Analytic Integral Solutions for Induced Gravitational Waves

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Abstract

We present analytic integral solutions for the second-order induced gravitational waves (GWs). After presenting all the possible second-order source terms, we calculate explicitly the solutions for the GWs induced by the linear scalar and tensor perturbations during matter- and radiation-dominated epochs.

\textit{Unified Astronomy Thesaurus concepts:} Cosmology (343); Large-scale structure of the universe (902); Gravitational waves (678)

1. Introduction

A series of detection of the gravitational waves (GWs) by the LIGO and Virgo collaborations (Abbott et al. 2016a, 2016b, 2017a, 2017b, 2017c, 2017d) has opened the era of multi-messenger astronomy led by GWs. The observed GW signals are originated from merging black holes and/or neutron stars, but there should be other events energetic enough to generate observable GWs. Such cosmological origins include cosmic strings (Vachaspati & Vilenkin 1985; Brandenberger et al. 1986), phase transition (Witten 1984; Hogan 1986), and preheating (Khlebnikov & Tkachev 1997). But the nonlinear nature of gravity tells us that there is a persistent source of GWs—they can be induced by other cosmological perturbations at nonlinear order (Mollerach et al. 2004; Ananda et al. 2007; Baumann et al. 2007); while the scalar-, vector-, and tensor-type perturbations are decoupled at linear order, they couple to each other and thus can be generated at nonlinear order (Noh & Hwang 2004; Hwang & Noh 2007). Especially, the second-order tensor perturbations, or GWs, induced by linear scalar perturbations\footnote{It should be noted that large scalar perturbations may well be induced by large tensor perturbations on small scales, leading to copious production of primordial black holes (Nakama & Suyama 2015, 2016). Such induced scalar perturbations subsequently can further induce tensor perturbations. This is an interesting possibility, but is beyond the scope of the present work.} may well be sizable if on small scales the primordial curvature perturbation is enhanced during inflation (Alabidi et al. 2012) or the density perturbation grows during an early matter-dominated (MD) epoch (Assadullahi & Wands 2009; Jedamzik et al. 2010; Alabidi et al. 2013).

In most of the literature, however, the study of second-order induced GWs has been focused on the MD epoch in one particular gauge choice for the scalar perturbations. The reason could be twofold. First, the (geometric) scalar perturbation remains constant even on sub-horizon scales during an MD epoch, but it decays quickly if it enters the horizon during a radiation-dominated (RD) epoch (see Equations (29) and (30)). Thus, even if the primordial curvature perturbation is enhanced on small scales, one would naively expect such an enhancement to disappear during an RD epoch so that there should be no sizable induced GWs from scalar perturbations (see, however, Inomata et al. 2019a, 2019b). Furthermore, there is an upper bound on the contribution of the primordial GWs from the observations on the cosmic microwave background (CMB): in terms of the so-called tensor-to-scalar ratio, $r < 0.07$ at the pivot scale $k = 0.05$ Mpc$^{-1}$ (BICEP2 Collaboration et al. 2016). This means on CMB scales the amplitude of the primordial GWs should be about 1/10 or even smaller than that of the primordial curvature perturbation. Thus, naturally scalar perturbations should be the most dominant source for the induced GWs compared to the other types of cosmological perturbations. For these reasons, it is very sensible to consider the induced GWs by scalar perturbations during an MD epoch.

Nevertheless, this does not mean at all that we have a complete and satisfactory understanding of second-order induced GWs. First, there is no a priori reason why scalar perturbations should be considered only in a particular gauge condition. Moreover, as the linear scalar perturbations depend on the choice of gauge, the second-order GWs induced by their quadratic combinations should be also dependent on the gauge conditions. This is obvious and indeed was noticed early (Arroja et al. 2009) but was explicitly shown only recently in Hwang et al. (2017). Second, while scalar perturbations are likely to be the most important source for induced GWs, other types of perturbations need not be neglected from the beginning. Especially, the linear tensor perturbations should be persistent. Furthermore, it is possible that the contributions of the tensor perturbations can be enhanced (Mukhanov & Vikman 2006); thus, the induced second-order GWs from them may well be significant accordingly.\footnote{More exactly, if the tensor-induced GWs are to be dominant, the scalar perturbations should not increase. Otherwise, the scalar-induced GWs are very likely to be more prominent than the tensor-induced ones. This condition allows certain models to be viable (e.g., Jain et al. 2009, 2010), but generally excludes models that include a break during inflation (e.g., Pi et al. 2019).} This indeed happens in certain concrete models beyond the standard slow-roll inflation (Brandenberger et al. 2007; Kobayashi et al. 2010; Gong 2014; Cai et al. 2015; Mylova et al. 2018). Third, as the universe has evolved through both RD and MD epochs, for a complete description of the induced GWs we need a proper understanding of the RD epoch as it is occurring. This was recognized early as well (Ananda et al. 2007; Assadullahi & Wands 2010), but an analytic approach has been taken only recently (Espinosa et al. 2018; Kohri & Terada 2018).

In this article, we provide analytic integral solutions for second-order induced GWs from both linear scalar and tensor...
perturbations. We also present the full second-order source terms with all three types of cosmological perturbations, so it should be straightforward to calculate the solutions from the sources with vector perturbations. This article is outlined as follows. In Section 2, we provide the full traceless evolution equation for the spatial metric tensor, including all the explicit second-order source terms. In Section 3, we solve the equations of motion for the linear cosmological perturbations that will be used in Section 4 to compute the analytic integral solutions for the second-order-induced GWs. We briefly summarize our results in Section 5. Some technical details are relegated to the appendix sections.

2. Second-order Equation

Our metric convention of a flat Friedmann universe including cosmological perturbations is

\[ ds^2 = -a^2(1 + 2\alpha)dt^2 - 2a^2B_i dx^i + a^2[(1 + 2\varphi)\delta_{ij} + 2\gamma_{ij} + 2C_{(i,j)} + 2h_{ij}]dx^i dx^j, \] (1)

where \( dt = dt/a \) is the conformal time and \( a(\eta) \) is the scale factor. The indices of the perturbation variables are raised and lowered by \( \delta_{ij} \). Further, the shear \( \chi_i \) is written as

\[ \chi_i = a(B_i + a\gamma_i + aC_i). \] (2)

The nonlinear equation necessary for the induced GWs can be obtained from the traceless evolution equation for the spatial metric. Up to second order, the full equation is given by Appendix A(A.4). Further, writing \( B_i = B_i^{(v)} + B_i^{(t)} \) with \( B_i^{(t)} = 0 \), the shear \( \chi_i \), anisotropic stress \( \Pi_{ij} \) and peculiar velocity \( v_i \) of the perfect fluid, whose energy density and pressure are written as \( \rho \) and \( p \), respectively, can be decomposed in terms of the scalar gradient, transverse vector, and transverse and traceless tensor components as

\[ \chi_i = a(\beta + a\gamma_i) + a(B_i^{(v)} + aC_i) = \chi_i^{(t)} \] (3)

\[ \Pi_{ij} = \frac{1}{a^2}\left(\Pi_{ij} - \frac{\delta_{ij}}{3}\Delta\Pi\right) + \frac{1}{a}\Pi_{ij}^{(v)} + \frac{\delta_{ij}}{3}\Pi^{(t)}_k, \] (4)

\[ v_i = -v_i^{(t)} \] (5)

where the superscripts \( (v) \) and \( (t) \) denote, respectively, the transverse vector and transverse and traceless tensor. Note that the last term of \( \Pi_{ij} \) is added because \( \delta_{ij} \Pi_{ij} = \Pi_{ii} \neq 0 \) at nonlinear order but is given by

\[ \Pi_{ij} = 2\delta_{ij} \Pi^v + \nu_i \nu_j \Pi_{ij} + \cdots. \] (6)

With these decompositions, the second-order traceless evolution equation (Equation (A.4)) can be written as

\[ \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\Delta}{a^2}h_{ij} - 8\pi G\Pi_{ij}^{(v)} + \frac{1}{a^2}\left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\Delta\right)\left[\frac{1}{a}\frac{d}{dt}(a\chi_i - \alpha - \varphi - 8\pi G\Pi_i) + \frac{1}{a}\left[\frac{1}{a^2}\frac{d}{dt}(a\chi_{i,j}^{(t)}) - 8\pi G\Pi_{i,j}^{(t)}\right]\right] = s_{ij}, \] (7)

where \( s_{ij} \) denotes all second-order terms, which serve as the second-order source. As there are scalar, vector, and tensor perturbations, at second order six combinations are possible:

1. Scalar–scalar source \( s_{ij}^{(ss)} \) denotes the collection of the products of two scalar perturbations:

\[
\begin{align*}
    s_{ij}^{(ss)} &= \frac{1}{a^2}\frac{d}{dt}\left[a(2\varphi\chi_{ij} + \varphi\chi_j + \varphi\chi_i)\right] + \frac{1}{a^2}(\kappa \chi_{ij} - 4\varphi \varphi_{ij} - 3\varphi_{ij}\varphi) + \frac{1}{a^2}\chi_{ij}\chi_{jk} \\& + \frac{1}{a^2}\left[2\alpha \chi_{ij} - H_{\alpha} \chi_{ij} + \alpha \chi_{ij} - 2(\alpha + \varphi)\alpha_{ij} - \alpha_{i,j} - 2\varphi_{i,j}\rho_{ij}\right] + 8\pi G(\rho + p)v_{i,j} - 16\pi G\varphi \frac{\Pi_{ij}}{a^2} \\& - \frac{\delta_{ij}}{3}\left[\frac{1}{a^2}\frac{d}{dt}\left[a(2\varphi \Delta \chi + 2\varphi \chi^{(t)})\right] + \frac{1}{a^2}(\kappa \Delta \chi - 4\varphi \Delta \varphi - 3\varphi^{(t)}\varphi_{ij}) + \frac{1}{a^2}\chi^{(t)}\chi_{ij}\right] \\& + \frac{1}{a^2}\left[2\alpha \Delta \chi - H_{\alpha} \Delta \chi + \alpha \Delta \chi - 2(\alpha + \varphi)\Delta \alpha - \alpha^{(t)}\alpha_{i,j} - 2\alpha^{(t)}\varphi_{i,j}\right] + 8\pi G(\rho + p)v^4 v_{i,j} - 16\pi G\varphi \frac{\Delta \Pi}{a^2}. \end{align*}
\] (8)
2. Scalar–tensor source $s_{ij}^{(st)}$ denotes the collection of the products of one of the scalar perturbations and tensor perturbations:

$$s_{ij}^{(st)} = \frac{d}{dt} \left[ \dot{h}_{ij} + 2 \left( \varphi \dot{h}_{ij} + \varphi h_{ij} + \frac{1}{a^2} h_{ij} + \frac{1}{a^2} \chi^{k}_{ij} (h_{ik,j} + h_{jk,i} - h_{ij,k}) \right) \right]$$

$$+ 3H \left[ \dot{h}_{ij} + 2 \left( \varphi \dot{h}_{ij} + \varphi h_{ij} + \frac{1}{a^2} h_{ij} + \frac{1}{a^2} \chi^{k}_{ij} (h_{ik,j} + h_{jk,i} - h_{ij,k}) \right) \right]$$

$$+ \alpha \frac{d}{dt} \left( \dot{h}_{ij} - \frac{1}{a^2} \chi^{k}_{ij} \dot{h}_{ik,j} + \frac{1}{a^2} \chi^{k}_{ij} (h_{ik,j} + h_{jk,i} - h_{ij,k}) \right)$$

$$- \frac{\delta_{ij}}{3} \left[ \frac{1}{a^2} 2 h^{kl} \chi^{kl}_{ij} \right] + 3H \left[ \frac{1}{a^2} 2 h^{kl} \chi^{kl}_{ij} \right]$$

$$+ \frac{1}{a^2} \chi^{k}_{ij} \dot{h}_{ik,j} - \frac{1}{a^2} \chi^{k}_{ij} \dot{h}_{ik,j}$$

$$- 16\pi G \left[ \varphi \Pi^{ij}_{ij} + \frac{1}{a^2} h_{ij} \Delta \Pi - \frac{\delta_{ij}}{3} h^{kl} \Pi^{kl}_{ij} \right]. \quad (9)$$

