New Cameron–Liebler line classes with parameter $\frac{q^2+1}{2}$

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Abstract

New families of Cameron–Liebler line classes of PG(3, q), $q \geq 7$ odd, with parameter $(q^2 + 1)/2$ are constructed.

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1 Introduction

In this paper we deal with Cameron–Liebler line classes of PG(3, q). The notion of Cameron–Liebler line class was introduced in the seminal paper \[4\] with the purpose of classifying those collineation groups of PG(3, q) having the same number of orbits on points and lines. On the other hand, a classification of Cameron–Liebler line classes would yield a classification of symmetric tactical decompositions of points and lines of PG(n, q) and that of certain families of weighted point sets of PG(3, q) \[13\]. Cameron–Liebler line classes are also related to other combinatorial structures such as two–intersection sets, strongly regular graphs and two–weight codes \[1\], \[14\].

In PG(3, q), a Cameron–Liebler line class $\mathcal{L}$ with parameter $x$ is a set of lines such that every spread of PG(3, q) contains exactly $x$ lines of $\mathcal{L}$, \[4\], \[13\]. There exist trivial examples of Cameron–Liebler line classes $\mathcal{L}$ with parameters $x = 1, 2$ and $x = q^2, q^2 - 1$. A Cameron–Liebler line class with parameter $x = 1$ is either the set of lines through a point or the set of lines in a plane. A Cameron–Liebler line class with parameter $x = 2$ is the union of the two previous example, if the point is not in the plane \[4\], \[13\]. In general, the complement of a Cameron–Liebler line class with parameter $x$ is
a Cameron–Liebler line class with parameter $q^2 + 1 - x$ and the union of two disjoint Cameron–Liebler line classes with parameters $x$ and $y$, respectively, is a Cameron–Liebler line class with parameter $x + y$.

It was conjectured that no other examples of Cameron–Liebler line classes exist [4], but Bruen and Drudge [3] found an infinite family of Cameron–Liebler line classes with parameter $x = (q^2 + 1)/2$, $q$ odd. Bruen–Drudge’s example admits the group $G = P\Omega^-(4, q)$, stabilizing an elliptic quadric $Q^-(3, q)$ of $PG(3, q)$, as an automorphism group.

In [10], Govaerts and Penttila found a Cameron–Liebler line class with parameter $x = 7$ in $PG(3, 4)$.

Infinite families of Cameron–Liebler line classes with parameter $(q^2 - 1)/2$ were found for $q \equiv 5$ or $9 \pmod{12}$ in [6], [7]. By construction, such a family $X$ never contains the lines $Y$ in a plane $\Pi$ and the lines $Z$ through a point $z \notin \Pi$. Therefore, $X \cup Y$ and $X \cup Z$ are both examples of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$.

For non–existence results of Cameron–Liebler line classes see [9], [12] and references therein.

Recently, by perturbating the Bruen–Drudge’s example, a new infinite family of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, $q \geq 5$ odd, has been constructed independently in [5] and [8]. Such a family admits the stabilizer of a point of $Q^-(3, q)$ in the group $G$, say $K$.

Here, we introduce a new derivation technique for Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, see Theorem 3.9. Applying such a derivation to the Bruen–Drudge’s example, we construct a new family of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, $q \geq 7$ odd, not equivalent to the examples known so far and admitting a subgroup of $K$ of order $q^2(q + 1)$.

Under the Klein correspondence between the lines of $PG(3, q)$ and points of a Klein quadric $Q^+(5, q)$, a Cameron–Liebler line class with parameter $i$ gives rise to a so–called $i$–tight set of $Q^+(5, q)$.

A set $\mathcal{T}$ of points of $Q^+(5, q)$ is said to be $i$–tight if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i(q + 1) + q^2 & \text{if } P \in \mathcal{T} \\ i(q + 1) & \text{if } P \notin \mathcal{T}. \end{cases}$$

where $\perp$ denotes the polarity of $PG(5, q)$ associated with $Q^+(5, q)$.

For more results on tight sets of polar spaces, see [1].

Throughout the paper $q$ is a power of an odd prime.
2 The Bruen–Drudge’s example

Let \( X_1, \ldots, X_4 \) be homogeneous projective coordinates in \( \text{PG}(3, q) \). Let \( \omega \) be a non–square element of \( \text{GF}(q) \). Let \( \mathcal{E} \) be the elliptic quadric of \( \text{PG}(3, q) \) with equation \( X_1^2 - \omega X_2^2 + X_3 X_4 = 0 \) and quadratic form \( Q \). Each point \( P \in \mathcal{E} \) lies on \( q^2 \) secant lines to \( \mathcal{E} \), and so lies on \( q + 1 \) tangent lines. Let \( \mathcal{L}_P \) be a set of \((q + 1)/2\) tangent lines to \( \mathcal{E} \) through \( P \), and let \( \mathcal{E} \) be the set of external lines to \( \mathcal{E} \); then the set

\[
\bigcup_{P \in \mathcal{E}} \mathcal{L}_P \cup \mathcal{E}
\]

has size \((q^2 + 1)(q^2 + q + 1)/2\) which is the number of lines of \( \text{PG}(3, q) \) in a Cameron–Liebler line class with parameter \((q^2 + 1)/2\). By suitably selecting the sets \( \mathcal{L}_P \), it is in fact a Cameron–Liebler line class \([3]\). Let \( G = \text{PGL}(4, q) \) be the commutator subgroup of the full stabilizer of \( \mathcal{E} \) in \( \text{PGL}(4, q) \). The group \( G \) has three orbits on points of \( \text{PG}(3, q) \), i.e., the points of \( \mathcal{E} \) and other two orbits \( \mathcal{O}_s \) and \( \mathcal{O}_n \), both of size \( q^2(q^2 + 1)/2 \). The two orbits \( \mathcal{O}_s \), \( \mathcal{O}_n \) correspond to points of \( \text{PG}(3, q) \) such that the evaluation of the quadratic form \( Q \) is a non–zero square or a non–square in \( \text{GF}(q) \), respectively. We say that a point \( X \) of \( \text{PG}(3, q) \) is a square point or a non–square point with respect to a given quadratic form \( F \) according as the evaluation \( F(X) \) is a non–zero square or a non–square in \( \text{GF}(q) \). In its action on lines of \( \text{PG}(3, q) \), the group \( G \) has four orbits: two orbits, say \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), both of size \((q + 1)(q^2 + 1)/2\), consisting of lines tangent to \( \mathcal{E} \) and two orbits, say \( \mathcal{L}_2 \) and \( \mathcal{L}_3 \), both of size \( q^2(q^2 + 1)/2 \) consisting of lines secant or external to \( \mathcal{E} \), respectively. The block-tactical decomposition matrix for this orbit decomposition is

\[
\begin{pmatrix}
1 & 1 & 2 & 0 \\
q & 0 & \frac{q-1}{2} & \frac{q+1}{2} \\
0 & q & \frac{q-1}{2} & \frac{q+1}{2} \\
q + 1 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2}
\end{pmatrix},
\]

and hence the point-tactical decomposition matrix is

\[
\begin{pmatrix}
\frac{q+1}{2} & \frac{q+1}{2} & q^2 & 0 \\
q + 1 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \\
0 & q + 1 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2}
\end{pmatrix}.
\]

Simple group–theoretic arguments show that a line of \( \mathcal{L}_0 \) (\( \mathcal{L}_1 \)) contains \( q \) points of \( \mathcal{O}_s \) (\( \mathcal{O}_n \)), that a line secant to \( \mathcal{E} \) always contains \((q - 1)/2\) points of
Let $E$ be a line that is either secant or external to $E$. Then the number of lines of $L_0$ (or of $L_1$) meeting $E$ equals $(q + 1)/2$.

