CORRELATIONS OF THE THUE–MORSE SEQUENCE

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Abstract. The pair correlations of the Thue–Morse sequence and system are revisited, with focus on asymptotic results on various means. First, it is shown that all higher-order correlations of the Thue–Morse sequence with general real weights are effectively determined by a single value of the balanced 2-point correlation. As a consequence, we show that all odd-order correlations of the balanced Thue–Morse sequence vanish, and that, for any even \( n \), the \( n \)-point correlations of the balanced Thue–Morse sequence have mean value zero, as do their absolute values, raised to an arbitrary positive power. All these results also apply to the entire Thue–Morse system. We finish by showing how the correlations of the Thue–Morse system with general real weights can be derived from the balanced 2-point correlations.

Dedicated to the memory of Uwe Grimm

1. Introduction

The study of (possibly hidden) long-range order of sequences over finite alphabets, in particular binary ones, has a long and interesting history; see [2, 8] and the references therein for background. For about 100 years, starting with the insight of Norbert Wiener, methods from harmonic analysis have been instrumental to detect all kinds of long-range correlations via spectral methods. While sequences with strong almost periodicity (and hence pure point spectrum) were the first to be analysed and understood, the ones with continuous spectra remained somewhat enigmatic. In particular, the classic Thue–Morse (or Prouhet–Thue–Morse) sequence with the singular continuous measure induced by it became a paradigm of a degree of order intermediate between pure point and absolutely continuous. First analysed by Mahler [20] in 1927 by direct means, it later saw a systematic reformulation by Kakutani [17] with dynamical systems methods, and has recently been analysed in a fractal geometric setting via the thermodynamic formalism [7] and in the context of hyperuniformity [9]. Many obvious generalisations are known [9, Sec. 5.2], and the progress in this direction has also triggered new research on absolutely continuous spectra [13, 14, 15] as well as general spectral considerations on the basis of renormalisation techniques [19, 11, 10, 6, 12].

While much of the current literature concerns either the autocorrelation of the Thue–Morse sequence (respectively system) or the maximal spectral measure in the orthocomplement of the point spectrum, compare [8] and [25], much less is known about the general correlation functions. It is the purpose of this paper to fill this gap by deriving further asymptotic
properties and a general approach to the (higher-order) correlation functions and determining some of their properties. As an added benefit, this provides extra insight into the invariant probability measure on the shift space that is induced by the Thue–Morse sequence. Let us also mention that the (balanced) Thue–Morse sequence is Gowers uniform for any of the standard uniformity norms \([18]\), which is to say that certain averages of \(n\)-point correlation functions, for \(n = 2^s\) with \(s \in \mathbb{N}\), decay asymptotically with a power-law upper bound. It is thus a natural question to also consider other averages of correlation functions and their asymptotic averages, as we shall do below.

This paper is organised as follows. In Section 2, we set the scene and give a brief summary of the Thue–Morse sequence and the dynamical system generated by it, together with some classical results on the two-point correlations (or autocorrelations). Here, we add some results on their asymptotic properties. We continue in Sections 3 and 4 with the \(n\)-point correlations of the balanced Thue–Morse system, which, via a general recurrence, are shown to be determined by the values on an \((n-1)\)-dimensional unit cube, and further that these values are determined by the value of the autocorrelation at zero, again, via the recurrence. This result is then used to show that all odd-order correlations vanish, and that, under a natural ordering, all even-order correlations have mean value zero, in various ways. In Section 5, we more generally show that all weighted correlations are fully determined once the balanced ones are known, and a general renormalisation structure is employed to achieve this.

2. Preliminaries

Let \((t_k)_{k \geq 0}\) be the (one-sided) Thue–Morse sequence, or word, taking the values \(\pm 1\), defined by \(t_0 = 1\) and, for \(k \geq 0\), by

\[
(2.1) \quad t_{2k} = t_k \quad \text{and} \quad t_{2k+1} = -t_k.
\]

This sequence is the fixed point, starting from the seed \(a\), of the substitution

\[
\varrho = \varrho_{TM} : \begin{cases} a \mapsto ab, \\ b \mapsto ba, \end{cases}
\]

where we specialise the values by \(a = -b = 1\).

Lemma 2.1. For \(m \in \mathbb{N}_0\), we have \((t_0, t_1, \ldots, t_{2^m-1}) = (1, -1)^{\otimes m}\).

With \(t_0 = 1\) and the the structure of the \(m\)-fold Kronecker product, one obvious way to prove the lemma is through induction in \(m\) via the action of the substitution. Here, we follow an alternative path, as it provides additional insight.

Proof. Note that (2.1) implies that \(t_n = (-1)^{s_2(n)}\), where \(s_2(n)\) is the number of 1s in the binary expansion of \(n\). To prove the claim for all \(m\), we only need to show that \(t_a = -t_{2^m+a}\) holds for all \(a \in \{0, \ldots, 2^m - 1\}\). This follows immediately since, for \(a \in \{0, \ldots, 2^m - 1\}\), we have \(s_2(2^m + a) = s_2(a) + 1\). \(\square\)
Now, we properly extend the Thue–Morse sequence to a bi-infinite sequence (or word) 
\( w := (w_n)_{n \in \mathbb{Z}} \) by defining 
\[
  w_n = \begin{cases} 
    t_n, & \text{for } n \geq 0, \\
    t_{-n-1}, & \text{for } n < 0.
  \end{cases}
\]

Similar to the above, the word \( w \) is the bi-infinite fixed point of the square of the Thue–Morse substitution, \( \varrho^2 \), starting from the seed \( a|a \), where as before \( a = -b = 1 \); see [8, Rem. 4.8].