3. Tensor–tensor source $s_{ij}^{(tt)}$ denotes the collection of the products of two tensor perturbations:

$$s_{ij}^{(tt)} = \frac{d}{dt} \left( 2 h^{kl}_{ij} \dot{h}_{kl} \right) + 3H \left( 2 h^{kl}_{ij} \dot{h}_{kl} \right) - \frac{\delta_{ij}}{3} \left[ \frac{1}{a^2} 2 h^{kl}_{ij} \dot{h}_{kl} \right]$$

$$+ \frac{1}{a^2} \left( 2 h^{kl}_{ij} (h_{kl,j} - h_{kl,i} - h_{ij,l} - h_{ij,k}) - 2 h^{k}_{ij} \Delta h_{kl} - h^{kl}_{ij} (2 h_{kl,j} - 3 h_{kl,l}) \right) - 16\pi G \left( h^{kl}_{ij} \Pi^{kl}_{ij} - \frac{\delta_{ij}}{3} h^{kl} \Pi^{kl}_{ij} \right). \quad (10)$$

4. Scalar–vector source $s_{ij}^{(sv)}$ denotes the collection of the products of one of the scalar perturbations and vector perturbations:

$$s_{ij}^{(sv)} = \frac{1}{a^2} \frac{d}{dt} \left[ a \left( \chi_{ij}^{(s)} \alpha + 2 \chi_{ij}^{(s)} \varphi + \varphi \chi_{ij}^{(s)} + \varphi \chi_{ij}^{(v)} \right) \right]$$

$$+ \frac{1}{a^2} \frac{d}{dt} \left( \frac{1}{a^2} \chi_{ij}^{(v)} \right) - \frac{1}{a^2} \chi_{ij}^{(s)} \chi_{ij}^{(v)} + \frac{1}{a^2} \chi_{ij}^{(s)} \chi_{ij}^{(v)} \chi_{ij}^{(v)}$$

$$- \frac{\delta_{ij}}{3} \left[ \frac{1}{a^2} 2 a \chi^{v}_{ij} \chi^{v}_{ij} \chi^{v}_{ij} \right]$$

$$+ \frac{1}{a^2} \chi_{ij}^{(v)} \chi_{ij}^{(v)} - 16\pi G \left( \varphi \Pi^{ij}_{ij} - \frac{\delta_{ij}}{3} h^{kl} \Pi^{kl}_{ij} \right). \quad (11)$$

5. Vector–vector source $s_{ij}^{(vv)}$ denotes the collection of the products of two vector perturbations:

$$s_{ij}^{(vv)} = -\frac{1}{a^2} \chi_{ij}^{(v)} \chi_{ij}^{(v)} + \frac{1}{a^2} \left( \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} + \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} - \frac{\delta_{ij}}{3} \left( \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} \right) \right)$$

$$+ \frac{1}{a^2} \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} \chi_{ij}^{(v)} + 8\pi G \left( \varphi \Pi^{ij}_{ij} - \frac{\delta_{ij}}{3} h^{kl} \Pi^{kl}_{ij} \right). \quad (12)$$

6. Vector–tensor source $s_{ij}^{(vt)}$ denotes the collection of the products of one of the vector perturbations and tensor perturbations:

$$s_{ij}^{(vt)} = \frac{d}{dt} \left[ h^{kl}_{ij} \chi_{ij}^{(v)} \right] - \frac{1}{a^2} \chi_{ij}^{(v)} \chi_{ij}^{(v)} - \frac{\delta_{ij}}{3} \left[ \frac{1}{a^2} 2 h^{kl}_{ij} \chi_{ij}^{(v)} \right] + 3H \left[ \frac{1}{a^2} 2 h^{kl}_{ij} \chi_{ij}^{(v)} \right]$$

$$+ \frac{1}{a^2} \chi_{ij}^{(v)} \chi_{ij}^{(v)} + 16\pi G \left( h^{kl}_{ij} \Pi^{kl}_{ij} - \frac{\delta_{ij}}{3} h^{kl} \Pi^{kl}_{ij} \right). \quad (13)$$

Having sorted out all possible second-order source terms, we can proceed to find the solution of the second-order induced GWs as follows. First, we solve the linear equations and obtain their solutions. Then these linear solutions can be used to obtain the explicit.
form of the sources. After the transverse-traceless projection of the source $s_{ij}$ (see Equation (43)), we can solve the inhomogeneous equation for the tensor perturbations and obtain the analytic integral solutions.

3. Linear Solutions

3.1. Vector Perturbations at Linear Order

The following linear equations for the vector-type perturbations are derived respectively from the momentum constraint, traceless evolution Equation (7) and momentum conservation equation (Hwang & Noh 2007):

$$\frac{\Delta}{2a^3} \chi^{(v)}_i + 8\pi G (\rho + p) v^{(v)}_i = 0,$$

$$\frac{1}{a^2} \frac{d}{dt} \left( a \chi^{(v)}_i \right) - 8\pi G \Pi^{(v)}_i = 0,$$

$$\frac{1}{a^4(\rho + p)} \frac{d}{dt} \left[ a^4(\rho + p) v^{(v)}_i \right] + \frac{\Delta}{2a^2} \Pi^{(v)}_i = 0.$$

A great simplification is made in the case of vanishing vector-type stress, $\Pi^{(v)}_i = 0$. Then all the linear vector perturbations always vanish:

$$\chi^{(v)}_i = v^{(v)}_i = 0. \quad (17)$$

So among the possible sources to the second-order GWs, scalar–vector, vector–vector, and vector–tensor contributions are absent, and we have only scalar–scalar, scalar–tensor, and tensor–tensor sources.

3.2. Scalar Perturbations at Linear Order

With the perturbation in the extrinsic curvature $\kappa$ being written as

$$\kappa = 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2} \chi,$$

the complete set of the linear equations for scalar perturbations is

$$4\pi G\delta \rho + H \chi + \frac{\Delta}{a^2} \varphi = 0,$$

$$\kappa + \frac{\Delta}{a^2} \chi - 12\pi G (\rho + p) a\alpha = 0,$$

$$\dot{k} + 2H\kappa - 4\pi G (\delta \rho + 3\delta p) + \left( 3H + \frac{\Delta}{a^2} \right) \alpha = 0,$$

$$\dot{\chi} + H \chi - \varphi - \alpha - 8\pi G \Pi \chi = 0,$$

$$\dot{\delta \rho} + 3H (\delta \rho + \delta p) - (\rho + p) \left( \kappa - 3H\alpha + \frac{\Delta}{a} \right) = 0,$$

$$\frac{1}{a^4(\rho + p)} \frac{d}{dt} \left[ a^4(\rho + p) \nu \right] - \frac{1}{a} \alpha - \frac{1}{a(\rho + p)} \left( \dot{\delta \rho} + \frac{2}{3} \frac{\Delta}{a^2} \Pi \right) = 0.$$

While $\dot{\delta \rho} = c_s^2 \delta \rho + c_s \delta S$ with $\delta S$ being the entropy perturbation, for a barotropic fluid in the absence of $\delta S$, simply $c_s^2 = w$. Then, assuming no anisotropic stress ($\Pi = 0$) for simplicity, these equations become even simpler and allow analytic solutions.

3.2.1. Solutions for Linear Scalar Perturbations during an MD Epoch

The solutions of the scalar perturbations during an MD epoch are already given in Hwang (1994) up to linear order, in Hwang et al. (2012) up to second order and in Yoo & Gong (2016) up to third order, respectively. The solutions can be written conveniently in terms of the curvature perturbation $\varphi$, which does not decay but remains constant during an MD epoch even on sub-horizon scales. Thanks to gauge transformations, the solutions in one gauge are enough to find those in other gauges.

We can readily solve the linear equations for the scalar perturbations during an MD epoch ($p = 0$) without anisotropic stress ($\Pi = 0$). Summarizing, in the comoving gauge for which $\nu = \gamma = 0$, we find

$$\varphi_v = R, \quad \alpha_v = 0, \quad \chi_v = \frac{2}{5H} R, \quad \kappa_v = \frac{2}{5} \frac{k^2}{a^2 H} R, \quad \delta_v = \frac{2}{5} \frac{k^2}{a^2 H^2} R.$$

Here, the subscript $v$ means the solutions are written in the comoving gauge. These solutions can be by gauge transformation used to obtain solutions in different gauge conditions, e.g., zero-shear gauge for which $\beta = \gamma = 0$; thus, as the name stands, $\chi = 0$ (thus a
subscript $\chi$) and we find

$$
\varphi_\chi = \frac{3}{5} \mathcal{R}, \quad \alpha_\chi = -\frac{3}{5} \mathcal{R}, \quad \kappa_\chi = -\frac{9}{5} H \mathcal{R}, \quad \nu_\chi = -\frac{2}{5aH} \mathcal{R}, \quad \delta_\chi = \frac{6}{5} \left(1 + \frac{k^2}{3a^2H^2}\right) \mathcal{R}. 
$$

Note that we can read easily the well-known relation during an MD epoch between the initial amplitude of the curvature perturbation in the comoving gauge $\mathcal{R}$ and that in the zero-shear gauge, or the gravitational potential $\Phi = -\alpha_\chi = \varphi_\chi$, as

$$
\Phi = \frac{3}{5} \mathcal{R}. 
$$

3.2.2. Solutions for Linear Scalar Perturbations during an RD Epoch

During an RD epoch, the linear equations of motion are most readily solvable in the zero-shear gauge. With $w = 1/3$ and $a \propto \eta$ during an RD epoch, the equation of motion for the curvature perturbation $\varphi_\chi$ is obtained from the trace evolution equation combined with the energy constraint and traceless evolution equation as (see, e.g., Mukhanov 2005)

$$
\frac{d^2 \varphi_\chi}{d\eta^2} + \frac{4}{\eta} \frac{d \varphi_\chi}{d\eta} - \frac{\Delta}{3} \varphi_\chi = 0. 
$$

Then, with $z \equiv k\eta/\sqrt{3}$ where $1/\sqrt{3}$ is the sound speed during the RD epoch, we can straightforwardly find the linear solutions for the scalar perturbations:

$$
\varphi_\chi = 2\mathcal{R} \frac{j_0(z)}{z}, \\
\nu_\chi = -\frac{1}{aH} \left[ j_0(z) - 2 \frac{j_1(z)}{z} \right] \mathcal{R}, \\
\kappa_\chi = -6H \left[ j_0(z) - 2 \frac{j_1(z)}{z} \right] \mathcal{R}, \\
\alpha_\chi = -\varphi_\chi, \\
\delta_\chi = 4\mathcal{R} \left[ -j_0(z) + 2 \frac{j_1(z)}{z} + z j_2(z) \right].
$$

where $j_n$ is the first-kind spherical Bessel function of order $n$. The solutions in other gauges can be obtained by appropriate gauge transformations, e.g., the curvature perturbation in the comoving gauge as $\varphi_\nu = \varphi - aH\nu$. In the comoving gauge, the linear solutions for the scalar perturbations are

$$
\varphi_\nu = \mathcal{R} j_0(z), \\
\chi_\nu = \frac{1}{H} \left[ j_0(z) - 2 \frac{j_1(z)}{z} \right] \mathcal{R}, \\
\kappa_\nu = 3H^2 \left[ j_0(z) - 2 \frac{j_1(z)}{z} \right] \mathcal{R}, \\
\alpha_\nu = \left[ 2j_0(z) - 4 \frac{j_1(z)}{z} + z j_2(z) \right] \mathcal{R}, \\
\delta_\nu = 4\mathcal{R} \left[ -2j_0(z) + 4 \frac{j_1(z)}{z} + z j_2(z) \right].
$$

Here, we have set the coefficients in such a way that the initial amplitude of the curvature perturbation in the comoving gauge is, as for the solution during an MD epoch, $\mathcal{R}$, i.e., $\lim_{z \to 0} \varphi_\nu(z) = \mathcal{R}$. Note that since $\lim_{z \to 0} j_0(z) = 1$ and $\lim_{z \to 0} j_1(z)/z = 1/3$, we can find the well-known relation during an RD epoch between the comoving curvature perturbation $\mathcal{R}$ and the gravitational potential $\Phi$ as $\Phi = 2\mathcal{R}/3$.

