LETTER

The Jarzynski relation, fluctuation theorems, and stochastic thermodynamics for non-Markovian processes

T Speck and U Seifert

II. Institut für Theoretische Physik, Universität Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany
E-mail: speck@theo2.physik.uni-stuttgart.de and useifert@theo2.physik.uni-stuttgart.de

Received 18 July 2007
Accepted 11 September 2007
Published 26 September 2007

Online at stacks.iop.org/JSTAT/2007/L09002
doi:10.1088/1742-5468/2007/09/L09002

Abstract. We prove the Jarzynski relation for general stochastic processes including non-Markovian systems with memory. The only requirement for our proof is the existence of a stationary state, therefore excluding non-ergodic systems. We then show how the concepts of stochastic thermodynamics can be used to prove further exact non-equilibrium relations like the Crooks relation and the fluctuation theorem on entropy production for non-Markovian dynamics.

Keywords: stochastic processes (theory), nonequilibrium fluctuations in small systems
1. Introduction

The Jarzynski relation \[ 1 \] connects non-equilibrium work values \( W \) spent in driving a system initially in equilibrium with the change of free energy \( \Delta F \equiv F_B - F_A \) between initial (A) and final state (B) through the non-linear average \( \langle e^{-\beta W} \rangle = e^{-\beta \Delta F} \). (1)

Here, \( \beta \equiv (k_B T)^{-1} \) is the inverse temperature of the heat bath that the system is coupled to and \( k_B \) is Boltzmann’s constant. The Jarzynski relation has found widespread application in both experiments and computer simulations (for reviews, see \[ 2,3 \]). While (1) was derived originally for deterministic dynamics \[ 1 \] and then generalized to stochastic Markov processes \[ 4 \], its extension to general non-Markovian dynamics is still open. The special case of a driven harmonic oscillator has been treated analytically and numerically in \[ 5 \]. In \[ 6 \], the case of non-equilibrium baths with memory is discussed as regards the entropy production.

In this paper, we will first give a general proof of the Jarzynski relation which basically extends a previous proof for Markovian dynamics \[ 7 \]. We then specialize to Gaussian noise and discuss the role of time reversal. This will allow us to show that further non-equilibrium relations like the Crooks relation \[ 8,9 \] and the detailed fluctuation theorem in non-equilibrium steady states \[ 10–12 \] hold as well. Together with their counterparts holding for deterministic thermostated dynamics \[ 13–15 \], all these non-equilibrium relations show a surprising robustness against the underlying dynamics.

2. Proof of the Jarzynski relation for general stochastic processes

Let the energy of the system be given by a Hamiltonian \( H(\Gamma, \lambda) \), where \( \Gamma \) is a point in phase space. We assume that we can control the system externally through a change of the parameter \( \lambda \) where the function \( \lambda(\tau) \) is called the protocol. We will consider trajectories
Γ(τ) in the interval
\[ t_0 \leq 0 \leq \tau \leq t \leq t_1 \]  (2)
involving four times. At the lower boundary \( t_0 \) we prepare the system such that it retains no memory of earlier times. We then observe single trajectories from \( t_0 \) to \( t_1 \). However, we will drive the system through a change of \( \lambda \) only during the inner interval \( 0 \leq \tau \leq t \) such that \( \lambda \) is constant outside. The change of energy identified as the work spent along a single trajectory \( \Gamma(\tau) \) is then
\[ W[\Gamma(\tau); \lambda(\tau)] = \int_{t_0}^{\tau} d\tau \lambda(\tau) \frac{\partial H}{\partial \lambda}(\Gamma(\tau), \lambda(\tau)). \]  (3)
Hence, the work depends only on the inner section of the trajectory, which is the first ingredient for the proof. In the following, we will drop the implicit dependence on the protocol in the argument of functionals.

