GROWTH OF POLYNOMIALS NOT VANISHING IN A DISK

KHANGEMBAM BABINA DEVI*, MAISNAM TRIVENI DEVI, N. REINGACHAN, BARCHAND CHANAM

Department of Mathematics, National Institute of Technology Manipur, Langol 795004, India

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Abstract. This paper deals with the problem of finding some upper bound estimates for the maximal modulus of a lacunary polynomial on a disk of radius \( R \), \( R \geq 1 \) under the assumption that the polynomial does not vanish in another disk with radius \( k \), \( k \geq 1 \). Our results sharpen as well as generalize a result recently proved by Hussain [Indian J. Pure Appl. Math., (https://doi.org/10.1007/s13226-021-00169-7)]. Further, these results generalize as well as sharpen some known results in this direction.

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1. INTRODUCTION

For a polynomial \( p(z) = \sum_{v=0}^{n} a_v z^v \) of degree \( n \) and \( R > 0 \), set \( M(p, R) = \max_{|z|=R} |p(z)| \). We denote \( M(p, 1) \) by \( \| \ p \ \| \), the uniform norm of a polynomial \( p \) on the unit disk \( |z| = 1 \). The study of inequalities that relate the norm of a polynomial on a larger disk to that of its norm on the unit disk and their various versions is a classical topic in analysis. Over a period, these inequalities have been generalized in different domains and in different norms. It is a simple deduction from the maximum modulus principle (see [13], p. 158) that for \( R \geq 1 \),

\[
M(p, R) \leq R^n \| p \| .
\]
Equality holds in (1) only for \( p(z) = \lambda z^n \), \( \lambda \neq 0 \) being a complex number. Noting that these extremal polynomials have all zeros at the origin, it is natural to seek improvements under appropriate condition on the zeros of \( p(z) \). It was shown by Ankeny and Rivlin [1] that if \( p(z) \) is a polynomial having no zero in \( |z| < 1 \), then inequality (1) can be replaced by

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \| p \| .
\]

Inequality (2) is sharp and equality holds for \( p(z) = \lambda + \mu z^n \) with \( |\lambda| = |\mu| \).

As a refinement of (2), it was shown by Govil [8] that if \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zero in \( |z| < 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \| p \| - \frac{n}{2} \left( \| p \|^2 - 4|a_n|^2 \right) \times \left[ \frac{(R - 1) \| p \|}{\| p \| + 2|a_n|} - \ln \left\{ 1 + \frac{(R - 1) \| p \|}{\| p \| + 2|a_n|} \right\} \right].
\]

Inequality (3) was sharpened by Dewan and Bhat [3], which was later generalized by Govil and Nyuydinkong [9] that if \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zero in \( |z| < 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^{n+k}}{1+k} \right) \| p \| - \left( \frac{R^{n-1}}{1+k} \right) m - \frac{n}{1+k} \left( \frac{(\| p \|-m)^2 - (1+k)^2|a_n|^2}{\| p \|-m} \right) \times \left[ \frac{(R-1)(\| p \|-m)}{\| p \|-m + (1+k)|a_n|} - \ln \left\{ 1 + \frac{(R-1)(\| p \|-m)}{\| p \|-m + (1+k)|a_n|} \right\} \right],
\]

where \( m = \min_{|z|=k} |p(z)| \).

Inequality (4) was generalized by Gardner et al. [5] in a different direction by considering polynomials of the form \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \). More precisely, Gardner et al. [5] proved the following result.

**Theorem 1.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \( |z| < k \), \( k \geq 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + k^\mu}{1+k^\mu} \right) \| p \| - \left( \frac{R^{n-1}}{1+k^\mu} \right) m - \frac{n}{1+k^\mu} \left( \frac{(\| p \|-m)^2 - (1+k^\mu)^2|a_n|^2}{\| p \|-m} \right) \times \left[ \frac{(R-1)(\| p \|-m)}{(\| p \|-m + (1+k^\mu)|a_n|)} - \ln \left\{ 1 + \frac{(R-1)(\| p \|-m)}{(\| p \|-m + (1+k^\mu)|a_n|)} \right\} \right],
\]

(5)
where $m = \min_{|z|=k}|p(z)|$.

Inequality (4) of Govil and Nyuydinkong [9] is a special case of Theorem 1, when $\mu = 1$. For $k = \mu = 1$, Theorem 1 reduces to the result of Dewan and Bhat [3], which is a sharpening of inequality (3). Very recently, Dalal and Govil [2] used a recurrence relation and proved the following sharpening of inequality (3).

**Theorem 2.** If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for $R \geq 1$ and any $1 \leq N \leq n$, 

$$M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \| p \| - n \left( \frac{2|a_n|}{\| p \|} \right) \left( 1 - \frac{n}{2} \right) h(N),$$

where

$$h(N) = (R - 1) - \left( 1 + \frac{2|a_n|}{\| p \|} \right) \ln \left\{ 1 + \frac{(R - 1)\| p \|}{\| p \| + 2|a_n|} \right\} \text{ for } N = 1,$$

$$h(N) = \left( \frac{R^n - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{n-v} - 1}{N - v} \right) (-1)^v \left( 1 + \frac{2|a_n|}{\| p \|} \right) \left( \frac{2|a_n|}{\| p \|} \right)^{v-1}$$

$$+ (-1)^N \left( 1 + \frac{2|a_n|}{\| p \|} \right) \left( \frac{2|a_n|}{\| p \|} \right)^{N-1} \ln \left\{ 1 + \frac{(R - 1)\| p \|}{\| p \| + 2|a_n|} \right\} \text{ for } N \geq 2.$$ 

For $k = 1$, it is obvious from Lemma 10 that $|a_n| \leq \frac{\| p \|}{2}$, and by Lemma 6, the function $h(N)$ is a non-negative increasing function of $N$, $1 \leq N \leq n$, it easily follows that the bound in (6) is sharper than that obtained from (3). In 2005, Gardner et al. [6] used the coefficients of the polynomial $p(z)$ and proved the following generalization and refinement of inequality (3).

**Theorem 3.** If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k, k \geq 1$, then for $R \geq 1$,

$$M(p, R) \leq \left( \frac{R^n + s_1}{1 + s_1} \right) \| p \| - \left( \frac{R^n - 1}{1 + s_1} \right) m - \frac{n}{1 + s_1} \left\{ \frac{(\| p \| - m)^2}{\| p \| - m} - (1 + s_1)^2 |a_n|^2 \right\},$$

$$\times \left[ \frac{(R - 1)(\| p \| - m)}{(\| p \| - m) + (1 + s_1)|a_n|} - \ln \left\{ 1 + \frac{(R - 1)(\| p \| - m)}{(\| p \| - m) + (1 + s_1)|a_n|} \right\} \right]$$

where

$$s_1 = k^{\mu + 1} \left\{ \frac{\mu}{n} \left( \frac{|a_\mu|}{|a_0| - m} k^{\mu - 1} + 1 \right) \right\}.$$
and \( m = \min_{|z|=k} |p(z)| \).

