A Novel Convolution-Based Algorithm for the Acyclic Network Symbolic Reliability Function Problem

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\textsuperset{1}ABSTRACT Acyclic (binary-state) networks are commonly implemented in many diverse disciplines and applications, including information systems, processes management, project management. These networks owe their popularity to the fact that they do not use directed cycles. Network reliability is the most commonly used tool for evaluating and managing systems modeled on acyclic networks, and minimal paths (MPs) play a significant role in evaluating this reliability. This study therefore proposes a new algorithm based on a novel convolution concept for evaluating acyclic network reliability. The proposed algorithm is able to find all convolution-based MP sets within polynomial time, and then obtain the symbolic function of the acyclic network reliability in terms of those convolution-based MP sets based on the pivotal decomposition. Its total time complexity is $O(2^n)$, which is the best among all existing MP algorithms which are at least $O(2^{|P|})$, where $O(|P|) = O(2^n)$, and $n$ and $|P|$ are the number of nodes and MPs, respectively. The correctness and time complexity of the proposed algorithm is proven and examined. An example is used to display the novel convolution-based algorithm is implemented to solve the acyclic network reliability problem.

\textsuperset{2}INDEX TERMS Network reliability, acyclic network, minimal path (MP), convolution, the pivotal decomposition.

\textsuperset{3}ACRONYMS

MP/MC Minimal path/cut.
BFS/DFS The breadth/depth first search

\textsuperset{4}NOTATION

\begin{itemize}
  \item $|\bullet|$ The number of elements in set $\bullet$.
  \item $\Pr(\bullet)$ The occurrence probability of set $\bullet$.
  \item $E$ An arc set.
  \item $V$ $V = \{1, 2, \ldots, n\}$ is a node set.
  \item $e_{ij}$ $e_{ij} \in E$ is an arc between nodes $i$ and $j$.
  \item $a_i$ The $i$th arc in $E$.
  \item $p_i$ $p_i = Pr(a_i)$.
  \item $G = G(V, E)$ An acyclic network with $V, E$, the source node 1, and the sink node $n$, respectively. In addition, all node labels are arranged in such a way that $e_{ij} \in E$ if $i < j$.
\end{itemize}

For example, Fig. 1a is an acyclic network.

$G_i = G(V - \{j|j < i\}, E - \{e_{h,l}\} | < i \text{ and for all } e_{h,l} \in E})$. For example, $G_4$ of the network in Fig. 1a is shown in Fig. 1b.

$G/\{e_{i,j}\} = G(V, E)$ with the exception that node $i$ and arc $e_{i,j}$ are merged into node $j$ and $e_{j,k}$ for all $e_{j,k} \in E$, respectively. For example, $G/\{e_{1,4}\}$ of the network in Fig. 1a is shown in Fig. 1c.

$G - \{e_{i,j}\} = G(V, E - \{e_{i,j}\})$. For example, $G - \{e_{1,4}\}$ of the network in Fig. 1a is shown in Fig. 1d.

$a_{i_1,i_2,\ldots,i_p}$ The new arc after merging arcs $a_{i_k}$ for $k = 1, 2, \ldots, \varphi$. 

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For example, $a_{2,5}$ in Fig. 1c is the new arc by merging $a_2$ and $a_3$ in Fig. 1a.

$m(a_i, a_j)$
The new arc $a_{ij}$ by merging $a_i$ and $a_j$ with new probability $Pr(a_{ij}) = 1 - (1 - p_i)(1 - p_j)$.

$r(a_i, a_j)$
The $a_i$ is replaced with $a_j$.

$Pr(P_k, m(a_i, a_j))$
All arcs $a_i$ in $P_k$ are merged with $a_j$.

$Pr(P_k, r(a_i, a_j))$
All arcs $a_i$ in $P_k$ are replaced with $a_j$.

$[e] \oplus A$
where $A$ is a path set. For example, $[e_{4,5}] \oplus P_5 = \{(e_{4,5}, e_{5,7}), (e_{4,5}, e_{5,6}, e_{6,7})\}$ if $P_5 = \{(e_{5,7}), (e_{5,6}, e_{6,7})\}$.

$E_i$
$E_i = \{e_{ij}\}$ for all $e_{ij} \in E \subseteq E$ for $i = 1, 2, \ldots, n$. For example, $E_2 = \{e_{2,3}, e_{2,4}\}$ in Fig. 1a. Note that $E_n = \emptyset$.

