Small asymptotic translation lengths of pseudo-Anosov maps on the curve complex

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Abstract

Let $M$ be a hyperbolic fibered 3-manifold with $b_1(M) \geq 2$ and let $S$ be a fiber with pseudo-Anosov monodromy $\psi$. We show that there exists a sequence $(R_n, \psi_n)$ of fibers and monodromies contained in the fibered cone of $(S, \psi)$ such that the asymptotic translation length of $\psi_n$ on the curve complex $\mathcal{C}(R_n)$ behaves asymptotically like $1/|\chi(R_n)|^2$. As applications, we can reprove the previous result by Gadre–Tsai that the minimal asymptotic translation length of a closed surface of genus $g$ asymptotically behaves like $1/g^2$. We also show that this also holds for the cases of hyperelliptic mapping class group and hyperelliptic handlebody group.

Keywords: pseudo-Anosov, curve complex, asymptotic translation length, fibered 3-manifold, hyperelliptic mapping class group, handlebody group

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1 Introduction

Let $S_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures. We will simply denote it by $S$. The mapping class group of $S$, denoted $\text{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of $S$. By Nielsen–Thurston classification theorem, each element of $\text{Mod}(S)$, called a mapping class, is either periodic, reducible, or pseudo-Anosov.

For a non-sporadic surface $S$, that is, the surface with $3g - 3 + n \geq 2$, the curve complex $\mathcal{C}(S)$ is defined to be a simplicial complex whose vertex set $\mathcal{C}^0(S)$ is the set of homotopy classes of essential simple closed curves in $S$, and whose $k$-simplices are formed by $k + 1$ distinct vertices whose representatives can be chosen to be pairwise disjoint. We will restrict our attention to the 1-skeleton $\mathcal{C}^1(S)$ of
Theorem A. For all sufficiently large $n$, $R_n = \tilde{S}/\langle h^n \tilde{\psi} \rangle$ is a fiber of $M$ with $|\chi(R_n)| \asymp n$ whose pseudo-Anosov monodromy $\psi_n$ satisfies

$$\ell_C(\psi_n) \lesssim \frac{1}{|\chi(R_n)|^2}.$$
The above family of fibers in a fibered 3-manifold was first considered by McMullen and he proved the following theorem providing short geodesics on the moduli space when $S$ is a closed surface.

**Theorem 1.1** (McMullen, Theorem 10.2 in [McM00]). For all $n$ sufficiently large,

$$R_n = \tilde{S}/\langle h^n \tilde{\psi} \rangle$$

is a closed surface of genus $g_n \simeq n$, and $h^{-1}: \tilde{S} \to \tilde{S}$ descends to a pseudo-Anosov mapping class $\psi_n \in \text{Mod}(R_n)$ with

$$\log \lambda(\psi_n) \asymp \frac{1}{g_n},$$

where $\lambda(\psi_n)$ is the stretch factor of $\psi_n$.

Although McMullen dealt with closed hyperbolic 3-manifolds in Theorem 1.1, we can adopt the same proof for the general case of fibers of cusped hyperbolic 3-manifolds. In such case, we have to say $\log \lambda(\psi_n) \asymp 1/|\chi(R_n)|$ and $|\chi(R_n)| \asymp n$.

As a consequence of Theorem A, we can determine the behaviour of minimal asymptotic translation lengths of a few subgroups of mapping class groups. First of all, the fact that $L_C(\text{Mod}(S_g)) \asymp 1/g^2$ also follows from Theorem A by considering genus 2 surface and any mapping class fixing a nontrivial cohomology class. For instance, consider the mapping class $\psi = T_{a_1} T_{a_2} T_{a_3} T_{b_1}^{-1} T_{b_2}^{-1}$ of the closed surface of genus 2 as in Figure 1, where $T_{\gamma}$ is the left-handed Dehn twist about a simple closed curve $\gamma$. In this figure, $[c]$ is a homology dual to the cohomology class fixed by $\psi$.

Furthermore, we improve the upper bound for the minimal asymptotic translation length for $S_g$.

**Proposition 1.2.** For closed surfaces $S_g$ with $g \geq 3$,

$$L_C(S_g) \leq \frac{1}{g^2 - 2g - 1}.$$ 

We remark that for $g \geq 3$, this is a sharper upper bound than Gadre–Tsai’s. The proof is contained in Appendix.

Valdivia [Val14] showed that fixing $g \geq 2$ as $n \to \infty$,

$$L_C(\text{Mod}(S_{g,n})) \asymp \frac{1}{n}.$$
For any fixed \( g \geq 2 \), \( \text{Mod}(S_{g,n}) \)
\( \cong \frac{1}{n^2} \) [Val14]
\( \leq \frac{1}{n} \) [GT11]

| \( L_C(\cdot) \) | \( \text{Mod}(S_{0,n}) \) | \( \text{Mod}(S_{1,2n}) \) | For any fixed \( g \geq 2 \), \( \text{Mod}(S_{g,n}) \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \geq \frac{1}{n^2} \) [Val14] | \( \geq \frac{1}{n^2} \) [GT11] | \( \geq \frac{1}{n} \) [Val14] |

Table 1: Minimal asymptotic translation lengths.

For the remaining cases of \( g = 0 \) and \( 1 \) as \( n \to \infty \), see Table 1.

Let \( D_n \) be the closed disk \( D \) with \( n \)-punctures and let \( \text{Mod}(D_n) \) be the mapping class group of \( D_n \) fixing the boundary \( \partial D \) of the disk \( D \) pointwise. We have a natural homomorphism
\[
c : \text{Mod}(D_n) \to \text{Mod}(S_{0,n+1})
\]
collapsing the boundary \( \partial D \) of the disk to the \( (n+1) \)th puncture of \( S_{0,n+1} \). By definition, \( c(\text{Mod}(D_n)) \) is the subgroup of \( \text{Mod}(S_{0,n+1}) \) which fixes one of the punctures of \( S_{n+1} \). Hence we have \( L_C(c(\text{Mod}(D_n)) \geq L_C(\text{Mod}(S_{0,n+1})) \). As an application of Theorem A, we have the following results.

**Theorem B** (cf. Table 1). We have

1. \( L_C(c(\text{Mod}(D_n))) \geq \frac{1}{n^2} \) and
2. \( L_C(\text{Mod}(S_{1,n})) \geq \frac{1}{n^2} \).

Let \( \mathcal{H}(S_g) \) be the hyperelliptic mapping class group and let \( \text{Mod}(\mathbb{H}_g) \) be the handlebody group of genus \( g \). We consider the hyperelliptic handlebody group
\[
\mathcal{H}(\mathbb{H}_g) = \text{Mod}(\mathbb{H}_g) \cap \mathcal{H}(S_g).\]

**Theorem C.** We have
\[
L_C(\mathcal{H}(\mathbb{H}_g)) \geq \frac{1}{g^2}.
\]

The following is an immediate corollary of the previous theorem.

