Symmetry Protected Weak Topological Phases in a Superlattice
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We explore novel topological phases realized in a superlattice system based on the Wilson-Dirac model. Our main focus is on a two-dimensional analogue of weak topological insulator phases. We find such phases as those characterized by gapless edge states that are protected by symmetry but sensitive to the orientation of the edge relative to the superlattice structure. We show that manifest and hidden reflection symmetries protect such weak topological phases, and propose bulk $\mathbb{Z}_2$ indices responsible for the topological protection of the edge states.

KEYWORDS: topological insulator, superlattice, Chern number, weak topological phase, Wilson-Dirac model

Topological classification is a new trend in the field of condensed matter. Although accepted only recently by the wide community in condensed matter physics, its position is very influential for determining future directions of the field. Beyond the Ginzburg-Landau paradigm, topological classification can be applicable to quantum liquids without fundamental symmetry breaking.¹,² Still, truly generic systems are not interesting and the symmetry again restricts the systems. Then, we have various physically interesting phases protected by symmetries. The periodic table of topological phases as an extension of the classical symmetry classes is situated at the heart of the idea.³–⁸ Recently, the extension of the standard classification scheme by including the diversity of topological phases protected by other types of symmetry has been investigated.⁹–¹⁶

Topological phases are mostly gapped; thus, their bulk is characterized by the absence of low-energy excitations. On the other hand, with boundaries or impurities, there exist peculiar localized modes as edge states. The emergence of the edge states is not accidental and is a fundamental property of topological phases, known as the bulk-edge correspondence.²⁷,²⁸ The edge states further reflect symmetries of topological phases and describe their variety beyond the bulk characterization.

In this study, we attempt to further extend the idea of the topological classification to a superlattice version of the Wilson-Dirac-type lattice model that exhibits hidden reflection symmetry. The motivation of this work is not purely academic. Now that basic understanding of simple topological insulator crystals has been established, a possible direction of not only theoretical research but also experimental research is to seek various topological quantum phenomena. Recently, multilayer heterostructures consisting of alternating layers of topological and ordinary insulators have been experimentally realized, exhibiting an interesting correlation of bulk and surface properties.¹⁸,¹⁹

In this letter, we highlight a two-dimensional (2D) analogue of such a superlattice system: a variant of the Wilson-Dirac type tight-binding model with a stripe structure (see Fig. 1). Without such a spatial nonuniformity, the simple 2D Wilson-Dirac model is known as the typical $\mathbb{Z}$-type model specified by the Chern number.²⁰ The extension of this model to the quantum spin Hall effect (QSHE)²¹ has been carried out,²² in which the $\mathbb{Z}_2$ invariant²³,²⁴ distinguishes between the QSHE and trivial phases. Here, we demonstrate that the superlattice version exhibits a richer phase diagram (see Fig. 2) that cannot be classified by a single topological invariant. Since the mass parameter $m$ controls the Chern number $c$ in a uniform system, the superlattice of the mass $(m_+, m_-)$ can be regarded as that of distinct Chern insulators ($c_+, c_-$). On the other hand, this model has the total Chern number $C$ in its own right. We find that, in the $C = 0$ sector, there appears a 2D analogue of weak topological insulating phases²⁵–²⁷ characterized by anisotropic topological properties. Let us mention here the system’s similarity to graphene. Although graphene is already gapless in the bulk, it exhibits direction-dependent boundary states with time reversal symmetry ($C = 0$).²⁸–³⁰

Figure 1 shows a schematic configuration of the superlattice model considered in this letter. The model is defined on a square lattice, and on each site $r = (x, y)$ of the lattice an electron is allowed to occupy two orbital (pseudo-spin) states with which a set of Pauli matrices $\sigma_\mu$ ($\mu = x, y, z$) is associated. The Hamiltonian is defined by

$$H = \sum_r \sum_{\mu = x, y} \left[ \langle r|\Gamma_\mu(r + \hat{\mu}) + |r + \hat{\mu}\rangle \Gamma_\mu^\dagger(r) \right] + |r\rangle V(r)\langle r|,$$