3.3. Tensor Perturbations at Linear Order

Decomposing the tensor perturbations in terms of the two polarization tensors $e^\pm$ and $e^\times$ in the Fourier space,

$$
h_{ij}(t, x) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} h_{ij}(t, k) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \{ h_{ij}(t, k) e^\pm(k) + h_{ij}(t, k) e^\times(k) \}.
$$

Since the polarization tensors are orthogonal to each other, i.e.,

$$
e^\pm_i e^\pm_j = 0 \quad \text{and} \quad e^\pm_i e^\times_j = e^\times_i e^\pm_j = 0, \\
e^\pm_i e^\times_j = e^\times_i e^\times_j = 1,
$$
we can invert this to find

\[ h_\lambda(t, k) = e^{i \mathbf{k} \cdot \mathbf{x}} \int d^3 x e^{-i \mathbf{k} \cdot \mathbf{x}} h_\lambda(t, x) \]  

(33)

for each polarization \( \lambda \). Then the linear equation of motion for each polarization mode is identical as

\[ \ddot{h} + 3H \dot{h} + \frac{k^2}{a^2} h = 0, \]  

(34)

where we have omitted the polarization index \( \lambda \). Introducing \( v \equiv ah \) and moving to the conformal time \( d\eta = dt/a \), the equation becomes

\[ \frac{d^2 v}{d\eta^2} + \left( k^2 - \frac{1}{a^2} \frac{d^2 a}{d\eta^2} \right) v = 0. \]  

(35)

**3.3.1. Solutions for Linear Tensor Perturbations during an MD epoch**

During an MD epoch, \( a \propto \eta^2 \) so that

\[ \frac{1}{a^2} \frac{d^2 a}{d\eta^2} = \frac{2}{\eta^2}. \]  

(36)

Thus, in terms of a new variable \( x \equiv k\eta \), Equation (35) becomes

\[ \frac{d^2 v}{dx^2} + \left( 1 - \frac{2}{x^2} \right) v = 0. \]  

(37)

The general solution of this equation is

\[ v = c_1 j_1(x) + c_2 y_1(x), \]  

(38)

where \( y_n \) is the second-kind spherical Bessel function of order \( n \). Since \( \lim_{x \to 0} j_1(x)/x = -\infty \) and \( \lim_{x \to 0} y_1(x)/x = 1/3 \), we choose \( j_1(x)/x \) as the proper solution, with the value at \( x \to 0 \) being the primordial value for the tensor perturbation \( h_\lambda^\lambda(k) \) for each polarization \( \lambda \):

\[ h_\lambda(\eta, k) = 3h_\lambda^\lambda(k) j_1(k\eta) \]  

(39)

**3.3.2. Solutions for Linear Tensor Perturbations during an RD Epoch**

The basic equation of motion for the tensor perturbations is essentially the same as that during an MD epoch. That is, with \( v \equiv ah \), we have the same linear equation for \( v \) given by Equation (35). The only difference is that since \( a \propto \eta \) during an RD epoch, with \( x \equiv k\eta \), Equation (35) is simply

\[ \frac{d^2 v}{dx^2} + v = 0, \]  

(40)

and the general solution is

\[ v = ah = c_1 j_0(x) + c_2 y_0(x). \]  

(41)

Since \( \lim_{x \to 0} j_0(x) \to -\infty \) and \( \lim_{x \to 0} y_0(x) = 1 \), we choose \( j_0(x) \) as the proper solution, with the value at \( x \to 0 \) being the primordial value for the tensor perturbation \( h_\lambda^\lambda(k) \) for each polarization \( \lambda \):

\[ h_\lambda(\eta, k) = h_\lambda^\lambda(k) j_0(k\eta). \]  

(42)

**4. Second-order Solutions for Induced GWs**

**4.1. Equation of Motion for Tensor Perturbations with Source**

To extract only the tensor parts from Equation (7), we apply the transverse-traceless projection so that \( s_{ij} \) on the right-hand side becomes what only sources tensor perturbations, \( s_{ij}^{(\text{tensor})} \) (Hwang & Noh 2007):

\[ s_{ij}^{(\text{tensor})} = s_{ij} - 3 \left( \delta_{ij} \partial_j \right) \Delta^{-2} \partial_k \partial^k s_{ijkl} - 2 \Delta^{-1} \partial_j \partial^k s_{ijkl} + 2 \Delta^{-2} \partial_k \partial_j \partial^k \partial^l s_{ijkl} \]  

\[ = s_{ij} - 2 \Delta^{-1} \partial_j \partial^k s_{ijkl} + \frac{1}{2} \Delta^{-2} \left( \partial_j \partial_j + \delta_{ij} \Delta \right) \partial_k \partial^k s_{ijkl}. \]  

(43)
Since the two traceless polarization tensors are orthogonal to each other, we can extract the individual equation of each polarization mode \( h_\lambda \) by multiplying the corresponding polarization tensor \( e_{ij}^\lambda \). Moreover, since \( e_{ij}^\lambda k_i = e_{ij}^\lambda k_j = 0 \), we have in the Fourier space

\[
e_{ij}^\lambda(k) \int d^3xe^{-ik\cdot x}s_{ij}^{(\text{tensor})}(x) = e_{ij}^\lambda(k) \int d^3xe^{-ik\cdot x}\int\frac{d^3q}{(2\pi)^3} e^{iq\cdot x}s_{ij}^{(\text{tensor})}(q)
\]

\[
= e_{ij}^\lambda(k) \int d^3q \left[s_{ij} - \frac{q_iq_j}{q^2}s_k^k + \frac{q_iq_k}{q^2}s_j^k + \frac{1}{2} \left( \frac{q_iq_jq_k}{q^4} + \frac{q_kq_j}{q^2}s_{ij}^k \right) s_k^k \right] \int \frac{d^3x}{(2\pi)^3} e^{-i(k-q)\cdot x}
\]

\[
= e_{ij}^\lambda(k) s_{ij}(k).
\]

Thus, for each polarization \( \lambda \) the equation of motion is

\[
\ddot{h}_\lambda(t, k) + 3\dot{H}h_\lambda(t, k) + \frac{k^2}{a^2}h_\lambda(t, k) = e_{ij}^\lambda(k) s_{ij}(k).
\]

One more simplification is ahead. Since the source term \( s_{ij} \) in the above equation is multiplied by the traceless polarization tensor \( e_{ij}^\lambda \), the terms proportional to \( \delta_{ij} \) in \( s_{ij} \) identically vanish on the right-hand side of Equation (45). Thus, in the absence of the anisotropic stress, the source terms that survive Equation (45) are

\[
s_{ij}^{(ax)} = \frac{1}{a^2} \frac{d}{dt} \left[ a(2\varphi\chi_{ij} + \varphi_j\chi_i + \varphi_i\chi_j) \right] + \frac{1}{a^2} \left( \kappa\chi_{ij} - 4\varphi\varphi_{ij} - 3\varphi_i\varphi_j \right) + \frac{1}{a^2} \chi_{ij}^k\chi_{jk}
\]

\[
+ \frac{1}{a^2} \left[ 2\dot{\alpha}\chi_{ij} - h\chi_{ij} + \dot{\alpha}\chi_{ij} - 2(\alpha + \varphi)\alpha_{ij} - \alpha\alpha_{ij} - 2\varphi(\alpha_{ij}) \right] + 8\pi G(\rho + p)\gamma_{ij},
\]

\[
s_{ij}^{(\text{ax})} = \frac{d}{dt} \left( 2h_k^i\dot{h}_{jk} \right) + 3H(2h_k^i\dot{h}_{jk})
\]

\[
+ \frac{1}{a^2} \left[ 2\dot{h}_{kk}(h_{ij,k} + h_{ij,k} - h_{ij,j}) - 2h_k^i\Delta h_{jk} - h_{kk}^j h_{kj,i} + 2h_k^i(k(h_{jk,k} - h_{jk,i})) \right],
\]

\[
s_{ij}^{(\text{ax})} = \frac{d}{dt} \left( h_{ij}\alpha + 2(\varphi h_{ij} + \varphi_{ij} + \frac{1}{a^2} h_k^k\chi_{jk} + \chi_{ij}^k(h_{kk,j} + h_{kk,j} - h_{kk,k}) \right)
\]

\[
+ 3H \left[ h_{ij}\alpha + 2(\varphi h_{ij} + \varphi_{ij} + \frac{1}{a^2} h_k^k\chi_{jk} + \chi_{ij}^k(h_{kk,j} + h_{kk,j} - h_{kk,k}) \right]
\]

\[
+ \alpha \frac{d}{dt}(h_{ij}) - \frac{1}{a^2} \chi_{ij}^k h_{jk,j} + n\dot{h}_{ij} + \frac{1}{a^2} \left[ -2h_k^k\alpha_{ij} - (h_{ij,j} + h_{ij,j} - h_{ij,k}) \alpha_{jk} \right] + \frac{1}{a^2} \chi_{ij}^k h_{jk} - \frac{1}{a^2} \chi_{ij}^j h_{jk} + \frac{1}{a^2} \left[ -2\varphi\Delta h_{ij} + h_k^k\varphi_{jk} - h_{ij}\Delta\varphi \right] + \varphi\varphi_{ij}(h_{jk,k} + h_{jk,j} - 3h_{jk,k})].
\]

To find the solution for this inhomogeneous equation, we use the Green’s function solution (for earlier attempts to apply the Green’s function solutions to cosmological perturbations, see Stewart & Gong 2001; Gong & Stewart 2002). That is, let \( L \) be a linear second-order differential operator, and the equation we want to solve is of the form

\[
Ly(x) = r(x),
\]

with the two homogeneous solutions being \( y_1 \) and \( y_2 \). Then, the full solution is given by

\[
y(x) = (\text{appropiate combination of } y_1 \text{ and } y_2 \text{ according to the boundary conditions})
\]

\[
+ \int d\bar{x}r(\bar{x}) \frac{y_1(\bar{x})y_2(x) - y_2(\bar{x})y_1(x)}{y_1(\bar{x})y_2'(\bar{x}) - y_2(\bar{x})y_1'(\bar{x})}G(s, \bar{x})_s
\]

We have seen that the homogeneous solutions during an MD epoch are given by Equation (38), then the Green’s function during an MD epoch becomes

\[
G_{\text{MD}}(\eta, \bar{\eta}) = \frac{x}{k}[J_0(\bar{x})y_1(x) - J_0(x)y_1(\bar{x})].
\]

With the homogeneous solutions during the RD epoch being given by Equation (41), the Green’s function during the RD epoch becomes

\[
G_{\text{RD}}(\eta, \bar{\eta}) = \frac{x}{k}[J_0(\bar{x})y_0(x) - J_0(x)y_0(\bar{x})].
\]

Comparing with Equation (51), the only difference is the order of the spherical Bessel functions inside the square brackets.
Thus, we expect the second-order induced GWs sourced by the product of two linear perturbations $X$ and $Y$ would be, during both MD and RD epochs, of the general integral form:

$$h_{\eta}(\eta, k) = \frac{1}{a} \int_0^\eta d\tilde{\eta} [a^2(k) e^{\eta}(k) s_\eta(k)] G(\eta, \tilde{\eta}) = \int \frac{d^3 q}{(2\pi)^3} [e^{\eta}(k) (\cdots)_{\eta}] X_0(k - q) Y_0(q) \int_0^\xi d\tilde{\xi} K(\tilde{\xi}, k, q).$$

(53)

Here, the integral over an internal momentum $q$ is because the source $s_\eta$ is the product of two perturbations $X$ and $Y$, it is written as a convolution in the Fourier space. The terms inside the square brackets constitute the (dimensionless) projection of the polarization tensor $e^{\eta}(k)$. $X_0(k)$ and $Y_0(k)$ denote, respectively, the initial amplitudes of $X$ and $Y$, i.e., $\mathcal{R}(k)$ and/or $h_{\eta}(k)$. Finally, the integral over $\tilde{\xi}$ is the kernel which is a function of both momenta as well as time. Our main concern in finding this analytic integral solution is to compute this kernel. In the following, we proceed with the sources Equations (46)–(48) to calculate the closed analytic form of the kernel.

4.2. Scalar–Scalar–induced GWs during an MD epoch

We first consider the scalar–scalar source Equation (46). The analytic integral solutions in various gauge conditions are given only very recently in Hwang et al. (2017) so we can check our results in this section. We consider only two gauges, comoving and zero-shear gauges in which the solutions of the linear scalar perturbations during an MD epoch are given, respectively, by Equations (25) and (26). We first compute the Fourier component of $s^{(\eta)}_{\eta}$ in the zero-shear gauge for which $\chi = 0$ and $\alpha = -\varphi$, so $s^{(\eta)}_{\eta}$ is greatly simplified. After straightforward calculations we find in the zero-shear gauge the source $s^{(\eta)}_{\eta}$ purely in terms of the initial perturbation $\mathcal{R}$ as

$$e^{\eta}(k) s^{(\eta)}_{\eta}(k) = \frac{1}{a^2} \int \frac{d^3 q}{(2\pi)^3} [e^{\eta}(k) q_\eta q_j] \mathcal{R}(k - q) \mathcal{R}(q).$$

(54)

Note that other than the overall $1/a^2$, there is no time dependence. Then for each polarization, the solution of the GWs induced by the scalar–scalar source in the zero-shear gauge is

$$h_{\eta}(\eta, k) = \frac{6}{5} \left[1 - 3 \frac{j_1(k\eta)}{k\eta}\right] \int \frac{d^3 q}{(2\pi)^3} \left[e^{\eta}(k) q_\eta q_j\right] \mathcal{R}(k - q) \mathcal{R}(q).$$

(55)

Compared with the general form of the solution Equation (53), the kernel is a function of $k$ and $\eta$ in the specific combination $k\eta$, and thus can be pulled out of the internal momentum integral.

