Classical Poincaré conjecture via 4D topology

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ABSTRACT

The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is confirmed by Perelman in arXiv papers solving Thurston’s program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by a method of 4D topology. For this proof, the spun torus-knot of every knot in every homotopy 3-sphere is observed to be a ribbon torus-knot in the 4-sphere, where Smooth 4D Poincaré Conjecture and Ribbonness of a sphere-link with (not necessarily meridian-based) free fundamental group are used. By examining a disk-chord system of a ribbon solid torus bounded by the spun torus-knot, it is proved that the knot belongs to a 3-ball in the homotopy 3-sphere. Then by Bing’s result, it is confirmed that the homotopy 3-sphere is diffeomorphic to the 3-sphere.

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1. Introduction

A homotopy 3-sphere is a smooth 3-manifold $M$ homotopy equivalent to the 3-sphere $S^3$. The following Poincaré Conjecture [22, 23] is positively shown by Perelman in arXiv papers [20, 21] solving positively Thurston’s program [24] on geometrizations of 3-manifolds (see [19] for detailed historical notes).

Poincaré Conjecture. Every homotopy 3-sphere $M$ is diffeomorphic to $S^3$. 

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A new confirmation of this result is presented here by combining Smooth 4D Poincaré Conjecture and Free Ribbon Lemma for an $S^2$-link in the 4-sphere $S^4$ with R. H. Bing’s result [2, 3] on Poincaré Conjecture. A homotopy 4-sphere is a smooth 4-manifold $X$ homotopy equivalent to the 4-sphere $S^4$. The following conjecture was a folklore conjecture.

**Smooth 4D Poincaré Conjecture.** Every smooth homotopy 4-sphere $X$ is diffeomorphic to $S^4$.

The positive proof of this conjecture is shown in [15]. A surface-link in $S^4$ is a smooth 4-manifold $X$ smoothly embedded in $S^4$. When $L$ is connected, it is a surface-knot. If all components of $L$ are 2-spheres, then it is an $S^2$-link. A surface-link $L$ in $S^4$ is trivial if $L$ bounds disjoint handlebodies in $S^4$, and a ribbon surface-link if $L$ is equivalent to a surface-link obtained from a trivial $S^2$-link $O$ by surgery along disjointly embedded 1-handles on $O$ in $S^4$. The following lemma is shown in [16] as Free Ribbon Lemma and used in Section 3.

**Free Ribbon Lemma.** Any $S^2$-link $L$ in $S^4$ with free fundamental group $\pi_1(S^4 \setminus L, b)$ is a ribbon $S^2$-link in $S^4$.

The proof of this lemma is moved from this preprint version to the paper [16] (for completeness of the argument), which is done by using Smooth 4D Poincaré Conjecture and Smooth Unknotting Conjecture explained as follows:

**Smooth Unknotting Conjecture.** Every smooth surface-link $L$ in $S^4$ with a meridian-based free fundamental group $\pi_1(S^4 - L, b)$ is a trivial surface-link.

The proof of this conjecture is shown by [12, 13, 14]. Artin’s spinning construction of a knot $k$ in $S^3$ in [1] to construct the spun $S^2$-knot $K(k)$ in the 4-sphere $S^4$ allows us to generalize to a connected graph $\gamma$ in every homotopy 3-sphere $M$ to construct the spun $S^2$-link $K(\gamma)$ in a homotopy 4-sphere $X(M)$ which is diffeomorphic to $S^4$ by Smooth 4D Poincaré Conjecture, so that $X(M)$ is identified with $S^4$. This construction is applied to a Heegaard graph $\gamma$ of $M$ (associated to a Heegaard splitting of $M$). Then the spun $S^2$-link $K(\gamma)$ is an $S^2$-link in $X(M)$ with free fundamental group (not always meridian-based free group). By Free Ribbon Lemma, the spun $S^2$-link $K(\gamma)$ is a ribbon $S^2$-link in $X(M)$. It is observed that for every knot $k$ in every homotopy 3-sphere $M$, there is a Heegaard graph $\gamma$ of $M$ such that $k$ is contained in the loop system of $\ell(\gamma)$ of $\gamma$. This means that the spun $S^2$-knot $K(k)$ of every knot $k$ in every homotopy 3-sphere $M$ is a ribbon $S^2$-knot in $X(M)$. Then, by definition,
the spun torus-knot $T(k)$ of every knot $k$ in every homotopy 3-sphere $M$ is a ribbon torus-knot in $X(M)$. Thus, the spun torus-knot $T(k)$ always bounds a ribbon solid torus $V_R$ in $X(M)$. By an argument of a disk-chord system of $V_R$ bounded by the spun torus-knot $T(k)$ in $X(M)$, the following result is shown.

**Theorem 1.1.** Every knot $k$ in every homotopy 3-sphere $M$ belongs to a 3-ball $D^3$ in $M$.

By combining Theorem 1.1 with the following result of Bing in [2, 3], it is proved that every homotopy 3-sphere $M$ is diffeomorphic to $S^3$. Thus, the proof of Poincaré conjecture is completed.

**Bing’s Theorem.** A homotopy 3-sphere $M$ is diffeomorphic to $S^3$ if every knot $k$ in $M$ belongs to a 3-ball in $M$.