(1)

where $\hat{\mu}$ stands for the unit vector in the $\mu$-direction, and the hopping and on-site potential terms are respectively specified.
where $d_i$ on the simplest superlattice structure shown in Fig. 1 in which $V$ is a vertical stripe. Taking into account the doubling of the unit cell one can block-diagonalize eq. (1) in the reciprocal space as

$$\langle k \sigma \rangle = \frac{i t}{2} \sigma_x + \frac{b}{2} \sigma_z,$$

$$V(r) = [m + (-1)^x \delta m - 2b] \sigma_z \equiv (m_{\pm} - 2b) \sigma_z. \quad (2)$$

Note that the lattice constant has been chosen to be unity; hence, $x$ and $y$ take only integral values. In this letter, we focus on the simplest superlattice structure shown in Fig. 1 in which $V(r)$ takes two alternating values on each column of the vertical stripe. Taking into account the doubling of the unit cell due to the stripe texture as a sublattice degree of freedom, one can block-diagonalize eq. (1) in the reciprocal space as $H = \sum_k \langle k \rangle \mathcal{H}(k)$ whose matrix element $\mathcal{H}(k)$ is given by

$$\mathcal{H}(k) = \begin{pmatrix}
M_x + t \sin k_x \sigma_y & \Gamma_x + e^{2ik_x} \Gamma_y \\
\Gamma_y + e^{2ik_y} \Gamma_x & M_y + t \sin k_y \sigma_x
\end{pmatrix}, \quad (3)$$

where different rows and columns specify the sublattice, i.e., whether the electron is on a blue column ($x$: even) or a red one ($x$: odd) in Fig. 1(a), and $M_{\pm} = [m_x + b(\cos k_y - 2)] \sigma_z$.

The uniform line $m_x = m_y = m (\delta m = 0)$ corresponds to the standard Wilson-Dirac mode\cite{20,22} $\mathcal{H}(k) = t \sum_{\mathbf{k}} \sin k_x \sigma_x + m(k) \sigma_z$, where $m(k) = [m + b \sum_{\mathbf{k}} (\cos k_y - 1)] \sigma_z$. The half-filled ground state of the model is classified by the Chern number\cite{31,32} that takes a nontrivial value of $C = 1$ when $0 < m/b < 2$, while $C = -1$ when $2 < m/b < 4$ (otherwise, $C$ is a trivial value ($C = 0$)); the change in the topological number corresponds to the closing of the gap [zeros of $m(k)$] at the Dirac point $k = (0, 0)$ and at its double’s points ($\pi, 0$), ($0, \pi$), ($\pi, \pi$). Away from the uniform line, it is still possible to compute the Chern number $C$ by averaging the method given in ref.\cite{33} to the present superlattice model. The phase diagram thus obtained is shown in Fig. 2.

This phase diagram has the following specific features: Topologically nontrivial phases with nonzero Chern numbers $C = \pm 1$ extend from the uniform line to a region of $m_x \neq m_{\pm}$. The regions of $C = \pm 1$ overlap (at least they appear to do so in the phase diagram) to form a finite domain of the $C = 0$ phase represented by A in Fig. 2. There appear other $C = 0$ phases in different parts of the phase diagram separated by topologically nontrivial phases. Are these $C = 0$ phases simply topologically trivial? Our answer is “No” in the phases represented by “A” and “B”. These phases exhibit gapless edge states in the ribbon geometry [panels (a) and (d) of Fig. 3]. Interestingly, the way these edge states appear depends on the way the system’s boundaries are introduced (compare the top and bottom rows of Fig. 3). The structure of the phase diagram, particularly the shape of regions A and B is strongly dependent on the hopping amplitude $t$ in contrast to the phase boundaries on the uniform line $m_x = m_y$.\cite{34} It should be emphasized that, although the concrete arrangement of distinct regions shown in Fig. 2 is not generic and varies continuously as a function of $t$, the behaviors of the edge states in the two types of $C = 0$ phases A and B are generic, implying that they are protected by some symmetry.

The concept of the bulk-edge correspondence is now established,\cite{17,35,37} indicating that the topological property of the bulk is fully reflected in the spectrum of the edge states. In the rest of this letter, we show that the gapless edge modes found in $C = 0$ phases A and B are indeed topologically protected, and interpret the corresponding $C = 0$ phases as a 2D analogue of the weak topological insulator in 3D.\cite{25,27} Figure 3 highlights the “weak” nature of phases A and B.

