Bundle Structure of Massless Unitary Representations of the Poincaré Group

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Reviewing the construction of induced representations of the Poincaré group of four-dimensional spacetime we find all massive representations, including the ones acting on interacting many-particle states. Massless momentum wavefunctions of non-vanishing helicity turn out to be more precisely sections of a $U(1)$-bundle over the massless shell, a property which to date was overlooked in bracket notation. Our traditional notation enables questions about square integrability and smoothness. Their answers complete the picture of relativistic quantum physics.

Frobenius’ reciprocity theorem prohibits massless one-particle states with total angular momentum less than the modulus of the helicity. There is no two-photon state with $J = 1$, explaining the longevity of orthopositronium.

Partial derivatives of the momentum wave functions are no operators which can be applied to massless states $\Psi$ with nonvanishing helicity. They allow only for covariant, noncommuting derivatives. The massless shell has a noncommutative geometry with helicity being its topological charge. A spatial position operator for $\Psi$, which constitutes Heisenberg pairs with the spatial momentum, is excluded by the smoothness requirement of the domain of the Lorentz generators.
1. Introduction

Some presentations of relativistic quantum physics, in particular textbooks on string theory, postulate the generators of the Poincaré group in Hilbert space to be given in terms of position operators

\[ X = (X^0, X^1, \ldots, X^{D-1}) \]

and momentum operators

\[ P = (P^0, P^1, \ldots, P^{D-1}) \]

which generate a 2D+1-dimensional Heisenberg group of operators

\[ U_a V_b = U_{a+b} \quad \text{and} \quad V_a U_b = V_{a+b} \quad \text{for} \quad a, b \in \mathbb{R}^D, \quad c \in \mathbb{C}, \]

where

\[ a \cdot b = \eta^{mn} a_m b_n = a_0 b_0 - a_1 b_1 - \cdots - a_{D-1} b_{D-1}. \]

On smooth and rapidly decreasing momentum wave functions \( \Psi : \mathbb{R}^D \rightarrow \mathcal{V}, p \mapsto \Psi(p) \), where \( \mathcal{V} \) is a sum of finite dimensional spaces with matrix representations of the Lorentz group and a non-degenerate scalar product, the operators \( P \) and \( X \) act by multiplication and differentiation

\[ (P^m \Psi)(p) = p^m \Psi(p), \quad (X^n \Psi)(p) = -i \eta^{nm} \partial_{p^n} \Psi(p), \]

hermitian with respect to the scalar product

\[ \langle \Phi | \Psi \rangle = \int d^Dp \ (\Phi(p) | \Psi(p) \rangle \rangle, \]

and represent the Heisenberg algebra, which in covariant string theory is part of its canonical quantization,

\[ [P^m, P^n] = 0 = [X^m, X^n], \quad [P^m, X^n] = i \eta^{mn} 1. \]

Hence, given finite dimensional matrices \( \Gamma_{mn} \) which commute with \( X \) and \( P \) and generate representations of the Lorentz group in \( \mathcal{V} \), the operators

\[ -i M_{mn} = -i (X_m P_n - X_n P_m) + \Gamma_{mn} \]

represent, analogous to angular momentum operators, the Lorentz algebra

\[ [\Gamma_{kl}, \Gamma_{mn}] = -\eta_{km} \Gamma_{ln} + \eta_{kn} \Gamma_{lm} + \eta_{lm} \Gamma_{kn} - \eta_{ln} \Gamma_{km}. \]

Prevalent as this representation is, it is fundamentally flawed: the Weyl relations (I) of \( X^0 \) require \( P^0 \) to be unitarily equivalent to \( P^0 + b, b \in \mathbb{R}, \)

\[ e^{ibX^0} P^0 e^{-ibX^0} = P^0 + b. \]

\[ ^1 \text{Notation: Let } T_a : x \mapsto x + a \text{ denote a translation in } \mathbb{R}^4, \quad T_\Lambda : x \mapsto \Lambda x \text{ a Lorentz transformation, and } \]
\[ T_{a, \Lambda} = T_a T_\Lambda \in \mathcal{G} \text{ a Poincaré transformation. We denote by } U_{a, \Lambda} \text{ its unitary representation with generators } \]
\[ P^m \text{ and } M_{mn} = -M_{nm}, \quad U_{a, \Lambda} = e^{iPa} e^{ia^m M_{mn} / 2}, \text{ in a Hilbert space of one-particle states.} \]
So the spectrum of $P^0$ is translation invariant, $\text{spec}(P^0) = \text{spec}(P^0 + b)$.

In all acceptable relativistic theories $\text{spec}(P^0)$ is non-negative. Hence they allow neither $X^0$ nor any operator $M_{0n}$ which employs $X^0$. Postulating constraints to restrict the spectrum to positive energy mass shells leads to contradictions [6]. The Heisenberg algebra is incompatible with a unitary representation of the Poincaré group $\mathfrak{P}$ on one-particle states.

Another reason to take the Heisenberg algebra as basis for the construction of Lorentz generators may be their action on fields $\Phi(x)$ with

$$U_{a,\Lambda}\Phi(x)U_{a,\Lambda}^{-1} = \Phi(\Lambda^{-1}(x-a)) .$$

(8)

But neither $x$ nor $\partial_x$ nor $\Phi(x)$ are operators in a Hilbert space. The fields are operator valued distributions. Integrated with smooth, rapidly decreasing testfunctions $f$

$$\Phi_f = \int d^4x f(x) \Phi(x)$$

and applied to the vacuum they generate a set of one-particle states $\Phi_f\Omega$ which, by the Reeh-Schlieder theorem [19], is dense already if all $f$ are restricted to vanish outside some arbitrarily chosen fixed open set. So the argument $x$ of the fields is not precisely the position of a particle, however suggestive this language about Feynman graphs may be.

To remind the reader we review the construction of induced representations. In appendices we provide the angle of the Wigner rotations in the massive and massless cases, filling the gap which prevented to date the straightforward determination of the Lorentz generators.

In the massive case they act on a sum of Hilbert spaces of fixed spin $s$, $(2s + 1) \in \mathbb{N}$, which all have the peculiarity to factorize

$$\mathcal{H} = \sum_s \mathcal{H}_s , \quad \mathcal{H}_s = \hat{\mathcal{H}}_s \otimes \mathcal{I}_s$$

(10)

into a Hilbert space $\hat{\mathcal{H}}_s$ which carries an irreducible spin-$s$ representation of $\mathfrak{P}$ with unit mass times a Hilbert space $\mathcal{I}_s$ which is pointwise invariant and is acted upon by the positive mass operator $M = \sqrt{P^2}$.

One-particle representations act on the span of eigenspaces of $M$ and are irreducible if and only if $\mathcal{I}_s = \mathbb{C}$.

This settles the question, what an interacting representation should be. For it necessary conditions have been specified [24] but no solution. For fixed spin massive representations of $\mathfrak{P}$ can differ from the free representation with mass $M$ only by employing an interacting mass $M'$. On many-particle states it acts in $\mathcal{I}_s$ as Hamiltonian of the relative motion [5].

Massless representations are not a continuous limit $m \to 0$ of massive representations. Their mass shell encloses a Lorentz fixed point $p = 0$ and
has the topology \( \mathbb{R} \times S^{D-2} \). Therefore the generators do not act on smooth wavefunctions of \( \mathbb{R}^{D-1} \) but, as it turns out, on sections of bundles with non-trivial transition functions. We derive these results by elementary analysis from the requirement of square integrability of rotated states.

The helicity of a photon cannot combine with its orbital angular momentum to a rotation invariant state: ‘There is no round photon’. Consequently there is no two-photon state which combines the helicities and the relative orbital angular momentum to \( j = 1 \), explaining why positronium with \( j = 1 \) cannot decay into two photons [13, 27].

We also show that no spatial position operator \( \vec{X} \) can exist which constitutes Heisenberg pairs with the spatial momenta and generates translations of spatial momentum.

### 2. Induced Representations

Mackey’s theorems [15] classify the unitary, strongly measurable\(^2\) representations \( U : g \mapsto U_g \) of the Poincaré group \( \mathcal{P} \) of a \( D \)-dimensional spacetime in a Hilbert space \( \mathcal{H} \), \( U_g : \mathcal{H} \to \mathcal{H} \), \( U_g U_{g'} = U_{gg'} \).

Each \( U \) decomposes, uniquely up to unitary equivalence, into a sum of representations which act on a product of \( \hat{\mathcal{H}}_s \) with an irreducible representation, and a pointwise invariant space \( \mathcal{I}_s \) (10).

To describe the irreducible representations in the cases of interest let \( G \) denote the cover of the restricted Lorentz group and \( H \subset G \) the cover of the stabilizer of some fixed timelike or lightlike momentum \( \underline{p} \in \mathbb{R}^D \).

\[
H = \{ h \in G : h\underline{p} = \underline{p} \} .
\]

The Lorentz orbit of \( \underline{p} \) is a mass shell \( \mathcal{M} \sim G/H \) with \( m > 0 \) or \( m = 0 \),

\[
\mathcal{M}_m = \left\{ \left( \sqrt{m^2 + \vec{p}^2} \right) : \vec{p} \in \mathbb{R}^{D-1} \right\} , \quad \mathcal{M}_0 = \left\{ \left( \sqrt{\vec{p}^2} \right) : \vec{p} \neq 0 \right\} .
\]

Their points correspond one-to-one to the left cosets \( gH \), as \( p = g\underline{p} = gh_p \).

As these cosets are either identical or disjoint, \( G \) is a bundle over a mass shell \( \mathcal{M} \) with fibers, which are each diffeomorphic to \( H \).

