We present a new approach for constructing covariant symplectic structures for geometrical theories, based on the concept of adjoint operators. Such geometric structures emerge by direct exterior derivation of underlying symplectic potentials. Differences and similarities with other approaches and future applications are discussed.

PACS numbers: 04.20.Jb, 04.40.Nr
Running title: On the symplectic....

* Permanent address

I. INTRODUCTION

Usually quantum field theories are studied by means of Feynman path integrals or by means of canonical quantization. Path integral quantization has the virtue to preserve all relevant symmetries, including Poincaré invariance; however, the resultant theory has not (unlike the canonical formalism) necessarily the standard interpretation in terms of quantum mechanical states and operators. On the other hand, the canonical formalism is considered to be the antithesis of a manifestly covariant treatment.

However, more recently, the essence of the canonical formulation has been developed independently by Witten et al [1, 2] and Suckerman [3] in such a way that manifestly preserves Poincaré invariance as well as other relevant symmetries. Such a formulation is based on a covariant description of Poisson brackets in terms of a symplectic structure defined on the manifold representing the
phase space of classical solutions; thus, quantization is carried out as the replacement of Poisson brackets with commutators, and the resultant quantum theory will be of the conventional type. Specifically, the Witten-et al approach requires the construction, a priori, of a bilinear product on variations of classical solutions. Subsequently, one needs to verify that such a bilinear form corresponds to a nondegenerate closed two-form on the phase space. Moreover, the bilinear form must be a covariantly conserved current in its spacetime dependence, as required for obtaining a symplectic structure manifestly covariant. More specifically, in such a description, the classical phase space is defined as the space of solutions of the classical equations of motion; such a definition is manifestly covariant. The construction of a covariantly conserved two-form $J^\mu$ on such phase space yields a symplectic structure $\omega$ defined as $\omega \equiv \int_\Sigma J^\mu d\Sigma_\mu$ (being $\Sigma$ an initial value hypersurface), independent of the choice of $\Sigma$ and, in particular, Poincaré invariant. Additionally, in terms of the symplectic structure $\omega$, the fact that Poisson brackets satisfy the Jacoby identity, is equivalent that $\omega$ to be a closed two-form on the phase space, which holds if $J^\mu$ itself is closed. With these properties, $J^\mu$ is known as the symplectic current. Such a quantization scheme has been applied, for example, for the analysis of two-dimensional gravity ([4] and references therein), and for the investigation of the Wess–Zumino–Witten model on a circle [5].

Although in the present article we shall obtain essentially the same geometric structures described above, the main novelty is that such structures emerge in a direct and natural way using the concept of adjoint operators. Particularly, the concept of self-adjoint operators shows that, in the cases considered here, there exist, in general, covariantly conserved currents, which correspond on the phase space, to zero-, one-, and, two-forms. Such differential forms are not independent, but that the two-form turns out to be the exterior derivative of the corresponding one-form, and correspond, thus, to an exact two-form (and automatically to a closed two-form, as required for the symplectic structure). In this manner, the present approach allows us to find, unlike the Witten–Zuckerman procedure, fundamental one-forms playing the role of symplectic potentials for the theory.

In the next section, we shall discuss the key concept of adjoint operators, and its consequences on the existence of covariantly conserved currents. The interpretation of the terms involved in such a definition as wedge products on the phase space is also discussed. In Section III only the non-Abelian gauge theories and pure general relativity are considered with the purpose of clarifying our
basic ideas and to have a direct comparison with previously known results, particularly with those
given in Reference [2]. In Section IV we shall finish with some concluding remarks on our results
and possible extensions of the present approach.

Concepts and definitions on differential forms, wedge products, exterior derivative, etc., come
entirely from Ref. [2].

II. ADJOINT OPERATORS AND CONSERVED CURRENTS

The general relationship between adjoint operators and covariantly conserved currents has been
already given in previous works ([7] and references cited therein), however, we shall discuss it in this
section for completeness.

