Finite-Time Consensus Learning for Decentralized Optimization with Nonlinear Gossiping

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Abstract

Distributed learning has become an integral tool for scaling up machine learning and addressing the growing need for data privacy. Although more robust to the network topology, decentralized learning schemes have not gained the same level of popularity as their centralized counterparts for being less competitive performance-wise. In this work, we attribute this issue to the lack of synchronization among decentralized learning workers, showing both empirically and theoretically that the convergence rate is tied to the synchronization level among the workers. Such motivated, we present a novel decentralized learning framework based on nonlinear gossiping (NGO), that enjoys an appealing finite-time consensus property to achieve better synchronization. We provide a careful analysis on its convergence and discuss its merits for modern distributed optimization applications, such as deep neural networks. Our analysis on how communication delay and randomized chats affect learning further enables the derivation of practical variants that accommodate asynchronous and randomized communications. To validate the effectiveness of our proposal, we benchmark NGO against competing solutions through an extensive set of tests, with encouraging results reported.

1 Introduction

The growth of data and the complexity of modern machine learning problems have often necessitated the deployment of distributed learning algorithms to scale up implementation in practice (Bottou 2010; Dean et al. 2012). Such learning schemes allow the workload to be split across multiple computing units, commonly known as workers, to parallelize the computation. This enables learning within a fraction of the time typically required for a single unit, and also adds robustness as failures of individual workers can be easily isolated. Apart from its efficiency, applications of distributed learning are also frequently driven by data-privacy (Balcan et al. 2012; Truex et al. 2019; Kairouz et al. 2019) or cost-performance (Kovalev et al. 2021; Haddadpour et al. 2021) considerations.

Crucial for all distributed-learning schemes is to establish a sense of consensus among individual workers. Two major strategies are employed to reinforce the desired coherence: centralized and decentralized synchronizations. As its name suggests, the former makes use of centralized nodes, commonly known as parameter servers (Li et al. 2014a), to aggregate model updates from and broadcast the latest parameters back to the workers, thereby enforcing model parallelism (i.e., exact synchronization). Centralized schemes are considered less scalable & more vulnerable due to communication bottlenecks & heavy reliance on central nodes, and more demanding for the network infrastructure.

In the decentralized setup, workers communicate only with their neighbors rather than the centralized server (Nedić et al. 2018). Parameter updates are executed on individual workers using local gradients, followed by a consensus update step to integrate information from their neighbors. Done properly, all workers will agree on a set of parameters that is close to the optimum (Wu et al. 2017). In other words, extra flexibility with the network topology can be achieved by settling with asymptotic or approximate synchronization.

For practical considerations, however, the efficiency afforded by distributed learning may be overshadowed by the cost of communication, thus preventing ideal linear scaling predicted by theory (Dean et al. 2012). The popularity of large-scale neural networks further aggravates this issue, in which the synchronization latency for big models may easily wash away the poten-
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As for the latter, it is always desirable to communicate frugally. One of the most successful strategies for synchronization under budget is gradient compression, which entails transmitting a succinct summary of the full gradients to reduce communications (Lin et al., 2018). A major motivation is that model gradients often manifest more redundancy (e.g., sparsity) compared with the full set of model parameters (Na et al., 2017 Ye and Abbe, 2018 Basu et al., 2020). Prominent examples from this category include gradient sparsification (Wangni et al., 2018) and quantized gradients (Wen et al., 2017 Alistarh et al., 2017 Wu et al., 2018 Reisizadeh et al., 2020).

This study focuses on decentralized learners, and addresses an issue orthogonal to the endeavors discussed above. Specifically, we seek answers on how to synchronize. Our investigation is motivated by the observation that, despite matching theoretical convergence rates, decentralized algorithms often fail to compete with centralized alternatives in practical settings. We hypothesize that this is due to the lack of synchrony among the collaborating workers in the decentralized setting. This is especially the case for over-parameterized models, where small discrepancies in the parameter space sometimes can lead to qualitatively different behavior. Consequently, we wish to keep the local models better synchronized with the same or fewer communications. One promising direction is to alter the communication protocol, a path we take in this work.

This paper revisits decentralized stochastic optimization from a consensus learning perspective, in the hope of better understanding the trade-offs involved and motivating new algorithms for improved practice. Our work yields the following contributions: (i) we propose a new family of consensus learning schemes based on nonlinear gossiping for decentralized optimization; (ii) we analyze the convergence of our solution, and discuss the computation-communication trade-offs under practical considerations; and (iii) we discuss extensions to more general learning settings, such as with randomized and time-varying network topology, and the communication delay case. Our theory generalizes the existing results for decentralized learning, and the proposed algorithms promise to better balance the tension between the communication bottleneck and learner synchrony.

We use the following notations in this paper: In the decentralized setup, the workers participate in peer-to-peer communications defined by the network topology, where each worker is only allowed to exchange information with the connecting workers. Formally, let $\mathcal{G} = \{\mathcal{V}, \mathcal{W}\}$ denote a weighted undirected graph with the set of nodes (or vertices) $\mathcal{V}$. $\mathbf{W}$ is a symmetric matrix called the weight matrix. We say that a set $v_i, v_j$ is an edge if $w_{ij} > 0$. For every node $v_i \in \mathcal{V}$, the degree $d(v_i)$ is the sum of the weights of the edges adjacent to $v_i$: $d(v_i) = \sum_{j=1}^{n} W_{ij}$. The degree matrix $D(\mathcal{G})$ is the diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$. The graph Laplacian is defined by $L = \mathbf{W} - \mathbf{W}$, whose eigenvalues encode information on how network topology affects convergence. $\lambda_i(\cdot)$ denotes the $i$-th smallest eigenvalue of a matrix.

2 Problem Setup

Consider the following distributed optimization problem: $\min_{x \in \mathbb{R}^d} \left[ \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right]$, where $x$ denotes the model parameters to be optimized and $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ for $i \in [n] \triangleq \{1, \ldots, n\}$ are the local objectives defined on each worker. For most modern machine learning applications, $f_i(x)$ is given by the empirical expectation $f_i(x) = \mathbb{E}_{\xi_i \sim D_i}[F_i(x, \xi_i)]$, where $D_i$ denotes the local data distribution, and $F_i(x, \cdot)$ is the local loss function of model parameters $x$ on the worker $i$.

We call the setting $i.i.d.$ if all $D_i$ are independent copies from the same underlying data distribution, and non-$i.i.d.$ otherwise. The non-$i.i.d.$ setting commonly arises when locally curated data differ distributionally (e.g., data collected from different countries), and especially the case when data privacy is of concern.

Before addressing the detailed analysis, we summarize the standard technical conditions and define the key synchronization index function throughout the paper.

**Assumption 2.1.** Local objectives $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ for all $i \in [n]$ are $L$-smooth and $\mu$-strongly convex, with the variance of their stochastic gradients uniformly bounded for all $x$ and $i$: \[ \mathbb{E}_{\xi_i \sim D_i} \| \nabla F_i(x, \xi_i) - \nabla f_i(x) \|^2 \leq \sigma_i^2, \quad (1) \]
\[ \mathbb{E}_{\xi_i \sim D_i} \| \nabla F_i(x, \xi_i) \|^2 \leq G^2, \forall x \in \mathbb{R}^d, i \in [n]. \quad (2) \]

**Assumption 2.2.** We assume that $W \in [0, 1]^{n \times n}$. $W$ is a symmetric $(W = W^\top)$ doubly stochastic ($W1 = 1, 1^\top W = 1^\top$) matrix.

**Definition 2.3** (Synchronization index (Olfati-Saber et al., 2007)). Denote $\delta_i = x_i - \frac{1}{n} \sum_{i=1}^{n} x_i$ as the dis-
agreement vector, and we define the Lyapunov function of $V$ as $V = \sum_i \|\delta_i\|^2$, which is also called the synchronization index.

For a small value of $V$, the parameters of different workers are similar to each other. When all workers are perfectly synchronized, $V$ function reaches zero. The dynamic of $V$ function depicts the convergence behavior of gossiping learning, and serves as an important role in decentralized learning.