**Lemma 2.2.** Let $E$ be a line of $L_0$ (resp. $L_1$). Then

- the number of lines of $L_0$ (resp. $L_1$) meeting $E$ in a point equals $q^2 + (q - 1)/2$;
- the number of lines of $L_1$ (resp. $L_0$) meeting $E$ in a point equals $(q + 1)/2$;

Let $L' = L_0 \cup L_3$. Using Lemma 2.1 and Lemma 2.2, it can be seen that $L'$ is the Cameron–Liebler line class constructed in $[3]$. In particular, $L'$ has the following three characters with respect to line–sets in planes of $\text{PG}(3, q)$:

\[
\frac{q + 1}{2}, \frac{q(q + 1)}{2}, \frac{q(q + 1)}{2} + 1,
\]

and

\[
\frac{q + 1}{2}, \frac{q(q + 1)}{2}, \frac{q(q + 1)}{2} + q + 1,
\]

with respect to line–stars of $\text{PG}(3, q)$.

Consider a point of $E$ (since $G$ is transitive on $E$ we can choose it as the point $U_3 = (0, 0, 1, 0)$) and let $\pi$ be the plane with equation $X_4 = 0$. Then $\pi$ is tangent to $E$ at the point $U_3$. The plane $\pi$ contains a set $L'_3$ consisting of $q^2$ lines of $L_3$. Let $L'_2$ be the set of $q^2$ lines of $L_2$ through $U_3$. In $[5]$, we showed that $L'' = (L' \setminus L'_3) \cup L'_3$ is a Cameron–Liebler line class of with parameter $(q^2 + 1)/2$, admitting the group $K$ as an automorphism group.

In particular, $L''$ has the following five characters with respect to line–sets in planes of $\text{PG}(3, q)$:

\[
\frac{q + 1}{2}, \frac{q(q - 1)}{2} - 1, \frac{q(q + 1)}{2}, \frac{q(q + 1)}{2} + q + 1, q^2 + \frac{q - 1}{2},
\]

and

\[
\frac{q + 3}{2}, \frac{q(q - 1)}{2}, \frac{q(q + 1)}{2} + 1, \frac{q(q + 1)}{2} + q + 2, q^2 + \frac{q + 1}{2},
\]

with respect to line–stars of $\text{PG}(3, q)$. It turns out that, if $q > 3$, these characters are distinct from those of a Bruen–Drudge Cameron–Liebler line class.
3 The new family

In this section we introduce a new derivation technique which will allow us to construct a new infinite family of Cameron–Liebler line classes with parameter \((q^2 + 1)/2, \ q \geq 7\).

With the notation introduced in the previous section, let \(E_\lambda\) be the elliptic quadric with equation \(X_1^2 - \omega X_2^2 + \lambda X_4^2 + X_3X_4 = 0, \ \lambda \in \text{GF}(q)\). Then the non–degenerate quadrics \(E = E_0, E_\lambda, \lambda \in \text{GF}(q) \setminus \{0\}\), together with \(\pi\), form a pencil \(\mathcal{P}\) of \(\text{PG}(3,q)\). The base locus of \(\mathcal{P}\) is the point \(U_3\). Let \(\perp (\text{resp. } \perp_\lambda)\) be the orthogonal polarity associated to \(E\) (resp. \(E_\lambda\)). Note that \(U_3 \perp_\lambda = \pi, \ \forall \lambda \in \text{GF}(q) \setminus \{0\}\).

With a slight abuse of notation we will denote in the same way a plane and the set of lines contained in it.

**Lemma 3.1.** If \(P \in O_s\), then

\[
|P^\perp \cap L_0| = \begin{cases} 
q + 1 & \text{if } q \equiv -1 \ (\text{mod } 4) \\
0 & \text{if } q \equiv 1 \ (\text{mod } 4)
\end{cases}
\]

\[
|P^\perp \cap L_1| = \begin{cases} 
0 & \text{if } q \equiv -1 \ (\text{mod } 4) \\
q + 1 & \text{if } q \equiv 1 \ (\text{mod } 4)
\end{cases}
\]

If \(P \in O_n\), then

\[
|P^\perp \cap L_0| = \begin{cases} 
0 & \text{if } q \equiv -1 \ (\text{mod } 4) \\
q + 1 & \text{if } q \equiv 1 \ (\text{mod } 4)
\end{cases}
\]

\[
|P^\perp \cap L_1| = \begin{cases} 
q + 1 & \text{if } q \equiv -1 \ (\text{mod } 4) \\
0 & \text{if } q \equiv 1 \ (\text{mod } 4)
\end{cases}
\]

**Proof.** It is enough to show that if \(t \in L_0\), then \(t^\perp\) belongs either to \(L_0\) or to \(L_1\), according as \(q \equiv -1 \ (\text{mod } 4)\) or \(q \equiv 1 \ (\text{mod } 4)\). Since \(G\) is transitive on elements of \(L_0\), let \(t\) be the line joining \(U_3\) with \(U_1 = (1,0,0,0)\). Then \(t^\perp\) is the line joining \(U_3\) with \(U_2 = (0,1,0,0)\), which belongs to \(L_0\) if and only if \(-1\) is not a square in \(\text{GF}(q)\), i.e., \(q \equiv -1 \ (\text{mod } 4)\). \(\square\)

**Lemma 3.2.** Every line of \(\text{PG}(3,q)\) not contained in \(\pi\) is tangent to exactly one elliptic quadric of \(\mathcal{P}\).
Proof. Let $\lambda \in \text{GF}(q)$ and let $P$ be a point of the elliptic quadric $E_\lambda$ of $\mathcal{P}$. Let $T_P$ be the pencil of lines through $P$ in $P^{-\lambda}$. In order to prove the result it is enough to show that a line $\ell$ of $T_P$ is either secant or external to a non-degenerate quadric of $\mathcal{P}$ distinct from $E_\lambda$. The plane $\pi$ meets $P^{-\lambda}$ in a line $r$ and $E_{\lambda'}$, $\lambda' \neq \lambda$, in a non-degenerate conic $C_{\lambda'}$, $\lambda' \in \text{GF}(q) \setminus \{\lambda\}$. Then $P$, $r$, $C_{\lambda'}$, $\lambda' \in \text{GF}(q) \setminus \{\lambda\}$, form a pencil of quadrics of $\pi$. From [11, Table 7.7], $r$ is the polar line of $P$ with respect to $C_{\lambda'}$. Hence, $P$ is an interior point with respect to $C_{\lambda'}$ and the result follows.