The second ingredient for the proof is the time evolution equation
\[ \partial_\tau p(\Gamma, \tau) = \dot{L}(\tau; t_0) p(\Gamma, \tau) \]  (4)
of the distribution \( p(\Gamma, \tau) \) determining the probability for finding the system in a specific region of phase space. The evolution of this distribution is governed by the operator \( \dot{L}(\tau; t_0) \). In the Markov case, the operator \( \dot{L}(\tau; t_0) \rightarrow \dot{L}_m(\lambda(\tau)) \) is the generator of a semi-group [17]. It then completely defines the stochastic process. It is independent of \( t_0 \) and therefore it does not depend on the details of preparation while it depends on time due to the change of the external parameter \( \lambda \).

It is somewhat surprising that the same, apparently time-local, equation (4) holds also for non-Markovian processes [16,17]. This can be understood by realizing that the complete information about processes with memory is contained in the transition probability depending on the whole history rather than in the single-point distribution \( p(\Gamma, \tau) \). We denote as \( \dot{U}(\tau' | \tau; t_0) \) the operator that propagates the system from time \( \tau < \tau' \) to the later time \( \tau' \). The propagator actually depends on the whole function \( \lambda(\tau) \) up to \( \tau' \) since any change of the protocol will have consequences for the following evolution. From the propagator, we can define the operator
\[ \dot{L}_a(\tau; t_0) \equiv \partial_\tau \dot{U}(\tau' | \tau; t_0)|_{\tau' = \tau^+} \]  (5)
describing a ‘substitute’, non-stationary Markov process, \( \dot{L}(\tau; t_0) \rightarrow \dot{L}_a(\tau; t_0) \), which leads to the same single-point distribution \( p(\Gamma, \tau) \) but to a different transition probability to the non-Markovian process [16]. In particular, knowledge of the operator (5) is not sufficient for calculating correlation functions. In contrast to the Markov case, the dependence on the control parameter \( \lambda \) of the operator (5) is implicit. In the appendix, we give an explicit example for such a substitute operator.

We restrict our proof to dynamics with a unique steady state, i.e., for fixed \( \lambda \) the system will relax towards a unique probability distribution \( p_\text{eq}(\Gamma, \lambda) \) depending on the control parameter, \( \lim_{\tau \rightarrow \infty} p(\Gamma, \tau) \rightarrow p_\text{eq}(\Gamma, \lambda) \). This is equivalent to ergodic processes with or without memory (see [18] for a discussion of non-Markovian processes which break ergodicity). In the absence of non-conservative driving, the stationary distribution must be the equilibrium Gibbs–Boltzmann distribution
\[ p_\text{eq}(\Gamma, \lambda) = [Z(\lambda)]^{-1} e^{-\beta H(\Gamma, \lambda)}, \]  (6)
where the partition function
\[ Z(\lambda) = \int d\Gamma \, e^{-\beta H(\Gamma, \lambda)} \] (7)
determines the free energy \( F(\lambda) = -\beta^{-1} \ln Z(\lambda) \). Then \( Z_{A,B} \) are the partition functions of the initial and the final state, respectively.

The third ingredient to the proof is the property that the equilibrium distribution is the stationary solution
\[ \hat{L}(\tau; t_0)p_{eq}(\Gamma, \lambda(\tau)) = 0 \] (8)
for the corresponding value \( \lambda = \lambda(\tau) \) of the control parameter. Whereas this is evident in the case of a Markovian operator, due to the implicit dependence on \( \lambda \) it is not so obvious in the non-Markovian case and we give a proof by contradiction. First we note that for a proper Markovian substitute process, the operator (5) must have a stationary solution. Now suppose that at time \( \tau' \) we stop the process and hold the parameter fixed with value \( \lambda = \lambda(\tau') \). Under very general conditions, which are fulfilled by any transition probability, the Perron–Frobenius theorem ensures that the propagator \( \hat{U}(\tau|\tau'; t_0) \) has an eigenstate \( p_1(\Gamma; \tau, \tau') \) corresponding to the eigenvalue 1 depending on \( \tau' \) and in principle also depending on \( \tau \), i.e.,
\[ \hat{U}(\tau|\tau'; t_0)p_1(\tau, \tau') = p_1(\tau, \tau'). \] (9)
Furthermore, this eigenstate \( p_1(\Gamma; \tau, \tau') \) is a normalized, non-negative probability distribution. From the definition (5), we calculate
\[ \hat{L}(\tau'; t_0)p_1(\tau, \tau') = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\hat{U}(\tau'|\tau'; t_0)p_1(\tau, \tau') - p_1(\tau, \tau')] \neq 0, \] (10)
which is non-zero for both arbitrary functions \( p_1 \) and for the eigenfunction \( p_1(\tau, \tau') \) of the propagator if the latter depends on \( \tau \) since \( \tau \) does not match the leading time argument of the propagator. This would mean that the substitute operator (5) has no stationary solution. This contradiction is resolved only if the eigenfunction \( p_1(\tau') \) is independent of \( \tau \). Moreover, taking then the limit \( \tau \to \infty \) in (9), we find from the ergodicity condition that \( p_1(\Gamma, \tau') = p_{eq}(\Gamma, \lambda) \). Finally, we note that due to causality, we do not have to actually stop the process at a \( \tau' \) since the system cannot depend on the future protocol and (8) must hold for all times \( \tau \).

