Prototype and reduced nonlinear integrable lattice systems with the modulated pulson behavior

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Abstract

A multi-component semi-discrete nonlinear integrable system associated with the relevant third-order auxiliary linear problem is claimed to be the prototype system for several reduced integrable systems formulated in terms of true dynamical field variables. The main conservation laws related to the prototype system are found in the framework of generalized recurrent approach. The two-fold Darboux–Bäcklund dressing technique as applied to the integration of prototype system is developed in details. The novel reduced complex-valued nonlinear integrable system embracing three coupled dynamical subsystems on a quasi-one-dimensional lattice is proposed and its concise Lagrangian and Hamiltonian representations are written down. The essentially nontrivial couplings between the two complex-valued Toda-like subsystems are shown to be mediated by the intermediate subsystem both in their kinetic and potential parts. The explicit multi-component solution of a modulated pulson type related to the reduced integrable nonlinear ternary system is presented and analyzed.

Keywords: Semi-discrete nonlinear integrable system, Conservation laws, Darboux–Bäcklund transformation, Lagrangian and Hamiltonian representations, Modulated pulson

1. Introduction

The seminal suggestion of one-dimensional dynamical lattice system with the exponential nonlinearity by Toda [1, 2, 3] and the subsequent discovery of its complete integrability by Manakov [4] and Flaschka [5] had undoubtedly been and continue to be the powerful motivating forces for the widespread development of numerous semi-discrete (i.e. continuous in time and discrete in spatial coordinate) integrable models [6, 7, 8, 9, 10, 11, 12, 13] including the multi-component ones [14, 15, 16, 17, 18, 19]. This stream of researches is accompanied by the recognized applicability of semi-discrete integrable nonlinear systems to the various branches of physics [20] inasmuch as the vast number of real physical objects are given on one or another type of quasi-one-dimensional lattices.

Among other semi-discrete nonlinear integrable systems the integrable systems generated by the various auxiliary linear problems of third [21, 22, 23, 24, 25, 26, 27, 28] or more higher [29, 30] order deserve a special attention as the systems comprising several subsystems of distinct physical origins.

Thus, in two our papers [22, 23] we have introduced the general integrable nonlinear system

\[ \dot{p}_{11}(n) = F_{12}(n)G_{21}(n-1) - F_{12}(n+1)G_{21}(n) \]  
\[ \dot{p}_{13}(n) = F_{12}(n)G_{23}(n-1) - F_{12}(n+1)G_{23}(n) \]  
\[ \dot{p}_{31}(n) = F_{32}(n)G_{21}(n-1) - F_{32}(n+1)G_{21}(n) \]  
\[ \dot{p}_{33}(n) = F_{32}(n)G_{23}(n-1) - F_{32}(n+1)G_{23}(n) \]
permitting a number of reductions from the so-called prototype $p_{11}(n), p_{13}(n), p_{33}(n), F_{12}(n), G_{21}(n), G_{23}(n), F_{32}(n)$ to the actual physical field variables. Here the overdot marks the derivative with respect to time $\tau$, while the discrete spatial coordinate $n$ is assumed to span all integers from minus to plus infinity. To integrate this system (1.1)–(1.8) we have developed rather complicated inverse scattering technique and found some simplest solutions related to the reduced system encompassing two real-valued Toda-like translational subsystems coupled with one orientational subsystem. Unfortunately, the orientational component of the obtained solution suffers to be unexcited [22, 23].

In present paper we try to fill in this gap in the framework of specially developed two-fold Darboux–Backlund dressing approach which allow to construct the nonlinear wave packet embracing all involved field components. In addition, we apply the general recurrent scheme capable to generate the infinite hierarchy of local conservation laws [29, 35, 36, 37, 38, 39]. Moreover, it might provide an indispensable starting tool in recurrent searching for an infinite such as the method of inverse scattering transform [34, 12, 22, 23] and the Darboux-Bäcklund dressing method [33, 36, 37, 38, 39].

To proceed with the most coherent consideration concerning the general nonlinear system (1.1)–(1.8) we begin with the confirmation of its complete integrability.

2. Complete integrability of the general nonlinear system

One can readily verify that the general semi-discrete nonlinear system of our interest (1.1)–(1.8) is presentable in the form of matrix-valued semi-discrete zero-curvature equation

$$\dot{L}(n[z]) = A(n + 1[z])L(n[z]) - L(n[z])A(n[z])$$

(2.1)

with the spectral $L(n[z])$ and evolution $A(n[z])$ operators given by the following $3 \times 3$ square matrices

$$L(n[z]) = \begin{pmatrix}
 p_{11}(n) + \lambda(z) & F_{12}(n) & p_{13}(n) \\
 G_{21}(n) & 0 & G_{23}(n) \\
 p_{31}(n) & F_{32}(n) & p_{33}(n) + \lambda(z)
\end{pmatrix}$$

(2.2)

$$A(n[z]) = \begin{pmatrix}
 0 & -F_{12}(n) & 0 \\
 -G_{21}(n - 1) & \lambda(z) & -G_{23}(n - 1) \\
 0 & -F_{32}(n) & 0
\end{pmatrix}$$

(2.3)

Here the symbols $\lambda(z)$ and $z$ denote the authentic and rationalizes time-independent spectral parameters, respectively. According to the commonly referred terminology [31, 32, 33], this representation (2.1)–(2.3) determines the complete integrability of the declared semi-discrete nonlinear system (1.1)–(1.8) in the Lax sense.

Let us remind that the zero-curvature equation itself (2.1) serves as the compatibility condition between two linear matrix-valued equations

$$X(n + 1[z]) = L(n[z])X(n[z])$$

(2.4)

$$\dot{X}(n[z]) = A(n[z])X(n[z])$$

(2.5)

referred to as the auxiliary linear problem. Here the notation $X(n[z])$ stands for the auxiliary matrix-function of discrete spatial coordinate $n$, continuous time $\tau$ and time-independent spectral parameter $z$. By and large, the auxiliary linear problem (2.4) and (2.5) is capable to lay the foundation for the development of various methods of system’s integration such as the method of inverse scattering transform [14, 12, 22, 23] and the Darboux-Bäcklund dressing method [33, 36, 37, 38, 39]. Moreover, it might provide an indispensable starting tool in recurrent searching for an infinite hierarchy of local conservation laws [39].
3. Natural constraints and the main local densities

It is well known that any nonlinear integrable system on an infinite regular chain possesses the infinite number of local conservation laws. The most straightforward way to isolate some of them is based upon the universal local conservation law

\[ \frac{d}{dt} \ln[\det L(n,z)] = SpA(n + 1|z) - SpA(n|z) \]  

appearing as a simple contraction of system’s zero-curvature representation \( (3.1) \).

Inasmuch as the determinant \( \det L(n,z) \) of spectral matrix \( L(n,z) \) depends on two distinct powers of spectral parameter \( \lambda(z) \) and the expression \( SpA(n + 1|z) - SpA(n|z) \) is equal to zero, the universal local conservation law \( (3.1) \) produces two uncellular conservation laws imposing two natural constraints onto the set of prototype field variables. The explicit record of natural constraints depends on a particular choice of boundary conditions for the prototype field variables and on an expected physical sense of reduced field variables. Assuming the underlying lattice being spatially uniform and pinned to the immovable frame of reference we take the natural constraints in the following form

\[ p_{11}(n)F_{32}(n)G_{23}(n) + p_{31}(n)F_{12}(n)G_{23}(n) - p_{33}(n)F_{12}(n)G_{23}(n) = 0 \]  

\[ F_{12}(n)G_{21}(n) + F_{32}(n)G_{23}(n) = -1 \]  

These constraints \( (3.2) \) and \( (3.3) \) are consistent with the following boundary conditions for the prototype fields:

\[ \lim_{n \to -\infty} p_{11}(n) \to 0, \lim_{n \to -\infty} p_{31}(n) \to 0, \lim_{n \to -\infty} p_{33}(n) \to 0, \lim_{n \to -\infty} F_{12}(n) \to F_{12}, \lim_{n \to -\infty} G_{21}(n) \to G_{21}, \lim_{n \to -\infty} F_{32}(n) \to F_{32}, \lim_{n \to -\infty} G_{23}(n) \to G_{23} \]  

where \( F_{12}G_{21} + F_{32}G_{23} = -1 \). As a consequence, the limiting eigenvalue problem

\[ L(z)X(z) = X(z)\xi(z) \]  

(3.4)

(where \( L(z) = \lim_{n \to -\infty} L(n,z) \)) prescribes the functional relationship \( \lambda(z) = z + 1/z \) in view of very simple resulting expressions

\[ \xi_1(z) = z \]  

\[ \xi_2(z) = \lambda(z) \equiv z + 1/z \]  

\[ \xi_3(z) = 1/z \]  

for the eigenvalues \( \xi_j(z) \). It is interesting to note that the functional dependence \( \lambda(z) = z + 1/z \) establishes a particular realization of the Zhukovsky transformation \( 40 \) well known in aerodynamics.

