A scheme for demonstration of fractional statistics of anyons in an exactly solvable model

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We propose a scheme to demonstrate fractional statistics of anyons in an exactly solvable lattice model proposed by Kitaev that involves four-body interactions. The required many-body ground state, as well as the anyon excitations and their braiding operations, can be conveniently realized through dynamic laser manipulation of cold atoms in an optical lattice. Due to the perfect localization of anyons in this model, we show that a quantum circuit with only six qubits is enough for demonstration of the basic braiding statistics of anyons. This opens up the immediate possibility of proof-of-principle experiments with trapped ions, photons, or nuclear magnetic resonance systems.

Anyons, as exotic quasiparticles living in two dimensions with fractional statistics [1], have attracted strong interest over the past two decades. The excitations in fractional quantum Hall systems have been predicted to be anyons [2], and fractional charges have been confirmed with some experimental evidence [3]. However, a direct observation of fractional statistics associated with anyon braiding is hard in this system and has recently attracted intriguing theoretical proposals [4]. Other systems with anyon excitations have also been proposed. In particular, in the context of topological quantum computation, Kitaev described two exactly solvable theoretical models [5, 6] which support anyons. An implementation scheme has been proposed for the second model Hamiltonian [5, 7, 8], which involves only two-body interactions. A detection method of anyons associated with this implementation scheme has been proposed recently [9]. This implementation scheme with an optical lattice requires achievement of very low temperature [7], which unfortunately is still somewhat beyond the current experimental capabilities.

In this paper, we propose a method to realize anyons and to observe their braiding statistics associated with the first Kitaev model that is based on four-body interactions. Four-body interactions are notoriously difficult to generate experimentally in a controllable fashion. Our key idea here is to generate dynamically the ground state and the excitations of this model Hamiltonian instead of direct ground-state cooling. Note that the anyon properties are associated with the underlying many-body entangled states. If we generate exactly the same ground and the excited states, we should be able to observe the same property. We propose several systems for implementation of this idea. First, we show that laser manipulation of ultra-cold atoms in an optical lattice provides a natural system to realize a large scale ground state of this model Hamiltonian. The braiding statistics of anyons can then be observed with a series of single-bit operations. Then, we show that the systems of trapped ions [10], photons [11], or nuclear magnetic resonance [12], provide a ready platform to realize a small scale system for proof-of-principle demonstration of the anyon braiding statistics. The anyons are perfectly localized quasiparticles in this model Hamiltonian, which means we do not need a large system for implementing their braiding operations. We show that as little as six qubits are enough for demonstration of the basic braiding statistics; and with nine qubits, one can further demonstrate robustness of the braiding operation to certain variations of the braiding path.

The first Kitaev model is a spin Hamiltonian for a two-dimensional square lattice [5] (more generally, we can extend the model from a square lattice to any planar graph). One associates each edge of the lattice with a qubit (a spin-1/2 particle). The model Hamiltonian is given by

$$H = -\sum_v A_v - \sum_f B_f,$$

where $A_v$ is defined for each vertex $v$ as $A_v = \prod_{j \in \text{star}(v)} X_j$, and $B_f$ is defined for each face $f$ as $B_f = \prod_{j \in \text{boundary}(f)} Z_j$ (See Fig. 1A for an illustration). The operators $X$ and $Z$ denote the standard Pauli matrix $\sigma_x$ and $\sigma_z$, respectively, and $A_v$ and $B_f$ are called the stabilizer operators in the context of quantum error correction. In a square lattice, each term of the Hamiltonian $H$ represents four-body interaction of local qubits. One can easily check that all the terms of this Hamiltonian commute, so it is straightforward to get the ground state $|\varphi\rangle$ of the Hamiltonian $H$, which is given by $A_v |\varphi\rangle = |\varphi\rangle$ and $B_f |\varphi\rangle = |\varphi\rangle$ for all vertexes and faces. The state $|\varphi\rangle$ is highly entangled [13].

A quasiparticle is generated on the vertex $v_0$ (or face $f_0$) if $A_{v_0}$ (or $B_{f_0}$), acting on the excited state $|\varphi_e\rangle$, yields an eigenvalue $-1$ instead of $+1$ for the ground state. The quasi-particles in these two cases are called e-particles (vertex) or m-particles (face), respectively. One can check that the e-particles and m-particles by themselves are bosons, but the mutual statistics between the e and m particles become the one for 1/2-anyons [3], as we
get a phase flip $e^{i\pi}$ if we move the e(m)-particle around the m(e)-particle along an arbitrary loop.

