QUANTUM FOLDING

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ABSTRACT. In the present paper we introduce a quantum analogue of the classical folding of a simply-laced Lie algebra $g$ to the non-simply-laced algebra $g^\sigma$ along a Dynkin diagram automorphism $\sigma$ of $g$. For each quantum folding we replace $g^\sigma$ by its Langlands dual $g^{\sigma\vee}$ and construct a nilpotent Lie algebra $n$ which interpolates between the nilpotent parts of $g$ and $g^{\sigma\vee}$, together with its quantized enveloping algebra $U_q(n)$ and a Poisson structure on $S(n)$. Remarkably, for the pair $(g, g^{\sigma\vee}) = (\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n})$, the algebra $U_q(n)$ admits an action of the Artin braid group $Br_n$ and contains a new algebra of quantum $n \times n$ matrices with an adjoint action of $U_q(\mathfrak{sl}_n)$, which generalizes the algebras constructed by K. Goodearl and M. Yakimov in [10]. The hardest case of quantum folding is, quite expectably, the pair $(\mathfrak{so}_8, G_2)$ for which the PBW presentation of $U_q(n)$ and the corresponding Poisson bracket on $S(n)$ contain more than 700 terms each.

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1. Introduction and main results

This work is motivated by the classical “folding” result for a simply laced semisimple Lie algebra $g$ and an admissible diagram automorphism $\sigma : g \to g$ (in the sense of [17, §12.1.1], see Section 2.1).

The fixed Lie algebra $g^\sigma = \{x \in g : \sigma(x) = x\}$ is also semisimple. (1.1)

Our goal is to find a quantum version of this result. Note, however, that the embedding of associative algebras $U(g^\sigma) \hookrightarrow U(g)^\sigma \subset U(g)$ induced by the inclusion $g^\sigma \hookrightarrow g$ does not admit a naive quantum deformation (see Appendix A). On the other hand, there exists a “crystal” version of the desired homomorphism. Namely, let $B_\infty(g)$ be the famous Kashiwara crystal introduced in [15]. The following result was proved by G. Lusztig in [17, Section 14.4].

**Proposition 1.1.** Let $\sigma$ be an admissible diagram automorphism of $g$. Then $\sigma$ acts on $B_\infty(g)$ and the fixed point set $B_\infty(g)^\sigma$ is naturally isomorphic to $B_\infty(g^{\sigma^\vee})$, where $g^{\sigma^\vee}$ is the Langlands dual Lie algebra of $g^\sigma$.

Note that one can identify (in many ways) the $\mathbb{C}(q)$-linear span of $B_\infty(g)$ with the quantized enveloping algebra $U_q^+(g^\sigma)$ of $g^\sigma$, where $g^\sigma$ stands for the “upper triangular” Lie subalgebra of $g$. This leads us to the following definition.

**Definition 1.2.** A quantum folding of $g$ is a $\mathbb{C}(q)$-linear embedding (not necessarily algebra homomorphism!)

$$i : U_q^+(g^{\sigma^\vee}) \hookrightarrow U_q^+(g)^\sigma \subset U_q^+(g)$$

(1.2)
(here $U_q^+(g^{\sigma^\vee})$ comes with powers of $q$ depending on $\sigma$; see Section 2.2 for the details).

We construct all relevant quantum foldings below (Proposition 1.19) and now focus on a rich algebraic structure that can be attached to each quantum folding.

**Definition 1.3.** We say that a $k$-algebra $A$ generated by a totally ordered set $X_A$ is Poincaré-Birkhoff-Witt (PBW) if the set $M(X_A)$ of all ordered monomials in $X_A$ is a basis of $A$. More generally, we say that $A$ is sub-PBW if $M(X_A)$ spans $A$ as a $k$-vector space (but $M(X_A)$ is not necessarily linearly independent).

For a given sub-PBW algebra $A$ we say that an algebra $U = U(A, X_A)$ is a $uberalgebra$ of $A$ if

(a) $U$ is generated by $X_A$ and is a PBW algebra with these generators;
(b) The identity map $X_A \to X_A$ extends to a surjective algebra homomorphism $U \to A$.

In general it is not clear whether a given sub-PBW algebra $A$ admits a uberalgebra. A criterion for uniqueness is based on the following notion of tameness of $(A, X_A)$. We need some notation. First, consider the natural filtration $k = A_0 \subset A_1 \subset \cdots$ given by $A_k = \text{Span}\{1, X_A, X_A^2, X_A^3, \ldots, (X_A)^k\}$, $k \in \mathbb{Z}_{\geq 0}$. Next, for each $X, X' \in X_A$ with $X < X'$ let $d(X, X')$ be the smallest number $d$ such that $X'X \in A_d$. We also denote by $d_0 = d_0(A, X_A)$ the maximum of all the $d(X, X')$.

**Definition 1.4.** We say that a sub-PBW algebra $(A, X_A)$ is *tame* if the set $M(X_A) \cap A_{d_0}$ is linearly independent.
Lemma 1.5. A tame sub-PBW algebra \((A, X_A)\) admits at most one (up to isomorphism) uberalgebra \(U(A, X_A)\).

In what follows, we will construct uberalgebras \(U(\iota)\) for several quantum foldings \(\iota\) as in (1.2), and these uberalgebras will depend on \(\mathfrak{g}\) and \(\sigma\) rather than on a particular choice of \(\iota\) and most algebras generated by the image of \(\iota\) will be tame. We need more notation.

Definition 1.6. Let \(A\) and \(B\) be PBW-algebras and let \(\iota : A \hookrightarrow B\) be an injective map (not necessarily an algebra homomorphism). We say that \(\iota\) is liftable if:

(i) For any ordered monomial \(X = X_1^{m_1} \cdots X_N^{m_N} \in M(X_A)\) one has
\[
\iota(X) = \iota(X_1)^{m_1} \cdots \iota(X_N)^{m_N}.
\]

(ii) There exists a finite subset \(Z_0 \subset \langle A\rangle_\iota\), where \(\langle A\rangle_\iota\) is the subalgebra of \(B\) generated by \(\iota(A)\), such that \(\langle A\rangle_\iota\) is sub-PBW with respect to \(\iota(X_A) \cup Z_0\) (with some ordering of \(\iota(X_A) \cup Z_0\) compatible with the ordering of \(X\)).

(iii) There exists a uberalgebra \(U(\iota) := U(\langle A\rangle_\iota, \iota(X_A) \cup Z_0)\) for \(\langle A\rangle\), and a surjective homomorphism \(\mu := \mu_\iota : U(\iota) \rightarrow A\) such that for all \(x \in X_A\), \(\mu(\iota(x)) = x\) and \(\mu(Z_0) = 0\).

If \(\iota\) is liftable and \((\langle A\rangle_\iota, \iota(X_A) \cup Z_0)\) is tame (hence \(U(\iota)\) is unique), in what follows we refer to an \(\iota\) as a tame liftable quantum folding.

For each liftable \(\iota\) we have a diagram
\[
\begin{array}{ccc}
U(\iota) & \xrightarrow{i} & B \\
\downarrow{\tilde{i}} & & \downarrow{\iota} \\
\tilde{i} \downarrow{\mu_\iota} & & \iota \\
A & \searrow & \end{array}
\tag{1.3}
\]
satisfying \(\mu_\iota \circ \tilde{i} = \text{id}_A\) and \(\iota \circ \tilde{i} = \iota\) where

- \(\tilde{i} : A \hookrightarrow U(\iota)\) is the canonical splitting of \(\mu\) given by \(\tilde{i}(X_1^{m_1} \cdots X_N^{m_N}) = \iota(X_1)^{m_1} \cdots \iota(X_N)^{m_N}\) in the notation of Definition 1.6.
- \(\iota : U(\iota) \rightarrow B\) is the structural algebra homomorphism given by \(\iota(X) = X\) for \(X \in \iota(X_A) \cup Z_0\) (e.g., the image of \(\tilde{i}\) is \(\langle A\rangle_\iota\)).

Note the following easy Lemma.

Lemma 1.7. In the notation of Definition 1.6, write \(X_A = \{X_1, \ldots, X_N\}\) as an ordered set. Fix \(1 \leq k \leq N\) and \(f \in \sum_{M \in \mathcal{M}(X_A \backslash \{X_k\})} \mathbb{K} M\). If the uberalgebra \(U(\iota)\) is optimal PBW (in the sense of Definition 2.17) then

(i) \((A, X_A')\), where \(X_A' = \{X_1, \ldots, X_k' := X_k + f, X_{k+1}, \ldots, X_N\}\), is a PBW algebra.

(ii) The injective linear map \(\iota' : A \rightarrow B\) given by the formula
\[
\iota'(X_1^{m_1} \cdots (X_k')^{m_k} X_{k+1}^{m_{k+1}} \cdots X_N^{m_N}) = \iota(X_1)^{m_1} \cdots \iota(X_k')^{m_k} \iota(X_{k+1}^{m_{k+1}}) \cdots \iota(X_N^{m_N})
\]
is liftable with \(\langle A\rangle_{\iota'} = \langle A\rangle_\iota\), \(U(\iota') = U(\iota)\) and \(\mu_{\iota} = \mu_{\iota'}\).

The following is our first main result (see Section 3 for greater details).
Theorem 1.8. For the pair \((\mathfrak{g}, \mathfrak{g}^{\mathbb{C}^+}) = (\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n})\), \(n \geq 3\) there exists a tame liftable quantum folding \(\iota : U_q^+(\mathfrak{sp}_{2n}) \hookrightarrow U_q^+(\mathfrak{so}_{2n+2})\). The corresponding algebra \(U(\iota)\) is isomorphic to \(S_q(\mathfrak{V} \otimes \mathfrak{V}) \times U_q^+(\mathfrak{sl}_n)\), where \(\mathfrak{V}\) is the standard \(n\)-dimensional \(U_q^+(\mathfrak{sl}_n)\)-module, and \(S_q(\mathfrak{V} \otimes \mathfrak{V})\) is a quadratic PBW-algebra in the category of \(U_q^+(\mathfrak{sl}_n)\)-modules. More precisely:

(i) The algebra \(S_q(\mathfrak{V} \otimes \mathfrak{V})\) is isomorphic to \(T(\mathfrak{V} \otimes \mathfrak{V})/(\langle \Psi - 1 \rangle)\), where \(\Psi : V^{\otimes 4} \rightarrow V^{\otimes 4}\) is a \(\mathbb{C}(q)\)-linear map given by:

\[
\Psi = \Psi_2 \Psi_1 \Psi_3 \Psi_2 + (q - q^{-1})(\Psi_1 \Psi_2 \Psi_1 + \Psi_1 \Psi_3 \Psi_2) + (q - q^{-1})^2 \Psi_1 \Psi_2
\]  

(1.4)

where \(\Psi_i : V^{\otimes 4} \rightarrow V^{\otimes 4}, 1 \leq i \leq 3\) is, up to a power of \(q\), the braiding operator in the category of \(U_q^+(\mathfrak{sl}_n)\)-modules that acts in the \(i\)-th and \((i + 1)\)st factors and satisfies the normalized Hecke equation \((\Psi_i - q^{-1})(\Psi_i + q) = 0\).

(ii) The covariant \(U_q^+(\mathfrak{sl}_n)\)-action on the algebra \(S_q(\mathfrak{V} \otimes \mathfrak{V})\) is determined by the natural action of the Hopf algebra \(U_q^+(\mathfrak{sl}_n)\) on \(\mathfrak{V} \otimes \mathfrak{V}\).

(iii) The algebra \(S_q(\mathfrak{V} \otimes \mathfrak{V})\) is PBW with respect to any ordered basis of \(\mathfrak{V} \otimes \mathfrak{V}\).

Remark 1.9. Strictly speaking, the cross product \(U_q^+(\mathfrak{V} \otimes \mathfrak{V}) \times U_q^+(\mathfrak{sl}_n)\) is “braided” in the sense of Majid [18] because \(U_q^+(\mathfrak{sl}_n)\) is a braided Hopf algebra (see [17] and Section 2.2).

We prove Theorem 1.8 in Section 3. In particular, the key ingredient in our proof of part (iii) is the following general fact that we failed to find in the literature, although numerous special cases are well-known (cf. for example [20, 11, 8]), settle Theorem 1.8(iii) (see Section 3 for details).

Proposition 1.10. The map \(\Psi\) satisfies:

(i) The braid equation in \((\mathfrak{V} \otimes \mathfrak{V})^{\otimes 3}\):

\[
(\Psi \otimes 1)(1 \otimes \Psi)(\Psi \otimes 1) = (1 \otimes \Psi)(\Psi \otimes 1)(1 \otimes \Psi).
\]  

(1.5)

(ii) The cubic version of the Hecke equation:

\[
(\Psi - 1)(\Psi + q^2)(\Psi + q^{-2}) = 0.
\]

In particular, \(\Psi\) is invertible and:

\[
\Psi^{-1} = \Psi_2 \Psi_1 \Psi_3 \Psi_2 + (q - q^{-1})(\Psi_2 \Psi_3 \Psi_2 + \Psi_1 \Psi_3 \Psi_2) + (q - q^{-1})^2 \Psi_3 \Psi_2.
\]

(iii) \(\dim(\Psi - 1)(\mathfrak{V} \otimes \mathfrak{V}) = \dim \Lambda^2 \mathfrak{V}\).

This and the following general fact that we failed to find in the literature, although numerous special cases are well-known (cf. for example [20, 11, 8]), settle Theorem 1.8(iii) (see Section 3 for details).

Theorem 1.11. Let \(Y\) be a finite-dimensional \(\mathbb{C}(q)\)-vector space and let \(\Psi\) be an invertible \(\mathbb{C}(q)\)-linear map \(Y \otimes Y \rightarrow Y \otimes Y\) satisfying the braid equation. Assume that:

(i) the specialization \(\Psi|_{q=1}\) of \(\Psi\) is the permutation of factors \(\tau : Y \otimes Y \rightarrow Y \otimes Y\),

(ii) \(\dim(\Psi - 1)(Y \otimes Y) = \dim \Lambda^2 Y\).

Then the algebra \(S_\Psi(Y) = T(Y)/(\langle(\Psi - 1)(Y \otimes Y)\rangle)\) is a flat deformation of the symmetric algebra \(S(Y)\) (hence \(S_\Psi(Y)\) is PBW for any ordered basis of \(Y\)).
We prove Theorem 1.11 in Section 2.6.
An explicit PBW presentation of both $S_q(V \otimes V)$ and $U(\iota)$ is more cumbersome, so we postpone it until Proposition 3.6. Below we provide a presentation of $U(\iota)$ by a minimal set of Chevalley-like generators satisfying Serre-like relations.

**Theorem 1.12.** The subalgebra $U(\iota) = S_q(V \otimes V) \rtimes U_q^+(\frak{sl}_n)$, $n \geq 2$ is generated by $u_1, \ldots, u_{n-1}$, $w$, and $z$ subject to the following relations (for all relevant $i, j$):

\[
[u_i, [u_i, u_j]]_{q^{-1}} = 0, \quad \text{if } |i - j| = 1, \quad u_i u_j = u_j u_i, \quad \text{if } |i - j| > 1,
\]

\[
u_i w = w u_i, \quad \text{if } i \neq 1, \quad u_i z = z u_i, \quad \text{if } i \neq 2, \quad zw = wz,
\]

\[
[u_1, [u_1, [u_1, w]]]_{q^{-2}} = 0, \quad [u_2, [u_2, z]]_{q^{-2}} = 0,
\]

\[
[w, [w, u_1]]_{q^{-2}} = -hwz, \quad [z, [u_2, [u_1, w]]]_{q^2} = [w, [u_1, [u_2, z]]]_{q^2},
\]

\[
2[z, [z, u_2]]_{q^{-1}} = h(z[u_1, u_2]w + w[u_2, u_1], -z + wu_1[z, u_2] + [u_2, z]_{q^{-1}} u_1 w),
\]

where $h = q - q^{-1}$ and we abbreviate $[a, b]_v = ab - vba$ and $[a, b]_1 = ab - ba$ (with the convention that $u_0 = 0$ if $n = 2$).

**Remark 1.13.** Under the decomposition $V \otimes V = S^2_q V \oplus \Lambda^2_q V$ in the category of $U_q(\frak{sl}_n)$-modules the generator $w$ (respectively $z$) of $U(\iota)$ is a lowest weight vector in the simple $U_q(\frak{sl}_n)$-module $S^2_q V$ (respectively $\Lambda^2_q V$), with the convention that $u_i$ equals the $(n - i)$th standard Chevalley generator $E_{n-i}$ of $U_q^+(\frak{sl}_n)$.

It is easy to show that $S_q(V \otimes V)/(\Lambda^2_q V) \cong S_q(S^2 V)$, $S_q(V \otimes V)/(S^2_q V) \cong S_q(\Lambda^2 V)$, where $S_q(S^2 V)$ and $S_q(\Lambda^2 V)$ are respectively the algebras of quantum symmetric and quantum exterior matrices studied in [19, 14, 22]. Due to this and the canonical identification $S(V \otimes V) = S(\Lambda^2 V \oplus S^2 V) = S(\Lambda^2 V) \otimes S(S^2 V)$, we can view $S_q(V \otimes V)$ as a deformation of the braided (in the category of $U_q(\frak{sl}_n)$-modules) tensor product $S_q(\Lambda^2 V) \otimes S_q(S^2 V)$ (see also Remark 1.16 for the Poisson version of this discussion).

This point of view is supported by the observation that our braiding operator $\Psi$ given by (1.4) is a deformation of the braiding $\Psi' := \Psi_2 \Psi_1 \Psi_2$ of $V \otimes V$ with itself in the category of $U_q(\frak{sl}_n)$-modules. Note, however, that latter braiding $\Psi'$ does not satisfy the condition (iii) of Theorem 1.11, therefore, the quadratic algebra $S_{\Psi'}(V \otimes V)$ (as defined in Theorem 1.11) is not a flat deformation of $S(V \otimes V)$.

**Remark 1.14.** In all quantum foldings we constructed so far the image of $\iota$ is contained in $U_q(\frak{g})^{gr}$, where $(-)^{gr}$ is the graded fixed point algebra defined for any graded algebra $A = \bigoplus_{\gamma \in \Gamma} A_{\gamma}$ and any automorphism $\sigma$ of $A$ by: $A^{gr} \sigma = \bigoplus_{\gamma \in \Gamma} \{ a \in A_{\gamma} : \sigma(a) = a \}$ (in our case, $\Gamma$ is the root lattice of $\frak{g}$). One can show that the subalgebra of $U_q^+(\frak{so}_{2n+2})$ generated by the image of $\iota : U_q^+(\frak{sp}_{2n}) \hookrightarrow U_q^+(\frak{so}_{2n+2})$ is isomorphic to $U_q^+(\frak{so}_{2n+2})^{gr}$, but we do not expect that this to happen in general (e.g., it fails for the pair $(\frak{g}, \frak{g}^{av}) = (\frak{so}_8, G_2)$). We will discuss the relationship between quantum foldings and graded fixed points of diagram automorphisms in a separate publication.

Theorem 1.18 implies that the “classical limit” $S(V \otimes V)$ of $S_q(V \otimes V)$ has a quadratic Poisson bracket which we present in the following

**Corollary 1.15.** In the notation of Theorem 1.18 let $\{X_i\}$, $i = 1, \ldots, n$ be the standard basis of $V$. Then the formulae (for all $1 \leq i \leq j \leq k \leq l \leq n$, where we
abbreviation $X_{ij} = X_i \otimes X_j$ for $1 \leq i, j \leq n$:

$$\{X_{ij}, X_{kl}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ij}X_{kl} - 2(X_{il}X_{kj} + X_{ki}X_{lj})$$

$$\{X_{ij}, X_{lk}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ij}X_{lk} - 2(X_{jk}X_{li} + X_{ki}X_{lj})$$

$$\{X_{ji}, X_{kl}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ji}X_{kl} - 2(X_{jl}X_{ki} + X_{kj}X_{li})$$

$$\{X_{ji}, X_{lk}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ji}X_{lk} - 2(X_{jk}X_{li} + X_{ki}X_{lj})$$

$$\{X_{ik}, X_{jl}\} = (\delta_{ij} + \delta_{il} - \delta_{ki} - \delta_{kl})X_{ik}X_{jl} - 2X_{jk}X_{li}$$

$$\{X_{ki}, X_{jl}\} = (\delta_{ij} + \delta_{il} - \delta_{ki} - \delta_{kl})X_{ki}X_{jl} - 2X_{jk}X_{li}$$

$$\{X_{il}, X_{jk}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{il}X_{jk} + 2(X_{ij}X_{lk} - X_{ji}X_{kl})$$

$$\{X_{il}, X_{kj}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{il}X_{kj} + 2(X_{ik}X_{lj} - X_{ji}X_{ki})$$

$$\{X_{ji}, X_{kj}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{ji}X_{kj}$$

$$\{X_{ij}, X_{jk}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{ij}X_{jk}$$

define a Poisson bracket on $S(V \otimes V)$.

Remark 1.16. In [10] K. Goodearl and M. Yakimov constructed quadratic Poisson brackets on $S(\Lambda^2 V)$ and $S(S^2 V)$. In parallel with Remark 1.13 one can show that the ideal of $S(V \otimes V)$ generated by $\Lambda^2 V = \text{Span}\{X_{ij} - X_{ji}\}$ (respectively by $S^2 V = \text{Span}\{X_{ij} + X_{ji}\}$) is Poisson hence the quotient of $S(V \otimes V)$ by this ideal is the Poisson algebra $S(S^2 V)$ (respectively $S(\Lambda^2 V)$) from [10]. Therefore, we can view the bracket given by Corollary 1.15 as a certain deformation of the Poisson bracket on $S(V \otimes V)$ obtained by lifting the brackets on $S(\Lambda^2 V)$ and $S(S^2 V)$.

We construct more liftable quantum foldings when $\sigma$ is an involution.

Theorem 1.17. If $(\mathfrak{g}, g^{\sigma_V}) = (\mathfrak{sl}_n \times \mathfrak{sl}_n, \mathfrak{sl}_n)$, $n = 3, 4$, then there exists a tame liftable quantum folding $\iota : U^+_q(\mathfrak{g}^{\sigma_V}) \hookrightarrow U^+_q(\mathfrak{g})$ such that $U(\iota)$ is a q-deformation of the universal enveloping algebra $U(V_n \rtimes (\mathfrak{sl}_n)^+)$, $n = 3, 4$, where $V_n$ is a finite-dimensional module (regarded as an abelian Lie algebra) over $(\mathfrak{sl}_n)^+$. More precisely,

(i) For $(\mathfrak{sl}_3 \times \mathfrak{sl}_3, \mathfrak{sl}_3)$, $V_3 = 1$ is the trivial one-dimensional $(\mathfrak{sl}_3)^+$-module and the enveloping algebra $U(\iota)$ is generated by $u_1, u_2$, and $z$ subject to the following relations

- $z$ is central;

- $u_i^2 u_j - (q^2 + q^{-2})u_i u_j u_i + u_j u_i^2 = (q - q^{-1})u_i z$ for $\{i,j\} = \{1,2\}$.

