New contractive conditions of integral type on complete $S$-metric spaces

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Abstract An $S$-metric space is a three-dimensional generalization of a metric space. In this paper our aim is to examine some fixed-point theorems using new contractive conditions of integral type on a complete $S$-metric space. We give some illustrative examples to verify the obtained results. Our findings generalize some fixed-point results on a complete metric space and on a complete $S$-metric space. An application to the Fredholm integral equation is also obtained.

Keywords Integral-type contractive conditions · Fixed point · $S$-metric

Mathematics Subject Classification Primary 47H10 · Secondary 54H25

Introduction

Recently, the notion of an $S$-metric has been introduced and studied as a generalization of a metric. This notion has been defined by Sedghi et al. [13] as follows:

Definition 1.1 [13] Let $X \neq \emptyset$ be any set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $u, v, z, a \in X$.

(S1) $S(u, v, z) = 0$ if and only if $u = v = z$.
(S2) $S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a)$.

Then the function $S$ is called an $S$-metric on $X$ and the pair $(X, S)$ is called an $S$-metric space.

Some fixed-point theorems have been given for self-mappings satisfying various contractive conditions on an $S$-metric space (see [4, 6, 8, 9, 13, 14]). One of the important results among these studies is the Banach’s contraction principle on a complete $S$-metric space.

Theorem 1.2 [13] Let $(X, S)$ be a complete $S$-metric space, $h \in (0, 1)$ and $T : X \to X$ be a self-mapping of $X$ such that

$$S(Tu, Tu, Tv) \leq hS(u, u, v),$$

for all $u, v \in X$. Then $T$ has a unique fixed point in $X$.

On the other hand some generalizations of the well-known Ćirić’s and Nemytskii-Edelstein fixed-point theorems obtained on $S$-metric spaces via some new fixed point results (see [8, 9, 13, 14] for more details).

Later, different applications of some contractive conditions have been constructed on an $S$-metric space such as differential equations, complex valued functions etc. (see [5, 7, 10, 11]).

In recent years, fixed-point theory has been examined for various contractive conditions. For example, contractive conditions of integral type were adapted into some studied fixed-point results. So more general fixed-point theorems were obtained.

Through the whole paper we assume that $\zeta : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative and such that for each $\varepsilon > 0$,

$$\int_0^\varepsilon \zeta(t)dt > 0. \quad (1)$$

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Branciari [1] studied a fixed-point theorem for a general contractive condition of integral type on a complete metric space as seen in the following theorem.

**Theorem 1.3** [1] Let $(X, \rho)$ be a complete metric space, \( h \in (0, 1) \), the function \( \zeta : [0, \infty) \to [0, \infty) \) be defined as in (1) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\rho(Tu,Tv) \leq h \int_0^1 \zeta(t) \, dt
\]

for all \( u, v \in X \), then \( T \) has a unique fixed point \( w \in X \) such that

\[
\lim_{n \to \infty} T^nu = w,
\]

for each \( u \in X \).

After the study of Branciari, some researchers have investigated new generalized contractive conditions of integral type satisfying some new generalized inequalities given in [6] on a complete \( S \)-metric space. Our results generalize some known fixed-point results on a complete metric space and on a complete \( S \)-metric space.

**Fixed-point results under some contractive conditions of integral type**

In this section we obtain new fixed-point theorems using some contractive conditions of integral type on a complete \( S \)-metric space. We construct three examples to show the validity of our results. At first we recall some basic results about \( S \)-metric spaces.

**Lemma 2.1** [13] Let \( (X, S) \) be an \( S \)-metric space. Then we have

\[
S(u, u, v) = S(v, v, u).
\]

The above Lemma 2.1 can be considered as a symmetry condition on an \( S \)-metric space. The following definition is related to convergent sequences on an \( S \)-metric space.

**Definition 2.2** [13] Let \( (X, S) \) be an \( S \)-metric space.

1. A sequence \( \{u_n\} \) in \( X \) converges to \( u \) if and only if \( S(u_n, u_n, u) \to 0 \) as \( n \to \infty \). That is, there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( S(u_n, u_n, u) < \varepsilon \) for each \( \varepsilon > 0 \). We denote this by

\[
\lim_{n \to \infty} u_n = u \text{ or } \lim_{n \to \infty} S(u_n, u_n, u) = 0.
\]

2. A sequence \( \{u_n\} \) in \( X \) is called a Cauchy sequence if \( S(u_n, u_m, u_m) \to 0 \) as \( n, m \to \infty \). That is, there exists \( n_0 \in \mathbb{N} \) such that for all \( n, m \geq n_0 \), \( S(u_n, u_m, u_m) < \varepsilon \) for each \( \varepsilon > 0 \).

