Higher Structures in Homotopy Type Theory

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Abstract. The intended model of the homotopy type theories used in Univalent Foundations is the $\infty$-category of homotopy types, also known as $\infty$-groupoids. The problem of higher structures is that of constructing the homotopy types needed for mathematics, especially those that aren’t sets. The current repertoire of constructions, including the usual type formers and higher inductive types, suffice for many but not all of these. We discuss the problematic cases, typically those involving an infinite hierarchy of coherence data such as semi-simplicial types, as well as the problem of developing the meta-theory of homotopy type theories in Univalent Foundations. We also discuss some proposed solutions.

1 Introduction

Homotopy type theory is at the same time a foundational endeavor, in which the aim is to provide a new foundation for mathematics, and an area of mathematics and logic, in which the aim is to provide tools for the mathematical analysis of homotopical (higher dimensional) structures. Let us call the former Univalent Foundations (UF) and the latter Homotopy Type Theory (HoTT), understanding that HoTT encompasses many different particular type theories.

In the present chapter we use the issue of higher structures as a lens with which to study both of these aims and their relations to other foundational approaches.

To motivate the problem of higher structures, we need to recall that the intended universe of UF, and the principal model of HoTT, is the realm of $\infty$-groupoids, a homotopical kind of algebraic structures that have elements, identifications, identifications between identifications, etc. ad infinitum, and these identifications behave sensibly in that we can invert them, compose them, and whisker by them, but the expected laws only hold up to higher identifications. Grothendieck’s homotopy hypothesis tells us that $\infty$-groupoids are the same as homotopy types, so we shall use these terms interchangeably, with a slight preference for the latter, as then homotopy type theories are both theories of homotopy types as well as homotopical type theories.
A common misconception is that higher homotopy types only occur in, or only are relevant to, homotopy theory. That is very far from the case, as even the type of sets, as used in most of mathematical practice, is a 1-type. And higher structures now feature prominently in many areas ranging from geometry, algebra, and number theory, to the mathematics of quantum field theories in physics and concurrency in computer science. An introduction to homotopy types and the homotopy hypothesis is given in Sect. 2.

It is a key point of difference between UF and earlier approaches to foundations inspired by category theory that the former takes $\infty$-groupoids rather than various notions of higher categories to be the basic objects of mathematics, from which the rest are obtained by adding further structure. This insight was due to Voevodsky\(^1\) who remarked that many natural constructions are not functorial in the sense of category theory. (Think for example of the center of a group.) However, every construction—if it is to have mathematical meaning—has to preserve the relevant notion of equivalence. (Isomorphic groups do have isomorphic centers, etc.)

Because UF aims to be a foundation for all of mathematics, it is necessary that its language, in the shape of the HoTT, provide the means of construction for all the homotopy types that are used in mathematics. For the construction of sets, this is not such a big problem, as most of the sets that occur in mathematics can be constructed from the type formers of Martin-Löf type theory. (But even here there are subtleties if we wish to remain in the constructive and predicative realm.)

The main problems appear when it comes to higher dimensional homotopy types. We discuss some positive results (structures that have already been constructed) as well as some open problems (structures that have not already been constructed) in Sect. 3.

We remark that although we expect some actual negative results (i.e., impossibility proofs) for some of the open problems with respect to some particular homotopy type theories, these have yet to appear. But anticipating that further means of construction will be necessary, we discuss potential solutions in Sect. 4.

For the remainder of this Introduction, we shall consider an analogy. Martin-Löf type theory can be considered as a formal system for making constructions. In fact, a variant with an impredicative universe was called the Calculus of Constructions (CoC), and a further extension, the Calculus of Inductive Constructions (CIC) is the basis for the proof-assistant Coq. And we shall be concerned with the question of the limits of the methods of construction available in constructive type theories. An obvious analogy presents itself, namely with euclidean geometry and the limits of the methods of geometric constructions using ruler and compasses. We shall (probably) find that, just as in the geometric case, certain objects are not constructible from the most basic constructions, and require further tools, such as the neusis, for their construction. However, we shall follow Pappus' prescription of parsimony and demand that everything that can be constructed with lesser means, should be so

\(^1\) In [54] he wrote: “The greatest roadblock for me was the idea that categories are ‘sets in the next dimension.’ I clearly recall the feeling of a breakthrough that I experienced when I understood that this idea is wrong. Categories are not ‘sets in the next dimension.’ They are ‘partially ordered sets in the next dimension’ and ‘sets in the next dimension’ are groupoids.”
constructed. As a corollary, since a proof is a special case of a construction, we demand that if something can be proved in a weaker system, then it should be so proved.

Obviously we can include among the list of further means of construction such well-known principles as the law of excluded middle (LEM), Markov’s principle (MP), the axiom of choice (AC), various kinds of transfinite induction (TI), as well as principles of impredicativity. Some of these, as well as weaker versions of these, are referred to as constructive taboos because admitting them is contrary to certain philosophical outlooks inspired by constructivism or intuitionism, and also because they cannot be mechanically executed at all, or only with greatly increased computational complexity.

A further aspect of the constructive taboos is that they reduce the number of models in which we can interpret the constructions. It is well-known that constructive systems admit many useful models, indeed, this is one reason why classical mathematicians may be interested in such systems. Non-homotopical constructive systems can often be modeled in toposes, more precisely, 1-toposes, which can be seen either as generalizations of Kripke models, as generalized spaces, or indeed as generalized worlds of sets.

It is suspected that HoTT can be modeled in higher toposes, more precisely, $(\infty, 1)$-toposes. These dramatically extend the usefulness of HoTT, for instance as explained in Schreiber’s Chapter. Earlier extensions of Martin-Löf type theory often imposed axioms, such as the uniqueness of identity proofs (UIP), that rule out higher dimensional models. These contradict the univalence axiom and may be called homotopical taboos. More refined axioms may hold in $(\infty, 1)$-toposes corresponding to 1-toposes (the 1-localic $(\infty, 1)$-toposes [35, Sect. 6.4]), but not in more general $(\infty, 1)$-toposes. These are called constructive-homotopical taboos.

