Finite-time scaling via linear driving

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Abstract. We propose systematic finite-time scaling and show that it can be used to determine both static and dynamic critical properties. The controllable finite time scale that affects the scaling of a system and that serves as an analogue of the system size in finite-size scaling is achieved simply but generally by manipulating the dynamics with an external field varying linearly with time. An analytic renormalization-group theory is developed to justify the scaling. Monte Carlo simulation results for two- and three-dimensional Ising models agree with existing ones and confirm the scaling and the method.

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1. Introduction

Finite-size scaling (FSS) is an important concept that has been developed into a powerful method to extract critical properties from numerical simulations of finite systems [1]–[3]. It is based on the ansatz that when the characteristic length $L$ of a system is smaller than the correlation length $\xi_\infty$ of the infinite system which diverges at the critical temperature $T_c$, $L$ becomes a relevant length scale and the order parameter $M$, for instance, obeys the FSS

$$M = L^{-\beta/\nu} g(L/\xi_\infty) = L^{-\beta/\nu} g'(\tau L^{1/\nu}),$$

where $g$ and $g'$ are scaling functions, $\beta$ and $\nu$ are the finite-size critical exponents that are assumed to be the same as those in the thermodynamic limit [4], and $\tau = |T - T_c|/T_c$. At $T_c$, $\tau = 0$ and hence $M \propto L^{-\beta/\nu}$. As a result, the exponent ratio $\beta/\nu$ can be determined by measuring $M$ for a series of $L$.

Noting the effectiveness of the method, it is quite surprising that very few works [5, 6] have been performed for her sister, finite-time scaling (FTS) in statistical physics, as the relaxation time $t_{eq}$ also diverges at $T_c$, leading to dynamic scaling. Here we propose a systematic realization of such a scaling and set up a method to show that it can also be powerful for measuring critical properties.

We now argue that the method of linear driving, namely of applying an external field varying linearly with time to a system near its criticality, is just such a realization. Note first that a direct analogue of the FSS by measuring variables at a series of time does not work, because for a statistical system studied numerically by the usual Monte Carlo (MC) approach, one needs a sufficiently long time to sample its configuration space in order for its variables to be measured correctly. Note also that a linear driving with a rate $R$ is readily realized in experiments by a triangular zigzag wave and is widely used, for example, in thermal analysis (e.g. differential scanning calorimeters (DSC)) [7]. In fact, we shall see later on that a linear driving is not only a simple but also a general method for implementing FTS. The driving imposes on the system an external time scale proportional to $R^{-1}$ [8]. Therefore, roughly for $c_1 t_{mic} \ll R^{-1} \ll c_2 t_{eq}$, where $t_{mic}$ denotes a microscopic time scale and the $c$’s set the dimension right (see below), the time scale of the system under driving is large enough, on the one hand, for microscopic details to be smeared out and scaling to emerge (or equivalently, the relevant correlation length large enough), and is not larger, on the other hand, than the intrinsic equilibrium relaxation time so that it becomes a relevant scale and thus an FTS regime emerges, within which varying $R$ is equivalent to varying $L$ in the FSS regime.

Within this regime, scaling can be readily derived. One assumes that even for a time-dependent $H$, $M$ obeys

$$M(t, \tau, H) = b^{-\beta/\nu} M(tb^{-z}, \tau b^{1/\nu}, H b^{\beta \delta/\nu}),$$

under a renormalization-group (RG) transformation of a length rescaling factor $b$ [2], where $t$ is the time, and $z$ and $\delta$ are critical exponents. Because $H = R t$, only two are independent out of the trio $H$, $R$ and $t$. One may thus replace $H$ in (2) with $R$ and assume the latter has an RG eigenvalue of $r_H$ [9] because it should also rescale near $T_c$ upon renormalization, where the subscript $H$ indicates that $H$ is varying in conformity with the paper by Zhong [10]. As $H = R t$, a scaling law