2.1 Gossiping SGD

Gossip averaging is one of the most popular techniques for decentralized learning (Koloskova et al., 2019; Lian et al., 2017b; Nedic et al., 2009; Tang et al., 2018). In this study, we restrict our discussion to the variants based on stochastic gradient descent (SGD).

In gossiping SGD, there are two major steps involved in iteration $t$: (i) $x_i(t) \rightarrow x_i(t+\frac{1}{2})$: SGD update (local model update) and (ii) $x_i(t+\frac{1}{2}) \rightarrow x_i(t+1)$: gossiping update (neighbor communication). The SGD step proceeds as regular SGD on a worker, with local models updated independently based on the gradient computed from a mini-batch of data curated on the individual workers. Under proper regularity conditions, the SGD updates push local models toward the local solution $x_i^* = \arg\min_{x} [f_i(x)]$.

The gains in scaling and the consensus on the global solution come from the gossiping updates. Now each worker exchanges messages with their neighbors and updates their models through localized averaging

$$x_i^{(t+1)} = x_i^{(t+\frac{1}{2})} + \gamma \sum_{j:(i,j) \in E} W_{ij} \Delta_{ij}^{(t+\frac{1}{2})}, \forall i \in [n], \quad (3)$$

where $\Delta_{ij}^{(t+\frac{1}{2})} = x_j^{(t+\frac{1}{2})} - x_i^{(t+\frac{1}{2})}$ is the update vector between worker $j$ and worker $i$ at iteration $t$, and $\gamma > 0$ controls the synchronization rate for local averaging.

Intuitively, similar to the outright averaging employed by their centralized counterparts, the gossiping update mollifies the noise term introduced from the SGD updates, thereby achieving linear speedup in terms of convergence. Without the SGD updates, all $x_i^{(t)}$ converge exponentially to the average $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i^{(0)}$ under gossiping updates. The convergence rate depends on the network topology: for a connected undirected network it is faster or equal to $1 - \gamma \lambda_2(L)$ (Olfati-Saber et al., 2007; Chen et al., 2017). $\lambda_2(L)$ is known as the algebraic connectivity of $W$.

2.2 Synchronization and convergence

Now let us compare centralized optimized and decentralized linear gossiping. Our goal is to underscore that better synchronization provides a faster overall convergence rate.

To motivate, consider the following hypothetical example: two populations of workers, with the same parameter mean $\bar{x}$ but different synchronization, i.e., $\text{Var}[x_i]$ differs. Typically $\bar{x}$ is identified as the current global solution that one seeks to optimize. For each local gradient $\partial x_i$, we could treat it as a noisy approximation $\partial \bar{x}$ of the target gradient $\partial x$ evaluated at $\bar{x}$. Let us execute one step of SGD and infinite steps of gossiping, then it essentially reduces to applying gradient descent with $\partial \bar{x}$ instead of $\partial x$. As $\text{Var}[x_i]$ becomes smaller (i.e., $\{x_i\}$ more concentrated around $\bar{x}$), the approximation to $\partial \bar{x}$ gets better (Figure 2). Consequently, one should expect better convergence provided the workers maintain a more synchronized status.

We verify this intuition experimentally in Figure 4. We use ring topology (left above in Figure 1) with $n = 5$ workers to solve a simple least square regression problem with i.i.d. data (i.e., each worker only gets data samples just from a non-overlapping partition of data distribution). We set different thresholds ($V_{th} = 1, 5, 10$) for the Lyapunov function to measure synchronization. After each SGD update, we execute gossiping updates until the tolerance is satisfied ($V \leq V_{th}$). We see that the synchronization level among workers strongly correlates with the distributed learning performance.

The following statement formalizes the intuition above, that the level of synchronization among the workers affects the convergence rate.

**Theorem 2.4** (Convergence and synchronization). Under Assumptions 2.7, centralized SGD and linear gossiping SGD with step-size $\eta_t = \frac{4}{\mu(t+1)}$, for parameter
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Figure 2: When workers are synced (close to each other in the state space, left plot), their gradients will be more aligned with that of the global average; if they are less coordinated (right), individual gradients will be more variable.

\[ f(x_{avg}^T) - f^* \leq \frac{\mu a^3}{8S_T} + \frac{4\sigma^2 T (T + 2a)}{\mu n S_T} + h(T) \]  (4)

where \( \sigma^2 = \frac{1}{n} \sum_i x_i^2 \), \( x_{avg}^T = \frac{1}{n} \sum_{t=0}^{T-1} w_t x_t \) for weights \( w_t = (a + 0.5)^2 \), and \( S_T = \sum_{t=0}^{T} w_t \geq \frac{1}{3} T^3 \). The synchronization-dependent terms are given by

\[ h(T) = \begin{cases} 0, & \text{Centralized SGD} \\ \left( \frac{2L+\mu}{n S_T} \sum_{t=0}^{T} w_t V(t) \right), & \text{Gossiping SGD} \end{cases} \]

where \( V(t) = \sum_i ||x_i(t) - \bar{x}||^2 \).

Now we are ready to discuss the interplay between synchronization and algorithmic convergence. After a sufficient number of iterations (i.e., a fairly large \( T \)), one recognizes the second term from the bracket in (4) is the leading term. For highly synchronized workers (small \( V(t) \)), the gossip SGD scheme matches its centralized counterpart. As such, provided a sufficient communication budget, one can maximally enforce the synchronization to expedite convergence.

3 Decentralized Learning with Nonlinear Gossiping

Above we have established the intuition that decentralized optimization is expected to benefit from improved consensus among workers. Inspired by this, we present NGO, a novel distributed optimization scheme based on Nonlinear Gossiping (NGO) consensus protocol. While most of the existing literature focuses on how to improve the communication efficiency for decentralized learning, we investigate a novel direction: how to improve synchronization by altering the communication updates. This implies our NGO can be combined with prior arts to further enhance learning efficiency. Below we elaborate our construction, relegating all technical proofs to the Supplementary Material (SM) Sec. A.

3.1 Finite-time consensus protocol

To motivate the new consensus updates, we first briefly review basic consensus schemes in the continuous time limit. Consider a network with \( n \) nodes, continuously coupled in the following manner:

\[ \dot{x}_i(t) = \sum_{j:(i,j) \in E} W_{ij} (x_j(t) - x_i(t)) \]  (5)

Note the discretized version of (5) corresponds to the consensus updates employed by linear gossiping \(^2\). One can readily show that following the dynamics described in (5), \( V(t) \) converges to zero in the asymptotic time limit if the graph is connected (\( t \to \infty \); see SM Sec. A), and the workers reaches consensus. For consensus learning, we desire that perfect synchronization can happen sooner, preferably within a finite time frame. The following statement verifies that this is possible through use of nonlinear couplings.

**Theorem 3.1** (Finite time consensus \(^3\)). Consider the following system with nonlinear coupling \( \phi(x) : \mathbb{R}^d \to \mathbb{R}^d \)

\[ \dot{x}_i(t) = \sum_{j:(i,j) \in E} W_{ij} \phi(x_j(t) - x_i(t)), \]  (6)

where \( \phi(z) = \text{sign}(z_1)|z_1|^{2p-1}, \cdots, \text{sign}(z_n)|z_n|^{2p-1} \) for \( p \in [\frac{1}{2}, 1) \). Then \( V(t) = 0 \) for all

\[ t > T^*(W, p) \triangleq \left( \frac{\sum_i (x_i(0) - \bar{x}(0))^2}{4\gamma(1-p)(\lambda_2(L(B)))^p} \right)^{1-p} \]  (7)

\( T^*(W, p) \) is a function of the nonlinear parameter \( p \) and the network topology \( W \), with \( B = \left[ W^\frac{1}{2} \right] \), \( \lambda_2(L(B)) \) is the algebraic connectivity of \( \mathcal{G}(B) \).

To illustrate, we randomly sample ten samples from Gaussian distribution to denote the initial state of ten workers. After evolving for 500 rounds according to Eqn (7), we plot the convergence results with \( p = 0.5 \) (sublinear), \( p = 1 \) (linear), \( p = 2 \) (superlinear), along with the corresponding synchronization index In Figure 3. We see that the sublinear coupling reaches consensus in finite time, follows by the linear coupling (exponentially) and finally the superlinear coupling.

