Abstract We explain how Regge theory and perturbative evolution may be made compatible at small $x$. The result not only gives striking support to the two-pomeron description of small-$x$ behaviour, but gives a rather clean test of perturbative QCD itself. When $x$ is very small, the proton’s gluon distribution function is significantly larger than is commonly believed. Perturbative evolution is invalid below $Q^2 \approx 5 \text{ GeV}^2$.

1 Introduction

There is a growing realisation\cite{1}\cite{2}\cite{3} that the conventional approach to perturbative evolution breaks down at small $x$. Certainly, when, at a given fixed small $x$, the structure function varies rapidly with $Q^2$, it is not sufficient to expand the DGLAP splitting function in powers of $\alpha_s$ and truncate the expansion after one or two terms. Rather, a resummation is needed and, while there are models of how to do this, so far no reliable method is available.

The Regge approach is apparently orthogonal to that of perturbative evolution. We have shown\cite{4} that Regge theory gives a very good description of the data, not only for the proton structure function $F_2(x, Q^2)$, but also for its charm component $F_2^c(x, Q^2)$ and for $J/\psi$ photoproduction. The data indicate rather clearly the need for a second pomeron, whose trajectory has intercept $(1 + \epsilon_0) \approx 1.4$, in addition to the familiar soft pomeron with intercept $(1 + \epsilon_1)$ about 1.08. We call the second pomeron the hard pomeron. While at one time it was hoped that one might calculate its intercept from the BFKL equation, it now seems likely\cite{5} that such a calculation is not within the scope of perturbative QCD. As we explain in this paper, perturbative QCD merely governs how the magnitude of the hard pomeron’s contribution to the structure function increases with $Q^2$.

Our belief is\cite{6}\cite{4} that the hard pomeron is already present in real-photon amplitudes and contributes with the same fixed power $(W^2)^{\epsilon_0}$ at all values of $Q^2$. That is, at sufficiently large $W$

$$F_2(x, Q^2) \sim f_0(Q^2)x^{-\epsilon_0}$$

for all values of $Q^2$ and, again at sufficiently large $W$,

$$\sigma^{\gamma p} \sim 4\pi^2\alpha_{\text{EM}}X_0(W^2)^{\epsilon_0}$$

with

$$X_0 = (Q^2)^{-1-\epsilon_0}f_0(Q^2)\bigg|_{Q^2=0}$$

There is no theoretical reason why the behaviour should be a power rather than something more complicated, but we have found\cite{4} that a power fits the data very well. If $X_0$ is to be finite, $f_0(Q^2)$ must vanish as $(Q^2)^{1+\epsilon_0}$ at $Q^2 = 0$. A very good fit to data, all the way from $Q^2 = 0$ to 5000 GeV$^2$, is provided by the economical parametrisation

$$f_0(Q^2) = X_0 \left(Q^2\right)^{1+\epsilon_0}/\left(1 + Q^2/Q_0^2\right)^{1+\epsilon_0}$$

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with $Q_0 \approx 3$ GeV. In this paper, we show that this form agrees remarkably well with what is obtained from DGLAP evolution, over a large range of $Q^2$. At present, it is not possible properly to apply perturbative evolution to the soft-pomeron component of $F_2(x, Q^2)$.

2 The DGLAP equation

The singlet DGLAP equation is

$$\frac{\partial}{\partial t} u(x, Q^2) = \int_1^x dz \ P(z, \alpha_s(Q^2)) u\left(\frac{x}{z}, Q^2\right)$$

where $P$ is the splitting matrix, $t = \log(\frac{Q^2}{\Lambda^2})$ and

$$u = \left(x \sum f(q_f + \bar{q}_f)\right)$$

Write the Mellin transforms

$$u(N, Q^2) = \int_0^1 dx \ x^{N-1} u(x, Q^2)$$

$$P(N, \alpha_s(Q^2)) = \int_0^1 dz \ z^N P(z, \alpha_s(Q^2))$$

Then

$$\frac{\partial}{\partial t} u(N, Q^2) = P(N, \alpha_s(Q^2)) u(N, Q^2)$$

A power contribution (1) to $F_2(x, Q^2)$ corresponds to a pole

$$\frac{f(Q^2)}{N - \epsilon_0} \ f(Q^2) = \left(\frac{f_q(Q^2)}{f_g(Q^2)}\right)$$

in $u(N, Q^2)$. More generally, consider a contribution $x^{-\epsilon_0} f(x, Q^2)$. We assume that $f(x, Q^2)$ vanishes at $x = 1$ and is differentiable, for all $Q^2$. Insert in the Mellin transform integral and integrate once by parts, to get

