CURL-CURL CONFORMING ELEMENTS ON TETRAHEDRA

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Abstract. In [24], we proposed $H(\text{curl}^2)$-conforming elements on both a triangle and a rectangle. This family of elements provides a brand new method to solve the quad-curl problem in 2 dimensions. In this paper, we turn our focus to 3 dimensions and construct an $H(\text{curl}^2)$-conforming tetrahedral finite element. The newly proposed element has been proved to have the optimal interpolation error estimate. Having tetrahedral elements, we can solve the quad-curl problem in any Lipschitz domain by conforming finite element method. We also provide several numerical examples of using our element to solve the quad-curl problem. The results of the numerical experiments show the effectiveness and correctness of our element.

1. Introduction

The quad-curl problem are involved in various practical problems, such as inverse electromagnetic scattering theory [3, 17, 21] or magnetohydrodynamics [27]. As its name implies, this problem involves a fourth-order curl operator which makes it much harder to solve than the lower-order electromagnetic problem [8, 12–16, 23]. The regularity of this problem was studied by Nicaise [19], Zhang [25], and Chen et al. [5]. As for the numerical methods, Zheng et al. developed a nonconforming finite element method for this problem in [27]. This method has low computational cost since it has small number of degrees of freedom (DOFs), but it bears the disadvantage of low accuracy. Based on Nédélec elements, a discontinuous Galerkin method and a weak Galerkin method were presented...

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in [11] and [22], respectively. In addition, error estimates for discontinuous Galerkin methods based on a relatively low regularity assumption of the exact solution are proposed in [4,5]. Another approach to deal with the quad-curl operator is to introduce an auxiliary variable and reduce the original problem to a second-order system [21]. Zhang proposed a different mixed scheme [25], which relaxes the regularity requirement in theoretical analysis. Very recently, Brenner et al. developed a 2 dimensional (2D) Hodge decomposition method in [1] by using Lagrange elements.

However, all above-mentioned methods have difficulties in implementing the boundary conditions. In this regard, the most natural way to solve this problem is the conforming finite element method. In [24], the authors constructed the curl-curl-conforming or $H(\text{curl}^2)$-conforming elements on rectangles and triangles to solve the quad-curl problem. In three dimensions (3D), the existing $H^2$-conforming (or $C^1$-conforming) elements [26] used for this problem may lead to false solutions when the exact solutions is not in $H^2(\Omega)$. Indeed, the treatment of boundary conditions is also an issue when using $H^2$-conforming elements to solve the quad-curl problem. Also, Neilan constructed a family of $H^1(\text{curl})$-conforming elements ($\mathbf{u} \in H^1$ and $\nabla \times \mathbf{u} \in H^1$) in [18] (see [10] for the 2D case). The family of elements can also lead to conforming approximations of the quad-curl problem. However, in this paper, we derive a conforming finite element space for $H(\text{curl}^2; \Omega)$ where the function regularity is weaker than the space $H^1(\text{curl})$. Such types of elements, to the best of the authors’ knowledge, are not available in the literature. Due to the large kernel space of the curl operator $\nabla \times$ (compared with the gradient operator $\nabla$), the construction of $H(\text{curl}^2)$-conforming elements is more difficult than that of $H^2$-conforming elements.

Our paper starts by describing the tetrahedral curl-curl-conforming finite element. The unisolvence and conformity of our $H(\text{curl}^2)$-conforming finite element can be verified by a rigorous mathematical analysis. Moreover, our new element has been proved to possess good interpolation properties. Although the involvement of normal derivatives to edges render the DOFs on a general element failing to relate to those on the reference element, we constructed an intermediate element whose DOFs can be related to those on the reference element and is close to our element. In this way, we prove the optimal error
estimate of the finite element interpolation. In our construction, the number of the degrees of freedom for the lowest-order element is 315. Because of the large number of DOFs, it’s hard to compute the Lagrange-type basis functions by the traditional method. Hence we employ the method proposed in [9] to obtain the basis functions on a general element.

The rest of the paper is organized as follows. In section 2 we list some function spaces and notations. Section 3 is the technical part, where we define the $H(\text{curl}^2)$-conforming finite element on a tetrahedron. In section 4 we give the error estimate for the interpolation. In section 5 we use our newly proposed elements to solve the quad-curl problem and give some numerical results to verify the correctness and efficiency of our method. Finally, some concluding remarks and possible future works are given in section 6. We present how to implement the finite element in Appendix and provide the code for it.

2. Preliminaries

Let $\Omega \in \mathbb{R}^3$ be a Lipschitz domain and $\mathbf{n}$ be the unit outward normal vector to $\partial \Omega$. We adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ or $W_0^{m,p}(D)$ on a simply-connected sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$. If $p = 2$, the space $W^{m,2}(D)$ is exactly the space $H^m(D)$ with the norm $\|\cdot\|_{m,D}$. If $m = 0$, the space $W^{0,p}(D)$ coincides with $L^p(D)$. When $D = \Omega$, we drop the subscript $D$. We use $W^{m,p}(D)$, $H^m(D)$, and $L^p(D)$ to denote the vector-valued Sobolev spaces $(W^{m,p}(D))^3$, $(H^m(D))^3$, and $(L^p(D))^3$.

Let $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{w} = (w_1, w_2, w_3)^T$, where the superscript $T$ denotes the transpose, then $\mathbf{u} \times \mathbf{w} = (u_2w_3 - w_2u_3, w_1u_3 - u_1w_3, u_1w_2 - w_1u_2)^T$ and $\nabla \times \mathbf{u} = (\partial_{x_2}u_3 - \partial_{x_3}u_2, \partial_{x_3}u_1 - \partial_{x_1}u_3, \partial_{x_1}u_2 - \partial_{x_2}u_1)^T$. For convenience, here and hereinafter we abbreviate the partial differential operators $\partial_{x_i}$ to $\partial_i$. We denote $(\nabla \times)^2 \mathbf{u} = \nabla \times \nabla \times \mathbf{u}$.

For $s = 1, 2$, we define

$$H(\text{curl}^s; D) := \{ \mathbf{u} \in L^2(D) : (\nabla \times)^j \mathbf{u} \in L^2(D), j = 1, s \}$$

with scalar products and norms are defined by

$$(\mathbf{u}, \mathbf{v})_{H(\text{curl}^s; D)} = (\mathbf{u}, \mathbf{v}) + \sum_{j=1}^{s} ((\nabla \times)^j \mathbf{u}, (\nabla \times)^j \mathbf{v})$$
and
\[ \|u\|_{H(\text{curl}_s; D)} = \sqrt{(u, u)_{H(\text{curl}_s; D)}}. \]

The spaces \( H_0(\text{curl}_s; D)(s = 1, 2) \) are defined as follows:

\[
H_0(\text{curl}; D) := \{ u \in H(\text{curl}; D) : n \times u = 0 \text{ on } \partial D \},
\]

\[
H_0(\text{curl}_2; D) := \{ u \in H(\text{curl}_2; D) : n \times u = 0 \text{ and } \nabla \times u = 0 \text{ on } \partial D \}.
\]

For a subdomain \( D \), a face \( f \), or an edge \( e \), we use \( P_k \) to represent the space of polynomials on them with a degree of no more than \( k \). We denote \( P_k = (P_k(D))^3 \). We also define

\[
\mathcal{R}_k = P_{k-1} \oplus S_k \text{ with } S_k = \{ p \in \tilde{P}_k \mid x \cdot p = 0 \},
\]

whose dimension is
\[
\dim \mathcal{R}_k = \frac{k(k+2)(k+3)}{2}.
\]

For the space \( P_k \), we have the following decomposition [7]

\[
P_k = \nabla P_{k+1} \oplus x \times P_{k-1}, \tag{2.1}
\]

\[
P_k = \nabla \times \mathcal{R}_{k+1} \oplus x P_{k-1}. \tag{2.2}
\]

The dimension of \( x \times P_{k-1} \) is \( \dim P_k - \dim P_{k+1} + 1 \) and the dimension of \( \nabla \times \mathcal{R}_{k+1} \) is \( \dim P_k - \dim P_{k-1} \).

We adopt the following Piola mapping [20] to relate the finite element function \( u \) on a general element \( K \) to a function \( \hat{u} \) on the reference element \( \hat{K} \) (the tetrahedron with vertices \((0,0,0), (1,0,0), (0,1,0), \) and \((0,0,1))):}

\[
u \circ F_K = B_K^{-T} \hat{u}, \tag{2.3}
\]

where the affine mapping
\[
F_K(x) = B_K \hat{x} + b_K. \tag{2.4}
\]

By a simple computation, we have
\[
(\nabla \times u) \circ F_K = \frac{B_K}{\det(B_K)} \hat{\nabla} \times \hat{u}, \tag{2.5}
\]
\[ \mathbf{n} \circ F_K = \frac{B_K^{-\mathbf{r}} \mathbf{n}}{\|B_K^{-\mathbf{r}} \mathbf{n}\|}, \quad \tau \circ F_K = \frac{B_K \mathbf{\hat{r}}}{\|B_K \mathbf{\hat{r}}\|}. \quad (2.6) \]

\[ \tau \circ F_K = \frac{B_K \mathbf{\hat{r}}}{\|B_K \mathbf{\hat{r}}\|}. \quad (2.7) \]

3. The Finite Element on a Tetrahedron

In this section, we will construct a family of finite elements, built on a tetrahedron, which is conforming in the space \( H(\text{curl}^2) \). To this end, we first introduce the following lemma which tells us the continuity conditions the finite element should satisfy.

**Lemma 3.1.** Let \( K_1 \) and \( K_2 \) be two non-overlapping Lipschitz domains having a common face \( \Lambda \) such that \( \overline{K_1} \cap \overline{K_2} = \Lambda \). Assume that \( \mathbf{u}_1 \in H(\text{curl}^2; K_1) \), \( \mathbf{u}_2 \in H(\text{curl}^2; K_2) \), and \( \mathbf{u} \in L^2(K_1 \cup K_2 \cup \Lambda) \) is defined by

\[
\mathbf{u} = \begin{cases} 
\mathbf{u}_1, & \text{in } K_1, \\
\mathbf{u}_2, & \text{in } K_2.
\end{cases}
\]

Then \( \mathbf{u}_1 \times \mathbf{n}_1 = -\mathbf{u}_2 \times \mathbf{n}_2 \) and \( \nabla \times \mathbf{u}_1 \times \mathbf{n}_1 = -\nabla \times \mathbf{u}_2 \times \mathbf{n}_2 \) on \( \Lambda \) implies that \( \mathbf{u} \in H(\text{curl}^2; K_1 \cup K_2 \cup \Lambda) \), where \( \mathbf{n}_i \ (i = 1, 2) \) is the unit outward normal vector to \( \partial K_i \), and note that \( \mathbf{n}_1 = -\mathbf{n}_2 \).

