Generic Rotation in a Collective $SD$ Nucleon-Pair Subspace

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Low-lying collective states involving many nucleons interacting by a random ensemble of two-body interactions (TBRE) are investigated in a collective $SD$-pair subspace, with the collective pairs defined dynamically from the two-nucleon system. It is found that in this truncated pair subspace collective vibrations arise naturally for a general TBRE hamiltonian whereas collective rotations do not. A hamiltonian restricted to include only a few randomly generated separable terms is able to produce collective rotational behavior, as long as it includes a reasonably strong quadrupole-quadrupole component. Similar results arise in the full shell model space. These results suggest that the structure of the hamiltonian is key to producing generic collective rotation.

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Recently Johnson and collaborators studied the low-lying spectra of many-body systems in the presence of random interactions. Surprisingly, they found that patterns of more orderly interactions persist, such as a predominance of total angular momentum $I^z = 0^+$ ground states with large gaps relative to excited states [1-3]. The subject of many-body systems interacting by a two-body random ensemble (TBRE) has been attracting much interest since this discovery. Many authors have tried to understand the regularities exhibited by a many-body system interacting randomly and to uncover other robust properties of many-body systems [4-22].

In Ref. [5] it was shown that both vibrational and rotational features arise in the interacting boson model (the IBM) [23] in the presence of random interactions. In contrast, as shown in many works, rotational behavior does not generically arise in fermion systems interacting via random interactions. It was suggested, therefore, that a special feature of the ensemble might be necessary to obtain a generic rotational behavior in fermion systems [3].

The remarkable rotational peak obtained in Ref. [5] is suggestive that a key to obtaining a rotational peak from a random hamiltonian might be to restrict the space to a collective subspace built up from the lowest $S$ and $D$ pairs, since these are the objects that are represented in the IBM by its $s$ and $d$ bosons [23]. To see whether this is true, we report here calculations in which random interactions are used in a truncated version of the shell model in which only collective $S$ and $D$ pairs are retained. We limit ourselves, however, to systems of identical nucleons.

Another possibility is that generic rotation requires the hamiltonian to have some specific features and is not just a consequence of the space. A similar philosophy was recently discussed by Velázquez and Zuker [18], who showed that a displaced random hamiltonian which is attractive on average usually leads to rotational motion. Here we consider an alternative way of modifying the hamiltonian – as a limited sum of separable interactions with random strengths – in order to study under which conditions it is possible to generate rotational features in the calculated spectra.
This can be readily implemented within the context of a truncated $SD$-pair space. A natural way to carry out such calculations is through the use of the nucleon pair approximation (NPA) [24,25]. The NPA is very similar in spirit to the generalized seniority scheme [26] and the broken pair approximation [27], which were used extensively in efforts to establish a microscopic foundation for the IBM [28]. The NPA, however, has several computational advantages over these earlier formulations, and is therefore the method we will employ in this work.

The NPA calculations we report here were carried out using the following strategy. First, we selected only those sets of random interactions/parameters for which the hamiltonian produced an $I^\pi = 0^+$ ground state and an $I^\pi = 2^+$ first excited state, when diagonalized for a system of two identical nucleons, and then assumed that the corresponding wave functions represent our collective $S$ and $D$ pairs, respectively. The $S$ and $D$ pairs obtained in such a way, though not fully self-consistent, should be good enough to permit a meaningful study of the regularities of the $SD$-pair approximation in the presence of random interactions. For each such case, the same two-body interaction was then used to calculate the spectrum for six identical nucleons, with the procedure iterated until there are acceptable statistics. In the analysis, we focused only on those cases for which the six particle system had a $0^+$ ground state, calculating for them the ratio $R = (E_{4i}^+ - E_{0i}^+)/(E_{2i}^+ - E_{0i}^+)$. For vibrational systems, $R = 2$, and for rotational systems $R = 10/3$. Rotational motion will be said to be generic for an $SD$-pair system if a clear peak with $R$ around $10/3 = 3.33$ appears.

