Modified Bosonic Gas Trapped in a Generic 3–dim Power Law Potential

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We analyze the consequences caused by an anomalous single–particle dispersion relation suggested in several quantum–gravity models, upon the thermodynamics of a Bose–Einstein gas trapped in a generic 3-dimensional power–law potential, within the semiclassical approximation. We show that the critical temperature must be corrected as a consequence of such deformation in the dispersion relation. Additionally, we prove that this shift in the critical temperature, in three different scenarios, namely, the thermodynamic limit, systems with a finite number of particles, and in a weakly interacting systems, can be described as a non–trivial function of the number of particles and the trap parameters. In other words, an appropriate manipulation of the number of particles, together with an adequate choice of the trap parameters, could be used, in principle, to enhance some quantum gravity manifestations.

I. INTRODUCTION

Several quantum gravity models suggest that the dispersion relation between energy and momentum for microscopic particles, must be modified as a consequence of the quantum structure of space–time [1–3].

A modified dispersion relation emerges as an adequate tool in the search for phenomenological consequences caused by this type of quantum gravity models. Nevertheless, the most difficult aspect in the search of quantum–gravity manifestations, is the smallness in the predicted effects [4]. If this kind of deformations are characterized for some Planck scale, then the quantum gravity effects becomes very small [1, 3]. In the non–relativistic limit, the deformed dispersion relation can be expressed as follows [2, 3] (in units where the speed of light $c = 1$)

$$E \simeq m + \frac{p^2}{2m} + \frac{1}{2M_p}\left(\xi_1 mp + \xi_2 p^2 + \xi_3 \frac{p^2}{m}\right).$$

(1)

Where $M_p (\simeq 1.2 \times 10^{28} eV)$ is the Planck mass, $m$ is the mass particle in question, and $p$ is the momentum. The three parameters $\xi_1$, $\xi_2$, and $\xi_3$, are model dependent [1, 3], and should take positive or negative values close to 1. Equation (1) is the starting point in the search for small manifestations of quantum gravity at low energies.

The use of Bose–Einstein condensation phenomenon as a tool in the search for some quantum–gravity manifestations in our low energy world, is not a new topic, and has produced an interesting series of studies in this direction [3, 7–11]. However, as mentioned above, the corrections caused by some quantum–gravity manifestation, in the relevant properties associated to the Bose–Einstein condensate, are still very small to be measured, then the main goal is search the possibility to enhance such possible manifestations.

The analysis of a Bose–Einstein condensation phenomena in the ideal case, weakly interacting, and with a finite number of particles, trapped in different types of potentials [12–23] shows that the main properties associated to the condensate, and in particular the critical temperature, depends strongly of the characteristics of the trapping potential in question, together with a non–trivial function of the number of particles. A logical step, is introduce a modified bosonic gas in this scenario and analyze the behavior of such systems in this quite general description. For this propose, let us define the next “modified Hamiltonian”

$$H = \frac{p^2}{2m} + \alpha p + U(\vec{r}).$$

(2)

Where $p$ is the momentum, $m$ is the mass of the particle and the term $\alpha p$ is the leading order modification in expression (1). The potential term

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\[ U(\vec{r}) = \sum_{i=1}^{d} A_i \left| \frac{r_i}{a_i} \right|^{s_i}, \tag{3} \]

in the Hamiltonian (2) is the so-called generic 3-dimensional power-law potential, where \( A_i \) and \( a_i \) are energy and length scales associated to the trap \([23]\). On the other hand, \( r_i \) are the \( d \) radial coordinates in the \( n_i \)-dimensional subspace of the 3-dimensional space. The sub-dimensions \( n_i \) satisfy the following expression in three spatial dimensions

\[ \sum_{i=1}^{d} n_i = 3. \tag{4} \]

If in equation (4) \( d = 3 \), \( n_1 = n_2 = n_3 = 1 \) then the potential becomes in the so-called cartesian trap, if \( d = 2 \), \( n_1 = 2 \) and \( n_2 = 1 \) then we obtain the cylindrical trap, and if \( d = 1 \), \( n_1 = 3 \) we have the spherical trap. On the other hand, if \( s_i \rightarrow \infty \) we have a free gas in a box. In this sense the potential included in the Hamiltonian (2) is quite general. Different combinations of these parameters gives different classes of potentials according to (3).

The outline of the present paper is the following one. In section 2 we deduce the shift in the critical temperature in the thermodynamic limit, in function of the number of particles. Section 3 studies the shift in the critical temperature caused by a finite number of particles. Section 4 analyzes the properties of a weakly interacting Bose-Einstein gas, for systems with a large number of particles. Finally, section 5 analyzes the main results and adds some comments concerning the perspectives of precision tests with Bose-Einstein condensates.

II. DEFORMED BOSONIC GAS TRAPPED IN A GENERIC 3-DIM POWER LAW POTENTIAL IN THE THERMODYNAMIC LIMIT

Let us calculate the number of microstates for the modified Hamiltonian (2). In the semiclassical approximation the number of microstates \([25]\) and the semiclassical energy associated with the modified Hamiltonian (2) are given respectively by

\[ \Sigma(\epsilon) = \frac{1}{(2\pi \hbar)^{3/2}} \int d^3 \vec{r} d^3 \vec{p}. \tag{5} \]

\[ \epsilon = \frac{p^2}{2m} + \alpha p + U(\vec{r}). \tag{6} \]

From equation (6) we obtain the relation between energy and momentum,

\[ p = (2m)^{1/2} \left( \epsilon + \frac{1}{2} m \alpha^2 - \sum_{i=1}^{d} A_i \left( \frac{r_i}{a_i} \right)^{s_i} \right)^{1/2} - \alpha m. \tag{7} \]