3.2 Decentralized nonlinear gossiping

Inspired from above, we consider the following nonlinear gossiping protocol after the SGD update

\[ x_i^{(t+1)} = x_i^{(t+\frac{1}{2})} + \gamma \sum_{j:(i,j) \in E} W_{ij} \phi(x_j(t) - x_i(t)) \]  (8)

We summarize the pseudocode of a simple implementation of decentralized NGO in Algorithm 1 and establish its convergence rate as follows.

\(^2\)We use \( f(t) \) to denote continuous dynamics.

\(^3\)We use consensus and synchronization interchangeably.
Figure 3: Convergence rate of nonlinear ($p \neq 1$) and linear ($p = 1$) couplings with ten workers. (a) Sub-linear coupling ($p = 0.5$). (b) Linear coupling ($p = 1$). (c) Super-linear coupling ($p = 2$). (d) Synchronization Index.

Finite time consensus is achieved with sub-linear coupling.

Algorithm 1 Nonlinear Gossiping (NGO) Learning

Require: Initialize value $\bar{x}^{(0)} \in \mathbb{R}^d$, $x_i^{(0)} = \bar{x}^{(0)}$ on each node $i \in [n]$, consensus step-size $\gamma$, learning rate $\eta$, communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, and number of total budget $T$.

for $t = 0, \ldots, T - 1$ do

In parallel for all workers $i \in [n]$

Sample $\xi^{(t)}_i$, compute gradient $g_i^{(t)} = \nabla F_i \left( x_i^{(t)}, \xi^{(t)}_i \right)$.

end for

for neighbors $j : \{i, j\} \in \mathcal{E}$ do

Send $x_j^{(t+1)}$ and receive $x_j^{(t+1)}$.

end for

end for

\begin{align*}
x^{(t+1)}_i &= x_i^{(t+1)} + \gamma \sum_{j: (i,j) \in \mathcal{E}} W_{ij} \phi \left( \Delta x_{ij}^{(t+1/2)} \right), \\
\text{where } \Delta x_{ij}^{(t+1/2)} &= x_j^{(t+1/2)} - x_i^{(t+1/2)}.
\end{align*}

Theorem 3.2 (Convergence of NGO). With hyperparameter $a \geq \max \left( \frac{3\kappa}{4}, 15\kappa \right)$, $\kappa = \frac{\mu}{\mu + \tau^2}$, and learning rate scheduling $\eta_t = \left( \frac{\mu \lambda}{\mu + \tau^2} \right)^t$, nonlinear gossiping learning converges at the rate

\begin{align*}
f \left( x^{(T)}_{\text{avg}} \right) - f^* &\leq \frac{\mu^3}{8ST} \left\| \bar{x}^{(0)} - x^* \right\|^2 + \frac{4T(T + 2a) \sigma^2}{\mu ST} \\
&+ \frac{2L + \mu}{nST} \sum_{t=0}^{T} w_t V(t)
\end{align*}

where $x^{(T)}_{\text{avg}} = \frac{1}{ST} \sum_{t=0}^{T-1} w_t x^{(t)}$ for weights $w_t = (a + t)^2$, and $S_T = \sum_{t=0}^{T-1} w_t \geq \frac{1}{4} T^3$, with $T^* (\mathcal{W}, p)$ is defined in [7].

3.3 Consensus with asynchronous or randomized communications

For practical considerations, it is often desirable that a solution should be robust to network latency and possible worker or network failures. For example, some workers may take longer to finish their local computation (commonly known as stragglers), which blocks the overall progress if faster workers have to wait for the synchronous updates. This is very common in heterogeneous computing environments, where decentralized learning schemes are mostly employed. On the other hand, the network may be subjected to a total communication budget (e.g., bandwidth), such that a worker can only afford to communicate with a pre-defined number of its neighbors [Koloskova et al. 2020].

We investigate (i) asynchronous consensus updates, where stale model parameters are used for consensus updates; and (ii) random communication graphs, where workers can only communicate to sub-sampled neighbors at each iteration. The following statement asserts that it suffices to prove the consensus protocols will achieve synchronization by themselves.

Lemma 3.3. The average $\bar{x}^t$ of the Gossiping-type SGD algorithm satisfies the following:

\begin{align*}
E[|x^{(t+1)}_i - x^*|^2] &\leq \left( 1 - \frac{\mu \eta}{2} \right) |x^{(t)}_i - x^*|^2 + \frac{\eta^2 \sigma^2}{n} \\
&- 2\eta (1 - 2L \eta) e_t + \eta_L^2 + L \eta + \mu V(t)
\end{align*}

where $e_t = E[f(x^{(t)}_i)] - f^*$.

Lemma 3.4. Gossiping-type SGD will converge provided the consensus protocol can reach synchronization.

Sketch of proof. Since the consensus protocol can reach synchronization, the Lyapunov function $V(t)$ can be bounded. Inserting this bound into [3.3], via an application of contraction mapping, the convergence of $e_t$ can be established, i.e., $f(x^{(t)}_i) \to f^*$ as $t \to \infty$. □

Nonlinear consensus with communication delays. To avoid technical clutter, we consider the following simplified nonlinear gossiping consensus problem, with delayed communications.

\begin{align*}
x_{i}^{(t+1)} = x_{i}^{(t)} + \sum_{j: (i,j) \in \mathcal{E}} W_{ij} \phi \left( x_{j}^{(t-\tau)} - x_{i}^{(t-\tau)} \right).
\end{align*}

The following statement gives the maximum tolerance of communication delays for consensus to work in terms of the spectrum of the graph Laplacian.

Theorem 3.5 (NGO convergence with stale updates). The consensus update defined by [7] asymptotically solves the average consensus problem with a uniform time-delay $\tau$ for all initial states if and only if $0 \leq \tau \leq \pi/2 \lambda_n(L)$. 
Nonlinear consensus on random graphs  We further study how the consensus protocol cope with communication uncertainties. In particular, we consider the following random graph model, as a tractable mathematical simplification of our problem. Let $u$ be the incidence rate of an edge, and the adjacency matrix $W^{(t)}(u)$ for graph $G \in \mathcal{G}(n, u)$. Define the following model

$$W_{ij}^{(t)}(u) = \begin{cases} 1, & \text{with prob. } u, \\ 0, & \text{with prob. } 1 - u. \end{cases}$$

For each time $t$, we have $W^{(t)}(u)$ as an i.i.d. draw from the above model, which poses NGO with randomized communications as the following random dynamic system:

$$x_i^{(t+1)} = x_i^{(t)} + \sum_{j: (i,j) \in E} W_{ij}^{(t)}(u) \phi \left( x_j^{(t)} - x_i^{(t)} \right).$$

Note this model subsumes cases with random communication failures or limited communications discussed at the beginning of this subsection. The following statement verifies its consensus property.

**Theorem 3.6.** The random dynamic system defined by (10)-(11) reaches consensus with probability 1.

4 Related Work

**Stochastic gradient descent** is the de facto practice in solving large-scale learning problems, which iteratively updates model parameters with noisy gradients (Zhang, 2004; Bottou, 2010; Ghadimi and Lan, 2013). Empirically, speedy convergence is expected from the SGD-type algorithm due to a better trade-off between faster iterations and a slightly slower convergence rate (Ruder, 2016; Li et al., 2014b). Scaling up SGD with parallel computation is receiving growing attention in recent years (Bordes et al., 2009; Lian et al., 2017; Jahanian et al., 2018; Qu and Li, 2019; Yu et al., 2019), and has witnessed great success (Goyal et al., 2017). Other progress in distributed SGD setting has been made, e.g., addressing the robustness to malicious inputs (Akhtar et al., 2018) and how to make performance-adaptive updates (Teng et al., 2019). The proposed NGO bears resemblance to recent nonlinear variants of SGD, such as sign-SGD (Bernstein et al., 2018). The difference lies in we apply nonlinearity to the consensus protocol, rather than model updates, yet synergies can be exploited.