Let \( R \) represent \( H \) unitarily in a Hilbert space \( \mathcal{V} \), e.g. in \( \mathbb{C}^{2s+1} \),

\[
R(h_2) R(h_1) = R(h_2 h_1) ,
\]

\(^2\)For each \( \Psi \in \mathcal{H} \) the map \( f_{\Psi} : \mathcal{P} \to \mathcal{H} \), \( g \mapsto U_g \Psi \) has to be measurable.
and denote by $\mathcal{H}_R$ the space of functions $\Psi : G \to \mathcal{V}$ which satisfy
\[ \Psi(gh) = R(h^{-1})\Psi(g) \quad \forall g \in G, h \in H. \tag{14} \]
$\mathcal{H}_R$ is mapped to itself by the multiplicative representation of translations $a$,
\[ (U_a \Psi)(g) = e^{ip_a} \Psi(g), \quad p = gp, \tag{15} \]
and by composition with the inverse Lorentz transformations
\[ (U_g \Psi)(g') = \Psi(g^{-1}g'), \quad U_g \Psi = \Psi \circ g^{-1}. \tag{16} \]

One easily confirms that $U_a$ and $U_g$ \( \text{(15-16)} \) represent $\mathcal{P}$,
\[ U_a U_b = U_{a+b}, \quad U_g U_{g'} = U_{g g'} \quad \text{and} \quad U_g U_a = U_b U_g \quad \text{with} \quad b = ga. \]

As $R$ is unitary, the scalar product of the values at $g$ of two functions $\Phi, \Psi \in \mathcal{H}_R, \Phi(g), \Psi(g) \in \mathcal{V}$,
\[ \langle \Phi | \Psi \rangle = \langle \Phi(g)|\Psi(g) \rangle = \langle R(h^{-1})\Phi(g)|R(h^{-1})\Psi(g) \rangle = \langle \Phi(gh)|\Psi(gh) \rangle \tag{17} \]
is constant in each left coset $gH$. So one can define a scalar product in $\mathcal{H}_R$
\[ \langle \Phi | \Psi \rangle = \int_{\mathcal{M}} \tilde{d}p \langle \Phi|\Psi \rangle_p \tag{18} \]
where $\tilde{d}p = \tilde{d}(gp)$ is a Lorentz invariant measure. We choose it with the conventional normalization factors of quantum field theory ($m \geq 0$),
\[ \tilde{d}p = \frac{d^3p}{(2\pi)^3 2\sqrt{m^2 + \vec{p}^2}}, \quad \tilde{d}(gp) = \tilde{d}p. \tag{19} \]

Equipped with this scalar product $\mathcal{H}_R$ becomes a Hilbert space and $U$ the unitary representation,
\[ \int_{\mathcal{M}} \tilde{d}(gp) (U_g \Phi|U_g \Psi)_{gp} = \int_{\mathcal{M}} \tilde{d}(gp) \langle \Phi|\Psi \rangle_p = \int_{\mathcal{M}} \tilde{d}p \langle \Phi|\Psi \rangle_p, \tag{20} \]
which one calls induced by the representation $R$ of the stabilizer $H$.

Two such induced representations are equivalent if and only if the inducing representations $R$ are equivalent and their masses coincide. They are irreducible only if $R$ is an irreducible representation of $H$.

The states, on which an irreducible representation acts, are the possible states of a single particle. In sufficient distance to other particles its time evolution is generated for an observer at rest by $P^0$ and completely determined by its mass, no matter whether it is composite or elementary and whether it is charged or neutral.
3. Sections of Bundles

The momentum wave function of a spin-$s$ particle in state $\psi$ has the physical meaning to give the probability

$$w(\Delta, P; \psi) = \int_{\Delta} d^3p \sum_{n=-s}^{s} |\psi_n(p)|^2$$

for measurements of the momentum $P$ to yield a result in $\Delta \subset \mathcal{M}$.

It is invertibly related to the more abstract functions $\Psi \in H_R$, which have a fixed dependence $\Psi(gh) = R^{-1}(h)\Psi(g)$ on right factors $h \in H$, by a section $\sigma$ of the bundle $G$ over the base $\mathcal{M}$, in physics parlance by a choice of a gauge. Mathematically $\sigma$ is a set of smooth maps

$$\sigma_\alpha : U_\alpha \to G,$$  \hspace{1cm} (22)

the local sections, with domains $U_\alpha \subset \mathcal{M}$ which cover $\mathcal{M} = \cup_\alpha U_\alpha$. Each $\sigma_\alpha$ has to cut each fiber over $U_\alpha$ once: it chooses for each $p \in U_\alpha$ a Lorentz transformation $\sigma_\alpha(p)$ which maps $p$ to $p$. This relates in $U_\alpha$ each function $\Psi \in H_R$ to its momentum wave function $\psi_\alpha : U_\alpha \to \mathbb{C}^{2s+1}$,

$$\psi_\alpha = \Psi \circ \sigma_\alpha.$$  \hspace{1cm} (23)

In their common domain two local sections differ by their transition function

$$h_{\alpha \beta} : U_\alpha \cap U_\beta \to H,$$

$$\sigma_\beta(p) = \sigma_\alpha(p) h_{\alpha \beta}(p) \quad \text{(no sum over } \alpha), \quad h_{\alpha \beta}^{-1}(p) = h_{\beta \alpha}(p).$$  \hspace{1cm} (24)

As $\Psi$ is in $H_R$, momentum wavefunctions are related in $U_\alpha \cap U_\beta$ by

$$\psi_\beta = R(h_{\beta \alpha}) \psi_\alpha \quad \text{(no sum over } \alpha).$$  \hspace{1cm} (25)

The other way round, to each set of functions $\psi_\alpha : U_\alpha \to \mathcal{V}$ with such transition functions there corresponds the function $\Psi \in H_R$ which for $g$ in a fiber over $U_\alpha$ is defined by

$$\Psi(g) = R(g^{-1} \sigma_\alpha(gp)) \psi_\alpha(gp) \quad \text{(no sum over } \alpha).$$  \hspace{1cm} (26)

In the fibers over $U_\alpha \cap U_\beta$ the local sections $\sigma_\alpha$ and $\sigma_\beta$ yield the same $\Psi$.

Left multiplication by $g$ maps $\sigma_\alpha(p)$ to $g \sigma_\alpha(p)$ in the fiber over some domain $U_\beta$ and related to $\sigma_\beta(gp)$ by an $H$-transformation, the Wigner rotation

$$W_{\beta \alpha}(g, p) = (\sigma_\beta(gp))^{-1} g \sigma_\alpha(p).$$  \hspace{1cm} (27)
It enters as follows the transformation of the momentum wave functions, (no sum over $\alpha$)

$$
(U_g \psi)_\beta (g p) = \frac{23}{14} (U_g \Psi)(g \sigma_\beta (g p)) \frac{27}{16} (U_g \Psi)(g \sigma_\alpha (p) W_{\beta \alpha}^{-1}) \frac{23}{27} = \frac{14}{16} (U_g \Psi)(g \sigma_\beta (g p)) W_{\gamma \alpha} (g_{2g_1} p),
$$

These transformations of momentum wave functions represent $G$, as for $p \in \mathcal{U}_\alpha$, $g_1 p \in \mathcal{U}_\beta$ and $g_2 g_1 p \in \mathcal{U}_\gamma$, the Wigner rotations satisfy by (27)

$$
W_{\gamma \alpha} (g_{2g_1} p) = W_{\gamma \beta} (g_2 g_1 p) W_{\beta \alpha} (g_1 p), \quad \text{(no sum over } \beta). \quad (29)
$$

One can drop the denomination of the neighbourhoods in (28)

$$
(U_g \psi)_\beta (g p) = R(W_{\beta \alpha} (g, p)) \psi_\alpha (p), \quad (30)
$$

if one agrees to read equations about values of sections to apply by definition in the neighbourhoods which contain the arguments.

In $D = 4$ for massless representations with non-trivial $R$ the complication of several local sections is unavoidable. The Lorentz group $G$ is a non-trivial bundle over $\mathcal{M}_0 = \{ p : p = e^a (1, \vec{n} ), a \in \mathbb{R}, \vec{n} \in S^2 \}$ which has the non-trivial topology of $\mathbb{R} \times S^2$. The subgroup $\text{SU}(2) \cong S^3 \subset G$ is a bundle over $S^2 \subset \mathcal{M}_0$ with fibers diffeomorphic to $S^1$, each winding once around each other fiber. This bundle, the Hopf bundle, does not allow to extend a

Figure 1: Stereographic Projection of the Hopf Fibration of $S^3$ \cite{20}
by the interlocked fibers, which one can cut once – all but one – in their intersection with a disk bordered by one chosen \( S^1 \). But this chosen \( S^1 \) is not cut in exactly one point. Mathematically: if a global section existed, then \( G \) would be a product manifold \( \mathcal{M}_0 \times H \) of the base and a fiber. But \( S^3 \subset G \) is simply connected – each closed path in \( S^3 \) can be continuously shrunk to a point – while \( S^1 \) is not, hence \( S^3 \neq S^2 \times S^1 \). So massless representations with nontrivial \( R \) have to employ at least two local sections.

4. Smooth Wavefunctions

Each \( \omega \) from the Lie algebra \( g \) of a finite dimensional Lie group \( G \) generates by the exponential map the elements \( g_t = e^{t \omega} \), \( t \in \mathbb{R} \), of a one-parameter group. The skew hermitian generators \(-iM_\omega \) of the one-parameter subgroups of a unitary representation \( g \mapsto U_g \), \( U_g : \mathcal{H}(\mathcal{M}) \rightarrow \mathcal{H}(\mathcal{M}) \), represent \( g \) on the subspace of states \( \Psi \) on which all \( U_{e^{t\omega}} \) act smoothly [21]

\[
-iM_\omega \Psi = \lim_{t \to 0} (U_{e^{t\omega}} \Psi - \Psi) / t .
\] (31)

They generate \( U_{e^{t\omega}} = e^{-itM_\omega} \) by their (\( \omega \)-dependent) spectral resolution \( E_\lambda \)

\[
M_\omega = \int dE_\lambda \, \lambda \, , \quad U_{e^{t\omega}} = e^{-itM_\omega} = \int dE_\lambda \, e^{-it\lambda} .
\] (32)

The unitary operators \( U_{e^{t\omega}} \) are defined in the complete Hilbert space \( \mathcal{H} \), while the power series \( \sum_k (-itM_\omega)^k \Psi / k! \) converges only in an analytic subspace which is unsuitably small for some purposes, e.g. it contains no state \( \Psi \) of compact support.

Because the maps \( g \mapsto U_g \Psi \) are measurable therefore the smoothened states

\[
\Psi_f = \int d\mu_g f(g) U_g \Psi
\] (33)

exist, which are averaged with smooth functions \( f : \mathbb{P} \rightarrow \mathbb{C} \) of compact support and with an invariant volume form \( d\mu_g \), \( d\mu_{g'} = d\mu_g \). These states transform by

\[
U_g \Psi_f = \Psi_{f \circ g^{-1}} .
\] (34)

Consequently they are infinitely differentiable with respect to the coordinates \( \omega \). They span the Gårding space, which is dense in \( \mathcal{H} \) and invariant under all \( U_g \) and their generators. It is the maximal common domain of the algebra of the skew hermitian generators [3, 21].