Remark 3.3. Note that $E_\lambda \subseteq O_s$ if and only if $-\lambda$ is a non-zero square in $\text{GF}(q)$.

Remark 3.4. Let $Q_\lambda$ be the quadratic form associated to $E_\lambda$ (then $Q_\lambda = Q_0$). For a point of $\pi$ the evaluation of $Q_\lambda$ is the same for all $\lambda \in \text{GF}(q)$.

Let $\pi_0 = O_s \cap \pi$ and $\pi_1 = O_n \cap \pi$. Then $|\pi_0| = |\pi_1| = q(q+1)/2$. We need the following result.

Lemma 3.5. Let $R$ be a point of $E_\lambda \setminus \{U_3\}$, $\lambda \neq 0$, let $\ell$ be a line meeting $E_\lambda$ exactly in $R$ and let $P = \pi \cap \ell$. If $P \in \pi_0$, then

$$|\ell \cap E| = \begin{cases} 0 & \text{if } \lambda \text{ is a non-square in } \text{GF}(q) \\ 2 & \text{if } \lambda \text{ is a square in } \text{GF}(q) \end{cases}.$$ 

If $P \in \pi_1$, then

$$|\ell \cap E| = \begin{cases} 2 & \text{if } \lambda \text{ is a non-square in } \text{GF}(q) \\ 0 & \text{if } \lambda \text{ is a square in } \text{GF}(q) \end{cases}.$$ 

Proof. Since there exists a subgroup of $K$ of order $q^2$ which permutes in a single orbit the $q^2$ points of an elliptic quadric of $\mathcal{P}$, w.l.o.g., we can choose the point $R$ as the point $(0,0,-\lambda,1) \in E_\lambda$. Then $\ell$ is contained in $R^{-\lambda}$. Assume that $P \in \pi_0$. Straightforward calculations show that $P$ is a point having coordinates $(x,y,0,0)$, where $x^2 - \omega y^2$ is a non-zero square in $\text{GF}(q)$ and the line $\ell$, apart from $P$, contains the points $(\mu x, \mu y, -\lambda, 1)$, $\mu \in \text{GF}(q)$. Note that $(\mu x, \mu y, -\lambda, 1) \in E$ if and only if $\lambda = \mu^2(x^2 - \omega y^2)$, that is if and only if $\lambda$ is a square in $\text{GF}(q)$. Analogously, if $P \in \pi_1$.

Let $\lambda_1 \neq 0$ be a fixed square in $\text{GF}(q)$ and let $\lambda_2$ be a fixed non-square in $\text{GF}(q)$. Consider the following sets of lines:

$$t_0 = \{r \in \mathcal{L}_0 : |r \cap \pi_0| = q\}, \quad t_1 = \{r \in \mathcal{L}_1 : |r \cap \pi_1| = q\},$$

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\[ T_{10} = \{ r \in L_2 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_0| = 1 \}, \]
\[ T_{11} = \{ r \in L_3 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_1| = 1 \}, \]
\[ T_{20} = \{ r \in L_3 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_0| = 1 \}, \]
\[ T_{21} = \{ r \in L_2 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_1| = 1 \}. \]

Then \(|t_0| = |t_1| = (q + 1)/2\) and \(|T_{10}| = |T_{11}| = |T_{20}| = |T_{21}| = q^2(q + 1)/2\).

Let \(A = T_{11} \cup T_{20}\) and \(B = T_{10} \cup T_{21}\). From Lemma 3.5, we have that \(A\) is a set consisting of \(q^2(q + 1)\) external lines to \(E\) and \(B\) is a set consisting of \(q^2(q + 1)\) secant lines to \(E\).

For a line \(\ell\) of PG(3, q), we define the following line–sets:
\[ A_\ell = \{ r \in A : |r \cap \ell| = 1 \}, \quad B_\ell = \{ r \in B : |r \cap \ell| = 1 \}. \]

**Remark 3.6.** Taking into account Remark 3.4, by construction, we have the following:

- the lines in \(T_{11} \cup t_1\) are all the \((q + 1)(q^2 + 1)/2\) tangent lines to \(\mathcal{E}_{\lambda_1}\) having \(q\) non–square points with respect to \(Q_{\lambda_1}\);
- the lines in \(T_{10} \cup t_0\) are all the \((q + 1)(q^2 + 1)/2\) tangent lines to \(\mathcal{E}_{\lambda_1}\) having \(q\) square points with respect to \(Q_{\lambda_1}\);
- the lines in \(T_{20} \cup t_0\) are all the \((q + 1)(q^2 + 1)/2\) tangent lines to \(\mathcal{E}_{\lambda_2}\) having \(q\) square points with respect to \(Q_{\lambda_2}\);
- the lines in \(T_{21} \cup t_1\) are all the \((q + 1)(q^2 + 1)/2\) tangent lines to \(\mathcal{E}_{\lambda_2}\) having \(q\) non–square points with respect to \(Q_{\lambda_2}\).

**Lemma 3.7.** Let \(\ell\) be a line of PG(3, q) such that \(\ell \not\in A \cup B\), then \(|A_\ell| = |B_\ell|\).

**Proof.** From Lemma 3.2, the line \(\ell\) either is not tangent neither to \(\mathcal{E}_{\lambda_1}\) nor to \(\mathcal{E}_{\lambda_2}\), or \(U_3 \in \ell \subset \pi\) and \(\ell\) is tangent to both \(\mathcal{E}_{\lambda_1}\) and \(\mathcal{E}_{\lambda_2}\). Observe that if \(\ell\) is not tangent neither to \(\mathcal{E}_{\lambda_1}\) nor to \(\mathcal{E}_{\lambda_2}\), then, from Remark 3.6 and Lemma 2.1, each of the following line–sets: \(T_{11} \cup t_1\), \(T_{20} \cup t_0\), \(T_{10} \cup t_0\), \(T_{21} \cup t_1\), contains \((q + 1)^2/2\) lines meeting \(\ell\) in a point. We consider several cases.

**Case 1:** \(|\ell \cap \mathcal{E}_{\lambda_1}| = |\ell \cap \mathcal{E}_{\lambda_2}| = 0\)

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There are two possibilities: either \( \ell \cap \pi_0 = 1 \). In this case, there is exactly one line of \( t_0 \) meeting \( \ell \) in a point and there are no lines of \( t_1 \) meeting \( \ell \) in a point. It follows that \( |A_\ell| = (q + 1)^2/2 + (q + 1)^2/2 - 1 \). Analogously, there are \((q + 1)^2/2 \) lines in \( T_{10} \cup t_0 \) meeting \( \ell \) and \((q + 1)^2/2 \) lines in \( T_{21} \cup t_1 \) meeting \( \ell \). Hence, \( |B_\ell| = (q + 1)^2/2 + (q + 1)^2/2 - 1 \).

b) If \( |\ell \cap \pi_1| = 1 \), then there is exactly one line of \( t_1 \) meeting \( \ell \) in a point and there are no lines of \( t_0 \) meeting \( \ell \) in a point. Hence, we get again \( |A_\ell| = |B_\ell| = q(q + 2) \).

c) If \( |\ell \cap \pi_0| = |\ell \cap \pi_1| = (q + 1)/2 \), then there are \((q + 1)/2 \) lines of \( t_1 \) meeting \( \ell \) in a point and \((q + 1)/2 \) lines of \( t_0 \) meeting \( \ell \) in a point. It follows that \( |A_\ell| = |B_\ell| = (q + 1)^2/2 + (q + 1)^2/2 - (q + 1)^2/2 - (q + 1)^2/2 = q(q + 1) \).

