Generalized Noncommutative Supersymmetry from a New Gauge Symmetry

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Abstract

Using the notion of a gauge connection on a flat superspace, we construct a general class of noncommutative \((D = 2, \mathcal{N} = 1)\) supertranslation algebras generalizing the ordinary algebra by inclusion of some new bosonic and fermionic operators. We interpret the new operators entering into the algebra as the generators of a \(U(1)\) (super) gauge symmetry of the underlying theory on superspace. These superalgebras are gauge invariant, though not closed in general. We then show that these type of superalgebras are naturally realized in a supersymmetric field theory possessing a super \(U(1)\) gauge symmetry. As the non-linearly realized symmetries of this theory, the generalized noncommutative (super)translations and super gauge transformations are found to form a closed algebra.

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1 Introduction

Deformation of the supersymmetry algebras in noncommutative spaces, or more generally in string theory in a non-trivial background, has drawn some attentions in the context of noncommutative field theories [1]-[8] (for a review on NCFT see [10], [11]). In particular, in this direction, the deformation of the $\mathcal{N} = 1, D = 2$ super Euclidean ($E^2$) algebra, as the simplest example of these algebras, has been studied in a recent paper [7]. The motivation of that work for looking for such a deformation was that, unlike in the ordinary case, translations of a noncommutative space form a noncommutative algebra [9]. So one naturally expects a corresponding deformation of the SUSY algebra when lifting this fact to the level of a noncommutative superspace [2, 3].

One of the main consequences of the above work was that a consistent deformation of a SUSY algebra associated to the noncommutativity of space naturally requires introduction of new bosonic and fermionic operators among the usual (super)translation (and rotation) generators. It was pointed out that these generators change only the states in the fundamental representation of the noncommutative super $E^2$ group [1]. As such, the deformation proposed in that (and also in this) paper has not a direct reflection on supersymmetric field theories on noncommutative spaces [1], because in those theories field operators naturally transform as in the adjoint representation [14]. Indeed, such a deformation is manifestly realized in a commutative theory whose field content belongs to the fundamental representation of the corresponding supergroup.

It was found that the new generators in the superalgebra of the noncommutative $E^2$ (briefly denoted as $E^2_\theta$) form a complete “basis” of expansion for functions of the Grassmann coordinates ($\theta^+, \theta^-$). This proposes the idea that these new operators must be interpreted as the generators of a “local” transformation acting on the “state superfield” $S(x, \theta)$ as follows

$$S \rightarrow f(\theta)S,$$

with $f$ being any complex valued function of the Grassmann coordinates. This idea has an immediate generalization if we take $f$ to be a function of both $x$ and $\theta$. So we are led to introduce a $U(1)$ super gauge transformation acting on a complex scalar superfield carrying this $U(1)$’s charge. A theory with this $U(1)$ gauge symmetry necessarily involves a gauge superfield $A_\alpha(x, \theta)$ which plays the role of a gauge connection on a flat superspace. This connection in
turn modifies the algebra of the supercharges in a way that they close only with the generators of this $U(1)$ super gauge transformations. In this paper we will use this idea for deriving a generalized version of the previous NC super $E^2$ algebra which provides also a unified framework for many other generalizations of the ordinary superalgebra such as the one with a central charge.

The paper is organized as follows: in section 2 we will present the definition of a super gauge symmetry and introduce the associated gauge superfield and show its application in generalization of the ordinary superalgebra together with two of its important special cases. Then in section 3 we study particular examples of this algebra by specializing the general form of the gauge superfield and investigate about the closure of the resulting algebras. Section 4 is devoted to the extension of the results of section 3 to the case of a general gauge superfield configuration. In section 5 we consider the issue of realization of this generalized type of superalgebras in 2D field theories possessing not only an ordinary supersymmetry but also a super gauge symmetry. In section 6 a purely bosonic analogue of the above models, involving only the translation sector of the complete supertranslation algebra, is constructed. The ideas of this purely bosonic case are further applied to the supersymmetric case in section 7 to show the closure of the general algebra, when interpreted as the algebra of the (non-linear) superfield transformations. We conclude the paper by a summary with some remarks in section 8.

2 Generalization of SUSY Using a Gauge Superfield

It is a well known fact that the noncommutativity of space in NCFT has a description in terms of a (suitably chosen) gauge field background on a commutative space \cite{3,4}. One is tempted to further generalize this idea by introducing the concept of a gauge superfield background and its associated (super) gauge transformation for theories defined on a commutative superspace. For theories in $D = 2$, $\mathcal{N} = 1$ superspace, we define a gauge superfield by a grassmann odd spinor superfield $A_{\alpha}(x, \theta)$ ($\alpha = \pm$) with a proper gauge transformation as dictated by the algebra as follows. The two components
of this spinor are related by conjugation,
\[(A_+)^\dagger = -A_- .\] (1)

The multiplet defined by \(A_\alpha\) consists of a (complex) scalar, a spinor and a vector field, which can be read easily from its expansion in powers of \(\theta^\pm\).

Let us denote the ordinary (undeformed) SUSY generators by \(Q_\alpha\) (see Appendix for our notations),
\[Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \partial_\pm , \] (2)

This is a linear differential operator obeying the usual Leibnitz rule, i.e.
\[Q_\pm (fg) = (Q_\pm f)g + (-1)^{\deg(f)} f(Q_\pm g), \] (3)
for any (odd or even) pair of superfields \(f, g\). Then we define the deformed generators \(Q_\alpha\) through introduction of \(A_\alpha\) as some superspace gauge connection (in a particular representation) as follows
\[Q_\alpha = Q_\alpha - iA_\alpha. \] (4)

The ordinary algebra of \(Q_\pm\) together with the grassmann property \(\{A_\alpha, A_\beta\} = 0\) then give rise to the algebra of \(Q_\pm\)
\[\{Q_+, Q_-\} = -i(Q_+A_- + Q_-A_+), \]
\[Q^2_\pm = i(\partial_\pm - Q_\pm A_\pm). \] (5)

where on the RHS, \(Q_\pm\) acts on \(A_\pm\) as a differential operator on a superfield. The expression on the RHS of the anticommutator of \(Q_+, Q_-\) is evidently invariant under the following (super)gauge transformations
\[A_\alpha \rightarrow A_\alpha + Q_\alpha \Lambda. \] (6)

Here \(\Lambda(x, \theta)\) is an arbitrary (parity even) real scalar superfield. This is a local gauge transformation in the sense that its parameter \(\Lambda\) is a function of the superspace coordinates. The gauge invariant quantity on the RHS of the first eq.(3) is called the superfield strength of \(A_\alpha\) which in this paper is denoted by \(T\);
\[T \equiv -i(Q_+A_- + Q_-A_+). \] (7)
Also the second eq. (5), compared to its counterpart in the ordinary superalgebra (first eq. (A.9)), hints on defining the generalized momenta as follows
\[ P_\pm \equiv -i(\partial_\pm - Q_\pm A_\pm). \] (8)

Note that the above representation of \( P, Q \) clearly depends on the gauge in which \( A_\pm \) is written. However, the algebra itself comes out to be gauge invariant! (see below).

2.1 Some Special Cases

Before studying further the general case, it is worthwhile to have a look at two of its important special cases.

**Case 1:** SUSY with a central charge (CE-SUSY)

Let us take an \( x \)-independent configuration as
\[ A_\pm(x, \theta) = c_\pm \theta^\mp, \] (9)
with \( c_\pm \) being complex numbers \( (c_+^* = -c_-) \). Then
\[ Q_+ A_+ = 0, \quad Q_+ A_- = c_-, \]
\[ Q_- A_+ = c_+, \quad Q_- A_- = 0, \] (10)

using which in the general expressions we find
\[ \{Q_+, Q_-\} = -i(c_+ + c_-), \]
\[ Q_\pm^2 = i\partial_\pm. \] (11)

This is the ordinary superalgebra modified by a central charge \( a \equiv -i(c_+ + c_-) \) (see [12]).

