On the Complexity of Hub Labeling

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Abstract

Hub Labeling (HL) is a data structure for distance oracles. Hierarchical HL (HHL) is a special type of HL, that received a lot of attention from a practical point of view. However, theoretical questions such as NP-hardness and approximation guarantee for HHL algorithms have been left aside. In this paper we study HL and HHL from the complexity theory point of view. We prove that both HL and HHL are NP-hard, and present upper and lower bounds for the approximation ratios of greedy HHL algorithms used in practice. We also introduce a new variant of the greedy HHL algorithm and a proof that it produces small labels for graphs with small highway dimension.

1 Introduction

The point-to-point shortest path problem is a classical optimization problem with many applications. The input to the problem is a graph \( G = (V,E) \), a length function \( f : E \rightarrow \mathbb{R} \), and a pair \( s,t \in V \). We define \( n = |V| \) and \( m = |E| \). The goal is to find \( \text{dist}(s,t) \), the length of the shortest \( s\text{-}t \) path in \( G \), where the length of a path is the sum of the lengths of its arcs. We assume that the length function is non-negative and that there are no zero-length cycles.

The hub labeling algorithm (HL) [8, 12] is a shortest path algorithm that computes vertex labels during preprocessing stage and answers \( s,t \) queries using only the labels of \( s \) and \( t \); the input graph is not used for queries [15]. For a directed graph a label \( \text{L}(v) \) for a vertex \( v \in V \) consists of the forward label \( \text{LF}(v) \) and the backward label \( \text{LB}(v) \). The forward label \( \text{LF}(v) \) consists of a sequence of pairs \( (w, \text{dist}(v,w)) \), where \( \text{dist}(v,w) \) is the distance (in \( G \)) from \( v \) to \( w \). The backward label \( \text{LB}(v) \) is similar, with pairs \( (u, \text{dist}(u,v)) \). Vertices \( w \) and \( u \) (for forward and backward labels, respectively) are called the hubs of \( v \). For an undirected graph \( \text{LF} = \text{LB} \), and we denote the labeling by \( L \), so \( \text{L}(v) \) itself is a set of pairs \( (w, \text{dist}(v,w)) \).

The labels must obey the cover property: for any two vertices \( s \) and \( t \), the set \( \text{LF}(s) \cap \text{LB}(t) \) must contain at least one hub \( v \) that is on a shortest \( s\text{-}t \) path (we say that \( v \) covers the \( [s,t] \) pair). Given the labels, HL queries are straightforward: to find \( \text{dist}(s,t) \), simply find the hub \( v \in \text{LF}(s) \cap \text{LB}(t) \) that minimizes \( \text{dist}(s,v) + \text{dist}(v,t) \).

Query time and space complexity depends on the label size. The size of a label \( |\text{L}(v)| \) is the number of hubs it contains. For a directed graph the size of a forward (backward) label, \( |\text{LF}(v)| \) (\( |\text{LB}(v)| \)), is the number of hubs it contains and the size of the full label of \( v \), \( \text{L}(v) = (\text{LF}(v), \text{LB}(v)) \), is \( |\text{L}(v)| = |\text{LF}(v)| + |\text{LB}(v)| \). Unless mentioned otherwise, preprocessing algorithms attempt to minimize the total labeling size \( |\text{L}| = \sum_v |\text{L}(v)| \).

Cohen et al. [8] give an \( O(\log n) \) approximation algorithm for HL preprocessing. This algorithm was generalized in [17] and sped up in [18]. These approximation algorithms compute small labels but, although polynomial, do not scale to large problems [18].

A special case of HL is hierarchical hub labeling (HHL) [4], where vertices are globally ranked by “importance” and the label for a vertex \( v \) can only have more important hubs than \( v \) and \( v \) itself.
implementations are faster in practice than general HL ones. For several important graph classes, such as road and complex networks, HHL implementations find small labelings and scale to large problems [3, 4, 6, 9]. However, for the algorithms used in practice such as hierarchical greedy (g-HHL) and hierarchical weighted greedy (w-HHL) there was no theoretical guarantee on the approximation ratio.

Most of the work on the computational complexity of HL (and HHL) algorithms is experimental. The exceptions are approximation algorithms for HL mentioned above, and upper bounds for HL in case of low highway dimension [5, 2, 3, 4]. However, there was no NP-completeness proof of for HL. NP-completeness was implicitly conjectured in [8]: this assumption motivates the $O(\log n)$-approximation algorithm. In addition, in [8] the authors prove that a more general problem, in which the paths to cover are part of the input, is NP-complete (which does not imply NP-hardness of the original problem).

In this paper we obtain the following results on HL and HHL complexity:

- We show that both the optimal HL and the optimal HHL problems are NP-complete.
- We show that in a network of highway dimension $h$ and diameter $D$, there is an HHL such that every label size is $O(h \log D)$, matching the HL bound of [2, 3, 1].
- We propose a variant of the greedy algorithm (called d-HHL), for which we prove
  - an $O(h \log n \log D)$ bound for every label size,
  - an $O(\sqrt{\pi} \log n \log D)$-approximation ratio compared to the optimal HL (and therefore the optimal HHL).
  - an $\Omega(\sqrt{\pi})$ lower bound on the approximation ratio for the optimal HHL.
- For g-HHL, we prove
  - an $O(\sqrt{\pi} \log n)$-approximation ratio compared to the optimal HL.
  - an $\Omega(\sqrt{\pi})$ lower bound on the approximation ratio for the optimal HHL.
- For w-HHL, we prove
  - an $O(\sqrt{\pi} \log n)$-approximation ratio compared to the optimal HL.
  - $\Omega(\sqrt{\pi})$ lower bound on the approximation ratio for the optimal HHL.
- We give an example showing that hierarchical labelings can be $\Omega(\sqrt{\pi})$ bigger than general labelings, improving and simplifying [13].

Our lower bounds on the greedy algorithms show that they do not give a poly-log approximation, leaving the question of the possibility of poly-log approximation open. This is an interesting theoretical problem that may have a practical impact as well.

2 Preliminaries

2.1 HL Approximation Algorithm

Cohen et al. obtain their $O(\log n)$ approximation algorithm for HL by formulating it as a weighted set cover problem and applying the well known greedy approximation algorithm for set cover. In the weighted set cover problem there is a universe set $U$, a family $\mathcal{F}$ of some subsets of $U$, a cost function $c : \mathcal{F} \rightarrow \mathbb{R}_+$, and the goal is to find a collection $\mathcal{C} \subseteq \mathcal{F}$ such that $\bigcup_{S \in \mathcal{C}} S = U$ and $\sum_{S \in \mathcal{C}} c(S)$ is minimized. The greedy set cover algorithm starts with an empty $\mathcal{C}$, then iteratively picks a set $S$ which maximizes the ratio of the number of newly covered elements in $U$ to the cost of $S$ and adds $S$ to $\mathcal{C}$.

The elements to cover in the equivalent set cover instance are vertex pairs $[u, v]$. For a directed graph pairs in $U$ are ordered and for an undirected graph pairs are unordered. We first discuss directed graphs, then undirected ones. Every possible set $P$ of vertex pairs such that there exists a vertex $u$ which hits a shortest path between every pair in $P$ is a set. (There are exponentially many sets, but they are not used explicitly.) The cost of a set $P$ is the number of vertices that appear in the first component of a pair in $P$ plus the number of vertices that appear in the second component of a pair in $P$.

