Quantum nonlocality for entanglement of quasiclassical states

Zhi-Rong Zhong, Jian-Qi Sheng, Li-Hua Lin and Shi-Biao Zheng
Fujian Key Laboratory of Quantum Information and Quantum Optics,
College of Physics and Information Engineering,
Fuzhou University, Fuzhou, Fujian 350108, China

Entanglement of quasiclassical (coherent) states of two harmonic oscillators leads to striking quantum effects and is useful for quantum technologies. These effects and applications are closely related to nonlocal correlations inherent in these states, manifested by the violation of Bell inequalities. With previous frameworks, this violation is limited by the size of the system, which does not approach the maximum even when the amount of entanglement approaches its maximum. Here we propose a new version of Bell correlation operators, with which a nearly maximal violation can be obtained as long as the associated entanglement approaches the maximum. Consequently, the revealed nonlocality is significantly stronger than those with previous frameworks for a wide range of the system size. We present a new scheme for realizing the gate necessary for measurement of the nonlocal correlations. In addition to the use in test of quantum nonlocality, this gate is useful for quantum information processing with coherent states.

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In their seminal paper [1], Einstein, Podolsky, and Rosen concluded that quantum-mechanical description of reality is not complete based on the assumption that the properties of an object is not affected by the measurement on another object that is far away from it, while quantum mechanics predicts that two or more spatially separated objects can be in an entangled state so that there exist nonlocal correlations between these objects. In 1964, Bell constructed an inequality imposed by local hidden theories, allowing experimental test of quantum nonlocality by measuring the correlations in the classical properties of entangled particles [2]. Since then, different versions of Bell inequalities have been proposed [3-5].

Though the Einstein-Podolsky-Rosen (EPR) paradox was formulated using entangled states of two systems with infinite-dimensional Hilbert space (e.g., harmonic oscillators), previous investigations on Bell nonlocality focused on systems with finite-dimensional space until 1998, when Banaszek and Wodkiewicz (BW) constructed the Bell inequality for two harmonic oscillators in terms of the two-mode Wigner function, which is associated with the joint quantum-number parity of these two systems under a displacement operation in phase space [6,7]. With this formulation and its generalization [8], the Bell signal for the original EPR state exceeds the bound imposed by local hidden theories, but cannot attain the maximum value of $2\sqrt{2}$ for two-partite entangled states, known as the Cirel’son bound [9]. Chen et al. [10] formulated a set of pseudospin operators, which allows correlations in the original EPR state to maximally violate the Bell inequality; however, measurement of these pseudospin observables is experimentally challenging.

Among various kinds of entangled states of harmonic oscillators, entangled coherent states are of particular interest [11,12]. On one hand, this kind of states, formed by coherent states with classical analogs, are a two-mode generalization of Schrödinger cat states [13] and can exhibit strong quantum interference effects, evidenced by fringes and negativity of Wigner quasiprobability distribution in phase space [14-16]. On the other hand, they can be considered two-qubit entangled states when the involved coherent states for each oscillator are approximately orthogonal to each other. In addition to fundamental interest, these states can be used for quantum information processing [11] and quantum metrology [17]. In particular, logic qubits encoded with cat states have inherent error-correctable functions, offering a promising possibility for implementation of fault-tolerant quantum computation [18]. With this encoding, implementation of a quantum algorithm corresponds to controlled creation and manipulation of entanglement of coherent states. These nonclassical effects and applications are closely related to the associated nonlocal correlations. Very recently, it was shown that quantum nonlocality is responsible for the advantage of quantum circuits over their classical counterparts [19]. Thus, the nonlocality of entangled coherent states are both of fundamental interest and of practical importance. This has been studied with different kinds of observables [8,14,20], but with which the Bell signal approaches $2\sqrt{2}$ only when the size of the system is large. As the decaying rate of the entanglement scales with the system size, it is highly desirable to construct a new version of Bell correlation operators, where the observables are experimentally detectable and with which stronger nonlocal correlations can be revealed.