Let \( S \) be the shift operator, \( (Sw)_i = w_{i+1} \), and \( \mathcal{X} = \mathcal{X}(w) := \{ S^i w : i \in \mathbb{Z} \} \subset \{ \pm 1 \}^\mathbb{Z} \) be the (discrete) hull. The space \( \mathcal{X} \) together with the \( \mathbb{Z} \)-action of the shift forms a topological dynamical system, denoted by \((\mathcal{X}, \mathbb{Z})\). It admits precisely one invariant probability measure, say \( \mu \). In other words, \((\mathcal{X}, \mathbb{Z})\) is uniquely ergodic. Since \( \mathcal{X} \) is also minimal (meaning that the \( \mathbb{Z} \)-orbit of every element is dense in \( \mathcal{X} \)), the system is even strictly ergodic; see [17, 8] for details. Using this measure \( \mu \), one has
\[
  \int_{\mathcal{X}} x_{m_0} x_{m_1} \cdots x_{m_{n-1}} \, d\mu(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} y_k + m_0 y_k + m_1 \cdots y_k + m_{n-1},
\]
where the choice of \( y = (y_i)_{i \in \mathbb{Z}} \in \mathcal{X} \) is arbitrary. This equality is a consequence of Birkhoff’s ergodic theorem for uniquely ergodic shift spaces, because the right-hand side is the Birkhoff average of a continuous function on \( \mathcal{X} \).

In this paper, we are interested in the correlations of the Thue–Morse system, both for the standard balanced weights described above and for more general real weights. Since the measure \( \mu \) is shift invariant, without loss of generality, we can fix one of the \( m_i \); we thus set \( m_0 = 0 \). Further, we let \( f : \{-1, 1\} \to \mathbb{R} \) and define the general \( n \)-point correlations of the \( f \)-weighted Thue–Morse sequence (and system) by
\[
  \eta_f(m_1, m_2, \ldots, m_{n-1}) := \int_{\mathcal{X}} f(x_0) f(x_{m_1}) \cdots f(x_{m_{n-1}}) \, d\mu(x).
\]
In the balanced case, that is, when \( f = \text{id} \) is the identity function, we suppress the subscript \( f \) and simply write \( \eta \) instead of \( \eta_{\text{id}} \). For a study towards a different generalisation, using multiple weight functions, see Aloui [3].

We note that \( \mu \) is the patch frequency measure of the system, as defined by its values on the cylinder sets defined via all finite words. Their frequencies can be extracted from the frequency-normalised Perron–Frobenius eigenvectors of the induced substitution matrices for Thue–Morse words of length \( n \); see [25, Sec. 5.4.3] or [8, Sec. 4.8.3] for details. The frequency module of the Thue–Morse system (that is, the Abelian group generated by all occurring frequencies) is given by
\[
  \mathcal{M}_\mu = \left\{ \frac{m}{3 \cdot 2^r} : r \in \mathbb{N}_0, m \in \mathbb{Z} \right\}.
\]
In particular, all word frequencies are integer linear combinations of letter frequencies and frequencies of words of length 2—which gives one way to prove (2.3). Indeed, the single 3 in the denominator emerges from the frequencies of words of length 2, while the powers of 2
reflect the substitution structure. One can then check that $\mathcal{M}_\mu$, in the parametrisation used, is an Abelian group, and the smallest one that contains all word frequencies.

While this view is, in some ways, satisfactory, it is still incomplete in the sense that the calculation of the frequencies is not trivial. This emphasises an alternative viewpoint via the correlations, which also completely determine the measure $\mu$ because they comprise the patch frequencies via suitable choices of the weight function $f$ and the number of points in (2.2). This is one of our motivations to study the correlations $\eta_f$ in some generality.

3. Pair correlations for balanced weights

Let us consider the correlations for balanced weights, $\{\pm 1\}$. The standard two-point correlation coefficients $\eta(m)$ of the Thue–Morse sequence (and system), which are also known as the autocorrelation coefficients, are usually introduced as

$$\eta(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} t_k t_{k+m},$$

which is consistent with our above definition. By symmetry, one has $\eta(-m) = \eta(m)$, which follows easily after dropping finitely many terms from the sum. Further, we clearly get $\eta(0) = 1$, and, for $m \geq 0$, one finds the repeatedly derived recursions \[20, 17, 8\]

$$\eta(2m) = \eta(m) \quad \text{and} \quad \eta(2m + 1) = -\frac{1}{2}(\eta(m) + \eta(m + 1)), \quad (3.1)$$

which are a direct consequence of the substitution structure. These recurrences allow one to compute all of the values of $\eta$ from $\eta(0)$. In particular, one has

$$\eta(1) = -\frac{\eta(0)}{3} = -\frac{1}{3}, \quad \text{(3.2)}$$

by solving the second equation in (3.1) with $m = 0$ for $\eta(1)$. We can write the pair of recurrence equations (3.1) in matrix form as

$$\begin{pmatrix} \eta(2m) \\ \eta(2m + 1) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \eta(m) \\ \eta(m + 1) \end{pmatrix},$$

which is valid for $m \geq 0$. As this rational matrix will be important for us later, we record a few points of interest here. Firstly, it has eigenvalues 1 and $-1/2$, with right eigenvectors $(1, -1/3)^T$ and $(0, 1)^T$, respectively. Next, set

$$E_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_1 := \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.3)$$

Note that $E_0$ and $J$ are involutions, while $E_1$ is an idempotent. For a $2 \times 2$ matrix $A$, define $A' := JAJ^{-1} = JAJ$. Then, one has $A'' = A$, while $E_0' = -E_0$. In particular, this yields the decomposition

$$\begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} = E_0 + J E_1 J = E_0 + E_1'.$
**Remark 3.1.** The recursion relations (3.1) define an infinite set of linear equations for the numbers \( \eta(m) \) with \( m \in \mathbb{N}_0 \). This set contains a (maximal) finite subset of equations that is closed, in the sense that they are equations for finitely many coefficients among themselves (and no others), while all remaining coefficients are then fully determined recursively from these ones. Here, they are the two equations for \( m = 0 \), namely

\[
\eta(0) = \eta(0) \quad \text{and} \quad \eta(1) = -\frac{1}{2}(\eta(0) + \eta(1)).
\]

The first one is a tautology, and simply an indication that the recursion relations alone do not specify the value of \( \eta(0) \). The second specifies \( \eta(1) \) as a function of \( \eta(0) \), as we saw in (3.2), while all \( \eta(m) \) with \( m \geq 2 \) are then determined recursively. In other words, the linear solution space to (3.1) is one-dimensional, and once \( \eta(0) \) is given, \( \eta \) is completely specified.