**Case 2:** SUSY in a noncommutative space (NC-SUSY)

Now consider the case with the background defined by
\[ A_\pm(x, \theta) = \mp \omega x^\mp \theta^\pm. \] (12)

For this configuration we have
\[ Q_+ A_+ = -\omega x^-, \quad Q_+ A_- = i\omega \theta^+ \theta^-, \]
\[ Q_- A_+ = i\omega \theta^+ \theta^-, \quad Q_- A_- = \omega x^+, \] (13)
from which we find
\[
\{Q_+, Q_-\} = 2\omega \theta^+ \theta^-,
\]
\[
Q_\pm^2 = i(\partial_\pm \pm \omega x^\mp).
\]  
(14)

This is the supertranslation part of the algebra referred to as a “noncommutative superalgebra” in the earlier work [7].

2.2 The General Case

From the above examples we conclude that several interesting cases are capable of a uniform description provided by introducing a gauge connection on superspace as in the eq.(4). Let us now return to the case of a general gauge superfield and study its associated supertranslation algebra, that is the algebra of the bosonic and fermionic generators \(P_\pm, Q_\pm\). For this aim the only things needed are the ordinary algebra of \(P_\pm, Q_\pm\) (eqs.(14)) together with the following simple rules

\[
[P_\pm, F] = -i\partial_\pm F
\]
\[
[Q_\pm, F]_{\text{deg} F} = Q_\pm F
\]  
(15)

Here \(F = F(x, \theta)\) denotes an arbitrary superfield with even or odd grassmann parity. In the second equation, the notation \([\cdot, \cdot]_{\text{deg} F}\) is a commutator or anticommutator depending on that \(F\) is even or odd. Using these rules, for instance, for the commutator of \(P_+, Q_+\) we find

\[
[P_+, Q_+] = \left[ -i(\partial_+ - Q_+ A_+), Q_+ - iA_+ \right]
\]
\[
= -\left( \partial_+ A_+ + iQ_+^2 A_+ \right)
\]
\[
= -\left( \partial_+ A_+ + i^2 \partial_+ A_+ \right) = 0.
\]  
(16)

By similar calculations we find that the non-trivial (anti)commutators between \(P_\pm, Q_\pm\) are written as

\[
\{Q_+, Q_-\} = -i(Q_+ A_- + Q_- A_+),
\]
\[
[P_+, P_-] = \partial_+ Q_- A_- - \partial_- Q_+ A_+,
\]
\[
[P_+, Q_-] = -(\partial_+ A_- - iQ_+ Q_- A_+),
\]
\[
[P_-, Q_+] = -(\partial_- A_+ - iQ_- Q_+ A_-).
\]  
(17)
which must be supplemented by the usual equations $Q_\pm^2 = -P_\pm$. The interesting point regarding the above equations is that despite $P, Q$’s themselves, which have gauge dependent representations, their algebra is invariant under the gauge transformations. This is because all the expressions on the RHS of these equations are gauge invariant combinations of $A_-^\pm$. The combination in the first line equation is manifestly invariant while the invariance of the other three combinations is easily proved. For example, the second combination under a gauge transformation $\Lambda$ changes as

$$
\delta_\Lambda(\partial_+ Q_- A_- - \partial_- Q_+ A_+) = \partial_+ Q_- (Q_- \Lambda) - \partial_- Q_+ (Q_+ \Lambda) = \partial_+ (i\partial_- \Lambda) - \partial_- (i\partial_+ \Lambda) = 0.
$$

(18)

A similar conclusion holds for the other three combinations of $A_-^\pm$. As such, we see that several representations of $P, Q$’s corresponding to different gauges of $A_-^\pm$ lead to the same algebra. Such representations are called gauge equivalent.

### 3 More special cases

Let us now focus on two special cases which are straightforward generalizations of the two cases in the previous section and are easy to investigate about such questions as the closure of the resulting algebra.

**Case 3:** (NC-SUSY with local noncommutativity parameter)

$$
A_-^\pm(x, \theta) = \theta^\pm A_-^\pm(x).
$$

(19)

where $A_-^\pm$ (related as $A_-^\pm = -A_-^\mp$) are components of a vector field. Simple computations like the previous ones using this gauge field gives

$$
\begin{align*}
[P_+, P_-] &= F(x), \\
\{Q_+, Q_-\} &= \theta^+ \theta^- F(x), \\
[P_+, Q_-] &= -\theta^- F(x), \\
[P_-, Q_+] &= \theta^+ F(x),
\end{align*}
$$

(20)

besides the other trivial relations including $Q_\pm^2 = -P_\pm$. Here $F(x) \equiv \partial_+ A_- - \partial_- A_+$ is the field strength of the ordinary gauge field $A_\pm(x)$. The fact that the two functions $A_-^\pm(x)$ appear in the algebra only through a particular
combination, $F(x)$, is a reflection of the gauge invariance of the algebra. The case of an ordinary NC space is a particular case of this in which the ordinary field strength is a constant (proportional to the inverse of the NC scale $\theta$). Hence we interpret $F(x)$ as some local (inverse) NC parameter. The four gauge invariant quantities in the general theory therefore reduce to a single one $F(x)$ in this special case. To examine the closure of the algebra, let us denote these gauge invariant operators by

$$F \equiv F(x), \quad O_\pm \equiv \theta^\pm F(x), \quad T \equiv \theta^+ \theta^- F(x).$$

(21)

Note that $F,T$ are hermitian operators while $(O_+^\dagger = O_-$. Next we form the (anti)commutators of these operators with $P,Q$’s:

$$[P_\pm, F] = -i\partial_\pm F(x),$$

$$\{Q_\pm, F\} = i\theta^\pm \partial_\pm F(x),$$

$$[P_\pm, O_+] = -i\theta^- \partial_\pm F(x),$$

$$\{Q_+, O_+\} = i\theta^+ \theta^- \partial_+ F(x),$$

$$\{Q_+, O_+\} = F(x),$$

$$\{Q_-, O_-\} = i\theta^- \theta^+ \partial_- F(x),$$

$$[P_\pm, O_-] = -i\theta^+ \partial_\pm F(x),$$

$$\{Q_-, O_-\} = F(x),$$

$$\{Q_-, O_-\} = i\theta^- \theta^+ \partial_+ F(x),$$

$$[P_\pm, T] = -i\theta^+ \theta^- \partial_\pm F(x),$$

$$[Q_\pm, T] = \pm \theta^\pm F(x).$$

(22)

For being a closed super Lie algebra, all functions on the RHS of this algebra must be written as linear combinations of $F(x), \theta^\pm F(x), \theta^+ \theta^- F(x)$ which are representations for the operators $F, O_\pm, T$ in this case. This is possible if and only if

$$\partial_\pm F(x) = \kappa_\pm F(x),$$

(23)

for some constant $\kappa_\pm \in \mathbb{C}$. It is solved to

$$F(x) = F_0\exp(\kappa_+ x^+ + \kappa_- x^-),$$

(24)
where $F_0$ is another constant and hermiticity of $F$ implies that

$$F_0^* = F_0, \quad \kappa_+^* = \kappa_-.$$  \hfill (25)

We note that the only rotationally symmetric solution is the one for $\kappa_\pm = 0$ corresponding to the case of a constant field strength $F = F_0$ (the NC-SUSY case).

The full algebra which is now closed is found by putting the above solution into the expressions for the (anti)commutators found in the above. The result is as follows

\[
\begin{align*}
[P_+, P_-] &= F, \quad \{Q_+, Q_-\} = T, \quad [P_\pm, Q_\mp] = \mp O_\mp, \\
[P_\pm, F] &= -i\kappa_\pm F, \quad [Q_\pm, F] = i\kappa_\pm O_\mp, \\
[P_\pm, O_\mp] &= -i\kappa_\pm O_\mp, \quad \{Q_+, O_+\} = i\kappa_+ T, \quad \{Q_-, O_-\} = F, \\
[P_\pm, O_\pm] &= -i\kappa_\pm O_\mp, \quad \{Q_+, O_-\} = F, \quad \{Q_-, O_+\} = -i\kappa_- T, \\
[P_\pm, T] &= -i\kappa_\pm T, \quad [Q_\pm, T] = \pm O_\mp,
\end{align*}
\]  \hfill (26)

besides the usual relations $Q_\pm^2 = -P_\pm$.

The reduction of this algebra to the NC-SUSY algebra in the $\kappa_\pm = 0$ case is almost evident, because in this case $F$ commutes with everything else (it belongs to the center of the algebra) and by Schure lemma we can take $F = F_0$, i.e. a constant times unity. The constant $F_0$ determines the inverse of the NC parameter as $\vartheta^{-1} \equiv \omega = \frac{1}{2} F_0$ [7].

