This paper deals with the old yet unsolved problem of defining and evaluating the stored electromagnetic energy—a quantity essential for calculating the quality factor, which reflects the intrinsic bandwidth of the considered electromagnetic system. A novel paradigm is proposed to determine the stored energy in the time domain leading to the method, which exhibits positive semi-definiteness and coordinate independence, i.e. two key properties actually not met by the contemporary approaches. The proposed technique is compared with an up-to-date frequency domain method that is extensively used in practice. Both concepts are discussed and compared on the basis of examples of varying complexity.

1. Introduction

In physics, an oscillating system is traditionally characterized [1] by its oscillation frequency and quality factor $Q$, which gives a measure of the lifetime of free oscillations. At its high values, the quality factor $Q$ is also inversely proportional to the intrinsic bandwidth in which the oscillating system can effectively be driven by external sources [2,3].

The concept of quality factor $Q$ as a single frequency estimate of relative bandwidth is most developed in the area of electric circuits [4] and electromagnetic radiating systems [3]. Its evaluation commonly follows
two paradigms. As far as the first one is concerned, the quality factor is evaluated from the knowledge of the frequency derivative of input impedance \([5–7]\). As for the second paradigm, the quality factor is defined as \(2\pi\) times the ratio between the cycle mean stored energy and the cycle mean lost energy \([5,8]\). Generally, these two concepts yield distinct results, but come to identical results in the case of vanishingly small losses, the reason being Foster’s reactance theorem \([9,10]\).

The evaluation of quality factor by means of frequency derivative of input impedance was made very popular by the work of Yaghjian & Best \([11]\) and is widely used in engineering practice \([12,13]\) thanks to its property of being directly measurable. Recently, this concept of quality factor has also been expressed as a bilinear form of source current densities \([14]\), which is very useful in connection with modern numerical software tools \([15]\). Regardless of the mentioned advantages, the impedance concept of quality factor suffers from a serious drawback of being zero in specific circuits \([16,17]\) and/or radiators \([17,18]\) with evidently non-zero energy storage. This unfortunately prevents its usage in modern optimization techniques \([19]\).

The second paradigm, in which the quality factor is evaluated via the stored energy and lost energy, is not left without difficulties either. In the case of non-dispersive components, the cycle mean lost energy does not pose a problem and may be evaluated as a sum of the cycle mean radiated energy and the cycle mean energy dissipated due to material losses \([20]\). Unfortunately, in the case of a non-stationary electromagnetic field associated with radiators, the definition of stored (non-propagating) electric and magnetic energies presents a problem that has not yet been satisfactorily solved \([3]\). The issue comes from the radiation energy, which does not decay fast enough in the radial direction, and is in fact infinite in the stationary state \([21]\).

In order to overcome the infinite values of total energy, the evaluation of stored energy in radiating systems is commonly accompanied by the technique of extracting the divergent radiation component from the well-known total energy of the system \([20]\). This method is somewhat analogous to the classical field renormalization \([22]\). Most attempts in this direction have been performed in the domain of time-harmonic fields. The pioneering work is the equivalent circuit method of Chu \([21]\), in which the radiation and energy storage are represented by resistive and reactive components of a complex electric circuit describing each spherical mode. This method was later generalized by several works of Thal \([23,24]\). Although powerful, this method suffers from the fundamental drawback of spherical harmonics expansion, which is unique solely in the exterior of the sphere bounding the sources. Therefore, the circuit method cannot provide any information on the radiation content of the interior region, nor on the connection of energy storage with the actual shape of the radiator.

The radiation extraction for spherical harmonics has also been performed directly at the field level. The classical work in this direction comes from Collin & Rothscild \([25]\). Their proposal leads to good results for canonical systems \([25–27]\) and has been analytically shown self-consistent outside the radian-sphere \([28]\). Similar to the work of Chu, this procedure is limited by the use of spherical harmonics to the exterior of the circumscribing sphere.

The problem of radiation extraction around radiators of arbitrary shape has been for the first time attacked by Fante \([29]\) and Rhodes \([30]\), giving the interpretation to Foster’s theorem \([10]\) in open problems. The ingenious combination of the frequency derivative of input impedance and the frequency derivative of far-field radiation pattern led to the first general evaluation of stored energy. Fante’s and Rhodes’ works have been later generalized by Yaghjian & Best \([11]\), who also pointed out an unpleasant fact that this method is coordinate-dependent. A scheme for minimization of this dependence has been developed \([11]\), but it was not until the work of Vandenbosch \([31]\) who, generalizing the expressions of Geyi for small radiators \([32]\) and rewriting the extraction technique into bilinear forms of current, was able to reformulate the original extraction method into a coordinate-independent scheme. A noteworthy discussion of various forms of this extraction technique can be found in the work of Gustafsson & Jonsson \([33]\). It was also Gustafsson et al., who emphasized \([19]\) that under certain conditions, this extraction technique fails, giving negative values of stored energy for specific current distributions. Hence the aforementioned approach also remains incomplete \([34]\).
The problem of stored energy has seldom been addressed directly in the time domain. Nevertheless, there are some interesting works dealing with time-dependent energies. Shlivinski expanded the fields into spherical waves in the time domain [35,36], introducing time domain quality factor that qualifies the radiation efficiency of pulse-fed radiators. Collarday [37] proposed a brute force method using the finite differences technique. In [38], Vandenbosch derived expressions for electric and magnetic energies in the time domain that, however, suffer from an unknown parameter called storage time. A notable work of Kaiser [39] then introduced the concept of rest electromagnetic energy, which resembles the properties of stored energy, but is not identical to it [40].