This structure is quite typical for a set of renormalisation equations. The infinite set of equations contains a (maximal) finite subset that is closed, often called the *self-consistency part*, while all other quantities are fixed recursively. This structure has recently been identified in more general inflation tilings, both in one and in higher dimensions, and gives access to various spectral properties; see [4, 6] and references therein. ♦

**Remark 3.2.** The coefficients \( \eta(m) \) from (3.1) define a function \( \eta: \mathbb{Z} \rightarrow \mathbb{R} \) which is positive definite and defines a measure \( \gamma = \eta \cdot \delta_\mathbb{Z} \) that is a positive definite pure point measure on \( \mathbb{Z} \). As such, it is Fourier transformable, which gives the positive measure

\[
\hat{\gamma} = \nu_{\text{TM}} * \delta_\mathbb{Z},
\]

where \( \nu_{\text{TM}} \) is a purely singular continuous probability measure on the torus \( T = \mathbb{R}/\mathbb{Z} \). Using the standard unit interval as a model for \( T \), one obtains the Riesz product representation

\[
\nu_{\text{TM}} = \prod_{\ell=0}^{\infty} \left( 1 - \cos(2^{\ell+1}\pi x) \right),
\]

which converges weakly. Here, the right-hand side is the standard notation for a sequence of absolutely continuous measures, each given by its Radon–Nikodym density, with the limit being singular continuous; see [8, Ch. 10.1] and references therein for details. In particular, the pointwise limit of the right-hand side vanishes on a set of full measure, while it diverges, or is ill-defined, on an uncountable null set. The details of such measures are best studied via the thermodynamic formalism [7].

The proof of the singular continuity of \( \nu_{\text{TM}} \) rests upon two properties of \( \eta \). First, by showing

\[
(3.4) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} \eta(m)^2 = 0,
\]

compare [8, Lemma 10.2], one excludes pure point contributions, via Wiener’s criterion. Then, since \( \eta(2^\ell) = -1/3 \) for all \( \ell \in \mathbb{N} \), one can employ the Riemann–Lebesgue lemma to show that no absolutely continuous component can exist; see [17, 8] for the details. ♦
Beyond (3.4), also the mean vanishes asymptotically,

\begin{equation}
\lim_{n \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} \eta(m) = 0.
\end{equation}

To see why this is true, set \( \Sigma(N) = \frac{1}{N} \sum_{m=0}^{N-1} \eta(m) \) and observe that, as \( N \to \infty \), one has \( \Sigma(2N+1) = \frac{2N}{2N+1} \Sigma(2N) + O(N^{-1}) \). Now, since \( |\eta(m)| \leq 1 \) for all \( m \in \mathbb{N}_0 \), one finds

\[
\Sigma(2N) = \frac{1}{2} \left( \frac{1}{N} \sum_{m=0}^{N-1} \eta(2m) + \frac{1}{N} \sum_{m=0}^{N-1} \eta(2m+1) \right) \\
= \frac{1}{2} \left( \frac{1}{N} \sum_{m=0}^{N-1} \eta(m) - \frac{1}{2N} \sum_{m=0}^{N-1} (\eta(m) + \eta(m+1)) \right) \\
= \frac{\Sigma(N)}{2} - \frac{\Sigma(N)}{4} - \frac{\Sigma(N)}{4} + \frac{\eta(0) - \eta(N)}{4N} \\
= \frac{1 - \eta(N)}{4N} = O(N^{-1})
\]

as \( N \to \infty \), where (3.1) was used in the second line, and (3.5) is obvious from here. In fact, the above derivation gives the stronger relation

\[
\frac{2N-1}{N} \sum_{m=0}^{N-1} \eta(m) = \frac{1 - \eta(N)}{2} \in [0, \frac{2}{3}].
\]

Further, using the result from (3.1) and (3.4) in conjunction with \( |\eta(n)| \leq 1 \), one finds the following consequence.

**Proposition 3.3.** For \( k \in \mathbb{N}_0 \), one has \( \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} \eta(m)^k = \delta_{k,0} \). Furthermore, one has \( \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)|^k = 0 \) for all \( k \geq 2 \). \( \square \)

Now, define \( \mu_\pm = \frac{1 \pm \eta}{2} \), which means \( \mu_+(n) + \mu_-(n) = 1 \) and \( \mu_+(n) - \mu_-(n) = \eta(n) \) for all \( n \in \mathbb{N}_0 \). From Proposition 3.3, one then finds

\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} \mu_\pm(m)^k = 2^{-k}
\end{equation}

for \( k \in \mathbb{N}_0 \). In fact, one obtains the coupled recursion relations

\[
\mu_\pm(2m) = \mu_\pm(m) \quad \text{and} \quad \mu_\pm(2m+1) = \frac{1}{2} \left( \mu_\pm(m) + \mu_\pm(m+1) \right)
\]

for \( m \in \mathbb{N}_0 \). These also follow from the general renormalisation relations in [4, Eq. (16)] by observing the letter exchange symmetry of the Thue–Morse system under \( a \leftrightarrow b \). This connection has the following immediate consequence.
Corollary 3.4. If $\nu_{\alpha \beta}(n)$ with $\alpha, \beta \in \{a, b\}$ and $n \in \mathbb{Z}$ denotes the relative frequency of occurrence of the distance $n$ in the Thue–Morse sequence $w$ between a letter of type $\alpha$ to the left and a letter of type $\beta$ to the right, with the obvious inversion for negative $n$, one has

$$
\nu_{aa}(n) = \nu_{bb}(n) = \mu_+(|n|) = \frac{1 + \eta(|n|)}{2} \quad \text{and} \quad \nu_{ab}(n) = \nu_{ba}(n) = \mu_-(|n|) = \frac{1 - \eta(|n|)}{2}
$$

for all $n \in \mathbb{Z}$. \qed

It remains to understand the mean value of $|\eta(m)|$. Via the Cauchy–Schwarz inequality in conjunction with $|\eta(m)| \leq 1$, it is elementary to derive the relation

$$
\left( \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)| \right)^2 \leq \frac{1}{N} \sum_{m=0}^{N-1} \eta(m)^2 \leq \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)|,
$$

so that (3.4) implies $\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)| = 0$, as expected. In fact, not only is the mean value of $|\eta(m)|$ equal to zero, we can say more as follows.