In this manuscript, we proposed a new version of Bell correlation operators for entangled coherent states, with which the uncovered nonlocality is significantly stronger than those with previous versions for a wide range of the system size. In our framework, when the coherent states
associated with each oscillator can approximately form a logic qubit, the operators used for constructing the correlations do not move this qubit out of the logic space. This is in contrast with previous frameworks, where the cat state qubits cannot remain in the logic space under the application of the correlation operators. This distinct feature enables a nearly maximal Bell inequality violation as long as the entangled coherent state can be approximately expressed as a two-qubit maximally entangled state. The size of the system required for approaching the upper bound $2\sqrt{2}$ is reduced by one order compared with that based on the BW formalism, which is of importance in view of decoherence. We propose a new scheme for implementing the phase gate required for measurement of the correlations. Besides fundamental interest, this gate has applications in quantum information processing with cat state qubits [21,22].

The entangled coherent states under consideration are defined as

$$\ket{\psi} = \mathcal{N} \ket{\alpha_1} \ket{\alpha_2} + e^{i\theta} \ket{-\alpha_1} \ket{-\alpha_2},$$

where $\mathcal{N} = \left[ 2 + 2 \cos \theta e^{-2(\alpha_1^2 + \alpha_2^2)} \right]^{-1/2}$ is the normalization factor, and $\ket{\alpha_j} = e^{-\alpha_j^2/2} \sum_{n=0}^{\infty} \frac{\alpha_j^n}{\sqrt{n!}} \ket{n}_j$ denotes the coherent state of amplitude $\alpha_j$ for the $j$th harmonic oscillator with $\ket{n}_j$ representing the Fock state with $n$ quanta. The observables we use to characterize the nonlocal correlations of these entangled states are the rotated parities, each combined by an effective cat state qubit rotation operator and the parity operator. The effective cat state qubit rotation operator associated with the $j$th oscillator is defined as

$$R_{j,z}(\phi_j) = D_j^\dagger(\alpha_j) G_j(\phi_j) D_j(\alpha_j),$$

where

$$G_j(\phi_j) = \ket{0}_j \bra{0} e^{i\phi_j} + \sum_{n=1}^{\infty} \ket{n}_j \bra{n}$$

is the phase gate for the $j$th oscillator that produces a phase shift $\phi_j$ if and only if this oscillator is in the ground state $\ket{0}_j$, and $D_j(\alpha_j) = e^{\alpha_j a_j^\dagger - \alpha_j^* a_j}$ denotes the displacement operator, with $a_j^\dagger$ and $a_j$ being the quantum-number rising and lowering operators. The operation $D_j(\alpha_j)$ translates the quantum state of the $j$th oscillator by an amount of $\alpha_j$ in phase space. The sequence of operations $D_j(\alpha_j), G_j(\phi_j)$, and $D_j^\dagger(\alpha_j)$ displaces $-\alpha_j$ to the vacuum state $\ket{0}_j$, produces a phase shift $\phi_j$ on $\ket{0}_j$, and finally transforms $\ket{0}_j$ back to $-\alpha_j$. On the other hand, $D_j(\alpha_j)$ evolves $\ket{\alpha_j}_j$ to $2\ket{\alpha_j}_j$, which is not affected by $G_j(\phi_j)$ and thus transformed back to $\ket{\alpha_j}_j$ by $D_j^\dagger(\alpha_j)$ when $|\ket{0} \alpha_j| < 2 |\alpha_j|$. Therefore, with the encoding $\ket{\alpha_j}_j, \ket{-\alpha_j}_j, \ket{0}_j \ket{L}_j = \ket{\alpha_j}_j$, $R_{j,z}(\phi_j)$ is equivalent to a rotation of the cat state qubit around the z axis by an angle $\phi_j$. On the other hand, the parity operator $P_j = (-1)^{a_j^\dagger a_j}$ corresponds to the Pauli spin operator $\sigma_j$. As a result, the rotated parity operator

$$P_{j,z}(\phi_j) = R_{j,z}(\phi_j) P_j R_{j,z}^\dagger(\phi_j)$$

is analogous to the spin operator along the axis with an angle $\phi_j$ to the axis on the xy plane, denoted as $\sigma_{j,x}$. We note that such an operator, like the pseudospin operators defined in Ref. [10], maps the infinite-dimensional Hilbert space onto a two-dimensional space [23].

