BRAID GROUPS AND SYMPLECTIC STEINBERG GROUPS

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Abstract. We construct a homomorphism \( f \) from the braid group \( B_{2n+2} \) on \( 2n+2 \) strands to the Steinberg group \( \text{St}(C_n, \mathbb{Z}) \) associated with the Lie type \( C_n \) and with integer coefficients. This homomorphism lifts the well-known symplectic representation of the braid groups. We also describe the image and the kernel of \( f \).

1. Introduction

In this article we provide a connection between low-dimensional topology and algebraic K-theory. More precisely, let \( B_{2n+2} \) be the braid group on \( 2n+2 \) strands \((n \geq 2)\). In [15], following work by Arnold, Magnus & Peluso, Birman, A’Campo et al. (see [1, 2, 4, 5, 19]), the second-named author investigated an action of \( B_{2n+2} \) on the free group \( F_{2n} \) on \( 2n \) generators obtained by viewing a twice-punctured surface of genus \( n \) as a double covering of the disk via an hyperelliptic involution. Linearizing this action, one obtains a homomorphism \( \bar{f} : B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z}) \) from the braid group to the symplectic modular group \( \text{Sp}_{2n}(\mathbb{Z}) \).

Now \( \text{Sp}_{2n}(\mathbb{Z}) \) is a Chevalley group of Lie type \( C_n \). By [22, 23, 24] it has a natural extension, its Steinberg group \( \text{St}(C_n, \mathbb{Z}) \), which is defined by means of a presentation by generators and relations. (Steinberg groups are basic ingredients in algebraic K-theory; see for instance [21].)

We show that we can lift \( \bar{f} \) to a homomorphism

\[
\tilde{f} : B_{2n+2} \to \text{St}(C_n, \mathbb{Z})
\]

from the braid group to the Steinberg group. We further describe the image and the kernel of \( f \). As an application we obtain a simple braid-like presentation of the image of \( f \) (resp. of the image of \( \bar{f} \)), which is a subgroup of finite index of \( \text{St}(C_n, \mathbb{Z}) \) (resp. of \( \text{Sp}_{2n}(\mathbb{Z}) \)).

The paper is a continuation of [16], which dealt with the case \( n = 2 \). It is organized as follows. In Section 2 we give a presentation of the Steinberg group \( \text{St}(C_n, \mathbb{Z}) \) and list a few properties of the special elements \( \gamma_n \). In Section 3 we construct the lifting \( \tilde{f} \) from the braid group to the Steinberg group. In general \( f \) is not surjective; in Section 4 we determine its image (see Theorem 4.1). Section 5 is devoted to a description of the kernel of \( f \); we highlight a braid \( \alpha_n \in B_{2n+2} \) which together with two other braids generate the kernel of \( f \) as a normal subgroup (see Theorem 5.8). In Section 6 we...
extend $f$ to an epimorphism $\tilde{f} : \tilde{B}_{2n+2} \to \text{St}(C_n, \mathbb{Z})$, where $\tilde{B}_{2n+2}$ is an Artin group (see Theorem 6.1) slightly bigger than the braid group $B_{2n+2}$.

2. The Steinberg group $\text{St}(C_n, \mathbb{Z})$

With any irreducible root system $\Phi$ Steinberg [23, 24] associated the so-called Steinberg group, which is an extension of the simple complex algebraic group of type $\Phi$. Later Stein [22] extended Steinberg’s construction over any commutative ring $R$, thus leading to the Steinberg group $\text{St}(\Phi, R)$. We are interested in the case when the root system $\Phi$ is of type $C_n$ ($n \geq 2$) and $R = \mathbb{Z}$ is the ring of integers. The corresponding Steinberg group $\text{St}(C_n, \mathbb{Z})$ is an extension of the symplectic modular group $\text{Sp}_{2n}(\mathbb{Z})$.

2.1. The symplectic modular group $\text{Sp}_{2n}(\mathbb{Z})$. Let $n$ be an integer $\geq 2$. Recall that $\text{Sp}_{2n}(\mathbb{Z})$ is the group of $2n \times 2n$ matrices $M$ with integral entries satisfying the relation $M^T J_{2n} M = J_{2n}$, where $M^T$ is the transpose of $M$, $J_{2n} = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}$ and $\text{Id}_n$ is the identity matrix of size $n$.

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the canonical basis of the free abelian group $\mathbb{Z}^n$. In Bourbaki’s notation the root system of type $C_n$ consists of the following elements (see [7, Chap. VI, § 4.6]): the elements $\pm \varepsilon_i \pm \varepsilon_j$ ($1 \leq i, j \leq n$, $i \neq j$) are the short roots and $\pm 2 \varepsilon_i$ ($1 \leq i \leq n$) are the long roots.

Denote by $E_{i,j}$ the $2n \times 2n$ matrix which has all entries equal to 0 except the $(i, j)$-entry which is equal to 1. The group $\text{Sp}_{2n}(\mathbb{Z})$ is generated by the following matrices (see [8]):

- $X_{i,j} = \text{Id}_{2n} + E_{i,j} - E_{j+i+n,n}$ for $1 \leq i, j \leq n$, $i \neq j$,
- $Y_{i,j} = \text{Id}_{2n} + E_{i,j+n} + E_{j+i+n} - E_{i+j+n,n}$ for $1 \leq i, j \leq n$, $i \neq j$,
- $Y'_{i,j} = Y'_{j,i}$ for $1 \leq i, j \leq n$, $i \neq j$,
- $Z_i = \text{Id}_{2n} + E_{i+i+n,n}$ for $1 \leq i \leq n$,
- $Z'_i = Z'_{i,i}$ for $1 \leq i \leq n$.

Note that $Y_{i,j} = Y'_{j,i}$ and $Y'_{i,j} = Y'_{j,i}$.

Each of these matrices generates a root subgroup corresponding to a root in the following way: $X_{i,j}$ corresponds to the root $\varepsilon_i - \varepsilon_j$, $Y_{i,j}$ to the root $\varepsilon_i + \varepsilon_j$, $Y'_{i,j}$ to $-\varepsilon_i - \varepsilon_j$, $Z_i$ to $2 \varepsilon_i$, and $Z'_i$ to $-2 \varepsilon_i$.

We now list the commutation relations between pairs of these matrices corresponding to non-opposite roots. In the following relations, the indices $i, j$ and $k$ are pairwise distinct and run over $\{1, \ldots, n\}$:

$[X_{i,j}, X_{j,k}] = X_{i,k}$, $[X_{i,j}, Y_{j,k}] = Y_{i,k}$,

$[X_{i,j}, Y'_{i,k}] = Y'_{j,k}^{-1}$, $[Y_{i,j}, Y'_{j,k}] = X_{i,k}$,

$[X_{i,j}, Y_{i,j}] = Z_j^2$, $[X_{i,j}, Y'_{i,j}] = Z'_j^{-2}$,

$[X_{i,j}, Z_j] = Z_j Y_{i,j} - Y_{i,j} Z_j$, $[X_{i,j}, Z'_j] = Z'_j Y'_{i,j}^{-1} - Y'_{i,j}^{-1} Z'_j$,

$[Y_{i,j}, Z'_j] = X_{j,i} Z'_j^{-1} = Z'_j^{-1} X_{j,i}$, $[Y'_{i,j}, Z_i] = X_{i,j}^{-1} Z'_i^{-1} = Z'_i^{-1} X_{i,j}^{-1}$.

The matrices commute for all other pairs of generators, except for $(X_{i,j}, X_{j,i})$, $(Y_{i,j}, Y'_{i,j})$ and $(Z_i, Z'_i)$, which are pairs corresponding to opposite roots.
2.2. A presentation of the Steinberg group. By [3, Sect. 3] and [22] the Steinberg group \( \text{St}(C_n, \mathbb{Z}) \) has a presentation with the same generators and relations as above, namely with generators \( x_{i,j}, y_{i,j}, y'_{i,j} \) \( (1 \leq i, j \leq n \) and \( i \neq j) \), \( z_i, z'_i \) \( (1 \leq i \leq n) \) subject to the following relations (where \( i, j, k \in \{1, \ldots, n\} \) are pairwise distinct):

\[
\begin{align*}
(2.1) & \quad y_{i,j} = y_{j,i}, \quad y'_{i,j} = y'_{j,i}, \\
(2.2) & \quad [x_{i,j}, x_{j,k}] = x_{i,k}, \\
(2.3) & \quad [x_{i,j}, y_{k,i}] = y_{i,k}, \\
(2.4) & \quad [x_{i,j}, y'_{k,i}] = y'_{j,k}, \\
(2.5) & \quad [y_{i,j}, y'_{j,k}] = x_{i,k}, \\
(2.6) & \quad [x_{i,j}, y_{i,j}] = z_i^2, \\
(2.7) & \quad [x_{i,j}, y'_{i,j}] = z'_j^2, \\
(2.8) & \quad [x_{i,j}, z_j] = z_i y_{i,j} = y_{i,j} z_i, \\
(2.9) & \quad [x_{i,j}, z'_j] = z'_j y'_{i,j} = y'_{i,j} z'_j, \\
(2.10) & \quad [y_{i,j}, z'_i] = x_{j,i} z_j^{-1} = z_j^{-1} x_{j,i}, \\
(2.11) & \quad [y'_{i,j}, z_i] = x_{i,j} z_j^{-1} = z_j^{-1} x_{i,j},
\end{align*}
\]

and all remaining pairs of generators commuting, except the pairs \((x_{i,j}, x_{j,i})\), \((y_{i,j}, y'_{i,j})\) and \((z_i, z'_i)\) for which we do not prescribe any relation.

Note that in view of (2.9) and (2.10) the generators \( x_{i,j} \) and \( y'_{i,j} \) can be expressed in terms of the other generators.

By construction there is a surjective homomorphism

\[ \pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z}) \]

sending each generator of \( \text{St}(C_n, \mathbb{Z}) \) represented by a lower-case letter to the symplectic matrix represented by the corresponding upper-case letter.

