Discrete quantum modes of the Dirac field in $AdS_{d+1}$ backgrounds

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Abstract

It is shown that the free Dirac equation in spherically symmetric static backgrounds of any dimensions can be put in a simple form using a special version of Cartesian gauge in Cartesian coordinates. This is manifestly covariant under the transformations of the isometry group so that the generalized spherical coordinates can be separated in terms of angular spinors like in the flat case, obtaining a pair of radial equations. In this approach the equation of the free field Dirac in $AdS_{d+1}$ backgrounds is analytically solved obtaining the formula of the energy levels and the corresponding normalized eigenspinors.

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1 Introduction

The interest in propagation of the quantum free fields on de Sitter or anti-de Sitter ($AdS$) spacetimes is due to the hope that these simple examples of matter fields minimally coupled with the gravitational field of the background could suggest new ways to further developments of the quantum field theory in curved spacetimes. Moreover, the discovery of the AdS/CFT correspondence [1] brings in a central position the theory of the free fields on the
(d+1)-dimensional AdS spacetimes (AdS_{d+1}) since their quantum modes can be related to the conformal field theories in the Minkowski-like boundaries of the AdS_{d+1} manifolds [2].

The quantum modes of the scalar free fields in central charts of the AdS_{d+1} backgrounds are well-studied either for d = 3 [3] or in the general case of any d [4, 5]. However, for the Dirac field we know only the solutions in the central charts of the AdS_{3+1} spacetime we have obtained few years ago [6]. These were derived using a special Cartesian tetrad-gauge which points out the central symmetry as a manifest one [7] helping us to simply separate the spherical variables of the Dirac equation in terms of the angular spinors defined in special relativity [8]. Recently, Gu, Ma and Dong have generalized the technique of separating spherical variables of the Dirac equation in Minkowski spacetime to flat (pseudo-Euclidean) spacetimes with generalized spherical coordinates of any dimensions [9]. This offers us the opportunity to generalize our method to any (d + 1)-dimensional curved spacetime with static charts having the SO(d)-symmetry, we call here (generalized) central charts. The idea is to write the Dirac equation in a suitable gauge allowing us to take over the whole procedure of the separation of generalized spherical variables worked out for the flat manifolds with central potentials [9]. In this way we have to obtain simple radial equations that may be solved for particular cases as that of the AdS_{d+1} spacetimes. The present article is devoted to this problem.

The main point of our approach is the mentioned Cartesian gauge which can be generalized to central backgrounds of any dimensions, where it will play a similar role in showing off the manifest SO(d) symmetry of the generalized central backgrounds. This leads to a simple covariant form of the Dirac equation in Cartesian coordinates and produces manifest covariant SO(d) generators of the spinor representation whose orbital and spin parts commute to each other [10, 7]. In these conditions the separation of the generalized spherical variables can be done in terms of the angular momentum eigenspinors defined in the (d + 1)-dimensional flat backgrounds [9]. After the separation of the angular variables there remain only two radial wave functions for which we write down the radial equations in the general case of any central chart with generalized spherical coordinates. These can be analytically solved for the AdS_{d+1} backgrounds in the same manner as in the case of d = 3 [6], finding only discrete quantum modes.

We start in the second section outlining a convenient version of gauge-
covariant Dirac theory in any dimensions. In Sec.3 we define the Cartesian
gauge in Cartesian coordinates and we bring the Dirac equation in a simpler
form, called reduced equation, which has similar properties as that in flat
spacetime, including the discrete symmetries. The next section is devoted
to the separation of variables in generalized spherical coordinates which gen-
erates the radial equations. In Sec.5 we obtain the quantum modes of the
Dirac field in the central charts with generalized spherical coordinates of the
AdS\(_{d+1}\) spacetime, deriving the normalized energy eigenspinors and the en-
ergy levels. The parity and charge-conjugation transformations we use are
defined in Appendix.

## 2 The gauge-covariant Dirac theory in \((d+1)\) dimensions

The theory of the Dirac field in curved spacetimes of any dimensions
involves three basic ingredients that can be chosen in different ways. These
are the local chart (i.e., natural frame), the gauge fields defining the local
frames and the Clifford algebra. In these conditions, the physical meaning of
the whole theory remains independent on the choice of these elements only if
we assume that this is gauge-covariant. On the other hand, the generalization
to a larger number of dimensions may be physically relevant in the sense of
the Kaluza-Klein theories only if the extra-dimensions are space-like. For
this reason we consider here the gauge-covariant theory of the Dirac spinors
in backgrounds with \(d+1\) dimensions among them only one is time-like.

