THE WEIERSTRASS CRITERION AND THE LEMAÎTRE–TOLMAN–BONDI MODELS WITH COSMOLOGICAL CONSTANT Λ

IVANA BOCHICCHIO
Dipartimento di Matematica ed Informatica, Università degli Studi di Salerno
Via Ponte Don Melillo, 84084, Fisciano (SA), Italy
ibochicchio@unisa.it

Salvatore capozziello
Dipartimento di Scienze Fisiche, Università di Napoli “Federico II” and INFN Sez. di Napoli,
Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy
Capozzie@na.infn.it

ETTORE LASERRA
Dipartimento di Matematica ed Informatica, Università degli Studi di Salerno
Via Ponte Don Melillo, 84084, Fisciano (SA), Italy
Elaserra@unisa.it

We analyze Lemaître–Tolman–Bondi models in presence of the cosmological constant Λ through the classical Weierstrass criterion. Precisely, we show that the Weierstrass approach allows us to classify the dynamics of these inhomogeneous spherically symmetric Universes taking into account their relationship with the sign of Λ.

Keywords: Spherically symmetrical models; Lemaître–Tolman–Bondi equations; Weierstrass criterion.

1. Introduction

The Weierstrass criterion is one of the most elegant and powerful tools of classical mechanics, in that it allows qualitative analysis of one-dimensional conservative motions, once the zeros of a suitable function, the Weierstrass function, are determined. The usefulness of this criterion also lies in the fact that it can also be successfully generalized to many problems outside of classical mechanics, see e.g. the recent papers [1, 2].

In particular, the focus of this paper is to study the qualitative behavior of Lemaître-Tolman-Bondi (LTB) models, endowed with a non-null cosmological constant Λ, by assuming a Weierstrass approach.\[ a \]

\[ a \] A similar approach has been developed in [3], but in that case only LTB models with null Λ were considered.
The importance of such Universes in relativistic cosmology is mainly based on the following observation: even if the Friedmann-Robertson-Walker (FRW) models\(^b\) are widely accepted to describe the cosmic evolution, there are several objections to such models which include singularities, horizons, observed inhomogeneities starting from galactic scales up to galaxy superclusters (e.g. Virgo supercluster) \(^3\). In order to overcome some of the difficulties faced by the FRW models, Lemaître \(^5\), Tolman \(^6\), Bondi \(^7\) and others have considered inhomogeneous spherically symmetric dust models. Hence, the main aim of the LTB approach is to encompass cosmic inhomogeneities, at both large and small scales, with the overall cosmic dynamics \(^8\). Besides, being the simplest inhomogeneous solutions of the Einstein equations, it is relatively easy to work with them. Furthermore, the interest in such models is recently increased due to the fact that some of them can be designed to satisfy several observational requirements \(^9,10,11\).

As such, through the Weierstrass method, we are able to qualitatively describe and classify the possible kinds of evolution of the LTB–models. In particular, in the peculiar case of the FRW–models, Friedmann discussion \(^12,13\) can be reduced to a straightforward application of the Weierstrass method.

The paper is organized as follows: in Sec. 2, the main features of LTB Universes are reviewed. A discussion on the qualitative study of the evolution of the \(r\)–shells in General Relativity through the Weierstrass method is given in Sec. 3. Section 4 is devoted to concluding remarks.

2. The Lemaître–Tolmann–Bondi Models

Let us give now a brief summary of the main features of LTB-models according to \(^14,15,16\). We will consider a dust system \(C\) which, during its evolution, generates a Riemannian manifold, with locally spatial spherical symmetry around a physical point \(O\). The metric can then be given the Levi Civita’s form \(^17\):

\[
d s^2 = A(t, r) d r^2 + R^2(t, r) (d \theta^2 + \sin^2 \theta d \phi^2) - c^2 d t^2 ,
\]

(1)

where \(t\) is the proper time of each particle and \(r, \theta, \varphi\) are co–moving Levi–Civita’s curvature spherical coordinates: we can interpret \(R(t, r)\) as the intrinsic radius of the \(O\)–sphere \(S(r)\) at time \(t\) so that \(\frac{1}{R^2}\) represents, at any point, the Gaussian curvature of the geodesic sphere with its centre at \(O\).