$$r_H = z + \beta \delta / \nu$$

is obtained.
follows, which may be regarded as a definition of \( r_H \), reflecting the rescaling of \( R \) with that of \( H \) and \( t \) since \( R = H/t \). Setting \( b = R^{-1/r_H} \), one finds
\[
M(t, \tau, R) = R^{\beta/\nu} m_1(t R^{z/r_H}, \tau R^{-1/vr_H}),
\]
(4)
or, in terms of the field \( H \),
\[
M(H, \tau, R) = R^{\beta/\nu} m_2(H R^{-\beta/s/r_H}, \tau R^{-1/vr_H}),
\]
(5)
where \( m_1 \) and \( m_2 \) are scaling functions. Equations (4) and (5) are the FTS analogue of (1). They appear more subtle but, together with the ansatz (2), can be derived analytically from a straightforward generalization of a dynamic RG theory \([10]\). The physics, however, is totally identical. The FTS regime is defined by
\[
\tau R^{-1/\nu} r_H \lesssim 1 \text{ or } R^{-1} \lesssim \tau^{-\nu} r_H \sim t_{eq}/H_{eq},
\]
which extends to all small enough rates for \( T = T_c \) because \( t_{eq} \propto \xi_{\infty} \) diverges.

In the following, we shall first provide an analytical justification of equation (2) and hence equations (4) and (5) by a straightforward extension of a previous RG theory at \( T_c \) \([10]\) to off-critical regions (section 2.1). It is remarkable that the assumption for critical exponents in FSS is not needed in FTS. To demonstrate the effectiveness of the FTS, we then develop a method for obtaining critical properties (section 2.2). We also discuss in section 2.3 the distinction of the FTS method to the usual critical dynamics and the advantages of the linear driving. To test the method, we then apply the method to the two-dimensional (2D) and 3D Ising models in section 3. Finally, a summary is presented in section 4.

2. Theory and method

In this section, we shall extend a previous nonequilibrium RG theory of linear driving at \( T_c \) to off-criticality to justify the FTS and develop an FTS method to determine critical properties.

2.1. RG theory of linear driving in the vicinity of \( T_c \)

Consider the usual \( \phi^4 \) model under an external field \( H \) conjugate to a nonconserved order parameter \( \phi \) with the free-energy functional
\[
F[\phi] = \int \mathrm{d}r \left\{ \frac{1}{2} \tau \phi^2 + \frac{1}{4!} g \phi^4 + \frac{1}{2} (\nabla \phi)^2 - H \phi \right\},
\]
(6)
where \( g \) is a coupling constant. For simplicity, we have used \( \tau \) defined in the introduction directly and shall not distinguish between the mean-field \( T_c \) and the real one. The dynamics is governed by \([11]\)
\[
\frac{\partial \phi}{\partial t} = -\lambda \frac{\delta F[\phi]}{\delta \phi} + \xi,
\]
(7)
where \( \xi \) is a Gaussian white noise satisfying
\[
\langle \xi(r, t) \rangle = 0, \quad \langle \xi(r, t) \xi(r', t') \rangle = 2\lambda \delta(r - r') \delta(t - t'),
\]
(8)
and \( \lambda \) is a kinetic coefficient. In order to use systematic field-theoretic methods, we recast the dynamics into an equivalent field theory with a dynamic functional \([12]\),
\[
I[\phi, \bar{\phi}] = \int \mathrm{d}r \mathrm{d}t \left\{ \bar{\phi} \left[ \phi + \lambda (\tau - \nabla^2) \phi + \frac{1}{3!} \lambda g \phi^3 - \lambda H \right] - \lambda \bar{\phi}^2 \right\},
\]
(9)
where $\tilde{\varphi}$ is an auxiliary response field [13]. Expectation values can then be obtained by taking appropriate derivatives of the generating functional

$$W[h, \tilde{h}] = \ln \int D(\varphi, \tilde{\varphi}) \exp \left[ -I[\varphi, \tilde{\varphi}] + \int \text{d}t (h \varphi + \tilde{h} \tilde{\varphi}) \right],$$

with respect to the external sources $h$ and $\tilde{h}$ that conjugate, respectively, to $\varphi$ and $\tilde{\varphi}$.

In the absence of a time-dependent external field, the mean-field theory of the $\varphi^4$ critical behavior is well known. A perturbation expansion around it works only for $d > 4$. For $d \leq 4$, however, the expansion is plagued with infrared divergences as $\tau \to 0$. These divergences are related to the ultraviolet ones, which can be removed by renormalization [2, 14]. In particular, the theory can be rendered finite for $d \leq 4$ by the renormalization factors defined as

$$\varphi \to \varphi_0 = Z_{\varphi}^{1/2} \varphi, \quad \tilde{\varphi} \to \tilde{\varphi}_0 = Z_{\tilde{\varphi}}^{1/2} \tilde{\varphi}, \quad g \to g_0 = N_d \mu^x Z_{\varphi}^{-2} Z_u u,$$

