CONSTANT CURVATURE MODELS IN SUB-RIEMANNIAN GEOMETRY

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Abstract. Each sub-Riemannian geometry with bracket generating distribution enjoys a background structure determined by the distribution itself. At the same time, those geometries with constant sub-Riemannian symbols determine a unique Cartan connection leading to their principal invariants. We provide cohomological description of the structure of these curvature invariants in the cases where the background structure is one of the parabolic geometries. As an illustration, constant curvature models are discussed for certain sub-Riemannian geometries.

1. Introduction

The central objects of Riemannian geometry are Levi-Civita connection and associated geodesics and curvature. There are three ways to define geodesic equation and study geodesics of a Riemannian manifold:

In the Hamiltonian approach, geodesic equation is the Hamiltonian equation in the cotangent bundle $T^*M$ with Hamiltonian $H(p) = \frac{1}{2} g^{-1}(p, p)$ and geodesics are projections of the integral curves of the Hamiltonian flow to $M$.

In the Lagrangian variational approach, the geodesic equation is the Euler-Lagrange equation for the length or energy functional in the space of curves.

In the geodesic approach by Levi-Civita, geodesics are defined as autoparallel curves $\gamma(t)$, such that the tangent field $\dot{\gamma}(t)$ is parallel with respect to Levi-Civita connection (unique torsionfree connection which preserves the metric).

Existence of such connection follows from the fact that Riemannian metric, considered as $SO(n)$-structure (i.e., a principal $SO(n)$-bundle $P \to M = P/\text{SO}(n)$ of frames) has trivial first prolongation. In other words, the bundle $P$ has a canonical $SO(n)$-equivariant trivialisation $\omega : TP \to \mathfrak{so}(n) + \mathbb{R}^n$ (Cartan connection). Projection to $M$ of integral curves of constant vector fields $\omega^{-1}v, v \in \mathbb{R}^n$ are geodesics.

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In sub-Riemannian geometry, which studies a manifold $M$ with a sub-Riemannian metric $g$ defined on a non-holonomic distribution $D$, all three approaches are working, but, as it was noted by A. M. Vershik and L.D. Faddeev [22], they lead to different equations for "sub-Riemannian geodesics". As they remark, in Riemannian geometry a geodesic may be defined as (locally) shortest curve and as a straightest curve, but in sub-Riemannain geometry these two notions are not equivalent. The definition of a sub-Riemannian geodesic as a shortest curve is based on the notion of Carnot-Carateodory metric $d(x,y)$ on a sub-Riemannian manifold $(M,D,g)$, defined as the infimum of length of horizontal curves, joining points $x,y \in M$. It is used in optimisation control theory. A. Agrachev et al., using a power series decomposition of the square of the distance function, define the curvature of sub-Riemannian metric and use it for infinitesimal variation of geodesics.

In non-holonomic mechanics, the geodesics are most important as straightest curves. The evolution of mechanical system with non-holonomic constrains is described by geodesic equation

$$\nabla \dot{\gamma} \dot{\gamma} = 0$$

for horizontal curve $\gamma(t)$, where $\nabla : \Gamma(D) \times \Gamma(D) \to \Gamma(G)$ is a partial connection, defined by the Koszul formula on a sub-Riemannian manifold $(M,D,g)$ with a fixed complementary to $D$ distribution, see [22], [13]. Moreover, V.V. Wagner proposed an extension of the sub-Riemannian metric to Riemannian one, by choosing some auxiliary complementary distributions and use it to define the curvature for the partial connection such that its vanishing is equivalent to the flatness of the partial connection $\nabla$.

Summarizing, there are diverse approaches applying different types of connections and curvature concepts in sub-Riemannian geometry. Depending on the problem one could

- construct a partial connection [6];
- use the notion of a connection over a bundle map [16, 17];
- extend sub-Riemannian metric to Riemannian metric [14, 11, 15];
- use a very general variational approach to the curvature [1].

The goal of this paper is to provide an efficient framework to tackle different equivalence problems in sub-Riemannian geometry. We use Cartan-Tanaka theory to construct a canonical Cartan connection and study the structure of the curvature tensor. The primary application we had in mind during the work on the paper was the study of constant curvature sub-Riemannian spaces. They possess the biggest possible symmetry algebra among the structures with the same metric symbol, see Section 6. We classify constant curvature sub-Riemannian structures in 2 particularly interesting cases: contact distributions and free 2-step distributions.

Consider a distribution $D$ on the manifold $M$. The sheaf $\mathcal{D}^{-1} = \mathcal{D}$ of vector fields with all values in $D$ generates the filtration by sheaves

$$\mathcal{D}^j = \{[X,Y], X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \ldots.$$  

We say that $D$ is a bracket generating distribution if for some $k$, $\mathcal{D}^{-k}$ is the sheaf of all vector fields on $M$. In particular, there is the corresponding
filtration of subspaces \( T_x M = D_x^{-k} \supset \cdots \supset D_x^{-1} \) at each \( x \in M \) and the associated graded tangent space
\[
\text{gr}(T_x M) = T_x M / D_x^{-k+1} \oplus \cdots \oplus D_x^{-1}
\]
comes equipped with the structure of a nilpotent Lie algebra.

A sub-Riemannian structure on \( M \) is given by a metric \( S \) which is defined only along \( D \). We say that \( (M, D, S) \) is a sub-Riemannian geometry with constant metric symbol if \( D \) is bracket generating, and the nilpotent algebra \( \text{gr} T_x M \), together with the metric, is isomorphic to a fixed graded Lie algebra
\[
\mathfrak{g}^{-} = \mathfrak{g}^{-k} \oplus \cdots \oplus \mathfrak{g}^{-1}
\]
with a fixed metric \( \sigma \) on \( \mathfrak{g}^{-1} \).

In the sequel we shall deal with sub-Riemannian geometries with constant metric symbols only. Under this assumption, we can employ the Cartan-Tanaka theory. We define
\[
\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}^{-})
\]
to be the Lie algebra of the Lie group \( G_0 \) of all automorphisms of the graded nilpotent algebra \( \mathfrak{g}^{-} \) preserving the metric \( \sigma \) on \( \mathfrak{g}^{-1} \), i.e. the algebra of certain derivations on \( \mathfrak{g}^{-} \). The action of the derivations extends the Lie algebra structure on \( \mathfrak{g}^{-} \) to the Lie algebra
\[
\mathfrak{g} = \mathfrak{g}^{-k} \oplus \cdots \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}_0.
\]

The graded bundle \( \text{gr}(TM) \) admits the canonical graded frame bundle with structure group \( \text{Aut}_\text{gr}(\mathfrak{g}^{-}) \). A sub-Riemannian structure is a particular type of a filtered \( G_0 \)-structure that is a reduction of the canonical graded frame bundle to the group \( G_0 \) ([8]). In Section 2 we show that the bundle \( \mathcal{G} \) of orthogonal sub-Riemannian frames admits a natural Cartan connection. In our particular example a Cartan connection is a form \( \omega : T\mathcal{G} \rightarrow \mathfrak{g} \) satisfying following properties:

- \( \omega_p : T_p \mathcal{G} \rightarrow \mathfrak{g} \) is an isomorphism for all \( p \in \mathcal{G} \);
- \( R^*_g(\omega) = \text{Ad}_{\mathfrak{g}^{-1}}(\omega) \) where \( R_g \) is a principal right action of an element \( g \in G_0 \);
- \( \omega(\zeta_X) = X \) where \( \zeta_X \) is the fundamental vector field corresponding to \( X \in \mathfrak{g}_0 \).

The curvature form \( \Omega \) of the Cartan connection \( \omega \) is a two-form defined by
\[
\Omega(\eta, \xi) = d\omega(\eta, \xi) + [\omega(\eta), \omega(\xi)], \quad \eta, \xi \in \mathfrak{X}(\mathcal{G}).
\]
It follows from the definition of a Cartan connection that \( \Omega \) is equivariant for the principal action, meaning \( R^*_g(\Omega) = \text{Ad}_{\mathfrak{g}^{-1}}(\Omega) \), and horizontal, i.e. \( \Omega(\zeta_X, \xi) = 0 \) for arbitrary \( X \in \mathfrak{g}_0 \). Therefore it is convenient to consider curvature function \( \kappa : \mathcal{G} \rightarrow \text{Hom}(\wedge^2(\mathfrak{g}/\mathfrak{g}_0), \mathfrak{g}) \) defined by the formula
\[
\kappa(X, Y) = \Omega(\omega^{-1}(X), \omega^{-1}(Y)), \quad X, Y \in \mathfrak{g}/\mathfrak{g}_0.
\]
It immediately follows that \( \kappa \) is equivariant function on \( \mathcal{G} \). Under reasonable normalization of the curvature, the whole curvature function could be expressed through its essential part, the so called harmonic curvature
\( \kappa_H : \mathcal{G} \to H^2(\mathfrak{g}_-, \mathfrak{g}) \), see Section 2 for the definition. Here \( H^2(\mathfrak{g}_-, \mathfrak{g}) \) is the second Lie algebra cohomology space of \( \mathfrak{g}_- \) with values in \( \mathfrak{g} \). Computation of \( H^2(\mathfrak{g}_-, \mathfrak{g}) \) is an essential step in understanding the structure of the curvature function.

The paper is organized as follows. In Section 2 we gather observations which explain why any sub-Riemannian structure with a constant metric symbol possesses a natural normal Cartan connection. All facts listed there are known [19, 24] and we summarize them for the convenience of the reader.

Section 3 is purely algebraic and shows how \( H^2(\mathfrak{g}_-, \mathfrak{g}) \) could be computed utilizing the information about cohomologies related to the underlying distribution. The results obtained in this section are of general nature. They may be applied for description of harmonic curvature of a Cartan connection under the assumption that the Tanaka prolongation of the non-positively graded Lie algebra \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \) is trivial. To give an example, according to the result by Yatsui [25] and independently by Cowling-Ottazzi [10] the Tanaka prolongation of \( \mathfrak{g} \) for a sub-conformal structure is either trivial, or \( \mathfrak{g}_- \) is the nilpotent Iwasawa component of a real rank 1 simple Lie algebra.

In Sections 4 and 5 we restrict our attention to contact sub-Riemannian structures and free 2-step distributions respectively. These two examples could be seen as a generalization of the extensively studied 3-dimensional contact case [3, 5, 14]. We compute cohomologies which reveal the structure of the harmonic curvature function. Then we show how to compute the harmonic curvature explicitly.

In the last Section 6 we illustrate how the algebraic information about harmonic curvature could be used for the classification of constant curvature spaces in sub-Riemannian geometry. The advantage of our method is that all computations are purely algebraic and reduce to the basic representation theory of semisimple Lie algebras. We provide the classification for contact case and free 2-step case in Theorem 9 and Theorem 10 respectively.