Integrating expression (6) in the momentum space, using the volume of a sphere, together with equation (7) at first order in \( \alpha \) leads to

\[ \Sigma(\epsilon; \alpha) = \frac{4\pi}{3(2\pi \hbar)^{3/2}} (2m)^{3/2} \int d^3 \vec{r} (\epsilon + \frac{1}{2} ma^2 - U(\vec{r}))^{3/2} \]

\[ - \alpha \frac{4\pi}{(2\pi \hbar)^{3/2}} m (2m) \int d^3 \vec{r} (\epsilon + \frac{1}{2} ma^2 - U(\vec{r})). \tag{8} \]

The integration over the space in equation (8) using the definition of the Beta function \( B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx = \Gamma(a)\Gamma(b)/\Gamma(a + b) \), after some algebraic manipulation, leads to the number of microstates associated to this system.
\[ \Sigma(\epsilon; \alpha) = \frac{4\pi}{3(2\pi\hbar)^3} (2m)^{3/2} \left( \epsilon + \frac{1}{2}m\alpha^2 \right)^{3/2 + \sum_{i=1}^{d} \frac{n_i}{s_i}} \left\{ \frac{\Gamma \left( \frac{3}{2} \right) \prod_{i=1}^{d} \left( A_i \frac{n_i}{s_i} \right)^{n_i} \Gamma \left( \frac{n_i}{s_i} + 1 \right)}{\prod_{i=1}^{d} \left( m \frac{n_i}{s_i} + 1 \right)} \right\} \] (9)

\[ - \alpha \frac{4\pi}{(2\pi\hbar)^3} 2m^2 \left( \epsilon + \frac{1}{2}m\alpha^2 \right)^{1+\sum_{i=1}^{d} \frac{n_i}{s_i}} \left\{ \frac{\Gamma \left( \frac{3}{2} \right) \prod_{i=1}^{d} \left( A_i \frac{n_i}{s_i} \right)^{n_i} \Gamma \left( \frac{n_i}{s_i} + 1 \right)}{\prod_{i=1}^{d} \left( m \frac{n_i}{s_i} + 1 \right)} \right\}. \] (10)

Where \( C \) is a constant that depends on the potential in question (in the case of a harmonic oscillator \( C = 8 \)), and \( \Gamma(x) \) is the Gamma function.

The number of microstates per energy unit can be obtained by taking the derivative with respect to \( \epsilon \) in equation (9)

\[ \Omega(\epsilon; \alpha) = \frac{d\Sigma(\epsilon; \alpha)}{d\epsilon} = \left( \epsilon + \frac{1}{2}m\alpha^2 \right)^{1/2 + \sum_{i=1}^{d} \frac{n_i}{s_i}} \Omega_1 - \alpha \left( \epsilon + \frac{1}{2}m\alpha^2 \right)^{\sum_{i=1}^{d} \frac{n_i}{s_i}} \Omega_2, \] (11)

where

\[ \Omega_1 = \frac{4\pi}{3(2\pi\hbar)^3} (2m)^{3/2} \left\{ \frac{\Gamma \left( \frac{3}{2} \right) \prod_{i=1}^{d} \left( A_i \frac{n_i}{s_i} \right)^{n_i} \Gamma \left( \frac{n_i}{s_i} + 1 \right)}{\prod_{i=1}^{d} \left( m \frac{n_i}{s_i} + 1 \right)} \right\}, \] (12)

Let us calculate the transition temperature. The number of particles in the continuum approximation is given by the following expression in the grand canonical ensemble [24]

\[ N = N_0 + \int_0^\infty d\epsilon \frac{\Omega(\epsilon; \alpha)}{z - \epsilon - 1}, \] (13)

where \( N_0 \) is the number of particles in the ground state, and \( z \) is the fugacity which is related to the chemical potential through the expression \( z = \exp(\beta\mu) \), here \( \beta = 1/\kappa T \), and \( \kappa \) is the Boltzmann constant.

Using equation (12) and equation (13), assuming that \( m\alpha^2/2 << \kappa T \), with the change of variables \( x = \beta\epsilon = \beta \left( \epsilon + \frac{1}{2}m\alpha^2 \right) \), allows us to express the number of particles as follows

\[ N = N_0 + \prod_{i=1}^{d} A_i^{-\frac{n_i}{s_i}} n_i! \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} g_\gamma(z_{eff})(\kappa T)\gamma - \alpha \left( \frac{m^2}{2\pi^2\hbar^3} \right) g_{\gamma-1/2}(z_{eff})(\kappa T)^{\gamma-1/2}. \] (14)

Where

\[ \gamma = \frac{3}{2} + \sum_{i=1}^{d} \frac{n_i}{s_i}, \] (15)

is the parameter that defines the shape of the potential, being \( z_{eff} = z \exp(\beta(\alpha^2/2)) \) an effective fugacity. Additionally \( \Gamma(x) \) is the gamma function.

The functions \( g_\nu(z) \) are the so-called Bose–Einstein functions defined by [24] [25]

\[ g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{1+e^x} - 1}. \] (16)

The Bose-Einstein function [10] diverges for \( z = 1 \) when \( \nu \leq 1 \) [24].

