ON GENERATING SOME KNOWN BLACK HOLE SOLUTIONS

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ABSTRACT: In this paper, we have presented an algorithm to generate various black hole solutions in general relativity and alternative theories of gravity. The algorithm involves few dimensional parameters that are assigned suitable values to specify the required black hole.
I. INTRODUCTION

A black hole, as known since several decades, is formed as a result of continued gravitational collapse of matter, consisting of a singularity covered by an event horizon. It represents a region of spacetime where a test particle is trapped in such a way that its escape speed exceeds the speed of light. A black hole is formed when a pressureless cloud of dust with a spherical symmetry, undergoes a collapse which is governed by gravity only. In this scenario, the pressure gradients are ignored and the collapse proceeds continuously till a singularity (refers to curvature singularity only, here and onwards) is formed. Under the assumptions of spherical symmetry, vanishing pressure gradients and asymptotic flatness, the formation of event horizon becomes inevitable. If any of these conditions is not satisfied, the collapse leads to a naked singularity where the horizon either never forms or it shrinks down to the singularity. The concept of a stationary black hole came from Einstein’s theory of general relativity. The very first solution of Einstein field equations derived by K. Schwarzschild represented a point source of infinite gravitational field in an empty asymptotically flat spacetime [1, 2]. Later on, people constructed other black hole solutions with the inclusion of extra parameters like charge and angular momentum [3] (see also [4] for a thorough collection and history of black hole solutions).

Since the genesis of black hole paradigm, people have been interested in the quest of answering whether naked singularities (singularities without horizons) could exist or not. Penrose [2, 5] put forth a ‘cosmic censorship conjecture’ which suggested that singularities must always be hidden inside horizon. Although this conjecture has neither been proved nor disproved yet has got much attention. If naked singularities exist, then these will be observable to distant observers and hence will be effective sources to test Einstein’s theory of relativity, in general and other theories of quantum gravity, in particular. In a typical gravitational collapse model, the horizon forms first and the singularity later, so that horizon encapsulates the singularity. A naked singularity could form if the collapse is so slow that the formation of a horizon is delayed [6]. Moreover, a naked singularity can also form if the gravitational collapse is not spherically symmetrical [7]. In the context of phantom cosmology, a naked singularity is also formed when a black hole accretes phantom
energy which possesses negative energy density. Due to its accretion, the mass of black hole decreases eventually leading to the appearance of a naked singularity \[8, 9\]. There has been pioneering work done by Virbhadra and his collaborators regarding the observational properties and detection of black holes and naked singularities using gravitational lensing techniques \[10, 11, 12, 13, 14, 15, 16\]. They have classified naked singularities into two kinds: ‘weakly naked’ which were hidden under single photon sphere and the ‘strongly naked’ which were hidden under no photon sphere. Interestingly, the qualitative features of gravitational lensing due to a weakly naked singularity turned out to be similar to the Schwarzschild black hole.

In this article, we are trying to provide a prescription how to get some known black holes existing in literature by monitoring transverse pressure only. That means one requires the knowledge of one of the components of energy momentum tensor, more exactly, the transverse pressure to generate some of the known black holes, namely, Schwarzschild, Schwarzschild de Sitter, Reissner-Nordström, Reissner-Nordström de Sitter black holes etc. Recently several authors have proposed several algorithms to obtain spherically symmetric solutions as well as black hole solutions \[17, 18, 19, 20, 21, 22, 23, 24\]. Our approach is simple and interesting because the algorithm involves few dimensional parameters that are assigned suitable values to generate the required black hole.

## II. BASIC EQUATIONS AND THE ALGORITHM

Let us consider an anisotropic matter distribution corresponding to the line element

\[
 ds^2 = -e^{\nu(r)} dt^2 + e^{-\nu(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\] (1)

The Einstein field equations for the above spherically symmetric metric are given by

\[
 - e^{\nu} \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right] + \frac{1}{r^2} = 8\pi \rho, 
\] (2)

\[
 - e^{\nu} \left[ \frac{1}{r^2} + \frac{\nu'}{r} \right] + \frac{1}{r^2} = 8\pi p_r, 
\] (3)

\[
 \frac{1}{2} e^{\nu} \left[ (\nu')^2 + \nu'' + \frac{2\nu'}{r} \right] = 8\pi p_t. 
\] (4)
Here $\rho$, $p_r$ and $p_t$ are energy density, radial and transverse pressures, respectively. The prime denotes derivative with respect to parameter $r$.

Eqs. (2) and (3) imply

$$\rho = p_r.$$  \hspace{1cm} (5)

Eq. (4) yields

$$2r^2(8\pi p_t) = \frac{d}{dr} \left( r^2 e^\nu \nu' \right).$$  \hspace{1cm} (6)

Integrating the above equation, one obtains

$$r^2(e^\nu)' = \int 2r^2(8\pi p_t)dr + D.$$  \hspace{1cm} (7)

Here $D$ is an integration constant. Integrating the above equation once more, one gets

$$e^\nu = E - D r + \int \left[ \frac{1}{r^2} \int 2r^2(8\pi p_t)dr \right] dr,$$  \hspace{1cm} (8)

where $E$ is an integration constant. Thus one gets black hole solutions by monitoring only one function $p_t$.