(ii) For $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$, the $(\mathfrak{sl}_4)^+$-module $V_4$ has a basis $z_{12, z_{13}, z_{23}, z_{12,3}, z_{12,23}}$, the action of Chevalley generators $e_1, e_2, e_3$ of $(\mathfrak{sl}_4)^+$ on $V_4$ is given by the following diagram

```
\begin{array}{ccc}
   e_1 & & e_2 \\
   \downarrow & & \downarrow \\
   z_{12,3} & & z_{23} \\
   \downarrow & & \downarrow \\
   z_{12,23} & & z_{12,23}
\end{array}
```
where an arrow from $z$ to $z'$ labeled by $e_i$ means that $e_i(z) = z'$, while $e_j(z) = 0$ for all $j \neq i$. The uberalgebra $U(i)$ is a quantized enveloping algebra of the Lie algebra $V_4 \times (\mathfrak{sll}_3)_+$ and it is generated by $u_1, u_2, u_3$, $z_{12} = z_{21}$, $z_{23} = z_{32}$, and $z_{13}$ subject to the relations:

- $u_iz_{ij} = z_{ij}u_i$, $i < j$, $u_1u_3 = u_3u_1$,
- $[u_i, [u_i, u_j]_{q^2}]_{q^{-2}} = hu_i z_{ij}$, $|i - j| = 1$,
- $[u_i, [u_i, z_{2j}]_{q^2}]_{q^{-2}} = hu_i z_{13}$ for $\{i, j\} = \{1, 3\}$,
- $(q + q^{-1})[z_{12}, z_{23}] = [u_2, [z_{12}, u_3]_{q^{-2}}]_{q^2} - [u_2, [z_{23}, u_1]_{q^{-2}}]_{q^2}$,
- $[u_2, z_{13}] + [z_{12}, z_{2j}] = h(z_{2j}u_1u_2 - u_2u_1z_{2j})$ for $\{i, j\} = \{1, 3\}$,
- $2[z_{12}, z_{2j}]_{q^2} = [z_{12}, [u_i, z_{2j}]_{q^2}]_{q^2} + [u_i, [z_{12}, z_{2j}]_{q^2}]_{q^2} = h\{z_{12}, z_{2j} + (q^2 + 1 + q^{-2})u_1u_2u_j - u_1u_2u_j - u_1u_ju_2 - u_2u_1u_j\}$ for $\{i, j\} = \{1, 3\}$,
- $[z_{12}, z_{2j}] - [u_i, [u_2, [u_j, z_{2j}]]] = h(u_3z_{12} - z_{12}z_{2j}u_3) + h^2(u_2u_1u_3z_{12} - z_{12}u_3u_2u_1)$ for $\{i, j\} = \{1, 3\}$, where we abbreviated $[a, b]_v = ab - vba$, $[a, b] = ab - ba$, $(a, b) = ab + ba$, and $h = q - q^{-1}$.

**Remark 1.18.** In case of $(\mathfrak{sl}_3 \times \mathfrak{sl}_3, \mathfrak{sl}_3)$ the uberalgebra $U(i)$ is PBW on the ordered set $\{u_1, u_2, u_21 = u_1u_2 - q^{-2}u_2u_1 - z, z\}$ subject to the following relations

- the element $z$ is central,
- $u_1u_2 = q^2u_2u_1$, $u_2u_21 = q^{-2}u_2u_1$,
- $u_1u_2 = q^{-2}u_2u_1 + u_2 + z$.

In particular, $S(1 \times \mathfrak{sl}_3)_+$ is generated by $\bar{u}_1, \bar{u}_2, \bar{u}_21, \bar{z}$ and the following

- $\{\bar{u}_1, \bar{z}\} = \{\bar{u}_2, \bar{z}\} = \{\bar{u}_21, \bar{z}\} = 0$,
- $\{\bar{u}_1, \bar{u}_2\} = -2\bar{u}_21\bar{u}_1 + 4\bar{u}_21 + 2\bar{z}$,
- $\{\bar{u}_1, \bar{u}_21\} = 2\bar{u}_21\bar{u}_1$, $\{\bar{u}_2, \bar{u}_21\} = -2\bar{u}_21\bar{u}_2$

defines a Poisson bracket on $S(1 \times \mathfrak{sl}_3)_+$.

The PBW-presentation of the uberalgebra $U(i)$ for the folding $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$ is more cumbersome (see Theorem 4.7). Similarly to the previous discussion, the PBW property of $U(i)$ defines a Poisson bracket on $S(V_3 \times (\mathfrak{sl}_3)_+)$, which, unlike that on $S(V_3 \times (\mathfrak{sl}_3)_+)$, includes cubic terms (Theorem 4.8). It would be interesting to construct both the uberalgebra and the corresponding Poisson bracket for the folding $(\mathfrak{sl}_n \times \mathfrak{sl}_n, \mathfrak{sl}_n)$, $n \geq 4$.

Now we will explicitly construct all tame liftable quantum foldings $i$ used in Theorems 1.8 and 1.7 along with their (yet conjectural) generalizations to all semisimple Lie algebras. We need some notation.

Given a semisimple simply laced Lie algebra $g$ with an admissible diagram automorphism $\sigma$, let $I$ be the set of vertices of the Dynkin diagram of $g$ and we denote by the same letter $\sigma$ the induced bijection $\sigma : I \to I$. Denote by $s_i$, $i \in I$ (respectively, by $s'_i$, $r \in I/\sigma$) the simple reflections of the root lattice of $g$ (respectively, of $g^{\sigma^\vee}$). Let $W(g) = \langle s_i : i \in I \rangle$ (respectively, $W(g^{\sigma^\vee}) = \langle s'_i : r \in I/\sigma \rangle$) be the corresponding Weyl group.

Denote by $\hat{w}_0$ (respectively, $w_0$) the longest element of $W(g)$ (respectively, of $W(g^{\sigma^\vee})$). Furthermore, denote by $R(w_0)$ the set of all reduced decompositions of $w_0$ i.e. of all sequences $i = (i_1, \ldots, i_m) \in (I/\sigma)^m$ where $m = \ell(w_0)$ is the Coxeter length of $w_0$ such that $s_{i_1} \cdots s_{i_m} = w_0$. Similarly, one defines the set $R(\hat{w}_0)$ of all reduced decompositions of $\hat{w}_0$.  


Note that each admissible diagram automorphism $\sigma$ defines an automorphism of $W(\mathfrak{g})$ via $s_i \mapsto s_{\sigma(i)}$ and its fixed subgroup $W(\mathfrak{g})^\sigma$ is isomorphic to $W(\mathfrak{g}^\sigma) = W(\mathfrak{g}^{\sigma'})$ via $s_i \mapsto \hat{s}_i = \prod_{r \in \mathcal{O}_i} s_i$ where $\mathcal{O}_i \subset I$ is the $r$-th $\sigma$-orbit in $I$ (see Proposition 2.4). We denote this natural isomorphism $W(\mathfrak{g}^{\sigma'}) \rightarrow W(\mathfrak{g})^\sigma$ by $w \mapsto \hat{w}$.

Thus, one can assign to each $i \in \{i_1, \ldots, i_m\} \in R(\mathfrak{w}_o)$ its lifting $\mathfrak{i} \in R(\hat{\mathfrak{w}}_o)$ via:

$$\mathfrak{i} = (\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_m})$$

(in fact, $\mathfrak{i}$ is unique up to reordering of each set $\mathcal{O}_{i_k}$).

Following Lusztig ([17, §40.2]), for each $\mathfrak{i} \in R(\hat{\mathfrak{w}}_o)$ (respectively, $\mathfrak{i} \in R(\mathfrak{w}_o)$ one defines a modified PBW-basis $M(X_i)$ of $U_q^+(\mathfrak{g})$ (respectively, $M(X_i)$ of $U_q^+(\mathfrak{g}^{\sigma'})$), see Section 2.4 for details (this modification will ensure the commutativity of the triangle in (1.3)).

One can show (see Lemma 2.11) that for any $\mathfrak{i} \in R(\hat{\mathfrak{w}}_o)$ the PBW basis $M(X_i)$ does not depend on the choice of a lifting $\mathfrak{i} \in R(\hat{\mathfrak{w}}_o)$ of $\mathfrak{i} \in R(\mathfrak{w}_o)$. Moreover, the action of $\sigma$ on $U_q^+(\mathfrak{g})$ preserves $M(X_i)$ for each such lifting $\mathfrak{i}$.

The following result serves as a definition of quantum folding for all $\mathfrak{g}$ and $\sigma$ (see Lemma 2.11 for details).

**Proposition 1.19.** Given an admissible diagram automorphism $\sigma$ of $\mathfrak{g}$, for each $\mathfrak{i} \in R(\mathfrak{w}_o)$ there is a natural injective $\mathbb{C}(q)$-linear map

$$\iota_1 : U_q^+(\mathfrak{g}^{\sigma'}) \hookrightarrow U_q^+(\mathfrak{g})^\sigma \subset U_q^+(\mathfrak{g})$$

which maps the modified PBW-basis $M(X_i)$ bijectively onto the fixed point set $M(X_i)^\sigma$ of $M(X_i)$.

In fact, the tame liftable foldings $\iota$ used in Theorems 1.8 and 1.17 were of the form $\iota_1, \mathfrak{i} \in R(\mathfrak{w}_o)$.

**Theorem 1.20.** Let $\mathfrak{g}$ be a simply laced semisimple Lie algebra and let $\sigma$ be its admissible diagram automorphism of order 2. Then for any reduced decompositions $\mathfrak{i}, \mathfrak{i}'$ of $\mathfrak{w}_o$ the subalgebras of $U_q^+(\mathfrak{g})$ generated by the images of $\iota_1$ and $\iota_1'$ are isomorphic.

This theorem is proved in Section 2.4.

However, if the order of $\sigma$ is at least 3, it frequently happens that the image of $\iota$ generates a non-sub-PBW algebra hence the uberalgebra $U(\iota_1)$ does not always exists (see Section 4.3). In order to restore the (sub-)PBW behavior of the algebras in question, we propose the modification, which we refer to as the enhanced uberalgebra $\hat{U}(\iota)$.

Indeed, in the assumptions of Definition 4.3 let us relax the assumption that $Z_0 \subset \langle A \rangle$, in Definition 1.6 Suppose that $B$ is PBW domain. Then we take $Z_0$ to be a finite subset of $\text{Frac}(\iota(A)) \cap B$, where $\text{Frac}(\iota(A)) \subset \text{Frac}(B)$ is the skew-subfield of the skew-filed $\text{Frac}(B)$ generated by $\iota(A)$ ($B$ is an Ore domain so its skew-field of fractions $\text{Frac}(B)$ is well-defined, see [3, Appendix A] for details).

We will refer to a map $\iota$ satisfying Definition 1.6 “relaxed” in such a way as enhanced liftable and to its uberalgebra (which we denote by $\hat{U}(\iota)$) as an enhanced uberalgebra of $\iota$. (A tame enhanced liftable $\iota$ is introduced accordingly). By construction, $\hat{U}(\iota)$ satisfies the diagram (1.3), however, it need not be generated by $A$ (unlike all known $U(\iota)$ for liftable $\iota$).
**Theorem 1.21.** Let \( n \geq 3 \) and let \((g, g^{\sigma \vee}) = (sl_3^{\times n}, sl_3)\) where \( sl_3^{\times n} = sl_3 \times \cdots \times sl_3 \) and \( \sigma \) is a cyclic permutation of factors. Then for both reduced decompositions \( i_1 = (121) \) and \( i_2 = (212) \) of \( w_0 \in W(g^{\sigma \vee}) \) the quantum folding \( u_r, r = 1, 2 \) is enhanced liftable and the enhanced uberalgebras \( \hat{U}(i_1) \) and \( \hat{U}(i_2) \) are isomorphic. Moreover, 

(i) \( \hat{U}(i_1) \) is generated by Chevalley-like generators \( u_1, u_2, \) and \( z_1, \ldots, z_{n-1} \), subject to Serre-like relations 

- \( z_k z_l = z_l z_k \) for \( k, l = 1, \ldots, n-1, \)
- \( u_i z_k = q^{-2k} z_k u_i \) for \( i = 1, 2, k = 1, \ldots, n-1, \)
- \( u_i^2 u_j - (q^n + q^{-n}) u_i u_j u_i + u_j u_i^2 = (q^{-1} - q) u_i \sum_{k=1}^{n-1} q^k z_{k,i} \) for \( \{i, j\} = \{1, 2\} \)

where we abbreviated \( z_{k,1} = z_{n-k,2} = z_k. \)

(ii) The enhanced uberalgebra \( \hat{U}(i_1) \) is a PBW algebra in the totally ordered set of generators \( \{u_2, u_{21}, u_1, z_1, \ldots, z_{n-1}\} \), where \( u_{21} = u_1 u_2 - q^{-n} u_2 u_1 - \sum_{k=1}^{n-1} \frac{q-q^{-1}}{q^n - q^{-n}} z_k, \)

subject to the commutation relations:

- \( z_k z_l = z_l z_k \) for all \( 1 \leq k, l \leq n-1, \)
- \( u_1 u_{21} = q^n u_{21} u_1, u_2 u_{21} = q^{-n} u_{21} u_2, \)
- \( u_1 z_k = q^{-2k} z_k u_1, u_2 z_k = q^{2k-2n} z_k u_2, u_{21} z_k = z_k u_{21} \) for all \( 1 \leq k \leq n-1, \)
- \( u_1 u_2 = q^{-n} u_{21} u_1 + u_2 + \sum_{k=1}^{n-1} \frac{q-q^{-1}}{q^n - q^{-n}} z_k. \)

(iii) \( \hat{U}(i_1) \) a quantum deformation of the enveloping algebra \( U(1^{n-1} \times (sl_3)_+) \), where \( 1 \) is the trivial one-dimensional \( (sl_3)_+ \)-module.

We prove Theorem 1.21 in Section 4.1.

**Remark 1.22.** It follows from Theorem 1.21 that the following defines a Poisson bracket on \( S(1^{n-1} \times (sl_3)_+) \)

- \( \{\tilde{u}_1, \tilde{u}_2\} = n \tilde{u}_1 \tilde{u}_2, \{u_2, \tilde{u}_2\} = -n \tilde{u}_1 \tilde{u}_2, \)
- \( \{\tilde{u}_1, \tilde{z}_k\} = (n - 2k) \tilde{u}_1 \tilde{z}_k, \{\tilde{u}_2, \tilde{z}_k\} = (k - 2n) \tilde{u}_2 \tilde{z}_k \) for \( k = 1, \ldots, n-1, \)
- \( \{\tilde{u}_1, \tilde{u}_2\} = n(2 \tilde{u}_21 - \tilde{u}_1 \tilde{u}_2) + 2 \sum_{k=1}^{n-1} \tilde{z}_k, \)

where \( \tilde{u}_1, \tilde{u}_2, \) and \( \tilde{z}_k, k = 1, \ldots, n-1 \) are PBW generators of \( S(1^{n-1} \times (sl_3)_+) \) obtained by certain specialization at \( q = 1 \) from generators of \( U(i_1). \) Note that the quotient by the Poisson ideal generated by \( \tilde{z}_1, \ldots, \tilde{z}_{n-1} \) is the Poisson algebra \( S(sl_3_+) \) with the standard Poisson bracket multiplied by \( n. \)

**Theorem 1.23.** Let \((g, g^{\sigma \vee}) = (so_8, G_2)\) where \( \sigma \) is a cyclic permutation of 3 vertices of Dynkin diagram of type \( D_4. \) Then for both reduced decompositions \( i_1 = (121212) \) and \( i_2 = (212121) \) of \( w_0 \in W(g^{\sigma \vee}) \) the quantum folding \( i_k, k = 1, 2 \) is enhanced liftable and the enhanced uberalgebras \( \hat{U}(i_1) \) and \( \hat{U}(i_2) \) are isomorphic to each other and to a quantum deformation of the universal enveloping algebra \( U(n_{G_2} \times (sl_2)_+) \), where \( n_{G_2} \) is a certain nonabelian nilpotent 13-dimensional Lie algebra with the covariant \( (sl_2)_+ \)-action. More precisely, 

(i) \( n_{G_2} \times (sl_2)_+ \) is generated by \( u = e_1, w, z_1, z_2 \) subject to the following relations

- \([u, [u, [u, w]]]] = [w, [w, u]] = 0; \)
- \([u, [u, z_i]] = [z_i, [z_i, [z_i, u]]] = [w, z_i] = 0, [w, [z_i, u]] = [z_1, z_2] \) for \( i = 1, 2, \)
- \([z_i, [u, z_i]] = [z_i, [u, z_2]] + [z_2, [u, z_i]] \) for \( i = 1, 2. \)
(ii) \( n_{G_2} \) is the Lie ideal in \( n_{G_2} \times (\mathfrak{sl}_2)_+ \) with the basis \( w_i, 1 \leq i \leq 5 \) and \( z_i, 1 \leq i \leq 8 \) and the multiplication table (only non-zero Lie brackets are shown):

\[
\begin{array}{ll}
[w_1, w_4] = -3w_5, & [w_2, w_3] = w_5, \\
[w_1, z_3] = [w_1, z_4] = [w_2, z_1] = [w_2, z_2] = [z_2, z_1] = -z_5, \\
[w_2, z_3] = [w_2, z_4] = [z_1, w_3] = [z_2, w_3] = 2z_6, \\
[w_3, z_3] = [w_3, z_4] = [z_3, z_4] = z_7, & [w_4, z_1] = [w_4, z_2] = -3z_7, \\
[z_1, z_3] = [z_2, z_4] = 2z_8, \\
[z_1, z_4] = z_6 + z_8, & [z_2, z_3] = -z_6 + z_8.
\end{array}
\]

(iii) \( \hat{U}(t_1) \) is generated by Chevalley-like generators \( u, w \) and \( z_1, z_2 \) and satisfies the following Serre-like relations (the list is incomplete):

\[
\begin{align*}
[u, [u, [u, w]_{q^{-3}}]_{q^{-1}}]_{q^{3}} &= 0, \\
[w, [w, u]_{q^{-3}}]_{q^{3}} &= [w, z_1]_{q^{3}} = [z_2, w]_{q^{3}} = q([z_1, w]_{q} + [w, z_2]_{q}), \\
[z_1, z_2] &= (q + q^{-1})[z_1, [w, u]_{q^{-3}}] - [z_2, [w, u]_{q^{-3}}], \\
[z_1, [u, w]_{q}] &= ([w, u]_{q}, z_2), \quad [z_1, [z_1, u]_{q^{-1}}]_{q} = [z_2, [z_2, u]_{q^{-1}}]_{q}, \\
[z_1, [z_1, u]_{q^{-1}}]_{q} &= q([z_1, [z_2, u]_{q^{-1}}]_{q} + [z_2, [z_1, u]_{q^{-1}}]_{q}) \\
&(+q - q^{-1})(q[z_1, [u, q^{-1}]_{q}z_2 - z_1[z_2, u]_{q^{-1}}]_{q})
\end{align*}
\]

We prove Theorem 1.23 in Section 5.

**Remark 1.24.** The non-tameness of the quantum folding assigned to \((\mathfrak{so}_4, G_2)\) causes serious computational problems for the corresponding superalgebra and the Poisson bracket on \(S(n_{G_2})\). At the moment the Poisson bracket involves around 700 terms and the PBW presentation of \(U(t_1)\) is even more complicated (they can be found at [http://ishare.ucr.edu/jacobg/G2.pdf](http://ishare.ucr.edu/jacobg/G2.pdf)).

This is one of the reasons why Theorem 1.23(iii) contains only a partial Serre-like presentation of \(U(t_1)\) in Chevalley-like generators \(u, w, z_1, z_2\). We dropped here the most notorious relations involving more than 30 terms each (see the above mentioned webpage).

Taking into account Theorems 1.8, 1.20, 1.21, and 1.23, we propose the following conjecture.

**Conjecture 1.25.** Let \( \sigma \) be any admissible diagram automorphism of \( \mathfrak{g} \) such that \( \mathfrak{g}^{\sigma^\vee} \) has no Lie ideals of type \( G_2 \). Then there exists a (unique) \( \mathfrak{g}_+ \)-module \( V_\sigma \) such that:

(i) for any \( i \in R(\mathfrak{w}) \), the folding \( t_i \) is tame enhanced liftable,

(ii) the corresponding enhanced superalgebra \( \hat{U}(t_i) \) is a flat deformation of both the universal enveloping algebra \( \hat{U}(\mathfrak{n} \rtimes \mathfrak{g}_+) \) and the symmetric algebra \( S(V_\sigma \rtimes \mathfrak{g}_+) \),

(iii) The skew field of fractions \( \mathcal{F}_{\text{Frac}}(U(t_i)) \) is generated by \( t_i(E_r), r \in I/\sigma \), where \( E_r \) are Chevalley generators of \( U_q^+(\mathfrak{g}^{\sigma_{\vee}}) \) (and \( t_i : U_q^+(\mathfrak{g}^{\sigma_{\vee}}) \hookrightarrow U(t_i) \) is the lifting of \( t_i \) given by \( (1.3) \)).

If \( \sigma \) is an involution, we drop “enhanced” in Conjecture 1.25 because we expect that \( \hat{U}(t_i) = U(t_i) \).

In particular, the conjecture implies that one can canonically assign to each simply laced Lie algebra \( \mathfrak{g} \) a finite-dimensional \( \mathfrak{g}_+ \)-module \( V_\sigma^{(k)} \) for each \( k \geq 2 \) (by taking \( \mathfrak{g}^\times k \) and its natural diagram automorphism \( \sigma \), the cyclic permutation of factors so that \((\mathfrak{g}^\times k)_{\sigma^\vee} = \mathfrak{g} \)). Theorem 1.17 implies that such a \( V_\sigma^{(k)} \) will be rather non-trivial.
even for \( \mathfrak{g} = \mathfrak{sl}_n \). It would be also interesting to explicitly compute the Poisson bracket on \( S(V_0^{(k)} \times \mathfrak{g}_+) \) predicted by Conjecture 1.25. It should be noted that if \( \mathfrak{g} \) has a diagram automorphism \( \sigma' \), then the corresponding uberalgebra also admits an automorphism extending \( \sigma' \). For example, in the notation of Theorems 1.21 and 1.17 the uberalgebra for the folding \( (\mathfrak{g}, \mathfrak{g}^{\sigma'}) = (\mathfrak{sl}_3^k, \mathfrak{sl}_3) \) has an automorphism \( \sigma' \) defined by \( u_1 \mapsto w_2, u_2 \mapsto u_1, z_i \mapsto z_{k-i}, 1 \leq i \leq k-1 \), while the uberalgebra for the folding \( (\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4) \) has an automorphism defined by \( e_1 \mapsto e_3, z_{12} \mapsto z_{32}, z_{32} \mapsto z_{12} \) and \( e_2, z_{13} \) are fixed.

Note also that the part (iii) of Conjecture 1.25 holds for all cases we considered so far, in particular, for the folding \( (\mathfrak{g}, \mathfrak{g}^{\sigma'}) = (\mathfrak{sl}_3^k, \mathfrak{sl}_3) \), the skew-field \( \mathcal{Frac}(U(\mathfrak{u})) \) is generated by \( u_1 \) and \( u_2 \) (one can show that each \( z_k, 1 \leq k \leq n-1 \) in Theorem 1.21 is a rational “function” of \( u_1 \) and \( u_2 \); see Lemma 4.4) and for \( (\mathfrak{g}, \mathfrak{g}^{\sigma'}) = (\mathfrak{so}_8, G_2) \) the skew-field \( \mathcal{Frac}(U(\mathfrak{u})) \) is generated by \( u \) and \( w \) (both \( z_1 \) and \( z_2 \) in Theorem 1.23 are rational “functions” of \( u \) and \( w \)).