For \( k = 1, \mu = 1 \), we have \( s_1 = 1 \), Theorem 3 reduces to the result of Dewan and Bhat [3]. Although the literature on polynomial inequalities involving growth is vast and growing over the last four decades many different authors produced a large number of research papers and monographs on such inequalities. One can see in the literature (for example, refer [2, 5, 6, 11]), the recent research and development in this direction.

2. **Lemmas**

We need the following lemmas for the proof of the theorem.

**Lemma 1.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \), then for \( |z| = R \geq 1 \),

\[
|p(z)| \leq R^n \left\{ 1 - \frac{(\|p\| - |a_n|)(R - 1)}{|a_n| + R \|p\|} \right\} \|p\|.
\]

**Lemma 2.** If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k > 0 \), then

\[
|p(z)| \geq m, \text{ for } |z| \leq k,
\]
where \( m = \min_{|z|=k} |p(z)| \).

The above Lemmas 1 and 2 are due to Gardner et al. [6].

**Lemma 3.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1 \), then

\[
\|p'\| \leq \frac{n}{1 + s_\mu} \|p\|,
\]
where

\[
s_\mu = \frac{k^{\mu+1}(\frac{\mu}{n} \frac{a_\mu}{a_0} |k^{\mu-1} + 1|)}{\frac{\mu}{n} \frac{a_\mu}{a_0} |k^{\mu+1} + 1|}.
\]

Inequality (9) is sharp and equality holds for the polynomial \( p(z) = (z^\mu + k^\mu)^\frac{n}{\mu} \), where \( n \) is a multiple of \( \mu \).

Lemma 3 is due to Qazi [14, Lemma 1].
**Lemma 4.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) and \( r \geq 1 \), then the function
\[
\left\{ 1 - \frac{(y-n|a_n|)(r-1)}{ry+n|a_n|} \right\} y
\]
is an increasing function of \( y \) for \( y > 0 \).

Lemma 4 is due to Mir et al. [12, Lemma H]

**Lemma 5.** Let
\[
h(N) = \int_{1}^{R} \frac{(r-1)r^{N-1}}{r+x} dr, \quad x > 0.
\]
Then, for \( N \geq 2 \),
\[
h(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{\nu=1}^{N-1} \left( \frac{R^\nu - 1}{N-\nu} \right) (-1)^\nu (x+1)x^{\nu-1}
\]
\[
+ (-1)^N (x+1)x^{N-1} \ln \left( \frac{R+x}{1+x} \right),
\]
(11)
and for \( N = 1 \),
\[
h(1) = (R-1) - (1+x) \ln \left( 1 + \frac{R-1}{1+x} \right).
\]
(12)

*Proof of Lemma 5.* Although the proof of this lemma was done by Dalal and Govil [2, Lemma 3.6], however, we include it here for the sake of completeness and for reader’s convenience.

For any \( x > 0 \), let
\[
u(N) = \int_{1}^{R} \frac{r^N}{r+x} dr,
\]
which is defined for \( N \geq 0 \). Then it is easy to see that \( h(N) = u(N) - u(N-1) \) for \( N \geq 1 \). Note that for \( N \geq 2 \),
\[
u(N) + xu(N-1) = \int_{1}^{R} \frac{r^N + x r^{N-1}}{r+x} dr
\]
\[
= \int_{1}^{R} \frac{r^{N-1}(r+x)}{r+x} dr
\]
\[
= \frac{R^N - 1}{N}
\]
\[
= w(N).
\]
Therefore,

\[(13) \quad u(N) = w(N) - xu(N - 1),\]

for \(N \geq 2\) and on solving the recurrence relation, we have

\[(14) \quad u(N) = \sum_{v=0}^{N-1} (-1)^v w(N-v)x^v + (-1)^N x^N u(0), \quad N \geq 1,\]

where

\[
u(0) = \int_1^R \frac{1}{r+x} dr = \ln \left( \frac{R+x}{1+x} \right).
\]

Substituting the value of \(u(0)\) in (14), we have

\[(15) \quad u(N) = \sum_{v=0}^{N-1} (-1)^v w(N-v)x^v + (-1)^N x^N \ln \left( \frac{R+x}{1+x} \right), \quad N \geq 1.
\]

Using \(h(N) = u(N) - u(N - 1)\) and value of \(w(N) = \frac{R^{N-1}}{N}\), we have for \(N \geq 2\),

\[
h(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N-v} \right) (-1)^v (1+x)x^{v-1}
\]

\[(16) \quad + \ (-1)^N (1+x)x^{N-1} \ln \left( \frac{R+x}{1+x} \right).
\]

For \(N = 1\), we have

\[
h(1) = \int_1^R \frac{(r-1)}{r+x} dr
\]

\[
= (R - 1) - (1+x) \ln \left( 1 + \frac{R - 1}{1+x} \right).
\]

This completes the proof of Lemma 5. \(\square\)

**Lemma 6.** The function \(h(N)\) defined in Lemma 5 is a non-negative increasing function of \(N\) for \(N \geq 1\).

**Proof of Lemma 6.** The proof of this lemma was done by Dalal and Govil [2, Lemma 3.7], however, a simple alternative proof is presented here.

Using the method of differentiation under the integral sign, we have

\[(17) \quad \frac{d}{dN} h(N) = \int_1^R \frac{(r-1)(r^{N-1})}{r+x} \ln r dr.
\]
Since, for \( r \in [1, R] \), \( \frac{(r-1)r^{N-1}}{r+x} \ln r \geq 0 \), therefore, we have
\[
\int_1^R \frac{(r-1)r^{N-1}}{r+x} \ln r dr \geq 0.
\]
From equality (17),
\[
\frac{d}{dN} h(N) \geq 0, \text{ for } N \geq 1.
\]
Therefore, \( h(N) \) is an increasing function of \( N \) for \( N \geq 1 \).

Further, since \( \frac{(r-1)r^{N-1}}{r+x} \geq 0 \) for \( N \geq 1 \), \( h(N) \geq 0 \) for \( N \geq 1 \).