It is useful at this point to briefly review the NPA formalism that has been used in these calculations. It begins with the introduction of an operator $A_{\mu}^{(r)\dagger}$ which creates a collective pair of angular momentum $r$ and $z$-projection $\mu$ and which is defined by

$$A_{\mu}^{(r)\dagger} = \sum_{ab} y(abr) \left( C_a^\dagger \times C_b^\dagger \right)_\mu^{(r)}. \quad (1)$$

Here $y(abr)$ are structure coefficients of the collective pair, and $r=0$ and 2 correspond to the $S$ or $D$ pairs, respectively. The $C_a^\dagger$ and $C_b^\dagger$ are single-particle creation operators with $a, b$ denoting the respective single-particle orbit, including its $j$ value. These pairs are coupled
stepwise to yield an $N$-pair basis $|\tau J_N\rangle = A^{(J_N)\dagger}(r_1, J_i)|0\rangle$ with

$$A^{(J_N)\dagger}(r_1r_2...r_N, J_1J_2...J_N) = (\ldots (A^{(r_1)\dagger} \times A^{(r_2)\dagger}(J_2) \times \ldots \times A^{(r_N)\dagger})^{(J_N)}) ,$$

where $J_1 = r_1$, and $J_N$ is the total angular momentum of the above $N$-pair operator.

The restricted separable hamiltonians that we consider in this work can be written in the form

$$H = H_0 + H_P + V_{ph} ,$$

where $H_0$, $H_P$ and $V_{ph}$ are the spherical single-particle energy term, generalized pairing, and particle-hole type interactions, respectively. The generalized pairing interaction, $H_P$, is defined as

$$H_P = V_0 + V_2 + \ldots ,$$

where $V_0$ is the monopole pairing interaction, defined as

$$V_0 = G_0 \mathcal{P}^{(0)\dagger}\mathcal{P}^{(0)} , \quad \mathcal{P}^{(0)\dagger} = \sum_a \frac{\hat{j}}{2} (C_a^\dagger \times C_a)^{(0)} ,$$

with $\hat{j} = (2j + 1)\frac{\hat{\tau}}{\hat{\tau}}$, and $V_2$ is the quadrupole pairing force, defined as

$$V_2 = G_2 \mathcal{P}^{(2)\dagger} \cdot \mathcal{P}^{(2)} , \quad \mathcal{P}^{(2)\dagger} = \sum_{ab} q(ab) \left( C_a^\dagger \times C_b^\dagger \right)^{(2)} .$$

The quantity $q(ab)$ appearing in the quadrupole pairing force is precisely the same quantity that appears in the quadrupole operator, namely

$$Q_M = \sum_{ab} q(ab) \left( C_a^\dagger \times \tilde{C}_b^\dagger \right)^{(2)}_M = \sum_{ambm'} \langle am|Q_M^{(2)}|bm'\rangle C_a^\dagger C_{bm'} ,$$

where $Q_M = r^2 Y_{2M}$, and

$$q(ab) = \frac{(-)^{j + 1/2} \tilde{j} \tilde{j}'}{\sqrt{20\pi}} C^{20}_{j_1/2,j'_1/2}\mathcal{R} .$$

$C^{20}_{j_1/2,j'_1/2}$ is a Clebsch-Gordan coefficient, and $\mathcal{R} = \langle nl|r^2|nl'\rangle$ [29]. The particle-hole interaction takes the form

$$V_{ph} = \kappa Q \cdot Q + \ldots .$$

The quadrupole-quadrupole piece is expressed in terms of the same quadrupole operator given above [30].
A general nuclear hamiltonian for one kind of particle can be written as

\[ H = H_0 - \frac{1}{4} \sum_{abcd,J} \sqrt{(1 + \delta_{ac})(1 + \delta_{bd})} \hat{J} \langle ab, J | V | cd, J \rangle \times \left[ (C_a^+ C_b^i)^{(J)} \times (C_c^i C_d^a)^{(J)} \right]^{(0)} . \]  

(9)

The various separable interactions can be readily transformed to the general form of a two-body interaction given in (9). The procedure for calculating matrix elements of one- and two-body interactions in the NPA is to first rewrite the matrix elements in terms of \( N \)-pair overlaps \( \langle \tau' J_N | \tau J_N \rangle \), and then to calculate these \( N \)-pair overlaps in terms of \( N - 1 \)-pair overlaps by using the Wick theorem for coupled clusters developed in [30]. Thus, the hamiltonian matrix elements are calculated in a recursive way in the NPA.

We now return to the problem motivating this study – to search for generic rotation in a collective \( SD \) nucleon-pair subspace. First we take a general two-body hamiltonian. The two-body matrix elements we use are defined as in Eq. (9) and the single-particle energies, \( H_0 \), are set to be zero.