By [20, Th.6.3] and [3, Kor.3.2] the kernel of the epimorphism \( \pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z}) \) is infinite cyclic generated by \( (x_{2\varepsilon_i}, x_{-2\varepsilon_i}, x_{2\varepsilon_i})^4 \), where \( x_{2\varepsilon_i} \) (resp. \( x_{-2\varepsilon_i} \)) is the generator corresponding to the long root \( 2\varepsilon_i \) (resp. to \(-2\varepsilon_i\)); this generator is independent of \( i \) and central (see Lemma 2.1 below; see also [18]).
2.3. The elements $w_\gamma$. For a root $\gamma$ let $x_\gamma$ be the generator of the Steinberg group corresponding to $\gamma$. Set

$$w_\gamma = x_\gamma x_\gamma^{-1} x_\gamma \in \text{St}(C_n, \mathbb{Z}).$$

In particular, we have

$$w_{2z_i} = z_i z_i' - 1 z_i \quad \text{and} \quad w_{-2z_i} = z_i z_i' - 1 z_i'.$$

For simplicity we write $w_i$ for $w_{2z_i}$. Since $z_i$ commutes with $z_j$ and with $z_j'$ when $i \neq j$, we have $w_i w_j = w_j w_i$ for all $(i, j) \in \{1, \ldots, n\}^2$.

The following equality holds for all roots $\gamma$:

$$w_\gamma = w_\gamma^{-1}$$

(for a proof, see [16, Lemma 2.2]). As a consequence, we have

$$w_\gamma x_\gamma w_\gamma^{-1} = w_{-\gamma} x_\gamma w_{-\gamma}^{-1} = x_{-\gamma}^{-1} x_\gamma x_{-\gamma}^{-1} x_\gamma x_{-\gamma}^{-1} = x_{-\gamma}^{-1}.$$  

Similarly,

$$w_\gamma x_{-\gamma} w_\gamma^{-1} = x_{-\gamma}^{-1}.$$

It follows from (2.15) and (2.16) that the square $w_\gamma^2$ commutes with $x_\gamma$ and with $x_{-\gamma}$ for all roots $\gamma$.

We also need the subsequent relation between an element $w_\gamma$ and the generator $x_\delta$ associated with a root $\delta$ such that $\gamma + \delta \neq 0$, namely

$$w_\gamma x_\delta w_\gamma^{-1} = x_\delta',$$

where $\delta'$ is the image of $\delta$ under the reflection $s_\gamma$ in the hyperplane orthogonal to $\gamma$ and $c = \pm 1$ (see Relation (R7) in [24, Chap. 3, p. 23]). Recall that $s_\gamma$ is given by

$$s_\gamma(\delta) = \delta - 2 \frac{\langle \gamma, \delta \rangle}{\langle \gamma, \gamma \rangle} \gamma,$$

where $(-, -)$ is the inner product of the Euclidean vector space of which the set $\{\varepsilon_1, \ldots, \varepsilon_n\}$ forms an orthonormal basis. To determine the sign $c$ (and the root $\delta'$) in (2.17) it is enough to compute the image $\pi(w_\gamma x_\delta w_\gamma^{-1})$ in $\text{Sp}_{2n}(\mathbb{Z})$.

In particular, for any long root $2\varepsilon_i$ the element $w_i = w_{2\varepsilon_i}$ commutes with all generators $x_{k,\ell}$, $y_{k,\ell}$ and $y'_{k,\ell}$ such that $k \neq i \neq \ell$. By contrast we have the non-trivial relations ($i \neq j$)

$$w_i y_{i, j} w_i^{-1} = x_{j, i}, \quad w_i x_{j, i} w_i^{-1} = y_{i, j}^{-1},$$

and

$$w_i y'_{i, j} w_i^{-1} = x_{j, i}, \quad w_i x_{j, i} w_i^{-1} = y'_{i, j}^{-1}$$

in the Steinberg group $\text{St}(C_n, \mathbb{Z})$. Hence, the conjugation by the square $w_i^2$ turns each generator $x_{i, j}$, $x_{j, i}$, $y_{i, j}$, $y'_{i, j}$ into its inverse, namely

$$w_i^2 y_{i, j} w_i^{-2} = y_{i, j}^{-1}, \quad w_i^2 x_{j, i} w_i^{-2} = x_{j, i}^{-1},$$

and

$$w_i^2 y'_{i, j} w_i^{-2} = y'_{i, j}^{-1}, \quad w_i^2 x_{i, j} w_i^{-2} = x_{i, j}^{-1}.$$  

The following lemma will be used in the sequel.
Lemma 2.1. For each \( i = 1, \ldots, n \) the element \( w_i \) is central in \( \text{St}(C_n, \mathbb{Z}) \) and we have \( w_i^4 = w_i^4 \).

Proof. The centrality of \( w_i \) follows from (2.15), (2.16), (2.20) and (2.21). Now all elements \( w_i \) are conjugate as a consequence of the following special cases of (2.17), where \( i \neq j \):

\[
w_{x_i - x_j} w_{x_i - x_j}^{-1} = z_j \quad \text{and} \quad w_{x_i - x_j} z_i w_{x_i - x_j}^{-1} = z_i'.
\]

The conclusion follows. \( \square \)

3. FROM THE BRAID GROUP TO THE STEINBERG GROUP

Let \( B_{2n+2} \) be the braid group on \( 2n+2 \) strands, where \( n \geq 2 \) is a fixed integer. It has a standard presentation with \( 2n+1 \) generators \( \sigma_1, \sigma_2, \ldots, \sigma_{2n+1} \) and the following relations \( (1 \leq i, j \leq 2n+1) \):

\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if} \quad |i - j| = 1,
\]

and

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{otherwise}.
\]

Let us now construct a homomorphism from \( B_{2n+2} \) to the symplectic Steinberg group \( \text{St}(C_n, \mathbb{Z}) \).

Theorem 3.1. There exists a unique homomorphism \( f : B_{2n+2} \to \text{St}(C_n, \mathbb{Z}) \) such that

\[
f(\sigma_1) = z_1, \quad f(\sigma_{2n+1}) = z_n,
\]

\[
f(\sigma_{2i}) = z_i^{-1} \quad \text{for} \quad i = 1, \ldots, n,
\]

\[
f(\sigma_{2i+1}) = z_i z_{i+1} y_{i,i+1}^{-1} \quad \text{for} \quad i = 1, \ldots, n-1.
\]

The homomorphism \( f \) is surjective if and only if \( n = 2 \).

Remarks 3.2. (a) By [15] the homomorphism \( \tilde{f} : B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z}) \) mentioned in the introduction is defined on the generators \( \sigma_i \) by

\[
\tilde{f}(\sigma_1) = Z_1, \quad \tilde{f}(\sigma_{2n+1}) = Z_n,
\]

\[
\tilde{f}(\sigma_{2i}) = Z_i^{-1} \quad \text{for} \quad i = 1, \ldots, n,
\]

\[
\tilde{f}(\sigma_{2i+1}) = Z_i Z_{i+1} Y_{i,i+1}^{-1} \quad \text{for} \quad i = 1, \ldots, n-1,
\]

where \( Y_{i,i+1} \), \( Z_i \) and \( Z_i' \) are the symplectic matrices defined in Section 2.1. It follows from these formulas and from the definition of \( f \) in Theorem 3.1 that the latter is a natural lifting of \( \tilde{f} \), i.e. we have \( \tilde{f} = \pi \circ f \), where \( \pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z}) \) is the natural projection.

(b) By definition of \( f \) and of \( w_1 = w_{2e_1} \) (see (2.13)) we have

\[
w_1 = f(\sigma_1 \sigma_2 \sigma_1).
\]

It follows from this equality and the remark at the end of Section 2.2 that the kernel of \( \pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z}) \), which is generated by \( w_1^4 \), belongs to the image of \( f \).

(c) Note that all three factors in the product \( z_i z_{i+1} y_{i,i+1}^{-1} \) expressing \( f(\sigma_{2i+1}) \) commute.
Proof. For the existence and the uniqueness of \( f \) it suffices to check that the values of \( f(\sigma_i) \) \((1 \leq i \leq 2n+1)\) in the Steinberg group \( \text{St}(C_n,\mathbb{Z}) \) satisfy the braid relations (3.1) and (3.2).

(i) Let us first check the trivial commutation relations (3.2).

- **Commutation of** \( f(\sigma_1) \) **with** \( f(\sigma_{2n+1}) \). This follows from the fact that \( z_1 \) and \( z_n \) commute.
- **Commutation of** \( f(\sigma_1) \) **with** \( f(\sigma_i) \) **when** \( i \geq 2 \). Indeed, \( z_1 \) commutes with \( z_i' \) when \( i \neq 1 \).
- **Commutation of** \( f(\sigma_1) \) **with all** \( f(\sigma_{2i+1}) \). This is implied by the commuting of \( z_1 \) with the other \( z_i \) and with the generators \( y_{i,j} \).

Similarly for the trivial braid relations involving \( f(\sigma_{2n+1}) = z_n \).

- **Commutation of** \( f(\sigma_2) \) **with** \( f(\sigma_{2i+1}) \). This follows from the fact that the generators \( z_i' \) commute with one another.
- **Commutation of** \( f(\sigma_{2i+1}) \) **with** \( f(\sigma_{2j+1}) \). Indeed, the \( z_i \)'s commute with one another, as do the \( y_{i,j} \)'s. Moreover, the \( z_i \)'s commute with the \( y_{i,j} \)'s.
- **Commutation of** \( f(\sigma_{2i}) \) **with** \( f(\sigma_{2j+1}) \) **when** \( i \notin \{i-1,i\} \). This is implied by the facts that \( z_i' \) commutes with \( y_{j,j+1} \) when \( i \neq j \) and that \( z_i' \) commutes with \( z_jz_{j+1} \) when \( j \notin \{i-1,i\} \).

(ii) The relation \( f(\sigma_1)f(\sigma_2)f(\sigma_1) = f(\sigma_2)f(\sigma_1)f(\sigma_2) \) reads as

\[
z_1z_1' = z_1'z_1,
\]

which is equivalent to \( w_2z_1 = w_1^{-2}z_1 \), where we use the notation of Section 2.3. The latter equality holds by (2.14).