The knowledge of the stored electromagnetic energy and the capability of its evaluation are also tightly connected with the question of its minimization [21,41–44]. Such lower bound of the stored energy would imply the upper bound to the available bandwidth, a parameter of great importance for contemporary communication devices.

In this paper, a scheme for radiation energy extraction is proposed following a novel line of reasoning in the time domain. The scheme aims to overcome the handicaps of the previously published works, and, furthermore, is able to work with general time-dependent source current distributed through a region of arbitrary shape. For the purpose of comparison, the most common frequency domain method based on the time-harmonic expressions of Vandenbosch [31] is presented as well. Both concepts are closely investigated and compared on the basis of examples of varying complexity. The working out of both concepts starts solely from the current flowing on a radiator, which is usually given as a result in modern electromagnetic simulators. This raises challenging possibilities of modal analysis [45] and optimization [46].

The paper is organized as follows. Two different concepts of quality factor \(Q\) that are based on electromagnetic energies (in both, the frequency and time domain), are introduced in §2. The following §3 presents numerical examples dealing with canonical radiators. The results are discussed in §4 and the paper is concluded in §5.

2. Energy concept of quality factor \(Q\)

In the context of energy, the quality factor is most commonly\(^1\) defined as

\[
Q = 2\pi \frac{\langle W_{sto}(t) \rangle}{W_{\text{lost}}} = 2\pi \frac{W_{sto}}{W_{\text{lost}}},
\]

where a time-harmonic steady state with angular frequency \(\omega_0\) is assumed, with \(W_{sto}(t)\) as the electromagnetic stored energy, \(\langle W_{sto}(t) \rangle = W_{sto}\) as the cycle mean of \(W_{sto}(t)\) and \(W_{\text{lost}}\) as the lost electromagnetic energy during one cycle [20]. In conformity with the font convention introduced above, in the following text, the quantities defined in the time domain are stated in calligraphic font, while the frequency domain quantities are indicated in the roman font.

A typical \(Q\)-measurement scenario is depicted in figure 1, which shows a radiator fed by a source current density \(J_{\text{source}}\). Assuming the time-harmonic steady state at the frequency \(\omega_0\) and the radiator made of conductors with ideal non-dispersive conductivity \(\sigma\) and lossless non-dispersive dielectrics, we can state that the lost energy during one cycle, needed for (2.1), can be evaluated as

\[
W_{\text{lost}} = -\int_{0}^{\alpha+T} \int_{V} E(r, t) \cdot J_{\text{source}}(r, t) \, dV \, dt = W_r + W_{\sigma},
\]

where \(W_r\) represents the cycle mean radiation loss and \(W_{\sigma}\) stands for the energy lost in one cycle via conduction. The part \(W_{\sigma}\) of (2.2) can be calculated as

\[
W_{\sigma} = \frac{\pi}{\omega_0} \int_{V} \sigma |E(r, \omega_0)|^2 \, dV,
\]

\(^1\)Throughout this paper, the so-called untuned quality factor [11] is used. The sole purpose of the quality factor \(Q\) in this paper is a dimensionless representation of the stored energy. The tuning of the quality factor by an ideal reactive lumped element can always be done in the post-processing stage.
with $V$ being the shape of radiator and $E$ being the time-harmonic electric field intensity under the convention $E(t) = \text{Re}(E(\omega) e^{i\omega t})$, $i = \sqrt{-1}$. At the same time, the near-field of the radiator [47] contains the stored energy $W_{sto}(t)$, which is bound to the sources and does not escape from the radiator towards infinity. The evaluation of the cycle mean energy $W_{sto}$ is the goal of the following §2a,b, in which the power balance [10] is going to be employed.

(a) Stored energy in time domain

This section presents a new paradigm of stored energy evaluation. The first step consists in imagining the spherical volume $V_1$ (figure 1) centred in the coordinate system, whose radius is large enough to lie in a far-field region [47]. The total electromagnetic energy content of the sphere (it also contains heat $W_\sigma$) is

$$W(V_1, t) = W_{sto}(t) + W_t(V_1, t), \quad (2.4)$$

where $W_t(V_1, t)$ is the energy contained in the radiation fields that have already escaped from the sources. Let us assume that the power source is switched on at $t = -\infty$, bringing the system into a steady state, and then switched off at $t = t_{\text{off}}$. For $t \in [t_{\text{off}}, \infty)$, the system is in a transient state, during which all the energy $W(V_1, t_{\text{off}})$ will either be transformed into heat at conductors or radiated through the bounding envelope $S_1$. Explicitly, Poynting’s theorem [10] states that the total electromagnetic energy at time $t_{\text{off}}$ can be calculated as

$$W(V_1, t_{\text{off}}) = \int_{t_{\text{off}}}^{\infty} \int_{V_1} \mathbf{E}(r, t) \cdot \mathbf{J}_{\text{ind}}(r, t) \, dV \, dt + \int_{t_{\text{off}}}^{\infty} \oint_{S_1} (\mathbf{E}_{\text{far}}(r, t) \times \mathbf{H}_{\text{far}}(r, t)) \cdot dS \, dt, \quad (2.5)$$

in which $S_1$ lies in the far-field region.