This completes the proof of Lemma 6. \( \square \)

**Lemma 7.** If \( p(z) = a_0 + \sum_{v=\mu}^n a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \( |z| < k \), \( k \geq 1 \), then the function
\[
f(u) = \frac{1 + \overline{a_\mu} u^{k+1}}{1 + k^{\mu+1} + \frac{n}{\mu} \overline{a_\mu} u^{k^{\mu+1} + k^{2\mu}}}
\]
is a non-increasing function of \( u \) for \( u > 0 \).

**Proof of Lemma 7.** Considering the first derivative of \( f \) w.r.t \( u \), we have
\[
f'(u) = \frac{\frac{n \overline{a_\mu}}{u} k^{2\mu} (1 - k^2)}{\left\{ 1 + k^{\mu+1} + \frac{n}{\mu} \overline{a_\mu} u^{k^{\mu+1} + k^{2\mu}} \right\}^2},
\]
which is non-positive, since \( (1 - k^2) \leq 0 \) for \( k \geq 1 \), and hence \( f(u) \) is a non-increasing function of \( u \). \( \square \)

**Lemma 8.** If \( p(z) = a_0 + \sum_{v=\mu}^n a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \( |z| < k \), \( k \geq 1 \), then for any real or complex number \( \lambda \) with \( |\lambda| < 1 \)
\[
|a_n| \leq \frac{1}{1 + s_2} (||p|| - |\lambda|m),
\]
where \( s_2 \) is as defined in (22) and \( m = \min_{|z|=k} |p(z)| \).

**Proof of Lemma 8.** Since \( p(z) = a_0 + \sum_{v=\mu}^n a_v z^v \), \( 1 \leq \mu \leq n \), then \( p'(z) = \sum_{v=\mu}^n va_v z^{v-1} \).

Hence, applying Cauchy’s inequality to \( p'(z) \) on the unit circle \( |z| = 1 \), we have
\[
\left| \frac{d^{n-1}}{dz^{n-1}} p'(z) \right|_{z=0} \leq (n-1)! \max_{|z|=1} |p'(z)|.
\]
That is,

\[(20) \quad |n a_n | \leq \| p' \| .\]

Combining inequality (27) in the proof of Theorem 4 and (20), we have inequality (19) and this completes the proof of Lemma 8. \(\square\)

The next lemma is due to Gardner et al. [4, Lemma 3].

**Lemma 9.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, \ 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1, \) then

\[ s_1 \geq k^\mu , \]

where \( s_1 \) is as defined in Theorem 3.

**Lemma 10.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) in \( |z| < k, k \geq 1, \) then

\[ |a_n| \leq \frac{1}{1 + k^\mu} (\|p\| - m) , \]

where \( m = \min_{|z|=k} |p(z)|. \)

Lemma 10 is due to Gardner et al. [5].

3. **MAIN RESULTS**

In this paper, we prove a result which is a generalization of Theorems 3 due to Gardner et al. [6]. More precisely, we prove

**Theorem 4.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1, \) then for any real or complex number \( \lambda \) with \( |\lambda| < 1, R \geq 1 \) and any positive integer \( N, 1 \leq N \leq n, \)

\[(21) \quad M(p, R) \leq \left( \frac{R^n + s_2}{1 + s_2} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_2} - n \left\{ \frac{\|p\| - |\lambda| m}{1 + s_2} - |a_n| \right\} S(N), \]

where

\[(22) \quad s_2 = \frac{k^{\mu+1} (\frac{\mu}{n} \frac{a_n}{a_0 + \lambda m} |k^{\mu-1} + 1|)}{\frac{\mu}{n} \frac{a_n}{a_0 + \lambda m} |k^{\mu+1} + 1|}. \]
and

\[
S(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + s_2)|a_n|}{\| p \| - |\lambda| m} \right\} \\
\times \ln \left\{ 1 + \frac{(R - 1)(\| p \| - |\lambda| m)}{\| p \| - |\lambda| m + (1 + s_2)|a_n|} \right\} \text{ for } N = 1,
\]

(23)

\[
S(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^N - v - 1}{N - v} \right) (-1)^v \left\{ 1 + \frac{(1 + s_2)|a_n|}{\| p \| - |\lambda| m} \right\} \left\{ (1 + s_2)|a_n| \right\}^{v-1} \\
+ (-1)^N \left\{ 1 + \frac{(1 + s_2)|a_n|}{\| p \| - |\lambda| m} \right\} \left\{ (1 + s_2)|a_n| \right\}^{N-1} \\
\times \ln \left\{ 1 + \frac{(R - 1)(\| p \| - |\lambda| m)}{\| p \| - |\lambda| m + (1 + s_2)|a_n|} \right\} \text{ for } N \geq 2,
\]

(24)

and \( m = \min_{|z|=k} |p(z)| \).

**Proof of Theorem 4.** Since the polynomial \( p(z) = a_0 + \sum_{\mu=1}^{n} a_\mu z^\mu \), \( 1 \leq \mu \leq n \), has no zero in \( |z| < k \), \( k \geq 1 \), therefore for every real or complex number \( \lambda \) with \( |\lambda| < 1 \), by Rouche’s Theorem the polynomial \( p(z) + \lambda m \) has no zero in \( |z| < k \), \( k \geq 1 \), where \( m = \min_{|z|=k} |p(z)| \).

Applying Lemma 3 to \( p(z) + \lambda m \), we have

\[
\| p' \| \leq \frac{n \| p(z) + \lambda m \|}{1 + s_2},
\]

(25)

where \( s_2 \) is as defined in (22).

Now, we choose the argument of \( \lambda \) suitably such that

\[
|p(z) + \lambda m| = |p(z)| - |\lambda| m.
\]

(26)

Using inequality (26) to (25), we have

\[
\| p' \| \leq \frac{n(\| p \| - |\lambda| m)}{1 + s_2}.
\]

(27)

Then, for each \( \theta \), \( 0 \leq \theta < 2\pi \) and \( 1 \leq r \leq R \), we have

\[
p(Re^{i\theta}) - p(e^{i\theta}) = \int_{1}^{R} e^{i\theta} p'(re^{i\theta}) dr,
\]

which implies

\[
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \int_{1}^{R} |p'(re^{i\theta})| dr.
\]
Now, applying Lemma 1 to the polynomial $p'(z)$ which is of degree $n - 1$, we get

\begin{equation}
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \int_{1}^{R} r^{n-1} \left\{ 1 - \frac{(\|p\| - n|a_n|)(r - 1)}{n|a_n| + r\|p'\|} \right\} \|p'\| dr.
\end{equation}

