**Fixed/Preassigned-Time Synchronization Control of Complex Networks With Time Varying Delay**

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**ABSTRACT** The problem of fixed-time (FXT) and preassigned-time (PAT) synchronization for delayed dynamic complex networks via developing control schemes is concerned. First, by comparing with the existing results, some new results for delayed networks model to ensure the synchronization of FXT and PAT are obtained. Second, some looser conditions are obtained for FXT synchronization, and several more accurate settling time (ST) estimates are established through several special functions. Third, by using some non-trivial control strategy, the PAT synchronization is investigated with limited control gains, and the synchronization time is able to specified in advance on the basis of practical need and is not relate to any parameter and any initial value. In particular, as a distinct form, the essential conditions for the FXT and PAT synchronization are obtained for the coupled neural networks. last, the improved the FXT and PAT synchronization results of dynamic delayed networks are demonstrated by one numerical example.

**INDEX TERMS** Complex delayed networks, preassigned-time synchronization, fixed-time synchronization, settling time.

**I. INTRODUCTION**

Complex networks are complex aggregates composed of interrelated and interactive basic units (nodes) with certain characteristics and functions. They usually contain a huge number of nodes and complex topological connections, showing rich and diverse statistical characteristics and dynamic evolution behavior [1]–[7]. As a matter of fact, complex networks are everywhere in the actual world, and they all affect and dominate the development process of modern human society at any time. In complex networks dynamic behavior and groups, more and more scholars and experts pay attention to synchronous behavior because of its realistic significance and universality. For various dynamic network, some interesting works on synchronization have been received [8]–[13].

In a word, synchronization is one type of overall harmonized dynamic action formed by the mutual coupling between external forces and dynamic systems. In the present research works of control theory, complex networks synchronization commonly takes an limitless time to attain. However, for real complex networks, we often hope that the complex networks can synchronize in a limited time as soon as possible. In addition, due to the fractional power term of the finite-time controller, the finite-time control has high robustness and anti-disturbance. At the moment, some scholars have started to investigated the finite-time synchronization problem of chaos system and achieved some worthy works [14]–[18].

Generally speaking, under the same condition, solutions starting from different initial values will achieve synchronization in different finite time which results in different synchronization settling time and estimation. However, due to the influence of external interference and other factors on the actual networks, it is difficult to know the initial value of the system, which will lead to the inability to prove the boundedness of the settling time of this kind of network. For the sake of overcoming the limitation and inconvenience caused by the relationship between initial value and settling time estimation, Polyakov brought forward the idea of FXT stability theory of chaos system in 2012 and provided the relevant criterion [19], which supplies a theoretical principle for investigating the FXT synchronization of dynamic systems.
FXT synchronization means that the network is first synchronized in finite-time, and the synchronization ST and its estimation are not rely on the initial value of the system and merely depend on system parameters and control parameters. Obviously, compared with finite-time synchronization, FXT synchronization is more convenient and extensive in practical application because the upper bound estimation of ST has not connection with the initial value [20]–[22]. In [20], Feng et al. considered FXT synchronization problem of complex-valued coupled memristive neural networks. Hu and Jiang studied FXT estimation and stabilization problem for chaos networks based on special functions in [21]. Cao and Li researched FXT synchronization problem of delayed memristive recurrent neural networks in [22].

It is worth noting that the ST is relevant to the parameters of controllers and systems in FXT synchronization problem. Nevertheless, in applications, it may be more preferable for the network to achieve synchronization in PAT which is determined by practical need and is unconcerned of any parameters and any initial value. Hence, PAT synchronization has received increasing attention because it can provide a solution that enables the network to reach synchronization within any specified time recently [23]–[26]. In [23], Hu et al. discussed FXT and PAT synchronization problem of coupled system by improving FXT stability theory. Liu et al. considered PAT synchronization problem of coupled networks via continuous activation function by designing delayed and unbounded control strategy in [24]. In [25], Ren et al. discussed PAT lag consensus control problem for leader-following systems. In [26], Liu et al. investigated PAT control problem for uncertain multi-agent systems of prescribed performance.

As we know, time delay is unavoidable because of the limited communication speed and likely causes undesirable dynamic actions, for instance instability behavior and oscillation. In order to overcome the negative effects of time delay, some practical control methods for complex networks are designed to solve the convergence or stability problems of time-delay systems. Thus, the influence of time varying delay should be considered in the synchronization of dynamic system. Recently, a lot of works for delays complex systems have been received. Yet, to the best of my knowledge, the FXT and PAT synchronization for complex delayed systems via feedback strategy obtain few attention. Therefore, we should make great hard to solve the interesting question in the paper.

Based on the above discussion, the problem of FXT and PAT synchronization for delayed dynamic complex networks via developing control schemes is concerned. FXT and PAT synchronization criteria and some corollaries are obtained in our paper which are very verifiable and useful in application. Compare with the previous results, our main results are more general and less conservative. The innovations of the paper are at least the following aspects. First, in consideration of the significance for time delay, we investigated the complex system with varying delay in this paper. Second, some improved requirements for FXT synchronization are obtained and a number of new estimates for ST are achieved in the light of a few special functions. Third, by designing controller without a linear sign function term, FXT synchronization of delayed systems is talked. Forth, several innovative control protocols with limit control strategy are explored to the PAT synchronization, where the synchronized time is not relevant with parameter and initial value of system and can be presupposed on the basis of actual need. Besides, as a distinct form, the essential conditions for the FXT and PAT synchronization are obtained for a type of coupled neural networks. In particular, some corollaries are given and compared with now available works, our works are less conservative.

The structure of the rest of the article is given as follows. Preliminaries and dynamical delayed complex networks are provided in Section 2. In Section 3, FXT synchronization of the investigated delayed model under a simpler controller is investigated. The PAT synchronization is considered by using some control strategies with finite control gains in Section 4. An example is provided to illustrate the feasibility of the given means in Section 5. Last, in Section 6, a simple summary is obtained.

**Notations:** Let $R$ represents the real numbers set, $R^n$ is $n$-dimensional real vectors set, and $R^{n \times n}$ denotes all $n \times n$ real matrices set, where $n$ is positive integer. For any $v \in R$, $\text{sign}(v)$ represents the sign function for $v$. For any $y = (y_1, y_2, \ldots, y^n)^T \in R^n$, the norm is denoted by $\|y\| = (y^T z)^{\frac{1}{2}}$, $|z|^\mu = (|z_1|^\mu, |z_2|^\mu, \ldots, |z^n|^\mu)^T$, $\text{sign}(z) = \text{sign}(z_1), \text{sign}(z_2), \ldots, \text{sign}(z^n))^T$. $y \circ z = (y_1 z_1, \ldots, y^n z^n)^T$. $I_n$ represents the $n$ dimensions identity matrix. $\lambda_{\max}(A)$ represents the maximum eigenvalues of matrix $A$.

