First Occurrence of Parity Vectors and the Regular Structure of $k$-Span Predecessor Sets in the Collatz Graph

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Abstract

We study finite paths in the Collatz graph, a directed graph with natural number nodes and where there is an edge from node $x$ to node $T(x) = T_0(x) = x/2$ if $x$ is even, or to node $T(x) = T_1(x) = (3x + 1)/2$ if $x$ is odd. Our first result is an algorithm that, when given a sequence of $n$ parity bits $p = b_0 b_1 \cdots b_{n-1} \in \{0, 1\}^n$, called a parity vector, finds the occurrences of this parity vector in the Collatz graph which are all the paths $o$, of length $n+1$, where the first $n$ nodes of $o$ have exactly the parities given by $p$. In particular, our algorithm can be used to find the first occurrence of such parity vectors $p$ (has smallest integer nodes out of all paths $o$), or indeed the $i$th for any $i \in \mathbb{N}$. In order to give this algorithm, we introduce $E(p)$, the Collatz encoding of a parity vector $p$, and the $(\alpha_{0, -1})$-tree, a binary tree which dictates the structure of first occurrence of parity vectors in the Collatz graph by using modular arithmetic in $\mathbb{Z}/3^i\mathbb{Z}$.

Our main result, which generalizes Colussi [TCS 2011], exploits the properties of first occurrence of parity vectors via their encoding $E(p)$ and the symmetries of the $(\alpha_{0, -1})$-tree in order to highlight some regular structure in the Collatz graph. We ask, informally speaking, what happens when we restrict the usage of the map $T_1$ to exactly $k \in \mathbb{N}$ applications? We show that the $k$-span predecessor set of $x \in \mathbb{N}$ in the Collatz graph, which contains any ancestor $y$ of $x$ that uses exactly $k$ times the map $T_1$ (and any number of times the map $T_0$ in order to reach $x$), can be defined, in binary, by a regular expression $\text{reg}_k(x)$. Hence, we exhibit a general regular structure in the Collatz graph.

Finally, we state three conjectures that are equivalent to the Collatz conjecture and are respectively related to the encoding function $E$, the $(\alpha_{0, -1})$-tree, and the family of regular expressions $\text{reg}_k(1)$.

1 Introduction

Let $\mathbb{N} = \{0, 1, \ldots\}$. The Collatz map,

$$T(x) = \begin{cases} T_0(x) = x/2 & \text{if } x \equiv 0 \mod 2 \\ T_1(x) = (3x + 1)/2 & \text{if } x \equiv 1 \mod 2 \end{cases} \quad \text{for } x \in \mathbb{N}$$

and the Collatz graph, the underlying directed graph generated by $T$ where nodes are all $x \in \mathbb{N}$ and there is an edge from $x$ to $y$ if $y = T(x)$, have been widely studied (see surveys [8] and [9]). The interest in these two objects is driven by a problem, open at least since the 60s: the Collatz conjecture. The conjecture states that the Collatz sequence of any $x > 0$, defined by $c_0 = x$ and $c_{n+1} = T(c_n)$, will, at some point reach 1, i.e., there exists $n_0 \in \mathbb{N}$ with $c_{n_0} = T^{n_0}(x) = 1$. Once the Collatz sequence of $x$ has reached 1, it enters the cycle $(1, 2, 1, 2, \ldots)$ which is the only non-zero cycle in the Collatz graph if the Collatz conjecture is true. As of 2017, the Collatz conjecture had been tested for all natural numbers below $87 \times 2^{60}$ without any counterexample found\footnote{\url{http://www.ERICR.nl/wondrous/}}.

One issue when working with Collatz sequences is that they seem to have very little structure or, more precisely, their structure seems very hard to understand. They are non-monotonic, in fact a given
sequence may contain multiple consecutive subsequences that switch between increasing and decreasing monotonicity and, experimentally, the number of steps one has to wait to reach 1 seems very hard to predict. Despite this apparent lack of structure, the goal of this paper, with our main result, Theorem 4.16, is to exhibit some regularity in the Collatz graph. We show:

**Theorem 4.16.** For all \( x \in \mathbb{N} \), for all \( k \in \mathbb{N} \) there exists a regular expression \( \text{reg}_k(x) \) which defines \( \mathcal{E}\text{Pred}_k(x) \).

Where the set \( \mathcal{E}\text{Pred}_k(x) \) is a binary encoding of \( \text{Pred}_k(x) \), the \( k \)-span predecessor set of \( x \) which is defined as containing all \( y \in \mathbb{N} \) that reach \( x \) by using \( k \) times the operator \( T_y \) and any number of times the operator \( T_0 \) in the Collatz process. The set \( \text{Pred}_k(x) \), which is infinite for all \( k \) as soon as \( x \) is not a multiple of three, also corresponds to predecessors of \( x \) at distance \( k \) in the odd Collatz graph, the underlying graph of an accelerated version of the Collatz process where even steps are ignored. In terms of \( \text{Pred}_k(x) \), the Collatz conjecture reformulates to: \( \forall y \geq 0, \exists k \in \mathbb{N}, y \in \text{Pred}_k(1) \). Our result, by using a different set of tools, generalizes the work of [2] which proved the case \( x = 1 \). Appendix E gives Python code [3] which implements the construction of Theorem 4.16 and automatically generates \( \text{reg}_k(x) \). Appendix E gives \( \text{reg}_1(1) \), which defines \( \mathcal{E}\text{Pred}_1(1) \) and is an example of the kind of regular expressions we can construct thanks to Theorem 4.16.

Parity vectors (see [11, 7, 12]), and their occurrences in the Collatz graph are our main object of study in order to get to Theorem 4.16. A parity vector of size \( n \) describes the parity of the first \( n \) elements of some finite Collatz trajectory of size \( n + 1 \) that we call “an occurrence” of this parity vector. For instance, \((118, 59, 89, 134, 67, 101)\) is an occurrence of the parity vector \((0, 1, 1, 0, 1, 1)\): 118 is even, 59 is odd, 89 is odd etc. In this way, we represent parity vectors with arrows instead of bits: the parity vector \((0, 1, 1, 0, 1, 1)\) becomes \(\downarrow\leftarrow\downarrow\leftarrow\). The first occurrence of this parity vector is \((22, 11, 17, 26, 13, 20)\) because it is the occurrence starting with the smallest integer. In Section 2, we first introduce some notation to work with occurrences of parity vectors: \( \alpha(p) \) is the set of all occurrences of a parity vector \( p \in \mathcal{P} \). Then, we reformulate some known results about parity vectors which will be crucial to our work, especially results given in [12] (Chapter 2). In particular, all parity vectors possess occurrences in the Collatz graph, and the set of occurrences of a parity vector is structured in an arithmetical progression (Theorem 2.10).

In Section 3, we focus on understanding the structure of the first occurrence of parity vectors in the Collatz graph. More precisely, for any parity vector \( p \), we show how the first occurrence of \( p \leftarrow \downarrow \) (the parity vector \( p \) followed by the arrow \( \downarrow \)) and the first occurrence of \( p \leftarrow \) recursively relate to the first occurrence of \( p \). This relation will involve \( \mathcal{E}(p) \), the Collatz encoding of \( p \) (Theorem 3.12) and working in the multiplicative group of \( \mathbb{Z}/3^2\mathbb{Z} \) with the construction of the \((\alpha_0, -1)\)-tree (Theorem 3.20 and Definition 3.22) a binary tree which dictates the structure of first occurrences in the Collatz graph. These two objects will give birth to Algorithm 1 an algorithm which, when inputed a parity vector \( p \) constructs its first (or its \( i^{\text{th}} \)) occurrence. In Section 4, we further exploit the results of Section 3 and the symmetries of the \((\alpha_0, -1)\)-tree in order to prove our main result, Theorem 4.10. Finally, throughout this paper, we give three equivalent reformulations of the Collatz conjecture, Theorem 3.16, Theorem 3.27 and Theorem 4.20 which are respectively related to Collatz encodings \( \mathcal{E}(p) \), the \((\alpha_0, -1)\)-tree and the family of regular expressions \( \text{reg}_k(1) \).

In conclusion, this paper shows that, even though \( k \)-span predecessor sets are typically infinite, they admit computationally simple descriptions as regular expressions. Hence, we have exhibited a general regular structure in the Collatz graph which may help to give a better understanding of Collatz sequences. In future work, we aim at using our main result in order to study the computational power of the Collatz process. Indeed, some generalisations of the Collatz map are known to be able to carry out any Turing machine computation by simulating two-counter machines [5, 6]. However, the Collatz conjecture, by characterizing the long-term behavior of all Collatz trajectories, seems to indicate that it is not possible (with a reasonable encoding) to embed arbitrary Turing computation in the Collatz process. We wish to know if the regular structure of \( k \)-span predecessor sets prevents the Collatz process of carrying such complex computations.

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2Here are the number of steps to wait for the Collatz sequence of \( x \) to reach one for \( 1 \leq x \leq 45 \): \([0, 1, 5, 2, 4, 6, 11, 3, 13, 5, 10, 7, 12, 14, 9, 14, 6, 6, 11, 11, 8, 16, 8, 70, 13, 13, 13, 67, 5, 18, 10, 10, 15, 15, 15, 23, 7, 69, 7, 20, 12] \)

3This code is also available and illustrated here: [https://github.com/tcosmo/CollatzRegular](https://github.com/tcosmo/CollatzRegular)
2 Parity Vectors and Occurrences of Parity Vectors

(a) $p_0 = \epsilon$

(b) $p_1 = \leftarrow\leftarrow\downarrow$

(c) $p_2 = \leftarrow\downarrow\leftarrow\leftarrow\downarrow$

Figure 1: Three examples of parity vectors in $P$

(a) An occurrence of $p_1 = \leftarrow\leftarrow\downarrow$

(b) An occurrence of $p_2 = \leftarrow\downarrow\leftarrow\leftarrow\downarrow$

(c) An occurrence of $p_3 = \downarrow\downarrow\downarrow\leftarrow$

Figure 2: Three occurrences of parity vectors

The concept of parity vector is introduced in [11] (under the name encoding vector) and used, for instance, in [7, 12, 10]. While we work with the same concept, we introduce a slightly different representation of parity vectors by using arrows $\downarrow$ and $\leftarrow$ instead of bits 0 and 1. The arrow $\downarrow$ will correspond to applying $T_0$ and the arrow $\leftarrow$ to applying $T_1$. This is done both because, in this format, parity vectors can be represented nicely in the plane (see Figure 1), and because binary strings will be omnipresent in Section 3 and we don’t want to confuse the reader with too many of them.

Definition 2.1 (Parity Vector). A parity vector is a word $p \in \{\downarrow, \leftarrow\}^*$, i.e. a finite word, possibly empty, over the alphabet $\{\downarrow, \leftarrow\}$. We call $P$ the set of all parity vectors. The empty parity vector is $\epsilon$. We define $\cdot$ to be the concatenation operation on parity vectors: $p = p_1 \cdot p_2$ is the parity vector consisting of the arrows of $p_1$ followed by the arrows of $p_2$. We use exponentiation in its usual meaning: $p^n = p \cdot p \cdots p$ $n$ times.

Example 2.2. Figure 1 shows three parity vectors in $P$: $p_0 = \epsilon$, $p_1 = \leftarrow\leftarrow\downarrow$ and $p_2 = \leftarrow\downarrow\leftarrow\leftarrow\downarrow\downarrow$. We have $p_1 \cdot p_2 = \leftarrow\leftarrow\downarrow\leftarrow\downarrow\leftarrow\downarrow\leftarrow\downarrow\leftarrow\downarrow\leftarrow\downarrow\downarrow$. We have: $||p_0|| = l(p_0) = 0$, $||p_1|| = 3$, $l(p_1) = 1$ and $||p_2|| = 6$, $l(p_2) = 4$.

Definition 2.3 (Norm and span). As in [12], we define two useful metrics on parity vectors:

1. The norm of $p$, $||p||$ is the total number of arrows in $p$.

2. The horizontal span $\text{span}$ of $p$, $l(p)$ is the number of arrows of type $\leftarrow$ in $p$.

Definition 2.4 (Occurrence of a parity vector). Let $p = a_0 \cdots a_{n-1} \in P$ be a parity vector with $a_i \in \{\downarrow, \leftarrow\}$ and $n = ||p||$. An occurrence of $p$ in the Collatz graph, or, for short, an occurrence of $p$, is a $(||p||+1)$-tuple, $(o_0, \ldots, o_{||p||}) \in \mathbb{N}^{||p||+1}$ such that:

$$o_{i+1} = \begin{cases} T_0(a_i) & \text{if } a_i = \downarrow \\ T_1(a_i) & \text{if } a_i = \leftarrow \end{cases}$$

With $0 \leq i < ||p||$. Note that $(o_0, \ldots, o_{||p||})$ is a path with $||p|| + 1$ nodes in the Collatz graph.

Example 2.5. The tuple $(1, 2)$ is an occurrence of the parity vector $p = \leftarrow$. We will say that is the first occurrence of $p$ because 1 is the smallest integer on whose $T_1$ can be applied. We will write $(1, 2) = \alpha_0(p)$, see Definition 2.6.

4Called length in [12]. We change terminology to avoid confusion with the notion of length of a word over an alphabet. However, we keep the same mathematical notation $l(p)$. 

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Definition 2.6 (Set of occurrences of a parity vector: \( \alpha(p) \)). Let \( p \in \mathcal{P} \). We call \( \alpha(p) \) the set of all the occurrences of the parity vector \( p \). We order this set, in increasing order, by the first number of each occurrence. Then, if it exists, \( \alpha_i(p) \in \mathbb{N}^{||p||+1} \) denotes the \( i \)-th occurrence of \( p \) within that order and \( \alpha_{i,j}(p) \) denotes the \( j \)-th term of the \( i \)-th occurrence. Note that if \( \alpha_i(p) \) exists then \( \alpha_{i,j}(p) \) is defined for \( 0 \leq j \leq ||p|| \). In order to facilitate reading, we will write \( \alpha_{i-1}(p) \) instead of \( \alpha_{i,||p||}(p) \) to refer to the last element of the occurrence \( \alpha_i(p) \). Finally, if the context clearly states the parity vector \( p \) we will abuse notation and write \( \alpha_{i,j} \) instead of \( \alpha_{i,j}(p) \).

Definition 2.7 (Feasibility). A parity vector \( p \in \mathcal{P} \) is said to be feasible if it has at least one occurrence, i.e. if \( \alpha_0(p) \) is defined. Furthermore, for \( x \in \mathbb{N} \):

- The parity vector \( p \) is said to be forward feasible for \( x \) if there exists \( i \) such that \( \alpha_{i,0}(p) = x \)
- The parity vector \( p \) is said to be backward feasible for \( x \) if there exists \( i \) such that \( \alpha_{i,-1}(p) = x \)

Lemma 2.8. Let \( p \in \mathcal{P} \) and \( i \in \mathbb{N} \). If the occurrence \( \alpha_i \) exists we have:

\[
\alpha_{i,-1} = T^{||p||}(\alpha_{i,0})
\]

Proof. Immediate from Definition 2.3. This Lemma is very similar to Lemma 2.17 in [12]. \( \square \)

Example 2.9. Figure 2 illustrates the concept of occurrence of a parity vector. Indeed we have:

- For \( p_1 = \leftrightarrow \ldots \ldots \leftrightarrow \), \( \alpha_0 = (3, 5, 8, 4) \), \( \alpha_{0,0} = 3 \), \( \alpha_{0,-1} = 4 \). The parity vector \( p_1 \) is forward feasible for 3 and backward feasible for 4.
- For \( p = \leftrightarrow \ldots \ldots \leftrightarrow \leftrightarrow \), \( \alpha_2 = (137, 206, 103, 155, 233, 350, 175) \), \( \alpha_{2,0} = 137 \), \( \alpha_{2,1} = 206 \), \( \alpha_{2,-1} = 175 \). The parity vector \( p_1 \) is forward feasible for 137 and backward feasible for 175.
- For \( p = \downarrow \downarrow \downarrow \downarrow \leftrightarrow \), \( \alpha_0 = (8, 4, 2, 1, 2) \), \( \alpha_{0,0} = 8 \), \( \alpha_{0,-1} = 2 \). Remark: in that example, the first occurrence of the parity vector \( p = \downarrow \downarrow \downarrow \downarrow \leftrightarrow \) encounters the cycle \((2, 1, 2, \ldots)\). The vocabulary of parity vectors and occurrences allows us to “unfold” cycles.