We consider now the initial space-like hypersurface \(V_3\) (with equation \(t = 0\)) and call \(r\)–shells the set of particles with co–moving radius \(r\) (i.e. the dust initially distributed on the surface of the geodesic sphere with center at \(O\) and radius \(r\) (\(O\)–sphere) \(S(r)\)). According to \(^14,15,16\), we assign, at each particle of a \(r\)–shell, the initial intrinsic radius \(r = R(r, 0)\) as the radial co–moving coordinate. Hence

\(^b\) That is the homogeneous spherically symmetric dust models.

\(^c\) See \(^17\) for a precise definition of locally spatial spherical symmetry around a physical point \(O\).
the metric of the initial spatial manifold $V^3$ takes the form
\[ d\sigma^2 = a^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \]
where $a(r) = A(r,0)$.

Now let us take into account the gravitational field equations with the cosmological constant $\Lambda$ and the conservation equations:
\[
\begin{align*}
G_{\alpha\beta} + \Lambda g_{\alpha\beta} &= -\frac{8\pi G_N}{c^4} T_{\alpha\beta}, \\
\nabla_\alpha T^\alpha_\beta &= 0,
\end{align*}
\]
where $G_N$ is the Newton gravitational constant.

It has been shown [18] that it is possible to break the corresponding Cauchy problem into two separate intrinsically formulated invariant problems: the problem of initial conditions and the restricted problem of evolution. By taking into account [14,7,18,15,16], we can immediately translate the restricted evolution problem into the following equations:
\[
\begin{align*}
A(t,r) &= a(r) R'(t,r) \\
\dot{R}^2 &= c^2 \left( \frac{1}{a^2(r)} - 1 \right) + \frac{2G_N M(r)}{R} + \frac{\Lambda}{3} R^2 \\
\mu(t,r) &= \frac{\mu_0(r) r^2}{R'(t,r) R^2(t,r)}
\end{align*}
\]
where a dot denotes differentiation with respect to $t$ and a prime differentiation with respect to $r$, $\mu(t,r)$ is the mass density ($\mu_0 = \mu(r,0)$) and $M(r)$ is the “Euclidean mass”
\[
M(r) = 4\pi \int_0^r \mu_0(s) s^2 \, ds.
\]

If the initial mass density is constant, equation 4 reduces to the Friedman equation and $M(r)$ is constant.

If, analogously to [19], we introduce the function
\[
\varepsilon(r) = \frac{1}{a^2(r)} - 1,
\]
which represents the percentage deviation of $a^2(r)$ from the Euclidean value $a^2 = 1$, Eq. 4 becomes
\[
\dot{R}^2 = \varepsilon(r) c^2 + \frac{2G_N M(r)}{R} + \frac{\Lambda}{3} R^2.
\]

\[d\] These equations differ from the previous [15,16] only for the additional term $\frac{\Lambda}{3} R^2$.
\[e\] In that $M(r)$ would be the mass of dust contained within $S(r)$, if the initial $O$–sphere $V^3$ was Euclidean.
Remark 1. The function $\varepsilon(r)$ and the initial metric coefficient $a^2(r)$ are connected to the initial data by the equation

$$\varepsilon(r) = \frac{1}{c^2} \left( \frac{\ddot{r}}{r} - \frac{2G_N M(r)}{r} - \frac{\Lambda}{3} r^2 \right)$$

which gives an initial constraint.

3. Generalization of the Weierstrass Criterion to the LTB–Models

3.1. The Weierstrass equation

We will use a generalized form of the Weierstrass criterion, which can be applied to many problems outside of classical mechanics too. Let’s consider the first order differential equation

$$\dot{x}^2 = \Phi(x)$$

that we may call Weierstrass equation with Weierstrass function $\Phi(x)$ (see [20]). Equations of the form (9) are frequently encountered in classical mechanics. For example, the natural motions of a material point of mass $m$, subjected to a conservative force deriving from the potential energy $V(x)$ and having mechanical energy $E$, are described by the Weierstrass equation

$$\dot{x}^2 = \frac{2}{m} [E - V(x)]$$

with the Weierstrass function

$$\Phi(x) = \frac{2}{m} [E - V(x)].$$

In addition the Weierstrass Eq. (9) translates into the double equation

$$\frac{dx}{dt} = \pm \sqrt{\Phi(x)},$$

which can be integrated by separating the variables

$$t(x) = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{\Phi(x)}} + t_0$$

where we choose the sign $\pm$ in agreement with the sign of the initial rate $\dot{x}_0$,

