SOLUTIONS OF SPDE’S ASSOCIATED WITH A STOCHASTIC FLOW

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Abstract. We consider the following stochastic partial differential equation,

\[ dY_t = L^*Y_t dt + A^*Y_t \cdot dB_t \]
\[ Y_0 = \psi, \]

associated with a stochastic flow \( \{X(t,x)\} \), for \( t \geq 0, x \in \mathbb{R}^d \), as in [Rajeev & Thangavelu, Probabilistic representations of solutions of the forward equations, Potential Anal. 28 (2008), no. 2, 139–162]. We show that the strong solutions constructed there are ‘locally of compact support’. Using this notion, we define the mild solutions of the above equation and show the equivalence between strong and mild solutions in the multi Hilbertian space \( S' \). We show uniqueness of solutions in the case when \( \psi \) is smooth via the ‘monotonicity inequality’ for \( (L^*, A^*) \), which is a known criterion for uniqueness.

1. Introduction

In this article we study the equation for the stochastic flow generated by a finite dimensional diffusion \( \{X(t,x)\} \) starting at \( x \in \mathbb{R}^d \) and satisfying a stochastic differential equation with smooth coefficients viz.

\[ dX_t = \sigma(X_t) \cdot dB_t + b(X_t)dt \]
\[ X_0 = x. \]  
(1.1)

We recall from [21] that the equation for the flow is given as

\[ dY_t = L^*Y_t dt + A^*Y_t \cdot dB_t \]
\[ Y_0 = \psi, \]  
(1.2)

where the operators \( L^*, A^* \) are adjoints of the operators \( L, A \) respectively, associated with the diffusion and defined in Section 2 below. There the solutions of (1.2) were constructed in the space of distributions with compact support and a fortiori, in some Hermite-Sobolev space \( S_p \) for some \( p \in \mathbb{R} \). Such solutions, given in terms of a representation of \( \psi \) as the derivative of continuous functions (see [22], [24]) and \( X(t,x) \) and its derivatives, can be shown to take values in \( S_q \) for some \( q < p \).

In particular, when \( \psi \) is given by a function, then the solution is given by

\[ Z_t(\psi) := \int_{\mathbb{R}^d} \psi(x) \delta_{X(t,x)} dx. \]  
(1.3)

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While the set up of the Hermite-Sobolev spaces facilitated the construction of solutions and thus settled the question of existence, these same spaces turn out to be difficult to handle when it comes to the question of uniqueness of the solutions.

It was shown in [21] that uniqueness of solutions for (1.2) follows from the so-called ‘Monotonicity inequality’ for the pair of operators $(L^*, A^*)$ (see (4.1) below). However, the multiplication operators that intervene in the definition of $L^*$ and $A^*$ create significant difficulties in proving these inequalities because of their non-self adjointness in these spaces. Uniqueness for the Gaussian case can however be handled by special methods (see [1]). For some background on the ‘Monotonicity inequality’ we refer to [2, 6, 7, 15, 19, 22]. In this paper we prove the Monotonicity inequality, in Section 4, in the self adjoint case i.e. when the inequality holds in $L^2(\mathbb{R}^d)$ or $q = 0$ and when the initial condition belongs to $S_p$, $p \geq 5$. In particular the solutions are unique when the initial condition $\psi$ is in $C^\infty_c(\mathbb{R}^d)$ ($\subset S$), the space of smooth functions with compact support.

We show that the solutions constructed in [21] have an important property viz. that they are ‘locally of compact support’ (see Definition 2.6 and Proposition 2.8). This means that up to a stopping time the supports of $Z_t(\psi)$ are contained in some compact set, almost surely. In particular, the supports grow slowly enough that such containment is possible. This contrasts sharply with the expected value of such solutions which need not be of compact support. Indeed this is precisely the behavior of the solutions of the Cauchy problem for the Laplacian with initial value $\delta_0$, the Dirac distribution at zero. The latter property is connected with the stochastic representation of the solutions to the Cauchy problem associated to $L^*$ (see [20, 21]).

While the above discussion relates to ‘strong solutions’ another notion of solution for stochastic PDE’s which is frequently used is the notion of ‘mild solution’. There is an extensive literature on solutions of SPDEs in the mild form in various function spaces (see [3,4,8,14]). However there seems to be very little on mild solutions in the dual of a countable Hilbertian space, the set up that we use (see however [9] equation (81), [11] Chapter 3, Section 5], [12, 13], [24 Theorem 6.1], [25 equation (3.11), p.314]). We show in Section 3 that the strong and mild solutions are equivalent in this set up. The mild solutions of (1.2) above, say $\{Y_t\}$, are defined in terms of the dual ($S_t^*$) of the semi-group $(S_t)$ associated with the diffusion $\{X_t\}$. This requires that the domain of $S_t^*$ be the distributions with compact support. More over we can obtain good bounds on the operator norms of $S_t^*$ only when the domain is restricted to distributions with support in a fixed compact set (see [24 Theorem 4.8]). Thus a term like $S_t^* A^* Y_t$ in the stochastic integral, in the definition of mild solution, is well defined if the support of the process $\{Y_t\}$ is contained in a fixed compact set. While this need not be the case in general, for stochastic integrals to make sense, it is enough if this property holds locally in time i.e. up to a stopping time. Thus the notion of a process which is ‘locally of compact support’ appears quite naturally in the definition of mild solutions in our set up.

We give two proofs that the strong solutions are also mild solutions. One of them goes through an integration by parts formula for the ‘product’ $S_{t-s}^* Y_s$ ($t$ fixed, $s \leq t$) while the other uses an Itô formula for the function $G(s, y) := < x, S_{t-s}^* y >$, $x \in S$ for fixed $x, t$ and $s \leq t$. The latter proof follows closely a well known computation in the finite dimensional case to prove the martingale representation for functionals of the form $f(X_t)$ by applying Itô formula to $g(s, y) := S_{t-s} f(y)$. 

Indeed, it follows in fact that the mild solution representation for finite dimensional diffusion is equivalent to the martingale representation (see Proposition 3.9).

2. Preliminaries

In this section, we describe the framework of our results which is the same as that of [21], and recall the main results from there. We also introduce the notion of distribution valued processes which are ‘locally of compact support’.

Let \( \Omega = C([0, \infty), \mathbb{R}^r) \) denote the set of continuous functions on \([0, \infty)\) with values in \( \mathbb{R}^r \). Let \( \mathcal{F} \) be the Borel \( \sigma \)-field on \( \Omega \) and \( P \) be the Wiener measure. We denote \( B_t(\omega) := \omega(t), \omega \in \Omega, t \geq 0 \) and recall that under \( P \), \( \{B_t\} \) is a standard \( r \) dimensional Brownian motion. Consider the following stochastic differential equation

\[
\begin{align*}
\frac{dX_t}{dt} & = \sigma(X_t) \cdot dB_t + b(X_t)dt \\
X_0 & = x
\end{align*}
\]

with \( \sigma = (\sigma_{ij}), i = 1, \ldots, d, j = 1, \ldots, r \) and \( b = (b_1, \ldots, b_d) \), where \( \sigma_{ij} \) and \( b_i \) are given \( C^\infty \) functions on \( \mathbb{R}^d \) with bounded derivatives. In particular, we have

\[
\|\sigma(x)\| + \|b(x)\| := \left( \sum_{i=1}^{d} \sum_{j=1}^{r} |\sigma_{ij}(x)|^2 \right)^{1/2} + \left( \sum_{i=1}^{d} |b_i(x)|^2 \right)^{1/2} \leq K(1 + |x|)
\]

for some \( K > 0 \) and \( |x|^2 := \sum_{i=1}^{d} x_i^2 \). Under the above assumptions on \( \sigma \) and \( b \), it is well known that a unique, non-explosive strong solution \( \{X(t, x, \omega)\}_{t \geq 0, x \in \mathbb{R}^d} \) exists on \((\Omega, \mathcal{F}, P)\) (see ref. [10]).

