any initial condition. Hence, \( \alpha > 3 \) and \( \beta = \frac{1}{2} \) are the best choices for parameters of (6).
TABLE I
ALGORITHM PERFORMANCE FOR DIFFERENT $\alpha$

| $\alpha > 3$ | $\beta = \frac{1}{2}$ | $0 < \alpha \leq 3$ | $\beta = \frac{3}{2\alpha}$ |
|-------------|------------------|------------------|------------------|
| convergence rates for $L(x(t), \lambda^*) - L(x^*, \lambda^*)$ and $\|Ax(t) - b\|^2$ | $O(1/t^2)$ | $O(1/t^{\frac{3}{2}})$ |
| convergence rates for $\|\dot{x}(t)\|$ and $\|\dot{\lambda}(t)\|$ | $O(1/t)$ | $O(1/t^{\frac{3}{2}})$ |
| $\|x(t) - x^*\|$ and $\|\lambda(t) - \lambda^*\|$ | bounded | not bounded |
| $\|\ddot{x}(t)\|$ and $\|t\dot{\lambda}(t)\|$ | bounded | not bounded |
| $\|\dddot{x}(t)\|$ and $\|\dddot{\lambda}(t)\|$ | bounded | not bounded |

The proposed method is a continuous-time ordinary differential equation, which may be discretized to obtain discrete-time algorithms. However, the discretization of the proposed method using explicit Euler scheme and modified symplectic scheme (see [32]) has been observed to be numerically instability by our simulation tests, which are omitted due to space limitations. The symplectic scheme [32] may be a potential scheme to find a properly designed discrete-time counterpart for the proposed method.

IV. APPLICATION TO NETWORK OPTIMIZATION

In this section, we apply the proposed method (6) to network optimization problems (4) and (5), and design distributed primal-dual accelerated algorithms.

A. Distributed Accelerated Algorithm for Scenario 1

Consider the distributed primal-dual accelerated algorithm

\[
\dddot{x}_i = -\frac{\alpha_i}{t} \dot{x}_i - \nabla f_i(x_i) - \sum_{j=1}^{n} a_{i,j} (x_i - x_j)
\]

\[
- \sum_{j=1}^{n} a_{i,j} \left( \lambda_i + \frac{1}{2} t \dot{\lambda}_i - \lambda_j - \frac{1}{2} t \dot{\lambda}_j \right),
\]

\[
\dddot{\lambda}_i = -\frac{\alpha_i}{t} \dot{\lambda}_i + \sum_{j=1}^{n} a_{i,j} \left( x_i + \frac{1}{2} t \ddot{x}_i - x_j - \frac{1}{2} t \ddot{x}_j \right),
\]

where $t \geq t_0 > 0$, $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$, $\lambda(t_0) = \lambda_0$, $\dot{\lambda}(t_0) = \dot{\lambda}_0$, $\alpha_i > 3$ is a parameter determined by agent $i \in \{1, \ldots, n\}$, and $a_{i,j}$ is the $(i,j)$th element of the adjacency matrix of graph $G$. 

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Define \( D_1 = \text{diag}\{[\alpha_1, \ldots, \alpha_n]\} \otimes I_q \) and \( L_{noq} = L_n \otimes I_q \), where \( L_n \) is the Laplacian matrix of \( G \). Then the algorithm (30) has a compact formula as follows

\[
\dot{x} = -\frac{1}{t}D_1 \dot{x} - \nabla f(x) - L_{noq}(\lambda + \frac{1}{2} \dot{\lambda}) - L_{noq}x, \quad (31a)
\]
\[
\dot{\lambda} = -\frac{1}{t}D_1 \dot{\lambda} + L_{noq}(x + \frac{1}{2} t \dot{x}). \quad (31b)
\]

By [11, Lemma 3.1], we have the following result.

**Lemma 4.1:** Let Assumption 2.1 hold. Then \( x^* \in \mathbb{R}^{nq} \) is a solution to (4) if and only if there exists \( \lambda^* \in \mathbb{R}^{nq} \) such that

\[
0_{nq} = \nabla f(x^*) + L_{noq} \lambda^* \text{ and } 0_{nq} = L_{noq} x^*. \quad (32)
\]

Next, we present the main results of the algorithm (31).