Theorem 2.10 and Algorithm 1 will give us the right tools to prove the claims of this example.

![Figure 3: Illustration of Theorem 2.10. Structure of the set of occurrences of the parity vector \( p = \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \), we have \( l(p) = 3 \) and \( ||p|| = 5 \). For this parity vector \( p \), we have \( \alpha_{0,0} = 11 \) and \( \alpha_{0,-1} = 10 \). As we can see, \( \alpha(p) \) has a simple structure](image)

The question “Are all parity vectors feasible?” is answered positively in [12] in Lemma 3.1. We reformulate this result in our context:

Theorem 2.10 (All parity vectors are feasible). Let \( p \in \mathcal{P} \). Then:

1. \( p \) is feasible i.e. \( \alpha_0 = (\alpha_{0,0}, \ldots, \alpha_{0,-1}) \in \mathbb{N}^{||p||+1} \) is defined
2. \( \alpha_{0,0} < 2^{||p||} \) and \( \alpha_{0,-1} < 3^l(p) \)
3. Finally we can completely characterize \( \alpha_{i,0} \) and \( \alpha_{i,-1} \) with:

\[
\alpha_{i,0} = 2^{||p||}i + \alpha_{0,0}
\]
\[
\alpha_{i,-1} = 3^l(p)i + \alpha_{0,-1}
\]
Proof. This Lemma is essentially a reformulation of Lemma 3.1 in [12]. We postpone the proof to Appendix A because the concepts which the proof needs (introduced in [12]) will not be used in the rest of this paper.

Remark 2.11. In Appendix B Theorem 5.2 we prove a generalization of Theorem 2.10. Indeed, for 0 ≤ j ≤ ||p||, we have:

\[ \alpha_{i,j}(p) = 2^{|u|} 3^{l(t)} + \alpha_{0,j}(p) \]

With t, u ∈ P uniquely defined by p = t · u and ||t|| = j. Furthermore we also have \( \alpha_{0,j}(p) < 2^{|u|} 3^{l(t)} \).

Theorem 2.10 is the special case where \( j = 0 \) or \( j = ||p|| \).

As a direct consequence of Theorem 2.10 Point 3 we have a simple criterion to determine if \( p \in \mathcal{P} \) is forward (or backward) feasible for \( x \in \mathbb{N} \) or not:

Corollary 2.12. Let \( p \in \mathcal{P} \) and \( x \in \mathbb{N} \). Then:

- \( p \) is forward feasible for \( x \) iff \( x \equiv \alpha_{0,0} \mod 2^{|p|} \)
- \( p \) is backward feasible for \( x \) iff \( x \equiv \alpha_{0,1} \mod 3^{|p|} \)

Proof. Immediate from Theorem 2.10 Point 3.

Example 2.13. Figure 3 illustrates the knowledge that Theorem 2.10 gives on the structure of \( \alpha(p) \), the set of occurrences of the parity vector \( p \).

Remark 2.14. Note that in [10], the author, using the tools of [12], proved that it is enough to prove the Collatz conjecture for any arithmetical progression. In particular, it is enough to prove the Collatz conjecture for \( \alpha_{i,j}(p) \) for all \( i \in \mathbb{N} \), for any \( p \in \mathcal{P} \), for any 0 ≤ j ≤ ||p||.

3 First Occurrence of Parity Vectors

While Theorem 2.10 gives a clear characterization of the structure of the set of occurrences of a parity vector \( p \), it remains quite unclear how to construct \( \alpha_0 \), the first occurrence of \( p \). In [12], through the proof of his Lemma 3.1 (c.f. Lemma A.7), the author offers an analytical way, given \( p \in \mathcal{P} \), to construct an \( x \) which is backward feasible for \( p \). By taking the reminder of \( x \) modulo \( 3^{|p|} \) we can deduce \( \alpha_{0,-1}(p) \) and from there the entire \( \alpha_0 \). However, this analytical method does not give a sense of the relationship between \( \alpha_0(p) \) and \( \alpha_0(p') \) where \( p' \) is a prefix of \( p \). In this Section, we aim at exhibiting this structural relationship which will be crucial to Section 4.

We exploit the “genealogy” of parity vectors in order to construct their \( \alpha_{0,0} \) and their \( \alpha_{0,-1} \). More precisely, in Theorem 3.12 we recursively relate \( \alpha_{0,0}(p \cdot 1) \) and \( \alpha_{0,0}(p \cdot -) \) to \( \alpha_{0,0}(p) \) through \( \mathcal{E}(p) \), the Collatz encoding of the parity vector \( p \). With Theorem 3.16 we reformulate the Collatz conjecture in terms of the encoding function \( \mathcal{E} \).

Similarly, with Theorem 3.20 we will recursively relate \( \alpha_{0,-1}(p \cdot 1) \) and \( \alpha_{0,-1}(p \cdot -) \) to \( \alpha_{0,-1}(p) \). This relation will involve the operators \( T_{0,k} \) and \( T_{1,k} \), defined on \( \mathbb{Z}/3^k \mathbb{Z} \), which are modular arithmetic versions of \( T_0 \) and \( T_1 \). Through the \( (\alpha_{0,-1}) \)-tree, which is the binary tree generated by these two operators, we will be able, in Theorem 3.27, to reformulate the Collatz conjecture in terms of \( \alpha_{0,-1} \). The symmetries of that tree will be at the root of our main result Theorem 4.10 stated in Section 4.2.

Finally, thanks to Theorem 3.12 and Theorem 3.20 we will be able to design Algorithm II an algorithm which constructs step by step \( \alpha_{0,0} \) and \( \alpha_{0,-1} \) and with them \( \alpha_0 \), the first occurrence of any parity vector \( p \).

Before we can start, let’s notice that first occurrence of parity vectors play an important role because any finite fragment of the Collatz sequence of \( x \) ending in 1 is almost always the first occurrence of the underlying parity vector:

Lemma 3.1. Let \( x \in \mathbb{N} \). Let \( (c_0, c_1, \ldots, c_n) = (x, T(x), \ldots, T^n(x)) \) be a finite fragment of the Collatz sequence of \( x \) such that \( c_n = 1 \). Let \( p \in \mathcal{P} \) be the parity vector of which \( (c_0, c_1, \ldots, c_n) \) is an occurrence of. Then, two cases:

- If \( l(p) > 0 \) then \( (c_0, c_1, \ldots, c_n) \) is the first occurrence of \( p \), i.e. \( \alpha_0(p) = (c_0, c_1, \ldots, c_n) \).
— If \( l(p) = 0 \) then \((c_0, c_1, \ldots, c_n)\) is the second occurrence of \( p \), i.e. \( \alpha_1(p) = (c_0, c_1, \ldots, c_n) \).

**Proof.** We know that there is \( p \in \mathcal{P} \) and \( i \in \mathbb{N} \) such that \((c_0, c_1, \ldots, c_n) = (\alpha_{i,0}(p), \alpha_{i,1}(p), \ldots, \alpha_{i,-1}(p))\). Two cases:

— If \( l(p) > 0 \), Theorem 2.10 Equation (2), enforces that \( i = 0 \) because \( c_n = 1 < 3^{l(p)} \).

— If \( l(p) = 0 \) then \( p = (\downarrow)^n \) and \( c_k = 2c_{k+1} \). We deduce \( x = 2^n \). Since \( 2^n \leq x < 2^{n+1} \), Theorem 2.10 Equation (1), enforces that \( i = 1 \).

\[\]

### 3.1 Constructing \( \alpha_{0,0} \) and the encoding function \( \mathcal{E} \)

The construction of \( \alpha_{0,0} \) relies on an observation similar to Corollary 1.3 of [11].

**Lemma 3.2.** Define \( \mathcal{P}_n = \{ p \in \mathcal{P} \text{ with } ||p|| = n \} \). Then the function \( f : \mathcal{P}_n \to \{0, \ldots, 2^n - 1\} \) defined by \( f(p) = \alpha_{0,0}(p) \) is a bijection.

**Proof.** By cardinality, because \( |\mathcal{P}_n| = |\{0, \ldots, 2^n - 1\}| = 2^n \), we just have to prove the injectivity of \( f \). Let \( p_1, p_2 \in \mathcal{P}_n \) such that \( f(p_1) = f(p_2) \). We write \( p_1 = a_0 \cdot \ldots \cdot a_{n-1} \) and \( p_2 = a'_0 \cdot \ldots \cdot a'_{n-1} \) with \( a_i, a'_i \in \{\downarrow, \uparrow\} \). Since \( \alpha_{0,0}(p_1) = \alpha_{0,0}(p_2) \) and that the Collatz process is deterministic we deduce:

\[
\begin{align*}
\alpha_{0,0}(p_1) &= \alpha_{0,0}(p_2) \\
\alpha_{0,1}(p_1) &= \alpha_{0,1}(p_2) \\
& \quad \vdots \\
\alpha_{0,-1}(p_1) &= \alpha_{0,-1}(p_2)
\end{align*}
\]

Thus, by Definition 2.4 we deduce that \( a_i = a'_i \) for \( 0 \leq i < n \). Thus \( p_1 = p_2 \) which ends the proof.

\[\]

In the following, with Theorem 3.12 we will gain a better understanding of the structure of the bijection \( f \) of Lemma 3.2 which will allow us to recursively construct \( \alpha_{0,0} \). This will involve looking at \( \alpha_{0,0} \) in base two and we need to introduce some notation:

**Definition 3.3 (The set \( B^* \)).** Let \( B^* \) be the set of finite (possibly empty) words written on the alphabet \( B = \{\downarrow, \uparrow\} \). We define \( \cdot \), the concatenation operator on these words and \( (B^*, \cdot) \) forms a monoid of which neutral element, the empty word, is denoted by \( \eta \). For \( \omega \in B^* \), \( |\omega| \) refers to the length (number of symbols) in the binary word \( \omega \).

**Definition 3.4 (The interpretation \( \mathcal{I} \) and \( \mathcal{I}^{-1} \)).** Each word \( \omega \in B^* \) can naturally be interpreted as the binary representation of a number in \( \mathbb{N} \). The function \( \mathcal{I} : B^* \to \mathbb{N} \) gives this interpretation. By convention, \( \mathcal{I}(\eta) = 0 \). Reciprocally, the partial function \( \mathcal{I}^{-1} : \mathbb{N} \to B^*_n \), where \( B^*_n \) is the set of \( \omega \in B^* \) with \( |\omega| = n \), gives the binary representation of \( x \in \mathbb{N} \) on \( n \) bits. The value of \( \mathcal{I}^{-1}(x) \) is defined only when \( n \geq \lceil \log_2(2x + 1) \rceil \). We set \( \mathcal{I}^{-1}(0) = \eta \). Finally, by \( \mathcal{I}^{-1}(x) \) we refer to the binary representation of \( x \in \mathbb{N} \) without any leading \( 0 \). Formally, \( \mathcal{I}^{-1}(x) = \mathcal{I}^{-1}_{\lceil \log_2(x) \rceil + 1}(x) \) if \( x \neq 0 \) and \( \mathcal{I}^{-1}(0) = \mathcal{I}^{-1}(0) = 0 \).

**Example 3.5.** We have \( \mathcal{I}(11) = \mathcal{I}(0011) = 3 \). We have \( \mathcal{I}^{-1}(3) = \mathcal{I}^{-1}_{2}(3) = 11 \) and \( \mathcal{I}^{-1}_{3}(3) = 000011 \).

We can now define the **Collatz encoding** of a parity vector \( p \in \mathcal{P} \):

**Definition 3.6 (Collatz encoding of a parity vector \( p \)).** We define \( \mathcal{E} : \mathcal{P} \to B^* \) the Collatz encoding function of parity vectors given by:

\[
\mathcal{E}(p) = \mathcal{I}^{-1}_{||p||}(\alpha_{0,0}(p))
\]

The function \( \mathcal{E} \) is well defined since \( \alpha_{0,0}(p) \) can be expressed, in binary, on \( ||p|| \) bits because \( \alpha_{0,0}(p) < 2^{||p||} \) by Theorem 2.10. The function \( \mathcal{E}^{-1} : B^* \to \mathcal{P} \) is naturally defined. Note that we have \( \mathcal{E}(\mathcal{P}_n) = B^*_n \).
Remark 3.7. $\mathcal{E}(p)$ is nothing more than the binary representation of $\alpha_{0,0}(p)$ on $||p||$ bits.

Lemma 3.8. For $p \in \mathcal{P}$, we have: $I(\mathcal{E}(p)) = \alpha_{0,0}(p)$ and $T(||p||)(I(\mathcal{E}(p))) = \alpha_{0,-1}(p)$.

Proof. Immediate from Definition 3.9 and Lemma 2.8.

Example 3.9. As outlined in Figure 11, we have for instance: $\mathcal{E}(\downarrow\downarrow) = 00$ or $\mathcal{E}(\downarrow\downarrow) = 101$.

We need one last concept and Lemma 3.11 before we can state Theorem 3.12, which recursively characterizes $\mathcal{E}$:

Definition 3.10 (Admissibility of an arrow). Let $a \in \{\downarrow, \leftarrow\}$. The arrow $a$ is said to be admissible for the number $x$ if and only if: $(a = \downarrow$ and $x$ is even) or $(a = \leftarrow$ and $x$ is odd).

Lemma 3.11. Let $p \in \mathcal{P}$ and $a \in \{\downarrow, \leftarrow\}$. Consider $\alpha_0(p \cdot a) = (\alpha_{0,0}(p \cdot a), \ldots, \alpha_{0,||p\cdot a||}(p \cdot a))$. Then two cases:

1. If $a$ is admissible for $\alpha_{0,-1}(p)$ then $(\alpha_{0,0}(p \cdot a), \ldots, \alpha_{0,||p||}(p \cdot a))$ is the first occurrence of $p$, i.e. we have: $\alpha_0(p) = (\alpha_{0,0}(p \cdot a), \ldots, \alpha_{0,||p||}(p \cdot a))$.

2. If $a$ is not admissible for $\alpha_{0,-1}(p)$, then $(\alpha_{0,0}(p \cdot a), \ldots, \alpha_{0,||p||}(p \cdot a))$ is the second occurrence of $p$, i.e. we have: $\alpha_1(p) = (\alpha_{0,0}(p \cdot a), \ldots, \alpha_{0,||p||}(p \cdot a))$.

Proof.

Theorem 3.12 (Recursive structure of $\mathcal{E}$). We have:

1. $\mathcal{E}(\epsilon) = \eta$

2. Let $p \in \mathcal{P}_n$ for some $n \in \mathbb{N}$ and $a \in \{\downarrow, \leftarrow\}$. We have:

$$\mathcal{E}(p \cdot a) = \begin{cases} 0 \cdot \mathcal{E}(p) & \text{if } a \text{ is admissible for } \alpha_{0,-1}(p) \\ 1 \cdot \mathcal{E}(p) & \text{if } a \text{ is not admissible for } \alpha_{0,-1}(p) \end{cases}$$

Proof. By Definition 3.6, we have $\mathcal{E}(\epsilon) = I_0^{-1}(\alpha_{0,0}(\epsilon)) = I_0^{-1}(0)$ and $I_0^{-1}(0) = \eta$ by Definition 3.3. Hence, $\mathcal{E}(\epsilon) = \eta$.

