THE SO(3) VORTEX EQUATIONS OVER ORBIFOLD RIEMANN SURFACES

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ABSTRACT. We study the general properties of the moduli spaces of SO(3) vortices over orbifold Riemann surfaces and use these to characterize the solutions of the SO(3) monopole equations on Seifert manifolds following in the footsteps of Mrowka, Ozsváth and Yu.

We also study the solutions to the SO(3) monopole equations on $S^1 \times \Sigma$ in order to motivate the construction of a version of monopole Floer homology, which we call framed monopole Floer homology, in analogy with the construction given by Kronheimer and Mrowka for the case of instanton Floer homology.

Finally, the SO(3) vortex moduli spaces provide a nice toy model for recent work due to Feehan and Leness regarding the study of a natural Morse-Bott function on the moduli space of SO(3) monopoles over Kahler manifolds. In particular, we compute the Morse-Bott indices of this function.

1. Introduction
1.1. SO(3) vortices and Morse-Bott Theory.

The study of vortices on Riemann surfaces has been so fruitful since the work of Jaffe and Taubes [49] that it would be entirely reasonable to assume that on this topic there is nothing new under the Sun. Indeed, the vortex equations have been studied and generalized in multiple directions (see for example [20, 19, 18, 44, 90, 77, 21, 12, 93, 17]), and they have proved incredibly useful in 3- and 4-manifold topology due to its interaction with the Seiberg-Witten equations [94, 85, 86, 87]. However, by scraping the bottom of the barrel one can still find some new things as we now explain.

First of all, an SO(3) vortex consists of a pair $(C, \Upsilon)$, where $C$ is a unitary connection on a $U(2)$ bundle $E$ over a Riemann surface $\Sigma$, and $\Upsilon$ is a section of $E$. The connection $C$ must induce a predetermined connection $C_{\text{det}}$ on $\text{det} E$, which is why we are really studying the SO(3) vortex equations instead of the $U(2)$ vortex equations. The equations $(C, \Upsilon)$ must satisfy are the equations (13), that is,

\[ *F_C^0 - i \left[ \Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E \right] = 0 \]
\[ \bar{\partial}_C \Upsilon = 0 \]

The meaning of these equations is explained in Section (4), but they probably already look familiar to most readers. By studying the solutions to the SO(3) vortex equations modulo gauge transformations one obtains the SO(3) vortex moduli space $M(\Sigma, E)$.

The study of these equations has a long history. For example, in his PhD thesis (see also [43 Section 5]), García-Prada studied the moduli space of non-abelian vortices from a gauge theoretic and algebraic geometric point of view. From the latter perspective, these moduli spaces can be identified with the moduli spaces of stable pairs, and García-Prada interpreted stable pairs of arbitrary rank on any compact Kahler manifold $X$ as stable bundles on $X \times \mathbb{C}P^1$, which allowed...
him to encode the stability parameter of the moduli space of stable pairs in terms of the stability condition in the sense of Mumford of the corresponding stable bundle.

Around the same time, Bradlow and Daskalopoulous \[19, 16\] studied the moduli spaces of stable pairs for the case of a Riemann surface from a more analytic point of view, and a bit later the moduli spaces for the rank two case were studied by Thaddeus \[90\] and Bertram \[111\] from the algebraic geometric point of view.

The case of the $U(2)$ vortex equations was also studied extensively from a gauge-theoretic perspective by Bradlow and Daskalopoulous \[19, 16\], as well as Bradlow, Daskalopoulous and Wentworth \[19\]. More importantly, the moduli space of $SO(3)$ vortices for Riemann surfaces without marked points is a particular case of the general framework developed in the paper by García-Prada, Gothen and Mundet i Riera \[45\].

For our purposes it was important to work with the more general setup of orbifold Riemann surfaces, using the ideas of Furuta and Steer \[42\] for the case of the Yang-Mills equations (which is the orbifold version of Atiyah and Bott's seminal work \[6\]). We use gauge-theoretic techniques instead of geometric invariant theory (GIT), which is also used to study these moduli spaces from the algebraic geometric point of view. In fact, Bertram describes briefly the construction of the moduli space of stable parabolic pairs \[11, Section 3\] (using GIT techniques).

One reason for studying the moduli space of $SO(3)$ vortices on Riemann surfaces is that they provide useful toy models complementing recent work of Feehan and Leness, who studied the $SO(3)$ monopole moduli spaces over Kahler surfaces from a Morse-theoretic point of view. Following the work of Hitchin \[48\], Feehan and Leness \[34\] studied the $L^2$ norm of the spinor of an $SO(3)$ monopole as a Morse-Bott function on the moduli space of $SO(3)$ monopoles over Kahler manifolds, which in particular means they had to compute the Morse-Bott indices of the critical sets of this function. We do the analogous computations for $SO(3)$ vortices over a Riemann surfaces. Since we are also finding these formulas for the case of orbifolds, the work \[70\] by Nasatyr and Steer also served as an important model. We now explain how the computation of these indices is done.

As mentioned earlier, $\mathcal{M}(\Sigma, E)$ is obtained by studying the solutions to the $SO(3)$ vortex equations modulo gauge. In this case, the gauge group used is the determinant-one gauge group $G^{\det}(E)$, which is not the gauge group of all unitary automorphisms of $E$. The upshot of doing this is that there is a residual symmetry on the moduli space of $SO(3)$ vortices modulo $G^{\det}(E)$.

More precisely, there is a circle action on the moduli space obtained by rescaling the section $T$, in other words, $T \rightarrow e^{it}T$. Associated to this circle action, there is a moment map $\mu$ which is essentially the $L^2$ norm of the section, that is, $\mu(C, T) = \frac{1}{2}\|T\|^2_{L^2(\Sigma)}$. Following Bradlow, Daskalopoulous and Wentworth \[19\], the idea is to study $\mu$ as a Morse-Bott function on the $SO(3)$ vortex moduli space (they studied this for the $U(2)$ vortex moduli space for the case of a smooth Riemann surface, i.e, without marked points).

The critical sets of $\mu$ can be identified with the fixed points of the circle action, which consists of the moduli space of projectively flat connections on $\Sigma$, and certain moduli spaces of abelian $U(1)$ vortices on $\Sigma$. The abelian vortices satisfy equations \[20\], which read

$$\ast F_{CL} - \frac{i}{2} |\alpha|^2 = \ast \frac{1}{2} F_{C^{\det}}$$

$$\bar{\partial}_{CL} \alpha = 0$$

Here $E$ has reduced as $E = L \oplus (L^* \otimes \det E)$, and $C_L$ is a connection on $L$, while $\alpha$ is a section of $L$. Via an easy Chern-Weil argument (still valid in the orbifold case, see lemma \[20\]) we must have $c_1(L) \leq \frac{1}{2} c_1(E)$, where in the orbifold case these Chern classes are in general rational cohomology.
classes, and Poincaré duality is used to regard these as rational numbers. Any SO(3) vortex which is not a projectively flat connection nor an abelian vortex will be called an irreducible SO(3) vortex, following the standard terminology in gauge theory.

In any case, a theorem due to Frankel [40] shows that the indices of these critical sets equals the dimension of the subspace on which the circle action acts with negative weight. Feehan and Leness recently [34] computed these indices for the analogue of $\mu$ in the case of SO(3) monopoles over a Kahler surface. In section Theorem (35) we compute these indices for the case of an orbifold Riemann surface.

Before giving the general formula for the index, we remark that if our Riemann surface is regarded as an orbifold Riemann surface we then write $\Sigma = (\Sigma, (p_1, a_1), \ldots, (p_n, a_n))$, where the marked points $p_1, \ldots, p_n$ have been assigned positive integers $a_1, \ldots, a_n$. Moreover, an orbifold $U(2)$ bundle $E$ over $\Sigma$ carries some isotropy data, which consists of integers $b_i^\pm$ at each marked points satisfying certain conditions which we recall in the next section.

**Theorem 1. General Properties of the SO(3) vortex moduli spaces and indices of the Morse-Bott function $\mu$:**

a) Suppose that $E$ is an orbifold $U(2)$ bundle over $\Sigma = (\Sigma, (p_1, a_1), \ldots, (p_n, a_n))$ with isotropy data $(b_1^+, \ldots, b_n^+)$. Then if $[C, \Gamma] \in M(\Sigma, E)$ is an irreducible SO(3) vortex, the moduli space is smooth at this point, and of dimension

$$\dim \mathcal{M}(\Sigma, E) = 2 \left( g - 1 + c_1(\det E) + n - n_0 - \sum_{i=1}^n \frac{b_i^- + b_i^+}{a_i} \right)$$

where $n_0 = \#\{i \mid b_i^- = b_i^+\}$.

b) There is an $S^1$ action on $\mathcal{M}(\Sigma, E)$ whose fixed point set consists of the moduli space of projectively flat connections on $ad E$ and the moduli spaces of abelian orbifold vortices associated to the reductions $E = \tilde{L} \oplus (\tilde{L}^* \otimes \det E)$, with $c_1(\tilde{L}) \leq \frac{1}{2} c_1(E)$.

If one assumes that $E$ is of odd degree in the smooth case or $E$ is an odd power of the fundamental orbifold line bundle $\tilde{L}_0$ in the case where the $a_i$ are mutually coprime, and also that

$$c_1(E) > 2c_1(K_E) = 2 \left( 2g - 2 + n - \sum_{i=1}^n \frac{1}{a_i} \right)$$

where $K_E$ represents the canonical orbifold line bundle, then $\mathcal{M}(\Sigma, E)$ is also smooth at the reducible SO(3) vortices. In fact, $\mathcal{M}(\Sigma, E)$ is a smooth closed Kahler manifold.

In particular, the function $\mu$ is a Morse-Bott function on $\mathcal{M}(\Sigma, E)$ and if the orbifold line bundle $\tilde{L}$ has isotropy $b_i$, the index of abelian vortex associated to the reduction $\tilde{E} = \tilde{L} \oplus (\tilde{L}^* \otimes \det \tilde{E})$ is

$$\text{ind}(\tilde{E}, \tilde{L}) = 2 \left[ g - 1 + c_1(\det \tilde{E}) - 2c_1(\tilde{L}) + \sum_{i: \epsilon_i = 1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i: \epsilon_i = -1} \frac{b_i^- - b_i^+}{a_i} \right]$$

where $\epsilon_i = 1$ if $b_i = b_i^+$, $\epsilon_i = -1$ if $b_i = b_i^-$. Here $n_- = \#\{i \mid b_i = b_i^- < b_i^+\}$.

**Remark 2.** a) The proof of the previous theorem is the main content of Sections 3, 4 and 5.

b) We point out that in general some assumption on the degree of $\tilde{E}$ seems necessary. For example, the cokernel of the linearized SO(3) vortex map at a projectively flat connection can basically be identified with $\text{coker} \partial C \cong H^1(\tilde{E})$ and the only way to guarantee the vanishing of $H^1(\tilde{E})$ for a semistable bundle $\tilde{E}$ is by taking $c_1(\tilde{E})$ sufficiently large.
c) Also, \( c_1(\mathcal{L}) \leq \frac{1}{2}c_1(\mathcal{E}) \) is a necessary but not sufficient condition for the appearance of a moduli space of abelian vortices. One also needs for \( \mathcal{L} \) to have the correct isotropy data and for the background degree of \( \mathcal{L} \) to be non-negative. We explain this terminology in Section \( \ref{sec:background} \) and illustrate this with the examples we do at the end of the paper.

d) An easy way to guarantee that the moduli space \( \mathcal{M}^*(\Sigma, \mathcal{E}) \) of irreducible \( SO(3) \) vortices is empty is by knowing that there are no orbifold line bundles \( \mathcal{L} \) compatible with the isotropy data of \( \mathcal{E} \) (this is more common that one might expect as the examples in Section \( \ref{sec:examples} \) illustrate). The reason for this is that the function \( \mu \) must achieve an absolute maximum on \( \mathcal{M}(\Sigma, \mathcal{E}) \), but if \( \mathcal{M}^*(\Sigma, \mathcal{E}) \subset \mathcal{M}(\Sigma, \mathcal{E}) \) is non-empty then the maximum must occur at an abelian vortex.

e) Finally, \( c_1(\mathcal{E}) > 2c_1(K_{\mathcal{E}}) \) does guarantee in the smooth case that the dimension of the moduli space of \( SO(3) \) vortices is positive [except when \( g = 0 \)]. Likewise, we will see in Section \( \ref{sec:examples} \) examples where this condition is not enough to guarantee positive dimensional moduli spaces.

In the smooth case (no marked points on the Riemann surface), the formula is quite clean and simply reads

\[
\text{ind}(E, L) = 2(g - 1 + \deg E - 2 \deg L)
\]

As an example of what one can do with the formula for the index, choosing \( \deg L = 0 \) we see that the index is \( 2(g - 1 + \deg E) \), which is the dimension formula for the moduli space of \( SO(3) \) vortices in the smooth case. This means that this moduli space of abelian vortices (which in fact ends up being a single point), corresponds to the maximum for the function \( \mu \).

Strictly speaking, these indices can be computed using the orbifold version of Riemann-Roch even in the case where we are not imposing conditions on \( \mathcal{E} \) which guarantee that the moduli spaces are smooth at the reducible points, so they are best interpreted (in the terminology of Feehan and Leness) as virtual Morse-Bott indices.

1.2. Framed Monopole Floer Homology and \( SO(3) \) monopoles on \( S^1 \times \Sigma \).

A second motivation for studying the moduli space of \( SO(3) \) vortices over Riemann surfaces which could be of more interest to low-dimensional topologists is the conjecture due to Kronheimer and Mrowka relating the framed instanton homology \( \text{HI}^\#(Y) \) of a 3-manifold and the tilde version \( \widetilde{HM}(Y) \) of monopole Floer homology (equivalently, the hat version of Heegaard Floer homology \( \widehat{HF}(Y) \)). More precisely, a special case of \([56] \) Conjecture 7.24] states the following:

**Conjecture 3.** \([56] \) Conjecture 7.24] Let \( Y \) denote a closed oriented 3-manifold. Then

\[
\text{HI}^\#(Y) \simeq \widehat{HF}(Y) \otimes \mathbb{C} \simeq \widetilde{HM}(Y) \otimes \mathbb{C}
\]

A definition of \( \widetilde{HM}(Y) = \bigoplus_{s \in \text{Spin}_c(Y)} \widetilde{HM}(Y, s) \) can be found in \([15] \) \([88] \) and by the work of Kutluhan, Lee and Taubes \([57] \), it is isomorphic to \( \widehat{HF}(Y) = \bigoplus_{s \in \text{Spin}_c(Y)} \widehat{HF}(Y, s) \). The Euler characteristic of these groups are

\[
\chi(\widetilde{HM}(Y)) = \begin{cases} \#H_1(Y, \mathbb{Z}) & b_1(Y) = 0 \\ 0 & b_1(Y) \neq 0 \end{cases}
\]

That \( \text{HI}^\#(Y) \) has the same Euler characteristic as \( \widetilde{HM}(Y) \) was shown by Scaduto in \([80] \) Corollary 1.4], and the isomorphism between the groups has been verified for certain cases in \([80] \) \([9] \) \([8] \).

The equality up to a sign between the Euler characteristics in the more general case of the sutured versions of monopole, Heegaard and instanton Floer homologies has been shown recently by Zhenkun Li and Fan Ye in their paper \([59] \) Theorem 1.2].
At this point it is useful to recall how $HI^\#(Y)$ is defined. One first considers the ordinary instanton Floer homology $HI(Y \# T^3, \gamma)$, where $\gamma$ represents the Poincaré dual for an admissible $U(2)$ bundle $E$ over $Y \# T^3$ (more precisely, $\gamma$ is a loop which represents one of the circle factors of $T^3$).

The group $HI(Y \# T^3, \gamma)$ is $\mathbb{Z}/8\mathbb{Z}$ graded, and $HI^\#(Y)$ is isomorphic to four consecutive summands of $HI(Y \# T^3, \gamma)$. In the special case of $Y = S^3$, it then follows that $HI^\#(S^3)$ corresponds to four summands of $HI(T^3, \gamma)$, which is two dimensional since it has two generators and no differential, so $HI^\#(S^3)$ ends up being one-dimensional [64 Section 4.1].

Notice that the Euler characteristic of these groups satisfies

\[
\chi(HI^\#(Y)) = \frac{1}{2} \chi(HI(Y \# T^3, \gamma)) = \frac{1}{2} \chi(CI(Y \# T^3, \gamma))
\]

where $CI(Y \# T^3, \gamma)$ denotes the chain complex used to define the Floer groups.

On the other hand, $\tilde{HM}(Y)$ can be defined in terms of a mapping cone construction which uses the monopole Floer chain complexes on $Y$ [13 67]. For us, the important fact is that $\tilde{HM}(Y)$ is defined in terms of data on $Y$, without using additional 3-manifolds like $T^3$ as in the case of $HI^\#(Y)$.

Thus, in order to try to compare $HI^\#(Y)$ and $\tilde{HM}(Y)$ it may be convenient to use an alternative construction of $\tilde{HM}(Y)$ so that it is also defined on $Y \# T^3$. As communicated to the author by Tom Mrowka, this alternative construction of $\tilde{HM}(Y)$ was known to him and Peter Kronheimer, and probably to the experts at large as well.

We motivate the construction in Section 7, but basically we consider $HM(Y \# T^3, \omega, \Gamma)$, that is, the monopole Floer homology on $Y \# T^3$ associated to a suitable non-balanced perturbation and a local coefficient system $\Gamma$, which is needed in order to obtain a well defined differential. We call this version of monopole Floer homology framed monopole Floer homology, and denote it as $HM^\#(Y)$ in analogy to the notation for the case of instantons.

It may not be clear why study $HM(Y \# T^3, \omega, \Gamma)$ in a paper that is about $SO(3)$ vortices over Riemann surfaces, but the main idea is that if we think of $T^3$ as $S^1 \times T^2$, then the perturbation term $\omega$ appears naturally from how the abelian vortices embed in the moduli space of $SO(3)$ vortices on $T^2$.

To be more specific, we first need to know how the solutions to the $SO(3)$ monopole equations on $S^1 \times \Sigma$ compare to the $SO(3)$ vortices on $\Sigma$. In Section 7 we prove (for the Seiberg-Witten case see [66, 68]):

**Theorem 4.** Suppose that $(B, \Psi)$ is an irreducible $SO(3)$ monopole for the spin-u structure

\[
V = (\mathbb{C} \oplus K_{\Sigma}^{-1}) \otimes E
\]

where $E$ is the pullback of a $U(2)$ bundle on $\Sigma$ under the obvious projection map $S^1 \times \Sigma \to \Sigma$. Write

\[
\Psi = \alpha \oplus \beta \in \Gamma(E) \oplus \Gamma(K_{\Sigma}^{-1} \otimes E)
\]

Then either $\beta$ vanishes identically or $\alpha$ vanishes identically.

a) The solutions with $\beta \equiv 0$ can be identified with the irreducible $SO(3)$ vortices for $(\Sigma, E)$.

b) The solutions with $\alpha \equiv 0$ can be identified (via Serre duality) with the irreducible $SO(3)$ vortices for $(\Sigma, K_{\Sigma}^{-1} \otimes E)$.

Moreover, if we assume that $c_1(E) > 2c_1(K_{\Sigma})$, then only solutions of type a) can occur.
Returning to our discussion of framed monopole homology, choose the $U(2)$ bundle $E$ over $T^2$ with $c_1(E) = 1$. The expected dimension of the moduli space of $SO(3)$ vortices is two, so after dividing by the circle action we get a manifold which is topologically an interval, the maximum of this interval corresponding to the unique abelian vortex inside this moduli space, and the minimum of this interval corresponding to the unique projectively flat connection on $T^2$.

This forces $HM(T^3, \omega)$ to be one dimensional (since Seiberg-Witten monopoles on $S^1 \times \Sigma$ are in bijection with abelian vortices on $\Sigma$). Therefore, we can think of the moduli space of $SO(3)$ vortices on $T^2$ as providing a canonical cobordism between the unique abelian vortex (which generates $HM^\#(S^3) = HM(T^3, \omega)$), and the unique projectively flat connection on $T^2$ (which can be taken as a generator for $HI^\#(S^3)$).

Thus, from this point of view, the isomorphism between $HM^\#(S^3)$ and $HI^\#(S^3)$ can be understood as a consequence of the fact that there is a canonical cobordism between the unique generator for $HM^\#(S^3)$ and the unique generator for $HI^\#(S^3)$. We should also point out that here we haven’t perturbed the $SO(3)$ monopole equations on $S^1 \times \Sigma$, since otherwise the moduli space would be zero dimensional, instead of an interval as we have in our situation.

We should also note that the chain complex $CM^\#(Y) = CM(Y \# T^3, \omega)$ which yields $HM(Y \# T^3, \omega, \Gamma)$ is finitely generated, thus the Euler characteristic of $HM^\#(Y)$ is well defined with respect to the Novikov field implicit in our choice of $\Gamma$. In particular,

$$\chi(HM^\#(Y)) = \chi(CM^\#(Y))$$

This immediately begs the questions:

1) How is $\chi(HM^\#(Y))$ related to $\chi(HM(Y))$?

2) How is $\chi(HM^\#(Y))$ related to $\chi(HI^\#(Y))$?

Regarding the first question, a Künneth formula proven by Kutluhan, Lee and Taubes in \[57\] relates $CM(Y \# T^3, \omega, \Gamma)$ to a mapping cone $S_\mathcal{U}((CM(Y \cup T^3), \omega, \Gamma))$. Since the right hand side still involves a local coefficient system, this is still not quite isomorphic to $HM(Y)$ (which is defined over integer coefficients for example), but rather a twisted version $HM(Y, \Gamma)$ (see the discussion preceding our theorem \[43\] for more details). But the Künneth formula implies that

$$\chi(HM^\#(Y)) = \chi(HM(Y))$$

Regarding the second question, thanks to work in progress by Aleksander Doan and Chris Gerig \[25\], which is based on ideas of Kronheimer and Mrowka, after appropriate choices of orientations

$$\chi(CM(Y \# T^3, \omega)) = \frac{1}{2} \chi(CI(Y \# T^3, \gamma))$$

so combining \[1\], \[2\] and \[3\] we obtain

$$\chi(HM(Y)) = \chi(HM^\#(Y)) = \chi(HI^\#(Y))$$

The interesting feature of the idea of Kronheimer and Mrowka used by Doan and Gerig is that the equality between both Euler characteristics is obtained by studying the $SO(3)$ monopole equations on the 3-manifold, so in a sense is a low dimensional version of Witten’s conjecture. One could then try to see if this can also be used for relating the Floer homologies $HM^\#(Y)$ and $HI^\#(Y)$, but that is outside the scope of this work.

\[1\] Recall that irreducible projectively flat connections on Riemann surfaces correspond to stable bundles, and in the case of an elliptic curve these were classified by Atiyah \[5\]. For the point of view of algebraic geometry, we are looking at the moduli space of stable bundles with a fixed determinant, which is why we get a single element.
1.3. $SO(3)$ monopoles on Seifert manifolds.

Finally, the fact that we studied the $SO(3)$ vortices over orbifold Riemann surfaces allows us to describe the solutions to the $SO(3)$ monopole equations on Seifert manifolds in terms of the $SO(3)$ vortex moduli spaces, which is the analogue of the computations done by Mrowka, Ozsváth and Yu \cite{MOY} for the case of the Seiberg-Witten equations.

Just as in their paper, one must work with connections which are spinorial with respect to a metric connection compatible with the Seifert structure (which is not the Levi-Civita connection). This connection can be thought of as an adiabatic connection \cite{Y}. In Section \(8\) we find:

**Theorem 5.** Suppose that $\pi : Y = S(L) \to \mathcal{S}$ is a Seifert manifold arising as the unit circle bundle of an orbifold line bundle $L \to \mathcal{S}$. Let $(B, \Psi)$ denote an irreducible $SO(3)$ monopole for the spin-$u$ structure

$$V = (\mathbb{C} \oplus \pi^*(K_{\mathcal{S}}^{-1})) \otimes \pi^*(\mathcal{E})$$

where $\mathcal{E}$ is an $U(2)$ bundle over $\mathcal{S}$. Write

$$\Psi = \alpha \oplus \beta \in \Gamma(\pi^*(\mathcal{E})) \oplus \Gamma(\pi^*(K_{\mathcal{S}}^{-1})) \otimes \pi^*(\mathcal{E})$$

Then either $\beta$ vanishes identically or $\alpha$ vanishes identically.

a) The solutions with $\beta \equiv 0$ can be identified with the irreducible $SO(3)$ vortices for $(\mathcal{S}, \mathcal{E}')$. Here $\mathcal{E}'$ is any $U(2)$ bundle over $\mathcal{S}$ satisfying $\det \mathcal{E}' \simeq \det \mathcal{E}$.

b) The solutions with $\alpha \equiv 0$ can be identified (via Serre duality) with the irreducible $SO(3)$ vortices for $(\mathcal{S}, K_{\mathcal{S}}^{-1} \otimes \mathcal{E}')$, where $\mathcal{E}'$ has the same meaning as in part a).

Moreover, if we assume that $c_1(\mathcal{E}) > 2c_1(K_{\mathcal{S}})$, then only solutions of type a) can occur.

**Remark 6.** a) We should point out that if one were interested in relating the flat $SU(2)$ or $SO(3)$ connections on $Y = S(L)$ with solutions to the Seiberg-Witten equations on $Y$, then the right $U(2)$ bundles to look at are those which satisfy $\det \mathcal{E} \simeq L^k$ for some integer $k$, since these are the ones Furuta and Steer used to relate the projectively flat connections on $Y$ with solutions to the Yang-Mills equations on $\mathcal{S}$ \cite[Theorem 3.7]{FS}.

b) At the end of Section \(8\) we work out the example of the Poincaré homology sphere. In this case one can choose $\det \mathcal{E} \simeq L_0$ where $L_0$ satisfies $c_1(L_0) = \frac{1}{2} \mathbb{Z}$ and generates the topological Picard group for orbifold lines bundles over $S^2(2, 3, 5)$.

There are six $U(2)$ bundles $\mathcal{E}'$ such that $\det \mathcal{E}' \simeq L_0$, but only two end up being interesting: call these $\mathcal{E}_1$ and $\mathcal{E}_2$. Each of these contain one projectively flat connection.

The expected dimension of the moduli space of $SO(3)$ vortices for $\mathcal{E}_1$ and $\mathcal{E}_2$ is 2 and 0 respectively, so after taking the quotient by the $S^1$ action, it is clear that the first one must be empty (in the sense that there are no irreducible $SO(3)$ vortices), while the first one becomes one dimensional, thus providing a cobordism between the unique abelian vortex and the unique projectively flat connection on the bundle $\mathcal{E}_1$.

So on the Poincaré homology sphere, the moduli space of $SO(3)$ vortices consists of:

i) The trivial $SU(2)$ connection,

ii) Two irreducible $SU(2)$ flat connections.
iii) One Seiberg-Witten monopole,

iv) A one-dimensional moduli space of irreducible $SO(3)$ monopoles which serves as a cobordism between the Seiberg-Witten monopole and one of the irreducible flat connections.

It is interesting to compare the case of the Poincaré homology sphere with that of $T^3$, since in both situations we have a one dimensional moduli of $SO(3)$ vortices connecting one of the flat connections which the unique Seiberg-Witten monopole.

In fact, thanks to the work of Taubes [84], it is known that the Casson invariant $\lambda_C(Y)$ [4] is morally $(1/2)$ the count of irreducible flat $SU(2)$ connections on $Y$.

