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**Higgs bosons in particle physics and in condensed matter**

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**Abstract** Higgs bosons – the amplitude modes – have been experimentally investigated in condensed matter for many years. An example is superfluid $^3\text{He}$-B, where the broken symmetry leads to 4 Goldstone modes and at least 14 Higgs modes, which are characterized by angular momentum quantum number $J$ and parity (Zeeman splitting of Higgs modes with $J = 2^+$ and $J = 2^-$ in magnetic field has been observed in 80’s). Based on the relation $E_{J+}^2 + E_{J-}^2 = 4\Delta^2$ for the energy spectrum of these modes, Yoichiro Nambu proposed the general sum rule, which relates masses of Higgs bosons and masses of fermions. If this rule is applicable to Standard Model, one may expect that the observed Higgs boson with mass $M_{H_1} = 125$ GeV has a Nambu partner – the second Higgs boson with mass $M_{H_2} = 325$ GeV. Together they satisfy the Nambu relation $M_{H_1}^2 + M_{H_2}^2 = 4M_{\text{top}}^2$, where $M_{\text{top}}$ is the top quark mass. Also the properties of the Higgs modes in superfluid $^3\text{He}$-A, where the symmetry breaking is similar to that of the Standard Model, suggest the possible existence of two electrically charged Higgs particles with masses $M_{H^+} = M_{H^-} \sim 245$ GeV, which together obey the Nambu rule $M_{H^+}^2 + M_{H^-}^2 = 4M_{\text{top}}^2$. A certain excess of events at 325 GeV and at 245 GeV has been reported in 2011, though not confirmed in 2012 experiments. Besides, we consider the particular relativistic model of top quark condensation that suggests the possibility that two twice degenerated Higgs bosons contribute to the Nambu sum rule. This gives the mass around 210 GeV for the Nambu partner of the 125 GeV Higgs boson. We also discuss the other possible lessons from the condensed matter to Standard Model, such as hidden symmetry, where light Higgs emerges as quasi Nambu-Goldstone mode, and the role of broken time reversal symmetry.

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1 Introduction

Condensed matter physics and particle physics use the same methods of quantum field theory and operate with similar phenomena. Typical example is the Anderson-Higgs mechanism of the formation of mass of gauge bosons, which has been discussed both in Standard Model of particle physics (SM) and in superconductors, where gauge symmetry is spontaneously broken. The gauge bosons become massive due to entanglement with Nambu-Goldstone (NG) bosons\textsuperscript{1,2,3,4}.

The Higgs amplitude modes – known as Higgs bosons – represent the other common objects. They have been first discovered in condensed matter: in superfluid $^3$He\textsuperscript{5,6} and later in superconductors\textsuperscript{7}. The discovery of the first Higgs boson in particle physics generated the new interest to their counterparts in condensed matter, see e.g. recent papers\textsuperscript{8,9,10,11,12} and references therein. We concentrate here mainly on Higgs bosons in superfluid $^3$He, which were studied for many years theoretically and experimentally and were served as inspiration for particle physics.

It was observed by Nambu\textsuperscript{13}, that in systems described by the BCS theory (superconductors, nuclear matter and especially superfluid $^3$He-B) there is a remarkable relation between the masses of the fermions and the masses of bosons. The collective bosonic modes emerging in the fermionic system – NG modes and Higgs amplitude modes – can be distributed into the pairs of Nambu partners. For each pair one has the relation,

$$M_1^2 + M_2^2 = 4M_f^2,$$

where $M_1$ and $M_2$ are gaps in the bosonic spectrum, and $M_f$ is the gap in the fermionic spectrum. In relativistic systems gaps in the energy spectrum corresponds to the mass of particles, which suggests that the masses of fermions and Higgs bosons in the relativistic theories, such as SM, can be related. Such relation exists for example in the Nambu - Jona - Lasinio (NJL) model\textsuperscript{14} of quantum chromo-dynamics, where it relates masses of the $\sigma$ - meson and of the constituent quark $M_\sigma \approx 2M_{\text{quark}}$.