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2. General properties of quantum foldings and PBW algebras

2.1. Folding of semisimple Lie algebras. Recall that each semisimple Lie algebra \( \mathfrak{g} = \langle e_i, f_i : i \in I \rangle \) is determined by its Cartan matrix \( A = (a_{ij})_{i,j \in I} \) (see e.g. [21]) via:

\[
(\text{ad} e_i)^{1-a_{ij}} e_j = 0 = (\text{ad} f_i)^{1-a_{ij}} f_j, \quad [e_i, f_j] = 0, \quad i \neq j
\]

and

\[
[e_i, f_i] = a_{ii} e_i, \quad [e_i, f_i] = -a_{ij} f_j, \quad i, j \in I
\]

Denote by \( \mathfrak{g}_+ \) the Lie subalgebra of \( \mathfrak{g} \) generated by the \( e_i, i \in I \).

We say that a bijection \( \sigma : I \to I \) is a diagram automorphism of \( \mathfrak{g} \) if \( a_{\sigma(i), \sigma(j)} = a_{ij} \) for all \( i, j \in I \). It is well-known that such \( \sigma \) defines a unique automorphism, which we also denote by \( \sigma \), of the Lie algebra \( \mathfrak{g} \) via

\[
\sigma(e_i) = e_{\sigma(i)}, \quad \sigma(f_i) = f_{\sigma(i)}, \quad i \in I.
\]

After [17, §12.1.1], a diagram automorphism \( \sigma \) is said to be admissible if for all \( i \in I, k \in \mathbb{Z}, a_{i, \sigma^k(i)} = 0 \), whenever \( \sigma^k(i) \neq i \).

In what follows we denote by \( I/\sigma \) the quotient set of \( I \) by the equivalence relation which consists of all pairs \( (i, \sigma^k(i)) \). In other words, we use \( I/\sigma \) as the indexing set for orbits of the cyclic group \( \langle \sigma \rangle = \{ 1, \sigma, \sigma^2, \ldots \} \) action on \( I \).

The following result is well-known (cf. for example [13, Proposition 7.9])

**Theorem 2.1.** Let \( \sigma \) be an admissible diagram automorphism of \( \mathfrak{g} \). Then the fixed Lie subalgebra \( \mathfrak{g}^{\sigma} = \{ x \in \mathfrak{g} : \sigma(x) = x \} \) of \( \mathfrak{g} \) is semi-simple, with:

- the Chevalley generators \( e'_r, f'_r \), \( r \in I/\sigma \) given by:

\[
e'_r = \sum_{i \in \mathcal{O}_r} e_i, \quad f'_r = \sum_{i \in \mathcal{O}_r} f_i,
\]

where \( \mathcal{O}_r \) is the \( r \)-th orbit of the \( \langle \sigma \rangle \)-action on \( I \).
• the Cartan matrix $A' = (a'_{r,s})$, $r, s \in I/\sigma$ given by

$$a'_{r,s} = \sum_{i \in O_r} a_{i,j}$$

for all $j \in O_s$, $r, s \in I/\sigma$.

### 2.2. Quantized enveloping algebras and Langlands dual folding.

For any indeterminate $v$ and for any $m \leq n \in \mathbb{Z}_{\geq 0}$, set

$$[n]_v := \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]_v! := \prod_{j=1}^{n}[j]_v, \quad \binom{n}{m}_v := \frac{[n]_v!}{[m]_v! [n-m]_v!}.$$  

For each semisimple Lie algebra $\mathfrak{g}$ we fix symmetrizers $d_i \in \mathbb{N}$ such that $d_i a_{ij} = a_{ij} d_j$ for all $i, j \in I$. Then denote by $C = (d_i a_{ij})$ the symmetrized Cartan matrix of $\mathfrak{g}$ (it depends on the choice of symmetrizers) and let $q_i := q_a$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ which is a $\mathbb{C}(q)$-algebra generated by the elements $E_i, F_i, K_i^{\pm 1}$, $i \in I$ subject to the relations

$$[E_i, F_j] = \delta_{ij} K_i - K_i^{-1} \frac{q_i}{q_i - q_i^{-1}}, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

as well as quantum Serre relations

$$\sum_{b=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{b}_{q_i} E_i^r E_j F_i^{1-a_{ij} - b} = 0 = \sum_{b=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{b}_{q_i} F_i^r F_j F_i^{1-a_{ij} - b}$$

for all $i \neq j$.

We denote by $U^+_q(\mathfrak{g})$ (resp. by $U^{\leq 0}_q(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the $E_i$, $i \in I$. (resp. by the $F_i, K_i^{\pm 1}$, $i \in I$). Note that $U_q(\mathfrak{g})$ and $U^+_q(\mathfrak{g})$ are completely determined by the symmetrized Cartan matrix $C$.

We now define the folding of symmetrized Cartan matrices for a given admissible diagram automorphism $\sigma$. For each $I \times I$ symmetric matrix $C$ and a bijection $\sigma : I \rightarrow I$ denote by $C^\sigma = (c^\sigma_{r,s})$ the $I/\sigma \times I/\sigma$ symmetric matrix with the entries:

$$c^\sigma_{r,s} = \sum_{i \in O_r, j \in O_s} c_{i,j}$$

for all $j \in O_s$, $r, s \in I/\sigma$.

**Lemma 2.2.** Let $C = A$ be the Cartan matrix of a simply-laced semisimple Lie algebra $\mathfrak{g}$ with an admissible diagram automorphism $\sigma$. Then $C^\sigma$ is a symmetrized Cartan matrix of $\mathfrak{g}^{\sigma,\vee}$, where $\mathfrak{g}^{\sigma,\vee}$ is the Langlands dual Lie algebra of the semisimple Lie algebra $\mathfrak{g}^{\sigma}$. More precisely, $C^\sigma = D^\sigma (A')^T$ where $A'$ is the Cartan matrix of $\mathfrak{g}^{\sigma}$ (given by (2.1)) and $D^\sigma$ is the diagonal matrix $\text{diag}([O_r], r \in I/\sigma)$.

**Proof.** By (2.1), (2.3) and the symmetry of $A$ we have for all $r, s \in I/\sigma$

$$c^\sigma_{r,s} = \sum_{i \in O_r, j \in O_s} a_{i,j} = \sum_{i \in O_r, j \in O_s} a_{j,i} = \sum_{i \in O_r} a'_{s,r} = [O_r] a'_{s,r}.$$  

This motivates the following notation. For each $\mathfrak{g}$ and $\sigma$ as above denote by $U_q(\mathfrak{g}^{\sigma,\vee})$ the quantized enveloping algebra determined by the matrix $C^\sigma$ from Lemma 2.2.
2.3. Braid groups and their folding. Given a semisimple Lie algebra $g$ with the Cartan matrix $A = (a_{ij})_{i,j \in I}$, let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the root lattice of $g$. Recall that the Weyl group $W(g)$ is generated by the simple reflections $s_i : Q \to Q$ given by:

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

for $i,j \in I$. It is well-known that $W(g)$ is a Coxeter group with the presentation

$$(s_is_j)^{m_{ij}} = 1,$$

where

$$m_{ij} = \begin{cases} 
1 & \text{if } i = j \\
2 & \text{if } a_{ij} = 0 \\
3 & \text{if } a_{ij} = a_{ji} = -1 \\
4 & \text{if } a_{ij}a_{ji} = 2 \\
6 & \text{if } a_{ij}a_{ji} = 3 
\end{cases}$$

(2.4)

For each $w \in W(g)$ denote by $R(w)$ the set of all reduced decompositions $i = (i_1, \ldots, i_m) \in I^m$ such that

$$w = s_{i_1} \cdots s_{i_m}$$

and $m$ is minimal (this $m$ is the Coxeter length $\ell(w)$). We denote by $w_o$ the longest element of $W(g)$.

The Artin braid group $Br_g$ is generated by the $T_i$, $i \in I$ subject to the relations (for all $i,j \in I$):

$$T_iT_j \cdots = T_jT_i \cdots .$$

To each $w \in W(g)$ one associates the element $T_w \in Br_g$ such that

$$T_w = T_{i_1} \cdots T_{i_m}$$

(2.6)

for each $i = (i_1, \ldots, i_m) \in R(w)$ (it follows from relations (2.5) that $T_w$ is well-defined).

Lemma 2.3 ([7, Theorem 4.21] and [5, Lemma 5.2]). For each $w \in W$ we have

$$T_{w_o}T_wT_{w_o}^{-1} = T_{w_o\sigma w_o} .$$

In particular, the element $C_g = \begin{cases} T_{w_o} & \text{if } w_o \text{ is in the center of } W \\
T_{w_o}^2 & \text{if } w_o \text{ is not in the center of } W \end{cases}$ is in the center of $Br_g$. Moreover, the center of $Br_g$ is generated by all $C_{g'}$, where $g'$ runs over the simple Lie ideals of $g$.

Now we return to the folding situation. Let $g$ be a semisimple Lie algebra and let $\sigma$ be its admissible diagram automorphism.

Note that $\sigma$ defines an automorphism of $W(g)$ (respectively of $Br_g$) via $\sigma(s_i) = s_{\sigma(i)}$ (respectively $\sigma(T_i) = T_{\sigma(i)}$) for $i \in I$. Denote by $\hat{w}_o$ (respectively, $w_o$) the longest element of $W(g)$ (respectively, of $W(g^{\sigma'})$). Since $\sigma$ preserves the Coxeter length, it follows that $\sigma(\hat{w}_o) = \hat{w}_o$. The following result provides a "folding" isomorphism of the corresponding Weyl and braid groups.

Proposition 2.4. For each semisimple simply laced Lie algebra $g$ and its admissible diagram automorphism $\sigma$ we have:
(i) The assignment
\[ s_r \mapsto \hat{s}_r = \prod_{i \in O_r} s_i, \quad r \in I/\sigma \]  
(2.7)
extends to an isomorphism of groups \( \hat{\varphi} : W(\mathfrak{g}^{\sigma^\vee}) \cong W(\mathfrak{g})^\sigma \subset W(\mathfrak{g}) \).

(ii) The assignment
\[ T_r \mapsto \hat{T}_r = \prod_{i \in O_r} T_i, \quad r \in I/\sigma \]  
(2.8)
extends to an isomorphism of groups \( \hat{\varphi} : Br_{\mathfrak{g}^{\sigma^\vee}} \cong (Br_{\mathfrak{g}})^\sigma \subset Br_{\mathfrak{g}} \). Under this isomorphism the element \( T_w \) of \( Br_{\mathfrak{g}^{\sigma^\vee}} \) is mapped to the element \( T_{\hat{w}} \) of \( Br_{\mathfrak{g}} \).

Proof. It is easy to see that (2.7) defines a group homomorphism because it respects the Coxeter relations (2.4). The injectivity also follows. Let us prove surjectivity, i.e., that each element \( w \in W(\mathfrak{g})^\sigma \) factors into a product of the \( \hat{s}_r \), \( r \in I/\sigma \). We proceed by induction on the Coxeter length of \( w \), the induction base being trivial. We need the following well-known result.

Lemma 2.5. Let \( w \in W(\mathfrak{g}) \) and \( i \neq j \) be such that \( \ell(s_iw) = \ell(s_jw) = \ell(w) - 1 \). Then there exists \( w' \) such that \( w = s_is_j\cdots w' \) and \( \ell(w') = \ell(w) - m_{ij} \). In particular, if \( I_0 \subset I \) satisfies
\[ \ell(s_iw) = \ell(w) - 1 \quad \text{for each} \quad i \in I_0, \]
\[ s_is'_j = s'_js_i, \quad \text{for all} \quad i, i' \in I_0, \]
then there exists \( w'' \) such that \( w = (\prod_{i \in I_0} s_i) \cdot w'' \) and \( \ell(w'') = \ell(w) - |I_0| \).

Indeed, let \( w \in W(\mathfrak{g})^\sigma \). Then there exists \( i \in I \) such that \( w = s_iw' \) for some \( w' \) with \( \ell(w') = \ell(w) - 1 \). Applying \( \sigma^k \), we obtain: \( s_is'_j = s'_js_j \), hence \( \ell(s_iw') = \ell(w') - 1 \). Thus the set \( I_0 = \{ i, \sigma(i), \sigma^2(i), \ldots \} \) satisfies the conditions of Lemma 2.5. Therefore, \( w = \hat{s}_iw'' \) with \( \ell(w'') = \ell(w) - |O_r| \). In particular, \( w'' \in W(\mathfrak{g})^\sigma \) and \( \ell(w'') < \ell(w) \) in which we finish the proof by induction. This proves (ii).

To prove (i) note that (2.8) defines a group homomorphism because it respects the Coxeter relations (2.5). The injectivity also follows. Let us prove surjectivity, i.e., that each element \( g \in (Br_{\mathfrak{g}})^\sigma \) factors into a product of the \( \hat{T}_r \), \( r \in I/\sigma \).

Following [5, 7], denote by \( Br_{\mathfrak{g}}^+ \) the positive braid monoid, i.e., the monoid generated by the \( T_r \), \( r \in I \) subject to (2.5).

Lemma 2.6 ([5 Proposition 5.5],[7 Proposition 4.17]).

(i) The assignment \( T_i \mapsto T_i \) defines an injective homomorphism of monoids \( Br_{\mathfrak{g}}^+ \hookrightarrow Br_{\mathfrak{g}} \). In other words, \( Br_{\mathfrak{g}}^+ \) is naturally a submonoid of \( Br_{\mathfrak{g}} \).

(ii) For each \( g \in Br_{\mathfrak{g}}^+ \) there exists an element \( g^+ \in Br_{\mathfrak{g}}^+ \) such that \( g = Cg^+ \) for some central element \( C \) of \( Br_{\mathfrak{g}} \).

Note that the central element \( C \) is the product of all generators \( C_{\mathfrak{g}} \) of the center of \( Br_{\mathfrak{g}}^+ \), where \( \mathfrak{g}^0 \) runs over simple Lie ideals of \( \mathfrak{g} \). This and Lemma 2.6 imply that for each \( g \in Br_{\mathfrak{g}}^+ \) there exists \( N \geq 0 \) such that \( C^N \cdot g \in Br_{\mathfrak{g}}^+ \). Taking into account that \( T_{\hat{w}} \) and hence \( C \hat{\mathfrak{g}} \) is fixed under \( \sigma \), it suffices to prove that any element \( g^+ \in (Br_{\mathfrak{g}}^+)^\sigma \) factors into a product of the \( \hat{T}_r \), \( r \in I/\sigma \).

We need the following result which is parallel to Lemma 2.5.
Lemma 2.7 ([5] Lemma 2.1). Let \( g^+ \in Br_0^+ \) and \( i \neq j \) be such that \( g^+ = T_i \cdot g_i^+ = T_j \cdot g_j^+ \) for some \( g_i^+, g_j^+ \in Br_0^+ \). Then there exists \( h^+ \in Br_0^+ \) such that \( g^+ = T_i T_j \cdots h^+ \).

In particular, if \( I_0 \subset I \) satisfies

- \( g^+ \in T_i \cdot Br_0^+ \) for each \( i \in I_0 \),
- \( T_i T_i' = T_i T_i' \) for all \( i, i' \in I_0 \),
then there exists \( h^+ \in Br_0^+ \) such that \( g^+ = (\prod_{i \in I_0} T_i) \cdot h^+ \).

We proceed by induction on length of elements in \( Br_0^+ \). Indeed, let \( g^+ \in (Br_0^+)^\sigma \). Then there exists \( i \in I \) such that \( g^+ = T_i \cdot g_i^+ \) for some \( g_i^+ \in Br_0^+ \). Applying \( \sigma^k \), we obtain: \( T_i \cdot g_i^+ = T_{\sigma^k(i)} \cdot \sigma^k(g^+) \), where \( \sigma^k(g^+) \in Br_0^+ \). Thus the set \( I_0 = \{ i, \sigma(i), \sigma^2(i), \ldots \} = \mathcal{O} \), satisfies the conditions of Lemma 2.7. Therefore, \( g^+ = T_i \cdot h^+ \) for some \( h^+ \in Br_0^+ \). In particular, \( h^+ \in (Br_0^+)^\sigma \) and is shorter than \( g^+ \) so we finish the proof by induction. This proves (ii) \( \square \)

2.4. PBW bases and quantum folding. G. Lusztig proved in [17] that \( Br_0 \) acts on \( U_q(\mathfrak{g}) \) by algebra automorphisms via:

\[
T_i(K_j^{\pm 1}) = K_j^{\pm 1} K_i^{\mp a_{ij}} , \quad T_i(E_j) = -K_i^{-1} F_i , \quad T_i(F_j) = -E_i K_i , \quad i, j \in I \n\]

\[
T_i(E_j) = \sum_{s+r = -a_{ij}} (-1)^r q_i^{-r} E_i^{(r)} E_j E_i^{(s)} , \quad T_i(F_j) = \sum_{s+r = -a_{ij}} (-1)^r q_i^{-r} F_i^{(s)} F_j F_i^{(r)} , \quad i \neq j \quad (2.9)
\]

where \( Y_i^{(k)} = \prod_{q_i^k} Y_i^k \) (these automorphisms \( T_i \) are denoted \( T_i^{(k)} \) in [17].) G. Lusztig also proved in [17] that \( T_w(E_i) \in U_q^+(\mathfrak{g}) \) if and only if \( \ell(ws_i) = \ell(w) + 1 \). Using this, for each \( i = (i_1, \ldots, i_m) \in R(w_0) \), define the ordered set \( X_i = \{ X_1, X_2, \ldots, X_m \} \subset U_q^+(\mathfrak{g}) \) by:

\[
X_k = X_{i,k} = c_k T_{i_1} \cdots T_{i_{k-1}} (E_{i_k}) ,
\]

where \( c_k = \gamma(s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k} - \alpha_{i_k})) \) and \( \gamma : Q \to \mathbb{C}(q)^\times \) is the unique group homomorphism defined by \( \gamma(\alpha_i) = q_i - q_i^{-1} \). It should be noted that for any \( w, w' \in W(\mathfrak{g}) \), \( i, i' \in I \) such that \( w_\alpha_i = w_\alpha_i' \), \( \gamma(w_\alpha_i - \alpha_i) = \gamma(w_\alpha_i' - \alpha_i') \).

We will need the following useful Lemma which is, most likely, well known.

Lemma 2.8. Suppose that \( i, j \in I \) and \( w \in W(\mathfrak{g}) \) satisfy \( w_\alpha_i = \alpha_j \). Then \( T_w(E_i) = E_j \).

Proof. We use induction on \( \ell(w) \), the induction base being trivial. Since \( w_\alpha_i = \alpha_j \) we have \( \ell(ws_i) = \ell(w) + 1 \) and \( ws_i w^{-1} = s_j \). Then by [2] Lemma 9.9], there exist \( k \in I \) and a \( i \in R(w) \) such that \( i \) terminates with \( (\ldots, i, k) \in R(w_0(i, k)s_i) \) where \( w_0(i, k) \) denotes the longest element of the subgroup of \( W(\mathfrak{g}) \) generated by \( s_i, s_k \). Since by [17] §§4.2.2–4

\[
T_{w_0(i, k)s_i}(E_i) = \begin{cases} E_k , & a_{ik} = a_{ki} = -1 \\ E_i , & \text{otherwise,} \end{cases} \quad (2.10)
\]

we conclude that either \( T_w(E_i) = T_{w'}(E_i) \) with \( w' \alpha_i = \alpha_j \) or \( T_w(E_i) = T_{w'}(E_k) \) with \( w' \alpha_k = \alpha_j \) and in both cases \( \ell(w') < \ell(w) \). \( \square \)
For each \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m \) define the monomial \( X_i^a \in U_q^+(\mathfrak{g}) \) by:
\[
X_i^a = X_i^{a_1} \cdots X_i^{a_m}.
\]

The following result is well-known.

**Proposition 2.9** ([17 Corollary 40.2.2]). The set \( M(X_i) \) of all monomials \( X_i^a \) is a PBW-basis of \( U_q^+(\mathfrak{g}) \).

**Remark 2.10.** The basis \( M(X_i) \) differs from Lusztig’s PBW basis from [17] in that we do not divide the monomials by \( q \)-factorials but rather by some factors which vanish at \( q = 1 \).

Let \( \sigma \) be an admissible diagram automorphism of a semisimple simply laced Lie algebra \( \mathfrak{g} \). For each \( r \in I/\sigma \) define the element \( \hat{E}_r := \prod_{i \in O_r} E_i \in U_q^+(\mathfrak{g}) \) and the set \( \hat{E}_r^* \subset U_q^+(\mathfrak{g}) \) of all monomials \( \prod_{i \in O_r} E_i^{a_i}, a_i \in \mathbb{Z}_{\geq 0} \). Note that \( \hat{E}_r \) is fixed under the action of \( \sigma \) on \( U_q^+(\mathfrak{g}) \) and the set \( \hat{E}_r^* \) is \( \sigma \)-invariant. Moreover, \( (\hat{E}_r^*)^\sigma = \{ \hat{E}_r^k | k \in \mathbb{Z}_{\geq 0} \} \). The following result is obvious.

**Lemma 2.11.** Assume that \( \sigma \) is an admissible diagram automorphism of \( \mathfrak{g} \) and let \( \mathbf{i} = (r_1, \ldots, r_m) \in R(\varpi) \). Let \( \hat{\mathbf{i}} \in R(\hat{\varpi}) \) be any lifting of \( \mathbf{i} \) (as defined in the Introduction). Then

1. \( M(X_{\hat{\mathbf{i}}}) = \hat{X}_1^* \cdots \hat{X}_m^* \) up to multiplication by non-zero scalars, where \( \hat{X}_1^* = T_{r_1} \cdots T_{r_{k-1}}(E_i^*) \), \( 1 \leq k \leq m \).
2. The basis \( M(X_{\hat{\mathbf{i}}}) \) is invariant under the action of \( \sigma \) on \( U_q^+(\mathfrak{g}) \) and the fixed point set \( M(X_{\hat{\mathbf{i}}})^\sigma \) coincides, up to scalars, with the set

\[
\hat{X}_1^a = \hat{X}_1^{a_1} \cdots \hat{X}_m^{a_m},
\]

where

\[
\hat{X}_k = c_k^{-1} T_{r_1} \cdots T_{r_{k-1}}(\hat{E}_{r_k})
\]

and \( c_k = \prod_{i \in O_{r_k}} \gamma(s_{r_i} \cdots s_{r_{k-1}}(\alpha_i) - \alpha_i) \).

This motivates the following definition.

**Definition 2.12** (i-th quantum folding). For each \( \mathbf{i} \in R(\varpi) \) define an injective linear map \( t_1 : U_q^+(\mathfrak{g}^{\sigma^\vee}) \hookrightarrow U_q^+(\mathfrak{g})^\sigma \subset U_q^+(\mathfrak{g}) \) by the formula
\[
t_1(X_i^a) = \hat{X}_i^a
\]
for \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m \).

