Probabilistically robust AC power flow in the presence of renewable sources and uncertain loads

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Abstract

The increasing penetration of renewable energy resources, paired with the fact that load can vary significantly, introduce a high degree of uncertainty in the behavior of modern power grids. Given that classical dispatch solutions are “rigid,” their performance in such an uncertain environment is in general far from optimal. For this reason, in this paper, we consider AC optimal power flow (AC-OPF) problems in the presence of uncertain loads and (uncertain) renewable energy generators. The goal of AC-OPF design is to guarantee that controllable generation is dispatched at minimum cost, while satisfying constraints on generation and transmission for almost all realizations of the uncertainty. We propose an approach based on a randomized technique recently developed, named scenario with certificates, which allows to tackle the problem without assuming any a-priori dependence of the voltages in the network on the uncertain generators/loads. The proposed solution can exploit the usually available probabilistic description of the uncertainty and variability, and provides solutions with a-priori probabilistic guarantees on the risk of violating the constraints on generation and transmission.

I. INTRODUCTION

Modern power grids are characterized by increasing penetration of renewable energy sources, such as solar photovoltaic and wind power. This trend is expected to increase in the near future, as also testified by strict commitments to large renewable power penetration being made by major countries worldwide; e.g., see [13], [15]. While the advantages of renewable energy in terms of environmental safeguard are indisputable, its introduction does not come without a cost. Indeed, renewable energy generation technologies are highly variable and not fully dispatchable, thus imposing novel challenges to the existing power system operational paradigm. As discussed in e.g. [5], when uncontrollable resources fluctuate, classical optimal power flow (OPF) solutions can provide very inefficient power generation policies, that result in line overloads and, potentially, cascading outrages.

Despite the increasingly larger investments, which are costly and subject to several regulatory and policy limitations, the frequency and scale of power outages are steadily growing. This situation clearly shows that a strategy...
based only on investments in technological improvements of the transmission lines and controllable generation capacity—as those discussed e.g. in [14]—is not sufficient anymore. Instead, radically new dispatch philosophies need to be devised, able to cope with the increasing uncertainty, due to unpredictable fluctuations in renewable output and time-varying loads.

Indeed, classical OPF dispatch is typically computed based on simple predictions of expected loads and generation levels for the upcoming time window. Although these predictions can be fairly precise for the case of traditional generators and loads, they may be highly unreliable in the case of renewable generators, thus explaining its failure in these latter situations.

It follows that one of the major problems in today’s power grids is the following: Given the high level of uncertainty introduced by renewable energy sources, design a dispatch policy that i) minimizes generation costs and ii) does not violate generation and transmission constraints for all admissible values of renewable power and variable demand. In other words, one would like to design an optimal dispatch policy that is robust against uncertainty.

However, such robust policies might be very conservative. Power networks can tolerate temporary violations of their generation and transmission constraints. Moreover, being robust with respect to any possible value of the uncertainty may lead to very inefficient policies, since some scenarios are very unlikely. Hence, in this paper we take a different approach: instead of requiring that the network constraints are satisfied for all possible values of uncertainty, we allow for a small well-defined risk of constraint violation. More precisely, we start by assuming that one can adapt the power generated by conventional generators, based on real-time information about current values of renewable power generation and demand. Under this assumption, we aim at minimizing cost of power generation while meeting demand and allowing a small well defined risk of violating network constraints. We re-formulate the problem by highlighting the fundamental difference between control and state variables. Then, to approximate the solution of the resulting (complex) optimization problem we leverage recent results on convex relaxations of the optimal power flow problem [23], and use a novel way of addressing probabilistic robust optimization problems known as scenario with certificates (SwC) [18]. In this way, we derive a convex problem that i) is efficiently solvable, ii) provides a good approximation of the optimal power generation under the above mentioned risk constraints and iii) for some special classes of networks, it provides an exact solution.

A. Previous Results

The optimal power flow problem is known to be NP-hard even in the absence of uncertainty; see e.g. the recent report [6] and references therein. Hence, several numerical approaches propose approximations of OPF, based e.g. on Newton methods [16], interior point based methods [38] or global optimizations heuristics [11], [22]. In particular, several approximations have been introduced to recover convexity and make the problem numerically tractable. The most common is DC approximation, see [12] and references therein, in which the AC OPF problem is linearized. However, the solution is in general sub-optimal and, more importantly, it may not be feasible, in the sense that it may not satisfy the original nonlinear power flow equations. Also, as noted in [12], the fact that DC approximation
fixes voltage magnitudes and ignores reactive power, makes the solution not applicable in several important practical situations.

Motivated by the above considerations, semidefinite programming (SDP) relaxations of the AC OPF problem have been recently introduced to alleviate the computational burden, in [21] for radial networks and in [4] for general networks—see [25] for a detailed review. These convex relaxations have recently received renewed interest thanks to [23], that analyzes the optimality properties of the relaxation, showing how in many practical situation these relaxations turn out to be nonconservative—in the sense that the solution of the relaxed problem coincides with that of the original nonlinear one. This result has sparked an interesting literature analyzing specific cases in which the SDP relaxations can be proven to be exact, see e.g. [17], and showing how graph sparsity can be exploited to simplify the SDP relaxation of OPF, see e.g. [3].

However, while these works have reached a good level of maturity, (see for instance the recent two-part tutorial [25], [26]), there is still very little research analyzing if and how these relaxations could be extended to the problem considered in this paper, namely AC OPF in the presence of possibly large and unpredictable uncertainties. Indeed, when it comes to robust OPF, most literature recurs to simple DC models, see e.g. [5], [27], with the drawbacks discussed before. In particular, there are recent results that use a scenario based approach to design dispatch policies, but they are also limited to the linearized DC case, see [33], [34], [35]. Notable exceptions are [37], in which a non-convex formulation and a randomized method for its solution are provided, and [32], where the relaxation in [23] together with a strict parameterization of the dependent variables is proposed, and a scenario approach is proposed. Both these works adopt the following philosophy for dealing in real-time with the uncertainties: at a slower time-scale (in our case, hourly-based), a dispatch strategy is designed to balance the power mismatch distributing it among all conventional generators through a deployment vector. This policy is then implemented to allow to counteract in real-time the measured power/load variations.

Inspired by these recent works, in this paper we provide a less conservative approach, that does not require dependent variable parameterization. Instead, we exploit a recently developed approach to probabilistic robust optimization scenario with certificates [18] to develop less conservative, efficient relaxations of the OPF in the presence of uncertainty.

B. The Sequel

The paper is organized as follows: In Section II, we precisely formulate the AC-OPF problem. We also describe the adopted dispatch policy to be used in real time to cope with uncertainty. Then, we introduce a precise formulation of the Robust AC-OPF problem, which divides the set of optimization variables into two distinct classes: i) independent/control variables (those that can be controlled by the operator), and ii) dependent/state variables (representing the state of the system). This reformulation represents one of the main contributions of our work, allowing to represent in a non-conservative way the problem of AC-OPF design in the presence of uncertainty. Also, in Section III, we show how this reformulation can be combined with the convex relaxation of [23], and we formally prove that this allows to recover its properties in terms of exactness of the relaxation in some special
cases. Moreover, in Section IV we show how this new formulation directly translates in the scenario with certificates paradigm. Numerical examples illustrating the performance of the proposed approach are provided in Section V. Finally, in Section VI some concluding remarks are presented.