**Decentralized coordination** is a naturally occurring phenomenon observed in many physical systems, such as flocking of birds and schooling of fish. Consequently, the study of network coordination with dynamic agents is an active topic in dynamical systems (Kempe et al., 2003; Xiao and Boyd, 2004). The seminal work of Olfati-Saber and Murray (2004) first established convergence analysis of non-trivial consensus protocols for a network of integrators with a directed information flow and fixed or switching topology. A comprehensive review on the topic can be found in Motter et al. (2013). More recently, Wang and Xiao (2010); Lu et al. (2016) investigated finite-time consensus problems for multi-agent systems and presented a framework for constructing effective distributed protocols. In this study, we explore how these more general consensus protocols can be leveraged to benefit distributed learning.

**Decentralized optimization** has attracted lots of attention due to the recent explosion of data and model complexity. Gossiping-type schemes combined with SGD, showing sub-linear convergence to optimality can be achieved via gradually diminishing step-size (Nedic et al., 2009). A number of later contributions (Shi et al., 2015; Johansson et al., 2008; Jakovetic et al., 2014; Matei and Baras, 2011) extended decentralized SGD to other settings, including stochastic networks, constrained problems and noisy environments. Under the non-i.i.d. setting, Tang et al. (2018) investigated how better performance can be achieved using decentralized SGD. Lian et al. (2017a) argued that decentralized algorithms are no compromise for their centralized counterparts, and on par performance can be expected. Our work complements existing studies via generalizing consensus protocols. Additionally, recent analysis on the learning dynamics of deep neural nets suggests distributed learners might be better synched in the nonlinear regime (Chizat et al., 2019).

**Worker asynchrony and random graph topology** promise to ameliorate the communication barrier in a distributed setup. Discussion on asynchronous SGD variants and their robustness to communication delays can be found in (Zheng et al., 2017; Mitlagkas et al., 2016; Zhou et al., 2018; Mania et al., 2017). Concerning the graph topology (Nedic et al., 2009), a number of later contributions (Shi et al., 2015; Johansson et al., 2008; Jakovetic et al., 2014; Matei and Baras, 2011) extended decentralized SGD to other settings, including stochastic networks, constrained problems and noisy environments. Under the non-i.i.d. setting, Tang et al. (2018) investigated how better performance can be achieved using decentralized SGD. Lian et al. (2017a) argued that decentralized algorithms are no compromise for their centralized counterparts, and on par performance can be expected. Our work complements existing studies via generalizing consensus protocols. Additionally, recent analysis on the learning dynamics of deep neural nets suggests distributed learners might be better synched in the nonlinear regime (Chizat et al., 2019).

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Another line of work is to reduce the communication frequency [Seide et al., 2014; Zinkevich et al., 2010; Stich, 2018]. Instead of transmitting a full gradient, methods with gradient compression, such as quantization and sparsification, can be a more efficient way to transmit useful information [Wen et al., 2017; Alistarh et al., 2017; Zhang et al., 2017; Koloskova et al., 2019]. This work investigates an orthogonal problem, which is how to achieve better synchrony under the same communication budget.

5 Experiments

To validate the utility of the proposed NGO algorithm and benchmark its performance against prior arts, we consider a wide range of synthetic and real-world tasks. All experiments are implemented with PyTorch, and our code is available from https://github.com/author_name/NGO. Details of our setup & more results are deferred to the SM Sec. B and C.

5.1 Experimental setup

Datasets. We consider the following real-world dataset: (i) MNIST (ii) CIFAR10 standard image classification tasks; (iii) TINYIMAGENET (Le and Yang, 2015): a scaled-down version of the classic natural image dataset IMAGENET, comprised of 200 classes, 500 samples per class and 10k validation images.

Data model. We distribute the $m$ data samples evenly across the $n$ workers, with one of the following two data models:

(i) i.i.d. setting, where each worker see a i.i.d. copy of data from the same distribution (all labels);

(ii) Non-i.i.d. setting, where each worker only sees data with a subset of labels.

We ensure non-overlapping data assignment, so each data point is assigned to only one worker. Note that learning with non-i.i.d. data is very challenging.

Baselines. The following competing baselines are considered to benchmark the proposed solution: (i) centralized learning (parameter server); (ii) decentralized SGD without compression [Lian et al., 2017b]; (iii) Local SGD [Stich, 2018].

Network topology. We consider the following network topologies: ring, random-connected and fully-connected (complete) graphs. Unless otherwise specified, we use the ring topology by default for illustrative purposes. This is the most constrained topology, as each worker can only interact with two neighbors.

Performance metrics. We are particularly interested in the level of synchronization, as measured by synchronization index $V$, in addition to the algorithmic convergence in terms of test accuracy and test loss (e.g., cross entropy for the supervised learning task and ELBO for the VI task).

5.2 Impact of nonlinearity

To investigate how the level of nonlinearity (parameter $p$) interacts with learning and synchronization rates to impact synchronization, convergence, and the robustness of training, we plot both synchronization and convergence performance of NGO with varying nonlinearity in Figure 4. As we increase the level of nonlinearity (i.e., by decreasing $p$), both performances improve. This is consistent with our theory: NGO enjoys the finite-time consensus property thus synchronizes better with a smaller $p$, which in turn reduces the synchronization index discussed in Theorem 2.3.

5.3 Random graph

We further evaluate NGO with random graphs. We systematically vary the edge probability (Eqn (10)) under the i.i.d. setup, and give the results in Figure 5. After some initial burn-in, the performance of NGO with a rather small $u = 0.4$ is on par with that of centralized SGD. We notice that the expectation of a random graph connecting edge numbers with $u = 0.2$ is equivalent to a ring topology for 10 workers. Compared to a fixed topology (e.g., ring), better performance is observed with the randomized communications with a similar communication budget (See Table 4 for more results). This can be explained by better information mixing can be expected when workers randomly sync their individual states.

5.4 Communication and convergence

We offer an alternative view that to alleviate the communication burden in distributed learning, one should optimize the communication efficiency rather than reducing the communication frequency (which is a practice in local SGD [Stich, 2018]), especially so for the non-i.i.d. setting. To confirm this, we reduce the consensus update frequency to implement a local SGD.
Finite-Time Consensus Learning for Decentralized Optimization with Nonlinear Gossiping

Table 1: Comparison of linear and nonlinear Gossiping for supervised learning (Acc)

| Dataset   | Topology | Gossip | NGO | Dataset   | Topology | Gossip | NGO |
|-----------|----------|--------|-----|-----------|----------|--------|-----|
| i.i.d.    | Centralized | 97.47  |     | i.i.d.    | Centralized | 83.87  |     |
| MNIST     | Ring     | 96.47  | 96.52| RANDOM    | Ring     | 62.30  | 64.84|
|           | Random   | 96.48  | 96.67| COMPLETE  | RANDOM   | 66.72  | 70.02|
|           | Complete | 96.35  | 96.70|          | COMPLETE | 71.28  | 73.61|

| Dataset   | Topology | Gossip | NGO | Dataset   | Topology | Gossip | NGO |
|-----------|----------|--------|-----|-----------|----------|--------|-----|
| i.i.d.    | Centralized | 74.52  |     | i.i.d.    | Centralized | 70.40  |     |
| CIFAR10   | Ring     | 70.48  | 72.38| RANDOM    | Ring     | 32.49  | 34.14|
|           | Random   | 72.24  | 73.57| COMPLETE  | RANDOM   | 50.11  | 51.10|
|           | Complete | 72.25  | 73.42|          | COMPLETE | 52.56  | 53.35|

| Dataset   | Topology | Gossip | NGO | Dataset   | Topology | Gossip | NGO |
|-----------|----------|--------|-----|-----------|----------|--------|-----|
| i.i.d.    | Centralized | 38.39  |     | i.i.d.    | Centralized | 36.79  |     |
| TINYIMAGENET | Ring   | 35.44  | 36.56| RANDOM    | Ring     | 30.67  | 31.25|
|           | Random   | 37.51  | 37.84| COMPLETE  | RANDOM   | 31.60  | 32.83|
|           | Complete | 37.67  | 37.95|          | COMPLETE | 31.93  | 32.78|

Figure 6: (left) Comparison of local SGD, linear and nonlinear gossiping SGD under the non-i.i.d. setting with complete graph. (right) Synchronization index.