As a special yet important example, let us assume a radiating device made exclusively of perfect electric conductors (PEC). In that case, the far-field can be expressed as [20]

$$\mathbf{H}_{\text{far}}(r, t) = -\frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{n_0 \times \hat{\mathbf{J}}(r', t')}{R} \, dV' \quad (2.6a)$$

and

$$\mathbf{E}_{\text{far}}(r, t) = -\frac{\mu}{4\pi} \int_{V'} \frac{\hat{\mathbf{J}}(r', t') - (n_0 \cdot \hat{\mathbf{J}}(r', t'))n_0}{R} \, dV' \quad (2.6b)$$
in which $c_0$ is the speed of light, $R = |r - r'|$, $n_0 = (r - r')/R$, $t' = t - R/c_0$ stands for the retarded time and the dot represents the derivative with respect to the time argument, i.e.

$$\mathbf{J}(r', t') = \frac{\partial \mathbf{J}(r', \tau)}{\partial \tau} \bigg|_{\tau = t'}.$$  

(2.7)

Since we consider the far-field, we can further write \[48\] $R \approx r$ for amplitudes, $R \approx r - r_0 \cdot r'$ for time delays, with $n_0 \approx n_0$ and $r = |r|$. Using (2.6a)-(2.7) and the above-mentioned approximations, the last term in (2.5) can be written as

\[
\int_{t_{\text{off}}}^{\infty} \int_{S_1} (E_{\text{far}}(r, t) \times H_{\text{far}}(r, t)) \cdot r_0 \, dS \, dt \\
= \frac{1}{Z_0} \int_{t_{\text{off}}}^{\infty} \int_{S_1} |E_{\text{far}}(r, t)|^2 \, dS \, dt \\
= \frac{\mu^2}{Z_0(4\pi)^2} \int_{t_{\text{off}}}^{\infty} \int_0^{\pi} \int_0^{2\pi} \left| \int_{V'} (\mathbf{J}(r', t') - (r_0 \cdot \mathbf{J}(r', t'))r_0) \, dV' \right|^2 \sin \theta \, d\phi \, d\theta \, dt,
\]

(2.8)

where $t' = t - r/c_0 + r_0 \cdot r'/c_0$, where $Z_0$ is the free space impedance and where the relation

$$H_{\text{far}}(r, t) = \frac{r_0 \times E_{\text{far}}(r, t)}{Z_0}$$

(2.9)

has been used. Using (2.5) and (2.8), we are thus able to find the total electromagnetic energy inside $S_1$, see figure 2 for graphical representation.

Note here that the total electromagnetic energy content of the sphere could also be expressed as

$$\mathcal{W}(V_1, t_{\text{off}}) = \frac{1}{2} \int_{V_1} (\mu|\mathcal{H}(r, t_{\text{off}})|^2 + \epsilon|E(r, t_{\text{off}})|^2) \, dV,$$

(2.10)

which can seem to be simpler than the aforementioned scheme. The simplicity is, however, just formal. The main disadvantage of (2.10) is that the integration volume includes also the near-field region, where the fields are rather complex (and commonly singular). Furthermore, contrary to (2.8), the radius of the sphere plays an important role in (2.10) unlike in (2.8), where it appears only via a static time shift $r/c_0$. In fact, it will be shown later on that this dependence can be completely eliminated in the calculation of stored energy.

In order to obtain the stored energy $\mathcal{W}_{\text{sto}}(t_{\text{off}})$ inside $S_1$ we, however, need to know the radiation content of the sphere at $t = t_{\text{off}}$. A thought experiment aimed at attaining it is presented in figure 3. It exploits the properties of (2.8). Consulting the figure, let us imagine that during the
Figure 3. Graphical representation of the radiated energy evaluation via (2.8) for a lossless radiator excited by ideal voltage source. Panel (a) shows a steady state just before \( t = t_{\text{off}} \), when the steady-state radiation (orange wavelets) as well as the steady-state stored energy (blue cloud) were maintained by the source. Panel (b) shows that at \( t > t_{\text{off}} \), the radiating current is modified so to inhibit any radiation, although they possibly create a new energy storage (green cloud). The radiation emitted before \( t = t_{\text{off}} \) (orange wavelets) is unaffected by this modification. Panel (c) depicts the time \( t \gg t_{\text{off}} \) when almost all radiation passed \( S_1 \). The radiation content of the sphere \( S_1 \) is evaluated via (2.8). The green stored energy does not participate as it is not represented by radiation and is consequently not captured by the integral (2.8). (Online version in colour.)

calculation of \( W(V_1, t_{\text{off}}) \) we were capturing the time course of the current \( J(r', t) \) at every point. In addition, let us assume that we define an artificial current \( J_{\text{freeze}}(r', t) \) as

\[
J_{\text{freeze}}(r', t) = \begin{cases} 
J(r', t), & t \leq t_{\text{off}} \\
J(r', t_{\text{off}}), & t > t_{\text{off}} 
\end{cases}
\]  

and use it inside (2.8) instead of the true current \( J(r', t) \). Expression (2.8) then claims that for \( t \leq t_{\text{off}} \) the artificial current \( J_{\text{freeze}}(r', t) \) is radiating in the same way as in the case of the original problem, but for \( t > t_{\text{off}} \) the generation of new radiation is instantly stopped (the radiation generated before \( t < t_{\text{off}} \) was not stopped). Therefore, if we now evaluate (2.8) over the new artificial current, it will give exactly the radiation energy \( W_r(V_1, t_{\text{off}}) \), which has escaped from the sources before \( t_{\text{off}} \). Subtracting it from \( W(V_1, t_{\text{off}}) \), we obtain the stored energy \( W_{\text{sto}}(t_{\text{off}}) \) and averaging over one period, we obtain the cycle mean stored energy

\[
W_{\text{sto}} = \langle W_{\text{sto}}(t_{\text{off}}) \rangle = \frac{1}{T} \int_{\alpha}^{\alpha+T} W_{\text{sto}}(t_{\text{off}}) \, dt_{\text{off}}. 
\]  

With respect to the freezing of the current, it is important to realize that this could mean an indefinite accumulation of charge at a given point. However, it is necessary to consider this operation as to be performed on the artificial impressed sources, which can be chosen freely.

