Abstract

We define the notion of A-model Lagrangian D-branes as introducing defects in the Calabi-Yau crystal. The crystal melting in the presence of these defects reproduces all genus string amplitudes as well as leads to additional non-perturbative terms.
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## 1 Introduction

In [1] a duality between topological A-type string theory and statistical model of a three dimensional melting crystal was proposed. In particular it was demonstrated that partition function $Z_{\text{crystal}}$ of the melting crystal corner computes the closed string partition function $Z_{\text{closed}}$ in the $\mathbb{C}^3$ background. More precisely,

$$Z_{\text{crystal}}(q) = Z_{\text{closed}}(q), \quad q = e^{x \rho(-g_s)} \tag{1}$$

where the string coupling constant $g_s$ is identified with the inverse of the temperature of the crystal. Also in [1] it was explained how to compute the topological vertex [2] in the crystal picture. In [4] this duality was interpreted as coming from quantum gravitational foam for target description of the A-model strings, which is a “quantum Kähler gravity”. Each geometry contributing to the quantum foam corresponds to a configuration of molten crystal (roughly speaking each atom removal from the crystal changes the quantum foam by blowing up of a corresponding point). It was proposed in [4] that in the limit $g_s \ll 1$ the quantum foam (the melting crystal) is a fundamental description of physics at short distances $\sim g_s^2 l_s$. 
String emerges in the high temperature limit \((1/T = g_s << 1)\) as an effective description of the theory at the string scale \(l_s\). A related mathematical proposal was also presented in [5]. Further developments have shed light on this duality: It was conjectured in [6] that A and B models on the same Calabi-Yau are S-dual to one another. Moreover it was shown in [7] that this S-duality follows from type IIB S-duality and that the atoms of the crystal can also be viewed as D(-1) instantons of the B-model. See also [9].

In this note we continue to investigate the proposed duality and would like to give the statistical interpretation for non-compact Lagrangian D-branes in the context of \(\mathbb{C}^3\). It turns out that the statistical interpretation of Lagrangian D-branes is introducing defects to the crystal. For the case of branes at some particular framing we find that the topological string amplitudes are identical to that of melting crystal in the presence of the defects introduced by D-branes. We also find the defect interpretation of the lattice for arbitrary framing and also in the presence of anti-D-branes. However for these cases the corresponding melting rule needs to be modified, similar to the melting rule in the context of local toric 3-folds (when we have more than one corner for the lattice). We will make some comments about what this melting rule would look like for these more general branes, but will not develop it in full detail in this paper.

The organization of this paper is as follows: In section 2 we review the results of [1]. In section 3 we present the geometry of modified crystal for branes and anti-branes and also give melting crystal picture for certain configurations of D-branes and anti-D-branes. In section 4 we discuss general brane and anti-brane configurations at arbitrary framing. In section 5 we show that our melting crystal picture of the D-branes is compatible with the semiclassical limit of small string coupling.

## 2 Review of the closed topological strings/melting crystal duality

In this section we review the duality between closed A-model topological string theory on \(\mathbb{C}^3\) and statistical model of melting crystal corner proposed in [1]. The A-model topological
string theory studies holomorphic maps from Riemann surfaces to the Kähler target space.

\[ \Phi : \Sigma \to X \]  

The partition function of this theory is given by

\[ Z_A = \exp\left( \sum g_s^{2g-2} F_g \right), \quad F_g = \int_{M_{\text{maps},g}} e^{-A_C} \]  

where \( A_C \) is the area of the holomorphic curve \( C = \Phi(\Sigma_g) \) of genus \( g \), and integration goes over the moduli space of maps \( M_{\text{maps},g} \) with a suitable measure. Here \( g_s \) denotes the string coupling constant. When the volume of \( X \) is large the leading contribution to \( Z_A \) comes from constant maps and the moduli space becomes a product \( M_{\text{maps},g} = M_g \times X \), where \( M_g \) is the moduli space of genus \( g \) curves. In this large volume limit the expressions for \( F_g \) were obtained in [11, 12]. As follows from the results of [11, 12] the closed string partition function on \( \mathbb{C}^3 \) (taking its Euler character to be 2) is given by the McMahon function:

\[ Z_{\text{closed}}(q) = M(q) := \prod_{n=1}^{\infty} (1 - q^n)^{-n}, \quad q = e^{-g_s} \]  

Now let us recall how the same function arises in the dual melting crystal picture [1]. Consider cubic lattice in the positive octant of \( R^3 \) with lattice spacing given by \( g_s \) and define the rules of melting. One can remove atom with coordinates \((x_0, y_0, z_0)\) from the lattice if and only if all the following sites are already vacant:

\[ (x, y_0, z_0) \quad x < x_0, \quad (x_0, y, z_0) \quad y < y_0, \quad (x_0, y_0, z) \quad z < z_0 \]  

With these rules, each configuration of \( k \) molten atoms corresponds to a “3d Young diagram” (a 3d partition) of \( k \) boxes in the positive octant of \( R^3 \). Now we sum over the random 3d partitions with the weight \( q^{\#\text{boxes}} \), where \( q = e^{-g_s} \) and \( g_s \) stands for inverse temperature of the crystal (measured in units of chemical potential for removing atoms). To compute

\[ Z_{\text{crystal}}(q) = \sum_{3d \text{ partition } \pi} q^{\#\text{boxes}} \]  

we recall that, as discussed in [10], a 3-dimensional partition \( \pi \) can be thought as a sequence of 2d partitions \( \{\mu(t)\}, t \in \mathbb{Z} \) obtained from diagonal slicing of \( \pi \) by planes \( x - y = t \). These 2d partitions obey the interlacing condition:

\[ \mu(t) < \mu(t + 1), \quad t < 0, \quad \mu(t + 1) < \mu(t) \quad t \geq 0 \]
where the 2d partitions $\mu$ and $\nu$ interlace, and we write $\mu > \nu$, if

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \ldots$$  \hspace{2cm} (8)

Here and below we let $\mu_k$ denote the number of squares in the k-th row of $\mu$. The diagonal slicing allows us to use a well known map from 2d partitions to states in the NS sector of a complex fermion:

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \{\psi_{n+rac{1}{2}}, \psi^*_{n+rac{1}{2}}\} = \delta_{m,n}$$  \hspace{2cm} (9)

The state corresponding to a partition $\mu$ is given by

$$|\mu\rangle = \prod_{i=1}^d \psi^*_{-a_i} \psi_{-b_i} |0\rangle$$  \hspace{2cm} (10)

where $d$ is the number of squares on the diagonal of $\mu$ and, denoting the transposed 2d partition by $\mu^T$, we define

$$a_i = \mu_i - i + \frac{1}{2}, \quad b_i = \mu^T_i - i + \frac{1}{2}$$  \hspace{2cm} (11)

The example of the state/2d partition correspondence is shown in Figure 1.