The greedy approximation algorithm for set cover as applied to this set cover instance is as follows. The algorithm maintains the set $U$ of uncovered vertex pairs: $[u, w] \in U$ if $L_f(u) \cap L_b(w)$ does not contain a vertex on a shortest $u$-$w$ path. Initially $U$ contains all vertex pairs $[u, w]$ such that $w$ is reachable from $u$. The algorithm terminates when $U$ becomes empty. Starting with an empty labeling, in each iteration, the algorithm adds a vertex $v$ to forward labels of vertices in a set $S' \subseteq V$ and to backward labels of
the vertices in \( S'' \subseteq V \) such that the ratio of the number of newly-covered pairs over the total increase in the size of the labeling is (approximately) maximized. Formally, let \( U(v, S', S'') \) be the set of pairs in \( U \) which are covered if \( v \) is added to \( L_f(u) : u \in S' \) and \( L_b(w) : w \in S'' \). The algorithm maximizes \( |U(v, S', S'')|/(|S'| + |S''|) \) over all \( v \in V \) and \( S', S'' \subseteq V \).

To find the triples \( (v, S', S'') \) efficiently the algorithm uses center graphs defined as follows. A center graph of \( v \), \( G_v = (X, Y, A_v) \), is a bipartite graph with \( X = V, Y = V \), and an arc \( (u, w) \in A_v \) if \([u, w] \in U \) and some shortest path from \( u \) to \( w \) goes through \( v \). The algorithm finds \( (v, S', S'') \) that maximizes \( |U(v, S', S'')|/(|S'| + |S''|) \) by computing a densest subgraph among all the subgraphs of the center graphs \( G_v \). The density of a graph \( G = (V, A) \) is \( |A|/|V| \). The maximum density subgraph (MDS) problem is the problem of finding an (induced) subgraph of a given graph \( G \) of maximum density. This problem can be solved in polynomial time using parametric flows (e.g., [11]). For a vertex \( v \), the arcs of a subgraph of \( G_v \) induced by \( S' \subseteq X \) and \( S'' \subseteq Y \) become covered if \( v \) is added to \( L_f(u) : u \in S' \) and \( L_b(w) : w \in S'' \). Therefore, the MDS of \( G_v \) maximizes \( |U(v, S', S'')|/(|S'| + |S''|) \) over all \( S', S'' \).

For undirected graphs we have \( L_f(v) = L_b(v) = L(v) \) by definition. Pairs \([u, v] \in U \) are unordered and the cost of a set \( P \) of unordered vertex pairs is the number of vertices that appear in a pair in \( P \). Let \( U(v, S) \) be the set of unordered vertex pairs that become covered if we add \( v \) to \( L(u) : u \in S \). We want to maximize \( U(v, S)/|S| \). To find such a tuple, we use another type of a center graph of \( G_v \). \( G_v \) is an undirected graph with vertex set \( V \) and with an edge \([u, w] \in E_v \) if \([u, w] \in U \) and some shortest path between \( u \) and \( w \) goes through \( v \). (For a pair \([v, w]\) there is a self-loop \([v, v]\) in \( E_v \).) Note that \( G_v \) is not necessarily bipartite. As in the directed case, MDS of \( G_v \) maximizes \( U(v, S)/|S| \) over all \( S \).

The following is a folklore lemma about the greedy set cover algorithm.

**Lemma 2.1.** If we run the greedy set cover algorithm where in each iteration we pick a set whose coverage to cost ratio is at least \( 1/f(n) \) fraction of the maximum coverage to cost ratio, then we get a cover of cost within an \( O\left(f(n) \log n\right) \) factor of optimal.

Cohen et al. [8] used this lemma and instead of finding the MDS exactly they used a linear-time 2-approximation algorithm [11]. The result is an \( O\left(\log n\right) \)-approximation algorithm running in \( O(n^5) \) time. Delling et al. [10] improve the running time to \( O(n^3 \log n) \).

### 2.2 Canonical HHL

Vertices are ordered if there is a bijection \( \pi : V \rightarrow \{1, \ldots, |V|\} \). We say that \( u \) is more important than \( v \) if \( \pi(u) < \pi(v) \). The labeling \( L \) is hierarchical if there is an order \( \pi \) such that \( u \in L_f(v) \cup L_b(v) \) implies \( \pi(u) \leq \pi(v) \). In this case we say that \( L \) respects \( \pi \).

Let \( P_{v,w} \) denote the set of all vertices on shortest paths from \( u \) to \( v \). For an order \( \pi \) we define a canonical HHL in the following way: \( u \in L_f(v) \) (resp. \( u \in L_b(v) \)) if and only if \( u \) is the most important vertex in \( P_{v,u} \) (resp. \( P_{v,w} \)). The following theorem is implicit in [4] and [10].

**Theorem 2.2.** For an order \( \pi \) the canonical HHL is the minimum HHL that respects \( \pi \).

**Proof.** We first show that the canonical HHL \( L \) obeys the cover property. For a pair \([v, w]\) let \( u \) be the most important vertex in \( P_{v,w} \). Consider any \( v-u \) shortest path. It is easy to see that it is a subpath of some \( v-w \) shortest path. Therefore by the definition of canonical labeling we have \( u \in L_f(v) \) and \( u \in L_b(w) \).

Now we show that \( L \) is a sublabeling of any HHL \( \tilde{L} \) that respects \( \pi \). Let \( u \in L_f(v) \) (resp. \( u \in L_b(v) \)). Then \( u \) is more important than any other vertex \( w \) on a \( v-u \) (resp. \( u-v \)) shortest path. Therefore \( \tilde{L}_b(u) \) (resp. \( \tilde{L}_f(u) \)) doesn’t have any such \( w \) except \( u \). Since \( \tilde{L} \) covers the \([v, u]\) (resp. \([u, v]\)) pair we have \( u \in L_f(v) \) (resp. \( u \in L_b(v) \)). So \( L \) is a sublabeling of \( \tilde{L} \).

### 2.3 Greedy HHL Algorithms

In this section we describe greedy HHL algorithms in terms of center graphs. For an alternative description and efficient implementation of these algorithms, see [4, 10].

A greedy HHL algorithm maintains the center graphs \( G_v = (X, Y, A_v) \) defined in Section 2.1 At each iteration, the algorithm selects a center graph of a vertex \( v \) and adds \( v \) to \( L_f(u) \) for all non-isolated
vertices \( u \in X \) and to \( L_0(w) \) for all non-isolated vertices \( w \in Y \). Note that after the labels are augmented this way, all vertex pairs \( [u, w] \) for which there is a \( u-w \) shortest path passing through \( v \) are covered. Therefore, the center graph of every vertex is chosen once, and the labeling is hierarchical.

Greedy algorithms differ by the criteria used to select the next center graph to process. The greedy HHL \((g\text{-HHL})\) algorithm selects the center graph with most edges. The weighted greedy HHL \((w\text{-HHL})\) algorithm selects a center graph with the highest density (the number of edges divided by the number of non-isolated vertices).

We propose a new distance greedy HHL \((d\text{-HHL})\) algorithm. To every vertex pair \([u, v]\) we assign a weight

\[
W(u, v) = \begin{cases} 
0, & \text{if } \text{dist}(u, v) = 0 \\
\alpha^2 \log_2(\text{dist}(u, v)), & \text{otherwise}
\end{cases}
\]

and use \( W \) to weight the corresponding edges in center graphs. At each iteration, \( d\text{-HHL} \) selects a center graph with the largest sum of edge weights.

We define the level of \([u, v]\) as \( \lfloor \log_2(\text{dist}(u, v)) \rfloor \) if \( \text{dist}(u, v) = 0 \) the level of \([u, v]\) is \(-\infty\). The definition of \( W \) insures that if \([u, v]\) is the maximum level uncovered vertex pair, \( W(u, v) \) is greater than the total weight of all lower-level uncovered pairs. Therefore \( d\text{-HHL} \) primarily maximizes the number of uncovered maximum level pairs that become covered, and other pairs that become covered are used essentially as tie-breakers.

We say that a vertex \( w \) has level \( i \) if at the iteration when \( w \) is selected by \( d\text{-HHL} \), the maximum level of an uncovered vertex pair is \( i \). As the algorithm proceeds, the levels of vertices it selects are monotony decreasing.