(iii) The relation \( f(\sigma_2)f(\sigma_{2i+1})f(\sigma_2) = f(\sigma_{2i+1})f(\sigma_2)f(\sigma_{2i+1}) \) reads for \( 1 \leq i \leq n-1 \) as

\[
z_i^{-1}z_i z_{i+1}^{-1} y_{i,i+1}^{-1} z_i' = z_i z_{i+1}^{-1} y_{i,i+1}^{-1} z_i' z_i z_{i+1}^{-1} y_{i,i+1}^{-1}.
\]

Let LHS (resp. RHS) be the element of \( \text{St}(C_n,\mathbb{Z}) \) represented by the left-hand (resp. right-hand) side of the previous equation.

Since \( z_i \) commutes with \( y_{i,i+1} \) and with \( z_{i+1} \), and the latter with \( z_i' \), we have

\[
\text{LHS} = z_i z_{i+1}^{-1} y_{i,i+1}^{-1} z_i' z_i^{-1}.
\]

By (2.13), (2.10), (2.14) and the trivial commutation relations we obtain

\[
\text{LHS} = z_i z_{i+1}^{-1} y_{i,i+1}^{-1} z_i' w_{-2i} = z_i y_{i,i+1}^{-1} z_i^{-1} z_i z_{i+1}^{-1} z_i' w_i = z_i y_{i,i+1}^{-1} z_i z_{i+1}^{-1} w_i = z_i y_{i,i+1}^{-1} z_i z_{i+1}^{-1} w_i = z_i y_{i,i+1}^{-1} x_{i+1,i}^{-1} w_i.
\]

Let us now deal with RHS. Since \( z_i \) commutes with \( y_{i,i+1} \), and \( z_{i+1} \) with \( z_i \), \( z_i' \) and \( y_{i,i+1} \), we have

\[
\text{RHS} = z_i y_{i,i+1}^{-1} z_i z_i^{-1} y_{i,i+1}^{-1} z_i' = z_i y_{i,i+1}^{-1} w_i y_{i,i+1}^{-1} = z_i y_{i,i+1}^{-1} x_{i+1,i}^{-1} w_i = \text{LHS}.
\]

For the third equality we have used (2.18).
(iv) Applying the automorphism $\sigma_i \mapsto \sigma_{2n+2-i}$ of $B_{2n+2}$, we reduce the relations $f(\sigma_{2i-1})f(\sigma_{2i})f(\sigma_{2i-1}) = f(\sigma_{2i})f(\sigma_{2i-1})f(\sigma_{2i})$ ($2 \leq i \leq n$) and $f(\sigma_{2n})f(\sigma_{2n+1})f(\sigma_{2n}) = f(\sigma_{2n+1})f(\sigma_{2n})f(\sigma_{2n+1})$ to the previous cases.

(v) The surjectivity for $n = 2$ was established in [16]. Let us now prove that $f$ is not surjective when $n \geq 3$. We remark that under the composition $B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_2)$ of $f$ with the reduction modulo 2 the images of the generators $\sigma_i$ of $B_{n+2}$ have order 2. Hence this morphism factors through the symmetric group $\mathfrak{S}_{2n+2}$ of all permutations of the set $\{1, \ldots, 2n+2\}$. If $f$ is surjective, then $\overline{f}$ is surjective too and we obtain a surjective morphism $\mathfrak{S}_{2n+2} \to \text{Sp}_{2n}(\mathbb{F}_2)$ since the reduction modulo 2 is surjective. Now for $n \geq 2$ the symmetric group $\mathfrak{S}_{2n+2}$ has no non-commutative proper quotient, which implies that the map $\mathfrak{S}_{2n+2} \to \text{Sp}_{2n}(\mathbb{F}_2)$ is an isomorphism. But this is impossible as $\text{Sp}_{2n}(\mathbb{F}_2)$ is a simple group when $n \geq 3$ and $\mathfrak{S}_{2n+2}$ is not. (See also [2, Proof of Statement B].) \qed

4. The image of the homomorphism $f$

As noted in Theorem 3.1, the homomorphism $f : B_{2n+2} \to \text{St}(C_n, \mathbb{Z})$ is not surjective when $n \geq 3$. We can nevertheless determine its image $f(B_{2n+2})$ inside $\text{St}(C_n, \mathbb{Z})$.

Consider the level 2 congruence subgroup $\text{Sp}_{2n}(\mathbb{Z})[2]$ defined as the kernel of the homomorphism $\text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_2)$ induced by reduction modulo 2. We lift $\text{Sp}_{2n}(\mathbb{Z})[2]$ to the Steinberg group by taking its preimage

\begin{equation}
\text{St}(C_n, \mathbb{Z})[2] = \pi^{-1}(\text{Sp}_{2n}(\mathbb{Z})[2])
\end{equation}

under the canonical projection $\pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z})$. Thus $\text{St}(C_n, \mathbb{Z})[2]$ is the kernel of the composition

\[ \text{St}(C_n, \mathbb{Z}) \xrightarrow{\pi} \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_2). \]

The group $\text{Sp}_{2n}(\mathbb{F}_2)$ being finite, $\text{St}(C_n, \mathbb{Z})[2]$ is of finite index in $\text{St}(C_n, \mathbb{Z})$.

Since by [3, Kor. 3.2] the kernel of the projection $\pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z})$ is the infinite cyclic group $\langle w_1^4 \rangle$ generated by $w_1^4$, we have the short exact sequence

\[ 1 \to \langle w_1^4 \rangle \to \text{St}(C_n, \mathbb{Z})[2] \xrightarrow{\pi} \text{Sp}_{2n}(\mathbb{Z})[2] \to 1. \]

Recall the surjective homomorphism $p : B_{2n+2} \to \mathfrak{S}_{2n+2}$ sending each generator $\sigma_i$ of $B_{2n+2}$ to the simple transposition $s_i \in \mathfrak{S}_{2n+2}$, where $s_i$ permutes $i$ and $i+1$ and leaves the remaining elements of $\{1, \ldots, 2n+2\}$ fixed. The kernel of $p$ is the pure braid group $P_{2n+2}$.

**Theorem 4.1.** Assume $n \geq 2$. (a) The images under $f$ of the pure braid groups $P_{2n+2}$ and $P_{2n+1}$ are both equal to $\text{St}(C_n, \mathbb{Z})[2]$: $f(P_{2n+2}) = f(P_{2n+1}) = \text{St}(C_n, \mathbb{Z})[2]$.

(b) The image of the full braid group $B_{2n+2}$ fits into the short exact sequence

\[ 1 \to \text{St}(C_n, \mathbb{Z})[2] \to f(B_{2n+2}) \to \mathfrak{S}_{2n+2} \to 1. \]

When $n = 2$ the homomorphism $f$ is surjective, that is $f(B_6) = \text{St}(C_2, \mathbb{Z})$. One recovers in this way the well-known isomorphism $\text{Sp}_4(\mathbb{F}_2) \cong \mathfrak{S}_6$. 

In general \( f(B_{2n+2}) \) is of finite index in \( \text{St}(C_n, \mathbb{Z}) \) with index \( i_n \) equal to

\[
i_n = \frac{\text{card } \text{Sp}_{2n}(\mathbb{F}_2)}{\text{card } \mathcal{S}_{2n+2}} = 2^{n^2} \frac{\prod_{i=1}^{n}(2^{2i} - 1)}{(2n + 2)!}
\]

(see [11, p. 64]). The values of \( i_n \) grow very rapidly; for low subscripts they are: \( i_2 = 1, i_3 = 36, i_4 = 13,056, i_5 = 51,806,208, i_6 = 2,387,230,064,640 \).

**Proof.** By Remark 3.2 (b) the kernel \( \langle w_i^1 \rangle \) of \( \pi : \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z}) \) is in the image of \( f \). Therefore, in view of the definitions of \( \text{St}(C_n, \mathbb{Z})[2] \) and of \( P_{2n+2} \) it is enough to check that the image of \( f = \pi \circ f : B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z}) \) satisfies the following two properties: (i) \( \bar{f}(P_{2n+2}) = \bar{f}(P_{2n+1}) = \text{Sp}_{2n}(\mathbb{Z})[2] \) and (ii) there is an isomorphism \( \bar{f}(B_{2n+2})/\text{Sp}_{2n}(\mathbb{Z})[2] \cong \mathcal{S}_{2n+2} \).

As we observed in Part (v) of the proof of Theorem 3.1, the composition

\[
B_{2n+2} \xrightarrow{\bar{f}} \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_2)
\]

factors through \( \mathcal{S}_{2n+2} \). Hence we have the inclusions \( \bar{f}(P_{2n+1}) \subset \bar{f}(P_{2n+2}) \subset \text{Sp}_{2n}(\mathbb{Z})[2] \).

The induced morphism \( \mathcal{S}_{2n+2} \to \text{Sp}_{2n}(\mathbb{F}_2) \) (which is non-trivial since the image of \( \sigma_1 \) in \( \text{Sp}_{2n}(\mathbb{F}_2) \) is not the identity) is injective since the images of \( \sigma_1 \) and \( \sigma_2 \) do not commute and \( \mathcal{S}_{2n+2} \) has no non-commutative proper quotient when \( n \geq 2 \). This implies that Property (ii) above follows from (i). Let us now prove the latter.

In order to establish the opposite inclusion \( \bar{f}(P_{2n+1}) \supset \text{Sp}_{2n}(\mathbb{Z})[2] \), it is sufficient to prove \( \bar{f}(B_{2n+1}) \supset \text{Sp}_{2n}(\mathbb{Z})[2] \). We appeal to [1]. In loc. cit. A’Campo considers the monodromy representation of \( B_{2n+1} \) in the free \( \mathbb{Z} \)-module \( V \) with basis \( \{ \delta_i, i = 1, \ldots, 2n \} \), endowed with an alternating form \( I \) satisfying \( I(\delta_i, \delta_{i+1}) = 1 \) for \( i = 1, \ldots, 2n - 1 \). This form is non-degenerate. The monodromy representation maps each generator \( \sigma_i \) of \( B_{2n+1} \) to the automorphism \( T_i \) of \( V \) defined by

\[
T_i(\delta_j) = \begin{cases} 
\delta_j & \text{if } j \neq i - 1, i + 1, \\
\delta_j + \delta_i & \text{if } j = i - 1, \\
\delta_j - \delta_i & \text{if } j = i + 1.
\end{cases}
\]

One thus obtains a representation of \( B_{2n+1} \) in the symplectic group \( \text{Sp}_{2n}(\mathbb{Z}) \).