The capability of universal local conservation law \( (3.1) \) in generating system’s local conservation laws is seen to be restricted only by two specimens.

In contrast, there exists the generalized direct procedure \( 29, 24, 27, 28 \) permitting to develop an infinite set of local conservation laws recursively without references to auxiliary spectral data. By definition, any local conservation law associated with some semi-discrete system given on an infinite quasi-one-dimensional lattice can be written in the form

\[ \rho(n) = J(n|n - 1) - J(n + 1|n) \]  

where the quantities \( \rho(n) \) and \( J(n + 1/2|n - 1/2) \) are referred to as the local density and the local current, respectively. Bearing in mind this general definition \( (3.8) \) we must find the recursive presentation \( (i.e. \) presentation based upon some proper expansion in spectral parameter \( z \) or inverse spectral parameter \( 1/z \) for the auxiliary quantities \( \Gamma_{j,k}(n|z) \) governed by the following set of spatial Riccati equations

\[ \Gamma_{j,k}(n + 1|z) \sum_{j=1}^{3} L_{j,k}(n|z)\Gamma_{j,k}(n|z) = \sum_{j=1}^{3} L_{j,k}(n|z)\Gamma_{j,k}(n|z) \]  

(3.9)
Then for the generating function $\ln M_i$, i.e., $\ln L$, quantities serve to generate the hierarchy of local densities and the hierarchy of local currents, respectively. In so doing, the terms with the same powers of spectral parameter in each of three ($j = 1, 2, 3$) generating series \textit{(3.11)} it is possible to recover any required number of local conservation laws from their infinite hierarchy.

The most productive is the second ($j = 2$) of generating series \textit{(3.11)}. To develop the second generating series it is sufficient to consider two auxiliary functions $\Gamma_{12}(n|z)$ and $\Gamma_{32}(n|z)$ since $\Gamma_{j}(n|z) \equiv 1$ in view of properties \textit{(3.10)}. Due to the evident symmetry $\lambda(z) = \lambda(1/z)$ of the functional spectral parameter $\lambda(z)$ we restrict ourselves only to serial expansions at the center $|z| \to 0$ of a rationalized complex spectral plane and seek the auxiliary functions $\Gamma_{12}(n|z)$ and $\Gamma_{32}(n|z)$ as follows:

\begin{equation}
\Gamma_{12}(n|z) = z \sum_{j=0}^{\infty} \gamma_{12}(n|m) z^m
\end{equation}

\begin{equation}
\Gamma_{32}(n|z) = z \sum_{j=0}^{\infty} \gamma_{32}(n|m) z^m.
\end{equation}

Then for the generating function $\ln M_{22}(n|z)$ written up to the second power in spectral parameter $z$ we obtain

\begin{equation}
\ln M_{22}(n|z) = \ln z - [p_{11}(n) + p_{33}(n)] z + \\
+ \left[ p_{11}^2(n)/2 + p_{13}^2(n)/2 + p_{13}(n)p_{31}(n) - G_{21}(n)F_{12}(n+1) - G_{23}(n)F_{32}(n+1) - 1 \right] z^2.
\end{equation}

By virtue of second generating equation \textit{(i. e. equation (3.11)) taken at $j = 2$} the quantities

\begin{equation}
p(n) = p_{11}(n) + p_{33}(n)
\end{equation}

\begin{equation}
h(n) = p_{11}^2(n)/2 + p_{13}^2(n)/2 + p_{13}(n)p_{31}(n) - G_{21}(n)F_{12}(n+1) - G_{23}(n)F_{32}(n+1) - 1
\end{equation}

acquire the status of local densities in two the most important local conservation laws.

The comprehensive analysis shows that the former \textit{(3.17)} of these local densities should be identified with the density of system’s momentum, whereas the latter one \textit{(3.18)} with the density of system’s energy.

By invoking the revealed natural constraints \textit{(3.2)} and \textit{(3.3)} it is possible to replace eight original prototype field variables by means of six properly chosen actual dynamical field variables. A particular realization of such a reduction is not unique. However, in any case we come to some reduced nonlinear lattice system comprising three coupled dynamical subsystems. As to the local density $h(n)$ its expression \textit{(3.18)} will be useful while writing the Hamilton function of reduced nonlinear dynamical system in terms actual fields.

We postpone both of above tasks up to the Section 9 and concentrate now on the development of Darboux–Bäcklund dressing integration scheme in terms of prototype field variables applicable to the general nonlinear system of our interest \textit{(1.1)–(1.3)}.\n
\[ \text{4} \]
4. Fundamentals of the Darboux–Bäcklund dressing technique

In this Section we itemize the main steps elucidating the essence of the Darboux–Bäcklund dressing scheme suitable for the integration of the prototype nonlinear system under study \([1.1]\)–\([1.8]\).

We start with the general definition of Darboux transformation \([37]\)

\[
\psi X(n|z) = ^\psi D(n|z) \psi X(n|z) .
\]  

(4.1)

The Darboux transformation \([4.1]\) connects the seed (\textit{a priori} known) \(\psi X(n|z)\) and crop (required) \(\psi X(n|z)\) solutions of the auxiliary linear problem \([2.4]\) and \([2.5]\) via the Darboux matrix \(\psi D(n|z)\), which should be properly chosen in accordance with the particular realizations \([2.2]\) and \([2.3]\) of auxiliary spectral \(L(n|z)\) and evolution \(A(n|z)\) operators. The Darboux matrix \(\psi D(n|z)\) must obey the set of matrix equations

\[
\psi D(n + 1|z) \psi L(n|z) = \psi L(n|z) \psi D(n|z)
\]  

(4.2)

\[
\psi D(n|z) = \psi A(n|z) \psi D(n|z) - \psi D(n|z) \psi A(n|z)
\]  

(4.3)

serving as the spatial \((4.2)\) and temporal \((4.3)\) compatibility conditions between the seed

\[
\psi X(n + 1|z) = \psi L(n|z) \psi X(n|z)
\]  

(4.4)

\[
\psi X(n|z) = \psi A(n|z) \psi X(n|z)
\]  

(4.5)

and crop

\[
\psi X(n + 1|z) = \psi L(n|z) \psi X(n|z)
\]  

(4.6)

\[
\psi X(n|z) = \psi A(n|z) \psi X(n|z)
\]  

(4.7)

embodiments of auxiliary linear problem \([2.4]\) and \([2.5]\). Here and later on the upper-front single index \(s\) or \(c\) indicates a quantity associated with the seed or crop solution, respectively. The upper-front double index \(cs\) marks a quantity related to or induced by the Darboux matrix. According to the general rules \([20, 38, 39]\) the spatial compatibility condition \((4.2)\) establishes the implicit Bäcklund transformation between the seed and crop solutions for the prototype fields, while the temporal compatibility condition \((4.3)\) allows to uncover the crucial spectral properties of Darboux matrix sufficient to restore explicitly its components indispensable for the development of the whole Darboux–Bäcklund dressing integration scheme.