We start with a such pseudo-Riemannian manifold, \(M_{d+1}\), whose pseudo-
Euclidean model has the flat metric \(\eta\) of signature \((1,d)\), corresponding
to the gauge group \(G(\eta) = SO(1,d)\). Since the Dirac theory involves lo-
cal orthogonal (non-holonomic) frames, we choose in \(M_{d+1}\) a local chart
with coordinates \(x^\mu, \alpha,...,\mu,\nu,... = 0,1,2,...,d\), and introduce local frames
using the gauge fields \(e_\alpha(x)\) and \(\hat{e}_\alpha(x)\) labeled by local (hated) indices,
\(\hat{\alpha},...\hat{\mu},\hat{\nu},... = 0,1,2,...,d\). Their components accomplish
\[e_\alpha^\mu \hat{e}_\nu^\mu = \delta_\nu^\mu, \quad e_\alpha^\mu \hat{e}_\mu^\nu = \delta_\alpha^\nu, \quad g_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta},\]
and give the components of the metric tensor of \(M_{d+1}\) in the natural frame,
\( g_{\mu \nu}(x) = \eta_{\alpha \beta} \hat{e}_{\mu}^\alpha(x) \hat{e}_{\nu}^\beta(x). \)

Bearing in mind that the irreducible representations of the Clifford algebra have only an odd number of dimensions, we consider a \((2m + 1)\)-dimensional Clifford algebra with \(2m \geq d\), acting on a spinor space with \(2^m\) dimensions. From the basis of this algebra we choose a suitable set of \(d + 1\) \(\gamma\)-matrices and match the phase factors in order to have

\[
\{ \gamma^\alpha, \gamma^\beta \} = 2\eta^{\alpha \beta}. \tag{2}
\]

Then \((\gamma^0)^\dagger = \gamma^0\) and \((\gamma^i)^\dagger = -\gamma^i\) \((i, j, k, ... = 1, 2, ..., d)\) which means that the Dirac adjoint of any matrix \(A\) can be defined in usual manner as \(\overline{A} = \gamma^0 A^\dagger \gamma^0\) so that the \(\gamma\)-matrices be self-adjoint, \(\overline{\gamma^\alpha} = \gamma^\alpha\). Notice that the point-dependent matrices \(\gamma^\mu(x) = e_i^\mu(x) \gamma^i\) are also self-adjoint with respect to the Dirac adjoint.

The group \(G(\eta)\) admits an universal covering group \(\tilde{G}(\eta)\) that is simply connected and has the same Lie algebra. The spin operators

\[
S^{\alpha \beta} = \frac{i}{4} [\gamma^{\alpha}, \gamma^{\beta}] \tag{3}
\]

represent the basis-generators of the spinor representation of \(\tilde{G}(\eta)\) which, in general, can be reducible. The real valued parameters of \(\tilde{G}(\eta)\) are the components \(\omega_{\alpha \beta} = -\omega_{\beta \alpha}\) of the skew-symmetric tensors, \(\omega\) defining the operators

\[
T(\omega) = e^{-iS(\omega)}, \quad S(\omega) = \frac{1}{2}\omega_{\alpha \beta} S^{\alpha \beta}, \tag{4}
\]

which transform the gamma-matrices according to the rule

\[
[T(\omega)]^{-1} \gamma^\alpha T(\omega) = \Lambda^{\alpha \beta}(\omega) \gamma^\beta \tag{5}
\]

where

\[
\Lambda^{\alpha \beta}(\omega) = \delta^{\alpha \beta} + \omega_{\alpha \beta} + \frac{1}{2} \omega_{\alpha \gamma} \omega_{\beta \gamma} + ... + \frac{1}{n!} \omega_{\alpha \gamma_1 \gamma_2 ... \gamma_n} \omega_{\beta \gamma_1 \gamma_2 ... \gamma_n} + ... \tag{6}
\]

is a transformation of \(G(\eta)\). We specify that the operators \(T(\omega) \in \tilde{G}(\eta)\) are unitary with respect to the Dirac adjoint satisfying \(\overline{T(\omega)} = [T(\omega)]^{-1}\).
Let us denote by $\psi$ the Dirac free field of mass $m$, and by $\overline{\psi} = \psi^\dagger \gamma^0$ its Dirac adjoint. In natural units (with $\hbar = c = 1$) its gauge invariant action [11] is

$$S[\psi, e] = \int d^4x \sqrt{g} \left\{ \frac{i}{2} (\overline{\psi} \gamma^\mu D_\mu \psi - (D_\mu \overline{\psi}) \gamma^\mu \psi) - m \overline{\psi} \psi \right\}$$  \hspace{1cm} (7)$$

where $g = |\det(g_{\mu\nu})|$ and $D_\mu = \nabla_\mu + \Gamma_\mu^{\text{spin}}$ are the covariant derivatives of the spinor field formed by the usual covariant derivatives $\nabla_\mu$ (acting in natural indices) and the spin connection $\Gamma_\mu^{\text{spin}} = \frac{i}{2} e^\beta_\alpha (e^\delta_\alpha \Gamma^\alpha_\beta_\mu - e^\delta_\beta_\mu) S^{\beta}_{\alpha \delta}$. The action of these covariant derivatives in the spinor sector is $D_\mu \psi = (\partial_\mu + \Gamma_\mu^{\text{spin}}) \psi$.