If $P$ is a linear partial differential operator that takes matrix-valued tensor fields into themselves,
then, the adjoint operator of $P$, is that operator $P^\dagger$, such that

$$
\text{Tr}\{f^{\rho\sigma\ldots}[P(g_{\mu\nu\ldots})]_{\rho\sigma\ldots} - [P^\dagger(f^{\rho\sigma\ldots})]_{\mu\nu\ldots}g_{\mu\nu\ldots}\} = \nabla_\mu J^\mu,
$$

where $\text{Tr}$ denotes the trace and $J^\mu$ is some vector field. From this definition, if $Q$ and $R$ are any
two linear operators, one easily finds the following properties:

$$(QR)^\dagger = R^\dagger Q^\dagger,$$

$$(Q + R)^\dagger = Q^\dagger + R^\dagger,$$

and in the case of a function $F$,

$$F^\dagger = F,$$

which will be used implicitly below.

From Eq. (1) we can see that this definition automatically guarantees that, if the field $f$ is a
solution of the linear system $P(f) = 0$, and $g$ a solution of the adjoint system $P^\dagger(g) = 0$, then we
obtain the continuity law $\nabla_\mu J^\mu = 0$, which establishes that $J^\mu$ is a covariantly conserved current
depending on the fields $f$ and $g$. This fact means that for any homogeneous equation system, one can
always construct a conserved current, taking into account the adjoint system. This general result
contains the self-adjoint case \( \mathcal{P}^\dagger = \mathcal{P} \) as a particular one, for which \( f \) and \( g \) correspond to two independent solutions (in fact, the cases treated in the present article are self-adjoint). Although this result has been established assuming only tensor fields and the presence of a single equation, such a result can be extended in a direct way to equations involving spinor fields, matrix fields, and the presence of more than one field \([6, 7]\).

Our main task in this work is to apply this very general result for the analysis of the symplectic forms on the phase space of the theories under consideration. Hence, it is important to clarify, in the first instance, what the fields \( f, g, J^\mu \) and the differential operators \( \mathcal{P}, \mathcal{P}^\dagger \), and \( \nabla_\mu \) will mean on the phase space at the level of Eq. (1). First, such operators will depend only on the background fields, and will correspond thus to zero-forms. Second, although in our previous works we have identified the fields \( f \) and \( g \) with solutions of the equations governing the first-order variations \([\mathcal{P}(f) = 0 = \mathcal{P}(g)]\), in the present work we shall see that it is possible to find simultaneously that \( \mathcal{P}(G) = 0 \), where \( G \) is some background field. Thus, since the background fields and the first-order variations correspond, on the phase space, to zero-forms and one-forms, respectively \([3]\), the left-hand side of Eq. (1) must be understood as a wedge product, \( \text{Tr}\{f \wedge \mathcal{P}(g) - \mathcal{P}^\dagger(f) \wedge g\} = \nabla_\mu J^\mu \), on such phase space, and something similar for the field \( J^\mu \) in its dependence on the fields \( f \) and \( g \). This subject will be clarified in the examples below.

### III. GEOMETRICAL THEORIES AND THEIR SYMPLECTIC STRUCTURES

In this section we shall see that the problem of finding the symplectic structures (and in some cases the symplectic potentials), is reduced to identify some fields satisfying some homogeneous linear equations.

#### A. Yang–Mills theory

Let us consider first the Yang–Mills equations:

\[
\mathcal{D}^\mu F_{\mu \nu} = 0, \tag{2}
\]
where \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the Yang-Mills curvature, \( A_{\mu} \) the background gauge connection, and \( \partial_{\mu} \equiv \partial_{\mu} + [A_{\mu}, \cdot] \), the gauge covariant derivative.

From Eq. (2), the variations of the background fields are governed by the equations:

\[
\delta F_{\mu\nu} + [\delta A_{\mu}, F_{\mu\nu}] = [\delta A_{\alpha}, \partial_{\mu} C_{\nu} - \partial_{\nu} C_{\mu}] + [F_{\mu\nu}, B_{\mu} \wedge C_{\nu}] = 0,
\]

where \( \delta F_{\mu\nu} = \partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu} \) is the variation of the curvature, \( \delta A_{\mu} \) the variation of the gauge connection, and the operator \( \mathcal{P} \) is a homogeneous linear operator depending only on the background fields. Up to here, the usual equations for the Yang–Mills fields and their variations.