**Proof.** The proof is similar with that of Lemma 3.1 in [24]. \( \blacksquare \)

From Lemma 3.1, we know that the element we will construct should satisfy the following continuity conditions:

- \( \mathbf{u}_1 \times \mathbf{n}_1 = -\mathbf{u}_2 \times \mathbf{n}_2 \).
- \( \nabla \times \mathbf{u}_1 \times \mathbf{n}_1 = -\nabla \times \mathbf{u}_2 \times \mathbf{n}_2 \).
- \( \nabla \times \mathbf{u}_1 \cdot \mathbf{n}_1 = \nabla \cdot (\mathbf{u}_1 \times \mathbf{n}_1) = -\nabla \cdot (\mathbf{u}_2 \times \mathbf{n}_2) = -\nabla \times \mathbf{u}_2 \cdot \mathbf{n}_2 \).

The last two conditions imply \( \nabla \times \mathbf{u}_1 \) and \( \nabla \times \mathbf{u}_2 \) are continuous across the face. Based on above continuity conditions, we give the definition of the element as follows.
**Definition 3.1** (Curl-curl-conforming element or $H(\text{curl}^2)$-conforming element on a tetrahedron). For any integer $k \geq 7$, the $H(\text{curl}^2)$-conforming element is defined by the triple:

- $K$ is a tetrahedron (see Figure 3.1)
- $P_K = \mathcal{R}_k$,
- $\Sigma_K = M_p(u) \cup M_e(u) \cup M_f(u) \cup M_K(u)$,

where $\Sigma_K$ is the set of the DOFs which are defined by the following.

- **Vertex DOFs:** $M_p(u)$
  - $\nabla \times u(p_i), \; i = 1, 2, 3, 4.$
  - $D(\nabla \times u)(p_i), \; i = 1, 2, 3, 4,$ except for $\partial_{x_3} (\nabla \times u)_{3}(p_i)$.
  - $D^2(\nabla \times u)(p_i), \; i = 1, 2, 3, 4,$ except for $\partial^2_{x_3 x_1} (\nabla \times u)_{1}(p_i), \; \partial^2_{x_2 x_2} (\nabla \times u)_{2}(p_i), \; \partial^2_{x_3 x_3} (\nabla \times u)_{3}(p_i)$.

  Here, we use $D\mathbf{v}$ to represent all the first-order derivatives of $\mathbf{v}$ and $D^2 \mathbf{v}$ all the second-order derivatives.

- **Edge DOFs:** $M_e(u)$
  - $\int_{e_i} \mathbf{u} \cdot \mathbf{\tau}_i q ds, \; \forall q \in P_{h-1}(e_i), \; i = 1, 2, \cdots, 6.$

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**Figure 3.1. A reference tetrahedron**
\[ - (\nabla \times \mathbf{u})(e_{ij}^0), \ i = 1, 2, \cdots, 6, \ j = 1, 2, \cdots, k - 6. \]
\[ - \nabla(\nabla \times \mathbf{u} \cdot \mathbf{v}_i)(e_{ij}^1) \cdot \mathbf{n}_i, \ i = 1, 2, \cdots, 6, \ j = 1, 2, \cdots, k - 5, \ \mathbf{v}_i = \mathbf{\tau}_i \mathbf{n}_i, \text{or} \ \mathbf{m}_i. \]
\[ - \nabla(\nabla \times \mathbf{u} \cdot \mathbf{v}_i)(e_{ij}^1) \cdot \mathbf{m}_i, \ i = 1, 2, \cdots, 6, \ j = 1, 2, \cdots, k - 5, \ \mathbf{v}_i = \mathbf{\tau}_i \text{or} \ \mathbf{n}_i. \]

where \( \mathbf{\tau}_i, \mathbf{n}_i, \mathbf{m}_i \) are the unit tangential vector and two unit normal vectors to the edge \( \mathbf{e}_i, \mathbf{e}_{ij}^0, j = 1, 2, \cdots, k - 6 \) are \( k - 6 \) distinct nodes on edge \( \mathbf{e}_i \) and \( \mathbf{e}_{ij}^1, j = 1, 2, \cdots, k - 5 \) are \( k - 5 \) distinct nodes on edge \( \mathbf{e}_i \).

- **Face DOFs:** \( M_f(\mathbf{u}) \)
  \[ - \frac{1}{\text{area}(f_i)} \int_{f_i} \mathbf{u} \cdot \mathbf{q} dS, \ \forall \mathbf{q} = B_K \hat{\mathbf{q}}, \ \hat{\mathbf{q}} \in P_{k-3}(\hat{f}_i)[\hat{\mathbf{x}} - (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}_i)\hat{\mathbf{n}}_i]|_{f_i}, \ i = 1, 2, 3, 4. \]
  \[ - \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \mathbf{u} \times \mathbf{n}_i \cdot \mathbf{q} dS, \ \forall \mathbf{q} \in [P_{k-7}(f_i)]^2, \ i = 1, 2, 3, 4. \]
  \[ - \int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{q} dS, \ \forall \mathbf{q} \in P_{k-7}(f_i)/\mathbb{R}, \ i = 1, 2, 3, 4, \]

where \( \mathbf{n}_i \) is the unit outward normal vector of the face \( f_i \).

- **Interior DOFs:** \( M_K(\mathbf{u}) \)
  \[ - \int_K \mathbf{u} \cdot \mathbf{q} dV, \ \forall \mathbf{q} = \det(B_K)^{-1} B_K \hat{\mathbf{q}}, \ \hat{\mathbf{q}} \in P_{k-4}(\hat{K})\hat{\mathbf{x}}. \]
  \[ - \int_K \nabla \times \mathbf{u} \cdot \mathbf{q} dV, \ \forall \mathbf{q} = \det(B_K) B_K^{-T} \hat{\mathbf{q}}, \ \forall \hat{\mathbf{q}} \in \hat{\mathbf{x}} \times P_{k-7}(\hat{K}). \]

**Remark 3.1.** For the DOF \( \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{n}_i \cdot \mathbf{q} dS \), we can replace it by \( \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{\tau}_i q dS \) and \( \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \mathbf{u} \cdot \mathbf{\tau}_2 q dS \). Here, \( q \in P_{k-7}(f_i) \) and \( \mathbf{\tau}_1, \mathbf{\tau}_2 \) are two non-collinear unit vectors in the face \( f_i \).

Now we have \((26 \times 4)\) node DOFs, \((6k + 18(k - 6) + 30(k - 5))\) edge DOFs, \((2(k - 2)(k - 1) + 6(k - 6)(k - 5) - 4)\) face DOFs, and \(((k - 3)(k - 2)(k - 1)/6 + (k - 5)(k - 4)(k - 3)/2 - (k - 4)(k - 3)(k - 2)/6 + 1)\) element DOFs, and therefore
\[
\dim(P_K) = 26 \times 4 + 6k + 18(k - 6) + 30(k - 5) + 2(k - 2)(k - 1) + 6(k - 6)(k - 5) - 4 \\
+ k^3/2 - 11k^2/2 + 21k - 26 = \frac{k(k + 2)(k + 3)}{2} = \dim(\mathcal{R}_k).
\]

Since \( k \geq 7 \), the minimum number of DOFs is 315.

**Theorem 3.1.** The DOFs defined in Definition 3.1 are well-defined for any \( \mathbf{u} \in H^{1/2+\delta}(K) \) with \( \delta > 0 \) and \( \nabla \times \mathbf{u} \in C^2(K) \).

**Proof.** Since \( \nabla \times \mathbf{u} \in C^2(K) \), the vertex DOFs and edge DOFs involving \( \nabla \times \mathbf{u} \) are well-defined. It follows from the Cauchy-Schwarz inequality that the face DOFs and interior
Figure 3.2. The vertex and edge DOFs of a 3D Argyris element and a 2D Argyris element

DOFs are well-defined since \( \mathbf{u}, \nabla \times \mathbf{u} \in H^{1/2+\delta}(K) \) and \( \mathbf{u}|_{\partial K}, (\nabla \times \mathbf{u})|_{\partial K} \in H^{\delta}(\partial K) \). By the argument in the proof of Lemma 5.38 in [16], the DOF \( \int_{e_i} \mathbf{u} \cdot \mathbf{\tau}_i q ds \) is well-defined if \( \nabla \times \mathbf{u} \in L^p(K) \) with \( p > 2 \). We has completed the proof since \( \nabla \times \mathbf{u} \in C^2(K) \subset L^p(K) \).

\[ \nabla \times \mathbf{u} = 0 \]

\[ D(\nabla \times \mathbf{u}) = 0 \]

**Theorem 3.2.** The finite element given by Definition 3.1 is unisolvent and conforming in \( H(\text{curl}^2) \).

**Proof.** (i). To prove the \( H(\text{curl}^2) \) conformity, it suffices to prove \( \mathbf{u} \times \mathbf{n}_i = 0 \) and \( \nabla \times \mathbf{u} = 0 \) on a face when all DOFs associated with this face vanish. Without loss of generality, we only consider the face \( f_4 \) in Figure 3.1 \( (x_3 = 0) \). By the first kind of edge DOFs,

\[ \mathbf{u} \cdot \mathbf{\tau}_i = 0 \] on each edge \( e_i \).

Furthermore, the node DOFs and edge DOFs involving \( \nabla \times \mathbf{u} \) perform like the 3D Argyris element (if restricted on a face, it will be like a 2D Argyris element, see Fig 3.2), we can get

\[ \nabla \times \mathbf{u} = 0 \] and \( D(\nabla \times \mathbf{u}) = 0 \) on each edge.
Restricted on the face $f_4$, we have
\[ \nabla \times \mathbf{u} \cdot \mathbf{n}_4 = 0 \quad \text{and} \quad D(\nabla \times \mathbf{u} \cdot \mathbf{n}_4) = 0 \] on each edge of this face.

