Some mean value results related to Hardy’s function

Xiaodong Cao¹, Yoshio Tanigawa²* and Wenguang Zhai³

¹Correspondence: tanigawa@math.nagoya-u.ac.jp; tanigawa_yoshio@yahoo.co.jp
²Nishisato 2-13-1, Meito, Nagoya 456-0084, Japan
Full list of author information is available at the end of the article
This work is supported by the National Natural Science Foundation of China (Grant No. 11971476)

Abstract
Let \( \zeta(s) \) and \( Z(t) \) be the Riemann zeta function and Hardy’s function respectively. We show asymptotic formulas for \( \int_0^T Z(t) \zeta(1/2 + it) dt \) and \( \int_0^T Z^2(t) \zeta(1/2 + it) dt \). Furthermore we derive an upper bound for \( \int_0^T Z^3(t) \chi^\alpha(1/2 + it) dt \) for \( -1/2 < \alpha < 1/2 \), where \( \chi(s) \) is the function which appears in the functional equation of the Riemann zeta function: \( \zeta(s) = \chi(s)\zeta(1 - s) \).

Keywords: Hardy’s function, Mean value theorems, Approximate functional equation, Exponential sum and integral

Mathematics Subject Classification: 11M06, 11N07

1 Introduction
Let \( Z(t) \) be Hardy’s function defined by
\[
Z(t) = \zeta(1/2 + it) \chi^{-1/2}(1/2 + it),
\]
where as usual \( \zeta(s) \) is the Riemann zeta-function and \( \chi(s) \) is the gamma factor appearing in the functional equation of \( \zeta(s) \):
\[
\zeta(s) = \chi(s)\zeta(1 - s).
\] (1)
The explicit form of \( \chi(s) \) is
\[
\chi(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s)
\] (2)
and its asymptotic behavior is given by
\[
\chi(\sigma + it) = \left( \frac{|t|}{2\pi} \right)^{1/2 - \sigma - it} e^{it(\pm \frac{\pi}{4})} \left( 1 + O \left( \frac{1}{|t|} \right) \right)
\] (3)
for \( |t| \geq t_0 > 0 \), where \( t = t \pm \frac{\pi}{4} = t + \text{sgn}(t) \frac{\pi}{4} \). See Ivic [10].

From (1), it follows that \( Z(t) \) is a real-valued even function for real \( t \) and \( |Z(t)| = |\zeta(1/2 + it)| \). Therefore the zeros of \( \zeta(s) \) on the critical line \( \text{Re} s = 1/2 \) coincide with the real zeros of \( Z(t) \). Historically, Hardy proved the infinity of the number of zeros of \( \zeta(s) \) on the critical line in 1914. A little later Hardy and Littlewood gave another proof by showing
that \( \int_0^T Z(t)dt \ll T^{7/8} \) and \( \int_0^T |Z(t)|dt \gg T \). See Chandrasekharan [3, Chapter II, §4 and Notes on Chapter II] or Titchmarsh [23, 10.5].

Since \( Z^2(t) = |\zeta(1/2 + it)|^2 \), 2k-th power moment of \( Z(t) \) is equivalent to 2k-th power moment of \( |\zeta(1/2 + it)| \). Hardy and Littlewood first showed the asymptotic formula in the case \( k = 1 \). In fact they showed that

\[
\int_0^T |\zeta(1/2 + it)|^2 dt \sim T \log T
\]

([5,6]). In 1926, Ingham [9] derived

\[
\int_0^T |\zeta(1/2 + it)|^2 dt = T \log \frac{T}{2\pi} + (2\gamma_0 - 1)T + E(T)
\]

with \( E(T) \ll T^{1/2} \log T \), where \( \gamma_0 \) is Euler’s constant. There are a lot of literatures on \( E(T) \) since then. For instance, Atkinson [1] gave an explicit formula for \( E(T) \), which becomes the fundamental tool of further researches on \( E(T) \). See Ivić [10] for more details. For \( k = 2 \), among other things, Ingham [9] showed that

\[
\int_0^T |\zeta(1/2 + it)|^4 dt = \frac{1}{2\pi^4} T \log^4 T + E_2(T)
\]

with \( E_2(T) = O(T \log^3 T) \) by applying the famous approximate functional equation of \( \zeta^2(s) \) of Hardy and Littlewood [7]. Ingham’s result was improved by Heath-Brown [8] to \( E_2(T) = T \sum_{n=0}^4 c_n \log^n T + O(T^{7/8+\varepsilon}) \). Motohashi [21] studied \( E_2(T) \) by the use of spectral theory of automorphic forms. See also Ivić [11] or Titchmarsh [23, 7.20]. Other mean value theorems (of even power) were studied by Hall [4] in connection with the distribution of consecutive zeros of \( Z(t) \).

As for odd power moments of \( Z(t) \), Ivić [12] proved in 2004 that

\[
\int_0^T Z(t)dt \ll T^{1/4+\varepsilon}.
\]

It shows that \( Z(t) \) changes sign quite often. Ivić’s result was sharpened to \( \int_0^T Z(t)dt \ll T^{1/4} \) by Jutila [17, 18] and Korolev [20] independently. Moreover they showed the Omega result \( \int_0^T Z(t)dt = \Omega_{\pm}(T^{1/4}) \) which was conjectured by Ivić [12]. It means that \( T^{1/4} \) is the true order of \( \int_0^T Z(t)dt \). Since there are a large amount of cancellations, it is expected that the cubic power moment has an exponent less than 1. In fact, Ivić showed that

\[
\int_T^{2T} Z^3(t)dt = 2\pi \sqrt{\frac{2}{3}} \sum_{\frac{T^2}{2\pi} \leq n \leq \frac{2T^2}{3\pi}} \frac{d_3(n)}{n^{1/6}} \cos \left( 3\pi n^{2/3} + \frac{1}{8} \pi \right)
\]

and conjectured that

\[
\int_0^T Z^3(t)dt \ll T^{3/4+\varepsilon}
\]

([14, Chapter 11]). Here \( d_3(n) \) denotes the number of triples \( (k_1, k_2, k_3) \) such that \( n = k_1 k_2 k_3, k_j \in \mathbb{Z}, k_j > 0 \). If we use (4), (5) and the Cauchy-Schwarz inequality we have
\[ \int_0^T Z^3(t) dt \ll T(\log T)^{5/2}. \] The best upper bound at present is due to Bettin, Chandee and Radziwill [2] who showed the second inequality of the following:

\[ \left| \int_0^T Z^3(t) dt \right| \leq \int_0^T |Z(t)|^3 dt \ll T(\log T)^{9/4}. \] (7)

It should be noted that \( T(\log T)^{9/4} \) is the correct order of \( \int_0^T |Z(t)|^3 dt \).