Case 2): \( |\ell \cap E_{\lambda_1}| = 2 \), \( |\ell \cap E_{\lambda_2}| = 0 \) or \( |\ell \cap E_{\lambda_1}| = 0 \), \( |\ell \cap E_{\lambda_2}| = 2 \)

Repeating the same argument as in Case 1), a), or Case 1), b), according as \( |\ell \cap \pi_0| = 1 \) or \( |\ell \cap \pi_1| = 1 \), we obtain \( |A_\ell| = |B_\ell| = q(q + 2) \).

Case 3): \( |\ell \cap E_{\lambda_1}| = 2 \), \( |\ell \cap E_{\lambda_2}| = 2 \)

Repeating the same argument as in Case 1), a), or Case 1), b), according as \( |\ell \cap \pi_0| = 1 \) or \( |\ell \cap \pi_1| = 1 \), we obtain \( |A_\ell| = |B_\ell| = q(q + 2) \). If \( U_3 \in \ell \), then there are \((q + 1)/2 \) lines of \( t_1 \) meeting \( \ell \) in a point and \((q + 1)/2 \) lines of \( t_0 \) meeting \( \ell \) in a point. It follows that \( |A_\ell| = |B_\ell| = q(q + 1) \).

Case 4): \( |\ell \cap E_{\lambda_1}| = |\ell \cap E_{\lambda_2}| = 1 \)

There are two possibilities: either \( \ell \in t_0 \) or \( \ell \in t_1 \).

a) \( \ell \in t_0 \). In this case, from Remark 3.6 and Lemma 2.2, each of the following line–sets: \( T_{10} \cup t_0 \), \( T_{20} \cup t_0 \), contains \( q^2 + (q-1)/2 \) lines meeting \( \ell \) in a point. Analogously, each of the following line–sets: \( T_{11} \cup t_1 \), \( T_{21} \cup t_1 \), has \( (q + 1)/2 \) elements meeting \( \ell \) in a point. On the other hand, \( t_0 \) contains \((q-1)/2 \) lines intersecting \( \ell \) in a point and \( t_1 \) contains \((q + 1)/2 \) lines intersecting \( \ell \) in a point. Hence, \( |A_\ell| = |B_\ell| = q^2 \).

b) \( \ell \in t_1 \). As in the previous case, we get again \( |A_\ell| = |B_\ell| = q^2 \).
The proof is now complete. \hfill \Box

**Lemma 3.8.** Let $\ell$ be a line of $PG(3, q)$.

- If $\ell \in A$, then $|A_{\ell}| = \frac{3q^2 + 3q - 2}{2}$ and $|B_{\ell}| = \frac{q^2 + 3q}{2}$;
- if $\ell \in B$, then $|A_{\ell}| = \frac{q^2 + 3q}{2}$ and $|B_{\ell}| = \frac{3q^2 + 3q - 2}{2}$.

**Proof.** Assume first that $\ell \in A$ and in particular that $\ell \in T_{11}$. From Remark 3.6 and Lemma 2.2, there are $q^2 + (q - 1)/2$ lines of $T_{11} \cup t_1$ meeting $\ell$ in a point, whereas, there are $(q + 1)/2$ lines of $T_{10}$ meeting $\ell$ in a point. Also, from Remark 3.6 and Lemma 2.1, each of the following line–sets: $T_{20} \cup t_0$, $T_{21} \cup t_1$ contains $(q + 1)/2$ lines meeting $\ell$ in a point. Since there is exactly one line of $t_1$ meeting $\ell$ in a point and there are no lines of $t_0$ meeting $\ell$ in a point, it follows that $|A_{\ell}| = q^2 + (q - 1)/2 - 1 + (q + 1)/2 = (3q^2 + 3q - 2)/2$ and $|B_{\ell}| = (q + 1)/2 + (q + 1)/2 - 1 = (q^2 + 3q)/2$. Similarly, if $\ell \in T_{20}$, repeating the previous arguments, we obtain the desired result. \hfill \Box

We are ready to prove the following result.

**Theorem 3.9.** Let $L$ be a Cameron–Liebler line class with parameter $(q^2 + 1)/2$ such that $A \subset L$ and $|B \cap L| = 0$. Then the set $\bar{L} = (L \setminus A) \cup B$ is a Cameron–Liebler line class with parameter $(q^2 + 1)/2$.

**Proof.** Since $L$ is a Cameron–Liebler line class with parameter $(q^2 + 1)/2$, we have that $|\{r \in L : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q + 1)(q^2 + 1)/2$ if $\ell \in L$, or $(q + 1)(q^2 + 1)/2$ if $\ell \notin L$.

Let $\ell$ be a line of $PG(3, q)$.

- If $\ell \in L \setminus (A \cup B)$, then $\ell \in \bar{L}$. From Lemma 3.7 it follows that $|\{r \in \bar{L} : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q + 1)(q^2 + 1)/2$.
- If $\ell \notin L \cup A \cup B$, then $\ell \notin \bar{L}$. From Lemma 3.7 it follows that $|\{r \in \bar{L} : |r \cap \ell| \geq 1\}|$ equals $(q + 1)(q^2 + 1)/2$.
- If $\ell \in A$, then $\ell \in L \setminus \bar{L}$. From Lemma 3.8 we have that $|\{r \in \bar{L} : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q + 1)(q^2 + 1)/2 - (3q^2 + 3q - 2)/2 - (q^2 + 3q)/2 - 1 = (q + 1)(q^2 + 1)/2$.
- If $\ell \in B$, then $\ell \in \bar{L} \setminus L$. From Lemma 3.8 we have that $|\{r \in \bar{L} : |r \cap \ell| \geq 1\}|$ equals $(q + 1)(q^2 + 1)/2 + (3q^2 + 3q - 2)/2 - (q^2 + 3q)/2 + 1 = q^2 + (q + 1)(q^2 + 1)/2$.

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The proof is now complete.

We consider \( \mathcal{L} \) being \( \mathcal{L}' \) and we denote by \( \mathcal{L}''' = (\mathcal{L}' \setminus \mathcal{A}) \cup \mathcal{B} \). Then, from Theorem 3.9, \( \mathcal{L}''' \) is a Cameron–Liebler line class with parameter \((q^2 + 1)/2\).