$$\dot{x}_0^2 = \Phi(x_0).$$

The importance of the Weierstrass approach is mainly based on the fact that it is possible to obtain the qualitative behavior of the solutions of a Weierstrass equation, without integrating it. Precisely, the zeros of the Weierstrass function have a leading role, in fact the solutions of the Weierstrass equation are confined in those regions of the $x$–axis where $\dot{x}^2 \geq 0$, hence the Weierstrass condition

$$\Phi(x) \geq 0$$

For a detailed exposition of the Weierstrass criterion see e.g. [20] § 1.3 p.23].
must be fulfilled: the solutions of the Weierstrass equation are confined in those regions where the Weierstrass function is non-negative.

These regions are unlimited or are limited by the extrema of the definition interval (eventually $+\infty$ or $-\infty$) and by the eventual zeros of the Weierstrass function, which are called barriers, because they cannot be crossed by the solution $x(t)$, so they split the range of possible values for $x$ into allowed and prohibited intervals.

Now let’s consider the zeros of $\Phi$.

If a barrier $x_B$ is a simple zero of $\Phi$, that is such that
\[
\Phi(x_B) = 0, \quad \Phi'(x_B) \neq 0,
\]
then it is called an inversion point $x_I$, because the motion reverses its course after reaching it (see e.g. [20,3]).

If a barrier $x_B$ is a multiple zero, that is such that
\[
\Phi(x_B) = 0, \quad \Phi'(x_B) = 0,
\]
then it separates two allowed intervals and it is called a soft barrier $x_S$.

A soft barrier is also called an asymptotic point, because it takes an infinite time to reach it. In fact, at a soft barrier the integral (12) diverges (see e.g. [20,3]). Finally we recall that an asymptotic point is also an equilibrium point (see e.g. [20]).

So, once these zeros are found, the qualitative behavior of the solutions of the Weierstrass equation (9) is completely determined.

3.2. The Weierstrass criterion for the evolving $r$-shells

In order to qualitatively study the behavior of $r$-shells in the case of non-null cosmological constant, let us analyze Eq. (12) which determines the evolution of the material continuum through the Weierstrass method.

Now we will focus our attention on a given single $r$-shell (that is we will consider $r$ as a fixed parameter), so we can regard the intrinsic radius $R$ as a function of the time only, $x(t) = R(t; r)$, and $\varepsilon(r), M(r)$ as constant; then Eq. (7) becomes a quadratic differential equation
\[
\dot{x}^2 = \varepsilon c^2 + \frac{2 G N M}{x} + \frac{\Lambda}{3} x^2, \tag{14}
\]
that is a Weierstrass equation with Weierstrass function
\[
\Phi(x; r) = \varepsilon(r) c^2 + \frac{2 G N M(r)}{x} + \frac{\Lambda}{3} x^2, \tag{15}
\]
depending on the parameter $r$. Eq. (13) translates into two equations, depending on the parameter $r$,
\[
\frac{dx}{dt} = \pm \sqrt{\Phi(x; r)} = \pm \sqrt{\varepsilon(r) c^2 + \frac{2 G N M(r)}{x} + \frac{\Lambda}{3} x^2}, \tag{16}
\]
\[\text{Once solved Eq. (12), Eqs. (11), (13) can be immediately solved.}\]
where we have to choose the sign ± in agreement with the sign of the initial rate \( \dot{x}_0 \),

\[
\dot{x}_0^2 = \Phi(x_0; r) = \varepsilon(r) c^2 + \frac{2 \, GN \, M(r)}{r} + \frac{\Lambda}{3} r^2 .
\]

**Remark 2.** In consequence of Poincaré’s theorem on the analytic dependence of a solution on a parameter, the solutions of each of the Eqs. (16) depend analytically on the parameter \( r \) in each point where \( \sqrt{\Phi} \) depends analytically on \( x \) and \( r \) (see e.g. [21]).