On the other hand, as Lim showed [60] (verifying a conjecture due to Kronheimer), on an integer homology sphere $Y$, the count of irreducible Seiberg-Witten monopoles, while not a topological invariant, can be made to agree with $\lambda_C(Y)$ after adding a suitable correction term.

So morally there are twice as many flat connections as there are Seiberg-Witten monopoles, and we find it is amusing to give examples where this “principle” holds on the nose.

This also suggests studying the monopole and instanton Floer homologies of the connected sum $Y \# \Sigma(2, 3, 5)$, in analogy to studying the monopole and instanton Floer homologies of $Y \# T^3$.

We also describe the case of the Brieskorn homology sphere $\Sigma(2, 3, 7)$ near the end of the paper. The computations are slightly more complicated, but one finds a similar picture to that of $\Sigma(2, 3, 5)$.

We finish this introduction by mentioning that in follow-up work we plan to study the flowlines for the $SO(3)$ Chern-Simons-Dirac functional on Seifert manifolds and $S^1 \times \Sigma$ in terms of the geometry of the moduli space of stable pairs on natural (orbifold) ruled surfaces associated to these 3-manifolds [47, 67, 78].

Outline of the paper:

In Section 2 we give a basic review of gauge theory on orbifolds. In many ways this material is already common knowledge, so this section is basically intended to fix some notation. If one is willing to invest some time in the algebraic geometry literature, it is not difficult to find very general formulas for the Hirzebruch-Grothendieck-Riemann-Roch index theorem on orbifolds (or even stacks!). However, usually these formulas are not fleshed out explicitly for the situations we have in mind, so we also work out some of them in this section.

Section 3 introduces the $SO(3)$ vortex equations over an orbifold Riemann surface. The discussion is pretty standard, and include things like the deformation theory of the moduli spaces, the conditions under which abelian vortices appear in the moduli space, and the Kahler structure on the moduli spaces. The fact that we are working over an orbifold only means we need to keep track of more topological data, but besides that it is almost indistinguishable from the smooth case. We are also quite explicit when describing the different structures on the moduli spaces, since many simplifications occur from the fact that we are in dimension two.

Section 4 discusses the well-known correspondence between $SO(3)$ vortices and stable pairs. Our use of the stable pairs point of view is very limited in this work, and we mostly wrote it to clarify certain things in case the reader is more acquainted with the $U(2)$ vortex equations.

In Section 5 we study the Morse-Bott function $\mu$ on the moduli space of $SO(3)$ vortices. This is essentially the version for Riemann surfaces of the work of Feehan and Leness [54], or the classical paper by Hitchin but now for the case of $SO(3)$ vortices [48].

Section 6 describes some basic facts about the $SO(3)$ monopole equations on 3-manifolds. This material is an adaptation of [37], but we found it useful to work it out explicitly because the reader may not be familiarized with certain features of the $SO(3)$ monopole equations. We also describe
broadly the $U(2)$ monopole equations on 3-manifolds, which could be of interest for technical reasons which we discuss in this section.

In Section 7 we describe the $SO(3)$ monopole equations on 3-manifolds of the form $S^1 \times \Sigma$, where $\Sigma$ is a Riemann surface. Our main interest is the case when $\Sigma$ is the 2-torus $T^2$, since this example is what motivated our definition of the framed monopole Floer homology of $Y$. We give its definition in this section, and discuss some of the properties mentioned in the introduction.

Finally, in Section 8 we describe the behavior of the $SO(3)$ monopole equations for Seifert manifolds and discuss the examples of $\Sigma(2,3,5)$ and $\Sigma(2,3,7)$.

Acknowledgement. The idea to study the $SO(3)$ vortex equations on Riemann surfaces arose from many helpful conversations with Paul Feehan and Tom Leonard surrounding their project [34] so the author would like to thank them for their encouragement and many valuable suggestions. The author would also like to thank Yi-Jen Lee for pointing out the Künneth formula in her paper [57] with Kutluhan and Taubes, as well as Chris Woodward, Andrei Teleman, Oscar García-Prada, Michael Thaddeus, Dinesh Valluri, Aleksander Doan, Tom Mrowka and Matthew Stoffregen for useful conversations and correspondence.

2. A Crash Course on Orbifolds

We now give a brief summary of gauge theory on orbifolds, which in the modern literature (e.g. [11] [3]) falls into the theory of analytic Deligne-Mumford stacks.

For our purposes, the main arena will be that of a Riemann surface $\Sigma$ with finitely many marked points $p_1, \cdots, p_n$. Since this material is standard, we refer to [72] [67] [71] [70] [23] [82] [83] [51] [53] [74] [75] [73] [14] [13] [2] [1] [58] [50] for more details and proofs of the main results we need.

We will follow the conventions of Kronheimer and Mrowka [54] and use a $\pi$ whenever we want to emphasize the orbifold structure of a geometric object. For example, $\hat{\Sigma}$ will denote an orbifold Riemann surface, while $\Sigma$ will denote the underlying topological space (i.e., we forget the marked points). So we can think of $\hat{\Sigma}$ as a shorthand notation for the data $(\Sigma, p_1, \cdots, p_n)$, with the isotropy at each of the marked points (which will be recalled soon), being implicit.

In general, a complex orbifold $\hat{\Sigma}$ consists of a connected paracompact complex space such that every point $p$ has an open neighborhood $U_p$ which is of the form $U_p \simeq \mathbb{V}_p/G_p$, where $\mathbb{V}_p$ is a suitable complex manifold and $G_p$ a finite group acting biholomorphically on $U_p$.

The germ $(\hat{X}, p)$ of an orbifold at $p$ is called the quotient germ [14] Definition 1.1. If $\dim \mathbb{C} \mathbb{X} = m$, we can assume that $G_p$ is a finite subgroup of $GL_m(\mathbb{C})$, unique up to conjugation, and $\mathbb{V}_p$ is an open neighborhood of $0 \in \mathbb{C}^m$ such that $g(V_p) = V_p$ for all $g \in G_p$. We can think of the quotient germ $(\hat{X}, p)$ as $(\hat{X}, p) = (\mathbb{C}^m, 0)/G_p$ and the map $\pi : (\mathbb{C}^m, 0) \to (\hat{X}, p)$ is called the local smoothing covering of $\hat{X}$ at $p$.

We will assume that there are finitely many points $p_1, \cdots, p_n$ of $\hat{X}$ where $G_p$ is non-trivial. In fact, we can assume that $G_p \simeq \mathbb{Z}/a_i\mathbb{Z}$, where the action is $z \in \mathbb{C}^m \to e^{2\pi i/a_i}z$. In this case the natural orbifold metric $\hat{g}$ on $\hat{X}$ is one with a conical singularity at each $p_i$ of cone angle $2\pi/a_i$. Integration on orbifolds can also be defined using a partition of unity, and the orbifold version of the de Rham complex yields cohomology groups isomorphic to those of the underlying topological space [55] Section 2.1. In other words,

$$H^k(\hat{X}; \mathbb{R}) \simeq H^k(X; \mathbb{R}) \simeq H^k_{DR}(X; \mathbb{R}), \quad k = 0, 1, \cdots, \dim_{\mathbb{R}} X$$
with a similar statement for \( C \) instead of \( R \). Moreover, the cup product of orbifold de Rham classes is induced by the wedge product. By the same token, any element of \( H^k(\tilde{X};R) \) can be represented by a unique orbifold harmonic \( k \) form.

**Orbi-bundles** can be defined in an analogous way to how we defined orbifolds. Namely, we have the data \( \pi : \tilde{E} \to \tilde{X} \), where both \( \tilde{E}, \tilde{X} \) are orbifolds and \( \pi \) is a holomorphic surjective map. If \( F \) is a complex manifold, \( \pi \) has fibre \( F \) if for every \( p \in \tilde{X} \) there is a local smoothing covering \( U_p = V_p / G_p \) with \( \tilde{E} \mid U_p \cong (V_p \times F) / G_p \) over \( U_p \).

In general the fibres of \( \pi \) are orbifolds, namely, \( \pi^{-1}(p) = F / G_p \) for some action of \( G_p \) on \( F \). In particular, notice that the projection \( \pi \) is not required to be locally trivial, as opposed to the usual vector bundle situation. Regardless, it still makes sense to talk about connections on vector orbi-bundles [58, Section 2.2].

For the particular case of an orbifold line bundle \( \tilde{L} \to \tilde{X} \), its tensor powers \( \tilde{L} \otimes q \) will continue to be orbifold line bundles. In fact, one can take \( q \) sufficiently large so that \( \tilde{L} \otimes q \) becomes locally trivial, in which case one can define the orbifold **Chern class** of \( \tilde{L} \) as

\[
c_1(\tilde{L}) \equiv \frac{1}{q} c_1(\tilde{L} \otimes q) \in H^2(\tilde{X};Q)
\]

Notice that in general this will be a class in the second cohomology of the orbifold with rational coefficients. Alternatively, one can also use Chern-Weil theory in order to define \( c_1(\tilde{L}) \) as

\[
c_1(\tilde{L}) = \left[ \frac{i}{2\pi} F_{\tilde{\nabla}} \right]
\]

where \([\cdot]\) denotes the orbifold de Rham class defined by the curvature of an orbifold connection \( \tilde{\nabla} \) on \( \tilde{L} \).

For higher rank orbi-bundles \( \tilde{E} \), the Chern classes \( c_i(\tilde{E}) \) can be defined in a similar way [14, Section 2]. Moreover, there is a natural notion of an **orbifold Euler characteristic** of \( \tilde{X} \), given by the formula

\[
e(\tilde{X}) = e(X) - \sum_{i=1}^n \left( 1 - \frac{1}{a_i} \right)
\]

In particular, this definition guarantees that the relation

\[
e(\tilde{X}) = (-1)^m c_m(\Omega^1_{\tilde{X}})
\]

continues to hold, where \( c_m(\Omega^1_{\tilde{X}}) \) is the orbifold Chern number of the orbifold holomorphic cotangent bundle.

When \( \mathcal{F} \) is a locally free orbifold sheaf of rank \( r \) on \( \tilde{X} \), the version of **Hirzebruch-Grothendieck-Riemann-Roch** reads in this context [14, Section 3]:

\[
\chi(\tilde{X}, \tilde{E}) = \int_{\tilde{X}} \text{ch}(\tilde{E}) \text{td}(\tilde{X}) + \sum_{i=1}^n \frac{1}{\#G_{p_i}} \left( \sum_{g \in G_{p_i}, \{1_m\}} \frac{\text{tr}(\rho(g))}{\det(1_m - g)} \right)
\]

where on the left hand side we mean the alternating sum of the dimensions of the cohomology groups \( H^i(\tilde{X}, \tilde{E}) \).

Here we are regarding \( g \in G_p \cong \mathbb{Z}/a_i\mathbb{Z} \) as an \( m \times m \) matrix, and \( \rho(g) \) denotes the action of \( g \) on the orbifold bundle \( \tilde{E} \).
In order to compare this formula with the one which appears in the paper by Furuta and Steer [12, Theorem 1.5], we take \( \tilde{X} = \Sigma \) and assume \( \tilde{E} = \tilde{L} \) is a line bundle. In this case,

\[
\begin{align*}
\text{ch}(\tilde{L}) &= \text{rk}(\tilde{L}) + c_1(\tilde{L}) = 1 + c_1(\tilde{L}) \\
\text{td}(\tilde{L}) &= 1 + \frac{c_1(\tilde{L})}{2}
\end{align*}
\]

Therefore we have

\[
\begin{align*}
\int_{\Sigma} \text{ch}(\tilde{L}) \text{td}(\Sigma) &= \int_{\Sigma} (1 + c_1(\tilde{L})) \left( 1 + \frac{c_1(\Sigma)}{2} \right) \\
&= \int_{\Sigma} \left( \frac{1}{2} \right) c_1(\tilde{L}) + c_1(\tilde{L}) \\
&= \frac{1}{2} \left( 2 - 2g - \sum_{i=1}^{n} \left( 1 - \frac{1}{a_i} \right) \right) + \int_{\Sigma} c_1(\tilde{L}) \\
&= (1 - g) - \sum_{i=1}^{n} \left( \frac{a_i - 1}{2a_i} \right) + \int_{\Sigma} c_1(\tilde{L})
\end{align*}
\]

Now we want to analyze the term \( \sum_{g \in G_{\mathcal{P}_i} \setminus \{1\}} \frac{\text{tr}(\rho(g))}{\det(1-g)} \). To understand how to compute \( \rho(g) \), it is useful to work with orbifold connections in a slightly more concrete way [51, section 1.(ii)].

For the case of an orbifold line bundle \( \tilde{L} \), near each marked point \( p \) we can choose polar coordinates \((r, \theta)\) on a disk neighborhood \( D_p \) of \( p \) so that we have the connection form

\[
A_\lambda = i\lambda d\theta
\]

where \( \lambda \) is a constant known the holonomy parameter. The reason for this name is that holonomy of a connection \( A \) on \( D_p \setminus \{p\} \) whose matrix connection coincides with \( A_\lambda \) near \( p \) on the positively-oriented small circles of constant \( r \) will have a limiting holonomy approximately equal to

\[
e^{-2\pi i \lambda}
\]

Strictly speaking, we want to study the previous situation up to conjugacy [81], but since \( U(1) \) is abelian this is not important, so it suffices to take \( 0 \leq \lambda < 1 \) in order to guarantee that \( e^{-2\pi i \lambda} \) exhausts all possible elements in \( U(1) \). We are interested in connections that differ from \( A_\lambda \) by a smooth one form on \( \Sigma \), and for the orbifold interpretation to hold, we need to take \( \lambda \in \mathbb{Q} \cap [0, 1) \). In fact, we will write \( \lambda = \frac{b}{a} \) for \( 0 < b < a \). The parameter \( b \) can then be identified with what Furuta and Steer call the isotropy data of the line bundle [12]. It is now clear that we can write

\[
\sum_{g \in G_{\mathcal{P}_i} \setminus \{1\}} \frac{\text{tr}(\rho(g))}{\det(1-g)} = \sum_{k=1}^{a_i-1} e^{-2\pi i kb/a_i} \left( \frac{a_i - 1}{1 - e^{2\pi i k/a_i}} \right)
\]

Now we need to find a more concrete version of the previous sum. A similar expression appears in [24, p.7], and the author would like to thank Dinesh Valluri for suggesting how to prove the following lemma.
Lemma 7. Let \( \zeta \) denote an \( a \)-th root of unity, which for simplicity we take as \( \zeta = e^{2\pi i/a} \). Then for any \( 0 < b < a \),
\[
\sum_{k=1}^{a-1} \frac{\zeta^{kb}}{1-\zeta^k} = \sum_{k=1}^{a-1} \frac{e^{2\pi ikb/a}}{1-e^{2\pi ik/a}} = b - \frac{a+1}{2}
\]

Proof. Step 1:
Since \( 1 = \zeta^a = (\zeta^j)^a \) we have that
\[
0 = (\zeta^j - 1)(1 + \zeta^j + \cdots + (\zeta^j)^{a-1})
\]
So for \( 0 < j < a \), we have
\[
\sum_{k=1}^{a-1} \zeta^{jk} = -1
\]
Step 2:
\[
\sum_{k=1}^{a-1} \frac{1}{\zeta^k - 1} = - \left( \frac{a-1}{2} \right)
\]
Observe that the left hand side of (6) equals
\[
\sum_{k=1}^{a-1} \frac{1}{\zeta^k - 1}
= \sum_{k=1}^{a-1} \frac{1 - \zeta^k + \zeta^k}{\zeta^k - 1}
= \sum_{k=1}^{a-1} \left[ -1 + \left( \frac{\zeta^k}{\zeta^k - 1} \right) \right]
= - (a-1) + \sum_{k=1}^{a-1} \frac{\zeta^k}{\zeta^k - 1}
= - (a-1) + \sum_{k=1}^{a-1} \frac{1}{1 - \zeta^{-k}}
\]
Now, notice that
\[
\sum_{k=1}^{a-1} \frac{1}{1 - \zeta^{-k}}
= \sum_{k=1}^{a-1} \frac{1}{1 - \zeta^{a-k}}
= \frac{1}{1 - \zeta^{a-1}} + \frac{1}{1 - \zeta^{a-2}} + \cdots + \frac{1}{1 - \zeta}
= \sum_{k=1}^{a-1} \frac{1}{1 - \zeta^k}
\]
Therefore, the proof of (6) becomes

\[ \sum_{k=1}^{a-1} \frac{1}{\zeta^k - 1} = -(a - 1) + \sum_{k=1}^{a-1} \frac{1}{1 - \zeta^k} \]

\[ \implies \sum_{k=1}^{a-1} \frac{1}{\zeta^k - 1} = -\frac{(a - 1)}{2} \]

**Step 3:**

(7)\[ \frac{\zeta^{kb} - 1}{\zeta^k - 1} = 1 + \zeta^k + \ldots + (\zeta^k)^{b-1} \]

This is obtained by factoring the numerator \( \zeta^{kb} - 1 = (\zeta^k)^b - 1 \).

**Proof of Lemma:**

\[ \sum_{k=1}^{a-1} \frac{\zeta^{kb} - 1}{\zeta^k - 1} = \sum_{k=1}^{a-1} \left( \sum_{j=0}^{b-1} \zeta^{kj} \right) \]

\[ = \sum_{j=0}^{b-1} \sum_{k=1}^{a-1} \zeta^{kj} \]

\[ = \text{Step (1)} (a - 1) + \sum_{j=1}^{b-1} \sum_{k=1}^{a-1} \zeta^{kj} \]

\[ = a - 1 - (b - 1) \]

\[ = a - b \]

Therefore,

(8)\[ \sum_{k=1}^{a-1} \frac{\zeta^{kb}}{\zeta^k - 1} - \sum_{k=1}^{a-1} \frac{1}{\zeta^k - 1} = a - b \]

Using step (2) we obtain

\[ \sum_{k=1}^{a-1} \frac{\zeta^{kb}}{\zeta^k - 1} = -\frac{(a - 1)}{2} + a - b = \frac{a + 1}{2} - b \]

Therefore

\[ \sum_{k=1}^{a-1} \frac{\zeta^{kb}}{1 - \zeta^k} = b - \frac{a + 1}{2} \]

\square
Therefore

\[
\sum_{i=1}^{n} \frac{1}{\# G_{p_i}} \left( \sum_{g \in G_{p_i} \setminus \{1\}} \frac{\text{tr}(\rho(g))}{\det(1-g)} \right) = \sum_{i=1}^{n} \frac{1}{a_i} \left( \sum_{k=1}^{a_i-1} \frac{e^{-2i\pi k b_i / a_i}}{1 - e^{2i\pi k / a_i}} \right) = \sum_{i=1}^{n} \frac{1}{a_i} \left( \sum_{k=1}^{a_i-1} \frac{e^{2i\pi k(a_i-b_i) / a_i}}{1 - e^{2i\pi k / a_i}} \right) = \sum_{i=1}^{n} \frac{1}{a_i} \left( a_i - b_i - \frac{a + 1}{2} \right) = -\sum_{i=1}^{n} \frac{b_i}{a_i} + \sum_{i=1}^{n} \frac{a_i - 1}{2a_i}
\]

So the orbifold version of Riemann Roch for line bundles is

\[
\dim_{\mathbb{C}} H^0(\hat{\Sigma}, \hat{L}) - \dim_{\mathbb{C}} H^1(\hat{\Sigma}, \hat{L}) = (1 - g) - \sum_{i=1}^{n} \left( \frac{a_i - 1}{2a_i} \right) + \int_{\hat{\Sigma}} c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i} + \sum_{i=1}^{n} \frac{a_i - 1}{2a_i} = (1 - g) + \int_{\hat{\Sigma}} c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i} = (1 - g) + c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i}
\]

which agrees with [42, Theorem 1.5], under the usual abuse of notation that identifies \(c_1(\hat{L})\) with a number.

Remark 8. The quantity \(c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i}\) is denoted the **background degree** in [67, Definition 2.7]. As explained in [42, p.44], if one thinks of \(H^*(\hat{\Sigma}, \hat{L})\) as \(H^*(\hat{\Sigma}, \mathcal{O}(\hat{L}))\), where \(\mathcal{O}(\hat{L})\) is the sheaf of holomorphic sections, then \(\mathcal{O}(\hat{L})\) is a locally free sheaf with degree \(c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i}\). We will write

(9)

\[
\deg_B(\hat{L}) = c_1(\hat{L}) - \sum_{i=1}^{n} \frac{b_i}{a_i}
\]

for this formula of the background degree.

One can also relate orbifold line bundles with divisors [70, Section 1B]. Namely, a divisor \(\hat{D}\) on \(\hat{\Sigma}\) is a finite linear combination

\[
\hat{D} = \sum_{p \in \hat{\Sigma}} \frac{n_p}{a_p} \cdot p
\]

where \(n_p \in \mathbb{Z}\), and \(a_p = 1\) if \(p\) is not one of the marked points on \(\hat{\Sigma}\). The degree of a divisor is then \(\deg \hat{D} = \sum \frac{n_p}{a_p}\).
Proposition 1.3 in \cite{70} then describes a bijection between equivalence classes of divisors and holomorphic prbifold line bundles. If \( \hat{L}_B \) is the line bundle corresponding to \( \hat{D} \), then \( c_1(\hat{L}_B) = \deg \hat{D} \).

For our purposes it is also necessary to have a formula of Riemann-Roch for \( U(2) \) orbifold bundles \( \hat{E} \). One option would be to apply formula \( \text{(4)} \) one more time, but we find it more elegant to bootstrap from Riemann-Roch for orbifold line bundles \( \hat{L} \). This requires a small detour into the classification of \( U(2) \) orbifold bundles, which can be found in \cite{12} Proposition 1.8, as well as \cite{70} Section 1A.

The isotropy data for a rank two \( U(2) \) orbifold bundle \( \hat{E} \) will be specified by the pairs of integers

\[
((b_1^{-}, b_1^{+}), \ldots, (b_n^{-}, b_n^{+}))
\]

where \( 0 \leq b_i^{-}, b_i^{+} < a_i \), for \( i = 1, \ldots, n \). In particular, we can find orbifold line bundles \( \hat{L}^{-}, \hat{L}^{+} \) with isotropy data \( (b_i^{-}) \) and \( (b_i^{+}) \) respectively. Then

\[
\hat{E} = \hat{L}^{-} \oplus \hat{L}^{+}
\]

is an \( U(2) \) orbifold bundle with isotropy data \( ((b_1^{-}, b_1^{+}), \ldots, (b_n^{-}, b_n^{+})) \). Notice that the notation seems to suggest that \( b_i^{-} \leq b_i^{+} \), in fact, this is assumed in \cite{70} p.599]. This convention is related to which group of automorphisms we are allowed to use.

Up to conjugacy, any matrix in \( U(2) \) takes the form

\[
\begin{pmatrix}
e^{-2\pi i \lambda_1} & 0 \\
0 & e^{-2\pi i \lambda_2}
\end{pmatrix}
\]

where we can assume \( 0 \leq \lambda_1 \leq \lambda_2 < 1 \). However, this classification requires conjugating by arbitrary matrices in \( U(2) \). In our case we are interested in using the determinant one gauge group, which means that the bundle \( \hat{L}^{-} \oplus \hat{L}^{+} \) needs to be distinguished from the bundle \( \hat{L}^{+} \oplus \hat{L}^{-} \), since an automorphism which permutes both factors would locally take the form \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and this matrix, while an element of \( U(2) \), is not an element of \( SU(2) \), because it has determinant \(-1\).

Therefore, with respect to the determinant one gauge group, we need to consider the isotropy data as ordered pairs, without any assumption regarding whether \( b_i^{-} \leq b_i^{+} \) or not. This point is briefly made on \cite{52} p. 602]. However, ultimately we are interested in studying the moduli spaces after taking the quotient by the residual \( S^1 \) action mentioned in the introduction. Since \( S^1 \times \{ \pm t_2 \} \) \( SU(2) \) can be identified with \( U(2) \) via the map \( (e^{i\theta}, A) \rightarrow e^{i\theta}A \), this point doesn’t end up making a big difference, so it is fine to assume \( b_i^{-} \leq b_i^{+} \) while reading this paper.

In any case, the important point is that the orbifold version of Hirzebruch-Grothendieck-Riemann-Roch states that the Euler characteristic \( \chi(\Sigma, \hat{E}) \) is topological, so it suffices to find the formula in the case of \( \chi(\Sigma, \hat{L}^{-} \oplus \hat{L}^{+}) \). By this we mean the following: the spaces \( H^0(\Sigma, \hat{E}) \) and \( H^1(\Sigma, \hat{E}) \) can be identified with the dimensions of the kernel and cokernel of an orbifold Dolbeault operator \( \partial_C : \Omega^0(\Sigma, \hat{E}) \rightarrow \Omega^{0,1}(\Sigma, \hat{E}) \), where \( C \) is an arbitrary \( U(2) \) orbifold connection on \( \hat{E} \).

In particular, we can assume that \( C \) is a \( U(2) \) connection compatible with the topological splitting \( \hat{E} = \hat{L}^{-} \oplus \hat{L}^{+} \), which we write as \( C = C^{-} \oplus C^{+} \). In this situation it is clear that \( \chi(\Sigma, \hat{E}) = \)}
\( \chi(\Sigma, L^-) + \chi(\Sigma, L^+) \) and thus we find that
\[
\dim \mathcal{C}H^0(\Sigma, \hat{L}) - \dim \mathcal{C}H^1(\Sigma, \hat{L}) = \chi(\Sigma, L^-) + \chi(\Sigma, L^+)
\]
\[
= (1 - g) + c_1(L^-) - \sum_{i=1}^{n} \frac{b_i^-}{a_i} + (1 - g) + c_1(L^+) - \sum_{i=1}^{n} \frac{b_i^+}{a_i}
\]
\[
= 2(1 - g) + c_1(\det \hat{E}) - \sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}
\]
In the special case of a SU(2) orbifold bundle \( \hat{E} \), we can assume that 0 \leq b_i^- \leq \lfloor a_i/2 \rfloor, and also take \( b_i^+ = a_i - b_i^- \). In particular, the previous formula reduces to
\[
2(1 - g) - n
\]
Summarizing our findings, we have found

**Theorem 9. Riemann-Roch for orbifold Riemann surfaces:**

a) If \( \hat{L} \) is an orbifold line bundle with isotropy data \((b_i)_{i=1}^n\), then
\[
\dim \mathcal{C}H^0(\Sigma, \hat{L}) - \dim \mathcal{C}H^1(\Sigma, \hat{L}) = (1 - g) + c_1(L) - \sum_{i=1}^{n} \frac{b_i^-}{a_i}
\]

b) If \( \hat{E} \) is an orbifold SU(2) bundle with isotropy data \((b_i^-, b_i^+) = (a_i - b_i^-)\)_{i=1}^n, then
\[
\dim \mathcal{C}H^0(\Sigma, \hat{E}) - \dim \mathcal{C}H^1(\Sigma, \hat{E}) = 2(1 - g) - n
\]
c) If \( \hat{E} \) is an orbifold U(2) bundle with isotropy data \((b_i^-, b_i^+)\)_{i=1}^n, and determinant line bundle \( \det \hat{E} \), then
\[
\dim \mathcal{C}H^0(\Sigma, \hat{E}) - \dim \mathcal{C}H^1(\Sigma, \hat{E}) = 2(1 - g) + c_1(\det \hat{E}) - \sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}
\]
Riemann-Roch is specially powerful when combined with Serre duality. Recall that **Serre duality** states that
\[
H^i(\Sigma, \hat{E}) \simeq H^{1-i}(\Sigma, K_\Sigma \otimes \hat{E}^*)
\]
Here \( K_\Sigma \) is the canonical bundle, which as an orbifold line bundle has Seifert invariants
\[
(2g - 2, a_1 - 1, \cdots, a_n - 1)
\]
in the sense that \( \deg_{\Sigma} K_\Sigma = 2g - 2 \) and
\[
c_1(K_\Sigma) = 2g - 2 + \sum_{i=1}^{n} \frac{a_i - 1}{a_i} = 2g - 2 + n - \sum_{i=1}^{n} \frac{1}{a_i}
\]
Thus, suppose we want to check which conditions on \( \hat{L} \) guarantee that \( H^1(\Sigma, \hat{L}) \) vanishes. First of all, by Serre duality,
\[
\dim H^1(\Sigma, \hat{L}) = \dim H^0(\Sigma, K_\Sigma \otimes \hat{L}^*)
\]
If $\tilde{L}$ has isotropy $b_i$ at the point $p_i$, we define
\[
\epsilon_i = \begin{cases} 
1 & 0 < b_i \\
0 & b_i = 0 
\end{cases}
\]
then the isotropy of $\tilde{L}^*$ at the point $p_i$ is $\epsilon_i(a_i - b_i)$, and the isotropy of $K_{\tilde{\Sigma}} \otimes \tilde{L}^*$ is
\[
\begin{cases} 
a_i - 1 & \text{if } b_i = 0 \\
a_i - 1 - b_i & \text{if } a_i - 1 > b_i \\
0 & \text{if } b_i = a_i - 1
\end{cases}
\]
Notice that the first case ends up being contained in the second one.