By Lemma 4, the quantity \( \left\{ 1 - \frac{(\|p\| - n|a_n|)(r - 1)}{n|a_n| + r\|p'\|} \right\} \|p'\| \) occurring in the integrand of (28) is an increasing function of \( \|p'\| \), and hence using inequality (27) for the value of \( \|p'\| \), we have for \( 0 \leq \theta < 2\pi \),

\begin{equation}
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \int_{1}^{R} r^{n-1} \left[ 1 - \frac{n}{1+s_2} \frac{(\|p\| - |\lambda|m) - n|a_n|}{n|a_n| + r(\frac{n}{1+s_2})(\|p\| - |\lambda|m)} (r - 1) \right] \frac{n}{1+s_2} (\|p\| - |\lambda|m) dr
\end{equation}

\begin{equation}
= \frac{n}{1+s_2} (\|p\| - |\lambda|m) \int_{1}^{R} r^{n-1} dr - \frac{n}{1+s_2} (\|p\| - |\lambda|m) \int_{1}^{R} r^{n-1} \left\{ \frac{((\|p\| - |\lambda|m) - (1 + s_2)|a_n|)}{(1 + s_2)|a_n| + r(\|p\| - |\lambda|m)} \right\} (r - 1) dr
\end{equation}

\begin{equation}
= \frac{R^n - 1}{1+s_2} (\|p\| - |\lambda|m) - \frac{n}{1+s_2} (\|p\| - |\lambda|m)(1 - f) \int_{1}^{R} \frac{(r - 1)r^{n-1}}{r + f} dr,
\end{equation}

where \( s_2 \) is as defined in (22) and \( f = \frac{|a_n|}{(\|p\| - |\lambda|m)} \).

Note that from Lemma 6, the integral \( \int_{1}^{R} \frac{(r - 1)r^{n-1}}{r + f} dr \) is a non-negative and increasing function of \( N \) for \( 1 \leq N \leq n \), therefore, we have

\begin{equation}
\int_{1}^{R} \frac{(r - 1)r^{n-1}}{r + f} dr \leq \int_{1}^{R} \frac{(r - 1)r^{n-1}}{r + f} dr.
\end{equation}

Noting from Lemma 8 that \( (1 - f) \geq 0 \) and using inequality (31) to (30), we have for every \( N, 1 \leq N \leq n \),

\begin{equation}
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \frac{R^n - 1}{1+s_2} (\|p\| - |\lambda|m) - \frac{n}{1+s_2} (\|p\| - |\lambda|m)(1 - f) \int_{1}^{R} \frac{(r - 1)r^{n-1}}{r + f} dr.
\end{equation}

Using Lemma 5 (on replacing \( x \) by \( f \)) for the value of integral in (32), we have for each \( 0 \leq \theta < 2\pi \),

\begin{equation}
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \frac{R^n - 1}{1+s_2} (\|p\| - |\lambda|m) - \frac{n}{1+s_2} (\|p\| - |\lambda|m)(1 - f)S(N),
\end{equation}
where $S(N)$ is as defined in (23) and (24).

Now, substituting the value of $f$ and using the obvious inequality

$$|p(Re^{i\theta})| \leq |p(Re^{i\theta}) - p(e^{i\theta})| + |p(e^{i\theta})|$$

(34)

$$\leq |p(Re^{i\theta}) - p(e^{i\theta})| + \|p\|$$

in (33), we get for $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$|p(Re^{i\theta})| \leq \left(\frac{R^n + s_2}{1 + s_2}\right)\|p\| - \frac{(R^n - 1)|\lambda|m}{1 + s_2} - \frac{n(\|p\| - |\lambda|m)}{1 + s_2} \left\{1 - \frac{(1 + s_2)|a_n|}{\|p\| - |\lambda|m}\right\} S(N),$$

which is inequality (21) and hence the proof of Theorem 4 is completed. □

**Remark 1.** By Lemma 6, it is noted that $S(N)$ given by (23) and (24) of Theorem 4 is an increasing function of $N$, $1 \leq N \leq n$, therefore, the bound in Theorem 4 improves most when $N = n$, the degree of the polynomial.

Further, when $p(z)$ is a polynomial of degree $n = 1$, then we have from inequality (30) in the proof of Theorem 4 that

$$|p(Re^{i\theta})| \leq \left(\frac{R + s_2}{1 + s_2}\right)\|p\| - \frac{R - 1}{1 + s_2} |\lambda|m - \frac{1}{1 + s_2} \left\{\|p\| - |\lambda|m - (1 + s_2)|a_1|\right\} \int_1^R \frac{r - 1}{r + f} dr.$$

Substituting the value of $f = \frac{(1 + s_2)|a_1|}{\|p\| - |\lambda|m}$ to the integral of (35) and using equality (12) of Lemma 5, we have

$$|p(Re^{i\theta})| \leq \left(\frac{R + s_2}{1 + s_2}\right)\|p\| - \frac{R - 1}{1 + s_2} |\lambda|m - \frac{1}{1 + s_2} \left\{\|p\| - |\lambda|m - (1 + s_2)|a_1|\right\} S(1),$$

(36)

where

$$S(1) = (R - 1) - \left\{1 + \frac{(1 + s_2)|a_1|}{\|p\| - |\lambda|m} \right\} \ln \left\{1 + \frac{(R - 1)(\|p\| - |\lambda|m)}{\|p\| - |\lambda|m + (1 + s_2)|a_1|}\right\}.$$

On the other hand, we have by a simple calculation

(37) $$M(p, R) = \max_{|z|=R} |p(z)| = \max_{|z|=R} |a_0 + za_1| = |a_0| + R|a_1|.$$

Hence, in particular, for a polynomial of degree 1, instead of using the bound given by (36) of $M(p, R)$, we prefer the exact value as given by (37).

From the above discussion, Theorem 4 in particular, assumes
Corollary 1. If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \(|z| < k, k \geq 1\), then for any real or complex number \( \lambda \) with \(|\lambda| < 1\) and \( R \geq 1\),

\[
M(p, R) = |a_0| + R |a_1| \text{ for } n = 1
\]

and

\[
M(p, R) \leq \left( \frac{R^n + s_2}{1 + s_2} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_2} - n \left\{ \frac{\|p\| - |\lambda| m}{1 + s_2} - |a_n| \right\} S(n),
\]

where

\[
S(n) = \left( \frac{R^n - 1}{n} \right) + \sum_{v=1}^{n-1} \left( \frac{R^{n-v} - 1}{n-v} \right) (-1)^v \left\{ 1 + \frac{(1 + s_2)|a_n|}{\|p\| - |\lambda| m} \right\} \left\{ \frac{1 + s_2)|a_n|}{\|p\| - |\lambda| m} \right\}^{v-1}
\]

\[+ (-1)^n \left\{ 1 + \frac{(1 + s_2)|a_n|}{\|p\| - |\lambda| m} \right\} \left\{ \frac{1 + s_2)|a_n|}{\|p\| - |\lambda| m} \right\}^{n-1}
\]

\[
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda| m)}{\|p\| - |\lambda| m + (1 + s_2)|a_n|} \right\} \text{ for } n \geq 2,
\]

\( s_2 \) is as defined in (22) and \( m = \min_{|z| = k} |p(z)| \).