In phase A, edge states appear at the boundaries parallel to the y-axis [Fig. 3(a)], whereas no edge states appear along those parallel to the x-axis [Fig. 3(c)]. As indicated in Fig. 3(a) a pair of counter-propagating modes, one localized at the left boundary (L) and the other at the right boundary (R), cross at zero energy and at a specific momentum (symmetric point), either at $k_x = 0$ or $k_x = \pi$. As we will show soon, this phase is reminiscent of the QShe, since $C = c_+ + c_- = 0$ for $c_+ = 1$ and $c_- = -1$, and one pair of edge states is due to $c_+ = 1$, and the other pair is due to $c_- = -1$. Note that, in the present model, the existence of particle-hole and inversion (or reflection) symmetries ensures the spectrum at each $k$ to be symmetric with respect to zero. This is also the case with the edge spectrum.

Phase B is, on the other hand, understood by considering the limit $m_{\pm} \to \infty$. We claim that the midgap states in Fig. 3(d) can be deformed into topologically protected flat bands in this limit. To be concrete, when $m_{\pm} \to \infty$, electron occupation at $m_{\pm}$ sites is suppressed, and the model reduces to just a set
of isolated one-dimensional ladders described by the reduced Hamiltonian

$$\mathcal{H}(k_z) = t \sin k_z \sigma_y + [m + b(\cos k_z - 2)] \sigma_z. \quad (4)$$

Owing to the chiral symmetry of this Hamiltonian, the Berry phase integrated over $k_z$ is quantized to $0$ or $\pi.29,30,38$ To see this, set $Y = t \sin k_z$ and $Z = m + b(\cos k_z - 2)$. Then, if the origin $(0, 0)$ is located inside the ellipse $(Y, Z)$ forms in the $Y-Z$ plane, i.e., $1 < m + b < 3$, the Berry phase is $\pi$ (nontrivial) in which zero-energy flat bands are expected.29,30,38 Thus, the isolated mid-gap states in Fig. 3(d) can be deformed into these topologically protected flat bands without the bulk gap closing. Indeed, in the phase diagram in Fig. 2, the blue and red regions become narrower as $m_1 \to \infty$, converging respectively to a linear region on $m_1 = 2$ and $m_1 = 6$.

What bulk topological invariant characterizes these weak topological phases embedded in $C = 0$? In contrast to the so-called $\mathbb{Z}_2$ topological insulator, the present system lacks time-reversal symmetry. Yet, as we demonstrate below, the proposed weak topological phases are protected by another type of $\mathbb{Z}_2$ invariant associated with manifest reflection symmetry as well as hidden reflection symmetry. To see this, it is convenient to introduce a unitary-transformed Hamiltonian:

$$\tilde{\mathcal{H}}(k) = U(k_z)\mathcal{H}(k)U^\dagger(k_z),$$
$$U(k_z) = \mathbb{I}_2 \otimes \text{diag}(1, e^{i k_z}), \quad (5)$$

where $\mathbb{I}_2$ operates on the Pauli matrices. The transformed Hamiltonian $\tilde{\mathcal{H}}(k)$ is represented simply as

$$\tilde{\mathcal{H}}(k) = a_\gamma \gamma_\mu + b_\gamma \gamma_\mu, \quad (6)$$

where $\gamma$-matrices are defined by $\gamma_1 = \sigma_x \otimes \sigma_y, \gamma_2 = \sigma_y \otimes \mathbb{I}_2, \gamma_3 = \sigma_x \otimes \sigma_y, \gamma_5 = \sigma_y \otimes \sigma_x$, and $\gamma_{\mu \nu} \equiv i[\gamma_\mu, \gamma_\nu]/2$. The coefficients $a_\gamma$ and $b_\gamma$ are listed in Table I.