Theorem 2.1 ([14]). For \( x \in \mathbb{R}^d \) and \( t \geq 0 \), let \( \{X(t, x, \omega)\} \) be the unique strong solution of (2.1). Then there exists a process \( \tilde{X}(t, x, \omega) \) such that

(i) For all \( x \in \mathbb{R}^d \), \( P\{\tilde{X}(0, x, \omega) = x, \forall t \geq 0\} = 1 \).

(ii) For a.e. \( \omega(P) \), \( x \to \tilde{X}(t, x, \omega) \) is \( C^\infty \) diffeomorphism for all \( t \geq 0 \).

(iii) Let \( \theta_t : \Omega \to \Omega \) be the shift operator i.e. \( \theta_t \omega(s) = \omega(s + t) \); then for \( s, t \geq 0 \), we have

\[
\tilde{X}(t + s, x, \omega) = \tilde{X}(s, \tilde{X}(t, x, \omega), \theta_t \omega)
\]

for all \( x \in \mathbb{R}^d \), a.e. \( \omega(P) \).

We denote the modification obtained in Theorem 2.1 again by \( \{X(t, x, \omega)\} \). For \( \omega \) outside a null set \( \tilde{N} \), the flow of diffeomorphisms induces, for each \( t \geq 0 \) a continuous linear map, denoted by \( X_t(\omega) \) on \( C^\infty \). The map \( X_t(\omega) : C^\infty \to C^\infty \) is given by \( X_t(\omega)(\varphi)(x) = \varphi(X(t, x, \omega)) \). This map is linear and continuous w.r.t. the topology on \( C^\infty \) given by the following family of seminorms: For \( K \subset \mathbb{R}^d \) a compact set, let \( \|\varphi\|_{n, K} := \max_{|\alpha| \leq n} \sup_{x \in K} |D^\alpha \varphi(x)| \) where \( \varphi \in C^\infty \) and \( n \geq 1 \) an integer and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \) and \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \). Let \( K_{t, \omega} \) denote the image of \( K \) under the map \( x \to X(t, x, \omega) \). Then using the chain rule we can show that there exists a constant \( C(t, \omega) > 0 \) such that

\[
\|X_t(\omega)(\varphi)\|_{n, K} \leq C(t, \omega)\|\varphi\|_{n, K_{t, \omega}}.
\]

Let \( X_t(\omega)^* \) denote the transpose of the map \( X_t(\omega) : C^\infty \to C^\infty \). Let \( \mathcal{E}' \) denote the space of distributions with compact support. We will denote the duality between
\( \mathcal{E}' \) and \( C^{\infty} \) by \( \langle \cdot, \cdot \rangle \). Below we will use the same notation for the \( L^2 \)-inner product. Then \( X_t(\omega)^* : \mathcal{E}' \to \mathcal{E}' \) is given by

\[
\langle X_t(\omega)^* \psi, \varphi \rangle = \langle \psi, X_t(\omega) \varphi \rangle
\]

for all \( \varphi \in C^{\infty} \) and \( \psi \in \mathcal{E}' \). For subsets \( K \) of \( \mathbb{R}^d \), we will denote by \( \mathcal{E}'(K) \) the set of \( \psi \in \mathcal{E}' \) with \( \text{supp} \, \psi \subseteq K \). Let \( K \) be a compact subset of \( \mathbb{R}^d \) and let \( \psi \in \mathcal{E}'(K) \). Let \( N = \text{order}(\psi) + 2d \). Then there exist continuous functions \( g_\alpha, |\alpha| \leq N, \text{supp} \, g_\alpha \subseteq V \) where \( V \) is an open set having compact closure, containing \( K \), such that

\[
\psi = \sum_{|\alpha| \leq N} \partial^\alpha g_\alpha.
\]

See [23]. Let \( \varphi \in C^{\infty} \). Let \( f_i \in C^{\infty} \) and \( f = (f_1, \ldots, f_d) \). Let \( \alpha \) be a multi index. We now describe each of the numbers \( \partial^\alpha (\varphi \circ f)(x) \), \( x \in \mathbb{R}^d \) as the result of a distribution (depending on \( x \in \mathbb{R}^d \)) acting on the test function \( \phi \). Let \( \beta^i, i = 1, \ldots, d \) be multi indices, each with \( d \) components. Using the chain rule for differentiation, we can verify that for each multi index \( \gamma \) with \( |\gamma| \leq |\alpha| \), there exist polynomials \( P_\gamma \), in a finite number of variables, with \( \deg P_\gamma = |\gamma| \), such that

\[
\partial^\alpha (\varphi \circ f)(x) = \sum_{|\gamma| \leq |\alpha|} \theta \sum_{|\gamma| \leq |\alpha|} \theta \sum_{|\gamma| \leq |\alpha|} \left( -1 \right)^{|\gamma|} P_\gamma (\partial^{\alpha_1} f_1, \ldots, \partial^{\alpha_d} f_d)(\beta^i) g_\alpha (x).
\]

For \( \omega \notin \mathcal{N} \), define \( Z_t(\omega) : \mathcal{E}' \to \mathcal{E}' \) by

\[
Z_t(\omega)(\psi) = \sum_{|\alpha| \leq N} (-1)^{|\alpha|} \sum_{|\gamma| \leq |\alpha|} (-1)^{|\gamma|} \int_V g_\alpha (x)
\]

\[
P_\gamma (\partial^{\alpha_1} X_1, \ldots, \partial^{\alpha_d} X_d)(\beta^i) \delta_X(t, \omega) dx
\]

Take \( Z_t(\omega) = 0 \) if \( \omega \in \mathcal{N} \).

Let \( \mathcal{S} \) be the space of smooth rapidly decreasing functions on \( \mathbb{R}^d \) with dual \( \mathcal{S}' \), the space of tempered distributions (see [11]). For \( p \in \mathbb{R} \), consider the increasing norms \( \| \cdot \|_p \), defined by the inner products

\[
\langle f, g \rangle_p := \sum_{|k|=0}^\infty (2^{|k|} + d)^{2p} \langle f, h_k \rangle \langle g, h_k \rangle, \quad f, g \in \mathcal{S}.
\]

Here, \( \{ h_k \}_{|k|=0}^\infty \) is an orthonormal basis for \( L^2(\mathbb{R}^d, dx) \) given by Hermite functions (for \( d = 1 \), \( h_k(t) = \left( 2^k k! \sqrt{\pi} \right)^{-1/2} \exp \{-t^2/2\} H_k(t) \), where \( H_k \) are the Hermite polynomials, see [11]), \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( L^2(\mathbb{R}^d, dx) \). We define the Hermite-Sobolev spaces \( \mathcal{S}_p, p \in \mathbb{R} \) as the completion of \( \mathcal{S} \) in \( \| \cdot \|_p \). Note that the dual space \( \mathcal{S}_p' \) is isometrically isomorphic with \( \mathcal{S}_{-p} \) for \( p \geq 0 \). We also have \( \mathcal{S} = \bigcap_{p \geq 0} \mathcal{S}_p \), \( \mathcal{S}' = \bigcup_{p > 0} \mathcal{S}_{-p} \) and \( \mathcal{S}_0 = L^2(\mathbb{R}^d) \). The space \( C_0(\mathbb{R}^d) \) of smooth functions with compact support is dense in \( \mathcal{S} \) (in \( \| \cdot \|_p \)) and hence in \( \mathcal{S}_p \), for \( p \in \mathbb{R} \).

**Theorem 2.2 ([21]).** Let \( \psi \) be a distribution with compact support having representation (2.2). Let \( p > 0 \) be such that \( \partial^\alpha \delta_x \in \mathcal{S}_{-p} \) for \( |\alpha| \leq N \). Then \( \{ Z_t(\psi) \}_{t \geq 0} \) is an \( \mathcal{S}_{-p} \)-valued adapted process such that for all \( t \geq 0 \),

\[
Z_t(\psi) = X_t^*(\psi) \quad \text{a.s.} \quad P.
\]
Example 2.3. We mention two examples corresponding to special initial values $\psi$ for which the process $\{Z_t(\psi)\}$ is the solution of the SPDE (1.2). Uniqueness of the solution in the case of the first example is one of the principal motivations for and the main application of, the results of this paper. We refer to the results of [19] for uniqueness in the case of the second example.

