Spinning ring wormholes:
a classical model for elementary particles?∗

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Abstract

Static horizonless solutions to the Einstein–Maxwell field equations, with
only a circular cosmic string singularity, are extended to exact rotating asymp-
totically flat solutions. The possible interpretation of these field configura-
tions as spinning elementary particles or as macroscopic rotating cosmic rings
is discussed.

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1 Introduction

The classical Einstein–Maxwell field equations coupling gravity to electromagnetism admit outside sources a variety of stationary axially symmetric solutions, among which the Kerr–Newman black hole solutions \([1]\) depending on three parameters \(M, Q\) and \(J\) which, from the consideration of the asymptotic behaviour of these field configurations, may be identified as their total mass, charge, and angular momentum. The Kerr–Newman metrics correspond to regular black-hole spacetimes only if

\[ M^2 \geq Q^2 + a^2, \]  

(1.1)

where \(a \equiv J/M\). A fourth physical characteristic, the total magnetic moment \(\mu\), is determined from the three others by the relation giving the gyromagnetic ratio

\[ g \equiv 2 \frac{M\mu}{JQ} = 2. \]  

(1.2)

In the case of elementary particles, the gyromagnetic ratio has the same relativistic (“anomalous”) value \(g = 2\). However their angular momentum and charge are of the order \(J \sim m_P^2\) (where \(m_P = (\hbar c/G)^{1/2}\) is the Planck mass) and \(Q \sim m_P\), so that

\[ a/Q \sim Q/M \sim m_P/m_e \sim 10^{22} \]  

(1.3)

(where \(m_e\) is the electron mass), and therefore

\[ M^2 < Q^2 + a^2. \]  

(1.4)

So the field configurations generated by elementary particles cannot be of the black hole type, but must necessarily exhibit naked singularities, contrary to the cosmic censorship paradigm \([2]\). It would therefore seem that there is no viable classical model for elementary particles (except possibly for neutral spinless particles) in the framework of the Einstein–Maxwell field theory.

In the charged spinless case \(J = 0\), the relation \(Q/M \sim m_P/m_e\) tells us that electromagnetism is preponderant, and that the naked point singularity in the spherically symmetric metric originates from that of the Coulomb central field. This point singularity is ultimately responsible for the divergences which plague both classical and quantum electrodynamics. A way to regularize these divergences is to replace the zero-dimensional point particles of traditional field theory by the one-dimensional fundamental objects of string theory.

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1 We use gravitational units \(G = 1, c = 1\).
String–like objects also occur as classical solutions to field theories with spontaneously broken global or local symmetries. Such symmetry breaking transitions are believed to have occurred during the expansion of the universe, leading to the formation of large, approximately straight cosmic strings. The long–range behavior of the metric generated by a straight cosmic string is given by an exact stationary solution of the vacuum Einstein equations with a line source carrying equal mass per unit length and tension. In the case of closed cosmic strings, or rings, this tension will cause the string to contract, precluding the existence of stationary solutions, unless the tension is balanced by other forces. A possibility is that of vortons, rotating loops of superconducting current–carrying string stabilized by the centrifugal force. Another possibility has been advocated by Bronnikov and co–workers, that of ring wormhole solutions to multi–dimensional field models. In the case of these static solutions, the gauge field energy–momentum curves space negatively to produce a wormhole, at the neck of which sits a closed cosmic string, which cannot contract because its circumference is already minimized.

In this talk, I first present a simple derivation of static ring wormhole solutions to the Einstein–Maxwell field equations, in the framework of the Ernst formalism. Then, using a recently proposed spin–generating method, I construct from these static solutions exact rotating Einstein–Maxwell cosmic ring solutions. Finally I discuss the spacetime geometry of these solutions, and their possible interpretation as elementary particles or as macroscopic cosmic rings.