Now by [1, Th. 1 (2)] the image of this monodromy representation contains the congruence subgroup \( \text{Sp}_{2n}(\mathbb{Z})[2] \). To conclude, it suffices to check that this representation is isomorphic to our \( \bar{f} \). Indeed, If we take

\[
\{ \delta_1, \delta_1 + \delta_3, \ldots, \delta_1 + \delta_3 + \cdots + \delta_{2n-1}, \delta_2, \delta_4, \ldots, \delta_{2n} \}
\]

as a basis of \( V \), the matrix of the form \( I \) becomes in \( V \cong \mathbb{Z}^{2n} \) the matrix \( J_{2n} \) considered in \( \S \, 2.1 \). One easily checks that in this new basis the matrix of \( T_i \) is equal to that of \( \bar{f}(\sigma_i) \) for \( i = 1, \ldots, 2n \). \( \square \)

**Remark 4.2.** The squares of all the generators \( x_{i,j}, y_{i,j}, y_{i,j}', z_{i}, z_{i}' \) of the Steinberg group \( \text{St}(C_n, \mathbb{Z}) \) lie in the image of \( f \), and even in the image of the pure braid group \( P_{2n+2} \) since they are in \( \text{St}(C_n, \mathbb{Z})[2] \).

**Remark 4.3.** In contrast with the case \( p = 2 \), the composite homomorphism \( B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_p) \) is surjective for all prime numbers \( p > 2 \) (see [1, Th. 1 (1)]).
Remark 4.4. Given a prime $p$, the level $p$ congruence subgroup $\text{Sp}_{2n}(\mathbb{Z})[p]$ is defined as the kernel of the homomorphism $\text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_p)$ induced by reduction modulo $p$. By [14, Prop. 6.7] it is torsion-free when $p \neq 2$. Let $\text{St}(C_n,\mathbb{Z})[p]$ be the kernel of the composite map $\text{St}(C_n,\mathbb{Z}) \xrightarrow{\pi} \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_p)$; it is of finite index in $\text{St}(C_n,\mathbb{Z})$. Now $\text{St}(C_n,\mathbb{Z})[p]$ is an extension of the torsion-free group $\text{Sp}_{2n}(\mathbb{Z})[p]$ by the infinite cyclic group $\langle w_4 \rangle$; therefore it is torsion-free as well ($p \neq 2$). It follows that the Steinberg group $\text{St}(C_n,\mathbb{Z})$ is virtually torsion-free, i.e. contains a finite-index torsion-free subgroup.

5. The kernel of $f$

In this section we determine the kernels of $\bar{f}: B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z})$ and of $f: B_{2n+2} \to \text{St}(C_n,\mathbb{Z})$.

5.1. The elements $\Delta_k$ and their images. We start with a few simple-looking elements of the kernel of $f$.

For $k \in \{2, \ldots, 2n+2\}$ let $\Delta_k \in B_k$ be the longest reduced positive braid in the group $B_k$ of braids with $k$ strands. It is defined inductively by $\Delta_2 = \sigma_1$ and $\Delta_{k+1} = (\sigma_1 \sigma_2 \cdots \sigma_k) \Delta_k$ for $k \geq 2$. It is well known (see [6] or [17, Sect. 1.3.3]) that its square $\Delta_k^2$ generates the center of $B_k$.

We now consider $\Delta_k$ as an element of $B_{2n+2}$ under the natural inclusion $B_k \to B_{2n+2}$ sending each generator $\sigma_i \in B_k$ ($1 \leq i \leq k - 1$) to $\sigma_i \in B_{2n+2}$. We may thus look for the image of $\Delta_k$ under the homomorphism $f: B_{2n+2} \to \text{St}(C_n,\mathbb{Z})$.

Let us first compute the image $f(\Delta^2_k)$ of the squares of some of the braids $\Delta_k$.

**Proposition 5.1.** Let $f: B_{2n+2} \to \text{St}(C_n,\mathbb{Z})$ be the homomorphism defined in Theorem 3.1. For each $i \in \{1, \ldots, n\}$ we have

$$f(\Delta^2_{2i+1}) = (w_1^4)^{(i-1)/2} \prod_{j=1}^{i} w_j^2.$$

Moreover,

$$f(\Delta^2_{2n+2}) = (w_1^4)^{n(n+1)/2}.$$

**Remark 5.2.** The element $\Delta^2_{2n+2}$ generating the center of $B_{2n+2}$ is in the kernel of $\bar{f}$, but not in the kernel of $f$, the order of $w_1^4$ being infinite in the Steinberg group $\text{St}(C_n,\mathbb{Z})$ (see [20, Th. 6.3]).

The following consequence of Proposition 5.1 provides us with non-trivial elements of the kernel of $f$.

**Corollary 5.3.** (a) If $1 \leq i \leq n$, then

$$f \left( \Delta^4_{2i+1} \Delta^{-4i^2}_3 \right) = 1.$$

(b) We also have

$$f \left( \Delta^2_{2n+2} \Delta^{-2n(n+1)}_3 \right) = 1.$$
Proof. By Proposition 5.1 and Lemma 2.1 we have $f(\Delta_{2i+1}^4) = (w_i^1)^2$. In the special case $i = 1$ we obtain $f(\Delta_4^1) = w_1^2$. Therefore

$$f(\Delta_{2i+1}^4) = (w_i^1)^2 = f(\Delta_3^{2i^2}).$$

Similarly, $f(\Delta_{2n+2}^2) = f(\Delta_3^{2(n+1)}). \Box$

To prove Proposition 5.1 we need two preliminary lemmas.

**Lemma 5.4.** We have

$$f(\sigma_1\sigma_2\cdots\sigma_{2i}) = \begin{cases} w_1z_1^{-1} & \text{if } i = 1, \\ (\prod_{j=1}^i w_j)(x_{1,2}x_{2,3}\cdots x_{i-1,i})z_i^{-1} & \text{if } 1 < i \leq n, \end{cases}$$

and

$$f(\sigma_1\sigma_2\cdots\sigma_{2i+1}) = \begin{cases} z_1 & \text{if } i = 0, \\ w_1y_{1,2}^{-1}z_2 & \text{if } i = 1, \\ (\prod_{j=1}^i w_j)(x_{1,2}x_{2,3}\cdots x_{i-1,i}) y_{i,i+1}^{-1}z_{i+1} & \text{if } 1 < i < n, \\ (\prod_{j=1}^i w_j)(x_{1,2}x_{2,3}\cdots x_{n-1,n}) & \text{if } i = n. \end{cases}$$

**Proof.** We proceed by induction on the length $k$ of the braid word $\sigma_1\cdots\sigma_k$.

For $k = 1, 2$ we have $f(\sigma_1) = z_1$ and $f(\sigma_1\sigma_2) = z_1z_1^{-1} = w_1z_1^{-1}$ by definition of $f$ and of $w_1$.

For $k \geq 3$, using the induction and (2.18), we obtain

$$f(\sigma_1\cdots\sigma_{2i}) = f(\sigma_1\cdots\sigma_{2i-1})f(\sigma_{2i}) = \left(\prod_{j=1}^{i-1} w_j\right)(x_{1,2}x_{2,3}\cdots x_{i-2,i-1}) y_{i-1,i}^{-1}z_i z_i^{-1} = \left(\prod_{j=1}^{i-1} w_j\right)(x_{1,2}x_{2,3}\cdots x_{i-2,i-1}) y_{i-1,i}^{-1}w_i z_i^{-1} = \left(\prod_{j=1}^{i-1} w_j\right)(x_{1,2}x_{2,3}\cdots x_{i-2,i-1}) w_i x_{i-1,i} z_i^{-1}.$$  

Observing that $w_i$ commutes with the $x_{j,j+1}$’s to its left, we obtain the desired formula.

To obtain the formula for $f(\sigma_1\cdots\sigma_{2i+1})$ it suffices to multiply the formula for $f(\sigma_1\cdots\sigma_{2i})$ on the right by $f(\sigma_{2i+1})$ which is equal to $z_i^{-1} z_{i,i+1} y_{i,i+1}^{-1} z_{i+1}$ if $1 < i < n$ and to $z_n$ if $i = n$ and to cancel the product $z_i^{-1} z_i$. \Box

A similar computation yields the following result.

**Lemma 5.5.** We have

$$f(\sigma_{2i}\cdots\sigma_{2i+1}) = \begin{cases} z_1^{-1}w_1 & \text{if } i = 1, \\ z_i^{-1} z_i \left(\prod_{j=1}^{i-1} w_j\right)(x_{i,i-1}\cdots x_{3,2}x_{2,1}) & \text{if } 1 < i \leq n, \end{cases}$$
and

\[
f(\sigma_{2i+1} \cdots \sigma_2 \sigma_1) = \begin{cases} 
z_{i+1} \left( \prod_{j=1}^{i} w_j \right) (x_{i+1,i} x_{i,i-1} \cdots x_{3,2} x_{2,1}) & \text{if } 1 \leq i < n, \\
\left( \prod_{j=1}^{n} w_j \right) (x_{n,n-1} x_{n-1,n-2} \cdots x_{3,2} x_{2,1}) & \text{if } i = n.
\end{cases}
\]

We can now prove Proposition 5.1.

**Proof of Proposition 5.1.** (a) For the first equality we argue by induction on \( i \). For \( i = 1 \) we have \( f(\Delta_2^3) = f(\sigma_1 \sigma_2 \sigma_1)^2 = w_1^2 \). The induction formula

\[
\Delta_{k+1}^2 = \Delta_k^2 (\sigma_k \sigma_{k-1} \cdots \sigma_2 \sigma_1)(\sigma_1 \sigma_2 \cdots \sigma_{k-1} \sigma_k)
\]

allows us to obtain \( f(\Delta_{2i+1}^2) \) from \( f(\Delta_{2i-1}^2) \) by multiplying the latter on the right by \( f(\sigma_{2i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i-1})f(\sigma_{2i} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i}) \).