3. The \( S \)-metric space \( (X, S) \) is called complete if every Cauchy sequence is convergent.

In the following lemma we see the relationship between a metric and an \( S \)-metric.

**Lemma 2.3** [4] Let \( (X, \rho) \) be a metric space. Then the following properties are satisfied:

1. \( S(u, v, z) = \rho(u, z) + \rho(v, z) \) for all \( u, v, z \in X \) is an \( S \)-metric on \( X \).
2. \( u_n \to u \) in \( (X, \rho) \) if and only if \( u_n \to u \) in \( (X, S) \).
3. \( \{u_n\} \) is Cauchy in \( (X, \rho) \) if and only if \( \{u_n\} \) is Cauchy in \( (X, S) \).
4. \( (X, \rho) \) is complete if and only if \( (X, S) \) is complete.

We call the function \( S_\rho \) defined in Lemma 2.3 (1) as the \( S \)-metric generated by the metric \( \rho \). It can be found an example of an \( S \)-metric which is not generated by any metric in [4, 9].

Now we give the following theorem.

**Theorem 2.4** Let \( (X, S) \) be a complete \( S \)-metric space, \( h \in (0, 1) \), the function \( \zeta : [0, \infty) \to [0, \infty) \) be defined as in (1) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\int_0^1 \zeta(t) \, dt \leq h \int_0^1 \zeta(t) \, dt,
\]

for all \( u, v \in X \). Then \( T \) has a unique fixed point \( w \in X \) and we have

\[
\lim_{n \to \infty} T^nu = w,
\]

for each \( u \in X \).

**Proof** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( T^{n_0}u_0 = u_n \).

Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (2), we obtain

\[
\int_0^1 \zeta(t) \, dt \leq h \int_0^1 \zeta(t) \, dt \leq \cdots \leq h^n \int_0^1 \zeta(t) \, dt.
\]

(3)

If we take limit for \( n \to \infty \), using the inequality (3) we get

\[
\lim_{n \to \infty} \int_0^1 \zeta(t) \, dt = 0,
\]

\[
\lim_{n \to \infty} S(\zeta(t)) = 0.
\]
since \( h \in (0, 1) \). The condition (1) implies
\[
\lim_{n \to \infty} S(u_n, u_n, u_{n+1}) = 0.
\]

Now we show that the sequence \( \{u_n\} \) is a Cauchy sequence. Assume that \( \{u_n\} \) is not Cauchy. Then there exists an \( \varepsilon > 0 \) and subsequences \( \{m_k\} \) and \( \{n_k\} \) such that
\[
m_k < n_k < m_{k+1}
\]
with
\[
S(u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon
\]
(4)
and
\[
S(u_{m_k}, u_{m_k}, u_{n_k}) < \varepsilon.
\]

Hence using Lemma 2.1, we have
\[
S(u_{m_k-1}, u_{m_k-1}, u_{n_k-1}) \leq 2S(u_{m_k-1}, u_{n_k-1}, u_{m_k})
\]
\[
+ S(u_{n_k-1}, u_{m_k-1}, u_{n_k})
\]
\[
< 2S(u_{m_k-1}, u_{n_k-1}, u_{m_k}) + \varepsilon
\]
and
\[
\lim_{k \to \infty} \int_0^\varepsilon \zeta(t) \, dt \leq \int_0^\varepsilon \zeta(t) \, dt.
\]
Using the inequalities (2), (4) and (5) we obtain
\[
\int_0^\varepsilon \zeta(t) \, dt \leq \int_0^\varepsilon \zeta(t) \, dt \leq h \int_0^\varepsilon \zeta(t) \, dt
\]
\[
\leq h \int_0^\varepsilon \zeta(t) \, dt,
\]
which is a contradiction with our assumption since \( h \in (0, 1) \). So the sequence \( \{u_n\} \) is Cauchy. Using the completeness hypothesis, there exists \( w \in X \) such that
\[
\lim_{n \to \infty} T^w u_0 = w.
\]
From the inequality (2) we find
\[
S(Tw, Tw, Tw) = S(Tw, Tw, Tw) \leq h \int_0^\varepsilon \zeta(t) \, dt.
\]
If we take limit for \( n \to \infty \), we get
\[
\int_0^\varepsilon \zeta(t) \, dt = 0,
\]
which implies \( Tw = w \).