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3 As Florian Oppermann pointed out to me \( \text{SO}(1,D-1) \sim \text{SO}(D-1) \times \mathbb{R}^{D-1} \) factorizes into \( \mathcal{M}_0 \times E(D-2) \sim S^{D-2} \times \text{SO}(D-2) \times \mathbb{R}^{D-1} \) only in the 4 cases \( D-1 = 1, 2, 4, 8 \) [11, page 292].
That such a common and dense domain exists for each unitary, strongly measurable representation of a finite-dimensional Lie group justifies in retrospect physicists who manipulate the unbounded generators algebraically not caring about domains.

The rough which one has to take with the smooth: an operator can be composed with a generator of a Lie group only if it maps smooth states to smooth states. Otherwise the algebraic product generator times operator is not defined. Rough operators with discontinuities or singularities in the group orbit, and be it only in a single point, cannot occur in the algebra of the Poincaré generators. Rough operators map physical states to unphysical states on which the Poincaré generators diverge.

5. Massive Representations

If one picks a momentum \( p \) from a massive shell \( \mathcal{M}_m, m > 0 \) \(^{[12]}\), then for an observer at relative rest it has the coordinates \( p = (m, 0, \ldots) \) and \( \mathcal{M}_m \) is its orbit under the restricted Lorentz group. The stabilizer of \( p \) is the group of rotations, \( \text{SO}(D-1) \) which in \( D = 4 \) is covered by \( H = \text{SU}(2) \). Each of its irreducible representations \( R \) is determined by its spin \( s \) where \( 2s + 1 \), the dimension of \( R \), is a natural number.

Each Lorentz transformation \( \Lambda = L_u O \) can be uniquely and continuously decomposed into a rotation \( O, O^T = O^{-1} \), and a boost \( L_u = L_u^T \) \(^{[4]}\),

\[
L_u = \left( \begin{array}{cc}
\sqrt{1 + \vec{u}^2} & \vec{u}^T \\
\vec{u} & 1 + \frac{\vec{u}\vec{u}^T}{1 + \sqrt{1 + \vec{u}^2}}
\end{array} \right)
\]

(35)

The boost \( L_u \) maps the four-momentum \( p = (m, 0, 0, 0) \) of a massive particle at rest to the momentum \( p = (\sqrt{m^2 + \vec{p}^2}, \vec{p}) = mu \) of a particle with four-velocity \( u \).

It provides the global section

\[
\sigma : \mathcal{M}_m \to \text{SO}(1, D-1)^+, \quad \sigma(p) = L_{p/m} \circ L_{p/m} \circ p = p \tag{36}
\]

This choice of a section allows to determine explicitly the generators of (28) by differentiation with respect to \( t \) at \( t = 0 \). The left side yields

\[
\omega^i(p) \partial_i \Psi - i M_{\omega} \Psi. \tag{36}
\]

\(^{4}\)The summation index \( i \) enumerates coordinates of the mass shell, in terms of which Lorentz transformations act smoothly e.g. the spatial components of the momentum. The derivatives \( \partial_{\mu}, n \in \{0, 1, 2, 3\} \), however, are not defined on wave functions of \( \mathcal{M}_m \). They are not functions of \( \mathbb{R}^{1,3} \).
Rotations $D = e^{i \omega}$ rotate four-velocities, $DL_{\alpha}D^{-1} = L_{D\alpha}$, thus they agree with their Wigner rotation, $W(D, p) = L_{D\alpha}^{-1}DL_{\alpha} = D$. Let $D_t = e^{t \omega}$ be unitarily represented by $R(e^{i \omega})\Psi(p) = (\exp \frac{t}{2} \omega^i_j \Gamma_{ij})\Psi(p)$ with skew hermitian matrices $\Gamma_{ij}$, which represent $\mathfrak{so}(3)$ or $\mathfrak{su}(2)$ ($i, j, k, l \in \{1, 2, 3\}$)

$$[\Gamma_{ij}, \Gamma_{kl}] = \delta_{ik} \Gamma_{jl} - \delta_{ik} \Gamma_{jl} - \delta_{jk} \Gamma_{il} + \delta_{jk} \Gamma_{il} , \quad \Gamma_{ij} = -\Gamma_{ij}^T = -\Gamma_{ji}.$$  

(37)

The derivative of $R(e^{i \omega})$ with respect to $t$ at $t = 0$ is $\frac{1}{2} \omega^i_j \Gamma_{ij} \Psi(p)$. Hence $\langle 28 \rangle$ implies

$$(-iM_{ij}\Psi)(p) = - (p^j \partial_{p^i} - p^i \partial_{p^j}) \Psi(p) + \Gamma_{ij} \Psi(p).$$

(38)

As $-iM_{ij}$ generate rotations, $(J^1, J^2, J^3) = (M_{23}, M_{31}, M_{12})$ are the components of the angular momentum operator $\vec{J}$. It consists of orbital angular momentum $\vec{L} = -ip \times \partial_p$ and spin $\vec{S}$ contributed by the matrices $i\Gamma$, $\vec{J} = \vec{L} + \vec{S}$, $[\vec{L}, \vec{S}] = 0$.

The component functions of the differential operator on the right side of $\langle 38 \rangle$ are the negative of the infinitesimal motion $\delta p$ of the points $p$. This motion occurs on the left side of $\langle 28 \rangle$ or as inverse transformation $M^{-1}_g$ in the adjoint transformation

$$\text{Ad}_g(f) = N_g \circ f \circ M^{-1}_g$$

(39)

of maps $f : \mathcal{M} \rightarrow \mathcal{N}$ of manifolds $\mathcal{M}$ to $\mathcal{N}$ on which $G$ is realized by transformations $M_g$ and $N_g, M_gM_g' = M_{gg'}, N_gN_g' = N_{gg'}, \text{Ad}_g\text{Ad}_g' = \text{Ad}_{gg'}$.

The Wigner rotation $W(L_{\alpha}, p)$ rotates from $p$ to $L_{\alpha}p$ by the deficit angle $\delta$ of the hyperbolic triangle with vertices $(L_{\alpha}p, p, p)$ and lengths $a = \text{dist} (\vec{L}, p)$ and $b = \text{dist} (L_{\alpha}p, p)$, and $\gamma$ the angle included by $\vec{p}$ and $\vec{u}$

$$\tan \frac{\delta}{2} = \frac{\sin \gamma}{(\coth \frac{a}{2} \coth \frac{b}{2}) + \cos \gamma}.$$  

(40)

We derive this result $\langle 109 \rangle$ in appendix A. Crucial as it is for the determination of the generators, I am unaware of a published derivation. For infinitesimal $L_{\alpha}$ and $b$ this yields

$$\frac{d \delta}{db}_{|b=0} = (\sin \gamma) \tanh \frac{a}{2} = (\sin \gamma) \frac{\sinh a}{\cosh a + 1} = (\sin \gamma) \frac{|\vec{p}|}{p^0 + m}.$$  

(41)

As the Wigner rotation of an infinitesimal boost $L_{0i}$ applied to $p$ rotates from $\vec{p}$ to $-e_i$, or from $e_i$ to $\vec{p}$ it is represented by $\Gamma_{ij}p^j/(p^0 + m)$. Combining both infinitesimal changes, we obtain the generators of boosts

$$(-iM_{0i}\Psi)(p) = p^0 \partial_{p^0} \Psi(p) + \Gamma_{ij} \frac{p^j}{p^0 + m} \Psi(p).$$  

(42)
The generators \((33, 42)\) are skew hermitian with respect to the scalar product \((19)\). They rotate and boost the four-vector \(P\)

\[
[-iM_{mn}, P^r] = -\delta^n_m P_n + \delta_m^n P_m ,
\]

and represent the Lorentz algebra \((6)\). For \([-iM_{ij}, -iM_{kl}]\) this is simple to check. It is manifest for \([-iM_{ij}, -iM_{0k}]\), because \(p^i, \partial_p^i\) and \(\Gamma_{ij}\) transform as vectors or products of vectors under the rotations of momentum and spin. The relations \([-iM_{0i}, -iM_{0j}] = iM_{ij}\) hold as if by miracle\(^5\)

\[
[p^i \partial_{p^j} + \Gamma_{ik} \frac{p^k}{p^0 + m}, p^0 \partial_{p^j} + \Gamma_{jl} \frac{p^l}{p^0 + m}] = \\
= \left( p^0 \frac{p^j}{p^0 + m} \partial_{p^j} + \frac{1}{(p^0 + m)^2} (p^0(p^0 + m)\delta^j_i - p^i p^j) \Gamma_{jl} \right) - (i \leftrightarrow j) + \\
+ \frac{p^k p^l}{(p^0 + m)^2} (\delta_{ij} \Gamma_{kl} - \delta_{il} \Gamma_{kj} - \delta_{kj} \Gamma_{il} + \delta_{kl} \Gamma_{ij}) \\
= - \left( - (p^i \partial_{p^j} - p^j \partial_{p^i}) + \Gamma_{ij} \right). \tag{44}
\]

Splitting the momenta \(p = mu\) of the wavefunctions into the mass \(m\) and the four-velocity \(u = (\sqrt{1 + u^2}, \vec{u})\) and allowing the mass \(m\) to depend on further invariant variables \(r\), as is the case for many-particle states, the generators of each massive spin-\(s\) representation, also each interacting one, are unitarily equivalent to

\[
P^m = U^m M, \quad 0 = [M, U^m] = [M, M_{mn}] = [U^m, U^n], \quad U^2 = 1,
\]

\[
(U^m \Psi)(u, r) = u^m \Psi(u, r), \quad (M \Psi)(u, r) = m(r) \Psi(u, r),
\]

\[
(-iM_{ij} \Psi)(u, r) = - (u^i \partial_{u^j} - u^j \partial_{u^i}) \Psi(u, r) + \Gamma_{ij} \Psi(u, r),
\]

\[
(-iM_{0i} \Psi)(u, r) = \sqrt{1 + u^2} \partial_u \Psi(u, r) + \Gamma_{ij} \frac{u^i}{\sqrt{1 + u^2}} \Psi(u, r).
\]

For fixed \(r\), the operators \(U^m\) and \(M_{mn}\) generate the irreducible unit mass spin-\(s\) representation of \(\mathcal{H}_s\) in \(\mathcal{H}_s\) \((10)\). The space of one-particle states is the span of the eigenspaces of \(M\).