II. OPTIMAL POWER FLOW ALLOCATION PROBLEM UNDER UNCERTAINTY

In this section, we briefly summarize the adopted power flow model, which takes into explicit account load uncertainties and variable generators, and we formalize the ensuing optimization problem, arising from the necessity of robustly guaranteeing that safety limits are not exceeded.

A. Nominal AC-OPF

We consider a power network with graph representation $G = \{\mathcal{N}, \mathcal{L}\}$, where $\mathcal{N} \equiv \{1, 2, \ldots, n\}$ denotes the set of buses (which can be represented as nodes of the graph), and $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ denotes the set of electrical lines connecting the different buses in the network (represented as edges of the graph). The set of conventional generator buses is denoted by $G \subseteq \mathcal{N}$, and its cardinality is $n_g$. As a convention, it is assumed that the bus indices are ordered so that the first $n_g$ buses are generators, i.e. $G \equiv \{1, 2, \ldots, n_g\}$. Each generator (we assume for ease of notation that no more than one generator is present on each generator bus) connected to the bus $k \in G$ provides complex power $P_{G}^{k} + Q_{G}^{k} i$, where $P_{G}^{k}$ is the active power generated by the $k$-th generator, and $Q_{G}^{k}$ is the corresponding reactive power. Goal of the network manager is to guarantee that the output of generators is such that the network operates safely and, if possible, at minimal cost.

We now elaborate on the different constraints that should be satisfied in order for the network to operate safely. First, consider the line $(l, m) \in \mathcal{L}$, i.e. the line connecting buses $l$ and $m$. Let $y_{lm}$ be the (complex) admittance of the line and $V_k = |V_k| \angle \theta_k$ be the (complex) voltage at bus $k$, with magnitude $|V_k|$ and phase angle $\theta_k$. Then, the following Balance Equations should be satisfied at all times

\begin{align}
P_{k}^{G} - P_{L}^{k} &= \sum_{l \in \mathcal{N}_k} \Re \{V_k(V_k - V_l)^* y_{kl}^*\}, \quad \forall k \in \mathcal{N}, \\
Q_{k}^{G} - Q_{L}^{k} &= \sum_{l \in \mathcal{N}_k} \Im \{V_k(V_k - V_l)^* y_{kl}^*\}, \quad \forall k \in \mathcal{N},
\end{align}

where $\mathcal{N}_k$ is the set of all neighboring buses directly connected to bus $k$, and $P_{L}^{k}$ and $Q_{L}^{k}$ denote the (known) active and reactive loads of the network bus $k$ (by convention, $P_{G}^{k}$ and $Q_{G}^{k}$ are set to zero in nongenerator nodes).

Also, at generator bus $k \in G$, one has the so-called Power Constraints, which restrict the active and reactive power, and have the form

\begin{align}
P_{k \min} \leq P_{G}^{k} \leq P_{k \max}, \quad \forall k \in G, \\
Q_{k \min} \leq Q_{G}^{k} \leq Q_{k \max}, \quad \forall k \in G.
\end{align}

The voltages should satisfy the following Voltage Constraints

\begin{align}
|V_k| \leq V_{\max}, \quad \forall k \in \mathcal{N}, \\
|V_l - V_m| \leq \Delta V_{\max}^{lm}, \quad \forall (l, m) \in \mathcal{L}
\end{align}
In [23], the constraints in (5b) have been proven to be practically equivalent to the more classical bounds
\[ |V_l(V_l - V_m) y_{lm}^*| \leq S_{lm \text{ max}}, \quad \forall (l, m) \in L, \]
where \( S_{lm \text{ max}} \) is the maximum apparent power flow which can path through the line \((l, m) \in L\).

B. Control and State Variables

In the power-systems literature, the variables appearing in formulation above are usually divided into two classes. Indeed, already in [10], the distinction between control and state variables is explicitly made. Control variables are those used by the network operator to set the operating condition of the network. State variables are dependent variables that represent the state of a power network, see also the recent survey [8].

In particular, control and state variables are defined differently depending on the type of bus. In a generator bus \( k \in \mathcal{G} \), usually referred to as PV bus (see e.g. [25, Remark 1]) active power \( P_k^G \) of the generator and magnitude \( |V_k| \) of the complex bus voltage represent the control variables, while phase angle \( \theta_k \) of bus voltage and generator reactive power \( Q_k^G \) are the state variables. In a load bus, or PQ bus, the active and reactive power of the load \( P_k^L, Q_k^L \) are given (their values are known to the network operator) while magnitude and phase angle of bus voltage \( |V_k|, \theta_k \) are state variables. A node to which both a generator and loads are connected is to be considered as a generator bus.

Remark 1 (Slack bus): In power flow studies, usually bus \( 0 \) is considered as reference or slack bus. The role of the slack bus is to balance the active and reactive power in the power grid. The slack bus must include a generator. In most cases, the voltage magnitude and phase are fixed at the slack bus, whereas active and reactive generator power are variables. Therefore, in a slack bus there is no control variable, while active and reactive generator power \( P_k^G \) and \( Q_k^G \) are state variables. For simplicity of notation, we do not include slack bus in the formulation. However, its introduction is seamless, and slack bus is considered in the numerical simulation.

To emphasize the inherent difference between control and state variables, we introduce the notation
\[
\mathbf{u} = \{ P_1^G \cdots P_{ng}^G, |V_1| \cdots |V_{ng}| \} \quad \text{(4)}
\]
\[
\mathbf{x} = \{ Q_1^G \cdots Q_{ng}^G, |V_{ng+1}|, \ldots, |V_n|, \theta_1, \ldots, \theta_n \} \quad \text{(5)}
\]
to denote respectively control and state variables. This allows to formulate the nominal OPF problem in the following way

Nominal AC-OPF
\[
\min_{\mathbf{u}} f(\mathbf{u}) \quad \text{s.t.: } \text{there exist } \mathbf{x} \text{ such that } g(\mathbf{x}, \mathbf{u}) = 0 \quad \text{(6)}
\]
\[
\text{and } h(\mathbf{x}, \mathbf{u}) \leq 0,
\]
where \( f = \sum_{k \in \mathcal{G}} f_k(P_k^G) \) with \( f_k(P_k^G) \) being the cost of the power generated by generator \( k \), the equality constraint \( g(\mathbf{x}, \mathbf{u}) = 0 \) defines the balance equations \((1)\), and the inequality constraint \( h(\mathbf{x}, \mathbf{u}) \leq 0 \) summarize the power and voltage constraints \((2)\) and \((3)\).
C. Generation/Demand Uncertainty

The problem described in the previous subsections is an idealized one, where it is assumed that the demand on the network is completely predictable (and thus known in the design phase), and there is no uncertainty in the amount of power being generated at all generator nodes. However, as discussed in the Introduction, these assumptions are rather strong, and unrealistic when dealing with modern power networks with high penetration of renewable sources and variable demands.