One can explicitly check that the commutator of the $SO(2)$ generator $J$ with $T, F, O_\pm$ is not closed unless for $\kappa_\pm = 0$ which, in agreement with the above statement, produces the only rotationally symmetric (i.e. the NC-SUSY) algebra in this category.

Case 4: (CE-SUSY with a local central charge)

$$A_\pm(x, \theta) = \theta^\mp f_\pm(x),$$  \hfill (27)

where $f_\pm$ (related as $f^*_\pm = -f_-^*$) are complex scalar fields. The (anti)commutators of $P, Q$'s in this case become

\[
\begin{align*}
\{Q_+, Q_-\} &= f(x), \\
[P_+, P_-] &= \theta^+ \theta^- \partial_+ \partial_- f(x), \\
[P_+, Q_-] &= -i\theta^+ \partial_+ f(x), \\
[P_-, Q_+] &= -i\theta^- \partial_- f(x),
\end{align*}
\]  \hfill (28)
where \( f \equiv -i(f_+ + f_-) \) is a real function. Again the appearance of only this particular combination of the two functions \( f_\pm(x) \) is a result of gauge invariance of the algebra. Let us as in the previous case identify the four functions of \((x, \theta)\) on the RHS of this algebra as the operators \( T, F, O_\pm \); i.e.

\[
T \equiv f(x), \quad F \equiv \theta^+ \theta^- \partial_+ \partial_- f(x), \quad O_\pm \equiv \pm i \theta^\pm \partial_\pm f(x)
\] (29)

Then by a simple calculation we find

\[
\begin{align*}
[P_\pm, F] &= i \theta^- \theta^+ \partial_\pm \partial_\mp f(x), \\
[Q_\pm, F] &= \pm \theta^\pm \partial_\pm \partial_\mp f(x),
\end{align*}
\]

\[
\begin{align*}
[P_\pm, O_+] &= \theta^+ \partial_\pm \partial_\mp f(x), \\
\{Q_+, O_+\} &= i \partial_+ f(x), \\
\{Q_-, O_+\} &= \theta^+ \theta^- \partial_\pm \partial_\mp f(x),
\end{align*}
\]

\[
\begin{align*}
[P_\pm, O_-] &= -\theta^- \partial_\pm \partial_\mp f(x), \\
\{Q_+, O_-\} &= \theta^+ \theta^- \partial_\pm \partial_\mp f(x), \\
\{Q_-, O_-\} &= -i \partial_- f(x),
\end{align*}
\]

\[
\begin{align*}
[P_\pm, T] &= -i \partial_\pm f(x), \\
[Q_\pm, T] &= i \theta^\pm \partial_\pm f(x).
\end{align*}
\] (30)

By the same logic as in the previous case, the closure condition for this algebra requires that \( f(x) \) satisfies the equations

\[
\partial_\pm f(x) = \kappa_\pm f(x) + c_\pm,
\] (31)

where for the reality of \( f \) (i.e. hermiticity of \( T \)) we need the constants \( \kappa_\pm, c_\pm \) satisfy

\[
\kappa_+^* = \kappa_-, \quad c_+^* = c_-.
\] (32)

The integrability condition, \( \partial_+(\partial_- f) = \partial_-(\partial_+ f) \), for the above equations further requires the relation

\[
\frac{c_+}{\kappa_+} = \frac{c_-}{\kappa_-},
\] (33)

Then the above equations are integrated to

\[
f(x) = a + b \exp(\kappa_+ x^+ + \kappa_- x^-)
\] (34)
where $a, b \in \mathbb{R}$ are arbitrary constants. Using this solution in the previous expressions, the closed algebra under consideration becomes

\[
\begin{align*}
[P_+, P_-] &= F, \quad \{Q_+, Q_-\} = T, \quad [P_\pm, Q_\mp] = \mp O_\pm, \\
[P_\pm, F] &= -i\kappa_\pm F, \quad [Q_\pm, F] = i\kappa_\pm O_\mp, \\
[P_\pm, O_+] &= -i\kappa_\pm O_+, \quad \{Q_+, O_+\} = i\kappa_+(T-a), \quad \{Q_-, O_+\} = F, \\
[P_\pm, O_-] &= -i\kappa_\pm O_-, \quad \{Q_+, O_-\} = F, \quad \{Q_-, O_-\} = -i\kappa_-(T-a), \\
[P_\pm, T] &= -i\kappa_\pm (T-a), \quad [Q_\pm, T] = \pm O_\pm.
\end{align*}
\] (35)

Obviously, for $a = 0$, this is the same as the algebra obtained in the previous case, though the generators now have a different (non-gauge equivalent) representations! For nonzero $a$ it is just a simple central extension of the same algebra as can be seen by putting $T' \equiv T - a$.

On the other hand, for $\kappa_\pm = 0$, we recover the NC-SUSY algebra written in new notations as

\[
\begin{align*}
[P_+, P_-] &= F, \quad \{Q_+, Q_-\} = T, \quad [P_\pm, Q_\mp] = \mp O_\pm, \\
\{Q_+, O_-\} &= F, \quad \{Q_-, O_+\} = F, \quad [Q_\pm, T] = \pm O_\pm.
\end{align*}
\] (36)

Since in this case $F$ commutes with everything else, by Schure lemma, we can treat it as a constant times the unity operator. To recover the previous form of the NC-SUSY algebra in [7], it is sufficient to put $F = 2\omega$ and then rescale $T, O_\pm$ as: $T \rightarrow 2\omega T$, $O_\pm \rightarrow 2\omega O_\pm$.

There is a class of representations of the general algebra for which $T = a$ is a constant. Indeed, assuming $T$ to be proportional to the unity operator in the above algebra, we find that for such representations we necessarily have $F = 0$, $O_\pm = 0$. The resulting algebra is a centrally extended version of the commutative algebra [12] whose only nontrivial relation is

\[
\{Q_+, Q_-\} = a.
\] (37)

## 4 General Algebra of $T, F, O_\pm$ with $P_\pm, Q_\pm$

We are now going to generalize the above results to the case with an arbitrary gauge superfield $A_\pm(x, \theta)$. In accordance with notations of the previous section we identify the operators $T, F, O_\pm$ by the RHS expressions of eq.(17)
as follows

\begin{align*}
T & \equiv -i(Q_+A_- + Q_-A_+), \\
O_+ & \equiv + \left( \partial_+A_- - iQ_+Q_+A_+ \right), \\
O_- & \equiv - \left( \partial_-A_+ - iQ_-Q_+A_- \right), \\
F & \equiv \partial_+Q_-A_- - \partial_-Q_+A_+.
\end{align*}

(38)

From the point of view of the $SO(2)$ rotations, $T, F$ are scalars while $O_\pm$ behave as components of a spinor. As mentioned earlier, these four quantities constitute a set of gauge invariant operators, among which only $T$ is first order in derivatives of $A_\pm$ while the others are of second order. So such, the most suitable candidate for a strength of the superfield $A_\pm$ is $T$, from which an invariant action can be built.

That these operators are gauge invariant can be seen more explicitly by observing that $O_\pm, F$ are related to $T$ by the following simple equations

\begin{align*}
O_\pm & = \pm Q_\mp T, \\
F & = Q_-Q_+T = -Q_+Q_-T.
\end{align*}

(39)

These are nothing but a consequence of the algebra of $\partial_\pm, Q_\pm$ in the above definitions of these quantities. Using these relations it is now easy to find the general form of the algebra which is written as

\begin{align*}
\{Q_+, Q_-\} & = T, \quad [P_+, P_-] = F, \quad [P_\pm, Q_\mp] = \mp O_\mp, \\
[P_\pm, F] & = -i\partial_\pm F, \quad [Q_\pm, F] = i\partial_\pm O_\mp, \\
[P_\pm, O_+] & = -i\partial_\pm O_+, \quad \{Q_+, O_+\} = i\partial_+ T, \quad \{Q_-, O_+\} = F, \\
[P_\pm, O_-] & = -i\partial_\pm O_-, \quad \{Q_+, O_-\} = F, \quad \{Q_-, O_-\} = -i\partial_- T, \\
[P_\pm, T] & = -i\partial_\pm T, \quad [Q_\pm, T] = \pm O_\mp.
\end{align*}

(40)

