Abstract. George Boros and Victor Moll’s masterpiece *Irresistible Integrals* does well to include a suitably-titled appendix, “The Revolutionary WZ Method,” which gives a brief overview of the celebrated Wilf–Zeilberger method of definite summation. Paradoxically, *Irresistible Integrals* does not contain the suitably-titled appendix, “The Revolutionary AZ Method,” which would have been an excellent place to give a brief overview of the Almkvist–Zeilberger method of definite integration! This omission can be forgiven, but once realized it must be rectified. The remarkable AZ machinery deserves to be more widely known to the general public than it is. We will do our part by presenting a series of case studies that culminate in an integral-based proof that \( e \) is irrational.

*Behold, I will stand before thee there upon the rock in Horeb; and thou shalt smite the rock and there shall come water out of it, that the people may drink.*

— Exodus 17:6

You have been up all night working out the ingenious solution to your latest problem. Your answer depends on the integral sequence

\[
I(n) = \int_{-\infty}^{\infty} \frac{x^{2n}}{(x^2 + 1)^{n+1}} \, dx,
\]

which you desperately need to evaluate. You know that you could break out special functions, contour integrals, or some other method, but you would really just like a quick answer without much fuss.

You run to download the file EKHAD from

https://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD

and read it into Maple with “\texttt{read EKHAD;}”. You type the command

\[
\texttt{AZd(x^\leftarrow(2 \times n) / (x^\leftarrow2 + 1)^\leftarrow(n + 1), x, n, N);}
\]

and hardly a second has passed when Maple produces the following:

\[-2 \ n - 1 + (2 \ n + 2) \ N, -x.\]

You cry out in joy, for the *Almkvist–Zeilberger algorithm* has told you that your integrand satisfies the “recurrence”

\[
(-2n - 1 + (2n + 2)N) \frac{x^{2n}}{(x^2 + 1)^{n+1}} = -\frac{d}{dx} x^{2n} \frac{x^{2n}}{(x^2 + 1)^{n+1}},
\]

where \( N \) is the shift operator defined by \( Nf_n(x) = f_{n+1}(x) \). Integrating this equation on \(-\infty, \infty\) gives the identity

\[
(-2n - 1 + (2n + 2)N)I(n) = 0,
\]
which would traditionally be written as

\[ I(n + 1) = \frac{2n + 1}{2(n + 1)} I(n). \]

Using the initial condition \( I(0) = \pi \), you crank out the first few terms of the sequence:

\[ \pi, \frac{\pi}{2}, \frac{3\pi}{8}, \frac{5\pi}{16}, \frac{35\pi}{64}, \frac{231\pi}{256}, \frac{429\pi}{1024}, \frac{6435\pi}{4096}, \frac{12155\pi}{16384}, \frac{46189\pi}{65536}, \frac{6435\pi}{262144} \]

The denominators look like powers of 2. After some experimentation, you let \( \pi = 1 \) and multiply by \( 4^n \). This produces some integers:

\[ 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756. \]

You visit the On-Line Encyclopedia of Integer Sequences [9] at https://oeis.org, type in your integers, and receive word that they are the central binomial coefficients \( \left( \begin{array}{c} 2n \\ n \end{array} \right) \).

You have just conjectured that

\[ I(n) = \int_{-\infty}^{\infty} \frac{x^{2n}}{(x^2 + 1)^{n+1}} \, dx = \frac{\pi}{4^n} \left( \begin{array}{c} 2n \\ n \end{array} \right). \]

To wrap up, you note that the final expression satisfies the same initial condition. You check the recurrence by typing the commands

\begin{verbatim}
L := Pi * binomial(2 * (n + 1), n + 1) / 4^(n + 1):
R := Pi * binomial(2 * n, n) / 4^n:
simplify(convert(L / R, factorial));
\end{verbatim}

and observing the output

\[ \frac{2n + 1}{2n + 2}. \]

This is a rigorous proof requiring minimal effort on your part. Such is a normal case study of the AZ algorithm.

In general, we often want to understand the sequence of definite integrals

\[ I(n) = \int F_n(x) \, dx. \]

Perhaps we would like to compute the first twenty or so terms to see what \( I(n) \) looks like. Sometimes we can ask a computer to churn these out, but other times \( F_n(x) \) is so complicated that even our electronic friends would struggle to keep up for large \( n \). What we need is an efficient algorithm to compute the terms of \( I(n) \). We need a recurrence.

