Abstract—We derive a hybrid feedback control law for the lateral leg spring (LLS) model so that the center of mass of a legged runner follows a curved path in horizontal plane. The control law enables the runner to change the placement and the elasticity of its legs to move in a desired direction. Stable motion along a curved path is achieved using curvature, bearing and relative distance between the runner and the curve as feedback. Constraints on leg parameters determine the class of curves that can be followed. We also derive an optimal control law that stabilizes the orientation of the runner's body relative to the velocity of the runner's center of mass.

I. INTRODUCTION

From tiny ants to large elephants, legged locomotion is the dominant method that animals use to move on the ground. Although the leg structures are vastly different across species, the mechanisms for walking, jumping or running obey strikingly similar principles. These similarities are captured by mathematical models such as the spring-loaded inverted pendulum (SLIP) model, see [1], and the lateral leg spring (LLS) model in [2]–[5]. The LLS model describes motion in the horizontal plane and the SLIP model describes motion in the sagittal (vertical) plane.

We use the LLS model in this paper in designing curve tracking control in the horizontal plane; the runner is modeled as a rigid body with two weightless springs attached to a point in the body called the center of pressure (COP). Each spring represents legs on one side of the body. The COP is usually not coincident with the center of mass (COM). Legged locomotions can be self-stabilized—the running or walking gaits stay close to being periodic under disturbances—without feedback control. As indicated by a recent review [6], the self-stabilized walking and running happen when the runner moves along a straight line.

There are many interests in engineering practice to design and build legged robots which are versatile on rough and uneven terrains. Legged robots are greatly appreciated in applications such as searching and rescuing missions and planet exploration. In most missions, the robots must be able to move along an arbitrary path. Feedback control is needed for tracking curved path as well as stabilizing the periodic gaits.

In this paper, we develop feedback control law for the LLS model so that the COM follows a curved path. The legged locomotion modeled by the LLS model is a hybrid system. Correspondingly, the tracking control contains a discrete tracking algorithm which guarantees convergence to the desired curve and a continuous law to control the leg parameters of the LLS model. We also develop an optimal control law to stabilize the posture of the rigid body. Section III serves as an introduction to the LLS model. The discrete tracking algorithm is developed in Section III In Section IV control laws are developed for the leg parameters to enable the discrete tracking algorithm. The constraints on the parameters and the effects on the tracking behavior are discussed in Section V and a modified continuous control law is introduced to handle the constraints. We then develop an optimal control law to achieve desired posture for the rigid body in Section VI. We provide simulation results in Section VII.

II. MOTION OF THE COM

In the horizontal plane, motion generated by the LLS model starts when the free end of one spring (or leg) is placed at a touchdown point $P$. At this starting moment, the spring is at its free length $L_0$. If the COM has a non-zero initial velocity $v$ that is not perpendicular to the spring, then the spring will be first compressed to a minimum length and later be restored to its free length. This process, starting and ending with the spring at its free length $L_0$, is called a stance phase or simply a stance. The COM moves from the starting position to an ending position after a stance. Suppose that mechanical energy is conserved during each stance, then the starting and ending speed of a stance is identical. As shown in Figure 1, the end of one stance serves as the beginning of the next stance with the touchdown point $P$ shifted from one side to the other. This allows us to distinguish left stances from right stances based on which leg is supporting the body.

The rigid body moves forward as a result of switching between left and right stances. As shown in Figure 1 we use $r_i$, $i = 1, 2, ..., $ to denote the position of the COM in a lab fixed coordinate frame at the beginning of the $i$th stance. We use $\alpha_i$ to denote the angle between the velocity $v_i$ and the spring at rest. For a right stance, the angle is measured counter-clockwise from the spring to the velocity vector. For a left stance, the angle is measured clockwise from the spring to the velocity vector. Under this convention for measuring angles, $\alpha_i$ has to be within the interval $(0, \pi/2)$ to generate forward locomotion.

We can view $r_i$ as points on a curve $\Gamma$ which is formed by straight line segments that connects $r_{i-1}$ with $r_i$ for all
i. This curve Γ is not the actual trajectory of the COM, but it intersects with the trajectory of the COM at the points r_i. We then let q_i = r_{i+1} - r_i, and define \( q_i = \| q_i \| \). We also define an angle \( \delta_i \) as the angle between the vector \( q_i \) and the horizontal axis of the lab frame, measured counter-clockwise from the axis. This angle describes the direction of curve Γ for the ith stance. The motion of the COM can now be described by a discrete system

\[
\mathbf{r}_{i+1} = \mathbf{r}_i + [q_i \cos \delta_i, q_i \sin \delta_i]^T. \tag{1}
\]

Next, we develop a boundary tracking algorithm for this discrete system.

III. TRACKING A DETECTED BOUNDARY

Suppose at the position \( r_i \), the runner is able to detect a segment of a boundary curve from sensor information, c.f., [7] and [8]. Suppose the runner is also able to estimate a point on the boundary curve that has the minimum distance to the COM. We call this point the closest point \( r_{c_i} \), shown in Figure 2. By selecting two extra points on the boundary near \( r_{c_i} \), the runner can estimate the tangent vector to the boundary curve \( \mathbf{x}_{c_i} \) and the curvature of the boundary curve \( \kappa_i \) using algorithms summarized in [7]. Here we suppose all estimates are perfect.