Remark 2. In particular, if \( k = 1 \), then \( s_2 = 1 \) and Theorem 4 reduces to the following interesting result which is a generalization of a result proved by Dalal and Govil [2].

Corollary 2. If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \(|z| < 1\), then for any real or complex number \( \lambda \) with \(|\lambda| < 1\), \( R \geq 1 \) and any positive integer \( N \), \( 1 \leq N \leq n \),

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \frac{R^n - 1}{2} |\lambda| m - n \left\{ \frac{\|p\| - |\lambda| m}{2} - |a_n| \right\} g^*(N),
\]

where

\[
g^*(N) = (R - 1) - \left( 1 + \frac{2|a_n|}{\|p\| - |\lambda| m} \right) \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda| m)}{\|p\| - |\lambda| m + 2|a_n|} \right\} \text{ for } N = 1,
\]

\[
g^*(N) = \left( \frac{R^n - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{n-v} - 1}{N-v} \right) (-1)^v \left\{ 1 + \frac{2|a_n|}{\|p\| - |\lambda| m} \right\} \left\{ \frac{2|a_n|}{\|p\| - |\lambda| m} \right\}^{v-1}
\]

\[+ (-1)^N \left\{ 1 + \frac{2|a_n|}{\|p\| - |\lambda| m} \right\} \left( \frac{2|a_n|}{\|p\| - |\lambda| m} \right)^{N-1}
\]

\[
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda| m)}{\|p\| - |\lambda| m + 2|a_n|} \right\} \text{ for } N \geq 2,
\]
and \( m = \min_{|z|=k} |p(z)|. \)

**Remark 3.** For \( \lambda = 0 \), Corollary 2 reduces to Theorem 2 proved by Dalal and Govil [2].

**Remark 4.** By Lemma 2, for \( |z| \leq k \) and \( m \neq 0 \),

\[
|p(z)| \geq m,
\]

where \( m = \min_{|z|=k} |p(z)|. \) In particular,

\[
|p(0)| \geq m,
\]

which implies

\[
|a_0| \geq m.
\]

For any real or complex number \( \lambda \) with \( |\lambda| < 1 \), inequality (45) further implies

\[
|a_0| \geq |\lambda| m.
\]

Now, using inequality (46), we have

\[
|a_0 + \lambda m| \geq |a_0| - |\lambda| m
\]

(47) \[ = |a_0| - |\lambda| m. \]

By Lemma 7, \( f(u) \) is a non-increasing function of \( u \), and hence

\[
f(|a_0 + \lambda m|) \leq f(|a_0| - |\lambda| m),
\]

which implies

\[
\frac{1}{1 + s_2} \leq \frac{1}{1 + s_3},
\]

(48) \[ \text{where } s_2 \text{ is as defined in (22) and}
\]

\[
s_3 = \frac{k^{\mu + 1} (\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - |\lambda| m} k^{\mu - 1} + 1)}{\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - |\lambda| m} k^{\mu + 1} + 1}.
\]

(49) \[ \text{Since } (\|p\| - |\lambda| m) \geq 0, \text{ inequality (48) gives}
\]

\[
\frac{n(\|p\| - |\lambda| m)}{1 + s_2} \leq \frac{n(\|p\| - |\lambda| m)}{1 + s_3}.
\]

(50)
Applying Lemma 4 to (50), we have for \( r \geq 1, \)

\[
1 - \left( \frac{\frac{n}{1+s_2}(\|p\| - |\lambda| m) - n|a_n|}{n|a_n| + \frac{m}{1+s_2}(\|p\| - |\lambda| m)} \right) (r - 1) \leq \frac{n}{1+s_2}(\|p\| - |\lambda| m)
\]

For \( r \geq 0, \) which is equivalent to

\[
r^{n-1} \left[ 1 - \left( \frac{\frac{n}{1+s_2}(\|p\| - |\lambda| m) - n|a_n|}{n|a_n| + \frac{m}{1+s_2}(\|p\| - |\lambda| m)} \right) (r - 1) \right] \leq \frac{n}{1+s_2}(\|p\| - |\lambda| m)
\]

Integrating both sides of (51) with respect to \( r \) from 1 to \( R \) and making use of the fact that the right hand side of inequality (29) simplifies exactly to that of (30) in the proof of Theorem 4, we have

\[
\frac{R^n - 1}{1+s_2}(\|p\| - |\lambda| m) - \frac{n}{1+s_2}(\|p\| - |\lambda| m)(1 - f) \int_1^R \frac{(r - 1)r^{n-1}}{r + f} dr
\]

\[
\leq \frac{R^n - 1}{1+s_3}(\|p\| - |\lambda| m) - \frac{n}{1+s_3}(\|p\| - |\lambda| m)(1 - g) \int_1^R \frac{(r - 1)r^{n-1}}{r + g} dr,
\]

where \( f = \frac{|a_n|(1+s_2)}{\|p\| - |\lambda| m} \) and \( g = \frac{|a_n|(1+s_3)}{\|p\| - |\lambda| m}. \)

We notice that the integral \( \int_1^R \frac{(r - 1)r^{n-1}}{r + g} dr \) occurring in the right hand side of (52) is a non-negative and increasing function of \( N \) for \( 1 \leq N \leq n \), therefore, we have

\[
\int_1^R \frac{(r - 1)r^{n-1}}{r + g} dr \leq \int_1^R \frac{(r - 1)r^{n-1}}{r + g} dr.
\]

Since \( \|p\| - |\lambda| m \geq 0 \), by Lemma 8, we have

\[
\frac{|a_n|(1+s_2)}{\|p\| - |\lambda| m} \leq 1,
\]

and hence

\[
1 - f = 1 - \frac{|a_n|(1+s_2)}{\|p\| - |\lambda| m} \geq 0.
\]
Similarly, $1 - g \geq 0$.