**Theorem 3.5.** For any $\alpha > \frac{\log(3)}{\log(4)} \approx 0.792481505$, we have

$$
\lim_{N \to \infty} \frac{1}{N^\alpha} \sum_{m=0}^{N-1} |\eta(m)| = 0.
$$

**Proof.** To start, note that, when $N \in [2^{2\ell}, 2^{2\ell+2}]$, one has the estimate

$$
\frac{1}{N^\alpha} \sum_{m=0}^{N-1} |\eta(m)| \leq \frac{1}{(2^{2\ell})^{\alpha}} \sum_{m=0}^{2^{2\ell+2}-1} |\eta(m)| \leq \frac{4^\alpha}{(2^{2\ell+2})^{\alpha}} \sum_{m=0}^{2^{2\ell+2}-1} |\eta(m)|,
$$

which implies that it suffices to show $2^{-(2\ell+2)\alpha} \sum_{m=0}^{2^{2\ell+2}-1} |\eta(m)| = o(1)$ as $\ell \to \infty$.

To this end, note that the recurrences (3.1) imply the estimates

$$
|\eta(4m)| = |\eta(m)|,
$$

$$
|\eta(4m + 1)| = \frac{1}{2} |\eta(2m) + \eta(2m + 1)| = \frac{1}{2} |\eta(m) - \frac{1}{2} (\eta(m) + \eta(m + 1))| \\
\leq \frac{1}{4} |\eta(m)| + \frac{1}{4} |\eta(m + 1)|,
$$

(3.7)

$$
|\eta(4m + 2)| = |\eta(2m + 1)| \leq \frac{1}{2} |\eta(m)| + \frac{1}{2} |\eta(m + 1)|,
$$

$$
|\eta(4m + 3)| = \frac{1}{2} |\eta(2m + 1) + \eta(2m + 2)| = \frac{1}{2} |\eta(m + 1) - \frac{1}{2} (\eta(m) + \eta(m + 1))| \\
\leq \frac{1}{4} |\eta(m)| + \frac{1}{4} |\eta(m + 1)|.
$$

As in the statement of the theorem, we let $\alpha > \frac{\log(3)}{\log(4)}$ and set $\Sigma'(N) = \frac{1}{N^\alpha} \sum_{m=0}^{N-1} |\eta(m)|$.

Arguing as above in (3.7), we have

$$
\Sigma'(4N) = \frac{1}{(4N)^\alpha} \sum_{r=0}^{3} \sum_{m=0}^{N-1} |\eta(4m + r)|
$$
\[ \leq \frac{1}{4^\alpha} \left( \frac{1}{N^\alpha} \sum_{m=0}^{N-1} |\eta(m)| + \frac{1}{4N^\alpha} \sum_{m=0}^{N-1} (|\eta(m)| + |\eta(m+1)|) \right. \\
\left. + \frac{1}{2N^\alpha} \sum_{m=0}^{N-1} (|\eta(m)| + |\eta(m+1)|) + \frac{1}{4N^\alpha} \sum_{m=0}^{N-1} (|\eta(m)| + |\eta(m+1)|) \right) \\
= \frac{3}{4^\alpha} \Sigma'_{N} + \frac{|\eta(N)| - |\eta(0)|}{4N^\alpha} \leq \frac{3}{4^\alpha} \Sigma'(N), \]

since \(|\eta(N)| - |\eta(0)| = |\eta(N)| - 1 \leq 0\). Now, letting \(N = 2^{2\ell+2}\), for \(\ell \geq 0\), we get
\[
\frac{1}{2(2\ell+2)^\alpha} \sum_{m=0}^{2^{2\ell+2}-1} |\eta(m)| = \Sigma'(2^{2\ell+2}) \leq \frac{3}{4^\alpha} \Sigma'(2^{2\ell}) \leq \left( \frac{3}{4} \right)^{\ell+1} = o(1). \]

**Remark 3.6.** The lower bound of \(\log(3)/\log(4)\) from Theorem 3.5 is reminiscent of the result that there is a \(C > 0\) such that
\[
sup_{\theta \in [0,1]} \left| \sum_{k \leq x} e^{2\pi ik\theta t_k} \right| \leq C x^{\log(3)/\log(4)},
\]

wherein the exponent \(\log(3)/\log(4)\) is optimal; see [16, 22, 23]. At first glance, one may conjecture that this exponent is also optimal in Theorem 3.5, but this is not the case. In fact, if, instead of partitioning the positive integers into arithmetic progressions with common difference 4 in (3.7), we use common difference 8, we have that Theorem 3.5 holds for any \(\alpha > \log(5)/\log(8) \approx 0.7739760313\). Using larger common differences, the bound for \(\alpha\) lowers. Computationally, we have gone up to common difference 2\(^{20}\), which shows that one may take any \(\alpha > 0.6526326476\) in Theorem 3.5. We leave the determination of the optimal value of \(\alpha\) for further investigation. \(\diamondsuit\)

Next, using Theorem 3.5 with the value \(\alpha = 1\) gives us the following result.

**Theorem 3.7.** For every real \(\beta > 0\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)|^\beta = 0.
\]

**Proof.** Hölder’s inequality gives
\[
(3.8) \quad \left( \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)|^\beta \right)^{1/\beta} \leq \left( \sum_{m=0}^{N-1} |\eta(m)|^{1+\beta} \right)^{\beta/\beta} \sum_{m=0}^{N-1} \frac{1}{N^{1+\beta}} = \left( \frac{1}{N} \sum_{m=0}^{N-1} |\eta(m)|^{1+\beta} \right)^{\beta}.
\]

Since \(|\eta(m)| \leq 1\) for all \(m\), we have \(|\eta(m)|^{1+\beta} \leq |\eta(m)|\) for all \(\beta \geq 0\). So, for \(\beta > 0\), the right-hand side (and then also the left-hand one) of (3.8) limits to zero as \(N\) grows. \(\square\)

We are now set to consider higher-order correlation functions.
4. General correlations for balanced weights

Analogously, we define the \( n \)-point correlations of the Thue–Morse system by

\[
\eta(m_1, m_2, \ldots, m_{n-1}) = \int_X x_0 \cdot (S^{m_1} x) \cdot (S^{m_2} x) \cdots (S^{m_{n-1}} x) \, d\mu(x).
\]

Similar to the above, these \( n \)-point correlations can be computed recursively.