Now the idea is to apply our present approach for obtaining all the symplectic structure for the theory, directly from the basic equations (2) and (3). For this purpose, let \( B_{\mu} \) and \( C_{\mu} \) be any two matrix-valued fields (which will be identified below as a pair of gauge connection variations in one case, and as the background gauge connection and its variation in the particular case of Abelian fields), and using the explicit form of the operator \( \mathcal{P} \) in Eq. (3), we have that

\[
B^{\nu} \wedge [\mathcal{P}(C_{\alpha})]_{\nu} - [\mathcal{P}(B^{\nu})]^{\alpha} \wedge C_{\alpha} = \partial_{\mu} \left[ B^{\nu} \wedge (\partial^{\nu} C_{\nu} - \partial_{\nu} C^{\nu}) + (\partial^{\mu} B^{\nu} - \partial^{\nu} B^{\mu}) \wedge C_{\nu} \right] + [F_{\mu\nu}, B_{\mu} \wedge C_{\nu}],
\]

where

\[
B^{\nu} \wedge [C_{\alpha}, F^{\alpha}_{\nu}] - [B_{\alpha}, F^{\alpha}_{\nu}] \wedge C^{\nu} = [F_{\mu\nu}, B^{\mu} \wedge C^{\nu}],
\]

and identities of the form \( B^{\nu} \wedge \partial^{\mu} \partial_{\nu} C^{\mu} \equiv \partial_{\mu} (B^{\nu} \partial^{\mu} C^{\nu} - \partial^{\mu} B^{\nu} \wedge C_{\nu}) + \partial_{\nu} \partial^{\nu} B^{\mu} \wedge C_{\alpha} \) have been used. Taking the trace of Eq. (4), we obtain

\[
\text{Tr}[B^{\nu} \wedge [\mathcal{P}(C_{\alpha})]_{\nu} - [\mathcal{P}(B^{\nu})]^{\alpha} \wedge C_{\alpha}] = \partial_{\mu} \text{Tr}[B_{\nu} \wedge (\partial^{\mu} C^{\nu} - \partial^{\mu} B^{\nu} \wedge C_{\nu})] = 0,
\]

which has the form of Eq. (1) with \( \mathcal{P} = \mathcal{P}^\dagger \). Thus, we can obtain the continuity equation:

\[
\partial_{\mu} J^{\mu} = 0, \quad J^{\mu} \equiv \text{Tr}[B_{\nu} \wedge (\partial^{\mu} C^{\nu} - \partial^{\mu} B^{\nu} \wedge C_{\nu})],
\]

provided that

\[
[\mathcal{P}(C_{\alpha})]_{\nu} = 0, \quad \text{and} \quad [\mathcal{P}(B^{\nu})]^{\alpha} = 0.
\]

As we shall see, the whole physical information about the covariant symplectic structure of the Yang–Mills theory is contained in Eq. (7), it remains only to identify the fields \( B_{\mu} \) and \( C_{\mu} \) satisfying Eqs. (8). In according to Eq. (3), the obvious case is to choose such fields as a pair of variations,
say $B_{\mu} = \delta A_1^{\mu}$, and $C_{\mu} = \delta A_2^{\mu}$ (they have not to correspond necessarily to the same variation). In this manner, $J^\mu$ in Eq. (7) corresponds, in this case, to the following (nondegenerate) two-form on the phase space:

$$J_{\mu} = \text{Tr}[\delta A_1^{\nu} \wedge \delta F_2^{\mu\nu} - \delta F_1^{\mu\nu} \wedge \delta A_2^{\nu}] = \frac{1}{2} \delta^2 \text{Tr}[A_1^{\nu} \delta F_2^{\mu\nu} - F_2^{\mu\nu} \delta A_1^{\nu} - F_1^{\mu\nu} \delta A_2^{\nu} + A_2^{\nu} \delta F_1^{\mu\nu}] = \delta \theta_{\mu}, \quad (9)$$