Outline of the proof of Poincaré Conjecture is as follows:

(1st Step) By using Smooth 4D Poincaré Conjecture, show that Artin’s spinning construction of every Heegaard graph $\gamma$ of every homotopy 3-sphere $M$ gives a spun $S^2$-link $K(\gamma)$ in $S^4$ with free fundamental group (not always meridian-based free group).

(2nd Step) By Free Ribbon Lemma, the spun $S^2$-link $K(\gamma)$ is a ribbon $S^2$-link in $S^4$.

(3rd Step) Show that every knot $k$ in $M$ is contained in a loop system $\ell(\gamma)$ of a Heegaard graph $\gamma$ of $M$, so that the spun $S^2$-knot $K(k)$ of $k$ is a ribbon $S^2$-knot in $S^4$.

(4th Step) By definition of a ribbon surface-knot, show that the spun torus-knot $T(k)$ of $k$ in $M$ is a ribbon torus-knot in $S^4$.

(5th Step) By using a ribbon solid torus $V_R$ bounded by the spun torus-knot $T(k)$ in $S^4$ and a disk-chord system of $V_R$, show that $K$ belongs to a 3-ball $D^3$ in $M$.

(6th Step) By Bing’s theorem, $M$ is diffeomorphic to $S^3$.

In Section 2, Artin’s spinning construction of a connected graph in a homotopy 3-sphere is explained. In Section 3, an argument of a disk-chord system of a ribbon solid torus bounded by a ribbon torus-knot is explained. In Section 4, the proof of Theorem 1.1 is done.
2. Artin’s spinning construction of a connected graph in a homotopy 3-sphere

Throughout this section, $M$ denotes a homotopy 3-sphere unless otherwise mentioned. For a homotopy 3-sphere $M$, let $M^{(o)}$ be the compact once-punctured manifold $\text{cl}(M \setminus B)$ of $M$ for a 3-ball $B$ in $M$. Let

$$S = \partial B = \partial M^{(o)}$$

be the boundary 2-sphere of $M^{(o)}$. The closed smooth 4-manifold $X(M)$ defined by

$$X(M) = M^{(o)} \times S^1 \cup S \times D^2$$

is called the spun manifold of $M$ with axis 4-submanifold $S \times D^2$. As a convention, the 3-submanifold $M^{(o)} \times 1$ of the product $M^{(o)} \times S^1$ is identified with $M^{(o)}$. In particular, a point $(q, 1) \in M^{(o)} \times 1$ is identified with the point $q \in M^{(o)}$. This 4-manifold $X(M)$ is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument and hence $X(M)$ is diffeomorphic to the 4-sphere $S^4$ by Smooth 4D Poincaré Conjecture.

From now on, the identification $X(M) = S^4$ is fixed.

A legged loop with base point $v$ is the union of a loop $k$ and an arc $\omega$ joining the base point $v$ with a point of $k$. The arc $\omega$ is called a leg. A legged loop system with base point $v$ is the union

$$\gamma = \bigcup_{i=1}^{n} \ell_i \cup \omega_i$$

of $n$ legged loops $\ell_i \cup \omega_i (i = 1, 2, \ldots, n)$ meeting only at the same base point $v$. Let $\ell(\gamma) = \bigcup_{i=1}^{n} \ell_i = \ell_*$ denote the loop system of the legged loop system $\gamma$. Let $\omega_* = \bigcup_{i=1}^{n} \omega_i$ and $v_* = \ell_* \cap \omega_*$. A regular neighborhood $B$ of $\omega_*$ in $M$ is taken as a 3-ball $B$ used for the compact once-punctured manifold $M^{(o)} = \text{cl}(M \setminus B)$ of $M$. Deform the subgraph $\gamma \cap B$ of $\gamma$ so that

$$\omega_* \subset B, \quad \omega_* \cap S = v_* \quad \text{and} \quad \ell_* \cap B = \ell_* \cap S = a'_*$$

for a regular neighborhood arc system $a'_*$ of $v_* \in \ell_*$. Let

$$a(\gamma) = \bigcup_{i=1}^{n} a_i = a_*$$

for a proper arc $a_i = \text{cl}(\ell_i \setminus a'_i) (i = 1, 2, \ldots, n)$ in $M^{(o)}$. Let

$$\dot{a}(\gamma) = \partial a_* = \partial a'_*$$

be the set of $2n$ points in the boundary 2-sphere $S$ of $M^{(o)}$. The spun $S^2$-link of the graph $\gamma$ is the $S^2$-link $K(\gamma)$ in the 4-sphere $X(M)$ defined by

$$K(\gamma) = a(\gamma) \times S^1 \cup \dot{a}(\gamma) \times D^2.$$
Lemma 2.1. The inclusion $M^{(o)} \setminus a(\gamma) \subset X(M) \setminus K(\gamma)$ induces an isomorphism
\[ \sigma : \pi_1(M \setminus \gamma, v^+) \to \pi_1(X(M) \setminus K(\gamma), v^+) \]
sending a meridian system of the proper arc system $a(\gamma)$ in $M^{(o)}$ to a meridian system of $K(\gamma)$, where the base point $v^+$ is taken in $S \setminus a_*$. 