**Theorem 4.1:** Suppose Assumption 2.1 holds. Let \((x(t), \lambda(t))\) be a trajectory of algorithm (31). Then

(i) \((x(t), \lambda(t), t\dot{x}(t), t\dot{\lambda}(t))\) is bounded for \( t \geq 0 \);

(ii) \(x(t)\) converges to the set of solutions to problem (4), and \((x(t), \lambda(t), t\dot{x}(t), t\dot{\lambda}(t))\) satisfies the convergence properties \( L_1(x(t), \lambda^*) - L_1(x^*, \lambda^*) = O(\frac{1}{t^2}) \), \( x^T(t)L_{noq} x(t) = O(\frac{1}{t^2}) \), \( \|\dot{x}(t)\| = O(\frac{1}{t}) \), and \( \|\dot{\lambda}(t)\| = O(\frac{1}{t}) \).

**Proof:** Let \((x^*, \lambda^*) \in \mathbb{R}^{nq} \times \mathbb{R}^{nq}\) satisfy (32). Define the augmented Lagrangian function of problem (4) as \( L_1(x, \lambda) = f(x) + \lambda^T L_{noq} x \). Define function

\[
V(t, x, \lambda, \dot{x}, \dot{\lambda}) = V_1(t, x) + V_2(t, x, \dot{x}) + V_3(t, \lambda, \dot{\lambda}) \quad (33)
\]

with \( V_1 = \frac{1}{2} t^2 \left[ L_1(x, \lambda) - L_1(x^*, \lambda^*) + \frac{1}{2} x^T L_{noq} x \right] \),
\[
V_2 = \|x + \frac{t}{2} \dot{x} - x^*\|^2 + \frac{1}{2} (x - x^*)^T (D_1 - 3I_{nq}) (x - x^*),
\]
\[
V_3 = \|\lambda + \frac{t}{2} \dot{\lambda} - \lambda^*\|^2 + \frac{1}{2} (\lambda - \lambda^*)^T (D_1 - 3I_{nq}) (\lambda - \lambda^*).\]

Clearly, function \( V \) is positive definite with respect to \((x, \lambda, t\dot{x}, t\dot{\lambda})\) for all \( t \geq t_0 > 0 \). One can prove the result using function \( V \) as the proof of Theorem 3.1. □

**Remark 4.1:** The algorithm (31) is an accelerated version of primal-dual algorithms in [5], [11]. Compared with [5], [11], the rate of convergence for algorithm (31) is accelerated to \( O(\frac{1}{t^2}) \). ◊
B. Distributed Optimization with Monotropic Constraint

Consider the problem (5). We first decompose the coupled information in constraint (5b). Define \( d \triangleq [d_1^T, \ldots, d_n^T]^T \in \mathbb{R}^{nm} \), \( z \triangleq [z_1^T, \ldots, z_n^T]^T \in \mathbb{R}^{nm} \), and \( \overline{W} = \text{diag}\{W_1, \ldots, W_n\} \in \mathbb{R}^{nm \times q} \). Let \( L_n \) be the Laplacian matrix of \( G \). Since \( G \) is connected and undirected by Assumption [2,1] \( \text{ker}(L_n) = \{v1_n : v \in \mathbb{R}\} \) and \( \text{range}(L_n) = \{w \in \mathbb{R}^n : w^T1_n = 0\} \) form an orthogonal decomposition of \( \mathbb{R}^n \) by the fundamental theorem of linear algebra [42]. Hence, \( \sum_{i=1}^n W_i y_i = \sum_{i=1}^n d_i \) if and only if there exists \( z \in \mathbb{R}^{nm} \) such that \( d - \overline{W} y - L_{nom} z = 0_{nm} \), where \( L_{nom} = L_n \otimes I_m \). It follows that problem (5) is equivalent to the following problem

\[
\min_{y \in \mathbb{R}^q, z \in \mathbb{R}^{nm}} h(y), \quad h(y) = \sum_{i=1}^n h_i(y_i),
\]

\[
\text{subject to } d - \overline{W} y - L_{nom} z = 0_{nm}.
\]