$V_i$
$V_i = \{j \in V \}$, there is an arc from nodes $i$ to $j$ in $E_i \subseteq V$ for $i = 1, 2, \ldots, n$. For example, $V_2 = \{3, 4\}$ in Fig. 1a.

$V_i^*$
$V_i^* = \{j \in V_i | e_{jk} \notin E_j\}$ for all $j, k \in V_i \subseteq V_i$ for $i = 1, 2, \ldots, n$. For example, $V_2^* = \{4\}$ in Fig. 1a.

$P_i$
The convolution-based MP set $P_i = \{e_{ij} \oplus P_j\}$ for all $j \in V_i$.

This will be proved in the set of all MPs from node $i$ to the sink node $n$ in Section 2. Note that $P_1 = P$ and $P_n = \emptyset$ if node 1 is the source node and node $n$ is the sink node.

For example, $P_5 = \{(e_{5,7}), (e_{5,6}, e_{6,7})\}$ since node 7 is the sink node and $(e_{5,7})$ and $(e_{5,6}, e_{6,7})$ are MPs from nodes 5 to 7 in Fig. 1a.

$P_k / \{e_{ij}\}$
The convolution-based MP set in $G_k / \{e_{ij}\}$.

$\deg(i)$
The out degree of node $i$.

**NOMENCLATURE**

**Reliability**
The success probability that there is at least a direct path from nodes 1 to $n$.

**MP/MC**
A special arc subset which is no longer an MP/MC after the removal of any of its arcs. For example, in Fig. 1a, $\{e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}, e_{5,7}\}$ is an MP and $\{e_{1,3}, e_{2,3}, e_{4,5}, e_{4,6}\}$ is an MC between nodes 1 and 7.

**Cycle**
A path from a node, returning to that node, e.g., the path from nodes 4 to 5 to 6, and back to node 4 is a cycle if $e_{4,4}$ exists in Fig. 1a.

**Acyclic network**
A network without any cycle is called an acyclic network.

**Convolution-based MP set**
A convolution-based MP set $P_i = \{e_{ij} \oplus P_j\}$ for all $j \in V_i$.

For example, $P_4 = \{e_{4,1} \oplus P_1\}$ for all $j \in V_4 = \{e_{4,5} \oplus P_5, (e_{4,6} \oplus P_6)\}$, where $P_5 = \{(e_{5,6}) \oplus P_6, (e_{5,7}) \oplus P_7\} = \{(e_{5,6}), (e_{5,7})\}$, and $P_7 = \emptyset$ in Fig. 1a.

**ASSUMPTIONS**
The acyclic network satisfies the following assumptions [16]–[21]:

1. No nodes ever fail, i.e., all nodes are perfectly reliable.
2. Each arc has two states: working or failed.
3. The acyclic network is connected, without parallel arcs, and free of self-loops.

**I. INTRODUCTION**
Digital communication networks have become essential to many diverse systems, such as computer and communication...
systems [1], distributed computing networks [2], [17], power electronic systems [3], oil/gas production systems [4], grid systems [5], repairable systems [6]–[8], human systems [9], the internet of things [10] and wireless sensor networks [11], etc [12]–[15]. All of these can be modeled as networks in system planning, design and control [1]–[13]. Of the many available networks, acyclic networks [16]–[21], [27], [28], [30], [32], [34] have been applied in many fields, including scheduling problems, data processing networks, causal structures, data compression, genealogy and version history, etc. References [16]–[21] primarily because they have no directed cycles, making acyclic networks an important element in our modern society.

Network reliability is measured as the success probability of a network functioning to meet its users’ requirements. For example, the (terminal pair) reliability of an acyclic network is the probability that there is at least one directed path from the source node to the sink node [16]–[21]. References [1]–[57] applied the reliability to evaluate the networks of different systems, and pursued more efficient or accurate solutions with the methods they proposed. Thus, network reliability is one of the most significant tools for the evaluation and validation of modern network performance.

A path is a sequence of arcs connecting a sequence of nodes. Finding the special path discussed above is based on the minimal path (MP). An MP is also an arc subset in network arc activity. It is no longer an MP after the removal of any of its arcs, which is why it is called minimal. In general, MPs play a central role in MP algorithms based on the graph theory in calculating the exact reliability of general networks.

There are two stages in calculating the network reliability based on MPs: 1) find all MPs, 2) calculate the network reliability in terms of MPs using the inclusion-exclusion technique (IET) [22], the inclusion-exclusion technique (SDP) [23]–[26] or the binary decision diagram [27]. The above two stages are both NP-hard problems [13]–[15]. The concept of an MP is intuitive and easier to realize than the others [16]–[36]. This study thus only focuses on the study of MPs and the calculation of acyclic network reliability using MPs.