We begin by demonstrating the following two properties:

(i) $\tilde{\mathcal{H}}(k)$ possesses not only particle-hole symmetry but also reflection (inversion) symmetry and (ii) $\tilde{\mathcal{H}}(k)$ is not periodic with respect to $k_z$, $\mathcal{H}(k_z + \pi, k_y) = U(\pi)\mathcal{H}(k_z, k_y)U^\dagger(\pi)$. Here, the form of $U(\pi) = \mathbb{I}_2 \otimes \sigma_x$ implies that a twisted boundary condition is imposed on the Hamiltonian $\tilde{\mathcal{H}}(k)$. Nevertheless, the half-filled ground states of $\tilde{\mathcal{H}}(k)$ and $\mathcal{H}(k)$ give the same Chern number on the same Brillouin zone $[0, \pi] \otimes [0, 2\pi]$.

Let us first note that the Hamiltonian (6) has particle-hole symmetry,

$$\Xi \tilde{\mathcal{H}}(-k) = -\tilde{\mathcal{H}}(k), \quad (7)$$

where $\Xi = -i \gamma_2 \gamma_3 K = \sigma_1 \otimes \mathbb{I}_2 K$ and $K$ is the complex conjugation operator. The presence of this symmetry is rather natural if one recalls that eq. (1) is a straightforward extension of the Wilson-Dirac Hamiltonian. The model has other symmetries described as

$$P_y \tilde{\mathcal{H}}(k_x, k_y)P_y^{-1} = \tilde{\mathcal{H}}(-k_x, k_y), \quad (8)$$

where $P_y = \gamma_3 K$ and $P_z = K$. These may be regarded as (anti-unitary) reflection symmetry with respect to the $y$- and $x$-directions, respectively, and therefore, the model has the inversion symmetry $P \tilde{\mathcal{H}}(k)P^{-1} = \mathcal{H}(-k)$, where $P = P_y P_z = \gamma_3$. The symmetries (7) and (8) are also manifest in $\mathcal{H}(k)$, but the symmetry (9) is hidden in $\mathcal{H}(k)$.

(ii) We introduce the Chern number $C$ for the transformed Hamiltonian. Let $\tilde{\psi}(k)$ be the negative energy multiplet of the Hamiltonian $\tilde{\mathcal{H}}(k)$ with a phase convention,

$$\tilde{\psi}(-k_x, k_y) = P_y \tilde{\psi}(k_x, k_y),$$
$$\tilde{\psi}(k_x, -k_y) = P_y \tilde{\psi}(k_x, k_y). \quad (10)$$

Note that the periodicity of the wave functions is such that $\tilde{\psi}(k_x + \pi, k_y) = U(\pi)\tilde{\psi}(k_x, k_y)$, where $U(\pi) = -i \gamma_1 \gamma_4$. Let $A_i = \tilde{\psi}^\dagger \partial_i \tilde{\psi}$ and $F_{12} = e_i \partial_i A_i$ be the Berry connection and curvature, respectively, where $\partial_i \equiv \partial / \partial k_i$. Because of eq. (10), these obey

$$\tilde{A}_j(-k_x, k_y) = (-)^{j-1} \tilde{A}_j(k_x, k_y),$$
$$\tilde{A}_j(k_x, -k_y) = (-)^j \tilde{A}_j(k_x, k_y),$$
$$\tilde{F}_{12}(-k_x, k_y) = \tilde{F}_{12}(k_x, k_y). \quad (11)$$

The Chern number $C$ of the half-filled states is given by the integration of $\tilde{F}_{12}(k)$ over the Brillouin zone. Let us verify $C = C$, where $C$ is defined in terms of the wave function $\psi(k)$ of the original Hamiltonian $\mathcal{H}(k)$ in eq. (3). The two wave functions can be related as $\tilde{\psi}(k) = U(k)\psi(k)$. This implies that $A_i = A_i + \psi^\dagger U^\dagger \partial_i \psi$, and hence, $F_{12} = \tilde{F}_{12} + e_i \partial_i (\text{tr} P_U U^\dagger \partial_i U)$, where $A_i$ and $F_{12}$ are the Berry connection and curvature defined through $\psi(k)$, respectively, and $P_{-}(k)$ is the projection operator for the occupied states, $P_{-}(k) = \psi(k)\psi^\dagger(k)$. Then, since $U^\dagger \partial_i U = -i \mathbb{I}_2 \otimes \text{diag}(0, 1)$ is a constant matrix and $P_{-}(k)$ is gauge-invariant as well as periodic on the Brillouin zone, the above difference between $\tilde{F}$ and $F$ vanishes if it is integrated over the Brillouin zone owing to the Stokes theorem on the torus. Thus, we reach $C = C$.