Now we show the uniqueness of the fixed point. Suppose that \( w_1 \) is another fixed point of \( T \). Using the inequality (2) we have
\[
S(w, w, w_1) = S(w, w, w_1) \leq h \int_0^\varepsilon \zeta(t) \, dt,
\]
which implies
\[
S(w, w, w_1) \leq 0.
\]

since \( h \in (0, 1) \). Using the inequality (1) we get \( w = w_1 \). Consequently, the fixed point \( w \) is unique. \( \square \)

**Remark 2.5**

(1) If we set the function \( \zeta : [0, \infty) \to [0, \infty) \) in Theorem 2.4 as
\[
\zeta(t) = 1,
\]
for all \( t \in [0, \infty) \), then we obtain the Banach’s contraction principle on a complete \( S \)-metric space.

(2) Since an \( S \)-metric space is a generalization of a metric space, Theorem 2.4 is a generalization of the classical Banach’s fixed-point theorem.

(3) If we set the \( S \)-metric as \( S : X \times X \times X \to \mathbb{R} \) and take the function \( \zeta : [0, \infty) \to [0, \infty) \) as
\[
\zeta(t) = 1,
\]
for all \( t \in [0, \infty) \) in Theorem 2.4, then we get Theorem 3.1 in [10] and Corollary 2.5 in [5] for \( n = 1 \).

**Example 2.6** Let \( X = \mathbb{R} \), \( k > 1 \) be a fixed real number and the function \( S : X \times X \times X \to [0, \infty) \) be defined as
\[
S(u, v, z) = \frac{k}{k+1}(|v - z| + |v + z - 2u|),
\]
for all \( u, v, z \in \mathbb{R} \). It can be easily seen that the function \( S \) is an \( S \)-metric. Now we show that this \( S \)-metric can not be generated by any metric \( \rho \). On the contrary, we assume that there exists a metric \( \rho \) such that
\[
S(u, v, z) = \rho(u, z) + \rho(v, z),
\]
(6)
for all \( u, v, z \in \mathbb{R} \). Hence we find
\[
S(u, u, z) = 2\rho(u, z) = -\frac{2k}{k+1}|u - z|
\]
and
\[
\rho(u, z) = \frac{k}{k+1}|u - z|.
\]
(7)
Similarly, we get
\[
S(v, v, z) = 2\rho(v, z) = \frac{2k}{k+1}|v - z|
\[ \rho(v, z) = \frac{k}{k+1} |v - z|, \]  

(8)

Using the equalities (6), (7) and (8), we obtain
\[ \frac{k}{k+1} (|v - z| + |v + z - 2u|) = \frac{k}{k+1} |u - z| + \frac{k}{k+1} |v - z|, \]

which is a contradiction. Consequently, \( S \) is not generated by any metric and \((\mathbb{R}, S)\) is a complete \( S \)-metric space.

Let us define the self-mapping \( T : \mathbb{R} \to \mathbb{R} \) as
\[ Tu = \frac{u}{6}, \]
for all \( u \in \mathbb{R} \) and the function \( \zeta : [0, \infty) \to [0, \infty) \) as
\[ \zeta(t) = 3t^2, \]
for all \( t \in [0, \infty) \). Then we get
\[ \int_0^\varepsilon \zeta(t) \, dt = \int_0^{3t^2} \, dt = \varepsilon^3 > 0, \]
for each \( \varepsilon > 0 \). Therefore \( T \) satisfies the inequality (2) in Theorem 2.4 for \( h = \frac{1}{2} \). Indeed, we have
\[ \frac{k^3}{27(k+1)^3} |u - v|^3 \leq \frac{4k^3}{(k+1)^3} |u - v|^3, \]
for all \( u, v \in \mathbb{R} \). Consequently, \( T \) has a unique fixed point \( u = 0 \).

Now we give the first generalization of Theorem 2.4.

**Theorem 2.7** Let \((X, S)\) be a complete \( S \)-metric space, the function \( \zeta : [0, \infty) \to [0, \infty) \) be defined as in (1) and \( T : X \to X \) be a self-mapping of \( X \) such that
\[ \lim_{n \to \infty} T^n u = w, \]
for each \( u \in X \).