For the convenience of the readers who are not familiar with the Cartan connections, the appendix explains a straightforward construction of the normalized connections in the case of trivial prolongations of \( \mathfrak{g} \), under the additional assumption of a fixed complement \( D' \) to \( D \) in \( TM \).

2. Normal Cartan connections associated with sub-Riemannian structures

Let us first remind the Tanaka prolongation procedure. Given a non-positively graded Lie algebra \( \mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \) with the entire negative part \( \mathfrak{g}_- \) generated by \( \mathfrak{g}_{-1} \), we inductively define the vector spaces \( \mathfrak{g}_{r+1} = \{ A \in \mathfrak{gl}_{r+1}(\mathfrak{g}_{\leq r}) ; \ A([X,Y]) = [A(X),Y] + [X,A(Y)], \ X,Y \in \mathfrak{g}_- \} \), i.e. we aim at the space of homogenous derivations of degree \( r + 1 \) on the previously defined algebra \( \mathfrak{g}_{\leq r} \). Notice the formula within the latter definition extends the Lie bracket to a bracket on \( \mathfrak{g}_{\leq r+1} \) whenever at least one of the arguments is from \( \mathfrak{g}_- \).
It turns out that if we obtain the trivial vector space $g_r = 0$ for some $r$, then all the subsequent spaces $g_s$, $s > r$, will be trivial, too. We might also proceed to $r = \infty$ with all $g_r \neq 0$ and in both cases, we call the resulting space $\hat{g} = \bigoplus_{r=-k}^{\infty} g_r$ the Tanaka prolongation of $g$. Finally, the brackets constructed on the prolongation $\hat{g}$ can be completed naturally to obtain the Lie algebra structure on $\hat{g}$ by requesting (again inductively)

$$[A, B](X) = [A(X), B] + [A, B(X)].$$

Notice, we might also start the above inductive prolongation procedure with $g_-$ and $g_0$ is then the entire algebra of graded derivations of homogeneity zero. We call the resulting prolongation the full Tanaka prolongation. Shrinking $g_0$ in the full prolongation to a smaller algebra $\tilde{g}_0$ ensures often the finiteness of the new Tanaka prolongation of $g_- \oplus \tilde{g}_0$.

Let us come back to the graded algebra $g$ in (1), i.e. $g_0$ is a subalgebra in $\mathfrak{so}(g_-)$. We are going to show that its Tanaka prolongation is trivial. Corollary 2 of theorem 11.1 in [21] reveals that the Tanaka prolongation of $g$ must be finite (indeed Tanaka shows that the finiteness can be reduced to the classical answer for $G$-structures when restricting to the $g_0$ action on $g_{-1}$ and this is clearly of finite type here).

Thus we may consider this prolongation $\hat{g} = g_- \oplus \cdots \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus \cdots$ and assume that $g_1 \neq 0$. Denote by $\hat{g}_{-1}$ a $g_0$-invariant complement to the centralizer of $g_1$ in $g_-$. The proposition 2.5 of [24] implies that the graded subalgebra $s \subset \hat{g}$ generated by $\hat{g}_{-1} \oplus [\hat{g}_{-1}, g_1] \oplus g_1$ is semisimple. By the general theory of graded semisimple Lie algebras, there should be a grading element $E \in s_0$ for $s$ which is impossible since $g_0$ is compact. Thus $g_1 = 0$ and $\hat{g} = g$.

Let us observe that the metric $\sigma$ extends canonically to the entire tensor algebra generated by $g_-$. Since the action of $g_0$ on the tensor algebra is reductive, we may identify the entire $g_-$ with an invariant component in the tensor algebra and thus the metric $\sigma$ uniquely extends to the entire $g_-$ making the individual components mutually orthogonal. This observation allows us to set suitable normalization conditions on the curvature of a Cartan connection of type $(g_-, G_0)$, where $G_0$ is a Lie group with Lie algebra $g_0$ such that the inclusion of $g_0$ into the derivations of $g_-$ integrates to a group homomorphism $G_0 \to \text{Aut}_{\mathfrak{g}_0}(g_-)$.

The Cartan connection of such a type is an affine connection on $M$ whose Cartan curvature function can be viewed as a function on the frame bundle valued in cochains $C^2(g_-, g)$ with the obvious differential $\partial$. Now, the adjoint mapping $\partial^*$, provides the complementary space $\ker \partial^*$ to the image $\text{im} \partial$. Thus the classical theory suggests the condition $\partial^* \kappa = 0$ as the right normalization for the Cartan connection. Additionally, the entire curvature $\kappa$ decomposes into its homogeneous components with respect to the grading of $g$.

This normalization satisfies the so called Condition (C) formulated in [18, Definition 3.10.1, p. 338] as a sufficient algebraic condition for the existence
of a unique Cartan connection with a normalized curvature for geometries on filtered manifolds. A more straightforward explanation of the construction of normalized Cartan connections has been recently published in [8] and [4]. See also [9] for the background on Cartan connections.

The harmonic part $\kappa_H$ of the curvature is the part lying in the kernel of $\partial$. It is annihilated by both $\partial$ and $\partial^*$. Due to the algebraic Hodge theory the kernel of the algebraic Laplace operator $\Delta = \partial \circ \partial^* + \partial^* \circ \partial$ corresponds to $H^2(g_-, g)$. As a result harmonic curvature function takes values in $H^2(g_-, g)$.

The Bianchi identity expresses the individual homogeneity components of $\partial \kappa$ by algebraic expressions in terms of lower homogeneities components and their derivatives. In particular, the entire curvature $\kappa$ vanishes if and only if its harmonic part is zero. We arrive at the following theorem ([19, Theorem 1] or [8, Example 3.4 and Theorem 4.8]):

**Theorem 1.** For each sub-Riemannian manifold $(M, D, S)$ with constant metric symbol, there is a unique Cartan connection $(\mathcal{G} \to M, \omega)$ of type $(\mathfrak{g}, G_0)$ with the curvature function $\kappa : \mathcal{G} \to \mathfrak{g} \otimes \Lambda^2 g^*_{-}$ satisfying $\partial^* \kappa = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection $\kappa_H$, i.e. the component with $\partial \kappa_H = 0$.

The distribution $D$ on $M$ itself often defines a nice finite type filtered geometry which enjoys a canonical Cartan connection, too. Many of them belong to the class of the parabolic geometries, for which the full Tanaka prolongation of $g_-$ is a semisimple Lie algebra $\bar{\mathfrak{g}}$ and $g_- = g_{-k} \oplus \cdots \oplus g_{-1}$ is the opposite nilpotent radical to the parabolic subalgebra $\mathfrak{p} = \bar{\mathfrak{g}}_0 \oplus \cdots \oplus \bar{\mathfrak{g}}_k \subset \bar{\mathfrak{g}}$, with $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$.

Let us fix a graded semisimple Lie algebra $\bar{\mathfrak{g}}$ and consider the graded frame bundle $\mathcal{G}_0 \to M$ of $\text{gr} TM$. Under some mild cohomological conditions which are listed in Section 3, the structure group of this bundle is the full group of graded automorphisms of $\mathfrak{g}_-$. Exactly as in the sub-Riemannian case, the algebraic Hodge theory and the corresponding normalization are available (though the theory behind is more complicated). In particular, the codifferential $\partial^*$ is constructed by means of the Killing form of $\bar{\mathfrak{g}}$.

Again, the harmonic part of the curvature correspond to the components isomorphic to the $H^2(g_-, \bar{\mathfrak{g}})$ and they can be computed equivalently either by means of $\partial$ or $\partial^*$.

See [9] for detailed background on the parabolic geometries and the following theorem on canonical Cartan connections with structure group $\mathbb{P}$, where $\mathbb{P}$ is a suitable Lie group with the Lie algebra $\mathfrak{p}$.

**Theorem 2.** Consider a bracket generating distribution $D$ on $M$ with the constant symbol equal to the negative part of a graded semisimple Lie algebra $\bar{\mathfrak{g}}$ and the corresponding frame bundle $\mathcal{G}_0 \to M$ of $\text{gr} TM$. Then there is a unique Cartan connection $(\mathcal{G} \to M, \omega)$ of type $(\bar{\mathfrak{g}}, \mathbb{P})$ with the curvature function $\bar{\kappa} : \mathcal{G} \to \bar{\mathfrak{g}} \otimes \Lambda^2 \bar{\mathfrak{g}}^*_{-}$ satisfying $\partial^* \bar{\kappa} = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection $\bar{\kappa}_H$, i.e. the component with $\partial \bar{\kappa}_H = 0$. 
Thus, starting with a parabolic geometry equipped additionally with the metric on the generating distribution $D$, there are the two curvatures $\kappa_H$ and $\bar{\kappa}_H$ there. The goal of this paper is to find relations between their algebraic properties.

As we shall see, the link between these two curvatures is quite tight from this point of view. This is due to the fact that in the setup of the latter two theorems, the cohomologies can be computed with respect to the standard Lie algebra cohomology differential $\partial : C^p(g, W) \to C^{p+1}(g, W)$, where the $g$-module $W$ is either $g$ or $\bar{g}$. The same cohomologies can be equally computed by the means of the adjoint codifferential $\partial^*$, but here the codifferentials are much more different in general. Thus also the normalizations of the curvatures $\kappa$ and $\bar{\kappa}$ are quite different and we cannot expect simple explicit links between the general values of $\kappa_H$ and $\bar{\kappa}_H$, without having the metric particularly well adjusted.

At the same time, clearly equivalence of sub-Riemannian structures must imply the equivalence of the distributions.

3. Cohomologies related to sub-Riemannian structures

Consider a non-positively graded Lie algebra $g = g_- \oplus g_0$ such that the Tanaka prolongation of $g$ is trivial. Let $\bar{g}$ be a graded Lie algebra such that $\bar{g}_- = g_-$ and $g_0 \subset \bar{g}_0$.

One particular choice of $\bar{g}$ we should keep in mind is the full Tanaka prolongation of $g_-$ when it is finite dimensional. For a sub-Riemannian structure $(M, D, S)$ this means that the distribution $D$ defines a finite type filtered geometry. In the case when $\bar{g}$ is semisimple the geometry admits a canonical Cartan connection. This is exactly the case for 2-step free distributions which we consider in Section 5.

If the full Tanaka prolongation of $g_-$ is infinite dimensional then the choice of Lie algebra $\bar{g}$ is not so obvious. Sub-Riemannian contact structures $(M^{2n+1}, D, S)$ which we consider in Section 4 are particular examples. We shall see that a sub-Riemannian contact structure with a constant metric symbol defines naturally a Levi-definite CR-structure on the manifold $M^{2n+1}$. Such CR-structures admit a canonical Cartan connection with values in $\mathfrak{su}(n+1,1)$ and we can consider $\bar{g} = \mathfrak{su}(n+1,1)$.