On the basis of the properties (i) and (ii), let us consider a topological invariant that characterizes the weak topological phase studied so far. Consider the Berry connection $A_i$. Because of eq. (10), the obstruction due to gauge fixing occurs mainly along the four symmetry lines $k_x = 0, \pi/2$ and $k_y = 0, \pi$. Namely, if the Berry connection is plotted on the Brillouin zone, vortices can appear on these lines. Moreover, they are always paired because of eq. (11). For example, on the Brillouin zone defined by $[0, \pi] \otimes [0, 2\pi]$, a pair on the $k_x = 0$ line sits on the points $(0, k_y^*)$ and $(0, 2\pi - k_y^*)$ with the same vorticity. Of course, at some other points away from these symmetry lines, obstructions can also occur. In this case, vortices appear “in quartets”, which are symmetric with respect to the four symmetry lines. These also have the same vorticity. Thus, we know that an even number of vortices always appear as long as the phase convention of eq. (10) is adopted. However, there are exceptions. Namely, the four crossing points of the four symmetry lines, i.e., $(0, 0), (\pi/2, 0), (\pi/2, \pi)$, and $(0, \pi)$, which will be referred to as $X_1, \ldots, X_4$ in this order. On these points, single vortices can appear. These single vortices cannot move away from these points even if one makes local gauge transformation, since if they did so,
they would need an odd number of partners, as discussed above. Indeed, in the phase \( C = \pm 1 \) in Fig. 2, an odd number of vortices are located on these four symmetry points \( \{ X_i \} \) in all gauges, as long as the phase convention of eq. (10) is used. See Table II.

This implies that unpaired vortices on these points can be used to reveal the topological properties of the present system. Indeed, these vortices inform us of the “parity” of the Chern number. Moreover, different configurations of vortices on \( \{ X_i \} \) imply topologically different phases, since the location of these vortices is gauge-invariant. Therefore, it is natural to expect that the \( C = 0 \) phase can be further distinguished by the obstruction on the four symmetry points. Note that the present Hamiltonian \( \tilde{\mathcal{H}} \) has inversion symmetry. Therefore, the obstruction on the four points \( \{ X_i \} \) is associated with the parity of the wavefunctions. With respect to the parity operator \( P \), eq. (10) implies that we choose

\[
\tilde{\psi}(-k) = P \tilde{\psi}(k). \tag{12}
\]

At the point \( X_1 = (0, 0) \), for example, this means that \( \tilde{\psi}(0) = P \tilde{\psi}(0) \). Let us assume that among the two occupied wavefunctions of \( \tilde{\psi}(0) \), \( n \) wavefunctions have a parity of \(-1 \) and the others have a parity of \(+1 \). Then, the former are the obstructions of the gauge-fixing condition (12). When \( n \) is even \( (n = 0 \) or \( 2 \)\), there appear an even number of vortices at \( X_1 \), which can move away from this point via suitable gauge transformation. However, if \( n \) is odd \( (n = 1) \), one vortex is forced to locate at \( X_1 \). The \( \mathbb{Z}_2 \) invariant in this context can be extracted from \( \det[\tilde{\psi}(-k)P \tilde{\psi}(0)] = \pm 1 \), where the case \(-1 \) is the \( \mathbb{Z}_2 \) obstruction of the condition (12).