With these three ingredients, the proof of the Jarzynski relation (1) becomes simple. We prepare the system at time \( t_0 \leq 0 \) in equilibrium and start to drive the system at \( \tau = 0 \) until \( \tau = t \) following the protocol \( \lambda(\tau) \). Inspecting the expression for the work (3), we see that its instantaneous change \( \dot{H}(\Gamma, \tau) \equiv \lambda_0H(\Gamma, \lambda) \) only depends on the actual state \( \Gamma \) that the system is in. Hence, the operator \( \hat{L}(\tau; t_0) \) is all we need to prove the Jarzynski relation including non-Markovian processes. To this end, we consider the joint probability \( \rho(\Gamma, w, \tau) \) for finding the system in state \( \Gamma \) at time \( \tau \) and for having accumulated an amount of work \( w \) up to this time [7, 19]. A change of the state or the control parameter \( \lambda \) will lead to a probability current \( j_w = \hat{H}\rho \) in the direction of the \( w \) coordinate; hence the equation of motion for the joint probability becomes
\[ \partial_{\tau}\rho(\Gamma, w, \tau) = [\hat{L}(\tau; t_0) - \dot{H}(\Gamma, \tau)\partial_w]\rho(\Gamma, w, \tau) \] (11)
Stochastic thermodynamics for non-Markovian processes

due to the conservation of probability. We can prove the Jarzynski relation (1) by first defining the function

$$\psi(\Gamma, \tau) \equiv \int_{-\infty}^{+\infty} dw \rho(\Gamma, w, \tau)e^{-\beta w}. \quad (12)$$

Since the probability for extreme work values $w \to \pm \infty$ vanishes, after one integration by parts the equation of motion becomes

$$\partial_\tau \psi(\Gamma, \tau) = [\hat{L}(\tau; t_0) - \beta \dot{H}(\Gamma, \tau)]\psi(\Gamma, \tau). \quad (13)$$

A solution of this equation is the equilibrium Boltzmann factor

$$\psi(\Gamma, \tau) = Z_A^{-1} e^{-\beta H(\Gamma, \lambda(\tau))} \quad (14)$$

provided $\hat{L}(\tau; t_0)\psi(\Gamma, \tau) = 0$ holds for all $\tau$ as discussed above; see (8). The solution (14) obeys both the initial equilibrium condition $\psi(\Gamma, 0) = p_{eq}(\Gamma, \lambda(0))$ and $\int d\Gamma \, \psi(\Gamma, t) = Z_B/Z_A$ which implies the Jarzynski relation (1).

We have thus proved that the Jarzynski relation (1) holds for any kind of non-Markovian noise by exploiting the existence of a time-local substitute operator (4) which annihilates the Gibbs–Boltzmann distribution $p_{eq}(\Gamma, \lambda)$ for any $\lambda$ reached during the process. This proof only involves the work definition (3) but no other thermodynamic notions like heat and entropy, which we will discuss now.