Since the spin connection satisfies $\Gamma_\mu^{\text{spin}} = -\Gamma_\mu^{\text{spin}}$, the quantity $\overline{\psi} \psi$ can be derived as a scalar, i.e. $\nabla_\mu (\overline{\psi} \psi) = D_\mu \overline{\psi} \psi + \overline{\psi} D_\mu \psi = \partial_\mu (\overline{\psi} \psi)$, while the quantities $\overline{\psi} \gamma^\alpha \gamma^\beta ... \psi$ behave as tensors of different ranks. Moreover, the use of covariant derivatives assures the covariance of the whole theory under the gauge transformations

$\hat{e}_{\mu}^\alpha (x) \rightarrow \hat{e}_{\mu}^\alpha (x) = \Lambda_{\hat{\alpha} / \beta}^\alpha [\omega (x)] \hat{e}_{\mu}^\hat{\alpha} (x)$,

$e_{\mu}^\alpha (x) \rightarrow e_{\mu}^\alpha (x) = \Lambda_{\hat{\beta} \alpha}^\alpha [\omega (x)] e_{\mu}^\hat{\beta} (x)$,  \hspace{1cm} (8)

$\psi (x) \rightarrow \psi' (x) = T [\omega (x)] \psi (x)$,

due to the point-dependent parameters $\omega_{\hat{\alpha} \hat{\beta}} (x)$, in the sense that $S(\psi', e') = S(\psi, e)$. Thus we reproduced the main features of the familiar tetrad gauge-covariant theories with spin in four dimensions from which we can take over now all the results arising from similar formulas. For example, in this way we find that

$$D_\mu (\gamma^\nu \psi) = \gamma^\nu D_\mu \psi$$,  \hspace{1cm} (9)

$$[D_\mu, D_\nu] \psi = \frac{1}{4} R_{\alpha \beta \mu \nu} \gamma^\alpha \gamma^\beta \psi$$,  \hspace{1cm} (10)$$

where $R$ is the Riemannian-Christoffel curvature tensor of $M_{d+1}$. The Dirac equation derived from the action (7), $\mathcal{E}_D \psi = m \psi$, involves the Dirac operator that can be written as

$$\mathcal{E}_D = i \gamma^\mu D_\mu = i \gamma^\alpha e^\mu_\alpha \partial_\mu + \frac{i}{2} \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} e^\mu_\alpha) \gamma^\alpha - \frac{1}{4} e^\mu_\alpha \{ \gamma^\alpha, \Gamma_\mu^{\text{spin}} \}.$$  \hspace{1cm} (11)$$

It is not difficult to verify that this is self-adjoint with respect to the Dirac adjoint.
A crucial problem is to find the generators of symmetries at the level of the relativistic quantum mechanics since these operators must commutes with the Dirac one. In the absence of other interactions these are produced only by the symmetries of the background. For explaining this mechanism it is worth reviewing some previous results we obtained in four dimensions and can be generalized to arbitrary dimensions. In [7] we defined the external symmetry group, $S(M)$, of a given manifold $M$ as the universal covering group of the isometry group, $I(M)$, pointing out that for each matter field there exists a specific representation of $S(M)$, induced by a linear representation of $\tilde{G}(\eta)$, that leaves the field equation invariant. Consequently, the generators of this representation are operators which commute with that of the field equation. They can be calculated starting with the Killing vectors corresponding to the isometries of $I(M)$ according to a rule obtained by Carter and McLenaghan for the Dirac field [10] which states that for any Killing vector $k^\mu$ there exists an operator

$$X_k = -ik^\mu D_\mu + \frac{1}{2} k_{\mu\nu} e^\mu_\alpha e^\nu_\beta S^\alpha^\beta$$

which commutes with $E_D$. We have shown that these operators are just the generators of the representation of $S(M)$ in the space of the spinors that is induced by the spinor representation of $\tilde{G}(\eta)$. Each generator (12) can be divided in an usual orbital part and a spin part, involving the spin matrices, whose forms depend on the choice of the gauge fields. When the spin parts commute with the orbital ones we say that the representation is manifest covariant. These results can be generalized for any $M_{d+1}$ manifold considering that the operators $S^\alpha^\beta$ are those defined by Eq.(3).

In other respects, from the conservation of the electric charge, one can deduce that when $e^0_i = 0$, $i = 1, 2, ..., d$, the relativistic scalar product of two spinors [11],

$$\langle \psi_1, \psi_2 \rangle = \int_D d^d x \frac{\mu(x) \bar{\psi}_1(x) \gamma^0 \psi_2(x)}{\sqrt{g(x) e^0_0(x)}}$$

defined on the $d$-dimensional space domain $D$ of the chart we use, has the weight function

$$\mu(x) = \sqrt{g(x) e^0_0(x)}.$$  

(14)

The coherence of the whole theory is guaranteed by the fact that the conserved quantity associated to the Killing vector $k^\mu$, given by the Noether theorem, is the expectation value $\langle \psi, X_k \psi \rangle$ of the operator (12) calculated using the scalar product (13) [12].
3 The Dirac equation in central backgrounds

In what follows we shall focus only on the static central manifolds. These have static central charts with Cartesian coordinates, the time \( x^0 = t \) and space Cartesian coordinates \( x^i, \ i = 1, 2, \ldots, d \), where the metric is time-independent and spherically symmetric. The isometry groups of these manifolds, \( I(M_{d+1}) \), include as a subgroup the group of the static central symmetry, \( I_c = T(1) \otimes SO(d) \), formed by time translations and \( d \)-dimensional orthogonal transformations of the Cartesian space coordinates seen as the components of the vector \( \mathbf{x} \equiv (x^1, x^2, \ldots, x^d) \). The central symmetry requires the metric tensor, \( g(\mathbf{x}) \), to transform manifestly covariant under the linear transformations of the space coordinates,

\[
x^i \rightarrow x'^i = R_{ij} x^j, \quad t' = t,
\]

produced by any \( R \in SO(d) \). In these conditions the corresponding line element has the general form

\[
ds^2 = g_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu = A(r) dt^2 - [B(r) \delta_{ij} + C(r) x^i x^j] dx^i dx^j
\]

where \( A, B \) and \( C \) are arbitrary functions of the Euclidean norm of \( \mathbf{x} \), \( r = |\mathbf{x}| = \sqrt{x^i x^i} \), which is invariant under \( SO(d) \) transformations. In applications it is convenient to replace these functions by new ones, \( u, v \) and \( w \), such that

\[
A = w^2, \quad B = \frac{u^2}{v^2}, \quad C = \frac{w^2}{r^2} \left( \frac{1}{u^2} - \frac{1}{v^2} \right).
\]

Then the metric appears as the conformal transformation of the simpler one with \( w = 1 \).