This is just similar to the algebra in case 3 if one replaces $\kappa_\pm$ by $\partial_\pm$. The requirement of the closure of this algebra means that $\partial_\pm T, \partial_\pm F, \partial_\pm O_\pm$ must be written as linear combinations of $T, F, O_\pm$, but not of course of $P_\pm, Q_\pm$. In the particular cases of the previous section, this leads to the same exponential configurations as we found there.
5 Realization of the Generalized SUSY in Field Theories with Super $U(1)$ Gauge Symmetry

5.1 Preliminaries

In this section we will consider the problem of constructing field theories realizing the generalized SUSY algebra as an algebra underlying their symmetries. These models turn out to be realizable by a class of supersymmetric field theories which, in addition to having a global (ordinary) SUSY, they also possess a local super gauge symmetry. We shall call such models as the “gauge superfield theories” or briefly as GSFT. Let us consider for convenience the simplest example of such models involving a gauge multiplet defined by $A_\alpha(x, \theta)$ as well as a scalar multiplet defined by $S(x, \theta)$. These superfields obey the following infinitesimal (super) gauge transformations

\begin{align}
\delta_\lambda A_\alpha &= Q_\alpha \Lambda, \\
\delta_\lambda S &= i\Lambda S,
\end{align}

(41) with $\Lambda(x, \theta)$ being a real valued scalar superfield. These superfields also transform under the ordinary SUSY transformations. The SUSY transformation of $S$ is

\[ \delta_Q S = (\tau Q) S. \]  

(42)

In order to obtain a SUSY invariant theory, the superfield $A_\alpha$ must also transform under SUSY but its transformation should be slightly different from that of $S$, because it does not directly enter into the action but after a “dualization” procedure (see below).

As is well known, for a superspace formulation of the ordinary SUSY field theories \cite{13}, we need to introduce the concept of supercovariant derivatives as follows

\[ D_\pm \equiv \frac{\partial}{\partial \theta^\pm} - i\theta^\pm \partial_\pm. \]  

(43)

These supercovariant derivatives have the important property that they anticommute with the SUSY generators

\[ \{D_\alpha, Q_\beta\} = 0, \]  

(44)

\footnote{In this section we denote several $\delta$-variations by a subscript on $\delta$ representing the appropriate generators. In the next section we will change this notation by replacing these subscripts with symbols denoting the corresponding transformation parameters.}
for all values of the spinor indices $\alpha, \beta$. Using these derivatives, for example, the kinetic Lagrangian of the complex scalar superfield $S$ is written as

$$\frac{1}{2} \mathcal{D}_\alpha S \mathcal{D}^\alpha \overline{S},$$

(45)

where the spinorial index $\alpha$ is raised with $\varepsilon^{\alpha\beta}$ and lowered with $\varepsilon_{\alpha\beta}$. This term is obviously invariant under the *global* $U(1)$ transformations defined by $S \to e^{i\Lambda} S$ but not under its local version. As in the ordinary gauge theory, however, we can make it local by introducing a gauge connection which in this case must be a spinor superfield $\mathcal{B}_\alpha$ transforming under the local $U(1)$ as

$$\mathcal{B}_\alpha \rightarrow \mathcal{B}_\alpha + \mathcal{D}_\alpha \Lambda.$$

(46)

This differs from the transformation property of $\mathcal{A}_\alpha$ in the replacement of $\mathcal{Q}_\alpha$ with $\mathcal{D}_\alpha$. Gauge invariant Lagrangians can then be obtained by replacing the ordinary (super)covariant derivatives $\mathcal{D}_\alpha$ with the “(super) gauge covariant” derivatives $\mathcal{D}_\alpha$ whose effect on a superfield with a $U(1)$ charge $e$ is defined as follows

$$\mathcal{D}_\alpha \equiv \mathcal{D}_\alpha - ie\mathcal{B}_\alpha.$$

(47)

In this way the (super)gauge invariant supersymmetric Lagrangian of $S$ becomes

$$\mathcal{L}_S = \frac{1}{2} \mathcal{D}_\alpha S \mathcal{D}^\alpha \overline{S}.$$

(48)

Here $S$ and $\overline{S}$ are assumed to have the $U(1)$ charges $+1$ and $-1$, respectively. So

$$\mathcal{D}_\alpha S \equiv (\mathcal{D}_\alpha - i\mathcal{B}_\alpha) S, \quad \mathcal{D}_\alpha \overline{S} \equiv (\mathcal{D}_\alpha + i\mathcal{B}_\alpha) \overline{S}.$$

(49)

The above Lagrangian $(\mathcal{L}_S)$ is manifestly supersymmetric provided $\mathcal{B}_\alpha$ transforms under SUSY similarly to $S$ as follows

$$\delta_\mathcal{Q} \mathcal{B}_\alpha = (\overline{\tau} \mathcal{Q}) \mathcal{B}_\alpha.$$

(50)

Then it can be seen that $\mathcal{D}_\alpha S$ transforms covariantly under the ordinary SUSY transformations; i.e.

$$\delta_\mathcal{Q}(\mathcal{D}_\alpha S) = -iS\delta_\mathcal{Q}\mathcal{B}_\alpha + \mathcal{D}_\alpha(\delta_\mathcal{Q}S)$$

$$= -i(\overline{\tau} \mathcal{Q})\mathcal{B}_\alpha + (\mathcal{D}_\alpha - i\mathcal{B}_\alpha)(\overline{\tau} \mathcal{Q})S$$

$$= (\overline{\tau} \mathcal{Q})(\mathcal{D}_\alpha - i\mathcal{B}_\alpha)S = (\overline{\tau} \mathcal{Q}) \mathcal{D}_\alpha S.$$

(51)
where we have used the eqs.(3),(44). This insures that any analytic function of $S, \overline{S}$ and their covariant derivatives changes covariantly under SUSY transformations and hence can be used for building a SUSY invariant action. However, other restrictions such as the gauge invariance and Lorentz invariance specially constrain the form of the physical Lagrangian. We emphasize that the gauge superfield $B_\alpha$ in the Lagrangian is not actually independent of $A_\alpha$ used to build the generalized SUSY algebra. In the following subsection we will relate them by a certain dualization map.

5.2 Realization of the Generalized SUSY in GSFT

We are now ready to show that our generalized superalgebra derived from the gauge dependent representations of the generators

$$\begin{align*}
Q_\pm &= Q_\pm - iA_\pm, \\
P_\pm &= -i(\partial_\pm - Q_\pm A_\pm),
\end{align*}$$

is realized in a GSFT. For this purpose, we first note that the transformations generated by $Q_\alpha$ on a scalar superfield are written as a combination of an ordinary SUSY and a gauge transformation as follows

$$\delta_Q S = (\tau Q - i\tau A)S = \delta_Q S + \delta_\Lambda S,$$

where the gauge parameter $\Lambda$ is “knitted” into the SUSY parameter $\epsilon$ as

$$\Lambda = \Lambda(\epsilon) \equiv -\tau A(x, \theta).$$

On the other hand, the invariance of the theory governing $(S, B_\alpha)$ requires a corresponding change of $B_\alpha$, which is expected (just as $\delta_Q S$) to be a combination of the ordinary SUSY and a gauge transformation. This can be seen explicitly by noting that $D_\alpha S$ must transform covariantly under the $Q$-transformations; i.e.

$$\delta_Q(D_\alpha S) = \delta_Q[(D_\alpha - iB_\alpha)S] = -iS\delta_Q B_\alpha + D_\alpha(\tau Q S) = \tau Q(D_\alpha S).$$

Using the anticommutator of $D, Q$'s

$$\{D_\alpha, Q_\beta\} = -iD_\alpha A_\beta - iQ_\beta B_\alpha,$$

14
the last equation becomes

\[ \delta Q(D\alpha S) = \bar{\epsilon}Q(D\alpha S) + iS[\bar{\epsilon}QB\alpha - D\alpha(\bar{\epsilon}\Lambda) - \delta QB\alpha] = \bar{\epsilon}Q(D\alpha S), \quad (57) \]

from which it follows

\[ \delta QB\alpha = \bar{\epsilon}QB\alpha - D\alpha(\bar{\epsilon}\Lambda) = \delta QB\alpha + \delta \Lambda B\alpha. \quad (58) \]

This is the expected result which again needs relating \( \Lambda \) to \( \epsilon \) (and \( A \)) as in the above.