### 2 Infinity groupoids and the homotopy hypothesis

The types in UF are supposed to be homotopy types, so let us dwell a bit on what they are, both from an intuitive point of view, and from the perspective of mathematics developed in set-theoretic foundations.

Intuitions are always hard to convey, and in the case of the notion of homotopy type, even more so. Intuition is, after all, best developed through practice and familiarity. One way to build an intuition for homotopy types is through working in a homotopy type theory, either on paper or with the help of a proof assistant. Many young workers in HoTT/UF did this before learning about homotopy theory from a classical point of view.

As a first approximation we can say that types $A$ are collections of objects together with for each pair of objects $a, b : A$, a type of identifications $p : a =_A b$, together with meaningful operations on these identifications, such as the ability to compose and invert them. And there should also be higher order operations that produce identifications between identifications, such as an identification $\alpha(p) : (p^{-1})^{-1} = p$.
for any \( p : a = b \). This description is meant to capture types in their incarnation as \( \infty \)-groupoids, and on this view, two types \( A, B \) can be identified if there is a (weak) functor \( F : A \to B \) that is an equivalence of \( \infty \)-groupoids.

Another intuition comes from describing types as (nice) topological spaces up to homotopy equivalence. The objects are the points of the space, and the identifications are the paths between points.

The homotopy hypothesis is the idea that these separate intuitions capture the same underlying concept. It grew out of Grothendieck’s homotopy hypothesis concerning a particular definition of \( \infty \)-groupoids [26]. The modern terminology is due to Baez [10].

In order to explain the subtlety of the situation, let us turn to the most common implementation of the idea of \( \infty \)-groupoids in the context of set-theoretic mathematics. Here these are represented by simplicial sets satisfying a certain filling condition. These simplicial sets are called Kan complexes in honor of [29]. A simplicial set is a functor \( X : \Delta^{op} \to \text{Set} \), where \( \Delta \) is the category of non-empty finite ordinals and order-preserving functions. This means concretely that a simplicial set consists of a set of \( n \)-simplices \( X_n \) for each \( n = 0, 1, \ldots \) together with face and degeneracy maps satisfying laws called the simplicial identities. We think of the 0 simplices as points, the 1-simplices as lines between points, 2-simplices as triangles, etc.

The Kan filling condition says that if we are given \( n \) compatible \( (n−1) \)-simplices in \( X \) in the sense that they could be \( n \) of the \( n+1 \) faces of an \( n \)-simplex, then there exists some such \( n \)-simplex. This condition is illustrated in Fig. 1 in some low-dimensional cases. In each case, we can think of the given data as a map from a horn, a sub-simplicial set \( \Lambda^n_k \subseteq \Delta^n \) of the standard \( n \)-simplex \( \Delta^n \) consisting of the union of all the faces opposite the \( k \)th vertex, into \( X \). A lift is some extension of this to a map from \( \Delta^n \) to \( X \), or equivalently, an \( n \)-simplex in \( X \) with the requisite faces.

For example, in Fig. 1b, if we are given two 1-simplices \( p \) and \( q \) in \( X \) with a common endpoint, then there exists some 2-simplex representing both a composite of \( p \) and \( q \) (the third face \( r \)) together with the interior representing the fact that \( r \) is the composite of \( p \) and \( q \).

Note that Kan complexes give a non-algebraic notion of \( \infty \)-groupoid: there exists composites and higher simplicial identifications, but there are no operations singling out a particular composite.
Here we come to a potential pitfall: we cannot say that homotopy types are Kan complexes, for they have different criteria of identity: In usual mathematical practice we identify two simplicial sets if they are isomorphic (this is already a weaker notion of identity than that provided by set theory!), whereas an identification between Kan complexes \( X \) and \( Y \) considered as homotopy types should be a homotopy equivalence.

And this is perhaps an appropriate point at which to give a type-theoretic take on Quine’s \cite{quine} famous slogans:\footnote{The second has also been discussed from a univalent perspective in \cite{hott-2013, hott-2017}.}

1. To be is to be the value of a variable, and
2. No entity without identity.

In 1 we require moreover that all variables be typed, so we say rather that to be an \( A \) is to be the value of a variable of type \( A \) and more importantly, to be is to be an element of a type, and in 2 we do not require any notion of identity between entities of different types, but we do require as an essential part of giving a type \( A \) that the identity type \( x =_A y \), for \( x, y : A \), is meaningful and correctly expresses the means of identifying elements of \( A \): no type without an identity type.

The discrepancy between the notion of identity between the model objects (here Kan complexes) and the desired notion of identity (here homotopy equivalence) is usually addressed using relative categories as a tool. A relative category consists of a category equipped with a wide subcategory of weak equivalences. This is often refined by adding more properties (e.g., the weak equivalences satisfy the two-out-of-three or the two-out-of-six properties) or structure, such as fibrations and/or cofibrations interacting nicely with the weak equivalences. A particularly well-behaved notion is that of a Quillen model category, which does indeed contain both fibrations and cofibrations in addition to weak equivalences, and is assumed to be complete and cocomplete.

The category of simplicial sets \( \text{sSet} \) can be equipped with the structure of a Quillen model category in which the fibrant objects are the Kan complexes (these are also cofibrant as all objects are cofibrant) and the weak equivalences between Kan complexes are the homotopy equivalences. The category of topological spaces, or more precisely, for technical reasons, the category of compactly generated topological spaces, \( \text{Top}_{cg} \), can likewise be equipped with a Quillen model structure in which the cofibrant objects are the nice spaces (technically, cell complexes; all objects are fibrant) and the weak equivalences between the nice spaces are the homotopy equivalences.

