Ballistic Coalescence Model

S. Ispolatov and P. L. Krapivsky

Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215

We study statistical properties of a one dimensional infinite system of coalescing particles. Each particle moves with constant velocity $\pm v$ towards its closest neighbor and merges with it upon collision. We propose a mean-field theory that confirms a $t^{-1}$ concentration decay obtained in simulations and provides qualitative description for the densities of growing, constant, and shrinking inter-particle gaps.

PACS numbers: 02.50-r, 01.75+m, 89.90+n

We introduce a simple deterministic model describing a coarsening dynamics of interacting domains. We arrived at this model in attempt to model the essential features of development of countries and civilizations. In our model, the civilizations are represented by domains that continuously cover one-dimensional “world” without gaps and overlaps. Neighboring civilizations are engaged in a permanent warfare: A bigger civilization invades the lesser one, so that the border (interface) moves with velocity $\pm v/2$ with $v$ being the same for all pairs of neighbors. The model is equivalent to an infinite particle aggregating system in which each particle (corresponding to the interface between domains) moves ballistically towards its closest neighbor. When a civilization shrinks to zero size and disappears, two particles, corresponding to its borders, collide. Such collisions between interfaces lead to coalescence, $I + I \rightarrow I$ where $I$ symbolizes an interface, and the emerging interface starts moving towards its nearest neighbor with velocity of the same magnitude $v/2$ (Fig. 1).

![Fig. 1. Schematic illustration of domain structure. Domains that grow, remain constant, and shrink at early time are denoted by G, N, and S, respectively.](image)

The gradual character and general nature of the model makes questionable its ability to reproduce specific details of human history that have been actually observed. However, it does provide some statistical characteristics like size distribution and predicts decrease of the total number of civilizations. An appealing simplicity of our model suggests that it might be relevant to description of other coarsening phenomena.

In our previous work [1], we studied a model with domains growing freely in otherwise empty space and engaging in instantaneous warfare upon contact. One can view the model of Ref. [1] and the present model as two limiting cases of the general growth-and-war process: The former growth-controlled process is limited by the growth rate while the latter process is war-controlled. Another model which has strong relevance to our system is the “cut-in-two” model of Ref. [2], introduced in the context of breath figures coarsening. Similarly to our model, it assumes the complete coverage without overlaps, but unlike our model, coarsening events are instantaneous. Namely, evolution proceeds by consecutive elimination of the current shortest domain.

In our model, which can be also called the ballistic coalescence model, there are three types of domains – growing, neutral (with constant length), and shrinking. A growing domain is bigger than both of its neighbors, so its length grows with velocity $v$. A neutral domain has one neighbor which is bigger and the other which is smaller, so its length remains constant though its location does change. Finally, a shrinking domain is smaller than both of its neighbors, and its length decreases in time with rate $v$. When such shrinking domain disappears, a change takes place in one of the two neighboring domains – a growing domain may become neutral, or a neutral domain may turn into a shrinking one. The number density of growing domains, $G(t)$, is always equal the number density of shrinking domains, $S(t)$, since the space remains continuously covered without gaps and overlaps throughout the process. A growing domain of initial length $\ell_0$ has the length $vt + \ell_0$ at time $t$, implying that $G(t) \sim t^{-1}$ and $S(t) \sim t^{-1}$. One might expect that the number density of neutral domains, $N(t)$, decays slower than $t^{-1}$, implying that the total domain density also exhibits anomalously slow decay. This indeed happens, e.g., in apparently similar model of ballistic annihilation, $I + I \rightarrow 0$, where $t^{-1/2}$ decay of the domain density is observed [3]. In both ballistic annihilation and ballistic coalescence models, growing domains cannot be neighbors, while an arbitrary number of consecutive neutral domains may coexist. However, in the ballistic annihilation model such a train of neutral domains has a good chance to live long while in our model this sequence of neutral domains can be eliminated by a single growing domain. Because of this relative vulnerability of neutral domains in the ballistic coalescence model, their num-
ber density scales similarly to the other two densities, $G(t)$ and $S(t)$. Moreover, we will show below that $N(t)$ becomes smaller than $G(t)$ and $S(t)$ as the process develops. Note that initially all the densities are the same, $G(0) = N(0) = S(0)$. Indeed, the initial sizes of domains are uncorrelated, so if we take three consecutive intervals the probabilities that the middle interval is the smallest, medium, or biggest are all the same.