Theorem 1 For large \( T \geq 0 \), we have

\[
\int_0^T Z(t) \zeta \left( \frac{1}{2} + it \right) dt = 2\sqrt{2\pi} e^{\frac{\gamma_0}{2}} \left( \frac{T}{2\pi} \right)^{3/4} \left( \frac{1}{2} \log \frac{T}{2\pi} + 2\gamma_0 - 2\log 2 - \frac{3}{2} \right) + O(T^{1/2} \log T).
\]

We recall that \( \gamma_0 \) is Euler’s constant which coincides with the 0-th coefficient of the Laurent expansion of \( \zeta(s) \) at \( s = 1 \).

Ivić’s conjecture (6) would follow from the bound of exponential sum

\[
\sum_{N \leq n \leq 2\sqrt{N}} \frac{d_3(n)}{n^{1/6}} e^{\frac{3\pi i n^2}{2}} \ll N^{1/2+\varepsilon},
\] (8)

or, as Ivić noted [15, (1.6)], from

\[
\sum_{N \leq n \leq 2N} d_3(n) e^{\frac{3\pi i n^2}{2}} \ll N^{2/3+\varepsilon}.
\] (9)

It seems that (9) (or (8)) is out of reach of the present method of exponential sums. However if we replace \( d_3(n) \) by \( d(n) \) (the divisor function \( d(n) = \sum_{d_1d_2 = n} 1 \)), we can prove the following theorem in the frame of Theorem 1.

Theorem 2 Let \( A \) be a parameter such that \( A \gg N^{-1/4} \). Then we have

\[
\sum_{N \leq k \leq 2\sqrt{N}} \frac{d(k)}{k^{1/6}} e^{\frac{3\pi i (Ak)^{1/2}}{2}} = \sqrt{3}A^{-4/3} \sum_{A^{4/3}N^{1/3} \leq k \leq \sqrt{3}A^{4/3}N^{1/3}} d(k)k^{1/2} e^{-\pi i (k/A)^2} + O(A^{-1/3}N^{1/2+\varepsilon}) + O(A^{1/3}N^{1/6} \log N) + O(A^{-1/9}N^{2/9+\varepsilon})
\]

\[ \ll A^{2/3}N^{1/2} \log N. \]

For another kind of mean value of \( Z(t) \) and \( \zeta(1/2 + it) \) we have

Theorem 3 For large \( T \geq 0 \) we have

\[
\int_0^T Z^2(t) \zeta(1/2 + it) dt = T \left\{ \frac{1}{2} \left( \log \frac{T}{2\pi} \right)^2 + a_1 \log \frac{T}{2\pi} + a_2 \right\} + O(T^{3/4} \log^2 T),
\]

where \( a_1 = 3\gamma_0 - 1, a_2 = 3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1, \gamma_j \) being the coefficients of the Laurent expansion of \( \zeta(s) \) at \( s = 1 \).
We note that the integral of the left-hand side has an asymptotic form. It may be interesting to compare with Ivić’s conjecture (6).

As for another mean value, we shall prove the following

**Theorem 4** Let $\alpha$ be a real fixed constant such that $-1/2 < \alpha < 1/2$. Then we have

$$
\int_{T}^{2T} Z^3(t) \chi^\alpha(1/2 + it) dt \ll \begin{cases} 
T^{1-\frac{1}{2} + \varepsilon} & \text{if } 0 \leq \alpha < 1/2, \\
T^{1+\frac{1}{2} + \varepsilon} & \text{if } -1/2 < \alpha \leq 0.
\end{cases}
$$

The cubic moment of Hardy’s function corresponds to $\alpha = 0$, but unfortunately this gives only $O(T^{1+\varepsilon})$.

### 2 Some Lemmas

**Lemma 1** Suppose that $f(x)$ and $\varphi(x)$ are real-valued functions on the interval $[a, b]$ which satisfy the conditions

1) $f^{(4)}(x)$ and $\varphi''(x)$ are continuous,
2) there exist numbers $H, A, U, 0 < H, A < U, 0 < b - a \leq U$, such that

$$
A^{-1} \ll f''(x) \ll A^{-1}, \quad f^{(3)}(x) \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2}
$$

$$
\varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}, \quad \varphi''(x) \ll HU^{-2},
$$

3) $f'(c) = 0$ for some $c, a \leq c \leq b$.

Then

$$
\int_{a}^{b} \varphi(x) \exp(2\pi if(x)) dx = \frac{1 + i\varphi(c) \exp(2\pi if(c))}{\sqrt{f''(c)}} + O(HAU^{-1}) + O \left( H \min(|f'(a)|^{-1}, \sqrt{A}) \right) + O \left( H \min(|f'(b)|^{-1}, \sqrt{A}) \right).
$$

This is Lemma 2 of Karatsuba and Voronin [19, p.71].

**Remark 1** Here we give an important remark. As is noted in Ivić and Zhai [16], the proof actually shows that if there is no $c$ which satisfies the condition 3, the term containing $c$ does not appear in the right-hand side. Moreover if $c = a$ or $c = b$, then the main term is to be halved.

**Lemma 2** For $\frac{1}{2} \leq \sigma < 1$ fixed, $1 \ll x, y \ll t^k, s = \sigma + it, xy = \left( \frac{t}{2\pi} \right)^k$, $t \geq t_0$ and $k \geq 1$ a fixed integer, we have

$$
\chi^k(s) = \sum_{m=1}^{\infty} \rho \left( \frac{m}{x} \right) d_k(m)m^{-s} + \chi^k(s) \sum_{m=1}^{\infty} \rho \left( \frac{m}{y} \right) d_k(m)m^{1-s-1} + O(t^{k(1-\sigma)/3}) + O(t^{k(1/2-\sigma)/2}y^{\sigma} \log^{k-1} t).
$$

Here $\chi(s)$ is the function defined by (2) and $\rho(u) \geq 0$ is a smooth function such that $\rho(u) + \rho(1/u) = 1$ for $u > 0$ and $\rho(u) = 0$ for $u \geq 2$.

This is Lemma 4 of [16]. See also [14, Theorem 4.16].

For the proof of Theorem 4 we need the following lemma.
Lemma 3 Let \( \alpha, \beta, \gamma \) be fixed real numbers such that \( \alpha(\alpha - 1)\beta\gamma \neq 0 \) and write \( e(x) = e^{2\pi ix} \). Let
\[
S = \sum_{k=H+1}^{2H} \sum_{n=N+1}^{2N} \left| \sum_{m=m+2M} \left( X\frac{m\alpha h\beta n\gamma}{M\alpha H\beta N\gamma} \right) \right|^*,
\]
where \(*\) means that
\[
\left| \sum_{N \leq n \leq N'} z_n \right|^* = \max_{N \leq N_1 \leq N_2} \left| \sum_{n=N_1}^{N_2} z_n \right|.
\]
Then we have
\[
S \ll (HN)_{1+\epsilon} \left( \frac{X}{HN^{3x}} \right)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{X}.
\]
This is Theorem 3 of Robert and Sargos [22].