In the case of the four-dimensional central backgrounds we proposed a Cartesian gauge in which the gauge fields are static and covariantly transform under transformations similar to (15). We adopt the same type of gauge here, in the Cartesian charts of the central manifolds \( M_{d+1} \), requiring the 1-forms \( d\hat{x}^\mu = \hat{e}^\mu_\alpha dx^\alpha \) to transform according to the rule

\[
d\hat{x}^{\hat{\mu}} \rightarrow d\hat{x}'^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_\alpha(x') dx'^\alpha = (Rd\hat{x})^{\hat{\mu}}.
\]

Then the gauge fields must have components of the form

\[
\hat{e}^0 = \hat{a}(r), \quad \hat{e}^0 = \hat{e}^i_0 = 0, \quad \hat{e}^i_j = \hat{b}(r) \delta_{ij} + \hat{c}(r) x^ix^j, \quad \hat{e}^i_0 = a(r), \quad e^0_i = e^i_0 = 0, \quad e^i_j = b(r) \delta_{ij} + c(r) x^ix^j,
\]

In applications it is convenient to replace these functions by new ones, \( u, v \) and \( w \), such that

\[
A = w^2, \quad B = \frac{u^2}{v^2}, \quad C = \frac{w^2}{r^2} \left( \frac{1}{u^2} - \frac{1}{v^2} \right).
\]

Then the metric appears as the conformal transformation of the simpler one with \( w = 1 \).
where, according to Eqs.(16) and (17), we find that
\begin{align}
\hat{a} &= w, \quad \hat{b} = \frac{w}{v}, \quad \hat{c} = \frac{1}{r^2} \left( \frac{w}{u} - \frac{w}{v} \right), \\
a &= \frac{1}{w}, \quad b = \frac{v}{w}, \quad c = \frac{1}{r^2} \left( \frac{u}{w} - \frac{v}{w} \right).
\end{align}
(21)
(22)

Since in this chart
\[
\sqrt{g} = B^{d-1}[A(B + r^2C)]^{\frac{1}{2}} = \frac{1}{ab^{d-1}(b + r^2c)} = \frac{w^{d+1}}{uv^{d-1}},
\]
(23)
we obtain the weight function (14) in our Cartesian gauge,
\[
\mu = \frac{1}{b^{d-1}(b + r^2c)} = \frac{w^d}{uv^{d-1}}.
\]
(24)

From Eqs.(21) and (22) we observe that \(w\) must be positively defined in order to keep the same sense of the time axes of the natural and local frames. In addition, it is convenient to consider that the function \(u\) is positively defined and to try to define the radial coordinate \(r\) so that \(u = 1\). However, the function \(v\) can be of any sign. When \(v > 0\) then the space axes of the local frame at \(x = 0\) are parallel with those of the natural frame, while for \(v < 0\) these are antiparallel.

Let us derive now the concrete form of the operators of the Dirac theory in this gauge. First we observe that if we replace the tetrad components in Eq.(11) then the last term of its left-hand side does not contribute. The remaining equation can be put in a simpler form introducing the reduced Dirac field, \(\tilde{\psi}\), defined as
\[
\psi(x) = \chi(r)\tilde{\psi}(x),
\]
(25)
where
\[
\chi = [\sqrt{g}(b + r^2c)]^{-1/2} = v^{\frac{d-1}{2}}w^{-\frac{d}{2}}.
\]
(26)
After this substitution we obtain the reduced Dirac equation, \(\tilde{\mathcal{D}}\tilde{\psi} = m\tilde{\psi}\), in which the reduced Dirac operator,
\[
\tilde{\mathcal{D}} = ia(r)\gamma^0\partial_t + ib(r)\gamma^i\partial_i + ic(r)(\gamma^i x^i) \left( \frac{d-1}{2} + x^i \partial_i \right),
\]
(27)
is independent on the derivatives of \(a, b\) and \(c\).
We observe that the reduced Dirac equation has a manifest symmetry similar to that of the Dirac equation in flat backgrounds with central potentials. This is the consequence of our version of Cartesian gauge which produces usual linear representation of the external symmetry group $S_c(M_{d+1})$, associated with $I_c(M_{d+1})$, showing off the central symmetry in a manifest covariant form. Indeed, taking into account that $S_c(M_{d+1}) = T(1) \otimes SO(d)$ involves the group $SO(d) \subset G(\eta)$ which is the universal covering group of $SO(d)$, we conclude that the spin operators $S^{ij}$ defined by Eq.(3) are the basis-generators of the spinor representation of $SO(d)$. Furthermore, following the method of Ref.[7], we find that the basis-generators of the induced representation of $S_c(M_{d+1})$ given by Eq.(12) are the $T(1)$ generator, $i\partial_t$, and the $\tilde{SO}(d)$ ones, $J_{ij} = L_{ij} + S_{ij} = -i(x^i \partial_j - x^j \partial_i) + S_{ij}$, (28) which play the role of total angular momentum. They commute with the operator (27) and, obviously, have a manifest covariant form.