**Corollary 1.3.** We have
\[
L_C(\mathcal{H}(S_g)) \geq \frac{1}{g^2} \text{ and } L_C(\text{Mod}(\mathbb{H}_g)) \geq \frac{1}{g^2}.
\]

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In this section, we begin with the following simple observation.

**Lemma 2.1.** Let \( f \in \text{Mod}(S) \) be a pseudo-Anosov mapping class and let \( \alpha \) be any essential simple closed curve in \( S \). If \( d_C(\alpha, f^m(\alpha)) = 1 \) for some \( m \in \mathbb{N} \), then \( \ell_C(f) \leq \frac{1}{m} \).

**Proof.** By the triangle inequality, we have

\[
\ell_C(f^m) = \liminf_{j \to \infty} \frac{d_C(\alpha, f^{jm}(\alpha))}{j} \\
\leq \liminf_{j \to \infty} \frac{\sum_{i=1}^{j} d_C(f^{(i-1)m}(\alpha), f^{im}(\alpha))}{j} \\
= \liminf_{j \to \infty} \frac{j \cdot d_C(\alpha, f^{m}(\alpha))}{j} = 1
\]

Since \( \ell_C(f^m) = m \ell_C(f) \), this completes the proof. \( \square \)

Now we prove our main theorem.

**Proof of Theorem A.** Since the lower bound was established by Gadre–Tsai, it is enough to show that there exists some constant \( C \) such that

\[
\ell_C(\psi_n) \leq \frac{C}{|\chi(R_n)|^2}.
\]

First, assume that \( S \) is a closed surface. Let \( c \) be a simple closed curve whose homology class is dual to the primitive cohomology class \( \xi_0 \) fixed by \( \psi \). Then the \( \mathbb{Z} \)-cover \( \tilde{S} \) corresponding to \( \xi_0 \) can be obtained by cutting \( S \) along \( c \) and pasting \( \mathbb{Z} \)-copies of \( S \setminus \{c\} \) together. Let \( \Sigma_i \) be the copies of \( S \setminus \{c\} \) on \( \tilde{S} \) such that the generator \( h : \tilde{S} \to \tilde{S} \) for the deck transformation group is given by \( h(\Sigma_i) = \Sigma_{i+1} \) for all \( i \) (See Figure 2).

Due to Theorem 1.1, \( h^{-1} \) induces a pseudo-Anosov mapping class in \( \text{Mod}(R_n) \). Choose a lift \( \tilde{\psi} \) and determine a constant \( k = k(\tilde{\psi}) \) such that

\[
\tilde{\psi}(\Sigma_0) \subset \Sigma_{-k} \cup \ldots \cup \Sigma_{k-1} \cup \Sigma_k.
\]
(One can choose a lift \(\tilde{\psi}\) so that \(k\) becomes minimal and such minimal \(k\) is an expanding rate in a sense that the number of copies of \(\Sigma_i\) in which \(\tilde{\psi}(\Sigma_0)\) is contained. In Figure 3, \(k = 1\).)

Let \(\alpha\) be a simple closed curve contained in \(\Sigma_0\) and let \([\alpha]\) be the \(\langle h^n \tilde{\psi}\rangle\)-orbit of \(\alpha\) in \(\tilde{S}\). By construction, if we can find \(m\) such that one of the representative curves in \([h^{-m}(\alpha)]\) is contained in \(\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_{n-k-1}\), then it is disjoint from both \(\alpha\) and \(h^n\psi(\alpha)\) in \(\tilde{S}\) and hence \([h^{-m}(\alpha)]\) is a disjoint curve from \([\alpha]\) in \(R_n\) (Notice that \(h\) and \(\tilde{\psi}\) commute with each other). Note that \(\tilde{\psi}^m(\alpha)\) is contained in \(\Sigma_{mk} \cup \ldots \cup \Sigma_{mk}\) and hence \(h^{mk+1} \tilde{\psi}^m(\alpha)\) lies in \(\Sigma_1 \cup \ldots \cup \Sigma_{2mk+1}\). Therefore the possible maximum \(m\) is determined by \(2km + 1 \leq n - k - 1\) and we have

\[
m = \left\lfloor \frac{n - k - 2}{2k} \right\rfloor.
\]

Since \([h^{mk+1} \tilde{\psi}^m(\alpha)] = [h^{-(n-k)m+1}(\alpha)]\) and \(h^{-1}\) descends to the pseudo-Anosov mapping class \(\psi_n\) in \(\text{Mod}(R_n)\), we have \(dC\left(\psi_n^{(n-k)m-1}(\alpha), [\alpha]\right) = 1\). By Lemma 2.1 and the fact that \(\chi(R_n)\) is a linear function in \(n\), we have

\[
\ell_C(\psi_n) \leq \frac{1}{(n-k)m-1},
\]

and hence \(\ell_C(\psi_n) \leq C/|\chi(R_n)|^2\) for some \(C > 0\).

For a punctured surface \(S\), the same argument as above also works except for the case when we cannot find a simple closed curve contained in one fundamental region, say \(\Sigma_0\). In such case, it is enough to choose a simple proper arc in \(\Sigma_0\) and compute the asymptotic translation length on the arc and curve complex \(\mathcal{AC}(R_n)\). It is because there is a retraction map \(r : \mathcal{AC}(R_n) \to C(R_n)\) which is 2-bilipschitz (see, for instance, [MM00, Lemma 2.2] or [HPW15]). This implies that the asymptotic translation lengths \(\ell_{\mathcal{AC}}(f)\) and \(\ell_C(f)\) of each pseudo-Anosov
mapping class \( f \) on the 1-skeletons \( \mathcal{AC}^1(S) \) and \( \mathcal{C}^1(S) \), respectively, have the same asymptotic behaviour, that is,

\[
\ell_{\mathcal{AC}}(f) \asymp \ell_{\mathcal{C}}(f).
\]

Then the same proof in the previous paragraph works for \( \mathcal{AC}(R_n) \) and this completes the proof.

\( \square \)

3 Applications

In this section we prove Theorems B and C by using Theorem A.

Consider a pseudo-Anosov mapping class \( \psi \in \text{Mod}(S) \). Let \( \Psi : S \to S \) be any representative of \( \psi \). The mapping torus \( M_\psi \) is defined by

\[
M_\psi = S \times [0,1]/\sim,
\]

where \( \sim \) identifies \((x,1)\) with \((\Psi(x),0)\) for each \( x \in S \). Then a fibered 3-manifold \( M_\psi \) is hyperbolic by Thurston’s hyperbolization theorem. Suppose that there is a primitive cohomology class \( \xi_0 \in H^1(S;\mathbb{Z}) \) fixed by \( \psi \). This implies that \( b_1(M_\psi) \geq 2 \). Then Theorem A says that for \( n \) sufficiently large, \( R_n = S/(\sim h^n \psi) \) is a fiber of \( M_\psi \) with \( \chi(R_n) \asymp 1/|\chi(R_n)|^2 \).