**Proposition 4.1.** For each \( i \in \{1, \ldots, n-1\} \), let \( r_i \in \{0, 1\} \) and set \( r = r_1 + \cdots + r_{n-1} \). Then, for any integers \( m_1, \ldots, m_{n-1} \geq 0 \), we have

\[
\eta(2m_1 + r_1, \ldots, 2m_{n-1} + r_{n-1}) = \frac{(-1)^r}{2} (\eta(m_1, \ldots, m_{n-1}) + (-1)^n \eta(m_1 + r_1, \ldots, m_{n-1} + r_{n-1})).
\]

**Proof.** Observe that, by Birkhoff’s ergodic theorem in conjunction with the absolute convergence and hence rearrangement invariance of all involved sums, we have

\[
\eta(2m_1 + r_1, \ldots, 2m_{n-1} + r_{n-1}) = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=0}^{2N-1} t_k t_{k+2m_1+r_1} \cdots t_{k+2m_{n-1}+r_{n-1}}
\]

\[
= \lim_{N \to \infty} \frac{1}{2N} \sum_{j=0}^{N-1} (t_{2j} t_{2j+2m_1+r_1} \cdots t_{2j+2m_{n-1}+r_{n-1}} + t_{2j+1} t_{2j+2m_1+r_1+1} \cdots t_{2j+2m_{n-1}+r_{n-1}+1})
\]

\[
= \lim_{N \to \infty} \frac{(-1)^r}{2N} \sum_{j=0}^{N-1} (t_j t_{j+m_1} \cdots t_{j+m_{n-1}} + (-1)^n t_j t_{j+m_1+r_1} \cdots t_{j+m_{n-1}+r_{n-1}})
\]

\[
= \frac{(-1)^r}{2} (\eta(m_1, \ldots, m_{n-1}) + (-1)^n \eta(m_1 + r_1, \ldots, m_{n-1} + r_{n-1})),
\]

which completes the argument. \( \square \)

**Remark 4.2.** The special case of Proposition 4.1 concerning 4-point correlations was proved by Ong. See [24] for this identity, and for some beautiful pictures using the values of the 4-point correlations of the Thue–Morse sequence. \( \diamond \)

**Remark 4.3.** It is well known that the *period doubling* (pd) substitution \( a \mapsto ab, b \mapsto aa \) defines a dynamical system that is a factor of the Thue–Morse system, but displays pure point spectrum and hence a higher degree of order; see [8, Thm. 4.7] for the details of the 2 : 1 covering relationship. Now, setting \( a = -b = -1 \), the pair correlations of the period doubling system with these weights appear here as a subset, via

\[
\eta_{pd}(m) = \eta(1, m, m+1)
\]

for \( m \in \mathbb{Z} \). This demonstrates that correlation functions with singular continuous averaging behaviour can still display perfect, almost-periodic order on thin subsets. This was also discussed in [26]. The existence of these highly structured thin subsets suggest that comparing partial sums of higher-order Thue–Morse correlations with \( N \) will not be enough, and that a higher power of \( N \) is necessary, a phenomenon we describe in what follows. \( \diamond \)
Proposition 4.1 is the multi-dimensional analogue of the recursions in (3.1). As above, in the case \( n = 2 \), this generalisation can be used to prove a zero-mean-value result analogous to (3.5), here over the nested \((n-1)\)-dimensional integer cubes in the first orthant. That is,

\[
\lim_{N \to \infty} \frac{1}{N^{n-1}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1, \ldots, m_{n-1}) = 0.
\]

To see this, we proceed as above by setting \( \mathcal{S}(N) = \frac{1}{N^{n-1}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1, \ldots, m_{n-1}) \).

Again, note that \( \mathcal{S}(2N+1) = \frac{(2N)^n}{(2N+1)^n} \mathcal{S}(2N) + O(N^{-1}) \) as \( N \to \infty \), where the error term is the result of the number of points on the surface of the cube growing like \( N^{n-1} \). It thus suffices to show that \( \mathcal{S}(2N) \to 0 \) as \( N \to \infty \). This, using Proposition 4.1, follows from

\[
\mathcal{S}(2N) = \frac{1}{(2N)^{n-1}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq 2N-1} \eta(m_1, \ldots, m_{n-1})
\]

\[
= \frac{1}{(2N)^{n-1}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \sum_{r_1, \ldots, r_{n-1} \in \{0,1\}} \eta(2m_1 + r_1, \ldots, 2m_{n-1} + r_{n-1})
\]

\[
= \frac{1}{2(2N)^{n-1}} \sum_{r_1, \ldots, r_{n-1} \in \{0,1\}} (-1)^{r_1 + \ldots + r_{n-1}}
\]

\[
\times \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1, \ldots, m_{n-1}) + (-1)^n \eta(m_1 + r_1, \ldots, m_{n-1} + r_{n-1})
\]

\[
= \frac{(-1)^n}{2(2N)^{n-1}} \sum_{r_1, \ldots, r_{n-1} \in \{0,1\}} (-1)^{r_1 + \ldots + r_{n-1}}
\]

\[
\times \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1 + r_1, \ldots, m_{n-1} + r_{n-1}),
\]

where, for the fourth equality, we have used that \( \sum_{r_1, \ldots, r_{n-1} \in \{0,1\}} (-1)^{r_1 + \ldots + r_{n-1}} = 0 \). Now, since \( |\eta(m_1, \ldots, m_{n-1})| \leq 1 \), we have

\[
\sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1 + r_1, \ldots, m_{n-1} + r_{n-1})
\]

\[
= \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1, \ldots, m_{n-1}) + O(N^{n-2}),
\]

so that \( \mathcal{S}(2N) \), as \( N \to \infty \), is equal to

\[
\frac{(-1)^n}{2(2N)^{n-1}} \sum_{r_1, \ldots, r_{n-1} \in \{0,1\}} (-1)^{r_1 + \ldots + r_{n-1}} \left( \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} \eta(m_1, \ldots, m_{n-1}) + O(N^{n-2}) \right),
\]

which overall is \( O(N^{-1}) \).