Up to now, we have not assumed any relation between the two gauge superfields \( A\alpha \) and \( B\alpha \), i.e. the one appearing in the representation of the algebra and the other in gauging of the field theory. They must be related, however, if we suppose that they should transform under the same gauge transformation with a parameter \( \Lambda \). Therefore, we must look for a map of the form \( B = B(A) \) with the property that transforming \( A \) like

\[ A\alpha \to A\alpha + Q\alpha \Lambda, \quad (59) \]

corresponds to a transformation of \( B \) as

\[ B\alpha \to B\alpha + D\alpha \Lambda. \quad (60) \]

This is indeed an easy task, if we keep in mind that every superfield \( \Lambda(x, \theta) \) has a unique decomposition of the form

\[ \Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(3)} + \Lambda^{(4)}, \quad (61) \]

where \( \Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)} \) stand for the terms proportional to \( 1, \theta^+, \theta^-, \theta^+\theta^- \) in the expansion of \( \Lambda \), respectively. The advantage of decomposing \( \Lambda \) in this way is that the effect of each of the operators \( Q_\pm \) or \( D_\pm \) on a term in each class \( \Lambda^{(i)} \) is in another definite class like \( \Lambda^{(j)} \). Therefore, we can easily relate the operations of \( Q \) and \( D \) on terms in each class and, as such, relate several terms of \( A_\pm \) to that of \( B_\pm \). The result of this analysis is summarized in the following:

\[
\begin{align*}
Q_+ \Lambda^{(1)} &= -D_+ \Lambda^{(1)}, & Q_- \Lambda^{(1)} &= -D_- \Lambda^{(1)}, \\
Q_+ \Lambda^{(2)} &= D_+ \Lambda^{(2)}, & Q_- \Lambda^{(2)} &= -D_- \Lambda^{(2)}, \\
Q_+ \Lambda^{(3)} &= -D_+ \Lambda^{(3)}, & Q_- \Lambda^{(3)} &= D_- \Lambda^{(3)}, \\
Q_+ \Lambda^{(4)} &= D_+ \Lambda^{(4)}, & Q_- \Lambda^{(4)} &= D_- \Lambda^{(4)}. \quad (62)
\end{align*}
\]
Applying a similar decomposition to $A_{\alpha}$,

$$A_{\alpha} = A_{\alpha}^{(1)} + A_{\alpha}^{(2)} + A_{\alpha}^{(3)} + A_{\alpha}^{(4)},$$  \hspace{1cm} (63)

and identifying the same class terms on both sides of $\delta_{\Lambda} A_{\alpha} = Q_{\alpha} \Lambda$, and using the above set of equations, we finally obtain

$$\delta_{\Lambda} (A_{\alpha}^{(1)} - A_{\alpha}^{(2)} + A_{\alpha}^{(3)} - A_{\alpha}^{(4)}) = D_{+} (\Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(3)} + \Lambda^{(4)}),$$

$$\delta_{\Lambda} (A_{\alpha}^{(1)} + A_{\alpha}^{(2)} - A_{\alpha}^{(3)} - A_{\alpha}^{(4)}) = D_{-} (\Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(3)} + \Lambda^{(4)}).$$ \hspace{1cm} (64)

This gives the identification of $B_{\pm}$ in terms of the components of $A_{\pm}$ as follows

$$B_{+} = A_{\alpha}^{(1)} - A_{\alpha}^{(2)} + A_{\alpha}^{(3)} - A_{\alpha}^{(4)},$$

$$B_{-} = A_{\alpha}^{(1)} + A_{\alpha}^{(2)} - A_{\alpha}^{(3)} - A_{\alpha}^{(4)},$$ \hspace{1cm} (65)

upon which the desired transformation property ($\delta_{\Lambda} B_{\alpha} = D_{\alpha} \Lambda$) results. We notice that the above map preserves the conjugation properties of the gauge superfields; i.e.

$$(A_{\pm})^\dagger = -A_{\mp} \quad \Leftrightarrow \quad (B_{\pm})^\dagger = -B_{\mp}. \hspace{1cm} (66)$$

It is seen that the map $B = B(A)$ is linear and hence can be written as

$$B_{+} = CA_{+}, \quad B_{-} = \overline{C}A_{-}.$$ \hspace{1cm} (67)

The operators $C, \overline{C}$ are defined by their action on an arbitrary superfield $\Lambda(x, \theta)$ as follows

$$C\Lambda \equiv \Lambda^{(1)} - \Lambda^{(2)} + \Lambda^{(3)} - \Lambda^{(4)},$$

$$\overline{C}\Lambda \equiv \Lambda^{(1)} + \Lambda^{(2)} - \Lambda^{(3)} - \Lambda^{(4)}.$$ \hspace{1cm} (68)

It is easy then to check that

$$C^2 = \overline{C}^2 = 1, \quad [C, \overline{C}] = 0.$$ \hspace{1cm} (69)

Hence the inverses of $C, \overline{C}$ are given by themselves. Also the combined effect of $C, \overline{C}$ on $\Lambda$ is given by

$$C\overline{C}\Lambda = \Lambda^{(1)} - \Lambda^{(2)} - \Lambda^{(3)} + \Lambda^{(4)}.$$ \hspace{1cm} (70)
We can use \( C, \overline{C} \) to relate \( Q_\pm \) to \( D_\pm \)

\[
D_+ = CQ_+, \quad D_- = \overline{C}Q_-.
\] (71)

These relations are easily proved by applying both sides of them on an arbitrary superfield \( \Lambda(x, \theta) \). These together with the eq. (67) propose a “duality” of the form

\[
A_\alpha \leftrightarrow B_\alpha, \\
Q_\alpha \leftrightarrow D_\alpha.
\] (72)

In a similar way, we can find the following rules for the commutation of \( C, \overline{C} \) with \( Q_\pm \)

\[
CQ_+ = -Q_+C, \quad \overline{C}Q_+ = Q_+\overline{C}, \\
CQ_- = Q_-C, \quad \overline{C}Q_- = -Q_-\overline{C}.
\] (73)

By replacing \( Q_\pm \) in terms of \( D_\pm \) in these relations, we find their dual relations

\[
CD_+ = -D_+C, \quad \overline{C}D_+ = D_+\overline{C}, \\
CD_- = D_-C, \quad \overline{C}D_- = -D_-\overline{C}.
\] (74)

As an application of these formulae, let us derive the \( Q \)-transformation of \( B_\alpha \) using that of \( A_\alpha \). We have seen that

\[
\delta_Q B_\alpha = \tau Q B_\alpha - D_\alpha(\tau A).
\] (75)

We can express this totally in terms of \( A \) using the above mentioned map between \( A \) and \( B \). In terms of the components, this becomes

\[
\delta_Q(CA_+) = (\epsilon^+ Q_+ + \epsilon^- Q_-)CA_+ - D_+(\tau A), \\
\delta_Q(CA_-) = (\epsilon^+ Q_+ + \epsilon^- Q_-)CA_- - D_-(\tau A).
\] (76)

By multiplication of the first (second) equation by \( C, \overline{C} \) and replacing \( D_+ \) (\( D_- \)) in terms of \( Q_+ \) (\( Q_- \)) and using all the above algebra, we finally obtain

\[
\delta_Q A_\pm = \epsilon^\mp(Q_+ A_- + Q_- A_+) = i\epsilon^\mp T.
\] (77)

After all, this shows that the \( Q \)-variation of \( A_\pm \) is a gauge invariant quantity! This is a general property of the \( Q \)-transformations: indeed one can check that the \( Q \)-transformations of \((S, A_\alpha)\)

\[
\delta_Q S = \epsilon^\alpha(Q_\alpha - iA_\alpha)S, \\
\delta_Q A_\pm = i\epsilon^\mp T,
\] (78)
commute with their gauge transformations $\Lambda$; i.e. in general

\[ [\delta_\Lambda, \delta_Q] = 0. \quad (79) \]

This is equivalent to saying that $\delta_Q$ of every superfield in the theory is a gauge covariant (or invariant) quantity.