Likewise, in the comoving gauge, we have $v = 0$ and $\alpha = 0$ during the MD epoch, so that the scalar–scalar source Equation (46) becomes

$$e^{\eta}(k) s^{(\eta)}_{\eta}(k) = \frac{1}{a^2} \int \frac{d^3 q}{(2\pi)^3} [e^{\eta}(k) q_\eta q_j] \left[1 - \frac{2}{25} \frac{k^2}{a^2 H^2}\right] \mathcal{R}(k - q) \mathcal{R}(q),$$

(56)

where the additional term comes from the spatial gradient of the shear. Then we can find trivially the solution as, using $aH = 2/\eta$ during the MD epoch,

$$h_{\eta}(\eta, k) = \frac{6}{5} \left[1 - 3 \frac{j_1(k\eta)}{k\eta} - \frac{(k\eta)^2}{60}\right] \int \frac{d^3 q}{(2\pi)^3} \left[e^{\eta}(k) q_\eta q_j\right] \mathcal{R}(k - q) \mathcal{R}(q).$$

(57)

Again, the kernel is a function of only $k\eta$. The reason why we have such a simple kernel for the scalar–scalar source is because the scalar perturbations can be written in terms of the constant $\mathcal{R}$, so the time integral is greatly simplified. These scalar–scalar-induced solutions in the zero-shear gauge (Equation (55)) and comoving gauge (Equation (57)) agree with Hwang et al. (2017). In Figure 1, we show the kernels barring the factor 6/5. As $k\eta$ becomes bigger, the zero-shear gauge kernel approaches 1 while that in the comoving gauge increases as $(k\eta)^2$. Thus, on small scales the amplitude of the induced GWs in the comoving gauge is much bigger than that in the zero-shear gauge. This shows clearly the gauge dependence of the scalar-induced GWs.

4.3. Scalar–Tensor–induced GWs during an MD epoch

Next we consider the GWs induced by the scalar–tensor source during an MD epoch. Considering first the zero-shear gauge in which $\chi = 0$ and $\alpha = -\varphi$, we find (omitting the subscript $\chi$ for the scalar perturbation $\varphi$)

$$s^{(\eta)}_{\eta}(k) = \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^3} \delta^{(3)}(k - q_1 - q_2) \frac{1}{2} \left[\left(\frac{d}{dt} + 3H\right)(\varphi_1 h_{1\eta} + 2\varphi_1 h_{2\eta}) - \varphi_1 \frac{d}{dt}(h_{1\eta}) + \kappa_{\eta} h_{2\eta} + \frac{2}{a^2} (2k^2 q_{2\eta} + q_1^2) \varphi_1 h_{2\eta}\right] e^{\eta}(q_1),$$

(58)

where the subscript 1 for a perturbation variable means it is a function of $q_1$, e.g., $\varphi_1 = \varphi(q_1)$ and so on. Using $\varphi_\chi = 3\mathcal{R}/5$ and $\kappa_\chi = -9H\mathcal{R}/5$ during the MD epoch with $\mathcal{R} = $ constant, the terms including the time derivative of the linear tensor perturbation
vanish and we have the following simple expression for the source:

$$e^{ij}_x(k) s^{(\alpha\beta)}_y(k) = \frac{1}{a^2} \int \frac{d^3q}{(2\pi)^3} \{e^{ij}_x(k)e^{ij}_y(q)\}(k^2 + q^2) \mathcal{R}(k-q) h_{\alpha\beta}(\eta, q). \tag{59}$$

It is very important to note that the linear order perturbation $h_{\alpha\beta}$ does possess time dependence as given by Equation (42). Thus, unlike the scalar–scalar source case, the time integral for the Green’s function solution includes another time-dependent function from $h_{\alpha\beta}$, namely, $j_i(k\eta)/(k\eta)$:

$$h_{\alpha\beta}(\eta, k) = \frac{18}{5} \int \frac{d^3q}{(2\pi)^3} \{e^{ij}_x(k)e^{ij}_y(q)\}(q^2 + k^2) \mathcal{R}(k-q) h_0^{\alpha\beta}(q) \frac{1}{kq} \int y_1(x) \int d\tilde{x}^2 \int \left( \frac{q}{k} \right) j_i(\tilde{x}) - j_i(x) \int \tilde{x}^2 ds \int \left( \frac{q}{k} \right) y_1(\tilde{x}). \tag{60}$$

The detail of the $\tilde{x}$ integrals is given in Appendix B.1.

In the comoving gauge where $v = 0$, using $\mathcal{R} = \text{constant}$ and

$$\frac{d}{dt} \left( \frac{1}{a^2} \right) = -\frac{1}{2a^2}, \tag{61}$$

which follows from $H = 2/(3t)$ during the MD epoch, straightforward calculations give

$$e^{ij}_x(k) s^{(\alpha\beta)}_y(k) = \frac{2}{a^3} \int \frac{d^3q}{(2\pi)^3} \{e^{ij}_x(k)e^{ij}_y(q)\}\mathcal{R}(k-q)h_0^{\alpha\beta}(q) \left[ \frac{1}{5H}(k^2 + q^2) h_{\alpha\beta}(q) + k^2 h_{\alpha\beta}(q) \right]. \tag{62}$$

Unlike the zero-shear gauge case, the time derivative of the linear tensor perturbation remains, which is given by

$$\dot{h}_{\alpha\beta}(q) = \frac{q}{a} \frac{d}{d(q\eta)} h_{\alpha\beta}(\eta, q) = \frac{q}{a} \left[ 3h_0^{\alpha\beta}(q) \left[ \frac{-3j_1(q\eta)}{(q\eta)^2} + \frac{j_2(q\eta)}{q\eta} \right] \right]. \tag{63}$$

We can find the solution in a manner very similar to Equation (60) but with different $s^{(\alpha\beta)}_y$ as

$$h_{\alpha\beta}(\eta, k) = \frac{6}{kq_0} \int \frac{d^3q}{(2\pi)^3} \{e^{ij}_x(k)e^{ij}_y(q)\}\mathcal{R}(k-q)h_0^{\alpha\beta}(q) \int d\tilde{x}^2 \int \left[ \frac{3q^2 + 7k^2 j_1(q\tilde{x}/k)}{q^2/10k} + \frac{k^2 - q^2}{10j_0(q\tilde{x})} \right]$$

$$\times \left[ j_i(\tilde{x})y_1(x) - j_i(x)y_1(\tilde{x}) \right]$$

$$= 6 \int \frac{d^3q}{(2\pi)^3} \{e^{ij}_x(k)e^{ij}_y(q)\}\mathcal{R}(k-q)h_0^{\alpha\beta}(q) \left[ \frac{q^2 + 5k^2}{10(q^2 - k^2)} j_1(k\eta) - \frac{5q^2 + 7k^2}{20(q^2 - k^2)} j_2(q\eta) + \frac{1}{10} j_0(q\eta) \right]. \tag{64}$$

This is the solution of the induced $h_{\alpha\beta}(k, \eta)$ from the scalar–tensor source in the comoving gauge. In Figure 2 we show the kernels in both gauges. For $k\eta \gtrsim 1$ with sizeable $q/k$, the comoving gauge kernel exhibits more rapid oscillations. But for small $q/k$, both kernels are approximated by $1/3 - j_1(k\eta)/(k\eta) + C q^2/k^2$ so they behave similarly. Compared to the scalar–scalar-induced GWs as we have seen previously, the difference in the zero-shear gauge and comoving gauge is not prominent. This is because the gradient of the shear, that gives rise to the huge difference on small scales for the scalar–scalar-induced GWs, is highly suppressed by the exponentially decaying linear GWs on small scales.
4.4. Tensor–Tensor-induced GWs during an MD epoch

Next we consider the tensor–tensor source during an MD epoch. Multiplying the polarization tensor $e^\ell_i(k)$ and separating the time-dependent part gives

$$e^\ell_i(k)\xi^{(\ell)}_k = \frac{1}{4\pi} \int d^3q d^3q' \delta^{(3)}(k - q_1 - q_2) \left\{ 9h_{\ell q_1 q_2}^i \xi^{(\ell)}_k (k) \xi^{(\ell)}_k (k) \xi^{(\ell)}_k (k) \xi^{(\ell)}_k (k) \right\} \frac{q_1 q_2}{k^2} \left( -\frac{3}{q_1^2 q_2^2} \delta_{ij} \bar{h}_j \right) \left( -\frac{3}{q_1^2 q_2^2} \delta_{ij} \bar{h}_j \right) \left( -\frac{3}{q_1^2 q_2^2} \delta_{ij} \bar{h}_j \right)$$

As can be seen, only the Bessel function terms contain time dependence. Then the solution for $h_\lambda$ can be written as

$$h_\lambda(t, k) = \int d^3q d^3q' \delta^{(3)}(k - q_1 - q_2) \left\{ C(q_1, q_2) + (q_1 \leftrightarrow q_2) \right\} \frac{9}{q_1^2 q_2^2} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \left( -\frac{3}{q_1^2 q_2^2} \delta_{ij} \bar{h}_j \right)$$

where $\bar{q}_1 \equiv q_1/k$ and $\bar{q}_2 \equiv q_2/k$, respectively.

Now, using the recurrence relation (B.4) for $n = 1$, we find

$$j_2(\bar{q}_2 x) = \frac{3}{\bar{q}_2 x} j_1(\bar{q}_1 x) - \bar{q}_2 j_0(\bar{q}_2 x) = \frac{3}{\bar{q}_2 x} \left[ \frac{j_1(\bar{q}_1 x)}{x} - \frac{\bar{q}_2 j_0(\bar{q}_2 x)}{3} \right]$$

so that the terms multiplied by $C(q_1, q_2)$ become very simple as

$$\left[ \frac{9}{q_1^2 q_2^2} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \delta_{ij} j_1(\bar{q}_1 x) j_1(\bar{q}_2 x) - \frac{3}{q_1^2 q_2^2} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \delta_{ij} j_0(\bar{q}_1 x) j_0(\bar{q}_2 x) - \frac{3}{q_1^2 q_2^2} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \delta_{ij} j_1(\bar{q}_1 x) j_1(\bar{q}_2 x) + \frac{1}{\bar{q}_1 \bar{q}_2} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \delta_{ij} j_1(\bar{q}_1 x) j_1(\bar{q}_2 x) \right] f_1(\bar{x})$$

where $f_1(\bar{x})$ denotes both first- and second-kind of the spherical Bessel functions. Since $\lim_{\bar{x} \to 0} \bar{x} = -1/\bar{x}^2 - 1/2 + \cdots$ and $\lim_{\bar{x} \to 0} \bar{x}^2 \bar{x} \sim \bar{x}^2$, we always have converging results. Thus, we can write Equation (66) as

$$h_\lambda(t, k) = \int d^3q d^3q' \delta^{(3)}(k - q_1 - q_2) \left\{ C(q_1, q_2) + (q_1 \leftrightarrow q_2) \right\} \frac{1}{x} \int_0^\infty d\bar{q}_1 d\bar{q}_2 \left[ j_1(\bar{q}_1 x) j_1(\bar{q}_2 x) j_1(\bar{q}_1 x) j_1(\bar{q}_2 x) \right]$$

Figure 2. The kernels in the (left) zero-shear gauge and (right) comoving gauge as a function of $k$. Since the kernels are also dependent on $q$, we set (solid lines) $q/k = 0.05$, (dashed lines) $q/k = 2$ and (dotted lines) $q/k = 20$ in both panels.
Integrals can be performed analytically, with the details given in Appendix B.2. Performing the momentum integral using the delta function, finally Equation (69) becomes

\[ h_0(\eta, k) = \int \frac{d^3q}{(2\pi)^3} 18\eta \phi_0(k) h_{\phi}(k-q) \left( e^{ik\phi}(k)e^{iq\phi}(k-q) F_{\text{MD}}(k, \eta) \right. \]
\[ + \frac{1}{k^2} \left( e^{ik\phi}(k)e^{iq\phi}(k)q_k e^{iq\phi}(k-q) + q_k k_q e^{iq\phi}(k-q) + k_k k_q e^{iq\phi}(k-q) \right) \]
\[ \left. + e^{ik\phi}(k) \left\{ \frac{1}{2} q_k q_{\phi} e^{iq\phi}(k-q) + q \cdot (k-q) e^{iq\phi}(k-q) - q_k k_{\phi} e^{iq\phi}(k-q) \right\} \right) G_{\text{MD}}(k, \eta), \]  

(70)

where the kernels \( F_{\text{MD}} \) and \( G_{\text{MD}} \) are given, respectively, by Equations (B.16) and (B.19). In Figure 3 we show \( F_{\text{MD}} \) and \( G_{\text{MD}} \).