We let \( \rho_i \) represent the distance between the COM at \( r_i \) and the closest point at \( r_{c_i} \). We also let

\[
\mathbf{x}_i = \frac{q_i}{\| q_i \|} \tag{2}
\]

be the unit vector in the direction of \( q_i \). We can then define two right handed frames—one at \( r_i \) and the other at \( r_{c_i} \)—by defining two unit normal vectors \( \mathbf{y}_i \) and \( \mathbf{y}_{c_i} \), as shown in Figure 2. The angle \( \theta_i \) between \( \mathbf{x}_i \) and \( \mathbf{x}_{c_i} \) is defined by letting

\[
\cos \theta_i = \mathbf{x}_i \cdot \mathbf{x}_{c_i} \quad \text{and} \quad \sin \theta_i = -\mathbf{x}_i \cdot \mathbf{y}_{c_i}. \tag{3}
\]

In Figure 2 the closest point on the boundary moves from \( r_{c_i} \) to \( r_{c_{i+1}} \) during a stance. We then assume that each stance is short enough so that the curvature of the boundary curve detected during this period can be approximated by a finite constant \( \kappa_i \). From Figure 2 we observe

\[
\beta = \frac{\pi}{2} - \theta_i. \tag{4}
\]

Therefore, according to the cosine law, we have

\[
(\rho_{i+1} + \frac{1}{\kappa_i})^2 = \left(\rho_i + \frac{1}{\kappa_i}\right)^2 - 2\left(\rho_i + \frac{1}{\kappa_i}\right)q_i \sin \theta_i + q_i^2. \tag{5}
\]

We can view \( \rho_i \) as state variables and consider \( q_i \) as the control over step size and \( \theta_i \) as steering control. Equation 5 describes the controlled relative motion between the COM and the detected boundary curve.

Suppose that the step size \( q_i \) has been determined. We need to find \( \theta_i \) such that \( \rho_i \) converges to a desired value \( \rho_c \) as \( i \rightarrow \infty \). If we can find a feedback control law \( f_i(\cdot) \) such that

\[
\rho_{i+1} - \rho_i = f_i(\rho_i - \rho_c), \tag{6}
\]

then we can choose the function \( f_i(\cdot) \) so that \( \rho_i - \rho_c \rightarrow 0 \). For example, we may let

\[
f_i(\rho_i - \rho_c) = -K_i \cdot (\rho_i - \rho_c) \tag{7}
\]

where \( 0 < K_i < 1 \). Observe that

\[
\rho_{i+1} - \rho_i = \rho_{i+1} - \rho_c - (\rho_i - \rho_c), \tag{8}
\]

and this leads to

\[
\rho_{i+1} - \rho_c = (1 - K_i)(\rho_i - \rho_c). \tag{9}
\]

Since \(-1 < 1 - K_i < 1\), it is true that \( \rho_i \rightarrow \rho_c \) as \( i \rightarrow \infty \).
Therefore, combining (16) and (19), we have found the necessary and sufficient condition for the existence of solution for $\theta_i$ as:

$$- (2\lambda_i + q_i) \leq f_i \leq q_i.$$  

(19)

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(19)

Then under the control of the fixed lab frame. Let $\lambda_i$ be the polar coordinates of the COM for the next stance from the discrete model. The size when being close to the curve. For a straight line, since a right stance is always followed by a left stance and vice versa, we use the convention that the stance $k$ is always a right stance and stance $k + 1$ is always a left stance. Therefore, stance $k + 2$ must be a right stance, etc. In the following, we will only show detailed derivation for a right stance; similar results for a left stance will be listed directly.

**IV. CONTROL THE LLS MODEL**

In order to generate desired $q_i$ and $\delta_i$ to control the COM movement, the runner needs to change its leg placements or the elasticity of the legs. We investigate the dynamics during each stance to establish the relations between the COM motion and the leg parameters.

Leg parameters for a right stance and a left stance often differ only by the sign. Since a right stance is always followed by a left stance and vice versa, we use the convention that the stance $k$ is always a right stance and stance $k + 1$ is always a left stance. Therefore, stance $k + 2$ must be a right stance, etc. In the following, we will only show detailed derivation for a right stance; similar results for a left stance will be listed directly.

We set up a polar coordinate system at the touchdown point $P_k$ with the horizontal $x$-axis parallel to the horizontal $X$-axis of the fixed lab frame. Let $(\eta, \psi)$ be the polar coordinates of the COM in this frame and let $\sigma$ describe the orientation of the rigid body. Then the total energy is:

$$E = \frac{1}{2} m \dot{\eta}^2 + \frac{1}{2} m \dot{\psi}^2 + \frac{1}{2} I \dot{\sigma}^2 + V(\eta)$$  

(25)
where we assume the spring has potential energy $V$ which depends only on its length.

During each stance, the dynamics can be described as a continuous nonlinear Hamiltonian system; the Hamilton equations are developed in [2]. The system is not integrable when the distance between the COM and the COP is nonzero. In this case numerical methods are necessary to compute the trajectory of the COM from knowledge of the states $(\eta, \psi, \sigma)$.

To illustrate analytical insights for the tracking problem, it is much easier to study the case when the COP and COM coincide. This is because the corresponding system is integrable. The Hamilton equations for the dynamics of the COM are

$$\begin{align*}
p_\eta &= m \dot{\eta}, \\
p_\psi &= m \eta^2 \dot{\psi}, \\
\dot{p}_\eta &= \frac{1}{m} \frac{\partial V}{\partial \eta}, \\
\dot{p}_\psi &= 0. \\
\end{align*}$$

(26)

Note that $(\zeta - \varphi)$ is the relative angle between $v_k$ and $x_{ek}$: the tangent vector to the boundary curve at the closest point. We know $\theta_k$ is the angle between vector $q_k$ and the same tangent vector $x_{ek}$. Therefore, $(\zeta - \varphi - \theta_k)$ is the angle between $v_k$ and $q_k$. From (28), we see that changing $\alpha_k$ will change the direction of the COM movement $q_k$. This is also true for a left stance where we have

$$\zeta_{k+1} - \varphi_{k+1} - \theta_{k+1} = -\left(\frac{\pi}{2} - \alpha_{k+1} - \frac{\phi_{k+1}}{2}\right).$$

(29)

Another relationship we can derive from Figure 3 is

$$q_k = 2\eta_k \sin \frac{\phi_k}{2}.$$ 

(30)

where $\eta_k$ represents the leg length at the moment of touch down.