Also each massive interacting representation has this form. Its mass \(M'\) commutes with all generators \(U^m\) and \(M_{mn}\) and is therefore invariant under translations \(V_a = e^{iU^a}\), generated by \(U^m\) and under Lorentz transformations. To yield a non-trivial \(S\)-matrix however, it must not commute with \(M\) \([5]\). Both \(M\) and \(M'\) are positive operators in the invariant Hilbert space \(\mathcal{H}_s\) of wavefunctions of \(r\).

\(^5\)Observe the summation range \(p^k p^k = (p^0)^2 - m^2 = (p^0 + m)(p^0 - m)\).

11
On many-particle states, $U^m$ generates translations of the center while $M'$ generates the interacting relative motion. The Hamiltonian $P^0 = U^0 M'$ separates not as a sum of Hamiltonians of the motion of the center and of the interacting relative motion, but factorizes as their product [5].

### 6. Massless Representations and Helicity

Picking arbitrarily from the massless shell

$$\mathcal{M}_0 = \{ (p^0, \vec{p}) : p^0 = |\vec{p}| > 0, \vec{p} \neq 0 \} ,$$

(46)

a momentum $p$, it has for a suitable observer coordinates $p = (1, 0, 0, \ldots 1)$ and $\mathcal{M}_0$ is its Lorentz orbit. It is the manifold $S^{D-2} \times \mathbb{R}$, as $\vec{p} \neq 0$ is specified by its direction and its nonvanishing modulus $|\vec{p}| = e^\alpha > 0$. Though the observer has adopted a unit of energy, this does not spoil dilational symmetry as dilations map $\mathcal{M}_0$ to itself and the transformations

$$(U e^\lambda \Psi)(p) = e^{-\lambda} \Psi(e^{-\lambda} p) ,$$

(47)

represent them unitarily with respect to the scalar product (19) with $m = 0$.

The stabilizer $H$ of $p_0$ is generated by infinitesimal Lorentz transformations $\omega, \eta \omega = - (\eta \omega)^T$, with $\omega p = 0$, $\omega^m_0 + \omega^m_z = 0$, thus of the form

$$\omega(\vec{a}, \vec{\omega}) = \begin{pmatrix} \vec{\omega}^T \\ \vec{a} \\ \vec{\omega} \\ \vec{\omega}^T \end{pmatrix} .$$

(48)

Here $\vec{\omega}$ is a skewsymmetric $(D-2) \times (D-2)$ matrix, which generates a rotation $w \in \text{SO}(D-2)$ and $\vec{a}$ is a $D-2$-vector. Because $\omega(\vec{a}, 0)$ and $\omega(\vec{b}, 0)$ commute they generate translations in $D-2$ dimensions. They are rotated by $w$. So $H$ is (the cover of) the Euclidean group $E(D-2)$ in $\mathbb{R}^{D-2}$.

By Mackey’s theorems [15] each massless unitary representation of the Poincaré group $\mathfrak{P}$ is a sum of irreducible representations which are induced by unitary irreducible representations of $E(D-2)$.

By the same theorems each such representation of $E(D-2)$ acts on the functions of the $\text{SO}(D-2)$ orbit, $S^{D-3}$ or $\{0\}$, of some $q \in \mathbb{R}^{D-2}$ and is characterized by a unitary irreducible representation of $q$’s stabilizer $\text{SO}(D-3)$ or, if $q = 0$, $\text{SO}(D-2)$.

In $D = 4$, the $\text{SO}(2)$-orbit of $q \neq 0$ is a circle, its stabilizer is trivial. The induced unitary representation of $E(2)$ acts on wave functions of a circle. Contrary to its denomination ‘continuous spin’ such a representation

12
contains only integer or half integer (infinitely many) helicities, the Fourier modes of the functions of the circle. Such a representation is not contained in the restriction of a finite-dimensional representation of the Lorentz group to \( E(2) \) but requires a field which in addition depends on additional continuous variables \[16\]. As these infinitely many massless states per given four-momentum make the specific heat of each cavity infinite \[26, p. 684\] they have to decouple from ordinary matter early enough after the hot big bang so as not to spoil our understanding of the thermodynamic evolution of the universe and its observed remnant, the microwave background radiation.

If \( q = 0 \in \mathbb{R}^2 \), then its orbit under \( SO(2) = E(2)/\mathbb{R}^2 \) only consists of \( q \).

All translations \( T_\delta \) in \( E(2) \) are represented trivially by \( T_\delta \equiv e^{i \mathbf{q} \cdot \mathbf{a}} = e^0 = 1 \). Each unitary irreducible representation \( R \) of \( SO(2) \) is one-dimensional and represents a rotation \( D_\delta \) by the angle \( \delta \) by multiplication with

\[
R_{D_\delta} = e^{-i h \delta}
\]

where in ray representations \( h \), the helicity, is some real number.

In \( D = 4 \) there is no global section with which to relate functions in \( \mathcal{H}_R \) \[14\] to momentum wave functions of \( \mathcal{M}_0 \) \[23\]. Therefore we use two local sections \( N_p \) and \( S_p \) to derive the transformation of momentum wave functions. They are differentiable in the north \( \mathcal{U}_N \) outside the negative 3-axis \( \mathcal{A}_- \) and in the south \( \mathcal{U}_S \) outside the positive 3-axis \( \mathcal{A}_+ \).

\[
\mathcal{U}_N = \{ p \in \mathcal{M}_0 : |\mathbf{p}| + p_z > 0 \}, \quad \mathcal{U}_S = \{ p \in \mathcal{M}_0 : |\mathbf{p}| - p_z > 0 \}, \quad \mathcal{A}_- = \{ (\lambda, 0, 0, -\lambda) : \lambda > 0 \}, \quad \mathcal{A}_+ = \{ (\lambda, 0, 0, \lambda) : \lambda > 0 \}.
\]

As \( \mathcal{U}_N \) and also \( \mathcal{U}_S \) cover \( \mathcal{M}_0 \) up to a set of vanishing measure, one need not distinguish integrals on these sets though as domains of differentiable functions one has to discriminate these sets carefully.

The northern section \( N_p = D_p B_p \) boosts \( p = (1, 0, 0, 1) \) in 3-direction to \( B_p p = (|\mathbf{p}|, 0, 0, |\mathbf{p}|) \) and rotates around \((-\sin \phi, \cos \phi, 0)\) by \( \theta, 0 \leq \theta < \pi \), to \( p = |\mathbf{p}|(1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = |\mathbf{p}|(1, n_x, n_y, n_z) \),

\[
D_p = \begin{pmatrix}
c_\phi & -s_\phi & c_\phi \\
s_\phi & c_\phi & s_\phi \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c_\theta & s_\theta & 0 \\
-s_\theta & c_\theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_\phi & s_\phi \\
-s_\phi & c_\phi \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
c_\phi & -s_\phi & c_\phi \\
s_\phi & c_\phi & s_\phi \\
-1 + n_z^2 & n_x & n_y \\
-n_x & 1 + n_z^2 & n_y \\
n_x & n_y & 1 + n_z^2
\end{pmatrix}
\]

For \( D_p \) to be smooth at \( p \) it is crucial that, different from the prevalent literature \[24, volume I, (2.5.47)\], \( D_p \) acts first by the inverse rotation by \( \phi \)
around $e_3$ and not only by the two rotations by the Euler angles $\theta$ around $e_2$ and $\phi$ around $e_3$. Their product is discontinuous at $\theta = 0$.

In place of the unhandy $4 \times 4$ Lorentz matrix $N_p$, one works more easily with its SL(2, C) representation $n_p$ (92, 93) using $p' = \sqrt{|p|}$, $\theta' = \theta/2$ and

$$\tan \theta' = \frac{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \frac{\sin \theta}{1 + \cos \theta} = \sqrt{\frac{p_x^2 + p_y^2}{|p| + p_z}} = \sqrt{\frac{|\vec{p}| - p_z}{|\vec{p}| + p_z}}, \quad (52)$$

$$n_p = \begin{pmatrix} \cos \theta' & -p' \sin \theta' e^{-i\phi} \\ \sin \theta' e^{i\phi} & p' \cos \theta' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{|\vec{p}| + p_z}}{|\vec{p}|} & -\frac{p_z - ip_y}{\sqrt{|\vec{p}| + p_z}} \\ \frac{p_x + ip_y}{|\vec{p}|\sqrt{|\vec{p}| + p_z}} & \frac{\sqrt{|\vec{p}| - p_z}}{|\vec{p}|} \end{pmatrix}. \quad (53)$$

Check that $n_p$ transforms $\hat{p} = (1 - \sigma^3)$ to $n_p \hat{p} (n_p)^\dagger = |\vec{p}| - \hat{p} \cdot \sigma = \hat{p}$ and for $b_p = 1$ leaves invariant the axis $-p_y \sigma_x + p_x \sigma_y$. In $\mathcal{U}_N$, $0 < \theta < \pi$, the section $n_p$ is smooth. It is discontinuous at $\mathcal{A}_-$ where the limit $\theta \to \pi$ depends on $\phi$.

The section

$$s_p = \begin{pmatrix} \cos \theta' e^{-i\phi} & -p' \sin \theta' \\ \sin \theta' & p' \cos \theta' e^{i\phi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p_z - ip_y}{|\vec{p}|\sqrt{|\vec{p}| - p_z}} & -\sqrt{|\vec{p}| - p_z} \\ \frac{p_x + ip_y}{|\vec{p}|\sqrt{|\vec{p}| - p_z}} & \frac{\sqrt{|\vec{p}| - p_z}}{|\vec{p}|} \end{pmatrix}. \quad (54)$$

is defined and smooth in the south, $0 < \theta \leq \pi$. The corresponding southern section $S_p$ rotates $B_p \vec{p}$ in the 1-3-plane by $\pi$ to $|\vec{p}|(1, 0, 0, -1)$ and then along a great circle by the smallest angle to $p$, namely by $\pi - \theta$ around the axis $(\sin \phi, -\cos \phi, 0)$.

