Finite Quantum Kinematics of the
Harmonic Oscillator

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Abstract

Arbitrarily small changes in the commutation relations suffice to transform the usual singular quantum theories into regular quantum theories. This process is an extension of canonical quantization that we call general quantization. Here we apply general quantization to the time-independent linear harmonic oscillator. The unstable Heisenberg group becomes the stable group SO(3). This freezes out the zero-point energy of very soft or very hard oscillators, like those responsible for the infrared or ultraviolet divergencies of usual field theories, without much changing the medium oscillators. It produces pronounced violations of equipartition and of the usual uncertainty relations for soft or

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hard oscillators, and interactions between the previously uncoupled excitation quanta of the oscillator, weakly attractive for medium quanta, strongly repulsive for soft or hard quanta.

1 Make it simple

The three main evolutions of physics in the twentieth century have a suggestive family resemblance. Each introduced a new kind of non-commutativity. The new non-commutativity in special relativity was that of boosts, in general relativity and the Standard Model gauge theories that of infinitesimal translations, and in quantum theory that of filter operations. The seminal work of Segal [17], which stimulated the present work, pointed out that further changes of this kind are necessary for stability and suggested one. Our main goal is finiteness, not stability, but the stabilizing changes Segal suggested lead ultimately to a finite quantum theory, including one of space-time. Such a theory has been sought by physicists since the formulation of quantum theory.

By gently modifying the commutation relations of an existing quantum theory one produces a simpler theory with the existing quantum theory as a suitable limiting case, and with nearly the same continuous symmetries. A special form of Segal’s general concept was applied retroactively to the relation between special relativity and Galileo relativity [12, 13]. More proactively, Snyder’s space-time quantization [19] was an attempted regularization and moved unwittingly toward simplicity but did not simplify the momentum algebra. Vilela made the first efforts to found a new particle theory on a simple algebra [21]. Something like the regularization of the harmonic oscillator proposed
by Segal is now under study by several groups from several points of
view [14, 15, 20].

For example, ’tHooft [20] studies the classical particles on a circle
and shows that under certain conditions, this system is equivalent to a
quantum harmonic oscillator. The work of Vilela Mendes differs from
others in presenting the quantum harmonic oscillator as a limit case
of a “more quantum” oscillator that has a more stable algebra in the
sense of Segal. We follow that line here.

Naturally one discards the unregularized theory in favor of the reg-
ularized one. This last step is overlooked in some older studies.

General quantization changes the quantization rules. It replaces
the usual quantization prescription by the following one:

*Make the commutator algebra of the generators a simple Lie alge-
bra.*

Briefly put: If the algebra is simple, keep it simple. If it is not
simple, make it simple by the least change possible.

We apply this strategy to all the algebras postulated in a physical
theory, on the grounds that they must all depend continuously on
experimental data that are subject to error. An algebra that is not
stable is not empirical but is at least partly based on ideology. It
is not possible to eliminate all such ideology-based hypotheses from
physics. But it is possible to reduce their number systematically.

This implies that canonical quantization and special and general
relativization are relatively small parts of a vast unifying drift toward
simplicity and unity of the groups beneath our physical theories. Gen-
eral quantization is an attempt to take part in that drift.

In the case of the algebra of variables, semisimplicity is as good as
simplicity, since it implies a direct sum that we can reduce to a simple term forever by a single measurement of the superselection variable that distinguishes these terms.

Canonical quantization introduced one quantum constant and stabilized the atom. General quantization introduces as many quantum constants as necessary to stabilize the group of the theory.

General quantization has no effect on (say) the rotator quantum algebra with given angular momentum $l$ ($\mathbf{L} \times \mathbf{L} = \mathbf{L}$, $\mathbf{L} \cdot \mathbf{L} = l(l+1)$), whose group is already isimple. But it changes the quantum dynamics of a free particle deeply, revising the theories of both space and time.

We test general quantization here on the kinematics of the linear harmonic oscillator, a ubiquitous constituent of all present field theories, and compare the finite quantum oscillators with the usual quantum oscillators, which are singular in several senses. The differences have profound consequences for extreme energy physics: the physics of both very high and very low energies.