Clearly if $\deg_B(K_{\tilde{\Sigma}} \otimes \tilde{L}^*) < 0$, then $H^1(\tilde{\Sigma}, \tilde{L})$ vanishes. Looking at the formula for the background degree, this requires
\[
c_1(K_{\tilde{\Sigma}} \otimes \tilde{L}^*) - \sum_{0 \leq b_i < a_i - 1} \frac{a_i - 1 - b_i}{a_i} < 0.
\]
This is the same as $2g - 2 + n - \sum_{i=1}^n \frac{a_i}{a_i} - \sum_{0 \leq b_i < a_i - 1} \frac{a_i - 1 - b_i}{a_i} < c_1(\tilde{L})$, or more succinctly
\[
2g - 2 + \sum_{b_i = a_i - 1} a_i - 1 - \frac{b_i}{a_i} < c_1(\tilde{L})
\]
But this ends up being equal to
\[
2g - 2 < \deg_B(\tilde{L})
\]
Therefore, we conclude that

\textbf{Lemma 10.} Suppose that $\tilde{L}$ is an orbifold line bundle such that $2g - 2 < \deg_B(\tilde{L})$. Then $H^1(\tilde{\Sigma}, \tilde{L})$ vanishes.

\textbf{Remark 11.} Observe that $c_1(K_{\tilde{\Sigma}} \otimes \tilde{L}^*) = c_1(K_{\tilde{\Sigma}}) - c_1(\tilde{L})$, so by \cite{70} Corollary 1.4 $H^1(\tilde{\Sigma}, \tilde{L})$ will also vanish whenever $c_1(K_{\tilde{\Sigma}}) < c_1(\tilde{L})$.

It is also important to mention the conditions an orbifold line bundle $\tilde{L}$ must satisfy in order to appear in a reduction of an orbifold $U(2)$ bundle $\tilde{E}$ \cite{70} \cite{42}. First of all, we must have the topological splitting
\[
\tilde{E} = \tilde{L} \oplus (\tilde{L}^* \otimes \det \tilde{E})
\]
The isotropy data of $\tilde{L}$ is determined by that of $\tilde{E}$ in the sense that $b_i \in \{b_i^-, b_i^+\}$ for all $i$. Define the vector $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ by the conditions
\[
\epsilon_i = \begin{cases} 
0 & b_i^- = b_i^+ \\
-1 & b_i = b_i^- \\
+1 & b_i = b_i^+
\end{cases}
\]
and the integers
\[
\begin{cases} 
n_0 = \# \{ i | \epsilon_i = 0 \} \\
n_\pm = \# \{ i | \epsilon_i = \pm 1 \}
\end{cases}
\]
Then $c_1(\tilde{L})$ satisfies $c_1(\tilde{L}) - \sum_{i=1}^n \frac{\epsilon_i(b_i^+ - b_i^-) + (b_i^+ + b_i^-)}{2a_i} \in \mathbb{Z}$.

The topological isomorphism classes of orbifold line bundles form a group under tensor product denoted $\text{Pic}_{U}^i(\tilde{\Sigma})$. According to \cite{42} Corollary 1.7, when $a_1, \cdots, a_n$ are mutually coprime, there is
a unique orbifold line bundle $\tilde{L}_0$ such that $c_1(\tilde{L}_0) = \frac{1}{\prod_{i=1}^n a_i}$, and any other orbifold line bundle $\tilde{L}$ has the form $\tilde{L}_0^k$, for some integer $k$.

Moreover, as explained in [42, p. 47], for $SO(3)$ orbifold bundles there is a commutative diagram

$$
\begin{array}{c}
\{U(2) - \text{orbifold bundles}\} \xrightarrow{\det} \text{Pic}_V^t(\tilde{\Sigma}) \\
\downarrow \text{Ad} \\
\{SO(3) - \text{orbifold bundles}\} \xrightarrow{w} \text{Pic}_{v_1}^t / (\text{Pic}_{v_1}^t)^2
\end{array}
$$

where the map $w$ is essentially the second Stiefel-Whitney class.

For the examples we will analyze at the end of this paper, it is important to know when an orbifold $U(2)$ bundle $\tilde{E}$ admits an irreducible projectively flat connection.

The proofs can be found in [42], but we found the summary from [70] especially useful. In particular, we need the following facts:

**Fact 12.** (See [42], [70, Lemmas 2.2 and 4.2])

a) Suppose $\tilde{E}$ is an orbifold $U(2)$ bundle with isotropy data $(b_1^\pm, b_2^\pm, \ldots, b_n^\pm)$ and let $n_0 = \# \{i \mid b_i^- = b_i^+ \}$. If $g = 0$ and $n - n_0 \leq 2$, then $\tilde{E}$ admits no irreducible projectively flat connections.

b) The space of irreducible projectively flat connections is connected and simply connected regardless of the genus of $\Sigma$.

c) The space of irreducible projectively flat connections is empty if and only if there exists a vector $(\epsilon_i)$, with $\epsilon_i = \pm 1$, such that $n_+ + \deg_{B}(\det \tilde{E}) \equiv 1 \mod 2$ and $n_+ - \sum_{i=1}^n \frac{\epsilon_i(b_i^+ - b_i^-)}{a_i} < 1 - g$.

Here $n_\pm = \# \{i \mid \epsilon_i = \pm 1 \}$.

### 3. The $SO(3)$ Vortex Moduli Space

Let $\tilde{\Sigma} = (\Sigma, p_1, \ldots, p_n)$ denote an orbifold Riemann surface and choose an orbifold $U(2)$ bundle $\tilde{E} \to \tilde{\Sigma}$. Our policy will be to not make any particular assumptions on $\tilde{E}$ at this point. For example, in the smooth case it is typical to assume $\deg E > 4g - 4$ [19, Assumption 2], but we will explain why this choice of degree is made in this section and the next.

Even if the bundle is an orbifold $SU(2)$ bundle, we will regard it as an orbifold $U(2)$ bundle, in the sense that we will make a choice of reference connection in $\det \tilde{E}$, though in the $SU(2)$ case we can just take it to be the trivial connection.

For notational ease, we will use $\cdot$ to remind the reader that our bundles and other geometric data should be interpreted in the orbifold sense, but for an orbisection of $\tilde{E}$, we will write $T$ instead of $\tilde{T}$. Similar remarks apply to our notation for orbifold connections.

It is a good moment to justify our unusual choice of notation at this point. Since we are eventually interested in using these moduli spaces for 3- and 4-manifolds, we want a notation that does not interfere with the one used in these contexts. In particular, we are trying to follow the notation of Kronheimer and Mrowka [55], so typically for us

- $X$ denotes a 4-manifold, $A$ a connection of some bundle defined on $X$, and $\Phi$ a section of some spinor-type bundle.
- $Y$ denotes a 3-manifold, $B$ a connection of some bundle defined on $Y$, and $\Psi$ a section of some spinor-type bundle.
- $\Sigma$ denotes a 2-manifold, $C$ a connection of some bundle defined on $\Sigma$, and $\Upsilon$ a section of some spinor-type bundle.

Further remarks regarding our notation will be explained as it becomes necessary.
**Definition 13.** Let $\mathcal{A}^{\text{det}}(\hat{E})$ denote the space of $U(2)$ connections on $\hat{E}$ which induce a fixed reference $U(1)$ connection $C^{\text{det}}$ on $\det \hat{E}$. The $SO(3)$ **vortex equations** are equations for a pair $(C, \Upsilon) \in \mathcal{A}^{\text{det}}(\hat{E}) \times \Gamma(\hat{E})$ which satisfy

$$
\left\{ \begin{array}{l}
\ast F_C^0 - i \left[ \Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E \right] = 0 \\
\bar{\partial}_C \Upsilon = 0
\end{array} \right.
$$

where $F_C^0$ denotes the traceless part of the curvature of $C$.

**Remark 14.** Observe that the first equation makes sense since $\Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E$ is a traceless hermitian endomorphism of $E$, so $i \left[ \Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E \right]$ is skew-hermitian, and thus can be regarded as an element of $\mathfrak{su}(2)$.

If we define $\mathcal{C}(\check{\Sigma}, \check{E}) = \mathcal{A}^{\text{det}}(\check{E}) \times \Gamma(\check{E})$ as the configuration space and $\mathcal{G}^{\text{det}}(\check{E})$ the gauge group (those automorphisms of $\check{E}$ with determinant one), we want to study the solutions to the $SO(3)$ vortex equations modulo gauge, i.e, the moduli space $\mathcal{M}(\check{\Sigma}, \check{E}) \subset \mathcal{C}(\check{\Sigma}, \check{E})/\mathcal{G}^{\text{det}}(\check{E})$. Recall that the gauge group $u \in \mathcal{G}^{\text{det}}(E)$ acts on $(C, \Upsilon)$ as

$$
u \cdot (C, \Upsilon) = (C - (d_C u) u^{-1}, u \Upsilon)
$$

The $SO(3)$ **vortex map** is the map

$$
\check{\mathfrak{g}}_{SO(3)} : \mathcal{A}^{\text{det}}(\check{E}) \times \Gamma(\check{E}) \to \Gamma(\check{\Sigma}; \mathfrak{su}(E)) \oplus \Gamma(K_E^{-1} \otimes \check{E})
$$

$$(C, \Upsilon) \to \left( \ast F_C^0 - i \left[ \Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E \right], \bar{\partial}_C \Upsilon \right)
$$

An easy computation shows that:

**Lemma 15.** The linearization of the $SO(3)$ vortex map is

$$
\mathcal{D} \check{\mathfrak{g}}_{SO(3), (C, \Upsilon)}(\check{\epsilon}, \check{\Upsilon}) = \left( \ast d_C \check{\epsilon} - i \left[ \check{\Upsilon} \check{\Upsilon}^* + \check{\Upsilon} \check{\Upsilon}^* - \Re \left( \check{\Upsilon}, \check{\Upsilon} \right) I_E \right], \bar{\partial}_C \check{\Upsilon} + \check{\epsilon}^* \otimes \check{\Upsilon} \right)
$$

where we have decomposed $\check{\epsilon} \in \Omega^1(\check{\Sigma}, \mathfrak{su}(E)) \subset \Omega^1(\check{\Sigma}, \mathfrak{su}(E) \otimes \mathbb{C})$ as $\check{\epsilon} = \check{\epsilon}' + \check{\epsilon}''$.

**Remark 16.** Recall that on a Riemann surface the Hodge star operator acting on one-forms satisfies $\ast^2 = -\text{Id}$, so it can be used to define an almost complex structure on $\Omega^1(\check{g}_E)$. In particular, whenever we write $c = c' + c''$, then $\ast c = -ic'$ + $ic''$.

**Proof.** It follows easily after observing that for $t \in \mathbb{R}$,

$$
\begin{bmatrix}
|\check{\Upsilon} + t \check{\Upsilon}|^2 - |\check{\Upsilon}|^2 + t^2 |\check{\Upsilon}|^2 + 2 \Re \left( \check{\Upsilon}, \check{\Upsilon} \right) \\
(\check{\Upsilon} + t \check{\Upsilon})(\check{\Upsilon} + t \check{\Upsilon})^* = t \check{\Upsilon} \check{\Upsilon} + t^2 \check{\Upsilon} \check{\Upsilon}^* + t \check{\Upsilon} \check{\Upsilon}^* + t \check{\Upsilon} \check{\Upsilon}^*
\end{bmatrix}
$$

\[ \square \]

**Lemma 17.** For $\xi \in \Omega^0(\check{\Sigma}; \mathfrak{su}(\check{E}))$, define

$$
\delta^0_{(C, \Upsilon)}(\xi) = (-d_C \xi, \xi \Upsilon)
$$

Then at each solution $(C, \Upsilon)$ to the $SO(3)$ vortex equations (14), there is a complex

$$
\Omega^0(\check{\Sigma}; \mathfrak{su}(\check{E})) \to \delta^0_{(C, \Upsilon)} \Omega^1(\check{\Sigma}; \mathfrak{su}(\check{E})) \oplus \Gamma(\check{E}) \to \mathcal{D} \check{\mathfrak{g}}_{SO(3), (C, \Upsilon)} \Omega^0(\check{\Sigma}; \mathfrak{su}(\check{E})) \oplus \Gamma(K_E^{-1} \otimes \check{E})
$$
\textbf{Proof.} Write $\mathfrak{g}_{SO(3)} = \left( \mathfrak{g}_{SO(3)}^1, \mathfrak{g}_{SO(3)}^2 \right)$. Then just as in [55, p.312], we have that

$$D\mathfrak{g}_{SO(3),(C,T)} \circ d_{(C,T)}^0(\xi) = \left[ [\xi, \mathfrak{g}_{SO(3)}^1(C,T)], \xi \mathfrak{g}_{SO(3)}^2(C,T) \right]$$

so the result follows since at a solution to the vortex equations we have $\mathfrak{g}_{SO(3)}(C,T) = (0,0)$. \hfill \Box

**Definition 18.** Define the $L^2$ \textbf{real} inner product. [55 Lemma 9.3.3]

\begin{equation}
\langle (\hat{c}_1, \hat{Y}_1), (\hat{c}_2, \hat{Y}_2) \rangle_{L^2} = \int_\Sigma \langle \hat{c}_1, \hat{c}_2 \rangle + \text{Re} \langle \hat{Y}_1, \hat{Y}_2 \rangle
\end{equation}

where the point-wise inner product on $\mathfrak{su}(E)$ is given by

$$\langle \hat{c}_1, \hat{c}_2 \rangle = \frac{1}{2} \text{Tr}(\hat{c}_1^\dagger \hat{c}_2)$$

Moreover, define

\begin{equation}
d_{(C,T)}^{0*}(\hat{c}, \hat{\tau}) = -d_{C}^* \hat{c} + [\hat{\tau} \hat{\tau}^* - \hat{\tau} \hat{\tau}^* - i \text{Re} < \hat{\tau}, \hat{\tau} > I_E]
\end{equation}

**Lemma 19.** If $d_{(C,T)}^{0*}(\hat{c}, \hat{\tau}) = 0$, then $(\hat{c}, \hat{\tau})$ is $L^2$ orthogonal to $\text{Im} d_{(C,T)}^0$.

\textbf{Proof.} Observe that

$$\langle (-d_{C} \xi, \xi \hat{T}), (\hat{c}, \hat{\tau}) \rangle_{L^2} = \langle -d_{C} \xi, \xi \hat{T} \rangle_{L^2} + \langle \xi \hat{T}, \hat{\tau} \rangle_{L^2} = \langle \xi, -d_{C} \xi \rangle_{L^2} + \langle \xi \hat{T}, \hat{\tau} \rangle_{L^2} = \int_\Sigma \langle \xi, -d_{C} \xi \rangle + \text{Re} \langle \xi \hat{T}, \hat{\tau} \rangle$$

So now we need to verify a pointwise identity. First of all, notice that for two matrices $M_1, M_2$ of $\mathfrak{su}(E)$, their hermitian inner product is real in the sense that

$$\langle M_1, M_2 \rangle = \frac{1}{2} \text{Tr}(M_1^\dagger M_2) = -\frac{1}{2} \text{Tr}(M_1 M_2^\dagger) = \frac{1}{2} \text{Tr}(M_2^\dagger M_1) = \langle M_2, M_1 \rangle$$

which justifies why there is no need to put Re in front of $\langle \xi, -d_{C} \xi \rangle$. On the other hand, we can re-write Re $\langle \xi \hat{T}, \hat{\tau} \rangle$ as

$$\frac{1}{2} \langle \xi \hat{T}, \hat{\tau} \rangle + \frac{1}{2} \langle \tau \xi \rangle$$

Therefore, the integrand we are considering becomes [recall $d_{(C,T)}^{0*}(\hat{c}, \hat{\tau}) = 0$ so we can substitute $d_{C}^* \hat{c}$]

$$\langle \xi, [-\hat{\tau} \hat{T}^* - \tau \hat{T}^* - i \text{Re} < \hat{\tau}, \hat{T} > I_E] \rangle + \frac{1}{2} \langle \xi \hat{T}, \hat{\tau} \rangle + \frac{1}{2} \langle \hat{T}, \xi \hat{T} \rangle$$

$$= \langle \xi, \hat{T} \hat{T}^* \rangle - \langle \xi, \hat{T} \hat{T}^* \rangle + \frac{1}{2} \langle \xi \hat{T}, \hat{\tau} \rangle + \frac{1}{2} \langle \hat{T}, \xi \hat{T} \rangle$$

$$= \frac{1}{2} \langle \xi \hat{T}, \hat{\tau} \rangle - \langle \xi, \hat{T} \hat{T}^* \rangle + \langle \xi, \hat{T} \hat{T}^* \rangle - \frac{1}{2} \langle \hat{T}, \xi \hat{T} \rangle$$

So really we need to compare the previous quantities. For example, we want to know whether

\begin{equation}
\frac{1}{2} \langle \xi \hat{T}, \hat{\tau} \rangle = \langle \xi, \hat{T} \hat{T}^* \rangle
\end{equation}
This would guarantee the vanishing of the first two terms, and the other two are essentially the same. We compute this locally. Observe that we can write

\[ \xi = \begin{pmatrix} it & -\bar{z} \\ z & -it \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \dot{\Upsilon} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \]

Start with the right hand side

\[ \langle \xi, \dot{\Upsilon} \dot{\Upsilon}^* \rangle = -\frac{1}{2} \text{tr} \left( \xi \dot{\Upsilon} \dot{\Upsilon}^* \right) = -\frac{1}{2} \text{tr} \left( \begin{pmatrix} it & -\bar{z} \\ z & -it \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right) = -\frac{1}{2} \text{tr} \left( \begin{pmatrix} it\phi_1 - \bar{z}\phi_2 \\ z\phi_1 - it\phi_2 \end{pmatrix} \begin{pmatrix} \bar{\Phi}_1 \\ \bar{\Phi}_2 \end{pmatrix} \right) = -\frac{1}{2} \text{tr} \left( \begin{pmatrix} * & it\phi_1 \\ \ast & \ast -it\phi_2 \end{pmatrix} \right) \]

where * are things we don’t care about since we are just computing a trace! On the other hand, the left hand side is

\[ \frac{1}{2} \langle \xi \Upsilon, \dot{\Upsilon} \rangle = \frac{1}{2} \dot{\Upsilon}^* \xi^* \dot{\Upsilon} = -\frac{1}{2} \left( \begin{pmatrix} \phi_1 & \Phi_2 \\ \bar{\Phi}_1 & \Phi_2 \end{pmatrix} \begin{pmatrix} it & -\bar{z} \\ z & -it \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right) = -\frac{1}{2} \left( \begin{pmatrix} it\phi_1 - \bar{z}\phi_2 \\ z\phi_1 - it\phi_2 \end{pmatrix} \right) \]

So the traces of both matrices will be the same. \( \square \)

**Definition 20.** Suppose that \((C, \Upsilon)\) is a solution to the SO(3) vortex equations. Define the cohomology groups

\[
\begin{align*}
H^0_{0, (C, \Upsilon)} &\equiv \ker d^0_{0, (C, \Upsilon)} \\
H^1_{0, (C, \Upsilon)} &\equiv \ker (d^0_{0, (C, \Upsilon)} \oplus \mathcal{D} \mathfrak{g}_{\text{SO}(3), (C, \Upsilon)}) \\
H^2_{0, (C, \Upsilon)} &\equiv \text{coker} \mathcal{D} \mathfrak{g}_{\text{SO}(3), (C, \Upsilon)}
\end{align*}
\]

**Lemma 21.** Suppose that \((C, \Upsilon)\) is a solution to the SO(3) vortex equations satisfying:

i) \( H^0_{0, (C, \Upsilon)} = 0 \).

ii) \( \partial_C : \Gamma(\bar{E}) \to \Gamma(K^{-1}_S \otimes \bar{E}) \) is surjective or alternatively \((C, \Upsilon)\) is irreducible, in the sense that \( \Upsilon \) is not identically zero and \( C \) is an irreducible connection.

Then \( H^2_{0, (C, \Upsilon)} = 0 \) and near \([C, \Upsilon] \in \mathcal{M}(\Sigma, \bar{E})\) the moduli space has the structure of a smooth manifold of real dimension

\[
\dim \mathcal{M}(\Sigma, \bar{E}) = \dim H^1_{0, (C, \Upsilon)} = 2(g - 1) + 2c_1(\det \bar{E}) + 2(n - n_0) - 2 \sum_{i=1}^{n} \frac{b_i^+ + b_i^-}{a_i}
\]
Proof. In order to show that
\[ \mathcal{D} \mathfrak{g}_{SO(3),(C,T)} : \Omega^1(\hat{S}; \mathfrak{su}(\hat{E})) \oplus \Gamma(\hat{E}) \to \Omega^0(\hat{S}; \mathfrak{su}(\hat{E})) \oplus \Gamma(K_{\Sigma}^{-1} \oplus \hat{E}) \]
is surjective we follow a similar argument to the one given by Hitchin in \cite[Section 5]{[48]}. First of all, we write
\[ \mathcal{D} \mathfrak{g}_{SO(3),(C,T)} = \left( \mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^1, \mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^2 \right) \]
A simple calculation shows that if we define \( J_{(C,T)}(c, \tilde{\gamma}) = (\ast c, i \tilde{\gamma}) \), then
\[ \mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^1(c, \tilde{\gamma}) = -d_{(C,T)}^0 \ast J(c, \tilde{\gamma}) \]
This computation is implicitly done in Lemma \([29]\), where we show that \( J \) defines an almost complex structure on the Zariski tangent space. Therefore, the formal adjoint of \( \mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^1 \) is (recall \( J^* = -J \))
\[ \mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^2 = J \circ d_{(C,T)}^0 \]
Now suppose that \( (\tilde{\xi}, \tilde{\gamma}) \) is such that
\[
\begin{cases}
J \circ d_{(C,T)}^0 \tilde{\xi} = 0 \\
\mathcal{D} \mathfrak{g}_{SO(3),(C,T)}^2 \tilde{\gamma} = 0
\end{cases}
\]
Since \( J^2 = -\text{Id} \), the first equation says that \( \tilde{\xi} \in \ker d_{(C,T)}^0 \), which vanishes by assumption i). Therefore \( \tilde{\xi} = 0 \).
Likewise, the second equation says that for all \( (c, \tilde{\gamma}) \in \Omega^1(\hat{S}; \mathfrak{su}(\hat{E})) \oplus \Gamma(\hat{E}) \),
\[ < \tilde{\partial}_C \tilde{\gamma} + \tilde{\gamma}'' \otimes \gamma, \tilde{\gamma} >_{L^2} = 0 \]
Taking \( \tilde{\gamma} = \gamma \) this means that \( < \tilde{\gamma}'' \otimes \gamma, \tilde{\gamma} >_{L^2} = 0 \) since \( \tilde{\partial}_C \gamma = 0 \). Hence, we obtain the condition
\[ < \tilde{\partial}_C \gamma, \gamma >_{L^2} = 0 \iff < \gamma, \tilde{\partial}_C^* \gamma >_{L^2} = 0 \iff \gamma \in \ker \tilde{\partial}_C^* \]
Therefore \( \tilde{\gamma} \) is zero if we assume that \( \tilde{\partial}_C^* \) is surjective. If we no longer assume that \( \tilde{\partial}_C \) is surjective, notice that from equation \([17]\) \( < \tilde{\gamma}'' \otimes \gamma, \tilde{\gamma} >_{L^2} = 0 \) is equivalent to \( < \tilde{\gamma}'' \ast \gamma \ast \gamma, 0 >_{L^2} = 0 \), thus we find that
\[ \tilde{\gamma} \gamma^* = \frac{1}{2} \text{tr}(\tilde{\gamma} \gamma) \]
Locally, we can write \( \gamma = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \) and \( \tilde{\gamma} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \), the previous equation is equivalent to
\[
\begin{pmatrix}
\phi_1 \phi_1 & \phi_1 \phi_2 \\
\phi_2 \phi_1 & \phi_2 \phi_2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} (\phi_1 \phi_1 + \phi_2 \phi_2) & 0 \\
\frac{1}{2} (\phi_1 \phi_1 + \phi_2 \phi_2) & \phi_1 \phi_1 + \phi_2 \phi_2
\end{pmatrix}
\]
Therefore, we conclude that
\[
\begin{cases}
\phi_1 \phi_2 = 0 \\
\phi_2 \phi_1 = 0
\end{cases}
\]
Now, if \( \gamma \) is not a section of some splitting holomorphic sub-bundle of \( \hat{E} \), then we must conclude that \( \phi_1 \) and \( \phi_2 \) must vanish on the open sets where \( \Phi_1, \Phi_2 \) are non-vanishing. But since \( \ast \tilde{\gamma} \) is a holomorphic section, this can only happen if \( \tilde{\gamma} \) vanishes.
To find the expected dimension of the moduli space at the solution \([C, \gamma]\), we need to compute
\[
\text{ind}(d_{(C,T)}^0, \mathcal{D} \mathfrak{g}_{SO(3),(C,T)})
\]
This operator is homotopic to the operator
\[(d_C^{0,*} \oplus d_C) \oplus \partial_C\]
so the index is equal to
\[\text{ind}(d_C^{0,*} \oplus d_C) + \text{ind}\partial_C\]

From the Riemann-Roch formula (11), we know that
\[\text{ind}\partial_C = 2(1-g) + c_1(\det \tilde{E}) - \sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}\]
To compute \(\text{ind}(d_C^{0,*} \oplus d_C)\), we use the Dolbeault model, and compute this index as
\[\chi(\tilde{\Sigma}; \mathfrak{su}(\tilde{E}) \otimes \mathbb{C})\]
Just as we found Riemann-Roch for \(U(2)\) orbifold bundles, suppose that 
\[\tilde{E} = \tilde{L}^- \oplus \tilde{L}^+\]
Then
\[\mathfrak{su}(\tilde{E}) \otimes \mathbb{C} \simeq \mathbb{C} \oplus (\tilde{L}^- \otimes \tilde{L}^{*+}) \oplus (\tilde{L}^{*-} \otimes \tilde{L}^+)\]
Therefore,
\[\chi(\tilde{\Sigma}; \mathfrak{su}(\tilde{E}) \otimes \mathbb{C}) = \chi(\tilde{\Sigma}; \mathbb{C}) + \chi(\tilde{\Sigma}; \tilde{L}^- \otimes \tilde{L}^{*+}) + \chi(\tilde{\Sigma}; \tilde{L}^{*-} \otimes \tilde{L}^+)\]
Recall that the orbifold cohomology groups with complex coefficients are isomorphic to the ordinary de Rham cohomology groups, in particular, this means that
\[\chi(\tilde{\Sigma}; \mathbb{C}) = 1 - g\]
To compute \(\chi(\tilde{\Sigma}; \tilde{L}^- \otimes \tilde{L}^{*+}) + \chi(\tilde{\Sigma}; \tilde{L}^{*-} \otimes \tilde{L}^+)\), we need to understand the isotropies of the line bundles \(L = \tilde{L}^- \otimes \tilde{L}^{*+}\) and \(L^* = \tilde{L}^{*-} \otimes \tilde{L}^+\).
Suppose that at a particular marked point, \(b_i^+ \neq b_i^-\). Then \(L\) and \(L^*\) have non-trivial isotropies. If \(L\) has isotropy \(b_{i,L}\) at the marked point \(p_i\), then we can take the isotropy \(b_{i,L^*}\) of \(L^*\) to be \(b_{i,L^*} = a_i - b_{i,L}\). Therefore, in the formula (10) for Riemann-Roch for orbifold line bundles, the terms \(\frac{b_{i,L^*}}{a_i}\) and \(\frac{b_{i,L}}{a_i}\) will add up to 1.
On the other hand, if one looks at a marked point where \(b_i^+ = b_i^-\), then, the isotropy of \(L\) and \(L^*\) become zero at this point.
In this way we obtain
\[\chi(\tilde{\Sigma}; L) + \chi(\tilde{\Sigma}; L^*) = 2(1-g) - (n - n_0)\]
adding the Euler characteristics we find
\[\chi(\tilde{\Sigma}; \mathfrak{su}(\tilde{E}) \otimes \mathbb{C}) = 3(1-g) - (n - n_0)\]
So the real expected dimension of the moduli space is
\[\dim_{\mathbb{R}} \mathcal{M}(\tilde{\Sigma}, \tilde{E}) = 6(g-1) + 2(n - n_0) + 4(1-g) + 2c_1(\det \tilde{E}) - 2\sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}\]
\[= 2(g-1) + 2c_1(\det \tilde{E}) + 2(n - n_0) - 2\sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}\]
\[\square\]
Lemma 23. A fixed point $[C, \Psi]$ to the circle action takes the form:

i) A projectively flat connection $[C, 0]$.

ii) A reducible solution $[C_L \oplus (C_L^* \otimes \text{det}), \alpha \oplus 0]$ compatible with a splitting $\mathcal{E} = \mathcal{L} \oplus (\mathcal{L}^* \otimes \text{det} \mathcal{E})$, where $(C_L, \alpha)$ satisfies the abelian vortex equation

\[
\begin{align*}
\text{abelian vortex equation} & \quad \begin{cases} 
*F_{C_L} - \frac{i}{2} |\alpha|^2 = *\frac{1}{2} F_{\text{det}} \\
\partial_{C_L} \alpha = 0
\end{cases} 
\end{align*}
\]

Moreover, $\alpha$ cannot vanish identically if either of the following conditions is satisfied:

I) In the smooth case, $c_1(\text{det} \mathcal{E})$ is of odd degree.