As we have seen in the previous section, it is natural to expect some strong relation between $H^2_{>0}(g_-, g)$ and $H^2_{>0}(\bar{g}_-, \bar{g})$.

**Theorem 3.** Let $\bar{g} = \bar{g}_- \oplus \bar{g}_0 \oplus \bar{g}_+$ be a graded Lie algebra and $g = g_- \oplus g_0$ be a non-positively graded Lie algebra such that $\bar{g}_- = \bar{g}_-$ and $g_0 \subset \bar{g}_0$. Assume that the Tanaka prolongation of $g$ is trivial, i.e. $H^1_{>1}(g_-, g) = 0$. The cohomology $H^2_{>0}(\bar{g}_-, g)$ as a $g_0$-submodule is isomorphic to a direct sum of 2 parts:

1. $H^1_{>0}(g_-, \bar{g}/g)/H^1_{>0}(g_-, \bar{g})$,

2. $\ker \pi_2 \subset H^2_{>0}(g_-, \bar{g})$, 

where \( \pi_2: H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g}) \) is the natural projection induced by the projection in cochains \( \pi: C^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to C^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g}) \).

We refer the two components of the harmonic curvature above as the \( H^1 \)-part and the \( H^2 \)-part respectively.

**Proof.** Let \( W \) be a \( \mathfrak{g}_- \)-submodule of \( \mathfrak{g}_- \)-module \( V \). Equivalently we can consider the following short exact sequence:

\[
0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.
\]

It induces the short exact sequence of differential complexes

\[
0 \longrightarrow C^\bullet(\mathfrak{g}_-, W) \overset{i}{\longrightarrow} C^\bullet(\mathfrak{g}_-, V) \overset{\pi}{\longrightarrow} C^\bullet(\mathfrak{g}_-, V/W) \longrightarrow 0
\]

and the long exact sequence in cohomologies

\[
\xymatrix{
\longrightarrow H^n(\mathfrak{g}_-, W) \overset{i}{\longrightarrow} H^n(\mathfrak{g}_-, V) \overset{\pi}{\longrightarrow} H^n(\mathfrak{g}_-, V/W) \\
\delta \ar@{^{(}->}[u] \ar@{^{(}->}[d] \\
h \ar@{^{(}->}[u] \ar@{^{(}->}[d] \\
H^{n+1}(\mathfrak{g}_-, W) \overset{i}{\longrightarrow} H^{n+1}(\mathfrak{g}_-, V) \overset{\pi}{\longrightarrow} H^{n+1}(\mathfrak{g}_-, V/W) \longrightarrow 0
}
\]

We are going to apply the long exact sequence (2) to the pair \( \mathfrak{g} \subset \bar{\mathfrak{g}} \).

Let us notice that the gradings on \( \mathfrak{g} \) and \( \bar{\mathfrak{g}} \) induce the gradings on the corresponding spaces of chains, and since the differential \( \partial \) respects this grading, we get grading on the cohomology spaces, too. Moreover, we may consider the sequences (2) for the individual homogeneous separately.

There is the general algebraic fact that \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\ell \) is the Tanaka prolongation of \( \mathfrak{g}_- \oplus \mathfrak{g}_0 \) if and only if \( H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0 \). Thus, in our case the second and the third rows of the long exact sequence (2) are

\[
\xymatrix{
\longrightarrow H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0 \longrightarrow H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \overset{\pi_1}{\longrightarrow} H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g}) \\
\delta \ar@{^{(}->}[u] \\
H^2_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) \overset{i_2}{\longrightarrow} H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \overset{\pi_2}{\longrightarrow} H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g}) \longrightarrow 0
}
\]

It is not hard to compute the connecting homomorphism \( \delta \). Let \( \alpha \in C^n(\mathfrak{g}_-, \bar{\mathfrak{g}}) \) such that \( \pi(\alpha) \) is a representative of \( h \in H^n(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g}) \). This in particular means that \( \partial(\alpha) \equiv 0 \mod \mathfrak{g} \). Then \( \delta(h) \in H^{n+1}(\mathfrak{g}_-, \mathfrak{g}) \) is represented by the \( \partial \alpha \in C^{n+1}(\mathfrak{g}_-, \mathfrak{g}) \).

From the exact sequence (3) one can see that \( H^2_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) \) consists of two parts. The first part is \( H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}} / \mathfrak{g})/H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \), which is mapped by \( \delta \) injectively into \( H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \). The second part is \( \text{im}\, i_2: H^2_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) \to H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \). Exactness of the sequence implies \( \text{im}\, i_2 = \ker \pi_2 \) which consists of such elements from \( H^2_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \) whose cohomology class allows a representative in \( C^2_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) \). \( \square \)
Let us now explain how the claim of the latter theorem simplifies when dealing with the case of semisimple algebras $\mathfrak{g}$, i.e., in the case of sub-Riemannian parabolic geometries. We present two general comments first and then we conclude this section with computation of the low homogeneities in $H^1_{>0}(\mathfrak{g}^{-}, \mathfrak{g}/\mathfrak{g})$.

Remark 1. Let us start with the $H^1$-part of the curvature in the theorem 3. Consider a semisimple graded Lie algebra $\mathfrak{g} = \mathfrak{g}^{-} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. As we noticed in the proof above, the first cohomology $H^1_{\geq 0}(\mathfrak{g}^{-}, \mathfrak{g})$ is answering the question whether the spaces $\mathfrak{g}_r$ are those appearing in the full Tanaka prolongation of $\mathfrak{g}^{-}$.

In particular, there are just two exceptions where $H^1_{>0}(\mathfrak{g}^{-}, \mathfrak{g}) \neq 0$ (we use the standard notation for the classification of parabolic subalgebras in semisimple algebras via the choice of some of the positive simple roots $\Delta^+$, cf. [9]):

- the projective geometries which are of type $A_l$ with gradings corresponding to $\{\alpha_1\} \subset \Delta^+$;
- the projective contact geometries which are of type $C_l$, $l \geq 3$, with gradings corresponding to $\{\alpha_1\} \subset \Delta^+$.

Thus we see that for semisimple $\mathfrak{g}$ in almost all cases the $H^1$-part of $H^2_{>0}(\mathfrak{g}^{-}, \mathfrak{g})$ is equivalent to $H^1_{>0}(\mathfrak{g}^{-}, \mathfrak{g}/\mathfrak{g})$.

Next, $\mathfrak{g}_0$ equals to all derivations on the graded algebra $\mathfrak{g}^{-}$, and consequently $\mathfrak{g}$ is the full prolongation of $\mathfrak{g}^{-}$, if and only if all the zero homogeneity cohomology $H^0_0(\mathfrak{g}^{-}, \mathfrak{g})$ vanishes. In all such cases, the distribution $\mathcal{D}$ itself completely determines the parabolic geometry in question and, thus, the canonical Cartan connection, too.

If $H^0_0(\mathfrak{g}^{-}, \mathfrak{g}) \neq 0$, then we need further reduction of the algebra of all derivations to $\mathfrak{g}_0$ in order to get $\mathfrak{g}$ as the Tanaka prolongation of $\mathfrak{g}^{-} \oplus \mathfrak{g}_0$. This is the case for the following geometries only:

- the length of the grading is $k = 1$;
- in all the contact cases (we remind that every complex simple Lie algebra admit a unique contact grading);
- $\mathfrak{g}$ is of type $A_l$ with $l \geq 3$ and the grading corresponds to $\{\alpha_1, \alpha_i\} \subset \Delta^+$, $2 \leq i \leq l$, where $\Delta^+$ is the the set of simple roots;
- $\mathfrak{g}$ is of type $C_l$ with $l \geq 2$ and the grading corresponds to $\{\alpha_1, \alpha_l\} \subset \Delta^+$,

see [9, Proposition 4.3.1, p. 426] or [23, Proposition 5.1, p. 473]).

Remark 2. We come to the $H^2$-part of the curvature in the theorem 3. The projection $\pi_2$ is zero whenever the cochains representing the cohomology are valued in $\mathfrak{g}$. In such a case, the $H^2$-part coincides with the entire cohomology $H^2(\mathfrak{g}^{-}, \mathfrak{g})$.

Actually, the structure of $H^2(\mathfrak{g}^{-}, \mathfrak{g})$ for semisimple graded $\mathfrak{g}$ is quite well known and positive homogeneities in the curvature are rather exceptional. A full list of them can be found in [23, Proposition 5.5, p. 477]. Moreover,
Proof. Applying the long exact sequence (2) to the pair \( \mathfrak{g}^0 / \mathfrak{g} \subset g / g \) we obtain:

\[
\cdots \longrightarrow H^1_1(\mathfrak{g}, \mathfrak{g}^0 / \mathfrak{g}) \longrightarrow H^1_1(\mathfrak{g}, \mathfrak{g} / \mathfrak{g}) \longrightarrow H^1_1(\mathfrak{g}, \mathfrak{g}^0 / \mathfrak{g}) = 0.
\]

In the formula above \( H^1_1(\mathfrak{g}, \mathfrak{g}^0 / \mathfrak{g}) = \text{Hom}(\mathfrak{g}^{-1}, \mathfrak{g}^0 / \mathfrak{g}) \) and \( \delta : \mathfrak{g}^1 / \mathfrak{g}^0 \rightarrow H^1_1(\mathfrak{g}, \mathfrak{g}^0 / \mathfrak{g}) \) is induced by \( \partial \) in \( C(\mathfrak{g}, \mathfrak{g} / \mathfrak{g}) \). Since \( \partial(\mathfrak{g}^0) = 0 \) and \( \partial(\mathfrak{g}_1) \subset \text{Hom}(\mathfrak{g}^{-1}, \mathfrak{g}^0 / \mathfrak{g}) \) in \( C(\mathfrak{g}, \mathfrak{g} / \mathfrak{g}) \) we get that

\[
\delta(\mathfrak{g}^1 / \mathfrak{g}^0) = \partial(\mathfrak{g}_1 + \mathfrak{g}^0) = \partial(\mathfrak{g}_1 + \mathfrak{g})
\]

and the formula (4) holds.
Assume now that \( \mathfrak{g} \) is semisimple with the grading element \( E \) and \( \mathfrak{g}_0 \subset \mathfrak{g}_0^{ss} \). For any \( A \in \mathfrak{g}_0^{ss} \) we have

\[
\text{tr} \left( (\text{ad} E \text{ ad} A)|_{\mathfrak{g}_0} \right) = i \cdot \text{tr} \left( \text{ad} A|_{\mathfrak{g}_0} \right) = 0.
\]

If we define by \( W \) the orthogonal complement to \( E \) in \( \mathfrak{g}_0 \) with respect to the Killing form then it follows that \( \mathfrak{g}_0^{ss} \subset W \). Since \( \mathfrak{g} \subset \mathfrak{g}_0^{ss} \subset W \) the space \( \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0/\mathfrak{g}) \) can be decomposed as

\[
\text{Hom}(\mathfrak{g}_{-1}, \mathbb{R}E + \mathfrak{g}) + \text{Hom}(\mathfrak{g}_{-1}, W/\mathfrak{g})
\]

Let \( B \) be the Killing form of \( \mathfrak{g} \). For every \( v \in \mathfrak{g}_1 \), \( w \in \mathfrak{g}_{-1} \) we have

\[
B([v, w], E) = B([v, w], E) = B(v, w),
\]

since \( [v, w] = -[E, w] = w \). Therefore \( \partial(v)(w) = B(v, w)E \mod W \).