Seeking for the Darboux matrix \(\psi D(n|z)\) we assume the following two-fold \((c = s + 2)\) ansatz

\[
\psi D(n|z) = \begin{bmatrix}
\psi K_{11}(n) & \psi C_{12}(n) & \psi T_{12}(n) & \psi K_{13}(n) \\
\psi E_{21}(n) & \psi C_{22}(n) & \psi U_{22}(n) & \psi E_{23}(n) \\
\psi K_{31}(n) & \psi C_{32}(n) & \psi T_{32}(n) & \psi K_{33}(n)
\end{bmatrix} ,
\]  

(4.8)

which proves to be consistent with the set of governing matrix equations \((4.2)\)–\((4.3)\). The notion “two-fold \((c = s + 2)\)” implies that each matrix element of adopted ansatz \((4.8)\) written as a polynomial of the spectral parameter \(\lambda(z)\) has the same leading term as the respective matrix element appearing in the product of two evolution matrices. The Darboux matrix taken in the two-fold form \((4.8)\) are able to generate nontrivial spatially finite solutions (from the vacuum one) embracing all three subsystems of a reduced nonlinear dynamical system.

The spectral properties of Darboux matrix \(\psi D(n|z)\) follow directly from the contracted form

\[
\frac{d}{dz} \ln[\psi \det(\psi D(n|z))] = \text{Sp} \psi A(n|z) - \text{Sp} \psi A(n|z)
\]  

(4.9)

of temporal compatibility condition \((4.3)\). Indeed, in view of the identity

\[
\text{Sp} \psi A(n|z) - \text{Sp} \psi A(n|z) \equiv 0 ,
\]  

(4.10)
dictated by the specific form \( [22] \) of evolution matrix \( A(n|z) \), the contracted equation \( [4.9] \) with the adopted ansatz \( [4.8] \) for the Darboux matrix \( c^s D(n|z) \) yields

\[
\det c^s D(n|z) = [\lambda(z) - \lambda(c^s z_+)] [\lambda(z) - \lambda(c^s z_-)] c^s W(n),
\]

where the quantity \( c^s W(n) \) and the spectral data \( c^s z_+, c^s z_- \) are proved to be time-independent \( c^s W(n) = 0, c^s z_+ = 0, c^s z_- = 0 \).

Evidently, \( \det c^s D(n|c^s z_+) = 0 \) and hence the Darboux transformation \( [4.1] \) yields \( \det c^s X(n|c^s z_+) = 0 \). Therefore the condition

\[
\sum_{k=1}^{3} c^s X_{jk}(n|c^s z_+) c^s \delta_{k} (c^s z_+) = 0 \tag{4.12}
\]

should be satisfied. The detailed form of this condition \( [4.12] \) is given by formula

\[
\sum_{j=1}^{3} \sum_{k=1}^{3} c^s D_{jk}(n|c^s z_+) c^s X_{ik}(n|c^s z_+) c^s \delta_{k} (c^s z_+) = 0. \tag{4.13}
\]

Here the functions \( c^s D_{jk}(n|c^s z_+) \) and \( c^s X_{ik}(n|c^s z_+) \) stand for the elements of respective matrices \( c^s D(n|z) \) and \( c^s X(n|z) \), while the time- and space-independent parameters \( c^s \delta_{k} (c^s z_+) \) serve as the spectral data. The invariability of spectral parameters \( c^s \delta_{k} (c^s z_+) \) in time and space has a status of rigorously proved theorem.

The obtained condition \( [4.13] \) encompasses six linear equations for fourteen Darboux functions \( c^s K_{11}(n), c^s C_{12}(n), c^s T_{12}(n), c^s K_{21}(n), c^s E_{21}(n), c^s V_{21}(n), c^s D_{22}(n), c^s U_{22}(n), c^s E_{22}(n), c^s V_{22}(n), c^s K_{31}(n), c^s C_{32}(n), c^s T_{32}(n), c^s K_{33}(n) \). Fortunately only six \( c^s C_{12}(n), c^s T_{12}(n), c^s D_{22}(n), c^s U_{22}(n), c^s C_{32}(n), c^s T_{32}(n) \) of them turn out to be truly independent. We prove this statement in Section 5 by invoking the implicit Bäcklund transformation \( [4.2] \) rewritten as twenty two detailed equations \( [5.1] - [5.22] \). In so doing, two Darboux functions \( c^s E_{21}(n) \) and \( c^s E_{22}(n) \) are proved to be determined explicitly by the \textit{a priori} known seed values \( c^s G_{21}(n) \) and \( c^s G_{22}(n) \) of the prototype functions \( G_{21}(n) \) and \( G_{22}(n) \). By dint of all these observations the above written formula \( [4.13] \) produces the set of six nonuniform linear equations serving to restore the unknown Darboux functions relying upon the known ones. In so doing, the seed solution \( c^s X(n|z) \) to the auxiliary linear problem \( [2.4] \) and \( [2.5] \) must be found beforehand.

Once the necessary elements of Darboux matrix have been found, the proper equations taken among the extended form \( [5.1] - [5.22] \) of implicit Bäcklund transformation \( [4.2] \) allow to obtain explicit crop solutions for the prototype field functions.

5. Peculiarities of the Darboux–Bäcklund dressing technique stipulated by the chosen two-fold Darboux matrix

As we have already mentioned, the first \( [4.2] \) of two matrix equations for the Darboux matrix should be treated as the implicit Bäcklund transformation between the seed \( c^s p_{11}(n), c^s p_{13}(n), c^s p_{31}(n), c^s p_{33}(n), c^s F_{12}(n), c^s G_{21}(n), c^s G_{22}(n) \), and crop \( c^s p_{11}(n), c^s p_{13}(n), c^s p_{31}(n), c^s p_{33}(n), c^s F_{12}(n), c^s G_{21}(n), c^s G_{22}(n) \), \( c^s F_{32}(n) \) solutions for the prototype fields. To justify this statement in the case of adopted two-fold ansatz \( [4.8] \) for the Darboux matrix it is sufficient to observe that the relevant matrix formula \( [4.2] \) comprises the following twenty two equations

\[
c^s K_{11}(n + 1) + c^s C_{12}(n + 1) c^s G_{22}(n) = c^s K_{11}(n) + c^s F_{12}(n) c^s E_{21}(n) \tag{5.1}
\]

\[
c^s K_{11}(n + 1) c^s p_{11}(n) + c^s T_{12}(n + 1) c^s G_{21}(n) + c^s K_{11}(n + 1) c^s p_{33}(n) =
= c^s p_{11}(n) c^s K_{11}(n) + c^s F_{12}(n) c^s V_{21}(n) + c^s p_{13}(n) c^s K_{31}(n) \tag{5.2}
\]

\[
c^s C_{12}(n) + c^s F_{12}(n) = 0 \tag{5.3}
\]

\[
c^s T_{12}(n) + c^s p_{11}(n) c^s C_{12}(n) + c^s F_{12}(n) c^s D_{22}(n) + c^s p_{13}(n) c^s C_{32}(n) = 0 \tag{5.4}
\]

\[
c^s K_{11}(n + 1) c^s F_{12}(n) + c^s K_{13}(n + 1) c^s F_{32}(n) =
= c^s p_{11}(n) c^s T_{12}(n) + c^s F_{12}(n) c^s U_{22}(n) + c^s p_{13}(n) c^s T_{32}(n) \tag{5.5}
\]
\( c^s C_{12}(n+1) \cdot s G_{23}(n) + c^s K_{13}(n+1) = c^s K_{13}(n) + c^s F_{12}(n) \cdot c^s E_{23}(n) \) 

(5.6)

\( c^s K_{11}(n+1) \cdot s p_{13}(n) + c^s T_{12}(n+1) \cdot s G_{23}(n) + c^s K_{13}(n+1) \cdot s p_{33}(n) = \)

\[ c^s p_{11}(n) \cdot c^s K_{13}(n) + c^s F_{12}(n) \cdot c^s V_{23}(n) + c^s p_{13}(n) \cdot c^s K_{33}(n) \]