In (30) and (28), $q_k$ is the distance the runner wants to travel in one stance, the angle $\theta_k$ can be solved from (10), and the angle $(\zeta_k - \varphi_k)$ is known. We want to solve for the leg parameters $\alpha_k$ and $\eta_k$, but $\phi_k$ is still unknown. This unknown can be solved from the continuous system equations (26).

At the starting position of the $k$th stance $r_k$ and the ending position of the $k$th stance $r_{k+1}$, by conservation of the angular momentum, we have

$$p_{\theta_{k+1}} = p_{\theta_k} = \eta_k^2 \psi = \eta_k v_k \sin \alpha_k.$$ 

(31)

Using the method of integration by quadrature, c.f. [9], we can compute the center swing angle $\phi_k$ for each stance as

$$\phi_k = 2 \int_{\eta_k}^{\eta_{min}} \frac{p_{\psi}}{\pm \sqrt{2E - \frac{p_{\psi}^2}{\eta^2} - 2V(\eta)}} \, d\eta.$$ 

(32)

where $\eta_{min}$ is the shortest length of the spring during the stance. When $\eta = \eta_{min}$, we have $\dot{\eta} = 0$. Thus we can solve $\eta_{min}$ from

$$2E - \frac{p_{\psi}^2}{\eta_{min}^2} - 2V(\eta_{min}) = 0.$$ 

(33)

Explicit formulas can be derived for $\phi_k$ when we use the linear spring potential $V = b_k (\eta - \eta_k)^2$ where $b_k$ is the spring constant for the $k$th stance. These formulas require the use of elliptic functions [2].

Since $\phi_k$ is now a known function of $\alpha_k$, $\eta_k$, and $b_k$, we can solve for any two of $\alpha_k$, $\eta_k$, and $b_k$ from (28) and (30) when keeping the other parameter constant. For a runner, controlling $\alpha_k$ means to find the appropriate angle between its leg and the direction of the COM motion. On the other hand, as reported by Jindrich and Full in [10], the cockroaches control the length $\eta_k$ by stretching or compressing their legs when turning. We see that changing $\eta_k$ will affect both $p_{\psi_k}$ and $V(\eta_k)$. This changes $\phi_k$ and hence controls $(\zeta_k - \varphi_k - \theta_k)$. Another means of steering is to change the potential energy $V(\eta)$, e.g., change the spring constant $b_k$, which also controls $\phi_k$.

If the conditions in Theorem 3.3 are satisfied, equation (10) always has a solution for $\theta_i$. The equations (28), (30) and (32) can be solved to implement the control $\theta_i$. We call this method the inverse method. Note that finding solutions for $\alpha_k$ and $\eta_k$ often requires numerical methods because $\phi_k$ is not a simple function of $\alpha_k$ and $\eta_k$.

We plot one right stance in Figure 3. Let $v_k$ be the velocity of the COM at the beginning of the $k$th stance. This $v_k$ provides initial value for the states $p_\eta$ and $p_\psi$ of the system (26). Because mechanical energy is conserved, the speed $v_k = \|v_k\|$ satisfies $v_{k+1} = v_k$. We let $\alpha_k$ represent the angle measured from the leg to the velocity $v_k$ at the moment of touchdown, and we call $\alpha_k$ the leg placement parameter. We use $\phi_k$ to measure the direction of velocity in the lab frame. By using simple geometric relationships in Figure 3 we have

$$\frac{\pi}{2} - \phi_k - \frac{\alpha_k}{2} = \delta_k - \varphi_k.$$ 

(27)

Therefore, using equation (24),

$$\zeta_k - \varphi_k - \theta_k = \frac{\pi}{2} - \alpha_k - \frac{\phi_k}{2}.$$ 

(28)

Fig. 3. The LLS model with COM coincide with COP. Angle $\phi_k$ is the center angle swept during the stance. Angle $\alpha_k$ is the angle between the leg and the velocity $v_k$ at the moment of touch down.
V. TRACKING BEHAVIOR UNDER CONSTRAINTS

For every stance, the LLS model generates the COM movement $q_k$ by controlling parameters such as $\alpha_k$, $\eta_k$ and $b_k$. In practice, these parameters all have to be bounded. These bounds post constraints on the possible $q_k$ that can be generated by the LLS model. In this section, we first discuss the constrained COM movement and investigate the constrained tracking behavior when a runner is running along a curve path. We then derive a new control law with proved convergence under the constraints.

A. the constraints

To generate forward locomotion, the relative angle $\alpha_k$ between the leg and the COM velocity $v_k$ should be bounded within the interval $(0, \pi/2)$. Figure 4 illustrates the possible $q_k$ that can be produced by changing $\alpha_k$ for a right stance when $q_k$ and $b_k$ are held constant. The parameters for the plotted LLS model are $m = 2.5g$, $v_k = 0.2m/s$, $\eta_k = 1.7cm$ and $b_k = 1.05 N/m$ which are typical for a cockroach. When $\alpha_{k} = 0$ and $\alpha_{k} = \pi/2$, we have $q_k = 0$. Therefore, in order to move forward effectively, the angle $\alpha_k$ must be within an interval $[\alpha_{\text{min}}, \alpha_{\text{max}}]$ with $\alpha_{\text{min}} > 0$ and $\alpha_{\text{max}} < \pi/2$. The solid segment in Figure 4 illustrates the possible $q_k$ between $\alpha_{\text{min}} = \pi/6$ and $\alpha_{\text{max}} = \pi/3$. We also plot the length $q_k$ as a function of the leg placement angle $\alpha_k$ in Figure 5. There the upper limit $q_{k}$ is 1.44cm. When $\alpha_k$ is within $[\pi/6, \pi/3]$, the minimum $q_k$ is 1.24cm. The changes in $q_k$ is not big for a wide range of $\alpha_k$. This is typical for LLS models.