Now, let $p \in \mathcal{P}_n$, $a \in \{\downarrow, \leftarrow\}$. Two cases:

1. If $a$ is admissible for $\alpha_{0,-1}(p)$, by Lemma 3.11, we have $\alpha_{0,0}(p \cdot a) = \alpha_{0,0}(p)$. Thus, we get that $I_{n+1}^{-1}(\alpha_{0,0}(p \cdot a)) = 0 \cdot I_{n}^{-1}(\alpha_{0,0}(p))$ since prepending a 0 to a binary string doesn’t change the number it represents. Hence, $\mathcal{E}(p \cdot a) = 0 \cdot \mathcal{E}(p)$.

2. If $a$ is not admissible for $\alpha_{0,-1}(p)$, by Lemma 3.11 and Theorem 2.10, we get $\alpha_{0,0}(p \cdot a) = \alpha_{1,0}(p) = 2^{||p||} + \alpha_{0,0}(p)$ which corresponds to prepending a bit 1 to the binary representation of $\alpha_{0,0}(p)$ on $n$ bits. We conclude that $I_{n+1}^{-1}(\alpha_{0,0}(p \cdot a)) = 1 \cdot I_{n}^{-1}(\alpha_{0,0}(p))$. Hence, $\mathcal{E}(p \cdot a) = 1 \cdot \mathcal{E}(p)$. 

$\square$
We know that \( \alpha \) as given by \( \text{arrow} \) a by an admissible arrow (in green) then by looking by parity vectors followed by a non admissible arrow. From Theorem 3.12 we deduce that a number.

**Remark 3.14.** From Theorem 3.12 we deduce that a number \( x \), through its binary representation \( \mathcal{I}^{-1}(x) \), gives an encoding for exactly one parity vectors of norm \( \lfloor \log_2(x) \rfloor + 1 \) (or norm 1 if \( x = 0 \)).

Thanks to Theorem 3.12 we give Corollary 3.15 which shows that some of the encodings given by \( \mathcal{E} \) are easy to decode:

**Corollary 3.15.** We have the following, for all \( n \in \mathbb{N} \):

1. \( \mathcal{E}^{-1}(0^n) = (\downarrow)^n \)
2. \( \mathcal{E}^{-1}(10^n) = (\downarrow)^n \leftarrow \)
3. \( \mathcal{E}^{-1}(10^{2n+1}) = \leftarrow (\downarrow)^{n+1} \downarrow \)
4. \( \mathcal{E}^{-1}(01^{n+1}) = \downarrow \downarrow (\downarrow)^{2n} \)
5. \( \mathcal{E}^{-1}(1^n) = (\leftarrow)^n \)

These statements allow to predict the first \( \lfloor \log_2(x) \rfloor + 1 \) arrows of the parity vector taken along the Collatz sequence of any \( x \in \mathbb{N} \) of which binary representation matches one of the different given expressions.

**Proof.** We prove Point \( 1 \) the other points are proven in Appendix \( C \). We proceed by induction on \( n \). We know that \( \mathcal{E}(\epsilon) = \eta \) (Theorem 3.12) hence \( \mathcal{E}^{-1}(0^n) = (\downarrow)^n \). Now, let’s suppose that \( \mathcal{E}^{-1}(0^n) = (\downarrow)^n \) for some \( n \in \mathbb{N} \). Let \( p = (\downarrow)^n \). By Lemma 3.8 \( \alpha_{0,0}(p) = \mathcal{I}(\mathcal{E}(p)) \). By induction hypothesis, \( \mathcal{I}(\mathcal{E}(p)) = \mathcal{I}(0^n) = 0 \). Hence, \( \alpha_{0,0}(p) = 0 \). By Lemma 2.3 we have \( \alpha_{0,0}(p) = T^n(\alpha_{0,0}(p)) = T^n(0) = 0 \). Hence, the arrow \( \downarrow \) is admissible for \( \alpha_{0,0}(p) = 0 \) and by Theorem 3.12 we deduce that \( \mathcal{E}(p \cdot \downarrow) = 0 \cdot \mathcal{E}(p) = 0^{n+1} \). Hence \( \mathcal{E}^{-1}(0^{n+1}) = (\downarrow)^{n+1} \) and we have the result.

---

Figure 4: Illustration of Theorem 3.12. Parity vectors \( p \) in \( \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \) ordered by \( \mathcal{E}(p) \), the binary representation of \( \alpha_{0,0}(p) \) on \( ||p|| \) bits. The number \( \alpha_{0,0} \) is written as a binary word, on \( n \) bits for \( \mathcal{P}_n \), as given by \( \mathcal{E}(p) \). Green arrows are admissible, brown arrows are non admissible. For instance, the last arrow \( a = \downarrow \) of \( p = \downarrow \downarrow \downarrow \downarrow \) is admissible because we read \( \alpha_{0,0}(\downarrow \downarrow \downarrow \downarrow) = 2 \) is even and \( a = \downarrow \). Numbers \( \alpha_{0,0} \) are written in base 10 and determine how we decide whether an arrow is admissible or not.
Finally, we can reformulate the Collatz conjecture in terms of the encoding function $E$:

**Theorem 3.16** (Collatz conjecture in terms of $\alpha_{0,0}$). The Collatz conjecture is equivalent to:

$$\forall \omega \in B^* \quad I(\omega) \neq 0 \Rightarrow \exists n \in \mathbb{N}, \, \alpha_{0,-1}(E^{-1}(0^n \cdot \omega)) = 1$$

**Proof.** Theorem 3.12 shows that keeping adding admissible arrows to a parity vector, here $E^{-1}(\omega)$, corresponds to prepending $0$s to the encoding given by $E$ while following the Collatz trajectory of its $\alpha_{0,0} = I(\omega)$. Hence, the Collatz conjecture is true if we ever reach one by doing so. Finally, Lemma 3.11 guarantees that looking at $\alpha_{0,-1}$ is enough and not $\alpha_{1,-1}$ for some $i$.  

### 3.2 Constructing $\alpha_{0,-1}$ and the $(\alpha_{0,-1})$-tree

The recursive construction of $\alpha_{0,0}$, given by Theorem 3.12, relies on knowing $\alpha_{0,-1}$ in order to decide the admissibility of the arrow which is being added at each step. We could of course compute the first $|p|\cdot a$ steps of the Collatz trajectory of $\alpha_{0,0}(p)$ to get to $\alpha_{0,-1}(p)$. However, in this Section we exhibit a recursive construction of $\alpha_{0,-1}$ which will be fruitful in consequences. More precisely, in Theorem 3.20 we show how $\alpha_{0,-1}(p \cdot a)$ relates to $\alpha_{0,-1}(p)$ with $a \in \{1, -1\}$. Then, we introduce the $(\alpha_{0,-1})$-tree, a binary tree driving the construction of $\alpha_{0,-1}$. The symmetries of the $(\alpha_{0,-1})$-tree will be at the root of the main result (Theorem 4.16).

The construction of $\alpha_{0,-1}$ involves working with the groups $\mathbb{Z}/3^k\mathbb{Z}$, $(\mathbb{Z}/3^k\mathbb{Z})^*$, and the operators $T_{0,k}$ and $T_{1,k}$ (see Definition 3.18). We first start by recalling some known facts about $\mathbb{Z}/3^k\mathbb{Z}$.

**Working with $\mathbb{Z}/3^k\mathbb{Z}$.** In the following, we identify $\mathbb{Z}/3^k\mathbb{Z}$ to $\{0, \ldots, 3^k - 1\} \subset \mathbb{N}$. This means that when we consider $x \in \mathbb{Z}/3^k\mathbb{Z}$ we are considering the smallest integer $n$ such that $n \equiv x \mod 3^k$ and thus we implicitly have $x < 3^k$. By $(\mathbb{Z}/3^k\mathbb{Z})^*$ we refer to the multiplicative group of $\mathbb{Z}/3^k\mathbb{Z}$, i.e.

$$(\mathbb{Z}/3^k\mathbb{Z})^* = \{ x \in \mathbb{Z}/3^k\mathbb{Z} \mid \exists y, xy \equiv 1 \mod 3^k \}.$$ 

From elementary group theory results we can deduce that

$$(\mathbb{Z}/3^k\mathbb{Z})^* = \{ x \mid x < 3^k \text{ and } x \text{ is not a multiple of } 3 \}.$$ 

The element $2$ is thus always invertible in $\mathbb{Z}/3^k\mathbb{Z}$. By $2^{-1}_k$ we refer to the modular inverse of $2$ in $\mathbb{Z}/3^k\mathbb{Z}$, this means that in $\mathbb{Z}/3^k\mathbb{Z}$ we have $2 \cdot 2^{-1}_k = 1$. Furthermore, it is known that $2^{-1}_k$ is a primitive root of $(\mathbb{Z}/3^k\mathbb{Z})^*$. This means that for all $x \in (\mathbb{Z}/3^k\mathbb{Z})^*$ there exists $n \in \mathbb{N}$ such that $x \equiv (2^{-1}_k)^n \equiv 2^{-n}_k \mod 3^k$. Finally, even though $(\mathbb{Z}/3^k\mathbb{Z})^* = \emptyset$ it will be useful, for the induction step of our main result (Theorem 4.16), to take the convention $(2^{-1}_k)^0 = 2^0 = 1$.

By Theorem 2.10 we know that $\alpha_{0,-1}(p) < 3^l(p)$ which means that $\alpha_{0,-1}(p)$ can always be seen as an element of $\mathbb{Z}/3^l(p)\mathbb{Z}$. With the next Lemma, we see that it is always also an element of $(\mathbb{Z}/3^l(p)\mathbb{Z})^*$:

**Lemma 3.17.** Let $p \in \mathcal{P}$. Then $l(p) \neq 0 \Leftrightarrow \alpha_{0,-1}(p) \in (\mathbb{Z}/3^l(p)\mathbb{Z})^*$. If $l(p) = 0$ then $\alpha_{0,-1}(p) = 0$.

**Proof.** We prove both directions:

— $\Rightarrow$: we suppose $l(p) \neq 0$. We know $\alpha_{0,-1}(p) < 3^l(p)$ (Theorem 2.10). We have to prove that $\alpha_{0,-1}(p)$ is not a multiple of three. The predecessor set of $y$, a multiple of 3, in the Collatz graph is reduced to $2^n y$ for $n \in \mathbb{N}$. Indeed, we know that all $2^n y$ are predecessors of $y$ by the operator $T_0$. Furthermore, the operator $T^{-1}_1(y) = (2y - 1)/3$ never yields to an integer if inputed a multiple of three and all $2^n y$ are. Hence no parity vector $p$ with $l(p) > 0$ can have an occurrence ending in a multiple of three and we have the result.

— $\Leftarrow$: if $l(p) = 0$ then $(\mathbb{Z}/3^l(p)\mathbb{Z})^* = \emptyset$ so we have the result.

If $l(p) = 0$ then $p$ has the form $p = 4^n$. By Corollary 3.15 Point 1 we have $\alpha_{0,0}(p) = 0$. Hence $\alpha_{0,-1}(p) = T^n(0) = 0$.  

Theorem 3.20 will be based on the use of two operators, $T_{0,k}$ and $T_{1,k}$, which are the equivalent of $T_0$ and $T_1$ in $\mathbb{Z}/3^k\mathbb{Z}$. We define them:

**Definition 3.18** ($T_{0,k}$ and $T_{1,k}$). The functions $T_{0,k} : \mathbb{Z}/3^k\mathbb{Z} \to \mathbb{Z}/3^k\mathbb{Z}$ and $T_{1,k} : \mathbb{Z}/3^k\mathbb{Z} \to \mathbb{Z}/3^k\mathbb{Z}$ are defined by:

$$T_{0,k}(x) = 2^{-1}_k x \quad T_{1,k}(x) = 2^{-1}_k (3x + 1)$$

In Appendix 13 we characterize the structure of $T_{0,k}$ and $T_{1,k}$. These characterizations will be important in the proof of Theorem 4.20.
Example 3.19. We have $2^{-1} = 5$ because $5 \cdot 2 \equiv 1 \text{ mod } 3^2$. Hence we have $T_{0,2}(3) = 2^{-1}3 = (5 \cdot 3 \text{ mod } 3^2) = 6$. We also have $T_{1,2}(4) = 2^{-1}(3 \cdot 4 + 1) = (3 \cdot 2 + 5 \text{ mod } 3^2) = 2$.

Theorem 3.20 (Recursive structure of $\alpha_{0,1}$). We have:

- $\alpha_{0,1}(\epsilon) = 0$
- Let $p \in P_n$ for some $n$ and $k = l(p)$. We have:

\begin{align*}
\alpha_{0,1}(p \cdot \downarrow) &= T_{0,k}(\alpha_{0,1}(p)) \quad (3) \\
\alpha_{0,1}(p \cdot \leftarrow) &= T_{1,k+1}(\alpha_{0,1}(p)) \quad (4)
\end{align*}

Proof. Since any $x \in \mathbb{N}$ is an occurrence of the parity vector $\epsilon$, we have $\alpha_0(\epsilon) = (0, \ldots)$ (tuple with one element). Hence $\alpha_{0,1}(\epsilon) = \alpha_{0,0}|_{\epsilon}|(\epsilon) = \alpha_{0,0}(\epsilon) = 0$. Now, let $p \in P_n$ for some $n \in \mathbb{N}$ and $k = l(p)$. Then notice that Equations (3) and (4) are well defined because of Theorem 2.10. Indeed, we know that $\alpha_{0,1}(p) < 3^k(p) = 3^k$ thus $\alpha_{0,1}(p) \in \mathbb{Z}/3^k\mathbb{Z}$ and $\alpha_{0,1}(p) \in \mathbb{Z}/3^k+1\mathbb{Z}$ and we can use the operators $T_{0,k}$ and $T_{0,k+1}$ on it.

Let’s consider $\alpha_{0,1}(p \cdot a)$ with $a \in \{\downarrow, \leftarrow\}$. Two cases:

- The arrow $a$ is admissible for $\alpha_{0,1}(p)$: in that case, by Lemma 3.11 we know that $(\alpha_{0,1}(p \cdot a), \alpha_{0,1}(p \cdot a), \ldots)_{(\alpha_{0,1}(p \cdot a))}$ is the first occurrence of $p$. Hence, $\alpha_{0,1}(p \cdot a) = \alpha_{0,1}(p \cdot a)$ and we have $\alpha_{0,1}(p \cdot a) = T_{0,0}(\alpha_{0,1}(p \cdot a)) = T_{0,1}(\alpha_{0,1}(p \cdot a))$. With $i = 0$ if $a = \downarrow$ or $i = 1$ if $a = \leftarrow$. Then two cases:

  1. If $a = \downarrow$ then $\alpha_{0,1}(p \cdot a)$ is even and $\alpha_{0,1}(p \cdot a) = T_{0,0}(\alpha_{0,1}(p \cdot a)) = T_{0,1}(\alpha_{0,1}(p \cdot a))$ by Lemma 3.12
  2. If $a = \leftarrow$ then $\alpha_{0,1}(p \cdot a)$ is odd and $\alpha_{0,1}(p \cdot a) = T_{1,0}(\alpha_{0,1}(p \cdot a)) = T_{1,1}(\alpha_{0,1}(p \cdot a))$ by Lemma 3.13

- The arrow $a$ is not admissible for $\alpha_{0,1}(p)$: in that case, by Lemma 3.11 we know that $(\alpha_{0,1}(p \cdot a), \alpha_{0,1}(p \cdot a), \ldots)_{(\alpha_{0,1}(p \cdot a))}$ is the second occurrence of $p$. Hence, by Theorem 2.10 $\alpha_{0,1}(p \cdot a) = 3^k + \alpha_{0,1}(p \cdot a)$. Now, $\alpha_{0,1}(p \cdot a) = T_{0,0}(\alpha_{0,1}(p \cdot a)) = T(3^k + \alpha_{0,1}(p \cdot a))$. Then two cases:

  1. If $a = \downarrow$ then $\alpha_{0,1}(p \cdot a)$ is odd and $\alpha_{0,1}(p \cdot a) = T_{0,0}(3^k + \alpha_{0,1}(p \cdot a)) = \frac{3^k + \alpha_{0,1}(p \cdot a)}{2}$ by Lemma 3.12
  2. If $a = \leftarrow$ then $\alpha_{0,1}(p \cdot a)$ is even and $\alpha_{0,1}(p \cdot a) = T_{1,0}(3^k + \alpha_{0,1}(p \cdot a)) = \frac{3^k + 3\alpha_{0,1}(p \cdot a)}{2} = T_{1,1}(\alpha_{0,1}(p \cdot a))$ by Lemma 3.13.