Now, $Z_{\text{crystal}}(q)$ can be easily recast in the operator language:

$$Z_{\text{crystal}}(q) = \langle 0 | \left( \prod_{t=0}^{\infty} q^{L_0} \Gamma_+(1) \right) q^{L_0} \left( \prod_{t=-\infty}^{-1} \Gamma_-(1) q^{L_0} \right) |0\rangle$$  \hspace{2cm} (12)

Here $|0\rangle$ is the state annihilated by all positive modes of $\psi$ and $\psi^*$ and operators $L_0, \Gamma_\pm(1)$ act in the following way.

$$L_0 |\mu\rangle = |\mu||\mu\rangle$$  \hspace{2cm} (13)

$$\Gamma_-(1) |\mu\rangle = \sum_{\nu > \mu} |\nu\rangle, \quad \Gamma_+(1) |\mu\rangle = \sum_{\nu < \mu} |\nu\rangle$$  \hspace{2cm} (14)

where $|\mu\rangle$ stands for the number of squares in 2d partition $\mu$. The final step in computing $Z_{\text{crystal}}(q)$ is to rewrite the correlator (12) in the form

$$Z_{\text{crystal}}(q) = \langle 0 | \prod_{n=1}^{\infty} \Gamma_+ \left( q^{n-\frac{1}{2}} \right) \prod_{m=1}^{\infty} \Gamma_- \left( q^{-m+\frac{1}{2}} \right) |0\rangle$$  \hspace{2cm} (15)
Figure 1: We draw 2d partition $\mu$ in a diagonal slice at $t = -2$. The numbers of rows are $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 1$. The fermion state is $|\mu> = \psi_{-7/2}^* \psi_{-5/2}^* \psi_{-1/2}^* \psi_{-1/2}|0>$. 

where

$$\Gamma_+(q^{n-\frac{1}{2}}) = q^{-(n-\frac{1}{2})L_o} \Gamma_+(1) q^{(n-\frac{1}{2})L_o}, \quad \Gamma_-(q^{-n+\frac{1}{2}}) = q^{-(n+\frac{1}{2})L_o} \Gamma_-(1) q^{(-n+\frac{1}{2})L_o} \quad (16)$$

We will say more about the operators $\Gamma_{\pm}(z)$ and their simple realization in the bosonized picture in the next section. Here we only need their commutation relation

$$\Gamma_+(z) \Gamma_-(z') = \left(1 - z/z'\right)^{-1} \Gamma_-(z') \Gamma_+(z) \quad (17)$$

which leads immediately to:

$$Z_{\text{crystal}}(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}, \quad q = e^{-gs} \quad (18)$$

Comparing (4) with (18) finishes the proof of the duality (1) for closed strings. In [1] it was shown how one can formulate the topological vertex amplitudes [2] in terms of crystal melting using the above operator formalism.


3 String/Crystal duality with D-branes

In this section we give the melting crystal interpretation of D-branes in topological A-type string theory on $\mathbb{C}^3$. We first recall the relevant results from string theory. In particular we explain how the D-branes affect the integral of Kähler form around 2-cycles surrounding D-branes and we use this to formulate the D-branes in the crystal melting problem as introducing defects. We show that the amplitudes for D-branes agree to all orders in string perturbation with the melting crystal picture. We also get additional non-perturbative pieces, suggesting a non-perturbative completion of string theory amplitudes.

3.1 D-branes in topological A-model string theory on $\mathbb{C}^3$

One may view $\mathbb{C}^3$ as a $T^3$ fibration. Let $(z_1, z_2, z_3)$ denote complex coordinates of $\mathbb{C}^3$ and let

$$(x, y, z) = (|z_1|^2, |z_2|^2, |z_3|^2)$$

be the coordinates on the base $\mathcal{O}^+$ which is the positive octant in $R^3$. We denote the coordinates on the fiber $T^3$ by $\theta_i$, $i = 1, 2, 3$. Now consider a Lagrangian D-brane which has the
geometry given by

\[ y = x + a = z + a, \quad \theta_1 + \theta_2 + \theta_3 = 0, \quad a > 0. \]  

(19)

The D-brane has topology \( S^1 \times \mathbb{R}^2 \) and ends on the y-axis at \( y = a \) (see Figure 2). As discussed in [2] there is back-reaction of the geometry to the presence of the D-brane. Namely, the periods of the Kähler form \( K \) are changed by the amount

\[ \Delta \int_{\Sigma} K = g_s \]

(20)

where \( \Sigma \) is any 2-cycle linking the D-brane world-volume. This back-reaction is going to be our guide in search for crystal interpretation of the brane.

Also, as explained in [14], in order to define quantum theory on a non-compact D-brane one has to specify boundary conditions at infinity called the “framing” of the D-brane. The trick is to modify the background by introducing extra direction \( \vec{f} \) in the toric base over which one of the cycles of the \( T^3 \) fiber degenerates along a “brane” in that direction at infinity. There is an integral ambiguity in the choice of \( \vec{f} \):

\[ \vec{f} \rightarrow \vec{f} - p\vec{v}_y \]

\( p \) is called the framing number. This leads to the notion of a preferred plane containing the D-brane. Let’s call this plane the D-brane plane \( P_D \). A choice of the framing is also equivalent to the choice of the plane \( P_D \) containing the D-brane and the choice of one of the two directions \( m_1, m_2 \). This follows from the orientation of \( \vec{f} \) versus \( -\vec{f} \). The concept of framing is illustrated in Figure 3. We will show that the notion of framing enters naturally in the crystal melting picture of the D-brane.