Using (53) to the right hand side of (52) and noting that $1 - g \geq 0$ and applying Lemma 5 for the values of integrals involved in the resulting inequality, we have

$$
\left( \frac{R^n - 1}{1 + s_2} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_2} - \frac{n(\|p\| - |\lambda|m)}{1 + s_2} \left\{ 1 - \frac{(1 + s_2)|a_n|}{\|p\| - |\lambda|m} \right\} S(n)
$$

(54)

$$
\leq \left( \frac{R^n - 1}{1 + s_3} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_3} - \frac{n(\|p\| - |\lambda|m)}{1 + s_3} \left\{ 1 - \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\} T(N),
$$

where $S(n)$ is as defined in (40) and

$$
T(N) = (R - 1) - \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\} \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda|m)}{\|p\| - |\lambda|m + (1 + s_3)|a_n|} \right\} \text{ for } N = 1,
$$

$$
T(N) = \left( \frac{R^n - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{n-v} - 1}{N - v} \right) (-1)^v \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\} \left\{ \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\}^{v-1}
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad + (-1)^{N} \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\} \left\{ \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda|m} \right\}^{N-1}
$$

(55)

$$
\times \ln \left( 1 + \frac{(R - 1)(\|p\| - |\lambda|m)}{\|p\| - |\lambda|m + (1 + s_3)|a_n|} \right) \text{ for } N \geq 2.
$$

Adding $\|p\|$ on both sides of (54), we have

$$
\left( \frac{R^n + s_2}{1 + s_2} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_2} - \frac{n}{1 + s_2} \{(\|p\| - |\lambda|m - (1 + s_2)|a_n|)\} S(n)
$$

$$
\leq \left( \frac{R^n + s_3}{1 + s_3} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_3} - \frac{n}{1 + s_3} \{(\|p\| - |\lambda|m - (1 + s_3)|a_n|)\} T(N),
$$

which clearly shows that Corollary 1 is an improvement of the following result which can be further deduced from Theorem 4.

**Corollary 3.** If $p(z) = a_0 + \sum_{v=1}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for any real or complex number $\lambda$ with $|\lambda| < 1$, $R \geq 1$ and any positive integer $N$, $1 \leq N \leq n$,

$$
M(p, R) \leq \left( \frac{R^n + s_3}{1 + s_3} \right) \|p\| - \frac{(R^n - 1)|\lambda| m}{1 + s_3} - \frac{n}{1 + s_3} \left\{ \frac{\|p\| - |\lambda|m}{1 + s_3} - |a_n| \right\} T(N),
$$

(56)
where

\[ s_3 = \frac{k^{\mu+1} \left( \frac{|a_\mu|}{\|a_0\| \lambda m} \right) k^{\mu-1} + 1}{\frac{\mu}{n} \left| \frac{|a_\mu|}{\|a_0\| \lambda m} \right| k^{\mu+1} + 1} \]

and

\[
T(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda m|} \right\}
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda m|)}{(\|p\| - |\lambda m| + (1 + s_3)|a_n|)} \right\}
\]

for \( N = 1 \),

\[
T(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N-v} \right) (-1)^v \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda m|} \right\} \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda m|}^{v-1}
\times (-1)^N \left\{ 1 + \frac{(1 + s_3)|a_n|}{\|p\| - |\lambda m|} \right\}^{N-1}
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - |\lambda m|)}{(\|p\| - |\lambda m| + (1 + s_3)|a_n|)} \right\}
\]

for \( N \geq 2 \),

and \( m = \min_{|z|=k} |p(z)| \).

Remark 5. In particular, for \( 0 \leq \lambda < 1 \), Corollary 3 becomes the following result of Hussain [10, Theorem 2].

Corollary 4. If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zero in \( |z| < k \), \( k \geq 1 \), then for any real or complex number \( \lambda \) with \( 0 \leq \lambda < 1 \), \( R \geq 1 \) and any positive integer \( N, 1 \leq N \leq n \),

\[
M(p, R) \leq \left( \frac{R^n + s_4}{1 + s_4} \right) \|p\| - \frac{(R^n - 1)\lambda m}{1 + s_4} - n \left\{ \frac{\|p\| - \lambda m}{1 + s_4} - |a_n| \right\} T^*(N),
\]

where

\[
s_4 = \frac{k^{\mu+1} \left( \frac{|a_\mu|}{\|a_0\| \lambda m} \right) k^{\mu-1} + 1}{\frac{\mu}{n} \left| \frac{|a_\mu|}{\|a_0\| \lambda m} \right| k^{\mu+1} + 1}
\]

and

\[
T^*(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + s_4)|a_n|}{\|p\| - \lambda m} \right\}
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - \lambda m)}{(\|p\| - \lambda m + (1 + s_4)|a_n|)} \right\}
\]

for \( N = 1 \),
\[ T^*(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N-v} \right)(-1)^v \left\{ 1 + \frac{(1+s_4)|a_n|}{\|p\| - \lambda m} \right\} \left\{ \frac{1}{\|p\| - \lambda m} \right\}^{v-1} \]
\[ + \ (-1)^N \left\{ 1 + \frac{(1+s_4)|a_n|}{\|p\| - \lambda m} \right\} \left\{ \frac{1}{\|p\| - \lambda m} \right\}^{N-1} \]
\[ \times \ln \left\{ 1 + \frac{(R-1)(\|p\| - \lambda m)}{\|p\| - \lambda m + (1+s_4)|a_n|} \right\} \text{ for } N \geq 2, \]

and \( m = \min_{|z|=k} |p(z)|. \)

**Remark 6.** In Theorem 2 of the paper of Hussain [10], the value of \( \Psi(N) \) is

\[ \Psi(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N-v} \right)(-1)^v \left\{ 1 + \frac{(1+S_{\lambda})|a_n|}{\|p\| - \lambda m} \right\} \left\{ \frac{1}{\|p\| - \lambda m} \right\}^{v-1} \]
\[ + \ (-1)^N \left\{ 1 + \frac{(1+S_{\lambda})|a_n|}{\|p\| - \lambda m} \right\} \left\{ \frac{1}{\|p\| - \lambda m} \right\}^{N-1} \]
\[ \times \ln \left\{ 1 + \frac{(R-1)(\|p\| - \lambda m)}{\|p\| - \lambda m + (1+S_{\lambda})|a_n|} \right\} \text{ for } N \geq 1. \]

However, we cannot get the value of \( \Psi(1) \) from inequality (64). This drawback has been resolved in Corollary 4 by separately highlighting the value \( T^*(N) \) for \( N = 1 \).

**Remark 7.** In the limit \( \lambda \to 1 \), Corollary 4 reduces to the following result which is a generalization of Theorem 3 due to Gardner et al. [6].