In their common domain local SL(2, C) sections differ by the multiplication from the right (24) with a momentum dependent SL(2, C) matrix which represent E(2)

$$w = \begin{pmatrix} e^{-i\delta/2} & 0 \\ a e^{i\delta/2} \end{pmatrix}. \quad (55)$$

In the case at hand, $s_p$ differs in $\mathcal{U}_N \cap \mathcal{U}_S$ from $n_p$ by a preceding rotation around the 3-axis by $2\phi(p)$. Geometrically this angle is the area of the spherical lune (spherical digon) with vertices $\hat{e}_z$ and $-\hat{e}_z$ and sides (meridians) through $\hat{e}_x$ and $\vec{p}/|\vec{p}|$,

$$s_p = n_p \begin{pmatrix} e^{-i\phi(p)} \\ e^{i\phi(p)} \end{pmatrix}, \quad e^{i\phi(p)} = \frac{p_x + ip_y}{\sqrt{(|\vec{p}| - p_z)(|\vec{p}| + p_z)}}. \quad (56)$$
The angle $\delta$ of the Wigner rotation $W(\Lambda, p)$ $\Lambda N_p = N_{\Lambda p} W(\Lambda, p)$ (27),
can be easily read off the Iwasawa decomposition (111) of the $\text{SL}(2, \mathbb{C})$
representation

$$\lambda n_p = n_{\Lambda p} w(\Lambda, p)$$

as twice the phase of $(n_{\Lambda p} w)_{22} = |(n_{\Lambda p} w)_{22}| e^{i\delta/2}$.

If $\Lambda$ is a rotation by $\alpha$ around an axis $\vec{n}$ then $\lambda$ is given by (92) and the
Wigner angle $\delta$ and its infinitesimal value are

$$\tan \frac{\delta}{2} = \frac{n_z |\vec{p}| + \vec{n} \cdot \vec{p}}{(|\vec{p}| + p_z)(\cot \frac{\delta}{2}) + n_x p_y - n_y p_x}, \quad \frac{d \delta}{d \alpha}_{|\alpha=0} = \frac{n_z |\vec{p}| + \vec{n} \cdot \vec{p}}{|\vec{p}| + p_z}. \quad (58)$$

As $R_{D_\delta} = e^{-i\delta}$, the representation $-i\hbar p_z / (|\vec{p}| + p_z)$ of the infinitesimal
Wigner rotation accompanies e.g. the infinitesimal rotation in the 3-1-plane.

Analogously one determines the Wigner angle of a boost (93)

$$\tan \frac{\delta}{2} = \frac{- (n_x p_y - n_y p_x)}{(|\vec{p}| + p_z)(\coth \frac{\delta}{2}) + n_z |\vec{p}| + \vec{n} \cdot \vec{p}}, \quad \frac{d \delta}{d \beta}_{|\beta=0} = \frac{- (n_x p_y - n_y p_x)}{|\vec{p}| + p_z}. \quad (59)$$

and, as $-iM_0$ boosts from $e_0$ to $-e_i$, obtains in $D = 4$ [18,14]

$$(-iM_{12} \Psi)_N(p) = - (p_x \partial_{p_x} - p_y \partial_{p_y}) \Psi_N(p) - i\hbar \Psi_N(p),$$
$$(-iM_{13} \Psi)_N(p) = - (p_z \partial_{p_z} - p_x \partial_{p_x}) \Psi_N(p) - i\hbar \frac{p_y}{|\vec{p}| + p_z} \Psi_N(p),$$
$$(-iM_{12} \Psi)_N(p) = - (p_y \partial_{p_y} - p_z \partial_{p_z}) \Psi_N(p) - i\hbar \frac{p_x}{|\vec{p}| + p_z} \Psi_N(p),$$

$$(-iM_{01} \Psi)_N(p) = |\vec{p}| \partial_{p_z} \Psi_N(p) - i\hbar \frac{p_y}{|\vec{p}| + p_z} \Psi_N(p),$$
$$(-iM_{02} \Psi)_N(p) = |\vec{p}| \partial_{p_z} \Psi_N(p) + i\hbar \frac{p_x}{|\vec{p}| + p_z} \Psi_N(p),$$
$$(-iM_{03} \Psi)_N(p) = |\vec{p}| \partial_{p_z} \Psi_N(p). \quad (60)$$

In $D$-dimensional spacetime the generators are\(^6\)

$$(-iM_{ij} \Psi)_N(p) = - (p^i \partial_{p_j} - p^j \partial_{p_i}) \Psi_N(p) + h_{ij} \Psi_N(p),$$
$$(-iM_{il} \Psi)_N(p) = - (p_z \partial_{p_l} - p^i \partial_{p_i}) \Psi_N(p) + h_{ik} \frac{p^k}{|\vec{p}| + p_z} \Psi_N(p),$$
$$(-iM_{0i} \Psi)_N(p) = |\vec{p}| \partial_{p_z} \Psi_N(p) + h_{ik} \frac{p^k}{|\vec{p}| + p_z} \Psi_N(p),$$
$$(-iM_{0z} \Psi)_N(p) = |\vec{p}| \partial_{p_z} \Psi_N(p), \quad (61)$$

\(^6\)Note the range of summation $p^k p^k = |\vec{p}|^2 - p^2_z = (|\vec{p}| + p_z)(|\vec{p}| - p_z)$. 

15
where \( p_z = p^{D-1} \), \( i, j, k \in \{1, \ldots, D - 2\} \) and \( h_{ij} = -h_{ji}, \ h_{ij}^T = -h_{ij} \), generate a representation of \( \text{SO}(D-2) \) \(^{(37)}\).

In \( D = 4 \) one has \( h_{ij} = -ih_{ij} \), where the real number \( h \), the helicity, is the angular momentum \( \vec{p} \cdot \vec{J}/|\vec{p}| \) in the direction of the momentum \( \vec{p} \neq 0 \),

\[
(p_x M_{23} + p_y M_{31} + p_z M_{12}) \Psi(p) = h|\vec{p}| \Psi(p). \tag{62}
\]

The mere fact, that differential operators satisfy a Lie algebra on some space of functions does not make them generators of a representation of the corresponding group. This is demonstrated by the operators \( -iM_{mn} \) \(^{(60)}\). On differentiable functions of the northern coordinate patch \( \mathcal{U} \) \(^{(50)}\) they satisfy the Lorentz Lie algebra \( (6) \) in \( D = 4 \) \([11, 8, 14]\) no matter which real value the helicity \( h \) has. The Lorentz Lie algebra does not restrict \( 2h \) to be an integer. The operators are formally skew hermitian with respect to the Lorentz invariant measure \( dp \), formally only, because the singularities at \( |\vec{p}| + p_z = 0 \) need closer investigation.

The operators \( (60) \) cannot generate the Lorentz group because the domain \( \mathcal{U}_N \) of the differentiable functions is too small: Lorentz generators act on smooth states, which have to be defined \( \textit{everywhere} \) on the massless shell \( \mathcal{M}_0 \). The group acts transitively and contains for each massless momentum \( p \) a rotation which maps it to \( p \in \mathcal{A}_- = \{ p : p^0 = |\vec{p}| = -p_z > 0 \} \).

For \( h \Psi_N(\hat{p}) \neq 0 \) the functions \( h(|\vec{p}| + p_z)^{-1} \Psi_N \) are not defined on \( \hat{p} \in \mathcal{A}_- \) and seem to contradict \( (16) \) by which smooth states in \( \mathcal{M}_R \) are transformed to smooth states. One cannot require all wave functions \( \Psi_N \) to vanish on \( \mathcal{A}_- \) because such functions do not span a space which is mapped to itself by rotations. One also cannot take comfort in the misleading argument \([9, 22]\) that the set of singular points of e.g. \( M_{13} \Psi \) has measure zero: The argument is irrelevant as not the measure of the singular set matters but the measure \( \tilde{\mu}(\Gamma_c) \) of the sets \( \Gamma_c = \{ p : |M_{13} \Psi(p)|^2 > c \} \), where \( \Psi \) is large. If the limit \( c \to \infty \) of \( c \tilde{\mu}(\Gamma_c) \) does not vanish then \( M_{13} \Psi \) is not square integrable.

That a set of measure zero does not count in quantum physics holds for the elements of each equivalence class of a wave function but \( \textit{not} \) for smooth states and their domain. Each smooth equivalence class contains one unique smooth function on which the generators are defined and act smoothly.

For negative \( p_z \) and with \( x = (p_x^2 + p_y^2)/p_z^2 \) one has

\[
|\vec{p}| + p_z = |\vec{p}| - |p_z| = |p_z|(\sqrt{1 + x} - 1) \leq |p_z| \frac{x}{2} \tag{63}
\]

because the concave function \( x \mapsto \sqrt{1 + x} \) is bounded by its tangent at \( x = 0 \). So

\[
\frac{1}{|\vec{p}| + p_z} \geq \frac{2|p_z|}{p_x^2 + p_y^2} \tag{64}
\]
diverges in a neighbourhood \( \mathcal{U} \) of \( \hat{p} \in \mathcal{A} \) at least like the inverse square of the axial distance \( r = \sqrt{p_x^2 + p_y^2} \).

If \( h \Psi_N \) does not vanish in a neighbourhood \( \mathcal{U} \) of \( \hat{p} \in \mathcal{A} \) then, for \( M_{31} \Psi_N \) to exist, it must not be differentiable there. Otherwise the multiplicative term of \( M_{31} \Psi_N \) dominates near \( \hat{p} \) where it scales as \(|p_z|/r\). Its squared modulus integrated over a sufficiently small \( \mathcal{U} \) in cylindrical coordinates is bounded from below by a positive number times an \( r \)-integral over \( r/r^2 \, dr \) which diverges at the lower limit \( r = 0 \). Hence the multiplicative term alone diverges.

Near \( \mathcal{A} \) the derivative term \( D \Psi_N = -p_z \partial_{p_x} \Psi_N \) in \( M_{31} \Psi_N \) has to cancel the multiplicative singularity \( M \Psi_N \) up to a function \( \chi \), which is smooth. This linear inhomogeneous condition \((D + M) \Psi_N = \chi\) has to cancel the multiplicative singularity \( \Psi_N = f \Psi_S \) where \( f \) satisfies the two homogeneous conditions

\[
|p_z|(\partial_{p_x} - 2i h \frac{p_y}{p_x^2 + p_y^2}) f = 0, \quad |p_z|(\partial_{p_y} + 2i h \frac{p_x}{p_x^2 + p_y^2}) f = 0,
\]

for both \( M_{31} \Psi_N \) and \( M_{32} \Psi_N \) to exist. They determine \( f(p) = e^{-2i h \varphi(p)} \) up to a factor.