(1) Let $\psi \in C_c^\infty(\mathbb{R}^d)$. Then $Z_t(\psi) = \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x)}\,dx$. This fact can be verified as follows.

\[
\langle Z_t(\psi), \phi \rangle = \int_{\mathbb{R}^d} \psi(x)\phi(X(t,x))\,dx = \int_{\mathbb{R}^d} \psi(x)(X_t(\phi))(x)\,dx = \langle \psi, X_t(\phi) \rangle.
\]

Moreover, $Z_t(\psi)$ is actually a function. To see this, let $J(t,x)$ denote the Jacobian obtained by the change of variables $x$ to $X(t,x)$. Since $x \mapsto X(t,x)$ is a diffeomorphism, $J(t,x)$ is non-zero, and in particular, $J(t,x)$ is either strictly positive or strictly negative. Now

\[
\langle Z_t(\psi), \phi \rangle = \int_{\mathbb{R}^d} \psi(X(t,\cdot)^{-1}x)(\phi(x)|J(t,x)|)\,dx.
\]

Therefore $Z_t(\psi)$ is given by the $C_c^\infty(\mathbb{R}^d)$ function $x \mapsto \psi(X(t,\cdot)^{-1}x)|J(t,x)|$. Note that the same computations go through if $\psi \in \mathcal{S}$. However, (2.3) need not hold.

(2) Take $\psi = \delta_x$ for some $x \in \mathbb{R}^d$. Then $Z_t(\psi) = \delta_{X(t,x)}$.

We now define the operators $A : C^\infty \to \mathcal{L}(\mathbb{R}^r, C^\infty)$ and $L : C^\infty \to C^\infty(\mathbb{R}^d)$ as follows: for $\varphi \in C^\infty,$ $x \in \mathbb{R}^d$,

\[
A\varphi = (A_1\varphi, \cdots, A_r\varphi),
\]

\[
A_i\varphi(x) = \sum_{k=1}^d \sigma_{ki}(x)\partial_k \varphi(x),
\]

\[
L\varphi(x) = \frac{1}{2} \sum_{i,j=1}^d \sigma_{ij}(x)\partial_{ij} \varphi(x) + \sum_{i=1}^d b_i(x)\partial_i \varphi(x).
\]

Remark 2.4. Since $\sigma$, $b$ are $C^\infty$ functions on $\mathbb{R}^d$ with bounded derivatives satisfying linear growth condition, therefore $L : \mathcal{S} \to \mathcal{S}$.

We define the adjoint operators $A^* : \mathcal{E}' \to \mathcal{L}(\mathbb{R}^r, \mathcal{E}')$ and $L^* : \mathcal{E}' \to \mathcal{E}'$

\[
A^*\psi = (A_1^*\psi, \cdots, A_r^*\psi),
\]

\[
A_i^*\psi = -\sum_{k=1}^d \partial_k (\sigma_{ki}\psi),
\]

\[
L^*\psi = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\sigma^4)_{ij}\psi) - \sum_{i=1}^d \partial_i (b_i\psi).
\]

Proposition 2.5 ([21 Proposition 3.2]). Let $\sigma_{ij} , i = 1, \cdots, d , j = 1, \cdots, r$ and $b_1, \cdots, b_d$ be $C^\infty$ functions on $\mathbb{R}^d$ with bounded derivatives. Let $p > 0$ and $q > [p] + 4$, where $[p]$ denotes the largest integer less than or equal to $p$. Let $K$ be a compact subset of $\mathbb{R}^d$. Then, $A^* : \mathcal{S}_{-p} \cap \mathcal{E}'(K) \to \mathcal{L}(\mathbb{R}^r, \mathcal{S}_{-q} \cap \mathcal{E}'(K))$ and
L^* : S_{-p} \cap \mathcal{E}'(K) \to S_{-q} \cap \mathcal{E}'(K). Moreover, there exists constants C_1(p) > 0, C_2(p) > 0 independent of the compact set K such that
\[ \|A^* \psi\|_{HS(-q)} \leq C_1(p)\|\psi\|_{-p}, \quad \|L^* \psi\|_{-q} \leq C_2(p)\|\psi\|_{-p} \]
where
\[ \|A^* \psi\|_{HS(-q)}^2 := \sum_{i=1}^{r} \left( \sum_{k=1}^{d} \partial_k (\sigma_{ki} \psi) \right)^2 \leq \sum_{i=1}^{r} \|A_i^* \psi\|_{-q}^2. \]

**Definition 2.6.** We say that an \( \mathcal{E}' \) valued process \( \{Y_t\} \) is locally of compact support if there exists an increasing sequence of stopping times \( \{\tau_n\} \) such that \( \tau_n \uparrow \infty \) and for each \( n \), a.s. \( supp(Y_t^{\tau_n}) \subset K, \forall t \) where \( \{K\} \) is some increasing family of compact sets.

**Proposition 2.7.** Let \( Y = (Y^1, \cdots, Y^r) \), where each \( \{Y_t^i\} \) is an \( S_{-p} \cap \mathcal{E}' \) valued adapted process with continuous paths in \( \| \cdot \|_{-p} \) norm and is locally of compact support. Then the local martingale \( \{\int_0^t Y_s^i \cdot dB_s\} \) is locally of compact support.

**Proof.** By our hypothesis, there exists an increasing sequence of stopping times \( \{\tau_n\} \) such that \( \tau_n \uparrow \infty \) and for each \( n \), a.s. \( supp((Y^i)_{\tau_n}) \subset K, \forall t \), where \( \{K\} \) is some increasing family of compact sets. Since \( \{Y_t^i\} \) has continuous paths, without loss of generality, we assume that \( \|Y_t^{\tau_n}\|_{HS(-p)} \leq n \) for \( t \geq 0 \). Hence we have the existence of the stochastic integral \( \{\int_0^{t \wedge \tau_n} Y_s^i \cdot dB_s\} \).

Suppose \( \phi \) is a \( C^\infty \) function such that the support of \( \phi \) and its derivatives are contained in the complement of \( K \). Then, a.s. for \( t \geq 0 \)
\[ \left\langle \int_0^{t \wedge \tau_n} Y_s^i \cdot dB_s, \phi \right\rangle = \sum_{i=1}^{r} \int_0^{t \wedge \tau_n} \langle (Y^i)_{\tau_n}^i, \phi \rangle dB_s = 0. \]
Since, by definition, \( \int_0^t Y_s^i \cdot dB_s = \int_0^{t \wedge \tau_n} Y_s^i \cdot dB_s, \) for \( t \leq \tau_n \), the result follows. \( \square \)

The open set \( V \) mentioned before (2.3) is bounded. Hence there exists \( \lambda > 0 \) such that \( V \) is a subset of the closed ball \( B(0, \lambda) \) of radius \( \lambda \) centered at the origin. For \( R \geq 0 \), define
\[ \tau_R := \inf\{t > 0 : \sup_{s \in [0,t]} \sup_{|x| \leq \lambda} |X_s(x)| \geq R\}. \]
Since \( \{\sup_{|x| \leq \lambda} |X_s(x)|\}_s \) is an adapted process, the process \( \{\sup_{s \in [0,t]} \sup_{|x| \leq \lambda} |X_s(x)|\}_t \) is adapted and increasing. Hence \( \tau_R \) is a stopping time for each \( R \).

**Proposition 2.8.** The process \( \{Z_t(\psi)\} \), defined by (2.4), is locally of compact support. Furthermore,
(1) \( t \leq \tau_R \Rightarrow supp(Z_t(\psi)) \subseteq B(0, R) \).
(2) As \( R \uparrow \infty \), \( \tau_R \uparrow \infty \).