3 Proofs of Theorem 1 and 2

Proof of Theorem 1

We consider the integral
\[
J = \int_T^{2T} Z(t) \xi \left( \frac{1}{2} + it \right) \, dt. \tag{10}
\]
By the definition of \( Z(t) \) and applying Lemma 2 we have
\[
Z(t) \xi \left( \frac{1}{2} + it \right) = t^2 \left( \frac{1}{2} + it \right) \chi^{-1/2} \left( \frac{1}{2} + it \right)
\]
\[
= \left( \sum_{k=1}^{\infty} \rho \left( \frac{k}{x} \right) \frac{d(k)}{k^{1/2 + it}} + \chi^2 \left( \frac{1}{2} + it \right) \sum_{k=1}^{\infty} \rho \left( \frac{k}{y} \right) \frac{d(k)}{k^{1/2 - it}} \right)
\]
\[
+ O \left( t^{-2/3} + O \left( t^{-2/3} \log t \right) \right) \chi^{-1/2} \left( \frac{1}{2} + it \right),
\]
where \( xy = (t/2\pi)^2 \). Substituting this expression to (10), we have
\[
J = J_1 + J_2 + O(T^{1/3}), \tag{11}
\]
where
\[
J_1 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) k^{-it} \chi^{-1/2} \left( \frac{1}{2} + it \right) \, dt \tag{12}
\]
and
\[
J_2 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) k^{-it} \chi^{3/2} \left( \frac{1}{2} + it \right) \, dt. \tag{13}
\]
We take
\[
x = 2 \left( \frac{t}{2\pi} \right), \quad y = \frac{1}{2} \left( \frac{t}{2\pi} \right),
\]
and put $K = \frac{T}{\pi}$. Then the ranges of $k$ in the sums in (12) and (13) are, in fact, $k \leq 4K$ and $k \leq K$, respectively.

We first consider $J_1$. Using (3) we find that

$$k^{-it} x^{-1/2} \left( \frac{1}{2} + it \right) = e^{-\frac{2i}{\pi} \pi^4 \int t^{-1/2} + it^{-1/2} + O(1/t),$$

hence we have

$$J_1 = e^{-\frac{2i}{\pi} \pi^4 \int t^{-1/2} + it^{-1/2} + O(1/t),$$

We evaluate the above integral by applying Lemma 1 with $\psi(t) = \rho \left( k \left( \frac{T}{\pi} \right) \right)$ and $f(t) = \frac{1}{\pi^2} (t \log \frac{t}{\pi^2} - t - t \log k^2)$. Note that $\psi(t)$ satisfies the conditions of Lemma 1 with $H = 1$, $U = T$. Since $f'(t_0) = 0$ if and only if $t_0 = 2\pi k^2$, the main term of the integral appears for $k$ such that

$$\left( \frac{T}{2\pi} \right)^{1/2} \leq k \leq \left( \frac{T}{\pi} \right)^{1/2}.$$

Thus we get

$$\int_{T} \rho \left( \frac{k}{x} \right) e^{\frac{1}{2}(t \log \frac{t}{2\pi} - t - t \log k^2)} dt$$

$$= M(k) + O \left( 1 + \min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T}{2\pi k^2} \right)} \right) + \min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T}{\pi k^2} \right)} \right) \right),$$

where

$$M(k) = e^{\frac{2i}{\pi} \rho \left( \frac{1}{2k} \right)} 2\sqrt{2\pi} ke^{-\pi i k^2} = 2\sqrt{2\pi} e^{\frac{2i}{\pi} k(-1)^k}$$

for $k$ satisfying the condition (14) and 0 otherwise. This yields that

$$J_1 = 2\sqrt{2\pi} e^{\frac{2i}{\pi} \sum' (-1)^k d(k) k^{1/2}}$$

$$+ \sum_{k \leq K} d(k) k^{1/2} \left( 1 + \min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T}{2\pi k^2} \right)} \right) + \min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T}{\pi k^2} \right)} \right) \right)$$

$$+ O(T^{1/2} \log T)$$

$$=: R_0 + R_1 + R_2 + R_3 + O(T^{1/2} \log T),$$

where $\sum'$ means that the terms for $k = (T/2\pi)^{1/2}$ and $k = (T/\pi)^{1/2}$ are to be halved if they are integers. It is clear that $R_1 \ll T^{1/2} \log T$. To estimate $R_2$, we divide the sum into four parts:

$$\sum = \sum_{1 \leq k < \frac{1}{2} \left( \frac{T}{\pi} \right)^{1/2}} + \sum_{\frac{1}{2} \left( \frac{T}{\pi} \right)^{1/2} \leq k < \left( \frac{T}{\pi} \right)^{1/2}} + \sum_{\left( \frac{T}{\pi} \right)^{1/2} \leq k \leq 2 \left( \frac{T}{\pi} \right)^{1/2}} + \sum_{2 \left( \frac{T}{\pi} \right)^{1/2} < k \leq 4K}$$

$$=: S_1 + S_2 + S_3 + S_4.$$

For $S_1$ and $S_4$ we have $\min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T}{2\pi k^2} \right)} \right) \ll \frac{1}{\log 4}$, hence we get $S_1 \ll T^{1/4} \log T$ and $S_4 \ll T^{1/2} \log T$. For $S_2$, we write $k = \left( \frac{T}{\pi} \right)^{1/2} - j$ for $k$ in this range and divide the
sum over \( j \) as \( S_2 = S_{2,1} + S_{2,2} \), where \( S_{2,1} \) is the sum for \( j = 0, 1, 2 \) and \( S_{2,2} \) is the sum for \( 3 \leq j \leq \lfloor \frac{T}{2\pi} \rfloor \). For \( S_{2,1} \) we use \( \min(\sqrt{T}, \frac{1}{\log(\log T)}) \leq \sqrt{T} \) and hence \( S_{2,1} \ll T^{1/4+\varepsilon} \) since the sum is finite. As for \( S_{2,2} \), from

\[
\log \left( \frac{T}{2\pi} \right)^{1/2} \cdot j = \left| \log \left( \frac{T}{2\pi} \right)^{1/2} \right| \ll \frac{j}{\left( \frac{T}{2\pi} \right)^{1/2}},
\]

we have

\[
S_{2,2} \ll T^{-1/4+\varepsilon} \sum_j \left( \frac{T}{2\pi} \right)^{1/2} j \ll T^{1/4+\varepsilon}.
\]