II. PRELIMINARIES

A type delayed complex system is considered.

\[
\dot{y}_i(t) = f(t, y_i(t), y_i(t - \tau(t))) + c \sum_{j=1}^{N} b_{ij} \Gamma(y_j(t)) - y_i(t)), \quad i \in \mathcal{I} = \{1, 2, \ldots, N\},
\]

where $y_i(t) = (y_i^1(t), y_i^2(t), \ldots, y_i^n(t))^T \in R^n$ is the state of the $i$th node, $f : \mathbb{R} \times R^n \times R^n \rightarrow R^n$ is a continuous nonlinear function, the time delay $\tau(t)$ represents the internal delay, $c > 0$ is the coupling strength, $\Gamma = \text{diag} \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is the inner matrix with $\gamma_i > 0$, $B = (b_{ij})_{N \times N}$ is the coupling connecting matrix which satisfies conditions as follows.

\[
b_{ij} \geq 0, \quad i \neq j, \quad b_{ii} = - \sum_{j=1}^{N} b_{ij}.
\]

In view of the condition (2), model (1) can be rewritten in the following.

\[
\dot{y}_i(t) = f(t, y_i(t), y_i(t - \tau(t))) + c \sum_{j=1}^{N} b_{ij} \Gamma(y_j(t)), \quad i \in \mathcal{I}.
\]
Definition 1 ([14]): The hyperplane
\[ \Lambda = \{y_1^T, \ldots, y_N^T \in \mathbb{R}^n \mid y_1(t) = \cdots = y_N(t) = z(t) \in \mathbb{R}^n \} \]
is said to be the synchronous manifold of (3), and \( z(t) \) is called the synchronization state of the (3). Assume that the directed network (1) is strongly connected and balanced

Obviously, on the basis of (3), \( z(t) \) satisfies the equation as follows.
\[ \dot{z}(t) = f(t, z(t), \dot{z}(t - \tau(t))), \quad i \in \mathcal{I}, \]
(4)
where \( z(t) \) may be a periodic orbit, an equilibrium point, or even a chaotic attractor.

Assumption 1 ([12]): For the vector-valued function \( f(t, y_1(t), y_i(t - \tau(t))) \), there exist constants \( l_1 > 0, l_2 > 0 \) such that
\[ (y_i(t) - z(t))^T [f(t, y_1(t), y_i(t - \tau(t))) - f(t, z(t), z(t - \tau(t)))] \leq l_1 (y_i(t) - z(t))^T (y_i(t) - z(t)) + l_2 (y_i(t) - z(t)) \]
\[ -z(t - \tau(t))^T (y_i(t) - z(t) - z(t - \tau(t))) \]
for any \( y_i(t), z(t) \in \mathbb{R}^n \).

Assumption 2: Time-delay function \( \tau(t) : [0, +\infty) \to [0, +\infty) \) is a real-valued continuous function and satisfies
\[ \dot{\tau}(t) \leq \sigma < 1. \]

Definition 2 ([23]): Dynamic systems (3) and (4) are said to realize fixed-time synchronization if there exists an FXT \( T_{\text{max}} \geq 0 \), which is not dependent of the initial values but may be relevant with control parameters and network parameters, and an ST function \( T(e(0)) \geq 0 \) such that
\[ \lim_{t \to T(e(0))} \|e(t)\| = 0, \quad e(t) = 0 \text{ for all } t \geq T(e(0)), \]
and \( T(e(0)) \leq T_{\text{max}} \) for any \( e(0) \in \mathbb{R}^{nN} \), where \( e(t) = (e^1(t), e^2(t), \ldots, e^N(t))^T \) and \( e_i(t) = y_i(t) - z(t) \) with \( i \in \mathcal{I} \).

Definition 3 ([23]): For a Pat Tp > 0, which is completely independent of system parameters and initial values, the networks (3) and (4) are said to be PAT synchronized within Tp if
\[ \lim_{t \to T_p} \|e(t)\| = 0, \quad e(t) = 0 \text{ for all } t \geq T_p. \]

Definition 4 ([27]): Define the incomplete beta function ratio as follows.
\[ I(x, p, q) = \frac{1}{B(p, q)} \int_0^x t^{p-1}(1-t)^{q-1} dt, \]
where \( 0 \leq x \leq 1, p > 0, q > 0 \), and \( B(p, q) \) is the beta function which is given by
\[ B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt. \]

Lemma 1 ([12]): Let \( b_1, b_2, \ldots, b_n \) are positive numbers and \( 0 \leq r_1 < r_2, \nu > 1 \), then
\[ \left( \sum_{i=1}^n b_i^{r_1} \right)^{\frac{1}{r_1}} \geq \left( \sum_{i=1}^n b_i^{r_2} \right)^{\frac{1}{r_2}}, \quad \sum_{i=1}^n b_i^{\nu} \geq n^{1-\nu} \left( \sum_{i=1}^n b_i \right)^\nu. \]

Lemma 2 ([12]): If \( Y \) and \( Z \) are real matrices, \( \varsigma > 0 \), then the following inequality holds.
\[ Y^T Z + Z^T X \leq \varsigma Y^T Y + \frac{1}{\varsigma} X^T X. \]

Lemma 3 ([23]): Assume that function \( V(e(t)) : \mathbb{R}^{nN} \to R \) is a positive definite and radially unbounded. If
\[ \dot{V}(e(t)) \leq k V(e(t)) - \alpha V^\beta(e(t)) - \beta V^\theta(e(t)), \]
e(t) \in \mathbb{R}^{nN}\{0\},
where \( k \in R, \alpha > 0, \beta > 0, 0 \leq \theta < 1, \delta > 1 \), then the results are true as follows.
1) If \( k \leq 0 \), the systems (3) and (4) are FXT synchronization with the ST \( T(e(0)) \) which is estimated by
\[ T(e(0)) \leq T_1 = \frac{\pi}{(\delta - \theta)^{\frac{1}{\beta}}} \frac{\csc(\epsilon \pi)}{\alpha^\delta \beta}, \]
where \( \epsilon = [(1 - \theta)/(\delta - \theta)]. \)
2) If \( 0 < k < \min(\alpha, \beta) \), the systems (3) and (4) are FXT synchronization with the ST \( T(e(0)) \) which is estimated by
\[ T(e(0)) \leq T_2 = \frac{\pi \csc(\epsilon \pi)}{\alpha(\delta - \theta) (\beta - k)} I \left( \frac{\alpha}{(\alpha - k)^\theta}, \frac{\beta}{(\beta - k)^\theta}, \epsilon \right), \]
\[ \times I \left( \frac{\beta}{(\alpha - k)} \right); 0, \epsilon, \epsilon \right). \]