4.5. Scalar–Scalar-induced GWs during an RD Epoch

Until now we have considered the second-order solutions for the induced GWs during an MD epoch. Now we consider the solutions during an RD epoch. Especially, after the MD epoch, the gauge dependence of the induced GWs during an RD epoch disappears. This is not the case, at least regarding the solutions in the two gauges as can be seen in Equations (74) and (76) (see also Tomikawa & Kobayashi 2020).

An important difference for the linear solutions for the scalar perturbations is that now the curvature perturbation \( \varphi \) does not stay constant, but decays once the mode enters the horizon. Thus, the scalar–scalar-induced GWs during an RD epoch do not behave simply as we have seen during the MD epoch, but exhibit rapid oscillations. We first consider the zero-shear gauge, in which the scalar–scalar source reads rather simply as

\[ s_{ij}^{(ss)}(k) = \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{a^2} q_i q_j \varphi_\alpha(q) \varphi_\alpha(k-q) + 8\pi G (\rho + p) q_i q_j \varphi_\alpha(q) \varphi_\alpha(k-q) \right\}. \]  

(71)

With the scalar solutions given by Equation (29) and defining

\[ \frac{q \eta}{\sqrt{3}} = \frac{q}{\sqrt{3}k} \equiv \tilde{q}_1 x \quad \text{and} \quad \frac{|k - q| \eta}{\sqrt{3}} = \frac{|k - q| k}{\sqrt{3}k} \equiv \tilde{q}_2 x, \]  

(72)

we can find

\[ e^{ik\phi}(k) s_{ij}^{(ss)}(k) = \frac{4}{a^2} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{k^2} e^{iq \phi}(k)q_i q_j \right\} R(q) R(k-q) \left\{ 6 \frac{j_0(\tilde{q}_1 x) j_1(\tilde{q}_2 x)}{\tilde{q}_1 x} - 2j_0(\tilde{q}_1 x) j_1(\tilde{q}_2 x) \tilde{q}_x \right\} \tilde{q}_x. \]  

(73)

Thus, the inhomogeneous solution during an RD epoch that contains the integral including \( G_{\text{RD}}(\eta, \tilde{\eta}) \) is

\[ h_0(\eta, k) = 4 \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{e^{ik\phi}(k) q_i q_j}{k^2} \right\} R(q) R(k-q) \int_0^\infty d\tilde{x} \tilde{x} \left\{ 6 \frac{j_0(\tilde{q}_1 \tilde{x}) j_1(\tilde{q}_2 \tilde{x})}{\tilde{q}_1 \tilde{x}} - 2j_0(\tilde{q}_1 \tilde{x}) j_1(\tilde{q}_2 \tilde{x}) \tilde{q}_\tilde{x} \right\} \tilde{q}_\tilde{x} - 2j_1(\tilde{q}_1 \tilde{x}) j_0(\tilde{q}_2 \tilde{x}) \right\} j_0(\tilde{\xi} \tilde{x}) \right\} \tilde{q}_x \tilde{q}_x \tilde{x} = \frac{1}{2} \int_0^\infty j_0(\tilde{\xi} \tilde{x}) \right\} \tilde{q}_x \tilde{q}_x \tilde{x}. \]  

(74)

The integral over \( \tilde{x} \) corresponds to the kernel for the scalar–scalar-induced GWs during an RD epoch and can be performed analytically, as given in Appendix B.3.
In the comoving gauge, with the same shorthanded notations as Equation (72), straightforward calculations give

\[
e_{ij}^{\eta}(k)s_{ij}^{(as)}(k) = \frac{1}{a^2} \int \frac{d^3q}{(2\pi)^3} e_{ij}^{\eta}(k)q_i q_j \mathcal{R}(\mathbf{q}) \mathcal{R}(\mathbf{k} - \mathbf{q}) \left\{ j_1(\bar{q}_1 x)j_1(\bar{q}_2 x) \left( \bar{q}_1 \hat{q}_2 x^2 + 2 \frac{\bar{q}_2}{\bar{q}_1} - \frac{2}{\bar{q}_1 \bar{q}_2} \right) \right\} + \left\{ j_1(\bar{q}_1 x)j_0(\bar{q}_2 x) + \bar{j}_0(\bar{q}_1 x) \bar{j}_1(\bar{q}_2 x) \right\} x^2 \left( 1 - 3\bar{q}_1^2 + \bar{q}_2^2 \right) + \frac{\bar{j}_0(\bar{q}_1 x)\bar{j}_0(\bar{q}_2 x)}{\bar{q}_2 \bar{q}_1} \left( 1 - \frac{x^2}{2} \left( 1 - 3\bar{q}_1^2 + \bar{q}_2^2 \right) \right)
\]

so that the solution for \( h_\lambda \) is written as

\[
h_\lambda(\eta, k) = \int \frac{d^3q}{(2\pi)^3} e_{ij}^{\eta}(k)q_i q_j \mathcal{R}(\mathbf{q}) \mathcal{R}(\mathbf{k} - \mathbf{q}) \left\{ j_1(\bar{q}_1 \hat{x})j_1(\bar{q}_2 \hat{x}) \left( -\bar{q}_1 \hat{q}_2 \hat{x}^2 \right) + 2 \frac{\bar{q}_2}{\bar{q}_1} - \frac{2}{\bar{q}_1 \bar{q}_2} \right\} + \left\{ j_1(\bar{q}_1 \hat{x})j_0(\bar{q}_2 \hat{x}) + \bar{j}_0(\bar{q}_1 \hat{x}) \bar{j}_1(\bar{q}_2 \hat{x}) \right\} x^2 \left( 1 - 3\bar{q}_1^2 + \bar{q}_2^2 \right) + \frac{\bar{j}_0(\bar{q}_1 \hat{x})\bar{j}_0(\bar{q}_2 \hat{x})}{\bar{q}_2 \bar{q}_1} \left( 1 - \frac{x^2}{2} \left( 1 - 3\bar{q}_1^2 + \bar{q}_2^2 \right) \right) \}
\]

The analytic results for the \( \hat{x} \)-integral terms are given in Appendix B.3. Comparing Equations (74) and (76), we see that they are clearly different and the gauge dependence of the scalar-induced GWs are persistent during an RD epoch. In Figure 4 we show the kernels in the both gauges.

4.6. Scalar–Tensor-induced GWs during an RD epoch

Next we consider the scalar–tensor-induced GWs during an RD epoch. Again, we work first in the zero-shear gauge for the scalar perturbations. From the linear solutions during an RD epoch, the time derivatives of \( \varphi_\chi \) and \( h_\lambda \) are

\[
\dot{\varphi}_\chi(\mathbf{q}) = -2H\mathcal{R}j_2 \left( \frac{\eta}{\sqrt{3}} \right),
\]

\[
\ddot{\varphi}_\chi(\mathbf{q}) = 2H^2 \mathcal{R} \left[ 5j_2 \left( \frac{\eta}{\sqrt{3}} \right) - \frac{\eta}{\sqrt{3}} \dot{j}_1 \left( \frac{\eta}{\sqrt{3}} \right) \right],
\]

\[
\dot{h}_\lambda(\mathbf{q}) = -Hq_\eta h_\lambda(\mathbf{q})j_1(\mathbf{q}),
\]

\[
\ddot{h}_\lambda(\mathbf{q}) = \frac{q^2}{a^2} \dot{h}_\lambda(\mathbf{q})j_2(\mathbf{q}).
\]

Then with \( \eta = \bar{q}_1 x \) and \( |\mathbf{k} - \mathbf{q}| = \bar{q}_2 x \), we can write

\[
s_{ij}^{(as)}(k) = \frac{k^2}{a^2} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{2}{x^2} j_2 \left( \frac{\bar{q}_1 x}{\sqrt{3}} \right) - \frac{\bar{q}_1 x}{\sqrt{3}} j_2 \left( \frac{\bar{q}_1 x}{\sqrt{3}} \right) \right\} j_0(\bar{q}_2 x) + \left\{ j_1(\bar{q}_1 x)j_0(\bar{q}_2 x) + \bar{j}_0(\bar{q}_1 x) \bar{j}_1(\bar{q}_2 x) \right\} \mathcal{R}(\mathbf{q}) \mathcal{R}(\mathbf{k} - \mathbf{q}) \]

\[
+ \frac{k^2}{a^2} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{2}{x^2} h_0(\bar{q}_1 x)j_2(\bar{q}_2 x) - \bar{q}_2 \bar{j}_0(\bar{q}_2 x) \right\} + \left\{ j_1(\bar{q}_1 x)j_0(\bar{q}_2 x) + \bar{j}_0(\bar{q}_1 x) \bar{j}_1(\bar{q}_2 x) \right\} \mathcal{R}(\mathbf{k} - \mathbf{q}) h_\lambda(\mathbf{q}) j_2(\mathbf{q}).
\]

(81)
The integral can be performed analytically and the individual integrations are given in Appendix B.4. In the comoving gauge, the details of each integration are given in Appendix B.4. In Figure 5, we show both kernels as a function of $k\eta$. Since the kernel is also dependent on the angle between $k$ and $q$, for simplicity we set in such a way that for both (solid lines) $q/k = 0.05$ and (dotted lines) $q/k = 20$ they are aligned perpendicular, i.e., $\cos(k \cdot q) = 0$, while for (dashed lines) $q/k = 2$ the angle between them is $2\pi/3$, $\cos(k \cdot q) = -1/2$.

We note that the first integral can be made identical to the second one upon defining $k - q \equiv p$, and then renaming the dummy integration variable $p$ as $q$. Thus, the scalar–tensor source term in the zero-shear gauge is, after multiplying the polarization tensor $e_{ij}^\Lambda(k)$,

$$e_{ij}^\Lambda(k) \delta_{\eta}(k) = \frac{4k^2}{a^2} \int \frac{d^3q}{(2\pi)^3} [e_{ij}^\Lambda(k) e_{ij}^\Lambda(q)] R(k - q) h_0^\Lambda(q) \times \left\{ \frac{1}{x^2} \left[ \frac{q_1^2}{\sqrt{3}} \right] [2j_2(q_2x) - \tilde{q}_2 x j_1(q_2x)] + \left( 1 + \frac{q_1^2}{k^2} \right) j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2x) \right\}. \quad (82)$$

Then the solution is

$$h_\lambda(\eta, k) = \int \frac{d^3q}{(2\pi)^3} [e_{ij}^\Lambda(k) e_{ij}^\Lambda(q)] R(k - q) h_0^\Lambda(q) \times 4 \int_0^x d\tilde{x}^2 \left\{ \frac{1}{x^2} j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) [2j_2(q_2\tilde{x}) - \tilde{q}_2 x j_1(q_2\tilde{x})] + \left( 1 + \frac{q_1^2}{k^2} \right) j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2\tilde{x}) \right\} [j_0(\tilde{x}) y_0(x) - j_0(x) y_0(\tilde{x})]. \quad (83)$$

The $\tilde{x}$ integral can be performed analytically and the individual integrations are given in Appendix B.4. In the comoving gauge, similarly upon changing the dummy integration variable we can find the source term as

$$e_{ij}^\Lambda(k) \delta_{\eta}(k) = \frac{2k^2}{a^2} \int \frac{d^3q}{(2\pi)^3} [e_{ij}^\Lambda(k) e_{ij}^\Lambda(q)] R(k - q) h_0^\Lambda(q) \left[ 2(1 - \tilde{q}_2^2) j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_0(q_2x) + \left( 1 + \frac{q_1^2}{3} + \tilde{q}_2^2 \right) \frac{q_1 x}{\sqrt{3}} j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_0(q_2x) \right. \left. + \frac{2\tilde{q}_2^2}{3} \tilde{q}_2 x j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2x) + 2 \frac{\tilde{q}_1/\sqrt{3}}{\tilde{q}_2} (1 - \tilde{q}_1^2) j_1 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2x) \right]. \quad (84)$$

so that the analytic integral solution for $h_\lambda$ is

$$h_\lambda(\eta, k) = \int \frac{d^3q}{(2\pi)^3} [e_{ij}^\Lambda(k) e_{ij}^\Lambda(q)] R(k - q) h_0^\Lambda(q) \times 2 \int_0^x d\tilde{x}^2 \left[ 2(1 - \tilde{q}_2^2) j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_0(q_2x) + \left( 1 + \frac{q_1^2}{3} + \tilde{q}_2^2 \right) \frac{q_1 x}{\sqrt{3}} j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_0(q_2x) \right. \left. + \frac{2\tilde{q}_2^2}{3} \tilde{q}_2 x j_0 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2x) + 2 \frac{\tilde{q}_1/\sqrt{3}}{\tilde{q}_2} (1 - \tilde{q}_1^2) j_1 \left( \frac{q_1 x}{\sqrt{3}} \right) j_1(q_2x) \right] [j_0(\tilde{x}) y_0(x) - j_0(x) y_0(\tilde{x})]. \quad (85)$$