In all the cases we get the result.

Example 3.21. On Figure 3 we are reading $\alpha_{0,1}(\downarrow \leftarrow) = 8$. On the other hand, Theorem 3.20 claims that $\alpha_{0,1}(\downarrow \leftarrow) = T_{1,2}(\alpha_{0,1}(\downarrow \leftarrow)) = T_{1,2}(2)$. Let’s verify that: $T_{1,2}(2) = 2^{-1}(3 \cdot 2 + 1) = 3 + 2^{-1} = 3 + \frac{3 + 2}{2} = 3 + 5 = 8$ as expected.

The $(\alpha_{0,1})$-tree. Theorem 3.20 implies that the operators $T_{0,k}$ and $T_{1,k}$ naturally give birth to a binary tree ruling the construction of $\alpha_{0,1}$. We call this tree the $(\alpha_{0,1})$-tree:

Definition 3.22 (The $(\alpha_{0,1})$-tree). We call the $(\alpha_{0,1})$-tree the binary tree with nodes in $\mathcal{N} \subset (\mathcal{P} \times \mathcal{N} \times \mathcal{N})$ constructed as follow, starting from node $x = (\epsilon, 0, 0)$:

1. The right son of $(p, x, k)$ is $((p \cdot \downarrow), T_{0,k}(x), k)$
2. The left son of $(p, x, k)$ is given by $((p \cdot \leftarrow), T_{1,k+1}(x), k + 1)$

Lemma 3.23. The set of nodes of the $(\alpha_{0,1})$-tree is given by: $\mathcal{N} = \{ (p, \alpha_{0,1}(p), l(p)) \text{ for } p \in \mathcal{P} \}$.

Proof. Immediate from Definition 3.22 and Theorem 3.20.
\[(\epsilon, 0, 0)^* \quad \cdots \quad (\epsilon, 2, 1)^* \quad (\downarrow, 0, 0)^* \]

\[
\downarrow\quad (\leftrightarrow, 8, 2) \quad \downarrow\quad (\leftrightarrow, 2, 2) \quad \downarrow\quad (\downarrow, 2, 1)^* \\
\downarrow\quad (\leftrightarrow, 26, 3) \quad \downarrow\quad (\downarrow, 4, 2) \quad (\downarrow, 1, 1) 
\]

Figure 5: First 4 levels of the \((\alpha_{0, -1})\)-tree. Two symmetries are highlighted by \(*\) and \(\star\)

**Structure of the \((\alpha_{0, -1})\)-tree.** Some elementary structure of the \((\alpha_{0, -1})\)-tree can be highlighted:

**Lemma 3.24** \((\alpha_{0, -1}((\leftarrow)^k))\). Considering only left sons from \((\epsilon, 0, 0)\) leads to the family of nodes \(((\leftarrow)^k, 3^k - 1, k) \in \mathcal{N}\) with \(k \in \mathbb{N}\). In other words, \(\alpha_{0, -1}((\leftarrow)^k) = 3^k - 1\) for \(k \in \mathbb{N}\).

**Proof.** By induction:

— **Base.** For \(k = 0\), by Theorem 3.20 we have \(\alpha_{0, -1}(\epsilon) = 3^0 - 1 = 0\).

— **Induction.** By Lemma 3.23 we have \(((\leftarrow)^{k+1}, x, k+1) \in \mathcal{N}\) with \(x = T_{1,k+1}(\alpha_{0, -1}((\leftarrow)^k))\). By induction hypothesis: \(x = T_{1,k+1}(3^k - 1) = 2^{k+1}(3^{k+1} - 2) = -2 \cdot 2^{k} = -1 = 3^{k+1} - 1\).

**Remark 3.25.** More generally, it can be proven than the first occurrence of \((\leftarrow)^k\), with \(k \in \mathbb{N}\) is:

\[
\alpha_0((\leftarrow)^k) = (2k^0 - 1, 2^j - 1, \ldots, 2^1 3^{k-1} - 1, 2^0 3^k - 1) 
\]

In other words, \(\alpha_{0, j}((\leftarrow)^k) = 2^j - 1 \cdot 3^j - 1\) for \(0 \leq j \leq \|((\leftarrow)^k\| = l((\leftarrow)^k) = k\).

**Lemma 3.26.** Let \(x \in \mathbb{N}\). We have the following equivalence:

\[
x = 0 \text{ or } (\exists k > 0, x \in (\mathbb{Z}/3^k\mathbb{Z})^*) \Leftrightarrow \exists p \in \mathcal{P}, (p, x, l(p)) \in \mathcal{N} 
\]

**Proof.** We prove both directions:

— \(\Rightarrow\): We can construct such a \(p\). If \(x = 0\) take \(p = \epsilon\). Let’s suppose \(x \in (\mathbb{Z}/3^k\mathbb{Z})^*\) with \(k > 0\). Then there is \(i_0 \in \mathbb{N}\) such that \(x = (2k^{i_0})^0 = 2k^{i_0}\). Now, by Lemma 3.21 we have that \((p_1, 3^k - 1, k)\), with \(p_1 = \leftarrow^k\), is a node of the \((\alpha_{0, -1})\)-tree. Since \(3^k - 1 \in (\mathbb{Z}/3^k\mathbb{Z})^*\), we consider \(i_1 \in \mathbb{N}\) such that \(3^k - 1 = 2_k^{i_1}\). There exists \(i_2 \in \mathbb{N}\) such that \(2_k^{i_1} = 1\) and from there we have \(2_k^{i_1} = x\). Hence, by Theorem 3.20 we have \(\alpha_{0, -1}(p_1 \cdot l^{i_2}) = x\). And, by Lemma 3.23 the node \((p, x, l(p))\) with \(p = (\leftarrow)^k(i_2)^{i_2}\) is in the \((\alpha_{0, -1})\)-tree.

— \(\Leftarrow\): let suppose there exists \(p \in \mathcal{P}\) such that \((p, x, l(p)) \in \mathcal{N}\). By definition we have \(x = \alpha_{0, -1}(p)\). Hence by Lemma 3.17 we deduce \(x = 0\) or \(x \in (\mathbb{Z}/l(p)\mathbb{Z})^*\) depending on whether \(l(p) = 0\) or \(l(p) > 0\).

The structure of \((\alpha_{0, -1})\)-tree will be more precisely studied in Secion 4.

**Symmetries of the \((\alpha_{0, -1})\)-tree.** Figure 5 illustrates the first four levels of the \((\alpha_{0, -1})\)-tree. Notice that, by construction of the \((\alpha_{0, -1})\)-tree, if two nodes \((p, x, k)\) and \((p', x, k)\) share the same \(x \) and \(k\) they will be the root of very similar sub-trees. This phenomenon is highlighted with the nodes \((\epsilon, 0, 0)\) and \((\downarrow, 0, 0)\), Figure 5 doesn’t show the sub-tree under \((\downarrow, 0, 0)\) as it can be entirely deduced from the sub-tree under \((\epsilon, 0, 0)\). The same would apply for the sub-trees under \((\leftarrow, 2, 1)\) and \((\downarrow, \leftarrow, 2, 1)\). These symmetries in the \((\alpha_{0, -1})\)-tree will be further exploited in Section 4 with the concept of “k-span equivalence” for parity vectors (see Definition 4.3). They will be central in the proof of Theorem 4.16.
Finally we can reformulate the Collatz conjecture in terms of the \((\alpha_{0,-1})\)-tree:

**Theorem 3.27** (Collatz conjecture in terms of \(\alpha_{0,-1}\)). The Collatz conjecture is equivalent to the following. Take any node \(n_0 = (p, x, k) \in \mathcal{N}\) of the \((\alpha_{0,-1})\)-tree with \(x \neq 0\), and do the following process:

- If \(x\) is even, branch to the right son of \(n_0\)
- If \(x\) is odd, branch to the left son of \(n_0\)

Repeat.

Then, there exists \(p' \in P\) such that this process will encounter a node \(n_1 \in \mathcal{N}\) of the form:

\[ n_1 = (p \cdot p', 1, l(p) + l(p')) \]

\[ . \]

**Proof.** We prove the two implications:

- Suppose that the Collatz conjecture is true. Let \((p, x, k) \in \mathcal{N}\) with \(x \neq 0\). Define the sequence of parity vectors \((p_n)\) which is constructed by the process: \(p_0 = p\) and \(p_{n+1} = p_n \cdot a\) with \(a\) the admissible arrow for \(\alpha_{0,-1}(p_n)\). Then define the sequence \(c_n = \alpha_{0,-1}(p_n)\). Then by **Theorem 3.20** because \(a\) is admissible we have \(c_{n+1} = T(c_n)\) and \((c_n)\) is the Collatz sequence of \(\alpha_{0,-1}(p) = x \neq 0\).

Hence, by the Collatz conjecture, we must have \(n^*\) such that \(c_{n^*} = 1\) which will correspond to the fact that the process defined in this Theorem has reached the node \((p_{n^*}, 1, l(p_{n^*}))\). Finally, we can write \(p_{n^*} = p \cdot p'\) with \(p'\) the collection of admissible arrows that were added in the process.

- Suppose that the claim on the \((\alpha_{0,-1})\)-tree is true. Let \(x \neq 0\), two cases:

1. If \(x\) is not a multiple of 3, by **Lemma 3.26** there exists \(p \in P\) and \(l(p)\) such that \((p, x, l(p)) \in \mathcal{N}\).

2. If \(x\) is a multiple of 3, let \(x' = \frac{x}{2^h}\) with \(h\) the number of factor 2 in \(x\), i.e. \(h = \nu_2(x)\). Then, \(x'\) is odd and \(x'' = T(x') = T_1(x')\) is not a multiple of three. By the previous argument, the Collatz conjecture is true for \(x''\) hence it is true for \(x\).

\[ \square \]

### 3.3 Constructing \(\alpha_0\)

**Algorithm 1** Constructing \(\alpha_0\). Auxiliary routines are given in Appendix \[E\]

```python
1 def get_first_occurrence(p):
2     alpha00 = 0
3     alpha0_minus_1 = 0
4     k = 0
5
6     for i, arrow in enumerate(p):
7         if not is_admissible(arrow, alpha0_minus_1):
8             alpha00 += 2**i
9
10        if arrow == LEFT_ARROW:
11            k += 1
12
13        alpha0_minus_1 = T_modular(arrow, k, alpha0_minus_1)
14
15     first_occurrence = [alpha00]
16     for i in range(len(p)):
17         first_occurrence.append(T(first_occurrence[-1]))
18
19 return first_occurrence
```

**Theorems 3.12 and 3.20** give all the theoretical material to design Algorithm 1 which computes \(\alpha_0(p)\) in \(||p||\) steps (Auxiliary routines are given in Appendix \[E\]). We prove the correctness and the complexity of the algorithm:

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Theorem 3.28. For \( p \in \mathcal{P} \) as input, the function \( \text{get\_first\_occurrence} \) of Algorithm 1 computes the first occurrence of \( p, \alpha_0(p) \). Algorithm 1 runs in \( O(||p||) \) steps.

Proof. Let write \( p = a_0 \ldots a_{||p||-1} \) with \( a_i \in \{\downarrow, \leftarrow\} \). By Theorem 3.12 and Theorem 3.20 at the end of the \( i \)th iteration of the first for loop (1.6) we have \( \alpha_{0,0} = \alpha_{0,0}(a_0 \ldots a_i) \) and \( \alpha_{0,\text{minus\_1}} = \alpha_{0,\text{minus\_1}}(a_0 \ldots a_i) \). Hence, after this loop is completed we have: \( \alpha_{0,0} = \alpha_{0,0}(p) \) and \( \alpha_{0,\text{minus\_1}} = \alpha_{0,\text{minus\_1}}(p) \). Therefore, the second for loop (1.17) fills the list \( \text{first\_occurrence} \) with \( \alpha_0(p) \) because \( \alpha_0(p) = (\alpha_{0,0}(p), T(\alpha_{0,0}(p)), \ldots, T^{||p||}(\alpha_{0,0}(p))) \). We have proved the correctness of the algorithm. The complexity in \( O(||p||) \) comes from the fact that the two loops of \( \text{get\_first\_occurrence} \) only perform elementar operations. \( \square \)

Remark 3.29. Thanks to Theorem 2.10 once we have the first occurrence of \( p \), we can easily extend Algorithm 1 to compute the \( i \)th: take the first \( ||p|| \) Collatz steps of the number \( 2||p||i + \alpha_0,0(p) \).

Remark 3.30. Note that the code of Algorithm 1 is valid Python code that you can run by for instance setting \( \text{BOTTOM\_ARROW}=0 \) and \( \text{LEFT\_ARROW}=1 \) which will make a parity vector \( p \in \mathcal{P} \) be a list of 0s and 1s corresponding to \( \downarrow \) and \( \leftarrow \).

4 \( k \)-Span Predecessor Sets are Regular

In this Section, we use the symmetries of the \((\alpha_{0,-1})\)-tree (see Definition 3.22 and the results of Section 3) in order to highlight regular structure in the Collatz graph. Indeed, with Theorem 4.10 we show that, in binary, for any \( x \in \mathbb{N} \), the set of \( y \) which reach \( x \) by using exactly \( k \) times the operator \( T_1 \) (and any number of times the operator \( T_0 \)) can be defined by a regular expression, \( \text{reg}_{\alpha}(x) \). This is a generalization of [2] which proved the case \( k = 1 \). However, we use a very different set of tools than in [2] in order to get our result. In Section 4.1 we define \( k \)-span predecessor sets, \( \text{Pred}_k(x) \), and their binary encodings \( \mathcal{E}\text{Pred}_k(x) \). Then, in Section 4.2 we exploit the symmetries of the \((\alpha_{0,-1})\)-tree in order to prove the regular structure of \( \mathcal{E}\text{Pred}_k(x) \) in Theorem 4.10, and we express this result in terms of \( \text{Pred}_k(x) \) in Corollary 4.11. Finally, with Theorem 4.20 we reformulate the Collatz conjecture in terms of the family of regular expressions \( \text{reg}_k(\alpha) \).

4.1 \( k \)-Span Predecessor Sets and their Encoding

Let first define properly what \( k \)-span predecessor sets are:

Definition 4.1. \( \text{Pred}_k(x) \). Let \( x \in \mathbb{N} \) and \( k \in \mathbb{N} \). The \( k \)-span predecessor set of \( x \), \( \text{Pred}_k(x) \), is the set of \( y \in \mathbb{N} \) which reach \( x \) by using exactly \( k \) times the operator \( T_1 \) and any number of times the operator \( T_0 \). Formally, the set of \( y \in \mathbb{N} \) such that there is \( n \in \mathbb{N} \) and \( (b_0,b_1,\ldots,b_{n-1}) \in \{0,1\}^n \) with:

\[
x = T(y) = T_{b_{n-1}}(\ldots T_{b_1}(T_{b_0}(x))) \text{ and } \sum_{i=0}^{n-1} b_i = k
\]

We will mainly use the following equivalent definition in terms of occurrences of parity vectors:

\[
\text{Pred}_k(x) = \{ y \in \mathbb{N} \mid \exists p \in \mathcal{P} \exists i \in \mathbb{N} \quad \alpha_{i,0}(p) = y \text{ and } \alpha_{i,-1}(p) = x \text{ and } l(p) = k \}
\] (5)

Remark 4.2. In a fast-forwarded version of the Collatz process where only odd steps are considered, \( k \)-span predecessor sets appear naturally. Indeed, consider the odd Collatz graph where nodes are all integers in \( 2\mathbb{N} + 1 \) and there is an edge from \( x \) to \( y \) iff \( y = (3x + 1)/2^h \) where \( h \) is the number of factor 2 in \( 3x + 1 \). Then, in this graph, ancestors of \( x \) at distance \( k \) are exactly the odd numbers in the set \( \text{Pred}_k(x) \). The set \( \text{Pred}_k(x) \) is a little more richer because it will also contain any even numbers of the form \( 2^h y \) with \( y \) odd in \( \text{Pred}_k(x) \). It is essentially this graph which is studied in [2].