It is hard to directly generate the interactions represented by the Hamiltonian $H$ and to cool the system to its ground state $|\varphi\rangle$. However, we note that the braiding statistics of anyons are directly associated with the entanglement properties of the underlying ground and excited states, whereas the Hamiltonian only plays an implicit role for nailing down the corresponding state. As long as we can create the state $|\varphi\rangle$ and generate the e and m-particles above this state, we should be able to demonstrate the fractional braiding statistics between these quasiparticles. Our task reduces to how to create the state $|\varphi\rangle$ and how to demonstrate the fractional statistics of quasiparticles excited from the state $|\varphi\rangle$. In the following, we propose two approaches, targeted at a large scale implementation and a small scale proof-of-principle demonstration, respectively.

Physically, the simplest method to generate a large scale state $|\varphi\rangle$ is to start from a two-dimensional (2D) cluster state. The cluster and the graph states, introduced in [14, 15], are defined as the co-eigenstate of a set of commuting stabilizer operators $S_i$ (with +1 eigenvalues). The 2D cluster state is associated with qubits on vertexes of a 2D square lattice, where a stabilizer operator is introduced for each vertex as $S_i = X_i \bigotimes_{j \in N(i)} Z_j$. Therein, $N(i)$ denotes the set of nearest neighbors of the vertex $i$. A 2D cluster state can be conveniently generated with ultra-cold atoms in a square optical lattice. Each atom is effectively two-level, which defines a qubit (or a spin-1/2 particle). One starts with all the atoms in equal superpositions of $|0\rangle$ and $|1\rangle$ states (co-eigenstates of $X_i$). Then, with control of the optical lattice potential, one turns on spin-dependent collisions [16] or tunneling [7] for a fixed amount of time (both give rise to an effective Ising interaction) to get the 2D cluster state [14]. Starting from a 2D cluster state, single-bit measurements of half of the qubits in the bases $X$ and $Z$ respectively will yield the desired state $|\varphi\rangle$ [17]. The measurement pattern is illustrated in Fig. 1B. One can check these measurements transfer the set of the stabilizer operators from $\{S_i\}$ to $\{\pm A_i, \pm B_f\}$ for the remaining qubits, as it is required for the ground state of the Hamiltonian (1). The sign factors $\pm 1$ in the stabilizer generators depend on the measurement outcomes from the measured qubits. All these sign factors can be readjusted to +1 by subsequent local Pauli operations.

With creation of the state $|\varphi\rangle$, one can demonstrate the fractional statistical phase of the 1/2-anyons through a Ramsey type interference experiment [9, 18]. First, by applying a single-bit rotation $Z$ to one of the edge qubits, one creates a pair of e-particles on the neighboring vertexes, as represented by an excited state $|\varphi_e\rangle$ of the Hamiltonian (1). Then, with half of the rotation $\sqrt{Z}$, we get the superposition $(|\varphi\rangle + |\varphi_e\rangle)/\sqrt{2}$. Similarly, with a single-bit rotation $X$ on another edge qubit, we create a pair of m-particles on the neighboring faces. Then, one can move one of the m-particles around one of the e-particles along a loop through successive $X$ rotations acting on the qubits in the loop (see Fig. 1C). After fusion of the m-particles (through another $X$ rotation), the underlying state becomes $(|\varphi\rangle - |\varphi_e\rangle)/\sqrt{2}$ due to the fractional phase $\pi$ acquired from braiding of the e and m-particles. This phase flip can be unambiguously detected. For instance, with another $\sqrt{Z}$ operation, $(|\varphi\rangle - |\varphi_e\rangle)/\sqrt{2}$ goes to $|\varphi\rangle$ (whereas without the statistics phase $\pi$, $(|\varphi\rangle + |\varphi_e\rangle)/\sqrt{2}$ would go to $|\varphi_e\rangle$). The states $|\varphi\rangle$ and $|\varphi_e\rangle$ can be distinguished by measuring the relevant stabilizer operators $\{A_v\}$ (measurement of the stabilizer operators only requires single-bit detection together with classical correlation of the measurement outcomes).

In the above implementation scheme, except for the initial step of preparation of the cluster state with spin-dependent atomic collisions, all the following steps are based on single-bit unitary operations or measurements.
With cold atoms in an optical lattice, one can indeed realize a large-scale system with millions of qubits [19], and thus can well demonstrate the robustness of the braiding operation: the statistics phase $\pi$ depends only on the topology of the loop and not on the detailed path. The only remaining experimental challenge in this scheme is the requirement of addressing of individual qubits for single-bit operations. The experiments on ultracold atoms are making progress towards the ability of single-bit addressing. On the other hand, there are also other experimental systems which start from the bottom up and have demonstrated the ability to fully control a small number of qubits. Along with that line, in the following we propose a different implementation scheme to provide proof-of-principle demonstration of anyons and their braiding statistics in small systems. This scheme is well within reach of the current experimental technology.

