MONOTONE HURWITZ NUMBERS AND THE HCIZ INTEGRAL II

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Abstract. Motivated by results for the HCIZ integral in Part I of this paper, we study the structure of monotone Hurwitz numbers, which are a desymmetrized version of classical Hurwitz numbers. We prove a number of results for monotone Hurwitz numbers and their generating series that are striking analogues of known results for the classical Hurwitz numbers. These include explicit formulas for monotone Hurwitz numbers in genus 0 and 1, for all partitions, and an explicit rational form for the generating series in arbitrary genus. This rational form implies that, up to an explicit combinatorial scaling, monotone Hurwitz numbers are polynomial in the parts of the partition.

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0. Introduction

This paper is a continuation of [13]. In [13], we studied the $N \to \infty$ asymptotics of the Harish-Chandra-Itzykson-Zuber matrix model on the group of $N \times N$ unitary matrices, and showed that the free energy of this matrix model admits an asymptotic expansion in powers of $N^{-2}$ whose coefficients are generating functions for a desymmetrized version of the double Hurwitz numbers [20, 24] which we called the monotone double Hurwitz numbers. The monotone double Hurwitz number $\bar{H}_g(\alpha,\beta)$ counts a combinatorially restricted subclass of the degree $d$ branched covers $f : C \to \mathbb{P}^1$ of the Riemann sphere by curves of genus $g$ which have ramification type $\alpha \vdash d$ over $\infty$, $\beta \vdash d$ over $0$, and $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$ additional simple branch points at fixed positions on $\mathbb{P}^1$, the number of which is determined by the Riemann-Hurwitz formula. The results of [13] thus prove the existence of an asymptotic expansion in the HCIZ matrix model and provide a topological interpretation of this expansion, thereby placing the HCIZ model on similar footing with the more developed theory of topological expansion in Hermitian matrix models [1, 2, 3, 9, 21]. See [6, 7] for previous results in this direction.

In this article, we give a thorough combinatorial analysis of the monotone single Hurwitz numbers $\bar{H}_g(\alpha) = \bar{H}_g(\alpha, (1^d))$, which count branched covers as above which are unramified over $0 \in \mathbb{P}^1$. As explained in our first paper [13], the fixed-genus generating functions of the monotone single Hurwitz numbers arise as the orders of the genus expansion in the “one-sided” HCIZ model. The one-sided HCIZ model is obtained when one of the two sequences of normal matrices which define the HCIZ potential has degenerate limiting moments.

Our study of the monotone single Hurwitz numbers $\bar{H}_g(\alpha)$ is motivated by the following result, which is a degeneration of the main theorem in our first paper [13, Theorem 0.1].

**Theorem 0.1.** Let $(A_N), (B_N)$ be two sequences of $N \times N$ normal matrices whose spectral radii are uniformly bounded, with least upper bound

$$M := \sup \{ \rho(A_N), \rho(B_N) : N \geq 1 \},$$

and which admit limiting moments

$$-\phi_k := \lim_{N \to \infty} \frac{1}{N} \text{tr}(A_N^k),$$

$$-\psi_k := \lim_{N \to \infty} \frac{1}{N} \text{tr}(B_N^k)$$

of all orders. Suppose furthermore that the limiting moments of $B_N$ are degenerate: $\psi_k = \delta_{1k}$. Let $0 \leq r < r_c$, where $r_c$ is the critical value

$$r_c = \frac{2}{27}.$$
Then, the free energy $F_N(z)$ of the HCIZ model with potential $V = zN \text{tr}(A_N U B_N U^*)$ admits an $N \to \infty$ asymptotic expansion of the form

$$F_N(z) \sim \sum_{g=0}^{\infty} \frac{C_g(z)}{N^{2g}},$$

which holds uniformly on the closed disc $\overline{D}(0, r M^{-2})$. Each coefficient $C_g(z)$ is a holomorphic function of $z$ on the open disc $D(0, r M^{-2})$, with Maclaurin series

$$C_g(z) = \sum_{d=1}^{\infty} C_{g,d} \frac{z^d}{d!},$$

where

$$C_{g,d} = \sum_{\alpha \vdash d} \bar{H}_g(\alpha) \phi_{\alpha}$$

and $\bar{H}_g(\alpha)$ is the number of $(r + 1)$-tuples $(\sigma, \tau_1, \ldots, \tau_r)$ of permutations from the symmetric group $S(d)$ such that

1. $\sigma$ has cycle type $\alpha$ and the $\tau_i$ are transpositions;
2. The product $\sigma \tau_1 \ldots \tau_r$ equals the identity permutation;
3. The group $\langle \sigma, \tau_1, \ldots, \tau_r \rangle$ acts transitively on $\{1, \ldots, d\}$;
4. $r = 2g - 2 + \ell(\alpha) + d$;
5. Writing $\tau_i = (s_i, t_i)$ with $s_i < t_i$, we have $t_1 \leq \cdots \leq t_r$.

Conditions (1)–(5) above may be taken as the definition of the monotone single Hurwitz numbers $\bar{H}_g(\alpha)$; note that they differ from the classical single Hurwitz numbers $H_g(\alpha)$ only in the constraint imposed by Condition (5). According to Theorem 0.1, the coefficients $C_g(z)$ in the $N \to \infty$ asymptotic expansion of the one-sided HCIZ free energy are generating functions for the monotone single Hurwitz numbers in fixed genus and all degrees. In this article, we study the Witten-type formal generating series of the monotone single Hurwitz numbers in all degrees and genera, i.e. we study the limit object associated to the one-sided HCIZ free energy directly.

0.1. **Main results and organization.** In this paper we study the structure of monotone Hurwitz numbers, and focus in particular on the striking similarities with classical Hurwitz numbers, which are present in almost every aspect of the theory. The classical Hurwitz numbers [22] have enjoyed renewed interest since emerging as central objects in recent approaches to Witten’s conjecture [26]. Our main results are stated without proof in this section. They are interleaved with the corresponding results in the theory of classical Hurwitz numbers in order to emphasize these similarities.

Introduce the generating function

$$\bar{H}(z, t, p_1, p_2, \ldots) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{r=0}^{\infty} t^r \sum_{\alpha \vdash d} \bar{H}^r(\alpha) p_\alpha,$$

\[\text{Note that the usual geometric definition of the Hurwitz numbers contains a further division by } d!, \text{ which is omitted here.}\]
where $\tilde{H}^r(\alpha) = \tilde{H}^r_{\mu}(\alpha)$ with $r = 2g - 2 + \ell(\alpha) + d$ and $z, t, p_1, p_2, \ldots$ are indeterminates. In Section 1, we provide a global characterization of $\tilde{H}$ in a manner akin to Virasoro constraints in random matrix theory: the monotone join-cut equation.

**Theorem 0.2.** The generating function $\tilde{H}$ is the unique formal power series solution of the partial differential equation

$$\frac{1}{2t} \left( z \frac{\partial \tilde{H}}{\partial z} - zp_1 \right) = \frac{1}{2} \sum_{i,j \geq 1} (i + j)p_i p_j \frac{\partial \tilde{H}}{\partial p_{i+j}} + ij p_i p_j \frac{\partial^2 \tilde{H}}{\partial p_i \partial p_j} + ij p_i p_j \frac{\partial \tilde{H}}{\partial p_i} \frac{\partial \tilde{H}}{\partial p_j}$$

with the initial condition $[z^0] \tilde{H} = 0$.

Note that this is almost exactly the same as the classical join-cut equation

$$\frac{\partial H}{\partial t} = \frac{1}{2} \sum_{i,j \geq 1} (i + j)p_i p_j \frac{\partial H}{\partial p_{i+j}} + ij p_i p_j \frac{\partial^2 H}{\partial p_i \partial p_j} + ij p_i p_j \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j}$$

which, together with the initial condition $[t^0] H = zp_1$, characterizes the generating function

$$H(z, t, p_1, p_2, \ldots) = \sum_{d=1}^{\infty} \sum_{r=0}^{\ell(\alpha)} \sum_{\alpha \vdash d} \frac{z^d}{d!} \sum_{r=0}^{\ell(\alpha)} \sum_{\alpha \vdash d} H^r(\alpha)p_\alpha$$

of the classical Hurwitz numbers, the only difference being that the left-hand side is a divided difference rather than a derivative with respect to $t$. This is a consequence of the fact that, in the monotone case, $t$ is an ordinary rather than exponential marker for the number $r$ of simple ramification points since the transpositions $\tau_i$ must be ordered as in Condition (5) of Theorem 0.1.

In Sections 3 and 4, we obtain explicit formulas for the low genus cases $\tilde{H}_0^r(\alpha)$ and $\tilde{H}_1^r(\alpha)$. The first of these is as follows.

**Theorem 0.3.** The genus zero monotone single Hurwitz numbers $\tilde{H}_0(\alpha)$, $\alpha \vdash d$ are given by

$$\frac{|\text{Aut } \alpha|}{d!} \tilde{H}_0(\alpha) = \left( \prod_{i=1}^{\ell(\alpha)} \frac{2\alpha_i}{\alpha_i} \right) (2d + 1)^{\ell(\alpha) - 3},$$

where

$$(2d + 1)^{\mathcal{F}} = (2d + 1)(2d + 2) \cdots (2d + k)$$

denotes a rising product with $k$ factors, and by convention

$$(2d + 1)^{\mathcal{F}} = \frac{1}{(2d + k + 1)^{-k}}$$

for $k < 0$.

In the extremal cases $\alpha = (1^d)$ and $\alpha = (d)$, this result was previously obtained by Zinn-Justin [27] and Gewurz and Merola [12], respectively. Theorem 0.3 should be compared with the well-known explicit formula for the genus zero Hurwitz numbers

$$\frac{|\text{Aut } \alpha|}{d!} H_0(\alpha) = (d - 2 + \ell(\alpha))! \left( \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i}{\alpha_1} \right)^{d(\alpha) - 3}$$
published without proof by Hurwitz [22] in 1891 and independently rediscovered and proved a century later by Goulden and Jackson [15].

The explicit formula for genus one monotone Hurwitz numbers is as follows.

**Theorem 0.4.** The genus one monotone single Hurwitz numbers $\vec{H}_1(\alpha)$, $\alpha \vdash d$ are given by

$$
\frac{|\text{Aut } \alpha|}{d!} \vec{H}_1(\alpha) = \frac{1}{24} \prod_{i=1}^{\ell(\alpha)} \frac{2\alpha_i}{\alpha_i!}
\times (2d + 1)^{\ell(\alpha)} - 3(2d + 1)^{\ell(\alpha)-1} - \sum_{k=2}^{\ell(\alpha)} (k-2)!/(2d+1)^{\ell(\alpha)-k} e_k(2\alpha + 1),
$$

where $e_k(2\alpha + 1)$ is the $k$th elementary symmetric polynomial in $2\alpha_i + 1$, $1 \leq i \leq \ell(\alpha)$.

This result should be compared with the explicit formula for the genus one classical Hurwitz numbers $H_1(\alpha)$,

$$
\frac{|\text{Aut } \alpha|}{d!} H_1(\alpha) = \frac{(d + \ell(\alpha))!}{24} \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!}
\times \left( (d^{\ell(\alpha)} - d^{\ell(\alpha)-1} - \sum_{k=2}^{\ell(\alpha)} (k-2)!/d^{\ell(\alpha)-k} e_k(\alpha) \right),
$$

which was conjectured in [18] and proved by Vakil [25], see also [16].

Via Theorem 0.1, theorems 0.3 and 0.4 yield explicit forms for the first two orders in the free energy of the one-sided HCIZ model. These results may be compared with the first two orders in the free energy of the Hermitian one-matrix model, which for example in the case of cubic vertices were conjectured by Brézin, Itzykson, Parisi and Zuber in [5] and rigorously verified in [2].

In Section 4 we obtain explicit forms for the fixed-genus generating functions

$$
\vec{H}_g(z, p_1, p_2, \ldots) = \sum_{d=1}^{\infty} \vec{H}_g(\alpha)p_\alpha z^d
$$

for the monotone single Hurwitz numbers in terms of an implicit set of Lagrangian variables.