This simple observation is extended to other symmetry points \( X_i \). Here, it should be noted that the wavefunction \( \psi(k) \) is not periodic with respect to \( k_x \) because of the twist operator \( U(k_x) \), as discussed below eq. (10). In particular, \( \psi(-k_x + \bar{k}_x, \bar{k}_y) = U(\bar{k}_y) \psi(-k_x, \bar{k}_y) = U(\bar{k}_y) P(k_x, \bar{k}_y) \), where \( \bar{k}_x = 0 \) or \( \pi \), and the phase convention at \( k_x = \pi/2 \) is thus modified. Note that \( P = \sigma_y \otimes s_2 \) and \( U(\bar{k}_y) = s_2 \otimes \sigma_3 \) and hence \( U(\bar{k}_y) P = \sigma_3 \otimes \sigma_3 \). Thus, by taking this periodicity into account, the condition (12) is explicitly written as

\[
\tilde{\psi}(X_i) = P_i \tilde{\psi}(X_i), \quad i = 1, \ldots, 4, \tag{13}
\]

where \( P_i \equiv \sigma_y \otimes s_2 \) \((i = 1, 4) \) and \( P_i \equiv \sigma_y \otimes \sigma_3 \) \((i = 2, 3) \). This leads to the \( \mathbb{Z}_2 \) obstruction

\[
\delta_i = \det \tilde{\psi}(X_i) P_i \tilde{\psi}(X_i), \tag{14}
\]

which can take \( \pm 1 \) only. Now let us define a set of four numbers \([n_1 n_2 n_3 n_4]\), where \( \delta_i = (-1)^{n_i} \). These numbers are gauge-invariant modulo 2, indicating that \( n_i = 1 \) \((0)\) implies obstruction \((no \text{ obstruction})\) at \( X_i \).

In Table II, we show several examples of this invariant. Note that, even in trivial \( C = 0 \) phases on the \( m_x = m_y \) line, the obtained invariant is \([0110]\), which seems nontrivial at first sight. This is, however, due to the effect of the twist operator \( U(\pi) \). Generally, \( U(\pi) = s_2 \otimes \sigma_3 \) implies that among four states, half are periodic and the other half are anti-periodic in \( k_z \); therefore, the half-filled ground states include one periodic state and one anti-periodic state. Therefore, we should define the relative \( \mathbb{Z}_2 \) invariant as “excitations” for this background obstruction such that \([ [\tilde{n}_1 n_2 n_3 n_4] ] = [n_1 n_2 n_3 n_4] - [0110] \) mod 2.

This is the bulk \( \mathbb{Z}_2 \) invariant that characterizes the weak topological phases in the present superlattice system. From Table II, we can interpret phase A as \([ [0011] ] = [ [1001] ] \) and thus the \( C = 1 + (-1) = 0 \) nature of this phase is established. It is also clear that the phase B, assigned \([ [0111] ] \), is distinguished from this phase.

Let us finally mention how the \( \mathbb{Z}_2 \) indices \( \tilde{n}_i \) are related to the edge spectrum in the two different ribbon geometries shown in Fig. 3. Define \( \tilde{\delta}_i = (-1)^{n_i} \). This is the relative \( \tilde{\delta}_i \) in eq. (14) with respect to the background. Then, as in ref. 39 we consider the product of two \( \tilde{\delta}_i 's \) \( [\tilde{\delta}_{1,1} \tilde{\delta}_{2,2}] \) on the line \( k_x = \bar{k}_x = 0, \pi \), while \( \bar{k}_x = 0, \pi/2 \) and define the quantity \( \pi_{k_x} = [\tilde{\delta}_{1,1} \tilde{\delta}_{2,2}] \). Again, \( \pi_{k_x} = \pm 1 \). If \( \pi_{k_x} = -1 \), the edge spectrum on the ribbon geometry directed to the \( \mu \)-axis becomes gapless at \( k_{\mu} = \varphi_{\mu} \); otherwise, the spectrum is gapped. Here, this statement can be verified empirically using the explicit values of \( \tilde{n}_i \) listed in Table II, while the proof of the statement involves the calculation of the Berry curvature integrated along the loop \( k_{\mu} = \varphi_{\mu} \) (Wilson loop). We leave detailed description of this proof to a forthcoming publication.