The choice of \( p = 0 \) framing in the convention of [14]' is the simplest one. The partition function of the topological A-model string theory on \( \mathbb{C}^3 \) in the presence of the D-brane (anti D-brane) with geometry [9] and framing \( p = 0 \) is given by [13]

\[ Z^{\text{string}}_{D}(q, a) = M(q) \prod_{n=1}^{\infty} \left( 1 - e^{-aq^{n-\frac{1}{2}}} \right), \quad q = e^{-g_s} \]

(21)

\(^1p = -1\) in the convention of [2]
Figure 3: The D-brane with zero framing. Plane $P_D$ contains the projection $D$ of the D-brane to $O^+$. 
\[ Z_{D}^{\text{string}}(q, a) = M(q) \prod_{n=1}^{\infty} \left( 1 - e^{-aq^{n-\frac{1}{2}}} \right)^{-1} \]  

where \( M(q) \) is the McMahon function (4). Note that since the brane amplitude at framing \( p \) is equivalent to anti-brane amplitude at framing \( 1 - p \) the above equations can also be viewed as statement for anti-brane (brane) with framing number 1.

Also, note that free energy \( F = -\log Z_{D}^{\text{string}}(q, a) \) has the following form:

\[ F = \sum_{n=1}^{\infty} \frac{e^{-na}}{2n\sinh(n g_s/2)} \]  

so that perturbative expansion \( F_g \sim (g_s/a)^{2g-1} \) breaks down for \( a \sim g_s \). Nevertheless, the partition function \( Z_{D}^{\text{string}}(q, a) \) is well defined for \( a \sim g_s \) and it is natural to ask what this sum computes. To find the answer to this question we are going to study the interpretation of D-branes in the melting crystal.

### 3.2 D-branes as defects in the melting crystal

How can one detect the presence of a D-brane in the crystal melting picture? To answer this question we would like to use the important information about the back-reaction of the geometry (20). Let us first recall the role of the Kähler form \( K \) in the crystal picture. It was discovered in [4] that the melting crystal partition function \( Z_{\text{crystal}}(q) \) arises from the sum over generalized\(^2\) Kähler geometries on \( \mathbb{C}^3 \) with Kähler forms \( K \) quantized in units of \( g_s \).

From this viewpoint the number of atoms removed from the corner can be thought of as the change in the number of holomorphic sections of the \( U(1) \) bundle whose curvature is \( K/g_s \).

Now we introduce a D-brane of geometry (19). The projection of the brane to the base and the framing vector define a plane \( P_D \) in \( O^+ \). This plane intersects \( z = 0 \) plane along the half-line \( m_1 : y = a + (1 + p)x \) and \( x = 0 \) plane along the half-line \( m_2 : y = a - pz \) (see Figure 3). There is a natural choice of a 2-cycle linking this D-brane which we denote by \( \Sigma \) dictated by the choice of framing: This cycle is a difference of planes whose projection to the toric base gives a pair of half-lines \( L_1, L_2 \) parallel to \( P_D \) in the direction of \( m_1 \) (fixed by the choice of framing). We postulate that this gives rise to a ‘closed cycle’ at infinity.

\(^2\)In a sense that the sum includes non-geometric excitations required for a consistent quantum theory
Figure 4: The half-lines $L_1$ and $L_2$ enter the definition of the 2-cycle $\Sigma$. A cross stands for vacancy and dot for atom.

$\Sigma = L_2 - L_1$, compatible with the fact that at infinity the framing of the D-brane chooses an extra vanishing direction of a circle. Here $L_1, L_2$ are given by

$$L_1 = (x, y_1 + (1 + p)x, z^*), \quad L_2 = (x, y_2 + (1 + p)x, z^*)$$

(24)

where $y_1, y_2$ and $z^*$ are fixed nonnegative integers such that $y_1 < a < y_2$. The lines $L_1$ and $L_2$ together with $\theta_1$ angle give rise to planes, which for general $p$ are not holomorphically embedded in $\mathbb{C}^3$. The period of the Kähler form $K$ along the 2-cycle $\Sigma$ is given by

$$\int_{\Sigma} K = g_s(|L_2| - |L_1|),$$

(25)

where $|L_i|$ is the number of atoms on the half-line $L_i$, $i = 1, 2$. From (25) follows that the change of the period by one in $g_s$ units, which is the characteristic signature of the D-brane, can be achieved by, for example, dropping a single atom from the half-line $L_1$. We illustrate this idea in Figure 4.

Recalling that parameters $y_1, y_2, z^*$ which specify a linking 2-cycle $\Sigma$ are arbitrary nonnegative integers (with the only constraint $y_1 < a < y_2$) we see that in the presence of a D-brane one layer of atoms is deleted from the $x = 0$ plane, in the region $0 \leq y < a$. In
other words the crystal develops an extra corner. Thus, we are led to the interpretation of D-branes as defects modifying the shape of the crystal. For many D-branes the crystal will develop many corners. Moreover the concavity or convexity of the corner is correlated with whether we have a D-brane or anti-D-brane. See Figure 5.

Note that the picture of modified crystal with D-branes at various framings makes sense as long as the corresponding $m_2$ lines do not intersect the crystal axis $z$ or the other corners developed. Thus only for these cases we can propose a simple dual crystal interpretation. Moreover the notion of linking cycle $\Sigma$ that we defined will only make sense for two choices of framings for each brane ($p = 0, -1$) otherwise the corresponding cycles intersect the boundary of the crystal. This already suggests that perhaps the rule of crystal melting may be different for general framings and this indeed is consistent with what we shall find. In fact the simplest rule is for framing $p = 0$ and that is the case we focus on.