**Theorem 0.5.** Let $s$ be the unique formal power series solution of the functional equation

$$
s = z (1 - \gamma)^{-2}
$$

in the ring $\mathbb{Q}[z, p_1, p_2, \ldots]$, where $\gamma = \sum_{k \geq 1} (2k!)/k^{2k}$. Also, define $\eta = \sum_{k \geq 1} (2k+1)(2k)!/k^{2k}$. Then, the genus one monotone single Hurwitz generating series is

$$
\vec{H}_1(z, p_1, p_2, \ldots) = \frac{1}{24} \log \frac{1}{1 - \eta} - \frac{\eta}{1 - \gamma} = \frac{1}{24} \log \left( \left( \frac{z}{s} \right)^2 \frac{\partial s}{\partial z} \right),
$$

and for $g \geq 2$ we have

$$
\vec{H}_g(z, p_1, p_2, \ldots) = -c_{g,(0)} + \frac{1}{(1 - \eta)^{2g-2}} \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha} \eta_\alpha}{(1 - \eta)^{\ell(\alpha)}},
$$
where
\[ \eta_j = \sum_{k \geq 1} k^j (2k+1) \binom{2k}{k} p_k s^k, \quad j \geq 1, \]
and the \( c_{g,\alpha} \) are rational constants. In particular, for the empty partition,
\[ c_{g,(0)} = -\frac{B_{2g}}{4g(g-1)} \]
where \( B_{2g} \) is a Bernoulli number.

These explicit forms for \( \vec{H}_g \) should be compared with the analogous explicit forms for the generating series
\[ H_g(z, p_1, p_2, \ldots) = \sum_{d=1}^{\infty} \sum_{\alpha \vdash d} H_g(\alpha) \frac{z^d}{d!} \]
for the classical single Hurwitz numbers. Adapting notation from previous works in order to highlight this analogy, let \( w \) be the unique formal power series solution of the functional equation
\[ w = ze^\delta \]
in the ring \( \mathbb{Q}[[z, p_1, p_2, \ldots]] \), where \( \delta = \sum_{k \geq 1} \frac{k^k}{k!} p_k w^k \). Also, define \( \phi = \sum_{k \geq 1} \frac{k^{k+1}}{k!} p_k w^k \).

Then, the genus one single Hurwitz generating series is [16]
\[ H_1(z, p_1, p_2, \ldots) = \frac{1}{24} \log \left( \frac{1}{1 - \phi} \right) = \frac{1}{24} \log \left( \left( \frac{z}{w} \right)^2 \frac{\partial w}{\partial z} \right), \]
and for \( g \geq 2 \) we have [19]
\[ H_g(z, p_1, p_2, \ldots) = \frac{1}{(1 - \phi)^{2g-2}} \sum_{d=1}^{3g-3} \sum_{\alpha \vdash d} a_{g,\alpha} \phi_\alpha \]
where
\[ \phi_j = \sum_{k \geq 1} \frac{k^{k+j+1}}{k!} p_k w^k, \quad j \geq 1, \]
and the \( a_{g,\alpha} \) are rational constants.

For genus \( g = 2, 3 \), these rational forms are given in the Appendix of this paper. The corresponding rational forms for the classical Hurwitz series can be found in [17]. Comparing these expressions, one may observe that in all cases
\[ c_{g,\alpha} = 2^{3g-3} a_{g,\alpha}, \quad \alpha + 3g - 3, \]
for \( g = 2, 3 \).

A key consequence of Theorem 0.5, also proved in Section 4, is that it implies the polynomiality of the monotone single Hurwitz numbers themselves.
Theorem 0.6. To each pair \((g,m)\) with \((g,m) \notin \{(0,1),(0,2)\}\) there corresponds a polynomial \(\vec{P}_g\) in \(m\) variables such that

\[
\frac{|\text{Aut } \alpha|}{|\alpha|!} \tilde{H}_g(\alpha) = \left( \prod_{i=1}^{m} \alpha_i \right) \vec{P}_g(\alpha_1, \ldots, \alpha_m)
\]

for all partitions \(\alpha\) with \(\ell(\alpha) = m\).

Theorem 0.6 is the exact analogue of polynomiality, originally conjectured in [18], for the classical Hurwitz numbers, which under the same hypotheses as in Theorem 0.6 asserts the existence of polynomials \(P_g\) in \(m\) variables such that

\[
(0.10) \quad \frac{|\text{Aut } \alpha|}{|\alpha|!} H_g(\alpha) = (2g - 2 + m + |\alpha|)! \left( \prod_{i=1}^{m} \frac{\alpha_i}{|\alpha_i|} \right) P_g(\alpha_1, \ldots, \alpha_m)
\]

for all partitions \(\alpha\) with \(m\) parts. The only known proof of this result relies on the ELSV formula [8]

\[
(0.11) \quad P_g(\alpha_1, \ldots, \alpha_m) = \int_{\overline{M}_{g,m}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_m \psi_m)}.
\]

Here \(\overline{M}_{g,m}\) is the (compact) moduli space of stable \(m\)-pointed genus \(g\) curves, \(\psi_1, \ldots, \psi_m\) are (complex) codimension 1 classes corresponding to the \(m\) marked points, and \(\lambda_k\) is the (complex codimension \(k\)) \(k\)th Chern class of the Hodge bundle. Equation (0.11) should be interpreted as follows: formally invert the denominator as a geometric series; select the terms of codimension \(\dim \overline{M}_{g,m} = 3g - 3 + m\); and “intersect” these terms on \(\overline{M}_{g,m}\). In contrast to this, our proof of Theorem 0.6 is entirely algebraic and makes no use of geometric methods.

A geometric approach to the monotone Hurwitz numbers would be highly desirable. The form of the rational expression given in Theorem 0.5, in particular its high degree of similarity with the corresponding rational expression (0.8) for the generating series of the classical Hurwitz numbers, suggests the possibility of an ELSV-type formula for the polynomials \(\vec{P}_g\). Further evidence in favour of such a formula is that the summation sets differ only by a contribution from the empty partition, which is itself a scaled Bernoulli number, of known geometric significance. Finally, observe that the ELSV formula implies that the coefficients \(a_{g,\alpha}\) in the rational form (0.8) are themselves Hodge integral evaluations, and for the top terms \(\alpha \vdash 3g - 3\) these Hodge integrals are free of \(\lambda\)-classes — the Witten case.

Equation (0.9), which deals precisely with the case \(\alpha \vdash 3g - 3\), might be a good starting point for the formulation of such a geometric result.

Section 1 closes with two results left unstated here, since they are of a more technical nature than those summarized above. These are a topological recursion in the style of Eynard and Orantin [11], and a join-cut equation for the monotone double Hurwitz series. Unlike the classical case [20], the join-cut equation for the monotone double Hurwitz numbers does not coincide with the join-cut equation for the single monotone Hurwitz numbers.

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1. Join-cut analysis

In this section, we analyse the effect of removing the last factor in a transitive monotone factorization. From this, we obtain a recurrence relation for the number of these factorizations, and differential equations which characterize some related generating series.

1.1. Recurrence relation. Let \( M_r(\alpha) \) be defined by

\[
H_r(\alpha) = |C_\alpha| M_r(\alpha).
\]

It follows from the centrality of symmetric functions of Jucys-Murphy elements, see [13], that \( M_r(\alpha) \) counts the number of \((r+1)\)-tuples \((\sigma_0, \tau_1, \ldots, \tau_r)\) satisfying conditions \((1) - (5)\) of Theorem 0.1, where \( \sigma_0 \) is a fixed but arbitrary permutation of cycle type \( \alpha \).

**Theorem 1.1.** The numbers \( M_r(\alpha) \) are uniquely determined by the initial condition

\[
M_0(\alpha) = \begin{cases} 
1 & \text{if } \alpha = \varepsilon, \\
0 & \text{otherwise,}
\end{cases}
\]

and the recurrence

\[
M_{r+1}(\alpha \cup \{k\}) = \sum_{k' \geq 1} k' m_{k'}(\alpha) M_r(\alpha \setminus \{k'\} \cup \{k+k'\})
\]

\[
+ \sum_{k'=1}^{k-1} M_r(\alpha \cup \{k', k-k'\})
\]

\[
+ \sum_{k'=1}^{k-1} \sum_{r'=0}^{r} \sum_{\alpha' \subseteq \alpha} M_{r'}(\alpha' \cup \{k'\}) M_{r-r'}(\alpha \setminus \alpha' \cup \{k-k'\}),
\]

where \( m_{k'}(\alpha) \) is the multiplicity of \( k' \) as a part in the partition \( \alpha \), and the last sum is over the \( 2^{l(\alpha)} \) subpartitions \( \alpha' \) of \( \alpha \).

**Proof.** As long as the initial condition and the recurrence relation hold, uniqueness follows by induction on \( r \). The initial condition follows from the fact that for \( r = 0 \) we must have \( \sigma = \text{id} \), and the identity permutation is only transitive in \( S_1 \).

To show the recurrence, fix a permutation \( \sigma \in S_d \) of cycle type \( \alpha \cup \{k\} \), where the element \( d \) is in a cycle of length \( k \), and consider a transitive monotone factorization

\[
(a_1 b_1)(a_2 b_2) \cdots (a_r b_r)(a_{r+1} b_{r+1}) = \sigma.
\]

The transitivity condition forces the element \( d \) to appear in some transposition, and the monotonicity condition forces it to appear in the last transposition, so it must be that \( b_{r+1} = d \). If we move this transposition to the other side of the equation and set \( \sigma' = \sigma(a_{r+1} b_{r+1}) \), we get the shorter monotone factorization

\[
(a_1 b_1)(a_2 b_2) \cdots (a_r b_r) = \sigma'.
\]
Depending on whether \(a_{r+1}\) is in the same cycle of \(\sigma'\) as \(b_{r+1}\) and whether (1.5) is still transitive, the shorter factorization falls into exactly one of the following three cases, corresponding to the three terms on the right-hand side of the recurrence.

**Cut:** Suppose \(a_{r+1}\) and \(b_{r+1}\) are in the same cycle of \(\sigma'\). Then, \(\sigma\) is obtained from \(\sigma'\) by cutting the cycle containing \(a_{r+1}\) and \(b_{r+1}\) in two parts, one containing \(a_{r+1}\) and the other containing \(b_{r+1}\), so \((a_{r+1} b_{r+1})\) is called a cut for \(\sigma'\), and also for the factorization (1.4). Conversely, \(a_{r+1}\) and \(b_{r+1}\) are in different cycles of \(\sigma\), and \(\sigma'\) is obtained from \(\sigma\) by joining these two cycles, so the transposition \((a_{r+1} b_{r+1})\) is called a join for \(\sigma\). Note that in the case of a cut, (1.5) is transitive if and only if (1.4) is transitive.

For \(k' \geq 1\), there are \(k' m_{k'}(\alpha)\) possible choices for \(a_{r+1}\) in a cycle of \(\sigma\) of length \(k'\) other than the one containing \(b_{r+1}\). For each of these choices, \((a_{r+1} b_{r+1})\) is a cut and \(\sigma'\) has cycle type \(\alpha \setminus \{k'\} \cup \{k + k'\}\). Thus, the number of transitive monotone factorizations of \(\sigma\) where the last factor is a cut is

\[
(1.6) \quad \sum_{k' \geq 1} k' m_{k'}(\alpha) M' (\alpha \setminus \{k'\} \cup \{k + k'\}),
\]

which is the first term in the recurrence.

**Redundant join:** Now suppose that \((a_{r+1} b_{r+1})\) is a join for \(\sigma'\) and that (1.5) is transitive. Then, we say that \((a_{r+1} b_{r+1})\) is a redundant join for (1.4).

The transposition \((a_{r+1} b_{r+1})\) is a join for \(\sigma'\) if and only if it is a cut for \(\sigma\), and there are \(k - 1\) ways of cutting the \(k\)-cycle of \(\sigma\) containing \(b_{r+1}\). Thus, the number of transitive monotone factorizations of \(\sigma\) where the last factor is a redundant join is

\[
(1.7) \quad \sum_{k' = 1}^{k-1} M' (\alpha \cup \{k', k - k'\}),
\]

which is the second term in the recurrence.

**Essential join:** Finally, suppose that \((a_{r+1} b_{r+1})\) is a join for \(\sigma'\) and that (1.5) is not transitive. Then, we say that \((a_{r+1} b_{r+1})\) is an essential join for (1.4). In this case, the action of the subgroup \(\langle (a_1 b_1), \ldots, (a_r b_r) \rangle\) must have exactly two orbits on the ground set, one containing \(a_{r+1}\) and the other containing \(b_{r+1}\). Since transpositions acting on different orbits commute, (1.5) can be rearranged into a product of two transitive monotone factorizations on these orbits. Conversely, given a transitive monotone factorization for each orbit, this process can be reversed, and the monotonicity condition guarantees uniqueness of the result.