Thus in the long time limit the densities behave according to

$$ G(t) = \frac{g}{t}, \quad N(t) = \frac{n}{t}, \quad S(t) = \frac{s}{t}, $$

(1)

with constants $g = s$ and $n$ to be determined. Consider an arbitrary shrinking domain of vanishing length. It may be surrounded by two neutral, two growing, or growing and neutral domains. Denote by $\lambda, \mu$, and $\nu$ the respective probabilities ($\lambda + \mu + \nu = 1$). The rate at which domains disappear, $vS(0, t)$, may be written as $\alpha g/t^2$ with some constant $\alpha$. Here $S(x, t)$ is the density of shrinking domains of length $x$. The normalization $\int_0^t dx S(x, t) = S(t) \sim t^{-1}$ suggests $S(0, t) \sim t^{-2}$ and explains the rate given above. Now it is straightforward to write down the rate equations

$$ \frac{d n}{d t} t = -\frac{\alpha g}{t^2} \mu, \quad \frac{d n}{d t} t = -\frac{\alpha g}{t^2} \mu, \quad \frac{d s}{d t} t = -\frac{\alpha g}{t^2} (1 - \mu - \lambda). $$

(2)

Summing Eqs. (2) gives $\alpha = (g+n+s)/g = 2+n/g$. The above equations are exact. To proceed further we need to know $\lambda, \mu, \nu$. We assume that the types of domains surrounding vanishing shrinking ones are uncorrelated, so the probability of picking up growing or neutral domain is proportional to its respective concentration. Mathematically, it means that

$$ \lambda = \left(\frac{n}{g+n}\right)^2, \quad \mu = \left(\frac{g}{g+n}\right)^2, \quad \nu = \frac{2gn}{(g+n)^2}. $$

(3)

Eqs. (3) imply $\alpha\mu = 1$ and $\alpha(1 - 2\mu) = n/g$. Substituting $\alpha = 2+n/g$ and $\mu = g^2(g+n)^{-2}$ into any of these equations allows us to find $n/g$.

$$ \frac{n}{g} = \frac{\sqrt{5} - 1}{2} \equiv \rho, $$

(4)

where $\rho$ is known as the “golden ratio”. We still need to determine one constant, say $g$. This can be accomplished by using the fact that the system is continuously covered with domains. We need to determine the length distributions $G(x, t)$, $N(x, t)$, and $S(x, t)$. In the limit of large time, we may ignore differences in sizes of growing domains. These differences are determined by initial size distribution and do not change with time. Thus the length distribution of growing domains is simply

$$ G(x, t) = \frac{g}{t} \delta(x - t). $$

(5)

Two other length distributions are expected to scale, so we write

$$ N(x, t) = \frac{g}{t^2} A \left(\frac{x}{t}\right), \quad S(x, t) = \frac{g}{t^2} B \left(\frac{x}{t}\right). $$

(6)

In the mean-field approximation (Eq. (3)), the governing equations for $N(x, t)$ and $S(x, t)$ read

$$ \frac{\partial N(x, t)}{\partial t} = -\frac{\alpha g}{t^2} \frac{N(x, t)}{N(t)} \left[\nu + 2\lambda \int_0^t dx' N(x', t)\right], $$

(7)

$$ \frac{\partial S(x, t)}{\partial t} - \frac{\partial S(x, t)}{\partial x} = -\frac{\alpha g}{t^2} \frac{N(x, t)}{N(t)} \left[\nu + 2\lambda \int_0^t dx' S(x', t)\right]. $$

(8)

The spatial derivative term in (8) accounts for continuous shrinking. The velocity of shrinking $v$ is set equal to one without loss of generality. We also have dropped two terms proportional to $\delta$-functions – the gain term in Eq. (8) relevant for conversion of growing domains into neutral, and the loss term in Eq. (8) accounting for removal of zero-size shrinking domains. These terms provide appropriate boundary conditions.

Using the scaling form (3) and relations (3) we simplify (8) and (8) to

$$ uA' = 2A \int_u^1 dv A(v) $$

(9)

and

$$ 2B + (1 + u)B' = -2A - 2A \int_u^1 dv A(v). $$

(10)

In these equations $u = x/t$, $A' = dA/du$, $B' = dB/du$. According to the definitions (1) and (3), the scaling functions $A(u)$ and $B(u)$ obey normalization conditions

$$ \int_0^1 du A(u) = \rho, \quad \int_0^1 du B(u) = 1. $$

(11)

Additionally, creation of the longest neutral domains $x = t$ and removal of the zero-size shrinking domains give $N(t, t) = \alpha g/t^2$ and $S(0, t) = \alpha g/t^2$. This provides the boundary conditions:

$$ A(1) = 1, \quad B(0) = \alpha. $$

(12)

Solving (1) gives

$$ A(u) = \frac{(2 + \rho^{-1})^2 u^{2\rho}}{(1 + \rho^{-1} + u^{2\rho+1})^2}. $$

(13)

Note that $A(0) = 0$, in agreement with intuitive expectation that there are no neutral domains of length zero.
To solve for the length distribution of shrinking domains we first multiply (8) on $1 + u$ so that the left-hand side becomes a complete derivative, $[(1 + u)^2 B]'$. The right-hand side contains already known functions. Integrating the resulting equation with the boundary condition $B(0) = \alpha = 2 + \rho$ we obtain

$$B(u) = \frac{2 + \rho}{(1 + u)^2} - \frac{1 + 2\rho}{1 + u^{2\rho+1}} - \frac{\rho(1 + \rho)(2 + \rho^{-1})^2}{1 + u} \frac{u^{2\rho+1}}{(1 + \rho^{-1} + u^{2\rho+1})^2}. \quad (14)$$

One can check that $B(1) = 0$, the result that agrees with intuitive expectation that there are no shrinking domains of maximal possible length $x = t$. We verified that the normalization condition of Eq. (11) is satisfied.