In what follows, we show that \( \mathcal{L}''' \) is left invariant by a group of order \( q^2(q + 1) \). We shall find it helpful to associate to a projectivity of \( \text{PGL}(4, q) \) a matrix of \( \text{GL}(4, q) \). We shall consider the points as column vectors, with matrices acting on the left.

**Proposition 3.10.** The Cameron–Liebler line class \( \mathcal{L}''' \) admits a subgroup of \( K \) of order \( q^2(q + 1) \).

**Proof.** Let \( \Psi \) be the subgroup of \( \text{PGL}(4, q) \) whose elements are associated to the following matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & -x \\
0 & 1 & 0 & -y \\
2x & -2wy & 1 & wy^2 - x^2 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

where \( x, y \in \text{GF}(q) \). Then \( \Psi \) is an elementary abelian group of order \( q^2 \).

Easy computations show that if \( g \in \Psi \) and \( P \in \mathcal{E}_\lambda \), then \( P^g \in \mathcal{E}_\lambda \) and \( \Psi \) acts transitively on \( \mathcal{E}_\lambda \setminus \{U_3\} \), \( \lambda \in \text{GF}(q) \); furthermore, the evaluation of \( Q_{\lambda'} \), \( \lambda' \neq \lambda \), is the same for both \( P \) and \( P^g \). Let \( \Phi \) be the subgroup of \( \text{PGL}(4, q) \) whose elements are associated to the following matrices:

\[
\begin{pmatrix}
z & wt & 0 & 0 \\
t & z & 0 & 0 \\
0 & 0 & u & 0 \\
0 & 0 & 0 & u
\end{pmatrix},
\]

where \( z, t, u \in \text{GF}(q) \) are such that \( z^2 - wt^2 = u^2 \). Note that the previous equation holds true if and only if the line of \( \text{PG}(2, q) \) joining the points \((0,0,1)\) and \((z,t,0)\) is secant to the conic \( D : X_1^2 - wX_2^2 - X_3^2 = 0 \). Since \((0,0,1)\) is an interior point with respect to \( D \), we have that up to a scalar factor there are exactly \( q + 1 \) triple \((z,t,u)\) such that \( z^2 - wt^2 = u^2 \). Hence \( |\Phi| = q + 1 \). Easy computations show that if \( g \in \Phi \) and \( P \in \mathcal{E}_\lambda \), then \( P^g \in \mathcal{E}_\lambda \); furthermore, the evaluation of \( Q_{\lambda'} \), \( \lambda' \neq \lambda \), for the point \( P \) is a square if and only if it is a square for \( P^g \). Since both, \( \Psi \) and \( \Phi \) stabilize \( U_3 \),
we have that the direct product $\Gamma = \Psi \times \Phi$ is a group of order $q^2(q + 1)$ fixing the point $U_3$ and the plane $\pi$. Hence $\Gamma$ is a subgroup of $K$.

The group $\Gamma$ has the orbits $\pi_0$, $\pi_1$ and $\{U_3\}$ on $\pi$. It follows that the stabilizer in $\Gamma$ of a point of $\pi \setminus \{U_3\}$ has order two. Let $t$ be a line of $PG(3, q)$ not contained in $\pi$. From Lemma 3.2, $t$ is tangent to exactly an elliptic quadric $E_\lambda$ at the point $R$. The stabilizer of $t$ in $\Gamma$ has to fix $R$ and $t \cap \pi$. Hence it has order at most two. On the other hand $(RU_3)^{-\lambda}$ does not depend on $\lambda$ and $\Gamma$ contains the involutory biaxial homology of $G$ fixing pointwise the lines $RU_3$ and $(RU_3)^{-\lambda}$. Hence $|t^\Gamma|$ equals $q^2(q + 1)/2$ and $\Gamma$ fixes $L''$.

$\square$

Let us denote by $O_s^i$ and $O_t^i$, the sets of size $q^2(q^2 + 1)/2$ corresponding to points of $PG(3, q)$ such that the evaluation of the quadratic form $Q_{\lambda_i}$, $i = 1, 2$, is a non–zero square or a non–square in $GF(q)$, respectively.

**Proposition 3.11.** The characters of $L''$, with respect to line–sets in planes of $PG(3, q)$ form a subset of:

$$\left\{ q^2 + \frac{q + 1}{2}, q^2 - \frac{3(q + 1)}{2}, \frac{q(q - 1)}{2} + 3(q + 1), \frac{q(q - 1)}{2} + 2(q + 1), \frac{q(q - 1)}{2} + q + 1, \frac{q(q - 1)}{2} - (q + 1), \frac{q(q + 1)}{2} - 2(q + 1) \right\}. $$

**Proof.** Note that if $\sigma$ is a plane distinct from $\pi$ and not containing $U_3$, then $\sigma = P^{\perp \lambda_i}$, for some $P \in E_{\lambda_i} \setminus \{U_3\}$. In particular we may assume that $P = (0, 0, -\lambda_1, 1)$.

The plane $\pi$ contains $q^2 + (q + 1)/2$ lines of $L''$.

Let $\sigma$ be a plane distinct from $\pi$.

If $\sigma \cap \pi \in L_0$, then $\sigma$ contains $q$ lines of $T_{20}$ and of $T_{10}$ and no line of $T_{11}$ and of $T_{21}$. Since $\sigma$ contains $q(q - 1)/2 + q + 1$ lines of $L'$, we have that $\sigma$ contains $q(q - 1)/2 + q + 1$ lines of $L''$.

If $\sigma \cap \pi \in L_1$, then $\sigma$ contains $q$ lines of $T_{11}$ and of $T_{21}$ and no line of $T_{20}$ and of $T_{10}$. Since $\sigma$ contains $q(q - 1)/2$ lines of $L'$, we have that $\sigma$ contains $q(q - 1)/2$ lines of $L''$.

Let $\sigma = P^{\perp}$, with $P \in E \setminus \{U_3\}$. Consider the tactical configuration whose points are the $q^2$ planes tangent to $E$ at some point of $E \setminus \{U_3\}$ and whose blocks are the $q^2(q + 1)/2$ lines contained in $T_{11}$ (in $T_{20}$). It turns out that $\sigma$ contains $q + 1$ lines of $T_{11}$ ($T_{20}$). Since $\sigma$ contains $q^2 + (q + 1)/2$ lines of $L'$, we have that $\sigma$ contains $q^2 - 3(q + 1)/2$ lines of $L''$.
Let $\sigma = P^\perp_{\lambda_1}, P \in \mathcal{E}_{\lambda_1} \setminus \{U_3\}$. Assume that $q \equiv 1 \pmod 4$. Taking into account Lemma 3.1 we have that $\sigma$ contains no line of $L_0$.

If $\lambda_1 - \lambda_2$ is a non-square, then $(P^\perp_{\lambda_1})^\perp_{\lambda_2} \in O^2_n$. The plane $\sigma$ contains $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{21}$ and $q + 1$ lines of $T_{20}$. Since $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}''$.