Thus we get \( S_2 \ll T^{1/4+\varepsilon} \). It is the same for \( S_3 \). Combining these estimates we find that \( R_2 \ll T^{1/4} \log T \). Similarly we have \( R_3 \ll T^{1/2} \log T \). As a result, we get

\[
J_1 = 2\sqrt{2\pi} e^{\frac{3i}{2}} \sum_{k \leq K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) e^{-i\pi(k\log \frac{k}{y} - t - \log k^{2/3})} dt + O(T^{1/2} \log T).
\]

Next we consider \( J_2 \). Similarly to the case of \( J_1 \), we have by (3) that

\[
J_2 = e^{\frac{3i}{2}} \sum_{k \leq K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) e^{-i\pi(k\log \frac{k}{y} - t - \log k^{2/3})} dt + O(T^{1/2} \log T).
\]

We apply Lemma 1 to the above integral with \( \varphi(t) = \rho(2k(2\pi/t)) \) and \( f(t) = -\frac{2}{t} (t \log \frac{k}{y} - t - \log k^{2/3}) \). In this case \( f'(t_0) = 0 \) if and only if \( t_0 = 2\pi k^{2/3} \) and \( t_0 \) is contained in the interval \( [T, 2T] \) if and only if

\[
\left( \frac{T}{2\pi} \right)^{3/2} \leq k \leq \left( \frac{T}{\pi} \right)^{3/2}.
\]

Since the range of the sum over \( k \) is \( 1 \leq k \leq K \), there are no such \( k \), that is, the integral in (16) does not have a main term. Considering the error term by Lemma 1 we find that

\[
J_2 \ll \sum_{k \leq K} \frac{d(k)}{k^{1/2}} \left( 1 + \min(\sqrt{T}, \frac{1}{\log \frac{2\pi}{k^{2/3}}}) + \min(\sqrt{T}, \frac{1}{\log \frac{2\pi}{k^{2/3}}}) \right) =: R'_1 + R'_2 + R'_3.
\]

We have clearly \( R'_1 \ll T^{1/2} \log T \). For \( R'_2 \) and \( R'_3 \) we note that \( |\log \frac{T}{2\pi} | \gg 1 \) since \( k \leq K \), which implies that \( R'_2, R'_3 \ll T^{1/2} \log T \). Hence

\[
J_2 \ll T^{1/2} \log T.
\]

From (11), (15) and (17), we get

\[
J = 2\sqrt{2\pi} e^{\frac{3i}{2}} \sum_{(\frac{T}{2\pi})^{1/2} \leq k \leq (\frac{T}{2\pi})^{1/2}} (-1)^k d(k) k^{1/2} + O(T^{1/2} \log T).
\]
Now dividing the interval \([0, T]\) as \(\bigcup_j [T/2^j, T/2^{j-1}]\) and summing the above evaluations we have
\[
\int_0^T Z(t) \zeta \left( \frac{1}{2} + it \right) dt = 2 \sqrt{2\pi} e^{\frac{\pi}{4}} \sum_{k \leq (\frac{T}{2\pi})^{1/2}} (-1)^k d(k) k^{1/2} + O(T^{1/2} \log T). \tag{18}
\]

To evaluate the sum on the right-hand side of (18) we recall that
\[
\sum_{k \leq x} (-1)^k d(k) = \frac{x}{2} (\log x + 2\gamma_0 - 1 - 2 \log 2) + O(x^{1/3 + \varepsilon})
\]
for \(x \gg 1\) (see, e.g., Ivić [13]), so by partial summation we have
\[
\sum_{k \leq x} (-1)^k d(k) k^{1/2} = \frac{1}{3} x^{3/2} \left( \log x + 2\gamma_0 - 2 \log 2 - \frac{2}{3} \right) + O(x^{5/6 + \varepsilon}).
\]

Substituting this form to (18) we finally get
\[
\int_0^T Z(t) \zeta \left( \frac{1}{2} + it \right) dt = \frac{2\sqrt{2\pi}}{3} e^{\frac{i\pi}{4}} \left( \frac{T}{2\pi} \right)^{3/4} \left( \frac{1}{2} \log \frac{T}{2\pi} + 2\gamma_0 - 2 \log 2 - \frac{2}{3} \right) + O(T^{1/2} \log T).
\]

This proves the assertion of Theorem 1. \(\square\)

**Proof of Theorem 2** Let \(A\) be a parameter such that \(T^{-1/2} \ll A \ll T^{3/2}\). We shall consider the integral
\[
I_A = \int_T^{2T} Z(t) \zeta \left( \frac{1}{2} + it \right) A^{it} dt
\]
by the same way as in the proof of Theorem 1. Applying Lemma 2 we get
\[
I_A = I_{A,1} + I_{A,2} + O(T^{1/3}), \tag{19}
\]
where
\[
I_{A,1} = \int_T^{2T} \chi^{-1/2} \left( \frac{1}{2} + it \right) \sum_{k=1}^{\infty} \rho \left( \frac{k}{A} \right) \frac{d(k)}{k^{1/2+it}} A^{it} dt \tag{20}
\]
and
\[
I_{A,2} = \int_T^{2T} \chi^{3/2} \left( \frac{1}{2} + it \right) \sum_{k=1}^{\infty} \rho \left( \frac{k}{xy} \right) \frac{d(k)}{k^{1/2-it}} A^{it} dt, \tag{21}
\]
where \(xy = (\frac{t}{2\pi})^2\). Hereafter we put \(K_0 = \left( \frac{T}{2\pi} \right)^{1/2} \). \(\square\)

Now we shall evaluate \(I_{A,1}\) and \(I_{A,2}\) by taking two different choices of \(x\) and \(y\), that is,

Case 1: we take \(x = 8A (\frac{t}{2\pi})^{1/2}\) and \(y = \frac{1}{2\pi} (\frac{t}{2\pi})^{3/2}\).

Case 2: we take \(x = \frac{A}{4} (\frac{t}{2\pi})^{1/2}\) and \(y = \frac{A}{4} (\frac{t}{2\pi})^{3/2}\)
3.1 Case 1

The ranges of the sums in $I_{A,1}$ and $I_{A,2}$ are at most $k \leq 16AK_0$ and $k \leq \frac{1}{4A}K_0^3$, respectively.