5.5 Scaling to large number of workers

To study the scaling properties of NGO, we train on 4, 16, 36 and 64 number of nodes. We give every worker the same amount of data regardless of the network size. The results are summarized in (Figure 7). Centralized learning has a good performance for all scales of networks. For decentralized protocols, our NGO consistently outperforms linear gossiping. Better performance is expected with the increase of workers as more data is utilized in training.

5.6 Synchronization rate and stability

While in general NGO delivers more favorable results compared to its linear counterparts, we notice NGO should be practiced with caution. This is because NGO can be more sensitive to the synchronization rate $\gamma_t$. When $\gamma_t$ is too high, some oscillations is expected for NGO (Figure 8 right).

6 Conclusions

We present a novel nonlinear gossiping SGD framework based on finite-time consensus protocol. We prove that NGO achieves a better convergence rate compared to linear gossiping, bridging the gap of centralized learning and decentralized learning. We highlight that our algorithm achieves better performance under a limited communication budget compared to gossiping SGD. Unlike prior-arts, we seek to improve the efficiency of communication with respect to synchronization rather than reducing the communication frequency or compressing messages. NGO can be easily adapted to work with asynchronous communications and randomized graph topology. In future work, we wish to establish NGO’s convergence under non-convex settings, and extend its applicability to heterogeneous learners.
References

Alistarh, D., Allen-Zhu, Z., and Li, J. (2018). Byzantine stochastic gradient descent. In NeurIPS, pages 4613–4623.

Alistarh, D., Grubic, D., Li, J., Tomioka, R., and Vojnovic, M. (2017). Qsgd: Communication-efficient sgd via gradient quantization and encoding. In NIPS, pages 1709–1720.

Balcan, M. F., Blum, A., Fine, S., and Mansour, Y. (2012). Distributed learning, communication complexity and privacy. In COLT, pages 26–1.

Basu, D., Data, D., Karakus, C., and Diggavi, S. N. (2020). Qsparse-local-sgd: Distributed sgd with quantization, sparsification, and local computations. IEEE Journal on Selected Areas in Information Theory, 1(1):217–226.

Bernstein, J., Wang, Y.-X., Azizzadenesheli, K., and Anandkumar, A. (2018). signsgd: Compressed optimisation for non-convex problems. In ICML.

Bordes, A., Bottou, L., and Gallinari, P. (2009). Sgd-qn: Careful quasi-newton stochastic gradient descent. Journal of Machine Learning Research, 10(Jul):1737–1754.

Bottou, L. (2010). Large-scale machine learning with stochastic gradient descent. In COMPSTAT, pages 177–186. Springer.

Boyd, S., Ghosh, A., Prabhakar, B., and Shah, D. (2006). Randomized gossip algorithms. IEEE/ACM Transactions on Networking, 14(SI):2508–2530.

Chen, J., Chen, T., and Lu, W. (2017). On second-order synchronization protocols of multi-agent systems. In CCC, pages 8330–8335. IEEE.

Chen, J., Lu, D., Xiu, Z., Bai, K., Carin, L., and Tao, C. (2021). Variational inference with holder bounds. arXiv preprint arXiv:2111.02947.

Chen, J., Xiu, Z., Henao, R., Goldstein, B., Carin, L., and Tao, C. (2020). Supercharging imbalanced data learning with energy-based contrastive representation transfer. arXiv preprint arXiv:2011.12454.

Chizat, L., Oyallon, E., and Bach, F. (2019). On lazy training in differentiable programming. In NeurIPS, pages 2933–2943.

Dean, J., Corrado, G., Monga, R., Chen, K., Devin, M., Mao, M., Senior, A., Tucker, P., Yang, K., Le, Q. V., et al. (2012). Large scale distributed deep networks. In NIPS, pages 1223–1231.

Ghadimi, S. and Lan, G. (2013). Stochastic first- and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368.

Goyal, P., Dollár, P., Girshick, R., Noordhuis, P., Wesolowski, L., Kyrola, A., Tulloch, A., Jia, Y., and He, K. (2017). Accurate, large minibatch sgd: Training imagenet in 1 hour. arXiv preprint arXiv:1706.02677.

Haddadpour, F., Kamani, M. M., Mokhtari, A., and Mahdavi, M. (2021). Federated learning with compression: Unified analysis and sharp guarantees. In AISTATS, pages 2350–2358. PMLR.

Hale, M. T. and Egerstedt, M. (2017). Convergence rate estimates for consensus over random graphs. In ACC, pages 1024–1029. IEEE.

Hatano, Y. and Mesbahi, M. (2005). Agreement over random networks. IEEE Transactions on Automatic Control, 50(11):1867–1872.

Jahani, M., He, X., Ma, C., Mokhtari, A., Mudigere, D., Ribeiro, A., and Takač, M. (2018). Efficient distributed hessian free algorithm for large-scale empirical risk minimization via accumulating sample strategy. In ICML.

Jakovetić, D., Xavier, J., and Moura, J. M. (2014). Fast distributed gradient methods. IEEE Transactions on Automatic Control, 59(5):1131–1146.

Johansson, B., Keviczky, T., Johansson, M., and Johansson, K. H. (2008). Subgradient methods and consensus algorithms for solving convex optimization problems. In CDC, pages 4185–4190. IEEE.

Kairouz, P., McMahan, H. B., Avent, B., Bellet, A., Bennis, M., Bhagoji, A. N., Bonawitz, K., Charles, Z., Cormode, G., Cummings, R., et al. (2019). Advances and open problems in federated learning. arXiv preprint arXiv:1912.04977.

Kempe, D., Dobra, A., and Gehrke, J. (2003). Gossip-based computation of aggregate information. In FOCS, pages 482–491. IEEE.

Koloskova, A., Loizou, N., Boreiri, S., Jaggi, M., and Stich, S. (2020). A unified theory of decentralized sgd with changing topology and local updates. In ICML, pages 5381–5393. PMLR.

Koloskova, A., Stich, S. U., and Jaggi, M. (2019). Decentralized stochastic optimization and gossip algorithms with compressed communication. In ICML.

Kovalev, D., Koloskova, A., Jaggi, M., Richtarik, P., and Stich, S. (2021). A linearly convergent algorithm for decentralized optimization: Sending less bits for free! In AISTATS, pages 4087–4095. PMLR.

Le, Y. and Yang, X. (2015). Tiny imagenet visual recognition challenge. CS 231N, 7(7):3.

Leblond, R., Pedregosa, F., and Lacoste-Julien, S. (2017). Asaga: asynchronous parallel saga. In AISTATS, pages 46–54. PMLR.
Lee, K., Lam, M., Pedarsani, R., Papailiopoulos, D., and Ramchandran, K. (2017). Speeding up distributed machine learning using codes. *IEEE Transactions on Information Theory*, 64(3):1514–1529.

Li, M., Andersen, D. G., Smola, A. J., and Yu, K. (2014a). Communication efficient distributed machine learning with the parameter server. In *NIPS*, pages 19–27.

Li, M., Zhang, T., Chen, Y., and Smola, A. J. (2014b). Efficient mini-batch training for stochastic optimization. In *SIGKDD*, pages 661–670.

Lian, X., Zhang, C., Zhang, H., Hsieh, C.-J., Zhang, W., and Liu, J. (2017a). Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. In *NIPS*, pages 5330–5340.

Lian, X., Zhang, W., Zhang, C., and Liu, J. (2017b). Asynchronous decentralized parallel stochastic gradient descent. In *ICML*.

Lin, Y., Han, S., Mao, H., Wang, Y., and Dally, W. J. (2018). Deep gradient compression: Reducing the communication bandwidth for distributed training. In *ICLR*.

Lu, W., Liu, X., and Chen, T. (2016). A note on finite-time and fixed-time stability. *Neural Networks*, 81:11–15.

Mania, H., Pan, X., Papailiopoulos, D., Recht, B., Ramchandran, K., and Jordan, M. I. (2017). Perturbed iterate analysis for asynchronous stochastic optimization. *SIAM Journal on Optimization*, 27(4):2202–2229.

Matei, I. and Baras, J. S. (2011). Performance evaluation of the consensus-based distributed subgradient method under random communication topologies. *IEEE Journal of Selected Topics in Signal Processing*, 5(4):754–771.