The above example suggests that it is possible to keep $\eta_k$, $q_k$ and $\phi_k$ constant for each stance. We control the spring constant $b_k$ and the leg placement angle $\alpha_k$. The advantage of this strategy is that the distance traveled by the COM is identical for every stance. This fact can help us analyze the tracking behavior later. For the LLS model plotted in Figure 4 and 5 in order to keep $q_k = 1.44cm$, we plot the spring constant $b_k$ as a function of the leg placement angle $\alpha_k$ in Figure 6. For $\alpha_k \in [\pi/6, \pi/3]$, $b_k$ lies between 0.78 N/m and 1.06 N/m. This range is not difficult to implement.

With $q_k$ constant for each stance, the possible movement of the COM can be depicted by a cone $C_k$. The two edges of the cone correspond to $\alpha_{k} = \alpha_{\text{min}}$ and $\alpha_{k} = \alpha_{\text{max}}$. The length of both edges are $q_k$. Figure 7 illustrates cone $C_{k+1}$ and cone $C_k$. $C_{k+1}$ grows from the end point of $q_k$ which lies in the circular arc of $C_k$. We found that $C_{k+1}$ is the mirror image of $C_k$ with $q_k$ being the axis of symmetry. This is because the velocity vector $v_{k+1}$ and $v_k$ are symmetric with respect to $q_k$, which can be proved by solving equations (26). Therefore, as the runner moves forward, the cone will flip from side to side.

B. running along a curve path and robustness

We use the index $i$ for all right and left stances. When running parallel to a desired curved path, the COM movement satisfies $\rho_i = \rho_c$ for all $i$. Therefore, we have $f_i = 0$ in equation (10). The following conditions are necessary:

A1) $q_i \in C_i$;
A2) $\theta_i = \sin^{-1}(q_i/2\lambda_i)$.
we let i.e., a right stance followed by a left stance. For convenience $\theta$ is the most robust. See the middle figure in Figure 7.

Condition A1 requires that the COM movement belongs to the cone that is feasible for the constrained model. Condition A2 requires that $q_i$ is parallel to the desired curve.

If the desired curve is a straight line segment, then $\theta_i = 0$ in condition A2. As the runner moving forward, the cone $C_i$ will be flipping side to side with respect to the straight line. In this case we have $\alpha_i = \alpha_{i+1}$. The robustness of this behavior is determined by the size of the cone and the value of $\alpha_i$. If we choose the value of $\alpha_i$ so that the COM movement $q_k$ is always in the middle of the cone, then the tracking behavior is the most robust. See the middle figure in Figure 2.

If the desired curve is convex to locally concave. No general conclusions can be drawn regarding which part of the cone $C_i$ is used for a stance.

If the desired curve is not convex, the tracking behavior is determined by the size of the cone and the value of $\alpha_i$. When running along a curve path in the counter clockwise direction.

The sub-cones $C^A_k$ and $C^B_k$, $q_k$ is the middle line of $C^A_k$, and $q_{k+1}$ is the middle line of $C^A_{k+1}$. The angle difference between $q_{k+1}$ and $q_k$ is $\gamma_i$.

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If the desired curve is convex, followed by a left stance: $\alpha_{i+2} - \alpha_{i+1} = -\gamma_{i+1}$. We can then write $\alpha_{i+1} - \alpha_i = \pm \gamma_i$ for all stances. This equation requires that $\gamma_i$ must be less than $(\alpha_{\text{max}} - \alpha_{\text{min}})$. This implies that $\lambda_i$, the instantaneous radius of the curve must satisfy

$$\lambda_i > \frac{q_i}{2 \sin \frac{\alpha_{\text{max}} - \alpha_{\text{min}}}{2}}.$$  

The middle lines of $C^A_k$ and $C^B_k$, $q_k$ is the middle line of $C^A_k$, and $q_{k+1}$ is the middle line of $C^A_{k+1}$. The angle difference between $q_{k+1}$ and $q_k$ is $\gamma_i$.

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much smaller than the radius of the curve path. Under these assumptions, \( \rho_{i+1} - \rho_i \ll 2(\rho_i + \frac{1}{\kappa}) \), and equation (10) can be simplified to
\[
\rho_{i+1} - \rho_c = \rho_i - \rho_c + \frac{q^2}{2\lambda_i} - q\sin \theta_i
\]  
(37)
where \( \lambda_i = (\rho_i + \frac{1}{\kappa}) \). We then view \( \sin \theta_i \) as the second state variable other than \( \rho_i - \rho_c \). It satisfies
\[
\sin \theta_i = \sin \left( \zeta_i - \varphi_i + \frac{\pi}{2} - \alpha_i - \frac{\phi_i}{2} \right)
\]  
(38)
when all the angles are mapped to the interval \((-\pi/2, \pi/2)\). We can apply the backstepping technique for discrete systems c.f. [11] to the system described by (37) and (38).