Planck’s quantum constant $\hbar$ froze out the very stiff oscillators responsible for the infinite heat capacity of cavity radiation in Maxwell’s theory, but the zero-point energy of the resulting quantum theory of electromagnetism still diverged, however, unless one arbitrarily replaced the local Lagrangian and Hamiltonian of Maxwell by non-local ones tailored to have some finite zero-point energy, usually set to 0 on grounds of Lorentz invariance. Indeed, the quantum theory of the harmonic oscillator carries the germs of all the main divergences of quantum field theory. Its basic operators of position $q$, momentum $p$ and Hamiltonian $H \sim \frac{1}{2}(p^2 + q^2)$ are undefined on almost every vector $\psi$ in its Hilbert space: $q\psi = p\psi = H\psi = \infty$. Such divergences occur in
a quantum theory if and only if its Hilbert space is infinite-dimensional.

The usual quantum oscillator theory is also unstable in the Segal sense detailed below. It is not as unstable as the classical theory, which has the operators $q$ and $p$ commute, and we have become accustomed to its foibles, but it is still not operational, in that its basic operations usually cannot be carried out. General quantization makes its Hilbert space finite-dimensional. The result is a finite quantum theory whose operations can in principle be carried out, with two Segal quantum constants $h', h''$ besides the usual Planck constant.

We find that introducing these constants freezes out even the offending zero-point oscillations of extremely hard or soft oscillators without greatly changing the zero-point energies of medium ones. The frozen oscillators have infinitesimal zero-point energies compared to the usual quantum values. They also grossly violate the usual equipartition and uncertainty relations.

This toy model illustrates how a finite quantum theory of the cavity might produce a finite zero-point energy without conflicting with the many finite predictions and symmetries of the usual quantum theory. We propose that the linear harmonic field oscillators considered fundamental in present quantum physics – those of supposedly fundamental fields, not those of elastic solids, say — are actually finite quantum oscillators near the bottoms of their energy spectra. The unobserved oscillators responsible for the infrared and ultraviolet divergencies of present quantum theories are frozen by finite quantum effects described here and contribute negligibly to the zero-point energy.

The change we carry out here is not enough to make quantum field theory finite. For that we must also simplify the Heisenberg algebra of
the space-time operators $x^\mu$ and $\partial_\mu$. This replaces the manifold theory of space-time, which assumes an infinity of events, by a simple quantum theory, with only a finite number of disjoint events, though the number may be arbitrarily large. Field theory has compound algebra on two levels, that of the underlying space-time and that of the overlying canonical commutation relations. In this paper we change only one level.

2 Algebra flexing and flattening.

A *semi-simple* group is a Lie group whose Lie algebra has no invariant solvable subalgebras; a Lie algebra $A$ being solvable if for some integer $n \geq 0$, $A^n = \{0\}$. Then its Lie algebra has no radical. A group that is not semisimple we call *compound*. General quantization reduces the radical and ultimately eliminates it.

Lie products $\times$ on a given vector space $A$, also called structure tensors, form a submanifold $\{\times\}$ of the tensor space $A \otimes A \dagger \otimes A \dagger$. A *regular* (stable, robust) algebra is one that is unchanged up to isomorphism by all sufficiently small changes in its structure tensor (Lie product) within the manifold $\{\times\}$. For example, the Lorentz algebra is stable against corrections to the speed of light. By algebra *flexing* we mean a homotopy of the structure tensor of a compound algebra that makes it semisimple. Algebra *flattening* is the inverse process. The well-known contraction process of Inönü and Wigner \[12\] is a special case of flattening accomplished by a one-parameter group of dilations of a coordinate system of the Lie algebra in a fixed direction. The inverse to group contraction is *group expansion* \[11\] and is a special
Several regularization processes have been used to remove unwanted infinities from physics. Unphysical regularizations cope with the divergencies of a theory without changing the finite results. They are regarded as giving the theory meaning rather than changing the theory. These include Pauli-Villars regularization [16], lattice regularization (lattice gauge theory) [22] and dimensional regularization [4, 5, 20]. They contain regularization parameters that go to singular limits, like the lattice spacing going to 0. Physical regularizations, on the contrary, change the finite predictions as well as making the infinite ones finite, and are intended as distinct physical theories in their own right. Their regularization parameters do not go to a singular limit but must be determined by experiment. The most famous example is Planck’s, which ultimately led to quantum theory. This was a simplification in that the associative algebra generated by the position and momentum variables has a Lie algebra with an infinite-dimensional radical for $h = 0$ but only a one-dimensional radical for $h > 0$.

