Entropy in the Thermal Model

Maciej Andrzej Stankiewicz

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Abstract

A brief review of the Hadron Gas model with reference to high energy heavy ion collisions is presented. The entropy dependence on temperature and baryonic chemical potential is numerically calculated, together with the entropy distribution between baryons and mesons. The theoretical entropy for a QGP with equivalent parameters is also calculated. It is shown that at low temperatures the dominant entropy contribution comes from baryons, while at high energies the dominant contribution comes from mesons. The turnover from baryon to meson domination occurs at $T \approx 140\text{MeV}$, which corresponds to an energy domain of the AGS at Brookhaven National Laboratory.
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Chapter 1

Introduction

Relativistic heavy ion collisions provide us with the opportunity to study strongly interacting matter at extremely high temperature and pressure. It is the current goal of collision experiments to attempt to create a new state of matter known as a Quark-Gluon Plasma (QGP). This refers to a state where the quarks and gluons, which were originally bound into nucleons, become deconfined and form a self-consistent plasma. This form of matter is believed to have existed in the very early universe (shortly after the Big Bang).

The Quark-Gluon Plasma was originally proposed as a byproduct of QCD. Quantum chromodynamics was developed as a theory of the strong interaction, but unlike QED, it is almost impossible to obtain analytic solutions. Lattice QCD (discrete points represent space-time) has been worked on numerically, and it suggests that a QGP forms when the temperature ($\sim 150 - 200 \text{ MeV}$) or density (up to $5-10 \rho_0$, $\rho_0 \sim 3 \times 10^{14} \text{g/cm}^3$, the nuclear density) becomes large enough.

The search for the QGP is currently ongoing. As the beam energies of heavy ion accelerators continue to increase, the systems resultant from the collisions have initial temperatures and densities close to the above thresholds.

Since direct observation of the QGP is impossible (it is very short-lived, and its component quarks and gluons become confined into hadrons before they reach the detectors), one has to infer its formation from the observed distributions of hadrons, leptons and photons. This is highly non-trivial, as most hadronic observables reflect conditions at freeze-out (when the hadrons stopped interacting – similar to the surface of last scattering.), by which time final state interactions may have caused loss of all information about the initial partonic state.

After the release of the latest data from RHIC, there has been some speculation at to whether data showed signs of existence of QGP, but the issue has not been settled yet. It is hoped that with still higher energies expected in the near future at LHC, results signalling the formation of QGP will be observed.
1.1 Ideal Hadron Gas Model

While QGP is expected to be the high-energy limit of a collection of hadronic particles, at lower temperatures there is a different model. The conventional, confined phase is usually referred to as Hadronic Gas (HG), consisting of hadrons of different types (including the short-lived resonances) such as $\pi$, $\rho$, $N$, $\Delta$, $K$, whose properties (mass, spin, degeneracy) are well known.

This phase does exist in heavy ion collisions. Even if a QGP is created, it will rapidly expand until the temperature drops below the critical value, and hadronization takes place. This means that the deconfined quarks and gluons of the QGP will form bound states of baryons, mesons and anti-baryons.

The system will then form a small volume that is filled with a huge variety of interacting hadrons. Assuming that a state of equilibrium is reached it seems plausible to treat the system (fireball) using the methods of statistical mechanics. The Thermal Model can be applied to a system at thermal equilibrium, although to find the particle multiplicities, chemical equilibrium also required.

1.1.1 Thermal and chemical freeze-out

Directly after the hadronization, the particles will be sufficiently close to interact strongly (distances of order 1fm), and it is expected that this form of interaction will cause the system to reach a state of “chemical equilibrium” (chemical referring to the composition of the fireball). Hence, chemical equilibrium means that the number of each form of particle will be constant - the rate of creation/annihilation for each type of particle will be exactly equal. This will then continue until the gas cools further (it is expanding), and the temperature drops below a critical value. This is called chemical freeze-out - the point at which inelastic collisions cease and all particle ratios are frozen.

A second form of freeze-out, known as thermal freeze-out is often also introduced. After the chemical freeze-out, once particle ratios become fixed, particles still interact elastically (eg. $\pi + N \rightarrow \Delta \rightarrow \pi + N$). These types of reactions have larger cross-sections and continue to occur, redistributing momentum in the system while unchanging the chemical composition. The fireball continues expanding and cooling until these interactions cease (thermal freeze-out). The particles then fly off towards the detectors.

By observing the particle abundances (which show the properties of the system at chemical freeze-out), one finds that (at CERN SPS energies) the distribution is characteristic of one at $T \sim 170$ MeV. By observing the particle momentum distributions (which characterize the system at thermal freeze-out), one can infer that (for CERN SPS), the thermal freeze-out occurs later at $T \sim 130$ MeV.
1.2 Theoretical Formulation of the Thermal Model

In the fireball, the number of particles is not fixed, and so one works with a Grand Canonical Ensemble. Firstly, consider the analysis of a static fireball. Hence the system lives for a period of time in a fixed volume $V$. Then for one type of particle:

$$\Omega_{GC} = -kT \ln Z$$

where $Z$ is the partition function, the form of which depends on whether the particle follows Bose-Einstein or Fermi-Dirac statistics. The specific quantities will be discussed in detail in the next section. The multiplicity of hadrons of species $i$ is given as:

$$N_i = V n_i = \frac{V g_i}{(2\pi \hbar)^3} \int f_i(p) d^3p$$

where $n_i$ is the number density, $g_i = 2J_i + 1$ is the spin degeneracy factor, and $f_i(p)$ is the momentum distribution function. In a thermodynamic equilibrium distribution, these functions have a relatively simple form:

$$f_i(p) = \left[ e^{\frac{E_i(p) - \mu_i}{kT}} + \epsilon \right]^{-1} = \left[ e^{\sqrt{p^2 + m_i^2} - \mu_i} + \epsilon \right]^{-1}$$

Here $k$ is Boltzmann’s constant, $T$ is the temperature of the system, and $\mu_i$ is the chemical potential of hadron $i$. The quantity $\epsilon$ equals $+1$ for fermion (FD statistics), and $-1$ for bosons (BE statistics). The limit $\epsilon \rightarrow 0$ corresponds to classical (Boltzmann) statistics.