As with redundant joins, there are \(k - 1\) choices for \(a_{r+1}\) to split the \(k\)-cycle of \(\sigma\) containing \(b_{r+1}\). Each of the other cycles of \(\sigma\) must be in one of the two orbits, so there are \(2^{\ell(\alpha)}\) choices for the orbit containing \(a_{r+1}\). Thus, the number of transitive monotone factorizations of \(\sigma\) where the last factor is an essential join is

\[
(1.8) \quad \sum_{k' = 1}^{k-1} \sum_{r' = 0}^{r} \sum_{\alpha' \subseteq \alpha} M'(\alpha' \cup \{k'\}) M'^{-r'}(\alpha \setminus \alpha' \cup \{k - k'\}),
\]

which is the third term in the recurrence. \(\square\)
1.2. **Operators.** To write the recurrence from Theorem 1.1 as a differential equation for some generating series, we introduce some operators. They will be used in later sections to manipulate this equation and solve it.

**Definition 1.2.** The three **lifting** operators are the \( \mathbb{Q}[[x,y,z,p_1,p_2,\ldots]] \)-linear differential operators

\[
\Delta_x = \sum_{k \geq 1} kx^k \frac{\partial}{\partial p_k}, \quad \Delta_y = \sum_{k \geq 1} ky^k \frac{\partial}{\partial p_k}, \quad \Delta_z = \sum_{k \geq 1} kz^k \frac{\partial}{\partial p_k}.
\]

The combinatorial effect of \( \Delta_x \), when applied to a generating series, is to pick a cycle marked by \( p_k \) in all possible ways and mark it by \( kx^k \) instead, that is, by \( x^k \) once for each element of the cycle. Note that \( \Delta_x x = 0 \), so that

\[
(1.9) \quad \Delta_x^2 = \sum_{i,j \geq 1} ijx^{i+j} \frac{\partial^2}{\partial p_i \partial p_j}.
\]

These operators are called lifting operators because of the corresponding projection operators:

**Definition 1.3.** The three **projection** operators are the operators

\[
\Pi_x = [x^0] + \sum_{k \geq 1} p_k[x^k], \quad \Pi_y = [y^0] + \sum_{k \geq 1} p_k[y^k], \quad \Pi_z = [z^0] + \sum_{k \geq 1} p_k[z^k],
\]

where \( \Pi_x \) is \( \mathbb{Q}[[y,z,p_1,p_2,\ldots]] \)-linear, \( \Pi_y \) is \( \mathbb{Q}[[x,z,p_1,p_2,\ldots]] \)-linear, and \( \Pi_z \) is \( \mathbb{Q}[[x,y,p_1,p_2,\ldots]] \)-linear. These operators commute and are idempotent, and \( \Pi_{xy} = \Pi_x \Pi_y, \Pi_{yx} = \Pi_x \Pi_y \Pi_z, \) etc. denote their compositions.

The combined effect of a lift and a projection when applied to a generating series in \( \mathbb{Q}[[p_1,p_2,\ldots]] \) is given by

\[
(1.10) \quad \Pi_x \Delta_x = \Pi_y \Delta_y = \Pi_z \Delta_z = \sum_{k \geq 1} kp_k \frac{\partial}{\partial p_k}.
\]

The identity

\[
(1.11) \quad \Delta_x \Pi_y f = \left[ y \frac{\partial}{\partial y} f \right]_{y=x} + \Pi_y \Delta_x f,
\]

which holds for \( f \in \mathbb{Q}[[x,y,z,p_1,p_2,\ldots]] \), will be useful later on, and can be checked by verifying it on elements of the form \( x^i y^j z^k p_\alpha \). We will also need some splitting operators.

**Definition 1.4.** The **splitting** operator is the \( \mathbb{Q}[[p_1,p_2,\ldots]] \)-linear operator defined by

\[
\text{Split}_{x \to y} (x^k) = x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1}
\]

for \( k \geq 2 \), and by \( \text{Split}_{x \to y}(1) = \text{Split}_{x \to y}(x) = 0 \). If \( f(x) \) is a power series in \( x \) over \( \mathbb{Q}[[p_1,p_2,\ldots]] \) with no constant term, then

\[
(1.12) \quad \text{Split}_{x \to y}(f(x)) = \frac{yf(x) - xf(y)}{x - y}.
\]
The combined effect of a lift, a split and a projection on a generating series in $\mathbb{Q}[[p_1, p_2, \ldots]]$ is

$$\Pi_y \text{Split} \Delta_x = \sum_{i,j \geq 1} (i + j)p_ip_j \frac{\partial}{\partial p_{i+j}}.$$  

1.3. Differential equations. With the lifting, projection and splitting operators, we can write the recurrence of Theorem 1.1 in terms of generating series.

**Theorem 1.5.** For $f = f(z, t, x, p) \in \mathbb{Q}[[z, t, x, p]]$, the differential equation

$$\frac{f - zx}{t} = \Pi_y \text{Split} f + \Delta_x f + f^2$$  

has the unique solution $f = \Delta_x \overrightarrow{H}(z, t, p)$.

**Proof.** The coefficient $[t^r]f$ of a solution $f \in \mathbb{Q}[[z, t, x, p]]$ can be computed recursively from coefficients $[t^{r'}]f$ for $r' < r$, with the base case $[t^0]f = zx$. Thus, by induction on $r$, such a solution exists and is unique.

To show that $\Delta_x \overrightarrow{H}$ is a solution, we use the fact that (1.14) is equivalent to the initial condition and recurrence from Theorem 1.1. Indeed, multiplying (1.14) by (1.15)

$$z^d \frac{d^r}{dr} p_\alpha k(m_k(\alpha) + 1)x^k \big|_{C_\alpha \cup \{k\}} = \frac{z^d t^r p_\alpha x^k}{\prod_{i \geq 1} \frac{m_i(\alpha)}{m_i(\alpha)!}!}$$

and summing over all choices of $d, r, \alpha, k$ with $d \geq k \geq 1, \alpha \vdash d - k$ and $r \geq 0$ gives

$$\frac{\Delta_x \overrightarrow{H} - zx}{t} = \Pi_y \text{Split} \Delta_x \overrightarrow{H} + \Delta_x^2 \overrightarrow{H} + (\Delta_x \overrightarrow{H})^2,$$

as shown by the following computations. We have

$$\Delta_x \overrightarrow{H} = \sum_{\substack{k \geq 1 \\alpha \vdash d \\alpha \vdash d \\alpha \vdash d}} z^d \frac{d^r}{dr} p_\alpha x^k \big|_{C_\alpha \cup \{k\}} M^r(\alpha)$$

$$= \sum_{\substack{k \geq 1 \\alpha \vdash d \\alpha \vdash d \\alpha \vdash d}} z^d \frac{d^r}{dr} p_\alpha x^k \big|_{C_\alpha \cup \{k\}} M^r(\alpha \cup \{k\})$$

$$= \sum_{\substack{k \geq 1 \\alpha \vdash d \\alpha \vdash d \\alpha \vdash d-k}} z^d t^r p_\alpha x^k M^r(\alpha \cup \{k\}) \big|_{C_\alpha \cup \{k\}} \prod_{i \geq 1} \frac{m_i(\alpha)}{m_i(\alpha)!}!,$$

so that

$$\frac{\Delta_x \overrightarrow{H} - zx}{t} = \sum_{\substack{k \geq 1 \\alpha \vdash d \\alpha \vdash d \\alpha \vdash d-k}} z^d t^r p_\alpha x^k M^{r+1}(\alpha \cup \{k\}) \big|_{C_\alpha \cup \{k\}} \prod_{i \geq 1} \frac{m_i(\alpha)}{m_i(\alpha)!}!.$$
which gives the left-hand side of (1.3) and (1.16). Next, we have

\[(1.21) \quad \Pi_y \text{Split } \Delta_x \tilde{H} = \sum_{k' \geq 1} \frac{z^{d \Delta x} t^{p \cup \{k'\}} x^{k-k'} M^r(\alpha \cup \{k\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!}
\]

\[(1.22) \quad = \sum_{k' \geq 1} \frac{z^{d \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \setminus \{k'\} \cup \{k + k'\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!}
\]

\[(1.23) \quad \text{where the second line is obtained from the first by reindexing, replacing } k \text{ by } k + k' \text{ and } \alpha \text{ by } \alpha \setminus \{k'\}. \text{ This is the first term of the right-hand side of (1.3) and (1.16). Also, we have}
\]

\[(1.24) \quad \Delta_x^2 \tilde{H} = \sum_{k' \geq 1} \frac{z^{d \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \cup \{k\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!}
\]

\[(1.25) \quad = \sum_{k' \geq 1} \frac{z^{d \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \setminus \{k'\} \cup \{k + k'\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!}
\]

\[(1.26) \quad (\Delta_x \tilde{H})^2 = \sum_{k' \geq 1} \sum_{d \geq k} \frac{z^{d + d' \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \cup \{k\}) M^r(\alpha \setminus \{k\})}{\prod_{i \geq 1} i^n(\alpha + \alpha') m_i(\alpha)! m_i(\alpha')!}
\]

\[(1.27) \quad = \sum_{k' \geq 1} \sum_{d \geq k} \frac{z^{d + d' \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \cup \{k\}) M^r(\alpha \setminus \{k\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!(m_i(\alpha')^{-1})(m_i(\alpha'))!}
\]

\[(1.28) \quad = \sum_{k' \geq 1} \sum_{d \geq k} \frac{z^{d + d' \Delta x} t^{p \prime} x^{k-k'} M^r(\alpha \cup \{k\}) M^r(\alpha \setminus \{k\})}{\prod_{i \geq 1} i^n(\alpha) m_i(\alpha)!}
\]

\[\text{where the second line is obtained from the first by reindexing, replacing } d \text{ by } d - d',
\]

\[k \text{ by } k - k' \text{ and } \alpha \text{ by } \alpha \setminus \alpha'; \text{ and the third line is obtained by replacing the summation}
\]
over $\alpha' + d' - k'$ by a summation over the $2^{\ell(\alpha)}$ subpartitions of $\alpha$, weighted by $\prod_{1 \leq i} \frac{m_i(\alpha)}{m_{i+1}(\alpha)}$ to account for the resulting overcount. This is the third term of the right-hand side of (1.3) and (1.16). □

As a corollary, we obtain a proof of the monotone join-cut equation.

**Proof of Theorem 0.2** This equation is the result of applying the projection operator $\frac{1}{2} \Pi_{z}$ to (1.16), and simplifying with (1.10), (1.13), (1.9), and noting that $\sum_{k \geq 1} \frac{\partial \bar{H}}{\partial p_k} \bar{H} = z \frac{\partial \bar{H}}{\partial z}$, so it is satisfied by $\bar{H}$. Except for $[z^0] \bar{H}$, uniqueness of the coefficients of $\bar{H}$ follows by induction on the exponent of the accompanying power of $t$. □

In what follows, it will be convenient to specialize $z = 1$ in our generating series for monotone Hurwitz numbers.

**Definition 1.6.** For $g \geq 0$, the genus $g$ generating series for monotone single Hurwitz numbers is

$$G_g = \bar{H}_g(1, p_1, p_2, \ldots) = \sum_{d \geq 1} \frac{p_{\alpha}}{d!} \bar{H}_g(\alpha),$$

and the genus-wise generating series for all monotone single Hurwitz numbers is

$$G = G(t, p_1, p_2, \ldots) = \sum_{g \geq 0} t^g G_g.$$

Note that this generating series is equivalent to $\bar{H}$, via the relations

$$G(t, p_1, p_2, \ldots, p_k, \ldots) = t \bar{H}(1, t^{1/2}, p_1, t^{-3/2} p_2, \ldots, t^{-(k+1)/2} p_k, \ldots)$$

$$\bar{H}(z, t, p_1, \ldots, p_k, \ldots) = t^{-2} G(t^2, z t^2 p_1, z^2 t^3 p_2, \ldots, z^k t^{k+1} p_k, \ldots)$$

In addition to replacing the marker for number of transpositions by a marker for genus, the marker $z$ for the size of the ground set has been removed from $G$; this is mainly to simplify the task of keeping track of whether the operators $\frac{\partial}{\partial p_k}$ are considered as $z$-linear or $s$-linear operators, where $z$ and $s$ are related by a functional relation, as in Theorem 0.5. See section 2 for the details.