where $F_1^{\mu\nu} = \partial_\mu A_1^{\nu} - \partial_\nu A_1^{\mu}$, $\delta F_1^{\mu\nu} = \partial_\mu \delta A_1^{\nu} - \partial_\nu \delta A_1^{\mu}$ (i = 1, 2), and we have used the Leibniz rule for the exterior derivative $\delta$, and the fact that $\delta^2 = 0$. In particular, if $\delta A_1^{\mu} = \delta A_2^{\mu}$, from Eq. (9) $J_{\mu} = 2 \text{Tr}(\delta A^\nu \wedge \delta F_{\mu\nu})$, which is essentially the Crncović-Witten current $[2]$. Furthermore, we have defined the one-form $\theta_{\mu}$ as

$$\theta_{\mu} \equiv \frac{1}{2} \text{Tr}[A_1^{\nu} \delta F_2^{\mu\nu} - F_2^{\mu\nu} \delta A_1^{\nu} - F_1^{\mu\nu} \delta A_2^{\nu} + A_2^{\nu} \delta F_1^{\mu\nu}], \quad (10)$$

in this manner, $\theta_{\mu}$ is the symplectic potential for the theory. Note that, according to Eq. (9), the symplectic potential is defined up to the exterior derivative of any matrix-valued field $\lambda_{\mu}$: $J_{\mu} = \delta(\theta_{\mu} + \delta \lambda_{\mu})$.

In the particular case of Abelian fields, from Eqs. (2) and (3) we have that $[P(A_\alpha)]_{\nu} = 0$, where $A_\alpha$ is the background gauge connection. In this manner, we can identify $B_\nu = \delta A_1^{\nu}$ and $C_\nu = A_2^{\nu}$ (a variation and a background gauge connection respectively), and then the symplectic potential $\theta_{\mu}$ given in Eq. (10) is [like the corresponding symplectic current in Eq. (9)] covariantly conserved. Moreover, we can identify for Abelian fields $B_\nu = A_1^{\nu}$, and $C_\nu = A_2^{\nu}$ (a pair of background fields) in Eq. (8); thus, from Eq. (7) $J_{\mu} = A_1^{\nu} F_2^{\mu\nu} - A_2^{\nu} F_1^{\mu\nu}$, which is a covariantly conserved zero-form on the phase space (a conserved current for the exact theory).

Since $J_{\mu}$ in Eq. (9) is an exact two-form (it comes from the variations of the symplectic potential $\theta_{\mu}$), corresponds automatically to a closed two-form ($\delta J^\mu = \delta^2 \theta^\mu = 0$), as required for the symplectic structure. Unlike the Crncović–Witten approach, we do not need to verify the covariant conservation of our symplectic current, such a property is guaranteed for Eqs. (7) and (8). Therefore, $\omega = \int_{\Sigma} J_{\mu} d\Sigma_\mu$ is the symplectic structure with the wanted properties for the Yang–Mills theory $[2]$. Moreover, since the present symplectic structure is essentially the Crncović-Witten result, has the same invariance properties under gauge transformations $[2]$. Specifically, as shown in Ref. $[4]$, under gauge transformations of the gauge connection $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \epsilon + [A_{\mu}, \epsilon]$, $\delta A_{\mu}$ and $\delta F_{\mu\nu}^i$ transform homogeneously, and then $J^\mu$ and $\omega$ are gauge invariant. Furthermore, following Ref. $[2]$, one can
verify easily that \( \omega \) has vanishing components in the gauge directions in field space [see Eq. (30) in such Reference], which allows us to construct the symplectic structure on the corresponding gauge-invariant space (reduced phase space).