A variety of tools and approaches have been proposed to find all MPs, including the depth-first search (DFS) [16]–[19], [28]–[31], [36], the universal generating function methods, [32], the algorithm based in the modified network [33], etc. The best-known algorithm for calculating acyclic network reliability was proposed by Yeh using a node-based DFS [16]. Yeh’s algorithm was also the first in the literature that was able to achieve the same time complexity \(O(n \cdot 2^n)\) as the MC based algorithm, which finds all MCs in an acyclic network reliability problem [16]. Note that the time complexity in finding MPs is still far less efficient than that for MCs in general networks [16].

The size of the modern network is increasing from year to year and application of the network reliability is broader and more flexible accordingly [13]–[15]. Hence, it is always a need to develop a more efficient algorithm than existing algorithms to calculate the exact acyclic network reliability [16]. Due to the number of MPs is \(O(2^n)\) in acyclic networks, the related time complexity is \(O(2|P|)\) in calculating the acyclic network reliability in terms of these MPs [16]. Thus, in the traditional MP algorithms, the major time complexity also the most difficult part is not from the searching for all MPs but from the last stage in calculating the reliability using SDP [23]–[26] or IET [22] in terms of MPs.

From the above discussion, if the number of MPs can be reduced, the time complexity will be dramatically improved. However, it is impossible to achieve this, since all MPs are required in the traditional MP algorithms to calculate the reliability at the final stage [16], [19]. According to our knowledge, the time complexity of the existing MP algorithms for finding all MPs in computing acyclic network reliability is all larger than \(O(2^n)\) [16]–[19], [22]–[33], [36]. This study therefore proposes a novel concept called convolution-based MP sets, which include all MPs in a convolution such that the number of the convolution-based MP sets is equal to the number of nodes. Thus, the runtime of the calculation of acyclic network reliability can be improved to \(O(2^n)\), where \(n\) is the number of nodes and also the number of the convolution-based MP sets.

The purpose of this paper is to develop a novel concept called convolution-based MP sets, together a new convolution-based algorithm to search for all convolution-based MP sets in polynomial time, which is much more efficient than existing algorithms which search for all MPs [16], [19]. Since the number of all convolution-based MP sets is only equal to the number of nodes, the runtime in calculating acyclic network reliability based on the pivotal decomposition [58] can be decreased in terms of these convolution-based MP sets.

This remainder of this paper is arranged as follows. The formal discussion of the novel convolution-based MP sets and the special arrangement of node labels are introduced in Section 2, together with a discussion of some important properties of the novel convolution-based MP sets which are helpful in searching for all convolution-base MP sets and in proving the correctness of this novel concept. In Section 3, in order to fully exploit the advantages of the proposed convolution-based MP sets, pivotal decomposition is applied to calculate the acyclic network reliability in terms of convolution-base MP sets. Some important properties of the convolution-based MP sets are also explored in Section 3. Section 4 presents the proposed convolution-based algorithm in detail, along with its time complexity. A numerical step-by-step example is implemented to demonstrate the proposed convolution-based algorithm’s performance in finding all convolution-based MP sets, as well as the calculation of the final reliability using obtained convolution-based MP sets. Concluding remarks on this pilot study are provided in Section 5.
II. NODE LABELS AND CONVOLUTION-BASED MP SETS

A number of useful properties and results are described in this section prior to discussing the proposed algorithm.

An important requirement before implementing the proposed convolution-based algorithm to search for all convolution-based MP sets in acyclic networks is that all node labels must be arranged such that each arc is from a node with a lower label to another node with a higher label. The following property shows that the above requirement can always be established in acyclic networks in polynomial time.

Property 1: In any acyclic network, all node labels can be renumbered with time complexity $O(|n|)$ such that $i < j$ if $e_{i,j} \in E$ for all nodes $i$ and $j$ in $V$.