The chemical potential $\mu_i$ is what ultimately distinguishes the hadrons from each other. The quantities which are conserved for particle interactions within a hadron gas (which is sufficiently short-lived and short-range that it only interacts strongly) are the baryon number, charge and strangeness. The chemical potential is then a linear combination of the three potentials:

$$\mu_i = \mu_B B_i + \mu_Q Q_i + \mu_S S_i$$

Here, $B_i$, $Q_i$ and $S_i$ are the baryon number, charge and strangeness of the $i$'th hadron respectively. This can be extended by adding the charm and bottomness potentials in a similar linear fashion. However, for the current calculations, only the particles composed of the $u$, $d$ and $s$ quarks have been used.

Introduction of the chemical potentials $\mu_B$, $\mu_Q$ and $\mu_S$ allows us to fulfill the appropriate conservation laws (for strong interactions). The strangeness of the system must be zero:

$$\sum_i S_i N_i = V \sum_i S_i n_i = 0$$

This can be implemented without knowing anything about the volume $V$. 

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Next, the total charge of the fireball must be the same as the total charge of the colliding nuclei. This calculation is however complex (volume modulated), but one needs to observe that the same factor exists in total baryon number. Dividing, we require the electric charge to baryon ratio to be:

\[
\left( \frac{\sum_i Q_i N_i}{\sum_i B_i N_i} \right)_{HG}^{\text{nuclei}} = \frac{Z}{A}
\]

This can be implemented for a general collision of two nuclei by varying \(\mu_Q\) until the above equation is satisfied. For Pb+Pb and Au+Au collisions, the ratios differ very slightly, but can be very well approximated by:

\[
\frac{N_p}{N_n} \approx 0.66 \quad \text{and} \quad \frac{Z}{A} \approx 0.40
\]

reflecting that there are slightly more neutrons than protons. A simplification I was encouraged to make was to fix the potential \(\mu_Q = 0\). As \(m_p \approx m_n\) to very good accuracy, setting the electric charge potential to zero introduces a charge-independence into the system, and as protons and neutrons have almost the same mass, the model will produce an equal number of them. Hence \(\frac{Z}{A} = 0.5\) for \(\mu_Q = 0\).

This may seem like an unnecessary simplification, and a deviation from real heavy ion collisions. However this is still exact for O+O and S+S collisions and it does produce a rather large simplification in the working. As it is, provided with \(T\) and \(\mu_B\), there is only one unknown parameter (\(\mu_S\)), and although there is no analytic method to solve for \(\mu_S\) to satisfy (1.5), one can do it iteratively. Without specifying \(\mu_Q\), there are two unknown parameters, and to satisfy both strangeness and charge conservation requires iterative methods in 2-D.

If one does not implement this simplification and for a given \(T\) and \(\mu_B\) does actually work out the value of \(\mu_Q\) for a heavy ion collision, one finds that it is usually very small and negative (\(\mu_Q < 0\), to make neutrons more favourable than protons). Typical values are \(-10\) MeV < \(\mu_Q\) < \(-4\) MeV, depending on the temperature.

Furthermore, the actual number of particles \(N_i\) and entropy \(S_i\) depend on the volume \(V\) of the fireball, and for this project that is an unknown quantity. I therefore always worked with the number and entropy densities \(n_i\) and \(s_i\). However all the particle proportionalities are still known.

The above is true for a static fireball - one that occupies a fixed volume \(V\). In a more general case, when the expansion of the system at freeze-out cannot be neglected, one needs to use the more complex Cooper-Frye formula to calculate the total yield of particles. However it has been shown [1] that in this more complex system, the particle ratios stay the same as in the static fireball case, as long as the thermodynamic parameters are constant along the freeze-out surface, so to get the particle ratios, one can work with a static fireball.
Chapter 2

Numerical Integration

The hadrons come in two varieties: baryons (anti-baryons) which obey Fermi-Dirac statistics, and mesons which obey Bose-Einstein statistics. Although the two behaviours are very different at low temperatures, the formalism is very similar – often just a change of sign. I’ll thus deal with both cases using $\pm$ and $\mp$ notation, with $^{FD}_{BE}$ convention (top sign for fermions).

2.1 Quantum Statistics Integrals

2.1.1 Number densities

As mentioned in the previous section, we define a momentum distribution function:

$$ f_i = f_i(p) = \left[ e^{\frac{E_i - \mu_i}{kT}} \pm 1 \right]^{-1} = \left[ e^{\frac{\sqrt{p^2 + m_{i}^2} - \mu_i}{kT}} \pm 1 \right]^{-1} $$

(2.1)

Then the number density of hadron $i$ is given by:

$$ n_i = g_i \int \frac{d^3p}{(2\pi\hbar)^3} f_i(p) $$

(2.2)

Assuming that the fireball is isotropic, the integral separates into angular and spatial parts:

$$ n_i = g_i \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos \theta \int_0^{\infty} \frac{p^2 dp}{(2\pi\hbar)^3} f_i(p) = \frac{g_i}{2\pi^2\hbar^3} \int_0^{\infty} f_i(p)p^2 dp $$

(2.3)

Going over to a system of units where $\hbar = 1$, the multiplicity of hadron $i$ becomes:

$$ n_i = \frac{g_i}{2\pi^2} \int_0^{\infty} f_i(p)p^2 dp = \frac{g_i}{2\pi^2} \int_0^{\infty} \frac{p^2 dp}{e^{\frac{E_i - \mu_i}{kT}} \pm 1} $$

(2.4)
2.1.2 Entropy densities

For a set Grand Canonical ensemble of particles $i$ in volume $V$, the partition function obeys:

$$\ln Z = \pm V g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \ln \left( 1 \pm e^{-\frac{E-\mu_i}{kT}} \right)$$  \hspace{1cm} (2.5)