Proof of Lemma 2.1. Note that there is a canonical isomorphism
\[ \pi_1(M^{(o)} \setminus a(\gamma), v^+) \cong \pi_1(M \setminus \gamma, v^+) \]
Then the desired isomorphism $\sigma$ is obtained by applying the van Kampen theorem between $(M^{(o)} \setminus a(\gamma)) \times S^1$ and $(S \setminus a(\gamma)) \times D^2$. This completes the proof of Lemma 2.1. □

Here is a note on Lemma 2.1.

Note 2.2. A general connected graph $\gamma$ with Euler characteristic $\chi(\gamma) = 1 - n$ in $M$ is deformed into a legged loop system $\gamma$ in $M$ by choosing a maximal tree to shrink to a base point $v$. Note that there are only finitely many maximal trees of $\gamma$ such that the loop systems $\ell(\gamma)$ of the resulting legged loop systems $\gamma$ are distinct as links. By Lemma 2.1, we can obtain finitely many distinct spun $S^2$-links in $S^4$ with isomorphic fundamental groups obtained by taking different maximal trees of the connected graph $\gamma$. This is a detailed explanation on the spun $S^2$-link of a connected graph associated with a maximal tree in [6, p.204] when $M = S^3$.

When a homotopy 3-sphere $M$ is given by a Heegaard splitting $V \cup V'$ pasting along a Heegaard surface $F = \partial V = \partial V'$ of genus $n$, a legged loop system $\gamma$ with loop system $\ell(\gamma)$ of $2n$ loops is constructed as follows. A spine of a handlebody $V$ of genus $n$ is a legged loop system $\gamma_V$ in $F = \partial V$ with base point $v$ such that the inclusion map $\gamma_V \to V$ induces an isomorphism $\pi_1(\gamma, v) \to \pi_1(V, v)$. A regular neighborhood $\hat{V}$ of $\gamma_V$ in $F$ is a planar surface in $F$. By [4, Theorem 10.2], there is a diffeomorphism $(\hat{V} \times [0, 1], \hat{V} \times 0) \to (V, \hat{V})$ sending every point $(x, 0) \in \hat{V} \times 0$ to $x \in V$. The surface $\hat{V}$ is called a spine surface of $V$. Let $\gamma_V$ and $\gamma_{V'}$ be spines of the handlebodies $V$ and $V'$ in $F$ with the same base point $v$, respectively. A Heegaard graph of $M$ is a legged loop system $\gamma = \gamma_M$ in $M$ with base point $v$ which is the union of legged loop systems $\gamma^+_V$ and $\gamma^+_{V'}$ obtained from $\gamma_V$ and $\gamma_{V'}$ by pushing $\gamma_V \setminus v$ and $\gamma_{V'} \setminus v$ into the interiors $\text{Int}V$ and $\text{Int}V'$, respectively. The following lemma is obtained.
Lemma 2.3. For every Heegaard graph $\gamma$ of every homotopy 3-sphere $M$, the fundamental group $\pi_1(X(M) \setminus K(\gamma), v^+)$ of the spun $S^2$-link $K(\gamma)$ in the 4-sphere $X(M)$ is a free group of rank $2n$.

Proof of Lemma 2.3. The closed complement $\text{cl}(M \setminus N(\gamma))$ for a regular neighborhood $N(\gamma)$ of $\gamma$ in $M$ is diffeomorphic to the handlebody $F^{(o)} \times [-1, 1]$ for the once-punctured surface $F^{(o)}$ of $F$. Since the fundamental group $\pi_1(F^{(o)} \times [0, 1], v^+)$ with base point $v^+$ taken in $(\partial F^{(o)}) \times [0, 1]$ is a free group of rank $2n$, the desired result is obtained from Lemma 2.1. □

It is noted that this free group in Lemma 2.3 is not necessarily a meridian-based free group. Here is an example.

Example 2.4. Let $\gamma$ be a legged loop system with base point $v$ in $M = S^3$ illustrated in Fig. with $\pi_1(M \setminus \gamma, v^+)$ a free group of rank 2. In fact, a trivial legged loop system is obtained by sliding an edge along another edge, so that $\pi_1(M \setminus \ell(\gamma), v^+)$ is a free group of rank 2. A regular neighborhood $V$ of $\gamma$ in $M$ and the closed complement $V' = \text{cl}(M \setminus V)$ constitute a genus 2 Heegaard splitting $V \cup V'$ of $M$ by noting that the 3-manifold $V'$ is a handlebody of genus 2 by the loop system theorem and the Alexander theorem (cf. e.g., [6]). Thus, the union $V \cup V'$ is a genus 2 Heegaard splitting of $M$. Since the legged loop system $\gamma$ with base point $v$ is a spine of $V$ by sliding the base point $v$ into $\partial V$, there is a Heegaard graph $\gamma_M$ of $M$ with $\gamma$ as $\gamma^+_V$. By Lemma 2.3, the spun $S^2$-link $K(\gamma_M)$ in the 4-sphere $X(M) = S^4$ has the free fundamental group $\pi_1(X(M) \setminus K(\gamma_M), v^+)$ of rank 4, which does not admit any meridian basis because the spun $S^2$-link $K(\gamma_M)$ in $S^4$ contains, as a component, the spun trefoil $S^2$-knot whose fundamental group is known to be not infinite cyclic.

