ON A SPECTRUM-LEVEL SPLITTING OF THE $BP(2)$-COOPERATIONS ALGEBRA

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Abstract. In the 1980s, Mahowald and Kane used Brown-Gitler spectra to construct splittings of $bo$ and $BP(1)$. These splittings helped make it feasible to do computations using the $bo$- and $BP(1)$-based Adams spectral sequences. In this paper, we construct an analogous splitting for $BP(2)$.

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1. Introduction

The main result of this paper is a splitting of $BP(2)$ in terms of Brown-Gitler spectra (Theorem 6.10). We will start by giving some context for this problem in Section 1.1. Then in Section 1.2, we give a summary of the rest of the paper, and outline our construction of this splitting.

1.1. Motivation.

1.1.1. The $n$-truncated Brown-Peterson spectra. Fix a prime $p$. The Brown-Peterson spectrum, denoted $BP$, is the complex-oriented cohomology theory associated to the universal $p$-typical formal group law. For each chromatic height $n$, there exists a closely related spectrum called the $n$-truncated Brown-Peterson spectrum, denoted $BP(n)$. The spectrum $BP(n)$ carries information about the formal group laws of height at most $n$.

At low heights, these spectra have some other familiar names. The height zero spectrum $BP(0)$ is the Eilenberg-MacLane spectrum for the integers localized at $p$, $\mathbb{Z}_{(p)}$. The height one spectrum $BP(1)$ is the connective $K$-theory spectrum $bu(2)$ when $p = 2$, and the Adams $l$-summand when $p$ is odd. The height two spectrum $BP(2)$ carries information about topological modular forms. Specifically, $tmf_1(3)$ is an $E_\infty$-form of $BP(2)$ at the prime 2 [LN12]. At the prime 3, a spectrum of topological automorphic forms with a certain level structure is an $E_\infty$-form of $BP(2)$ [HL10]. Although it is not known whether there are any $E_\infty$ structures for $BP(2)$ at primes $p \geq 5$, the spectrum $BP(2)$ is a summand of $tmf$, the connective spectrum for topological modular forms.
One of the tools that can be used to extract information from $BP(n)$ and related spectra about the homotopy groups of the sphere is called the $BP(n)$-based Adams spectral sequence. The $BP(n)$-based Adams spectral sequence is a member of a family of spectral sequences called the $R$-based Adams spectral sequences.

1.1.2. The $R$-based Adams spectral sequence. The Adams spectral sequence is a tool that can be used to approximate the homotopy groups of a connective spectrum $X$, such as the sphere. If $R$ is a ring spectrum satisfying certain conditions, then there is an $R$-based Adams spectral sequence

$$E_1^{*,*}(X, R) = \pi_*(R \wedge R^* \wedge X) \Rightarrow \pi_* X_R,$$

where $X_R$ denotes the $R$-completion of $X$. If $R$ is a flat ring spectrum (e.g., if $R_* R$ splits as a wedge sum of suspensions of $\pi_* R$ itself), then the $E_2$-page has a nice algebraic description, specifically $E_2^{*,*}(X, R) = E(R, R)(R_*, R, X)$.

Both the classical Adams spectral sequence and the Adams-Novikov Spectral Sequence are examples of such spectral sequences, for $R = H\mathbb{F}_p$ and $R = MU$, respectively. Although their $E_2$-pages are algebraic objects, they are still very complex and do not have a closed-form description. Furthermore, it becomes very difficult to compute differentials when $X = S^n$. Despite the difficulty of computing those differentials, these two spectral sequences are essential tools for studying the homotopy groups of the spheres. (A list of references and computations can be found in [Rav03]). More recently, Isaksen, Wang, and Xu have extended those computations through the 90-stem at the prime 2.

1.1.3. Background on $v_2$-periodicity. Another approach is to study these homotopy groups one height at a time. The $K(n)$-local homotopy groups of the sphere give an approximation of $\pi_* S$ containing information about $v_\alpha$-periodicity. At height 2, this approach has been well-developed. In [SW02], Shimomura and Wang computed the homotopy groups of the $K(2)$-local sphere at the prime $p = 3$. In [GHMR07], Goerss, Henn, Mahowald, and Rezk improved the conceptual framework and organization of these computations by constructing a resolution of the $K(2)$-local sphere at the prime 3, and explicitly describing the homotopy groups of its fibers. In [SY95], Shimomura and Yabe computed the homotopy groups of the $E(2)$-local sphere for primes $p > 3$. In [Beh12], Behrens gave a more conceptual description of these groups, and used this description to compute the $K(2)$-local homotopy groups of the sphere at these primes. These $K(2)$-local homotopy groups provide a partial understanding of $v_2$-periodicity, but more is left to understand about the $v_2$-periodic structure of $\pi_* S$, and its interactions with the $v_1$ and $v_0$-periodicity. The $BP(2)$-based Adams spectral sequence is one tool that could potentially be used to improve this understanding.

1.1.4. The bo-based and $BP(n)$-based Adams spectral sequences. Consider the $BP(n)$-based Adams spectral sequence

$$E_r(X, BP(n)) \Rightarrow \pi_* \widehat{X}_{BP(n)}.$$

Note that $\widehat{X}_{BP(2)}$ is actually equivalent to the $p$-localization $\widehat{X}_{(p)}$ [Rav03 Thm. 2.2.13]. The $BP(1)$-based Adams spectral sequence highlights the $v_1$-periodicity and makes it easier to extract information about the $v_1$-periodic structure. It is expected that the $BP(2)$-based Adams spectral sequence will display $v_2$-periodicity in a similar way. Throughout the rest of this paper, we will assume that everything is $p$-completed.

Initially, using a $BP(n)$-based Adams spectral sequence rather than the classical or Adams-Novikov spectral sequence might not seem like such a good idea; the spectra $BP(n)$ are not flat, and computing the $E_2$-page is very difficult. However, Mahowald’s work on the bo-based Adams
spectral sequence [Mah81] at the prime \( p = 2 \) indicated that such spectral sequences could be very useful, despite the difficulty in finding their \( E_2 \)-pages. The spectrum \( bo \) is also not flat, and describing the \( E_2 \)-page of the \( bo \)-based Adams spectral sequence is very difficult. However, Mahowald showed that, in a certain range, the \( bo \)-based Adams spectral sequence collapses at the \( E_2 \)-page. These computations gave a great deal of insight into the \( v_1 \)-periodic structure of the homotopy groups of the sphere, and were sufficient to prove the Telescope Conjecture at height one for the prime 2. Gonzalez showed in [Gon00] that the \( BP(1) \)-based Adams spectral sequence behaves in a similar way at odd primes, making it a useful tool for describing \( v_1 \)-periodicity at odd primes.

In further work, Lellman and Mahowald computed the \( v_1 \)-periodic portion of the \( E_2 \)-page of the \( bo \)-based Adams spectral sequence [LMS77]. In [Dav87], Davis computed the \( v_1 \)-torsion piece of the \( bo \)-based Adams spectral sequence through the 20-stem. Combined with Lellman and Mahowald’s computations, this gave a complete description of the \( bo \)-based Adams spectral sequence through the 20-stem. Recent work by Beaudry-Behrens-Bhattacharya-Culver-Xu [BBB+20] has extended these computations through the 40-stem.

In order to use a \( BP(n) \)-based Adams spectral sequence, we need a good understanding of \( BP(n) \wedge BP(n) \). One way to do this is to split \( BP(n) \wedge BP(n) \) into more manageable pieces, specifically as a sum of finitely generated \( BP(n) \)-modules. These splittings were essential for the computations described above. This is where Brown-Gitler spectra come into the picture.

1.1.5. Brown-Gitler spectra. In the 1970’s, Brown and Gitler constructed a family of spectra realizing certain sub-comodules of the dual Steenrod algebra at the prime \( p = 2 \) [BJG73]. Cohen then constructed the corresponding family for odd primes. Subsequently, Cohen [Coh81], Goerss-Jones-Mahowald [GJM86], Klippenstein [Kli88], and Shimamoto [Shi84] constructed finite spectra realizing the analogous family of sub-comodules of \( H_*BP(n) \), for \( 0 \leq n \leq 2 \). These spectra are collectively known as generalized Brown-Gitler spectra. The original Brown-Gitler spectra were constructed for studying immersions of manifolds, but these spectra and their generalizations have been used for many other interesting applications in the years since. One application is to produce decompositions of \( BP(n) \wedge BP(n) \), as well as smash products of other related spectra, into simpler pieces.

We use a filtration on the dual Steenrod algebra \( A_* \), known as the weight filtration and denoted \( wt \), to define the Brown-Gitler sub-comodules of the dual Steenrod algebra. At the prime 2, the Steenrod algebra has the form

\[
A_* \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots].
\]

At odd primes, the dual Steenrod algebra has the form

\[
A_* \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots).
\]

At any prime, we can define the weight filtration, by setting \( wt(\xi_i) = wt(\tau_i) = p^i \) for all \( i \), and \( wt(xy) = wt(x)wt(y) \). The comultiplication \( \Delta : A_* \to A_* \otimes A_* \) does not increase the weight.

The Brown-Gitler comodule \( B_0(k) \) is defined to be the sub-comodule of \( A_* \) generated by monomials of weight at most \( k \). Since \( H_*BP(n) \) is a comodule subalgebra of \( A_* \), the weight filtration extends to \( H_*BP(n) \) for each \( n \). Similarly, there is a family of height \( n \) Brown-Gitler comodules \( \{ B_n(k) \mid k \in \mathbb{N} \} \), where \( B_n(k) \) is the sub-comodule of \( H_*BP(n) \) generated by monomials of weight at most \( k \). (Note that in this choice of notation, the Brown-Gitler comodules \( B_n(pk), \ldots, B_n(pk+p−1) \) are the same for a fixed \( n \geq 1 \).)
The original Brown-Gitler spectra, which we will denote $H_k$, are finite spectra such that $H_k = B_{-1}(k)$. These spectra were constructed by Brown-Gitler at the prime $p = 2$ [BJG73] and by Cohen at odd primes [Coh81]. In [Shi84], Shimamoto constructed the integral Brown-Gitler spectra $\{HZ_k\}$, finite spectra such that $H_k HZ_k \cong B_0(k)$. Goerss-Jones-Mahowald constructed the height one Brown-Gitler spectra in [GJM86]. The height one Brown-Gitler spectra include the family of finite spectra $\{l_k\}$ such that $H_0 l_k \cong B_1(k)$. At the prime 2, there is another family of Brown-Gitler spectra at height one, the $bo$-Brown-Gitler spectra. The homology $H_0 bo$ is also a subalgebra of the dual Steenrod algebra, and so the weight filtration can be applied to $H_0 bo$. The finite spectrum $bo_k$ realizes the weight $\leq k$ component of $H_0 bo$. In [Kli88], Klippenstein constructed a family of finite spectra $\{BP_0(2)\}$, realizing the comodules $\{B_2(k)\}$. It is not known whether spectra realizing $B_n(k)$ exist for higher heights $n \geq 3$.

1.1.6. Splittings. At each prime $p$ and each height $n \geq 0$, there exists an isomorphism of $A_*$-comodules

\[ H_0 BP(n) \otimes H_0 BP(n) \cong \bigoplus_{k=0}^{\infty} H_0 BP(n) \otimes \Sigma^q B_{n-1}(k). \]

Analogous isomorphisms exist for $bo$ and $tmf$ at the prime 2, specifically

\[ H_0 bo \otimes H_0 bo \cong \bigoplus_{k=0}^{\infty} H_0 bo \otimes \Sigma^q B_0(k) \]

\[ H_0 tmf \otimes H_0 tmf \cong \bigoplus_{k=0}^{\infty} H_0 tmf \otimes \Sigma^q B_1(k). \]

At low heights, the Brown-Gitler spectra have been used to realize these splittings. It is well-known that there is a splitting

\[ BP(0) \wedge BP(0) \cong \bigvee_{k=0}^{\infty} BP(0) \wedge \Sigma^q H_k, \]

where $q = 2(p - 1)$. In the 1980’s, Mahowald constructed the following splitting of $bo \wedge bo$ in terms of integral Brown-Gitler spectra [Mah81, Thm 2.4]:

\[ bo \wedge bo \cong \bigvee_{k=0}^{\infty} bo \wedge \Sigma^q HZ_k. \]

Subsequently, Kane used Mahowald’s methods to construct an analogous splitting for $BP(1) \wedge BP(1)$ at odd primes [Kan81, Thm 11.1]:

\[ BP(1) \wedge BP(1) \cong \bigvee_{k=0}^{\infty} BP(1) \wedge \Sigma^q HZ_k. \]

This might lead one to guess that all of these splittings are realizable when the relevant Brown-Gitler spectra exist. However, things get more interesting at height 2: Davis and Mahowald demonstrated that the isomorphism for $tmf$ is not realizable, by showing that the homotopy groups...
of \( tmf \wedge tmf \) and \( \bigvee_{k=0}^{\infty} \Sigma^k tmf \wedge bo_k \) are not isomorphic \cite{DM10}. In spite of this, it was still conjectured that at the height \( n = 2 \), the splitting \cite{1} could be realized for all primes.

In particular, Culver provided strong evidence towards the 2-primary version of this conjecture in \cite{Cul19}, as well as the odd-primary version in \cite{Cul20}. He showed that \( \text{BP}(2) \wedge \text{BP}(2) \) splits as

\[
\text{BP}(2) \wedge \text{BP}(2) \simeq C \vee V,
\]

where \( V \) is a wedge sum of Eilenberg-Maclane spectra realizing \( HF_p \), and \( \pi_* C \) is \( v_2 \)-torsion free and concentrated in even degrees \cite{Cul19,Cul20}. Furthermore, he used this splitting to show the following isomorphism of homotopy groups:

\[
\pi_* \left( \text{BP}(2) \wedge \text{BP}(2) \right) \cong \pi_* \left( \bigvee_{k=0}^{\infty} \Sigma^k \text{BP}(2) \wedge l_k \right).
\]

The main result of this paper is that we can indeed realize the splitting \cite{7}.

**Theorem 6.10.** At all primes \( p \), there exists a splitting (up to \( p \)-completion)

\[
\text{BP}(2) \wedge \text{BP}(2) \simeq \bigvee_{k=0}^{\infty} \Sigma^k \text{BP}(2) \wedge l_k.
\]

1.2. **Summary of the paper.** Now we will outline our approach to the construction of the splitting of Theorem 6.10. A very brief summary is as follows. In Section 2, we start by recalling some helpful algebraic results. In Section 3, we recall Baker-Lazarev’s relative Adams spectral sequence in the category of \( \text{BP}(2) \)-modules, and then we reduce the problem of constructing the splitting of Theorem 6.10 to showing that a certain family of classes survive this spectral sequence. In Section 4, we discuss a universal coefficient spectral sequence and a hypercohomology spectral sequence. In Section 5, we construct a square of spectral sequences relating these auxiliary spectral sequences to the relative Adams spectral sequence. We then use this square in Section 6 to show that the necessary classes do indeed survive the relative Adams spectral sequence, proving Theorem 6.10. The remainder of this section gives a more detailed outline.

We start by recalling some well-known algebra results in Section 2.1 that will be useful throughout this paper. In Section 2.2, we recall the homology of \( \text{BP}(n) \). In Section 2.3, we recall the following splitting of \( H_* \text{BP}(n) \) in terms of Brown-Gitler comodules.

**Theorem 2.14.** \cite{Cul20} Lemma 4.8, \cite{Cul19} Prop 3.3 Let \( n \geq 0 \). Then there exists a family of maps

\[
\{ \theta_k : \Sigma^k B_{n-1}(k) \to H_* \text{BP}(n) | k \in \mathbb{N} \}
\]

such that their sum

\[
\bigoplus_{k=0}^{\infty} \theta_k : \bigoplus_{k=0}^{\infty} \Sigma^k B_{n-1}(k) \to H_* \text{BP}(n)
\]

is an isomorphism of \( E(n)_* \)-comodules.

Note that this is only an isomorphism of \( E(n)_* \)-comodules (not an \( A_* \)-comodule isomorphism). So we cannot realize this exact decomposition on the level of spectra. However, at low heights, we can use Baker-Lazarev’s relative Adams spectral sequence to leverage this decomposition to the spectrum-level splitting of Theorem 6.10 at the height \( n = 2 \).
Before discussing the relative Adams spectral sequence, we begin Section 3.1 by recalling the category of $R$-module spectra as discussed in \[EKM97, \text{Section III.3}\]. Let $R$ be an $S$-algebra ($A_\infty$ ring spectrum). Let $M$ be a right $R$-module, and let $N$ be a left $R$-module. We let $M \wedge_R N$ denote the coequalizer \[EKM97, \text{Def III.3.1}\]

$$M \wedge R \wedge N \longrightarrow M \wedge N \longrightarrow M \wedge_R N.$$

Let $[M,N]^R$ denote the group of homotopy classes of $R$-module maps from $M$ to $N$, and let $H^R_*M = \pi_*(H \wedge_R M)$. In \[BL01\], Baker-Lazarev introduce the following Adams spectral sequence in the category of $R$-modules:

**Proposition 3.6.** \[BL01\] Prop 2.1| Let $L,M$ be $R$-modules, and let $E$ be a commutative ring spectrum with $E^0_E$ flat as a left or right $E_\ast$-module. If $E^R_0L$ is projective as an $E_\ast$-module, then there is an Adams spectral sequence with

$$E_2^{s,t} = \text{Ext}_E^{s,t}(E^R_0L, E^R_0M).$$

In the case $R = BP(2)$, $E = H$, $M = BP(2) \wedge l_k$, and $N = BP(2) \wedge BP(2)$, we can use the other results of Section 3.1 to show that this spectral sequence is strongly convergent, and can be written in the following form.

**Proposition 3.11** For all $k \in \mathbb{N}$, there exists a strongly convergent relative Adams spectral sequence

$$\text{ASS} E_2^{s,t} = \text{Ext}_{E(2)}(H_* \Sigma^k l_k, H_* BP(2)) \Longrightarrow [BP(2) \wedge \Sigma^k l_k, BP(2) \wedge BP(2)]^{BP(2)}.$$

In Section 3.2, we explain how we will use this Adams spectral sequence to lift the $E(2)$-module splitting of Theorem 2.14 to the spectrum-level splitting of Theorem 6.10. Note that we can think of the map

$$\theta_k : H_* l_k \longrightarrow H_* BP(2) \wedge BP(2)$$

as a class in $\text{ASS} E_2^{0,0}$. To lift the map, we need to show that $\theta_k$ survives the spectral sequence. For dimension reasons, it is impossible for any differential to hit $\theta_k$. So if we can show that the differential leaving $\theta_k$ is always zero, then we will be able to conclude that $\theta_k$ survives the spectral sequence. Note that the Adams differential $d_2$ has degree $(r,r-1)$. We call classes in degree $E_2^{r,r-1}$ the potential obstructions to lifting $\theta_k : H_* B_{n}(k) \rightarrow H_* BP(\langle n \rangle)$.

Suppose that for each $k$, the map $\theta_k$ does indeed survive the relative Adams spectral sequence. Then we will have constructed a family of $BP(2)$-module maps

$$\tilde{\theta}_k : BP(2) \wedge l_k \rightarrow BP(2) \wedge BP(2)$$

whose sum induces the isomorphism $\bigoplus_{k=0}^\infty \theta_k$ on $H_*^{BP(2)}$.homology. Then it will follow immediately from Proposition 3.2 that

$$\sqrt{\tilde{\varphi}_k} : \bigvee_{k=0}^\infty BP(2) \wedge l_k \rightarrow BP(2) \wedge BP(2)$$

is a homotopy equivalence up to $p$-completion.

It is interesting to note that at height $n = 1$, there are no potential obstructions to lifting the analogous maps $\bar{\theta}_k : H_* B_0(k) \rightarrow H_* BP(1)$ to maps $\bar{\theta}_k : BP(1) \wedge B_0(k) \rightarrow BP(1) \wedge BP(1)$. This recovers Kane’s splitting \[7\]. However, at height 2, there are many potential obstructions, and we
will spend the remainder of the paper demonstrating that they are not actual obstructions. (There is an analogous spectral sequence to Equation \((9)\) at height \(n = 3\). However, we do not yet know whether the potential obstructions at that height survive the spectral sequence. At height \(n > 3\), it is not yet known whether the necessary Brown-Gitler spectra exist. 