We see that the projection of \( \partial(\mathfrak{g}_1 + \mathfrak{g}) \) onto \( \text{Hom}(\mathfrak{g}_{-1}, \mathbb{R}E + \mathfrak{g}) \) is bijective. Therefore

\[
H^1_1(\mathfrak{g}_{-1}, \mathfrak{g}) = \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0/\mathfrak{g})/\text{Hom}(\mathfrak{g}_{-1}, \mathbb{R}E + \mathfrak{g}) = \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0/(\mathfrak{g}_0 \oplus \mathbb{R}E))
\]

\[
\square
\]

**Theorem 5.** Let \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \) be a graded Lie algebra. Let \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \) be a non-positively graded Lie algebra with \( \mathfrak{g}_- = \mathfrak{g}_- \) and \( \mathfrak{g}_0 \subset \mathfrak{g}_0 \). Then

1. For \( i > 0 \) we have
   \[
   H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}_0) = \ker \delta : H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}_0/\mathfrak{g}^{i-1}) \to H^2_{i+1}(\mathfrak{g}_-, \mathfrak{g}_0/\mathfrak{g}^{i-1})
   \]
   where \( \delta \) is induced by \( \partial \) in \( C^\bullet_{i+1}(\mathfrak{g}_-, \mathfrak{g}_0/\mathfrak{g}) \) and
   \[
   H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}_0/\mathfrak{g}^{i-1}) = \text{Hom}(\mathfrak{g}_-, \mathfrak{g}_0^i/\mathfrak{g}^{i-1})/\partial(\mathfrak{g}_i + \mathfrak{g}^{i-1}).
   \]

2. If \( k \) is the highest homogeneity in \( \mathfrak{g} \) then \( H^1_{k+1}(\mathfrak{g}_-, \mathfrak{g}_0) = 0 \).

3. If \( \mathfrak{g} \) is a semisimple graded Lie algebra with the grading element \( E \) and \( \mathfrak{g}_0 \subset \mathfrak{g}_0^{ss} \), where \( \mathfrak{g}_0^{ss} \) is a semisimple part of \( \mathfrak{g}_0 \) then in homogeneity \( 2 \) we have
   \[
   H^1_2(\mathfrak{g}_-, \mathfrak{g}_0) \subset \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1)
   \]
   where \( \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \subset \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \) denotes the subspace of symmetric tensors with respect to the Killing form of \( \mathfrak{g} \).

Moreover if \( \mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathbb{R}E \) or \( \partial(\mathfrak{g}_2) = \wedge^2 \mathfrak{g}_1 \) then

\[
H^1_2(\mathfrak{g}_-, \mathfrak{g}_0) = \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1).
\]

**Proof.** Applying the long exact sequence (2) to \( \mathfrak{g}^{i-1}/\mathfrak{g} \subset \mathfrak{g} \) we obtain:

\[
H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}^{i-1}/\mathfrak{g}) \xrightarrow{\delta} H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) \xrightarrow{\delta} H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^{i-1})
\]

(5)

We show first that \( H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}^{i-1}/\mathfrak{g}) = 0 \). Let \( \alpha \in C^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}^{i-1}/\mathfrak{g}) \). Such \( \alpha \) has a form \( \sum_{\tau} w^{\tau}_{\mathfrak{g}} \cdot v^{\tau}_{\mathfrak{g}^{i-1}} \) where \( w^{\tau}_{\mathfrak{g}} \in \mathfrak{g}_- \) and \( v^{\tau}_{\mathfrak{g}^{i-1}} \in \mathfrak{g}_{i+1-t} \). Denote
by \( t_0 \) the minimal \( t \) such that \( u^j_{i+1-t} \neq 0 \) for some \( j \). Due to degree \( t_0 \geq 2 \) and therefore
\[
\partial(\alpha) = \sum_j \partial(\omega_j^{t_0}) \otimes u^j_{i+1-t_0} \mod (\mathfrak{g}^{i-t_0}/\mathfrak{g})
\]
can not be equal to zero since \( \partial(\omega_j^{t_0}) \neq 0 \).

The computation of \( H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^{i-1}) \) is straightforward. Due to degree the only elements in \( C^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^{i-1}) \) are \( \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}^i/\mathfrak{g}^{i-1}) \). All of them are obviously closed. By factoring out \( \partial(\mathfrak{g}_{i+1} + \mathfrak{g}^{i-1}) \) we obtain that
\[
H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^{i-1}) = \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}^i/\mathfrak{g}^{i-1})/\partial(\mathfrak{g}_{i+1} + \mathfrak{g}^{i-1}).
\]

To finish the proof of (1) it remains to apply exact sequence (5) keeping in mind that \( H^1_{i+1}(\mathfrak{g}_-, \mathfrak{g}^{i-1}/\mathfrak{g}) = 0 \). The second statement of the theorem immediately follows from the first one.

We proceed with the proof of (3). According to (1) the space \( H^1_{2}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) \) is equal to the kernel of
\[
\delta: H^1_{2}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^0) \to H^2_{2}(\mathfrak{g}_-, \mathfrak{g}^0/\mathfrak{g}) = H^2_{2}(\mathfrak{g}_- \otimes (\mathfrak{g}^0/\mathfrak{g})
\]
where
\[
H^1_{2}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}^0) = \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}^1/\mathfrak{g}^0)/\partial(\mathfrak{g}^2 + \mathfrak{g}^0).
\]

Let \( W \) be the orthogonal complement to \( E \) in \( \mathfrak{g}_0 \) with respect to the Killing form \( B \). As in the proof of Theorem 4 we have the canonical splitting
\[
C^2_{2}(\mathfrak{g}_-, \mathfrak{g}^0/\mathfrak{g}) = \text{Hom}(\wedge^2 \mathfrak{g}_{-1}, \mathbb{R}E + \mathfrak{g}) + \text{Hom}(\wedge^2 \mathfrak{g}_{-1}, \mathfrak{W}/\mathfrak{g}).
\]

which induces the canonical splitting in cohomologies
\[
H^2_{2}(\mathfrak{g}_-, \mathfrak{g}^0/\mathfrak{g}) = (\wedge^2 \mathfrak{g}_{-1}/\partial(\mathfrak{g}^*_{-2}) \otimes (\mathbb{R}E + \mathfrak{g}) + (\wedge^2 \mathfrak{g}^*_{-1}/\partial(\mathfrak{g}^*_{-2}) \otimes (\mathfrak{W}/\mathfrak{g}).
\]

Consider the map
\[
\pi \circ \partial: \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}^1/\mathfrak{g}) \to \text{Hom}(\wedge^2 \mathfrak{g}_{-1}, \mathbb{R}E + \mathfrak{g}),
\]
where \( \pi \) is defined by the splitting (6). The kernel of \( \pi \circ \partial \) is exactly \( \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \). Moreover the image of the projection of \( \partial(\mathfrak{g}_2) \) onto \( \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \) is mapped invectively to \( \partial(\mathfrak{g}^*_{-2}) \otimes E \). Therefore the induced map
\[
\pi \circ \delta: \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}^1/\mathfrak{g}^0)/\partial(\mathfrak{g}^2 + \mathfrak{g}^0) \to (\wedge^2 \mathfrak{g}^*_{-1}/\partial(\mathfrak{g}^*_{-2}) \otimes (\mathbb{R}E + \mathfrak{g})
\]

has \( \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \) as its kernel. As a result \( H^1_{2}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) \subseteq \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1) \). Finally, if \( \mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathbb{R}E \) or \( \partial(\mathfrak{g}^*_{-2}) = \wedge^2 \mathfrak{g}^*_{-1} \) the second component in (7) vanishes. This ensures that in these particular cases
\[
H^1_{2}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) = \ker \pi \circ \delta = \text{Sym}(\mathfrak{g}_{-1}, \mathfrak{g}_1).
\]

\[\square\]
4. Contact sub-Riemannian structures

Let \((M, H, g)\) be a contact sub-Riemannian manifold of dimension \(2n + 1\). An arbitrary contact form \(\theta\) defines a non-degenerate symplectic form \(\omega = d\theta|_H\) on \(H\). Both \(\theta\) and \(\omega\) are defined up to a conformal factor.

The symbol Lie algebra \(\mathfrak{g}_-\) of the contact structure \(H\) (that is the graded tangent space
\[
\text{gr}(T_xM) = H_x \oplus (T_xM/H_x)
\]
with the induced Lie bracket) is isomorphic to the Heisenberg Lie algebra
\[
\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathbb{R}e_0 + \mathbb{R}^{2n}
\]
with the non-trivial Lie bracket \([u, v] = \omega(u, v)e_0\), \(u, v \in \mathfrak{g}_{-1}\). Metric symbols of sub-Riemannian contact structures (i.e. the symbol algebra \(\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}\) together with a metric \(g\) on \(\mathfrak{g}_{-1}\)) is parametrized by skew-symmetric non-degenerate endomorphisms \(I_{\mathfrak{g}}\) defined up to rescaling. With respect to the \(g\)-orthogonal Fitting decomposition \(\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}_{-1}^{\lambda_k}\), the endomorphism can be written as
\[
I_{\mathfrak{g}} = \lambda_1 J_{\mathfrak{g}_{-1}}^{\lambda_1} \oplus \cdots \oplus \lambda_k J_{\mathfrak{g}_{-1}}^{\lambda_k}
\]
where \(J_j\) is a complex structure in the (even-dimensional) vector space \(\mathfrak{g}_{-1}^{\lambda_j}\) and \(\lambda_1 < \lambda_2 < \cdots < \lambda_k\). We may rescale \(I_{\mathfrak{g}}\) such that \(\lambda_k = 1\). Such \(I_{\mathfrak{g}}\) is canonically associated with the sub-Riemannian structure \((H, g)\) and define the complex structure \(J = J_{\mathfrak{g}_{-1}}^{\lambda_1} \oplus \cdots \oplus J_{\mathfrak{g}_{-1}}^{\lambda_k}\) on the contact distribution \(H\), i.e. an almost CR structure.