**Proof.** Suppose \( \varphi \) is a \( C^\infty \) function such that the support of \( \varphi \) is contained in the complement of \( B(0, R) \). Then (2.3) and \( t \leq \tau_R \) imply
\[ \langle Z_t(\psi), \varphi \rangle = \sum_{|\alpha| \leq N} (-1)^{|\alpha|} \sum_{|\gamma| \leq |\alpha|} (-1)^{|\gamma|} \int_V g_\alpha(x) P_t((\partial^\beta X_1, \cdots, \partial^\beta X_d)_{|\beta| \leq |\alpha|})(t, x) \langle \partial^\gamma \delta X(t, x), \varphi \rangle dx \]
for each compact set \( Y \) of equation (2.7).

**Remark 2.10.** We note that the process in the third term in the right hand side satisfies the following equation in strong solution of (2.6).

**Theorem 2.11**. The proof of Proposition 2.7.

**Remark 2.12.** \( \partial \) such that \( S \) compact support and satisfies the following equation in

\[
\text{Definition 2.9.} \quad \text{Let } p \in \mathbb{R} \text{ and } \psi \in S_p \cap \mathcal{E}'. \text{ Let } q \in \mathbb{R} \text{ be such that } A^* : S_p \cap \mathcal{E}'(K) \to \mathcal{L}(\mathbb{R}^r, S_q \cap \mathcal{E}'(K)) \text{ and } L^* : S_p \cap \mathcal{E}'(K) \to S_q \cap \mathcal{E}'(K) \text{ are bounded linear operators for each compact set } K \text{ in } \mathbb{R}^d. \text{ We say that } \{Y_t\} \text{ is a } (p,q) \text{ strong solution of (2.6) if it is an } S_p \cap \mathcal{E}'-valued \text{ adapted process, has continuous paths in } S_p, \text{ is locally of compact support and satisfies the following equation in } S_q, \text{ a.s.,}

\[
Y_t = \psi + \int_0^t A^* Y_s \cdot dB_s + \int_0^t L^* Y_s ds
\]

for all \( t \geq 0 \).

**Remark 2.10.** We note that the process in the third term in the right hand side of equation (2.7) is also locally of compact support. The proof is the same as in the proof of Proposition 2.7.

**Theorem 2.11** (21). Let \( \psi \in \mathcal{E}' \) have the representation (2.2). Let \( p > 0 \in \mathbb{R} \) be such that \( \partial^\gamma \delta_x \in S_{-p}, \ |\gamma| \leq N \). Let \( q > p \) be as in Proposition 2.7. Then the \( S_{-p} \)-valued continuous, adapted process \( \{Z_t(\psi)\}_{t \geq 0} \) defined by (2.4), is a \( (-p,-q) \) strong solution of (2.6).

**Remark 2.12.** It was noted in [21, Theorem 4.1] that \( p > \frac{d}{4} + \frac{|\gamma|}{2} \) is a sufficient condition for \( \partial^\gamma \delta_x \in S_{-p}, \) for any multi-index \( \gamma \). Thus in the previous theorem, we can state an explicit condition on \( p \) as \( p > \frac{d}{4} + \frac{N}{2} \).

**Proposition 2.13** (21). Let \( \psi \in \mathcal{E}' \) with representation (2.2). Let \( p > \frac{d}{4} + \frac{N}{2} \) where \( N = \text{order}(\psi) + 2d \). Let \( \{Z_t(\psi)\} \) be the \( S_{-p} \)-valued continuous adapted process defined by (2.4). Then for all \( T > 0 \),

\[
\sup_{t \leq T} \mathbb{E}\|Z_t(\psi)\|_{-p}^2 < \infty.
\]

3. Mild solutions

Let \( \{S_t\}_{t \geq 0} \) be the semigroup corresponding to \( \{X(t,x)\} \) solving (2.1) i.e. for \( f \in \mathcal{S}, \ S_tf(x) := \mathbb{E}f(X(t,x)). \) Then,

\[
\mathcal{C}_s \langle S_t f, \psi \rangle_{\mathcal{E}'} = \langle S_t f \circ X_t, \psi \rangle = \mathbb{E}\langle X_t(f), \psi \rangle = \mathbb{E}(f, Z_t(\psi)) = \mathbb{E}(f, \mathbb{E}Z_t(\psi))_{\mathcal{S}}.
\]

Consider the map \( S_t^* : \mathcal{E}' \to \mathcal{S} \) defined by \( S_t^* \psi := \mathbb{E}Z_t(\psi) \).
Theorem 3.1 (21). The following are the properties of the operators $S_t$ and $S_t^*$.

(a) We have $S_t : \mathcal{S} \to C^\infty$. The map $S_t^* : \mathcal{E}' \to \mathcal{S}'$ is adjoint to $S_t$ in the sense that

$$s^* \langle S_t^* \psi, \phi \rangle_S = \epsilon' \langle \psi, S_t \phi \rangle_{C^\infty}$$

for all $\psi \in \mathcal{E}'$ and $\phi \in \mathcal{S}$.

(b) Let $K \subset \mathbb{R}^d$ be a compact set and $p > 0$. Then for $q > \frac{d}{2} + [p] + 1$, $S_t^* : \mathcal{S}_p \cap \mathcal{E}'(K) \to \mathcal{S}_{-q}$ is a bounded linear operator. Further, for any $T > 0$, there exists a constant $C(T), 0 < C(T) < \infty$ such that

$$\sup_{t \leq T} \| S_t^* \|_H < C(T)$$

where $\| \cdot \|_H$ is the operator norm on the Banach space $H$ of bounded linear operators from $\mathcal{S}_p \cap \mathcal{E}'(K)$ to $\mathcal{S}_{-q}$.

As a consequence of Proposition 2.7, we get the next result.

Corollary 3.2. Let $Y = (Y^1, \cdots, Y^r)$, where each $\{Y^i_t\}$ is an $\mathcal{S}_p \cap \mathcal{E}'$ valued adapted process with continuous paths in $\mathcal{S}_p$ and is locally of compact support. Let $q$ be as in Theorem 2.1. Then for each $i$, $\{S_{t-s}^* Y^i_t \}_{s \in [0, t]}$ is an $\mathcal{S}_{-q}$ valued continuous adapted process and the process $\{ \int_0^t S_{t-s}^* Y^i_s \cdot dB_s \}$ is an $\mathcal{S}_{-q}$ valued continuous local martingale. Here the term $\int_0^t S_{t-s}^* Y^i_s \cdot dB_s$ denotes the sum $\sum_{r=1}^r \int_0^t S_{t-s}^* Y^i_r dB_r^i$, where $B^i, i = 1, \cdots, r$ are the components of the Brownian motion $\{B_t\}$.

In what follows, $p$ will denote an arbitrary but fixed non-negative real number. We also associate two positive real numbers $p'$ and $q$ to this $p$. By Proposition 2.5 we can choose $p' > [p] + 4$ such that

$$L^* : \mathcal{S}_p \cap \mathcal{E}'(\overline{B(0, R)}) \to \mathcal{S}_{-p'} \cap \mathcal{E}'(\overline{B(0, R)})$$

and

$$A^* : \mathcal{S}_p \cap \mathcal{E}'(\overline{B(0, R)}) \to \mathcal{L}(\mathbb{R}^r, \mathcal{S}_{-p'} \cap \mathcal{E}'(\overline{B(0, R)}))$$

are bounded linear operators for any $R > 0$. Now, by Theorem 3.1 we can choose $q > \frac{d}{2} + [p'] + 1$ such that

$$S_t^* : \mathcal{S}_{-p'} \cap \mathcal{E}'(\overline{B(0, R)}) \to \mathcal{S}_{-q}$$

is a bounded linear operator for any $R > 0$. Note that $0 < p < p' < q$.

Lemma 3.3. For $x \in \mathcal{S}_p \cap \mathcal{E}'$

(i) $L^* x = \lim_{t \to 0^+} \frac{S_t^* x - x}{t}$;

(ii) $\frac{d}{dt} S_t^* x = S_t^* L^* x$;

(iii) $S_t^* x - S_s^* x = \int_s^t S_u^* L^* x du$.