**Theorem 1.7.** For $f = f(t, x, p_1, p_2, \ldots) \in \mathbb{Q}[t, x, p_1, p_2, \ldots]$, the differential equation

$$f = \Pi_y \text{Split}_x f + t \Delta_x f + f^2 + x$$

has a unique solution with no constant term, $f = \Delta_x G(t, p_1, p_2, \ldots)$. Furthermore, the series $\Delta_x G_0 \in \mathbb{Q}[t, p_1, p_2, \ldots]$ is uniquely determined by the equation

$$\Delta_x G_0 = \Pi_y \text{Split}_x \Delta_x G_0 + (\Delta_x G_0)^2 + x$$

and the requirement that it have no constant term. For $g \geq 1$, the series $G_g \in \mathbb{Q}[t, p_1, p_2, \ldots]$ is uniquely determined by the equation

$$\Delta_x G_0 = \Pi_y \text{Split}_x \Delta_x G_0 + \sum_{g' = 1}^{g-1} \Delta_x G_{g'} \Delta_x G_{g-g'}.$$
Theorem 1.8. For e.g. H Equations (2.11) and (2.12). H a fixed number of parts, is analogous to the series This series for monotone Hurwitz numbers, which collects only the terms for α x (1.36) after moving some terms to the left-hand side to solve for ∆ Equation (1.34) can be obtained directly from the recurrence in Theorem 1.1.

Proof. Equation (1.34) can be obtained directly from the recurrence in Theorem 1.1 by using the weight

\[ e^{(r-\ell(a)-d+2)/2}p_\alpha x^k \prod_{i \geq 1} \frac{m_i(a)}{m_i(a)!!}, \]

or from (1.16) by using the substitutions (1.32). Extracting the coefficient of t^0 in (1.34) gives (1.35). The uniqueness comes from the fact that (1.34) is equivalent to the recurrence in Theorem 1.1. Extracting the coefficient of t^g for g ≥ 1 gives (1.36) after moving some terms to the left-hand side to solve for ∆_x G_g.

1.4. Topological recursion. For the purposes of this section, let α = (α_1, α_2, ..., α_\ell) be a composition of d instead of a partition; that is, we still have α_1 + α_2 + ⋯ + α_\ell = d, but we no longer require α_1 ≥ α_2 ≥ ⋯ ≥ α_\ell. Also for this section only, for given g ≥ 0 and \ell ≥ 1, consider the generating series

\[ \vec{H}_g(x_1, x_2, \ldots, x_\ell) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_\ell \geq 1} \frac{\vec{H}_g(\alpha)}{|C_\alpha|} x_1^{\alpha_1} x_2^{\alpha_2-1} \cdots x_\ell^{\alpha_\ell-1}. \]

This series for monotone Hurwitz numbers, which collects only the terms for α with a fixed number of parts, is analogous to the series H_g(x_1, x_2, ..., x_\ell) for Hurwitz numbers considered by Bouchard and Mariño [4 Equations (2.11) and (2.12)].

One form of recurrence for Hurwitz numbers is expressed in terms of the series H_g(x_1, x_2, ..., x_\ell). This is sometimes referred to as “topological recursion” (see, e.g., [4 Conjecture 2.1]; [10 Remark 4.9]; [11 Definition 4.2]).

**Theorem 1.8.** For g ≥ 0 and \ell ≥ 1, we have

\[ \vec{H}_g(x_1, x_2, \ldots, x_\ell) = \delta_{g,0} \delta_{\ell,1} + x_1 \vec{H}_{g-1}(x_1, x_2, \ldots, x_\ell) \]

\[ + \sum_{j=2}^\ell \frac{\partial}{\partial x_j} \left( \frac{x_1 \vec{H}_g(x_1, \ldots, \hat{x}_j, \ldots, x_\ell) - x_j \vec{H}_g(x_2, \ldots, x_\ell)}{x_1 - x_j} \right) \]

\[ + \sum_{g' = 0}^g \sum_{S \subseteq \{2, \ldots, k\}} x_1 \vec{H}_{g'}(x_1, x_S) \vec{H}_{g-g'}(x_1, x_{\overline{S}}), \]

where x_1, ..., \hat{x}_j, ..., x_\ell is the list of all variables x_1, ..., x_\ell except x_j, x_S is the list of all variables x_j with j ∈ S, and x_{\overline{S}} is the list of all variables x_j with j ∈ \{2, ..., k\} \ S.

**Proof.** The result follows routinely from the monotone join-cut recurrence in Theorem 1.1. □

**Remark 1.9.** Note that as a consequence of its combinatorial interpretation, the unique solution of the recurrence in Theorem 1.8 is symmetric in the variables
For \((g, \ell) = (0, 1)\), the recurrence above reduces to the equation \(\vec{H}_0(x_1) = 1 + x_1 \vec{H}_0(x_1)^2\), so we have

\[
\vec{H}_0(x_1) = 1 - \sqrt{1 - 4x_1} \quad \frac{2}{2x_1} = \sum_{k \geq 0} \frac{1}{k+1} \binom{2k}{k} x_1^k.
\]

After some calculation we obtain

\[
\vec{H}_0(x_1, x_2) = \frac{4}{\sqrt{1 - 4x_1}(1) - 4x_2(1 + \sqrt{1 - 4x_1})^2}.
\]

If we define \(y_i \) by \(y_i = 1 + x_i y_i^2\) for \(i \geq 1\), then

\[
\vec{H}_0(x_1) = y_1,
\]

\[
\vec{H}_0(x_1, x_2) = \frac{x_1 y_1 x_2 y_2 (x_2 y_2 - x_1 y_1)^2}{(y_1 - 1)(y_2 - 1)(x_2 - x_1)^2},
\]

where \(y_i'\) denotes \(\frac{\partial y_i}{\partial x_i}\). Thus, in the context of Eynard and Orantin \([11]\), we have the spectral curve \(y\), where \(y = 1 + xy^2\), but it is unclear to us what the correct notion of Bergmann kernel should be in this case.

1.5. Monotone double Hurwitz numbers. In this section, we give a combinatorial description of the boundary conditions of the monotone join-cut equation for monotone double Hurwitz numbers. The monotone double Hurwitz numbers were dealt with extensively in \([13]\); \(H^r(\alpha, \beta)\) counts \((r + 2)\)-tuples \((\sigma, \rho, \tau_1, \ldots, \tau_r)\) with \(\rho \in C_\beta\), satisfying conditions (1) – (5) of Theorem 0.1 suitably modified.

**Theorem 1.10.** The generating series

\[
\vec{H} = \vec{H}(z, t, p, q) = \sum_{d \geq 1 \atop \alpha, \beta \vdash d} \frac{z^d}{d!} t^r p^\alpha q^\beta \vec{H}^r(\alpha, \beta)
\]

for the monotone double Hurwitz numbers satisfies the equation

\[
\frac{1}{2t} \left( z \frac{\partial \vec{H}}{\partial z} - z p_1 q_1 - z^2 \frac{\partial}{\partial z} \sum_{d \geq 1 \atop r \geq 0 \atop \alpha, \beta \vdash d} \frac{z^d}{d!} t^r p^\alpha p_{r+1} q^\beta q_{r+1} N^r(\alpha, i; \beta, j) \right)
\]

\[
= \frac{1}{2} \sum_{i, j \geq 1} (i + j)p_i p_j \frac{\partial^2 \vec{H}}{\partial p_i \partial p_j} + ij p_i p_j \frac{\partial^2 \vec{H}}{\partial p_i \partial p_j} + ij p_i p_j \frac{\partial \vec{H}}{\partial p_i} \frac{\partial \vec{H}}{\partial p_j},
\]

where \(N^r(\alpha, i; \beta, j)\) is the number of transitive monotone solutions of

\[
\sigma \rho(a_1 b_1)(a_2 b_2) \cdots (a_r b_r) = \text{id}
\]

where \(\sigma \in C_{\alpha \cup \{i\}}\), \(\rho \in C_{\beta \cup \{j\}}\), and the element \(d\) is in a cycle of length \(i\) of \(\sigma\) and a cycle of length \(j\) of \(\rho\).
Proof. The proof is very similar in spirit to the proofs of Theorems 1.1 and 0.2, so we only sketch the combinatorial join-cut analysis.

For a fixed choice of \( \sigma \in C_{\alpha \cup \{i\}} \) with the element \( d \) in a cycle of length \( i \) and \( \rho \in C_{\beta \cup \{j\}} \) with \( d \) in a cycle of length \( j \), consider the monotone factorizations of \( \rho^{-1}\sigma^{-1} \) of length \( r \) which correspond to solutions of \( (1.46) \) counted by \( \tilde{H}^r(\alpha,\beta) \). These factorizations are counted with a weight of \( \frac{q^d}{d!}p_{\alpha}p_{\beta}q_{j} \). They are not necessarily the transitive monotone factorizations of \( \rho^{-1}\sigma^{-1} \), as the subgroup generated by the transpositions may be a proper subgroup of the subgroup generated by the transpositions and \( \sigma \) and \( \rho \). However, the join-cut analysis performed for Theorem 1.1 in the case where last factor of the monotone factorization involves the element \( d \) (that is, the cases of cuts, redundant joins, and essential joins) applies here essentially unchanged anyway, and gives the second term of the left-hand side of \( (1.45) \). When doing this analysis, \( \rho \) stays fixed while \( \sigma \) is modified, which corresponds to the fact that the variables \( q_i \) do not appear explicitly on the left-hand side of \( (1.45) \).

If \( d > 1 \) and the element \( d \) is not involved in any transpositions, then it must be a fixed point of \( \sigma \rho \), so there is some element \( k \) with \( \rho(d) = \sigma^{-1}(d) = k \). In this case, the subgroup\( \langle \sigma, \rho, (a_1 b_1), (a_2 b_2), \ldots, (a_r b_r) \rangle \) acts transitively on the ground set \( \{1, 2, \ldots, d\} \) if and only if \( k \neq d \) and the subgroup\( \langle \sigma(k d), (k d) \rho, (a_1 b_1), (a_2 b_2), \ldots, (a_r b_r) \rangle \) acts transitively on the subset \( \{1, 2, \ldots, d - 1\} \). Let \( \sigma' = \sigma(k d) \) and \( \rho' = (k d) \rho \). Then, \( \sigma' \in C_{\alpha \cup \{i-1\}} \) with \( k \) in a cycle of length \( i - 1 \), \( \rho' \in C_{\beta \cup \{j-1\}} \) with \( k \) in a cycle of length \( j - 1 \), and \( \sigma' \rho'(a_1 b_1)(a_2 b_2)\ldots(a_r b_r) = id \) is a transitive monotone solution of \( (1.46) \) counted by \( N^r(\alpha, i - 1; \beta, j - 1) \). This gives the second term on the right-hand side of \( (1.45) \).

The last case is where \( d = 1 \), and then \( \tilde{H}^r(\varepsilon, \varepsilon) = \delta_{r,0} \), which gives the first term on the right-hand side of \( (1.45) \). \( \square \)

2. VARIABLES AND OPERATORS

To obtain the series given in Theorems 0.5 from the join-cut differential equation, we perform a Lagrangian change of variables. That is, the generating series \( \bar{H}_g \) are defined in terms of the variable \( z \), while the expressions in these theorems are written in terms of the variable \( s \), and these two variables are related by the functional relation

\[
(2.1) \quad s = z \left( 1 - \sum_{k \geq 1} \binom{2k}{k} p_k s^k \right)^{-2}\,.
\]

However, it is more convenient to work with the quantities \( z^k p_k \) and \( s^k p_k \) instead of working with \( z \) and \( s \) directly. Therefore, we work with the generating series \( \bar{G}_g \), which correspond to \( \bar{H}_g \) with \( z^k p_k \) replaced by \( p_k \), and introduce another set of variables, \( q_1, q_2, \ldots \), to replace \( s^k p_k \).

In this section, we describe the relation between the variables \( p_1, p_2, \ldots \) and the variables \( q_1, q_2, \ldots \) and introduce the basic power series used to express \( \bar{G}_g \).
succinctly. We also collect a few computational lemmas. The notation of this section is used for the rest of this paper.