Lemma 4 ([23]): Assume that function \( V(e(t)) : \mathbb{R}^{nN} \to R \) is a positive definite and radially unbounded. If
\[ \dot{V}(e(t)) \leq k V(e(t)) - \alpha V^\beta(e(t)) - \beta V^\theta(e(t)), \]
e(t) \in \mathbb{R}^{nN}\{0\},
where constants \( 0 < k < 2\sqrt{\alpha \beta}, \alpha > 0, \beta > 0, 0 \leq \theta < 1, \delta > 1 \) and \( \delta + \theta = 2 \), then the systems (3) and (4) are FXT synchronization with the ST \( T(e(0)) \) which is estimated by
\[ T(e(0)) \leq T_3 = \frac{1}{\delta - 1} \frac{2}{4 \alpha \beta - k^2} \left( \frac{\pi}{2} + \arctan \left( \frac{k}{4 \alpha \beta \beta - k^2} \right) \right). \]

Lemma 5 ([23]): Assume that function \( V(e(t)) : \mathbb{R}^{nN} \to R \) is a positive definite and radially unbounded. If
\[ \dot{V}(e(t)) \leq \frac{\hat{T}}{T_p} (\beta V(e(t)) + \alpha V^\beta(e(t)) + \beta V^\theta(e(t))), \]
e(t) \in \mathbb{R}^{nN}\{0\},
where constants \( k, \alpha > 0, \beta > 0, 0 \leq \theta < 1, \delta > 1 \), and \( T_p > 0 \), then the systems (3) and (4) are FXT synchronization within \( T_p \), which is estimated by
\[ \hat{T} = \begin{cases} 
T_1, & k \leq 0 \\
T_2, & 0 < k < \min(\alpha, \beta) \\
T_3, & 0 < k < 2\sqrt{\alpha \beta}, \delta + \theta = 2.
\end{cases} \]
Let \( e_i(t) = (e^1_i(t), e^2_i(t), \ldots, e^N_i(t))^T = y_i(t) - z(t) (i \in \mathcal{I}) \) be synchronization errors. For the sake of making the states of
system (3) FXT or PAT synchronize with \(z(t)\), then controlled delayed system is given in the following.

\[
\dot{y}_i(t) = \frac{f(t, y_i(t), y_j(t - \tau(t))) + c \sum_{j=1}^{N} b_{ij} \Gamma y_j(t) + u_i(t)}{\eta}.
\]

(5)

where \(u_i(t)\) is controller strategy.

### III. FIXED-TIME SYNCHRONIZATION

With the assistance of Lemmas 3 and 4, we will design appropriate \(\eta_1, \eta_2, \eta_3, \eta_4\) and \(\mu, \delta, \gamma\), such that the system (8) can realize FXT synchronization. The major results are presented in this section.

For the sake of obtaining the main works, we design the following feedback control.

\[
u_i(t) = -\eta_1 e_i(t) - \eta_2 \text{sign}(e_i(t)) \circ |e_i(t)|^\mu
\]

\[
-\eta_3 \text{sign}(e_i(t)) \circ |e_i(t)|^\delta
\]

\[
-\eta_2 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\mu}{\mu}} e_i(t)
\]

\[
-\eta_3 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\delta}{\delta}} e_i(t)
\]

(6)

where \(\eta_1, \eta_2, \eta_3, \eta_4 > 0, 0 \leq \mu < 1 \) and \(\delta > 1\).

According to the controller (6), then the error network is given as follows.

\[
e_i(t) = \tilde{f}(t, y_i, z, y_i^T, z^T) + c \sum_{j=1}^{N} b_{ij} \Gamma e_j(t) - \eta_1 e_i(t)
\]

\[
-\eta_2 \text{sign}(e_i(t)) \circ |e_i(t)|^\mu - \eta_3 \text{sign}(e_i(t)) \circ |e_i(t)|^\delta
\]

\[
-\eta_2 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\mu}{\mu}} e_i(t)
\]

\[
-\eta_3 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\delta}{\delta}} e_i(t)
\]

(7)

where \(\tilde{f}(t, y_i, z, y_i^T, z^T) = f(t, y_i, y_i(t - \tau(t))) - f(t, z(t), z(t - \tau(t)))\).

**Theorem 1:** Under Assumptions 1-2 and the control law (6), \(2 \lambda_1 - \eta_4 \leq 0\), with the synchronized time \(T(e(0))\) which is given by

\[
T(e(0)) \leq \tilde{T}_1 = \frac{\pi \csc(\epsilon \pi)}{\eta_2 (\delta - \mu)} \left( \frac{\eta_3}{\alpha} \right)^{\frac{\epsilon}{\epsilon - 1}} \left( \frac{\alpha}{\alpha + \eta_2 - \lambda} \right)^{\frac{1}{\epsilon - 1}} I \left( \frac{\alpha}{\alpha + \eta_2 - \lambda} \right)
\]

where \(\alpha = \eta_3 (n + 1) \frac{\lambda^{n+1}}{\lambda^n}, \epsilon = (1 - \mu)/(\delta - \mu)\).

1) if \(0 < \lambda \leq \min[\alpha, \eta_2]\), the controlled system (5) is the FXT synchronized with the synchronized time \(T(e(0))\) which is estimated by

\[
T(e(0)) \leq \tilde{T}_2 = \frac{\pi \csc(\epsilon \pi)}{\alpha (\delta - \mu)} \left( \frac{\alpha}{\eta_2 - \lambda} \right)^{1-\epsilon} I \left( \frac{\alpha}{\alpha + \eta_2 - \lambda} \right)^{\frac{1}{\epsilon - 1}} I \left( \frac{\alpha}{\alpha + \eta_2 - \lambda} \right)
\]

3) particularly, when \(\mu + \delta = 2\) in control law (6), the controlled system (5) is the FXT synchronized if \(4 \eta_2 \lambda > \lambda^2\) and the synchronized time \(T(e(0))\) is given by

\[
T(e(0)) \leq \tilde{T}_3 = \frac{\pi}{(\delta - 1)\sqrt{\gamma}} + \frac{2}{(\delta - 1)\sqrt{\gamma}} \arctan \left( \frac{\lambda}{\sqrt{\gamma}} \right),
\]

where \(\gamma = 4 \eta_2 \lambda - \lambda^2\).