3. Stochastic thermodynamics

The Jarzynski relation (1) can be embedded into the larger framework of stochastic thermodynamics. The crucial idea is to extend the notion of work and heat to small, stochastic systems coupled to a heat bath in the following way: one first identifies the energy change caused externally as the work and then the heat dissipated due to the interaction with the bath follows from the first law [20]. The first law thus reads

$$Q[\Gamma(\tau)] \equiv W[\Gamma(\tau)] - \Delta H \quad (15)$$

with the change of internal energy $\Delta H \equiv H(\Gamma(t_1), \lambda(t)) - H(\Gamma(t_0), \lambda(0))$ along the specific trajectory $\Gamma(\tau)$. The sign of the heat is a convention; here we choose it to be positive if energy is dissipated into the bath. Using the definition of the work (3), we write the right-hand side of equation (15) under one integral sign. By inserting the total derivative of the energy, the heat becomes the functional

$$Q[\Gamma(\tau)] = - \int_{t_0}^{t_1} d\tau \, \dot{\Gamma}(\tau) \cdot \frac{\partial H}{\partial \Gamma}(\Gamma(\tau), \lambda(\tau)). \quad (16)$$

So far, we have not made any assumptions about the bath and the dynamics of the system and hence the expressions for work and heat should hold for both Markovian and non-Markovian dynamics.

doi:10.1088/1742-5468/2007/09/L09002
4. Time reversal

The identification of the heat allows for a second route for deriving the Jarzynski relation (1) via time reversal [9, 12, 21]. For notational simplicity, we restrict our discussion to one overdamped degree of freedom \( \Gamma = x \) moving in the potential \( H = V(x, \lambda) \). We define the functional

\[
R[x(\tau)] \equiv \ln \frac{P[x(\tau)]}{P[\tilde{x}(\tau)]} = \ln \frac{P[x(\tau)|x_0]p_{\text{eq}}(x_0)}{P[\tilde{x}(\tau)|\tilde{x}_1]p_{\text{eq}}(x_1)}
\]

which fulfills the relation \( \langle \exp[-R] \rangle = 1 \) by definition since \( P[x(\tau)] \) is the probability of a trajectory \( x(\tau) \). In the second step, we have separated the initial state \( x_0 \equiv x(t_0) \) from the conditional probability \( P[x(\tau)|x_0] \) times the probability distribution of the initial state. The path \( \tilde{x}(\tau) \equiv \bar{x}(t_0 + t_1 - \tau) \) denotes the time reversal starting at \( x_1 \equiv x(t_1) \).

We model the dynamics of a system coupled to a non-Markovian bath with Gaussian noise \( \eta \) through the generalized Langevin equation

\[
\gamma(t - \tau) \circ \dot{x}(\tau) = -V'(x(t), \lambda(t)) + \eta(t)
\]

with friction kernel \( \gamma(\tau) \). The prime denotes derivation with respect to \( x \). For convenience, we define the operation

\[
g(\tau) \circ h(\tau) \equiv \int_{t_0}^{t_1} \, d\tau \, g(\tau)h(\tau)
\]

where integration is carried out over the free variable \( \tau \) analogously to Einstein’s sum convention. If one or both of the functions depend on \( x \) then by this short notation we mean \( g(\tau) \equiv g(x(\tau), \lambda(\tau)) \).

Equation (18) includes the Markov case through choosing a time-local friction kernel \( \gamma(\tau) = 2\gamma \delta(\tau) \) with friction coefficient \( \gamma \). The bath correlation function is defined as

\[
C(\tau_1 - \tau_2) \equiv \langle \eta(\tau_1)\eta(\tau_2) \rangle
\]

and the system is guaranteed to equilibrate with the heat reservoir through Kubo’s second fluctuation-dissipation theorem [22]

\[
\gamma(\tau) = \begin{cases} 
\beta C(\tau) & \tau \geq 0, \\
0 & \tau < 0.
\end{cases}
\]