One would naturally propose that the modification due to the D-brane in the A-model partition function is the same as the statistical mechanical model of melting crystal but now with the modified crystal $O^+/\mathcal{M}_\nu$ studied in [1, 10]. Here $\mathcal{M}_\nu$ is a cylinder with the 2d partition $\nu$ as its base. In other words, one considers 3d partitions in $O^+$ and regards the
two 3d partitions as identical if they differ only inside $\mathcal{M}_\nu$. The weight of the 3d partition is determined by the number of boxes outside $\mathcal{M}_\nu$. We will see that this is indeed the correct rule for the case of framing $p = 0$. For general framing the crystal geometry is modified as we have found, but the melting rules also need to be modified. Some aspects of this will be discussed in the next section. In the remainder of this section we concentrate on the case with $p = 0$ and show how to recover the D-brane amplitudes from the melting crystal.

In the operator language the 3d partition in the modified container is obtained from alternating evolution with $\Gamma_-$ and $\Gamma_+$ whose action on diagonal slices provide appropriate interlacing pattern. We illustrate the evolution giving “room with the corners” in Figure 5. Now we would like to check the proposed interpretation for the D-branes by explicit computation using realization of $\Gamma_\pm$ as operators in fermion Hilbert space. It turns out that this realization of D-branes is consistent with the intuition developed in the B-model topological string theory [2] that non-compact D-branes are fermions.

Let us recall [10] the representation of operators $\Gamma_\pm$ in the bosonized picture. One introduces chiral boson $\phi(z)$ such that

$$\psi^*(z) = e^{-\phi(z)} : , \quad \psi(z) = e^{\phi(z)} :$$

and expand $\phi(z)$ into zero, positive and negative modes

$$\phi(z) = \phi_0(z) + \phi_+(z) + \phi_-(z), \quad [\phi_+(z), \phi_-(w)] = \log(1 - \frac{z}{w})$$

Then $\Gamma_\pm(z)$ are given by

$$\Gamma_\pm(z) = e^{\mp \phi_0(z)}$$

Note that $\psi(z)$ and $\psi^*(z)$ are related to $\Gamma_\pm(z)$:

$$\psi^*(z) = e^{-\phi_0(z)} \Gamma_-(z) \Gamma_+^{-1}(z), \quad \psi(z) = e^{\phi_0(z)} \Gamma_-^{-1}(z) \Gamma_+(z)$$

Now we propose that introducing the D-brane of the geometry at $a = g_s(N_0 + \frac{1}{2})$ with framing $p=0$ amounts to inserting the operator

$$\Psi_D(z) = \Gamma_-^{-1}(z) \Gamma_+(z)$$
at \( z = q^{-(N_0 + \frac{1}{2})} \) in the correlator (15). We choose not to include the zero mode part \( e^{\phi_0} \) in the definition of the D-brane operator \( \Psi_D(z) \) since from the crystal viewpoint it is more natural to deal with zero fermion number correlators. As discussed in [8] disregarding zero modes corresponds to having compensating (fermion number) flux at infinity.

To verify our proposal for the D-brane operator let us compute

\[
Z^{\text{crystal}}_D(q, N_0) := \langle 0 | \prod_{n=1}^{\infty} \Gamma_+ \left( q^{n-\frac{1}{2}} \right) \prod_{m=1}^{N_0 + 1} \Gamma_- \left( q^{-m+\frac{1}{2}} \right) \Psi_D \left( q^{-(N_0 + \frac{1}{2})} \right) \prod_{m=1}^{\infty} \Gamma_- \left( q^{-m+\frac{1}{2}} \right) | 0 \rangle
\] (31)

Using our definition (30) we have

\[
Z^{\text{crystal}}_D(q, N_0) = \langle 0 | \prod_{n=1}^{\infty} \Gamma_+ \left( q^{n-\frac{1}{2}} \right) \prod_{m=1}^{N_0} \Gamma_- \left( q^{-m+\frac{1}{2}} \right) \Gamma_+ \left( q^{-(N_0 + \frac{1}{2})} \right) \prod_{m=1}^{\infty} \Gamma_- \left( q^{-m+\frac{1}{2}} \right) | 0 \rangle
\] (32)

Note that insertion of a D-brane \( \Psi_D \) creates an anticipated disorder at \( q^{-(N_0 + \frac{1}{2})} \) in a sequence of \( \Gamma_- \) operators. The commutation relation (17) gives the result

\[
Z^{\text{crystal}}_D(q, N_0) = M(q) \xi(q) \prod_{n=1}^{\infty} \left( 1 - e^{-g_s \left( N_0 + \frac{1}{2} \right)} q^{n-\frac{1}{2}} \right), \quad q = e^{-g_s} \] (33)

where \( \xi(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \). Let us compare string answer (21) with (33):

\[
Z^{\text{crystal}}_D(q, N_0) = \xi(q) Z^{\text{string}}_D(q, a), \quad a = g_s (N_0 + \frac{1}{2})
\] (34)

The renormalization factor \( \xi(q) \) is important for the melting crystal picture of the D-brane. Due to \( \xi(q) \) factor we can write \( Z^{\text{crystal}}_D(q, N_0) \) in the simple form:

\[
Z^{\text{crystal}}_D(q, N_0) = M(q) \prod_{n=1}^{N_0} (1 - q^n)^{-1} = P \left( 0, 0, \nu(N_0) \right) (q)
\] (35)

Here \( \nu(N_0) \) stands for the 2d partition with single row of length \( N_0 \) and \( P \left( 0, 0, \nu \right) (q) \) was defined in [1] as the partition function of the 3d partitions in the modified container \( \mathcal{O}^+ / \mathcal{M}_\nu \).

\( P \left( 0, 0, \nu(N_0) \right) (q) \) can also be viewed as the partition function of the melting crystal when melting starts from the cubic lattice \( \Lambda \in \mathcal{O}^+ \) with all the sites

\( (0, y, z), \quad y < N_0 g_s \)

already vacant (see Figure 6).
Figure 6: Modified lattice for a D-brane with \( p = 0 \), \( N_0 = 3 \). The crosses stand for vacancies and dots for atoms.

