CELL STRUCTURES ON THE BLOB ALGEBRA

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Abstract. We consider the $r = 0$ case of the conjectures by Bonnafé, Geck, Iancu and Lam on cellular structures on the Hecke algebra of type $B$. We show that this case induces the natural cell structure on the blob algebra $b_n$ by restriction to one-line bipartitions.

1. Introduction

The purpose of this article is to continue the investigation, initiated in [RH], of the relationship between the representation theories of the Hecke algebra $H_n = H_n(Q, q)$ of type $B$ and of the blob algebra $b_n = b_n(q, m)$. The Hecke algebra $H_n$ of type $B$ is a well-known two-parameter deformation of the hyperoctahedral group whereas the blob algebra $b_n$, introduced in [MS] from motivations in statistical mechanics, is a diagram algebra of marked (blobbed) Temperley-Lieb diagrams. A main point of our work, already present in [RH], is that $b_n$ can also be realized as a quotient of $H_n$ thus making the $b_n$-representations $H_n$-representations by inflation. Viewing $b_n$ as a quotient of $H_n$ is analogous to viewing the Temperley-Lieb algebra $TL_n$ as a quotient of the Hecke algebra of type $A$, and indeed $b_n$ is also sometimes called the Temperley-Lieb algebra of type $B$.

Dipper-James-Murphy introduced in [DJM] for each bipartition $(\lambda, \mu)$ of total degree $n$ a Specht module $S_n(\lambda, \mu)$ for $H_n$. Let $J_n$ be the kernel of the quotient map $H_n \to b_n$. We then showed in [RH] that $J_n S_n(\lambda, \mu) = 0$ as long as $(\lambda, \mu)$ is a one-line bipartition and so these $S_n(\lambda, \mu)$ factor over the quotient map to become $b_n$-modules. One might now suspect that $S_n(\lambda, \mu)$ is a standard module for the quasi-hereditary algebra $b_n$. Indeed, we showed that many properties of the standard modules are shared by the $S_n(\lambda, \mu)$, but somewhat surprisingly we could prove in [RH] that they do not verify the relevant universal property and so do not identify with standard modules, except in trivial cases.

Recall that G. Lusztig’s monograph [Lu2] on the representation theory of Hecke algebras with unequal parameters contains a construction of cells in the associated Weyl group, generalizing the construction for one-parameter Hecke algebras from [KL]. As a matter of fact, he gives for each choice of a total order on an Abelian group $\Gamma$, such that $a := \log q, b := \log Q \in \Gamma$, a construction of a Kazhdan-Lusztig type basis of $H_n$ that induces a corresponding cell partition of the Weyl group. To each of these cells there is an associated cell module of the Hecke algebra. In [BGIL] a series a conjectures were formulated for type $B$ which, if true, would put a high degree of structure on this. Assume that $b \not\in \{a, 2a, \ldots, (n-1)a\}$ and that $a$ and $b$ are positive in $\Gamma$. According to the conjectures, the setting should give rise to a cellular algebra datum on $H_n$ in the sense of Graham and Lehrer,

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where the underlying poset \( \Lambda \) should be the set of bipartitions \( \text{Bip}(n) \) of total degree \( n \) with partial order and map \( \Lambda \times \Lambda \rightarrow \mathcal{H}_n \) defined in terms of a certain domino insertion algorithm, depending on \( \Gamma \). Furthermore, by the work of Bonnafé and Jacon \([BJ]\), the different cellular algebra structures on \( \mathcal{H}_n \) should account for the different ways of parameterizing the simple modules for \( \mathcal{H}_n \) that are given by Ariki’s Theorem in \([A]\).

These conjectures have only been fully proved in the so-called asymptotic case \( b > (n - 1)a \), see \([BI]\), where the cell modules turn out to be the ones given by Dipper-James-Murphy. In this work we focus on the case \( \Gamma := \mathbb{Z}, a := 2 \) and \( b = 1 \). This is another extreme case since \( b < a \) and so \( r = 0 \) in the \([BGIL]\) notation. We show that the poset structure on \( \text{Bip}(n) \) in this case is compatible with the quasi-hereditary order on the category of \( b_n \)-modules when restricted to one-line bipartitions, the map being given by \( (\lambda, \mu) \mapsto k-l \) where \( \lambda = (k) \) and \( \mu = (l) \). We show that the ideal \( J_n \) is generated by the set of Kazhdan-Lusztig elements \( C_w \) for which \( w \) does not correspond to a one-line bipartition. We moreover show that the cell module given by the one-line bipartition \( (\lambda, \mu) \) is isomorphic to the \( b_n \) standard module \( \Delta_n(k-l) \) where \( \lambda = (k) \) and \( \mu = (l) \). To summarize our findings: the \( a = 2, b = 1 \) case of the \([BGIL]\) conjectures induces the blob algebra category when restricted to one-line bipartitions.

This given, the algorithm described in \([Ja]\) can be used to answer the question that was raised in \([RH]\), namely to describe the Kleshchev bipartition that corresponds to the simple \( b_n \)-module \( L_n(\lambda) \).

Let us indicate the layout of the article. The first section contains a combinatorial analysis of the domino insertion algorithm already mentioned above. The main result is a characterization of the elements \( W_b \) of the Weyl group \( W_n \) of type \( B \) that go to two-line tableaux under domino insertion. This characterization uses the Coxeter presentation of \( W_n \). The section relies on results of Taskin, \([T]\).

In the next section we recall the presentation of \( b_n \) as a quotient of \( \mathcal{H}_n \) and show that the defining ideal is given by the Kazhdan-Lusztig type elements \( C_w \in \mathcal{H}_n \) where \( w \notin W_b \). In the following section we show our main results, identifying the cell modules with the standard modules. To be more precise, we show that the cell modules verify the universal property for the standard modules, given within the framework of the globalization-localization formalism. For this to work we rely on Lusztig’s results in \([Lu1]\) that we combine with the results of Fan and Green \([FG]\) on type \( A \).

Finally, in the last section we show how the Fock space approach to the representation theory of \( \mathcal{H}_n \) can be used to reprove the main results of \([MW]\) and to obtain the Kleshchev bipartitions of the simple modules of \( b_n \).

2. Basic notation and domino insertion

In this section we first fix some basic notation that shall be used throughout the article. We then investigate the domino insertion algorithm for the Weyl group of type \( B \). We describe the elements that are mapped to two-line partitions, that is domino tableaux in less than two lines.
Let $W_n$ be the Weyl group of type $B_n$. It is a Coxeter group on generators $s_0, s_1, \ldots, s_{n-1}$ with relations

\[
  s_i^2 = 1 \quad \text{for } i = 0, \ldots, n - 1 \\
  (s_i s_{i+1})^3 = 1 \quad \text{for } i = 1, \ldots, n - 2 \\
  (s_i s_j)^2 = 1 \quad \text{for } |i - j| > 2 \\
  (s_0 s_1)^4 = 1
\]

Let $I_n := I_n^+ \cup I_n^-$ where $I_n^+ := \{1, 2, \ldots, n\}$ and $I_n^- := \{-1, -2, \ldots, -n\}$. Then $W_n$ can also be described as the subgroup of the symmetric group on the elements $I_n$ generated by $s_0 := (-1, 1)$ and

\[
  s_i := (i, i + 1)(-i, -i - 1)
\]

in cycle notation. We shall adopt the convention that cycles are multiplied from right to left. The subgroup of $W_n$ generated by $s_1, s_2, \ldots, s_{n-1}$ is the symmetric group $S_n$.

For elements $w \in W_n$ we shall also use word notation

\[
  w = i_1 i_2 i_3 \ldots i_n
\]

where $i_k \in I_n$. By this we mean that $w$ acts in the following way on $I_n$

\[
  w : 1 \mapsto i_1, 2 \mapsto i_2, \ldots, n \mapsto i_n
\]

and then also necessarily $-1 \mapsto -i_1, -2 \mapsto -i_2, \ldots, -n \mapsto -i_n$. In this setting we use the standard notation $\bar{i} := -i \in I_n^-$ for $i \in I_n^+$. Thus $i$ appears in $w = i_1 i_2 i_3 \ldots i_n \in W_n$ if and only if $\bar{i}$ does not not appear.

It is normally clear whether a given $w \in W_n$ is written as a product of Coxeter generators or as a word over $I_n$ and we shall therefore generally not mention explicitly the chosen form. For example

\[
  w = 23\bar{T} = s_0 s_1 s_2 \in W_3
\]

We denote by $<_B$ the Bruhat-Chevalley order on $W_n$ where by convention the neutral element $1 \in W_n$ is the smallest of all. Assume that $w = i_1 i_2 i_3 \ldots i_n \in W_n$. Then the following conditions describe the right descent set of $w$ with respect to $<_B$ see eg. [BB]

\[
  ws_k < w \text{ iff } i_k > i_{k+1} \quad \text{for } k = 1, 2, \ldots, n - 1 \\
  ws_0 < w \text{ iff } i_1 < 0
\]

If $w \in W_n$ is written in word from, its right descent set can be used to write it as a reduced expression in the Coxeter generators $s_i$.

**Example.** Assume that $w = 3\bar{T}24$. Then $s_1 s_0 s_1 s_0 s_2 s_1$ is a reduced expression for $w$ obtained from the above description of the right descent set. Indeed, $s_2 s_1$ moves $3$ past $\bar{T}2$, then $s_0$ changes $\bar{T}$ to $1$ and finally $s_1 s_0 s_1$ changes $2$ to $2$.

We shall throughout be specially interested in $W_b = W_{n,b}$ which we define as the subset of $W_n$ consisting of those $w$ that have no reduced expressions $w = s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_N}$ that contain a subexpression $s_{i_k} s_{i_{k+1}} s_{i_{k+2}}$ of the form

\[
  s_i s_{i \pm 1} s_i \quad \text{for } i = 1, 2, \ldots, n - 2 \quad \text{or } s_{n-1} s_{n-2} s_{n-1}
\]

for $k = 1, 2, \ldots, N - 2$. Thus the subexpression $s_0 s_1 s_0$ is allowed whereas $s_1 s_0 s_1$ is not.
Our aim is to describe the image of \( W_b \) under the domino insertion correspondence described for example in [BGIL]. In order to do so we first need a description of \( W_b \) in terms of words. This description will only be indirect, but for our purposes this will be sufficient.

**Lemma 1.** Assume that \( w \in W_n \) and assume that it can be written as follows

\[
w = i_1 i_2 i_3 \ldots i_k \overline{a}_1 i_{k+1} \ldots i_n
\]

where \( i_1, i_2, \ldots, i_k, a_1 > 0 \). Then \( w \in W_b \) if and only if

\[
a_1 < i_1 < i_2 < \ldots < i_k \quad \text{and} \quad a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots i_n \in W_b
\]

**Proof.** Suppose that \( w \in W_b \) and define \( w_1 \in W_n \) by

\[
w_1 = a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots i_n
\]

Using the above description of the right descent set we get that \( w \) has a reduced expression of the form

\[
w = w_1 s_0 s_1 \ldots s_{k-2} s_{k-1} s_k
\]

and the second statement follows, since any reduced expression for \( w_1 \) can be extended to a reduced expression for \( w \).