$$\lambda \to \lambda_0 = (Z_{\varphi} / Z_{\tilde{\varphi}})^{1/2} Z\lambda, \quad \tau \to \tau_0 = Z_{\varphi}^{-1} Z_\lambda \tau + \tau_c,$$

where $\epsilon = 4 - d$, $N_d = 2 / [(4\pi)^{d/2} \Gamma(d/2)]$, $\Gamma$ is the Euler gamma function, $\mu$ is an arbitrary momentum scale, $\tau_c$ is the fluctuation shift of the mean-field $T_c$, which can be neglected if dimension regulations [15] are employed, and the subscripts 0 indicate bare variables. These factors will then give rise to the usual critical exponents, which we shall show shortly [2, 14].

In the presence of a spatially uniform and linearly time-dependent external field $H$ with a small rate $R$, one notes that no new divergence except the extrinsic one at $t \to \infty$ or $\omega \to 0$ in the frequency domain is generated. As a result, no new renormalization factor $Z$ besides (11) has to be introduced to cure the divergences, except the possible initial slip [16] that does not contribute because we start with an equilibrium state (see section 3 below). In order to deal with the linear driving [10], we perform the renormalization at the critical point $\tau = 0$ and $H = 0$ and take $\tau$ and $H$ as the sources of the composite operators $\tilde{\varphi} \varphi$ and $\tilde{\varphi}$, respectively. This requires the renormalization factor of $H$, which is

$$H \to H_0 = Z_{\varphi}^{-1/2} H,$$

because $H \varphi$ is not renormalized [14]. By exploiting the fact that the bare quantities are independent of $\mu$ and expanding the averaged order parameter

$$M(\tau, H) = \langle \varphi(\tau, H) \rangle = G_{10,0}(\tau, H) = \sum_{N,N'} \frac{1}{N!N'} ! \lambda^{N+N'} \tau^{N'} H^N G_{1N,N'}(0, 0)$$

in a Taylor’s series in $\tau$ and $H$ at every definite time instant, where the Green function $G_{1N,N'}$ is defined as

$$G_{1N,N'} = \frac{\delta^{1+N+N'} W[h, \tilde{h}, \tau]}{\delta h \delta \tilde{h} \delta^{N+N'} \tau},$$

the RG equation is thus

$$\left( \mu \partial_\mu + \xi \lambda \partial_\lambda + \beta \partial_\beta + \gamma_{\varphi^2} \tau \partial_\tau + \frac{1}{2} \gamma H \partial_H + \frac{1}{2} \gamma \right) M = 0,$$

with the Wilson’s functions being defined as derivatives at fixed bare parameters,

$$\zeta(u) = \mu \partial_\mu \ln \lambda, \quad \gamma(u) = \mu \partial_\mu \ln Z_{\varphi}, \quad \gamma_{\varphi^2}(u) = \mu \partial_\mu \ln \tau, \quad \beta(u) = \mu \partial_\mu u.$$

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where \( \partial_i \) indicates the partial derivative with respect to \( i \). No new Wilson’s function due to the driving has to be introduced.

At the fixed point \( u = u^* \), \( \beta(u^*) = 0 \), the solution of (15) is

\[
M(\lambda, \tau, H, u, \mu) = \rho^{\nu^*/2}M(\lambda, \rho^{\nu^*}, \tau \rho^{\nu^*}, H \rho^{\nu^*/2}, u^*, \mu \rho),
\]

(17)

where \( \rho \) is a running variable and starred quantities denote corresponding values at the fixed point. On the other hand, from the naive dimensions of various variables

\[
[r] \sim \mu^{-1}, \quad \tau \sim \mu^2, \quad u \sim \mu^0, \quad \lambda t \sim \mu^{-2},
\]

\[\phi \sim \mu^{(d-2)/2}, \quad H \sim \mu^{(d+2)/2}, \quad \bar{\phi} \sim \mu^{(d+2)/2},\]

(18)

one obtains a homogeneous form

\[
M(\lambda, \tau, H, u, \mu) = \rho^{-(d-2)/2}M(\lambda, \tau \rho^{-2}, \tau \rho^{1/v}, H \rho^{1/v}),
\]

(19)

Combining (17) and (19) leads to

\[
M(t, \tau, H) = \rho^{-\beta/v}M(t \rho^{-z}, \tau \rho^{1/v}, H \rho^{1/v}),
\]

(20)

with the critical exponents given by

\[
\eta = \gamma^*, \quad v^{-1} = 2 - \gamma^* - \beta/v = (d - 2 + \eta)/2,
\]

\[\delta = (d + 2 - \eta)/(d - 2 + \eta), \quad z = 2 + \gamma^*,\]

(21)

where we have chosen \( \lambda \) as the time unit.