## 2.4 Highway Dimension

In this section we review the definition of highway dimension (HD) and related concepts. As HD is defined for undirected graphs, when we talk about HD we assume that all graphs are undirected and connected.

**Definition 2.3.** Given a shortest path \( P = (v_1, \ldots, v_k) \) and \( r > 0 \), a shortest path \( P' \) is an \( r \)-witness for \( P \) if and only if \( \ell(P') > r \) and one of the following conditions holds:

1. \( P' = P; \) or
2. \( P' = (v_0, v_1, \ldots, v_k); \) or
3. \( P' = (v_1, \ldots, v_k, v_{k+1}); \) or
4. \( P' = (v_0, v_1, \ldots, v_k, v_{k+1}). \)

**Definition 2.4.** A shortest path \( P \) is \( r \)-significant if it has an \( r \)-witness path.

Let \( \mathcal{P}_r \) denote the set of all \( r \)-significant paths. Given a vertex \( v \) and a path \( P \), we define the distance from \( v \) to \( P \) by \( \text{dist}(v, P) = \min_{w \in P} \text{dist}(v, w) \).

**Definition 2.5.** A shortest path \( P \) is \((r, d)\)-close to a vertex \( v \) if \( P \) is \( r \)-significant with an \( r \)-witness path \( P' \) such that \( \text{dist}(v, P') \leq d \).

Note that if \( P \) is \((r, d)\)-close to \( v \), then \( P \) is also \((r', d')\)-close to \( v \) for any \( 0 < r' \leq r \) and \( 0 \leq d \leq d' \).

Let the \( r \)-neighborhood of \( v \), denoted by \( S_r(v) \), be the set of all \( P \in \mathcal{P}_r \) that are \((r, 2r)\)-close to \( v \).

Given a set of paths \( \mathcal{P} \), we say that \( H \subseteq V \) is a hitting set for \( \mathcal{P} \) if every path in \( \mathcal{P} \) contains a vertex in \( H \).

**Definition 2.6.** A network \((G, \ell)\) has highway dimension (HD) \( h \) if \( h \) is the smallest integer such that for any \( r > 0 \) and any \( v \in V \), there exists a hitting set \( H \) for \( S_r(v) \) (that depends on \( v \) and \( r \)) with \( |H| \leq h \).

Given \( r \geq 0 \) and \( v \in V \), we define the ball of radius \( r \) centered at \( v \), \( B_r(v) \), to be the set of all vertices within distance at most \( r \) from \( v \).

A notion related to highway dimension is that of a sparse shortest-path hitting set (SPHS).

**Definition 2.7.** For \( r > 0 \), an \((h, r)\)-SPHS is a hitting set \( C \subseteq V \) for \( \mathcal{P}_r \) such that \( \forall v \in V \), \( |B_{2r}(v) \cap C| \leq h \).

Abraham et al. [12, 1] show:

**Theorem 2.8.** If the highway dimension of a network \((G, \ell)\) is \( h \), then (1) for any \( r > 0 \), a minimum hitting set for \( \mathcal{P}_r \) is an \((h, r)\)-SPHS and (2) if shortest paths are unique one can find an \((h \log h, r)\)-SPHS in polynomial time.
3 HHL and Highway Dimension

Abraham et al. [5] show that a network with HD $h$ and diameter $D$ has an HHL with $|L(v)| = O(h \log D)$, and that in polynomial time one can find an HHL with $|L(v)| = O(h \log h \log D)$. We show similar results for HHL.

Assume that edge lengths are at least 1 and let $D$ be the diameter of the network $(G, \ell)$. A multiscale SPHS of $(G, \ell)$ is a collection of sets $C_i$ for $0 \leq i \leq \log D$, where each $C_i$ is a $(h, 2^{i-1})$-SPHS. In particular, note that $C_0 = V$, since every vertex is an $(1/2)$-significant path. For $0 \leq i \leq \log D$, let $Q_i = C_i \setminus \bigcup_{j=i+1}^{\log D} C_j$.

**Theorem 3.1.** A network with HD $h$ and diameter $D$ has an HHL with $|L(v)| = O(h \log D)$ for all $v \in V$, and if shortest paths are unique one can find in polynomial time an HHL with $|L(v)| = O(h \log h \log D)$.

**Proof.** Consider the ordering $r$ such that for $i < j$ each $v \in Q_i$ is less important than each $v \in Q_j$ (i.e. $r(w) < r(v)$), and vertices within each $Q_i$ are ordered arbitrarily. For each $v \in Q_i$, define

$$L(v) = \{v\} \cup \{r(w) > r(v), \ w \in C_j \cap B_{2^i}(v)\}.$$ 

Consider a shortest $s$–$t$ path $P$ and let $i$ be such that $2^{i-1} < \ell(P) \leq 2^i$. Assume, w.l.g., that $r(s) < r(t)$. Let $s \in Q_2$ and $t \in Q_\ell$; we have $x \leq y$.

If $y \geq i$, then $t \in B_{2^i}(s)$ so $t \in C_y \cap B_{2^i}(s)$ and therefore $t \in L(s)$. If $y < i$, then since $x \leq y < i$ there must be a vertex $w \neq s, t$ such that $w \in P \cap C_i$. By the definition of $i$, $w \in B_{2^i}(s)$ and $w \in B_{2^i}(t)$. Therefore $w \in L(s) \cap L(t)$. In both cases, the cover property holds.

Using the multiscale SPHS provided by Theorem 2.8 we get that there exists an HHL such that $|L(v)| = O(h \log D)$ and if shortest paths are unique we can compute in polynomial time an HHL such that $|L(v)| = O(h \log h \log D)$.

Next we discuss the distance greedy $d$-HHL algorithm (defined in Section 2.3).

**Theorem 3.2.** In a network with HD $h$ and diameter $D$ $d$-HHL finds a labeling with $|L(v)| = O(h \log n \log D)$, for all $v \in V$.

**Proof.** We show that for every vertex $v$ and level $i$, $L(v)$ contains $O(h \log n)$ hubs at level $i$.

Consider the (consecutive) iterations of the algorithm that select vertices at level $i$. Consider $v \in V$ and $B_{2^i}(v)$. Since $d$-HHL already covered all vertex pairs of level greater than $i$, $v$ can accumulate hubs of level $i$ only from vertices in $B_{2^i}(v)$.

Suppose at some step the algorithm chooses a level $i$ vertex $w$ in $B_{2^i}(v)$. Every $x$–$y$ shortest path of length $\geq 2^i$ hit by $w$ is in $S_{2^i}(v)$. By the definition of highway dimension, there is a hitting set $H$ for $S_{2^i}(v)$ with $|H| \leq h$.

We call a yet uncovered vertex pair $[x, y]$ relevant if there is a $x$–$y$ shortest path in $S_{2^i}(v)$ and $\text{dist}(x, y) \geq 2^i$. Since $H$ is a hitting set for $S_{2^i}(v)$, $H$ is also a hitting set for the set of relevant vertex pairs (it hits a shortest path between each such pair). It follows that there is a vertex $u \in H$ which covers at least $1/h$ relevant vertex pairs. By the greedy choice of $w$, $w$ hits at least the same number of relevant pairs as $u$ does.

After $h$ consecutive vertices from $B_{2^i}(v)$ are selected, the number of relevant vertex pairs is at most $(1 - 1/h)^h \leq 1/e$ fraction of the original, i.e., is reduced by a factor of $e$. The initial number relevant vertex pairs is bounded by $n^2$, therefore the algorithm chooses $O(h \log n)$ vertices in $B_{2^i}(v)$ before all relevant vertex pairs are hit. Once all the relevant vertex pairs are hit, the algorithm will not choose any level $i$ vertices in $B_{2^i}(v)$. □

4 Upper Bounds

In Sections 4 and 5, we assume that isolated vertices are deleted from the center graphs, so their density is the number of edges divided by the number of (non-isolated) vertices.
4.1 Greedy

We show that g-HHL finds an HHL of size that is within an \( O(\sqrt{n} \log n) \) factor of the optimal HL size. We prove this by bounding the ratio of the density of the center graph picked by g-HHL and the density of the MDS of a center graph.