There are plenty of ad hoc methods to find a recurrence for \( I(n) \). You could integrate by parts or differentiate under the integral sign, for example. But these all require ingenuity, insight, and hard work. As Sir Alfred Whitehead once remarked, such ingenuity is overrated. No one wants to work hard—we want answers!

The painless way to discover these recurrences for large classes of integrals is the Almkvist–Zeilberger algorithm. This is the direct analog of the celebrated Wilf–Zeilberger method of automatic definite summation, but it has received less attention than its discrete counterpart. Our goal here is to explore the Almkvist–Zeilberger algorithm with a few case studies, leaving the door open for more experimentation.
1. A QUICK START GUIDE TO THE AZ ALGORITHM. The Wilf–Zeilberger method of definite summation is a breakthrough in automatic summation techniques. Roughly, the Wilf–Zeilberger method can automatically prove (and semi-automatically discover) most commonly occurring summation identities of the form

\[ S(n) = \sum_k f(n, k) = RHS(n). \]

One piece of the puzzle is that, whenever \( f(n, k) \) is a “suitable” function, it satisfies a special type of inhomogeneous linear recurrence with polynomial coefficients in \( n \). Specifically, there exists a nonnegative integer \( d \) and polynomials \( p_j(n) \) such that

\[ \sum_{j=0}^{d} p_j(n) f(n + j, k) = G(n, k + 1) - G(n, k), \]

where \( G(n, k) \) is some function with \( G(n, \pm \infty) = 0 \). Summing over \( k \) yields the recurrence

\[ \sum_{j=0}^{d} p_j(n) S(n + j) = G(n, \infty) - G(n, -\infty) = 0. \]

This method, also known as creative telescoping, has been (rightly) advertised from here to the Moon and back. See the article [11], the book [10], the lecture notes [14], and the lively Monthly article [8].

The Almkvist–Zeilberger algorithm is to definite integrals what the Wilf–Zeilberger method is to definite sums. The input to the algorithm is a “suitable” function \( F_n(x) \) with a discrete parameter \( n \). The output is a nonnegative integer \( d \), polynomials \( p_k(n) \), and a rational function \( R(x) \) such that

\[ \sum_{k=0}^{d} p_k(n) F_{n+k}(x) = \frac{d}{dx} R(n, x) F_n(x). \]

The left-hand side is independent of \( x \) except for the \( F_n(x) \), so integrating this equation on \([0, 1]\), say, gives

\[ \sum_{k=0}^{d} p_k(n) \int_0^1 F_{n+k}(x) = R(n, 1) F_n(1) - R(n, 0) F_n(0). \]

If \( F_n(0) = F_n(1) = 0 \) and \( R(n, x) \) is well-behaved, then \( I(n) = \int_0^1 F_n(x) \) satisfies

\[ \sum_{k=0}^{d} p_k(n) I(n + k) = 0, \]

meaning that we have discovered a recurrence for the sequence of integrals \( I(n) \). The only thing to verify is that \( F_n(x) \) is “suitable,” and that \( R(n, x) \) is well-behaved on the region of integration.

What functions are “suitable”? The requirement is that \( F_n(x) \) is hypergeometric in \( n \) and \( x \), meaning that there exist fixed rational functions \( R_1(n, x) \) and \( R_2(n, x) \) such that
This is all that the algorithm needs to produce its identity.

The version of the Almkvist–Zeilberger algorithm that we will use is implemented in the procedure $AZd(f, x, n, N)$ in the Maple package EKHAD referenced in the introduction. It takes an expression $f$ in the continuous variable $x$ and discrete parameter $n$. The symbol $N$ stands for the “shift” operator $N$ on the set of sequences by

$$Na(n) = a(n + 1).$$

For example, the Fibonacci numbers $F(n)$ satisfy

$$(N^2 - N - 1)F(n) = 0.$$ 

Now, let us get on to the case studies.

2. FACTORIALS. Let us begin humbly, by evaluating an integral that we already know.

**Proposition 1.** For each integer $n \geq 0$,

$$I(n) = \int_0^\infty e^{-x}x^ndx = n!.$$ 

**Proof.** Typing the command

$$AZd(\exp(-x) \ast x^n, x, n, N);$$

into Maple produces:

$$N - n - 1, -x.$$ 

That is, the Almkvist–Zeilberger algorithm has told us that

$$(N - (n + 1))f_n(x) = -\frac{d}{dx}e^{-x}x^{n+1}.$$ 

Since the antiderivative of the right-hand side vanishes for $x = 0$ and $x = \infty$, integrating on $[0, \infty)$ gives

$$(N - (n + 1))I(n) = 0,$$

and since $I(0) = 1$, we have $I(n) = n!$.