As we perform the renormalization at the critical point and utilize the scheme of dimension regulations and minimal subtraction with \( \varepsilon \) expansion [15], a scheme in which dynamics decouples from statics [17], all the \( Z \) factors can be chosen to be identical to the usual \( \varphi^4 \) model. As a result, all the static critical exponents and the dynamic critical exponent \( z \) determined from (21) are identical to those of the usual scalar Model A [10, 14].

Now the linearly varying field \( H = Rt \) can be complemented to (20) [10]. Equation (20) implies \( H \) and \( t \) transform as \( H' = H \rho^{\beta/v} \) and \( t' = t \rho^{-z} \), respectively. In the vicinity of \( T_c \), \( R \) should also scale upon renormalization. Suppose that it transforms as \( R' = R \rho^{\varepsilon} \), since \( H' = R t' \), then the scaling law (3) follows. Replacing the time variable \( t \) with \( R \) in (20) gives rise to (2), which agrees, of course, with the result of [10] when \( \tau = 0 \). Setting \( \rho = R^{-1/v} \) further, one then arrives at (5). As only two out of the trio \( t, R \) and \( H \) are independent, one can obtain similarly,

\[
M(t, \tau, R) = t^{-\beta/v}m_3(R t^1/v, \tau t^1/v),
\]

(22)

where \( m_3 \) is also a scaling function.

Crossover between the FTS and the equilibrium behavior can be readily analyzed. For equations (4) and (5), as pointed out in section 1, the FTS regime is defined by \( \tau R^{-1/v} H \lesssim 1 \) or \( R^{-1} \lesssim t_{eq}/H_{eq} \), while for large \( \tau R^{-1/v} H \) or small \( R \ll |\tau|^{1/v} H \sim H_{eq}/t_{eq} \), the field varies so slowly that although it is changing, before it changes, the system has already equilibrated so that the usual equilibrium scaling

\[
M(\tau, H) = \tau^\beta f_1(H \tau^{-\delta})
\]

(23)

emerges, where \( f_1 \) is another scaling function. For (22), note that it can be written as

\[
M(\tau, t, R) = t^{-\beta/v}m_3[R t^1/v, (t/t_{eq})^1/v].
\]

(24)
It is then clear that for \( t < t_{\text{eq}} \), the FTS regime appears, while for \( t > t_{\text{eq}} \), equilibrium behavior (23) appears essentially. Therefore, all \( m \)'s have similar asymptotic behavior as,

\[
m_i(x, y) \rightarrow \begin{cases} 
m_4(x), & \text{for } y \to 0, \\
y^{\beta} f_i(xy^{-\beta_3}), & \text{for } y \to \infty, \end{cases} (i = 1, 2, 3)
\]

(25)

where \( m_4 \) is a scaling function.

We have therefore justified (2). It is remarkable in this formulation that the critical exponents are naturally identical with those of the usual infinite time systems. Also, no new independent exponent has to be introduced. In fact, since an expansion of the partition function in terms of a space–time-dependent magnetic field generates correlation functions, the scaling properties of thermodynamic functions of a time-dependent magnetic field such as equations (4) and (5) follow naturally once the field is so small that one still remains in the critical region.

2.2. An FTS method

We develop in this section an FTS method to determine critical properties and point out its relation to existing methods. As there is one more variable in (5) compared to (1), the usual FSS method cannot be utilized. However, in the FTS regime, the external time scale dominates and drives the system out of equilibrium. Hysteresis then emerges even at criticality. Thus, one may invoke the hysteresis loop area \( A = \int M dH \), which reflects the energy loss in the nonequilibrium process by scanning \( H \) back and forth with the same rate as \( H = \pm H_0 \mp R t \) to form a hysteresis loop and integrating over \( H \), where \( H_0 > 0 \) is a constant that is chosen to be sufficiently large to make \( M \) saturated but does not affect the results otherwise. Note that in this case the critical point at \( H = 0 \) is shifted from \( t = 0 \) to \( t_0 \equiv H_0/R \) and so \( H = \mp R(t - t_0) \) and hence the only change is to replace \( t \) in (2) with \( t - t_0 \). This shows the advantage of the linear driving. From (5), one then obtains the area scaling with respect to a single variable as

\[
A(\tau, R) = R^{\beta(\delta+1)/\nu H} f(\tau R^{-1/\nu H})
\]

(26)

after integrating out \( H \), where \( f \) is a scaling function. No dependence on \( H_0 \) appears as borne out by numerical results.