The probability of a certain noise history \( \eta(\tau) \) obeys \( P[\eta(\tau)] > 0 \) for any continuous path \( \eta(\tau) \) and normalization \( \int [d\eta(\tau)] P[\eta(\tau)] = 1 \) with functional measure \( [d\eta(\tau)] \). Gaussian noise is completely defined by its (zero) mean and correlations \( C(\tau) \). We therefore write \( P[\eta(\tau)] = \exp\{-A[\eta(\tau)]\} \) introducing the quadratic ‘action’ functional

\[
A[\eta(\tau)] = \frac{1}{2} \eta(\tau_1) \circ K(\tau_1 - \tau_2) \circ \eta(\tau_2).
\]

The symmetric noise kernel \( K(\tau) = K(-\tau) \) is the operator inverse of the bath correlation function \( C(\tau) \),

\[
C(\tau_1 - \tau) \circ K(\tau - \tau_2) = \delta(\tau_1 - \tau_2).
\]

We change variables from \( \eta \) to \( x \) with the probability of a single trajectory

\[
P[x(\tau)|x_0] = J[x(\tau)] \exp \{-A_s[x(\tau)] - A_a[x(x)]\}.
\]
This change of variables makes it necessary to consider the conditional probability since the noise history does not determine the initial state $x_0$. The total action becomes a sum of two terms defined as

$$A_a[x(\tau)] \equiv \frac{1}{2} V'(\tau_1) \circ K(\tau_1 - \tau_2) \circ V'(\tau_2)$$

$$+ [\gamma(\tau_1 - \tau) \circ \dot{x}(\tau)] \circ K(\tau_1 - \tau_2) \circ [\gamma(\tau_2 - \tau) \circ \dot{x}(\tau)]$$

and

$$A_a[x(\tau)] \equiv [\gamma(\tau_1 - \tau) \circ \dot{x}(\tau)] \circ K(\tau_1 - \tau_2) \circ V'(\tau_2),$$

where we have replaced the noise through the generalized Langevin equation (18). This change of variables involves further the Jacobian $J[x(\tau)] \equiv \det[\delta \eta(t)/\delta x(\tau)]$ given by the functional determinant. For time reversal, we run along the trajectory $\tilde{x}(\tau)$ in the opposite direction from $t_1$ to $t_0$. In the integrals, this amounts both to the substitution $\dot{x} \rightarrow -\dot{x}$ and to inverting the time argument of the kernels. Under these operations, both the symmetric action $A_s$ and the Jacobian $J$ stay invariant. The antisymmetric action of the time-reversed trajectory becomes

$$A_s[\tilde{x}(\tau)] = -[\gamma(\tau - \tau_1) \circ \dot{x}(\tau)] \circ K(\tau_1 - \tau_2) \circ V'(\tau_2)$$

and hence the sum is

$$-A_a[x(\tau)] + A_a[\tilde{x}(\tau)] = -\dot{x}(\tau_1) \circ M(\tau_1 - \tau_2) \circ V'(\tau_2)$$

with kernel

$$M(\tau_1 - \tau_2) = [\gamma(\tau_1 - \tau) + \gamma(\tau_1 - \tau)] \circ K(\tau - \tau_2).$$

The sum in the square brackets equals the symmetric bath correlation function $C(\tau_1 - \tau)$ through use of the fluctuation-dissipation theorem (21). Following (23), the kernel then reduces to the $\delta$-function. The functional $R$ finally reads

$$\frac{1}{\beta} R[x(\tau)] = Q[x(\tau)] + \Delta V - \Delta F = W[x(\tau)] - \Delta F$$

with the heat (16) independent of the actual bath correlation function $C(\tau)$. The only requirement is that the bath itself is and stays in equilibrium as in the case of Markov processes.