**Proof** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( T^n u_0 = u_n \).

Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (9), the condition (S2) and Lemma 2.1 we get
\[ S(u_n, u_{n+1}, u_{n+1}) \leq \max\{s(u_n, u_{n+1}, u_{n+1}), S(u_n, u_{n+1}, u_{n+1})\}, \]
which implies
\[ \zeta(t) dt \leq \left( \frac{h_1 + h_3 + h_4}{1 - 2h_3 - h_4} \right) \int_0^\infty \zeta(t) \, dt, \]
If we put \( h = \frac{h_1 + h_3 + h_4}{1 - 2h_3 - h_4} < 1 \), then we find \( h < 1 \) since \( h_1 + h_3 + h_4 < 1 \). Using the inequality (10) we have
Taking limit for \( n \rightarrow \infty \), using the inequality (11) we get

\[
\lim_{n \to \infty} S(u_n, u_{n+1}) = \frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt = 0,
\]

since \( h \in (0, 1) \). The condition (1) implies

\[
\lim_{n \to \infty} S(u_n, u_{n+1}) = 0.
\]

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence \( \{u_n\} \) is Cauchy. Then there exists \( w \in X \) such that

\[
\lim_{n \to \infty} T^n u_0 = w,
\]

since \( (X, S) \) is a complete \( S \)-metric space. From the inequality (9) we find

\[
S(u_n, u_{n+2}) \leq S(u_n, u_{n+1}) + S(u_{n+1}, u_{n+2}) + \max \{S(u_n, u_{n+1}), S(u_{n+1}, u_{n+2})\}.
\]

Taking limit for \( n \to \infty \) and using Lemma 2.1 we get

\[
\frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt \leq (h_3 + h_4) \int_0^{T^n u_0} \zeta(t) \, dt,
\]

which implies \( Tw = w \) since \( h_3 + h_4 < 1 \).

Now we show the uniqueness of the fixed point. Let \( w_1 \) be another fixed point of \( T \). Using the inequality (9) and Lemma 2.1, we get

\[
\frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt = \frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt \leq h_1 \int_0^{T^n u_0} \zeta(t) \, dt + h_2 \int_0^{T^n u_0} \zeta(t) \, dt + h_3 \int_0^{T^n u_0} \zeta(t) \, dt + h_4 \int_0^{T^n u_0} \zeta(t) \, dt,
\]

which implies

\[
\frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt \leq (h_1 + h_2 + h_3) \int_0^{T^n u_0} \zeta(t) \, dt.
\]

Then we obtain

\[
\frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt = 0,
\]

that is, \( w = w_1 \) since \( h_1 + h_2 + h_3 < 1 \). Consequently, \( T \) has a unique fixed point \( w \in X \).

**Remark 2.8**

(1) If we set the function \( \zeta : [0, \infty) \to [0, \infty) \) in Theorem 2.7 as

\[
\zeta(t) = 1,
\]

for all \( t \in [0, \infty) \), then we obtain Theorem 3 in [6].

(2) Theorem 2.7 is a generalization of Theorem 2.4 on a complete \( S \)-metric space. Indeed, if we take \( h_1 = h \) and \( h_2 = h_3 = h_4 = 0 \) in Theorem 2.7, then we get Theorem 2.4.

(3) Since Theorem 2.7 is a generalization of Theorem 2.4, Theorem 2.7 generalizes the classical Banach’s fixed-point theorem.

(4) If we set the \( S \)-metric as \( S : X \times X \times X \to \mathbb{R} \) and take the function \( \zeta : [0, \infty) \to [0, \infty) \) as

\[
\zeta(t) = 1,
\]

for all \( t \in [0, \infty) \) in Theorem 2.7, then we get Theorem 3.1 in [7].

Now we give the second generalization of Theorem 2.4.

**Theorem 2.9** Let \( (X, S) \) be a complete \( S \)-metric space, the function \( \zeta : [0, \infty) \to [0, \infty) \) be defined as in (1) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\frac{1}{2} \int_0^{T^n u_0} \zeta(t) \, dt \leq h_1 \int_0^{T^n u_0} \zeta(t) + h_2 \int_0^{T^n u_0} \zeta(t) \, dt + h_3 \int_0^{T^n u_0} \zeta(t) \, dt + h_4 \int_0^{T^n u_0} \zeta(t) \, dt + h_5 \int_0^{T^n u_0} \zeta(t) \, dt,
\]

which implies

\[
(12)
\]
for all \( u, v \in X \) with nonnegative real numbers \( h_i \) \((i \in \{1, 2, 3, 4, 5, 6\})\) satisfying \( \max \{h_1 + h_2 + 3h_3 + h_5 + 3h_6, h_1 + h_3 + h_4 + h_6\} < 1 \). Then \( T \) has a unique fixed point \( w \in X \) and we have

\[
\lim_{n \to \infty} T^n u = w,
\]

for each \( u \in X \).

