Achievability and Impossibility of Exact Pairwise Ranking

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Abstract

We consider the problem of recovering the rank of a set of $n$ items based on noisy pairwise comparisons. We assume the SST class as the family of generative models. Our analysis gave sharp information theoretic upper and lower bound for the exact requirement, which matches exactly in the parametric limit. Our tight analysis on the algorithm induced by the moment method gave better constant in Minimax optimal rate than Shah and Wainwright [2017] and contribute to their open problem. The strategy we used in this work to obtain information theoretic bounds is based on combinatorial arguments and is of independent interest.

1 Introduction

Pairwise ranking focuses on the task of deciding the rank within an unordered set with $n$ items. The decision process is built upon the observation over the outcomes of comparison between every two parties in multi-round fashion. This problem stems from the identification of the best player in the tournament and the estimation of relative rank among teams. In the more recent literature, a variety of novel applications, including recommender systems and peer grading, applies the algorithms for this problem. Motivated by these active examples, we study the information theoretic and computational limits of a general class of parametric models (e.g., the SST class) in this problem.

In the problem of interest, we consider a collection of $n$ parties and assume that the observation consists of outcomes on the multiple rounds of pairwise comparisons. We assume the outcome is Bernoulli, where party $i$ wins over $j$ with probability $M_{i,j} \in (0,1)$ that relies on the relative rank between them and the outcome across different pairs to be independent of each other. Alike Shah and Wainwright [2017], we also considered the random design observation model, where the observation of any pairs in each round is observed with probability $p$. In particular, we focus on the problem of exact recovery, where the estimator converges almost surely to the ground truth as the scale of the problem goes to infinity. Under this requirement, we formally discuss the fundamental limits in this problem. A typical question in this problem is When do there exists estimators that guarantee weak consistency? and When do there exists no estimator that guarantees strong consistency?

We note that the impossibility and achievability for exact recovery are closely related to the All or Nothing phenomenon. This phenomenon is fundamental and is a direct result of the Borel-Cantelli lemmas. It states that when the number of correlated or weakly correlated random variables goes to infinity, the probability of events concerning a countable union of them will be either 1 (e.g., All) or 0 (e.g., Nothing). However, to decide the probability being 0 or 1 is nontrivial, which involves the estimation of boundary conditions. The information theoretic impossibility and achievability establish the threshold of this phase transition. Generally, by impossibility, we mean that no estimator exists while achievability argues for its converse. Another notion termed computational achievability is also used where the estimator is assumed to be constructed out of polynomial complexity procedure.

The major contribution in this work can be concluded in two folds. First, we established the sharp information theoretic limits that converge exactly in parametric limits. This result comes from the idea of finding the combinatorial events that result in the failure and is necessary in the success of MAP estimator, which yield a sharp estimate on the limits. Shah and Wainwright [2017] has established the Minimax rate for this problem whereas our work is the Bayes rate under uni-
form prior. Our result is not implied by theirs and is definitely stronger, since Bayes rate is strictly upper bounded by Minimax rate. Secondly, we gave an improved efficient algorithm with polynomial complexity. This yields a better rate and, in the meanwhile, yields a better constant between computational achievability and its converse, which has also been posted as an open problem in Shah and Wainwright [2017].

**Organization:** We organize this work through multiple sections. Section 2 introduces the general literature of this problem. Section 3 introduces the mathematical formulation of the problem, with extra background knowledge for information theoretic limits. Section 4 gives the formal result of our analysis together with the proofs. Section 5 gives sharp guarantees of moment method.

## 2 Related Work

Many historical and recent works have made remarkable progress in the many forms of this problem. The problem was first studied in the parametric models like BTL [Bradley and Terry 1952] and Thurstone [Thurstone 1927], where they gave Minimax optimal rate as well as an efficient algorithm for the ranking in SST class. Chatterjee [2015] studied the problem of estimating the probability matrix itself. Rajkumar and Agarwal [2014], Rajkumar et al. [2015] gave several result on the pairwise ranking problem under different models. Their result guarantees the consistency of the counting algorithm, but the result is loose compared with Shah and Wainwright [2017]. Our work further strengthened the bound via a simple but effective analysis of the moment method.

Several works also considered the top-$k$ recovery problem, which is a weaker form than the full recovery problem considered in this work. Chen et al. [2017] give an efficient algorithm for the top $K$ ranking algorithm in the SST class. Chen and Suh [2015] gives a spectral MLE algorithm that can achieve exact ranking with high probability as regularity conditions. However, their parametric model is BTL, which is shown to be not as effective in characterizing the real tournament result. Shah and Wainwright [2017] considered both the full recovery and top $k$ recovery problem and proved that the counting algorithm is Minimax optimal. Instead of focusing on the Minimax rate, we considered the Bayes rate, which is stated as the limit over the prior. This is a harder problem as the standard theory suggests Bayes risk is upper bounded by Minimax risk and they almost never converge. Heckel et al. [2019] studies the active ranking, where their result suggests that a logarithmic factor can be dropped from the sample complexity by switching from passive learning regime to active learning regime. We stick to the multi-round complete observation model.