The zeros of the Weierstrass function (15) depend not only on the sign of \( \Lambda \), but obviously on the sign of \( \varepsilon \) too, so the evolution of the \( r \)-shells will depend on both signs, the sign of \( \varepsilon \) and the sign of \( \Lambda \). These different situations will be analyzed in details in the following sections, where the zeros of the Weierstrass function are obtained by finding the positive real roots of the third degree equation

\[
\Lambda x^3 + 3\varepsilon c^2 x + 6GNM = 0 .
\]

To highlight the role of the cosmological constant, in the following we will consider Eq. (14) in the form:

\[
\dot{x}^2 = \frac{1}{3} x^2 [\Lambda + W(x)] ,
\]

where

\[
W(x) = \frac{3\varepsilon c^2}{x^2} + \frac{6GNM}{x^3} .
\]

Then the Weierstrass condition (13) holds when

\[
\Lambda \geq -W(x).\tag{20}
\]

**Remark 3.** Once \( W(x) \) is introduced, a barrier \( x_B (\neq 0) \) is a soft barrier \( x_S \) iff \( W'(x_B) = 0 \).

In fact from Eqs. (9) and (18) it follows that

\[
\Phi'(x) = \frac{2}{3} x [\Lambda + W(x)] + \frac{1}{3} x^2 W'(x).
\]

Supposing \( \Phi(x_B) = 0 \) then \( \Lambda + W(x_B) = 0 \); so, since \( \Phi'(x_B) = \frac{1}{3} x^2 W'(x_B) \), \( \Phi'(x_B) = 0 \Rightarrow W'(x_B) = 0 \) and viceversa.

We remark that, by considering the Weierstrass function \( \Phi(x) \) in the peculiar case of a FRW–Universe, it is easy to re–obtain the Friedmann discussion [12].
3.2.1. Null cosmological constant $\Lambda$

For sake of completeness, in this section we briefly recall the qualitative behavior of LTB–models with $\Lambda = 0$, since they have already been studied in [1,2].

For negative values of $\varepsilon$, there is only one barrier 

$$x_I = \frac{-2GM}{\varepsilon c^2},$$

which is a simple zero of the Weierstrass function, that is an inversion point. So, if a $r$–shell is initially expanding, it will go on expanding until the intrinsic radius reaches the value $x_I$; then it will contract back from $x_I$ towards the center of symmetry $O$ until it collapses in a finite time (see [1]).

When $\varepsilon$ is null (Euclidean case) the inversion point goes to infinity, so if a $r$–shell is initially expanding, it will go on expanding without limit, approaching the null expansion rate (see [1]).

Finally, a similar result can be achieved for positive $\varepsilon$: there are no barriers, so if the $r$–shell is initially expanding it will go on expanding with decreasing rate approaching the limit value $\dot{x}_l = \sqrt{\varepsilon c^2}$ (see [1]).

Fig. 1. The Weierstrass function $\Phi(x)$ for $\Lambda = 0$ in three different cases. More precisely

i) when $\varepsilon < 0$, the function $\Phi(x)$ has a simple zero at $x_I$ (i.e. there is one inversion point $x_I$) and the motion $\forall 0 < x \leq x_I$ is possible;

ii) when $\varepsilon = 0$, the inversion point goes to infinity and the motion is possible $\forall x > 0$;

iii) when $\varepsilon > 0$, there are no barriers: the motion $\forall x > 0$ is possible. In this case the Weierstrass function admits the red line $\Phi(x) = \varepsilon c^2$ as asymptote.
3.2.2. Positive cosmological constant $\Lambda$

The positivity of the cosmological constant implies an open model for null and positive values $\varepsilon(r)$, while a different and more complex situation is obtained for negative values of $\varepsilon(r)$.

$\Lambda$ is positive and $\varepsilon$ is positive or null

The evolution of LTB–models in these two cases is very similar.

In fact when $\varepsilon = 0$ (Euclidean case), the function (19) becomes

$$W(x) = \frac{6G_N M}{x^3};$$

so, from (20), if $\Lambda > 0$ then $\Phi(x) > 0 \forall x > 0$.

Instead when $\varepsilon > 0$, the function (19) becomes

$$W(x) = \frac{3\varepsilon c^2}{x^2} + \frac{6G_N M}{x^3} > 0$$

and, from (20), if $\Lambda > 0$ then $\Phi(x) > 0 \forall x > 0$.

Hence, in both cases, there are no barriers and we have a monotonic expansion: if the $r$–shell is initially expanding, it will go on expanding without limit, otherwise it will collapse.