If now \( a_1 < i_1 < i_2 < \ldots < i_k \) is not satisfied then by the description of the right descent set there will be an index \( 1 \leq j \leq k \) such that \( w_1 s_j < w_1 \). But by formula (●) this contradicts the assumption that \( w \in W_b \).

To show the other implication we assume that \( a_1 < i_1 < \ldots < i_k \) holds, that \( w_1 = a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots i_n \in W_b \) and that \( w \notin W_b \). Since \( s_0 s_1 s_2 \ldots s_{k-1} s_k \) is a unique presentation of \( w_1^{-1} w \) and since \( w_1 \in W_b \) we conclude that \( w_1 \) must have a reduced expression of the form \( w_1 := w_2 s_j \), for an index \( j \) such that \( 0 \leq j \leq k \). But then \( s_j \) belongs to the right descent set for \( w_1 \), contradiction. \( \Box \)

Suppose that \( w = i_1 i_2 i_3 \ldots i_n \in W_n \). A decreasing subsequence of \( w \) of length \( k \) is a subsequence \( i_{t_1}, i_{t_2}, \ldots, i_{t_k} \) of \( w \) with \( i_j < i_{j+1} \) and \( i_{t_{j+1}} > i_{t_j} \) for \( j = 1, \ldots, k \). Define \( W_c := W_b \cap S_n \). Then it is known that \( W_c \) corresponds under the Robinson-Schensted correspondence to pairs of partitions of at most two lines. Hence \( W_c \) can also be described as the words over \( I_n^+ \) that have no decreasing subsequences of length strictly greater than two.

**Theorem 1.** Suppose \( w \in W_n \) and write it as

\[
w = i_1 i_2 \ldots i_{k_1} \overline{a}_1 i_{k_1+1} \ldots i_{k_2} \overline{a}_2 i_{k_2+1} \ldots i_{k_3} \overline{a}_3 i_{k_3+1} \ldots i_n
\]

where \( \overline{a}_1, \ldots, \overline{a}_t \) are the only negative numbers that occur in \( w \). Define

\[
w^l := a_1 a_{l-1} a_{l-2} \ldots a_1 i_1 i_2 i_3 \ldots i_l \ldots i_n.
\]

Then \( w \in W_b \) if and only if

\[
a_1 < a_{l-1} < a_{l-2} < \ldots < a_1 < i_1 < i_2 < i_3 < \ldots < i_{k_1}
\]

and \( w^l \) has no decreasing subsequences of length strictly greater than 2.
Proof. Suppose first that \( w \in W_b \). We generalize \( w' \) as follows
\[
\begin{align*}
 w^1 &= a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots \overline{i}_{k_2} i_{k_2+1} \ldots \overline{i}_{k_3} \overline{i}_{k_3+1} \ldots i_n \\
 w^2 &= a_2 a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots \overline{i}_{k_2} i_{k_2+1} \ldots \overline{i}_{k_3} \overline{i}_{k_3+1} \ldots i_n \\
 & \vdots \\
 w^l &= a_l \ldots a_2 a_1 i_1 i_2 \ldots i_k i_{k+1} \ldots \overline{i}_{k_2} i_{k_2+1} \ldots \overline{i}_{k_3} \overline{i}_{k_3+1} \ldots i_n
\end{align*}
\]
By the proof of the previous Lemma we have \( w^k \in W_b \) for all \( k \) and so we get the inequalities
\[
 a_l < a_{l-1} < a_{l-2} < \ldots < a_1 < i_1 < i_2 < i_3 < \ldots < i_{k_l}
\]
by using the previous Lemma recursively. But \( w^l \in W_b \cap S_n \) and so we have proved one implication of the Theorem.

The other implication follows in a similar way from the previous Lemma.

The notion of domino tableaux shall be important to us. A domino tableau is the Young diagram of an integer partition of \( 2n \) with node set partitioned into dominoes, that is horizontally or vertically neighboring nodes. The dominoes are labeled with numbers \( 1, 2, \ldots, n \). A domino tableau is called standard if the labeling is increasing from left to the right and from top to bottom. Let \( SDT(n) \) denote the set of standard domino tableaux in \( n \) dominoes. Below is an example from \( SDT(6) \).

\[
\begin{array}{ccc}
 & & 4 \\
1 & 2 & 5 \\
3 & 6 & \\
\end{array}
\]

We define \( SDT := \bigcup_n SDT(n) \). For \( S \in STD \) we let \( Sh(S) \) denote the shape of its underlying partition. Let \( SDT^2(n) \) be the set
\[
SDT^2(n) := \{ (S, T) \in SDT(n) \times SDT(n) \mid Sh(S) = Sh(T) \}.
\]
The domino insertion algorithm establishes a bijection between \( W_n \) and \( SDT^2(n) \). It can be viewed as a generalization of the Robinson-Schensted algorithm. We shall not here give a precise description of the algorithm, but refer the reader to for instance [BGIL].

Let us denote by \( (P(w), Q(w)) \) the pair of domino tableaux associated with \( w \in W_n \) under domino insertion. We say that \( w \) and \( w_1 \) belong to the same Knuth (plactic) class, or \( w \sim_\mathcal{K} w_1 \), if \( P(w) = P(w_1) \). Dually, we say that \( w \) and \( w_1 \) belong to the same dual Knuth (coplactic) class, or \( w \sim_\mathcal{K}^* w_1 \), if \( Q(w) = Q(w_1) \).

In this setting, Taskin considers in [T] the following relations on the elements of \( W_n \) in word form, generalizing the Knuth relations
\[
\begin{align*}
 \ldots f(2) f(3) f(1) \ldots & \sim_\mathcal{K} \ldots f(2) f(1) f(3) \ldots \quad (1) \\
 \ldots f(1) f(3) f(2) \ldots & \sim_\mathcal{K} \ldots f(3) f(1) f(2) \ldots \quad (2) \\
 i_1 i_2 \ldots & \sim_\mathcal{K} \overline{i}_1 \overline{i}_2 \ldots \quad \text{if } |i_1| > |i_2| \quad (3)
\end{align*}
\]
where \( f : I_n \to I_n \) is any bijection such that \( f(1) < f(2) < f(3) \).
He proves the following Theorem.

**Theorem 2.** Suppose \( w, z \in W_n \). Then they belong to the same plactic class if and only if there is a sequence \( w_1, w_2, \ldots, w_k \in W_n \) such that \( w = w_1, z = w_k \) and \( w_i K \sim w_{i+1} \) or \( w_{i+1} K \sim w_i \) for \( i = 1, 2, \ldots, k-1 \).

The dual Knuth relations are defined by \( w^{DK} w_1 \) if \( w^{-1} K \sim w_1^{-1} \). If \( w \) and \( w_1 \) are written in word form, they do not act on neighboring elements, and as a matter of fact, they do not admit as simple a description as in the symmetric group case. On the other hand, since \( Q(w) = P(w^{-1}) \) the previous Theorem has an obvious dual version:

**Theorem 3.** Suppose \( w, z \in W_n \). Then they belong to the same coplactic class if and only if there is a sequence \( w_1, w_2, \ldots, w_k \in W_n \) such that \( w = w_1, z = w_k \) and \( w_i^{DK} \sim w_{i+1} \) or \( w_{i+1}^{DK} \sim w_i \) for \( i = 1, 2, \ldots, k-1 \).

The following Lemma will be useful.

**Lemma 2.** \( W_b \) is stable under the Knuth relations (1), (2) and (3).

**Proof.** Assume that \( w \in W_b \) and write it in the form
\[
w = \overline{i_1 a_1} \overline{i_2 a_2} \cdots \overline{i_l a_l} w_1
\]
where \( w_1, \overline{i_j} \) are words, possibly empty, over \( I_n^+ \) for \( j = 1, 2, \ldots, l \) and \( a_j > 0 \) for \( j = 1, 2, \ldots, l \). Write
\[
\overline{i_1 i_2} \cdots \overline{i_l} w_1 = \overline{i_1} \overline{i_2} \cdots \overline{i_k}
\]
Let us first verify that the Knuth relations (1) and (2) map \( w \) to another element of \( W_b \). Assume first that (1) acts in the part of \( w \). We know from Theorem 1 that all \( \overline{i_j} \) are increasing sequences over \( I_n^+ \) and that
\[
a_l < a_{l-1} < a_{l-2} < \ldots < a_1 < \overline{i_l} < \overline{i_{l-1}} < \ldots < \overline{i_1}
\]
where the inequalities hold for all elements of the subsequences, and so the pattern \( f(2) f(3) f(1) \) can only occur if \( f(1) = \overline{s} \) for some \( 1 \leq r \leq l \) and \( f(3) = i_s \) for some \( s \). But then clearly (1) takes \( w \) to another element of \( W_b \). Likewise we see that (1) acting in the pattern \( f(2) f(1) f(3) \) of \( \overline{i_1 a_1} \overline{i_2 a_2} \cdots \overline{i_l a_l} \) takes \( w \) to another element of \( W_b \).

In the case of the Knuth relation (2) acting in
\[
\overline{i_1 a_1} \overline{i_2 a_2} \cdots \overline{i_l a_l}
\]
we argue similarly. By the inequalities (4), the only decreasing subsequences of \( \overline{i_1 a_1} \overline{i_2 a_2} \cdots \overline{i_l a_l} \) are of the form \( \overline{i_r a_r} \) for some \( r, s \) and so in the pattern \( f(1) f(3) f(2) \) we have that \( f(3) = i_r \) for some \( r \) whereas \( f(2) = i_s \) for some \( s \). But since \( f(1) \) is less than \( f(2) \) it must be \( \overline{s} \) for some \( t \) and so changing \( f(1) f(3) f(2) \) to \( f(3) f(1) f(2) \) gives another element of \( W_b \). We argue similarly in case of the pattern \( f(3) f(1) f(2) \).

Certainly, only the Knuth relations (1) and (2) can act in the \( w_1 \) part of \( w \). But by the theory of the usual Robinson-Schensted algorithm, the Knuth relations (1) and (2) preserve the length of the longest decreasing
subsequence when acting on words over $I_n^+$. Hence we get from Theorem 1 that they map $w$ to $W_b$.

We finally consider the case where the action of the Knuth relations involves both $\bar{i}_1 \bar{a}_1 \bar{i}_2 \bar{a}_2 \ldots \bar{i}_l \bar{a}_l$ and $w_1$. The relevant Knuth relations are then only $(1)$ and $(2)$ and $\bar{a}_l$ must occur in first or second position of the relation.

**Case** $f(2)f(3)f(1)$: This case does not occur since $f(1)$ would belong to $w_1$ and would be less than $\bar{a}_l$, which contradicts the fact that $w_1$ is a word over $I_n^+$.

**Case** $f(2)f(1)f(3)$: Using once more that the only decreasing subsequences of $\bar{i}_1 \bar{a}_1 \bar{i}_2 \bar{a}_2 \ldots \bar{i}_l \bar{a}_l$ are of the form $i_r \bar{a}_s$, we get in this case that $f(1) = \bar{a}_l$ whereas $f(2)$ is unbarred. Applying the Knuth relation $(1)$ yields $f(2)f(3)f(1)$, and hence $\bar{i}_l$ changes to $\bar{i}_l f(3)$, which is still increasing.