The scaled length distributions of neutral and shrinking domains are plotted on Figs. 2 and 3.

Now we can compute the constant $g$. The requirement that the space is completely covered,

$$\int_0^t dx x [G(x, t) + N(x, t) + S(x, t)] = 1, \quad (15)$$

gives

$$g = \left(1 + \int_0^1 du u [A(u) + B(u)]\right)^{-1} \approx 0.625837. \quad (16)$$

We performed molecular dynamic simulations for systems of $2 \cdot 10^6$ particles with periodic boundary conditions. Results were averaged over 500 initial configurations. We found that the ratio $n/g$ of neutral to growing particle densities decreases slowly from 1 to approximately 0.66. This value is 7% larger than the theoretical prediction $n/g = \rho \approx 0.618033989$; the statistical error for the measured value is less than 3%.

We also determined numerically the probabilities $\lambda, \mu, \nu$ and found more pronounced difference with mean-field predictions of Eq. (3): $\lambda \approx 0.1, \mu \approx 0.5, \nu \approx 0.4$ vs. $\lambda_{\text{MFT}} \approx 0.145898034, \mu_{\text{MFT}} \approx 0.381966011, \nu_{\text{MFT}} \approx 0.472135955$. We found that scaling (3) for the neutral and shrinking domain length distribution functions holds indeed, although the shape of the experimental curves is different from mean-field predictions (Figs. 2 and 3). The discrepancy is particularly drastic for the size distribution of neutral domains indicating that correlations, not accounted by the mean-field theory, are especially important for them.

![Fig. 2. Plot of scaled domain size distribution for neutral domains, $A(x/t)$. Time is incremented by 2.25: $t = 2192$, $t = 4932$, $t = 11097$, $t = 24968$. Mean-field prediction (Eq. 13) is shown by dashed line.](image)

![Fig. 3. Plot of scaled domain size distribution for shrinking domains, $B(x/t)$. Time is incremented by 2.25: $t = 2192$, $t = 4932$, $t = 11097$, $t = 24968$. Mean-field prediction (Eq. 14) is shown by dashed line.](image)

![Fig. 4. Schematic illustration of domain structure for a system with interface annihilation](image)
Since a domain length can instantaneously increase upon each shrinking event, such events often lead to changes in as many as four domains surrounding the shrinking one (central group of domains on Fig. 4). The model is thus different from the ballistic annihilation model \[3\]. Furthermore, correlations become more important than in our original ballistic coalescence model. However, this interface annihilation model exhibits the same scaling as the original interface coalescence model, \textit{i.e.}, the length of a typical growing, neutral, or shrinking domain grows linearly in time, and domain concentration decays as \(1/t\).

Scaled plots of length distribution for growing, neutral, and shrinking domains are presented of Fig. 5. Note that unlike the original model, the length distribution of growing domain remains unsingular and unbounded.

![Figure 5](image)

**Fig. 5.** Plots of scaled domain size distribution for growing domains (○), neutral domains (○), and shrinking domains (+) for interface annihilation model. Simulation included averaging over 500 configurations of 2 million initial domains each.

Coarsening rules in the above models are similar to those in one-dimensional Potts models with zero-temperature Glauber dynamics. Indeed, in the infinite-state Potts model interfaces aggregate upon collisions while in the 2-state Potts model, \textit{i.e.} the Ising model, interfaces annihilate upon collisions. The difference lies in interface dynamics – in the Potts models domain walls usually undergo random walk rather than ballistic motion.

In summary, we have analyzed a model of ballistic aggregation of particles where each particle is approaching its nearest neighbor with constant and universal velocity. It can be considered as an idealized model of a world in which all neighboring countries are engaged in permanent warfare with larger countries advancing and smaller receding. We found a universal linear in time scaling for all types of domains and developed a mean-field like approach to the domain size distribution function.

We are thankful to L. Frachebourg and S. Redner for helpful discussions. This work was partially supported by grants from ARO and NSF.

[1] S. Ispolatov, P. L. Krapivsky, and S. Redner, Phys. Rev. E 54, 1274 (1996).
[2] B. Derrida, C. Godrèche, and I. Yekutieli, Phys. Rev. A 44, 6241 (1991).
[3] Y. Elskens and H. L. Frisch, Phys. Rev. A 31, 3812 (1985).
[4] E. Ben-Naim, S. Redner, and P. L. Krapivsky, J. Phys. A 29, L561 (1996).