When \( x \) is a multiple of 3, \( \text{Pred}_k(x) \) is simple to characterize:

Lemma 4.3. Let \( x \in \mathbb{N} \). If \( x \) is a multiple of three then: \( \forall k > 0, \text{Pred}_k(x) = \emptyset \). If \( x \) is not a multiple of three then: \( \forall k \geq 0, \text{Pred}_k(x) \neq \emptyset \).
Proof. More generally, if \( x \) is a multiple of three, the set of ancestors of \( x \) in the Collatz graph is reduced to \( \text{Pred}_0(x) = \{2^n x \text{ for } n \in \mathbb{N} \} \). Indeed, \( T_{3}^{-1}(x) = \frac{2x-1}{3} \) cannot be an integer if \( x \equiv 0 \mod 3 \) and \( x \equiv 0 \mod 3 \Rightarrow \forall n \in \mathbb{N}, \ 2^n x \equiv 0 \mod 3 \).

If \( x \) is not a multiple of three, we still have \( \text{Pred}_0(x) = \{2^n x \text{ for } n \in \mathbb{N} \} \). For \( k > 0 \), two cases:

- If \( x < 3^k \) then by Lemma \( \[3.26\] \) there exists \( p \in \mathbb{P} \) such that \( x = \alpha_{0,-1}(p) \) hence \( \alpha_{0,0}(p) \in \text{Pred}_k(x) \) and \( \text{Pred}_k(x) \neq \emptyset \).

- If \( x \geq 3^k \), we have: \( x = 3^k q + \tilde{x} \) with \( \tilde{x} < 3^k \). Note that \( \tilde{x} \) cannot be a multiple of three since \( k > 0 \), otherwise \( x \) would be. Hence \( \tilde{x} \in \mathbb{Z}/3^k\mathbb{Z}^* \) and by the same argument as above \( \text{Pred}_k(\tilde{x}) \neq \emptyset \).

There is \( y \in \text{Pred}_k(\tilde{x}) \) and \( p \in \mathbb{P} \) such that \( \alpha_{0,0}(p) = y \). By Theorem \( \[2.10\] \) we have \( i = 0 \) because \( \tilde{x} < 3^k \). Thus we deduce \( \alpha_{q,0}(p) = 2^{|p|}q + y \in \text{Pred}_k(x) \neq \emptyset \).

\[ \square \]

**Encodings of \( k \)-span predecessor sets.** We will prove the regular structure of \( k \)-span predecessor sets by working with their encoding in base two as given by the encoding function \( \mathcal{E} \). Before we can formally introduce \( \mathcal{E}\text{Pred}_k(x) \), the encoded version of \( \text{Pred}_k(x) \), we need the following result which characterizes more the \( i \) of Definition \( \[4.4\] \) Equation \( \[5\] \).

**Lemma 4.4.** Let \( x \in \mathbb{N} \) and \( k \in \mathbb{N} \). Then:

\[ \text{Pred}_k(x) = \{ y \in \mathbb{N} | \exists p \in \mathbb{P} \ \alpha_{1,0}(p) = y \ \text{and} \ \alpha_{i,-1}(p) = x \ \text{and} \ l(p) = k \} \]

With \( i \) the quotient in the euclidian division of \( x \) by \( 3^k \). In other words, \( i = \lfloor \frac{x}{3^k} \rfloor \).

**Proof.** By Definition \( \[4.4\] \) Equation \( \[5\] \), we have \( x = \alpha_{i,-1}(p) \). Hence, by Theorem \( \[2.10\] \) we have: \( x = 3^k i + \alpha_{0,-1}(p) \) with \( \alpha_{0,-1}(p) < 3^k \). Hence \( i \) the quotient in the euclidian division of \( x \) by \( 3^k \) and we have \( i = \lfloor \frac{x}{3^k} \rfloor \).

\[ \square \]

**Definition 4.5 \((\mathcal{E}\text{Pred}_k(x))\).** Let \( x \in \mathbb{N} \) and \( k \in \mathbb{N} \). We define the set \( \mathcal{E}\text{Pred}_k(x) \subset \mathcal{B}^* \) according to two cases:

- If \( x < 3^k \) then:

\[ \mathcal{E}\text{Pred}_k(x) = \{ \mathcal{E}(p) \mid p \in \mathbb{P} \text{ such that } \alpha_{0,-1}(p) = x \text{ and } l(p) = k \} \]

- If \( x \geq 3^k \) then:

\[ \mathcal{E}\text{Pred}_k(x) = \{ \mathcal{I}^{-1}(i) \bullet \mathcal{E}(p) \mid p \in \mathbb{P} \text{ such that } \alpha_{i,-1}(p) = x \text{ and } l(p) = k \} \]

With \( i = \lfloor \frac{x}{3^k} \rfloor \).

The following Lemma highlights the tight link between \( \text{Pred}_k(x) \) and \( \mathcal{E}\text{Pred}_k(x) \). Binary strings in \( \mathcal{E}\text{Pred}_k(x) \) represent numbers in \( \text{Pred}_k(x) \), with potential leading 0s, in a one-to-one correspondence:

**Lemma 4.6.** Let \( x > 0 \) and \( k \in \mathbb{N} \). The following function defines a bijection between \( \mathcal{E}\text{Pred}_k(x) \) and \( \text{Pred}_k(x) \):

\[ g : \mathcal{E}\text{Pred}_k(x) \rightarrow \text{Pred}_k(x) \]

\[ \omega \mapsto \mathcal{I}(\omega) \]

**Proof.** 1. The function \( g \) is well defined. Indeed, for any \( \omega \in \mathcal{E}\text{Pred}_k(x) \)

\[ \omega \in \mathcal{E}\text{Pred}_k(x) \Leftrightarrow \exists p \in \mathbb{P} \ \mathcal{I}(\omega) = 2^{|p|}i + \mathcal{I}(\mathcal{E}(p)) \text{ with } \alpha_{0,-1}(p) = x \text{ and } l(p) = k \] (Equation \( \[6\] \))

\[ \Leftrightarrow i = 2^{|p|}i + \alpha_{0,0}(p) \] (Lemma \( \[3.26\] \))

\[ \Leftrightarrow i = \alpha_{0,0}(p) \] (Lemma \( \[2.10\] \))

\[ \Leftrightarrow i = g(\omega) \in \text{Pred}_k(x) \] (Lemma \( \[4.4\] \))

With \( i = \lfloor \frac{x}{3^k} \rfloor \).
2. The function $g$ is injective. Let $\omega_1, \omega_2 \in \mathcal{E} \text{Pred}_k(x)$ with $\omega_1 = \omega \cdot \mathcal{E}(p_1)$ and $\omega_2 = \omega \cdot \mathcal{E}(p_2)$ with $\omega = \eta$ if $i = \lfloor \frac{\omega}{x} \rfloor = 0$ else $\omega = \mathcal{I}^{-1}(i)$. We have $x = \alpha_{i-1}(p_1) = \alpha_{i-1}(p_2)$ by hypothesis. Let suppose $g(\omega_1) = g(\omega_2)$. We get $2^{|p_1|}i + \mathcal{I}(\mathcal{E}(p_1)) = 2^{|p_2|}i + \mathcal{I}(\mathcal{E}(p_2))$. By Lemma 4.3 we get $2^{|p_1|}i + \alpha_{0,0}(p_1) = 2^{|p_2|}i + \alpha_{0,0}(p_2)$. By Theorem 2.11 we get $\alpha_{i,0}(p_1) = \alpha_{i,0}(p_2)$. If $|p_1| \neq |p_2|$, for instance $|p_1| < |p_2|$, we have $p_2 = p_1 \cdot (\lambda)^{|p_2| - |p_1|}$. Indeed, by determinism of the Collatz process, $p_1$ must be a prefix of $p_2$ as they both are forward feasible for $y = \alpha_{i,0}(p_1) = \alpha_{i,0}(p_2)$. Furthermore, we can’t add any more arrows of type $\rightarrow$ because $l(p_1) = l(p_2)$. But, $\alpha_{i-1}(p_1) = x \neq 0$ thus $\alpha_{i-1}(p_2) = x / (2^{|p_2|} - |p_1|) = x$ which contradicts $\alpha_{i-1}(p_1) = \alpha_{i-1}(p_2)$. Hence we have $|p_1| = |p_2|$ and thus $p_1 = p_2$ because, by determinism of the Collatz process, there is only one path of a given norm between $\alpha_{i,0}(p_1)$ and $\alpha_{i-1}(p_1)$ and thus one corresponding parity vector. Hence, $\omega_1 = \omega_2$.

3. The function $g$ is surjective. Let $y \in \text{Pred}_k(x)$. By Lemma 4.4 there exists $p \in \mathcal{P}$ with $\alpha_{i,0} = y$ and $\alpha_{i,-1} = x$ with $i = \lfloor \frac{\omega}{x} \rfloor$. Similarly to the proof of Point 1 the reader can verify that $\mathcal{E}(p) \in \mathcal{E} \text{Pred}_k(x)$ is a valid antecedent of $y$ in the case $i = 0$ and that $\mathcal{I}^{-1}(i) \cdot \mathcal{E}(p) \in \mathcal{E} \text{Pred}_k(x)$ is a valid antecedent of $y$ otherwise.

\hfill \Box

**Remark 4.7.** If $x = 0$, then for all $k > 0$ we have $\mathcal{E} \text{Pred}_k(x) = \text{Pred}_k(x) = \emptyset$. However, $\mathcal{E} \text{Pred}_0(x) = \{ 0, 00, 00, \ldots \} = \{ 0 \}^*$ but $\text{Pred}_0(x) = \{ 0 \}$. Remark as well than when $x$ is a multiple of three, by Lemma 4.3 $\text{Pred}_k(x) = \mathcal{E} \text{Pred}_k(x) = \emptyset$ for all $k > 0$. In that case the above result trivially holds.

### 4.2 Exploiting Symmetries of the $(\alpha_{0,-1})$-tree

The proof of our main result, Theorem 4.16 will boil down to studying the structure of parity vectors $p \in \mathcal{P}$ such that, at fixed $k$ and $x \in \mathbb{Z} / 3^k \mathbb{Z}^*$, the triple $(p, x, k)$ is a node of the $(\alpha_{0,-1})$-tree. We will see that this structure can be entirely characterized by a regular expression which depends on binary strings $\Pi_k$ (see Definition 4.9). We first start by formalising the idea of symmetry in the $(\alpha_{0,-1})$-tree which occurs when two different nodes share the same $x$ and $k$, with the definition of “$k$-span equivalence” on parity vectors:

**Definition 4.8 (k-span equivalence).** Two parity vectors $p_1, p_2 \in \mathcal{P}$ are said to be $k$-span equivalent if $l(p_1) = l(p_2) = k$ and $\alpha_{0,-1}(p_1) = \alpha_{0,-1}(p_2)$. We write $p_1 \simeq_k p_2$. Note that $\simeq_k$ is an equivalence relation.

The following binary strings will play a central role in how we will describe the equivalent classes of $k$-span equivalence:

**Definition 4.9 (Parity sequence of $(\mathbb{Z} / 3^k \mathbb{Z}^*)^*$).** For $k > 0$, we define $\Pi_k \in B^*$, the parity sequence of $(\mathbb{Z} / 3^k \mathbb{Z}^*)^*$ as follow:

$$\Pi_k = b_0 \ldots b_{\pi_k-1} \text{ with } |\Pi_k| = \pi_k = |(\mathbb{Z} / 3^k \mathbb{Z}^*)^*| = 2 \ast 3^{k-1} \text{ and: }$$

$$b_{\pi_k-1-i} = \begin{cases} 0 & \text{if } 2_{\pi_k-1}^{-1} \text{ is even} \\ 1 & \text{if } 2_{\pi_k-1}^{-1} \text{ is odd} \end{cases}$$

By convention, we fix $\pi_0 = 1$.

**Example 4.10.** For $k = 3$, we have $2_3^{-1} = 14$. The sequence of powers of $2_3^{-1}$ in $(\mathbb{Z} / 3^3 \mathbb{Z}^*)^*$ is: $[1, 14, 7, 17, 22, 11, 19, 23, 25, 26, 13, 20, 10, 5, 16, 8, 4, 2]$. The associated parity sequence $(0 \text{ when even and } 1 \text{ when odd})$ is: $101101111101001000$. Finally, $\Pi_3$ is the mirror image of this: $\Pi_3 = 00010010111101101$. We have:

- $\Pi_1 = 01$
- $\Pi_2 = 000111$
- $\Pi_3 = 00010010111101101$
- $\Pi_4 = 00000011001000100101100001111111001101011011101100111$
**Remark 4.11.** By using the main result of [2] (Theorem 1) and our main result, Theorem 4.14, it can be shown that the strings \( \Pi_k \) correspond exactly to the “seeds” of [2]. However, the “seeds” are defined in a very different way in [2]. Hence, we inherit all the results of [2] on the “seeds” for our \( \Pi_k \). For instance we have:

\[
- \quad \mathcal{I}(\Pi_k) = \mathcal{I}((\Pi_k)^{-1})^3/3 = 7\mathcal{I}(\Pi_k)^{-1}4^{\pi_k-1} \quad \text{(Definition of the “seeds” in [2])}
- \quad \mathcal{I}(\Pi_k) = \frac{4^{\pi_k-1}}{\pi_k+1} \quad \text{(Lemma 1 in [2])}
\]

Thanks to the result of [4] we also learn that \( \Pi_k \) is the repetent of \( 1/3^k \) in binary.

We now give, for any \( p \in \mathcal{P} \), a very straightforward way, which involves strings \( \Pi_k \), to build an infinite family of parity vectors \( (p_n) \) which are all \( k \)-span equivalent to \( p \):

**Definition 4.12** (Rotation operator \( \mathcal{R} \)). Let \( \omega = b_0 \ldots b_{n-1} \in \mathcal{B}^* \) with \( |\omega| = n \). Then, for \( 0 \leq i < n \) we define the \( i \)-th rotation to the right of \( \omega \) to be: \( \mathcal{R}_i(\omega) = b_{(n-i \mod n)}b_{(n-(i-1) \mod n)} \ldots b_{(n-(i-(n-1))) \mod n} \).

**Example 4.13.** We have \( \mathcal{R}_2(000111) = 110001 \).

**Lemma 4.14.** Let \( p \in \mathcal{P} \) and \( k = l(p) > 0 \). Define \( p_n = p \cdot \alpha^{n\pi_k} \), i.e. the parity vector \( p \) followed by \( n\pi_k \) arrows of type \( \downarrow \), where \( \pi_k = |\Pi_k| \). Then, for all \( n \in \mathbb{N} \) we have \( p \simeq_k p_n \). Furthermore we can characterize \( \alpha_{0,0}(p_n) \) through \( \mathcal{E}(p_n) \) with:

\[
\mathcal{E}(p_{n+1}) = \mathcal{R}_{i_0}(\Pi_k) \bullet \mathcal{E}(p_n) \iff \mathcal{E}(p_n) = (\mathcal{R}_{i_0}(\Pi_k))^n \bullet \mathcal{E}(p)
\]

With \( 0 \leq i_0 < \pi_k \) such that \( \alpha_{0,0}(p) = 2^{-i_0} \) in \( (\mathbb{Z}/3^k\mathbb{Z})^* \).