We discuss the Nambu sum rule in $^3$He-B and in thin films of $^3$He-A in Sec.\textsuperscript{2} with application to SM Higgs bosons. In Sec.\textsuperscript{3} we discuss the effect of hidden symmetry, which leads to the relatively small mass of Higgs boson, which emerges as a quasi NG mode. The role of flat directions in the Higgs potential is discussed in Sec.\textsuperscript{4}. In Sec.\textsuperscript{5} the Nambu sum rule is extended to 3D $^3$He-A, where the spectrum of fermions is anisotropic and gapless. The role of broken time reversal symmetry in transformation of NG boson to the Higgs boson is discussed in Sec.\textsuperscript{6} on example of spin and orbital waves in ferromagnets and Kelvin waves on quantized vortices. Sec.\textsuperscript{7} is devoted to the consideration of the relativistic NJL model of top - quark condensation, where the Nambu sum rule naturally arises.
2 Nambu sum rule for SM Higgs bosons: hints from superfluid $^3$He

2.1 Higgs field and Higgs potential in superfluid $^3$He

Superfluidity in liquid $^3$He and superconductivity are based on the mechanism of Cooper pairing. The Higgs field appears as a composite object made of two fermions – two $^3$He atoms in superfluid $^3$He or two electrons in superconductors. The order parameter is the vacuum expectation value of the creation operator of two fermions, such as $\langle ee \rangle$ for Cooper pairing of electrons in superconductors.

In superfluid $^3$He the condensate is formed by Cooper pairs in the spin-triplet $p$-wave state. The order parameter (Higgs field) is $3 \times 3$ complex matrix $A_{ai}$, it transforms as a vector under spin rotation for given orbital index $(i)$ – and as a vector under an orbital rotation for given spin index $(a)$. The Ginzburg-Landau free energy functional – the Higgs potential – is invariant under the group $G = SO_3(3) \otimes SO_3(3) \otimes U(1)$ of spin, orbital and gauge rotations.  

$$F = -\alpha A_{ai}^* A_{ai} + \beta_i A_{ai}^* A_{ai}^* A_{aj} A_{bj} + \beta_2 A_{ai}^* A_{ai} A_{aj} A_{bj} + \beta_3 A_{ai}^* A_{ai}^* A_{aj} A_{aj} + \beta_2 A_{ai} A_{aj} A_{bj} A_{aj}.$$  

The approximate symmetry $SO_3(3) \otimes SO_3(3)$ with respect to separate spin and orbital rotations is similar to the so-called custodial symmetry in particle physics. It gives extra NG bosons in $^3$He-A and in $^3$He-B, which become Higgs bosons with a relatively small mass (Leggett frequency) due to a tiny spin-orbit interaction.

2.2 Higgs bosons in $^3$He-B

The B-phase of $^3$He is characterized by the quantum numbers $S = 1$, $L = 1$, $J = 0$ of spin, orbital momentum and total angular momentum respectively. This corresponds to the symmetry breaking scheme $G \rightarrow H$, where the symmetry of the degenerate vacuum states is $H = SO_J(3)$. The collective modes in the vicinity of an equilibrium degenerate state, chosen as $A_{ai}(\text{eq}) = \Delta \delta_{ai}$ with $\Delta$ being the gap in the fermionic spectrum, are propagating deviations of the Higgs field

$$A_{ai} - A_{ai}(\text{eq}) = u_{ai} + iv_{ai}.$$  

Altogether there are 18 real variables $u$ and $v$, and correspondingly 18 collective bosonic modes. They are classified by quantum numbers $J = 0, 1, 2$. Four modes are gapless NG bosons resulting from the symmetry breaking $G \rightarrow H$. This satisfies the conventional wisdom that the total number of NG modes = the number of broken symmetry generators $(7 - 3 = 4)$. The rest 14 bosons are amplitude modes – Higgs bosons with non-zero gaps. The energy gaps of bosons are given by:

$$E_{a,i}^{(J)} = \sqrt{2\Delta^2(1 \pm \eta^{(J)})},$$  

where parameters $\eta^{(J)}$ are determined by the symmetry of the system, $\eta^{J = 0} = \eta^{J = 1} = 1$, and $\eta^{J = 2} = \frac{1}{2}$. Eq. (4) illustrates the Nambu conjecture for $^3$He-B: the gaps of real and imaginary modes in each sector $J$ are related by equation

$$[E_{a,i}^{(J)}]^2 + [E_{i,j}^{(J)}]^2 = 4\Delta^2.$$  

(5)
The sector $J = 0$ contains one pair of the Nambu partners (the Higgs amplitude mode with gap $2\Delta$ – the pair-breaking mode, and the NG mode – sound wave):

$$E_u^{(0)} = 2\Delta, \quad E_v^{(0)} = 0.$$ (6)

For $J = 1$ there are 3 pairs (3 NG modes – spin waves, and 3 Higgs modes):

$$E_u^{(1)} = 0, \quad E_v^{(1)} = 2\Delta.$$ (7)

The sector $J = 2$ contains 10 Higgs bosons which form 5 Nambu pairs (5 real squashing modes + 5 imaginary squashing modes):

$$E_u^{(2)} = \sqrt{2/5} (2\Delta), \quad E_v^{(2)} = \sqrt{3/5} (2\Delta).$$ (8)

The 5-fold Zeeman splitting of the Higgs modes with $J = 2$ in magnetic field has been observed in 80's, for the latest experiments see 19.