Quillen \cite{quillen} proved that these two model categories give rise to equivalent homotopy categories. For this purpose he introduced the notion of (what is now called) a Quillen equivalence between model categories. Given a nice space \( X \), the corresponding singular Kan complex \( \Pi_{\infty}(X) \) has as \( n \)-simplices the continuous maps from the topological \( n \)-simplex into \( X \), and given a Kan complex \( A \), the corresponding space is the geometric realization \( |A| \) given by gluing together topological simplices according to the face and degeneracy maps in \( A \).

That the homotopy categories are equivalent is a first step towards getting what we actually want. We would actually like to show that the Quillen model categories
of simplicial sets and topological spaces give rise to equivalent homotopy types (in both cases restricting to the objects that lie in a fixed Grothendieck universe). And which notion of homotopy type should we use here? It turns out not to matter, but it is easiest to make (large) Kan complexes out of either one.

The way this is achieved is by enhancing both $\text{sSet}$ and $\text{Top}_{cg}$ to simplicially enriched categories (the latter via the singular Kan complex construction on the level of mapping spaces) such that they become simplicial model categories, and then taking the homotopy coherent nerves of the subcategories of homotopy equivalences between bifibrant objects, i.e., between the model objects on both sides.

Notice that to get a good theory of homotopy types in the classical set-up we seem to need also a good theory of $(\infty, 1)$-categories, that is, categories (weakly) enriched in homotopy types, in order to also get a good hold on the universe of homotopy types, which is another name of course anticipating the type-theoretic notion of a universe, and which consists of the homotopy types that are small relative to some Grothendieck universe.

There is another model structure on simplicial sets whose bifibrant objects are the quasi-categories, those that satisfy a weakening of the Kan filling conditions that make them suitable as models of $(\infty, 1)$-categories. This notion was introduced by Boardman and Vogt [12] and the resulting theory of $(\infty, 1)$-categories has been studied extensively by Joyal [28] and Lurie [35] (see also the appendix of [35] for details on simplicial model categories as discussed above).

My point in bringing out these technicalities is not only to explain how homotopy types are defined and handled in set-theoretic mathematics, but also to give a sense of the subtleties involved. It has taken many years to give a good account of how to treat higher structure in set-theoretic mathematics (often by working in a 1-category-theoretic layer), and there are still many open questions about which constructions and properties are invariant under weak equivalences inside a model category and under Quillen equivalences between model categories. For instance, it was just recently established that a Quillen adjunction always induces an adjunction between underlying quasi-categories, and hence an adjunction of the presented $(\infty, 1)$-categories [38]. Another line of open questions concern the possibility of algebraic models for $\infty$-groupoids, where composition, inverses, etc., are given by operations rather than merely assumed to exist. It is quite possible that type theory will be influential in this area, see for instance the suggestion of Brunerie [13, Appendix].

Thus it should come as no surprise that there are still open questions about how to treat higher structures in HoTT/UF, which is a much younger endeavor. These are the matters we shall now turn to.

3 Higher Structures in HoTT/UF

When Voevodsky proposed using type theory as a foundation for mathematics, he based this on the insight that higher structures in mathematics are not always
naturally objects of a higher *category*, but they *are* always naturally objects of a higher *groupoid*.

Among the ∞-groupoids we find *truncated* higher groupoids, those whose structure is concentrated in a finite range of dimensions. At the lowest level (truncation level −2) we find the contractible types, those that only have one element up to identifications. Secondly, we have propositions. These are types all of whose identity types are contractible.

Moving up in the dimensions, we find next the sets, all of whose identity types are propositions, and the 1-groupoids, all of whose identity types are sets, and so on. We recall from Altenkirch’s Chapter that these truncation levels have a natural formalization in HoTT in terms of predicates

\[ \text{hasDimension} : \mathbb{N}_{-2} \to \text{Type} \to \text{Prop}, \]

and that we have corresponding types of *n*-truncated types, *n*-Type.

Not all types are truncated. The 2-sphere, for example, has structure in all dimensions, so it’s not an *n*-type, for any *n*.

The *n*-types are related to the universe of all types, Type, via the truncation construction that maps a type *X* to its closest *n*-type \( \|X\|_n \). There is a construction \(|-|_n : X \to \|X\|_n \) giving rise to an equivalence

\[ - \circ |-|_n : (\|X\|_n \to Y) \to (X \to Y) \]

for any *n*-type *Y*.

When we go to discuss higher structure, it is often the untruncated types that are the hardest to construct. The principal reason is that we can often construct truncated types in a top-to-bottom fashion, dimensionwise. To construct a proposition, we can just specify the type of evidence *P* that the proposition is true and then if necessary take the propositional truncation \( \|P\|_{-1} \).

I want to emphasize at this point that the sets we discussed above (and in the Chapters of Altenkirch and Ahrens-North) are *not* the sets of set theory! Following Quine’s dictum, these are different notions because they have different notions of identity. Let us temporarily use subscripts to differentiate, and write set\(_1\) for a set theorist’s set and set\(_2\) for a structuralist/homotopy theorist’s set (this is also the model theorist’s notion). There is even a third notion of set, set\(_0\), which is arguably more fundamental than either set\(_1\) or set\(_2\), and which is the one taught in elementary education.

From a type-theoretic point of view, a set\(_0\) is simply a subset of a fixed universal set\(_2\), *X*. That is, we have the type Set\(_0\)(*X*) := \( P(*X*) := (X \to \text{Prop}) \) representing the powerset of *X*. We have an elementary membership relation, \( \varepsilon_0 : X \times \text{Set}_0(X) \to \text{Prop} \), and two sets\(_0\) are equal if they have the same elements in this sense.