3.1 Fibered 3-manifolds from braids

Let \( B_n \) be the the braid group with \( n \) strands. In this paper braids are depicted vertically. We define the product \( \beta \beta' \) of \( \beta, \beta' \in B_n \) in the usual way, namely, we stuck \( \beta \) on \( \beta' \) and concatenate the bottom \( i \)th end point of \( \beta \) with the top \( i \)th end point of \( \beta' \) for each \( i = 1, \cdots, n \). Then we obtain \( n \) strands. The product \( \beta \beta' \) is the resulting \( n \)-braid after rescaling.

Here we briefly review a relation between \( B_n \) and \( \text{Mod}(D_n) \). To do this we assign an orientation for each \( n \)-braid from the bottom endpoints to the top endpoints (see Figure 5(2)). We take a natural basis \( t_i \in H_1(D_n;\mathbb{Z}) \), where a representative of \( t_i \) is a small oriented loop in \( D_n \) centered at the \( i \)th puncture of \( D_n \) for \( i = 1, \cdots, n \). Let \( c_i \) be a simple proper arc in \( D_n \) which connects the \( i \)th puncture of \( D_n \) to the boundary \( \partial D \) as in Figure 5(1). Then there is an isomorphism

\[
\Gamma : B_n \to \text{Mod}(D_n)
\]

which sends the generator \( \sigma_i \) of \( B_n \) to the left-handed half twist \( h_i \) (see Figure 5(2) and (3)). The orientation of braids as we described above induces the motion of \( n \) punctures in the disk, which defines the above map \( \Gamma \).

Let us recall the homomorphism \( \epsilon : \text{Mod}(D_n) \to \text{Mod}(S_{0,n+1}) \) defined in Section 11. We sometimes identify \( f \in \text{Mod}(D_n) \) with \( \epsilon(f) \in \text{Mod}(S_{0,n+1}) \). We simply denote by \( \beta \), both mapping classes \( \Gamma(\beta) \in \text{Mod}(D_n) \) and \( \epsilon(\Gamma(\beta)) \in \text{Mod}(S_{0,n+1}) \).
Figure 5: (1) Arcs $c_i$ in the $n$-punctured disk $D_n$. (2) Generators $\sigma_i$. (3) Half twist $h_i$: ($c'_i = h_i(c_i)$ and $c'_{i+1} = h_i(c_{i+1})$.)

The closure $\overline{\beta}$ of $\beta \in B_n$ is a knot or link in the 3-sphere $S^3$. Let $\mathcal{A}$ be a braid axis of $\beta$ which is an unknot in $S^3$. Then $\overline{\beta}$ runs around the unknot $\mathcal{A}$ in a monotone manner. We set $\text{br}(\beta) = \overline{\beta} \cup \mathcal{A}$ which is a link in $S^3$ whose number of the components is greater than or equal to 2, and let us set $M_\beta = S^3 \setminus \text{br}(\beta)$. The 3-manifold $M_\beta$ is homeomorphic to the interior of the mapping torus of the monodromy $\beta \in \text{Mod}(D_n)$, and $b_1(M_\beta) \geq 2$. A spanning disk by the unknot $\mathcal{A}$ embedded in $M_\beta$ has $n$ punctures, and it is a fiber of $M_\beta$ with monodromy $\beta$.

3.2 Subgroups of mapping class groups

The hyperelliptic mapping class group $\mathcal{H}(S_g)$ is the subgroup of $\text{Mod}(S_g)$ consisting of isotopy classes of orientation preserving homeomorphisms of $S_g$ that commute with some fixed hyperelliptic involution $S : S_g \to S_g$ (Figure 6(1)). The handlebody group $\text{Mod}(\mathbb{H}_g)$ is the subgroup of $\text{Mod}(S_g)$ consisting of isotopy classes of orientation preserving homeomorphisms of $S_g$ that extend to homeomorphisms on the handlebody $\mathbb{H}_g$ of genus $g$. We let

$$\mathcal{H}(\mathbb{H}_g) = \text{Mod}(\mathbb{H}_g) \cap \mathcal{H}(S_g)$$

and call it the hyperelliptic handlebody group.

Let $SB_m$ be the spherical $m$-braid group. We now introduce the subgroup $SW_{2n}$ of $SB_{2n}$. Let $A_1, A_2, \ldots, A_n$ be $n$ disjoint unknotted arcs properly embedded in the 3-ball $D^3$ so that $A = A_1 \cup \cdots \cup A_n$ is unlinked, see Figure 7(1). The boundary $\partial A$ is the set of $2n$ points in the 2-sphere $\partial D^3$.

For $b \in SB_{2n}$, we stick $b$ on $A$, and concatenate the bottom endpoints of $b$ with the endpoints of $A$. As a result we obtain $n$ disjoint (knotted) arcs $^bA$ properly embedded in $D^3$, see Figure 7(2). The wicket group $SW_{2n}$ is the subgroup of $SB_{2n}$ generated by braids $^bA$’s such that $^bA$ is isotopic to $A$ relative to $\partial A$. It is easy to see that the braid $w \in SB_6$ as shown in Figure 7(3) is an element of $SW_6$. 
There is a spherical version of the isomorphism $\Gamma : B_n \to \text{Mod}(D_n)$, namely we have a surjective homomorphism $SB_m \to \text{Mod}(S_{0,m})$ which sends the generator $\sigma_i$ of $SB_m$ to the left-handed half twist between the $i$th and $(i+1)$st punctures (cf. Figure 5(2)(3)). We also denote this homomorphism by

$$\Gamma : SB_m \to \text{Mod}(S_{0,m})$$

Its kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by a full twist $\Delta^2 \in SB_m$, where $\Delta$ is a half twist (also known as Garside element). When $m = 2n$ the image $\Gamma(SW_{2n})$ of $SW_{2n}$ under the map $\Gamma$ is a subgroup of $\text{Mod}(S_{0,2n})$ which is so-called Hilden group, denoted by $SH_{2n}$, and

$$SH_{2g+2} \simeq SW_{2g+2}/\langle \Delta^2 \rangle$$

holds (see [SE17]).