II) In the orbifold case $\Sigma = (\Sigma, p_1, \cdots, p_n)$ with the multiplicities $a_i$ of $p_i$ mutually coprime, we have $\text{det} \mathcal{E}$ is an odd integer power of the fundamental orbifold line bundle $L_0$ satisfying $c_1(L_0) = \frac{1}{a_1 \cdots a_n}$.

Remark 24. When we have a splitting $\mathcal{E} = \mathcal{L} \oplus (\mathcal{L}^* \otimes \text{det} \mathcal{E})$, it is assumed implicitly that $\mathcal{L}$ has the correct isotropy as a subbundle of $\mathcal{E}$ as discussed before.

Proof. Case i) is clear.

ii) If $C$ is reducible, then there is a splitting $\mathcal{E} = \mathcal{L} \oplus (\mathcal{L}^* \otimes \text{det} \mathcal{E})$ of the $U(2)$ bundle compatible with the reduction $C = C_L \oplus (C_L^* \otimes \text{det})$, where $C_L$ denotes a $U(1)$ connection on $\mathcal{L}$. Likewise, we can decompose $\Psi$ into components $\Psi = \alpha \oplus \beta$. The curvature decomposes as

\[
F_C = \begin{pmatrix} F_{C_L} & 0 \\ 0 & F_{\text{det}} - F_{C_L} \end{pmatrix}
\]

so the traceless part is

\[
F_C^0 = F_C - \frac{1}{2} \text{tr}(F_C)1_E = \begin{pmatrix} F_{C_L} - \frac{1}{2} F_{\text{det}} & 0 \\ 0 & \frac{1}{2} F_{\text{det}} - \frac{1}{2} F_{C_L} \end{pmatrix}
\]
The other term of the curvature equation is

\[
\begin{pmatrix} \alpha \\
\beta \\
\bar{\alpha} \\
\bar{\beta} \end{pmatrix} \begin{pmatrix} |\alpha|^2 & \alpha \bar{\beta} \\
\bar{\beta} & |\beta|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 \\
\frac{1}{2} \bar{\beta}^2 - \frac{1}{2} |\alpha|^2 \end{pmatrix}
\]

So

\[
\Upsilon - \frac{1}{2} |\Upsilon|^2 I_E = \begin{pmatrix} \frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 \\
\frac{1}{2} \bar{\beta}^2 - \frac{1}{2} |\alpha|^2 \end{pmatrix}
\]

Therefore, the \(SO(3)\) vortex equations at reducible solutions can be expressed as

\[
\begin{cases}
* F_{C_L} - * \frac{i}{2} F_{C_{\text{det}}} - \frac{i}{2} |\alpha|^2 + \frac{i}{2} |\beta|^2 = 0 \\
\partial_{C_L} \alpha = 0 \\
\partial_{C_L} \otimes C_{\text{det}} \beta = 0
\end{cases}
\]

The last two equations say that \(\alpha, \beta\) are holomorphic sections, so by the unique continuation principle, if they vanish on an open set, they must vanish identically. Using the second equation we can conclude that at least one of the two sections must vanish. Without loss of generality we can assume that \(\beta\) vanishes (see next remark). Therefore, \(\alpha\) must solve the vortex equation

\[
U(1) \text{ vortex equations}
\begin{cases}
* F_{C_L} - \frac{i}{2} |\alpha|^2 = \frac{1}{2} F_{C_{\text{det}}} \\
\partial_{C_L} \alpha = 0
\end{cases}
\]

Up to this point, it is not impossible for \(\alpha\) to vanish identically. If this happens, then we obtain \(2 F_{C_L} = F_{C_{\text{det}}}\), and so Chern-Weil theory says that

\[
c_1(\det E) = 2 c_1(\bar{L})
\]

Recall that in this case we are referring to the orbifold Chern classes, which are rational cohomology classes in general.

a) In the smooth case, the condition \((21)\) is not satisfied if we assume that \(c_1(\det E)\) is of odd degree.

b) In the orbifold case, the classification of topological line bundles is more complicated as explained before (see also \([42, \text{Proposition 1.4}]\)). However, if we assume that \(a_1, \cdots, a_n\) are coprime, then we know that any line bundle has the form \(\bar{L}_0^k\) for some integer \(k\). Therefore, \(\det E = \bar{L}_0^k\) for some \(k\) and \(\bar{L} = L_0^k\) for some \(l\). Since \(c_1\) continues to be a homomorphism, \(c_1(\det E) = k c_1(L_0)\) and \(c_1(L) = l c_1(\bar{L}_0)\), so \((21)\) is equivalent to

\[
k = 2l
\]

which again does not have any solution if we assume that \(k\) is an odd integer. \(\square\)

Remark 25. Notice that in the proof of the previous lemma we said that we could assume \(\beta\) to vanish. In reality this depends on the fact that we are ultimately interested in studying the solutions up to gauge.

As mentioned before, a gauge transformation which swaps the factors \(\bar{L}, (\bar{L}^* \otimes \det E)\) is locally of the form \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), which is not of determinant one. However, the section \(\begin{pmatrix} \alpha \\ 0 \end{pmatrix}\) is gauge equivalent to the section \(\begin{pmatrix} 0 \\ -\alpha \end{pmatrix}\) if we use the matrix \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), which is of determinant one. Since eventually we are interested in the moduli space after the quotient by the residual \(S^1\) action...
is taken, then \( \begin{pmatrix} 0 & -\alpha \\ -\alpha & 0 \end{pmatrix} \) will be equivalent to \( \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \), so it is possible to swap the factors after the full symmetry is taken into account. Therefore, for our purposes no generality is lost by assuming that \( \beta \) is the section which vanishes.

**Lemma 26.** If \((C, \Upsilon)\) is a solution to the \(SO(3)\) vortex equations \([14]\) then \(H^0_{(C, \Upsilon)}\) vanishes identically except when \(C\) is a reducible connection and \(\Upsilon\) vanishes identically.

**Proof.** Recall that \(d^0_{(C, \Upsilon)}(\xi) = (-d_C \xi, \xi \Upsilon)\) and that \(H^0_{(C, \Upsilon)} = \ker d^0_{(C, \Upsilon)}\). It is clear that when \(C\) is a reducible connection and \(\Upsilon\) vanishes identically, then \(\dim H^0_{(C, \Upsilon)} \geq 1\) [it has at least an \(S^1\) stabilizer].

If \(C\) is an irreducible connection, then \(\ker d_C = \{0\}\) so \(H^0_{(C, \Upsilon)}\) vanishes.

If \(C\) reduces to \(C = C_L \oplus (C^*_L \otimes C^{\text{det}})\), then \(C_L\) has at least an \(S^1\) stabilizer which acts on \(\Upsilon = \alpha \oplus 0\) by multiplication. Therefore, if \(\alpha\) does not vanish identically, the action of \(S^1\) on \((C_L, \alpha)\) will be trivial, and the result follows. \(\square\)

In the next sections the following function (moment map) will be very important.

**Definition 27.** Define the Higgs strength function

\[
\mu : \mathbb{A}^{\text{det}}(\tilde{E}) \times \Gamma(\tilde{E}) \rightarrow \mathbb{R}
\]

\[
(C, \Upsilon) \rightarrow \frac{1}{2} \|\Upsilon\|_{L^2(\Sigma)}^2
\]

**Lemma 28.** Suppose that \((C_L, \alpha)\) is an abelian vortex, i.e, it solves equation \([20]\). Then

\[
\mu(C_L, \alpha) = \pi(c_1(\tilde{E}) - 2c_1(\tilde{L}))
\]

In particular, an abelian vortex can only appear in a reduction of the bundle \(\tilde{E}\) if \(c_1(\tilde{L}) \leq \frac{1}{2} c_1(\tilde{E})\).

**Proof.** The curvature equation \(\ast F_{C_L} - \frac{i}{4} |\alpha|^2 = \ast \frac{1}{2} F_{C^{\text{det}}}\) is equivalent to

\[
\frac{\ast i F_{C_L}}{2\pi} + \frac{1}{4\pi} |\alpha|^2 = \frac{\ast i F_{C^{\text{det}}}}{4\pi}
\]

and after integrating of \(\tilde{\Sigma}\) we conclude that

\[
\frac{1}{4\pi} \int |\alpha|^2 = \frac{1}{2} c_1(\tilde{E}) - c_1(\tilde{L})
\]

from which the result follows. \(\square\)

Now we discuss the interpretation of \(\mu\) as a moment map. For this we need to introduce other structures on the moduli space. Recall that in equation \([15]\) we defined the real inner product

\[
g_{(C, \Upsilon)}((\dot{c}_1, \dot{\Upsilon}_1), (\dot{c}_2, \dot{\Upsilon}_2)) = \int_{\Sigma} \langle \dot{c}_1, \dot{c}_2 \rangle + \text{Re} \langle \dot{\Upsilon}_1, \dot{\Upsilon}_2 \rangle
\]

With respect to this inner product we have:

**Lemma 29.** The endomorphism of \(H^1_{(C, \Upsilon)} = \ker(d^0_{(C, \Upsilon)} \oplus D\tilde{S}_{SO(3), (C, \Upsilon)})\) given by

\[
J_{(C, \Upsilon)}(\dot{c}, \dot{\Upsilon}) = (\ast \dot{c}, i \dot{\Upsilon})
\]

defines an almost complex structure on \(H^1_{(C, \Upsilon)}\) which is \(g\) -orthogonal, i.e, \(g(J\bullet, J\bullet) = g(\bullet, \bullet)\).
Proof. Checking that $J^2 = -\text{Id}$ is straightforward as well as the compatibility with $g$. The only interesting thing to verify is that $H^1_{(C,T)}$ is preserved by this almost-complex structure. Recall that $H^1_{(C,T)}$ consisted of the equations

$$
\begin{cases}
-d^*c \cdot [\hat{\mathcal{Y}}^* - \mathcal{Y}(\mathcal{Y})^* - i\text{Re} \langle \mathcal{Y}, \hat{\mathcal{Y}} \rangle I_E] = 0 \\
* d_c \hat{c} - i \left[ \hat{\mathcal{Y}}^* + \mathcal{Y}^* - \text{Re} \langle \mathcal{Y}, \hat{\mathcal{Y}} \rangle I_E \right] = 0 \\
\hat{\mathcal{Y}} + \hat{c}'' \otimes \mathcal{Y} = 0
\end{cases}
$$

Since $*c'' = i\hat{c}''$, it is straightforward to check that the last equation is preserved under $J$. For the first equation,

$$
\begin{align*}
&- d^*c \cdot [(i\hat{\mathcal{Y}})^* - \mathcal{Y}(i\hat{\mathcal{Y}})^* - i\text{Re} \langle \mathcal{Y}, i\hat{\mathcal{Y}} \rangle I_E] \\
&= * d_c \cdot [(i\hat{\mathcal{Y}})^* + \mathcal{Y}^* - \text{Re} \langle \mathcal{Y}, i\hat{\mathcal{Y}} \rangle I_E] \\
&= - \left( * d_c \cdot [\hat{\mathcal{Y}}^* + \mathcal{Y} \cdot \mathcal{Y}^* - \text{Re} \langle \mathcal{Y}, \hat{\mathcal{Y}} \rangle I_E] \right)
\end{align*}
$$

Conversely,

$$
\begin{align*}
&* d_c \cdot [(i\hat{\mathcal{Y}})^*] - i \left[ \mathcal{Y}(i\hat{\mathcal{Y}})^* + (i\hat{\mathcal{Y}})^* - \text{Re} \langle \mathcal{Y}, i\hat{\mathcal{Y}} \rangle I_E \right] \\
&= - d^*c \cdot [i\hat{\mathcal{Y}} + i\mathcal{Y} \cdot \mathcal{Y}^* - \text{Re} \langle i\mathcal{Y}, \hat{\mathcal{Y}} \rangle I_E] \\
&= - d^*c \cdot [i\hat{\mathcal{Y}} + \mathcal{Y} \cdot \mathcal{Y}^* - \text{Re} \langle i\mathcal{Y}, \hat{\mathcal{Y}} \rangle I_E] \\
&= 0
\end{align*}
$$

So the claim is verified.

Therefore, we can define the 2-form

$$\Omega(\bullet, \bullet) = g(\bullet, J \bullet)$$

Since the formulas for $g, J, \Omega$ are base-point independent, it is clear that $\Omega$ will be a closed two-form, and $J$ being integrable implies that we have a Kahler form on the moduli space.

Now consider the vector field

$$\partial_{\theta,(C,T)} = (0, i\mathcal{Y})$$

This is clearly the vector field associated to the $S^1$ action defined before in equation (19).

Moreover, notice that

$$
(1_{\theta}, \Omega)(\hat{c}, \hat{\mathcal{Y}}) = \Omega((0, i\mathcal{Y}), (\hat{c}, \hat{\mathcal{Y}})) = \int \text{Re} \langle i\mathcal{Y}, i\hat{\mathcal{Y}} \rangle = \int \text{Re} \langle \mathcal{Y}, \hat{\mathcal{Y}} \rangle = d\mu_{(C,T)}(\hat{c}, \hat{\mathcal{Y}})
$$

Where $\mu$ was the Higgs strength function. This is precisely the condition that guarantees that $\mu$ is the moment map associated to the $S^1$ action [16, Chapter 2].
However, notice that $(0, i\mathcal{T})$ does not belong to $H^1_{(C, \mathcal{T})}$. Indeed, if one looks at whether $(0, i\mathcal{T})$ satisfies the first equation for $H^1_{(C, \mathcal{T})}$, we see that
\[(i\mathcal{T})^* - \mathcal{T}(i\mathcal{T})^* - i\text{Re} \mathcal{T} > I_E\]
and this clearly need not vanish. However, since $(0, i\mathcal{T})$ does satisfy the other two equations defining $H^1_{(C, \mathcal{T})}$, this just means that the representative for $(0, i\mathcal{T})$ in $H^1_{(C, \mathcal{T})}$ is of the form $(0, i\mathcal{T}) + d^0(C, \mathcal{T})\xi$, for some $\xi \in \Omega^0(\mathcal{E}, \mathfrak{su}(E))$.

Therefore, in order to check that the moment map condition holds with respect to the vector $(0, i\mathcal{T}) + d^0(C, \mathcal{T})\xi$, it suffices to know that for any $(\hat{c}, \hat{\mathcal{T}}) \in H^1_{(C, \mathcal{T})}$, we have
\[
\Omega(d^0(C, \mathcal{T})\xi, (\hat{c}, \hat{\mathcal{T}})) = 0
\]
See also the discussion in [27 Section iv), page 294]. In our case, this condition is satisfied, since
\[
\Omega(d^0(C, \mathcal{T})\xi, J(\hat{c}, \hat{\mathcal{T}})) = 0
\]

The last equality is true since $J(\hat{c}, \hat{\mathcal{T}}) \in H^1_{(C, \mathcal{T})}$, and therefore it satisfies the Coulomb condition. Summarizing our findings we have found:

**Theorem 30.** The two-form $\Omega$ is a Kahler form on the smooth points of the $SO(3)$ vortex moduli space, and $\mu$ is a moment map for the $S^1$ action defined in [19].

In the next few sections we will study some properties of the map $\mu$ using this moment map interpretation.

4. $SO(3)$ Vortices and Stable Pairs

In order to study the moment map $\mu$ more easily, we will now compare in this section and the next the $SO(3)$ vortex equations to the stable pairs equations [90 18 19 20 11].

A stable pair $(\tilde{\partial}_C, \mathcal{Y})$ consists for us of a holomorphic structure $\tilde{\partial}_C$ on the orbifold bundle $\tilde{E}$, together with a holomorphic section $\mathcal{Y}$ of $\tilde{E}$, that is, $\tilde{\partial}_C\mathcal{Y} = 0$. When convenient, we will sometimes write the stable pair as $(\mathcal{E}, \mathcal{T})$, where $\mathcal{E}$ refers to the bundle $\tilde{E}$ endowed with the holomorphic structure given by $\tilde{\partial}_C$.

The complex gauge group $G^\text{det}_0(\tilde{E})$ acts on the space of stable pairs via
\[
u \cdot (\tilde{\partial}_C, \mathcal{Y}) = (\nu \circ \tilde{\partial}_C \circ u^{-1}, u\mathcal{Y}) = (\tilde{\partial}_Cu - (\tilde{\partial}_Cu)u^{-1}, u\mathcal{Y})
\]
Following [30 Section 6.4], we can identify the tangent space to the space of $\tilde{\partial}$ operators with $\Omega^{0,1}(End_0\tilde{E})$, where as usual $End_0\tilde{E}$ represents the bundle of trace free endomorphisms of $\tilde{E}$ (this is because we are working with bundles having a fixed determinant). The linearization of the derivative of the complex gauge group action is $\tilde{\partial}_C$, and so we have:

**Lemma 31.** The deformation theorem for the stable pair equation is determined by the cohomology groups of the complex:
\[
0 \rightarrow \Omega^0(\mathcal{S}, End_0\tilde{E}) \rightarrow^{d_1} \Omega^{0,1}(\mathcal{S}, End_0\tilde{E}) \oplus \Omega^0(\mathcal{S}, \tilde{E}) \rightarrow^{d_2} \Omega^{0,1}(\tilde{E}) \rightarrow 0
\]
where
\[
\begin{align*}
    d_1 \xi &= (-\bar{\partial}_C \xi, \xi \Upsilon) \\
    d_2 (\dot{c}', \dot{\Upsilon}) &= \bar{\partial}_C \dot{\Upsilon} + \dot{c}' \Upsilon
\end{align*}
\]

Proof. This is a simple computation, and essentially it is the orbifold version of [19, Proposition 2.1] □

Now we review the relevant version of the Hitchin-Kobayashi correspondence between solutions to the \(SO(3)\) vortex equations and stable pairs. This correspondence has found different instantiations, and the orbifold version is a trivial extension of the argument given in [63, 31, Theorem 1.4.2], since integration by parts and the Kahler identities continue to work on orbifolds.

Observe first of all that if \((C, \Upsilon)\) is an \(SO(3)\) vortex, then \(C\) naturally defines a holomorphic structure \(\bar{\partial}_C\), and \((\bar{\partial}_C, \Upsilon)\) is a stable pair. Since \(G^\text{det}(\hat{E})\) is a subgroup of \(G^\text{det}_C(\hat{E})\), this means that there is a well-defined map
\[
\mathcal{M}^\text{cor}(\hat{E}, \hat{E}) \to \mathcal{M}^\text{st}(\hat{E}, \hat{E})
\]
\[
[C, \Upsilon]_R \to [\bar{\partial}_C, \Upsilon]_\text{C}
\]
where \([\cdot]_R\) denotes the gauge equivalence class with respect to the “real” gauge group \(G^\text{det}(\hat{E})\), while \([\cdot]_\text{C}\) denotes the gauge equivalence class with respect to the complexified gauge group \(G^\text{det}_C(\hat{E})\). We need to understand the surjectivity of this map.

We will focus in the case that \(C\) is irreducible and \(\Upsilon\) is nowhere vanishing, since the case of \((C, 0)\) can be found in [83], while the case of a \(U(1)\) vortex on an orbifold can be found in [67, Theorem 5].

We recall first the definition of stability for a pair \((\bar{\partial}_C, \Upsilon)\) [31, Section 1.3]. [63, Chapter 1, p.3]. Let \(\hat{E}\) denote the bundle \(\hat{E}\) with the holomorphic structure induced by \(\bar{\partial}_C\). Denote by \(R(\hat{E})\) the set of subsheaves of \(\hat{E}\) with torsion free quotients. For any sheaf \(\hat{F}\) define the slope \(\mu_\beta(\hat{F})\) as
\[
\mu_\beta(\hat{F}) = \frac{\int_{\hat{E}} c_1(\hat{F})}{\text{rk}(\hat{F})} \in \mathbb{Q}
\]

Definition 32. The pair \((\hat{E}, \Upsilon)\) is stable if \(\mu_\beta(\hat{F}) < \mu_\beta(\hat{E})\) whenever \(\hat{F} \in R(\hat{E})\) and \(\Upsilon \in H^0(\hat{F})\). In other words,
\[
c_1(\hat{L}) < \frac{1}{2} c_1(\hat{E})
\]
whenever \(\Upsilon \in H^0(\hat{L})\).

The same arguments to those in [31, Section 2.3] show that a pair \((\bar{\partial}_C, \Upsilon)\) which solves the projective vortex equation
\[
*F_C^0 - i \left[ \Upsilon \Upsilon^* - \frac{1}{2} |\Upsilon|^2 I_E \right] = 0
\]
must be stable, provided that \(C\) is irreducible and \(\Upsilon\) non-vanishing. Moreover, if the pair \((\bar{\partial}_C, \Upsilon)\) is stable then it will solve the projective vortex equations (up to gauge of course). Therefore, we have

Theorem 33. The irreducible solutions to the \(SO(3)\) vortex equations \((C, \Upsilon)\) [so \(C\) is irreducible and \(\Upsilon\) non-vanishing], are in bijection with the stable pairs. In other words, the Kobayashi-Hitchin map \(\mathfrak{m}\) is a bijection when restricted to the irreducible \(SO(3)\) vortices.
The reader may be more familiarized with a version of the stable pair equations where a stability parameter explicitly appears [18, 19, 90, 20]. This occurs if one uses the $U(2)$ vortex equations instead of the $SO(3)$ vortex equations, so we now proceed to explain their relationship. We can think of the $U(2)$ vortex equations as equations for a pair $(C, \Upsilon)$ which now satisfy
\begin{equation}
\begin{aligned}
&\ast F_C - i\Upsilon\Upsilon^* + i\tau \text{Id}_E = 0 \\
&\partial_C\Upsilon = 0
\end{aligned}
\end{equation}

The main difference with equations (13) is that we are no longer requiring that $C$ induce a fixed connection $C_{\text{det}}$ on $\det \hat{E}$, which is why we now have an equation for the entire curvature of $F_C$ and not just its trace-free part. Here $\tau$ is a constant which is related to a stability parameter. Clearly we can break the curvature equation into its traceless and trace parts, so that the $U(2)$ equations read
\begin{equation}
\begin{aligned}
2 \ast F_{C_{\text{det}}} - i|\Upsilon|^2 + 2i\tau &= 0 \\
\ast F_C^0 - i \left[ \Upsilon\Upsilon^* - \frac{1}{2}|\Upsilon|^2 I_E \right] &= 0 \\
\partial_C\Upsilon &= 0
\end{aligned}
\end{equation}

Notice that if we integrate the first equation, we obtain the relation
\[ \text{vol}(\Sigma)\tau = \mu(\Upsilon) + 2\pi c_1(\det \hat{E}) \]

Therefore, fixing a value of $\tau$ is equivalent to fixing a level set for the moment map $\mu$. Clearly, any solution $(C, \Upsilon)$ to the $U(2)$ vortex equations will yield a solution to the $SO(3)$ vortex equations, provided $C$ happened to induce our predetermined connection $C_{\text{det}}$ on $\det \hat{E}$. On the other hand, if we start with a solution $(C, \Upsilon)$ to our $SO(3)$ vortex equations, there is a priori no reason why the first equation in (23) has to be satisfied, so the moduli space of $SO(3)$ vortices is not just the moduli space of $U(2)$ vortices which induce a particular connection on $\det \hat{E}$.

### 5. A Morse-Bott Function on the Moduli Space

As explained in Theorem (30), the Higgs strength function can be interpreted as a moment map for the $S^1$ action on the $SO(3)$ vortex moduli space.

This means that whenever we choose the $U(2)$ bundle $\hat{E}$ with the property that it guarantees the moduli space to be smooth (at the reducibles), Frankel’s theorem holds [40], which implies in our case the $\mu$ is a Morse-Bott function with the critical set being equal to the fixed points of the $S^1$ action. In other words, the critical sets of $\mu$ consist of the abelian vortices or the moduli space of flat connections.