This is of course not the set-theorist’s notion of set, according which sets\(_1\) are elements (rather than subsets) of a universe of discourse *U* (itself a set\(_2\)) that is equipped with a membership relation \( \varepsilon_1 : U \times U \to \text{Prop} \) satisfying the axiom of extensionality (and preferably many other set-theoretic axioms).
The naive set-theoretical hope would be to solve the equation $U = \mathcal{P}(U)$ (as an identification of $\text{sets}_2$, i.e., an isomorphism). This is impossible because of Cantor’s diagonal argument, but it can be approximated by the cumulative hierarchy $V$, a construction that can be performed in HoTT via a higher inductive type [52, Sect. 10.5]. Here $V$ is a large set that is the least solution of the equation $V = \mathcal{P}_{\text{small}}(V)$, where $\mathcal{P}_{\text{small}}(V) \coloneqq \Sigma A : \text{Set}. \Sigma f : A \to V. \text{isInjective}(f)$ is the type of small subsets of $V$. Such sets can be thought of as certain well-founded trees, and their study has a quite combinatorial flavor.

The default notion of set in HoTT/UF is $\text{set}_2$ given by the 1-type $\text{Set}$, and this seems to be the one most often used in mathematical practice outside of set theory. For instance, in almost all mathematical contexts, each set can be replaced by an isomorphic copy without changing the meaning of anything. Of course, $\text{sets}_0$ as elements of powerset $0$-types/sets $\text{sets}_2$ also occur throughout mathematical practice, but for these, the set theorist’s and the structuralist’s notion coincide.

Likewise the notion of category splits into several distinct notions: I will denote by $\text{precategory}$ the notion defined in Sect. 4.4 of the Chapter by Ahrens-North, and leave the unadorned term $\text{category}$ for a univalent precategory. Indeed, in most category-theoretic contexts, each category can be replaced by an equivalent while preserving the meaning. It is also useful to have the term $\text{strict category}$ [52, Sect. 9.6] for a precategory whose type of objects is a set. From the perspective of set-theoretic mathematics, the 2-type of categories arise from a Quillen model category structure on the 1-category of strict categories.

Most of category theory can be formalized in HoTT/UF using the univalent definition of category. A precategory can be thought of as a category with extra structure, namely equipped with a functor from an $\infty$-groupoid. For a strict category, this functor has as domain a 0-dimensional homotopy type. In set-theoretic foundations, it will automatically be the case that every category can be equipped with such a strict structure, but in UF this is an extra assumption, indeed a constructive-homotopical taboo.

### 3.1 Analytic and synthetic aspects of HoTT/UF

A foundational theory must be synthetic, in that it describes how to construct and reasons with its fundamental objects in terms of postulated rules. It couldn’t be otherwise, for if it described the “fundamental” objects in terms of other, more fundamental, objects, and derived its rules from the properties of those, it would hardly be foundational.

Homotopy type theories are synthetic theories of $\infty$-groupoids. The approach is deeply logical, where we think of logic as invariant theory as pioneered by Mautner [37] and later developed by Tarski in a 1966 lecture [50]. Both Mautner and Tarski were inspired by the approach to geometry given in Klein’s Erlangen Program [30]. The idea is that the logical notions are those that are invariant under that maximal notion of symmetries of the universes of discourse. If the universe of
discourse is a set, then the corresponding symmetry group is the symmetric group consisting of bijections of the set with itself, but if it is a higher homotopy type, then it is the (higher) automorphism group consisting of all self-homotopy equivalences.

In analogy with the synthetic theories of various notions of geometry (euclidean, affine, projective, etc.), homotopy type theories are synthetic theories of homotopy types (and set theories are synthetic theories of sets), cf. also [9].

The analytic aspect is that all the rest of mathematics, all mathematical objects, their types, and their structure, needs to be developed in terms of homotopy types. And a key criterion for success of a formalized notion is that it satisfies what Ahrens-North call the principle of equivalence, and which I linked to Quine’s dictum above: that the identity type captures the intended notion of identifications between the mathematical objects that we are modeling.

One novel aspect of doing this analytical work in HoTT is when defining a structured object, it can be a challenge already to get the correct carrier type. In set-theoretic foundations, any carrier set of the correct cardinality will do, but in HoTT we are more discerning.

We do reap some benefits of this extra care. For instance, any construction (which, remember, could be proving a proposition, inhabiting a set, etc.) we perform on a generic category is guaranteed to be invariant under equivalence of categories, and we can use the rules of identity types to transport the construction along any equivalence.

Compare this to the situation in set-theoretic foundations: there we have to prove invariance under equivalence for any construction on types of dimension greater than 0. For sets, this is not necessary, because if we are given a set for which the notion of identification between the elements is given by an equivalence relation, we can take the quotient. In this way, the situation in set theory is marginally better than that of type theory pre-HoTT, where the set-quotient construction was not generally available, leading to what some practitioners have called “setoid hell”. But in set theory, the same problem arises for any mathematical type of dimension greater than zero, so we may surmise that formalizations based on set theory will run into “higher groupoid hell”.

### 3.2 Some constructions that are possible

Let us finally take a look at some constructions that are possible in HoTT. Many of these are already discussed in [52]; references are provided in other cases. I will structure this discussion according to the means of construction used. Firstly, there are those that only use the basic constructions in Martin-Löf type theory, namely Σ- and Π-types, identity types, universes, as well as (finitary) inductive types such as the natural numbers, disjoint unions, the empty type, and the unit type.

Next, there are those that use in addition the univalence axiom. Following that, there are those that can be reduced to one particular higher inductive type, the (homotopy) pushout.
Finally, we find those constructions that seem to require more advanced higher inductive types, and in the next subsection I shall discuss those for which there is no known construction at the time of writing.

In *Basic Martin-Löf type theory* (MLTT) we can already define many important notions such as homotopy fibers and other pullbacks, the predicate `hasDimension : ℕ → Type → Type` and the types `n-Type`. We have the types of categories and †-categories (cf. [52, Sect. 9.7]), as well as many other types of mathematical objects occurring outside of homotopy theory. But we are severely limited in our ability to construct inhabitants of these types, or prove properties about them. For instance, we cannot prove that hasDimension is valued in propositions (this requires function extensionality), nor can we prove that anything is not a set (such as the type `Set` itself), since there are models of MLTT in which every type is a set. We cannot construct set quotients, and, perhaps most embarrassingly, we cannot even define the logical operations of disjunction and existential quantification, as these require propositional truncation!