The algebra of infinitesimal automorphisms of the metric symbol algebra \((\mathfrak{g}_-, g)\) is the direct sum of the unitary algebras
\[
\text{aut}(\mathfrak{g}_-, g) = \text{u}(\mathfrak{g}_{-1}^{\lambda_1}) \oplus \cdots \oplus \text{u}(\mathfrak{g}_{-1}^{\lambda_k}) = \text{u}(n_1) \oplus \cdots \oplus \text{u}(n_k)
\]
where \(\dim \mathfrak{g}_{-1}^{\lambda_j} = 2n_j\).

Since we assume that the metric symbol is constant, the sub-Riemannian structure induces a \(g\)-orthogonal decomposition
\[
H = V_1 \oplus \cdots \oplus V_k
\]
of the contact distribution into a sum of \(I_{\mathfrak{g}} = g^{-1} \circ \omega\)-invariant distributions s.t. \(I_{\mathfrak{g}}|_{V_j} = \lambda_j J_{V_j}\) and determines a complex structure \(J = J_{V_1} \oplus \cdots \oplus J_{V_k}\) on the contact distribution \(H\), i.e. an almost CR structure.

The symbol algebra of almost CR contact structure \((H, J)\) is well known:
\[
\mathfrak{g}_- + \bar{\mathfrak{g}}_0 = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + (u(n) + \mathbb{R}E),
\]
where \(E\) is the grading element. It is well known that the full prolongation of this Lie algebra is
\[
\bar{\mathfrak{g}} = \text{su}(n+1, 1) = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1 + \bar{\mathfrak{g}}_2.
\]
In matrix notations
\[
\bar{\mathfrak{g}} = \text{su}(n+1, 1) = \begin{pmatrix}
\lambda + i\mu & -y^* & i\beta \\
x & A - \frac{2}{n}i\mu I_n & y \\
i\alpha & -x^* & -\lambda + i\mu
\end{pmatrix},
\]
where \(\lambda, \alpha, \beta, x, y\) are real numbers.
where $A \in \mathfrak{su}(n)$, $x, y \in V = \mathbb{R}^{2n} = \mathbb{C}^n$, $\alpha, \beta, \lambda, \mu \in \mathbb{R}$. The entry $i\alpha$ has the homogeneity $-2$, $x$ has the homogeneity $-1$, $\lambda, \mu$ and $A$ have the homogeneity zero, $y$ has the homogeneity one, and $i\beta$ has the homogeneity two.

Now we describe the cohomology $H^2(\mathfrak{g}, \bar{\mathfrak{g}})$ and relate it with the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ associated with contact sub-Riemannian structure with maximally symmetric metric symbol $\mathfrak{g}$ that is in the case when $I_\mathfrak{g} = J$ and $\mathfrak{g} = \mathfrak{g}_- + u(n)$.

The harmonic curvature $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$ has two components, of homogeneity 1 and 2. If $n > 1$ then the complexification of homogeneity one component gives two conjugate complex representations. They correspond to bilinear maps $\Lambda^2 V^{1,0} \to V^{0,1}$ and $\Lambda^2 V^{0,1} \to V^{1,0}$. This torsion is the obstruction to the integrability of the $CR$-structure and it is proportional to the Nijenhuis tensor. The other cohomology component takes values in $\Lambda^{1,1} V \otimes \mathfrak{su}(n)$. We see that

$$\ker \pi_2 : H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to H^2_{>0}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) = H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}).$$

In the following theorem we shall associate $\mathfrak{g}^*_{-1}$ with $\mathfrak{g}_{-1}$ with the help of the canonical hermitian form. Then $u(n)$ is generated by $u \wedge J \, v = u \wedge v + J u \wedge J v$, where $u, v \in \mathfrak{g}_{-1}$.

Let $e_i$ be an orthonormal basis of $\mathfrak{g}_{-1}$ with respect to the canonical hermitian form. Then $\mathfrak{g} - [e_i, Je_i] \in \mathfrak{g}_{-2}$ does not depend on the choice of $i$.

**Theorem 6.** The cohomology of sub-Riemannian contact structure with maximally symmetric symbol is a direct sum of $H^2_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$ and the component generated by the homogeneity 2 symmetric tensors

$$(9) \quad \alpha_{pq} = \sum \alpha_t (e_p \wedge_J e_t \otimes e^*_t \wedge e^*_q + e_q \wedge_J e_t \otimes e^*_t \wedge e^*_p) + (J e_p \otimes e^*_q + J e_q \otimes e^*_p) \wedge z^*.$$

**Proof.** Apply Theorems 4 and 5. The formula (9) is obtained by applying differential to the space of tensors from $\text{Hom}(\mathfrak{g}_{-1}, \bar{\mathfrak{g}})$ symmetric with respect to the Killing form. It remains to show that $H^1_{>0}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = 0$.

Let $y$ be a generator of $\mathfrak{g}_2$. Consider an arbitrary element $\beta \in C^1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$:

$$\beta = A^i z^* \otimes e_i + B_j e^*_j \otimes y.$$ 

For arbitrary $e_k, e_i \in \mathfrak{g}_{-1}$ such that $[e_k, e_j] = 0$ we have

$$\partial \beta(e_k, e_j) = B_j[e_k, y].$$

As a result $B_j = 0$ for arbitrary $j$. Finally

$$\partial \beta = A^i \partial(z^*) \otimes e_i \mod \mathfrak{g}^0/\mathfrak{g}$$

implies that $A^i = 0$ for arbitrary $i$ as well. \qed
Let us observe, that the latter symmetric tensors $\alpha_{(pq)}$ are cochains generating the cohomology space. The generators in the common kernel of $\partial$ and $\partial^*$ may look more messy in general.

5. Free 2-step sub-Riemannian structures

5.1. Generalities. Let $M$ be a manifold of dimension $\frac{m(m+1)}{2}$ with $m \geq 3$. We say that distribution $D$ of dimension $m$ is a free distribution on $M$ if $D+[D,D]=TM$. The general parabolic geometry theory implies that there exists a natural regular normal Cartan connection of type $(\bar{G},\bar{P})$ where $\bar{G}$ is a connected component of $SO(m+1,m)$ and $\bar{P}$ is a stabilizer of an isotropic $m$-dimensional subspace in $\mathbb{R}^{2m+1}$, cf [9, Section 4.3.2]. The Lie algebras of $\bar{G}$ and $\bar{P}$ are

$$\bar{g} = \left\{ \begin{pmatrix} A & X & Y \\ Z & 0 & -X^t \\ T & -Z^t & -A^t \end{pmatrix} \right\}, \quad \bar{p} = \left\{ \begin{pmatrix} A & 0 & 0 \\ Z & 0 & 0 \\ T & -Z^t & -A^t \end{pmatrix} \right\},$$

where $A,Y,T \in \text{Mat}(m,\mathbb{R})$, $X,Z \in \mathbb{R}^n$, $Y+Y^t = T+T^t = 0$. We introduce the following basis in $\bar{g}$

$$e^{[ij]} = T^i_j - T^j_i, \quad e^j = Z^j, \quad e^j_i = A^j_i, \quad e_j = X_j, \quad e_{[ij]} = Y^i_j - Y^j_i.$$

The metric $g$ defines a reduction of $\bar{P}$-principal bundle $\bar{G}$ to $G_0 = SO(m,\mathbb{R})$-principal bundle $\mathcal{G}$ of orthogonal frames. The sub-Riemannian structure on top of the distribution $D$ could be given in terms of orthonormal frame $X_1,\ldots,X_m$ on $D$. Let’s define $X_{[ij]} = [X_i,X_j]$. Due to the fact that $D$ is a free distribution the graded symbol of $\{X_i,X_{[jk]}\}$ is given by $e_i$, $e_{[jk]}$ with the same relations as in $\bar{g}$. The Lie algebra $\mathfrak{g}_0$ is $\mathfrak{so}(m,\mathbb{R})$ generated by $s^i_j = e^i_j - e^j_i$. The infinitesimal model of the corresponding sub-Riemannian structure is given by

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \langle e_{[ij]} \rangle \oplus \langle e_k \rangle \oplus \langle s^i_j \rangle.$$

The structure of the cohomology group $H^2(\mathfrak{g}_-,\bar{g})$ depends on $m$. For $n>3$ the cohomology is the highest weight component in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$. This means that the complete obstruction to local flatness of a free step-2 distribution for $n>3$ is given by a torsion. On the contrary the obstruction to local flatness for $n=3$ is given by a curvature. The cohomology group $H^2(\mathfrak{g}_-,\bar{g})$ in this case is the highest weight component in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0$.

The following 2 theorems give explicit description of the harmonic curvature $H^2_{\partial\bar{\partial}}(\mathfrak{g}_-,\bar{g})$. Again, notice that the theorems provide generators at the level of cochains generating the cohomology space. The generators in the common kernel of $\partial$ and $\partial^*$ would look much more messy.
Theorem 7. The $H^2$-part of $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ coincides with $H^2(\mathfrak{g}_-, \mathfrak{g})$ for $n > 3$. If $n = 3$ then $H^2$-part of $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ is a 1-dimensional subspace of 27-dimensional $H^2(\mathfrak{g}_-, \mathfrak{g})$. It is generated by

$$\sum_{(i,j,k) \in \mathfrak{g}_+} (-1)^{sgn((i,j,k))} s^i_j \otimes e^*_j \wedge e^*_{[jk]} + \sum_{\{i,j,k\}} e_t \otimes e^*_t \wedge e^*_{[ik]}$$

Proof. The first statement of the theorem is obvious since $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ takes values in $\mathfrak{g}_-$ for $n > 3$. For $n = 3$ note that according to the next theorem $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g}) = 0$ which implies that $\partial : C^3(\mathfrak{g}_-, \mathfrak{g}) \to C^2(\mathfrak{g}_-, \mathfrak{g})$ never takes values in $C^2(\mathfrak{g}_-, \mathfrak{g})$. Now the direct check shows that the only elements in the highest weight component of $\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ which take values in $\mathfrak{g}_0$ are proportional to (10). 

\[\square\]

Theorem 8. The $H^1$-part of $H^2_{>0}(\mathfrak{g}_-, \mathfrak{g})$ consists of 2 subspaces:

- homogeneity 1 subspace is generated by tensors

$$\alpha_{(ij)}^k = \left( e_j \otimes e_i^* + e_i \otimes e_j^* + \sum_t \left( e_{[it]} \otimes e_{[it]}^* + e_{[it]} \otimes e_{[it]}^* \right) \right) \wedge e_k^*$$

symmetric and traceless in $(i, j)$;

- homogeneity 2 subspace is generated by tensors

$$\alpha_{(pq)} = \sum_t e_t \otimes (e^*_t \wedge e^*_q + e^*_t \wedge e^*_p) + \sum_{t,r} e_{[tr]} \otimes e_{[tr]}^* \wedge e_{[qr]}^*$$

symmetric in $(p, q)$.

