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Boundary integral equation for electromagnetic wave scattering by a homogeneous body of arbitrary shape

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Abstract
Boundary integral equation is derived for the problem of scattering of electromagnetic waves by 3D homogeneous body of arbitrary shape.

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1 Introduction

Let \( D \subset \mathbb{R}^3 \) be a bounded domain with a smooth connected boundary \( S \). The exterior domain \( D' = \mathbb{R}^3 \setminus D \) is filled with a homogeneous material with parameters \((\epsilon_0, \sigma = 0, \mu_0)\), \( D \) is filled with a material with parameters \((\epsilon, \sigma, \mu_0)\), \( \epsilon, \epsilon_0, \sigma, \mu_0 \) are constants. Let \( \epsilon' = \begin{cases} \epsilon + \frac{i\omega}{\mu}, & \text{in } D, \\ \epsilon_0, & \text{in } D'. \end{cases} \) Here \( \omega \) is the frequency, \( \mu_0 \) is magnetic parameter, \( \epsilon, \epsilon_0 \) are dielectric parameters, \( \sigma \geq 0 \) is conductivity.

The governing equations in \( \mathbb{R}^3 \) are

\[
\nabla \times E = i\omega \mu_0 H, \quad \nabla \times H = -i\omega \epsilon' E.
\]

The boundary conditions on \( S \) are continuity of the tangential components of \( E \) and normal components of \( \epsilon' E \) across \( S \):

\[
[N, E^+] = [N, E^-],
\]
and
\[ N \cdot (\epsilon' E^+) = N \cdot (\epsilon_0 E^-). \]  \hfill (3)

Here \( N \) is the unit normal to \( S \), pointing into \( D' \), \([N, E] (E \cdot N)\) is the cross (dot) product of vectors, and \( E^+ (E^-) \) is the limiting value of \( E \) as \( x \to s \in S, x \in D(D') \).

We assume that the incident field \((E_0, H_0)\) satisfies equations (1) in \( \mathbb{R}^3 \) with \( \epsilon' \) replaced by \( \epsilon_0 \). For example one may take the incident field to be a plane wave: \( E_0 = e_1 e^{ikx_3} \), where \( e_i \cdot e_j = \delta_{ij} \), \( \{e_j\}_{j=1}^3 \) is the standard Euclidean basis, \( \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \)

If \( E \) is found then
\[ H = (i\omega\mu_0)^{-1} \nabla \times E. \]  \hfill (4)

From (1) one gets
\[ \nabla \times \nabla \times E - K^2 E = 0 \quad \text{in } \mathbb{R}^3, \]  \hfill (5)

where
\[ K^2 = \begin{cases} K^2 \text{ in } D, \quad K^2 = \omega^2 \epsilon' \mu_0, \\ k^2 \text{ in } D', \quad k^2 = \omega^2 \epsilon_0 \mu_0. \end{cases} \]  \hfill (6)

The scattering problem consists of finding the solution of equations (5), (2), (3), such that
\[ E = E_0 + V, \]  \hfill (7)
\[ V_r - ikV = o \left( \frac{1}{r} \right), \quad r := |x| \to \infty. \]  \hfill (8)

Assumption (8) means that \( V \) satisfies the radiation condition.

## 2 Derivation of the boundary integral equations

Let us look for the solution to (5), (2), (3), (7), (8), of the form:
\[ E = \begin{cases} \nabla \times \int_S G(x, t)J(t)dt, & x \in D, \\ \nabla \times \int_S g(x, t)j(t)dt + E_0(x), & x \in D', \end{cases} \]  \hfill (9)

where
\[ g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad G(x, y) = \frac{e^{iK|x-y|}}{4\pi|x-y|}. \]  \hfill (10)

The currents \( j \) and \( J \) are vector fields tangential to \( S \).

Thus, there are four scalar unknowns: the two unknown vector fields \( j \) and \( J \), tangential to \( S \).
For any \( j \) and \( J \) vector \( E \) solves equation (5) in \( D \) and in \( D' \) and satisfies conditions (7) and (8) because \( g \) satisfies the radiation condition (8). Thus, (10) is the solution to the scattering problem if \( j \) and \( J \) can be chosen so that the boundary conditions (2), (3) are satisfied.