To construct the Bell inequality, we define the rotated parity-parity correlation as

$$E(\phi_1, \phi_2) = \langle P_{1,z}(\phi_1) \otimes P_{2,z}(\phi_2) \rangle.$$  

$E(\phi_1, \phi_2)$ corresponds to the mean value of the product of the spin-like observables of the two cat state qubits, $\sigma_{1,x}$ and $\sigma_{2,x}$. The Bell inequality, based on correlations defined in this way, is

$$S_{RP} = \left| E(\phi_1, \phi_2) + E(\phi_1, \phi_2') + E(\phi_1', \phi_2) - E(\phi_1', \phi_2') \right| \leq 2.$$  

For the entangled coherent state of Eq. (1), we have

$$E(\phi_1, \phi_2) = N^2 \left\{ e^{-2|\alpha_j|^2|\alpha_j'|^2} + K_{\alpha_1,\alpha_2} e^{i\phi_j} + c.c. \right\},$$

where

$$K_{\alpha_1,\alpha_2}(\phi_j) = e^{-2|\alpha_j|^2 (2 \cos \phi_j - 1) + 2 e^{-6|\alpha_j|^2} (1 - \cos \phi_j)}.$$  

The first two terms of $E(\phi_1, \phi_2)$ are respectively contributed by the components $-\alpha_j \alpha_j'$ and $\alpha_j \alpha_j'$, while the last two terms arise from their quantum coherence, which is responsible for the entanglement between the two harmonic oscillators. The values of $|\alpha_1|$ and $|\alpha_2|$ determine the accuracy of the effective cat state qubit rotations. With the increase of $|\alpha_j|$, the rotated parity operator $P_{j,z}(\phi_j)$ is closer to the spin operator $\sigma_{j,x}$. When $|\alpha_j| \rightarrow \infty$ ($j = 1,2$), $E(\phi_1, \phi_2) = \cos(\theta - \phi_1 - \phi_2)$, which is in perfect analogy with the correlation between two photons in a maximally polarization-entangled states [24]. In this limit, the optimized Bell signal reaches the maximum $2\sqrt{2}$. As will be shown, a nearly maximal Bell violation can be obtained even with quite moderate values of $|\alpha_1|$ and $|\alpha_2|$. We note that the Bell violation discussed here arises from the quantum entanglement between two distinct bosonic modes, which is in stark contrast with situation investigated in a recent experiment [25], where the Bell violation results from the classical correlation between the polarization and amplitude of the same optical field.

To show the degree of the Bell inequality violation for limited values of $\alpha_1$ and $\alpha_2$, we perform numerical
simulations of the Bell signal $S_{\text{RP}}$. For simplicity, we here set $\theta = 0$. For this setting, the choice $\phi_1 = 0$, $\phi_2 = \pi/4$, $\phi_1' = -\pi/2$, and $\phi_2' = -\pi/4$ ensures $S_{\text{RP}}$ to reach the maximum when $|\alpha_j| \to \infty$. As shown in Eqs. (7) and (8), the correlation $E(\phi_1, \phi_2)$ has nothing to do with the phases of $\alpha_1$ and $\alpha_2$, which will be taken to be positive real numbers in our simulations. To clearly show the difference between the Bell violation with the present correlation operators and that with the BW formalism, we first take $\alpha_1 = \alpha_2 = \alpha$. The four correlations $E(0, \pi/4)$, $E(0, -\pi/4)$, $E(-\pi/2, \pi/4)$, and $E(-\pi/2, -\pi/4)$ as functions of the value of $\alpha$, are displayed in Fig. 1. As expected, each of the theses correlations is 1 when $\alpha = 0$. This is due to the fact that for this case $|\psi\rangle = |0\rangle_1 |0\rangle_2$ and $P_{j,z}(\phi_j) |0\rangle_j = |0\rangle_j$, so that $\langle P_{j,z}(\phi_j) \otimes P_{2,z}(\phi_2) \rangle = 1$. With the increases of $\alpha$, the parity of each oscillator drops quickly with the other being traced out and the entanglement contributes more and more to the correlations; $E(-\pi/2, -\pi/4)$ quickly approaches $-\sqrt{2}/2$, while the other correlations tend to $\sqrt{2}/2$.