Hence we can rewrite it as,
\[ \nabla \times \mathbf{u} \cdot \mathbf{n}_4 = x_1^2 x_2^2 (1 - x_1 - x_2)^2 \varphi, \quad \varphi \in P_{k-7}(f_4). \]

By the integration by parts and $\mathbf{u} \cdot \tau|_{e_i} = 0$,
\[ \int_{f_4} \nabla \times \mathbf{u} \cdot \mathbf{n}_4 C dS = \int_{\partial f_4} C \mathbf{u} \cdot \tau_i ds + \int_{f_4} \mathbf{u} \times \mathbf{n}_4 \cdot \nabla C dS = 0, \]
which together with the third kind of face DOFs, we get
\[ \nabla \times \mathbf{u} \cdot \mathbf{n}_4 = 0. \]

Hence there exits a $\phi \in P_k(K)$ s.t.
\[ \mathbf{u} \cdot t_1 = \nabla \phi \cdot t_1, \]
\[ \mathbf{u} \cdot t_2 = \nabla \phi \cdot t_2, \]
where $t_1$ and $t_2$ are two mutually orthogonal unit vectors in the face $f_4$. In fact, since
\[ 0 = \nabla \times \mathbf{u} \cdot \mathbf{n}_4 = \nabla \cdot (\mathbf{u} \times \mathbf{n}_4), \]
we have $\mathbf{u} \times \mathbf{n}_4 = \nabla f \times \phi = (\nabla \phi \cdot \mathbf{t}_2) \mathbf{t}_1 + (-\nabla \phi \cdot \mathbf{t}_1) \mathbf{t}_2$
which implies $\mathbf{u} \cdot \mathbf{t}_1 = \nabla \phi \cdot \mathbf{t}_1$ and $\mathbf{u} \cdot \mathbf{t}_2 = \nabla \phi \cdot \mathbf{t}_2$ since $\mathbf{u} \times \mathbf{n}_4 = -(\mathbf{u} \cdot \mathbf{t}_1) \mathbf{t}_2 + (\mathbf{u} \cdot \mathbf{t}_2) \mathbf{t}_1$.

According to $\mathbf{u} \cdot \tau|_{e_i} = 0$, we have $\nabla \phi \cdot \tau_i = 0$ on $\partial f_4$, which implies that $\phi$ can be chosen as
\[ \phi = x_1 x_2 (1 - x_1 - x_2) \psi \quad \text{for some} \quad \psi \in P_{k-3}(f_4). \]

Applying integration by parts, we obtain, for a $\mathbf{q}$ such that $\mathbf{q} \cdot \mathbf{n}_4 = 0$,
\[ (\mathbf{u}, \mathbf{q})_{f_4} = (\phi, \nabla \cdot \mathbf{q})_{f_4} = (x_1 x_2 (1 - x_1 - x_2) \psi, \nabla \cdot \mathbf{q})_{f_4}. \]

By the first kind of face DOFs, we pick a $\mathbf{q}$ s.t. $\nabla \cdot \mathbf{q} = \psi$ and arrive at $\psi = 0$, i.e. $\mathbf{u} \times \mathbf{n}_4 = 0$. Recall that $D(\nabla \times \mathbf{u} \times \mathbf{n}_4) = 0$ and $\nabla \times \mathbf{u} \times \mathbf{n}_4 = 0$ on each edge of $f_4$, i.e.,
\[ (\nabla \times \mathbf{u})_1 = x_1^2 x_2^2 (1 - x_1 - x_2)^2 \varphi_1, \quad \varphi_1 \in P_{k-7}(f_4), \]
\[ (\nabla \times \mathbf{u})_2 = x_1^2 x_2^2 (1 - x_1 - x_2)^2 \varphi_2, \quad \varphi_2 \in P_{k-7}(f_4). \]
According to the second kind of face DOFs, $\nabla \times \mathbf{u} \times \mathbf{n}_4 = 0$ which together with $\nabla \times \mathbf{u} \cdot \mathbf{n}_4 = 0$ leads to $\nabla \times \mathbf{u} = 0$ on the face $f_4$.

(ii). Now, we consider the unisolvence. We only need to prove that vanishing all DOFs for $\mathbf{u} \in P_K$ yields $\mathbf{u} = 0$. By virtue of the fact that $\nabla \times \mathbf{u} = 0$ on $\partial K$, we can rewrite $\nabla \times \mathbf{u}$ as:

$$\nabla \times \mathbf{u} = x_1 x_2 x_3 (1 - x_1 - x_2 - x_3) \Phi$$

with $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in P_{k-5}(K)$, and hence

$$\partial_{x_1}(\nabla \times \mathbf{u})_1 = x_2 x_3 (1 - 2 x_1 - x_2 - x_3) \Phi_1 + (1 - x_1 - x_2 - x_3) x_1 \partial_{x_1} \Phi_1,$$

$$\partial_{x_2}(\nabla \times \mathbf{u})_2 = x_1 x_3 (1 - 2 x_2 - x_1 - x_3) \Phi_2 + (1 - x_1 - x_2 - x_3) x_2 \partial_{x_2} \Phi_2,$$

$$\partial_{x_3}(\nabla \times \mathbf{u})_3 = x_1 x_2 (1 - 2 x_3 - x_1 - x_2) \Phi_3 + (1 - x_1 - x_2 - x_3) x_3 \partial_{x_3} \Phi_3.$$

When $x_1 = 0$, $\partial_{x_2}(\nabla \times \mathbf{u})_2 + \partial_{x_3}(\nabla \times \mathbf{u})_3 = 0$ which leads to $\partial_{x_1}(\nabla \times \mathbf{u})_1 = 0$ because $\nabla \cdot \nabla \times \mathbf{u} = \partial_{x_1}(\nabla \times \mathbf{u})_1 + \partial_{x_2}(\nabla \times \mathbf{u})_2 + \partial_{x_3}(\nabla \times \mathbf{u})_3 = 0$. It implies $\Phi_1$ has a factor $x_1$. Similarly, $\Phi_2$ has a factor $x_2$ and $\Phi_3$ has a factor $x_3$. Then

$$\nabla \times \mathbf{u} = x_1 x_2 x_3 (1 - x_1 - x_2 - x_3) [x_1 \Phi_1, x_2 \Phi_2, x_3 \Phi_3]^T$$

with $\Phi = [\Phi_1, \Phi_2, \Phi_3] \in P_{k-6}(K), i = 1, 2, 3$. Due to the second kind of interior vanishing DOFs and the fact that

$$(\nabla \times \mathbf{u}, \nabla q)_K = (\mathbf{u} \times \mathbf{n}, \nabla q)_{\partial K} = 0, \forall q \in P_{k-5}(K),$$

we have

$$(\nabla \times \mathbf{u}, q)_K = 0, \forall q \in P_{k-6}(K), \quad (3.1)$$

here we used the decomposition (2.1). By setting $q = \Phi$ in (3.1), we get

$$\nabla \times \mathbf{u} = 0 \text{ in } K.$$ 

Therefore, we can choose a $\Psi = x_1 x_2 x_3 (1 - x_1 - x_2 - x_3) \Psi$ with $\Psi \in P_{k-4}(K)$ such that

$$\mathbf{u} = \nabla \Psi.$$
Again, by applying integration by parts,
\[(u, q)_K = (\nabla \Psi, q)_K = (\Psi, \nabla \cdot q)_K = (x_1 x_2 x_3 (1 - x_1 - x_2 - x_3) \bar{\Psi}, \nabla \cdot q)_K.
\]
Using the first kind of interior DOFs and choosing a \( q \) s.t. \( \nabla \cdot q = \bar{\Psi} \), we get \( \bar{\Psi} = 0 \) and hence \( u = 0. \)

4. Error Estimate of Finite Element Interpolation

Provided \( u \in H^{1/2+\delta} (K) \) with \( \delta > 0 \) and \( \nabla \times u \in C^2 (K) \), we can define an \( H(\text{curl}^2) \) interpolation operator on \( K \) denoted as \( \Pi_K \) by
\[ M_p(u - \Pi_K u) = 0, M_e(u - \Pi_K u) = 0, M_f(u - \Pi_K u) = 0, \text{and } M_K(u - \Pi_K u) = 0, \]
where \( M_p, M_e, M_f, \) and \( M_K \) are the sets of DOFs in Definition 3.1.

The DOFs of the finite element defined in Definition 3.1 involves normal derivatives to edges, it’s hard to relate the interpolation \( \Pi_K \) on a general element \( K \) to \( \Pi_{\hat{K}} \) on a reference element \( \hat{K} \). To estimate the interpolation error, we introduce a finite element slightly different from our element, but the corresponding interpolation on \( K \) and that on \( \hat{K} \) can be related easily.

Definition 4.1. For any integer \( k \geq 7 \), an \( H(\text{curl}) \)-conforming element is defined by the triple:

\( K \) is a tetrahedron (see Figure 3.1)
\[ P_K = \mathcal{R}_k, \]
\[ \hat{\Sigma}_K = \hat{M}_p(u) \cup \hat{M}_e(u) \cup \hat{M}_f(u) \cup \hat{M}_K(u), \]
where \( \hat{\Sigma}_K \) are the DOFs obtained by the following slight changes to \( \Sigma_K \) in Definition 3.1 with the other DOFs staying the same.

- Edge DOFs: \( \hat{M}_e(u) \)
  \[ - \left( \nabla (\nabla \times u \cdot (v_m^i \times v_n^j)) \cdot v_l^i \right) (e_{ij}^l), i = 1, 2, \cdots, 6, j = 1, 2, \cdots, k-5, \{l, [m, n]\} = \{2, [1, 2]\}, \{2, [1, 3]\}, \{2, [2, 3]\}, \{3, [1, 2]\}, \text{or } \{3, [1, 3]\}, \]
where $v^i_n (n = 1, 2, 3)$ are the three edges intersected at the beginning point of $e_i$ with $v^i_1 = e_i$ and $e^i_{1j} (j = 1, 2, \cdots, k - 5)$ are $k - 5$ distinct nodes on edge $e_i$.

- Face DOFs: $\tilde{M}_f(u)$
  \[- \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times u \cdot (v^i_m \times v^i_n) q dS, \forall q \in P_{k-7}(f_i), i = 1, 2, 3, 4, [m, n] = \{[1, 3], [2, 3]\}.
\]
where $v^i_l (l = 1, 2, 3)$ are the three edges intersected at the beginning point of the face $f_i$ with $v^i_1 (l = 1, 2)$ are the two edges in the face $f_i$.

**Remark 4.1.** With these small changes, the element is no longer conforming in $H(\text{curl}^2)$. It’s only conforming in $H(\text{curl})$.