Physical regularizations are subtler than unphysical ones but their consequences for human thought have been more dramatic. General quantization is proposed as a physical regularization.

Compound algebras are unstable with respect to a small change in their structures [17]. Flexing stabilizes them.

Conversely, flattening destabilizes. Approximating a circle by a tangent line or a sphere by a tangent plane are well-known flattenings. The circle and sphere are finite and their flattened form is infinite. Finite dimensional representations of the group of the sphere — such as spherical harmonic polynomials — form a complete set on the sphere, and
all the operators of an irreducible representation have finite bounded spectra. On the other hand the tangent plane is not compact and requires infinite-dimensional representations of the translation group for a complete set, and its algebra generators have unbounded spectra.

3 Simplifying the Heisenberg Algebra

The Heisenberg Lie algebra \( \mathbf{H}(1) \) is defined by the commutation relations:

\[
\begin{align*}
[p, x] & = -i\hbar \\
[i, x] & = 0 \\
[x, i] & = 0
\end{align*}
\] (1)

It is compound and the imaginary unit \( i \) generates its radical. Segal proposed to simplify \( \mathbf{H}(1) \) by introducing two more quantum constants, which we designate here by \( \hbar' \) and \( \hbar'' \). His expanded commutation relations are, except for notation,

\[
\begin{align*}
[p, x] & = -\hbar i \\
[i, p] & = -\hbar' x \\
[x, i] & = \hbar'' p
\end{align*}
\] (2)

\[\text{[6, 21, 17].} \] The irreducible unitary representations of this group are infinite-dimensional. To avoid possible divergences and other problems, we instead use the SO(3) regularization \[\text{[2, 9, 10].} \]

\[
\begin{align*}
[p, x] & = -i\hbar \\
[i, p] & = -x\hbar' \\
[x, i] & = -\hbar'' p
\end{align*}
\] (3)
Ultimately we will need an indefinite metric for physical reasons, but not for the time-independent harmonic oscillator.

Regularizing the Heisenberg algebra means changing the role of $i$ in the theory from constant central element to quantum variable operator on the same footing as $p$ and $q$. We call this $i$-activation. The new variable that it introduces is called a regulator. A previous exploration in quaternion quantum theory activated an $i$ that served as the electromagnetic axis $\eta(x)$ that resolves the electroweak gauge boson into electromagnetic and weak bosons [7], and gives mass to the charged partner of the photon through the Stückelberg-Higgs effect. This led to a natural SU(2) that was interpreted as isospin. That theory was dropped because it did not leave room for color SU(3). Here we activate $i$ on more principled grounds, namely the principle of simplicity. There is now plenty of room for internal groups like color SU(3), though they do not arise for the harmonic oscillator.

General quantization leads to the same kind of factor-ordering problems as the special case of canonical quantization. To reduce these we regularize not Hermitian observables directly but skew-Hermition generators

$$\hat{q} = iq, \quad \hat{p} = -ip.$$ \hfill (4)

The usual quantum commutation relations are then

$$[\hat{q}, \hat{p}] = \hbar i$$
$$[i, \hat{q}] = 0,$$
$$[\hat{p}, i] = 0,$$
$$i^2 = -1.$$ \hfill (5)
The regularized generators $\check{q}, \check{p}, \check{i}$ obey

$$
\begin{align*}
[\check{q}, \check{p}] &= \hbar \check{i}, \\
[\check{i}, \check{q}] &= \hbar' \check{p}, \\
[\check{p}, \check{i}] &= \hbar'' \check{q},
\end{align*}
$$

(6)

We suppose $\hbar, \hbar', \hbar'' > 0$ so the orthogonal group is SO(3). The quantities with a breve “ˇ” are the new expanded quantum operators. In this way the simplification process introduces a new dynamically variable generator $\check{i}$, somewhat as general-relativization introduced the new dynamical variable $g_{\mu\nu}$ the gravitational metric tensor field. The most primitive theory with a dynamical variable like $\check{i}$ is quaternion quantum field theory [7]. There $\check{i}$ generates rotations about the electric (or electromagnetic) axis in isospin space, defining a natural Higgs field. We suppose that the present generator $\check{i}$ is also a Higgs field.