The solid line of Fig. 2(a) shows the Bell signal $S_{\text{RP}}$ calculated with the rotated parity operators as a function of $\alpha$. The dashed line represents the maximized Bell signal obtained by the generalized BW formalism [14], defined as

$$S_{\text{BW}} = \left| E(\beta_1, \beta_2) + E(\beta_1, \beta_2') + E(\beta_1', \beta_2) - E(\beta_1', \beta_2') \right|,$$

where

$$E(\beta_1, \beta_2) = \left\langle D_1(\beta_1) P_1 D_1^\dagger(\beta_1) \otimes D_2(\beta_2) P_2 D_2^\dagger(\beta_2) \right\rangle.$$

FIG. 1: (Color online) Numerical simulations of correlations for the 3-mode cat state as functions of $\alpha$: (a), $E_3(0, -\pi/4, \pi/4)$; (b), $E_3(0, \pi/4, -\pi/4)$; (c), $E_3(\pi/2, -\pi/4, -\pi/4)$; (d), $E_3(\pi/2, \pi/4, \pi/4)$. The solid and dotted lines correspond to the correlations for the cat state and the equally-weighted classical mixture of the two components $|\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3$ and $|\alpha\rangle_1 |-\alpha\rangle_2 |-\alpha\rangle_3$, respectively.

FIG. 2: (Color online) MK signal for the 3-mode cat state as a function $\alpha$. The solid line represents the numerical result calculated by Eq. (12), and the dotted line denotes the maximum allowed by local realism.

can be interpreted as follows. In this formalism, for a certain value of $\alpha$, all the four correlations are fixed. In the range $0 \leq \alpha < 0.32$, the correlations $E(-\pi/2, \pi/4)$ and $E(-\pi/2, -\pi/4)$ are almost equal and cancel out each other in the Bell signal, so that $S_{\text{RP}}$ approximates to the sum of $E(0, \pi/4)$ and $E(0, -\pi/4)$, which drop as $\alpha$ increases. On the other hand, in the BW formalism one can choose a suitable set of displacement parameters for each value of $\alpha$, so that $E(\beta_1, \beta_2)$ drops faster than the sum of the other three correlations, and consequently $S_{\text{BW}}$ increases slowly with $\alpha$.

When $\alpha$ exceeds 0.32, the discrepancy between the correlations $E(-\pi/2, -\pi/4)$ and $E(-\pi/2, -\pi/4)$ enlarges quickly, so that the contribution of their difference to $S_{\text{RP}}$ improves fast. For $\alpha > 0.47$, $S_{\text{RP}}$ surpasses 2. When $\alpha > 0.55$, the Bell inequality violation obtained by our formalism is stronger than that by the BW formalism, and $S_{\text{RP}}$ quickly approaches the bound $2\sqrt{2}$. The qualitative interpretation is as follows. Under the application of $R_{j,z}(\phi)$, logic state $|1\rangle_j$ undergoes a phase shift $\phi_j$, while $|0\rangle_j$ evolves to $|0\rangle_j + e^{-2|\alpha|^2} (e^{i\phi_j} - 1) |1\rangle_j$, whose overlap with $|0\rangle_j$ approaches unity even for a moderate value of $\alpha$. For example, this overlap is above 0.999 for $\alpha = \sqrt{2}$. As a result, the operation $R_{j,z}(\phi_j)$ acts as a nearly perfect rotation around the $z$ axis for the cat state qubit, which remains in the logic space $\{ |0\rangle_j, |1\rangle_j \}$. After this operation. On the other hand, in the BW formalism the qubit rotations are replaced by displacement operations, which move the corresponding harmonics oscillators out of the logic space for a limited value of $\alpha$ [8]. Consequently, the value of $\alpha$ required for nearly maximal Bell violation is much larger than that with our formalism. For example, with the BW formalism the optimized Bell signal is $S_{\text{BW}} = 2.77$ for $\alpha = 3$; based on our formalism, the Bell signal of the same value can be obtained for $\alpha \simeq 1$, corresponding to a two-mode cat state with the size reduced by one order [26]. Since the quantum coherence of a cat state decays at a rate increasing with its size [26], our formalism is important for experimental investigation of the quantum nonlocality for entanglement of quasiclassical states. We note
that the pseudospin flip operators proposed in Ref. [10] also move the cat state qubits out of the logic space when \(|\alpha|\) is not large [8]. Another important feature of our formalism is that the obtained Bell signal \(S_{RP}\) is close to the bound \(2\sqrt{2}\) when the concurrence \(C\), characterizing the parity-parity entanglement, approaches its maximum 1. For example, with the choice \(\alpha = 1.1\) and \(\theta = 0\), \(C = \frac{1 - |\alpha|^2}{\sqrt{1 + |\alpha|^2}} \approx 0.984\), and \(S_{RP} \approx 2.799\), while the Bell signal obtained with the BW formalism is only 2.589.