Figure 1: A legged loop system $\gamma$ in $S^3$ with free fundamental group of rank 2
Given a proper arc system $a_*$ in $M^{(o)}$, there is a legged loop system $\gamma$ in $M$ with the proper arc system $a(\gamma) = a_*$ in $M^{(o)}$. The spun $S^2$-link $K(\gamma)$ in $X(M)$ is uniquely determined by the arc system $a_*$ and thus denoted by $S(a_*)$. The following lemma is used toward the final step of the proof of Poincare conjecture.

**Lemma 2.5.** Let $a_*$ be a proper arc system in a compact once-punctured manifold $M^{(o)} = \text{cl}(M \setminus B)$ of a homotopy 3-sphere $M$. If the spun $S^2$-link $S(a_*)$ in the 4-sphere $X(M)$ is a trivial $S^2$-link, then the proper arc system $a_*$ is in a boundary-collar $S \times [0, 1]$ of $M^{(o)}$.

**Proof of Lemma 2.5.** By Lemma 2.1, the fundamental group $\pi_1(M^{(o)} \setminus a(\gamma), v^+)$ is a meridian-based free group. Consider the 2-sphere $S$ as the boundary

$$\partial(d \times [0, 1]) = d \times 0 \cup (\partial d) \times [0, 1] \cup d \times 1$$

of the product $d \times [0, 1]$ for a disk $d$ so that $d \times 0$ contains one end of the proper arc system $a_*$ and $d \times 1$ contains the other end of the proper arc system $a_*$. Let $(E; E_0, E_1)$ be the triplet obtained from $(M^{(o)}, d \times 0, d \times 1)$ by removing a tubular neighborhood of $a_*$ in $M^{(o)}$. For $v^+ \in E_0$, the inclusion $E_0 \subset E$ induces an isomorphism

$$\pi_1(E_0, v^+) \rightarrow \pi_1(E, v^+).$$

By [4, Theorem 10.2], $E$ is diffeomorphic to the connected sum of the product $E_0 \times [0, 1]$ and a homotopy 3-sphere. This means that the proper arc system $a_*$ is in a boundary-collar $S \times [0, 1]$. This completes the proof of Lemma 2.5. □

3. A ribbon surface-link and a disk-chord system of a ribbon handlebody system

By combining Lemmas 2.3 with Free Ribbon Lemma in Section 1, the following lemma is obtained.

**Lemma 3.1.** The spun $S^2$-links $K(\gamma)$ of every Heegaard link $\gamma$ of every homotopy 3-sphere $M$ is a ribbon $S^2$ link in $X(M)$.

The following lemma makes a connection between a knot in $M$ and a Heegaard graph of $M$.

**Lemma 3.2.** For every knot $k$ in every homotopy 3-sphere $M$, there is a Heegaard graph $\gamma$ of $M$ such that the knot $k$ is equivalent to a component of the loop system $\ell(\gamma)$ of $\gamma$ in $M$. 

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Proof of Lemma 3.2. By considering $k$ as a polygonal loop in $M$, there is a triangulation $\mathcal{T}$ of $M$ whose 1-skeleton $\mathcal{T}^{(1)}$ contains the knot $k$. The graph $\mathcal{T}^{(1)}$ is deformed into a legged loop system $\gamma'$ in $M$ so that $k$ is a component of the loop system $k(\gamma')$. Let $V'$ be a regular neighborhood of $\gamma'$ in $M$ which is a handlebody. The legged loop system $\gamma'$ is deformed into a spine $\gamma_V$ of the handlebody $V'$. The closed complement $V = \text{cl}(M \setminus V)$ is also a handlebody, so that there is a Heegaard splitting $V \cup V'$ of $M$. Hence there is a Heegaard graph $\gamma$ of $M$ obtained from $\gamma_V$ and $\gamma_V'$ such that $k$ is equivalent to a component of the loop system $\ell(\gamma)$. □

By Lemma 3.2, there is a Heegaard graph $\gamma$ of $M$ whose loop system contains the knot $k$. By Lemma 3.1, the spun $S^2$-link $K(\gamma)$ is a ribbon $S^2$-link in $X(M)$, so that the spun $S^2$-knot $K(k)$ is a ribbon $S^2$-knot in $X(M)$ because any component of a ribbon $S^2$-link in $S^4$ is a ribbon $S^2$-knot in $S^4$ by definition. Thus, the following result is obtained.

Lemma 3.3. For every knot $k$ in every homotopy 3-sphere $M$, the spun $S^2$-knot $K(k)$ is a ribbon $S^2$-knot in $X(M)$.

For a knot $k$ in the interior of $M^{(0)} = \text{cl}(M \setminus B)$ for a 3-ball $B$, the spun torus-knot of $k$ is a torus-knot $T(k)$ in $X(M)$ given by the inclusions

$$T(k) = k \times S^1 \subset M^{(0)} \times S^1 \subset M^{(0)} \times S^1 \cup S \times D^2 = X(M).$$

The spun torus-knot $T(k)$ in $X(M)$ is uniquely constructed up to choices of a 3-ball $B$. The following lemma is important to our purpose.

Lemma 3.4. For every knot $k$ in every homotopy 3-sphere $M$, the spun torus-knot $T(k)$ is a ribbon torus-knot in $X(M)$.