2 Static Einstein–Maxwell ring wormholes

Let me first review the Ernst reduction of the stationary Einstein–Maxwell field equations. Under the assumption of a timelike Killing vector field $\partial_t$, these solutions may be parametrized by the metric

$$ds^2 = f (dt - \omega_i dx^i)^2 - f^{-1} d\sigma^2, \quad d\sigma^2 = h_{ij} dx^i dx^j,$$

and the electromagnetic fields

$$F_{i0} = \partial_i v, \quad F^{ij} = f h^{-1/2} \epsilon^{ijk} \partial_k u.$$  (2.2)

where the scalar potentials $f$, $v$, $u$, the vector potential $\omega_i$ and the reduced spatial metric $h_{ij}$ depend only on the three space coordinates $x^i$. Using the Einstein–Maxwell equations, the vector potential $\omega_i$ may be dualized to the scalar twist potential $\chi$ defined by

$$\partial_i \chi = -f^2 h^{-1/2} h_{ij} \epsilon^{jkl} \partial_k \omega_l + 2(u \partial_i v - v \partial_i u).$$  (2.3)
The complex Ernst potentials are defined in terms of the four real scalar potentials $f$, $\chi$, $v$ and $u$ by

$$E = f + i\chi - \bar{\psi}\psi, \quad \psi = v + iu.$$  \hfill (2.4)

The stationary Einstein–Maxwell equations then reduce to the three–dimensional Ernst equations

\begin{align*}
  f\nabla^2 E &= \nabla E \cdot (\nabla E + 2\bar{\psi}\nabla \psi), \hfill (2.5) \\
  f\nabla^2 \psi &= \nabla \psi \cdot (\nabla E + 2\bar{\psi}\nabla \psi), \hfill (2.6) \\
  f^2 R_{ij}(h) &= \text{Re} \left[ \frac{1}{2} E_{(i} \bar{E}_{j)} + 2\psi E_{(i} \bar{\psi}_{j)} - 2\bar{\psi} \psi_{(i} \bar{\psi}_{j)} \right], \hfill (2.7)
\end{align*}

where the scalar products and Laplacian are computed with the metric $h_{ij}$. These equations, which are invariant under an SU(2,1) group of transformations \[11\], are those of a gravitating SU(2,1) $\sigma$ model. Electrostatic solutions correspond to real potentials $E$ and $\psi$. In this case it is well known \[12\] that if $E$ and $\psi$ are functionally related, this relation is necessarily linear and can be reduced, by a gauge transformation $$(E, \psi) \rightarrow (E_0, \psi_0),$$

to

$$E_0(x) = E_0$$  \hfill (2.8)

constant; then equation (2.5) is identically satisfied, while Eq. (2.6) reduces to

$$\nabla^2 \psi_0 = 2 \frac{\psi_0}{E_0 + \psi_0^2} (\nabla \psi_0)^2.$$  \hfill (2.9)

Because the metric of the target space SU(2,1)/SU(2)×U(1)) is indefinite, the electrostatic case can be divided in three equivalence classes $E_0 < 0$, $E_0 = 0$, and $E_0 > 0$. Representative solutions of these three classes depending on a single real potential are

$$
E_0 = -1, \quad \psi_0 = \coth(\sigma), \quad f_0 = 1/\sinh^2 \sigma \\
E_0 = 0, \quad \psi_0 = 1/\sigma, \quad f_0 = 1/\sigma^2 \\
E_0 = +1, \quad \psi_0 = \cot(\sigma), \quad f_0 = 1/\sin^2 \sigma$$  \hfill (2.10)

where the potential $\sigma(x)$ is harmonic

$$\nabla^2 \sigma = 0.$$  \hfill (2.11)

We note that the electric and gravitational potentials (2.10) are singular for $\sigma = 0$ if $E_0 = -1$ or 0, and for $\sigma = n\pi$ (n integer) if $E_0 = +1$. Other electrostatic
solutions depending on a single potential may be obtained from these by SU(2,1) transformations.

As we wish to obtain axisymmetric ring–like solutions, we choose oblate spheroidal coordinates \([x, y]\), related to the usual Weyl coordinates \((\rho, z)\) by
\[
\rho = \nu (1 + x^2)^{1/2} (1 - y^2)^{1/2}, \\
z = \nu xy.
\] (2.12)

In these coordinates, the three–dimensional metric
\[
d\sigma^2 = \nu^2 \left[ e^{2k} (x^2 + y^2) \left( \frac{dx^2}{1 + x^2} + \frac{dy^2}{1 - y^2} \right) + (1 + x^2)(1 - y^2) d\varphi^2 \right]
\] (2.13)
depends on the single function \(k(x, y)\). Now, following Bronnikov et al. [7], we assume the harmonic potential \(\sigma\) to depend only on the variable \(x\), which yields
\[
\sigma = \sigma_0 + \alpha \arctan x,
\] (2.14)
where \(\sigma_0\) and \(\alpha\) are integration constants, and
\[
e^{2k} = \left( \frac{1 + x^2}{x^2 + y^2} \right) \mathcal{E}_0 \alpha^2.
\] (2.15)