## 3 Problem Formulations

Here we give a formal statement of the problem. Given an integer $n \geq 2$, consider the collections of $n$ parties, indexed by $[n] := \{1, \ldots, n\}$ each with their own ‘quality’ value $w_1, \ldots, w_n \in \mathbb{R}^+$. Since the ranking is agnostic of the order in the generative model, without loss of generality, we can assume that their quality is monotonically decreasing with indices. e.g. $w_1 > \ldots > w_n$. Let $M_{i,j} \in (0, 1)$ be the probability that the $i$-th strongest party wins over the $j$-th in a single round of competition.

A model in the SST class is defined by a strictly increasing function $F$ with $\sup(F) \in [0, 1]$ such that

$$M_{i,j} = F(w_i - w_j)$$

In this work, we denote $\gamma_{\text{max}} = \sup_i M_{i,j}$ and $\gamma_{\text{min}} = \inf_i M_{i,j}$. We denote $M(\mathcal{P}) = \mathcal{P}(A|\mathcal{P})$ by the induced measure for random matrix $A \in \mathcal{A} = \{0, 1\}^{n \times n}$. $\mathcal{P}$ is the state space on which prior is defined. We let $\mathcal{P} : [n] \rightarrow [n]$ be the symmetric group of permutations on $[n]$. Let $\Pi \in \mathcal{P}$ be the random variables defined by $P(\Pi = \pi) = \mathcal{P}(\pi)$ for all $\pi \in \mathcal{P}$, where $\mathcal{P}$ is the prior, which we assumed to be uniform. The ordered outcome graphs are written as $G(V, A)$, where $V$ is the finite set of vertices and $A$ is the adjacency matrix. Moreover, we say a graph $G$ is induced by some $A$ if its adjacency matrix is $A$.

The problem is stated as follows: We assume that first $\Pi$ is drawn from $\mathcal{P}$ according to $\mathcal{P}$, without loss of generality, let $\Pi = \pi^*$. The generative model permutes the index of parties according to $\pi^*$ and generates a series of total $m$ observations, denoted by $A_m^\pi = \{A_1^\pi, \ldots, A_m^\pi\}$ that is i.i.d. drawn from $\mathcal{A}$ according to $M(\Pi = \pi^*)$. We denote the normalized ensemble by $A_m = \frac{1}{m} \sum_{i=1}^{m} A_i$. The objective is to estimate $\pi^*$ based on $A_m$. Moreover, we denote $M(\Pi = \pi^*)$ by $M_{\pi^*}$ for simplicity. Denote $M_{\pi^*, i,j} = \text{Bernoulli}(M(\pi^*)^{-1}(i), (\pi^*)^{-1}(j))$ be the Bernoulli measure parameterized by the ground truth probability of ground truth $i$ wins over $j$. The notations used throughout this work is that we denote random variables by capital letters and their value by lower-case ones.
3.1 Model Specification

We consider the random design of experiments. Without loss of generality, we use $A$ to denote $A'$ that is assumed to be of identical probability measure. We studied the random design matrix, assuming that with probability $1 - p$ two parties will cancel their comparison. This model is also studied in Shah and Wainwright [2017], which can be stated as follows

**Assumption.** For any two parties $i, j$ such that $i < j$, the random design gives

$$A_{i,j} \sim M_{\pi^*, i, j}, A_{i,j} = -A_{j,i} \quad \text{with probability } p$$
$$A_{j,i} = A_{i,j} = 0 \quad \text{with probability } 1 - p$$

In the sequel, we omit $\Pi = \pi^*$ and use $P(\cdot)$ to denote $P(\cdot | \Pi = \pi^*)$ where no confusions are made.

3.2 Prelimnaries

We review some classical notations throughout this work, which follow the standard ones in the network inference problems. The following definition of disagreement is analogous to the Hamming distance in this problem, defined by

**Definition 1** (Disagreement). The disagreement between two permutations $\pi_1, \pi_2 \in \mathcal{P}$ is defined by

$$R(\pi_1, \pi_2) = \frac{1}{n} \sum_{i=1}^{n} 1_{\pi_1(i) \neq \pi_2(i)}$$

Based on the consistency of estimators, we give definitions on the exact ranking:

**Definition 2.** We say the estimator $\hat{\pi}$ satisfies exact ranking if the following requirement is achieved:

$$P_{M, \mathcal{P}}(R(\hat{\pi}, \pi^*) = 0) = 1 - o(1)$$

where $\pi^*$ is the ground truth. This also reminds us of the strong consistency or convergence a.e.

Then the standard information theoretical limits are listed for completeness

**Definition 3** (Impossibility). Assuming the parametric space is $\Theta$, we say the exact pairwise ranking is impossible in $\Theta$ if no estimator achieves exact ranking for any $\theta \in \Theta$.

Similarly, contrary to the impossibility, we also have achievability in both the information theoretic and computational sense.

**Definition 4** (Achievability). Assuming the parametric space is $\Theta$, we say exact pairwise ranking is (information theoretically) achievable if for any $\theta \in \Theta$, there is an estimator $\hat{\theta}$ that satisfies exact ranking. Moreover, we say that exact pairwise ranking is computationally achievable if the procedure to obtain $\hat{\theta}$ is with polynomial complexity.