**Case** $f(1)f(3)f(2)$: In this case we have that $f(1) = \bar{a}_l$ and $f(3)$ and $f(2)$ are unbarred, since $f(2) \in w_1$ and $f(3) > f(2)$. Thus also $f(3) \in w_1$. The application of the Knuth relation $(2)$ changes $f(1)f(3)f(2)$ to $f(3)f(1)f(2)$ and hence $\bar{i}_l$ changes to $\bar{i}_l f(3)$. But no element of $\bar{i}_l$ can be bigger than $f(3)$ for if $i_r$ were such an element than we may assume it is the last one of $\bar{i}_l$ and $i_r f(3) f(2)$ would be a decreasing subsequence longer than three, inside $a_l a_{l-1} a_{l-2} \ldots a_1 \bar{i}_l \bar{a}_2 \ldots \bar{i}_l w_1$. Thus $\bar{i}_l f(3)$ is increasing and we are done in this case as well.

**Case** $f(3)f(1)f(2)$: We have $f(1) = \bar{a}_l$. Using the Knuth relation $(2)$ it changes to $f(1)f(3)f(2)$ and thus $\bar{i}_l$ changes to $\bar{i}_l \backslash f(3)$ which is clearly increasing.

We now finally check that also the third Knuth relation $(3)$ takes $w$ to an element of $W_b$. Using Theorem 1 we see that it only acts in $w$ if the first two elements of $w$ are either on the form $i_1 \bar{a}_2$ with $i_1 > a_2$ or $\bar{a}_1 \bar{a}_2$ with $a_1 > a_2$. But the relation $(3)$ interchanges these two, thus finishing the proof of the Lemma.

**Corollary 1.** $W_b$ is a union of plactic classes and also a union of coplactic classes.

**Proof.** The previous Lemma amounts to saying that $W_b$ is a union of plactic classes. But $Q(w) = P(w^{-1})$ and $W_b$ is stable with respect to $w \mapsto w^{-1}$, hence $W_b$ is also a union of coplactic classes. 

For $w \in W_n$ we define $Sh(w)$ by $Sh(P(w))$ or, equivalently, by $Sh(Q(w))$. Define

$$STD_{\leq 2} := \{(S, T) \in STD^2 \mid Sh(S) \text{ has less than two lines} \}.$$

We are now in position to prove the main Theorem of this section.

**Theorem 4.** Suppose that $w \in W_n$. Then $w \in W_b$ if and only if $Sh(w)$ is a Young diagram of at most two lines. In other words, $W_b$ is in correspondence with $STD_{\leq 2}$ under domino insertion.

**Proof.** Assume first that $Sh(w)$ has at most two lines. Using Theorems 2 and 3 there is $w_1 \in W_n$ related to $w$ through a series of Knuth or dual Knuth relations such that $P(w_1)$ and $Q(w_1)$ both have one of the forms
depending on the parity of the first line of $Sh(w)$. Under the domino insertion algorithm, the first tableau corresponds to

$$\begin{array}{cccccc}
1 & 2 & \ldots & k \sqcup & k+1 & \ldots \\
3 & \ldots & k & \sqcup & \ldots & n
\end{array}$$

whereas the second tableau corresponds to

$$\begin{array}{cccccc}
1 & 3 & \ldots & k & k+1 & \ldots \\
2 & 4 & \ldots & k & \sqcup & \ldots & n
\end{array}$$

Since they both belong to $W_b$ we deduce from Lemma 2 that $w$ also belongs to $W_b$ and one implication of the Theorem is proved.

To prove the other implication we take $w \in W_b$ and show that $P(w)$ has at most two lines. Write first $w$ in the form

$$w = \iota_1 a_1 \iota_2 a_2 \ldots \iota_u a_u w_1$$

where $w_1, \iota_j$ are words over $I_n^+$ and $a_j > 0$. We set

$$i_1 i_2 i_3 \ldots i_k := \iota_1 i_2 \ldots \iota_u.$$

By Theorem 1 there is now a $t$ such that $P := P(\iota_1 a_1 \iota_2 a_2 \ldots \iota_u a_u)$ is the following domino tableau

$$\begin{array}{ccccccc}
a & a & \ldots & a & i_1 & \ldots & i_k
\end{array}$$

Let $w_1 = j_1 j_2 \ldots j_{n-k-u}$ and let $j_{i_1} j_{i_2} \ldots j_{i_r}$ be the subsequence of $w_1$ consisting of those elements $j_i$ that are less than $i_k$. Then by Theorem 1 we have that $j_{i_1} j_{i_2} \ldots j_{i_r}$ is an increasing subsequence. Let $j_{i_1} j_{i_2} \ldots j_{i_s}$ be the subsequence of $w_1$ consisting of those elements that are positioned before $j_{i_r}$ in $w_1$ and are bigger than $i_k$. By Theorem 1 this is also an increasing subsequence. Setting $z_1 := j_1 j_2 \ldots j_{i_r}$ and $z_2 := j_{i_r+1} j_{i_r+2} \ldots j_{n-k-u}$ we have obviously that $w_1 = z_1 z_2$. Moreover $z_1$ is a shuffle of its subsequences $j_{i_1} j_{i_2} \ldots j_{i_r}$ and $j_{i_1} j_{i_2} \ldots j_{i_s}$. Let us first assume that this shuffle is trivial in the sense that $z_1 = j_{i_1} j_{i_2} \ldots j_{i_r} j_{i_1} j_{i_2} \ldots j_{i_s}$.

Let us consider the insertion of $z_1$ in $P$. If $j_{i_1}$ must be entered in the two-line part of $P$, say if $a_1 < j_{i_1} < a_2$, the resulting domino will be

$$\begin{array}{cccccc}
a_1 & j_{i_1} & \ldots & a_2 & \ldots & i_1 \quad i_1 \ldots \quad i_k
\end{array}$$

that is, one vertical domino in $P$ become horizontal, and the first horizontal domino becomes vertical. If $j_{i_2}$ must also be entered in the two-line part of the tableau, the resulting tableau will look as follows

$$\begin{array}{ccccccc}
1 & 2 & \ldots & k & k+1 & \ldots & n
\end{array}$$
where once again a vertical domino becomes horizontal and a horizontal becomes vertical. Since the sequence \( j_1, j_2 \ldots j_r \) is increasing this pattern is repeated until arriving at the elements that must be inserted in the one-line part of the tableau. These are inserted by bumping horizontal dominoes to the second line, giving tableaux of the form

```
[ ]    [ ]    [ ]    [ ]
```

We next describe the insertion of the other elements of \( z_1 \), those from \( j_1 \), \( j_2 \ldots j_s \). But this is much simpler, since the element to be inserted will always be bigger than those so far inserted. It is therefore inserted as a horizontal domino at the end of the first line, without bumping.

This last description also shows that in general, when \( z_1 \) is a more complicated shuffle of \( j_1, j_2 \ldots j_r \) and \( j_1, j_2 \ldots j_s \), the insertion of the elements of \( j_1, j_2 \ldots j_s \) does not influence the insertion of the elements of \( j_1, j_2 \ldots j_r \). We have thus proved that the insertion of all elements of \( z_1 \) gives a two-line domino tableau.

Finally, we consider the insertion of the elements of \( z_2 \). But the elements of \( z_2 \) are all bigger than the elements of \( P_1 = P(\lambda_1, \lambda_2, \ldots, \lambda_u, \mu_1, \mu_2, \ldots, \mu_v) \) and so they are inserted as horizontal dominoes at the end of \( P_1 \). To be precise, the resulting domino tableau is simply the concatenation of the lines of \( P_1 \) and \( P(z_2) \). The Theorem is proved. \( \square \)

In the remaining part of this section, we formulate a result which is a first strong indication of the connection between the empty core case of the \([BGIL]\) conjectures and the representation theory of \( b_n \), where \( b_n \) is the blob algebra mentioned in the introduction.

Let \( \text{Par}_0(n) \) denote the set of integer partitions of degree \( n \) with empty core and set \( \text{Par}_0 := \bigcup_{n \geq 0} \text{Par}_0(n) \). Similarly, let \( \text{Bip}(n) \) denote the set of bipartitions \( (\lambda, \mu) \) of total degree \( n \) and set \( \text{Bip} := \bigcup_{n \geq 0} \text{Bip}(n) \). We denote by \( \text{ST}_0(n), \text{ST}_\emptyset, \text{SBT}(n) \) and \( \text{SBT} \) the set of standard (bi)tableaux with underlying shape in \( \text{Par}_0(n), \text{Par}_0, \text{Bip}(n) \) and \( \text{Bip} \). For \( \lambda \) a partition we denote by \( Q(\lambda) \) its two-quotient. Then \( Q(\lambda) \in \text{Bip}(m) \) if \( \lambda \in \text{Par}_0(2m) \) and \( Q \) induces a bijection

\[
Q : \text{Par}_0 \rightarrow \text{Bip}.
\]

Following \([BGIL]\) we define a partial order on \( \text{Bip} \) by the rule

\[
(\lambda, \mu) < (\tau, \nu) \quad \text{iff} \quad Q^{-1}(\lambda, \mu) < Q^{-1}(\tau, \nu)
\]

where \( < \) refers to the usual dominance order on partitions.

Let \( \text{Bip}_1(n) \) denote the set of one-line bipartitions of total degree \( n \). An element of \( \text{Bip}_1(n) \) is of the form \( (a, n-a) \) for some positive integer \( a \) with \( 0 \leq a \leq n \). We shall use the shorthand notation \( (a) \) for such \( (\lambda, \mu) \) but reserve the notation \( (a, n-a) \) for a conventional (two-line) partition. Set \( \text{Bip}_1 := \bigcup_{n \geq 0} \text{Bip}_1(n) \).

Define \( \Lambda_n := \{-n, -n+2, \ldots, n-2, n\} \). Then there is a bijection

\[
f : \text{Bip}_1(n) \rightarrow \Lambda_n, \quad (a) \mapsto a - b.
\]
Let \( < \) (also) denote the order on \( \Lambda_n \) induced by \( f \), that is, for \( \lambda, \mu \in \Lambda_n \), \( \lambda < \mu \) iff \( f^{-1}(\lambda) < f^{-1}(\mu) \).

Note that \( \Lambda_n \) is the parameterizing set for the quasi-hereditary category \( b_n\text{-mod} \) of \( b_n \)-modules. The hereditary order is given by \( \lambda <_{qh} \mu \) iff \( |\lambda| > |\mu| \) for \( \lambda, \mu \in \Lambda_n \). We now have the following result.

**Theorem 5.** a) \( \text{Bip}_1(n) \) is a coideal in \( \text{Bip} \) with respect to \( < \).

b) The order \( < \) on \( \Lambda_n \) is a refinement of \( <_{qh} \).