**Proof:** Construct the Lyapunov function

\[
V(t) = \left( \sum_{i=1}^{N} e_i^T(t) e_i(t) + \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1}{2}}.
\]

(8)

Then, its derivative can be presented as follows.

\[
\dot{V}(t) = \frac{1}{2} V(t) \left[ 2 \sum_{i=1}^{N} e_i^T(t) \left( \tilde{f}(t, y_i, z, y_i^T, z^T) + c \sum_{j=1}^{N} b_{ij} \Gamma e_j(t) - \eta_1 e_i(t) - \eta_2 \text{sign}(e_i(t)) \circ |e_i(t)|^\mu
\]

\[
-\eta_3 \text{sign}(e_i(t)) \circ |e_i(t)|^\delta
\]

\[
\eta_2 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\mu}{\mu}} e_i(t)
\]

\[
-\eta_3 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\delta}{\delta}} e_i(t)
\]

\[
+ \frac{\eta_4}{1-\sigma} \sum_{i=1}^{N} e_i^T(t) e_i(t) - \eta_4 \sum_{i=1}^{N} e_i^T(t)
\]

\[
- \tau(t) e_i(t - \tau(t)) \right)
\]

\[
\leq \frac{1}{2} V(t) \left[ 2 \lambda_1 \sum_{i=1}^{N} e_i^T(t) e_i(t) + 2 \lambda_2 \sum_{i=1}^{N} e_i^T(t - \tau(t))
\]

\[
- \eta_1 \sum_{i=1}^{N} e_i^T(t) e_i(t)
\]

\[
- \eta_2 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\mu
\]

\[
- \eta_3 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\delta
\]

\[
\eta_2 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\mu}{\mu}} e_i(t)
\]

\[
-\eta_3 \left( \eta_4 \frac{\int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds}{||e_i||^2} \right)^{\frac{1+\delta}{\delta}} e_i(t)
\]

where \(\lambda_1, \lambda_2 \leq 0\), with the synchronized time \(T(e(0))\) which is given by

\[
T(e(0)) \leq \tilde{T}_3 = \frac{\pi}{(\delta - 1)\sqrt{\gamma}} + \frac{2}{(\delta - 1)\sqrt{\gamma}} \arctan \left( \frac{\lambda}{\sqrt{\gamma}} \right),
\]
\[
\begin{align*}
\sum & + \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} e_i^T(t)e_i(t) - \eta_4 \sum_{i=1}^{N} e_i^T(t) \\
& - \tau(t)e_i(t - \tau(t)) \bigg] \\
\leq & \frac{1}{2V(t)} \left[ \sum_{i=1}^{N} \sum_{k=1}^{n} e_i^T(t)(2I_1 - 2\eta_1 + \frac{\eta_4}{1 - \sigma})e_i^T(t) \\
& + 2c \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \gamma_k e_i^T(t) b_{ij} + b_{ij} e_j^T(t) \\
& - 2\eta_2 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\mu \\
& - 2\eta_3 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\delta \\
& - 2\eta_2 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\mu} \\
& - 2\eta_3 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\delta} \right] \\
\leq & \frac{1}{2V(t)} \left( \sum_{i=1}^{N} e_i^T(t) \right) \left( (2I_1 - 2\eta_1 + \frac{\eta_4}{1 - \sigma})I_N \\
& + 2c\gamma_k \frac{B^T + B}{2} e_i^T(t) - 2\eta_2 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \\
& \times \circ |e_i(t)|^\mu - 2\eta_3 \sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\delta \\
& - 2\eta_2 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\mu} \\
& - 2\eta_3 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\delta} \right), \quad (9)
\end{align*}
\]

where $\hat{e}_k = (e_{k1}, e_{k2}, \ldots, e_{kn})^T$ for $k = 1, 2, \ldots, n$. Since
\[
\sum_{i=1}^{N} e_i^T(t) \text{sign}(e_i(t)) \circ |e_i(t)|^\mu = \sum_{i=1}^{N} e_i^T(t)||e_i(t)||^\mu = \sum_{i=1}^{N} \sum_{k=1}^{n} |e_i^T(t)|^{1+\delta}
\]
and using Lemma 1, it can be obtained that
\[
\sum_{i=1}^{N} \sum_{k=1}^{n} |e_i^T(t)||^{1+\delta} + \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\delta} \\
\geq (nN + 1)^{1+\delta} \left( \sum_{i=1}^{N} \sum_{k=1}^{n} |e_i^T(t)|^2 + \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{\tau(t)}^{t} e_i^T(s)e_i(s)ds \right)^{1+\delta} \\
\geq (nN + 1)^{1+\delta} V^{1+\delta}(t).
\]

Then, it can be gotten that
\[
\hat{V}(t) \leq \lambda V(t) - \eta_2 V^\mu(t) - \eta_3 (nN + 1)^{1+\delta} V^\delta(t). \quad (10)
\]

where $\lambda_k = \lambda_{\text{max}} \left( (I_1 - 1 + \frac{\eta_4}{1 - \sigma})I_N + c\gamma_k \frac{B^T + B}{2} \right)$, $\lambda = \text{max}(\lambda_k)$.

If $\lambda \leq 0$, for $e(t) \in R^{nN} \setminus \{0\}$
\[
\hat{V}(t) \leq -\eta_2 V^\mu(t) - \eta_3 (nN + 1)^{1+\delta} V^\delta(t).
\]

Hence, from Lemmas 3 and 4, the system (5) is FXT synchronization. This proves Theorem 1. \hfill \Box

Especially, a type of controlled network is considered.
\[
\hat{y}_i(t) = -A_1 y_i(t) + Dg(y_i(t)) + Cg(y_i(t - \tau(t))) \\
+ c \sum_{j=1}^{N} b_{ij} \Gamma y_j(t) + u_i(t), \quad (11)
\]

where $i \in \mathcal{S}$, $b_{ij}$ is defined as (2), $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in R^n$ is the state vector of the $i$th neuron, $A = \text{diag}(a_1, a_2, \ldots, a_n)$ is the decay matrix with $a_i > 0$, $D = (d_{ij})_{n \times n}$ and $C = (c_{ij})_{n \times n}$ are the configuration matrix and delayed configuration matrix, respectively, $g(y_i(t)) = (g_1(y_{i1}(t)), g_2(y_{i2}(t)), \ldots, g_n(y_{in}(t)))^T$ is the activation nonlinear function.

For the sake of ensuring the FXT synchronization for the network. The statement is provided in the following.

Assumption 3: Suppose $s > 0$, then
\[
(g(y_i) - g(z))^T (g(y_i) - g(z)) \leq s(y_i - z)^T (y_i - z),
\]

for any $y_i, z \in R^n$.

The synchronization state $z(t)$ is given by
\[
\hat{z}(t) = -A z(t) + Dg(z(t)) + Cg(z(t - \tau(t))). \quad (12)
\]

Corollary 1: Under Assumptions 2-3 and the controller (6), $s - \eta_4 \leq 0$.

1) the controlled system (11) is the FXT synchronized if $\lambda \leq 0$ with the synchronization time $T(e(0))$ which is given by
\[
T(e(0)) \leq \tilde{T}_1 = \frac{\pi}{\eta_2(\delta - \mu)} \left( \frac{\eta_2}{\alpha} \right)^{\frac{\epsilon}{\delta}} \csc(\epsilon \pi),
\]

where $\alpha = \eta_3(nN + 1)^{1+\delta}$, $\epsilon = (1 - \mu)/(\delta - \mu)$.