The gap between these two types of achievabilities results in an information-computational gap. Intuitively, the impossibility and achievability stay at the two ends of phase transition. The former indicates uniform failure in the probability family, while the latter indicates uniform success. In simpler problems like detection with binary stochastic block model (SBM), the number of parameters is small, and these two notations converge. However, in a general problem, the two thresholds are not the same. The gap between the two thresholds governs the width of phase transition and will be an essential characteristic in our problem.

Then the following claim establishes a sufficient condition of impossibility.

**Claim 1** (Sufficiency of Impossibility). When the parameter space $\Theta$ is finite, and the Bayes estimator fails exact ranking, the exact ranking is impossible. In particular, when the loss of the Bayes risk is $0/1$ loss, the Bayes estimator can be achieved by maximizing a posteriori.

By its definition, the Bayes estimator is optimal w.r.t. the prior on the disagreement metric. Hence, any estimator will have a higher probability of nonzero disagreement than the Bayes estimator, which implies that the Bayes estimator is on the limit of impossibility.

Previous literature Shah and Wainwright [2017] considers the exact ranking with Minimax optimality. However, Bayes estimator is equivalent to Minimax estimator when only a single estimator exists. Their results can be seen as an attempt to close the threshold of achievability and its converse, which differs from the problem studied here. It is also important to note that in this problem, the impossibility is not a converse argument for achievability, we discussed this phenomena in the next section. Hence, the contribution is largely exclusive.

4 Fundamental Limits of Ranking

The threshold staying between sharp impossibility and achievability resulted in the phase transition phenomena. In this section, we discuss the two ends of it. Our result shows that using the condition over $\Theta(N)$ parameters, two sides of the threshold are closed at the limit.
4.1 Information Theoretic Lower Bound

In this subsection, we present the discussion over the impossibility requirement. The method used here originates in multiple hypothesis testing. Recall that, for each vertex \( i \in [n] \), the state space contains all possible outcomes of comparisons with the rest \( n-1 \) parties. And we can see this problem as trying to distinguish the order between any two parties, whose scale goes in \( O(n^2) \). Intuitively, if two elements are too close to each other, the estimation will fail due to the impossibility of doing so. Although a similar argument is formed in Shah and Wainwright [2017], who discuss the testing over \( n-1 \) adjacent pairs of parties to obtain Minimax lower bound, our discussion is different as we are focusing on lower bounding the risk of the test problem itself. This estimate hence provides a lower bound for the original problem. Shah and Wainwright [2017] considers \( \Delta := \inf_{i \in [n-1]} \frac{1}{n} \sum_{k=1}^{n} (M_{i,k} - M_{i+1,k}) \) as an indicator of success for the top-\( k \) discovery. Our result suggests that this indicator will be loose for the information theoretic threshold. Instead, we resort to focusing on lower bounding the risk of the test problem itself.

Here we state the following sufficient conditions on the impossibility of exact ranking, which is analogous to the idea of Signal to Noise Ratio (SNR) in signal processing. When the SNR is too low, the noise will abuse the original signal, making it impossible to be recovered. The following two conditions are sufficient for the failure.

1. **Connectivity**: If the undirected random graph induced by \( \mathcal{A}^m \) is unconnected with \( \Theta(1) \) probability, the exact ranking fails

2. **Failure of the MAP Estimator**: If the MAP estimator failed with probability \( \Theta(1) \), then the exact ranking fails

4.1.1 Connectivity

In the seminal work by Erdos et al. [1960], the connectivity of a single Erdos–Renyi graph \( G(n,p) \) satisfies that when \( p = \frac{\log n + c}{n} \)

\[
P(G(n,p) \text{ is connected }) \to e^{-e^{-c}}
\]

Then we analogously establish the result for the ensemble random graphs \( G^m(n,p) \) induced by \( \mathcal{A}^m \)

**Theorem 1.** When \( p = \frac{\log n + c}{mn} \)

\[
P(G^m(n,p) \text{ is connected }) \to e^{-e^{-c}}
\]

**Proof.** The proof is straightforward once we observe that \( G^m(n,p) \) is the same as \( G(n, 1-(1-p)^m) \)

And the above theorem directly implies the following:

**Corollary 1.1.** When \( c > 1 \) and \( p \geq \frac{\log n}{mn} \), then the graph \( G^m(n,p) \) is connected.

In the sequel, we only considered the situation where the graph is connected, where \( p \geq \frac{\log n}{mn} \)

4.1.2 Failure of MAP Estimator

Here we discuss an estimate on the upper bound of exact ranking. The result we established is stated as

**Theorem 2** (Impossibility). Assume that \( M_{i,j} = \Theta(1) \) as \( n \to \infty \). Let \( \Delta_{i,j,l} = M_{i,l} - M_{j,l} \). Let \( K_0 := \frac{1}{\gamma_{max}} + \frac{1}{\gamma_{min}} \). Assume that \( p > \frac{\log n}{mn} \). Then the following implies impossibility of exact ranking:

\[
\inf_{i,j \in [n]} \Delta_{i,j,l}^2 \leq \frac{4 \log n}{K_0 m p}
\]

or, under a stricter condition

\[
\Delta \leq \frac{2}{n} \sqrt{\log n} \frac{1}{K_0 m p}
\]

First, we decouple the random design matrix into a pointwise product between a Bernoulli filter with the original random matrix. Without loss of generality, we can always denote \( \pi^* \) by the identity mapping since the exact order in the random matrix does not affect the procedure of MAP.