In Fig. 1 we plot the distribution of \( R \) values for six identical nucleons in the \( sd \), \( pf \) and \( sdg \) shells, respectively. We first note that the distribution of \( R \) values in the \( sd \) shell within an \( SD \) subspace is similar to that obtained in the full shell model space [3] – a broad distribution extending to \( R \sim 1.3 \). When one goes to larger shells, the distributions become sharper, and shift to the right from the \( sd \) shell \( (R \sim 1.3) \) to the \( sdg \) shell \( (R \sim 1.91) \). Nevertheless, no sharp peak at \( R \sim 3.33 \) appears, even though the distribution does extend to that region. From this we conclude that generic rotational collectivity in the shell model does not seem to emerge from pair truncation of the space alone. Furthermore, statistically the full shell model space and the \( SD \) truncated subspace defined here give essentially the same results for a general two-body interaction.

Since rotations do not seem to arise in a collective \( SD \)-pair subspace if a TBRE hamiltonian is assumed, we turn now to the second possibility, that generic rotation might appear if we use a more-restrictive hamiltonian. As suggested earlier, we will consider the possibility of using pairing plus particle-hole type interactions, with their strength parameters
In Figure 2, we show results for six identical nucleons in the $sd$ shell based on a sum of three terms, monopole pairing, quadrupole pairing and quadrupole-quadrupole. All are defined according to (3-8). When all three interaction strengths are treated on the same footing – except that $G_2$ and $\kappa$ are in units of $MeV/fm^4$, whereas $G_0$ is in units of $MeV$ – we arrive at the distribution of $R$ values shown in Fig. 2a. In this case, no sharp rotational peak is observed. Instead, a peak appears around $R \sim 1.3$, with a long tail extending to $R \sim 3.1$. If we artificially enhance the $Q \cdot Q$ strength parameter $\kappa$ by a factor $\epsilon$, we arrive at the results shown in Figs. 2b-d. As $\epsilon$ is increased, i.e. as the quadrupole-quadrupole strength is enhanced, a peak at $R \sim 3.1$ gradually appears. On the other hand, the probability of $R > 3.1$ remains very small.

As we progressively increase the size of the shell, the peak at $R \sim 1.3$ gradually disappears and another peak, very close to $R \sim 3.3$, emerges. This is illustrated in Fig. 3 where we show results for six identical nucleons in the $pf$, $sdg$, $pfh$, and $sdgi$ shells with $\epsilon = 1.0$. For a large shell, the peak at $R \sim 3.3$ becomes very well pronounced. When we examine the calculated results more carefully, we find that when $R \sim 3.33$, the ratio of $(E_{6^+} - E_{0^+})/(E_{2^+} - E_{0^+})$ is also very compact and close to 7, the value in the rotational limit.

It is interesting to see what happens if we use the same restricted phenomenological hamiltonian in the full shell model space. This can be readily done for the $sd$ shell, for which we show the corresponding full shell model results in Fig. 4. In Fig. 4a, we limit ourselves to monopole pairing and a $Q \cdot Q$ interaction, while in Fig. 4b we include quadrupole pairing as well. One gets predominantly a rotational peak with $R \sim 10/3$. When $\epsilon = 10$, we obtain a rotational peak at $R \sim 10/3$, but this peak disappears as we gradually reduce to $\epsilon = 1$.

Based both on the pair-truncated results – which we were able to perform for many different shells – and the full shell-model results – limited to the $sd$ shell – we conclude that rotational motion is related closely to the form of the two-body interaction. In particular, for systems of identical nucleons there must be a strong quadrupole-quadrupole component.
in the interaction for a rotational peak to emerge.

To check the role of higher multipole interactions in producing the $R \sim 3.3$ peak, in Fig. 5 we plot the distribution of $R$ values of six identical nucleons in the $pfh$ shell interacting by a monopole pairing, quadrupole pairing, quadrupole-quadrupole force, hexadecapole pairing and hexadecapole-hexadecapole force. The hexadecapole interaction that we use has no $r$ dependence (i.e., the radial matrix element is assumed to be $R = 1$), so we introduce a renormalization factor, 40, as ‘compensation’. To isolate the role of the hexadecapole interactions, we introduce an additional multiplicative factor $\epsilon$ for the hexadecapole interactions. We note that the hexadecapole forces suppress the $R \sim 3.3$ peak. If one artificially enlarges the hexadecapole forces, the $R \sim 3.3$ peak is further quenched.