In Section 4.1 we will introduce Robinson’s universal coefficient spectral sequence. We will use this auxiliary spectral sequence to show that the potential obstructions survive the Adams spectral sequence. This is an adaptation of the technique used by Klippenstein to construct a splitting of \(bu \wedge BP(n)\) in terms of finitely generated \(bu\)-modules \((3.12)\). 

Specifically, we will compare the universal coefficient spectral sequence

\[(8) \quad E_2^{u,v} = \text{Ext}_B P(2), (BP(2), l_k, BP(2)) \Rightarrow [l_k, BP(2) \wedge BP(2)] \]

to the relative Adams spectral sequence

\[(9) \quad E_2^{s,t} = \text{Ext}_{E(2), k} (H_* l_k, H_* BP(2)) \Rightarrow [BP(2) \wedge l_k, BP(2) \wedge BP(2)]^{BP(2)} \]

to show that the potential obstructions survive the relative Adams spectral sequence. Klippenstein explicitly computed the \(E_2\)-pages of the universal coefficient spectral sequence and the Adams spectral sequences needed to construct the splitting of Theorem 3.12. However, both the \(E_2\)-pages of the Adams spectral sequence \((9)\) and the universal coefficient spectral sequence \((8)\) are more complicated to compute than their height one analogues. So instead of completely computing these \(\text{Ext}\) terms and directly comparing them, we will use another spectral sequence called the hypercohomology spectral sequence as a bridge between them.

In Section 4.2, we will review Cartan-Eilenberg’s hypercohomology spectral sequence. Let \(G(-)\) denote the right-derived functor \(G(-) = \mathbb{R} \text{Hom}_{E(2)} (\mathbb{F}_p, -)\), and let \(P(2) = G(\mathbb{F}_p)\). We will use the equivalence of categories of Corollary 2.2 to present the following special case of the universal coefficient spectral sequence

\[(10) \quad \bigoplus_{r=t_2-t_1 \atop t=t_2-t_1} \text{Ext}^{r, r}_{p,2}((\text{Ext}^{t_1, t_1}_{E(2), l_k}(\mathbb{F}_p, H_* l_k), \text{Ext}^{t_2, t_2}_{E(2), l_k}(\mathbb{F}_p, H_* BP(2))) \rightarrow H^{r+q, t} \mathbb{R} \text{Hom}_P (G(H_* l_k), G(H_* BP(2))). \]

Then we will use Koszul duality to show the following corollary.

**Corollary 4.13.** There exists an isomorphism

\[H^{u, v} \mathbb{R} \text{Hom}_P (G(H_* l_k), G(H_* BP(2))) \cong \text{Ext}^{u, v}_{E(2), l_k}(H_* l_k, H_* BP(2)). \]

So we can think of this particular hypercohomology spectral sequence as one that converges to the \(E_2\)-page of the Adams spectral sequence.

We begin Section 5.1 by recalling Culver’s splitting of \(BP(2) \wedge BP(2)\) \((4)\). It follows immediately from his analysis that \(BP(2) \wedge l_k\) splits in an analogous way, so we can state the following theorem.

**Theorem 5.3** If \(X = BP(2)\) or \(X = l_k\), then there exists a splitting

\[BP(2) \wedge X \simeq C_X \vee V_X, \]

such that \(V_X\) is a sum of suspensions of \(\mathbb{F}_p\)-Eilenberg-Maclane spectra, and \(\text{Ext}_{\mathbb{A}_s}(\mathbb{F}_p, H_* C_X)\) is \(v_2\)-torsion free and concentrated in even \((t-s)\)-degrees. This splitting is unique up to equivalence.
We will use a relative Adams spectral sequence to upgrade this to a $BP(2)$-module splitting.

**Proposition 5.5.** The splitting of Theorem 5.3 is a $BP(2)$-module splitting.

In Section 5.2 we will use this splitting to extend Culver’s analysis [Cul20] of the Adams spectral sequence converging to $BP(2), BP(2)$. In Section 5.3 we will use this analysis to show that the $E_2$-page of the hypercohomology spectral sequence (9) and the $E_2$-page of the universal coefficient spectral sequence (8) are isomorphic, as stated in the following lemma.

**Lemma 5.47.** There exists an isomorphism

$$
\bigoplus_{s=r_2-r_1 \atop t=t_2-t_1} \Ext^{u,t}_{E_2} \left( \Ext^{r_1,t_1}_{E_2}(\mathbb{F}_p, H_*l_k), \Ext^{r_2,t_2}_{E_2}(\mathbb{F}_p, H_*BP(2)) \right) \xrightarrow{\HSS} \Ext^{u,t}_{E_2}(H_*l_k, H_*BP(2)) \xrightarrow{\ASS} [l, BP(2) \wedge BP(2)]_{t-r-u}
$$

We can express these two results as the following square relating the Adams spectral sequence and the universal coefficient spectral sequence. (This square is not necessarily commutative, but that is not an issue— it will still provide sufficient information about the relationship between the Adams spectral sequence (9) and the universal coefficient spectral sequence (8) for our purposes.)

In Section 6 we will use the square to analyze the potential obstructions. We will start by using Culver’s splitting from Section 5.1 to show that we can actually look at a smaller square: Let $C, V$ denote the summands $C_{BP(2)}, V_{BP(2)}$ and let $C_k, V_k$ denote the summands $C_l, V_l$ in the splitting of Theorem 5.3. We can use these $BP(2)$-module splittings to decompose the $E_2$-page of the Adams spectral sequence (9) as as a sum of four components. For degree reasons, any potential obstructions must be contained in the summand $\Ext^{u,t}_{E_2}(H_*^{BP(2)}C, H_*^{BP(2)}C)$. So we will restrict our attention to the following square.

$$
\bigoplus_{s=r_2-r_1 \atop t=t_2-t_1} \Ext^{u,t}_{BP(2)} \left( \Ext^{r_1,t_1}_{E_2}(\mathbb{F}_p, H_*l_k), \Ext^{r_2,t_2}_{E_2}(\mathbb{F}_p, H_*BP(2)) \right) \xrightarrow{\HSS} \Ext^{u,t}_{E_2}(H_*l_k, H_*BP(2)) \xrightarrow{\ASS} [l, BP(2) \wedge BP(2)]_{t-r-u}
$$

We start to analyze this square by identifying a vanishing line in the hypercohomology spectral sequence.

**Proposition 6.5.** For all $u > 2$,

$$
\Ext^{u,*}_{BP(2)} \left( \Ext^{u,*}_{E_2}(\mathbb{F}_p, H_*^{BP(2)}C), \Ext_{E_2}(\mathbb{F}_p, H_*^{BP(2)}C) \right) = 0.
$$
It follows that every class on the line $\text{Ext}^{u=1,*}_{BP(2)}$ survives the universal coefficient spectral sequence. We can combine this observation with the isomorphism between the $E_2$-pages of the hypercohomology spectral sequence and the universal coefficient spectral sequence (Lemma 5.47) to prove the following.

**Proposition 6.7.** The odd $(t-s)$-degree component of the $E_2$-page of the Adams spectral sequence is isomorphic to the $(u=1)$-line of the universal coefficient spectral sequence, that is,

$$\bigoplus_{t-s \text{ odd}} \text{Ext}^{*,t}_{E(2)}\left(H_*^{BP(2)}C_k, H_*^{BP(2)}C\right) \cong \text{Ext}^{u=1,*}_{BP(2)}\left(\pi_*C_k, \pi_*C\right).$$

Proposition 6.5 and the isomorphism of Lemma 5.47 imply that $\text{Ext}^{u=1,*}_{BP(2)}$ must survive the universal coefficient spectral sequence. Combined with Proposition 6.7, this implies the following theorem.

**Theorem 6.8.** Let $x \in \bigoplus_{t-s \text{ odd}} \text{Ext}^{*,t}_{E(2)}\left(H_*^{BP(2)}C_k, H_*^{BP(2)}C\right)$. Then $x$ survives the Adams spectral sequence

$$\text{Ext}^{*,*}_{E(2)}\left(H_*^{BP(2)}C_k, H_*^{BP(2)}C\right) \Longrightarrow [C_k, C]^{BP(2)}.$$  

Note that the potential obstructions to lifting $\theta_k$ are all contained in odd $(t-s)$-degree, so the following corollary is immediate.

**Corollary 6.9.** Let $x$ be a potential obstruction to lifting $\theta_k$ in the Adams spectral sequence

$$\text{ASS} \text{E}_2^{*,t} = \text{Ext}^{*,t}_{E(2)}\left(H_*l_k, H_*BP(2)\right) \Longrightarrow [BP(2) \land l_k, BP(2) \land BP(2)]^{BP(2)}.$$  

Then $x$ is not a boundary, that is, there is no $y \in \text{ASS} \text{E}_r$ such that $d_r(y) = x$.

So indeed the maps $\theta_k$ can be lifted, and we can construct our splitting.

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## 2. Background

In Section 2.1 we will start by reviewing some basic facts about comodules over finite graded coalgebras which will be used throughout this paper. In Section 2.2 we will recall the homology of $BP(n)$. In Section 2.3 we will discuss the well-known $E(n)_*$-comodule splitting

$$H_*BP(n) \cong \bigoplus_{k=0}^{\infty} \Sigma^k B_n(k).$$
2.1. **Equivalence of categories.** Let \( \Gamma \) be a graded coalgebra over a field \( k \), and let \( \Gamma^* \) denote its dual graded algebra. Let \( \text{Comod}_\Gamma \) denote the category of graded left \( \Gamma \)-comodules, and let \( \text{Mod}_{\Gamma^*} \) denote the category of graded left \( \Gamma^* \)-modules. We will use the following functor from \( \text{Comod}_\Gamma \) to \( \text{Mod}_{\Gamma^*} \).

**Lemma 2.1.** \cite{Pal01} Lemma 0.3.3] Let \( \Gamma \) be a graded coalgebra over a field \( k \), and let \( \Gamma^* \) be its dual graded algebra. Let \( \langle , \rangle : \Gamma^* \otimes \Gamma \to k \) be the evaluation map. That is, if \( \alpha \in \Gamma^* \) and \( x \in \Gamma \), then \( \langle \alpha, x \rangle = \alpha(x) \). Let \( (M, \psi : M \to \Gamma \otimes M) \) be a graded left \( \Gamma \)-comodule. Let \( \mu_M : \Gamma^* \otimes M \to M \) be the composition

\[
\mu_M : \Gamma^* \otimes M \xrightarrow{1 \otimes \psi} \Gamma^* \otimes \Gamma \otimes M \xrightarrow{\langle , \rangle \otimes 1} k \otimes M \cong M.
\]

Then \( (M, \mu_M) \) is a graded left \( \Gamma^* \)-module. Furthermore, the assignment \( (M, \psi_M) \mapsto (M, \mu_M) \) defines a functor \( \varphi : \text{Comod}_\Gamma \to \text{Mod}_{\Gamma^*} \).

When \( \Gamma \) is finite, the functor \( \varphi \) is an equivalence of categories.

**Corollary 2.2.** \cite{Pal01} Lemma 0.3.3] Suppose that \( \Gamma \) is a finite graded coalgebra over a field \( k \). Then \( \varphi : \text{Comod}_\Gamma \to \text{Mod}_{\Gamma^*} \) is an equivalence of categories.

We are particularly interested in the case where \( \Gamma = E(2)_* := E(\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2) \), and \( \Gamma^* \cong E(2) := E(Q_0, Q_1, Q_2) \). The equivalence of categories between \( E(2) \)-modules and \( E(2)_* \)-comodules will be used throughout the rest of the paper.

We will also use the following results relating projective, free, and injective modules and comodules. The first is immediate from the equivalence of categories.

**Lemma 2.3.** \cite{Doi81} Prop 4] A module \( M \) is projective (resp. injective) as an \( E(2)_* \)-comodule if and only if \( M \) is projective (resp. injective) as an \( E(2)_* \)-comodule.

**Lemma 2.4.** \cite{Mar83} Prop. 12.8] Let \( M \) be a module over \( E(2) \) which is finitely generated in each degree. Then the following are equivalent:

1. \( M \) is free;
2. \( M \) is projective;
3. \( M \) is injective.

The following corollary is immediate.

**Corollary 2.5.** Let \( M, N \) be \( E(2)_* \)-comodules. Then

\[
\text{Ext}_{E(2)}^{s,t}(M, N) \cong \text{Ext}_{E(2)}^{s,t}(M, N).
\]

**Proof.** Consider a projective resolution

\[
0 \leftarrow M \leftarrow P^0 \leftarrow P^1 \leftarrow \cdots
\]

of \( E(2)_* \)-comodules. By Lemma 2.3 each \( P^i \) is also a projective \( E(2) \)-module. By the equivalence of categories,

\[
\text{Hom}_{E(2)}(P^i, N) \cong \text{Hom}_{E(2)}(P^i, N).
\]

So indeed \( \text{Ext}_{E(2)}^{s,t}(M, N) \cong \text{Ext}_{E(2)}^{s,t}(M, N) \). \( \square \)
2.2. The action of $E(n)$ on $H_*BP(n)$. Let $A_*$ denote the dual Steenrod algebra, i.e., $A_* = H_*H$. Let $Q_i$ denote the $i^{th}$ Milnor primitive, and let $E(n)$ denote the subalgebra $E(Q_0, Q_1, \ldots, Q_n)$ of the Steenrod algebra.

We will start by reviewing the action of $E(n) = E(Q_0, Q_1, \ldots, Q_n)$ on $A_*/E(n)_*$. In Section 2.2.1, we explicitly describe the action at odd primes, and in Section 2.2.2, we explicitly describe the action at $p = 2$. Then we will recall the well-known fact that $H_*BP(n) \cong A_*/E(n)_*$ at all primes.

2.2.1. The odd-primary case. At odd primes, $A_* \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$, where $|\bar{\xi}_i| = 2p^i - 2$ and $|\bar{\tau}_i| = 2p^i - 1$.

The homology of $BP(n)$ is most easily described using the conjugates of these usual generators. Let $\bar{\xi}_k$ denote the conjugate of $\xi_k$, and let $\bar{\tau}_k$ denote the conjugate of $\tau_k$.

Let $\Delta : A_* \to A_* \otimes A_*$ be the comultiplication map. Then for all $k \geq 0$,

\begin{equation}
\Delta(\bar{\xi}_k) = 1 \otimes \bar{\xi}_k + \sum_{j=1}^{k-1} \bar{\xi}_j \otimes \bar{\xi}^{p^j}_{k-j} + \bar{\xi}_k \otimes 1 \tag{13}
\end{equation}

\begin{equation}
\Delta(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{j=1}^{k-1} \bar{\tau}_j \otimes \bar{\xi}^{p^j}_{k-j} + \bar{\tau}_k \otimes 1 \tag{14}
\end{equation}

Let $Q_i$ denote the $i^{th}$ Milnor primitive, and let $E(n)$ denote the subalgebra $E(Q_0, Q_1, \ldots, Q_n)$ of the Steenrod algebra.

At odd primes, $Q_i$ is the dual of $\bar{\tau}_i$.

For all $n \geq 0$, we can define an $E(n)_*$-coaction

$$\psi : A_* \to E(n)_* \otimes A_*$$

to be the composition

$$\psi : A_* \xrightarrow{\Delta} A_* \otimes A_* \longrightarrow E(n)_* \otimes A_*$$

where the second map is the natural projection.

Specifically,

\begin{equation}
\psi(\bar{\xi}_k) = 1 \otimes \bar{\xi}_k \tag{15}
\end{equation}

\begin{equation}
\psi(\bar{\tau}_k) = \sum_{i=0}^{\min(k,n)} \bar{\tau}_i \otimes \bar{\xi}^{p^i}_{k-i} + 1 \otimes \bar{\tau}_k \tag{16}
\end{equation}

Recall from Corollary 2.2 that there is an equivalence of categories $\varphi : Comod_{E(n)_*} \to Mod_{E(n)}$ for all $n \geq 0$. So we can instead consider the $E(n)$-module structures induced by $\psi$, which are sometimes easier to deal with.

**Proposition 2.6.** Let

$$\mu : E(n) \otimes A_* \to A_*$$

denote the $E(n)$ action induced by $\psi$ (Lemma 2.7).

At odd primes,

$$Q_j \bar{\tau}_k = \begin{cases} 
\bar{\xi}^{p^j}_{k-j} & \text{if } j \leq k \\
0 & \text{if } j > k
\end{cases}$$
\[ Q_j \xi_k = 0 \text{ for all } k \in \mathbb{N}. \]

**Proof.** Recall that for all \( j \), \( Q_j \) is the dual of \( \bar{\tau}_k \).

So

\[
Q_j(\xi_k) = \langle Q_j, 1 \rangle \otimes \xi_k = 0
\]

(17)

\[
Q_j(\bar{\tau}_k) = \sum_{i=0}^{\min(k,n)} \langle Q_j, \bar{\tau}_i \rangle \bar{\tau}_{k-i} + \langle Q_j, 1 \rangle \otimes \bar{\tau}_k = \begin{cases} 
\bar{\xi}^{p^i}_{k-j} & \text{if } j \leq k \\
0 & \text{if } j > k
\end{cases}
\]

(18)

\[ \square \]

Recall that \( A//E(i)_* \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes E(\bar{\tau}_{i+1}, \bar{\tau}_{i+2}, \ldots) \)

is a sub-comodule algebra of \( A_* \). So the action of \( E(n) \) on \( A//E(i)_* \) is also given by the formula above.

### 2.2.2. The 2-primary case.

At the prime \( p = 2 \),

\[ A_* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots], \]

where \( |\xi_i| = 2^{p^i} - 1 \). As in the odd primary case, we will work with the conjugates \( \bar{\xi}_i \) of the usual generators. The comultiplication map \( \Delta : A_* \to A_* \otimes A_* \) has the same formula as in the odd-primary case. That is,

\[
\Delta(\bar{\xi}_k) = 1 \otimes \bar{\xi}_k + \sum_{j=1}^{p^k-1} \bar{\xi}_j \otimes \bar{\xi}_{k-j} + \bar{\xi}_k \otimes 1
\]

At the prime \( p = 2 \), \( Q_i \) is the dual of \( \bar{\xi}_{i+1} \). Using the same technique as in the odd case, we arrive at the action of \( Q_i \) on \( A//E(n)_* \) at the prime \( p = 2 \).

At the prime 2,

\[ A//E(n)_* \cong \mathbb{F}_p[\xi_1^2, \ldots, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \ldots]. \]

**Proposition 2.7.** Let

\[ \mu : E(n) \otimes A_* \to A_* \]

denote the \( E(n) \) action induced by \( \psi \) (Lemma 2.7).

At the prime \( p = 2 \),

\[
Q_j \bar{\xi}_k = \begin{cases} 
\bar{\xi}^{p^j}_{k-j} & \text{if } j < k \\
0 & \text{if } j \geq k
\end{cases}
\]

2.2.3. The homology of \( BP(n) \). Next we will recall the homology of \( BP(n) \). We start with the following result of S. Wilson.

**Theorem 2.8.** [Wil75, Prop. 1.7] Let \( n \geq 0 \). Then

\[ H^*BP(n) \cong A//E(Q_0, Q_1, \ldots, Q_n). \]

**Lemma 2.9.** Let \( \Sigma \) be a Hopf algebra. Let \( \psi_N : N \to \Sigma \otimes N \) be a left \( \Sigma \)-comodule which is finitely generated in each degree, and let \( \psi_M : M \to M \otimes \Sigma \) be a right \( \Sigma \)-comodule which is also finitely generated in each degree. Then \( (M \square \Sigma N)^* \cong M^* \otimes N^* \).
Proof. First, note that $M \square_N N$ is defined to be the equalizer

$$M \square_N N \longrightarrow M \otimes N \xrightarrow{\psi_M \otimes 1} M \otimes \Sigma \otimes N.$$

Let $\mu_M : \Sigma \otimes M^* \rightarrow M^*$ and $\mu_\Gamma : \Gamma^* \otimes \Sigma^* \rightarrow \Sigma^*$ be the dual actions induced by $\psi_M$ and $\psi_\Gamma$. Taking the dual of the equalizer diagram above, we get a coequalizer

$$(M \square_N N)^* \longleftarrow M^* \otimes N^* \xleftarrow{\mu_M \otimes \mu_N} M^* \otimes \Sigma^* \otimes N^*.$$

This is exactly the coequalizer diagram that defines $M^* \otimes \Sigma^* N^*$. 

\[\square\]

Remark 2.10. Let $E(n)$ denote the $A$-subalgebra $E(Q_0, Q_1, \ldots, Q_n)$. Recall that $A/E(n)$ is defined to be $A \otimes_{E(n)} \mathbb{F}_p$. So Lemma 2.9 tells us that the dual of $A/E(n)$, denoted $A/E(n)_*$, is $A/E(n)_* \cong A \square_{E(n)} \mathbb{F}_p$.