We define the random design matrix by \( \bar{A}_1^m = \{ \bar{A}^1, \ldots, \bar{A}^m \} \) and decouple the generative procedure. Assume that we have a Bernoulli random matrix \( Q \in [0,1]^{n \times m} \) defined by

\[
Q_{i,j} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}
\]

subjecting to \( Q_{i,j} = Q_{j,i} \). Then we can decouple the \( \bar{A} = A \otimes Q \) where \( \otimes \) is the element-wise product. Moreover, we have that for any \( a > 0 \)

\[
P(\bar{A}_{i,j} = a) = P(A_{i,j} = a, Q_{i,j} > 0) = (1-p)P(A_{i,j} = a)
\]

and \( P(\bar{A}_{i,j} = 0) = 1-p \). We decouple the entry sum of \( \mathcal{A}^m \) into positive and negative part:

\[
\mathcal{A}_{i,j}^m = \sum_{k=1}^{m} Q_{i,j} = 1 \mathcal{A}_{i,j}^{m,+} = 1
\]

\[
\mathcal{A}_{i,j}^{m,-} = \sum_{k=1}^{m} Q_{i,j} = 1 \mathcal{A}_{i,j}^{m,-} = -1
\]
And we rewrite the maximum a posteriori criteria as

$$\hat{\pi}_{\text{MAP}} = \arg \max_{\pi \in \mathcal{P}} P(\Pi = \hat{\pi} | M, \hat{A})$$

$$= \arg \max_{\pi \in \mathcal{P}} \prod_{k=1}^{m} P(\hat{A}^k | M, p, \Pi = \hat{\pi}) P(\Pi = \hat{\pi})$$

$$= \arg \max_{\pi \in \mathcal{P}} \sum_{k=1}^{m} \log P(\hat{A}^k | M, p, \Pi = \hat{\pi})$$

$$= \arg \max_{\pi \in \mathcal{P}} \sum_{i<j}^{m} \log \left( P(\hat{A}_{\hat{\pi}(i),\hat{\pi}(j)} | M_{i,j}, p, \Pi = \hat{\pi}) \right)$$

According to the likelihood function of Binomial random variables, we have

$$\log P(\hat{A}_{i,j} | M_{i,j}) = A_{i,j} \cdot \log(M_{i,j})$$

$$+ (1 - A_{i,j}) \cdot \log(1 - M_{i,j})$$

Then we look into the objective that we maximize, recall that we denote the ground truth $\pi^*$ to be the identity bijection. We define $F : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ as the score function

$$F(i,j) = \sum_{k=1}^{m} \sum_{l \in [n]} 1_{Q_{\hat{\pi}(i),=1} \cap \log P(\hat{A}_{\hat{\pi}(i),l} | M_{j,l}, \Pi = \hat{\pi})}$$

$$= \sum_{l \in [n]\setminus \{i_1, i_2\}} A_{\hat{\pi}(i),l} \log(M_{j,l}) + A_{\hat{\pi}(i),l} \log(1 - M_{j,l})$$

Hence, terms in the log likelihood function concerning $\hat{\pi}(i_1)$ and $\hat{\pi}(i_2)$ are termed by $F(i_1, i_1)$ and $F(i_2, i_2)$.

We can quickly identify an event that will cause the MAP procedure fails to return correct estimator:

$$F(i_1, i_2) + F(i_2, i_1) \leq F(i_1, i_1) + F(i_2, i_2)$$

, given by

$$\sum_{l \in [n]\setminus \{i_1, i_2\}} \left( A_{\hat{\pi}(i),l} \log(M_{j,l}) + A_{\hat{\pi}(i),l} \log(M_{i,l}) \right)$$

$$+ A_{\hat{\pi}(i),l} \log(1 - M_{j,l}) + A_{\hat{\pi}(i),l} \log(1 - M_{i,l}) \geq$$

$$\sum_{l \in [n]\setminus \{i_1, i_2\}} \left( A_{\hat{\pi}(i),l} \log(M_{j,l}) + A_{\hat{\pi}(i),l} \log(M_{i,l}) \right)$$

$$+ A_{\hat{\pi}(i),l} \log(1 - M_{j,l}) + A_{\hat{\pi}(i),l} \log(1 - M_{i,l})$$

which implies that

$$E_{i_1, i_2} := \sum_{l \in [n]\setminus \{i_1, i_2\}} \left( A_{\hat{\pi}(i_1),l}^{m_+} - A_{\hat{\pi}(i_2),l}^{m_+} \right) \log \left( \frac{M_{j,l}}{M_{i,l}} \right)$$

$$+ \left( A_{\hat{\pi}(i_1),l}^{m_-} - A_{\hat{\pi}(i_2),l}^{m_-} \right) \log \left( \frac{1 - M_{j,l}}{1 - M_{i,l}} \right) \geq 0$$

The idea to approach the upper bound for MAP estimator is to explore over the event such that MAP fails, namely $P(E_{i_1, i_2} \geq 0)$.