To compute all of the \( n \)-point correlations for a given \( n \), we only need to know the values of \( \eta \) at the \( 2^{n-1} \) corners of the \((n-1)\)-dimensional unit hypercube, because all other correlations
are recursively determined from these finitely many values. They can be calculated in two different ways as follows.

First, for a point \((r_1, \ldots, r_{n-1}) \in \{0, 1\}^n\), with \(r = r_1 + \ldots + r_{n-1}\) as above, we get

\[
\eta(r_1, r_2, \ldots, r_{n-1}) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} t_k t_{k+r_1} \cdots t_{k+r_{n-1}}
\]

(4.2)

\[
= \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
1, & \text{if } n \text{ is even and } r \text{ is even}, \\
-\frac{1}{3}, & \text{if } n \text{ is even and } r \text{ is odd},
\end{cases}
\]

which follows simply from the fact that \((t_m)^2 = 1\) for any \(m\) in conjunction with the fact that 1 and \(-1\) occur equally frequently and with bounded gaps in \((t_k)_{k \in \mathbb{N}_0}\). In particular, one has

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=n}^{n+N-1} t_k = 0,
\]

uniformly in \(n \in \mathbb{N}_0\). Combining this with Proposition 4.1 gives the following immediate consequence.

**Corollary 4.4.** All odd-order correlations of the balanced Thue–Morse system vanish. □

The second approach generalises Remark 3.1 in realising that the recursions in Proposition 4.1 can once again be seen as an infinite set of linear equations for the coefficients \(\eta(m_1, \ldots, m_{n-1})\). It is clear that the \(2^{n-1}\) equations with all \(m_i \in \{0, 1\}\) form a closed subset, as explained in Remark 3.1, and that all other coefficients are recursively determined from the solution of these equations, which are the \(\eta\)-coefficients at the \(2^{n-1}\) corners of the unit hypercube.

**Corollary 4.5.** The solution space of the linear recursion equations from Proposition 4.1 is one-dimensional. Consequently, for any \(n \in \mathbb{N}\), the \(n\)-point correlations of the balanced Thue–Morse system are uniquely specified by a single number, namely by \(\eta(0, \ldots, 0)\).

**Proof.** The recursive structure is clear from Proposition 4.1. In particular, all coefficients are fully determined once the \(\eta(r_1, \ldots, r_{n-1})\) for all \(r_i \in \{0, 1\}\) are known. Choosing all \(m_i = 0\) in the recursion relations of Proposition 4.1, one gets

\[
\eta(r_1, \ldots, r_{n-1}) = \frac{(-1)^r}{2} (\eta(0, \ldots, 0) + (-1)^n \eta(r_1, \ldots, r_{n-1}))
\]

with \(r = r_1 + \ldots + r_{n-1}\) as before. This simply gives

\[
\eta(r_1, \ldots, r_{n-1}) = \frac{(-1)^r}{2 + (-1)^n r} \eta(0, \ldots, 0)
\]

(4.3)

for all \((r_1, \ldots, r_{n-1}) \in \{0, 1\}^{n-1}\), which establishes our claim. □
Remark 4.6. It is clear from (4.2) that \( \eta(0, \ldots, 0) \) is either 0 (for \( n \) odd) or 1 (for \( n \) even). The balanced odd-order correlations vanish accordingly, while the even-order correlations are thus fully determined by the value \( \eta(0) = 1 \) from the 2-point correlations. \( \diamond \)

The result from Eq. (4.1) can be extended to get an analogue of Theorems 3.5 and 3.7 as follows. First, observe that (4.3) implies the relations

\[
\eta(4m_1, \ldots, 4m_{n-1}) = \eta(m_1, \ldots, m_{n-1})
\]

\[
\eta(4m_1 + 1, \ldots, 4m_{n-1} + 1) = \frac{1}{4}(\eta(m_1 + 1, \ldots, m_{n-1} + 1) - (-1)^n \eta(m_1, \ldots, m_{n-1}))
\]

\[
\eta(4m_1 + 2, \ldots, 4m_{n-1} + 2) = -\frac{1}{2}(\eta(m_1 + 1, \ldots, m_{n-1} + 1) + (-1)^n \eta(m_1, \ldots, m_{n-1}))
\]

\[
\eta(4m_1 + 3, \ldots, 4m_{n-1} + 3) = \frac{1}{4}(\eta(m_1, \ldots, m_{n-1}) - (-1)^n \eta(m_1 + 1, \ldots, m_{n-1} + 1)).
\]

Now, one arrives at the following result.

**Theorem 4.7.** Let \( n \geq 2 \) be fixed. Then, for any \( \alpha > \frac{\log(3)}{\log(4)} \) and \( \beta > 0 \), one has

\[
\lim_{N \to \infty} \frac{1}{N^{\alpha(n-1)}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} |\eta(m_1, \ldots, m_{n-1})| = 0 \quad \text{and}
\]

\[
\lim_{N \to \infty} \frac{1}{N^{\alpha(n-1)}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} |\eta(m_1, \ldots, m_{n-1})|^\beta = 0.
\]

**Proof.** The above recursions, via the triangle inequality, give the analogue of (3.7) for general \( n \). Now, setting

\[
\Sigma(N) = \frac{1}{N^{\alpha(n-1)}} \sum_{0 \leq m_1, \ldots, m_{n-1} \leq N-1} |\eta(m_1, \ldots, m_{n-1})|^\beta,
\]

we can repeat our previous estimates, with minor, but obvious variations. Indeed, first setting \( \beta = 1 \), we can repeat the proof of Theorem 3.5, which gives the first claim. Then, setting \( \alpha = 1 \), the second claim follows in complete analogy to the proof of Theorem 3.7. \( \square \)

**Example 4.8.** As an example, beyond the standard 2-point correlations, we consider the 4-point correlations. In this case, let us define

\[
\eta(m_1, m_2, m_3) := \begin{pmatrix}
\eta(m_1, m_2, m_3) \\
\eta(m_1, m_2, m_3 + 1) \\
\eta(m_1, m_2 + 1, m_3) \\
\eta(m_1, m_2 + 1, m_3 + 1) \\
\eta(m_1 + 1, m_2, m_3) \\
\eta(m_1 + 1, m_2, m_3 + 1) \\
\eta(m_1 + 1, m_2 + 1, m_3) \\
\eta(m_1 + 1, m_2 + 1, m_3 + 1)
\end{pmatrix}.
\]