**Corollary 2.11.** The homology of $BP\langle n \rangle$ is $H_*BP\langle n \rangle \cong A_*E(n)_*$.

2.3. The homology-level splitting. Fix a prime $p$, and let $q = 2(p - 1)$. Recall that for all $i \geq 0$, $A/E(i)_*$ is a sub-comodule algebra of $A_*$, We will let $A/E(-1)_*$ denote $A_*$. Define a weight function $wt$ on the monomials of $A_*$ by setting $wt(\xi_j) = wt(\tilde{\xi}_j) = p^j$, and let $wt(xy) = wt(x) + wt(y)$.

Let the $j^{th}$ Brown-Gitler comodule $B_i(j)$ be the subspace of $A/E(i)_*$ generated by monomials of weight at most $j$, and let $M_i(j)$ be the subspace of $A/E(i)_*$ generated by monomials of weight exactly $j$. These subspaces have the following submodule structures.

**Lemma 2.12.** [Cul20 Prop 4.6], [Cul19 Prop 3.3]

Let $i \geq -1$. Then for all $k \geq 0$, $M_i(k)$ is an $E(i)_*$-submodule of $A/E(i)_*$, and $B_i(k)$ is an $E(i + 1)_*$-subcomodule of $A/E(i)_*$.

**Lemma 2.13.** [Cul20 Lemma 4.8][Cul19 Prop 3.3] Let $i \geq -1$. Then for all $k \geq 0$, there exists an $E(i + 1)_*$-module isomorphism

$$\theta_k : \Sigma^k B_i(k) \rightarrow M_{i+1}(pk).$$

Note that $A/E(i)_* \cong \bigoplus_{k=0}^\infty M_i(pk)$, so by Lemma 2.13 we have the following isomorphism.

**Theorem 2.14.** [Cul20 Lemma 4.8][Cul19 Prop 3.3] Let $n \geq 0$. Then there exists a family of maps

$$\{\theta_k : \Sigma^k B_{n-1}(k) \rightarrow H_*BP\langle n \rangle \mid k \in \mathbb{N}\}$$

such that their sum

$$\bigoplus_{k=0}^\infty \theta_k : \bigoplus_{k=0}^\infty \Sigma^k B_{n-1}(k) \rightarrow H_*BP\langle n \rangle$$

is an isomorphism of $E(n)_*$-comodules.

In Section 3.1 we will discuss Baker-Lazarev’s relative Adams spectral sequence in the category of $BP\langle 2 \rangle$-modules (Proposition 3.10). Then in Section 3.2 we will explain how this spectral sequence can be used at height $n = 2$ to lift Theorem 2.14 to the splitting

$$BP\langle 2 \rangle \wedge BP\langle 2 \rangle \simeq \bigoplus_{k=0}^\infty BP\langle 2 \rangle \wedge \Sigma^k B_1(k).$$
3. The relative Adams spectral sequence

3.1. The category of $BP(2)$-modules.

3.1.1. Definitions and basic results. In this section we recall Baker-Lazarev’s results on the category of $R$-modules, and in particular their adaptation of the Adams spectral sequence to this setting. We also use Kato-Tilson’s construction of the relative K"unneth spectral sequence. Let $R$ be a homotopy commutative ring spectrum. We start by recalling the category of $R$-module spectra as discussed in EKMM. Let $R$ be an $S$-algebra $(A_\infty$ ring spectrum). Let $M$ be a right $R$-module, and let $N$ be a left $R$-module. We let $M \wedge_R N$ denote the coequalizer $[EKMM97, \text{Def III.3.1}]$

$$M \wedge_R N \longrightarrow M \otimes_R N \longrightarrow M \wedge_R N.$$

Note that by definition,

$$M \cong M \wedge_{BP(2)} BP(2)$$

$$N \cong BP(2) \wedge_{BP(2)} N.$$

This $R$-module smash product commutes with the ordinary smash product, so that if $M$ is a right $R$-module, $N$ is an $R$-bimodule and $L$ is a left $R$-module,

$$M \wedge_R N \wedge L \cong M \otimes_R N \wedge R L.$$

Let $E$ be an $R$-algebra, and let $M$ be an $R$-module spectrum. Then the $E$-homology in the category of $R$-modules is defined to be

$$E^*_R(M) = \pi_*(E \wedge_R M).$$

It will be helpful to note that

$$E^*_R(R \wedge M) \cong EM.$$

We will use the following special case of the K"unneth formula for $R$-modules.

**Proposition 3.1.** (K"unneth formula for $R$-modules) Let $Y$ be an $R$-module spectrum. Then

$$H_* R \otimes H_*^R Y \cong H_* Y.$$

**Proof.** Let $X, Y$ be any $R$-module spectra. By [EKMM97, Thm IV.4.1], there exists a K"unneth spectral sequence

$$E^{*,*}_2 \cong Tor^H_*(H_* X, H_*^R Y) \implies H_*(X \wedge_R Y).$$

Note that $H_* R$ is flat over $H_*$, so the spectral sequence collapses at the $E_2$-page, with $E^{*,*}_2 \cong H_* X \otimes H_*^R Y$. In the case $X = R$, this yields

$$H_* R \otimes H_*^R Y \cong H_* Y.$$

We will also need to use a version of the Whitehead theorem in the category of $R$-modules.

**Proposition 3.2.** (Whitehead Theorem for $R$-modules) Let $X, Y$ be $p$-complete spectra which are also $BP(2)$-modules. If $\varphi : X \rightarrow Y$ is a $BP(2)$-module map such that $H_*^{BP(2)} \varphi$ is an isomorphism, then $\varphi$ is a homotopy equivalence.
Proof Suppose that $X$ is a $p$-complete spectrum. Recall that $p$-complete spectra are $H$-local in the category of $S$-module spectra. We will show that $X$ is also $H$-local in the category of $BP(2)$-modules.

Suppose $A$ is $H$-acyclic in the category of $BP(2)$-modules, that is $H_*^{BP(2)} A = 0$. Recall from Proposition 3.1 that

$$H_* A \cong H_* BP(2) \otimes H_*^{BP(2)} A,$$

so $H_* A = 0$. By [Bou79] Lemma 1.5, $BP(2) \wedge A$ is also $H$-acyclic in the ordinary category of $S$-modules. So

$$[A, X]^{BP(2)} \simeq [BP(2) \wedge A, X] = 0.$$

So $X$ is indeed $H$-local in the category of $BP(2)$-modules.

So our map $\varphi : X \to Y$ of $p$-complete modules is in fact a map of $H$-local modules in the $BP(2)$-module category. Let $F$ denote the fiber of $\varphi$. We are assuming that $H_*^{BP(2)} \varphi$ is an isomorphism, so $F$ must be $H$-acyclic in the category of $BP(2)$-modules. By definition, $[F, X]^{BP(2)} = 0$, and so $\varphi : X \to Y$ is an equivalence in the category of $BP(2)$-modules. It follows that $\varphi$ is still an equivalence after forgetting the $BP(2)$-module structure.

\[ \square \]

3.1.2. The relative Adams spectral sequence. Let $X$ be an $E_3$ algebra, and let $A$ and $B$ be commutative $S$-algebras over $R$ which are cofibrant $R$-modules. Then the K"unneth spectral sequence

$$Tor_*^R (A_*, B_*) \implies \pi_*(A \wedge B)$$

is multiplicative.

Baker-Lazarev used the multiplicativity of the K"unneth spectral sequence to prove the following theorem. The commutativity of the K"unneth spectral sequence was originally stated as Theorem 2.12 of [BL01]. Although a flaw was found in the proof of [BL01] Thm 2.12, Tilson addressed this issue and gave an alternative proof for the case where $R$ is an $E_\infty$-algebra in [Til16]. Katth"an and Tilson then generalized this result to $E_3$-commutative algebras in [KT17] Thm 2.12.

Baker-Lazarev's statement, [BL01] Prop 1.2, is given only for $R$ an $E_\infty$-commutative algebra. However, their proof of [BL01] Prop 1.2 relies solely on the multiplicativity of the K"unneth spectral sequence, so [KT17] Thm 2.12 allows us to restate their theorem here for $R$ an $E_3$-commutative algebra.

Proposition 3.4. [BL01] Prop 1.2 Let $R$ be an $E_3$-commutative algebra. Let $E = R/I$ be a regular quotient such that $R/I$ is a commutative $S$-algebra, and $I_*$ is generated by the regular sequence $u_0, u_1, \ldots$. Then as an $E_*$-algebra, $E_*^R E = E(\alpha_i | i \geq 0)$, where $|\alpha_i| = |u_i| + 1$.

Recall that the mod-$p$ Eilenberg-Maclane spectrum $H$ can be described as a quotient of $BP(n)$ by the regular sequence $(p, v_1, \ldots, v_n)$. In [HW22], Hahn and Wilson showed that at each prime $p$ and for each height $n \in \mathbb{N}$, there exists an $E_3 BP$-algebra structure on $BP(n)$. So we can apply Proposition 3.4 to describe $H_*^{BP(2)} H$ as follows.

Proposition 3.5. As an $F_p$-algebra, $H_*^{BP(n)} H \cong E(n)_*$. 
Proof. By Proposition [3.3] \( H_*^{BP(n)} H \cong E(\beta_j) \ j > n \), where \( |\beta_j| = |v_j| + 1 \). Recall that \( |v_j| = 2p^j - 1 \).

At odd primes, \( E(n)_* = E(\bar{r}_0, \ldots, \bar{r}_n) \), with \( |\bar{r}_i| = 2p^j - 1 = |v_j| + 1 \). At the prime \( p = 2 \), \( E(n)_* = E(\xi_1, \ldots, \xi_{i+1}) \) with \( |\xi_j| = 2p^j - 1 = |v_j| + 1 \).

Note that each prime, the dimensions of the generators of \( E(n) \) and \( H_*^{BP(2^n)} H \) match. So indeed \( H_*^{BP(n)} H \cong E(n)_* \).

We will use Baker-Lazarev’s adaptation of the Adams spectral sequence, the relative Adams spectral sequence in the category of \( R \)-modules. (In their paper, Baker and Lazarev work in the category of commutative \( S \)-algebras. However, their construction of the relative Adams spectral sequence only uses homotopy commutativity).

**Proposition 3.6.** [BL01 Prop 2.1] Let \( L, M \) be \( R \)-modules, and let \( E \) be a homotopy commutative ring spectrum with \( E^* R \) flat as a left or right \( E_* \)-module. If \( E^* R L \) is projective as an \( E_* \)-module, then there is an Adams spectral sequence with

\[
E_2^{s,t} = Ext^{s,t}_{E_* R}(E_* R L, E_* R M).
\]

So for the case \( R = BP(2) \) and \( E = H \), we have the following Adams spectral sequence:

\[
E_2^{s,t} = Ext^{s,t}_{E(2)}(H_*^{BP(2)} L, H_*^{BP(2)} M)
\]

for any \( BP(2) \)-modules \( L \) and \( M \). This spectral sequence weakly converges to \( [L, \tilde{M}^R_E]^{BP(2)} \), where \( \tilde{M}^R_E \) denotes the \( E \)-nilpotent completion of \( M \) in the category of \( R \)-modules.

Let \( E^* \) be the cosimplicial spectrum with \( E^k = E^\wedge k \). The coface maps are induced by the unit maps \( S^0 \to E \), and the degeneracies by the multiplication \( E \wedge E \to E \). The ordinary \( E \)-nilpotent completion of \( M \) can be defined as the totalization of \( E^* \wedge M \). The \( E \)-nilpotent completion of \( M \) in the category of \( R \)-modules is defined analogously.

**Definition 3.7.** Let \( E^* R \) be the cosimplicial spectrum with \( E^k = E^\wedge k \). The coface maps are induced by the unit maps \( R \to E \), and the degeneracies by the multiplication \( E \wedge R E \to E \). The \( E \)-nilpotent completion of \( M \) in the category of \( R \)-modules is

\[
X_E^R = Tot(E^* R \wedge X).
\]

(If we take \( R \) to be the sphere spectrum \( S \), we recover the ordinary \( E \)-nilpotent completion of \( M \).)

Let \( \alpha : A \to B \to E \) be a diagram of \((−1)\)-connected homotopy commutative \( S \)-algebras, and let \( M \) be a \( B \)-module spectrum. Let \( \hat{M}^B_E \) denote the \( E \)-nilpotent completion of \( M \) in the category of \( B \)-modules. Note that \( \alpha : A \to B \) induces a natural \( A \)-module structure on \( M \), so we let \( \hat{M}^A_E \) denote the \( E \)-nilpotent completion of \( M \) in the category of \( A \)-modules. Note that for any pair of \( B \)-modules \( M, N \), there is a natural homomorphism \( M \wedge_A N \to M \wedge_B N \) induced by \( \alpha \). So we can consider the induced homomorphism \( \hat{M}^A_E \to \hat{M}^B_E \), and compare the two \( E \)-nilpotent completions.

**Remark 3.8.** In Carlsson’s paper [Car08], he works in the category of \( E_\infty \)-commutative \( S \)-algebras. However, his proof of [Car08 Thm 6.10] only relies on homotopy commutativity, so we restate the result here for homotopy commutative spectra.

**Theorem 3.9.** [Car08 Thm 6.10] Let \( \alpha : A \to B \to E \) be a diagram of \((−1)\)-connected homotopy commutative \( S \)-algebras, and let \( M \) be a \( B \)-module spectrum. Suppose that the homomorphism
\( \pi_0 \alpha : \pi_0 A \to \pi_0 B \) is an isomorphism. Then the natural homomorphism \( \hat{M}_E^A \to \hat{M}_E^B \) is a weak equivalence of spectra.

It follows that the \( H \)-nilpotent completion of a \( BP(2) \)-module \( M \) \( \hat{M}_H^{BP(2)} \) is weakly equivalent to the ordinary \( H \)-nilpotent completion \( \hat{M}_H^S \). Since we are working up to \( p \)-completion throughout this paper, we will simply say that the spectral sequence

\[
E_2^{s,t} = Ext_{E(2)}(H^s_{BP(2)}L, H^t_{BP(2)}M)
\]

weakly converges to \( BP(2) \)-module maps from \( L \) to \( M \), denoted \([L, M]^{BP(2)}\).

We wish to consider the case \( L = BP(2) \wedge l_k, M = BP(2) \wedge BP(2) \). Recall that for any \( BP(2) \)-module \( X \),

\[
H^*_{BP(2)}BP(2) \wedge X \cong H_*X,
\]

so we can write the \( E_2 \)-page of this spectral sequence as

\[
E_2^{s,t} \cong Ext_{E(2)}(H_*l_k, H_*BP(2)).
\]

Furthermore, note that \( H_*l_k \) is a finite module, while \( H_*BP(2) \) is connective finitely generated in each degree. So by Boardman’s convergence criterion [Boa99, Thm 7.1], we have the following strongly convergent spectral sequence.

**Proposition 3.10.** Let \( n \geq 0 \). The for all \( k \in \mathbb{N} \), there exists a strongly convergent Adams spectral sequence

\[
E_2^{s,t} = Ext_{E(2)}(H_*l_k, H_*BP(2)) \implies [BP(2) \wedge l_k, BP(2) \wedge BP(2)].
\]

**3.2. How we will use the relative Adams spectral sequence.** Recall from Theorem 2.14 that there exists a family of \( E(2)_* \)-comodule maps

\[
\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^k H_*l_k \to H_*BP(2)
\]

such that \( \bigoplus_{k=0}^{\infty} \theta_k \) is an isomorphism.

By Equation (23), we can think of this as a family of \( E(2)_* \)-comodule maps

\[
\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^k H_*^{BP(2)}(BP(2) \wedge l_k) \to H_*^{BP(2)}(BP(2) \wedge BP(2)).
\]

We will use the following relative Adams spectral sequence to lift these maps to the level of spectra.

**Proposition 3.11.** Let \( n \geq 0 \). Then for all \( k \in \mathbb{N} \), there exists a strongly convergent Adams spectral sequence

\[
E_2^{s,t} = Ext_{E(n)}(H_*B_n(k), H_*BP(n)) \implies [BP(n) \wedge B_{n-1}(k), BP(n) \wedge BP(n)]^{BP(n)}.
\]

**Proof.** By Proposition 3.6, there exists an Adams spectral sequence

\[
E_2^{s,t} = Ext_{E(2)}(\Sigma^k H_*l_k, H_*BP(2)) \implies [BP(2) \wedge l_k, BP(2) \wedge BP(2)]^{BP(2)}.
\]

Note that \( H_*l_k \) is finite, while \( H_*BP(2) \wedge BP(2) \) is connective and finitely generated in each degree. It follows that \( Ext_{E(2)}^{s,t}(H_*l_k, H_*BP(2)) \) is finite for each pair \((s, t)\). So by [Boa99, Thm 7.1], the spectral sequence strongly converges to \([BP(2) \wedge l_k, BP(2) \wedge BP(2)]^{BP(2)}\). \( \square \)
We can think of $\theta_k$ as a class in degree $E_2^{0,0}$. Suppose that we can show that each map $\theta_k$ survives its spectral sequence. Then we will have constructed a family of $BP(2)$-module maps

$$\tilde{\theta}_k : BP(2) \land l_k \to BP(2) \land BP(2)$$

whose sum induces the isomorphism $\bigoplus_{k=0}^{\infty} \theta_k$ on $H_\ast^{BP(2)}$-homology. Then it will follow immediately from Proposition 3.2 that

$$\bigvee_{k=0}^{\infty} \tilde{\varphi}_k : \bigvee_{k=0}^{\infty} BP(2) \land l_k \to BP(2) \land BP(2)$$

is a homotopy equivalence up to $p$-completion. So we will spend the rest of the paper demonstrating that $\theta_k$ survives the spectral sequence.

Note that $\theta_k$ is in degree $(s,t) = 0$, while the differential has degree $|d_r| = (r, r + 1)$. So the potential obstructions to lifting $\varphi_k$ are those classes found in $Ext_{E(2)}^{s,s+1}(\Sigma^q H_\ast l_k, H_\ast BP(2))$, where $s \geq 2$. We will show that all of these potential obstructions survive the spectral sequence.

It is interesting to note that the analogous situation for $BP(1)$, there are no potential obstructions to lifting these maps. So one can immediately recover Kane’s splitting of $BP(1) \land BP(1)$ (Equation (7)) using this method. However, there are many potential obstructions to lifting our family of maps

$$\{\theta_k : \Sigma^q H_\ast l_k \to H_\ast BP(2)\}.$$

To deal with these obstructions, we will adapt the method used by Klippenstein to construct a decomposition of $bu \land BP(n)$ in terms of $bu$-modules generated by integral Brown-Gitler spectra. Specifically, he constructed the following splitting.

**Theorem 3.12.** [Kli89 Thm 11] Let $n \geq 0$, and let $p$ be an odd prime. Then there exists a splitting

$$bu \land BP(n) \lor HV \cong \bigvee_{I \in \mathbb{N}^n} \Sigma^{d(I)} bu \land HZ_{[i_n]} \lor HV'$$

where $I = (i_1, \ldots, i_n)$, $d(I) = |\xi_1^2 \xi_2 \cdots \xi_n^2|$, and $HV, HV'$ are Eilenberg-Maclane spectra of $\mathbb{F}_p$-vector spaces.