We note that the reflection \(x \leftrightarrow -x\) is an isometry for the three–dimensional metric (2.13), which has two points at infinity \(x = \pm \infty\). The full four–dimensional metric (2.1) is quasi–regular (i.e. regular except on the ring \(x = y = 0\), see below) for \(x \in \mathbb{R}\) if
\[
|\sigma_0| > |\alpha| \pi/2 \quad \text{for } \mathcal{E}_0 = -1 \text{ or } 0, \\
(n + |\alpha|/2)\pi < \sigma_0 < (n + 1 - |\alpha|/2)\pi \quad (|\alpha| < 1) \quad \text{for } \mathcal{E}_0 = +1
\] (2.16)
for some integer \(n\). If these conditions are fulfilled, this metric describes a wormhole spacetime with two asymptotically flat regions connected through the disk \(x = 0\) \((z = 0, \rho < \nu)\). There is no horizon. The point singularity of the spherically symmetric (Reissner–Nordström) solution is here spread over the ring \(x = y = 0\) \((z = 0, \rho = \nu)\), near which the behavior of the spatial metric
\[
d\sigma^2 = \nu^2 [(x^2 + y^2)^{1-\mathcal{E}_0 \alpha^2} (dx^2 + dy^2) + d\varphi^2]
\] (2.17)
is that of a cosmic string with deficit angle \(\pi(\mathcal{E}_0 \alpha^2 - 1)\), which is negative in all cases of interest (it can be positive only for \(\mathcal{E}_0 = +1, |\alpha| > 1\), corresponding to a singular
solution). This ring singularity disappears in the limit of a vanishing deficit angle ($E_0 = +1, |\alpha| \to 1$), where the solution reduces to a Reissner–Nordström solution with naked point singularity. The asymptotic behaviours of the gravitational and electric potentials at the two points at infinity are those of particles with masses and charges

$$M_* = \mp \alpha \nu \frac{\psi_0(\pm \infty)}{\sqrt{f_0(\pm \infty)}}, \quad Q_* = \pm \alpha \nu; \quad (2.18)$$

the three cases (2.10) lead respectively to $Q_*^2 < M_*^2$ for $E_0 = -1$, $Q_*^2 = M_*^2$ for $E_0 = 0$, and $Q_*^2 > M_*^2$ for $E_0 = +1$. The vanishing of the sum of the outgoing electric fluxes at $x = \pm \infty$ shows that the ring $x = y = 0$ is uncharged.

In the case $E_0 = -1$, the charged ring solution may be transformed by SU(2,1) transformations to the neutral ring solution \[13\]

$$E_0 = f_0 = e^{-2\sigma}, \quad \psi_0 = 0. \quad (2.19)$$

Another case of special interest is $E_0 = +1$, $\sigma_0 = \pi/2$, corresponding to a symmetrical wormhole metric, which can be interpreted as describing a massive charged particle living in a two-sheeted spacetime, the two sheets of which are related by charge conjugation \[14\]. The mass of this particle $M_* = \alpha \nu \sin(\alpha \pi/2)$ does not depend on the point at infinity considered, and is positive, even though the deficit angle is negative. For the physical characteristics of this particle to be those of a spinless electron, we should take $|\alpha| \sim m_e/m_P$, and $\nu \sim m_P^2/m_e$, of the order of the classical electron radius.

### 3 Generating spinning ring wormholes

A simple procedure which generates from any asymptotically flat static axisymmetric solution of the Einstein–Maxwell equations a family of asymptotically flat spinning solutions has recently been proposed \[8\]. As generalized in \[13\], this procedure $\Sigma$ involves three successive transformations:

1) The electrostatic solution (real potentials $E, \psi, e^{2k}$) is transformed to another electrostatic solution ($\hat{E}, \hat{\psi}, e^{2k}$) by the SU(2,1) involution $\Pi$:

$$\hat{E} = \frac{-1 + \mathcal{E} + 2\psi}{1 - \mathcal{E} + 2\psi}, \quad \hat{\psi} = \frac{1 + \mathcal{E}}{1 - \mathcal{E} + 2\psi}, \quad e^{2k} = e^{2k}. \quad (3.1)$$