McMahan, H. B., Moore, E., Ramage, D., Hampson, S., et al. (2017). Communication-efficient learning of deep networks from decentralized data. In *AISTATS*.

Mitsiagkas, I., Zhang, C., Hadjis, S., and Ré, C. (2016). Asynchrony begets momentum, with an application to deep learning. In *Allerton*, pages 997–1004. IEEE.

Motter, A. E., Myers, S. A., Anghel, M., and Nishikawa, T. (2013). Spontaneous synchrony in power-grid networks. *Nature Physics*, 9(3):191.

Na, T., Ko, J. H., Kung, J., and Mukhopadhyay, S. (2017). On-chip training of recurrent neural networks with limited numerical precision. In *IJCNN*, pages 3716–3723. IEEE.

Nedic, A., Olshevsky, A., Ozdaglar, A., and Tsitsiklis, J. N. (2009). On distributed averaging algorithms and quantization effects. *IEEE Transactions on Automatic Control*, 54(11):2506–2517.

Nedić, A., Olshevsky, A., and Rabbat, M. G. (2018). Network topology and communication-computation tradeoffs in decentralized optimization. *Proceedings of the IEEE*, 106(5):953–976.

Nedic, A., Olshevsky, A., and Shi, W. (2017). Achieving geometric convergence for distributed optimization over time-varying graphs. *SIAM Journal on Optimization*, 27(4):2597–2633.

Olfati-Saber, R., Fax, J. A., and Murray, R. M. (2007). Consensus and cooperation in networked multiagent systems. *Proceedings of the IEEE*, 95(1):215–233.

Olfati-Saber, R. and Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on automatic control*, 49(9):1520–1533.

Qu, G. and Li, N. (2019). Accelerated distributed nesterov gradient descent. *IEEE Transactions on Automatic Control*, 65(6):2566–2581.

Recht, B., Re, C., Wright, S., and Niu, F. (2011). Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *NIPS*, pages 693–701.

Reisizadeh, A., Mokhtari, A., Hassani, H., Jadbabaie, A., and Pedarsani, R. (2020). Fedpaq: A communication-efficient federated learning method with periodic averaging and quantization. In *AISTATS*, pages 2021–2031. PMLR.

Ren, W. and Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on automatic control*, 50(5):655–661.

Ruder, S. (2016). An overview of gradient descent optimization algorithms. arXiv preprint arXiv:1609.04747.

Seide, F., Fu, H., Droppo, J., Li, G., and Yu, D. (2014). 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns. In *INTERSPEECH*.

Shamir, O. (2014). Fundamental limits of online and distributed algorithms for statistical learning and estimation. In *NIPS*, pages 163–171.

Shi, W., Ling, Q., Wu, G., and Yin, W. (2015). Extra: An exact first-order algorithm for decentralized consensus optimization. *SIAM Journal on Optimization*, 25(2):944–966.

Stich, S. U. (2018). Local sgd converges fast and communicates little. In *ICLR*.

Tandon, R., Lei, Q., Dimakis, A. G., and Karampatziakis, N. (2017). Gradient coding: Avoiding strag-
glers in distributed learning. In *ICML*, pages 3368–3376.

Tang, H., Gan, S., Zhang, C., Zhang, T., and Liu, J. (2018). Communication compression for decentralized training. In *NeurIPS*, pages 7652–7662.

Tao, C., Chen, L., Dai, S., Chen, J., Bai, K., Wang, D., Feng, J., Lu, W., Bobashev, G., and Carin, L. (2019). On fenchel mini-max learning. In *NeurIPS*, volume 32, pages 10427–10439.

Teng, Y., Gao, W., Chalus, F., Choromanska, A. E., Goldfarb, D., and Weller, A. (2019). Leader stochastic gradient descent for distributed training of deep learning models. In *NeurIPS*, pages 9821–9831.

Truex, S., Baracaldo, N., Anwar, A., Steinke, T., Ludwig, H., Zhang, R., and Zhou, Y. (2019). A hybrid approach to privacy-preserving federated learning. In *ACL*, pages 1–11.

Wang, J., Sahu, A. K., Yang, Z., Joshi, G., and Kar, S. (2019). Matcha: Speeding up decentralized sgd via matching decomposition sampling. In *ICC*.

Wen, W., Xu, C., Yan, F., Wu, C., Wang, Y., Chen, Y., and Li, H. (2017). Terngrad: Ternary gradients to reduce communication in distributed deep learning. In *NIPS*, pages 1509–1519.

Wu, J., Huang, W., Huang, J., and Zhang, T. (2018). Error compensated quantized sgd and its applications to large-scale distributed optimization. In *ICML*.

Xiao, L. and Boyd, S. (2004). Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78.

Ye, M. and Abbe, E. (2018). Communication-computation efficient gradient coding. In *ICML*, pages 5610–5619. PMLR.

Yu, C., Tang, H., Renggli, C., Kassing, S., Singla, A., Alistarh, D., Zhang, C., and Liu, J. (2018). Distributed learning over unreliable networks. In *ICML*.
Appendix

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A Technical Support

A.1 Theoretical Results of Synchronization

Before stating the theoretical results, we need to define an important class of digraphs that appear frequently throughout this section.

**Definition S1** (balanced digraphs Olfati-Saber and Murray (2004)). A digraph $\mathcal{G}$ is called balanced if $\sum_{j \neq i} W_{ij} = \sum_{j \neq i} W_{ji}$ for all $i \in \mathcal{V}$.

We first revisit the continuous version of gossip algorithm.

**Lemma S2** (Olfati-Saber et al. (2007)). Let $\mathcal{G}$ be a connected undirected graph. Then,

$$\dot{x}_i = \sum_{j : (i,j) \in \mathcal{E}} W_{ij} (x_j(t) - x_i(t))$$

(12)

asymptotically reaches synchronization for all initial states. Moreover, the synchronization value is $\frac{1}{n} \sum_i x_i(0)$ that is equal to the average of the initial values.

**Proof.** The dynamics of system (12) can be expressed in a compact form as

$$\dot{x} = -Lx$$

(13)

where $L$ is known as the graph Laplacian of $\mathcal{G}$. By definition, $L$ has a right eigenvector of $1$ associated with the zero eigenvalue because of the identity $L1 = 0$. For the case of undirected graphs, graph Laplacian satisfies the following property:

$$x^\top L x = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} W_{ij} (x_j - x_i)^2$$

(14)

By defining a quadratic disagreement function as

$$\phi(x) = \frac{1}{2} x^\top L x$$

(15)
it becomes apparent that algorithm (12) is the same as

\[ \dot{x} = -\nabla \phi(x) \]  

(16)

or the gradient-descent algorithm. This algorithm globally asymptotically converges to the agreement space provided that two conditions hold: 1) \( \mathbf{L} \) is a positive semidefinite matrix; 2) the only equilibrium of (12) is \( \alpha_1 \) for some \( \alpha \). Both of these conditions hold for a connected graph and follow from the SOS property of graph Laplacian in (14). Therefore, an average-synchronization is asymptotically reached for all initial states. \( \square \)

The discrete-time convergence result is almost identical to its continuous-time counterpart.

**Theorem S3** ([Ren and Beard 2005]). Consider a network of workers with topology \( \mathcal{G} \) applying the distributed consensus algorithm

\[ x_i^{(t+1)} = x_i^{(t)} + \gamma \sum_{j: (i,j) \in \mathcal{G}} W_{ij}(x_j^{(t)} - x_i^{(t)}) \]  

(17)

where \( 0 < \gamma < 1/\Delta \), and \( \Delta \) is the maximum degree of the network. Let \( \mathcal{G} \) be a strongly connected digraph. Then an average-synchronization is asymptotically reached if \( W \) is doubly stochastic.

Suppose that worker \( i \) receives a message sent by its neighbor \( j \) after a time-delay of \( \tau \). This is equivalent to a network with a uniform one-hop communication time delay.