Let \( \lambda \triangleq [\lambda_1^T, \ldots, \lambda_n^T]^T \in \mathbb{R}^{nm} \). We design the distributed primal-dual accelerated algorithm as

\[
\ddot{y}_i = -\frac{1}{t} \alpha_i \dot{y}_i - \nabla h_i(y_i) + W_i^T(\lambda_i + \frac{t}{2} \dot{\lambda}_i),
\]

\[
\ddot{\lambda}_i = -\frac{1}{t} \alpha_i \dot{\lambda}_i + d_i - W_i(y_i + \frac{t}{2} \dot{y}_i) - \sum_{j=1}^n a_{i,j}(\lambda_i - \lambda_j) - \sum_{j=1}^n a_{i,j}(z_i + \frac{t}{2} \dot{z}_i - z_j - \frac{t}{2} \dot{z}_j),
\]

\[
\ddot{z}_i = -\frac{1}{t} \alpha_i \dot{z}_i + \sum_{j=1}^n a_{i,j}(\lambda_i + \frac{t}{2} \dot{\lambda}_i - \lambda_j - \frac{t}{2} \dot{\lambda}_j),
\]

where \( t \geq t_0 > 0 \), \( \alpha_i > 3 \), \( y_i(t_0) = y_{i,0} \), \( \dot{y}_i(t_0) = \ddot{y}_{i,0} \), \( \lambda_i(t_0) = \lambda_{i,0} \), \( \dot{\lambda}_i(t_0) = \ddot{\lambda}_{i,0} \), \( z_i(t_0) = z_{i,0} \), \( \dot{z}_i(t_0) = \ddot{z}_{i,0} \), and \( i \in \{1, \ldots, n\} \).

Define \( D_2 \triangleq \text{diag}\{\alpha_1 I_{q_1}, \ldots, \alpha_n I_{q_n}\} \), \( D_3 \triangleq \text{diag}\{[\alpha_1, \ldots, \alpha_n]\} \otimes I_m \), and the modified Lagrangian function \( L_2 : \mathbb{R}^q \times \mathbb{R}^{nm} \times \mathbb{R}^{nm} \) as

\[
L_2(y, z, \lambda) = h(y) + \lambda^T(d - \overline{W} y - L_{nom} z) - \frac{1}{2} \lambda^T L_{nom} \lambda.
\]

The term \(-\frac{1}{2} \lambda^T L_{nom} \lambda\) does not change the saddle points of the function due to (38), and will help proving the convergence of \( \lambda_i \)'s. Hence, it is often used in the design of distributed algorithms for resource allocation problems (see [2, 3]). Then the algorithm (35) is equivalent to the saddle point dynamics of (36) given by

\[
\ddot{y} = -\frac{1}{t} D_2 \ddot{y} - \nabla y L_2(y, z, \lambda + \frac{t}{2} \dot{\lambda}),
\]

\[
\ddot{\lambda} = -\frac{1}{t} D_3 \ddot{\lambda} + \nabla \lambda L_2(y + \frac{t}{2} \dot{y}, z + \frac{t}{2} \dot{z}, \lambda),
\]
\[ \ddot{z} = -\frac{1}{t} D_3 \dot{z} - \nabla_z L_2(y, z, \lambda + \frac{t}{2} \dot{\lambda}). \]  

(37c)

where \( y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0, \lambda(0) = \lambda_0, \dot{\lambda}(0) = \dot{\lambda}_0, z(t_0) = z_0, \) and \( \dot{z}(t_0) = \dot{z}_0. \)

Similar to Lemma 4.3 of [2], [3], the following result holds.

**Lemma 4.2:** Let Assumption 2.1 hold. Then \( (y^*, z^*) \in \mathbb{R}^{nq} \times \mathbb{R}^{nm} \) is a solution to the problem (34) if and only if there exists \( \lambda^* \in \mathbb{R}^{nq} \) such that

\[
0_q = \nabla g(y^*) - WW^\top \lambda^*, \quad 0_{nm} = L_{nom} \lambda^*,
\]

\[
0_{nm} = d - WW^\top y^* - L_{nom} z^*.
\]

(38)

The following theorem shows the convergence rate.