On the other hand, the reduced Dirac equation can be put in Hamiltonian form $i\partial_t \tilde{\psi} = \mathcal{H} \tilde{\psi}$ as in the case of the flat manifolds using the operators

$$\mathcal{H} = -i \frac{u(r)}{r^2} (\gamma^0 \gamma^i x^i) \left( \frac{d-1}{2} + x^i \partial_i \right) - i \frac{v(r)}{r^2} (\gamma^0 \gamma^i x^i) \mathcal{K} + w(r) \gamma^0 m, \quad (29)$$

$$\mathcal{K} = \gamma^0 \left( S^{ij} L_{ij} + \frac{d-1}{2} \right). \quad (30)$$

Since $\gamma^0$ commutes with $J_{ij}$ and $\mathcal{K}$ we have

$$[\mathcal{H}, J_{ij}] = 0, \quad [\mathcal{H}, \mathcal{K}] = 0. \quad (31)$$

Consequently, all the properties related to the conservation of the angular momentum, including the separation of variables in spherical coordinates, will be similar to those of the usual Dirac theory in the flat spacetimes with $d + 1$ dimensions [9]. Moreover, we note that the discrete transformations of parity and charge conjugation can be also defined. These leave Eq.(27) invariant and have the same significance as in special relativity [13, 8]. Thus, for example, the charge conjugation transforms each particular solution of positive frequency into the corresponding one of negative frequency (see the Appendix).
4 Separation of spherical coordinates

The next step is to introduce the generalized spherical coordinates, $r, \theta_1, \theta_2, ..., \theta_{d-1}$, [14] associated with the space coordinates of our natural Cartesian frame,

$$\begin{align*}
x^1 &= r \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{d-1}, \\
x^2 &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-1}, \\
&\vdots \\
x^d &= r \cos \theta_{d-1}.
\end{align*}$$

(32)

Then from Eqs.(16) and (17) one obtains the line element

$$ds^2 = w^2 dt^2 - w^2 u^2 dr^2 - w^2 v^2 r^2 d\theta^2,$$

(33)

where $d\theta^2$ is the usual line element on the sphere $S^{d-1}$ [14],

$$d\theta^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 + \cdots.$$  

(34)

Consequently, in this chart we have

$$\sqrt{g_s} = \sqrt{g} r^{d-1} \prod_{a=1}^{d-1} (\sin \theta_a)^{a-1},$$

(35)

and similarly for the weight function (14). The special form of the reduced Dirac equation allows one to separate the spherical variables as in the case of the central motion in flat spacetimes, using the angular spinors $\phi_{\kappa,(j)}(\hat{x})$ defined in Ref.[9]. These depend only on the spherical variables represented by the unit vector $\hat{x} = \mathbf{x}/r$, being determined by a set of weights $(j)$ of an irreducible representation of $SO(d)$ and the eigenvalue $\kappa = \pm|\kappa|$ of the operator $K$ which concentrates all the angular operators of $\mathcal{H}$.

The separation procedure is complicated being different for even or odd values of $d$ [9]. For odd $d$ the particular solution of given energy, $E$, and positive frequency corresponding to the irreducible representation $(j)$ may have the form

$$\tilde{\psi}_{E,\kappa,(j)}(t, \mathbf{x}) = r^{\frac{d-1}{2}} e^{-iEt} \left( \begin{array}{c} f_{E,\kappa,(j)}^+(r) \phi_{\kappa,(j)}(\hat{x}) \\ i f_{E,\kappa,(j)}^-(r) \phi_{-\kappa,(j)}(\hat{x}) \end{array} \right),$$

(36)
where \( \kappa \) can take positive or negative values. However, if \( d \) is even then there are pairs of associated irreducible representations, \( (j_1) \) and \( (j_2) \), giving the same eigenvalue of the first Casimir operator, among them one takes the first one for positive values of \( \kappa \) and the second one for the negative values, \( \kappa = -|\kappa| \). Consequently, the particular solutions read

\[
\tilde{\psi}_{E,|\kappa|,(j_1)}(t, x) = r^{-\frac{d-1}{2}} e^{-iEt} \times \left[ f^+_{E,|\kappa|}(r) \phi_{|\kappa|,(j_1)}(\hat{x}) + if^-_{E,|\kappa|}(r) \phi_{-|\kappa|,(j_1)}(\hat{x}) \right], \tag{37}
\]

\[
\tilde{\psi}_{E,-|\kappa|,(j_2)}(t, x) = r^{-\frac{d-1}{2}} e^{-iEt} \times \left[ f^+_{E,-|\kappa|}(r) \phi_{-|\kappa|,(j_2)}(\hat{x}) + if^-_{E,-|\kappa|}(r) \phi_{-|\kappa|,(j_2)}(\hat{x}) \right]. \tag{38}
\]