When subtracting the radiated energy from the total energy, it is important to take into account that for \( t \leq t_{\text{off}} \), the currents were the same in both situations. Thus defining \( D = \max \{|r'|\} \), we can state that for \( t \leq t_{\text{off}} + (r - D)/c_0 \), the integrals (2.8) will exactly cancel during the subtraction, see figure 4. The relation (2.8) can then be safely evaluated only for \( t' = t - D/c_0 + r_0 \cdot r'/c_0 \) (the worst-case scenario depicted in figure 4b), which means that the current need to be saved only for \( t > t_{\text{off}} - 2D/c_0 \). It is crucial to take into consideration that this is equivalent to say that, after all, the bounding sphere \( S_1 \) does not need to be situated in the far-field. It is sufficient (and from the computational point of view also advantageous), if \( S_1 \) is the smallest circumscribing sphere centred in the coordinate system, for the rest of the far-field is cancelled anyhow (figure 4a).

As a final note, we mention that even though the above-described method relies on the integration on a spherical surface, the resulting stored energy properly takes into account the actual geometry of the radiator, representing thus a considerable generalization of the time domain prescription for the stored energy proposed in [28] which is able to address only the
regions outside the smallest circumscribing sphere. Further properties of the method are going to be presented on the numerical results in §3 and will be detailed in §4.

(b) Stored energy in frequency domain

This section rephrases the stored energy evaluation by Vandenbosch [31], which approaches the issue in the frequency domain, using the complex Poynting’s theorem that states [20] that

$$-\frac{1}{2} \langle E, J \rangle = P_m - P_e + 2i\omega (W_m - W_e) = P_{in},$$  \hspace{1cm} (2.13)

in which $P_{in}$ is the cycle mean complex power, the terms $P_m$ and $P_e$ form the cycle mean radiated power $P_m - P_e$ and $2\omega (W_m - W_e)$ is the cycle mean reactive net power, and

$$\langle u, v \rangle = \int_V u(r) \cdot v^*(r) \, dV$$  \hspace{1cm} (2.14)

is the inner product [49]. In the classical treatment of (2.13), $W_m$ and $W_e$ are commonly taken [20] as $\mu |H|^2 / 4$ and $\epsilon |E|^2 / 4$ that are integrated over the entire space. Both of them are infinite for the radiating system. Nonetheless, when electromagnetic potentials are used [50], the complex power balance (2.13) can be rewritten as

$$P_{in} = P_m^A - P_e^\phi + 2i\omega (W_m^A - W_e^\phi) = \frac{i\omega}{2} ((A, J) - (\phi, \rho)),$$  \hspace{1cm} (2.15)

where $A$ represents the vector potential, $\phi$ represents the scalar potential and $\rho$ stands for the charge density. As an alternative to the classical treatment, it is then possible to write

$$W_m^A - i\frac{P_m^A}{2\omega} = \frac{1}{4} (A, J)$$  \hspace{1cm} (2.16)

and

$$W_e^\phi - i\frac{P_e^\phi}{2\omega} = \frac{1}{4} (\phi, \rho)$$  \hspace{1cm} (2.17)

without altering (2.13). However, it is important to stress that in such case, $W_m^A$ in (2.16) and $W_e^\phi$ in (2.17) generally represent neither stored nor total magnetic and electric energies [20]. Some attempts have been undertaken to use (2.16) and (2.17) as stored magnetic and electric energies even in non-stationary cases [51]. These attempts were however faced with extensive criticism [52,53], mainly due to the variance of separated energies under gauge transformations.
Regardless of the aforementioned issues, (2.16) and (2.17) were modified [31] in an attempt to obtain the stored magnetic and electric energies. This modification reads

\[ \tilde{W}_m \equiv W_m^A + \frac{W_{\text{rad}}}{2} \] (2.18a)

and

\[ \tilde{W}_e \equiv W_e^\phi + \frac{W_{\text{rad}}}{2}, \] (2.18b)

where the particular term

\[ W_{\text{rad}} = \text{Im} \left\{ k^2 (L_{\text{rad}} \mathbf{J}, \mathbf{J}) - (L_{\text{rad}} \nabla \cdot \mathbf{J}, \nabla \cdot \mathbf{J}) \right\} \] (2.19)

is associated with the radiation field, and the operator

\[ L_{\text{rad}} U = \frac{1}{16\pi \varepsilon \omega^2} \int_{V'} U(r') e^{-ikR} dV' \] (2.20)

is defined using \( k = \omega / c_0 \) as the wavenumber. The electric current \( \mathbf{J} \) is assumed to flow in a vacuum. For computational purposes, it is also beneficial to use the radiation integrals for vector and scalar potentials [47], and rewrite (2.16), (2.17) as [14]

\[ W_m^A - \frac{p_m}{2\omega} = k^2 \langle \mathbf{L} \mathbf{J}, \mathbf{J} \rangle \] (2.21)

and

\[ W_e^\phi - \frac{p_e^\phi}{2\omega} = \langle \mathbf{L} \nabla \cdot \mathbf{J}, \nabla \cdot \mathbf{J} \rangle, \] (2.22)

with

\[ L U = \frac{1}{16\pi \varepsilon \omega^2} \int_{V'} U(r') e^{-ikR} \frac{1}{R} dV'. \] (2.23)

It is suggested in [31] that \( \tilde{W}_{\text{sto}} = \tilde{W}_m + \tilde{W}_e \) is the stored energy \( W_{\text{sto}} \). Yet this statement cannot be considered absolutely correct, since as it was shown in [19,54], \( \tilde{W}_{\text{sto}} \) can be negative. Consequently, it is necessary to conclude that \( \tilde{W}_{\text{sto}} \), defined by the frequency domain concept [31], can only approximately be equal to the stored energy \( W_{\text{sto}} \), resulting in

\[ \tilde{W}_{\text{sto}} \approx W_{\text{sto}}, \] (2.24)

and then by analogy with (2.1)

\[ \tilde{Q} = 2\pi \frac{\tilde{W}_{\text{sto}}}{W_{\text{lost}}} = 2\pi \frac{\tilde{W}_m + \tilde{W}_e}{W_{\text{lost}}} \approx Q \] (2.25)

is defined.