2.1. Two sets of variables. Let $q_1, q_2, \ldots$ be a countable set of indeterminates, and let

$$\gamma = \sum_{k \geq 1} \binom{2k}{k} q_k, \quad \eta = \sum_{k \geq 1} (2k + 1) \binom{2k}{k} q_k$$

be formal power series in these indeterminates. If we set

$$p_k = q_k (1 - \gamma)^{2k}$$

for all $k \geq 1$, then $p_1, p_2, \ldots$ are power series in $q_1, q_2, \ldots$. Since these power series have no constant term, and the linear term in $p_k$ is simply $q_k$, they can be solved recursively to write $q_1, q_2, \ldots$ as power series in $p_1, p_2, \ldots$. Thus, we can identify the rings of power series in these two sets of variables, and write

$$R = \mathbb{Q}[[p_1, p_2, \ldots]] = \mathbb{Q}[[q_1, q_2, \ldots]].$$

Using the multivariate Lagrange Implicit Function Theorem (see [14, Theorem 1.2.9]), we can relate the coefficient extraction operators $[p_\alpha]$ and $[q_\alpha]$ as follows.

**Theorem 2.1.** Let $\alpha + d \geq 0$ be a partition and $f \in R$ be a power series. Then

$$[p_\alpha] f = [q_\alpha] f \frac{1 - \eta}{(1 - \gamma)^{2d+1}}.$$  

**Proof.** Let $\phi_k = (1 - \gamma)^{-2k}$, so that $q_k = p_k \phi_k$. Then, from [14, Theorem 1.2.9], we get

$$[p_\alpha] f = [q_\alpha] f \phi_\alpha \det \left( \delta_{ij} - q_j \frac{\partial}{\partial q_j} \log \phi_i \right)_{i,j \geq 1}.$$  

We have $\phi_\alpha = (1 - \gamma)^{-2d}$, and using the fact that $\det(I - AB) = \det(I - BA)$ for matrices $A$ and $B$, we can compute the determinant as

$$\det \left( \delta_{ij} - q_j \frac{\partial}{\partial q_j} \log \phi_i \right)_{i,j \geq 1} = \det \left( \delta_{ij} - \frac{2iq_j}{1 - \gamma} \binom{2j}{j} \right)_{i,j \geq 1}$$

$$= 1 - \sum_{k \geq 1} \frac{2kq_k}{1 - \gamma} \binom{2k}{k}$$

$$= \frac{1 - \eta}{1 - \gamma}. \quad \Box$$

Let

$$D_k = p_k \frac{\partial}{\partial p_k}, \quad D = \sum_{k \geq 1} k D_k$$

be differential operators on $R$. Note that they all have the set $\{p_\alpha\}_{\alpha + d \geq 0}$ as an eigenbasis, with eigenvalues given by

$$D_k p_\alpha = m_k p_\alpha, \quad \mathcal{D} p_\alpha = |\alpha| p_\alpha,$$

and consequently commute with each other. Let

$$E_k = q_k \frac{\partial}{\partial q_k}, \quad \mathcal{E} = \sum_{k \geq 1} k E_k$$

as eigenbasis, with eigenvalues given by
be the corresponding differential operators on \( R \) for the basis \( \{ q_\alpha \}_{\alpha \geq 0} \). This second set of differential operators commutes with itself, but not with the first. By using the relation (2.3) and computing the action of \( E_k \) on \( p_j \), we can verify that

\[
E_k = D_k - \frac{2q_k}{1 - \gamma} \binom{2k}{k} D
\]

as operators. It follows that

\[
E = \frac{1 - \eta}{1 - \gamma} D
\]

and

\[
D_k = E_k + 2 \frac{q_k}{1 - \eta} \binom{2k}{k} E,
\]

so that we can write each set of differential operators in terms of the other. Using the fact that \( 2E \gamma = \eta - \gamma \), it will also be useful to note that for any integer \( i \) and any power series \( A \in R \),

\[
(1 - \gamma)^i (2E - i) (1 - \gamma)^i A = 1 - \eta (2D - i) A,
\]

which reduces to the relation (2.13) when \( i = 0 \).

### 2.2. Projecting and splitting.

For the series \( \Delta_x G_g \), the change of variables from \( p_1, p_2, \ldots \) to \( q_1, q_2, \ldots \) also corresponds to the change of variables from \( x, y, z \) to \( \hat{x}, \hat{y}, \hat{z} \) defined by the relations

\[
x = \hat{x} (1 - \gamma)^2, \quad y = \hat{y} (1 - \gamma)^2, \quad z = \hat{z} (1 - \gamma)^2.
\]

Since the power series \( (1 - \gamma)^2 \) is invertible in the ring \( R \), we can identify the \( R \)-algebras \( R[[x]] \) and \( R[[\hat{x}]] \), and similarly for any subset of \( \{ x, y, z \} \) and the corresponding subset of \( \{ \hat{x}, \hat{y}, \hat{z} \} \). In terms of these variables, we have

\[
\Pi_x = [x^0] + \sum_{k \geq 1} p_k [x^k] = [x^0] + \sum_{k \geq 1} q_k [\hat{x}^k],
\]

and similarly for \( \Pi_y, \Pi_z \), and

\[
\text{Split} (f(x)) = \frac{yf(x) - xf(y)}{x - y} = \frac{\hat{y}f(x) - \hat{x}f(y)}{\hat{x} - \hat{y}}
\]

for a series \( f(x) \in R[[x]] \) with no constant term in \( x \).

### 2.3. Series and polynomials.

In addition to the series

\[
\gamma = \sum_{k \geq 1} \binom{2k}{k} q_k, \quad \eta = \sum_{k \geq 1} (2k + 1) \binom{2k}{k} q_k,
\]

we define the series

\[
\eta_j = \mathcal{E}^j \eta = \sum_{k \geq 1} k^j (2k + 1) \binom{2k}{k} q_k
\]

for \( j \geq 0 \). Note that \( \eta_0 = \eta \). Our solutions to the join-cut equation will be expressed in terms of these. In particular, for genus 2 and higher, \( G_g \) will lie in the subring

\[
Q = \mathbb{Q}[ (1 - \eta)^{-1}, \eta_1 (1 - \eta)^{-1}, \eta_2 (1 - \eta)^{-1}, \ldots ]
\]
of $R$. The further subring

$$P = \mathbb{Q}[\eta_1(1 - \eta)^{-1}, \eta_2(1 - \eta)^{-1}, \ldots] \subset \mathbb{Q}$$

will also be useful, especially in the context of the $\mathbb{Q}$-vector space decomposition

$$Q = \bigoplus_{i \geq 0} (1 - \eta)^{-i} P.$$ 

Let $u, v, w$ be defined by

$$u = (1 - 4\hat{x})^{-\frac{1}{2}}, \quad v = (1 - 4\hat{y})^{-\frac{1}{2}}, \quad w = (1 - 4\hat{\zeta})^{-\frac{1}{2}}.$$ 

Then, we have

$$u = \sum_{k \geq 0} \binom{2k}{k} \hat{x}^k,$$

and

$$\hat{x} \frac{\partial}{\partial \hat{x}} = \frac{u^3 - u}{2} \frac{\partial}{\partial u},$$

so

$$\gamma = \Pi_x(u - 1), \quad \eta = \Pi_x(u^3 - 1), \quad \eta_1 = \Pi_x \frac{3}{2}(u^5 - u^3), \quad \ldots$$

In fact, if we define

$$\eta_j^u = \left(\hat{x} \frac{\partial}{\partial \hat{x}}\right)^j (u^3 - 1),$$

so that $\eta_j = \Pi_x \eta_j^u$, then it can be seen that for $j \geq 1$, $\eta_j^u$ is an odd polynomial in $u$ of degree $2j + 3$ divisible by $(u^5 - u^3)$. Thus, the set $\{\eta_j^u\}_{j \geq 1}$ is a $\mathbb{Q}$-basis for the vector space $(u^5 - u^3)\mathbb{Q}[u^2]$. This will be useful to show that various expressions project down to the subring $Q$ of $R$, or to a particular subspace of $Q$.

2.4. Lifting. With this notation, it is useful to compute the image of the lifting operator $\Delta_x$ on some elements of $R[[\hat{x}, \hat{y}, \hat{\zeta}]]$.

**Lemma 2.2.**

$$\Delta_x(q_k) = k\hat{x}^k + \frac{k(u^3 - u)q_k}{1 - \eta}$$

$$\Delta_x(\hat{x}) = \frac{\hat{x}(u^3 - u)}{1 - \eta}$$

$$\Delta_x(\hat{y}) = \frac{\hat{y}(u^3 - u)}{1 - \eta}$$

$$\Delta_x(\eta_j) = \eta_j^u + \frac{(u^3 - u)\eta_{j+1}}{1 - \eta}.$$

From this list, we can also conclude that $\Delta_x$ preserves some important subspaces of $Q[u]$.

**Corollary 2.3.** For each $i, j \geq 0$, the operator $\Delta_x$ restricts to a function

$$\Delta_x : (u^3 - u)^i (1 - \eta)^{-j} P[u^2] \rightarrow (u^3 - u)^{i+1} (1 - \eta)^{-j-1} P[u^2].$$
3. Genus zero

In this section, we prove Theorem 0.3. Our strategy is to define the series $G_0$ by

\[(3.1) \quad G_0 = \sum_{d \geq 1} \sum_{\alpha \vdash d} (2d + 1)(2d + 2) \cdots (2d + \ell - 3) \frac{|C_{\alpha}|}{d!} p_\alpha \prod_{k=1}^\ell \alpha_k \left(\frac{2\alpha_k}{\alpha_k}\right)\]

and then to show that the series $\Delta_x G_0$ satisfies the genus zero join-cut equation (1.35) of Theorem 1.7. This involves performing a Lagrangian change of variables to get a closed form for $\Delta_x G_0$.

**Theorem 3.1.** With the definition above for $G_0$,

\[(3.2) \quad (2D - 2)(2D - 1)(2D)G_0 = \frac{(1 - \gamma)^3}{1 - \eta} - 1.\]

**Proof.** For $\alpha \vdash d \geq 0$, $p_\alpha$ is an eigenvector of the operator $D$ with eigenvalue $d$, so we can use this operator to transform the expression for $G_0$ into a negative binomial. For $\alpha \vdash d \geq 1$, using Theorem 2.1, we have

\[(3.3) \quad [p_\alpha](2D - 2)(2D - 1)(2D)G_0
\]

\[= (2d - 2)(2d - 1)(2d + 1) \cdots (2d + \ell - 3) \frac{|C_{\alpha}|}{d!} \prod_{k=1}^\ell \alpha_k \left(\frac{2\alpha_k}{\alpha_k}\right)\]

\[= (-1)\ell \left(\frac{2 - 2d}{\ell}\right) \left(\frac{\ell}{m_1, m_2, \ldots}\right) \prod_{j \geq 1} \left(\frac{2j}{j}\right)^{m_j} p_j^{m_j}\]

\[= [q_\alpha](1 - \gamma)^{2 - 2d}
\]

\[= [p_\alpha](1 - \gamma)^3
\]

\[= \frac{(1 - \gamma)^3}{1 - \eta}.\]

**Remark 3.2.** This formula is the main motivation for the definition of $s$ through the functional relation

\[(3.4) \quad s = z \left(1 - \sum_{k \geq 1} \left(\frac{2k}{k}\right) s^k p_k\right)^{-2}\]

and the change of variables from $z$ to $s$ or, equivalently, from $p_1, p_2, \ldots$ to $q_1, q_2, \ldots$.

After computing the coefficient of $p_\alpha$ when $d = 0$ separately, we get

\[(3.5) \quad (2D - 2)(2D - 1)(2D)G_0 = \frac{(1 - \gamma)^3}{1 - \eta} - 1. \quad \square\]

**Theorem 3.3.** Let

\[(3.6) \quad G'_0 = (2D)G_0, \quad G''_0 = (2D - 1)G'_0, \quad G'''_0 = (2D - 2)G''_0.\]
Then,
\begin{equation}
G''_0 = \frac{1}{2} - \frac{1}{2}(1 - \gamma)^2,
\end{equation}
\begin{equation}
D_k G''_0 = \frac{(1 - \gamma)^2}{1 - \eta} \binom{2k}{k} q_k,
\end{equation}
\begin{equation}
D_k G'_0 = (1 - \gamma) \frac{1}{2k - 1} \binom{2k}{k} q_k,
\end{equation}
\begin{equation}
D_k G_0 = \frac{1}{2k(2k - 1)} \binom{2k}{k} q_k - \sum_{j \geq 1} \frac{2j + 1}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} q_j q_k.
\end{equation}

Proof. By (2.15) with $i = 2$, we have
\begin{equation}
G''_0 = (2D - 2)^{-1} G''_0
\end{equation}
\begin{equation}
= \frac{1}{2} + (2D - 2)^{-1} (G''_0 + 1)
\end{equation}
\begin{equation}
= \frac{1}{2} + (1 - \gamma)^2 (2E - 2)^{-1}
\end{equation}
\begin{equation}
= \frac{1}{2} - \frac{1}{2}(1 - \gamma)^2.
\end{equation}
Note that the kernel of $(2D - 2)$ is spanned by $p_\alpha$ where $d = 1$, that is, $p_1$. Since the constant and linear terms for power series in $p_1, p_2, \ldots$ and series in $q_1, q_2, \ldots$ are equal, it is easy to check that this expression for $G''_0$ agrees with the definition of $G_0$ on the coefficient of $p_1$. This establishes (3.7). Using the relation (2.13) to convert between $D$ and $E$, we then have
\begin{equation}
D_k G''_0 = (1 - \gamma) D_k \gamma = \frac{(1 - \gamma)^2}{1 - \eta} \binom{2k}{k} q_k,
\end{equation}
which establishes (3.8).