There is a close connection between the wicket group and the hyperellitic handlebody group which we explain below. We first recall a fundamental theorem by Birman and Hilden which relates $\mathcal{H}(S_g)$ to $\text{Mod}(S_{0,2g+2})$. Each homeomorphism
on $S_g$ which commutes with the hyperelliptic involution $S : S_g \to S_g$ preserves the set of fixed points of $S$ consisting of $2g + 2$ points. Such a homeomorphism induces a homeomorphism on a sphere $S_g/S$ which preserves these fixed points (Figure 6(2)). Thus we have a map

$$q : \mathcal{H}(S_g) \to \text{Mod}(S_{0,2g+2})$$

by choosing a representative of each mapping class of $\mathcal{H}(S_g)$ which commutes with $S$. It is shown in [BH71] that the map $q$ is well-defined and it is a surjective homomorphism whose kernel is generated by $\iota = [S] \in \mathcal{H}(S_g)$. In particular we have

$$\mathcal{H}(S_g)/\langle \iota \rangle \simeq \text{Mod}(S_{0,2g+2}) \simeq SB_{2g+2}/\langle \Delta^2 \rangle.$$

On the other hand, it is proved in [SE17] that there is a surjective homomorphism

$$Q : \mathcal{H}(H_g) \to SH_{2g+2}$$

whose kernel is generated by $\iota$. The map $Q$ is given by the restriction

$$q|_{\mathcal{H}(H_g)} : \mathcal{H}(H_g) \to SH_{2g+2} < \text{Mod}(S_{0,2g+2}).$$

Putting all things together, we have

$$\mathcal{H}(H_g)/\langle \iota \rangle \simeq SH_{2g+2} \simeq SW_{2g+2}/\langle \Delta^2 \rangle.$$

Thus an element $f \in SH_{2g+2}$ can be described by a braid $v \in SW_{2g+2}$, i.e., $f = \Gamma(v)$. Moreover a lift $\hat{f}$ of $f$ under the map $q|_{\mathcal{H}(H_g)} = Q$ is an element of $\mathcal{H}(H_g)$.

We simply denote by $v$, the element $\Gamma(v)$ in the Hilden group $SH_{2g+2}$.

**Lemma 3.1.** Let $f \in \text{Mod}(S_{0,2g+2})$ and let $\hat{f} \in \mathcal{H}(S_g)$ be a lift of $f$ under the map $q : \mathcal{H}(S_g) \to \text{Mod}(S_{0,2g+2})$. We take any $\alpha \in \mathcal{A}^0(S_{0,2g+2})$, i.e., $\alpha$ is a homotopy class of an arc or simple closed curve in $S_{0,2g+2}$. Suppose that $d_\mathcal{A}(\alpha, f^m(\alpha)) = 1$ for some $m \in \mathbb{N}$, where $d_\mathcal{A}$ is the path metric on $\mathcal{A}(S_{0,2g+2})$. Then

$$\ell_C(\hat{f}) \leq \frac{1}{m}.$$

It is well-known and not hard to see that if $f \in \text{Mod}(S_{0,2g+2})$ is pseudo-Anosov, then $\hat{f} \in \mathcal{H}(S_g)$ is also pseudo-Anosov. We use Lemma 3.1 for some pseudo-Anosov elements of $SH_{2g+2} < \text{Mod}(S_{0,2g+2})$ in the proof of Theorem [C].

**Proof of Lemma 3.1** By abuse of the notation, a representative of $\alpha \in \mathcal{A}^0(S_{0,2g+2})$ is denoted by the same $\alpha$. Let $\hat{\alpha} \subset S_g$ be a lift of a simple arc or simple closed curve $\alpha$ in $S_{0,2g+2}$ under the map $q$. Then $\hat{\alpha} \in \mathcal{A}(S_g)$, that is $\hat{\alpha}$ is a simple closed curve, and the assumption implies that $d_\mathcal{C}(\hat{\alpha}, (\hat{f})^m(\hat{\alpha})) = 1$. The claim follows from Lemma 2.1.

□
Figure 8: (1) $\beta = \sigma_1^{-2}\sigma_2 \in B_3$, (2) $\mathbb{Z}$-cover $\tilde{S}$ over $S = D_3$ corresponding to the dual to $c = c_1$. (3) $c$, $\beta(c)$ and $\beta^2(c)$.

Figure 9: Illustration of $h^2\tilde{\psi} : \tilde{S} \to \tilde{S}$. Shaded regions in (1)(2) and (3) are $\Sigma_i$, $\tilde{\psi}(\Sigma_i)$ and $h^2\tilde{\psi}(\Sigma_i)$ respectively.
3.3 Proofs of Theorems B and C

Proof of Theorem B (1). We consider the pseudo-Anosov braid $\beta = \sigma_1^{-2}\sigma_2 \in B_3$ (Figure 3(1)) and the fibered hyperbolic 3-manifold $M_\beta$. We take a fiber $S = D_3$ with monodromy $\psi = \beta$ of $M_\beta$. Let $\xi_0 \in H^1(S; \mathbb{Z})$ be the primitive cohomology class which is dual to the homology class of the proper arc $c = c_1$ in $S$ (Figure 3(1)). Observe that the induced homomorphism $\psi_* : H_1(D_3; \mathbb{Z}) \to H_1(D_3; \mathbb{Z})$ maps the generator $t_1$ to itself. This tells us that $\xi_0$ is fixed by $\psi$. Figure 3(2) illustrates the $\mathbb{Z}$-cover $\tilde{S}$ corresponding to $\xi_0$. We consider the canonical lift $\tilde{\psi} : \tilde{S} \to \tilde{S}$ of $\psi$ which means that $\tilde{\psi}$ fixes the preimage $p^{-1}(\partial D)$ of the (outer) boundary of the 3-punctured disk pointwise. (In Figure 3(1)(2), the set $p^{-1}(\partial D) \cap \Sigma_i$ is thickened.) Choose a lift $\tilde{c}(i)$ of $c$ under the projection $p : \tilde{S} \to S$ so that $\tilde{c}(i) = \Sigma_{i-1} \cap \Sigma_i$, see Figure 3(2). In other words, $\tilde{c}_i$ and $\tilde{c}_{i+1}$ bound the copy $\Sigma_i$. The proper arc $\beta(c)$ (see Figure 3(3)) determines the image $\tilde{\psi}(\tilde{c}_i)$ for $i \in \mathbb{Z}$. (Observe (from Figure 3(1) and (2))) that

$$\tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i \quad \text{and} \quad \tilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_i \cup \Sigma_{i+1}.$$ 

Hence for each $n \geq 0$

$$h^n\tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1+n} \cup \Sigma_{i+n} \quad \text{and} \quad (h^n\tilde{\psi})^{-1}(\Sigma_i) = h^{-n}\tilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_{i-n} \cup \Sigma_{i-n+1}.$$ 

For $\ell > 0$, we have

$$(h^n\tilde{\psi})^\ell(\Sigma_i) \subset \Sigma_{i-\ell+\ell n} \cup \cdots \cup \Sigma_{i+\ell n} \cup \Sigma_{i+\ell n},$$

$$(h^n\tilde{\psi})^{-\ell}(\Sigma_i) \subset \Sigma_{i-\ell n} \cup \Sigma_{i-\ell n+1} \cup \cdots \cup \Sigma_{i-\ell n+\ell}.$$ 