The function \( \Psi_S \) is smooth in the southern coordinate patch \( \mathcal{U}_S \) and related in \( \mathcal{U}_N \cap \mathcal{U}_S \) by the transition function \( f^{-1} = h_{SN} \) to \( \Psi_N \)

\[
\Psi_S(p) = h_{SN}(p) \Psi_N(p), \quad h_{SN}(p) = e^{2i h \varphi(p)} = \left( \frac{p_x + i p_y}{\sqrt{p_x^2 + p_y^2}} \right)^{2h}.
\]

Each state \( \Psi \) is given by local sections \( \Psi_N \) and \( \Psi_S \) of a bundle over \( S^2 \times \mathbb{R} \) with transition function \( h_{SN} \) which is defined and smooth in \( \mathcal{U}_N \cap \mathcal{U}_S \) only if \( 2h \) is integer. This is why the helicity of a massless particle is integer or half integer. Multiplying (66) with \( h_{SN} \) one obtains from (66)

\[
\begin{align*}
(-iM_{12} \Psi)_S(p) &= -(p_x \partial_{p_y} - p_y \partial_{p_x}) \Psi_S(p) + i h \Psi_S(p), \\
(-iM_{31} \Psi)_S(p) &= -(p_z \partial_{p_x} - p_x \partial_{p_z}) \Psi_S(p) - i h \frac{p_y}{|\vec{p}| - p_z} \Psi_S(p), \\
(-iM_{32} \Psi)_S(p) &= -(p_z \partial_{p_y} - p_y \partial_{p_z}) \Psi_S(p) + i h \frac{p_x}{|\vec{p}| - p_z} \Psi_S(p), \\
(-iM_{01} \Psi)_S(p) &= |\vec{p}| \partial_{p_x} \Psi_S(p) + i h \frac{p_y}{|\vec{p}| - p_z} \Psi_S(p), \\
(-iM_{02} \Psi)_S(p) &= |\vec{p}| \partial_{p_y} \Psi_S(p) - i h \frac{p_x}{|\vec{p}| - p_z} \Psi_S(p), \\
(-iM_{03} \Psi)_S(p) &= |\vec{p}| \partial_{p_z} \Psi_S(p).
\end{align*}
\]

The same generators result if, along the lines of the derivation of (66), one reads the Wigner angle \( \delta \) from the phase of \((\lambda \lambda)_{12} = -|s_{\lambda p} w_{12}| e^{i \delta/2}\). All \( M_{mn} \Psi \) are square integrable, rapidly decreasing and smooth if \( \Psi \) is.
For all $\omega$ in the Lorentz Lie algebra the operators $-iM_\omega = -i/2 \omega^{mn}M_{mn}$ are by construction the derivatives of unitary one-parameter groups

$$-iM_\omega(U_\omega \psi) = \partial_t(U_\omega \psi) ,$$

which act on the dense and invariant domain $\mathcal{D}$ of smooth states, where the transformations $U_\omega$ and their products represent unitarily the Lorentz group. So $-iM_\omega$ not only satisfy the Lorentz algebra but they are skew-adjoint (by Stone’s theorem) and generate a unitary representation of the Lorentz group.

In $D > 4$ dimensions

$$\Psi_S(p) = R_p^2 \Psi_N(p),$$

where $R_p$ represents the rotation from $\vec{p}_T/|\vec{p}_T|$ to $\vec{e}_x$ which leaves vectors orthogonal to the $\vec{e}_x - \vec{p}_T$-plane pointwise invariant.

For helicity $h \neq 0$ each continuous momentum wave function $\Psi$ has to vanish along some line, a Dirac string in momentum space, from $\ln |\vec{p}| = -\infty$ to $\ln |\vec{p}| = \infty$. Namely, if one removes the set $\mathcal{N}$, where $\Psi$ vanishes, from the domains $\mathcal{U}_N$ and $\mathcal{U}_S$ then the remaining sets $\mathcal{U}_N$ and $\mathcal{U}_S$ cannot both be simply connected. In $\mathcal{U}_N$ the phase of $\Psi_N$ is continuous and its winding number along a closed path, being integer, does not change under deformations of the path. For a contractible path this winding number vanishes, as the phase along the path becomes constant upon its contraction. If $\mathcal{U}_N$ is simply connected then it contains a contractible path around $\phi_+$ which also lies in $\mathcal{U}_S$. On this path the phase of $\Psi_N$ has vanishing winding number but the phase of $\Psi_S(p) = e^{2i\phi(p)} \Psi_N(p)$ winds $2h$-fold. So the path cannot be contractible in $\mathcal{U}_S$.

For $h \neq 0$ and if $\Psi$ does not vanish on the 3-axis then the partial derivatives of $\Psi_N$ and $\Psi_S = e^{2i\phi(p)} \Psi_N$ are not both square integrable. Well defined in $\mathcal{M}_0$ and skew hermitian with respect to the measure $\tilde{dp}$ are the covariant derivatives

$$D_i = i |\vec{p}|^{-1/2} M_{0i} |\vec{p}|^{-1/2} = \partial_{p^i} + A_i - \frac{p^i}{2|\vec{p}|^2},$$

with the connection $\tilde{A}$ in $\mathcal{U}_N$ and $\mathcal{U}_S$ given by

$$\tilde{A}_N(p) = \frac{i \hbar}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} -p_y & -p_x \\ p_x & 0 \end{pmatrix}, \quad \tilde{A}_S(p) = \frac{-i \hbar}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} -p_y & p_x \\ 0 & 0 \end{pmatrix},$$

and related in the $\mathcal{U}_N \cap \mathcal{U}_S$ by the transition function

$$D_{Si} = e^{2i\phi(p)} D_{Ni} e^{-2i\phi(p)}.$$
The covariant derivative $D_j$ and the momentum $P^i$ do not constitute Heisenberg pairs as the commutator $[D_i, D_j]$ yields the field strength of a momentum space monopole of charge $\hbar$ at $p = 0$,

\[
\begin{align*}
[P^i, P^j] &= 0, \\
[D_i, D_j] &= F_{ij} = \partial_i A_j - \partial_j A_i = i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|^3}.
\end{align*}
\]

(73)

The geometry of the massless shell of particles with nonvanishing helicity is noncommutative.

In terms of the covariant derivative the generators of Lorentz transformations (60, 67) take the rotation equivariant form

\[
\begin{align*}
-i M_{ij} &= -(P^i D_j - P^j D_i) - i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|}, \\
-i M_{0j} &= -|\vec{p}|^{1/2} D_j |\vec{p}|^{1/2}.
\end{align*}
\]

(74)

They satisfy the Lorentz algebra on account of (73) for arbitrary real $\hbar$. However, the covariant derivative $D$ is a skew hermitian operator only if $2\hbar$ is integer.

The integrand $\mathcal{F} = \frac{1}{2} \int d^3 p \left( p^i F_{ij} \right)$, the first Chern class, is a topological density: Integrated on each surface $\mathcal{S}$ which is diffeomorphic to a sphere around the apex $p = 0$ of the cone $p^0 = |\vec{p}|$ it yields a value

\[
\frac{1}{4\pi} \int_{\mathcal{S}} \mathcal{F} = i \hbar
\]

which depends only on the transition function of the bundle. The integral remains constant under smooth, local changes of the connection $A_i(p)$ of the covariant derivative as the integral of $\mathcal{F}(A) = d(p^i A_i)$ on coordinate patches changes by boundary terms only and they vanish for local changes.

7. Angular Momentum

The massless shell is foliated in spheres with radius $|\vec{p}|$, $\mathcal{M}_0 = \mathbb{R}_+ \times S^2$. Hence the Hilbert space $\mathcal{H}(\mathcal{M}_0)$ is a tensor product $\mathcal{L}^2(\mathbb{R}_+) \otimes \mathcal{H}_h(S^2)$ and $\mathcal{L}^2(\mathbb{R}_+)$, the space of wave functions of the energy $E = |\vec{p}|$, is left pointwise invariant under rotations. In $\mathcal{H}_h(S^2)$ the $\text{SO}(3)$-representation is induced by the representation $R_D \delta = e^{-i \hbar \delta}$ of rotations around the 3-axis. To be induced by an irreducible representation does not make the representation of $\text{SO}(3)$ irreducible. Rather $\mathcal{H}_h$ decomposes into angular momentum multiplets, each characterized by its total angular momentum $j$. 

19
Such a multiplet contains a state $\Lambda$ which is annihilated by $J_+ = M_{23} + iM_{31}$ and by $M_{12} - j$,

$$(M_{12} - j)\Lambda = 0, \quad (M_{23} + iM_{31})\Lambda = 0.$$  \hfill (76)

By (60) these are differential equations for $\Lambda_N$. They become easily solvable if we consider $\Lambda_N$ as a function of the complex stereographic coordinates

$$w = u + iv = \frac{p_x + ip_y}{|\vec{p}| + p_z}, \quad \bar{w} = u - iv = \frac{p_x - ip_y}{|\vec{p}| + p_z},$$  \hfill (77)

which map the northern domain to $\mathbb{C}$. Then the differential equations read

$$(w \partial_w - \bar{w} \partial_{\bar{w}} + h - j)\Lambda_N = 0, \quad (w^2 \partial_w + \partial_{\bar{w}} + hw)\Lambda_N = 0.$$  \hfill (78)

Recollecting that $w \partial_w$ measures the homogeneity in $w$, $w \partial_w w^r = r w^r$ (where superscripts denote exponents), the first equation is solved by $\Lambda_N = w^{j-h} g(|w|^2)$ and by the second equation $g$ is homogeneous in $(1 + |w|^2)$ of degree $-j$.

$$( (j-h)g + (|w|^2 + 1)g') w^{j-h+1} = 0, \quad \Lambda_N(w,\bar{w}) = \frac{w^{j-h}}{(1+|w|^2)^j}.$$  \hfill (79)

The state $\Lambda$ is smooth only if $j-h$ is a nonnegative integer. It is square integrable with respect to the rotation invariant measure

$$d\Omega = \frac{4\, du\, dv}{(1+u^2 + v^2)^2}$$  \hfill (80)

if also $j+h$ is nonnegative which is just the restriction to be smooth also in southern stereographic coordinates. They are related to the northern coordinates in their common domain by inversion at the unit circle,

$$w' = \frac{p_x + ip_y}{|\vec{p}| - p_z} = \frac{1}{w}, \quad \bar{w}' = \frac{p_x - ip_y}{|\vec{p}| - p_z} = \frac{1}{w},$$  \hfill (81)

and $\Lambda_S$ is smooth only if $j+h$ is a nonnegative integer

$$\Lambda_S(w',\bar{w'}) = (\frac{w}{|w|})^{2j} \Lambda_N = \frac{w'^{j+h}}{(1+|w'|^2)^j}.$$  \hfill (82)

So $j \geq |h|$: There is no round photon with $j = 0$. This follows also from (62) which for $h \neq 0$ excludes that all angular momenta $M_{ij}$ vanish.