Eq.(8), which supports the Nambu conjecture (1), may serve as a hint for SM. If the symmetry breaking in SM is related to the top quark condensate $\langle \bar{t}t \rangle$, then one may expect that the discovered Higgs boson with mass $M_{H_1} = 125$ GeV has the Nambu partner with mass $M_{H_2} \approx \sqrt{4M_{top}^2 - M_{H_1}^2} \sim 325$ GeV. In 2011 the CDF collaboration 20 has announced the preliminary results on the excess of events in $ZZ \rightarrow llll$ channel at the invariant mass $\approx 325$ GeV. CMS collaboration also reported a small excess in this region 21. In 22,23 it was argued that this may point out to the possible existence of a new scalar particle with mass $M_{H_2} \approx 325$ GeV.

2.3 Higgs bosons in superfluid phases in 2+1 films

The Higgs field $A_{\alpha i}$ in 2D thin films contains $3 \times 2 \times 2 = 12$ real components. There are two possible phases: the A-phase and the planar phase. Both phases have isotropic gap $\Delta$ in the 2D case. The degenerate vacuum state of the A-phase, $A_{\alpha i}(eq) = \Delta \hat{e}_\alpha (\hat{x}_i + i\hat{y}_i)$, corresponds to the symmetry breaking $G = SO_L(2) \otimes SO_S(3) \otimes U(1) \rightarrow H = U(1)_Q \otimes SO_S(2)$, where the combined symmetry $U_Q(1)$ is similar to the electromagnetic symmetry of SM. The 12 collective modes are classified in terms of the “electric” charge $Q$ and include 5 – 2 = 3 NG bosons + 9 Higgs amplitude modes. Their energies obey Eq.4, with quantum number $Q$ instead of $J$. This is another example, where the Nambu sum rule works. The parameters $\eta$ are determined by the symmetry of the system. Both in the A-phase and in the planar phase they get three possible values $\eta = 1, \eta = -1, and \eta = 0$. In the A-phase, these modes form two pairs of Nambu partners (triply degenerated), with $Q = 0$ and $|Q| = 2$ (see also Ref.24):

$$E_1^{(Q=0)} = 0, \quad E_2^{(Q=0)} = 2\Delta,$$

$$E_1^{(Q=\pm 2)} = \sqrt{2}\Delta, \quad E_2^{(Q=\pm 2)} = \sqrt{2}\Delta.$$ (9)

Since masses of $Q = \pm 2$ and $Q = -2$ modes are equal, the Nambu rule necessarily leads to the definite value of the masses of the “charged” Higgs bosons. Because
of the common symmetry breaking scheme in SM and in $^3$He-A, Eq.(10) may serve as a hint for existence of two Higgs bosons in SM with equal masses

$$M_{H^+} = M_{H^-} = \sqrt{2}M_{\text{top}}.$$  

(11)

This mass is about 245 GeV. A certain excess of events in this region has been observed by ATLAS in 2011 (see, for example,\textsuperscript{25}).

3 Hidden symmetry: light Higgs as quasi Nambu-Goldstone mode

The mass of the observed Higgs boson is rather small compared to the characteristic electroweak scale of order 1 TeV. This may indicate an existence of some approximate (custodial symmetry). We have already mentioned the custodial symmetry of separate spin and orbital rotations in superfluid $^3$He, which leads to Higgs bosons with small mass originating from the quasi-NG modes – spin waves.

Here we consider the hidden symmetry emerging in the BCS theory of superfluid $^3$He-A, which corresponds to the weak coupling approximation. Application of this hidden symmetry to the structure of the topological defects in $^3$He-A was discussed in\textsuperscript{26}. In the BCS approximation, there are the following relations between the $\beta$-parameters of quartic terms in Higgs potential (2):

$$-2\beta_1 = \beta_2 = \beta_3 = \beta_4 = -\beta_5.$$  

These relations have a crucial effect for bosons in $^3$He-A: they give rise to 3 extra NG bosons due to hidden symmetry and one more NG bosons due to flat direction\textsuperscript{27,28}.