It is remarkable that both these types of solutions involve the same set of radial wave functions that depend on \( E \) and

\[
\kappa = \pm \left( \frac{d-1}{2} + l \right), \quad l = 0, 1, 2, \ldots, \tag{39}
\]

where \( l \) is an auxiliary orbital quantum number defining the representations \( (j), (j_1) \) and \( (j_2) \) \[9\]. The radial wave functions obey a pair of radial equations similar to those found in the case of \( d = 3 \) \[6\]. For a given \( \kappa \), we embed these radial equations in the eigenvalue problem

\[
H_{E,\kappa} f_{E,\kappa} = E f_{E,\kappa} \tag{40}
\]

of the radial Hamiltonian operator

\[
H_{\kappa} = \begin{pmatrix} mw & u \frac{d}{dr} + \kappa \frac{v}{r} \\ -u \frac{d}{dr} + \kappa \frac{v}{r} & -mw \end{pmatrix} \tag{41}
\]

acting in the two-dimensional space with elements \( f = (f^+, f^-)^T \). After the separation of variables the scalar product (13) splits in angular and radial terms. Supposing that the angular spinors are normalized with respect to an angular scalar product, we remain with the radial scalar product

\[
\langle \psi_1, \psi_2 \rangle = \langle f_1, f_2 \rangle = \int_{D_r} \frac{dr}{u} f_1^* f_2. \tag{42}
\]

The radial weight function \( \mu = 1/u \), resulted from Eqs.(24) and (26), is just that we need in order to have \( (u \partial_r)^\dagger = -u \partial_r \) such that \( H_{E,\kappa} \) be Hermitian with respect to the scalar product (42). Thus we obtain an independent radial problem which has to be solved in each particular case separately using appropriate methods.
5 Quantum modes in $AdS_{d+1}$ spacetimes

The $AdS_{d+1}$ spacetime is the hyperboloid $\eta_{AB}Z^AZ^B = R^2$ of radius $R = 1/\omega$ in the $(d + 2)$-dimensional flat spacetime of coordinates $Z^{-1}$, $Z^0$, $Z^1$, ..., $Z^d$ and metric $\eta_{AB} = \text{diag}(1, 1, -1, ..., -1)$. Here we consider the static chart of Cartesian coordinates $(t, x)$ defined as

$$
Z^{-1} = R \sec \omega r \cos \omega t,
Z^0 = R \sec \omega r \sin \omega t,
Z = R \frac{x}{r} \tan \omega r,
$$

and the corresponding static chart with generalized spherical coordinates where the line element reads \[ds^2 = \eta_{AB}dZ^AdZ^B = \sec^2 \omega r \left( dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r d\theta^2 \right).\] In this chart $r \in D_r = [0, \pi/2\omega)$ and, therefore, the whole space domain is $D = D_r \times S^{d-1}$. In addition, we specify that the time of $AdS_{d+1}$ must satisfy $t \in [-\pi/\omega, \pi/\omega)$ while $t \in (-\infty, \infty)$ defines the universal covering spacetime of $AdS_{d+1}$ ($CAdS_{d+1}$) [3].

Furthermore, from Eq.(44) we identify

$$u(r) = 1, \quad w(r) = \sec \omega r, \quad v(r) = \omega r \csc \omega r,$$

and, introducing the notation $k = m/\omega$ (i.e. $mc^2/\hbar \omega$ in usual units), we find the Hamiltonian of the radial problem

$$H_\kappa = \begin{pmatrix}
\omega k \sec \omega r & \frac{d}{dr} + \omega \kappa \csc \omega r \\
-\frac{d}{dr} + \omega \kappa \csc \omega r & -\omega k \sec \omega r
\end{pmatrix}.\]$$

Its form suggests us to perform the local rotation $f \rightarrow \hat{f} = U(r) f = (f^+, f^-)^T$ where

$$U(r) = \begin{pmatrix}
\cos \frac{\omega r}{2} & \sin \frac{\omega r}{2} \\
-\sin \frac{\omega r}{2} & \cos \frac{\omega r}{2}
\end{pmatrix}.$$ 

The transformed Hamiltonian,

$$\hat{H}_\kappa = U(r)H_\kappa U^+(r) = \frac{\omega}{2}1_{2\times2} = \begin{pmatrix}
\nu & \frac{d}{dr} + W_\kappa \\
-\frac{d}{dr} + W_\kappa & -\nu
\end{pmatrix} (48)$$

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giving the new eigenvalue problem
\[ \hat{H}_\kappa \hat{f} = \left( E - \frac{\omega}{2} \right) \hat{f}, \]  
has supersymmetry since it has the requested specific form with \( \nu = \omega(k + \kappa) \) and the superpotential
\[ W_\kappa(r) = \omega(\kappa \cot \omega r - k \tan \omega r). \]