2) if $0 < \lambda < \text{min}(\alpha, \eta_2)$, the controlled system (11) is the FXT synchronized with the synchronization time $T(e(0))$ which is given by
\[
T(e(0)) \leq \tilde{T}_2 = \frac{\pi \csc(\epsilon \pi)}{\alpha(\delta - \mu)} \left( \frac{\alpha}{\eta_2} \right)^{1-\epsilon} I \left( \frac{\alpha}{\alpha + \eta_2 - \lambda}, 1 - \epsilon, \epsilon \right),
\]

\[
\csc(\epsilon \pi) = \frac{\eta_2}{\alpha(\delta - \mu)} \left( \frac{\alpha}{\alpha + \eta_2 - \lambda}, 1 - \epsilon, \epsilon \right).
\]
3) particularly, when $\mu + \delta = 2$ in control law (6), the controlled system (11) is the FXT synchronized if $4\eta_2 \hat{\alpha} > \lambda^2$ and the synchronization time $T(e(0))$ is given by

$$T(e(0)) \leq \tilde{T}_3 = \frac{\pi}{(\delta - 1) \sqrt{\gamma}} + \frac{2}{(\delta - 1) \sqrt{\gamma}} \arctan \left( \frac{\lambda}{\sqrt{\gamma}} \right),$$

where $\gamma = 4\eta_2 \hat{\alpha} - \lambda^2$.

**Proof:** Under assumption 2 and Lemma 2, we can obtain

$$(y(t) - z(t))^T \left( f(t, y(t), y(t - \tau(t))) - f(t, z(t), z(t - \tau(t))) \right)$$

where $\lambda = \lambda_{\text{max}}(-A - A^T + DD^T + CC^T + sI_N)$ which indicates assumption 1 establishes and $l_1 = \frac{\lambda}{2}, l_2 = \frac{\lambda}{4}$. Hence, the coupled system (11) is FXT synchronized to (12) from Theorem 1. This proves Corollary 1.

Assume matrix $C = 0$, the network (11) can be provided as follows.

$$\dot{y}_i(t) = -Ay_i(t) + Dg(y_i(t)) + e_j \sum_{j=1}^{N} b_{ij} \Gamma y_j(t) + u_i(t). \quad (13)$$

Hence, the synchronization state $z(t)$ is represented by

$$\dot{z}(t) = -Az(t) + Dg(z(t)), \quad i \in \mathcal{I}. \quad (14)$$

**Corollary 2:** Under Assumption 3 and the controller (6), 1) the controlled system (13) is the FXT synchronized if $\lambda \leq 0$ and the synchronization time $T(e(0))$ is given by

$$T(e(0)) \leq \tilde{T}_1 = \frac{\pi}{\eta_2(\delta - \mu)} \left( \frac{\eta_4}{\eta_2} \right)^{\epsilon} \csc(\epsilon \pi),$$

where $\hat{\alpha} = \eta_3(nN + 1)^{\frac{1}{4}}$, $\epsilon = (1 - \mu)/(\delta - \mu)$.

2) if $0 < \lambda < \min[\hat{\alpha}, \eta_2]$, the controlled delayed system (13) is the FXT synchronized with the synchronization time $T(e(0))$ which is estimated by

$$T(e(0)) \leq \tilde{T}_2 = \frac{\pi \csc(\epsilon \pi)}{\hat{\alpha}(\delta - \mu)} \left( \frac{\hat{\alpha}}{\eta_2} \right)^{1-\epsilon} \left( \frac{\hat{\alpha}}{\alpha + \eta_2 - \lambda} \right)^{\epsilon} \left( \frac{\hat{\alpha}}{\alpha + \eta_2 - \lambda} \right)^{1-\epsilon} \frac{1}{2} + \frac{\eta_4}{\eta_2} \frac{1}{\alpha + \eta_2 - \lambda} \left( \frac{\eta_4}{\eta_2} \right)^{\epsilon} \left( \frac{\eta_4}{\eta_2} \right)^{1-\epsilon} \frac{1}{2} \epsilon$$

where $\tilde{T}_2 = f(t, y(t), z(t)) = f(t, y(t), z(t) - \tau(t)) - f(t, z(t), z(t) - \tau(t))$.

**Theorem 2:** Under Assumptions 1-2 and the control strategy (15), then the controlled system (5) is the PAT synchronized within the PAT $T_p$ if $\lambda \leq 0$, $2l_2 = \eta_4 \leq 0$.

**Proof:** Define the following Lyapunov function as

$$V(t) = \left( \sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds \right)^{\frac{1}{2}}.$$
Then, the derivative of \( V(t) \) is represented in the following.

\[
\dot{V}(t) \leq \frac{1}{2V(t)} \left( \sum_{k=1}^{n} (\alpha^{k}(t))^T \left( 2(l_k - 2\eta_1 + \frac{\eta_4}{1-\sigma})I_N \right. \right.
\]
\[
\left. \left. + 2\gamma_2 \frac{B^T + B}{2} \right) \dot{\alpha}(t) - \frac{\dot{T}_3}{T_p} \eta_2 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\mu \right. \\
\left. - \frac{\dot{T}_1}{T_p} \eta_3 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\delta \right) \\
\left. \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \dot{\alpha}(t)(\alpha(t) - \alpha_i(t)) \right) \dot{\alpha}(t) \right) + \eta_2 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\mu \\
\left. - \frac{\dot{T}_1}{T_p} \eta_3 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\delta \right)
\]
\[
\sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \dot{\alpha}(t)(\alpha(t) - \alpha_i(t)) \right) \dot{\alpha}(t) = \frac{\dot{T}_1}{T_p} \eta_3 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\delta
\]
\[
\sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \dot{\alpha}(t)(\alpha(t) - \alpha_i(t)) \right) \dot{\alpha}(t) = \frac{\dot{T}_1}{T_p} \eta_3 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\delta
\]
\[
\sum_{i=1}^{N} \int_{t-\tau(t)}^{t} \dot{\alpha}(t)(\alpha(t) - \alpha_i(t)) \right) \dot{\alpha}(t) = \frac{\dot{T}_1}{T_p} \eta_3 \sum_{i=1}^{N} \dot{\alpha}(t) \times \text{sign}(\alpha_i(t)) \circ |\alpha_i(t)|^\delta
\]