**Theorem 4.1.** g-HHL is an \( O(\sqrt{n} \log n) \)-approximation algorithm for HL.

**Proof.** Suppose that at some iteration, the algorithm picks a center graph with \( m' \) arcs and \( n' \) vertices. Then by the definition of g-HHL all center graphs have at most \( m' \) arcs, so the density of the maximum density subgraph (over all center graphs) is at most \( \sqrt{m'} \). The density ratio of the maximum density subgraph to that of the chosen center graph is at most

\[
\frac{\sqrt{m'}}{m'/n'} \leq \frac{n'}{\sqrt{m'/2}} = \sqrt{2n'} \leq \sqrt{2n}.
\]

Here we use the fact that the chosen graph has no isolated vertices, so \( m' \geq n'/2 \). It follows that the density of the chosen center graph is a \( \sqrt{2n} \)-approximation of the maximum density of any subgraph. By Lemma 2.1 we have that the labeling size is larger than the size of the optimal HL by at most \( O(\sqrt{n} \log n) \) factor. 

Since HHL is a special case of HL we have

**Corollary 4.2.** g-HHL is an \( O(\sqrt{n} \log n) \)-approximation algorithm for HHL.

4.2 Distance Greedy

We show that d-HHL finds an HHL of size within an \( O(\sqrt{n} \log n \log D) \) factor of the optimal HL size. But first we need to extend our concept of hub labels.

Cohen et al. [8] defined a more general notion of hub labels for a given set \( U \) of vertex pairs. Such labels are required to have a vertex \( u \in L(u) \cap L(v) \) which is on a shortest path between \( u \) and \( v \) for each \( [u, v] \in U \). The \( O(\log n) \) approximation algorithm described in Section 2.1 works for this more general notion of HL; Lemma 2.1 and Theorem 4.1 hold.

**Theorem 4.3.** d-HHL is an \( O(\sqrt{n} \log n \log D) \)-approximation algorithm for HL.

**Proof.** Let OPT denote the size of the optimal HL. Let \( U_i \) be a set of vertex pairs at level \( i \) which are not covered by vertices at higher levels when we run d-HHL. Let \( H_{L_i} \) be the optimal HL to cover vertex pairs from \( U_i \) and let \( OPT_i \) be size of \( H_{L_i} \). Since \( U_i \) is a subset of all vertex pairs, \( OPT_i \) doesn’t exceed OPT. By Theorem 4.1 we can use the g-HHL algorithm to find \( O(\sqrt{n} \log n) \) approximation for \( H_{L_i} \).

Now let’s return to d-HHL. Since every two pairs at the same level have the same weight and weights of all lower-level vertex pairs are negligible, at the consecutive set of iterations in which d-HHL covers \( U_i \) it picks the same vertices as g-HHL when we run it on \( U_i \).

So the labels found by d-HHL have size

\[
\sum_{i=0}^{\lfloor \log D \rfloor} O(\sqrt{n} \log n)OPT_i \leq \sum_{i=0}^{\lfloor \log D \rfloor} O(\sqrt{n} \log n)OPT = O(\sqrt{n} \log n \log D)OPT.
\]

**Corollary 4.4.** d-HHL is an \( O(\sqrt{n} \log n \log D) \)-approximation algorithm for HHL.

4.3 Weighted Greedy

Although w-HHL is motivated by the approximation algorithm of Cohen et al., it does not achieve \( O(\log n) \) approximation. We show that w-HHL finds an HHL of size larger than the size of the optimal HL by an \( O(\sqrt{n} \log n) \) factor. The key to the analysis is the following lemma.

**Lemma 4.5.** If \( (V, E) \) is a graph with no isolated vertices, then \( G \) is an \( O(\sqrt{n}) \)-approximation of the maximum density subgraph of \( G \).
Proof. Consider a subgraph \((V', E')\) of \(G\). Let \(|V| = n\), \(|E| = m\), \(|V'| = n'\), \(|E'| = m'\). Then

\[
m' \leq \min(m, n'^2) = n' \min\left(\frac{m}{n'}, n'\right) \leq n' \sqrt{m}.
\]

where the last step follows since if \(n' \leq \sqrt{m}\), \(\min\left(\frac{m}{n'}, n'\right) = n' \leq \sqrt{m}\), and if \(n' > \sqrt{m}\), \(\min\left(\frac{m}{n'}, n'\right) = \frac{m}{n'} \leq \sqrt{m}\).

Since \(G\) goes not have isolated vertices, \(m \geq n/2\), so we have

\[
\frac{m'}{n'} \leq \sqrt{m} = \frac{m}{n} \frac{n}{\sqrt{n}} \leq \frac{m}{n} n/\sqrt{2n} \leq \frac{m}{n} 2\sqrt{n}.
\]

\(\square\)

**Theorem 4.6.** \(w\)-HHL is an \(O(\sqrt{n} \log n)\)-approximation algorithm for HL.

Proof. At each iteration, \(w\)-HHL picks the center graph with the maximum ratio of the number of edges divided by the number of vertices. By Lemma 4.5 the density of this graph is smaller than the density of the densest subgraph of a center graph by at most \(O(\sqrt{n})\). Therefore by Lemma 2.1 \(w\)-HHL produces an HHL of size within an \(O(\sqrt{n} \log n)\) factor of the size of the optimal HHL. \(\square\)

**Corollary 4.7.** \(w\)-HHL is an \(O(\sqrt{n} \log n)\)-approximation algorithm for HHL.

5 Lower Bounds

In this section we show that \(g\)-HHL, \(d\)-HHL and \(w\)-HHL do not give a poly-log approximation. We present graphs for which these algorithms find a labeling worse than the optimal HHL by a polynomial factor. We also show that our upper bounds are fairly tight.

5.1 Greedy

We show that for a graph in Figure 1a \(g\)-HHL finds a labeling larger by a factor of \(\Omega(\sqrt{n})\) than the optimal HHL (and therefore the optimal HL).

**Lemma 5.1.** There is a graph family for which \(g\)-HHL finds HHL of size \(\Omega(n^{3/2})\) while the optimal HHL size is \(O(n)\).

Proof. Consider the directed graph \(G = (V, A)\) in Figure 1a. The graph \(G\) has \(n = \Theta(k^2)\) vertices \(V = \{a_1, \ldots, a_k, b_1, \ldots, b_{k+1}\} \cup \{c_{ij} \mid 1 \leq i \leq k + 1, 1 \leq j \leq k\}\). The arcs are \(A = \{(a_i, b_j) \mid 1 \leq i \leq k, 1 \leq j \leq k + 1\} \cup \{(b_i, c_{ij}) \mid 1 \leq i \leq k + 1, 1 \leq j \leq k\}\) all of length 1.