If $\lambda_1 - \lambda_2$ is a square, then $(P^\perp_{\lambda_1})^\perp_{\lambda_2} \in O^2_n$. The plane $\sigma$ contains $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{21}$ and $q + 1$ lines of $T_{20}$. Since $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}''$.

If $\lambda_1 - \lambda_2$ is a square, then $(P^\perp_{\lambda_1})^\perp_{\lambda_2} \in O^2_n$. The plane $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}''$.

Let $\sigma = P^\perp_{\lambda_2}, P \in \mathcal{E}_{\lambda_2} \setminus \{U_3\}$. Assume that $q \equiv 1 \pmod 4$. Taking into account Lemma 3.1 we have that $\sigma$ contains $q + 1$ lines of $L_0$.

If $\lambda_2 - \lambda_1$ is a non-square, then $(P^\perp_{\lambda_2})^\perp_{\lambda_1} \in O^1_n$. The plane $\sigma$ contains $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{11}$ and $q + 1$ lines of $T_{10}$. Since $\sigma$ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2 + 2(q + 1)$ lines of $\mathcal{L}''$.

If $\lambda_2 - \lambda_1$ is a square, then $(P^\perp_{\lambda_2})^\perp_{\lambda_1} \in O^1_n$. The plane $\sigma$ contains $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{10}$ and $q + 1$ lines of $T_{11}$. Since $\sigma$ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}''$.

Assume that $q \equiv -1 \pmod 4$. Taking into account Lemma 3.1 we have that $\sigma$ contains $q + 1$ lines of $L_0$.

If $\lambda_2 - \lambda_1$ is a non-square, then $(P^\perp_{\lambda_2})^\perp_{\lambda_1} \in O^1_n$. The plane $\sigma$ contains $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{10}$ and $q + 1$ lines of $T_{11}$. Since $\sigma$ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that $\sigma$ contains $q(q-1)/2 - (q + 1)$ lines of $\mathcal{L}''$.

If $\lambda_2 - \lambda_1$ is a square, then $(P^\perp_{\lambda_2})^\perp_{\lambda_1} \in O^1_n$. The plane $\sigma$ contains $(q + 1)/2$
lines of \( T_{20} \) and of \( T_{21} \), no line of \( T_{11} \) and \( q + 1 \) lines of \( T_{10} \). Since \( \sigma \) contains \( q(q - 1)/2 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L'' \).

Let \( \sigma = P^{\perp_{\lambda_3}}, P \in \mathcal{E}_{\lambda_3} \setminus \{U_3\}, \lambda_3 \in \text{GF}(q) \setminus \{0, \lambda_1, \lambda_2\} \).

Taking into account Lemma 3.1, we have that \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_s \) if and only if \( \lambda_3 \) is a square in \( \text{GF}(q) \).

Assume that \( \lambda_3 \) is a square in \( \text{GF}(q) \). The following possibilities arise:

- \( \lambda_3 - \lambda_1, \lambda_3 - \lambda_2 \) are squares in \( \text{GF}(q) \) and then \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_1 \) and \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_2^2 \). If \( q \equiv -1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{20} \) and of \( T_{10} \) and no line of \( T_{21} \) and of \( T_{11} \). Since \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L'' \). If \( q \equiv 1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{11} \) and of \( T_{21} \) and no line of \( T_{20} \) and of \( T_{10} \). Since \( \sigma \) contains \( q(q - 1)/2 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 \) lines of \( L'' \);

- \( \lambda_3 - \lambda_1 \) is a non–square and \( \lambda_3 - \lambda_2 \) is a square in \( \text{GF}(q) \), then \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_1 \) and \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_2^2 \). If \( q \equiv -1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{20} \) and of \( T_{11} \) and no line of \( T_{21} \) and of \( T_{10} \). Since \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 -(q + 1) \) lines of \( L'' \). If \( q \equiv 1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{21} \) and of \( T_{10} \) and no line of \( T_{20} \) and of \( T_{11} \). Since \( \sigma \) contains \( q(q - 1)/2 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 + 2(q + 1) \) lines of \( L'' \);

- \( \lambda_3 - \lambda_1 \) is a square and \( \lambda_3 - \lambda_2 \) is a non–square in \( \text{GF}(q) \), then \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_1 \) and \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_2^2 \). If \( q \equiv -1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{21} \) and of \( T_{10} \) and no line of \( T_{20} \) and of \( T_{11} \). Since \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 + 3(q + 1) \) lines of \( L'' \). If \( q \equiv 1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{20} \) and of \( T_{11} \) and no line of \( T_{21} \) and of \( T_{10} \). Since \( \sigma \) contains \( q(q - 1)/2 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 - 2(q + 1) \) lines of \( L'' \);

- \( \lambda_3 - \lambda_1, \lambda_3 - \lambda_2 \) are non–squares in \( \text{GF}(q) \) and then \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_1 \) and \( (P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_2^2 \). If \( q \equiv -1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{11} \) and of \( T_{21} \) and no line of \( T_{20} \) and of \( T_{10} \). Since \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L' \), we have that \( \sigma \) contains \( q(q - 1)/2 + q + 1 \) lines of \( L'' \). If \( q \equiv 1 \pmod{4} \), \( \sigma \) contains \( q + 1 \) lines of \( T_{20} \) and of \( T_{10} \).
and no line of $T_{21}$ and of $T_{11}$. Since $\sigma$ contains $q(q - 1)/2$ lines of $L'$, we have that $\sigma$ contains $q(q - 1)/2$ lines of $L''$.

Assume that $\lambda_3$ is a non-square in GF$(q)$. Arguing as above, we have the following possibilities:

- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are squares in GF$(q)$. In this case $\sigma$ contains $q(q - 1)/2$ or $q(q - 1)/2 + q + 1$ lines of $L''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1$ is a non-square and $\lambda_3 - \lambda_2$ is a square in GF$(q)$. In this case $\sigma$ contains $q(q - 1)/2 - 2(q + 1)$ or $q(q - 1)/2 + 3(q + 1)$ lines of $L''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1$ is a square and $\lambda_3 - \lambda_2$ is a non-square in GF$(q)$. In this case $\sigma$ contains $q(q - 1)/2 - 2(q + 1)$ or $q(q - 1)/2 - (q + 1)$ lines of $L''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are non-squares in GF$(q)$. In this case $\sigma$ contains $q(q - 1)/2$ or $q(q - 1)/2 + q + 1$ lines of $L''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.

The proof is now complete.

Proposition 3.12. The characters of $L''$, with respect to line-stars of $\text{PG}(3, q)$ form a subset of:

$$\left\{ \frac{q + 1}{2}, \frac{5(q + 1)}{2}, \frac{q(q + 1)}{2} - 2(q + 1), \frac{q(q + 1)}{2} - (q + 1), \right.$$  

$$\frac{q(q + 1)}{2}, \frac{q(q + 1)}{2} + q + 1, \frac{q(q + 1)}{2} + 2(q + 1), \frac{q(q + 1)}{2} + 3(q + 1) \right\}.$$  

Proof. Through the point $U_3$, there pass $(q + 1)/2$ lines of $L''$.