Equation (26) is now similar to (1). At \( \tau = 0 \), exact power-law behavior emerges, namely

\[
A|_{\tau=0} \propto R^{\beta(\delta+1)/\nu H}.
\]

(27)

Consequently, one can utilize a series of different time scales \( R^{-1} \) and find relevant exponents similar to the case of FSS. When \( \tau \neq 0 \), there are corrections by the scaling function \( f(\tau R^{-1/\nu H}) \). This can be used to determine \( T_c \); it is the temperature at which deviations from the power-law behavior between \( A \) and \( R \) at various \( T \)'s minimize.

Another quantity that can be easily measured is the coercivity \( H_c \), that is the field \( H \) at which \( M = 0 \). One solves from (5) \( H \) at \( M = 0 \), giving \( H_c = R^{\beta/\nu H} h(\tau R^{-1/\nu H}) \), where \( h \) is also a scaling function. Again at \( \tau = 0 \), exact power law follows:

\[
H_c|_{\tau=0} \propto R^{\beta/\nu H}.
\]

(28)

From \( n_H = \beta/\nu H \) and \( n'_H = \beta(\delta+1)/\nu H \), \( \delta \) and \( \beta/\nu H \) or \( \beta/\nu z \) can be determined. The hyperscaling law \( \beta(\delta+1) = d \nu \) and (3) can then yield \( z \).

To determine other exponents, we differentiate \( A \) with respect to \( \tau \) at \( \tau = 0 \), leading to

\[
\frac{\partial A}{\partial \tau}|_{\tau=0} \propto R^0,
\]

(29)

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where $a_1 = \beta(\delta + 1)/\nu r_H - 1/\nu r_H$. From $n_H, n'_H$ and $a_1$, all the exponents can thus be expressed in them as

\[
\begin{align*}
\delta &= n_H/(n'_H - n_H), \\
\beta/\nu &= (n'_H - n_H)/n'_H, \\
z &= d(1-n_H)/n'_H, \\
r_H = d/n'_H, \\
\beta &= (n'_H - n_H)/(n'_H - a_1), \\
\nu &= n'_H/d(n'_H - a_1).
\end{align*}
\]  

(30)

Note that in (30) the first two lines require only $n_H$ and $n'_H$, whereas the last line needs $a_1$.

This FTS method has got some benefit from the techniques of the short-time critical dynamics [18, 19] in determination of $T_c$ and using derivative, although the approach is different. The latter takes advantage of the results that initial evolution may show scaling [16, 20] and can thus obtain $T_c$ and critical exponents with high accuracy without being subject to critical slowing down. The dynamics of the system remains, however, unchanged. For the present method, we avoid the critical slowing down via manipulating the dynamics of a system near $T_c$ by applying to it a linearly varying external field. There are many other methods that may either measure a similar amount of critical information or avoid critical slowing down. However, few methods [21] can achieve both and we shall not list them here. Yet, methods with linear driving have been employed to measure critical properties [10, 22] and study first-order phase transitions [9, 23]. One of those methods is to combine the linear driving with MCRG [22]. Another method is to use the known static exponents to determine the dynamic ones at a known $T_c$ [10]. In this latter case, the linear driving of either temperature at $H = 0$ or external field at $T_c$ was studied independently. The influence of temperature below $T_c$ on the field driving has been discussed [9], but without an analytical theory. Here, we consider both temperature and external field simultaneously and mix them with the idea of FTS into an effective and systematic nonequilibrium method to determine both static and dynamic critical properties.

2.3. Discussions

We discuss in this subsection the distinction of the FTS method from the usual critical dynamics [11, 24, 25] including the short-time dynamics [16], [18]–[20] and the advantages of the linear driving.

First, we would point out that the FTS method is physically distinct from the usual critical dynamics [11, 24, 25] including the short-time dynamics [16], [18]–[20] and the advantages of the linear driving.

One might call the short-time New Journal of Physics 12 (2010) 043036 (http://www.njp.org/)}
dynamics a form of FTS if one likes, but the present one is more similar in spirit to the FSS in that there is an external characteristic scale in both cases to play with.