Notice that if we fix $n \geq 2$, then $(h^n\tilde{\psi})^\ell(\Sigma_i) \cap \Sigma_i = \emptyset$ for each $\ell > 0$, and hence $R_n = \tilde{S}/(h^n\tilde{\psi})$ is a surface. In fact $R_n$ is a disk with $2n$ punctures, and hence we can think of $R_n$ as a sphere with $2n + 1$ punctures. See Figures 9 and 10. Note that one of the punctures of $R_n$, say $p_\infty$ comes from the preimage of the boundary of the disk under the projection $p : \tilde{S} \to S = D_3$. By Theorem 1.1 we know $h^{-1}$ descends to the monodromy $\psi_n$, and we see that $\psi_n$ maps $p_\infty$ to itself. Thus $\psi_n \in \epsilon(\text{Mod}(D_3))$. By Theorem 4 we have $\ell_C(\psi_n) \leq C/n^2$ for some constant $C$, and hence $L_C(\epsilon(\text{Mod}(D_2n))) \leq C/n^2$.

We turn to the pseudo-Anosov braid $\phi = \beta^2 \in B_3$. The hyperbolic fibered 3-manifold $M_\phi$ has a fiber $S = D_3$ with monodromy $\phi$. The dual to $c = c_1$ is the primitive cohomology class fixed by $\phi$. Consider $\mathbb{Z}$-cover $\tilde{S}$ corresponding to this cohomology class. We set $\tilde{\phi} = (\tilde{\psi})^2 : \tilde{S} \to \tilde{S}$ which is the canonical lift of $\phi$. By using the proper arc $\phi(c) = \beta^2(c)$ (see Figure 3(3)), we find how each copy $\Sigma_i$ maps on $\tilde{S}$ under $\tilde{\phi}$. By the same argument as above, we see that $\tilde{S}/(h^n\tilde{\phi})$ is a sphere with $2n + 2$ punctures which is a fiber of $M_\phi$ for $n$ large. Also we see that $\phi_n$ fixes one of the punctures of the fiber (which comes form the preimage of the boundary of the disk). Thus $\phi_n \in \epsilon(\text{Mod}(D_2n+1))$. By Theorem 4 we have $\ell_C(\phi_n) \leq C'/n^2$ for some constant $C' > 0$. This tells us that $L_C(\epsilon(\text{Mod}(D_2n+1))) \leq C'/n^2$. This completes the proof. \qed
Figure 10: (1) Shaded region descends to $R_2 \simeq S_{0.5}$. See also Figure 9. (2) Shaded region descends to $R_3 \simeq S_{0.7}$. (Note that $[\tilde{c}_i] = [h^n\tilde{\psi}(\tilde{c}_i)]$ in $R_n$.)

Figure 11: Two small circles indicate punctures of $S_{1.2}$. (1) A basis $\alpha, \beta, \gamma \in H_1(S_{1.2};\mathbb{Z})$. (2) $m, \ell$ in $S_{1.2}$. (3) Image of $c$ under $\psi = T_m^{-1}f_\ell$. 

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and 11(3). By the same argument as in the proof of Theorem B(1), we verify that we prove it by using Theorem A for the convenience of readers. Let \( L \) be as before, i.e., \( L = \mathbb{Z} \) and \( \xi \) be a simple closed curve in \( S \). Consider the mapping torus \( M = \psi \S \). The induced map \( \psi \) is pseudo-Anosov and fixed by \( \psi \). Then the cohomology class \( a \) is dual to \( \xi \). For more details.

We set \( \tilde{\psi} : \tilde{S} \to \tilde{S} \) of the lift \( \tilde{\psi} \) with \( \tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i \). A copy \( \Sigma_i \) and \( \tilde{\psi}(\Sigma_i) \) are shaded.

**Proof of Theorem B(2).** Let \( L_W \) be the Whitehead link in \( S^3 \). The complement \( S^3 \setminus L_W \) is a fibered hyperbolic 3-manifold with a fiber \( S_{1,2} \). We now recall its pseudo-Anosov monodromy \( \psi \) defined on the fiber \( S_{1,2} \), see [KR, Appendix B] for more details.

We use a basis \( \alpha, \beta, \gamma \in H_1(S_{1,2}; \mathbb{Z}) \) in Figure 11(1). Let \( m \) be a be simple closed curve in \( S_{1,2} \), and \( \ell \) an oriented loop based at one of the punctures of \( S_{1,2} \). Let \( c \) be a representative of the generator \( \beta \in H_1(S_{1,2}; \mathbb{Z}) \), see Figure 11. We set \( \psi = T_m^{-1}f_\ell \in \text{Mod}(S_{1,2}) \) where \( f_\ell \) is the mapping class which represents the point-pushing map along \( \ell \), see Figure 11(3). Then \( \psi \) is the monodromy of a fibration on \( S^3 \setminus L_W \), i.e., \( M_\psi \) is homeomorphic to \( S^3 \setminus L_W \). In particular \( \psi \) is pseudo-Anosov since \( S^3 \setminus L_W \) is hyperbolic. Observe that the induced map \( \psi_* : H_1(S_{1,2}; \mathbb{Z}) \to H_1(S_{1,2}; \mathbb{Z}) \) sends \( a, \beta \) and \( \gamma \) to \( \alpha - \beta - \gamma, \beta + \gamma \) and \( \gamma \) respectively. Then the cohomology class \( \xi_0 \in H^1(S_{1,2}; \mathbb{Z}) \) which is dual to \( c \) is primitive and fixed by \( \psi \). We consider \( \mathbb{Z} \)-cover \( \tilde{S} \) over \( S = S_{1,2} \) corresponding to \( \xi_0 \), and we take a lift \( \tilde{\psi} : \tilde{S} \to \tilde{S} \) such that \( \tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i \), see Figures 12 and 11(3). By the same argument as in the proof of Theorem B(1), we verify that \( R_n \) is a torus with \( 2n + 1 \) punctures if \( n \geq 2 \). By Theorem A we conclude that \( L_c(\text{Mod}(S_{1,2n+1})) \leq C/n^2 \) for some constant \( C > 0 \).