We first compute \( f(\sigma_{2i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i-1}) \). By Lemmas 5.4 and 5.5,

\[
f(\sigma_{2i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i-1})
= z_i \left( \prod_{j=1}^{i-1} w_j \right) x_{i,i-1} x_{i-1,i-2} \cdots x_{3,2} x_{2,1} \left( \prod_{j=1}^{i-1} w_j \right) x_{1,2} x_{2,3} \cdots x_{i-2,i-1} y_{i-1,i}^{-1} z_i.
\]

Using (2.18), we have \( x_{k+1,k} w_k w_{k+1} = w_k w_{k+1} x_{k,k+1}^{-1} \) for all \( k = 1, \ldots, i-1 \). These relations allow us to push all the \( w_j \)'s to the left, yielding

\[
f(\sigma_{2i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i-1})
= z_i \left( \prod_{j=1}^{i-2} w_j \right) w_{i-1,i-1} w_{i-1} \times \left( x_{i-2,i-1}^{-1} x_{i-1,i}^{-1} x_{2,3}^{-1} x_{1,2}^{-1} x_{1,2} x_{2,3} \cdots x_{i-2,i-1}^{-1} \right) y_{i-1,i}^{-1} z_i.
\]

\[
= z_i \left( \prod_{j=1}^{i-1} w_j \right) (y_{i-1,i} y_{i-1,i}^{-1}) z_i = z_i \left( \prod_{j=1}^{i-1} w_j \right) z_i = \left( \prod_{j=1}^{i-1} w_j \right) z_i^2
\]

since \( z_i \) commutes with the other \( w_j \)'s.

In the same way we obtain

\[
f(\sigma_{2i} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i})
= z_i^{-1} z_i \left( \prod_{j=1}^{i-1} w_j \right) x_{i,i-1} \cdots x_{3,2} x_{2,1} \left( \prod_{j=1}^{i} w_j \right) x_{1,2} \cdots x_{i-1,i} z_i^{-1}
\]

\[
= z_i^{-1} z_i \left( \prod_{j=1}^{i-1} w_j \right) w_{i} \left( x_{i-1,i}^{-1} x_{i-1,i}^{-1} x_{2,3}^{-1} x_{1,2}^{-1} x_{1,2} \cdots x_{i-1,i} \right) z_i^{-1}
\]

\[
= z_i^{-1} z_i z_i' \left( \prod_{j=1}^{i-1} w_j \right) w_i,
\]

the last equality resulting from (2.15) and the fact that \( z_i' \) commutes with \( w_j \) when \( j \neq i \).
Using the previous calculations, Equation (2.16) and Lemma 2.1, we deduce
\[ f(\Delta_{2i+1}^2) = f(\Delta_{2i-1}^2) f(\sigma_{2i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i-1}) f(\sigma_{2i} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2i}) \]
\[ = f(\Delta_{2i-1}^2) \left( \prod_{j=1}^{i-1} w_j^2 \right) z_i^2 z_i' z_i^{-1} z_i'^{-1} \left( \prod_{j=1}^{i-1} w_j^2 \right) w_i \]
\[ = f(\Delta_{2i-1}^2) \left( \prod_{j=1}^{i-1} w_j^4 \right) z_i w_i z_i' w_i = f(\Delta_{2i-1}^2) \left( \prod_{j=1}^{i-1} w_j^4 \right) z_i z_i^{-1} w_i^2 \]
\[ = f(\Delta_{2i-1}^2) \left( \prod_{j=1}^{i-1} w_j^4 \right) w_i^2 = f(\Delta_{2i-1}^2) w_1^4(i-1) w_i^2. \]

Using the induction hypothesis, we obtain
\[ f(\Delta_{2i+1}^2) = (w_1^4)^{(i-1)(i-2)/2} \left( \prod_{j=1}^{i-1} w_j^2 \right) w_1^{4(i-1)} w_i^2 = (w_1^4)^i(i-1)/2 \prod_{j=1}^{i} w_j^2, \]
which is the desired formula.

(b) We now prove the second formula of Proposition 5.1. By Lemmas 5.4 and 5.5 we have
\[ f(\sigma_{2n+1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2n+1}) \]
\[ = \left( \prod_{j=1}^{n} w_j \right) (x_{n,n-1} \cdots x_{2,1}) \left( \prod_{j=1}^{n} w_j \right) (x_{1,2}x_{2,3} \cdots x_{n-1,n}). \]

Pushing the \( w_i \)'s to the left as above, we obtain
\[ f(\sigma_{2n+1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2n+1}) = \prod_{j=1}^{i=n} w_j^2. \]

Hence, by the induction formula, the first formula of the proposition for the case \( i = n \), and Lemma 2.1 we have
\[ f(\Delta_{2n+2}^2) = f(\Delta_{2n+1}^2) f(\sigma_{2n+1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{2n+1}) \]
\[ = (w_1^4)^{n(n-1)/2} \left( \prod_{j=1}^{n} w_j^2 \right) = (w_1^4)^{n(n-1)/2} \prod_{j=1}^{n} w_j^4 \]
\[ = (w_1^4)^{n(n-1)/2}(w_1^4)^n = (w_1^4)^{n(n+1)/2}. \]

\[ \square \]

5.2. A special element of the kernel of \( f \). Recall from [16] that for \( n = 2 \) the kernel of \( f : B_6 \to \text{St}(C_2, \mathbb{Z}) \) is the normal closure of the braid
\[ (\sigma_1 \sigma_2 \sigma_1)^2(\sigma_1 \sigma_3^{-1} \sigma_5)(\sigma_1 \sigma_2 \sigma_1)^{-2}(\sigma_1 \sigma_3^{-1} \sigma_5). \]

We now exhibit a similar element of the kernel of \( f : B_{2n+2} \to \text{St}(C_n, \mathbb{Z}) \) when \( n \geq 3 \).
For any $n \geq 2$ set
\begin{equation}
(5.3) \quad \beta_n = \prod_{i=0}^{n} \sigma_{2i+1}^{(-1)^i} = \sigma_1 \sigma_3^{-1} \cdots \sigma_{2n+1}^{-1} \in B_{2n+2}.
\end{equation}

For $n = 2$, the braid $\beta_2$ is the element $\sigma_1 \sigma_3^{-1} \sigma_5$ appearing in (5.2). Since by Relations (3.1) the odd-numbered generators $\sigma_{2i+1}$ commute with one another, the product in (5.3) can be taken in any order. Observe also that the square $\beta_n^2$ of $\beta_n$, being the product of squares of generators of the braid group, belongs to the pure braid group $P_{2n+2}$.

As the $z_j$’s commute with one another and with the $y_{i,i+1}$’s, for all $n \geq 2$ we obtain
\begin{equation}
(5.4) \quad f(\beta_n) = \prod_{i=1}^{n-1} y_{i,i+1}^{(-1)^{i+1}} = y_{1,2} y_{2,3} \cdots y_{n-1,n}^{(-1)^n}.
\end{equation}

Using the elements $\Delta_k$ introduced in Section 5.1, we set
\begin{equation}
(5.5) \quad \gamma_n = \prod_{i=1}^{n-1} \Delta_{2i+1}^2 = \Delta_2^2 \Delta_4^2 \cdots \Delta_{2n-1}^2 \in B_{2n+2}.
\end{equation}

We have $\gamma_2 = \Delta_3^2 = (\sigma_1 \sigma_2 \sigma_1)^2$. It follows from the centrality of each $\Delta_k^2$ in the braid group $B_k$ that the product defining $\gamma_n$ can be taken in any order. Clearly, $\gamma_n$ is a pure braid for all $n \geq 2$. Actually, $\gamma_n \in P_{2n-1} \subset P_{2n+2}$.

By analogy with (5.2) we consider the element
\begin{equation}
(5.6) \quad \alpha_n = \gamma_n \beta_n \gamma_n^{-1} \beta_n \in B_{2n+2}.
\end{equation}

Note that $\alpha_n$ belongs to the pure braid group $P_{2n+2}$ as it can be expressed as the product $\alpha_n = \gamma_n (\beta_n \gamma_n^{-1} \beta_n)^{-2} \gamma_n^2$ of pure braids. Moreover, $\alpha_n$ is non-trivial since it is the product of $(\sigma_{2n+1})^{-1}$ with an element of $B_{2n}$. Clearly, $\alpha_2$ is the braid appearing in (5.2).

**Proposition 5.6.** For all $n \geq 2$ we have $f(\alpha_n) = 1$ in the Steinberg group $\text{St}(C_n, \mathbb{Z})$.

**Proof.** Since the conjugation by the central element $w_1^2$ is trivial, it follows from Proposition 5.1 that the conjugation by $f(\Delta_{2i+1}^2)$ is equal to the conjugation by $w_1^2 w_2^2 \cdots w_i^2$. Therefore, the conjugation by $f(\gamma_n)$ is equal to the conjugation by $\prod_{k=1}^{n-1} w_k^{2(n-k)}$. In the previous product we may omit each factor whose exponent $2(n-k)$ is divisible by 4. Thus the conjugation by $f(\gamma_n)$ is equal to the conjugation by $w_{n-1}^2 w_{n-3}^2 w_{n-5}^2 \cdots$. Now the conjugation by $w_{n-2k+1}^2$ is non-trivial only on the factors $y_{n-2k,n-2k+1}$ and $y_{n-2k+1,n-2k+2}$ of $f(\beta_n)$ and turns each of them into its inverse. Therefore, $f(\gamma_n) f(\beta_n) f(\gamma_n)^{-1} = f(\beta_n)^{-1}$, from which one deduces the desired result. \qed

**Question 5.7.** Is there a geometric interpretation, for instance in terms of Dehn twists, for the braid $\alpha_n \in B_{2n+2}$?
5.3. The kernels of $f$ and of $f$. We now state our main result on these kernels.

In case $n = 2$ the kernel $\text{Ker}(f)$ is the normal closure of $\alpha_2$ and $\text{Ker}(f')$ is the normal closure of $\alpha_2$ and $(\sigma_1\sigma_2\sigma_1)^4 = (\sigma_1\sigma_2)^6 = \Delta_3^2$ (see [16, Th. 4.1 and Cor. 4.2]).

We now turn to the general case. Note that by Part (v) of the proof of Theorem 3.1, the kernel of $f'$, hence also the kernel of $f$, is contained in the pure braid group $P_{2n+2}$.

**Theorem 5.8.** Assume $n \geq 3$.

(a) The kernel of $f : B_{2n+2} \to \text{Sp}_{2n}(\mathbb{Z})$ is the normal closure of the set consisting of the three braids $\Delta_3^2, \Delta_3^4$ and $\alpha_n$.