**Proof.** We have \( l(p_n) = l(p) = k > 0 \). By Lemma 3.17 and Theorem 3.20 we know that \( \alpha_{0,0}(p) \in (\mathbb{Z}/3^k\mathbb{Z})^* \). Furthermore, by Theorem 3.20 \( \alpha_{0,0}(p_n) = T^{n\pi_k}_{\alpha_{0,0}}(\alpha_{0,0}(p)) = (2^k)^{n\pi_k}\alpha_{0,0}(p) = 1 \cdot \alpha_{0,0}(p) = \alpha_{0,0}(p) \) since \( \pi_k \) is the order of the group \( (\mathbb{Z}/3^k\mathbb{Z})^* \). Hence we have \( p_n \simeq_k p \). Furthermore, by Theorem 5.12 we know that \( \mathcal{E}(p_{n+1}) = \omega \bullet \mathcal{E}(p_n) \) with \( \omega = b_0 \ldots b_{\pi_k-1} \in \mathcal{B}^* \) such that:

\[
b_{\pi_k-1} = \begin{cases} 0 & \text{if } 2^{-i_0-1} \text{ is even} \\ 1 & \text{if } 2^{-i_0-1} \text{ is odd} \end{cases}
\]

With \( 0 \leq i_0 < \pi_k \) such that \( \alpha_{0,0}(p) = 2^{-i_0} \) in \( (\mathbb{Z}/3^k\mathbb{Z})^* \).

By definition, the string \( \omega = b_0 \ldots b_{\pi_k-1} \) is exactly \( \mathcal{R}_{i_0}(\Pi_k) \) and we have the result.

---

**Figure 6:** Illustration of Lemma 4.14. How the parity vector \( p = \downarrow \downarrow \rightarrow \rightarrow \) (in blue), with \( l(p) = 2 \), distributes on the elements of \( (\mathbb{Z}/3^2\mathbb{Z})^* \). The first occurrence of \( p \) is such that \( \alpha_{0,1} = 8 = 2^{2^0} = 2^3 \). The parity vector \( p \) is \( k \)-span equivalent to the parity vector \( p' = \downarrow \downarrow \rightarrow \rightarrow \) (in brown).

**Remark 4.15.** The result of Lemma 4.14 is illustrated in Figure 6. The parity vector \( p \) distributes in a “spiral” around the elements of \( (\mathbb{Z}/3^2\mathbb{Z})^* \). When \( \pi_k \) arrows of type \( \downarrow \) have been added to \( p \), a full “turn” has been done and we get a path \( k \)-span equivalent to \( p \).

\[ \text{Figure 6: Illustration of Lemma 4.14. How the parity vector } p = \downarrow \downarrow \rightarrow \rightarrow \text{ (in blue), with } l(p) = 2, \text{ distributes on the elements of } (\mathbb{Z}/3^2\mathbb{Z})^*. \text{ The first occurrence of } p \text{ is such that } \alpha_{0,1} = 8 = 2^{2^0} = 2^3. \text{ The parity vector } p \text{ is } k\text{-span equivalent to the parity vector } p' = \downarrow \downarrow \rightarrow \rightarrow \text{ (in brown).} \]

\[ \text{Remark 4.15.} \text{ The result of Lemma 4.14 is illustrated in Figure 6. The parity vector } p \text{ distributes in a “spiral” around the elements of } (\mathbb{Z}/3^2\mathbb{Z})^*. \text{ When } \pi_k \text{ arrows of type } \downarrow \text{ have been added to } p, \text{ a full “turn” has been done and we get a path } k\text{-span equivalent to } p. \]

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Theorem 4.16 ($\varepsilon\text{Pred}_k(x)$ is regular). For all $x \in \mathbb{N}$, for all $k \in \mathbb{N}$ there exists a regular expression $\text{reg}_k(x)$ which defines $\varepsilon\text{Pred}_k(x)$.

Proof. The regular expressions we use are words in $B^*$ together with parenthesis (), the disjunction operator | and Kleene star *. For instance, the expression $(01)^*(00)(11)$ recognises any word of the for $(01)^n$ or $(01)^n11$.

Let $x \in \mathbb{N}$. Let suppose $x$ is a multiple of 3. If $x = 0$ we have $\varepsilon\text{Pred}_0(0) = \{(0)^n \mid n \in \mathbb{N}\}$ (see Remark 4.7), hence we take $\text{reg}_0(0) = (0)^*$ otherwise, by Lemma 4.3 $\text{reg}_k(x) = \eta$ for $k > 0$ since $\varepsilon\text{Pred}_k(x)$ is empty in that case. When $k = 0$ and $x \neq 0$, we can take $\text{reg}_0(x) = (I^{-1}(x)) \bullet (0)^*$ since $\varepsilon\text{Pred}_0(x) = \{(x^{-1}(x)) \bullet (0)^n \mid n \in \mathbb{N}\}$ by Definition 4.5 $(x \geq 3^k)$.

Let suppose that $x$ is not a multiple of 3. In that case, by Lemma 4.3 the set $\varepsilon\text{Pred}_k(x)$ is never empty. Let suppose that $\text{reg}_k(x)$ exists for all $x$ such that $x < 3^k$. Now, if $k \in \mathbb{N}$ is such that $x \geq 3^k$, by Definition 4.3 we can take $\text{reg}_k(x) = (I^{-1}((\frac{x}{3^k}))) \bullet \text{reg}_k((x \mod 3^k))$ in order to define $\varepsilon\text{Pred}_k(x)$. Indeed, by Theorem 2.10 $\alpha_{i, 1}(p) = x \iff \alpha_{i, 0} = (x \mod 3^k)$ with $i = [\frac{x}{3^k}]$.

Hence we just have to prove that $\text{reg}_k(x)$ exists for all $x$, non multiple of three such that $x < 3^k$, i.e. $x \in \mathbb{N}/(\mathbb{Z}/3^k\mathbb{Z})^*$. We prove by induction on $k$ the following result:

$$H(k) = \forall x \in \mathbb{N}/(\mathbb{Z}/3^k\mathbb{Z})^* \text{ there exists } \text{reg}_k(x) \text{ which defines } \varepsilon\text{Pred}_k(x)$$

Note that, in this case, when $x \in \mathbb{N}/(\mathbb{Z}/3^k\mathbb{Z})^*$, characterizing $\varepsilon\text{Pred}_k(x) = \{E(p) \mid p \in P \text{ such that } \alpha_{0, k}(p) = x \text{ and } l(p) = k\}$, boils down, by Definition 3.6 to characterizing $\{p \mid p \in P \text{ such that } \alpha_{0, k}(p) = x \text{ and } l(p) = k\}$ which is a $k$-span equivalence class.

— Base $k = 0$. Trivially true because $(\mathbb{Z}/3^k\mathbb{Z})^* = \emptyset$. Note that the following induction step will rely on knowing $\text{reg}_k(0)$. We have shown above that $\text{reg}_0(0) = (0)^*$.

— Inductive step. Let $k \in \mathbb{N}$ such that $H(k)$ holds. We show that $H(k + 1)$ holds. Let $x \in \mathbb{N}/(\mathbb{Z}/3^{k+1}\mathbb{Z})^*$ such that $x = 2^{-i_0}k$ with $0 \leq i_0 < \pi_{k+1}$.

Let $p \in P$ such that $(p, x, k + 1)$ is in the $(\alpha_{0, k+1})$-tree. Let’s analyse the structure of $p$. In all generality, in the $(\alpha_{0, k+1})$-tree, we are in the following context:

$$\begin{align*}
(p_2, x_2) &= 2^{-i_2}k, \\
(p_1, x_1) &= T_{1, k+1}(x_2) = 2^{-i_0}k, k + 1, \\
(p', x) &= 2^{-i_0}k, k + 1, \\
(p, x, k + 1)
\end{align*}$$

In order to prove the generality of this situation, three points:

1. Since $l(p) = k + 1 \geq 1$ we can decompose $p = p_2 \cdot \downarrow^m$ with $m \in \mathbb{N}$ and $p_2 \in P$ such that $l(p_2) = k$. We can write $m = n\pi_{k+1} + r$ with $r < \pi_{k+1}$ and $p_2 = p_2 \cdot \downarrow^r \downarrow^{\pi_{k+1}}$. We call the number $n$ the “repeating value”. By Lemma 4.11 we know that $p \simeq_{k+1} p' = p_2 \cdot \downarrow^r$, hence $\alpha_{0, k+1}(p') = \alpha_{0, k+1}(p) = x$ and $E(p) = (\mathcal{R}_{i_0}((\Pi_{k+1}))^n \bullet E(p')$. It remains to characterize $E(p') = E(p_2 \cdot \downarrow^r = E(p_2 \cdot \downarrow^r)$ with $p_1 = p_2 \cdot \downarrow$.

2. Let’s consider $x_2 = \alpha_{0, k+1}(p_2)$. If $k \neq 0$, by Lemma 3.17 we know that $x_2 \in (\mathbb{Z}/3^k\mathbb{Z})^*$ and we can write $x_2 = 2^{-i_2}k$ with $0 \leq i_2 < \pi_k$. Note that if $k = 0$, by the same Lemma, we have $x_2 = 0 = 2^{-i_2}$ by convention. Thus in all case we can write $x_2 = 2^{-i_2}k$ with $0 \leq i_2 < \pi_k$. By Theorem 3.12 we deduce that $E(p_1) = b_{i_2} \cdot E(p_2)$ where $b_{i_2} = 0$ if $2^{-i_2}$ is odd and $1$ if $2^{-i_2}$ is even.
3. Let’s consider now \( x_1 = \alpha_{0,-1}(p_2 \cdot \leftarrow) \). By Theorem 3.20 we have \( x_1 = T_{1,k+1}(x_2) = T_{1,k+1}(2_k^{i_2}) \). Furthermore, by the same Theorem 3.20 we have \( x = \alpha_{0,-1}(p_1') = \alpha_{0,-1}(p_1 \cdot \leftarrow) = T_{0,k+1}^{(}1(x_2) = T_{0,k+1}^{(}1(x_1) \). Hence \( x = 2^{i_1}_k x_1 \) and thus \( x_1 = 2^{i_1}_k + 1 \) with \( 0 \leq i_1 = -i_0 + r < \pi_k+1 \). By Theorem 3.12 we deduce that:

\[
\mathcal{E}(p' = p_1 \cdot \leftarrow) = \omega \cdot \mathcal{E}(p_1)
\]

With \( \omega = j_0 \ldots j_r \in B^* \), \( |\omega| = r \), and \( j_r = -1 \).

We refer to such \( \omega \) by \( \text{join}_{\omega} \) because it is uniquely determined by \( i_2 \) such that \( x_1 = T_{1,k+1}(x_2) = T_{1,k+1}(2_k^{i_2}) \). Indeed, Lemma 1.23 shows that \( T_{1,k+1} \) is injective on \( \mathbb{Z}/3^k\mathbb{Z} \) hence \( i_2 \neq i'_2 \Rightarrow T_{1,k+1}(2_k^{-i_2}) \neq T_{1,k+1}(2_k^{-i'_2}) \). Different values of \( i_2 \) will yield to different \( x_1 \) and thus different \( i_1, r \) and \( \text{join}_{\omega} \). The name \( \text{join} \) refers to the fact that this parity sequence arises from “joining”, in the \((\alpha_{0,-1})\)-tree, \( x_1 = 2^{i_1}_k + 1 \) to \( x = 2^{i_0}_k + 1 \) with \( r = i_0 - i_1 \geq 0 \) arrows of type \( \downarrow \). Notice that we can have \( \text{join}_{\eta} = \eta \) in the case where \( r = 0 \).

Over all, from Points 1, 2, 3, we deduce that:

\[
\mathcal{E}(p) = (\mathcal{R}_{\omega_0}(\Pi_{k+1}))^n \cdot (\text{join}_{\omega}) \cdot (b_{\iota_2}) \cdot \mathcal{E}(p_2)
\]

We have \( l(p_2) = k \). If \( k \neq 0 \), we will be able to reduce to the induction hypothesis since \( x_2 \in (\mathbb{Z}/3^k\mathbb{Z})^* \). If \( k = 0 \) we have \( x_2 = 0 = 0 \) and we use \( \text{reg}_{0}(0) \) previously constructed.

As a synthesis, notice that any value of \( 0 \leq i_2 < \pi_k \), any node \((p_2, x_2 = 2^{i_2}_k, k) \) and any repeating value \( n \in \mathbb{N} \) will lead to the construction of a \((p, x, k + 1) \) with different \( p \) for each choice of \( i_2, p_2 \) and \( n \). Hence, we have completely characterized the structure of nodes of the form \((p, x, k + 1) \).

From the above analysis, we can deduce the recursive expression of \( \text{reg}_{k+1}(x) \), we have:

\[
\text{reg}_{k+1}(x) = (\mathcal{R}_{\omega_0}(\Pi_{k+1}))^r (\text{join}_{\eta_0} (\alpha_{0,-1}) (\text{reg}_{k}(2^{i_0}_k)) (\text{join}_{\eta_0} (\alpha_{0,-1}) (\text{reg}_{k}(2^{i_1}_k)) (\text{join}_{\eta_0} (\alpha_{0,-1}) (\text{reg}_{k}(2^{i_2}_k)) \ldots (\text{join}_{\eta_0} (\alpha_{0,-1}) (\text{reg}_{k}(2^{i_{\pi_k-1}_k}))
\]

Notice the fact that for \( k > 0 \) the word \( b_{\pi_k-1}b_{\pi_k-2} \ldots b_0 \) is the binary complement of \( \Pi_k \).

\[\square\]

**Remark 4.17.** Theorem 4.16 generalises Theorem 1 in [2] which handled the case \( x = 1 \). Note that the author of [2] used very different techniques.

**Example 4.18.** Appendix E gives Python code (also available and illustrated at https://github.com/tcosmo/CollatzRegular) which implements the construction of Theorem 4.16. For instance for \( \text{reg}_1(8) \), we get:

\[
\text{reg}_2(8) = (111000) * (((1110)(0)(01) * (((0)(1)((0)*))) | (((1)(1)(10) * (((1) (0)*)')))'))
\]

Where \( 0 \) stands for \( \eta \). While this regular expression follows exactly the format given by Theorem 4.16 it is better visualised in a more compact tree:

We can challenge this regular expression. We see in Figure 4 that for \( p = -c^2 \) we have \( \mathcal{E}(p) = 11, \alpha_{0,-1} = 8 \) and \( l(p) = 2 \). Thus, \( \mathcal{E}(p) \in \mathcal{E}\text{Pred}_1(8) \). By Theorem 4.16 \( \text{reg}_3(8) \) should recognize \( \mathcal{E}(p) = 11 \). We see from the above tree that it is the case. In a case where \( x \geq 3^k \) such as \( x = 8 \) and \( k = 1 \), the Theorem gives:

\[
\text{reg}_1(8) = (10)((10) * (((1)(1)(0)*)))
\]

We see that this expression recognizes 101 which is the binary representation of 5 and we know that \( 8 = T_1(5) \). Appendix E gives \( \text{reg}_1(1) \) which gives a sense of the exponential growth of the representation of \( \mathcal{E}\text{Pred}_1(x) \) when \( k \) gets bigger.
Going back to Pred_k(x). Theorem 4.10 characterizes EPred_k(x). We now want to characterize Pred_k(x). By Lemma 4.19 we know that there is a tight link between EPred_k(x) and Pred_k(x): EPred_k(x) contains binary representations of numbers in Pred_k(x), with potential leading 0s, in a one-to-one correspondence (when x ≠ 0). Thanks to Theorem 4.16 we can give an upper bound on the number of leading 0s we have to add to I^−1(y) in order to know whether y ∈ Pred_k(x) or not:

Corollary 4.19. Let x ∈ N and k ∈ N then:

\[ y \in \text{Pred}_k(x) \iff \exists n \leq B(x, k), \ 0^n \cdot I^−1(y) \text{ is recognised by } \text{reg}_k(x) \]

With \( B(x, k) = \begin{cases} 0 & \text{if } x \geq 3^k \\ (k + 1)^2 & \text{otherwise} \end{cases} \).