Let $P \in E \setminus \{U_3\}$. Consider the tactical configuration whose points are the $q^2$ points of $E \setminus \{U_3\}$ and whose blocks are the $q^2(q + 1)/2$ lines contained in $T_{10}$ (in $T_{21}$). It turns out that through $P$ there pass $q + 1$ lines of $T_{10}$ ($T_{21}$). Since $P$ is on $q + 1$ lines of $L'$, we have that $P$ is on $5(q + 1)/2$ lines of $L''$.

Let $P \in \pi_0$. Through $P$ there pass $q$ lines of $T_{10}$ and of $T_{20}$ and no line of $T_{11}$ and of $T_{21}$. Since $P$ is on $q(q + 1)/2 + q + 1$ lines of $L'$, we have that $P$ is on $q(q + 1)/2 + q + 1$ lines of $L''$.  

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Let $P \in \pi_1$. Through $P$ there pass $q$ lines of $T_{11}$ and of $T_{21}$ and no line of $T_{10}$ and of $T_{20}$. Since $P$ is on $q(q + 1)/2$ lines of $L'$, we have that $P$ is on $q(q + 1)/2$ lines of $L''$. 

Let $P \in \mathcal{E}_{\lambda_1} \setminus \{U_3\}$. 

Assume that $q \equiv 1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_1} \subseteq O_s$. If $\lambda_2 - \lambda_1$ is a square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through $P$ there pass $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{21}$ and $q + 1$ lines of $T_{20}$. Since $P$ is on $q(q + 1)/2 + q + 1$ lines of $L'$, we have that $P$ is on $q(q + 1)/2$ lines of $L''$. If $\lambda_2 - \lambda_1$ is a non–square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through $P$ there pass $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{20}$ and $q + 1$ lines of $T_{21}$. Since $P$ is on $q(q + 1)/2 + q + 1$ lines of $L'$, we have that $P$ is on $(q + 1)/2 + 2(q + 1)$ lines of $L''$. 

Assume that $q \equiv -1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_1} \subseteq O_s$. If $\lambda_2 - \lambda_1$ is a square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through $P$ there pass $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{21}$ and $q + 1$ lines of $T_{20}$. Since $P$ is on $q(q + 1)/2 + q + 1$ lines of $L'$, we have that $P$ is on $q(q + 1)/2 - (q + 1)$ lines of $L''$. If $\lambda_2 - \lambda_1$ is a non–square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through $P$ there pass $(q + 1)/2$ lines of $T_{11}$ and of $T_{10}$, no line of $T_{20}$ and $q + 1$ lines of $T_{21}$. Since $P$ is on $q(q + 1)/2$ lines of $L'$, we have that $P$ is on $(q + 1)/2 - (q + 1)$ lines of $L''$. 

Let $P \in \mathcal{E}_{\lambda_2} \setminus \{U_3\}$. 

Assume that $q \equiv 1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_2} \subseteq O_s$. If $\lambda_1 - \lambda_2$ is a square, then $\mathcal{E}_{\lambda_2} \subseteq O_s^1$. Through $P$ there pass $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{11}$ and $q + 1$ lines of $T_{10}$. Since $P$ is on $q(q + 1)/2$ lines of $L'$, we have that $P$ is on $q(q + 1)/2 + q + 1$ lines of $L''$. If $\lambda_1 - \lambda_2$ is a non–square, then $\mathcal{E}_{\lambda_2} \subseteq O_s^1$. Through $P$ there pass $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{10}$ and $q + 1$ lines of $T_{11}$. Since $P$ is on $q(q + 1)/2$ lines of $L'$, we have that $P$ is on $(q + 1)/2 - (q + 1)$ lines of $L''$. Assume that $q \equiv -1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_2} \subseteq O_s$. If $\lambda_1 - \lambda_2$ is a square, then $\mathcal{E}_{\lambda_2} \subseteq O_s^1$. Through $P$ there pass $(q + 1)/2$ lines of $T_{20}$ and of $T_{21}$, no line of $T_{11}$ and $q + 1$ lines of $T_{10}$. Since $P$ is on $q(q + 1)/2 + q + 1$ lines of $L'$, we have that $P$ is on $(q + 1)/2 + 2(q + 1)$ lines of $L''$. 

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If \( \lambda_1 - \lambda_2 \) is a non–square, then \( E_{\lambda_2} \subseteq O^1_n \). Through \( P \) there pass \( (q + 1)/2 \) lines of \( T_{20} \) and of \( T_{21} \), no line of \( T_{10} \) and \( q + 1 \) lines of \( T_{11} \). Since \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L' \), we have that \( P \) is on \( q(q + 1)/2 \) lines of \( L'' \).

Let \( P \in E_{\lambda_3} \setminus \{ U_3 \}, \lambda_3 \in GF(q) \setminus \{ 0, \lambda_1, \lambda_2 \} \).

Taking into account Remark 3.3, we have that \( P \in O_s \) if and only if \( -\lambda_3 \) is a square in GF(\( q \)). Assume that \( -\lambda_3 \) is a square in GF(\( q \)). The following possibilities arise:

- \( \lambda_1 - \lambda_3, \lambda_2 - \lambda_3 \) are squares in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). Through \( P \) there pass \( q + 1 \) lines of \( T_{20} \) and of \( T_{10} \) and no line of \( T_{21} \) and of \( T_{11} \). Since \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L' \), we have that \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L'' \);

- \( \lambda_1 - \lambda_3 \) is a square and \( \lambda_2 - \lambda_3 \) is a non–square in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). Through \( P \) there pass \( q + 1 \) lines of \( T_{21} \) and of \( T_{10} \) and no line of \( T_{20} \) and of \( T_{11} \). Since \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L' \), we have that \( P \) is on \( q(q + 1)/2 + 3(q + 1) \) lines of \( L'' \);

- \( \lambda_1 - \lambda_3 \) is a non–square and \( \lambda_2 - \lambda_3 \) is a square in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). Through \( P \) there pass \( q + 1 \) lines of \( T_{11} \) and of \( T_{20} \) and no line of \( T_{21} \) and of \( T_{10} \). Since \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L' \), we have that \( P \) is on \( q(q + 1)/2 - (q + 1) \) lines of \( L'' \);

- \( \lambda_1 - \lambda_3, \lambda_2 - \lambda_3 \) are non–squares in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). Through \( P \) there pass \( q + 1 \) lines of \( T_{11} \) and of \( T_{21} \) and no line of \( T_{20} \) and of \( T_{10} \). Since \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L' \), we have that \( P \) is on \( q(q + 1)/2 + q + 1 \) lines of \( L'' \).