Proof: It is not significant that the sink node is the one with the largest label. Let $V_i = \{j \in V | i \}$ There is an arc from nodes $i$ to $j$ in $E_i$ and $V^* = \{j \in V | e_{i,k} \notin E_j \}$ for all $j,k \in V_i$ for $i = 1, 2, \ldots, n$. Note that both $V_i$ and $V^*$ are non-empty node subsets since an acyclic network is a connected network for $i = 1, 2, \ldots, n$, i.e., so there is at least a path from node $1$ to any other nodes. Renumber the label of the $k^{th}$ node in $V^*_n$ to be $(n+|\sum_{i=0}^{k-1} V^*_i| - k)$. Repeat the above process until $h = 1$. From the above, all node labels can be renumbered such that $e_{i,j} \notin E$ if $i > j$ for all nodes $i$ and $j$ in $V$. Moreover, each node in the above procedure is only visited twice. Thus, the time complexity is $O(n)$ in the above process.

The following property shows that the characteristic of the node label discussed in Property 1, i.e., that $i < j$ if $e_{i,j} \in E$ for all nodes $i$ and $j$ in $V$, is also held in any sub-MP:

Property 2: If $p$ is a sub-MP from nodes $i$ to $j$, then $i < j$.

Proof: Follows directly from Property 1.

For example, in Fig. 1a, $\{e_{2,3}, e_{3,4}, e_{4,5}, e_{5,7}\}$ is an MP from nodes 2 to 7 and $2 < 7$.

The relationships between the MPs and sub-MPs are explored in the following two properties. These two properties are helpful to prove the correctness of the novel convolution-based MP sets concept discussed in Property 5.

Property 3: If $p^* \in P_i$, $p = (p^* - \{e_{i,j}\}) \in P_j$, where $e_{i,j}$ is the first arc in $p$.

Proof: Any sub-path from nodes $i$ to $k$ in $P$ is still an MP from nodes $i$ to $k$.

For example, in Fig. 1a, $\{e_{2,3}, e_{3,4}, e_{4,5}, e_{5,7}\}$ is an MP from nodes 2 to 7 and $2 < 7$.

The relationships between the MP set and the convolution-based MP set are explored next. This is also the most important property in this study.

Property 4: If $p$ is an MP from nodes $j$ to $n$ and $e_{i,j} \in E$, then $\{e_{i,j}\} \cup p \in P_k$.

Proof: Since $i < j$ and $j < n$ from Properties 1-3, node $i$ is impossible in $p^*$. This property must therefore be true.

For example, let $p \in P_3$ and $p = \{e_{3,4}, e_{4,5}, e_{5,7}\}$ in Fig. 1a. Since $\{e_{2,3}\} \in E$, then $\{e_{2,3}\} \cup \{e_{3,4}, e_{4,5}, e_{5,7}\} = \{e_{2,3}, e_{3,4}, e_{4,5}, e_{5,7}\} \in P_2$.

The relationships between the MP set and the convolution-based MP set are explored next. This is also the most important property in this study.

Property 5: $P_i = \{|e_{i,j}| \cup p | \text{all } j \text{ in } V_i \text{ and all } p \in P_j \}$ for $i = 1, 2, \ldots, n$.

Proof: From Property 3, any $p^* \in P_i$ has $p^* = \{|e_{i,j}| \cup p \}$, where $e_{i,k} \in E_j = \{e_{i,j} | \text{all } e_{i,j} \in E \}$ is the first arc in $p^*$ and $p = (p^* - \{e_{i,k}\}) \in P_i$, i.e., $P_i \subseteq \{|e_{i,j}| \cup p \}$ for all $j \in V_i$ and for all $p \in P_j$. Let $p^* = \{|e_{i,j}| \cup p \}$, where $p \in P_j$. Then $p^* \in P_i$ from Property 4, i.e., $\{|e_{i,j}| \cup p | \text{all } j \text{ in } V_i \text{ and all } p \in P_j \} \subseteq P_i$. Thus, from above, $P_i = \{|e_{i,j}| \cup p \}$ for all $j \in V_i$ and for all $p \in P_j$.

The next property shows that all MPs are included in $P_i$.

Property 6: All MPs can be found in convolution-based MPs.

Proof: $P_i = \{|e_{i,j}| \cup p | \text{all } j \text{ in } V_i \}$ and for all $p \in P_j$.

Duplicate MPs increase computation complexity, and must therefore be removed before calculating network reliability [16], [34], [35]. The following property confirms that there are no duplicate MPs, i.e., no such burden, in the proposed convolution-based MP sets.
Property 7: There are no duplicate MP sets in $P_i$ if there are no duplicates in $E_i = \{e_{ij}\}$ for all $e_{ij} \in E$ for $i = 1, 2, \ldots, n$.