With univalence we get function extensionality (as shown by Voevodsky), and we can now prove many structural properties. Besides hasDimension landing in `Prop`, we can prove that `n-Type` is an `(n + 1)`-type, and we get the equivalence principle for the types of algebraic structures and for categories as mentioned in the Chapter by Ahrens-North. (See also [8].) We can also prove that the `n`th universe is not an `n`-type for any external natural number `n` [32].

At this point we can explore an intermediate route: instead of adding higher inductive types, we can assume the propositional resizing axiom [52, Axiom 3.5.5], stating that the inclusion map `Prop_i → Prop_{i+1}` of propositions in the `i`th universe into the propositions in the `(i + 1)`st universe is an equivalence. This makes the theory impredicative, and it allows us to mimic many impredicative tricks known from (constructive) set theory. For instance, the propositional truncation can be defined as `∥A∥ := ΠP : Prop. (X → P) → P`. (If we have the law of excluded middle, then we can just define `∥A∥ := ¬¬A`.)

The (homotopy) pushout type is a simple, but versatile example of a higher inductive type. It generalizes the disjoint union. Its inputs are three types `A`, `B`, and `C`, together with functions `f : C → A` and `g : C → B`. (Such a configuration is called a span.) The pushout is a new type `D := A ⊔_f^g B` (often written `A ⊔^C B` if `f` and `g` can be deduced from the context) together with injections `left : A → D` and `right : B → D` fitting together in a square

\[
\begin{array}{ccc}
C & \xrightarrow{f} & B \\
\downarrow & & \downarrow^{\text{right}} \\
A & \xleftarrow{\text{left}} & D
\end{array}
\]

whose commutativity is given by a constructor `\(\text{glue} : \Pi x : C. \text{left}(f \ x) = \text{right}(g \ x)\)`.

(See [52, Sect. 6.8] for the elimination and computation rules.)

Now a quite remarkable phenomenon appears. Most of the higher inductive types that are commonly used can be constructed just from pushouts and the other con-
struc-tions in MLTT with univalent universes. These include joins and suspensions (and therefore, spheres), cofibers (and thus smash products), sequential colimits, the propositional truncation [22, 31] and all the higher truncations [43], set quotients, and in fact, all non-recursive HITs specified using point-, 1-, and 2-constructors by a con-struction due to van Doorn [24]. We also get cell complexes [15], Eilenberg-MacLane spaces [33], and projective spaces [17], and so a lot of algebraic topology can be developed on this basis, and even a theory of \(\infty\)-groups [14] and spectra (and thus homology and cohomology theory), culminating in a proof that \(\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}\) [13], and a formalized proof of the Serre spectral sequence for cohomology [23].

Another important construction enabled by pushouts is the Rezk completion, which turns a precategory into the category it represents. This can in fact be done using univalence alone [1], at the cost of going to a larger universe. This can be avoided either using pushouts or using the propositional resizing axiom.

Because so many things can be developed on the basis of univalence, pushouts, and propositional resizing, Shulman suggested that we define an \(\text{elementary} (\infty, 1)\)-topos to be a finitely complete and cocomplete, locally cartesian closed \((\infty, 1)\)-category with a subobject classifier and object classifiers [49]. Let me correspondingly intro-duce the term \(\text{elementary HoTT}\) for MLTT with univalence, pushouts, and proposi-tional resizing.

Finally, let me mention some of the known constructions that seem to require more than the above means, but that can nonetheless be effected via more general higher inductive types. First, there is the cumulative hierarchy as mentioned above [52, Sect. 10.5], the Cauchy-complete real numbers [52, Sect. 11.3], as well as the partiality monad [3]. I won’t say more about these, since this Chapter is supposed to be about higher structure. For these it is more relevant to mention localizations at a family of maps [44]. For example, if we localize at a family of maps of the form \(P(a) \to 1\), for \(a : A\), where each type \(P(a)\) is a proposition, then we obtain a inner model of type theory in itself, in this case a topological localization.

Of course, the number of things that have been constructed and proved in HoTT grows every day, so undoubtedly I’ll have left some out. Many of these constructions have already been formalized in proof assistants for HoTT. In the next section we move to constructions that we don’t yet know how to perform (and which may perhaps require new means of construction).

3.3 Some constructions that seem impossible

Because proving propositions is in HoTT/UF a special case of making constructions, any currently open problems count as constructions we don’t yet know how to perform.\(^3\) But for some of these, it is expected that the difficulty is not just that the construction is tricky to perform with the currently available means of construc-

\(^3\) See https://ncatlab.org/homotopytypetheory/show/open+problems for an up-to-date list of open problems.
tion, but rather that we conjecture that entirely new means of construction will be necessary.

The prime example that I will focus on is that of \((\infty,1)\)-categories, and the related notions of (semi-)simplicial types. Intuitively, an \((\infty,1)\)-category \(C\) consists of a type of objects \(C_0\), for every pair of objects \(a,b : C_0\) a type of morphisms \(C_1(a,b)\), operations for identities and composition, operations that witness the unit- and associative laws, operations that witness higher laws that these must satisfy, and so on \textit{ad infinitum}. The problem is to come up with a way of specifying all these higher coherence operations in a single type.

The basic example of an \((\infty,1)\)-category from the point of view of type theory is the category of types \(\text{Type}\), and as morphisms from \(A\) to \(B\) the type of functions from \(A\) to \(B\), \(S_1(A,B) \equiv A \to B\). Here, the evident identity and composition operations satisfy all the laws and higher laws \textit{definitionally}, so it ought to be particularly easy to show that \(\mathcal{S}\) is an \((\infty,1)\)-category, that is, if we knew how to define the type of \((\infty,1)\)-categories, \((\infty,1)\)-\textbf{Cat}.