3. “A COMPLICATED INTEGRAL.” This is from Section 3.8 of [5].

**Proposition 2.**

$$I(n) = \int_0^\infty \frac{x^n}{(x + 1)^{n+r+1}} \, dx = \left[ r \binom{r + n}{n} \right]^{-1}.$$ 

Proof. Typing the command

\[ AZd(x^n / (x + 1)^{(n + r + 1)}, x, n, N); \]

into Maple produces:

\[ (n + 1) + (-n - r - 1) N, x. \]

And for \( r > 0 \), integrating the implied identity

\[
((n + 1) - (n + r + 1)N) \frac{x^n}{(x + 1)^{n+r+1}} = \frac{d}{dx} \frac{x^n}{(x + 1)^{n+r+1}}
\]

yields

\[
((n + 1) - (n + r + 1)N) I(n) = 0.
\]

The sequence \( (r^{(r+n)})^{-1} \) satisfies the same recurrence and initial condition (check!).

\[ \square \]

4. CENTRAL BINOMIAL COEFFICIENTS.

Proposition 3. The integral sequence

\[ I(n) = \int_0^1 (x(1-x))^n \, dx \]

satisfies

\[
(N - \frac{n + 1}{2(2n + 3)})I(n) = 0.
\]

Proof. Typing the command

\[ AZd((x * (1 - x))^n, x, n, N); \]

into Maple produces:

\[ n + 1 + (-4 n - 6) N, (-1 + 2 x) (-1 + x) x. \]

Integrating the implied identity

\[
(N - \frac{n + 1}{2(2n + 3)})(x(1-x))^n = \frac{d}{dx} (2x - 1)(x - 1)x(x(1-x))^n,
\]

on \([0, 1]\) yields the result, since the antiderivative of the right-hand side vanishes at \( x = 0 \) and \( x = 1 \).

\[ \square \]
The recurrence implies that $I(n)$ begins as follows:

\[
\frac{1}{6}, \frac{1}{30}, \frac{1}{140}, \frac{1}{630}, \frac{1}{2772}, \frac{1}{12012}, \frac{1}{51480}, \ldots
\]

Corollary 1.

\[
I(n) = \frac{1}{(2n + 1)\binom{2n}{n}}.
\]

Proof. Both sequences satisfy the same recurrence and initial condition (check!).

Integral evaluation is an area full of unintended consequences. Here we have an example, since one way to try and evaluate $I(n)$ is by applying the binomial theorem to the integrand:

\[
I(n) = \int_0^1 x^n(1 - x)^n \, dx
\]

\[
= \int_0^1 \sum_{k=0}^n \binom{n}{k}(-1)^k x^{n+k} \, dx
\]

\[
= \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{n+k+1}.
\]

This remaining sum is complicated, but we can pair it with our previous corollary to get another.

Corollary 2.

\[
\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{n+k+1} = \frac{1}{(2n + 1)\binom{2n}{n}}.
\]

5. IRRATIONALITY. Our final case study is a slightly more complicated sequence of integrals. We will derive, not a closed form, but a wealth of other information.

Proposition 4. The integral sequence

\[
I(n) = \int_0^1 (x(1 - x))^n e^{-x} \, dx
\]

satisfies

\[
(N^2 + 2(2n + 3)(n + 2)N - (n + 1)(n + 2))I(n) = 0.
\]

Proof. Let $f_n(x)$ be the integrand. The Almkvist–Zeilberger algorithm produces the “calculus exercise”

\[
(N^2 + 2(2n + 3)(n + 2)N - (n + 1)(n + 2))f_n(x)
\]

\[
= \frac{d}{dx}(-2nx^3 - x^4 + 3nx^2 - 2x^3 - nx + 5x^2 - 2x) f_n(x),
\]

and integrating this proves the proposition.
This recurrence is hopelessly complicated. We almost surely cannot solve it, but it does produce the following initial terms:

\[-1 + \frac{3}{e}, \quad 14 - \frac{38}{e}, \quad -426 + \frac{1158}{e}, \quad 24024 - \frac{65304}{e}, \ldots\]