In the case of asymptotically flat fields with the large distance ($r \to +\infty$) monopole behavior $f \simeq f(\infty)(1 - 2\sqrt{f(\infty)}M/r)$, $\psi \simeq \psi(\infty) + f(\infty)Q/r$ ($r$ being the radial distance associated with the reduced spatial metric $h_{ij}$), if the gauge is chosen so
that \( f(\infty) = (1 + \psi(\infty))^2 \), then the asymptotic behaviors of the resulting gravitational potential

\[
\hat{f} = \frac{f}{|F|^2}, \quad F = \frac{1}{2}(1 - \mathcal{E} + 2\psi),
\]

and electric potential \( \hat{\psi} \) are those of the Bertotti–Robinson solution \([14]\), \( \hat{\psi} \approx r/f(\infty)(M + Q) \), \( \hat{f} \approx r^2/f(\infty)^2(M + Q)^2 \).

2) The static solution \((\hat{\mathcal{E}}, \hat{\psi}, e^{2k})\) is transformed to a uniformly rotating frame by the global coordinate transformation

\[
d\phi = d\phi' + \Omega dt', \quad dt = dt',
\]
leading to the gauge–transformed complex fields

\[
\hat{\mathcal{E}}' = \hat{\mathcal{E}} - \Omega^2 \left( \frac{\rho^2}{f} + \hat{\phi}^2 \right) + 2i\Omega (z + \hat{\mathcal{F}} + \hat{\psi}\hat{\phi}),
\]

\[
\hat{\psi}' = \hat{\psi} + i\Omega \hat{\phi}, \quad e^{2k'} = \left( 1 - \Omega^2 \frac{\rho^2}{f^2} \right) e^{2k},
\]

where \( \rho \) and \( z \) are Weyl coordinates,

\[
d\sigma^2 = e^{2k} (d\rho^2 + dz^2) + \rho^2 d\phi'^2,
\]

and \( \hat{\mathcal{F}}, \hat{\phi} \) are the dual Ernst potentials defined in Weyl coordinates by

\[
\hat{\mathcal{F}}_{,m} = \frac{\rho}{f} \epsilon_{mn} \hat{\mathcal{E}}_{,n}, \quad \hat{\phi}_{,m} = \frac{\rho}{f} \epsilon_{mn} \hat{\psi}_{,n}
\]

\((m,n = 1,2, \text{with } x^1 = \rho, x^2 = z)\). While such a transformation on an asymptotically Minkowskian metric leads to a non–asymptotically Minkowskian metric, it does not modify the leading asymptotic behavior of the Bertotti–Robinson metric, so that the fields \((3.4)\) are again asymptotically Bertotti–Robinson.

3) The solution \((\hat{\mathcal{E}}', \hat{\psi}', e^{2k'})\) is transformed back by the involution \( \Pi \) to a solution \((\mathcal{E}', \psi', e^{2k'})\) which is, by construction, asymptotically flat, but now has asymptotically dipole magnetic and gravimagnetic fields. As shown in \([3]\), the combined transformation \( \Sigma \) transforms the Reissner–Nordström family of solutions into the Kerr–Newman family.

The static ring wormhole solutions of the preceding section have two distinct asymptotically flat regions \( x \to \pm \infty \). Clearly, the application of the general spin–generating procedure \( \Sigma \) to such wormhole spacetimes \((\mathcal{E}_0, \psi_0(x))\) requires first selecting a particular region at infinity, e. g. \( x \to +\infty \), and carrying out a gauge
transformation

\[ \mathcal{E}(x) = c^2 \mathcal{E}_0 - 2cd\psi_0(x) - d^2, \quad \psi(x) = c\psi_0(x) + d \quad (3.7) \]

\((f(x) = c^2 f_0(x))\), depending on two parameters \(c\) and \(d\) constrained so that \(f(+\infty) = (1 + \psi(+\infty))^2\), i.e.

\[ c^2 \mathcal{E}_0 - 2b\psi_0(+\infty) - b^2 = 0 \quad (3.8) \]

\((b \equiv d+1)\). The limiting values of the Ernst potentials will be generically different at the other point at infinity \(x \to -\infty\), so that the function \(F\) in (3.2) will not go there to zero but to a constant value. Consequently, for \(x \to -\infty\) the fields \((\hat{\mathcal{E}}, \hat{\psi})\) will not be asymptotically Bertotti–Robinson, and therefore the final fields \((\mathcal{E}', \psi')\) will no longer be asymptotically Minkowskian. A perturbative approach to the generation of slowly rotating ring wormhole solutions from the static symmetrical wormhole [1] shows, independently of the present construction, that this asymmetry between the two points at infinity is a necessary feature of spinning ring wormholes.