We now easily generalize the above result to give melting crystal picture for the M D-branes of the geometry (19) with \( a_i = g_s(N_i + \frac{1}{2}) \), \( i = 1, \ldots, M \) and \( N_M < N_{M-1} < \ldots N_1 \). In the operator language we compute

\[
Z^{\text{crystal}}_{\text{MD-branes}}(q, N_i) := <0 | \prod_{n=1}^{\infty} \Gamma^+(q^n-\frac{1}{2}) \prod_{m=1}^{N_{M+1}} \Gamma^-(q^{-m}+\frac{1}{2}) \Psi_D(q^{-(N_M+\frac{1}{2})}) \prod_{m=N_{M+1}} \Gamma^-(q^{-m}+\frac{1}{2}) \ldots \prod_{m=N_{M+1}} \Gamma^-(q^{-m}+\frac{1}{2}) \Psi_D(q^{-(N_1+\frac{1}{2})}) \prod_{m=N_1+1}^{\infty} \Gamma^-(q^{-m}+\frac{1}{2}) |0> \tag{36}
\]

Evaluating the above correlator using (17) we find

\[
Z^{\text{crystal}}_{\text{MD-branes}}(q, N_i) = M(q) \prod_{i<j} \left(1-q^{N_i-N_j}\right) \prod_{i=1}^{M} \prod_{n_i=1}^{N_i} \left(1-q^{n_i}\right)^{-1} = P\left(0, 0, \nu\left(N_i\right)\right) \tag{37}
\]

where \( \nu\left(N_i\right) \) stands for the 2d partition such that \( \nu_i = N_i - M + i \). The last equality in (37) follows from the relation established in [1]

\[
P\left(0, 0, \nu\right)(q) = M(q)q^{-\frac{||\nu||^2}{2}}C_{\nu}(q^{-1}) \tag{38}
\]
where $||\lambda||^2 = \sum_i \lambda_i^2$ and $C_{\lambda,\nu}(q^{-1})$ is the topological vertex defined in [3]. In the crystal language one is melting the cubic lattice in the positive octant with all the sites

$$(g_s(i-1), y, z), \quad y < g_s(N_i - M + i), \quad i = 1, \ldots, M$$

already vacant (see Figure 7 for the case of two D branes).

We have considered D-branes which end on the y-axis. Let us now insert an anti-D-brane ending at $x = g_s(N_0 + \frac{1}{2})$. We claim that the anti-D-brane operator is given by

$$\Psi_{D,x} (z) = \Gamma_-(z) \Gamma_+^{-1}(z)$$

inserted at $z = q^{(N_0 + \frac{1}{2})}$ where subscript $x$ indicates that we are dealing with anti-D-branes ending at x-axis. This is clear from the jumping rule for the Kähler class. We can now check this claim by computing the amplitude:

$$Z_{\text{crystal}}^{D,x}(q, N_0) := \langle 0 | \prod_{m=N_0+2}^{\infty} \Gamma_+ (q^{m-\frac{1}{2}}) \Psi_{D,x} (q^{(N_0 + \frac{1}{2})}) \prod_{m=1}^{N_0+1} \Gamma_+ (q^{m-\frac{1}{2}}) \prod_{n=1}^{\infty} \Gamma_- (q^{-n+\frac{1}{2}}) | 0 \rangle \quad (40)$$

We use (39) to recast $Z_{\text{crystal}}^{D,x}(q, N_0)$ as

$$Z_{\text{crystal}}^{D,x}(q, N_0) = \langle 0 | \prod_{m=N_0+2}^{\infty} \Gamma_+ (q^{m-\frac{1}{2}}) \Gamma_- (q^{(N_0 + \frac{1}{2})}) \prod_{m=1}^{N_0} \Gamma_+ (q^{m-\frac{1}{2}}) \prod_{n=1}^{\infty} \Gamma_- (q^{-n+\frac{1}{2}}) | 0 \rangle \quad (41)$$
Note that insertion of $\Psi_{D,x}$ creates a disorder at $q^{(N_0+\frac{1}{2})}$ in a sequence of $\Gamma_+$ operators. Computing (41) we find

$$Z_{\text{crystal}}^{D,x}(q, N_0) = M(q)\xi(q) \prod_{n=1}^{\infty} \left(1 - e^{-g_s(N_0+\frac{1}{2})q^{n-\frac{1}{2}}}\right), \quad q = e^{-g_s}$$

(42)

where $\xi(q) = \prod_{n=1}^{\infty}(1 - q^n)^{-1}$. Note that (42) is equal (up to the factor $\xi(q)$) to the string answer (34) for anti-D-brane with framing $p = 1$.

$Z_{\text{crystal}}^{D,x}(q, N_0)$ can also be viewed as the partition function of the melting crystal when melting starts from the lattice $\Lambda_x \in O^+$ with all the sites

$$(x, 0, z), \quad x < N_0 g_s$$

already vacant (see Figure 8).

We would like to end this section with a comment about extra factor $\xi(q)$ which appears in the crystal melting picture as compared with the string theory answer (34). To ensure that this extra factor cannot be detected in perturbative stringy computations we propose to modify the definition of a D-brane operator in crystal picture

$$\tilde{\Psi}_D = q^{-1/24}\Psi_D$$

(43)
Then, extra factor would be $\eta(q)^{-1}$ and the change in the free energy due to this factor is non-perturbative in the $g_s \to 0$ limit.

$$\Delta F = -\log \eta(q) = -\log \eta(\tilde{q}) + \frac{1}{2} \log \frac{g_s}{2\pi}, \quad \tilde{q} = \exp\left(-\frac{(2\pi)^2}{g_s}\right)$$ (44)

In the S-dual language [7] the factor $\eta(q)^{-1}$ may be thought as coming from extra $D(-1)$ instantons in the presence of Lagrangian NS2-brane. It would be interesting to determine analogs of $\xi(q)$ for arbitrary framings.