**Remark 2.13.** Combinatorially, \( t_1 \) is a bijection \( M(X_i) \xrightarrow{\sim} M(X_i)^\sigma \), which can be interpreted as a certain bijection of Kashiwara crystals. More precisely, we can view \( t_1 \) as the composition of the canonical isomorphism \( B_{\infty}(\mathfrak{g}^{\sigma^\vee}) \cong B_{\infty}(\mathfrak{g}^\vee) \) with the Lusztig’s \( \sigma \)-equivariant identification \( \hat{L}_1 : B_{\infty}(\mathfrak{g}) \xrightarrow{\sim} M(X_i) \) so that the following
diagram commutes, in view of Proposition 1.11 and Lemma 2.11

\[ B_\infty(g) \xrightarrow{L_i} M(X_i) \]

\[ B_\infty(g^{\sigma \vee}) \xrightarrow{L_\sigma} M(X_i) \]

where the vertical arrows are natural inclusions of \( \sigma \)-fixed point subsets.

**Remark 2.14.** The Definition 2.12 makes sense for any \( i \in R(w) \) where \( w \) is any element of the Weyl group. In that case we replace \( U_q^+(g) \) by the algebra \( U_q(w) \) introduced in [6, 17] and extensively studied in [24, 25]. One can also define quantum foldings for some pairs \( (g, g') \) not related by a diagram automorphism (e.g., for \( (sp_6, G_2) \)). Namely, following [16], one can embed \( B_\infty(g') \) into \( B_\infty(g) \) and then extend the embedding linearly to \( i: U_q(g') \hookrightarrow U_q(g) \) using Lusztig’s identifications \( B_\infty(g') \cong M(X_i) \) and \( B_\infty(g) \cong M(X_1) \).

We conclude this section with the proof of Theorem 1.20.

**Proof of Theorem 1.20.** It is sufficient to prove the statement for two reduced decompositions \( i, i' \in R(w_0) \) which differ by one braid relation involving \( r, s \in I/\sigma \) and thus suffice to consider the rank 2 case. We have the following two possibilities:

1°. \( |O_r| = 2, |O_s| = 1 \) and \( i = (r, s, r, s), i' = (s, r, s, r) \). Let \( O_r = \{i, j\}, O_s = \{k\} \), with \( a_{i,k} = a_{j,k} = -1 \). Then \( \hat{i} = (i, j, k, i, j, k), \hat{i}' = (k, i, j, k, i, j) \) and the elements \( \hat{X}_s \) for these two decompositions are, respectively,

\[ \hat{X}_1 = E_i E_j, \hat{X}_2 = \frac{[E_i, [E_j, E_k]_{q^{-1}}]_{q^{-1}}}{(q - q^{-1})^2}, \hat{X}_3 = \frac{[E_i, E_k]_{q^{-1}}[E_j, E_k]_{q^{-1}}}{(q - q^{-1})^2}, \hat{X}_4 = E_k, \]

while \( \hat{X}_1' = \hat{X}_4, \hat{X}_2' = \hat{X}_1', \hat{X}_3' = \hat{X}_2' \) and \( \hat{X}_4' = \hat{X}_1' \), where \( * \) is the unique anti-automorphism of \( U_q^+(g) \) such that \( E_i^* = E_i \) for \( i \in I \). A straightforward computation shows that

\[ \hat{X}_2' = q^{-2}(\hat{X}_3 + (q - q^{-1})^{-1}[\hat{X}_4, \hat{X}_1]\hat{X}_4), \hat{X}_3' = \hat{X}_2 + q^{-1}(q - q^{-1})^{-1}[\hat{X}_4, \hat{X}_1]. \]

Therefore, the subalgebra of \( U_q^+(g) \) generated by \( \hat{X}_r \) contains all elements \( \hat{X}_r' \). Applying \( * \) we obtain the opposite inclusion.

2°. \( |O_r| = |O_s| = 2 \) and \( i = (r, s, r), i' = (s, r, s) \). Let \( O_r = \{i, j\}, O_s = \{k, l\} \) with \( a_{i,k} = a_{j,l} = -1 \) and \( a_{i,l} = a_{j,k} = 0 \). Then we have \( \hat{X}_1 = E_i E_j = \hat{X}_3, \hat{X}_3 = E_k E_l = \hat{X}_1' \) and

\[ \hat{X}_2 = \frac{[E_i, E_k]_{q^{-1}}[E_j, E_l]_{q^{-1}}}{(q - q^{-1})^2}, \hat{X}_2' = \frac{[E_k, E_l]_{q^{-1}}[E_i, E_j]_{q^{-1}}}{(q - q^{-1})^2}. \]

It is easy to check that

\[ \hat{X}_2' = \hat{X}_2 + q^{-1}(q - q^{-1})^{-1}[\hat{X}_3, \hat{X}_1], \]

hence \( \hat{X}_2' \) is contained in the subalgebra of \( U_q^+(g) \) generated by \( \hat{X}_k, 1 \leq k \leq 3 \). Interchanging the role of \( r \) and \( s \) completes the proof. \( \square \)
2.5. **Diamond Lemma and specializations of PBW algebras.** We will use the following version of Bergman’s Diamond Lemma ([4]). Let $A$ be an associative $k$-algebra and suppose that $A$ is generated by a totally ordered set $X_A$. Let $M(X_A)$ be the set of ordered monomials on the $X_A$.

**Proposition 2.15.** Assume that the defining relations for $(A, X_A)$ are

$$X'X = \sum_{M \in M(X_A)} c^M_{X,X'} M,$$

where $c^M_{X,X'} \in k^\times$ and for any $M \in M(X_A)$, $c^M_{X,X'} \neq 0$ implies that $M < X'X$ in the lexicographic order. If for all $X < X' < X''$ there exist a unique $S \subset M(X_A)$ and a unique $\{a_M : M \in S\} \subset k^\times$ such that

$$X''X'X = \sum_{M \in S} a_M M,$$

then $(A, X_A)$ is a PBW algebra.

Note that, unlike [20], we do not require $(A, X_A)$ to be quadratic. In fact, in most cases where we will need to apply the Diamond Lemma, this will not be the case.

We will now list some elementary properties of specializations which will be needed later. The simplest instance of specialization is given by the following definition. Throughout this subsection, let $k = \mathbb{C}(t)$ (later on, we set $t = q - 1$) and denote by $k_0$ the set of all $f = f(t) \in k$ such that $f(0)$ is defined. Clearly, $k_0$ is a (local) subalgebra of $k$ and for each non-zero $f \in k$ either $f \in k_0$ or $f^{-1} \in k_0$.

**Definition 2.16.** Let $U$ and $V$ be $k$-vector spaces with bases $B_U$ and $B_V$ respectively. Let $F : U \to V$ be a $k$-linear map that all matrix coefficients $c_{b,b'} = c_{b,b'}(t) \in k$ defined by:

$$F(b) = \sum_{b' \in B_V} c_{b,b'} b'$$

for $b \in B_U$, belong to $k_0$. Then the specialization $F_0$ of $F$ is a $k$-linear map $U \to V$ given by:

$$F_0(b) = \sum_{b' \in B_V} c_{b,b}(0) b'$$

for $b \in B_U$ (here, unlike in the literature on deformation theory, we preserve the ground field $k = \mathbb{C}(t)$ after the specialization because it is more convenient to view both $F$ and $F_0$ as $k$-linear maps $U \to V$).

Similarly, let $(A, X_A)$ be a PBW algebra. Then, in the notation of Definition 1.3, it has a unique presentation:

$$X'X = \sum_{M \in M(X_A)} c^M_{X,X'} M$$

for all $X, X' \in X_A$ such that $X < X'$, where all $c^M_{X,X'} \in k$. 
Definition 2.17. We say that the PBW algebra \((A, X_A)\) is specializable if all the \(c^M_{X,X'}\) belong to \(k_0\) and in that case define the specialization \((A_0, X_A)\) to be the associative \(k\)-algebra with the unique presentation:

\[
X'X = \sum_{M \in M(X_A)} c^M_{X,X'}(0)M
\]

for all \(X, X' \in X_A\) with \(X < X'\), where all \(c^M_{X,X'} \in k\).

We say that a specializable PBW algebra \((A, X_A)\) is optimal if \((A_0, X_A)\) is just the polynomial algebra \(k[X_A]\), that is, if \(c^M_{X,X'}(0) = \delta_{M,XX'}\) for all relevant \(X, X', M\) (i.e., the defining relations in \((A_0, X_A)\) are \(X'X = XX'\)). In that case we define a bi-differential bracket \(\{\cdot, \cdot\}\) on \(k[X_A]\) by:

- (Leibniz rule) \(\{xy, z\} = x\{y, z\} + y\{x, z\}, \{x, yz\} = xy\cdot y\cdot z\) for all \(x, y, z \in k[X_A]\).
- (skew symmetry) \(\{x, y\} = -\{y, x\}\) for all \(x, y \in k[X_A]\).
- for all \(X, X' \in X_A\) with \(X < X'\)

\[
\{X', X\} = \sum_{M \in M(X_A)} \left. \frac{\partial c^M_{X,X'}}{\partial t} \right|_{t=0} M. \tag{2.11}
\]

Proposition 2.18. Let \((A, X_A)\) be a specializable PBW-algebra. Then:

(i) Its specialization \((A_0, X_A)\) is also PBW.

(ii) If, additionally, \((A, X_A)\) is optimal, then the bracket \(\{\cdot, \cdot\}\) on \(A_0 = k[X_A]\) is Poisson.

Proof. Denote \(A'_0 = \sum_{M \in M(X_A)} k_0 M\). Clearly, this is a \(k_0\)-subalgebra of \(A\) and a free \(k_0\)-module. Taking into the account that \((t)\) is a (unique) maximal ideal in \(k_0\), we see that the specialization \(A_0\) of \(A\) is canonically isomorphic to \(k \otimes_{k_0} A'_0/tA'_0\).

To prove (ii), suppose that in \(A_0\) we have \(\sum_{M \in M(X_A)} c_M M = 0\), where all \(c_M \in \mathbb{C}\). This implies that \(\sum_{M \in M(X_A)} c_M M \in tA'_0\), hence \(\sum_{M \in M(X_A)} c_M M = \sum_{M \in M(X_A)} tc'_M M\) for some \(c'_M \in k_0\). Since \(A'_0\) is a free \(k_0\)-module, this implies that \(c_M = c'_M = 0\) for all \(M\) and completes the proof of (ii).

Now we prove (i). Optimality of \((A, X_A)\) implies that the commutator \([a, b]\) of any \(a, b \in A'_0\) belongs to the ideal \(tA_0\). For any \(a_0, b_0 \in A'_0/tA'_0\) denote:

\[
\{a_0, b_0\} = \pi\left(\frac{[a,b]}{t}\right)
\]

where \(\pi : A'_0 \rightarrow A'_0/tA'_0\) is the canonical projection and \(a, b \in A'_0\) are any elements such that \(\pi(a) = a_0, \pi(b) = b_0\). Clearly, the bracket \(\{a_0, b_0\}\) is well-defined (i.e., it does not depend on the choice of representatives \(a\) and \(b\)). This bracket is Poisson because the original commutator bracket was skew-symmetric, and satisfied both the Liebniz rule and Jacobi identities.

It remains to verify (2.11). Indeed, let \(X, X' \in X_A\) with \(X < X'\). We have

\[
\{X', X\} = \pi\left(\frac{X'X - XX'}{t}\right) = \pi\left(\frac{c^M_{XX'} - 1}{t}\right)XX' + \sum_{M \neq XX'} \pi\left(\frac{c^M_{XX'}}{t}\right) M.
\]
This gives (2.11) because \( \pi(f) = f(0) \) and \( \pi\left( \frac{f - f(0)}{t} \right) = \frac{\partial f}{\partial t} \bigg|_{t=0} \) for any \( f \in \mathbb{k}_0 \). The proposition is proved.

2.6. Nichols algebras and proof of Theorem 1.11. We will now prove Theorem 1.11 which allows to establish the PBW property when an algebra is quadratic and is defined in terms of a braiding. Retain the notation of Section 2.5.

Proof. Let \( Y \) be a \( \mathbb{k} \)-vector space and \( \Psi : Y \otimes Y \to Y \otimes Y \) be linear map. For each \( k \geq 2 \) we define the linear maps \( \Psi_i : Y^\otimes k \to Y^\otimes k \), \( i = 1, \ldots, k - 1 \) by the formula:

\[
\Psi_i = 1^{\otimes i-1} \otimes \Psi \otimes 1^{\otimes k-i-1}.
\]

If \( \Psi \) is invertible and satisfies the braid equation (1.3), then for each \( k \geq 2 \) one obtains the representation of the braid group \( Br_{\mathbb{k}} \) on \( Y^\otimes k \) (in the notation of Section 2.4) via \( T_i \mapsto \Psi_i \) for \( i = 1, \ldots, k - 1 \).

Therefore, one can define the braided factorial \( [k]!_{\Psi} : Y^\otimes k \to Y^\otimes k \) by the formula:

\[
[k]!_{\Psi} = \sum_{w \in S_k} \Psi_w
\]

where \( \Psi_w \) is the image of \( T_w \) (given by (2.6)) in \( \text{End}_k(Y^\otimes k) \).

It is well-known (see e.g. [23]) that \( I_{\Psi} := \bigoplus_{k \geq 2} [k]!_{\Psi} \) is a two-sided ideal in the tensor algebra \( T(Y) \). The quotient algebra \( B_{\Psi}(Y) := T(Y)/I_{\Psi} \) is called the Nichols-Woronowicz algebra of the braided vector space \( (Y, \Psi) \).

For each linear map \( \Psi : Y \otimes Y \to Y \otimes Y \) denote \( A_{\Psi}(Y) := T(Y)/\langle \ker(\Psi + 1) \rangle \).

We need the following result.

Proposition 2.19. Let \( Y \) be a \( \mathbb{k} \)-vector space and let \( \Psi : Y \otimes Y \to Y \otimes Y \) be an invertible \( \mathbb{k} \)-linear map satisfying the braid equation. Assume that:

(i) The specialization \( \Psi_0 = \Psi|_{t=0} \) of \( \Psi \) is a well-defined (with respect to a basis of \( Y \)) invertible linear map \( Y \otimes Y \to Y \otimes Y \) satisfying the braid equation.

(ii) The Nichols-Woronowicz algebra \( B_{\Psi_0}(Y) \) is isomorphic to \( A_{\Psi_0}(Y) \) as a graded vector space.

(iii) \( \dim \ker(\Psi + 1) = \dim \ker(\Psi_0 + 1) \).

Then \( B_{\Psi}(Y) \cong B_{\Psi_0}(Y) \) as a graded vector space and one has an isomorphism of algebras

\[
B_{\Psi}(Y) \xrightarrow{\sim} A_{\Psi}(Y).
\]

Proof. We will need two technical results.

Lemma 2.20. Let \( U \) and \( V \) be a finite-dimensional \( \mathbb{C}(t) \) vector spaces and \( F : U \to V \) be a linear map such that its specialization \( F_0 \) at \( t = 0 \) is a well-defined map \( U \to V \) (with respect to some bases of \( B_U \) and \( B_V \)). Then

(a) \( \dim F_0(U) \leq \dim F(U) \) and \( \dim \ker F \leq \dim \ker F_0 \).

(b) Assume that \( \dim \ker F = \dim \ker F_0 \). Then there is a linear map \( G : V \to V \) such that:

(i) \( G(V) = \ker F \),

(ii) the specialization \( G_0 \) of \( G \) at \( t = 0 \) is well-defined,

(iii) \( G_0(V) = \ker F_0 \).
Proof. Fix bases $B_U$ and $B_V$ and identify $U$ with $k^n$, $V$ with $k^m$, and $F : k^n \to k^n$ with its $m \times n$ matrix.

It is well-known (and easy to show) that for each non-zero $F \in \text{Mat}_{m \times n}(k)$ there exist $g_t \in GL_m(k_0)$ and $h_t \in GL_n(k_0)$ such that

$$F = g_t P h_t ,$$

(2.12)

where $P$ is an $m \times n$-matrix such that $P_{ij} = 0$ unless $(i, j) \in \{(1, 1), \ldots, (r, r)\}$, and $P_{ii} = t^{\lambda_i}$ for $i = 1, \ldots, r$, where $r = \text{rank}(F)$ and $\lambda_i \in \mathbb{Z}$. Therefore, $F_0$ is well-defined if and only if all $\lambda_i \geq 0$.

In particular, $F_0 = g_0 P h_0$ and $\text{rank}(F_0) \leq k$, where $P_0$ is the specialization of $P$ at $t = 0$. That is,

$$\dim F_0(U) = \text{rank}(M_0) \leq \text{rank}(M_t) = \dim F(U) .$$

This in conjunction with the equality $\dim \ker F + \text{rank}(F) = \dim V$ proves (a).

Now we prove (b). Clearly, the condition $\dim \ker F = \text{dim ker } F_0$ is equivalent to $\text{rank}(P_t) = r$, i.e., in the decomposition 2.12 one has $\lambda_1 = \cdots = \lambda_r = 0$, i.e., $P = P_0$ is the matrix (not depending on $t$) of the standard projection $k^n \to k^r \subset k^m$.

Let $P^\perp \in \text{Mat}_{n \times n}(\mathbb{C})$ be the standard projection $k^n \to \text{Span}\{e_{r+1}, \ldots, e_n\} \subset k^n$ (e.g., $PP^\perp = 0$). Denote $G := h_t^{-1}P^\perp$ so that the specialization $G_0$ of $G$ at $t = 0$ is well-defined and given by $G_0 = h_0^{-1}P^\perp$.

Clearly,

$$G(k^n) = h_t^{-1}(\text{Span}\{e_{r+1}, \ldots, e_n\}) = \ker P h_t = \ker F .$$

Similarly, $G_0(k^n) = \ker F$. This proves (b).

Lemma 2.21. Let $F$ be a free $k$-algebra on $y_i$, $i \in I$ where $I$ is a finite set. Fix a grading on $F$ with $\deg y_i \in \mathbb{Z}_{\geq 0}$. Fix any finite subset $B_t$ of specializable (with respect to the natural monomial basis of $F$) homogeneous elements in $F$. Then $\dim \langle B_t \rangle_n \geq \dim \langle B_0 \rangle_n$, where $\langle B_t \rangle_n$ (respectively, $\langle B_0 \rangle_n$) is the $n$th homogeneous component of the ideal in $F$ generated by $B_t$ (respectively, by the specialization $B_{t=0}$ of $B_t$).

Proof. Clearly

$$\langle B_t \rangle_n = \bigoplus_{i+j+k=n} \sum_{b \in B_t : \deg b = j} F_i b F_k .$$

Define $\langle B_t \rangle_n = \bigoplus_{i+j+k=n} \sum_{b \in B_t : \deg b = j} F_i b F_k$ and let $F : \langle B_t \rangle_n \to F_n$ be the natural map which is the identity on each summand. Clearly the specialization of $F_0$ at $t = 0$ with respect to the natural monomial basis in both spaces is well-defined and the image of $F$ (respectively, of $F_0$) is $\langle B_t \rangle_n$ (respectively, $\langle B_0 \rangle_n$). Then the assertion follows from Lemma 2.20(a).

The algebra $B_{\Psi}(Y)$ is graded and $B_{\Psi}(Y)_k$ is isomorphic to $[k]_\Psi!(Y^\otimes k)$ as a vector space. Therefore,

$$\dim(B_{\Psi}(Y)_k) \leq \dim(B_{\Psi})_k$$

(2.13)

by Lemma 2.20(a) with $V = Y^\otimes k$ and $F = [k]_\Psi!$.

On the other hand, note that $\ker(\Psi + 1) = \ker[2]_\Psi$ and so we have a structural homomorphism of graded algebras $A_{\Psi}(Y) \to B_{\Psi}(Y)$. In particular, we obtain a surjective homomorphism of vector spaces $Y^\otimes k/\langle B_t \rangle_k \to B_{\Psi}(Y)_k$ where $B$ is the
image of natural basis in $Y^\otimes k$ under the map $G$ from Lemma 2.20. It follows from Lemma 2.21 that $\dim \mathcal{A}_\psi(Y)_k \geq \dim \mathcal{A}_\psi(Y)_k$. Combining this with (2.13) and the obvious inequality $\dim \mathcal{A}_\psi(Y)_k \geq \dim \mathcal{B}_\psi(Y)_k$ we obtain $\dim \mathcal{A}_\psi(Y)_k \geq \dim \mathcal{A}_\psi(Y)_k$ which implies $\dim \mathcal{A}_\psi(Y)_k = \dim \mathcal{B}_\psi(Y)_k$ for all $k$. This completes the proof of Proposition 2.19.

Now we can complete the proof of Theorem 1.11. First, in the notation of Proposition 2.19 with $Y^*$ and $-\Psi^*: Y^* \otimes Y^* \rightarrow Y^* \otimes Y^*$.

Taking $t = q-1$ and $\Psi_0^* = \tau$ in Proposition 2.19 we see that $\mathcal{B}_{-\Psi_0^*}(Y^*) = \Lambda(Y^*) = \mathcal{B}_{-\Psi_0^*}(Y^*)$ hence $\mathcal{A}_{-\Psi_0^*}(Y^*) \cong \mathcal{B}_{-\Psi_0^*}(Y^*)$ is a flat deformation of the exterior algebra $\Lambda(Y^*)$. Taking into account that $(\ker(-\Psi^* + 1))^{\perp} = (\Psi - 1)(Y \otimes Y)$ we see that the quadratic dual $\mathcal{A}_{-\Psi_0^*}(Y^*)^{\perp} = S(Y)$ is a flat deformation of $S(Y)$.

Theorem 1.11 is proved.

2.7. Module algebras and semi-direct products. It is well-known that $U_q(g)$ is a Hopf algebra with:

- The coproduct $\Delta: U_q(g) \rightarrow U_q(g) \otimes U_q(g)$ given by:
  $$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i.$$ 
- The counit $\varepsilon: U_q(g) \rightarrow \mathbb{C}(q)$ given by:
  $$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^{\pm1}) = 1.$$
- The antipode $S: U_q(g) \rightarrow U_q(g)$ given by
  $$S(F_i) = -F_i K_i, \quad S(E_i) = -K_i^{-1} E_i, \quad S(K_i) = K_i^{-1}.$$ 

In particular, $U_q(g)$ admits the (left) adjoint action on itself, which we denote by $(u, v) \mapsto (\text{ad} u)(v)$. The action is given by:

$$\text{ad} u(v)(v) = u_{(1)} v S(u_{(2)}),$$
where $\Delta(u) = u_{(1)} \otimes u_{(2)}$ in the Sweedler notation. By definition,

$$(\text{ad} K_i)(u) = K_i u K_i^{-1}, \quad (\text{ad} E_i)(u) = E_i u - K_i u K_i^{-1} E_i, \quad (\text{ad} F_i)(u) = (F_i u - u F_i) K_i.$$ 

In particular, the quantum Serre relations can be written as

$$(\text{ad} E_i)^{(1-a_{ij})}(E_j) = 0.$$ 

We will also need the right action of $U_q(g)$ on itself. Let $*$ be the unique anti-automorphism of $U_q(g)$ defined by $E_i^* = E_i, F_i^* = F_i$ and $K_i^* = K_i^{-1}$. Then we define \(\text{ad}^* u = * \circ \text{ad} u \circ *\). In particular, we have

$$(\text{ad}^* K_i)(u) = K_i u K_i^{-1}, \quad (\text{ad}^* E_i)(u) = u E_i - E_i K_i u K_i^{-1}, \quad (\text{ad}^* F_i)(u) = K_i^{-1}[u, F_i].$$

and

$$T_i(E_j) = (\text{ad}^* E_i)^{(1-a_{ij})}(E_j).$$

It is easy to see that $\text{ad}^* u$ is in fact the right adjoint action for a different co-product on $U_q(g)$. Note that for all $i, j \in I$

$$(\text{ad} E_i)(E_j) = (\text{ad}^* E_j)(E_i),$$

while for all $i, j \in I$ and $w \in W$ such that $w \alpha_i = \alpha_j$ we have, by Lemma 2.8

$$T_w((\text{ad} E_i)(u)) = (\text{ad} E_j)(T_w(u)), \quad T_w((\text{ad}^* E_i)(u)) = (\text{ad}^* E_j)(T_w(u)).$$
Given a bialgebra $U$, refer to an algebra in the category $U$-mod as a module algebra over $U$. The following Lemma is obvious.