Proof of Lemma 3.4. From construction, the spun $S^2$-knot $K(k)$ in $X(M)$ is obtained from $T(k)$ by the unique 2-handle surgery, so that the spun torus-knot $T(k)$ is obtained from the spun $S^2$-knot $K(k)$ by the converse 1-handle surgery. By definition, the spun torus-knot $T(k)$ is a ribbon torus-knot, completing the proof. □

Assume that a ribbon surface-link $L$ is obtained from a trivial oriented $S^2$-link $O$ by surgery along a 1-handle system $h_s$ of disjointly embedded oriented 1-handles $h_j (j = 1, 2, \ldots, s)$ (for some $s$) on $O$ in $S^4$. A ribbon handlebody system bounded by a ribbon surface-link is discussed here (see [17 II.3.61]). Let $B_i$ be a system of disjoint 3-balls $B_i (i = 1, 2, \ldots, m)$ in $S^4$ bounded by $O$. The intersection $h_j \cap O$
Figure 2: Two arcs of $k$ near a disk $d_i$ drawn as thick lines

consists of two disks, called the **attaching disks** of $h_j$ to $O$. A **meridian disk** of the 1-handle $h_j$ is a proper disk in $h_j$ parallel to any one of the attaching disks. By an isotopic deformation of the 1-handle system $h_\ast$, the intersection $h_\ast \cap \text{Int} B_i$ can be assumed to be a meridian disk system (possible empty) in $h_\ast$, whose number of meridian disks is called the **ribbon index** of $h_\ast$ in $B_i$. A **ribbon handlebody system** of a ribbon surface-link $L$ is the union

$$V_R = B_\ast \cup h_\ast,$$

which is an immersed handlebody system bounded by $L$ in $S^4$. The **ribbon index** of $V_R$ is the total number of the ribbon indexes of $h_\ast$ in $B_i$ for all $i$. The **disk-chord system** of a ribbon surface-link $L$ is the pair $(d_\ast, \alpha_\ast)$ of a disk system $d_\ast$, called a **based disk system**, and an arc system $\alpha_\ast$, called a **chord system**, in $S^4$ obtained from the ribbon handlebody system $V_R = B_\ast \cup h_\ast$ by shrinking the 3-ball $B_i$ into a disk $d_i$ for every $i$ and then shrinking the 1-handle $h_j$ into a core arc $\alpha_j$ of $h_j$ spanning the loop system $o_\ast = \partial d_\ast$, called a **based loop system**, for every $j$. See Fig. 2 (1) for a situation around a disk in a based disk system. From construction, the ribbon index of $h_\ast$ in $B_i$ is equal to the number of the transverse intersection points $\alpha_\ast \cap \text{Int} d_i$, called the **chord index** of $\alpha_\ast$ in $d_i$. The **chord index** of the disk-chord system $(d_\ast, \alpha_\ast)$ is the total number of the chord indexes of $\alpha_\ast$ in $d_i$ for all $i$. By the orientations of $L$ and $S^4$, the based disk system $d_\ast$ can be uniquely oriented, and the ribbon handlebody system $V_R$ and the ribbon surface-link $L$ are uniquely recovered from the disk-chord system $(d_\ast, \alpha_\ast)$ by thickening the chord system $\alpha_\ast$ and the based disk system $d_\ast$, where an argument in [5] is needed for uniqueness of the embedded 1-handle system. Let

$$\Delta^2 \subset \Delta^3 \subset \Delta^4$$

be the inclusions such that $\Delta^4$ is a 4-ball in $S^4$, $\Delta^3$ is a proper 3-ball of $\Delta^4$ and $\Delta^2$
is a proper disk of $\Delta^3$. A disk-chord system $(d_*, \alpha_*)$ of $L$ in $S^4$ can be moved into $\text{Int}\Delta^3$ isotopically by first moving a neighborhood of the based disk system $d_*$ into $\text{Int}\Delta^3$ and then moving the remaining part of the arc system $\alpha_*$ into $\text{Int}\Delta^3$ (see [17, II.3.61]). So, assume that a disk-chord system $(d_*, \alpha_*)$ of $L$ is in $\text{Int}\Delta^3$. The ribbon handlebody system $V_R$ and the ribbon surface-link $L$ are uniquely realized from a disk-chord system $(d_*, \alpha_*)$ of $L$ in $\text{Int}\Delta^3$. A chord graph of $L$ is a diagram $C(o_*, \alpha_*)$ in $\text{Int}\Delta^2$ for a chord graph $o_* \cup \alpha_*$ of $L$ in $\text{Int}\Delta^3$. A chord diagram of $L$ is a diagram $C(o_*, \alpha_*)$ in $\text{Int}\Delta^2$ for a chord graph $o_* \cup \alpha_*$ of $L$ in $\text{Int}\Delta^3$. A ribbon handlebody system $V_R$ of $L$ cannot be uniquely recovered because in general a disjoint disk system $d_*$ in the interior of $\Delta^3$ with $\partial d_* = o_*$ is not unique (see [17, Lemma I.1.4]). So, to fix a ribbon handlebody system $V_R$ of $L$, every loop of the based loop system $o_*$ should be fixed as it is shown in of Fig. 2 (2). The following observation is obtained from the above argument.