Then the entropy is defined by:

$$S_i = \frac{\partial}{\partial T}(kT \ln Z)$$

$$= k \ln Z + kT \frac{\partial Z}{\partial T}$$

$$= k \ln Z + kT \cdot \frac{Z}{Z} \frac{\partial}{\partial T} \left( \pm V g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \ln \left( 1 \pm e^{-\frac{E-\mu_i}{kT}} \right) \right)$$

$$= kT g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \frac{\partial}{\partial T} \left[ \ln \left( 1 \pm e^{-\frac{E-\mu_i}{kT}} \right) \right]$$

$$= \pm kV g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ \ln \left( 1 \pm e^{-\frac{E-\mu_i}{kT}} \right) + T \frac{\pm e^{-\frac{E-\mu_i}{kT}}(1 \pm e^{-\frac{E-\mu_i}{kT}})^{\frac{1}{2}}}{1 \pm e^{-\frac{E-\mu_i}{kT}}} \right]$$

$$= \pm kV g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ \ln \left( \frac{e^{-\frac{E-\mu_i}{kT}}}{1 \pm e^{-\frac{E-\mu_i}{kT}}} \right) \pm \left( \frac{E-\mu_i}{kT} \right) \frac{1}{e^{-\frac{E-\mu_i}{kT}} + 1} \right]$$

$$= \pm kV g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ - \ln f_i + \ln \left( \frac{e^{-\frac{E-\mu_i}{kT}}}{1 \pm e^{-\frac{E-\mu_i}{kT}}} \right) \left( 1 \pm f_i \right) \right]$$

$$= \pm kV g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ - \ln f_i + \ln \left( \frac{e^{-\frac{E-\mu_i}{kT}}}{1 \pm e^{-\frac{E-\mu_i}{kT}}} \right) \left( 1 \pm f_i \right) \right]$$

$$= \pm kV g_i \int \frac{d^3 p}{(2\pi \hbar)^3} \left[ - \ln f_i + \ln \left( \frac{1 \mp f_i}{f_i} \right) \left( 1 \pm f_i \right) \right]$$

Integrating through the angular parts, one obtains the entropy density ($\hbar = k = 1$):

$$s_i = \frac{g_i}{2\pi^2} \int_0^{\infty} p^2 dp \left[ - f_i \ln f_i \mp (1 \mp f_i) \ln(1 \mp f_i) \right]$$  \hspace{1cm} (2.6)
2.1.3 Change of variables

The integrals for \( n_i \) and \( s_i \) are in \( p \), but for \( p \gg m \):

\[
f_i(p) = \left[ e^{\frac{\sqrt{p^2 + m^2} - \mu_i}{2}} \mp 1 \right]^{-1} \approx \left[ e^{\frac{(p + m^2/2p) - \mu_i}{2}} \pm 1 \right]^{-1} \approx \left[ e^{\frac{p}{p + m^2 / 2p}} \right]^{-1} = e^{-\frac{p}{p + m^2 / 2p}} \tag{2.7}
\]

So the integrand is modulated by a decreasing exponential of form \( e^{-p/T} \). This is not desirable, as one would like the integrals to not be dominated by terms involving the temperature (to be able to observe qualitative behaviour). This is easily fixed by introducing \( x = p/T \), \( \Rightarrow p^2 dp = T^3 x^2 dx \), and the limits of integration remain 0 to \( \infty \). The two densities are then:

\[
n_i = T^3 \frac{g_i}{2\pi^2} \int_0^\infty f_i(x)x^2 dx \tag{2.8}
\]

\[
s_i = T^3 \frac{g_i}{2\pi^2} \int_0^\infty \left[ -f_i(x) \ln f_i(x) \mp (1 \mp f_i(x)) \ln(1 \mp f_i(x)) \right] x^2 dx
\]

where we now have:

\[
f_i(x) = \frac{1}{e^{\sqrt{x^2 + (m_i/T)^2 - \mu_i/T}} \pm 1} \tag{2.9}
\]

The quantities \( m_i/T \) and \( \mu_i/T \) are constants for each particle and parameter set.

2.2 Gauss-Laguerre Integration Technique

The above integrals can be evaluated using a standard Simpson method of dividing the domain into strips and approximating the function on each strip by a parabola; the total integral being the sum of the areas of all the strips. This method is very accurate (provided the strip widths are small), but also time-consuming. I did implement this method, integrating the domain from 0 to 20, with \( \Delta x = 0.002 \). However there exists a more efficient and far more subtle integration technique.

2.2.1 Gaussian quadratures

The method of Gaussian quadratures is not a very well known method of calculating integrals, as it is only readily applicable to a limited set of integrands. This topic can be found in the literature, although the proofs of relevant theorems are not presented, and I’ve been unable to reproduce them. The following arguments are taken from *Numerical Recipes* [6], combined with my own working.

Consider the class of functions that take the form of a polynomial multiplied by some known function \( W(x) \). Then given the function \( W(x) \) and some integer \( N \),
one can create a set of $N$ weights $w_i$ and abscissas $x_i$, to approximate the integral by a finite summation:

$$\int_a^b W(x) f(x) dx \approx \sum_{i=1}^N w_i f(x_i)$$  \hspace{1cm} (2.10)

such that the approximation is exact if $f(x)$ is a polynomial. Now for a given $N$, there are $2N$ parameters that can be chosen at will (the weights and abscissas). It then follows that (2.10) can only be exact for $2N$ linearly independent polynomials, and it is usually chosen such that it holds for all $f(x)$ with degree $D \leq 2N - 1$.

For such a choice, approximating a more general integral by a finite sum is only accurate if $f(x)$ can be “well approximated by a polynomial”. However almost all well-behaved (infinitely-differentiable) functions can be well approximated by a polynomial of sufficiently high degree. The value of $N$ ought to be chosen sufficiently large to accommodate for slightly more complicated functions.

**Orthogonal polynomials**

For a specified function $W(x)$ and limits of integration, define a “scalar product”:

$$\langle f | g \rangle \equiv \int_a^b W(x) f(x) g(x) dx$$  \hspace{1cm} (2.11)

Then one can find a set of polynomials that satisfy (i) there exists exactly one polynomial of order $j$, called $(p_j(x))$, and (ii) all the polynomials are mutually orthogonal over the specified weight function $W(x)$. As a further requirement one can require that the polynomials $p_j(x)$ be normalized – the scalar product with themselves giving unity. Then:

$$\langle p_i(x) | p_j(x) \rangle = \delta_{ij}$$  \hspace{1cm} (2.12)

A set of such orthonormal polynomials can constructed through a recurrence relation (Grand-Schmidt procedure), although it may not be the most efficient way.

It can be shown that the polynomial $p_j(x)$ has exactly $j$ distinct roots in the interval $(a, b)$. This becomes relevant when one uses the fundamental theorem of Gaussian quadratures: The abscissas $x_i$ of the $N$-point Gaussian quadrature formula (2.10) with weighting function $W(x)$ on $(a, b)$ are precisely the roots of the orthogonal polynomial $p_N(x)$.