The hidden symmetry can be visualized in the following way. The A-phase Higgs field $A_{\alpha i}(\text{eq}) = \Delta \hat{x}_\alpha (\hat{x}_i + i\hat{y}_i)$ can be represented as a sum of two terms

$$A_{\alpha i}(\text{eq}) = \Delta \frac{1}{2} (\hat{x}_\alpha + i\hat{y}_\alpha) (\hat{x}_i + i\hat{y}_i) + \Delta \frac{1}{2} (\hat{x}_\alpha - i\hat{y}_\alpha) (\hat{x}_i + i\hat{y}_i).$$  

(12)

The first term represents the subsystem with quantum numbers $S_z = L_z = +1$ (spin-up component), while the second subsystem has $S_z = -L_z = -1$ (spin-down component). In the BCS theory of $^3$He-A, the spin-up and spin-down components of Higgs field are independent: they may have different phases and different directions of orbital quantization axis, $\hat{l}_+$ and $\hat{l}_-$. Together with 2 degrees of freedom for the choice of spin quantization axis, the vacuum states of the Higgs field have $(2 + 1) \times 2 + 2 = 8$ degrees of freedom. According to conventional wisdom, this suggests 8 NG bosons instead of 5 NG modes in the absence of custodial symmetry. Thus the hidden symmetry should lead to $8 - 5 = 3$ extra NG bosons, which acquire small mass due to quantum corrections and become the Higgs fields.

This rule of counting of the number of NG bosons is obeyed for all $^3$He-A vacua with one exception: on the sub-manifold of the vacuum states where the orbital vectors $\hat{l}_+$ and $\hat{l}_-$ of the two spin subsystems are equal as in Eq.(12), the number of NG modes is 9 instead of 8, thus violating the conventional wisdom.

The theorems concerning the number of NG modes in the broken symmetry states are discussed in recent literature (see Refs.\textsuperscript{29,30,31,32} and references therein). With some nondegeneracy assumption about the low-energy effective action, the total number of NG bosons (or quasi-NG bosons, if the symmetry is hidden) adds up to the number of broken symmetry generators. Typically this is the difference
between the number of generators of \( G \) and \( H \) groups. The number of NG modes can be smaller, e.g. if the time-reversal symmetry is violated, see Sec. 6.

However, \(^3\)He-A provides an example where the number of NG modes exceeds the number of broken symmetry generators. Due to this example, the counting rule has been reformulated by S.P. Novikov: the number of NG modes coincides with the dimension of the ’tangent space’ \( 33 \). The mismatch between the total number of NG bosons and the number of broken symmetry generators equals the number of extra flat directions in the Higgs potential. The Novikov theorem is general, it is applicable irrespective of whether the symmetry is true or approximate (hidden), i.e. irrespective of whether the NG bosons are genuine or pseudo.

The Higgs potential, which is ’flat’ along some directions (i.e. there are rays in field space along which the potential vanishes) has been discussed in relation to cosmological inflation and in supersymmetric theories, see e.g. review. As distinct from the other theories of flat directions, in \(^3\)He the quartic terms in the Higgs potential in Eq. (2) are non-zero. Nevertheless, for some sub-manifold of vacuum states, the extra flat direction leads to 9 NG modes for 8 broken symmetry generators. The flat directions are ‘lifted’ when the hidden symmetry is violated, as a result the quasi NG modes acquire mass and become the Higgs bosons.

4 Flat directions in Higgs potential and extended \( SO(6) \) symmetry

Here we demonstrate, how the extra flat directions lead to substantial extension of the symmetry of tangent space. For that we add two components of spin singlet s-wave Higgs field \( \Psi \) to 18 components of spin-triplet p-wave Higgs field \( A_{\alpha i} \), and introduce the set of 15+1 generators of transformation or 15+1 operators:

\[
S \ , \ L \ , \ r^a \ , \ p_{\alpha i}^a
\]

The extended group has 9 more generators \( p_{\alpha i}^a \), which act on Higgs fields as

\[
L A_i^a = i e_{ijk} A_k^a \ , \ r^a A_i = i e^{\alpha \beta \gamma} A_i^\gamma \ , \ L A_i = A_i^\gamma \ , \ r^a \Psi = \Psi.
\]

