A note on interval edge-colorings of graphs

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An edge-coloring of a graph $G$ with colors 1, 2, \ldots, $t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. In this paper we prove that if a connected graph $G$ with $n$ vertices admits an interval $t$-coloring, then $t \leq 2n - 3$. We also show that if $G$ is a connected $r$-regular graph with $n$ vertices has an interval $t$-coloring and $n \geq 2r + 2$, then this upper bound can be improved to $2n - 5$.

Keywords: edge-coloring, interval coloring, bipartite graph, regular graph

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. An $(a, b)$-biregular bipartite graph $G$ is a bipartite graph $G$ with the vertices in one part all having degree $a$ and the vertices in the other part all having degree $b$. A partial edge-coloring of $G$ is a coloring of some of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a partial edge-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors of colored edges incident to $v$.

An edge-coloring of a graph $G$ with colors 1, 2, \ldots, $t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable, if there is $t \geq 1$ for which $G$ has an interval $t$-coloring. The set of all interval colorable graphs is denoted by $\mathcal{N}$. For a graph $G \in \mathcal{N}$, the greatest value of $t$ for which $G$ has an interval $t$-coloring is denoted by $W(G)$.

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The concept of interval edge-coloring was introduced by Asratian and Kamalian [2]. In [2, 3] they proved the following theorem.

**Theorem 1** If $G$ is a connected triangle-free graph and $G \in \mathcal{N}$, then

$$W(G) \leq |V(G)| - 1.$$ 

In particular, from this result it follows that if $G$ is a connected bipartite graph and $G \in \mathcal{N}$, then $W(G) \leq |V(G)| - 1$. It is worth noting that for some families of bipartite graphs this upper bound can be improved. For example, in [1] Asratian and Casselgren proved the following

**Theorem 2** If $G$ is a connected $(a, b)$-biregular bipartite graph with $|V(G)| \geq 2(a + b)$ and $G \in \mathcal{N}$, then

$$W(G) \leq |V(G)| - 3.$$ 

For general graphs, Kamalian proved the following

**Theorem 3** [6]. If $G$ is a connected graph and $G \in \mathcal{N}$, then

$$W(G) \leq 2|V(G)| - 3.$$ 

The upper bound of Theorem 3 was improved in [5].

**Theorem 4** [5]. If $G$ is a connected graph with $|V(G)| \geq 3$ and $G \in \mathcal{N}$, then

$$W(G) \leq 2|V(G)| - 4.$$ 

On the other hand, in [7] Petrosyan proved the following theorem.

**Theorem 5** For any $\varepsilon > 0$, there is a graph $G$ such that $G \in \mathcal{N}$ and

$$W(G) \geq (2 - \varepsilon)|V(G)|.$$ 

For planar graphs, the upper bound of Theorem 3 was improved in [4].

**Theorem 6** [3]. If $G$ is a connected planar graph and $G \in \mathcal{N}$, then

$$W(G) \leq \frac{11}{6}|V(G)|.$$ 

In this note we give a short proof of Theorem 3 based on Theorem 1. We also derive a new upper bound for the greatest possible number of colors in interval edge-colorings of regular graphs.
2. Main results

Proof of Theorem 3. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\alpha$ be an interval $W(G)$-coloring of the graph $G$. Define an auxiliary graph $H$ as follows:

$$V(H) = U \cup W,$$

where

$$U = \{u_1, u_2, \ldots, u_n\}, W = \{w_1, w_2, \ldots, w_n\}$$

and

$$E(H) = \{u_iw_j, u_jw_i \mid v_iv_j \in E(G), 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{u_iw_i \mid 1 \leq i \leq n\}.$$

Clearly, $H$ is a connected bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring $\beta$ of the graph $H$ in the following way:

1. $\beta(u_iw_j) = \beta(u_jw_i) = \alpha(v_iv_j) + 1$ for every edge $v_iv_j \in E(G)$,

2. $\beta(u_iw_i) = \max S(v_i, \alpha) + 2$ for $i = 1, 2, \ldots, n$.

It is easy to see that $\beta$ is an edge-coloring of the graph $H$ with colors $2, 3, \ldots, W(G) + 2$ and $\min S(u_i, \beta) = \min S(w_i, \beta)$ for $i = 1, 2, \ldots, n$. Now we present an interval $(W(G) + 2)$-coloring of the graph $H$. For that we take one edge $u_iw_i$ with $\min S(u_i, \beta) = \min S(w_i, \beta) = 2$, and recolor it with color 1. Clearly, such a coloring is an interval $(W(G) + 2)$-coloring of the graph $H$. Since $H$ is a connected bipartite graph and $H \in \mathfrak{N}$, by Theorem 1 we have

$$W(G) + 2 \leq |V(H)| - 1 = 2|V(G)| - 1,$$

thus

$$W(G) \leq 2|V(G)| - 3.$$

$\square$

Theorem 7. If $G$ is a connected $r$-regular graph with $|V(G)| \geq 2r + 2$ and $G \in \mathfrak{N}$, then

$$W(G) \leq 2|V(G)| - 5.$$

Proof. In a similar way as in the prove of Theorem 3 we can construct an auxiliary graph $H$ and to show that this graph has an interval $(W(G) + 2)$-coloring. Next, since $H$ is a connected $(r + 1)$-regular bipartite graph with $|V(H)| \geq 2(2r + 2)$ and $H \in \mathfrak{N}$, by Theorem 2 we have

$$W(G) + 2 \leq |V(H)| - 3 = 2|V(G)| - 3,$$

thus

$$W(G) \leq 2|V(G)| - 5.$$

$\square$
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