The above results are obtained displaying explicitly the variation of the gauge connection \( \delta A_\alpha \) in Eq. (3). However, it is not the only way for obtaining such results. One can consider Eq. (3) in its original form \( \partial / \mu \delta F_{\mu \nu} + [\delta A^\alpha, F_{\mu \nu}] = 0 \), and the relation \( \delta F_{\mu \nu} = \tilde{\phi}_\mu \delta A_\nu - \tilde{\phi}_\nu \delta A_\mu \), as a system of equations governing the field variations \( \delta F_{\mu \nu} \), and \( \delta A_\mu \), considering them as independent field variables:

\[
\begin{bmatrix}
\tilde{\phi}^\mu & -[F_\alpha^\nu, ] \\
1 & (\delta^\alpha_\beta \tilde{\phi}_\nu - \delta^\alpha_\nu \tilde{\phi}_\beta)
\end{bmatrix}
\begin{bmatrix}
\delta F_{\mu \nu} \\
\delta A_\alpha
\end{bmatrix} = 0,
\]

and using again the definition (1) with \( \mathcal{P} \) now being the matrix operator in the preceding equation, one obtains essentially the same results.

**B. General relativity**

The variations of the vacuum Einstein equations \( R_{\mu \nu} = 0 \) are

\[
\nabla_\alpha \delta \Gamma^\alpha_{\mu \nu} - \nabla_\mu \delta \Gamma^\alpha_{\nu \alpha} = 0,
\]

where \( \nabla_\alpha \) is the covariant derivative compatible with the background metric \( g_{\mu \nu} \), and \( \delta \Gamma^\alpha_{\mu \nu} = \frac{1}{2} g^{\alpha \beta}(\nabla_\mu \delta g_{\nu \beta} + \nabla_\nu \delta g_{\mu \beta} - \nabla_\beta \delta g_{\mu \nu}) \), the variation of the metric connection \( \Gamma^\alpha_{\mu \nu} \). Displaying explicitly the metric variations \( \delta g_{\mu \nu} \), Eqs. (11) take the form

\[
[g^\alpha_\mu \nabla^\beta \nabla_\mu + g^\alpha_\mu \nabla^\beta \nabla_\nu - g^\alpha_\mu g^\beta_\nu \nabla_\rho - g^{\alpha \beta} \nabla_\mu \nabla_\nu + g_{\mu \nu}(g^{\alpha \beta} \nabla_\rho \nabla_\rho - \nabla_\beta \nabla_\alpha)] \delta g_{\alpha \beta} = 0,
\]

which can be written in a compact form as

\[
[\mathcal{E}(\delta g_{\alpha \beta})]_{\mu \nu} = 0,
\]

where \( \mathcal{E} \) is the linear operator (depending only on the background fields) appearing in Eq. (12).

With the same idea of the above case, let \( A_{\mu \nu} \) and \( B_{\mu \nu} \) be any two 2-index (symmetric) tensor fields (in the first case these fields will be identified as a pair of metric variations for constructing the symplectic current, and as the background metric and a metric variation in the second case for
obtaining the corresponding symplectic potential), and using the explicit form of the operator \( E \), we have that

\[
B^{\mu \nu} \wedge [\mathcal{E}(A_{\alpha \beta})]_{\mu \nu} - [\mathcal{E}(B^{\mu \nu})]_{\alpha \beta} \wedge A^{\alpha \beta} = \nabla_\mu S^{\alpha \beta \lambda \rho \gamma}(B_{\alpha \beta} \wedge \nabla_\lambda A_{\rho \gamma} - \nabla_\lambda B_{\rho \gamma} \wedge A_{\alpha \beta}),
\]

where

\[
S^{\alpha \beta \lambda \rho \gamma} = g^{\mu (\rho} g^{\mu)}(\alpha g^{\beta})\lambda - \frac{1}{2}g^{\mu \lambda} g^{\alpha (\beta} g^{\gamma)}\lambda g^{\rho \gamma} - \frac{1}{2}g^{\alpha \beta} g^{\mu \lambda} g^{\rho \gamma} + \frac{1}{2}g^{\alpha \beta} g^{\mu \lambda} g^{\rho \gamma}.
\]

Like the Yang–Mills case, Eq. (14) has the form of Eq. (1) with \( E = E^\dagger \). Then, we obtain the local continuity equation:

\[
\nabla_\mu J^\mu = 0, \quad J^\mu \equiv S^{\alpha \beta \lambda \rho \gamma}(B_{\alpha \beta} \wedge \nabla_\lambda A_{\rho \gamma} - \nabla_\lambda B_{\rho \gamma} \wedge A_{\alpha \beta}),
\]

provided that

\[
[\mathcal{E}(A_{\alpha \beta})]_{\mu \nu} = 0, \quad \text{and} \quad [\mathcal{E}(B^{\mu \nu})]_{\alpha \beta} = 0.
\]