(b) The kernel of $f : B_{2n+2} \to \text{St}(C_n, \mathbb{Z})$ is the normal closure of the set consisting of the commutator $[\sigma_3, \Delta_3^4]$ and of the elements $\Delta_3^4 \Delta_3^{-16}$ and $\alpha_n$.

As a consequence, the finite-index subgroup $\bar{f}(B_{2n+2})$ of $\text{Sp}_{2n}(\mathbb{Z})$ has a presentation with $2n + 1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}$ subject to the braid relations (3.1), (3.2) and the three additional relations

\[(5.7) \quad \Delta_3^4 = \Delta_3^2 = \alpha_n = 1.\]

Similarly, $f(B_{2n+2})$ has a presentation with the same generators subject to the braid relations and the relations $[\sigma_3, \Delta_3^4] = \Delta_3^4 \Delta_3^{-16} = \alpha_n = 1$.

**Proof.** (a) As we observed in the proof of Theorem 4.1, the restriction of $f$ to the subgroup $B_{2n+1}$ of $B_{2n+2}$ is the monodromy representation considered in [1]. The kernel of this restriction is the **hyperelliptic Torelli group** $ST_n^1$ investigated in [9]. In **loc. cit.** Brendle, Margalit, Putman prove that $ST_n^1$ is isomorphic to a subgroup $\mathcal{B}L_{2n+1}$ of the mapping class group of the disk with $2n + 1$ marked points. By Theorem C of [9] and the comments thereafter the subgroup $\mathcal{B}L_{2n+1}$ is generated by squares of Dehn twists about curves in the disk surrounding exactly 3 or 5 marked points. Now under the standard identification of the mapping class group with the braid group $B_{2n+1}$ (see e.g. [17, §1.6]), the square of a Dehn twist about a curve surrounding 3 (resp. 5) marked points corresponds in the braid group to a conjugate of the element $\Delta_3^2$ (resp. of $\Delta_3^4$). Thus, $\text{Ker}(f') \cap B_{2n+1} = \text{Ker}(f') \cap P_{2n+1}$ is the normal closure of $\Delta_3^2$ and $\Delta_3^4$.

To determine the whole kernel inside $P_{2n+2}$, we use the fact that $P_{2n+2}$ is the semi-direct product of a normal free group and of $P_{2n+1}$ (see [17, §1.3]). This free group has $2n + 1$ generators $A_i, 2n+2$ $(i = 1, \ldots, 2n + 1)$ defined by $A_{2n+1, 2n+2} = \sigma_2^{2n+1}$ and

\[(5.8) \quad A_i, 2n+2 = (\sigma_2^{2n+1} \sigma_2 \cdots \sigma_i \cdots) \sigma_i^2 (\sigma_2^{2n+1} \sigma_2 \cdots \sigma_i \cdots)^{-1}, \quad \text{when } 1 \leq i \leq 2n.\]

The following expression for $A_i, 2n+2$ will be used in the sequel:

\[(5.9) \quad A_i, 2n+2 = (\sigma_2 \sigma_2^{2n+1} \cdots \sigma_i \cdots) \sigma_i \sigma_2^{2n+1} \sigma_2 \cdots \sigma_i. \quad (1 \leq i \leq 2n)\]

To derive (5.9) from (5.8) use the braid relations or draw a picture.

Recall the special element $\gamma_n = \gamma_n \beta_n \gamma_n^{-1} \beta_n$ given by (5.6). The element $\gamma_n$ belongs to $P_{2n}$ and $\beta_n$ to $B_{2n-1}$. Since $\sigma_2^{2n+1}$ commutes with $B_{2n}$, we have $\alpha_n \in B_{2n} \sigma_2^{2n+1} \cap P_{2n+2} = P_{2n} \sigma_2^{2n+1}$. Consequently, $A_{2n+1, 2n+2} = \sigma_2^{2n+1}$ belongs to $P_{2n} \sigma_2^{2n+1}$.\[14\] FRANÇOIS DIGNE AND CHRISTIAN KASSEL
Let us deal with $A_{i,2n+2}$ when $1 \leq i \leq 2n$. Since $\sigma_2 \sigma_{2n-1} \cdots \sigma_i$ belongs to $B_{2n+1}$, we see from (5.9) and what we established for $\sigma_{2n+1}^2$ that $A_{i,2n+2}$ is conjugate under $B_{2n+1}$ of an element of $P_{2n}^1 \alpha_n$ or of $P_{2n}^1 \alpha_n^{-1}$.

So, at the cost of adding $\alpha_n$, we have reduced Ker($f$) to its intersection with $P_{2n+1}$. In view of the above considerations, this completes the proof of Part (a).

(b) Let $N$ be the normal closure of $[\sigma_3, \Delta_3^4]$, $\Delta_2^4 \Delta_3^{-16}$ and $\alpha_n$. We have $N \subset$ Ker($f$): this follows from the centrality of $f(\Delta_3^4) = w_4^1$, from Corollary 5.3 (a) applied to $i = 2$, and from Proposition 5.6.

Let us next remark that $N$ contains the commutator $[\beta, \Delta_3^4]$ for each braid $\beta$. Indeed, since $\Delta_3^4$ is a product of $\sigma_1$ and $\sigma_2$, it commutes with all generators $\sigma_i$ and their inverses with $4 \leq i \leq 2n+1$. On the other hand $\Delta_3^4$ is central in $B_3$; therefore it commutes with $\sigma_1$ and $\sigma_2$ and their inverses. Since by definition $N$ contains $[\sigma_3, \Delta_3^4]$ and we have $[\sigma_3^{-1}, \Delta_3^4] = \sigma_3^{-1}[\sigma_3, \Delta_3^4]^{-1}\sigma_3$, it contains all commutators of the form $[\sigma_i^\pm 1, \Delta_3^4]$. Using the commutator identities

$$[\beta_1, \beta_2, \Delta_3^4] = \beta_1[\beta_2, \Delta_3^4] \beta_1^{-1} [\beta_1, \Delta_3^4]$$

for $\beta_1, \beta_2 \in B_{2n+2}$, we conclude by induction on the length of expression of a braid in the generators $\sigma_i^{\pm 1}$.

The kernels of $f$ and of $\bar{f}$ are connected by the short exact sequence

$$1 \to \text{Ker}(f) \longrightarrow \text{Ker}(\bar{f}) \xrightarrow{f} \langle w_4^1 \rangle \to 1. \tag{5.10}$$

Indeed, clearly Ker($f$) sits inside Ker($\bar{f}$) as a normal subgroup. If $\bar{f}(\beta) = 1$, then $f(\beta)$ belongs to the kernel of $\pi: \text{St}(C_n, \mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{Z})$, which we know to be infinite cyclic generated by $w_4^1$. The homomorphism $f: \text{Ker}(\bar{f}) \to \langle w_4^1 \rangle$ is surjective since $w_4^1 = f(\Delta_3^4)$ and $\bar{f}(\Delta_3^4) = 1$. Moreover, since $\langle w_4^1 \rangle \cong \mathbb{Z}$, the short exact sequence (5.10) is split with splitting $\langle w_4^1 \rangle \to \text{Ker}(\bar{f})$ given by $w_4^1 \mapsto \Delta_3^4$.

The short exact sequence (5.10) induces the quotient short exact sequence

$$1 \to \text{Ker}(f)/N \longrightarrow \text{Ker}(\bar{f})/N \xrightarrow{\bar{f}} \langle w_4^1 \rangle \to 1.$$ 

To conclude that Ker($f$) = $N$, it suffices to check that $f: \text{Ker}(\bar{f})/N \to \langle w_4^1 \rangle$ is injective.

Now, by Part (a) each element of Ker($\bar{f}$) is a product of conjugates of the braids $\Delta_3^4, \Delta_2^4, \alpha_n$ and their inverses. Since $\alpha_n \equiv 1$ and $\Delta_3^4 \equiv (\Delta_3^4)^4$ modulo $N$, each element of Ker($\bar{f}$) is equal modulo $N$ to a product of conjugates of $\Delta_3^4$ and of its inverse. Since $[\beta, \Delta_3^4] \in N$ for any braid $\beta$, we have $\beta \Delta_3^4 \beta^{-1} \equiv \Delta_3^4$ modulo $N$. Hence, Ker($\bar{f}$)/$N$ is generated by $\Delta_3^4$ mod $N$. The image of $\Delta_3^4$ being $w_4^1$, the morphism $f: \text{Ker}(f)/N \to \langle w_4^1 \rangle$ is an isomorphism. \hfill \square

**Remark 5.9.** As Benjamin Enriquez pointed to us for $n = 2$, the element $\alpha_2$ can be rewritten as $\alpha_2 = \Delta_3^4 \Delta_3^{-2} \sigma_5^2$. For $n \geq 3$ this led us to consider the element

$$\alpha' = \Delta_3^{-2} \Delta_2^{-2} \sigma_{2n+1} \in B_{2n+2}.$$

Using the computations of Section 5.1, it is easy to check that $\alpha_n'$ is in the kernel of $f$. By its very definition $\alpha_n'$ belongs to $P_{2n}^2 \sigma_{2n+1}^2$, an additional feature it shares with $\alpha_n$. Reasoning as in the previous proof, we deduce
that Theorem 5.8 also holds with $\alpha'_n$ instead of $\alpha_n$. Accordingly, the relation $\alpha_n = 1$ of (5.7) can be replaced by the relation $\sigma_{2n+1}^2 = \Delta_{2n}^3$.

5.4. Restriction to $B_{2n+1}$. We conclude this section with a result which will be used in Section 6.2. We denote by $f_{2n+1} : B_{2n+1} \to St(C_n, \mathbb{Z})$ (resp. $\bar{f}_{2n+1} : B_{2n+1} \to Sp_{2n}(\mathbb{Z})$) the restriction of $f$ (resp. of $\bar{f}$) to $B_{2n+1}$.

**Proposition 5.10.** The kernel of $f_{2n+1} : B_{2n+1} \to St(C_n, \mathbb{Z})$ is the normal closure of $[\sigma_3, \Delta_3^4]$ and $\Delta_3^4 \Delta_3^{-16}$.