Hence, according to Lemma 5, the system (5) is PAT synchronized under the PAT 
controller (18), the controlled system (5) is the PAT synchronized 
within the PAT 

\[
\dot{V}(t) = -\eta_1 \alpha(t) - \frac{\dot{T}_3}{T_p} \eta_2 \text{sign}(\alpha(t)) \\
\times \circ |\alpha(t)|^\mu - \frac{\dot{T}_3}{T_p} \eta_3 \text{sign}(\alpha(t)) \circ |\alpha(t)|^\delta \\
- \frac{\dot{T}_3}{T_p} \eta_2 \left( \frac{\eta_4}{1-\sigma} \right) e_i(t) s_i(t) ds \\
\times e_i(t) \times |e(t)|^\mu \\
- \frac{\dot{T}_3}{T_p} \eta_3 \left( \frac{\eta_4}{1-\sigma} \right) e_i(t) s_i(t) ds \\
\times e_i(t) \times |e(t)|^\delta.
\]

Particularly, via the following control scheme

\[
\dot{u}(t) = -\eta_1 \alpha(t) - \frac{\dot{T}_3}{T_p} \eta_2 \text{sign}(\alpha(t)) \\
\times \circ |\alpha(t)|^\mu - \frac{\dot{T}_3}{T_p} \eta_3 \text{sign}(\alpha(t)) \circ |\alpha(t)|^\delta \\
- \frac{\dot{T}_3}{T_p} \eta_2 \left( \frac{\eta_4}{1-\sigma} \right) e_i(t) s_i(t) ds \\
\times e_i(t) \times |e(t)|^\mu \\
- \frac{\dot{T}_3}{T_p} \eta_3 \left( \frac{\eta_4}{1-\sigma} \right) e_i(t) s_i(t) ds \\
\times e_i(t) \times |e(t)|^\delta.
\]

where \( \eta_1, \eta_2, \eta_3, \eta_4 > 0 \) are the control strengths, \( T_p > 0 \) is a PAT, and \( \delta \) satisfies \( \delta > 1, \dot{T}_3 \) is defined in Theorem 1, and the PAT synchronization of the system (5) is achieved as follows.

\[
\dot{V}(t) \leq -\alpha V(t) - \beta V^\theta(t), \quad e(t) \in R^n \setminus \{0\},
\]

where \( \alpha > 0, \beta > 0, 0 < \theta < 1, \delta > 1 \), is widely utilized to guarantee FXT synchronization. Different from these,

\[
\dot{V}(t) \leq -\alpha V^\delta e(t) - \beta V^\theta e(t), \quad e(t) \in R^n \setminus \{0\},
\]

where \( \alpha > 0, \beta > 0, 0 < \theta < 1, \delta > 1 \), is introduced in this article. Evidently, the second equation is

Proof: For \( e(t) \in R^n \setminus \{0\} \), it can be obtained that

\[
\dot{V}(t) \leq \lambda V(t) - \frac{\dot{T}_2}{T_p} \eta_2 V^\mu(t) + \eta_3 (nN + 1) \frac{\dot{T}_2}{T_p} V^\delta(t).
\]

As a result of \( 0 < T_p \leq \dot{T}_2 \), if \( \lambda > 0 \),

\[
\dot{V}(t) \leq -\lambda V(t) - \frac{\dot{T}_2}{T_p} \eta_2 V^\mu(t) + \eta_3 (nN + 1) \frac{\dot{T}_2}{T_p} V^\delta(t).
\]

Hence, according to Lemma 5, the system (5) is PAT synchronized. This proves Theorem 3.

Remark 2: In most previous FXT synchronization results,
less conservative and more flexible compared with the first equation. In Lemma 3, in order to guarantee the FXT synchronization, the condition \( 0 < k < \min(\alpha, \beta) \) is essential. We replace condition \( 0 < k < \min(\alpha, \beta) \) with condition \( 0 < k < 2^{1/2} \alpha \beta \) in Lemma 4. It is obvious that the later is less rigorous and more relaxed.

**Remark 3:** In controller design, the linear part \(-k_i \text{sign}(\cdot)\) is essential and indispensable in most previous results of FXT synchronization [28]–[32]. As everyone knows, the sign function will cause the oscillation behavior of the network. Distinct in these control laws, a simpler controller without sign function term (6) is designed to realize FXT complex networks synchronization in this article. Besides, based on the Lyapunov functions as follows:

\[
V(t) = \sum_{i=1}^{N} e_i^T(t) e_i(t),
\]

FXT synchronization of coupled systems has been investigated in [13], [17], [33]. Different from these works, a distinct Lyapunov function of (8) is provided in the paper in order to investigate FXT synchronization.

**Remark 4:** So far as we know, PAT synchronization are few investigated of dynamic delayed systems. In [24], by proposing an unbounded and time varying control strategy, Liu studied cluster synchronization with a PAT of complex networks. Distinct in the result [24], new control strategy (15), (18) and (21) are provided to analyze the PAT synchronization. Significantly, the control gains of these controllers can be effectively realized and are limited in practice. Especially, PAT synchronization time can not be dependent of any parameter and any initial value. Therefore, the application prospect of PAT synchronization is more broader than FXT synchronization.

**Remark 5:** Using Lemmas 3-5, the network can realize fixed-time synchronization under linear feedback control in this paper. Feedback control is continuous control approach. For discontinuous control approach, such as adaptive intermittent control, a lot of results have been obtained. However, the fixed time synchronization of networks cannot be received by using Lemmas 3-5 under adaptive intermittent control. Hence, it is meaningful to concern this problem in our recent research topics.

**Remark 6:** In [23], the model

\[
\dot{z}_i(t) = Az_i(t) + f(z_i(t)) + \Delta(t) + \alpha \sum_{j=1}^{M} c_{ij} \Gamma \dot{z}_j(t), \quad i \in \mathcal{M}
\]

has been considered. In our paper, the model

\[
\dot{y}_i(t) = f(t, y_i(t), y_i(t - \tau(t))) + c \sum_{j=1}^{N} b_{ij} \Gamma y_j(t), \quad i \in \mathcal{F}
\]

has been considered. The model in [23] is a special form of the model in this paper. In addition, the model we considered in this paper has time delay. In the proof of Theorem 1, the treatment of time delay is very complex. It is embodied in the construction of Lyapunov function and controller. In the constructed Lyapunov function, the integral term \( \int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds \) can eliminate the influence of delayed error nonlinearity about the states of delayed complex networks. In order to FXT and PAT synchronization,

\[
-\eta_2 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds \right)^{1+4} \frac{e_i(t)}{||e_i(t)||^2}
\]

and

\[
-\eta_3 \left( \frac{\eta_4}{1 - \sigma} \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) e_i(s) ds \right)^{1+4} \frac{e_i(t)}{||e_i(t)||^2}
\]

have been contained in controller (6).

**V. NUMERICAL SIMULATIONS**

In the section, a coupled network is considered to show the availability of the outcomes achieved in the article.