### 4 Crystal amplitudes for more general D-brane configurations

So far we have discussed crystal melting interpretation for special configuration of D-branes. Here we would like to consider more general configuration of branes and anti-branes at various framings.

#### 4.1 Anti-D-brane amplitudes

Let us consider anti-D-brane ending on the $y$ axis and zero framing. We will take into consideration that the jump of the Kähler period along the 2-cycle linking anti-D-brane is opposite in sign to the corresponding jump around D-brane. Then, we can naturally propose the shape of the melting crystal as shown in Figure 9. Note that the corners are now concave, as opposed to the case for branes ending on the $y$-axis.

To verify this proposal and find the rules of melting we again use operator language. We suggest that anti-D-brane ending on $y$-axis at $y = g_s(N_0 + \frac{1}{2})$ is described by insertion of the operator

$$\Psi_D(z) = \Gamma_-(z)\Gamma_+^{-1}(z)$$ (45)

at $z = q^{-(N_0 + \frac{1}{2})}$.

$$Z^{\text{Crystal}}_{\bar{D}}(q, N_0) := <0| \prod_{n=1}^{\infty} \Gamma_+\left(q^{-n+\frac{1}{2}}\right) \prod_{m=1}^{N_0+1} \Gamma_-\left(q^{-m+\frac{1}{2}}\right) \Psi_D(q^{-(N_0 + \frac{1}{2})}) \prod_{N_0+2}^{\infty} \Gamma_-\left(q^{-m+\frac{1}{2}}\right)|0>$$ (46)
We use (45) to recast $Z_{\text{crystal}}^{\text{D}}(q, N_0)$ as

$$<0| \prod_{n=1}^{\infty} \Gamma_+(q^{n+\frac{1}{2}}) \prod_{m=1}^{N_0} \Gamma_-(q^{-m+\frac{1}{2}}) \Gamma_2^{-1}(q^{-(N_0+\frac{1}{2})}) \prod_{N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}})|0> \quad (47)$$

Note that insertion of an anti-D-brane $\Psi_D$ creates a much more complicated disorder at $q^{-\left(N_0+\frac{1}{2}\right)}$ in a sequence of $\Gamma_-$ operators than D-brane does. The commutation relation (17) gives the result for anti-D-brane

$$Z_{\text{crystal}}^{\text{D}}(q, N_0) = M(q) \xi^{-1}(q) \prod_{n=1}^{\infty} \left(1 - e^{-gs\left(N_0+\frac{1}{2}\right)} q^{n-\frac{1}{2}} \right)^{-1} = M(q) \prod_{n=1}^{N_0} (1 - q^n), \quad q = e^{-gs} \quad (48)$$

Let us compare string answer (22) with (48):

$$Z_{\text{crystal}}^{\text{D}}(q, N_0) = \xi^{-1}(q) Z_{\text{string}}^{\text{D}}(q, a), \quad a = gs(N_0 + \frac{1}{2}) \quad (49)$$

Now we would like to give a crystal interpretation of (48). A natural guess which gives the correct jump of the Kähler class is to start melting the cubic lattice $\Lambda$ in the positive octant with vacant sites

$$(0, y, z), \quad y \geq gsN_0.$$ 

It turns out that in addition to the choice of this modified lattice, one also has to modify the melting rule to reproduce the correct answer predicted from the operator formulation (and...
known string amplitudes). To formulate the modified rule we use
\[
\prod_{n=1}^{N_0} (1 - q^n) = \sum_{\nu \in \mathcal{P}_{N_0}} (-)^{\sum_{i=1}^{N_0} n_i} q^{\sum_{i=1}^{N_0} i n_i}
\]
(50)
where \(\mathcal{P}_{N_0}\) is a set of column strict\(^3\) 2d partitions with maximum column length \(l_{\text{max}} \leq N_0\)
and \(n_i = 0, 1\) counts if a column of length \(i\) is present in \(\nu\). Here is a tentative proposal for the
modified rule. The melting in \(x = 0\) plane starts from \((0, (N_0 - 1) g_s, 0)\) and is going in the
direction of decreasing \(y\) and increasing \(z\). The allowed configurations for this new melting
are collections of atoms forming column strict 2d partitions. So for example, a configuration
\[
\left\{ (0, (N_0 - 1) g_s, 0), (0, (N_0 - 1) g_s, g_s) \right\}
\]
is not allowed but a configuration
\[
\left\{ (0, (N_0 - 1) g_s, 0), (0, (N_0 - 1) g_s, 0), (0, (N_0 - 2) g_s, 0), (0, (N_0 - 1) g_s, g_s) \right\}
\]
is allowed. The sign of configuration is the number of columns, which is two in the example
(51). We give a more nontrivial example of an allowed configuration molten in the \(x = 0\)
plane in Figure 10. The melting in the rest of the positive octant goes in a standard way
independently of the melting in the \(x = 0\) plane. For this reason a total configuration
of molten atoms has the form \((A; B)\) where \(A\) is a configuration obtained from melting in
\(O^+\) and \(B\) is a configuration (with signs) in the \(x = 0\) plane. For more anti-D-branes or
mixtures of D-branes and anti-D-branes, one naturally expects that the modified lattice be
consistent with the change in Kähler class. Viewing D-brane as fermion and anti-D-brane
as anti-fermion, this is also consistent with the fact that \(\oint \partial \phi\) measures the Kähler class of
the crystal and its jump across the insertion of fermion or anti-fermion is consistent with the
expected geometry of the lattice. However, the precise rule for the melting crystal remains
to be worked out for this more general melting. Of course the operator expressions always
exist for such amplitudes. In this regard, we would like to emphasize that the global rule
of melting is also absent for closed strings on toric Calabi-Yau manifolds with more than
one fixed point of the toric action. The rules given in [4] are local\(^4\) near each of the fixed

\(^3\)Column strict means that the lengths of columns are strictly decreasing

\(^4\)There are also modifications which account for world sheet instantons wrapped on compact 2-cycles
between fixed points
Figure 10: An example of an allowed configuration molten in $x = 0$ plane. The number of columns is three, $N_0 = 8$, the contribution to $Z$ is $(-1)^3 q^{11}$. 
Figure 11: \( N_1 = 3, \quad N_2 = 11 \). 6 atoms are molten near each of the branes on top of the three molten strings.

point. This is related to the fact that what appears as brane to one corner of the crystal will appear as anti-brane to another corner. In the spirit of [4] we can get a rough idea about the melting crystal interpretation of the D-brane and anti-D-brane when they are far apart. In this case we can regard our models of melting as local ones which are valid in the vicinity of each brane.