**Proof.** In [CL] a bijection \( \mathcal{Q} : \text{SDT} \to \text{SBT} \) is described. It induces \( \mathcal{Q} : \text{Par}_0(2n) \to \text{Bip}(n) \) by taking shapes. One then checks the following formulas

\[
\mathcal{Q}^{-1} : (a), (b) \mapsto (2a, 2b) \quad \text{for} \ a \geq b
\]

\[
\mathcal{Q}^{-1} : (a), (b) \mapsto (2b - 1, 2a + 1) \quad \text{for} \ a > b
\]

We deduce that \( \mathcal{Q}^{-1}(\text{Bip}_1(n)) \) consists of all partitions of \( 2n \) of at most two lines and thus \( \text{Bip}_1(n) \) indeed is a coideal in \( \text{Bip} \) with respect to \( < \) as claimed in a).

In order to prove b) we note that the above formulas give

\( (n), (\emptyset) \succ (\emptyset), (n) \succ (n - 1), (1) \succ (1), (n - 1) \succ (n - 2), (2) \succ \ldots \)

The last term is \( (\frac{n}{2}), (\frac{n}{2}) \) or \( (\frac{n-1}{2}), (\frac{n+1}{2}) \) depending on the parity of \( n \). The statement of b) follows from this. In fact we see that the only difference between \( < \) and \( <_{qh} \) is that \( -\lambda < \lambda \) if \( \lambda \in \Lambda_n \) and \( \lambda > 0 \), whereas they are noncomparable with respect to \( <_{qh} \).

\( \square \)

### 3. Cell theory in \( \mathcal{H}_n \)

Let \( \Gamma \) be a finitely generated free Abelian group containing the elements \( a, b \). We use exponential notation for the elements of \( \Gamma \), writing \( e^g \) for \( g \in \Gamma \).

Define \( q := e^a \) and \( \mathcal{Q} := e^b \). Let \( \mathcal{A} \) be the \( \mathbb{C} \)-algebra \( \mathcal{A} := \mathbb{C}[\Gamma] \). The Hecke algebra \( \mathcal{H}_n = \mathcal{H}_n(\mathcal{Q}, \mathcal{Q}) \) of type \( B \) is the \( \mathcal{A} \)-algebra generated by \( T_0, T_1, \ldots, T_{n-1} \) subject to the relations

\[
T_i T_{i-1} T_i = T_{i-1} T_i T_{i-1} \quad \text{for} \ i = 2, 3, \ldots, n - 1
\]

\[
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0
\]

\[
T_i T_j = T_j T_i \quad \text{for} \ \{i - j \} > 1
\]

\[
(T_i - q)(T_i + q^{-1}) = 0, \quad (T_0 - \mathcal{Q})(T_0 + \mathcal{Q}^{-1}) = 0
\]

The above definition of the Hecke algebra of type \( B \) is convenient for dealing with cell theory. It is the one used for instance in [Lu1], [Lu2] and [BGIL]. The frequently used ground ring in the literature \( \mathbb{C}[\mathcal{Q}, \mathcal{Q}^{-1}, q, q^{-1}] \) is obtained as a special case of the above by setting \( \Gamma := \mathbb{Z}a \oplus \mathbb{Z}b \). The Hecke algebra defined over this ground ring is denoted the generic Hecke algebra.

Assume that \( f : \Gamma \to \mathbb{C}^* \) is a group homomorphism. Then \( f \) extends canonically to an algebra homomorphism \( f : \mathcal{A} \to \mathbb{C} \) and we can define the specialized Hecke algebra \( \mathcal{H}_{n,f} := \mathcal{H}_n \otimes_\mathcal{A} \mathbb{C} \). For example \( f(g) = 1, \forall g \) gives the group algebra \( \mathcal{H}_{n,f} = \mathbb{C}W_n \).

Define elements \( C_i \) of \( \mathcal{H}_n \) by \( C_0 := T_0 - \mathcal{Q} \) and \( C_i := T_i - q \) for \( i = 1, 2, \ldots, n - 1 \). Let \( J_n \) be the following ideal of \( \mathcal{H}_n \)

\[
J_n := \langle C_1 C_2 C_1 - C_1, C_1 C_0 C_1 - [2] q C_1 \rangle
\]
where \([n]_x := x^{n-1} + x^{n-3} + \ldots + x^{-n+3} + x^{-n+1}\) is the usual Gaussian integer. We then define the Temperley-Lieb algebra of type \(B\) as

\[
TLB_n := \mathcal{H}_n/J_n
\]

In the case of the generic Hecke algebra, this definition already appears in [GL1] where \(TLB_n\) is also referred to as the blob algebra, but actually it differs slightly from the presentation of the blob algebra \(b_n\) that is used in eg. [MR] and [RH]. Let us be more specific about the relationship.

Let \(k\) be a field and assume that \(q \in k^\times\), \(q \neq 1, -1\) and \(m \in \mathbb{Z}\). In [RH] and other references \(b_n = b_n(q, m)\) is defined as the \(k\)-algebra on generators \(U_0, U_1, U_2 \ldots, U_{n-1}\) and relations

\[
\begin{align*}
U_i U_{i+1} U_i &= U_i \quad \text{for } i = 1, 2 \ldots, n-2 \\
U_{i+1} U_i U_{i+1} &= U_{i+1} \quad \text{for } i = 1, 2 \ldots, n-2 \\
U_i U_0 U_1 &= [m-1] U_1 \\
U_0^2 &= -[2] U_i \quad \text{for } i = 1, 2 \ldots, n-1 \\
U_0^2 &= -[m] U_0, \quad U_i U_j = U_j U_i \quad \text{for } |i-j| > 1
\end{align*}
\]

where \([a] = \frac{q^n - q^{-n}}{q - q^{-1}}\). The following Lemma relates this to \(TLB_n\).

**Lemma 3.** Suppose \(k := \mathbb{C}\). Assume \(q \in \mathbb{C}^\times \setminus \{1, -1\}\) and set \(Q := iq^m\). Define \(TLB_{n,q,Q} := TLB_n \otimes_A \mathbb{C}\) where \(\mathbb{C}\) is made into an \(A\)-algebra via \(f : \Gamma \rightarrow \mathbb{C}^\times\) such that \(f(a) = q, f(b) = Q\). Then the rules

\[
\begin{align*}
C_i &\mapsto U_i, \quad i = 1, 2, \ldots, n-1, \quad C_0 \mapsto i(q - q^{-1}) U_0
\end{align*}
\]

define an isomorphism \(g : TLB_{n,q,Q} \rightarrow b_n(q, m)\).

**Proof.** This is just a matter of verifying the relations. \(\square\)

For \(w \in W_n\) we define \(T_w := T_{i_1} T_{i_2} \ldots T_{i_N}\) where \(w = s_{i_1} s_{i_2} \ldots s_{i_N}\) is a reduced expression. By the relations, \(T_w\) is independent of the reduced expression. Moreover, \(T_w\) is invertible since \(T_i\) is invertible for all \(i\); indeed we have

\[
T_0^{-1} = T_0 - Q + Q^{-1}, \quad T_i^{-1} = T_i - q + q^{-1} \quad \text{for } i = 1, 2, \ldots, n-1
\]

The bar involution \(h \mapsto \overline{h}\) on \(\mathcal{H}_n\) is the ring automorphism given by

\[
T_w \mapsto T_w^{-1}, \quad q \mapsto q^{-1}, \quad Q \mapsto Q^{-1}
\]

Suppose now that we have fixed a total order \(<\) on \(\Gamma\), making it into an ordered group. Then Lusztig has associated a Kazhdan-Lusztig type basis \(C_w, w \in W_n\) to \(\mathcal{H}_n\). It is uniquely defined by the conditions

\[
\overline{C_w} = C_w, \quad C_w - T_w = \bigoplus_{w \in W_n} A_{>0} T_w
\]

where \(A_{>0} := \sum_{\gamma \in \Gamma, \gamma > 0} \mathbb{C} e^\gamma\).

Associated with the basis \(C_w\) there is a preorder \(\leq_L\) on \(W_n\), generated by \(y \leq_L w\) if \(C_y\) appears in the expansion of \(C_s C_y\) in the \(C_w\)-basis. The associated equivalence relation is denoted \(\sim_L\) and its classes left cells. Thus, \(z \sim_L w\) if \(z \leq w\) and \(w \leq z\). Similarly we define the preorders \(\leq_R\) and \(\leq_{LR}\) and the equivalence relations \(\sim_R\) and \(\sim_{LR}\). The associated classes are called right cells and two-sided cells.
We shall always assume that \( a \) and \( b \) are positive in \( \Gamma \) and so we get by the equations (5) the following formulas
\[
C_s^0 = T_0 - Q, \quad C_{s_i} = T_i - q \quad \text{for} \quad i = 1, 2, \ldots, n - 1
\]
In other words, we have that \( C_{s_i} = C_i \).

Assume that \( b \notin \{ a, 2a, \ldots, (n-1)a \} \). Let \( r \in \mathbb{N} \cup \{0\} \cup \{\infty\} \) be given by \( ra < b < (r + 1)a \) or \( r := \infty \) if \( b > (n - 1)a \). According to the conjectures of [BGIL], the representation theory of \( \mathcal{H}_n \) should only depend on \( \Gamma, a \) and \( b \) through \( r \).

Let us consider the following \( A \)-submodule of \( \mathcal{H}_n \):
\[
\mathcal{J}_n := \text{span}_A \{ C_w \mid w \notin W_b \}
\]

The following is the main Theorem of this section.

**Theorem 6.** Assume that \( r = 0 \) and assume that part \( c^+ \) of Conjecture A of [BGIL] is valid for \( r = 0 \). Then we have that \( \mathcal{J}_n = \mathcal{J}_n \).

**Proof.** Since \( c^+ \) is assumed to be true we have that \( \leq_{LR} \) is given by dominance order under domino insertion. Combining with Theorem 4 we get that \( \mathcal{J}_n \) is an ideal in \( \mathcal{H}_n \).

In order to show that \( \mathcal{J}_n \subset \mathcal{J}_n \) it is then enough to verify that the generators of \( \mathcal{J}_n \) belong to \( \mathcal{J}_n \). Now we have
\[
C_1 C_2 C_1 = (T_1 - q)(T_2 - q)(T_1 - q) = T_1 T_2 T_1 - q T_1 T_2 - q T_2 T_1 + q^2 T_1 + q^2 T_2 - q^3 + T_1 - q
\]
and hence
\[
C_{s_1 s_2 s_1} = C_1 C_2 C_1 - C_1
\]
On the other hand, \( P(s_1 s_2 s_1) \) has the following form
\[
\begin{array}{cccccc}
1 & 4 & 5 & \cdots & a \\
2 \\
3 
\end{array}
\]
and so \( C_{s_1 s_2 s_1} \in \mathcal{J}_n \). Similarly we have
\[
C_1 C_0 C_1 = (T_1 - q)(T_0 - Q)(T_1 - q) = T_1 T_0 T_1 - q T_0 T_1 + Q q^{-1} T_1 - Q - q T_1 T_0 + q^3 T_0 + q Q T_1 - q^2 Q
\]
But \( -a + b < 0 \) and so \( Q q^{-1} \notin A_{>0} \) and we must subtract \( [2] q^{-1} \) \( C_1 \) to get \( C_{s_1 s_0 s_1} \). Hence
\[
C_{s_1 s_0 s_1} = C_1 C_0 C_1 - [2] q^{-1} C_1
\]
But \( P(s_1 s_0 s_1) \) is as follows
\[
\begin{array}{cccccc}
1 & 3 & 4 & \cdots & a \\
2 
\end{array}
\]
and so indeed \( C_{s_1 s_0 s_1} \in \mathcal{J}_n \).