Consequently, the transformed radial wave functions, \( \hat{f}^\pm \), satisfy the second order equations
\[ \left( -\frac{1}{\omega^2} \frac{d^2}{dr^2} + \frac{k(k \mp 1)}{\cos^2 \omega r} + \frac{\kappa(\kappa \mp 1)}{\sin^2 \omega r} \right) \hat{f}^{(\pm)}(r) = \epsilon^2 \hat{f}^{(\pm)}(r), \]
where \( \epsilon = E/\omega - 1/2 \). The general solutions can be written in terms of Gauss hypergeometric functions [15],
\[ \hat{f}^{\pm}(r) = N_{\pm} \sin^{2s_{\pm}} \omega r \cos^{2p_{\pm}} \omega r \]
\[ \times F \left( s_{\pm} + p_{\pm} - \frac{\epsilon}{2}, s_{\pm} + p_{\pm} + \frac{\epsilon}{2}, 2s_{\pm} + 1, 2 \sin^2 \omega r \right), \]
depending on parameters that must satisfy
\[ 2s_{\pm}(2s_{\pm} - 1) = \kappa(\kappa \mp 1), \]
\[ 2p_{\pm}(2p_{\pm} - 1) = k(k \mp 1), \]
and on the normalization factors \( N_{\pm} \). The next step is to select the suitable values of these parameters and to calculate \( N_+/N_- \) such that the functions \( \hat{f}^\pm \) should be solutions of the transformed radial problem (49) with a good physical meaning. This can be achieved only when \( F \) is a polynomial selected by a suitable quantization condition since otherwise \( F \) is strongly divergent for \( \sin^2 \omega r \rightarrow 1 \). Then the functions \( \hat{f}^\pm \) will be square integrable with normalization factors calculated according to the condition
\[ \langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle = \int_{D_r} dr \left( |\hat{f}^+(r)|^2 + |\hat{f}^-(r)|^2 \right) = 1, \]
resulted from the fact that the matrix (47) is orthogonal.
The discrete energy spectrum is given by the particle-like CAdS quantization conditions
\[ \epsilon = 2(n_+ + s_+ + p_+) \quad , \quad \epsilon > 0 \quad , \]
that must be compatible with each other, i.e.
\[ n_+ + s_+ + p_+ = n_- + s_- + p_- \quad . \quad (57) \]
Hereby we see that there is only one independent radial quantum number, \( n_r = 0, 1, 2, ... \). In other respects, if we express the solutions (52) in terms of Jacobi polynomials, we observe that these functions remain square integrable for \( 2s_+ > -1/2 \) and \( 2p_+ > -1/2 \). Since from Eq.(39) we have \( |\kappa| \geq \frac{d-1}{2} \), we are forced to consider only the positive solutions of Eqs.(53). However, the Eqs.(54) admit either the solutions \( 2p_+ = k \) and \( 2p_- = k + 1 \) defining the boundary conditions of regular modes or other possible values, \( 2p_+ = -k + 1 \) and \( 2p_- = -k \), giving the irregular modes for which \( k < 1/2 \) [6]. Here we restrict ourselves to write down only the energy eigenspinors of the regular modes on CAdS.

Let us consider first \( \kappa = |\kappa| \) and choose \( 2s_+ = \kappa \) and \( 2s_- = \kappa + 1 \) for which Eq.(57) is accomplished provided \( n_+ = n_r \) and \( n_- = n_r - 1 \). Then the functions \( \hat{f}^\pm \) given by Eqs.(52) represent a solution of the transformed radial problem (49) only if
\[ \frac{N_-}{N_+} = \frac{2n_r}{2|\kappa| + 1} \quad . \quad (58) \]
These can be expressed in terms of Jacobi polynomials as
\[
\hat{f}^+_{n_r,|\kappa|}(r) = N \sin^{2|\kappa|} \omega r \cos^k \omega r P_{n_r}^{(|\kappa|+\frac{1}{2},\frac{1}{2})}(\cos 2\omega r) ,
\]
\[
\hat{f}^-_{n_r,|\kappa|}(r) = N \sin^{2|\kappa|+1} \omega r \cos^{k+1} \omega r P_{n_r-1}^{(|\kappa|+\frac{1}{2},k+\frac{1}{2})}(\cos 2\omega r) ,
\]
We specify that from Eq.(58) we understand that the second equation of (59) gives \( \hat{f}^- = 0 \) when \( n_r = 0 \). For \( \kappa = -|\kappa| \) we use the same procedure finding that \( 2s_+ = |\kappa| + 1, 2s_- = |\kappa|, n_+ = n_- = n_r \) and
\[ \frac{N_-}{N_+} = -\frac{2|\kappa| + 1}{2n_r + 2k + 1} \quad . \quad (60) \]
The corresponding normalized radial wave functions are
\[
\hat{f}^+_{n_r,-|\kappa|}(r) = N \left[ \frac{n_r + k + \frac{1}{2}}{n_r + |\kappa| + \frac{1}{2}} \right]^\frac{1}{2}...
\]

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\[
\hat{f}_{n_r, |\kappa|}(r) = -N \left[ \frac{n_r + |\kappa| + \frac{1}{2}}{n_r + k + \frac{1}{2}} \right]^{\frac{1}{2}} \\
\times \sin^{|\kappa|} \omega r \cos^{k+1} \omega r P^{|\kappa|+\frac{1}{2}, k+\frac{1}{2}}_{n_r}(\cos 2\omega r).
\]

In both cases, the general normalization factor calculated from Eq.(55) is

\[
N = \sqrt{2\omega} \left[ \frac{n_r! \Gamma(n_r + k + |\kappa| + 1)}{\Gamma(n_r + |\kappa| + \frac{1}{2}) \Gamma(n_r + k + \frac{1}{2})} \right]^{\frac{1}{2}}.
\]