6 A Purely Bosonic Analogue

The above constructions have a purely bosonic counterpart in the ordinary (abelian) gauge theory. Let us for simplicity work with the case of a single charged (complex) scalar field $\phi(x)$ coupled to a gauge field $A_\mu(x)$ living in an arbitrary dimensional (Minkowski) spacetime. This theory is defined by the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi^*, \quad (80) \]

which is a Poincare and gauge invariant quantity. The gauge covariant derivative $\nabla_\mu$ on $\phi, \phi^*$ (with charges $\pm 1$) is defined as usual as

\[ \nabla_\mu \phi \equiv (\partial_\mu - iA_\mu)\phi, \quad \nabla_\mu \phi^* \equiv (\partial_\mu + iA_\mu)\phi^*. \quad (81) \]

The above Lagrangian is invariant under the translations

\[ \delta_\alpha \phi = a^\nu \partial_\nu \phi, \quad \delta_\alpha A_\mu = a^\nu \partial_\nu A_\mu, \quad (82) \]

as well as under the gauge transformations

\[ \delta_\Lambda \phi = i\Lambda \phi, \quad \delta_\Lambda A_\mu = \partial_\mu \Lambda. \quad (83) \]

Obviously, a general gauge transformation does not commute with translations

\[ [\delta_\alpha, \delta_\Lambda] \neq 0. \quad (84) \]

However, we can generalize translations to a new (non-linearly realized) symmetry of the theory commuting with gauge transformations. This is obtained by “knitting” the (local) gauge transformations and the (global) translations through relating their parameters in a Lorentz invariant way as

\[ \Lambda(a) = -a^\mu A_\mu(x). \quad (85) \]
Let us show the combined effect of these two transformations by

$$\tilde{\delta}_a \equiv \delta_a + \delta_{\Lambda(a)}.$$  \hspace{1cm} (86)

Then the effect of this new (global) transformation on \((\phi, A_\mu)\) is written as

$$\tilde{\delta}_a \phi = a^\mu (\partial_\mu - i A_\mu) \phi = a^\mu \nabla_\mu \phi,$$
$$\tilde{\delta}_a A_\mu = a^\nu (\partial_\nu A_\mu - \partial_\mu A_\nu) = a^\nu F_{\nu\mu}. \hspace{1cm} (87)$$

Obviously, these are written in the form of gauge covariant quantities which implies that the transformations defined by \(\tilde{\delta}_a\) and \(\delta_\Lambda\) commute,

$$[\tilde{\delta}_a, \delta_\Lambda] = 0. \hspace{1cm} (88)$$

We now consider the commutator of two \(\tilde{\delta}_a\)'s. For simplicity it is better to work with \(\tilde{\delta}_\mu\)'s, instead of \(\tilde{\delta}_a\), which are related to the latter by

$$\tilde{\delta}_a \equiv a^\mu \tilde{\delta}_\mu. \hspace{1cm} (89)$$

For the above purpose, we first compute \(\tilde{\delta}_\mu(\tilde{\delta}_\nu \phi)\) and \(\tilde{\delta}_\mu(\tilde{\delta}_\nu A_\rho)\)

$$\tilde{\delta}_\mu(\tilde{\delta}_\nu \phi) = \tilde{\delta}_\mu(\nabla_\nu \phi) = -i \tilde{\delta}_\mu A_\nu \phi + \nabla_\nu(\tilde{\delta}_\mu \phi) = -i F_{\mu\nu} \phi + \nabla_\nu \phi,$$
$$\tilde{\delta}_\mu(\tilde{\delta}_\nu A_\rho) = \tilde{\delta}_\mu F_{\nu\rho} = \partial_\nu(\tilde{\delta}_\mu A_\rho) - \partial_\rho(\tilde{\delta}_\mu A_\nu) = \partial_\nu F_{\mu\rho} - \partial_\rho F_{\mu\nu} = \partial_\nu F_{\mu\rho} \hspace{1cm} (90)$$

where in the last step we used the Bianchi identities on \(F_{\mu\nu}\). Antisymmetrizing with respect to \(\mu, \nu\) then gives

$$[\tilde{\delta}_\mu, \tilde{\delta}_\mu] \phi = -2i F_{\mu\nu} \phi - [\nabla_\mu, \nabla_\nu] \phi = -i F_{\mu\nu} \phi,$$
$$[\tilde{\delta}_\mu, \tilde{\delta}_\mu] A_\rho = \partial_\nu F_{\mu\rho} - \partial_\rho F_{\mu\nu} = -\partial_\rho F_{\mu\nu}. \hspace{1cm} (91)$$

where in the first line we have used of \([\nabla_\mu, \nabla_\nu] = -i F_{\mu\nu}\) and in the last line of the Bianchi identities. The expressions on the RHS are obviously in the form of gauge transformations on \((\phi, A_\rho)\) with a parameter equal to \(-F_{\mu\nu}\).

Thus for general transformations \(\tilde{\delta}_a, \tilde{\delta}_b\) with parameters \(a^\mu, b^\mu\) we conclude

$$[\tilde{\delta}_a, \tilde{\delta}_b] = \delta_{\Lambda(a,b)}, \hspace{1cm} (92)$$

where the field dependent parameter of the gauge transformations \(\Lambda(a,b)\) is given by

$$\Lambda(a,b) = -a^\mu b^\nu F_{\mu\nu}(x). \hspace{1cm} (93)$$
We notice that this $\Lambda$ itself is a gauge invariant quantity. The algebra of the variations $\delta_a, \delta_\Lambda$ which is summarized by the eqs.(88),(92) is obviously closed. This is in contrast to the algebra of the generators of these variations naively obtained by taking their commutators; i.e.

$$[\nabla_\mu, \nabla_\nu] = -iF_{\mu\nu}, \quad [\nabla_\rho, F_{\mu\nu}] = \partial_\rho F_{\mu\nu}. \quad (94)$$

This is a closed algebra only for those configurations of $F_{\mu\nu}(x)$ satisfying differential equations of the form

$$\partial_\rho F_{\mu\nu} = C_{\rho\mu\nu}^\kappa\lambda F_{\kappa\lambda} + D_{\rho\mu\nu}, \quad (95)$$

for some constant $C, D$ parameters. In particular, constant $F_{\mu\nu}$ configurations lead to a closed Heisenberg algebra.

7 The Algebra of the Superfields Transformations

In this section, we give direct generalizations of the results of the previous section on the closure of the algebra of the fields variations to a supersymmetrized version of it appearing in the case of a GSFT. In this case we find that the variations of the superfields $(S, A_\alpha)$ under the non-linear “generalized” versions of the translations, supersymmetry, and gauge transformations (i.e., $\delta_a, \delta_\epsilon, \delta_\Lambda$) form a closed algebra.

Let us begin this analysis by computing the effect of the commutator $[\delta_\epsilon, \delta_{\epsilon'}]$ on $S$ for two different spinor parameters $\epsilon, \epsilon'$. Initially, we have

$$\delta_\epsilon S = \tau Q S = (\tau Q + i\Lambda(\epsilon))S, \quad (96)$$

where $\Lambda(\epsilon)$ is defined by the eq.(54). The $\delta_{\epsilon'}$ variation of this expression gives

$$\delta_{\epsilon'}(\delta_\epsilon S) = iS\delta_{\epsilon'}\Lambda(\epsilon) + (\tau Q)(\bar{\tau}Q)S. \quad (97)$$

Now, we have

$$\delta_{\epsilon'}\Lambda(\epsilon) = -\delta_{\epsilon'}(\epsilon^+ A_++\epsilon^- A_-) = -(\epsilon^+\epsilon'^+ + \epsilon^-\epsilon'^-)iT, \quad (98)$$

where we used the previous result for $\delta_\epsilon A_\pm$, eq.(77). Note that $\delta_{\epsilon'}\Lambda(\epsilon) = -\delta_\epsilon\Lambda(\epsilon')$. As a result, we obtain

$$[\delta_\epsilon, \delta_{\epsilon'}]S = -[\tau Q, \bar{\tau}Q]S - 2(\epsilon^+\epsilon'^+ + \epsilon^-\epsilon'^-)TS. \quad (99)$$
The first term in this equation is easily computed using the algebra of $Q$’s, eq.(5), as follows
\[
[\tau Q, \tau' Q] = -2\epsilon^+ \epsilon'^+ Q_+^2 - 2\epsilon^- \epsilon'^- Q_-^2 - (\epsilon^+ \epsilon'^- + \epsilon^- \epsilon'^+)\{Q_+, Q_-\}
= 2\epsilon^+ \epsilon'^+ p_+ + 2\epsilon^- \epsilon'^- p_- - (\epsilon^+ \epsilon'^- + \epsilon^- \epsilon'^+) T.
\] (100)