The details of each integration are given in Appendix B.4. In Figure 5, we show both kernels (Equations (83) and (85)).
4.7. Tensor–Tensor-induced GWs during an RD epoch

Next we consider the tensor–tensor-induced GWs during an RD epoch. Working in a similar manner to the MD epoch, multiplying the polarization tensor \( k_{eij} \) gives

\[
\begin{align*}
&\frac{1}{a^2} \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^3} \delta^{(3)}(k - q_{12}) \left( h_0^{\lambda_1}(q_1) h_0^{\lambda_2}(q_2) e^{\mu_1}_{ij}(k) q_1 q_2 e^{{k_1}_{e1}}_{\mu_1} e^{{k_2}_{e2}}_{\mu_2} \right) f_j(q_1 \eta) f_j(q_2 \eta) \\
&\quad + h_0^{\lambda_1}(q_1) h_0^{\lambda_2}(q_2) e^{\mu_1}_{ij}(k) \left[ e_\lambda_{ij}(\eta_{12}) (-q_{23} q_{23} e^{{k_2}_{e2}}_{\mu_2} - q_{23} q_{23} e^{{k_2}_{e2}}_{\mu_2} + q_{23} q_{23} e^{{k_2}_{e2}}_{\mu_2}) \\
&\quad + \frac{1}{2} q_{11} q_{11} e^{{k_1}_{e1}}_{\mu_1} e^{{k_2}_{e2}}_{\mu_2} + (q_1 \cdot q_2) e^{{k_1}_{e1}}_{\mu_1} e^{{k_2}_{e2}}_{\mu_2} - q_1 q_2 e^{{k_1}_{e1}}_{\mu_1} e^{{k_2}_{e2}}_{\mu_2} \right] \hat{j}_0(q_1 \eta) \hat{j}_0(q_2 \eta) \right) \\
&\quad + (q_1 \leftrightarrow q_2).
\end{align*}
\]

Thus, with \( x = k\eta, \tilde{q}_1 \equiv q_1/k \) and \( \tilde{q}_2 \equiv q_2/k, \)

\[
\begin{align*}
&h_\lambda(\eta, k) = \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^3} \delta^{(3)}(k - q_{12}) \frac{1}{k^2} \left\{ \left[ \tilde{L}_\lambda(q_1, q_2) + (q_1 \leftrightarrow q_2) \right] \int_0^\infty d\tilde{x} \tilde{x}^2 j_1(q_1 \tilde{x}) j_1(q_2 \tilde{x}) [j_0(\tilde{x}) y_0(\tilde{x}) - j_0(\tilde{x}) y_0(\tilde{x})] \\
&\quad + \left[ \tilde{D}_\lambda(q_1, q_2) + (q_1 \leftrightarrow q_2) \right] \int_0^\infty d\tilde{x} \tilde{x}^2 j_0(\tilde{x}) j_0(\tilde{x}) [j_0(\tilde{x}) y_0(\tilde{x}) - j_0(\tilde{x}) y_0(\tilde{x})] \right\}.
\end{align*}
\]

The \( \tilde{x} \) integrals can be performed analytically and finally we can write

\[
\begin{align*}
&h_\lambda(\eta, k) = \int \frac{d^3 q}{(2\pi)^3} h_0^{\lambda_1}(q) h_0^{\lambda_2}(k - q) e^{\mu_1}_{ij}(k) e^{k_1}_{\mu_1}(q) e^{k_2}_{\mu_2}(k - q) F_{\text{RD}}(k, q, \eta) \\
&\quad + \frac{1}{k^2} \left( e_\lambda^{(k)}(k) e^{k_1}_{\mu_1}(q) q_{k} e^{k_2}_{\mu_2}(k - q) + q_{k} e_\lambda^{(k)}(q) q_{k} e^{k_2}_{\mu_2}(k - q) + q_{k} e_\lambda^{(k)}(q) q_{k} e^{k_2}_{\mu_2}(k - q) \right) \\
&\quad + e_\lambda^{(k)}(k) \left[ \frac{1}{2} q_{k} q_{k} e^{k_1}_{\mu_1}(q) e^{k_2}_{\mu_2}(k - q) + q \cdot (k - q) e^{k_1}_{\mu_1}(q) e^{k_2}_{\mu_2}(k - q) - q_{k} e^{k_1}_{\mu_1}(q) e^{k_2}_{\mu_2}(k - q) \right] G_{\text{RD}}(k, q, \eta),
\end{align*}
\]

where the kernels \( F_{\text{RD}} \) and \( G_{\text{RD}} \) are given, respectively, by Equations (B.41) and (B.42). In Figure 6 we show \( F_{\text{RD}} \) and \( G_{\text{RD}}. \)

5. Conclusions

In this article, we have presented the equation of motion for the tensor perturbations up to second order in perturbations, including all possible quadratic combinations of different types of cosmological perturbations. These terms serve as sources to generate second-order GWs. Given that the universe is filled with a perfect fluid matter with vanishing anisotropic stress, only linear scalar and tensor perturbations contribute to the source terms. And we have found the analytic integral solutions for the second-order GWs during both the MD and RD epochs induced by the scalar–scalar, scalar–tensor, and tensor–tensor sources. The transition between the MD and
RD epochs can be considered by separating the time integral of Equation (53). That is, for the transition from the epoch $A$ to the epoch $B$ at $\eta = \eta_*$, we may write (Kohri & Terada 2018)

$$h_\lambda(\eta, \kappa) = \frac{1}{a} \left\{ \int_0^{\eta_*} d\eta' \frac{a^3(\eta')}{a^3(\eta)} [a^3(\eta) \epsilon_\lambda^\kappa(\kappa) s^A_\lambda(\kappa)] G_{A-B}(\eta, \eta) + \int_{\eta_*}^{\eta} d\eta [a^3(\eta) \epsilon_\lambda^\kappa(\kappa) s^A_\lambda(\kappa)] G_{B}(\eta, \eta) \right\},$$

(89)

where the first term represents the change of the propagation of the GWs produced during $A$ through $B$, and the second term denotes the modification of the source. These terms can be found by matching, for the first term, the solutions and, for the second term, the kernels. However given the complicated source terms, it is a formidable task to compute the effects of the transition analytically, so we do not proceed any further but are satisfied with the above schematic form, which can in principle be further manipulated.

Since the primary tensor perturbations are persistent irrespective of the sources, it is interesting to discuss if the tensor-induced GWs could ever be observationally significant. To address this question quantitatively, it is necessary to study further the relation between the tensor-induced GWs and the primary GWs. For example, if the primary GWs exhibit a sharp peak around at a certain scale $k$, the corresponding tensor-induced GWs are likely to be more prominent than the primary ones over a broader range of the wavenumber other than the peak. In this regards, the tensor-induced GWs may serve as a useful tool to probe the shape of the primary GWs and the relevant underlying physics.

As the kernels that involve rapid oscillations are analytically specified, the solutions given in this article should be useful for analytic and/or numerical studies of the second-order induced GWs.

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**Appendix A**

**Traceless Evolution Equation for the Spatial Metric**

The traceless evolution equation for the spatial metric is written compactly using the Arnowitt–Deser–Misner formulation (Arnowitt et al. 2008). With the metric

$$ds^2 = -N^2(dx^0)^2 + \gamma_{ij} (N^i dx^0 + dx^i)(N^j dx^0 + dx^j),$$

(A.1)

where $N$, $N^i$, and $\gamma_{ij}$ denote, respectively, the lapse function, shift vector, and spatial metric, the dynamics of the spacetime is described by the spatial metric $\gamma_{ij}$ through the curvature variables of the spatial hypersurfaces. Along with the matter contents residing in the spacetime, the geometric equations for the curvature variables constitute a complete set of the equations of motion. The extrinsic curvature $K_{ij}$ is introduced as

$$K_{ij} \equiv \frac{1}{2N} (N_{ij} + N_{ji} - \gamma_{ij,0}),$$

(A.2)

where a vertical bar denotes a covariant derivative with respect to $\gamma_{ij}$. The evolution equation for the traceless part $\mathcal{K}_{ij} = K_{ij} - \gamma_{ij} K / 3$ with $K \equiv K^i_i$ is

$$\mathcal{K}_{i,j\ell} \equiv \frac{K_{i,j\ell} N^\ell}{N} + \frac{K_{j,i\ell} N^\ell}{N} - \frac{K_{\ell,ij} N^\ell}{N} = K \mathcal{K}_{ij} - \frac{1}{N} \left( N^{\ell}_{ij} - \frac{\delta^\ell_{ij}}{3} N^{\ell}_{jk} \right) + \mathcal{R}_{ij} - 8\pi G T_{ij},$$

(A.3)

where $\mathcal{R}_{ij}$ is the traceless part of the intrinsic curvature tensor $R_{ij}$ constructed from $\gamma_{ij}$. This is fully nonlinear, geometric equation. To incorporate cosmological perturbations, we expand it perturbatively up to desired accuracy (Noh & Hwang 2004; Hwang & Noh 2007), or write the exact nonlinear equation for cosmological perturbations (Gong et al. 2017). Using Equation (1) and expanding up
to second order in perturbations, this equation is written as

\[
\ddot{h}^i_j + 3H\dot{h}^i_j - \frac{\Delta}{a^2}h^i_j + \frac{1}{a^2} \left[ a^2 \frac{d}{dt} \left( h^{(i)}_{j} \right) + 3Hh^{(i)}_{j} \right] - \frac{\delta^i_j}{3} \left[ a^2 \frac{d}{dt} \left( \chi^{(i)}_{j,k} \right) + 3H\chi^{(i)}_{j,k} \right] \\
- \frac{1}{a^2} \left( \partial^i \partial^j - \frac{\delta^i_j}{3} \Delta \right) (\alpha + \varphi) - 8\pi G \left( \Pi^i_j - \frac{\delta^i_j}{3} \Pi^{k,k} \right) \\
= \frac{d}{dt} \left( \hat{h}^i_j + \frac{\chi^{(i)}_{j}}{a^2} \right) \alpha + 2(\varphi^{\\beta k} + h^{k}) \hat{h}^{(i)}_{k,j} + \frac{\chi^{(i)}_{j}}{a^2} + 2h^{k} \varphi_{k,j} \right) + 2h^{k} \varphi_{k,j} \left( \alpha + \varphi \right) - \varphi_{k,j} \right) + h^{k,j} \right)
+ \frac{\delta^i_j}{3} \left[ -\alpha \Delta \alpha + \Delta \left( -\alpha + \frac{1}{a^2} \chi^{(i)}_{j,k} \right) - 2(\varphi^{\\beta k} + h^{k}) \chi^{(i)}_{j,k} + 2(\varphi^{\\beta k} + h^{k}) \chi^{(i)}_{j,k} + \varphi^{k} \chi^{(i)}_{j,k} \right] \\
+ \frac{1}{a^2} \left[ -3\varphi^{i,j} + 4\varphi^{i,j} - 4\varphi \Delta h^{i} - 2h^{i,j} \Delta \varphi + \varphi h^{i,j} + \varphi h^{i,j} - 3\varphi^{i,j} + 2\varphi^{k}h^{j,k} \\
- 2h^{k}h^{i,j} - h^{i,j}h^{j,k} - h^{i,j}h^{k,l} - h^{i,k}h^{j,l} - 2h^{i,j}h^{j,k} + h^{i,k}h^{j,l} + h^{i,j}h^{j,k} + h^{i,j}h^{j,k} \\
- \frac{3}{3} (-3\varphi^{i,j} - 4\varphi \Delta \varphi + 2\varphi^{k}h^{i,j} - 4h^{i,j} \Delta h^{i} - 3h^{i,j}h^{j,k} + 2h^{i,j}h^{j,k} + h^{i,j}h^{j,k} + 2h^{i,j}h^{j,k} + 2h^{i,j}h^{j,k}) \\
- 16\pi G \left[ (\varphi^{\\beta k} + h^{k}) \Pi^{i,j} + \frac{\delta^i_j}{3} (\varphi^{\\beta k} + h^{k}) \Pi^{i,j} \right] + 8\pi G (\rho + p) \left( v^{i,j} - \frac{\delta^i_j}{3} v^{i,j} \right).
\]

(A.4)

Appendix B
Integrals of Bessel Functions.