3.2.1. Why we chose to use a relative Adams spectral sequence. To construct this splitting, Klippenstein used an Adams spectral sequence

(24) $E_2^{s,t} = Ext_{A_\ast}(\Sigma^{d(I)} H_\ast HZ_{[i_n]}, bu \land BP(n)) \Rightarrow [HZ_{[i_n]}, bu \land BP(n)]$

To construct maps

$$\varphi_I : HZ_k \to bu \land BP(n)$$

and then showed that the composite

$$bu \land BP(n) \lor HV \cong \bigvee_{I \in \mathbb{N}^n} \Sigma^{d(I)} bu \land HZ_{[i_n]} \lor HV'$$

induces an isomorphism in homology. Note that after a change-of-rings,

$$E_2^{s,t} = Ext_{A_\ast}(\Sigma^{d(I)} H_\ast HZ_{[i_n]}, bu \land BP(n)) \cong Ext_{E(1)_\ast}(\Sigma^{d(I)} H_\ast HZ_{[i_n]}, H_\ast BP(n)).$$

We could have taken an analogous approach for constructing our splitting. Specifically, the $E(2)_\ast$-comodule homomorphism $\theta_k : H_\ast \Sigma^q l_k \to H_\ast BP(2)$ extends to an $A_\ast$-comodule homomorphism

$$\overline{\theta}_k : H_\ast \Sigma^q l_k \to H_\ast BP(2) \otimes H_\ast BP(2).$$
It can then be shown algebraically that the composite

\[ \bigoplus_{k=0}^{\infty} H_* \langle \Sigma^k l_k \rangle \xrightarrow{1 \otimes \theta_k} H_* \langle \langle \Sigma^k l_k \rangle \rangle \rightarrow H_* \langle \langle \Sigma^k l_k \rangle \rangle \]

induces an isomorphism. We found it more efficient to use a relative Adams spectral sequence instead for several reasons. First, our approach makes it more straightforward to employ the \( BP(2) \)-module structure of \( BP(2) \wedge I_k \) when analyzing the potential obstructions. Furthermore, using a relative Adams spectral sequence saves us the trouble of confirming that the composite above really does induce an isomorphism. Additionally, we found it necessary to use other relative Adams spectral sequences to prove Proposition 5.5 at the primes 2 and 3, so we needed to include most of the discussion and background in Section 3.1 regardless of which approach we chose.

To construct the splitting of Theorem 4.1, Klippenstein used an Adams spectral sequence to lift the \( \langle \Sigma^d(I) \rangle \) to the level of spectra. Just as in our situation, there are many potential obstructions to lifting Theorem 4.1. The universal coefficient spectral sequence.

4. Auxiliary spectral sequences

4.1. The universal coefficient spectral sequence. We will start by recalling Robinson’s universal coefficient spectral sequence.

**Theorem 4.1.** [Rob87, p. 9] Let \( R \) be a ring spectrum, and let \( G \) be an \( R \)-module spectrum. Then for any connective ring spectrum \( X \), there exists a universal coefficient spectral sequence

\[ \text{Ext}^*_R(Z;X,G) \Rightarrow [X,G]. \]

Note that \( BU \wedge BP(n) \) is naturally a \( BU \)-module, so Klippenstein was able to use a universal coefficient spectral sequence of the following form:

\[ E_2^{u,v} = \text{Ext}^*_{BU}(\Sigma^d(I)BUH\Sigma I_n, BU BP(n)) \Rightarrow [BP(2) \wedge H\Sigma I_n, BU BP(n)]_{BP(2)}. \]

Recall that

\[ [H\Sigma I_n, BU BP(n)] \cong [BP(2) \wedge H\Sigma I_n, BU BP(n)]_{BP(2)}, \]

so the spectral sequence converges to the same target as \( \langle \Sigma^d(I) \rangle \). Although they converge to the same target, these two spectral sequences have very different filtrations. The \( E_2 \)-page of the Adams spectral sequence has nonzero classes in arbitrarily high filtration. However, the ring \( BU \) has global dimension 2, which tells us that \( Ext^*_{BU} = 0 \) for all \( u \geq 3 \). Note that the differential \( d^u_{CSS} \) increases the homological degree \( u \) by \( r \). This implies that every class on the line \( Ext^*_{BU} \) survives the spectral sequence.

In turn, this tells us that for each nonzero class on the \( (u = 1) \)-line of the universal coefficient spectral sequence, there must be a corresponding nonzero class in \( Ext^*_{BU}(\Sigma^d(I)H_* H\Sigma I_n, H_* BU \wedge HBP(n)) \) which survives the Adams spectral sequence. By directly computing both \( E_2 \)-pages, Klippenstein showed that the the the potential obstructions to lifting \( \varphi_l \) must be among the corresponding classes that survive the Adams spectral sequence.
We will adapt this technique, comparing the universal coefficient spectral sequence

\[ E_2^{u,v} = \text{Ext}_*^{B^*P} (B^*P, k, B^*P, k) \rightarrow \text{Ext}_*^{B^*P} (B^*P, k, B^*P, k) \]

to the Adams spectral sequence \([\mathbb{S}]\) to show that the potential obstructions in \([\mathbb{S}]\) survive the spectral sequence. However, our situation is immediately more complicated than the \([\mathbb{S}]\)-analogue for two reasons. First, \(B^*P\) has global dimension 3 rather than 2, leaving more room for non-zero differentials in the universal coefficient spectral sequence. Secondly, both \(\text{Ext}_*^{*,*} \) and \(\text{Ext}_*^{*,n} \) are far more complicated to compute than their height one analogues. So instead of computing these \(\text{Ext}\) terms and directly comparing them, we will use another spectral sequence called the *hypercohomology spectral sequence* as a bridge between them.

4.2. The hypercohomology spectral sequence. We will start by recalling the hypercohomology spectral sequence, a construction of Cartan-Eilenberg [Car99]. Let \(T\) be an additive functor such that \(\text{Ext}^{*,*}(\Lambda)\) is covariant, \(\text{Ext}^{*,1}(\Lambda)\) is contravariant, and \(\text{Ext}^{1,*}(\Lambda)\) is covariant. Then we can consider the double complex \(T^{*,*}(X,Y)\) where

\[ T^{p,q}(X,Y) = \bigoplus_{p,q \geq 0} T(X^{p,q}, Y^{p,q}). \]

The double complex has two filtrations: the first filtration

\[ F^p_I := \bigoplus_{q \geq 0} T^{q,p}(X,Y) \]

and the second filtration

\[ F^q_{II} := \bigoplus_{p \geq 0} T^{q,p}(X,Y). \]

In certain cases, spectral sequences can be associated to these filtrations.

**Definition 4.2.** [Car99] p.324 A filtration \(F\) of a complex \(D\) is said to be regular if there exists a function \(u(n)\) such that \(H^p F^n(D) = 0\) for all \(p > u(n)\).

By definition, \(X^{p,q}\) and \(Y_{p,q}\) are both zero whenever \(q < 0\). So \(H^{p+q}T^{p,*}(X,Y) = 0\) when \(p > n\). Therefore, the first filtration is regular with respect to the total cohomology of the complex [Car99] p.369].

If the second filtration is also regular, then each filtration induces an associated spectral sequence converging to \(H^*\mathbb{R}T(A,C)\). Specifically, these are the spectral sequences associated to the double complex \(T(X,Y)\). We will use the spectral sequence associated to the second filtration.

**Proposition 4.3.** [Car99] p.370

Let \(A, C\) be complexes as described above. If the second filtration \(F^q_{II}\) is regular, then the following spectral sequence is convergent:

\[ \bigoplus_{p_1 + p_2 = p} H^q \mathbb{R}T(H^{p_1}(A), H^{p_2}(C)) \Rightarrow H^{p+q} \mathbb{R}T(A,C). \]
Consider the functor $G(-) = \mathbb{R}Hom_{E(m)}(\mathbb{F}_p, -)$. Let $P(m)$ denote the derived functor $P(m) = \mathbb{F}_p[v_0, v_1, \ldots, v_m]$. Then the right derived functor $\mathbb{R}Hom_{E(m)}(\mathbb{F}_p, -)$ can be represented by the chain complex $Hom_{E(m)}(Q^*, -)$, where $Q^*$ is a projective resolution of $\mathbb{F}_p$ over $E(m)$.

This chain complex is a module over the ring $\mathbb{R}Hom_{E(m)}(\mathbb{F}_p, \mathbb{F}_p)$. Note that $P(m) \cong \mathbb{R}Hom_{E(m)}(\mathbb{F}_p, \mathbb{F}_p)$.

Proof. By the chain complex homomorphism pictured below.

Let $G : \mathcal{D}(E(m)) \to \mathcal{D}(P(m))$ denote the derived functor $G(-) = \mathbb{R}Hom_{E(m)}(\mathbb{F}_p, -)$. We will need to use the following property of $P(m)$.

**Proposition 4.4.** [Eis13 Cor 19.7] Let $L$ be an ordinary module over $P(m)$. Then the global dimension of $L$ is at most $m$.

We also need to note that the functor $G : \mathcal{D}(E(m)) \to \mathcal{D}(P(m))$ preserves the internal grading $t$.

**Proposition 4.5.** The functor $G : \mathcal{D}(E(m)) \to \mathcal{D}(P(m))$ preserves the internal degree $t$.

**Proof.** Consider the functor $G : \mathcal{D}(E(m)) \to \mathcal{D}(P(m))$. Let $f : M \to N$ be an $E(m)$-module map. Let $Q^*$ be a projective resolution of $\mathbb{F}_p$ over $E(m)$. Then $G(f) : G(M) \to G(N)$ can be represented by the chain complex homomorphism pictured below.

\[
\begin{array}{cccccccc}
\cdots & \to & Hom_{E(m)}(Q^r, M) & \to & Hom_{E(m)}(Q^{r+1}, M) & \to & \cdots \\
& & \downarrow f_* & & \downarrow f_* & & \\
\cdots & \to & Hom_{E(m)}(Q^r, N) & \to & Hom_{E(m)}(Q^{r+1}, N) & \to & \cdots \\
\end{array}
\]

Each $f_* : Hom_{E(m)}(P^r, M) \to Hom_{E(m)}(P^{r+1}, M)$ has degree $|f|$, so $G(f)$ has degree $|f|$.

Now we can state a special case of Proposition 4.3.

**Lemma 4.6.** Let $M, N$ be $E(2)_*$-comodules. Let $G(-)$ denote the right derived functor $\mathbb{R}Hom_{E(2)}(-, \mathbb{F}_p)$, and let $P(2)$ denote $G(\mathbb{F}_p)$. Then there exists a spectral sequence

\[
\bigoplus_{r \geq r_1, t \geq t_2 - t_1} Ext^u_{P(2)}(Ext^{r_1, t_1}_{E(2)}(\mathbb{F}_p, M), Ext^{r_2, t_2}_{E(2)}(\mathbb{F}_p, N)) \Rightarrow H^{r+u} \mathbb{R}Hom_{P(2)}(G(M), G(N)).
\]

**Proof.** Recall that $M$ and $N$ have an induced $E(2)$-module structure as described in Corollary 2.2. Let $X$ be a projective resolution of the complex $G(M)$, and let $Y$ be an injective resolution of $G(N)$. By Proposition 4.4, the resolutions $X$ and $Y$ can be chosen so that $X^{q, q}$ and $Y^{q, q}$ are both zero for all $q > m$. So $H^n F_1^{q, q} = 0$ for all $q > m$, and the filtration is regular. So we have the following spectral sequence described in Proposition 4.3.

\[
\bigoplus_{p_1 - p_2 = p} H^q \mathbb{R}Hom_{P(m)}(H^{p_1} G(M), H^{p_2} G(N)) \Rightarrow H^{p+q} \mathbb{R}Hom_{P(m)}(G(M), G(N)).
\]
Finally, we can apply Corollary 2.2 to exchange $\text{Ext}_{E(m)}$ for $\text{Ext}_{E(m)}$. So we can express the hypercohomology spectral sequence in the form in which we will use it:

$$
\bigoplus_{r=r_2-r_1 \atop t=t_2-t_1} \text{Ext}^q_{P(2)} \left( \left( \text{Ext}^{t_1}_{E(2)}, \mathbb{F}_p, H, l_k \right), \text{Ext}^{t_2}_{E(2)}, \mathbb{F}_p, H, BP(2) \right)
\Rightarrow H^{r+q, t} R \text{Hom}_{P(2)} \left( \left( \mathbb{G}(H, l_k), \mathbb{G}(H, BP(2)) \right) \right).
$$

(28)

□

Now we will use Koszul duality to show that we can rewrite $H^{s,t} R \text{Hom}_{P(m)}(\mathbb{G}(M), \mathbb{G}(N))$ as $\text{Ext}^{s,t}_{E(m)}(\mathbb{G}(M), \mathbb{G}(N))$ under certain conditions. First we will recall some definitions.

**Definition 4.7.** Let $R$ be a differential graded algebra, and let $D(R)$ denote the derived category of differential graded modules over $R$. A derived module $C$ over $R$ is said to be coherent if $H_* C$ is finite. Let $D(R)^f$ denote the full subcategory of $D(R)$ generated by coherent derived $E$-modules.

**Definition 4.8.** [ABIM10, p.9] A full subcategory $\mathcal{C}$ of $D(R)$ is said to be thick if it satisfies the following properties.

1. $\mathcal{C}$ is additive.
2. $\mathcal{C}$ is closed under direct summands.
3. If $C' \to C \to C''$ is an exact triangle in $D(R)$ such that two of the three modules are contained in $\mathcal{C}$, then so is the third.

**Definition 4.9.** [ABIM10 2.1.4] If $C$ is a derived module over $D(R)$, then $\text{Thick}_R(C)$ denotes the intersection of all thick subcategories containing the derived module $C$.

Koszul duality can be used to compare modules over $E(m)$ to modules over $P(m)$. Specifically, we will use the following two results.

**Lemma 4.10.** [ABIM10] Remark 7.5 Let $R$ be a derived algebra over a field $k$. Then the subcategories $\text{Thick}_R(k)$ and $D(R)^f$ are equivalent.

**Theorem 4.11.** [ABIM10] Remark 7.4 [CI15 Thm 4.1] The assignment

$$G : D(E(m)) \to D(P(m))$$

$$C \mapsto \mathbb{R} \text{Hom}_{E(m)}(\mathbb{F}_p, C)$$

is an exact functor. Furthermore, $G$ restricts to an equivalence

$$G : \text{Thick}_{E(m)}(k) \to \text{Thick}_{P(m)}(P(m)).$$

We can use Lemma 4.10 to restate Theorem 4.11 as follows.

**Theorem 4.12.** The assignment

$$G : D(E(m)) \to D(P(m))$$

$$C \mapsto \mathbb{R} \text{Hom}_{E(m)}(\mathbb{F}_p, C)$$

is an exact functor. Furthermore, $G$ restricts to an equivalence.
Corollary 4.13. There exists an isomorphism
\[ H^{u,t} \mathbb{R}\text{Hom}_{P(2)}(G(H_* l_k), G(H_* BP(2))) \cong \text{Ext}^{u,t}_{E(2)}(H_* l_k, H_* BP(2)). \]

Proof. Recall from Theorem 2.14 that \( H_* BP(2) \) splits as a sum of finite \( E(2) \)-modules, that is,
\[ H_* BP(2) \cong \bigoplus_{m=0}^{\infty} \Sigma^q H_* l_m. \]

Note that \( H_* BP(2) \) is finite in each degree \( t \), so \( G(H_* BP(2)) \cong G\left( \bigoplus_{k=0}^{\infty} H_* l_k \right) \cong \bigoplus_{k=0}^{\infty} G(H_* \Sigma^q l_k). \) It follows that
\[ \mathbb{R}\text{Hom}^{u,t}(G(H_* l_k), G(H_* BP(2))) \cong \mathbb{R}\text{Hom}^{u,t}(G(H_* l_k), \bigoplus_{n=0}^{\infty} \Sigma^n G(H_* l_n)). \]

Likewise, note that \( G^s(H_* l_k) \) is connective and finite for each degree \( s \). It follows that
\[ \mathbb{R}\text{Hom}^{u,t}(G(H_* l_k), \bigoplus_{n=0}^{\infty} \Sigma^n G(H_* l_n)) \cong \bigoplus_{n=0}^{\infty} \Sigma^n \mathbb{R}\text{Hom}^{u,t}(G(H_* l_k), G(H_* l_n)). \]

By Theorem 4.12
\[ \mathbb{R}\text{Hom}^{s,t}(G(H_* l_k), G(H_* l_n)) \cong \text{Ext}^{s,t}_{E(2)}(H_* l_k, H_* l_n). \]

So by Theorem 2.14
\[ \mathbb{R}\text{Hom}^{s,t}(G(H_* l_k), G(H_* BP(2))) \cong \text{Ext}^{s,t}_{E(2)}(H_* l_k, H_* BP(2)). \]

\qed

So indeed the hypercohomology spectral sequence (28) converges to the \( E_2 \)-page of the Adams spectral sequence [10].

5. Constructing a square of spectral sequences

5.1. Culver’s splitting. In this section, we start by recalling Culver’s splitting of \( BP \wedge BP(2) \), as well as the analogous splittings of \( BP(2) \wedge l_k \). Then we will use Baker-Lazarev’s relative Adams spectral sequence in the category of \( BP(2) \)-modules to show that these are in fact \( BP(2) \)-module splittings.

We start by recalling the following theorem of Margolis.

Theorem 5.1. [Mar74, Thm 2] Let \( Y \) be a bounded below, locally finite spectrum. Then there exists a pair of spectra unique up to equivalence \( C, V \) such that \( V \) is the Eilenberg-Maclane spectrum of an \( \mathbb{F}_p \)-vector space, \( H^C \) contains no free \( A \)-summands, and

\[ Y \simeq C \vee V. \]
Culver combined Theorem 5.1 above with an analysis of the Adams spectral sequence
\[ \text{Ext}_{A_*}(F_p, H_*(BP(2) \wedge BP(2))) \longrightarrow \pi_*BP(2) \wedge BP(2) \to \] 
to show the following result.

**Theorem 5.2.** [Cul20 Cor 3.22] At each prime \( p \), there exists a splitting unique up to equivalence
\[ BP(2) \wedge BP(2) \simeq C \vee V \]
where
\[
\begin{align*}
(1) & \quad V \text{ is the Eilenberg-Maclane spectrum of a mod-} p \text{ vector space,} \\
(2) & \quad \text{and } \text{Ext}_{A_*}(F_p, H_*C) \text{ is } v_2\text{-torsion free.}
\end{align*}
\]
Substituting \( BP(2) \wedge l_k \) for \( BP(2) \wedge BP(2) \) in the proof of Corollary 3.22 of [Cul20] yields an analogous splitting for \( BP(2) \wedge l_k \), as stated in the following theorem.

**Theorem 5.3.** If \( X = BP(2) \) or \( X = l_k \), then there exists a splitting unique up to equivalence
\[ BP(2) \wedge X \simeq C(X) \vee V_X \]
such that \( V_X \) is a sum (of finite type) of suspensions of \( F_p \)-Eilenberg-Maclane spectra, and \( \text{Ext}_{A_*}(F_p, H_*C_X) \) is \( v_2\)-torsion free and concentrated in even \((t-s)\)-degrees.

Now we will use the relative Adams spectral sequence of Proposition 3.6 to show that these are in fact \( BP(2) \)-module splittings.

**Proposition 5.4.** The \( E(2)_* \)-comodule \( H_*^{BP(2)}V_X \) is free and of finite type.

**Proof.** The summand \( V_X \) is a sum of mod-\( p \) Eilenberg-MacClane spectra (of finite type), so \( H_*V_X \) is a free \( A_* \)-comodule of finite type. By Proposition 3.3,
\[ A/E(2)_* \otimes H_*^{BP(2)}V_X \cong H_*V_X. \]
It follows that \( H_*^{BP(2)}V_X \) is a free \( E(2)_* \)-comodule of finite type. \( \Box \)

**Proposition 5.5.** The splitting
\[ BP(2) \wedge X \simeq C(X) \vee V_X \]
is \( BP(2) \)-module splitting. We will denote the summands \( C_{BP(2)} \) (resp. \( V_{BP(2)} \)) by \( C \) (resp. \( V \)), and we will denote \( C_{l_k} \) (resp. \( V_{l_k} \)) by \( C_k \) (resp. \( V_k \)).