It is crucial for the success of UF that we have a working definition and theory of \((\infty,1)\)-categories. Even if they are not \textit{fundamental} in the sense that everything is built out of them, they are still \textit{fundamental} in the sense that they are a key tool in the development of modern higher algebra, geometry, and topology.

The problem of defining \((\infty,1)\)-categories is equivalent to the problem of defining the type of simplicial types, \(\text{sType}\). A \textit{simplicial type} is just a functor \(X : \Delta^\text{op} \to \mathcal{S}\), so if we know how to define \((\infty,1)\)-\textbf{Cat} and the type of functors between any two \(C,D : (\infty,1)\)-\textbf{Cat}, then we can define \(\text{sType}\). On the other hand, \((\infty,1)\)-\textbf{Cat} itself can be defined as the subtype of \(\text{sType}\) consisting of \textit{complete Segal types} (also called \textit{Rezk types}) [42].

The problem of defining \((\infty,1)\)-\textbf{Cat} can also be reduced to another, apparently simpler problem, namely that of defining the type of semi-simplicial types, \(\text{ssType}\). A \textit{semi-simplicial type} is a functor \(X : \Delta^\text{op}_+ \to \mathcal{S}\), where \(\Delta_+\) is the subcategory of \(\Delta\) with the same objects but only injective functions. However, \(\Delta_+\) is a \textit{direct category}, viz. a 0-truncated category where the relation “\(x\) has a nonidentity arrow to \(y\)” is a well-founded relation on the \textit{set} of objects, so we can give a more direct description as follows: A semi-simplicial type \(X\) consists of:

- a type of 0-simplices \(X_0\), and
- for every pair of 0-simplices \(a_0,a_1 : X_0\), a type of 1-simplices from \(a_0\) to \(a_1\), \(X_1(a_0,a_1)\), and
- for every triple of 0-simplices \(a_0,a_1,a_2 : X_0\) and 1-simplices \(a_0 \in X_1(a_0,a_1)\), \(a_0, a_{21} \in X_1(a_0,a_2)\), and \(a_{12} \in X_1(a_1,a_2)\), a type of 2-simplices with boundary \(a_{01}, a_{02}, a_{12}\),
- and so on . . .

Again we have the problem that it seems that infinitely much data is needed, but here it seems more plausible that an inductive (or coinductive) approach could work. Only, no-one has figured out how to do it, and at an informal poll of HoTT-researchers in Warsaw in 2015 a majority believed that it is impossible.

We can define \((\infty,1)\)-categories in terms of semi-simplicial types, as the complete semi-Segal types [19]. This work was inspired by analogous work in the classical
setting [27]. Thus, the problems of defining $(\infty, 1)$-categories, simplicial types, and semi-simplicial types are equivalent, but currently just beyond reach.

It is interesting to contrast the case of $(\infty, 1)$-categories with that of $\infty$-groups:

An $(\infty, 1)$-category structure on a pointed, connected type of objects is the same as an $\infty$-monoid (the type of which we also don’t know how to construct). But the type of $\infty$-groups is simply that of pointed, connected types, with the type of group elements being the identity type $\Omega A \equiv (pt =_{A} pt)$, with $pt : A$ the designated point.

While the above problems concern “large” types, there are also problems of higher structure concerning “small” types. For example, the three-sphere as a type $S^{3}$ should carry the structure of an $\infty$-group, because it is the homotopy type of the Lie group $SU(2)$. Thus, we should be able to construct the homotopy type of the classifying space $BSU(2)$ with $\Omega BSU(2) = S^{3}$, but so far we’ve not been able to do so. (We have the H-space structure, which is a first approximation [18].) In this case, however, we expect that no new means of construction are needed.

An obvious approach would be to construct in the usual way a simplicial set whose homotopy type is $BSU(2)$. If we then had the realization operation $|\cdot| : s\text{Set} \to \text{Type}$, that turns a simplicial set into the homotopy type it represents, then we’d be done. But such a realization operation itself seems impossible to construct in elementary HoTT!

As a final, important, but more open-ended problem, let me mention the problem of developing the meta-theory of HoTT inside HoTT/UF. This has two sides, one relatively easy, and one quite hard. The relatively easy side is the syntactic one, but even here there are difficulties. We can represent extrinsic untyped syntax and corresponding transformations familiar from compiler theory: picking a surface syntax, lexing and parsing this syntax, and then type-checking it. The result should be intrinsic syntax containing only well-typed elements. The intrinsic syntax can be modeled by quotient inductive types (QIT) [4], already mentioned in Altenkirch’s Chapter. The main difficulty here is one of software-engineering: how do we structure both the intrinsic and extrinsic syntax, and the transformations between them, in sufficient generality to cover all the kinds of type theory we are interested in.

The more difficult side is the semantic one. We want to define interpretations of the intrinsic syntax in inner models, first of all the canonical model where syntactic types $\vdash A$ are mapped to types $\llbracket A \rrbracket$, syntactic terms $\vdash a : A$ are mapped to terms $\llbracket a \rrbracket : \llbracket A \rrbracket$, and so on. (For proof-theoretic reasons, we expect to only be able to represent the interpretation locally, for instance type theory with $n$ universes inside the $(n + 1)$st universe, or using stronger principles in the target type theory than are in the source type theory.) Shulman has called this problem “making HoTT eat itself” [48], for which the term autophagy suggests itself.

The problem is that if we use the QIT intrinsic syntax, then everything syntactic is a (homotopy) set and the elimination rule will only allow us to eliminate into sets, whereas for the canonical model we’re eliminating into $\text{Type}$. And if we try to formalize the intrinsic syntax using a non-truncated HIT, then it seems we need infinitely many layers of coherence (reminding us of our problems above with semi-simplicial types and the homotopical realization of simplicial sets).
4 Possible further means of construction

Now that we have seen concretely both the range of constructions that are currently possible in (elementary) HoTT, and some prominent problems that seem out of reach, let us take stock.