Then, Proposition 4.1 implies that

\[
(4.4) \quad \eta(2m_1 + r_1, 2m_2 + r_2, 2m_3 + r_3) = B_{(r_1, r_2, r_3)} \eta(m_1, m_2, m_3),
\]
where

\[
B_{(0,0,0)} := \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
B_{(0,0,1)} := \frac{1}{2} \begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

\[
B_{(0,1,0)} := \frac{1}{2} \begin{pmatrix}
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

\[
B_{(0,1,1)} := \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

Further, if \(J_8\) is the 8×8 anti-diagonal matrix with all ones on the anti-diagonal, one has

\[
B_{(1,0,0)} := J_8 B_{(0,1,1)} J_8, \quad B_{(1,0,1)} := J_8 B_{(0,1,0)} J_8
\]

\[
B_{(1,1,0)} := J_8 B_{(0,0,1)} J_8, \quad B_{(1,1,1)} := J_8 B_{(0,0,0)} J_8.
\]

The implied relations from (4.4) are really a matrix version of the general recursions from (4.3). Further, the sum matrix is

\[
B := \sum_{r_1,r_2,r_3 \in \{0,1\}} B_{(r_1,r_2,r_3)} = \frac{1}{2} \begin{pmatrix}
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\
\end{pmatrix},
\]

which is an idempotent, that is, \(B^2 = B\). There is a lot of Thue–Morse structure in the explicit form of the matrices, as the interested readers will have noticed. Here, we state a nice connection with the matrices \(E_0, E_1\) and \(J\) defined in (3.3). It is quite clear that \(J_8 = J^{\otimes 3}\). Somewhat less clear is the relationship between \(B_{(i,j,k)}\) and Kronecker products of the matrices from (3.3). One can check that

\[
B_{(i,j,k)} = \frac{1}{2} \left( E_i \otimes E_j \otimes E_k + J^{\otimes 3} (E_{1-i} \otimes E_{1-j} \otimes E_{1-k}) J^{\otimes 3} \right) = \frac{1}{2} \left( E_i \otimes E_j \otimes E_k + E'_{1-i} \otimes E'_{1-j} \otimes E'_{1-k} \right).
\]

We leave further details, including the validity of the generalisation

\[
B_{(i_1,\ldots,i_{n-1})} = \frac{1}{2} \left( E_{i_1} \otimes \cdots \otimes E_{i_{n-1}} + (-1)^n E'_{1-i_1} \otimes \cdots \otimes E'_{1-i_{n-1}} \right),
\]

to the curious reader. \(\diamondsuit\)
The relationship in Eq. (4.4) suggests that one can study the correlations \( \eta(m_1, \ldots, m_{n-1}) \) via a related regular sequence [1]. To do this, we start with the generalisation of the relationship in (4.4),

\[
(4.5) \quad \eta(2m_1 + r_1, \ldots, 2m_{n-1} + r_{n-1}) = B_{(r_1, \ldots, r_{n-1})} \eta(m_1, \ldots, m_{n-1}).
\]

For each \( i \in \{0, 1, \ldots, 2^{n-1} - 1 \} \) with binary expansion \((i)_2 = r_1 r_2 \cdots r_{n-1}\), we set

\[
B_i = B_{(r_1, r_2, \ldots, r_{n-1})}.
\]

Now, we define the sequence \( \eta_n := (\eta_n(m))_{m \geq 0} \) by

\[
\eta_n(m) = e_1^t B_{i_0} B_{i_1} \cdots B_{i_m} e_1,
\]

where \((m)_n = i_s \cdots i_1 i_0\) is the base-\( n \) expansion of \( m \), and \( e_1 \) is the standard column basis vector of length \( 2^n \) with the 1 in the first position.

Using the related sequence \( \eta_n \), we record the following result as a refinement of (4.1). Here, instead of taking the mean value over the \( n \)-dimensional integer cubes of the first orthant as in (4.1), we traverse the integer points in that orthant according to the order imposed by the relationship (4.5).

**Theorem 4.9.** Let \( n \geq 2 \). Then, \( \eta_n \) has mean value zero, that is,

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{j=0}^{m-1} \eta_n(j) = 0.
\]

**Proof.** Note that, since all odd correlations vanish, without loss of generality, we assume that \( n \) is even. Set \( \Sigma(m) := \frac{1}{m} \sum_{j=0}^{m-1} \eta_n(j) \). Proposition 4.1 gives that \( \eta_n(j) = O(1) \), so that

\[
\Sigma(2^{n-1}m) \sim \Sigma(2^{n-1}m + a) \quad \text{as } m \to \infty
\]

for any \( a \in \{0, 1, \ldots, 2^{n-1} - 1\} \). To prove the result, it thus suffices to show that we have \( \Sigma(2^{n-1}m) \to 0 \) as \( m \to \infty \). We use that, given by Proposition 4.1, for even \( n \) and any \( a \in \{0, 1, \ldots, 2^{n-1} - 1\} \),

\[
\eta_n(2^{n-1}m + a) = \frac{t_a}{2}(\eta_n(2^{n-1}m) + \eta_n(2^{n-1}m + a)),
\]

where \( t_a \) is the value of the Thue–Morse sequence at \( a \) from (2.1). These recurrences are the direct generalisations of those in (3.1). With these in hand, we simply compute

\[
\Sigma(2^{n-1}m) = \frac{1}{2^{n-1}} \sum_{a=0}^{2^{n-1}-1} \frac{1}{m} \sum_{j=0}^{m-1} \frac{t_a}{2}(\eta_n(j) + \eta_n(j + a))
\]

\[
= \frac{1}{2^{n-1}} \sum_{a=0}^{2^{n-1}-1} t_a \left( \frac{1}{2m} \sum_{j=0}^{m-1} (\eta_n(j) + \eta_n(j + a)) \right)
\]

\[
= O(m^{-1}) + \frac{\Sigma(m)}{2^{n-1}} \sum_{a=0}^{2^{n-1}-1} t_a,
\]
where we have again used that \( \eta_n(j) \) is bounded to give the last equality. Since \( \sum_{a=0}^{2^{n-1}-1} t_a = 0 \), we have that \( \Sigma(2^{n-1}m) = O(m^{-1}) \), which proves the result. \( \square \)

5. Correlations for general weights

We now consider the Thue–Morse system for general real weights, \( f(-1) \) and \( f(1) \). Two quantities will be of paramount importance here. First, we have

\[
\mathbb{E}(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} f(w_m) = \int_{\mathcal{X}} f(x_0) \, d\mu(x) = \frac{f(1) + f(-1)}{2},
\]

since the frequencies of 1 and \(-1\) in \( w \) are both equal to \( 1/2 \). Second, set

\[
h_f := \frac{f(1) - f(-1)}{2},
\]

so that \( f(\pm 1) = \mathbb{E}(f) \pm h_f \).