**Proof.** By Proposition 5.4, \( H_*^{BP(2)}V_X \) is a free summand of \( H_*V_X \), with natural injection and projection maps
\[ i : H_*^{BP(2)}V_X \rightarrow H_*X \]
and
\[ j : H_*X \rightarrow H_*^{BP(2)}V_X. \]

We can use Baker-Lazarev’s adaptation of the Adams spectral sequence (Proposition 3.6) to lift these maps to the level of spectra. By Proposition 3.6, there exist conditionally convergent spectral sequences
\[ \text{Ext}_{E(2)_*}^{s,t}(H_*^{BP(2)}V_X, H_*X) \Rightarrow [V_X, BP(2) \wedge X]^{BP(2)} \]
\[ \text{Ext}_{E(2)_*}^{s,t}(H_*X, H_*^{BP(2)}V_X) \Rightarrow [BP(2) \wedge X, V_X]^{BP(2)}. \]
Note that $H_{BP}^{2} V_X$ is a free $E(2)_*$-comodule, so by Lemma 2.4, $H_{BP}^{2} V_X$ is also injective. So both spectral sequences are concentrated on the $s = 0$ line. By Boa99 Thm 7.1, this implies that the spectral sequences are strongly convergent. So the maps $i$ and $j$ indeed lift to $BP(2)$-module maps

$$ V_X \rightarrow BP(2) \wedge X \rightarrow V_X. $$

Note that $\overline{j} \circ \overline{i}$ induce an isomorphism on $H_*^{BP(2)}$-homology. By Proposition 3.2, this implies that $j \circ i$ is an equivalence up to $p$-completion. So indeed

$$ BP(2) \wedge X \simeq C_X \vee V_X $$

is a $BP(2)$-module splitting.

\[ \square \]

5.2. Further analysis of the relative Adams spectral sequence for $BP(2)_*X$. The goal of this section is to show that there are no hidden extensions in the Adams spectral sequences converging to $BP(2)_*BP(2)$ and $BP(2)_*l_k$.

Consider $BP(2) \wedge X$, where $X$ is $BP(2)$ or $l_k$. Recall from Theorem 5.3 that $BP(2) \wedge X$ splits as

$$ BP(2) \wedge X \simeq C_X \vee V_X, $$

where $V_X$ is a sum of suspensions mod-$p$ Eilenberg-MacLane spectra, and $\pi_* C_X$ is $v_2$-torsion free. We will denote the summands $C_{BP(2)}$ (resp. $V_{BP(2)}$) by $C$ (resp. $V$), and we will denote $C_k$ (resp. $V_k$) by $C$ (resp. $V$).

The goal is to show that there exists an Adams spectral sequence

$$ Ext^t_{H_*^{BP(2)} H_*^{BP(2)} BP(2)_* BP(2) \wedge X} \Rightarrow BP(2)_* X. $$

Using Proposition 3.1, we can rewrite this as

$$ Ext^t_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} X) \Rightarrow BP(2)_* X. $$

(Note that this is just the $E_2$-page of the ordinary Adams spectral sequence for $BP(2)_* X$, after the classical change-of-rings).

The splitting above implies that the Adams spectral sequence converging to $BP(2)_*BP(2)$ splits as the following two separate spectral sequences:

(30) $$ E_2^{s,t} = Ext^{s,t}_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} C) \Rightarrow \pi_{t-s} C $$

and

(31) $$ E_2^{s,t} = Ext^{s,t}_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} V) \Rightarrow \pi_{t-s} V. $$

We already know that $\pi_* V \cong \oplus \Sigma^r H \mathbb{F}_p$, so no hidden extensions are possible in $\pi_*$. However, showing that there are no hidden extensions in the spectral sequence converging to $\pi_* C$ is a lot more work. In Cul19 Thm 2.1 and Cul20 Thm 3.1, Culver showed that $Ext^t_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} C)$ is $v_2$-torsion free, that is, $v_2^r x \neq 0$ for all $x \neq 0 \in Ext^t_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} C)$ and $r \in \mathbb{N}$. While there is certainly torsion in the form of equations like $v_0 x + v_1 y + v_2 z = 0$, we can adapt his proof to show that $Ext^t_{E(2)_*} (\mathbb{F}_p, H_*^{BP(2)} C)$ is $v_0$ and $v_1$-torsion free in the same manner (that is, $v_0^r x \neq 0$
and \( v_1^t x \neq 0 \) for all \( x \neq 0 \in \text{Ext}_{E(2)}(\mathbb{F}_p, H_*^{BP(2)} C) \) and \( r \in \mathbb{N} \). This will imply that there is no room for hidden extensions in \( [30] \).

Recall that by the equivalence of categories of Corollary 2.2

\[
\text{Ext}_{E(2)}(\mathbb{F}_p, H_*^{BP(2)} C) \cong \text{Ext}_{E(2)}(\mathbb{F}_p, H_*^{BP(2)} C).
\]

It will be easier to look at things from the \( E(2) \)-module perspective in this section, since we will use Margolis homology extensively later on. First we will discuss the \( v_1 \)-Bockstein spectral sequences.

**Proposition 5.6.** Let \( M \) be an \( E(2) \)-module. Let \((i,j,h)\) be any permutation of \((0,1,2)\). There exists a \( v_1 \)-Bockstein spectral sequence

\[
v_{i,BSS} E_{1}^{s,t,r} = \text{Ext}_{E(Q_j,Q_h)}(\mathbb{F}_p, M)[v_1] \Rightarrow \text{Ext}_{E(2)}(\mathbb{F}_p, M)
\]

where

\[
v_{i,BSS} E_{1}^{s,t,r} = \text{Ext}_{E(Q_j,Q_h)}(\mathbb{F}_p, M)\{v_1^t\}
\]

and the differential \( d_m \) has the form

\[
d_m : E_{m}^{s,t,r} \to E_{m}^{s-k+1,t-|Q_i|k,r+k}.
\]

**Proof.** This can be proved in exactly the same way as Culver’s [Cul20 Corollary 3.11], substituting \( v_1 \) for \( v_2 \) and \( E(Q_j,Q_h) \) for \( E(Q_0,Q_1) \) in the appropriate places.

We will use the \( v_1 \)-Bockstein spectral sequence to show that there is no \( v_1 \)-torsion in \( \text{Ext}_{E(2)}^{s,t}(\mathbb{F}_p, H_*^{BP(2)} C) \) if \( \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, H_*^{BP(2)} C) \) is concentrated in even \( t-s \) degrees.

**Proposition 5.7.** Let \( M \) be an \( E(2) \)-module. Suppose that \( \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, M) \) is concentrated in even \( t-s \) degrees. Then there is no \( v_1 \)-torsion in \( \text{Ext}_{E(2)}^{s,t}(\mathbb{F}_p, M) \).

**Proof.** Observe that the differential \( d_m \) has \( t-s \) degree \((1-|Q_i|)k-1\), which is always odd. So if \( \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, M) \) is concentrated in even \((t-s)\)-degrees, then there are no nonzero differentials in the \( v_1 \)-BSS, and so the spectral sequence collapses at the \( E_1 \)-page. Thus \( v_1^n x \neq 0 \) for all \( n \in \mathbb{N} \) and \( x \neq 0 \in \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, M) \).

So if we can show that \( \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, H_*^{BP(2)} C) \) is concentrated in even \((t-s)\)-degrees, then we will have proven that there is no \( v_1 \)-torsion in \( \text{Ext}_{E(2)}^{s,t}(\mathbb{F}_p, H_*^{BP(2)} C) \). We will spend most of the remainder of this section proving that \( \text{Ext}_{E(Q_j,Q_h)}^{s,t}(\mathbb{F}_p, H_*^{BP(2)} C) \) is indeed concentrated in even \((t-s)\)-degrees.

In Section 5.2.1 we will produce a splitting

\[
H_*^{BP(2)} C \cong E(Q_j,Q_h) S_i \oplus R_i
\]

such that \( S_i \) is a free \( E(Q_j,Q_h) \)-module and and \( R_i \) has no free summands, for each permutation \((i,j,h)\) of \( \{0,1,2\} \) (Corollary 5.12). It follows quickly that \( \text{Ext}_{E(Q_j,Q_h)}(\mathbb{F}_p, S_i) \) is concentrated in even \( t-s \) degree. This reduces the problem to showing that \( \text{Ext}_{E(Q_j,Q_h)}(\mathbb{F}_p, R_i) \) is also concentrated in even degrees. In Section 5.2.2 we will recall Adams-Priddy’s usage of Margolis homology to classify invertible modules over an exterior algebra on two generators. In Section 5.2.3 we will recall Adams-Priddy’s computation of the Ext groups of these invertible modules. Next we will analyze the Margolis homology of particular submodules of \( H_* BP(2) \); in Section 5.2.4 we will
handle the odd-primary case, and in Section 5.2.6, we will present the analogous results at the prime 2. These computations will allow us to decompose $H_\ast BP(2)$ in terms of invertible modules. In Section 5.2.6, we will combine the results of Section 5.2.2 and Section 5.2.3 with our computations from Section 5.2.4 and Section 5.2.5 to show that $Ext_{E(Q_0, Q_0)}(F_p, R_i)$ is indeed concentrated in even $t - s$ degrees. In Section 5.2.7, we will combine these computations with Proposition 5.7 to conclude that there are no hidden extensions in the Adams spectral sequence converging to $BP(2)_\ast BP(2)$, as well as the one converging to $BP(2)_\ast l_k$ for all $k \in \mathbb{N}$.

5.2.1. Decomposing $H_\ast^{BP(2)}C$. We start by recalling the notion of length.

**Definition 5.8.** [Cul20, Definition 3.1] Let $x$ be a monomial in $A_\ast$. If $p$ is odd, then we can write $x$ in the form

$$x = \bar{\xi}_1^1 \bar{\xi}_2^2 \cdots \bar{\xi}_r^r \bar{x}_0^0 \bar{x}_1^1 \cdots \bar{x}_s^s \in A_\ast.$$ 

Then the length of the monomial, denoted $\ell(x)$, is defined to be

$$\ell(x) = \epsilon_0 + \epsilon_1 + \cdots + \epsilon_s.$$ 

If $p = 2$, then we can write $x$ in the form

$$x = \xi_1^{2j_1 + \epsilon_1} \xi_2^{2j_2 + \epsilon_2} \cdots \xi_r^{2j_r + \epsilon_r},$$ 

where $0 \leq \epsilon_i \leq 1$. Then the length of the monomial, denoted $\ell(x)$, is defined to be

$$\ell(x) = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_r.$$ 

**Remark 5.9.** Consider a monomial $x$. Note that the length of every nonzero summand of $Q_j x$ is exactly $\ell(x) - 1$.

**Proposition 5.10.** [Cul20, Proposition 3.7, Corollary 3.22] [Cul19, Proposition 2.22, Section 2.6] Let $S$ be the $E(2)$-submodule of $H_\ast BP(2)$ generated by monomials of length at least 3, ie

$$S := E(2)\{x \in H_\ast BP(2) | \ell(x) \geq 3\}.$$ 

Then

$$H_\ast^{BP(2)} V \cong S.$$ 

So if $x$ is a monomial in $H_\ast^{BP(2)} C$, then $\ell(x) \leq 2$.

**Proposition 5.11.** Let $(i, j, h)$ be a permutation of $\{0, 1, 2\}$. Let $S_i$ be the $E(Q_j, Q_h)$-submodule of $H_\ast^{BP(2)} C$ generated by monomials of length exactly 2, that is,

$$S_i := E(Q_j, Q_h)\{x \in H_\ast BP(2) | \ell(x) = 2\}.$$ 

Then $S_i$ is a free $E(Q_j, Q_h)$ module.

**Proof.** This is completely analogous to the proof of Proposition 3.12 of [Cul20] (for odd primes) and Corollary 2.24 of [Cul19] (for $p = 2$). To prove the $S_i$ case, substitute $E(Q_j, Q_h)$ for $E(Q_0, Q_1)$. \(\square\)

**Corollary 5.12.** Let $R_i$ be the quotient of $H_\ast C$ by $S_i$, that is $R_i$ is an $E(Q_j, Q_h)$ module such that the following sequence is exact:

$$0 \to S_i \to H_\ast C \to R_i \to 0.$$ 

Then there exists a splitting of $E(Q_j, Q_h)$-modules $H_\ast C \cong_{E(Q_j, Q_h)} S_i \oplus R_i$. 
Furthermore, the summand \( R_i \) contains no free summands.

Proof. The module \( S_i \) is free, so Lemma 2.4 tells us that \( S_i \) is also injective. So the exact sequence splits. Consider a monomial \( x \in R_i \). The length of \( x \) is at most one, so Remark 5.9 tells us that \( Q_j Q_h x = 0 \). So there are no monomials that could generate a free summand in \( R_i \). \( \Box \)

Now we will analyze \( \text{Ext}^{s,t}_{E(Q_j,Q_h)}(\mathbb{F}_p, S_i) \).

**Proposition 5.13.** The term \( \text{Ext}^{s,t}_{E(Q_j,Q_h)}(\mathbb{F}_p, S_i) \) is concentrated in degrees \((s,t)\) such that \( s = 0 \) and \( t \) is even.

Proof. Recall that \( S_i \cong \bigoplus_{x \in J} E(Q_j,Q_h)\{x\} \), where \( J \) is the set of monomials in \( H_*^{BP(2)} C \) having length exactly 2. It follows that

\[
\text{Ext}^{s,t}_{E(Q_j,Q_h)}(\mathbb{F}_p, S_i) \cong \bigoplus_{x \in J} \Sigma^{|Q_0 Q_1 x|} \mathbb{F}_p.
\]

Monomials of even length have even dimension, so \( |Q_0 Q_1 x| \) is also even. So indeed \( \text{Ext}^{s,t}_{E(Q_j,Q_h)}(\mathbb{F}_p, S_i) \) is concentrated in degrees \((s,t)\) such that \( s = 0 \) and \( t \) is even. \( \Box \)

In order to show that \( \text{Ext}^{s,t}_{E(Q_j,Q_h)}(\mathbb{F}_p, R_i) \) is also concentrated in even degrees, we will first need to recall some results of Adams and Priddy on the classification of invertible modules over an exterior algebra on two generators.

**5.2.2. The classification of invertible modules.** We will start by recalling Margolis’s stable category of modules over an exterior algebra [Mar83, p.205]. Let \( M, N \) be graded modules over an exterior algebra \( E \). A homomorphism \( f : M \rightarrow N \) is said to be stably trivial if there exists a projective \( E \)-module \( P \) such that \( f \) factors through \( P \). A pair of homomorphisms \( g, h : M \rightarrow N \) is said to be stably equivalent (denoted \( g \sim h \)) if \( g - h \) is stably trivial.

**Definition 5.14.** [Mar83, p.205] Let \( \text{Stab}(E) \) denote the stable category of \( E \)-modules. That is, \( \text{Stab}(E) \) is the category whose objects are the \( E \)-modules, and whose morphisms are \( \text{Hom}_E(M,N)/\sim \).

An equivalence class \( M \) in \( \text{Stab}(E) \) is said to be invertible if there exists an equivalence class \( N \) such that \( M \otimes N \) is stably equivalent to \( \mathbb{F}_p \). The invertible stable equivalence classes and their tensor product form the Picard group of \( \text{Stab}(E) \), denoted \( \text{Pic}(\text{Stab}(E)) \).

**Theorem 5.15.** [AP76, Theorem 3.6] Let \( E(\alpha, \beta) \) be a graded exterior algebra on two generators over \( \mathbb{F}_p \) such that \( |\alpha| < |\beta| \). Let \( \Sigma \) denote the graded \( E(\alpha, \beta) \)-module such that

\[
\Sigma_t = \begin{cases} 
\mathbb{F}_p & \text{if } t = 1 \\
0 & \text{if } t \neq 1
\end{cases}.
\]

Let \( I \) denote the augmentation ideal of \( E(\alpha, \beta) \). Then the Picard group \( \text{Pic}\left(\text{Stab}(E(\alpha, \beta))\right) \) is

\[
\text{Pic}\left(\text{Stab}(E(\alpha, \beta))\right) \cong \mathbb{Z}\{\Sigma\} \oplus \mathbb{Z}\{I\}.
\]

We can use Margolis homology to identify when a module is invertible.

**Definition 5.16.** Let \( M \) be a module over \( E(\alpha, \beta) \). The Margolis homology of \( M \) with respect to \( \alpha \) is denoted by \( M_*(M, \alpha) \), and is defined to be

\[
M_*(M, \alpha) = \ker \alpha : M \rightarrow M
\]

\[
\text{im} \alpha : M \rightarrow M.
\]
Margolis homology has a Whitehead Theorem and a Künneth Theorem, both of which we will use in this section.

**Proposition 5.17. (Künneth Theorem)** [Mar83, p.315] Let $M$ and $N$ be $E(\alpha, \beta)$-modules. Then $\mathcal{M}_*(M \otimes N, \alpha) \cong \mathcal{M}_*(M, \alpha) \otimes \mathcal{M}_*(N, \alpha)$.

**Proposition 5.18. (Whitehead Theorem)** [Mar83, Thm 18.3] Let $M$ and $N$ be connective $E(\alpha, \beta)$-modules. Consider an $E(\alpha, \beta)$-module homomorphism $f : M \to N$. The map $f$ is a stable equivalence if and only if $f$ induces an isomorphism on both $\alpha$ and $\beta$-Margolis homology.

**Lemma 5.19.** [AP76, Proposition 3.5] Consider a graded exterior algebra on two generators $E(\alpha, \beta)$ over a field $\mathbb{F}_p$. Let $M$ be a finitely generated graded module over $E(\alpha, \beta)$. Then $M$ is stably equivalent to an invertible module if and only if both $\mathcal{M}_*(M, \alpha)$ and $\mathcal{M}_*(M, \beta)$ are one-dimensional vector spaces over $\mathbb{F}_p$.

So if a finitely generated graded module $M$ has one-dimensional Margolis homology, then $M \cong \Sigma^a \otimes I^{\otimes b} \oplus F$, where $a, b \in \mathbb{Z}$ and $F$ is a free module. Furthermore, we can use the Margolis homology of $M$ to determine $a$ and $b$.

**Proposition 5.20.** [AP76, Pf. of Theorem 3.6] The Margolis homology of $\Sigma^a I^{\otimes b}$ is

$$
\mathcal{M}_*(\Sigma^a I^{\otimes b}, \alpha) \cong \Sigma^{a+|\alpha|} \mathbb{F}_p
$$

$$
\mathcal{M}_*(\Sigma^a I^{\otimes b}, \beta) \cong \Sigma^{a+|\beta|} \mathbb{F}_p
$$

**Proof.** The algebra $E(\alpha, \beta)$ acts trivially on the module $\Sigma$, so the Margolis homology of $\Sigma$ is

$$
\mathcal{M}_*(\Sigma, \alpha) \cong \Sigma \mathbb{F}_p
$$

$$
\mathcal{M}_*(\Sigma, \beta) \cong \Sigma \mathbb{F}_p.
$$

Consider the long exact sequence

$$
\cdots \to \mathcal{M}_*(E(\alpha, \beta), \alpha) \to \mathcal{M}_*(\mathbb{F}_p, \alpha) \to \Sigma^{-|\alpha|} \mathcal{M}_*(I, \alpha) \to \Sigma^{-|\alpha|} \mathcal{M}_*(E(\alpha, \beta), \alpha) \to \cdots
$$

Since the Margolis homology of $E(\alpha, \beta)$ is trivial, it follows that the Margolis homology of the augmentation ideal $I$ is

$$
\mathcal{M}_*(I, \alpha) \cong \Sigma |\alpha| \mathbb{F}_p
$$

$$
\mathcal{M}_*(I, \beta) \cong \Sigma |\beta| \mathbb{F}_p.
$$

By the Künneth formula (Proposition 5.17),

$$
\mathcal{M}_*(\Sigma^a I^{\otimes b}, \alpha) \cong \Sigma^{a+|\alpha|} \mathbb{F}_p
$$

$$
\mathcal{M}_*(\Sigma^a I^{\otimes b}, \beta) \cong \Sigma^{a+|\beta|} \mathbb{F}_p.
$$

□

It will be useful to have an explicit description of $I^{-1}$.