### 3. Results

While it is straightforward to prove that \( W_{\text{sto}} = \tilde{W}_{\text{sto}} \) for RLC circuits made of lumped (non-radiating) elements [55], the evaluation of the stored energy, and, thereby, of the quality factor \( Q \), for radiating structures is far more involved. This is due to the fact that the radiating energy should be subtracted correctly.

The method proposed in §2a was implemented according to the flowchart depicted in figure 5. The evaluation is done in Matlab [56]. The current density \( \mathbf{J}(r', t) \) and the current \( i_{R_0}(t) \) flowing through the resistance \( R_0 \) are the only input quantities used, see figure 5. Note here, that in the general formulation of the method, the power lost at the resistance \( R_0 \) is contained in the first integral on the RHS of (2.5). In particular cases treated in this section, we use the ideal voltage source \( i_{R_0}(t) = 0 \) and perfectly conducting bodies, and thus the first integral in RHS of (2.5) vanishes.

In order to verify the proposed approach, several types of radiators are going to be calculated, namely the centre-fed dipole, Yagi-Uda antenna, bowtie antenna and spherical helix antenna.
All these radiators are made of an infinitesimally thin PEC and operate in vacuum background. The stored energy $W_{sto}$ calculated with the help of the novel method is going to be compared with the results of the classical approach detailed in §2b leading to $\tilde{W}_{sto}$. The comparison is performed via the corresponding quality factors $Q$ and $\tilde{Q}$. For the sake of completeness, the well-known approximation [11] of the quality factor $Q$ based on the frequency derivative of the input impedance, denoted by $Q_{Z}$, is evaluated as well.

All essential steps of the method are going to be explained using the example of a centre-fed dipole in §3a. Subsequently, in §3b–d, the method is going to be directly applied to more complicated radiators. The most important properties of the novel method are going to be examined in the subsequent discussion §4.

(a) Centre-fed thin-strip dipole

The first structure to be calculated is a canonical radiator: a dipole of the length $L$ and width $w = L/200$. The dipole is fed by a voltage source [47] located in its centre.

The calculation starts in FEKO commercial software [57] in which the dipole is simulated. The dipole is fed by a unitary voltage and the current $J(r', \omega)$ is evaluated within the frequency span from $ka = 0$ to $ka \approx 325$ for 8192 samples. The resulting current is imported into Matlab. We define the normalized time $t_n = \omega_0/(2\pi)$ (see $x$-axis of figures 6 and 7), where $\omega_0$ is the angular frequency that the quality factor $Q$ is going to be calculated at. Then, iFFT over $S(\omega)J(r', \omega)$, with

$$S(\omega) = \frac{i\pi}{2} (\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) + \frac{e^{-i\omega t_{off}}}{\omega - \omega_0} \left( \frac{e^{i\omega_0 t_{off}}}{\omega_0 - \omega_0} - \frac{e^{-i\omega_0 t_{off}}}{\omega_0 - \omega_0} \right)$$

(3.1)

is applied, and the time domain current $J(r', t)$ with $\Delta t_n = 0.02$ for $t_n \in (0, 163)$ obtained. Note that (3.1) is just the Fourier transform of $s(t) = \sin(\omega_0 t)H(t_{off} - t)$ with $H(t)$ being the Heaviside unit-step function. The implementation details of iFFT, which must also contain singularity extraction of the source spectrum $S(\omega)$, are not discussed here, as they are not of importance to the method.
of the stored energy evaluation itself. The next step consists in the evaluation of (2.8) for both, the original current \( J(r', t) \) and frozen current \( J_{\text{freeze}}(r', t) \) (figure 5).

At this point, it is highly instructive to explicitly show the time course of the current at the centre of the dipole (figure 6), as well as the time course of the power passing through the surface \( S_1 \) in both aforementioned scenarios (original and frozen currents; figure 7). The source was switched off at \( t_n = 0 \). During the following transient (blue lines in figures 6 and 7), all energy content of the sphere is lost by the radiation. Within the second scenario, with the current constant for \( t > t_{\text{off}} \), the generation of new radiation is instantaneously stopped at \( t_{\text{off}} = 0 \). The power radiated for \( t_{\text{off}} > 0 \) (red line in figure 7) then represents the radiation that existed at \( t = t_{\text{off}} \) within the sphere, but needed some time to leave the volume. Subtracting the blue and red curves
in figure 7 and integrating in time for \( t > t_{\text{off}} \) then gives the stored energy at \( t = t_{\text{off}} \). In order to construct the course of \( W_{\text{sto}}(t_{\text{off}}) \), the stored energy is evaluated for six different switch-off times \( t_{\text{off}} \). The resulting \( W_{\text{sto}}(t_{\text{off}}) \) is then fitted by

\[
W_{\text{sto}}(t_{\text{off}}, \omega_0) = A + B \sin(2\omega_0 t_{\text{off}} + \beta). \tag{3.2}
\]

The fitting was exact (within the used precision) in all fitted points, which allowed us to consider (3.2) as an exact expression for all \( t_{\text{off}} \). The constant \( A \) then leads to \( W_{\text{sto}}(\omega_0) \) and thus to \( Q(\omega_0) \).