Let \( K \) be the kernel of the projection map \( \pi : \mathcal{H}_n/\mathcal{J}_n \to \mathcal{H}_n/\mathcal{J}_n \). We need to show that \( K = 0 \). Since \( \pi \) is surjective, it is enough to prove that \( \mathcal{H}_n/\mathcal{J}_n \) and \( \mathcal{H}_n/\mathcal{J}_n \) are free over \( A \) of the same rank.
It is shown in Proposition 2.1 of [CGM] that $TLB_n = H_n/J_n$ is free, at least over the ground ring $\mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$, but the proof is easily seen to work over $A$ as well. The rank of $TLB_n$ is given by the cardinality of the diagram basis and can also be read off from the Bratelli diagram for $TLB_n$. It is
\[ \text{rank } H_n/J_n = \sum_{i=0}^{n} \binom{n}{i}^2 \]
On the other hand, since $\{C_w\}$ is a basis of $H_n$ we have that $H_n/J_n$ is free over $A$ with rank
\[ \text{rank } H_n/J_n = |W_b| \]
Recall the bijection $\mathcal{Q} : SDT \to SBT$ from [CL]. By the proof of Theorem 5, it restricts to a bijection between standard domino tableaux in $STD(n)$ with less than two lines and one-line standard bitableaux with shape in $\text{Bip}_1(n)$. The number of pairs of one-line bitableaux of shape $(i, n-i)$ is $\binom{n}{i}^2$ and so we conclude that $\text{rank } H_n/J_n = \text{rank } H_n/J_n$, as needed. □

**Remark.** Recall that $c^+$ of Conjecture A of [BGIL] is the statement
\[ y \leq \mathcal{LR} w \iff Sh(y) \leq Sh(w) \]
It is useful to observe that for the above proof to work, actually only '$\implies$' is needed.

**Corollary 2.** Assume that $\Gamma = \mathbb{Z}$ with the standard order and that $b = 1$ and $a = 2$. Then $J_n = J_n$.

**Proof.** By Remark 4.1 of [BJ], which on the other hand relies on [Lu1], we get that $c^+$ of Conjecture A of [BGIL] is valid under the assumptions. We then apply the Theorem. □

In order to apply the Corollary, we shall from now on assume that $\Gamma := \mathbb{Z}$ with the standard order, and that $b := 1$, $a = 2$. Although this does not cover all of the $r = 0$ case of [BGIL] we shall, somewhat misleadingly, refer to it that way.

We need both versions of the blob algebra. Hence, in order for Lemma 3 and the Corollary to work we impose the following conditions on $q, Q$
\[ q \neq \pm 1, q^l = 1, Q := iq^m, q = -q^{2m} \] (6)
These conditions will be satisfied for example if $q$ is a primitive $l$’th root of unity such that $l = 2(2m - 1)$.

We choose from now on $q, Q, m, l$ satisfying (6). We use the notation $H_{n,q,Q}$ for the specialized Hecke algebra $H_f$ with respect to these choices.

**Corollary 3.** Let $J_{n,q,Q}$ denote the canonical image of $J$ in $H_{n,q,Q}$. Then we have $TLB_{n,q,Q} = H_{n,q,Q}/J_{n,q,Q} = b_n(q, m)$.

**Proof.** This follows from the Theorem and Lemma 3. □
4. Representation theory

In this section we use the results of the previous sections to study the representation theory of $b_n$. Our main result is that the cell modules in the $r=0$ case are the standard modules for $b_n$.

Recall that $[2] \neq 0$ so that we can define $e = -\frac{1}{[2]}U_{n-1}$. This is an idempotent of $b_n$, and we have that $eb_ne \cong b_{n-2}$. Hence it gives rise to the localization functor

$$F : b_n\text{-mod} \to b_{n-2}\text{-mod}, \quad M \mapsto eM$$

$F$ is exact, it has as left adjoint functor the globalization functor $G$

$$G : b_{n-2}\text{-mod} \to b_n\text{-mod}, \quad M \mapsto b_ne \otimes_{b ne} M$$

Recall that $\Lambda_n := \{-n, -n + 2, \ldots, n - 2, n\}$ is the parameterizing set for the quasi-hereditary category $b_n\text{-mod}$. Let $\Delta_n(\lambda) \in b_n\text{-mod}$ denote the standard module associated with $\lambda \in \Lambda$. We have that

$$F\Delta_n(\lambda) \cong \begin{cases} 
\Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_n \setminus \{\pm n\} \\
0 & \text{otherwise}
\end{cases}$$

and $\Delta_n(\pm n) \cong L_n(\pm n)$ where $L_n(\lambda)$ is the simple module given by $\lambda$. This implies the universal property for $\Delta_n(\lambda)$ as the projective cover of $L_n(\lambda)$ in the truncated subcategory of $b_n\text{-mod}$ consisting of modules with composition factors of the form $L_n(\mu)$ with $\mu \leq \lambda$.

Let now $w_n \in W_n$ and denote by $\mathcal{C} = \mathcal{C}_{w_n} \subseteq W_n$ its left cell. Consider the following ideals of $\mathcal{H}_n$

$$I_{\leq \mathcal{C}} w_n := \text{span}_\mathbb{C}\{C_{w} | w \leq_L w_n\}, \quad I_{< \mathcal{C}} w_n := \text{span}_\mathbb{C}\{C_{w} | w \leq_L w_n, w \not\in \mathcal{C}\}$$

and define the cell module

$$\mathcal{V}_{w_n} := I_{\leq \mathcal{C}} w_n/I_{< \mathcal{C}} w_n$$

Since conjecture A of [BGIL] is true in the $r=0$ case, we get by the results of the previous section that $\mathcal{V}_{w_n}$ is a $b_n$-module. A basis for $\mathcal{V}_{w_n}$ is given by the classes of $C_w$ for $w \in \mathcal{C}$.

Recall from the previous sections that $W_n$ is realized as the subgroup of the symmetric group on the elements $I_n$ generated by $s_0 := (-1, 1)$ and $s_i := (i, i+1)(-i, -i-1)$. Let us denote by $\iota$ the associated injection of groups $\iota : W_n \to S_{I_n} = S_{2n}$:

$$\iota(s_0) = (1, -1), \quad \iota(s_i) = (i, i+1)(-i, -i+1)$$

According to [Lu1], each left cell $\mathcal{C}$ of $W_n$ is now of the form $\mathcal{C} = \iota^{-1}(\overline{\mathcal{C}}) = \overline{\mathcal{C}} \cap W_n$ where $\overline{\mathcal{C}}$ is a left cell of $S_{I_n}$; this relies heavily on $r = 0$.

The left cells on $S_{I_n} = S_{2n}$ can be described using the usual Robinson-Schensted correspondence when we use the natural order on $I_n$, that is

$$\overline{\pi} < \ldots < \overline{\tau} < 1 < 2 < \ldots < n.$$ 

We need the following Lemma.
Lemma 4. Let $\mathcal{C}$ be a left cell in $W_n$. Assume that $\mathcal{C} \subseteq W_b$ and that $\mathcal{C} = \bar{\mathcal{C}} \cap W_n$ where $\bar{\mathcal{C}}$ is a left cell of $S_{I_n}$. Then under the Robinson-Schensted bijection on $S_{I_n}$ with respect to the above order on $I_n$, $\bar{\mathcal{C}}$ corresponds to a tableau in at most two lines.

Proof. Let $P'$ and $Q'$ denote the $P$ and $Q$-parts of the Robinson-Schensted correspondence on $S_{2n}$. For $z, z_1 \in \bar{\mathcal{C}}$ we have $Q'(z) = Q'(z_1)$ and $P'(z)$ and $Q'(z)$ have the same shape. Assume now that $w \in W_b$ and write it in word form as $w = i_1 i_2 \ldots i_n$ with $i_j \in I_n$. We then have

$$\iota(w) = \overline{w}^{op} w$$

where $\overline{w}^{op} := \overline{w}_n \overline{w}_{n-1} \ldots \overline{w}_1$ and so $P'(\iota(w)) = P'(\overline{w}^{op} w)$.

We now appeal to the description of $W_b$ given in Theorem 1. Using it, there are no decreasing subsequences of $\overline{w}^{op} w$ of length three or more, and thus $P'(\overline{w}^{op} w)$ has at most two lines, as claimed.

The figure illustrates $\overline{w}^{op} w$ where $l = 4$.

\[\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\end{array}\]

Lemma 5. a) Assume that $U_{n-1} C_{w_n} \neq 0$. Then there exists $w_{n-2} \in \mathcal{C}_{w_n} \cap W_{n-2}$ and a scalar $a \in \mathbb{C} \setminus \{0\}$ such that $U_{n-1} C_{w_n} = a U_{n-1} C_{w_{n-2}}$.

b) Assume $U_i C_{w_n} \in \mathcal{V}_{w_n} \setminus \{0\}$ for some $i > 0$. Then there exists $z \in \mathcal{C}_{w_n}$ and a scalar $a \in \mathbb{C} \setminus \{0\}$ such that $U_i C_{w_n} = a C_z$.

c) Assume that $U_{n-1} \mathcal{V}_{w_n} = 0$. Then $U_i \mathcal{V}_{w_n} = 0$ for all $i > 0$. Moreover $\mathcal{V}_{w_n} \simeq \Delta_n(\pm n)$, specially dim $\mathcal{V}_{w_n} = 1$.

Proof. Take $w_n \in \mathcal{C}_{w_n} = \mathcal{C}$ and let $C_{w_n} \in \mathcal{H}_n$ be the associated Kazhdan-Lusztig element. Then we have

$$U_{n-1} C_{w_n} = C_{s_{n-1} C_{w_n}} = \sum_{z \in \mathcal{W}_n} N_{n-1, w_n, z} C_z \quad (8)$$

where $N_{n-1, w_n, z}$ are the structure constants in $\mathcal{H}_n$ with respect to the $C$-basis. Let $\mathcal{H}_{2n}$ be the Hecke algebra associated to $S_{2n}$, with parameter $q$, and let us denote by $\tilde{C}_w$ the usual one-parameter Kazhdan-Lusztig element.
for \( w \in S_{2n} \). If \( w \in W_n \) we write \( \tilde{C}_w := \tilde{C}_{\iota(w)} \). Then we have

\[
\tilde{C}_{s_{n-1}} \tilde{C}_{w_n} = \sum_{z \in S_{2n}} \tilde{N}_{n-1,w_n,z} \tilde{C}_z
\]

(9)

where \( \tilde{N}_{n-1,w_n,z} \) are the structure constants in \( \mathcal{H}_{2n} \) with respect to its \( \tilde{C} \)-basis. Lusztig shows in this setting in [Lu1] that

\[
\text{if } z \in W_n \text{ and } N_{n-1,w_n,z} \neq 0 \text{ then } \tilde{N}_{n-1,w_n,z} \neq 0
\]

(10)

Now we have

\[
\tilde{C}_{s_{n-1}} = (T_{(n-1,n)} - q)(T_{(-n+1,-n)} - q) = U_{(n-1,n)} U_{(-n+1,-n)}
\]

Reducing (8) modulo \( I_{\leq C} w_n \) we get the corresponding equation in \( \mathcal{V}_{w_n} \):

\[
U_{n-1} C_{w_n} = \sum_{z \in \mathcal{C}} N_{n-1,w_n,z} C_z \text{ modulo } I_{\leq C} w_n
\]

(11)

But \( \mathcal{C} = \tilde{\mathcal{C}} \cap W_n \) and so by (10) any \( z \) occurring in this sum with \( N_{n-1,w_n,z} \neq 0 \) gives a nonzero \( \tilde{N}_{n-1,w_n,z} \) in

\[
\tilde{C}_{s_{n-1}} \tilde{C}_{w_n} = U_{(n-1,n)} U_{(-n+1,-n)} \tilde{C}_{w_n} = \sum_{z \in S_{2n}} \tilde{N}_{n-1,w_n,z} \tilde{C}_z \text{ modulo } \mathfrak{J}
\]

(12)

where

\[
\mathfrak{J} := \text{span}_C \{ \tilde{C}_w | w \in S_{2n}, \ w \leq_L \tilde{\mathcal{C}}, \ w \notin \tilde{\mathcal{C}} \}
\]

But using the previous Lemma we may consider (12) as an equation in a cell module \( \Delta_{2n}(k) \) for the Temperley-Lieb algebra \( TL_{2n} \).