**Lemma 5.21.** [AP76, Proof of Lemma 3.5] Let $J$ be the module defined by the short exact sequence

$$
0 \to \mathbb{F}_p \to E(\alpha, \beta) \to J \to 0.
$$

Then $J = I^{-1}$. 

Proof. Applying Margolis homology to the short exact sequence (32) induces a long exact sequence
\[ \cdots \to \Sigma^{[\alpha]} M_\ast(J, \alpha) \to M_\ast(F_p, \alpha) \to M_\ast(E(\alpha, \beta), \alpha) \to \cdots. \]
Recall that \( M_\ast(E(\alpha, \beta), \alpha) = 0 \). So
\[ \Sigma^{[\alpha]} M_\ast(J, \alpha) \cong M_\ast(F_p, \alpha). \]
Thus \( M_\ast(J, \alpha) \cong \Sigma^{-[\alpha]} F_p \). By the same argument, \( M_\ast(J, \beta) \cong \Sigma^{-[\beta]} F_p \). It follows from Lemma 5.19 that \( J \) is an invertible module (up to stable equivalence). Observe that \( J \) is a quotient of \( E(\alpha, \beta) \), so it has no free summands. By Theorem 5.15 \( J \) is isomorphic to \( \Sigma^a I^b \) for some integers \( a \) and \( b \). It follows from Proposition 5.20 that \( a = 0 \) and \( b = -1 \). So indeed \( J \) is \( I^{-1} \).

5.2.3. The \( \text{Ext} \) groups of invertible modules. Now we will recall some results on the \( \text{Ext} \) groups of invertible modules.

**Corollary 5.22.** If \( b \leq 0 \), then
\[ \text{Ext}^{s,t}_{E(\alpha, \beta)}(F_p, I^\otimes b) \cong \text{Ext}^{s-b,t}_{E(\alpha, \beta)}(F_p, F_p) \quad \text{for } s > 0. \]

**Proof.** The short exact sequence (32) induces a long exact sequence
\[ \cdots \to \text{Ext}^s_{E(\alpha, \beta)}(F_p, E(\alpha, \beta)) \to \text{Ext}^s_{E(\alpha, \beta)}(F_p, J) \to \text{Ext}^{s+1}_{E(\alpha, \beta)}(F_p, F_p) \to \cdots. \]
By Lemma 2.4 the module \( E(\alpha, \beta) \) is injective, so \( \text{Ext}^s_{E(\alpha, \beta)}(F_p, E(\alpha, \beta)) = 0 \) for all \( s > 0 \). So
\[ \text{Ext}^s_{E(\alpha, \beta)}(F_p, J) \cong \text{Ext}^{s+1}_{E(\alpha, \beta)}(F_p, F_p) \]
for all \( s > 0 \). Tensoring the short exact sequence (32) with \( J^\otimes k \) yields a short exact sequence
\[ 0 \to J^\otimes k \to F \to J^\otimes k+1 \to 0, \]
where \( F := E(\alpha, \beta) \otimes J^\otimes k \) is a free module. So by induction,
\[ \text{Ext}^s_{E(\alpha, \beta)}(F_p, J^\otimes k+1) \cong \text{Ext}^{s+1}_{E(\alpha, \beta)}(F_p, J^\otimes k) \]
for \( s > 0 \), for all \( k \in \mathbb{N} \). That is, for all \( k \geq 0 \)
\[ \text{Ext}^s_{E(\alpha, \beta)}(F_p, J^\otimes k) \cong \text{Ext}^{s+b}_{E(\alpha, \beta)}(F_p, F_p) \quad \text{for } s > 0. \]
By Lemma 5.21 it follows that for all \( b \leq 0 \),
\[ \text{Ext}^s_{E(\alpha, \beta)}(F_p, I^\otimes b) \cong \text{Ext}^{s-b}_{E(\alpha, \beta)}(F_p, F_p) \quad \text{for } s > 0. \]

We will use the following criteria for showing that the \( \text{Ext}^{s,t} \) groups of a module over \( E(Q_j, Q_h) \) are zero when \( s > 0 \) and \( (t - s) \) is odd.

**Proposition 5.23.** Consider a graded \( E(Q_j, Q_h) \)-module \( M \). Suppose without loss of generality that \( j < h \). If the Margolis homology of \( M \) is
\[ M_\ast(M, Q_j) \cong F_p\{x\} \]
\[ M_\ast(M, Q_h) \cong F_p\{y\} \]
where \( |x| \) is even and \( |y| > |x| > 0 \), then \( \text{Ext}^{s,t}_{E(Q_j, Q_h)}(F_p, M) \) is concentrated in even \( (t - s) \)-degrees for all \( s > 0 \).
Proof. By Lemma \textbf{5.19} and Theorem \textbf{5.15} the module $M$ is stably equivalent to $\Sigma^a I^b$ for some $a, b \in \mathbb{N}$. By Proposition \textbf{5.20}

\begin{equation}
\begin{aligned}
a + b |Q_j| &= |x| \\
a + b |Q_h| &= |y|. 
\end{aligned}
\end{equation}

It follows that

\begin{equation}
b(|Q_h| - |Q_j|) = |y| - |x|
\end{equation}

\begin{equation}
a = |x| - b |Q_j|.
\end{equation}

Note that $|Q_h| < |Q_j| < 0$, so $|Q_h| - |Q_j|$ is negative. Since $|y| - |x|$ is positive, Equation \textbf{(33)} implies that $b$ must also be negative. So we can apply Corollary \textbf{5.22} to get

\[ Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, \Sigma^a I^b) \cong Ext_{E(Q_j, Q_h)}^{s-b,t-a}(\mathbb{F}_p, \mathbb{F}_p) \text{ for all } s > 0. \]

Recall that

\[ Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[v_{s,t}]. \]

Note that $|v_{s,t}| = (1, 2^p - 1)$, so $Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$ is concentrated in even $(t - s)$-degrees. Furthermore, since $|x|$ is even and $|Q_j|$ is odd, Equation \textbf{(34)} tells us that $b$ has the same parity as $a$.

Thus $Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, \Sigma^a I^b)$ must also be concentrated in even $(t - s)$ degrees. Recall that

\[ Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, M) \cong Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, \Sigma^a I^b) \]

for all $s > 0$, so indeed $Ext_{E(Q_j, Q_h)}^{s,t}(\mathbb{F}_p, M)$ is concentrated in even $(t - s)$-degrees for all $s > 0$. $\square$

Next we will analyze the Margolis homology of $H_*BP(2)$ and use it to decompose $H_*BP(2)$ in terms of invertible $E(Q_j, Q_h)$-modules. The description differs slightly at $p = 2$ from other primes, so we will first describe the odd-primary situation and then do $p = 2$.

5.2.4. Analyzing the Margolis homology of $H_*BP(2)$: the odd-primary case. Let $W_1$ denote the $E(Q_1, Q_2)$-subalgebra of $H_*BP(2)$ generated by the exterior elements, that is,

\[ W_1 = E(\bar{\tau}_3, \bar{\tau}_4, \bar{\tau}_5, \ldots) \otimes \mathbb{F}_p[\bar{\xi}_2^p, \bar{\xi}_3^p, \ldots]. \]

Let $W_o$ be the $E(Q_0, Q_2)$-subalgebra of $H_*BP(2)$ generated by exterior elements $\bar{\tau}_k$ such that $k$ is odd, that is,

\[ W_o = E(\bar{\tau}_3, \bar{\tau}_5, \ldots) \otimes \mathbb{F}_p[\bar{\xi}_2^p, \bar{\xi}_3^p, \ldots]. \]

Likewise, let $W_e$ be the $E(Q_0, Q_2)$-subalgebra of $H_*BP(2)$ generated by exterior elements $\bar{\tau}_k$ such that $k$ is even:

\[ W_e = E(\bar{\tau}_4, \bar{\tau}_6, \ldots) \otimes \mathbb{F}_p[\bar{\xi}_2^p, \bar{\xi}_4, \bar{\xi}_6, \ldots]. \]

Let $T_k(x_1, x_2, \ldots) = \mathbb{F}_p[x_1, x_2, \ldots] / (x_1^{p^k}, x_2^{p^k}, \ldots)$. The following lemma is an adaptation of Lemma 3.15 of \cite{Cul20}.

\begin{lemma}
As an $E(Q_1, Q_2)$-module,
\[ H_*BP(2) \cong W_1 \otimes T_2(\bar{\xi}_1) \otimes T_1(\bar{\xi}_2, \bar{\xi}_3, \ldots). \]

As an $E(Q_0, Q_2)$-module,
\[ H_*BP(2) \cong W_e \otimes W_o \otimes T_2(\bar{\xi}_1, \bar{\xi}_2). \]
\end{lemma}
We will need to use the Margolis homology of $H_*BP(2)$.

**Theorem 5.25.** [Cul20] Theorem 2.18] The Margolis homology of $H_*BP(2)$ is

$$\mathcal{M}_*(H_*BP(2), Q_0) \cong \mathbb{F}_p[\tilde{\xi}_1, \tilde{\xi}_2]$$

$$\mathcal{M}_*(H_*BP(2), Q_1) \cong \mathbb{F}_p[\tilde{\xi}_1] \otimes T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots)$$

$$\mathcal{M}_*(H_*BP(2), Q_2) \cong T_2(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \ldots).$$

Next we will determine the Margolis homology of $W_1$.

**Proposition 5.26.** The Margolis homology of $W_1$ is

$$\mathcal{M}_*(W_1, Q_1) \cong \mathbb{F}_p[\tilde{\xi}_1^p]$$

$$\mathcal{M}_*(W_1, Q_2) \cong T_1(\tilde{\xi}_2^p, \tilde{\xi}_3^p, \ldots).$$

**Proof.** By the Künneth formula for Margolis homology (Proposition 5.17),

$$\mathcal{M}_*(H_*BP(2), Q_1) \cong \mathcal{M}_*(W_1, Q_1) \otimes \mathcal{M}_*(T_2(\tilde{\xi}_1), Q_1) \otimes \mathcal{M}_*(T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots), Q_1).$$

Since the $Q_1$-action on $T_2(\tilde{\xi}_1)$ and $T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots)$ is trivial, it follows that

$$\mathcal{M}_*(T_2(\tilde{\xi}_1), Q_1) \cong T_2(\tilde{\xi}_1)$$

$$\mathcal{M}_*(T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots), Q_1) \cong T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots).$$

Combining this with Theorem 5.25, we see that

$$\mathcal{M}_*(H_*BP(2), Q_1) \cong \mathcal{M}_*(W_1, Q_1) \otimes T_2(\tilde{\xi}_1) \otimes T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots) \cong \mathbb{F}_p[\tilde{\xi}_1] \otimes T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots).$$

It follows that

$$\mathcal{M}_*(W_1, Q_1) \cong \mathbb{F}_p[\tilde{\xi}_1^p].$$

Likewise, note that the $Q_2$-action on $T_2(\tilde{\xi}_1)$ and $T_1(\tilde{\xi}_2, \tilde{\xi}_3, \ldots)$ is trivial. So by the same argument,

$$\mathcal{M}_*(W_1, Q_2) \cong T_1(\tilde{\xi}_2^p, \tilde{\xi}_3^p, \ldots).$$

\[\Box\]

Note that both the $Q_0$ and $Q_2$-actions on $T_2(\tilde{\xi}_1) \otimes T_1(\tilde{\xi}_3, \tilde{\xi}_5, \ldots)$ and on $T_2(\tilde{\xi}_2) \otimes T_1(\tilde{\xi}_4, \tilde{\xi}_6, \ldots)$ are trivial. So we can use the same argument as in the proof of Proposition 5.26 to compute the Margolis homology of $W_o$ and $W_e$.

**Proposition 5.27.** The Margolis homology of $W_e$ and $W_o$ is

$$\mathcal{M}_*(W_e, Q_0) \cong \mathbb{F}_p[\tilde{\xi}_2^p]$$

$$\mathcal{M}_*(W_e, Q_2) \cong T_2(\tilde{\xi}_4, \tilde{\xi}_6, \ldots)$$

$$\mathcal{M}_*(W_o, Q_0) \cong \mathbb{F}_p[\tilde{\xi}_1^p]$$

$$\mathcal{M}_*(W_o, Q_2) \cong T_2(\tilde{\xi}_3, \tilde{\xi}_5, \ldots).$$
Note that all monomials in $W_1$ have weight divisible by $p^3$. Let $W_1(k)$ denote the weight $p^3k$ component of $W_1$. The actions of $Q_1$ and $Q_2$ on $H_*BP(2)$ are weight-preserving, so

\[(35)\quad \mathcal{M}_*(W_1, Q_j) \cong \bigoplus_{k=0}^{\infty} \mathcal{M}_*(W_1(k), Q_j)\]

for $j = 1, 2$.

Likewise, the actions of $Q_0$ and $Q_2$ on $H_*BP(2)$ are weight-preserving, and every monomial in $W_o$ is divisible by $p^3$, so we will let $W_o(n)$ denote the weight $p^3n$ component of $W_o$. Note that every monomial in $W_o(n)$ is divisible by $p^3$, so it will be more convenient to let $W_e(n)$ denote the $p^4n$ component of $W_o(n)$. Then we can use the decompositions

\[(36)\quad \mathcal{M}_*(W_e, Q_j) \cong \bigoplus_{n=0}^{\infty} \mathcal{M}_*(W_e(n), Q_j)\]

\[(37)\quad \mathcal{M}_*(W_o, Q_j) \cong \bigoplus_{n=0}^{\infty} \mathcal{M}_*(W_o(n), Q_j)\]

for $i = 0, 2$.

Next we will compute the Margolis homology of these submodules.

**Proposition 5.28.** The Margolis homology of $W_1(n)$ is

\[\mathcal{M}_*(W_1(n), Q_1) \cong \mathbb{F}_p\{\xi_1^{2n}\}\]

\[\mathcal{M}_*(W_1(n), Q_2) \cong \mathbb{F}_p\{(\xi_2^{n_0}\xi_3^{n_1}\cdots\xi_{r+2}^{n_r})^p\}\]

where $n_0 + n_1p + \cdots + n_r p^r$ is the $p$-adic expansion of $n$.

**Proof.** The decomposition \((35)\) tells us that the Margolis homology of $W_1(n)$ should consist of the weight $p^3n$ component of the Margolis homology of $W_1$. Note that $wt(\xi_1^{p^3m}) = p^3m$. Combining this with Proposition 5.26 we see that

\[\mathcal{M}_*(W_1(n), Q_1) \cong \mathbb{F}_p\{\xi_1^{2n}\}\]

Suppose that $x$ is a nonzero monomial in $\mathcal{M}_*(W_1, Q_2)$. Then $x$ has the form

\[x = \xi_2^{m_0p}\xi_3^{m_1p}\cdots\xi_{r+2}^{m_rp}\]

where $0 \leq m_i < p$ for all $0 \leq i \leq r$.

Note that

\[wt(\xi_2^{m_0p}\xi_3^{m_1p}\cdots\xi_{r+2}^{m_rp}) = p^3(m_0 + m_1p + \cdots + m_rp^r),\]

so the weight of $x$ is exactly $p^3m$ where $m_0 + m_1p + \cdots + m_r p^r$ is the $p$-adic expansion of $m$.

So by Equation \((35)\) and Proposition 5.26

\[\mathcal{M}_*(W_1(n), Q_2) \cong \mathbb{F}_p\{(\xi_2^{n_0}\xi_3^{n_1}\cdots\xi_{r+2}^{n_r})^p\}\]

where $n_0 + n_1p + \cdots + n_r p^r$ is the $p$-adic expansion of $n$. \(\square\)
Proposition 5.29. The $Q_0$-Margolis homologies of $W_0(n)$ and $W_e(n)$ are

$$\mathcal{M}_*(W_0(n), Q_0) \cong \mathbb{F}_p \{ \bar{\xi}_1^{p^n} \}$$

$$\mathcal{M}_*(W_e(n), Q_0) \cong \mathbb{F}_p \{ \bar{\xi}_2^{p^n} \}.$$  

Suppose that $n \in \mathbb{N}$, with $p$-adic expansion $n = n_0 + n_1 p + \cdots + n_r p^r$. Let $m_i = n_{2i} + pn_{2i+1}$, and $s = \lfloor r/2 \rfloor$. Then

$$\mathcal{M}_*(W_e(n), Q_2) \cong \mathbb{F}_p \{ \bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+4}^{m_s} \}.$$  

Likewise,

$$\mathcal{M}_*(W_0(n), Q_2) \cong \mathbb{F}_p \{ \bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+3}^{m_s} \}.$$  

Proof. First note that $wt(\bar{\xi}_1^{p^n}) = p^3 n$ and $wt(\bar{\xi}_2^{p^n}) = p^4 n$, so it follows immediately from combining Proposition 5.27 with (30) and (37) that indeed

$$\mathcal{M}_*(W_0(n), Q_0) \cong \mathbb{F}_p \{ \bar{\xi}_1^{p^n} \}$$

$$\mathcal{M}_*(W_e(n), Q_0) \cong \mathbb{F}_p \{ \bar{\xi}_2^{p^n} \}.$$  

Consider a monomial $x \in \mathcal{M}_*(W_e, Q_2)$. It follows from Proposition 5.27 that $x$ is of the form

$$x = \bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+4}^{m_s},$$

where $0 \leq m_i < p^2$ for all $0 \leq i \leq s$. Note that $wt(x)$ must be divisible by $p^4$, so $wt(x) = p^4 n$ for some $n \in \mathbb{N}$. So

$$m_0 + m_1 p^2 + \cdots + m_s p^2 = n.$$  

Suppose that $n$ has $p$-adic expansion $n = n_0 + n_1 p + \cdots + n_r p^r$. Then $m_i = n_{2i} + pn_{2i+1}$. So there is exactly one monomial of weight $p^4 n$ for each $n$, and

$$\mathcal{M}_*(W_e(n), Q_2) \cong \mathbb{F}_p \{ \bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+4}^{m_s} \}.$$  

Likewise,

$$wt(\bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+3}^{m_s}) = p^3 n,$$

and so

$$\mathcal{M}_*(W_0(k), Q_2) \cong \mathbb{F}_p \{ \bar{\xi}_3^{m_0} \bar{\xi}_5^{m_1} \cdots \bar{\xi}_{2s+3}^{m_s} \}. \quad \square$$

5.2.5. Analyzing the Margolis homology of $H_*BP(2)$: the 2-primary case. Here, we record the 2-primary analogues of the results in Section 5.2.4. Each computation and proof is exactly analogous to its odd-primary version, so our exposition is significantly less detailed than in the previous subsection.

Recall from Section 2.2.2 that at the prime 2,

$$H_*BP(2) \cong \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4, \ldots].$$

At the prime 2, the action of $E(2)$ on $H_*BP(2)$ is given by

$$Q_j \bar{\xi}_k = \begin{cases} \bar{\xi}_2^{2j+1} & \text{if } j < k \\ \bar{\xi}_k^{2j-1} & \text{if } j \geq k \end{cases}.$$  

As in the previous section, let $T_k(x_1, x_2, \ldots) = \mathbb{F}_2[x_1, x_2, \ldots]/(x_1^{2k}, x_2^{2k}, \ldots)$. 
Let $W_1$ denote the $E(Q_1, Q_2)$-algebra
\[ W_1 = T_1(\xi_1, \xi_5, \ldots) \otimes F_2[\xi_2^4, \xi_3^4, \ldots] \otimes F_2[\xi_6^8]. \]

Let $W_o$ denote the $E(Q_0, Q_2)$-algebra
\[ T_1(\xi_5, \xi_7, \ldots) \otimes F_2[\xi_2^2, \xi_3^2, \ldots] \otimes F_2[\xi_6^8], \]
and let $W_e$ denote the $E(Q_0, Q_2)$-algebra
\[ T_1(\xi_4, \xi_6, \ldots) \otimes F_2[\xi_2^2, \xi_3^2, \ldots] \otimes F_2[\xi_6^8]. \]

The following lemma is an adaptation of [Cul19] Lemma 2.31. It is the 2-primary versions of Lemma 5.24.