Once the abscissas $x_1...x_n$ are known, the weights $w_i$ can be found by solving:

$$\sum_{i=1}^N p_k(x_i) w_i = \int_a^b W(x) p_k(x) dx = \int_a^b W(x) p_k(x) dx \cdot \delta_{0k}$$  \hspace{1cm} (2.13)

for each value of $k$ from 0 to $N - 1$ (all integrands except for $k = 0$ are zero, due to orthogonality). The parameters $x_i$ and $w_i$ will make (2.10) exact for all any function that can be written as a linear combination of $p_i(x)$. 
2.2.2 Laguerre polynomials

The form of integrals that are useful for this problem are polynomials modulated by $e^{-x}$, integrated over the positive real axis. For this purpose take $W(x) = e^{-x}$, and the limits of integration from 0 to $\infty$. The orthogonal polynomials can be defined in terms of a recurrence relation, although there is a simpler method:

- Define $L_0(x) \equiv 1$. Then $\int_0^\infty L_0(x)W(x) = 1$ as required.
- For all other $n$ define $L_n \equiv \frac{e^x}{n!} \frac{d}{dx}^n (xe^{-x})$, from which one gets:

$$\int_0^\infty L_n(x)W(x) dx = \int_0^\infty \frac{1}{n!} \frac{d}{dx}^n (x^n e^{-x}) dx = \frac{1}{n!} \left[ \frac{d}{dx^{n-1}} (x^n e^{-x}) \right]_0^\infty = 0$$

Directly showing that (2.13) is satisfied, and also giving the first orthonormality condition (as $L_0 = 1$), if one rewrites the equation as $\int_0^\infty L_0(x)L_n(x)W(x) = \delta_{0n}$.

To see the form of the Laguerre polynomials, the first four have been calculated:

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = \frac{1}{2}(2 - 4x + x^2)$$
$$L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3)$$

A general formula can be guessed, and shown to be true. To find the coefficient of $x^k$ from the expression $\frac{d}{dx^n}(xe^{-x})$, the product rule is used repetitively, and the derivative must be applied $k$ times to the exponential and $n-k$ times to the monomial. This produces a factor of $(-1)^k \cdot \frac{n!}{k!}$. As there are $n$ derivatives applied in total, and $k$ are to the exponential, there is an additional factor of $\binom{n}{k}$. Hence:

$$L_n(x) = \frac{1}{n!} \sum_{k=0}^n (-1)^k \frac{n!}{k!} \binom{n}{k} x^k = \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)! (k!)^2} x^k \quad (2.14)$$

Orthonormality

The orthonormality of Laguerre polynomials is mentioned in all the literature, although I have been unable to find a proper proof of this property. The normalization $\int_0^\infty L_i^2(x)W(x) dx = 1$ can be derived from (2.14) by calculating the coefficient of each $x^k$, and using the property $\int_0^\infty x^k e^{-x} dx = k!$. This method is however very long and algebraically intensive, and not shown in this project. To show that distinct Laguerre polynomials are orthogonal is a harder task, one that I’ve been unable to complete myself, or indeed find a proof of this relation.

If one takes it for granted that Laguerre polynomials are orthonormal, then one can expand any polynomial $f(x)$ of degree less that $2N$ as a linear combination of $L_i(x)$, and as integrals preserve associativity, (2.10) will be exact for all such $f(x)$. 

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2.2.3 An explicit construction

For \( N = 2 \) it is possible to construct the abscissas and weights directly, by simply assuming (2.10) and \( \int_0^{\infty} x^k e^{-x} \, dx = k! \). This gives a system of four equations:

\[
\begin{align*}
    w_1 + w_2 &= 1 \\
    x_1 w_1 + x_2 w_2 &= 1 \\
    x_1^2 w_1 + x_2^2 w_2 &= 2 \\
    x_1^3 w_1 + x_2^3 w_2 &= 6
\end{align*}
\]

Multiplying the second equation by \((x_1 + x_2)\) yields:

\[
x_1 + x_2 = (x_1 + x_2)(x_1 w_1 + x_2 w_2) = x_1^2 w_1 + x_1 x_2(w_1 + w_2) + x_2^2 w_2 = 2 + x_1 x_2
\]

While multiplying the third equation by \((x_1 + x_2)\) yields:

\[
2(x_1 + x_2) = (x_1 + x_2)(x_1^2 w_1 + x_2^2 w_2) = x_1^3 w_1 + x_1 x_2(x_1 w_1 + x_2 w_2) + x_2^2 w_2 = 6 + x_1 x_2
\]

giving two equations for \((x_1 + x_2)\) and \(x_1 x_2\). It follows that \(x_1 x_2 = 2\) and \(x_1 + x_2 = 4\).

Solving for the individual \(x\)'s (ordering is not defined – take \(x_1 < x_2\)) gives

\[
x_1 = 2 - \sqrt{2} \quad \text{and} \quad x_2 = 2 + \sqrt{2}
\]

Note that the \(x_i\)'s are roots of the quadratic \(x^2 - 4x + 2\), which is (up to a constant factor) \(L_2(x)\). Hence the abscissae have been shown to correspond to the zeroes of the relevant Laguerre polynomial.

The problem is now essentially solved. Putting the found values for the \(x_i\)'s into the original system leaves a set of linear equations for \(w_i\)'s, which can be trivially solved. The solution:

\[
w_1 = \frac{1}{4}(2 + \sqrt{2}) \quad \text{and} \quad w_2 = \frac{1}{4}(2 - \sqrt{2})
\]

A little algebra will show that the integral equations for (2.10) are satisfied for \(x^k\) for \(k = 0, 1, 2, 3\). Hence the two Laguerre point problem is solved. This result, although simple to derive algebraically, allows one to integrate (from zero to infinity) any exponentially modulated polynomial up to cubic by simply evaluating the function at two points.