5. Discussion and the path to proving further non-equilibrium relations

We can now discuss the role of the two times $t_0$ and $t_1$. In equilibrium, fluctuations have no memory. Since the trajectory $\dot{x}(\tau)$ is just the mirror image of $x(\tau)$, their probabilities must then be equal, $P[x(\tau)] = P[\tilde{x}(\tau)]$, with $R = 1$. We can therefore choose an arbitrary interval $t_0 \leq \tau \leq t_1$ during which we observe the trajectory, and the heat fulfils $Q = -\Delta V$. When we now drive the system through the manipulation of $\lambda$, we can actually choose the driving interval as the observation interval, $t_0 \rightarrow 0^-$ and $t_1 \rightarrow t^+$. However, the noise kernel $K(\tau)$ defined through (23) then depends on the two times $t_0$ and $t_1$. This reflects the fact that both the forward and the time-reversed trajectory are cut off although the force at the boundary still ‘remembers’ the velocity of earlier times.
Although we have shown the relation (30) only for Gaussian noise, the fact that in the first part of the paper we have proven the Jarzynski relation without assuming a specific type of noise suggests that the relation (30) is also valid more generally. However, a direct treatment of non-Gaussian noise within the path integral formalism seems technically challenging.

The fact that (30) holds also for non-Markovian dynamics implies the validity of other non-equilibrium relations. First, from the definition (17), one can derive the Crooks relation \[ \frac{p_R(-W)}{p_F(+W)} = e^{-\beta(W - \Delta F)} \] (31) if we distinguish between the probability distribution of the work \( p_{F,R}(W) \) spent in the forward and time-reversed processes, respectively. Second, for an equilibrated bath, we can still identify the dissipated heat as entropy change in the heat bath, \( \Delta s_m = \beta Q \). With the change of entropy of the system,

\[
\Delta s \equiv s(t_1) - s(t_0), \quad s(\tau) \equiv -\ln p(\Gamma(\tau), \tau),
\] (32)

the fluctuation theorem for the total entropy production \( \Delta s_{tot} = \Delta s_m + \Delta s \) [12]

\[
\langle e^{-\Delta s_{tot}} \rangle = 1
\] (33)

remains valid.

Finally, our analysis of non-Markovian processes can be easily extended to systems driven by non-conservative forces which for constant \( \lambda \) reach a non-equilibrium steady state with probability distribution \( p_s(\Gamma) \). In this case, we have to include the non-conservative forces \( f \) in the external work, leading to

\[
W[\Gamma(\tau)] = \int_0^\tau d\tau \left[ \dot{\lambda}(\tau) \frac{\partial H}{\partial \lambda}(\Gamma(\tau), \lambda(\tau)) + f(\tau) \cdot \dot{\Gamma}(\tau) \right].
\] (34)

The heat is still determined through the first law (15). If we generalize the functional \( R \) from equation (17) by replacing the equilibrium distribution \( p_{eq} \) with the stationary distribution \( p_s \), it is straightforward to derive the relation

\[
R[\Gamma(\tau)] = \beta Q[\Gamma(\tau)] + \ln \frac{p_s(\Gamma(t_0))}{p_s(\Gamma(t_1))},
\] (35)

from which the integral fluctuation theorem for entropy production (33) follows. Moreover, in the case of stationary driving (\( \dot{\lambda} = 0 \)) also the detailed fluctuation theorem [10]–[12]

\[
\frac{P(-\Delta s_{tot})}{P(+\Delta s_{tot})} = e^{-\Delta s_{tot}}
\] (36)

follows for a finite time interval, where \( P(\Delta s_{tot}) \) is the probability distribution of the total entropy production. The relation (36) is also found in deterministic steady state systems in the long time limit [13]–[15].

6. Summary and outlook

As our main result, we have shown that the Jarzynski relation holds for general ergodic systems governed by stochastic dynamics including non-Markovian processes.
further confirmed in the case of Gaussian noise that the relations (30) and (35) between the heat $Q$ and the functional $R$, which serve as a convenient starting point for deriving further exact non-equilibrium relations, still hold for non-Markovian processes. Therefore this class of exact non-equilibrium relations, of which the Jarzynski relation is arguably the most prominent, shows a surprising robustness against the underlying dynamics. An open question which will require further investigation is to what extent the concepts discussed in this paper can be generalized to non-ergodic systems.