**Observation 3.5.** A ribbon surface-link $L$ and a ribbon handlebody system $V_R$ in $S^4$ are uniquely realized in $\text{Int}\Delta^4$ from a disk-chord system $(d_*, \alpha_*)$ in $\text{Int}\Delta^3$, and also from a chord graph $o_* \cup \alpha_*$ in $\text{Int}\Delta^3$ or a chord diagram $C(o_*, \alpha_*)$ in $\text{Int}\Delta^2$ by fixing every loop of the based loop system $o_*$ as it is shown in Fig. 2 (2).

A chord diagram has the advantage of being easy to handle. For example, the moves on chord diagrams for equivalent ribbon surface-links are known in [7, 8, 9, 10]. A ribbon handlebody $V_R$ bounded by a ribbon torus-knot $T$ is called a ribbon solid torus. The following lemma is an easy exercise of the moves on chord diagrams in [7] and used in Section 4.

**Lemma 3.6.** Every ribbon solid torus of ribbon index $n$ bounded by a ribbon torus-knot $T$ in $\text{Int}\Delta^4$ is deformed into a ribbon solid torus $V_R$ with $\partial V_R = T$ which is realized by a disk-chord system $(d_*, \alpha_*)$ in $\text{Int}\Delta^3$ of $\text{Int}\Delta^4$ where

$$d_* = \{d_i | i = 1, 2, \ldots, n\}, \quad \alpha_* = \{\alpha_i | i = 1, 2, \ldots, n\} \quad \text{and} \quad o_* = \partial d_*$$

such that

1. the chord $\alpha_i$ connects $o_i$ to $o_{i+1}$ for every $i (i = 1, 2, \ldots, n)$ with $o_{n+1} = o_1$, and
2. the chord index of $\alpha_*$ to $d_i$ is equal to 1 for every $i$.  

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The disk-chord system \((d_*, \alpha_*)\) in Lemma 3.6 is called a *circular primitive disk-chord system* or briefly a *CP disk-chord system* (see Fig. 3 (1), (2) for examples). The spine of a disk-chord system \((d_*, \alpha_*)\) is a graph \(\Gamma\) obtained from \(d_* \cup \alpha_*\) by shrinking every disk \(d_i\) into a vertex \(v_i\) for every \(i\). A *regular maximal tree* of \(\Gamma\) is a tree \(\tau^+\) in \(\Gamma\) obtained from a maximal tree \(\tau\) of \(\Gamma\) by taking a regular neighborhood of \(\tau\) in \(\Gamma\). A *regular maximal tree* of a disk-chord system \((d_*, \alpha_*)\) is a disk-chord system \(\tau^+(d_*, \alpha_*)\) obtained from a regular maximal tree \(\tau^+\) of the spine \(\Gamma\) by making every vertex \(v_i\) in \(\tau^+\) back to the original disk \(d_i\) for every \(i\). Let \(\dot{\tau}^+(d_*, \alpha_*) = \dot{\tau}^+\) be the set of all the degree 1 vertexes of \(\tau^+\). The arc system

\[ e_* = \text{cl}(\Gamma \setminus \tau^+) = \text{cl}((d_* \cup \alpha_*) \setminus \tau^+(d_*, \alpha_*)) \]

is called the *complementary arc system* of a regular maximal tree \(\tau^+(d_*, \alpha_*)\) in a disk-chord system \((d_*, \alpha_*)\).

![Figure 3: CP disk-chord systems of ribbon solid tori (1), (2) bounded by the spun torus-knot of the trefoil knot (3)](image)

**4. Main result: Proof of Theorem 1.1**

Throughout this section, the proof of Theorem 1.1 is done. Let \(k\) be a knot in a homotopy 3-sphere \(M\). If \(k\) is a trivial knot in \(M\), then the knot \(k\) belongs to a 3-ball \(D^3\) in \(M\). So, assume that \(k\) is a non-trivial oriented knot in \(M\). Since the spun torus-knot \(T(k)\) is a ribbon torus-knot in \(X(M)\) by Lemma 3.4, there is a ribbon solid torus \(V_R\) of some ribbon index \(n\) with \(\partial V_R = T(k)\) in \(\text{Int}\Delta^4\) which is realized by a CP disk-chord system \((d_*, \alpha_*)\) of chord index \(n\) in \(\text{Int}\Delta^3\) and a chord diagram \(C(d_*, \alpha_*)\) in \(\text{Int}\Delta^2\) by Observation 3.5. Since there is a meridian-preserving isomorphism \(\pi_1(M \setminus k, v^+) \to \pi_1(X(M) \setminus T(k), v^+)\) by the van Kampen theorem, the longitude of \(k\) in \(M\) represents an infinite order element in the fundamental group \(\pi_1(X(M) \setminus T(k), v^+)\). This implies that an oriented meridian loop of \(V_R\) is a uniquely
determined loop in $T(k)$ up to isotopies of $T(k)$, and the CP disk-chord system $(d_*, \alpha_*)$ is assumed that $k$ meets $d_i$ with just one boundary arc and just one interior point transversely for every $i$, as in Fig.2(1) (see also Fig.3(1), (2) for examples). Assume that $k$ is in $\text{Int}M^{(o)}$. The following lemma is obtained.

**Sublemma 4.1.** The disk system $d_i (i = 1, 2, \ldots, n)$ is deformed into $\text{Int}M^{(o)}$ by an isotopy of $X(M)$ keeping the knot $k$ fixed.