A result not really evident for the above case is the following. Suppose we use \(N\) Laguerre points. Then ANY polynomial \(P(x)\) which has zeroes at the Laguerre points will have a zero integral. This despite the extra \(N - 1\) supposed degrees of freedom. An extreme case is a polynomial of the form:

\[
P(x) = (x_N - x) \times \Pi_{i=1}^{N-1}(x - x_i)^2
\]

which is positive for all \(x \leq x_N\) (without loss of generality \(x_N > x_i\)), and only negative once \(x \geq x_N\), by which time \(P(x)e^{-x}\) is expected to be heavily suppressed. This is not actually true: \(P(x)\) grows as \(x^{2N-1}\) at \(x \sim x_N\), which is much faster than the negative exponential at the same point (for \(N = 15, x_N \approx 48 \gg e^{x_N/(2N)}\)).
2.2.4 Applicability of Laguerre integration

Unfortunately, this neat form of integration is only exactly applicable to integrals of the form (2.10) - polynomials modulated by a decreasing exponential. However the integrals that are needed for quantum statistics, are not like that (2.9). For large values of $x$, $f_i(x) \to e^{-x}$. However, there is the $\epsilon = \pm 1$ factor, which creates slight problems for $x \approx 0$, and also the $m_i/T$ factor which deforms the exponential factor.

The method can however be salvaged by taking a "large" number of points. For $n = 15$ (as used), the method works for any polynomial up to degree 29. Although I have no analytic proof, it is rather intuitive that the method should still give a good approximate answer. There are 15 points where the function is evaluated. The integral is completely determined by the 15 data points. Now consider an exponentially modulated polynomial function passing through the same 15 points. There are still 14 degrees of polynomial freedom, by using which one can approximate $f_i(x)$ very well over it’s domain.

Numerical results for performing the integrals using the Gauss-Laguerre and Simpson method (considered almost exact) show a difference between the two methods of less than 5 ppm (parts per million). For more complex functions (such as $s_i$), the Laguerre method appears even more doomed, as $s_i$ involves terms proportional to $(1 \mp f_i) \ln(1 \mp f_i)$, where the exponential fall-off is not clear.

Consider the part of the expression for $x \gg \frac{m}{T}$. Then the function $f$ of form (2.9) can be reasonably well approximated by: $f_i(x) \simeq \left[e^{x - \mu/T} \pm 1\right]^{-1} \ll 1$. Then using the Taylor expansion for the natural logarithm $\ln(1 + x) \approx x$, the entropy integrand becomes:

$$- f_i \ln f_i \mp (1 \mp f_i) \ln(1 \mp f_i) \approx - f_i \ln f_i + (1 \mp f_i) \mp f_i$$
$$= - f_i \ln f_i + (1 \mp f_i) f_i$$
$$= f_i (1 \mp f_i - \ln f_i)$$
$$\approx f_i (1 - \ln f_i)$$
$$\approx (1 + (x - \mu/T)) e^{\mu/T - x}$$
$$= e^{\mu/T} (x + 1 - \mu/T) e^{-x} \quad (2.15)$$

where I have approximated the distribution function by the Maxwellian limit for large $x$. So the entropy integrand, although complicated for $x \ll m/T$, is in the required form of a polynomial modulated by an exponential for large $x$. So Laguerre integration can be applied here as well, although more cautiously than before. The integrals remain accurate (for the worst data set) to within a half of a percent.
Chapter 3

Entropy Calculations

The derived equation (2.6) gives an exact value for the entropy density of hadron $i$. To calculate the total entropy density of the system, one is required to sum over the contributions of all the hadrons. To this end I was provided with a list of all known hadrons composed of the $u, d$ and $s$ quarks that have a mass below 2.6GeV. This contains all the unflavoured and strange mesons, baryons and antibaryons. Charmed particles were not included, but the program can be easily extended to cater for these additional hadrons. The particle listing was presented in the following format (corresponding to $\pi^+, \rho^0, K^0, p, \Delta^{++}$)

1. 0.140 -1. 0. 0. 1.
3. 0.770 -1. 0. 0. 0.
1. 0.498 -1. 1. 0. 0.
2. 0.938 1. 0. 1. 1.
4. 1.232 1. 0. 1. 2.

The first column corresponds to the degeneracy factor $g_s = 2s + 1$. The second column gives the mass of the hadron in GeV. The third column gives $\epsilon = \pm 1$, depending on whether the particle obeys Fermi-Dirac or Bose-Einstein statistics. The next three columns give the strangeness $S$, the baryon number $B$ and the charge $Q$ of the particle. These three parameters will be used in the calculation of the particle’s chemical potential (1.4). These are all the parameters that are required to be able to work out the entropy. The input file contained 358 hadrons.

The chemical potential depends on the strangeness, baryon number and charge of each particle, as well as the values of the corresponding potentials. Charge independence has been assumed ($\mu_Q = 0$), which essentially disregards the last column of input. The strangeness potential is fixed by (1.5), as the total strangeness of the system must be zero.
This leaves two free parameters: temperature $T$ and baryonic chemical potential $\mu_B$. By analysis of past heavy ion collisions, and applying the thermal model to the particle ratios observed, one can fit the parameters $T$, $\mu_B$ and the other potentials to get the best agreement with experiment. These values correspond to the system parameters at chemical freeze-out.

Figure 3.1: Relationship between the temperature $T$ and chemical potential $\mu_B$ at chemical freeze-out, as found by observing heavy ion collisions.

There are 25 data points shown in Fig 3.1, largely concentrated at $T \approx 0.165$GeV. The points furthest to the right correspond to the lowest energy collisions, and as the beam energies increase, the temperature $T$ at freeze-out increases, while $\mu_B$ decreases. As the baryonic chemical potential can never become negative (always excess baryons), and the temperature at the highest energy collisions appears to tend to an almost constant value, it is expected that when LHC becomes operational, the parameters will have values $\mu_B \approx 0^+$, and $T \approx 0.17$ GeV.

In all further calculations, the 25 data points describing the freeze-out curve are used. All plots shown are always against one of $T$ or $\mu_B$, the other being implicit.
For all the points on the $T$-$\mu_B$ curve, it has been pointed out by Cleymans and Redlich [4] that the average energy per hadron at the chemical freeze-out is almost constant throughout. In particular $\frac{\epsilon}{n} \approx 1$ GeV.

Further, it has been shown by Cleymans [3], that the baryonic potential can be fitted to the beam energy using a rather simple formula:

$$\mu_B(s) \approx \frac{a}{1 + \sqrt{s}/b}$$  \hspace{0.5cm} (3.1)

with $a \approx 1.27$ GeV, and $b \approx 4.3$ GeV. The temperature dependence on the beam energy can be implemented using $\frac{\epsilon}{n} \approx 1$ GeV. Plots of the temperature and baryonic potential as a function of $\sqrt{s}$ are provided in [3] – here I only reproduce the curve for $\mu_B$, due to the unavailability of an analytic form of the $T$ dependence.