Using this in the previous equation gives finally
\[
[\delta \epsilon, \delta \epsilon'] S = -2(\epsilon^+ \epsilon'^+ p_+ + \epsilon^- \epsilon'^- p_-) S - (\epsilon^+ \epsilon'^- + \epsilon^- \epsilon'^+) T S.
\] (101)

The RHS of this equation is evidently written as a combination of a generalized translation (generated by $P_\pm$) with a gauge transformation, i.e.
\[
[\delta \epsilon, \delta \epsilon'] S = \delta a S + \delta \Lambda S \equiv i(a^+ p_+ + a^- p_-) S + i\Lambda S,
\] (102)

where the parameters $a^\pm$, $\Lambda$ of these two transformations depend on $(\epsilon, \epsilon')$ as follows
\[
a^\pm(\epsilon, \epsilon') \equiv 2i\epsilon^\pm \epsilon'^\pm,
\Lambda(\epsilon, \epsilon') \equiv i(\epsilon^+ \epsilon'^- + \epsilon^- \epsilon'^+) T.
\] (103)

We note that the parameter of gauge transformations $\Lambda(x, \theta)$ is a gauge invariant quantity proportional to $T(x, \theta)$. We now consider the effect of $[\delta \epsilon, \delta \epsilon']$ on $A_\pm$. Taking the $\delta \epsilon'$-variation of $\delta a A_\pm$, we find
\[
\delta \epsilon'(\delta \epsilon A_\pm) = \delta \epsilon(i e^\mp T) = e^\mp[Q_+ (\delta \epsilon A_-) + Q_- (\delta \epsilon A_+)]
= -i e^\mp(\epsilon^+ \epsilon'^+ Q_+ + \epsilon^- \epsilon'^- Q_-) T.
\] (104)

Antisymmetrizing this expression with respect to $(\epsilon, \epsilon')$, we obtain
\[
[\delta \epsilon, \delta \epsilon'] A_\pm = 2ie^\mp e'^\mp Q_\mp T + i(\epsilon^+ \epsilon'^- + \epsilon^- \epsilon'^+) Q_{\mp} T.
\] (105)

The second term on the RHS is the gauge variation of $A_\pm$ with a parameter $\Lambda(\epsilon, \epsilon')$ as defined for $S$ by eq.(103). The first term which is proportional to $a^\mp(\epsilon, \epsilon')$ is the definition of $\delta a A_\pm$ (see also below)
\[
\delta a A_\pm \equiv a^\mp Q_{\mp} T.
\] (106)

As such, the last equation is in the same form as expected from the eq.(102); i.e.
\[
[\delta \epsilon, \delta \epsilon'] A_\pm = \delta a A_\pm + \delta \Lambda A_\pm.
\] (107)
Note that the above definition for $\delta_a A_\pm$ is consistent with what we may expect from the purely bosonic theory introduced in the last section. Indeed, the purely bosonic subalgebra of the generalized SUSY algebra underlying the GSFT is the one obtained from $P_\pm$ using their representations by

$$P_\pm \equiv -i \nabla_\pm \equiv -i(\partial_\pm - iA_\pm),$$

(108)

where $A_\pm$ is defined as

$$A_\pm \equiv -iQ_\pm A_\pm.$$  

(109)

This mimics the ordinary gauge field $A_\mu(x)$ in the bosonic theory, though here $A_\pm(x, \theta)$ is not a field but a superfield. Then from the bosonic theory the $\delta_a$-variations (there denoted as $\hat{\delta}_a$) of $S, A_\pm$ become

$$\delta_a S = a^+ \nabla_+ S + a^- \nabla_- S,$$

$$\delta_a A_\pm = \pm ia^\mp F,$$

(110)

where $F$ is the analogue of the ordinary field strength; i.e.

$$F \equiv i(\partial_+ A_- - \partial_- A_+) = \partial_+ Q_- A_- - \partial_- Q_+ A_+.$$  

(111)

Assuming now the previous definition for $\delta_a A_\pm$, we see that $\delta_a A_\pm$ takes precisely the expected form

$$\delta_a A_\pm = -iQ_\pm (\delta_a A_\pm) = -iQ_\pm (a^\mp Q_\pm T) = \pm ia^\mp F,$$

(112)

where we have used the relation $F \equiv Q_- Q_+ T$.

Similar to the purely bosonic case, the gauge invariances of the $\delta_a$-variations of the superfields (as seen from the eqs. (106),(110)) imply that these variations commute with their gauge variations; i.e.

$$[\delta_\Lambda, \delta_a]S = [\delta_\Lambda, \delta_a]A_\pm = 0.$$  

(113)

A similar statement holds for $\delta_\epsilon$, as we have seen

$$[\delta_\Lambda, \delta_\epsilon]S = [\delta_\Lambda, \delta_\epsilon]A_\pm = 0.$$  

(114)

Let us now consider the commutator of two $\delta_a$'s. Firstly, for $A_\pm$ we have

$$\delta_a(\delta_a' A_\pm) = \delta_a(a'^\mp Q_\pm T)$$

$$= -ia'^\mp Q_\pm (Q_+ a^+ Q_+ T + Q_- a^- Q_- T)$$

$$= a'^\mp Q_\pm (a^+ \partial_+ + a^- \partial_-) T,$$  

(115)
where we used repeatedly of the eq.(106) and the definition of $T$. As a result we obtain

$$[\delta_a, \delta_{a'}]A_\pm = \pm(a^+ a'^- - a^- a'^+)\partial_\pm Q_\pm T = Q_\pm[i(a^+ a'^- - a^- a'^+)Q_- Q_+ T],$$

(116)

where we used some algebra of $Q$’s. We see that the commutator of two $\delta_a$’s is a gauge transformation with a parameter $\Lambda(a, a') \equiv i(a \wedge a')Q_- Q_+ T = Q_\pm i(a \wedge a')F$. Where $a \wedge a' \equiv (a^+ a'^- - a^- a'^+)$. Note that the gauge parameter itself is a gauge invariant quantity proportional to $F$. To be complete, we prove the same statement also for $S$. We have

$$\delta_a(\delta_{a'} S) = \delta_a(a^+ \nabla_+ S + a^- \nabla_- S) = a'^+ (\nabla_+ \delta_a S - iS \delta_a A_+) + a'^- (\nabla_- \delta_a S - iS \delta_a A_-).$$

(118)

Applying the previous expressions for $\delta_a S$ and $\delta_a A_\pm$ to this equation and using the algebra of $\nabla_\pm$, we find

$$[\delta_a, \delta_{a'}] S = -(a^+ a'^- - a^- a'^+) [\nabla_+, \nabla_-] S - 2(a^+ a'^- - a^- a'^+) FS$$

$$= (a \wedge a') FS - 2(a \wedge a') FS = i\Lambda(a, a') S. $$

(119)

This is the expected gauge transformation of $S$ with the same parameter as found for $A_\pm$.

Finally, we consider the commutator of $\delta_\epsilon$ and $\delta_a$ on $S, A_\pm$. On $A_\pm$ we have

$$\delta_\epsilon(\delta_a A_\pm) = a^\mp Q_\pm (\delta_a T) = -a^\mp Q_\pm (\epsilon^+ Q_+ + \epsilon^- Q_-) T,$$

$$\delta_a(\delta_\epsilon A_\pm) = i \epsilon^\mp \delta_\epsilon T = i \epsilon^\mp (a^+ \partial_+ + a^- \partial_-) T. $$

(120)

In the first line we have used of $\delta_a T = -\tau QT$, which has a minus sign contrary to the naive expectation. Hence using the algebra of $Q$’s we find

$$[\delta_\epsilon, \delta_a]A_\pm = a^\mp (\epsilon^+ Q_+ Q_+ + \epsilon^- Q_+ Q_-) T - \epsilon^\mp (a^+ Q_+^2 + a^- Q_-^2) T$$

$$= Q_\pm (a^+ \epsilon^- Q_- + a^- \epsilon^+ Q_+) T \equiv Q_\pm \Lambda(a, \epsilon),$$