The first conclusion is that we’d very much like to prove that the problem of semi-simplicial types cannot be solved in elementary HoTT. But assuming that, the next conclusion is that elementary HoTT is by itself too incomplete to serve its foundational role as the basis for UF. Further means of constructions need to be added. But which ones, and how do we decide which to add?

In some sense the situation is analogous to the question of new axioms for set theory, but there are two main differences: First, we want to use type theory as a programming language and that means that for any proposed extension, we should say how the new constructions compute when combined with the other constructions of type theory. The univalence axiom is a sore point in this regard, as it has been a long-standing open problem to give it a computational meaning. This is now close to being solved via various cubical type theories [6, 11, 20], but there remains a question of whether the corresponding model structures on various categories of cubical sets model ∞-groupoids (we know that the test model structures do [16]). It is still completely open whether the propositional resizing axiom can be given computational meaning.

Secondly, we want to use HoTT also in other models than ∞-groupoids. It is conjectured that elementary HoTT can be interpreted in any (∞, 1)-topos: a left-exact localization of the functor category $C^{op} \to S$ for a small (∞, 1)-category $C$. (It would take us too far afield to give the exact formulation and up-to-date status of this conjecture; see [46].)

Some of the most interesting targets are given by cohesive (∞, 1)-toposes, whose objects can be thought of as geometrically structured ∞-groupoids. (See also Schreiber’s Chapter.) For example, in the cohesive (∞, 1)-topos of smooth ∞-groupoids, Smooth∞Gpd, we find all smooth manifolds among the 0-truncated objects. And we certainly want to be able to do this “at the top level”. But however we solve this problem, it should probably be with computationally meaningful (constructive) means, so that we’re able to do proofs by reflection inside these models. (Of course not all models of interest will be definable constructively.)

Summing up, we expect there to be a stratification of homotopy type theories,
• \text{HoTT} \text{ua} : \text{MLTT} \plus \text{the univalence axiom and propositional resizing},

• \text{HoTT} \text{el} : \text{HoTT} \text{ua} \plus \text{pushouts},

• \text{HoTT} \text{el}++ : \text{HoTT} \text{el} \plus \text{constructions needed for} (\infty, 1)-\text{Cat},

• \text{HoTT} \text{UF} : \text{HoTT} \text{el}++ \plus \text{reflective constructions}.

Here, \text{HoTT} \text{ua} is the basis for Voevodsky’s UniMath formalization effort [55, 56]. We know that \text{HoTT} \text{el} is strictly stronger (in the sense of having fewer models; not in the sense of proof-theoretic strength), because it is consistent with \text{HoTT} \text{ua} that the \( n \)th universe is an \( n \)-type, while if the universes are closed under pushouts, then they are not truncated. Since we don’t have impossibility proofs regarding the constructions of \((\infty, 1)\)-\text{Cat} nor for autophagy, it is still conceivable that we can take \text{HoTT} \text{el}++ and \text{HoTT} \text{UF} to be \text{HoTT} \text{el}.

For each of these we can consider adding classical axioms such as the law of the excluded middle (LEM) or the axiom of choice (AC). These can be seen as constructions that we don’t know how to perform in general, but that an omniscient being would be able to perform. I will not here take any sides in the debate about whether mathematics is better done with or without these principles—I’ll only say that it seems to me that a foundational theory should give its users the choice. (And in any case, we want theories without these constructive taboos for reasoning about other toposes of interest.)

We can also remove the resizing axiom to get (generalized) predicative systems, and for both the predicative and impredicative systems we may add various generalized inductive types to increase the proof-theoretic strength if needed, while keeping the systems constructive and without changing their class of \((\infty, 1)\)-topos models. Thus, the stratification above is meant primarily to distinguish the (intended) models of the theories, and not their proof-theoretic strength, which is an orthogonal concern.

With all that in mind, let us discuss some possible further means of construction that we might add either for \text{HoTT} \text{el} or \text{HoTT} \text{UF}.

### 4.1 Simplicial type theory

For any \((\infty, 1)\)-topos \( C \), we can consider the simplicial objects in \( C \), i.e., functors \( \Delta^\text{op} \to C \), and this is again an \((\infty, 1)\)-topos. As mentioned above, we find therein a full subcategory of \((\infty, 1)\)-categories relative to \( C \). This is the basis for the suggestion by Riehl and Shulman [42] for a synthetic type theory for \((\infty, 1)\)-categories. In their type theory, let’s call it sHoTT, types are interpreted as simplicial types, and they give definitions for Segal and Rezk types with the latter representing \((\infty, 1)\)-categories. They can also define a type of discrete simplicial types, representing ordinary types/\(\infty\)-groupoids, but this type is not a Rezk type and so not the representation of the \((\infty, 1)\)-category of ordinary types, \( S \). Indeed, it is not clear at this moment, whether this is at all representable in their system.

Much work remains before we can judge how useful this type theory is for reasoning about \((\infty, 1)\)-categories. But from a philosophical point of view it cannot
be satisfactory to view sHoTT as a foundational theory, for instance playing the role of HoTT$_{UF}$. (And it is of course not intended as such!) Because simplicial types are begging to be analyzed as such: simplicial objects in a category of types, and not to be taken as unanalyzed in themselves. We really want to be able to define simplicial types inside a theory where types are the fundamental objects.

4.2 Two-level type theories

Another approach to solving the problem of defining simplicial types is to have another layer above the univalent type theory in which to reason about infinitary strict constructions, including (semi-)simplicial types. One proposal is Voevodsky’s Homotopy Type System (HTS) [53]. This system has a distinction between fibrant and non-fibrant types, and it has two identity types: the usual homotopical identity type that only eliminates into fibrant types as well as a new non-fibrant strict equality type that satisfies the reflection rule: if $e : a =_A a'$ is an inhabitant of the strict equality type, then $a$ and $a'$ are definitionally equal. This rule of course makes type-checking undecidable, necessitating a further language of evidence for typing derivations.