**Proposition 5.1.** For any \( f : \{-1, 1\} \to \mathbb{R} \), we have \( \eta_f(m) = h_f^2 \eta(m) + \mathbb{E}(f)^2 \) and

\[
\eta_f(m_1, m_2) = h_f^2 \mathbb{E}(f)(\eta(m_1) + \eta(m_2) + \eta(m_1 - m_2))) + \mathbb{E}(f)^3.
\]

**Proof.** We use the identity

\[
f(w_n) f(w_{n+m}) = (f(w_n) - \mathbb{E}(f))(f(w_{n+m}) - \mathbb{E}(f)) + \mathbb{E}(f)(f(w_n) + f(w_{n+m})) - \mathbb{E}(f)^2
\]

to give (with \( S^m f := f \circ S^m \))

\[
\eta_f(m) = \int_{\mathcal{X}} (f - \mathbb{E}(f))(S^m f - \mathbb{E}(f)) \, d\mu + \mathbb{E}(f) \int_{\mathcal{X}} (f + S^m f) \, d\mu - \mathbb{E}(f)^2
\]

\[
= h_f^2 \eta(m) + \mathbb{E}(f)(\mathbb{E}(f) + \mathbb{E}(f)) - \mathbb{E}(f)^2 = h_f^2 \eta(m) + \mathbb{E}(f)^2,
\]

which is the first desired result. The second result follows similarly, recalling that \( \eta(m_1, m_2) \) vanishes by Corollary 4.4. \( \square \)

The point of Proposition 5.1 is to show that, in order to calculate \( \eta_f \), one only needs to know the values of \( \eta \) and so, by Corollary 4.5 and Remark 4.6, one only needs to know the value \( \eta(0) \). Indeed, in general, we have

\[
f(w_n) \prod_{j=1}^{n-1} f(w_{k+m_j}) = (f(w_k) - \mathbb{E}(f)) \prod_{j=1}^{n-1} (f(w_{k+m_j}) - \mathbb{E}(f))
\]

\[
- \sum_{j=0}^{n-1} (-\mathbb{E}(f))^{n-j} p_j(f(w_k), f(w_{k+m}), \ldots, f(w_{k+m_{n-1}})),
\]

where \( p_j(z_1, \ldots, z_m) \) is the elementary symmetric polynomial of degree \( j \) in \( m \) variables. Thus

\[
\eta_f(m_1, \ldots, m_{n-1}) = h_f^2 \eta(m_1, \ldots, m_{n-1}) - \sum_{j=0}^{n-1} (-\mathbb{E}(f))^{n-j} \int_{\mathcal{X}} p_j(f, S^m f, \ldots, S^{m_{n-1}} f) \, d\mu.
\]
Noting that, for any $n \geq 2$, we have
\[ \eta_f(m_1, \ldots, m_{n-1}) = \int_X f \cdot (S^{m_1}f) \cdots (S^{m_{n-1}}f) \, d\mu \]
gives the following result.

**Theorem 5.2.** For any $n \geq 2$, the $n$-point correlations of the $f$-weighted Thue–Morse system can be calculated from the balanced correlations. Consequently, they are ultimately derived from the single value $\eta(0, \ldots, 0)$, which is 0 for $n$ odd and 1 for $n$ even. $\square$

This result can be made explicit as follows. First, observe that we have
\[ (5.1) \int_X x_0 x_{\ell_1} \cdots x_{\ell_{n-1}} \, d\mu(x) = \eta(\ell_1 - \ell_0, \ell_2 - \ell_0, \ldots, \ell_{n-1} - \ell_0) = 0 \]
for all even $n$, as a result of Corollary 4.4, because this refers to an odd-order correlation. Now, with $f(x_i) = \mathbb{E}(f) + x_i h_f$ and $m_0 := 0$, observe
\[ \eta_f(m_1, \ldots, m_{n-1}) = \int_X f(x_0) f(x_{m_1}) \cdots f(x_{m_{n-1}}) \, d\mu(x) = \int_X \prod_{i=0}^{n-1} (\mathbb{E}(f) + x_i h_f) \, d\mu(x) = \mathbb{E}(f)^n + \sum_{i=1}^{n} \mathbb{E}(f)^{n-i} h_f^i \int_X p_i(x_{m_0}, x_{m_1}, \ldots, x_{m_{n-1}}) \, d\mu(x), \]
with the elementary symmetric polynomials as above. Here, the integral over $p_i$ vanishes whenever $i$ is odd, as a result of (5.1). A simple calculation now gives that the general correlation coefficient $\eta_f(m_1, \ldots, m_{n-1})$ is equal to
\[ \mathbb{E}(f)^n + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} h_f^{2r} \mathbb{E}(f)^{n-2r} \sum_{0 \leq i_0 < i_1 < \cdots < i_{2r-1} \leq n-1} \eta(m_{i_1} - m_{i_0}, m_{i_2} - m_{i_0}, \ldots, m_{i_{2r-1}} - m_{i_0}), \]
which expresses the general coefficients in terms of the balanced ones.

In view of our above analysis, two comments are in order. On the one hand, the various generalisations to Thue–Morse-like sequences [5, 8] can and should be analysed, expecting analogous results. On the other hand, it will be interesting to also look at higher-order correlations in systems with absolutely continuous spectrum, such as the Rudin–Shapiro (or Golay–Rudin–Shapiro) sequence and its various generalisations [13, 14, 15], and to identify any crucial difference from the singular continuous cases. See the work of Mazáč [21] for a first study in this direction.

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