When general quantization introduces new group generators in this way for simplification, we call these regularization operators or “regulators.” The physical constants to which regulators reduce in the singular theory we call regularization constants or “regulants.” Examples of a regulator in present physics are the Riemann curvature of space-time (as the commutator of covariant transports) and the gravitational field itself (as the anticommutator of unit Clifford vectors). Examples of regulants are $\hbar$ and $c$.

Except for scale factors the simplified commutation relations are those of an SO(3) quantum skew-angular-momentum operator-valued vector $\check{\mathbf{L}} = \mathbf{L} \times \mathbf{L}$ for a dipole rotator in three dimensions. We assume an irreducible representation with

$$
\check{\mathbf{L}}^2 = -l(l + 1)
$$

(7)
where \( l \) can have any non-negative half-integer eigenvalue. In the present work it suffices to consider only integer values of \( l \). Then the \( \hat{L}_x, \hat{L}_y, \hat{L}_z \) are represented by \((2l + 1) \times (2l + 1)\) matrices obeying

\[
\begin{align*}
[\hat{L}_1, \hat{L}_2] &= \hat{L}_3, \\
[\hat{L}_2, \hat{L}_3] &= \hat{L}_1, \\
[\hat{L}_3, \hat{L}_1] &= \hat{L}_2, \\
(\hat{L}_1)^2 + (\hat{L}_2)^2 + (\hat{L}_3)^2 &= -l(l + 1).
\end{align*}
\]

(8)

We fix the scale factors with

\[
\begin{align*}
\hat{q} &= Q\hat{L}_1, \\
\hat{p} &= P\hat{L}_2, \\
\hat{i} &= J\hat{L}_3,
\end{align*}
\]

(9)

By (6)

\[
\begin{align*}
J &= \sqrt{\hbar'\hbar''} = 1/l, \\
Q &= \sqrt{\hbar'}, \\
P &= \sqrt{\hbar''}.
\end{align*}
\]

(10)

The commutation relations (8) and the angular momentum quantum number \( l \) determine a simple (associative) enveloping algebra \( \text{Alg}(L, l) \). The spectral spacing of the \( \hat{L}_3 \) is 1, so the finite quantum constants \( Q, P, J \) serve as quanta of position, momentum and \( \hat{i} \). Since \( q, p \) are supposed to have continuous spectra in quantum theory, the constants \( Q, P \) must be very small on the ordinary quantum scale. It follows that \( J = QP/\hbar \) is also very small on that scale and \( l \gg 1 \).

For \( l \gg \sqrt{l} \gg 1 \), variations \( \delta(i^2) \leq O(l^{-1/2}) \ll 1 \) about \((i)^2 = -1\) can be negligible at the same time as the spectral intervals \( \delta p \leq P\sqrt{l} \)
and $\delta q \leq Q\sqrt{l}$ for quasicontinuous $p, q \approx 0$. This simulates the usual oscillator.

4 Finite Linear Harmonic Oscillator

Now we specialize to the oscillator by fixing a Hamiltonian. For given finite-quantum constants $P, Q$ the finite harmonic oscillator has a Hamiltonian of the form

$$H = \frac{P^2 L_x^2}{2m} + \frac{kQ^2 L_y^2}{2} := \frac{K}{2} (L_x^2 + \kappa^2 L_y^2)$$

(11)

where

$$K = \frac{P^2}{m}, \quad \kappa^2 = \frac{\hbar^2 m k}{\hbar''}.$$ 

(12)

For fixed $\hbar, \hbar', \hbar''$, all finite oscillators are divided into three kinds with ill defined boundaries: medium, where kinetic and potential terms in $H$ are of comparable size ($\kappa \sim 1$); soft, when the potential energy term is dominant ($\kappa \to 0$); and hard, when the kinetic energy term is dominant ($\kappa \to \infty$). Examination of the Hamiltonian of a spin-zero scalar field (Klein-Gordon field) in quantum field theory shows that the possibilities $\kappa \ll 1$ and $\kappa \gg 1$ are also important. The oscillators that give rise to infrared divergencies of the quantum field theory correspond to soft oscillators of the finite quantum theory. Those that feed ultraviolet divergencies correspond to hard oscillators.
5 Medium oscillators

The case $\kappa = 1$ is symmetric under rotations about the $z$ axis, and so is especially simple [20]. Since

$$ (\hat{L}_1)^2 + (\hat{L}_2)^2 + (\hat{L}_3)^2 = (\hat{L})^2, \quad (13) $$

$$ \hat{H} = \frac{K}{2} \left( l(l+1) + (\hat{L}_3)^2 \right) \quad (14) $$

The oscillator quantum number $n$ that labels the energy level is now

$$ n = l + m. \quad (15) $$

The expanded energy spectrum is

$$ E_n = \frac{K}{2} (l(l+1) - (n-l)^2) = lK \left( n + \frac{1}{2} - \frac{n^2}{2l} \right) \quad (16) $$

For $n \ll \sqrt{l} \ll l$ this reproduces the usual uniformly-spaced oscillator energy spectrum as closely as desired, but with multiplicity 2 for each level instead of 1.