Provided $u \in H^{1/2+\delta}(K)$ with $\delta > 0$ and $\nabla \times u \in C^2(K)$, we can define an $H(\text{curl})$ interpolation operator on $K$ denoted as $\Lambda_K$ by

\[
\tilde{M}_p(u - \Lambda_K u) = 0, \quad \tilde{M}_e(u - \Lambda_K u) = 0, \quad \tilde{M}_f(u - \Lambda_K u) = 0, \quad \text{and} \quad \tilde{M}_K(u - \Lambda_K u) = 0,
\]

(4.2)

where $\tilde{M}_p$, $\tilde{M}_e$, $\tilde{M}_f$, and $\tilde{M}_K$ are the sets of DOFs in Definition 4.1.

**Lemma 4.1.** Assume that $\Lambda_K$ is well-defined. Then under the transformation (2.3), we have $\Lambda_K u \circ F_K = B_K^{-T} \Lambda_{\hat{K}} \hat{u}$.

Proof. By the transformations (2.3), (2.5), (2.6), and (2.7), we have all the DOFs defined in $\Sigma_K$ except for the vertex DOFs and the second kind of edge DOFs are equivalent with those for $\hat{u}$ on $\hat{K}$. The vertex DOFs and the second kind of edge DOFs are liner combinations of those for $\hat{u}$ on $\hat{K}$. According to Proposition 3.4.7 in [2], we complete the proof. $\blacksquare$

To get the error estimate of the interpolation operator $\Lambda_K$, we introduce an interpolation $I_K w \in P_{k-1}(k \geq 7)$ for $w$ s.t.

- Vertex DOFs: $i = 1, 2, 3, 4$
  \[- (I_K w)(p_i) = w(p_i).
  \]
  \[- D(I_K w)(p_i) = D w(p_i).
\]
\[ D^2(I_K \mathbf{w})(\mathbf{p}_i) = D^2 \mathbf{w}(\mathbf{p}_i). \]

- **Edge DOFs:** \( i = 1, 2, \ldots, 6 \)
  \[ (I_K \mathbf{w})(e^0_{ij}) = \mathbf{w}(e^0_{ij}), \quad j = 1, 2, \ldots, k - 6. \]
  \[ (\nabla(I_K \mathbf{w} \cdot (\mathbf{v}_m \times \mathbf{v}_n))_i \cdot \mathbf{v}_i)(e^1_{ij}) = (\nabla(\mathbf{w} \cdot (\mathbf{v}_m \times \mathbf{v}_n))_i \cdot \mathbf{v}_i)(e^1_{ij}), \quad j = 1, 2, \ldots, k - 5, \]
  \{i, [m, n]\} = \{2, [1, 2]\}, \{2, [1, 3]\}, \{2, [2, 3]\}, \{3, [1, 2]\}, \{3, [1, 3]\}, or \{3, [2, 3]\}.

where \( \mathbf{v}_n^i (n = 1, 2, 3) \) are the three edges intersected at the beginning point of \( e_i \) with \( \mathbf{v}_1^i = e_i, \mathbf{e}_{ij}^0 (j = 1, 2, \ldots, k - 6) \) are \( k - 6 \) distinct nodes on edge \( e_i \) and \( \mathbf{e}_{ij}^1 (j = 1, 2, \ldots, k - 5) \) are \( k - 5 \) distinct nodes on edge \( e_i \).

- **Face DOFs:** \( i = 1, 2, 3, 4 \)
  \[ - \frac{1}{\text{area}(f_i)} \int_{f_i} I_K \mathbf{w} \cdot (\mathbf{v}_m \times \mathbf{v}_n)q \mathbf{d}S = \frac{1}{\text{area}(f_i)} \int_{f_i} \mathbf{w} \cdot (\mathbf{v}_m \times \mathbf{v}_n)q \mathbf{d}S, \quad \forall q \in P_{k-7}(f_i), \]
  \[ [m, n] = \{1, 3\}, \{2, 3\}. \]
  \[ - \int_{f_i} I_K \mathbf{w} \cdot \mathbf{n} q \mathbf{d}S = \int_{f_i} \mathbf{w} \cdot \mathbf{n} q \mathbf{d}S, \quad \forall q \in P_{k-7}(f_i). \]
  \[ - \det(B_K) \int_{f_i} \nabla \cdot I_K \mathbf{w} q \mathbf{d}S = \det(B_K) \int_{f_i} \nabla \cdot \mathbf{w} q \mathbf{d}S, \quad \forall q \in P_{k-5}(f_i). \]

where \( \mathbf{v}_l^i (l = 1, 2, 3) \) are the three edges intersected at the beginning point of the face \( f_i \) with \( \mathbf{v}_1^i (l = 1, 2) \) are the two edges in the face \( f_i \).

- **Interior DOFs:**
  \[ - \int_K I_K \mathbf{w} \cdot \mathbf{q} \mathbf{d}V = \int_K \mathbf{w} \cdot \mathbf{q} \mathbf{d}V, \quad \forall \mathbf{q} = \det(B_K)B_K^{-T} \hat{\mathbf{q}}, \quad \forall \hat{\mathbf{q}} \in \hat{\mathbf{x}} \times P_{k-7}(\hat{K}). \]
  \[ - \int_K \nabla \cdot I_K \mathbf{w} q \mathbf{d}V = \int_K \nabla \cdot \mathbf{w} q \mathbf{d}V, \quad \forall q = \det(B_K)^{-1} \hat{q}, \quad \forall \hat{q} \in P_{k-6}(K)/\mathbb{R}. \]

**Remark 4.2.** All the DOFs used to define \( I_K \) are those DOFs in Definition 4.1 involving \( \nabla \times \mathbf{u} \) together with all the missing DOFs because of \( \nabla \cdot \nabla \times \mathbf{u} = 0 \).

**Lemma 4.2.** If \( \mathbf{w} \in H^s(K) \cap C^2(K) \) and there exists a pair \( \{m, q\} \) s.t. \( H^s(K) \hookrightarrow W^{m,q}(K) \), then we have the following error estimates for the interpolation \( I_K \).

\[
\| \mathbf{w} - I_K \mathbf{w} \|_{m,q,K} \leq C |K|^{1/2} h_K^{s-m} \| \mathbf{w} \|_{s,K}.
\]

**Proof.** The proof is standard, c.f. Theorem 3.1.4 in [6]. \( \blacksquare \)

**Lemma 4.3.** If \( \nabla \times \mathbf{u} \in H^s(K) \cap C^2(K) \), then

\[
\| I_K \nabla \times \mathbf{u} - \nabla \times \Lambda_K \mathbf{u} \|_{m,q,K} \leq |K|^{1/2} h_K^{s-m} \| \nabla \times \mathbf{u} \|_{s,K}.
\]
Proof. For simplicity of notations, we let \( w = I_K \nabla \times u - \nabla \times \Lambda_K u \). Since \( w \in P_{k-1} \), we have \( w = I_K w = \sum c_i(w)N_i \), where \( c_i(w) \) are the DOFs and \( N_i \) are the corresponding basis functions. We first show all the DOFs except for \( \nabla (w \cdot (v^i_2 \times v^i_3)) \cdot v^i_3 \) vanish. Some of those are obvious 0 by the definition of \( I_K \) and \( \Lambda_K \). We only need to check the others.

At first,

\[
\begin{align*}
  w_{3z} &= (I_K \nabla \times u - \nabla \times \Lambda_K u)_{3z} = (I_K \nabla \times u)_{3z} + (\nabla \times \Lambda_K u)_{1x} + (\nabla \times \Lambda_K u)_{2y} \\
  &= (\nabla \times u)_{3z} + (\nabla \times u)_{1x} + (\nabla \times u)_{2y} = 0.
\end{align*}
\]

Similarly, \( w_{1xx} = w_{2yy} = w_{3zz} = 0 \).

Applying integration by parts as well as the definition of \( I_K \) and \( \Lambda_K \), we have

\[
\int_{f_i} w \cdot n_i C dS = \int_{f_i} \nabla \times u \cdot n_i C dS - \int_{f_i} (\nabla \times \Lambda_K u) \cdot n_i C dS
= \int_{f_i} \nabla \times u \cdot n_i C dS - \int_{\partial f_i} \Lambda_K u \cdot \tau C ds = \int_{f_i} \nabla \times u \cdot n_i C dS - \int_{\partial f_i} u \cdot \tau C ds = 0.
\]

Finally, using the definition of \( I_K \) and the fact \( \nabla \cdot \nabla \times = 0 \), we have

\[
\begin{align*}
  \int_{f_i} \nabla \cdot w q dS &= \int_{f_i} \nabla \cdot (\nabla \times u - \nabla \times \Lambda_K u) q dS = 0, \\
  \int_{K} \nabla \cdot w q dV &= \int_{K} \nabla \cdot (\nabla \times u - \nabla \times \Lambda_K u) q dV = 0.
\end{align*}
\]

Now we estimate the non-vanishing term. By the definition of \( I_K \), we have

\[
\nabla (w \cdot (v^i_2 \times v^i_3)) \cdot v^i_3 = \nabla ((\nabla \times (u - \Lambda_K u)) \cdot (v^i_2 \times v^i_3)) \cdot v^i_3.
\]

Since the divergence of \( \nabla \times (u - \Lambda_K u) \) is 0, we can find 8 constants \( C^l_{[m,n]} (1 \leq l \leq 3, 1 \leq m < n \leq 3 \) except for the case \( l = 3, m = 2 \), and \( n = 3 \) \) independent of \( h_K \) s.t.

\[
\nabla ((\nabla \times (u - \Lambda_K u)) \cdot (v^i_2 \times v^i_3)) \cdot v^i_3
= \sum_{\{l,[m,n]\} \neq \{3,[2,3]\}} C^l_{[m,n]} \nabla ((\nabla \times (u - \Lambda_K u)) \cdot (v^i_m \times v^i_n)) \cdot v^i_i,
\]
which can be finished by mapping to the reference element, finding the constant and then mapping back. Furthermore, we have

\[
\sum_{\{l[m,n]\} \neq \{3[2,3]\}} C^l_{[m,n]} \nabla ((\nabla \times (u - \Lambda_K u)) \cdot (v^i_m \times v^i_n)) \cdot v^i_l
\]

\[
= \sum_{\{1[m,n]\}} C^l_{[m,n]} \nabla ((\nabla \times u - I_K \nabla \times u)) \cdot (v^i_m \times v^i_n)) \cdot v^i_1, \quad \text{the definition of } \Lambda_K u
\]

the last equal sign is because \( \nabla \times \Lambda_K u \) restricted on the edge \( e_i \) is a polynomial vector of order \( k - 1 \) which can be determined by all the vertex DOFs

\[
\nabla((\nabla \times (u - \Lambda_K u)) \cdot (v^i_m \times v^i_n)) \cdot v^i_l = \sum_{1 \leq m<n \leq 3} C^l_{[m,n]} \nabla ((\nabla \times u - I_K \nabla \times u)) \cdot (v^i_m \times v^i_n)) \cdot v^i_1,
\]

\[
\leq Ch^3_K |\nabla \times u - I_K (\nabla \times u)|_{1,\infty,K}
\]

\[
\leq C|K|^{-1/2} h^3_K h^{-1}_K |\nabla \times u|_{s,K}. \tag{4.3}
\]

Suppose \( N_i \) is the basis functions associated with the non-vanishing DOFs. Then

\[
\|N_i\|_{m,q,K} \leq Ch^{-2-m+3/q} \|\hat{N}_i\|_{m,q,K}, \tag{4.4}
\]

where \( \hat{N}_i = |B_K|B_K^{-1}N_i \) are the basis functions on the reference element.