**Theorem S4.** The algorithm

\[ \dot{x}(t) = \sum_{j: (i,j) \in \mathcal{G}} W_{ij}(x_j(t - \tau) - x_i(t - \tau)) \]

asymptotically solves the average synchronization problem with a uniform one-hop time-delay \( \tau \) for all initial states if and only if \( 0 \leq \tau < \pi/2\lambda_n \), where \( \lambda_n \) is the largest eigenvalue of the graph Laplacian \( \mathbf{L} \).

A switching network can be modeled using a dynamic graph \( \mathcal{G}_s \), parameterized with a switching signal \( s_t : \mathbb{R} \to J \) that takes its values in an index set \( J = \{1, 2, \cdots, n\} \).

**Theorem S5** ([Olfati-Saber and Murray 2004]). Consider a network of workers with algorithm \( \dot{x} = \mathbf{L}_{\mathcal{G}_k} \) with topologies \( \mathcal{G}_k \in \Gamma \). Suppose every graph in \( \Gamma \) is a balanced digraph that is strongly connected and let \( \lambda^*_2 = \min_{k \in J} \lambda_2(\mathcal{G}_k) \). Then, for any arbitrary switching signal, the workers asymptotically reach an average-synchronization for all initial states with a speed faster or equal to \( \lambda^*_2 \).

**A.2 Proof of Theorem 3.1**

*Proof.* Recall the definition of Lyapunov function

\[ V(t) = \sum_{i=1}^{n} (x_i(t) - \bar{x})^2, \]

after differentiating we have:

\[ \dot{V}(t) = 2 \sum_{i} (x_i(t) - \bar{x})(\dot{x}_i(t) - \dot{\bar{x}}) \]

\[ = 2\gamma \sum_{i} (x_i(t) - \dot{\bar{x}}(t)) \sum_{j: (i,j) \in \mathcal{E}} W_{ij}\phi(x_j(t) - x_i(t)) \]  

(18)

Here we use the fact that \( \sum_{(i,j) \in \mathcal{E}} W_{ij}\phi(x_j(t) - x_i(t)) = 0, \forall t > 0 \), thus \( \dot{\bar{x}}(t) = \bar{x} \) is invariant.
The average \( \bar{x}^{(t)} \) of the iterates of Algorithm 2 and Algorithm 3 satisfy the following:

\[
\mathbb{E}_{\xi_1^{(t)}, \ldots, \xi_n^{(t)}} \left\| \bar{x}^{(t+1)} - x^* \right\|^2 \\
\leq \left(1 - \frac{\mu \eta_h}{2}\right) \left\| \bar{x}^{(t)} - x^* \right\|^2 + \frac{\eta_h^2 \sigma^2}{n} - 2\eta_h \left(1 - 2L\eta_h\right) \left(f(\bar{x}^{(t)}) - f^*\right) \\
+ \frac{2\eta_h^2 L^2}{n} + Ln\eta_h + \mu \eta_h \sum_{i=1}^{n} \left\| x_i^{(t)} - x_i^{(t)} \right\|^2
\]

where \( \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \).

**A.3 Convergence rate**

The following lemma for decentralized learning is derived in Koloskova et al. (2019).

Lemma S6. The average \( \bar{x}^{(t)} \) of the iterates of Algorithm 2 and Algorithm 3 satisfy the following:

\[
\sum_{i} \sum_{j: (i,j) \in E} W_{ij} |x_i(t) - x_j(t)| \phi \left(|x_j(t) - x_i(t)|\right)
= \sum_{(i,j) \in E} W_{ij} (x_i(t) - x_j(t))^{2p}
\geq \left[ \sum_{(i,j) \in E} W_{ij}^p (x_i(t) - x_j(t))^2 \right]^p
\]  \hspace{1cm} (19)

where \( B = [W_{ij}^p] \in \mathbb{R}^{n \times n} \). Notice that \( \sum_{i=1}^{n} W_{ij}^p (\delta_i - \delta_j)^2 = 2\delta^T \mathbf{L}(B) \delta \), and \( \sum_i \delta_i = 0 \), we have

\[
\frac{\sum_{i,j=1}^{n} W_{ij}^p (\delta_i - \delta_j)^2}{V(t)} = \frac{2\delta^T \mathbf{L}(B) \delta}{\delta^T \delta} \geq 2\lambda_2(\mathbf{L}(B))
\]

where \( \mathbf{L}(B) \) is the graph Laplacian of \( \mathcal{G}(B) \) and \( \lambda_2(\mathbf{L}(B)) \) is the algebraic connectivity of \( \mathcal{G}(B) \), \( \lambda_2(\mathbf{L}(B)) \) is positive since the graph is connected.

Therefore,

\[
\dot{V}(t) = 2\gamma \sum_i (x_i(t) - \bar{x}) \sum_{j: (i,j) \in E} W_{ij} \phi(x_j(t) - x_i(t))
= 2\gamma \sum_{(i,j) \in E} W_{ij} x_i(t) \phi(x_j(t) - x_i(t))
= 2\gamma \sum_{(i,j) \in E} W_{ij} (x_i(t) - x_j(t)) \phi(x_j(t) - x_i(t))
\leq -4\gamma (\lambda_2(\mathbf{L}(B)))^p V(t)^p
\]

Therefore \( V(t) = 0 \), when

\[
t > T^* = \frac{V^{1-p}(0)}{4\gamma (1-p) (\lambda_2(\mathbf{L}(B)))^p}
\]

\( \square \)
Similarly, for centralized learning, we have
\[
E_{\xi_1^{(t)}, \ldots, \xi_n^{(t)}} \| x^{(t+1)} - x^* \|^2 \leq \left( 1 - \frac{\mu \eta_t}{2} \right) \| x^{(t)} - x^* \|^2 + \frac{\eta_t^2 \sigma^2}{n} - 2 \eta_t \left( 1 - 2L \eta_t \right) f(x^{(t)}) - f^*
\]

According to Lemma 21 in Koloskova et al. (2019), for linear gossip algorithm we have:
\[
\| x^{(t+1)} - \bar{x}^{(t+1)} \|^2 \leq 40\eta_t^2 \frac{1}{\beta^2} n G^2
\]

where \( \beta = 1 - \gamma \lambda_2(L(W)) \). For nonlinear gossip algorithm, according to Theorem 3.1, the term \( \| x^{(t+1)} - \bar{x}^{(t+1)} \|^2 \leq 40\eta_t^2 \frac{1}{\beta^2} n G^2 \) vanishes in finite time \( T^* \).

**Theorem S7.** With stepsize \( \eta_t = \frac{4}{\mu(a + 1)} \), for parameter \( a \geq \max\{ \frac{2}{\beta}, 15\kappa \} \), \( \kappa = \frac{L}{\mu} \), centralized learning converges at the rate
\[
f \left( x^{(T)}_{\text{avg}} \right) - f^* \leq \frac{\mu a^3}{8S_T} \| x^{(0)} - x^* \|^2 + \frac{4T(T + 2a) \bar{\sigma}^2}{\mu S_T} + \frac{2L + \mu}{S_T} \sum_{t=0}^{T^*} \sum_{i=1}^{n} (a + t)^2 \| x_i^{(t)} - \bar{x} \|^2
\]

where \( x^{(T)}_{\text{avg}} = \frac{1}{S_T} \sum_{t=0}^{T-1} w_t x^{(t)} \) for weights \( w_t = (a + t)^2 \), and \( S_T = \sum_{t=0}^{T-1} w_t \geq \frac{1}{3} T^3 \), finite time
\[
T^* = \left( \frac{\sum_{i=1}^{n} \left( x_i^{(0)} - \bar{x}^{(0)} \right)^2}{4\gamma(1 - p) (\lambda_2(L(B)))^{p-1}} \right)^{1-p}
\]

with \( B = \left[ W_{ij}^{1/2} \right] \), \( \lambda_2(L(B)) \) is the algebraic connectivity of \( G(B) \).