By (3) and the trivial estimate for the error term we get

$$I_{A,1} = e^{-\frac{3}{8}} \sum_{k \leq 16AK_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) e^{\frac{1}{2}(t \log \frac{t}{2\pi} - t - t \log (\frac{t}{A}))} \, dt$$

$$+ O \left( A^{1/2}T^{1/4+\varepsilon} \right). \tag{22}$$

We shall evaluate the integral by Lemma 1. Let $f(t) = \frac{1}{4\pi} (t \log \frac{t}{2\pi} - t - t \log (\frac{t}{A}))$. Then $f'(t_0) = 0$ if and only if $t_0 = 2\pi (\frac{k}{A})^2$ and $T \leq t_0 \leq 2T$ if and only if

$$A \left( \frac{T}{2\pi} \right)^{1/2} \leq k \leq A \left( \frac{T}{\pi} \right)^{1/2} \tag{23}.$$ 

We find that all $k$ satisfying (23) are contained in the range $k \leq 16AK_0$. Therefore the integral in (22) has a main term which is given by

$$M_A(k) = e^{\frac{3}{8}} \rho \left( \frac{1}{8} \right) 2\sqrt{2\pi} \frac{k}{A} e^{-\pi(k/A)^2}$$

for $A \left( \frac{T}{2\pi} \right)^{1/2} \leq k \leq A \left( \frac{T}{\pi} \right)^{1/2}$ and $M_A(k) = 0$ otherwise. We note that $\rho(1/8) = 1$ in the above formula. It follows from Lemma 1 and (22) that

$$I_{A,1} = e^{-\frac{3}{8}} \sum_{A \left( \frac{T}{2\pi} \right)^{1/2} \leq k \leq A \left( \frac{T}{\pi} \right)^{1/2}} \frac{d(k)}{k^{1/2}} M_A(k)$$

$$+ \sum_{k \leq 4AK_0} \frac{d(k)}{k^{1/2}} O \left( 1 + \min \left( \sqrt{T}, \frac{1}{|\log (\frac{T}{\pi k/A})|} \right) + \min \left( \sqrt{T}, \frac{1}{|\log (\frac{T}{\pi k/A})|} \right) \right)$$

$$+ O(A^{1/2}T^{1/4+\varepsilon}).$$

Similarly to the proof of Theorem 1, we see that the contributions from the $O$-terms are bounded by $O(A^{1/2}T^{1/4+\varepsilon} + A^{-1/2}T^{1/4+\varepsilon})$. Hence we get

$$I_{A,1} = e^{\frac{3}{8}} \frac{2\sqrt{2\pi}}{A} \sum_{A \left( \frac{T}{2\pi} \right)^{1/2} \leq k \leq A \left( \frac{T}{\pi} \right)^{1/2}} d(k)k^{1/2} e^{-\pi(k/A)^2}$$

$$+ O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{1/4+\varepsilon}). \tag{24}$$

Next we consider $I_{A,2}$. Similarly to $I_{A,1}$ we have

$$I_{A,2} = e^{\frac{3}{8}} \sum_{k \leq \frac{1}{4A}K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) e^{\frac{1}{2}(t \log \frac{t}{2\pi} - t - t \log (Ak)^{2/3})} \, dt$$

$$+ O(A^{-1/2}T^{3/4+\varepsilon}).$$

If we put $f(t) = -\frac{3}{4\pi} (t \log \frac{t}{2\pi} - t - t \log (Ak)^{2/3})$ this time, $f'(t_0) = 0$ if and only if $t_0 = 2\pi (Ak)^{2/3}$ and so $T \leq t_0 \leq 2T$ if and only if

$$\frac{1}{A} \left( \frac{T}{2\pi} \right)^{3/2} \leq k \leq \frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2} \tag{25}.$$
Since \( k \) runs in the range \( 1 \leq k \leq \frac{1}{4\pi}K_0^2 \), there is no main term in the integral of \( J_{A,2} \).

Hence by Lemma 1, we get similarly that

\[
J_{A,2} \ll \sum_{k \leq \frac{1}{4\pi}K_0^2} \frac{d(k)}{k^{1/2}} \left( 1 + \min \left( \sqrt{T}, \frac{1}{|\log \left( \frac{(T/2\pi)^{3/2}}{AK} \right)|} \right) \\
+ \min \left( \sqrt{T}, \frac{1}{|\log \left( \frac{(T/2\pi)^{3/2}}{AK} \right)|} \right) \right)
\ll A^{-1/2}T^{-3/4+\varepsilon} + A^{1/2}T^{-1/4+\varepsilon}.
\]  

(26)

From (19), (24) and (26), we obtain

\[
J_A = e^{\frac{2i\pi}{A}} \sum_{\frac{A}{2} \leq k \leq A} d(k)k^{1/2}e^{-\pi i(k/A)^2} \\
+ O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{3/4+\varepsilon}) + O(T^{1/3}).
\]

(27)

### 3.2 Case 2

In this choice of \( x \) and \( y \), the sums in (20) and (21) are actually over \( k \leq \frac{1}{2}AK_0 \) and \( k \leq \frac{1}{2}K_0^3 \), respectively. Thus

\[
J_{A,1} = e^{-\frac{2\pi i}{A}} \sum_{k \leq \frac{1}{2}K_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) e^{\frac{i\pi \log x}{\pi} - t - t \log (\frac{k}{x})^2} dt \\
+ O(A^{1/2}T^{1/4+\varepsilon})
\]

and

\[
J_{A,2} = e^{\frac{2\pi i}{A}} \sum_{k \leq \frac{1}{2}K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) e^{\frac{-2\pi i(t \log (\frac{k}{y})^2 + t \log (AK)^{2/3})} dt \\
+ O(A^{-1/2}T^{-3/4+\varepsilon}).
\]

As for \( J_{A,1} \), the integral has a main term if and only if \( k \) satisfies (23). Since \( k \) runs over \( 1 \leq k \leq \frac{1}{2}K_0 \), there are no such \( k \). The contribution from the error term of the integral is the same as in the previous case since the range of the sum has the same order, hence we get

\[
J_{A,1} \ll A^{1/2}T^{-1/4+\varepsilon} + A^{-1/2}T^{-1/4+\varepsilon}.
\]

(28)

On the other hand, the integral of \( J_{A,2} \) has a main term if and only if \( k \) satisfies (25), and in fact all \( k \) are in the range \( k \leq \frac{8}{A}K_0^2 \). Hence by Lemma 1, \( J_{A,2} \) has the following form:

\[
J_{A,2} = e^{\frac{2\pi i}{A}} \sum_{\frac{1}{2} \leq k \leq \frac{1}{2}(\frac{T}{2\pi})^{3/2}} \frac{d(k)}{k^{1/2}}M_A(k) \\
+ \sum_{k \leq \frac{8}{A}K_0^2} \frac{d(k)}{k^{1/2}} \left( 1 + \min \left( \sqrt{T}, \frac{1}{|\log \left( \frac{(T/2\pi)^{3/2}}{AK} \right)|} \right) \right)
\]