We are typically interested in the course of \( Q \) with respect to the frequency. Repeating the above-explained procedure for varying \( \omega_0 \), we obtain the red curve in figure 8. In the same figure, the comparison with \( \tilde{Q} \) from §2b (blue curve) and \( Q_Z \) [11] (green curve) is depicted. To calculate \( \tilde{Q} \) by means of (2.21), (2.22), (2.19), (2.24) in the frequency domain, we used the currents \( J'(r', \omega) \) from FEKO. The quality factor \( Q_Z \) has been evaluated from the input impedance supplied by the same software.

(b) Yagi-Uda antenna

Yagi-Uda antenna was selected as a representative of quite complex structure. The antenna has the same dimensions as in [11] and is depicted in the inset in figure 10. As this antenna has
non-unique phase centre, it can serve as an ideal candidate for verification of the coordinate independence of the novel method. The results were calculated in the same way as in the previous examples and are indicated in figures 9 and 10. The comparison between the results in figure 9 and those related to the dipole in figure 7 clearly reveals that the transient state is remarkably longer in the case of Yagi-Uda antenna, which means that the longer integration time is required. Furthermore, it can be seen (red curve for $t > t_{off}$) that the bounding sphere contains a considerable amount of radiation that should be subtracted. The accuracy of this subtraction is embodied in figure 10, which shows the quality factors $Q$, $\tilde{Q}$ and $Q_Z$. Notice the similarity between $Q$ and $\tilde{Q}$.

(c) Bowtie antenna

The next example to be treated is a bowtie antenna, a representative of ultra-wideband antennas which exhibit very low value of the quality factor [47]. The proportions of the bowtie are depicted in figure 11 and are taken identical to those from [18] allowing thus for direct comparison. Owing to the computationally demanding evaluation of the time-domain method, the quality factor $Q$...
has been calculated at only eight frequency points represented by red markers in figure 11. Good agreement with the quality factors $\tilde{Q}$ and $Q_Z$ can be observed.

(d) Spherical helix antenna

The last example given in this paper is a three-dimensional thin-wire spherical helix antenna of radius $a$, which is fed in its centre by the voltage gap (figure 12). This antenna represents ultimate and computationally challenging example, both because of number of discretization segments and because of relatively complicated transient response. The results for all three quality factors are depicted in figure 12. Excellent agreement of all methods can be observed.

4. Discussion

Based on the previous sections, important properties of the novel time domain technique can be isolated and discussed. This discussion also poses new and so far unanswered questions that can be addressed in future.

The coordinate independence/dependence constitutes an important issue of many similar techniques evaluating the stored electromagnetic energy. Contrary to the radiation energy subtraction of Fante [29], Rhodes [30], Yaghjian & Best [11] or Gustafsson & Jonsson [33], the new time-domain method can be proved to be coordinate-independent. It means that the same results are obtained irrespective of the position and rotation of the coordinate system. Owing to the explicit reference to coordinates, this statement in question may not be completely obvious from (2.8). However, it should be noted that any potential spatial shift or rotation of coordinate system emerges only as a static time shift of the received signal at the capturing sphere. Such static shift is irrelevant to the energy evaluation due to the integration over semi-infinite time interval.

The positive semi-definiteness represents another essential characteristic. It should be immanent in all theories concerning the stored energy. Although (2.8) contains the absolute value, it is difficult to mathematically prove the positive semi-definiteness of the stored energy evaluation as a whole, because it is not automatically granted that the integration during the second run integrates smaller amount of energy than the integration during the first run. Despite that, we can anticipate the expected behaviour from the physical interpretation of the method, which stipulates that the energy integrated in the second run must have been part of the first run as well. At worst, the subtraction of both runs can give null result. This observation is in perfect agreement with the numerical results. Nevertheless, the exact and rigorous proof admittedly remains an unresolved issue that is to be addressed in the future.
Unlike the methods of Fante [29], Rhodes [5] or Collin & Rothschild [25], the obvious benefit of the novel method consists in its ability to account for a shape of the radiator, not being restricted to the exterior of the circumscribing sphere.

Finally, it is crucial to realize that the novel method is not restricted to the time-harmonic domain, but can evaluate the stored energy in any general time-domain state of the system. This raises new possibilities for analysing radiators in the time domain, namely the ultra-wideband radiators and other systems working in the pulse regime.

5. Conclusion

Two different concepts aiming to evaluate the stored electromagnetic energy and the resulting quality factor $Q$ of the radiating system were investigated. The novel time domain scheme constitutes the first one, while the second one uses time-harmonic quantities and classical radiation energy extraction. Both methods were subject to theoretical and numerical comparison. For completeness, the quality factor $Q$ resulting from the two aforementioned methods has also been compared to the common approximation by the normalized frequency derivative of the input impedance.

It was shown that the most practical scheme based on the frequency derivative of the input impedance generally fails to give the correct quality factor, but may serve as a very good estimate. By contrast, the frequency domain concept with far-field energy extraction was found to work correctly in the case of general RLC circuits and simple radiators. Unlike the newly proposed time domain scheme, it could however yield negative values of stored energy, which is actually known to happen for specific current distributions. In this respect, the novel time domain method proposed in this paper could be denoted as reference, since it exhibits the coordinate independence, positive semi-definiteness, and most importantly, takes into account the actual shape of the radiator. Another virtue of the novel scheme is constituted by the possibility to use it out of the time-harmonic domain, e.g. in the realm of radiators excited by general pulse.

The follow-up work should focus on the radiation characteristics of separated parts of radiators or radiating arrays, the investigation of different time domain feeding pulses and their influence on performance of ultra-wideband radiators and, last but not least, on the theoretical formulation of the stored energy density generated by the new time domain method.