The transformation (3.1) on the static solution (3.7) leads to the transformed Ernst potentials and associated dual potentials,

\[ \hat{\mathcal{E}} = \frac{2 - b}{b} + \frac{2(k - 1)}{cb\tilde{\psi}_0}, \quad \hat{\psi} = \frac{1 - b}{b} + \frac{k}{cb\tilde{\psi}_0}, \]

\[ \hat{F} = -\nu \frac{b}{c} 2(k - 1)y, \quad \hat{\phi} = -\nu \frac{b}{c} ky, \quad (3.9) \]

with \(k \equiv 1 + \psi(+\infty) = (c^2 \mathcal{E}_0 + b^2)/2b\) and \(\tilde{\psi}_0(x) = \psi_0(x) - \psi_0(+\infty) = (\psi(x) + 1 - k)/c\). The steps 2 and 3 above then lead to the spinning Ernst potentials

\[ 1 + \mathcal{E}' = \frac{1 + \mathcal{E} - 2\Omega \nu \alpha \Delta \tilde{\psi}_0 y}{\Delta}, \quad 1 + \psi' = \frac{1 + \psi - i\Omega \nu \alpha \Delta \tilde{\psi}_0 y}{\Delta}, \]

\[ \Delta = 1 + \Omega^2 \nu^2 cb\tilde{\psi}_0((\alpha^2 k^2 b^2/2c^2 + \xi(1 - y^2)) - i\Omega \nu cb\eta \tilde{\psi}_0 y) \quad (3.10) \]

\((\Delta = \hat{F}'/\hat{F})\) where \(\hat{f}(x) = f_0(x)/b^2 \tilde{\psi}_0^2(x)\), and

\[ \xi = (1 + x^2)/2\hat{\phi} - \alpha^2 k^2 b^2/2c^2, \quad \eta = x + \alpha(2b - k)/c - \alpha k^2/c^2 \tilde{\psi}_0. \quad (3.11) \]

To recover the spacetime metric from the Ernst potentials (3.10), we must solve the duality equation (2.3) relating the metric function \(\omega\) to \(\chi' = \text{Im}\mathcal{E}'\). This is achieved by computing according to (2.3) the partial derivative \(\partial_\nu \omega(x, y)\), which is a rational function of \(y\), and integrating it with the boundary condition \(\omega(x, \pm 1) = 0,\)
which ensures regularity on the axis \( \rho = 0 \). The resulting spacetime metric is of the form (2.1), (2.13) with the metric functions

\[
\begin{align*}
    f' &= |\Delta|^{-2} (1 - \Omega^2 \rho^2 / f^2) f, \\
    \omega'_\varphi &= \Omega \nu^2 (1 - y^2) \\
    \omega'_\rho &= \frac{|\Delta_0|^{3/2} (1 + x^2)}{c^2 b^2 \psi_0^2 (f^2 - \Omega^2 \rho^2)} - \frac{f^2 \xi^2}{1 + x^2} + \eta (\eta - \frac{b}{c}),
\end{align*}
\]

(3.12)

where \( \Delta_0(x) = \Delta(x, y_0(x)) \) with \( 1 - y_0^2 = \hat{f}^2 / \Omega^2 \nu^2 (1 + x^2) \). It is easily checked that this metric is invariant under the combined parameter rescaling

\[
c \to \lambda c, \ b \to \lambda b, \ k \to \lambda k, \ \Omega \nu \to \lambda^{-2} \Omega \nu, \ \nu \to \lambda \nu
\]

together with a time rescaling \( t \to \lambda^{-1} t \). Hence we can choose without loss of generality \( k = 1 \) (\( \psi(+\infty) = 0, \ f(+\infty) = 1 \)) so that the parameters \( b \) and \( c \) are determined by \( \alpha \) and \( \sigma_0 \) (\( c^2 = 1/f_0(+) = b = 1 - c \psi_0(+\infty) \)), and in the above \( \tilde{\psi}_0(x) = c^{-1} \psi(x) \).