The ground-state energy for this oscillator is

$$ E_0 = \frac{1}{2} Kl = \frac{1}{2} 1/2h\omega, \quad (17) $$

exactly the usual oscillator ground energy, since $Kl = h\omega$.

The main new feature is that this finite oscillator has an upper energy limit

$$ E_{\text{max}} = \frac{1}{2} Kl(l+1) \quad (18) $$

as required by a finite quantum theory.

In the general case of $\kappa \sim 1$ we obtain an upper bound for the ground energy by a variational approximation with the trial function
$|L_z = \pm l\rangle$. This reproduces our previous result (17), now as an upper bound for the ground energy of a medium FLHO:

$$E_0 \leq \frac{1}{2} Kl. \quad (19)$$

Medium oscillators have many states with $m$-value close to its extremum value $m = \pm l$. The usual Heisenberg uncertainty principle

$$(\Delta p)^2(\Delta q)^2 \geq \frac{1}{4}(i[p, q])^2 = \frac{\hbar^2}{4}. \quad (20)$$

becomes

$$(\Delta L_x)^2(\Delta L_y)^2 \geq \frac{\hbar^2}{4}(L_z)^2|_{L_z \approx \pm l} \quad (21)$$

for a low-lying energy level of a medium oscillator. By (9) and (10),

$$(\Delta p)^2(\Delta y)^2 \geq \frac{\hbar^2}{4} \quad (22)$$

for large $l$. So medium oscillator states in low-lying energy levels have uncertainties at or above the lower limit set by the Heisenberg uncertainty principle.

### 6 Soft oscillators

Recall our finite quantum oscillator Hamiltonian

$$\hat{H} = \frac{K}{2}(\hat{L}_x^2 + \kappa^2 \hat{L}_y^2) \quad (23)$$

When $\kappa \ll 1$ we can estimate the spectrum of $\hat{H}$ using perturbation theory. The unperturbed Hamiltonian for our problem is the kinetic energy

$$H_0 = \frac{K}{2} \hat{L}_x^2 \quad (24)$$
and the unperturbed eigenvectors are $|L_x = m\rangle$ so the unperturbed energy levels are

$$E_m(0) = \frac{K}{2} m^2.$$  \hfill (25)

The first-order shifts are

$$\delta E_m = \frac{K}{2} \langle L_x = m|L_y^2|L_x = m \rangle.$$  \hfill (26)

Due to the axial symmetry of $|L_x = m\rangle$,

$$\langle L_x = m|L_y^2|L_x = m \rangle = \langle L_x = m|L_z^2|L_x = m \rangle.$$  \hfill (27)

Therefore the energy shift is, to lowest order in $\kappa^2$,

$$\frac{K}{2} \langle L_x = m|\kappa^2L_y^2|L_x = m \rangle = \frac{K}{4} \kappa^2 \langle m|L_x^2 + L_y^2|m \rangle$$

$$= \frac{K}{4} \kappa^2 \langle m|L^2 - L_z^2|m \rangle$$

$$= \frac{K}{4} \kappa^2 l(l+1) - m^2.$$  \hfill (28)

The new energy spectrum is then

$$E_m \approx \frac{K}{2} m^2 + \Delta E_m$$

$$= \frac{K}{2} m^2 + \frac{1}{4} K \kappa^2 [l(l+1) - m^2]$$  \hfill (29)

The estimated upper bound for the energy is

$$E_{max} \approx \frac{1}{2} K l^2 (1 + \frac{\kappa^2}{2l})$$  \hfill (30)

For $\kappa \to 0$ this reproduces the upper bound for the unperturbed hamiltonian $L_z^2$, as it should. The zero-point energy $E_0$ of first-order perturbation theory is

$$E_0 \approx \frac{1}{4} \kappa^2 K l(l+1)$$  \hfill (31)