Expanding around \( \alpha = 0 \) to first in order \( \alpha \), using the properties of the Bose–Einstein functions [24]
\[ x \frac{\partial}{\partial x} g_\nu(x) = g_{\nu-1}(x), \] (17)

allows us to write expression (14) as

\[ N = N_0 + C \Pi_{l=1}^d \frac{n_l}{n_l} a_l^{n_l} \Gamma \left( \frac{n_l}{s_l} + 1 \right) \left[ \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} g_{\gamma}(z)(\kappa T)^\gamma - \alpha \left( \frac{m^2}{2\pi^2\hbar^3} \right) g_{\gamma-1/2}(z)(\kappa T)^{\gamma-1/2} \right]. \] (18)

Let us define

\[ V_{\text{char}} = C \Pi_{l=1}^d \frac{n_l}{n_l} a_l^{n_l} \Gamma \left( \frac{n_l}{s_l} + 1 \right), \] (19)

as the characteristic volume associated with the system. We can see that from expression (19), when \( s_i \to \infty \), then \( V_{\text{char}} \) becomes the volume associated with a free gas in a box. In this sense \( V_{\text{char}} \) can be interpreted as the available volume occupied by the gas [18, 22].

When the number of particles, \( N \), is large, we may neglect the zero-point energy contributions and we can equate the lowest energy to zero [26]. Hence, for the onset of the condensation in the thermodynamic limit the number of particles in the ground state is negligible \( N_0 = 0 \) and the fugacity \( z = 1 \), with these assumptions the Bose–Einstein functions \( g_{\gamma}(z) \) are given by the Riemann Zeta functions \( \zeta(\gamma) \) [24]. It is noteworthy to mention that the phenomenon of condensation for a modified bosonic gas in the thermodynamic limit is possible when \( \gamma > 3/2 \) according to the properties of the Bose–Einstein functions (16). Expression (18) at the transition temperature now becomes in

\[ N = V_{\text{char}} \left[ \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \zeta(\gamma)(\kappa T_0)^\gamma - \alpha \left( \frac{m^2}{2\pi^2\hbar^3} \right) \zeta(\gamma - 1/2)(\kappa T_0)^{\gamma-1/2} \right], \] (20)

where \( T_0 \) is the critical temperature.

If we set \( \alpha = 0 \) in equation (20) we obtain the usual result [23]

\[ N = V_{\text{char}} \left[ \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \zeta(\gamma)(\kappa T_0)^\gamma \right]. \] (21)

From which we obtain the usual definition of the critical temperature \( T_0 \) in the thermodynamic limit

\[ T_0 = \left[ \frac{N}{V_{\text{char}} \zeta(\gamma)} \left( \frac{2\pi\hbar^2}{m} \right)^{3/2} \right]^{1/\gamma} \frac{1}{\kappa}. \] (22)

At this point it is noteworthy to mention that the most general definition of thermodynamic limit can be expressed as

\[ N \to \infty, \quad V_{\text{char}} \to 0, \] (23)

keeping the product \( NV_{\text{char}} \to const \), and is valid for all power law potentials in any spatial dimensionality [21].

Using these facts, the critical temperature in the thermodynamic limit is well defined.

Equations (20) and (22) implies

\[ \frac{\Delta T_c}{T_0} \approx \alpha \frac{\zeta(\gamma - 1/2)}{\gamma \zeta(\gamma)} \left( \frac{2m}{\pi} \right)^{1/2} \left( \frac{(2m\hbar^2)^{3/2}}{V_{\text{char}} m^{3/2} \zeta(\gamma)} \right)^{-1/2\gamma} N^{-1/2\gamma}. \] (24)

It is noteworthy to mention that the correction in the critical temperature (24) depends strongly on the functional form between the number of particles and the parameters of the potential.
Let us suppose now that our condensate is trapped in an anisotropic three-dimensional harmonic-oscillator potential. For this trap, the value of the shape parameter is given by $\gamma = 3$, with $A_i = \hbar \omega_i/2$, $a_i = \sqrt{\hbar/m\omega_i}$ (see expression (19)), and with the definition $\tilde{\omega} = (\omega_1 \omega_2 \omega_3)^{1/3}$, we obtain from expression (24)

$$\frac{\Delta T_c}{T_0} \simeq \frac{\frac{1}{3} \zeta(5/2)}{\zeta(3)} \left( \frac{8m}{\pi \hbar^2} \right)^{1/2} N^{-1/6}. \tag{25}$$

If we set $\alpha = 0$ in equation (25) then there is not shift in the critical temperature, $\Delta T_c/T_0 = 0$. In typical experiments the number of particles vary from a few thousand to several millions, and frequencies $\omega_i$ are bounded i.e., $\omega_i < 87$ of Hertz [5]. For instance, in the case of $^8^7$Rb, with say, $\omega_1 = 2\pi \times 50$Hz, $\omega_2 = 2\pi \times 40$Hz, $\omega_3 = 2\pi \times 20$Hz, and $\alpha = \frac{m \omega_1^2 c}{2 \hbar} > 0$, $c$ is the speed of light and $M_p$ is the Planck’s mass, with say, $N = 10^{18}$ and $N = 10^{20}$ we obtain from (25) that the shift in the critical temperature is given respectively by $\Delta T_c/T_0 \simeq \xi_1 9.7 \times 10^{-9}$ and $\Delta T_c/T_0 \simeq \xi_1 3 \times 10^{-8}$.