Moreover, the index of a particular critical set is the dimension of the subspace on which the circle action acts with negative weight. As mentioned before, a version of Frankel’s theorem for almost hermitian manifolds was used recently byFeehan andLeness in order to compute the Morse indices of the analogue of $\mu$ in the case of the $SO(3)$ monopole moduli spaces (see Theorem 1 in [34] and the discussion that follows). In the gauge theory context, probably the first use of this idea appeared in Hitchin’s paper [48, Section 7]. For the case of $U(2)$ vortices some calculations in this spirit are done in [20, Sections 3 and 4], and Thaddeus also computes the dimensions of the subspaces with positive and negative weights in [91, Section 8].

First we understand the deformation theory at an abelian vortex. Recall that the linearization of the $SO(3)$ vortex map was given in Lemma 19
\[ D\mathfrak{H}_{SO(3), (C, \Upsilon)}(\dot{c}, \dot{\Upsilon}) = \left( *d_C \dot{c} - i \left[ \Upsilon\Upsilon^* + \Upsilon^*\Upsilon^* - \text{Re} \langle \Upsilon, \Upsilon \rangle I_E \right], \partial_C \partial + \mathcal{C}'' \otimes \Upsilon \right) \]
and likewise from equation \([16]\)
\[
\frac{d}{dt}^*_{\mathcal{C},\mathcal{Y}}(\hat{c}, \hat{\mathcal{Y}}) = -d_{\mathcal{C}}^* \hat{c} + [\hat{\mathcal{Y}} \mathcal{Y}^* - \mathcal{Y} \hat{\mathcal{Y}}^* - i \text{Re} < i \mathcal{Y}, \hat{\mathcal{Y}> I_{\mathcal{E}}}] 
\]
At an abelian vortex we can write \(C = C_L \oplus (C_L^* \otimes \text{Cdet})\) and \(\mathcal{Y} = \alpha \oplus 0\). Since \(\mathcal{E} = \hat{L} \oplus (\hat{L}^* \otimes \text{det} \hat{E})\), we have \(\bar{g}_{\mathcal{E}} = i \mathbb{R} \oplus (\hat{L}^2 \otimes (\text{det} \hat{E})^{-1})\). If \(\hat{\mathcal{Y}} = (\hat{T}_t \oplus \hat{T}_n)\) and \(\hat{c} = \hat{c}_t \oplus \hat{c}_n\) (here \(t\) and \(n\) stand for tangential and normal respectively) then
\[
\begin{align*}
\hat{\mathcal{Y}} \mathcal{Y}^* &= \left( \begin{array}{cc}
\hat{T}_t & 0 \\
0 & \hat{T}_n
\end{array} \right) \\
\mathcal{Y} \hat{\mathcal{Y}}^* &= \left( \begin{array}{cc}
\alpha^{*} & 0 \\
0 & \alpha
\end{array} \right) \\
\hat{\mathcal{Y}}^* \mathcal{Y}^* - \text{Re} \left\langle \mathcal{Y}, \hat{\mathcal{Y}} \right\rangle |_{\mathcal{E}} &= \left( \begin{array}{cc}
\text{Re} \left\langle \alpha, \hat{T}_t \right\rangle & \alpha \hat{T}_n^* \\
\hat{T}_n \alpha^* & -\text{Re} \left\langle \alpha, \hat{T}_t \right\rangle
\end{array} \right)
\end{align*}
\]
We can interpret \(\hat{c}\) as a matrix by writing \(\hat{c} = \left( \begin{array}{cc}
\hat{c}_t & \hat{c}_n \\
-\hat{c}_n & \hat{c}_t
\end{array} \right)\). Then if we write \(\hat{N} = \hat{L}^2 \otimes (\text{det} \hat{E})^{-1}\) so that \(\hat{c}_n \in \Omega^1(\bar{\Sigma}; \hat{N})\), and \(C_N\) represents the induced connection \(C_L^* \otimes \text{Cdet}\),
\[
*_{\mathcal{C}} \hat{c} - i \left[ \mathcal{Y} \hat{\mathcal{Y}}^* + \hat{\mathcal{Y}} \mathcal{Y}^* - \text{Re} \left\langle \mathcal{Y}, \hat{\mathcal{Y}} \right\rangle |_{\mathcal{E}} \right] = \left( \begin{array}{c}
*_{\mathcal{C}} \hat{c}_t - i \text{Re} \left\langle \alpha, \hat{T}_t \right\rangle \\
*_{\mathcal{C}} \hat{c}_n - i\alpha \hat{T}_n^*
\end{array} \right)
\]
Likewise,
\[
\partial_C \hat{\mathcal{Y}} + \hat{c}'' \otimes \mathcal{Y} = \left( \begin{array}{c}
\partial_C \hat{T}_t + \hat{c}_t'' \otimes \alpha \\
\partial_C \hat{T}_n + \hat{c}_n'' \otimes \alpha
\end{array} \right)
\]
It is easy to recognize the first row in each of these vectors as corresponding to the linearization to the \(U(1)\) vortex equations. The remaining rows together with the Coulomb conditions leads us to define the normal operator
\[
\mathcal{D}_{(C_L, \alpha)}^n(\hat{c}, \hat{\mathcal{Y}}) = \left( \begin{array}{c}
-\partial_C \hat{c}_n - \alpha \hat{T}_n^* \\
*_{\mathcal{C}} \hat{c}_n - i\alpha \hat{T}_n^*
\end{array} \right)
\]
Observe that the normal bundle to the abelian vortex moduli space inside the \(SO(3)\) vortex moduli space then corresponds to \(\mathcal{D}_{(C_L, \alpha)}^n\). For the case of the projectively flat connections, the story is a bit different. Here the normal operator is easily seen to be
\[
\mathcal{D}_{(C, 0)}^n(\hat{c}, \hat{\mathcal{Y}}) = \left( \begin{array}{c}
-\partial_C \hat{c}_n \\
\partial_C \hat{c}
\end{array} \right)
\]
A simple adaptation of \([21]\) shows:

**Lemma 34.** Suppose that \(c_1(\hat{E}) > 2c_1(K_{\bar{E}})\) and that for the smooth case \(\hat{E}\) is of odd odd degree or when all the \(a_1, \ldots, a_n\) are co-prime we are in the case where \(\text{det} \hat{E}\) is an odd power of \(\hat{L}_0\). Then the moduli space \(\mathcal{M}(\bar{\Sigma}, \hat{E})\) is smooth at the abelian vortices and the projectively flat connections.
Proof. It is easier to explain why $\mathcal{D}^n_{(\mathcal{L},\alpha)}$ is surjective. Since $\ker d_C = \{0\}$ by assumption, we just need to worry about the surjectivity of $\partial_C$. Thus we need to guarantee that $H^1(\hat{E})$ vanishes.

If $H^1(\hat{E}) \neq 0$, then $H^0(K_{\hat{E}}^*E^*) \neq 0$ by Serre duality, so following the argument in [90, 1.10], we conclude that there is an injection

$$0 \to K_{\hat{E}}^{-1} \otimes \det \hat{E} \otimes L_D \to \hat{E}$$

for some effective divisor $D$ (recall $L_D$ was the orbifold line bundle corresponding to $D$).

By the famous theorem of Narasimhan and Seshadri [64, 69] in the orbifold (or more generally parabolic) version, the bundle $\hat{E}$ equipped with the holomorphic structure provided by $\partial_C$ is stable. Thus

$$c_1 \left( K_{\hat{E}}^{-1} \otimes \det \hat{E} \otimes L_D \right) < \frac{1}{2}c_1(\hat{E})$$

At the same time, the left hand side is greater than or equal to $c_1(\hat{E}) - c_1(K_{\hat{E}})$, so we conclude that

$$\frac{1}{2}c_1(\hat{E}) < c_1(K_{\hat{E}})$$

and thus

$$c_1(\hat{E}) < 2c_1(K_{\hat{E}})$$

The converse $c_1(\hat{E}) > 2c_1(K_{\hat{E}})$ implies that $\partial_C$ is surjective follows from the same argument given by Thaddeus in [93, 1.10].

The argument given for the surjectivity of $\mathcal{D}^n_{(\mathcal{L},\alpha)}$ is indistinguishable from the one we gave in Lemma [21]. Alternatively, one can check that

$$H^1(\hat{L}^* \otimes \det \hat{E}) \simeq H^0(K_{\hat{E}} \otimes \hat{L} \otimes (\det \hat{E})^*)$$

and since $c_1(\hat{L}) < \frac{1}{2}c_1(\hat{E})$ and we are assuming $\frac{1}{2}c_1(\hat{E}) > c_1(K_{\hat{E}})$ we have

$$c_1(K_{\hat{E}} \otimes \hat{L} \otimes (\det \hat{E})^*) = c_1(K_{\hat{E}}) + c_1(\hat{L}) - c_1(\hat{E}) < c_1(K_{\hat{E}}) - \frac{1}{2}c_1(\hat{E}) < 0$$

and thus $H^1(\hat{L}^* \otimes \det \hat{E})$ vanishes, guaranteeing the surjectivity of the $\partial_{\mathcal{L}} \otimes C_{\det}$ operator.

From the stable pair perspective, the Zariski tangent space can be identified with the hypercohomology

$$H^1(\text{End}_0 \hat{E} \to \hat{T})$$

where $\text{End}_0 \hat{E} = \mathfrak{su}(2) \otimes \mathbb{C}$. The reason for this is that the deformation theory at $(\partial_{\mathcal{L}}, \mathcal{T})$ is obtained from the complex

$$0 \to \Omega^0(\hat{\Sigma}, \text{End}_0 \hat{E}) \to d_1 \Omega^{0,1}(\hat{\Sigma}, \text{End}_0 \hat{E}) \oplus \Omega^0(\hat{\Sigma}, \hat{E}) \to d_2 \Omega^{0,1}(\hat{\Sigma}, \hat{E})$$

where

$$\begin{cases} d_1 \xi = (-\partial_{\mathcal{L}} \xi, \xi \mathcal{T}) \\ d_2 (c''', \hat{T}) = (\partial_{\mathcal{L}} \hat{T} + c''' \mathcal{T}) \end{cases}$$

When $\mathcal{T} = \alpha \otimes 0$, we know that the matrices [37, Section 3.1]

$$G(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\theta} \end{pmatrix}$$

belong to the stabilizer of $(\hat{E}, \alpha)$. Thus $G(\theta)$ acts with trivial weight on the $L$ factor, and with positive weight on the $\hat{M} \equiv \hat{L}^* \otimes \det \hat{E}$ factor. Since

$$\mathfrak{su}(2) \simeq \mathbb{R} \oplus (\hat{L} \otimes \hat{M}^*)$$
After complexifying and writing an element \( A \in \mathfrak{sl}(2, \mathbb{C}) \) as

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}
\]

with

\[
\begin{align*}
& a_{11} \in \text{hom}(\hat{L}, \hat{L}) \simeq \mathbb{C} \\
& a_{12} \in \text{hom}(\hat{M}, \hat{L}) \simeq \hat{L} \otimes \hat{M}^* \\
& a_{21} \in \text{hom}(\hat{L}, \hat{M}) \simeq \hat{M} \otimes \hat{L}^*
\end{align*}
\]

we get the decomposition

\[
\text{End}_0 \hat{E} \simeq \mathbb{C} \oplus (\hat{L} \otimes \hat{M}^*) \oplus (\hat{M} \otimes \hat{L}^*)
\]

Decompose the maps \( d_1, d_2 \) as

\[
\begin{align*}
d_1 \xi &= \begin{pmatrix} -\tilde{\partial}_{11} & -\tilde{\partial}_{L \otimes \hat{M}} \cdot \xi_{12} \\ -\tilde{\partial}_{M \otimes \hat{L}} \cdot \xi_{21} & \tilde{\partial}_{12} \end{pmatrix}, & \left( \begin{array}{c} \xi_{11} \\ \xi_{21} \end{array} \right) \\
d_2 \left( \begin{array}{c} a_{11} \\ a_{21} \end{array} \right) &= \begin{pmatrix} \tilde{\partial}_{L} \hat{Y}_L + a_{11}\alpha \\ \tilde{\partial}_{M} \hat{Y}_M + a_{21}\alpha \end{pmatrix}
\end{align*}
\]

Comparing the equations, we see that for a stable pair the **tangential complex** is

\[
\Omega^0(\hat{\Sigma}, \mathbb{C}) \quad \Omega^0(\hat{\Sigma}, \mathbb{C}) \oplus \Omega^0(X, \hat{L}) \quad \Omega^0(\hat{\Sigma}, \hat{L})
\]

\[
\xi_{11} \quad \rightarrow (-\tilde{\partial}_{\xi_{11}}, \xi_{11}\alpha) \quad (a_{11}, \hat{Y}_L) \quad \rightarrow \tilde{\partial}_{L} \hat{Y}_L + a_{11}\alpha
\]

while the **normal complex** is the direct sum of the **negative weight complex**

\[
\Omega^0(\hat{\Sigma}, \hat{L} \otimes \hat{M}^*) \quad \Omega^0(\hat{\Sigma}, \hat{L} \otimes \hat{M}^*) \quad \Omega^0(\hat{\Sigma}, \hat{M})
\]

\[
\xi_{12} \quad \rightarrow (-\tilde{\partial}_{M \otimes \hat{L}} \cdot \xi_{21}, \xi_{21}\alpha) \quad (a_{21}, \hat{Y}_M) \quad \rightarrow \tilde{\partial}_{M} \hat{Y}_M + a_{21}\alpha
\]

and the **positive weight complex**

\[
\Omega^0(\hat{\Sigma}, \hat{M} \otimes \hat{L}^*) \quad \Omega^0(\hat{\Sigma}, \hat{M} \otimes \hat{L}^*) \oplus \Omega^0(X, \hat{M}) \quad \Omega^0(\hat{\Sigma}, \hat{M})
\]

\[
\xi_{21} \quad \rightarrow (-\tilde{\partial}_{\xi_{21}}, \xi_{21}\alpha) \quad (a_{21}, \hat{Y}_M) \quad \rightarrow \tilde{\partial}_{M} \hat{Y}_M + a_{21}\alpha
\]

Since a slice for the gauge group action \( G^\text{det}_C(\hat{E}) \) can be taken to be \( \ker \tilde{\partial}_C \) [H1 Section 4.1], [89], the normal bundle at \( (\hat{\Sigma}_C^\text{L}, \alpha) \) can be identified with \( \ker \tilde{\partial}_C^* \) together with the kernel of the map \( \hat{Y}_M \rightarrow \tilde{\partial}_M \hat{Y}_M + a_{21}\alpha \).

So to compare the normal bundles we complexify \( \mathcal{D}_{n_{C\text{L},\alpha}} \). This allows us to write \( \hat{c}_n \) in terms of its \((1,0)\) and \((0,1)\) types. For example, the second equation of \( \mathcal{D}_{n_{C\text{L},\alpha}} \) becomes

\[
\begin{align*}
* d_{C\text{S}} \hat{c}_n - i\alpha \hat{Y}_n^* &= -*d_{C\text{S}} \cdot * \hat{c}_n - i\alpha \hat{Y}_n^* \\
&= d_{C\text{S}}^* \cdot \hat{c}_n - i\alpha \hat{Y}_n^* \\
&= d_{C\text{S}}^* (-i\hat{c}_n + ic_n) - i\alpha \hat{Y}_n^*
\end{align*}
\]
while the first equation can be rewritten as

$$-d^*_C (c'_n + c''_n) - \alpha \tilde{Y}_n$$

So the first two equations defining $\ker \mathcal{D}_{(C, \alpha)} \otimes \mathbb{C}$ become equivalent to

$$\begin{cases}
d^*_C (c'_n - c''_n) + \alpha \tilde{Y}_n = 0 \\
d^*_C (c'_n + c''_n) + \alpha \tilde{Y}_n = 0
\end{cases}$$

which is the same as

$$\begin{cases}
d^*_C c'_n + \alpha \tilde{Y}_n = 0 \\
d^*_C c''_n = 0 \iff c'_n \in \ker \partial \mathcal{C}_N
\end{cases}$$

Since $\ker \tilde{\partial} \mathcal{C}_N$ was one of the equations defining the normal bundle for the stable pair equation, this means that the Zariski tangent space for the moduli space of $SO(3)$ vortices can be regarded as a subspace of the Zariski tangent space to the moduli space of stable pairs. Since both moduli spaces have the same dimensions, there is an isomorphism between the Zariski tangent spaces, so the moduli spaces are isomorphic as well (not just homeomorphic).

Once we know this, we can compute the dimension of the negative weight complex (24). A simple application of the index formula shows that this is the same as

$$\dim \mathbb{H}^1 (L \otimes M^* \to 0) = \dim \mathbb{H}^1 (\Sigma, \tilde{L} \otimes M^*) = -2 \chi \Sigma, \tilde{L}^2 \otimes (\det \tilde{E})^{-1}$$

where in the last step we used that $\mathbb{H}^0 (\Sigma, \tilde{L} \otimes M^*)$ vanishes. This happens since

$$\deg_B (\tilde{L} \otimes M^*)$$
$$\leq c_1 (\tilde{L} \otimes M^*)$$
$$= 2 \deg \tilde{L} - \deg \tilde{E}$$
$$< 0$$

where we used Lemma (28) (strictly speaking we need to assume that $\tilde{E}$ was chosen so that there are no abelian vortices of degree $\frac{1}{2} \deg \tilde{E}$).

To apply Riemann-Roch (10) we need to know what the isotropy of $\tilde{L}^2 \otimes (\det \tilde{E})^{-1}$ is. Recall that if the $U(2)$ bundle $\tilde{E}$ has isotropy $\left( \begin{array}{cc} \sigma^b_i & b^+_i \\ \sigma^b_i & b^-_i \end{array} \right)$, then the isotropy of the determinant line bundle is $\sigma^b_i + b^+_i$, while the isotropy of $\tilde{L}^2$ will be $\sigma^2 b_i$, where $b_i = \frac{\epsilon_i (b^+_i - b^-_i) + (b^+_i + b^-_i)}{2}$, $\epsilon_i = 1$ if $b_i = b^+_i$ and $\epsilon_i = -1$ if $b_i = b^-_i$. Therefore the isotropy of $\tilde{L}^2 \otimes (\det \tilde{E})^{-1}$ is

$$\epsilon_i (b^+_i - b^-_i) + (b^+_i + b^-_i) - (b^-_i + b^+_i) = \epsilon_i (b^+_i - b^-_i)$$

if $\epsilon_i = 1$ then this gives $b^+_i - b^-_i$ and this is already between 0 and $a_i$ so there is nothing to worry about.

If $\epsilon_i = -1$ then this gives $b^-_i - b^+_i$ which is non-positive (smaller or equal to 0 and definitely bigger than $-a_i$), so we should in fact have to consider a shift by $a_i$, i.e., the isotropy is

$$\begin{cases}
b^+_i - b^-_i & \epsilon_i = 1 \\
a_i + b^-_i - b^+_i & \epsilon_i = -1
\end{cases}$$
Therefore, (minus) the Euler characteristic according to Riemann-Roch is

\[ g - 1 + c_1(\det E) - 2c_1(\hat{L}) + \sum_{i|\epsilon_i=1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1} \frac{b_i^- - b_i^+}{a_i} \]

and thus the index is twice this amount, i.e., we found

**Theorem 35.** Suppose that a bundle \( \hat{E} \) is chosen on \( \hat{\Sigma} \) so \( c_1(\hat{E}) > 2c_1(K_{\hat{E}}) \) and that for the smooth case we are in the case \( \hat{E} \) with odd degree or when all the \( a_1, \cdots, a_n \) are co-prime we are in the case where \( \det \hat{E} \) is an odd power of \( L_0 \).

Consider the moduli space of abelian vortices associated to the reduction \( \hat{E} = \hat{L} \oplus \hat{L}^* \otimes \det \hat{E} \). Then the Morse-Bott index of \( \mu \) at this submanifold of abelian vortices is given by the formula

\[
\text{index}(\hat{E}, \hat{L}) = 2 \left[ g - 1 + c_1(\det \hat{E}) - 2c_1(\hat{L}) + \sum_{i|\epsilon_i=1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1} \frac{b_i^- - b_i^+}{a_i} \right]
\]

where \( \hat{L} \) has isotropy \( b_i \), and \( \epsilon_i = 1 \) if \( b_i = b_i^+ \), \( \epsilon_i = -1 \) if \( b_i = b_i^- \). Here \( n_- = \# \{ i \mid b_i = b_i^- < b_i^+ \} \).

**Remark 36.** From the previous formula we can observe the following:

In the smooth case the formula for the index reduces to

\[
2(g - 1 + \overbrace{\deg E - 2 \deg L}^{\geq 0})
\]

Therefore the index of the abelian vortices will always be non-negative whenever \( g \geq 1 \). If we choose \( \deg E \) odd, then the index is strictly positive whenever \( g \geq 1 \) so this means that the minima of \( \mu \) is not achieved at the abelian vortices.

Moreover, observe that when \( \deg L = 0 \), then the index gives \( 2g - 2 + 2 \deg E \), which is the dimension formula for the moduli space of stable pairs we found in [18]. This vortex moduli space thus corresponds to the critical set which is the absolute maximum for \( \mu \), and since the vortex moduli space can be identified with \( \text{Sym}^{2 \deg L} \hat{\Sigma} \), this means that the maximum occurs at a single abelian vortex (it cannot be more than one since the level sets of \( \mu \) are connected [7, Theorem IV.3.1]).

In the orbifold case when the \( a_i \) are coprime, if \( \det \hat{E} = \hat{L}_0^k \) and \( \hat{L} = \hat{L}_0 \) then the previous quantity is the same as

\[
2 \left[ g - 1 + \frac{k - 2l}{a_1 \cdots a_n} + \sum_{i|\epsilon_i=1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1} \frac{b_i^- - b_i^+}{a_i} \right]
\]

So the only chance this index has of being non-positive is if we take \( g = 0 \). This is consistent with the fact that on \( S^2 \) with marked points the moduli space of parabolic stable bundles can be empty depending on the choice of weights [69, 64].

**Example 37.** To illustrate the index calculation, consider on \( T^2 \) the \( U(2) \) bundle \( E \) with \( \deg E = 1 \). The expected dimension of the \( SO(3) \) monopole moduli space is given in equation [18] and happens to be \( 2 \) in this case. Since \( \deg E > 4g - 4 \) the moduli space of stable pairs is smooth [90].

The abelian moduli spaces can only appear when \( \deg L \leq \frac{1}{2} \deg E = \frac{1}{2} \), so \( \deg L = 0 \) and hence there is a unique solution up to gauge (because the abelian vortex can be interpreted as a
holomorphic function and must thus be constant). The index of this abelian monopole is 2, so it corresponds to the absolute maximum. Since the level sets of \( \mu \) are connected \([7, \text{Theorem IV.3.1}]\), there is only a minimum, which must be a flat connection. After we take the quotient by the \( S^1 \) action, this gives an interpretation of the moduli space of stable pairs as a cobordism between the abelian vortex and the flat connections.

6. \( SO(3) \) Monopoles on 3-Manifolds

We now review the \( SO(3) \) monopole equations on 3-manifolds, which can be regarded as the dimensional reduction of the \( SO(3) \) monopole equations on 4-manifolds introduced by Pidstrigach and Tyurin and studied extensively by Feehan and Leness \([35, 33, 37, 36, 38, 39]\). In particular, we will follow mostly their conventions \((\text{see [37, Section 2.1.2]}))\).

In the same way that the Seiberg-Witten equations require a choice of spin-c structure in order to be defined on a 3 or 4-manifold, the \( SO(3) \) monopole equations require the choice of a spin-\( u \) structure \( t \). The most expedient way to define them on 3-manifolds \( Y \) is as follows.

Consider a spin-c structure \( s = (S, \rho) \) which is represented by a spinor bundle \( S \) with corresponding Clifford multiplication map \( \rho \). Let \( E \) denote a \( U(2) \) bundle on \( Y \). Then \( (S, \rho) \) and \( E \) determine the spin-\( u \) structure

\[
t = (V, \rho_V) = (S \otimes E, \rho \otimes 1_E)
\]

By abuse of notation we will write \( \rho \) instead of \( \rho_V \). The above construction shows that spin-\( u \) structures always exist on a 3-manifold, so the next question is the topological classification of these. Notice first of all that if \( L \) is an arbitrary complex line bundle then

\[
S \otimes E \simeq (S \otimes L) \otimes (E \otimes L^{-1})
\]

and since \( s_L = (S \otimes L, \rho \otimes 1_L) \) represents another spin-c structure, the decomposition of \( V \) as \( S \otimes E \) is not intrinsic. Since

\[
\begin{align*}
c_1(s) + c_1(E) &= c_1(s_L) + c_1(E \otimes L^{-1}) \\
\omega_2(\text{su}(E)) &= \omega_2(\text{su}(E \otimes L^{-1}))
\end{align*}
\]

the three-dimensional version of the discussion in \([37, \text{Section 2.1.3]}\) shows that the spin-\( u \) structure \( t \) determines the characteristic classes

\[
\begin{align*}
c_1(t) &\equiv c_1(s) + c_1(E) \\
\omega_2(t) &\equiv \omega_2(\text{su}(E))
\end{align*}
\]

On 3-manifolds \( \omega_2(\text{su}(E)) \) can always be lifted to an integral cohomology class because of the Bockstein long exact sequence and the fact that \( H^2(Y; \mathbb{Z}) \simeq \mathbb{Z} \) has no two-torsion, so any pair \((c, \omega) \in H^2(Y; \mathbb{Z}) \times H^2(Y; \mathbb{Z}_2)\) determines a spin-\( u \) structure.

If we think of the \( SO(3) \) monopole equations as a system of equations used to relate the monopole and instanton Floer homologies on a specific 3-manifold \([29]\), we should regard \( \omega_2(t) \) as being the characteristic class provided the \( SO(3) \) bundle used to define the instanton Floer homology of that 3-manifold.

Since every \( SO(3) \) bundle can be lifted to a \( U(2) \) bundle on a 3-manifold, there is no harm in choosing a reference spin-c structure to be a torsion spin-c structure, i.e., one for which \( c_1(s) = 0 \), so that we can think of the spin-\( u \) structure \( t \) as being completely specified by a choice of a \( U(2) \) bundle and nothing else.

Choosing the reference spin-c structure \( s \) to be torsion also has the added advantage that there is a \textit{canonical} lift of the Levi-Civita connection on \( TY \) to a spin-c connection on \( \hat{S} \) \([61, \text{Section} \dots]\).
5.1. With this reference spin-c connection, we can define the configuration space where the $SO(3)$ monopole equations are defined as

$$C(Y, t) = A_{\text{det}}(E) \times \Gamma(S \otimes E)$$

where the notation $A_{\text{det}}(E)$ indicates that we are considering the space of unitary connections $B$ on $E$ which induce a fixed connection $B^{\text{det}}$ on $E$. The unperturbed $SO(3)$ monopole equations are then equations for a pair $(B, \Psi)$ which satisfy the equations:

$$\begin{align*}
\text{(26)} & \quad \text{SO}(3) \text{ monopole equations} \\
& \quad \begin{cases} 
*F_B^0 + 2\rho^{-1}(\Psi \Psi^*)_{00} = 0 \\
D_B \Psi = 0
\end{cases}
\end{align*}$$

Here $(\Phi \Phi^*)_{00}$ represents the component of $\Phi \Phi^* \in \Gamma(S \otimes E \otimes S^* \otimes E^*) \simeq \Gamma(\text{End}S \otimes \text{End}E)$ which is traceless on each of the factors of $\text{End}(S) \otimes \text{End}(E)$, that is, $(\Phi \Phi^*)_{00}$ is a section of $\text{End}_0(S) \otimes \text{End}_0(E)$. We are using $\rho : TY \to \text{hom}(S, S)$ to identify $TY$ isometrically with the sub-bundle $su(S)$ of traceless, skew-adjoint endomorphisms equipped with the inner product $\frac{i}{2}\text{tr}(a^*b)$.