Proof. In homogeneity 4 we apply Theorem 5 to obtain that $H^1_{>0}(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) = 0$. An arbitrary element of homogeneity 3 has the form

$$\beta = A^{|r|}_p e^p \otimes e^*_{[r]} + B^p_{[r]} e^*_{[r]} \otimes e_p^*.$$

To compute the differential of $\beta$ we need the following commutation relations:

$$[e_i, e^*_j] = e^*_i, \quad [e^*_{[ij]}, e_{[k]i}] = \begin{cases} e^*_j, & i \neq j \\ e^*_i + e^*_j, & i = j \end{cases}$$

Using the relations above we obtain

$$\partial \beta(e_{[ij]} \wedge e_k) = A^{|ij|}_p [e^p, e_k] + B^k_{[ij]} e^*_{[ij]}, e_{[r]}^p$$

$$= A^{|ij|}_p e^p_k + \sum_{i < t} B^k_{[it]} e_{[ij]}, e_{[it]}^p + \sum_{j < t} B^k_{[jt]} e_{[ij]}, e_{[jt]}^p$$

$$+ \sum_{r < i} B^k_{[r]} e_{[ij]}, e_{[r]}^p + \sum_{r < j} B^k_{[r]} e_{[ij]}, e_{[r]}^p$$

$$= A^{|ij|}_p e^p_k + \sum_{i} B^k_{[it]} e_{[ij]}, e_{[it]}^p + \sum_{t} B^k_{[ij]} e_{[ij]}, e_{[ij]}^p$$

$$= A^{|ij|}_p e^p_k - B^k_{[ij]} (e^*_j + \delta^*_i e^*_j) - B^k_{[ij]} (e^*_i + \delta^*_j e^*_i).$$
The coefficient in front of $e_i^j$ is equal to

$$\delta^k_i A_i^{[ij]} - 2B_{[ij]}^k$$

and should be equal to 0 since we want $\partial \beta(e_{[ij]} \wedge e_k) \in \mathfrak{so}_n(\mathbb{R})$. Therefore if $k \neq i$ then $B_{[ij]}^k = 0$ which implies that all $B_{[ij]}^k$ are equal to 0. This yields

$$\partial \beta(e_{[ij]} \wedge e_k) = e_k^r A_r^{[ij]}$$

and therefore $A_r^{[ij]} = 0$ follows immediately from the requirement $\partial \beta(e_{[ij]} \wedge e_k) \in \mathfrak{so}_n(\mathbb{R})$. We conclude that $H^1_3(g_-, \tilde{g}/g) = 0$.

Using Theorem 4 we obtain that $H^1_1(g_-, \tilde{g}/g)$ is a traceless in $(i, j)$ part of the space generated by elements

$$\beta_{(ij)} = (e_j^i + e_i^j) \otimes e_k^*,$$

and that elements from $H^1_2(g_-, \tilde{g}/g)$ correspond to symmetric tensors from $\text{Hom}(g_{-1}, \tilde{g}_1)$. The explicit formula for generators is:

$$\beta_{(pq)} = \frac{1}{2} \left( e^q \otimes e^*_p + e^p \otimes e^*_q - \sum_i \left( e_i^q \otimes e^*_p|_i + e_i^p \otimes e^*_q|_i \right) \right).$$

In order to obtain $H^1$-part of $H^2(g_-, g)$ it remains to take differential of representatives in $H^1(g_-, \tilde{g}/g)$. We have

$$\alpha_{(ij)} = \partial \beta_{(ij)} = \left( e_j^i \otimes e^*_i + e_i^j \otimes e^*_j + \sum_t \left( e_{[jt]} \otimes e^*_t|_{[ij]} + e_{[it]} \otimes e^*_t|_{[ij]} \right) \right) \wedge e^*_k,$$

$$\alpha_{(pq)} = \partial \beta_{(pq)} = \sum_t e_t \otimes (e^*_t|_{[pq]} \wedge e^*_q + e^*_t|_{[pq]} \wedge e^*_p) + \sum_{t,r} e_{[tr]} \otimes e^*_t|_{[pr]} \wedge e^*_r.$$  

\[\square\]

5.2. The Cartan connections. The latter theorem provides helpful information how to proceed when building distinguished connections via the classical exterior calculus. Let us illustrate this by following the computations along the lines of [12, Section 4].

Suppose $n > 3$. We start by fixing any local orthonormal frame $X_1, \ldots, X_n$ of $D$ and complete it to a frame of $TM$ by adding the vector fields $X_{[ij]} = [X_i, X_j]$. Let $\{\theta^i, \theta^{[jk]}\}$ be the corresponding coframe. Clearly, the annihilator $D^\perp \subset T^*M$ is generated by the forms $\theta^{[jk]}$, and by the very construction,

$$d\theta^{[jk]} = -\theta^j \wedge \theta^k \mod D^\perp.$$  

The canonical Cartan connection $\omega$ will consist of components $\omega^i$ and $\omega^{[jk]}$ (building the soldering form on $M$), and the connection forms $\omega^i$ (providing a matrix of forms, valued in $\mathfrak{so}(g_{-1})$). While the components $\omega^{[jk]} = \theta^{[jk]}$ can stay fixed, the other ones have to be adjusted according to our normalization.
Now, assume we have got the canonical connection \( \omega \) and deal with the data of homogeneity one. In the chosen frame, we may write
\[
\omega^i = \theta^i + C^i_{[jk]}\omega^{[jk]},
\]
\[
\omega^i_j = A^i_{kj}\omega^k \mod D^\perp.
\]
We want to find the right coefficients \( C^i_{[jk]} \) which provide the complement \( D' \) to \( D \) in \( TM \), uniquely determined by the co-closed curvature normalization. At the same time the coefficients \( A^i_{jk} \) are the Christoffel coefficients of the restriction of the wanted metric connection to \( D \). Actually, the latter Christoffel symbols uniquely extend to a metric connection on the entire \( TM \) and we know that there is exactly one such metric connection for each choice of the torsion. Thus we have to focus on the conditions on \( C^i_{[jk]} \) and \( A^i_{jk} \), leading to the proper normalization of the torsion.

We know that the splitting \( TM = D \oplus D' \) has to be respected by the connection and this itself fixes the symmetric parts of the torsion components \( D \times D' \mapsto D \) and \( D' \times D \mapsto D' \). Further we know that the \( H^2 \)-part of the torsion is the completely tracefree component in \( D'^* \wedge D^* \otimes D' \).

Since our algebra \( \mathfrak{g}_0 \) provides much smaller freedom in normalizations than the entire \( \mathfrak{g}_{\geq 0} \), we cannot require to kill all other components of the homogeneity one torsion. But the description of the generators from the Theorem 8 implies that the homogeneity one torsion component \( D'^* \wedge D^* \otimes D' \) can be uniquely normalized to be symmetric (with respect to the metric on \( D' \)) and including only the one trace component aside the completely tracefree part. This pins down the choice of \( D' \), i.e. the coefficients \( C^i_{[jk]} \).

The latter requirements, including the vanishing of the antisymmetric part, also define the derivative on \( D' \) in the \( D \) directions, which is a homogeneity one data, too.

To conclude the homogeneity one part of the torsion, we have to fix its \( D'^* \wedge D^* \otimes D \) component. Although Theorem 8 suggests there should be some non-trivial link with the trace part of the torsion \( D'^* \wedge D^* \otimes D' \), we may also choose this component to vanish (although the curvature of the resulting Cartan connection will not be co-closed any more finally). Any such choice fixes the derivative on \( D \) in the \( D \) directions by the Koszul formula.

In the homogeneity one step of our construction, we have used only part of the freedom in our normalization of the forms \( \omega^i_j \). Now we have to write
\[
\omega^i_j = A^i_{kj}\omega^k + E^i_{j[kl]}\omega^{[kl]},
\]
with already known coefficients \( A^i_{kj} \), but with \( E^i_{j[kl]} \) to be still determined by exploiting the homogeneity two data.

We may proceed the same way as in homogeneity one with \( D \) and \( D' \) swapped. First, the remaining freedom is used to ensure that torsion component \( D \times D^\perp \mapsto D \) will be symmetric with respect to the metric on \( D \) (remember the symmetric part is fixed by the splitting, but we can kill the
anti-symmetric part). This component of the torsion already contributes to the expected curvature components, cf. the homogeneity 2 cohomology generator in the $H^1$-part of the harmonic curvature in the above theorem. Finally, the torsion of the restriction of the connection to $D'$ would be strictly related to the latter symmetric component, but we may also choose this to vanish. Any choice of this torsion component again defines the remaining part of the connection by the Koszul formula. This complete procedure exactly uses all our freedom.

Let us observe that although we could explicitly construct the normalized Cartan connection with the co-closed curvature, the simplified choices might be even more useful in practice.

6. Constant curvature spaces

Constant curvature Cartan geometries are generalizations of the classical Riemannian constant sectional curvature spaces. They could be regarded as the most simple examples of Cartan geometries of a particular type.

**Definition 1.** Let $(G \to M, \omega)$ be a Cartan connection of a sub-Riemannian structure on $M$. We say that $\omega$ has a constant curvature if the curvature function $\kappa : \mathcal{G} \to \mathfrak{g} \otimes \Lambda^2 \mathfrak{g}^*$ is constant.

It follows from the definition immediately that if $\omega$ has a constant curvature then $\kappa$ takes values in $G_0$-invariant part of $\mathfrak{g} \otimes \Lambda^2 \mathfrak{g}^*$. In particular, a Cartan connection of a sub-Riemannian structure has constant curvature only if harmonic curvature takes values in $G_0$-invariant part of $H^2(\mathfrak{g}_-, \mathfrak{g})$. Moreover, harmonic curvature completely defines the whole curvature function. Therefore to every element in $\text{Inv}_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})$ corresponds at most one constant curvature space. In other words, there exists an injective map:

$$\{\text{classes of locally equivalent constant curvature spaces}\} \to \text{Inv}_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g}).$$

This simple fact motivates our classification strategy for constant curvature sub-Riemannian manifolds: first we compute $G_0$-invariant part of $H^2(\mathfrak{g}_-, \mathfrak{g})$ and then provide models corresponding to elements of $H^2(\mathfrak{g}_-, \mathfrak{g})$.

In this section we obtain a full classification of constant curvature spaces for maximally symmetric contact sub-Riemannian structures and free 2-step sub-Riemannian structures. We start with the maximally symmetric contact case.

**Theorem 9.** Every non-flat $(2n+1)$-dimensional constant curvature space with the maximally symmetric contact symbol is locally equivalent to the sphere in $\mathbb{C}^{n+1}$ or pseudo-sphere in $\mathbb{C}^{1,n}$. The distribution in this case is a maximal complex subspace in the tangent space to the (pseudo)-sphere. Up to scale, the metric coincides with the one which is induced from the ambient space.