Another proposal is the two-level type theory (2LTT) [2, 7]. This is similar to HTS in that it distinguishes between fibrant and non-fibrant types (the latter are called pretypes, but instead of the reflection rule for the strict equality type it adds the rule for uniqueness of equality proofs (Streicher’s K) and function extensionality as an axiom. Thus, type checking is decidable, but the function extensionality axiom breaks computation.

In order to define simplicial types in 2LTT an extra principle beyond the basic set-up is needed. This can be the assumption that the fibrant and non-fibrant natural numbers coincide, or the more technical assumption that Reedy fibrant diagrams of fibrant types indexed by a strict Reedy category have fibrant limits. The former limits the class of available models severely, while the latter does not.

However useful the two-level type theories may turn out to be, they also seem unsatisfactory from a foundational perspective. Because what is a pretype? Pre-types can only be motivated via the models of HoTT as described in set-theoretic mathematics, where they arise as the objects of a model category presenting an $(\infty, 1)$-topos. But they are not preserved by an equivalence of $(\infty, 1)$-toposes, so they don’t have a presentation-independent meaning. They seem to be merely a tool of convenience.

4.3 Computational type theories

If we limit ourselves to one constructive model, then there is a principled way of making sense of new constructions. This is via the paradigm of computational type
theories in the Nuprl tradition [21]. Here we consider a particular model to give meaning explanations for the judgments of type theory. For a certain notion of cubical sets, this has been done by Harper’s group [6, 5]. The benefit of the approach is that it guarantees that all constructions are computationally meaningful and make sense in the model. The downside is that it is tied to a particular model (though by being judicious with which primitives are added to the computational language this downside can be minimized) and that it also leads to a type theory without decidable type checking, so a separate proof theory or language of evidence is needed.

4.4 Presentation axioms

It may have perhaps occurred to some readers that the problems discussed in Sect. 3 should be solved in the same way that they are solved in homotopical mathematics based in set-theory, namely by working with set-based presentations.

We already mentioned geometric realization, an operation that produces the underlying homotopy type of a given simplicial set or topological space. We could consider adding $|·| : \mathbf{sSet} \to \mathbf{Type}$ as a basic construction and the axiom stating that $|·|$ is surjective, meaning that every type is merely equivalent to the geometric realization of some simplicial set. And perhaps we should add a further axiom stating that every function $A \to B$ between types arises (merely) as the geometric realization of a function between representing simplicial sets.

Something like this may indeed be appropriate at the level of HoTT$_{UF}$ if it could be given a computational meaning. But certainly not at the level of HoTT$_{el^{++}}$, because the axioms would severely restrict the range of models (they are constructively-homotopical taboos).

Even the much weaker axiom sets cover (SC), stating that every type admits a surjection from a set admits a simple counter-model $(\infty, 1)$-topos [47].

On the other hand, SC (or something like it) is necessary in order to describe the semantics of HoTT (with universes) in presheaf toposes. Indeed, the universe in a presheaf topos is built from certain sets covering the 1-types of presheaves of small sets.

5 Conclusion

Higher structures are at once the raison d’être and so far, the Achilles’ heel, of HoTT/UF from a foundational perspective. HoTT can handle with ease many important higher structures, such as the 1-type of sets and the 2-type of categories, that can only imperfectly be represented in other foundational systems. But so far it cannot define the (untruncated) type of $(\infty, 1)$-categories, and this is a major impediment to the foundational aspirations of HoTT/UF. Of course, HoTT can be (and has
been) used successfully to reason about (structured) homotopy types. In this way, the various homotopy type theories function as domain specific languages (DSLs).

To be foundational, however, we need to find a compelling construction of, and theory of, $(\infty, 1)$-categories, and of the semantics of HoTT-DSLs, inside homotopy type theory itself. It appears that new methods of construction are needed, but it is at this time not clear what they should be.

A dramatic possibility, not mentioned in Sect. 4, is that we should take $(\infty, 1)$-categories to be fundamental after all, and build a synthetic type theory where the types are $(\infty, 1)$-categories rather than $\infty$-groupoids. This would be a directed type theory. Such a thing would undoubtedly be quite complicated due to the need to keep track of variances (see [34, 39] for some preliminary attempts), and it would represent a return to the old ways of thinking about categorical foundations, albeit updated to account for a homotopical perspective. We would carve out the $\infty$-groupoids as those $(\infty, 1)$-categories all of whose morphisms are invertible rather than trying to build $(\infty, 1)$-categories out of $\infty$-groupoids. In any case, directed type theories should be useful also as DSLs for reasoning about $(\infty, 2)$-toposes.

Personally, I think we’ll find some solution that allows us to stay at the level of $\infty$-groupoids for the foundational theory. Perhaps there is a kind of two-level type theory that allows us to capture the strict nature of the $(\infty, 1)$-category of types without postulating a bunch of meaningless pretypes.

An analogy can perhaps be made with the foundations of stable homotopy theory. The $(\infty, 1)$-category of spectra is a symmetric monoidal stable $(\infty, 1)$-category, and from a foundational point of view, this is the correct viewpoint, since spectra should be identified when they are weakly equivalent. However, it was discovered that this $(\infty, 1)$-category can be presented by symmetric monoidal Quillen model categories (i.e., very strict structures), and this has been very important in facilitating computations in stable homotopy theory [25, 36]. (I should mention that there is work-in-progress by Finster-Licata-Morehouse-Riley on developing a HoTT-DSL for stable homotopy theory targeting the cohesive $(\infty, 1)$-topos of parametrized spectra: this captures the strict monoidal structure of spectra in a type theory.)

And so it may be, that in order to realize the foundational potential of HoTT/UF, we shall need to capture the strict structure of type theory itself, perhaps by reflecting more of judgmental structure at the level of types.

I’m confident that a good solution will be eventually found. The field is still young, and it will be exciting to see what the future brings.

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