New elements of symmetry mix triplet and singlet amplitudes of Higgs field. The nonzero commutators of these 16 operators are

\[
[L_i, L_j] = i e_{ijk} L_k \ , \ [r^a, r^\beta] = i e^{\alpha \beta \gamma} r^\gamma
\]

\[
[r^a, p_{\alpha i}^\beta] = i e^{\alpha \beta \gamma} r^\gamma \ , \ [L_i, r^a] = i e_{ijk} p_{\alpha k}^a
\]

\[
[p_{\alpha i}^a, p_{\beta j}^\beta] = i \left( S^a e_{ijk} L_k + \delta_{ij} e^{\alpha \beta \gamma} r^\gamma \right)
\]
The 15 generators in Eq. (13) form the $SO(6)$ group:

$$[\lambda_{ab}, \lambda_{bc}] = i\lambda_{ca} \tag{19}$$

where $\lambda_{ab}$ is antisymmetric $6 \times 6$ matrix with components:

$$\lambda_{12} = \mathcal{L}_z \!, \; \lambda_{23} = \mathcal{L}_x \!, \; \lambda_{31} = \mathcal{L}_y \!, \; \lambda_{45} = \mathcal{F}_z \!, \; \lambda_{54} = \mathcal{F}_x \!, \; \lambda_{64} = \mathcal{F}_y \!, \; \lambda_{14} = \mathcal{P}_x \!, \; \lambda_{41} = \mathcal{P}_x \!, \; \lambda_{16} = \mathcal{P}_x \!, \; \lambda_{24} = \mathcal{P}_x \!, \; \lambda_{25} = \mathcal{P}_y \!, \; \lambda_{26} = \mathcal{P}_y \!, \; \lambda_{34} = \mathcal{P}_y \!, \; \lambda_{35} = \mathcal{P}_y \!, \; \lambda_{36} = \mathcal{P}_y \! \tag{20}$$

Together with the gauge group $U(1)$, the hidden symmetry group in BCS regime is $G_h = SO(6) \otimes U(1)$, which transforms the Higgs field as

$$\langle A_\alpha^i, \Psi \rangle \rightarrow e^{i\phi} e^{i\theta^a \alpha^i} e^{i\theta^b \psi} e^{i\Omega^\beta} (A_\alpha^i, \Psi) \tag{24}$$

Here $\theta^a$ and $\theta^b$ are rotation angles in spin and orbital spaces correspondingly, $\phi$ is the parameter of the phase rotations, while 9 other parameters $\Omega^\beta$ are angles of additional rotations of $SO(6)$ group. Thus 10 complex components of the triplet + singlet Higgs form 10 dimensional representation of the $SU(4)$ or $SO(6)$ group.

This extended symmetry group describes the properties of the $^3$He-A in the BCS approximation, if $\mathbf{1}_x = \mathbf{1}_z$ and the vacuum state of the Higgs field is $A_\alpha^i(\text{eq}) = \Delta_0 \delta^{i0}(\mathbf{1}_x + i\mathbf{1}_y)$. The expansion of the Higgs potential in terms of the deviations of the Higgs field from its equilibrium value in Eq. (3) is (in dimensionless units):

$$\delta F = \sum_a [(u^a_1 - v^a_1)^2 + (u^a_2 - v^a_2)^2] + 2[(u^a_1 + v^a_1)^2 + (u^a_2 - v^a_2)^2 + (u^a_2 - v^a_2)^2] \tag{25}$$

This quadratic form is exactly zero if the deviations of the order parameter are obtained by the action of all elements of $G_h$, i.e. if $\delta A_\alpha^i(G_h) = G_h A_\alpha^i(\text{eq}) - A_\alpha^i(\text{eq})$. Thus $G_h$ is the extended symmetry of the Higgs potential in tangent space, and this symmetry leads to the flat directions. Its subgroup $H_h$ – the symmetry group of the vacuum state ($H_h A_\alpha^i(\text{eq}) = 0$) – has 5 generators:

$$H_h = SU(2) \otimes U(1) \otimes U(1), \tag{26}$$

$$\left( \mathcal{J}^x - \mathcal{P}_z^x, \mathcal{J}^y - \mathcal{P}_z^y, \mathcal{J}^z - \mathcal{P}_z^z \right); \mathcal{J}^x + \mathcal{P}_z^x; \mathcal{J} - \mathcal{L}_z = Q, \tag{27}$$

where $Q$ is again the analog of electric charge in SM. So, the BCS model of $^3$He-A contains $16 - 5 = 11$ NG bosons (two of them correspond to oscillations of the scalar condensate $\Psi$) and $20 - 11 = 9$ Higgs modes.

