A Zero-Curvature Representation of Electromagnetism and the Conservation of Electric Charge

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Abstract
We show that the laws of electromagnetism in \((D + 1)\)-dimensional Minkowski space-time \(\mathcal{M}\), explicitly for \(D = 1, 2\) and 3, can be obtained from an integral representation of the zero-curvature equation in the corresponding loop space \(\mathcal{L}^{(D-1)}(\mathcal{M})\). The conservation of the electric charge can be seen as the result of a hidden symmetry in this representation of the dynamical equations.

Keywords Loop space · Zero-curvature representation · Electromagnetism · Conserved charges

1 Introduction
Hidden symmetries play a special role in the construction of soliton solutions and their conserved charges in integrable field theories [1]. In \((1 + 1)\)-dimensional space-time these theories will generally admit what is called a zero-curvature representation [2, 3] in which their dynamical equations become equivalent to the flatness of a Lie algebra-valued 1-form connection whose components are functions of the physical fields and their derivatives.

This representation lies in the core of the construction of conserved charges [4] which are obtained from the holonomy operator associated to that 1-form connection: its flatness implies the path independence of the holonomy in space-time and leads to an isospectral evolution of the holonomy evaluated over space; the preserved eigenvalues of this operator are the conserved charges.

In integrable field theories, some of these charges may coincide with those obtained from Noether’s theorem but the symmetries here are hidden in the gauge invariance of the zero-curvature equation, which also defines the ground for the development of algebraic methods to construct soliton solutions.

The possibility of extending the zero-curvature formulation to field theories in higher-dimensional space-time as an attempt to understand integrability in this context was explored in [5]. The crucial step toward this approach was the generalization of the holonomy operator through a non-abelian Stokes theorem and the interpretation of this construction in loop space [6]. In [7, 8] it was shown how this non-abelian Stokes theorem can be used to define the integral version of Yang–Mills equations leading to dynamically gauge-invariant conserved charges. Some consequences of this formulation of gauge theories in loop space were discussed in [9, 10].

In the present paper we show that the integral equations of electromagnetism can be represented in loop space by an integral equation for a flat connection \(\mathcal{A}\). In a Minkowski \((D + 1)\)-dimensional space-time \(\mathcal{M}\), this connection is constructed in the corresponding loop space which is defined by the maps \(\mathcal{L}^{(D-1)}(\mathcal{M}) \equiv \{ \Gamma : S^{D-1} \to \mathcal{M} | \Gamma(0) = x_R \}\), taking \((D - 1)\)-dimensional spheres in space-time, based at \(x_R\), to points in this loop space. The connection is defined in terms of an exact \(D\)-form in space-time evaluated on the loop \(S^{D-1}\). The flatness of the loop space connection, \(\delta \mathcal{A} = 0\), will give the local conservation of the electric charge and the charge itself can be obtained as the eigenvalues of the generalized \(D\)-holonomy evaluated over the space as a consequence of the (hidden) symmetry of the integral equation in loop space: its invariance under homotopic transformations of the path.

The integral equation in loop space is written in space-time as a Lorentz scalar integral equation and by choosing...
appropriately a space-time slicing with \( D \)-dimensional hyper-volumes one can recover the usual integral expressions, namely Gauss’, Faraday and Ampère–Maxwell laws.

2 The Integral Equations of Electromagnetism in \( 1 + 1 \)Dimensions

The Maxwell equations in 2-dimensional space-time, in Gaussian coordinates, are given by

\[
\partial_{\mu} F^{\mu\nu} = \frac{2}{c} J^\nu \quad \mu, \nu = 0, 1
\]

(1)

where \( J^\mu = (c\rho, j) \) is the covariant electric current density and the Faraday tensor defined in terms of the gauge potential \( A^\mu \) is given by \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) having \( F_{01} = E = -F_{10} \) the only non-vanishing components, with \( E \) the electric field.

We consider the loop space \( \mathcal{L}^{(0)}(\mathcal{M}) \). This is the space of maps \( \Gamma \) from the sphere \( S^0 \) into the space-time manifold \( \mathcal{M} \) defined as \( \mathcal{L}^{(0)}(\mathcal{M}) = \{ \Gamma : S^0 \to \mathcal{M} | \Gamma(0) = x_R \} \) where \( x_R \) is a fixed point in \( \mathcal{M} \) which we call the reference point. The image of the map \( \Gamma \) will be, in this case, also points in the loop space \( \mathcal{L}^{(0)}(\mathcal{M}) \).

Let us define a 1-form connection \( \mathcal{A} \) in this loop space [6] as

\[
\mathcal{A} = A_{\mu} \delta x^\mu.
\]

(2)

We want to find an integral representation of the Maxwell equations in \( \mathcal{L}^{(0)}(\mathcal{M}) \) based on a flat connection in this space. This can be done if we write the field in space-time \( A_{\mu} \) as the components of the exact 1-form \( \mathcal{A} = df = \partial_{\mu} f^\mu dx^\mu \), since then \( \delta \mathcal{A} = \frac{i}{2} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) \delta x^\mu \wedge \delta x^\nu = 0 \).

Let us write \( f \) in terms of the physical fields as

\[
f = \frac{i}{2} \beta \epsilon_{\mu\nu\rho} F_{\mu\nu}.
\]

(3)

where \( \beta \) is an arbitrary constant.

Using the equations of motion we obtain the components \( A_{\mu} \) as

\[
A_{\mu} = \frac{i}{c} \beta \epsilon_{\mu\nu} J^\nu
\]

and consequently the loop space connection is given by

\[
\mathcal{A} = \frac{i}{c} \beta \epsilon_{\mu\nu} J^\nu \delta x^\mu.
\]

(4)

(5)

Given the flatness of the connection we can associate its integral over a path \( \Gamma \) in loop space with the value of a potential \( \varphi \) evaluated at the borders of this path. This is the integral representation of the zero-curvature equation for the connection in loop space:

\[
\Delta \varphi = \int_{\Gamma} \mathcal{A}(\sigma) \, d\sigma
\]

(6)

where \( \mathcal{A}(\sigma) = A_{\mu} \frac{dx^\mu}{d\sigma} \), with \( \sigma \) parameterizing the path \( \Gamma \). Clearly, by construction we have that \( \varphi = f \) and \( A_{\mu} = \partial_{\mu} f \).