In the framework considered in this paper, we assume both the existence of renewable sources and uncertain load. A renewable energy generator connected to bus $k \in \mathcal{N}$ provides an uncertain complex power

$$P^R_k(\delta^R_k) + Q^R_k(\delta^R_k)i = P^{R,0}_k + Q^{R,0}_k i + \delta^R_k,$$

with $P^{R,0}_k + Q^{R,0}_ki$ being the nominal (predicted) power generated by the renewable energy source\(^1\) and $\delta^R_k \in \Delta^R_k \subset \mathbb{C}$ representing an uncertain complex fluctuation, which mainly depends on the environmental conditions, such as wind speed in the case of wind generators. The uncertain demand in bus $k \in \mathcal{N}$ is also represented as

$$P^L_k(\delta^L_k) + Q^L_k(\delta^L_k)i = P^{L,0}_k + Q^{L,0}_ki + \delta^L_k,$$

where $P^{L,0}_k$ and $Q^{L,0}_k$ denote the expected active and reactive load and $\delta^L_k \in \Delta^L_k \subset \mathbb{C}$ is the complex fluctuation in the demand at bus $k \in \mathcal{N}$. The support set is the point $\{0\}$ if no uncertainty (i.e. no renewable generator or variable load) is present in bus $k$. To simplify the notation, we collect the different sources of uncertainty by introducing an uncertainty vector

$$\delta = [\delta^L_1 \cdots \delta^L_n \delta^R_1 \cdots \delta^R_n]^T,$$

which varies in the set $\Delta = \Delta^L_1 \times \cdots \times \Delta^L_n \times \Delta^R_1 \times \cdots \times \Delta^R_n$.

To deal in a rigorous way with the uncertainties introduced above, we adopt a modification of the so-called frequency control (primary and secondary control), similar to that discussed in [5] in the context of DC-OPF. In classical OPF, this approach is used to distribute to the generators the difference between real-time (actual) and predicted demand, through some coefficients which are generator specific. However, these coefficients are in general decided a priori in an ad-hoc fashion. This approach worked well in cases where the amount of power mismatch was not significant; however, once renewable generators are in the power network, this difference may become large, thus leading to line overloads in the network. The approach presented in [5] specifically incorporates these distribution parameters in the OPF optimization problem. In the setup proposed in this paper, we follow a similar strategy, and formally introduce a deployment vector $\alpha = [\alpha_1, \ldots, \alpha_n]^T$, with $\sum_{k \in \mathcal{G}} \alpha_k = 1$, $\alpha_k \geq 0$ for all $k \in \mathcal{G}$, whose purpose is to distribute among the available generators the power mismatch created by the uncertain generators and loads. To the best of our knowledge, the use of a deployment vector was originally introduced in the context of DC-approximations in [33] and [35], and further improvements on this concept are described in [34], [36]. During operation, the active generation output of each generator is modified according to the realizations of

\(^1\)We use the convention that $P^R_k = 0$, $Q^R_k = 0$ if no renewable generator is connected to node $k$. 

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the uncertain loads and power (which are assumed to be measured on-line) as follows

\[
\bar{P}_G^k = P_G^k + \alpha_k \left( \sum_{j \in \mathcal{N}} \text{Re}\{\delta^L_j\} + \sum_{k \in \mathcal{R}} \text{Re}\{\delta^R_k\} \right)
\]

(9)

\[
= P_G^k + \alpha_k s^T \text{Re}\{\delta\}, \quad \forall k \in \mathcal{G}
\]

with \( s^T = [1^T_n, -1^T_n] \). Note also that, with the introduction of the uncertain renewable energy generator and load into the power network, summarized by the vector \( \delta \), both Balance Equations (1) and Power Generation Constraint (2) become uncertain. Hence, the equality constraints are rewritten as

\[
P_G^k + \alpha_k s^T \text{Re}\{\delta\} + P_R^k(\delta) - P_L^k(\delta) = \sum_{i \in \mathcal{N}_k} \text{Re}\{V_k(V_k - V_i)^* y_{ki}\}, \quad \forall k \in \mathcal{N}
\]

(10a)

\[
Q_G^k + Q_R^k(\delta) - Q_L^k(\delta) = \sum_{i \in \mathcal{N}_k} \text{Im}\{V_k(V_k - V_i)^* y_{ki}\}, \quad \forall k \in \mathcal{N},
\]

(10b)

\[
1^T \alpha = 1
\]

(10c)

and the power inequality constraints become

\[
P_{k\min} \leq P_G^k + \alpha_k s^T \text{Re}\{\delta\} \leq P_{k\max}, \quad \forall k \in \mathcal{G}
\]

(11a)

\[
Q_{k\min} \leq Q_G^k \leq Q_{k\max}, \quad \forall k \in \mathcal{G},
\]

(11b)

\[
\alpha_k \geq 0, \quad \forall k \in \mathcal{G}.
\]

(11c)

Following the discussion in Section II-B, we note that the deployment vector \( \alpha \) should also be considered as a control variable. Hence, the set of control variables is redefined as

\[
u = \{P_1^G, \ldots, P_n^G, |V_1|, \ldots, |V_n|, \alpha_1, \ldots, \alpha_n\}.
\]

(12)

More importantly, it should be remarked that, with the introduction of the uncertainties, the value of the state variables will depend on the specific realization of the uncertainty. To emphasize this dependence on the uncertainty vector \( \delta \), we use the following notation

\[
x_{\delta} \doteq \{Q_1^G, \ldots, Q_{n_{\delta}}^G, |V_{n_{\delta}+1}|, \ldots, |V_n|, \theta_1, \ldots, \theta_n\}.
\]

(13)

We are now ready to provide a formal statement of the robust version of the optimal power flow problem. To this end, denote by \( g(u, x_{\delta}, \delta) = 0 \) the uncertain equality constraints collected in (10), and by \( h(u, x_{\delta}, \delta) \leq 0 \) the uncertain inequalities collected in (11).

**Robust AC-OPF**

\[
\min_u f(u) \text{ s.t.: for all } \delta \in \Delta, \text{ there exist } x_{\delta} \text{ such that }
\]

\[
g(u, x_{\delta}, \delta) = 0 \text{ and } h(u, x_{\delta}, \delta) \leq 0.
\]

(14)
In the above formulation of the robust OPF problem, originally introduced in [37], the objective is to optimize the values of the nominal generated power, the bus voltage magnitude at generator node and the deployment vector so that i) the network operates safely for all values of the uncertainty and ii) the generation cost of the network is minimized. In fact, if a solution to the problem above exists, we guarantee that for any admissible uncertainty, there exists a network state $x_\delta$ satisfying the operational constraints. Note that problem (14) is computationally very hard, since the uncertainty set $\Delta$ is an infinite set (actually, uncountable). Hence, the optimization problem (14) is a nonlinear/nonconvex semi-infinite optimization problem. Moreover, in (14) the constraints are enforced to hold for “all” possible values of the uncertain parameters. This is in many cases excessive, and leads to conservative results, with consequent degradation of the cost function (in our case, higher generation cost).