**Lemma 2.22.** Let $A$ be a left module algebra over $U_q(g)$. Then the action of Chevalley generators on $A$ satisfies:

$$K_i(ab) = K_i(a)K_i(b), \quad E_i(ab) = E_i(a)b + K_i(a)E_i(b), \quad F_i(ab) = F_i(a)K_i^{-1}(b) + aF_i(b)$$

for all $a, b \in A$ and $i \in I$.

**Definition 2.23.** For any bialgebra $B$ and its module algebra $A$ define the cross product $A \rtimes B$ to be the the vector space $A \otimes B$ with the associative product given by:

$$(a \otimes b)(a' \otimes b') = a \cdot (b_1(1)) \otimes b_2, \cdot b'$$

for all $a, b \in A$, $b \in B$ (where $\Delta(b) = b_1 \otimes b_2$ in Sweedler notation). In what follows, we suppress tensors and will write $a \cdot b$ instead of $a \otimes b$ and $b \cdot a$ instead of $(1 \otimes b)(a \otimes 1)$ in the algebra $A \rtimes B$.

Similarly, one can replace $B$ by a braided bialgebra, i.e., a bialgebra in a braided category $\mathcal{C}$ and $A$ by a module algebra over $B$ in $\mathcal{C}$. Our main example is when $\mathcal{C}_Q$ is the category of $Q$-graded vector spaces with the braiding $\Psi_{U,V} : U \otimes V \to V \otimes U$ for $U, V \in \text{Ob} \mathcal{C}_Q$ given by

$$\Psi_{U,V}(u \otimes v) = q^{(\mu,\nu)}v \otimes u, \quad u \in U_\mu, \quad v \in V_\nu$$

where $(\cdot, \cdot)$ is the inner product on $Q$ given by

$$(\alpha_i, \alpha_j) = d_i a_{ij} \quad (2.18)$$

where $C = (d_i a_{ij})_{i,j \in I}$ is a symmetrized Cartan matrix of a semisimple lie algebra $g$. Let $Y = C(q) \otimes_Z Q$. As G. Lusztig proved in [17], the Nichols-Woronowicz algebra $B_{\Psi,Y}(Y)$ (see Section 2.6) is naturally isomorphic to $U_q^+(g)$. The following obvious fact is parallel to Lemma 2.22.

**Lemma 2.24.** Let $A = \bigoplus_{\nu \in Q} A_\nu$ be a module algebra over (the braided bialgebra) $U_q^+(g)$. Then

(i) For any $a \in A_\nu$, $b \in A$, $i \in I$ one has

$$E_i(ab) = E_i(a)b + q^{(\alpha_i,\nu)}aE_i(b).$$

(ii) The braided cross product $A \rtimes U_q^+(g)$ is the algebra generated by $A$ and $U_q^+(g)$ (and isomorphic to $A \otimes U_q^+(g)$ as a vector space) subject to the relations

$$E_i \cdot a = E_i(a) + q^{(\alpha_i,\nu)}a \cdot E_i$$

for all $a \in A_\nu$, $i \in I$. In particular, if $A$ is a PBW algebra, then so is $A \rtimes U_q^+(g)$.

**Remark 2.25.** In fact if $A = \bigoplus_\nu A_\nu$ is a $U_q(g)$-module algebra with $K_i|_{A_\nu} = q^{(\alpha_i,\nu)}$, then the braided cross product $A \rtimes U_q^+(g)$ is simply the subalgebra of the ordinary cross product $A \rtimes U_q(g)$ generated by $A$ and $U_q^+(g)$. Moreover, $A \rtimes U_q(g) \cong A \rtimes U_q^+(g) \otimes U_q^{\le 0}(g)$ as a vector space.
Using Lemma 2.22 and \[17\] §3.1.5 we obtain for any \(U_q(\mathfrak{g})\)-module algebra \(A\) and \(a, b \in A\)

\[
E_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{p(r-p)} E_i^{(r-p)}(K_i^p(a)) E_i^{(p)}(b)
\]

(2.19)

\[
F_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{p(r-p)} F_i^{(p)}(a) K_i^{-p}(F_i^{(r-p)}(b)),
\]

where \(Y_i^{(r)} := ([r]_{q_i})^{-1} Y_i^r\). In particular, if \(K_i(a) = v_i a\), then

\[
E_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{p(r-p)} v_i^p E_i^{(r-p)}(a) E_i^{(p)}(b)
\]

(2.20)

Given \(i = (i_1, \ldots, i_k) \in I^k\) and \(m = (m_1, \ldots, m_k) \in \mathbb{Z}_{\geq 0}^k\), let \(E_i^{(m)} = E_{i_1}^{(m_1)} \cdots E_{i_k}^{(m_k)}\). Using an obvious induction, we immediately obtain from (2.20)

\[
E_i^{(m)}(ab) = \sum_{m', m'' \in \mathbb{Z}_{\geq 0}^k : m' + m'' = m} q_i^{\frac{1}{2} \sum_{1 \leq r, s \leq k} m'_r m''_s (\alpha_r, -\alpha_s)} v^{m''} E_i^{(m')} (a) E_i^{(m'')} (b),
\]

(2.21)

where \(v^{m''} = \prod_{r=1}^{k} v_{i_r}^{m''_r}\).

Following M. Kashiwara and G. Lusztig (\[17\] \[15\]), define for all \(i \in I\) and for all \(u \in U_q^+(\mathfrak{g})\) the elements \(r_i(u), s_r(u) \in U_q^+(\mathfrak{g})\) by

\[
[u, F_i] = \frac{K_i \cdot s_r(u) - r_i(u) \cdot K_i^{-1}}{q_i - q_i^{-1}}.
\]

(2.22)

**Lemma 2.26** (\[17\] Lemma 1.2.15, Proposition 3.1.6). For all \(i, j \in I, u, v \in U_q^+(\mathfrak{g})\)

(i) \(s_r(u) = r_i(u^*)^*\) and \(r_i(s_r(u)) = s_r(r_i(u))\);

(ii) \(r_i(E_j) = \delta_{ij}\) and \(r_i(uv) = r_i(u) K_i^{-1} v K_i + u r_i(v)\);

(iii) If \(u \in \ker r, r\text{ then } (ad F_i)(u) = (q_i - q_i^{-1})^{-1} r_i(u)\);

(iv) \(s_r(ad E_j(u)) = q_i^{-a_{ij}} E_j \cdot s_r(u) - q_i^{a_{ij}} K_j \cdot s_r(u) \cdot K_j^{-1} \cdot E_j\);

(v) \(\bigcap_{i \in I} \ker r_i = \bigcap_{i \in I} \ker r_i = \mathbb{C}(q)\).

Recall that for any \(J \subset I\) the parabolic subalgebra \(\mathfrak{p}_J\) of \(\mathfrak{g}\) is the Lie subalgebra generated by \(\mathfrak{g}_+\) and \(f_J, j \in J\). Let \(U_q(\mathfrak{p}_J)\) be the subalgebra of \(U_q(\mathfrak{g})\) generated by \(U_q^+(\mathfrak{g})\) and by the \(F_j, K_j^\pm 1, j \in J\). Let \(U_q(\mathfrak{g}_J) = \langle E_j, F_j, K_j^\pm 1 : j \in J \rangle\) and define

\[U_q(\mathfrak{r}_J) = \{ u \in U_q^+(\mathfrak{g}) : s_r(u) = 0, \forall j \in J \} \]

Clearly \(U_q(\mathfrak{p}_J)\) is a Hopf subalgebra of \(U_q(\mathfrak{g})\), \(U_q(\mathfrak{g}_J)\) is a Hopf subalgebra of \(U_q(\mathfrak{p}_J)\), and \(U_q(\mathfrak{r}_J)\) is a subalgebra of \(U_q(\mathfrak{p}_J)\). The following corollary is an immediate consequence of Lemma 2.26

**Corollary 2.27** (Quantum Levi factorization).

(i) \(U_q(\mathfrak{r}_J)\) is preserved by the adjoint action of \(U_q(\mathfrak{g}_J)\) on \(U_q(\mathfrak{p}_J)\). In particular, \(U_q(\mathfrak{r}_J)\) is a \(U_q(\mathfrak{g}_J)\)-module algebra.

(ii) \(U_q(\mathfrak{p}_J) = U_q(\mathfrak{r}_J) \rtimes U_q(\mathfrak{g}_J)\).
3. Folding ($\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n}$) and Proof of Theorems [1.8] and [1.12]

3.1. The algebras $U^+_{q,n}$ and $U_{q,n}$. In what follows we take $I = \{-1, 0, \ldots, n-1\}$ for $\mathfrak{g} = \mathfrak{so}_{2n+2}$ so that $\sigma$ interchanges $-1$ and $0$ and fixes each $i = 1, \ldots, n-1$. Accordingly, we set $I/\sigma := \{0, 1, \ldots, n-1\}$ for $\mathfrak{g}^{\mathfrak{sp}} = \mathfrak{sp}_{2n}$.

Let $U_{q,n}$ be the associative $\mathbb{C}(q)$-algebra generated by $U_q(\mathfrak{sl}_n)$ and $w, z$ subject to the following relations

\begin{align*}
[F_i, w] &= 0 = [F_i, z], \quad K_i w K_i^{-1} = q^{-2\delta_{i,1}} w, \quad K_i z K_i^{-1} = q^{-\delta_{i,2}} z \quad (3.1a) \\
[E_i, w] &= 0, \quad i \neq 1, \quad [E_i, z] = 0, \quad i \neq 2, \quad [z, w] = 0 \quad (3.1b) \\
[E_1, [E_1, [E_1, w]_{q^{-2}}]]_{q^2} &= 0 = [E_2, [E_2, z]_{q^2}]_{q^2}, \quad (3.1c) \\
[w, [w, E_1]_{q^2}]_{q^{-2}} &= -hwz, \quad [z, [E_2, [E_1, w]_{q^2}]]_{q^2} = [w, [E_1, [E_2, z]_{q^2}]]_{q^2}, \quad (3.1d) \\
2[z, [E_2]_{q^2}]_{q^{-1}} &= h[z, E_1, E_2]_{q} + w[E_2, E_1]_{q^2} + w[E_1, E_2]_{q} + [E_2, z]_{q^{-1}}E_1w \quad (3.1e)
\end{align*}

where $h = q - q^{-1}$ and we abbreviate $[a, b]_v = ab - vba$ and $[a, b]_1 = ab - ba$ (with the convention that $E_2 = 0$ if $n = 2$) and the $E_i, F_i, K_i^{\pm 1}, 1 \leq i \leq n-1$, are Chevalley generators of $U_q(\mathfrak{sl}_n)$. Let $U_{q,n}^+$ be the associative $\mathbb{C}(q)$-algebra generated by the $E_i, 1 \leq i \leq n-1$ and by $w, z$ subject to the relations (3.1a)–(3.1e). Let $V$ be the standard $U_{q}(\mathfrak{sl}_n)$-module. We denote $S_q(V \otimes V) := S_q(V \otimes V)$ in the notation of Theorem [1.11] where $\Psi : V^{\otimes 4} \rightarrow V^{\otimes 4}$ is the $U_q(\mathfrak{sl}_n)$-equivariant map given by (1.4).

The following theorem is the main result of the section.

**Theorem 3.1.**

(i) $S_q(V \otimes V)$ is a PBW algebra on any ordered basis of $V \otimes V$.

(ii) The algebra $U_{q,n}^+$ is isomorphic to the braided cross product $S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n)$ and in particular is PBW.

(iii) The algebra $U_{q,n}$ is isomorphic to the cross product $S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$ and also to the tensor product of $U_{q,n}^+$ and the subalgebra of $U_q(\mathfrak{sl}_n)$ generated by the $F_i, K_i^{\pm 1}, i \in I$. In particular, $U_{q,n}^+$ is a subalgebra of $U_{q,n}$.

(iv) The assignment $w \mapsto E_0, z \mapsto 0$ defines a homomorphism $\mu : U_{q,n} \rightarrow U_q(\mathfrak{sp}_{2n})$. Its image is the (parabolic) subalgebra of $U_q(\mathfrak{sp}_{2n})$ generated by $U_q^+(\mathfrak{sp}_{2n})$ and $K_i^{\pm 1}, F_i, 1 \leq i \leq n-1$.

(v) The assignment $w \mapsto E_0 E_{-1}$ and

\[ z \mapsto \frac{1}{q^h}([E_{-1}, [E_1, E_0]_{q^2}]_{q} + [E_0, [E_1, E_{-1}]_{q^2}]_{q}) \]

defines an algebra homomorphism $\iota : U_{q,n} \rightarrow U_q(\mathfrak{so}_{2n+2})$. Its image is contained in the (parabolic) subalgebra of $U_q(\mathfrak{so}_{2n+2})$ generated by $U_q^+(\mathfrak{so}_{2n+2})$ and $K_i^{\pm 1}, F_i, 1 \leq i \leq n-1$.

(vi) The assignments

\[ T_i(w) = \begin{cases} (q + q^{-1})^{-1}[w, E_i]_{q^{-2}}, & i = 1 \\ w, & i \neq 1 \end{cases} \]

\[ T_i(z) = \begin{cases} [z, E_2]_{q^{-1}}, & i = 2 \\ z, & i \neq 2 \end{cases} \quad (3.2) \]

extend Lusztig’s action (2.3) of the braid group $Br_{\mathfrak{sl}_n}$ on $U_q(\mathfrak{sl}_n)$ to an action on $U_{q,n}$ by algebra automorphisms. Moreover, $\mu$ and $\iota$ are $Br_{\mathfrak{sl}_n}$-equivariant.
This theorem is proved in the rest of Section 3.

Remark 3.2. It is interesting to observe a complement to Theorem 3.1(iv): the quotient algebra $U_q(n)/(w)$ is isomorphic to the (parabolic) subalgebra of $U_q(so_{2n})$ generated by $U_q^+(so_{2n})$ and $K_i, K_i^{-1}, F_i, i = 1, \ldots, n - 1$.

Remark 3.3. Note that the subalgebra $S_q(V \otimes V)$ of $U_q(n)$ is not preserved by the action of $Br_{sl_n}$. For example, $T_3^2(w), T_2^2(z) \notin S_q(V \otimes V)$.

Remark 3.4. The image of $S_q(V \otimes V)$ under $\mu$ is isomorphic to $U_q(\mathfrak{r}_J) \subset U_q(\mathfrak{sp}_{2n})$, as defined in Section 2.7, where $J = \{1, \ldots, n - 1\}$. Furthermore, $i(S_q(V \otimes V))$ is a quantum deformation of the coordinate ring of $M_{\leq 2}$, where $M_{\leq 2}$ is the variety of all matrices with the symmetric part of rank at most 2. Moreover, both homomorphisms are compatible with the cross-product structure, e.g. $i(U_q) = i(S_q(V \otimes V)) \times U_q(sl_n)$.

3.2. Structure of algebra $S_q(V \otimes V)$. Let $\{v_i\}, 1 \leq i \leq n$ be the standard basis of the $n$-dimensional $U_q(sl_n)$ module $V$. Let $X_{ij} = v_i \otimes v_j$ be the standard basis of $V \otimes V$. In particular, we have

$$E_i(v_j) = \delta_{ij-1}v_{j-1}, \quad E_i(X_{j,k}) = \delta_{ij-1}X_{j-1,k} + \delta_{i,k}q^{\delta_{ij}}X_{j+1,k-1} - \delta_{ij}X_{j,k-1}$$

$$F_i(v_j) = \delta_{ij}v_{j+1}, \quad F_i(X_{j,k}) = \delta_{ij}q^{\delta_{ij}}X_{j+1,k} + \delta_{i,k}X_{j,k+1}$$

(3.3)

for all $1 \leq i < n$ and for all $1 \leq j, k \leq n$. Let $T : V \otimes V \to V \otimes V$ be the $\mathbb{C}(q)$-linear map defined by

$$T(X_{ij}) = q^{-\delta_{ij}}X_{ji}, \quad T(X_{ji}) = q^{\delta_{ij}}X_{ij} - (q - q^{-1})X_{ji}$$

for all $1 \leq i \leq j \leq n$. It is well-known that $T$ is an isomorphism of $U_q(sl_n)$-modules, satisfies $(T - q^{-1})(T + q) = 0$ and the braid equation on $V^{\otimes 3}$. Define $\Psi_i : V^{\otimes k} \to V^{\otimes k}$ by $\Psi_i = 1^{\otimes i-1} \otimes T \otimes 1^{\otimes k-i-1}$. Then $\Psi_i$ are isomorphisms of $U_q(gl_n)$-modules. Let

$$\Psi = \Psi_2 \Psi_3 \Psi_2 + (q - q^{-1})(\Psi_1 \Psi_2 \Psi_1 + \Psi_1 \Psi_3 \Psi_2) + (q - q^{-1})^2 \Psi_1 \Psi_2.$$  

It will be convenient for us to regard $\Psi$ as an element of the Hecke algebra $H(S_n)$. Recall that $H(S_n)$ is the quotient of the group algebra over $\mathbb{C}(q)$ of the braid group $Br_{sl_n}$ by the ideal generated by $(T_i - q^{-1})(T_i + q), 1 \leq i \leq n - 1$. In particular, $V^{\otimes n}$ is an $H(S_n)$-module. A well-known result of Jimbo ([12], Proposition 3]) provides a quantum analogue of Schur-Weyl duality, namely the image of $U_q(gl_n)$ in $End V^{\otimes n}$ is the centralizer of the image of $H(S_n)$ and vice versa. It is also well-known that the Hecke algebra $H(S_n)$ is semi-simple.

Proof of Proposition 1.10. Since $H(S_n)$ is semi-simple, to prove part (ii) (respectively, part (iii)) of Proposition 1.10, it is sufficient to show that these identities hold in any simple finite dimensional representation of the Hecke algebra $H(S_6)$ (respectively, $H(S_4)$). For, we use a realization of the multiplicity free direct sum of all simple finite dimensional $H(S_n)$-modules, known as the Gelfand model, constructed in [11, Theorem 1.2.2], which we briefly review for the reader’s convenience.

Let $I_n$ be the set of involutions in the symmetric group $S_n$ and let $I_{n,k} \subset I_n$ be the set of all involutions containing $k$ cycles of length 2 so each $I_{n,k}$ is an orbit for the action of $S_n$ on $I_n$. Given $w \in I_{n,k}$, one defines $\hat{\ell}(w) = \min\{\ell(v) : vwv^{-1} = w\}$.
\[
\prod_{i=1}^k s_{2i-1}. \text{ Let } \mathcal{V}^{(k)}_n = \text{Span}\{C_w : w \in J_{n,k}\} \text{ and set } \mathcal{V}_n = \bigoplus_{0 \leq k \leq n/2} \mathcal{V}^{(k)}_n. \text{ Then }
\]
\[
T_i(C_w) = \begin{cases} 
-qC_w, & s_iw_i = w, \ell(ws_i) < \ell(w) \\
q^{-1}C_w, & s_iw_i = w, \ell(w) < \ell(ws_i) \\
qC_{s_iws_i} - (q - q^{-1})C_w, & s_iw_i \neq w, \ell(w) < \ell(s_iw_i) \\
q^{-1}C_{s_iws_i}, & s_iw_i \neq w, \ell(s_iw_i) < \ell(w)
\end{cases}
\]
defines a representation of the Hecke algebra \(H(S_n)\) on \(\mathcal{V}_n\) which realizes the Gelfand model for \(H(S_n)\). Clearly, \(\mathcal{V}^{(k)}_n\) is an \(H(S_n)\)-submodule of \(\mathcal{V}_n\) and \(\mathcal{V}^{(0)}_n\) is the trivial \(H(S_n)\)-module. A straightforward computation then shows that the matrix of \(\Psi\) on \(\mathcal{V}^{(1)}_4\) with respect to the basis \(C_{(i,j)}, 1 \leq i < j \leq 4\), is
\[
\begin{pmatrix}
0 & q^3h & -q^3h & 0 & 0 & q^4 \\
0 & -q^2 & 0 & 0 & 0 & 0 \\
-q^3h & -q^{-2}h^2 & -h^2 & -q^{-2} & 0 & qh \\
0 & 0 & -q^2 & 0 & 0 & 0 \\
-q^3h & -q^{-2}h^2 & -h^2 & 0 & -q^{-2} & -q^{-1}h \\
q^{-4} & q^{-3}h & q^{-1}h & 0 & 0 & 0
\end{pmatrix}
\]
while \(\Psi|_{\mathcal{V}^{(2)}_4} = \text{id}\). Here we abbreviate \(h = q - q^{-1}\). Part (iii) is now straightforward. Part (iv) is checked similarly and we omit the details.

It remains to prove (iii). Let \(\tau = \tau_{V \otimes V, V' \otimes V}\) be the permutation of factors. Note that by the quantum Schur-Weyl duality, the vector subspace \((V^{\otimes m})^+\) of \(U_q(\mathfrak{sl}_n)\)-highest weight vectors in \(V^{\otimes m}\) is isomorphic to the direct sum of simple \(H(S_m)\)-modules \(S^\lambda\), where \(\lambda\) runs over the set of all partitions of \(m\) with at most \(\dim V\) nonzero parts. In particular, if \(\dim V \geq m\) then \((V^{\otimes m})^+ \cong \mathcal{V}_m \cong \bigoplus_{\lambda} S^\lambda\). To complete the argument, we need the following result, which is an immediate consequence of Schur-Weyl duality.

**Lemma 3.5.** Let \(\Psi' \in H(S_m)\) be such that \(\Psi'\) is specializable at \(q = 1\) on \(\mathcal{V}_m\) with respect to the basis \(C_w, w \in J_m\) and suppose that \(\dim \Psi'(S^\lambda) = \dim \Psi'|_{q=1}(S^\lambda)\) for all partitions \(\lambda\) of \(m\). Then for any \(V\), \(\dim \Psi'(V^{\otimes m}) = \dim \Psi'|_{q=1}(V^{\otimes m})\).

Recall that \(\mathcal{V}^{(0)}_4 = S^{(4)}\) and it is easy to see that \(\mathcal{V}^{(1)}_4 = S^{(2,1^2)} \oplus S^{(3,1)}\) while \(\mathcal{V}^{(2)}_4 = S^{(2,2)} \oplus S^{(1^4)}\). Therefore, \((\Psi - 1)(S^\lambda) = (\tau - 1)(S^\lambda) = 0\) for \(\lambda \in \{(4), (2,2), (1^4)\}\). Finally, one can easily show that \(\dim(\Psi - 1)(S^\lambda) = \dim(\tau - 1)(S^\lambda) = 2\) for \(\lambda \in \{(2,1^2), (3,1)\}\).

We can now prove the first part of Theorem 3.1.