From the definitions of \( f \) and \( A_{\mu} \) in terms of the physical fields and their local relations given by Eq. (1) this equation becomes

\[
E(x) - E(x_R) = \frac{2}{c} \int_{0}^{2\pi} \epsilon_{\mu\nu\rho} \frac{dx^\mu}{d\sigma} \, d\sigma.
\]

(7)

This is the Lorentz scalar integral equation of electromagnetism in \( 1 + 1 \) dimensional space-time. In order to obtain the usual version of the integral equations, here equivalent to the Gauss’ and Ampère’s laws for the electric field, we need to specify the curve \( \gamma \) is space-time where we integrate the dual of the electric current (see Fig. 1).

When \( \gamma \) is considered to be purely spatial at a constant time \( t \) we have, from Eq. (7)

\[
E(t,x) - E(t,x_R) = \frac{2}{c} \int_{x_R}^{x} \epsilon_{\mu\nu\rho} \frac{dx^\mu}{d\sigma} \, d\sigma = 2 \int_{x_R}^{x} \rho \, dx'
\]

(8)

which can be recognized as the Gauss law for the electric field.

Next, the Ampère–Maxwell law follows from the integral Eq. (7) when we consider the curve \( \gamma \) to have constant spatial coordinate \( x \):

\[ \text{Fig. 1} \] With an appropriate choice for the paths in space-time the Lorentz scalar integral equations will give the usual integral expressions for the laws of electrodynamics.
Fig. 2. The curve in the (1 + 1)-dimensional space-time is equivalent to the path defined in the corresponding loop space.

\[ E(t, x) - E(0, x) = \frac{2}{c} \int_0^t \epsilon_j J^j \, dx^0 \, d\sigma = -2 \int_0^t j \, dt. \quad (9) \]

We notice that if the curve \( \gamma \) is taken to be infinitesimal, i.e., \( x = x_R + \delta x \), then the differential Eq. (1) can be recovered from Eqs. (8) and (9):

\[
E(t, x) - E(t, x_R) \approx 2\rho \Delta x \implies \frac{\partial E}{\partial x} = 2\rho
\]
\[
E(t, x) - E(0, x) \approx -2j \Delta t \implies \frac{\partial E}{\partial t} = -2j.
\]

2.1 The Conservation of the Electric Charge in 2 Dimensions

The loop space representation of the integral dynamical equations defines a relation between the 0-form \( \varphi \) at the borders of the path \( \Gamma \) with the integral of the 1-form connection \( A \) along this path.

Considering an infinitesimal variation of \( \Gamma \) keeping its borders fixed, we find from Eq. (6) that

\[ \delta(\Delta \varphi) = \int_{\Gamma} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) \frac{dx^\nu}{d\sigma} \delta x^\mu \, d\sigma. \]

Since the border of \( \Gamma \) remains fixed, \( \Delta \varphi \) should not change and consequently the l.h.s of the equation above vanishes. On the other hand, given that \( A \) is exact, also the r.h.s above vanishes and we conclude that Eq. (6) remains invariant under homotopic deformations of the path \( \Gamma \). In other words, \( \varphi \) is path-independent and equivalently \( A \) is flat. Writing the loop space connection in terms of the physical fields we have

\[ \delta A = \partial_{\mu} A_{\nu} \delta x^\nu \wedge \delta x^\mu = \frac{\sqrt{2} \beta}{c} \epsilon^{\nu\mu\lambda} \partial_\lambda J^0 \delta x^0 \delta x^1 = \frac{i4\beta}{c} \partial_{\nu} J^\nu \delta x^0 \delta x_1 \]

(10)

and consequently \( \delta A = 0 \) implies \( \partial_{\mu} J^\mu = 0 \), i.e., the local conservation of the electric charge is obtained as a consequence of the zero-curvature of the loop space connection.

This path independence of the integral Eq. (6) in loop space is the hidden symmetry behind this conservation law.

The conserved charges associated to this symmetry can be obtained from the holonomy operator defined by the parallel transport equation along a curve \( \gamma \) in space-time parameterized by \( \sigma \in [0, 2\pi] \).

\[ \frac{dW}{d\sigma} + A_{\mu} \frac{dx^\mu}{d\sigma} W = 0, \]

(11)
whose solution can be formally written as

\[ W_\gamma = e^{-\int_{\gamma_0}^{\gamma} A_{\mu} \frac{dx^\mu}{d\sigma}} W_0, \]

(12)
where \( W_0 \) is obtained from the initial conditions.

We consider the paths given in Fig. 2 joining the points \( x_R \) and \( x = (ct, L) \). We assume that the two paths \( \gamma_L \circ \gamma_0 \) and \( \gamma_r \circ \gamma_s \) can be deformed into each other by continuous transformations \( x^\mu \rightarrow x^\mu + \delta x^\mu \) which make the holonomy \( W \) calculated over \( \gamma_r \circ \gamma_s \), changes as \(^2\)

\[ \delta W = \int_0^{2\pi} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) \frac{dx^\nu}{d\sigma} \delta x^\mu \, d\sigma. \]

(13)

The r.h.s of Eq. (13) vanishes since \( A \) is an exact 1-form, giving \( \delta W = 0 \), which means that the holonomy is in fact path-independent. So, the holonomy calculated over \( \gamma_L \circ \gamma_0 \) is identical to that calculated over \( \gamma_r \circ \gamma_s \) and we have the identity

\[ W_{\gamma_r} \cdot W_{\gamma_s} = W_{\gamma_L} \cdot W_{\gamma_0}. \]

(14)

In terms of the physical fields, using Eq. (12) with \( W_0 = 1 \) for simplicity, the operator on the r.h.s. is given by

\(^2\) See Appendix 1
\[ W_{tL} \cdot W_{t0} = e^{i2\beta \int_{t_0}^{t_1} d\tau} e^{-i2\beta \partial_0 t} \]

and similarly for the l.h.s.

\[ W_{t_1} \cdot W_{t_{R}} = e^{-i2\beta \partial_1 t} e^{i2\beta \cdot 0} \]

where \( Q \) is the electric charge

\[ Q = \int_{x_R}^{x_L} \partial \, dx. \quad (15) \]

We can rewrite (14) as

\[ W_{t_1} = W_{tL} \cdot W_{t_{R}} \cdot W^{-1}_{t_{R}} \quad (16) \]

and assuming that \( \beta(t, x) \to 0 \) in the limit where \( x_R \) and \( L \) go to infinity, the operators \( W_{tL} \) and \( W_{t_{R}} \) become the identity and we remain with

\[ e^{-i2\beta \partial_0 t} = e^{-i2\beta \partial_0 t(0)} \Rightarrow e^{i2\beta \Delta Q} = 1 \quad (17) \]

where \( \Delta Q \equiv Q|_{t>0} - Q|_{t=0} \). Since \( \beta \) is arbitrary this identity implies

\[ \Delta Q = 0 \quad (18) \]

giving us the conservation of electric charge in time. These charges are defined as the eigenvalues of the holonomy restricted to space

\[ W_{t} = e^{-i2\beta \partial_0 t}. \quad (19) \]

This conservation law is a consequence of the path invariance of the holonomy operator or equivalently, of the flatness of the connection in loop space.