20
The SO(2) representation $R_{D_j} = e^{-ih\delta}$, $2h \in \mathbb{Z}$, induces in the space of sections over the sphere no SO(3) multiplet with $j < |h|$ and one multiplet for $j = |h|$, $|h| + 1, \ldots$, i.e. the representation $j$ is induced with multiplicity

$$n_h(j) = \begin{cases} 0 & \text{if } j < |h| \\ 1 & \text{if } j \in \{ |h|, |h| + 1, |h| + 2, \ldots \} \end{cases} \quad (83)$$

Vice versa, the restriction of the SO(3) representation $j$ contains the SO(2) representation $h$ with the multiplicity

$$m_j(h) = \begin{cases} 0 & \text{if } j < |h| \\ 1 & \text{if } j \in \{ |h|, |h| + 1, |h| + 2, \ldots \} \end{cases} \quad (84)$$

These multiplicities exemplify Frobenius’ reciprocity [15]: The representation $h$ of the subgroup $H$ induces on sections over $G/H$ each representation $j$ of the group $G$ with the same multiplicity with which the restriction of $j$ to the subgroup $H$ contains $h$,

$$m_j(h) = n_h(j). \quad (85)$$

Photons have helicity $+1$ or $-1$. By Bose symmetry two-photon states to satisfy $\Psi_{ij}(p_1, p_2) = \Psi_{ji}(p_2, p_1)$ or

$$\chi^{ij}(u, q) = \chi^{ji}(u, -q), \quad i, j \in \{ +1, -1 \}, \quad (86)$$

where $\chi^{ij}(u, q)$ is the wave function $\Psi_{ij}(p_1, p_2)$ in terms of the center variables $u = (p_1 + p_2)/\sqrt{(p_1 + p_2)^2}$ and the relative momentum $[5]$.

$$q = L_u^{-1}(p_1 - u(u\cdot p_1)) = -L_u^{-1}(p_2 - u(u\cdot p_2)). \quad (87)$$

$L_u^{-1}$ boosts $u$ to $u = (1, 0, 0, 0)$ such that $q \in \mathbb{R}^3$. The helicity $i$ is the angular momentum $m_q$ of the first photon in the direction of $q$ and $j$ is the angular momentum of the second photon in the opposite direction $-q$ [62]. Hence the helicities add to angular momentum $m = i - j$ and $\chi^{++}$ and $\chi^{-}$ induce rotation multiplets from $m = 0$. As they are two even functions of $q$ they induce two multiplets with even spin $j = 0, 2, 4, \ldots$

The function $\chi^{++}$ determines $\chi^{-}$. Its helicities combine to angular momentum $m = 2$ which by $[83]$ induces one multiplet with spin $|j| = 2, 3, \ldots$.

So there is no spin-1 multiplet of two photons. This conclusion is the Landau Yang theorem [13, 27]. It clarifies why positronium with $j = 1$ does not decay into two photons: the interaction is Poincaré invariant and therefore preserves the total momentum $P$ and the total spin. But there is no two-photon state with $j = 1$ into which $j = 1$ positronium can decay. There is a three photon state with $j = 1$. But the decay rate to three photons is suppressed roughly by a factor $\alpha \sim 1/137$ for the production of an additional photon which is why one finds positronium predominantly in its stablest state as orthopositronium with $j = 1$. 
8. Position of a Massless Particle

For a massless particle there cannot exist a spatial position operator $\vec{X}$ which shifts the momentum, $e^{i\vec{b} \cdot \vec{X}} \vec{P} e^{-i\vec{b} \cdot \vec{X}} = \vec{P} + \vec{b}$, and generates together with $P^0 = \sqrt{\vec{P}^2}$ an algebra. Such an algebra would contain

$$[X^i, [X^i, P^0]] = -\frac{D - 2}{|\vec{P}|}, \quad (88)$$

thus, for $D \neq 2$, all inverse powers of $|\vec{P}|$. Hence the domain of the algebra contained only states $\Psi$ which decreased near $\vec{p} = 0$ faster than any power of $|\vec{p}|$. As the domain of the generators is invariant under the group, it is invariant under shifts and contained all shifted states. So $(e^{i\vec{b} \cdot \vec{X}} \Psi)(\vec{p}) = \Psi(\vec{p} + \vec{b})$ would have to vanish for all $\vec{b}$ at $\vec{p} = 0$. But this means $\Psi = 0$. The Heisenberg algebra of $\vec{X}$ and $\vec{P}$ is incompatible with the Hamiltonian $\sqrt{\vec{P}^2}$, which is not a smooth operator but rough at $\vec{p} = 0$.

In the plane of spatial momenta of the massless shell the Lorentz fixpoint $\vec{p} = 0$ breaks the translation symmetry of the plane. There the dispersion $p^0 = \sqrt{\vec{p}^2}$ is not smooth. This lack of translational symmetry does not occur on massive shells and frustrates all attempts [12, 17, 18, 25] to construct a position operator for massless particles.

9. Conclusions

We corrected widespread misconceptions about the hermitian generators of Poincaré transformations. Using the traditional notation of quantum physics we reconciled the smoothness requirements of physical states with the singularities which the Lorentz generators, acting on massless states of nonvanishing helicity, develop on the negative $z$-axis (common representations are even singular on the complete $z$-axis). The singularity vanishes on the negative $z$-axis in a different gauge but reappears on the positive $z$-axis: the smooth wave function is a section of a bundle which does not allow globally partial derivatives with respect to the momenta, but only noncommuting covariant derivatives. This constitutes arguably the most elementary example of noncommutative geometry in physics.

The modulus of the helicity is a lower bound of the total angular momentum of one-photon states. It excludes a two-photon states with $J = 1$ confirming the Landau Yang theorem, which prevents orthopositronium to decay into two photons.
At the fixpoint $p = 0$ of Lorentz transformations the Hamiltonian $H = \sqrt{p^2}$ is only continuous, not smooth. This is why one cannot extend the algebra of Poincaré generators by a spatial position operator, which generates translations of spatial momentum.

**A. The Wigner Rotation of Massive Particles**

To derive the angle of the Wigner rotations, we use the fact that the Lorentz group $\text{SO}(1,3)$ represents $\text{SL}_2(\mathbb{C})$. For definiteness and to fix our notation we review these well-known relations. We use the $1$-matrix $\sigma^0$ and the Pauli matrices $\sigma^i, i = 1, 2, 3$, with matrix elements $\sigma^m_{\alpha \dot{\beta}}$, $m \in \{0, 1, 2, 3\}$, $\alpha, \dot{\beta} \in \{1, 2\}$,

$$
\sigma^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$

(89)

and write products

$$
\sigma^i \sigma^j = \delta^{ij} 1 + i \varepsilon_{ijk} \sigma^k, \quad i, j, k \in \{1, 2, 3\}; \quad \varepsilon_{123} = 1
$$

(90)

of their linear combinations concisely as

$$
(\vec{m} \cdot \vec{\sigma})(\vec{n} \cdot \vec{\sigma}) = (\vec{m} \cdot \vec{n}) 1 + i(\vec{m} \times \vec{n}) \cdot \vec{\sigma}.
$$

(91)

If $\vec{m} \parallel \vec{n}$ then $\vec{m} \cdot \vec{\sigma}$ and $\vec{n} \cdot \vec{\sigma}$ commute, they anticommute if $\vec{m} \perp \vec{n}$.

For each unit vector $\vec{n}$ one has $(\vec{n} \cdot \vec{\sigma})^2 = 1$ and series of $\vec{n} \cdot \vec{\sigma}$ simplify

$$
U_{\alpha \vec{n}} = \exp(-i\alpha/2 \vec{n} \cdot \vec{\sigma}) = (\cos \alpha/2) 1 - i (\sin \alpha/2 \vec{n} \cdot \vec{\sigma}),
$$

(92)

$$
V_{\beta \vec{n}} = \exp(-\beta/2 \vec{n} \cdot \vec{\sigma}) = (\cosh \beta/2) 1 - (\sinh \beta/2 \vec{n} \cdot \vec{\sigma}).
$$

(93)

The matrices $\sigma^m$ are a basis of the real four dimensional vectorspace of hermitian $2 \times 2$-matrices

$$
\hat{k} = \sigma^m k^m \eta_{mn} = \begin{pmatrix} k^0 - k^3 & -k^1 + i k^2 \\ -k^1 - i k^2 & k^0 + k^3 \end{pmatrix}.
$$

(94)

Their determinant $\det \hat{k} = k^m k^n \eta_{mn} = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2$ is invariant for all $D$ with $\det D = 1$ under the linear transformation ($D^T := D^*\Psi$)

$$
\hat{k} \mapsto \hat{k}' = D \hat{k} D^\dagger, \quad k' = \Lambda_D k.
$$

(95)
So $k \mapsto k' = \Lambda_D k$ is a Lorentz transformation and $D \mapsto \Lambda_D$ is a representation of $\text{SL}(2, \mathbb{C})$ in $\text{SO}(1, 3)$, $\Lambda_D \Lambda_D' = \Lambda_{D'D'}$. Physics parlance reverses the relation and claims sloppily that $D$ represents $\Lambda$.

Explicitly $U_{\tilde{a}\tilde{a}}' = U_{\tilde{a}\tilde{a}}^{-1}$ is unitary and effects a rotation by $\alpha$ around $\tilde{n}$. To see this decompose $\tilde{k} = (\tilde{k} \cdot \tilde{n}) \tilde{n} + \tilde{k}_\perp$ into parallel and transverse parts and split $k = k_\parallel + k_\perp$ with $k_\parallel = (k^0, (\tilde{k} \cdot \tilde{n}) \tilde{n})$ and $k_\perp = (0, \tilde{k}_\perp)$.