Summarizing, we have analyzed in this paper the conditions for the appearance of rotational motion for random hamiltonians in the context of a system of identical nucleons restricted to collective $S$ and $D$ pairs. We find that in such a truncated pair subspace vibrations arise generically for a general TBRE hamiltonian but rotations do not. With appropriate restrictions on the form of the hamiltonian, we are able to generate collective rotations, as had been found earlier for $sd$ boson systems [4]. In Ref. [4], a TBRE ensemble with 7 or 8 independent parameters was used in a space of $s$ and $d$ bosons, while here we take a TBRE hamiltonian with only 3 parameters for nucleons in many-$j$ shells. Not surprisingly, the quadrupole-quadrupole interaction seems to play a key role in obtaining a peak at $R \sim 3.33$. It was found, that all other interaction terms tend to wipe out the rotational peak. It is also noted that the results are not effected appreciably by the inclusion of single-particle splittings.

It is interesting to ask why the interacting boson model with fully random boson parameters is able to give rise to rotations, while the shell model truncated to $SD$ pairs cannot. The answer may lie in the fact that the interacting boson model should not be thought of as simply a pair truncation of the model space, but rather as a truncation of the model space that arises from the dominance of quadrupole correlations. Thus, the interacting boson model, whether modelled by a random hamiltonian or not, is already a consequence
of quadrupole and pairing correlations. There is no inconsistency, therefore, between the
results of [5] on the interacting boson model and those reported here. It would be interesting
to see whether these conclusions hold up in the presence of both protons and neutrons.

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FIG. 1  The distribution of $R$ values for six identical nucleons interacting by a general TBRE hamiltonian in a) the $sd$ shell, b) the $pf$ shell, and c) the $sdg$ shell.

FIG. 2  The distribution of $R$ values for six identical nucleons in the $sd$ shell interacting by a random monopole pairing, quadrupole pairing, and quadrupole-quadrupole force. The strength $\kappa$ of the quadrupole-quadrupole interaction is multiplied by a factor $\epsilon$ to assess the importance of the quadrupole-quadrupole interaction.

FIG. 3  The distribution of $R$ values for six identical nucleons interacting by a random monopole pairing, quadrupole pairing, and quadrupole-quadrupole force for a) the $pf$ shell, b) the $sdg$ shell, c) the $pfh$ shell, d) the $sdgi$ shell. In all cases, $\epsilon = 1.0$.

FIG. 4  The distribution of $R$ values for six identical nucleons in a $sd$ shell interacting by a) monopole pairing plus a quadrupole-quadrupole force, and b) monopole and quadrupole pairing plus a quadrupole-quadrupole force. The c) and d) are the same as a) and b), respectively, except that a quadrupole enhancement of $\epsilon = 10$ was used.

FIG. 5  The distribution of $R$ of six identical nucleons in the $pfh$ shell interacting by a monopole pairing, quadrupole pairing, quadrupole-quadrupole force, hexadecapole pairing and hexadecapole-hexadecapole force. The hexadecapole interactions have no $r$ dependence, so we use 40 as a renormalization factor. To see the effect of hexadecapole interactions, we multiply an adjustable factor, $\epsilon$, on them. a) $\epsilon = 1$, and b) $\epsilon = 25$. 
FIGURE 1  Y.M. ZHAO  
March 2nd/2002
FIG. 2

Y.M.Zhao  April 2nd/2002
Y.M. ZHAO  FIGURE 3
April 5th/2002
\[ G_4 = 1: \text{monopole pairing plus QQ force} \]

\[ G_4 = 10: \text{monopole pairing plus QQ force} \]

\[ R \]

FIGURE 4  Y.M. ZHAO, MAY 10th/2002
FIG. 5

(a) the pfh shell
\[ P_0 + P_2 + QQ + P_4 + HH \]
Renormalization of
\[ (P_4 + HH) : 40, \varepsilon = 1 \]

(b) the pfh shell
\[ P_0 + P_2 + QQ + P_4 + HH \]
Renormalization of
\[ (P_4 + HH) : 40, \varepsilon = 25 \]