Proof. The proof follows from standard duality arguments.
Definition 3.4. Let \( \psi \in S_{-p} \cap E' \). We say that \( \{Y_t\} \) is a \((-p, -q)\) mild solution of \((2.3)\) if it is an \(S_{-p} \cap E'\)-valued adapted process, with continuous paths in \(S_{-p}\) and is locally of compact support and satisfies the following equation in \(S_{-q}\), a.s.,

\[
Y_t = S^*_t \psi + \int_0^t S^*_{t-s} A^* Y_s \cdot dB_s
\]

for all \(t \geq 0\).

Remark 3.5. As mentioned in Remark \((2.10)\), if \(\{Y_t\}\) is a strong solution, then all the terms in \((2.7)\) are locally of compact support (see Proposition \((2.7)\)). For an arbitrary \(\psi \in E'\), the distribution \(S^*_t \psi\) need not be compactly supported. To see this, take \(r = d, b(\cdot) \equiv 0, \psi = \delta_x\) and \(\sigma(\cdot) \equiv I_d\), the identity matrix. Then \(Z_t(\psi) = \delta_{X(t,x)}\). \(S^*_t \psi\) is not compactly supported. As such, if \(\{Y_t\}\) is a mild solution, then the terms on the right hand side of \((3.4)\) need not be locally of compact support.

Proposition 3.6. For each \(R > 0\), the map \(t \mapsto S^*_t\) is of finite variation in the operator norm \(\| \cdot \|_{\mathcal{L}(S_{-p} \cap E'([0,R]), S_{-q})}\). In particular, For all \(x \in S_{-p} \cap E'\), the map \(t \mapsto S^*_t x\) is of finite variation in \(\| \cdot \|_{-q}\) norm.

Proof. Let \(\Pi = \{0 = t_0 < t_1 < t_2 \cdots < t_n = t\}\) be a partition of \([0,t]\). Observe that

\[
\sum_{i=0}^{n-1} \left\| S^*_{t_{i+1}} - S^*_{t_i} \right\|_{\mathcal{L}(S_{-p} \cap E'([0,R]), S_{-q})} = \sum_{i=0}^{n-1} \sup_{\|x\|_{-p}, x \in E'} \left\| S^*_{t_{i+1}} x - S^*_{t_i} x \right\|_{-q}
\]

\[
= \sum_{i=0}^{n-1} \sup_{\|x\|_{-p}, x \in E'} \left\| S^*_{t_{i+1}} x \right\|_{-q} \left\| S^*_{t_i} x \right\|_{-q}
\]

\[
\leq \sum_{i=0}^{n-1} \sup_{\|x\|_{-p}, x \in E'} \int_{t_i}^{t_{i+1}} \left\| S^*_{t} x \right\|_{-q} \, ds
\]

\[
\leq \sum_{i=0}^{n-1} \sup_{\|x\|_{-p}, x \in E'} C(T) \| L^* x \|_{-p} \int_{t_i}^{t_{i+1}} ds
\]

\[
\leq \sum_{i=0}^{n-1} \sup_{\|x\|_{-p}, x \in E'} C(T) C_2(p) \int_{t_i}^{t_{i+1}} ds
\]

\[
\leq \sum_{i=0}^{n-1} C(T) C_2(p) (t_{i+1} - t_i) \| x \|_{-p}
\]

\[
\leq C(T) C_2(p) \sum_{i=0}^{n-1} (t_{i+1} - t_i)
\]
\[ C(T)C_2(p)t. \]

Hence the proof. \hfill \Box

Let \( R > 0 \). Let \( t \in [0, \infty) \mapsto y_t \) be an \( S_{-p} \cap \mathcal{E}'(\overline{B(0, R)}) \) valued continuous map. Let \( \Pi_m = \{ 0 = t_0^m < t_1^m < t_2^m \cdots < t_n^m = t \} \) be a sequence of partitions of \([0, t]\) such that \( |\Pi_m| = \max |t_{i+1}^m - t_i^m| \to 0 \) as \( m \to \infty \). Let us consider the simple functions \( s \mapsto y_s^m := \sum_{i=0}^{n-1} 1_{(t_i^m, t_{i+1}^m]}(s)y_{t_i^m} \). Define

\[
\int_0^t dS^*_s y_s := \lim_{m \to \infty} \int_0^t dS^*_s y_s^m := \lim_{m \to \infty} \sum_{i=0}^{n-1} |S^*_{t_{i+1}^m} - S^*_{t_i^m}|[y_{t_i^m}]
\]

**Proposition 3.7.** The limit in (3.5) exists as an element of \( \mathcal{S}_{-q} \) and is independent of the sequence of partitions chosen.

**Proof.** Fix \( T > 0 \). The map \( s \mapsto y_s \) is uniformly continuous on \([0, T]\). Therefore given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |y_u - y_v|_p < \epsilon \) whenever \( u, v \in [0, T] \) with \( |u - v| < \delta \).

Choose \( m, l \) sufficiently large such that \( |\Pi_m| < \delta, |\Pi_l| < \delta \). Let us denote

\[ h^m := \int_0^t dS^*_s y_s^m \quad \text{and} \quad h^l := \int_0^t dS^*_s y_s^l. \]

Let \( \Pi := \{ 0 = r_0 < r_1 < r_2 \cdots < r_k = t \} \) be the refinement of the two partitions. Note that \( |\Pi| < \delta \). In particular \( |y^m - y^l|_p < \epsilon \). Now, we show that \( ||h^m - h^l||_{-q} \to 0 \) as \( m, l \to \infty \).

\[
||h^m - h^l||_{-q} = \left\| \int_0^t dS^*_s [y_s^m - y_s^l] \right\|_{-q}
\]

\[ = \left\| \sum_{i=0}^{k-1} \left( S^*_{r_{i+1}} - S^*_{r_i} \right) [y_{r_i} - y_{r_i}] \right\|_{-q}
\]

\[ \leq \sum_{i=0}^{k-1} \left\| S^*_{r_{i+1}} - S^*_{r_i} \right\|_{L(S_{-p} \cap \mathcal{E}'(\overline{B(0, R)}), \mathcal{S}_{-q})} ||y_{r_i} - y_{r_i}||_{-p}
\]

\[ \leq \epsilon \sum_{i=0}^{k-1} \left\| S^*_{r_{i+1}} - S^*_{r_i} \right\|_{L(S_{-p} \cap \mathcal{E}'(\overline{B(0, R)}), \mathcal{S}_{-q})}
\]

where \( TV_{[0,t]}(S^*_s) \) denotes the total variation of the map \( s \mapsto S^*_s \) on \([0, t]\), which is finite from Proposition 3.4. Since \( \epsilon \) was arbitrary, the sequence \( \{h^m\} \) is Cauchy and hence \( \lim_{m \to \infty} h^m \) exists.

By standard arguments, we can show the limit is independent of the sequence of partitions chosen. \hfill \Box

**Theorem 3.8.** Let \( L^*, A^* \) and \( S^*_s \) satisfy (3.1), (3.2) and (3.3) respectively. Let \( \{Y_t\} \) be a \((-p, -q)\) strong solution of (2.6). Then it is also a \((-p, -q)\) mild solution.

**First proof of Theorem 3.8.** Since \( \{Y_t\} \) is locally of compact support, there exists an increasing sequence of stopping times \( \{\tau_n\} \) such that \( \tau_n \uparrow \infty \) and for each \( n \), a.s. \( \text{supp}(Y_t^{\tau_n}) \subset B(0, R_n) \) for some \( R_n > 0 \).
Fix \( t \geq 0 \) and \( x \in S \). Fix a natural number \( n \). Consider the function \( G : [0, t] \times (S_{-p'} \cap \mathcal{L}^r(B(0, R_n))) \rightarrow \mathbb{R} \), where \( G \in C^{1,2}_r([0, t] \times (S_{-p'} \cap \mathcal{L}^r(B(0, R_n)))) \) defined by

\[
G(s, y) := s_q \langle x, S_{t-s}^* y \rangle_{S_{-q}}.
\]

Recall that by Theorem 3.1, localization (on \( G \)), and for all \( t \geq 0 \), we get the required relation.