$$-\frac{1}{N - \epsilon_0} \int_0^1 dx \ x^{N-\epsilon_0} f_x(x, Q^2)$$

So there is a pole at $N = \epsilon_0$ with residue

$$-\int_0^1 dx \ f_x(x, Q^2) = f(0, Q^2)$$

With 4 active quark flavours and a flavour-blind hard pomeron, $f_q(Q^2) = \frac{18}{5} f_0(Q^2)$. We find$[^{6}]$, on taking the residue of the pole at $N = \epsilon_0$ on each side of the Mellin transform (8) of the DGLAP equation,

$$\frac{\partial}{\partial t} f(Q^2) = P(N = \epsilon_0, \alpha_s(Q^2)) f(Q^2)$$

We have previously$[^{4}]$ fitted accurate ZEUS and H1 data in the range $x < 0.001$, $0.045 \leq Q^2 \leq 35$ GeV$^2$, together with data for $\sigma^{\gamma p}$. The result is

$$\epsilon_0 = 0.437$$
The error on each of these quantities is large, but we have given their values to this accuracy because their errors are strongly correlated.

If we include 4 flavours of quark and antiquark in the sum in (6), then at $Q^2 = 20$ GeV$^2$ the singlet quark distribution $x \sum_f (q_f + \bar{q}_f) \sim 0.095x^{-\epsilon_0}$ at sufficiently small $x$. We found also that the charmed-quark component $F^c_2$ of $F_2$ is apparently governed almost entirely by hard-pomeron exchange at small $x$, even at small values of $Q^2$, and that, within the experimental errors, its magnitude is consistent with the hard pomeron being flavour-blind. The hard-pomeron description of $F^c_2$ is shown in figure 1. According to perturbative QCD, the charmed quark originates from a gluon in the proton and the two distributions are proportional to each other to a good approximation over a wide range of $x$ and $Q^2$[9]. So this implies that the gluon distribution also is hard-pomeron dominated. The conventional approach to QCD evolution is correct if $x$ is not too small, because it does not then probe the splitting

\[ Q_0^2 = 9.11 \text{ GeV}^2 \quad X_0 = 0.00146 \]
matrix at $z = 0$, which is where its expansion in powers of $\alpha_s$ is illegal. We assume that at $x = 0.01$ and $Q^2 = 20$ GeV$^2$ the value of $g(x, Q^2)$ extracted by the HERA experiments is reasonably close to the correct value. At $Q^2 = 20$ GeV$^2$ and $x = 0.01$, an NLO fit\cite{10} to the combined ZEUS and H1 data gives $xg(x, Q^2) = 5.7 \pm 0.7$. Other authors\cite{11}\cite{12}\cite{13} find much the same value. This is $8 \pm 1$ times hard-pomeron component of the singlet quark distribution.

An unresummed perturbation expansion of the splitting matrix $P(N, \alpha_s)$ is not valid\cite{6} for small values of $N$ because it introduces a spurious singularity at $N = 0$. According to (12) and (13), we need $P(N, \alpha_s)$ at a value of $N$ far from 0, and so it is reasonable to hope that resummation is not needed. The numerical values of the elements of the matrix $P(N, \alpha_s)$ in one and two-loop order are plotted in figure 4.9 of the book by Ellis, Stirling and Webber\cite{7} for the value $\alpha_s = 2\pi/30$. From these we may evaluate the running-coupling splitting matrix $P(N, \alpha_s(Q^2))$ in one and two-loop order\footnote{It is not made clear in the book that the gg plot includes the necessary factor $2n_f$.}. We use the one and two-loop forms for the running coupling\cite{7}:

\begin{align}
\alpha_s^{LO}(Q^2) &= \frac{1}{bt} \\
\alpha_s^{NLO}(Q^2) &= \frac{1}{bt} \left[ 1 - \frac{b' \log t}{bt} \right]
\end{align}

(15a)

where

\begin{equation}
b = \frac{33 - 2n_f}{12\pi} \quad b' = \frac{153 - 19n_f}{2\pi(33 - 2n_f)}
\end{equation}

(15b)

In each case, we choose $\Lambda$ such that $\alpha_s(M_Z^2) = 0.116$. This gives

$\Lambda^{LO} = 140$ MeV \quad $\Lambda^{NLO} = 400$ MeV

(16)

We use four flavours throughout as, at the energies we are considering, the charm contribution is active and the beauty contribution is so small that its omission has a negligible effect.

We integrate the differential equation (12). The result for the singlet quark distribution is shown in figure 2a, where the solid curve is the result of the two-loop-order perturbative QCD evolution according to (12), and the broken curve is the fit to the data given by (4). We have taken the ratio of the gluon distribution to the hard-pomeron component of the singlet quark distribution to be 8.0 at $Q^2 = 20$ GeV$^2$. Note that the gluon/quark ratio is a parameter which in principle we could change. However it turns out that the value of 8 we have obtained from the NLO fit to the combined ZEUS and H1 data works well. Figure 2b shows how the gluon distribution

$xg(x, Q^2) = f_g(Q^2)x^{-\epsilon_0}

(17)

evolves. Figure 3 shows that there is very little difference between one-loop-order and two-loop-order evolution, except at small $Q^2$. This is because we have used the splitting function for a value of $N$ safely away from $N = 0$ and it encourages the hope that resummation, if we knew how to perform it, would make little difference to these results.