Authors' contributions. All authors contributed to the formulation, did numerical simulations and drafted the manuscript. All authors gave final approval for publication.

Competing interests. We declare we have no competing interests.

Funding. The authors would like to acknowledge the support of project 15-10280Y funded by the Czech Science Foundation.

Acknowledgements. The authors are grateful to Ricardo Marques (Department of Electronics and Electromagnetism, University of Seville) and Raul Berral (Department of Applied Physics, University of Seville) for many valuable discussions that stimulated some of the core ideas of this contribution. The authors are also grateful to Jan Eichler (Department of Electromagnetic Field, Czech Technical University in Prague) for his help with the simulations.

References

1. Morse PM, Feshbach H. 1953 Methods of theoretical physics. New York, NY: McGraw-Hill.
2. Hallen E. 1962 Electromagnetic theory. London, UK: Chapman & Hall.
3. Volakis JL, Chen C, Fujimoto K. 2010 Small antennas: miniaturization techniques & applications. New York, NY: McGraw-Hill.
4. Collin RE. 1992 Foundations for microwave engineering, 2nd edn. New York, NY: John Wiley.
5. Rhodes DR. 1976 Observable stored energies of electromagnetic systems. J. Franklin Inst. 302, 225–237. (doi:10.1016/0016-0032(79)90126-1)
6. Kajfez D, Wheless WP. 1986 Invariant definitions of the unloaded Q factor. IEEE Antennas Propag. Mag. 34, 840–841. (doi:10.1109/TMTT.1986.1133452)
7. Harrington RF. 1958 On the gain and beamwidth of directional antennas. *IRE Trans. Antennas Propag.* 6, 219–225. (doi:10.1109/TAP.1958.1144605)
8. IEEE Standard Definitions of Terms for Antennas. Piscataway, NJ: IEEE Press, IEEE Std 145-1993.
9. Foster RM. 1924 A reactance theorem. *Bell System Tech. J.* 3, 259–267. (doi:10.1098/rspla.1977.0018)
10. Harrington RF. 2001 *Time-harmonic electromagnetic fields*, 2nd edn. New York, NY: John Wiley.
11. Yaghjian AD, Best SR. 2005 Impedance, bandwidth and Q of antennas. *IEEE Trans. Antennas Propag.* 53, 1298–1324. (doi:10.1109/TAP.2005.844443)
12. Best SR, Hanna DL. 2010 A performance comparison of fundamental small-antenna designs. *IEEE Antennas Propag. Mag.* 52, 47–70. (doi:10.1109/MAP.2010.5466398)
13. Sievenpiper DF, Dawson DC, Jacob MM, Kanar T, Sanghoon K, Jiang L, Quarfoth RG. 2012 Experimental validation of performance limits and design guidelines for small antennas. *IEEE Trans. Antennas Propag.* 60, 8–19. (doi:10.1109/TAP.2011.2167938)
14. Capek M, Jelinek L, Hazdra P, Eichler J. 2014 The measurable Q factor and observable energies of radiating structures. *IEEE Trans. Antennas Propag.* 62, 311–318. (doi:10.1109/TAP.2013.2287519)
15. Jin J-M. 2010 *Theory and computation of electromagnetic fields*. New York, NY: John Wiley.
16. Gustafsson M, Nordebo S. 2006 Bandwidth, Q factor and resonance models of antennas. *Prog. Electromagn. Res.* 62, 1–20. (doi:10.2528/PIER06033003)
17. Capek M, Jelinek L, Hazdra P. 2015 On the functional relation between quality factor and fractional bandwidth. *IEEE Trans. Antennas Propag.* 63, 2787–2790. (doi:10.1109/TAP.2015.2414472)
18. Gustafsson M, Jonsson BLG. 2015 Antenna Q and stored energy expressed in the fields, currents, and input impedance. *IEEE Trans. Antennas Propag.* 63, 240–249. (doi:10.1109/TAP.2014.2368111)
19. Gustafsson M, Cismasu M, Jonsson BLG. 2012 Physical bounds and optimal currents on antennas. *IEEE Trans. Antennas Propag.* 60, 2672–2681. (doi:10.1109/TAP.2012.2194658)
20. Jackson JD. 1998 *Classical electrodynamics*, 3rd edn. New York, NY: John Wiley.
21. Chu LJ. 1948 Physical limitations of omni-directional antennas. *J. Appl. Phys.* 19, 1163–1175. (doi:10.1063/1.1715038)
22. Dirac PAM. 1938 Classical theory of radiating electrons. *Proc. R. Soc. Lond. A* 167, 148–169. (doi:10.1098/rspl.1938.0124)
23. Thal HL. 1978 Exact circuit analysis of spherical waves. *IEEE Trans. Antennas Propag.* 26, 282–287. (doi:10.1109/TAP.1978.1141822)
24. Thal HL. 2012 Q bounds for arbitrary small antennas: a circuit approach. *IEEE Trans. Antennas Propag.* 60, 3120–3128. (doi:10.1109/TAP.2012.2196920)
25. Collin RE, Rothschild S. 1964 Evaluation of antenna Q. *IEEE Trans. Antennas Propag.* 12, 23–27. (doi:10.1109/TAP.1964.1138151)
26. McLean JS. 1996 A re-examination of the fundamental limits on the radiation Q of electrically small antennas. *IEEE Trans. Antennas Propag.* 44, 672–675. (doi:10.1109/8.496253)
27. Manteghi M. 2010 Fundamental limits of cylindrical antennas. Technical Report. 1, Virginia Tech.
28. Collin RE. 1998 Minimum Q of small antennas. *J. Electromagn. Waves Appl.* 12, 1369–1393. (doi:10.1163/156939398X01457)
29. Fante RL. 1969 Quality factor of general ideal antennas. *IEEE Trans. Antennas Propag.* 17, 151–157. (doi:10.1109/TAP.1969.1139411)
30. Rhodes DR. 1977 A reactance theorem. *Proc. R. Soc. Lond. A* 353, 1–10. (doi:10.1098/rspla.1977.0018)
31. Vandenbosch GAE. 2010 Reactive energies, impedance, and Q factor of radiating structures. *IEEE Trans. Antennas Propag.* 58, 1112–1127. (doi:10.1109/TAP.2010.2041166)
32. Geyi W. 2003 A method for the evaluation of small antenna Q. *IEEE Trans. Antennas Propag.* 51, 2124–2129. (doi:10.1109/TAP.2003.814755)
33. Gustafsson M, Jonsson BLG. 2015 Stored electromagnetic energy and antenna Q. *Prog. Electromagn. Res.* 150, 13–27. (doi:10.2528/PIER14111502)
34. Vandenbosch GAE. 2013 Reply to ‘Comments on “Reactive energies, impedance, and Q factor of radiating structures”’. *IEEE Trans. Antennas Propag.* 61, 6268. (doi:10.1109/TAP.2013.2281573)
35. Shlivinski A, Heyman E. 1999 Time-domain near-field analysis of short-pulse antennas—part I: spherical wave (multipole) expansion. *IEEE Trans. Antennas Propag.* **47**, 271–279. (doi:10.1109/8.761066)