(5.7)

\( c^s E_{21}(n+1) + s G_{23}(n) = 0 \)

(5.8)

\( c^s E_{21}(n+1) \cdot s p_{11}(n) + c^s V_{21}(n+1) + c^s D_{22}(n+1) \cdot s G_{23}(n) + c^s E_{21}(n+1) \cdot s p_{31}(n) = 0 \)

(5.9)

\( c^s V_{21}(n+1) \cdot s p_{11}(n) = c^s U_{22}(n+1) \cdot s G_{21}(n) + c^s V_{23}(n+1) \cdot s p_{31}(n) = \)

\[ c^s G_{21}(n) \cdot c^s K_{11}(n) + c^s G_{23}(n) \cdot c^s K_{31}(n) \]

(5.10)

\( c^s E_{21}(n+1) \cdot s F_{12}(n) + c^s E_{21}(n+1) \cdot s F_{32}(n) = c^s G_{23}(n) \cdot c^s C_{12}(n) + c^s G_{23}(n) \cdot c^s C_{32}(n) \)

(5.11)

\( c^s V_{21}(n+1) \cdot s F_{12}(n) + c^s V_{23}(n+1) \cdot s F_{32}(n) = c^s G_{21}(n) \cdot c^s T_{12}(n) + c^s G_{23}(n) \cdot c^s T_{32}(n) \)

(5.12)

\( c^s E_{21}(n+1) + s G_{23}(n) = 0 \)

(5.13)

\( c^s E_{21}(n+1) \cdot s p_{13}(n) + c^s D_{22}(n+1) \cdot s G_{23}(n) + c^s E_{21}(n+1) \cdot s p_{33}(n) + c^s V_{23}(n+1) = 0 \)

(5.14)

\( c^s V_{21}(n+1) \cdot s p_{13}(n) = c^s U_{22}(n+1) \cdot s G_{23}(n) + c^s V_{23}(n+1) \cdot s p_{33}(n) = \)

\[ c^s G_{21}(n) \cdot c^s K_{13}(n) + c^s G_{23}(n) \cdot c^s K_{33}(n) \]

(5.15)

\( c^s K_{31}(n+1) + c^s C_{32}(n+1) \cdot s G_{21}(n) = c^s F_{32}(n) \cdot c^s E_{21}(n) + c^s K_{31}(n) \)

(5.16)

\( c^s K_{31}(n+1) \cdot s p_{11}(n) + c^s T_{32}(n+1) \cdot s G_{21}(n) + c^s K_{33}(n+1) \cdot s p_{31}(n) = \)

\[ c^s p_{11}(n) \cdot c^s K_{11}(n) + c^s F_{32}(n) \cdot c^s V_{21}(n) + c^s p_{33}(n) \cdot c^s K_{31}(n) \]

(5.17)

\( c^s C_{32}(n) + s F_{32}(n) = 0 \)

(5.18)

\( c^s p_{31}(n) \cdot c^s C_{12}(n) + c^s F_{32}(n) \cdot c^s D_{22}(n) = c^s T_{32}(n) + c^s p_{33}(n) \cdot c^s C_{32}(n) \)

(5.19)

\( c^s K_{31}(n+1) \cdot s F_{12}(n) = c^s K_{33}(n+1) \cdot s F_{32}(n) = \)

\[ c^s p_{31}(n) \cdot c^s T_{12}(n) + c^s F_{32}(n) \cdot c^s U_{22}(n) + c^s p_{33}(n) \cdot c^s T_{32}(n) \]

(5.20)

\( c^s C_{32}(n) \cdot s G_{23}(n) + c^s K_{33}(n+1) = c^s F_{32}(n) \cdot c^s E_{23}(n) + c^s K_{33}(n) \)

(5.21)

\( c^s K_{31}(n+1) \cdot s p_{13}(n) + c^s T_{32}(n+1) \cdot s G_{23}(n) + c^s K_{33}(n+1) \cdot s p_{33}(n) = \)

\[ c^s p_{11}(n) \cdot c^s K_{13}(n) + c^s F_{23}(n) \cdot c^s V_{23}(n) + c^s p_{33}(n) \cdot c^s K_{33}(n) \].

(5.22)

Provided all fourteen Darboux functions \( c^s K_{11}(n), c^s C_{12}(n), c^s T_{12}(n), c^s K_{13}(n), c^s E_{21}(n), c^s V_{21}(n), c^s D_{22}(n), c^s U_{22}(n), c^s E_{23}(n), c^s V_{23}(n), c^s K_{31}(n), c^s C_{32}(n), c^s T_{32}(n), c^s K_{33}(n) \) in these detailed relationships (5.1)–(5.22) have been excluded, we come to just eight equations establishing explicit Bäcklund transformation between eight seed and eight crop prototype field functions.

Though having been very impractical for the actual calculations of solitonic or any other solutions the explicit Bäcklund transformation as such provides a strong argument for the original implicit Bäcklund transformation (5.1)–(5.22) to be naturally incorporated into the Darboux–Bäcklund integration procedure as its crucial and intrinsically noncontradictory element.
First of all, two Bäcklund relations (5.8) and (5.13) clearly indicate that two Darboux functions $c^s E_{21}(n)$ and $c^s E_{23}(n)$ are actually known $c^s E_{21}(n) = -G_{21}(n-1)$, $c^s E_{23}(n) = -G_{23}(n-1)$. As a consequence, the formulas (5.1), (5.6), (5.16), (5.21) accompanied by another two Bäcklund relations (5.3) and (5.18) yield

\[
\begin{align*}
  c^s K_{11}(n) &= c^s K_{11} - c^s C_{12}(n) \, t_{21}(n-1) \\
  c^s K_{13}(n) &= c^s K_{13} - c^s C_{12}(n) \, t_{23}(n-1) \\
  c^s K_{31}(n) &= c^s K_{31} - c^s C_{32}(n) \, t_{31}(n-1) \\
  c^s K_{33}(n) &= c^s K_{33} - c^s C_{32}(n) \, t_{33}(n-1) .
\end{align*}
\]

(5.23) (5.24) (5.25) (5.26)

Here the coordinate-independent parameters $c^s K_{jk}$ are proved to be time-independent too. To confirm their time-independence it is sufficient to combine the expressions (5.23)–(5.26) for $c^s K_{jk}(n)$ with the proper evolution equations selected from the matrix-valued evolution equation for the Darboux matrix (4.3) and to use the equalities (5.3), (5.8), (5.13), (5.18). On the other hand, the formulas (5.9), (5.12) yield

\[
\begin{align*}
  c^s V_{21}(n) &= t_{21}(n-1) \, s_{p_{11}(n-1)} + t_{23}(n-1) \, s_{p_{31}(n-1)} - c^s D_{22}(n) \, t_{21}(n-1) \\
  c^s V_{23}(n) &= t_{23}(n-1) \, s_{p_{31}(n-1)} + t_{23}(n-1) \, s_{p_{33}(n-1)} - c^s D_{22}(n) \, t_{23}(n-1) .
\end{align*}
\]

(5.27) (5.28)

These six relationships (5.23)–(5.28) accompanied by equalities (5.8) and (5.13) say that only six independent Darboux functions $c^s C_{12}(n)$, $c^s T_{12}(n)$, $c^s D_{22}(n)$, $c^s U_{22}(n)$, $c^s C_{32}(n)$, $c^s T_{32}(n)$ have to be found relying upon the spectral properties of Darboux matrix.