![Fig. 2. The Weierstrass function $\Phi(x)$ for positive $\Lambda$. Cases $\varepsilon = 0$ and $\varepsilon > 0$ are analyzed. Here there are not inversion points: the motion $\forall x > 0$ is possible.](image-url)
Positive $\Lambda$ and negative $\varepsilon$

When $\varepsilon$ is negative, the function (19) becomes

$$W(x) = -\frac{3|\varepsilon|c^2}{x^2} + \frac{6G_NM}{x^3},$$

and it has a minimum at

$$x = x_M = \frac{3G_NM}{|\varepsilon|c^2}.$$

Moreover, from Remark 3, it is clear that when $\Lambda$ assumes a suitable critical value $\Lambda_c$ defined as

$$\Lambda_c = -W(x_M) = \frac{|\varepsilon|^3c^6}{9G_N^2M^2},$$

the Weierstrass function (15) has a soft barrier in the point $x_M$, that we can also write as

$$x_M = \left(\frac{3G_NM}{\Lambda_c}\right)^{\frac{1}{3}}.$$

Hence when $\Lambda$ assumes its critical value $\Lambda_c$, the evolution of $r$–shells in LTB–models is static and stable.

Precisely the behavior of each $r$–shell depends by the initial conditions: their evolution is really static if the initial intrinsic radius equals $x_M$; on the other hand, the static situation is a limiting situation: if $x_0 \neq x_M$, the $r$–shell go on expanding asymptotically approaching to the static model with $x_M$ as intrinsic radius.

If $\Lambda > \Lambda_c$, there are no barriers, hence if the $r$–shell is initially expanding ($\dot{x}_0 > 0$), it will go on expanding without limit.

Finally, when $0 < \Lambda < \Lambda_c$, the Weierstrass function (15) admits two simple zeros

$$\tau_1 = 2\sqrt{\frac{|\varepsilon|c^2}{\Lambda} \cos \frac{\alpha + 4\pi}{3}}$$
and
$$\tau_2 = 2\sqrt{\frac{|\varepsilon|c^2}{\Lambda} \cos \frac{\alpha}{3}},$$

where $\alpha$ is defined by

$$\tan \alpha = \frac{\sqrt{-\Delta}}{-q},$$

with $q = \frac{3G_NM}{\Lambda}$ and $\Delta = \frac{9G_N^2M^2|\varepsilon|^3c^6}{\Lambda^3}$. 

So when the initial intrinsic radius is less that $\tau_1$ and the $r$–shell is initially expanding, it will go on expanding until the intrinsic radius reaches the maximal expansion

\h
In fact its first $x$–derivative $\frac{dW(x)}{dx} = \frac{6|\varepsilon|c^2}{x^3} - \frac{18G_NM}{x^4}$ is null when $x = x_M$. In addition, evaluating the second derivative of $W(x)$ in $x_M$, we obtain the positive value $\frac{2|\varepsilon|^3c^{10}}{27G_N^2M^4}$. Hence $x_M$ is a minimum point for the function $W(x)$.

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Note that $\Delta$ is negative since $0 < \Lambda < \Lambda_c$. Moreover, since the tangent of $\alpha$ is negative, $\alpha \in (\frac{\pi}{2}, \pi)$, hence $\tau_1 < \tau_2$. 


point $\mathcal{F}_1$; then it will contract back from $\mathcal{F}_1$ towards the center of symmetry $O$ until it collapses in a finite time; on the other hand, when the initial intrinsic radius is greater than $\mathcal{F}_2$ and if the $r$–shell is initially expanding, it will go on expanding.

Note that the open or closed evolution of the $r$–shell strongly depends from the initial conditions.

Moreover, a particular situation is observed when the initial intrinsic radius $x_0$ is such that $\mathcal{F}_1 < x_0 < \mathcal{F}_2$, since in this case the Weierstrass function is negative: \textit{the initial intrinsic radius can’t belong to the interval $(\mathcal{F}_1, \mathcal{F}_2)$}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{weierstrass.png}
\caption{The Weierstrass function $\Phi(x)$ for positive $\Lambda$ and negative $\varepsilon$. In this case the behavior of the function depends on $\Lambda$. More precisely
i) when $\Lambda = \Lambda_c$, the function $\Phi(x)$ has a multiple zero at $x_M$. An equilibrium position or a limiting position corresponds to the soft barrier. In the first case, $x = x_M$ is the unique possible position.
In the second case, the motion $\forall 0 < x < x_M$ is possible;
ii) when $0 < \Lambda < \Lambda_c$, the function $\Phi(x)$ has two simple zeros at $\mathcal{F}_1$ and $\mathcal{F}_2$ (i.e. there are two inversion points $\mathcal{F}_1$ and $\mathcal{F}_2$). The motion is possible $\forall 0 < x \leq \mathcal{F}_1$ and $\forall x \geq \mathcal{F}_2$;
iii) when $\Lambda > \Lambda_c$ the Weierstrass function has no zeros. The motion $\forall x > 0$ is possible.}
\end{figure}