36. Shlivinski A, Heyman E. 1999 Time-domain near-field analysis of short-pulse antennas—part II: reactive energy and the antenna Q. *IEEE Trans. Antennas Propag.* **47**, 280–286. (doi:10.1109/8.761067)

37. Collardey S, Sharaiha A, Mahdjoubi K. 2006 Calculation of small antennas quality factor using FDTD method. *IEEE Antennas Wireless Propag. Lett.* **5**, 191–194. (doi:10.1109/LAWP.2006.873947)

38. Vandenbosch GAE. 2013 Radiators in time domain-part II: Finite pulses, sinusoidal regime and Q factor. *IEEE Trans. Antennas Propag.* **61**, 4004–4012. (doi:10.1109/TAP.2013.2261045)

39. Kaiser G. 2011 Electromagnetic inertia, reactive energy and energy flow velocity. *J. Phys. A.: Math. Theor.* **44**, 1–15. (doi:10.1088/1751-8113/44/34/345206)

40. Capek M, Jelinek L. 2015 Various interpretations of the stored and the radiated energy density. (http://arxiv.org/abs/1503.06752)

41. Harrington RF. 1960 Effects of antenna size on gain, bandwidth, and efficiency. *J. Nat. Bur. Stand. 64-D*, 1–12. (doi:10.6028/jres.064d.003)

42. Yaghjian AD, Gustafsson M, Jonsson BLG. 2013 Minimum Q for lossy and lossless electrically small dipole antenna. *Prog. Electromagn. Res.* **143**, 641–673. (doi:10.2528/PIER13103107)

43. Jonsson BLG, Gustafsson M. 2015 Stored energies in electric and magnetic current densities for small antennas. *Proc. R. Soc. A* **471**, 1–23. (doi:10.1098/rspa.2014.0897)

44. Jelinek L, Capek M, Hazdra P, Eichler J. 2015 An analytical evaluation of the quality factor $Q_Z$ for dominant spherical modes. *IET Microw. Antennas Propag.* **9**, 1096–1103. (doi:10.1049/iet-map.2014.0302)

45. Capek M, Hazdra P, Eichler J. 2012 A method for the evaluation of radiation Q based on modal approach. *IEEE Trans. Antennas Propag.* **60**, 4556–4567. (doi:10.1109/TAP.2012.2207329)

46. Gustafsson M, Nordebo S. 2013 Optimal antenna currents for Q, superdirectivity, and radiation patterns using convex optimization. *IEEE Trans. Antennas Propag.* **61**, 1109–1118. (doi:10.1109/TAP.2012.2227656)

47. Balanis CA. 1989 *Advanced engineering electromagnetics*. New York, NY: John Wiley.

48. Balanis CA. 2005 *Antenna theory analysis and design*, 3rd edn. New York, NY: John Wiley.

49. Akhiezer NI, Glazman IM. 1993 *Theory of linear operators in Hilbert space*, 2nd edn. New York, NY: Dover.

50. Morgenthaler FR. 2011 *The power and beauty of electromagnetic fields*. New York, NY: John Wiley.

51. Carpenter CJ. 1989 Electromagnetic energy and power in terms of charges and potentials instead of fields. *Proc. IEE A* **136**, 55–65.

52. Uehara M, Allen JE, Carpenter CJ. 1992 Comments to ‘Electromagnetic energy and power in terms of charges and potentials instead of fields’. *Proc. IEE A* **139**, 42–44.

53. Endean VG, Carpenter CJ. 1992 Comments to ‘Electromagnetic energy and power in terms of charges and potentials instead of fields’. *Proc. IEE A* **139**, 338–342.

54. Jelinek L, Capek M, Hazdra P, Eichler J. 2014 Lower bounds of the quality factor $Q_Z$. In *Proc. 2014 IEEE Int. Symp. on Antennas and Propagation and North America Radio Science Meeting*, pp. 53–54. (doi:10.1109/APS.2014.6904358)

55. Capek M, Jelinek L, Vandenbosch GAE, Hazdra P. 2014 A scheme for stored energy evaluation and a comparison with contemporary techniques. (http://arxiv.org/abs/1403.0572)

56. The MathWorks. 2015 Matlab. Natick, MA: The Mathworks, Inc.

57. EM Software & Systems. South Africa: FEKO.