This data is very suggestive! It leads us to conjecture that

\[I(n) = a_n + b_ne^{-1}\]

for some integers \(a_n\) and \(b_n\). We can check this immediately with the recurrence: if \(I(n) = a_n + b_ne^{-1}\) and \(I(n + 1) = a_{n+1} + b_{n+1}e^{-1}\), then

\[
I(n + 2) = -2(2n + 3)(n + 2)I(n + 1) + (n + 1)(n + 2)I(n)
= -2(2n + 3)(n + 2)(a_{n+1} + b_{n+1}e^{-1}) + (n + 1)(n + 2)(a_n + b_ne^{-1})
= a_{n+2} + b_{n+2}e^{-1}.
\]

where we take

\[
a_{n+2} = -2(2n + 3)(n + 2)a_{n+1} + (n + 1)(n + 2)a_n
b_{n+2} = -2(2n + 3)(n + 2)b_{n+1} + (n + 1)(n + 2)b_n.
\]

That is, \(a_n\) and \(b_n\) are sequences of integers which satisfy the same recurrence that \(I(n)\) satisfies, only the initial conditions are different:

\[
a_1 = -1 \quad a_2 = 14
b_1 = 3 \quad b_2 = -38.
\]

Better yet, note that

\[
\frac{-a_4}{b_4} = \frac{24024}{65304}
= 0.36787945 \ldots
\approx e^{-1}.
\]

That is, \(-a_n/b_n\) seems to be a good approximation to \(e^{-1}\)!

To see why this is, we must go back to the initial integral. For \(0 \leq x \leq 1\), we have \(x(1-x) \leq 1/4\), therefore

\[
0 \leq I(n) = \int_0^1 e^{-x}(x(1-x))^n \, dx \leq \frac{1}{4^n} \int_0^1 e^{-x} \, dx,
\]

which shows that \(I(n)\) goes to zero exponentially quickly. Therefore

\[
|a_n + b_ne^{-1}| \to 0
\]

exponentially quickly, meaning that \(-a_n/b_n\) gives an exponentially-good rational approximation of \(e^{-1}\). To double check, we can use our recurrence to compute \(a_{20}/b_{20}\):
\[
\frac{a_{20}}{b_{20}} = \frac{493294164866383351699429534601141833239920640000}{1340912564441170249019237618446466016434749440000} = 0.3678794411714423215955237701614608674 \ldots
\]
\[
\approx e^{-1}.
\]

Better still, this remarkable approximation \(-a_n/b_n \approx e^{-1}\) is too good to be true in the following sense.

**Proposition 5.** Let \(\alpha\) be a real number. If there exist sequences of integers \(a_n\) and \(b_n\) such that \(|b_n| \to \infty\) and

\[
|\alpha - \frac{a_n}{b_n}| \leq \frac{C}{|b_n|^{1+\delta}}
\]

for some positive constants \(C\) and \(\delta\), then \(\alpha\) is irrational.

**Proof.** If \(\alpha = a/b\) is rational, then

\[
|\alpha - \frac{a_n}{b_n}| = \left| \frac{(b_n a - b a_n)}{|b_n|} \right| \geq \frac{C}{|b_n|}
\]

for some positive constant \(C\). But the inequality

\[
\frac{C}{|b_n|} \leq \frac{C}{|b_n|^{1+\delta}}
\]

is impossible if \(|b_n| \to \infty\). \(\square\)

This fact together with our approximation \(-a_n/b_n \approx e^{-1}\) gives us a proof that \(e\) is irrational.

**Proposition 6.** \(e\) is irrational with \(\delta = 1\).

**Proof.** Let \(a_n\) and \(b_n\) be the approximating sequences induced by

\[
I(n) = \int_0^1 e^{-x}(x(1-x))^n \, dx.
\]

We have

\[
|a_n + b_n e^{-1}| \leq \frac{1}{4^n} \int_0^1 e^{-x} = \frac{C}{4^n}.
\]

The sequence \(b_n\) satisfies the recurrence

\[(N^2 + 2(2n + 3)(n + 2)N - (n + 1)(n + 2))b_n = 0.\]

It turns out—see [13]—that this reveals considerable asymptotic information about \(b_n\). In particular, if we rewrite the recurrence as a polynomial in \(n\), the leading coefficient is \(4N - 1\). The only solution to \(4N - 1 = 0\) is \(N = 1/4\), and this implies that \(1/4^n \leq C' \frac{1}{|b_n|}\) for some constant \(C'\). Thus

\[
|a_n + b_n e^{-1}| \leq \frac{C'}{|b_n|},
\]

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or
\[
\left| \frac{a_n}{b_n} + e^{-1} \right| \leq \frac{C'}{|b_n|^{1+\delta}},
\]
where \(\delta = 1\). The claim follows from the previous proposition.