### III. DEFORMED BOSONIC GAS AND FINITE NUMBER OF PARTICLES

In this section let us calculate the leading correction in the critical temperature caused for a finite number of particles in our modified bosonic gas. For this purpose we must take into account the associated zero–point energy $\Delta E_0$, the number of particles vary from a few thousand to several millions, and frequencies $\bar{\omega}$ from tens to hundreds of Hertz [5]. For instance, in the case of $^8^7$Rb, with say, $\omega_1 = 2\pi \times 50$Hz, $\omega_2 = 2\pi \times 40$Hz, $\omega_3 = 2\pi \times 20$Hz, and $\alpha = \frac{m \omega_1^2 c}{2 \hbar} > 0$, $c$ is the speed of light and $M_p$ is the Planck’s mass, with say, $N = 10^{18}$ and $N = 10^{20}$ we obtain from (25) that the shift in the critical temperature is given respectively by $\Delta T_c/T_0 \simeq \xi_1 9.7 \times 10^{-9}$ and $\Delta T_c/T_0 \simeq \xi_1 3 \times 10^{-8}$.

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$$N = N_0 + V_{\text{char}} \left[ \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} g_{\gamma}(z_{\text{eff}}(\kappa T)) - \alpha \left( \frac{m^2}{2\pi \hbar^3} \right) g_{\gamma-1/2}(z_{\text{eff}}(\kappa T))^{\gamma-1/2} \right]. \tag{26}$$

where $z_{\text{eff}} = e^{\beta(\mu + m^2 a^2/2)}$ is an effective fugacity, and $g_{\nu}(x)$ is the Bose–Einstein function.

We can expand the last equation using the behavior of Bose–Einstein functions about zero [24]

$$g_{\nu}(e^{-\delta}) = \frac{\Gamma(1 - \nu)}{\delta^{1-\nu}} + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \zeta(\nu - i) \delta^i \tag{27}$$

$$g_{m}(e^{-\delta}) = \frac{(-1)^{m-1}}{(m-1)!} \left[ \sum_{i=1}^{m-1} \frac{1}{i} - \ln(\delta) \right] \delta^{m-1} + \sum_{i=0, i\neq m-1}^{\infty} \frac{(-1)^i}{i!} \zeta(m - i) \delta^i \tag{28}$$

In expressions (27) and (28), $\Gamma(x)$ is the gamma function and $\zeta(y)$ is the Riemann Zeta function. Expression (27) is valid for all $\nu < 1$, and to all non–integral $\nu > 1$, and expression (28) is valid for positive integers $m$, both expressions are valid in the limit $\delta \rightarrow 0$. The case $\nu = 1$ is rather simple, in this case we have

$$g_{1}(e^{-\delta}) = -\ln(1 - e^{-\delta}) \rightarrow \ln(1/\delta), \tag{29}$$

when $\delta \rightarrow 0$.

With this assumptions, we can express equation (26) at $T_c$, with $z_{\text{eff}} = e^{(-\kappa T_c)}$ as follows (at $T_c$ the chemical potential $\mu$ is bounded i.e., $\mu = \epsilon_0$)

$$N = V_{\text{char}} \left[ \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} g_{\gamma}(e^{-\kappa T_c}) - \alpha \left( \frac{m^2}{2\pi \hbar^3} \right) g_{\gamma-1/2}(e^{-\kappa T_c})^{\gamma-1/2} \right]. \tag{30}$$

Equation (30) implies

$$\left( \kappa T_c \right)^\gamma = \frac{N}{V_{\text{char}} g_{\gamma}(e^{-\kappa T_c})^{3/2}} \left[ 1 + \frac{\alpha}{N} \left( \frac{m^2}{2\pi \hbar^3} \right) g_{\gamma-1/2}(e^{-\kappa T_c})^{\gamma-1/2} \right]. \tag{31}$$
Using the result for the critical temperature $T_0$ in the thermodynamic limit expression \(22\) and equation \(31\), we obtain

$$
\left( \frac{T_c}{T_0} \right)^\gamma = \frac{\zeta(\gamma)}{g_\gamma(e^{-\delta_e})} \left[ 1 + \frac{\alpha}{N} \left( \frac{m^2}{2\pi^2\hbar^3} \right) g_{\gamma-1/2}(e^{-\delta_e})(\kappa T_c)^{-1/2} \right].
$$

(32)

If we set $\alpha = 0$ in equation \(32\) we recover the “long range effects” corrections given in \(23\). For an anisotropic three-dimensional harmonic oscillator potential ($\gamma = 3$ and using the expression \(32\) together with expressions \(27\) and \(28\), we obtain that the relative correction in the critical temperature first order in $\alpha$ and $\epsilon_0$, in function of the number of particles is given approximately by

$$
\frac{\Delta T_c}{T_0} \simeq -\frac{\zeta(2)}{3\zeta(3)^{2/3}} \left( \frac{1}{\hbar \omega} \right)^{1/3} \epsilon_0 N^{-1/3} + \alpha \left[ \frac{\zeta(5/2)}{3\zeta(3)^{5/6} \pi \hbar \omega} \right]^{1/2} N^{-1/6} + \left( \frac{\zeta(3/2)}{3\zeta(3)^{1/2}} - \frac{\zeta(5/2)\zeta(2)}{9\zeta(3)^{3/2}} \right) \left( \frac{2}{\hbar \omega \pi^{1/3}} \right)^{3/2} m \epsilon_0 N^{-1/2} + ... 
$$

(33)

If in equation \(33\) we set $\alpha = 0$, we obtain the first correction caused by a finite number of particles \(13, 15, 16, 26\). In fact, the term proportional to the zero–point energy $\epsilon_0$, is the leading correction caused by the finite size effects of the system.