$\tilde{k}_\parallel$ commutes with each power series in $\tilde{n} \cdot \vec{\sigma}$. Hence it is invariant

$$U \tilde{k}_\parallel U^\dagger = \tilde{k}_\parallel U U^\dagger = \tilde{k}_\parallel .$$

As $\tilde{k}_\perp$ is orthogonal to $\tilde{n}$ the matrix $\tilde{k}_\perp$ anticommutes with $\tilde{n} \cdot \vec{\sigma}$

$$\hat{k}_\perp \tilde{n} \cdot \vec{\sigma} = -\tilde{n} \cdot \vec{\sigma} \tilde{k}_\perp .$$

Therefore $U (\tilde{k}_\perp \cdot \vec{\sigma}) U^\dagger = U^2 (\tilde{k}_\perp \cdot \vec{\sigma})$ and $U_{\tilde{a}\tilde{a}}^2 = U_{2\tilde{a}\tilde{a}}$ implies

$$U (\tilde{k}_\perp \cdot \vec{\sigma}) U^\dagger = (\cos \alpha - i \sin \alpha \tilde{n} \cdot \vec{\sigma}) (\tilde{k}_\perp \cdot \vec{\sigma}) = (\cos \alpha \tilde{k}_\perp + \sin \alpha \tilde{n} \times \tilde{k}_\perp) \cdot \vec{\sigma} .$$

So $U_{\tilde{a}\tilde{a}} = e^{-i\tilde{a} \tilde{n} \cdot \vec{\sigma}}$ causes by $k \mapsto U_{\tilde{a}\tilde{a}} \hat{k} U_{\tilde{a}\tilde{a}}^\dagger$ the rotation $k \mapsto D_{\tilde{a}\tilde{a}} k$ of four-vectors $k$ by the angle $\alpha$ around the axis $\tilde{n}$

$$D_{\tilde{a}\tilde{a}} : \ k_\parallel + k_\perp \mapsto k_\parallel + \cos \alpha k_\perp + \sin \alpha (0, \tilde{n} \times \tilde{k}_\perp) .$$

Vice versa, to the rotation $D_{\tilde{a}\tilde{a}}$ by $\alpha$ around the axis $\tilde{n}$ there corresponds the pair of unitary matrices $U = e^{-i\tilde{a} \tilde{n} \cdot \vec{\sigma}}$ and $-U = e^{-i\tilde{a} \tilde{n} \cdot \vec{\sigma}}$.

As $V_{\tilde{b}\tilde{a}} = V_{\tilde{b}\tilde{a}}^\dagger$ is hermitian and because $\tilde{k}_\perp$ anticommutes with $\tilde{n} \cdot \vec{\sigma}$, the transverse part is invariant

$$V (\tilde{k}_\perp \cdot \vec{\sigma}) V^\dagger = V V^{-1} (\tilde{k}_\perp \cdot \vec{\sigma}) = \tilde{k}_\perp \cdot \vec{\sigma}$$

while the longitudinal part $\tilde{k}_\parallel = k^0 - (\tilde{k} \cdot \tilde{n}) \tilde{n} \cdot \vec{\sigma}$ commutes and by $V_{\tilde{b}\tilde{a}}^2 = V_{2\tilde{b}\tilde{a}}$ transforms as

$$V_{\tilde{b}\tilde{a}} \tilde{k}_\parallel V_{\tilde{b}\tilde{a}}^\dagger = \tilde{k}_\parallel V_{2\tilde{b}\tilde{a}} = (k^0 \mathbf{1} - (\tilde{k} \cdot \tilde{n}) \tilde{n} \cdot \vec{\sigma}) (\text{ch} \beta \mathbf{1} - (\text{sh} \beta \tilde{n} \cdot \vec{\sigma})$$

$$= ((\text{ch} \beta) k^0 + (\text{sh} \beta) (\tilde{k} \cdot \tilde{n})) \mathbf{1} - ((\text{sh} \beta) k^0 + (\text{ch} \beta) (\tilde{k} \cdot \tilde{n})) \tilde{n} \cdot \vec{\sigma}$$

$$= k'^0 \mathbf{1} - (\tilde{k}' \cdot \tilde{n}) \tilde{n} \cdot \vec{\sigma} .$$

We conclude

$$\tilde{k}' = \tilde{k}_\perp, \quad \left( \begin{array}{c} k^0 \\ (\tilde{k} \cdot \tilde{n}) \tilde{n} \end{array} \right) = \left( \begin{array}{cc} \text{ch} \beta & \text{sh} \beta \tilde{n} \tilde{n}^T \\ \text{sh} \beta \tilde{n} & \text{ch} \beta \tilde{n} \tilde{n}^T \end{array} \right) \left( \begin{array}{c} k^0 \\ (\tilde{k} \cdot \tilde{n}) \tilde{n} \end{array} \right) .$$
So $V_{\beta \bar{\beta}}$ effects the boost $L_a$ \((55)\) in direction of $\vec{n} = \vec{u}/|\vec{u}|$ with rapidity $\beta$, $\sh{\beta} = |\vec{u}|$. To each boost $L_a$ there corresponds the pair of $V_{\beta \bar{\beta}}$ and $-V_{\beta \bar{\beta}}$ of SL\((2, \mathbb{C})\) matrices. SL\((2, \mathbb{C})\) is a double cover of SO\((1, 3)\)\(^\dagger\).

To calculate the Wigner rotation for massive particles

$$W(\Lambda, p) = L_{\Lambda u}^{-1} L_a, \quad p = m u$$

(103)

is a ‘Herculean task’ \cite{23} and requires ‘tedious manipulations’ \cite{22}. As we are unaware of a citable derivation, we present here a derivation. It simply reads the Wigner rotation from the product of SL\((2, \mathbb{C})\) matrices $\lambda$, $l_a$ and $w$

$$\lambda l_a = l_{\Lambda u} w(\Lambda, p)$$

(104)

which correspond to the Lorentz transformation $\Lambda$, the boosts $L_{\Lambda u}$, $L_{\Lambda u}$ and the Wigner rotation. Using the notation

$$l_a = \ch{a} - \sh{a} \vec{n}_a \cdot \sigma, \quad w = \cos \delta' - i \sin \delta' \vec{n} \cdot \sigma, \quad a' = \frac{a}{2}, \quad \delta' = \frac{\delta}{2}$$

(105)

the products of two boosts $l_{b l}$ and of a boost $l_{-c}$ with a rotation $w$ yield

$$\begin{align*}
(\ch{b'} - \sh{b'} \vec{n}_b \cdot \sigma) (\ch{a'} - \sh{a'} \vec{n}_a \cdot \sigma) &= \\
(\ch{b'} \ch{a'} + \sh{b'} \sh{a'} \vec{n}_a \cdot \vec{n}_b) - (\ch{a'} \sh{b'} \vec{n}_b + \ch{b'} \sh{a'} \vec{n}_a) \cdot \sigma + i \sh{b'} \sh{a'} (\vec{n}_b \times \vec{n}_a) \cdot \sigma, \\
(\ch{c'} + \vec{n}_c \cdot \sigma) (\ch{c'} + \vec{n}_c \cdot \sigma) &= \\
(\ch{c'} \cos \delta' - i \sh{c'} \sin \delta' \vec{n}_c \cdot \vec{n}) (\ch{c'} + \vec{n}_c \cdot \sigma) + \sh{c'} (\cos \delta' \vec{n}_c + \sin \delta' \vec{n}_c \times \vec{n}) \cdot \sigma + \\
- i \sh{c'} \sin \delta' \vec{n} \cdot \sigma.
\end{align*}$$

(106)

(107)

Both products are equal, $l_{b l} l_a = l_{-c} w$, if the complex coefficients of the linearly independent matrices $\mathbf{1}$ and the $\sigma$-matrices match. Comparing the coefficients determines $w$ without knowing $l_{-c}$ explicitly.

The coefficients of $\mathbf{1}$ agree only if $\vec{n} \cdot \vec{n}_c = 0$. The axis $\vec{n}$ of the Wigner rotation is orthogonal to the direction $\vec{n}_c$ of the resulting boost; $\vec{n}_c$ lies in the plane spanned by $\vec{n}_a$ and $\vec{n}_b$, as it must: $l_b$ and $l_a$ are in the subgroup SO\((1, 2)\) of boosts and rotations in this plane.

With $\vec{n}_b \cdot \vec{n}_a = \cos \gamma$ and $\vec{n}_a \times \vec{n}_b = \vec{n} \sin \gamma$, where $\gamma$ is the angle included by $\vec{n}_a$ and $\vec{n}_b$, the comparison of the coefficients of $\mathbf{1}$ and of $\vec{n} \cdot \sigma$ yields

$$\begin{align*}
(\ch{c'}) (\cos \delta') &= (\ch{b'}) (\ch{a'}) + (\sh{b'}) (\sh{a'}) (\cos \gamma), \\
(\ch{c'}) (\sin \delta') &= (\sh{b'}) (\sh{a'}) (\sin \gamma).
\end{align*}$$

(108)
The ratio of both equations determines the looked for angle $\delta = 2\delta'$ \[23\]

$$\tan \frac{\delta}{2} = \frac{\sin \gamma}{(\coth \frac{a}{2} \coth \frac{b}{2}) + \cos \gamma}.$$ \hspace{1cm} (109)

It has the same sign as $\gamma$, the Wigner rotation $W(L_u, p)$ rotates from $\vec{p}$ to $\vec{u}$. In the hyperbolic plane\[7\] the area of the hyperbolic triangle is this deficit angle $\delta = \pi - \alpha - \beta - \gamma$[10]. The area is bounded by $\pi$ though the sides of the hyperbolic triangle can be arbitrarily long.

### B. Iwasawa Decomposition

By the Schmidt orthogonalization procedure each matrix $g \in \text{SL}(d, C)$ decomposes continuously into a product of 3 factors, $g = kan$, where $k$ is from $K = \text{SU}(d)$, which is compact, $a$ is real and diagonal and from a noncompact abelian subgroup $A$ and $n$ is lower triangular from a subgroup $N$ of exponentials of nilpotent matrices. Elementary algebra confirms such an Iwasawa decomposition of $g \in \text{SL}(2, C)$ to be

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \delta^* & \beta \\ -\beta^* & \delta \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$ \hspace{1cm} (110)

where $ad - bc = 1$ and

$$r = \sqrt{|c|^2 + |d|^2}, \quad \beta = \frac{c}{r}, \quad \delta = \frac{d}{r}, \quad z = \frac{ac^* + bd^*}{r^2}.$$ \hspace{1cm} (111)

The first factor in the decomposition corresponds to a rotation, the second to a boost in 3-direction and the third matrix represents an E(2)-translation \[55\].

Decompositions of the group $\text{SL}(d, C) = KAN$ into into a product of a maximal compact subgroup $K$, an abelian subgroup $A$ and a nilpotent subgroup $N$ are equivalent, $K' = sKs^{-1}, A' = sKs^{-1}$ and $N' = sNs^{-1}$. For fixed groups $K, A$ and $N$ each group element $g = kan$ factorizes uniquely.

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