Consider the center graphs when \(g\)-HHL starts and the labeling is empty. Shortest paths containing \(a_i\) include the path from \(a_i\) to itself, the paths from \(a_i\) to \(b_x\), and the paths from \(a_i\) to \(c_{xy}\), so number of edges in the center graph of \(a_i\) is

\[
1 + (k + 1) + k(k + 1) = (k + 1)^2 + 1.
\]

Shortest paths containing \(b_i\) include the path from \(b_i\) to itself, \(k\) paths from \(a_j\) to \(b_i\), another \(k\) paths from \(b_i\) to \(c_{ij}\), and the \(k^2\) paths from \(a_j\) to \(c_{ij}\) for a total of

\[
1 + k + k^2 = (k + 1)^2.
\]

Shortest paths containing \(c_{ij}\) include the path from \(c_{ij}\) to itself, from \(c_{ij}\) to \(b_i\), and the \(k\) paths from \(c_{ij}\) to \(a_x\) for a total of

\[
1 + 1 + k = k + 2.
\]

So \(g\)-HHL will pick an \(a_i\) vertex for some \(i\) first. Note that if when \(g\)-HHL picks an \(a_i\) vertex, the center graph of \(a_j, j \neq i\) does not change, and the center graphs of the \(b_i\)'s and the \(c_{ij}\)'s loose edges. Therefore \(g\)-HHL will continue picking \(a\)-vertices until there are none left. After that, a center graph of some \(b_i\) has \(k + 1\) edges and a center graphs of \(c_{ij}\) has 2 edges. So \(g\)-HHL will pick all \(b\) vertices next, and then all the \(c\) vertices.
The order found by g-HHL is \(a_1, \ldots, a_k, b_1, \ldots, b_{k+1}\) followed by \(c\) vertices and the labeling it produced is as follows. \(|L_f(a_i)| = |L_0(a_i)| = 1\), \(|L_f(b_i)| = 1 + k\). \(L_0(b_i) = 1 + k\), \(|L_f(c_{ij})| = 1\), and \(|L_0(c_{ij})| = 2 + k\).

Therefore the total size of the labeling is \(\Omega(k^3) = \Omega(n^{3/2})\).

A better order for this graph is \(b_1, \ldots, b_{k+1}, a_1, \ldots, a_k\) followed by \(c\) vertices. The canonical labeling corresponding to this order is as follows. \(|L_f(a_i)| = (k + 1) + 1\), \(|L_0(a_i)| = 1\), \(|L_f(b_i)| = |L_0(b_i)| = 1\), \(|L_f(c_{ij})| = 1\), and \(|L_0(c_{ij})| = 2\). The total size of this labeling is \(O(k^2) = O(n)\).

We have shown that for \(G = (V,E)\), g-HHL produces a labeling larger than the optimal one by an \(\Omega(\sqrt{n})\) factor, so our \(O(\sqrt{n} \log n)\) upper bound on the approximation ratio of g-HHL of Section 4.1 is tight up to a logarithmic factor.

### 5.2 Distance Greedy

We show that for a graph in Figure 1a d-HHL finds a labeling larger by a factor of \(\Omega(\sqrt{n})\) than the optimal HHL (and therefore the optimal HL).

**Lemma 5.2.** There is a graph family for which d-HHL finds HHL of size \(\Omega(n^{3/2})\) while the optimal HHL size is \(O(n)\).

**Proof.** Consider the directed graph \(G = (V,A)\) depicted in Figure 1a. There are paths of length 0, 1, and 2. While there are some paths of length 2 yet uncovered, d-HHL selects a vertex to hit the maximum number of paths with length 2. Weights of all paths of length 0 and 1 matter only when d-HHL chooses between two vertices which hit exactly the same number of paths of length 2.

At the beginning \(a_i\) hits \(k(k + 1)\) paths of length 2 and \(b_i\) hits \(k^2\) paths of length 2. So d-HHL selects \(a_i\). As d-HHL proceeds, the number of paths of length 2 hit by \(b_i\) decreases and the number of paths of length 2 hit by \(a_i\) remains the same \(k(k + 1)\).

Therefore the order found by d-HHL is \(a_1, \ldots, a_k, b_1, \ldots, b_{k+1}\) followed by all the \(c\) vertices. Exactly the same order is produced by g-HHL. From Lemma 5.1 we know that the size of the canonical labeling of this order is \(\Omega(n^{3/2})\) while the size of optimal HHL is \(O(n)\).

So d-HHL can also produce a labeling of size \(\Omega(\sqrt{n})\) away from optimal. This makes a fairly good match with the \(O(\sqrt{n} \log n \log D)\) upper bound established in Section 4.2.

Theorem 3.2 gives \(O(h \log n \log D)\) bound for the maximum label size produced by d-HHL. The graph in Figure 1a gives us a good lower bound on the maximum label size as the following lemma specifies.

**Lemma 5.3.** There is a graph family for which d-HHL finds HHL with maximum label size \(\Omega(h \log D)\).

**Proof.** Consider the directed graph \(G = (V,A)\) in Figure 1a. The diameter of \(G\) is 2. Let’s find the highway dimension \(h\) of \(G\).

Abraham et al. [11] Lemma 3.5] show that the maximum degree of a vertex is a lower bound on the HD. Thus \(h\) is at least \(k + 1\). Note that all \(b_i\) form a hitting set of size \(k + 1\) for all paths of length greater than 0. So any \(S_i(v)\) has a hitting set with at most \(k + 2\) vertices and thus \(h = \Theta(k)\).

In the labels found by d-HHL we have \(|L(c_i)| = k + 2\) (cf. the proof of Lemma 5.1). Since \(h = \Theta(k)\) and \(D = 2\) we have \(|L(c_i)| = \Theta(h \log D)\).

Lemma 5.3 shows that the upper bound of Theorem 3.2 is right up to a \(O(\log n)\) factor.
We show that for a graph in Figure 1b w-HHL finds a labeling of size larger than the size of the optimal HHL by a factor of $\Omega(\sqrt{n})$ (and therefore the optimal HL).

**Lemma 5.4.** There is a graph family for which w-HHL finds HHL of size $\Omega(n^{4/3})$ while the optimal HHL size is $O(n)$.

**Proof.** Consider the undirected graph $G = (V, E)$ in Figure 1b. The vertices of $G$ are $V = \{a, b\} \cup \{c_i \mid 1 \leq i \leq k\}$ so $n = |V| = \Theta(kl)$. The edges are $E = \{(a, d_{ij}) \mid 1 \leq i \leq k, 1 \leq j \leq l\} \cup \{(b, c_i) \mid 1 \leq i \leq k\} \cup \{(c_i, d_{ij}) \mid 1 \leq i \leq k, 1 \leq j \leq l\}$ All edges have length 2 except for those adjacent to $a$, which have length 3. The lengths of the edges are set so that shortest paths between distinct $d$ vertices adjacent to the same $c$ vertex go through the $c$ vertex.

We set $l = 2k^2$; so $k = \Theta(\sqrt{n})$. As we shall see, this is large enough to make w-HHL choose the $c$ vertices before choosing $b$. However, this causes the $c$ vertices to be added to the labels of many $d$ vertices and leads to a large total label size.

Consider the center graphs when w-HHL starts and the labeling is empty. Since the graph is connected, all center graphs have no isolated vertices, so all the denominators of the densities of the center graphs are the same and equal $n$.

Consider now the numerators (number of pair covered) by the different vertices. Vertex $a$ covers the shortest paths between the $d$ vertices adjacent to different $c$'s. Therefore the center graph of $a$ has $\Omega((kl)^2)$ edges, which is asymptotically more than the number of edges in the other center graphs. So w-HHL chooses $a$ to be the most important vertex.

Following this first choice of $a$, all vertex pairs consisting of $a$ and $d$'s are covered, except for the pairs of $d$'s of the form $d_{ij}$ and $d_{ir}$ (both adjacent to $c_i$). The vertex $d_{ij}$ is an endpoint of every uncovered shortest path containing it and therefore the density of the center graph of $d_{ij}$ is constant. As we shall see, the density of other center graphs is higher, so the $d$ vertices are chosen last.

We show that after choosing $a$, w-HHL chooses $c$ vertices until there are no $c$ vertices left. Suppose the number of remaining $c$ vertices is $t$: $1 \leq t \leq k$. We show that the density of the center graph of each of the remaining $c$'s is larger than the density of the center graph of $b$. First we observe that at this point the number of vertices in the center graph of $b$ and in the center graph of each of the remaining $c$ vertices is the same, namely $1 + t + tl$.