This stationary solution is singular if the denominator \( \Delta \) in (3.10) vanishes. The imaginary part of this function vanishes on the surface \( y = 0 \), where its real part takes the form

\[
\Delta(x, 0) = 1 + \frac{\Omega^2 \nu^2 b^3 (1 + x^2) \psi^3(x)}{2 f(x)}.
\]

(3.13)

The zeroes of this function thus correspond to strong ring singularities (similar to the ring singularity of the Kerr metric) of our solution. From the asymptotic behavior

\[
\psi(x) \simeq \frac{\alpha}{cx} \quad (x \to +\infty)
\]

(3.14)

we see that (3.13) is dominated for \( x \to +\infty \) by its first term +1, while for \( x \to -\infty \), \( \psi(x) \) goes to a constant value, and (3.13) is dominated by its second term, which is of the sign of \( abc \). Hence, a necessary condition for the absence of such singularities is

\[
abc > 0.
\]

(3.15)

In the cases \( E_0 = -1 \) or 0, it follows from Eq. (3.8) that \( sign \ bc = -sign \ \psi_0(+\infty) \), so that the necessary regularity condition (3.13) is satisfied if the static mass \( M_s \) is positive, which also ensures that the static regularity condition (2.16) is satisfied, and hence, using

\[
\frac{d \psi_0}{dx} = -\frac{\alpha \nu f_0}{1 + x^2},
\]

(3.16)
that $\psi(x)$, and thus also $\Delta(x,0)$, keep a constant sign over the whole real axis. Therefore the positivity of the static mass ensures the quasi–regularity of the spinning solution (3.10).

In the case $\xi_0 = +1$, the product of the two roots of Eq. (3.8) for the ratio $b/c$ is negative, so that the sign of $bc$ can always be chosen to be equal to that of $\alpha$. If $|\alpha| > 1$ (the case for which the static solution is always singular), the range of $\sigma(x)$ is larger than $\pi$, so that $c/\psi(x) = (\cot \sigma(x) - \cot \sigma(\infty))^{-1}$ varies from $+\infty$ at $(x \to +\infty)$ to $-\infty$ (for some finite value of $x$), leading (whatever the sign of $bc$) to a simple zero of (3.13) and thus to a strong ring singularity of the spinning solution. If $|\alpha| < 1$, the range of $\sigma(x)$ is smaller than $\pi$. Then (choosing $abc > 0$), the quasi–regularity condition (2.16) also ensures, as in the cases $\xi_0 = -1$ or 0, the quasi–regularity of the spinning solution. However, even if this condition is not satisfied so that $\psi(x)$ changes sign somewhere, $\Delta(x,0)$ is positive both for $x \to +\infty$ and for $x \to -\infty$, so that it can have two zeroes or none, depending on the value of $\Omega^2$. A singular static solution therefore leads to a quasi–regular spinning solution if $\Omega^2$ becomes larger than a certain critical value $\Omega_c^2(\alpha, \sigma_0)$.

4 Discussion

It is obvious from the form of the metric (3.12) that it is still singular on the rotating cosmic ring $x = y = 0$, with the same deficit angle $\pi(\xi_0 \alpha^2 - 1)$ as in the static case. The gravitational potential $f'$ vanishes on the stationary limit surfaces $\hat{f}(x) = \pm \Omega \rho$, where the full metric is regular. However the spinning solution (3.12) is horizonless, just as the corresponding static solution [15].

This spinning solution is by construction asymptotically flat for $x \to +\infty$, but (as expected) not for $x \to -\infty$, where the metric has the asymptotic behavior

$$ds^2 \simeq -l^{-2} \rho^{-2} (dt + (\Omega/4)(\rho^2 + 4z^2) d\varphi)^2 - 16\Omega^{-2} l^6 \rho^4 (d\rho^2 + dz^2) + l^4 \rho^4 d\varphi^2$$

(4.1)

(with $l^2 = \Omega/4 \hat{f}(-\infty)$), which can be viewed as the asymptotic form of a rotating Melvin–like [18] solution.

To elucidate the nature of the apparent singularity for $x \to -\infty$ ($\rho \to \infty$), we study geodesic motion in the exact metric (3.12). The first–integrated geodesic equation of motion may be written as

$$h^2(\rho^2 + z^2) + V = 0,$$

(4.2)

with

$$h^2(\rho, z) = f'^{-1} e^{2k'} > 0,$$
\[ V(\rho, z) = (L + E\omega')^2 \frac{f'}{\rho^2} + \eta - \frac{E^2}{f'}, \quad (4.3) \]