**Proof of Sublemma 4.1.** For every $i$, let $c_i$ be a simple arc in $d_i$ connecting the point $k \cap \text{Int}(d_i)$ to a point in the arc $k \cap \partial d_i$. The arc system $c_i (i = 1, 2, \ldots, n)$ is deformed into a bi-collar neighborhood $M^{(o)} \times [-1, 1]$ of $M^{(o)}$ with $M^{(o)} \times 0 = M^{(o)}$ in $X(M)$ by an isotopy keeping $M^{(o)}$ fixed. Then the arc system $c_i (i = 1, 2, \ldots, n)$ is projected into $M^{(o)}$ by a general position argument. A deformed disk system $d_i (i = 1, 2, \ldots, n)$ in $M^{(o)}$ is obtained from the arc system $c_i (i = 1, 2, \ldots, n)$ in $M^{(o)}$ by widening them as a small disk system, completing the proof of Sublemma 4.1. □

By Sublemma 4.1, consider that the CP disk-chord system $(d_*, \alpha_*)$ of $V_R$ is in $M^{(o)}$. The spine $\Gamma$ of $(d_*, \alpha_*)$ is a degree 4 graph in $M^{(o)}$. For every regular maximal tree $\tau^+$ of $\Gamma$, there is a disk $\delta^2$ in $M^{(o)}$ with $\hat{\tau}^+ = \tau^+ \cap \partial \delta^2$ such that a neighborhood of every degree 4 vertex of $\tau^+$ in $\delta^2$ gives Fig.2(1) in $\tau^+(d_*, \alpha_*)$. The disk $\delta^2$ is called a **regular support disk** for $\tau^+(d_*, \alpha_*)$. This disk $\delta^2$ is moved into the 2-sphere $S = \partial M^{(o)}$. Let $\delta^3 = \delta^2 \times [0, 1]$ be a collar of $\delta^2$ in $M^{(o)}$ which is a 3-ball with $\delta^3 \cap S = \delta^2 \times 0 = \delta^2$. Let $e_*$ be the complementary arc system of $\tau^+(d_*, \alpha_*)$ in $(d_*, \alpha_*)$ consisting of arcs $e_i (i = 1, 2, \ldots, n+1)$, where $n$ is the chord index of the CP disk-chord system $(d_*, \alpha_*)$ which is determined by the Euler characteristics $\chi(\Gamma) = -n$. The knot $k$ in $M^{(o)}$ is deformed in $M^{(o)}$ so that the intersection $t = k \cap \delta^3$ is a tangle in $\delta^3$ whose projection image under the canonical projection

$$\delta^3 = \delta^2 \times [0, 1] \to \delta^2$$

is the regular maximal tree $\tau^+$ in the regular support disk $\delta^2$ by pushing $\tau^+(d_*, \alpha_*) \setminus \hat{\tau}^+(d_*, \alpha_*)$ into $\delta^2 \times (0, 1)$ and then by creating a crossing point by the move from (1) to (3) in Fig.2. Then the regular maximal tree $\tau^+$ in $\delta^2$ can be regarded as a tangle diagram of $t$ in $\delta^3$. Let $[t, \tau^+]$ be the disk union between the tangle $t$ and the graph $\tau^+$ in the preimage of $\tau^+$ under the canonical projection $\delta^3 \to \delta^2$. The following sublemma is essentially observed in [11] Theorem 2.3 (3)] for an inbound arc diagram.

**Sublemma 4.2.** The spun $S^2$-link $T(t)$ of a tangle $t$ in $\delta^3$ in the 4-disk

$$U^4 = \delta^3 \times [0, 1] \times S^1 \cup \delta^2 \times D^2 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M)$$
bounds a ribbon 3-ball system
\[ V'_R = [t, \tau^+] \times S^1 \cup \tau^+ \times D^2 \]
which extends to a ribbon solid torus \( V_R \) of the spun torus-knot \( T(k) \) such that the compact complement \( \text{cl}(V_R \setminus V'_R) \) is a disjoint 3-ball system bounded by the spun \( S^2 \)-link \( S(e_s) \) in \( X(M) \).

**Proof of Sublemma 4.2.** If \( t \) is a 1-string tangle with \( \tau^+ \) a simple arc, then \( V'_R = [t, \tau^+] \times S^1 \cup \tau^+ \times D^2 \) is a 1-handle thickening \( t \), that is a ribbon 3-ball with ribbon index 0. If \( t \) is a 2-string tangle with \( \tau^+ \) just one degree 4 vertex graph, then \( t \) is the 2-tangle in Fig. 2 (3) and \( V'_R \) is a ribbon 3-ball system giving a disk-chord system \( \tau^U(d_s, \alpha_s) \) in the 4-ball \( U^4 \) such that \( \tau^U(d_s, \alpha_s) \) is diffeomorphic to the regular maximal tree \( \tau^+(d_s, \alpha_s) \) of \( (d_s, \alpha_s) \) in \( \delta^3 \). Let \( \delta^3 \) be a 4-ball in \( U \) with \( \delta^3 \) as a proper 3-ball. The following sublemma is needed.

**Sublemma 4.3.** There is an orientation-preserving diffeomorphism of \( X(M) \) sending \((U^4, \tau^U(d_s, \alpha_s)) \) to \((\delta^3, \tau^+(d_s, \alpha_s)) \).