For $\kappa \to 0$ this is infinitesimal compared to the usual QLHO.
A soft oscillator shows no resemblance to the usual quantum oscillator. Its energy levels do not have uniform spacing. Its kinetic energy dwarfs its potential energy, so equipartition is grossly violated. The low energy states are near $|L_x = 0\rangle$ instead of $|L_z = \pm l\rangle$. Its $p$ degree of freedom is frozen out. It is “too soft to oscillate.” There is not enough energy in the $q$ degree of freedom, even at its maximum excitation, to produce one quantum of $p$. The uncertainty relation reads

$$\left(\Delta L_x\right)^2 \left(\Delta L_y\right)^2 \geq \frac{\hbar^2}{4} \langle L_z \rangle^2_{|L_x = 0\rangle} \approx 0$$

(32)

Therefore

$$\Delta p \Delta q \ll \frac{\hbar}{2},$$

(33)

which violates the Heisenberg uncertainty principle grossly.

7 Hard oscillators

The story is just reversed for hard oscillators but the gross violations of usual quantum principles remain the same. A hard oscillator has much greater potential than kinetic energy. Its low energy states are now near $|L_y = 0\rangle$ instead of $|L_z = \pm l\rangle$ (the medium case) or $|L_x = 0\rangle$ (the soft case). Its $q$ degree of freedom is frozen out. It is “too hard to oscillate.” There is not enough energy in the $p$ degree of freedom, even at maximum excitation, to arouse one quantum of $q$.

A hard oscillator can also be treated by perturbation theory. The kinetic energy is the perturbation. We may carry all the of the main results in the previous section for soft FLHO oscillators to the hard ones simply by replacing $\kappa$ with $1/\kappa$ and $K$ with $K\kappa^2$. A hard FLHO shows no resemblance to the usual QLHO. Its zero-point energy $E_0$ is
now

\[ E_0 \approx \frac{K}{4} l(l + 1) \]  

(34)

For \( \kappa \to \infty \) this is infinitesimal compared to the usual quantum oscillator zero-point energy. Its energy levels of a hard oscillator are not uniformly spaced. Its uncertainty relation reads

\[ (\Delta L_x)^2 (\Delta L_y)^2 \geq \frac{\hbar^2}{4} \langle L_z \rangle |_{L_y = 0} \approx 0 \]  

(35)

Therefore

\[ \Delta p \Delta q \ll \frac{\hbar}{2}, \]  

(36)

which seriously violates the Heisenberg uncertainty principle again.

## 8 Unitary Representations

Variables \( p \) and \( q \) do not have finite-dimensional unitary representations in classical and quantum physics. They are continuous variables and generate unbounded translations of each other. But since in the finite quantum theory, all operators become finite and quantized, we expect all translations to become rotations with simple finite-dimensional unitary representations.

The canonical group of a classical oscillator becomes the unitary group of an infinite-dimensional Hilbert space for a quantum oscillator, and the unitary group of a \( 2l+1 \) dimensional Hilbert space for the finite oscillator.

The Lie algebra generated by momentum and position as infinitesimal symmetry generators is \( \mathbf{H}(1) \) for the classical and quantum oscillator and the \( \mathbf{SO}(3) \) angular momentum algebra for the finite oscillat-
tor. The corresponding Lie groups are the Heisenberg group $\mathbf{H}(1)$ and $\mathbf{SO}(3)$.

Unitarily inequivalent unitary representations of the canonical commutation relations are forbidden in quantum mechanics but present and important in quantum field theory, but general quantization eliminates them. After general quantization the commutation relations become those of a large simple group, and we presently explore the orthogonal groups. Once its invariants are fixed, as by measurement, the finite-dimensional unitary representations of this group are uniquely determined up to unitary equivalence. Yet the general quantized theory approaches the usual singular theory in an appropriate limit, where the dimension of the representations grow without bound and the group becomes compound. The inequivalent representations of quantum field theory must return in that singular limit. More than that we cannot say at this stage in the development.