Shortest paths through $b$ include the paths between $c_i$ and $c_j$ for $i < j$, the shortest paths from $c_i$ to $d_{jr}$ for $i \neq j$ and paths from $b$ to all the vertices that have not been picked yet. So the number of edges in the center graph of $b$ is

$$t(t - 1)/2 + t(t - 1)l + (1 + t + tl) = \frac{tl}{2} + tl + t^2l.$$  \(1\)

Shortest paths through $c_i$ include the paths between $d_{ij}$ and $d_{ir}$ for $j < r$, the paths from $d_{ir}$ to $c_j$ for $j \neq i$, the paths from $b$ to $d_{ij}$, and the paths from $c_i$ to all the vertices that have not been picked yet. So the number of edges in the center graph of a remaining $c$ vertex is

$$l(l - 1)/2 + (l + t)l + (1 + t + tl) = \frac{tl}{2} + tl + t^2l.$$  \(2\)

Subtracting Equation 1 from Equation 2, and using the facts that $l = 2k^2$ and $k \geq t$, we get

$$l(t - 1)/2 + tl - t(t - 1)/2 - t(t - 1)l = \frac{tl^2}{2} + 2lt + t/2 - l/2 - t^2l/2 - t^2l \geq t^2/2 - t^2l = 2k^4 - 2k^3 > 0.$$  \(3\)

So w-HHL chooses $a$ first, followed by all $c$ vertices, then $b$ and all $d$ vertices. The size of the corresponding canonical labeling is

$$n + \sum_{i=1}^{k} (1 + t + tl) + 1 + kl = \Omega(lk^2) = \Omega(n^{4/3}).$$
A better ordering is the one which puts $a$ is first, followed by $b$, the $c$ vertices, and finally the $d$ vertices. The size of the corresponding canonical labeling is

$$n + (1 + k + kl) + k(1 + l) + kl = O(n).$$

Therefore on the graph in Figure 13, the size of the labeling produced by $w$-HHL is larger than the optimal by a factor of $\Omega(\sqrt{n})$. There is a factor of $\Omega(\sqrt{n} \log n)$ gap between this lower bound and the $O(\sqrt{n} \log n)$ upper bound of Section 4.3.

6   NP-Completeness

6.1 Undirected Graphs

In this section we prove that the problems of finding an optimal HL and an optimal HHL are NP-hard by a reduction from Vertex Cover (VC). The reduction takes an instance of VC consisting of a graph $G$ and an integer $k$ and produces an undirected graph $G'$ and an integer $k'$ such that the following conditions are equivalent

1. There is an HL of size $k'$ in $G'$.
2. There is an HHL of size $k'$ in $G'$.
3. There is a VC of size $k$ in $G$.

Our results imply NP-completeness of HL and HHL in undirected graphs.

Before presenting the reduction we prove the following useful lemma.

Lemma 6.1. Let $G = (V, E)$ be a graph and $S$ be a star graph, distinct from $G$, with a root $s$ and $|V|$ leaves. Let $G'$ be the union of the graphs $G$ and $S$, with additional edges between $s$ and some vertices of $V$. If $G'$ is connected then there are optimal HL and HHL for $G'$ such that $s \in L(x)$ for every vertex $x$.

Proof. Let $L$ be an optimal HL (or HHL) labeling of $G'$. First, assume that for a leaf $u \in S$ we have that $s \not\in L(u)$. Since $(s, u) \in G'$ we must have that $u \in L(s)$, and the only pair of vertices covered by $u \in L(s)$ is $[s, u]$. So if we add $s$ to $L(u)$ and remove $u$ from $L(s)$, we get a valid labeling of the same size as $L$ which is optimal. Therefore we may assume that $s \in L(u)$ and $u \not\in L(s)$ for every leaf $u \in S$.

Next, assume that for some $v \in V$, and a leaf $u \in S$ we have that $u \in L(v)$. Since $u \in L(v)$ is used only to cover the pair $[u, v]$, we can remove $u$ from $L(v)$ and add $s$ to $L(v)$ if it is not already there while keeping the labeling valid. This way we can transform $L$, without increasing its size, to a labeling such that the labels of $v \in V$ do not contain leaves of $S$.

Finally, assume that there is a vertex $v \in V$ such that $s \not\in L(v)$. Then $L(v) \cap S = \emptyset$. Since the pair $[u, v]$ for every leaf $u \in S$ has to be covered, $L(u)$ must contain a vertex of $V$. Remove vertices of $V$ from $L(u)$ for all $u \in S$ and add $s$ to $L(v)$ for all vertices $v \in X$ such that $s$ is not in $L(v)$ already. This keeps the labeling valid and cannot increase its size. \qed

Now we describe the reduction. We reduce the problem of deciding whether there is a VC of size at most $k$ in $G$ to the problem of deciding whether there is an HL of size at most $k'$ in a graph $G'$. Lemma 6.7 shows that $G'$ has an HL of size at most $k'$ iff it has an HHL of size at most $k'$, so it follows that our reduction also proves that deciding whether there is an HHL of a given size is also NP-complete. We construct $G' = (V', E')$ from $G = (V, E)$ as follows.

1. For each vertex $v \in V$ we add three vertices, $v_1, v_2,$ and $v_3$ to $V'$ and two edges $\{v_1, v_2\}$ and $\{v_2, v_3\}$ to $E'$.
2. For each edge $\{u, v\} \in E$ we add an edge $\{u_1, v_1\}$ to $E'$.
3. We add a star $S$ with $3|V'|$ leaves and a root $s$ to $G'$ and add $\{s, v_1\}$ to $E'$ for every $v \in V$.

The graph $G'$ is shown in Figure 2a. All edges have length 1. By Lemma 6.1 we can assume w.l.g. that in an optimal labeling all vertices have $s$ in their labels. Therefore, all paths between $v_i$ and $u_j$ such that $\{u, v\} \notin E$ are covered (hereinafter when we write $v_i$ we mean $v_i$ for $i = 1, 2, 3$).
For each vertex \( v \in V \) we have a subgraph \( G'_v \) in \( G' \) which is a path \((v_1, v_2, v_3)\). Any labeling must cover all \([v_1, v_2]\) pairs of \( G'_v \). We show that w.l.o.g the labeling covers these paths either as in Figure 2b in which case we say that \( v \) is a type 1 vertex or as in Figure 2c in which case we say that \( v \) is a type 2 vertex. Note that a type 2 vertex uses one more hub in the labeling, so to reduce the labeling size we want to minimize the number of type 2 vertices. We will show, however, that the type 2 vertices must form a vertex cover for the labeling to be valid.

**Lemma 6.2.** There is an optimal labeling \( L \) of \( G' \) such that for each vertex \( v \in V \) if \( v_1 \in L(v_2) \), then \( v \) is a type 2 vertex, otherwise \( v \) is a type 1 vertex.

**Proof.** Vertex \( v_3 \in L(v_2) \) can cover only the pair \([v_2, v_3]\). So if \( v_3 \in L(v_2) \) we can remove \( v_3 \) from \( L(v_2) \) and put \( v_2 \) in \( L(v_3) \) instead. Similarly \( v_3 \in L(v_1) \) can cover only the pair \([v_1, v_3]\). So if \( v_3 \in L(v_1) \) we can remove \( v_3 \) from \( L(v_1) \) and put \( v_1 \) in \( L(v_3) \) instead. Now if \( v_1 \in L(v_3) \) then we don’t need \( v_2 \) in \( L(v_1) \) and can replace \( v_2 \in L(v_1) \) by \( v_1 \in L(v_2) \) keeping \( L \) optimal and making \( v \) a type 2 vertex.