The conventional symmetry breaking pattern in $^3$He-A, $G = SO_3(3) \otimes SO_3(3) \otimes U(1) \rightarrow H = SO_3(2) \otimes U(1)$, gives 7 – 2 = 5 NG bosons. The flat directions emerging in the BCS model lead to 6 additional NG bosons, or to 4 if one neglects the oscillations of the scalar Higgs field $\Psi$. When the explicit corrections to the weak coupling approximation are introduced, or the quantum corrections are taken into account, these 4 modes become Higgs bosons with small masses. See Ref.35 for experiments with massive Higgs modes in $^3$He-A.
5 Nambu sum rule for gapless fermions

Similar to the 2D case in Eqs. (9) and (10), in 3D $^3$He-A the modes with “electric charge” $Q = \pm 2$ and $Q = 0$ obey the Nambu sum rule, but in a modified form. In 3D $^3$He-A, the gap in the fermionic spectrum is anisotropic and vanishes in the direction of $\mathbf{l}$. The nodes in spectrum demonstrate another possible scenario of the symmetry breaking in SM, which leads to splitting of the degenerate Fermi point instead of formation of the fermionic mass gap. The lesson from $^3$He-A is that in such case, the term $M^2_f$ in the Nambu sum rule (1) must be substituted by the angle average of the square of anisotropic gap $\bar{\Delta}^2$. For $^3$He-A one obtains

$$E_1(Q=0) = 0, \quad E_2(Q=0) = 2\bar{\Delta}, \quad E^{(Q=+2)} = E^{(Q=-2)} = \sqrt{2}\bar{\Delta},$$

$$\bar{\Delta}^2 \equiv \langle \Delta^2(\theta) \rangle = \frac{2}{3}\Delta^2_0.$$ (28)

6 Broken time reversal symmetry: Higgs from NG boson

As is well known in condensed matter community, the violation of time reversal symmetry $T$ leads to splitting of NG boson to the mode with quadratic spectrum and the mode with the gapped spectrum, the Higgs mode. In particular, this happens for spin waves in ferromagnets, where $T$ is spontaneously broken, and for the Kelvin waves propagating along quantized vortex in superfluids, where the circulating flow around the vortex breaks the $T$-symmetry.

In ferromagnets, the symmetry breaking pattern is $SO_3(3) \to SO_3(2)$. Typically this leads to $3 - 1 = 2$ NG modes with linear spectrum $\omega_{1,2} = ck$ (spin waves). But the broken $T$ symmetry splits the two branches into one NG mode with quadratic spectrum and the Higgs mode with gap. For small $k$ one has

$$\omega_1 = \frac{k^2}{M}, \quad \omega_2 = M c^2 + \frac{k^2}{M}, \quad k \ll M c.$$ (30)

Superfluid $^3$He-A has orbital angular momentum and thus represents the liquid orbital ferromagnet. Splitting of the linear spectrum of orbital waves in $^3$He-A according to Eq. (30) can be found in Eqs. (6.52-54) in. (28). Orbital waves in $^3$He-A are analogs of photons. However, in SM such effect would be possible only if the CPT and Lorentz symmetries are violated.

A vortex line in superfluids breaks translational symmetry in two transverse directions. The similar linear topological defect without violation of $T$-symmetry would have two NG modes propagating along the line. But the broken $T$ symmetry of the vortex combines two NG modes with linear spectrum into one NG mode with quadratic spectrum – Kelvin waves – according to Eq. (30). Recent discussion of the NG modes on vortices and strings see in. (37,38).

7 Nambu sum rules in the relativistic models of top quark condensation

We consider the NJL model of general type that involves all 6 quarks and all 6 leptons (neutrino is supposed to be of Dirac type). Let us consider the particular
form of the four-fermion action. It is obtained under the supposition that the tensor of coupling constants standing in front of the four-fermion term is factorized and under the supposition that lepton number originates from the fourth color in the spirit of Pati-Salam models. The action of the NJL model has the form

\[ S = \int d^4x \left\{ \bar{\chi} \left[ i \gamma \partial \right] \chi + \frac{8\pi^2}{\Lambda^2} (\bar{\chi}_{A}C_{A}L_{B}X_{R}^{J,\beta,B}(\bar{\chi}_{L(\beta)}B_{R}R_{L}X_{L}) + \bar{\chi}_{L(\beta)}B_{R}R_{L}X_{L})W_{\chi}W_{\chi}^\dagger R_{P}R_{P}^\dagger B_{B} \right\} \] (31)