**Corollary 5.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1, \) then for \( R \geq 1 \) and any positive integer \( N, 1 \leq N \leq n, \)

\[ M(p,R) \leq \left( \frac{R^n + s_1}{1 + s_1} \right) \|p\| - \left( \frac{R^n - 1}{1 + s_1} \right) n \left\{ \frac{\|p\| - m}{1 + s_1} - |a_n| \right\} R(N), \]

where \( s_1 \) is as defined in Theorem 3 and

\[ R(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1+s_1)|a_n|}{\|p\| - m} \right\} \]
\[ \times \ln \left\{ 1 + \frac{(R-1)(\|p\| - m)}{\|p\| - m + (1+s_1)|a_n|} \right\} \text{ for } N = 1, \]
\[ R(N) = \left( \frac{R^N - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^N v - 1}{N - v} \right) (-1)^v \left\{ 1 + \left( \frac{1 + s_1 |a_n|}{\| p \| - m} \right) \right\} \left\{ \frac{1 + s_1 |a_n|}{\| p \| - m} \right\}^{v-1} \]

\[ + (-1)^N \left\{ 1 + \left( \frac{1 + s_1 |a_n|}{\| p \| - m} \right) \right\} \left\{ \frac{1 + s_1 |a_n|}{\| p \| - m} \right\}^{N-1} \]

\[ \log \left\{ 1 + \left( \frac{R - 1)(\| p \| - m)}{(\| p \| - m) + (1 + s_1 |a_n|} \right) \right\} \text{ for } N \geq 2, \]

\[ and \quad m = \min_{|z|=k} |p(z)|. \]

**Remark 8.** From Lemma 6, we have \( T^*(N) \) is a non-negative increasing function of \( N \) for \( N \geq 1 \) and hence \( T^*(1) \leq T^*(N), 1 \leq N \). Using this and Lemma 8, Corollary 4 also reduces to the result of Hussain [10, Corollary 1].

**Corollary 6.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1 \), then for any real or complex number \( \lambda \) with \( 0 \leq \lambda < 1 \) and for \( R \geq 1, \)

\[ M(p,R) \leq \left( \frac{R^n + s_4}{1 + s_4} \right) \| p \| - \frac{R^n - 1}{1 + s_4} \lambda m - \frac{n}{1 + s_4} \left\{ \frac{(\| p \| - \lambda m)^2 - (1 + s_4)^2 |a_n|^2}{\| p \| - \lambda m} \right\} \]

\[ \times \left[ \frac{(R - 1)(\| p \| - \lambda m)}{(\| p \| - \lambda m) + (1 + s_4 |a_n|} - \ln \left\{ 1 + \left( \frac{(R - 1)(\| p \| - \lambda m)}{(\| p \| - \lambda m) + (1 + s_4 |a_n|} \right) \right\} \right], \]

where \( s_4 \) is as defined in (61) and \( m = \min_{|z|=k} |p(z)|. \)

**Remark 9.** For \( \lambda = 1 \), Corollary 6 assumes a result of Gardner et al. [6].

**Remark 10.** For \( \lambda = 0, s_4 \) as defined in Corollary 6 becomes

\[ s_0 = \frac{k^{\mu+1} \left( \frac{\mu |a_n|}{\mu} \right)^{\mu-1}}{\mu |a_n| k^{\mu+1} + 1}, \]

and hence Corollary 6 also becomes

**Corollary 7.** If \( p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1 \), then for \( R \geq 1, \)

\[ M(p,R) \leq \left( \frac{R^n + s_0}{1 + s_0} \right) \| p \| - \frac{n}{1 + s_0} \left\{ \frac{(\| p \|)^2 - (1 + s_0)^2 |a_n|^2}{\| p \|} \right\} \]

\[ \times \left[ \frac{(R - 1)\| p \|}{\| p \| + (1 + s_0)|a_n|} - \ln \left\{ 1 + \left( \frac{(R - 1)\| p \|}{\| p \| + (1 + s_0)|a_n|} \right) \right\} \right], \]

where \( s_0 \) is as defined in (68).
Remark 11. For \( k = 1 \) and \( \mu = 1 \), we have \( s_0 = 1 \), then Corollary 7 reduces to inequality (3).

Remark 12. By Lemma 9, \( s_1 \geq k^\mu \), therefore

\[
\frac{n}{1 + s_1} \left( \|p\| - m \right) \leq \frac{n}{1 + k^\mu} \left( \|p\| - m \right),
\]

where \( m = \min_{|z|=k} |p(z)| \) and \( s_1 \) is as defined in Theorem 3.

Applying Lemma 4 to (69), we have for \( r \geq 1 \)

\[
1 - \frac{n}{1 + s_1} \left( \|p\| - m \right) - n|a_n| \left( r - 1 \right) + \frac{n}{1 + k^\mu} \left( \|p\| - m \right)
\]

\[
\leq 1 - \frac{n}{1 + k^\mu} \left( \|p\| - m \right) - n|a_n| \left( r - 1 \right) + \frac{n}{1 + s_1} \left( \|p\| - m \right).
\]

For \( r \geq 0 \), the above inequality is equivalent to

\[
\frac{R^n - 1}{1 + s_1} \left( \|p\| - m \right) - \frac{n}{1 + s_1} \left( \|p\| - m \right)(1-c) \int_1^R \frac{(r-1)r^{n-1}}{r+c} dr
\]

\[
\leq \frac{R^n - 1}{1 + k^\mu} \left( \|p\| - m \right) - \frac{n}{1 + k^\mu} \left( \|p\| - m \right)(1-a) \int_1^R \frac{(r-1)r^{n-1}}{r+a} dr,
\]

where \( c = \frac{|a_n|(1+s_1)}{\|p\| - m} \) and \( a = \frac{|a_n|(1+k^\mu)}{\|p\| - m} \).