3.2.3. \textit{Negative cosmological constant}

Let us focus now on the qualitative behavior of $r$–shells for negative value of $\Lambda$.
In this case, the zeros of the Weierstrass function may only be simple zeros, which correspond to inversion points. Even if the expression of the maximum value reached by the intrinsic radius depends by the sign of $\varepsilon$, for each analyzed situation ($\varepsilon = 0$, $\varepsilon < 0$ and $\varepsilon > 0$) the model is closed.
Table 1. Evolution of LTB $r$–shells for positive cosmological constant and different signs of $\varepsilon(r)$.

| Sign of $\varepsilon(r)$ | Evolution of $r$–shells |
|--------------------------|-------------------------|
| $\varepsilon = 0$        | Open                    |
| $\varepsilon < 0$, $\Lambda = \Lambda_c$ | Static solution as effective or limiting solution. In the second case the evolution is open |
| $\Lambda > 0$, $\varepsilon < 0$, $\Lambda > \Lambda_c$ | Open |
| $\varepsilon < 0$, $0 < \Lambda < \Lambda_c$ | Closed if $x_0 \leq x_1 = 2\sqrt{\frac{6G_NM}{\Lambda}} \cos \frac{\pi + \frac{4\pi}{3}}{2}$, Open if $x_0 \geq x_2 = 2\sqrt{\frac{6G_NM}{\Lambda}} \cos \frac{\pi}{2}$ |
| $\varepsilon > 0$        | Open                    |

Table 2. Evolution of LTB $r$–shells for negative cosmological constant and different signs of $\varepsilon(r)$.

| Sign of $\varepsilon(r)$ | Evolution of $r$–shells |
|--------------------------|-------------------------|
| $\varepsilon = 0$        | Closed                  |
| $\Lambda < 0$, $\varepsilon < 0$ | Closed |
| $\varepsilon > 0$        | Closed                  |

The maximum value of the intrinsic radius depends also by $\Lambda$.

**Negative $\Lambda$ and null $\varepsilon$**

When $\varepsilon = 0$ (Euclidean case), we find the barrier

$$x_S = \left(\frac{6G_NM}{-\Lambda}\right)^{\frac{1}{2}},$$

which is a simple zero of the Weierstrass function \[13\] and corresponds to an inversion point. We have $\Phi(x) \leq 0$ for $x \geq x_S$, hence the evolution is possible for all values $0 \leq x \leq x_S$.

When $\varepsilon = 0$ and $\Lambda < 0$, if the $r$–shell is initially expanding, it will go on expanding until the intrinsic radius reaches the inversion point $x_S$; then it will contract back from $x_S$ toward the centre of symmetry $O$ until it collapses in a finite time.
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The Weierstrass function $\Phi(x)$ for negative $\Lambda$ in three different cases. More precisely

i) when $\varepsilon = 0$, the function $\Phi(x)$ has a simple zero at $x_S$ (i.e. there is one inversion point $x_S$) and the motion $\forall 0 < x \leq x_S$ is possible;

ii) when $\varepsilon < 0$, the function $\Phi(x)$ has a simple zero at $x_I$ (i.e. there is one inversion point $x_I$) and the motion $\forall 0 < x \leq x_I$ is possible;

iii) when $\varepsilon > 0$ the Weierstrass function has a simple zero. The value of this zero depends on $\Lambda$ as above specified. It can be individuated by $x_{I_1}$ or by $x_{I_2}$ (i.e. there is one inversion point $x_{I_1}$ or $x_{I_2}$) and the motion $\forall 0 < x \leq x_{I_1}$ (or clearly $\forall 0 < x \leq x_{I_2}$) is possible;

finally, when $\varepsilon \leq -1$ it is represented the limiting case $\varepsilon = -1$ when $x_{Limit}$ is the lower bound for the set of inversion points $x_I$.