**Proof of part (iii) of Theorem 3.1.** By Proposition 1.10 \(\Psi\) satisfies the braid relation and condition (ii) of Theorem 1.11. Since \(\Psi_i\) specializes to the transposition of factors with respect to the standard basis of \(V^{\otimes 4}\), it follows that \(\Psi\) specializes to the permutation of factors in \((V^{\otimes 2})^{\otimes 2}\). It remains to apply Theorem 1.11. \(\Box\)
Proposition 3.6. The algebra $S_q(V \otimes V)$ is generated by the elements $X_{ij}$, $1 \leq i, j \leq n$, subject to the following relations for all $1 \leq i < j \leq k \leq l \leq n$

\[
X_{ij}X_{kl} = q_{ikj}q_{ijl}X_{kl}X_{ij} + hq^{-d_{ij}}(q^{-d_{ij}}X_{kl}X_{ij} + q_{ikj}q_{ijl}X_{kl}X_{ij})
+ h^2q_{ijl}q_{ljl}X_{kl}X_{ij} + h^2q_{ikj}q_{ijk}X_{kl}X_{ijl} + h^3q^{-d_{il}}q_{ijk}X_{kl}X_{lij}
\]

\[
X_{jk}X_{ik} + hq^{-d_{il}}(q^{-d_{il}}X_{jk}X_{ik} + q_{ijl}q_{ljl}X_{jk}X_{ikl}) + h^2q_{jik}q_{ikj}X_{jk}X_{lil} + h^2q^{-d_{ik}}q_{ijkl}X_{jk}X_{lil}
\]

\[
X_{jl}X_{ij} = q_{ikj}l_{jkl}X_{jl}X_{ij} + hq^{-d_{il}}(q^{-d_{il}}X_{jl}X_{ij} + q_{ikj}l_{jkl}X_{jl}X_{ijl}) + h^2q_{ijl}l_{ljl}X_{jl}X_{ijl} + h^2q^{-d_{il}}q_{ijkl}X_{jl}X_{ljl}
\]

\[
X_{ij}X_{kl} = q_{ikj}l_{jkl}X_{ij}X_{kl} + hq^{-d_{il}}(q^{-d_{il}}X_{ij}X_{kl} + q_{ikj}l_{jkl}X_{ij}X_{kl}) + h^2q_{ijl}l_{ljl}X_{ij}X_{ljl} + h^2q^{-d_{il}}q_{ijkl}X_{ij}X_{ljl}
\]

where $h = q - q^{-1}$, $q_{abc} = q^{\delta_{ab}-\delta_{bc}}$, $q_{abc} = (q_{abc})^{-1}$ and $q_{abc} = q^{\delta_{ab}-\delta_{bc}}$.

Proof. One can show that for all $1 \leq i < j \leq k \leq l \leq n$

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kl}X_{ij} + q^{-d_{ij}}X_{kl}X_{ijl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{ijl}
\]

\[
\Psi(X_{ij}X_{lk}) = q_{ikj}l_{jkl}X_{lk}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{lk}X_{ij} + q^{-d_{il}}X_{lk}X_{ijkl}) + h^2q_{ikj}l_{jkl}X_{lk}X_{ijkl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

Proof. One can show that for all $1 \leq i < j \leq k \leq l \leq n$

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kl}X_{ij} + q^{-d_{ij}}X_{kl}X_{ijl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{ijl}
\]

\[
\Psi(X_{ij}X_{lk}) = q_{ikj}l_{jkl}X_{lk}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{lk}X_{ij} + q^{-d_{il}}X_{lk}X_{ijkl}) + h^2q_{ikj}l_{jkl}X_{lk}X_{ijkl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]

\[
\Psi(X_{ij}X_{kl}) = q_{ikj}l_{jkl}X_{kl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{il}}X_{kl}X_{ij} + q^{-d_{il}}X_{kl}X_{kl}) + h^2q_{ikj}l_{jkl}X_{kl}X_{kl}
\]

\[
\Psi(X_{ij}X_{kjl}) = q_{ikj}l_{jkl}X_{kjl}X_{ij} + hq_{ikj}l_{ijl}(q^{-d_{ij}}X_{kjl}X_{ij} + q^{-d_{ij}}X_{kjl}X_{kjl}) + h^2q_{ikj}l_{jkl}X_{kjl}X_{kjl}
\]
Remark 3.7. It is easy to check that the quotient of

\[ \Psi(X_{ki}X_{ij}) = q_{ikj}^+q_{ij}^+X_{ij}X_{ki} + hq^{\delta_{ij}}q_{ij}^-X_{ki}X_{kj} \]

\[ \Psi(X_{kj}X_{il}) = q_{klj}^+q_{lj}^+X_{ij}X_{il} + hq^{\delta_{ij}}q_{lj}^-X_{ki}X_{il} \]

\[ - h^2(X_{ki}X_{jl} + q_{ji}^-X_{lj}X_{ki}) \]

\[ \Psi(X_{ki}X_{ij}) = q_{ikj}^+q_{ij}^+X_{ij}X_{ki} + hq^{\delta_{ij}}q_{ij}^-X_{kj}X_{il} \]

\[ \Psi(X_{kj}X_{li}) = q_{klj}^+q_{lj}^+X_{ij}X_{li} + hq^{\delta_{ij}}q_{ij}^-X_{ki}X_{kl} \]

\[ - h^2(X_{ki}X_{jl} + q_{ji}^-X_{lj}X_{ki}) \]

\[ - h^2(q_{ikl}^+q_{kl}^+X_{li}X_{kj} + q_{klj}^+X_{ij}X_{ki}) \]

Since the quotient \( S = (V \otimes V)/(\Psi - 1)(V \otimes V) \) is a flat deformation of \( S^2(V \otimes V) \),
the canonical images of the \( X_{ab}X_{cd} \) with \( (a,b) \not\approx (c,d) \) (where the order is defined by \( (a,b) \not\approx (c,d) \) if \( \min(c,d) < \min(a,b) \) or \( \min(a,b) = \min(c,d) \) and \( \max(c,d) < \max(a,b) \), while \( (i,j) \not\approx (j,i) \), for all \( i \leq j \)) form a basis of \( S \). Using this basis we obtain the formulae in the Proposition from the above formulae for \( \Psi \).

Remark 3.7. It is easy to check that the quotient of \( S_q(V \otimes V) \) by the ideal generated by the elements \( X_{ij}qX_{ji} \), \( 1 \leq i < j \leq n \) (respectively, by the elements \( X_{ij}q^{-1}X_{ji} \), \( 1 \leq i < j \leq n \) and \( X_{ii} \), \( 1 \leq i \leq n \)) is isomorphic to the algebra \( S_q(S^2V) \) (respectively, \( S_q(A^2V) \)); cf. [14, Theorem 0.2] and [22, (1.1)], respectively, and also [19, 9, 26]).

We can now prove Corollary 3.15.

\[ \text{Proof.} \] The algebra \( S_q(V \otimes V) \) is clearly optimal specializable with respect to its PBW basis on the \( X_{ij} \), \( 1 \leq i, j \leq n \) with the total order defined as in Proposition 3.6. It remains to apply Propositions 2.18(ii) and 3.6. \qed

3.3. Cross product structure of \( U_{q,n}^+ \) and \( U_{q,n}^* \). In this section we will use the usual numbering of nodes in the Dynkin diagram of \( \mathfrak{so}_{2n+2} \), that is, the simple root \( \alpha_{n-1} \) corresponds to the triple node. Retain the notations of Section 3.2.

Proposition 3.8 (Theorem 3.1(ii)).

(i) The natural homomorphism \( U_{q,n}^+ \rightarrow U_{q,n}^* \) is injective and as vector spaces \( U_{q,n}^* \cong U_{q,n}^+ \otimes U_{q,n}^{\leq 0} \), where \( U_{q,n}^{\leq 0} \) is defined as in Section 2.2.
(ii) The assignment \( w \mapsto w' := X_{n,n}, \ z \mapsto z' := q(X_{n-1,n} - qX_{n,n-1}) \) defines isomorphisms of algebras \( \psi : U_{q,n} \to S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n) \) and \( \psi_+ : U_{q,n}^+ \to S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n) \).

Proof. First we prove that the elements \( w', z' \) satisfy the relations (3.1b–3.1c). It follows from (3.3) that \( w' \) (respectively, \( z' \)) is a lowest weight vector of the \( U_q(\mathfrak{sl}_n) \)-submodule of \( V \otimes V \) isomorphic to \( V_{2\varpi_i} \) (respectively, \( V_{\varpi_i} \)), where \( \varpi_i \) is the \( i \)th fundamental weight of \( \mathfrak{sl}_n \). In particular, we have

\[
E_i^{2\delta_i,n-1+1}(w') = 0 = E_i^{\delta_i,n-2+1}(z'), \quad F_i(w') = F_i(z') = 0. \tag{3.4}
\]

Using Lemma 2.24(ii) we immediately conclude that \( w' \) and \( z' \) satisfy (3.1a), (3.1c) and the first two relations in (3.1b). To prove the last relation in (3.1b) note that

\[
[w', z'] = [X_{n,n}, qX_{n-1,n} - q^2X_{n,n-1}] = 0
\]

since \( X_{n,n-1}X_{n,n} = q^2X_{n,n}X_{n,n-1} \) and \( [X_{n-1,n}, X_{n,n}] = q^2(q^{-1})X_{n,n}X_{n,n-1} \) by Proposition 3.6. To prove the first relation in (3.1d), note that

\[
[w', w'; E_{n-1}]_{q^2} = [[E_{n-1}, w'], w']_{q^2} = [X_{n-1,n} + q^{-1}X_{n,n-1}, X_{n,n}]_{q^2} = (1 - q^2)(X_{n,n}X_{n-1,n} - qX_{n,n}X_{n,n-1}) = (q^{-1} - q)w'z'.
\]

The remaining identities are checked similarly. Using Lemma 2.24(ii), we rewrite them in the form \( \sum Y_i m_i \), where \( m_i \in \{1, E_{n-1}, E_{n-2}, E_{n-1}E_{n-2}, E_{n-2}E_{n-1}\} \) and in particular are linearly independent and \( Y_i \in S_q(V \otimes V) \). Then we check that \( Y_i = 0 \) which can be done either using the presentation from Proposition 3.6 or by observing that \( \text{Im}(\Psi - 1) = \ker((\Psi + q^2)(\Psi + q^{-2})) \). This is a rather tedious, albeit simple, computation, which was performed on a computer.

Thus, we proved that \( \psi : U_{q,n} \to S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n) \) is a surjective homomorphism of algebras. The same argument shows that we have a surjective homomorphism of algebras \( \psi_+ : U_{q,n}^+ \to S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n) \).

To complete the proof of the proposition, we prove first that \( \psi_+ \) is an isomorphism. Let \( \mathcal{F} \) be the free algebra on the \( E_i, 1 \leq i \leq n-1 \), \( w \) and \( z \) and define a grading on \( \mathcal{F} \) by \( \deg E_i = \deg w = 1, \deg z = 2 \). Let \( \mathcal{I}_q \) be the kernel of the structural homomorphism \( \mathcal{F} \to U_{q,n}^+ \). It is easy to see that \( \mathcal{I}_q \) is homogeneous with respect to this grading. Regard \( S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n) \) as a graded algebra with the grading induced by the homomorphism \( \psi_+ \). By Lemma 2.24 we have \( \dim(U_{q,n}^+)_k \leq \dim(\mathcal{F}/\mathcal{I}_q)_k \) for all \( k \) where \( \mathcal{I}_q \) is the specialization of \( \mathcal{I}_q \) at \( q = 1 \). On the other hand, it is easy to see that \( \mathcal{F}/\mathcal{I}_q \) is isomorphic to \( U(n) \) where \( n = (V \otimes V) \rtimes (\mathfrak{sl}_n)_+ \), which we can regard as a graded Lie algebra with the grading compatible with that on \( U_{q,n}^+ \). Since both \( U(n) \) and \( S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n) \) are PBW algebras on the set of the same cardinality, it follows that \( \dim(U(n))_k = \dim(S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n))_k \) for all \( k \). This and the obvious inequality \( \dim(U_{q,n}^+_k) \geq \dim(S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n))_k \) proves the second assertion in part (iii).

Let \( U_{q,n}' \) be the subalgebra of \( U_{q,n} \) generated by the \( E_i, i \in I \) and by \( w, z \). Clearly, we have a canonical surjective homomorphism \( \pi : U_{q,n}^+ \to U_{q,n}' \) and \( \psi_+ = \psi \circ \pi \). Since \( \psi_+ \) is an isomorphism and both \( \psi \) and \( \pi \) are surjective, it follows that \( \pi \) is an isomorphism and proves the first assertion in part (iii). To establish the remaining assertions, we need the following easy Lemma.
Proof of Theorem 3.1 follows from (2.22) and (3.5), (3.6). Furthermore, observe that Lemma 3.9.



3.4. Structural homomorphisms. In this section we prove parts (iv) and (v) of Theorem 3.1. We use the numeration of nodes in the Dynkin diagram of \( \mathfrak{so}_{2n+2} \) and \( \mathfrak{sp}_{2n} \) introduced in Section 3.1. Note first that part (iv) of Theorem 3.1 is trivial since modulo the ideal generated by \( z \) its defining relations are precisely the defining relations of \( U_q(\mathfrak{sp}_{2n}) \) where \( w \) corresponds to \( E_0 \).

To prove part (v) of Theorem 3.1 let \( W = E_0 E_{-1} \) and

\[
Z = (q^2 - 1)^{-1}([E_0, [E_1, E_{-1}]]_q + [E_{-1}, [E_1, E_0]]_q)_q^{\circ}
\]

\[
= (1 - q^{-2})^{-1}((\text{ad}^* E_0)(\text{ad}^* E_1)(E_{-1}) + (\text{ad}^* E_{-1})(\text{ad}^* E_1)(E_0))
\]

\[
= (1 - q^{-2})^{-1}((\text{ad}^* E_0)(\text{ad} E_1)(E_1) + (\text{ad}^* E_{-1})(\text{ad} E_0)(E_1))
\]

be the images of \( w \) and \( z \) in \( U_q(\mathfrak{so}_{2n+2}) \). Clearly

\[
i_r(W) = r_i(W) = 0, \quad i > 0, \quad 0r(W) = E_{-1}, \quad -1r(W) = E_0.
\]

Using Lemma 2.27 we obtain

\[
i_r(Z) = 0, \quad i > 0, \quad 0r(Z) = q(\text{ad}^* E_1)(E_{-1}), \quad -1r(Z) = q(\text{ad}^* E_1)(E_0).
\]

It is easy to check that \( Z^* = Z \), hence \( r_i(Z) = 0 \) for all \( i > 0 \). Finally, we have

\[
Z = q[W, E_1]q^{-2} - q^{-1}[2q]_q^{-1}(\text{ad} E_0)(\text{ad} E_{-1})(E_1)
\]

\[
= q[E_1, W]q^{-2} - q^{-1}[2q]_q^{-1}(\text{ad}^* E_0)(\text{ad}^* E_{-1})(E_1).
\]

Proof of Theorem 3.1(v). We need to show that the elements \( W \) and \( Z \) satisfy the relations (3.1a)-(3.1c). The last two identities in (3.1a) are trivial, while the first follows from (2.22) and (3.5), (3.6). Furthermore, observe that

\[
[E_i, W] = (\text{ad} E_i)(W), \quad i > 1, \quad [E_1, [E_1, [E_1, W]]_q^{2}]_q^{2} = (\text{ad} E_1)^3(W),
\]

while

\[
[E_i, Z] = (\text{ad} E_i)(Z), \quad i > 0, i \neq 2, \quad [E_2, [E_2, Z]]_q^{2} = (\text{ad} E_2)^2(Z).
\]

The first two identities in (3.1b) are now immediate from (2.10). The first identity in (3.1c) follows from (2.20) since \( (\text{ad} E_1)^2(E_i) = 0, i \in \{-1, 0\} \) by quantum Serre’s relations. The second is also a consequence of quantum Serre relations since \( \text{ad} E_2 \) commutes with \( \text{ad}^* E_i \), \( \text{ad} E_j \), \( i, j \in \{0, -1\} \). To prove the last relation in (3.1b), note that since

\[
(\text{ad}^* E_i)^2(\text{ad}^* E_j)(E_j) = 0, \quad (\text{ad} E_i)(\text{ad}^* E_j)(\text{ad} E_i)(E_1) = 0, \quad \{i, j\} = \{0, 1\}
\]
by quantum Serre relations, it follows that

\begin{align*}
E_i(\text{ad}^* E_i)(\text{ad}^* E_1)(E_j) &= q(\text{ad}^* E_i)(\text{ad}^* E_j)(E_i), \\
E_i(\text{ad}^* E_j)(\text{ad}^* E_1)(E_i) &= q^{-1}(\text{ad}^* E_j)(\text{ad}^* E_1)(E_i),
\end{align*}

hence

\[ WZ = ZW. \]

To prove the first identity in \((3.1d)\), notice that since \([Z, W] = 0\) we obtain from \((3.7)\)

\[ [W, [W, E_1]_q]_q^2 - (q^{-1} - q)WZ = [W, [W, E_1]_q]_q^2 - q^{-1}Z]_q^2 = [2]_q(q - q^{-1})^{-1} [W, (\text{ad} E_0)(\text{ad} E_1)]_q. \]

Since for all \(x \in U_q^+(\mathfrak{so}_{2n+2})\)

\[ [W, x]_q^2 = E_0[E_{-1}, x]_q + q[E_0, x]_q E_{-1} \]

and \([E_i, x]_q = (\text{ad} E_i)(x)\) for \(x = (\text{ad} E_0)(\text{ad} E_{-1})(E_1)\) and \(i \in \{0, -1\}\), it follows from the quantum Serre relations that

\[ [W, (\text{ad} E_0)(\text{ad} E_{-1})(E_1)]_q = 0, \]

which together with \((3.7)\) implies the first relation in \((3.1d)\). To prove the remaining identities, we use Lemma 2.26(v). Note that

\[ i_r([a, b]_v) = [r(a), b]_v q^{-(\gamma, \alpha_i)} [a, i_r(b)]_v q^{(\gamma, \alpha_i)}, \]

for all \(a \in U_q^+(\mathfrak{g}), b \in U_q^+(\mathfrak{g}), \gamma, \delta \in Q\) and \(v \in \mathbb{C}(q)^*\).

To prove \((3.1d)\), note that

\[ [E_2, [E_1, W]_q]_q = q^3(\text{ad}^* E_2)(\text{ad}^* E_1)(W), \]

while

\[ [E_1, [E_2, Z]_q]_q = q^2(\text{ad}^* E_1)(\text{ad}^* E_2)(Z). \]

Thus, we want to show that \(i_r(x) = 0\) for all \(i \in I\) where

\[ x = q[Z, (\text{ad}^* E_2)(\text{ad}^* E_1)]_q - [W, (\text{ad}^* E_1)(\text{ad}^* E_2)]_q. \]

This is trivial if \(i > 2\). For \(i = 2\) we obtain

\[ 2r(x) = (q - q^{-1}) q^2 (Z, (\text{ad}^* E_1)(W)]_q - [W, (\text{ad}^* E_1)(Z)]_q, \]

\[ = (q - q^{-1}) q^2 (Z, (\text{ad}^* E_1)(W]) + (\text{ad}^* E_1)(Z)W - (\text{ad}^* E_1)(W)Z + q^{-2}W(\text{ad}^* E_1)(Z)) \]

\[ = (q - q^{-1}) q^2 ((\text{ad}^* E_1)(ZW) - (\text{ad}^* E_1)(WZ)) = (q - q^{-1})(\text{ad}^* E_1)[Z, W] = 0, \]

where we used already established \((3.1b)\) and \((2.20)\). Similarly

\[ 1r(x) = (q - q^{-1}) q[Z, (\text{ad}^* E_2)]_q - q^2 [W, (\text{ad}^* E_2)]_q \]

\[ = (q - q^{-1}) q^2 (\text{ad}^* E_2)(W, Z)]_q = 0. \]

The computation of \(i_r(x)\) for \(i \in \{-1, 0\}\) and the ones for the last identity, are rather tedious and where performed on a computer. \(\square\)
Remark 3.10. It can be shown that the kernel of the homomorphism \( \hat{i} : U_{q,n}^+ \rightarrow U_q(\mathfrak{so}_{2n+2}) \) is generated by an element of degree 3 in \( S_q(V \otimes V) \) which is a lowest weight vector of a simple \( U_q(\mathfrak{sl}_n) \)-submodule of \( (V \otimes V)^{\otimes 3} \) isomorphic to \( V_{2wz} \). On the other hand, the image of \( \hat{i} \) equals to the subalgebra of \( \sigma \)-invariant elements in \( U_q(\mathfrak{so}_{2n+2}) \) graded by \( Q^* \).

3.5. Braid group action on \( U_{q,n} \).

Proof of part (VI) of Theorem 3.1. Let \( \hat{U}_{q,n} \) be the algebra generated by \( U_q(\mathfrak{sl}_n) \) and \( w, z \) subjects to the relations (3.1a), (3.1b), (3.1c), except the commutativity relation \([w, z] = 0\). Clearly that \( \hat{U}_{q,n} \) is a quotient of \( U_{q,n} \). First we prove the following

Proposition 3.11. The formulae (3.2) extend the action of \( Br_{sl_n} \) on \( U_q(\mathfrak{sl}_n) \) to an action on \( \hat{U}_{q,n} \) by algebra automorphisms.