### 3 The Integral Equations of Electromagnetism in 2+1 Dimensions

In 3-dimensional space-time the differential equations of electrodynamics are given by [11]

\[ \partial_\mu F^{\mu\nu} = \frac{2\pi}{e} J^\nu \quad (20) \]

\[ \partial_\mu \tilde{F}^{\mu} = 0, \quad \mu; \nu = 0, 1, 2 \quad (21) \]

where \( J^\nu = (c, j^1, j^2) \) is the electric 3-current density, \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field whose components are the electric vector field \( E \) and the magnetic scalar field \( B \) given as \( F_{0i} = E_i \) and \( F_{ij} = -\varepsilon_{ij} B \), and \( \tilde{F}^{\mu} = \frac{1}{2} e^{i\nu\lambda} F_{\nu\lambda} \) is the Hodge dual of the electromagnetic field\(^3\).

The integral equations of electromagnetism will be represented in the loop space \( L^{(1)}(\mathcal{M}) \) which is defined by the maps \( L^{(1)}(\mathcal{M}) = \{ \Gamma : S^1 \to \mathcal{M} | \Gamma(0) = x_R \} \). We consider a reference point \( x_R \) in space-time \( \mathcal{M} \) and a family of loops (closed curves) based at this point. Then, each of these loops will correspond to a point in loop space, which is the image of the map defined above. So, a collection of homotopic loops scanning a 2-dimensional surface will define, in the loop space, a path \( \Gamma \) (Fig. 3). We define a 1-form connection in the loop space as

\[ A = \oint_t G_{\mu\nu} \partial x^\mu / \partial \sigma \delta x^\nu d\sigma \]

where \( G_{\mu\nu} \) is an antisymmetric tensor which we integrate over each loop based at \( x_R \) parameterized by \( \sigma \in [0, 2\pi] \) and labelled by \( \tau \in [0, 2\pi] \). So, this connection is defined, in space-time, on each loop and thus, it takes values at the points of the loop space \( L^{(1)}(\mathcal{M}) \).

Taking \( G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \), the components of the exact 2-form \( G = dC \), this connection becomes flat, i.e., its curvature vanishes: \( \delta A = 0 \). So, in order to find an integral representation of the zero-curvature equation as the integral equations of the electromagnetism in loop space we write the components of the 1-form \( C = C_\mu dx^\mu \) in terms of the physical fields as

\[ C_\mu = i(eA_\mu + \beta \tilde{F}_\mu) \]

where \( \beta \) is an arbitrary constant and \( e \) the elementary electric charge.

Using the dynamical Eqs. (20) and (21) the components of \( G \) can be written as

\[ G_{\mu\nu} = ieF_{\mu\nu} + \frac{i2\pi\beta}{e} \varepsilon_{\mu\lambda\nu} J^\lambda \]

and the connection in loop space reads

\[ A = \oint_t F_{\mu\nu} \partial x^\mu / \partial \sigma \delta x^\nu d\sigma + \frac{i2\pi\beta}{e} \oint_t \varepsilon_{\mu\lambda\nu} J^\lambda \partial x^\mu / \partial \sigma \delta x^\nu d\sigma. \quad (25) \]

\(^3\) We use \( e_{021} = 1 \) and \( \varepsilon_{ij} \equiv e_{ij}^{021} \).
As in the previous case, the representation of the integral equations in loop space will be defined by an equation like Eq. (6),

$$\Delta \varphi = \int_{\Gamma} A(\tau) d\tau$$  \hspace{1cm} (26)

where $\Gamma$, the path in loop space, stands now for the scanning of the 2-dimensional surface $\Sigma$ in space-time with homotopically equivalent loops, based at $x_\mu$ which is located at the border $\partial \Sigma$ and

$$A(\tau) \equiv \int_0^{2\pi} G_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma.$$  \hspace{1cm} (27)

This integral equation in loop space is in fact a representation of the Stokes theorem in space-time for the 1-form $C$, which is in this case the potential $\varphi$:

$$\int_{\partial \Sigma} C_\mu \frac{dx^\mu}{d\sigma} d\sigma = -\int_{\Sigma} G_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau,$$  \hspace{1cm} (28)

where $\Sigma$ is a 2-dimensional surface in space-time and $\tau \in [0, 2\pi]$ parameterizes the loops scanning this surface such that the loop with $\tau = 0$ is the infinitesimal one (or point-loop) around the reference point $x_\mu$ and the loop with $\tau = 2\pi$ is that which defines the border $\partial \Sigma$ of the surface $\Sigma$.

The integral equations of electromagnetism are a consequence of the Stokes theorem and the differential equations of motion, so, writing the fields in Eq. (28) as defined by Eqs. (23) and (24), given the arbitrariness of $\beta$ we have the set of equations

$$\int_{\partial \Sigma} F_\mu \frac{dx^\mu}{d\sigma} d\sigma = -\int_{\Sigma} \epsilon_{\mu\nu\lambda} F^\lambda \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau$$  \hspace{1cm} (29)

and defining the normal vector to the curve $\partial \Sigma$ as $\hat{n} dr = e_\beta dx^\beta$ we can write the above result as

$$\int_{\partial \Sigma} \tilde{F}_\mu \frac{dx^\mu}{d\sigma} d\sigma = -\int_{\Sigma} \epsilon_{\mu\nu\lambda} \tilde{F}^\lambda \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$  \hspace{1cm} (30)

Let us show that this set of Lorentz scalar integral equations imply those usually presented, namely, the Gauss law, the Maxwell–Ampère law and the Faraday law.