By another words, the number of nonuniform linear equations encoded in expression (4.13) indeed coincides with the number of unknown independent Darboux functions. Hence, all six independent Darboux functions $c^s C_{12}(n)$, $c^s T_{12}(n)$, $c^s D_{22}(n)$, $c^s U_{22}(n)$, $c^s C_{32}(n)$, $c^s T_{32}(n)$ can be calculated in terms of seed solution $t X(n) z$ to the auxiliary linear problem (2.4) and (2.5) supplemented by the spectral characteristics $c^s z_{ka}$ and $c^s s_{c^s z_{ka}}$ of requested Darboux matrix (4.2). In so doing, the relevant set of six nonuniform linear equations is split into three separate subsets given by formulas

\[
\begin{align*}
  c^s C_{12}(n) \left[ t_{21}(n-1) \, s_{Y_{1}(n) c^s z_{ka}} - \lambda (c^s z_{ka}) \, s_{Y_{2}(n) c^s z_{ka}} + t_{23}(n-1) \, s_{Y_{3}(n) c^s z_{ka}} \right] - \\
  c^s T_{12}(n) \left[ t_{21}(n-1) \, s_{Z_{1}(n) c^s z_{ka}} + t_{23}(n-1) \, s_{Z_{3}(n) c^s z_{ka}} \right] - \\
  c^s D_{22}(n) \left[ t_{21}(n-1) \, s_{Y_{1}(n) c^s z_{ka}} - \lambda (c^s z_{ka}) \, s_{Y_{2}(n) c^s z_{ka}} + t_{23}(n-1) \, s_{Y_{3}(n) c^s z_{ka}} \right] - \\
  c^s U_{22}(n) \left[ t_{21}(n-1) \, s_{Z_{1}(n) c^s z_{ka}} - \lambda (c^s z_{ka}) \, s_{Z_{3}(n) c^s z_{ka}} \right] - \\
  c^s C_{32}(n) \left[ t_{21}(n-1) \, s_{Y_{1}(n) c^s z_{ka}} + t_{23}(n-1) \, s_{Y_{3}(n) c^s z_{ka}} \right] - \\
  c^s T_{32}(n) \left[ t_{21}(n-1) \, s_{Z_{1}(n) c^s z_{ka}} + t_{23}(n-1) \, s_{Z_{3}(n) c^s z_{ka}} \right] .
\end{align*}
\]

(5.29) (5.30) (5.31)

Here the shorthand notation

\[
  s_{Y_{j}(n) c^s z_{ka}} = \sum_{k=1}^{3} s_{X_{jk}(n) c^s z_{ka}} c^s s_{c^s z_{ka}}
\]

(5.32)

has been introduced. The first set (5.29) is seen to produce the expressions for $c^s C_{12}(n)$ and $c^s T_{12}(n)$. The second set (5.30) gives rise to the expressions for $c^s D_{22}(n)$ and $c^s U_{22}(n)$. At last, the third set (5.31) allows to obtain the expressions for $c^s C_{32}(n)$ and $c^s T_{32}(n)$. By dint of relationships (5.23)–(5.28) between the dependent and independent Darboux functions the dependent Darboux functions $c^s K_{11}(n)$, $c^s K_{13}(n)$, $c^s K_{31}(n)$, $c^s K_{33}(n)$ and $c^s V_{21}(n)$, $c^s V_{23}(n)$ are also readily restorable.
Thus, the procedure of obtaining the expressions for Darboux functions relying upon the spectral properties of Darboux matrix is seen to be complete.

The next step in our task is to select eight formulas among the implicit Bäcklund transformation suitable to convert the expressions for Darboux functions into the expressions for crop field functions. Below we summarize the recipes for relevant calculations.

1. The expressions for $^c F_{12}(n)$ and $^c F_{32}(n)$ are given directly by formulas (5.3) and (5.18).
2. The expressions for $^c G_{21}(n)$ and $^c G_{23}(n)$ follow from the set of two linear equations (5.10) and (5.15).
3. The expressions for $^c p_{11}(n)$ and $^c p_{13}(n)$ are obtainable from the set of two linear equations (5.2) and (5.7) with the relationship (5.3) for $^c F_{12}(n)$ taken into account.
4. At last, the expressions for $^c p_{31}(n)$ and $^c p_{33}(n)$ are obtainable from the set of two linear equations (5.17) and (5.22) with the relationship (5.18) for $^c F_{32}(n)$ taken into account.

6. Solution to the auxiliary linear problem versus the vacuum seed solution to the prototype nonlinear system

In order to complete the development of Darboux–Bäcklund dressing scheme we must rely upon some known solution to the prototype nonlinear system (1.1)–(1.8) as the seed one. In practice the most preferable seed solution is chosen to be a trivial (presumably vacuum) solution ensuring the respective seed solution to the auxiliary linear problem (2.4) and (2.5) being presentable in an explicit closed form.

The most suitable seed solution to the prototype nonlinear system of our interest (1.1)–(1.8) looks as follows

$$^s p_{11}(n) = 0, \quad ^s p_{13}(n) = 0, \quad ^s p_{31}(n) = 0, \quad ^s p_{33}(n) = 0, \quad ^s F_{12}(n) = 0, \quad ^s G_{21}(n) = 0, \quad ^s G_{23}(n) = 0, \quad ^s F_{32}(n) = 0, \quad \text{where}$$

$$0 F_{12}, 0 G_{21}, 0 G_{23}, 0 F_{12} \text{ are coordinate- and time-independent quantities subjected to the natural condition}$$

$$0 F_{12} 0 G_{21} + 0 F_{32} 0 G_{23} = -1. \quad (6.1)$$

Consequently, the seed spectral matrix $^s L(n|z)$ and the seed evolution matrix $^s A(n|z)$ become coordinate- and time-independent $^s L(n|z) = 0 L(z)$ and $^s A(n|z) = 0 A(z)$. Moreover, the zero-curvature equation (2.1) yields the strict commutativity relation

$$0 A(z) 0 L(z) - 0 L(z) 0 A(z) = 0 \quad (6.2)$$

and hence the matrices $0 L(z)$ and $0 A(z)$ must possess the same set of eigenfunctions. However, the eigenvalues $0 \xi_k(z)$ of matrix $0 L(z)$ are proved to be distinct from the eigenvalues $0 \eta_k(z)$ of matrix $0 A(z)$. Precisely we have

$$0 \xi_1(z) = z \quad (6.3)$$

$$0 \xi_2(z) = \lambda(z) \equiv z + 1/z \quad (6.4)$$

$$0 \xi_3(z) = 1/z \quad (6.5)$$

and

$$0 \eta_1(z) = 1/z \quad (6.6)$$

$$0 \eta_2(z) = 0 \quad (6.7)$$

$$0 \eta_3(z) = z. \quad (6.8)$$

As for the seed solution $^s X(n|z) = 0 X(n|z)$ to the auxiliary linear problem (2.4) and (2.5) we come to the expression given by matrix

$$0 X(n|z) = \begin{pmatrix}
-0 F_{12} z^n \exp(\tau/z) & +i 0 G_{23} [z + 1/z]^n & -0 F_{12} z^{-n} \exp(\tau z) \\
0 & 0 & 0 \\
-0 F_{32} z^n \exp(\tau/z) & -i 0 G_{21} [z + 1/z]^n & -0 F_{32} z^{-n} \exp(\tau z)
\end{pmatrix}. \quad (6.9)$$
7. Symmetries between the prototype field functions and their implications for the Darboux–Bäcklund scheme

The symmetries between the dynamical fields of an integrable nonlinear dynamical system are known to stipulate the symmetries between the scattering data of relevant auxiliary spectral problem [12, 38, 39]. The practical role of these symmetries for the explicit analytical integration of nonlinear systems is also indisputable [12, 38, 39].