(121)

which is a gauge transformation on $A_\pm$ with the parameter

$$\Lambda(a, \epsilon) \equiv (a^+ \epsilon^- Q_+ + a^- \epsilon^+ Q_-) T$$

$$= a^+ \epsilon^- O_+ - a^- \epsilon^+ O_-.$$
Thus in this case the gauge parameter is a linear combination of the two gauge invariants $O_{\pm}$. Accordingly, on $S$ we have

$$\delta_\epsilon(\delta_a S) = -iS(a^+\delta_\epsilon A_+ + a^-\delta_\epsilon A_-) + (a^+\nabla_+ + a^-\nabla_-)\delta_\epsilon S. \quad (123)$$

Using the definition of $A_\pm$ (eq.(103)) we find

$$\delta_\epsilon A_\pm = -\epsilon^\pm Q_\pm T, \quad (124)$$

upon using which the last expression becomes

$$\delta_\epsilon(\delta_a S) = iS(a^+\epsilon^- Q_+ T + a^-\epsilon^+ Q_- T) + i(a^+ P_+ + a^- P_-)(\epsilon^+ Q_+ + \epsilon^- Q_-)S. \quad (125)$$

Similarly, we find

$$\delta_a(\delta_\epsilon S) = -iS(\epsilon^+ a^- Q_- T + \epsilon^- a^+ Q_+ T) + i(\epsilon^+ Q_+ + \epsilon^- Q_-)(a^+ P_+ + a^- P_-)S. \quad (126)$$

Subtracting the last two expressions gives the expected result

$$[\delta_\epsilon, \delta_a]S = 2iS(a^+\epsilon^- Q_+ T + a^-\epsilon^+ Q_- T) + i[a^+ P_+ + a^- P_-, \epsilon^+ Q_+ + \epsilon^- Q_-]S$$

$$= 2iS(a^+\epsilon^- Q_+ T + a^-\epsilon^+ Q_- T) - iS(a^+\epsilon^- Q_+ T + a^-\epsilon^+ Q_- T)$$

$$= i(a^+\epsilon^+ Q_+ T + a^-\epsilon^- Q_- T)S = i\Lambda(a, \epsilon)S, \quad (127)$$

where we have used the previous expressions for the commutators of $P_\pm$ with $Q_\pm$. This concludes our proof of the closure of the algebra of supertranslations and super gauge symmetries in a GSFT.

The above computations reveal a generic structure: when evaluating a generic commutator of two non-gauge transformations like $[\delta G_1, \delta G_2]$ on $S$, we encounter two types of terms: one is due to the variation of $S$ only, which has the form $-\delta [G_1, G_2]S$, while the other is due to the variation of $A_\alpha$ having the form $2[G_1, G_2]S$. The two terms have intriguingly similar forms so that they add up to $[G_1, G_2]S$; i.e. minus the expression would be obtained if we had considered the variation of $S$ only. In other words

$$[\delta G_1, \delta G_2] = \delta [G_1, G_2]. \quad (128)$$

Now, since $[G_1, G_2]$ is a gauge invariant quantity, the parameters of the transformations on the RHS become also gauge invariant.

Another consequence of these calculations is that the gauge invariant operators $T, O_\pm, F$ appearing on the RHS of the algebra of $P_\pm, Q_\pm$ may be
interpreted as a basis of expansion for the parameter $\Lambda(x, \theta)$ of the gauge transformations and, hence, as the generators of these transformations. This is because, as we have seen, the parameters of the transformations on the RHS are gauge invariant quantities proportional to these gauge invariant operators.

8 Conclusion

In this paper, using the notion of a gauge superfield $A_\alpha$ as a gauge connection on superspace, we have constructed a generalized class of SUSY algebras, which besides including the usual (super)translation generators $P_\pm, Q_\pm$, they involve the new generators $T, F, O_\pm$ corresponding to the generators of the $U(1)$ gauge symmetry associated to $A_\alpha$. We found that this algebra is gauge invariant, but in general not closed in the sense of a super Lie algebra, if we treat $A_\alpha$ as some fixed superfield. Nevertheless, it is closed for very specific configurations of $A_\alpha$ (or more precisely, its field strength $T$). Two examples within this class are the centrally extended supersymmetry (CE-SUSY) and the non-commutative supersymmetry (NC-SUSY) algebras. These are indeed the only rotationally symmetric configurations for which the algebra also closes with the $SO(2)$ rotation generator $J$. There are, however, other non-rotationally symmetric configurations of the gauge superfield on which also the algebra closes (but of course without the $SO(2)$ generator).

We showed that the generalized SUSY algebra is realized in a gauge superfield theory (GSFT) which, in addition to a global supersymmetry, it also possesses a local superspace gauge symmetry. In this case, the generalized supertranslation symmetries are realized as a result of knitting these (local and global) symmetries of the theory in a particular way. As transformations changing both the gauge and matter superfields $(A_\alpha, S)$, the supertranslations and super gauge transformations were found to form a closed algebra. We noted that, in this interpretation, the generalized supersymmetry is realized by a non-linear realization of the corresponding transformations $\delta_a, \delta_\epsilon, \delta_\Lambda$ on $(A_\alpha, S)$. The algebra is then as follows

$$
[\delta_\epsilon, \delta_\epsilon'] = \delta_a + \delta_\Lambda, \quad [\delta_a, \delta_a'] = \delta_\Lambda, \quad [\delta_\epsilon, \delta_\alpha] = \delta_\Lambda,
$$
$$
[\delta_\Lambda, \delta_\epsilon] = 0, \quad [\delta_\Lambda, \delta_\alpha] = 0, \quad [\delta_\Lambda, \delta_\Lambda] = 0.
$$

(129)
The last commutation relation accounts for the fact that the gauge symmetry under consideration is an abelian one (in the superspace sense it is a $U(1)$ gauge symmetry). It would be interesting to find the non-abelian generalization of the above constructions. As might be expected, the above algebra generalizes the ordinary supertranslation algebra by replacing any vanishing commutator in the ordinary case by a gauge transformation. We computed the dependences of the transformation parameters on the RHS of these equations to the parameters on their LHS.

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9 Appendix

In this appendix we collect some definitions, conventions and properties which are required in the course of this paper.

The complex coordinates $x^\pm$ and their derivatives are defined in terms of the real coordinates $(x^1, x^2)$ as

$$
x^\pm \equiv \frac{1}{2}(x^1 \pm ix^2), \quad \partial_x \equiv \partial_1 \mp i\partial_2. \quad (A.1)
$$

We use the complex spinors $\theta$ with the components $\theta^\pm$ related by conjugation

$$(\theta^+)^* = \theta^-.$$  \quad (A.2)

The inner product of two spinors $\epsilon$ and $\theta$ is defined in terms of their components as follows

$$\bar{\epsilon}\theta \equiv \epsilon^+\theta_+ + \epsilon^-\theta_-.$$  \quad (A.3)

The spinorial indices are raised with $\varepsilon^{\alpha\beta}$ and lowered with $\varepsilon_{\alpha\beta}$, which means that

$$\theta_+ = \theta^-, \quad \theta_- = -\theta^+.$$  \quad (A.4)

We can check that the above inner product is always real; i.e.

$$(\bar{\epsilon}\theta)^* = \bar{\epsilon}\theta.$$  \quad (A.5)

26
Under a $SO(2)$ rotation with an angle $\alpha$, the spinor and the vector components with higher indices are transformed as

$$\theta^\pm \to e^{\pm i\alpha/2} \theta^\pm, \quad x^\pm \to e^{\pm i\alpha} x^\pm,$$

(A.6)

while those with lower indices are changed by the inverses of these transformations.

Supertranslations are defined as

$$\delta \theta^\pm = \epsilon^\pm, \quad \delta x^\pm = i\epsilon^\pm \theta^\pm.$$

(A.7)

The corresponding ordinary translation and supertranslation generators are

$$P_\pm = -i\partial_\pm, \quad Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \partial_\pm.$$

(A.8)

The ordinary supertranslation algebra is then given by

$$Q_\pm^2 = -P_\pm, \quad \{Q_+, Q_-\} = 0,$$

$$[P_+, P_-] = 0, \quad [P_\pm, Q_\pm] = 0.$$  

(A.9)

The conjugation properties of these operators are

$$(P_\pm)^\dagger = P_\mp, \quad (Q_\pm)^\dagger = Q_\mp,$$

(A.10)

while those for their action on a complex (grassmann even) superfield $S$ are

$$(P_\pm S)^* = -P_\mp S^*, \quad (Q_\pm S)^* = -Q_\mp S^*.$$

(A.11)

For a grassmann odd superfield the sign on the RHS of the second equation is reversed.

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