The Gauss law is obtained from Eq. (30) when we take the surface $\Sigma$ to be completely spatial. For the l.h.s of this equation we get

$$\int_{\partial \Sigma} \tilde{F}_\mu \frac{dx^\mu}{d\sigma} d\sigma = \int_{\gamma_0} \tilde{F}_\mu d\mu - \int_{\gamma_0} \tilde{F}_\mu d\mu + \int_{\gamma_1} \tilde{F}_\mu d\mu - \int_{\gamma_1} \tilde{F}_\mu d\mu - \int_{\gamma} \tilde{F}_\mu d\mu$$

$$= \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma_1} \epsilon_\beta E_\beta d\mu - \int_{\gamma} \epsilon_\beta E_\beta d\mu$$

$$= \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma} \epsilon_\beta E_\beta d\mu$$

$$= \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma} \epsilon_\beta E_\beta d\mu$$

$$= \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_0} \epsilon_\beta E_\beta d\mu - \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma_1} \epsilon_\beta E_\beta d\mu + \int_{\gamma} \epsilon_\beta E_\beta d\mu$$

Fig. 4 In 2 + 1 dimensions we consider a 2-dimensional surface in space-time in order to obtain the Ampère–Maxwell law from the Lorentz scalar integral equation

$$\int_{\partial \Sigma} \tilde{F}_\mu \frac{dx^\mu}{d\sigma} d\sigma = \int_{\Sigma} \epsilon_{\mu\nu\lambda} \tilde{F}^\lambda \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau,$$  \hspace{1cm} (32)

where $dS = \epsilon_\beta \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau$ is the area element.

Then the integral Eq. (30) becomes the usual Gauss’ law for the electric field:

$$\int_{\partial \Sigma} \tilde{F}_\mu \frac{dx^\mu}{d\sigma} d\sigma = 2\pi \int_{\Sigma} \rho dS.$$  \hspace{1cm} (33)

Now, consider the 2-dimensional surface $\Sigma$ with a component in the $x^0$ direction as depicted in Fig. 4. In this case, the l.h.s. of Eq. (30) reads

$$\int_{\partial \Sigma} \tilde{F}_\mu \frac{dx^\mu}{d\sigma} d\sigma = 2\pi \int_{\Sigma} \rho dS.$$  \hspace{1cm} (34)
If we consider an infinitesimal time lapse in the results above then the Eq. (30) becomes
\[
\int_{\gamma_t} \mathbf{E} \cdot d\mathbf{r} - \int_{\gamma_0} \mathbf{E} \cdot d\mathbf{r} - c\Delta t (B[b]_b - B[a]_a) = -2\pi\Delta t \int_a^b \mathbf{j} \cdot d\mathbf{r}.
\]
and in the limit where \(\Delta t \rightarrow 0\) we have finally the usual Ampère–Maxwell law
\[
B[b]_b - B[a]_a = \frac{1}{c} \frac{d}{dt} \int_a^b \mathbf{E} \cdot d\mathbf{r} + 2\pi \int_a^b \mathbf{j} \cdot d\mathbf{r}.
\]

The Faraday law is in fact a mathematical identity and this is clear from the integral version given by the Lorentz scalar expression (Eq. 29). In order to obtain the usual formula of this law we consider a 2-dimensional spatiotemporal surface defined by folding the previously used open surface such that \(\gamma_a \sim \gamma_b^{-1}\).

The l.h.s. of Eq. (29) reads
\[
\oint A_\mu dx^\mu = \int_{\gamma_0} A_\mu dx^\mu - \int_{\gamma_t} A_\mu dx^\mu
\]
and using Eq. (29) again for each of the terms above at constant time we get
\[
\oint A_\mu dx^\mu = \int_{S_0} B|_{t=0} dS - \int_{S_t} B|_{t>0} dS
\]
where \(S_0\) and \(S_t\) are the areas enclosed by \(\gamma_0\) and \(\gamma_t\), respectively.

Now, for the r.h.s of Eq. (29) we have
\[
-\int_{\Sigma} \mathbf{F} \cdot d\sigma = -\int_{\gamma_t} \mathbf{E} \cdot d\mathbf{r} - c\Delta t \int_{\gamma_0} \mathbf{j} \cdot d\mathbf{r}.
\]
\[
\Delta t \rightarrow 0
\]
which, in the limit of \(\Delta t \rightarrow 0\) becomes (Fig. 5)
\[
\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c} \frac{d}{dt} \int_{S_0} B dS.
\]

### 3.1 The Conservation of the Electric Charge in 3 Dimensions

The loop space connection is flat and this result is a direct consequence of the fact that \(G\) is exact so \(dG = 0\) and
\[
\delta A = \oint (\partial_\lambda G_{\mu\nu} + \partial_\mu G_{\nu\lambda} + \partial_\nu G_{\lambda\mu}) \frac{\partial x^\mu}{\partial \sigma} \delta x^\nu \wedge \delta x^\sigma = 0.
\]

In terms of the physical fields, we find that
\[
\delta A = \frac{i2\pi\beta}{c} \int \partial_\mu J^\mu \ d^3x = 0
\]
and consequently the electric charge is conserved.
These conserved charges can be obtained from the generalization of the holonomy operator defined by the 2-holonomy $V$: a parallel transport operator in loop space $L^{(1)}(\mathcal{M})$ obeying the equation\(^4\)

\[
\frac{dV}{d\tau} - \left( \int_0^{2\pi} G_{\mu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma} d\sigma \right) V = 0
\]  

(43)

whose solution can be formally written as

\[
V_{\Sigma} = e^{\int_{\Sigma} G_{\mu} \frac{\partial x^\mu}{\partial \tau} d\tau} V_0
\]  

(44)

where $\Sigma$ is a 2-dimensional surface and $V_0$ is obtained from the initial conditions.