Hence, in the remainder of this paper, we follow a chance-constrained approach, in which a probabilistic description of the uncertainty is assumed, and a solution is sought which is valid for the entire set of uncertainty except for a (small) subset having probability measure smaller than a desired (small) risk level $\varepsilon$. This approach is suitable for problems where “occasional” violation of constraints can be tolerated. One can argue that this is the case in power networks, since violation of line flow constraints does not necessarily lead to immediate line tripping. Rather, the line gradually heats up until a critical condition is reached and only then the line is disconnected. Therefore, if line overload happens with low probability, this will not lead to line tripping nor it will damage the network.

Also, in the robust case, one clearly needs to assume that the uncertainty set $\Delta$ is bounded, and to know this bound. In a probabilistic setup, however, one can also consider cases where $\Delta$ is unbounded, that is where the distribution of $\delta$ has infinite support, as e.g. the Gaussian case. Formally, in the sequel, we assume that the uncertainty vector $\delta$ is random with possibly unbounded support $\Delta$. Then, given a (small) risk level $\varepsilon \in (0, 1)$, the chance constrained version of the optimal power flow problem is stated as follows

**Chance-constrained AC-OPF**

$$\min_u f(u) \text{ s.t.: } \Pr \left\{ \delta \in \Delta \text{ for which } \exists x_\delta \text{ such that } g(u, x_\delta, \delta) = 0 \text{ and } h(u, x_\delta, \delta) \leq 0 \right\} \leq \varepsilon.$$ (15)

III. EFFICIENT NUMERICAL RELAXATION

A. Approximation of Non-Convex Terms

In this subsection, we extend the relaxation technique discussed in [4], [23] to the robust case circumventing the non-convexity associated with the optimal power flow problem (14). We stress that the only source of non-convexity of (14), and also of the original problem (6), is due to non-linear terms $V_kV_l$’s appearing in Balance Equations (1), (10) and Voltage Constraints (3). However, the quadratic constraints can be reformulated as linear ones by introducing a new variable $W = VV^*$ where $V$ is the vector of complex bus voltages $V \doteq [V_1, \ldots, V_n]^T.$ In order to replace $VV^*$ with the new variable $W$, two additional constraints need to be included: i) the matrix $W$ needs to be positive semi definite, i.e. the following Positivity Constraint should hold $W \succeq 0$, and ii) its rank should be

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one, i.e. the following *Rank Constraint* should hold \( \text{rank}(W) = 1 \). An important observation is that, by introducing matrix \( W \), the only source of nonconvexity is captured by the rank constraint. Indeed, as shown first in [23] and subsequently in [29], in most cases this constraint can be dropped without affecting the OPF solution.

To formally define the *convexified version* of the Robust AC-OPF, we note that bus voltage \( V \) appears in *Voltage Constraints* (3) and *Balance Equations* (10). Therefore, these constraints are redefined in terms of the new variable \( W \) as

\[
(V_k \text{ min})^2 \leq W_{kk} \leq (V_k \text{ max})^2, \quad \forall k \in N
\]  

\[
W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{lm} \text{ max})^2, \quad \forall (l, m) \in L
\]  

respectively. Recalling that the first \( n_g \) buses are generator buses and the remaining ones are load buses, we observe that the matrix \( W \) can be written as

\[
W = VV^* = 
\begin{bmatrix}
|V_1|^2 & V_1V_2^* & \cdots & V_1V_{n_g}^* & \cdots & V_1V_n^* \\
|V_2|^2 & V_2V_3^* & \cdots & \cdots & \cdots & V_2V_n^* \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
|V_{n_g}|^2 & \cdots & \cdots & \cdots & \cdots & |V_n|^2
\end{bmatrix}
\]  

(18)

It is immediately noticed that some elements of \( W \) involve the control variables \(|V_1|, \ldots, |V_{n_g}|\), while others are dependent variables corresponding to the voltage magnitude \(|V_{n_g+1}|, \ldots, |V_n|\) at non-generator nodes, and the voltage phases \( \theta_1, \ldots, \theta_n \). In order to distinguish between control and state variables appearing in it, we “decompose” \( W \) into the sum of two submatrices \( W^u \) and \( W^x \) as follows

\[
W^u \doteq \text{diag}(|V_1|^2, \ldots, |V_{n_g}|^2, 0, \ldots), \quad W^x = W - W^u
\]  

(19)

In this decomposition, we have a matrix \( W^u \) that includes the diagonal elements of \( W \) corresponding to the generator nodes only, while the remaining elements of \( W \) are collected in \( W^x \). With this in mind, the control and state variables are redefined as \( u \doteq \{P^G, \alpha, W^u\} \) and \( x_\delta \doteq \{Q^G, W^x\} \) respectively. With this notation settled, we are in the position to formally introduce the convexified version of the robust AC optimal power flow problem as follows.
Convexified Robust AC-OPF (CR-AC-OPF)

\[
\text{minimize} \quad \sum_{k \in G} f_k(P^G_k) \quad (20)
\]

subject to: for all \( \delta \in \Delta \), there exist \( Q^G = Q^G(\delta) \), \( W^x = W^x(\delta) \) such that

\[
W = W^u + W^x, \quad 1^T \alpha = 1, \quad W \succeq 0, \quad \alpha_k \geq 0, \quad \forall k \in G
\]

\[
P_k^G + \alpha_k s^T \text{Re}\{\delta\} + P^R_k(\delta) - P^L_k(\delta) = \sum_{l \in N_k} \text{Re}\{(W_{kk} - W_{kl})^* y_{kl}\}, \quad \forall k \in N
\]

\[
Q_k^G - Q_k^L(\delta) - Q_k^L(\delta) = \sum_{l \in N_k} \text{Im}\{(W_{kk} - W_{kl})^* y_{kl}\}, \quad \forall k \in N
\]

\[
P_{k \min} \leq P_k^G + \alpha_k s^T \text{Re}\{\delta\} \leq P_{k \max}, \quad \forall k \in G
\]

\[
Q_{k \min} \leq Q_k^G \leq Q_{k \max}, \quad \forall k \in G
\]

\[
(V_k \min)^2 \leq W_{kk} \leq (V_k \max)^2, \quad \forall k \in N
\]

\[
W_{ml} + W_{nm} - W_{lm} - W_{ml} \leq (\Delta V_{lm} \max)^2, \quad \forall (l, m) \in L
\]

To the best of the authors knowledge, the formulation of the CR-AC-OPF is original, and it represents the first main contribution of the present paper. First, the approach clearly differs from the formulations based on DC power flow. Second, it improves upon the formulation based on convex AC-formulation in [32]. Indeed, in that work, the authors cope with the need of guaranteeing the existence of a different value of \( W \) for different values of the uncertainty \( \delta \) by imposing a specific dependence on \( W \) from the uncertainty. In our notation, [32] introduces the following finite (linear) parameterization

\[
W(\delta) = A + \sum_k B_k \delta_k \quad (21)
\]

where \( A, B_1, \ldots, B_n \) become design variables in the optimization problem. With respect to this formulation, we remark the following: i) the formulation is surely more conservative than the CR-AC-OPF one, since it imposes a specific dependence on \( W \), while we do not impose any a-priori form. This gain in terms of performance is shown in Section V ii) the parameterization in (21) requires modification of the voltage magnitude at the generators during operation. Indeed, as previously noted, \( W \) involves not only the state variables, but also the control variables corresponding to generator voltages \( |V_1|, \ldots, |V_{ng}| \) which are not supposed to change during the operation. In the CR-AC-OPF, contrary, all control variables—including generators voltage magnitude—are designed during the optimization phase and do not need to be changed when demand/renewable power fluctuations occur.