**Proof of Sublemma 4.3.** For the regular maximal tree \( \tau^+ \) in the regular support disk \( \delta \), find a 2-disk \( \delta^2_0 \subset \text{Int}\delta \) such that \( \tau' = \delta^2_0 \cap \tau^+ \) has \( \text{cl}(\tau^+ \setminus \tau') \cong \tau^+ \times [0, 1] \) and construct a 4-ball \( \delta^4_0 \subset \text{Int}U \) with \( \delta^2_0 \) as a trivial proper disk. Then construct a proper 3-ball \( \delta^3_0 \subset \delta^3 \) with \( \delta^3_0 \) as a proper disk. Note that there is an orientation-preserving diffeomorphism of \( S^4 \) sending the triad \((\delta^4_0, \delta^3_0, \delta^2_0) \) to the triad \((\delta^4, \delta^3, \delta^2) \) and the regular maximal tree \( \tau^+(d_s, \alpha_s) \) of \( (d_s, \alpha_s) \) given by \( \tau' \) in \( \delta^3_0 \) to \( \tau^+(d_s, \alpha_s) \) in \( \delta^3 \). Since \( \text{cl}(U^4 \setminus \delta^4_0) \) is diffeomorphic to \( S^3 \times [0, 1] \) (see [15]), there is an orientation-preserving diffeomorphism
\[ (\text{cl}(U^4 \setminus \delta^4_0), \text{cl}(U^4 \setminus \delta^4_0) \cap \tau^+) \to (S^3, \tau^+) \times [0, 1]. \]

Then there is a triad \((U^4, U^3, U^2) \) with \( U^3 \) a proper 3-ball in \( U^4 \) and \( U^2 \) a proper 2-disk in \( U^3 \) such that there is an orientation-preserving diffeomorphism of \( S^3 \) sending the triad \((U^4, U^3, U^2) \) to the triad \((\delta^4_0, \delta^3_0, \delta^2_0) \) and \( \tau^U(d_s, \alpha_s) \) in \( U^3 \) to \( \tau'(d_s, \alpha_s) \) in \( \delta^3_0 \). Thus, there is an orientation-preserving diffeomorphism of \( S^4 \) sending the triad \((U^4, U^3, U^2) \) to the triad \((\delta^4, \delta^3, \delta^2) \) and \( \tau^U(d_s, \alpha_s) \) in \( U^3 \) to \( \tau^+(d_s, \alpha_s) \) in \( \delta^3 \). This completes the proof of Sublemma 4.3. □

By Sublemma 4.3, the ribbon 3-ball system \( V'_R \) realizing \( \tau^U(d_s, \alpha_s) \) in \( U^4 \) extends to a ribbon solid torus \( V_R \) in \( S^4 \). This means that the spun \( S^2 \)-link \( S(e_s) \) in \( X(M) \) bounds the disjoint 3-ball system \( \text{cl}(V_R \setminus V'_R) \). This completes the proof of Sublemma 4.2. □
By Lemma 2.5 and Sublemma 4.2, the proper arc system $e_*$ and hence $k$ are in the 3-ball $D^3$ which is a regular neighborhood of $\delta^2 \times [0, 1]$ in $M^{(o)}$. This completes the proof of Theorem 1.1. □

5. Conclusion

A general problem arising from this paper is how any given ribbon solid torus bounded by the spun torus-knot $T(k)$ of a knot $k$ relates to a knot diagram $D(k)$ of $k$. For example, the CP disk-chord system $(d_*, \alpha_*)$ in Fig. 3 (1) is seen to represent a ribbon solid torus bounded by the spun torus-knot $T(k)$ of the trefoil knot $k$ in Fig. 3 (3). In fact, the ribbon torus-knot given by Fig. 3 (1) is equivalent to the ribbon torus-knot given by Fig. 3 (2) by moves on chord diagrams in [7, 8, 9, 10] and by Sublemma 4.2 the CP disk-chord system of Fig. 3 (2) is the CP disk-chord system of the spun ribbon solid torus of the trefoil knot diagram $D(k)$ shown in Fig. 3 (3). It would be interesting to point out that the CP disk-chord system $(d_*, \alpha_*)$ in Fig. 3 (1) is not the CP disk-chord system of the spun ribbon solid torus of any knot diagram $D'(k)$ of the trefoil knot $k$. To see this, the cross-index in [18] is used. If $(d_*, \alpha_*)$ is obtained from the spun ribbon solid torus of a trefoil knot diagram $D'(k)$, then the complementary arc system $e_*$ of any regular maximal tree $\tau^+(d_*, \alpha_*)$ in $(d_*, \alpha_*)$ in a regular support disk $\delta$ must have the cross-index 0 in the annulus $A$ given by any extended disk $\delta^+$ such that $\text{Int} \delta^+ \supset \delta$ and $e$ is an immersed arc system in the annulus $A = (\delta^+ \setminus \delta)$. However, the cross-index of $e_*$ in an annulus $A$ is 1 for the diagram given in Fig. 3 (1). This means that the CP disk-chord system $(d_*, \alpha_*)$ in Fig. 3 (1) is not the CP disk-chord system of the spun ribbon solid torus of any trefoil knot diagram $D'(k)$.

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