We consider the path $\Gamma_{L} \cdot \phi \cdot \Gamma_{O}$ in loop space, as in Fig. 6: the spatial surface $\Sigma_0 \sim \Gamma_{O}$ at constant time $t = 0$ is scanned with loops starting at the point-loop at $x_R$, until the loop which defines the border $\partial \Sigma_0$, and then, moving from this last loop forward in time up to the loop $\partial \Sigma_t$. The 2-holonomy along this path (over this surface) is given by (considering, for simplicity, $V_0 = 1$)

\[
V_{\Gamma_{L} \cdot V_{\Gamma_{O}}} = e^{\int_{\Sigma_0 \times \mathbb{R}} G_{\mu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma} d\sigma d\tau} V_0
\]  

(45)

From Eq. (24) we have that

\[
\int_{\Sigma \times \mathbb{R}} G_{\mu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau = \frac{-ie}{c} \int \mathbf{E} \cdot d\mathbf{r} + i2\pi \beta \int \mathbf{j} \cdot d\mathbf{r}.
\]

Assuming that $\|\mathbf{E}\|$ falls off quickly enough and that the electric current is localized, the quantity above should vanish in the limit where the radius of $\partial \Sigma_0$ goes to infinity. What remains is then

\[
V_{\Gamma_{L} \cdot V_{\Gamma_{O}}} = e^{\int S_{\Sigma_0} F_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau} e^{\int s_{\Sigma_0} \mathbf{E} \cdot d\mathbf{r}} = e^{-ie \int_{\Sigma_0} \mathbf{B} \cdot d\mathbf{r}} e^{2\pi \beta \int_{\Sigma_0} d\mathbf{r}}.
\]

(46)

In the construction of the 2-holonomy over the path $\Gamma_{L} \cdot \Gamma_{x_R}$, we notice that $V_{\Gamma_{x_R}}$ becomes trivial as the radius of the point-loop around $x_R$ becomes zero and we get

\[
V_{\Gamma_{L} \cdot V_{\Gamma_{x_R}}} = e^{-ie \int_{\Sigma_0} \mathbf{B} \cdot d\mathbf{r}} \cdot e^{2\pi \beta \int_{\Sigma_0} d\mathbf{r}}.
\]

(47)

The path independence of the 2-holonomy is a direct consequence of the flatness of the connection in loop space and it implies that, once $\Gamma_{L} \cdot \Gamma_{x_R}$ can be obtained from continuous deformations of $\Gamma_{L} \cdot \phi \cdot \Gamma_{O}$, we have the relation

\[
V_{\Gamma_{L} \cdot V_{\Gamma_{x_R}}} = V_{\Gamma_{L} \cdot V_{\Gamma_{O}}} \Rightarrow V_{\Gamma_{L} \cdot V_{\Gamma_{O}}} = V_{\Gamma_{L} \cdot V_{\Gamma_{x_R}}}^{-1}.
\]

(48)

Given that the operators $V_{\Gamma_{L}}$ and $V_{\Gamma_{x_R}}$, as discussed above, become unity, the path independence of the 2-holonomy gives us that the electric charge $Q$, defined by the eigenvalues of

\[
V_{\Gamma_{L}} = e^{2\pi \beta |Q|},
\]

(49)

and the magnetic flux over the entire space, defined by the eigenvalues of

\[
V_{\Gamma_{L}} = e^{-ie \int_{\Sigma_0} \mathbf{B} \cdot d\mathbf{r}}
\]

(50)

are conserved in time.

### 4 The Integral Equations of Electromagnetism in 3 + 1Dimensions

In 3 + 1 dimensions, the electric and magnetic vector fields $\mathbf{E}$ and $\mathbf{B}$ can be written as the components of the electromagnetic field strength $F_{\mu\nu}$. $\nu = 0, \ldots, 3$ as $E^t = F_{t\nu} = -F_{\nu t}$ and $B^k = -\frac{1}{2} \varepsilon_{ijk} F_{ij}^t$.

Maxwell equations are given by

\[
\frac{\partial F_{\mu\nu}}{\partial \tau} = \frac{4\pi}{c} J^\mu
\]

(51)

\[
\frac{\partial B^k}{\partial \tau} = 0
\]

(52)

where $J^\mu = (c \rho, \mathbf{j})$ is the Lorentz covariant current and $\mathbf{F}^\mu = \frac{1}{2} \varepsilon_{ijk} F_{ij}^t$ is the Hodge dual of the electromagnetic field.

The construction of the integral representation of the Maxwell equation is done in the loop space $L^{(2)}(\mathcal{M}) = \{ \Gamma : S^2 \rightarrow \mathcal{M} | \Gamma(0) = x_R \}$. This mapping will relate closed 2-dimensional surfaces in space-time, which are all based at the reference point $x_R$, with points in the loop space.

We define the 1-form connection in $L^{(2)}(\mathcal{M})$ by

\[
A = \int_\Sigma H_{\mu\nu\lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \sigma} d\sigma d\tau
\]

(53)

where $H_{\mu\nu\lambda}$ is a 3-form in space-time which is integrated over the 2-dimensional closed surface $\Sigma$, parameterized by $\sigma \in [0, 2\pi]$, $\tau \in [0, 2\pi]$. By scanning a 3-dimensional volume in space-time with these closed surfaces as in Fig. 7 we define a path in the loop space.

\[\square\]
Let us take $H = dB$, i.e., we consider it to be an exact 3-form. This is a sufficient condition for $A$ to be flat, i.e., $\delta A = 0$. We look for an integral representation in loop space of this zero-curvature equation such that it will be equivalent to Maxwell integral equations of electromagnetism. Such an

$$\partial_3 B_{\mu \nu} + \partial_\mu B_{\nu 3} + \partial_\nu B_{\mu 3} = i \alpha (\partial_\mu F_{\nu 3} + \partial_\nu F_{\mu 3} + \partial_3 F_{\mu \nu}) + i \beta \epsilon_{\mu \nu \lambda} \partial_\lambda F_{\rho \tau} = \frac{4 \pi \beta}{c} \epsilon_{\mu \nu \lambda} F_{\rho \tau}^\rho \tau.$$  \hfill (58)

and the mathematical relation Eq. (57) defines the Lorentz scalar integral equations of electrodynamics:

$$\tilde{F} = 0.$$ \hfill (61)

These are Lorentz scalar equations for the flux of the electromagnetic field strength and its Hodge dual through 2-dimensional surfaces in Minkowski space-time.

We now proceed to show that these equations imply the usual integral laws of electrodynamics when the 3-dimensional volumes in space-time are appropriately chosen.