(29)
\[
+ \min \left( \sqrt{T}, \frac{1}{\log \left( \frac{T/\pi^{3/2}}{ Ak} \right)} \right)
+ O(A^{-1/2}T^{3/4+\varepsilon}),
\]

where

\[
\tilde{M}_A(k) = e^{-\frac{2\pi}{\rho} \left( \frac{1}{4} \right) \frac{2\sqrt{2\pi}}{\sqrt{3}} (Ak)^{1/3} e^{3\pi i(Ak)^{2/3}}}
\]

for \( \frac{1}{A} \left( \frac{T}{2\pi} \right)^{3/2} \leq k \leq \frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2} \) and 0 otherwise. We see that the contribution from the \( O \)-term is the same as the previous case, therefore

\[
J_{A,2} = e^{\frac{2\sqrt{2\pi}}{\sqrt{3}} A^{1/3}} \sum_{\frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2} \leq k \leq \frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2}} \frac{d(k)}{k^{1/6}} e^{3\pi i(Ak)^{2/3}}
+ O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{-1/4+\varepsilon}).
\]

From (28) and (29) we obtain that

\[
J_A = e^{\frac{2\sqrt{2\pi}}{\sqrt{3}} A^{1/3}} \sum_{\frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2} \leq k \leq \frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2}} \frac{d(k)}{k^{1/6}} e^{3\pi i(Ak)^{2/3}}
+ O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{1/4+\varepsilon}) + O(T^{1/3}).
\]

Now we have two expressions of \( J_A \): (27) and (30). Comparing these expressions we obtain

\[
\sum_{\frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2} \leq k \leq \frac{1}{A} \left( \frac{T}{\pi} \right)^{3/2}} \frac{d(k)}{k^{1/6}} e^{3\pi i(Ak)^{2/3}}
= \sqrt{3}A^{-4/3} \sum_{A \left( \frac{T}{\pi} \right)^{1/2} \leq k \leq A \left( \frac{T}{\pi} \right)^{1/2}} d(k)k^{1/2} e^{-\pi i(k/A)^2}
+ O(A^{-5/6}T^{3/4+\varepsilon}) + O(A^{1/6}T^{1/4+\varepsilon}) + O(A^{-1/3}T^{1/3+\varepsilon})
\ll A^{1/6}T^{3/6} \log T,
\]

where the last inequality is obtained by the trivial estimate. In (31), we take \( T = 2\pi (AN)^{2/3} \). Then (31) is transformed to
\[
\sum_{N \leq k \leq 2\sqrt{N}} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}} = \sqrt{3} A^{-4/3} \sum_{A^{4/3} N^{1/3} \leq k \leq \sqrt{A^{4/3} N^{1/3}}} d(k) k^{1/2} e^{-\pi i (k/A)^2} + O(A^{-1/3} N^{1/6 + \varepsilon}) + O(A^{1/3} N^{1/6 + \varepsilon}) + O(A^{-1/9} N^{2/9 + \varepsilon}) \ll A^{2/3} N^{1/4} \log N
\]

defined for \(A \gg N^{-1/4}\). This proves the assertion of Theorem 2.

4 Proof of Theorem 3

Since the method is similar to Theorem 1, we shall only give an outline of proof. Let

\[
I = \int_T^{2T} Z^2(t) \zeta \left( \frac{1}{2} + it \right) dt.
\]

This time we have

\[
Z^2(t) \zeta \left( \frac{1}{2} + it \right) = \zeta^3 \left( \frac{1}{2} + it \right) \chi^{-1} \left( \frac{1}{2} + it \right)
\]

\[
= \chi^{-1} \left( \frac{1}{2} + it \right) \sum_{k=1}^\infty \rho \left( \frac{k}{x} \right) \frac{d_3(k)}{k^{1/2 + it}} + \chi^2 \left( \frac{1}{2} + it \right) \sum_{k=1}^\infty \rho \left( \frac{k}{y} \right) \frac{d_3(k)}{k^{1/2 - it}} + O \left( t^{-1/2} \right) + O \left( t^{-2} y^{1/2} \log^2 t \right),
\]

(32)

where \(xy = \left( \frac{T}{2\pi} \right)^3\).

We take \(x = 2 \left( \frac{T}{2\pi} \right)^3/2 \) and \(y = \frac{1}{2} \left( \frac{T}{2\pi} \right)^3/2 \) in (32) and put \(K_3 = \left( T/\pi \right)^3/2 \). Then the ranges of \(k \) in the above two sums are at most \(k \leq 4K_3 \) and \(k \leq K_3 \), respectively. Hence

\[
I = \sum_{k \leq 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) k^{-it} \chi^{-1} \left( \frac{1}{2} + it \right) dt + \sum_{k \leq K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) k^{it} \chi^2 \left( \frac{1}{2} + it \right) dt + O(T^{1/2}) =: I_1 + I_2 + O(T^{1/2}),
\]

(33)

As for \(I_2 \), using (3), we get

\[
I_2 = e^{\frac{3i}{2}} \sum_{k \leq 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) e^{-2i(t \log \frac{4T}{\pi} - t \log \sqrt{k})} dt + O(T^{3/4} \log^2 T).
\]

As in the previous case, we apply Lemma 1 to the above integral with \(\varphi(t) = \rho \left( 2k \left( \frac{2\pi}{T} \right)^3/2 \right) \) and \(f(t) = -\frac{1}{2} (t \log \frac{4\pi}{T} - t - t \log \sqrt{k}) \). Then we find that \(f'(t_0) = 0 \) if and only if \(t_0 = 2\pi \sqrt{k} \), and this \(t_0 \) is contained in the interval \([T, 2T]\) if and only if

\[
\left( \frac{T}{2\pi} \right)^2 \leq k \leq \left( \frac{T}{\pi} \right)^2.
\]

(34)
Since \( k \) runs over the range \( 1 \leq k \leq K_3 \), there is no \( k \) which satisfies (34), hence the main term does not appear in this integral. On the other hand, the error term of this integral is given by

\[
1 + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{2\pi k}|} \right) + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{\pi k}|} \right) \ll 1,
\]

hence

\[
I_2 \ll \sum_{k \leq K_3} \frac{d_3(k)}{k^{1/2}} + T^{3/4} \log^2 T \ll T^{3/4} \log^2 T.
\] (35)

Next we treat \( I_1 \). By (3) again, we have

\[
I_1 = e^{-\frac{\pi i}{4}} \sum_{T \leq k \leq 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) e^{i(t \log \frac{T}{x^2} - t - t \log k)} dt + O(T^{3/4} \log^2 T).
\]