Assume that \( -\lambda_3 \) is a non–square in GF(\( q \)). Arguing as above, we have the following possibilities:

- \( \lambda_1 - \lambda_3, \lambda_2 - \lambda_3 \) are squares in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). In this case \( P \) is on \( q(q + 1)/2 \) lines of \( L'' \);

- \( \lambda_1 - \lambda_3 \) is a square and \( \lambda_2 - \lambda_3 \) is a non–square in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). In this case \( P \) is on \( q(q + 1)/2 + 2(q + 1) \) lines of \( L'' \);

- \( \lambda_1 - \lambda_3 \) is a non–square and \( \lambda_2 - \lambda_3 \) is a square in GF(\( q \)) and then \( P \in O^1_s \cap O^2_n \). In this case \( P \) is on \( q(q + 1)/2 - 2(q + 1) \) lines of \( L'' \);

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• $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are non–squares in $\text{GF}(q)$ and then $P \in O^1_n \cap O^2_n$. In this case $P$ is on $q(q + 1)/2$ lines of $L''$.

The proof is now complete. \hfill \Box

**Theorem 3.13.** If $q \geq 7$ odd, the Cameron–Liebler line class $L''$ is not equivalent to one of the previously known examples.

*Proof.* From the proof of Proposition 3.12 if $P \in E \setminus \{U_3\}$, through $P$ there pass $5(q + 1)/2$ lines of $L'''$. Since, for $q \geq 7$, the value $5(q + 1)/2$ does not appear among the characters of $L'$ and $L''$, we may conclude that $L'''$ is distinct from $L'$ and $L''$. On the other hand, both examples $X \cup Y$ and $X \cup Z$ admit $q^2 + q + 1$ as a character, but from Proposition 3.11 and Proposition 3.12 such a value does not appear as a character of $L'''$. \hfill \Box

**Remark 3.14.** Let $\square_q$ denote the non–zero square elements of $\text{GF}(q)$. From the proof of Proposition 3.11 and of Proposition 3.12 if there exist $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \text{GF}(q) \setminus \{0, \lambda_1, \lambda_2\}$ such that $a_1, a_2, a_3, a_4 \in \square_q$, $b_1, b_2, b_3, b_4 \notin \square_q$ and $a_1 - \lambda_1 \notin \square_q, a_1 - \lambda_2 \notin \square_q, a_2 - \lambda_1 \notin \square_q, a_2 - \lambda_2 \notin \square_q, a_3 - \lambda_1 \notin \square_q, a_3 - \lambda_2 \notin \square_q, a_4 - \lambda_1 \notin \square_q, a_4 - \lambda_2 \notin \square_q, b_1 - \lambda_1 \notin \square_q, b_1 - \lambda_2 \notin \square_q, b_2 - \lambda_1 \notin \square_q, b_2 - \lambda_2 \notin \square_q, b_3 - \lambda_1 \notin \square_q, b_3 - \lambda_2 \notin \square_q, b_4 - \lambda_1 \notin \square_q, b_4 - \lambda_2 \notin \square_q$, then $L'''$ has exactly eight characters with respect to line–sets in planes of $\text{PG}(3, q)$ (line–stars of $\text{PG}(3, q)$).

**Remark 3.15.** Note that both, $L'$ and $L''$, are Cameron–Liebler line classes satisfying the requirements of Theorem 3.9 and that, starting from $L'$ or $L''$, the replacement technique described in Theorem 3.9 can be iterated $(q - 1)/2$ times.

**Remark 3.16.** Computations performed with Magma [2] suggest that starting from $L'$ and applying Theorem 3.9 (multiple derivation is allowed), apart from Bruen–Drudge’s example and the example described in [5] and [8], we get what follows. The notation $\alpha^i$ in the character strings below stands for: there are $i$ planes containing $\alpha$ lines of the Cameron–Liebler line class or there are $i$ line–stars containing $\alpha$ lines of the Cameron–Liebler line class.

$q = 7$

A new example arises having the characters:

1) $13^{49}, 21^{126}, 29^{77}, 37^{98}, 45^{49}, 53$ with respect to line–sets in planes of $\text{PG}(3, 7)$ and $4, 12^{49}, 20^{98}, 28^{77}, 36^{126}, 44^{49}$ with respect to line–stars of $\text{PG}(3, 7)$. 

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Three new examples arise having the characters:

\[ q = 9 \]

i) \(16^{81}, 36^{207}, 46^{288}, 56^{81}, 66^{162}, 86\) with respect to line–sets in planes of \(\text{PG}(3,9)\) and \(5, 25^{162}, 35^{81}, 45^{288}, 55^{207}, 75^{81}\) with respect to line–stars of \(\text{PG}(3,9)\);

ii) \(26^{162}, 36^{207}, 46^{126}, 56^{162}, 66^{162}, 86\) with respect to line–sets in planes of \(\text{PG}(3,9)\) and \(5, 25^{162}, 35^{162}, 45^{126}, 55^{207}, 65^{162}\) with respect to line–stars of \(\text{PG}(3,9)\);

iii) \(26^{162}, 36^{126}, 46^{288}, 56^{162}, 66^{162}, 86\) with respect to line–sets in planes of \(\text{PG}(3,9)\) and \(5, 15^{81}, 35^{162}, 45^{288}, 55^{126}, 65^{162}\) with respect to line–stars of \(\text{PG}(3,9)\).

Five new examples arise having the characters:

\[ q = 11 \]

i) \(31^{121}, 43^{121}, 55^{308}, 67^{429}, 79^{242}, 91^{121}, 103^{121}, 127\) with respect to line–sets in planes of \(\text{PG}(3,11)\) and \(6, 30^{121}, 42^{121}, 54^{242}, 66^{429}, 78^{308}, 90^{121}, 102^{121}\) with respect to line–stars of \(\text{PG}(3,11)\), see Remark 3.14

ii) \(43^{242}, 55^{550}, 67^{187}, 79^{121}, 91^{242}, 103^{121}, 127\) with respect to line–sets in planes of \(\text{PG}(3,11)\) and \(6, 30^{121}, 42^{242}, 54^{121}, 66^{187}, 78^{550}, 90^{242}\) with respect to line–stars of \(\text{PG}(3,11)\);

iii) \(43^{242}, 55^{429}, 67^{308}, 79^{363}, 115^{121}, 127\) with respect to line–sets in planes of \(\text{PG}(3,11)\) and \(6, 18^{121}, 54^{363}, 66^{308}, 78^{429}, 90^{242}\) with respect to line–stars of \(\text{PG}(3,11)\);

iv) \(31^{121}, 43^{242}, 55^{187}, 67^{187}, 79^{484}, 91^{242}, 127\) with respect to line–sets in planes of \(\text{PG}(3,11)\) and \(6, 42^{242}, 54^{484}, 66^{187}, 78^{187}, 90^{242}, 102^{121}\) with respect to line–stars of \(\text{PG}(3,11)\);

v) \(19^{121}, 55^{249}, 67^{308}, 79^{363}, 91^{242}, 127\) with respect to line–sets in planes of \(\text{PG}(3,11)\) and \(6, 42^{242}, 54^{363}, 66^{308}, 78^{429}, 114^{121}\) with respect to line–stars of \(\text{PG}(3,11)\).

Interestingly, if we start from \(L''\), we get the complements of the abovementioned examples. In general, it seems a difficult task to determine how many inequivalent examples of Cameron–Liebler line classes arise from Theorem 3.9 and we leave it as an open problem.
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