**Theorem 4.2:** Let Assumption 2.1 hold and \( h(\cdot) \) be strictly convex. Suppose \( (y(t), \lambda(t), z(t)) \) is a trajectory of algorithm (37). Then

(i) the trajectory of \( (y(t), z(t), \lambda(t), t\dot{y}(t), t\dot{z}(t), t\dot{\lambda}(t)) \) is bounded for \( t \geq t_0 > 0; \)

(ii) the trajectory of \( y(t) \) converges to the solution of problem (5) and the trajectory satisfies

\[
L_2(y(t), z^*, \lambda^*) - L_2(y^*, z^*, \lambda(t)) = O(\frac{1}{t}), \quad \lambda^\top(t)L_{nom}\lambda(t) = O(\frac{1}{t}), \quad \|\dot{y}(t)\| = O(\frac{1}{t}), \quad \|\dot{\lambda}(t)\| = O(\frac{1}{t}), \quad \|\dot{z}(t)\| = O(\frac{1}{t}).
\]

**Proof:** Let \( (y^*, z^*, \lambda^*) \) satisfy (38). Define \( V(t, y, \lambda, z, \dot{y}, \dot{z}, \dot{\lambda}) = V_1(t, y) + V_2(t, y, \dot{y}) + V_3(t, \lambda, \dot{\lambda}) + V_4(t, z, \dot{z}) \) with \( V_1 = \frac{1}{2} t^2 [L_2(y, z^*, \lambda^*) - L_2(y^*, z^*, \lambda)], \)

\[
V_2 = \|y + \frac{t}{2} \dot{y} - y^*\|^2 + \frac{1}{2} (y - y^*)^\top (D_2 - 3 I_{nm})(y - y^*),
\]

\[
V_3 = \|\lambda + \frac{t}{2} \lambda^* - \lambda\|^2 + \frac{1}{2} (\lambda - \lambda^*)^\top (D_3 - 3 I_{nm})(\lambda - \lambda^*),
\]

\[
V_4 = \|z + \frac{t}{2} \dot{z} - z^*\|^2 + \frac{1}{2} (z - z^*)^\top (D_3 - 3 I_{nm})(z - z^*),
\]

where \( L_2(\cdot) \) is defined in (36). Then \( V \) is positive definite with respect to \((y, z, \lambda, t\dot{y}, t\dot{z}, t\dot{\lambda})\) for all \( t \geq t_0 > 0. \) Using \( V \) as the Lyapunov function, one can prove this theorem similarly to that of Theorem 3.1.

**Remark 4.2:** The algorithm (37) is a modified version of the design in [2] by using the proposed accelerated method. Compared with results in [2] which have \( O(\frac{1}{t}) \) convergence rate, the algorithm (37) has the \( O(\frac{1}{t^2}) \) convergence rate.

\[ \diamond \]

**C. Numerical Simulation**

In this subsection, we present numerical simulations for the network optimization problem (5). Each local function is a log-sum-exp function \( h_i(y_i) = \rho \log \left[ \sum_{j=1}^m \exp((c_{i,j} y_i - b_{i,j})/\rho) \right], \)
where $n = 20$, $m = 4$, $\rho = 20$, $q = 2$, $d_0 = [30, 50]^T \in \mathbb{R}^2$, $W_i \in \mathbb{R}^{2 \times 2}$, $c_{i,j} \in \mathbb{R}^2$ and $b_{i,j} \in \mathbb{R}$ are random vectors and scalars generated from a uniform distribution on the interval $[0, 1]$.

Simulation results of algorithm (37) are shown in Figs. 1 and 2. Because the $y$-axis of Fig. 1 is presented in log-scale, the slope of a trajectory shows the order of the convergence rate. Fig. 1 shows that the accelerated design in this paper has a faster performance than that of algorithms in [2] and [19]. In Fig. 2, it is shown that the dual variables will reach a consensus and the constraint (5b) is satisfied.
V. Conclusion

We have developed a primal-dual Nesterov’s accelerated method for a class of convex optimization problems with affine equality constraints and applied the method to two types of network optimization problems. In particular, via a Lyapunov approach, we have analyzed critical choices of parameters in the algorithm design and proved that the convergence rate of the Lagrangian function is $O(\frac{1}{t^2})$, which is faster than the rate $O(\frac{1}{t})$ of standard primal-dual methods. We further designed distributed accelerated primal-dual algorithms for optimization problems with consensus constraints and extended monotropic optimization problems using the proposed method.

Directions of future work include finding suitable discretization themes and designing convergent discrete-time counterparts for the proposed continuous-time method. The analysis of high-resolution ODE models and better measures of convergence rates to the accelerated primal-dual method is also of great importance. Moreover, the convergence to a minimizer rather than the set of solutions for primal-dual accelerated methods is also worth investigating in future.

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