Let us assume the form of $p_t$ as

$$8\pi p_t = A + \frac{Q}{r^n},$$  \hspace{1cm} (9)

where $A$, $Q$ and $n$ are arbitrary constants. Using the above form of $p_t$ in equation (8), one gets

$$e^\nu = E - \frac{D}{r} + \frac{A}{3} r^2 + \frac{2Q}{(n-2)(n-3)r^{n-2}}.$$  \hspace{1cm} (10)

Eq. (2) yields

$$8\pi \rho = -A + \frac{(1 - E)}{r^2} + \frac{2Q}{(n-2)r^n}.$$  \hspace{1cm} (11)

Thus we obtain a general class of black hole solutions supported by the anisotropic fluid distribution given in (9) and (11). Here one can note that $n \neq 2, 3$. For $n = 2$ and $n = 3$, one has to use equation (8) directly to obtain metric coefficient.

**A. Schwarzschild Black hole**

A Schwarzschild black hole represents a point source of mass $M$ of infinite gravitation in an empty spacetime. The spacetime is spherically symmetric and possesses a curvature
singularity at $r = 0$ which is hidden inside an event horizon located at $r = 2M$. This solution is obtained from our algorithm if we put $A = Q = 0$, $D = 2M$ and $E = 1$, then the above solutions imply $p_r = p_t = \rho = 0$ and $e^\nu = 1 - \frac{2M}{r}$. Vanishing pressures and densities imply that the exterior of a black hole is nothing but vacuum. This solution is highly idealized since most black holes are rotating and possibly surrounded by matter like stars and dust.

B. Schwarzschild - de Sitter Black hole (SdS)

A SdS spacetime represents a Schwarzschild black hole immersed in cosmological constant (generally denoted by $\Lambda$) dominated universe. The parameter $\Lambda$ is an ad hoc term added in the Einstein field equations to obtain a matter distribution having negative pressure. This matter is usually termed as the dark energy which is homogeneous and isotropic perfect fluid. In the presence of this fluid, the spacetime outside the black hole expands with acceleration. If we choose $Q = 0$, $D = 2M$ and $E = 1$, then the above solution implies $8\pi p_r = -8\pi p_t = 8\pi \rho = -A$ and $e^\nu = 1 - \frac{2M}{r} + \frac{4}{3}r^2$. Here, $A$ plays the role of cosmological constant.

C. Reissner-Nordström Black hole (RN)

The RN black hole represents a spherically symmetric spacetime containing a mass $M$ and charge $Q$. The spacetime is singular at $r = 0$ hidden under two horizons $r_{h\pm} = M \pm \sqrt{M^2 - Q^2}$. If $Q^2 = M^2$, then the spacetime represents an extreme RN black hole while if $Q^2 > M^2$, it yields a naked singularity at $r = 0$. If we adopt $A = 0$, $D = 2M$, $n = 4$, $Q > 0$ and $E = 1$, then the above solutions imply $8\pi p_r = 8\pi p_t = 8\pi \rho = \frac{Q}{r}$ and $e^\nu = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. One can note that the above solutions correspond to braneworld black hole for $Q < 0$. From the astrophysical point of view, charged black holes are not of much interest since these get neutralized by interacting with the neighboring charged clouds and plasmas.
D. Reissner-Nordström - de Sitter Black hole (RNdS)

The RNdS spacetime represents a RN black hole in a cosmological constant dominated universe. The spacetime is singular at \( r = 0 \) hidden under multiple horizons. If we take \( D = 2M, n = 4 \) and \( E = 1 \), then the above solutions imply \( 8\pi p_t = A + \frac{Q}{r} \) and \( 8\pi p_r = 8\pi \rho = -A + \frac{Q}{r} \) and \( e^\nu = 1 - \frac{2M}{r} + \frac{Q}{r^2} + \frac{A}{3} r^2 \). For \( A < 0 \), case 1 and 2 indicate anti de Sitter type solutions. Thus sign of \( A \) gives dS/AdS black holes.

E. Black hole surrounded by quintessence

Recently, Kiselev \[25\] have obtained a new black solution surrounded by quintessence. The quintessence is represented by a homogeneous and time dependent scalar field. The equation of state parameter \( \omega_q \) of quintessence varies between \(-1\) and \(-1/3\). Since \( \omega_q < 0 \), it serves as an alternative candidate for dark energy. If we insert \( D = 2M, A = 0, n = 3(\omega_q+1), Q = \frac{6c\omega_q(3\omega_q+1)}{2} \) and \( E = 1 \), then the above solutions imply \( 8\pi p_t = \frac{3c\omega_q(3\omega_q+1)}{2r(3\omega_q+1)} \) and \( 8\pi p_r = 8\pi \rho = \frac{c(3\omega_q+2)}{r(3\omega_q+1)} \) and \( e^\nu = 1 - \frac{2M}{r} + \frac{c}{r(3\omega_q+1)} \).