**Proof.** For \( \eta_t \leq \frac{1}{4T} \) it holds \( 2L \eta_t - 1 \leq -\frac{1}{2} \) and \( (2 \eta_t L^2 + L + \mu) < (2L + \mu) \), hence
\[
E\| x^{(t+1)} - x^* \|^2 \leq \left( 1 - \frac{\mu \eta_t}{2} \right) E\| x^{(t)} - x^* \|^2 + \frac{\eta_t^2 \sigma^2}{n} - \eta_t \epsilon_t + \eta_t \frac{2L + \mu}{n} V^{(t)}
\]

From Lemma 23 in Koloskova et al. (2019), in nonlinear gossip algorithm we get
\[
\frac{1}{S_T} \sum_{t=0}^{T-1} w_t \epsilon_t \leq \frac{\mu a^3}{8S_T} a_0 + \frac{4T(T + 2a) \bar{\sigma}^2}{\mu S_T} + \frac{2L + \mu}{n S_T} \sum_{t=0}^{T^*} V^{(t)}
\]

for weights \( w_t = (a + t)^2 \) and \( S_T \triangleq \sum_{t=0}^{T-1} w_t = \frac{T}{6} (2T^2 + 6a T - 3T + 6a^2 - 6a + 1) \geq \frac{1}{4} T^3 \). The last term equals zero in centralized learning. \( \square \)

**B Toy Model Experiments**

We set the target model as \( y = \sin x + \epsilon \), where \( \epsilon \sim N(0,0.1) \) is a random noise added to the data generation. In the toy model, the 6400 datapoints randomly assigned to 5 workers, and the partition of data is non-overlapping for all the workers. The network on each worker is set as a 3-layer linear network: \( 1 \xrightarrow{\text{Linear}+\text{ReLU}} 64 \xrightarrow{\text{Linear}+\text{ReLU}} 64 \xrightarrow{\text{Linear}+\text{ReLU}} 1 \).
Finite-Time Consensus Learning for Decentralized Optimization with Nonlinear Gossiping

Figure S1: Consensus result on nonlinear gossip with and without time delay.

![Figure S1](image1.png)

Figure S2: The data distribution of each worker using non-i.i.d. data partition. The color bar denotes the number of data samples. Each rectangle represents the number of data samples of a specific class on a worker.

C Real-world Experiments

C.1 Data Partition

We distribute the $m$ data samples evenly across the $n$ workers, with one of the following two data models:

(i) **i.i.d.** setting, where data are randomly assigned to each worker, such that each worker sees i.i.d. copies from the same distribution;

(ii) **Non-i.i.d.** setting, where each worker only sees data with one, or a few, of particular labels.

Specifically, under the non-i.i.d. setting, we split MNIST into 300 shards, and each worker takes two shards of images from the data pool without replacement. In non-i.i.d. CIFAR10, the data is split into 40 shards and each worker selects 4 shards from the pool without replacement. We use Dirichlet distribution to generate the non-i.i.d. TINYIMAGENET data partition among workers (Chen et al., 2020). We sample $p_k \sim Dir_N(\beta)$ and allocate a $p_{k,j}$ proportion of the instances of class $k$ to worker $j$, where $Dir(\beta)$ is the Dirichlet distribution with a concentration parameter $\beta$ (0.5 by default). The data distributions among workers in default settings are shown in Figure [C.1](image2.png).

C.2 Supervised Learning

**Scalability** To further study the scaling properties of NGO, we train on 4, 16, 36 and 64 number of nodes. We notice that under this setting, we distribute the whole dataset evenly to nodes, which is different from Sec. 5.5. The corresponding parameters are listed in Table [S1](image3.png). Gossip type algorithm slows down due to the influence...
Table S1: Summary of communication topologies

| Topology     | Node Degree | Spectral Gap | $n = 4$ | $n = 16$ | $n = 36$ | $n = 64$ |
|--------------|-------------|--------------|--------|--------|--------|--------|
| Ring         | 2           | 0.67         | 0.05   | 0.01   | 0.003  |
| Random($u$)  | $un$        | -            | -      | -      | -      |
| Complete     | $(n - 1)$   | 1            | 1      | 1      | 1      |

Figure S3: Scalability results on i.i.d. CIFAR10

of the graph topology which is consistent with the spectral gaps order (see Tab. S1). Nonlinear gossip SGD consistently outperforms linear gossip, and the improvement is more obvious when number of nodes increases.

C.3 Unsupervised Learning

We now compare the performance of linear and nonlinear gossiping for unsupervised learning problem. Here we adopt the VAE setting, where one seeks to maximize the lower bound to the data likelihood by optimizing the stochastic autoencoder $q(z)$ and conditional data likelihood $p(x|z)$ (Chen et al., 2021; Tao et al., 2019). We compare the result of linear and nonlinear gossiping, together with the centralized learning result under i.i.d. setting and non-i.i.d. setting. In Table C.4 we compare the evidence lower bound on MNIST dataset and the results show that NGO outperforms Gossiping under both settings, and NGO performs almost as good as centralized learning. For Cifar 10 and MNIST, we provide a few reconstruction with the average model trained with NGO compared to their samples in Figure S5.

C.4 Network Architectures

Table S2: Comparison of NGO and Gossip for unsupervised learning (VAE)

| Setting      | Gossip | NGO  |
|--------------|--------|------|
| i.i.d. Centralized | 113.73 |      |
| i.i.d. Ring    | 124.35 | 113.27 |
| i.i.d. Random  | 124.83 | 114.62 |
| i.i.d. FC      | 121.79 | 114.41 |
| non i.i.d. Centralized | 93.18  |      |
| non i.i.d. Ring    | 113.05 | 111.56 |
| non i.i.d. Random  | 113.37 | 110.59 |
| non i.i.d. FC      | 113.43 | 110.82 |
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Figure S4: (left) Correlation between synchronization and test accuracy.

Figure S5: (left) Samples reconstructed from CIFAR10 dataset with average model trained with nonlinear gossip algorithm. (right) Samples generated from MNIST dataset with model on different workers trained with nonlinear gossip algorithm.

Table S3: MNIST classification experiment network architecture.

| Network | Architecture |
|---------|--------------|
| Classifier | conv2d(unit=10, kernel=5, stride=1) + maxpool2d(stride=2) + ReLU + dropout2d(p=0.5) + conv2d(unit=20, kernel=5, stride=1) + maxpool2d(stride=2) + ReLU + fc(unit=50) + ReLU + dropout2d(p=0.5) + fc(unit=10) + logsoftmax |

Table S4: MNIST VAE experiment network architecture.

| Network | Architecture |
|---------|--------------|
| Encoder | fc(unit=784) + ReLU + fc(unit=512) + ReLU + fc(unit=256) + fc(unit=2) |
| Decoder | fc(unit=2) + ReLU + fc(unit=256) + ReLU + fc(unit=512) + ReLU + fc(unit=784) + sigmoid |

Table S5: Cifar10 classification experiment network architecture.

| Network | Architecture |
|---------|--------------|
| Network | conv2d(unit=6, kernel=5, stride=1) + ReLU + maxpool2d(stride=2) + conv2d(unit=16, kernel=5, stride=1) + ReLU + maxpool2d(stride=2) + fc(unit=120) + ReLU + fc(unit=84) + ReLU + fc(unit=10) |
Table S6: Cifar10 VAE experiment network architecture.

| Network  | Architecture                                      |
|----------|---------------------------------------------------|
| **Encoder** | conv2d(unit=16, kernel=3, stride=1) +BN+ReLU   |
|          | + conv2d(unit=32, kernel=3, stride=2)+BN+ReLU   |
|          | + conv2d(unit=32, kernel=3, stride=1)+BN+ReLU   |
|          | + conv2d(unit=16, kernel=3, stride=2)+BN+ReLU   |
|          | + FC(unit=512)+BN+ReLU                           |
| **Decoder** | FC(unit=512)+BN+ReLU                              |
|          | FC(unit=1024)+BN+ReLU                            |
|          | deconv(unit=32, kernel=3, stride=2)+BN+ReLU      |
|          | deconv(unit=32, kernel=3, stride=1)+BN+ReLU      |
|          | deconv(unit=16, kernel=3, stride=2)+BN+ReLU      |
|          | deconv(unit=768, kernel=3, stride=2)             |

Table S7: Tiny Imagenet classification experiment network architecture.

| Network  | Architecture                                      |
|----------|---------------------------------------------------|
| **Network** | resnet18[1] + FC(unit=200)                      |

\[1\text{Resnet 18 without last layer}\]