To approach this problem, we construct a set of random variables $C^{(i_1, i_2)} \in \mathbb{R}^{2 \times n}$ such that when $l \notin \{i_1, i_2\}$,

$$C_{i_1, l}^{(i_1, i_2)} = \left\{ \begin{array}{ll} \log \left( \frac{M_{j,l}}{M_{i,l}} \right) & \text{with probability } M_{i,l}p \\ \log \left( \frac{1 - M_{j,l}}{1 - M_{i,l}} \right) & \text{with probability } (1 - M_{i,l})p \end{array} \right. \right.$$  

and

$$C_{i_2, l}^{(i_1, i_2)} = \left\{ \begin{array}{ll} \log \left( \frac{M_{j,l}}{M_{i,l}} \right) & \text{with probability } M_{i,l}p \\ \log \left( \frac{1 - M_{j,l}}{1 - M_{i,l}} \right) & \text{with probability } (1 - M_{i,l})p \end{array} \right. \right.$$  

Hence, the error probability can be rewritten through sampling $m$ independent copies of $C^{(i_1, i_2)}$, denote by $\{C^{(1)}, \ldots, C^{(m)}\}$, and we have

$$P(E_{i_1, i_2} \geq 0) = P( \sum_{s \in [m]} \sum_{l \in [n]\setminus \{i_1, i_2\}} C_{i_1, l}^{(s)} \leq 0)$$

To approach this transformed version of problem, we start by reviewing a few technical lemmas.

**Lemma 3** (Paley-Zygmund). For random variable $Z > 0$, for $\theta \in (0, 1)$ we have

$$P(Z \geq \theta \mathbb{E}[Z]) \geq (1 - \theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}$$

which directly implies

**Corollary 3.1** (Second Order Method). $P(Z \geq 0) \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}$

The following lemma gives a sufficient condition of impossibility. The idea in proving this lemma comes from Borel-Cantelli, which can be proved by the above corollary.

**Lemma 4.** Assume that $M_{i,j} = \Theta(1)$ as $n \rightarrow \infty$ (e.g. non-vanishing) and $\sup_k (M_{i,i+1} - M_{i-1,i}) = \Delta \rightarrow 0$ as $n \rightarrow \infty$. If $P(E_{i, i+1} > 0) = \Omega(\frac{1}{n^2})$ for all $i \in [n-1]$, then the exact ranking will be impossible.

The above lemma argues for an sufficient condition of impossibility. Hence if we can argue for the condition to be asymptotically almost surely (a.a.s) , then we can obtain a bound of impossibility. This leads to the formal proof of impossibility.
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**Proof.** The proof follows from Paley-Zygmund inequality. Let \( \theta = \frac{1}{\log(\exp(tZ))} \), we have:

\[
P(Z > 0) = P(\exp(Z) > 1) \\
\geq \left( 1 - \frac{1}{\mathbb{E} [\exp(tZ)]} \right)^2 \frac{\mathbb{E} [\exp(tZ)]^2}{\mathbb{E} [\exp(2tZ)]}
\]

For the simplicity of notation, we denote

\[
\mathcal{A}_i = \frac{M_{i2,l}}{M_{i1,l}}, \quad \mathcal{B}_l = 1 - \frac{M_{i2,l}}{M_{i1,l}}
\]

And we note that \( \log \mathcal{A}_i \cdot \log \mathcal{B}_l < 0 \). Without loss of generality, we assume \( \log \mathcal{A}_i > 0, \log \mathcal{B}_l < 0 \) and let \( X_i \sim \text{Bernoulli}(M_{i1,l}), Y_i \sim \text{Bernoulli}(M_{i2,l}) \) in what follows.

The moment generating function for random variable \( C_{i1,i2}^{(t)} \) by

\[
\psi_{C_{i1,i2}^{(t)}}(t) = \mathbb{E} [\exp(tC_{i1,i2}^{(t)})] \\
= \left( \frac{M_{i2,l}}{M_{i1,l}} \right)^t p_{M_{i1,l}} + 1 - p + \left( \frac{1 - M_{i2,l}}{1 - M_{i1,l}} \right)^t p(1 - M_{i1,l}) \\
+ \left( \frac{1 - M_{i2,l}}{1 - M_{i1,l}} \right)^t p(1 - M_{i1,l}) \\
= \mathcal{A}_i^t p_{M_{i1,l}} + 1 - p + \mathcal{B}_l^t p(1 - M_{i1,l}) \\
= 1 - p + pD_{f_i}(X_i||Y_i)
\]

where for \( X_i \sim P, Y_i \sim Q \), we have that

\[
D_{f_i}(X_i||Y_i) = D_f(P||Q) = \int \frac{dp}{dq} f_i(q) dx
\]

is the f-divergence with \( f_i(x) = x^t \).