**Proof** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as\n
\[
T^n u_0 = u_n.
\]

Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (12), the condition (S2) and Lemma 2.1 we get

\[
\lim_{n \to \infty} \int_0^1 \zeta(t) \, dt = \left( \int_0^1 \zeta(t) \, dt \right) \cdot \lim_{n \to \infty} \int_0^1 \zeta(t) \, dt = \left( \int_0^1 \zeta(t) \, dt \right) \cdot \lim_{n \to \infty} \int_0^1 \zeta(t) \, dt = \left( \int_0^1 \zeta(t) \, dt \right) \cdot \lim_{n \to \infty} \int_0^1 \zeta(t) \, dt.
\]

which implies

\[
\int_0^1 \zeta(t) \, dt \leq \left( \frac{h_1 + h_2 + 4h_5 + h_6}{1 - 2h_4 - h_3 - 2h_6} \right) \int_0^1 \zeta(t) \, dt.
\]

If we put \( h = \frac{h_1 + h_2 + h_3 + h_4 + h_5 + 3h_6}{1 - 2h_4 - h_3 - 2h_6} < 1 \) then we get \( h < 1 \) since \( h_1 + h_2 + 3h_3 + h_5 + 3h_6 < 1 \). Using the inequality (13) we have

\[
\int_0^1 \zeta(t) \, dt \leq h^n \int_0^1 \zeta(t) \, dt.
\]

If we take limit for \( n \to \infty \), using the inequality (14) we get

\[
\lim_{n \to \infty} \int_0^1 \zeta(t) \, dt = 0,
\]

since \( h \in (0, 1) \). The condition (1) implies

\[
\lim_{n \to \infty} S(u_n, u_n, u_{n+1}) = 0.
\]

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence \( \{u_n\} \) is Cauchy. Then there exists \( w \in X \) such that

\[
\lim_{n \to \infty} T^n u_0 = w,
\]
\[ S(Tu, Tu, u) \]
\[ \int_0^1 \zeta(t)dt \leq (h_1 + h_3 + h_4 + h_6) \]
\[ S(Tu, Tu, u) \]
\[ \int_0^1 \zeta(t)dt, \]
which implies \( Tw = w \) since \( h_1 + h_3 + h_4 + h_6 < 1 \).

Now we show the uniqueness of the fixed point. Let \( w_1 \) be another fixed point of \( T \). Using the inequality (12) and Lemma 2.1, we get
\[ S(w, w, w_1) \]
\[ \int_0^1 \zeta(t)dt = \int_0^1 \zeta(t)dt \leq h_1 \]
\[ S(w, w, w_1) \]
\[ + h_2 \]
\[ S(w, w_1, w_1) \]
\[ + h_4 \]
\[ \max\{S(w, w, w_1)S(w, w_1, w), S(w, w_1, w), S(w_1, w_1, w_1), S(w_1, w_1, w_1), S(w_1, w_1, w_1)\} \]
\[ + h_6 \]
\[ \int_0^1 \zeta(t)dt, \]
which implies
\[ S(w, w, w_1) \]
\[ \int_0^1 \zeta(t)dt \leq (h_1 + h_3 + h_4 + h_6) \]
\[ S(w, w, w_1) \]
\[ \int_0^1 \zeta(t)dt, \]
Then we obtain
\[ S(w, w, w_1) \]
\[ \int_0^1 \zeta(t)dt = 0, \]
that is, \( w = w_1 \) since \( h_1 + h_3 + h_4 + h_6 < 1 \). Consequently, \( T \) has a unique fixed point \( w \in X \).

**Remark 2.10**

(1) In Theorem 2.9, if we set the function \( \zeta : [0, \infty) \rightarrow [0, \infty) \) as
\[ \zeta(t) = 1, \]
for all \( t \in [0, \infty) \), then we obtain Theorem 4 in [6].

(2) Theorem 2.9 is a generalization of Theorem 2.4 on a complete S-metric space. Indeed, if we take \( h_1 = h \) and \( h_2 = h_3 = h_4 = h_5 = h_6 = 0 \) in Theorem 2.9, then we get Theorem 2.4.

(3) Since Theorem 2.9 is another generalization of Theorem 2.4, Theorem 2.9 generalizes the classical Banach’s fixed-point theorem.

(4) If we set the S-metric as \( S : X \times X \times X \rightarrow C \) and take the function \( \zeta : [0, \infty) \rightarrow [0, \infty) \) as
\[ \zeta(t) = 1, \]
for all \( t \in [0, \infty) \) in Theorem 2.9, then we get Theorem 3.4 in [7].

In the following example we give a self-mapping satisfying the conditions of Theorems 2.7 and 2.9, respectively, but does not satisfy the condition of Theorem 2.4.