**Proof.** The normal closure $N'$ of the proposition sits inside $\text{Ker} f_{2n+1}$. By [9] any element of $\text{Ker} f_{2n+1}$ is a product of conjugates of $\Delta_3^4$ and $\Delta_3^{-4}$. We have $\Delta_3^4 \equiv \Delta_3^{-16}$ modulo $N'$. Reasoning as in the proof of Theorem 5.8 (b), we conclude that $\Delta_3^4$ is central modulo $N'$. As above, there is a short exact sequence

$$1 \to \text{Ker} f_{2n+1} \to \text{Ker}(\bar{f}_{2n+1}) \to \langle w_1^4 \rangle \to 1.$$ 

The induced quotient short exact sequence is

$$1 \to \text{Ker} f_{2n+1}/N' \to \text{Ker}(\bar{f}_{2n+1})/N' \to \langle w_1^4 \rangle \to 1.$$ 

As in the proof of Theorem 5.8 (b), the map onto $\langle w_1^4 \rangle$ is bijective and we deduce that the kernel of $f_{2n+1}$ is $N'$.

$\square$

6. Extending $f$ to an epimorphism

By Theorem 3.1 the homomorphism $f : B_{2n+2} \to St(C_n, \mathbb{Z})$ is not surjective when $n \geq 3$. We shall now extend $f$ to a surjective homomorphism $\bar{f} : \hat{B}_{2n+2} \to St(C_n, \mathbb{Z})$, which we define in the following subsection. Let us assume that $n$ is a fixed integer $\geq 3$.

6.1. **The Artin groups $\hat{B}_k$.** For $k \geq 5$ let $\Gamma_k$ be the graph with $k$ vertices labeled $0, 1, \ldots, k - 1$ and with unique edges between the vertices labeled $i$ and $i + 1$ for $i \in \{1, \ldots, k - 2\}$ plus a unique edge between the vertices labeled $0$ and $4$.

![The graph $\Gamma_k$](image)

Let $\hat{B}_k$ be the Artin group (also called Artin–Tits group or generalized braid group) associated with the graph $\Gamma_k$; see [10, 12] or [17, Sect. 6.6]. It has a presentation with generators $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ and with the standard braid relations (3.1) and (3.2) between $\sigma_1, \ldots, \sigma_{k-1}$ together with the following additional relations involving the generator $\sigma_0$:

\begin{equation}
\sigma_0 \sigma_3 \sigma_0 = \sigma_4 \sigma_0 \sigma_4 \quad \text{and} \quad \sigma_0 \sigma_i = \sigma_i \sigma_0 \quad (i \neq 4).
\end{equation}

The corresponding Coxeter groups are infinite unless $5 \leq k \leq 7$. They are finite of type $\tilde{A}_5$ if $k = 5$, of type $\tilde{D}_6$ if $k = 6$, and of type $\tilde{E}_7$ if $k = 7$ (it is affine of type $E_7$ if $k = 8$).

In the same way as there are natural homomorphisms $B_k \to B_{k+1}$, there are natural (injective) homomorphisms $\hat{B}_k \to \hat{B}_{k+1}$ and $j : B_k \to \hat{B}_k$. The latter is defined by $j(\sigma_i) = \sigma_i$ for all $i \in \{1, \ldots, k - 1\}$.

Here is the reason why we introduce the Artin groups $\hat{B}_k$. 
Theorem 6.1. There exists a unique homomorphism \( \hat{f} : \hat{B}_{2n+2} \to \text{St}(C_n, \mathbb{Z}) \) such that \( \hat{f}(\sigma_0) = z_2 \) and \( \hat{f}(\sigma_i) = f(\sigma_i) \) for all \( i = 1, \ldots, 2n + 1 \). The restriction of the homomorphism \( \hat{f} \) to \( \hat{B}_{2n+1} \) is surjective.

Consequently, the homomorphism \( \hat{f} : \hat{B}_{2n+2} \to \text{St}(C_n, \mathbb{Z}) \) is surjective as well. It extends the non-surjective homomorphism \( f : B_{2n+2} \to \text{St}(C_n, \mathbb{Z}) \) in the sense that \( f = \hat{f} \circ \sigma \).

Proof. (a) Let us first check that \( \hat{f} \) is well-defined. Since it extends \( f \), we have only to deal with the relations (6.1) involving the additional generator \( \sigma_0 \). Now \( \hat{f}(\sigma_0) \) commutes with \( \hat{f}(\sigma_i) = f(\sigma_i) \) for all \( i \neq 4 \) in view of the defining relations of the Steinberg group (see Section 2.2). The image under \( \hat{f} \) of the relation \( \sigma_0 \sigma_4 \sigma_0 = \sigma_4 \sigma_0 \sigma_4 \) in (6.1) is equivalent to \( z_2 z_2' z_2 = z_2 z_2' z_2 z_2' \), which holds in \( \text{St}(C_n, \mathbb{Z}) \), as explained in the proof of Theorem 3.1.

(b) By definition of \( \hat{f} \) it suffices to establish that the image \( f(B_{2n+1}) \) of \( f \) together with \( z_2 \) generates the Steinberg group \( \text{St}(C_n, \mathbb{Z}) \).

To this end, it is sufficient to prove that the images of \( \hat{f}(B_{2n+1}) \) and \( z_2 \) in \( \text{Sp}_{2n}(\mathbb{F}_2) \) generate the latter finite group. Indeed, by Theorem 4.1 the subgroup \( H \) of \( \text{St}(C_n, \mathbb{Z}) \) generated by \( \hat{f}(B_{2n+1}) \) and \( z_2 \) contains the kernel \( \text{St}(C_n, \mathbb{Z})[2] \) of the surjective morphism \( \text{St}(C_n, \mathbb{Z}) \twoheadrightarrow \text{Sp}_{2n}(\mathbb{Z}) \to \text{Sp}_{2n}(\mathbb{F}_2) \). Hence, if the image of \( H \) is the whole group \( \text{Sp}_{2n}(\mathbb{F}_2) \), we obtain the equality \( H = \text{St}(C_n, \mathbb{Z}) \).

Let us denote the images in \( \text{Sp}_{2n}(\mathbb{F}_2) \) of the generators \( x_{i,j}, y_{i,j}, y'_{i,j}, z_i, z'_i \) of \( \text{St}(C_n, \mathbb{Z}) \) by \( \bar{x}_{i,j}, \bar{y}_{i,j}, \bar{y}'_{i,j}, \bar{z}_i, \bar{z}'_i \), respectively. Note that the latter generate \( \text{Sp}_{2n}(\mathbb{F}_2) \) and that each of them is of order 2, hence equal to its inverse.

Define \( E_n \) to be the set of the images of \( \{ \hat{f}(\sigma_1), \hat{f}(\sigma_2), \ldots, \hat{f}(\sigma_{2n}) \} \) in \( \text{Sp}_{2n}(\mathbb{F}_2) \). We have

\[
E_n = \{ \bar{z}_1, \bar{z}'_1, \bar{z}_1 \bar{z}_2 \bar{y}_1, \bar{z}_2, \bar{z}_2 \bar{z}_3 \bar{y}_2, \bar{z}_3, \ldots, \bar{z}_n \}.
\]

In order to prove Theorem 6.1, it is sufficient to check that \( E_n \cup \{ \bar{z}_2 \} \) generates \( \text{Sp}_{2n}(\mathbb{F}_2) \). We will establish this assertion by induction on \( n \).

Note that the assertion holds for \( n = 2 \): indeed, \( z_2 = \hat{f}(\sigma_5) \) so that the set \( E_2 \cup \{ \bar{z}_2 \} \) generates the image of \( \hat{f} \), which by the surjectivity result of Theorem 3.1 is equal to the whole group \( \text{Sp}_4(\mathbb{F}_2) \).

We now prove the assertion for \( n = 3 \).

Lemma 6.2. The set \( E_3 \cup \{ \bar{z}_2 \} \) generates \( \text{Sp}_6(\mathbb{F}_2) \).

Proof. Let \( G_3 \) be the subgroup of \( \text{Sp}_6(\mathbb{F}_2) \) generated by

\[
E_3 \cup \{ \bar{z}_2 \} = \{ \bar{z}_1, \bar{z}'_1, \bar{z}_1 \bar{z}_2 \bar{y}_1, \bar{z}_2, \bar{z}_2 \bar{z}_3 \bar{y}_2, \bar{z}_3, \bar{z}_2 \}.
\]

Obviously, \( G_3 \) contains \( \bar{y}_1, \bar{y}_2, \bar{z}_2 \bar{z}_3 \). We have the following relations deduced from Equations (2.2), (2.3), (2.8) and (2.10):

\[
[\bar{y}_1, \bar{z}'_2] = \bar{x}_1 \bar{z}_1, \quad \text{whence} \quad \bar{x}_1, \bar{z}_1 \in G_3;
\]

\[
[\bar{x}_1, \bar{z}_2 \bar{z}_3] = [\bar{x}_1, \bar{y}_2, \bar{z}_3] = \bar{y}_1, \quad \text{whence} \quad \bar{y}_1 \in G_3;
\]

\[
[\bar{y}_1, \bar{z}_3] = \bar{x}_3 \bar{z}_3, \quad \text{whence} \quad \bar{x}_3, \bar{z}_3 \in G_3;
\]

\[
[\bar{y}_1, \bar{z}'_2] = \bar{x}_2 \bar{z}_2, \quad \text{whence} \quad \bar{x}_2, \bar{z}_2 \in G_3;
\]
lies in the kernel of \( \hat{\alpha} \), whence \( \hat{x}_{1,3} \in G_3 \):

\[
[\hat{y}_{1,3}, \hat{z}_3'] = \hat{x}_{1,3} \hat{z}_3', \quad \text{whence } \hat{x}_{1,3} \in G_3; \\
[\hat{x}_{3,1} \hat{z}_3, \hat{x}_{1,2}] = [\hat{x}_{3,1}, \hat{x}_{1,2}] = \hat{x}_{3,2}, \quad \text{whence } \hat{x}_{3,2} \in G_3; \\
[\hat{x}_{3,2}, \hat{x}_{2,1}] = \hat{x}_{3,1}, \quad \text{whence } \hat{x}_{3,1} \in G_3.
\]

Since \( \hat{x}_{3,1} \) and \( \hat{x}_{3,1} \hat{z}_3 \) belong to \( G_3 \), so does \( \hat{z}_3 \). Now we know that all \( \hat{y}_{i,j} \), \( \hat{z}_i \) and \( \hat{z}_i' \) with \( i, j \in \{1, 2, 3\} \) and \( i \neq j \) belong to \( G_3 \). As follows from a remark in Section 2.2, these elements generate the whole group \( \text{Sp}_6(\mathbb{F}_2) \).