![Figure 3.2: Value of baryonic chemical potential $\mu_B$ as a function of beam energy $\sqrt{s}$, with parameter results from four experiments put it.](image)

Various features of the thermal model are calculated (next sections) in terms of $T$ and $\mu_B$, and to find the corresponding beam energy $\sqrt{s}$ for a given $(T, \mu_B)$, the above relationship will be used.
3.1 Determining $\mu_S$

The strangeness chemical potential $\mu_S$ is uniquely defined by $T$ and $\mu_B$, but there it however no analytic way of determining it directly. It needs to be solved numerically. To do so, it is essential to first know approximately what value is expected.

3.1.1 First estimate

Every strange particle has a corresponding anti-particle, which has exactly opposite strangeness, while identical mass and degeneracy factor. They have opposite charges, but with $\mu_Q = 0$ this is irrelevant. In the current discussion, a meson’s chemical potential depends only on the strangeness of the particle, so of the two groups: strange mesons and anti-strange mesons, the one with the greater chemical potential will exist in large quantities.

For baryons however, there is $\mu_B$, which is always positive. Hence there will always be more baryons than anti-baryons, and in the baryon section of the fireball, there will be an excess of strange quarks.

To keep total strangeness zero, there must be an excess of anti-strange quarks in the mesons. Hence $\mu_S \cdot S(\bar{\pi}) > 0$. As strangeness of $\pi$ is +1, we deduce that $\mu_S > 0$.

For an upper bound, I presumed that $\mu_S$ will scale not faster than linearly with $\mu_B$, and have taken $\mu_S < \frac{1}{3} \mu_B$. Although I have no analytical way of showing that this will suffice, numerically it was shown to work.

Combining two bounds, the original interval was taken as: $0 < \mu_S < \frac{1}{3} \mu_B$.

3.1.2 Numerics

The momentum distribution function was originally mentioned in the first chapter. It is defined as:

$$f_i(p) = \left[ e^{\frac{\sqrt{p^2+m_i^2}}{\pi T} - \mu_i} + e^{-\frac{\sqrt{p^2+m_i^2}}{\pi T} - \mu_i} \right]^{-1}$$  \hspace{1cm} (3.2)

Then the number of particles of species $i$ that will be produced is:

$$n_i = \frac{g_i}{2\pi^2} \int_0^\infty f_i(p)p^2 dp = \frac{g_i}{2\pi^2} \int_0^\infty \frac{p^2 dp}{e^{\frac{E-\mu_i}{kT}} \pm 1}$$  \hspace{1cm} (3.3)

Combining this with (1.5), for a given $T$ and $\mu_B$:

$$\mathcal{F}(\mu_S) = \sum_i S_i n_i = \sum_i \frac{S_i g_i}{2\pi^2} \int_0^\infty \frac{p^2 dp}{e^{\frac{E-\mu_i}{kT}} \pm 1} = 0$$  \hspace{1cm} (3.4)
The problem has been transformed into one of finding a zero of the function \( F(\mu_S) \). Starting with the bounds \( 0 < \mu_S < \frac{1}{3}\mu_B \), it turns out that the \( F \) is monotonically growing over this interval \( (F(0) < 0 \text{ and } F(\frac{1}{3}\mu_B) > 0) \), and an iterative solution (such as the bisection method) can be implemented to find a zero of \( F \).

Importantly here, each evaluation of \( F \) requires the evaluation of almost 200 integrals (one only needs to find \( n_i \) for strange particles) of a non-standard form. This can be done using the Simpson method, but that is extremely time consuming, and the program takes a very long time to find a value of \( \mu_S \) to a sufficient accuracy. I have calculated \( n_i \) for a few typical parameter sets, and have found that for the \( \pi \)'s the Simpson and Laguerre methods differ by around 20 ppm, while for all other particles the absolute difference was of order \( 10^{-8} \) or less. As pions are non-strange, all the necessary integrals can be computed using the very fast Laguerre method (15 points), and the bisection algorithm can be implemented.

### 3.1.3 Results

![Graph showing the relationship between strangeness \( \mu_S \) and baryonic potential \( \mu_B \).](image)

Figure 3.3: Relationship between the strangeness \( \mu_S \) and baryonic potential \( \mu_B \), calculated numerically along the freeze-out curve.
The result is rather neat, although not too surprising. At very low $\mu_B$, when the temperature is almost constant, the strangeness potential goes linearly with $\mu_B$. As the baryonic potential increases, so does $\mu_S$, but more slowly, and the plot curves downwards. The little twist at the last point is rather unexpected, possibly due to uncertainty in $T$.

![Figure 3.4: Ratio of strangeness to baryonic potential ($\mu_S/\mu_B$) dependance on $\mu_B$](image)

This plot is essentially the same as Fig 3.3, although it does show more clearly the linear dependence of $\mu_S$ on $\mu_B$ as the baryonic chemical potential is small, together with the fall-off for larger values of $\mu_B$ (corresponding to lower temperatures). The limiting value for the ratio of the chemical potentials is:

$$\lim_{\mu_B \to 0} \frac{\mu_S}{\mu_B} \approx 0.236 \quad (3.5)$$

For a QGP, there are no hadrons, and for total strangeness to be zero, $\mu(s) = \mu(\bar{s})$, and as the strange quark has baryon number $\frac{1}{3}$ and strangeness $-1$ (opposites signs for $\bar{s}$), it is required that $\mu_S = \frac{1}{3}\mu_B$ in a QGP. The limiting value found does not correspond to that for a QGP, as it is calculated for a hadron gas at freeze-out, which differs from the hadron gas at formation (when QGP disassociates), which in turn which differs from QGP through a phase transition, breaking limit continuity.
3.2 Entropy in a Hadron Gas

Once the strangeness potential $\mu_S$ has been determined, the chemical potential for each hadron becomes known, and the integrals can be computed. Recalling that the entropy density for one hadron (2.3) can be written as:

$$s_i = T^3 \frac{g_i}{2\pi^2} \int_0^\infty \left[-f_i(x)\ln f_i(x) \mp (1 \mp f_i(x))\ln(1 \mp f_i(x))\right] x^2 dx$$

it is expected that $s_i$ will eventually go as $T^3$ for sufficiently large temperatures. Also, as the $T^3$ term is common for all hadrons at any given temperature, one can calculate the integrals without it, and insert it later. For this reason, I define the normalized entropy density for hadron $i$:

$$f_i \equiv \frac{g_i}{2\pi^2} \int_0^\infty \left[-f_i(x)\ln f_i(x) \mp (1 \mp f_i(x))\ln(1 \mp f_i(x))\right] x^2 dx$$

Then the entropy density of each group of hadrons can be written as:

$$s_M = T^3 S_M = T^3 \sum_{\text{Mesons}} f_i$$
$$s_B = T^3 S_B = T^3 \sum_{\text{Baryons}} f_i$$
$$s_T = T^3 S_T = T^3 \sum_{\text{Hadrons}} f_i$$

which defines the group normalized entropy densities $S_M$, $S_B$ and $S_T$.