For instance, in the case of the $^{87}$Rb, with $\alpha = \xi_1 (\frac{\hbar \omega}{2 M_p} c) > 0$, $c$ is the speed of light and $M_p$ is the Planck’s mass, we obtain that the leading correction in the shift \(33\) caused by the deformation for $N = 10^6$ particles, is given by

$$
\xi_1 \left( 4.5 \times 10^{-7} (\frac{\hbar}{2 M_p} c)^{1/2} + 3.7 \times 10^{-20} (\frac{\hbar}{M_p}) \right) 
$$

and for $N = 10^5$ we obtain $\xi_1 \left( 1.4 \times 10^{-6} (\frac{\hbar}{2 M_p} c)^{1/2} + 1.11 \times 10^{-16} (\frac{\hbar}{M_p}) \right)$,

where $\omega_0 = \frac{1}{2} (\omega_1 + \omega_2 + \omega_3)$.

This correction, unfortunately is small and show the requirement not only of a finite system, but also with a small number of particles, in which the approach followed in this paper is an approximation, due to problems of fluctuations and non-equivalence of the ensembles when the system contain both, a finite and small number of particles \(5, 6\).

**IV. WEAKLY INTERACTING MODIFIED BOSONIC GAS**

A realistic system does include interaction among its constituents. In this aim let us estimate the correction in the critical temperature for a deformed bosonic gas taking into account interactions among its constituents within the Hartree-Fock theory in the semi–classical approximation, which basically consist in the assumption that the constituents of the gas behave like a non–interacting bosons, moving in a self–consistent mean field and is valid when $\epsilon(\vec{r}, \vec{p}) \gg \mu$ for dilute gases \(3, 26\).

Let us propose a particularly simple modified Hartree-Fock type spectrum,

$$
\epsilon(\vec{r}, \vec{p}) = \frac{p^2}{2m} + \alpha p + U(\vec{r}) + 2U_0 n(\vec{r}).
$$

(34)

Where the term $2U_0 n(\vec{r})$ is a mean field generated by the interactions with the other constituents of the bosonic gas \(5\). $U_0$ is the coupling constant which is related with the s–wave scattering length $a$ through the next expression

$$
U_0 = \frac{4\pi \hbar^2}{m} a.
$$

(35)

On the other hand $n(\vec{r})$ is the spatial density of the cloud.

Let us calculate the spatial density associated with our modified Hartree–Fock spectrum \(34\). In the semiclassical approximation, the single–particle phase–space distribution is given by \(5, 26\)

$$
n(\vec{r}, \vec{p}) = \frac{1}{e^{\beta (\epsilon(\vec{r}, \vec{p}) - \mu)} - 1}
$$

(36)

The number of particles in the 3–dimensional space obey the normalization condition,
to obtain an expression for the number of particles in function of the chemical potential over the momentum space, allows us to obtain the spatial distribution associated with the modified Hartree–Fock ground state above the critical temperature is negligible. Using the expression (34), and integrating expression (36)

\[ N = \frac{1}{(2\pi \hbar)^3} \int d^3 \vec{r} \int d^3 \vec{p} n(\vec{r}, \vec{p}) = \int d^3 \vec{r} n(\vec{r}) = \int d^3 \vec{p} n(\vec{p}), \quad (37) \]

are the spatial and momentum densities, respectively.

Expression (37) is equivalent to expression (13). In expression (37) we assume that the number of particles in the ground state above the critical temperature is negligible. Using the expression (34), and integrating expression (36) over the momentum space, allows us to obtain the spatial distribution associated with the modified Hartree–Fock spectrum (34)

\[ n(\vec{r}) = \int d^3 \vec{p} n(\vec{r}, \vec{p}), \quad n(\vec{p}) = \int d^3 \vec{r} n(\vec{r}, \vec{p}) \quad (38) \]

Using the properties of the Bose–Einstein functions, equation (17), we can expand expression (39) around \( U_0 = 0 \), with the result

\[ n(\vec{r}) = n_0(\vec{r}) + U_0(2\kappa T)^{-1} \lambda^{-6} \left[g_{3/2}(Z)g_{1/2}(Z) - \alpha \left(\frac{m}{\pi \hbar}\right)^2 g_1(\lambda) + g_1(Z)g_{1/2}(Z)\right], \quad (41) \]

where

\[ Z = e^{\beta(\mu_{eff} - U(\vec{r}))}. \quad (42) \]

Being \( n_0(\vec{r}) \) the space density distribution for the ideal case \( U_0 = 0 \),

\[ n_0(\vec{r}) = \lambda^{-3}g_{3/2}(Z) - \alpha \lambda^{-2} \left(\frac{m}{\pi \hbar}\right)^2 g_1(Z). \quad (43) \]

Integrating the normalization condition (37) using expression (11) with the corresponding potential (3), allows us to obtain an expression for the number of particles in function of the chemical potential \( \mu \), the temperature \( T \), the coupling constant \( U_0 \), and the deformation parameter \( \alpha \), given by

\[ N = V_{char} \left(\frac{m}{g_1} + 1\right) \left[\left(\frac{m}{2\pi \hbar^2}\right)^{3/2} g_1(\mu_{eff})(\kappa T)\right]^{\gamma} \]

\[ - \alpha \left(\frac{m^2}{2\pi \hbar^3}\right) g_1(\mu_{eff})(\kappa T)^{-1/2} - U_0 \left(\frac{m}{2\pi \hbar^2}\right)^3 g_{3/2,1/2,\gamma-3/2}(\mu_{eff})(\kappa T)^{-3/2} \]

\[ + \alpha U_0 \left(\frac{m}{2\pi \hbar^2}\right)^{5/2} \left(\frac{m}{\pi \hbar}\right)(\kappa T)^\gamma \left[G_{3/2,0,\gamma-3/2}(\mu_{eff}) + G_{1,1/2,\gamma-3/2}(\mu_{eff})\right]. \quad (44) \]

Where

\[ G_{\eta,\sigma,\gamma-3/2}(\mu_{eff}) = \sum_{ij=1}^{\infty} \frac{z_{\epsilon_{eff}}^{i+j}}{i^\eta j^\sigma (i+j)^{\gamma-3/2}}. \quad (45) \]
Where, $z_{eff} = e^{\beta(\mu + ma^2/2)}$. If we set $U_0 = 0$ in equation (43), we recover our previous result for a modified ideal bosonic gas in the thermodynamic limit [13]. Additionally, setting $\alpha = 0$ and $U_0 = 0$, we recover the usual result for an ideal bosonic gas in the thermodynamic limit [21].