Moreover, although the CR-AC-OPF formulation represents a relaxation of the robust optimal power flow problem, we can show that there are special cases in which this relaxation is indeed exact. In particular, we introduce next a class of networks where CR-AC-OPF does provide the optimal solution to the Robust AC-OPF problem. To this end, we first briefly recall some concepts from graph theory that are central for the results to follow. For a more detailed discussion on the following definitions and their interpretation in the context of power
networks the reader is referred to [29].

**Definition 1:** A network/graph is called weakly cyclic if every edge belongs to at most one cycle.

**Definition 2:** A network is said to be lossless if Re\{y_{lm}\} = 0, \forall (l, m) \in \mathcal{L}.

With these definitions at hand, we can now describe a class of networks for which the relaxed formulation above provides an exact solution. This theorem represents a natural extension of the results presented in [29] to the robust case and the proof is given in appendix.

**Theorem 1:** Consider a lossless weakly-cyclic network with cycles of size 3, and assume $Q_k^{\text{min}} = -\infty$ for every $k \in \mathcal{G}$. Then, the convex relaxation CR-AC-OPF is exact.

**Proof:** See Appendix A.

We should remark that the class considered in Theorem 1 is not fully realistic. However, we feel that this result is important for several reasons. First, it shows that the exactness of the relaxation for a particular class of networks—proven for the nominal case in [29]—carries over to the robust formulation we introduced in (20). This shows that CR-AC-OPF represents indeed the right way to formulate the robust counterpart of the AC-OPF problem. Second, several results of the same spirit have been obtained in the literature for deterministic (no uncertainty) networks, see for instance [26]: we believe that these results can be extended to the convexified robust formulation introduced here. This will the subject of further research.

### B. Penalized and robust costs

Note that, besides the configurations where the relaxation has been proven to be exact, in the general case it is important to obtain solutions $W$ with low rank. To this end, we adopt a penalization technique similar to that introduced in [28], [29]. More precisely, it is observed there that maximizing the weighted sum of off-diagonal entries of $W$ often results in a low-rank solution. To this end, one can augment the objective function with a weighted sum of generators’ reactive power—for lossless network—and apparent power loss over the series impedance of some of the lines of the network (the so-called problematic lines $\mathcal{L}_{\text{prob}}$)—for lossy network—to increase the weighted sum of the real parts of off-diagonal elements of $W$ and hence obtain a low rank solution. Formally, let $L_{lm} = |S_{lm} + S_{ml}|$ denote the apparent power loss over the line $(l, m)$ with $S_{lm} = (W_{ll} - W_{lm})^* y_{lm}^*$. Given nonnegative factors $\gamma_b$ and $\gamma_\ell$, in [28], [29] the following penalized cost was considered

$$f_{\text{pen}}(P^G, Q^G, W) = \sum_{k \in \mathcal{G}} f_k(P_k^G) + \gamma_b \sum_{k \in \mathcal{G}} Q_k^G + \gamma_\ell \sum_{(l,m) \in \mathcal{L}_{\text{prob}}} L_{lm}.$$  

Since, in the proposed approach, $Q^G$ and $W$ depend on the value of the uncertainty $\delta$, the formulation of the robust optimal power flow problem needs to reflect this fact. More precisely, the right cost function to consider is

$$\max_{\delta \in \mathcal{D}} f_{\text{pen}}(P^G, Q^G, W) = \max_{\delta \in \mathcal{D}} \sum_{k \in \mathcal{G}} f_k(P_k^G) + \gamma_b \sum_{k \in \mathcal{G}} Q_k^G + \gamma_\ell \sum_{(l,m) \in \mathcal{L}_{\text{prob}}} L_{lm}.$$  

The constraints remain the same as in (20). This modification of the cost provides a way of “encouraging” low rank solutions and it has been shown to work well in practice.
IV. A RANDOMIZED APPROACH TO CR-AC-OPF

In this section, we first briefly summarize the main features of the scenario with certificates problem, and then show how this approach can be used to tackle in an efficient way the CR-AC-OPF problem introduced in the previous section.

A. Scenario with Certificates

In [18] the design with certificates approach was introduced, in which a clear distinction is made between design variables $x$ and certificates $\xi$. In particular, in scenario with certificates (SwC) the function $f(\theta, \xi, \delta)$ is jointly convex in $\theta$ and $\xi$ for given $\delta \in \Delta$. Then, the following robust optimization problem with certificates is introduced

$$\min_{\theta} \quad c^T \theta$$

subject to $\forall \delta \in \Delta \exists \xi = \xi(\delta)$ satisfying $f(\theta, \xi, \delta) \leq 0$. \hfill (22)

Then, it is shown that problem (22) can be approximated by introducing the following scenario optimization with certificates problem, based on the extraction of $N$ random samples $\delta^{(1)}, \ldots, \delta^{(N)}$ of the uncertainty,

$$\theta_{SwC} = \arg \min_{x, \xi_1, \ldots, \xi_N} \quad c^T \theta$$

subject to $f(\theta, \xi_i, \delta^{(i)}) \leq 0, \ i = 1, \ldots, N$. \hfill (23)

Note that contrary to the classical scenario problem, in SwC a new certificate variable $\xi_i$ is created for every sample $\delta^{(i)}$. In this way, one implicitly constructs an “uncertainty dependent” certificate, without assuming any a-priori explicit functional dependence on $\delta$.

To present the properties of the solution $\theta_{SwC}$, we first introduce the violation probability of a given design $\theta$ as follows

$$\text{Viol}(\theta) = \Pr \left\{ \exists \delta \in \Delta | \exists \xi \text{ satisfying } f(\theta, \xi, \delta) \leq 0 \right\}.$$ 

Then, the main result regarding the scenario optimization with certificates is recalled next. The result was derived in [18] as an extension to the classical scenario approach developed in [7].

**Theorem 2:** Assume that, for any multisample extraction, problem (23) is feasible and attains a unique optimal solution. Then, given an accuracy level $\varepsilon \in (0, 1)$ and a confidence level $\beta \in (0, 1)$, if the number of samples is chosen as

$$N \geq N_{SwC} = \frac{e}{\varepsilon (e-1)} \left( \ln \frac{1}{\beta} + n_{\theta} - 1 \right) \hfill (24)$$

where $n_{\theta}$ is the dimension of $\theta$, and $e$ is the Euler number. Then, with probability at least $1 - \beta$, the solution $\theta_{SwC}$ of problem (23) satisfies $\text{Viol}(\theta_{SwC}) \leq \varepsilon$.

Note that, in the above theorem, we guaranteed with high confidence $(1 - \beta)$ that the solution returned by the SwC has a risk of violation of constraints less than the predefined (small) risk level $\varepsilon$. 