Let \( f_i = -K_i(\rho_i - \rho_c) \). According to Theorem 3.3, we can select \( K_i \) such that (10) can be solved for \( \theta_i \). We let this solution be \( \tilde{\theta}_i \) i.e.
\[
\sin \tilde{\theta}_i = \frac{-f_i^2 - 2\lambda_i f_i + q^2}{2\lambda_i q} = \frac{q}{2\lambda_i} - \frac{f_i}{q} - \frac{f_i^2}{2\lambda_i q},
\]  
(39)
Using the fact that
\[
f_i^2 = (\rho_{i+1} - \rho_i)^2 \ll 2(\rho_i + \frac{1}{\kappa}) = 2\lambda_i,
\]  
(40)
we have
\[
q\sin \tilde{\theta}_i = \frac{q^2}{2\lambda_i} - f_i.
\]  
(41)
Note that \( \tilde{\theta}_i \) is different from the state variable \( \theta_i \) in (37) and (38). Here \( \tilde{\theta}_i \) is a function of \( \rho_i \) and can be viewed as the desired value for the state \( \theta_i \).

We can then solve for the term \( \frac{q^2}{2\lambda_i} \) from (41) and substitute this term in the right hand side of (37). This yields
\[
\rho_{i+1} - \rho_c = \rho_i - \rho_c + f_i - q(\sin \theta_i - \sin \tilde{\theta}_i).
\]  
(42)
We define
\[
\tilde{\rho}_{i+1} = \rho_i - \rho_c + f_i - (1 - K_i)(\rho_i - \rho_c).
\]  
(43)
This \( \tilde{\rho}_{i+1} \) can be viewed as the desired value for the state \( \rho_{i+1} - \rho_c \). With the help of \( \tilde{\theta}_i \) and \( \tilde{\rho}_{i+1} \), we rewrite (42) as
\[
\rho_{i+1} - \rho_c - \tilde{\rho}_{i+1} = -q(\sin \theta_i - \sin \tilde{\theta}_i).
\]  
(44)
We design \( \alpha_i \) as a feedback law so that the right hand side of (38) satisfies
\[
\sin \left( \zeta_i - \varphi_i + \frac{\pi}{2} - \alpha_i - \frac{\phi_i}{2} \right) = \sin \tilde{\theta}_i + \frac{\tilde{K}_i}{q}(\rho_i - \rho_c - \tilde{\rho}_i)
\]  
(45)
where \( \tilde{K}_i \) is a scalar which will be determined later. Thus (38) becomes
\[
\sin \theta_i - \sin \tilde{\theta}_i = \frac{\tilde{K}_i}{q}(\rho_i - \rho_c - \tilde{\rho}_i).
\]  
(46)
We show that the closed loop system given by (44) and (46) converges to the state where \( \theta_i = \tilde{\theta}_i \) and \( \rho_i = \rho_c \).

**Lemma 5.1:** Consider the system given by (37) and (38). Let \( K_i \) and \( \tilde{K}_i \) be such that \( |1 - K_i| < 1 \) and \( |\tilde{K}_i| < 1 \).

Suppose (45) has a solution for \( \alpha_i \). Then as \( i \to \infty \), we must have \( \theta_i \to \tilde{\theta}_i \) and \( \rho_i \to \rho_c \).

**Proof:** If (45) has a solution for \( \alpha_i \), then (44) and (46) hold. Therefore
\[
\sin \theta_i - \sin \tilde{\theta}_i = -\tilde{K}_i(\sin \theta_{i-1} - \sin \tilde{\theta}_{i-1}).
\]  
(47)
Since \( |\tilde{K}_i| < 1 \), it is true that \( \sin \theta_i - \sin \tilde{\theta}_i \to 0 \) as \( i \to \infty \). Meanwhile, from (44), we conclude \( \rho_{i+1} - \rho_c - \tilde{\rho}_{i+1} \to 0 \) as \( i \to \infty \). This implies that \( \rho_{i+1} - \rho_c \to (1 - K_i)(\rho_i - \rho_c) \).

Since \( |1 - K_i| < 1 \), we conclude \( \rho_i \to \rho_c \) as \( i \to \infty \).

If we allow \( |\tilde{K}_i| \) to be arbitrary large then (45) always has solutions for \( \alpha_i \). By selecting proper value for \( K_i \), we can find a solution that satisfies the constraints for \( \alpha_i \). However, Lemma 5.1 requires that \( |\tilde{K}_i| < 1 \) to achieve asymptotic convergence. We want to find out when (45) fails to have a solution for \( \alpha_i \) if \( |\tilde{K}_i| < 1 \).

If we can find \( \alpha_i \in [\alpha_{\min}, \alpha_{\max}] \) such that \( \sin \theta_i - \sin \tilde{\theta}_i = 0 \). Then we can let \( \tilde{K}_i = 0 \) and a solution for \( \alpha_i \) exists. This is exactly the inverse method.

If we can not find a constrained \( \alpha_i \) such that \( \sin \theta_i - \sin \tilde{\theta}_i = 0 \), then let
\[
\tilde{M}_i = \min_{\alpha_i \in [\alpha_{\min}, \alpha_{\max}] \} |\sin \theta_i - \sin \tilde{\theta}_i|.
\]  
(48)
We let
\[
\tilde{K}_i = \frac{\tilde{M}_i q}{\rho_i - \rho_c - \tilde{\rho}_i}.
\]  
(49)
If
\[
|\rho_i - \rho_c - \tilde{\rho}_i| > \tilde{M}_i q
\]  
(50)
is true, then \( |\tilde{K}_i| < 1 \). Equation (45) becomes \( \sin \theta_i - \sin \tilde{\theta}_i = \tilde{M}_i \), and a solution for \( \alpha_i \) can be found. Therefore, when a solution for \( \alpha_i \) can not be found for \( K_i \) given in (49), we must have
\[
|\rho_i - \rho_c - \tilde{\rho}_i| \leq \tilde{M}_i q.
\]  
(51)
We argue that this implies that the distance between the runner and the desired path becomes sufficiently small as \( i \to \infty \).