Let us start by considering Eq. (61), which is the integral version of the Bianchi identity, when $\Omega$ is a 3-dimensional spatial volume at a given instant of time. The l.h.s of that equation becomes

$$\oint_{\partial \Omega} F_{\mu \nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau = \oint_{\Omega} \left( \partial_\nu B_{\mu 3} + \partial_\mu B_{\nu 3} + \partial_3 B_{\mu \nu} \right) \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$

where $\alpha$ and $\beta$ are arbitrary constants. Once the Maxwell equations (Eq. 51) are satisfied by these fields, the integrand in the r.h.s of Eq. (57) can be written as

$$\int_{\Omega} \epsilon_{\mu \nu \lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$  \hfill (60)

The arbitrariness of $\alpha$ and $\beta$ implies that the following two equations hold simultaneously

$$\int_{\partial \Omega} F_{\mu \nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau = 0.$$  \hfill (61)

$$\int_{\partial \Omega} \tilde{F}_{\mu \nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau = \frac{4 \pi}{c} \int_{\Omega} \epsilon_{\mu \nu \lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau.$$  \hfill (62)

We now define the components $B_{\mu \nu}$ in terms of the electromagnetic field strength and its Hodge dual

$$B_{\mu \nu} = i(\alpha F_{\mu \nu} + \beta \tilde{F}_{\mu \nu}).$$

We can continuously deform into each other

$$\oint_{\partial S} B \cdot dS.$$  \hfill (57)

and the r.h.s being equal to zero, we recover the Gauss law for the magnetic field.

Next we consider the same Eq. (61) but now with $\Omega$ as a 3-dimensional volume in space-time which we shall take as the cylinder $\Omega = D^2 \times \mathbb{R}$. The l.h.s of Eq. (61) is now decomposed in three parts, corresponding
to the flux of the electromagnetic field strength across
the three surfaces which form the border of the cylinder:
\( \partial \Omega = (D^0_t)^{-1} \cup D^2_t \cup (S^1 \times \mathbb{R}) \). The
time direction is taken at the axis of symmetry of this cylinder and we
reverse the orientation of the bottom disk \( D^0_t \), at \( t = 0 \), so that we
can consider it as a closed orientable surface (Fig. 8).

Then we have, for the flux of the electromagnetic field
on the cylinder:

\[
\frac{4\pi}{c} \int_{\Omega} J^\mu \epsilon_{\mu\nu\lambda} \frac{\partial \chi^\mu}{\partial \tau} \frac{\partial \chi^\nu}{\partial \zeta} \frac{\partial \chi^\lambda}{\partial \varphi} \, d\sigma d\tau d\zeta = \frac{4\pi}{c} \int_{\Omega} J^\mu \epsilon_{\mu\nu\lambda} \frac{\partial \chi^\mu}{\partial \tau} \frac{\partial \chi^\nu}{\partial \zeta} \frac{\partial \chi^\lambda}{\partial \varphi} \, d\sigma d\tau d\zeta = -4\pi \int_{\Omega} \rho d^3 x
\]

and Eq. (62) gives the Gauss law for the electric field.

Finally, considering \( \Omega = D^2 \times \mathbb{R} \), that cylindrical volume
in space and time, with the same parameterization as before, the l.h.s
of Eq. (62) becomes

\[
\int_{\partial \Omega} F^\mu_\nu \frac{\partial \chi^\mu}{\partial \tau} \frac{\partial \chi^\nu}{\partial \zeta} \, d\sigma d\tau = \Phi(E, D^1) - \Phi(E, D^0)
\]

and Eq. (62) gives the Gauss law for the electric field.

Finally, considering \( \Omega = D^2 \times \mathbb{R} \), that cylindrical volume
in space and time, with the same parameterization as before, the l.h.s
of Eq. (62) becomes

\[
\frac{4\pi}{c} \int_{\Omega} J^k \epsilon_{ijk} \frac{\partial \chi^i}{\partial \varphi} \frac{\partial \chi^j}{\partial \zeta} \frac{\partial \chi^k}{\partial \tau} \, d\sigma d\tau d\zeta = \frac{4\pi}{c} \int_{\Omega} J^k \epsilon_{ijk} \frac{\partial \chi^i}{\partial \varphi} \frac{\partial \chi^j}{\partial \zeta} \frac{\partial \chi^k}{\partial \tau} \, d\sigma d\tau d\zeta = 4\pi \int_{D^2} \mathbf{B} \cdot \sigma d\tau d\zeta.
\]
For an infinitesimal time lapse, Eq. (62) gives
\[-\frac{1}{c} \frac{\Delta \Phi(E, D^2)}{\Delta t} + \oint_{\partial D^2} B \cdot dx = \frac{4\pi}{c} \Phi(i, D^2),\]
which in the limit $\Delta t \to 0$ defines the Ampère–Maxwell law of induction:
\[-\frac{1}{c} \frac{d\Phi(E, D^2)}{dt} + \oint_{\partial D^2} B \cdot dx = \frac{4\pi}{c} \Phi(i, D^2).\] (64)

### 4.1 The Conservation of the Electric Charge in 4 Dimensions

By construction, the loop space connection is flat, i.e., $\delta A = 0$, since it is defined in terms of an exact field, $H = dB$ integrated over closed surfaces. This implies locally that the electric charge is conserved when we consider the definition of $B$ in terms of the physical fields and take into account the differential Maxwell equations:

\[A = \frac{i4\pi\beta}{c} \int_{\Sigma} J^r \epsilon_{\mu\nu\lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau\] (65)

and

\[\delta \int Ad\zeta = -\frac{i4\pi\beta}{c} \int \partial_\mu J^\mu d^4x = 0.\] (66)

In order to obtain the conserved charges we consider the generalization of the holonomy operator given by the 3-holonomy $U$, satisfying the parallel transport equation

\[\frac{dU}{d\zeta} - \int_0^{2\pi} \int_0^{2\pi} H_{\mu\lambda\zeta} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau U = 0\] (67)

whose solution can be formally written as

\[U_\Omega = e^{i\alpha H_{\rho\sigma} \frac{\partial x^\rho}{\partial \sigma} \frac{\partial x^\sigma}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau d\zeta} U_0,\] (68)

where $U_0$ is defined by the initial conditions.

Clearly, the flatness of the connection implies that the 3-holonomy operator is path-independent in loop space.

Now, we split space-time into space and time and construct a volume whose border changes in time from $t = 0$. Since the 3-holonomy operator is path independent, let us use the following convenient path: a composition $\Gamma = \Gamma_L \circ \Gamma_0$ where $\Gamma_L$ starts at $\Sigma_0$ and goes to $\partial \Omega_0$, the border of the completely spatial volume $\Omega$ at constant time $t = 0$ and $\Gamma_L$ a path which starts at $\partial \Omega_0$ and changes only in the time direction ending at $\partial \Omega_1$, the purely spatial volume at time $t$ (Fig. 9).