Numerical simulations have been an important integrated part to study critical phenomena [3]. However, usual numerical methods including the FSS suffer from the notoriously critical slowing down because a system takes, to equilibrate, a long time that diverges at the critical point where the methods work [26, 27]. To overcome critical slowing down, one method is the short-time critical dynamics [16, 20], which concentrates only on the initial evolution of observables and can thus avoid it. As critical properties can be derived from the short-time scaling, this method has been widely employed to obtain $T_c$ and critical exponents with high accuracy [19, 21]. The dynamics of the system remains, however, unchanged. Another recent approach [28] is to sum over the energy distribution that depends in principle only on the Hamiltonian instead of the state distribution that relies heavily on evolution and thus may obtain equilibrium properties without suffering from critical slowing down. Cluster algorithms can also effectively avoid critical slowing down but the price to pay is that the original dynamics is completely altered and thus cannot by studied within the methods [26, 27]. The FTS, on the other hand, explores a different strategy to deal with the problem of critical slowing down, namely, a strategy to manipulate the dynamics of a system near its criticality by applying to it an external field varying linearly with a rate $R$, the inverse of which serves as a finite time scale to dismiss the problem. In particular, for the present method, a system only needs roughly a time of $(c_1 R)^{-1}$ to be stationary and then is dominated by the external time scale even at $T_c$. Although $R^{-1}$ needs to be large in order to probe asymptotic behavior, the time, albeit long, required to reach a prescribed $H$ such as $H_c$ is predictable. Consequently, critical slowing down does not constitute a problem since the linear driving has converted it into visible processes of running small $R$’s. This may thus give rise to more accurate results. Moreover, contrary to the cluster methods [26, 27], information about the original dynamics survives and can be determined within the method as shown in last subsection.

We have established in section 2.1 that there is a single-parameter FTS with the field rate $R$, which combines the usual scaling transformations of $H$ and $t$. One can then understand why the linear driving with a single rate parameter is the simplest but most general method.

Firstly, it is the simplest method to relate $H$ and $t$. Consider a driving of the form $H = R t^2$. If the critical point is at $H = 0$ and $t = 0$, there are two methods to obtain the RG eigenvalue called $r_1$ of $R_1$ in this case. The first one makes direct use of the transformations of $H$ and $t$ as we used in obtaining $r_H$. If $t' = t b^{-2}$, then $r^2$ also rescales as $(r^2)' = (t^2) b^{-2z}$. Thus $r_1 = 2z + \beta \delta / \nu$, similar to the scaling law (3). The second method is to use the rate of the field, denoted by $W$, which is $W = 2 R t$, a time-dependent function. Assuming that the RG eigenvalue of $W$ satisfies again the scaling law (3), one sees that $r_H = r_1 - z$ since $W = 2 R t$. Consequently, one again has $r_1 = 2z + \beta \delta / \nu$. One sees therefore that a driving of the form $R t^2$ only introduces an unnecessary factor of 2.

However, as pointed out above, this is only true when $t = 0$ is the critical point. When one wants to obtain the hysteresis area $A$, one has to use $H = H_0 + R_1 t^2$ to obtain a saturated hysteresis loop (see section 3 below). In this case, the critical point at $H = 0$ is not at $t = 0$ but at $t = t_0$ because $H = R_1 (t^2 - t_0^2)$. As a result, the scale transformations should become complicated and one should no longer have the simple area scaling as (26) displayed. This therefore shows the advantage of the linear driving; its initial value can be translated arbitrarily without changing its form.

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We note in passing that in the RG theory, section 2.1, time is introduced through dimension analysis (19). Also, because we have ignored possible initial slip, the time should extend from $-\infty$ to $+\infty$. Consequently, $t$ itself should scale in principle. But when it is related to $H$, its initial value matters since scaling only exists in the vicinity of $H = 0$, the critical point.

Moreover, for a usual two-parameter driving $H = H_0 \sin \omega t$, one might think that the inverse frequency $\omega^{-1}$ of the sinusoidal field would be a better time scale, which was assumed previously [6]; this could work, however, only when the initial climbing part of a field cycle matters, namely, when $H \sim H_0 \omega t$ and thus $R \sim \omega$ for a large time scale or small frequency. In this case, the two parameters just reduce essentially to a single one. In general, when a full period of the field is needed, the amplitude $H_0$ then matters, because the rate is $R = dH/dt = H_0 \omega \cos \omega t$, a complicated function of both $H_0$ and $\omega$ as well as $t$ similar to the above example! This may have profound consequences on experiments that apply a sinusoidal field to probe scaling behavior.