We define \( M = \sup_i \tilde{M}_i \). Since \( |\sin \theta_i - \sin \tilde{\theta}_i| \leq 2 \) regardless of the value of \( \theta_i \) and \( \tilde{\theta}_i \), it is true that \( M \leq 2 \). The following theorem claims that the controlled movement of the COM converges to a small neighborhood of the desired curve.

**Theorem 5.2:** Consider the system given by (37) and (38) controlled by the approximation method. Suppose the gain \( K_i \) is constantly \( K \) for all \( i \) and \( |1 - K| < 1 \). We determine the value for leg placement angle \( \alpha_i \) by solving (45). For every \( i \), we let \( \tilde{K}_i \) be given by (49). Then we must have \( \lim_{i \to \infty} |\rho_i - \rho_c| \leq M q/(1 - |1 - K|) \).

**Proof:** If \( |\tilde{K}_i| < 1 \) for all \( i \), Lemma 5.1 claims that the system converges to \( \rho_i = \rho_c \). Then \( \lim_{i \to \infty} |\rho_i - \rho_c| = 0 < \tilde{M}_i q/(1 - |1 - K|) \).

If for some time indices \( j \), \( |\tilde{K}_j| \geq 1 \), then condition (51) must be true. We must have
\[
|\rho_j - \rho_c - \tilde{\rho}_j| \leq \tilde{M}_j q \leq \tilde{M} q
\]  
(52)
This implies that
\[
|\rho_j - \rho_c| \leq |1 - K| \cdot |\rho_{j-1} - \rho_c| + \tilde{M} q.
\]  
(53)
For the rest of time indices $i$ where $|\tilde{K}_i| < 1$, equation (44) implies that
\[ |\rho_i - \rho_c| \leq |1 - K| \cdot |\rho_{i-1} - \rho_c| + |\sin \theta_{i-1} - \sin \tilde{\theta}_{i-1}| q \] (54)
Since $\tilde{K}_{i-1}$ is given by (49), we must have $|\sin \theta_{i-1} - \sin \tilde{\theta}_{i-1}| = M_{i-1}$. This implies that (53) is satisfied for indices $i$. Therefore (53) is satisfied for all $i$ and $j$. Because $|1 - K| < 1$, we conclude that $\lim_{i \to \infty} |\rho_i - \rho_c| \leq \tilde{M} q/(1 - |1 - K|)$.

Remark 5.3: From Theorem 5.2, we conclude that larger step size (bigger $q$) and faster convergence rate in relative distance (bigger $K$) may cause larger tracking error. This agrees with intuition. The Theorem is conservative because we can make $K_i$ adaptive. When $|\rho_i - \rho_c|$ small, we can let $K_i$ to be close to 1 to reduce the possible tracking error. Also, $\tilde{M}$ does not have to be the supremum of $M_i$ for all $i$. Instead, we can let $i$ be larger than any finite time. This may reduce $\tilde{M}$. Another observation is that by increasing $\alpha_{\max} - \alpha_{\min}$, we can reduce $\tilde{M}$, hence reduce the tracking error. This also agrees with intuition.

Remark 5.4: Equation (48) is used to compute $\alpha_i$ that minimizes the difference between $\theta_i$ and $\tilde{\theta}_i$. This is why the method is called the approximation method. Such a method can be devised heuristically without applying the backstepping technique. But the backstepping technique helps to justify convergence. One can also derive other convergent method which have different performance than the current one. But from the proof of Lemma 5.1 we conclude that the approximation method gives the fastest convergence rate among all such methods derived from the backstepping procedure.

VI. CONTROLLED RIGID BODY DYNAMICS

Unlike stable running along a straight line, the rigid body angular momentum $p_\sigma$ should not be zero for stable running along a curve; otherwise no turning can happen. Since our goal is to control the runner to the desired curve, even if the final stable running is along a straight line, the runner need to turn in order to move to that line from an arbitrary initial position. The LLS model dynamics for the rigid body is
\[
\begin{align*}
\dot{p}_\sigma &= I \dot{\alpha} \\
\dot{\sigma}_\sigma &= \tau
\end{align*}
\] (55)
where $\tau$ represents the torque to produce turning for the rigid body. Runners can produce this torque by using muscles connecting legs and body or by changing force distribution over multiple legs.

As indicated in Figure 9, the angle between the body axis and the velocity of the COM is $\phi_k(\sigma_k - \varphi_k)$. For stable running, we want to achieve the following gaits: after the $k$th stance, the ending angle $(\sigma_k - \varphi_k)$ differs from the starting $(\sigma_{k+1} - \varphi_{k+1})$ only by the sign; accordingly, the direction of the angular momentum will be reversed, i.e., $p_{\sigma_{k+1}} = -p_{\sigma_k}$. This stable running requires that as $k \to \infty$,
\[
\begin{align*}
\sigma_k - \varphi_k &\to C_1 \text{ and } p_{\sigma_k} \to C_2 \\
\sigma_{k+1} - \varphi_{k+1} &\to -C_1 \text{ and } p_{\sigma_{k+1}} \to -C_2
\end{align*}
\] (56)
for right stances and
\[
\begin{align*}
\sigma_k - \varphi_k &\to -C_1 \text{ and } p_{\sigma_k} \to -C_2 \\
\sigma_{k+1} - \varphi_{k+1} &\to C_1 \text{ and } p_{\sigma_{k+1}} \to C_2
\end{align*}
\] (57)
for left stances where $C_1$ and $C_2$ are pre-selected constants.