**Lemma 5.30.** As an $E(Q_1, Q_2)$-module,
\[ H_\ast BP(2) \cong W_1 \otimes T_3(\xi_1^2) \otimes T_1(\xi_2^2, \xi_3^2, \ldots). \]

As an $E(Q_0, Q_2)$-module,
\[ H_\ast BP(2) \cong W_e \otimes W_o \otimes T_3(\xi_1^2, \xi_2^2). \]

**Lemma 5.31.** [Cul19] Proposition 2.14] At the prime 2, the Margolis homology of $H_\ast BP(2)$ is
\[ \mathcal{M}_\ast(H_\ast BP(2), Q_0) \cong F_2[\xi_1^2, \xi_2^2] \]
\[ \mathcal{M}_\ast(H_\ast BP(2), Q_1) \cong F_2[\xi_1^2] \otimes E(\xi_2^2, \xi_3^2, \ldots) \]
\[ \mathcal{M}_\ast(H_\ast BP(2), Q_2) \cong T_2(\xi_1^2, \xi_2^2, \ldots). \]

Combining Lemma 5.30 and Lemma 5.31 we can determine the Margolis homology of $W_1$, $W_e$, and $W_o$. The proof is completely analogous to that of Proposition 5.26.

**Proposition 5.32.** At the prime 2, the Margolis homology of $W_1$ is
\[ \mathcal{M}_\ast(W_1, Q_1) \cong F_p[\xi_1^8] \]
\[ \mathcal{M}_\ast(W_1, Q_2) \cong T_1(\xi_2^4, \xi_3^4, \ldots). \]

**Proposition 5.33.** At the prime 2, the Margolis homology of $W_e$ and $W_o$ is
\[ \mathcal{M}_\ast(W_e, Q_0) \cong F_p[\xi_1^8] \]
\[ \mathcal{M}_\ast(W_e, Q_2) \cong T_2(\xi_2^4, \xi_3^4, \ldots) \]
\[ \mathcal{M}_\ast(W_o, Q_0) \cong F_p[\xi_2^8] \]
\[ \mathcal{M}_\ast(W_o, Q_2) \cong T_2(\xi_4^2, \xi_6^2, \ldots). \]

Note that all monomials in $W_1$ have weight divisible by 8. So we will define $W_1(k)$ to be the weight $8k$ component of $W_1$. Note also that $W_e$ consists solely of monomials of weight divisible by 8 and $W_o$ of monomials divisible by 16, so we will let $W_e(k)$ be the weight $8k$ component and $W_o(k)$ the weight $16k$ component of $W_o$. Recall that the action of $E(2)$ on $H_\ast BP(2)$ is weight-preserving, so we get a decomposition of the form
\[ \mathcal{M}_\ast(W_1, Q_j) \cong \bigoplus_{n=0}^{\infty} \mathcal{M}_\ast(W_1(n), Q_j) \]
for \( j = 1, 2 \), as well as decompositions

\[
\mathcal{M}_*(W_e, Q_j) \cong \bigoplus_{n=0}^{\infty} \mathcal{M}_*(W_e(n), Q_j)
\]

\[
\mathcal{M}_*(W_o, Q_j) \cong \bigoplus_{n=0}^{\infty} \mathcal{M}_*(W_o(n), Q_j)
\]

for \( j = 0, 2 \).

Next we will compute the Margolis homology of these submodules. The computations are exactly analogous to those for the odd-primary versions (Proposition 5.28, Proposition 5.29).

**Proposition 5.34.** At the prime 2, the Margolis homology of \( W_1(n) \) is

\[
\mathcal{M}_*(W_1(n), Q_1) \cong F_2\{\xi_1^n\}
\]

\[
\mathcal{M}_*(W_1(n), Q_2) \cong F_2\{\bar{\xi}_2^n, \xi_3^n, \cdots, \xi_{r+2}^n\}
\]

where \( n = n_0 + 2n_1 + \cdots + 2^rn_r \) is the 2-adic decomposition of \( n \).

**Proposition 5.35.** At the prime 2, the \( Q_0 \)-Margolis homologies of \( W_o(n) \) and \( W_e(n) \) are

\[
\mathcal{M}_*(W_o(n), Q_0) \cong F_p\{\bar{\xi}_2^n\}
\]

\[
\mathcal{M}_*(W_e(n), Q_0) \cong F_p\{\bar{\xi}_2^n\}
\]

Suppose that \( n \in \mathbb{N} \), with \( p \)-adic expansion \( n = n_0 + n_1p + \cdots n_rp^r \). Let \( m_i = n_{2i} + pm_{2i+1} \), and \( s = \lfloor r/2 \rfloor \). Then

\[
\mathcal{M}_*(W_e(n), Q_2) \cong F_p\{\xi_1^{2m_0}, \xi_3^{2m_1}, \cdots, \xi_{2s+4}^{2m_s}\}.
\]

Likewise,

\[
\mathcal{M}_*(W_o(n), Q_2) \cong F_p\{\xi_2^{2m_0}, \xi_3^{2m_1}, \cdots, \xi_{2s+3}^{2m_s}\}.
\]

5.2.6. The \( \Ext \) groups of \( H_\ast BP(2) \) on subalgebras of \( E(2) \). Now we can return to working at an arbitrary prime \( p \).

**Proposition 5.36.** For all \( s > 0 \) and \( t - s \) odd, \( \Ext_{E(Q_1, Q_2)}^{s,t}(F_p, W_1) = 0 \).

**Proof.** We will state the proof for odd primes. The argument at \( p = 2 \) is exactly analogous. Recall from Equation (35) that

\[
\mathcal{M}_*(W_1(n), Q_j) \cong \bigoplus_{k=0}^{\infty} \mathcal{M}_*(W_1(k), Q_j)
\]

for \( j = 0, 2 \). The Margolis homology of \( W_1(n) \) is of the form

\[
\mathcal{M}_*(W_1(n), Q_1) \cong F_p\{\bar{\xi}_1^n\}\]

\[
\mathcal{M}_*(W_1(n), Q_2) \cong F_p\{\bar{\xi}_2^n, \bar{\xi}_3^n, \cdots, \bar{\xi}_{r+2}^n\}
\]

where \( n_0 + n_1p + \cdots + n_rp^r \) is the \( p \)-adic expansion of \( k \). Note that \( |\bar{\xi}_1^n| \) is even, and \( |\bar{\xi}_2^n, \bar{\xi}_3^n, \cdots, \bar{\xi}_{r+2}^n| > |\bar{\xi}_1^n| \). So by Proposition 5.23, \( \Ext_{E(Q_1, Q_2)}^{s,t}(F_p, W_1(n)) \) is concentrated in even \( t - s \) degrees for all \( n \in \mathbb{N} \) and \( s > 0 \). It follows that at odd primes, \( \Ext_{E(Q_1, Q_2)}^{s,t}(F_p, W_1) \) is concentrated in even \( t - s \) degrees for all \( s > 0 \). \( \square \)
We have an analogous result for \(W_e \otimes W_o\).

**Proposition 5.37.** For all \(s > 0\) and \(t - s\) odd, \(\text{Ext}^{s,t}_{E(Q_0, Q_2)}(\mathbb{F}_p, W_e \otimes W_o) = 0\).

**Proof.** We will state the proof for odd primes. The argument at \(p = 2\) is exactly analogous. Equation (36) and Equation (37) tells us that
\[
W_e \otimes W_o \cong \bigoplus_{n,n' \in \mathbb{N}} W_e(n) \otimes W_o(n').
\]

Consider a pair \(n, n' \in \mathbb{N}\), with \(p\)-adic expansions \(n = n_0 + n_1 p + \cdots + n_r p^r\) and \(n' = n'_0 + n'_1 p + \cdots + n'_s p^s\). Proposition [5,29] tells us that the Margolis homology of \(W_e(n) \otimes W_o(n')\) is
\[
\mathcal{M}_*(W_e(n) \otimes W_o(n'), Q_0) \cong \mathbb{F}_p \{ \xi_1^{n_0} \xi_2 \cdots \xi_1^{n_r} \}
\]
where \(m_i = n_i + pm_{i+1}\). Note that \(\xi_1^{n_0} \xi_2 \cdots \xi_1^{n_r}\) is even, and
\[
|\xi_1^{n_0} \xi_2 \cdots \xi_1^{n_r}| > |\xi_2^{n'} \xi_1^{n'}|.
\]

So it follows from Proposition [5,29] that \(\text{Ext}^{s,t}_{E(Q_0, Q_2)}(\mathbb{F}_p, W_e(n) \otimes W_o(n')) = 0\) for all \(t - s\) odd and \(s > 0\). \(\square\)

**Proposition 5.38.** Let \(0 \leq j < h \leq 2\). Then for all \(s > 0\) and \(t - s\) odd,
\[
\text{Ext}^{s,t}_{E(Q_0, Q_2)}(\mathbb{F}_p, H_*\text{BP}(2)) = 0.
\]

**Proof.** We will state the proof for odd primes, but the arguments at \(p = 2\) are exactly analogous. The case \(j = 0, h = 1\) follows from [Cul20, 3.18].

Next we will consider the case \(j = 1, h = 2\). Recall that
\[
H_*\text{BP}(2) \cong E(Q_1, Q_2) W_1 \otimes T_2(\xi_1) \otimes T_1(\xi_2, \xi_3, \ldots).
\]
Note that \(E(Q_1, Q_2)\) acts trivially on \(T_2(\xi_1) \otimes T_1(\xi_2, \xi_3, \ldots)\), so
\[
H_*\text{BP}(2) \cong E(Q_1, Q_2) \bigoplus_{x \in \mathcal{J}} \Sigma^{|x|} W_1,
\]
where \(\mathcal{J}\) is the set of all monomials in \(T_2(\xi_1) \otimes T_1(\xi_2, \xi_3, \ldots)\). Note that all monomials in \(\mathcal{J}\) have even degree, so indeed by Proposition [5,36] \(\text{Ext}^{s,t}_{E(Q_0, Q_2)}(\mathbb{F}_p, H_*\text{BP}(2))\) is concentrated in even \(t - s\) degrees for \(s > 0\).

Finally we will check the \(j = 0, h = 2\) case. Recall from Lemma 5.24 that
\[
H_*\text{BP}(2) \cong E(Q_0, Q_2) W_0 \otimes W_e \otimes T_2(\xi_1, \xi_2).
\]
Note that \(E(Q_0, Q_2)\) acts trivially on \(T_2(\xi_1, \xi_2)\), so
\[
H_*\text{BP}(2) \cong E(Q_0, Q_2) \bigoplus_{x \in \mathcal{I}} \Sigma^{|x|} W_1,
\]
where \(\mathcal{I}\) is the set of all monomials in \(T_2(\xi_1, \xi_2)\). Note that all monomials in \(\mathcal{I}\) have even degree, so indeed by Proposition 5.37 \(\text{Ext}^{s,t}_{E(Q_0, Q_2)}(\mathbb{F}_p, H_*\text{BP}(2))\) is concentrated in even \(t - s\) degrees for \(s > 0\). \(\square\)
Theorem 5.39. Let \((i,j,h)\) be any permutation of \((0,1,2)\). Then for all \(t - s\) odd,
\[
\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,H^*_{BP(2)}C) = 0.
\]

Proof. Recall that \(H_*BP(2) \cong E(2)H^*_{BP(2)}C \oplus H_*V\), and that \(H_*V\) is a free \(E(2)\)-module. So
\[
\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,H^*_{BP(2)}C) \cong \text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,H_*BP(2))
\]
for all \(s > 0\). By Proposition 5.38, \(\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,H^*_{BP(2)}C)\) is concentrated in even degrees for all \(s > 0\). So all that is left to check is that there are no classes in odd \(t\)-degree on the \((s = 0)\)-line.

Recall from Corollary 5.12 that
\[
H^*_{BP(2)}C \cong E(Q_i,Q_h)S_i \oplus R_t.
\]

Proposition 5.13 tells us that \(\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,S_i)\) is concentrated in even \((t-s)\)-degree. So all that is left is to show that \(\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,R_t)\) contains no classes in odd \(t\)-degree.

Suppose towards a contradiction that there was a nonzero class \(x \in \text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,R_t)\) such that \(|x|_t\) is odd. Note that \(|v_j|_{t-s}\) and \(|v_k|_{t-s}\) are even. Since we have already shown that there are no classes in odd \(t-s\) degree above the \((s = 0)\)-line in \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\), it follows that \(v_jx = v_kx = 0\). Such a class could only come from a free summand in \(R_t\). But we already know from Corollary 5.12 that \(R_t\) contains no free summands. So indeed there are no classes in odd \(t\) degree on the \((s = 0)\)-line of \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\). \(\square\)

5.2.7. Checking for hidden extensions.

Theorem 5.40. There are no hidden extensions in the Adams spectral sequence
\[
\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H_*BP(2)) \implies BP(2)_{t-s}BP(2),
\]

Proof. Recall that \(H_*BP(2) \cong E(2)H^*_{BP(2)}C \oplus H_*BP(2)V\). First we will check that there are no hidden extensions in \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\). Let \(0 \leq i < j \leq 2\). Theorem 5.39 tells us that for all \(t-s\) odd,
\[
\text{Ext}^{s,t}_{E(Q_i,Q_h)}(\mathbb{F}_p,H^*_{BP(2)}C) = 0.
\]

By Proposition 5.41, this implies that there is no \(v_i\)-torsion in \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\). That is, if \(x \in \text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\), then \(v_i^r x \neq 0\) for \(i = 0,1,2\) and \(r \in \mathbb{N}\). So indeed there are no hidden extensions in \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}C)\). Recall that \(V\) is the Eilenberg-Maclane space for an \(\mathbb{F}_p\)-vector space, so there are also no hidden extensions in \(\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H^*_{BP(2)}V)\). \(\square\)

We also want to show that there are no hidden extensions in the Adams spectral sequence
\[
\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H_*l_k) \implies BP(2)_{t-s}l_k.
\]

Theorem 5.41. There are no hidden extensions in the Adams spectral sequence
\[
\text{Ext}^{s,t}_{E(2)}(\mathbb{F}_p,H_*l_k) \implies BP(2)_{t-s}l_k.
\]

Proof. Recall from Proposition 5.5 that \(BP(2) \wedge l_k\) splits as
\[
BP(2) \wedge l_k \simeq C_k \vee V_k.
\]
where $V_k$ is a sum of mod-$p$ Eilenberg-MacLane spectra and $C_k$ is $v_2$-torsion free. So the Adams spectral sequence converging to $BP(2)_* l_k$ splits as the following two spectral sequences:

$$\text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} C_k) \implies \pi_* C_k$$

$$\text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} V_k) \implies \pi_* V_k.$$  

Since $V_k$ is a sum of mod-$p$ Eilenberg-MacLane spectra, there are no hidden extensions in the spectral sequence converging to $\pi_* V_k$. All that is left to check is that there are no hidden extensions in the component converging to $\pi_* C_k$.

Recall from Theorem 5.3 that $H_* BP(2) \cong H_*^{BP(2)} C_k \oplus H_*^{BP(2)} V_k$, where $H_*^{BP(2)} V_k$ is a free $E(2)$-module and $H_*^{BP(2)} C_k$ contains no free summands.

Recall from Theorem 2.14 that $H_* BP(2) \cong E(2)_\infty \bigoplus_{k=0}^{\infty} \Sigma^k H_* l_k$. Since $H_*^{BP(2)} C_k$ and $H_*^{BP(2)} V_k$ both contain no free summands (and $H_*^{BP(2)} V_k$ and $H_*^{BP(2)} C_k$ consist entirely of free summands), it follows that

$$H_*^{BP(2)} C_k \cong \bigoplus_{k=0}^{\infty} H_*^{BP(2)} C_k.$$  

So

$$\text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} C_k) \cong \bigoplus_{k=0}^{\infty} \text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} C_k).$$  

It follows from Theorem 5.39 that there is no $v_1$-torsion in $\text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} C_k)$. So there is no room for hidden extensions in $\text{Ext}_{E(2)}^s (\mathbb{F}_p, H_*^{BP(2)} C_k)$. □

5.3. An isomorphism of $E_2$-pages.

5.3.1. The Adams spectral sequence is multiplicative. Next, we will show that the $E_2$-pages of the hypercohomology spectral sequence and universal coefficient spectral sequences are isomorphic in the cases that we are interested in. We will need to use the following multiplicative properties of the Adams spectral sequence.

Specifically, we will use the following theorem of Adams.

**Proposition 5.42.** [Rav03 Thm 2.3.3] Let $Y'$, $Y''$, and $Y$ be connective spectra whose $H\mathbb{F}_p$-homology is of finite type, with a pairing

$$\alpha : Y' \wedge Y'' \to Y.$$  

Then $\alpha$ induces a natural pairing on the Adams spectral sequences

$$\alpha : E_r^{ASS} (S^0, Y') \otimes E_r^{ASS} (S^0, Y'') \implies E_r^{ASS} (S^0, Y)$$  

with the following properties:

1. The pairing on $E_r$ induces the pairing on $E_{r+1}$.
2. The pairing on $E_\infty$ corresponds to

$$\alpha_\infty : \pi_* Y' \otimes \pi_* Y'' \to \pi_* Y.$$
(3) Let $c : \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y') \otimes \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y'') \to \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y' \otimes H_* Y'')$ denote the external cup product. Let $\beta$ be the composite

$$\beta : H_* Y' \otimes H_* Y'' \to H_* (Y' \wedge Y'') \xrightarrow{\alpha} E_* Y,$$

where the first map is the K"unneth isomorphism. Then the composite

\[
\text{Ext}_{A_2}(\mathbb{F}_p; H_* Y') \otimes \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y'') \xrightarrow{c} \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y' \otimes H_* Y'') \xrightarrow{\beta} \text{Ext}_{A_2}(\mathbb{F}_p; H_* Y)
\]

is the pairing induced on $E_2$ by $\alpha$.

Let $\mu : BP(2) \wedge BP(2) \to BP(2)$ denote the usual multiplication. It follows that the $BP(2)$-module action

$$BP(2) \wedge BP(2) \wedge X \xrightarrow{\mu \otimes 1} BP(2) \wedge X$$

induces a natural pairing on the Adams spectral sequence

$$E_r(S^0, BP(2)) \otimes E_r(S^0, BP(2) \wedge X) \to E_r(S^0, BP(2) \wedge X),$$

such that the action on the $E_2$-terms is the composite described in (3), and the action of the $E_\infty$-page corresponds to the action

$$BP(2)_* \otimes BP(2)_* X \xrightarrow{\mu \otimes 1} BP(2)_* X.$$

Recall that the $E_2$-page of the Adams spectral sequence $E_r(S^0, BP(2))$ is isomorphic to $\text{Ext}_{E_2}(\mathbb{F}_p; \mathbb{F}_p)$, and that the $E_2$-page of the Adams spectral sequence $E_r(S^0, BP(2) \wedge X)$ is isomorphic to $\text{Ext}_{E_2}(\mathbb{F}_p, H_* X)$. We will use the following property of the pairing discussed above.

**Proposition 5.43.** Let $X = l_k$ or $BP(2)$. Consider the Adams spectral sequences

$$\text{Ext}_{E_2}(\mathbb{F}_p; \mathbb{F}_p) \Rightarrow BP(2)_*,$$

$$\text{Ext}_{E_2}(\mathbb{F}_p, H_* X) \Rightarrow BP(2)_* X.$$ The action on the $E_2$-page

$$\text{Ext}_{E_2}(\mathbb{F}_p; \mathbb{F}_p) \otimes \text{Ext}_{E_2}(\mathbb{F}_p, H_* X) \to \text{Ext}_{E_2}(\mathbb{F}_p, H_* X)$$

determines the action

$$BP(2)_* \otimes BP(2)_* X \to BP(2)_* X$$

up to hidden extensions.

5.3.2. Comparing the associated graded modules as vector spaces. Consider the Adams spectral sequence

$$\text{Ext}_{E_2}(\mathbb{F}_p, H_* BP(2)) \Rightarrow BP(2)_*.$$ The associated graded algebra to the Adams filtration on $BP(2)_*$ is isomorphic to $\mathcal{P}(2)$. Let $gr_* : \text{Mod}_{BP(2)} \to \text{Mod}_{\mathcal{P}(2)}$ denote the functor sending a module over $BP(2)_*$ to its associated graded module over $\mathcal{P}(2)$.