Proof: Since $E_{n-1} = \{e_{n-1,j}\}$ if $e_{n-1,j} \in E$ or $E_{n-1} = \emptyset$ if $e_{n-1,j} \notin E$, then $P_{n-1} = \{\{e_{n-1,j}\} \uplus P_i\}$ for all $e_{n-1,j} \in E_{n-1}$. If $e_{n-1,j} \notin E$ and $P_{n-1} = \emptyset$ if $E_{n-1} = \emptyset$. There are therefore no duplicates in $P_{n-2} = \{\{e_{n-2,j}\} \uplus p\}$ for all $j \in V_{n-2}$ and for all $p \in P_{n-1}$ since there are no duplicates in $P_{n-1}$. Similarly, there can be no duplicates in $P_{n-3} = \{\{e_{n-3,j}\} \uplus p\}$ for all $j \in V_{n-3}$ and for all $p \in P_j$ since there are no duplicates in $P_j$ for all $i$ with $e_{n-3,i} \in E_{n-3}$. Repeating the above procedure, it is concluded that there are no duplicate MP sets in $P_i$ if there are no duplicates in $E_i$ for $i = 1, 2, \ldots, n$, therefore, this property is true.

The time complexity in obtaining the $P_i$ is discussed below:

Property 8: It takes the time complexity $O(\text{deg}(i))$ to find $P_i$ if all $P_j$ are known in advance for all $e_{ij} \in E_i$ and for $i = 1, 2, \ldots, n$.

Proof: Since $P_i = \{\{e_{ij}\} \uplus p\}$ for all $j \in V_i$ and for all $p \in P_j$, $\text{deg}(i) = |\{e_{ij}\} \in E_i|$, and all $P_j$ are known in advance if $e_{ij} \in E_i$.

From Property 8, the following corollary is immediately obtained:

Corollary 1: All convolution-based MP sets are found in $O(|E|)$.

Proof: The total degree of all nodes is $|E|$.

III. THE PROBABILITY OF CONVOLUTION-BASED MPS

Once all convolution-based MP sets are found, the problem is how to calculate the acyclic network reliability using the found convolution-based MP sets. It is a trivial matter to expand all convolution-based MP sets to obtain all MPs, and then calculate the reliability using those MPs, as with traditional MP algorithms. However, it would be pointless to develop the novel convolution concept proposed in this study if it remains necessary to use all MPs to calculate the final reliability. Therefore, this study first explores the probability of each convolution-based MP set, and then uses pivotal decomposition to obtain the symbolic reliability function by fully exploiting the convolution structure of convolution-based MP sets.

The following property is the basis for calculating the probability of the convolution MP sets:

Property 9: $\Pr(\{e_{ij}\} \uplus P_i) = \Pr(\{e_{ij}\}) \cdot \Pr(P_i)$.

Proof: $\{e_{ij}\} \notin P_i$ for all $p \in P_i$.

For example, $\Pr(\{e_{4,5}\} \uplus P_3) = \Pr(\{e_{4,5}\}) \cdot \Pr(P_3)$.

The next property shows the how the network reliability is calculated in terms of the convolution-based MP sets in this study:

Property 10: $\Pr(P_i) = \Pr(\bigcup_{\forall j \in V_i} \{e_{ij}\} \uplus p) = \Pr(G) = \Pr(P_1)$.

Proof: $P_i = \{e_{ij}\} \uplus P_j$ for all $j \in V_i$.

For example, in Fig. 1a, $\Pr(P_4) = \Pr(\{e_{4,5}\} \uplus P_3) \cup \{e_{4,6}\} \uplus P_0$ since $P_4 = \{e_{4,5}\} \uplus P_3 \cup \{e_{4,6}\} \uplus P_0$ since $P_4 = \{e_{4,5}\} \uplus \{e_{4,6}\}$ for all $j \in V_4 = \{e_{4,5}\} \uplus P_3 \cup \{e_{4,6}\} \uplus P_0$, since $P_4 = \{e_{4,5}\} \uplus P_3 \cup \{e_{4,6}\} \uplus P_0$. In addition, $\Pr(P_1) = \Pr(\{e_{1,2}\} \uplus P_2) \cup \{e_{1,3}\} \uplus P_3 \cup \{e_{1,4}\} \uplus P_4)$, and this is the reliability of the network in Fig. 1a.