We have always assumed that the size of the system $L$ is so large that there are no finite-size effects. Formally the size effects can be readily taken into account by including $L$ in (2). However, one may need a knowledge of the dependence of the relaxation time on the size [29]. We shall not pursue this issue further here but instead leave it for future investigation. Nevertheless, one can simply test the validity of FTS by checking the size effects numerically.

To conclude this section, we would emphasize again that linear driving is amenable to experiments and thus the FTS method may be used as a possible convenient tool to determine critical properties experimentally.

3. Simulation results

Now we test the method using MC simulations on the dynamic Ising model. The Hamiltonian of the model under an external field $H$ is

$$\mathcal{H} = -\frac{J}{k_B T} \sum_{i,j} S_i S_j + H \sum_i S_i,$$

(32)

where all symbols are standard. For simplicity, we shall adopt a unit in which $k_B = J = 1$; and the time unit is MC steps per spin. The usual Metropolis algorithm is applied to 2D and 3D lattices, whose $T_c$ are known to be 2.269 185 and 4.511 529 [30], respectively. The lattice sizes used are $500^2$ and $100^3$. We have checked that for sufficiently large lattice sizes, smaller lattices only enlarge fluctuations but yield nearly identical results. However, too small lattices might show finite-size effects as pointed out above and extrapolations might be needed for accurate results. To speed up the simulations, we use sequential instead of random sampling applied in [10]. The results below are, however, independent of this choice and thus provide direct evidence for universality in dynamics. For each rate $R$, we start with all spins pointing down at $-H_0$ at which several MC steps are sufficient to equilibrate the system due to off-criticality, and then increase $H$ to $H_0$ and decrease it to $-H_0$ again to form a sample of the hysteresis loop. Fifty such independent samples are used on average. For this sample size, the smallest rate $10^{-6}$ used in 3D, which takes up most of the time needed for all the simulations in 3D, needs less than $10^6$ time steps, though the smaller the rate, the smaller the hysteresis loop, as below. Larger sample sizes increase can be seen from figures 1 and 2 the statistical precision of the results, while sufficiently small rates are essential for accuracy because, firstly, $R$ must meet
Table 1. Standard deviations (SD×10⁻⁵) from the power-law behavior as $T$ varies around $T_c$ for 2D and 3D.

|   | 2D     | 2D     | 2D     | 2D     | 2D     | 2D     | 2D     | 2D     | 2D     | 2D     |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|   | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    |
|   | 2.219  | 2.259  | 2.267  | 2.268  | 2.269  | 2.269  | 2.270  | 2.271  | 2.273  | 2.279  | 2.299  |
| SD| 1216   | 544    | 367    | 334    | 321    | 315    | 351    | 384    | 450    | 638    | 2072   |

|   | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     | 3D     |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|   | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    | $T$    |
|   | 4.4615 | 4.5015 | 4.509  | 4.510  | 4.511  | 4.511  | 4.512  | 4.513  | 4.516  | 4.5215 | 4.5415 |
| SD| 3034   | 1875   | 732    | 625    | 564    | 530    | 583    | 749    | 1054   | 2057   | 31855  |

Figure 1. The hysteresis area $A$ plotted with error bars versus $R$ in logarithmic scales for (a) 2D and (b) 3D. Lines are linear fits at $T_c = 2.2692$ and $4.5115$ for 2D and 3D, respectively. Large errors for small rates and $T > T_c$ are due to their vanishingly small hysteresis.

$R^{-1} \gg c_t t_{\text{mic}}$ and, secondly, the large hystereses with their large standard errors for large rates overwhelm the small ones and thus affect the fitting.

3.1. Determination of $T_c$

Dependences of $A$ on $R$ and $T$ for 2D and 3D are presented in logarithmic scale in figure 1. One sees that $A$ decreases reasonably with increasing $T$ and the time scale $R^{-1}$, as there is a longer time to equilibrate. Near $T_c$, the relation between $A$ and $R$ is almost linear, but it curves up and down below and above $T_c$, respectively, for small $R$’s. Deviations from the power law between $A$ and $R$ when $T$ varies are given in table 1. The minimum deviation appears at $T = 2.2692$ for 2D and $T = 4.5115$ for 3D, which are consistent with the known $T_c$’s. Their precision is 0.002, within which the measured values of $A$ at the same $R$ overlap together due to the limited sample size.