Since the operator $D_k$ commutes with $(2D - 1)$, we can use (2.15) again, this time with $i = 1$, to get
\begin{equation}
D_k G'_0 = (2D - 1)^{-1} D_k G''_0
\end{equation}
\begin{equation}
= (1 - \gamma) (2E - 1)^{-1} \binom{2k}{k} q_k
\end{equation}
\begin{equation}
= (1 - \gamma) \frac{1}{2k - 1} \binom{2k}{k} q_k.
\end{equation}
This establishes (3.9).

Finally, since $D_k$ also commutes with $(2D)$ and we know that $D_k G_0$ has no constant term, by (2.15) with $i = 0$, we have
\begin{equation}
D_k G_0 = (2D)^{-1} D_k G'_0
\end{equation}
\begin{equation}
= (2E)^{-1}(1 - \eta) \frac{1}{2k - 1} \binom{2k}{k} q_k
\end{equation}
\begin{equation}
= (2E)^{-1} \frac{1}{2k - 1} \binom{2k}{k} q_k - \sum_{j \geq 1} \frac{2j + 1}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} q_j q_k
\end{equation}
\begin{equation}
= \frac{1}{2k(2k - 1)} \binom{2k}{k} q_k - \sum_{j \geq 1} \frac{2j + 1}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} q_j q_k.
\end{equation}
This establishes (3.10).
With this expression for the partial derivatives of $G_0$, we can compute $\Delta_x G_0$.

To get a concise expression, we introduce the following power series.

**Lemma 3.4.** The power series

$$F(\hat{x}, \hat{y}) = \sum_{j \geq 0} \sum_{k \geq 1} \frac{(2j + 1)k}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} \hat{x}^k \hat{y}^j$$

can be expressed as

$$F(\hat{x}, \hat{y}) = \frac{(u^2 - 1)v^2}{2u(u + v)}.$$  

**Proof.**

$$F(\hat{x}, \hat{y}) = \sum_{j \geq 0} \sum_{k \geq 1} \frac{(2j + 1)k}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} \hat{x}^k \hat{y}^j$$

$$= \int_0^1 \hat{x}t(1 - 4\hat{t}t)^{-\frac{1}{2}}(1 - 4\hat{y}t)^{-\frac{1}{2}} \frac{dt}{t}$$

$$= \left[ \frac{1}{2(\hat{y} - \hat{x})} (1 - 4\hat{x}t)^{\frac{1}{2}} (1 - 4\hat{y}t)^{-\frac{1}{2}} \right]_{t=0}^{t=1}$$

$$= \frac{\hat{x}}{2(\hat{y} - \hat{x})} \left( (1 - 4\hat{x})^{\frac{1}{2}} (1 - 4\hat{y})^{-\frac{1}{2}} - 1 \right)$$

$$= \frac{(u^2 - 1)v^2}{2u(u + v)}. \quad \Box$$

**Theorem 3.5.** If $G_0$ is defined by (3.1), then

$$\Delta_x G_0 = 2F(\hat{x}, 0) - \Pi_y F(\hat{x}, \hat{y}),$$

where $F(\hat{x}, \hat{y})$ is as defined in Lemma 3.4 and this series satisfies the genus zero join-cut equation (1.35) of Theorem 1.7. Therefore, $G_0$ is the generating series for genus zero monotone single Hurwitz numbers.

**Proof.** By (3.10), we have

$$\Delta_x G_0 = \sum_{k \geq 1} \frac{k\hat{x}^k}{q_k} D_k G_0$$

$$= \sum_{k \geq 1} \frac{1}{2(2k - 1)} \binom{2k}{k} \hat{x}^k - \sum_{j \geq 1} \sum_{k \geq 1} \frac{(2j + 1)k}{2(j + k)(2k - 1)} \binom{2j}{j} \binom{2k}{k} q_j \hat{x}^k$$

$$= 2F(\hat{x}, 0) - \Pi_y F(\hat{x}, \hat{y}).$$

To verify that $\Delta_x G_0$ satisfies equation (1.35), we need to check that the expression

$$\Delta_x G_0 - \Pi_y \text{Split}_{x \rightarrow y} \Delta_x G_0 - (\Delta_x G_0)^2 - x$$
is zero. We can rewrite each of these terms as
\begin{equation}
\Delta_x G_0 = \Pi_{yz} \left( 2F(\hat{x}, 0) - F(\hat{x}, \hat{y}) \right),
\end{equation}
\begin{equation}
\Pi_y \text{ Split } \Delta_x G_0 = \Pi_{yz} \left( \frac{\hat{y}(2F(\hat{x}, 0) - F(\hat{x}, \hat{z}))}{\hat{x} - \hat{y}} - \hat{x}(2F(\hat{y}, 0) - F(\hat{y}, \hat{z})) \right),
\end{equation}
\begin{equation}
(\Delta_x G_0)^2 = \Pi_{yz} \left( (2F(\hat{x}, 0) - F(\hat{x}, \hat{y})) (2F(\hat{x}, 0) - F(\hat{x}, \hat{z})) \right),
\end{equation}
\begin{equation}
x = \hat{x}(1 - \gamma)^2 = \Pi_{yz} \left( \frac{1}{2}(1 - u^{-2})(2 - v)(2 - w) \right),
\end{equation}
to get an expression of the form
\begin{equation}
\Pi_{yz} W(u, v, w),
\end{equation}
where \( W(u, v, w) \) is a rational function of \( u, v, w \). This rational function itself is not zero, but a straightforward computation shows that its symmetrization with respect to \( y \) and \( z \), that is, \( \frac{1}{2}(W(u, v, w) + W(u, w, v)) \), is zero. Thus,
\begin{equation}
\Pi_{yz} W(u, v, w) = \Pi_{yz} \left( \frac{1}{2}(W(u, v, w) + W(u, w, v)) \right) = 0,
\end{equation}
which completes the verification. \( \square \)

**Proof of Theorem 0.3** The result follows immediately from Theorem 3.5 and (3.1). \( \square \)

To compute the right-hand side of the join-cut equation (1.36) for genus one, we also need the following corollary.

**Corollary 3.6.**
\begin{equation}
\Delta^2_x G_0 = \frac{1}{16}(u^2 - 1)^2.
\end{equation}

**Proof.** This follows from (3.29) by applying the identity (1.11) and simplifying. \( \square \)

### 4. Higher Genera

In this section, we prove Theorem 0.5, giving expressions for \( \vec{H}_g \) for \( g \geq 1 \). We do so by solving the higher genus join-cut equation (1.36) of Theorem 1.7 and keeping track of the form of the solution. This gives an expression for the generating series \( \Delta_x G_g \), which we then integrate to obtain \( G_g \), or equivalently, \( \vec{H}_g \). To establish the value of the constant of integration which must be used, we use a formula of \cite{23} for the one-part case of the monotone single Hurwitz numbers.

To establish degree bounds on the solutions, we will need a notion of degree on the spaces \( Q \) and \( (u^3 - u)Q[u^2] \). Recall from section 2 that \( Q = Q[(1 - \eta)^{-1}, \eta_1(1 - \eta)^{-1}, \eta_2(1 - \eta)^{-1}, \ldots] \) and \( (u^3 - u)Q[u^2] \) has basis over \( Q \) given by \( (u^3 - u), \eta_1^u, \eta_2^u, \ldots \).

**Definition 4.1.** For an element
\begin{equation}
A = \sum_{\alpha \vdash d \geq 0} \sum_{j \geq 0} \frac{a_{j, \alpha}}{(1 - \eta)^{j+\ell}} \in Q,
\end{equation}
its weight is
\begin{equation}
\nu(A) = \max \{ d : a_{j, \alpha} \neq 0 \}.
\end{equation}
For an element
\[(4.3) \quad B = \sum_{\alpha^* \cdot d \geq 0} \sum_{j \geq 0} \left( b_{j, \alpha, 0}(u^3 - u) + \sum_{k \geq 1} b_{j, \alpha, k} \eta_k^u \right) \frac{\eta_k}{(1 - \eta)^{j+\ell}} \in (u^3 - u)Q[u^2], \]
its weight is
\[(4.4) \quad \nu(B) = \max\{d + k : b_{j, \alpha, k} \neq 0\}. \]

In particular, note that
\[(4.5) \quad \nu(\eta_k(1 - \eta)^{-1}) = \nu(\eta_k^u) = \nu(u^{2k+3} - u^{2k+1}) = k \]
for \(k \geq 1\), and
\[(4.6) \quad \nu((1 - \eta)^{-1}) = \nu(u^3 - u) = 0, \]
so that the weight of a polynomial in \(u\) can be determined from the weights of the coefficients of \(u^3, u^5, u^7, \ldots\).

When solving the higher genus equation, we will also need the \(R\)-linear operator \(T\) defined by
\[(4.7) \quad T: (u^3 - u)R[u^2] \rightarrow (u^3 - u)R[u^2], \]
\[(4.8) \quad (u^3 - u)f(u^2) \rightarrow u^3 - u \frac{u^5 - u^{3j}(f(u^2) - f(v^2))}{1 - \eta^2}. \]

**Lemma 4.2.** The operator \(T\) is locally nilpotent, meaning that for every \((u^3 - u)f(u^2) \in (u^3 - u)R[u^2]\), there is some \(n \geq 0\) with \(T^n ((u^3 - u)f(u^2)) = 0\). Furthermore, for each \(j \geq 0\), the operator \(T\) restricts to a function
\[(4.9) \quad T: (u^3 - u)(1 - \eta)^{-j}P[u^2] \rightarrow (u^3 - u)(1 - \eta)^{-j}P[u^2],\]
and on these subspaces, \(\nu(T((u^3 - u)f(u^2))) \leq \nu((u^3 - u)f(u^2))\).

**Proof.** The operator \(T\) strictly reduces the degree in \(u\) of every nonzero element of \((u^3 - u)R[u^2]\), so repeated application of \(T\) will always eventually produce zero.

Now, suppose that \((u^3 - u)f(u^2) \in (u^3 - u)Q[u^2]\), where \(f(u^2)\) has degree \(2k\) in \(u\), so that \(\nu((u^3 - u)f(u^2)) = k\). The expression \((u^3 - u)^j(f(u^2) - f(v^2))/u^2 - v^2\) can be written as a polynomial in \(u\), where the coefficient of \(u^{2i}\) is a polynomial in \((v^3 - v^3)(1 - \eta)^{-j}Q[u^2]\) of degree at most \(2k - 2i + 3\) in \(v\). Then, as noted in section 2, the coefficient of \(u^{2i}\) is a linear combination of \(\eta_1, \eta_2, \ldots, \eta_{k-1}\) with coefficients in \(Q\). Applying \(\Pi_y\) and multiplying by \((u^3 - u)(1 - \eta)^{-1}\) gives a linear combination of terms of the form \((u^{2i+3} - u^{2i+1})\eta_j(1 - \eta)^{-1}\) for \(j = 1, 2, \ldots, k - i\), which lie in \((u^3 - u)P[u^2]\) and have weight at most \(k\).