The relevant quantity in the construction of the 3-holonomy operator $U_T = U_{\Gamma_L} \cdot U_{\Gamma_0}$ is the integral of the connection in $\zeta$.

Then we have

\[
\int_{\Gamma_0} A(\zeta) d\zeta = \frac{i4\pi\beta}{c} \int_{\Omega_0} \int_0^{\zeta} \epsilon_{\mu\nu\lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau d\zeta = i4\pi\beta Q(0)
\]

\[
\int_{\Gamma_L} A(\zeta) d\zeta = i4\pi\beta \int_0^{\Gamma_L} \int_0^{\zeta} \epsilon_{\mu\nu\lambda} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} \frac{\partial x^\lambda}{\partial \zeta} d\sigma d\tau d\zeta = i4\pi\beta \int_0^{\Gamma_L} \Phi(i, \partial \Omega_0) dt.
\]

where $Q(0)$ stands for the electric charge at time $t = 0$.

Now, assuming the charge distribution to be localized, then at $t \to \infty$ we have $|j| \sim r^{-(2+\epsilon)}$, $\epsilon > 0$ and this means that $U_{\Gamma_L} = 1$ if the border $\partial \Omega_0$ is far enough from the charges. So, we end up with (considering $U_0 = 1$ for simplicity)

\[U_T = e^{i4\pi\beta Q(0)}.\] (71)

Next we consider a similar path joining the same two points in loop space: $\Gamma' = \Gamma_L \circ \Gamma_{Rg}$ where now $\Gamma_L$ is the path joining the closed surface $\Sigma_0$ at time $t = 0$ to the surface $\Sigma_r$ at a later time and $\Gamma_{Rg}$ is the purely spatial path joining the surface $\Sigma_r$ to the surface $\partial \Omega_1$, at constant time $t$.

The operator $U_{\Gamma'_{Rg}}$ will become the unity when the radius of the sphere at the reference point goes to zero and we remain with

\[U_{\Gamma'} = e^{i4\pi\beta Q(t)}.\] (72)

Now, the path independence of $U$ dictates that $U_{\Gamma'} = U_{\Gamma}$, given that the two paths $\Gamma$ and $\Gamma'$ are homotopically equivalent in loop space. With the considerations above for the behavior of the fields at spatial infinity, this relation gives

\[e^{i4\pi\beta \Delta Q} = 1 \quad \Delta Q \equiv Q(t) - Q(0)\] (73)

defining the conservation of the electric charge, which is given by the eigenvalues of the operator

\[U_{\Gamma} = e^{i4\pi\beta Q(t)}.\] (74)
5 Conclusions

We have shown that it is possible to formulate the integral equations of electrodynamics in \((D+1)\)-dimensional spacetime \(\mathcal{M}\) as the integral version of the zero-curvature equation in the loop space \(\mathcal{L}^{(D-1)}(\mathcal{M})\):

\[
\Delta \varphi = \int_\gamma A(s) ds \tag{75}
\]

where \(\varphi\) defines the integral of a \((D-1)\)-form over a \((D-1)\)-dimensional hypersurface in space-time and \(A\) is the corresponding flat connection given in terms of an exact \(D\)-form. In particular, for \(D = 1, 2, 3\) we have respectively

\[
\varphi = f, \quad \varphi = \int \mathcal{C}_\mu \frac{dx^\mu}{d\sigma} ds, \quad \varphi = \int \mathcal{B}_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau} d\sigma d\tau \tag{76}
\]

and

\[
A = A_\mu \delta x^\mu, \quad A = \int \mathcal{G}_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \delta x^\nu ds, \quad A = \int \mathcal{H}_{\mu_1...\mu_D} \frac{\partial x^\mu_1}{\partial \sigma} \frac{\partial x^\mu_2}{\partial \tau} ... \frac{\partial x^\mu_D}{\partial \tau} d\sigma d\tau \tag{77}
\]

with

\[
A = df, \quad G = dC, \quad H = dB. \tag{78}
\]

Generally speaking, the connection defined in \(\mathcal{L}^{(D-1)}(\mathcal{M})\) is given in terms of the components of the \(D\)-form in space-time \(\omega = \mathcal{C}_\mu dx^\mu \wedge ... \wedge dx^{D-1}\) as

\[
A = \mathcal{C}_\mu dx^\mu \tag{79}
\]

This integral representation in loop space is equivalent to the differential equation

\[
A = \delta \varphi \tag{80}
\]

so that \(A\) is a pure gauge connection. The local conservation law for the electric charge becomes a consequence of the mathematical identity \(\delta A = \delta^2 \varphi = 0\) in loop space. The flatness of the connection in loop space implies that the generalized holonomy operator defined by

\[
\frac{dW}{ds} + (-1)^{(D-1)}A(s) W = 0 \tag{81}
\]

whose solution can be formally written as

\[
W = e^{(-1)^D\int A(s) ds} W_0 \tag{82}
\]

is path-independent.

A homotopic variation of \(\gamma\) implies a variation of \(W\) which depends on the curvature of \(A\) and so, \(\delta W = 0\). Therefore, the equations of electrodynamics have a symmetry under the homotopic transformations of the path in loop space and the conserved charges are obtained as the eigenvalues of the generalized holonomy, evaluated at constant time.

A. The Construction of the Generalized Holonomies and Their Path Independence in Loop Space

In what follows we present the reasoning behind the construction of the generalized holonomies which define the parallel transport operators in \(\mathcal{L}^{(1)}(\mathcal{M})\) and \(\mathcal{L}^{(2)}(\mathcal{M})\). This is a review but for the abelian case of what was first introduced in [5] and later discussed in [6] and [7].

Consider the holonomy \(W\) defined by the parallel transport equation

\[
\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0 \tag{83}
\]

along a curve \(\gamma\). Suppose this curve to be closed and obtained by continuous deformations \(x^\mu \rightarrow x'^\mu + \delta x^\mu\) from a curve \(\gamma_0\) sharing a common point \(x_R\) with \(\gamma\).

We label the set of homotopic closed curves with \(\tau \in [0, 2\pi]\) such that \(\gamma\) has \(\tau = 2\pi\) and \(\gamma_0\), \(\tau = 0\).

Let us assume that we know \(W\) on \(\gamma_0\). For instance, if \(\gamma_0\) is the point-loop at \(x_R\), \(W\) can be taken as \(W_0\), an initial value for Eq. (83).