We notice that the integral \( \int_1^R \frac{(r-1)r^{n-1}}{r+a} dr \) occurring in the right hand side of (71) is a non-negative and increasing function of \( N \) for \( 1 \leq N \leq n \), therefore, we have

\[
\int_1^R \frac{(r-1)r^{n-1}}{r+a} dr \leq \int_1^R \frac{(r-1)r^{n-1}}{r+a} dr.
\]
Using (72) to the right hand side of (71) and noting from Lemma 10 that $1 - a \geq 0$ and applying Lemma 5 for the values of integrals involved in the resulting inequality, we have

\[
\left( \frac{R^n - 1}{1 + s_1} \right) \|p\| - \left( \frac{R^n - 1}{1 + s_1} \right)^m - n \left( \frac{\|p\| - m}{1 + s_1} - |a_n| \right) R(n) \\
\leq \left( \frac{R^n - 1}{1 + k^\mu} \right) \|p\| - \left( \frac{R^n - 1}{1 + k^\mu} \right)^m - n \left( \frac{\|p\| - m}{1 + k^\mu} - |a_n| \right) H(N).
\]

Adding $\|p\|$ on both sides of (73), we have

\[
\left( \frac{R^n + s_1}{1 + s_1} \right) \|p\| - \left( \frac{R^n - 1}{1 + s_1} \right)^m - n \left( \frac{\|p\| - m}{1 + s_1} - |a_n| \right) R(n) \\
\leq \left( \frac{R^n + k^\mu}{1 + k^\mu} \right) \|p\| - \left( \frac{R^n - 1}{1 + k^\mu} \right)^m - n \left( \frac{\|p\| - m}{1 + k^\mu} - |a_n| \right) H(N),
\]

where $R(n)$ is as defined in (66) and (67) for $N = n$ and

\[
H(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\} \times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + k^\mu)|a_n|} \right\} \text{ for } N = 1,
\]

\[
H(N) = \left( \frac{R^n - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{n-v} - 1}{N-v} \right) (-1)^v \left\{ 1 + \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\} \left\{ \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\}^{v-1} \\
+ (-1)^N \left\{ 1 + \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\} \left\{ \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\}^{N-1} \\
\times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + k^\mu)|a_n|} \right\} \text{ for } N \geq 2.
\]

Since by Lemma 5, the function $R(N)$ of Corollary 5 is sharpest when $N = n$, and by inequality (74), this particular bound is sharper than the bound given by the following result.

**Corollary 8.** If $p(z) = a_0 + \sum_{v=1}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$ and any positive integer $N$, $1 \leq N \leq n$,

\[
M(p, R) \leq \left( \frac{R^n + k^\mu}{1 + k^\mu} \right) \|p\| - \left( \frac{R^n - 1}{1 + k^\mu} \right)^m - n \left\{ \frac{\|p\| - m}{1 + k^\mu} - |a_n| \right\} H(N),
\]

where

\[
H(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\} \times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + k^\mu)|a_n|} \right\} \text{ for } N = 1,
\]

\[
M(p, R) \leq \left( \frac{R^n + k^\mu}{1 + k^\mu} \right) \|p\| - \left( \frac{R^n - 1}{1 + k^\mu} \right)^m - n \left\{ \frac{\|p\| - m}{1 + k^\mu} - |a_n| \right\} H(N),
\]

\[
H(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + k^\mu)|a_n|}{\|p\| - m} \right\} \times \ln \left\{ 1 + \frac{(R - 1)(\|p\| - m)}{(\|p\| - m) + (1 + k^\mu)|a_n|} \right\} \text{ for } N = 1,
\]
Remark 13. In Theorem 1 of the paper of Hussain [10], the value of \( H(N) \) is

\[
H(N) = (R^N - 1) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N-v} \right) (1+R) \left\{ 1 + \frac{(1+k^\mu)|a_n|}{\|p\|-m} \right\} \left\{ \frac{(1+k^\mu)|a_n|}{\|p\|-m} \right\}^{v-1} + (-1)^N \left\{ 1 + \frac{(1+k^\mu)|a_n|}{\|p\|-m} \right\} \left\{ \frac{(1+k^\mu)|a_n|}{\|p\|-m} \right\}^{N-1} \times \ln \left\{ 1 + \frac{(R-1)(\|p\|-m)}{\|p\|-m + (1+k^\mu)|a_n|} \right\} \text{ for } N \geq 1.
\]

(78)

However, we cannot get the value of \( H(1) \) from inequality (78). This disadvantage has been resolved in Corollary 8 by giving separately the value of \( H(N) \) for \( N = 1 \). Thus for \( N \geq 2 \), Corollary 8 gives the bound as given by the bound of Theorem 1 of Hussain [10].

Remark 14. By Lemma 5, \( H(1) \leq H(N) \). Using this and Lemma 10 in Corollary 8, we have

\[
M(p,R) \leq \left( \frac{R^n + k^\mu}{1+k^\mu} \right) \|p\| - \left( \frac{R^n - 1} {1+k^\mu} \right) \left\{ \frac{\|p\|-m}{1+k^\mu} - n \left\{ \frac{\|p\|-m}{1+k^\mu} - |a_n| \right\} H(1) \right\},
\]

where \( m = \min_{|z|=k} |p(z)| \) and

\[
H(1) = \left( R - 1 \right) - \left\{ 1 + \frac{(1+k^\mu)|a_n|}{\|p\|-m} \right\} \times \ln \left\{ 1 + \frac{(R-1)(\|p\|-m)}{\|p\|-m + (1+k^\mu)|a_n|} \right\}.
\]

(80)

Hence, when \( N = 1 \), Corollary 8 reduces to Theorem 1 which further deduces to inequality (4) for \( \mu = 1 \).

Remark 15. It is easy to observe that for \( \mu = 1 \), Corollary 8 gives a generalization of Theorem 2 due to Dalal and Govil [2].
Remark 16. For $\lambda = 0$ and using the fact of inequality (68), Corollary 4 is a generalization of Theorem 2.

Corollary 9. If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$ and any positive integer $N$, $1 \leq N \leq n$,

$$M(p, R) \leq \left( \frac{R^n + s_0}{1 + s_0} \right) \|p\| - n \left\{ \frac{\|p\|}{1 + s_0} - |a_n| \right\} \phi(N), \tag{81}$$

where $s_0$ is as defined in (68) and

$$\phi(N) = \left( R - 1 \right) - \left\{ 1 + \frac{(1 + s_0)|a_n|}{\|p\|} \right\}$$

$$\times \ln \left\{ 1 + \frac{(R - 1)\|p\|}{\|p\| + (1 + s_0)|a_n|} \right\} \text{ for } N = 1, \tag{82}$$

$$\phi(N) = \left( \frac{R^n - 1}{N} \right) + \sum_{v=1}^{N-1} \left( \frac{R^{N-v} - 1}{N - v} \right) (-1)^v \left\{ 1 + \frac{(1 + s_0)|a_n|}{\|p\|} \right\} \left\{ \frac{(1 + s_0)|a_n|}{\|p\|} \right\}^{v-1}$$

$$+ (-1)^N \left\{ 1 + \frac{(1 + s_0)|a_n|}{\|p\|} \right\} \left\{ \frac{(1 + s_0)|a_n|}{\|p\|} \right\}^{N-1}$$

$$\times \ln \left\{ 1 + \frac{(R - 1)\|p\|}{\|p\| + (1 + s_0)|a_n|} \right\} \text{ for } N \geq 2. \tag{83}$$

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.
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