For \( u \in [0, t], y \in S_{-p'} \cap \mathcal{L}^r(B(0, R_n)), \) using Lemma 3.3 we have

\[
\langle x, - \int_0^u S_{t-s}^* L^* y \, ds \rangle = \langle x, - \int_0^t S_{t-s}^* L^* y \, ds \rangle = \langle x, S_{t-u}^* y - S_t^* y \rangle = G(u, y) - G(0, y).
\]

Then the partial derivative \( G_s(s, y) = -s_q \langle x, S_{t-s}^* L^* y \rangle_{S_{-q}}, y \in S_{-p'} \cap \mathcal{L}^r(B(0, R_n)). \)

Now for \( y, z \in S_{-p'} \cap \mathcal{L}^r(B(0, R_n)), \) we have

\[
G(s, y + z) - G(s, y) = \langle x, S_{t-s}^* z \rangle.
\]

Since \( z \mapsto \langle x, S_{t-s}^* z \rangle \) is a bounded linear functional on \( S_{-p'} \), we get the Fréchet derivative \( G_y(s, y) = \langle x, S_{t-s}^* \rangle : S_{-p'} \rightarrow \mathbb{R} \). Consequently, \( G_{yy}(s, y) = 0. \)

Since

\[
Y_s^{\tau_n} = \psi + \int_0^{s \wedge \tau_n} A^* Y_{r}^{\tau_n} \, dB_r + \int_0^{s \wedge \tau_n} L^* Y_{r}^{\tau_n} \, dr, \forall s \in [0, t]
\]

applying Itô’s formula (see [41][52]) we get a.s., for all \( t \geq 0, \)

\[
\langle x, Y_s^{\tau_n} \rangle - \langle x, S_t^* \psi \rangle = G(t, Y_s^{\tau_n}) - G(0, Y_0^{\tau_n})
\]

\[
= \int_0^{t \wedge \tau_n} G_s(s, Y_s^{\tau_n}) \, ds + \int_0^{t \wedge \tau_n} G_y(s, Y_s^{\tau_n}) L^* Y_s^{\tau_n} \, ds
\]

\[
+ \int_0^{t \wedge \tau_n} G_{y^2}(s, Y_s^{\tau_n}) A^* Y_s^{\tau_n} \cdot dB_s
\]

\[
= -\int_0^{t \wedge \tau_n} \langle x, S_{t-s}^* L^* Y_s^{\tau_n} \rangle \, ds + \int_0^{t \wedge \tau_n} \langle x, S_{t-s}^* L^* Y_s^{\tau_n} \rangle ds
\]

\[
+ \int_0^{t \wedge \tau_n} \langle x, S_{t-s}^* A^* Y_s^{\tau_n} \rangle \cdot dB_s
\]

\[
= \langle x, \int_0^{t \wedge \tau_n} S_{t-s}^* A^* Y_s^{\tau_n} \cdot dB_s \rangle.
\]

Since \( x \in S \) was arbitrary, we have a.s., for all \( t \geq 0, \)

\[
Y_s^{\tau_n} - S_t^* \psi = \int_0^{t \wedge \tau_n} S_{t-s}^* A^* Y_s^{\tau_n} \cdot dB_s.
\]

Letting \( n \) go to \( \infty \), we get the required relation.

\[\square\]

Second proof of Theorem 3.8 We first claim that (3.6) holds.

(3.6) \[
\int_0^t S_{t-s}^* \circ dB_s = \int_0^t S_{t-s}^* L^* Y_s \, ds + \int_0^t S_{t-s}^* A^* Y_s \, dB_s.
\]

Here \( S_t^* : S_{-p'} \cap \mathcal{L}^r(B(0, R)) \rightarrow S_{-q} \) is a bounded linear operator for every \( R > 0 \) and \( \{Y_t\}_{t \geq 0} \) is a continuous semimartingale, which is locally of compact support.
The integral on the left hand side of (3.7), i.e. \( \int_0^t S^*_t \cdot dY_s \) is well defined (see \[17, \text{Chapter 4}\]).

Since \( \{Y_t\} \) is locally of compact support, there exists an increasing sequence of stopping times \( \{\tau_n\} \) such that \( \tau_n \downarrow \infty \) and for each \( n \), a.s. \( \text{supp}(Y^\tau_{t_n}) \subset B(0, R_n), \forall t \) for some \( R_n > 0 \). Since \( \{Y_t\} \) has continuous paths in \( S_{-p} \), without loss of generality, we assume \( \|Y^\tau_n\|_{-p} \leq n \). Let us also consider the partition, \( \mathcal{T} := \{0 = s_0 < s_1 < \cdots < s_n = t\} \), where \( |\mathcal{T}| := \max |s_{i+1} - s_i| \). Now, we can show

\[
\sum_{\{s_i \in \mathcal{T}\}} \left\| \int_{s_i}^{s_{i+1}} S^*_t L^* Y^\tau_{s} \, dr - S^*_t L^* Y^\tau_{s_i}(s_{i+1} - s_i) \right\|_q \to 0
\]

as \( |\mathcal{T}| \to 0 \). Similarly for the stochastic integral,

\[
\sum_{\{s_i \in \mathcal{T}\}} E \left\| \int_{s_i}^{s_{i+1}} S^*_t A^* Y^\tau_{s} \cdot dB_r - S^*_t A^* Y^\tau_{s_i}(B_{s_{i+1}} - B_{s_i}) \right\|_q^2 \to 0
\]

as \( |\mathcal{T}| \to 0 \). Hence,

\[
\begin{align*}
\int_0^{\tau_k} S^*_t \cdot dY^\tau_s \\
= \lim_{|\mathcal{T}| \to 0} \sum_{\{s_i \in \mathcal{T}\}} S^*_t Y^\tau_{s_i+1} - Y^\tau_{s_i} \\
= \lim_{|\mathcal{T}| \to 0} \sum_{\{s_i \in \mathcal{T}\}} \int_{s_i}^{s_{i+1}} S^*_t L^* Y^\tau_{s} \, dr + \lim_{|\mathcal{T}| \to 0} \sum_{\{s_i \in \mathcal{T}\}} \int_{s_i}^{s_{i+1}} S^*_t A^* Y^\tau_{s} \cdot dB_r \\
= \sum_{\{s_i \in \mathcal{T}\}} S^*_t L^* Y^\tau_{s_i}(s_{i+1} - s_i) \\
+ \lim_{|\mathcal{T}| \to 0} \sum_{\{s_i \in \mathcal{T}\}} S^*_t A^* Y^\tau_{s_i}(B_{s_{i+1}} - B_{s_i}).
\end{align*}
\]

where in the last but one equality, the second limit in the right hand side is taken in \( L^2(\Omega) \). Letting \( \tau_k \to \infty \), we get (3.6). Then,

\[
\begin{align*}
\int_0^t S^*_t A^* Y_s \, dB_s &= \int_0^t S^*_t \cdot dY_s - \int_0^t S^*_t L^* Y_s \, ds \\
(3.7) \quad &= \int_0^t S^*_t \cdot dY_s + \int_0^t dB^*_t Y_s.
\end{align*}
\]

Here, we have used the fact that, if \( x \in S_{-p} \cap \mathcal{E}'(B(0, R)) \) then \( \frac{d}{dx} S^*_t x = S^*_t L^* x \). Note that the second integral on the right hand side of (3.7), \( \int_0^t dS^*_t Y_s \) was defined in (2.5).