In accord with Eq. (13), an obvious identification for the fields \( A_{\mu \nu} \), and \( B_{\mu \nu} \) satisfying Eqs. (17) is

\[
A_{\alpha \beta} = \delta g^1_{\alpha \beta}, \quad \text{and} \quad B_{\mu \nu} = \delta g^2_{\mu \nu},
\]

we mean, a pair of variations. In this manner, from Eq. (16),

\[
J^\mu = S^{\alpha \beta \lambda \rho \gamma}(\delta g^2_{\alpha \beta} \wedge \nabla_\lambda \delta g^1_{\rho \gamma} - \nabla_\lambda \delta g^2_{\rho \gamma} \wedge \delta g^1_{\alpha \beta}),
\]

corresponds to a covariantly conserved two-form on the phase space. The last expression can be rewritten, using Eq. (15), in terms of the variations of the metric connection:

\[
J^\mu = (\delta \Gamma^\mu_{\alpha \beta})_1 \wedge \left[ \delta g^2_{\alpha \beta} + \frac{1}{2}g^{\alpha (\beta} (\delta \ln g)_{2)} - (\delta \Gamma^\nu_{\alpha \nu})_1 \wedge \left[ \delta g^2_{\alpha \nu} + \frac{1}{2}g^{\alpha \nu} (\delta \ln g)_{2}\right] - (1 \leftrightarrow 2),
\]

where \( (\delta \Gamma^\mu_{\alpha \beta})_1 = \frac{1}{2}g^{\mu \rho} \left[ \nabla_\alpha \delta g^1_{\beta \rho} + \nabla_\beta \delta g^1_{\alpha \rho} - \nabla_\rho \delta g^1_{\alpha \beta}\right], \) \( \left( \delta \ln g\right)_{2} = g^{\mu \nu} \delta g^2_{\mu \nu} = -g_{\mu \nu} \delta g^2_{\mu \nu}, \) and \( (1 \leftrightarrow 2) \) means a term similar to the first one, just interchanging the subscripts 1 and 2, such as Eq. (19). If we set \( \delta g^1_{\mu \nu} = \delta g^2_{\mu \nu} = \delta g_{\mu \nu} \), \( J^\mu \) in Eq. (20) reduces exactly to the Crnčović–Witten current [see Eq. (34) of Ref. [3]].

However, the choice (18) for the fields \( A_{\alpha \beta} \), and \( B_{\mu \nu} \), is not the unique one for satisfying Eqs. (17). We can keep \( A_{\alpha \beta} = \delta g^1_{\alpha \beta}, \) but to identify \( B_{\mu \nu} \) as the background metric \( g_{\mu \nu} \), since \( \nabla_\lambda g_{\mu \nu} = 0. \)
and the explicit form of $E$ in Eq. (12), we have that

$$[E(g_{\mu\nu})]_{\alpha\beta} = 0.$$  \hspace{1cm} (21)

Therefore, from Eq. (16), we have that the one-form

$$\theta^\mu \equiv S^\mu_{\alpha\beta\lambda\rho\gamma} g_{\alpha\beta} \nabla_\lambda \delta g^1_{\rho\gamma},$$  \hspace{1cm} (22)

is also a covariantly conserved current on the phase space. $\theta^\mu$ can also be rewritten in terms of the variations of the metric connection:

$$\theta^\mu = g^{\mu\alpha} (\delta \Gamma^\nu_{\alpha\nu})_1 - g^{\alpha\beta} (\delta \Gamma^\mu_{\alpha\beta})_1.$$  \hspace{1cm} (23)