Next, Boole’s expansion theorem is applied to solve the equation given in Property 10 and fully exploit the convolution-based MP sets. Boole’s expansion theorem is often referred to as pivotal decomposition, and is described below [58]:

**Theorem 1:** $R(G) = \Pr(ak) \cdot R(G/\{ak\}) + (1 - \Pr(ak)) \cdot R(G - \{ak\})$, where $ak = e_{ij}, G - \{ak\}$ is the new network by removing $ak$ from $G(V, E); G/\{ak\}$ is a new network by merging nodes $i$ and $j$ to a new node, say $v, e_{ij}$ and $e_{jk}$ to $e_{v,k}$ if $e_{ij}, e_{jk} \in E, e_{ij}$ and $e_{jk}$ to $e_{v,k}$ if $e_{v,k} \in E$. Then let $\Pr(ek) = 1 - (1 - \Pr(e_{ij})) \cdot (1 - \Pr(e_{jk})) + \Pr(e_{v,k}) = 1 - (1 - \Pr(e_{ij}))$.

Theorem 1 is implemented repeatedly by considering node 1 (or the node merged with node 1) from node 1 to node $i$ with the smallest label with $e_{ij} \in E$. For simplicity, let $a_{1,2,...,10}$ be the new arc after merging $a_{1,2,...,10}$. Before giving an example to explain the pivotal decomposition, the next property is provided in advance since it demonstrates the advantage of convolution MP sets.

**Property 11:** $R(G/\{e_{i+1}\}) = R(G/\{e_{i+1,k}\}$ for all $e_{i+1,k} \in E_{i+1}$ if node $i$ is the node with smallest label.

**Proof:** There is no long path from node $i$ via node $i + 1$ to the other nodes after removing $e_{i,i+1}$ and also there is no node can reach node $i$.

For example, in Fig. 1a, $a_3$ is the first arc to be considered in the pivotal decomposition as it is the arc from node 1 to node 2, node 2 has the smallest label, and $e_{1,2} \in E$. Because $R(G) = \Pr(P_1) = \Pr(\{e_{1,2}\} \uplus P_2) \cup \{e_{1,3}\} \uplus P_3 \cup \{e_{1,4}\} \uplus P_4)$ and $R(G) = \Pr(\{a_3\}) \cdot R(G/\{a_3\}) + (1 - \Pr(\{a_3\})) \cdot R(G - \{a_3\})$. Then $R(G/\{a_3\}) = \Pr(P_2/\{a_3\})$ from Property 11. Thus, $R(G)$ can be reduced to:

$R(G) = \Pr(\{a_3\}) \cdot R(P_2/\{a_3\}) + (1 - \Pr(\{a_3\})) \cdot R(G - \{a_3\})$.

Since $P_2$ is known already before $P_1$, the pivotal decomposition will be implemented to $G - \{a_3\}$ next. The above procedure is repeated until the final reliability is obtained. Note that $G/\{a_3\}$ and $G - \{a_3\}$ are depicted in Fig. 1b and Fig. 1c.

The superior performance of proposed algorithm over existing MP algorithms is demonstrated below.

**Property 12:** The time complexity is $O(2^n)$ to calculate the network reliability using Property 11.

**Proof:** There are at most $n$ arcs to be considered in Property 11, and each arc has only two states.

IV. THE PROPOSED CONVOLUTION-BASED ALGORITHM

The overall procedure for the proposed convolution-based algorithm for generating all convolution-based MP sets and finding the reliability in terms of these found convolution-based MP sets from the source node 1 to the sink node $n$ in acyclic networks is introduced in this section.

A. THE OVERALL PROCEDURE FOR THE PROPOSED CONVOLUTION-BASED ALGORITHM

**Input:** An acyclic network $G(V, E)$ with node set $V$, edge set $E$, source node 1, and sink node $n$. 
Output: All convolution-based MP sets in $G(V, E)$.

STEP 0. Let $P_0 = \emptyset$ and $i = n - 1$.

STEP 1. Find $V_i$.

STEP 2. Let $P_i = \{e_{i,j} \cup p_j \} \cup \{j \in V_i$ and for all $j \in P_i\}$, and calculate the symbolic reliability of $Pr(P_i)$ using pivotal decomposition.

STEP 3. If $i > 0$, let $i = i - 1$ and go to STEP 1. Otherwise, halt, and $P_1$ is the complete MP set.

The correctness and time complexity of the above algorithm are demonstrated below based on the properties discussed in Sections 2 and 3.

Theorem 2: The proposed convolution-based algorithm calculates the acyclic network reliability with time complexity $O(2^n)$.

Proof: It takes $O(n)$ to locate all convolution-based MP sets without duplication from Corollary 1, and $O(2^n)$ to calculate the acyclic network reliability in terms of the found convolution-based MP sets.