We resume Part (b) of the proof of Theorem 6.1. Assume that the assertion above holds for \( n \geq 3 \) and let us prove it for \( n + 1 \).

Let \( G_{n+1} \) be the subgroup of \( \text{Sp}_{2n+2}(\mathbb{F}_2) \) generated by \( E_n+1 \cup \{ \hat{z}_3 \} \). By the induction hypothesis, since \( E_n \) is a subset of \( E_{n+1} \), the group \( G_{n+1} \) contains \( \text{Sp}_{2n}(\mathbb{F}_2) \), viewed as the group of matrices in \( \text{Sp}_{2n+2}(\mathbb{F}_2) \) with entries equal to 0 for the indices \((i, n+1)\) and \((n+1, i)\) with \( i \neq n + 1 \) and for the indices \((i, 2n+2)\) and \((2n+2, i)\) with \( i \neq 2n+2 \). In particular, \( G_{n+1} \) contains \( \hat{z}_3 \) by Lemma 6.2.

Now consider the subgroup of \( G_{n+1} \) generated by

\[
\{ \hat{z}_2, \hat{z}_2', \hat{z}_2 \hat{z}_3 \hat{y}_{2,3}, \hat{z}_3, \hat{z}_3 \hat{z}_4 \hat{y}_{3,4}, \hat{z}_4', \ldots, \hat{z}_4'_{n+1} \} \cup \{ \hat{z}_3 \}.
\]

Since \( \hat{z}_3 \in G_{n+1} \), this subgroup is a subgroup of \( G_{n+1} \).

By the induction hypothesis applied to all subscripts increased by 1, this subgroup is isomorphic to the symplectic group \( \text{Sp}_{2n}(\mathbb{F}_2) \), now viewed as the group of matrices in \( \text{Sp}_{2n+2}(\mathbb{F}_2) \) with entries equal to 0 for the indices \((i, 1)\) and \((1, i)\) with \( i \neq 1 \) and for the indices \((i, n+2)\) and \((n+2, i)\) with \( i \neq n+2 \). It follows that all generators \( \hat{x}_{i,j} \), \( \hat{y}_{i,j} \), \( \hat{y}_i' \), \( \hat{z}_i \) of \( \text{Sp}_{2n+2}(\mathbb{F}_2) \) belong to \( G_{n+1} \), except possibly \( \hat{x}_{1,n+1} \), \( \hat{x}_{n+1,1} \), \( \hat{y}_{1,n+1} \) and \( \hat{y}_{n+1,1} \). But the latter also belong to \( G_{n+1} \) in view of the commutator relations

\[
[\hat{x}_{1,n}, \hat{x}_{n,n+1}] = \hat{x}_{1,n+1}, \quad [\hat{x}_{n+1,n}, \hat{x}_{n,n+1}] = \hat{x}_{n+1,1}, \\
[\hat{x}_{1,n}, \hat{y}_{n,n+1}] = \hat{y}_{1,n+1}, \quad [\hat{x}_{n+1,n}, \hat{y}_{n,n+1}] = \hat{y}_{n+1,1},
\]

which follow from Relations (2.2)–(2.4).

6.2. **Elements of the kernel of \( \hat{f} \).** Let \( \hat{f}_{2n+1} : \hat{B}_{2n+1} \to \text{St}(C_n, \mathbb{Z}) \) be the restriction of \( \hat{f} \) to \( \hat{B}_{2n+1} \). Recall the restrictions \( f_{2n+1} : B_{2n+1} \to \text{St}(C_n, \mathbb{Z}) \) and \( \hat{f}_{2n+1} : B_{2n+1} \to \text{Sp}_{2n}(\mathbb{Z}) \) defined in Section 5.4.

We consider the element

\[
\alpha_0 = (\sigma_1 \sigma_2 \sigma_1)^2 (\sigma_1 \sigma_3^{-1} \sigma_3)^{-2} (\sigma_1 \sigma_3^{-1} \sigma_3)
\]

of the Artin group \( \hat{B}_{2n+1} \) of type \( \Gamma_{2n+1} \). This element lies in the kernel of \( \hat{f}_{2n+1} \) since

\[
\hat{f}_{2n+1}(\alpha_0) = w_1^2 y_{1,2} y_{1,2}^{-1} y_{1,2}^{-1} = y_{1,2}^2 y_{1,2}^{-2} = 1.
\]

Note that mapping \( \sigma_0 \) to \( \sigma_3 \) and \( \sigma_i \) to \( \sigma_1 \) for \( i = 1, \ldots, 4 \), we obtain an isomorphism \( j \) from the subgroup \( \hat{B}_5 \) of \( \hat{B}_{2n+1} \) generated by \( \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0\} \) to the standard braid group \( B_6 \). The composed morphism

\[
\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0 \rangle \overset{j}{\to} B_6 \overset{\hat{f}}{\to} \text{St}(C_2, \mathbb{Z})
\]

is equal to the restriction of \( \hat{f}_{2n+1} \) and \( j \) maps \( \alpha_0 \) to \( \alpha_2 \), which by Theorem 5.6 lies in the kernel of \( f : B_6 \to \text{St}(C_2, \mathbb{Z}) \). We thus recover the fact that \( \alpha_0 \) belongs to \( \text{Ker}(\hat{f}_{2n+1}) \).
Theorem 6.3. The kernel of the restriction of \( \hat{f}_{2n+1} \) to the pure Artin group \( \hat{P}_{2n+1} \) of type \( \Gamma_{2n+1} \) is the normal closure of \( \alpha_0, [\sigma_3, \Delta_3^4] \) and \( \Delta_3^4 \Delta_3^{-16} \) in \( \hat{B}_{2n+1} \).

Proof. It follows from Proposition 5.10 that
\[
\text{Ker}(\hat{f}_{2n+1}) \cap P_{2n+1} = \text{Ker}(f_{2n+1}) \cap P_{2n+1}
\]
is the normal closure of \([\sigma_3, \Delta_3^4]\) and \(\Delta_3^4 \Delta_3^{-16}\) in \(B_{2n+1}\).

To determine the kernel in \(\hat{P}_{2n+1}\) we apply [13, Corollary 8] (and its proof). This result states that if \(\Gamma\) is a Coxeter graph and \(i\) a vertex of \(\Gamma\), the pure Artin group of type \(\Gamma\) is the semi-direct product of a normal subgroup generated by conjugates of the squares \(\sigma_j^2\) with \(j \in \Gamma\) and of the pure Artin group of type \(\Gamma \setminus \{i\}\).

We apply this to \(\Gamma = \Gamma_k\) and \(i = 0\). All generators \(\sigma_j\) are conjugate in \(\hat{B}_{2n+1}\); in particular their squares are conjugate to \(\sigma_0^2\). Consequently, \(\hat{P}_{2n+1}\) is the semi-direct product of a normal subgroup generated by elements conjugate to \(\sigma_0^2\) in \(\hat{B}_{2n+1}\) and of \(P_{2n+1}\). Since \(\alpha_0\) lies in \(P_4\sigma_0^2 \subset P_{2n+1}\sigma_0^2\) (see the proof of Theorem 5.8) and \(\alpha_0 \in \text{Ker}\hat{f}_{2n+1}\), we deduce the theorem. \(\square\)

Remark 6.4. By Theorem 6.1 the morphism \(\hat{f}_{2n+1} : \hat{B}_{2n+1} \to \text{St}(C_n, Z)\) is surjective. Composing it with the natural surjections \(\text{St}(C_n, Z) \to \text{Sp}_{2n}(Z) \to \text{Sp}_{2n}(\mathbb{F}_2)\), we obtain a surjective morphism \(\hat{B}_{2n+1} \to \text{Sp}_{2n}(\mathbb{F}_2)\). The latter epimorphism factors through \(W(\Gamma_{2n+1}) \to \text{Sp}_{2n}(\mathbb{F}_2)\), where \(W(\Gamma_{2n+1})\) is the Coxeter group associated with the graph \(\Gamma_{2n+1}\), since the image of each generator \(\sigma_i\) has order 2 for \(i = 0, \ldots, 2n\).

Remark 6.5. We can say a little more about the kernel of \(\hat{f}_{2n+1}\) when \(n = 3\). In this case the Coxeter group \(W(\Gamma_7)\) is of type \(E_7\) and its center has order 2. Let \(w_0\) be its non-trivial central element. Since the center of \(\text{Sp}_6(\mathbb{F}_2)\) is trivial, the element \(w_0\) has to be in the kernel of the above surjective morphism \(W(\Gamma_7) \to \text{Sp}_6(\mathbb{F}_2)\). Since the order of \(W(\Gamma_7)\) is twice the order of \(\text{Sp}_6(\mathbb{F}_2)\), this kernel is exactly \(\{1, w_0\}\). We deduce that any element of \(\text{Ker}\hat{f}_7\) is either in the pure Artin group \(\hat{P}_7\) of type \(\Gamma_7\), or in the coset \(w_0\hat{P}_7\), where \(w_0\) is a fixed preimage of \(w_0\) in \(\hat{B}_7\).

Since we know \(\text{Ker}\hat{f}_7 \cap \hat{P}_7\) by Theorem 6.3, it remains to determine the intersection of \(\text{Ker}\hat{f}_7\) with the coset \(w_0\hat{P}_7\). A computation shows that the image of \(w_0\) in \(\text{Sp}_6(\mathbb{Z})\) is trivial. Thus \(\hat{f}_7(w_0) = w_0^{4k} \in \text{St}(C_3, Z)\) for some integer \(k\). Consequently, \(w_0\Delta_3^{-4k}\) belongs to \(\text{Ker}\hat{f}_7\) and the full kernel \(\text{Ker}\hat{f}_7 : \hat{B}_7 \to \text{St}(C_3, Z)\) is the normal closure of \(\alpha_0, [\sigma_3, \Delta_3^4], \Delta_3^4 \Delta_3^{-16}\) and \(w_0\Delta_3^{-4k}\). We have not been able to determine the exponent \(k\).

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