Moreover, the conserved currents $J^\mu$ and $\theta^\mu$ given in Eqs. (19)–(20) and (22)–(23) respectively, are not independent. Considering that $\delta \sqrt{g} = \frac{1}{2} \sqrt{g} \delta \ln g$, from Eq. (23), we have that

$$\delta (\sqrt{g} \theta^\mu) = \sqrt{g} \left[ \delta g^\mu_{\alpha\rho} \wedge (\delta \Gamma^\nu_{\alpha\nu})_1 - \delta g^{\alpha\beta}_{\rho\gamma} \wedge (\delta \Gamma^\mu_{\alpha\beta})_1 \right] - \frac{1}{2} \sqrt{g} \left[ g^{\mu\alpha} (\delta \Gamma^\nu_{\alpha\nu})_1 - g^{\alpha\beta} (\delta \Gamma^\mu_{\alpha\beta})_1 \right] \wedge (\delta \ln g)_2,$$  \hspace{1cm} (24)

where we have considered also that $\delta^2 = 0$, the Leibniz rule for the exterior derivative, and a variation of the background metric appearing in Eq. (23) in general different of $\delta g^1_{\mu\nu}$, and denoted conveniently by $\delta g^2_{\mu\nu}$. Making a direct comparison, the right-hand side of Eq. (24) corresponds, by a factor of $\sqrt{g}$, to the first term on the right-hand side of Eq. (20). With an interchange of the superscripts 1 and 2 in Eq. (24) (which corresponds to identify $A_{\alpha\beta}$ with the background metric and $B_{\mu\nu}$ with the metric variation), we obtain essentially the second term on the right-hand side of Eq. (20). In this manner, we can rewrite

$$\theta^\mu = S^\mu_{\alpha\beta\lambda\rho\gamma} (g^2_{\alpha\beta} \nabla_\lambda \delta g^1_{\rho\gamma} + g^1_{\alpha\beta} \nabla_\lambda \delta g^2_{\rho\gamma}),$$  \hspace{1cm} (25)

and then,

$$\delta (\sqrt{g} \theta^\mu) = \sqrt{g} J^\mu,$$  \hspace{1cm} (26)

which means that $\sqrt{g} J^\mu$ is an exact two-form, and $\sqrt{g} \theta^\mu$ is then the symplectic potential for the theory (which is defined up to the exterior derivative of any vector field). Since $\nabla_\lambda g_{\mu\nu} = 0$ and $g = g(g_{\mu\nu})$, $\sqrt{g} \theta^\mu$ and $\sqrt{g} J^\mu$ are, like $\theta^\mu$ and $J^\mu$, also covariantly conserved.
In the Crncović-Witten approach, one needs to show that $\nabla_\mu J^\mu = 0$; in the present approach $J^\mu$ comes directly from the continuity equation (16). Moreover, from Eq. (26), $\sqrt{g} J^\mu$ is an exact two-form, and automatically a closed two-form, as required for the symplectic structure $\omega = \int_\Sigma \sqrt{g} J^\mu d\Sigma_\mu$, which has the wanted properties. Since $\omega$ is essentially that given in Ref. [2], has the same invariance properties under gauge transformations described in such reference.

If we choose $A_{\mu\nu} = g^1_{\mu\nu}$, and $B_{\alpha\beta} = g^2_{\alpha\beta}$ (a pair of background solutions), both satisfying Eq. (21), then from the local equation (16), we have that $J^\mu = 0$, which means that there no exist a (local) conserved current for the exact theory different to the trivial one.

Finally, if we consider Eq. (11) and the relation between $\delta \Gamma$ and $\delta g_{\mu\nu}$ as a system for these field variations (considering them as independent), one obtains essentially the same results.

IV. CONCLUDING REMARKS

As we have seen, the present approach based on the concept of (self-)adjoint operators leads, in a rigorous way, to local continuity laws for the theory under study. Such continuity equations disclose the existence of different conserved currents, in particular those associated with a covariant description of the corresponding symplectic structure.