At the end, we show that the cross variation \([S^*, Y]_t = 0 \) for \( t \geq 0 \), where

\[
[S^*, Y]_t := \lim_{\max |t_{i+1} - t_i| \to 0} \sum_{i=0}^{n-1} (S^*_{t_{i+1}} - S^*_{t_i})(Y_{t_{i+1}} - Y_{t_i}).
\]
This follows from the fact that $\|\langle S^*, Y \rangle_{\tau_i} \|_{-q} = 0$, which can be verified as in Proposition 3.7. Now, from (3.7), we write the following integration by parts formula
\[
\int_0^t S_{t-s}^* A^* Y_s \cdot dB_s = \int_0^t S_{t-s}^* dY_s + \int_0^t dS_{t-s}^* Y_s = Y_t - S_t^* Y_0,
\]
which implies $Y_t = S_t^* \psi + \int_0^t S_{t-s}^* A^* Y_s \cdot dB_s$. This completes the proof. \(\Box\)

**Proposition 3.9.** [From mild solutions to martingale representations] Fix $x \in \mathbb{R}^d$ and consider the initial condition $\psi = \delta_x$ in (2.6). Then the mild solution representation (3.4) is equivalent to the martingale representation of square integrable functionals of the diffusion $\{X(t, x)\}$.

**Proof.** If the martingale representation holds, we have in particular, for every $f \in \mathcal{S}$, the explicit representation (see, for example [18]),
\[
(3.8) \quad f(X(t, x)) = \mathbb{E}f(X(t, x)) + \sum_{i=1}^r \int_0^t A_i S_{t-s} f(X(s, x)) dB_s^i.
\]
To see this, consider the function $g : [0, t] \times \mathbb{R}^d \to \mathbb{R}$ given by $g(s, x) := S_{t-s} f(x) = \mathbb{E}f(X(t-s, x))$. Then $g \in C^{1,2}([0, t] \times \mathbb{R}^d)$ and Itô formula gives
\[
\begin{align*}
\frac{d}{ds} g(s, x) &= A_i g(s, x) \partial_j S_{t-s} f(X(s, x)) \sigma_{ji}(X(s, x)) dB_s^i,
\end{align*}
\]
since $\partial_i g(s, x) + L g(s, x) = 0, \forall s \in [0, t], x \in \mathbb{R}^d$. But, from the definition of the operators $A_i$ in Section 2,
\[
\sum_{j=1}^d \partial_j S_{t-s} f(X(s, x)) \sigma_{ji}(X(s, x)) = A_i S_{t-s} f(X(s, x)),
\]
which implies (3.8).

Since $Z_t(\psi) = \delta_{X(t, x)}$ (see Example 2.3), we have by duality
\[
(3.9) \quad \langle f, \delta_{X(t, x)} \rangle = \mathbb{E}\langle f, \delta_{X(t, x)} \rangle + \sum_{i=1}^r \int_0^t \langle f, S_{t-s}^* A^*_i \delta_{X(s, x)} \rangle dB_s^i.
\]
Thus (3.4) holds. Conversely, if (3.4) holds then the strong solution $\{Z_t(\psi)\}$, given by (2.3) is also a mild solution and $Z_t(\psi) = \delta_{X(t, x)}$ (see Example 2.3). Hence
\[
\delta_{X(t, x)} = S_t^* \delta_x + \int_0^t S_{t-s}^* A^*_i \delta_{X(s, x)} \cdot dB_s
\]
\[
= \mathbb{E}\delta_{X(t, x)} + \sum_{i=1}^r \int_0^t S_{t-s}^* A^*_i \delta_{X(s, x)} dB_s^i.
\]
Now for any $f \in \mathcal{S}$, we get from the identity $f(X(t, x)) = \langle f, \delta_{X(t, x)} \rangle$ that (3.9) holds.

Using the above representation and Markov property, one can get martingale representations for functionals of the form $f_1(X(t_1, \cdot))f_2(X(t_2, \cdot)) \cdots f_k(X(t_k, \cdot))$, where $f_i \in \mathcal{S}$, as in [3][18]. Using density arguments, one can get representations for all square integrable functionals of the diffusion $\{X(t, x)\}$. \(\Box\)
Theorem 3.10. Let \( \{Y_t\} \) be a \((-p,-q)\) mild solution of (2.6). Then there exists \( q' > q \) such that the mild solution is a \((-p,-q')\) strong solution.

Proof. From (3.4), we have the equality in \( S_{-q} \)
\[
Y_s = S^*_s \psi + \int_0^s S^*_{s-r} A^* Y_r \cdot dB_r.
\]

Note that, from Proposition 2.5 we have the boundedness of the linear operator \( L^* : S_{-q} \cap E'(K) \to S_{-q'} \cap E'(K) \), for any \( q' > |q| + 4 \), where \( K \) is some compact set in \( \mathbb{R}^d \). In fact the same argument gives the boundedness of \( L^* : S_{-q} \to S_{-q'} \), for any \( q' > |q| + 4 \).

Hence, operating on both sides of the above equation by the linear operator \( L^* \), and integrating from 0 to \( t \) we obtain an equality in \( S_{-q'} \)
\[
\int_0^t L^* Y_s ds = \int_0^t L^* S^*_s \psi ds + \int_0^t \int_0^s L^* S^*_{s-r} A^* Y_r \cdot dB_r ds.
\]

Now, by applying stochastic Fubini and integration by parts formulas respectively on the R.H.S. of the above equation we get
\[
\int_0^t L^* Y_s ds = \int_0^t L^* S^*_s \psi ds + \int_0^t \int_0^s L^* S^*_{s-r} A^* Y_r ds \cdot dB_r
\]
\[
= S^*_t \psi - \psi + \int_0^t S^*_{t-s} A^* Y_s \cdot dB_s - \int_0^t A^* Y_s \cdot dB_s
\]
\[
= (S^*_t \psi + \int_0^t S^*_{t-s} A^* Y_s \cdot dB_s) - \psi - \int_0^t A^* Y_s \cdot dB_s
\]
\[
= Y_t - \psi - \int_0^t A^* Y_r \cdot dB_r.
\]

Hence \( Y_t = \psi + \int_0^t A^* Y_s \cdot dB_s + \int_0^t L^* Y_s ds \) in \( S_{-q'} \). This completes the proof. \( \Box \)

4. Uniqueness

We now consider the uniqueness of strong and mild solutions of (2.6). The uniqueness condition, viz. the Monotonicity inequality, involves both the domain and the range of the operators \( L^*, A^* \). For a compact subset \( K \) of \( \mathbb{R}^d \), it was shown in [21] that \( A^* : S_p \cap E'(K) \to L(\mathbb{R}^d, S_q \cap E(K)) \) and \( L^* : S_p \cap E'(K) \to S_q \cap E(K) \) are bounded linear operators, first when \( p, q \) are both positive satisfying \( p > |q| + 4 \) and then by duality when \( p, q \) are negative satisfying \( -q > [-p] + 4 \).

Definition 4.1 (Monotonicity inequality, [21 equation (4.2)]). Fix \( p, q \) both positive or both negative, such that \( A^* : S_p \cap E' \to L(\mathbb{R}^d, S_q \cap E(K)) \) and \( L^* : S_p \cap E' \to S_q \cap E(K) \) are linear operators. Say that the pair of operators \( (L^*, A^*) \) satisfies the \((p,q)\) Monotonicity inequality if
\[
2\langle \phi, L^* \phi \rangle_q + \sum_{i=1}^r \| A^*_i \phi \|^2_{q'} \leq C_K \| \phi \|^2_q, \forall \phi \in S_p \cap E'(K),
\]
for all compact subsets \( K \) of \( \mathbb{R}^d \). Here \( C_K \) is some positive constant depending on the set \( K \).
Theorem 4.2 ([21] Theorem 4.4). Let $p \geq 0$ and $q > \lceil p \rceil + 4$. If the $(-p, -q)$ Monotonicity inequality holds, then we have the uniqueness of $(-p, -q)$ strong solutions.

As a consequence of Theorem 4.2, we have the next result.