From this, the result follows by \(Q\)-linearity. \(\square\)

**Theorem 4.3.** For \(g \geq 1\),
\[(4.10) \quad \Delta_x G_g \in (u^3 - u)(1 - \eta)^{1-2g}P[u^2],\]
and \(\nu(\Delta_x G_g) \leq 3g - 2\). In particular,
\[(4.11) \quad \Delta_x G_1 = \frac{u^5 - 2u^3 + u}{16(1 - \eta)} + \frac{(u^3 - u)\eta_1}{24(1 - \eta)^2} \]
Proof. Recall that the higher genus join-cut equation \((1.36)\) is
\[
(4.12) \quad (1 - 2\Delta_x G_0 - \text{Split}_{x\to y}) G_g = \Delta_x^2 G_{g-1} + \sum_{g' = 1}^{g-1} \Delta_x G_{g'} \Delta_x G_{g-g'}.
\]
Note that the left-hand side operator is \(R\)-linear. Using \((3.29)\) with Lemma 3.4 and expressing \(\text{Split}_{x\to y}\) in terms of \(u\) and \(v\), we have
\[
(4.13) \quad (1 - 2\Delta_x G_0 - \text{Split}_{x\to y}) (u^3 - u)f(u^2)
= \Pi_y \left( (2 - v^3)(u^2 - 1)f(u^2) - \frac{(u^2 - 1)(v^2 - v^3)(f(u^2) - f(v^2))}{u^2 - v^2} \right)
= \frac{(1 - \eta)}{u} (1 - T) (u^3 - u)f(u^2)
\]
for \(f(u^2) \in R[u^2]\). Since \(T\) is locally nilpotent, it follows that this operator is invertible on \((u^2 - 1)R[u^2]\), with inverse given by
\[
(4.14) \quad (1 - 2\Delta_x G_0 - \text{Split}_{x\to y})^{-1} (u^2 - 1)f(u^2)
= (1 + T + T^2 + \cdots) \left( \frac{(u^3 - u)f(u^2)}{1 - \eta} \right).
\]
Thus, we have
\[
(4.15) \quad \Delta_x G_g = \frac{1}{1 - \eta} \left( 1 + T + T^2 + \cdots \right) \left( u\Delta_x^2 G_{g-1} + \sum_{g' = 1}^{g-1} u\Delta_x G_{g'} \Delta_x G_{g-g'} \right).
\]

Then, using Corollary 3.6 for the value of \(\Delta_x^2 G_0\), we can compute \(\Delta_x G_1\) directly. Using and Corollary 2.3 and Lemma 4.2 it follows that \(\Delta_x G_g \in (u^3 - u)(1 - \eta)^{-2g}P[u^2]\) by induction on \(g\).

Using Lemma 2.2, a straightforward computation shows that for \(A, B \in (u^3 - u)Q[u^2]\), we have
\[
(4.16) \quad \nu(u\Delta(A)) \leq \nu(A) + 3, \quad \nu(uAB) \leq \nu(A) + \nu(B) + 2,
\]
and since \(T\) does not increase weights, the bound \(\nu(\Delta_x G_g) \leq 3g - 2\) also follows by induction on \(g\). \(\square\)

Having solved for \(\Delta_x G_g = \Delta_x G_g\), we now need to solve for \(G_g\) and show that it has the right form. To do so, we first compute \(\sum_{k \geq 1} E_k G_g\) from \(\Delta_x G_g\), and then invert the operator \(\sum_{k \geq 1} E_k\).

**Theorem 4.4.** We have
\[
(4.17) \quad \sum_{k \geq 1} E_k G_1 \leq \frac{\eta}{24(1 - \eta)} + \frac{\gamma}{8(1 - \gamma)},
\]
and for \(g \geq 2\),
\[
(4.18) \quad \sum_{k \geq 1} E_k G_g \in (1 - \eta)^{-2}Q.
\]
Proof. It follows from (2.12) that

\[
\sum_{k \geq 1} E_k G_g = \left( \sum_{k \geq 1} D_k - \frac{2\gamma}{1 - \gamma} \right) G_g
\]

(4.19)

\[
= \Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - \frac{2\gamma}{1 - \gamma} \right) \Delta_x G_g
\]

(4.20)

\[
= \Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - \frac{2\gamma}{1 - \gamma} \right) \Delta_x G_g.
\]

(4.21)

We can replace the term \( \Pi_x \left( \frac{2\gamma}{1 - \gamma} \right) \Delta_x G_g \) in this expression by using the operator identity

\[
\Pi_x = [u^0] \left( 1 - 2\Delta_x G_0 - \text{Split}_{x \rightarrow y} \right) - (1 - \gamma)[u^1],
\]

which can be checked on the \( Q \)-linear space \((u^3 - u)Q[u^2]\) by verifying it on the basis \((u^3 - u), \eta_1^u, \eta_2^u, \ldots\). Then, we have

\[
\sum_{k \geq 1} E_k G_g = \Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - 2(u - 1)[u^1] \right) \Delta_x G_g
\]

(4.23)

\[
- \frac{2\gamma}{1 - \gamma} [u^0] \left( 1 - 2\Delta_x G_0 - \text{Split}_{x \rightarrow y} \right) \Delta_x G_g.
\]

(4.24)

For \( g = 1 \), we can use the expression for \( \Delta_x G_1 \) from Theorem 4.3 to compute

\[
\sum_{k \geq 1} E_k G_1 = \frac{\eta}{24(1 - \eta)} + \frac{\gamma}{8(1 - \gamma)}
\]

(4.25)

from this. For \( g \geq 2 \), we have

\[
\left( 1 - 2\Delta_x G_0 - \text{Split}_{x \rightarrow y} \right) \Delta_x G_g = \Delta_x^2 G_{g - 1} + \sum_{g' = 1}^{g - 1} \Delta_x G_{g'} \Delta_x G_{g - g'}
\]

(4.26)

\[
\in (u^3 - u)^2(1 - \eta)^{2 - 2g} P[u^2],
\]

so in particular, this has no constant term as a polynomial in \( u \). Thus, for \( g \geq 2 \),

\[
\sum_{k \geq 1} E_k G_g = \Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - 2(u - 1)[u^1] \right) \Delta_x G_g.
\]

(4.28)

Also, we have

\[
\Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - 2(u - 1)[u^1] \right) (u^3 - u) = 0,
\]

(4.29)

\[
\Pi_x \left( \left( \frac{x}{\partial x} \right)^{-1} - 2(u - 1)[u^1] \right) \eta_j^u = \eta_{j - 1},
\]

(4.30)

for \( j \geq 1 \). This determines the action of this \( Q \)-linear operator on \((u^3 - u)Q[u^2]\).

Since

\[
\Delta_x G_g \in (u^3 - u)(1 - \eta)^{-3} Q[u^2]
\]

(4.31)
for \( g \geq 2 \) and
\[
\eta_0 \frac{1}{1-\eta} = 1 - \frac{1}{1-\eta},
\]
we get
\[
\sum_{k \geq 1} E_k G_g \in (1-\eta)^{-2} Q.
\]

**Theorem 4.5.** We have
\[
G_1 = \frac{1}{24} \log(1-\eta)^{-1} - \frac{1}{8} \log(1-\gamma)^{-1},
\]
and for \( g \geq 2 \),
\[
G_g = -c_g(0) + \frac{1}{(1-\eta)^{2g-2}} \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} c_{g,\alpha} \eta_\alpha \eta \frac{1}{(1-\eta)^{\ell+\ell}},
\]
where \( c_g(0) = -\frac{B_{2g}}{4g(g-1)} \).

**Proof.** Note that \( \gamma, \eta, \eta_1, \eta_2, \ldots \) are all eigenvectors of the differential operator \( \sum_{k \geq 1} E_k \) with eigenvalue 1, since they are purely linear in \( q_1, q_2, \ldots \). Thus, up to a constant, we have
\[
\left( \sum_{k \geq 1} E_k \right)^{-1} \frac{\eta}{1-\eta} = \int_0^1 \frac{\eta t}{1-\eta t} \frac{dt}{t} = \log(1-\eta)^{-1}
\]
and
\[
\left( \sum_{k \geq 1} E_k \right)^{-1} \frac{\gamma}{1-\gamma} = \int_0^1 \frac{\gamma t}{1-\gamma t} \frac{dt}{t} = \log(1-\gamma)^{-1}
\]
Together with the constraint that \( G_1 \) has no constant term, we get
\[
G_1 = \frac{1}{24} \log(1-\eta)^{-1} - \frac{1}{8} \log(1-\gamma)^{-1}.
\]
When \( j \geq 2 \), we have
\[
\left( \sum_{k \geq 1} E_k \right)^{-1} \eta_\alpha \frac{1}{(1-\eta)^{j+\ell}} = \int_0^1 \frac{\eta_\alpha t^\ell}{(1-\eta)^{j+\ell}} \frac{dt}{t} \in (1-\eta)^{-1} Q,
\]
so for some constant \( c_g \), for \( g \geq 2 \), we get
\[
G_g - c_g \in (1-\eta)^{-1} Q.
\]
In fact, since the kernel of \( \Delta_x \) on \( Q \) is \( Q \) and \( \Delta_x G_g \in (u^3 - u)(1-\eta)^{1-2g} P[u^2] \), it follows from [Corollary 2.3] that \( G_g - c_g \in (1-\eta)^{2g-2g} P \). Since \( \nu(\Delta_x G_g) \leq 3g - 3 \), it follows from [Lemma 2.2] that \( \nu(G_g) \leq 3g - 3 \). Thus, we can write
\[
G_g = c_g + \frac{1}{(1-\eta)^{2g-2}} \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} c_{g,\alpha} \eta_\alpha \frac{1}{(1-\eta)^{\ell}},
\]
where the coefficients \( c_{g,\alpha} \) are rational numbers. All that remains is to compute the value of \( c_g \). The only other term of the sum which contributes a constant term
is $c_{g,e}(1-\eta)^{2-2g}$, so we must have $c_g = -c_{g,e}$. If we expand $G_g$ as a power series in $p_1, p_2, \ldots$ and keep only the constant and linear terms, we get

\begin{equation}
G_g = c_g + c_{g,e} + (2g-2)c_{g,e}\eta + \sum_{k=1}^{N_g} c_{g,(k)}\eta^k + O(p_ip_j) \tag{4.42}
\end{equation}

\begin{equation}
= \sum_{d \geq 1} \left( (2g-2)c_{g,e} + \sum_{k=1}^{N_g} c_{g,(k)}d^k \right) (2d+1)\left( \frac{2d}{d} \right) p_d + O(p_ip_j).
\end{equation}

For fixed $g \geq 2$, the expression

\begin{equation}
f(d) = (2g-2)c_{g,e} + \sum_{k=1}^{N_g} c_{g,(k)}d^k
\end{equation}

is a polynomial in $d$, and $f(0)/(2-2g) = c_g$.

By [23], the number of transitive monotone factorizations of genus $g$ of the cycle $(1 \ 2 \ldots d)$, or any other cycle of length $d$, is

\begin{equation}
\frac{1}{d!} \left( \frac{2d-2}{d-1} \left( \frac{2d-2+2g}{2d-2} \right) \left( \frac{z^{2g}}{(2g)!} \right) \left( \frac{\sinh(\frac{z}{2})}{z/2} \right) \right)^{2d-2}.
\end{equation}

There are $(d-1)!$ of these long cycles, so the coefficient of $p_d$ in $G_g$ is

\begin{equation}
\frac{(d-1)!}{d!} \frac{1}{d} \left( \frac{2d-2}{d-1} \left( \frac{2d-2+2g}{2d-2} \right) \left( \frac{z^{2g}}{(2g)!} \right) \left( \frac{\sinh(\frac{z}{2})}{z/2} \right) \right)^{2d-2}.
\end{equation}

Comparing this to the expression above and expanding the binomial coefficients into factorials gives

\begin{equation}
f(d) = \frac{(2d-2+2g)!}{(2-2g)(2d+1)(2d+2)!} \left( \frac{z^{2g}}{(2g)!} \right) \left( \frac{\sinh(\frac{z}{2})}{z/2} \right)^{2d-2}.
\end{equation}

For fixed $g$, the coefficient of $z^{2g}/(2g)!$ extracted here is a polynomial in $d$, and setting $d = 0$, we get

\begin{equation}
c_g = \frac{f(0)}{2-2g} = \frac{1}{4g(g-1)(1-2g)} \left( \frac{z^{2g}}{(2g)!} \right) \left( \frac{\sinh(\frac{z}{2})}{z/2} \right)^{-2}
\end{equation}

\begin{equation}
= \frac{1}{4g(g-1)(1-2g)} \left( \frac{z^{2g}}{(2g)!} \right) \left( \frac{z^2e^z}{(e^z-1)^2} \right)
\end{equation}

\begin{equation}
= \frac{1}{4g(g-1)(1-2g)} \left( \frac{z^{2g}}{(2g)!} \right) \left( 1 - \frac{z}{e^z-1} \right)
\end{equation}

\begin{equation}
= \frac{B_{2g}}{4g(g-1)},
\end{equation}

since $z/(e^z-1)$ is the generating series for the Bernoulli numbers. \hfill \Box

**Proof of Theorem 0.5** The result follows immediately from Theorem 4.5 and Definition 1.6. \hfill \Box
5. Lagrange inversion and polynomiality

In this section, we use Lagrange inversion to derive an explicit formula for the genus one monotone single Hurwitz numbers (Theorem 0.4) from the generating series given in Theorem 0.5 and to establish a polynomiality result for higher genera (Theorem 0.6).