**Proof.** The result without the bound on \( n \) is an immediate consequence of Theorem 4.10.

If \( x = 0, y \in \text{Pred}_k(x) \iff k = 0 \) and \( y = 0 \iff \exists n \in \mathbb{N} \ 0^n \cdot I^−1(y) \) is recognised by \( \text{reg}_0(0) = (0)^* \) since any \( n \in \mathbb{N} \) will work.

If \( x ≠ 0 \), by Lemma 4.19 we have:

\[ y \in \text{Pred}_k(x) \iff g^−1(y) \in E\text{Pred}_k(x) \iff g^−1(y) \text{ is recognised by } \text{reg}_k(x) \]

We have \( g^−1(y) = \omega \in \mathbb{B}^* \) such that \( I(\omega) = y \). Hence, there exists \( n \in \mathbb{N} \) such that \( g^−1(y) = 0^n \cdot I^−1(y) \) and we get the result without the bound on \( n \).

The stake is now to bound \( n \), the number of leading 0s one has to prepend to the binary representation of \( y \) in order to test whether \( y \in \text{Pred}_k(x) \). We derive this bound from analysing \( \text{reg}_k(x) \) hence we review all the cases of the proof of Theorem 4.10.

Let \( x \in \mathbb{N} \). Let suppose \( x \) is a multiple of 3. If \( x = 0 \) we have \( \text{reg}_0(0) = (0)^* \) and \( \text{Pred}_0(0) = \{0\} \) hence any bound will work. If \( k > 1 \) we have \( \text{reg}_k(x) = \eta \) hence any bound will also work. When \( k = 0 \) and \( x ≠ 0 \), we have \( \text{reg}_0(x) = (\overline{I−1(x)} \cdot (0)^* \text{ thus } \text{reg}_0(x) \text{ starts with a bit equal to } 1 \text{ hence it will not recognise any leading } 0 \text{ and taking } B(x, k) = 0 \text{ will work (we are in a special case of } \geq 3^k) \).

Let suppose that \( x \) is not a multiple of 3. If \( x \geq 3^k \) we have \( \text{reg}_k(x) = (\overline{I−1((1/}\sqrt{3})) \cdot \text{reg}_k((x \mod 3^k)), \) again, \( \text{reg}_k(x) \) starts with a bit equal to 1 and the bound \( B(x, k) = 0 \) will work.

Now, the last case is \( x \in (\mathbb{Z}/3^k\mathbb{Z})^* \) with \( k > 0 \). We prove by induction on \( k > 0 \) the following result:

\[ \overline{H}(k) = \forall x \in (\mathbb{Z}/3^k\mathbb{Z})^*, \text{ the number of leading } 0s \text{ that } \text{reg}_k(x) \text{ can accept is bounded by } (k + 1)^2^n \]

— **Base** \( k = 1 \). In that case we only have \( x = 1 \) and \( x = 2 \) to consider. We have:

\[ \text{reg}_1(1) = (01)\ast(((0)(1)((0)\ast))) \]
\[ \text{reg}_1(2) = (10)\ast(((1)(1)((0)\ast))) \]

In both case the maximum number of leading 0s that can be accepted is bounded by \( 1 \leq (1 + 1)^2 = 4 \) hence the bound \( B(x, k) = (k + 1)^2 \) is valid.

— **Inductive step.** Let suppose the result true for some \( k > 0 \). By Theorem 4.16 we have:

\[ \text{reg}_{k+1}(x) = (\mathcal{R}_{\omega}(\Pi_{k+1}))^\ast( \ (\text{join}_0)(b_0)(\text{reg}_{k}(2^0)) \mid \ (\text{join}_1)(b_1)(\text{reg}_{k}(2^1)) \mid \ (\text{join}_2)(b_2)(\text{reg}_{k}(2^2)) \mid \ \ldots \mid \ (\text{join}_{\pi_k-1})(b_{\pi_k-1})(\text{reg}_{k}(2^-(\pi_k-1)))) \) \]
The longest trail of consecutive 0s in the string \( \Pi_{k+1} \) (and thus in \( R_{i_0}(\Pi_{k+1}) \)) will correspond to the 0s given by the parity of the sequence \( 2^m, 2^{m-1}, \ldots, 2^1 \) with \( m \) the largest integer such that \( 2^m < 3^{k+1} \). We have \( m = \lfloor (k+1) \log_2(3) \rfloor \). The same is true for the maximum number of leading 0s in the expression \( \text{join}_{i_2} \). The bit \( b_i \) can also be equal to 0s. Let's call \( z_{k+1} \) (resp. \( z_k \)) the number of leading 0s that \( \text{reg}_{k+1}(x) \) (resp. \( \text{reg}_k(x) \)) can accept. Since \( R_{i_0}(\Pi_{k+1}) \) is not all 0s, we can't use Kleene star to have an unbounded number of leading 0s. We deduce:

\[
z_{k+1} \leq \max((k+1) \log_2(3), (k+1) \log_2(3) + 1 + z_k) \leq 2(k+1) + 1 + z_k.
\]

By induction hypothesis we deduce,

\[
z_{k+1} \leq 2(k+1) + 1 + (k+1)^2 = (k+2)^2.
\]

Hence we have the result and the bound \( B(x, k) = (k+1)^2 \) is valid when \( x < 3^k \).

We can finally state a conjecture equivalent to the Collatz conjecture in terms of the family of regular expressions \( \text{reg}_k(1) \):

**Theorem 4.20** (The Collatz conjecture and \( \text{reg}_k(1) \)). The Collatz conjecture is equivalent to:

\[
\forall y > 0, \exists k \in \mathbb{N}, \exists n \leq (k+1)^2, 0^n \cdot I^{-1}(y) \text{ is recognised by } \text{reg}_k(1)
\]

**Proof.** Direct consequence of Corollary 4.19 (the bound \( (k+1)^2 \) works in all the cases) since the Collatz conjecture is equivalent to: \( \forall y > 0, \exists k \in \mathbb{N}, y \in \text{Pred}_k(1) \).

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**References**

[1] J. Capco. Odd Collatz Sequence and Binary Representations. Preprint, Mar. 2019.
[2] L. Colussi. The convergence classes of Collatz function. *Theor. Comput. Sci.*, 2011.
[3] J. Conway. Unpredictable iterations. *Number Theory Conference*, 1972.
[4] P. C. Hew. Working in binary protects the repetends of 1/3h: Comment on Colussi’s 'The convergence classes of Collatz function’. *Theor. Comput. Sci.*, 618:135–141, 2016.
[5] P. Koiran and C. Moore. Closed-form analytic maps in one and two dimensions can simulate universal Turing machines. *Theoretical Computer Science*, 210(1):217–223, Jan. 1999.
[6] S. A. Kurtz and J. Simon. The Undecidability of the Generalized Collatz Problem. In *TAMC 2007*, pages 542–553, 2007.
[7] J. C. Lagarias. The 3x + 1 problem and its generalizations. *The American Mathematical Monthly*, 92(1):3–23, 1985.
[8] J. C. Lagarias. The 3x+1 problem: An annotated bibliography (1963–1999) (sorted by author), 2003.
[9] J. C. Lagarias. The 3x+1 problem: An annotated bibliography, ii (2000-2009), 2006.
[10] K. Monks. The sufficiency of arithmetic progressions for the 3x + 1 conjecture. *Proceedings of the American Mathematical Society*, 134, 10 2006.
[11] R. Terras. A stopping time problem on the positive integers. *Acta Arithmetica*, 30(3):241–252, 1976.
[12] G. J. Wirsching. *The dynamical system generated by the 3n + 1 function*. Springer, Berlin New York, 1998.
A Feasible Vectors

In this Section we present the formalism used in [12] in order to prove Theorem 2.10.

These results are based on a compact representation of parity vectors called feasible vectors in [12]:

Definition A.1 (Feasible vectors). The set of feasible vectors is \( \mathcal{F} = \bigcup_{k=0}^{\infty} \mathbb{N}^{k+1} \). For a feasible vector \( s = (s_0, \ldots, s_k) \in \mathcal{F} \), the length of \( s \) is written \( l(s) \) and the \emph{length} is \( k \). The \emph{norm} of \( s \) is \( ||s|| = l(s) + \sum_{i=0}^{l(s)} s_i \).

Example A.2. A feasible vector is a compact way to represent a parity vector. For instance, the parity vector \( p = \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow = (\downarrow)^1 \leftarrow (\downarrow)^3 \leftarrow (\downarrow)^0 \leftarrow (\downarrow)^2 \). Will represent by the feasible vector \( s = (1, 3, 0, 2) \). We have \( ||p|| = ||s|| \) and \( l(p) = l(s) \).

Definition A.3 (Backtracing Function). Let \( s = (s_0, \ldots, s_k) \in \mathcal{F} \), the backtracing function of \( s \) is \( v_s : \mathbb{N} \rightarrow \mathbb{Q} \) defined by:

\[
v_s(x) = T_0^{−s_0} \circ T_1^{−s_1} \circ \ldots \circ T_1^{−s_1} \circ T_0^{−s_k}
\]

If \( v_s(x) \in \mathbb{N} \) then we say that \( s \) is backward feasible for \( x \).

Lemma A.4 (Lemma 2.17 in [12]). Let \( s \in \mathcal{F} \) and \( x \in \mathbb{N} \) such that \( s \) is backward feasible for \( x \). Then we have: \( T^{|s|}(v_s(x)) = x \).

Example A.5. In Figure 2, we have \( p_3 = (\downarrow)^3 \leftarrow (\downarrow)^0 \). Hence the corresponding feasible vector is \( s = (3, 0) \). We read on Figure 2: \( v_s(2) = 8 \). Hence, \( T^{|s|}(v_s(2)) = 2 \).

Being a composition of affine functions, \( v_s \) is affine. The author of [12] completely characterises the structure of \( v_s \):

Lemma A.6 (Lemma 2.13 in [12]). For \( s = (s_0, \ldots, s_k) \in \mathcal{F} \) define:

\[
c(s) = \frac{2||s||}{3^{l(s)}} \quad \text{and} \quad r(s) = \sum_{j=0}^{k-1} \frac{2^{j+s_0+\ldots+s_j}}{3^{j+1}}
\]

Then for any \( x \in \mathbb{N} \) we have: \( v_s(x) = c(s)x - r(s) \).

Finally the following lemma of [12] will essentially give the proof of Theorem 2.10.

Lemma A.7 (Lemma 3.1 in [12]). Let \( s \in \mathcal{F} \). Then there is exactly one \( a < 3^{|s|} \) such that for any \( b \in \mathbb{N} \):

\( s \) is backward feasible for \( b \Leftrightarrow b \equiv a \mod 3^{|s|} \)

Proof. We know that: \( s \) is backward feasible for \( b \Leftrightarrow v_s(b) \in \mathbb{N} \). Lemma A.6 gives: \( v_s(b) = c(s)b - r(s) = \frac{1}{3^{l(s)}} \left( 2^{|s|}b - 3^{|s|}r(s) \right) \). Hence, with \( d = 3^{|s|}r(s) \in \mathbb{N} \):

\( s \) is backward feasible for \( b \Leftrightarrow b \equiv 2^{-|s|}d \mod 3^{|s|} \)

Because \( 2^{|s|} \) is invertible in \( \mathbb{Z}/3^{|s|}\mathbb{Z} \).

\( \square \)

Finally we can prove Theorem 2.10:

Theorem 2.10 (All parity vectors are feasible).

Let \( p \in \mathcal{P} \). Then:

1. \( p \) is feasible i.e. \( a_0 = (a_{0,0}, \ldots, a_{0,-1}) \in \mathbb{N}^{|p|+1} \) is defined
2. \( a_{0,0} < 2^{|p|} \) and \( a_{0,-1} < 3^{|p|} \)
3. Finally we can completely characterize \( a_{i,0} \) and \( a_{i,-1} \) with:

\[
a_{i,0} = 2^{|p|}i + a_{0,0} \quad (1)
\]

\[
a_{i,-1} = 3^{|p|}i + a_{0,-1} \quad (2)
\]
Proof. Let \( p \in \mathcal{P} \) and \( s \) his associated feasible vector. By Lemma \ref{A.7} we deduce that Point \ref{A.1} holds with the existence of \( \alpha_{0,-1} < 3^{\|p\|} \). From the same Lemma, we get Equation \ref{A.1}. Equation \ref{A.1} comes from Lemma \ref{A.6} \( \alpha_{i,0} = v_s(\alpha_{i,-1}) = v_s(3^{\|p\|}i + \alpha_{0,-1}) = 2^{\|s\|}i + v_s(\alpha_{0,0}) = 2^{\|s\|}i + \alpha_{0,0} \). Finally, the bound \( \alpha_{0,0} < 2^{\|p\|} \) can also be derived from Lemma \ref{A.6} \( \alpha_{0,0} = v_s(\alpha_{0,-1}) = \frac{2^{\|s\|}}{3^{|s|-1}} \alpha_{0,-1} - r(s) < 2^{\|s\|} \) since \( r(s) \geq 0 \) and \( 2^{\|s\|} = 2^{\|p\|} \).

**B Generalised Occurrence Theorem**

In this Section, we generalise Theorem \ref{2.10} with Theorem \ref{B.2} which characterizes the relationship between \( \alpha_{i,j} \) and \( \alpha_{0,j} \) for all \( j \). We use the notation introduced in Section \ref{2} We thank Jose Capco for his precious help in proving this result.

**Lemma B.1.** Let \( p \in \mathcal{P} \) and suppose that \( 0 \leq j \leq \|p\| \) then there exists a unique couple \( (t, u) \in \mathcal{P}^2 \) such that \( p = t \cdot u \) and \( ||t|| = j \).

**Proof.** Immediate from Definition \ref{2.1} The parity vector \( t \) is composed by the first \( j \) arrows of \( p \) and the parity vector \( u \) by the last \( \|p\| - j \) ones.

**Theorem B.2** (Generalized occurrence theorem). Let \( p \in \mathcal{P} \) and \( \alpha \) be the set of the occurrences of \( p \). Suppose that \( 0 \leq j \leq \|p\| \) and \( t, u \in \mathcal{P} \) are as in the Lemma above: \( p = t \cdot u \) with \( ||t|| = j \). Then for any \( i \in \mathbb{N} \) we have:

\[
\alpha_{i,j} = 3^{\|t\|} 2^{\|u\|} i + \alpha_{0,j}
\]

With \( 0 \leq \alpha_{0,j} < 3^{\|t\|} 2^{\|u\|} \).