Corollary 4.3. Let $p, p', q$ be as in (3.1), (3.2), (3.3). Let $q'$ be as in Theorem 3.1. If the $(-p, -q')$ Monotonicity inequality holds, then we have the uniqueness of $(-p, -q)$ mild solutions.

Proof. If $\{Y_t^1\}$ and $\{Y_t^2\}$ are two $(-p - q)$ mild solutions of (2.6), then by Theorem 3.10 they are also $(-p, -q')$ strong solutions. If the $(-p, -q')$ Monotonicity inequality holds, then we have a.s. $Y_t^1 = Y_t^2$, $\forall t \geq 0$ in $S_{-q'}$. Since both $\{Y_t^1\}$ and $\{Y_t^2\}$ are $S_{-p}$ valued, we get the required uniqueness. □

We now describe a situation where the Monotonicity inequality holds. See [7, Theorem 2.1], [2, Theorem 4.6] for other cases where this inequality holds.

Theorem 4.4. Let $\sigma, b$ be as in (2.1). Fix $p \geq 5$. Then $(L^*, A^*)$ satisfies the $(p, 0)$ Monotonicity inequality.

Proof. From the remark about the boundedness of $A^*$ and $L^*$ made before Definition 4.1, it is easy to verify the inequality (4.1) for $\phi \in C_c^\infty(K) \subset S_p \cap E'(K)$, for any compact set $K$. Consider the operator $L^*_1 \psi := -\sum_{i=1}^d \partial_i (b_i \psi)$. Recall from Section 2 that we also use $\langle \cdot, \cdot \rangle$ for the inner product in $L^2$. We have

$$\langle L^*_1 \phi, \phi \rangle = -\sum_{i=1}^d \langle \partial_i b_i \phi, b_i \phi \rangle = \left\langle \left( -\sum_{i=1}^d \partial_i b_i \right) \phi, \phi \right\rangle - \sum_{i=1}^d \langle \partial_i \phi, b_i \phi \rangle$$

and hence

$$\langle L^*_1 \phi, \phi \rangle = -\frac{1}{2} \left\langle \left( \sum_{i=1}^d \partial_i b_i \right) \phi, \phi \right\rangle \leq C_b \|\phi\|^2;$$

where $C_b$ is a positive constant depending on $b$. Recall that $A^* \phi = (A^*_1 \phi, \ldots, A^*_r \phi)$ with $A^*_1 \psi = -\sum_{j=1}^d \partial_j (\sigma_{j1} \psi)$. Also define $L^*_2 \phi := \frac{1}{2} \sum_{i,j=1}^d \partial^2_{ij}((\sigma \sigma^T)_{ij} \phi)$. For any $1 \leq i, j \leq d$, integration by parts yields

$$\langle \partial_j (\sigma_{jk}) \phi, \sigma_{ik} \partial_i \phi \rangle = \int_{\mathbb{R}^d} \partial_j (\sigma_{jk}) \phi \sigma_{ik} \partial_i \phi$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \partial_j (\sigma_{jk}) \sigma_{ik} \partial_i (\phi^2) = -\frac{1}{2} \int_{\mathbb{R}^d} \partial_i (\partial_j (\sigma_{jk}) \sigma_{ik}) \phi^2$$

Using above observation, we have

$$\sum_{k=1}^r \| A^*_k \phi \|^2$$

$$= \sum_{k=1}^r \left\langle \sum_{j=1}^d \partial_j (\sigma_{jk}) \phi, \sum_{i=1}^d \partial_i (\sigma_{ik} \phi) \right\rangle$$
\[
= \sum_{k=1}^{r} \sum_{i,j=1}^{d} \left[ \langle \partial_j (\sigma_{jk}) \phi, \partial_i (\sigma_{ik}) \phi \rangle + \langle \partial_j (\sigma_{jk}) \phi, \sigma_{ik} \partial_i \phi \rangle + \langle \sigma_{jk} \partial_j \phi, \partial_i (\sigma_{ik}) \phi \rangle \right]
+ \sum_{k=1}^{r} \left\langle \sum_{j=1}^{d} \sigma_{jk} \partial_j \phi, \sum_{i=1}^{d} \sigma_{ik} \partial_i \phi \right\rangle
\]
\[
= \sum_{k=1}^{r} \sum_{i,j=1}^{d} \left[ \langle \partial_j (\sigma_{jk}) \phi, \partial_i (\sigma_{ik}) \phi \rangle + 2 \langle \partial_j (\sigma_{jk}) \phi, \sigma_{ik} \partial_i \phi \rangle \right]
+ \sum_{k=1}^{r} \left\langle \sum_{j=1}^{d} \sigma_{jk} \partial_j \phi, \sum_{i=1}^{d} \sigma_{ik} \partial_i \phi \right\rangle
\]
\[
= \sum_{k=1}^{r} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left[ \partial_j (\sigma_{jk}) \partial_i (\sigma_{ik}) - \partial_i (\partial_j (\sigma_{jk}) \sigma_{ik}) \right] \phi^2 + \sum_{k=1}^{r} \left\langle \sum_{j=1}^{d} \sigma_{jk} \partial_j \phi, \sum_{i=1}^{d} \sigma_{ik} \partial_i \phi \right\rangle
\]
\[
= - \sum_{k=1}^{r} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \partial_{ij}^2 (\sigma_{jk}) \sigma_{ik} \phi^2 + \sum_{k=1}^{r} \left\langle \sum_{j=1}^{d} \sigma_{jk} \partial_j \phi, \sum_{i=1}^{d} \sigma_{ik} \partial_i \phi \right\rangle
\]

Another integration by parts argument yields,

\[
\langle L^*_2 \phi, \phi \rangle = \frac{1}{2} \sum_{i,j=1}^{d} \left\langle \partial_{ij}^2 \left( \sum_{k=1}^{r} \sigma_{ik} \sigma_{jk} \phi \right), \phi \right\rangle
\]
\[
= - \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \langle \partial_j (\sigma_{ik} \sigma_{jk}) \phi + \sigma_{ik} \sigma_{jk} \partial_j \phi, \partial_i \phi \rangle
\]
\[
= - \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \int_{\mathbb{R}^d} \partial_j (\sigma_{ik} \sigma_{jk}) \partial_i \phi \partial_i \phi - \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \langle \sigma_{ik} \sigma_{jk} \partial_j \phi, \partial_i \phi \rangle
\]
\[
= - \frac{1}{4} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \int_{\mathbb{R}^d} \partial_j (\sigma_{ik} \sigma_{jk}) \partial_i (\phi^2) - \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \langle \sigma_{ik} \sigma_{jk} \partial_j \phi, \partial_i \phi \rangle
\]
\[
= - \frac{1}{4} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \int_{\mathbb{R}^d} \partial_{ij}^2 (\sigma_{jk}) \sigma_{ik} \phi^2 - \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{r} \langle \sigma_{ik} \sigma_{jk} \partial_j \phi, \partial_i \phi \rangle
\]

Then

\[
2 \langle L^*_2 \phi, \phi \rangle + \sum_{k=1}^{r} \| A^*_k \phi \|^2 \leq C_{\sigma,K} \| \phi \|^2,
\]

where \( C_{\sigma,K} \) is a positive constant depending on \( \sigma \) and \( K \). Now adding (4.2) and (4.3) together, we get the required inequality. \( \square \)

As an application of Theorem 4.4 we get the next result, wherein we note that the initial value \( \psi \) need no longer be of compact support.

**Corollary 4.5.** Let \( \psi \in S \) and \( p \geq 5 \). We have the existence and uniqueness of \( (p,0) \) strong solutions of (2.4).

**Proof.** Let \( \psi \) be a \( C^\infty_c(\mathbb{R}^d) \) function. Recall that \( \{ Z_i(\psi) \} \) is \( S_p \cap E' \) valued for \( p \geq 5 \), is locally of compact support and solves (2.6) ((see Example 2.2) and Theorem 2.11).

From Theorem 4.4 we get the uniqueness.
Since $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}$ (in $\| \cdot \|_p$), by density arguments, the result follows.

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