As usual, $*$ denotes the Hodge star operator on $Y$. Finally, $D_B$ represents the twisted Dirac operator acting on sections of $S \otimes E$, which strictly speaking should be written as $D_{B_S \otimes B}$, where $B_S$ denotes the canonical spin-c connection we are using on $S$, coming from the fact that we are using a torsion spin-c structure for describing $t$.

One can first consider the solutions to the $SO(3)$ monopole equations on $Y$ modulo the gauge group $G_{\text{det}}(E)$. We will denote such equivalence classes as pairs $[B, \Psi]$. In the space

$$B(Y, t) = C(Y, t)/G_{\text{det}}(E)$$

there is still a residual $S^1$ action given by

$$e^{i\theta} \cdot [B, \Psi] \to [B, e^{i\theta} \Psi]$$

**Lemma 38.** The fixed points of this circle action which solve (26) are of two types:

a) The pairs $[B, 0]$ with vanishing spinor, which correspond to the projectively flat connections.

b) The pairs of the form $[B_L \oplus (B^{\text{det}} \otimes B^*_L), \Psi = \psi \oplus 0]$ corresponding to a reduction of the bundle $E$ into two line bundles $E = L \oplus (\text{det } E \otimes L^*)$, which solve the perturbed Seiberg-Witten equations

$$\begin{align*}
\text{(27)} & \quad \text{perturbed Seiberg Witten equations} \\
& \quad \begin{cases} 
*F_{B_L} + \rho^{-1}(\psi \psi^*)_{0} - \frac{1}{2} * F_{B^{\text{det}}} = 0 \\
D_B \psi = 0
\end{cases}
\end{align*}$$

associated to the spin-c structure $S \otimes L$.

**Proof.** This is basically a restatement of [87] Lemma 3.12, with some simplifications in the case of a 3-manifold, given the choice of our reference spin-c structure. The only thing we really need to check is our choice of conventions for the constants. We have chosen them so that they agree with [55] Eq 4.4. At a reducible connection $B = B_L \oplus (B^{\text{det}} \otimes B^*_L)$, the traceless part of the curvature $F_B^0$ can be written as

$$\begin{pmatrix}
F_{B_L} & 0 \\
0 & \frac{1}{2} F_{B^{\text{det}}} - F_{B_L}
\end{pmatrix}$$

so the curvature equation becomes

$$*F_{B_L} - \frac{1}{2} * F_{B^{\text{det}}} + \rho^{-1}(\psi \psi^*)_{0} = 0$$
We used the fact that for $\Psi = \psi \otimes 0$, we have [37, Eq. 3.13]

$$(\Psi \Psi^*)_{00} = (\psi \psi^*)_0 \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Notice that $\det(s_L) = \det(S \otimes L) \approx L^{\otimes 2}$, so in the notation of Kronheimer and Mrowka, $B^t \equiv B_L^{\otimes 2}$ is the induced connection on $\det(s_L)$. In particular, $F_{B^t} = 2F_{B_L}$ and so the previous equation can be rewritten as

$$\frac{1}{2} \ast F_{B^t} + \rho^{-1}(\psi \psi^*)_0 = \frac{1}{2} \ast F_{B^{\text{Int}}},$$

This agrees with the conventions used in [55, Eq 4.11].

It is very important to notice that when the Seiberg-Witten monopoles appear as fixed points of the $S^1$ action, the equations they solve are “automatically” perturbed by the term $F_{B^{\text{Int}}}$. These perturbations can have dramatic consequences, the first one being the following:

**Lemma 39.** Suppose that $(B_L, \psi)$ solves the perturbed Seiberg-Witten equations [27], where $B_L \in A(L)$ is a $U(1)$ connection, and $\psi \in \Gamma(S \otimes L)$. Then $\psi$ vanishes identically if and only if $c_1(E) \in H^2(Y; \mathbb{Z})$ is an even class, i.e., $w_2(\mathfrak{su}(E))$ vanishes. In other words, if $c_1(E)$ is an odd class, then there are no “reducible” solutions to the equations [27].

**Proof.** Observe that a solution of the form $(B_L, 0)$ must solve the equation

$$F_{B^{\text{Int}}} = 2F_{B_L}$$

and by Chern-Weil theory we conclude that

$$c_1(E) = 2c_1(L)$$

from which the assertion in the lemma follows immediately.

**Remark 40.** Another consequence of these perturbations is that in general the symmetry between the solutions to the Seiberg-Witten equations for $\mathfrak{s}$ and its conjugate spin-c structure $\bar{\mathfrak{s}}$ is lost. This is analogous to a comment Donaldson makes in [28, Section 6].

In fact, if define

$$\omega \equiv \frac{1}{4} F_{B^{\text{Int}}}$$

then $\omega$ is a harmonic representative of $-\frac{i}{2}F_{B^{\text{Int}}} \in H^2(Y; i\mathbb{R})$ and using $F_{B^t} = 2F_{B_L}$, the equations [27] can be rewritten as [55, Eq. 29.2]

$$\begin{cases}
\frac{1}{2} \rho(F_{B^t} - 4\omega) - (\psi \psi^*)_0 = 0 \\
D_B \psi = 0
\end{cases}$$

(28)

In particular, if $c_1(E)$ is an odd class, then the term $\omega$ corresponds to a non-balanced perturbation, in the terminology of Kronheimer and Mrowka [55, Definition 29.1.1]. A consequence of working with a non-balanced perturbation is that the definition of the monopole Floer homology groups $HM(Y, s, \omega)$ require the choice of a local coefficient system in order to make sense of the differential, since in general there are monotonicity issues when defining these groups. Some choices of local coefficient systems are described in [55, Section 30.2].

It remains a daunting challenge to try to define a suitable version of $SO(3)$ monopole Floer homology, because of the presence of reducibles (i.e., abelian $U(1)$ monopoles and flat connections).

As a historical analogy, before the construction of monopole Floer homology in full generality by Kronheimer and Mrowka [55], the simplest case where it could be defined was for three manifolds...
with \( b_1(Y) > 0 \) and for non-torsion spin-c structures, since in this case reducibles could be avoided by using suitable perturbations. The difficult case was that of torsion spin-c structures, which was solved thanks to the blow-up model introduced by Kronheimer and Mrowka.

Likewise, we can think of defining \( SO(3) \) monopole Floer homology as trying to define monopole Floer homology in the torsion case, which begs the question of whether there is an analogue of the non-torsion case where reducibles can be avoided.

Such an analogue may be provided by a construction of an \( U(2) \) monopole Floer homology. We explain the basic ideas behind this construction: assuming all the technical hurdles can be solved, it can be regarded as giving rise to a family of Floer homologies, indexed by subintervals of the real line. So in a sense this would be the Floer-theoretic analogue of the typical wall-crossing phenomenon for the various moduli spaces which appear in Algebraic Geometry.

In this case we no longer fix a reference connection on \( \det E \), so we are working with the enlarged configuration space

\[
C_{U(2)}(Y, t) = \mathcal{A}(E) \times \Gamma(S \otimes E)
\]

Suppose that \( \omega \) is a harmonic representative of \( c_1(E) \) [before it was chosen to be a representative of \( -\frac{1}{2} \pi c_1(E) \)].

The \( U(2) \) monopole equations \( SW_{U(2)}(Y, t, \tau) \) are equations for a pair \((B, \Psi)\) which satisfy

\[ U(2) \text{ monopole equations } SW_{U(2)}(Y, t, \tau) \begin{cases} *F_B + 2\rho^{-1}(\Psi \Psi^*)_0 + i\tau(*\omega)I_E = 0 \\ D_B \Psi = 0 \end{cases} \]

The main differences with the \( SO(3) \) monopole equations \( [20] \) are the following:

1. As mentioned before, we are no longer fixing a reference connection on \( \det E \).
2. Since we dropped the reference connection, we now need an equation which involves the entire curvature \( F_B \) of \( B \), instead of just its traceless part. Subsequently, \((\Psi \Psi^*)_0 \) denotes the component of \( \Psi \Psi^* \) which is traceless only on the “spinorial” component, that is, \((\Psi \Psi^*)_0 \in \Gamma(\text{End}_0S \otimes \text{End}E)\).
3. Notice that the traceless part of the curvature equation in \( [29] \) gives us back the \( SO(3) \) monopole equations \( [20] \).

It is not difficult to identify the “critical values” of \( \tau \), that is, those values where the flat connections and Seiberg-Witten solutions can appear.

**Lemma 41.** a) Suppose that \((B, \Psi = 0)\) solves the \( U(2) \) monopole equations \( [29] \). Then \( \tau = \pi \) and \( B \) is a flat \( U(2) \) connection. If we assume \( c_1(E) \) is odd, then \( B \) is irreducible, i.e., not compatible with a decomposition \( E = L_1 \oplus L_2, B = B_1 \oplus B_2 \).

b) Suppose that \((B = B_1 \oplus B_2, \Psi = \psi \oplus 0)\) solves the equation \( [29] \), where \( E = L_1 \oplus L_2 \) is compatible with the reduction of \( B \). Then \( \tau = 2\pi n \) for an arbitrary integer, and \((B_1, \psi)\) solves the perturbed Seiberg-Witten equations \( [24] \) for the spin-c structure \( S \otimes L_1 \) with perturbation term \( F_{B_2} \). Moreover, \( c_1(L_1) \) and \( c_1(L_2) \) are both integer multiples of \( c_1(E) \), i.e., their Chern classes belong to the ray spanned by \( c_1(E) \) inside the integer lattice \( H^2(Y; \mathbb{Z}) \).

**Proof.** a) This is a Chern-Weil argument once again. Taking traces we find that \( \text{tr}(F_B) = -2i\tau \omega \) and therefore \( \tau = \pi \). If \( B \) were compatible with a reduction then

\[
F_B = \begin{pmatrix} F_{B_1} & 0 \\ 0 & F_{B_2} \end{pmatrix} = -i\tau \omega I_E
\]

we would conclude that \( F_{B_1} = F_{B_2} \). Therefore \( L_1 \simeq L_2 \) and \( c_1(E) \) must represent an even class, which cannot be true by assumption.
b) In this case the equations become

\[
\begin{aligned}
*F_{B_1} + 2\rho^{-1}(\psi\psi^*)_0 + i\tau(*\omega) &= 0 \\
F_{B_2} &= -i\tau\omega \\
D_{B_1}\psi &= 0
\end{aligned}
\]

Substituting the second equation into the first one we recognize the perturbed Seiberg Witten equation \[\text{[28]}\]

\[
*F_{B_1} + 2\rho^{-1}(\psi\psi^*)_0 - *F_{B_2} = 0
\]

By Chern-Weil theory the second equation says that \(-2\pi ic_1(L_2) = -i\tau c_1(E)\), so \(\tau\) is of the form \(\tau = 2\pi n\), for \(n \in \mathbb{Z}\) an integer. Since

\[
c_1(E) = c_1(L_1 \oplus L_2) = c_1(L_1) + c_1(L_2) = c_1(L_1) + nc_1(E)
\]

this forces \(c_1(L_1)\) to be

\[
c_1(L_1) = (1 - n)c_1(E)
\]

\[\square\]

Remark 42. a) Notice the second part of the previous lemma implies that the only monopole Floer groups that the \(U(2)\) monopole Floer equations can “see” are those whose spin-c structure satisfy \(c_1(\gamma) = c_1(S \otimes L_1) = 2(1 - n)c_1(E)\), for \(n\) an arbitrary integer. In particular, notice that all spin-c structures which are torsion can appear in this picture. Also, in the specific case that \(Y\) is an integer homology \(S^1 \times S^2\), and we choose \(c_1(E)\) as a generator of \(H^2(Y; \mathbb{Z}) \cong \mathbb{Z}\), then all spin-c structures can arise in this case as well.

b) Implicitly we are also saying that solutions to equation \[\text{[29]}\] of the form \((B = B_1 \oplus B_2, \Phi = \Psi_1 \oplus \Psi_2), E = L_1 \oplus L_2\), do not arise unless one of the spinors \(\Psi_1\) or \(\Psi_2\) vanish identically. An easy way to see that otherwise by taking the traceless part of the curvature equation, we would obtain a pair \((B = B_1 \oplus B_2, \Phi = \Psi_1 \oplus \Psi_2)\) which satisfies the \(SO(3)\) monopole equations \[\text{[29]}\], where none of the spinors is identically zero. However, \[\text{[55]}\] Lemma 5.22 shows that this cannot be the case. A quick look at the statement of Feehan and Leness shows that they stated this already for a perturbed version of the \(SO(3)\) monopole equations, which had to satisfy specific properties so that this would not occur. Therefore, this already shows one needs to be very careful about which properties will continue to hold after adding (holonomy) perturbations.

Having introduced the \(U(2)\) monopole equations, it makes sense to state the following conjecture:

Conjecture 43. Suppose that \(Y\) is a closed oriented 3-manifold, and \(E\) is an \(U(2)\) bundle which satisfies the admissibility condition that \(w = c_1(E) \in H^2(Y; \mathbb{Z})\) is a non-torsion odd class. Choose a spin-c structure \(\gamma\) and consider the bundle \(V = S \otimes E\) representing the spin-u structure \(t\), where \((S, \rho)\) is a spinor bundle representing \(\gamma\).

Then for \(\tau \neq \pi, 2\pi n\) for \(n \in \mathbb{Z}\), it is possible to define a Seiberg-Witten \(U(2)\) monopole Floer homology \(HM_{U(2)}(Y, t, \tau)\). Some expected features of these groups are the following:

a) The chain complex of \(HM_{U(2)}(Y, t, \tau)\) will be built out of a suitably perturbed version of the equations \[\text{[29]}\]. These solutions can be identified with the critical points of a perturbed \(U(2)\) Chern-Simons-Dirac functional \(CS_{t, \tau}\), morally a Morse function on the space \(B(Y, t) = (A(E) \times \Gamma(V)) / G(E)\).

b) The differential involves the count of gradient flow lines mod \(\mathbb{R}\)-translations of \(CS_{t, \tau}\) as in every other Floer homology. In general, there will be monotonicity issues [a bound on the dimension of flow lines will not imply a uniform energy bound on the corresponding moduli spaces], so the
groups $HM_{U}(Y,t,\tau)$ will require a Novikov ring (local coefficient system) so that the differential is well defined.

c) The groups will be $\mathbb{Z}/2\mathbb{Z}$ graded in general.

d) Write $\mathbb{R}\setminus\{\pi, 2\pi n \mid n \in \mathbb{Z}\} = \bigcup I_{i}$ as a disjoint union of open intervals. If $\tau, \tau'$ belong to the same “chamber” $I_{i}$, then $HM_{U}(Y,t,\tau) \simeq HM_{U}(Y,t,\tau')$, i.e, the groups are isomorphic to each other as long as one does not cross a “wall”.

Clearly behind this conjecture there is a great deal of work that will require verification, but it could serve as a toy model before a general construction of $SO(3)$ monopole Floer homology, which would prove even more difficult in general.

7. $SO(3)$ Monopoles on $S^{1} \times \Sigma$: Framed Monopole Homology

Now we want to understand the solutions to the $SO(3)$ monopole equations on $Y = S^{1} \times \Sigma$ in terms of the solutions to the $SO(3)$ vortex equations on $\Sigma$.

Before doing this, it is useful to understand how the spin-c structures on $Y$ are related to the spin-c structures on $\Sigma$. More details can be found in [68 Section 3]. Throughout our discussion we will use a product metric on $S^{1} \times \Sigma$.

Suppose $s_{Y}$ is a spin-c structure on $Y$. Since $H^{2}(S^{1} \times \Sigma; \mathbb{Z})$ has no 2-torsion, we can specify $s_{Y}$ by its element $c_{1}(s_{Y}) \in H^{2}(S^{1} \times \Sigma; \mathbb{Z})$. Let $S_{Y}$ denote a spinor bundle representing $s_{Y}$ and $L_{Y} = \det S_{Y}$ the corresponding line bundle.

If we think of $S^{1} \times \Sigma$ as $[0, 1] \times \Sigma$ with the boundaries identified, then $L_{Y}$ is constructed as the pullback under the projection $[0, 1] \times \Sigma \to \Sigma$ of a line bundle $L_{\Sigma}$ by gluing along the boundaries with an isomorphism $u \in \text{Map}(\Sigma, S^{1})$. Then

$$c_{1}(s_{Y}) = c_{1}(L_{Y}) = c_{1}(L_{\Sigma}) + [S^{1}] \cup [u]$$

where $[u]$ is the class of $u$ in $[\Sigma; S^{1}] \simeq H^{1}(\Sigma; \mathbb{Z})$. The spin-c structure $s_{Y}$ induces a spin-c structure $s_{\Sigma}$ with determinant line bundle $L_{\Sigma}$. If $S_{\Sigma}$ is a spinor bundle for $s_{\Sigma}$, we can assume that it is of the form

$$S_{\Sigma} = (\mathbb{C} \oplus K_{\Sigma}^{-1}) \otimes (L_{\Sigma} \otimes K_{\Sigma})^{1/2}$$

where $K_{\Sigma} = \Omega^{1,0}(\Sigma)$ is the canonical bundle of $\Sigma$, and $(L_{\Sigma} \otimes K_{\Sigma})^{1/2}$ denotes a line bundle whose square is the line bundle $L_{\Sigma} \otimes K_{\Sigma}$. In particular, if $L_{\Sigma} \simeq \mathbb{C}$ is the trivial line bundle, then we obtain the canonical spin-c structure on $\Sigma$

$$S_{\text{can}} \simeq K_{\Sigma}^{1/2} \oplus K_{\Sigma}^{-1/2}$$

Notice that an implicit choice of spin structure on $\Sigma$ was made so that we could find a square root of the canonical bundle, so we will assume a fixed choice once and for all. Moreover, if $[u] = 0$, then $c_{1}(L_{Y}) = 0$ which means that we can regard the torsion spin-c structure on $Y$ as being of the form

$$S_{\text{tor}} = K_{\Sigma}^{1/2} \oplus K_{\Sigma}^{-1/2}$$

Now suppose that $E$ is a $U(2)$ bundle on $\Sigma$. We can then consider the spin-u structure

$$V = (K_{\Sigma}^{1/2} \oplus K_{\Sigma}^{-1/2}) \otimes E = (\mathbb{C} \oplus K_{\Sigma}^{-1}) \otimes (K_{\Sigma}^{1/2} \otimes E)$$
Therefore, it is more elegant to reabsorb $K^{1/2}_\Sigma$ into the bundle $E$ we are considering and work with the canonical spin-$u$ structure.

$$V_{can} = (\mathbb{C} \oplus K^{-1}_\Sigma) \otimes E$$

It suffices to define the Clifford map $\rho$ on $S = \mathbb{C} \oplus K^{-1}_\Sigma$. First of all, notice that

$$\rho(d\theta)^2 = -|d\theta|^2 g_\Sigma^{-1} S = -1_S$$

so $\rho(d\theta)$ is an endomorphism of $S$ which squares to $-1$. We will thus identify $\rho(d\theta)$ with the matrix

$$\rho(d\theta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Decompose a one form $\eta \in T^*(S^1 \times \Sigma)$ as

$$\eta = t d\theta + \varpi$$

where $\varpi$ is identified with the pullback of a $(0, 1)$ form on $\Sigma$. Notice that our conventions are such that

$$|\eta|_{g^\varpi} = t^2 + 2|\varpi|^2_{g^\Sigma}$$

and that we are regarding $\Omega^{0,1}(\Sigma)$ only as a real vector space, so that we have the isomorphism $\Omega^{0,1}(\Sigma) \simeq \Omega^1(\Sigma)$.

Finally, we define for $(\alpha, \beta) \in \mathbb{C} \oplus \pi^* K^{-1}_\Sigma$ the Clifford map

$$\rho(\eta) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} it\alpha - \sqrt{2} \langle \beta, \varpi \rangle \\ \sqrt{2} \alpha \varpi - it\beta \end{pmatrix}$$

Notice that $K^{-1}_\Sigma$ is being interpreted as a complex vector space, and that $\langle \beta, \varpi \rangle$ refers to the hermitian inner product on $\Omega^{0,1}(\Sigma)$, complex linear in the first entry. Slightly abusing notations we can write this as

$$\rho(\eta) = \begin{pmatrix} i \langle d\theta, \eta \rangle & -\sqrt{2} \varpi^* \\ \sqrt{2} \varpi & -i \langle d\theta, \eta \rangle \end{pmatrix}$$

It is also useful to have local expressions for these formulas. Consider a coframe $e^1, e^2$ of $T^* \Sigma$ and define

$$\begin{cases} \epsilon = \frac{1}{\sqrt{2}} (e^1 + ie^2) \\ \bar{\epsilon} = \frac{1}{\sqrt{2}} (e^1 - ie^2) \end{cases}$$

Under an almost complex structure $J$ of $\Sigma$, we have that $J(e^1)$ is mapped to $e^2$, thus

$$J(\epsilon) = \frac{1}{\sqrt{2}} (e^2 - ie^1) = -i \sqrt{2} \epsilon$$

In other words, $\epsilon$ is of type $(1, 0)$. Likewise, $\bar{\epsilon}$ is of type $(0, 1)$ since $J(\bar{\epsilon}) = i\bar{\epsilon}$. It is an easy check that

$$\rho(\epsilon)(\alpha, \beta) = (-\sqrt{2} \langle \beta, \bar{\epsilon} \rangle, 0) \iff \begin{pmatrix} 0 & -\sqrt{2} \bar{\epsilon} \\ 0 & 0 \end{pmatrix}$$

$$\rho(\bar{\epsilon})(\alpha, \beta) = (0, \sqrt{2} \epsilon \alpha) \iff \begin{pmatrix} 0 & 0 \\ \sqrt{2} \epsilon & 0 \end{pmatrix}$$

1\text{Notice that } \mathbb{C} \oplus K^{-1}_\Sigma \text{ is not a torsion spin-c structure in general, but this does not make a big difference for our arguments, since } \text{ad}(K^{1/2}_\Sigma \otimes E) \simeq \text{ad}(E). \text{ Moreover, the Chern connection is being implicitly used on } K^{-1}_\Sigma, \text{ and the trivial connection on the } \mathbb{C} \text{ factor.}
Following the discussion in [26, Section 3.2], we can write a connection $B$ on $S^1 \times \Sigma$ in the form

$$B = C(\theta) + c(\theta)d\theta$$

where $C(\theta)$ is a one parameter periodic family of $U(2)$ connections on $\Sigma$ and $c(\theta) \in \Omega^0(\Sigma, \mathfrak{u}(2))$ is also another periodic family. The curvature becomes

$$(30) \quad F_B = F_{C(\theta)} + d\theta \wedge \left( \frac{\partial C(\theta)}{\partial \theta} + d_{C(\theta)c(\theta)} \right)$$

Moreover, the Dirac operator can be written as [compare with [26, Lemma 3.5], [67, Eq 5.12], [62, Section 3.1]]

$$D_B = \left( \begin{array}{cc} i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) & \sqrt{2}\partial^*_{C(\theta)} \\ \sqrt{2}\partial_{C(\theta)} & -i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) \end{array} \right)$$

Therefore, the Dirac equation for $(\alpha, \beta)$ reads

$$\begin{cases} 
\left( i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) \alpha + \sqrt{2}\partial^*_{C(\theta)} \beta = 0 \\
\sqrt{2}\partial_{C(\theta)} \alpha - i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) \beta = 0
\end{cases}$$

where $\alpha \in \Omega^0(E)$ and $\beta \in \Omega^{0,1}(E)$.

In order to understand the curvature equation, decompose the curvature in terms of a co-frame as

$$F_B^0 = F_{\theta,e^1}d\theta \wedge e^1 + F_{\theta,e^2}d\theta \wedge e^2 + F_{e^1,e^2}e^1 \wedge e^2$$

$$*F_B^0 = F_{\theta,e^1}e^2 - F_{\theta,e^2}e^1 + F_{e^1,e^2}d\theta$$

Since $\frac{e^1}{\sqrt{2}} = e^1$, $\frac{e^2}{\sqrt{2}} = e^2$ then $\rho(e^1) = \left( \begin{array}{cc} 0 & -\epsilon \\
\epsilon & 0 \end{array} \right)$, $\rho(e^2) = \left( \begin{array}{cc} 0 & i\epsilon \\
i\epsilon & 0 \end{array} \right)$

$$\rho(*F_B^0) = \left( \begin{array}{cc} iF_{\theta,e^1}e^2 & (iF_{\theta,e^1} + F_{\theta,e^2})e \\
(iF_{\theta,e^1} - F_{\theta,e^2})\epsilon & -iF_{e^1,e^2}\epsilon \end{array} \right)$$

Thus the curvature equation reads (compare with [31, p.10])

$$\begin{cases} 
iF_{e^1,e^2} + (\alpha \otimes \alpha^*)_0 - (A\beta \otimes \beta^*)_0 = 0 \\
(iF_{\theta,e^1} - F_{\theta,e^2})\epsilon + 2(\beta \otimes \alpha^*)_0 = 0
\end{cases}$$

The expression $(A\beta \otimes \beta^*)_0$ denotes that a contraction with the symplectic form on $\Sigma$ is implicit.

**Theorem 44.** Suppose $(B, (\alpha, \beta))$ satisfies the SO(3) vortex equations on $S^1 \times \Sigma$, for the spin-u structure $V_{can} = (\mathbb{C} \oplus K^{-1}) \otimes E$. Here $\alpha \in \Omega^0(E)$ and $\beta \in \Omega^{0,1}(E)$. Write $B = C(\theta) + c(\theta)d\theta$. Then

$$\begin{align*}
i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) \alpha + \sqrt{2}\partial^*_{C(\theta)} \beta &= 0 \\
\sqrt{2}\partial_{C(\theta)} \alpha - i \left( \frac{\partial}{\partial \theta} + c(\theta) \right) \beta &= 0 \\
iF_{e^1,e^2} + (\alpha \otimes \alpha^*)_0 - (A\beta \otimes \beta^*)_0 &= 0 \\
(iF_{\theta,e^1} - F_{\theta,e^2})\epsilon + 2(\beta \otimes \alpha^*)_0 &= 0
\end{align*}$$

In fact, all solutions are of the form $(B, (\alpha, 0))$ or $(B, (0, \beta))$. 
In the first case, we get a solution of the form
\[ \partial_C \alpha = 0 \]
\[ F_C - i(\alpha \otimes \alpha^*)_0 = 0 \]
which can be identified with an \( \text{SO}(3) \) vortex on the bundle \( E \).

In the second case, we get a solution of the form
\[ \partial^*_C \beta = 0 \]
\[ F_C + i(\Lambda \beta \otimes \beta^*)_0 = 0 \]
which can be identified via Serre duality with an \( \text{SO}(3) \) vortex on the bundle \( K_S \otimes E^{-1} \).

In particular, if we assume that \( \deg E > 2 \deg K_S \), then this second type of moduli space of \( \text{SO}(3) \) vortices is empty.

Remark 45. The previous theorem states that on \( S^1 \times \Sigma \) the solutions to the \( \text{SO}(3) \) monopole equations decompose into the union of two moduli spaces of \( \text{SO}(3) \) vortices on the Riemann surface. This is the analogue of what happens for the solutions to the \( \text{SO}(3) \) monopole equations on a Kahler surface, where the moduli space also decomposes into the union of two moduli spaces of stable pairs on the Kahler surface (see [63, Theorem 6.3.10] and [31, Section 1.2]).