If \( v_1 \not\in L(v_3) \) then we have \( v_2 \in L(v_1) \) to cover the pair \([v_1, v_3]\). So either \( v \) is a type 1 or there an additional hub \( v_1 \in L(v_3) \). In the latter case we can remove \( v_2 \) from \( L(v_1) \) and put \( v_1 \) into \( L(v_3) \), making \( v \) a type 2 vertex. \( \square \)

For an edge \( \{u, v\} \in E \) let \( G'_{uv} \) be the subgraph of \( G' \) corresponding to this edge as shown in Figure 2d. \( G_{uv} \) contains all shortest paths between \( u_1 \) and \( u_j \). Note that no vertex of \( G' \) other than \( u_i \) and \( v_j \) hits these paths. We say that a hub \( u_j \in L(v_j) \) or \( v_i \in L(u_j) \) is a \( \{u, v\} \)-crossing hub.

**Lemma 6.3.** If there is an edge \( \{u, v\} \in E \) then the labels of \( u_i, v_i, 1 \leq i \leq 3 \) contain at least 3 \( \{u, v\} \)-crossings.

**Proof.** Consider three pairs: \([u_1, v_1], [u_2, v_2] \) and \([u_3, v_3] \). To cover each \([u_i, v_i] \) pair for \( i = 1, 2, 3 \) we need a \( \{u, v\} \)-crossing hub. So \( L(u_i) \cup L(v_i) \) contains a \( \{u, v\} \)-crossing hub. Since all three \( L(u_i) \cup L(v_i) \) are disjoint, \( L \) has at least 3 \( \{u, v\} \)-crossing hubs. \( \square \)

The following lemma shows that the type 2 vertices must form a VC.

**Lemma 6.4.** There is an optimal labeling \( L \) for \( G' \) such that for each edge \( \{u, v\} \in E \) there is at least one type 2 vertex among \( u \) and \( v \).

**Proof.** By Lemma 6.2 we can assume that every vertex is either a type 1 or a type 2 vertex in \( L \).

Suppose \( \{u, v\} \in E \) and both \( u \) and \( v \) are type 1 vertices. A partial labeling is shown in Figure 2e. Since \( u_i \not\in L(u_2) \), \( u_1 \) cannot cover the pair \([v_1, v_2] \). Similarly, \( v_1 \) cannot cover the pair \([u_1, v_2] \) and neither \( u_1 \) nor \( v_1 \) can cover the pair \([u_2, v_2] \). With one more hub to cover the pair \([u_1, v_1] \) it follows that we need
Theorem 6.8. The problem of deciding whether an undirected graph has an

Theorem 6.6. The problem of deciding whether an undirected graph has an

NP-complete.

with two arcs \( \{u, v\} \) which there is a simple reduction from the undirected case.

\[ \{ \{ u, v \}, \{ w, x \} \} \]

with unit lengths. If we change length of edges

HHL

The labels in Figure 2e and Figure 2g respect this order. Thus the

\( v \)

we put the triple \( \{ v, s, t \} \) in labels of \( u \), so all \( \{ u, w, j \} \) pairs for \( w \neq v \) remain covered. Also all \( \{ v, w, j \} \) pairs remain covered, since

\( v_2 \) can’t be used as a hub for any \( \{ v_1, w \} \) pair. \( \square \)

The following lemma gives a reduction from VC to HL.

Lemma 6.5. The graph \( G \) has a VC of size \( k \) if and only if \( G' \) has an HL of size \( 12|V| + 1 + 3|E| + k \).

Proof. Assume \( G \) has a vertex cover of size at most \( k \). We construct an HL of \( G' \) as follows. We put \( s \) and \( v \) in itself in \( L(v) \) for every \( v \in V' \). Since there are \( 6|V| + 1 \) vertices in \( G' \) this contributes \( 12|V| + 1 \) hubs. Then we make each vertex of the vertex cover a type 2 vertex and each vertex which is not in the vertex cover a type 1 vertex. We use 2 hubs to cover \( G' \) for a type 1 vertex and 3 hubs for a type 2 vertex, for the total of \( 2|V| + k \) hubs. For each edge \( \{u, v\} \in E \) we use 3 \( \{u, v\} \)-crossing hubs to cover

\( G_{uv} \) as shown in Figure 2e and Figure 2g. So the total labeling size is \( 12|V| + 1 + 3|E| + k \).

Now assume that \( L \) is an optimal HL of \( G' \) of size \( 12|V| + 1 + 3|E| + k \). By Lemma 6.1 we know that any vertex \( w \in G' \) has \( s \) in its label and by Lemma 6.2 we know that there exists such an \( L \) that makes every vertex \( v \in V' \) either a type 1 or a type 2 vertex. By Lemma 6.3 we know that there are at least 3 \( \{u, v\} \)-crossing hubs for any edge \( \{u, v\} \in E \). Since the size of \( L \) is at most \( 12|V| + 1 + 3|E| + k \) it follows that there are at most \( k \) type 2 vertices in \( L \). Lemma 6.4 implies that these \( k \) vertices form a vertex cover. \( \square \)

Theorem 6.6. The problem of deciding whether an undirected graph has an HL of size at most \( k \) is NP-complete.

The following lemma shows that our reduction is in fact also a valid reduction from VC to finding an optimal HHL.

Lemma 6.7. The graph \( G' \) has an HL of size \( k' \) if and only if it has an HHL of size \( k' \).

Proof. The ‘if’ part follows from the fact that every HHL is an HL. For the ‘only if’ part consider an optimal HL \( L \) of size at most \( k' \). By Lemma 6.2 each vertex is either of type 1 or of type 2. Consider the following order of the vertices of \( G' \). The most important vertex is \( s \) followed by all the leaves of \( S \). Then we put the triple \( v_1, v_2, v_3 \) for all type 2 vertices where for each \( v, v_1 \) is more important than \( v_2 \) which is more important than \( v_3 \) and the order of the triples corresponding to different vertices is arbitrary. Finally put a triple \( v_2, v_1, v_3 \) for all type 1 vertices where for each \( v, v_2 \) is more important than \( v_1 \) which is more important than \( v_3 \) and the order of the triples corresponding to different vertices is arbitrary. The labels in Figure 2e and Figure 2g respect this order. Thus the HHL \( L \) corresponding to this order has exactly 3 \( \{u, v\} \)-crossings for each \( \{u, v\} \in E \). Therefore by Lemma 6.3 \( L \) has the same size as \( L \). \( \square \)

Lemma 6.7 and Lemma 6.5 immediately imply the following

Theorem 6.8. The problem of deciding whether an undirected graph has an HHL of size at most \( k \) is NP-complete.

Theorem 6.6 and Theorem 6.8 show that both HL and HHL are NP-Complete in undirected graphs with unit lengths. If we change length of edges \( \{s, v_1\} \) for \( v \in V \) from 1 to 0.9 our proof is not affected. However, the shortest paths in \( G' \) become unique. So HL and HHL are NP-Complete in undirected graphs even when shortest paths are unique.

6.2 Directed Graphs

Here we show that both optimal HL and HHL are NP-hard in directed graphs. We begin with HHL, for which there is a simple reduction from the undirected case.

Let \( G \) be an undirected graph. We transform \( G \) to directed graph \( G' \) by replacing each edge \( \{u, v\} \) with two arcs \( (u, v) \) and \( (v, u) \). Now we present the reduction.

Lemma 6.9. The graph \( G \) has an HHL of size \( k \) if and only if \( G' \) has an HHL of size \( 2k \).
Theorem 6.10. The problem of deciding whether a directed graph has an HHL of size at most k is NP-complete.

Proof. To show the “only if” part, we take the labeling $\tilde{L}$ constructed from $L$ as follows $\tilde{L}_f(v) := L(v)$ and $\tilde{L}_b(v) := L(v)$. Now we show the “if” part. We can assume that $\tilde{L}_f, \tilde{L}_b$ is a canonical labeling (or replace the labeling by a smaller canonical one). Since $G'$ is symmetric, from the definition of canonical labeling it follows that for any vertex v the forward label has exactly the same hubs as the backward label. Moreover the distances from v to and from the hubs are the same. So $L$ defined as $L(v) := \tilde{L}_f(v)$ is a valid labeling for $G$. \qed

Remark 6.11. For a directed graph a minimum HHL need not be symmetric.