Similarly, for the random variable \( C_{i2,i} \), we have

\[
\psi_{C_{i2,i}^{(t)}}(t) = \left( \frac{M_{i1,l}}{M_{i2,l}} \right)^t p_{M_{i2,l}} + 1 - p + \left( \frac{1 - M_{i1,l}}{1 - M_{i2,l}} \right)^t p(1 - M_{i2,l}) \\
= \mathcal{A}_i^t p_{M_{i2,l}} + 1 - p + \mathcal{B}_l^t p(1 - M_{i2,l}) \\
= 1 - p + pD_{f_i}(Y_i||X_i)
\]

Assembling pieces, we have

\[
\mathbb{E} [\exp(tE_{i1,i2})] = \mathbb{E} [\exp(\sum_{i \in \{1,2\}} \sum_{l \in \{1\}} tC_{i1,l})] \\
= \prod_{i \in \{1,2\}} \prod_{l \in \{1\}} \mathbb{E} [\exp(tC_{i1,l})]
\]

which implies

\[
P(E_{i1,i2} > 0) > \frac{\left( \mathbb{E} [\exp(tE_{i1,i2})] - 1 \right)^2}{\mathbb{E} [\exp(2tE_{i1,i2})]}
\]

From Lemma 4, we note that impossibility is implied by

\[
P(E_{i1,i2} > 0) = \Omega \left( \frac{1}{n} \right)
\]

for all \( i_2 - i_1 = 0 \). This is implied by

\[
\left( \prod_{s \in \{1\}} \prod_{l \in \{1\}} \prod_{i \in \{1\}} \mathbb{E} [\exp(tC_{i,l})] - 1 \right)^2
\]

\[
= \Omega \left( \frac{1}{n} \prod_{s \in \{1\}} \prod_{l \in \{1\}} \prod_{i \in \{1\}} \mathbb{E} [\exp(2tC_{i,l})] \right)
\]

taking the logarithm over both side, we have

\[
\sum_{s \in \{1\}} \sum_{l \in \{1\}} \sum_{i \in \{1\}} (2\log(\mathbb{E} [\exp(tC_{i,l}^{(s)})] - 1) \\
- \log(\mathbb{E} [\exp(2tC_{i,l}^{(s)})] - 1) \geq -\log n
\]

Note that \( \mathbb{E} [\exp(tC_{i1,i2})] \to 1 \) as \( n \to \infty \), the above can be implied by

\[
\frac{1}{n} \sum_{l \in \{1\}} \sum_{i \in \{1\}} 2\log(\mathbb{E} [\exp(tC_{i1,l} + tC_{i2,l})] - 1) \\
- \log(\mathbb{E} [\exp(2tC_{i1,l} + 2tC_{i2,l})] \leq \log n
\]

By the definition of \( f_i \)-divergence, let \( \Delta_{i1,i2} = M_{i1,l} - M_{i2,l} \), we have

\[
D_{f_i}(X_i||Y_i) = \left( \frac{M_{i2,l}}{M_{i1,l}} \right)^t M_{i1,l} + \left( \frac{1 - M_{i2,l}}{1 - M_{i1,l}} \right)^t (1 - M_{i1,l}) \\
= (M_{i1,l})^{-(t-1)}(M_{i1,l} + \Delta_{i1,i2} t)^t \\
+ (1 - M_{i1,l})^{-(t-1)}(1 - M_{i1,l} - \Delta_{i1,i2} t)^t \\
= (M_{i1,l})^{-(t-1)} \\
\cdot (M_{i1,l} + t\Delta_{i1,i2,1} M_{i1,l}^{-1} + \frac{t}{2}(t-1) \Delta_{i1,i2,1}^2 t^{-1} - M_{i1,l} - \Delta_{i1,i2,1} t)^t \\
+ (1 - M_{i1,l})^{-(t-1)}(1 - M_{i1,l} - \Delta_{i1,i2,1} t)^t \\
+ \frac{1}{2} t(t-1) \Delta_{i1,i2,1}^2 (1 - M_{i1,l} - \Delta_{i1,i2,1} t)^{-2} + o(\Delta_{i1,i2,1}^2) \\
= 1 + \frac{1}{2} t(t-1)(M_{i1,l}^{-1} + (1 - M_{i1,l})^{-1}) \Delta_{i1,i2,1}^2 \\
+ o(\Delta_{i1,i2,1}^2)
\]

Collecting pieces, we concluded impossibility can be implied by

\[
\frac{1}{n} \sum_{l \in \{1\}} (-2t(t-1))(\frac{1}{M_{i1,l}} + \frac{1}{1 - M_{i1,l}}) \Delta_{i1,i2,1}^2 \leq \log n
\]

Let \( t = \frac{1}{2} \) and note that

\[
K_0 := 1 + \frac{1}{\gamma_{\text{max}}} - \frac{1}{\gamma_{\text{min}}}
\]
we finally conclude by
\[ \frac{K_0}{4} \sum_{i \in [n] \setminus \{i_1, i_2\}} \Delta^2_{i_1, i_2, l} \leq \frac{\log n}{mp} \]
which completes the proof. \( \Box \)

### 4.2 Information Theoretic Upper Bound

Then we discuss an estimate on the lower bound of exact ranking. The result we established as the information theoretic achievability is stated as

**Theorem 5.** Assume that the probability is non-vanishing, Let \( \Delta_{i_1, i_2} = M_{i_1} - M_{i_2} \), then the following implies that there exists algorithm that rank exactly.

\[ \inf_{i_1, i_2 \in [n]} \sum_{l \in [n]} \Delta^2_{i_1, i_2, l} \geq \frac{K_0 \log n}{4mp} \]

Or under stricter condition:

\[ \tilde{\Delta} \geq \sqrt{\frac{K_0 \log n}{4mn}} \]

To obtain the information theoretic upper bound, we rely on the following lemma.