It is observed by Gadre and Tsai [GT11] that \( L_c(\text{Mod}(S_{1,2n})) \propto 1/n^2 \). We prove it by using Theorem A for the convenience of readers. Let \( a \) and \( b \) be simple closed curves in \( S_{1,2} \) as in Figure 13(1), and let \( c \) be as before, i.e., \( \beta = [c] \). Consider \( \psi = T_{-a} \in \text{Mod}(S_{1,2}) \) which is pseudo-Anosov by Penner’s criterion. The induced map \( \psi_* \) maps a basis \( a, \beta \) and \( \gamma \) of \( H_1(S_{1,2}; \mathbb{Z}) \) to \( \alpha + \beta + \gamma \), and \( \gamma \), respectively. Thus \( \psi \) fixes a primitive cohomology class \( \xi_0 \in H^1(S_{1,2}; \mathbb{Z}) \) which is dual to \( a \). Consider \( \mathbb{Z} \)-cover \( \tilde{S} \) over \( S \) corresponding to \( \xi_0 \) (Figure 13(2)) and pick a lift of \( \tilde{\psi} : \tilde{S} \to \tilde{S} \) of \( \psi \). We can apply Theorem A for the fiber \( (S_{1,2}, \psi) \) of the mapping torus \( M_\psi \) together with \( \xi_0 \in H^1(S_{1,2}; \mathbb{Z}) \) fixed by \( \psi \). Theorem 1.1 says that for all \( n \) sufficiently large, \( R_n \) is a fiber of \( M_\psi \). In this case \( R_n \) is a torus with \( 2n + n_0 \) punctures, where \( n_0 \) is an even number which depends on the choice of the lift \( \tilde{\psi} \). By Theorem A we conclude that \( L_c(\text{Mod}(S_{1,2n})) < C'/n^2 \) for some
constant $C' > 0$. This completes the proof. \hfill $\square$

**Proof of Theorem C.** First of all we introduce spherical braids $x_{2k}, y_{2k} \in SB_{2k}$ for $k \geq 5$ as shown in Figure 14. It is easy to verify that they are elements of $SW_{2k}$. We define $w_{2k} \in SW_{2k}$ for each $k \geq 5$ as follows.

\[
\begin{align*}
  w_{4n+8} &= x_{4n+8}(y_{4n+8})^n & \text{if } 2k = 4n + 8 \text{ for some } n \geq 1, \\
  w_{4n+10} &= (x_{4n+10})^2(y_{4n+10})^n & \text{if } 2k = 4n + 10 \text{ for some } n \geq 0.
\end{align*}
\]

Consider an element in the Hilden group $SH_{2k}$ corresponding to $w_{2k}$ (see Section 3.2) and its mapping torus $M_{w_{2k}}$. In [SE17] it is shown that when $2k = 4n + 8$ for $n \geq 1$, $M_{w_{2k}}$ is homeomorphic to the mapping torus $M_w$ of the pseudo-Anosov element in $SH_6$ corresponding to the pseudo-Anosov braid $w \in SW_6$ (Figure 7(3)). In other words $M_w$ is hyperbolic and it has a fiber $S_{0,2k}$ with pseudo-Anosov monodromy $w_{2k}$ when $2k = 4n + 8$. We see that from the construction in [SE17] of these fibers of $M_w$, a sequence of fibers $(S_{0,4n+10}, w_{4n+10})$ of $M_w$ comes from Theorem A. More precisely, if we remove the 6th strand of $w$, then we obtain a spherical braid with 5 strands. Regarding such a braid as the one on the disk, we have a 5-braid, say $\psi \in B_5$. Clearly $M_{\psi}$ is homeomorphic to $M_w$. We consider a fiber $S = D_5$ with monodromy $\psi$ of the mapping torus $M_\psi \simeq M_w$. Since $\psi_5$ maps the generator $t_5$ to itself (see the 5th strand of the braid $w$ in Figure 7(3)), the cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ which is dual to the proper arc $c = c_5$ is fixed by $\psi$. Let $\tilde{S}$ be the $\mathbb{Z}$-cover of $S$ corresponding to $\xi_0$. We consider the canonical lift $\tilde{\psi} : \tilde{S} \to \tilde{S}$ of $\psi$. Then $R_n = \tilde{S}/(h^n\tilde{\psi})$ is a fiber of $M_\psi$ with monodromy $\psi_n$ for $n$ large. In this case $R_n$ is a sphere with $4n + 8$ punctures and the monodromy $\psi_n$ is given by the braid $w_{4n+8} \in SW_{4n+8}$. By the proof of Theorem A there exist
Figure 14: (1) $x_{2k} \in SW_{2k}$. (2) $y_{2k} \in SW_{2k}$.

$\alpha \in \mathcal{AC}(R_n)^0$ and $m \asymp n^2$ such that $d_{\mathcal{AC}}(\alpha, (\psi_n)^m(\alpha)) = 1$. Then by Lemma 3.1 a lift $\hat{\psi} \in \mathcal{H}(\mathbb{H}_{2n+3})$ satisfies $\ell_c(\hat{\psi}) \leq 1/m$, which implies $\ell_c(\hat{\psi}) \leq C/n^2$ for some constant $C > 0$. Thus we have $L_c(\mathcal{H}(\mathbb{H}_{2n+3})) \leq C/n^2$.

To obtain the upper bound $L_c(\mathcal{H}(\mathbb{H}_{2n+4})) \leq C'/n^2$ for some $C' > 0$, we take the second power $\psi^2 \in B_5$ of the above $\psi$ and we set $\phi = \psi^2$. We consider a fiber $S = D_5$ with monodromy $\phi$ in the mapping torus $M_\phi$. Note that $\phi$ fixes the same $\xi_0 \in H^1(S; \mathbb{Z})$. Let $\tilde{S}$ be the $\mathbb{Z}$-cover over $S$ as before and let $\tilde{\phi} = (\tilde{\psi})^2 : \tilde{S} \to \tilde{S}$ which is the canonical lift of $\phi$. Now we apply Theorem A for the fiber $(S, \phi)$ of $M_\phi$ together with $\xi_0$. By using the same argument as in [SE17], we find that for $n$ large, $\tilde{S}/(h^n\tilde{\phi})$ is a fiber of $M_\phi$ which is the sphere with $4n+10$ punctures and its monodromy is described by the braid $w_{4n+10} \in SW_{4n+10}$. In the same manner as above, we obtain the desired upper bound of $L_c(\mathcal{H}(\mathbb{H}_{2n+4}))$. This completes the proof.