**Negative $\Lambda$ and negative $\varepsilon$**

When $\Lambda$ and $\varepsilon$ are both negative, there is a barrier

$$x_I = \left( \frac{3G_NM}{|\Lambda|} + \sqrt{\frac{9G_N^2M^2}{|\Lambda|^2} + \frac{|\varepsilon|^3c^6}{|\Lambda|^3}} \right)^{\frac{1}{2}} + \left( \frac{3G_NM}{|\Lambda|} - \sqrt{\frac{9G_N^2M^2}{|\Lambda|^2} + \frac{|\varepsilon|^3c^6}{|\Lambda|^3}} \right)^{\frac{1}{2}}. \quad (24)$$

The barrier $x_I$ is a simple zero of the Weierstrass function, that is an inversion point, so the model is again closed: *if the $r$–shell is initially expanding, it will continue to expand until the intrinsic radius reaches its maximum value $x_I$; then it will contract back from $x_I$ toward the center of symmetry $O$ until it collapses in a finite time.*
Negative $\Lambda$ and positive $\varepsilon$

Let’s put, for sake of commodity,

$$\Delta = 9G_N^2M^2 - \frac{c^6}{|\Lambda|^3}. $$

When

$$|\Lambda| \geq \frac{\varepsilon^3c^6}{9G_N^2M^2},$$

we have $\Delta \geq 0$ and there is a barrier

$$x_{I_1} = \left(\frac{3G_NM}{|\Lambda|} + \Delta^\frac{1}{3}\right)^\frac{1}{2} + \left(\frac{3G_NM}{|\Lambda|} - \Delta^\frac{1}{3}\right)^\frac{1}{2},$$

which is a simple zero of the Weierstrass function, so corresponding to an inversion point.

Instead, when

$$|\Lambda| < \frac{\varepsilon^3c^6}{9G_N^2M^2},$$

$\Delta$ is strictly negative; let’s put, for sake of commodity,

$$\alpha = \arctan\left(\frac{\sqrt{-\Delta}|\Lambda|}{3G_NM}\right),$$

then there is a barrier

$$x_{I_2} = 2\sqrt{\frac{3c^2}{|\Lambda|}\cos\frac{\alpha}{3}},$$

which is again a simple zero of the Weierstrass function, so corresponding to an inversion point.

Hence, if the $r$–shell is initially expanding, it will continue to expand until the intrinsic radius reaches the value $x_{I_1}$ or $x_{I_2}$; then it will contract back from $x_{I_1}$ or $x_{I_2}$ toward the center of symmetry $O$ until it collapses in a finite time.

4. Concluding Remarks

In this paper, we have analyzed a cosmological scenario based on the spherically symmetric dust solutions of the Einstein equations, that is LTB models with non–null cosmological constant, through an approach strictly related to the Weierstrass method of Classical Mechanics. In our case, it allows a systematic analysis of LTB models according to the different signs and values of $\varepsilon(r)$ and $\Lambda$.

In particular, while we can easily classify the LTB evolution with null cosmological constant [12], namely the Universe is open when $\varepsilon \geq 0$ and closed if $\varepsilon < 0$, the situation in presence of cosmological constant is much more complex. First of all, one has to study the evolution for positive and negative values of $\Lambda$ and, for each of these, relate the evolution to the sign of $\varepsilon$. In particular, when $\Lambda > 0$, for $\varepsilon \geq 0$, the
same global behavior of LTB models with null cosmological constant is achieved:
the Universe is spatially open. In the other cases ($\varepsilon < 0$) the Universe can be open
or closed and this feature strongly depends on $\Lambda$ and on the initial conditions. The
various dynamical cases are summarized in Table I.

On the other hand, when $\Lambda < 0$, for any value of $\varepsilon$, LTB Universes present the
same evolution: they are always spatially closed. In particular, the maximum value
reached by the intrinsic radius depends only on $\varepsilon$ when $\varepsilon \leq 0$ and also on $\Lambda$ when
$\varepsilon > 0$. The dynamical behavior is more simple and summarized in Table II.

In addition note that, in the cases corresponding to spatially closed Universes,
we are not only able to select such a characterizing feature, but also to obtain the
exact value of the maximum that the $r$-shell can reach.

As concluding remark, it is worth noticing that the method outlined here can
be, in principle, applied any time that cosmological dynamical system can be recast
along the Weierstrass function and an effective cosmological constant can be defined.
For example, it could be possible to apply this approach to cosmological models
including scalar fields \cite{22}.

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