Further simulations confirm that for \(\alpha_1 \neq \alpha_2\) the Bell signal can also approach \(2\sqrt{2}\) with moderate values of \(\alpha_1\) and \(\alpha_2\), as shown in Fig. 2(b). For example, when \(\alpha_1 = 1.1\) and \(\alpha_2 = 1.3\), \(S_{RP} = 2.812\). We note that, for \(\theta \neq 0\), the rotation angles \((\phi_1, \phi_2, \phi_1', \phi_2')\) need to be adjusted so that \(C(\phi_1, \phi_2) = C(\phi_1, \phi_2') = C(\phi_1', \phi_2) = -C(\phi_1', \phi_2') = \mp \sqrt{2}/2\), where \(C(\phi_1, \phi_2) = \cos(\theta - \phi_1 - \phi_2)\). With this adaption, for different values of \(\theta\) the Bell signal, as a function of \(\alpha_1\) and \(\alpha_2\), behaves similarly, approximating to \(2\sqrt{2}\) when \(e^{-2(|\alpha|^2 + |\alpha|^2)} \leq 1\) \((j, k = 1, 2)\).

The phase gate \(G_{j}(\phi_j)\) required to detect the Bell signal can be realized using a qubit dispersively coupled to the \(j\)th oscillator and driven by two subsequent pulses, each producing a \(\pi\) rotation on the qubit conditional on the oscillator state \(|0\rangle_j\) [21, 22]. With suitable choice of the axes of the two rotations, the phase gate \(G_{j}(\phi_j)\) with a desired phase shift can be obtained, as has been experimentally demonstrated in circuit quantum electrodynamics architectures [21, 22]. We here show that such gates can be implemented with a single square-shaped pulse. In the interaction picture, the Hamiltonian for such a system is

\[
H_I = \hbar \Omega_j a_j^\dagger a_j |e\rangle_j \langle e| + \hbar \Omega_j e^{i\delta_j t} |e\rangle_j \langle g| + h.c.,
\]  

(11)

where \(|e\rangle_j\) and \(|g\rangle_j\) denote the excited and ground states of the qubit, \(\chi\) is the qubit frequency shift induced by one quantum of the oscillator due to the dispersive coupling, \(\delta_j\) is the detuning between the qubit and the drive with Rabi frequency \(\Omega_j\), and h.c. denotes the Hermitian conjugate. The qubit is initialized in the ground state \(|g\rangle_j\).

When the oscillator is in the vacuum state \(|0\rangle_j\), the system evolves as

\[
|\psi_0(\tau)\rangle = e^{-i\delta_j \tau/2} \left(\cos(\varepsilon_j \tau) + \frac{i}{2\varepsilon_j} \sin(\varepsilon_j \tau)\right) |g\rangle_j |0\rangle_j,
\]  

(12)

where \(\varepsilon_j = \sqrt{\Omega_j^2 + \delta_j^2}/4\), \(\tau\) is the duration of the pulse. With the choice \(\delta_j = 2\Omega_j \sqrt{\pi - \phi_2}/|\phi_2 - 2\pi - \phi_1|\) \((\phi_2 > 0)\) or \(\delta_j = -2\Omega_j \sqrt{\phi_2}/|\phi_2 - 2\pi + \phi_1|\) \((\phi_2 < 0)\) and \(\varepsilon_j \tau = \pi\), \(|\psi_0(\tau)\rangle = e^{i\phi_j} |g\rangle_j |0\rangle_j\).

Under the condition \(\delta_j, \Omega_j \ll |\chi|\), the qubit almost remains in the state \(|g\rangle_j\) throughout the pulse if the oscillator is in a Fock state \(|n\rangle_j\) \((n \neq 0)\) due to large detunings [27]. As a consequence, the oscillator under-
implementation of the phase gate necessary for realizing the cat state qubit rotation with a qubit dispersively coupled to the corresponding oscillator. In addition to nonlocality test, this gate has applications in quantum information processing with coherent states.

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