where \(E\) and \(L\) are the constants of motion associated with the Killing vectors \(\partial_t\) and \(\partial_\varphi\), and \(\eta = +1, -1\) or 0 for timelike, null or spacelike geodesics. For \(E \neq 0\), the effective potential \(V\) has a pole at the stationary limit \(f'(x, y) = 0\), reflecting test particles coming from the asymptotically flat region \(x \to +\infty\), and (from (4.1)) a parabolic barrier behavior for \(x \to -\infty\). These potential barriers disappear for \(E = 0\), \(\eta \leq 0\), in which case all geodesics extend to the sphere \(x \to -\infty\), which is at infinite affine distance. In the case \(E = 0\), however, timelike or null geodesics (\(\eta \geq 0\)) do not extend to \(x \to +\infty\). The conclusion is that all test particles coming from \(x \to +\infty\) are eventually reflected back to \(x \to +\infty\), so that there is no loss of information to \(x \to -\infty\).

The mass, angular momentum, charge, and magnetic dipole moment associated with the rotating ring solution may be read from the multipole expansion of the Ernst potentials near \(x \to +\infty\)

\[
\begin{align*}
\mathcal{E}' &= 1 - \frac{2M}{\nu x} - \frac{2iJy}{\nu x^2} + \cdots \quad (x \to +\infty), \\
\psi' &= \frac{Q}{\nu x} + \frac{iy}{\nu x^2} + \cdots \quad (x \to +\infty),
\end{align*}
\]

A careful computation leads to the values of these parameters

\[
\begin{align*}
M &= (\nu\alpha/c)(b - 1 + \tau), & J &= \nu\beta(M + \delta), \\
Q &= (\nu\alpha/c)(1 - \tau), & \mu &= \nu\beta(Q - \delta),
\end{align*}
\]

with \(\beta = \Omega\nu\alpha^2b^2/c^2\), \(\tau = \beta^2c^2/2\alpha^2b\), \(\delta = \nu c(1 - \mathcal{E}_0/3ab)\).

Under what conditions can these values correspond to those of elementary particles? Combining the above values we obtain

\[ M^2 - Q^2 - a^2 = \nu^2(\beta^2 - \mathcal{E}_0\alpha^2) - a^2. \quad (4.6) \]

The quasi-regularity condition (3.13) implies \(\delta > 0\), so that \(|a| = |J|/M > \nu|\beta|\), hence a sufficient condition for the inequality (1.4) to be satisfied is \(\mathcal{E}_0 \geq 0\). The gyromagnetic ratio

\[ g = 2\frac{M(Q - \delta)}{Q(M + \delta)} \quad (4.7) \]

is never equal to 2, but can be very close to 2 for very small values of \(\delta\). One would then expect that the values of the independent free parameters \(\nu, \alpha, \beta\) and \(\tau\) may be adjusted so that the four physical parameters (4.5) take their elementary particle
values. However, the regularity constraint (2.16) strongly restricts the range of allowed parameter values. Specifically, the requirement \( g \simeq 2 \) can be satisfied if \( \mathcal{E}_0 \alpha^2 - 1 \simeq 0 \), implying \( \mathcal{E}_0 = +1 \) and \(|\alpha| \simeq 1\). The regularity constraint then implies \( c \approx \sin^2 \sigma (+\infty) \) (from Eq. (3.8)), must satisfy \( c < \varepsilon \pi \), with \( \varepsilon \equiv 1 - |\alpha| \). 

The physical parameters of such spinning ring “particles”, approximately given by

\[
M \simeq -Q \simeq \frac{\nu \alpha}{c} (\beta^2 - 1), \quad a \simeq \frac{\mu}{M} \simeq \nu \beta, \quad (4.8)
\]

are generically all of the same order, in contradistinction with the case (1.3) of ordinary elementary particles.

In the case of large quantum numbers (\(|J| \gg m_P^2\)), our classical solutions (3.12) describe macroscopic closed cosmic strings with negative deficit angle, but positive total mass. These cosmic strings satisfy the elementary particle constraint (1.4) if \( E_0 \geq 0 \). The exotic line source \( x = y = 0 \) is spacelike if its proper rotation velocity \( v_0 \) is smaller than 1. This velocity is determined by writing the line element as

\[
ds^2 = \kappa^2 dt^2 - \frac{\rho^2}{\kappa^2} (d\varphi + \frac{\kappa^2}{\rho} v dt)^2 - f'^{-1} e^{2 \kappa'} \nu^2 (x^2 + y^2) \left( \frac{dx^2}{1 + x^2} + \frac{dy^2}{1 - y^2} \right), \quad (4.9)
\]

with \( \kappa^2 = \rho^2 f'/(\rho^2 - f'^2 \omega'^2), v = f' \omega' / \rho \), which yields

\[
v_0 = \frac{f' \omega'}{\nu} (x = y = 0). \quad (4.10)
\]