**Example 2.11** Let \( \mathbb{R} \) be the complete S-metric space with the S-metric defined in Example 1 given in [9]. Let us define the self-mapping \( T : \mathbb{R} \rightarrow \mathbb{R} \) as
\[ Tu = \begin{cases} u + 80 & \text{if } u \in \{0, 2\} \\ 75 & \text{otherwise} \end{cases}, \]
for all \( u \in \mathbb{R} \) and the function \( \zeta : [0, \infty) \rightarrow [0, \infty) \) as
\[ \zeta(t) = 2t, \]
for all \( t \in [0, \infty) \). Then we get
\[ \int_0^e \zeta(t)dt = \int_0^e 2tdt = e^2 > 0, \]
for each \( e > 0 \). Therefore \( T \) satisfies the inequality (9) in Theorem 2.7 for \( h_1 = h_2 = h_3 = 0, h_4 = \frac{1}{2} \) and the inequality (12) in Theorem 2.9 for \( h_1 = h_3 = h_4 = h_5 = h_6 = 0, h_2 = h_6 = \frac{1}{4} \). Hence \( T \) has a unique fixed point \( u = 75 \). But \( T \) does not satisfy the inequality (2) in Theorem 2.4. Indeed, if we take \( u = 0 \) and \( v = 1 \), then we obtain
\[ \int_0^1 2tdt = 100 \leq \frac{2}{0} \int_0^1 2tdt = 4h, \]
which is a contradiction since \( h \in (0, 1) \).

Finally, we give another generalization of Theorem 2.4.

**Theorem 2.12** Let \( (X, S) \) be a complete S-metric space, the function \( \zeta : [0, \infty) \rightarrow [0, \infty) \) be defined as in (1) and \( T : X \rightarrow X \) be a self-mapping of \( X \) such that
\[ S(Tu, Tu, v) \]
\[ \int_0^1 \zeta(t)dt \leq h_1 \]
\[ S(u, u, v) \]
\[ \int_0^1 \zeta(t)dt \]
\[ S(Tu, Tu, u) \]
\[ \int_0^1 \zeta(t)dt \]
\[ h_2 \]
\[ S(Tv, Tv, v) \]
\[ \int_0^1 \zeta(t)dt \]
\[ h_3 \]
\[ \int_0^1 \zeta(t)dt \]
\[ h_4 \]
\[ \int_0^1 \zeta(t)dt, \]
(15)
for all \( u, v \in X \) with nonnegative real numbers \( h_i \ (i \in \{1, 2, 3, 4\}) \) satisfying \( h_1 + h_2 + h_3 + 3h_4 < 1 \). Then \( T \) has a unique fixed point \( w \in X \) and we have
\[ \lim_{n \to \infty} Tu = w, \]
for each \( u \in X \).
Proof Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( T^n u_0 = u_n \).

Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (15), the condition (S2) and Lemma 2.1 we get

\[
\begin{align*}
S(u_n, u_{n+1}) &\leq S(T^{n-1}u_n, T^{n-1}u_n) + h_2 \int_0^1 \zeta(t) \, dt + h_3 \int_0^1 \zeta(t) \, dt \\
&\quad + \max \{S(u_n, u_{n-1}), S(u_{n-1}, u_{n+1})\} + h_4 \int_0^1 \zeta(t) \, dt \\
&= (h_1 + h_2 + h_3) \int_0^1 \zeta(t) \, dt \\
&\quad + \max \{S(u_n, u_{n-1}), S(u_{n-1}, u_{n+1})\} + h_4 \int_0^1 \zeta(t) \, dt,
\end{align*}
\]

which implies

\[
\begin{align*}
S(u_n, u_{n+1}) &\leq \left( h_1 + h_2 + h_3 \right) \int_0^1 \zeta(t) \, dt \\
&\quad + \max \{S(u_n, u_{n-1}), S(u_{n-1}, u_{n+1})\} + h_4 \int_0^1 \zeta(t) \, dt.
\end{align*}
\]

If we put \( h = h_1 + h_2 + h_3 \), then we find \( h < 1 \) since \( h_1 + h_2 + h_3 + 3h_4 < 1 \). Using the inequality (16) and mathematical induction, we have

\[
\begin{align*}
S(u_n, u_{n+1}) &\leq h^n \int_0^1 \zeta(t) \, dt.
\end{align*}
\]

Taking limit for \( n \to \infty \) and using the inequality (17) we find

\[
\begin{align*}
\lim_{n \to \infty} \int_0^1 \zeta(t) \, dt &= 0,
\end{align*}
\]

since \( h \in (0, 1) \). The condition (1) implies

\[
\lim_{n \to \infty} S(u_n, u_n, u_{n+1}) = 0.
\]