Analyzing the prototype nonlinear system of our interest (1.1)–(1.8) one can readily observe that its equations are consistent with the symmetries of complex conjugation

\[
p_{11}(n) = p_{13}^*(n) \\
p_{13}(n) = p_{31}^*(n) \\
p_{33}(n) = p_{11}^*(n) 
\]

between the prototype field functions. In view of these symmetries (7.1)–(7.4) we inevitably come to the following symmetry relations

\[
\Omega [L(n|1/z^*)]^T \Omega = L(n|z) \\
\Omega [A(n|1/z^*)]^T \Omega = A(n|z)
\]

for the spectral \(L(n|z)\) and evolution \(A(n|z)\) matrices written in terms of prototype fields. Here the involutory matrix \(\Omega\) is given by formula

\[
\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

(7.7)

We impose the similar symmetries

\[
\Omega [X(n|1/z^*)]^T \Omega = X(n|z) \\
\Omega [^cD(n|1/z^*)]^T \Omega = ^cD(n|z)
\]

(7.8) (7.9)

to be valid also for the matrix-solution \(X(n|z)\) to the auxiliary linear problem (2.4) and (2.5) as well as for the Darboux matrix \(^cD(n|z)\) (4.8).

By the way, the expression (6.2) for the zero-labeled seed matrix \(^0X(n|z)\) has been purposely arranged to be consistent with the just adopted symmetries.

By means of the last two symmetry relations (7.5) and (7.8) one can establish the symmetry relations

\[
^c_{z_+} = ^c_{z_+} = 1 = ^c_{z_-} \quad ^c_{z_-}
\]

(7.10)

and

\[
[ ^c_{E1(}^c_{z_+)} ] / ^c_{E1(}^c_{z_-)} = 1 = [ ^c_{E3(}^c_{z_-)} ] / ^c_{E1(}^c_{z_+)} \\
[ ^c_{E2(}^c_{z_+)} ] / ^c_{E2(}^c_{z_-)} = 1 = [ ^c_{E2(}^c_{z_-)} ] / ^c_{E2(}^c_{z_+)} \\
[ ^c_{E3(}^c_{z_+)} ] / ^c_{E1(}^c_{z_-)} = 1 = [ ^c_{E1(}^c_{z_-)} ] / ^c_{E3(}^c_{z_+)}
\]

(7.11) (7.12) (7.13)

for the spectral data \(^c_{z_+}\) and \(^c_{E3(}^c_{z_+}\). When tackling these symmetries (7.10)–(7.13) the expression (4.11) for \(\det ^cD(n|z)\) and the condition (4.12) supporting the property \(\det ^cX(n|}^c_{z_+} = 0\) have also to be invoked.

10
8. Prototype crop solution generated by the two-fold Darboux–Bäcklund transformation

In this section we consider the practical implementation of two-fold Darboux–Bäcklund integration scheme with regard to the prototype integrable nonlinear system (1.5)–(1.8). The last four equations (1.5)–(1.8) of this system clearly indicate that the basic calculations can be safely concentrated only on four prototype functions \( F_{12}(n), G_{23}(n), G_{23}(n), F_{32}(n) \).

To implement our task let us summarize the relevant steps to be made in the framework of two-fold Darboux–Bäcklund integration scheme already reported in the previous Sections. Thus, three sets (5.20)–(5.31) of nonuniform linear equations provide us with the solutions for the Darboux functions \( c^iC_{12}(n), c^iT_{12}(n), c^iD_{22}(n), c^iU_{22}(n), c^iC_{32}(n), c^iT_{32}(n) \). By virtue of relationships (5.23)–(5.26) the obtained expressions for \( c^iC_{12}(n) \) and \( c^iC_{32}(n) \) give rise to the solutions for the Darboux functions \( c^iK_{11}(n), c^iK_{13}(n), c^iK_{31}(n), c^iK_{33}(n) \). The expressions for another two Darboux functions \( c^iV_{21}(n) \) and \( c^iV_{23}(n) \) follow from the relationships (5.27) and (5.28) by the use of already known expression for \( c^iD_{22}(n) \). Once the explicit expressions for the Darboux functions \( c^iC_{12}(n) \) and \( c^iC_{32}(n) \) have been found, the prototype crop functions \( c^iF_{12}(n), c^iF_{32}(n) \) are given by two proper formulas (5.21) and (5.18) of implicit Bäcklund transformation. Another two prototype crop functions \( c^iG_{21}(n) \) and \( c^iG_{23}(n) \) are obtainable as the solutions to the set of two nonuniform linear equations (5.10) and (5.15) of implicit Bäcklund transformation inasmuch as all involved Darboux functions \( c^iK_{11}(n), c^iK_{13}(n), c^iK_{31}(n), c^iK_{33}(n) \) and \( c^iV_{21}(n), c^iU_{22}(n), c^iV_{23}(n) \) have been restored beforehand.

Though just outlined dressing procedure is valid to start with the prototype seed solution \( s_{p11}(n), s_{p13}(n), s_{p31}(n), s_{p33}(n), s_{F12}(n), s_{G21}(n), s_{G23}(n), s_{F32}(n) \) of an arbitrary order \((s = 0, 1, 2, 3, \ldots)\), we apply it to \textit{a priori} known trivial solution \( s_{p11}(n) = 0, s_{p13}(n) = 0, s_{p31}(n) = 0, s_{p33}(n) = 0, s_{F12}(n) = 0, s_{F12}, s_{G21}(n) = 0, s_{G23}(n) = 0, s_{F32}(n) = 0, s_{F32} \) labeled by index \( s = 0 \). Then, taking into account the explicit formula (6.9) for the zeroth seed solution \( s_{X(nz)} \) to the auxiliary linear problem (2.4) and (2.5) we find the prototype crop functions \( s_{F12}(n), s_{F32}(n), s_{G21}(n), s_{G23}(n) \) in the form

\[
2F_{12}(n) = \left(20K_{11}F_{12} + 20K_{13}F_{32}\right) \times \\
- \left(20K_{12}F_{12} + 20K_{13}F_{32}\right)
\]

\[
2F_{32}(n) = \left(20K_{12}F_{12} + 20K_{31}F_{32}\right) \times \\
+ \left(20K_{13}F_{12} + 20K_{31}F_{32}\right)
\]

\[
2G_{21}(n) = \left(20K_{31}G_{21} - 20K_{23}G_{31}\right) \times \\
- \left(20K_{32}G_{21} - 20K_{23}G_{31}\right)
\]

\[
2G_{23}(n) = \left(20K_{31}G_{21} - 20K_{23}G_{31}\right) \times \\
+ \left(20K_{32}G_{21} - 20K_{23}G_{31}\right)
\]
Here
\[ R(n) = (20z^n + 1/20z^n)^\alpha \exp(\tau/20z^n) + (1/20z^n)^\alpha \exp(\tau/20z^n) \]
\[ S(n) = 20^n \exp(\tau/20z^n) \]
(8.5)

In what follows we take
\[ K_{11} = K_{33} \]
(8.7)
and
\[ K_{13} = K_{31} \]
(8.8)
without the loss of generality.

By virtue of the symmetry relations (7.10)–(7.13) it is reasonable to parameterize the spectral data by formulas
\[ z_\pm = \exp(\pm\mu + ik) \]
(8.9)
\[ e_1(20z^n) = \exp(\pm\alpha + i\beta) \]
(8.10)
\[ e_2(20z^n) = g \exp(\pm i\delta) \]
(8.11)
\[ e_3(20z^n) = \exp(\mp\alpha - i\beta) \]
(8.12)
where \( k, \alpha, \beta, g, \delta \) are arbitrary real parameters, and \( \mu \) is the real positive parameter. To suppress systematic divergences of crop field functions at spatial and temporal infinities we are obliged to impose two restrictions
\[ \mu > \ln(\sqrt{2}) \]
(8.13)
\[ k = \sigma \pi/2 \]
(8.14)
on the parameters \( \mu \) and \( k \) with the coefficient \( \sigma \) being defined by the equality
\[ \sigma^2 = 1. \]
(8.15)

In order to apply the general results of this Section to an actual nonlinear integrable dynamical system we must adopt some proper parametrization of prototype field variables compatible both with the natural constraints (3.17), (3.18) and the earlier adopted symmetries (7.1)–(7.4).