Let us now show a). We have \( N_{n-1,w_n,z} \neq 0 \) and so \( \tilde{N}_{n-1,w_n,z} \neq 0 \). But by [FG] we know that \( \tilde{C}_z = U_\iota(z) \) modulo \( \mathfrak{J} \), where as usual \( U_w := U_{i_1} \ldots U_{i_r} \) for \( w = s_{i_1} \ldots s_{i_r} \). Using the diagram presentation of \( \Delta_{2n}(k) \) we now deduce that \( \iota(z) = s_{(-n+1,-n)} z_1 s_{(n-1,n)} \) where \( z_1 \in S_{I_n-2} \) and hence

\[
z_1 = s_{(-n+1,-n)} \iota(z) s_{(n-1,n)} = \iota(z s_{n-1}) \in \iota(W_n) \cap S_{I_n-2} = \iota(W_n-2)
\]

and a) is proved.

We then show b). For each \( z \) with \( N_{n,w_n,z} \neq 0 \) we have by (10) that \( \tilde{N}_{n-1,w_n,z} \neq 0 \). But using [FG] once more, at most one \( z \) can give \( \tilde{N}_{n-1,w_n,z} \neq 0 \), proving b).

Let us then show c). By the previous sections, \( \mathcal{V}_{w_n} \) is a module for \( b_n \). Since \( F \mathcal{V}_{w_n} = U_{n-1} \mathcal{V}_{w_n} = 0 \), it follows from the general representation theory of \( b_n \) that

\[
\mathcal{V}_{w_n} = \Delta_n(n)^k \oplus \Delta_n(-n)^l
\]

for certain multiplicities \( k,l \). Since \( \mathcal{V}_{w_n} \) is a cell module, the products \( C_{s_{i_1}} C_{s_{i_2}} \ldots C_{s_{i_k}} C_{w_n} \) generate \( \mathcal{V}_{w_n} \). But by assumption only \( C_{s_0} C_{w_n} = U_0^k C_{w_n} \in \mathcal{V}_{w_n} \) can be nonzero and since \( U_0^k \) is a scalar multiple of \( U_0 \) we conclude that \( k = 1 \), \( l = 0 \) or \( k = 0 \), \( l = 1 \) and so \( \dim \mathcal{V}_{w_n} = 1 \). The Lemma is proved.

We are now in position to prove our main Theorem.
Theorem 7. Assume that $q$ is a primitive $l$th root of unity such that $q = -q^{2m}$ and $Q := iq^m$. Let $\mathfrak{c} = \mathfrak{c}_{w_n}$ be a left cell for $W_n$ and let $\mathcal{V}_{w_n}$ be the corresponding cell module. Then we have an isomorphism of $b_n$-modules

$$\mathcal{V}_{w_n} \simeq \Delta_n(\lambda)$$

where $\lambda = a - b$ for $Q(Sh(w_n)) = (a), (b)$.

Proof. Assume that $F\mathcal{V}_{w_n} \neq 0$ and consider the adjointness map $\varphi = \varphi_{w_n} : G \circ F\mathcal{V}_{w_n} \to \mathcal{V}_{w_n}$. It is given concretely by multiplication

$$\varphi : b_n e \otimes_{eb_n e} \mathcal{V}_{w_n} \to \mathcal{V}_{w_n}, \quad U \otimes ev \mapsto U ev$$

Using b) of the previous Lemma and the definition of left cells, we see that $\varphi$ is surjective.

We now prove that $\ker \varphi$ is zero. Recall from [MR] that $U_i U_{i+1} \ldots U_{n-1}$, where $i = 0, 1, \ldots, n - 1$ generate $b_n e$ as an $eb_n e$-module. Using this and part a) of the previous Lemma we can write any $k \in b_n e \otimes_{eb_n e} \mathcal{V}_{w_n}$ in the form

$$k = \sum_{i=0,1,\ldots,n-1} \sum_{w_{n-2} \in \mathfrak{c}_{w_n} \cap W_{n-2}} \lambda_{i,w_{n-2}} U_i U_{i+1} \ldots U_{n-1} \otimes_{eb_n e} U_{n-1} C_{w_{n-2}}$$

where $\lambda_{i,w_{n-2}} \in \mathbb{C}$. Since $U_{n-1}$ and $C_{w_{n-2}}$ commute we have

$$U_{n-1} C_{w_{n-2}} = -\frac{1}{[2]} U_{n-1} C_{w_{n-2}} U_{n-1} = -\frac{1}{[2]} e C_{w_{n-2}} e$$

Assume now that $k \in \ker \varphi$. We then get

$$k = -\frac{1}{[2]} \sum_{i=0}^{n-2} \sum_{w_{n-2}} \lambda_{i,w_{n-2}} U_i U_{i+1} \ldots U_{n-1} \otimes_{eb_n e} e C_{w_{n-2}} e = -\frac{1}{[2]} \sum_{i=0}^{n-2} \sum_{w_{n-2}} \lambda_{i,w_{n-2}} U_i U_{i+1} \ldots U_{n-1} U_{n-1} C_{w_{n-2}} \otimes_{eb_n e} U_{n-1}$$

which is zero since $k \in \ker \varphi$. This proves that $\varphi$ is an isomorphism.

Using a) of the previous Lemma once again, we now deduce that

$$F\mathcal{V}_{w_n} \simeq \mathcal{V}_{w_{n-2}} \text{ for } w_n = w_{n-2}s_{n-1}, w_{n-2} \in \mathfrak{c}_{w_n} \cap W_{n-2}$$

By Corollary 3.8 of [BGIL], $w_{n-2}$ is independent of the choice of $w_n$. Under domino insertion, $Sh(w_n)$ is obtained from $Sh(w_{n-2})$ by adding two horizontal dominoes, one at the end of each line. Hence, using the formulas for $Q$ given in the proof of Theorem 5, we find that

$$Q(Sh(w_{n-2})) = (a - 1), (b - 1) \text{ if } Q(Sh(w_n)) = (a), (b)$$

and hence the difference is the same.

If $F\mathcal{V}_{w_n} = 0$ we get by c) of the previous Lemma that $\mathcal{V}_{w_n} \simeq \Delta_n(\pm n)$ and hence that $\dim \mathcal{V}_{w_n} = 1$. But then the combinatorial description of left cells in terms of domino tableaux gives $w_n = 1$ or $w_n = s_0$. For $w_n = 1$ we have $Q(Sh(w_n)) = (n), (\emptyset)$ whereas for $w_n = s_0$ we have $Q(Sh(w_n)) = (\emptyset), (n)$, compatible with the actions of $U_0$ in $\mathcal{V}_{w_n}$. The Theorem is proved.

Remark. We think that the Theorem is valid for more general choices of $q$ and $Q$ within $r = 0$. 
5. The Fock space

In this section we give two applications of Theorem 7 that both rely on the Fock space approach to the representation theory of $\mathcal{H}_n$. The first gives a new proof of the main results of [MW] using Ariki’s Theorem and the second settles the question of determining the Kleshchev bipartition that corresponds to the simple $b_n$-module $L_n(\lambda)$. To set this up we first need the following Theorem.

**Theorem 8.** In the Grothendieck group of $b_n$-modules the equality $\Delta_n(\lambda) = S_n(k, l)$ holds where $\lambda = k - l$ and $S_n(k, l)$ is the Dipper-James-Murphy Specht module for $\mathcal{H}_n$ corresponding to the bipartition $(k), (l)$.

**Proof.** This follows basically from Theorem 3 and Theorem 6 of [RH]. On the other hand, since $[RH]$ is based on a realization of $b_n$ as a quotient of the Ariki-Koike algebra $\mathcal{A}\mathcal{K}_n(\lambda_1, \lambda_2, q)$ and a realization of $S_n(k, l)$ as a permutation module in the Ariki-Yamada-Terasoma tensor space for $\mathcal{A}\mathcal{K}_n(\lambda_1, \lambda_2, q)$, we still give a few details on how to convert from one situation to the other.

Recall that we have $TLB_{n,q,Q} = \mathcal{H}_{n,q,Q}/J_{n,q,Q}$ where $J_{n,q,Q}$ is the image of

$$J_n = \langle C_1C_2C_1 - C_1, \, C_1C_0C_1 - [2]Q^qC_1 \rangle$$

in the specialized Hecke algebra. By Lemma 3 this algebra is also isomorphic to the blob algebra $b_n(q, m)$, that on the other hand was realized in [RH] as $\mathcal{A}\mathcal{K}_n(\lambda_1, \lambda_2, q)/G_n$ where $\lambda_1 = \frac{q^m}{q - q}, \lambda_2 = \frac{q^{-m}}{q - q}$ and $G_n$ is the ideal of $\mathcal{A}\mathcal{K}_n(\lambda_1, \lambda_2, q)$ generated by $(X_1X_2 - \lambda_1\lambda_2)(g_1 - q)$. The last realization requires the conditions $q^d \neq 1, \lambda_1 \neq \lambda_2, \lambda_1 \neq q^2\lambda_2$ and these conditions are imposed throughout [RH].

But instead of converting directly between the two settings by choosing appropriate $\lambda_1, \lambda_2$, we prefer to proceed as follows.

The Hecke algebra $\mathcal{H}_n$ is an Ariki-Koike with parameters $\lambda_1 = Q, \lambda_2 = -1/Q$ and so we can develop the theory of [RH] entirely from the $\mathcal{H}_n$ point of view, once we have proved that $J_n$ acts trivially in the Ariki-Terasoma-Yamada tensor space $V^\otimes n$ when dim $V = 2$, corresponding to Theorem 1 of [RH].