**Lemma 6 (Information Theoretic Achievability).** If MAP estimator does not recover the ground truth, then \( \Theta(1) \) probability there exists \( i_1, i_2 \in [n] \) such that

\[ F(i_1, i_2) + F(i_2, i_1) \leq F(i_1, i_1) + F(i_2, i_2) \]

By the above lemma, we found out a necessary event of success. If this event is asymptotically impossible, then we can argue for the success of MAP estimator, which finally establish the achievability. We then give formal proof of achievability.

**Proof.** Let \( X_{i, l} \sim \text{Bernoulli}(M_{i, l}) \). We recall that the centered moment generating function of \( C_{i_1, i_2}^{(i_1, i_2)} + C_{i_2, i_1}^{(i_1, i_2)} \) is

\[ \psi_{C_{i_1, i_2}^{(i_1, i_2)} + C_{i_2, i_1}^{(i_1, i_2)}}(t) \]

\[ = \mathbb{E}[\exp(C_{i_1, i_2}^{(i_1, i_2)} + C_{i_2, i_1}^{(i_1, i_2)} - \mathbb{E}[C_{i_1, i_2}^{(i_1, i_2)} + C_{i_2, i_1}^{(i_1, i_2)}])] \]

\[ = \frac{M_{i_2, l}(1 - M_{i_1,l})}{M_{i_1, l}(1 - M_{i_2, l})} \exp(M_{i_2, l} - M_{i_1, l}) \]

\[ \cdot (1 - p + pD_{i_1}(X_{i_1, l} || X_{i_2, l}) \cdot (1 - p + pD_{i_1}(X_{i_2, l} || X_{i_1, l})) \]

taking the logarithm over both side, we have

\[ \log (\psi_{C_{i_1, i_2}^{(i_1, i_2)} + C_{i_2, i_1}^{(i_1, i_2)}}(t)) \]

\[ \leq \frac{t^2\Delta_{i_1, i_2, l}^2}{2} \left( \frac{1}{M_{i_1, l}} + \frac{1}{1 - M_{i_1, l}} + \frac{1}{M_{i_2, l}} + \frac{1}{1 - M_{i_2, l}} \right) \]

\[ + o(\Delta_{i_1, i_2, l}^2) \]

\[ \leq p\Delta_{i_1, i_2, l}^2 t^2 \left( \frac{1}{\gamma_{\text{max}}} + \frac{1}{\gamma_{\text{min}}} \right) + o(\Delta_{i_1, i_2, l}^2) \]

By Bennet’s inequality, we have for \( x \geq 0 \), we have

\[ P\left( \sum_{s \in [n]} \sum_{l \in [n]} C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)} - \mathbb{E}[C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)}] \geq x \right) \]

\[ \leq \prod_{l \in [n] \setminus \{i_1, i_2\}} \exp(tx) C_{i_1, l}^{(s)}(t) + C_{i_2, l}^{(s)}(t) \]

Let \( K_0 = \frac{1}{\gamma_{\text{max}}} + \frac{1}{\gamma_{\text{min}}} \). Minimizing the R.H.S. we have

\[ P\left( \sum_{s \in [n]} \sum_{l \in [n]} C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)} - \mathbb{E}[C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)}] \geq x \right) \]

\[ \leq \exp\left( - \frac{x^2}{2pm \sum_{l \in [n] \setminus \{i_1, i_2\}} \Delta_{i_1, i_2, l}^2 K_0} \right) \]

which implies that

\[ P(E_{i_1, i_2} \geq 0) \]

\[ = P\left( \sum_{s \in [n]} \sum_{l \in [n] \setminus \{i_1, i_2\}} C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)} - \mathbb{E}[C_{i_1, l}^{(s)} + C_{i_2, l}^{(s)}] \right) \]

\[ \geq -m \sum_{l \in [n] \setminus \{i_1, i_2\}} \mathbb{E}[C_{i_1, l}^{(i_1, i_2)} + C_{i_2, l}^{(i_1, i_2)}] \]

\[ \leq \exp\left( - \frac{m^2 (\sum_{l \in [n] \setminus \{i_1, i_2\}} \mathbb{E}[C_{i_1, l}^{(i_1, i_2)} + C_{i_2, l}^{(i_1, i_2)}])^2}{2pm \sum_{l \in [n] \setminus \{i_1, i_2\}} \Delta_{i_1, i_2, l}^2 K_0} \right) \]

where we applied the fact that when \( \Delta_{i_1, i_2} \to 0 \) and note that when \( M_{i_1, l} \) and \( M_{i_2, l} \) are on the same side of \( \frac{1}{2} \)

\[ \mathbb{E}[C_{i_1, l}^{(i_1, i_2)} + C_{i_2, l}^{(i_1, i_2)}] = p(M_{i_2, l} - M_{i_1, l}) \]

\[ \log \left( \frac{M_{i_2, l}(1 - M_{i_1, l})}{M_{i_1, l}(1 - M_{i_2, l})} \right) \]

\[ \geq 4 \Delta_{i_1, i_2, l}^2 p \]

We upperbound the joint event by the union bound:

\[ P\left( \bigcup_{i_1, i_2 \in [n]} E_{i_1, i_2} \right) \]

\[ \leq \sum_{i_1, i_2 \in [n]} P(E_{i_1, i_2}) \]

\[ \leq \frac{n(n - 1)}{2} \exp\left( - \frac{8mp \inf_{i_1, i_2 \in [n] \setminus \{i_1, i_2\}} \Delta_{i_1, i_2, l}^2 K_0} \right) \]