A Entropies of pseudo-Anosov mapping classes

A.1 Minimal pseudo-Anosov entropies

Let $T(S)$ be the Teichmüller space of $S$ with the Teichmüller metric $d_T$. The mapping class group $\text{Mod}(S)$ acts on $T(S)$, and the translation length of $f \in \text{Mod}(S)$ on $T(S)$ is defined by

$$\ell_T(f) = \inf_{X \in T(S)} d_T(X, f(X)).$$

Each pseudo-Anosov element $f \in \text{Mod}(S)$ has a representative $\Phi : S \to S$ which satisfies the followings. There exist a pair of transverse measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ and a constant $\lambda > 1$ such that

$$\Phi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u) \quad \text{and} \quad \Phi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s).$$

Such a representative is called a pseudo-Anosov homeomorphism. The constant $\lambda > 1$ does not depend on the choice of a representative $\Phi$, and $\lambda = \lambda(f)$ is called
the stretch factor (or dilatation) of $f$. The logarithm of $\lambda(f)$ is often called the entropy of $f$. By the result of Bers [Ber78],
\[ \ell_T(f) = \log(\lambda(f)). \]

Arnoux and Yoccoz observed that if we fix $S$, then the set of entropies
\[ \{ \ell_T(f) \mid f \in \text{Mod}(S) \text{ is pseudo-Anosov} \} \]
is a closed and discrete subset of $\mathbb{R}$, see [Iva92, Lemma 2.12] for example. In particular for any subgroup $H$ of $\text{Mod}(S)$, there exists a minimum of $\ell_T(f)$ over all pseudo-Anosov elements $f \in H$. We denote the minimum by $L_T(H)$. Penner [Pen91] proved that
\[ L_T(\text{Mod}(S_g)) \asymp \frac{1}{g}. \]

Table 2 shows other asymptotic behaviors of minimal entropies in the case when $g$ is fixed as $n \to \infty$. For asymptotic behaviors of minimal entropies in the cases of the previous subgroups $\mathcal{H}(S_g)$, $\mathcal{H}(\mathbb{H}_g)$ and $\mathcal{H}(\mathbb{H}_g)$, see Table 3.

### A.2 Lipschitz constant to the curve complex

Let $S = S_g$ and let
\[ \text{sys} : \mathcal{T}(S) \to C^1(S) \]
be the systole map which assigns a hyperbolic metric one of its shortest simple closed curves. Then syst is $(K,C)$-coarsely Lipschitz for some $K,C > 0$, that is, for all $X$ and $Y$ in $\mathcal{T}(S)$, we have
\[ d_c(\text{sys}(X),\text{sys}(Y)) \leq Kd_T(X,Y) + C. \]

See Masur–Minsky [MM99] for detail. Gadre, Hironaka, Kent and Leininger [GHKL13] study the optimal Lipschitz constant of the systole map
\[ \kappa_g = \inf\{K \geq 0 \mid \text{sys is } (K,C)\text{-coarsely Lipschitz for some } C > 0\}, \]
and they establish
\[ \kappa_g \asymp \frac{1}{\log(g)}. \]
To prove this, the authors give the upper bound \( \kappa_g \leq C/\log(g) \) for some constant \( C > 0 \). For the lower bound of \( \kappa_g \), they consider the ratio
\[ r(f) = \frac{\ell_C(f)}{\ell_T(f)} \]
and prove the inequality
\[ r(f) \leq \kappa_g \]
for any pseudo-Anosov \( f \in \text{Mod}(S_g) \). Then they construct a sequence of pseudo-Anosov mapping classes \( f_g \in \text{Mod}(S_g) \) which satisfies \( \ell_C(f_g) \asymp 1/g \) and \( \ell_T(f_g) \asymp \log(g)/g \). These imply that \( r(f_g) \asymp 1/\log(g) \).

Now consider a sequence \((R_n, \psi_n)\) of fibers in Theorem A under the assumption that the fibered hyperbolic 3-manifold \( M \) is closed. In this case \( R_n \) is a closed surface with the genus \( g(R_n) \asymp n \). By Theorems A and 1.1, for all \( n \) sufficiently large, we have \( \ell_C(\psi_n) \asymp 1/n^2 \) and \( \ell_T(\psi_n) \asymp 1/n \), and hence
\[ r(\psi_n) \asymp \frac{1}{n}. \]
This means that the ratio \( r(\psi_n) \) is strictly smaller than the optimal constant \( \asymp 1/\log(n) \) of the systole map \( \text{sys} : T(R_n) \to C^1(R_n) \).

**B Proof of Proposition 1.2**

In this appendix, we improve the upper bound of \( L_C(S_g) \).

**Proposition 1.2** For \( g \geq 3 \), we have
\[ L_C(S_g) \leq \frac{1}{g^2 - 2g - 1}. \]

In the proof of Theorem B1, we showed that \( M_\beta = M_{\sigma_1^{-2}, \sigma_2} \), so called the magic manifold, admits a sequence \((R_n, \psi_n)\) of the fiber \( R_n = D_{2n} \) and the monodromy \( \psi_n \) for \( n \geq 2 \). We use this sequence for the proof. Terminology related to train tracks can be found in [BH95] or [FM12] for example.

We think of \( R_n \) as a sphere with \( 2n + 1 \) punctures. An invariant train track \( \tau_n \) and a train track representative \( p_n : \tau_n \to \tau_n \) of \( \psi_n : S_{0,2n+1} \to S_{0,2n+1} \) are studied in [Kim15, Example 4.6]. Figure 13 shows the train track \( \tau_n \subset S_{0,2n+1} \) and its image \( \psi_n(\tau_n) \). Each of monogons of \( S_{0,2n+1} \setminus \tau_n \) (bounded by loop edges of \( \tau_n \)) contains a puncture of \( S_{0,2n+1} \), the \((n-1)\)-gon of \( S_{0,2n+1} \setminus \tau_n \) contains another puncture, and the other connected component of \( S_{0,2n+1} \setminus \tau_n \) contains the other puncture \( p_\infty \) in the proof of Theorem B1. Recall that \( \psi_n \) maps \( p_\infty \) to
Figure 15: Small circles indicate punctures of $S_{0, 2n+1}$. (1) Train track $\tau_n$. (2) $\psi_n(\tau_n)$, where $e' = \psi_n(e)$. ($p_\infty$ is not drawn here.)

itself. Figure 16 gives the directed graph $\Gamma_n$ of $\psi_n(\tau_n)$. We first prove the following.

**Proposition B.1.** For $n \geq 4$, we have

$$L_C(\epsilon(\text{Mod}(D_{2n-1}))) \leq \frac{1}{n^2 - 4n + 2} \quad \text{and} \quad L_C(\epsilon(\text{Mod}(D_{2n}))) \leq \frac{1}{n^2 - 4n + 2}.$$  

In particular for $n \geq 4$,

$$L_C(S_{0, 2n}) \leq \frac{1}{n^2 - 4n + 2} \quad \text{and} \quad L_C(S_{0, 2n+1}) \leq \frac{1}{n^2 - 4n + 2}.$$  

**Proof.** We assume $n \geq 4$. Let $\mathcal{N}(\tau_n) \subset S_{0, 2n+1}$ be a fibered neighborhood of $\tau_n$ (see [PP87, page 360] for the definition) equipped with a retraction $\mathcal{N}(\tau_n) \searrow \tau_n$. For a connected subset $\tau' \subset \tau_n$, we define a fibered neighborhood $\mathcal{N}(\tau')$ of $\tau'$ as follows.