In order to obtain the leading correction in the shift for the critical temperature caused by the interactions in our deformed bosonic gas, let us expand the expression (44) at first order in $T = T_0$, $\mu = 0$, $U_0 = 0$, and $\alpha = 0$, $T_0$ is the critical temperature in the thermodynamic limit given by expression (22), with the result

$$N = V_{char} \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \zeta(\gamma)(\kappa T_0)^{\gamma - 1}$$

$$+ \left[ T - T_0 \right] \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \zeta(\gamma)(\kappa T_0)^{\gamma - 1}$$

$$- U_0 \left( \frac{m}{2\pi\hbar^2} \right)^3 G_{3/2, 1/2, \gamma - 3/2}(1)(\kappa T_0)^{\gamma + 1/2}$$

$$+ \mu \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \zeta(\gamma - 1)(\kappa T_0)^{\gamma - 1} - \alpha \frac{m^2}{\pi^2\hbar^4} \zeta(\gamma - 1/2)(\kappa T_0)^{\gamma - 1/2} \right].$$

At the critical temperature $T_c$ for large $N$, in the mean field approach $\mu_c = 2U_0 n_0(\vec{r} = \vec{0})$ [2], which means that the critical density at the center of the trap is the same as for the non–interacting model. In the usual case when $\alpha = 0$, $n_0(\vec{r} = \vec{0}) = \lambda^{-3}_c \zeta(3/2)$ in the large $N$ limit [5], but in our case we have to modified the value for $\mu_c$ at the critical temperature according to expression (43), because of the divergent behavior of the Bose-Einstein functions related to $n_0(\vec{r} = \vec{0})$, namely $g_1(Z)$ and $g_{1/2}(Z)$ when $Z = 1$. When the integrals associated with the Bose–Einstein functions converges, the value $ma^2/2$ is negligible and can be replaced by zero. However when the integral associated to the Bose–Einstein functions can diverge at $Z \to 1$ the minimal energy associated with the corresponding system must be taken into account [21].

In order to avoid the divergent behavior of these functions, we have to take into account the minimal energy of this system. We are interested in the corrections due to $\alpha$ in the large $N$ limit, so we will take as a minimum energy of the system $ma^2/2$, ignoring the corrections due to the corresponding ground state energy or finite–size effects, which have already been calculated in equation (22).

Let us define $n_0(\vec{r} = \vec{0})$ at the critical temperature using expression (43) as

$$n_0(\vec{r} = \vec{0}) = \lambda^{-3}_c g_{3/2}(e^{\beta_c ma^2/2}) - 2\alpha U_0 \lambda^{-2}_c \left( \frac{m}{\pi\hbar} \right) g_1(e^{\beta_c ma^2/2}).$$

With the help of expressions (27), (28), and (29) [24], we can define the Bose–Einstein functions $g_{3/2}(e^{\beta_c ma^2/2})$ and $g_1(e^{\beta_c ma^2/2})$, when $(\beta_c ma^2/2) \to 0$ as

$$g_{3/2}(e^{\beta_c ma^2/2}) \simeq \zeta(3/2) + \Gamma(-1/2) \left( \frac{ma^2}{2\kappa T_c} \right)^{1/2}$$

$$g_1(e^{\beta_c ma^2/2}) \simeq \ln \left( \frac{2\kappa T_c}{ma^2} \right).$$

Neglecting the second order terms in $U_0$ and $\alpha$, allows us to write $\mu_c$ using expression (47) as follows

$$\mu_c \simeq 2U_0 \lambda^{-3}_c \zeta(3/2) - 2\alpha U_0 \lambda^{-2}_c \left( \frac{m}{\pi\hbar} \right) \ln \left( \frac{2\kappa T_c}{ma^2} \right).$$

If in equation (50) we take the limit when $\alpha \to 0$ we recover the usual value for $\mu_c$ at the critical temperature [2]. Inserting (50) in (46), we finally obtain that the shift in the critical temperature caused by the interactions in a modified bosonic gas in function of the number of particles is given approximately by
\[
\frac{\Delta T_c}{T_0} \approx -a R_0^{\frac{1}{2}} \left( \frac{m}{2\pi \hbar^2} \right)^{\frac{1}{2}} \left[ \frac{2\zeta(3/2)\zeta(\gamma - 1) - G_{3/2,1/2,3/2}(1)}{\zeta(\gamma)} \right] N^{\frac{1}{12}} \\
+ \alpha (8\pi m)^{1/2} \zeta(\gamma - 1/2) (R_0 N)^{-\frac{1}{12}} \\
+ \alpha a \frac{4m(\pi \hbar)^{-1}\zeta(\gamma - 1)}{\zeta(\gamma)} \ln \left( \frac{(R_0 N)^{\frac{1}{2}}}{m a^2} \right),
\]

where \( R_0 \) is given by
\[
R_0 = \left( \frac{2\pi \hbar^2}{m} \right)^{3/2} \left[ \frac{\zeta(\gamma)^{-1}}{\nu_{\text{char}}} \right].
\]

Setting \( \alpha = 0 \) in equation \((51)\) we obtain the usual shift in the critical temperature caused by weakly interactions.