Here, \(a, b,\) and \(c\) denote arbitrary positive constants.

B.1. Integrals for Scalar–Scalar and Scalar–Tensor-Induced GWs during an MD epoch

For \(s_{ij}^{(\alpha)}\) during an MD epoch, we have for the zero-shear gauge the following integrals for the spherical Bessel functions:

\[
\int_0^x d\xi \xi^2 j_i(x) = 3x^2 j_i(x) - x^3 j_0(x),
\]

(B.1)

\[
\int_0^x d\xi \xi^2 y_i(x) = 3 + 3x^2 y_i(x) - x^3 y_0(x),
\]

(B.2)

along with the following identities of the spherical Bessel functions:

\[
\left( \frac{1}{x} \frac{d}{dx} \right)^m \left[ x^{n+1} f_n(x) \right] = x^{n-m+1} f_{n-m}(x),
\]

(B.3)

\[
f_{n-1} + f_{n+1} = \frac{2n + 1}{x} f_n.
\]

(B.4)

Also, we have the following integrals for the comoving gauge:

\[
\int_0^x d\xi \xi^2 j_i(x) = 5x^2 (x^2 - 6) j_i(x) - x^3 (x^2 - 10) j_0(x),
\]

(B.5)

\[
\int_0^x d\xi \xi^2 y_i(x) = -30 + 5x^2 (x^2 - 6) y_i(x) - x^3 (x^2 - 10) y_0(x).
\]

(B.6)
For \( s^x_j(x) \) during the MD epoch, for the zero-shear gauge,
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) \chi_j(\tilde{x}) = \frac{1}{a^2 - 1} [-x^2 j_x(\tilde{x}) - ax^2_j(\tilde{x})] ,
\]
(B.7)
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) y_j(\tilde{x}) = \frac{1}{a^2 - 1} [-a + x^2 j_x(\tilde{x}) y_j(\tilde{x})] .
\]
(B.8)
These give a rather simple result:
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) \chi_j(\tilde{x}) y_j(\tilde{x}) = \frac{1}{a^2 - 1} [-a j_j(\tilde{x}) - a j_j(\tilde{x})] .
\]
(B.9)
For the comoving gauge, we have the following integrals:
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) \chi_j(\tilde{x}) y_j(\tilde{x}) = \frac{1}{(a^2 - 1)^2} [3 - a^2 + x^2 j_x(\tilde{x})[(a^2 - 1) j_x(\tilde{x}) - (a^2 - 3) j_x(\tilde{x})]]
\]
\[
+ ax^2 j_x(\tilde{x})[-2 y_j(\tilde{x}) + (a^2 - 1) x y_j(\tilde{x})] .
\]
(B.10)
Again, we find a rather simple result from these integrals:
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) \chi_j(\tilde{x}) y_j(\tilde{x}) = \frac{1}{(a^2 - 1)^2} [(a^2 - 3) j_j(\tilde{x}) + 2 a j_x(\tilde{x}) - (a^2 - 1) x j_x(\tilde{x})] .
\]
(B.12)

### B.2. Integrals for Tensor–Tensor-induced GWs during an MD epoch

They can be arranged as, for the first two integrals,
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) f_x(\tilde{x})
\]
\[
= (a^2 + b^2 - c^2)(a^2 - b^2 - c^2)x^2 j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) - (a^2 + b^2 - c^2)(a^2 - b^2 + c^2)x^2 j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
- \frac{(a^2 + b^2 - c^2)x^2}{16a^2b^2c} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) + \frac{x}{2a} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) + \frac{x}{2b} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
- \frac{c}{2ab} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
+ (a - b - c)(a + b + c)(a^2 + b^2 - c^2)
\]
\[
\times [\text{Si}[a - b - c] - \text{Si}[a + b - c] - \text{Si}[(a - b + c)] + \text{Si}[(a + b + c)] ,
\]
(B.13)
\[
\int_0^x d\tilde{x}^2 j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
= (a^2 + b^2 - c^2)(a^2 - b^2 - c^2)x^2 j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) - (a^2 + b^2 - c^2)(a^2 - b^2 + c^2)x^2 j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
- \frac{(a^2 + b^2 - c^2)x^2}{16a^2b^2c} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) + \frac{x}{2a} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x}) + \frac{x}{2b} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
- \frac{c}{2ab} j_x(\tilde{x}) j_y(\tilde{x}) j_z(\tilde{x}) y(\tilde{x})
\]
\[
+ (a - b - c)(a + b + c)(a^2 + b^2 - c^2)
\]
\[
\times [\text{Ci}[a - b - c] - \text{Ci}[a + b - c] - \text{Ci}[(a - b + c)] + \text{Ci}[(a + b + c)]
\]
\[
- \frac{3a^2 + 3(b^2 - c^2)(a^2 + b^2 + c^2)}{48a^2b^2c^2} ,
\]
(B.14)
Note that with the definition
\[
\text{Ci}(x) \equiv \int_0^x \frac{1 - \cos t}{t} dt = \gamma + \log x - \text{Ci}(x) ,
\]
(B.15)
where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant, while both $\text{Ci}(x)$ and $\log x$ are diverging as $x \to 0$, $\lim_{x \to 0} \text{Ci}(x) = 0$. Then we find

$$
\frac{1}{x} \int_0^x d\tilde{x} \tilde{j}_2((a\tilde{x})j_2(b\tilde{x})[j_1(\tilde{x})y_1(x) - j_1(x)y_1(\tilde{x})]) = F_{\text{MD}}(a, b, x)
$$

$$
= \left(\frac{a^2 + b^2 - 1}{16b^2} \frac{\tilde{j}_1(ax)}{ax} j_0(bx) + \frac{\gamma}{ax} \frac{\tilde{j}_1(bx)}{bx} + \frac{1}{2} \frac{\tilde{j}_1(ax)}{ax} j_0(bx) + \frac{1}{2} \frac{\tilde{j}_1(bx)}{bx} \right)
$$

$$
+ \frac{3a^2 + b^2 + 3 - 6a^2 - 6b^2 - 2a^2b^2}{48a^2b^2} \frac{j_1(ax)}{x} \frac{j_0(bx)}{x} + \frac{(a - b - 1)(a + b - 1)(a + b + 1)(a^2 + b^2 - 1)}{64a^2b^2} x.
$$

Likewise, for the next two integrals,

$$
\int_0^x d\tilde{x} \tilde{j}_1(ax)j_1(bx)j_1(cx)
$$

$$
= -x^2 \frac{1}{4} \frac{\tilde{j}_1(ax)j_0(bx)j_1(cx)}{ax} - \left(\frac{a^2 + b^2 - c^2}{8ab} \frac{\tilde{j}_0(ax)j_0(bx)j_1(cx)}{ax} - \frac{(a^2 - b^2 + c^2)x^2}{8ab} \frac{\tilde{j}_0(ax)j_0(bx)j_0(cx)}{ax} - \frac{(a^2 - b^2 + c^2)x^2}{8ab} \frac{\tilde{j}_0(ax)j_0(bx)j_0(cx)}{ax} \right)
$$

$$
+ \frac{(a - b - c)(a + b - c)(a - b + c)(a + b + c)}{32a^2b^2c^2} \left[ \frac{\text{Si}[(a - b - c)x] - \text{Si}[(a + b - c)x] - \text{Si}[(a - b + c)x] + \text{Si}[(a + b + c)x]}{x} \right] \text{.}
$$

Thus,

$$
\frac{1}{abx} \int_0^x d\tilde{x} \tilde{j}_1(ax)j_1(bx)j_1(cx) \text{Si}[j_1(\tilde{x})y_1(x) - j_1(x)y_1(\tilde{x})] = G_{\text{MD}}(a, b, x)
$$

$$
= \left(\frac{-a^2 + b^2 + 1}{8b^2} \frac{\tilde{j}_0(ax)}{ax} j_0(bx) + \frac{(a^2 - b^2 + 1)}{8a^2} \frac{\tilde{j}_0(ax)}{ax} j_0(bx) + \frac{a^2 + b^2 - 1}{8a^2b^2} \frac{\tilde{j}_0(ax)}{ax} j_0(bx) \right)
$$

$$
+ \frac{(a - b - 1)(a + b - 1)(a - b + 1)(a + b + 1)}{32a^2b^2} x.
$$

Notice that comparing with Equation (B.16), $F_{\text{MD}}(a, b, x)$ and $G_{\text{MD}}(a, b, x)$ are related by

$$
\frac{a^2 + b^2 - 1}{2} G_{\text{MD}}(a, b, x) = F_{\text{MD}}(a, b, x) - \frac{1}{2} \frac{\tilde{j}_1(ax)}{ax} j_0(bx) + \frac{1}{6} \frac{\tilde{j}_1(ax)}{ax} j_0(bx) \text{.}
$$

**B.3. Integrals for Scalar–Scalar-induced GWs during an RD epoch**

We find

$$
\int_0^x d\tilde{x} \tilde{j}_0(ax)j_0(bx)j_0(cx) = \frac{1}{4abc} \left[ \text{Si}[(a - b - c)x] - \text{Si}[(a + b - c)x] - \text{Si}[(a - b + c)x] - \text{Si}[(a + b + c)x] \right] \text{,}
$$

$$
\int_0^x d\tilde{x} \tilde{j}_0(ax)j_0(bx)j_0(cx) = \frac{1}{4abc} \left[ \text{Ci}[(a - b - c)x] - \text{Ci}[(a + b - c)x] + \text{Ci}[(a - b + c)x] - \text{Ci}[(a + b + c)x] \right] \text{.}
$$
\[
\int_0^x d\xi j_0(x) j_0(bx) j_0(cx) = \frac{x}{a^4 b^2 c} [j_0[(a - b - c)x] - j_0[(a - b + c)x] + j_0[(a + b + c)x]]
\]
\[
+ \left[ \frac{1}{a^2 b^2 c} \{ \Sigma[(a - b - c)x] + \Sigma[(a + b - c)x] + \Sigma[(a - b + c)x] - \Sigma[(a + b + c)x] \} \right].
\]
\[
\int_0^x d\xi j_0(x) j_0(bx) j_0(cx) = \frac{x}{a^4 b^2 c} [j_0[(a - b - c)x] + j_0[(a - b + c)x] + j_0[(a + b + c)x]]
\]
\[
+ \left[ \frac{1}{a^2 b^2 c} \{ \Sigma[(a - b - c)x] + \Sigma[(a + b - c)x] + \Sigma[(a - b + c)x] - \Sigma[(a + b + c)x] \} \right].
\]
\[
\int_0^x d\xi j_0(x) j_0(bx) j_0(cx) = \frac{x}{a^4 b^2 c} [j_0[(a - b - c)x] + j_0[(a - b + c)x] + j_0[(a + b + c)x]]
\]
\[
+ \left[ \frac{1}{a^2 b^2 c} \{ \Sigma[(a - b - c)x] + \Sigma[(a + b - c)x] + \Sigma[(a - b + c)x] - \Sigma[(a + b + c)x] \} \right].
\]

For the comoving gauge, we can find
\[
\int_0^x d\xi j_0(x) j_0(bx) j_0(cx) = \frac{x}{a^4 b^2 c} [j_0[(a - b - c)x] - j_0[(a - b + c)x] + j_0[(a + b + c)x]]
\]
\[
+ \left[ \frac{1}{a^2 b^2 c} \{ \Sigma[(a - b - c)x] - \Sigma[(a - b + c)x] + \Sigma[(a + b + c)x] \} \right].
\]
\[ \int_0^x d\xi^2 j_{\alpha}(ax)j_{\beta}(bx) \gamma_0(cx) = \frac{x^2}{4abc} \left[ y_1[(a - b - c)x] - y_1[(a + b - c)x] + y_1[(a - b + c)x] - y_1[(a + b + c)x] \right] \]

\[ + \frac{2(a^4 + 3a^2b^2 + 6ab^2 + 2b^2c^2)}{(a - b - c)^2(a - b + c)^2(a + b + c)^2}, \]  