Now, if we want to get \(W\) over \(\gamma\), then instead of integrating Eq. (83) directly we can compute the change of \(W\) which is calculated over \(\gamma_0\) while this path is deformed into \(\gamma\).

For an infinitesimal deformation of the path we have \(W \rightarrow W + \delta W\) and the variation \(\delta W\) can be obtained from Eq. (83) as follows. We start by considering the variation of the equation as a whole:

\[
\delta \frac{dW}{d\sigma} + \delta \left( A_\mu \frac{dx^\mu}{d\sigma}\right) W + A_\mu \frac{dx^\mu}{d\sigma} \delta W = 0.
\]

Then we multiply this expression by \(W^{-1}\) and rewrite the first term getting

\[
\frac{d}{d\sigma} (W^{-1} \delta W) + W^{-1} \frac{dW}{d\sigma} W^{-1} \delta W + \delta \left( A_\mu \frac{dx^\mu}{d\sigma}\right) + W^{-1} A_\mu \frac{dx^\mu}{d\sigma} \delta W = 0
\]

and using Eq. (83) the second and fourth terms cancel each and what remains is
\[
\frac{d}{d\sigma} \left( W^{-1} \delta W \right) + \delta \left( A_\mu \frac{dx^\mu}{d\sigma} \right) = 0.
\]

This equation can be integrated in \( \sigma \in [0, 2\pi] \) giving

\[
\delta W = - \int_0^{2\pi} \delta \left( A_\mu \frac{dx^\mu}{d\sigma} \right) d\sigma
\]

and finally, calculating the variation of the term in the r.h.s:

\[
\delta W = - \int_0^{2\pi} \left( \partial_\tau A_\mu \frac{dx^\mu}{d\sigma} \partial x^\tau - \partial_\tau A_\mu \frac{dx^\mu}{d\sigma} \partial x^\tau \right) d\sigma - A_\mu \partial x^\mu d^2\tau.
\]

Considering that for the deformations of the loop, \( \partial x^\mu(\sigma = 0) = \partial x^\mu(\sigma = 2\pi) = 0 \) and defining \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \), we get that the change of the holonomy due to a deformation of the loop is

\[
\delta W = \left( \int_0^{2\pi} F_{\mu\nu} \frac{dx^\mu}{d\sigma} \partial x^\nu d\sigma \right) W. \quad (84)
\]

This shows that if the connection \( A = A_\mu dx^\mu \) is flat, i.e., if \( F_{\mu\nu} = 0 \), then the holonomy is independent of the path over which it is calculated if these paths can be deformed into each other while their end-points remain fixed.

Now, since the set of loops is parameterized by \( \tau \in [0, 2\pi] \), we can write the expression above for the variation of the holonomy into a differential equation for \( W \)

\[
\frac{dW}{d\tau} = \left( \int_0^{2\pi} F_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} d\sigma \right) W = 0 \quad (85)
\]

and the holonomy over \( \gamma \) can be obtained by integrating this equation from \( \tau = 0 \) up to \( \tau = 2\pi \).

Now if we consider the loop space \( \mathcal{L}^{(1)}(M) \), the quantity

\[
A(\tau) \equiv \int_0^{2\pi} F_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} d\sigma
\]

defines a connection evaluated at each loop in space-time which correspond to points in the loop space and by varying these loops with continuous transformations we defined a path \( \Gamma \) in \( \mathcal{L}^{(1)}(M) \).

The above equation which defines \( W \) can be seen as the parallel transport equation in this loop space and therefore we consider the generalization of this equations and define the 2-holonomy \( V \) as satisfying

\[
\frac{dV}{d\tau} = \left( \int_0^{2\pi} G_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} d\sigma \right) V = 0, \quad (86)
\]

where \( G_{\mu\nu} \) is an antisymmetric tensor.

Now, let us consider a closed 2-dimensional surface \( \Sigma \) which can be obtained by continuous deformations from another closed surface \( \Sigma_0 \) sharing a common point \( x_0 \) with each other. The 2-holonomy \( V \) can be calculated over \( \Sigma \) by direct integration of \( (86) \) but can also be obtained as the result of the deformation of the surface \( \Sigma_0 \) into \( \Sigma \), once \( V \) over \( \Sigma_0 \) is known.

In order to get \( V \) following this second approach we need to find how it varies when we deform the closed surface. This is done in a similar way as it was done before to find \( \delta W \): We start by considering the variation of the Eq. \( (86) \), then multiply the result by \( V^{-1} \) and finally compute the variation of the term containing \( G_{\mu\nu} \) explicitly and integrate the expression in \( \tau \). This shall leave us with

\[
\delta V = \int_0^{2\pi} \int_0^{2\pi} \left( \partial_\mu G_{\mu\nu} + \partial_\nu G_{\tau\lambda} + \partial_\lambda G_{\mu\nu} \right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} d\sigma d\tau = 0, \quad (87)
\]

From here we see that if \( G \) is an exact 2-form, then the 2-holonomy is surface independent.

Now, parameterizing this variation with \( \zeta \in [0, 2\pi] \), such that \( \zeta = 0 \) labels the surface \( \Sigma_0 \) and \( \zeta = 2\pi \), the surface \( \Sigma \), we get the differential equation

\[
\frac{dV}{d\zeta} = \int_0^{2\pi} \int_0^{2\pi} \left( \partial_\tau G_{\mu\nu} + \partial_\nu G_{\tau\lambda} + \partial_\lambda G_{\mu\nu} \right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} d\sigma d\tau = 0, \quad (88)
\]

and integrating this equation in \( \zeta \) will give us the desired \( V \) on the surface \( \Sigma \).

In the loop space \( \mathcal{L}^{(2)}(M) \) the above equation defines the parallel transport through a 3-dimensional volume in space-time, which corresponds to a path \( \Gamma \) in the loop space, parameterized by \( \zeta \). This leads us to introduce the 3-holonomy \( U \) defined by

\[
\frac{dU}{d\zeta} - \int_0^{2\pi} \int_0^{2\pi} H_{\lambda\mu\nu} \frac{dx^\lambda}{d\zeta} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\sigma d\tau = 0, \quad (89)
\]

with \( H_{\lambda\mu\nu} \) a completely antisymmetric tensor.

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Declarations

Conflict of Interest G. Lucchini and V. B. Zaché state that there are no conflicts of interest.

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