The symplectic structures described in Ref. [1-3], are always related to a pair of solutions of the equations governing the variations of classical solutions. In the present scheme, the self-adjoint case corresponds, as we have seen in the examples, to that case. Nevertheless, as discussed in Sec. II, there exists a more general case, which establishes the possibility of constructing a (nondegenerate) two-form related to a solution of the equations governing the variations, and a solution of the corresponding adjoint system. No such possibility was previously known in the literature. However, such a two-form is not necessarily closed, remaining to study under what conditions this two-form represents a symplectic structure. In fact, there are several cases in physics involving operators that are not self-adjoint, where the present approach will be useful: usual free massless fields equations of spin greater that one on a curved spacetime, equations for first-order variations coming from string-
inspired actions, etc. Works along these lines are in progress and will be the subject of forthcoming communications.

On the other hand, the Zuckerman formalism, unlike the present one, requires an explicit extension for covering fermionic fields [3]. Even though in the present article we have limited our discussion to bosonic field theories, the adjoint operator formalism allows us to treat bosonic and fermionic fields (and the simultaneous presence of both) on the same footing, since the fundamental definition (1), which is our starting point, extends for spinor fields [4]. In this case, we are particularly interested in superstring theory, and works along these lines are also in progress.

Finally, the connection between adjoint operators and conserved currents used in the present article, has been also used in Ref. [7], although for a different purpose: for obtaining conserved quantities from non-Hermitian systems. In this manner, a scheme based on adjoint operators has different ramifications of wide interest in physics, whose applications also will be the aim of future investigations.

**ACKNOWLEDGMENT**

This work was supported by CONACyT and the Sistema Nacional de Investigadores (México). The author wants to thank Professor Robert Wald for the kind hospitality provided at the Enrico Fermi Institute, University of Chicago.

**APPENDIX: NO ‘PUZZLE’ FOR THE SYMPLECTIC CURRENT**

In the Crnković–Witten approach [3], unlike the present scheme, there is not a procedure for obtaining the explicit form for the symplectic structure (or for the symplectic potential). In fact, it may be very difficult to guess such an explicit form for more general and complicated cases. However, without invoking the concept of adjoint operators used in the present scheme, one may be able of
obtaining the explicit form of the potential symplectic, starting directly from the basic equations for the variations. For example, in the general relativity case, the equation (11) for the variations can be rewritten in the form

$$\nabla_\alpha (\delta \Gamma^\alpha_{\mu\nu} - \delta^\alpha_\mu \delta \Gamma^\lambda_{\nu\lambda}) = 0,$$

which implies that the tensor field

$$T^\alpha_{\mu\nu} \equiv \delta \Gamma^\alpha_{\mu\nu} - \delta^\alpha_\mu \delta \Gamma^\lambda_{\nu\lambda}$$

is a covariantly conserved one-form on the phase space. Since $\nabla_\lambda g_{\mu\nu} = 0$, and $g = g(g_{\mu\nu})$, the one-form $T^\alpha \equiv \sqrt{g}g^{\mu\nu}T^\alpha_{\mu\nu}$ is also covariantly conserved: $\nabla_\alpha T^\alpha = 0$. Using Eq. (23), it is very easy to find that $g^{\mu\nu}T^\alpha_{\mu\nu} = \theta^\alpha$, thus $T^\alpha = \sqrt{g}\theta^\alpha$. In this manner, $T^\alpha$ coming from Eq. (A1), is the symplectic potential, whose variations generate automatically a closed two-form. However, regardless of adjoint operators, one must verify the covariant conservation of such a two-form in order to obtain a covariant description.

References

[1] E. Witten, N. Phys. B276, 291 (1986).
[2] C. Crncović and E. Witten, in Three Hundred Years of Gravitation, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
[3] E. Zuckerman, in Mathematical Aspects of String Theory, edited by S. T. Yau (World Scientific, Singapore, 1986), p. 259.
[4] Kwang-Sup Soh, Phys. Rev. D. 49, 1906 (1994).
[5] M. Chu, P. Goddard, I. Halliday, D. Olive, and A. Schwimmer, Phys. Lett. B 266, 71 (1991).
[6] G. F. Torres del Castillo, Gen. Relativ. Gravit 22, 1085 (1990).
[7] R. Cartas-Fuentevilla, J. Math. Phys. (N.Y.) 41, 7521 (2000).