9. Reduced ternary system in the framework of Lagrangian and Hamiltonian formulations

As we have already mentioned in Section 3, the natural constraints (3.17) and (3.18) are empowered to reduce eight prototype field variables to six actual dynamical ones. In so doing, the dynamics of the reduced system will be governed by three nonlinear Lagrange equations or alternatively by six nonlinear Hamilton equations. The key idea to introduce appropriate dynamical variables is to invent parametrization formulas converting both of natural constraints (3.2) and (3.3) into identities with the symmetry relations (7.1)–(7.4) taken into account. A particular realization of such parameterizations is not unique \[22,23\] and here we consider the following one

\[ F_{12}(n) = + \exp[\pm q_-(n)] \sqrt{1 + i\sigma(n)}/2 \]
(9.1)
\[ G_{21}(n) = - \exp[-q_-(n)] \sqrt{1 + i\sigma(n)}/2 \]
(9.2)
\[ G_{23}(n) = - \exp[-q_+(n)] \sqrt{1 - i\sigma(n)}/2 \]
(9.3)
\[ F_{32}(n) = + \exp[\pm q_+(n)] \sqrt{1 - i\sigma(n)}/2 \]
(9.4)
actual nonlinear lattice system governed by the set of three Lagrange equations

\[ q = \frac{1}{2} \dot{q}_\pm(n)[1 + s^2(n)] + \frac{1}{2} \dot{q}_\mp(n)[1 + is(n)] \]  

(9.5)

\[ p_{13}(n) = \frac{\exp[ q_\pm(n) - q_\mp(n)]}{4 \sqrt{1 + s^2(n)}} \{ \dot{q}_\pm(n)[1 + s^2(n)] + \dot{q}_\mp(n)[1 - s^2(n)][1 - is(n)] + 2is(n) \} \]  

(9.6)

\[ p_{31}(n) = \frac{\exp[ q_\pm(n) - q_\mp(n)]}{4 \sqrt{1 + s^2(n)}} \{ \dot{q}_\pm(n)[1 + s^2(n)] + \dot{q}_\mp(n)[1 + s^2(n)][1 + is(n)] - 2is(n) \} \]  

(9.7)

\[ p_{33}(n) = \frac{1}{4} [ \dot{q}_\pm(n) - \dot{q}_\mp(n)][1 + s^2(n)] + \frac{1}{2} \dot{q}_\mp(n)[1 - is(n)] \]  

(9.8)

These parametrization formulas (9.1)—(9.8) are seen to support the complex conjugate symmetries \( q_\pm(n) = q_\pm^*(n) \) and \( q_\mp(n) = q_\mp^*(n) \) between the field variables \( q_\pm(n) \) and \( q_\mp(n) \) supplemented by the reality \( s(n) = s^*(n) \) of field variable \( s(n) \). Evidently, the same symmetries take place for the time derivatives of field variables.

By dint of the adopted parameterizations (9.1)—(9.8), the prototype nonlinear system (1.1)—(1.8) is reduced to an actual nonlinear lattice system governed by the set of three Lagrange equations

\[ \frac{d}{dt} [\partial L/\partial \dot{q}_\pm(n)] = \partial L/\partial q_\pm(n) \]  

(9.9)

\[ \frac{d}{dt} [\partial L/\partial \dot{s}(n)] = \partial L/\partial s(n) \]  

(9.10)

\[ \frac{d}{dt} [\partial L/\partial \dot{q}_\mp(n)] = \partial L/\partial q_\mp(n) \]  

(9.11)

with the Lagrange function given by expression

\[ L = \frac{1}{4} \sum_{m=-\infty}^{\infty} [1 - is(m)] q_\pm^2(m) + \frac{1}{4} \sum_{m=-\infty}^{\infty} [1 + is(m)] q_\mp^2(m) + \frac{1}{8} \sum_{m=-\infty}^{\infty} [1 + s^2(m)][\dot{q}_\pm(m) - \dot{q}_\mp(m)]^2 + \frac{1}{4} \sum_{m=-\infty}^{\infty} s^2(m) \frac{1}{1 + s^2(m)} - \frac{1}{2} \sum_{m=-\infty}^{\infty} \exp[+q_\pm(m + 1) - q_\mp(m)] \sqrt{[1 - is(m + 1)][1 - is(m)]} \]  

\[ - \frac{1}{2} \sum_{m=-\infty}^{\infty} \exp[+q_\mp(m + 1) - q_\pm(m)] \sqrt{[1 + is(m + 1)][1 + is(m)]} + \sum_{m=-\infty}^{\infty} 1 \]  

(9.12)

The relevant Hamiltonian formulation of reduced nonlinear system in question is based upon the set of Hamilton equations

\[ \dot{q}_\pm(n) = \partial H/\partial p_\pm(n) \]  

\[ \dot{p}_\pm(n) = -\partial H/\partial q_\pm(n) \]  

\[ \dot{s}(n) = \partial H/\partial \dot{n}(n) \]  

\[ \dot{r}(n) = -\partial H/\partial s(n) \]  

\[ \dot{q}_\mp(n) = \partial H/\partial p_\mp(n) \]  

\[ \dot{p}_\mp(n) = -\partial H/\partial q_\mp(n) \]  

(9.13)

(9.14)

(9.15)

with the Hamilton function given by expression

\[ H = \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{p_\pm^2(m)}{1 - is(m)} + \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{p_\mp^2(m)}{1 + is(m)} + \frac{1}{4} \sum_{m=-\infty}^{\infty} [ p_\pm(m) + p_\mp(m) ]^2 + \frac{1}{2} \sum_{m=-\infty}^{\infty} [1 + s^2(m)] \frac{r^2(m)}{1 + s^2(m)} + \frac{1}{2} \sum_{m=-\infty}^{\infty} \exp[ q_\pm(m + 1) - q_\mp(m)] \sqrt{[1 - is(m + 1)][1 - is(m)]} \]  

\[ + \frac{1}{2} \sum_{m=-\infty}^{\infty} \exp[ q_\mp(m + 1) - q_\pm(m)] \sqrt{[1 + is(m + 1)][1 + is(m)]} - \sum_{m=-\infty}^{\infty} 1 \]  

(9.16)
The standard Hamiltonian form (9.13)–(9.15) of above dynamical equations points out on the standard form of related Poisson structure.

In either of its incarnations (9.9)–(9.12) or (9.13)–(9.16) the reduced nonlinear dynamical system comprises three dynamical subsystems coupled both in their kinetic and potential parts and it can be referred to as the nonlinear integrable ternary system with the combined inter-mode couplings. Two of its subsystems are seen to be qualified as the subsystems of complex-valued Toda-like type.

One can readily verify that the Lagrange \( \mathcal{L} \) (9.12) and Hamilton \( \mathcal{H} \) (9.16) functions are related via the Legendre transformation

\[
\mathcal{H} + \mathcal{L} = \sum_{m=-\infty}^{\infty} [p_+(m)\dot{q}_+(m) + r(m)\dot{s}(m) + p_-(m)\dot{q}_-(n)]
\]  

with the quantities \( p_+(n), r(n), p_-(n) \) being defined by the conventional relationships

\[
p_+(n) = \partial \mathcal{L} / \partial \dot{q}_+(n)
\]

\[
r(n) = \partial \mathcal{L} / \partial \dot{s}(n)
\]

\[
p_-(n) = \partial \mathcal{L} / \partial \dot{q}_-(n).
\]