Combining (4.3) and (4.4), we complete the proof. \( \blacksquare \)

**Theorem 4.1.** If \( u \in H^s(K) \) and \( \nabla \times u \in H^s(K) \cap C^2(K) \) with \( s > 1 \), then we have the following error estimates for the interpolation \( \Lambda_K \),

\[
\|u - \Lambda_K u\|_K \leq Ch^s_K (\|u\|_{s,K} + \|\nabla \times u\|_{s,K}), \tag{4.5}
\]

\[
\|\nabla \times (u - \Lambda_K u)\|_K \leq Ch^s_K \|\nabla \times u\|_{s,K}. \tag{4.6}
\]
\[ \| (\nabla \times)^2 (u - \Lambda_K u) \|_{K} \leq Ch_K^{s-1} \| \nabla \times u \|_{s,K}. \]  

(4.7)

**Proof.** Because of the relationship \( \Lambda_K u \circ F_K = B_K^T \Lambda_K \hat{u} \) obtained in Lemma 4.1, the proof is standard, cf, Theorem 3.11 in [24], where Lemma 4.2 will be used. \( \blacksquare \)

**Theorem 4.2.** If \( u \in H^s(K) \) and \( \nabla \times u \in H^s(K) \cap C^2(K) \) with \( s > 1 \), then we have the following error estimates for the interpolation \( \Pi_K \),

\[ \| u - \Pi_K u \|_{K} \leq Ch_K^s(\| u \|_{s,K} + \| \nabla \times u \|_{s,K}), \]  

(4.8)

\[ \| \nabla \times (u - \Pi_K u) \|_{K} \leq Ch_K^s \| \nabla \times u \|_{s,K}, \]  

(4.9)

\[ \| (\nabla \times)^2 (u - \Pi_K u) \|_{K} \leq Ch_K^{s-1} \| \nabla \times u \|_{s,K}. \]  

(4.10)

**Proof.** Since \( u - \Pi_K u = u - \Lambda_K u + \Lambda_K u - \Pi_K u \), it remains to estimate \( \Lambda_K u - \Pi_K u \) in three different norms or semi-norms. We denote \( \Delta = \Lambda_K u - \Pi_K u \) which is a polynomial with a degree of no more than 7. Also, the DOFs in \( \tilde{\Sigma}_K \) for \( \Delta \) except for \( (\nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i)) \cdot v_i^i(e_{ij}) + \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot (v_m^i \times v_n^i) \text{qdS}) \) are all 0. Then

\[ \Delta = \sum_{i=1}^{6} \sum_{j=1}^{k-5} \sum_{l=2}^{3} (\nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i)) \cdot v_i^i(e_{ij}) N_{ijkl}^{[m,n]} + \sum_{i=1}^{4} \sum_{[m,n]} \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot (v_m^i \times v_n^i) \text{qdS} N_{i}^{[m,n]}, \]

where \( N_{ijkl}^{[m,n]} \) and \( N_{i}^{[m,n]} \) are the basis functions of the finite element defined in Def. 4.1 which are associated with the DOFs \( (\nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i)) \cdot v_i^i(e_{ij}) + \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot (v_m^i \times v_n^i) \text{qdS}) \). We write \( v_m^i \times v_n^i \) and \( v_i^i \) as a linear combination of \( \tau, m, n \) and get

\[ \nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i)) \cdot v_i^i \]

\[ = \nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i \cdot \tau) + (v_m^i \times v_n^i \cdot n)n + (v_m^i \times v_n^i \cdot m)m) \]

\[ = \nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i \cdot \tau) + (v_m^i \times v_n^i \cdot n)n + (v_m^i \times v_n^i \cdot m)m) \]

\[ = \nabla(\nabla \times \Delta \cdot (v_m^i \times v_n^i \cdot \tau) + (v_m^i \times v_n^i \cdot n)n + (v_m^i \times v_n^i \cdot m)m) \]

\[ \cdot \left( (v_i^i \cdot n)n + (v_i^i \cdot m)m \right) \]
\[
\begin{align*}
&\nabla (\nabla \times (u - \Lambda_K u) \cdot (v^i_m \times v^i_n \cdot \tau) \tau + (v^i_m \times v^i_n \cdot n)n + (v^i_m \times v^i_n \cdot m)m) \\
&\quad \cdot ((v^i_i \cdot n)n + (v^i_i \cdot m)m).
\end{align*}
\]

Each term has the following estimate. We only show the first term
\[
\begin{align*}
\nabla (\nabla \times (u - \Lambda_K u) \cdot ((v^i_m \times v^i_n \cdot \tau) \tau) \cdot (v^i_i \cdot n)n \\
\leq Ch^3_K |\nabla \times (u - \Lambda_K u)|_{1,\infty,K} \\
\leq C|K|^{-1/2}h^3_K \frac{h^s_K}{\rho_K} |\nabla \times u|_{s,K}.
\end{align*}
\]

Similarly,
\[
\begin{align*}
\frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot (v^i_m \times v^i_n)q dS \\
= \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot ((v^i_m \times v^i_n \cdot \tau_1) \tau_1 + (v^i_m \times v^i_n \cdot \tau_2) \tau_2 + (v^i_m \times v^i_n \cdot n)n)q dS \\
= \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot ((v^i_m \times v^i_n \cdot \tau_1) \tau_1 + (v^i_m \times v^i_n \cdot \tau_2) \tau_2)q dS,
\end{align*}
\]

where \(\tau_1\) and \(\tau_2\) are two unit orthogonal vectors in the face \(f_i\). Each term has the following estimate
\[
\begin{align*}
\frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \Delta \cdot (v^i_m \times v^i_n \cdot \tau_1) \tau_1 q dS \\
= \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times (u - \Lambda_K u) \cdot (v^i_m \times v^i_n \cdot \tau_1) \tau_1 q dS \\
\leq Ch^2_K |\nabla \times (u - \Lambda_K u)|_{0,\infty,K} \\
\leq C|K|^{-1/2}h^2_K h^s_K |\nabla \times u|_{s,K}.
\end{align*}
\]

According to the mapping (2.3), both \(N_{[m,n]}^{[m,n]}\) and \(N_i^{[m,n]}\) satisfy
\[
\begin{align*}
\|N_{[m,n]}\| &\leq Ch^{1/2}_K \|\tilde{N}_{[m,n]}\|, \\
\|\nabla \times N_{[m,n]}\| &\leq Ch^{-1/2}_K \|\tilde{\nabla} \times \tilde{N}_{[m,n]}\|, \\
\|\nabla \times \nabla \times N_{[m,n]}\| &\leq Ch^{-3/2}_K \|\tilde{\nabla} \times \tilde{\nabla} \times \tilde{N}_{[m,n]}\|,
\end{align*}
\]
where \( \hat{N}^{[m,n]} \) are the corresponding basis functions on \( \hat{K} \) and they satisfy \( N^{[m,n]} = B_K^T \hat{N}^{[m,n]} \). By combining the above estimates, we complete the proof.

5. Numerical Experiments

In this section, we use the \( H(\text{curl}^2) \)-conforming finite element developed in Section 3 to solve the quad-curl problem which is introduced as: For \( f \in H(\text{div}^0; \Omega) \), find \( u \) s.t.

\[
(\nabla \times)^4 u + u = f \quad \text{in } \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega,
\]

\[
u \times n = 0 \quad \text{on } \partial \Omega,
\]

\[
\nabla \times u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \in \mathbb{R}^3 \) is Lipschitz domain and \( n \) is the unit outward normal vector to \( \partial \Omega \). Divergence-free condition \( \nabla \cdot u = 0 \) satisfies automatically, since we have the lower-order term \( u \) in the equation (5.1). The variational formulation is to find \( u \in H_0(\text{curl}^2; \Omega) \), s.t.

\[
a(u, v) = (f, v) \quad \forall v \in H_0(\text{curl}^2; \Omega),
\]

with \( a(u, v) = (\nabla \times \nabla \times u, \nabla \times \nabla \times v) + (u, v) \).

Let \( T_h \) be a partition of the domain \( \Omega \) consisting of tetrahedra. We denote by \( h_K \) the diameter of each element \( K \in T_h \) and by \( h \) the mesh size of \( T_h \). We define

\[
V_h = \{ v_h \in H(\text{curl}^2; \Omega) : v_h|_K \in \mathcal{R}_k, \ \forall K \in T_h \}.
\]

\[
V_h^0 = \{ v_h \in V_h, \ n \times v_h = 0 \ \text{and} \ \nabla \times v_h = 0 \ \text{on} \ \partial \Omega \}.
\]

Now we define a global interpolation operator \( \Pi_h \) which is defined piecewise:

\[
\Pi_h|_K = \Pi_K.
\]

The global interpolation has the following error estimates.
**Theorem 5.1.** If \( u \in H^s(\Omega) \) and \( \nabla \times u \in H^s(\Omega) \cap C^2(\Omega) \) with \( s > 1 \), then we have the following error estimates for the interpolation \( \Pi_h \),

\[
\| u - \Pi_h u \| \leq C h^s (\| u \|_s + \| \nabla \times u \|_s), \tag{5.3}
\]
\[
\| \nabla \times (u - \Pi_h u) \| \leq C h^s \| \nabla \times u \|_s, \tag{5.4}
\]
\[
\| (\nabla \times)^2 (u - \Pi_h u) \| \leq C h^{s-1} \| \nabla \times u \|_s. \tag{5.5}
\]

The Theorem is proved by the fact \( \| (\nabla \times)^i (u - \Pi_h u) \| = \sum_{K \in T_h} \| (\nabla \times)^i (u - \Pi_K u) \|_K \) and Theorem 4.2.