Let us therefore prove how the analogue of Theorem 1 of [RH] is proved. Let $V$ be a complex vector space of dimension two and let $v_1, v_2$ be a basis. Let $R \in \text{End}_C(V \otimes V)$ be given by

$$R(v_i \otimes v_j) = qv_i \otimes v_j \quad \text{if } i = j$$

$$R(v_2 \otimes v_1) = v_1 \otimes v_2$$

$$R(v_1 \otimes v_2) = v_2 \otimes v_1 + (q - q^{-1})v_1 \otimes v_2$$

For $i = 1, 2, \ldots, n - 1$, we let $T_i \in \mathcal{H}_n$ act in the tensor space $V^\otimes n$ by

$$T_i := Id^{\otimes i-1} \otimes R \otimes Id^{\otimes n-i-1}$$

For $v = v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_k} \otimes v_{i_{k+1}} \ldots \otimes v_{i_n}$, we define $S_k \in \text{End}_C(V^\otimes n)$ by

$$S_k(v) = \begin{cases} qv_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_{k-1}} \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \ldots \otimes v_{i_n} & \text{if } i_{k-1} = i_k \\ v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_k} \otimes v_{i_{k-1}} \otimes \ldots \otimes v_{i_n} & \text{otherwise} \end{cases}$$
and let $\varpi \in \text{End}_C(V^\otimes n)$ be given by
\[
\varpi(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := \begin{cases} Qv & \text{if } i_1 = 1 \\ -Q^{-1}v & \text{if } i_1 = 2 \end{cases}
\]
Then $T_0$ acts in the Ariki-Terasoma-Yamada tensor space of [ATY] through
\[
T_0 := T_1^{-1} \ldots T_{n-2}^{-1}T_{n-1}^{-1}S_{n-1}S_{n-2} \ldots S_1 \varpi.
\]
Let us now show that the ideal $J_n$ is annihilated under this action. This is well-known for the generator $C_1 C_2 C_1 - C_1$ so we concentrate on $C_1 C_0 C_1 - [2] C_1$. Since $C_1$ acts semisimply in $\text{span}\{ v_i \otimes v_j \mid i, j = 1, 2 \}$ with eigenvalue 0 of multiplicity three and eigenvalue $-2$ of multiplicity one, it is enough to check the relation on vectors of the form $C_1 v$ where $v = v_2 \otimes v_1 \otimes v_3 \otimes \ldots \otimes v_n$ since $C_1 v \neq 0$ for such $v$. But $C_1 v = (v_1 \otimes v_2 - q v_2 \otimes v_1) \otimes v_3 \otimes \ldots \otimes v_n$ is an eigenvector for $C_1$ of eigenvalue $-2$ and hence it is enough to show that
\[
C_1 C_0 (v_1 \otimes v_2 - q v_2 \otimes v_1) \otimes v_n = [2] q (v_1 \otimes v_2 - q v_2 \otimes v_1) \otimes v_n
\]
where $v = v_1 \otimes \ldots \otimes v_n$. Let us consider the left hand side of this equation. Using Lemma 1 of [RH], which is a reformulation of a result of [ATY], we find that
\[
C_1 C_0 q v_2 \otimes v_1 \otimes v_n = q^2 (Q + Q^{-1}) C_1 v_1 \otimes v_2 \otimes v_n
\]
We then consider $C_1 C_0 v_1 \otimes v_2 \otimes v_n$ which we rewrite as follows
\[
C_1 C_0 v_1 \otimes v_2 \otimes v_n = (T_1 - q)(T_0 - Q) v_1 \otimes v_2 \otimes v_n = (T_1 - q) T_0 v_1 \otimes v_2 \otimes v_n - Q C_1 v_1 \otimes v_2 \otimes v_n
\]
We here consider the first term $(T_1 - q) T_0 v_1 \otimes v_2 \otimes v_n$ which we rewrite as follows
\[
(T_1 - q) T_0 v_1 \otimes v_2 \otimes v_n = -q (T_1 - q) T_1 T_0 v_1 \otimes v_2 \otimes v_n = -q Q(T_1 - q) v_2 \otimes v_1 \otimes v_n - q^2 Q C_1 v_1 \otimes v_2 \otimes v_n
\]
where we for the second equality used the argument given in the proof of Theorem 1 of [RH]. Summing up, the LHS of (13) equals
\[
(-Q - q^2 Q^{-1}) C_1 v_1 \otimes v_2 \otimes v_n
\]
which coincides with the RHS.

We can now develop the theory of [RH] from the Hecke algebra point of view. Especially, for $\lambda \in \Lambda_n$ we define the permutation module
\[
M_n(\lambda) := \text{span}_C\{ v_1 \otimes v_2 \otimes \ldots \otimes v_n \mid \# \{ k : i_k = 1 \} - \# \{ k : i_k = 2 \} = \lambda \}
\]
and get that $M_n(\lambda)$ satisfies the functorial properties for $F$ of (7).

Theorem 3 of [RH] is proved by induction. One checks that the inductive step works for all choices of the parameters satisfying $\lambda_1 \neq \lambda_2$. But $\lambda_1 = Q = iq^m$ and $\lambda_2 = -Q^{-1} = iq^{-m}$ and so we have $\lambda_1 / \lambda_2 = q^{-2m} = -q \neq 1$, as needed. The induction basis is based on Lemma 3 of [RH]. The proof of that Lemma works provided that $\lambda_1 (q - q^{-1}) \neq q (\lambda_1 - \lambda_2)$. But this is equivalent to $-q \neq q^2$ that is $q \neq -1$, as needed.

Finally the proof of Theorem 6 of [RH] claiming that $M_n(\lambda) \cong S_n(k,l)^{\otimes}$ is independent of the choices of the parameters and goes directly over. But in the Grothendieck group of $b_n$-modules, $S_n(k,l)$ is equal to its contragredient dual $S_n(k,l)^{\otimes}$, and so the proof of the Theorem is finished. \qed
Remark. In view of Theorem 7, an alternative proof might have been obtained using the results of section 4 of \cite{P1}.

Remark. At this point we may remark that combining Theorem 7 with Lemma 2 of \cite{RH}, we get many examples of cells modules for different choices of \( r \) that are not isomorphic. Indeed Lemma 2 of \cite{RH} gives many examples of the adjointness map \( G \circ F M_n(\lambda) \to M_n(\lambda) \) failing to be an isomorphism. Note that the condition in that Lemma 2, that \( q \) be an odd order root of unity, is not needed for showing that the adjointness map is not surjective – as is indeed mentioned in the proof of that Lemma 2.

We now recall the Fock space approach to the representation theory of \( \mathcal{H}_n \). Let \( s = (s_1, s_2) \in \mathbb{Z}^2 \) and let \( \mathfrak{F}^{s} \) be the associated Fock space of level two. As a \( \mathbb{C}(v) \)-vector space it is given by

\[
\mathfrak{F}^{s} = \bigoplus_{\lambda \in \text{Bip}} \mathbb{C}(v) | \lambda, s \rangle
\]

where \( | \lambda, s \rangle \) is a symbol. Let us briefly recall how it becomes an integrable module for the quantum group \( U_q(\hat{sl}_n) \) where \( e = l/2 \). Since \( U_q(\hat{sl}_n) \) is the \( \mathbb{C}(v) \)-algebra generated by \( e_i, f_i, i = 0, 1, \ldots, e - 1 \) and \( k_h, h \in \mathfrak{h} \) subject to certain well-known relations, it is enough to explain how these generators act in \( \mathfrak{F}^{s} \).

To any bipartition \((\lambda^{(1)}, \lambda^{(2)})\) we associate its diagram

\[
\{(i, j, c) \mid c = 1, 2 \text{ and } 1 \leq j \leq \lambda_{c}^{(i)}\}
\]

For a node \( \gamma = (i, j, c) \) of \((\lambda^{(1)}, \lambda^{(2)})\) we define its \( c \)-residue by \( \text{res}_c(\gamma) = j - i + s_c \mod e \). We define a total order on the nodes of \((\lambda^{(1)}, \lambda^{(2)})\) by \( \gamma = (i, j, c) < \gamma' = (i', j', c') \) if \( j - i + s_c < j' - i' + s_{c'} \) or if \( j - i + s_c = j' - i' + s_{c'} \) and \( c' < c \) (notice this last inequality!). If \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) and \( \mu = (\mu^{(1)}, \mu^{(2)}) \) are bipartitions such that \( \lambda \subseteq \mu \) and \( \gamma = \mu \setminus \lambda \) is an \( i \)-node we say that \( \gamma \) is a removable \( i \)-node of \( \mu \) and an addable \( i \)-node of \( \lambda \) and we set

\[
N_{i}^{>}(\lambda, \mu) := \#\{ \text{addable } i \text{-nodes } \gamma' \text{ of } \lambda \text{ such that } \gamma' > \gamma \}
\]

\[
N_{i}^{<}(\lambda, \mu) := \#\{ \text{removable } i \text{-nodes } \gamma' \text{ of } \lambda \text{ such that } \gamma' > \gamma \}
\]

\[
N_{i}^{<}(\lambda, \mu) := \#\{ \text{addable } i \text{-nodes } \gamma' \text{ of } \lambda \text{ such that } \gamma' < \gamma \}
\]

\[
N_{i}^{<}(\lambda, \mu) := \#\{ \text{removable } i \text{-nodes } \gamma' \text{ of } \lambda \text{ such that } \gamma' < \gamma \}
\]

The actions of \( f_i, e_i \) on a basis vector of \( \mathfrak{F}^{s} \) are now as follows

\[
f_i | \lambda, s \rangle = \sum_{\mu, \text{res}(\mu, \lambda) = i} v^{N_{i}^{>}(\lambda, \mu)} | \mu, s \rangle
\]

\[
e_i | \mu, s \rangle = \sum_{\lambda, \text{res}(\mu, \lambda) = i} v^{-N_{i}^{<}(\lambda, \mu)} | \lambda, s \rangle
\]

There are similar formulas for the other generators. It is one of the important issues of the Fock space approach to the representation theory of \( \mathcal{H}_n \) that \( \mathfrak{F}^{s} \) with this action not only depends on the classes \( s_1 \mod e \) and \( s_2 \mod e \), but on \( s \) itself.