It is interesting to ask what the smallest system size is for demonstration of the braiding statistics of anyons. The system size has to be significantly larger than the size of anyons for the braiding operation. For any systems with anyons, the quasiparticles are necessarily localized in space (otherwise it is impossible to define the braiding). For the first Kitaev model, the quasiparticles are particularly well localized: the e–particle is on a single vertex and the m-particle is on a single face. Due to this perfect localization, we do not need a large system for a basic braiding operation. It turns out that the graph shown in Fig. 2A is the smallest system for implementation of the anyon braiding operation, which just requires manipulation of six qubits.

The graph in Fig. 2A corresponds to the ground state of the Hamiltonian $H_6 = -A_1 - A_2 - B_1 - B_2 - B_3 - B_4$ of six edge qubits, where

$$A_1 = X_1 X_2 X_3, \quad A_2 = X_3 X_4 X_5 X_6, \quad B_1 = Z_1 Z_2 Z_4, \quad B_2 = Z_2 Z_3 Z_5, \quad B_3 = Z_4 Z_6, \quad B_4 = Z_5 Z_6.$$ 

One can check that its ground state $|\phi\rangle_6$ is equivalent under local single-bit operations to a graph state shown in Fig. 2B (see Ref. [17] for definition of the graph state). One can use five controlled phase flip (CPF) gates to prepare the graph state of Fig. 2B and thus also $|\phi\rangle_6$. Figure 2C shows the detailed preparation circuit for the state $|\phi\rangle_6$ which involves five CPF gates as well as a few Hard-mard (H) gates. Then, as also shown in Fig. 2C, a $\sqrt{Z_3}$ operation on the qubit 3 generates the superposition state $|\phi\rangle_6 + |\varphi\rangle_6)/\sqrt{2}$, where $|\varphi\rangle_6$ has a pair of e-particles on two vertices of the edge 3. Another $X_4$ operation on the qubit 4 further generates a pair of m-particles on two neighboring faces of the edge 4. With four $X$ operations on the qubits 6-5-3-4, we achieve the braiding by moving an m-particle around a e-particle along the loop shown in Fig. 2A. Finally, we fuse (annihilate) all the quasiparticles (with another $\sqrt{Z_3}$ and $X_4$ operation), and the resultant state is detected with single-bit measurements (in $X$ or $Z$ basis) of the six stabilizer operators.

In the above six-qubit minimum implementation, the braiding loop is unique and there is no way to show the robustness of the topological braiding operation by moving the anyons along different paths. To have demonstration of some robustness of the anyon operation to different braiding paths, we give another implementation in Fig. 3 which uses nine qubits. The ground state of this nine-qubit Hamiltonian represented by Fig. 3A is locally equivalent to graph state shown in Fig. 3B, and the detailed implementation circuit is given in Fig. 3C. In this plane geometry (see Fig. 3A), there are several different loops for the quasiparticles. The anyonic operation only depends on the topological character of the loop, i.e., whether the different types of quasiparticle (e and m) “braid” with each other. For instance, both the loops 6-5-3-4 and 9-8-5-3-4-7 of the m-particles (the dashed lines in Fig. 3A) give the same result of the statistical phase, while the loop 9-8-6-7 (the solid line in Fig 3A) has no effect since it does not enclose a e-particle. Braiding along different loops can be implemented with single-bit $X$ operations on different sets of individual qubits. See Fig. 3C for an example circuit implementing braiding of a m-particle with an e-particle. The m-particle is taken along

![Diagram](image-url)
In the context of quantum computing, entangling gates of six to eight qubits have been demonstrated in with trapped ions \[11\], six-photon entanglement in the linear optics system \[11\], and manipulation of a dozen of nuclear spins in the nuclear magnetic resonance system \[12\]. With such experimental capabilities, the above six-bit and nine-bit implementation schemes can be expected to be realized in the very near future. This kind of proof-of-principle demonstration of anyons in small and relatively simple systems will represent an important step towards the long pursued goal to demonstrate fractional statistics of quasiparticles in a macroscopic material. Such abilities, properly extended to large systems, will also be critical for future implementation of fault-tolerant topological quantum computation.

In summary, we have proposed a method to demonstrate fractional braiding statistics of anyons in an exactly solvable spin model. Two types of implementation schemes are described, respectively targeted at a large-scale realization with cold atoms in an optical lattice and a small-scale realization with some available qubit systems, such as trapped ions, photons, or nuclear spins in liquids of molecules. The proposed schemes open the prospect of experimental demonstration of anyons in the near future.

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