Let us analyze the expression \((51)\). The first right-hand side term is associated to the usual correction caused by interactions, the second term is a correction caused by the deformation term, and the last one, corresponds a correction caused by interactions and the deformation term. A decreasing or increasing in the temperature depends of the sign of \( a \) and \( \alpha \). For \( a > 0 \) (repulsive interactions) and \( \alpha > 0 \), the correction in the shift caused by the deformation is a positive shift. On the other hand for \( a < 0 \) (attractive interactions) and \( \alpha > 0 \) the shift caused by the deformation term is negative. In the case of \( a > 0 \) and \( \alpha < 0 \) we obtain a negative shift, and for \( a < 0 \) and \( \alpha < 0 \), the second term is negative and the third is positive. An increasing or decreasing corrections, depends of the sign of \( \alpha \) and \( a \) and its combinations. The second term in expression \((51)\) proportional to \( N^{-\frac{2}{12}} \), becomes smaller when the number of particles is large but, the third term proportional to the scattering length becomes bigger.

In the case of an anisotropic three-dimensional harmonic oscillator potential \( \gamma = 3 \), with \( A_i = \hbar \omega_i/2, a_i = \sqrt{\hbar/m\omega_i} \), then from expression \((51)\) we obtain
\[
\frac{\Delta T_c}{T_0} \approx \left( \frac{a}{a_{ho}} \right) \left[ \frac{2\zeta(3/2)\zeta(2) - G_{3/2,1/2,3/2}(1)}{(2\pi)^{1/2}3\zeta(3)^{5/6}} \right] N^{1/6} \\
+ \alpha \left( \frac{2^{3/2}\pi^{-1/2}\zeta(5/2)}{3\zeta(3)^{5/6}} \right) \left( \frac{1}{(\hbar \omega)^3} \right)^{1/6} m^{1/2} N^{-1/6} \\
+ \alpha a \left( \frac{4m(\pi \hbar)^{-1}\zeta(2)}{3\zeta(3)} \right) \ln \left( \frac{2(\hbar \omega)N^{1/3}}{\zeta(3)ma^2} \right).
\]

Where we used the usual definitions
\[
\bar{\omega} = (\omega_1\omega_2\omega_3)^{1/3}, \quad a_{ho} = (\frac{\hbar}{m\omega})^{1/2}.
\]

By analyzing expression \((53)\), we obtain for \(^{87}\)Rb, with say \( N = 10^{18}, \ N = 10^9, \) and \( 10^6 \) particles, the corrections caused by the deformation term are given respectively by \( \xi_1 \left( \frac{1}{\omega_{\text{char}}} 4.5 \times 10^{-8} + 9.12 \times 10^{-9} \ln(3.7 \times 10^{14} \bar{\omega}) \right), \)
\[\xi_1 \left( \frac{1}{\omega_{\text{char}}} 1.42 \times 10^{-7} + 9.12 \times 10^{-9} \ln(3.7 \times 10^{11} \bar{\omega}) \right), \]
\[\xi_1 \left( \frac{1}{\omega_{\text{char}}} 4.5 \times 10^{-7} + 9.12 \times 10^{-9} \ln(3.7 \times 10^{10} \bar{\omega}) \right), \]
with the scattering length \( a = 5.77 \times 10^{-9} \) m and \( \bar{\omega} = (\omega_1\omega_2\omega_3)^{1/3} \). On the other hand, with say, \( \bar{\omega} \approx 2\pi \times 300 \) Hz, for \(^{87}\)Rb, the third term proportional to the product \( aa \) becomes bigger than the first one proportional to \( \alpha \) in equation \((53)\), when the number of particles \( N \) is approximately \( N \geq 10^7 \), which means a change of sign when \( a < 0 \), assuming \( \xi_1 \approx -1 \).

If we set \( \alpha = 0 \) in expression \((53)\) we recover the result given in \((14)\).

V. CONCLUSIONS

Using the formalism of the semiclassical approximation, we have analyzed the Bose–Einstein condensation for a modified bosonic gas trapped in a 3–D power law potential in three regimes, namely in the thermodynamic limit, in
systems with a finite number of particles, and in a weakly interacting systems in the large \( N \) limit. We deduced the density of states for the aforementioned system in the ideal case and we have deduced the parameter that characterizes the shape of the potential, which are directly related to the existence of the condensate in the thermodynamic limit. We also deduced the shift in the critical temperature in the thermodynamic limit, in finite systems, and for a weakly interacting bosonic gases in function of the number of particles, equations (24), (32), and (51), which are valid for any potential defined by the generic 3-dimensional power-law potential (3) within the semiclassical approximation. Nevertheless, the case of an ideal Bose–Einstein gas in a box in three dimensions, deserves a separately analysis.

In the case of an uniform Bose–Einstein gas in a box \( s_i \to \infty \), and hence \( \gamma = 3/2 \), then, according to equation (18) the number of particles for this system is given by

\[
N = N_0 + \left( \kappa T \right)^{3/2} \frac{1}{\Omega_1} g_{3/2} - \alpha \left( \kappa T \right) \frac{1}{\Omega_2} g_1(z) + \alpha^2 \left( \kappa T \right)^{3/2} \frac{1}{\Omega_3} g_{1/2}(z) \tag{55}
\]

Apparently Bose–Einstein condensation is not possible for this kind of systems in three dimensions in the thermodynamic limit, because of the divergent behavior of Bose–Einstein functions \( g_1(z) \) and \( g_{1/2}(z) \), when \( \mu = 0 \) at the transition temperature [11]. In the usual case, when \( \alpha = 0 \), we have only the first term in expression (55), in this situation the condensation is possible in three dimensions for a bosonic gas in a box, and assuming that the minimal energy for this system is zero, we can fix the value of \( \mu = 0 \) safely without problems of divergence and \( g_{3/2}(1) \) is just \( \zeta(3/2) \). Nevertheless, the minimal value of the energy for a modified bosonic gas in a box is not zero, in fact, this value is \( ma^2/2 \).