For right stances, we let
\[
\begin{align*}
\sigma_{k+1} - \varphi_{k+1} &= (1 - K_4)(\sigma_k - \varphi_k - C_1) \\
p_{\sigma_{k+1}} + C_2 &= (1 - K_5)(p_{\sigma_k} - C_2)
\end{align*}
\] (58)
where $0 < K_4, K_5 < 1$. For left stances, we may let
\[
\begin{align*}
\sigma_{k+2} - \varphi_{k+2} &= (1 - K_4)(\sigma_k - \varphi_k + C_1) \\
p_{\sigma_{k+2}} - C_2 &= (1 - K_5)(p_{\sigma_k} + C_2).
\end{align*}
\] (59)
Note that $\varphi_k$ and $\varphi_{k+1}$ are known from (66) and (67) in Appendix I. It is not difficult to see that the discrete systems (58) and (59) achieve the desired convergence specified by (56) and (57).

Therefore, for the $k$th stance, we want to design the control torque $\tau$ so that starting from $p_{\sigma_k}$ and $\sigma_k$, the runner will reach the state $p_{\sigma_{k+1}}$ and $\sigma_{k+1}$ given by
\[
\begin{align*}
p_{\sigma_{k+1}} &= (1 - K_5)p_{\sigma_k} + (2 - K_5)C_2 \\
\sigma_{k+1} &= (1 - K_4)\sigma_k + K_4\varphi_k + \pi - \varphi_k - 2\alpha_k + (2 - K_4)C_1.
\end{align*}
\] (60)

We formulate an optimal control problem with the starting state given by $(p_{\sigma_k}, \sigma_k)$ and ending state given by $(p_{\sigma_{k+1}}, \sigma_{k+1})$ in equation (60). The ending state should be achieved within the duration $T_k$ for the $k$th stance with the cost function $\int_0^{T_k} \tau^2(t) dt$ minimized. The solution for $\tau_k(t)$ for the $k$th stance can be obtained by applying the maximum principle. We have
\[
\tau_k(t) = \frac{1}{2}(A_{2k} - A_{3k} - t)
\] (61)
where $t \in [0, T_k]$, $I$ is the moment of inertia, and
\[
\begin{align*}
A_{1k} &= \frac{24 I^2}{T_k^3}(\sigma_{k+1} - \sigma_k) - \frac{12 I}{T_k}(p_{\sigma_{k+1}} + p_{\sigma_k}) \\
A_{2k} &= \frac{12 I}{T_k^2}(\sigma_{k+1} - \sigma_k) - 8 \frac{I}{T_k} p_{\sigma_{k+1}} - 8 \frac{I}{T_k} p_{\sigma_k}.
\end{align*}
\] (62)
The detail of this derivation is included in Appendix [II]. The duration $T_k$ can be computed in the similar way as $\phi_k$ in (32) as

$$T_k = 2 \int_{q_{k-1}}^{q_{k}} 1 \sqrt{2E - \frac{1}{\eta^2} - 2V(\eta)} \, d\eta. \quad (63)$$

In equations (60) and (62), although $\tau_k(t)$ is open-loop control during each stance for $t \in [0, T_k]$, feedback is achieved through $p_{\alpha_k}$ and $\sigma_k$ when switching from one stance to another.

VII. SIMULATIONS

We present simulation results to demonstrate tracking a circle centered at the origin with radius 0.02 m. The parameters for the LLS model are the same as in section V. The desired distance to the circle is 0.03 m. Initially, the runner start from (0.1, 0) outside the circle. The speed of the COM is 0.2 m/s. The initial direction of the velocity is $\pi/3$. The leg placement angle $\alpha_i$ is constrained to be within the interval $(\pi/3, \pi/3)$. We change the spring constant $b_i$ so that $q_i$ is always equal to 1.53 cm, which is 90% of the leg length at rest. The gain $K_i$ is selected to be 0.5. When $\vartheta_i$ can not be achieved by $\alpha_i$, we simply use the value for $\alpha_i$ that will minimize the differences between desired $\vartheta_i$ and achievable $\vartheta_i$; hence implemented the approximation method.

The trajectory of the COM and the distance between the COM and the circle are plotted in Figure 10. We see that the convergence is achieved after 12 stances which take less than one second.

VIII. SUMMARY AND FUTURE WORK

We have analyzed the control of LLS model and designed a hybrid curve tracking control law for legged locomotion. Using measurements of the curve for feedback, the discrete algorithm guarantees convergence to the desired curve path. During each stance, the controlled continuous dynamics is analyzed. The parameters of the LLS model is determined to implement the discrete algorithm at the beginning of each stance. We have also investigated the effects of parameter constraints. These constraints limited tracking ability. For straight lines and convex curves, a steady state can be reached. The robustness of these steady states depends on the range for the parameters.

Interesting results regarding wall following behaviors of cockroaches are reported by Camhi and Johnson in [12]. When its antenna touches the wall, a cockroach turns away from the wall but keeps the antenna in contact with the wall for a certain time period. The experiments there performed near a raffled piecewise linear wall, not a smooth curved wall. Using a smooth curved wall with convex shape will also be interesting since a steady state can be reached.

Recently, a wall following wheeled robot using antenna like tactile sensor was reported in [8]; curve tracking for atomic force microscope is discussed in [13]; a general boundary tracking control law is derived for Newtonian particles in [14]. Our work, although intended for legged locomotion, may be adapted to handle other cases regarding curve tracking for platforms with hybrid motion dynamics.

Fig. 10. Tracking a circle with radius 0.02 m. In (a), the cross symbols indicate the touchdown points. The units for both horizontal and vertical axis are meters. In (b), the distance (in meters) between the COM and the circle is plotted as a function of time (in seconds). We can see it converges to the desired separation 0.03 m.