3.2. Critical exponents

Having determined $T_c$, we can then measure $n_H'$ and $n_H$ from the slopes of $A$ and $H_c$ versus $R$, respectively, at $T_c = 2.2692$ for 2D and $T_c = 4.5115$ for 3D in logarithmic scales (figures 1 and 2(a)). To determine $a_1$, we use differences at $T = 2.259$ and 2.279 for 2D and $T = 4.5015$ and 4.5215 for 3D to approximate the derivatives. As $A$ is smaller than one, we choose $\partial A/\partial \tau$ instead of $\partial \ln A/\partial \tau$ employed in [19] in order to reduce error through error propagations. The

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Table 2. Measured and derived exponents compared with available results.

| $d$ | $n_H'$ | $n_H$ | $a_1$ | $r_H$ | $\delta$ | $z$ | $\beta$ | $\nu$ |
|-----|--------|-------|-------|-------|--------|-----|-------|------|
| 2   | 0.4965(18) | 0.4653(19) | 0.244(7) | 4.028(15) | 14.9(1.3) | 2.154(11) | 0.124(10) | 0.983(28) |
| 3   | 0.6650(28) | 0.5466(22) | 0.314(7) | 4.511(19) | 4.62(15) | 2.045(13) | 0.337(11) | 0.632(13) |

$^a$ Calculated from other exponents in the same row [10].
$^b$ Exact results.
$^c$ From [33].
$^d$ From [31].
$^e$ Average of [32] and [36].

The coercivity $H_c$ (a) and the difference $\Delta A/\Delta T$ (b) plotted with error bars versus $R$ in logarithmic scales at $T_c = 2.2692$ and $4.5115$ for 2D and 3D lattices, respectively.

The resultant exponents in table 2 (the second and fourth rows) compare satisfactorily to available results listed in the third and fifth rows [10]. Note that the present $n_H$ and $n_H'$ are measured directly but the available ones were calculated from the existing exponents and therefore had smaller errors. The error of $a_1$ is the most serious one in our method similar to the short-time dynamics [18, 19], as it involves derivative. Yet, the relative errors are generically less than 4% except $\delta$ and $\beta$ in 2D. This is because, for the former, due to its large value, $(n_H' - n_H)$ is small and comes up as a denominator, and for the latter, its small value enlarges its error. Nevertheless, they are quite close to their respective exact values. The measured 3D $\delta$ is smaller than 4.789(2) [31], because it is a monotonically increasing function of $n_H$, which is slightly smaller than the available value. Our 3D $\beta$ and $\nu$ agree well with the available values. So does the 2D $\nu$. These are remarkable because it has been notoriously difficult to determine $\nu$ [3]. Our 2D $z$ agrees with $z = 2.172(6)$ from damage spreading and heat bath dynamics [32], $z = 2.1665(12)$ from FSS of correlation times obtained from eigenvalues.

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of the dynamic matrix \([33]\), and \(z = 2.158(17)\) derived from \(n_H\) with \(\beta, \delta\) and \(\nu\) as inputs \([10]\). So does the 3D one that is close to \(z = 2.022\) from \(\epsilon\) expansion \([34]\), \(z = 2.032(4)\) from \([32]\), \(z = 2.04(3)\) from time-displaced correlation functions \([35]\), \(z = 2.030(4)\) from the statistical dependence time \([36]\) and \(z = 2.036(11)\) from \([10]\).

4. Summary

We have proposed a systematic realization of FTS and developed an analytic RG theory for it. The finite time scales are achieved by manipulating the dynamics with a linearly varying external field. Note that the present FTS is distinct from the usual critical dynamics including the short time one in which there is scaling with the time or frequency in that here one has instead of the natural involution time a controllable external time scale that affects the scaling behavior of the system. This scaling also shows the simplicity and generality of the single-parameter linear driving. We have also derived a method of FTS that can determine both static and dynamic critical exponents as well as the critical point of a system. Agreement of the results for 2D and 3D Ising models with existing ones confirms that FTS can be as effective as its spatial counterpart. It is appreciated that no higher order cumulants ought to be measured that incur large fluctuations and all exponents are determined directly from the slopes in double logarithmic plots without searching an exponent for data collapsing. More importantly, critical slowing down has been dismissed and converted into visible processes. For illustration, we have used only quite limited samples and rates, and hence the precision is not so high, but should be readily improved straightforwardly.

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