Let \( U_q(\hat{sl}_n) \to U_q(\hat{sl}_n), u \mapsto \overline{u} \) be the bar involution given by

\[
\overline{v} := v^{-1}, \quad \overline{f_i} := f_i, \quad \overline{e_i} := e_i, \quad \overline{k_h} := k_{-h}
\]

and let \( \mathfrak{F}^{s} \to \mathfrak{F}^{s}, x \mapsto \overline{x} \) be the bar involution of the Fock space constructed by Uglov in \cite{U}. It satisfies \( \overline{\emptyset, s} = \emptyset, s \rangle \) and is compatible with the bar
involution on \( \mathcal{U}_e(\tilde{\mathfrak{sl}_n}) \), that is \( u\overline{u} = \overline{u}u \) for \( u \in \mathcal{U}_e(\tilde{\mathfrak{sl}_n}) \) and \( x \in \mathfrak{g}^* \). By the results of [U] we get for \( \lambda \in \text{Bip} \) a unique \( G(\lambda, s) \in \mathfrak{g}^* \) such that
\[
G(\lambda, s) = G(\lambda, s) = G(\lambda, s) \equiv [\lambda, s] \mod v\mathbb{C}[v] \mathfrak{g}^*
\]
Write for \( \mu \in \text{Bip} \)
\[
G(\mu, s) = \sum_{\lambda \in \text{Bip}} d_{\lambda, \mu}^s(\nu) |\lambda, s\rangle
\]
Set \( \mathcal{M}[s] := \mathcal{U}_e(\tilde{\mathfrak{sl}_n}) |\emptyset, s\rangle \). Then \( \mathcal{M}[s] \) is an integrable module for \( \mathcal{U}_e(\tilde{\mathfrak{sl}_n}) \) and so the crystal/canonical basis theory applies to it. In fact, there is a subset \( \text{Bip}_e \subset \text{Bip} \) such that \( G(\lambda, s) \) for \( (\lambda, s) \in \text{Bip}_e \) is the canonical basis/global crystal basis of \( \mathcal{M}[s] \). Set \( \text{Bip}_e(n) := \text{Bip}_e \cap \text{Bip}(n) \). Assume that \( m \equiv s_1 - s_2 \). Then by the deep Theorem of Ariki in [A], we have that \( \text{Bip}_e(n) \) parameterizes the irreducible modules for \( \mathcal{H}_n \) with corresponding decomposition numbers \( d_{\lambda, \mu}^s(1) \).

The proof of our next Theorem is essentially the same as the proof of Theorem 4.7 of [BJ], but notice that Theorem 4.7 of [BJ] requires the validity of the Conjectures A, B and B’ of [BJ]. As already mentioned, Conjecture A holds in the \( r = 0 \) case whereas, as we shall see, we can replace Conjecture B by Theorem 7 and Conjecture B’ by our previous Theorem 8.

**Theorem 9.** Let \( m, l, e \) be as above and let \( p \) be the largest integer such that \( m + pe \leq 0 \) and set \( s := (m + pe, 0) \). Then for \( \mu \in \text{Bip}_e(n) \) we have
\[
G(\mu, s) = [\mu, s] + \sum_{\lambda \in \text{Bip}(n), \lambda \prec \mu} d_{\lambda, \mu}^s(\nu) |\lambda, s\rangle
\]
Moreover, identifying \( \tau = (t_1, t_2) \in \text{Bip}_1(n) \) with \( f(\tau) = t_1 - t_2 \in \Lambda_n \) we have for \( \lambda, \mu \in \text{Bip}_1 \) that
\[
[\Delta_n(\lambda), L_n(\mu)] = d_{\lambda, \mu}^s(1)
\]
**Proof.** By the choice of \( s \) we have formula (14) as in the proof of Theorem 4.7 of [BJ]. Notice now that \( m + pe \neq 0 \). Thus we have that \( \lambda \) and \( \mu \) of (15) are FLOTW bipartitions, that is they belong to \( \text{Bip}_e(n) \), see [BJ].

Take now \( \nu = (n_1, n_2) \in \text{Bip}_1(n) \) corresponding to \( \nu \in \Lambda_n \). According to Ariki’s Theorem there exists \( \mu \in \text{Bip}_e(n) \) such that the decomposition number \( d_{\lambda, \nu} := [S_n(\lambda), L_n(\nu)] \) satisfies
\[
[\Delta_n(\lambda), L_n(\nu)] = d_{\lambda, \nu} = d_{\lambda, \mu}^s(1)
\]
for all \( \lambda \in \text{Bip}(n) \) where we used Theorem 8 for the first equality. Setting \( \lambda = \nu \) we get that \( \nu \preceq \mu \) and setting \( \lambda = \mu \) we get that \( \mu \preceq \nu \). Hence \( \mu = \nu \) and the Theorem is proved.

The next step is now to calculate the numbers \( d_{\lambda, \mu}^s(1) \) for \( \lambda, \mu \in \text{Bip}(n) \). Uglov’s proof of the existence of \( G(\lambda, s) \) is not straightforward, but still constructive; notice that the algorithm has been simplified by Yvonne in [Y]. On the other hand, since we only focus on bipartitions in \( \text{Bip}_1(n) \) actually the properties of \( G(\lambda, s) \) already mentioned are sufficient to calculate \( G(\lambda, s) \) and hence \( d_{\lambda, \mu}^s(1) \).

Indeed, set \( m_- := -(m + (p + 1)e) \) and recall from [MW] that the choices of \( e \) and \( m \) determine an alcove geometry in \( \mathbb{R} \) with zero dimensional walls in the integral points \( \mathcal{M} := \{m_- + ke \mid k \in \mathbb{Z} \} \) and fundamental alcove \( A_0 \) being
the one that contains 0. The associated Weyl group \( W \) is infinite dihedral, generated by \( s_+ \) and \( s_- \) where \( s_+ \) (\( s_- \)) is the reflection in the right (left) wall of the fundamental alcove. Set \( \Lambda_n^{\text{reg}} := \Lambda_n \setminus \mathcal{M} \) and for \( \lambda \in \Lambda_n^{\text{reg}} \) write \( A_\lambda \) for the alcove containing \( \lambda \). For \( \lambda \in \Lambda_n^{\text{reg}} \) we define \( w_\lambda \in W \) by the condition \( w_\lambda A_0 = A_\lambda \). Thus \( w_\lambda < w_\mu \) in the Bruhat-Chevalley order implies \( \lambda > \mu \) in the quasi-hereditary order. We can now formulate the next Theorem. The second part of it was proved in [MW] using completely different methods.

**Theorem 10.** Let \( \lambda, \mu \in \Lambda_n^{\text{reg}} \). Then we have

\[
d_{\lambda,\mu}(v) = \begin{cases} 
  v^{|(w_\lambda) - l(w_\mu)|} & \text{if } w_\lambda \leq w_\mu \\
  0 & \text{otherwise}
\end{cases}
\]  

(16)

\[
[\Delta_n(\lambda), L_n(\mu)] = \begin{cases} 
  1 & \text{if } w_\lambda \leq w_\mu \\
  0 & \text{otherwise}
\end{cases}
\]  

(17)

**Proof.** Following [MW] we enumerate the elements of \( W \) as follows

\[
w_i = \begin{cases} 
  1 & \text{if } i = 0 \\
  s-s_+s_i \ldots (-i \text{ terms}) & \text{if } i < 0 \\
  s_+s_-s_+ \ldots (i \text{ terms}) & \text{if } i > 0
\end{cases}
\]

and define \( A_i := w_i A_0 \). Then, \( A_i \) is the alcove at distance \( i \) from \( A_0 \), positioned to the right if \( i \) is positive and to the left if \( i \) is negative.

Write \( s_1 := m + pe \) such that \( s = (s_1, 0) \). Set furthermore \( m_+ := m_- + e \). Then the fundamental alcove is limited by \( m_- \) and \( m_+ \). Assume now that \( \lambda = (k_1, k_2) \) belongs to \( A_i \cap \Lambda_n^{\text{reg}} \) with \( i \geq 0 \). Let \( r_1, r_2 \in \{0, 1, \ldots, e-1\} \) be the residues modulo \( e \) of \( k_1 + s_1 + k_2 \).

We now act with elements of the form \( f_{r_1+j} \ldots f_{r_1+1}f_{r_1} \) in \( |\lambda, s\rangle \) and consider the images in \( \mathfrak{F}^{s, \geq 2} := \mathfrak{F}^{s}/I^{\geq 2} \) where \( I^{\geq 2} := \text{span}\{\nu, s\} | \nu \notin \text{Bip}_1 \} \). These images move towards the right wall of \( A_i \). The wall will be reached when \( r_1 + j = r_2 \mod e \) and the image will be \( |\mu, s\rangle \) where \( \mu = (k_1 + r_1 - r_2, k_2) \), i.e. with \( v \) power equal to \( v^k \) since \( k \geq 0 \). Notice here that the wall \( m_+ \) of \( A_0 \) corresponds exactly to the second case in the definition of the order relation on the nodes.

In the formalism of translation functors, as exposed for example in [S], the process just described corresponds to translation upwards on the wall.

Acting with \( f_{r_1} \) in \( |\mu, s\rangle \) and considering the images in \( \mathfrak{F}^{s, \geq 2} := \mathfrak{F}^{s}/I^{\geq 2} \) the result is

\[
|\mu^{\text{up}}, s\rangle + v|\mu^{\text{down}}, s\rangle
\]

where \( \mu^{\text{up}} = (k_1 + 1, k_2 + r_1 - r_2) \) and \( \mu^{\text{down}} = (k_1, k_2 + r_1 - r_2 + 1) \) and once again we get correspondence with the translation functor formalism.

Similarly, we go through the other cases and find that translation upwards through the wall behaves as above whereas translation downwards through the wall \( |\mu, s\rangle \) is given by

\[
v^{-1}|\mu^{\text{up}}, s\rangle + |\mu^{\text{down}}, s\rangle
\]

where \( \mu^{\text{up}} \) and \( \mu^{\text{down}} \) are chosen analogously to the first case.

Using these rules, together with (14) and Theorem 5 it us now straightforward to calculate \( G(\lambda, s) \) modulo \( I^{\geq 2} \) for \( \lambda \in \text{Bip}_1 \) to obtain formula (16). Finally, formula (17) then follows from the previous Theorem. \( \square \)
Let us finish by mentioning another application of our results. Recall that the Kleshchev bipartitions are those of Bip_e where \( s = (d + qe, 0) \) and \( d + qe > n - 1 - e \), this is the so-called asymptotic case. The Kleshchev bipartitions give the simple modules when we use the Dipper-James-Murphy Specht modules to parameterize.

The question raised in [RH] of determining the Kleshchev bipartition \( \lambda = (l_1, l_2) \) that corresponds to the simple \( b_n \)-module with parameter \( \tau = (t_1, t_2) \) can now be solved by applying Kashiwara’s operators to the crystal graphs of the Fock spaces. Consider as an example \( e = 3, m = 2 \). Then \( s = (-1, 0) \).

In the crystal graph of \( M(-1,0) \) we have

\[
\tilde{f}_0\tilde{f}_1\tilde{f}_0\tilde{f}_2\tilde{f}_1\tilde{f}_0\tilde{f}_0\tilde{f}_2(\emptyset, \emptyset) = (6, 4)
\]

whereas the same sequence of crystal operators sends \((\emptyset, \emptyset)\) to \((6, 3), (1)\) in \( M(11,0) \).

Jacon has constructed in [Ja] an algorithm for converting between such crystal graphs. The following tables have been calculated using an implementation of his algorithm in the GAP system. They convert between the bipartitions in Bip_1(10) and the corresponding Kleshchev bipartitions, denoted KBip_1(10).

It can be seen that the correspondence between Bip_1(10) and KBip_1(10) works as the identity in the top \( m \) lines of all of these tables. This is no coincidence. In fact it follows from Lemma 2 of [RH] that the standard modules and the Dipper-James-Murphy Specht modules coincide in these cases and hence also the simple modules. Notice that the conditions of that Lemma on \( l \) to be odd and \( n_2 \neq m \) mod \( l \) can be replaced by \( n_2 \neq m \) mod \( e \), as can easily be seen from the proof of the Lemma.

Unfortunately, in general we do not have a non-recursive description of the elements of KBip_1(\( n \)).
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