Here \( \chi_{A}^{T} = \{(u_k, d_k); (c_k, s_k); (t_k, b_k)\} \) for \( k = 1, 2, 3 \) is the set of the quark doublets with the generation index \( \alpha \) while \( \chi_{A}^{T} = \{(v_e, e); (v_\mu, \mu); (v_\tau, \tau)\} \) is the set of the lepton doublets. \( \Lambda \) is the dimensional parameter. Hermitian matrices \( L, R, I, W \) contain dimensionless coupling constants. The form of action Eq. (31) with \( W = \text{diag}(1 + \frac{1}{2}W_{\mu\tau}, 1, 1, 1) \) is fixed by the requirement that there is the \( SU(3) \otimes SU(2) \otimes U(1) \) symmetry. We imply that all eigenvalues of matrices \( L, R, I \) are close to each other. We assume the existence of an approximate symmetry: at the zero order of a perturbation theory the eigenvalues of \( L, R, I \) are all equal to each other, and \( W_{\mu\tau} = 0 \). For example, the action of the corresponding form appears in the model with the gauge field of Lorentz group. Any small corrections to this equality gives the eigenvalues of \( L, R, I \) that only slightly deviate from each other, and the value of \( W_{\mu\tau} \) that only slightly deviates from 0. (After the suitable rescaling \( \Lambda \) plays the role of the cutoff, while the eigenvalues of \( L, R, I \) are all close to 1.)

Bosonic spectrum of this model is formally given by the expressions for the bosonic spectrum of the model suggested in and calculated in one-loop approximation in . It is implied that in vacuum the composite scalar fields \( h_q = \bar{q}q \) are condensed for all fermions \( q = u, d, c, s, t, b, e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau \).

There are two excitations in each \( \bar{q}q \) channel with masses \( M_{\bar{q}q}^{P} \) and \( M_{\bar{q}q}^{S} \), and four excitations (i.e. two doubly degenerated excitations) in each \( \bar{q}_1\bar{q}_2 \) channel. (Pairings of leptons and quarks are also allowed and give the colored scalar fields.) We denote the masses \( M_{\bar{q}_1\bar{q}_2}^{P}, M_{\bar{q}_1\bar{q}_2}^{S} \). It is worth mentioning that each of the scalar quark - antiquark and lepton - antiquark bosons carries two color indexes. In the absence of the \( SU(3) \) gauge field each of these channels represents the degenerate nonet. When the color interactions are turned on we are left with the singlet and octet states. Traceless octet states as well as the color scalar excitations of the quark - lepton channels cannot exist as distinct particles due to color confinement.

Instead of the trivial Nambu sum rule of the simplest models of top - quark condensation \( M_{t} = 2M_{L} \) we have the sum rule :\n
\[
\begin{align*}
[M_{\bar{q}_1\bar{q}_2}^{+}]^2 + [M_{\bar{q}_1\bar{q}_2}^{-}]^2 + [M_{\bar{q}_1\bar{q}_2}^{+}]^2 + [M_{\bar{q}_1\bar{q}_2}^{-}]^2 & \approx 4(M_{\bar{q}_1\bar{q}_2}^{P} + M_{\bar{q}_1\bar{q}_2}^{S}), \quad (q_1 \neq q_2); \\
[M_{\bar{q}_1\bar{q}_2}^{P}]^2 + [M_{\bar{q}_1\bar{q}_2}^{S}]^2 & \approx 4M_{\bar{q}_1}\]
\end{align*}
\] (32)

In the case when the \( t \)-quark contributes to the formation of the given scalar excitation, its mass dominates, and in each channel \((\bar{t}, \bar{t}, \ldots)\) we come to the relation \( \Sigma M_{\bar{q}_1\bar{q}_2}^{P} \approx 4M_{\bar{q}_1} \), where the sum is over scalar excitations in the given channel.

It is important, that although the corrections to the eigenvalues of \( L, R, I, W \) are small, this does not mean that the corrections to the masses are small. Instead, the large difference between masses may appear in this way. The symmetry breaking pattern of the considered model is \( U_{ud,L}(2) \otimes \cdots \otimes U_{eL}(2) \otimes U(1)_{\mu} \otimes \cdots \otimes U(1)_{\mu} \).
$U(1)_e \to U(1)_u \otimes \ldots \otimes U(1)_e$. Among the mentioned Higgs bosons there are 24 Goldstone bosons that are exactly massless (in the channels $t(1 \pm \gamma^5)\bar{b}, t\gamma^5\bar{t}, c(1 \pm \gamma^5)\bar{s}, c\gamma^5\bar{c}, u(1 \pm \gamma^5)\bar{d}, u\gamma^5\bar{u}, b\gamma^5\bar{b}, s\gamma^5\bar{s}, d\gamma^5\bar{d}$ and in the similar lepton-lepton channels). There are Higgs bosons with the masses of the order of the t-quark mass ($t(1 \pm \gamma^5)\bar{t}, t(1 \mp \gamma^5)\bar{b}, t\gamma^5\bar{t}, t(1 \pm \gamma^5)\bar{s}, t\gamma^5\bar{c}, t(1 \pm \gamma^5)\bar{d}, t\gamma^5\bar{u}$, and similar quark-lepton states). The other Higgs bosons have masses much smaller than the t-quark mass. That’s why a lot of physics is to be added in order to make this model realistic. Extra light Higgs bosons should be provided with the masses of the order of $M_t$. In principle, this may be achieved if the new gauge symmetries are added, that are spontaneously broken. Then the extra light Higgs bosons may become massive via the Higgs mechanism.