9 Conclusion

We suggest that algebra flattening causes the infinities of present physics. Since quantum theory began as a regularization procedure of Planck, it is rather widely accepted that further regularization of present quantum physics calls for further quantization, but what to quantize and how to quantize it remains at least a bit unclear. If we regard quantization as another step in group regularization, the rest of the path becomes clear. It is blazed with radicals ripe for relativization. General quantization of the linear harmonic oscillator results in a finite quantum theory with three quantum constants $\hbar, \hbar', \hbar''$ instead of the
usual one. The finite quantum oscillator is isomorphic to a dipole rota-
tor with $N = l(l + 1) \sim 1/(\hbar') \gg 1$ states and bounded Hamiltonian $H = A(L_x)^2 + B(L_y)^2$. Its position and momentum variables are quantized with uniformly spaced bounded finite spectra and supposedly universal quanta of position and momentum. For fixed quantum con-
stants and large $N \gg 1$ there are three broad classes of finite oscillator, soft, medium, and hard. The field oscillators responsible for infra-red and ultraviolet divergences are soft and hard respectively. Medium oscillators have $\sim \sqrt{N}$ low-lying states having nearly the same zero-point energy and level spacing as the quantum oscillator and nearly obeying the Heisenberg uncertainty principle and the equipartition principle.

The corresponding rotators are nearly polarized along the $z$ axis with $L_z \sim \pm l$.

The soft and hard oscillators have infinitesimal 0-point energy, and grossly violate both equipartition and the Heisenberg uncertainty re-
lation. They do not resemble the quantum oscillator at all. Their low-lying energy states correspond to rotators with $L_x \sim 0$ or $L_y \sim 0$ instead of $L_z \sim \pm l$. Soft oscillators have frozen momentum $p \approx 0$ because their maximum potential energy is too small to produce one quantum of momentum. Hard oscillators have frozen position $q \approx 0$ because their maximum kinetic energy is too small to produce one quantum of position.

The zero-point energy of a physical oscillator likely contributes to its gravitational field. It will be interesting to estimate its contribu-
tion to astronomical gravitational fields. For a consistent estimate we should regularize the quantum field theory, not just one of its oscilla-
tors. This changes not only the structure of the individual oscillators,
as considered here, but also the number and distribution of the oscillators. We leave this study for later, but it is already easy to say how it will proceed, and what form it will take.

Field theory has compound algebras on two levels, that of the underlying space-time and that of the overlying canonical commutation relations. In this paper we change only the top level, but to simply field theory we must also simplify the Heisenberg algebra of the lower-level space-time operators $x^\mu$ and $\partial_\mu$. This replaces the manifold theory of space-time, which assumes an infinity of events, by a simple quantum space-time theory, with only a finite number of disjoint events, though that number may be arbitrarily large.

We must then combine two finite-dimensional algebras, that of the local field variables and that of the space-time-energy-momentum variables, to make the finite-dimensional algebra of the field theory. In c discrete theories, the combination process is exponentiation $S^T$ where $S$ is the local field-variable state-set and $T$ is the space-time set. In the q/c theories that work best today, where the numerator $q$ indicates that $S$ is quantum and the denominator $c$ indicates that $T$ is still classical, an exponential still exists and is used. General quantization leads to q/q theories. In that case the usual exponential $S^T$ becomes basis-dependent, and the most economical invariant construct that includes all the special cases is the exterior algebra over $S \otimes T$, but this is still finite-dimensional. Since general quantization gives time too a beginning and an end, the time-development is certainly not unitary. As a result the problem of reconciling unitarity, causality, and Lorentz invariance is eliminated. On the other hand, since the Lorentz group is already simple, Lorentz invariance is unaffected by general
quantization.

General quantization modifies low- and high-energy physics. Because the low-lying energy levels of medium oscillators have nearly uniform spacing, the energy of two excitations is but slightly less than the sum of their separate energies. The corresponding quanta nearly do not interact, and the small interaction that they have is attractive. For soft or hard oscillators, the energy level varies quadratically with the energy quantum number. The energy of two quanta of oscillation is twice the sum of their separate energies, for example. The corresponding quanta have a repulsive interaction of great strength; the interaction energy is equal to the total energy of the separate quanta. Thus the simplest regularization leads to interactions between the previously uncoupled excitation quanta of the oscillator, weakly attractive for medium quanta, strongly repulsive for soft or hard quanta.

Like Dirac’s theory of the “anomalous” magnetic moment of the relativistic electron, these extreme-energy effects depend on factor ordering. They can be adjusted to fit the data by re-ordering factors and so are not crucial tests of the theory. A group regularization of a time-dependent free Dirac equation has been carried out \[10\] and the extension to interactions is under study.

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