Appendix. Substitute operator for a moving trap

As an illustration, we calculate the substitute operator in the case of a particle moving in one dimension with position $x$ which is trapped in a harmonic potential $V(x, \lambda) = (k/2)(x - \lambda)^2$. The generalized Langevin equation (18) then becomes linear and can be solved by Laplace transformation as

$$x(t) = G_1(t)x_0 + \int_0^t d\tau G_2(t - \tau)[k\lambda(\tau) + \eta(\tau)],$$

where the two kernels are given as the inverse Laplace transforms of $\hat{m}(s)$ and $\hat{G}_2(s) = [s\gamma(s) + k]^{-1}$. The system is prepared at time $t = 0$ in equilibrium with initial position $x_0$ drawn from $p_{eq}(x, 0)$. Due to the change of the external parameter $\lambda$, the mean

$$m(t) \equiv \langle x(t) \rangle = \int_0^t d\tau kG_2(t - \tau)\lambda(\tau)$$

is a functional of $\lambda(\tau)$. Without loss of generality, we have set $\lambda(0) = 0$ and hence $\langle x_0 \rangle = 0$.

The substitute operator for one-dimensional Gaussian processes has been worked out explicitly in [16], reading in general

$$\tilde{L}_\omega(t) = -\partial_x[\hat{\chi}(t)x + \hat{\mu}(t) - \frac{1}{2}\hat{\sigma}(t)\partial_x].$$

The functions $\hat{\mu}(t)$ and $\hat{\sigma}(t)$ are determined through the differential equations

$$\dot{m}(t) = \dot{\mu}(t) + \chi(t)m(t), \quad \dot{v}(t) = \dot{\sigma}(t) + 2\dot{\chi}(t)v(t)$$

with time-dependent mean $m(t)$ and variance $v(t)$. The correlation function

$$\chi(t, t') \equiv \frac{\langle[x(t) - m(t)][x(t') - m(t')]\rangle}{v(t')}$$

with $\chi(t, t) = 1$ determines $\dot{\chi}(t) \equiv \partial_\tau \chi(\tau, t)|_{\tau=t}$.

To be more specific, we choose an exponential friction kernel

$$\gamma(t) = ke^{-\kappa t} \Rightarrow \hat{\gamma}(s) = \frac{\kappa}{s + \kappa} \Rightarrow G_2(t) = (\tilde{\kappa}/k)^2e^{-\tilde{\kappa}t} + \frac{\delta(t)}{\kappa + k}$$

with inverse timescale $\tilde{\kappa} \equiv \kappa k/\kappa + k$. In the Markov limit, $\kappa \to \infty$ yields $\tilde{\kappa} \to k$ as expected. Using the explicit expression for the kernel $G_2(t)$, we calculate the mean

$$m(\tau) = e^{-\tilde{\kappa}(\tau - \tau')}m(\tau') + \lambda[1 - e^{-\tilde{\kappa}(\tau - \tau')}]$$

where we have stopped the process at $\tau'$ with parameter $\lambda = \lambda(\tau')$. This equation shows the basic features of ergodic non-Markovian processes. For fixed $\lambda$, the mean
\( m(\tau \to \infty) \to \lambda \) relaxes towards this value. It is a functional of \( \lambda(\tau) \) up to \( \tau' \) and afterwards depends on the time difference \( \tau - \tau' \) only. The time derivative yields \( \dot{m}(\tau) = -\bar{\kappa}m(\tau) + \bar{\kappa}\lambda \) and indeed a straightforward calculation of (A.1) confirms \( \dot{\chi} = -\bar{\kappa} \).

Therefore, we have \( \dot{\mu} = \bar{\kappa}\lambda \) and since we do not change the strength of the trap, the variance is \( v = 1/(\beta k) \) leading to \( \dot{\sigma} = 2\bar{\kappa}/(\beta k) \). Hence, the substitute operator for fixed \( \lambda \) becomes

\[
\hat{L}_s = \bar{\kappa}\partial_x \left[ (x - \lambda) + \frac{1}{\beta k} \partial_x \right]
\]

with stationary solution \( p_{eq}(x, \lambda) \propto \exp\left[-\beta(k/2)(x - \lambda)^2\right] \) for all times \( \tau \geq \tau' \).

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