To guarantee the consistency of MAP, we will need:

\[ \inf_{i_1, i_2 \in [n]} \sum_{l \in [n] \setminus \{i_1, i_2\}} \Delta_{i_1, i_2, l} \geq \frac{K_0 \log n}{4mp} \]

**Discussion:** The result given in those two bounds exactly equal at the limit of \( \gamma_{\text{min}} = \gamma_{\text{max}} \). Recall that in this case \( K_0 = 4 \). The reason that those two bounds are not converging in the general case (e.g. \( \gamma_{\text{max}} > \frac{1}{2} \)) is the concavity of information theoretic achievable region in the space spanned by \( \Theta(n^2) \) random variable. When only \( \Theta(n) \) random variables are considered, the gap in the bound is not sharp in general. \( \Box \)
5 Efficient Algorithm

In this section, we present the guarantee for moment
method, which is efficient and has a finer guarantee
than the analysis of counting algorithm by Shah and
Wainwright [2017]. In particular, our algorithm recov-
ers rank based on the estimated first moment of row
sum in the ensembled random matrix. Our result is
formally stated as:

**Theorem 7.** The proposed efficient algorithm
achieves exact recovery when

$$\bar{\Delta} \geq \sqrt{\frac{\log n}{nmp}}$$

To obtain this result, we first state the moment
method. The first moment estimate of entries of prob-
ability matrix is given by

$$\hat{M}_{i,j} = \frac{A_{i,j}^n}{2p} + \frac{1}{2} = \frac{1}{2pm} \sum_{t \in [m]} A_{i,j,t}^n + \frac{1}{2}$$

with $$\hat{M}_{i,i} = \frac{1}{2}$$ And it is easy to see that the unbiased-
ness

$$E[\hat{M}_{i,j}] = M_{i,j}$$

Although $$p$$ is explicitly included here, the estimation
can be carried out when $$p$$ is unknown a prior, since
the goal is ranking instead of estimation. In that case,
we will be estimating $$pM_{i,j}$$ instead.

Then the algorithm ranked the party based on the en-
try sum of estimator :

$$M_i = \sum_{j \in [n]} \frac{\hat{M}_{i,j}}{n}$$

Here we introduced the sufficient and necessary con-
tions on the success of pairwise ranking with this method.

**Claim 2.** Let $$\hat{G}_{i,j} = M_i - M_j$$. Then the exact re-
covery is equivalent to:

$$\hat{G}_{i,i+1} > 0 \text{ for all } i \in [n-1] \text{ a.a.s.}$$

In the sequel, we also denote

$$G_{i,j} = \frac{1}{n} \sum_{k=1}^{n} (M_{i,k} - M_{j,k})$$

as the popularity gap.

Based on the above claim, we can establish the guar-
antees for the moment methods. The technical details
are delayed to the appendix.

Our method is within the tractable computational re-
region because the sorting and the mean estimation can
all be conducted in polynomial complexity.

**Discussion:** Our upper bound gave a sharper rate
than Shah and Wainwright [2017] and closed the con-
stant from 8 to 1.

1. The technique used in this analysis is Bernett’s in-
equality and the first-order Taylor approximation
     to achieve sharpness. As Berstein’s inequality is a direct result of Bernett’s, our analysis yields a
     sharper bound than Shah and Wainwright [2017].

2. The multiple hypothesis testing problems only en-
gaged $$\Theta(n)$$ pairs while Shah and Wainwright
    [2017] engaged $$\Theta(n^2)$$ pairs.

On the other hand, this bound does not contradict
what we have for the information theoretic upper
bound, which follows that theorem 7 is a condition
on the 1-norm. If we state the sufficient condition in
the form analogous to the information theoretic up-
per bound. It will become $$\inf_{i,j} \sum_{t \in [n]} \Delta_{i,j,t}^2 \geq \frac{n \log n}{m p}$$,
which is over pessimistic.

6 Future Work

The open problem that has not been addressed here is
whether the constant gap can be closed at the minimax
optimal rate. Shah and Wainwright [2017] gave a $$\frac{1}{70}$$
constant for this problem via Fano’s method on mul-
tiple hypothesis testing constructing on a $$\Theta(n)$$ pairs.
Hence, our central question will be whether a sharper
rate can be established for the Minimax lower bound.

The second problem is if we can relax the require-
ment to almost exact recovery. In this case, MAP is not
admissible. Shah and Wainwright [2017] studied a
notion called approximated recovery and gave guaran-
tees for them under constant error gap in Hamming
distance. However, their argument does not argue for
vanishing error, and the estimator that works for the
non-vanishing threshold does not imply almost exact
recovery. Hence, it will be interesting to see if we can
establish the impossibility condition on the almost ex-
act recovery.

7 Conclusion

This work studies the problem of pairwise ranking un-
der the requirement of exact ranking. We established
the information theoretic upper and lower bound that
is exact in the limit. We performed a tight analysis
on the moment method for this problem and gave a
sharper guarantee for the Minimax optimal rate. Open
question on whether the lower bound can also be im-
proved is raised.
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