**Example:** The dynamic networks model for variable delay is considered in the following.

\[
\dot{y}_i(t) = -A y_i(t) + D g_1(y_i(t)) + C g_2(y_i(t - \tau(t))) + c \sum_{j=1}^{M} b_{ij} \Gamma y_j(t) + u_i(t), \quad i = 1, 2, \ldots, 8, \quad (22)
\]

where \( y_i(t) = (y_1^i(t), y_2^i(t)) \in \mathbb{R}^2, \quad i = 1, 2, \ldots, 8, \quad g_1(u) = g_2(u) = \tan(h(u_1), \tan(h(u_2))), \quad \tau(t) = \epsilon^2/(1 + \epsilon^2), \quad c = 2, \) and \( A, \Gamma, \) and \( B, \) as shown at the bottom of the next page.

In order to synchronize the network (22), that is, \( \lim_{t \to \infty} ||y_i(t) - z(t)|| = 0, \quad i = 1, 2, \ldots, 8, \) the synchronized equation is given in the following.

\[
\dot{z}(t) = -A z(t) + D g_1(z(t)) + C g_2(z(t - \tau(t))). \quad (23)
\]

The dynamic property of network (23) is revealed in Fig. 1 with the initial values \((z_1(s), z_2(s))^T = (0.8, 0.6)^T\) of

![FIGURE 1. The dynamic of (22) with the initial conditions](image)
Let \( \eta_1 = 13, \eta_2 = 2.5, \eta_3 = 3, \eta_4 = 1, s = 1 \). By calculation, \( \hat{\lambda} = \lambda_{\text{max}}(-A - A^T + DD^T + CC^T + \eta N) = 28.9614, \hat{\lambda} = 2.1492 \). In the following, two different parameters are selected in the controller \((6)\): 1) \((\mu, \delta) = (0.5, 1.1)\) and 2) \((\mu, \delta) = (0.9, 1.1)\). It can be obtained by calculation that \( \hat{\alpha} = 2.6037, \gamma = 21.4179 \). By Corollary 1, the system \((22)\) can be synchronized via the controller \((6)\) with FXT \( T < \hat{T}_2 = 18.4466, T < \hat{T}_3 = 8.6671 \). The simulation results are presented in Figs. 2 and 3. In the following, we provide a comparison between 2) and 3) in Corollary 1 when \((\mu, \delta) = (0.9, 1.1)\). Then it can be obtained that the conditions 2) and 3) of Corollary 1 are satisfied. By Corollary 1, the system \((22)\) can be synchronized via the controller \((6)\) with FXT \( T < \hat{T}_2 = 23.7101, T < \hat{T}_3 = 8.6671 \) which is revealed in Fig 4. As can be seen from the value of \( \hat{T}_3, \hat{T}_3 \), the later is less rigorous and more relaxed under the identical conditions.

In the following, the PAT synchronization of the network \((22)\) will be verified. The rest of control parameters is determined to the following two sets: 1) \((\mu, \delta, T_p) = (0.5, 1.1, 15)\) and 2) \((\mu, \delta, T_p) = (0.9, 1.1, 7)\). Evidently, all conditions in Corollary 4 and 5 are satisfied and the PAT synchronization is guaranteed under the control schemes \((18)\)
and (21). The corresponding numerical results are illustrated in Figs. 5 and 6.

**Remark 7:** In numerical simulations, two different parameters are selected in the controller (6): 1)\((\mu, \delta) = (0.5, 1.1)\) and 2)\((\mu, \delta) = (0.9, 1.1)\). Through simple analysis, it can be seen that the first case satisfies the second condition of Corollary 1. The second case satisfies the second and the third conditions of Corollary 1. By Corollary 1, the system (22) can be synchronized via \((\mu, \delta) = (0.5, 1.1)\) with settling time \(\tilde{T}_2 = 18.4466\). When \((\mu, \delta) = (0.9, 1.1)\), the system (22) can be synchronized with settling time \(\tilde{T}_2 = 23.7101\) and \(\tilde{T}_3 = 8.6671\) which is revealed in Fig. 4. As can be seen from the values of \(\tilde{T}_2, \tilde{T}_3\), the later is less rigorous and more relaxed under the identical conditions. In the following, the PAT synchronization of the network (22) has been verified. The rest of control parameters is determined to the following two sets: 1)\((\mu, \delta, T_p) = (0.5, 1.1, 15)\) and 2)\((\mu, \delta, T_p) = (0.9, 1.1, 7)\). The corresponding numerical results are illustrated in Figs. 5 and 6.

**VI. CONCLUSION**

In this essay, the FXT and PAT synchronization are investigated for a type coupled system with delay. Firstly, the coupled dynamical network model we investigated has time varying delay. Secondly, the established conditions are more relaxed in this paper and comparison with the existing results, the values for ST are more precise. Thirdly, the designed controller without \(-k_1\text{sign}(\cdot)\) in FXT synchronization is simpler and more effective, and the given controllers with limit control gains in PAX synchronization are distinct from the controller in [24] with limit gains. Moreover, when \(\tau(t) = 0\), a exceptional case which investigated in [11], [15], [16], [25] can be obtained. In this sense, the works achieved in the paper are more common. Finally, the numerical results are obtained to demonstrate the effectiveness of the existing scheme.

Recently, intermittent control has attracted increasing interest naturally in a variety of applications because this control method is more economic and can reduce the amount of the transmitted information. For adaptive control, the control parameters can automatically adjust themselves according to some proper updating laws. As we know, fixed-time synchronization of delayed dynamic networks under adaptive intermittent control are few investigated. Hence, it is meaningful to concern this problem in our recent research topics.

**REFERENCES**

[1] L. Chua and T. Roska, *Cellular Neural Networks and Visual Computing: Foundation and Applications*. New York, NY, USA: Cambridge Univ. Press, 2002.

[2] M. Faloutsos, P. Faloutsos, and C. Faloutsos, “On power-law relationships of the internet topology,” *Comput. Commun. Rev.*, vol. 29, no. 4, pp. 251–262, Oct. 1999.

[3] A. Cardillo, S. Scellato, V. Latora, and S. Porta, “Structural properties of planar graphs of urban street patterns,” *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 73, no. 6, Jun. 2006, Art. no. 066107.

[4] M. E. J. Newman, “Scientific collaboration networks. I. Network construction and fundamental results,” *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 64, no. 1, Jun. 2001, Art. no. 016131.

[5] N. Guelzim, S. Bottani, P. Bourgine, and F. Képès, “Topological and causal structure of the yeast transcriptional regulatory network,” *Nature Genet.*, vol. 31, no. 1, pp. 60–63, May 2002.

[6] Q. Wang and C. Sun, “Distributed asymptotic consensus in directed networks of nonaffine systems with nonvanishing disturbance,” IEEE/CAA J. Autom. Sinica, vol. 8, no. 6, pp. 1133–1140, Jun. 2021.