Proof. We note the following useful Lemma

Lemma 3.12. In \( \hat{U}_{q,n} \) we have

\[
[F_i, T_1(w)] = -\delta_{i,1}[w, E_1]_{q^{-2}}K_1, \quad [F_i, T_2(z)] = -\delta_{i,2}q^{-1}zK_2
\]

\[
[T_1(E_1), T_1(w)]_{q^{-2}} = [w, E_1]_{q^{-2}}, \quad [T_2(E_2), T_2(z)]_{q^{-1}} = z
\]

\[
[T_2(w), w]_{q^{-2}} = [[E_1, w]_{q^{-2}}, E_2]_{q^{-1}}, \quad [T_1(E_2), z]_{q^{-1}} = [[E_2, z]_{q^{-1}}, E_1]_{q^{-1}}.
\]

Clearly, \([T_i(F_j), T_i(w)] = 0\) (respectively, \([T_i(F_j), T_i(z)] = 0\)) for all \( i \) and for all \( i \neq 1 \) (respectively, for all \( i \neq 2 \)). Since

\[
[T_1(F_1), T_1(w)] = -[2]_q^{-1}[E_1, [E_1, W]_{q^{-2}}]_{q^2}K_1 = 0
\]

\[
[T_2(F_2), T_2(z)] = [E_2, [E_2, z]_{q^{-1}}]_{q}K_2 = 0.
\]

we conclude that \([T_1(F_i), T_i(w)] = 0\) unless \( i = 2 \), while by Lemma 3.12

\[
[T_1(F_2), T_1(w)] = [[F_1, F_2]_{q}, T_1(w)] = [[F_1, T_1(w)], F_2]_{q} = -[w, E_1]_{q^{-2}}[K_1, F_2]_{q} = 0
\]

\[
[T_2(F_j), T_2(z)] = [[F_2, F_j]_{q}, T_2(z)] = [[F_2, T_2(z)], F_j]_{q} = -q^{-1}z[K_2, F_j]_{q} = 0, \quad j = 1, 3
\]

The remaining identity in (3.1a) is clearly preserved. Similarly, for all \( i \) and for all \( j \neq 1 \) (respectively \( j \neq 2 \)) we obtain \([T_i(F_j), T_i(w)] = 0\) (respectively \([T_i(F_j), T_i(z)] = 0\)). The remaining identities follow from Lemma 3.12 and direct computations. For example, for \( i = 1, 3 \)

\[
[T_2(E_i), T_2(z)] = q^{-2}E_2E_iE_2z - q^{-2}E_2zE_2E_i - q^{-1}E_iE_2E_2z
\]

\[
+ E_iE_2zE_2 + q^{-1}zE_2E_2E_i - zE_2E_iE_2
\]

\[
= [z, (\text{ad } E_2)^{(2)}(E_i)]_{q^{-2}} = 0,
\]

where we used (3.1c) and quantum Serre relations. It is not hard to check, using the above Lemma, that the maps \( T_i \) are invertible with their inverses given on \( w \) and \( z \) by

\[
T_i^{-1}(w) = [2]_{q^{-1}}[E_1, [E_1, w]_{q^{-2}}], \quad T_i^{-1}(z) = [E_2, z]_{q^{-1}}
\]

while \( T_i^{-1}(w) = w \) if \( i \neq 1 \) and \( T_i^{-1}(z) = z \) if \( i \neq 2 \). Thus, we conclude that the \( T_i \) are automorphisms of \( \hat{U}_{q,n} \). Finally, the only braid relations that need to be checked are \( T_1T_2T_1(w) = T_2T_1T_2(w) = T_2T_1(w) \) and \( T_2T_1T_2(z) = T_1T_2T_1(z) = T_1T_2(z) \), where \( i = 1, 3 \). This is done by a direct computation. \( \square \)
To complete the proof of part (vi) of Theorem 3.13 it suffices to show that the kernel of the canonical map \( U_{q,n} \to U_{q,n} \) is preserved by the \( T_i, 1 \leq i \leq n - 1 \). For example, consider (3.1d). Note that [\( w, [w, E_1]_{q-2} \]_q^2 = [[E_1, w]_{q-2}, w]_q^2. Using Lemma 3.12 we obtain

\[
[[T_i(E_1), T_i(w)]_{q-2}, T_i(w)]_q^2 = \begin{cases} 
[[E_1, w]_{q-2}, w]_q^2, & i > 2 \\
[[w, [w, E_1]_{q-2} q^2], E_2]_{q-1}, & i = 2 \\
[2]_{q-1}^{-1} [[w, [w, E_1]_{q-2} q^2], E_1]_{q-2}, E_1], & i = 1,
\end{cases}
\]

while \( T_i(w)T_i(z) = wz, i > 2 \),

\[
T_1(w)T_1(z) = [2]_{q-1}^{-1} [[wz, E_1]_{q-2}, E_1]
\]

and

\[
T_2(w)T_2(z) = [wz, E_2]_{q-1}.
\]

Thus, the first relation in (3.1d) is preserved. The computations for the remaining relations are rather tedious and where performed on a computer. The relations can be checked in many different ways; perhaps, the simplest is to use the isomorphism \( U_{q,n} \cong S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n) \), which allows us to write any element of \( U_{q,n} \) as \( \sum_i Y_i m_i \), where \( Y_i \in S_q(V \otimes V) \) and the \( m_i \) are linearly independent elements of \( U_q(\mathfrak{sl}_n) \). Writing a relation in this form, we then check that \( (\Psi + q^2)(\Psi + q^{-2})(Y_i) = 0 \) hence \( Y_i \in \operatorname{Im}(\Psi - 1) \).

3.6. Liftable quantum foldings and \( U_{q,n}^+ \) as a uberalgebra. In this section we use the standard numbering of the nodes of all Dynkin diagrams.

**Theorem 3.13.** In the notation of Theorem 3.1, \( i_1 \) for any \( i \in R(w_0) \) is a tame liftable folding with \( U(i_1) = U_{q,n}^+ \) and \( \mu_{i_1} = \mu \). In particular, \( \tilde{i}_1 \) splits \( \mu \) and we have a commutative diagram

\[
\begin{array}{ccc}
U_{q,n}^+ & \xrightarrow{i} & U_q^+(\mathfrak{so}_{2n+2}) \\
\downarrow{\tilde{i}_1} & & \downarrow{\iota_1} \\
U_q^+(\mathfrak{sp}_{2n}) & & \\
\end{array}
\]

where all maps commute with the right multiplication with \( U_q(\mathfrak{sl}_n) \).

**Proof.** Let \( w'_0 \) be the longest element in \( W(\mathfrak{sl}_n) = W((\mathfrak{sp}_{2n}), J) \), where \( J = \{1, \ldots, n-1\} \) and let \( i = (n-1, n-2, n-1, \ldots, 1, \ldots, n-1) \in R(w'_0) \). Set

\[
i_r = (n, n-1, n-2, n-1, n, \ldots, r, \ldots, n), \quad 1 \leq r \leq n.
\]

First, we prove the Theorem for \( i = i_0 \) where \( i_0 \) is the concatenation \( i_1 i'_1 \).

Given \( j = (r_1, \ldots, r_k) \in (I/\sigma)^k \), write \( w_j = s_{r_1} \cdots s_{r_k} \) and \( T_j = T_{w_j} \) (respectively, \( T_j = T_{w_j} \)). It is easy to check that

\[
w_i(\alpha_s) = \alpha_{r+n-s-1}, \quad r < s \leq n - 1, 1 \leq r \leq n.
\]

In particular, \( T_{i_1}(E_i) = E_{n-i}, 1 \leq i \leq n - 1 \) by Lemma 2.8 hence \( T_{i_1} \) acts as the diagram automorphism \( \tau \) of \( U_q^+(\mathfrak{sl}_n) = U_q^+(\mathfrak{sp}_{2n}) \). Define the elements \( y_{i,j}^+ \in \mathbb{C}(\mathfrak{sp}_{2n}) \) as
Proposition 3.14. The elements \( x_{ij}^+ \in U_q(\mathfrak{so}_{2n+2}) \) and \( y_{ij}^+ \in U_q(\mathfrak{sp}_{2n}) \) defined in (3.10) are given by the following formulae

\[
\begin{align*}
\gamma \beta_{ij}^{-1}(q-q^{-1})^{1+\delta_{ij}} & \hat{T}_{i+1} \cdots \hat{T}_{i+n-1} E_{i+n-j}^{\delta_{ij}} E_{i+n-j}^{\delta_{ij}} E_{i+n-j+1}(E_{i+n-j+1}), & 1 \leq i < j \leq n \\
\beta_{ij}^{-1}(q-q^{-1})^{1+\delta_{ij}} & \hat{T}_{i+1} \cdots \hat{T}_{i+n-1} E_{i+n-j}^{\delta_{ij}} E_{i+n-j+1}(E_{i+n-j+1}), & 1 \leq i \leq n \\
\gamma \beta_{ij}^{-1}(q-q^{-1})^{1+\delta_{ij}} & \hat{T}_{i+1} \cdots \hat{T}_{i+n-1} E_{i+n-j}^{\delta_{ij}} E_{i+n-j+1}(E_{i+n-j+1}), & 1 \leq i \leq j \leq n
\end{align*}
\]

where \( \hat{T} \) is the \( \gamma \beta_{ij}^{-1}(q-q^{-1})^{1+\delta_{ij}} \) term in (3.9), that \( \beta_{ij} = \alpha_i + \cdots + \alpha_{i-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n + \alpha_{n+1} \) and hence \( \gamma \beta_{ij}^{-1}(q-q^{-1})^{1+\delta_{ij}} = \hat{T}_{i+1} \cdots \hat{T}_{i+n-1} E_{i+n-j}^{\delta_{ij}} E_{i+n-j+1}(E_{i+n-j+1}) \).

Since

\[
\hat{w}_i = s_{i+1} s_i \cdots s_{i+n-1} w_i, \quad 1 \leq i \leq n - 1
\]

and \( \ell(w_i) = \ell(w_{i+1}) + n - i + 2 \), we have for all \( 1 \leq i \leq n \)

\[
\hat{T}_{i+1} \cdots \hat{T}_{i+n-1} E_{i+n-j}^{\delta_{ij}} E_{i+n-j+1}(E_{i+n-j+1}) = T_{n+1} \cdots T_{i+1} T_{i+2}(E_{i}) = T_{n+1} \cdots T_{i+1}(E_{i}).
\]

Then it is easy to see, using (2.15), (2.16) and the obvious observation that \( \text{ad}^* E_r \) and \( \text{ad}^* E_{s} \) commute if \( ar_s = 0 \), that

\[
\hat{c}_{i+1}^{+} \cdots E_{n+1}(E_{i}) = \hat{E}_{i+1}^{+} \cdots E_{i+1}(E_{i}) = \hat{E}_{i} \cdots \hat{E}_{n+1}(E_{i+1}).
\]
To establish (3.12) for 1 ≤ i < j < n, note that by (3.9) and Lemma 2.8 we have
\( \tilde{T}_{i+1}(E_{i+n-j}) = E_j, i < j < n. \) Therefore,
\[
\tilde{c}^+_x = \tilde{T}_{i+1} \cdots T_{i+n-j}(E_{i+n-j}) = \tilde{T}_{i+1} \cdots \tilde{E}_{i+n-j}(E_{i+n-j})
\]
where we used [2, Lemma 3.5], (3.16) and (2.17).

Finally, we prove by a downward induction on \( i \) that
\[
\tilde{T}_{i+1} \cdots T_{n-1}(E_n) = T_{\sigma^{n-i}(n)} T_{n-1} \cdots T_{i+1}(E_i).
\]
(3.17)

If \( i = n - 1 \), it follows from Lemma 2.8 that \( T_nT_{n+1}T_{n-1}(E_n) = T_{n+1}(E_{n-1}) \) so the induction begins. For the inductive step, suppose that \( n - i \) is odd, the case of \( n - i \) even being similar. Using (3.15), we can write
\[
\tilde{w}_{n+1}s \cdot s \cdots s = s_{n+1}s \cdots s \cdot s \tilde{w}_{n+1}s \cdots s
\]
and since both expressions are reduced the induction hypothesis yields
\[
\tilde{T}_{i+1} \cdots T_{n-1}(E_n) = T_{n+1} \cdots T_i T_{n-1} \cdots T_{i+1}(E_i).
\]

The inductive step now follows from the braid relations and Lemma 2.8 Using [2, Lemma 3.5] we obtain from (3.17) that
\[
\tilde{c}^+_x = (\tilde{E}_{n+1} \cdots \tilde{E}_{i+1}(E_i)) \tilde{E}_{n-1} \cdots \tilde{E}_{i+1}(E_i)
\]
and similarly \( \tilde{E}_{i+1} \cdots \tilde{E}_{n-1}(E_r) = 0, r \in \{n, n+1\} \), (3.13) follows immediately from (2.21).

We can now complete the proof of the Theorem. First, observe that (3.3) implies that \( X_{ii} = E_{i}^{(2)} \cdots E_{n-1}(X_{nn}) \), 1 ≤ i ≤ n − 1 while
\[
X_{ji} = E_j \cdots E_{n-2}(X_{n,n-1}), \quad 1 ≤ i < j ≤ n.
\]

Since \( X_{n,n-1} = [2]^{-1}(E_{n-1}(w) - q^{-1}z) \), we obtain
\[
\tilde{c}_i(y_{ij}^+) = \frac{E_j \cdots E_{n-2}(E_i)}{(q - q^{-1})^{2n-i-j-1}} \left( \frac{E_{n-1}(w) - q^{-1}z}{q^2 - q^{-2}} \right), \quad 1 ≤ i < j ≤ n
\]
\[
\tilde{c}_i(y_{ii}^+) = (c_i^+)^{-1} E_{i}^{(2)} \cdots E_{n-1}(w), \quad 1 ≤ i ≤ n.
\]

Note that for all \( u \in S_q(V \otimes V), 1 ≤ i ≤ n - 1 \) we have \( \mu(E_i(u)) = \tilde{E}_i(\mu(u)) \) and similarly \( i(E_i(u)) = \tilde{E}_i(u) \). This, together with (3.7), Proposition 3.14 and the multiplicativity of \( \tilde{c}_i \) and \( c_i \) immediately implies the assertion for \( i = i_0 \).

To complete the proof, it remains to apply Lemma 1.7 and the argument from the proof of Theorem 1.20.

This completes the proof of Theorems 1.8 and 1.12.
4. Diagonal foldings

4.1. **Folding** \((\mathfrak{sl}_3^\times, \mathfrak{sl}_3)\). Consider the algebra \(\mathfrak{s}_n := \mathfrak{sl}_3^\times\) with the diagram automorphism \(\sigma\) which is a cyclic permutation of the components.

Let \(\mathcal{A}_{q,3}^{(n)}\) be the associative \(\mathbb{C}(q)\)-algebra generated by \(u_1, u_2\) and \(z_k, 1 \leq k \leq n - 1\) subjects to relations given in Theorem 1.21(i).

**Theorem 4.1.** (i) The algebra \(\mathcal{A}_{q,3}^{(n)}\) is PBW on the totally ordered set

\[\{u_2, u_{21}, u_1\} \cup \{z_k : 1 \leq k \leq n - 1\},\]

where

\[u_{21} = u_1 u_2 - q^{-n} u_2 u_1 - \sum_{k=1}^{n-1} \frac{q - q^{-1}}{q^k - q^{-k}} z_k.\]

(ii) The assignment \(u_i \mapsto E_i, z_k \mapsto 0\) defines a surjective algebra homomorphism \(\mu: \mathcal{A}_{q,3}^{(n)} \to U_q(\mathfrak{s}_n^{\sigma^\vee}).\)

**Proof.** Since \(u_{21} = \left[u_1, u_2\right]_q - \sum_{r=1}^{n-1} [r]_q^{-1} z_r\), we have

\[\left[u_1, u_{21}\right]_q^n = \left[u_1, \left[u_1, u_2\right]_q^{-n}\right]_q - \sum_{r=1}^{n-1} \frac{u_1}{[r]_q} \left[u_1, z_r\right]_q = \sum_{r=1}^{n-1} \left(q^r (q^{-1} - q) + \frac{q^{2r} - 1}{[r]_q}ight) u_1 z_r = 0.\]

Similarly, we can write

\[u_{21} = -q^{-n} \left[u_2, u_1\right]_q^n - \sum_{r=1}^{n-1} z_{n-r} = -q^{-n} \left[u_2, u_1\right]_q^n - \sum_{r=1}^{n-1} [n-r]_q^{-1} z_{n-r}\]

hence

\[\left[u_2, u_{21}\right]_q^n = -q^{-n} \left[u_2, \left[u_2, u_1\right]_q^n\right]_q - \sum_{r=1}^{n-1} [n-r]_q^{-1} \left[u_2, z_{n-r}\right]_q q^{-n}\]

\[= -q^{-n} (q^{-1} - q) \sum_{r=1}^{n-1} q^r u_2 z_{n-r} - \sum_{r=1}^{n-1} \frac{1 - q^{2(r-n)}}{[n-r]_q} u_2 z_{n-r}\]

\[= (q - q^{-1}) \sum_{r=1}^{n-1} \left(q^r - \frac{q^{2(r-n)} - 1}{q^{n-r} - q^{-n}}\right) u_2 z_{n-r} = 0.\]

Since clearly \(u_{21}\) commutes with the \(z_r\), we obtain the PBW relations from Theorem 1.21(iii). The above computations also show that PBW relations imply Serre-like relations. To prove that \(\mathcal{A}_{q,3}^{(n)}\) is PBW, we use Diamond Lemma (Proposition 2.15). It is easy to see that the only situation which needs to be checked is the monomial \(u_1 u_{21} u_2\). We have

\[(u_1 u_{21}) u_2 = q^n u_{21} u_1 u_2 = q^n (u_2 u_{21} u_1 + u_{21}^2 + \sum_{k=1}^{n-1} [k]_q^{-1} u_2 z_k) = u_1 (u_{21} u_2).\]

The second assertion is obvious. \(\square\)
We now proceed to prove that $A_{q,3}^{(n)}$ is the desired enhanced uber algebra for this folding.

Given $x_\alpha \in U_q(\mathfrak{sl}_3)$ we denote its copy in the $i$th component of $U_q(\mathfrak{sl}_3)^{\otimes n}$ by $x_{\alpha,i}$. Let $i_1 = (121)$ and $i_2 = (212)$. Define the elements $y_i \in U_q(\mathfrak{s}_q^n)$ and $Y_{i,n} \in U_q(\mathfrak{s}_n)$, $i \in \{1, 2, 12, 21\}$ by

$$X_{i_1} = \{y_1, y_2, y_3\}, \quad \hat{X}_{i_1} = \{Y_{1,n}, Y_{21,n}, Y_{2,n}\}$$

as ordered sets, in the notation of Section 2.4 and similarly for $i_2$. It is immediate that $y_1 = E_1$, $y_2 = E_2$ and

$$y_{12} = (q^n - q^{-n})^{-1}(E_1E_2 - q^{-n}E_2E_1), \quad y_{21} = (q^n - q^{-n})^{-1}(E_2E_1 - q^nE_1E_2)$$

while $Y_{1,n} = \prod_{i=1}^n E_{1,i}$, $Y_{2,n} = \prod_{i=1}^n E_{2,i}$ and

$$Y_{21,n} = \prod_{i=1}^n E_{21,i}, \quad Y_{12,n} = \prod_{i=1}^n E_{12,i},$$

where

$$E_{12,i} = \frac{E_{2,i}E_{1,i} - q^{-1}E_{1,i}E_{2,i}}{q - q^{-1}}, \quad E_{21,i} = \frac{E_{1,i}E_{2,i} - q^{-1}E_{2,i}E_{1,i}}{q - q^{-1}}.$$

In particular, $t_{i_1}$ (respectively, $t_{i_2}$) is given by

$$t_{i_1}(y_1^a y_2^b y_3^c) = Y_{1,n}^a Y_{21,n}^b Y_{2,n}^c, \quad t_{i_1}(y_2^a y_{12}^b y_3^c) = Y_{2,n}^a Y_{12,n}^b Y_{1,n}^c.$$

We will need some identities for the elements $E_{\alpha,i}$. Clearly

$$qE_{21,i} + E_{12,i} = E_{1,i}E_{2,i}, \quad qE_{12,i} + E_{21,i} = E_{2,i}E_{1,i} \quad (4.1)$$

It follows from quantum Serre relations that

$$E_{i,r}E_{ij,s} = q^{-b_{r,s}}E_{ij,s}E_{i,r}, \quad E_{j,r}E_{ij,s} = q^{b_{r,s}}E_{ij,s}E_{j,r}, \quad i \neq j \in \{1, 2\}. \quad (4.2)$$

In particular, this implies that

$$E_{12,i}E_{21,i} = E_{21,i}E_{12,i}.$$

Let $Z_{0,1} = E_{21,1}$, $Z_{1,1} = E_{12,1}$ and define inductively

$$Z_{i,k} = Z_{i,k-1}E_{21,k} + Z_{i-1,k-1}E_{12,k}, \quad 0 \leq i \leq k, \quad (4.3)$$

where we use the convention that $Z_{i,k} = 0$ if $i < 0$ or $i > k$. In particular, $Z_{0,n} = Y_{21,n}$ and $Z_{n,n} = Y_{12,n}$.

**Lemma 4.2.** We have for all $0 \leq k, l \leq n$

$$Y_{1,n}Z_{k,n} = q^{n-2k}Z_{k,n}Y_{1,n}, \quad Y_{2,n}Z_{k,n} = q^{n+2k}Z_{k,n}Y_{2,n}, \quad Z_{k,n}Z_{l,n} = Z_{l,n}Z_{k,n} \quad (4.4)$$

$$Y_{1,n}Y_{2,n} = \sum_{k=0}^n q^{n-k}Z_{k,n}, \quad Y_{2,n}Y_{1,n} = \sum_{k=0}^n q^kZ_{k,n}. \quad (4.5)$$

**Proof.** The first relation is just (12) for $n = 1$. Then, using induction on $n$, we obtain

$$Y_{1,n}Z_{k,n} = Y_{1,n-1}E_{1,n}(Z_{k,n-1}E_{21,n} + Z_{k-1,n-1}E_{12,n})$$

$$= q^{n-1-2k}Z_{k,n-1}E_{1,n}E_{21,n}Y_{1,n-1} + q^{n+1-2k}Z_{k-1,n-1}E_{1,n}E_{12,n}Y_{1,n-1}$$

$$= q^{n-2k}Z_{k,n}Y_{k,n}.$$
The second identity in (4.4) is proved similarly while the last is obvious since $E_{21,r}, E_{12,s}$ commute for all $1 \leq r, s \leq n$. To prove (4.5), we again use the inductive definition of the $Z_{k,n}$. For $n = 1$ this relation coincides with (4.1), while

$$Y_{1,n}Y_{2,n} = Y_{1,n-1}Y_{2,n-1}E_{1,n}E_{2,n} = \sum_{k=0}^{n-1} q^{n-k-1}Z_{k,n-1}(E_{12,n} + qE_{21,n})$$

$$= \sum_{k=0}^{n} q^{n-k-1}Z_{k,n-1}E_{12,n} + \sum_{k=0}^{n} q^{n-k}Z_{k,n-1}E_{21,n} = \sum_{k=0}^{n} q^{n-k}Z_{k,n}.$$

The remaining identity is proved similarly. □

As an immediate corollary, we obtain

$$Y_{1,n}Y_{2,n} = q^{-n}Y_{2,n}Y_{1,n} + (q^n - q^{-n})Y_{21,n} + \sum_{k=1}^{n-1} (q^{n-k} - q^{k-n})Z_{k,n} \quad (4.6)$$

**Example 4.3.** Let $n = 3$ and take $i = i_1$. Then in $\langle U_q^+(\mathfrak{s}\mathfrak{l}_3) \rangle_i$ we have

$$Y_{1,3}Y_{2,3} = q^{-3}Y_{2,3}Y_{1,3} + (q^3 - q^{-3})Y_{21,3} + Y_{21,3}'.$$

Since the terms in $Y_{21,3}'$ quasi-commute with $Y_{1,3}$ with different powers of $q$ and are linearly independent, we obtain an infinite family of generators by taking $q$-commutators of $Y_{1,3}$ with $Y_{21,3}'$. Thus, this algebra cannot be sub-PBW, since it clearly has polynomial growths.

**Lemma 4.4.** The elements $Z_{k,n}$, $1 \leq k \leq n - 1$, are contained in $\langle U_q^+(\mathfrak{s}\mathfrak{l}_3) \rangle_i \cap \text{Frac} U_q^+(\mathfrak{s}\mathfrak{l}_3)^{\otimes n}$ for both reduced expressions $i$ of the longest element in the Weyl group of $\mathfrak{s}\mathfrak{l}_3$.