3 Discussion

We have a number of comments on these results:

1 It is evident from figure 2a that two-loop perturbative QCD describes the evolution of the strength of the hard-pomeron contribution to $F_2(x, Q^2)$ extremely well for $Q^2$ greater than about 5 GeV$^2$, up to values of $Q^2$ beyond where data exist. The perturbative QCD differential equation is a large-$Q^2$
equation and is not valid when $Q^2$ is small; it even makes the hard-pomeron contribution to $F_2$ become negative below 2.5 GeV$^2$.

2 Our fit to the data, from which we extracted the hard-pomeron contribution, used data from $Q^2 = 0$ to 35 GeV$^2$. Over most of this range, the hard-pomeron contribution is only a small part of $F_2(x, Q^2)$, though its relative magnitude has increased by $Q^2 = 35$ GeV$^2$: see figure 4. So, although it is well-established$^{[11,12]}$ that the complete $F_2(x, Q^2)$ obeys perturbative QCD evolution beyond $Q^2 = 5$ GeV$^2$ for the range of $x$ where data exist, it is not at all trivial that the hard-pomeron part of it does. This successful link with perturbative QCD provides rather striking verification of the hard-pomeron concept and of our having extracted it correctly from the data.

3 It also provides a clean test of perturbative QCD itself, one that genuinely tests evolution rather than being a global fit$^{[11,12]}$. It depends on just one parameter, the ratio of the gluon distribution to the hard-pomeron part of the singlet quark distribution at some value of $Q^2$, which we chose to be 20 GeV$^2$. The curve in figure 2a uses the value 8.0 for this ratio but, as we have said, analysis of the HERA data gives an error $\pm 1$ on this number. Changing it within this range has a fairly small effect
Figure 4: The thick curves are fits to $F_2(x,Q^2)$ at $Q^2 = 5$ and 35 GeV$^2$, with the H1 data used at those values of $Q^2$. The thin curves are the hard-pomeron contributions.

at large $Q^2$, about 10% at 5000 GeV$^2$, but causes quite a large change in the details of how it breaks away from the phenomenological curve when we evolve back to small $Q^2$. Similar remarks apply to changing $\Lambda$ within its allowed range of values.

4 Our phenomenological fit behaves at large $Q^2$ as the power $Q^{\epsilon_0}$, while the solution to the perturbative QCD differential equation (12) rather behaves asymptotically as a power of $\log Q^2$, approximately $(\log Q^2)^2$; nevertheless the two agree well over a very large range of $Q^2$. According to figure 2a, they begin to diverge from each other only at values of $Q^2$ beyond the data, so in our fit we could instead use the perturbative QCD form. However, it is very much more complicated to parametrise and, more importantly, using the perturbative QCD form at small $Q^2$ is incorrect. Using different forms for different $Q^2$ ranges would lead to matching problems and it would be almost impossible to achieve the required analyticity in $Q^2$ at the join.

5 The conventional approach to evolution expands the splitting matrix $P(N, \alpha_s)$ in powers of $\alpha_s$. Successive terms in the expansion are increasingly singular at $N = 0$. This is a signal that the expansion is illegal$^6$ for small values of $N$, since the complete splitting matrix is regular at $N = 0$. Because an unresummed expansion that needs the splitting matrix at small $N$ makes the splitting function larger than it really is, a gluon distribution of a given magnitude apparently gives stronger evolution than it really should. That is, the conventional approach will tend to under-estimate the magnitude of $xg(x,Q^2)$ in certain regions of $(x,Q^2)$ space. This is verified by our results for the evolution of $xg(x,Q^2)$: figure 5 shows the proton’s gluon structure function at two values of $Q^2$, according to our calculations, which do not use the splitting matrix at small $N$, and compares it with what is extracted from the data by conventional means.

6 We cannot use a similar approach to the evolution of the contribution from the soft pomeron, because this needs $P(N, \alpha_s)$ at $N \approx 0.08$, dangerously close to $N = 0$. Handling this will need resummation$^{14\,3\,5}$, so as to tame the singularity at $N = 0$, but we do not yet have a reliable way to perform this resummation. As we have explained, experiment finds that the charm-quark distribution does not contain a soft-pomeron term; we are assuming that our inability to handle the soft pomeron is therefore not relevant for the gluon distribution.