3.2.1 Meson-Baryon distribution

The normalized entropy density $f_i$ was evaluated for each hadron. This was done using both the Simpson and Gauss-Laguerre integration techniques. It was observed, that although for several particles there was a “large” discrepancy (up to 0.5%) between the two methods, these occurred only for the heaviest particles, where $\sqrt{x^2 + m_i^2} \approx x$ is a very bad approximation for all but very large $T$. However, the hadron gas is always dominated by particles of lowest mass. In the case of mesons these are the $\pi$, $\eta$, $\rho$, $K$’s. For baryons it is $p$, $n$ and light resonances, and for all of the mentioned particles, the integration methods do not differ by more that 10 ppm. If dealing with the (normalized) entropy density for mesons, baryons and the total, the two methods give very similar results (the relative differences never exceeded 20 ppm), which cannot be separated on a graph.

The $T-\mu_B$ relationship has been shown earlier (Fig 3.1), so all plots from here onwards will be against temperature.
It is expected that for systems with low energy (and temperature), the hadron gas will be dominated by the protons and neutrons (baryons), and that the mesons will exist only in small quantities. Here, I put anti-baryons together with baryons. For very high temperatures (as observed at RHIC) the system is however dominated by pions, and the entropy is expected to be mostly among the mesons. This is indeed observed:

Figure 3.5: Calculated normalized entropy density $S$ for mesons, baryons, and total. The baryon and meson entropies are seen to cross when $T \approx 140\text{MeV}$.

The plot contains many interesting features. For low temperatures, as was expected, the baryon entropy is far higher than that for mesons. Indeed, it appears to be far greater than the entropy at higher temperatures - this is however not true. The plot shows $s_i$ divided by the cube of the temperature. The baryon and meson curves intersect very near to $T = 140\text{MeV}$. This corresponds to $\mu_B \approx 0.406 \text{ GeV}$, and $\sqrt{s} \approx 9.3 \text{ GeV}$. This falls within the domain of the Alternating Gradient System (AGS) at the Brookhaven National Lab. For the high temperatures (lots of points, as $T$ seems to approach a constant value as $\mu_B \to 0$), the mesons dominate as expected, while the baryons continue to contribute substantially less for the increasing values of $T$. 

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The total normalized entropy density $S_T$ exhibits a very neat behaviour. For $T < 100\,\text{MeV}$, it decreases almost hyperbolically as the baryon contribution goes down. It then flattens out, has a minimum at $T = 130\,\text{MeV}$, and then increases very slowly. For the high $T$, the value of $S_T$ is almost constant, as if it was approaching a limit. In that region $S_T \approx 6.15$

This limiting behaviour is not totally surprising. It was expected that as $T \gg m$ for the hadrons, $s_T \sim T^3$, which is the same as having $S_T$ constant. However this limiting behaviour is observed already at $T \sim 130\,\text{MeV}$, which corresponds to the pion mass, and is a lot less than the masses of all the baryons.

For completeness, one may also look at a plot of the proper entropy densities $s$ and their dependance on temperature:

![Figure 3.6: Calculated entropy density $s$ for mesons, baryons, and the total.](image)

Here it can be clearly seen that the baryon entropy density, unlike $S_B$ does not decrease initially. It increases (as one would expect), although slower than $T^3$. It does however reach a maximum at $T = 150\,\text{MeV}$, when $\mu_B$ starts decreasing fast relative to the increase in $T$. For mesons, the entropy density simply increases very fast with $T$. A power law fit gives $s_M \sim T^{5.29 \pm 0.04}$. 
3.2.2 Baryon - Anti-baryon distribution

Although this was not originally planned, it was an automatic consequence of doing the baryon-meson distributions, and though it does not give any new results, it is a simple check on the intuitive expectations for a hadronic gas. At low temperatures, baryons are favoured greatly over anti-baryons. At higher temperatures (and beam energies), $\mu_B \to 0$, and anti-particles are produced in sizeable quantities. Although there is still a bias towards baryons, it is far smaller.

To do the plot, I originally planned to show $S_B$ and $S_{\overline{B}}$, but that plot is not clear. Recall that $S$ for baryons plus anti-baryons is maximal for low $T$, and in that region $S_B \approx 0$. Only at high $T$ will the anti-baryons contribute significantly, and so I plotted the proper entropy densities $s_B$ and $s_{\overline{B}}$.

![Figure 3.7: Calculated entropy densities $s_B$ and $s_{\overline{B}}$](image)

Although maybe more pronounced at high $T$, the plot is as expected. The anti-baryons always contribute less than the baryons, but as $\mu_B \to 0$, they do limit to the same value as the baryons. $s_B$ drops sharply as $T \approx 165.5$ MeV remains almost constant while $\mu_B$ decays almost linearly in the same region.
3.2.3 Entropy of strange particles

The distribution of the entropy among the mesons, baryons and anti-baryons has been demonstrated. I have mentioned that for mesons, the entropy contributions come mainly from the $\pi$, $\eta$ and $K$ particles, while baryon contributions come mainly from nucleons and light resonances. There are also lots of strange particles with slightly higher masses, which for higher $T$ are expected to contribute significantly. This does indeed happen:

![Figure 3.8: Fractional contribution to the entropy of mesons/baryons/total from strange particles (the $\phi = s\bar{s}$ is considered unflavoured)](image)

The result is rather interesting. For the mesons, even at low $T$, there is a little contribution from the kaons, and this rises steadily with increasing $T$. For the baryons, at low $T$ there are almost no strange particles (as expected), but for higher $T$ this rises very sharply, and for $T > 160\text{MeV}$, the strange particles ($\Lambda, \Sigma$ etc.) dominate. This may be accidental, but the meson and baryon contributions cross around 0.5, corresponding to equal strange and non-strange contributions.
3.3 Entropy in a QGP

The calculations of entropy in a QGP are simpler than those in a hadron gas, as there are far fewer "particles". In this case, the only contributors to the entropy are the quarks, anti-quarks and gluons. As for the hadrons, I only used unflavoured and strange quarks (u, d, s and the anti-quarks). Gluons can be treated in the same fashion.