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B. SwC Solution to CR-AC-OPF

Clearly, problem CR-AC-OPF represents a robust optimization problem with certificates, in which the design variables are those “controllable” by the network manager, i.e. $\theta \equiv u$ while the certificates represent the quantities that can be "adjusted" to guarantee constraint satisfaction, i.e. $\xi \equiv x_\delta$. Based on this consideration, in this paper we propose the following solution strategy to the CR-AC-OPF problem:

SwC-AC-OPF Design Procedure

i) Given probabilistic levels $\varepsilon$, and $\beta$ compute $N_{SwC}$ according to (24).

ii) Generate $N \geq N_{SwC}$ sampled scenarios $\delta^{(1)}, \ldots, \delta^{(N)}$, where the uncertainty is drawn according to its known probability density.

iii) Solve the following convex optimization problem, which returns the control variables $P^G, W^u, \alpha$.

\textbf{SwC-AC-OPF}

\begin{equation}
\begin{aligned}
\text{minimize} & \quad P^G, W^u, \alpha, Q^{G, [1]}, \ldots, Q^{G, [N]}, W^x, [1], \ldots, W^x, [N] \\
\text{subject to:} & \quad \forall i = 1, \ldots, N \ \\
W^u[i] & = W^u[i] + W^x[i], \quad W^x[i] \geq 0, \quad \alpha_k \geq 0, \quad \forall k \in \mathcal{G} \\
L^m_{lm} & = \left| (W^u_{lm} - W^x_{lm}) y^*_m \right| + \left| (W^x_{lm} - W^x_{ml}) y^*_l \right| \\
\sum_{k \in \mathcal{G}} f_k(P^G_k) + \gamma \sum_{k \in \mathcal{G}} Q^{G, [i]}_k + \gamma \ell \sum_{(l,m) \in \mathcal{G}^{lm}} L^m_{lm} & \leq \gamma \\
P^G_k + \alpha_k s^T \text{Re}\{\delta^{(i)}\} + P^{R, [i]}(\delta^{(i)}) - P^{L, [i]}(\delta^{(i)}) & = \\
\sum_{l \in N_k} \text{Re}\left\{ (W^x_{kl} - W^x_{kl}) y^*_l \right\}, \quad \forall k \in \mathcal{N} \\
Q^{G, [i]}_k + Q^{R, [i]}(\delta^{(i)}) - Q^{L, [i]}(\delta^{(i)}) & = \\
\sum_{l \in N_k} \text{Im}\left\{ (W^x_{kl} - W^x_{kl}) y^*_l \right\}, \quad \forall k \in \mathcal{N} \\
P_{k \min} & \leq P^G_k + \alpha_k s^T \text{Re}\{\delta^{(i)}\} \leq P_{k \max}, \quad \forall k \in \mathcal{G} \\
Q_{k \min} & \leq Q^{G, [i]}_k \leq Q_{k \max}, \quad \forall k \in \mathcal{G} \\
(V_{k \min})^2 & \leq P^G_{kk} \leq (V_{k \max})^2, \quad \forall k \in \mathcal{N} \\
W^x_{ll} + W^x_{mn} & - W^x_{lm} - W^x_{ml} \leq (\Delta W^x_{lm})^2, \quad \forall (l, m) \in \mathcal{L}
\end{aligned}
\end{equation}

iv) During operation, measure uncertainty in generations and loads $\delta$, and accommodate the $k$-th controllable generator as $\bar{P}^G_k = P^G_k + \alpha_k s^T \text{Re}\{\delta\}, \quad |V_k| = \sqrt{W_{kk}}, \quad k \in \mathcal{G}$.

We remark that Theorem [2] guarantees that the SwC-AC-OPF design procedure is such that the probabilistic constraints of the Chance-Constrained AC-OPF [15] are satisfied with high confidence $(1 - \beta)$. In other words, one has an \textit{a-priori} guarantee that the risk of violating the constrains is bounded, and one can accurately bound
this violation level by choosing the probabilistic parameters $\varepsilon, \beta$. The above described procedure represents the main result of this paper. In Section V we demonstrate how this procedure outperforms existing ones in terms of guarantees of lower line-violations.

C. Handling Security Constraints

An important observation is that the approach introduced in this paper can be readily extended to handle security constraint. In particular, the popular $N-1$ security requirements discussed, for instance, in [32] can be directly included in the robust optimization problem (14), and in its subsequent derivations. We recall that, in the $N-1$ security constrained OPF framework, only the outages of a single component are taken into account. A list of $N_{\text{out}}$ possible outages, $I_{\text{out}} = \{0, 1, \ldots, N_{\text{out}}\}$, is formed (with 0 corresponding to the case of no outages). Then, a large optimization problem is constructed, with $N_{\text{out}}$ instances of the constraints, where the $i$-th instance corresponds to removing the $i$-th component form the equations, with $i \in I_{\text{out}}$. The exact same approach can be replicated here.

However, we remark that the SwC formulation allows for a more precise and realistic handling of the possible outages. Indeed, we note that in a real network some buses may have a larger probability of incurring into an outage, for instance because they are located in a specific geographical position, or because they employ older technologies. To reflect this scenario, we associate to each element $i \in I_{\text{out}}$ in the network a given probability of outage

$$p_i^{\text{out}} \in [0, 1],$$

which can be different for each component, and is supposed to be known to the network manager. Large values of $p_i$ correspond to large probability of an outage occurring in the $i$-th component. Then, in our framework, outages can be treated in the same way as uncertainties. Formally, we can associate to each component a random variable $\delta_i^{\text{out}}, i \in I_{\text{out}}$, with Bernoulli density with mean $1 - p_i^{\text{out}}$. Then, the scenario with certificates design is applied considering the extended uncertainty $\{\delta, \delta^{\text{out}}\}$.

In practice, when constructing the SwC-AC-OPF problem, for each sampling instance the $i$-th component is removed with probability $p_i^{\text{out}}$. It is important to note that this approach goes beyond the standard $N-1$ security constrained setup, since: i) multiple simultaneous outages are automatically taken into account, ii) it allows weighting differently the different component.

V. Numerical Examples

In order to examine the effectiveness of the proposed method, we perform extensive simulations using New England 39-bus system case adopted form [39]. The network has 39 buses, 46 lines and 10 conventional generation units, and it is modified to include 4 wind generators connected to buses 5, 6, 14 and 17. The renewable energy generators and all loads connected to different buses are considered to be uncertain. In total, there are 46 uncertain parameters in the network. The goal is to design active power and voltage amplitude of all controllable generators as well as the distribution vector $\alpha$ such that the generation cost is minimized while all the constraint of the network i.e. line flow, bus voltage and generators output constraint are respected with high probability.
A. Numerical Results

We use the methodology presented in Section IV to solve the robust optimal power flow problem for New England 39 bus system. We consider a 24 hour demand pattern shown in Fig. 1(a) and solve the optimal power flow problem for each hour by minimizing the nominal cost. The error probability distribution for wind generator and load is chosen based on Pearson system [31]—as suggested in [19], [20]. Pearson system is represented by the mean \( \mu \) (first moment), variance \( \sigma^2 \) (second moment), skewness \( \gamma \) (third moment) and kurtosis \( \kappa \) (forth moment). In the simulation, we set \( \sigma = 0.2 \times \) (predicted value), \( \gamma = 0 \), and, \( \kappa = 3.5 \) leading to a leptokurtic distribution with heavier tail than Gaussian. The selected probabilistic accuracy and confidence levels \( \varepsilon \) and \( \beta \) are 0.02 and \( 1 \times 10^{-15} \) respectively resulting in 5,105 number of scenario samples. We note that the number of design (control) variables is 31 in our formulation. The penetration level is chosen to be 30%. The penetration level indicates how much of the total demand is provided by the wind generators. A 30% penetration level means that, in total, the wind generators provide 30% of the nominal load. We also assume that each renewable generator contributes equally to provide this power. The optimization problem is formulated and then solved using YALMIP [24] and Mosek [2] respectively.