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence \( \{u_n\} \) is Cauchy. Then there exists \( w \in X \) such that

\[
\lim_{n \to \infty} T^n u_0 = w,
\]

since \( (X, S) \) is a complete \( S \)-metric space. From the inequality (15) we find

\[
\begin{align*}
S(u_n, u_{n+1}, w) &\leq S(T^{n-1}u_n, T^{n-1}u_n, Tw) + h_2 \int_0^1 \zeta(t) \, dt \\
&\quad + h_3 \int_0^1 \zeta(t) \, dt + \max \{S(u_n, u_{n-1}, w), S(Tw, u_{n+1})\} + h_4 \int_0^1 \zeta(t) \, dt.
\end{align*}
\]

If we take limit for \( n \to \infty \), using Lemma 2.1 we get

\[
\begin{align*}
S(Tw, Tw, w) &\leq (h_3 + h_4) \int_0^1 \zeta(t) \, dt,
\end{align*}
\]

which implies \( Tw = w \) since \( h_3 + h_4 < 1 \).

Now we show the uniqueness of the fixed point. Let \( w_1 \) be another fixed point of \( T \). Using the inequality (15) and Lemma 2.1, we get

\[
\begin{align*}
S(w, w, w_1) &\leq S(Tw, Tw, w_1) + h_2 \int_0^1 \zeta(t) \, dt \\
&\quad + h_3 \int_0^1 \zeta(t) \, dt + \max \{S(w, w, w_1), S(w_1, w, w_1)\} + h_4 \int_0^1 \zeta(t) \, dt.
\end{align*}
\]

which implies

\[
\begin{align*}
S(w, w, w_1) &\leq (h_1 + h_4) \int_0^1 \zeta(t) \, dt.
\end{align*}
\]

Then we obtain

\[
\begin{align*}
S(w, w, w_1) &\leq 0,
\end{align*}
\]

which implies \( w = w_1 \).
that is, \( w = w_1 \) since \( h_1 + h_4 < 1 \). Consequently, \( T \) has a unique fixed point \( w \in X \). \( \square \)

**Remark 2.13**

1. If we set the function \( \zeta : [0, \infty) \to [0, \infty) \) in Theorem 2.12 as
   \[
   \zeta(t) = 1,
   \]
   for all \( t \in [0, \infty) \), then we obtain Theorem 2 in [6].

2. Theorem 2.12 is another generalization of Theorem 2.4 on a complete \( S \)-metric space. Indeed, if we take \( h_1 = h \) and \( h_2 = h_3 = h_4 = 0 \) in Theorem 2.12, then we get Theorem 2.4.

3. Since Theorem 2.12 is another generalization of Theorem 2.4, Theorem 2.12 generalizes the classical Banach’s fixed-point theorem.

Let us consider the self-mapping \( T : \mathbb{R} \to \mathbb{R} \) and the function \( \zeta : [0, \infty) \to [0, \infty) \) defined in Example 2.11.

Then \( T \) satisfies the contractive condition (15) in Theorem 2.12 and so \( u = 75 \) is a unique fixed point of \( T \). Notice that \( T \) does not satisfy the inequality (2) in Theorem 2.4.

### An application to the Fredholm integral equation

In this section, we give an application of the contraction condition (2) to the Fredholm integral equation

\[
y(u) = l(u) + \lambda \int_a^b k(u, t) y(t) dt,
\]

where \( y : [a, b] \to \mathbb{R} \) with \(-\infty < a < b < \infty, k(u, t) \) which is called the kernel of the integral equation (18) is continuous on the squared region \([a, b] \times [a, b] \) with \(|k(u, t)| \leq M (M > 0) \) and \( l(u) \) is continuous on \([a, b] \).

Let \( C[a, b] = \{ f \mid f : [a, b] \to \mathbb{R} \text{is continuous} \} \).

Now we define the function \( S : C[a, b] \times C[a, b] \times C[a, b] \to [0, \infty) \) by

\[
S(f, g, h) = \sup_{u \in [a, b]} |f(u) - h(u)| + \sup_{u \in [a, b]} |f(u) + h(u) - 2g(u)|,
\]

(19)

for all \( f, g, h \in C[a, b] \). Then the function \( S \) is an \( S \)-metric. Now we show that this \( S \)-metric can not be generated by any metric \( \rho \). We assume that this \( S \)-metric is generated by any metric \( \rho \), that is, there exists a metric \( \rho \) such that

\[
S(f, g, h) = \rho(f, h) + \rho(g, h),
\]

(20)

for all \( f, g, h \in C[a, b] \). Then we get

\[
S(f, f, h) = 2\rho(f, h) = 2 \sup_{u \in [a, b]} |f(u) - h(u)|
\]

and

\[
\rho(f, h) = \sup_{u \in [a, b]} |f(u) - h(u)|.
\]