Let us consider a D-brane at \( a = g_s(N_1 + \frac{1}{2}) \) and an anti-D-brane at \( b = g_s(N_2 + \frac{1}{2}) \) so that \( N_2 \gg N_1 \). Note that in the presence of D-brane and anti-D brane there are essentially new types of objects (one dimensional lines) to be added to the melting picture [4]. These are “strings of atoms “ of length \( 5 N_2 - N_1 \). The new configurations account for the factor

\[ \text{the number of atoms in a string} \]
(1 − q^{N_2 − N_1})^{-1} present in the partition function:

\[ Z_{D_D} = M(q)(1 − q^{N_2 − N_1})^{-1} \prod_{n=1}^{N_1} (1 − q^n)^{-1} \prod_{n=1}^{N_2} (1 − q^m) \]  

(52)

We show in the Figure 11 a typical configuration of molten atoms on top of melting three string-like objects.

### 4.2 The D-brane framing in crystal picture

Here we would like to discuss the crystal amplitudes for D-branes in arbitrary framing. We will show, very much like the case for the anti-D-branes, that the operator formulation of the amplitude is compatible with the modified lattice we had obtained before. Though a direct interpretation of the statistical mechanical model encoded by the operator formulation is unclear.

Let us recall the string theory partition function in the presence of a D-brane (anti-D-brane) of the geometry (19) with framing \( [2] \):

\[ Z_{D\text{string}}^\text{string}(q, a, p) = M(q) \sum_{k=0}^{\infty} e^{-ka}(-1)^{(1-p)k} q^{(1-p)k(k-1)/2} q^{k/2} \prod_{n=1}^{k} (1 - q^n)^{-1}, \quad q = e^{-gs} \]  

(53)

\[ Z_{D\text{string}}^\text{string}(q, a, p) = M(q) \sum_{k=0}^{\infty} e^{-ka}(-1)^{pk} q^{pk(k-1)/2} q^{k/2} \prod_{n=1}^{k} (1 - q^n)^{-1}, \]  

(54)

Now we are going to illustrate how string answers can be reproduced from the operator formulation on the modified crystal (up to the renormalization factors \( \xi^{\pm 1} \)). We define the fermion operators \( \psi^{(p)} \) and \( \psi^{*(p)} \):

\[ \psi^{(p)}(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{p}{2}}(-)q^{p(n+1)/2} q^{n-1}, \quad \psi^{*(p)}(z) = -\sum_{n \in \mathbb{Z}} \psi_{n+\frac{p}{2}}(-)q^{p(n+1)/2} q^{-n-1} \]  

(55)

and in analogy with (31,46)\(^7\) insert them at \( q^{-(N_0+\frac{1}{2})} \)

\[ Z_{D\text{crystal}}^\text{crystal}(q, N_0, p) := \langle v_{N_0+1} \prod_{n=1}^{\infty} \Gamma_{+}(q^{n-\frac{1}{2}}) \prod_{m=1}^{N_0+1} \Gamma_{-}(q^{-m+\frac{1}{2}}) \psi^{(p)}(q^{-(N_0+\frac{1}{2})}) \prod_{N_0+2}^{\infty} \Gamma_{-}(q^{-m+\frac{1}{2}}) \rangle \]  

(56)

\(^6\)We use convention for framing as in [14]

\(^7\)It is more convenient here to insert operators with nonzero fermion number and thus change the vacuum at \( t = \infty \). For a single D-brane (anti-D-brane) this is equivalent to our treatment in section 3.
\[ Z_{\text{anti-D}}^{\text{crystal}}(q, N_0, p) := \langle v_{-1} \prod_{n=1}^{\infty} \Gamma_+ \left( q^{n-\frac{1}{2}} \right) \prod_{m=1}^{N_0+1} \Gamma_+ \left( q^{-m+\frac{1}{2}} \right) \psi^*(p) \left( q^{-\left( N_0+\frac{1}{2} \right)} \right) \prod_{N_0+2}^{\infty} \Gamma_+ \left( q^{-m+\frac{1}{2}} \right) |0 \rangle \]  

(57)

where \( \langle 0 | = \langle v_1 | e^{\phi_0} \) and \( \langle 0 | = \langle v_{-1} | e^{-\phi_0}. \) In order to compute (56) we first find correlators involving modes \( \psi_{n+\frac{1}{2}}: \)

\[ P_n = \langle v_1 \prod_{n=1}^{\infty} \Gamma_+ \left( q^{n-\frac{1}{2}} \right) \prod_{m=1}^{N_0+1} \Gamma_+ \left( q^{-m+\frac{1}{2}} \right) \psi_{n+\frac{1}{2}} \prod_{N_0+2}^{\infty} \Gamma_+ \left( q^{-m+\frac{1}{2}} \right) |0 \rangle \]  

(58)

These can be extracted from the known expression for \( Z_D^{\text{crystal}}(q, N_0, p = 0) \) (35) to be

\[ P_n = (-1)^{n+k} \frac{M(q)\xi(q)(-)^k q^{k^2/2}}{2} \prod_{s=1}^{k} (1 - q^s)^{-1}, \quad k \geq 0, \quad P_n = 0, \quad n \leq -2 \]  

(59)