The \( H(\text{curl}^2) \)-conforming finite element method seeks \( u_h \in V_0^h \), s.t.

\[
a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_0^h. \tag{5.6}
\]

To implement the boundary conditions, we let all the DOFs which yields homogenous boundary conditions be 0.

**5.1. Example 1.** We consider the problem (5.1) on a unit cube \( \Omega = (0, 1) \times (0, 1) \times (0, 1) \) with exact solution

\[
u = \begin{pmatrix}
0 \\
3\pi \sin^3(\pi x) \sin^3(\pi y) \sin^2(\pi z) \cos(\pi z) \\
-3\pi \sin^3(\pi x) \sin^3(\pi z) \sin^2(\pi y) \cos(\pi y)
\end{pmatrix}, \tag{5.7}
\]

Then the source term \( f \) can be obtained by a simple calculation. Denote

\[ e_h = u - u_h. \]

We partition the unit cube into \( N^3 \) small cubes and then partition each small cube into 6 congruent tetrahedra. Varying \( h = 1/N \) from \( 1/2 \) to \( 1/8 \), Table 5.1 illustrates the errors and convergence rates of \( u_h \) with \( k = 7 \) in several different norms. We can observe the convergence rates of 7 in \( H(\text{curl}) \) norm and of 6 in \( H(\text{curl}^2) \) norm which coincide with the convergence orders of the interpolation \( \Pi_h \).
Table 5.1. Example 1: Numerical results by the lowest-order tetrahedral $H$(curl$^2$) element

| $h$  | $\|\mathbf{u} - \mathbf{u}_h\|$ | rates | $\|\nabla \times \mathbf{u} - \nabla \times \mathbf{u}_h\|$ | rates | $\| (\nabla \times)^2 \mathbf{u} - (\nabla \times)^2 \mathbf{u}_h\|$ | rates |
|------|-------------------------------|-------|---------------------------------|-------|---------------------------------|-------|
| 1/2  | 3.8334785395e+00              | 10.8753 | 8.0089356298e-01               | 5.1543 | 1.6715185815e+01               | 4.0572 |
| 1/3  | 4.6617638169e-02              | 6.6651  | 9.9072060818e-02               | 5.1588 | 3.2621165763e-02               | 4.4177 |
| 1/4  | 6.8520104719e-03              | 9.6538  | 2.2460507680e-02               | 6.7446 | 9.0519796164e-01               | 5.5650 |
| 1/5  | 7.9482178822e-04              | 9.7577  | 4.9865729850e-03               | 7.3648 | 8.7363073645e-02               | 5.9716 |
| 1/6  | 1.3416712567e-04              | 9.7979  | 1.2897568262e-03               | 7.3648 | 8.7363073645e-02               | 5.9716 |
| 1/7  | 2.9628521344e-05              | 9.4904  | 4.1443815436e-03               | 7.1025 | 3.479494022e-02                | 5.8304 |
| 1/8  | 8.3433920597e-06              | 1.6053643598e-04 | 1.5974611799e-02 |

5.2. Example 2. In this example, we consider the problem (5.1) on a unit cube with the source term $\mathbf{f} = (1, 1, 1)^T$. In this case, we can not express the exact solution explicitly, so we seek an approximation of $\|\mathbf{u} - \mathbf{u}_h\| = \sqrt{a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h)}$. Due to the orthogonality $a(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h) = 0$, we have

$$
\|\mathbf{u} - \mathbf{u}_h\|^2 = \|\mathbf{u}\|^2 - 2a(\mathbf{u} - \mathbf{u}_h, \mathbf{u}) + \|\mathbf{u}_h\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}_h\|^2.
$$

Since $\mathbf{u}_h \in V_h \subset V_{h/2}$, $a(\mathbf{u} - \mathbf{u}_{h/2}, \mathbf{u}_h) = 0$ and then

$$
\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2}\|^2 - 2a(\mathbf{u}_{h/2}, \mathbf{u}_h) + \|\mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2}\|^2 - 2a(\mathbf{u}, \mathbf{u}_h) + \|\mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2}\|^2 - \|\mathbf{u}_h\|^2.
$$

Thanks to $a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_{h/2}) = a(\mathbf{u} - \mathbf{u}_{h/2}, \mathbf{u} - \mathbf{u}_{h/2})$ since $a(\mathbf{u}_h, \mathbf{u} - \mathbf{u}_{h/2}) = a(\mathbf{u}_{h/2}, \mathbf{u} - \mathbf{u}_{h/2}) = 0$, we have

$$
\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2} - \mathbf{u}\|^2 + 2a(\mathbf{u}_h - \mathbf{u}, \mathbf{u} - \mathbf{u}_{h/2}) + \|\mathbf{u} - \mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2} - \mathbf{u}\|^2 + \|\mathbf{u} - \mathbf{u}_h\|^2 \approx \|\mathbf{u} - \mathbf{u}_h\|^2.
$$

Now we can treat $\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 = \|\mathbf{u}_{h/2}\|^2 - \|\mathbf{u}_h\|^2$ as an approximation of $\|\mathbf{u} - \mathbf{u}_h\|^2$. The Table 5.2 shows that the convergence rates in energy norm are about 2. The convergence rate are deteriorated due to the poor solution regularity.
We also draw the graph of the numerical solution at $y = 0.1$ (which is close to the boundary). From the graph 5.1, we do not observe any oscillation phenomenon, which indicates the boundary conditions are implemented correctly.

Table 5.2. Example 2: Numerical results by the lowest-order tetrahedral $H(\text{curl}^2)$ element

| $h$     | $\|u_h\|$ | $\|\nabla \times u_h\|$ | $\|\nabla \times u_h\|^2$ | $\|u - u_h\|$ | rates |
|---------|-----------|------------------------|-----------------|-------------|-------|
| 1/1     | 4.0503711308e-04 | 2.1012866605e-03 | 2.201941906e-02 | 1.9004476086e-02 |       |
| 1/2     | 6.8754227877e-04 | 3.4074245801e-03 | 2.8957231505e-02 | 1.9895511952e-03 | 3.2558 |
| 1/4     | 6.8874370251e-04 | 3.4044210424e-03 | 2.9025822581e-02 | 3.6939587847e-04 | 2.4292 |
| 1/8     | 6.8880221694e-04 | 3.4044374844e-03 | 2.9028170036e-02 | 5.3. Example 3. We also consider the problem (5.1) on an L-shape domain $\Omega = (0, 1) \times (0, 1) / [0.5, 1) \times (0, 0.5) \times [0, 1]$ with source term $f = (1, 1, 1)^T$.

Table 5.3 illustrates errors and convergence rates of $u_h$ in this case. Due to the singularity of the domain, convergence rates deteriorate to around 4/3.

We should note that it is not very precise to use the mesh sizes 1, 1/2, 1/4, 1/5 for estimating the errors in energy norm. However, we only can refine the mesh to 1/5 because of the huge number of DOFs. We just use these data to estimate the errors approximately.

Table 5.3. Example 3: Numerical results by the lowest-order tetrahedral $H(\text{curl}^2)$ element

| $h$     | $\|u_h\|$ | $\|\nabla \times u_h\|$ | $\|\nabla \times u_h\|^2$ | $\|u - u_h\|$ | rates |
|---------|-----------|------------------------|-----------------|-------------|-------|
| 1/2     | 1.4343168282e-04 | 8.3323524761e-03 | 1.2019789956e-02 | 2.6363756085e-02 | 1.0795 |
| 1/3     | 8.7241298710e-05 | 5.0404132111e-04 | 9.4396047316e-03 | 1.7018568232e-03 | 1.2624 |
| 1/4     | 5.8426817327e-05 | 3.3808222459e-04 | 1.2795582951e-03 | 1.1836017614e-03 |       |
| 1/5     | 4.1850682102e-05 | 2.4212569094e-04 | 6.5995367635e-03 |               |       |
In this paper, we construct and analyze, for the first time, the tetrahedral $H(\text{curl}^2)$-conforming element. We employ our new elements to solve the quad-curl problem in numerical experiments. It turns out that our new element works well for solving quad-curl problem. However, this element has a great number of DOFs, which makes it difficult to get a family of very accurate basis functions and expensive to refine the grid further. In our future research, we will construct hierarchical basis functions of curl-curl conforming elements and try to decrease the number of DOFs.
Since the finite element proposed in this article involves normal derivatives to edges, we can not relate the basis functions on the general element and those on the reference element by mapping (2.3). In addition, it is difficult to obtain the basis functions on each general element by solving a large-scale (for the lowest-order case, it is 315×315) and ill-conditioned matrix. Hence we apply the method proposed in [9] for Argyris element to construct our basis functions on a general element. In this section, we will demonstrate this method for the lowest-order case.

Suppose $x_\alpha, \alpha = 1, 2, 3, 4, e_\alpha, \alpha = 1, 2, 3, 4$ are the 4 vertex, 6 edges, and 4 faces of an general element $K$. Suppose also $m_\alpha$ is the midpoint of the edge $e_\alpha$ and $e_{\alpha,j}, j = 1, 2$ are the two tripartite points of the edge $e_\alpha$.