In this case \( \varphi(t) = \rho(k(2\pi T)^{1/2}/2) \) and \( f(t) = \frac{1}{2\pi t^2} (t \log \frac{T}{x^2} - t - t \log k) \). We see that \( f'(t_0) = 0 \) if and only if \( t_0 = 2\pi k \) and this \( t_0 \) is contained in \([T, 2T]\) if and only if

\[
\frac{T}{2\pi} \leq k \leq \frac{T}{\pi}.
\]

Therefore we have

\[
\int_T^{2T} \rho \left( \frac{k}{x} \right) e^{i(t \log \frac{T}{x^2} - t - t \log k)} dt
= M(k) + O \left( 1 + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{2\pi k}|} \right) + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{\pi k}|} \right) \right),
\]

where \( M(k) \) is the main term given by

\[
M(k) = e^{\frac{\pi i}{4}} \rho \left( \frac{1}{2\sqrt{k}} \right) (2\pi t_0)^{1/2} e^{-2\pi ik} = 2\pi e^{\frac{\pi i}{4}} k^{1/2}
\]

for \( k \) such that \( \frac{T}{2\pi} \leq k \leq \frac{T}{\pi} \) and 0 otherwise. Therefore we get

\[
I_1 = 2\pi \sum_{\frac{T}{2\pi} \leq k \leq \frac{T}{\pi}}' d_3(k)
+ \sum_{T \leq k \leq K_3} \frac{d_3(k)}{k^{1/2}} \left( 1 + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{2\pi k}|} \right) + \min \left( \sqrt{T}, \frac{1}{|\log \frac{T}{\pi k}|} \right) \right)
= 2\pi \sum_{\frac{T}{2\pi} \leq k \leq \frac{T}{\pi}}' d_3(k) + O(T^{3/4} \log^2 T).
\] (36)

Here we can get the last \( O \)-term by the same way as in the previous case. Combining (33), (35) and (36), we obtain

\[
I = 2\pi \sum_{\frac{T}{2\pi} \leq k \leq \frac{T}{\pi}}' d_3(k) + O(T^{3/4} \log^2 T).
\]
Now dividing the interval \([0, T]\) as \(\cup_j [T/2^j, T/2^{j-1}]\) and summing the above estimate we obtain that
\[
\int_0^T Z^2(t) \zeta \left( \frac{1}{2} + it \right) dt = 2\pi \sum_{k \leq T} d_3(k) + O(T^{3/4} \log^2 T).
\]

Theorem 3 follows from the well-known formula:
\[
\sum_{n \leq x} d_3(n) = x \left( \frac{1}{2} \log^2 x + (3\gamma_0 - 1) \log x + 3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1 \right) + O(x^{1/2}),
\]
where \(\gamma_j\) is the coefficients of the Laurent expansion of \(\zeta(s)\) at \(s = 1\).

5 Proof of Theorem 4

To prove Theorem 4, we put
\[
I(\alpha) = \int_T^{2T} Z^3 \left( \frac{1}{2} + it \right) \chi(\alpha \left( \frac{1}{2} + it \right) dt
\]
where \(\alpha\) is a fixed constant such that \(-1/2 < \alpha < 1/2\). This time we have
\[
I = I_1 + I_2 + O(T^{1/2}),
\]
where
\[
I_1(\alpha) = \sum_{k = 1}^{\infty} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{x} \right) k^{-it} \chi(\alpha) \left( \frac{1}{2} + it \right) dt
\]
and
\[
I_2(\alpha) = \sum_{k = 1}^{\infty} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho \left( \frac{k}{y} \right) k^{-it} \chi(\alpha + \frac{1}{2}) \left( \frac{1}{2} + it \right) dt
\]
where \(xy = \left( \frac{T}{2\pi} \right)^3\). We only sketch an outline of evaluations of \(I_1(\alpha)\).

Assume that \(0 \leq \alpha < \frac{1}{2}\). We take \(x = 2T(\frac{T}{2\pi})^{1/2}\) and \(y = \frac{1}{2}(\frac{T}{2\pi})^{1/2}\) and put \(K_4 = \left( \frac{T}{2\pi} \right)^{3/2}\). Then the range of \(k\) in the sum of (37) and (38) are at most \(1 \leq k \leq 4K_4\) and \(1 \leq k \leq K_4\), respectively.

The integral in (37) becomes
\[
e^{\frac{\alpha i}{2}} \left( a - \frac{1}{2} \right) \int_T^{2T} \rho \left( \frac{k}{x} \right) e^{(\frac{3}{2}-a)it} \left( t \log \frac{T}{2\pi} - t \log k + \frac{T}{2\pi} \right) dt + O(1).
\]
The main term of this integral appears only when
\[
\left( \frac{T}{2\pi} \right)^{\frac{3}{2}-a} \leq k \leq \left( \frac{T}{\pi} \right)^{\frac{3}{2}-a}
\]
in which case it is given by
\[
M_\alpha(k) = e^{\frac{\alpha i}{2}} \rho \left( \frac{k \pi^2}{x} / 2 \right) e^{2\pi k \frac{1}{\pi} \sqrt{3/2 - \alpha}} e^{-\left( \frac{1}{2} - a \right) \frac{1}{2} k \pi i \frac{1}{\pi} \sqrt{3/2 - \alpha}}.
\]
By Lemma 1 again, we get
\[
I_1(\alpha) = e^{\frac{N\alpha}{2} - \frac{1}{2}} \frac{2\pi}{\sqrt{3/2 - \alpha}} \sum_{\frac{T}{2\pi} \leq k \leq \frac{T}{2\pi}} \frac{d_3(k)}{k^{3/2 - \alpha}} e^{\frac{i\pi}{2} \frac{k^{1/2} \alpha}{2}} e^{\frac{1}{2} \frac{1}{(k/2)^{1/2} \alpha}} \\
+ \sum_{k \leq k_4} d_3(k) \frac{1}{k^{1/2}} O \left(1 + \min \left(\sqrt{T}, \frac{1}{\log (T/2\pi)^{3/2 - \alpha}} \right) \right) + \min \left(\sqrt{T}, \frac{1}{\log (T/2\pi)^{3/2 - \alpha}} \right).
\]

(39)

Just in the same way as the previous cases, we can see easily that the above O-term is estimated as \(O(T^{3/4} \log^2 T)\).

On the other hand, for \(I_2(\alpha)\), the main term does not appear from the integral by the assumption \(0 \leq \alpha < 1/2\) and the sum over \(k\) is estimated as \(O(T^{3/4} \log^2 T)\).