**Proof.** Using Lemma 4.2 we obtain for all $s > 0$

$$Y_{1,n}^sY_{2,n} = \left( \sum_{k=0}^{n} q^{(s-1)(n-2k)+n-k}Z_{k,n} \right) Y_{1,n}^{s-1}.$$  

Note that one of the $Z_{0,n}$, $Z_{n,n}$ is contained in $\langle U_q^+(\mathfrak{s}\mathfrak{l}_3) \rangle_i$. Taking $1 \leq s \leq n$ yields a system of linear equations for the $Z_{k,n}$ in $\langle U_q^+(\mathfrak{s}\mathfrak{l}_3) \rangle_i \cap \text{Frac} U_q^+(\mathfrak{s}\mathfrak{l}_3)^{\otimes n}$ with the matrix $(q^{ns+k(1-2s)})$ where $1 \leq s \leq n + 1$, $0 \leq k \leq n$. This matrix is easily seen to be non-degenerate. □

**Proposition 4.5.** The assignment

$$u_i \mapsto Y_{i,n}, \quad i = 1, 2, \quad z_k \mapsto [k]q(q^{n-k} - q^{k-n})Z_{k,n}$$

defines an algebra homomorphism $i : A_{q,3}^{(n)} \to U_q(\mathfrak{s}\mathfrak{l}_n)^{\otimes n}$. In particular, the subalgebra of $U_q^+(\mathfrak{s}\mathfrak{l}_3)^{\otimes n}$ generated by $u_i(U_q^+(\mathfrak{s}\mathfrak{l}_3)^{\otimes n})$ and $Z_{0} = \{Z_{k,n} : 1 \leq k \leq n - 1\}$ is independent of $i$ and is sub-PBW.

**Proof.** It is sufficient to prove that $Y_{1,n}$, $Y_{2,n}$ and $Z_{k,n}$, $1 \leq k \leq n - 1$ satisfy the relations given in Theorem 4.21(i). The first two relations are already obtained in
Lemma 4.2 Furthermore (4.4) implies

\[
[Y_{1,n}, [Y_{1,n}, Y_{2,n}]_{q^{-n}}]_{q^n} = \sum_{k=0}^{n-1} h_{n-k}[Y_{1,n}, Z_{k,n}]_{q^n} = -\sum_{k=1}^{n-1} q^k h_{n-k} h_k Y_{1,n} Z_{k,n}.
\]

where \( h_k = q^k - q^{-k} \). Similarly,

\[
[Y_{2,n}, [Y_{2,n}, Y_{1,n}]_{q^{-n}}]_{q^n} = \sum_{k=0}^{n-1} h_k [Y_{2,n}, Z_{k,n}]_{q^n} = -\sum_{k=0}^{n-1} q^{-k} h_{n-k} h_k Y_{2,n} Z_{k,n}.
\]

The remaining assertions are trivial. \( \square \)

Define \( \tilde{i}_1 : U_q(\mathfrak{sl}_n) \to A_{q,3}^{(n)} \) by extending multiplicatively the assignments

\[
y_1 \mapsto u_1, \quad y_2 \mapsto u_2, \quad y_{21} \mapsto (q^n - q^{-n})^{-1} [u_1, u_2]_{q^n} - \sum_{k=1}^{n-1} [k]^{-1} z_k.
\]

The map \( \tilde{i}_2 \) is defined similarly.

Proposition 4.6. The maps \( \tilde{i}_r, r = 1, 2 \) split \( \mu \) and for each of them the diagram (1.3) commutes with \( \iota = \iota_4 \). In particular, \( A_{q,3}^{(n)} \) is the unique uberalgebra for this quantum folding.

Proof. We only show this for \( \iota = \iota_1 \), the argument for \( \iota_2 \) being similar. Since \( \iota_4 \) is multiplicative on the modified PBW basis, it is enough to check that the diagram (1.3) commutes on \( y_1, y_2 \) and \( y_{21} \), which is straightforward from Lemma 4.2 and from the definition of \( \mu, i \) and \( \tilde{i}_4 \). \( \square \)

4.2. Folding \( (\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4) \). Now we turn our attention to the folding \( (\mathfrak{sl}_4^\times, \mathfrak{sl}_4) \) with \( \sigma \) being the permutation of the components. Let \( A_{q,4} = A_{q,4}^{(2)} \) be the algebra with generators \( u_i, 1 \leq i \leq 3, z_{12} = z_{21}, z_{13} = z_{31} \) and \( z_{23} = z_{32} \) and relations given in Theorem 1.17(ii).

Theorem 4.7. The algebra \( A_{q,4} \) is PBW on the totally ordered set

\[
\{Y_2, Y_{21}, Y_{23}, Z_{21}, Z_{23}, Y_{13}, Z_{123}, Z_{321}, Z_{1232}, Z_{13}, Y_3, Y_1\},
\]

where \( Y_i = u_i, 1 \leq i \leq 3, Z_{ij} = (1 - q^2)^{-|i-j|} z_{ij}, 1 \leq i \neq j \leq 3, \) and

\[
Y_{2i} = \frac{u_i u_2 - q^{-2} u_2 u_i + q^{-1} z_{12}}{q^2 - q^{-2}}, \quad Z_{i2j} = \frac{[u_j, z_{21}]_{q^{-2}} + q^{-1} z_{13}}{(1 - q^4)(1 - q^{-2})},
\]

\[
Y_{213} = \frac{[u_j, Y_{21}]_{q^{-2}} - Z_{2j}}{q^2 - q^{-2}} - \frac{Z_{213} [Z_{2j}, Y_{21}]_{q^{-2}}}{[2]_q}, \quad Z_{2132} = \frac{q Z_{2j} Y_{2i} - q^{-1} Y_{2i} Z_{2j}}{q - q^{-1}}.
\]
where \( \{i, j\} = \{1, 3\} \). The PBW-type relations are given by the following formulae (where \( i \in \{1, 3\} \) and \( \{i, j\} = \{1, 3\} \))

\[
\begin{align*}
Y_{21}Y_2 &= q^2Y_2Y_{21}, & Z_{21}Y_2 &= Y_2Z_{21}, & Z_{1232}Y_2 &= q^2Y_2Z_{1232}, \\
Y_{13}Y_2 &= Y_2Y_{13} + h_2Y_{21}Y_{23} + h_1Z_{1232}, \\
Y_{2}Z_{13} &= q^2Z_{13}Y_2 + h_1(Y_2Z_{123} + q^2Y_2Z_{213}) - q^{-2}Y_{23}Z_{121} - q^{-2}Z_{21}Z_{23} - q^{-1}Z_{21}Z_{23} - h_1h_2Z_{1232}, \\
Y_{1}Y_2 &= q^{-2}Y_{21}Y_2 + h_2Y_{21} + h_1Y_{21}, & Y_{23}Z_{13} &= Z_{13}Y_{21} + h_1(Y_2Z_{123} - Z_{21}Y_{13}), \\
Y_{23}Y_{21} &= Y_{21}Y_{23}, & Z_{21}Y_{21} &= Y_{21}Z_{21}, & Y_{13}Y_{21} &= q^2Y_{21}Y_{13}, \\
Z_{21}Z_{21} &= Y_{21}Z_{21} + h_1(Y_2Z_{123} - Y_2Z_{231} + q^{-2}Y_{21}Z_{23} - q^{-2}Y_{21}Z_{23}), \\
Z_{13}Z_{21} &= Z_{21}Z_{13} + h_1(Y_2Z_{123} - Y_2Z_{231} + q^{-2}Y_{21}Z_{23} - q^{-2}Y_{21}Z_{23}), & Y_{13}Z_{21} &= Z_{21}Y_{13} + qh_1Y_{21}Z_{21}, \\
Z_{21}Z_{21} &= q^2Z_{21}Z_{21} + h_1(Z_{21}Y_{13} - Y_{21}Z_{21} - Z_2Y_{21}), \\
Z_{13}Z_{21} &= Z_{21}Z_{13} + h_1q^{-2}(qY_{21}Z_{21}Y_i + q^{-1}Y_{21}Z_{21}Y_i - Y_{21}Z_{13} + Z_{21}Z_{21}), \\
Z_{13}Z_{13} &= Z_{13}Z_{13} + h_1q^{-2}(qY_{21}Z_{21}Y_i + q^{-1}Y_{21}Z_{21}Y_i - Y_{21}Z_{13} + Z_{21}Z_{21}), \\
Y_1Z_{21} &= q^{-2}Z_{21}Y_i + h_1Z_{13} + h_2Z_{21}, & Y_1Z_{21} &= q^2Y_{13}Z_{13} - qh_1Z_{13}Z_{21}, \\
Y_1Z_1 &= Z_{21}Y_1, & Z_{13}Y_1 &= Z_{13}Y_{13}, & Z_{1232}Y_1 &= q^{-2}Y_{13}Z_{123}, \\
Y_1Y_1 &= q^3Y_{13}Y_1, & Z_{21}Z_{21} &= Z_{1232}Z_{21}, \\
Z_{13}Z_{21} &= Z_{13}Z_{21} + h_1(Y_2Y_{23}Z_{121} - Y_2Z_{21}Y_{23} + q^{-2}Y_{21}Z_{123}), \\
Z_{1232}Z_{21} &= Z_{1232}Z_{21} + h_1(Y_2Y_{23}Z_{121} - Y_2Z_{21}Y_{23} + q^{-2}Y_{21}Z_{123}), \\
Z_{21}Z_{21} &= Z_{1232}Z_{21} + h_1(Y_2Y_{23}Z_{121} - Y_2Z_{21}Y_{23} + q^{-2}Y_{21}Z_{123}), & Y_1Z_{13} &= Z_{13}Y_{13} + Z_{13}Y_{13} + Z_{13}Y_{13}, \\
Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), & Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), \\
Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), & Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), \\
Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), & Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), \\
Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}), & Y_1Z_{21} &= Z_{21}Y_1 - h_1q(Y_2Z_{13} - Z_{21}Y_{13}),
\end{align*}
\]

where we abbreviate \( h_k = q^k - q^{-k} \).

**Proof.** Since the proof is rather computational, we only provide a sketch. First, we define an algebra \( \mathcal{A} \) with generators \( Y_\alpha, \alpha \in \{1, 2, 3, 21, 23, 13\} \) and \( Z_\beta, \beta \in \{21, 23, 13, 123, 321, 1232\} \) and relations as above. Using the Diamond Lemma (see Proposition 2.13) we show that \( \mathcal{A} \) is PBW on these generators with the total order as defined in the theorem. Next, we introduce generators \( u_i = Y_1, 1 \leq i \leq 3, z_{ij} = (1 - q^2)^{-|i-j|}Z_{ij} 1 \leq i, j \leq 3 \) and show that they satisfy the relations in Theorem 1.17.11. In particular, this yields a surjective homomorphism of algebras \( A_{q,4} \to \mathcal{A} \). To prove that it is an isomorphism, we use Lemma 2.2.1. We define a grading on \( A_{q,4} \) by \( \deg u_i = 1, \deg z_{12} = \deg z_{23} = 2 \) and \( \deg z_{13} = 3 \). It is easy to see that specializations of defining relations of \( A_{q,4} \) are defining relations in the universal enveloping algebra of a nilpotent Lie algebra \( n \) of dimension 12 generated by \( u_i, 1 \leq i \leq 3 \) and \( z_{ij} = z_{ji}, 1 \leq i < j \leq 3 \) subject to the relations

\[
\begin{align*}
[u_i, u_j] &= 0, & |i - j| &= 1, & [z_\alpha, z_\beta] &= 0, \alpha, \beta \in \{12, 13, 23\} \\
[u_i, z_{i2}] &= [u_2, z_{i2}] = [u_i, z_{13}] = 0, & i & \neq 2
\end{align*}
\]
formule define a Poisson structure on its specialization (only non-zero brackets are non-zero).

Theorem 4.8. In particular, \( \dim U(n)_k = \dim A_n^k \) for all \( k \), where we endow \( A_n^k \) with the induced grading. It remains to apply Lemma 2.21.

Using the above and Proposition 2.18, we immediately obtain

**Theorem 4.8.** The algebra \( A_{q,4} \) is optimal specializable. In particular, the following formulæ define a Poisson structure on its specialization (only non-zero brackets are shown), where \( i \in \{ 1, 3 \} \) and \( \{ i, j \} = \{ 1, 3 \} \).

\[
\begin{align*}
[Y_2, Y_j] &= 2Y_2Y_i - 4Y_2i - 2Z_{23} \\
[Y_2, Y_{21}] &= -2Y_{12}Y_2 \\
[Y_2, Y_{13}] &= -4Y_{12}Y_{32} - 2Z_{2132} \\
[Y_2, Z_{i2}] &= -2Z_{2i}Y_{2j} + 2Z_{2132} \\
[Y_2, Z_{13}] &= 2Y_2Z_123 + 2Y_2Z_{13} + 2Y_2Z_{321} - 2Y_{21}Z_{23} - 2Y_{23}Z_{21} - 2Z_{21}Z_{23} \\
[Y_2, Z_{2132}] &= -2Y_2Z_{2132} \\
[Y_1, Y_{2i}] &= 2Y_iY_{2i} \\
[Y_1, Y_{21}] &= -2Y_2Y_j + 4Y_{13} + 2Z_{2j} \\
[Y_1, Y_{13}] &= -2Y_{12}Y_13 \\
[Y_1, Z_{i2}] &= 2Y_{12}Z_{2j} - 2Z_{2132} \\
[Y_1, Z_{i3}] &= -2Y_{12}Z_{2j} \\
[Y_1, Z_{13}] &= 2Y_{2i}Z_{13} - 2Z_{2i}Y_{13} \\
[Y_1, Y_{13}] &= 2Y_{13}Y_i \\
[Y_{13}, Z_{i3}] &= -2Y_{13}Z_{13} + 2Z_{123}Z_{321} \\
[Y_{13}, Z_{1232}] &= 2Y_{13}Z_{2132} \\
[Y_1, Z_{2j}] &= -2Z_{2j}Y_i + 4Z_{j2i} + 2Z_{13} \\
[Y_{13}, Z_{i2}] &= 2Y_{12}Z_{2j} \\
[Z_{21}, Z_{2j}] &= -2Y_2Z_{123} + 2Y_2Z_{321} - 2Y_{21}Z_{23} + 2Y_{23}Z_{21} \\
[Z_{i2}, Z_{i3}] &= 2Z_{1232}Y_i + 2Y_{2i}Z_{2j} - 2Z_{2i}Z_{i2} - 2Z_{2i}Y_{13} \\
[Z_{i2}, Z_{i3}] &= 2Y_2Z_{2j}Y_i - Y_{2i}Z_{2j}Y_i + Y_{2i}Z_{13} - Z_{2i}Z_{321} \\
[Z_{2i}, Z_{2j}] &= -2Y_{2i}Z_{2j} - 2Y_{2i}Z_{13} + 2Z_{2i}Y_{13} + 2Z_{2i}Z_{2j} \\
[Z_{2i}, Z_{1232}] &= -2Y_2Z_{2i}Z_{2j} + 2Y_{2i}Z_{1232} + 2Y_{2i}Y_{23}Z_{2i} \\
[Z_{13}, Z_{i2}] &= 2Z_{2i}Y_{13}Y_i - 2Y_{23}Z_{321}Y_i + 2Z_{123}Z_{321} - 2Y_{13}Z_{13} \\
[Z_{13}, Z_{2132}] &= 2Y_{21}Y_{23}Z_{13} - Y_{21}Y_{23}Z_{123} - Y_{21}Z_{13}Z_{321} - Y_{21}Y_{13}Z_{321} + Z_{21}Z_{23}Y_{13} + Y_{21}Y_{23}Z_{321} \\
[Y_1, Z_{2i}] &= 2Y_{12}Z_{2j} \\
[Z_{1232}, Y_i] &= 2Y_{2i}Z_{2j} - 2Y_{13}Z_{2i} \\
[Z_{1232}, Z_{i2}] &= 2Y_{2i}Y_{23}Z_{2i} - 2Y_{2i}Z_{2j}Y_i + 2Y_{13}Z_{1232} 
\end{align*}
\]

It remains to prove that \( A_{q,4} \) is the uberalgebra for our quantum folding. For, let \( i = \{ 2, 1, 3, 2, 1, 3 \} \in R(w_o) \) and define the elements \( x_\alpha \in U_q^+(16) \) by

\[
X_i = \{ x_2, x_{21}, x_{23}, x_{13}, x_1, x_3 \}
\]
as ordered sets, in the notation of Section 2.4. We identify the \( x_\alpha \) with the elements of the first copy of \( U_q^+(16) \) in \( U_q^+(16) \) and denote by \( x'_\alpha \) the corresponding elements of...
Lemma 4.9. Let $\tilde{y}_a = x_{\alpha}x'_a$ and let
\[
\tilde{z}_{i_1} = q^{-1}(x_{2i_2}x'_{i_1} + x_{2i_1}x'_{2i_2} - 2y_{i_2}), \\
\tilde{z}_{i_1} = q^{-2}(4y_{i_3} - 2x_{2i_2}x'_{i_1} - 2x_{2i_2}x'_{i_2} - 2x_{2i_2}x'_{i_3} - 2x_{2i_2}x'_{2i_3}x'_{i_1} \\
+ x_{2i_3}x'_{2i_2}x'_{i_1} + x_{2i_3}x'_{2i_2}x'_{i_2} + x_{2i_3}x'_{2i_2}x'_{i_3} + x_{2i_3}x'_{2i_2}x'_{i_3}), \\
\tilde{z}_{i_2} = q^{-1}(x_{13}x'_{i_2}x'_{i_1} + x_{2j}x_{13}x'_{i_1} - 2y_{i_1}) \\
\tilde{z}_{i_123} = x_{2i_3}x_{2i_2}x'_{i_3} + x_{2i_3}x_{2i_2}x'_{i_3} - 2q^{-1}y_{1i}y_{i_2}.
\]
Then the assignment $Y_{\alpha} \mapsto \tilde{y}_a, Z_{\alpha} \mapsto \tilde{z}_a$ defines a surjective algebra homomorphism $A_{q,4} \rightarrow \langle U^+_q(\mathfrak{sl}_4) \rangle_{\iota_i}$. Moreover, the folding $\iota_i$ is tame liftable.

In this case as well it can be shown that the diagram (1.3) commutes for a suitable choice of $\tilde{t}_i$. We conclude this section with the following problem.

Problem 4.10. Construct the uberalgebra for the quantum folding $(\mathfrak{sl}_n^k, \mathfrak{sl}_n)$ for all $n$ and $k \geq 2$.

5. Folding $(\mathfrak{so}_8, G_2)$

In this section we let $\mathfrak{g} = \mathfrak{so}_8$ with $I = \{0,1,2,3\}$ so that $\sigma$ is a cyclic permutation of $\{1,2,3\}$ and $I/\sigma = \{0,1\}$. In this numbering we have $(\alpha_0, \alpha_0) = 2, (\alpha_1, \alpha_1) = 6$ in $\mathfrak{g}^{\sigma^\vee}$ which we abbreviate as $\mathfrak{g}^\sigma$ since its Langlands dual is obtained by renumbering the simple roots. Let $U_{q,G_2}$ be the associative $\mathbb{C}(q)$-algebra generated by $U_q(\mathfrak{sl}_2)$ with Chevalley generators $E_0, F_0, K_{0}^{\pm 1}$ and $w, z_1, z_2$ satisfying the relations given in Theorem (1.23) (with $u$ replaced by $E_0$) as well as
\[
[F_0, w] = 0 = [F_0, z_j], \quad K_0 w K_0^{-1} = q^{-3}w, \quad K_0 z_j K_0^{-1} = q^{-1}z_j, \quad j = 1, 2.
\]

Theorem 5.1.

(i) The algebra $U_{q,G_2}$ is isomorphic to the cross product $A_q \rtimes U_q(\mathfrak{sl}_2)$, where $A_q$ is a flat deformation of the symmetric algebra of the nilpotent Lie algebra $\mathfrak{n}_{G_2}$ defined in Theorem (1.23).

(ii) The assignment $w \mapsto E_1, z_j \mapsto 0, j = 1, 2$ defines a homomorphism $\hat{\iota} : U_q(\mathfrak{g}^\sigma) \rightarrow U_q(\mathfrak{g})$. Its image is the (parabolic) subalgebra of $U_q(\mathfrak{g}^\sigma)$ generated by $U_q^+(\mathfrak{g})$ and $K_0^{\pm 1}, F_0$.

(iii) The assignments $w \mapsto E_1E_2E_3$ and
\[
\begin{align*}
z_1 & \mapsto [E_1E_2E_3, E_0]q^{-3} - \frac{q^2 + 1 + q^{-2}}{(q - q^{-1})^2} [E_1, [E_2, [E_3, E_0]q^{-1}]]q^{-1}, \\
z_2 & \mapsto [E_0, E_1E_2E_3]q^{-3} - \frac{q^2 + 1 + q^{-2}}{(q - q^{-1})^2} [[E_0, E_1]q^{-1}, E_2]q^{-1}, E_3]q^{-1}
\end{align*}
\]
define an algebra homomorphism $\mu : U_{q,G_2} \rightarrow U_q(\mathfrak{g})$. Its image is contained in the (parabolic) subalgebra of $U_q(\mathfrak{g})$ generated by $U_q^+(\mathfrak{g})$ and $K_0^{\pm 1}, F_0$. 
(iv) The assignments
\[ T_0(w) = ((q^2 + 1 + q^{-2})(q + q^{-1}))^{-1}[[w, E_0]_{q^{-1}}]_{q^{-1}}, \quad T_0(z_j) = [z_j, E_0]_{q^{-1}} \]
extend Lusztig’s action (2.9) of the braid group \( Br_{sl_2} \) on \( U_q(sl_2) \) to an action on \( U_{q,G_2} \) by algebra automorphisms. Moreover, \( \mu \) and \( \hat{i} \) are \( Br_{sl_2} \)-equivariant.

Most of the computations necessary to prove this theorem were performed on a computer and were involving rather heavy computations (for example, it took about 22 hours for the UCR cluster to check that the Diamond Lemma holds). Otherwise, the structure of the proof is rather similar to the ones discussed above.

Appendix A. Naive quantum folding does not exist

In this appendix, we show that the classical additive folding does not admit a quantum deformation even in the simplest possible case of \( sl_4 \). We use the standard numbering of the nodes of its Dynkin diagram. Let \( u_1 = E_1 + E_3, u_2 = E_2 \). We obviously have
\[ u_2^2u_1 - (q + q^{-1})u_2u_1u_2 + u_1u_2^2 = 0. \]
On the other hand, suppose that we have a relation
\[ \sum_{j=0}^{3} c_j u_1^j u_2 u_1^{3-j} = 0. \] (A.1)
Retain the notation of 2.7. Applying \( r_2 \) and \( r_2r_1 \) to the left hand side of (A.1) and considering the coefficients of linearly independent monomials we obtain a system of linear equations for the \( c_i, 0 \leq i \leq 3 \) with the matrix
\[
\begin{pmatrix}
1 & q^{-1} & q^{-2} & q^{-3} \\
3q^2 & q(q^2 + 2) & 2q^2 + 1 & 3q \\
q^2 + 1 + q^{-2} & q + 2q^{-1} & 2 + q^{-2} & q + q^{-1} + q^{-3} \\
3(q^2 + 1) & 5q + q^{-1} & q^2 + 5 & 3(q + q^{-1})
\end{pmatrix}
\]
of determinant \(-(q - q^{-1})^6\). Thus, there is no relation of the form (A.1). Clearly, replacing \( u_1 \) and \( u_2 \) by \( E_1 + E_3 + (q-1)a, E_2 + (q-1)b \), where \( a, b \) are \( \sigma \)-invariant elements of degree greater than one will not affect the above calculation. Thus, there is no embedding of \( U_q^+(so_5) \) into \( U_q^+(sl_4) \) which deforms the embedding of \( so_5 \) into \( sl_4 \).

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