**Proof.** As it was shown in the discussion preceding Theorem 6 the space $H^2(\mathfrak{g}_-, \mathfrak{g})$ splits into two non-trivial irreducible representations of $\mathfrak{u}(n)$. 
Therefore elements which are invariant with respect to $g_0 = u(n)$ could exist only in $H^1$-part of $H^2(g_-, g)$. According to Theorem 5

$$H^1(g_-, g/\bar{g}) = S^2(g^*_{-1})$$

as $g_0$-module. In order to compute $u(n)$-invariant subspace in $S^2(g^*_{-1})$ we can restrict our attention to the $J$-invariant bilinear forms since $J \in u(n)$. This space can be identified with the space of hermitian operators on $C^n$. Any $u(n)$-invariant hermitian operator is proportional to identity. This implies that $\text{Inv}_{g_0} H^1(g_-, g/\bar{g})$ is 1-dimensional. We conclude that the family of constant curvature spaces with the maximally symmetric contact symbol is no more than 1-dimensional. Depending on the sign of the curvature function every non-zero element in $H^1$-part of $H^2(g_-, g)$ can be realized either as the sphere in $C^{n+1}$ or pseudo-sphere in $C^{1,n}$. □

In terms of Lie algebras the homogeneous infinitesimal model for the sphere in $C^{n+1}$ is a factor of

$$u(n+1) = \begin{pmatrix} iz & -u + iv \\ u + iv & U \end{pmatrix}, \quad z \in \mathbb{R}, \; v, u \in \mathbb{R}^n; U \in u(n)$$

by the subalgebra

$$u(n) = \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}$$

with the distribution $D$ given by

$$\left( \begin{array}{cc} 0 & -u + iv \\ u + iv & * \end{array} \right)$$

and sub-Riemannian norm $\|u + iv\|^2 = \|u\|^2 + \|v\|^2$. The description for pseudo-sphere in $C^{n,1}$ is analogous.

In order to attack the same problem for free 2-step distributions we need some well known results on representation theory of complex semisimple Lie algebras. We refer here to the appendix of [20] and use notations from there.

Consider a rank $l$ simple Lie algebra. Let $\pi_i$, $1 \leq i \leq l$ be the fundamental weights. For the convenience we assume that $\pi_0 = \pi_{l+1} = 0$. We denote by $R(\lambda)$ the irreducible representation with the highest weight $\lambda$. Then $R(\lambda_1)R(\lambda_2)$ defines a tensor product of representations $R(\lambda_1)$ and $R(\lambda_2)$, $S^k(\lambda)$ is a $k$-th symmetric power of $R(\lambda)$ and $\Lambda^k(\lambda)$ is a $k$-th wedged power of $R(\lambda)$. For Lie algebras of type $B_l$, $l \geq 2$ ($n = 2l + 1$) we use the notation

$$\Lambda^p(\pi_1) = \hat{\pi}_p = \begin{cases} \pi_p & 1 \leq p \leq l - 1, \\ 2\pi_l & p = l, l + 1, \\ \pi_{n-p} & l + 2 \leq p \leq 2l, \end{cases} \hat{\pi}_0 = \hat{\pi}_n = 0.$$
For Lie algebras of type $D_l$, $l \geq 3$ ($n = 2l$) we use the notation

$$\Lambda^p(\pi_1) = \pi_p = \begin{cases} 
\pi_p & 1 \leq p \leq l - 2, \\
\pi_{l-1} + \pi_l & p = l - 1, l + 1, \\
\pi_{n-p} & l + 2 \leq p \leq 2l,
\end{cases}$$

and $R(\pi_l + \lambda) = R(2\pi_{l-1} + \lambda) + R(2\pi_l + \lambda)$. Finally we write $R_{\mathfrak{sl}}(\lambda)$ and $R_{\mathfrak{so}}(\lambda)$ for modules over $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{so}(n, \mathbb{C})$ respectively.

**Theorem 10.** Every non-trivial sub-Riemannian constant curvature space modeled on free 2-step distribution of rank $n$, $n \geq 4$, is locally equivalent to $SO(n + 1)$ or $SO(1, n)$. In this case the left-invariant distribution on the corresponding Lie algebra is

$$\begin{pmatrix} 0 & \pm v \\
v & 0 \end{pmatrix}, \quad v \in \mathbb{R}^n$$

with the norm $||v||^2 = \sum_{i=1}^{n} v_i^2$.

For free 2-step distributions of rank 3 there exists a 2-dimensional family of constant curvature models.

**Proof.** We are going to check $\mathfrak{g}_0$-invariance of three different components of $H^2(\mathfrak{g}_-, \mathfrak{g})$. The computations for the generic case are different from cases $n = 3$, $n = 4$. We consider the generic case first.

1. The complexification of $H^2$-part in $H^2(\mathfrak{g}_-, \mathfrak{g})$ is the highest weight $\mathfrak{sl}(n, \mathbb{C})$-module in Hom$(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g}_-)$. Its weight as $\mathfrak{sl}(n, \mathbb{C})$-module is $\pi_{n-1} + \pi_{n-2} + \pi_2$. We need to compute its decomposition as $\mathfrak{so}(n, \mathbb{R})$ module. First, we have

$$R_{\mathfrak{sl}}(\pi_{n-1}) R_{\mathfrak{sl}}(\pi_{n-2}) = R_{\mathfrak{sl}}(\pi_{n-1} + \pi_{n-2}) + R_{\mathfrak{sl}}(\pi_{n-3}),$$

$$R_{\mathfrak{so}}(\pi_{n-1}) R_{\mathfrak{so}}(\pi_{n-2}) = R_{\mathfrak{so}}(\pi_{n-1} + \pi_{n-2}) + R_{\mathfrak{so}}(\pi_{n-1}) + R_{\mathfrak{so}}(\pi_{n-3}).$$

Therefore

$$R_{\mathfrak{sl}}(\pi_{n-1} + \pi_{n-2}) = R_{\mathfrak{so}}(\pi_{n-1} + \pi_{n-2}) + R_{\mathfrak{so}}(\pi_{n-1}).$$

Next we have

$$R_{\mathfrak{sl}}(\pi_{n-1} + \pi_{n-2} + \pi_2) \subset R_{\mathfrak{sl}}(\pi_{n-1} + \pi_{n-2}) R_{\mathfrak{sl}}(\pi_2)$$

$$= (R_{\mathfrak{so}}(\pi_{n-1} + \pi_{n-2}) + R_{\mathfrak{so}}(\pi_{n-1})) R_{\mathfrak{so}}(\pi_2)$$

$$= (R_{\mathfrak{so}}(\pi_{n-1} + \pi_2) + R_{\mathfrak{so}}(\pi_{n-1})) R_{\mathfrak{so}}(\pi_2).$$

Finally,

$$R_{\mathfrak{so}}(\pi_1) R_{\mathfrak{so}}(\pi_2) = R_{\mathfrak{so}}(\pi_1 + \pi_2) + R_{\mathfrak{so}}(\pi_1) + R_{\mathfrak{so}}(\pi_3).$$

The last expression shows that the complexification of $H^2(\mathfrak{g}_-, \mathfrak{g})$ doesn’t contain any trivial submodules.
2. As \(\mathfrak{so}(n, \mathbb{C})\)-module the homogeneity 1 component in \(H^1(\mathfrak{g}_-, \tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \mathbb{C}\) is a part of:
\[
S^2(\pi_1)R(\pi_1) = (R(2\pi_1) + R(\pi_0))R(\pi_1) = R(3\pi_1) + R(\pi_1 + \pi_2) + 2R(\pi_1).
\]
As we see this component doesn’t contain any trivial submodule.

3. The homogeneity 2 component in \(H^1(\mathfrak{g}_-, \tilde{\mathfrak{g}}/\mathfrak{g})\) is a symmetric 2-tensor and we know:
\[
S^2_{\mathfrak{so}}(\pi_1) = R_{\mathfrak{so}}(2\pi_1) + R_{\mathfrak{so}}(\pi_0).
\]
Therefore the family of constant curvature structures is not more than 1-dimensional in the generic case.

The corresponding cohomology element invariant under the action of \(\mathfrak{so}(n, \mathbb{R})\) is
\[
\alpha = \sum_{i=1}^{n} \alpha_{i(ii)} = \sum_{i,t=1}^{n} \left( e_t \otimes e^*_t \wedge e^*_t + \sum_{r=1}^{n} e_{[tr]} \otimes e^*_{[tr]} \wedge e^*_{[tr]} \right).
\]
One could check that constant curvature models with curvature \(k\alpha\) are isomorphic to \(SO(n+1)\) if \(k > 0\) and to \(SO(1, n)\) if \(k < 0\).

4. In the case \(n = 4\) the computations are slightly different since \(\mathfrak{so}_4(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})\). The fundamental weights are \((i\pi_1, j\pi_1)\). We use the notation
\[
R(\tilde{\pi}_1) = R(\pi_1, \pi_1), \quad R(\tilde{\pi}_2) = \Lambda^2(\tilde{\pi}_1) = R(\pi_0, 2\pi_1) + R(2\pi_1, \pi_0), \quad R(\tilde{\pi}_3) = \Lambda^3(\tilde{\pi}_1) = R(\pi_1, \pi_1).
\]
As in previous computations \(H^2\)-part of the harmonic curvature belongs to
\[
(R(\tilde{\pi}_1 + \tilde{\pi}_2) + R(\tilde{\pi}_1)) R(\tilde{\pi}_2) = (R(\pi_1, 3\pi_1) + R(3\pi_1, \pi_1) + R(\pi_1, \pi_1))(R(\pi_0, 2\pi_1) + R(2\pi_1, \pi_0))
\]
\[
= R(\pi_1, 5\pi_1) + R(5\pi_1, \pi_1) + 2R(3\pi_1, 3\pi_1)
\]
\[
+ 3R(\pi_1, 3\pi_1) + 3R(3\pi_1, \pi_1) + 4R(\pi_1, \pi_1)
\]
and doesn’t have trivial components. Homogeneity 1 component in \(H^1(\mathfrak{g}_-, \tilde{\mathfrak{g}}/\mathfrak{g})\) belongs to
\[
S^2(\tilde{\pi}_1)R(\tilde{\pi}_1) = R(3\pi_1, 3\pi_1) + R(\pi_1, 3\pi_1) + R(3\pi_1, \pi_1) + 2R(\pi_1, \pi_1)
\]
and doesn’t have any trivial components as well. Homogeneity 2 component in \(H^1(\mathfrak{g}_-, \tilde{\mathfrak{g}}/\mathfrak{g})\) has 1-dimensional trivial submodule
\[
S^2(\tilde{\pi}_1) = R(2\tilde{\pi}_1) + R(\tilde{\pi}_0).
\]
Explicit check shows that again we obtain unique up to scaling constant curvature models on \(SO(5)\) and \(SO(4, 1)\).