It is a cruel irony that almost every real is irrational, yet we are often helpless to prove that any naturally occurring constant such as \(e\pi\), \(e + \pi\), or \(\gamma\) (the Euler–Mascheroni constant) is irrational. The “constructive irrationality” method we have just used gives us a possible framework to approach irrationality: find an approximation, check that \(\delta > 0\). Because irrationality results are so difficult, any hint or direction is worth investigating.

This constructive style of proof was made very famous by Roger Apéry [2, 12]. In 1978, during an infamous talk at Marseille, Apéry proved that
\[
\zeta(2) = \sum_{k \geq 1} \frac{1}{k^2} \quad \text{and} \quad \zeta(3) = \sum_{k \geq 1} \frac{1}{k^3}
\]
are irrational. Euler knew that \(\zeta(2) = \pi^2/6\), so the irrationality of \(\zeta(2)\) was no big deal. The irrationality of \(\zeta(3)\) was stunning.

Apéry’s proof relied on writing down (seemingly at random) the recurrence
\[
(n + 2)^3N^2 - (2n + 3)(17n^2 + 51n + 39)N + (n + 1)^3 = 0.
\]
and choosing two solutions \(a_n\) and \(b_n\) with different initial conditions, just as we did for \(e^{-1}\). After considerable checking, it turned out that \(a_n/b_n\) converged to \(\zeta(3)\) sufficiently quickly to prove its irrationality with \(\delta \approx 0.080529\).

The proof left the audience with many questions. Where did that recurrence come from? Why did those particular solutions work? Would Apéry’s argument generalize to other constants, such as \(\zeta(5)\)?

Shortly after Apéry’s proof, Fritz Beukers [4] cleared up some of the mystery when he elegantly reproved that \(\zeta(2)\) and \(\zeta(3)\) are irrational by considering integrals of the form
\[
\int_0^1 \int_0^1 \frac{x^n y^n (1-x)^n (1-y)^n}{1-xy} \, dx \, dy
\]
and
\[
\int_0^1 \int_0^1 \int_0^1 \frac{x^n y^n z^n (1-x)^n (1-y)^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \, dx \, dy \, dz,
\]
respectively, which is similar to what we have done here.

Despite the efforts of many experts, it remains unclear how to generalize either Apéry’s or Beukers’ arguments to prove that any odd-zeta value other than \(\zeta(3)\) is irrational. Simply put, we do not properly understand how \(\zeta(3)\) is related to the ratio \(a_n/b_n\) of distinct solutions to a single recurrence, so we cannot construct similar recurrences for \(\zeta(5)\) and beyond. The relevant term here is Apéry limit [6].

6. IRRATIONALITY MEASURES. Proving that \(e\) is irrational is an easy exercise, but our constructive proof gives more: a quantitative measure on the irrationality of \(e\).
For any real $\alpha$, the *irrationality measure* of $\alpha$, denoted $\mu(\alpha)$, is defined to be the smallest real $\mu$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{\mu+\epsilon}}$$

holds for any $\epsilon > 0$ and for all integers $p$ and $q$ with $q$ sufficiently large. If no such $\mu$ exists, then we set $\mu(\alpha) = \infty$.

Irrationality measures are intimately tied to constructive irrationality proofs. If we can construct a sequence of rationals $a_n/b_n$ such that

$$|\alpha - \frac{a_n}{b_n}| \leq \frac{C}{b_n^{1+\delta}},$$

then $\mu(\alpha) \geq 1 + \delta$. In many cases—see [12]—this also implies the upper bound $\mu(\alpha) \leq 1 + \frac{1}{\delta}$. Our proof happens to be one of these cases, and we get both $\mu(e) \geq 1 + 1 = 2$ and $\mu(e) \leq 1 + \frac{1}{1} = 2$, which implies $\mu(e) = 2$.

Though we have used integrals to construct our approximations, irrationality measures are also tightly linked with simple continued fractions. See Section 11.3 of the classic book *Pi and the AGM* by the great masters Jonathan and Peter Borwein for another proof of the irrationality measure of $e$, along with much more discussion about irrationality measures in general [3].