III. FURTHER OBSERVATIONS

Our approach is interesting in the sense that the transverse pressure would give similar black hole spacetimes like modified theories of Einstein’s theory give. We provide few examples:
A. Example

Beato et al. \[26\] have obtained a new black hole solution using nonlinear electrodynamics as
\[
e^\nu = 1 - \frac{2m[1 - \tanh(\frac{q^2}{2mr})]}{r}.
\] (12)

Here \(m\) and \(q\) are respectively the mass and charge of the black hole. Using our approach, one will see the following form of the transverse pressure gives the very similar black hole spacetime.
\[
8\pi p_t = \frac{1}{\cosh^2(\frac{q^2}{2mr})r^4} \left[ q^2 + \frac{q^4}{4mr} \tanh(\frac{q^2}{2mr}) \right].
\] (13)

The other components of the energy stress tensor are given by
\[
8\pi p_r = 8\pi \rho = \frac{1}{r^2} - \frac{q^2}{r^4[\cosh(\frac{q^2}{2mr})]^2}.
\] (14)

B. Example

Bardeen \[27\] have obtained non singular black hole solution as
\[
e^\nu = 1 - \frac{mr^2}{(r^2 + q^2)^\frac{3}{2}}.
\] (15)

As above \(m\) and \(q\) are respectively the mass and charge of the black hole. Using our approach, one will see the following form of the transverse pressure provides the very similar black hole spacetime.
\[
8\pi p_t = \frac{m}{2(r^2 + q^2)^\frac{3}{2}} \left[ 10r^4 - 5q^2r^2 - 6q^4 \right].
\] (16)

Here, the other components of the energy stress tensor are found to be
\[
8\pi p_r = 8\pi \rho = \frac{3mq^2}{2(r^2 + q^2)^\frac{3}{2}}.
\] (17)
C. Example

Dymnikova [28] have obtained another non singular black hole solution as

$$e^\nu = 1 - \frac{2m[1 - e^{(-\frac{r^3}{2mr_0^3})}]}{r}. \quad (18)$$

Here, $m$ is mass and $r_0^2 = \frac{3}{\Lambda}$, $\Lambda$ is related to the positive cosmological constant. Using our approach, one will see the following form of the transverse pressure gives the very similar black hole spacetime.

$$8\pi p_t = \frac{3}{r^2} e^{-\frac{r^3}{2mr_0^3}} \left[ \frac{3r^3}{4mr_0^2} - 1 \right]. \quad (19)$$

Here, the other components of the energy stress tensor are obtained as

$$8\pi p_r = 8\pi \rho = \frac{3}{r^2} e^{-\frac{r^3}{2mr_0^3}}. \quad (20)$$

D. Example

Chamblin et al [29] have obtained charged brane world black hole solution as

$$e^\nu = 1 - \frac{2GM}{r} + \frac{Q^2 + \beta}{r^2} + \frac{l^2Q^4}{20r^6}. \quad (21)$$

Here, $M$ and $Q$ are respectively the mass and charge of the black hole. $l$, $\beta$ are related to the bulk cosmological constant and five dimensional mass parameter respectively. Using our approach, one will see the following form of the transverse pressure gives birth the very similar black hole spacetime

$$8\pi p_t = \frac{Q^2 + \beta}{r^4} - \frac{3l^2Q^4}{4r^8}. \quad (22)$$

The other components of the energy stress tensor are

$$8\pi p_r = 8\pi \rho = \frac{Q^2 + \beta}{r^4} - \frac{l^2Q^4}{4r^8}. \quad (23)$$
E. Example

Virbhadra et al. [30] provided a conformal scalar dyon black hole which is an exact solution of Einstein - Maxwell field equations and is characterized by the scalar charge ($q_s$), electric charge ($q_e$) and magnetic charge ($q_m$) as

$$e^\nu = (1 - \frac{Q_{CSD}}{r})^2,$$

where $Q_{CSD}^2 = q^2_s + q^2_e + q^2_m$.

Our approach shows that the following form of the transverse pressure gives the very similar black hole spacetime.

$$ (8\pi p_t) = \frac{Q_{CSD}^2}{r^4}. $$

Here, the remaining components of the energy stress tensor are

$$ 8\pi p_r = 8\pi \rho = \frac{Q_{CSD}^2}{r^4}. $$

IV. FINAL REMARKS

In this paper, we have presented a scheme of generating some known spherically symmetric black hole solutions satisfying Eq. (1). Other interesting solutions like Kerr, Kerr-Neumann and Janis-Newman-Winicour solutions are not obtained from our scheme. Though we do not provide any new black hole solutions, despite we give a clue how one can get some black hole spacetimes. We would also mention that the solutions discussed in this paper may not necessarily black holes and could be naked singularities. The differentiation between these solutions can decisively be obtained through more rigorous analysis. Our approach may be applied to generate various black hole solutions in general relativity and alternative theories of gravity [32].
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