In order to examine robustness of the obtained solutions, we run an a-posteriori analysis based on Monte Carlo simulation. To this end, we generated 10,000 random samples from the uncertainty set—corresponding to uncertain active and reactive generated wind power and load—and for each sample, modified the network to include wind power generators, replaced the load vector by its uncertain counterpart, computed the power mismatch, distributed the mismatch to all conventional generators by using (9) and solved the power flow problem using runpf command of MATPOWER [39] to derive line flows. The empirical violation of line flows is computed by counting the number of times each line exceeds its limit in the Monte Carlo simulation and dividing this value by 10,000. Figure 1(b) shows the 24 hour empirical violation of line flow for all lines of the network. The mean value of generation cost associated with the 24 hour demand is also shown in Fig. 1(a). We note that this graph is obtained by averaging the generation cost over all 10,000 samples.

In Fig. 2, we compare line loading percentage for scenario with certificates approach against standard optimal power flow design where no uncertainty is taken into account while designing the control parameters. Demand level is 6,406 MVA. This proves the necessity of adopting a robust strategy for the new power grids containing renewable generators. In such a network if we rely on the classical OPF design where no uncertainty is taken into account in designing the control variables, it results in very frequent overloads, as shown in Fig. 2(b), and hence frequent line tripping or even cascading outage. On the other hand, the robust strategy presented in the current paper successfully designs the control variables such that only very occasional violation happens during the operation of the network in the presence of large number of uncertain parameters.

We also compare the proposed strategy against the one presented in [32]. In this comparison, we only consider uncertainty in renewable energy generators. Loads are considered to be known exactly. This is because [32] can only handle uncertainty in renewable energy generator. In [32], the number of design variables appearing in the
Fig. 1: (a): 24 hours demand pattern and average generation cost computed in the posteriori analysis based on Monte Carlo simulation. (b): 24 hours empirical violation of line flow computed in the posteriori analysis based on Monte Carlo simulation.

Fig. 2: (a): Line loading percentage for 10,000 random samples extracted from the uncertainty set, results of scenario with certificates approach, (b) results of standard OPF design where no uncertainty is taken into account in designing the control parameters; In the boxplots, the red line represents the median value, edges of each box correspond to the 25th and 75th percentiles, whereas the whiskers extend to 99% coverage. The magenta marks denote the data outliers.

optimization problem is much larger than SwC approach. This is due to the linear parametrization (21) where some additional design variables are introduced. Since the number of scenario samples $N_{\text{SwC}}$ depends on the number of design variables $n_{\theta}$, additional design variables lead to significant growth in the number of scenario samples as compared to SwC approach. The worst-case cost is minimized in the OPF formulation—see subsection III-B. The penetration level is assumed to be 33% and demand level is 7,112 MVA corresponding to the peak demand in Fig. 1.

Choosing $\varepsilon = 0.1$, $\beta = 1 \times 10^{-10}$, the number of scenario samples for [32] is 11,838—with 896 design variables—and for scenario with certificates is 839—with only 31 design variables. Large number of scenario samples associated with the approach presented in [32] results in much higher computational cost compared to
Fig. 3: The probability of line flow violation compared with [32]. The number of scenario samples for SwC is 839 while [32] requires 11,838 samples.

TABLE I: Comparison between the average generation cost and average computation time – over 10,000 Monte Carlo simulation – achieved using SwC with the one obtained by [32].

| Approach | Generation Cost [$] | Computation Time [min] |
|----------|---------------------|------------------------|
| SwC      | 25,280              | 49                     |
| [32]     | 25,359              | 3,074                  |

SwC approach. Indeed, the optimization problem formulated based on [32] takes more than 51 hours to be solved on a workstation with 12 cores and 48 GB of RAM, while the solution of the SwC approach takes less than 49 minutes. We remark that the computational time refers to a non-optimized implementation of the problem. We also remark that, in this case, the alternative approach proposed in [32], based on computing a probabilistically guaranteed hyper-rectangle, is not viable, since it would amount at imposing the constraints on $2^{46} \approx 7 \times 10^{13}$ vertices! Finally, we note that sequential techniques as those proposed in [11] may be adopted to significantly reduce computation times.

Finally, we run an *a-posteriori* analysis based on Monte Carlo simulation to estimate the probability of line flow violation. In the posteriori analysis, we use exactly the same set of samples—different from the design samples of course—to evaluate performance of the two approaches. Figure 3 compares the probability of line flow violation for the two approaches. This comparison shows that the linear parameterization adopted in [32] is conservative leading to larger violation level compared to SwC approach. The average—over 10,000 samples—generation cost is also compared in Table I. Therefore, the paradigm presented in this paper improves upon [32] in computational complexity, violation probability, and generation cost.
VI. Concluding Remarks

In this paper, we proposed a novel approach to the AC optimal power flow problem in the presence of uncertain renewable energy sources and uncertain load. Assuming that the probability distribution of the uncertainty is available, we aim at optimizing the nominal power generation subject to a small well defined risk of violating generation and transmission constraints. To tackle this complex NP-hard problem, we propose a randomized algorithm based on the novel concept of scenario with certificates and on convex relaxations of power flow problems. The effectiveness of the proposed solution is illustrated via numerical examples, where it is shown that one can significantly decrease the probability of constraint violation without a significant impact on the nominal power generation cost. Moreover, the approach is shown to be very efficient from a computational viewpoint. Future research aims at exploiting the flexibility of the methodology to extend its application. For instance, we can consider the case of optimizing with respect to the slack bus voltage, to improve the objective function and even the safety. Since the slack bus voltage is a discrete variable, the resulting OPF problem will be of mixed-integer type, as in [30].