Finally, using the inverse transformation of (47) we obtain the radial wave functions

\[
f_{n_r, \kappa}^\pm(r) = \hat{f}_{n_r, \kappa}^\pm(r) \cos \frac{\omega r}{2} + \hat{f}_{n_r, \kappa}^\mp(r) \sin \frac{\omega r}{2}
\]

of the particular solutions of positive frequency (36), (37) and (38) of the reduced Dirac equation. The last step is to restore the form of the field \(\psi\), according to Eqs.(25), (26) and (45), writing down the particular solutions of positive frequency of the original Dirac equation,

\[
\psi_{n_r, \kappa, (j)}^{pa}(t, x) = \left( \frac{\omega r}{\sin \omega r} \right)^{\frac{d+1}{2}} \cos^{\frac{d}{2}} \omega r \tilde{\psi}_{n_r, \kappa, (j)}(t, x).
\]

If these are interpreted as particle-like solutions then the antiparticle-like ones can be derived directly through the charge conjugation as

\[
\psi_{n_r, \kappa, (j)}^{ap}(t, x) = C \left( \psi_{n_r, \kappa, (j)}^{pa} \right)^T,
\]

where the matrix \(C\) is defined by Eq.(70).

The energy levels result from Eq.(56) where we must take into account that \(\omega k = m\) and \(\omega e = E - \omega/2\). Thus we obtain

\[
E_{n_r, \kappa} = \begin{cases} 
  m + \omega(2n_r + \kappa + \frac{1}{2}) & \text{for } \kappa = |\kappa| \\
  m + \omega(2n_r + |\kappa| + \frac{3}{2}) & \text{for } \kappa = -|\kappa| 
\end{cases}
\]

which suggests us to introduce the principal quantum number

\[
\tilde{n} = \begin{cases} 
  2n_r + |\kappa| - \frac{d-1}{2} = 2n_r + l & \text{for } \kappa = |\kappa| \\
  2n_r + |\kappa| - \frac{d-1}{2} + 1 = 2n_r + l + 1 & \text{for } \kappa = -|\kappa| 
\end{cases}
\]
taking the values $0, 1, 2, \ldots$ since $l$ ranges as in Eq.(39) and $n_r = 0, 1, 2, \ldots$. With its help we can write the compact formula of the energy levels

\[ E_n = m + \omega \left( n + \frac{d}{2} \right), \quad n = 0, 1, 2, \ldots \]  

(68)

which is similar to that of the $d$-dimensional homogeneous harmonic oscillator but having a relativistic rest energy. These energy levels are deeply degenerated and the problem of determining the degree of degeneracy seems to be complicated requiring a special study.

6 Conclusions

We demonstrated that our Cartesian gauge in central charts leads to simple reduced Dirac equations in Cartesian coordinates and similar radial problems in spherical coordinates for any central background $M_{d+1}$. These reduce to a pair of radial equations that depend on $d$ only through the quantum number $\kappa$. This property may allow one to solve the radial problem for large sets of spacetimes of the same type and to study how depend the quantum modes on $d$.

In the case of the $CAdS_{d+1}$ spacetimes we found that the energy spectrum of the Dirac field minimally coupled with gravity is discrete and equidistant like that of the scalar field but the ground state energy is quite different. We remind the reader that the energy levels of the scalar free field are [4]

\[ E_{sc}^{n} = \sqrt{m^2 + \frac{d^2}{4} \omega^2 + \omega \left( n + \frac{d}{2} \right)}, \quad n = 0, 1, 2, \ldots . \]  

(69)

The difference is due to the fact that in curved spacetimes the free Dirac equation is no more the square root of the Klein-Gordon equation with the same mass. This gives rise to new problems concerning the physical interpretation of the mass terms in theories where the matter fields are minimally coupled with gravity. In other respects, our result has consequences even in holography [2] since this indicates that the conformal dimensions of the boundary Klein-Gordon and Dirac (or Majorana) fields of the ADS/CFT conjecture may be different.
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A Parity and charge conjugation

In the theory of the Dirac field in curved spacetimes $M_{d+1}$ the discrete transformations can be defined in a similar way as in special relativity. Thus the parity changes $x \rightarrow -x$ and $\psi(x,t) \rightarrow \gamma^0 \psi(-x,t)$ leaving the reduced Dirac equation invariant. The form of the charge conjugation depends on the representation of the Dirac matrices. These satisfy Eqs.(2) and, consequently, they must be either symmetric or skew-symmetric. Let us assume that there are $s+1$ symmetric $\gamma$-matrices, namely $\gamma^0$ and $s$ matrices with space-like indices, $\gamma^{\hat{\alpha}_1}, \gamma^{\hat{\alpha}_2}, ..., \gamma^{\hat{\alpha}_s}$. Then the matrix

$$C = (-1)^{\frac{s(s+1)}{2}} \gamma^0 \gamma^{\hat{\alpha}_1} \gamma^{\hat{\alpha}_2}, ..., \gamma^{\hat{\alpha}_s}$$

has the properties $C^{-1} = C^T = (-1)^s C^T$ and $C \gamma^{\hat{\mu}} C^{-1} = (-1)^s (\gamma^{\hat{\mu}})^T$. With its help one can define the charge conjugated spinor $\psi^c = C \overline{\psi}^T$ of the spinor $\psi$ and verify that the reduced Dirac equation given by the operator (27) remains invariant under the charge conjugation $\psi \rightarrow \psi^c$. Note that this transformation is point-independent which suggests that the vacuum state could be stable (or invariant [11]) in quantum field theories based on field equations invariant under this type of charge-conjugation. Particularly, it is known that the Euclidean vacuum of the $AdS$ spacetime is invariant [3, 11].

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