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Similarly, subtract (29) yields Equation (28) must satisfy (28) and (29).

\[ \theta_{k+1} - \theta_k = (\zeta_{k+1} - \varphi_{k+1}) - (\zeta_k - \varphi_k) + \pi - \left( (\alpha_k + \alpha_{k+1}) \right) - \frac{\phi_k + \phi_{k+1}}{2}. \]  

(64)

The angle \((\zeta_k - \varphi_k)\) is measured between the COM velocity vector and the tangent vector of the desired curve. In a similar fashion, we find

\[ \theta_{k+2} - \theta_{k+1} = (\zeta_{k+2} - \varphi_{k+2}) - (\zeta_{k+1} - \varphi_{k+1}) + \pi - \left( (\alpha_{k+2} + \alpha_{k+1}) + \frac{\phi_{k+2} + \phi_{k+1}}{2} \right). \]  

(65)

We now establish a difference equation which describes the change of \((\zeta_k - \varphi_k)\).

Comparing the COM velocity \(v_k\) and \(v_{k+1}\), we notice that they are reflectively symmetric to each other with \(q_k\) as the axis of symmetry, as shown in Figure 7. This can be proved from the solution of equations (26). From Figure 3, we conclude that

\[ \varphi_{k+1} - \varphi_k = 2\pi - \phi_k - \alpha_k \]  

(66)

from a right stance to a left stance and

\[ \varphi_{k+2} - \varphi_{k+1} = -2\pi - \phi_{k+1} - \alpha_{k+1} \]  

(67)

from a left stance to a right stance. From Figure 2, we can derive the change of \(\zeta_k\) as

\[ \zeta_{k+1} - \zeta_k = \gamma_k. \]  

(68)

The angle \(\gamma_k\) can be determined using the sine law

\[ \sin \gamma_k = \frac{\sin \beta}{\rho_{k+1} + 1}. \]  

(69)

We use \(\beta = \pi/2 - \theta_k\) and \(\rho_{k+1} = \rho_k + f_k\) to obtain

\[ \sin \gamma_k = \frac{q_k}{\lambda_k + f_k} \cos \theta_k. \]  

(70)

From (66) and (68), we obtain

\[ (\zeta_{k+1} - \varphi_{k+1}) - (\zeta_k - \varphi_k) = \gamma_k - (\pi - \phi_k - 2\alpha_k). \]  

(71)

Similarly,

\[ (\zeta_{k+2} - \varphi_{k+2}) - (\zeta_{k+1} - \varphi_{k+1}) = \gamma_{k+1} + (\pi - \phi_{k+1} - 2\alpha_{k+1}). \]  

(72)

Using (71) and (64), we deduce that

\[ \theta_{k+1} - \theta_k = -(\alpha_{k+1} - \alpha_k) - \frac{\phi_{k+1} + \phi_k}{2} + \gamma_k. \]  

(73)

Using (72) and (65), we have

\[ \theta_{k+2} - \theta_{k+1} = (\alpha_{k+2} - \alpha_{k+1}) + \frac{\phi_{k+2} + \phi_{k+1}}{2} + \gamma_{k+1}. \]  

(74)

Equations (73) and (74) must be satisfied for all feasible \(\theta_k\), \(\theta_{k+1}\), and \(\theta_{k+2}\).

APPENDIX II

OPTIMAL CONTROL FOR RIGID BODY DYNAMICS

The system equations for the rigid body dynamics is

\[ \dot{\sigma} = \frac{p_\sigma}{I}, \quad \dot{p}_\sigma = \tau. \]  

(75)

We want to minimize the cost function \(\int_0^T \tau^2(t)dt\). Applying the maximum principle, we define the controlled Hamiltonian as

\[ H_{\lambda, \tau} = \lambda_1 \dot{\sigma} + \lambda_2 \dot{p}_\sigma - \tau^2 \]  

(76)

where \(\lambda_1\) and \(\lambda_2\) are the adjoint variables. The optimal control \(\tau\) that minimizes \(H_{\lambda, \tau}\) is computed by letting \(\partial H_{\lambda, \tau}/\partial \tau = 0\). This yields \(\tau = \lambda_2/2\). The Hamiltonian for the system under this control is

\[ H_{\lambda} = \lambda_1 \frac{p_\sigma}{I} + \frac{1}{4} \lambda_2^2. \]  

(77)

This Hamiltonian induces the following Hamilton’s equations:

\[ \dot{\sigma} = \frac{p_\sigma}{I}, \quad \dot{p}_\sigma = \lambda_2 \]  

\[ \lambda_1 = 0, \quad \lambda_2 = -\frac{1}{I} \lambda_1. \]  

(78)

From these equations we first observe that \(\lambda_1(t) = \lambda_1(0)\).

Then we can solve for \(\lambda_2\) as

\[ \lambda_2(t) = \lambda_2(0) - \frac{\lambda_1(0)}{I} t. \]  

(79)

In order to determine \(\lambda_1(0)\) and \(\lambda_2(0)\), we integrate

\[ p_\sigma = \frac{1}{2} (\lambda_2(t) - \lambda_1(0)) t \]  

(80)

from 0 to \(T_k\). This yields

\[ p_{\sigma_{k+1}} = p_{\sigma_k} + \frac{1}{2} \lambda_2(0) T_k - \frac{1}{4} \lambda_1(0) T_k^2. \]  

(81)

We then integrate

\[ \dot{\sigma} = \frac{p_\sigma}{I} = \frac{1}{4} \left( p_{\sigma_k} + \frac{1}{2} \lambda_2(0) t - \frac{1}{4} \lambda_1(0) t^2 \right). \]  

(82)

This gives us

\[ \sigma_{k+1} = \sigma_k + \frac{1}{4} I \lambda_2(0) T_k^2 - \frac{1}{12} \lambda_1(0) T_k^3. \]  

(83)

From (81) and (83), we can solve for \(\lambda_1(0)\) and \(\lambda_2(0)\). One can verify that \(\lambda_1(0) = A_{1_k}\) and \(\lambda_2(0) = A_{2_k}\), where \(A_{1_k}\) and \(A_{2_k}\) are given in (62), are the solutions.