(B.30)

\[ \int_0^x d\xi^2 j_{\alpha}(ax)j_{\beta}(bx)j_{\alpha}(cx) = \frac{x^2}{4abc} \left[ -y_1[(a - b - c)x] - y_1[(a + b - c)x] + y_1[(a - b + c)x] + y_1[(a + b + c)x] \right] \]

\[ + \frac{x^2}{4abc} \left[ (a^2 - 3ab + b^2 - ac + bc)j_1[(a - b - c)x] - y_0[(a - b - c)x] \right] \]

\[ - (a^2 + 3ab + b^2 - ac - bc)j_1[(a + b - c)x] - y_0[(a + b - c)x] \]

\[ - (a^2 - 3ab + b^2 + ac - bc)j_1[(a - b + c)x] - y_0[(a - b + c)x] \]

\[ + (a^2 + 3ab + b^2 + ac + bc)j_1[(a + b + c)x] - y_0[(a + b + c)x] \]

\[ + \frac{1}{4abc} \left[ -Si[(a - b - c)x] + Si[(a + b - c)x] + Si[(a - b + c)x] - Si[(a + b + c)x] \right], \]

(B.31)

We find

\[ \int_0^x d\xi j_{\alpha}(ax) j_{\beta}(bx) j_{\alpha}(cx) = \frac{1}{6ab} \gamma_0(ax)j_{\alpha}(bx)j_{\alpha}(cx) - \frac{a}{3b} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{b}{3a} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) + \frac{c^2}{6ab} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) \]

\[ - \frac{1}{6b} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{1}{6a} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{1}{3} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{c}{6ab} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) \]

\[ + \frac{1}{24a^2b^2c} \left[ (a - b - c)(2a^2 - 2ab + 2b^2 - ac + bc - c^2)Cin[(a - b - c)x] \right] \]

\[ -(a + b - c)(2a^2 - 2ab + 2b^2 - ac - bc - c^2)Cin[(a + b - c)x] \]

\[ -(a - b + c)(2a^2 + 2ab + 2b^2 + ac - bc + c^2)Cin[(a - b + c)x] \]

\[ +(a + b + c)(2a^2 - 2ab + 2b^2 + ac + bc - c^2)Cin[(a + b + c)x] \]  

\[ - \frac{1}{6ab}, \]  

(B.31)

\[ \int_0^x d\xi j_{\alpha}(ax) j_{\beta}(bx) j_{\alpha}(cx) = \frac{1}{6ab} \gamma_0(ax)j_{\alpha}(bx)j_{\alpha}(cx) - \frac{a}{3b} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{b}{3a} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) + \frac{c^2}{6ab} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) \]

\[ - \frac{1}{6b} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{1}{6a} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{1}{3} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) - \frac{c}{6ab} x^2 j_{\alpha}(ax) j_{\alpha}(bx) j_{\alpha}(cx) \]

\[ + \frac{1}{24a^2b^2c} \left[ (a - b - c)(2a^2 + 2ab + 2b^2 - ac - bc - c^2)Si[(a - b - c)x] \right] \]

\[ -(a + b - c)(2a^2 - 2ab + 2b^2 - ac + bc - c^2)Si[(a + b - c)x] \]

\[ +(a - b + c)(2a^2 + 2ab + 2b^2 + ac - bc + c^2)Si[(a - b + c)x] \]

\[ -(a + b + c)(2a^2 - 2ab + 2b^2 + ac + bc - c^2)Si[(a + b + c)x] \],

(B.34)
\[
\int_0^\infty d\xi x j_0(a\xi) j_0(b\xi) j_0(c\xi) \\
\quad = -\frac{1}{a} xj_0(ax)j_0(bx)j_0(cx) - \frac{1}{2} x^2j_0(ax)xj_0(bx)j_0(cx) + \frac{b}{2a} x^2j_0(ax)j_0(bx)j_0(cx) + \frac{c}{2a} x^2j_0(ax)j_0(bx)j_0(cx) \\
\quad + \frac{1}{8a^2bc} \{-[a^2 - (b + c)^2]Si[(a - b - c)x] + [a^2 - (b - c)^2]Si[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Si[(a - b + c)x] - [a^2 - (b + c)^2]Si[(a + b + c)x]\}, \tag{B.35}
\]

\[
\int_0^\infty d\xi x j_0(a\xi) j_0(b\xi) y_0(c\xi) \\
\quad = -\frac{1}{a} xj_0(ax)j_0(bx)y_0(cx) - \frac{1}{2} x^2j_0(ax)xj_0(bx)y_0(cx) + \frac{b}{2a} x^2j_0(ax)j_0(bx)y_0(cx) + \frac{c}{2a} x^2j_0(ax)j_0(bx)y_0(cx) \\
\quad + \frac{1}{8a^2bc} \{[a^2 - (b + c)^2]Cin[(a - b - c)x] - [a^2 - (b - c)^2]Cin[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Cin[(a - b + c)x] - [a^2 - (b + c)^2]Cin[(a + b + c)x]\} - \frac{1}{2ac}. \tag{B.36}
\]

\[
\int_0^\infty d\xi j_2(a\xi) j_0(b\xi) j_0(c\xi) \\
\quad = \frac{b^2}{2a^2} xj_0(ax)j_0(bx)j_0(cx) + \frac{c^2}{2a^2} xj_0(ax)j_0(bx)j_0(cx) - \frac{3}{2a} xj_0(ax)j_0(bx)j_0(cx) - \frac{a}{8} x^2j_0(ax)j_0(bx)j_0(cx) \\
\quad + \frac{b^2}{8a} x^2j_0(ax)j_0(bx)j_0(cx) + \frac{c^2}{8a} x^2j_0(ax)j_0(bx)j_0(cx) + \frac{b}{8} x^2j_0(ax)j_0(bx)j_0(cx) - \frac{b^3}{8a^2} x^2j_0(ax)j_0(bx)j_0(cx) \\
\quad - \frac{3b^2c}{8a^2} x^2j_0(ax)j_0(bx)j_0(cx) + \frac{b}{2a} xj_0(ax)j_0(bx)j_0(cx) + \frac{c}{8} x^2j_0(ax)j_0(bx)j_0(cx) - \frac{3b^2c}{8a^2} x^2j_0(ax)j_0(bx)j_0(cx) \\
\quad - \frac{c^3}{8a^2} x^2j_0(ax)j_0(bx)j_0(cx) + \frac{c}{2a} xj_0(ax)j_0(bx)j_0(cx) - \frac{bc}{4a} x^2j_0(ax)j_0(bx)j_0(cx) \\
\quad + \frac{1}{32a^3bc} \{-[a^2 - (b + c)^2]Si[(a - b - c)x] + [a^2 - (b - c)^2]Si[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Si[(a - b + c)x] - [a^2 - (b + c)^2]Si[(a + b + c)x]\} + \frac{1}{32a^3bc} \{[a^2 - (b + c)^2]Cin[(a - b - c)x] - [a^2 - (b - c)^2]Cin[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Cin[(a - b + c)x] - [a^2 - (b + c)^2]Cin[(a + b + c)x]\}, \tag{B.37}
\]

\[
\int_0^\infty d\xi j_2(a\xi) j_0(b\xi) y_0(c\xi) \\
\quad = \frac{b^2}{2a^2} xj_0(ax)j_0(bx)y_0(cx) + \frac{c^2}{2a^2} xj_0(ax)j_0(bx)y_0(cx) - \frac{3}{2a} xj_0(ax)j_0(bx)y_0(cx) - \frac{a}{8} x^2j_0(ax)j_0(bx)y_0(cx) \\
\quad + \frac{b^2}{8a} x^2j_0(ax)j_0(bx)y_0(cx) + \frac{c^2}{8a} x^2j_0(ax)j_0(bx)y_0(cx) + \frac{b}{8} x^2j_0(ax)j_0(bx)y_0(cx) - \frac{b^3}{8a^2} x^2j_0(ax)j_0(bx)y_0(cx) \\
\quad - \frac{3b^2c}{8a^2} x^2j_0(ax)j_0(bx)y_0(cx) + \frac{b}{2a} xj_0(ax)j_0(bx)y_0(cx) + \frac{c}{8} x^2j_0(ax)j_0(bx)y_0(cx) - \frac{3b^2c}{8a^2} x^2j_0(ax)j_0(bx)y_0(cx) \\
\quad - \frac{c^3}{8a^2} x^2j_0(ax)j_0(bx)y_0(cx) + \frac{c}{2a} xj_0(ax)j_0(bx)y_0(cx) - \frac{bc}{4a} x^2j_0(ax)j_0(bx)y_0(cx) \\
\quad + \frac{1}{32a^3bc} \{-[a^2 - (b + c)^2]Cin[(a - b - c)x] - [a^2 - (b - c)^2]Cin[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Cin[(a - b + c)x] - [a^2 - (b + c)^2]Cin[(a + b + c)x]\} + \frac{1}{32a^3bc} \{[a^2 - (b + c)^2]Cin[(a - b - c)x] - [a^2 - (b - c)^2]Cin[(a + b - c)x] \\
\quad + [a^2 - (b - c)^2]Cin[(a - b + c)x] - [a^2 - (b + c)^2]Cin[(a + b + c)x]\}, \tag{B.38}
\]

\[\text{B.5. Integrals for Tensor–Tensor-induced GWs during an RD epoch}\]

We can perform the integrals analytically to find

\[
\int_0^\infty d\xi x^2j_1(a\xi)j_1(b\xi)j_0(c\xi) \\
\quad = \frac{c}{2ab} x^2j_0(ax)j_0(bx)j_1(cx) - \frac{1}{2a} x^2j_0(ax)j_0(bx)j_0(cx) - \frac{1}{2b} x^2j_1(ax)j_0(bx)j_0(cx) \\
\quad - \frac{a^2 + b^2 - c^2}{8a^2b^2c} \{Si[(a - b - c)x] - Si[(a + b - c)x] - Si[(a - b + c)x] + Si[(a + b + c)x]\}, \tag{B.39}
\]
\[
\int_0^\infty d\tilde{x} \tilde{x}^2 j_1(\tilde{x}) j_1(b \tilde{x}) y_0(c \tilde{x}) = \frac{c}{2ab} x^2 y_0(ax) y_0(bx) y_0(cx) - \frac{1}{2a} x^2 j_1(ax) j_1(bx) y_0(cx) - \frac{1}{2b} x^2 j_1(ax) j_1(bx) y_0(cx) + \frac{1}{2abc} + \frac{a^2 + b^2 - c^2}{8a^2 b^2 c} \{ \sin[(a - b - c)x] - \sin[(a + b - c)x] + \sin[(a - b + c)x] - \sin[(a + b + c)x] \}. \tag{B.40}
\]

Thus,
\[
\int_0^\infty d\tilde{x} \tilde{x}^2 j_1(\tilde{x}) j_1(b \tilde{x}) y_0(x) - j_0(x) y_0(\tilde{x}) \right\} = \frac{1}{2ab} F_{RD}(a, b, x)
\]
\[
= \frac{1}{2ab} \left[ j_0(ax) j_0(bx) - j_0(x) + \frac{a^2 + b^2 - 1}{4ab} \right] \left\{ \left( -\sin[(a - b - 1)x] y_0(x) - \sin[(a - b - 1)x] j_0(x) \right) - \left( -\sin[(a + b - 1)x] y_0(x) + \sin[(a + b - 1)x] j_0(x) \right) \right. \\
+ \left. \left( -\sin[(a + b + 1)x] y_0(x) + \sin[(a + b + 1)x] j_0(x) \right) \right\}. \tag{B.41}
\]
\[
\int_0^\infty d\tilde{x} \tilde{x}^2 j_0(ax) j_0(bx) j_0(\tilde{x}) y_0(x) - j_0(x) y_0(\tilde{x}) \right\} = \frac{1}{2} G_{RD}(a, b, x)
\]
\[
= \frac{1}{4ab} \left\{ \left( -\sin[(a - b - 1)x] y_0(x) - \sin[(a - b - 1)x] j_0(x) \right) + \left( -\sin[(a + b - 1)x] y_0(x) + \sin[(a + b - 1)x] j_0(x) \right) \right. \\
+ \left. \left( -\sin[(a + b + 1)x] y_0(x) + \sin[(a + b + 1)x] j_0(x) \right) \right\}. \tag{B.42}
\]

Note that
\[
a^2 + b^2 - \frac{1}{2} G_{RD}(a, b, x) = F_{RD}(a, b, x) - j_0(ax) j_0(bx) + j_0(x). \tag{B.43}
\]

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