Proof. We follow the proof in [68, Proposition 3.1], [26, Proposition 3.6], which applies with cosmetic changes to our situation.

Suppose that \((B, (\alpha, \beta))\) is irreducible in that \( B \) is irreducible and \( (\alpha, \beta) \) non-vanishing. This forces the the stabilizer of this \( \text{SO}(3) \) monopole to be \( \text{Id} \) under the determinant one gauge group. Moreover, one can use a gauge transformation to assume the connection is in “temporal” gauge, which means that \( B = C(\theta) \) instead of the more general case \( B = C(\theta) + c(\theta) d\theta \). So we can assume that \( c(\theta) = 0 \) in the equations (31)

Since \( i\partial^*_C(\theta) = \Lambda \partial_C(\theta) \) on \( (0, 1) \) forms, the first equation (31) can be written as
\[ -\frac{\partial}{\partial \theta} \alpha + \sqrt{2} \Lambda \partial_C(\theta) \beta = 0 \]

Differentiate this with respect to \( \theta \) to find
\[ 0 = -\frac{\partial^2}{\partial \theta^2} \alpha + \sqrt{2} \left( \frac{\partial}{\partial \theta} \Lambda \partial_C(\theta) \right) \beta + \sqrt{2} \Lambda \partial_C(\theta) \frac{\partial \beta}{\partial \theta} \]

Because of the second equation in (31), \( \frac{\partial}{\partial \theta} \beta = -\sqrt{2} i \partial_C(\theta) \alpha \), so we obtain
\[ 0 = -\frac{\partial^2}{\partial \theta^2} \alpha + \sqrt{2} \left( \frac{\partial}{\partial \theta} \Lambda \partial_C(\theta) \right) \beta - 2i \Lambda \partial_C(\theta) \underbrace{\partial_C(\theta) \alpha}_{(0,1) \text{ form}} \]

Using the Kahler identity \( i\partial^*_C(\theta) = \Lambda \partial_C(\theta) \) one more time we end up with
\[ 0 = -\frac{\partial^2}{\partial \theta^2} \alpha + \sqrt{2} \left( \frac{\partial}{\partial \theta} \Lambda \partial_C(\theta) \right) \beta + 2 \partial_C(\theta) \partial_C(\theta) \alpha \]

Take inner product with \( \alpha \), and use integration by parts and the third equation in (31) to conclude
\[ 0 = -\int_{\Sigma} \langle \frac{\partial^2}{\partial \theta^2} \alpha, \alpha \rangle + \sqrt{2} \int_{\Sigma} \langle (\alpha \otimes \beta^*)_0 \beta, \alpha \rangle + 2 \int_{\Sigma} |\partial_C(\theta) \alpha|^2 \]
The integrand of the term in the middle is
\[ \left( |\beta|^2 |\alpha| - \frac{1}{2} (\langle \beta, \alpha \rangle)_{\tilde{E}} \right) = |\beta|^2 |\alpha| - \frac{1}{2} < \beta, \alpha >_{\tilde{E}} \]
Integrate from 0 to 1 to obtain (recall that we are thinking of $S^1 \times \Sigma$ as $[0, 1] \times \Sigma$ with the ends identified)
\[ \int_0^1 \left( |\beta|^2 |\alpha| - \frac{1}{2} < \beta, \alpha >_{\tilde{E}} \right) = 0 \]
So $\alpha$ is $\theta$ independent and by Cauchy Schwarz since $| < \beta, \alpha >_{\tilde{E}} |^2 \leq |\beta|^2 |\alpha|^2$ we conclude that either $\alpha$ or $\beta$ vanishes identically. Because of the curvature decomposition (30) we also find that $\frac{\partial C}{\partial \theta} = 0$ thus all the data pulls back from the surface $\Sigma$, and the claim of the theorem is now straightforward.

For the last claim, suppose that $\deg E > 2K_{\Sigma}$. Then $\deg(K_\Sigma \otimes E^{-1}) < 0$ so if $(C, \beta)$ denotes an $SO(3)$ vortex on the bundle $K_\Sigma \otimes E^{-1}$, we have $c_1(L) < \frac{1}{4} \deg(K_\Sigma \otimes E^{-1}) < 0$ for any holomorphic sub-line bundle of $E$ for which $\beta \in H^0(L)$ and this is clearly impossible. Thus the moduli space of $SO(3)$ vortices must be empty.

Now we specialize to the case that $\Sigma = T^2$. The canonical bundle of an elliptic curve is trivial, and choose the $U(2)$ bundle over $T^2$ with $\deg E = 1$. Now we discuss the solutions to the $SO(3)$ monopole equations $(B, \Psi)$ on $S^1 \times T^2$.

- For the irreducible $SO(3)$ monopoles, where $B$ is an irreducible $SO(3)$ connection, and $\Psi$ non-vanishing, by the previous theorem (11), $\Psi = \alpha \oplus 0$ or $\Psi = 0 \oplus \beta$, and $B$ pulls back from a connection $C$ on $T^2$. Moreover, $(C, \alpha)$ is an $SO(3)$ vortex on $E$ and there are no $SO(3)$ vortices of the second kind since $\deg E = 1 > 0 = 2 \deg K_{T^2}$. As described in Example (37), the dimension of the moduli space of $SO(3)$ vortices on $E$ is two, which becomes one dimensional after taking the quotient by the $S^1$ action.

- The moduli space of flat connections on $T^3$. As described in (54) Section 4.1], there are two flat connections on $T^3$. One corresponds to the flat connection (representation) on $T^2$, while the other is obtained from this one by applying a twist to the representation on $T^2$.

- The Seiberg-Witten solutions which appear associated to our choice of spin-u structure must satisfy the perturbed Seiberg Witten equations (27):

\[
\begin{align*}
*F_{B_L} + \rho^{-1}(\psi \psi^*)_0 - \frac{1}{2} * F_{B_{\text{det}}} &= 0 \\
D_{\tilde{B}} \psi &= 0
\end{align*}
\]

Here $\psi$ corresponds to a section of $K_{\Sigma}^{1/2} \otimes L \subset K_{\Sigma}^{1/2} \otimes E$ for a splitting of $E = L \oplus (L^* \otimes \det E)$.

Thus $\psi$ satisfies the abelian vortex equations
\[
\begin{align*}
*_{T^2} F_C - \frac{i}{2} |\psi|^2 &= \frac{i}{2} *_{T^2} F_{C_{\text{det}} E} \\
\partial_C \psi &= 0
\end{align*}
\]

The usual Chern-Weil argument says that
\[ \deg L + \frac{1}{4\pi} \int_{T^2} |\psi|^2 = \frac{1}{2} \deg E = \frac{1}{2} \]
Since \( \deg L \geq 0 \), this forces \( \deg L = 0 \) and thus \( \psi \) is a holomorphic function on \( T^2 \). Thus it is constant and up to gauge there is a unique solution. In this way we find that there is a unique solution to these perturbed Seiberg-Witten equations on \( T^3 \).

In particular, for this choice of \( U(2) \) bundle on \( T^3 \), the moduli space of \( SO(3) \) monopoles on \( T^3 \) consists of a one-dimensional moduli space (an interval) with one endpoint corresponding to the Seiberg-Witten monopole, the other endpoint corresponding to the flat connection which pulls back from the flat connection on \( T^2 \), and an additional flat connection not connected to this one dimensional moduli space. Based on this picture, we define:

**Definition 46.** Suppose that \( Y \) is a closed oriented 3-manifold. The **framed monopole Floer** groups of \( Y \), denoted \( HM^#(Y) \), is defined as the monopole Floer homology \( HM(Y \# T^3, \omega, \Gamma) \). Here \( \omega \) is a non-balanced perturbation which is a harmonic representative of \(-\frac{1}{2}\pi c_1(E) \in H^2(T^2;\mathbb{R}) \subset H^2(T^3;\mathbb{R}) \) for the \( U(2) \) bundle \( E \) over \( T^2 \) of degree one. Since \( \omega \) is a non-balanced perturbation, the definition of the monopole Floer homology requires the use of a local coefficient system \( \Gamma \).

**Theorem 48.** The framed monopole Floer group \( HM^#(Y) \) is isomorphic to the group \( \bar{HM}(Y, \Gamma) \). Moreover, the Euler characteristic of \( HM^#(Y) \) (hence \( HM(Y, \Gamma) \)) is the same as the one of \( H^#(Y) \).

**Remark 47.** Some choices of local systems are described in [54 Chapter VIII] as well as in the author’s paper [32].
Remark 49. The equality of the Euler characteristics was explained near the end of the introduction to this work.

8. \textit{SO}(3) Monopoles on Seifert Manifolds

Now we proceed to analyze the \textit{SO}(3) monopole equations on a Seifert manifold $Y$. We will follow [74] and [67]. Unfortunately, our notation and some of our conventions are not quite isomorphic to those used in either reference, so we have to redo some of their computations.

For the identities we require we need to have very good control on all the constants appearing in our formulas, so at the cost of redundancy we will reprove some of the results in each paper.

First of all, for an \textit{SO}(3) connection $\partial\nabla$ on the cotangent bundle $T^*Y$ which is not necessarily the Levi-Civita connection, we say that a connection $\nabla_{\psi}$:

$$\Gamma(S) \rightarrow \Gamma(T^*Y \otimes S)$$

is a \textit{spinorial connection} with respect to $\partial\nabla$ if all vector fields $v$ on $Y$, sections $\Phi$ of $S$ and one forms $\theta$ we have

$$\nabla_{\psi} v(\rho(\theta)\Phi) = \rho(\partial\nabla v(\theta))\Phi + \rho(\theta)(\nabla_{\psi} \Phi)$$

If $(e^i)$ is a coframe and the connection matrix of $\partial\nabla$ is given by $\omega$ then the connection matrix of $\nabla_{\psi}$ is

$$\nabla_{\psi} e^i = \omega e^i = \sum_{j=1}^{3} \omega_{ji} \otimes e^j$$

where $\omega \in \Omega^1(Y) \otimes \mathfrak{so}(3)$ then the matrix of $\nabla_{\psi}$ is

$$-\frac{1}{4} \sum_{i,j=1}^{3} \omega_{ij} \otimes \rho(e^i \wedge e^j) + bId_S$$

where $b$ is any imaginary valued one form.

With this baggage out of the way we can start defining some special geometric structures on $Y = S(L)$. Recall that $S(L)$ is a principal $S^1$ bundle. As such, we can find a connection $i\eta$ for $S(L)$ and we assume that it is of constant curvature. We give $Y$ the metric $g_Y = \eta \otimes \eta + \pi^*(g_{\Sigma})$.

This is a locally homogeneous Riemann metric on $Y = S(L)$. The global 1 form induces a reduction in the structure group of $TY$ to $SO(2)$; the kernel of $\eta$ is naturally identified with the pull back of the orbifold tangent bundle of $\Sigma$, so that we have orthogonal splittings

$$TY \simeq \mathbb{R}\frac{\partial}{\partial \phi} \oplus \pi^*(T\Sigma)$$

$$T^*Y \simeq \mathbb{R}\eta \oplus \pi^*(T^*\Sigma)$$

Here $\frac{\partial}{\partial \phi}$ is the vector field dual to $\eta$ with respect to $g_Y$. Notice that $\frac{\partial}{\partial \phi}$ has unit length with respect to this metric and in fact it is a Killing vector field for the metric $g_Y$, that is, the Lie derivative of the metric $g_Y$ with respect to $\frac{\partial}{\partial \phi}$ vanishes.

\textsuperscript{3}Our convention is chosen in such a way that if $\nabla_{\psi}^TY$ is the induced connection on the tangent bundle then $\nabla_{\psi}^TY e_i = \sum_{j=1}^{3} \omega_{ji} e_j$ where $e_i$ an orthonormal frame
Moreover, for any vector field $v_\Sigma$ dual to the pull back of a 1 form from $\tilde{\Sigma}$, we have that 
\[
\left[v_\Sigma, \frac{\partial}{\partial \phi}\right] = 0.
\]
If $\nabla^LC_\Sigma$ is the Levi-Civita connection on $T^*\Sigma$, we give $T^*Y$ the so called adiabatic connection
\[
(34) \quad \nabla^\infty = d \oplus \pi^*(\nabla^LC_\Sigma)
\]
In particular,
\[
\begin{cases}
\nabla^\infty \eta = 0 \\
\nabla^\infty \pi^*(\tilde{\theta}) = \pi^*(\nabla^LC_\Sigma \tilde{\theta})
\end{cases}
\]
where $\tilde{\theta}$ is any 1 form on $\tilde{\Sigma}$. Since $d\eta$ is constant, let $c_\eta$ be the constant determined by
\[
(35) \quad d\eta = 2c_\eta \pi^*(\mu_{\Sigma})
\]
Using Chern-Weil theory one deduces that
\[
(36) \quad c_\eta = -\frac{\pi \int_{\Sigma} c_1(L)}{\text{vol}(\Sigma)} = -\frac{\pi \deg(L)}{\text{vol}(\Sigma)}
\]
The following estimate will allows us to compare two Dirac operators associated to the adiabatic and Levi-Civita connections.

**Lemma 50.** (Comparing Dirac Operators, [67, Lemma 5.5]) Let $(S, \rho)$ be a spin-c structure over $Y = S(L)$ and $\nabla^S, \nabla^S_{LC}$ a pair of connections which are spinorial with respect to $\nabla^\infty$ and $\nabla^LC$ respectively, which induce the same connection on the determinant line bundle of $S$. Then, for any $\theta \in \Omega^1(Y, \mathbb{R})$, we have
\[
\nabla^S_{LC, \thetaY} = \nabla^S_{\thetaY} + \frac{c_\eta}{2} (\rho(\theta) - 2 \langle \theta, \eta \rangle \rho(\eta))
\]
where $\thetaY$ denotes the vector field which is $g_Y$ dual to $\theta$. If $D_{LC}$ and $D^\infty$ are the corresponding Dirac operators then
\[
D_{LC} = D^\infty - \frac{1}{2} c_\eta
\]
In particular, this implies that the Dirac operator $D^\infty$ is self adjoint, since $D_{LC}$ and multiplication by a real scalar valued function are self adjoint.

**Proof.** Recall the Cartan’s structure equations [22, Section 4.3]. Let $\{e^0, e^1, e^2\}$ be a (local) orthonormal frame of $T^*Y$. Define the connection matrix for the Levi Civita connection $\nabla^LC$ by
\[
\nabla^LC e^i = \sum_{j=0}^{2} \omega^LC_{ji} \otimes e_j
\]
then
\[
de^i = \sum_{j=0}^{2} e^j \wedge \omega^LC_{ij} = -\sum_{j=0}^{2} \omega^LC_{ij} \wedge e^j
\]
These can be conveniently rewritten as
\[
d \begin{bmatrix}
e^0 \\
e^1 \\
e^2
\end{bmatrix} = - \begin{bmatrix}
0 & \omega^LC_{01} & \omega^LC_{02} \\
\omega^LC_{10} & 0 & \omega^LC_{12} \\
\omega^LC_{20} & \omega^LC_{21} & 0
\end{bmatrix} \wedge \begin{bmatrix}
e^0 \\
e^1 \\
e^2
\end{bmatrix}
\]
Choose $e^0 = \eta$, and $e^1, e^2$ the pull back from an orthonormal frame on $\tilde{\Sigma}$. We obtain the system of equations

\begin{align}
\frac{d\eta}{d\epsilon^0} &= -\omega_{10}^{LC} \wedge e^0 - \omega_{12}^{LC} \wedge e^2 \\
\frac{d\epsilon^1}{d\epsilon^2} &= -\omega_{10}^{LC} \wedge e^0 - \omega_{21}^{LC} \wedge e^1
\end{align}

(37)

Given that

\begin{align}
d\eta = 2c_\eta \mu_{\tilde{\Sigma}} = 2c_\eta e^1 \wedge e^2
\end{align}

our first equation in (37) becomes

\begin{align}
2c_\eta e^1 \wedge e^2 = e^1 \wedge \omega_{01}^{LC} - \omega_{02}^{LC} \wedge e^2
\end{align}

Therefore we have

\begin{align}
\omega_{01}^{LC} &= c_\eta e^2 \\
\omega_{02}^{LC} &= -c_\eta e^1
\end{align}

Also, if we let $\omega_{\Sigma}^{LC}$ be the connection form for the Levi-Civita connection, since pull back commutes with exterior differentiation then the structure equations for $\omega_{\Sigma}^{LC}$ are

\begin{align}
d \begin{bmatrix} e^1 \\ e^2 \end{bmatrix} = - \begin{bmatrix} 0 & \omega_{12}^{LC} \\ \omega_{21}^{LC} & 0 \end{bmatrix} \wedge \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}
\end{align}

Comparing (38) with (37) and using the antisymmetry of the $\omega_{ij}$ we obtain the equations

\begin{align}
c_\eta e^2 \wedge e^0 - \omega_{12}^{LC} \wedge e^2 &= -\omega_{12}^{LC} \wedge e^2 \implies (-c_\eta e^0 - \omega_{12}^{LC} + \omega_{12}^{\Sigma}) \wedge e^2 = 0 \\
-c_\eta e^1 \wedge e^0 - \omega_{21}^{LC} \wedge e^1 &= -\omega_{21}^{LC} \wedge e^1 \implies e^1 \wedge (-c_\eta e^0 + \omega_{21}^{LC} - \omega_{21}^{\Sigma}) = 0
\end{align}

Clearly we obtain

\begin{align}
\omega_{21}^{LC} = c_\eta e^0 + \omega_{21}^{\Sigma}
\end{align}

Therefore the connection matrix for the Levi-Civita connection $\omega_{LC}^{Y}$ becomes

\begin{align}
\left( \begin{array}{cc}
0 & \omega_{01}^{LC} \\
\omega_{10}^{LC} & 0 \\
\omega_{20}^{LC} & \omega_{21}^{LC} \\
0 & 0
\end{array} \right)
= \left( \begin{array}{cc}
0 & c_\eta e^2 \\
-c_\eta e^2 & 0 \\
c_\eta e^1 & c_\eta e^0 + \omega_{12}^{LC} \\
c_\eta e^1 & \omega_{21}^{LC}
\end{array} \right)
\end{align}

The connection matrix for $\nabla^{\infty}$ is easier to find since it is a reducible connection and trivial in the first summand; it is simply

\begin{align}
\left( \begin{array}{cc}
0 & 0 \\
0 & -\omega_{12}^{LC} \\
0 & \omega_{21}^{LC} \\
0 & \omega_{21}^{\Sigma}
\end{array} \right)
\end{align}

Therefore the difference between connection matrices is

\begin{align}
\nabla^{LC} - \nabla^{\infty} \iff \left( \begin{array}{cc}
0 & c_\eta e^2 \\
-c_\eta e^2 & 0 \\
c_\eta e^1 & c_\eta e^0 \\
c_\eta e^1 & \omega_{21}^{\Sigma}
\end{array} \right) = (\tilde{\omega}_{ij})
\end{align}

As mentioned in [67], we are writing the connection matrix for $TY$ which differs by a sign from the connection matrix for $T^*Y$, hence explaining the difference sign with their paper.
Using formula \( \Box \) the corresponding spin connections differ by

\[
\nabla^S_{LC} - \nabla^S = \frac{1}{4} \sum_{i,j=0}^2 \hat{\omega}_{ij} \otimes \rho(e^i)\rho(e^j)
\]

\[
= \frac{1}{2} \left( c_2 e^2 \otimes \rho(0)\rho(e^1) - c_2 e^1 \otimes \rho(0)\rho(e^2) - c_2 e^0 \otimes \rho(e^1)\rho(e^2) \right)
\]

\[
= \frac{c_n}{2} \left( e^0 \otimes \rho(e^1)\rho(e^2) + e^1 \otimes \rho(0)\rho(e^2) - e^2 \otimes \rho(0)\rho(e^1) \right)
\]

\[
= \frac{c_n}{2} \left( -e^0 \otimes \rho(e^0) + e^1 \otimes \rho(e^1) + e^2 \otimes \rho(e^2) \right) \rho(\mu_Y)
\]

where we used \( 1_S = \rho(\mu_Y) = \rho(e_0)\rho(e_1)\rho(e_2) \). To check the first formula in the lemma \( \nabla^S_{LC,\theta^\#} = \nabla^S_{\theta^\#} + \frac{c_n}{2} (\rho(\theta) - 2 (\theta, \eta) \rho(\eta)) \) notice that it is linear in \( \theta^\# \) and so we just need to verify it on the coframe for each covector \( e^0, e^1, e^2 \).

We will just check it for \( \theta = e^0 \) and \( \theta = e^1 \) since \( \theta = e^2 \) is entirely analogous to the second case. In the case \( \theta^\# = e_0 \) because of the previous calculation

\[
\nabla^S_{LC,e_0} - \nabla^S_{e_0} = \frac{c_n}{2} \left( -e^0 \otimes \rho(e^0) + e^1 \otimes \rho(e^1) + e^2 \otimes \rho(e^2) \right) e_0
\]

\[
= -\frac{c_n}{2} \rho(e^0)
\]

On the other hand

\[
\frac{c_n}{2} (\rho(e^0) - 2 (e^0, \eta) \rho(\eta)) = \frac{c_n}{2} (\rho(e^0) - 2 \rho(\eta))
\]

\[
= -\frac{c_n}{2} \rho(e^0)
\]

and so both formulas agree. When \( \theta = e^1 \) our long calculation implies that

\[
\nabla^S_{LC,e_1} - \nabla^S_{e_1} = \frac{c_n}{2} \left( -e^0 \otimes \rho(e^0) + e^1 \otimes \rho(e^1) + e^2 \otimes \rho(e^2) \right) e_1
\]

\[
= \frac{c_n}{2} \rho(e^1)
\]

On the other hand

\[
\frac{c_n}{2} (\rho(e^1) - 2 (e^1, \eta) \rho(\eta)) = \frac{c_n}{2} \rho(e^1)
\]

Which verifies the formula. The statement about the Dirac operators \( D_{LC} = D^\infty - \frac{1}{2} c_n \) also follows because

\[
D_{LC} = \nabla_{LC} - D^\infty = \frac{c_n}{2} \left( -\rho(e^0)\rho(e^0) + \rho(e^1)\rho(e^1) + \rho(e^2)\rho(e^2) \right)
\]

\[
= \frac{c_n}{2} (1 - 1 - 1)
\]

\[
= -\frac{c_n}{2}
\]

As a corollary, we also have
Corollary 51. ([67] Corollary 5.7) The anticommutator of the Dirac operator with Clifford multiplication by a one form $\theta$ is given by the formula

$$
\{D^\infty, \rho(\theta)\} = -\rho((d + d^*)\theta) - 2\nabla^S_{\theta^*} + 2c_\eta \eta \rho(\eta)
$$

Proof. This follows from the analogous formula for the Levi-Civita operator [10 Proposition 3.45]

$$
\{D_{LC}, \rho(\theta)\} = \rho((d + d^*)\theta) - 2\nabla^2_{LC,\theta^*} + \frac{c_\eta}{2} \rho(\theta) - 2\langle \theta, \eta \rangle \rho(\eta)
$$

Since $D_{LC} = D^\infty - \frac{c_\eta}{2}$ and $\nabla^S_{LC,\theta^*} = \nabla^S_{\theta^*} + \frac{c_\eta}{2} (\rho(\theta) - 2\langle \theta, \eta \rangle \rho(\eta))$ we find that

$$
\{D^\infty, \rho(\theta)\} - c_\eta \rho(\theta) = \rho((d + d^*)\theta) - 2\nabla^S_{\theta^*} - c_\eta (\rho(\theta) - 2\langle \theta, \eta \rangle \rho(\eta))
$$

from which the result is now obvious. \qed

For our purposes we are interested in the case where we twist our spinor bundle with a rank two hermitian bundle $E$, endowed with a $U(2)$ connection $B$. Then we have a twisted connection

$$
\nabla^{S,B} : \Gamma(S \otimes E) \rightarrow \Gamma(T^*Y \otimes S \otimes E)
$$

defined in the usual way

$$
\nabla^{S,B}(s \otimes s_E) = (\nabla^S s) \otimes s_E + s \otimes (\nabla^B s_E)
$$

Here $s$ is a section of $S$, while $s_E$ a section of $E$. The twisted Dirac operator is then

$$
D_{S,B} = \sum_{i=0}^2 \rho(e^i)\nabla_{e_i}^{S,B} = \rho(\eta)\nabla_{e_1}^{S,B} + \rho(e^1)\nabla_{e^1_{e_1}}^{S,B} + \rho(e^2)\nabla_{e^2_{e_2}}^{S,B}
$$

Observe first of all that since $D_S$ [called $D^\infty$ before] is self-adjoint thanks to Lemma [60], we know that $D_{S,B}$ is a self-adjoint operator as well. Hence

$$
D^2_{S,B} D_{S,B} = D^2_{S,B}
$$

and our objective is to find an useful decomposition for the square of this operator, in other words, a Weitzenbock-type formula. Tautologically we define

$$
D_{2,S,B} = D_{S,B} - \rho(\eta)\nabla_{e_1}^{S,B}
$$

Then

$$
D^2_{S,B} = \left(\rho(\eta)\nabla_{e_1}^{S,B}\right)^2 + D^2_{2,S,B} + \left\{\rho(\eta)\nabla_{e_1}^{S,B}, D_{2,S,B}\right\}
$$

decomposes $D^2_{S,B}$ as a sum of three operators, the first two which are evidently non-negative. The third term is an anti-commutator, which we need to analyze explicitly.

Lemma 52. The term $\left\{\rho(\eta)\nabla_{e_1}^{S,B}, D_{2,S,B}\right\}$ is equal to

$$
\left\{\rho(\eta)\nabla_{e_1}^{S,B}, D_{2,S,B}\right\} = \rho(\eta)\rho(e^1)F_{S,B}(\partial_{\varphi}, e_1) + \rho(\eta)\rho(e^2)F_{S,B}(\partial_{\varphi}, e_2)
$$

Remark. This computation should be compared with the formula given in [67 Lemma 5.8]. There $E$ would be a rank complex line bundle, and the formula would read in terms of an orthonormal frame

$$
\left\{\rho(\eta)\nabla_{e_1}^{S,B}, D_{2,S,B}\right\} = \rho(\eta)\left(\rho(e^1)(db)_{\eta,e_1} + \rho(e^2)(db)_{\eta,e_2}\right)1_S
where \( b \) is the imaginary valued one form which appears in \( \Sigma \) and we decomposed it as
\[
d\bar{b} = (d\bar{b})_{\eta,e_1} \eta \wedge e^1 + (d\bar{b})_{\eta,e_2} \eta \wedge e^2 + (d\bar{b})_{e_1,e_2} e^1 \wedge e^2
\]

**Proof.** Clearly,
\[
\left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, D_{2,S,B} \right\} = \left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, \rho(e^1)\nabla^S_{e_1} \right\} + \left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, \rho(e^2)\nabla^S_{e_2} \right\}
\]
where \( \partial_\varphi = \partial_\varphi^\Sigma \), and it suffices to find the first term since the other one is analogous.

For a spinor \( \Psi \) of \( S \otimes E \) we have
\[
\left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, \rho(e^1)\nabla^S_{e_1} \right\} \Psi = \rho(\eta)\nabla^S_{\partial_\varphi} (\rho(e^1)\nabla^S_{e_1} \Psi) + \rho(e^1)\nabla^S_{e_1} (\rho(\eta)\nabla^S_{\partial_\varphi} \Psi)
\]
Now we use the fact that the connection is spinorial, i.e., \( \nabla^S_{e_1} (\rho(\eta)) = \rho(\nabla^\infty \eta)\Psi + \rho(\eta)(\nabla^S_{e_1} \Psi) \).