A traditional acyclic network needs $O(2^{|P_i|})$ to calculate the reliability after knowing the complete MP set $P_i$, then number of which is equal to $O(2^n)$. Thus, the proposed algorithm outperforms existing algorithms in calculating the acyclic network reliability theoretically.

B. EXAMPLE

For convenience and ease of understanding the proposed algorithm, it is easier to demonstrate the proposed convolution-based algorithm with examples. It is illustrated using the acyclic network shown in Fig. 1a to demonstrate the step-by-step procedure below to find all convolution-based MP sets and calculate the reliability in terms of these convolution-based MP sets.

Solution:

STEP 0. Let $P_7 = \emptyset$ and $i = n - 1 = 6$.

STEP 1. $V_6 = \{7\}$ and $P_7 = \emptyset$, let $P_6 = \{e_{6,7} \cup P_7\} = \{e_{6,7}\}$.

STEP 2. Let $Pr(P_6) = Pr(\{e_{6,7}\}) = p_{12}$.

STEP 3. $i = 6 > 1$, let $i = i - 1 = 5$ and go to STEP 1.

STEP 1. Since $V_5 = \{6,7\}$ and $P_7 = \emptyset$, let $P_5 = \{e_{5,7}, e_{5,6} \cup P_6\}$.

STEP 2. Let $Pr(P_5) = Pr(\{e_{5,7}, e_{5,6} \cup P_6\}) = Pr(\{e_{5,7}\}) + Pr(\{e_{5,6}, e_{6,7}\}) - Pr(\{e_{5,7}, e_{5,6}, e_{6,7}\}) = p_{10} + p_{11} \cdot p_{12} - p_{10} \cdot p_{11} \cdot p_{12}$.

STEP 3. $i = 5 > 1$, let $i = i - 1 = 4$ and go to STEP 1.

STEP 1. Since $V_4 = \{5,6\}$, let $P_4 = \{e_{4,5} \cup P_5, e_{4,6} \cup P_5\}$.

STEP 2. $R(P_4) = Pr(G_4) = Pr(\{e_{4,5}\}) \cdot R(G_4/[e_{4,5}]) + (1 - Pr(\{e_{4,5}\})) \cdot R(G_4 - \{e_{4,5}\}) = Pr(\{e_{4,5}\}) \cdot R(P_5, m(p_9, p_{11})) + (1 - Pr(\{e_{4,5}\})) \cdot Pr(\{e_{4,6}\}) \cdot R(P_6) = p_8 \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_8) \cdot p_{9p12}$.

STEP 3. $i = 4 > 1$, let $i = i - 1 = 3$ and go to STEP 1.

STEP 1. Since $V_3 = \{4,5\}$, let $P_3 = \{e_{3,4} \cup P_4, e_{3,5} \cup P_5\}$.

### TABLE 1. Final Results of the illustrated example and the comparison with MPs.

| $k$ | $\psi(k)$ | $P_k$ | MPs in $P_k$ |
|-----|-----------|-------|-------------|
| 7   | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 6   | $\{7\}$ | $e_{6,7}$ | $e_{6,7}$ |
| 5   | $\{6,7\}$ | $\{e_{6,5}\}$ | $e_{6,5}$ |
| 4   | $\{5,6\}$ | $\{e_{5,4}\}$ | $e_{5,4}$ |
| 3   | $\{4,5\}$ | $\{e_{4,3}\} \cup P_3$ | $\{e_{4,3}\}, e_{4,5}$ |
| 2   | $\{3,4\}$ | $\{e_{3,2}\} \cup P_2$ | $\{e_{3,2}\}, e_{3,4}$ |
| 1   | $\{2,3,4\}$ | $\{e_{2,1}\} \cup P_1$ | $\{e_{2,1}\}, e_{2,3}, e_{2,4}$ |

**STEP 2.** $R(P_3) = Pr(G_3) = Pr(\{e_{3,4}\}) \cdot R(G_3/[e_{3,4}]) + (1 - Pr(\{e_{3,4}\})) \cdot R(G_3 - \{e_{3,4}\}) = Pr(\{e_{3,4}\}) \cdot R(P_4/\{e_{3,4}\}) + (1 - Pr(\{e_{3,4}\})) \cdot Pr(\{e_{3,5}\})R(P_5) = p_8 \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_{7,8}) \cdot p_{9p12} + (1 - p_8) \cdot p_7 \cdot (p_{10} + p_{9,11} \cdot p_{12}) - p_{10} - p_{11} \cdot p_{12}$.