It is unusual that we know $\mu(e)$ exactly. Irrationality measures fall into three “regions”:

| $\alpha$ is ... | $\mu(\alpha)$ |
|-----------------|---------------|
| rational        | 1             |
| algebraic with degree $> 1$ | 2             |
| transcendental  | $\geq 2$      |

So $\mu(e) \geq 2$ is automatic by its transcendence, but $\mu(e) = 2$ is a surprise. Normally the best we can do is give an upper bound on $\mu(\alpha)$ for specific transcendental $\alpha$.

In fact, since it is so hard to establish irrationality, we have invented a new game: finding better and better upper bounds for the irrationality measure of famous constants. If you can find the best upper bound for $\mu(\pi)$, or $\mu(\zeta(3))$, then you get to hold the world record for a few months until someone beats you. For example, the current “world record” upper bound on $\mu(\pi)$ is held by Zeilberger and Zudilin [16], who showed that

$$\mu(\pi) \leq 7.103205334137\ldots$$

Ignoring technical details, their proof is very similar to ours. The basic idea is to find a rapidly-decaying sequence of integrals $I(n)$ such that $I(n) = a_n + \pi b_n$ for integers $a_n$ and $b_n$, then show that $a_n$ and $b_n$ have nice asymptotic properties.

To find their approximating sequences, Zeilberger and Zudilin tweaked integrals similar to the integrals Beukers used in his $\zeta(3)$ proof. They added parameters to the integrands and performed an exhaustive computer search to find those parameters which gave the empirically best upper bound, then went back and checked the details. This computer search method continues to provide possible avenues for constructive irrationality proofs; see [7] and [15].
7. CONCLUSIONS. It is too late for us to become famous proving that $\zeta(3)$ is irrational. We should be content just to have some new tools to play with. We should certainly show students how to use the AZ algorithm.

But you never know—one day you might just plug the right integrand into the Almkvist–Zeilberger algorithm to prove that

(FAMOUS CONSTANT)
is irrational.

Until then, have fun!

REFERENCES

[1] Almkvist, G., Zeilberger, D. (1990). The method of differentiating under the integral sign. J. Symb. Comput. 10(6): 571–592.
[2] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. Astérisque. 61: 11–13.
[3] Borwein, J., Borwein, P. (1987). Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity. Hoboken: Wiley-Interscience.
[4] Beukers, F. (1979). A note on the irrationality of $\zeta(2)$ and $\zeta(3)$. Bull. Lond. Math. Soc. 11(3): 268–272.
[5] Boros, G., Moll, V. (2004). Irrresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge: Cambridge University Press.
[6] Chamberland, M., Straub, A. (2020). Apéry limits: Experiments and proofs. https://arxiv.org/abs/2011.03400
[7] Dougherty-Bliss, R., Koutschan, C. and Zeilberger, D. (2021). Tweaking the Beukers integrals in search of more miraculous irrationality proofs a la Apéry. Available at: https://arxiv.org/abs/2101.08308
[8] Nemes, I., Petkovšek, M., Wilf, H., Zeilberger, D. (1997). How to do Monthly problems with your computer. Amer. Math. Monthly. 104(6): 505–519.
[9] OEIS Foundation Inc. (2021), The on-line encyclopedia of integer sequences. Available at: http://oeis.org.
[10] Petkovšek, M., Wilf, H., Zeilberger, D. (1997). A=B. Wellesley: AK Peters.
[11] Tefera, A. (2004). What is...a Wilf–Zeilberger pair. AMS Notices. 57(4): 508–509.
[12] Van der Poorten, A. (1979). A proof that Euler missed... Math. Intell. 1(4): 195–203.
[13] Wimp, J., Zeilberger, D. (1985). Resurrecting the asymptotics of linear recurrences. J. Math. Anal. Appl. 111(1): 162–176.
[14] Zeilberger, D. (1995). Three recitations on holonomic systems and hypergeometric series. J. Symb. Comput. 20(5/6): 699–724.
[15] Zeilberger, D., Zudilin, W. (2019). Automatic discovery of irrationality proofs and irrationality measures. Available at: https://arxiv.org/abs/1912.10381
[16] Zeilberger, D., Zudilin, W. (2020). The irrationality measure of $\pi$ is at most 7.103205334137... Mosc. J. Comb. Number Theory. 9(4): 407–419.

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