In our case the anticommutator becomes
\[
\rho(\eta)\rho(\nabla^\infty e_1) (\nabla^S_{e_1} \Psi) + \rho(\eta)\rho(e^1)\nabla^S_{\partial_\varphi} (\nabla^S_{e_1} \Psi)
\]
\[
+ \rho(e^1)\rho(\nabla^\infty \eta) (\nabla^S_{\partial_\varphi} \Psi) + \rho(e^1)\rho(e^1)\nabla^S_{e_1} (\nabla^S_{\partial_\varphi} \Psi)
\]
Now, for our adiabatic connection we have that \( \nabla^\infty \eta = 0 \) and \( \nabla^\infty \pi^* (\tilde{\theta}) = \pi^*(\nabla^\infty C \tilde{\theta}) \) so the first term in each row of the previous expression disappears. Also, using the Clifford relations \( \rho(e^1) \rho(\eta) = -\rho(\eta)\rho(e^1) \) the computation for the anticommutator becomes
\[
\left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, \rho(e^1)\nabla^S_{e_1} \right\} \Psi = \rho(\eta)\rho(e^1)\nabla^S_{\partial_\varphi} \nabla^S_{e_1} \Psi
\]
To compute the commutator recall that because \( \nabla^S_{e_1} \) is a connection on a vector bundle it has a curvature \( F_{S,B} \) given by [76, Section 3.3.3]
\[
[\nabla^S_{v}, \nabla^S_{w}] - \nabla^S_{[v,w]} = F_{S,B}(v,w)
\]
Since \( \partial_\varphi \) commutes with any vector field which is dual to a form that pulls back from \( \Sigma \) we have that \( [\partial_\varphi, e_1] = 0 \) and so
\[
[\nabla^S_{\partial_\varphi}, \nabla^S_{e_1}] = F_{S,B}(\partial_\varphi, e_1)
\]
Since a similar result holds for \( \partial_\varphi \) and \( e_2 \) we have just found that
\[
\left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, D_{2,S,B} \right\} = \rho(\eta)\rho(e^1)F_{S,B}(\partial_\varphi, e_1) + \rho(\eta)\rho(e^2)F_{S,B}(\partial_\varphi, e_2)
\]
\[\square\]

**Remark 53.** Notice that as [67] point out before Lemma 5.8, the term \( \left\{ \rho(\eta)\nabla^S_{\partial_\varphi}, D_{2,S,B} \right\} \) ends up being a zeroth-order operator, not a first order operator.

Now we write the \( SO(3) \) monopole equations on a Seifert manifold. Locally, we can decompose the curvature in the same way as in the previous section, namely
\[
F_B^0 = F_{\eta,e_1} \eta \wedge e^1 + F_{\eta,e_2} \eta \wedge e^2 + F_{e_1,e_2} e^1 \wedge e^2
\]
\[
*F_B^0 = F_{\eta,e_1} e^2 - F_{\eta,e_2} e^1 + F_{e_1,e_2} d\eta
\]
and we obtain:
Theorem 54. Suppose $(B, \Psi)$ satisfies the $SO(3)$ vortex equations on a Seifert manifold $Y$, for the spin-$u$ structure $V_{an} = (\mathbb{C} \oplus \pi^*(K_{\Sigma}^{-1})) \otimes \pi^*(\hat{E})$. Write $\Psi = \alpha + \beta$ with $\alpha \in \Omega^0(\pi^*(\hat{E}))$ and $\beta \in \Omega^{0,1}(\pi^*(K_{\Sigma}^{-1} \otimes \hat{E}))$.

Then with respect to the $SO(3)$ monopole equations
\begin{align}
if_{e^1,e^2} + (\alpha \otimes \alpha^*)_0 - (A\beta \otimes \beta^*)_0 &= 0 \\
(iF_{\eta,e^1} - F_{\eta,e^2}) \xi + 2(\beta \otimes \alpha^*)_0 &= 0 \\
\rho(\eta)\nabla_{\hat{g}}(S,B)\Psi + D_{\hat{g}}(S,B)\Psi &= 0
\end{align}
(42)
all solutions are of the form $(B,(\alpha,0))$ or $(B,(0,\beta))$.

In the first case, we get a solution of the form
\begin{align}
\partial_C\alpha &= 0 \\
F_C - i(\alpha \otimes \alpha^*)_0 &= 0
\end{align}
which can be identified with a $SO(3)$ vortex on a bundle $E'$ which satisfies that $\det E' \simeq \det E$.

In the second case, we get a solution of the form
\begin{align}
\partial_C\beta &= 0 \\
F_C + i(A\beta \otimes \beta^*)_0 &= 0
\end{align}
which can be identified via Serre duality with an $SO(3)$ vortex on the bundle $K_{\Sigma} \otimes E'^{-1}$, where again $E$ is a $U(2)$ bundle satisfying $\det E' \simeq \det E$.

In particular, if we assume that $c_1(E) > 2c_1(K_{\Sigma})$, then this second type of moduli space of stable pairs is empty.

Remark 55. a) Notice that implicitly we chose a connection $C^{\text{det}}$ on $\det E$, and we are considering all $U(2)$ connections on $\pi^*(\hat{E})$ which induce the predetermined connection $\pi^*(C^{\text{det}})$ on $\pi^*(\det E)$. Using [67, Proposition 5.3], there is a bijection between bundles such connections on orbifold line bundles over $\Sigma$, and the usual connections on line bundles over $Y$, whose curvature pulls up from $\Sigma$ and whose fiberwise holonomy is trivial.

Thus, this is why the bundle $\pi^*(\hat{E})$ “remembers” the choice of connection we made downstairs (i.e., on the orbifold $\Sigma$). Since there are several $U(2)$ orbifold bundles with the same $\det E$ (up to isomorphism), we need to consider all the distinct choices, which differ in their isotropy data. But at least there are only finitely many isomorphism classes to consider.

b) One could also be interested in understanding how the abelian vortices that appear in these moduli spaces are related to those appearing in MOY [67]. As Theorem 1 of MOY states, the Seiberg-Witten solutions that appear there can be identified with two copies of the space of effective orbifold divisors over $\Sigma$ with orbifold degree less than $-\frac{1}{\chi(\Sigma)} = \frac{\alpha(K_{\Sigma})}{2}$. For simplicity suppose that the $a_i$ are coprime and that $\Sigma$ admits orb-spin bundles [67, Definition 5.13].

This means that there is a square-root $K_{\Sigma}^{1/2}$ of $K_{\Sigma}$ in the sense that $2c_1(K_{\Sigma}^{1/2}) = c_1(K_{\Sigma})$. In our setup this would require all the $a_i$ to be odd integers and moreover $K_{\Sigma}^{1/2}$ is unique (see the proof of Corollary 5.17 in MOY). Thus, if we take for example $\det E \simeq K_{\Sigma} \otimes L_0$ this is a valid choice in the sense that $K_{\Sigma}$ is an even power of $L_0$ thus $\det E$ is an odd power of $L_0$. However, this choice for $\det E$ is not within the vanishing condition stated in our theorem thus we obtain:

1) Moduli spaces of $SO(3)$ vortices on $E'$: now abelian vortices can arise provided $c_1(L) < \frac{1}{2}c_1(K_{\Sigma} \otimes L_0) = \frac{1}{2}c_1(K_{\Sigma}) + \frac{1}{2a_1-\cdots-a_n}$. 
2) Moduli spaces of $SO(3)$ vortices on $K_S \otimes \hat{E}^{-1}$: now abelian vortices can arise provided $c_1(L) < \frac{1}{2} c_1(K_S) - \frac{1}{2} \sum_{i=1}^{\alpha_0}$. This seems the closest one can get to the constraint that appears on MOY. However, because of this choice for $\det \hat{E}$ it is not clear that the moduli space of $SO(3)$ vortices is smooth at the reducible so it seems better to assume $c_1(E) > 2 c_1(K_S)$ instead.

**Proof.** We follow the strategy of [67, Theorem 4]. If we start with the equation $D_S B \Psi = 0$, and apply $D_{S,B}$ to it, we obtain

$$0 = \left( \rho(\eta) \cdot \nabla^{S,B} \right)^2 \Psi + D_{S,B}^2 \Psi + \rho(\eta) \rho(\epsilon_1) F_{S,B}(\partial_\varphi, c_1) \Psi + \rho(\eta) \rho(\epsilon_2) F_{S,B}(\partial_\varphi, c_2) \Psi$$

Now take inner product with $\Psi$ and integrate over $Y$ in order to obtain

$$0 = \left| \nabla^{S,B}_{\partial_\varphi} \Phi \right|^2 + |D_{S,B} \Phi|^2 + \frac{1}{2} \langle \rho(\eta) \rho(\epsilon_1) F_{\eta,e_1} \Psi + \rho(\eta) \rho(\epsilon_2) F_{\eta,e_2} \Psi, \Psi \rangle$$

(43) where $F_{\eta,e_1} = F_{S,B}(\partial_\varphi, c_1)$ and $F_{\eta,e_2} = F_{S,B}(\partial_\varphi, c_2)$.

Under a slight abuse of notation, recall from last section that

$$\rho(\eta) = \begin{pmatrix} 0 & 0 \\ -i & -i \end{pmatrix} \quad \rho(\epsilon_1) = \begin{pmatrix} 0 & -\epsilon \\ \bar{\epsilon} & 0 \end{pmatrix} \quad \rho(\epsilon_2) = \begin{pmatrix} 0 & i\epsilon \\ \bar{\epsilon} & 0 \end{pmatrix}$$

Thus

$$\rho(\eta) \rho(\epsilon_1) = \begin{pmatrix} 0 & -i\epsilon \\ -i\bar{\epsilon} & 0 \end{pmatrix} \quad \rho(\eta) \rho(\epsilon_2) = \begin{pmatrix} 0 & -\epsilon \\ \bar{\epsilon} & 0 \end{pmatrix}$$

Recall that if we were to write $\Psi$ locally as $\Psi = \sum_{i,j=1}^{2} c_{ij} \Psi^S_i \otimes \Psi^E_j$, where $\Psi^S_1, \Psi^S_2$ is an orthonormal basis for $S$ and $\Psi^E_1, \Psi^E_2$ an orthonormal basis for $E$, then for any one form $\rho(\theta) \Psi = \sum_{i,j=1}^{2} c_{ij} \rho(\theta) \Psi^S_i \otimes \Psi^E_j$, while for any section $\xi$ of $\mathfrak{su}(E)$, $\xi \Psi = \sum_{i,j=1}^{2} c_{ij} \xi \Psi^S_i \otimes (\xi \Psi^E_j)$. In other words, the Clifford multiplication action on spinors $\Psi$ is uncoupled from the action of $\mathfrak{su}(E)$ on spinors.

Therefore, we can write

$$\rho(\eta) \rho(\epsilon_1) F_{\eta,e_1} \Psi = \begin{pmatrix} 0 & -i\epsilon \\ -i\bar{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \rho(\eta) \rho(\epsilon_2) F_{\eta,e_2} \Psi = \begin{pmatrix} 0 & -\epsilon \\ \bar{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\epsilon \\ -i\bar{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} F_{\eta,e_1} \alpha \\ \bar{\epsilon} F_{\eta,e_1} \beta \end{pmatrix} \quad = \begin{pmatrix} 0 & -\epsilon \\ \bar{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} F_{\eta,e_2} \alpha \\ \bar{\epsilon} F_{\eta,e_2} \beta \end{pmatrix}$$
And so we must understand
\[
\left\langle \left( -\epsilon (iF_{\eta,e_1} + F_{\eta,e_2})\beta \right), \left( \alpha \right) \right\rangle
= 2\left\langle \left( (\alpha \otimes \beta^*)\alpha \beta \right), \left( \alpha \beta \right) \right\rangle
= 2\left\langle \left( |\beta|^2\alpha - \frac{1}{2} < \beta, \alpha > E \beta \right), \left( \alpha \beta \right) \right\rangle
= 2\left( |\beta|^2|\alpha|^2 - \frac{1}{2} < \beta, \alpha > |E| < \beta, \alpha > |2| \right)
\]

Just as in the case of $S^1 \times \Sigma$, by Cauchy-Schwarz this is non-negative so going back to the equality (43) we conclude that
\[
\begin{cases} 
\nabla_{S} \alpha \equiv 0 \\
\nabla_{S} \beta \equiv 0 \\
D_{2,S,B}\Psi \equiv 0 \\
\alpha \equiv 0 \text{ or } \beta \equiv 0
\end{cases}
\]

Once we know this, the argument is identical to the one given by [67, Theorem 4]. Namely, the first two equations say that $\alpha, \beta$ actually pullback from the orbifold Riemann surface. Likewise, the connection $B$ will pullback from an orbifold bundle $\tilde{E}'$ over $\tilde{\Sigma}$ which satisfies $\det \tilde{E}' \simeq \det \tilde{E}$. Finally, the operator $D_2$ can then be identified with a twisted $\bar{\partial} \oplus \bar{\partial}^*$ operator [67, p. 709], so this is why $D_{2,S,B}\Psi \equiv 0$ together with the equation for $F_{e_1,e_2}$ gives a solution means that we have a solution to the $SO(3)$ vortex equations.

The vanishing result uses and the identification with the stable pairs on $\tilde{\Sigma}$ is exactly the same argument as the one we used for $S^1 \times \Sigma$. □

**Example 56.** The Poincaré Homology Sphere

As is well known the Poincaré homology sphere can be identified with the Brieskorn homology sphere $\Sigma(2, 3, 5)$, and as a Seifert manifold, the corresponding orbifold is $S^2(2, 3, 5)$.

In this case the generator for the topological Picard group is an orbifold line bundle with $c_1(\tilde{L}_0) = \frac{1}{30}$. The isotropy invariants consists of three integers $b_1, b_2, b_3$ satisfying

\[
1 \leq b_1 < 2 \quad 1 \leq b_2 < 3 \quad 1 \leq b_3 < 5
\]

These numbers are found by the requirement that

\[
\frac{1}{\prod \alpha_i} - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} = \frac{1}{30} - \frac{15b_1 + 10b_2 + 6b_3}{30}
\]

be an integer (the background degree of $\tilde{L}_0$). It is easy to check that the only choice which works is

\[
b_1 = b_2 = b_3 = 1
\]

Thus we will write $\tilde{L}_0(1,1,1)$ to emphasize the isotropy data.

Recall that the Seifert invariants for the canonical bundle were given in equation (12). Thus, our non-vanishing condition reads in this case

\[
c_1(\tilde{E}) > 2c_1(K_{\tilde{E}}) = 2 \left( 0 - 2 + 3 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \right) = 2 \left( \frac{30 - 15 - 10 - 6}{30} \right) = -\frac{1}{15}
\]
In particular, it suffices to take \( \det \tilde{E} \simeq \tilde{L}_0 \). Now suppose that \( \tilde{E}' \) is a \( U(2) \) bundle with \( \det \tilde{E}' \simeq \tilde{L}_0 \). Write the isotropy invariants of \( \tilde{E}' \) as \( ((b_i^{-}, b_i^{+}), (b_2^{-}, b_2^{+}), (b_3^{-}, b_3^{+})) \). The conditions these integers must satisfy are

\[
0 \leq b_i^{-} \leq b_i^{+} < a_i
\]

and also

\[
b_i^{-} + b_i^{+} \equiv 1 \mod a_i
\]

since they must give the isotropy data of \( \tilde{L}_0 \).

We find by trial and error that the only options which work are

\[
((0, 1), (0, 1), (0, 1)) \quad ((0, 1), (0, 1), (2, 4)) \quad ((0, 1), (0, 1), (3, 3))
\]

\[
((0, 1), (2, 2), (0, 1)) \quad ((0, 1), (2, 2), (2, 4)) \quad ((0, 1), (2, 4), (3, 3))
\]

In order to analyze these bundles in a systematic way we recall the Facts \( [12] \), as well as some other important things to keep in mind when analyzing the different moduli spaces:

1. Because \( g = 0, n = 3 \), if \( n_0 = \# \{ i \mid b_i^{-} = b_i^{+} \} \geq 1 \), then \( \tilde{E}' \) admits no projectively flat connections. Moreover, the space of projectively flat connections is connected and for three marked points of expected dimension zero (this can be seen from the proof of our Lemma \( [21] \)).
2. If the moduli space \( \mathcal{M}(\tilde{E}, \tilde{E}') \) admits no moduli space of abelian vortices, then there are no irreducible vortices inside this moduli space.
3. The abelian vortices which can arise must satisfy \( c_1(\tilde{L}) < \frac{1}{2} c_1(\tilde{L}_0) = \frac{1}{2} \det(\tilde{E}) \). Since \( \tilde{L} = \tilde{L}_0 \) for some integer \( l \) and \( l \geq 0 \) because otherwise \( H^0(\tilde{L}) \) vanishes, we must have \( \tilde{L} \) is the trivial line bundle (isotropy \( (0, 0, 0) \) and \( c_1(\tilde{L}) = 0 \). Thus in this case any isotropy data which does not contain a \( 0 \) in each of the three pairs \( (b_i^{-}, b_i^{+}) \) will have an empty moduli space of irreducible \( SO(3) \) vortices.
4. Moreover, the dimension of the moduli space of irreducible \( SO(3) \) vortices can be read from \( [18] \)

\[
2 \left( g - 1 + c_1(\det \tilde{E}) + (n - n_0) - \sum_{i=1}^{n} \frac{b_i^{-} + b_i^{+}}{a_i} \right) = 2 \left( -1 + \frac{1}{30} + 3 - n_0 - \sum_{i=1}^{n} \frac{b_i^{-} + b_i^{+}}{a_i} \right)
\]

5. The index for an abelian monopole will be according to equation \( [25] \)

\[
2 \left( g - 1 + c_1(\det \tilde{E}) - 2c_1(\tilde{L}) + \sum_{i|c_i=1} b_i^{+} - b_i^{-} \right) + n_- + \sum_{i|c_i=-1} \frac{b_i^{-} - b_i^{+}}{a_i} \right) =
\]

\[
2 \left[ -1 + \frac{1}{30} + \sum_{i|c_i=1} \frac{b_i^{+} - b_i^{-}}{a_i} + n_- + \sum_{i|c_i=-1} \frac{b_i^{-} - b_i^{+}}{a_i} \right]
\]

With this information, we can make the following table [the dimensions refer to those before we take the quotient by the circle action]:
The Brieskorn Homology Sphere

The Brieskorn Homology Sphere consists of: a) the trivial $SU(2)$ connection, b) two irreducible flat $SU(2)$ connections, c) one Seiberg-Witten monopole, d) a one dimensional moduli space of $SO(3)$ monopoles which serves as a "cobordism" between the Seiberg-Witten monopole and one of the irreducible flat connections.

A more realistic example to consider is the following:

**Example 57.** The Brieskorn Homology Sphere $\Sigma(2, 3, 7)$

In this case we are looking at $\mathbb{S}^2(2,3,7)$. The generator $L_0$ now satisfies $c_1(L_0) = \frac{1}{42}$ and its isotropy must satisfy that

$$\frac{1}{42} = \frac{1}{42} (b_1 + 14 b_2 + 6 b_3)$$

is an integer. By trial and error we find that the only choice is

$$b_1 = 1, b_2 = 2, b_3 = 6$$

Write $L_0(1,2,6)$. Now the vanishing condition reads

$$c_1(\mathcal{E}) > 2 c_1(K_{\Sigma}) = 2 \left( 0 - 2 + 3 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} \right) = \frac{1}{21}$$

So we need to look at the moduli spaces of $SO(3)$ vortices with $\det \mathcal{E} \simeq L_0^3$. Notice that the isotropy is now $L_0^3(1,0,4)$. In this case the expected dimension would need to be

$$2 \left( 2 + \frac{1}{14} - \frac{b_1^- + b_1^+}{2} - \frac{b_2^- + b_2^+}{3} - \frac{b_3^- + b_3^+}{7} \right)$$

Notice that in this case $b_1^- + b_1^+ = 1$ and $b_2^- + b_2^+ = 3$ so this forces the expected dimension to be

$$2 \left( \frac{1}{2} + \frac{1}{14} - \frac{b_3^- + b_3^+}{7} \right) = 1 + \frac{1 - 2b_3^- - 2b_3^+}{7}$$

and since one of the $b_3^\pm$ must be non-trivial, this forces the dimension after taking the quotient by the $S^1$ action to be negative, thus we end up with empty moduli space of $SO(3)$ vortices, which is not interesting.

Thus, we need to increase $c_1(\det \mathcal{E})$ in order to make the moduli spaces of $SO(3)$ vortices to have positive expected dimension. The next choice is thus $\det \mathcal{E} \simeq L_0^5$. Notice that the isotropy is now $L_0^5(1,1,2)$. In this case the formula for the expected dimension reads

$$2 \left( 2 + \frac{5}{42} - \frac{b_1^- + b_1^+}{2} - \frac{b_2^- + b_2^+}{3} - \frac{b_3^- + b_3^+}{7} \right)$$

| Isotropy                  | $\mathcal{M}^*(\Sigma, E')$ | # Proj. Flat | # Abelian Vortices |
|---------------------------|------------------------------|--------------|--------------------|
| $((0,1),(0,1),(0,1))$     | non empty, of dim 2          | one          | one vortex of index 2 |
| $((0,1),(0,1),(2,4))$     | empty, exp. dim=0            | one (isolated)| no vortices (wrong isotropy) |
| $((0,1),(0,1),(3,3))$     | empty, exp. dim=2            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0,1),(2,2),(0,1))$     | empty, exp. dim=2            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0,1),(2,2),(2,4))$     | empty, exp. dim=4            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0,1),(2,4),(3,3))$     | empty, exp. dim=4            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
Again, $b_1^- + b_1^+ = 1$, but now we can have $b_2^- + b_2^+ = 1$ and $b_3^- + b_3^+ = 2$ in which case the dimension would be
\[
2 \left( 2 + \frac{5}{42} - \frac{1}{2} - \frac{1}{3} - 2 \frac{2}{7} \right) = 2
\]
so that means we will find positive dimensional moduli spaces of $SO(3)$ vortices!

The isotropy $((b_1^-, b_1^+), (b_2^-, b_2^+), (b_3^-, b_3^+))$ of our $U(2)$ bundles must satisfy
\[
\begin{cases}
0 \leq b_1^- \leq b_1^+ \leq 1 & b_1^- + b_1^+ - 1 \equiv 0 \mod 2 \\
0 \leq b_2^- \leq b_2^+ \leq 1 & b_2^- + b_2^+ - 1 \equiv 0 \mod 3 \\
0 \leq b_3^- \leq b_3^+ \leq 2 & b_3^- + b_3^+ - 2 \equiv 0 \mod 7
\end{cases}
\]

Again after trial and error one finds the isotropies
\[
\begin{align*}
((0, 1), (0, 1), (0, 2)) & \quad ((0, 1), (0, 1), (1, 1)) & \quad ((0, 1), (0, 1), (3, 6)) & \quad ((0, 1), (0, 1), (4, 5)) \\
((0, 1), (2, 2), (0, 2)) & \quad ((0, 1), (2, 2), (1, 1)) & \quad ((0, 1), (2, 2), (3, 6)) & \quad ((0, 1), (2, 2), (4, 5))
\end{align*}
\]

In addition to the facts used for the case of the Poincaré homology sphere, notice that:

1. The moduli spaces of abelian vortices which can arise are associated to $\tilde{L}_0(1, 2, 6)$, $\tilde{L}_0^2(0, 1, 5)$ and the trivial one $\tilde{L}_{\text{triv}}$. The background degrees are
\[
\begin{align*}
\deg_B \tilde{L}_{\text{triv}} &= 0 \\
\deg_B \tilde{L}_0(1, 2, 6) &= \frac{1}{12} - \frac{3}{4} - \frac{2}{3} - \frac{6}{5} = -2 \\
\deg_B \tilde{L}_0^2(0, 1, 5) &= \frac{2}{12} - \frac{9}{4} - \frac{1}{3} - \frac{5}{7} = -1
\end{align*}
\]
so the space of abelian vortices to consider is really just $\tilde{L}_{\text{triv}}$, since recall from MOY that these moduli spaces of abelian vortices are isomorphic to $\text{Sym}^{\deg_B} \tilde{L}(\Sigma)$, thus they are empty if the background degree is negative.

2. From Facts [12], the space of irreducible projectively flat connections is empty if and only if there exists a vector $(\epsilon_i)$, with $\epsilon_i = \pm 1$, such that $n_+ + \deg_B(\det \hat{E}) \equiv 1 \mod 2$ and $n_+ - \sum_{i=1}^n \frac{\epsilon_i(b_i^+ - b_i^-)}{a_i} < 1 - g$. Here $n_\pm = \# \{ i \mid \epsilon_i = \pm 1 \}$.

3. Moreover, the dimension of the moduli space of irreducible $SO(3)$ vortices can be read from [13]
\[
2 \left( (g - 1) + c_1(\det \hat{E}) + (n - n_0) - \sum_{i=1}^n \frac{b_i^- + b_i^+}{a_i} \right) = 2 \left( -1 + \frac{5}{42} + 3 - n_0 - \sum_{i=1}^n \frac{b_i^- + b_i^+}{a_i} \right)
\]

4. The index for an abelian monopole will be according to equation [25]
\[
2 \left( g - 1 + c_1(\det \hat{E}) - 2c_1(\hat{L}) + \sum_{i|\epsilon_i=1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1} \frac{b_i^- - b_i^+}{a_i} \right) = \left[ -1 + \frac{1}{42} + \sum_{i|\epsilon_i=1} \frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1} \frac{b_i^- - b_i^+}{a_i} \right]
\]

With this information, we can make the following table [the dimensions refer to those before we take the quotient by the circle action]:
| Isotropy          | $\mathcal{M}^*(\Sigma, E')$ | # Proj. Flat | # Abelian Vortices |
|------------------|-------------------------------|-------------|--------------------|
| $((0, 1), (0, 1), (0, 2))$ | non empty, of dim 2           | one         | one vortex of index 2 |
| $((0, 1), (0, 1), (1, 1))$ | empty, exp. dim= 0            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0, 1), (0, 1), (3, 6))$ | empty, exp. dim= 0            | one         | no vortices (wrong isotropy) |
| $((0, 1), (0, 1), (4, 5))$ | empty, exp. dim=0             | empty (see below) | no vortices (wrong isotropy) |
| $((0, 1), (2, 2), (0, 2))$ | empty, exp. dim=−2            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0, 1), (2, 2), (1, 1))$ | empty, exp. dim=−4            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0, 1), (2, 2), (3, 6))$ | empty, exp. dim=−4            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |
| $((0, 1), (2, 2), (4, 5))$ | empty, exp. dim=−4            | empty since $n_0 \geq 1$ | no vortices (wrong isotropy) |

The reason why the bundle with isotropy $((0, 1), (0, 1), (4, 5))$ is empty is that we can take the isotropy vector $\epsilon = (1, −1, −1)$, so $n_+ = 1$ and $\frac{3}{2} − \frac{1}{2} − \frac{1}{2} > 0$ which verifies the criterion mentioned before for the emptiness of the space of irreducible projectively flat connections.

Thus one more time our picture is similar to the case of the Poincaré homology sphere. On $\Sigma(2, 3, 7)$ we have the trivial connection, two irreducible flat connections, one Seiberg-Witten monopole, and a “cobordism” of $SO(3)$ monopoles between this Seiberg-Witten monopole and one of the irreducible flat connections.

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