7 The fact that we can fit the data so well with the only singularities of $u(N, Q^2)$ in $N \geq 0$ identified with the hard and soft pomeron contributions, supports our belief$^6$ that $P(N, \alpha_s)$ does not have any singularities in this region. This again contrasts with the conventional attitude: we maintain that the relevant
singularities of $u(N,Q^2)$ in the complex $N$-plane are not generated by the perturbative evolution, which merely governs how the strength of their contribution increases with $Q^2$. They are already present at $Q^2 = 0$ and their position does not vary with $Q^2$. We have explained before\cite{6} that there are strong theoretical reasons to believe this: one cannot simply start perturbative QCD evolution at some value of $Q^2$ and ignore how this joins on to what is found for smaller values.

8 We stress again how very few parameters we need to fit the data for $F_2(x,Q^2)$ for $x < 0.001$. The hard-pomeron contribution is parametrised in terms of the three parameters $X_0, \epsilon_0$ and $Q_0$ introduced in (4), and for the soft pomeron we have similarly\cite{4} $X_1, \epsilon_1$ and $Q_1$:

$$f_1(Q^2) = X_1 \left( Q_1^2 \right)^{1+\epsilon_1} / \left( 1 + Q^2 / Q_1^2 \right)^{1+\epsilon_1}$$

Of these, $\epsilon_1 \approx 0.08$ is determined from soft hadronic interactions, and $X_1$ from the fit to $\sigma^{pp}$. The latter requires\cite{4} also a contribution from $f_2$ and $a_2$ exchange, but this does not contribute very much to $F_2(x,Q^2)$ for $x < 0.001$.

9 We have remarked that, although the data for $F_2(x,Q^2)$ are now highly accurate, the value of $\epsilon_0$ is still quite poorly determined. It depends\cite{4} on what is assumed for the large-$Q^2$ behaviour of the soft-pomeron coefficient function $f_1(Q^2)$. In this paper we have used $\epsilon_0 = 0.437$, which corresponds to assuming that $f_1(Q^2)$ has the form (18) and so goes to a constant at large $Q^2$. But one may obtain an equally good fit to the data for $F_2(x,Q^2)$ by assuming that $f_1(Q^2)$ vanishes like $1/Q$ at large $Q^2$. Figure 6a shows the equivalent of figure 2a for this case, with $\epsilon_0 = 0.394$, and figure 6b is the equivalent of figure 5a, from which it is seen that the gluon structure function needs to be slightly larger at $Q^2 = 20$ GeV$^2$ than in the previous case. By $Q^2 = 2000$ GeV$^2$ it is quite a lot larger: see figure 7. The lower curve in this figure is the CTEQ4M prediction, based on conventional unresummed NLO evolution\cite{12}.

10 The LO or NLO perturbative QCD evolution makes the coefficient $f_0(Q^2)$ of the leading power of $1/x$ increase indefinitely with increasing $Q^2$. In order not to conflict with the momentum sum rule, at any fixed $x$ nonleading powers of $x$ must become progressively more important, so that the largest value of $x$ for which the leading power alone gives a good approximation to $F_2(x,Q^2)$ decreases as
Figure 6: Equivalents of (a) figure 2a and (b) figure 5a for the case \( \epsilon_0 = 0.394 \)

Figure 7: Gluon distribution at \( Q^2 = 2000 \text{ GeV}^2 \). The upper line is the perturbative QCD-evolved distribution for \( \epsilon_0 = 0.394 \), the middle line for \( \epsilon_0 = 0.437 \), and the lower line is the MRST20011 two-loop prediction\cite{11,13}

\( Q^2 \) increases\textsuperscript{†}. This is why, while our plots of the gluon distribution at \( Q^2 = 20 \text{ GeV}^2 \) extend up to \( x = 0.01 \), for \( 200 \text{ GeV}^2 \) we stop at \( x=0.001 \), and for \( 2000 \text{ GeV}^2 \) at 0.0001. We guess that these values for the limits of the validity of the single-term approximation are safe.

To summarise, what we have achieved is a genuine evolution, in contrast to global fits\cite{11,12} which use data at large \( Q^2 \) to help constrain the fit already at small \( Q^2 \). We used data up to \( Q^2 = 35 \text{ GeV}^2 \) to determine the hard-pomeron component of \( F_2(x,Q^2) \); in this fit we did not assume perturbative QCD evolution, but just made a numerical fit to the data\cite{4}. Now we find that, for the hard-pomeron contribution, we have almost perfect agreement with perturbative QCD evolution up to \( Q^2 = 5000 \text{ GeV}^2 \). This is a striking verification that the hard-pomeron concept is correct, as well as being a success for perturbative QCD itself. It leads us to conclude that, at very small \( x \), the gluon structure function is somewhat larger than has until now been believed.

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