[7] Q. Wang, H. E. Psilakis, C. Sun, and F. L. Lewis, “Adaptive NN distributed control for time-varying networks of nonlinear agents with antagonistic interactions,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 6, pp. 2573–2583, Jun. 2021, doi: 10.1109/TNNLS.2020.3006840.

[8] S. H. Strogatz and I. Stewart, “Coupled oscillators and biological synchronization,” *Sci. Amer.*, vol. 269, no. 6, pp. 102–109, 1993.

[9] C. M. Gray, “Synchronous oscillations in neuronal systems: Mechanisms and functions,” *J. Comput. Neurosci.*, vol. 1, nos. 1–2, pp. 11–38, 1994.

[10] R. Wang and L. Chen, “Synchronizing genetic oscillators by signaling molecules,” *J. Biol. Rhythms*, vol. 20, no. 3, pp. 257–269, Jun. 2005.

[11] C. Hu, H. He, and H. Jiang, “Edge-based adaptive distributed method for synchronization of intermittently coupled spatiotemporal networks,” *IEEE Trans. Autom. Control*, early access, Jun. 14, 2021, doi: 10.1109/TAC.2021.3088805.

[12] M. Liu, H. Jiang, C. Hu, Z. Yu, and Z. Li, “Pinning synchronization of complex delayed dynamical networks via generalized intermittent adaptive control strategy,” *Int. J. Robust Nonlinear Control*, vol. 30, no. 1, pp. 421–442, Jan. 2020.

[13] M. Liu, H. Jiang, and C. Hu, “Synchronization of hybrid-coupled delayed dynamical networks via aperiodically intermittent pinning control,” *J. Franklin Inst.*, vol. 353, pp. 2722–2742, Aug. 2016.

[14] J. Li, H. Jiang, C. Hu, and A. Alsaedi, “Finite/ Fixed-time synchronization control of coupled memristive neural networks,” *J. Franklin Inst.*, vol. 356, no. 16, pp. 9928–9952, Nov. 2019.
[15] G. Ji, C. Hu, J. Yu, and H. Jiang, “Finite-time and fixed-time synchronization of discontinuous complex networks: A unified control framework design,” J. Franklin Inst., vol. 355, no. 11, pp. 4665–4685, Jul. 2018.
[16] M. Liu, H. Jiang, and C. Hu, “Aperiodically intermittent strategy for finite-time synchronization of delayed neural networks,” Neurocomputing, vol. 310, pp. 1–9, Oct. 2018.
[17] M. Liu, H. J. Jiang, and C. Hu, “Finite-time synchronization of delayed dynamical networks via aperiodically intermittent control,” J. Franklin Inst., vol. 354, pp. 5374–5397, Sep. 2017.
[18] D. Van Vu, M. H. Trinh, and H.-S. Ahn, “Distance-based formation tracking with unknown bounded reference velocity,” in Proc. 20th Int. Conf. Control, Autom. Syst. (ICCAS), Oct. 2020, pp. 524–529.
[19] A. Polyakov, “Nonlinear feedback design for fixed-time stabilization of linear control systems,” IEEE Trans. Autom. Control., vol. 57, no. 8, pp. 2106–2110, Aug. 2012.
[20] L. Feng, C. Hu, J. Yu, H. Jiang, and S. Wen, “Fixed-time synchronization of coupled memristive complex-valued neural networks,” Chaos, Solitons Fractals, vol. 148, Jul. 2021, Art. no. 110993.
[21] C. Hu and H. Jiang, “Special functions-based fixed-time estimation and stabilization for dynamic systems,” IEEE Trans. Syst., Man, Cybern. Syst., early access, Mar. 12, 2021, doi: 10.1109/TSMC.2021.3062206.
[22] J. Cao and R. Li, “Fixed-time synchronization of delayed memristor-based recurrent neural networks,” Sci. China Inf. Sci., vol. 60, no. 3, Mar. 2017, Art. no. 032201.
[23] C. Hu, H. He, and H. Jiang, “Fixed/preassigned-time synchronization of complex networks via improving fixed-time stability,” IEEE Trans. Cybern., vol. 51, no. 6, pp. 2882–2892, Jun. 2021, doi: 10.1109/TBCYB.2020.2977934.
[24] X. Liu, D. W. C. Ho, and C. Xie, “Prespecified-time cluster synchronization of complex networks via a smooth control approach,” IEEE Trans. Cybern., vol. 50, no. 4, pp. 1771–1775, Apr. 2020.
[25] Y. Ren, W. Zhou, Z. Li, L. Liu, and Y. San, “Prescribed-time cluster lag consensus control for second-order non-linear leader-following multiagent systems,” ISA Trans., vol. 109, pp. 49–60, Mar. 2021.
[26] D. Liu, Z. Liu, C. L. P. Chen, and Y. Zhang, “Prescribed-time containment control with prescribed performance for uncertain nonlinear multi-agent systems,” J. Franklin Inst., vol. 358, no. 3, pp. 1782–1811, Feb. 2021.
[27] C. Thompson, E. Pearson, L. Connor, and H. Hartley, “Tables of percentage points of the incomplete beta-function,” Biometrika, vol. 32, no. 2, pp. 151–181, Oct. 1941.
[28] C. Chen, L. Li, H. Peng, Y. Yang, L. Mi, and B. Qiu, “Fixed-time projective synchronization of memristive neural networks with discrete delay,” Phys. A, Stat. Mech. Appl., vol. 534, Nov. 2019, Art. no. 122248.
[29] H. Lu, W. He, Q.-L. Han, and C. Peng, “Fixed-time synchronization for coupled delayed neural networks with discontinuous or continuous activations,” Neurocomputing, vol. 314, pp. 143–153, Nov. 2018.
[30] J. Xiao, Z. Zeng, A. Wu, and S. Wen, “Fixed-time synchronization of delayed Cohen–Grossberg neural networks based on a novel sliding mode,” Neural Netw., vol. 128, pp. 1–12, Aug. 2020.
[31] C. Chen, L. Li, H. Peng, and Y. Yang, “Fixed-time synchronization of inertial memristor-based neural networks with discrete delay,” Neural Netw., vol. 109, pp. 81–89, Jan. 2019.
[32] R. Wei, J. Cao, and A. Alsaedi, “Fixed-time synchronization of memristive Cohen–Grossberg neural networks with impulsive effects,” Int. J. Control, Autom. Syst., vol. 16, no. 5, pp. 2214–2224, Oct. 2018.
[33] C. Chen, L. Li, H. Peng, Y. Yang, L. Mi, and H. Zhao, “A new fixed-time stability theorem and its application to the fixed-time synchronization of neural networks,” Neural Netw., vol. 123, pp. 412–419, Mar. 2020.

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