**STEP 3.** $i = 3 > 1$, let $i = i - 1 = 2$ and go to STEP 1.

**STEP 1.** Since $V_2 = \{3,4\}$, let $P_2 = \{\{e_{2,3}\} \cup P_3, \{e_{2,4}\} \cup P_3\}$.

**STEP 2.** $R(P_2) = Pr(G_2) = Pr(\{e_{2,3}\}) \cdot R(G_2/[e_{2,3}]) + (1 - Pr(\{e_{2,3}\})) \cdot R(G_2 - \{e_{2,3}\}) = Pr(\{e_{2,3}\}) \cdot R(P_3, m(p_9, p_{11})) + (1 - Pr(\{e_{2,3}\})) \cdot Pr(\{e_{2,4}\}) \cdot R(P_4) = p_4 \cdot (p_{5,6} \cdot p_{7,8} \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_{7,8}) \cdot p_{9p12} + (1 - p_8) \cdot p_7 \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_8) \cdot p_{9p12}$.

**STEP 3.** $i = 2 > 1$, let $i = i - 1 = 1$ and go to STEP 1.

**STEP 1.** Since $V_1 = \{2,3,4\}$, let $P_1 = \{\{e_{1,2}\} \cup P_2, \{e_{1,3}\} \cup P_3, \{e_{1,4}\} \cup P_4\}$.

**STEP 2.** $R(P_1) = Pr(G_1) = Pr(\{e_{1,2}\}) \cdot R(G_1/[e_{1,2}]) + (1 - Pr(\{e_{1,2}\})) \cdot R(G_1 - \{e_{1,2}\}) = Pr(\{e_{1,2}\}) \cdot R(P_2, m(p_{5,6}, p_{2,5,6})) + (1 - Pr(\{e_{1,2}\})) \cdot Pr(\{e_{1,3}\} \cup P_3, \{e_{1,4}\} \cup P_4), \text{ and Pr}(e_{1,2}) \cdot R(P_2/[e_{1,2}]) = p_3 \cdot (p_{4,5} \cdot p_{5,6} \cdot p_{7,8} \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_{7,8}) \cdot p_{9p12} + (1 - p_8) \cdot p_7 \cdot (p_{10} + p_{9,11} \cdot p_{12} - p_{10} \cdot p_{9,11} \cdot p_{12}) + (1 - p_8) \cdot p_{9p12})$.

**STEP 3.** $i = 1 > 1$, let $i = i - 1 = 0$ and go to STEP 1.
The final results of the proposed algorithm in finding all convolution-based MP sets and the related MPs in each convolution-based MP set are summarized in Table 1. From Table 1, we can observe that the way to represent the results of the convolution-based MP set, \( p_1 \) (column 3), is simpler and more concise than that of the traditional MPs set, MPs in \( p_i \) (column 4).

**V. CONCLUSIONS AND FUTURE WORK**

This pilot study proposes a novel and efficient convolution-based algorithm to calculate acyclic network reliability in terms of convolution-based MP sets.

Convolution-based MP sets is a novel concept which includes all MPs in a convolution manner, which is why it only requires polynomial time to find all convolution-based MP sets. By fully exploiting this convolution structure, pivotal decomposition is able to calculate the acyclic reliability in a more efficient way than traditional methods.

From the discussion and proofs above, the time complexity of the proposed convolution-based algorithm is \( O(n) \) in finding convolution-based MP sets and to calculate the acyclic network reliability using the pivotal decomposition after finding all convolution-based sets. Compared with existing algorithms, whose time complexity is at least \( (m2^n) \) already in finding all MPs before calculating the reliability [16], the proposed algorithm is the first in the literature (to the best of the authors’ knowledge) with lower time complexity, i.e., \( O(2^n) \), in calculating acyclic network reliability.

From a general and theoretical point of view, the proposed convolution-based algorithm is more attractive than existing MP algorithms based on time complexity. Furthermore, by applying an example, the acyclic network shown in Fig. 1a, our proposed convolution-based algorithm is simpler and more concise to represent all convolution-based MP sets and related reliability. Therefore, our proposed convolution-based algorithm has the convinced result and is in a dominant position on time complexity and the way to represent the results.

Future work with the proposed algorithm will have it compete fairly with the other existing MP algorithms using more networks (e.g. the open source data set) for advanced comparisons in order to establish the superiority of the proposed algorithm.

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