For $k \geq 1$, let $\Theta_k : \mathbb{R} \rightarrow \mathbb{R}$ be the differential operator

$$\Theta_k = \left(\frac{2k}{k}\right) - 1 \frac{\partial}{\partial q_k}.$$  \hspace{1cm} (5.1)

Then, we have

$$\Theta_k(\gamma) = 1, \quad \Theta_k(\eta) = 2k + 1, \quad \Theta_k(\eta_i) = (2k + 1)k^i.$$ \hspace{1cm} (5.2)

The following lemma shows how this operator is related to the quantities described in Theorems 0.4 and 0.6.

**Lemma 5.1.** If $f \in \mathbb{R}$ is a power series and $\alpha \vdash d \geq 0$, then

$$\frac{d!|q_\alpha|f}{C_\alpha \prod_{j=1}^\ell \alpha_j (2\alpha_j)} = \Theta_\alpha f \bigg|_{q_1=q_2=\ldots=0}.$$ \hspace{1cm} (5.3)

**Proof.** Since

$$\frac{d!}{|C_\alpha|} = \prod_{j=1}^\ell \alpha_j \cdot \prod_{k \geq 1} m_k!,$$ \hspace{1cm} (5.4)

where $m_1, m_2, \ldots$ are the part multiplicities of $\alpha$, we have

$$\frac{d!|q_\alpha|f}{|C_\alpha| \prod_{j=1}^\ell \alpha_j (2\alpha_j)} = \prod_{j=1}^\ell \left(\frac{2\alpha_j}{\alpha_j}\right)^{-1} \left[ \frac{q_1^{m_1} q_2^{m_2} \ldots}{m_1! m_2! \ldots} \right] f = \Theta_\alpha f \bigg|_{q_1=q_2=\ldots=0}. \hspace{1cm} \square$$

**Proof of Theorem 0.4.** To show that

$$\frac{24\tilde{H}_1(\alpha)}{|C_\alpha| \prod_{i=1}^\ell \alpha_i (2\alpha_i)} = (2d+1)^7 - 3(2d+1)^{\ell-1} - \sum_{k=2}^{\ell} (k-2)! (2d+1)^{\ell-k} e_k(2\alpha+1),$$ \hspace{1cm} (5.6)

we apply Lagrange inversion to the generating series

$$G_1 = \frac{1}{24} \log(1-\eta)^{-1} - \frac{1}{8} \log(1-\gamma)^{-1}.$$ \hspace{1cm} (5.7)
from Theorem 4.5, expand the result as a power series in \( \gamma \) and \( \eta \), and apply \( \Theta_\alpha \) to each term. By Theorem 2.1, we have

(5.8) \[
[p_\alpha] \log(1 - \eta)^{-1} = [q_\alpha] \log(1 - \eta)^{-1}(1 - \eta)(1 - \gamma)^{-2d-1} \\
= [q_\alpha] \left( \eta - \sum_{k \geq 2} \frac{\eta^k}{k(k-1)} \right) \left( \sum_{j \geq 0} \left(-2d - 1\right)^j (-1)^j \gamma^j \right) \\
= [q_\alpha] \left( \eta - \sum_{k \geq 2} (k - 2)! \frac{\eta^k}{k!} \right) \left( \sum_{j \geq 0} (2d + 1) \gamma^j \right) \\
= [q_\alpha] \left( (2d + 1)^{\ell} \gamma^{\ell-1} \left( \frac{\eta}{\ell-1} \right) - \sum_{k=2}^{\ell} (k - 2)! (2d + 1)^{\ell-k} \frac{\eta^k}{k!} \gamma^{\ell-k} \right).
\]

Note that, by iterating the product rule, we get

(5.9) \[
\Theta_\alpha \left( \frac{\eta^k \gamma^{\ell-k}}{k! \ell!} \right) = \sum_{1 \leq i_1 < \cdots < i_k \leq \ell} \frac{\Theta_{\alpha_{i_1}}(\eta) \cdots \Theta_{\alpha_{i_k}}(\eta)}{\Theta_{\alpha_{i_1}}(\gamma) \cdots \Theta_{\alpha_{i_k}}(\gamma)} \Theta_{\alpha_1}(\gamma) \cdots \Theta_{\alpha_{\ell}}(\gamma)
\]

(5.10) \[
= \sum_{1 \leq i_1 < \cdots < i_k \leq \ell} (2\alpha_{i_1} + 1)(2\alpha_{i_2} + 1) \cdots (2\alpha_{i_k} + 1)
\]

(5.11) \[
= e_k(2\alpha + 1),
\]

and this is constant as a power series in \( q_1, q_2, \ldots \), and hence unaffected by evaluation at \( q_1 = q_2 = \cdots = 0 \). Using Lemma 5.1, we have

(5.12) \[
\frac{d![p_\alpha] \log(1 - \eta)^{-1}}{[C_\alpha] \prod_{i=1}^{\ell} \alpha_i(\frac{2\alpha_i}{\alpha_i})} \\
= \Theta_\alpha \left( (2d + 1)^{\ell} \gamma^{\ell-1} \left( \frac{\eta}{\ell-1} \right) - \sum_{k=2}^{\ell} (k - 2)! (2d + 1)^{\ell-k} \frac{\eta^k}{k!} \gamma^{\ell-k} \right) \\
= (2d + 1)^{\ell} e_1(2\alpha + 1) - \sum_{k=2}^{\ell} (k - 2)! (2d + 1)^{\ell-k} e_k(2\alpha + 1)
\]

since \( e_1(2\alpha + 1) = (2d + \ell) \). This gives most of the terms on the right-hand side of the genus one formula.
For the remaining term, using Theorem 2.1 again and the fact that $p_\alpha, q_\alpha$ are eigenvectors for the operators $D, E$ respectively, we have

\begin{align}
[p_\alpha] \log(1 - \gamma)^{-1} &= \frac{1}{d} [p_\alpha] D \log(1 - \gamma)^{-1} \\
&= \frac{1}{d} [q_\alpha] (1 - \gamma)^{-2d} \frac{1 - \eta}{1 - \gamma} D \log(1 - \gamma)^{-1} \\
&= \frac{1}{d} [q_\alpha] (1 - \gamma)^{-2d} E \log(1 - \gamma)^{-1} \\
&= \frac{1}{d} [q_\alpha] E \left( \frac{1}{2d} (1 - \gamma)^{-2d} \right) \\
&= [q_\alpha] \frac{1}{2d} (1 - \gamma)^{-2d} \\
&= [q_\alpha] (2d + 1)^{\ell - 1} \gamma^\ell \ell!.
\end{align}

Applying Lemma 5.1, we get

\begin{align}
d! [p_\alpha] \log(1 - \gamma)^{-1} |_{C_\alpha} \prod_{i=1}^\ell \alpha_i(2\alpha_i) &= \Theta_\alpha \left( (2d + 1)^{\ell - 1} \gamma^\ell \ell! \right) \\
&= (2d + 1)^{\ell - 1}.
\end{align}

This gives the remaining term on the right-hand side of the genus one formula. \hfill \Box

Using similar techniques, we can get a polynomiality result for monotone single Hurwitz numbers for fixed genus and number of parts.

**Proof of Theorem 0.6.** Noting that $d = \sum_{j=1}^\ell \alpha_j$, the explicit formulas for genus zero and one from Theorems 0.3 and 0.4 show that the expression

\begin{equation}
\bar{H}_g((\alpha_1, \alpha_2, \ldots, \alpha_\ell)) |_{C_\alpha} \prod_{i=1}^\ell \alpha_i(2\alpha_i)
\end{equation}

is a polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ when $g = 0$ and $\ell \geq 3$, and when $g = 1$ and $\ell \geq 1$. For $g \geq 2$, we have the rational form

\begin{equation}
G_g = -c_{g,0} + \frac{1}{(1 - \eta)^{2g-2}} \sum_{d=0}^{3g-3} \sum_{\alpha^d} c_{g,\alpha} \eta^\alpha (1 - \gamma)^2 d + 1.
\end{equation}

As in the proof of Theorem 0.4, we can apply Lagrange inversion to each of the terms, expand the result as power series in $\gamma, \eta, \eta_1, \eta_2, \ldots$, and extract coefficients using Lemma 5.1.

By Theorem 2.1 we have

\begin{equation}
[p_\alpha] \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_k^{\alpha_k} = [q_\alpha] \frac{\eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_k^{\alpha_k}}{(1 - \eta)^{2d+1}(1 - \gamma)^{2d+1}}.
\end{equation}

This can be expanded as an infinite linear combination of terms of the form

\begin{equation}
[q_\alpha] \frac{\eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_k^{\alpha_k}}{(1 - \gamma)^{2d+1}},
\end{equation}
but only the finitely many terms with $a_0 + a_1 + \cdots + a_k \leq \ell$ have a nonzero contribution. For these terms, applying Lemma 5.1 gives

\begin{equation}
\frac{d!}{|C_\alpha| \prod_{i=1}^\ell \alpha_i(2\alpha_i)} \left( \frac{\eta_0^{a_0} \cdots \eta_k^{a_k}}{(1 - \gamma)^{2d+1}} \right)
= \frac{d!}{|C_\alpha| \prod_{i=1}^\ell \alpha_i(2\alpha_i)} \left( (2d + 1)^{\ell - \sum \alpha_i} \eta_0^{a_0} \cdots \eta_k^{a_k} \cdot \frac{(\ell - \sum \alpha_i)!}{\gamma^{\ell - \sum \alpha_i}} \right),
\end{equation}

which is a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. For fixed $\ell$,

\begin{equation}
\tilde{H}_g((\alpha_1, \alpha_2, \ldots, \alpha_\ell)) = \frac{d!}{|C_\alpha| \prod_{i=1}^\ell \alpha_i(2\alpha_i)} \frac{G_g}{|C_\alpha| \prod_{i=1}^\ell \alpha_i(2\alpha_i)}
\end{equation}

is a finite linear combination of these polynomials, so it is a polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. 

\section*{Appendix A. Rational forms for $g = 2$ and $3$}

The following equations give the rational forms for the genus two and three generating series for the monotone single Hurwitz numbers, as described in Theorem 0.3.

Genus two:

\begin{equation}
720 \tilde{H}_2 = -3 + \frac{3}{(1 - \eta)^2} + \frac{5\eta_3 - 6\eta_2 - 5\eta_1}{(1 - \eta)^3} + \frac{29\eta_2\eta_1 - 10\eta_1^2}{(1 - \eta)^4} + \frac{28\eta_1^3}{(1 - \eta)^5}.
\end{equation}

Genus three:

\begin{equation}
90720 \tilde{H}_3 = 90 + \frac{-90}{(1 - \eta)^4} + \frac{70\eta_6 + 63\eta_5 - 377\eta_4 - 189\eta_3 + 667\eta_2 + 126\eta_1}{(1 - \eta)^5}
+ \frac{1078\eta_5\eta_2 + 2012\eta_4\eta_2 + 1209\eta_3\eta_2 + 1214\eta_2^2}{(1 - \eta)^6}
+ \frac{1998\eta_2\eta_3 - 3914\eta_1\eta_3 - 2627\eta_2^2 - 2577\eta_1\eta_2 + 1967\eta_1^2}{(1 - \eta)^7}
+ \frac{8568\eta_1^2\eta_4 + 26904\eta_1\eta_2\eta_3 + 10092\eta_1^2\eta_2 + 58300\eta_2^3}{(1 - \eta)^8}
+ \frac{13440\eta_1\eta_2^2 - 20322\eta_1^2\eta_2 + 4352\eta_1^3}{(1 - \eta)^9}
+ \frac{44520\eta_1^3\eta_3 + 86100\eta_1^2\eta_2 + 499800\eta_1^3\eta_2 - 15750\eta_1^4}{(1 - \eta)^9}
+ \frac{162120\eta_1^4\eta_2 + 31080\eta_1^3\eta_2^2}{(1 - \eta)^9}
+ \frac{6860\eta_1^6}{(1 - \eta)^10}.
\end{equation}

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