$$\mathcal{N}(\tau') = \mathcal{N}(\tau_n) \cap U(\tau'),$$

where $U(\tau')$ is a small neighborhood of $\tau'$ in the 2-sphere $S^2$. We denote by $r$, $p_1$, $q_1$, $\cdots$, $p_{n-1}$, $q_{n-1}$, the non-loop edges of $\tau_n$ as shown in Figure 15. We take

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n points \( v_0, v_1, v_2, \ldots, v_{n-1} \subset \tau_n \), each of which lies on an infinitesimal edge of the \((n-1)\)-gon, see Figure 17(1). For \( 1 \leq i < j \leq n-1 \), let \( \tau(i, j) \) be the connected component of \( \tau_n \setminus \{v_i, v_j\} \) containing \( p_i, q_i, p_{i+1}, q_{i+1}, \ldots, p_j, q_j \), see Figure 17(2). Let \( \mathcal{N}(p_i q_i p_{i+1} q_{i+1} \cdots p_j q_j) = \mathcal{N}(\tau(i, j)) \)

For \( 1 \leq j \leq n-2 \), let \( \tau(j) \) be the connected component of \( \tau_n \setminus \{v_j, v_{n-1}\} \) containing \( r, p_1, q_1, \ldots, p_j, q_j \), see Figure 17(3). Let \( \mathcal{N}(rp_1 q_1 \cdots p_j q_j) = \mathcal{N}(\tau(j)) \).

We take an essential arc \( c \) connecting the two punctures as in Figure 18(1). Then \( c \) is carried by \( \tau_n \). Notice that if \( i \geq 2 \), then \( \mathcal{N}(p_i q_i p_{i+1} q_{i+1} \cdots p_{n-1} q_{n-1}) \) is disjoint from \( c \). Since \( c \subset \mathcal{N}(rp_1 q_1) \), we have

\[
\psi_n^1(c) \subset \mathcal{N}(p_1 q_1 p_2 q_2), \\
\psi_n^2(c) \subset \mathcal{N}(p_2 q_2 p_3 q_3), \\
\vdots \\
\psi_n^{1+(n-3)}(c) = \psi_n^{n-2}(c) \subset \mathcal{N}(p_{n-2} q_{n-2} p_{n-1} q_{n-1})
\]

(see Figures 16 and 18). Observe that \( \psi_n^2(\psi_n^{n-2}(c)) = \psi_n^n(c) \subset \mathcal{N}(rp_1 q_1 p_2 q_2) \). We have

\[
\psi_n^{n+1}(c) \subset \mathcal{N}(p_1 q_1 p_2 q_2 p_3 q_3), \\
\vdots \\
\psi_n^{(n+1)+(n-4)}(c) = \psi_n^{2n-3}(c) \subset \mathcal{N}(p_{n-3} q_{n-3} p_{n-2} q_{n-2} p_{n-1} q_{n-1})
\]

In the same manner, we have for \( 2 \leq k \leq n-2 \),

\[
\psi_n^{(k-1)n-k}(c) \subset \mathcal{N}(p_{n-k} q_{n-k} \cdots p_{n-1} q_{n-1})
\]
Figure 17: (1) Points $v_0, v_1, \ldots, v_{n-1}$. (2) $\tau(2, n-1) \subset N(p_2q_2 \cdots p_{n-1}q_{n-1})$. (3) $\tau(1) \subset N(rp_1q_1)$.

When $k = n-2$,

$$\psi_n^{(n-3)n-(n-2)} = \psi_n^{n^2-4n+2} \subset N(p_2q_2 \cdots p_{n-1}q_{n-1}).$$

Hence

$$d_{AC}(c, \psi_n^{n^2-4n+2}) = 1.$$

If we consider a regular neighborhood of $c$ in $S^2$, then we obtain an essential simple closed curve $\alpha$ in $S^0_{2n+1}$ as the boundary of the neighborhood in question. Notice that $\alpha$ is also carried by $\tau_n$ and $\alpha \subset N(rp_1q_1)$. The above argument shows that $\psi_n^{n^2-4n+2}(\alpha) \subset N(p_2q_2 \cdots p_{n-1}q_{n-1})$ and $\alpha$ is disjoint from $\psi_n^{n^2-4n+2}(\alpha)$. Recall that $\psi_n$ is defined on $R_n = D_{2n}$. This together with Lemma 2.1 implies that

$$L_C(\psi(\text{Mod}(D_{2n}))) \leq \frac{1}{n^2-4n+2}.$$

To show $L_C(\psi(\text{Mod}(D_{2n-1}))) \leq \frac{1}{n^2-4n+2}$, we fill the hole (i.e., puncture) in the $(n-1)$-gon of $S_{0,2n+1} \setminus \tau_n$. The assumption $n-1 \geq 3$ ensures that $\tau_n$ extends to a train track $\tau_n$ in $S_{0,2n}$ and $\psi_n : S_{0,2n+1} \to S_{0,2n+1}$ extends to $\psi_n : S_{0,2n} \to S_{0,2n}$ which is still pseudo-Anosov. In particular $\psi_n$ maps the puncture $p_\infty$ to itself. By abuse of notation, we can think of $\psi_n : S_{0,2n} \to S_{0,2n}$ as an element of $\text{Mod}(D_{2n-1})$. The train track representative $p_n : \tau_n \to \tau_n$ also extends to a train track representative $\overline{p}_n : \overline{\tau}_n \to \overline{\tau}_n$ of $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$. All non-loop edges of $\overline{\tau}_n$ are coming from those of $\tau_n$, and hence the directed graph $\Gamma_n$ for $\overline{p}_n : \overline{\tau}_n \to \overline{\tau}_n$ is the same as $\Gamma_n$ for $p_n : \tau_n \to \tau_n$. For the arc $\overline{\tau}$ and the simple
closed curve $\gamma$ in $S_{0,2n}$ coming from $c$ and $\alpha$ in $S_{0,2n+1}$, the above argument on $c$ and $\alpha$ tells us that

$$d_{AC}(\gamma, \psi_n^{2-4n+2}(\gamma)) = 1 \quad \text{and} \quad d_C(\gamma, \psi_n^{2-4n+2}(\gamma)) = 1. \quad (B.1)$$

The last equality in (B.1) together with Lemma 2.1 gives the desired upper bound.

\[\square\]

**Proof of Proposition 1.2.** By Lemma 3.1 together with the first equality in (B.1) about $\psi_n : S_{0,2n} \to S_{0,2n}$, we have $L_C(S_{n-1}) \leq \frac{1}{n-4n+2}$ for $n \geq 4$. Thus for $g \geq 3$,

$$L_c(S_g) \leq \frac{1}{(g+1)^2 - 4(g+1) + 2} = \frac{1}{g^2 - 2g - 1}.$$

\[\square\]

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