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1. X. G. Wen: Phys. Rev. B 40 (1989) 7387.
2. Y. Hatsugai: J. Phys. Soc. Jpn. 75 (2006) 123601.
3. M. R. Zirnbauer: J. Math. Phys. 37 (1996) 4986.
4. A. Altland and M. R. Zirnbauer: Phys. Rev. B 55 (1997) 1142.
5. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig: Phys. Rev. B 78 (2008) 195125; AIP Conf. Proc. 1134 (2009) 10.
6. A. Kitaev: AIP Conf. Proc. 1134 (2009) 22.
7. M. Z. Hasan and C. L. Kane: Rev. Mod. Phys. 82 (2010) 3045.
8. X.-L. Qi and S.-C. Zhang: Rev. Mod. Phys. 83 (2011) 1057.
9. M. Sato: Phys. Rev. B 81 (2010) 220504.
10. X. Chen, Z.-C. Gu, and X.-G. Wen: Phys. Rev. B 83 (2011) 035107.
11. L. Fidkowski and A. Kitaev: Phys. Rev. B 83 (2011) 075103.
12. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa: Phys. Rev. B 85 (2012) 075125.
13. L. Fu: Phys. Rev. Lett. 106 (2011) 106802.
14. R.-J. Slager: A. Mesarov, V. Juričič and J. Zaanen: Nat. Phys. 9 (2013) 98.
15. Y. Ueno, A. Yamakage, Y. Tanaka, and M. Sato: arXiv:1303.0202.
16. C.-K. Chiu, H. Yao, and S. Ryu: arXiv:1303.1843.
17. Y. Hatsugai: Phys. Rev. Lett. 71 (1993) 3697.
18) K. Nakayama, K. Eto, Y. Tanaka, T. Sato, S. Souma, T. Takahashi, K. Segawa, and Y. Ando: arXiv:1206.7043.
19) T. Valla, H. Ji, L. M. Schoop, A. P. Weber, Z.-H. Pan, J. T. Sadowski, E. Vescevco, A. V. Fedorov, A. N. Caruso, Q. D. Gibson, L. Müchler, C. Felser, and R. J. Cava: arXiv:1208.2741.
20) X.-L. Qi, T. L. Hughes, and S.-C. Zhang: Phys. Rev. B 78 (2008) 195424.
21) C. L. Kane and E. J. Mele: Phys. Rev. Lett. 95 (2005) 226801.
22) B. A. Bernevig, T. L. Hughes, and S.-C. Zhang: Science 314 (2006) 1757.
23) C. L. Kane and E. J. Mele: Phys. Rev. Lett. 95 (2005) 146802.
24) L. Fu and C. L. Kane: Phys. Rev. B 74 (2006) 195312.
25) L. Fu, C. L. Kane, and E. J. Mele: Phys. Rev. Lett. 98 (2007) 106803.
26) J. E. Moore and L. Balents: Phys. Rev. B 75 (2007) 121306.
27) R. Roy: Phys. Rev. B 79 (2009) 195322.
28) M. Fujita, K. Wakabayashi, K. Nakada, and K. Kusakabe: J. Phys. Soc. Jpn. 65 (1996) 1920.
29) S. Ryu and Y. Hatsugai: Phys. Rev. Lett. 89 (2002) 077002.
30) Y. Hatsugai: Solid State Commun. 149 (2009) 1061.
31) D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs: Phys. Rev. Lett. 49 (1982) 405.
32) M. Kohmoto: Ann. Phys. 160 (1985) 343.
33) T. Fukui, Y. Hatsugai, and H. Suzuki: J. Phys. Soc. Jpn. 74 (2005) 1674.
34) In the presence of a superlattice structure, it is observed that the structure of the phase diagram is determined by the competition of two length scales: the interval of the stripe structure and the penetration depth of the edge wave function, particularly in the $x$-direction; the latter being a function of $t$. See, e.g., K.-I. Imura, A. Yamakage, S. Mao, A. Hotta, and Y. Kuramoto: Phys. Rev. B 82 (2010) 085118, and references therein.
35) G. E. Volovik: The Universe in a Helium Droplet (Oxford University Press, Oxford, 2003, Sect. 22).
36) A. M. Essin and V. Gurarie: Phys. Rev. B 84 (2011) 125132.
37) T. Fukui, K. Shiozaki, T. Fujiwara, and S. Fujimoto: J. Phys. Soc. Jpn. 81 (2012) 114602.
38) M. Sato, Y. Tanaka, K. Yada, and T. Yokoyama: Phys. Rev. B 83 (2011) 224511.
39) L. Fu and C. L. Kane: Phys. Rev. B 76 (2007) 045302.