Proof. Consider the 4-cycle graph $C_4 = (V, E)$, $V = \{v_0, v_1, v_2, v_3\}$, $E = \{v_i, v_{i+1 \text{mod } 4} \mid 0 \leq i \leq 3\}$ and the corresponding directed graph $C'_4$. An HHL $L$ of size 16 for $C'_4$ is shown in Figure 3 (for example $L_f(v_0)$ contains $v_0$ and $v_3$ and $L_b(v_0)$ contains $v_0$ and $v_1$). Note that it is not symmetric as for example $L_f(v_0) \neq L_b(v_0)$. Any labeling in $C'_4$ satisfying $L_f = L_b$ correspond to a labeling in $C_4$ of half the size. So in order to show that there is no symmetric labeling of $C'_4$ of size 16 we show that there is no labeling of $C_4$ of size at most 8. Indeed we need 4 hubs to cover the pairs $[v_1, v_2]$ and 4 hubs to cover the pairs $[v_1, v_{i+1 \text{mod } 4}]$. This already counts for 8 hubs. Therefore no $v_i$ is in $L(v_{i+2 \text{mod } 4})$. To cover the $[v_0, v_2]$ pair we need $v_1$ (or $v_3$, the case is similar) to be in both $L(v_0)$ and $L(v_2)$ and therefore $L(v_1)$ contains only $v_1$. But now we have no common hub for the $[v_1, v_3]$ and therefore it is uncovered. So there is no HHL of size 8 for $C_4$. \qed

Now we present another reduction from VC to HHL in a directed graph. For a VC instance $G = (V, E)$ we construct an HHL instance $G' = (V', A')$, $V' = \{w\} \cup \{v_1, v_2 \mid v \in V\} \cup \{e \mid e \in E\}$, $A' = \{(w, v_1), (v_1, v_2), v \in V\} \cup \{(u_1, v_2), (v_1, u_2), (u_2, e), (v_2, e) \mid e = \{u, v\} \in E\}$. All arcs have length 1. For each edge $e = \{u, v\}$ from $G$ we have a gadget as shown in Figure 4 (consider only straight arcs).

For any labeling we have $x$ in both $L_f(x)$ and $L_b(x)$ for all vertices $x$ and either $x \in L_b(y)$ or $y \in L_f(x)$ for all arcs $(x, y)$. Let us call such hubs mandatory and all other hubs non-mandatory. Mandatory hubs cover all pairs $\{x, y\}$ such that $\text{dist}(x, y) \leq 1$. Any labeling for $G'$ has at least $M(G') = 2|V'| + |A'|$ mandatory hubs.

Lemma 6.12. The graph $G$ has a VC of size $k$ if and only if $G'$ has an HHL of size $M(G') + k$.

Proof. We claim that mandatory hubs are enough to cover all pairs in $G'$ except $\{w, e\}$ for $e \in E$, which means all pairs $\{x, y\}$ with $\text{dist}(x, y) \leq 2$. The sufficient labeling is shown in Figure 4 by curly arcs: a solid curly arc $(x, y)$ means $y \in L_f(x)$ and a dashed curly arc $(x, y)$ means $y \in L_b(x)$. Indeed, for a pair $\{x, y\}$ with $\text{dist}(x, y) = 2$ we have either $x = w$ or $y = e$ for some $e \in E$. In the former case $y = u_2$ for some $u \in V$ and the common hub is $u_1$. In the latter case $x = u_1$ for some $u \in V$ and either $e = \{u, v\}$ or $e = \{v, v'\}$ for some neighbor $v \in V$ of $u$. In both cases $v_2$ is the common hub.
Since \( \text{dist}(w,e) = 3 \) for a \( e \in E \) we need a non-mandatory hub to cover a \([w,e]\) pair. The non-mandatory hubs correspond to the vertex cover in \( G \). If there is a VC of size \( k \) in \( G \) then it is sufficient to use exactly \( k \) non-mandatory hubs: add \( v_2 \) to \( L_f(w) \) for every \( v \) in VC.

Suppose there is an HL with at most \( k \) non-mandatory hubs. We build a VC of size at most \( k \). For a non-mandatory hub \( e \in L_f(w) \) and any non-mandatory hub in \( L_b(e) \) for an edge \( e = \{u,v\} \in E \), add \( u \) to the VC. For a non-mandatory hub \( v_2 \in L_f(w) \) for some \( v \in V \), add \( v \) to the VC. It is easy to see that this is indeed the vertex cover.

\[ \text{Theorem 6.13.} \quad \text{The problem of deciding whether a directed graph has an HL of size at most} \ k \ \text{is NP-complete.} \]

\section{HL vs. HHL}

In [13], it is shown that the gap between the size of the optimal HHL and the size of the optimal HL can be \( \Omega(n^{0.26}) \). We show that for a graph in Figure 5 the gap is \( \Omega(n^{0.5}) \).

\[ \text{Theorem 7.1.} \quad \text{There is a graph family for which the optimal HHL size is \( \Omega(\sqrt{n}) \) times larger than the optimal HL size.} \]

\textbf{Proof.} Consider the undirected graph shown in Figure 5. The graph consists of \( k \) distinct stars each with \( k-1 \) leaves. The centers of the stars are connected such that they form a clique. Finally, there is an additional vertex \( s \) connected to the leaves of all stars. The total number of vertices is \( n = k^2 + 1 \). The length of every edge is 1.

Consider the following HL for this graph. The vertex \( s \) is in every label. A center \( v \) of a star \( S \) is in the labels of all of the vertices of \( S \). Finally, every star-center has every other star-center in its label. It is easy to verify that the cover property holds for this labeling. Each leaf \( u \) of some star \( S \) has a label of size \( O(1) \). The label of \( s \) is of size \( O(1) \). The size of the label of each star-center is \( k + 1 \). It follows that the total size of this labeling is \( O(n) \).

To construct an HHL, we need to order the centers of the stars. Fix such an order. Consider a leaf \( u \) of some star with center \( c(u) \), and let \( i \) be the number of star-centers which are more important than \( c(u) \). For each star-center \( v \) that is more important than \( c(u) \), \( (u, c(u), v) \) is the shortest path between \( u \) and \( v \), so either \( v \) is in \( L(u) \) or \( u \) is in \( L(v) \). This accounts for \( i \) hubs in the labels due to the pair \( u, v \). The total contribution of such hubs to the size of the labeling is

\[ (k-1) \sum_{i=1}^{k-1} i = k(k-1)^2/2 = \Omega(n^{3/2}). \]

If follows that the total size of any hierarchical labeling is \( \Omega(n^{3/2}) \). This yields an \( \Omega(\sqrt{n}) \) gap between the optimal HL and the optimal HHL.

\[ \text{The results of Section 4 imply that the gap in Theorem 7.1 is within} \ O(\log n) \ \text{factor of the best possible.} \]
8 Concluding Remarks

Our lower bounds for greedy algorithms show that in contrast with HL the greedy algorithm does not give a poly-log approximation for HHL. This motivates the question of whether a poly-log approximation algorithm for HHL exists. Our lower bound for w-HHL is Ω(√n) factor away from the upper bound, which leaves the open question to determine the polynomial factor for the w-HHL algorithm approximation guarantee.

On many problem classes g-HHL and w-HHL find labelings of size close to that found by the O(log n)-approximation algorithm for HL [10]. It would be interesting to get a theoretical explanation of this phenomena, for example by proving a better approximation ratio for g-HHL or w-HHL on natural classes of graphs.

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