Condition (2) is a vector equation, which is equivalent to three scalar equations, and equation (3) is a scalar equation. Therefore equations (2) and (3) together are equivalent to four scalar equations for four unknown scalar functions, the coordinates of the tangential to \( S \) vector fields \( j \) and \( J \).

Equation (2) can be written as a Fredholm-type integral equation. We have

\[
\int_S [N_x, [\nabla_x G(x, t), J]]|_{x \to s, x \in D} = -\frac{A^+ J + J}{2} + \int_S \nabla_s G(s, t) N_s \cdot J(t) dt,
\]

(11)

where the known formula for the limiting value of the normal derivative of the single-layer potential was used (see, e.g., [2]).

Since \( J(s) \cdot N_s = 0 \) (because \( J(s) \) is a tangential vector field) and \( J(s) \) is assumed Lipschitz, the last integral in (11) converges absolutely. Since we assume the surface sufficiently smooth, e.g., \( S \in C^{1, a} a > 0 \), and the incident field is smooth, the currents \( j \) and \( J \) are as smooth as the data, in particular, they are Lipschitz. The class of surfaces, satisfying the condition \( S \in C^{1, a} \), consists of surfaces whose graph in local coordinates is differentiable and its derivative satisfies the H"older condition with the exponent \( a \in (0, 1] \).

The operator \( A^+ \) in (11) is defined as

\[
A^+ J = \int_S \frac{\partial}{\partial N_s} g(s, t) J(t) dt.
\]

(12)

Thus, equation (2) can be written as:

\[
-\frac{A^+ J + J}{2} + \int_S \nabla_s G(s, t) N_s \cdot J(t) dt = -\frac{A^- j + j}{2} + \int_S \nabla_s g(s, t) N_s \cdot j(t) dt + [N, E_0],
\]

(13)

where

\[
A^- j = \int_S \frac{\partial}{\partial N_s} g(s, t) j(t) dt.
\]

(14)
Equation (13) is of Fredholm type: the integral operators in (13) are compact in $C(S)$ and in $L^2(S)$. Equation (3) yields:

$$N_s \cdot \int_S [\nabla_x G(x, t), J(t)]_{x \to s, x \in D} dt = N_s \int_S [\nabla_x g(x, t), j(t)]_{x \to s, x \in D'} dt + N_s \cdot E_0.$$  \hfill (15)

Equation (15) is singular.

**Claim:** The integrals in (15) exist as Cauchy principal values.

Let us verify this claim. One has

$$N_s \cdot [\nabla_x G(x, t), J(t)]_{x \to D, x \in D} = N_s \cdot \left[ e^{iK \frac{r_{ts}}{r_{st}}} - \frac{1}{r_{st}} \right] r_{ts}^0, J(s) \right]$$

$$+ O \left( \frac{1}{r_{st}} \right), \quad r_{ts}^0 := \frac{r_{ts}}{r_{st}}, \quad r_{st} = |\vec{r}_{ts}|,$$  \hfill (16)

because $|J(s) - J(t)| \leq c|s - t|$.

The singular term in (16) is

$$\frac{1}{4\pi r_{st}^2} N_s \cdot [r_{ts}^0, J(s)] = \frac{|J(s)|}{4\pi r_{st}^2} \sin \theta,$$  \hfill (17)

where $\theta = \theta(s, t)$ is the angle between the $x$-axis and the vector $\vec{r}_{ts}$. We choose the $x$-axis in the plane tangential to $S$ at the point $s$ so that it is directed along the vector $J(s)$.

Since

$$\int_0^\pi \sin \theta d\phi = 0, \quad t = e^{i\phi}, \quad 0 \leq \phi < 2\pi,$$

the **Claim** follows from Theorem 1.1 on p. 221 in [1]. This theorem says that a singular integral $\int_{\mathbb{R}^m} \frac{f(x, \theta)}{|x - y|^m} u(y) dy$ exists as a Cauchy principal value if

$$\int_{S^{m-1}} f(x, \theta) ds = 0, \quad \theta = \frac{y - x}{|y - x|}.$$

Numerical methods for solving Fredholm equations and singular integral equations are well developed ([1]). They are not discussed here.

**References**

[1] S. Mikhlin, S. Prössdorf, Singular integrals; operators, Springer-Verlag, Berlin, 1986.

[2] C. Müller, Foundations of the mathematical theory of electromagnetic waves, Springer-Verlag, Berlin, 1969.