Now it remains to evaluate the sum over \(k\) in (39). Let
\[
S = \sum_{\frac{T}{2\pi} \leq k \leq \frac{T}{2\pi}} \frac{d_3(k)}{k^{3/2 - \alpha}} e^{\frac{i\pi}{2} \frac{k^{1/2} \alpha}{2}} e^{\frac{1}{2} \frac{1}{(k/2)^{1/2} \alpha}}.
\]

By partial summation we may have
\[
S \ll T^{\frac{1}{2} - \frac{1}{4}} \max_{\frac{T}{2\pi} \leq k \leq \frac{T}{2\pi}} \left| \sum_{\frac{T}{2\pi} \leq k \leq k'} d_3(k) e^{\frac{1}{2} \frac{1}{(k/2)^{1/2} \alpha}} \right|.
\]

(40)

Considering the definition of \(d_3(k)\), it is reduced to the estimate of the sum of the form
\[
S_1 := \sum_{T_1 \leq k_1 k_2 k_3 \leq 2T_1} e^{2\pi i c(k_1 k_2 k_3)^2},
\]

where \(\delta = \frac{1}{3/2 - \alpha}\) and \((\frac{T}{2\pi})^{3/2 - \alpha} \leq T_1 \leq \frac{1}{2\pi} (\frac{T}{2\pi})^{3/2 - \alpha}\). Since \(\delta \neq 0, 1\) we can apply Lemma 3. Divide the summation condition in \(S_1\) into \(O(\log^3 T)\) subintervals of the form \((k_1, k_2, k_3) \in [H, 2H] \times [N, 2N] \times [M, 2M]\). By symmetry of \(k_j\), we can assume that \(M\) is the largest, hence \(M \gg T_1^{1/3}\). Now applying Lemma 3 to the sum \(S_1\) by taking \(X = (HNM)^\delta \gg T_1^\delta\), we find that
\[
S_1 \ll T_1^{1+\epsilon} (T_1^{(\frac{3}{2} - \frac{1}{2})/4} + T_1^{-1/6} + T_1^{-\delta}) \ll T_1^{2/3 + \delta/4 + \epsilon}.
\]

(41)

Here the last inequality follows from the assumption \(0 \leq \alpha < 1/2\). By (40), (41) and \(T_1 \asymp T^{3/2 - \alpha}\), \(\delta = \frac{1}{3/2 - \alpha}\) we find that
\[
S \ll T_1^{1 - \frac{\alpha}{6} + \epsilon}.
\]

This proves the assertion in the case \(0 \leq \alpha < 1/2\).

In the case \(-1/2 < \alpha \leq 0\), we take \(x = \frac{1}{2}(\frac{T}{2\pi})^{3/2}\) and \(y = 2(\frac{T}{2\pi})^{3/2}\). Then the main term arises from the integral corresponding \(I_2(\alpha)\) and the assertion is proved similarly. We omit the details in this case.
Authors’ contributions
The authors are very grateful to the referee for his/her valuable comments and suggestions. Furthermore he/she kindly pointed out the authors the reference [2], where (7) is established, and the possibility to prove Theorem 3 more quickly from stating with \( \int_{0}^{T} |\zeta(1/2 + it)|^2 n^{-\sigma} dt \).

Author details
1 Department of Mathematics and Physics, Beijing Institute of Petro-Chemical Technology, Beijing 102617, People’s Republic of China, 2 Nishisato 2-13-1, Meito, Nagoya 456-0084, Japan, 3 Department of Mathematics, China University of Mining and Technology, Beijing 100083, People’s Republic of China.

Received: 11 August 2020 Accepted: 8 March 2021 Published online: 16 April 2021

References
1. Atkinson, F.V.: The mean value of the Riemann zeta-function. Acta Math. 81, 353–376 (1949)
2. Bettin, S., Chandee, V., Radziwiłł, M.: The mean square of the product of the Riemann zeta-function with Dirichlet polynomials. J. Reine Angew. Math. 729, 51–79 (2017)
3. Chandrasekharan, K.: Arithmetical Functions. Springer, New York (1970)
4. Hall, R.R.: The behaviour of the Riemann zeta-function on the critical line. Mathematika 46, 281–313 (1999)
5. Hardy, G.H., Littlewood, J.E.: Contributions to the theory of the Riemann zeta-function and the distribution of primes. Acta Math. 41, 119–196 (1918)
6. Hardy, G.H., Littlewood, J.E.: The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz. Proc. Lond. Math. Soc. 21(2), 39–74 (1922)
7. Hardy, G.H., Littlewood, J.E.: The approximate functional equation of \( \zeta(s) \) and \( \zeta^{2}(s) \). Proc. Lond. Math. 29(2), 81–97 (1929)
8. Heath-Brown, D.R.: The fourth power moment of the Riemann zeta-function. Proc. Lond. Math. Soc. 38(3), 385–422 (1979)
9. Ingham, A.E.: Mean value theorems in the theory of the Riemann zeta-function. Proc. Lond. Math. Soc. 27(2), 273–300 (1926)
10. Ivić, A.: The Theory of the Riemann Zeta-Function. Wiley, New York (1985) (2nd edn. Dover, Mineora, (2003))
11. Ivić, A.: Lectures on Mean Values of the Riemann Zeta-Function, Tata Inst. Fund. Res. Lec. Math. Phy. 82, Bombay (1991)
12. Ivić, A.: On the integral of Hardy’s function. Arch. Math. 83, 41–47 (2004)
13. Ivić, A.: On the divisor function and the Riemann zeta-function in short intervals. Ramonjan J. 19, 207–224 (2009)
14. Ivić, A.: The Theory of Hardy’s Z-Function. Cambridge University Press, Cambridge (2013)
15. Ivić, A.: On a cubic moment of Hardy’s function with a shift. In: Montgomery, H.L., Nikeghbali, A., Rassias, M.T. (eds.) Exploring the Riemann Zeta Function, 99–112. Springer, Berlin (2017)
16. Ivić, A., Zhai, W.: On certain integrals involving the Dirichlet divisor problem. Functiones et Approximatio 62(2), 247–267 (2020)
17. Jutila, M.: Atkinson’s formula for Hardy’s function. J. Number Theory 129, 2853–2878 (2009)
18. Jutila, M.: An asymptotic formula for the primitive of Hardy’s function. Ark. Mat. 49, 97–107 (2011)
19. Karatsuba, A.A., Vinogradov, S.M.: The Riemann Zeta-Function. Walter de Gruyter, Berlin (1992)
20. Korolev, M.A.: On the integral of Hardy’s function. Izv. Math. 72, 429–478 (2008)
21. Motohashi, Y.: Spectral Theory of the Riemann Zeta-Function. Cambridge University Press, Cambridge (1997)
22. Robert, D., Sargos, P.: Three-dimensional exponential sums with monomials. J. Reine Angew. Math. 591, 1–20 (2006)
23. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Oxford University Press, Oxford (1986)

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.