5. Consider the case \(n = 3\). In this case the complexification of \(\mathfrak{g}_0 = \mathfrak{so}(3, \mathbb{R})\) is \(\mathfrak{sl}(2, \mathbb{C})\). As \(\mathfrak{sl}(2, \mathbb{C})\)-module the complexification of \(\mathfrak{g}_{-1}\) and \(\mathfrak{g}_{-2}\) are \(R(2\pi_1)\).
First of all, $H^2$-part of the harmonic curvature is 1-dimensional and therefore is trivial under the action of $\mathfrak{so}(3)$. Secondly, homogeneity 1 component in $H^1(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) \otimes \mathbb{C}$ belongs to

$$S^2(2\pi) R(2\pi) = R(6\pi) + R(4\pi) + 2 R(2\pi)$$

and doesn’t have any trivial components. Homogeneity 2 component in $H^1(\mathfrak{g}_-, \mathfrak{g}/\mathfrak{g}) \otimes \mathbb{C}$ has 1-dimensional trivial submodule

$$S^2(2\pi) = R(4\pi) + R(\pi_0).$$

To sum up the family of constant curvature structures is 2-dimensional. □

**Remark 3.** One can use Theorems 7 and 8 in order to write down homogeneous models explicitly for free sub-Riemannian structures of rank 3. The corresponding family is a cone over a circle with the vertex of the cone being a flat model. All models apart 2 lines passing through the vertex of the cone have 9-dimensional semisimple symmetry Lie algebras. The symmetry algebras of models on the remaining 2 lines have a 6-dimensional semisimple part in the Levi decomposition.

7. **Appendix — Admissible Cartan connections for filtered $G_0$-structures with trivial prolongation**

The goal of this section is to provide a simple and straightforward construction of canonical Cartan connections under algebraic assumptions relevant for the situations dealt with in the paper. In particular we show that such construction is almost identical to the well known theory of classical $G$-structures.

Exactly as in Section 2 above, let $D$ be a bracket generating $n$-dimensional distribution on an $m$-dimensional manifold $M$ and $D_{-1} = D \subset D_{-2} \subset \cdots \subset D_{-k} = TM$ the associated filtration. As always, we assume that the distribution $D$ has constant symbol, that is for any $x \in M$ the negatively graded Lie algebra

$$\text{gr} T_x M = (T_x M)_{-k} + \cdots + (T_x M)_{-1} = (D_{-k}/D_{-k+1})_x + \cdots + (D_{-1})_x$$

is isomorphic to some fixed symbol algebra $\mathfrak{g}_- = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1}$.

Further, we fix a subalgebra $\mathfrak{g}_0 \subset \text{der}\mathfrak{g}_-$ of (graded) derivations, which integrates to a Lie group $G_0 \subset \text{Aut}(\mathfrak{g}_-)$ of automorphisms and we denote by

$$\pi : \mathcal{F} \to M$$

the associated filtered $G_0$-structure (or graded $G_0$ structure), i.e. $G_0$-principal bundle of graded frames $f : \mathfrak{g}_- \to \text{gr} T_x M$ (isomorphisms of the graded Lie algebras) in the graded tangent space $\text{gr} TM$. Let us assume that the Tanaka prolongation $\mathfrak{g}^{(1)}$ of the non positively graded Lie algebra $\mathfrak{g} := \mathfrak{g}_{<0} = \mathfrak{g}_- + \mathfrak{g}_0$ is trivial. As detailed in Section 2, it is the case if $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$, and then there is a unique normal Cartan connection for this filtered $G_0$-structure.
To simplify the exposition, we will assume that the depth $k = 2$, i.e. the symbol algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ is metaabelian. Then a complementary distribution $D'$ to $D$, such that $TM = D \oplus D'$, allows to identify graded frames with tangent frames and identify the filtered $G_0$-structure with a standard $G_0$-structure.

Recall that a $G$-structure on a principal $G$-bundle $\pi : \mathcal{F} \to M$ over $m$-dimensional manifold $M$ is defined by a soldering 1-form $\theta : T\mathcal{F} \to \mathbb{R}^m$, that is strictly horizontal ($\ker \theta = T^\text{vert}\mathcal{F}$) $G$-equivariant vector-valued 1-form.

Note that for the filtered $G_0$-structure there is the natural projection $T_xM \to (T_xM)_{-2} = T_xM/D_x$, $X \mapsto X_{-2}$ which defines the partial $\mathfrak{g}_{-2}$-valued soldering form $\theta_{-2} : T\mathcal{F} \to \mathfrak{g}_{-2}$ on the filtered $G_0$-structure $\mathcal{F}$ defined by $\theta_{-2}(X_f) = f^{-1}(\pi_*X)_{-2}$.

A $G_0$-structure in $\pi : \mathcal{F} \to M$ defined by an extension of the partial soldering form $\theta_{-2}$ to a $\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$-valued soldering form is called admissible. A complementary distribution $D'$ defines an isomorphism

$$\text{gr}_{D'} : TM \to \text{gr} TM, X \mapsto \text{gr}(X) = X_{-2} \oplus X_{-1}$$

where $X = X_{-2} + X_{-1} \in T_xM = D'_x \oplus D$ is the natural decomposition of $X$ and we identify $D'$ with $(TM)_{-2} = TM/D$ and extends graded frames to an admissible frames. We get

**Lemma 1.** Any complementary distribution $D'$ to $D$ (s.t. $TM = D \oplus D'$) defines an admissible $G_0$-structure on the principal bundle $\pi : \mathcal{F} \to M$ and any admissible $G_0$-structure is associated with some $D'$. The soldering form $\theta = \theta_{D'}$ of such $G_0$-structure is given by

$$\theta_f(X) = f^{-1}\text{gr}(\pi_*X) \in \mathfrak{g}_-.$$ 

Let us observe that there always is the distinguished choice of $D'$ given by the co-closeness curvature normalization for the sub-Riemannian structures, as explained in detail in Section 2. In particular examples, the identification of $D'$ usually amounts to minimizing traces of individual components of the curvature. We have illustrated this in the detailed discussion of the 2-step free distributions in the end of Section 5.

**7.1. Construction of a Cartan connection.** Let $\pi : \mathcal{F} \to M$ be a filtered $G_0$-structure. We consider the $G_0$-structure on $M$ defined by a complementary distribution $D'$. Each element $f \in \mathcal{F}$ is identified with an admissible frame $f : \mathfrak{g}_- \to \text{gr}T_xM = T_xM$. We denote by $f^{-1} : T_xM \to \mathfrak{g}_-$ the dual coframe.

**Algebraic assumptions.**

i) We assume that the first prolongation $\mathfrak{g}_0^{(1)} = 0$ of the linear Lie algebra $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}_-)$ is trivial. It is the case if the first prolongation of $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_0$ is trivial.
ii) We assume that there is a fixed $G_0$-invariant complement $\mathcal{T}$ to the subspace $\delta(g_0 \otimes g_-^\ast)$ in the vector space of torsions $\text{Tor} := g_- \otimes \Lambda^2 g_-^\ast$ (here $\delta$ means the usual Spencer differential).

According to theory of $G$-structures, under these assumptions there is a unique connection $\omega : T\mathcal{F} \rightarrow g_0$ on the principal bundle $\pi : \mathcal{F} \rightarrow M$ whose torsion function $t : \mathcal{F} \rightarrow \text{Tor}$ take values in $D$.

If $H = \ker \omega_f$ is the horizontal space of the connection and $\pi_H := (\pi^\ast)_H : H \rightarrow T_xM$ is the natural isomorphism then the value $t_f \in \text{Tor}$ is given by

$$t_f(u,v) = d\theta_f(\pi_H^{-1}fu, \pi_H^{-1}fv) \in g_-$$

for any $u,v \in g_-$. In this case, 1-form

$$\varpi = \omega + \theta : T\mathcal{F} \rightarrow g = g_0 + g_-$$

is a Cartan connection modelled on the affine space $g_- = (G_0 \rtimes g_-)/G_0$ that is a $G_0$-equivariant map without kernel (such that $\varpi_f : T_f\mathcal{F} \rightarrow g$ is a vector space isomorphism) which extends the vertical parallelism $T_{\text{vert}}\mathcal{F} \rightarrow g_0$.

Note that we may consider the same 1-form $\varpi = \omega + \theta$ as a Cartan connection in $\mathcal{F}$ modelled on the homogeneous manifold $G_- = G/G_0 = (G_0 \rtimes G_-)/G_0$ where $G = G_0 \rtimes G_-$ is a semidirect product of the nilpotent Lie group, generated by the Lie algebra $g_-$ and a group $G_0 \subset \text{Aut}(G_-)$ of its automorphisms. Such Cartan connections $\varpi$ on the filtered $G_0$-structure $\mathcal{F} \rightarrow M$ are called admissible.

We may summarize:

**Proposition 1.** Any admissible Cartan connection is obtained in this way and it is associated with a complementary distribution $D'$.

The curvature

$$\kappa = d\varpi + \frac{1}{2}[\varpi, \varpi]$$

of this connection is a 2-form on $\mathcal{F}$ valued in $g = (g_- + g_0)$, which describes the deviation of the bundle $\pi : \mathcal{F} \rightarrow M$ from the model bundle $G \rightarrow G/G_0$.

Here $[\phi, \psi]$ denote the bracket of $g$-value 1-forms, defined as 2-form by

$$[\phi, \psi](X,Y) = [\phi(X), \psi(Y)] - [\psi(Y), \phi(X)].$$

The curvature form is decomposed into $g_0$-valued form $\Omega$ called the curvature and the $g_-$-valued part $\Theta$, called torsion. The general Cartan structure equations split nicely, too:

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

$$\Theta = d\theta + \frac{1}{2}[\theta, \theta] + [\omega, \theta].$$

The last equation may be decomposed into $g_{-1}$ and $g_{-2}$-part as follows:

$$\Theta_{-1} = d\theta_{-1} + [\omega, \theta_{-1}]$$

$$\Theta_{-2} = d\theta_{-2} + \frac{1}{2}[\theta_{-1}, \theta_{-1}] + [\omega, \theta_{-2}]$$
Note that the 2-forms $\kappa, \Omega, \Theta$ can be considered as the $G_0$-invariant functions

$$K : \mathcal{F} \to \Lambda^2(g)^* \otimes g, \quad R : \mathcal{F} \to \Lambda^2(g)^* \otimes g_0, \quad T : \mathcal{F} \to \Lambda^2(g)^* \otimes g^-$$

called Cartan curvature function, curvature function and torsion function respectively.

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