For anti-D-brane the procedure is analogous. Then we immediately find the relation between crystal and string partition functions

\[ Z_D^{\text{crystal}}(q, N_0, p) = \xi(q) Z_D^{\text{string}}(q, a, p), \quad a = g_s(N_0 + \frac{1}{2}) \]  

(60)

\[ Z_D^{\text{crystal}}(q, N_0, p) = \xi^{-1}(q) Z_D^{\text{string}}(q, a, p), \quad a = g_s(N_0 + \frac{1}{2}) \]  

(61)

where \( \xi(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \) Now an important point is that we can write operators \( \psi(p) \) and \( \psi^*(p) \) as [2,8]

\[ \psi(p) = U_p \psi U_{-p}, \quad \psi^*(p) = U_p \psi^* U_{-p} \]  

(62)

Here \( U_p = (-1)^{pL_0} q^{pC_2} \) and

\[ L_0 = \sum_{k \in \mathbb{Z}} (k + 1) : \psi_{k+\frac{1}{2}} \psi^*_{-k-\frac{1}{2}} : , \quad C_2 = - \sum_{k \in \mathbb{Z}} \frac{k(k+1)}{2} : \psi_{k+\frac{1}{2}} \psi^*_{k-\frac{1}{2}} : \]  

(63)

Recall that \( L_0 \) and \( C_2 \) act on the state \( |\mu> \) corresponding to 2d partition \( \mu \) as follows:

\[ L_0 |\mu> = |\mu||\mu>, \quad C_2 |\mu> = \frac{1}{2} \kappa_\mu |\mu> \]  

(64)

where \( |\mu| = \sum_i \mu_i \) and \( \frac{1}{2} \kappa_\mu = \sum_{(i,j) \in \mu} i - j. \) As was discussed in [1] operator \( U_{-p} \) is responsible for the change of slicing of the 3d partition. More precisely, \( U_{-p} \) maps a 2d partition in a
diagonal slice \( x - y = t \) to the 2d partition in the plane intersecting \( z = 0 \) plane along the line \( y = -t + (1 + p)x \) and \( x = 0 \) plane along the line \( y = -t - pz \). We illustrate the action of \( U_{-p} \) in Figure 12. We regard (62) as an indication that in the melting crystal picture the initial lattice \( \Lambda(p) \in \mathcal{O}^+ \) has vacancies in the \( y-z \) plane at the sites:

\[
(0, y, z) \quad y < N_0 - pz
\]

Initial lattice for the case of \( p = -2 \) is shown in Figure 13.

\section{D-branes in the crystal picture in the limit \( g_s \rightarrow 0 \)}

In this section we demonstrate that our proposal for D-branes in the crystal picture is consistent with string theory in the semiclassical limit \( g_s \rightarrow 0 \). As shown in \cite{10, 15} the lattice corner disappears in the limit \( g_s \rightarrow 0 \) and smooth geometry, called the limiting shape, emerges. The average configuration contains many molten atoms and is given by the volume above the 3d graph\(^8\)

\[
(x, y, z) = (R, U + R, V + R), \quad U = y - x, \quad V = z - x, \quad R = R(U, V)
\]

\(^8\)There is also \( x, y, z \) symmetric parametrization of the limiting shape \cite{10}
Figure 13: Initial lattice $\Lambda_{(-2)}$ for a D-brane with $p = -2, N_0 = 3$.

$R(U, V)$ is defined as

$$R(U, V) = \frac{1}{4\pi^2} \int \int_{0}^{2\pi} \log|F(U + i\alpha, V + i\beta)|d\alpha d\beta$$ (67)

where

$$F(u, v) = e^{-u} + e^{-v} + 1$$ (68)

It is quite remarkable [1] that the same function $F(u, v)$ enters the equation for the Calabi Yau 3-fold mirror to $\mathbb{C}^3$

$$Z_1 Z_2 = F(u, v)$$ (69)

Note that the 3d graph intersects $(y, z)$ plane in $\mathcal{O}^+$ over the curve

$$e^{-y} + e^{-z} = 1, \quad y \in [0, \infty]$$ (70)

In the absence of D-branes the partition function behaves as:

$$Z_{\text{crystal}}(q) \to \exp\left(-\frac{V_0}{g_s^2}\right)$$ (71)

where $V_0$ is the volume of the complement of the limiting shape (66). Now let us include a zero-framing D-brane of the geometry (19). We treat this as a small perturbation to the melting crystal and view the additional D-brane defect in the background of the molten
crystal. We have proposed in section 3 that in this case the melting starts from the lattice in $\mathcal{O}^+$ with all the sites $(0, y, z)$, $y < N_0 g_s$ already vacant. This implies that the deleted region in the presence of the D-brane $V_D$ differs from $V_0$ by the amount

$$V_D = V_0 - g_s A(N_0, 0)$$

(72)

where $A(N_0, 0)$ stands for the area of the figure in $yz$-plane in $\mathcal{O}^+$ below the curve (see Figure 14)

$$e^{-y} + e^{-z} = 1, \quad y \in [0, N_0 g_s]$$

(73)

But it is exactly $A(N_0, 0)$ that was shown in [14] to be the genus zero open string free energy in the A-model on $\mathbb{C}^3$ in the presence of a zero-framing D-brane. This proves that our proposal in section 3 for the zero-framing D-branes in crystal melting picture is consistent with the string theory results.

From [14] we also get evidence for the crystal interpretation of framing proposed in section 4. As explained in [14] the genus zero open string free energy in the presence of a D-brane
with framing $p$ is given by the area $A(N_0, p)$ of the figure in $yz$-plane in $O^+$ below the curve (see Figure 15)

$$e^{-y} + e^{-z} = 1, \quad y \in [0, N_0g_s - pz]$$

(74)

This is consistent with our proposal in section 4 that the melting starts from the lattice in $O^+$ with all the sites $(0, y, z)$, $y < N_0g_s - pz$ already vacant. Indeed, in this case the deleted region would have the volume

$$V_{D,p} = V_0 - g_s A(N_0, p)$$

(75)

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