We define
\[ L_{v,\alpha,i}^v(\phi) = (\nabla \times \phi)_1(x_\alpha), \quad i = 1, 2, 3, \ \alpha = 1, 2, 3, 4. \]
\[ L_{o,v,\alpha,i}^v(\phi) = \partial_o((\nabla \times \phi)_1(x_\alpha)), \quad i = 1, 2, 3, \ \alpha = 1, 2, 3, 4, \ \rho \in \{x, y, z\} \text{ if } i = 1, 2, \text{ otherwise, } \rho \in \{x, y\}. \]
\[ L_{o,v,\alpha,i}^o(\phi) = \partial_o^2((\nabla \times \phi)_1(x_\alpha)), \quad i = 1, 2, 3, \ \alpha = 1, 2, 3, 4, \ \rho \in \begin{cases} \{yy, zz, xx, xy, yz\}, & i = 1, \\ \{xx, zz, xx, xy, yz\}, & i = 2, \\ \{xx, yy, xx, xy, yz\}, & i = 3. \end{cases} \]
\[ L_{1,e,\alpha,i}^v(\phi) = (\nabla \times \phi)_1(m_\alpha), \quad i = 1, 2, 3, \ \alpha = 1, 2, \cdots, 6. \]
\[ L_{a,n,\alpha,i,j}^m(\phi) = (\nabla((\nabla \times \phi \cdot a_i) \cdot n)(e_{\alpha,j}), \quad i = 1, 2, 3, \ \alpha = 1, 2, \cdots, 6, \ j = 1, 2, \text{ and } a_1 = \tau, \ a_2 = n, \ a_3 = m, \]
where $n, m$ and $\tau$ are two unit normal vectors and the unit tangent vector to $e_\alpha$.
\[ L_{a,m,\alpha,i,j}^m(\phi) = (\nabla((\nabla \times \phi \cdot a_i) \cdot m)(e_{\alpha,j}), \quad i = 1, 2, \ \alpha = 1, 2, \cdots, 6, \ j = 1, 2, \]
where $a_1 = \tau$ and $a_2 = n$.
\[ L_{1,f,\alpha,i}^v(\phi) = \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \times \phi \cdot \tau_i dS, \quad i = 1, 2, \ \alpha = 1, 2, 3, 4, \text{ and } \tau_i \text{ is the unit tangent} \]
vector in the face $f_\alpha$.

$L_{\alpha,i}^{0,e}(\phi) = \int_{e_\alpha} \phi \cdot \tau_\alpha q_i ds, \ \alpha = 1, 2, \cdots, 6, \ i = 1, 2, \cdots, 7.$

$L_{\alpha,i}^{0,f}(\phi) = \frac{1}{\text{area}(f_\alpha)} \int_{f_\alpha} \phi \cdot q_i dS, \ \alpha = 1, 2, 3, 4, \ i = 1, 2, \cdots, 15.$

$L_{i}^{0,K}(\phi) = \int_{K} \phi \cdot q_i dV, \ i = 1, 2, \cdots, 20.$

$L_{i}^{1,K}(\phi) = \int_{K} \nabla \cdot \phi \cdot q_i dV, \ i = 1, 2, 3.$

We list these functionals as $L_j$ for $j = 1, 2, \cdots, 315$ in the following order:

$L_{1}^{v,i}, i = 1, 2, 3, \ L_{2}^{v,i}, i = 1, 2, 3, \ L_{3}^{v}, i = 1, 2, 3, \ L_{4}^{v,i}, i = 1, 2, 3,$

$L_{1}^{o,1}(\text{the elements in o are ordered as } x, y, z),$ 

$L_{1}^{o,2}(\text{the elements in o are ordered as } x, y, z),$ 

$L_{1}^{o,3}(\text{the elements in o are ordered as } x, y),$ 

$L_{2}^{o,1}, L_{2}^{o,2}, L_{2}^{o,3}, L_{3}^{o,1}, L_{3}^{o,2}, L_{3}^{o,3}, L_{4}^{o,1}, L_{4}^{o,2}, L_{4}^{o,3},$ 

$L_{1}^{o,1}(\text{the elements in o are ordered as } y, z, x, y, z),$ 

$L_{1}^{o,2}(\text{the elements in o are ordered as } y, z, x, y, z),$ 

$L_{1}^{o,3}(\text{the elements in o are ordered as } x, y, z, x, y, z),$ 

$L_{2}^{o,1}, L_{2}^{o,2}, L_{2}^{o,3}, L_{3}^{o,1}, L_{3}^{o,2}, L_{3}^{o,3}, L_{4}^{o,1}, L_{4}^{o,2}, L_{4}^{o,3},$ 

$L_{1}^{l,e,i}, i = 1, 2, 3, \ L_{2}^{l,e,i}, i = 1, 2, 3, \ L_{3}^{l,e,i}, i = 1, 2, 3,$ 

$L_{4}^{l,e,i}, i = 1, 2, 3, \ L_{5}^{l,e,i}, i = 1, 2, 3,$ 

$L_{1}^{T,n}, L_{1}^{n,n}, L_{1}^{n,m}, L_{1}^{m,n}, L_{1}^{m,m}, L_{2}^{n,n}, L_{2}^{n,m}, L_{2}^{m,n}, L_{2}^{m,m}, L_{3}^{n,n}, L_{3}^{n,m}, L_{3}^{m,n}, L_{3}^{m,m}, L_{4}^{n,n}, L_{4}^{n,m}, L_{4}^{m,n}, L_{4}^{m,m}, L_{5}^{n,n}, L_{5}^{n,m}, L_{5}^{m,n}, L_{5}^{m,m},$
obtain the basis functions on a general element. We introduce a new set of functionals

\[ L^{1,f}_i, i = 1, 2, \ L^{1,f}_{2,i}, i = 1, 2, \ L^{1,f}_{3,i}, i = 1, 2, \ L^{1,f}_{4,i}, i = 1, 2, \]

\[ L^{0,e}_1, i = 1, 2, \cdots, 7, \ L^{0,e}_{2,i}, i = 1, 2, \cdots, 7, \ L^{0,e}_{3,i}, i = 1, 2, \cdots, 7, \]

\[ L^{0,f}_1, i = 1, 2, \cdots, 15, \ L^{0,f}_{2,i}, i = 1, 2, \cdots, 15, \ L^{0,f}_{3,i}, i = 1, 2, \cdots, 15, \ L^{0,f}_{4,i}, i = 1, 2, \cdots, 15, \]

\[ L^{0,K}_i, i = 1, 2, \cdots, 20, \ L^{1,K}_i, i = 1, 2, 3. \]

The functionals \( \tilde{L}_i, i = 1, 2, \cdots, 315 \) are the counterparts on the reference element. The basis functions \( \{N_j\}_{j=1}^{315} \) for the finite element on a general element \( K \) satisfy

\[ L_i(N_j) = \delta_{ij}, i, j \in 1, 2, \cdots, 315. \]

The basis functions \( \{\tilde{N}_j\}_{j=1}^{315} \) for the finite element on the reference element \( \hat{K} \) satisfy

\[ \tilde{L}_i(\tilde{N}_j) = \delta_{ij}, i, j \in 1, 2, \cdots, 315. \]

Note that \( N_j \) and \( \tilde{N}_j \) can not be related with mapping (2.3). We define

\[ \tilde{L}_i(\phi) = \hat{L}_i(B^T_K \phi \circ F). \]

Since both sets \( L_i \) and \( \tilde{L}_i \) are bases of \( \mathcal{R}^*_7 \), the dual space to \( \mathcal{R}_7 \), there exists a matrix \( C = (c_{ij}) \) such that

\[ \tilde{L}_i = \sum_{j=1}^{315} c_{ij} L_j \text{ in } \mathcal{R}^*_7. \]

By elementary transposition argument, it follows that

\[ N_i \circ F = \sum_{k=1}^{315} c_{ki} \tilde{N}_k \text{ in } \mathcal{R}_7. \]

If we have obtained the basis functions on the reference element and the matrix \( C \), we then obtain the basis functions on a general element. We introduce a new set of functionals \( L^*_i, i = 1, 2, \cdots, 383 \), which are listed in order as follows.

\[ L^*_1, i = 1, 2, 3, \ L^*_2, i = 1, 2, 3, \ L^*_3, i = 1, 2, 3, \ L^*_4, i = 1, 2, 3, \]

\[ L^0_{1,1} (\text{ the elements in } o \text{ are ordered as } x, y, z), \]
$L^o_{1,2}$ (the elements in $o$ are ordered as $x, y, z$),

$L^o_{1,3}$ (the elements in $o$ are ordered as $x, y, z$),

$L^o_{2,1}, L^o_{2,2}, L^o_{2,3}, L^o_{3,1}, L^o_{3,2}, L^o_{3,3}, L^o_{4,1}, L^o_{4,2}, L^o_{4,3}$,

$L^o_{1,1}$ (the elements in $o$ are ordered as $xx, yy, zz, xz, xy, yz$),

$L^o_{1,2}$ (the elements in $o$ are ordered as $xx, yy, zz, xz, xy, yz$),

$L^o_{1,3}$ (the elements in $o$ are ordered as $xx, yy, zz, xz, xy, yz$),

$L^o_{2,1}, L^o_{2,2}, L^o_{2,3}, L^o_{3,1}, L^o_{3,2}, L^o_{3,3}, L^o_{4,1}, L^o_{4,2}, L^o_{4,3}$,

$L^e_{1,i}, i = 1, 2, 3, L^e_{2,i}, i = 1, 2, 3, L^e_{3,i}, i = 1, 2, 3,$

$L^e_{4,i}, i = 1, 2, 3, L^e_{5,i}, i = 1, 2, 3,$

$L^m_{1,1}, L^m_{1,2}, L^m_{1,3}, L^m_{2,1}, L^m_{2,2}, L^m_{2,3}, L^m_{3,1}, L^m_{3,2}, L^m_{3,3}, L^m_{4,1}, L^m_{4,2}, L^m_{4,3}, L^m_{5,1}, L^m_{5,2}, L^m_{5,3}, L^m_{6,1}, L^m_{6,2}, L^m_{6,3}, L^m_{7,1}, L^m_{7,2}, L^m_{7,3}.$
where \( L_{\alpha,n}^{f} = \int_{f_{\alpha}} \nabla \times \phi \cdot n \, dS \), \( \alpha = 1, 2, 3, 4 \), and \( n \) is the unit normal vector of the face \( f_{\alpha} \).