(21)

Similarly, we obtain

\[
S(g, g, h) = 2\rho(g, h) = 2 \sup_{u \in [a, b]} |g(u) - h(u)|
\]

and

\[
\rho(g, h) = \sup_{u \in [a, b]} |g(u) - h(u)|.
\]

(22)

Using the equalities (20), (21) and (22), we find

\[
\begin{align*}
\sup_{u \in [a, b]} |f(u) - h(u)| + \sup_{u \in [a, b]} |f(u) + h(u) - 2g(u)|
= & \sup_{u \in [a, b]} |f(u) - h(u)| + \sup_{u \in [a, b]} |g(u) - h(u)|,
\end{align*}
\]

which is a contradiction. Hence this \( S \)-metric is not generated by any metric \( \rho \). Consequently, \( (C[a, b], S) \) is a complete \( S \)-metric space.

**Proposition 3.1** Let \( (C[a, b], S) \) be a complete \( S \)-metric space with the \( S \)-metric defined in (19) and \( \lambda \) be a real number with

\[
|\lambda| < \frac{1}{M(b - a)}.
\]

Then the Fredholm integral equation (18) has a unique solution \( y : [a, b] \to \mathbb{R} \).

**Proof** Let us define the function \( T : C[a, b] \to C[a, b] \) as

\[
Ty(u) = l(u) + \lambda \int_a^b k(u, t) y(t) dt.
\]

Now we show that \( T \) satisfies the contractive condition (2). We get

\[
S(Ty_1, Ty_1, Ty_2) = 2 \sup_{u \in [a, b]} |Ty_1(u) - Ty_2(u)|
\]

\[
= 2 \sup_{u \in [a, b]} \left| \lambda \int_a^b k(u, t) (y_1(u) - y_2(u)) dt \right|
\]

\[
\leq 2|\lambda| M \sup_{u \in [a, b]} \left| \int_a^b (y_1(u) - y_2(u)) dt \right|
\]

\[
\leq 2|\lambda| M \sup_{u \in [a, b]} \int_a^b |y_1(u) - y_2(u)| dt
\]

\[
\leq 2|\lambda| M \sup_{u \in [a, b]} \int_a^b |y_1(u) - y_2(u)| dt
\]

\[
\leq |\lambda| M(b - a) S(y_1, y_1, y_2)
\]

\[
< S(y_1, y_1, y_2),
\]

\[
\square
\]
which implies
\[ s(Ty_1, Ty_2, Ty_3, Ty_4) < s(y_1, y_2, y_3, y_4) \]
\[ \int_0^1 \zeta(t) \, dt < \int_0^1 \zeta(t) \, dt. \]

Consequently, the contractive condition (2) is satisfied and the Fredholm integral equation (18) has a unique solution \( y \).

Now we give an example of Proposition 3.1.

**Example 3.2** Let us consider the Fredholm integral equation defined as
\[ y(u) = e + \lambda \int_1^e \frac{\ln t}{t} y(t) \, dt. \]  
(23)

Now we find a solution of the Fredholm integral equation (23) with the initial condition \( y_0(u) = 0 \). We solve this equation for \( |\lambda| < \frac{1}{e-1} \) since \( |\ln t| < 1 \) for all \( 1 \leq u, t \leq e \). We obtain
\[ y_1(u) = e, \]
\[ y_2(u) = e + \lambda \int_1^e \frac{\ln t}{t} \, dt = e + \lambda e \ln u, \]
\[ y_3(u) = e + \lambda \int_1^e \frac{\ln t}{t} (e + \lambda e \ln t) \, dt \]
\[ = e + \lambda e \ln u + \frac{\lambda^2}{2} e \ln u, \]
\[ y_4(u) = e + \lambda \int_1^e \frac{\ln t}{t} (e + \lambda e \ln t + \frac{\lambda^2}{2} e \ln t) \, dt \]
\[ = e + \lambda e \ln u + \frac{\lambda^2}{2} e \ln u + \frac{\lambda^3}{2} e \ln u, \]
\[ \ldots \]
\[ y_n(u) = e + \lambda e \ln u \left[ 1 + \frac{\lambda}{2} + \frac{\lambda^2}{4} + \cdots + \frac{\lambda^n}{2^n} \right] \]
\[ - e + \frac{2\lambda}{2 - \lambda} e \ln u. \]

Consequently, this is a solution of the Fredholm integral equation (18) for \( |\lambda| < \frac{1}{e-1} < 1. \)

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