For baryon numbers, one takes \( B(q) = \frac{1}{3} \) and \( B(\bar{q}) = -\frac{1}{3} \). Charge independence gives \( \mu_Q = 0 \). To ensure that strangeness is conserved, \( \mu(s) = \mu(\bar{s}) \), and hence \( \mu_S = \frac{1}{3} \mu_B \). This deals with the three chemical potentials.

The \( u \) and \( d \) quarks are assumed massless, and the strange quark was given a mass of 150MeV. Gluons are massless, and have no charge, baryon number or strangeness, so \( \mu(g) = 0 \). For spin degeneracies \( g_s \), I included colour, so \( g_s(q) = 6 \) and \( g_s(g) = 8 \times 2 = 16 \). This is all the information required to do the calculations.

Figure 3.9: Normalized entropy densities \( S \) for a QGP. The total is plotted, together with the individual contributions from the quarks, anti-quarks and gluons.
I had taken the $T-\mu_B$ set of parameters, and calculated the entropy density that would result from a QGP at those values (although $\mu_S$ is different). The result:

As can be seen in Fig 3.9, the plot for the total entropy density is very similar to that of the total entropy density of a hadronic gas. Starting very large, it decreases almost hyperbolically, and then levels off at a value - presumably it will limit to a constant. There is however no little dip as in the HG case, and for large $T$, while seemingly limiting to a constant, the normalized density $S$ is slowly decreasing for a QGP, whereas it was slightly increasing in the case of a HG. The expected limiting value for the QGP being $S_{QGP} \approx 20.45$.

Next, it is observed that the gluon normalized entropy density does not depend on temperature. This is exactly what is expected, as the gluons have no mass, and no chemical potential, so all the $T$ factors drop out. For $T \geq 165.5\text{MeV}$, I found that $S(g) > S(q) > S(\overline{q})$. It is required that the quarks dominate the anti-quarks, but at high temperatures the values become very close, and get exceeded by the gluon entropy. This was expected, although maybe only at higher $T$.

The strangeness contribution to the total was also calculated:

Figure 3.10: Contribution of total entropy due to $s$ and $\overline{s}$ quarks
The plot shown in Fig 3.10 does have a similar shape to the strangeness contribution of mesons in a hadron gas. There is however a large difference in that for mesons, the strange particle contribution went to a little more than a half, whereas here the strange (anti-strange) quark contributions are never more than 0.21 of the total. This can be easily explained – for a HG, the strange (anti-strange) quarks can bond with an $u$ or $d$ quark to form light kaons. In QGP, strange quarks exist alone, and as they are heavy ($m(s) \approx 150\text{MeV}$), their entropy will be exceeded by the $u$ quarks, the $d$ quarks, AND the gluons at high energies. So the upper bound for $s, \bar{s}$ contribution is $\frac{1}{4} = 0.25$. Although this is not reached, the contribution is in the expected range.

### 3.4 Comparison of HG and QGP

The limiting value for the normalized entropy density in a QGP has been calculated to be a little more than three times greater than the corresponding value in a hadron gas. This is true approximately true for all $T$. A comparison plot for the total entropy densities $s$ of the two phases is presented:

![Figure 3.11: Total entropy densities $s$ for a Hadron Gas and for a QGP](image)

Figure 3.11: Total entropy densities $s$ for a Hadron Gas and for a QGP
From Fig 3.11 one can clearly see that the entropy in a QGP is higher by a factor of between 3 and 5 than the entropy in a hadron gas with the same temperature and baryon chemical potential. This is due to the QGP consisting of essentially free quarks and gluons, allowing for far greater degrees of freedom, and hence directly increasing the entropy of the system.

It is expected that the transition between a hadron gas and a QGP will be through a phase transition. If one could take a hadron gas and increase the temperature until a QGP formed, the entropy would originally follow the HG curve, and at some critical temperature “jump” to the QGP curve. As the entropy curves differ by an almost constant factor (∼ 3), a jump discontinuity is expected, signifying a first-order phase transition.

### 3.5 Conclusion

For an ideal hadron gas model of heavy ion collisions, the entropy contributions have been calculated at freeze-out for a set of parameter values satisfying the Cleymans-Redlich freeze-out criterion. The entropy densities have been calculated for mesons and baryons as well as the total (Fig 3.5 and Fig 3.6). It has been demonstrated that at low temperatures (energies) the entropy of the system is baryon dominated, while at high T the mesons dominate. The baryon-meson cross-over occurs for \( T \approx 140 \text{ MeV} \), which corresponds to \( \sqrt{s} \approx 9.3 \text{ GeV} \). This falls within the domain of the AGS at Brookhaven.

For each of the considered parameter values (\( T \) and \( \mu_B \)) the entropy density of a corresponding QGP has been calculated and presented (Fig 3.11). Due to “free” quarks in a QGP and extra degrees of freedom, the entropy in each corresponding QGP is significantly higher than that of the HG. This shows that for a transition from a hadron gas to a QGP, the entropy would have a jump discontinuity, characteristic of a first order phase transition.

At very low temperatures \( T \approx 0.05 \text{ GeV} \), the strange particle contributions towards the entropy (in both HG and QGP) are almost negligible. As the energies increase, the strange particle contributions increase monotonically with \( T \), up to 55% in a hadron gas, and almost 25% in a QGP.

Also, for a hadron gas the baryon – anti-baryon dependence was investigated. At medium and low temperatures (\( T \leq 140 \text{ MeV} \)), the contributions from anti-baryons remain negligible. Only at very high energies (such as at RHIC), when \( T \approx 165 \text{ MeV} \), the multiplicities of anti-baryons become significant, and the ratio \( \frac{\Xi}{p} \approx 0.8 \) for the highest \( T \) data points. For LHC it is expected that \( \frac{\Xi}{p} \rightarrow 1^- \).
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