References

[1] MA. Abido. Optimal power flow using particle swarm optimization. Int. J. of Electrical Power & Energy Systems, 24(7):563–571, 2002.
[2] E. Andersen and K. Andersen. The Mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In High performance optimization, 197–232. Springer, 2000.
[3] X. Bai and H. Wei. A semidefinite programming method with graph partitioning technique for optimal power flow problems. Int. J. of Electrical Power & Energy Systems, 33(7):1309–1314, 2011.
[4] X. Bai, H. Wei, K. Fujisawa, and Y. Wang. Semidefinite programming for optimal power flow problems. Int. J. of Electrical Power & Energy Systems, 30(6-7):383 – 392, 2008.
[5] D. Bienstock, M. Chertkov, and S. Harnett. Chance-constrained optimal power flow: risk-aware network control under uncertainty. SIAM Review, 56(3):461–495, 2014.
[6] M. B. Cain, R. P. O’Neill, and A. Castillo. History of optimal power flow and formulations. Technical report, US Federal Energy Resource Commission (FERC), December 2012.
[7] G. C. Calafiore and M. C. Campi. The scenario approach to robust control design. IEEE Trans. on Automatic Control, 51(5):742–753, 2006.
[8] F. Capitanescu. Critical review of recent advances and further developments needed in AC optimal power flow. Electric Power Systems Research, 136:57–68, 2016.
[9] J. Carpentier. Contribution a l’étude du dispatching économique. Bull. Soc. Francaise des Electriciens, 3:431–447, 1962.
[10] J. Carpentier. Optimal power flows. Int. J. of Electrical Power and Energy Systems, 1(1):3–15, 1979.
[11] M. Chamanbaz, F. Dabbene, R. Tempo, V. Venkataramanan, and Q. G. Wang. Sequential randomized algorithms for convex optimization in the presence of uncertainty. IEEE Trans. on Automatic Control, 61:2565–2571, 2016.
[12] C. Coffrin and P. Van Hentenryck. A linear-programming approximation of ac power flows. INFORMS J. on Computing, 26(4):718–734, 2014.
[13] European Commission. Strategic energy technology plan. http://ec.europa.eu/energy/en/topics/technology-and-innovation/strategic-energy-technology-plan 2016.
[14] A. J. Conejo, M. Carrión, and J. M. Morales. Decision making under uncertainty in electricity markets, volume 1. Springer, 2010.
[15] Global Wind Energy Council. Global wind statistics 2015. http://www.gwec.net/wp-content/uploads/eip/GWEC-PRstats-2015_LR.pdf 2016.
[16] H.W. Dommel and W.F. Tinney. Optimal power flow solutions. IEEE Trans. on Power Apparatus & Systems, 10:1866–1876, 1968.
[17] M. Farivar and S. H. Low. H. Low. Branch flow model: relaxations and convexification (Parts I, II). \textit{IEEE Trans. on Power Systems}, 28(3):2554–2572, 2013.

[18] S. Formentin, F. Dabbene, R. Tempo, L. Zaccarian, and S. M. Savaresi. Robust linear static anti-windup with probabilistic certificates. \textit{IEEE Trans. on Automatic Control}, 62(4):1575–1589, 2017.

[19] B. Hodge, D. Lew, and M. Milligan. Short-term load forecast error distributions and implications for renewable integration studies. In \textit{Proc. IEEE Green Technologies Conf.}, pages 435–442. IEEE, 2013.

[20] B. Hodge, D. Lew, M. Milligan, H. Holttinen, S. Sillanpää, E. Gómez-Lázaro, R. Scharff, L. Söder, X. G. Larsén, G. Giebel, D. Flynn, and J. Dobchinsky. Wind Power Forecasting Error Distributions: An International Comparison. In \textit{Proc. 11th Annual Int. Workshop on Large-Scale Integration of Wind Power into Power Systems}, 2012.

[21] R.A. Jabr. Radial distribution load flow using conic programming. \textit{IEEE Trans. on Power Systems}, 21(3):1458–1459, 2006.

[22] L.L. Lai, J.T. Ma, R. Yokoyama, and M. Zhao. Improved genetic algorithms for optimal power flow under both normal and contingent operation states. \textit{Int. J. of Electrical Power & Energy Systems}, 19(5):287–292, 1997.

[23] J. Lavaei and S. H. Low. Zero duality gap in optimal power flow problem. \textit{IEEE Trans. on Power Systems}, 27(1):92–107, Feb 2012.

[24] J. Lofberg. Yalmip: A toolbox for modeling and optimization in MATLAB. In \textit{Proc. IEEE Int. Symposium on Computer Aided Control Systems Design}, pages 284–289. IEEE, 2004.

[25] S. H. Low. Convex relaxation of optimal power flow; Part I: formulations and equivalence. \textit{IEEE Trans. on Control of Network Systems}, 1(1):15–27, March 2014.

[26] S. H. Low. Convex relaxation of optimal power flow; Part II: exactness. \textit{IEEE Trans. on Control of Network Systems}, 1(2):177–189, June 2014.

[27] R. Madani, M. Ashraphijuo, and J. Lavaei. Promises of conic relaxation for contingency-constrained optimal power flow problem. In \textit{Proc. of 52nd Annual Allerton Conf.}, pages 1064–1071, 2014.

[28] R. Madani, S. Sojoudi, and J. Lavaei. Convex relaxation for optimal power flow problem: Mesh networks. \textit{IEEE Trans. on Power Systems}, 30(1):199–211, 2015.

[29] E. Mohagheghi, A. Gabash, and P. Li. A framework for real-time optimal power flow under wind energy penetration.. \textit{Energy}, 10(4):535, 2017.

[30] R. Zimmerman, C. Murillo-Sánchez, and R. Thomas. Matpower: Steady-state operations, planning, and analysis tools for power systems research and education. \textit{IEEE Trans. on Power Systems}, 26(1):12–19, 2011.
Let $P_{G,\text{opt}}$ and $W_{u,\text{opt}}$ be achievers of the solution of CR-AC-OPF. Now, take any $\delta \in \Delta$ and consider the following optimization problem

$$\max \sum_{k \in G} Q_k$$

subject to:

$$P_{G,\text{opt}} + \alpha_{k,\text{opt}} s^T \Re\{\delta\} + \sum_{l \in N_k} \Re \{(W_{kk} - W_{kl})^* y_{kl}\} \leq P_{G}^C + \alpha_{k,\text{opt}} s^T \Re\{\delta\}, \forall k \in N$$

$$Q_{k \min} \leq -Q_{k \min}^R(\delta) + Q_{k \max}^L(\delta) = \sum_{l \in N_k} \Im \{(W_{kk} - W_{kl})^* y_{kl}\} \leq Q_{k \max}, \forall k \in N$$

$$(W_{u,\text{opt}})_{kk} \leq W_{kk} \leq (W_{u,\text{opt}})_{kk}, \forall k \in G;$$

$$(V_{k \min})^2 \leq W_{kk} \leq (V_{k \max})^2, \forall k \in N/G$$

$$W_{ll} + W_{mm} - W_{lm} - W_{ml} \leq (\Delta V_{\max})^2, \forall (l, m) \in L.$$ 

Note that, for the given value of the uncertainty, the solution of the optimization problem above has exactly the same generation cost as the solution of the CR-AC-OPF. The optimization problem above is of the same form as that used in the proof of part (a) of Theorem 2 in [29]. Hence, the same reasoning can be applied to show that there exists a rank one solution for the problem above. Since, as mentioned before, the solution of the convex relaxation and the one of the original optimal power flow problem coincide when $\operatorname{rank}(W) = 1$, this implies that the values of control variables obtained using CR-AC-OPF are a feasible power allocation for any value of the uncertainty $\delta \in \Delta$. Moreover, for all $\delta \in \Delta$, the generation cost is equal to the optimal value of CR-AC-OPF. This, together with the fact that the solution of CR-AC-OPF is optimal for at least one $\delta^* \in \Delta$, leads to the conclusion that CR-AC-OPF does provide the best worst-case solution or, in other words, the robustly optimal power generation allocation. \qed