**Proposition 5.44.** Let $gr BP(2)_* X$ denote the associated graded module to the Adams filtration on $BP(2)_* X$. If $X = BP(2)$ or $l_k$, then there exists an $\mathbb{F}_p$-vector space isomorphism

$$\varphi : \text{Ext}_{E_2}(\mathbb{F}_p, H_* X) \to BP(2)_* X.$$
Lemma 5.47. There exists an isomorphism
\[ \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, H_*BP^{'(2)}) \cong BP^{'(2)}_*BP^{'(2)} \]
\[ \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, l_k) \cong BP^{'(2)}_*l_k \]

It follows that for each nonzero class \( x \in E^{'(2)}_* \), there exists a unique nonzero class \( \tilde{x} \in grBP^{'(2)}_*X \), and vice versa. \( \square \)

5.3.3. Comparing the multiplicative structures.

Proposition 5.45. Let \( x, y \in \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, H_*X) \). Let \( \tilde{x}, \tilde{y} \) denote classes in \( BP^{'(2)}_*X \) such that
\[ gr(\tilde{x}) = \varphi(x) \quad \text{and} \quad gr(\tilde{y}) = \varphi(y). \]
Then \( p^i v_1^j v_2^k \tilde{x} = \tilde{y} \) if and only if \( v_0^i v_1^j v_2^k x = y \).

Proof. As remarked above, these spectral sequences are multiplicative. So the \( \mathcal{P}(2) \cong \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, \mathbb{F}_p) \)-module action on the \( E_2 \)-page determines the \( BP^{'(2)}_* \)-action on the target, up to hidden extensions. It is well-known that the \( v_0 \)-action extends so that if \( v_0 x = y \), then \( p \tilde{x} = \tilde{y} \). Furthermore, we have shown in Theorem 5.41 and Theorem 5.40 that there is no room for hidden extensions in either of these spectral sequences. So when \( X = BP^{'(2)} \) or \( l_k \), the \( BP^{'(2)}_* \)-module multiplication on \( BP^{'(2)}_*X \) is completely determined by the \( \mathcal{P}(2) \)-module multiplication on \( \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, H_*X) \). That is, for any \( x, y \in E^n_* \), \( v_0^i v_1^j v_2^k x = y \) if and only if \( p^i v_1^j v_2^k \tilde{x} = \tilde{y} \). \( \square \)

Note that \( gr(BP^{'(2)}_*) \cong \mathcal{P}(2) \), so we can define a functor \( gr_* : \text{Mod}_{BP^{'(2)}} \rightarrow \text{Mod}_{BP^{'(2)}} \) which sends a \( BP^{'(2)}_* \)-module to its associated graded module.

Proposition 5.46. The functor \( gr_* : \text{Mod}_{BP^{'(2)}} \rightarrow \text{Mod}_{BP^{'(2)}} \) restricts to an equivalence of categories on finitely generated free modules.

Proof. First we confirm that \( gr_* \) is essentially surjective on the subcategory of finitely generated free \( \mathcal{P}(2) \)-modules. Any finitely generated free \( \mathcal{P}(2) \)-module can be written in the form \( \bigoplus_{i \in I} \mathcal{P}(2)\{x_i\} \), where \( I \) is a finite set. The module \( \bigoplus_{i \in I} BP^{'(2)}_*\{x_i\} \) is also a free finitely generated \( BP^{'(2)}_* \)-module, and
\[ gr_* \left( \bigoplus_{i \in I} BP^{'(2)}_*\{x_i\} \right) = \bigoplus_{i \in I} \mathcal{P}(2)\{x_i\}. \]
So indeed \( gr_* \) is essentially surjective on the subcategory of finitely generated free \( \mathcal{P}(2) \)-modules.

Now we will check that \( gr_* \) is fully faithful on the subcategory of finitely generated free \( \mathcal{P}(2) \)-modules. First, note that
\[ \text{Hom}_{BP^{'(2)}_*}(BP^{'(2)}_*, BP^{'(2)}_*) \cong \text{Hom}_{\mathcal{P}(2)}(gr_*BP^{'(2)}_*, gr_*BP^{'(2)}_*). \]
It follows that
\[ \text{Hom}_{BP^{'(2)}_*} \left( F, G \right) \cong \text{Hom}_{\mathcal{P}(2)} \left( gr_*F, gr_*G \right) \]
for all free and finitely generated \( BP^{'(2)}_* \)-modules \( F, G \). \( \square \)

Now we are ready to prove the following theorem.

Lemma 5.47. There exists an isomorphism
\[ \text{Ext}^{*\mathbb{N}}_{\mathcal{P}(2)} \left( \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, H_*l_k), \text{Ext}^*_{E(2)_*}(\mathbb{F}_p, H_*BP^{'(2)}) \right) \cong \text{Ext}^{*\mathbb{N}}_{BP^{'(2)}} \left( BP^{'(2)}_*l_k, BP^{'(2)}_*BP^{'(2)} \right). \]
Proof. Consider the module $BP(2)^*l_k$. We can construct a projective resolution $P^*$ over $BP(2)^*$ of the following form

$$0 \leftarrow BP(2)^*l_k \leftarrow \bigoplus_j \Sigma^r(j)BP(2)^* \leftarrow d_0 \cdots \leftarrow d_3 \bigoplus_j \Sigma^r(j)BP(2)^* \leftarrow 0.$$

Consider the chain complex $Q^*$ obtained by applying $gr_* : Mod_{BP(2)^*} \to Mod_{P^*}$ to $P^*$. By Proposition 6.1, $Q^*$ is a projective resolution of $Ext_{E(2)}(F_p, H_*l_k)$. Furthermore, $Hom_{BP}(P^*, BP(2)^*BP(2))$ is quasi-isomorphic to $Hom_{E(2)}(Q^*, Ext_{E(2)}(F_p, H_*BP(2)))$. So indeed the $Ext$ terms are isomorphic.

So we have constructed the following square.

$$\begin{array}{cccc}
Ext_{BP}(Ext_{E(2)}(F_p, H_*l_k), Ext_{E(2)}(F_p, H_*BP(2))) & \xrightarrow{HSS} & Ext_{E(2)}(H_*l_k, H_*BP(2)) \\
\downarrow & & \downarrow \\
Ext_{BP}(BP(2)^*l_k, BP(2)^*, BP(2)) & \xrightarrow{ASS} & [l_k, BP(2)^*BP(2)]
\end{array}$$

(39)

6. Analyzing the potential obstructions

Recall that a class $x \in Ext_{E(2)}(H_*l_k, H_*BP(2))$ is said to be a potential obstruction (to lifting the map $\theta_k$) if $x$ has degree $|x|_{s,t}$ such that $s \geq 2$ and $t-s$ is odd. In this section, we will show that all these potential obstructions must generate a nonzero class on the $E_\infty$-page of the spectral sequence. This implies that none of the potential obstructions can be boundaries for any differential $d_r$, and so $d_r(\theta_k) = 0$ for all $r$. Thus $\theta_k$ survives the Adams spectral sequence as well, allowing us to lift the necessary maps to construct the splitting of Theorem 6.10.

Recall the Adams spectral sequence (3)

$$Ext_{E(2)}^{s,t}(H_*l_k, H_*BP(2)) \implies [BP(2)^*l_k, BP(2)^*BP(2)]BP(2).$$

Likewise, we can use the the $BP(2)^*$-module splitting $BP(2)^*l_k$ constructed in Proposition 5.5 to further divide this spectral sequence into a summand of two more spectral sequences.

Proposition 6.1. The Adams spectral sequence

$$Ext_{E(2)}^{s,t}(H_*l_k, H_*BP(2)) \implies [BP(2)^*l_k, BP(2)^*BP(2)]BP(2)$$

splits as a sum of the following four spectral sequences:

$$Ext_{E(2)}(H_*BP(2)C_k, H_*BP(2)C) \implies [C_k, C]BP(2)$$

$$Ext_{E(2)}(H_*BP(2)C_k, H_*BP(2)V) \implies [C_k, V]BP(2)$$

$$Ext_{E(2)}(H_*BP(2)V_k, H_*BP(2)V) \implies [V_k, V]BP(2)$$

$$Ext_{E(2)}(H_*BP(2)V_k, H_*BP(2)C) \implies [V_k, C]BP(2).$$
Furthermore, any potential obstructions to lifting $\theta_k$ will be contained in the summand
\[ \text{Ext}_{E(2)}^s \left( H^*_n R, H^{BP(2)}_* C, H^{BP(2)}_* C \right). \]

Proof. The $BP(2)$-module splittings of $BP(2) \wedge BP(2)$ and $BP(2) \wedge I_k$ constructed in Proposition 5.3 allows us to split this Adams spectral sequence as a sum of four separate Adams spectral sequences:
\[ \text{Ext}_{E(2)}^s \left( H^*_n R, H^{BP(2)}_* C, H^{BP(2)}_* C \right) \implies [C, C]^{BP(2)} \]
\[ \text{Ext}_{E(2)}^s \left( H^*_n R, H^{BP(2)}_* C, H^{BP(2)}_* V \right) \implies [C, V]^{BP(2)} \]
\[ \text{Ext}_{E(2)}^s \left( H^*_n R, H^{BP(2)}_* V, H^{BP(2)}_* V \right) \implies [V, V]^{BP(2)} \]
\[ \text{Ext}_{E(2)}^s \left( H^*_n R, H^{BP(2)}_* V, H^{BP(2)}_* C \right) \implies [V, C]^{BP(2)} \]

Recall from Proposition 5.4 that $H^*_n R$ and $H^*_n V$ are both free $E(2)_*$-comodules of finite type. Lemma 2.4 tells us that $H^*_n R$ is therefore also injective, and $H^*_n V$ is also projective. So each of the summands, except for $\text{Ext}_{E(2)}^s \left( H^*_n C, H^{BP(2)}_* C \right)$, is concentrated on the $(s = 0)$-line. Recall that any potential obstructions to lifting the map $\theta_k : BP(2) \wedge I_k \to BP(2) \wedge BP(2)$ must have $s$-degree at least 2. So to analyze the potential obstructions, we can restrict our attention to the summand
\[ \text{Ext}_{E(2)}^s \left( H^*_n C, H^{BP(2)}_* C \right) \implies [C, C]^{BP(2)}. \]

\[ \Box \]

Since the splittings of $BP(2) \wedge BP(2)$ and $BP(2) \wedge I_k$ that were used above are $BP(2)$-module splittings, we can apply the same splittings to the universal coefficient spectral sequence. So instead of needing the entire square (39), we can restrict our attention to the square below in analyzing the potential obstructions.

\[
\begin{array}{ccc}
\text{Ext}_{BP(2)}^s \left( \pi_* C, \pi_* C \right) & \xrightarrow{\text{UCSS}} & \text{Ext}_{BP(2)}^s \left( \pi_* C, \pi_* C \right) \\
\text{Ext}_{E(2)}^s \left( F_p, H^{BP(2)}_* C \right) & \xrightarrow{HSS} & \text{Ext}_{E(2)}^s \left( H^{BP(2)}_* C, H^{BP(2)}_* C \right) \\
\end{array}
\]

Now we will analyze the $E_2$-page $\text{Ext}_{BP(2)}^s \left( \pi_* C, \pi_* C \right)$ of the universal coefficient spectral sequence. We will need to use a classical result of Auslander-Buchsbaum about Noetherian local rings and their modules.

**Definition 6.2.** Let $R$ be a commutative ring, $I$ an ideal in $R$, and $M$ an $R$-module. The $I$-depth of $M$ is
\[ \text{depth}_I(M) = \min \{ s | \text{Ext}_R^s(R/I, M) \neq 0 \} \].

If $R$ has a maximal ideal $m$, then the depth of $R$ is defined as
\[ \text{depth}(R) = \text{depth}_m(R). \]

The depth of $R$ is at most equal to the dimension of the ring $R$. When $\text{depth}(R) = \text{dim}(R)$, the ring is said to be Cohen-Macaulay.
Proposition 6.3. \cite{Eis13} p.451] Let $R$ be a regular local ring. Then $R$ is Cohen-Macaulay.

Lemma 6.4 (Auslander-Buchsbaum formula). \cite{Eis13} Thm 19.9] Let $R$ be a Noetherian local ring with maximal ideal $I$. If $M$ is a nonzero finitely generated $R$-module with finite projective dimension $\text{pd}(M)$, then $\text{pd}(M) + \text{depth}(M) = \text{depth}(R)$.

This formula yields a vanishing line on the $E_2$-page of the hypercohomology spectral sequence.

Proposition 6.5. For all $u > 2$,

$$\text{Ext}^{u,*}_{(2)}(E_2, (\mathbb{F}_p, H^{BP(2)}_* C_k), \text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C)) = 0.$$ 

Proof. Note that $\mathcal{P}(2)$ is a regular local ring of dimension 3 with maximal ideal $(v_0, v_1, v_2)$, and so for any finitely generated $2$-module $M$ over $E(2)$,

$$\text{depth}(M) = 3 - \text{pd}(M).$$

Culver’s inductive computations in \cite{Cul19} Section 3.3, \cite{Cul20} Section 5.1 tell us that the $\mathcal{P}(2)$-module $\text{Ext}_{(2)}(\mathbb{F}_p, H_* l_k)$ is finitely generated, so its submodule $\text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C_k)$ must be finitely generated as well. So we can apply the Auslander-Buchsbaum formula (Lemma 6.4). Note that if $M$ is $v_2$-torsion free, then $\text{Hom}_{\mathcal{P}(2)}(\mathbb{F}_p, M) = 0$. Recall from Theorem \[Eis13] that $H^{BP(2)}_* C_k$ is $v_2$-torsion free. So

$$\text{depth}(\text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C_k)) \geq 1.$$ 

Thus the projective dimension of $\text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C_k)$ over $\mathcal{P}(2)$ is at most 2.

So indeed

$$\text{Ext}^q_{\mathcal{P}(2)}(\text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C_k), \text{Ext}_{(2)}(\mathbb{F}_p, H^{BP(2)}_* C)) = 0$$ for all $q > 2$. 

Since the universal coefficient spectral sequence and hypercohomology spectral sequence have isomorphic $E_2$-pages, we can use this result to show that certain classes must survive the universal coefficient spectral sequence.

Proposition 6.6. Let $x \in \text{Ext}^{1,*}_{BP(2)}(\pi_* C_k, \pi_* C)$. Then $x$ survives the universal coefficient spectral sequence

$$\text{UCSS} E^{v,*}_{2}(C_k, C) = \text{Ext}^{1,*}_{BP(2)}(\pi_* C_k, \pi_* C) \Rightarrow [C_k, C]^{BP(2)}.$$ 

Proof. Recall from Lemma 5.47 that

$$\text{HSS} E_2(H^{BP(2)}_* C_k, H^{BP(2)}_* C) \cong \text{UCSS} E_2(C_k, C).$$

So by Proposition 6.5

$$\text{UCSS} E^{v,*}_{2}(C_k, C) = 0$$ for all $u > 2$.

The differential $\text{UCSS} E^{v,*}_{r} : \text{UCSS} E^{v,*}_{u} \rightarrow \text{UCSS} E^{u+r,*}_{u}$ increases the $u$-degree of the differential by $r$. So $x$ is neither the target nor the source of a differential in the universal coefficient spectral sequence.

Now we are ready to compare the $E_2$-page of the Adams spectral sequence to the $E_2$-page of the universal coefficient spectral sequence.
Proposition 6.7. The odd \((t - s)\)-degree component of the \(E_2\)-page of the Adams spectral sequence is isomorphic to the \((u = 1)\)-line of the universal coefficient spectral sequence, that is,

\[
\bigoplus_{t-s \text{ odd}} \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \cong \Ext^{u=1,s}_{UCSS}(\pi_*C_k, \pi_*C).
\]

Proof. Consider the hypercohomology spectral sequence

\[\text{HSS} E_2^{u,r,t}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \cong \bigoplus_{r=r_2-r_1, t=t_2-t_1} \Ext^{r_2,r_1}_{HSS}(\mathbb{F}_p, H^*_{BP(2)}C), \Ext^{t_2,t_1}_{HSS}(\mathbb{F}_p, H^*_{BP(2)}C)) \rightleftharpoons \Ext^{r-u,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C).
\]

Let \(x \in \Ext^{r,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C)\), and let \(\theta\) denote an element that detects \(x\) in \(\text{HSS} E_2^{u,*}(H^*_{BP(2)}C_k, H^*_{BP(2)}C)\). By Theorem 5.3 both \(\Ext^{r_1,t_1}_{E(2)}(\mathbb{F}_p, H^*_{BP(2)}C_k)\) and \(\Ext^{t_2,t_1}_{E(2)}(\mathbb{F}_p, H^*_{BP(2)}C)\) are concentrated in degrees such that \(t_2 - s_2\) and \(t_1 - s_1\) are even. So \(\theta\) is odd if and only if \(|x|_u\) is odd.

By Proposition 6.7 \(\text{HSS} E_2^{u,*} = 0\) for all \(u > 2\). So if \(|x|_{1-s}\) is odd, then \(x\) must lift to the \((u = 1)\)-line of \(\text{HSS} E_2^{u,*}(H^*_{BP(2)}C, H^*_{BP(2)}C)\). Furthermore, there is no room for differentials to leave or enter the \((u = 1)\)-line. Since these classes are all located in the same filtration, there is no room for hidden extensions. So

\[
\bigoplus_{t-s \text{ odd}} \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \cong \text{HSS} E_2^{s,t}(H^*_{BP(2)}C_k, H^*_{BP(2)}C).
\]

Furthermore, recall from Lemma 5.4 that

\[
\text{HSS} E_2^{s,t}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \cong \text{UCSS} E_2^{u=1,*}(C_k, C).
\]

So indeed

\[
\bigoplus_{t-s \text{ odd}} \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \cong \text{UCSS} E_2^{u=1,*}(C_k, C).
\]

Recall that the potential obstructions in the Adams spectral sequence to lifting \(\theta_k\) are all located in odd \((t - s)\) degree, so proving the following theorem will demonstrate that the potential obstructions survive.

Theorem 6.8. Let \(x \in \bigoplus_{t-s \text{ odd}} \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C)\). Then \(x\) survives the Adams spectral sequence

\[
\Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C) \implies [C_k, C]_{BP(2)}.
\]

Proof. By Proposition 6.7 any nonzero class \(y\) on the \((u = 1)\)-line of the universal coefficient spectral sequence \(\text{UCSS} E_2^{u,*}(C_k, C)\) must lift to a nonzero class \(\tilde{y}\) in \([C_k, C]_{BP(2)}\). Recall that \(\pi_*C_k\) and \(\pi_*C\) are concentrated in even degree, so \(\tilde{y}\) will have odd \(t - s\) degree. So \(\tilde{y}\) must be detected by a nonzero class \([\tilde{y}] \in \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C)\) having odd \(t - s\) degree. Proposition 6.7 tells us that there is at most one nonzero odd-degree class in \(\text{ASS} E_2^{s,*}\) for each nonzero class in \(\text{UCSS} E_2^{u,*}\). So the entire summand

\[
\bigoplus_{t-s \text{ odd}} \Ext^{s,t}_{E(2)}(H^*_{BP(2)}C_k, H^*_{BP(2)}C)
\]
Corollary 6.9. Let \( x \) be a potential obstruction to lifting \( \theta_k \) in the Adams spectral sequence
\[
\text{ASS} E_2^{s,t} = \text{Ext}_{E(2)}^{s,t}(H_* l_k, H_* BP(2)) \implies \{ BP(2) \wedge l_k, BP(2) \wedge BP(2) \}^{BP(2)}.
\]
Then \( x \) is not a boundary, that is, there is no \( y \in \text{ASS}_r \) such that \( d_r(y) = x \).

It follows that \( d_r(\theta_k) = 0 \) for all \( k \). So we can indeed lift the family of maps \( \{ \theta_k : H_* l_k \to H_* BP(2) \} \) to maps of spectra \( \{ \tilde{\theta}_k : BP(2) \wedge l_k \to BP(2) \wedge BP(2) \} \), and we have arrived at the following theorem.

Theorem 6.10 (Main Theorem). For all primes \( p \geq 5 \), there exists a splitting
\[
BP(2) \wedge BP(2) \simeq \bigvee_{k=0}^{\infty} \Sigma^{q_k} BP(2) \wedge l_k.
\]
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