On the stationary limit surfaces \( \hat{f} = \pm \Omega \rho \), where \( f' \) vanishes while \( \omega' \) has a pole, \( v = \Omega \rho / \hat{f} = \pm 1 \). So a necessary condition for the string to be spacelike is that it lie outside the stationary limit surfaces. There is a priori no reason for this condition to be consistent with the requirement \( g \simeq 2 \).

A specially interesting case is \( \tau = 1 \), corresponding to a neutral spinning cosmic ring (\( Q = 0 \)). We shall mainly consider the subcase \( \mathcal{E}_0 = 0, \tau = 1 \) which is more simple to study. The special case \( \mathcal{E}_0 = 0 \) is invariant under rescalings of the scalar potential \( \sigma \), so that the \( \tau = 1 \) solution actually depends only on two parameters, the scale \( \nu \) and the dimensionless parameter \( \beta \), with \( b = 2, \Omega = 1/\nu \beta \), and the physical parameters

\[
M = \nu |\beta|, \quad Q = 0, \quad J = \pm \frac{4\nu^2}{3} \beta^2, \quad \mu = \mp \frac{\nu^2}{3}, \quad (4.11)
\]

with \( \pm = \text{sign} \beta \). Numerical computation shows that the string is spacelike only in the range \( 0.29 < |\beta| < 0.40 \). The upper limit corresponds to the case \( |\beta| = 4/\pi^2 \) where the string lies on the stationary limit surface, with \( \kappa \simeq 2.15 \), while \( \kappa \) goes to
infinity at the lower limit. The string rotation velocity is always very close to 1 in this interval, \( v_0 \approx 0.992 \), and the string proper perimeter \( 2\pi\nu/k \approx 10M \) is of the order or smaller than its Schwarzschild radius. The value (4.11) of the magnetic moment can be understood as arising from a current intensity flowing through the superconducting cosmic ring, according to the classical formula \( \mu = IS \). Taking the classical area \( S \) spanned by the ring to be of the order of \( \nu^2 \), we estimate this intensity \( I \) to be of the order of the Planck intensity; a more precise value could be obtained by computing the source of the electromagnetic field deriving from the complex potential \( \psi' \) in (3.10). The subcase \( \mathcal{E}_0 = -1, \tau = 1 \) leads to somewhat similar results.

## 5 Conclusion

Starting from static ring wormhole solutions, we have generated quasi–regular, horizonless axisymmetric solutions to the Einstein–Maxwell field equations, with only a rotating cosmic ring singularity. These solutions depend on four parameters, the values of which can be chosen such that the elementary particle constraints (1.2) and (1.4) are satisfied. However, it turns out that for the “elementary” orders of magnitude \( J \sim Q^2 \sim m^2 P \), the mass of these objects cannot be small, but is also of the order of the Planck mass.

In the case of large quantum numbers \( (J \gg m^2 P) \), these solutions describe macroscopic, charged or neutral, current–carrying rotating cosmic rings or, in other words, self–gravitating vortons. The deficit angle at the ring is negative, but nevertheless the net total mass is positive. The constraints that the solution be quasi–regular, and that the exotic matter source on the ring be space–like, severely restrict the solution parameters, leading to rapidly rotating rings with a gyromagnetic ratio significantly different from 2.

This work should be extended in several directions. An important question, albeit one difficult to answer, is that of the linearization stability of the axisymmetric solutions presented here. Also, one would expect, from the analysis of [8], that static cosmic ring solutions should also exist in other gravitating field theories, such as Kaluza–Klein theory, or dilaton–axion gravity. Because the stationary field equations of these theories have a high degree of symmetry, it might then in principle [8] be possible to generate rotating cosmic ring solutions to these theories. Finally, it would certainly be interesting to obtain quasi–regular self–gravitating ring solutions with a positive deficit angle. In this respect, the mechanism discussed in Sect. 3, according to which a singular static ring solution can lead to a quasi–regular rotating ring solution if the (unphysical) angular “velocity” \( \Omega \) is large
enough — which recalls the mechanism responsible for the stability of vortons — might prove to be useful, and should be further investigated.

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