In principle, all Higgs bosons $h$ in the channels $t\bar{t}, b\bar{b}, \tau\bar{\tau}, \nu\bar{\nu}, \ldots, d\bar{d}, t(1 \pm \gamma^5)\bar{b}, \ldots$ are coupled to the fields of the Standard Model in a similar way. However, already at the tree level the corresponding coupling constants are different for different Higgs bosons. (The form of the Higgs boson decay Lagrangian is given in \textsuperscript{45}.) The cross-sections of the processes (that may be observed at the LHC) like $pp \to h \to WW, ZZ, gg, \gamma\gamma$ for the $\bar{t}t$ Higgs bosons are much larger than for the other Higgs Bosons and are close to that of the Standard Model. This means, in particular, that the scalar boson of the present model in the $\bar{t}t$ channel with mass $\approx 350$ GeV is excluded by the LHC data. Therefore, some additional physics is necessary that either suppresses the corresponding cross-section or makes this state much heavier. The decays of the other Higgs bosons to $ZZ, WW, \gamma\gamma, gg$ are suppressed compared to that of $\bar{t}t$. Therefore, these scalar states are not excluded by the LHC data. In the processes like $pp \to h \to c\bar{c}, b\bar{b}, \tau\bar{\tau}$ the scalar states $c\bar{c}, \bar{t}t, \tau\bar{\tau}$ dominate at the tree level. At the present moment we do not comment on the possible exclusion of these states by the LHC data.

8 Conclusions

Experience with the Higgs and NG bosons in condensed matter allows us to suspect, that the observed Higgs boson is not fundamental: it may come as a composite object emerging in the fermionic vacuum. If so, there can be several species of Higgs bosons with different quantum numbers and with hierarchy of masses related to the hierarchy of hidden symmetries. Some particular analogies with condensed matter allows us even to predict the possible values of masses of extra Higgs bosons using the sum rule proposed by Nambu. The hint from superfluid $^3$He-B suggests the mass $\sim 325$ GeV, while the hint from superfluid $^3$He-A suggests two degenerated Higgs bosons with mass $\sim 245$ GeV. However, in the particular relativistic model of top quark condensation the four (two pairs) Higgs bosons contribute to the sum rule of Eq. (32). This pattern suggests the mass 210 GeV for the Nambu partner of the 125 GeV Higgs boson. In relation to cosmology, the thermodynamics of quantum liquids allows us to explain why the huge vacuum energy related to the Higgs field does not contribute to cosmological constant in the equilibrium vacuum\textsuperscript{39}.

It is worth mentioning that the Nambu relation between the masses of Higgs bosons and the fermion masses is valid only in the one-loop approximation. Formally, this approximation works in the relativistic NJL model only, when the
higher loop quadratic divergences are subtracted. At the present moment the source of such a subtraction remains unclear. However, there exists the theory, where in the similar situation it does takes place. In quantum hydrodynamics there formally exist the divergent contributions to various quantities (say, to vacuum energy) due to the quantized sound waves. The quantum hydrodynamics is to be considered as a theory with finite cutoff $\Lambda$. The loop divergences in the vacuum energy are to be subtracted just like we do for the case of the NJL model. In hydrodynamics the explanation of such a subtraction is that the microscopic theory to which the hydrodynamics is an approximation works both at the energies smaller and larger than $\Lambda$, and this microscopic theory contains the contributions from the energies larger than $\Lambda$. These contributions exactly cancel the divergences appeared in the low energy effective theory. This exact cancellation occurs due to the thermodynamical stability of vacuum. In\textsuperscript{47} it was suggested that a similar pattern may provide the mechanism for the cancellation of the divergent contributions to vacuum energy in quantum gravity and divergent contributions to the Higgs boson mass in the Standard Model. We suppose, that in our case of the NJL model the contributions of the trans-$\Lambda$ degrees of freedom cancel the dominant divergences in the bosonic and fermionic masses leaving us with the one-loop approximation as an effective tool for the evaluation of physical quantities.

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