These assumptions allows us to write expression (55) at the critical temperature as follows

\[
N = \left( \kappa T_c \right)^{3/2} \frac{1}{\Omega_1} g_{3/2}(e^{\beta_0 ma^2/2}) - \alpha \left( \kappa T_c \right) \frac{1}{\Omega_2} g_1(e^{\beta_0 ma^2/2}) + \alpha^2 \left( \kappa T_c \right)^{3/2} \frac{1}{\Omega_3} g_{1/2}(e^{\beta_0 ma^2/2}) \tag{56}
\]

With the help of expressions (18), we may now express equation (56) as follows

\[
N = \left( \kappa T_c \right)^{3/2} \frac{1}{\Omega_1} \zeta(3/2) + \alpha \left( \kappa T_c \right) \left[ \frac{\Omega_1 \Gamma(-1/2)m^{1/2}}{2^{1/2}} - \frac{\Omega_2 \ln \left( \frac{2\kappa T_c}{m\alpha^2} \right)}{m^{1/2}} + \frac{2^{1/2} \Omega_3 \Gamma(1/2)}{m^{1/2}} \right]. \tag{57}
\]

If we set \( \alpha = 0 \) in expression (57), we obtain the usual result for a bosonic gas in a box [24]. Using the definition of the critical temperature for a bosonic gas in a box in three dimensions \( T_0 = \frac{\hbar^2}{\pi m} \left( \frac{N}{V \zeta(3/2)} \right)^{2/3} \) in the thermodynamic limit [24, 26] and with the expression (57), we obtain the shift in the critical temperature caused by the deformation term in function of the number of particles

\[
\frac{\Delta T_c}{T_0} \approx \alpha \left[ \frac{2^{5/2} m V^{1/3} \zeta(3/2)}{\pi \hbar} \ln \left( \frac{4\pi \hbar^2}{(m\alpha)^2 V^{2/3} \zeta(3/2)^2} N^{2/3} \right) \right] + \frac{m V^{1/3} \zeta(3/2)^{-2/3}}{(2\pi)^{1/2}} \left( \frac{4}{3(2\pi)^{1/2} - 2(2\pi)^{1/2}/9} \right) N^{-1/3}. \tag{58}
\]

Where \( V \) is the volume of the container. Expression (58) shows that the condensation for a modified bosonic gas in a box is possible, taking into account the minimal energy of the system in question. Nevertheless, this system is unstable, because the fluctuations in the number of particles becomes anomalous [11].

In the case of the different values for the shape parameter, we notice that for spherical traps \( d = 1, n_1 = 3 \), and setting, say \( s_1 = 2 \) then, \( \gamma = 3 \), hence from equation (24) in the thermodynamic limit we obtain

\[
\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/6}, \tag{59}
\]

On the other hand for cylindrical traps \( d = 2, n_1 = 2, \) and \( n_2 = 1 \), setting, say \( s_1 = 2, \) and \( s_2 = 1 \), the trap parameters are given respectively by \( \gamma = 9/2 \). From equation (24) in the thermodynamic limit we obtain

\[
\frac{\Delta T_c}{T_0} \sim \alpha N^{-1/9}, \tag{60}
\]
These two simple cases, illustrate how the tiny contributions caused by the quantum–structure of the space–time, could be enhanced, varying the number of particles.

A very important consequence, is that for values between $3/2 < \gamma \leq 5/2$ we have condensation, but the system is unstable [11]. In this range of values of the shape parameter $\gamma$ we have, for example, potentials of the type $V(\vec{r}) \sim x^3 + y^3 + z^3$ in the case of a Cartesian potential, $V(\vec{r}) \sim \rho^3 + z^3$ in the case of cylindrical traps, and $V(\vec{r}) \sim r_3$ in the case of spherical traps. We have noticed also, that for potentials of the type $V(\vec{r}) \sim \rho^4 + z^2$, $V(\vec{r}) \sim r^6$ or any other combination such that $3/2 < \gamma \leq 5/2$ we have condensation, but the system is apparently unstable (assuming that the zero point energy is negligible), in the thermodynamic limit, because the fluctuations in the number of particles becomes anomalous [11]. When the fluctuations in the number of particles becomes anomalous, in the thermodynamic limit, could leads to an immediate collapse or implosion of the condensate [21]. This last assertion, allows us, in principle, obtain a criterium to discriminate when the quantum–gravity manifestations are more feasible to be measured, by analyzing the stability for these systems [11]. On the other hand, we prove that, in the case of stable systems (for example, the harmonic oscillator), the tiny contribution due to the quantum–structure of the space–time is enhanced by the number of particles $N$.

The present work shows that the effects caused for the quantum structure of space–time could be, in principle, amplified taking into account corrections caused by a finite number of particles in an ideal bosonic gas, or a large number of particles in an interacting bosonic gas, together with an adequate election of the parameters in the corresponding potential. The shift in the critical temperature of a condensate could provide bounds for the parameters associated to the deformation suggested in several quantum gravity models.

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