If we can find two matrices \( D = (d_{ij}) \) and \( E = (e_{ij}) \) such that
\[
\bar{L}_{i} = \sum_{j=1}^{383} d_{ij} L_{j}^{*}, \quad \text{in} \, \mathbb{R}_{7}^{*}, \, i = 1, 2, \ldots, 315,
\]
\[
L_{i}^{*} = \sum_{j=1}^{315} e_{ij} L_{j}, \quad \text{in} \, \mathbb{R}_{7}^{*}, \, i = 1, 2, \ldots, 383,
\]
then \( C = DE \). By the transformation (2.5) and the chain rule, we have
\[
\left( \bar{L}_{\alpha,1}^{v} \bar{L}_{\alpha,2}^{v} \bar{L}_{\alpha,3}^{v} \right)^{T} = \det(B_{K})B_{K}^{-1}\left( L_{\alpha,1}^{*,v} L_{\alpha,2}^{*,v} L_{\alpha,3}^{*,v} \right)^{T},
\]
\[
\left( \bar{L}_{\alpha,1}^{x} \bar{L}_{\alpha,1}^{y} \bar{L}_{\alpha,1}^{z} \bar{L}_{\alpha,2}^{x} \bar{L}_{\alpha,2}^{y} \bar{L}_{\alpha,2}^{z} \bar{L}_{\alpha,3}^{x} \bar{L}_{\alpha,3}^{y} \bar{L}_{\alpha,3}^{z} \right)^{T} = W\left( L_{\alpha,1}^{*,x} L_{\alpha,1}^{*,y} L_{\alpha,1}^{*,z} L_{\alpha,2}^{*,x} L_{\alpha,2}^{*,y} L_{\alpha,2}^{*,z} L_{\alpha,3}^{*,x} L_{\alpha,3}^{*,y} L_{\alpha,3}^{*,z} \right)^{T},
\]
\[
\left( \bar{L}_{\alpha,1}^{yy} \bar{L}_{\alpha,1}^{zz} \bar{L}_{\alpha,1}^{xy} \bar{L}_{\alpha,2}^{yy} \bar{L}_{\alpha,2}^{zz} \bar{L}_{\alpha,2}^{xy} \bar{L}_{\alpha,3}^{yy} \bar{L}_{\alpha,3}^{zz} \bar{L}_{\alpha,3}^{xy} \right)^{T} = V\left( L_{\alpha,1}^{*,yy} L_{\alpha,1}^{*,zz} L_{\alpha,1}^{*,xy} L_{\alpha,2}^{*,yy} L_{\alpha,2}^{*,zz} L_{\alpha,2}^{*,xy} \ldots L_{\alpha,3}^{*,yy} L_{\alpha,3}^{*,zz} L_{\alpha,3}^{*,xy} \right)^{T},
\]
where \( W \) is the first 8 rows of the matrix \( \det(B_{K})B_{K}^{-1} \bigotimes B_{K}^{T} \) and \( V \) is the matrix
\[
\det(B_{K})B_{K}^{-1} \bigotimes H \quad \text{without first, eighth, fifteen rows with}
\]
\[
H = \left( \begin{array}{cccccc}
B_{11}^{2} & B_{21}^{2} & B_{31}^{2} & 2B_{11}B_{31} & 2B_{11}B_{21} & 2B_{31}B_{21} \\
B_{12}^{2} & B_{22}^{2} & B_{32}^{2} & 2B_{12}B_{32} & 2B_{12}B_{22} & 2B_{32}B_{22} \\
B_{13}^{2} & B_{23}^{2} & B_{33}^{2} & 2B_{13}B_{33} & 2B_{13}B_{23} & 2B_{33}B_{23} \\
B_{11}B_{13} & B_{21}B_{23} & B_{31}B_{33} & B_{11}B_{33} + B_{13}B_{31} & B_{11}B_{23} + B_{13}B_{21} & B_{31}B_{23} + B_{33}B_{21} \\
B_{11}B_{12} & B_{21}B_{22} & B_{31}B_{32} & B_{11}B_{32} + B_{12}B_{31} & B_{11}B_{22} + B_{12}B_{21} & B_{31}B_{22} + B_{32}B_{21} \\
B_{12}B_{13} & B_{22}B_{23} & B_{32}B_{33} & B_{12}B_{33} + B_{13}B_{32} & B_{12}B_{23} + B_{13}B_{22} & B_{32}B_{23} + B_{33}B_{22}
\end{array} \right).
\]
Similarly,

\[
\begin{pmatrix}
\tilde{L}_{\alpha,1}^{1,e} \\
\tilde{L}_{\alpha,2}^{1,e} \\
\tilde{L}_{\alpha,3}^{1,e}
\end{pmatrix} = \text{det}(B_K)B_K^{-1}
\begin{pmatrix}
L_{\alpha,1}^{s,1,e} \\
L_{\alpha,2}^{s,1,e} \\
L_{\alpha,3}^{s,1,e}
\end{pmatrix},
\]

If we represent \( B_K^{-T}\hat{\tau}_\alpha \) and \( B_K \hat{n}_\alpha \) by using a group of bases \( \tau_\alpha, n_\alpha, m_\alpha \), we’ll get

\[
\begin{pmatrix}
\tilde{L}_{\alpha,i}^{s,\tau.n} \\
\tilde{L}_{\alpha,i}^{s,n.\tau} \\
\tilde{L}_{\alpha,i}^{s.m.\tau} \\
\end{pmatrix} = G_\alpha \begin{pmatrix}
L_{\alpha,i}^{s,\tau.n} \\
L_{\alpha,i}^{s.n.\tau} \\
L_{\alpha,i}^{s.m.\tau}
\end{pmatrix}^T
\]

with

\[
G_\alpha = \text{det}(B_K) \begin{pmatrix}
[[\tau_\alpha,n_\alpha,m_\alpha]B_K^{-T}\hat{\tau}_\alpha] \\
[[\tau_\alpha,n_\alpha,m_\alpha]B_K^{-T}\hat{n}_\alpha] \\
[[\tau_\alpha,n_\alpha,m_\alpha]B_K^{-T}\hat{m}_\alpha]
\end{pmatrix}^T.
\]

Similarly,

\[
\begin{pmatrix}
\tilde{L}_{\alpha,1}^{1,f} \\
\tilde{L}_{\alpha,2}^{1,f} \\
\tilde{L}_{\alpha,3}^{1,f}
\end{pmatrix} = C_\alpha \begin{pmatrix}
L_{\alpha,1}^{s,1,f} \\
L_{\alpha,2}^{s,1,f} \\
L_{\alpha,3}^{s,1,f}
\end{pmatrix} \quad \text{with} \quad C_\alpha = \text{det}(B_K) \begin{pmatrix}
[[\tau_1^0,\tau_2^0,n_\alpha]^{-1}B_K^{-T}\hat{\tau}_1^\alpha] \\
[[\tau_1^0,\tau_2^0,n_\alpha]^{-1}B_K^{-T}\hat{\tau}_2^\alpha] \\
[[\tau_1^0,\tau_2^0,n_\alpha]^{-1}B_K^{-T}\hat{m}_\alpha]
\end{pmatrix}^T.
\]

Since \( L_{\alpha,i}^{0,e} = \tilde{L}_{\alpha,i}^{0,e}, \quad L_{\alpha,i}^{0,f} = \tilde{L}_{\alpha,i}^{0,f}, \quad L_{\alpha,i}^{1,K} = \tilde{L}_{\alpha,i}^{1,K} \), and \( L_{\alpha,i}^{0,K} = \tilde{L}_{\alpha,i}^{0,K} \), we have

\[
\begin{align*}
\tilde{L}_{\alpha,i}^{0,e} &= L_{\alpha,i}^{s,0,e}, & \tilde{L}_{\alpha,i}^{0,f} &= L_{\alpha,i}^{s,0,f}, \\
\tilde{L}_{\alpha,i}^{0,K} &= L_{\alpha,i}^{s,0,K}, & \tilde{L}_{\alpha,i}^{1,K} &= L_{\alpha,i}^{s,1,K}.
\end{align*}
\]

Now we are in the position to giving the explicit expression of \( D \),

\[
D = \text{diag}(\text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, \text{det}(B_K)B_K^{-1}, G_1, G_1, G_2, G_2, G_3, G_3, G_4, G_4, G_5, G_5, G_6, G_6, C_1, C_2, C_3, C_4, I_{125 \times 125}).
\]
Next, we express $L^*$ by $L$ and some other functionals which actually can also be represented by $L$.

It’s trivial to represent $L_{a,i}^*, L_{a,1}^*, L_{a,2}^*, L_{a,3}^*$. Now we show how to represent $L_{\tau,\tau}^*$. Let $\varphi(t) = (\nabla \times \phi \cdot \tau)(tx_\beta + (1-t)x_\alpha) \in P_6(t)$. Then

$$L_{\gamma,1}^* \phi = \varphi(t) \frac{1}{3} = -\frac{248}{81} \varphi(0) - \frac{8}{81} \varphi(1) + \frac{256}{81} \varphi(1/2) - \frac{40}{81} \varphi'(0) + \frac{1}{81} \varphi'(1) - \frac{2}{81} \varphi''(0),$$

$$L_{\gamma,2}^* \phi = \varphi(t) \frac{2}{3} = \frac{8}{81} \varphi(0) + \frac{248}{81} \varphi(1) - \frac{256}{81} \varphi(1/2) + \frac{1}{81} \varphi'(0) - \frac{40}{81} \varphi'(1) + \frac{2}{81} \varphi''(1).$$

If we represent $\varphi(0), \varphi(1), \varphi'(0), \varphi'(1), \varphi''(0), \varphi''(1)$ by some $L_i$, we can obtain $L_{\gamma,3}^*$ in the form of a linear combination of $L_{i}, i = 1, 2, \cdots, 315$. Similarly, we can represent $L_{*n,\tau}$ and $L_{*m,\tau}$. As for $L_{*m,m}$, we can find 8 constants such that

$$\nabla(\nabla \times \phi \cdot m) \cdot m = C_1 \nabla(\nabla \times \phi \cdot \tau) \cdot \tau + C_2 \nabla(\nabla \times \phi \cdot \tau) \cdot n + C_3 \nabla(\nabla \times \phi \cdot \tau) \cdot m$$

$$+ C_4 \nabla(\nabla \times \phi \cdot n) \cdot \tau + C_5 \nabla(\nabla \times \phi \cdot n) \cdot n + C_6 \nabla(\nabla \times \phi \cdot n) \cdot m$$

$$+ C_7 \nabla(\nabla \times \phi \cdot m) \cdot \tau + C_8 \nabla(\nabla \times \phi \cdot m) \cdot n,$$

since $\nabla \cdot (\nabla \times \phi) = 0$. Furthermore, $\nabla(\nabla \times \phi \cdot \tau), \nabla(\nabla \times \phi \cdot n)$ and $\nabla(\nabla \times \phi \cdot \tau) \cdot m$ can be determined by the values of $\nabla \times \phi$ and its up to second derivatives at two endpoints since they are in $P_6$. So far, we can represent $L_{*m,m}$ in terms of $L$.

Since $(\nabla \times \phi \cdot n, 1)_f = (\phi \cdot \tau, 1)_{\partial f}$, we can represent $L_{*1,f}$ as a linear combination of $L_{0,e}$, and hence $L_i$.

Finally, we can express $L^*$ by $L$. Then we obtain $C = DE$. Because of the large number of degrees of freedom, it’s tedious to implement this process in Matlab. We provide the code for basis functions at https://github.com/QianZhangMath/3D-curl-curl-conforming-FE.

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