10. Modulated pulson solution to the nonlinear integrable ternary system

According to the Lagrange representation (9.9)–(9.12) the integration of reduced nonlinear system can be safely restricted only by calculations of coordinate field functions \( q_+(n), q_-(n) \). To accomplish this purpose we must invert earlier obtained results (8.1)–(8.4) for \( \, \, 2F_{12}(n), \, \, 2G_{21}(n), \, \, 2G_{23}(n), \, \, 2F_{32}(n) \) into formulas for \( \, \, 2q_+(n), \, \, 2s(n), \, \, 2q_-(n) \) by means of respective parametrization formulas (9.1)–(9.4). In so doing, the specification formulas (8.5)–(8.15) supplemented by the parametrization formulas

\[
0F_{12} = + \exp(+q_-) \sqrt{(1 + is)/2}
\]

\[
0G_{21} = - \exp(-q_-) \sqrt{(1 + is)/2}
\]

\[
0G_{23} = - \exp(-q_+) \sqrt{(1 - is)/2}
\]

\[
0F_{32} = + \exp(+q_+) \sqrt{(1 - is)/2}
\]

for the prototype seed quantities \( \, \, 0F_{12}, \, \, 0G_{21}, \, \, 0G_{23}, \, \, 0F_{32} \) turn out to be useful. The final result for the crop solution to the reduced integrable nonlinear lattice system of our interest (9.9)–(9.12) obtainable in the framework of two-fold Darboux–Bäcklund dressing technique reads as follows

\[
q_+(n) = q_+ + \frac{1}{2} \ln \left[ \frac{\Phi^2(n)}{\Phi^2(n - 1)} \right] + \frac{1}{4} \ln \left\{ \left[ 1 + s \frac{\exp(-q_+ - q_-) \Theta(n)}{\Phi(n)} \right]^2 + \frac{\exp(-2q_+ - 2q_-) \Theta(n)}{1 + s^2} \frac{\Phi^2(n)}{\Phi^2(n)} \right\} - \frac{i}{2} \arctan \left[ \frac{\exp(-q_+ - q_-) \Theta(n)}{\Phi(n) + s \exp(-q_+ - q_-) \Theta(n)} \right]
\]

\[
s(n) = s + \exp(-q_+ - q_-) \sqrt{1 + s^2} \frac{\Theta(n)}{\Phi(n)}
\]
\[ q_-(n) = \frac{1}{2} \ln \left[ \frac{\Phi^2(n)}{\Phi^2(n-1)} \right] + \frac{1}{4} \ln \left\{ \frac{\exp(-q_+ - q_-)}{\Phi(n)} + \frac{\exp(-2q_+ - 2q_-)}{\Phi(n)} \right\} + \frac{1}{2} \arctan \left[ \frac{\exp(-q_+ - q_-)}{\Phi(n) + s \exp(-q_+ - q_-)} \right]. \] (10.7)

Here the upper front indices in field functions \( \tilde{q}_+(n) \), \( \tilde{s}(n) \), \( \tilde{q}_-(n) \) have been omitted for the stylistic purposes and the shorthand notations

\[ \Phi(n) = 2\sigma \sinh(2\mu - \mu + 2\alpha) + 2 \sinh(\mu) \sin[\sigma n - \sigma \pi/2 + 2\beta - 2\tau \sigma \cosh(\mu)] \] (10.8)

\[ \Theta(n) = g \exp(\mu n - \mu + \alpha)[2 \sinh(\mu)]^n \sin[\sigma n/2 + \delta - \beta + \tau \sigma \exp(\mu)] + g \exp(-\mu n + \mu - \alpha)[2 \sinh(\mu)]^n \sin[\sigma n - \sigma \pi/2 + \delta + \beta - \tau \sigma \exp(\mu)] \] (10.9)

for the typical functional combinations have been introduced. The obtained solution (10.5)–(10.7) embraces all three coupled subsystems to be excited. Considering the expression (10.8) for the function \( \Phi(n) \) appearing in denominators of all three components \( q_+(n), s(n), q_-(n) \) of our solution (10.5)–(10.7), we clearly see that it has distinct signs at opposite spatial infinities. Consequently, at some unfortunate values of parameter \( \alpha \) the quantity \( \Phi(n) \) as the function of time \( \tau \) can change its sign even at some admissible (i.e. integer) values of spatial coordinate \( n \). At such spatial coordinates all three components \( q_+(n), s(n), q_-(n) \) as the functions of time \( \tau \) are inflicted by the infinite discontinuities emerging periodically in time with the cyclic frequency equal to \( 2 \cosh(\mu) \).

In order to eliminate such nonphysical solutions we must impose the additional condition \( \alpha = -\mu \) onto the parameter \( \alpha \), where \( l \) is an arbitrary integer. In this adjusted case we have \( \sinh(2\mu - \mu - 2\mu) \leq -\sinh(\mu) \) at \( n \leq l \) and \( \sin(2\mu - \mu - 2\mu) \geq + \sinh(\mu) \) at \( n \geq l + 1 \). Therefore, the modified quantity

\[ \Phi(n) = 2\sigma \sinh(2\mu - \mu - 2\mu) + 2 \sinh(\mu) \sin[\sigma n - \sigma \pi/2 + 2\beta - 2\tau \sigma \cosh(\mu)] \] (10.10)

taken as the function of time \( \tau \) can acquire zero values only in two neighboring spatial points \( n = l \) and \( n = l + 1 \) never changing its sign in either of them.

As a result, the components \( q_+(n) \) and \( q_-(n) \) demonstrate splashes of infinite amplitudes appearing periodically in time with the cyclic frequency equal to \( 2 \cosh(\mu) \) in three neighboring spatial points \( n = l, n = l + 1, n = l + 2 \), while the component \( s(n) \) demonstrates splashes only in two points \( n = l \) and \( n = l + 1 \). In so doing, all three components remain being continuous functions of time \( \tau \). Evidently, the splashes taking place in three distinct spatial points \( n = l, n = l + 1, n = l + 2 \) occur in different instances of time. Moreover, in each of these analyzed formulas (10.5)–(10.7) we observe contributions modulated by the function \( \Theta(n) \) characterized by two low cyclic frequencies \( \exp(\mu) \) and \( \exp(-\mu) \). For this reason the nonlinear wave packet described by the obtained solution (10.5)–(10.7) could be referred to as the modulated pulson. In general, all three cyclic frequencies \( 2 \cosh(\mu) \) and \( \exp(\mu), \exp(-\mu) \) are seen to be incommensurable. Hence, the spatial pattern of modulated pulson observed at one instant of time can never be repeated at any other instant.

11. Conclusion

By writing this paper we have tried to attract attention to the broad investigation of multi-component semi-discrete nonlinear integrable systems associated with the auxiliary linear problems of third and more higher orders. In view of very sophisticated dynamics prospective for the description of various physical phenomena such systems require the development of adequate and understandable methods of their integration. The first fundamental steps in this direction had been made by Caudrey in his famous work [34] suggested the most general inverse scattering approach to the integration of semi-discrete nonlinear integrable systems generated by the auxiliary linear problems of arbitrary orders. Yet, up to now his pioneer attempts appear to be underestimated. Maybe such an attitude is caused by the tedious analytical calculations inevitably arising at practical implementation of Caudrey’s integration scheme [22,23].
In present paper we have considered the alternative approach of integration as applied to the particular prototype semi-discrete nonlinear integrable system associated with the relevant auxiliary linear problem of third order. The approach is based upon the two-fold Darboux transformation for the seed auxiliary matrix function accompanied by the implicit Bäcklund transformation for a supposedly known seed solution to the prototype semi-discrete nonlinear integrable system itself. The majority of calculations have been made in terms of prototype field functions in order that their results to be suitable for any reduced semi-discrete nonlinear integrable system compatible with the so called natural constraints and the adopted symmetries for the prototype field functions.

We have also used the generalized version of direct recurrent approach to find the main conservation laws applicable to any integrable system associated with the chosen form of third order auxiliary linear problem. One of the obtained conservation laws originates the Hamilton function of any reduced semi-discrete nonlinear integrable system comprising three coupled dynamical subsystems. This fact is readily verifiable on the reduced ternary system explicitly written down in terms of two complex-valued Toda-like subsystems and one intermediate subsystem. The coupling between the Toda-like subsystems is mediated by the intermediate subsystem both in the kinetic and potential parts of Hamilton function.

Having taken into account the general findings of two-fold Darboux–Bäcklund dressing scheme we have obtained the explicit solution to the reduced ternary system in the form of modulated pulson wave packet.

As the final remark, it is necessary to stress that the complex-valued dynamical fields in the reduced ternary system can be rearranged into the purely real fields responsible for the translational and orientational modes typical of long macromolecules both synthesized and natural origins.

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