**Proof.** Let \( \alpha \) be the occurrences of \( p \), \( \beta \) be the occurrences of \( t \) and \( \gamma \) the occurrences of \( u \). We know that there exists \( a, b, k, m \in \mathbb{N} \) such that:

\[
\begin{align*}
x_0 &:= \alpha_{0,0} = \beta_{a,0} \\
x &:= \alpha_{i,0} = \beta_{a+k,0} \\
y_0 &:= \alpha_{0,j} = \beta_{a,j} = \gamma_{b,0} \\
y &:= \alpha_{i,j} = \beta_{a+k,j} = \gamma_{b+m,0} \\
z_0 &:= \alpha_{0,\|p\|} = \gamma_{b,||t||} \\
z &:= \alpha_{i,\|p\|} = \gamma_{b+m,||t||}
\end{align*}
\]

By Theorem \ref{2.10} we have the following identities:

\[
\begin{align*}
x &= 2^{\|p\|}i + x_0 \\
&= 2^{\|p\|}(a + k) + \beta_{0,0} \\
&= 2^{\|t\|}k + 2^{\|t\|}a + \beta_{0,0} \\
&= 2^{\|t\|}k + \beta_{a,0} \\
&= 2^{\|t\|}k + x_0
\end{align*}
\]

Similarly:

\[
\begin{align*}
y &= 2^{\|u\|}m + y_0 = 3^{\|t\|}k + y_0 \\
z &= 3^{\|s\|}i + z_0 = 3^{\|u\|}m + z_0
\end{align*}
\]

From the equation equating to \( x \) we find:

\[
k = 2^{\|p\| - ||t||}i = 2^{\|u\|}i
\]

The equation for \( y \) yields the desired result:

\[
y = 3^{\|t\|}2^{\|u\|}i + y_0 \Leftrightarrow \alpha_{i,j} = 3^{\|t\|}2^{\|u\|}i + \alpha_{0,j}
\]
We now want to prove the inequality:

\[ \alpha_{0,j} < 3^{\ell(t)}2^{\|u\|} \]

For the two extreme cases \( j = 0 \) or \( j = \|t\| \), this is just the result of Theorem 2.10. So we assume that \( 0 < j < \|t\| \) and we prove recursively assuming that \( \alpha_{0,j} < 2^{\|u\|}3^{\ell(t)} \). Let \( t', u' \in \mathcal{P} \) such that \( \|t'\| = j + 1 \) and \( p = t' \cdot u' \). We have \( \|u'\| = \|u\| - 1 \) and \( \|t'\| = \|t\| + 1 \). We now have two cases:

Case \( \alpha_{0,j} \downarrow \alpha_{0,j+1} \): In this case we have \( l(t') = l(t) \) because no arrow of type \( \leftarrow \) is added on that step. Further more we have:

\[
\alpha_{0,j+1} = (\alpha_{0,j})/2 \\
< 2^{\|u\| - 1}3^{\ell(t)} \quad \text{(IH)} \\
= 2^{\|u'\|}3^{\ell(t')} 
\]

Which is what we want.

Case \( \alpha_{0,j+1} \leftarrow \alpha_{0,j} \): In this case we have \( l(t') = l(t) + 1 \) because one arrow of type \( \leftarrow \) is added on that step. Further more we have:

\[
\alpha_{0,j+1} = (3\alpha_{0,j} + 1)/2 \\
\leq \frac{3(2^{\|u\|}3^{\ell(t)} - 1) + 1}{2} \quad \text{(IH)} \\
\leq 2^{\|u\| - 1}3^{\ell(t)+1} - 1 \\
< 2^{\|u'\|}3^{\ell(t')} 
\]

Which is what we want.

C Decoding \( \mathcal{E}(p) \)

In this Section, we prove the remaining points of Corollary 3.15 (Section 3.1). In order to prove Points 5 and 6, we will need results from Section 3.2.

**Corollary 3.15** We have the following for all \( n \in \mathbb{N} \):

1. \( \mathcal{E}^{-1}(0^n) = (\downarrow)^n \)
2. \( \mathcal{E}^{-1}(1(0)^n) = (\downarrow)^n \leftarrow \)
3. \( \mathcal{E}^{-1}(1(0)^{2n+1}) = (\downarrow\leftarrow)^n \leftarrow \)
4. \( \mathcal{E}^{-1}(1(0)^{2n+1}1) = (\leftarrow)^{n+1} \downarrow \)
5. \( \mathcal{E}^{-1}(01)^{n+1} = \downarrow \downarrow (\downarrow)^{2n} \)
6. \( \mathcal{E}^{-1}(1^n) = (\leftarrow)^n \)

Note that these statements allow to predict the first \( \lceil \log_2(x) \rceil + 1 \) arrows of the parity vector taken along the Collatz sequence of any \( x \in \mathbb{N} \) of which binary representation matches one of the different given expressions.

**Proof.**

1. Proved in main text

2. By Theorem 3.12 we know that \( \mathcal{E}(\downarrow)^n \leftarrow = b \cdot \mathcal{E}(\downarrow)^n \) with \( b \in \{0,1\} \). From Point 1 of this Corollary we have \( \mathcal{E}(\downarrow)^n = 0^n \) and from the proof of this Point we know that \( \alpha_{0,-1}(\downarrow)^n = 0^n \). Since the arrow \( \leftarrow \) is not admissible for \( b = 0 \) and \( \mathcal{E}(\downarrow)^n \leftarrow = 1(0)^n \) and we have the result.
3. We show by induction on $n$ that, for $p_n = \langle \downarrow \rangle^n$, we have $E(p_n) = (0)2^n$ and $\alpha_0(p_n) = 2$. For $n = 0$, $p_0 = \langle \rangle$, we know that $\alpha_0(\langle \rangle) = (1,2)$ since 1 is the smallest integer on which $T_1$ can be applied. Hence, $E(p_0) = I_{-1}(\alpha_0(p_0)) = I_{-1}(1) = 1$ and $\alpha_0(p_0) = 2$. Let suppose the result true for some $n \in \mathbb{N}$. By Theorem 3.12, we have $E(p_{n+1}) = E(p_n \downarrow \langle \rangle = b_2 b_1 E(p_n)$ with $b_1, b_2 \in \{0,1\}$. We know that $b_1 = 0$ since $\alpha_0(p_n) = 2$ and $b_1$ corresponds to the arrow $\downarrow$. Now, by admissibility of $\downarrow$, we have $\alpha_0(p_n \downarrow \langle \rangle) = T(\alpha_0(p_n)) = T(2) = 1$. Hence we have $b_2 = 0$ because $\downarrow$ is admissible for 1. By the same argument, we have $\alpha_0(p_n \downarrow \langle \rangle) = T(\alpha_0(p_n \downarrow \langle \rangle)) = T(2) = 1$ and we conclude $E(p_{n+1}) = 1(0)2^{(n+1)}$ and $\alpha_0(p_{n+1}) = 2$ and the induction is proved. Now, we know that $E(\langle \downarrow \rangle^n) = 1 \bullet E(\langle \downarrow \rangle^n) = 1 \bullet E(p_n)$ because $\langle \rangle$ is not admissible for $\alpha_0(p_n) = 2$. Hence we have the result.

4. Very similar to Point 3.

5. By induction, for $p_n = \langle \downarrow \langle \rangle^{2^n}$, let’s prove $E(p_n) = (01)^{n+1}$ and $\alpha_0(p_n) = 1$. For $n = 0$, we have $E(\langle \rangle) = 0 \bullet E(\langle \rangle) = 01$ since $\downarrow$ is admissible for 2. Hence we have $\alpha_0(p_0) = 0$ and $E(\langle \rangle) = 1$. By Theorem 3.20, for $n = 0$, we have $E(\langle \rangle) = T_{0,1}(\alpha_0(\langle \rangle)) = T_{0,1}(2) = 1$.

Now, $E(\langle \downarrow \rangle^{2^{n+1}}) = b_2 b_1 \bullet E(\langle \downarrow \rangle^{2^n})$. We have $b_1 = 1$ since $\downarrow$ is not admissible for $\alpha_0(p_n) = 1$. By Theorem 3.20, for $n = 0$, we have $\alpha_0(p_n \downarrow \langle \rangle) = T_{0,1}(\alpha_0(p_n \downarrow \langle \rangle)) = T_{0,1}(2) = 2$. Hence we have $b_0 = 0$ since $\downarrow$ is admissible for $\alpha_0(p_n) = 1$. Hence we have $E(p_{n+1}) = 01(01)^{n+1} = (01)^{n+1}$ which gives the result.

6. Let’s have $p_n = \langle \rangle^n$. We have $E(p_n) = \eta$. For $n = 1$, by Lemma 3.22, we know that $\alpha_0(p_n) = 3^n$ is 3-periodic. Hence we have $E(p_n) = 1 \bullet E(p_n) = (1)^{n+1}$ since the arrow $\langle \rangle$ is never admissible for an even number. 

\[ 2k^{-1} = \frac{3^k + 1}{2} \]

**Proof.** Let $z_k = \frac{2k+1}{2} \in \mathbb{N}$. We have $z_k < 3^k$ thus $z_k \in \mathbb{Z}/3^k \mathbb{Z}$. We also have, $2z_k = 3^k + 1 \equiv 1 \mod 3^k$. Hence $z_k$ meets all the requirements to be $2k^{-1}$.

**Lemma D.2** (Structure of $T_{0,k}$). Let $k \in \mathbb{N}$. For $x \in \mathbb{Z}/3^k \mathbb{Z}$ we have:

$$T_{0,k}(x) = \begin{cases} x/2 = T_0(x) & \text{if } x \text{ is even} \\ (3^k + x)/2 & \text{if } x \text{ is odd} \end{cases}$$

**Proof.** Let $x \in \mathbb{Z}/3^k \mathbb{Z}$, we have $x < 3^k$. Two cases:

- Case $x$ even. We have $x/2 < 3^k$ and $2 \ast (x/2) = x$. This shows that $x/2 = 2k^{-1}x$ and thus $T_{0,k}(x) = x/2$ for x even.

- Case $x$ odd. We have $2k^{-1}x \equiv \frac{3^k + 1}{2}x \mod 3^k$ by Lemma D.1. Because $x = 2y + 1$, we have $2k^{-1}x \equiv \frac{3^k + 1}{2}(2y + 1) \equiv y + \frac{3^k + 1}{2} \equiv \frac{3^k + 2y + 1}{2} \equiv (3^k + x)/2 \mod 3^k$. We also have $(3^k + x)/2 < 3^k$ thus, in $\mathbb{Z}/3^k \mathbb{Z}$, $2k^{-1}x = \frac{3^k + x}{2}$ and $T_{0,k}(x) = (3^k + x)/2$.

**Lemma D.3** (Structure of $T_{1,k}$). Let $k \in \mathbb{N}$. The function $T_{1,k+1}$ is $3^k$-periodic. Hence we simply have to characterize the behavior of $T_{1,k+1}$ on $\mathbb{Z}/3^k \mathbb{Z}$.

For $x \in \mathbb{Z}/3^k \mathbb{Z}$ we have:

$$T_{1,k+1}(x) = \begin{cases} (3^k + 3x + 1)/2 & \text{if } x \text{ is even} \\ (3x + 1)/2 = T_1(x) & \text{if } x \text{ is odd} \end{cases}$$
Proof. For $x \in \mathbb{Z}/3^k\mathbb{Z}$ we have: $T_{1,k+1}(x + 3^k) = 2_{k+1}^{-1}(3^{k+1} + 3x + 1) \equiv 2_{k+1}^{-1}(3x + 1) \mod 3^{k+1}$. Thus the function $T_{1,k+1}$ is $3^k$-periodic. Now, two cases:

— Case $x$ odd. We have $(3x + 1)/2 < 3^{k+1}$ and $2 \cdot ((3x + 1)/2) = 3x + 1$. This shows that $(3x + 1)/2 = 2_{k+1}^{-1}(3x + 1)$ and thus $T_{1,k+1}(x) = (3x + 1)/2$ for $x$ odd.

— Case $x$ even. We have $x = 2y$ and $2_{k+1}^{-1}(3 \cdot 2y + 1) \equiv 3y + 2_{k+1}^{-1} \mod 3^{k+1}$. We have $3y + 2_{k+1}^{-1} = 3y + \frac{3^k}{3^{k+1}} \equiv 3y + 3x + 1/2$ by Lemma D.1. Furthermore, $(3^k + 3x + 1)/2 < 3^{k+1}$ so we can conclude that $T_{1,k+1}(x) = (3^k + 3x + 1)/2$ when $x$ is even.

E Code to Generate $\text{reg}_k(x)$ and Auxiliary Routines

Algorithm 2 gives Python code which implements the construction of Theorem 4.16 and generates $\text{reg}_k(x)$. This code, together with the code of Algorithm 1 is available and illustrated at: https://github.com/tcosmo/CollatzRegular.

### Algorithm 2 Generating $\text{reg}_k(x)$

```python
1 def reg_k(x, k):
2     if k == 0:
3         return "( 0 )" if x == 0 else "({})(0)*".format(int_to_binary(x))
4     if x%3 == 0:
5         return ""
6     if x >= 3**k:
7         return "({})( )".format(int_to_binary(x//(3**k)), reg_k(x%(3**k), k))
8     i = log2_k(x,k)
9     Pik = get_Pi_k(k)
10    reg = "{}\(*\{}\".format(rotation(Pik, i))
11    inv = 0 if k-1 == 0 else inv2(k-1)
12    for j in range(get_pi_k(k-1)):
13        curr = 0 if k-1 == 0 else (inv**(j))%(3**k(k-1))
14        reg += "({})\({}\)\({}\)".format(join_k(T1_k(curr, k), x, k), 1-(curr%2), reg_k(curr, k-1))
15    reg += " )"
16    reg += "|
17    return reg
```

Warning. If you wish to test these regex in Python we strongly recommend not to use the standard re package which does not parse our expressions correctly. This issue may come from known bugs in the package before Python 3.7. An alternative that worked for us is redone.

The following are routines used in Algorithm 1 or Algorithm 2:

```python
1 # # # General routines
2 def T(x):
3     return T1(x) if x%2 else T0(x)
4 def T0(x):
5     return x/2
6 def T1(x):
7     return (3*x+1)/2
8 def T0_k(k,x):
9     return (inv2(k)*x%(3**k))
```

5Error with zero-width matches, see https://pypi.org/project/regex/
6https://github.com/cyphar/redone
def $T_1(k, x)$:
    return $(\text{inv}2(k) \cdot (3 \cdot x + 1)) \% (3 \cdot 2^k)$

def $\text{inv}2(k)$:
    return $(3 \cdot 2^k + 1) / 2$

### Specific to Algorithm 1

def $\text{is} \_ \text{admissible}(arrow, x)$:
    return $(\text{arrow} == \text{BOTTOM} \_ \text{ARROW} \text{ and } x \% 2 == 0) \text{ or } (\text{arrow} == \text{LEFT} \_ \text{ARROW} \text{ and } x \% 2 == 1)$

def $\text{T} \_ \text{modular}(arrow, k, \text{alpha} \_ \text{minus}1)$:
    if $\text{arrow} == \text{BOTTOM} \_ \text{ARROW}$:
        return $T_0(k, \text{alpha} \_ \text{minus}1)$
    return $T_1(k, \text{alpha} \_ \text{minus}1)$

### Specific to Algorithm 2

def $\text{get} \_ \pi(k)$:
    if $k == 0$:
        return 1
    return $2 \cdot (3 \cdot 2^{k-1})$

def $\text{get} \_ \pi \_ k(k)$:
    inv = $\text{inv}2(k)$
    Pik = ""  
    for $i$ in range(get_pi(k)):
        nb = $(\text{inv}^i) \% (3 \cdot 2^k)$
        Pik += $\text{str}(\text{nb} \% 2)$
    return Pik[:: -1]

def $\text{log} 2^k(x, k)$:
    inv = $\text{inv}2(k)$
    for $i$ in range(get_pi(k)):
        nb = $(\text{inv}^i) \% (3 \cdot 2^k)$
        if nb == x:
            return $i$

def $\text{join} \_ k(x, y, k)$:
    if $(x \% 3) \times (y \% 3) == 0$:
        return ""
    if not $(x < 3 \cdot 2^k \text{ and } y < 3 \cdot 2^k)$:
        return None
    s = ""
    i0 = $\text{log} 2^k(x, k)$
    inv = $\text{inv}2(k)$
    for $i$ in range(i0, i0 + get_pi(k)):
        nb = $(\text{inv}^i) \% (3 \cdot 2^k)$
        if nb == y:
            return s[:: -1]
    s += $\text{str}(\text{nb} \% 2)$
    return "".join([s[(n-(i-1))\%n] for i in range(n)])

F $\text{reg}_4(1)$

The following is the regular expression which defines $\mathcal{E}[\text{Pred}_4(1)$, i.e. it recognises the binary representation (with enough leading 0s) of any number $y$ that uses 4 times the operator $T_1$ and any number of times the operator $T_0$ in order to reach 1 in the Collatz process. For a fixed $x \in \mathbb{N}$ not a multiple of three, the size of the regular expression, given by Theorem 4.16 which defines $\mathcal{E}[\text{Pred}_2(x)$ grows exponentially with $k \in \mathbb{N}$.

$\text{reg}_4(1) =$

$(000000011001010010001111111111011011011011011101111111111111111111111111111111111111111111111111111111111111)$

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